Full-program induction: verifying array programs sans loop invariants

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Abstract
Arrays are commonly used in a variety of software to store and process data in loops. Automatically proving safety properties of such programs that manipulate arrays is challenging. We present a novel verification technique, called full-program induction, for proving (a sub-class of) quantified as well as quantifier-free properties of programs manipulating arrays of parametric size $N$. Instead of inducting over individual loops, our technique inducts over the entire program (possibly containing multiple loops) directly via the program parameter $N$. The technique performs non-trivial transformations of the given program and pre-conditions during the inductive step. The transformations assist in effectively reducing the assertion checking problem by transforming a program with multiple loops to a program which has fewer and simpler loops or is loop free. Significantly, full-program induction does not require generation or use of loop-specific invariants. To assess the efficacy of our technique, we have developed a prototype tool called V Ajra. We demonstrate the performance of V Ajra vis-a-vis several state-of-the-art tools on a large set of array manipulating benchmarks from the international software verification competition (SV-COMP) and on several programs inspired by algebraic functions that perform polynomial computations.

Keywords Full-program induction · Inductive proof · Hoare triple · Array programs · Difference program · Difference pre-condition · Quantified assertions · Loop invariant

1 Introduction

Use of software-controlled systems in industrial and household appliances is constantly increasing. Functionalities of such software are programmed with extensive use of loops and conditional statements that manipulate different data structures such as arrays, lists, and vectors to store and process data during its operation. Programs with loops manipulating arrays are quite common in many such applications. These programs are expected to be of immensely high-quality as their erroneous functioning can cause adversities to businesses as well as human lives. Thus, ensuring correctness of these programs is of paramount importance. Unfortunately, assertion checking in such programs is, in general, undecidable. Existing tools therefore use a combination of verification techniques that work well for certain classes of programs and assertions, and yield conservative results otherwise.

In this paper, we present a new verification technique, called full-program induction, to add to this arsenal of techniques. Specifically, we focus on programs with loops manipulating arrays, where the size of each array is a symbolic integer parameter $N$ ($> 0$). We allow (a sub-class of) quantified and quantifier-free pre- and post-conditions that may depend on the symbolic parameter $N$. Thus, the problem we wish to solve can be viewed as checking the validity of a parameterized Hoare triple $\{\phi(N)\} P_N \{\psi(N)\}$ for all values of $N$ ($> 0$), where the program $P_N$ computes with arrays of size $N$, and $N$ is a free variable in $\phi(\cdot)$ and $\psi(\cdot)$.

Like earlier verification approaches [1], our technique also relies on mathematical induction to reason about programs with loops. However, the way in which the inductive claim is formulated and proved differs significantly from the previ-
ous techniques. Specifically, (i) we induct on the full program (possibly containing multiple loops) with parameter $N$ and not on iterations of individual loops in the program, (ii) we perform non-trivial correct-by-construction transformation of the given program and the pre-condition, whenever feasible, to simplify the inductive step of reasoning, (iii) we strengthen the pre- and post-condition simultaneously during the inductive step using the auxiliary inductive predicates obtained by employing Dijkstra’s weakest pre-condition computation, (iv) we recursively apply the technique to prove the inductive step and most importantly (v) we do not require explicit or implicit loop-specific inductive invariants to be provided by the user or generated by a solver (viz. by constrained Horn clause solvers [2–4] or recurrence solvers [5,6]). The combination of these factors often reduces reasoning about a program with multiple loops to reasoning about one with fewer (sometimes even none) and “simpler” loops, thereby simplifying proof goals. In this paper, we demonstrate this, focusing on programs with sequentially composed, but non-nested loops.

1.1 Motivating examples

We present a couple of examples to illustrate our technique and showcase its salient features. The first example presents the basic ideas behind full-program induction. The second example highlights various nuanced features of the technique and is used as a running example in this paper.

Figure 1 shows an example of a Hoare triple, where the pre- and post-conditions are specified using assume and assert statements. This triple effectively verifies the formula $\sum_{i=0}^{N-1}(1 + \sum_{k=0}^{i-1}6 \cdot (k + 1)) = i^3$ for all $i \in \{0 \ldots N - 1\}$, and for all $N > 0$. Although each loop in

```c
// assume(True)
1. A[0] = 6;
2. B[0] = 1;
3. C[0] = 0;
// assert((C[0] = 0^3) ∧
// (B[0] = 1^3 - 0^3) ∧
// (A[0] = 2^3 - 2×1^3 + 0^3))
```

Fig. 2 Base-case Hoare triple

Fig. 1 is simple, their sequential composition makes it difficult even for state-of-the-art tools like VIAP [5], VÉRIABS [7], FREQHORN [4], TILER [8], VAPHOR [9], or BOOSTER [10] to prove the post-condition correct. In fact, none of the above tools succeed in automatically proving the quantified post-condition in Fig. 1. In contrast, our technique full-program induction proves the post-condition in Fig. 1 correct within a few seconds.

Full-program induction reduces checking the validity of the Hoare triple in Fig. 1 to checking the validity of two “simpler” Hoare triples, represented in Figs. 2 and 3. The base case of our inductive reasoning is shown in Fig. 2, where every loop in the program is statically unrolled a fixed number of times after instantiating the program parameter $N$ to a small constant value (here $N = 1$). As the induction hypothesis, we assume that the Hoare triple $\{\phi(N - 1)\} \mathcal{P}_{N-1} \{\psi(N - 1)\}$ holds for values of $N > 1$. Note that this assumption does not relate to a specific loop in the program, but to the entire program $\mathcal{P}_N$. For the motivating example, the induction hypothesis states that the entire Hoare triple in Fig. 1, after substituting $N$ with $N - 1$, holds. Notice that the induction hypothesis is on the entire program including all three loops and not on individual loops. The inductive step of the reasoning shown in Fig. 3 proves the post-condition, by automatically generating the computation to be performed after the program with parameter $N - 1$ has executed and strengthening the pre- and post-conditions using auxiliary predicates. Note that all the program statements in Fig. 3 have syntactic counterparts in Fig. 1, but this may not be the case in general. We conceptualize the computation in the inductive step using the notions of difference program and difference pre-condition in the later sections. Effectively, we reasoned about three sequentially composed loops in Fig. 1 together, without the need for any implicitly or explicitly specified loop invariants. We defer a discussion of how our technique computes these Hoare triples and how auxiliary predicates are generated to iteratively strengthen the pre- and post-conditions to Sect. 5, where we present the details of our algorithms.

It is important to mention a few things here to highlight the simplifications illustrated by the Hoare triples in Figs. 2 and 3 that resulted from the application of the full-program
Fig. 3 Inductive-step Hoare triple

// assume(  
// (N > 1) ∧ (C_Nm1[N-2] = (N-2)³) ∧  
// (B_Nm1[N-2] = (N-1)³ - (N-2)³) ∧  
// (A_Nm1[N-2] = N³ - 2×(N-1)³ + (N-2)³) 
// )
1. A[N-1] = A_Nm1[N-2] + 6;
2. B[N-1] = B_Nm1[N-2] + A_Nm1[N-2];
3. C[N-1] = C_Nm1[N-2] + B_Nm1[N-2];

// assert(  
// (C[N-1] = (N-1)³) ∧  
// (B[N-1] = N³ - (N-1)³) ∧  
// (A[N-1] = (N+1)³ - 2×N³ + (N-1)³) 
// )

Fig. 4 Running example

// assume(∀i∈[0,N) A[i] = 1)  
1. S = 0;
2. for(i=0; i<N; i++) {
  3.  S = S + A[i];
  4. }
5. for(i=0; i<N; i++) {
  6.  A[i] = A[i] + S;
  7. }
8. for(i=0; i<N; i++) {
  9.  S = S + A[i];
  10. }

// assert(S = N × (N+2))

induction technique on the problem in Fig. 1. First, the programs in Figs. 2 and 3 are loop free. Second, their pre- and post-conditions are quantifier free. Third, the validity of these Hoare triples (Figs. 2 and 3) can be easily proved, e.g., by bounded model checking [11] with a back-end SMT solver like Z3 [12]. Fourth, the value computed in each iteration of each loop in Fig. 1 is data dependent on previous iterations of the respective loops as well as on the value computed in previous loops. Even though none of these loops can be trivially translated to a set of parallel assignments, our method still succeeds in automating the inductive step of the analysis. Last, we did not require any specialized constraint solving techniques like recurrence solving, theory of uninterpreted functions, or constrained Horn clause solving to verify these Hoare triples, thus making our technique orthogonal to these approaches when proving properties of array programs.

Now consider the Hoare triple shown in Fig. 4. The program updates a scalar variable $S$ and an array variable $A$. The first loop adds the value of each element in array $A$ to variable $S$. The second loop adds the value of $S$ to each element of $A$. The last loop aggregates the updated content of $A$ in $S$. The pre-condition $\varphi(N)$ is a universally quantified formula on array $A$ stating that each element has the value 1. We need to establish the post-condition $\psi(N)$, which is a predicate on $S$ and $N$. Note that the post-condition has nonlinear terms making it quite challenging to prove. We will use the Hoare triple in Fig. 4 as our running example to illustrate important aspects of the full-program induction technique.

Since the program $P_N$ updates the same scalar variable $S$ and the array $A$ in multiple sequentially composed loops, we rename these scalars and arrays such that each loop in $P_N$ updates its own copy of scalar variables and arrays. This ensures that when $P_{N-1}$ terminates we have access to the values of these variables after each loop in the program. The renamed program is shown in the Hoare triple in Fig. 5a.

In the base case of our inductive reasoning, we instantiate the parameter $N$ to a small constant value (say $N = 1$). As a result, every loop in the program $P_N$ in Fig. 5a can be statically unrolled a fixed number of times. The resulting Hoare triple can be easily compiled to a first-order logic formula and verified using an SMT solver. As the induction hypothesis, we assume that the Hoare triple $\{\varphi(N - 1)\} P_{N-1} \{\psi(N - 1)\}$, shown in Fig. 5b, holds for values of $N > 1$. This Hoare triple is obtained by substituting $N$ with $N - 1$ in the entire Hoare triple in Fig. 5a.

The Hoare triple in Fig. 5c is computed for the inductive step. Intuitively, the difference program $\partial P_N$ recovers the effect of the computation in $P_N$ on all scalar variables and arrays after the computation in $P_{N-1}$ has been performed. It includes the iterations of a loop in $P_N$ that are missed by $P_{N-1}$. When program statements are impervious to the value of $N$, the values computed in such statements are the same in $P_N$ and $P_{N-1}$, and hence, they may not need any modification. However, $\partial P_N$ may contain code to “rectify” values of variables and arrays that have different values at corresponding statements in $P_N$ vis-a-vis $P_{N-1}$. The code, possibly consisting of loops, to rectify the values of variables and arrays is further simplified whenever possible. Consequently, not all program statements of $P_N$ in Fig. 5a may have a syntactic counterpart in Fig. 5c and vice versa. We present in detail the algorithms for the computation and simplification of the difference program in Sects. 4.5 and 4.6. The inductive step may not be immediately established, in which case we strengthen the pre- and post-conditions using automatically inferred auxiliary predicates as shown in Fig. 5c.

1.2 Beyond loop-invariant-based proofs

Techniques based on synthesis and use of loop invariants are popularly used to reason about programs with loops. These
techniques have been successfully applied to verify different classes of array manipulating programs, viz. [4,13–20]. If we were to prove the assertion in Fig. 1 using such techniques, it would be necessary to use appropriate loop-specific invariants for each of the three loops in Fig. 1. The weakest loop invariants needed to prove the post-condition in this example are:

\[ \forall i \in [0, t_1) (A[i] = 6i + 6) \] for the first loop (lines 1-4),
\[ \forall j \in [0, t_2) (B[j] = 3j^2 + 3j + 1) \land (A[j] = 6j + 6) \] for the second loop (lines 5-8), and
\[ \forall k \in [0, t_3) (C[k] = k^3) \land (B[k] = 3k^2 + 3k + 1) \] for the third loop (lines 9-12).

Notice that these invariants are quantified and have nonlinear terms. Similarly, the weakest loop invariants needed to prove the post-condition for the program in Fig. 5a are:

\[ \forall j \in [0, i) (A[j] = 1) \land (S = j) \] for the first loop (lines 2-4),
\[ \forall k \in [0, i) (A1[k] = N + 1) \land (A[k] = 1) \land (S = N) \] for the second loop (lines 5-7), and
\[ \forall l \in [0, i) (A1[l] = N + 1) \land (S1 = l \times (N + 1) + N) \] for the third loop (lines 9-11).

Unfortunately, automatically deriving such quantified nonlinear inductive invariants for each loop is far from trivial. Template-based invariant generators, viz. [21,22], are among the best-performers when generating such complex invariants. However, their abilities are fundamentally limited by the set of templates from which they choose. We therefore choose not to depend on inductive loop invariants at all in our work. Instead, we make use of inductive pre- and post-conditions—a notion that is related to, yet significantly different from loop-specific invariants. Specifically, inductive pre- and post-conditions are computed for the entire program, possibly consisting of multiple loops, instead of for each loop in the program.

As is clear from the discussion above, the primary difference between a proof generated by an invariant synthesis technique and the proof generated by our method is that we no longer need loop-specific safe inductive invariants. Instead, we generate and verify the Hoare triples shown in Figs. 2 and 3 considering the entire program \( P_N \) in Fig. 1 and the Hoare triple shown in Fig. 5c considering the entire program \( P_N \) in Fig. 5a. Automatically generating these Hoare triples in some cases may be more difficult than automatically generating inductive invariants for each loop and vice versa. However, as demonstrated by the motivating examples, there are several complex programs, where it may be easier to generate these Hoare triples than compute safe inductive invariants for individual loops. It is a considerable challenge for verification techniques to be able to automatically generate these invariants and to the best of our knowledge none of the current state-of-the-art techniques do so.

1.3 Effectiveness of full-program induction

We have implemented the full-program induction technique in a prototype tool called VAJRA. Written in C++, the tool is built on top of a compiler framework (LLVM/CLANG [23]) and uses an off-the-shelf SMT solver (Z3 [12]) at the back-end. Our experiments show that the full-program induction technique is able to solve several difficult problem instances, which other techniques either fail to solve, or can solve only with the help of sophisticated recurrence solvers. VAJRA is significantly more efficient as compared to other tools on a set of benchmarks.
Needless to say, each approach has its own strengths and limitations, and the right choice always depends on the problem at hand. Full-program induction is no exception, and despite its several strengths, it has its own limitations, which we discuss in detail in Sect. 7.1.

The full-program induction technique is orthogonal to other verification approaches proposed in the literature, making it suitable to be a part of an arsenal of verification techniques. It has already been incorporated within a verification tool, namely VERIABS [7]. Since the 2020 edition of the international software verification competition (SV-COMP), VERIABS invokes full-program induction (via our tool VAJRA) in its pipeline of techniques for verifying programs with arrays from the set of benchmarks in the verification competition (refer [7]).

1.4 Primary contributions of our work

This paper is a revised and extended version of [24]. Our main contributions can be summarized as follows:

1. We introduce full-program induction as a technique for reasoning about assertions in programs with loops manipulating arrays with parametric size bounds. Full-program induction does not need loop-specific invariants in order to prove assertions, even when the program contains multiple sequentially composed loops.

2. We describe practical algorithms for performing full-program induction. We elaborate the generalized algorithms for computing the difference program and the difference pre-condition.

3. We present a new algorithm to compute a progress measure, based on the characteristics of the difference program. This gives a measure of how easy it is to prove the inductive step of our technique using constraint solving based techniques like bounded model checking.

4. We give rigorous proofs of correctness for the presented algorithms. We demonstrate these algorithms using a running example.

5. We present generalizations of the full-program induction technique to programs with multiple parameters and loops with increasing and/or decreasing loop counters.

6. We describe a prototype tool VAJRA that implements the algorithms for performing full-program induction, using (i) the compiler framework LLVM/CLANG for analysis and transformation of the input program and (ii) an off-the-shelf SMT solver, viz. Z3, at the back-end to discharge verification conditions.

7. We present an extensive experimental evaluation on a large suite of benchmarks that manipulate arrays. VAJRA outperforms the state-of-the-art tools VIAP, VERIABS, BOOSTER, VAPHOR, and FREQHORN, on the set of benchmark programs.

Several contributions listed above are beyond those presented in [24]. These include the contributions 2, 3, 4, 5, and 7.

The remainder of the paper is structured as follows. In Sect. 2, we give a formal overview of the full-program induction technique. Sect. 3 presents the syntactic restrictions on the program as well as the pre- and post-conditions and the representation of programs as control flow graphs. Sect. 4 discusses the algorithms for computing the difference program and the difference pre-condition, as well as the prerequisite analyses and transformations. In Sect. 5, we present the algorithms for full-program induction, prove their correctness and demonstrate each algorithm on the running example. In Sect. 6, we give an algorithm to check whether the recursive application of our technique will eventually be able to verify the given program. Sect. 7 talks of the generalizations of our technique in different settings. Sect. 8 presents the implementation of our technique in VAJRA, its evaluation on a set of benchmarks and comparison vis-a-vis state-of-the-art tools. In Sect. 9, we discuss the related techniques from the literature. Finally, Sect. 10 presents concluding remarks on our work and possible future directions.

