Regularizing transformations of polygons

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Abstract. We start with a generic \( n \)-gon \( Q_0 \) with vertices \( q_{j,0} (j = 0, \ldots, n - 1) \) in the \( d \)-dimensional Euclidean space \( \mathbb{E}^d \). Additionally, \( m + 1 \) real numbers \( u_0, \ldots, u_m \in \mathbb{R} (m < n) \) with \( \sum_{\mu=0}^{m} u_{\mu} = 1 \) are given. From these initial data we iteratively define generations of \( n \)-gons \( Q_k \) in \( \mathbb{E}^d \) for \( k \in \mathbb{N} \) with vertices \( q_{j,k} := \sum_{\mu=0}^{m} u_{\mu} q_{j+\mu,k-1} \). We can show that this affine iteration generally regularizes in an affine sense.

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1. Introduction

Schoenberg [6], Ziv [7], Nicoller [2] and Donisi et al. [1] studied geometric iteration processes starting with a generic \( n \)-gon \( Q_0 \) in \( \mathbb{E}^2 \). They use homotheties to construct vertices of a next generation polygon \( Q_1 \). Reiterating this process creates a series of generations \( Q_k \). This iteration, in general, has a regularizing effect on the polygon. Surprisingly, the result for \( n \)-gons in the plane \( \mathbb{E}^2 \) presented by Roeschel in [5] is also valid for \( n \)-gons in higher dimensions. In [5] the proof for \( \mathbb{E}^2 \) is based on the fact that the space of planar \( n \)-gons is spanned by the planar prototype \( n \)-gons of \( \mathbb{E}^2 \). As this does not hold for higher dimensions the proof for \( \mathbb{E}^d \) with \( d \geq 3 \) demands another approach with different arguments. We prove an affine regularization theorem: these iterations in higher dimensions also deliver generations \( Q_k \) approaching the affine shape of regular planar polygons.

2. The spatial affine iteration

We use vectors in \( \mathbb{R}^d \) to describe points of the \( d \)-dimensional Euclidean space \( \mathbb{E}^d (d > 2) \) with respect to a Cartesian coordinate frame \( \{O; x_1, \ldots, x_d\} \). We start with some spatial \( n \)-gon \( Q_0 \subset \mathbb{E}^d \) with vertices \( \{q_{0,0}, q_{1,0}, \ldots, q_{n-1,0}\} \).
Figure 1 An example for $n = 8$ and $m = d = 3$: the polygon $Q_0$ with vertices of a cube and the first generation polygon $Q_1$ for $(u_0, u_1, u_2, u_3) = (0.2, -0.35, 0.75, 0.4)$. (n > 2, $q_{j,0} \in \mathbb{R}^d$). Our starting polygon $Q_0$ shall be called polygon of generation 0.

On the other hand in an $m$-dimensional affine space $\mathbb{R}^m$ ($0 < m < n$) with a simplex $S := \{a_0, \ldots, a_m\}$ we choose a reference point $z^*$ with respect to $S$: Let $z^* := \sum_{\mu=0}^{m} u_\mu a_\mu$ be given by its barycentric coordinates $(u_0, \ldots, u_m) \in \mathbb{R}^{1 \times (m+1)}$ with $\sum_{\mu=0}^{m} u_\mu = 1$.

Let $\alpha_{j,1}$ be the affine mappings from the ordered reference simplex vertex set $S$ to ordered sets of $m$ consecutive vertices $q_{j,0}, \ldots, q_{j+m,0}$ of $Q_0$ ($j \in \mathbb{J} := \{0, \ldots, n-1\}$; first index mod $n$). Each of these $n$ affine mappings is applied to the reference point $z^*$; this way we get $n$ image points $q_{j,1} := \alpha_{j,1}(z^*) = \sum_{\mu=1}^{m} u_\mu q_{j+\mu-1,0}$ which form a new $n$-gon $Q_1$ called the generation 1 polygon.

The same process can now be applied, in turn, to the polygon $Q_1$ with the same reference simplex $S$ and the same reference point $z^*$, creating a subsequent polygon $Q_2$. Iteration yields a series of polygons. $Q_k := \{q_{0,k}, \ldots, q_{n-1,k}\}$ is the $k$th generation polygon with vertices

$$q_{j,k} = \sum_{\mu=0}^{m} u_\mu q_{j+\mu,k-1} \in \mathbb{R}^d \quad (j \in \mathbb{J}, \, k \in \mathbb{N}\{0\}). \quad (2.1)$$

The procedure is a $d$-dimensional generalisation of the geometric iteration presented in [5]. Figure 1 shows the first iteration step for an example with $n = 8$ and $m = d = 3$.

3. The iteration process

We describe the polygons $Q_k$ by $d \times n$-matrices $Q_k := (q_{0,k}, \ldots, q_{n-1,k})$ in $\mathbb{R}^{d \times n}$ with $q_{j,k}$ (2.1). Formula (2.1) can be rewritten as a product of matrices $Q_k := Q_{k-1} \cdot M$ with the circulant $n \times n$-matrix $M \in \mathbb{R}^{n \times n}$.
The \( n \)th complex roots of unity \( \zeta_j := \exp(i\frac{2j\pi}{n}) = \cos \frac{2j\pi}{n} + i \sin \frac{2j\pi}{n} \) \((j \in \mathbb{Z})\). We define the vectors
\[
P_j := (\zeta_j^0, \ldots, \zeta_j^{n-1}) \in \mathbb{C}^{1 \times n} \quad (j \in \mathbb{Z})
\]
and have \( P_j \cdot M = P_j \sum_{\mu=0}^{m} u_\mu \zeta_j^\mu \) and \( M \cdot P^t_{n-j} = (\sum_{\mu=0}^{m} u_\mu \zeta_j^\mu) \cdot P^t_{n-j} \). Thus, the vectors \( P_j \) and \( P^t_{n-j} \) \((j \in \mathbb{J})\) are left and right eigenvectors of \( M \). The corresponding eigenvalue is
\[
\lambda_j := \sum_{\mu=0}^{m} u_\mu \zeta_j^\mu \quad (j \in \mathbb{J}).
\]

As \((u_0, \ldots, u_m) \in \mathbb{R}^{1 \times (m+1)}\) and \( \zeta_j^\mu \) we have \( \bar{\lambda}_j = \lambda_{n-j} \) for all \( j \in \mathbb{J}\setminus\{0\} \).

We now regard two matrices out of \( \mathbb{C}^{n \times n} \)
\[
L := \frac{1}{\sqrt{n}} \begin{pmatrix} P_0 \\ \vdots \\ P_{n-1} \end{pmatrix} \quad \text{and} \quad R := \frac{1}{\sqrt{n}} \begin{pmatrix} P_0 \\ \vdots \\ P_1 \end{pmatrix}.
\]

\( L \) and \( R \) are symmetric and regular for \( n > 1 \) (see [3,5,6] and [7]). We have: \( L = \overline{R} \) and \( L \cdot R = I_{n,n} \) with the \( n \times n \)-unit matrix \( I_{n,n} \); the matrices \( L \) and \( R \) are unitary \( n \times n \)-matrices in \( \mathbb{C}^{n \times n} \). We have \( L \cdot M \cdot R = D(\lambda_0, \ldots, \lambda_{n-1}) \) with the diagonal matrix \( D(\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{C}^{n \times n} \) containing the eigenvalues \( \lambda_j \) of \( M \) as its elements in the main diagonal. This yields \( M = R \cdot D(\lambda_0, \ldots, \lambda_{n-1}) \cdot L \) and
\[
Q_k \cdot R = Q_{k-1} \cdot R \cdot D(\lambda_0, \ldots, \lambda_{n-1}) \quad \text{and} \quad Q_k \cdot R = Q_0 \cdot R \cdot D(\lambda_0, \ldots, \lambda_{n-1})^k \quad \text{for} \quad k \in \mathbb{N}\setminus\{0\}.
\]

We get \( Q_k \cdot R = \frac{1}{\sqrt{n}} \left( \sum_{\nu=0}^{n-1} q_{\nu,k} \zeta_n^\nu, \sum_{\nu=0}^{n-1} q_{\nu,k} \zeta_{n-1}^\nu, \ldots, \sum_{\nu=0}^{n-1} q_{\nu,k} \zeta_1^\nu \right) \).