2 Overview of full-program induction

We now elaborate on the core idea behind the full-program induction technique. Our goal is to check the validity of the parameterized Hoare triple \( \{ \phi(N) \} \ P_N \ \{ \psi(N) \} \) for all \( N > 0 \). A visual representation of this Hoare triple is shown in Fig. 6, where the clouds represent (possibly quantified) formulas and boxes represent programs/code fragments.

Intuitively, at a conceptual level, our approach works like any other inductive reasoning technique. However, the induction is over the entire program, via the program parameter \( N \), and not on the individual loops in the program.

We first check the base case, where we verify that the parameterized Hoare triple holds for some small values of \( N \), say \( 0 < N \leq M \). We rely on an important, yet reasonable, assumption that can be stated as follows: For every value of \( N (\geq 0) \), every loop in \( P_N \) can be statically unrolled a number (say \( f(N) \)) of times that depends only on \( N \), to yield a loop-free program \( P_N \) that is semantically equivalent.
to $P_N$. Note that this does not imply that reasoning about loops can be translated into loop-free reasoning. In general, $f(N)$ is a non-constant function, and hence, the number of unrollings of loops in $P_N$ may strongly depend on $N$. In our experience, loops in a vast majority of array manipulating programs (including Figs. 1 and 4 and all our benchmarks) satisfy the above assumption. Consequently, the base case of our induction reduces to checking a Hoare triple for a loop-free program. Checking a Hoare triple for a loop-free program is easily achieved by compiling the pre-condition, program and post-condition into an SMT formula, whose (un)satisfiability can be checked with an off-the-shelf backward SMT solver.

Next, we hypothesize that $\{\phi(N - 1)\} P_{N-1} \{\psi(N - 1)\}$ holds for some $N > M$, visually depicted in Fig. 7. A few things are worth mentioning here. First, the entire Hoare triple is assumed not just the formula in the post-condition. Second, the assumption is not on a specific loop in the program, but the entire program $P_N$. Third, the change in the parameter from $N$ to $N - 1$ is uniform across the entire Hoare triple and not on a specific part there-off.

We then try to show that the induction hypothesis implies $\{\phi(N)\} P_N \{\psi(N)\}$. While this sounds simple in principle, there are several technical difficulties en-route. Our contribution lies in overcoming these difficulties algorithmically for a large class of programs and assertions, thereby making full-program induction a viable and competitive technique for proving properties of array manipulating programs.

The inductive step is the most complex one, and is the focus of the rest of the paper. Recall that the inductive hypothesis asserts that $\{\phi(N - 1)\} P_{N-1} \{\psi(N - 1)\}$ is valid. To make use of this hypothesis in the inductive step, we must relate the validity of $\{\phi(N)\} P_N \{\psi(N)\}$ to that of $\{\phi(N - 1)\} P_{N-1} \{\psi(N - 1)\}$. We propose doing this, whenever possible, via two key notions—that of “difference” program and “difference” pre-condition.

Given a parameterized program $P_N$, intuitively the “difference” program $\partial P_N$ is one such that $\{\phi(N)\} P_N \{\psi(N)\}$ holds iff $\{\phi(N)\} P_{N-1} \{\psi(N)\}$ holds, where “;” denotes sequential composition. Refer to Fig. 8 for a visual representation of the Hoare triple after the decomposition of $P_N$ into $P_{N-1}$ and $\partial P_N$. We will use this interpretation of a “difference” program in the subsequent parts of this paper.

A simple way of ensuring the correctness of this transformation is by having a difference program $\partial P_N$ such that the sequential composition $P_{N-1}; \partial P_N$ is semantically equivalent to $P_N$. Decomposition of $P_N$ into $P_{N-1}$ and $\partial P_N$ is visually depicted in Fig. 9. It ensures that, upon termination, same program state is reached by both $P_N$ and $P_{N-1}; \partial P_N$. The given post-condition may not be impacted by the entire program state, and hence, the semantic equivalence alluded to here may not be required, in general. Thus, the semantic equivalence of the decomposition is a strong condition. It is referred here only for the ease of explaining the inductive setup and for an intuitive demonstration of soundness of the decomposition. For the purposes of full-program induction semantic equivalence is not really necessary, and we do not refer to this interpretation of the “difference” program further due to its restrictive nature.

The “difference” pre-condition $\partial \phi(N)$ is a formula such that the following conditions hold.

1. $\phi(N) \rightarrow (\phi(N - 1) \odot \partial \phi(N))$, where the Boolean operator $\odot$ is $\land$ when $\phi(N)$ is a universally quantified formula and it is $\lor$ when $\phi(N)$ is an existentially quantified formula. We depict this decomposition of $\phi(N)$ into $\phi(N - 1)$ and $\partial \phi(N)$ in Fig. 10.

2. The execution of $P_{N-1}$ does not affect the truth of $\partial \phi(N)$. This can be visualized using Fig. 8, where the dashed line
Computing the “difference” program $\partial P_N$ and the “difference” pre-condition $\partial \varphi(N)$ is not easy in general. In Sect. 5, we discuss ways to overcome these problems and challenges.

Assuming we have $\partial P_N$ and $\partial \varphi(N)$ with the properties stated above, the proof obligation $\{\varphi(N)\} P_N \{\psi(N)\}$ can now be reduced to proving the Hoare triples $\{\varphi(N - 1)\} P_{N-1} \{\psi(N - 1)\} \land \partial \varphi(N)$ and $\{\psi(N - 1) \land \partial \varphi(N)\} \partial P_N \{\psi(N) \land \text{Pre}(N)\}$. While this is somewhat reminiscent of loop invariants, observe that $\text{Pre}(N)$ is not really a loop-specific invariant. Instead, it is analogous to computing an invariant for the entire program, possibly containing multiple loops. Specifically, the above process strengthens both the pre- and post-condition of the resulting Hoare triple may, in turn, require a new pre-condition $\text{Pre}'(N - 1)$ to be satisfied. This process of strengthening the pre- and post-conditions of the Hoare triple involving $\partial P_N$ can be iterated until a fix-point is reached, i.e., no further pre-conditions are needed for the parameterized Hoare triple to hold. While the fix-point was quickly reached for all benchmarks we experimented with, we also discuss how to handle cases where the above process may not converge easily. Note that since we effectuatively strengthen the pre-condition of the Hoare triple in the inductive step, for the overall induction to go through, it is also necessary to check that the strengthened assertions hold at the end of each base-case check. Automatically computing $\text{Pre}(N)$ to strengthen the pre- and post-condition of the Hoare triple may not always be straightforward, especially when the difference program $\partial P_N$ has loops. In such cases, we recursively apply our technique on the generated Hoare triple $\{\psi(N - 1) \land \partial \varphi(N)\} \partial P_N \{\psi(N)\}$. This helps our technique converge when the generated difference program has one or more loops. We check if the recursive invocation of our technique will yield beneficial results using a progress measure influenced by several characteristics of the difference program.

The technique outlined above is called full-program induction, and the following theorem is the basis for the soundness of full-program induction.

**Theorem 1** Given $\{\varphi(N)\} P_N \{\psi(N)\}$, suppose the following are true:

1. For $N > 1$, $\{\varphi(N)\} P_{N-1}; \partial P_N \{\psi(N)\}$ holds iff $\{\varphi(N)\} P_N \{\psi(N)\}$ holds.
2. For $N > 1$, there exists a formula $\varphi(N)$ such that
   (a) $\varphi(N)$ doesn’t refer to any program variable or array element modified in $P_{N-1}$, and
   (b) $\varphi(N) \rightarrow \varphi(N - 1) \land \partial \varphi(N)$.
3. There exists an integer $M \geq 1$ and a parameterized formula $\text{Pre}(M)$ such that
   (a) $\{\psi(N)\} P_N \{\psi(N)\}$ holds for $0 < N \leq M$,
   (b) $\{\psi(M)\} P_M \{\psi(M) \land \text{Pre}(M)\}$ holds, and
   (c) $\{\psi(N - 1) \land \text{Pre}(N - 1) \land \partial \varphi(N)\} \partial P_N \{\psi(N) \land \text{Pre}(N)\}$ holds for $N > M$. 

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![Fig. 10](image1.png)

**Fig. 10** Difference pre-condition

![Fig. 11](image2.png)

**Fig. 11** Inductive step

![Fig. 12](image3.png)

**Fig. 12** Strengthening pre- and post-conditions
Then \(\{\varphi_N\} P_N \{\psi_N\}\) holds for all \(N \geq 1\).

**Proof** For \(0 < N \leq M\), condition 3(a) (the base case) ensures that \(\{\varphi(N)\} P_N \{\psi(N)\}\) holds. For \(N > M\), note that by virtue of conditions 1 and 2(b), \(\{\varphi(N)\} P_N \{\psi(N)\}\) holds if \(\varphi(N-1) \land \varphi(N)\) \(P_{N-1}; \partial P_N \{\psi(N) \land \text{Pre}(N)\}\) holds. With \(\psi(N-1) \land \text{Pre}(N-1)\) as a mid-condition, and by virtue of condition 2(a), the latter Hoare triple holds for \(N > M\) if \(\{\varphi(M)\} P_M \{\psi(M) \land \text{Pre}(M)\}\) holds and \(\{\psi(N-1) \land \text{Pre}(N-1) \land \varphi(N)\}\) \(\partial P_N \{\psi(N) \land \text{Pre}(N)\}\) holds for all \(N > M\). Both these triples are seen to hold by virtue of conditions 3(b) and (c).

\[\square\]

### 3 Preliminaries

We consider array manipulating programs generated by the grammar shown in Fig. 13 (adapted from [8]). This grammar restricts programs to have non-nested loops. Specifically, programs generated starting from StLF are loop free. The non-terminal St can generate programs with loops but their bodies are generated from StLF, thereby forbidding nesting of loops. Note also that expressions for indexing arrays are generated from the non-terminal IndE, and such expressions cannot refer to other array elements. However, this is not really a restriction on the expressive power of programs since every array index expression that depends on other array elements, say \(A[e]\), can be replaced by an array index expression that depends on temporary variables, say \(v\), that are pre-assigned to the respective array elements, viz. \(A[e]\). For example, \(A[B[i]] = C[D[i]]\); can be rewritten as \(v1 = B[i]; v2 = D[i]; A[v1] = C[v2]\). Finally, note that loop bound expressions are generated using the non-terminal Lbnd, and such expressions can only involve constants and the parameter \(N\). While the above restrictions limit the class of programs to which our technique currently applies, there is still a large collection of useful programs, with possibly long sequences of loops, that are included in the scope of our work. In reality, our technique also applies to a sub-class of programs with nested loops and with loop bound expressions that involve scalar variables. However, characterizing this sub-class of programs through a grammar is a bit unwieldy, and we avoid doing so for reasons of clarity.

A program \(P_N\) is a tuple \((V, L, A, PB, N)\), where \(V\) is a set of scalar variables, \(L \subseteq V\) is a set of loop counter variables, \(A\) is a set of array variables, \(PB\) is the program body, and \(N\) is a special symbol denoting a positive integer parameter. In the grammar shown above, we assume \(A \subseteq A, v \in V \setminus L, \ell \in L\) and \(c \in \mathbb{Z}\). Furthermore, “relop” is assumed to be one of the relational operators and “op” is an arithmetic operator. We discuss more about the operators supported by our technique in Sect. 4.5. We also assume that each loop \(L\) has a unique loop counter variable \(\ell\) that is initialized at the beginning of \(L\) and is incremented by 1 at the end of each iteration. Assignments in the body of \(L\) are assumed not to update \(\ell\). Finally, for each loop with termination condition \(\ell < L\text{bnd}\), we assume that \(L\text{bnd}\) is an expression in terms of \(N\). We denote by \(k_L(N)\) the number of times loop \(L\) iterates in the program with parameter \(N\).

We admit Hoare triples of the form \(\{\varphi(N)\} P_N \{\psi(N)\}\), where \(\varphi(N)\) and \(\psi(N)\) are either universally quantified, existentially quantified or quantifier-free formulas of the form \(\forall I (\Phi(A, V, I, N))\), \(\exists I (\Phi(I, N) \land \Psi(A, V, I, N))\) and \(\Upsilon(A, V, I, N)\), respectively. In the above, \(I\) is an array index variable, \(\Phi\) is a quantifier-

### 3.1 Tracking control flow

We represent a program \(P_N\) using its control flow graph (or CFG) \(G_C = (Locs, CE, \mu)\), where \(Locs\) denotes the set of control locations (nodes) of the program, \(CE \subseteq Locs \times Locs \times \{tt, ff, U\}\) are the control flow edges, and \(\mu : Locs \rightarrow AssignSt \cup BoolE\) annotates every node in \(Locs\) with either an assignment statement (of the form \(v = E\) or \(A[\text{IndE}] = E\)) from those represented by AssignSt, or a Boolean expression from those represented by BoolE. Two distinguished control locations, called \(n_{\text{start}}\) and \(n_{\text{end}}\) in \(Locs\) represent the entry and exit points of the program. An edge \((n_1, n_2, label)\) represents flow of control from \(n_1\) to \(n_2\) without any other intervening node. It is labeled \(tt\) or \(ff\) if \(\mu(n_1)\) is a Boolean condition, and is labeled \(U\) otherwise. If \(\mu(n_1)\) is a Boolean condition, there are two outgoing edges from \(n_1\), labeled \(tt\) and \(ff\), respectively, and control flows from \(n_1\) to \(n_2\) along \((n_1, n_2, label)\) only if \(\mu(n_1)\) evaluates to \(label\). If \(\mu(n_1)\) is an assignment statement, there is a single outgoing edge from \(n_1\), and it is labeled \(U\). Henceforth, we use CFG to refer to a control flow graph, and use \(P_N\) to refer to both a program and its CFG, when there is no confusion.

A CFG may have cycles due to the presence of loops in the program. A back-edge of a loop is an edge from the node corresponding to the last statement in the loop body to the node representing the loop head. An exit-edge is an edge from the loop head to a node outside the loop body. An incoming-edge is an edge to the loop head from a node outside the loop body. We assume that every loop has exactly one back-edge, one incoming-edge, and one exit-edge.

A node \(m\) in a control flow graph strictly post-dominates a node \(n\) if all control flow paths from node \(m\) pass through \(n\) before reaching the exit node and \(m\) is not the same as
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Fig. 13  Program grammar

The immediate post-dominator of node $n$ is a node that strictly post-dominates $n$ but does not strictly post-dominate any other node that strictly post-dominates $n$.

For every node $n$ in the CFG, we use $\text{def}(n)$ and $\text{uses}(n)$ to refer to the set of scalar variables and arrays (not loop counter variables) that are defined and used, respectively, in the statement or Boolean expression at $n$. We include the symbolic parameter $N$ in the set $\text{uses}(n)$ if the statement at node $n$ makes use of $N$. Since the parameter $N$ cannot be re-defined by any program generated according to the grammar in Fig. 13, it never appears in $\text{def}(n)$ for any node $n$. If $A$ represents an array in $\text{def}(n)$, we use $\text{defIndex}(A, n)$ to refer to the index expression of the element of $A$ updated at $n$. Similarly, if $A \in \text{uses}(n)$, we use $\text{useIndexSet}(A, n)$ to refer to the set of index expression(s) of element(s) of $A$ read at $n$.

Example 1  The CFG of the program in Fig. 4 is shown in Fig. 14. The nodes are numbered such that they coincide with the line numbers in the program. The graph has three cycles each corresponding to a loop in the given program. \{ (1, 2), (2, 5), (5, 8) \} are incoming-edges, \{ (4, 2), (7, 5), (10, 8) \} are back-edges and \{ (2, 5), (5, 8), (8, End) \} are exit-edges.

Node 2 strictly post-dominates nodes 1, 4 and Start. Node 2 is an immediate post-dominator of 1 and 4. Node 8 strictly post-dominates all nodes except itself and End. Node 8 is an immediate post-dominator of 5 and 10. End node strictly post-dominates all nodes except itself. Start node does not strictly post-dominate any other node. Similarly, the post-domination relations for other nodes can be computed.

The set of scalars and arrays defined at nodes 1, 3, and 9 is $\text{def}(n) := \{S\}$ and the set at node 6 is $\text{def}(n) = \{A\}$. The index of $A$ updated at node 6 is $\text{defIndex}(A, 6) = i$. The set of scalars and arrays used at nodes 3, 6 and 9 is $\text{uses}(n) = \{S, A\}$. For node 1, as there are no uses of scalars or arrays, $\text{uses}(n) = \emptyset$. The set of indices of array $A$ used at nodes 3, 6 and 9 is $\text{useIndexSet}(A, n) = \{i\}$.