Then (3.5) yields
\[
\sum_{\nu=0}^{n-1} q_{\nu,k} \zeta_{n-j}^\nu = \lambda_j^k \sum_{\nu=0}^{n-1} q_{\nu,0} \zeta_{n-j}^\nu \quad \forall j \in \mathbb{J}.
\]
Due to \( \lambda_0 = \sum_{\mu=0}^{m} u_\mu \zeta_0^\mu = 1 \) and \( \zeta_n^{\nu} = 1 \), the index \( j = 0 \) in (3.6) delivers \( \sum_{\nu=0}^{n-1} q_{\nu,k} = \sum_{\nu=0}^{n-1} q_{\nu,0} \) for all \( k \in \mathbb{N} \setminus \{0\} \): All polygons \( Q_k \) have the same center of gravity.

From now on let the initial polygon \( Q_0 \) have its center of gravity in the origin \( O := (0, \ldots, 0)^t \). So we can be sure that for all \( k \in \mathbb{N} \)

\[
\frac{1}{n} \sum_{\nu=0}^{n-1} q_{\nu,k} = o_d := (0, \ldots, 0)^t. \quad (3.7)
\]

As the matrix \( R \) is regular the initial polygon \( Q_0 \) can explicitly be retrieved from the \( d \times n \)-matrix

\[
Q_0 \cdot R =: B = (b_0, \ldots, b_{n-1}) \in \mathbb{C}^{d \times n} \quad \text{with} \quad b_j = \frac{1}{\sqrt{n}} \sum_{\nu=0}^{n-1} q_{\nu,0} \zeta_{n-j}^{\nu} \in \mathbb{C}^d. \quad (3.8)
\]

From \( q_{\nu,0} \in \mathbb{R}^d \) and \( \zeta_{n-j}^{\nu} = \overline{\zeta}_j^{\nu} \) we get \( b_j = b_{n-j} \) for all \( j \in \mathbb{J}^* := \{1, \ldots, n-1\} \). Because of (3.7) the first column vector is zero: \( b_0 = o_d \). Equation (3.5) yields

\[
Q_k \cdot R = B \cdot D(\lambda_0, \ldots, \lambda_{n-1})^k = (o_d, \lambda_1^k b_1, \ldots, \lambda_{n-1}^k b_{n-1}). \quad (3.9)
\]

Thus, we do not alter the recursion in any way if we replace the diagonal matrix \( D(\lambda_0, \ldots, \lambda_{n-1}) \) in (3.5) by the diagonal matrix \( D^* := D(0, \lambda_1, \ldots, \lambda_{n-1}) \).

With this in mind, the iteration process can be described by

\[
Q_k = B \cdot D^{*k} \cdot L = \frac{1}{\sqrt{n}} \sum_{\nu=1}^{n} \lambda_{\nu}^k b_\nu P_\nu \iff q_{j,k} = \frac{1}{\sqrt{n}} \sum_{\nu=1}^{n} \lambda_{\nu}^k b_\nu \zeta_j^{\nu} \quad (3.10)
\]

for \( j \in \mathbb{J} \). Note that \( b_\nu P_\nu \in \mathbb{C}^{d \times n} \) for \( \nu \in \mathbb{J}^* \).

4. Prototype polygons

The Gaussian plane of complex numbers \( \mathbb{C} \) can be interpreted as a Euclidean plane \( \mathbb{E}^2 \) with a Cartesian coordinate frame \( \{O, 1, i\} \). We embed \( \mathbb{E}^2 \) into \( \mathbb{E}^d \) by identifying 1 and \( i \) with the \( d \)-dimensional unit vectors \( e_1 := (1, 0, 0, \ldots, 0)^t \) and \( e_2 := (0, 1, 0, \ldots, 0)^t \), respectively. The elements of \( P_j \) (3.2) can be viewed as a collection of \( n \) points \( \zeta_j^{\nu} (\nu \in \mathbb{J}) \) equally distributed on the unit circle of \( \mathbb{E}^2 \subset \mathbb{E}^d \) centered in \( O \) with \( j \in \mathbb{J}^* := \{1, \ldots, n-1\} \). Its points can be written as

\[
T_j = e_1 \frac{P_j + \overline{P}_j}{2} + e_2 \frac{P_j - \overline{P}_j}{2i} = e_1 \frac{P_j + P_{n-j}}{2} + e_2 \frac{P_j - P_{n-j}}{2i}. \quad (4.1)
\]