4 Difference computation

In this section, we focus on the generation of two crucial components for performing full-program induction, namely (i) the difference program $\partial P_N$ and (ii) the difference precondition $\partial \varphi(N)$.

Computing the difference program $\partial P_N$ is a non-trivial endeavor. Figure 15 presents a high-level overview of the sequence of steps involved in the generation of a difference program. We first carefully rename the variables and arrays such that each loop in the renamed program refers to its own copy of variables/arrays. Note that this is similar in spirit to SSA renaming, although there are important differences that will become clear in Sect. 4.1. We next peel the last (in some cases the last few) iteration(s) of each loop in the program such that the remaining part of each loop in the peeled version of $P_N$ iterates exactly the same number of times as the corresponding loop in $P_{N-1}$. Throughout this paper, we use the term peel to denote the last (or last few as the case may be) iteration(s) of a loop that have been removed from...
the loop. The motivation for this peeling is that the difference program can often be constructed by moving the peels of individual loops to the end of the program and stitching them up in appropriate ways, as will be discussed in detail in Sects. 4.5.4 and 4.5.5. In order to ensure that the semantics of the program is preserved even after moving the peels to the end of the program, we need to do a careful analysis of the data dependencies between variables and array elements updated/read in statements within loops and those updated/read in the peeled iterations. This is achieved by computing a customized data dependence graph, details of which are presented in Sect. 4.3. In general, variables and array elements in the program \( P \) can have data/control dependencies on the parameter \( N \) beyond those attributable to the iteration counts of loops being possibly determined by \( N \). We call such variables/array elements as “affected” by \( N \) and identify them using a special data-flow analysis and the data dependencies computed above. Details of this analysis are presented in Sect. 4.4. Finally, we move the peels of loops to the end of the program and use the information about data dependencies and affected variables computed above to appropriately stitch and modify them to obtain an unoptimized version of the difference program. In general, this modification may involve adding carefully constructed loops in the difference program itself. It turns out that the difference program obtained in this way can often be significantly optimized using simple optimization techniques. This includes things like pruning superfluous computational steps and accelerating loops among others. We include this optimization as the last step in our flow for generating the difference program.

Our empirical studies show that for the success of full-program induction, it is very important that the difference program \( \partial P_N \) be “simpler” (defined more precisely later) than the original program \( P_N \). The optimizations enabled by the affected variable analysis and the simplification of the difference program are crucial to actualize this requirement. Each of the above steps, depicted in Fig. 15, is elaborated in Sects. 4.1–4.6.

### 4.1 Renaming variables and arrays

Recall that our proposed approach requires us to construct a difference program \( \partial P_N \) such that \( \{ \psi(N) \} P_N \{ \psi(N) \} \) holds iff \( \{ \psi(N) \} P_{N-1} \{ \psi(N) \} \) holds (condition 1 of Theorem 1). A natural (though not necessary) way to do this is to construct \( \partial P_N \) such that both \( P_N \) and \( P_{N-1} \) induce all relevant scalar variables and arrays in exactly the same way. Note, however, that \( P_N \) may update scalar variables or array in multiple sequentially composed loops. Therefore, when \( P_{N-1} \) terminates and \( \partial P_N \) starts executing (in \( P_{N-1} \); \( \partial P_N \)), we may no longer have access to the values of scalar variables and arrays that resulted after individual loops in \( P_{N-1} \) terminated. In general, this makes it difficult to construct \( \partial P_N \) compositionally from the peels of individual loops while ensuring that \( P_{N-1} \); \( \partial P_N \) has the same effect as \( P_N \) on all relevant scalar variables and arrays. To circumvent this problem, we propose to pre-process \( P_N \) such that each loop in \( P_N \) updates its own “private” copy of scalar variables and arrays. We add glue code to copy the values of these scalar variables and arrays after one loop ends and before the next one begins. We also rename the variables/arrays referred in the post-condition \( \psi(N) \) to their versions corresponding to the last loop in the program. As we show later, this eases the construction of \( \partial P_N \), and also helps in inductive strengthening of the pre- and post-conditions.

It is important to note here that static single assignment (SSA) [25] is a well-known technique for renaming scalar variables such that a variable is updated at most once in a program. Similarly, array SSA renaming has been studied earlier in the context of compilers to achieve similar goals [26]. Unlike SSA renaming, we do not have the stringent requirement of a single update in the whole program. For our method to function successfully, we only require each loop to update its own copy of a scalar/array variable. We note that these well-studied techniques can be easily adapted for our purposes.

In the following discussion, we define the **collapsed CFG** of a program \( P_N \) as the CFG obtained by collapsing all nodes and edges in the body of each loop of \( P_N \) into a single node identified with the loop head. Given the syntactic restrictions
on the input programs as discussed in Sect. 3, the collapsed CFG is a finite directed acyclic graph (DAG). This DAG has finitely many paths, and along each such path, there is a total ordering of all collapsed loops appearing along the path. For notational clarity, we henceforth use $vA$ (as opposed to $v$ for a scalar variable and $A$ for an array) as a combined symbolic name to refer to a scalar variable or array, depending on the context. At each node $n$ of the collapsed CFG, we rename each scalar/array $vA$ to $vAN$. Note that when $n$ is a loop head, this amounts to renaming all scalars/arrays in the body of the loop as well (due to collapsing of nodes in the loop body). To ensure the correct flow of data values between nodes of the collapsed CFG, we create fresh nodes called glue nodes whenever required, and add program statements in these glue nodes that effectively copy values of the appropriate scalars/arrays from one node of the collapsed CFG to another.

We assume the availability of a function $\text{RENAME}$, that generates a program with the necessary renaming as described, while ensuring correct data flow. For notational convenience, we call the renamed program corresponding to $P_N$ as $N_P$. The interested reader can find a detailed algorithm for $\text{RENAME}$ in [27].

**Lemma 1** Let $n$ be a node in the collapsed CFG of $P_N$. In every execution of the renamed program $N_P$ in which control flows through $n$, the scalar variable/array $vAN$ is not updated after the execution exits $n$.

**Proof** Since the collapsed CFG of $N_P$ is acyclic, once control flow exits node $n$, it cannot come back to either $n$ or to any node $n'$ that has a control flow path to $n$. The proof now follows from the observation that renaming ensures that any scalar variable/array renamed $vAN$ can only be updated in glue nodes immediately leading to node $n$ or in node $n$ itself.

For convenience of exposition, we will henceforth refer to the property formalized in Lemma 1 as the “no-overwriting” property of renamed programs. For a node $n$ that corresponds to a collapsed loop in the collapsed CFG of $P_N$, we will also use the notation $\mu(n)$ to denote the entire loop represented by $n$ in the subsequent discussion.

**Lemma 2** $\text{\{\psi(N)\}} P_N \text{\{\psi(N)\}}$ holds iff $\{\psi'(N)\} P_N \text{\{\psi'(N)\}}$ holds.

**Proof** (Sketch) Suppose the Hoare triple $\{\psi(N)\} P_N \text{\{\psi(N)\}}$ holds. Then, at every node $n$ in the collapsed CFG of $P_N$ there exists a pre-condition invariant $\text{inv}^\text{pre}_n$ and a post-condition invariant $\text{inv}^\text{post}_n$, such that (i) the Hoare triple $\{\text{inv}^\text{pre}_n\} \mu(n) \{\text{inv}^\text{post}_n\}$ at node $n$ holds (ii) the composition of these Hoare triples entails $\{\psi(N)\} P_N \text{\{\psi(N)\}}$. Note that once renaming is done, the Hoare triple obtained by renaming variables/arrays in $\text{inv}^\text{pre}_n$ and $\text{inv}^\text{post}_n$ and by replacing $\mu(n)$ with the corresponding renamed program statement(s) holds iff $\{\text{inv}^\text{pre}_n\} \mu(n) \{\text{inv}^\text{post}_n\}$ holds. Composing these renamed Hoare triples at all the nodes in the collapsed CFG of $N_P$ proves the forward direction of the lemma. The case in the reverse direction is similar.

We end this subsection with an illustration of the program transformation accomplished by applying the renaming strategy mentioned above. For convenience, we replicate our running example from Fig. 4 in Fig. 16a.

**Example 2** Consider the program shown in Fig. 16a. This program has multiple sequentially composed loops that update a scalar $S$ and an array $A$. The transformed program after renaming the scalar and array variables using function $\text{RENAME}$ is shown in Fig. 16b, where we have used simple names for the renamed versions of $S$ and $A$ for ease of readability. Notice that we rename the array $A$ in the second loop to $A1$, and rename the variable $S$ in the third loop to $S1$. The statement at line 8 in Fig. 16b is the glue code to copy values from one version of the renamed scalar/array (variable $S$ in our program) to another version of the same scalar/array (version $S1$). We avoid creating new versions of $S$ and $A$ for statements and loops that do not update them. Values are read directly from the version of $S$ and $A$ that reaches the access location. This helps in reducing the glue code required for renaming quite significantly.

### 4.2 Peeling the loops

Recall from Sect. 2 that our induction strategy requires us to use $P_{N-1}$; $\partial P_N$ in place of $P_N$ when proving the Hoare triple $\{\psi(N)\} P_N \text{\{\psi(N)\}}$. In general, the parameter $N$ may determine the number of times each loop in $P_N$ iterates (see, for example, Fig. 16b). Therefore, the count of iterations of a loop in $P_{N-1}$ may differ from the corresponding count in $P_N$. Relating $P_N$ and $P_{N-1}$ requires taking into account such differences of loop iterations. Toward this end, we transform $P_N$ by peeling the last few iterations of each loop as needed, so that corresponding loops in $P_{N-1}$ and the transformed $P_N$ iterate the same number of times. This is done by function $\text{PEELALLLOOPS}$ shown in Algorithm 1. The algorithm first makes a copy, viz. $P^p_{N}$, of the non-collapsed input CFG $P_N$. Let $\text{LOOPS}(P^p_N)$ denote the set of loops of $P^p_N$, and let $k_L(N)$ and $k_L(N-1)$ denote the number of times loop $L$ iterates in $P^p_N$ and $P_{N-1}$, respectively. The difference $k_L(N) - k_L(N-1)$, computed in line 5, gives the extra iteration count of loop $L$ in $P^p_N$. If this difference is not a constant, we currently report a failure of our technique (line 7). For example, consider a loop in $P_N$ with the counter $i$ initialized to 0 and the loop termination condition “$i < N^2$.” The corresponding loop in $P_{N-1}$ has the same initialization but the termination condition is “$i < (N-1)^2$.” Thus, $k_L(N) = N^2$...
Algorithm 1 PEELALLLOOPS((Locs, CE, μ) : program P_N)

1: P^p_N := (Locsp, CE^p, μ^p), where Locsp = Locs, CE^p = CE, μ^p = μ; ▷ P^p_N is a copy of P_N
2: PeelNodes := ∅;
3: for each loop L ∈ LOOPS(P^p_N) do
4:     Let k_L(N) be the expression for iteration count of L in P^p_N;
5:     PeelCount := SIMPLIFY(k_L(N) − k_L(N − 1));
6:     if PeelCount is not a constant then
7:         throw “Failed to peel non-constant number of iterations”;
8:     end if
9:     (P^p_{N′}, Locs′) := PEELSINGLELOOP(P^p_N, L, k_L(N − 1), PeelCount);
▷ We assume availability of function PeelSingleLoop, for example, from a compiler framework
▷ like LLVM.
▷ It transforms loop L so that last PeelCount iterations of L are peeled.
▷ Updated CFG and newly created CFG nodes for the peeled iterations are returned.
10: PeelNodes := PeelNodes ∪ Locs′;
11: end for
12: return (P^p_N, PeelNodes);

and k_L(N − 1) = (N − 1)^2 and the difference of these iteration counts is “2 × N + 1.” Our technique is unable to handle such cases currently. Note that such cases cannot arise if the upper bounds of all loops in P_N are linear functions of N.

The routine PEELSINGLELOOP transforms loop L of P^p_N as follows: it replaces the termination condition (ℓ < k_L(N)) of L by (ℓ < k_L(N − 1)). It also peels the last (k_L(N) − k_L(N − 1)) iterations of L and adds control flow edges such that the peeled iterations are executed immediately after the loop body is iterated k_L(N − 1) times. Effectively, PEELSINGLELOOP peels the last (k_L(N) − k_L(N − 1)) iterations of loop L in P^p_N. The transformed CFG is returned as the updated P^p_{N′} in line 9. In addition, PEELSINGLELOOP also returns the set Locs′ of all CFG nodes newly added while peeling the loop L. We accumulate these newly added nodes for loops in the set PeelNodes in line 10. Henceforth, we call all nodes in PeelNodes as peeled nodes, all other nodes in the CFG as non-peeled nodes, and the CFG resulting from the invocation of PEELALLLOOPS as a peeled program. This function PEELALLLOOPS returns the peeled program P^p_N and the set of peeled nodes PeelNodes in line 12.

We now state several useful properties of peeled programs.

**Lemma 3** Let n ∈ Locs be a node in the peel of loop L, and let n_h be the loop head of loop L. For every n′ ∈ Locs that is not in the peel, if there is a control flow path in P^p_N from n′ to n, the path necessarily passes through n_h.

**Proof** The proof follows from the observation that the peel of a loop L must necessarily execute after the loop L has itself executed k_L(N − 1) times. Hence, the sole predecessor of the first node in the peel must be the loop-head node n_h. It follows that every control flow path from n′ to n where n′ is not in the peel must pass through n_h. □

Peeling of loops can destroy the no-overwriting property (as mentioned in Lemma 1), since the same variable/array can get updated in a loop L and also in its peel. However, a
weaker variant of the no-overwriting property continues to hold, as described below.

**Lemma 4** Let \( n \) be a node in the collapsed CFG of \( P_N \). In every execution of the renamed and peeled program \( P^p_N \) in which control flows through \( n \), the following hold.

1. If \( n \) is not a collapsed node, the no-overwriting property as in Lemma 1 holds for all scalar variables/arrays \( vA^n \).
2. If \( n \) is a node representing a collapsed loop \( L \), the scalar variable/array \( vA^n \) is not updated at any subsequent node along the execution, except possibly in the peel of \( L \).

**Proof** Follows from the same reasoning as used in the proof of Lemma 1. □

We will henceforth refer to the property formalized in Lemma 4 as the “no-overwriting” property of the renamed and peeled program \( P^p_N \).

**Lemma 5** In the peeled program \( P^p_N \), each conditional branch node in a peel of a loop has an immediate post-dominator in the same peel.

**Proof** The syntactic restrictions on the input program, imposed by the grammar shown in Fig. 13, do not admit `break`, `continue`, `goto`, `exit`, and `return` statements. Since a loop body is also syntactically a complete program, conditional branch nodes in the body of the loop, if any, always have an immediate post-dominator node within the body of the same loop. The peel of a loop is obtained by creating a copy of the loop body (using the function `PeelSingleLoop` invoked on line 9 of routine `PeelAllLoops` in Algorithm 1). Thus, the conditional branch nodes, if any, in the peel have an immediate post-dominator node within the same peel. □

Finally, the following lemma asserts that peeling does not change the Hoare semantics of programs.

**Lemma 6** \( \{\varphi_N\} P_N \{\psi_N\} \) holds iff \( \{\varphi_N\} P^p_N \{\psi_N\} \) holds.

**Proof** Follows immediately from the observation that peeling each loop preserves the semantics of the program. □

**Example 3** We execute function `PeelAllLoops` on the renamed version of our running example, shown in Fig. 16b. The resulting program is shown in Fig. 17. The algorithm first computes the number of iterations to be peeled from a loop in the program, given by `PeelCount`. The upper bound expression of each loop in the program is \( N \). Hence, the number of iterations to be peeled is \( N - (N - 1) = 1 \). In other words, only the last iteration is to be peeled from each loop. The function appends the statements in the peeled iteration after each loop and updates the upper bound expressions of each loop in the resulting program, as shown in Fig. 17. The algorithm also returns the set of peeled nodes, i.e., CFG nodes corresponding to the statements at lines 5, 9, and 14.

### 4.3 Tracking data dependencies

As discussed in Sect. 3.1, the flow of control in a program can be conveniently represented by a CFG. A CFG, however, does not immediately provide information about data dependencies between program statements. We use a separate data dependence graph, or DDG, to summarize data dependencies between relevant statements in a program. Our primary purpose in constructing such a DDG is to understand the dependencies of and from statements that are executed in \( P_N \) but not in \( P^p_N \). These are related to the peeled statements described in Sect. 4.2, and determine what must eventually go into the difference program \( \partial P_N \), so that \( P_N \) and \( P^p_N \) have the same effect on arrays and scalar variables.