\( T_j \) is represented by a matrix \( t_{\nu,j} \in \mathbb{R}^{d \times n} \) with columns

\[
t_{\nu,j} := \left( \cos \frac{2\pi \nu j}{n}, \sin \frac{2\pi \nu j}{n}, 0, \ldots, 0 \right)^t \in \mathbb{R}^d (\nu \in \mathbb{J}). \quad (4.2)
\]

\( T_j \) forms the so-called ‘regular prototype \( n \)-gon of \( j \)th kind’. The regular \( n \)-gon \( T_{n-j} \) is symmetric to \( T_j \) w.r.t. the axis \( e_1 \) and thus affinely equivalent to \( T_j \). If \( j \) and \( n \) are relatively prime the polygon \( T_j \) is either a regular \( n \)-gon or an
n-sided regular star. If $j$ is a divisor of $n$ with $n = jp$ the polygon $T_j$ is either a regular $p$-gon or an ordinary regular star with $p$ vertices, each of the vertices being multiply counted ($j$ times).

5. The concept of affine regularization

An affine mapping of $\mathbb{E}^d$ keeping the origin $O$ in its place is described by

$$\beta : \mathbb{E}^d \rightarrow \mathbb{E}^d, \; x \mapsto \beta(x) = Cx \quad \text{with} \quad C \in \mathbb{R}^{d \times d}. \quad (5.1)$$

The affine image of the polygon $Q_k = (q_{0,k}, \ldots, q_{n-1,k})$ is $\beta(Q_k) := C \cdot Q_k$. Our iteration (2.1) seems to regularize for certain $Q_j$ of $j$th kind irrespectively of the choice of the starting polygon $Q_0$. In order to examine this interesting peculiarity we compare the $n$-gons $Q_k$ with a regular prototype $n$-gon $T_j$ (4.1) of $j$th kind$^1$:

**Definition 5.1.** We call the iteration (2.1) affinely regularizing of kind $j$ with $1 \leq j \leq n/2$ if, for any generic initial polygon $Q_0$, there exist affine mappings $\beta_k : \mathbb{E}^d \rightarrow \mathbb{E}^d$ transforming $Q_k = (q_{0,k}, \ldots, q_{n-1,k})^t$ into polygons $\beta_k(Q_k)$ with the property that the series $\Delta_k$ of sums of the squared distances

$$\Delta_k := \sum_{\nu=0}^{n-1} \|\beta_k(q_{\nu,k}) - t_{\nu,j}\|^2 = \text{tr} \left( (T_j - \beta_k(Q_k))^t \cdot (T_j - \beta_k(Q_k)) \right) \quad (5.2)$$

of respective vertices of $\beta(Q_k)$ and of the regular prototype polygon $T_j$ of $j$th kind is a null series: $\lim_{k \rightarrow \infty} \Delta_k = 0$.

6. The affine regularization theorem

The shape of the polygons $Q_k$ depends on the input data set $Q_0$ and on the barycentric coordinates $(u_0, \ldots, u_m)$ of the reference point $z^*$ with $\sum_{\mu=0}^{m} u_\mu = 1$. The latter determine the matrix $M$ (3.1), the eigenvalues $\lambda_j$ and the diagonal matrix $D^* = D(0, \lambda_1, \ldots, \lambda_{n-1})$. The norms $n_j := |\lambda_j|$ of $\lambda_j$ for $j \in \mathbb{J}^*$ are given by

$$n_j^2 = \lambda_j \lambda_j = \sum_{\mu,\nu=0}^{m} u_\mu u_\nu \zeta_j^{\mu-\nu}. \quad (6.1)$$

We put $N := \max \{n_1, \ldots, n_{n-1}\}$. Let the barycentrics $(u_0, \ldots, u_m)$ be chosen generally such that not all $\lambda_1, \ldots, \lambda_{n-1}$ vanish. $N = 0$ is equivalent with $\lambda_1 = \cdots = \lambda_{n-1} = 0$ and can only occur if $m = n - 1$ and, additionally, $(u_0, \ldots, u_{n-1}) = (1/n, \ldots, 1/n)$. This case of iterated series of ‘degenerate $n$-gons’ $Q_k$, all collapsing into the center of gravity $O$ shall be excluded further on. For $0 < N < 1$ the series $Q_k$ gradually contracts for increasing $k$ and tends towards the center of gravity $O$. For $N = 1$ the series $Q_k$ remains finite,

$^1$As the prototypes $T_j$ and $T_{n-j}$ are affinely equivalent, an iteration regularizing of $j$th kind will also be regularizing of kind $n - j$ and we can confine ourselves to $1 \leq j \leq n/2$. 
but in general still may change its shape and its position from generation to
generation. For \( N > 1 \) the series \( Q_k \) gradually expands for increasing \( k \).