While there are several notions of data dependence used in the literature (see [28,29] for details), we use a fairly simple notion that best serves our purpose. We say that there is a read-after-write data dependence from \( n_1 \) to \( n_2 \) if the statement at \( n_2 \) uses a data value that is potentially generated by the statement at \( n_1 \). There is another kind of data dependence that is peculiar to our approach that also needs special handling. It may so happen that the glue code inserted between two nodes during renaming has a loop, say \( L_1 \), that updates an array \( A \) that is also subsequently updated in another loop, say \( L_2 \) in non-glue code. If the peel of \( L_1 \) potentially updates an element of \( A \) that is also updated in the non-peeled part of \( L_2 \), then we have a write-after-write dependence between

// assume(\( \forall i \in [0,N] \) \( A[i] = 1 \))
1. \( S = 0; \)
2. for(\( i=0; i<N-1; i++ \) \{ \)
3. \( S = S + A[i]; \)
4. \} \)
5. \( S = S + A[N-1]; \)
6. for(\( i=0; i<N-1; i++ \) \{ \)
7. \( A[i] = A[i] + S; \)
8. \} \)
9. \( A[N-1] = A[N-1] + S; \)
10. \( S1 = S; \)
11. for(\( i=0; i<N-1; i++ \) \{ \)
12. \( S1 = S1 + A1[i]; \)
13. \} \)
14. \( S1 = S1 + A1[N-1]; \)

// assert(\( S1 = N \times (N+2) \))
a statement in the peel of a (glue) loop and subsequent statement in the non-peeled part of another (non-glue) loop. We call such a dependence non-peeled-write-after-peeled-write dependence. Since we intend to move peels to the end of the program to construct a difference program, this kind of dependence poses a problem. Therefore, we explicitly identify such non-peeled-write-after-peeled-write dependencies below. Given the way our renaming operates, it is easy to see that such a dependence can only arise for arrays and not for scalars.

Note that in the above case when we have a glue loop followed by a non-glue loop updating the same array, there may also be write-after-write dependencies between the non-peeled (resp. peeled) part of the glue loop and the non-peeled (resp. peeled) part of the non-glue loop. However, such dependencies are preserved if we move peels of all loops to the end of the program to construct a difference program. Therefore, such write-after-write dependencies do not pose any problem for our purposes, and hence we do not keep track of them. Furthermore, due to the way our renaming operates, it can be seen that write-after-read dependencies can never arise between nodes of the collapsed CFG.

Formally, a DDG is a directed graph \( G_D = (\text{Locs}, \text{DE}, \mu) \), where \( \text{Locs} \) and \( \mu \) are exactly as in the definition of a CFG, and \( \text{DE} \subseteq \text{Locs} \times \text{Locs} \) represents read-after-write and non-peeled-write-after-peeled-write dependencies between statements in the program. Since our primary interest is in using data dependencies to identify parallelism, the grammar in Fig. 13 allows only loop counter to be updated in a loop-head node, it suffices to restrict our attention to data dependencies between distinct non-loop-head nodes in the peeled program.

Several existing compilers generate program dependence graph, or PDG from a given input program, and a DDG can be extracted from such a PDG [30]. Standard data-flow analysis techniques are usually used to identify data dependencies when constructing a PDG [30, 31]. One needs to be particularly careful when identifying dependence between statements updating and accessing array elements, since it is not only the same array name that must be involved in the update and access, but also the same element in the array. The problem is further compounded by the fact that array indices can be arbitrary expressions in general. While vectorizing compilers can compute precise dependencies with array index expressions using sophisticated dependence tests [32], it is not always the case that these are implemented in non-vectorizing compilers. A conservative generation of DDG may contain spurious data dependence edges, which, in our context, can lead to the construction of a difference program that is more complex than what is needed.

Let \( n \) and \( n' \) be two CFG (hence also DDG) nodes. A conservative way of generating DDG edges is to add the edge \((n, n')\) to \( \text{DE} \) if there is a control flow path \( \pi : (n = n_1, n_2, \ldots, n_{r-1}, n_r = n') \) in the CFG such that one of the following conditions hold.

\[ \text{D1: } \left( \text{def}(n) \cap \text{uses}(n') \right) \setminus \bigcup_{i=2}^{r-1} \text{def}(n_i) \text{ contains a scalar variable } v, \text{ or} \]

\[ \text{D2: } \text{def}(n) \text{ contains an array } A \text{ such that} \]

(a) Either of the following conditions hold:

(i) \( A \in \text{uses}(n') \) and there is a common value that the index expression \( \text{defIndex}(A, n) \) and some index expression in \( \text{useIndexSet}(A, n') \) can have.

(ii) \( n \in \text{PeelNodes} \) and \( n' \notin \text{PeelNodes} \) and \( A \in \text{def}(n') \) and there is a common value that both the index expressions \( \text{defIndex}(A, n) \) and \( \text{defIndex}(A, n') \) can have.

(b) Some elements of \( A \) are potentially not updated along the path \( \pi \).

Lemma 7 For \( n, n' \in \text{Locs} \) such that \( n' \) is reachable from \( n \) in the CFG, if neither condition D1 nor condition D2 holds, then there is no read-after-write or non-peeled-write-after-peeled-write dependence from \( n \) to \( n' \).

Proof We prove the lemma by contradiction. Suppose, if possible, neither D1 nor D2 holds and yet there is a read-after-write dependence due to the data value generated at \( n \) being potentially used at \( n' \). There are two cases to consider.

- If the data value pertains to a scalar variable \( v \) that is updated at \( n \) and accessed at \( n' \), then there must be a control flow path \( \pi \) from \( n \) to \( n' \) along which \( v \) is not updated at any intermediate node. This implies condition D1 is satisfied—a contradiction!

- Suppose the data value pertains to an element of array \( A \) that is updated at \( n \) and accessed at \( n' \). Let the index expression of the array element updated at \( n \) be \( e \), and let the corresponding index expression of the same element accessed at \( n' \) be \( e' \). Clearly, both \( e \) and \( e' \) can assume the same value (the concrete index of the element of \( A \) under consideration), and there is a control flow path \( \pi \) from \( n \) to \( n' \) along which this specific array element has not been updated. This implies that both the conditions D2(a)(i) and D2(b) are satisfied, and hence condition D2 is satisfied—a contradiction!

Suppose, if possible, neither D1 nor D2 holds and yet there is a non-peeled-write-after-peeled-write dependence from \( n \) to \( n' \). Suppose the data value pertains to an element of array \( A \) that is updated at nodes \( n \) and \( n' \) where \( n \in \text{PeelNodes} \) and \( n' \notin \text{PeelNodes} \). Let the index expression of the array element updated at \( n \) be \( e \), and let the corresponding index expression
of the same element updated at \( n' \) be \( e' \). Clearly, both \( e \) and \( e' \) can assume the same value (the concrete index of the element of \( A \) under consideration), and there is a control flow path \( \pi \) from \( n \) to \( n' \) along which this specific array element has not been updated. This implies that both the conditions D2(a)(ii) and D2(b) are satisfied, and hence, condition D2 is satisfied—a contradiction! \( \square \)

Condition D2(b) above is not easy to check in general. However, for programs generated by the grammar in Fig. 13, it is possible to detect that condition D2(b) is violated in special cases. As an example, if there is a loop that updates all elements of array \( A \) in every control flow path from \( n \) to \( n' \), then indeed condition D2(b) is violated. For purposes of this paper, we use this special case as a sufficient condition to detect violation of condition D2(b), and conservatively assume that the condition is potentially satisfied in all other cases. Needless to say, a more precise analysis can be done with additional computational effort to reduce the degree of conservativeness in the above check for condition D2(b). We defer such an improved analysis to future work. We now look at how condition D2(a) is checked. Recall from Sect. 3 that the array indices in our programs can only be expressions in terms of constants, scalar variables, the loop counter variables, and the parameter \( N \). Furthermore, our programs do not have nested loops. Therefore, at most one loop counter variable can appear in an array index expression. Specifically, if \( e \) is the index expression \( \text{defIndex}(A, n) \), and if node \( n \) is part of a loop \( L \) with loop counter \( \ell \), then \( e \) depends in general on \( \ell \), \( N \) and a set of scalar constants. Otherwise, i.e., if node \( n \) is not part of a loop, \( e \) depends on \( N \) and a set of scalar constants. A similar reasoning applies for array index expression(s) in \( \text{useIndexSet}(A, n') \) as well. Condition D2(a) is satisfied if the constraint \( (e = e') \) has a model, i.e., is satisfiable, for some index expression \( e' \in \text{useIndexSet}(A, n') \), subject to the following conditions:

- Loop counters \( \ell \) and \( \ell' \) must have values within their respective loop bounds.
- If both \( n \) and \( n' \) are part of the same loop, then \( \ell \leq \ell' \) (update at \( n \) cannot happen in an iteration after access at \( n' \)).
- Every scalar variable \( v \) that appears in both \( e \) and \( e' \) and is updated along some control flow path from \( n \) to \( n' \) is renamed in \( e' \) to a fresh variable (since the values of \( v \) in \( e \) and \( e' \) may be different).

Function COMPUTE-DDG, shown in Algorithm 2, constructs the DDG for an input program \( P_A \) represented using its CFG (\( \text{Locs}, CE, \mu \)). We use the notation \( n \rightarrow^X n' \) to denote that there is a control flow path from \( n \) to \( n' \) in the CFG that passes through intermediate nodes in \( X \subseteq \text{Locs} \). COMPUTE-DDG proceeds by initializing the set of data dependence edges \( DE \) to \( \emptyset \), and by checking for every pair of distinct non-loop-head nodes \( (n, n') \) such that \( n \rightarrow^\text{Locs}(n') \), whether condition D1 or D2 referred to above is satisfied. If either one of the conditions is satisfied, it adds \( (n, n') \) to \( DE \) (lines 42–43).

The set \( S \) of scalar variables and arrays that potentially introduce data dependence from \( n \) to \( n' \) is initialized to \( \text{def}(n) \cap \text{uses}(n') \) in line 4. If \( n \) is a peeled node and \( n' \) is a non-peeled node, then we append \( S \) with \( \text{def}(n) \cap \text{def}(n') \) in line 5. Subsequently, the check for D1 is done in the loop in lines 7–12. In each iteration of this loop, we choose a scalar variable \( v \) from the set \( S \) and check whether there exists a control flow path from \( n \) to \( n' \) such that no intermediate node along the path updates \( v \). The latter check is implemented by first collecting all nodes \( n'' \) (other than \( n \) and \( n' \) that doesn’t update \( v \) in the set \( N\text{DefV} \) (line 8). If there is a control flow path from \( n \) to \( n' \) that passes through intermediate nodes in \( N\text{DefV} \), the value of \( v \) updated at \( n \) can reach the use of \( v \) at \( n' \). In this case, there is a potential data dependence of \( n' \) on \( n \) through \( v \) and condition D1 is satisfied. We therefore set \( D1\text{Sat} \) to True and abort the search over additional scalar variables \( v \) in \( S \) (line 10). Otherwise, there is no dependence of \( n' \) on \( n \) through \( v \).

If the flag \( D1\text{Sat} \) is not set to True even after iterating over all scalar variables in \( S \), we turn to checking if condition D2 can be satisfied. Toward this end, we iterate over all array names \( A \) remaining in \( S \), and formulate a constraint to check if condition D2(a) is satisfied (lines 15–33). This condition effectively checks if it is possible for the index expression \( \text{defIndex}(A, n) \) to have the same value as any index expression \( e' \in \text{useIndexSet}(A, n') \) or if it is possible for the index expression \( \text{defIndex}(A, n) \) to have the same value as the index expression \( \text{defIndex}(A, n') \) when \( n \) is a peeled node and \( n' \) is a non-peeled node. If not, the update/read of array \( A \) at \( n \) and \( n' \) cannot be for the same element, and hence, there is no data dependence \( (n, n') \) through \( A \). As discussed above, to check if condition D2(a) is satisfied, we must conjoin loop bound constraints for loop counter variables in case \( n \) or \( n' \) is present in a loop (lines 15–21), and rename every scalar variable \( v \) in expressions \( e' \in \text{useIndexSet}(A, n') \) that also appears in the index expression \( \text{defIndex}(A, n) \), if \( v \) is potentially re-defined in a control flow path from \( n \) to \( n' \). The renaming of scalar variables, if needed, is done in lines 22–31. The call to \( \text{IsSat} \) in line 32 is an invocation of an SMT solver that tells us whether the constraint fed to it as argument is satisfiable, i.e., has a model. If not, condition D2(a), and hence D2, is violated. Otherwise, we check in lines 34 and 35 if there exists a loop \( L'' \) not containing \( n \) and \( n' \) that necessarily executes as control flows from \( n \) to \( n' \) (\( \rightarrow^\text{Locs}(n') \) checks this), and in which all elements of the array \( A \) are updated. Recall from the grammar in Fig. 13 that all loops in our programs are for loops with a loop counter that increments by 1 in each operation, and cannot be updated in the body of the loop. For
such programs, it is sometimes easy to identify if a loop \( L'' \) is indeed updating all elements of an array \( A \). If we cannot determine whether \( L'' \) necessarily updates all elements of \( A \), we conservatively assume that it does not and the check in line 35 fails. If both the checks in lines 34 and 35 succeed, we conclude that condition D2(b), and hence D2, has been violated. In all other cases, we conservatively assume that D2 is satisfied, and set \( D2Sat \) to True in line 39. In such cases, we also abort the search over additional array variables \( A \) in \( S \).

**Lemma 8** Given a program represented as \((Locs, CE, \mu)\), let \((Locs, DE, \mu)\) be the DDG computed by COMPUTEDDG. For every pair of distinct non-loop-head nodes \( n, n' \in Locs \), if \((n, n') \notin DE\), there is no read-after-write or non-peeled-write-after-peeled-write data dependence from \( n \) to \( n' \).
Full-program induction: verifying array programs sans loop invariants

Since function ComputeDDG in Fig. 17 we execute function A1 to represent data dependence through the array variable A. The CFG for this program is shown using solid edges in Fig. 18. For convenience of exposition, we have executed function ComputeDDG on this CFG, we obtain the data dependence edges shown using dashed edges in Fig. 18. For ease of understanding, each DDG edge \((n, n')\) is also labeled by a scalar variable/array that is responsible for the data dependence of \(n'\) on \(n\). Thus, DDG edges \((n_1, n_3), (n_1, n_5), (n_3, n_5), (n_5, n_7), (n_5, n_9)\) and \((n_5, n_{10})\) represent data dependence through the scalar variable \(S\) and edges \((n_{10}, n_{12}), (n_{10}, n_{14}), (n_{12}, n_{14})\) represent data dependence through the scalar variable \(S_1\). In all these cases, condition D1 holds. Similarly, DDG edges \((n_7, n_{12}), (n_9, n_{14})\) represent data dependence through the array A1, since conditions D2(a) and D2(b) hold in these cases. Note that edge \((n_7, n_{14})\) (resp. \((n_9, n_{12})\)) is not added although A1 \(\in\) def\((n_7)\) uses\((n_{14})\) (resp. \(\in\) def\((n_9)\) uses\((n_{12})\)) because condition D2(a) fails in this case. To see why D2(a) fails, notice that \(\phi_7 := 0 \leq i < N - 1\) and \(\phi_{14} := \text{True}\). The expressions used to define and access array A are defIndex\((A, 7) := i\) and useIndexSet\((A, 14) := (N - 1)\). The constraint \(0 \leq i < N - 1\) and \(i = N - 1\), computed in line 32 of Algorithm 2, is unsatisfiable. This violates D2(a)(i). Note that, in this example there are no non-peeled-write-after-peeled-write dependencies.

### 4.4 Identifying “affected” variables

Recall that every loop \(L\) originally present in \(P_N\) iterates \(k_L(N - 1)\) times in \(P_{N-1}\) and \(k_L(N)\) times in \(P_N\). The iterations missed by \(P_{N-1}\) are represented by the peeled statements computed by function PeelAllLoops. It is natural to expect the difference program \(\partial P_N\) to contain the peeled statements, perhaps with some adaptations, if \(P_{N-1}; \partial P_N\) is to have the same effect as \(P_N\) on all relevant scalar variables and arrays. However, statements that are present in both \(P_N\) and \(P_{N-1}\) may also differ in their semantics, and therefore require “rectification” in \(\partial P_N\). For example, statements like \(x = N; \; \text{if} (x > N)\) in \(P_N\) become \(x = N - 1; \; \text{if} (x > N - 1)\), respectively, in \(P_{N-1}\). Clearly, the corresponding statements in \(P_N\) and \(P_{N-1}\) in the above examples have different semantics. We say that such statements, though present in both \(P_{N-1}\) and \(P_N\), are “affected” by the parameter \(N\), and potentially need to be “rectified” in \(\partial P_N\). Our goal in this subsection is to identify all relevant scalar variables/arrays that are potentially affected in this sense, i.e., they are updated by versions of the same statement in \(P_N\) and \(P_{N-1}\) but can potentially result in different values being assigned due to the change in the parameter \(N\). We use the data dependence information computed in the previous subsection to identify such variables and arrays, which we also call affected variables/arrays. Once these variables/arrays are identified, we can proceed to generate the difference program \(\partial P_N\) for effecting any rectification that may be needed.