We will prove that for any \( N > 0 \), the algorithm is—in general—affinely regularizing. We divide the set of indices into two distinct subsets:

\[
\mathbb{J}_1 := \{ j \in \mathbb{J}^* / |\lambda_j| = N \} \neq \emptyset \quad \text{and} \quad \mathbb{J}_2 := \mathbb{J}^* \setminus \mathbb{J}_1. \tag{6.2}
\]

According to (3.3), for any \( j^* \in \mathbb{J}_1 \) the index \( n - j^* \) is also contained in \( \mathbb{J}_1 \); for even \( n \) and \( j^* = n/2 \) these two indices coincide. We have

\[
\frac{|\lambda_j|}{N} = 1 \quad \forall j \in \mathbb{J}_1 \quad \text{and} \quad 0 \leq \frac{|\lambda_j|}{N} < 1 \quad \forall j \in \mathbb{J}_2. \tag{6.3}
\]

Equations (3.10) yield

\[
Q_k = \frac{N^k}{\sqrt{n}} \left( \sum_{\nu \in \mathbb{J}_1} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu} \right) \Leftrightarrow q_{j,k} = \frac{N^k}{\sqrt{n}} \left( \sum_{\nu \in \mathbb{J}_1} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} \zeta_{j,\nu} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} \zeta_{j,\nu} \right). \tag{6.4}
\]

Regardless of the input data \( b_j \) (3.8) the coefficients \( (\frac{\lambda_{\nu}}{N})^k \) form null series for all \( \nu \in \mathbb{J}_2 \) and \( k \to \infty \); the coefficients \( (\frac{\lambda_{\nu}}{N})^k \) for all \( \nu \in \mathbb{J}_1 \) are complex numbers of norm 1 for all \( k \in \mathbb{N} \).

\( Q_k \) and any homothetic image \( \rho_k(Q_k) \) have the same affine shape. Following Definition 5.1 we can apply homotheties \( \rho_k : \mathbb{E}^d \to \mathbb{E}^d \) with \( x \mapsto x \frac{\sqrt{n}}{N} \). These homotheties \( \rho_k \) turn (6.4) into

\[
\rho_k(Q_k) = \sum_{\nu \in \mathbb{J}_1} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu} \Leftrightarrow \rho_k(q_{j,k}) = \sum_{\nu \in \mathbb{J}_1} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} \zeta_{j,\nu} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} \zeta_{j,\nu}. \tag{6.5}
\]

With reference to the cardinal number of the index set \( \mathbb{J}_1 \) we have three cases:

**Case A:** The index set \( \mathbb{J}_1 \) contains just one element. This can only happen if \( n \) is an even integer and the barycentrics \( (u_0, \ldots, u_m) \) lead to \( \mathbb{J}_1 = \{ n/2 \} \). We have \( \zeta_{n/2} = -1 \), and \( \lambda_{n/2} = \sum_{\mu=0}^{m} u_\mu (-1)^\mu \in \mathbb{R} \). As \( N = |\lambda_{n/2}| > 0 \) and therefore \( \lambda_{n/2} = \pm N \neq 0 \) formula (6.5) reads as

\[
\rho_k(Q_k) = (\pm 1)^k b_{n/2} P_{n/2} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu}. \tag{6.6}
\]

For every \( k \) we apply a further homothety \( \sigma_k : \mathbb{E}^d \to \mathbb{E}^d \) with

\[
\sigma_k(x) = (\pm 1)^k x \Rightarrow \sigma_k(\rho_k(Q_k)) = b_{n/2} P_{n/2} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\pm \lambda_{\nu}}{N} \right)^k b_{\nu} P_{\nu}. \tag{6.7}
\]

We have \( b_{n/2} = \sum_{\nu=0}^{n-1} q_{\nu,0} \zeta_{n/2} = \sum_{\nu=0}^{n-1} (-1)^\nu q_{\nu,0} \in \mathbb{R}^d \). For a generic input polygon \( Q_0 \) we can assume \( b_{n/2} \neq 0 \). In this case we choose an affine mapping \( \tau \) with fixed point \( O \) and \( b_{n/2} \mapsto e_1 \in \mathbb{R}^d \). The mapping \( \tau \) induces an affine mapping \( \mathbb{C}^d \to \mathbb{C}^d \) transforming \( b_{\nu} \ (\nu \in \mathbb{J}_2) \) into \( b_{\nu}^* := \tau(b_{\nu}) \in \mathbb{C}^d \); the
vectors \( b^*_\nu \) do not depend on \( k \). The affine mapping \( \beta_k := \tau \circ \sigma_k \circ \rho_k \) places the \( k \)th generation polygon \( Q_k \) into

\[
\beta_k(Q_k) = e_1 P_{n/2} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\pm \lambda_{\nu}}{N} \right)^k b^*_\nu P_\nu. \tag{6.8}
\]

The distance vectors \( d_{j,k} \) of the vertices of \( \beta_k(Q_k) \) to the respective vertices of the prototype polygon \( T_{n/2} = e_1 P_{n/2} \) (4.1) are the columns of \( D_k = (d_{0,k}, \ldots, d_{n-1,k}) \) with

\[
D_k = \sum_{\nu \in \mathbb{J}_2} \left( \frac{\pm \lambda_{\nu}}{N} \right)^k b^*_\nu P_\nu \iff d_{j,k} = \sum_{\nu \in \mathbb{J}_2} \left( \frac{\pm \lambda_{\nu}}{N} \right)^k b^*_\nu \zeta_j^\nu (j \in \mathbb{J}). \tag{6.9}
\]

The vectors \( b^*_\nu \zeta_j^\nu \) are independent from \( k \). As the norms of \( \left( \frac{\lambda_{\nu}}{N} \right)^k \) form null series for all \( \nu \in \mathbb{J}_2 \) we can be sure that \( \lim_{k \to \infty} d_{j,k} = 0_d \) for all \( j \in \mathbb{J} \). The sum of the squared distances \( \Delta_k := \sum_{j=0}^{n-1} ||d_{j,k}||^2 \) is a null series: \( \lim_{k \to \infty} \Delta_k = 0 \).

Thus, according to our Definition 5.1 the iteration process in case A is affinely regularizing of kind \( n/2 \). For generic input \( Q_0 \) the polygons \( Q_k \) approach the shape of the \( n \)-gon \( T_{n/2} \). The straight lines approximating the polygons \( Q_k \) tend towards the straight line through \( O \) with direction vector \( b_{n/2} \).

**Case B:** The index set \( \mathbb{J}_1 \) contains exactly two different elements: \( \mathbb{J}_1 = \{ j^*, n-j^* \} \) with \( 1 \leq j^* < n/2 \). In a way, this could be considered the general case. We put \( \lambda_{j^*} = Ne^{i\phi} \) and \( \lambda_{n-j^*} = Ne^{-i\phi} \) with some real angle \( \phi \in [0,2\pi) \) and define \( W := \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k b^*_\nu P_\nu \). Then (6.5) yields

\[
\rho_k(Q_k) = e^{ik\phi} b_{j^*} P_{j^*} + e^{-ik\phi} b_{n-j^*} \overline{P}_{j^*} + W. \tag{6.10}
\]

Let \( b_{j^*} := x + iy \) with \( x, y \in \mathbb{R}^d \). We then have \( b_{n-j^*} = \overline{b}_{j^*} = x - iy \) and

\[
\rho_k(Q_k) = x(e^{ik\phi} P_{j^*} + e^{-ik\phi} \overline{P}_{j^*}) + iy(e^{ik\phi} P_{j^*} - e^{-ik\phi} \overline{P}_{j^*}) + W. \tag{6.11}
\]