Function ComputeAffected, shown in Algorithm 3, computes the set of affected scalar variables/arrays of a peeled program \(P^p_N\) represented by its CFG \((\text{Locs}^p, \text{CEP}^p, \mu^p)\). Besides the CFG, the function also takes as input the set of CFG nodes corresponding to peeled statements, denoted PeelNodes. Recall that such a set is obtained when function PeelAllLoops is invoked. ComputeAffected starts by constructing the data dependence graph using function ComputeDDG (line 1). The set of data dependence edges thus obtained is represented by \(DE^p\). We use AffectedVars to denote the set of affected variables and arrays of \(P_N\), and initialize it to the empty set in line 2. We also maintain a list of nodes \(n\) in WorkList such that the semantics of the program statement in \(\mu(n)\) is potentially affected (directly...
Algorithm 3 COMPUTEFFECTED((Locs^p, CE^p, μ^p); peeled program P_N^p, PeelNodes: peeled statements)

1: ((Locs^p, DE^p, μ^p) := COMPUTEDDG((Locs^p, CE^p, μ^p), PeelNodes);
2: AffectedVars := ∅;
3: WorkList := {n | n ∈ Locs^p \ PeelNodes, N ∈ uses(n) or ∃n', n' ∈ PeelNodes ∧ (n', n) ∈ DE^p};
4: ProcessedNodes := ∅;
5: while WorkList is not empty do
6: Remove a node n from the head of WorkList;
7: ProcessedNodes := ProcessedNodes ∪ {n};
8: if μ(n) is an assignment statement then
9: AffectedVars := AffectedVars ∪ def(n);
10: for all nodes n' s.t. n' ∈ Locs^p \ PeelNodes, n' ∈ ProcessedNodes and (n, n') ∈ DE^p do
11: WorkList := APPENDToList(WorkList, n');
12: end for
13: else if μ(n) is a branch condition then
14: for all nodes n' s.t. n' ∈ Locs^p \ PeelNodes, n' ∈ ProcessedNodes and n Locs^p \ n' do
15: WorkList := APPENDToList(WorkList, n');
16: end for
17: else continue;
18: end if
19: end while
20: return AffectedVars;

or indirectly by N. This worklist is initialized in line 3 with all non-peeled nodes n that either (i) have N in uses(n), or (ii) are potentially data dependent on a peeled node, i.e., ∃n'. n' ∈ PeelNodes and (n', n) ∈ DE^p. The exclusion of peeled nodes from the worklist is justified by the observation that these statements are present in P_N but not in P_{N−1}. Therefore, these must necessarily appear (possibly with modifications) in ∂N, and no additional analysis is needed to identify these statements or variables/arrays updated by them. We also keep track of all non-peeled nodes that have been processed so far in the set ProcessedNodes, initialized in line 4.

The loop in lines 5–19 iterates over the worklist, processing one node at a time to identify affected scalar variables and arrays. We remove the node n at the head of the worklist and add it to ProcessedNodes in lines 6–7. If μ^p(n) is an assignment statement, we conservatively consider the scalar variable or array updated at n to be potentially affected (marked in line 9). We also add all as-yet unprocessed nodes n' that have a data dependence on n to the worklist in line 11. This accounts for nodes that are potentially affected because they use a value that is generated at node n. If node n corresponds to a conditional branch statement, we conservatively consider all non-peeled nodes n' that are reachable from n in the CFG of P_N^p as potentially affected by N. If such a node n' hasn’t been processed yet, we add it to the worklist in line 15. Finally, if n corresponds to a loop head, we skip the identification of affected variables from n (line 17). This is justified since the special form of loops allowed by the grammar in Fig. 13 permits only loop counter variables to be updated in a loop head, and loop counter variables are not relevant for the post-conditions we wish to prove. The overall set of affected scalar variables/arrays is iteratively computed until there are no nodes left in the worklist to process. Since the CFG of P_N^p has only a finite number of nodes, and since no node is processed more than once (thanks to the book-keeping done using the set ProcessedNodes), the loop in lines 5–19 is guaranteed to terminate.

Lemma 9 Let P_N^p be a peeled program fed as input to the function COMPUTEFFECTED. Let n be a node in P_N^p such that some scalar variable/array in uses(n) is transitively data/control dependent on N or on a peeled node in P_N^p. Then n is added to WorkList during the execution of function COMPUTEFFECTED.

Proof A transitive data/control dependence as referred to in the lemma can be represented by a sequence of nodes n_{i1}, n_{i2}, ..., n_{ik} (= n), where either (i) k = 1 and N ∈ uses(n) or (ii) k > 1 and uses(n_{i_k}) is data/control dependent on n_{i_{k−1}}, for 2 ≤ j ≤ k. By Lemma 8, in case (ii), there is an edge from n_{i_{k−1}} to n_{i_j}, for 2 ≤ j ≤ k, in the data dependence graph computed at line 1 of function COMPUTEFFECTED. We now prove the claim by induction on k.

We consider two base cases of the induction. If k = 1, we know that N ∈ uses(n). Hence, n is added to WorkList in line 3 of the function COMPUTEFFECTED. If k = 2, either n_{i_1} is a peeled node or N ∈ uses(n_{i_1}). In the former case, n = n_{i_2} is added to WorkList in line 3 of COMPUTEFFECTED. Otherwise, n_{i_1} is added to WorkList in line 3 and must be removed from WorkList in a later iteration before function COMPUTEFFECTED terminates. In the iteration in which n_{i_1} is removed from WorkList, the node n = n_{i_2} is
added to WorkList either due to data dependence (line 11 of COMPUTEAffected) or control dependence (line 15 of COMPUTEAffected) of uses(n) on n_{i_k}.

We next hypothesize that, for every transitive data/control dependence on N or on a peeled node, represented by the sequence of nodes (n_{i_1}, n_{i_2}, ..., n_{i_k}), where 2 \leq j \leq k - 1, the node n_{i_j} is added to WorkList during the execution of COMPUTEAffected.

For the inductive step, consider a transitive data/control dependence on N or on a peeled node, represented by the sequence of nodes (n_{i_1}, n_{i_2}, ..., n_{i_k}), where n_{i_k} = n. This implies that uses(n_{i_{k-1}}) is also data/control dependence on N or on a peeled node. Now by the inductive hypothesis, n_{i_{k-1}} must be added to WorkList in some iteration of the loop in lines 5–19 of COMPUTEAffected. In the iteration in which n_{i_{k-1}} is removed from WorkList, the node n = n_{i_k} is added to WorkList either due to data dependence (line 11 of COMPUTEAffected) or control dependence (line 15 of COMPUTEAffected) of uses(n_{i_k}) on n_{i_{k-1}}. □

In order to study additional properties of function COMPUTEAffected, we need to introduce some additional notation. Given a peeled program P^t_N generated by PEELALLLOOPS, let P^*_N denote the program obtained by removing the peels of all loops from P^t_N. Clearly, P^*_N is identical to the un-peeled program P_N (fed as input to PEELALLLOOPS), but with all instances of N in upper bound expressions of loops replaced by N − 1. The program P^*_N is also closely related, but not identical, to P_{N-1}. Indeed, P_{N-1} has all occurrences of N (not just those appearing in upper bound expressions of loops) in P_N replaced by N − 1, whereas only upper bound expressions of loops are modified to obtain P^*_N. As an example, the loop in Fig. 19a in the program P_N transforms to the loop in Fig. 19b in the program P^*_N and to the loop in Fig. 19c in the program P_{N-1}.

Since there is a bijection between the nodes in the CFGs of P_N and P_{N-1}, and similarly between the nodes in the CFGs of P_N and P^*_N, there exists a bijection between the nodes in the CFGs of P_{N-1} and P^*_N as well. It is also easy to see that since programs generated by the grammar in Fig. 13 only allow loop bound expressions that depend on constants and N, if L and L' are corresponding loops in P_{N-1} and P^*_N, respectively, then both L and L' iterate exactly the same number, i.e., k_L(N − 1), times.

Let n_0, n_1, n_2, ..., n_l denote nodes in the CFG of P_{N-1}, and let n'_{0}, n'_{1}, n'_{2}, ..., n'_{l} denote the corresponding nodes (per the bijection) in the CFG of P^*_N. For notational convenience, we let n_0 and n_1 be the start and end nodes, respectively, of the CFG of P_{N-1}, and similarly for n'_0 and n'_1. Let σ denote an arbitrary initial state, i.e., valuation of all scalar variables and arrays, from which we wish to start executing P_{N-1} and P^*_N. Since programs generated by the grammar in Fig. 13 are deterministic, there is exactly one control flow path, say π, in the CFG of P_{N-1} that corresponds to the execution of P_{N-1} starting from σ. A similar argument holds for P^*_N, and let π' be the corresponding path in its CFG. In the following discussion, we use n_{i_j} to denote the j-th node starting from n_0 in π, where i_0 = 0 and 0 ≤ j < |π|.

The interpretation of n'_{i_j} in the context of π' is analogous.

**Lemma 10** Let π (resp. π') be the path in the CFG of P_{N-1} (resp. P^*_N) that corresponds to the execution of P_{N-1} (resp. P^*_N) starting from the state σ. Let  \hat{\pi} : (n_0, n_1, ..., n_{i_j}) be a prefix of the path π. Suppose, upon termination of COMPUTEAffected, no conditional branch node in  \hat{\pi} is present in ProcessedNodes. Then, (n'_{0}, n'_{1}, ..., n'_{i_j}) must be a prefix of π'.

**Proof** Recall that the CFGs of P_{N-1} and P^*_N are identical (including all loop bounds) except possibly for the usage of N or an expression involving N in some conditional branches and/or assignment statements. Note that by definition, none of these CFGs contain any peeled nodes. Since π and π' start at corresponding nodes n_0 and n'_0 in the respective CFGs, every subsequent node n_{i_j} in π must be matched by the corresponding node n'_{i_j} in π', until a branch node is encountered along one of the paths and the branch condition potentially depends on N. To prove the lemma, it therefore suffices to show that no conditional branch node n_{i_j} in  \hat{\pi} has any transitive data/control dependence on N. Let n_{i_k} be a conditional branch node in  \hat{\pi}. By Lemma 9, if any scalar variable/array in uses(n_{i_k}) is transitive data/control dependent on N, then n_{i_k} must be added to WorkList at some point during the execution of function COMPUTEAffected. Consequently, n_{i_k} must also be removed from WorkList (line 6) and added to ProcessedNodes (line 7) before COMPUTEAffected terminates. However, this violates the premise of the claim, i.e., n_{i_k} is not present in ProcessedNodes on termination of COMPUTEAffected. Therefore, no scalar variable/array in uses(n_{i_k}) can be transitive data/control dependent on N. This completes the proof of the lemma. □
Lemma 11 Let $P_N^p$ be a peeled program fed as input to the function ComputeAffected. Let $vA$ be a scalar variable/array that is absent in AffectedVars when ComputeAffected terminates. If $P_{N-1}$ and $P_N^p$ are executed starting from the same state $\sigma$, then $vA$ has the same value on termination of both programs.

Proof We prove the lemma by contradiction. If possible, let $\sigma$ be a state (i.e., valuation of variables and arrays) such that $vA$ has different values on termination of $P_{N-1}$ and $P_N^p$, when both programs are executed starting from $\sigma$. As before, we use $\pi$ and $\pi'$ to denote the paths in the CFGs of $P_{N-1}$ and $P_N^p$, respectively, that correspond to the execution of the respective programs starting from $\sigma$. Note that by definition, none of these CFGs contain any peeled nodes. We consider the following cases.

- If none of $\pi$ and $\pi'$ updates $vA$, the value of $vA$ at the end of execution of the two programs is the same as the value it had in $\sigma$. Clearly, the lemma holds in this case.
- Suppose node $n_{ij}$ in $\pi$ updates $vA$. We define $\text{Branch}(\pi, n_{ij})$ to be the set of all nodes $n_{ik}$ in the prefix of $\pi$ ending at $n_{ij}$ such that $n_{ik}$ corresponds to a conditional branch statement. Similarly, $\text{Dep}(\pi, n_{ij})$ is defined to be the set of all nodes $n_{ik}$ in the same prefix of $\pi$ such that there is a path through data dependency edges in $\text{Dep}(\pi)$ from node $n_{ik}$ to either $n_{ij}$ or to one of the nodes in $\text{Branch}(\pi, n_{ij})$.

If possible, let $n_{ik}$ be a node in $\text{Branch}(\pi, n_{ij})$ such that the branch condition in $\mu^n(n_{ik})$ has a (possibly transitive) data dependence on $N$. By Lemma 9, $n_{ik}$ must be added to WorkList sometime during the execution of ComputeAffected. Since $n_{ij}$ is reachable from $n_{ik}$ along $\pi$, this further implies that $n_{ij}$ must be added to WorkList, and subsequently $vA \in \text{def}(n_{ij})$ must be added to AffectedVars during the execution of ComputeAffected. However, we know that $vA$ is not present in AffectedVars on termination of ComputeAffected. Therefore, no branch condition in any node $n_{ik}$ in $\text{Branch}(\pi, n_{ij})$ can be transitively data dependent on $N$. It follows that no node in $\text{Branch}(\pi, n_{ij})$ can be added to WorkList during the execution of ComputeAffected. Hence, none of them can be present in ProcessedNodes on termination of ComputeAffected. By Lemma 10, it now follows that if $(n_{il_0}, n_{i_1}, \ldots, n_{i_l})$ is a prefix of $\pi$, then $(n'_{il_0}, n'_{i_1}, \ldots, n'_{i_l})$ must be a prefix of $\pi'$.

Since the scalar variable/array $vA$ is updated at $n_{ij}$ in $\pi$, the statement at $n'_{ij}$ in $\pi'$ must also update $vA$. Therefore, (i) $vA$ is updated at $n'_{ij}$ in $\pi'$, and (ii) for every node $n_{ik}$ in $\text{Dep}(\pi, n_{ij}) \cup \text{Branch}(\pi, n_{ij})$, the corresponding node $n'_{ik}$ is present in $\text{Dep}(\pi', n'_{ij}) \cup \text{Branch}(\pi', n'_{ij})$.

Finally, we argue that no node in $\text{Dep}(\pi, n_{ij})$ can be transitively data dependent on $N$. Indeed, if this was not the case, by Lemma 9, $n_{ij}$ would be added to WorkList during the execution of ComputeAffected, and hence $vA$ would be added to AffectedVars. However, this violates the premise that $vA$ is absent in AffectedVars. Combining this with the result obtained above, we find that as far as path $\pi$ is concerned, no node in $\text{Dep}(\pi, n_{ij}) \cup \text{Branch}(\pi, n_{ij})$ is transitively data dependent on $N$. Since modifications, if any, in statements at corresponding nodes of $P_{N-1}$ and $P_N^p$ only involve replacing $N - 1$ by $N$, such modifications preserve the dependence of every node on $N$. Therefore, no node in $\text{Dep}(\pi', n'_{ij}) \cup \text{Branch}(\pi', n'_{ij})$ transitively depends on $N$. This implies that the statements labeling nodes in $\text{Dep}(\pi, n_{ij}) \cup \text{Branch}(\pi, n_{ij})$ in $\pi$ are identical to the statements labeling the corresponding nodes in $\text{Dep}(\pi', n'_{ij}) \cup \text{Branch}(\pi', n'_{ij})$.

Since both $P_{N-1}$ and $P_N^p$ start from the same state $\sigma$, the values of $vA$ computed by $P_{N-1}$ after executing the sequence of statements corresponding to $(n_{i_0}, n_{i_1}, \ldots, n_{i_l})$ must therefore be identical to that computed by $P_N^p$ after executing the sequence of statements corresponding to $(n'_{i_0}, n'_{i_1}, \ldots, n'_{i_l})$. This proves the lemma.

- The case when node $n'_{ij}$ in $\pi'$ updates $vA$ is analogous to the above case. \qed

For a variable/array $vA$ that is not identified as affected, $vA$ cannot be in the def set of a non-peeled node that either (i) has a transitive data dependence on $N$ or on a peeled node, or (ii) has a control flow path from a branch node that, in turn, has a transitive data dependence on $N$ or on a peeled node. The following lemma formalizes this property.

Lemma 12 Let $P_N^p$ be a peeled program fed as input to the function ComputeAffected. Let $vA$ be a scalar variable/array that is absent in AffectedVars after ComputeAffected terminates. Then, $vA \notin \text{def}(n)$ for every non-peeled node $n$ in $P_N^p$ such that some scalar variable/array in uses($n$) is transitively data/control dependent on $N$ or on a peeled node in $P_N^p$.

Proof Consider an arbitrary non-peeled node $n$ in $P_N^p$ such that some scalar variable/array in uses($n$) has a transitive data/control dependence on $N$ or on a peeled node. Then, by Lemma 9, $n$ must be added to WorkList sometime during the execution of ComputeAffected. Consequently, $n$ must also be removed from WorkList (line 6 of ComputeAffected). If $n$ is an assignment node, then def($n$) is added to the set AffectedVars (line 9 of ComputeAffected). But since $vA$ is absent in AffectedVars, it follows that $vA \notin \text{def}(n)$. \qed

For clarity of exposition, we will henceforth refer to the property formalized in Lemma 12 as the “not-affected” property in our arguments.
Example 5 Consider the peeled program in Fig. 17 along with its DDG in Fig. 18. Recall that nodes in the DDG are named such that node \( n_i \) corresponds to the statement at line \( i \) of the peeled program. The value of variable \( S \) computed in the peeled node (line 5 in Fig. 17) of the first loop is used to define array \( A1 \) in the body of the second loop (line 7). Furthermore, the value of variable \( S \) computed in line 5 is used to initialize the value of \( S1 \) in line 10. Thus, the algorithm initializes the worklist with the non-peeled nodes \( n_7 \) and \( n_10 \) that are data dependent on the peeled node \( n_5 \). Array \( A1 \) and variable \( S1 \) updated at \( n_7 \) and \( n_10 \), respectively, are marked as affected in line 9 of \( \text{COMPUTE}_\text{AFFECTED} \). We then add non-peeled nodes that have a data dependence on \( n_7 \) and \( n_10 \) to the worklist in line 11 of the algorithm. Since array \( A1 \) is used in node \( n_{12} \) in the third loop to define the variable \( S1 \), \( n_{12} \) is added to the worklist. Subsequently, the variable \( S1 \) updated at \( n_{12} \) is marked as affected. No further non-peeled nodes have any data dependence on \( n_{12} \) and the worklist therefore becomes empty. Function \( \text{COMPUTE}_\text{AFFECTED} \) therefore terminates with \([A1, S1] \) as the set of potentially affected variables/arrays. Note that variable \( S \) updated in the first loop (line 3 of Fig. 17) is not marked as affected since its value doesn’t transitively depend on \( N \) or on any variable/array updated in peeled nodes.

4.5 Generating the difference program \( \partial P_N \)

We now have most of the ingredients to generate \( \partial P_N \) from \( P_N \) such that \( P_N \) and \( P_{N-1} \) differ by \( \partial P_N \) have the same effect on scalar variables and arrays of interest. For notational convenience, in the remainder of this subsection, we use \( P_N \) to denote the renamed version of a given program, and \( P_N^p \) to denote the peeled version of the renamed program.

Generating the difference program \( \partial P_N \) from the renamed and peeled program using the set of affected variables computed previously is still a daunting task. In order to help the reader better visualize the computation of difference program as well as to simplify the proof of correctness, we present steps involved in the computation of \( \partial P_N \) as a sequence of simple program transformations. Figure 20 presents a high level overview of this sequence of transformations. We start with a peeled program \( P_N^p \). We first canonicalize it to a program \( T_N^p \) that consists of a sequence of statements of a specific form (explained in Sect. 4.5.1). The statements in \( T_N^p \) corresponding to the peels of loops in \( P_N^p \) are then moved to the end of \( T_N^p \) to obtain the program \( T_N^m \). The resulting program \( T_N^m \) can be viewed as the program \( P_N^m \) followed by the peels of all loops in \( P_N^p \). We call the block of statements corresponding to the peels of all loops as \( \text{Peel}(P_N) \). Finally, if variables/arrays of interest are not those identified as affected by \( \text{COMPUTE}_\text{AFFECTED} \), \( P_N^p \) can be replaced with \( P_{N-1}^p \). This allows us to obtain the difference program as \( \text{Peel}(P_N) \). In the subsequent sections, we present each transformation in detail, describe the programs generated by them, and prove that they preserve the overall semantics of the program as far as the variables/arrays of interest are concerned. This allows us to show that for a large class of programs wherein the variables/arrays of interest have specific properties, it is possible to use just the peels of loops in \( P_N^p \) as the difference program. This simplifies the computation of \( \partial P_N \) significantly.

We continue to use \( vA \) to denote a scalar variable or an array depending on the context. If \( vA \) is an array, the discussion below applies to every individual element \( vA[j] \), where \( j \) is an index in the allowed range of indices of array \( vA \). However, for notational convenience, we use \( vA \) (and not \( vA[j] \)) to refer to such an array element in the lemmas below. Note that this implies that the proof, once completed, applies to an arbitrary element of the array \( vA \), and hence to the whole of \( vA \).

4.5.1 Canonicalizing the program

In this section, we describe a simple transformation of the program \( P_N^p \) that allows us to view the program as a linear sequence of statements of a specific form. The transformation is only meant for purposes of simplifying the proofs of lemmas in the subsequent subsections and making them more approachable.

For every program \( P_N \) that can be generated by the grammar shown in Sect. 3, we rewrite the corresponding peeled program \( P_N^p \) as a linear sequence of statements of the form:

\[
\text{if}(C) \text{ then } S \text{ else skip,}
\]

where \( \text{skip} \) is shorthand for the assignment statement \( x = x \); for an arbitrary scalar variable \( x \) in \( P_N^p \). The program fragment \( S \) is either (i) a loop, or (ii) a peel of a loop, or (iii) an assignment statement outside loops and peels in \( P_N^p \). The conditional expression \( C \) is a conjunction of Boolean expressions along the True (resp. False) branches of all the conditional branch nodes \( b \) within the scope of which the program fragment \( S \) occurs in the program \( P_N^p \). Since skip
does not change values of any variables or arrays, we omit the else part in our subsequent discussion for notational clarity. Henceforth, we refer to statements in this form as guarded statements.

We now describe how to construct a program $T_N^p$ consisting of a sequence of guarded statements starting from a given peeled program $P_N^p$. We have already seen earlier how to construct the collapsed CFG of $P_N^p$. For each loop that has been peeled, we additionally collapse all nodes in the peel of the loop to a single node to obtain an even more collapsed CFG $G_C$. By the restrictions imposed by our grammar, $G_C$ is necessarily a directed acyclic graph. We first assign a topological index to all nodes of $G_C$ such that the index of a node $n$ is strictly larger than the indices of all nodes that have edges to $n$ in $G_C$. For each node $n$ in order of its topological index, we now construct a guarded statement for it as follows. If $n$ does not lie within the scope of any branch statements, we construct the guarded statement “if(True) then $S$” where $S$ corresponds to the loop, peel or assignment statement at node $n$. Otherwise, we conjoin the conditional expressions along the True (resp. False) branches of all the conditional branch nodes $b$ within the scope of which node $n$ lies in $G_C$. If this conjoined expression is $C$, then we generate the guarded statement “if($C$) then $S$” where $S$ is as previously defined. We illustrate the above construction on an example program with loops and branches.

**Example 6** Consider the peeled program $P_N^p$ shown in Table 1(a). The variables and arrays of the input program are renamed (as described in Sect. 4.1) ensuring that along each path of the program the value of a variable or an array element is accessible till the end of the path and all the loops in the program are peeled (as described in Sect. 4.2). We number the lines in the program such that the statements in loops (and in peels) have the same line number but with an alphabet appended when multiple statements are present. This numbering allows us to refer to the program statements with a consistent line number even after collapsing loops and peels. $P_N^p$ has three peeled loops, $L_1$ (lines 4a and 4b), $L_2$ (lines 8a and 8b) and $L_3$ (lines 11a and 11b). Loops $L_1$ and $L_2$ along with their peels are within the scope of the conditional branch statement in line 2. Loop $L_3$ and its peel are not within the scope of any branch statement.

Table 1(b) shows the program $T_N^p$ generated by our transformation. To distinguish the statements in $P_N^p$ from those in $T_N^p$, the line numbers of statements in $T_N^p$ are suffixed with a prime symbol. As can be seen, $T_N^p$ is a linear sequence of guarded statements. The line numbers of each guarded statement in $T_N^p$ match the line number of the corresponding statement in $P_N^p$. Notice that every assignment statement in $P_N^p$ appears exactly at one unique location in $T_N^p$.

Note that if a variable/array element is used in the conditional expression at a branch node $b$, then it must have been updated (if at all) in an assignment node that has a control flow path to $b$. From the no-overwriting property as stated in Lemma 4, it now follows that the same variable/array element cannot be updated in any node that has a control flow path from $b$. This includes all nodes within the scope of branch $b$. This interesting property allows us to prove the following lemma.

**Lemma 13** Let $T_N^p$ be the canonicalized version of $P_N^p$. Let $n$ be a node corresponding an assignment statement in the CFG of $P_N^p$ (and hence $T_N^p$). Let $\pi$ (resp. $\pi'$) be a path in the collapsed CFG of $P_N^p$ (resp. $T_N^p$) starting from the state $\sigma$. Then the following hold.

1. $n$ is reached along $\pi$ iff $n$ is also reached along $\pi'$.
2. The program state $\sigma_n$ is computed at node $n$ along $\pi$ iff the program state $\sigma_n$ is computed at node $n$ along $\pi'$.

**Proof (Sketch)** We consider a path $\pi$ in the collapsed CFG of $P_N^p$ corresponding to an execution starting from $\sigma$. Let $n$ be a node corresponding to an assignment statement along path $\pi$. Let $\pi'$ be the prefix of $\pi$ that ends at $n$. We prove by induction on the length of $\pi'$ that $n$ is also reached along $\pi'$ and the program state computed at $n$ along $\pi'$ is the same as the state at $n$ along $\pi$. The proof crucially uses the fact that the guards of all statements in $T_N^p$ that do not correspond to statements in nodes along $\pi$ evaluate to False. The reasons for this are (i) every such guard has a conjunct that is the negation of some branch condition $b$ that evaluates to True along $\pi$, (ii) the consequence of the no-writing property stated above, and (iii) sequencing of statements in topological index order in $T_N^p$. In particular, (ii) and (iii) above ensure that the values of all variables/array elements used in the branch node $b$ along $\pi$ are the same as the corresponding values used in $b$ along $\pi'$.

The converse direction of the proof is similar. □

4.5.2 Reordering the peels

We now reorder the statements in the program $T_N^p$ such that all guarded statements corresponding to peels of loops are executed after all other guarded statements. However, the relative ordering among the guarded statements corresponding to peels is preserved. We use $T_N^p$ to denote the program obtained after this reordering.

**Example 7** Continuing with the canonicalized program $T_N^p$ shown in Table 1(b), the reordered program $T_N^p$ is shown in Table 1(c). The line numbers follow the same pattern described in Example 6. Notice that in $T_N^p$, the guarded statements corresponding to peels at lines 5', 9' and 12' appear after all other statements in the program.

Let $vA$ be a variable/array in $P_N^p$ that is not identified as affected by function COMPUTEFFECTED. The following
lemmas establish that if programs $T_p^N$ and $T_o^N$ are executed from the same state $\sigma$, they always compute the same value of $v_A$. Specifically, Lemma 14 shows that all data dependencies that potentially have a bearing on the value of $v_A$ are the same in $T_p^N$ and $T_o^N$. Lemma 15 uses this to show that the value of $v_A$ computed by $T_p^N$ and $T_o^N$ are the same. For clarity of exposition in the following discussion, when we say that there is a data dependence path from $n_i$ to $n_j$ in a program, we mean that there is a path from $n_i$ to $n_j$ in the DDG of the program. Similarly, when we say that there is a control dependence path from $n_i$ to $n_j$, we mean that there is a data dependence path from $n_i$ to a conditional branch node within whose scope $n_j$ lies.

**Lemma 14** Let $v_A$ be a scalar variable/array that is absent in $\text{AffectedVars}$ when $\text{ComputeAffected}$ is invoked on $P_p^N$. Let $n$ be a node in $P_p^N$ such that $v_A \in \text{def}(n)$. For every node $n'$ in $P_p^N$, there is a data/control dependence path from $n'$ to $n$ in $T_p^N$ if there exists such a path in $T_o^N$.

**Proof** Consider nodes $n$ and $n'$ in $P_p^N$ (hence also in $T_p^N$ and $T_o^N$). We consider two cases.

1. Suppose there is a data/control dependence path from $n'$ to $n$ in $T_p^N$. From our construction of $T_p^N$, we know that there exists such a data/control dependence path in $P_p^N$ as

---

Table 1 (a) Program $P_p^N$, (b) Transformed Program $T_p^N$, (c) Transformed & Reordered Program $T_o^N$ and (d) Program $P_o^N$ / Program $P_p^N$: Peel($P_p^N$)

| (a) Program $P_p^N$ | (b) Transformed Program $T_p^N$ | (c) Transformed & Reordered Program $T_o^N$ | (d) Program $P_o^N$ / Program $P_p^N$: Peel($P_p^N$) |
|---------------------|-------------------------------|------------------------------------------|------------------------------------------|
| 1. $S = 10$;       | 1'. if(True) $S = 10$;       | 1. $S = 10$;                            |
| 2. if($S > 5$) {    | 3'. if($S > 5$) $S_1 = 1$;   | 2a. if($S > 5$) {                       |
| 3. $S_1 = 1$;      | 4'. if($S > 5$) Loop L1;    | 3. $S_1 = 1$;                           |
| 4a. for($i = 0$; $i < N - 1$; $i ++$) //Loop L1 | 5'. if($S > 5$) Peel of L1;  | 4a. for($i = 0$; $i < N - 1$; $i ++$) //Loop L1 |
| 4b. $A_1[i] = A[i] + 1$; | 6'. if(!($S > 5$)) $S_1 = 20$; | 4b. $A_1[i] = A[i] + 1$; |
| 5. $A_1[N-1] = A[N-1] + 1$; //Peel of L1 | 7'. if(!($S > 5$)) Loop L2; | 6a. } else { |
| 6. } else {       | 8'. if(!($S > 5$)) Loop L2; | 7. $S_1 = 20$; |
| 7. $S_1 = 20$;     | 9'. if(!($S > 5$)) Peel of L2; | 8a. for($i = 0$; $i < N - 1$; $i ++$) //Loop L2 |
| 8a. for($i = 0$; $i < N - 1$; $i ++$) //Loop L2 | 10a. } | 8b. $A_1[i] = A[i]$; |
| 8b. $A_1[i] = A[i]$; //Peel of L2 | 11a. for($i = 0$; $i < N - 1$; $i ++$) //Loop L3 | 10. } |
| 11a. for($i = 0$; $i < N - 1$; $i ++$) //Loop L3 | 12a. $A_2[N-1] = A[N-1] + S_1$; //Peel of L3 | 11b. $A_2[i] = A_1[i] + S_1$; |
| 12. $A_2[N-1] = A[N-1] + S_1$; //Peel of L3 |
well. We now show that such a data/control dependence path also exists in \( T'_{N} \) by considering two sub-cases.

(a) Suppose \( n \) is a non-peeled node in \( P'_{N} \). Since \( vA \) is not identified as affected, by the not-affected property (Lemma 12), we have that \( n' \) is not a peeled node. Therefore, both \( n \) and \( n' \) are non-peeled nodes. Since the relative ordering of all non-peeled nodes is preserved by our reordering transformation, the data/control dependence between \( n \) and \( n' \) continues to exist in \( T'_{N} \) as well.

(b) Suppose \( n \) is a peeled node in \( P'_{N} \). If \( n' \) is also a peeled node, then since the relative ordering among the peeled nodes is preserved by reordering, the data/control dependence between \( n \) and \( n' \) continues to exist in \( T'_{N} \). On the other hand, if \( n' \) is a non-peeled node, since all non-peeled nodes precede all peeled nodes after reordering, the data/control dependence exists in \( T'_{N} \) in this case as well.

2. Suppose there is a data/control dependence path from \( n' \) to \( n \) in \( T'_{N}^{o} \). We show that such a dependence path exists in \( T'_{N}^{p} \) by considering the following sub-cases.

(a) If both \( n' \) and \( n \) are non-peeled (resp. peeled) nodes, then since reordering does not change the relative ordering of the non-peeled (resp. peeled) nodes, the dependence is present in \( T'_{N}^{o} \).

(b) The case where \( n \) is a non-peeled node and \( n' \) is a peeled node cannot arise, since all non-peeled nodes appear before peeled nodes in \( T'_{N}^{o} \).

(c) If \( n' \) is a non-peeled node and \( n \) is a peeled node, there are two further sub-cases. If \( n' \) is ordered before \( n \) in \( T'_{N}^{o} \) then the dependence is present in \( T'_{N}^{p} \) as well. Otherwise, we ask if the dependence from \( n' \) to \( n \) in \( T'_{N}^{p} \) is a read-after-write or write-after-write. Since \( n \) is ordered before \( n' \) in \( T'_{N}^{p} \), by the no-overwriting property (Lemma 4), both \( n \) and \( n' \) cannot update the same renamed variable. Therefore, the dependence from \( n' \) to \( n \) in \( T'_{N}^{p} \) cannot be read-after-write, and hence must be write-after-write. This requires a variable/array element in \( \text{uses}(n) \) to also be present in \( \text{def}(n') \). Such a variable/array element must have been updated (if at all) prior to its use in node \( n \). Once again, by the no-overwriting property, this variable/array element cannot be updated by \( n' \), which appears after \( n \) in \( T'_{N}^{p} \). This completes the proof.