For a generic input \( n \)-gon \( Q_0 \) the two vectors \( x, y \in \mathbb{R}^d \) are linearly independent. Let \( \sigma : \mathbb{E}^d \longrightarrow \mathbb{E}^d \) be any affine mapping that maps the two vectors \( x, y \) into \( \sigma(x) := e_1/2 \) and \( \sigma(y) := -e_2/2 \). \( \sigma \) induces an affine mapping \( \mathbb{C}^d \longrightarrow \mathbb{C}^d \) transforming \( b_\nu(\nu \in \mathbb{J}_2) \) into \( \sigma(b_\nu) \). We have

\[
\sigma(\rho_k(Q_k)) = \left( e_1 \cos k\phi + e_2 \sin k\phi \right) \frac{P_{j^*} + \overline{P}_{j^*}}{2} + (-e_1 \sin k\phi + e_2 \cos k\phi) \frac{P_{j^*} - \overline{P}_{j^*}}{2i} + \sum_{\nu \in \mathbb{J}_2} \left( \frac{\lambda_{\nu}}{N} \right)^k \sigma(b_\nu) P_\nu. \tag{6.12}
\]

We define the complex numbers \( \theta_{\mu,\nu} := \sigma(b_\mu)^t \overline{\sigma(b_\nu)} \) for \( \mu, \nu \in \mathbb{J}_2 \). The matrices

\[
R_k := \left( \begin{array}{cccccc}
\cos k\phi & \sin k\phi & 0 & \ldots & 0 \\
-\sin k\phi & \cos k\phi & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{array} \right), \tag{6.13}
\]
describe rotations $\tau_k$ in $\mathbb{E}^d$. The induced mappings $\tau_k$ in $\mathbb{C}^d$ transform the vectors $\sigma(b_\nu)$ into vectors $\tau_k(\sigma(b_\nu)) \in \mathbb{C}^d(\nu \in \mathbb{J}_2)$. The mappings $\beta_k := \tau_k \circ \sigma \circ \rho_k$ are affine mappings from $\mathbb{E}^d$ into $\mathbb{E}^d$ and deliver

$$\beta_k(Q_k) = e_1 \frac{P_{j*} + \overline{P}_{j*}}{2} + e_2 \frac{P_{j*} - \overline{P}_{j*}}{2i} + \sum_{\nu \in \mathbb{J}_2} \left(\frac{\lambda_\nu}{N}\right)^k \tau_k(\sigma(b_\nu)) P_\nu. \quad (6.14)$$

As every $\tau_k$ preserves scalar products we have $\tau_k(\sigma(b_\mu))^t \tau_k(\sigma(b_\nu)) = \theta_{\mu,\nu}$ for all $\mu, \nu \in \mathbb{J}_2$. According to (5.2) we compute the sum of squared distances of the vertices of $\beta_k(Q_k)$ to the respective vertices of the prototype polygon $T_{j*}$ and arrive at

$$\Delta_k = \text{tr} \left( (T_{j*} - \beta_k(Q_k))^t \cdot (T_{j*} - \beta_k(Q_k)) \right) = n \sum_{\mu \in \mathbb{J}_2} \left(\frac{\lambda_\mu \lambda_\mu^\ast}{N^2}\right)^k \theta_{\mu, n - \mu}. \quad (6.15)$$

As the values $\theta_{\mu, n - \mu}$ are independent from $k$ and $0 \leq \frac{\lambda_\mu \lambda_\mu^\ast}{N^2} < 1$ for all $\mu \in \mathbb{J}_2$ the values $\Delta_k$ ($k \in \mathbb{N}$) form a null series. Accordingly, the corresponding iteration process in case B is regularizing of kind $j^*$ with $1 \leq j^* < n/2$. For generic input $n$-gons $Q_0$ the two vectors $x$ and $y$ determine a plane $\varepsilon^*$ through $O$. The planes $\varepsilon_k$ approximating the polygon $Q_k$ tend towards $\varepsilon^*$.

**Case C:** The index set $\mathbb{J}_1$ contains more than two different elements. We have $j^*, j^{**}, n - j^* \in \mathbb{J}_1$ with $1 \leq j^* < j^{**} \leq n/2$. According to (6.1) this is characterized by

$$\sum_{\mu, \nu=0}^m u_\mu u_\nu \zeta_j^{\mu - \nu} = \sum_{\mu, \nu=0}^m u_\mu u_\nu \zeta_j^{\mu - \nu}. \quad (6.16)$$

The coefficients of $u_\mu u_\nu$ in (6.16) are $\zeta_j^{\mu - \nu} + \zeta_j^{\nu - \mu} - \zeta_j^{\mu - \nu} - \zeta_j^{\nu - \mu} \in \mathbb{R}$. The corresponding barycentrics $(u_0, \ldots, u_m)$ denote points $z^* \in \mathbb{R}^m$ which, in general, are positioned on an $(m - 1)$-dimensional quadric of $\mathbb{R}^m$ containing the vertices of the simplex $S$. In this case we cannot prove any regularizing effect of the affine iteration.