**Proof** Follows from Lemma 14 and the fact that reordering does not change the individual guarded statements in the canonicalized program \( T'_{N}^{p} \).

4.5.3 De-canonicalizing the reordered program

Recall that our aim is to decompose the program \( P_{N} \) into two program fragments, the program \( P_{N-1} \) and the difference program \( \partial P_{N} \). We have seen above that the reordering step already achieves the purpose of moving the guarded statements corresponding to peels to the end of the program, providing a good candidate for the difference program \( \partial P_{N} \). However, the part of the reordered program that precedes the guarded statements corresponding to peels may not have syntactic similarity with \( P_{N-1} \) in general. In order to remedy this situation, we now “undo” the canonicalization process (as described in Sect. 4.5.1) that allowed us to view the program \( P_{N} \) as a linear sequence of guarded statements. Specifically, we transform the guarded statements back to statements of the form that were present in \( P_{N}^{o} \) to begin with. We do this separately for the guarded statements corresponding to peels, and for the part of \( T'_{N} \) that precedes these, so that we obtain a program fragment that is syntactically similar to \( P_{N-1} \) followed by a difference program. In the subsequent discussion, we call the resulting de-canonicalized program \( P'_{N} \).

**Example 8** Consider the reordered program \( T'_{N} \) from the example shown in Table 1(c). The program \( P'_{N} \) shown in Table 1(d) is obtained by de-canonicalization. The line numbers follow the pattern similar to the program \( P_{N} \) as described in Example 6. It is worth noticing that, the statements corresponding to peels of loops appear after all other statements in \( P_{N}^{o} \) and part of program \( P_{N}^{o} \) that precedes the peels is syntactically similar to \( P_{N-1} \).

Notice that after de-canonicalization, there may be more conditional branch nodes in the CFG of \( P'_{N} \) as compared to the CFG of \( P_{N}^{o} \) but fewer conditional branch nodes as compared to the CFG of \( T'_{N}^{o} \).

**Lemma 16** Let \( P'_{N} \) be the de-canonicalized version of \( T'_{N}^{o} \). Let \( n \) be a node corresponding an assignment statement in the CFG of \( T'_{N}^{o} \) (and hence \( P'_{N} \)). Let \( \pi \) (resp. \( \pi' \)) be a path in the collapsed CFG of \( T'_{N}^{o} \) (resp. \( P'_{N} \)) starting from the state \( \sigma \). Then the following hold.

1. \( n \) is reached along \( \pi \) iff \( n \) is also reached along \( \pi' \).
2. The program state \( \sigma_{n} \) is computed at node \( n \) along \( \pi \) iff the program state \( \sigma_{n} \) is computed at node \( n \) along \( \pi' \).

**Proof** The proof is similar to that shown in Lemma 13.
4.5.4 Peels of loops as the difference program

Recall from Sect. 4.4 that \( P_N^* \) is effectively \( P_N \) with the peels removed. This is exactly what we get by de-canonicalizing the part of \( T_N^* \) that precedes the guarded statements corresponding to peels. If we call the de-canonicalized version of the guarded statements corresponding to peels as \( \text{Peel}(P_N^p) \) then \( P_N^p \) can be written as \( P_N^* \); \( \text{Peel}(P_N^p) \).

It turns out that \( \text{Peel}(P_N^p) \) can be constructed directly from \( P_N \) without having to go through canonicalization, reordering and de-canonicalization. We now describe how to do this. Recall from Sect. 4.2 that \( \text{PeelNodes} \) denotes the set of peeled nodes in \( P_N^p \). Let \( \text{CondNodes} \) be the set of all non-peeled conditional branch nodes \( b \) such that there is a peeled node \( n \) within the scope of the branch \( b \). In other words, if \( d \) denotes the immediate post-dominator of \( b \), there is a path from \( b \) to \( d \) that passes through \( n \). We define \( \text{Locs} \) to be the set \( \text{PeelNodes} \cup \text{CondNodes} \). Only these nodes in the CFG of \( P_N^p \) are relevant for the construction of \( \text{Peel}(P_N^p) \). Therefore, we construct \( \text{Peel}(P_N^p) \) by replacing the labels of all other nodes in the CFG of \( P_N^p \) by skip. Recall that skip is a syntactic shorthand for \( x = x \); for a variable \( x \), as discussed before. Since a sequence of skip statements can be collapsed without changing the program semantics, we use \( \text{Peel}(P_N^p) \) to denote the program obtained after this optimization.

**Example 9** Consider the peeled program \( P_N^p \) shown in Table 1(a). The program has a non-peeled conditional branch statement on line 2. The peeled statements on lines 6 and 13 are within the scope of the conditional branch statement on line 2. Thus, \( \text{CondNodes} = \{2\} \) and \( \text{Locs} = \{2b, 5, 6b, 9, 10b, 12\} \). The program \( \text{Peel}(P_N^p) \) is the program fragment consisting of the nodes in the set \( \text{Locs} \) in Table 1(d). Notice that the non-peeled conditional branch node in \( P_N^p \) (on line 2) that has the peeled nodes within its scope is retained in \( \text{Peel}(P_N^p) \) along with the peels of loops.

**Lemma 17** Let \( P_N^p \) be a peeled program and let \( vA \) be a scalar variable/array in \( P_N^p \) that is absent in \( \text{AffectedVars} \) when \( \text{COMPUTEAFFFECTED} \) is executed on \( P_N^p \). If \( P_N^p \) and \( P_N^* \); \( \text{Peel}(P_N^p) \) are executed from the same state \( \sigma \), then \( vA \) has the same value on termination of both programs.

**Proof** We break the proof in two parts. We first show that if \( P_N^p \) and \( P_N^* \); \( \text{Peel}(P_N^p) \) are executed starting from the same state \( \sigma \), then \( vA \) has the same value on termination of both programs. This follows easily from Lemmas 13, 15, and 16.

Next, we show that if \( P_N^* \); \( \text{Peel}(P_N^p) \) and \( P_N^* \); \( \text{Peel}(P_N^p) \) are executed from the same state \( \sigma \), then \( vA \) has the same value on termination of both programs. We prove this part by case analysis.

Suppose the last update to \( vA \) in \( P_N^* \); \( \text{Peel}(P_N^p) \) happens in a non-peeled node \( n \) in \( P_N^* \). Then, the proof follows immediately from Lemma 11.

Suppose the last update to \( vA \) in \( P_N^* \); \( \text{Peel}(P_N^p) \) happens in a peeled node \( n \) in \( P_N^p \). Let \( S \) denote the set of variables/arrays \( vA \) such that the updated value of \( vA \) at node \( n \) depends on the values of each \( vA' \in S \) on termination of \( P_N^* \). There are two sub-cases to consider.

If \( vA' \in S \) is identified as affected by \( \text{COMPUTEAFFFECTED} \), then by Lemma 11 the value of every such \( vA' \) is the same after termination of \( P_N^* \) and \( P_N^* \). This implies that the value of \( vA \) is also same after termination of \( P_N^p \); \( \text{Peel}(P_N^p) \) and \( P_N^* \); \( \text{Peel}(P_N^p) \).

Now consider the case where some \( vA' \in S \) is identified as affected by \( \text{COMPUTEAFFFECTED} \). Let \( L \) be a loop in \( P_N^p \) from which the node \( n \) is peeled. From the construction of peeled nodes, we know that for every node \( n \) in the peel of \( L \) there is a corresponding node \( n' \) in the “uncollapsed” body of loop \( L \) such that the \( \text{def} \) and \( \text{uses} \) sets of the two nodes \( n \) and \( n' \) coincide. Since the update to \( vA \) at node \( n \) depends on \( vA' \) that is identified as affected, the update to \( vA \) at node \( n' \) in loop \( L \) must also depend on the affected variable/array \( vA' \).

However, this would cause \( \text{COMPUTEAFFFECTED} \) to identify \( vA \) as an affected variable. This leads to a contradiction since we know \( vA \) is not affected. This completes the proof.

**Lemma 17** allows us to use \( \text{Peel}(P_N^p) \) as the difference program \( \partial N \) if none of the scalar variables and arrays of interest are identified as affected by \( \text{COMPUTEAFFFECTED} \). This holds true in the case where the post-condition \( \psi^r(N) \) does not refer to any affected variable/array. Note that using \( \text{Peel}(P_N^p) \) as the difference program \( \partial N \) works even if there are other variables/arrays (not of interest) that are affected. However, if some of our variables/arrays of interest are indeed identified as affected by \( \text{COMPUTEAFFFECTED} \), we must include additional code in the difference program that effectively “rectifies” the values of affected variables as computed by \( \partial N \). We elaborate on this in the next subsection.

4.5.5 Generalized computation of difference programs

Recall that \( \text{Peel}(P_N^p) \) was constructed by replacing some of the nodes in the collapsed CFG of \( P_N^p \) with skip and by simplifying the resulting CFG. As seen above, this suffices to serve as the difference program \( \partial N \) if none of the variables/arrays of interest are identified as affected by function \( \text{COMPUTEAFFFECTED} \). If, however, some variables/arrays of interest are identified as affected, \( \text{Peel}(P_N^p) \) as computed above may no longer serve as a correct difference program. To see an example of this, consider the peeled program in Fig. 17. If we were to compute \( \text{Peel}(P_N^p) \) for this program we would get the program fragment in lines 12–14 shown in Fig. 21. In this program, array \( A1 \) and scalar variable \( S1 \) are identified as affected. Notice that, the program \( P_N \); \( \text{Peel}(P_N) \) in Fig. 21 computes incorrect values of \( A1 \) at line 6 and \( S1 \) at line 14.
// assume(∀ i ∈ [0, N) A[i] = 1)
1. S = 0;
2. for (i = 0; i < N − 1; i++) {
3.   S = S + A[i];
4. }
5. for (i = 0; i < N − 1; i++) {
6.   A[i] = A[i] + S;
7. }
8. S1 = S;
9. for (i = 0; i < N − 1; i++) {
10.  S1 = S1 + A[i];
11. }
12. S = S + A[N − 1];
13. A[N − 1] = A[N − 1] + S;
14. S1 = S1 + A[N − 1];

// assert(S1 = N × (N+2))

Fig. 21 Hoare triple with program 𝑃_{N−1}; Peel(𝑃_{N})

Interestingly, even in cases like the above example, a correct difference program can often be constructed by modifying the way in which Peel(𝑃_{N}) is constructed. To prevent confusion, we do not call the program resulting from this modified construction as Peel(𝑃_{N}). Instead we call it ∂𝑃_{N}. The modification referred to above concerns which statements are replaced by skip and which are retained but possibly with a change. Specifically, all assignment statements that update an affected variable/array but were earlier (while constructing Peel(𝑃_{N})) replaced by skip are retained with a possibly changed expression in the right hand side of the assignment. Since no new nodes are added to the CFG of 𝑃_{N} in this way of constructing ∂𝑃_{N}, there is a natural injective mapping, say β, from the nodes in the CFG of ∂𝑃_{N} to those in the CFG of 𝑃_{N}.

To understand how the right hand side expressions of assignments may need to be changed when constructing ∂𝑃_{N}, consider an execution of each of 𝑃_{N} and 𝑃_{N−1}; ∂𝑃_{N} starting from the same initial state σ.

Definition 1 For every node 𝑛 in the CFG of ∂𝑃_{N} and for every variable/array element 𝑣A we say that 𝑣A has a rectified value at 𝑛 if its value at 𝑛 matches the value of 𝑣A at β(𝑛). Otherwise, we say that 𝑣A has an unrectified value at 𝑛.

For every node 𝑛 in ∂𝑃_{N} that updates an affected variable/array 𝑣A of interest, we modify the right hand side of the assignment (if necessary) such that the right hand side expression evaluates to the rectified value of 𝑣A at 𝑛. This expression is constructed in such a manner that it uses the unrectified value of 𝑣A at 𝑛 (if the assignment statement was replaced by a skip) in its computation of the rectified value.

This construction allows us to establish an important property of the resulting program ∂𝑃_{N}: every variable/array 𝑣A of interest has its rectified value at every node 𝑛 in ∂𝑃_{N}.

We now elaborate on how we construct the modified right hand side expression of an assignment statement at node 𝑛 in ∂𝑃_{N} that updates the affected variable/array 𝑣A. We assume that we have access to the rectified and unrectified values of all variables/arrays 𝑣A replaced by a possibly changed expression in the right hand side of the assignment statement at node β(𝑛) in 𝑃_{N}. The easiest way to do this would be to construct the right hand side expression exclusively in terms of the rectified values of 𝑣A at node 𝑛. Note that, this results in a difference program that is as complex as the original program 𝑃_{N}. This defeats our purpose, since full-program induction can succeed only if ∂𝑃_{N} is “simpler” than 𝑃_{N}. Therefore, we do not use this naive method and present an operator algebra to compute of the rectified value of 𝑣A in the assignment statement in terms of its unrectified value and the “difference” between the rectified and unrectified values of other variables/arrays that have a data dependence to 𝑣A.

Let ◦ be a binary operator on a set 𝑆 that denotes the domain of values of variables/arrays in 𝑃_{N}. We say that 𝑒 is the right identity element of ◦ if 𝑣 ◦ 𝑒 = 𝑣 and 𝑒 is the left identity element if 𝑒 ◦ 𝑣 = 𝑣 for each 𝑣 ∈ 𝑆. We call 𝑣⁻◦ the right inverse element of 𝑣 under ◦ if 𝑣 ◦ 𝑣⁻◦ = 𝑒 and we call it the left inverse element if 𝑣⁻◦ ◦ 𝑣 = 𝑒 for each 𝑣 ∈ 𝑆. We say that ◦ is an associative operator if (𝑢 ◦ 𝑣) ◦ 𝑤 = 𝑢 ◦ (𝑣 ◦ 𝑤). We say that ◦ is a commutative operator if 𝑢 ◦ 𝑣 = 𝑣 ◦ 𝑢.

For the following lemmas, we assume that ◦ is an associative operator, there exists a left identity element under ◦ in 𝑆 and each element in 𝑆 has a right inverse under ◦.

Lemma 18 Let 𝑛 be a node in ∂𝑃_{N} such that the statement at β(𝑛) in 𝑃_{N} is 𝑢 := 𝑣 ◦ 𝑤. Suppose 𝑤 is an affected variable/array of interest. Let 𝑤_{N−1}, 𝑢_{N−1}, 𝑣_{N−1} denote the values of 𝑤, 𝑢 and 𝑣 at the end of execution of 𝑃_{N−1}. Let 𝑤_{N}, 𝑢_{N}, 𝑣_{N} be the rectified values of 𝑤, 𝑢 and 𝑣 at node 𝑛. Then the rectified value of 𝑤 is computed as 𝑤_{N} := 𝑤_{N−1} ◦ ((𝑣_{N−1})⁻◦ ◦ ((𝑢_{N−1})⁻◦ ◦ 𝑢_{N}) ◦ 𝑣_{N}).

Proof We proceed as follows:
1. 𝑤_{N} := 𝑒 ◦ 𝑤_{N}
2. 𝑤_{N} := (𝑤_{N−1} ◦ (𝑤_{N−1})⁻◦) ◦ 𝑤_{N}
3. 𝑤_{N} := 𝑤_{N−1} ◦ ((𝑤_{N−1})⁻◦ ◦ 𝑤_{N})
4. 𝑤_{N} := 𝑤_{N−1} ◦ ((𝑢_{N−1} ◦ 𝑣_{N−1})⁻◦ ◦ (𝑢_{N} ◦ 𝑣_{N}))
5. 𝑤_{N} := 𝑤_{N−1} ◦ ((𝑣_{N−1})⁻◦ ◦ (𝑢_{N−1})⁻◦ ◦ (𝑢_{N} ◦ 𝑣_{N}))
6. 𝑤_{N} := 𝑤_{N−1} ◦ ((𝑣_{N−1})⁻◦ ◦ ((𝑢_{N−1})⁻◦ ◦ 𝑢_{N}) ◦ 𝑣_{N})

Suppose ◦ is additionally a commutative operator. Then the following lemmas hold.
Lemma 19  Under the assumptions stated in Lemma 18, the rectified value of \( w \) is computed as \( w_N := w_{N-1} \circ (u_N \circ (u_{N-1})^{-\circ}) \circ (v_N \circ (v_{N-1})^{-\circ}). \)

Proof  We continue from the proof of Lemma 18 and proceed as follows:

\[
\begin{align*}
6. & \quad w_N := w_{N-1} \circ ((v_{N-1})^{-\circ} \circ ((u_{N-1})^{-\circ} \circ u_N) \circ v_N) \\
7. & \quad w_N := w_{N-1} \circ (((u_{N-1})^{-\circ} \circ u_N) \circ (v_{N-1})^{-\circ} \circ v_N) \\
8. & \quad w_N := w_{N-1} \circ ((u_{N-1})^{-\circ} \circ u_N) \circ ((v_{N-1})^{-\circ} \circ v_N) \\
9. & \quad w_N := w_{N-1} \circ (u_N \circ (u_{N-1})^{-\circ}) \circ (v_N \circ (v_{N-1})^{-\circ}) \quad \Box
\end{align*}
\]

To use the equation from Lemma 19 for statements with non-commutative operators such as \( \{ -, + \} \) often used in practice, we perform a simple transformation that allows us to use commutative operators in place of non-commutative ones. As an example of this transformation, consider the expressions \( u + (-v) \) and \( u \times (1/v) \). This allows us to use the equation in Lemma 19 when for every element \( v \in S \), the elements \(-v\) and \( 1/v \) are also in \( S \).