We call the barycentrics $(u_0, \ldots, u_m)$ ‘generic’ if they do not lead to Case C or, for $m = n - 1$, they are different from $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. Overall, we have

**Theorem 6.1.** Affine Regularization Theorem. For generic barycentrics the iteration process (2.1) is affinely regularizing according to Definition 5.1. The barycentrics $(u_0, \ldots, u_m)$ $(m < n)$ determine the eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ given by (3.3) and their maximal norm $N > 0$. If there is exactly one index $j^*$ (with $1 \leq j^* \leq n/2$) with eigenvalue $\lambda_j$ of norm $N$ the iteration is regularizing of kind $j^*$.

If the iteration is affinely regularizing of kind $j^*$ then, for a generic input $n$-gon $Q_0$, the shape of $Q_k$ gradually approaches the shape of an affinely transformed prototype $n$-gon $T_{j*}$.
Figure 2 An example with $n = 8$, $m = d = 3$ with $\mathbb{J}_1 = \{1, 7\}$

Figure 3 An exceptional example for $n = 8$, $m = d = 3$ with $b_4 = \mathbf{0}_d$ and $\mathbb{J}_1 = \{4\}$. For this specific $n$-gon $Q_0$ the algorithm works as if it was affinely regularizing of kind 3

Figure 2 shows an example for the same initial octagon $Q_0$ as in Fig. 1 ($n = 8$, $m = d = 3$). Here the barycentrics of $z^*$ are $(u_0, u_1, u_2, u_3) = (0.4, 0.5, 0.3, -0.2)$. We get $n_1 \approx 1.03, n_2 \approx 0.71, n_3 \approx 0.13, n_4 \approx 0.4$ and therefore we have $\mathbb{J}_1 = \{1, 7\}$. The algorithm is regularizing of kind 1 (case B). The figure shows $Q_0$ and the following generations up to $Q_{16}$.

7. Remarkable exceptions

For specific initial polygons $Q_0$ the algorithm may deliver unexpected results. If the coefficient vectors $b_\nu$ of the regarded eigenvalues $\lambda_\nu$ for $\nu \in \mathbb{J}_1$ in (6.4)
vanish the respective eigenvalues have no influence on the regularizing process. So, for such a specific \( n \)-gon \( Q_0 \), the algorithm works in the same way as if in (3.10) these eigenvalues \( \lambda_\nu \) had been replaced by \( \lambda_\nu = 0 \). The remaining eigenvalues deliver another maximum norm \( N^* < N \) and a different set \( \mathcal{J}_1 \). Now our classification (Sect. 6) reveals the affine shape of the series \( Q_k \).

Figure 3 shows such an example for \( n = 8 \), \( m = d = 3 \) where we have \((u_0, u_1, u_2, u_3) = (0.5, -0.25, 0.5, 0.25)\). We get \( n_1 \approx 0.52 \), \( n_2 = 0.5 \), \( n_3 \approx 0.99 < 1 \), \( n_4 = 1 \). Hence \( N = 1 \) and we conclude that the algorithm is affinely regularizing of kind 4; \( Q_k \) is expected to approach the shape of the prototype \( T_4 \) which is a line segment. The special initial octagon \( Q_0 \), however, yields \( b_4 = o_d \); we put \( \lambda_4 := 0 \) and perform a new case study. The affine shape of \( Q_k \) tends towards the prototype \( T_3 \). Figure 3 displays \( Q_0 \) and the following generations up to \( Q_6 \).

8. Conclusion

We studied affine iterations transforming an initial \( n \)-gon \( Q_0 \) in \( \mathbb{E}^d \) \((d > 1)\) into successive generations of \( n \)-gons \( Q_k \). The Affine Regularization Theorem in this paper does not only extend the results in [5] to dimensions \( d > 2 \); surprisingly, even for dimensions \( d > 2 \) the regularization leads to planar, regular prototypes no matter which generic input \( n \)-gon \( Q_0 \) we start with. For very specific input \( n \)-gons \( Q_0 \), though, the same algorithm seems to regularize in a different way. The understanding of this phenomenon completes the results.

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