Lemma 20  Let \( n \) be a node in \( \partial P_N \) such that the statement at \( \beta(n) \) in \( P_N \) is \( w := w \circ v \). Then the rectified value of \( w \) is computed as \( w_N := w_N \circ (w_N \circ (w_{N-1})^{-\circ}) \) along with the presumption \( w \circ v = w_{N-1} \).

Proof  Using the given presumption \( w_N = w_{N-1} \), we have the definition of the identity element as: \( e = w_N \circ (w_N)^{-\circ} = w_N \circ (w_{N-1})^{-\circ}. \)

As the first step, we use the result from Lemma 19 and proceed as follows:

\[
\begin{align*}
1. & \quad w_N := w_{N-1} \circ (w_N \circ (w_{N-1})^{-\circ}) \circ (v_N \circ (v_{N-1})^{-\circ}) \\
2. & \quad w_N := w_{N-1} \circ (w_N \circ (w_{N-1})^{-\circ}) \\
3. & \quad w_N := w_{N-1} \circ (u_N \circ (u_{N-1})^{-\circ}) \\
4. & \quad w_N := w_{N-1} \circ (v_N \circ (v_{N-1})^{-\circ}) \quad \Box
\end{align*}
\]

It is worth noting that the rectification described in Lemmas 18, 19, and 20 applies not only when the set \( S \) is integers, i.e., integers are stored as array elements but even when the set consists of matrices, vectors, and polynomials. When matrices are stored as array elements, such arrays are called tensors. These are extensively used in machine learning algorithms. Further, it applies to interesting operators such as \( 0, + \text{ mod } x, \times \text{ mod } y \) as well as to other interesting algebraic structures. It is also worth mentioning that for a restricted class of programs our technique extends to computing the differences of programs that manipulate heaps.

The routine PROGRAMDIFF presented in Algorithm 4 shows how the difference program is computed. In line 1, we peel each loop in the program \( P_N \) and collect the list of peeled nodes using Algorithm 1. We then compute the set of affected variables using Algorithm 3 (line 2). The difference program \( \partial P_N \) inherits the skeletal structure of the peeled program \( P'_N \) after peeling each loop (line 3). Next, we collapse all nodes and edges in the body of each loop into a single node identified with the loop head in the CFG of \( P'_N \) using the function COLLAPSELOOPBODY in line 4. The collapsed CFG of the resulting program \( \partial P_N \) is a DAG with finitely many paths. We then initialize a worklist of CFG nodes with \( n_{start} \) in line 5.

The while loop in lines 7–31 performs a breadth-first top-down traversal over the DAG of \( \partial P_N \) starting from the node \( n_{start} \) and processes one node at a time. We first remove a node \( n \) from the worklist in line 8. We store the nodes that are already processed by our algorithm in ProcessedNodes (that is initialized in line 6). We add the node \( n \) removed from the worklist to ProcessedNodes in line 9. Next, the loop in lines 10–12 appends each successor \( n' \) of \( n \) to the worklist that is not already processed. We use the routine SUCC to obtain the list of successors of node \( n \).

If \( n \) is a peeled node, then we retain it as is in the difference program (line 13). Otherwise, we check if any scalar variable/array used at node \( n \) is affected at line 14. We have defined the sub-routine HASAFFECTEDVARS that checks if the scalar variable/array defined at node \( n \) is affected.

For nodes \( n \) that refer to an affected variable/array, we do the following. We check if a node \( n \) is a glue node that refers to an affected variable/array in line 15 and retain such nodes as is in the difference program. Otherwise, we check if the node \( n \) corresponds to a loop head in line 16. We uncollapse the nodes corresponding to a loop head that represent the entire loop in line 17. We assume that the sub-routine NODES(L) returns the set of CFG nodes in loop \( L \). Next, the loop in lines 18–23 iterates over all nodes \( n' \) in the body of \( L \) and process one node at a time. In line 19, we check if the variable/array updated at node \( n' \) is affected using function HASAFFECTEDVARS, and compute its rectified value in line 20, using the function NODEDIFF. If the variable/array defined at \( n' \) is not identified as affected, then we remove from \( \partial P_N \) nodes \( n' \) that do not update an affected variable/array using the routine REMOVENODE in line 21. For a non-peeled node \( n \) that does not correspond to a loop head, we compute the rectified value of an affected variable/array defined at \( n \) in line 24, using the function NODEDIFF.

For nodes \( n \) that are not peeled nodes and do not update an affected variable/array, we do the following. We compute the set CondNodes of conditional branch nodes that have at least one peeled node within its scope in line 27. In line 28, we remove from \( \partial P_N \) nodes \( n \) (including collapsed loop nodes) that do not update an affected variable/array and are not in the set CondNodes, using the routine REMOVENODE, as they do not need any rectification.

The sub-routine NODEDIFF computes the statements that rectify values of variables/arrays updated at a node. It determines the type of statement (assignment, aggregation, or
Algorithm 4 \textsc{ProgramDiff}(\textit{Locs}, \textit{CE}, \mu): renamed program \textit{P}_N, \textit{GlueNodes}: set of glue nodes

\begin{algorithmic}[1]
\State {\((\text{Locs}^\prime, \text{CE}^\prime, \mu^\prime), \text{PeelNodes}\) := \textsc{PeelAllLoops}(\textit{Locs}, \textit{CE}, \mu);}
\State \textit{AffectedVars} := \textsc{ComputeAffected}(\textit{Locs}^\prime, \textit{CE}^\prime, \mu^\prime), \text{PeelNodes};
\State \partial\textit{P}_N := (\textit{Locs}^\prime, \textit{CE}^\prime, \mu^\prime), \text{where \textit{Locs}^\prime := Locs}^\prime, \textit{CE}^\prime := \textit{CE}^\prime, \text{and} \mu^\prime := \mu^\prime;
\State \partial\textit{P}_N := \textsc{CollapseLoopBody}(\partial\textit{P}_N);  \quad \triangleright \text{Collapse nodes and edges of each loop into its loop-head}
\State \textit{WorkList} := (\partial\textit{P}_N);  \quad \triangleright \text{Add the start node to the worklist}
\State \textit{ProcessedNodes} := \emptyset;
\While {\text{\textit{WorkList}}} is not empty
\State Remove a node \textit{n} from head of \textit{WorkList};
\State \textit{ProcessedNodes} := \textit{ProcessedNodes} \cup \{n\};
\For {each node \textit{n}’ \in \textit{Succ}(n) \setminus \textit{ProcessedNodes}}
\State \textit{WorkList} := \textsc{AppendToList}(\textit{WorkList}, \textit{n}’);
\EndFor
\If {\textit{n} \in \textit{PeelNodes}} \textbf{continue}; \quad \triangleright \text{Difference computation not required}
\ElseIf {\textsc{HasAffectedVars}(n, \textit{AffectedVars})} \textbf{continue}; \quad \triangleright \text{Retain the glue loop}
\ElseIf {\textit{n} is a loop-head}
\State \textit{L} := \textsc{CollapseLoopBody}(\textit{n}); \quad \triangleright \text{Uncollapse the loop-head}
\For {each node \textit{n}’ \in \textit{Nodes}(\textit{L})}
\If {\textsc{HasAffectedVars}(n’, \textit{AffectedVars})} \textbf{then}
\State \mu’(n’) := \textsc{NodeDiff}(n’, \mu, \textit{AffectedVars});
\Else \partial\textit{P}_N := \textsc{RemoveNode}(n’, \partial\textit{P}_N); \quad \triangleright \text{No affected variables at node \textit{n}’}
\EndIf
\EndFor
\Else \mu’(n) := \textsc{NodeDiff}(n, \mu, \textit{AffectedVars});
\EndIf
\EndIf
\EndIf
\EndWhile
\State \textbf{return} \partial\textit{P}_N;
\EndAlgorithm

\textsc{HasAffectedVars}( \textit{n}: node, \textit{AffectedVars}: set of affected variables )
\begin{algorithmic}[1]
\If {\exists \textit{A} such that \textit{v} \textit{A} \in \textit{def}(\textit{n}) and \textit{v} \textit{A} \in \textit{AffectedVars}} \textbf{return} \textbf{True};
\Else \textbf{return} \textbf{False};
\EndIf
\EndAlgorithm

\textsc{NodeDiff}( \textit{n}: node, \mu: node labelling function, \textit{AffectedVars}: set of affected variables )
\begin{algorithmic}[1]
\If {\mu(n) is of the form \textit{w}_N := \textit{r}_N \circ \textit{r}_N} \textbf{then}
\State \textbf{return} \textit{w}_N := \textit{w}_{N-1} \circ (\textit{r}_N \circ (\textit{r}_N^{-1})) \circ (\textit{r}_N \circ (\textit{r}_N^{-1}))^{-1}; \quad \triangleright \text{Refer Lemma 19}
\ElseIf {\mu(n) is of the form \textit{w}_N := \textit{w}_N \circ \textit{r}_N \circ \textit{r}_N^{-1}} \text{wherein \textit{w}_N is a scalar} \textbf{then}
\State \textbf{return} \textit{w}_N := \textit{w}_N \circ (\textit{r}_N \circ (\textit{r}_N^{-1}))^{-1}; \quad \triangleright \text{Refer Lemma 20}
\Else\If {\exists \textit{v} \in \textit{use}(\textit{n}) \text{ and } \textit{v} \in \textit{AffectedVars} \lor (C_N \neq C_N^{-1} is satisfiable) \textbf{then}}
\State \textbf{throw} “Branch conditions in \textit{P}_N and \textit{P}_{N-1} may not evaluate to same value”;
\Else \textbf{return} \mu(n); \textbf{end if}
\EndIf
\EndIf
\EndAlgorithm

To explain the intuition behind the steps of Algorithm 4, we use the convention that all variables and arrays of \textit{P}_{N-1} have the suffix _\textit{N}m1 (for N-minus-1), while those of \textit{P}_N have the suffix _\textit{N}. This allows us to express variables/arrays of \textit{P}_N in terms of the corresponding variables/arrays of \textit{P}_{N-1} in a systematic way in \partial\textit{P}_N, given that the intended composition is \textit{P}_{N-1} \circ \partial\textit{P}_N.

For assignment statements, we compute the rectified values as follows. For every assignment statement of the form \( v = E \) \textit{in} \textit{L}, a corresponding statement is generated in \partial\textit{P}_N.

Branch condition (at the given node and acts accordingly. For assignment statements, we compute the rectified value as shown in Lemma 19 and for aggregating statements, we compute the rectified value as shown in Lemma 20. For the nodes representing a conditional branch in \partial\textit{P}_N, we determine if its conditional expression evaluates to the same value in \textit{P}_N and \textit{P}_{N-1}. If so, the conditional branch is retained as is in \partial\textit{P}_N. Otherwise, currently our technique cannot compute \partial\textit{P}_N and we report a failure using the \textbf{throw} statement.
that expresses \( v_N \) in terms of \( v_{N-1} \) and the difference (or ratio) between versions of variables/arrays that appear as sub-expressions in \( E \) in \( P_{N-1} \) and \( P_N \).

While the implementation is currently restricted to simple arithmetic operators (\(+\), \(-\), \(\times\), \(\div\)), specifically for the ease of implementation and its use in practice, as previously stated, our rectification method is general and applies to several operators beyond the ones mentioned here. The following example shows the computation of rectified values of variables/arrays updated in simple program statements.

**Example 10** The statement \( A_N[i] = B_N[i] + v_N \), in \( P_N \) gives rise to the statement \( A_N[i] = A_{N-1}[i] + (B_N[i] + (-B_{N-1}[i]) + (v_N + (-v_{N-1})) \); in \( \partial P_N \) that rectifies the value of \( A_N[i] \). Similarly, the statement \( A_N[i] = B_N[i] \times v_N \); in \( P_N \) gives rise to the statement \( A_N[i] = A_{N-1}[i] \times (B_N[i] \times (1/B_{N-1}[i]) \times (v_N \times (1/v_{N-1})) \); under the assumption \( B_{N-1}[i] \times v_{N-1} \neq 0 \).

The program \( P_N \) may have statements that aggregate/accumulate values in scalars. This kind of statement requires special processing when generating the difference program \( \partial P_N \). The next example shows the computation of rectified values of variables/arrays in statements that accumulate values in scalar variables.

**Example 11** Consider the loop \( \text{for}(i=0; i<N; i++) \) \{ \( \text{sum}_N = \text{sum}_N + A_N[i]; \) \} in program \( P_N \). The difference \( A_N[i] + (-A_{N-1}[i]) \) is aggregated over all indices from 0 through \( N-2 \). In this case, the loop in \( \partial P_N \) that rectifies the value of \( \text{sum}_N \) has the following form: \( \text{sum}_N = \text{sum}_{N-1}; \text{for} (i=0; i<N-1; i++) \{ \text{sum}_N = \text{sum}_N + (A_N[i] + (-A_{N-1}[i])); \} \). A similar aggregation for multiplicative ratios can also be shown.

Conditional branch statements pose a considerable challenge to the computation of difference programs. A branch condition may evaluate to different outcomes in \( P_N \) and \( P_{N-1} \), for the same value of \( N \). When this happens, programs \( P_N \) and \( P_{N-1} \) execute totally unrelated blocks of statements. In such situations, it is immensely difficult to rectify the values of variables/arrays computed along the unrelated branches, and hence, our algorithm avoids doing so. Only when we can determine that the condition evaluates to the same value in \( P_N \) and \( P_{N-1} \), we rectify values of variables/arrays computed along the corresponding branches. Next, we present examples with branch conditions to highlight this.

**Example 12** Consider the conditional branch statement \( \text{if}(t3 == 0) \) in line 10 of Fig. 1. The branch condition evaluates to the same value in \( P_N \) and \( P_{N-1} \) because the condition has no dependence on \( N \). Therefore, the branch statement is used as is during the computation of the difference program. However, recall that since the arrays accessed in the program are not affected, none of the loops are retained in the difference program shown in Fig. 3.

Consider another conditional branch statement \( \text{if}(A[i] == N) \) in \( P_N \). The corresponding statement in \( P_{N-1} \) is \( \text{if}(A[i] == N-1) \). Clearly, the conditions in these statements do not evaluate to the same value in \( P_N \) and \( P_{N-1} \). Thus, our algorithm flags a failure to compute the difference program and terminates.

There are programs where conditional branch statements with dependence on \( N \) evaluate to the same value. For example, consider the program \( P_N \) in Fig. 22a. The program \( P_{N-1} \) is shown in Fig. 22b. While the branch condition (indirectly) depends on the value of \( N \), it evaluates to the same value in \( P_N \) and \( P_{N-1} \), since the amount of change in the value of variables \( x \) and \( y \) used in the branch condition is same. Our algorithm can successfully compute the difference program in such cases.

The restriction on branch conditions that use affected variables/arrays can be further relaxed by handling the case when the condition evaluates to \( \text{True} \) in \( P_{N-1} \) and to \( \text{False} \) in \( P_N \) by restoring the values of variables/arrays to their values at the predecessor of the branch node. However, when a branch condition evaluates to \( \text{False} \) in \( P_{N-1} \) and to \( \text{True} \) in \( P_N \), the entire computation within the branch has to be performed again in \( \partial P_N \) instead of just executing the rectification code. For example, consider the branch statement, \( \text{if}(i < N) \) \text{Loop};. If the branch condition \( i < N-1 \) in \( P_{N-1} \) evaluates to \( \text{False} \), then the condition \( i < N \) in \( P_N \) definitely evaluates to \( \text{True} \). This will require the difference program to execute the entire computation performed by the code fragment \text{Loop}; and not just the difference program corresponding to \text{Loop};. This will hamper the progress guarantees on the class of programs that our technique can.
verify. Hence, we currently avoid handling these cases in the algorithms and consider them as a part of future work.

We now prove the soundness of the routine PROGRAMDIFF from Algorithm 4. For the following lemma, we assume that \( \partial P_N \) is the difference program computed when function PROGRAMDIFF is invoked on the renamed program \( P_N \). Suppose both \( P_N^p \) and \( P_{N-1} \); \( \partial P_N \) are executed from the same initial state \( \sigma \). We assume \( \pi : (n_0, n_1, \ldots, n_k) \) to be the path in the CFG of \( \partial P_N \) corresponding to the execution of the difference program from the state obtained after \( P_{N-1} \); \( \partial P_N \). We assume \( \pi' : (n_0', n_1', \ldots, n'_k) \) to be the corresponding path in the CFG of the peeled program \( P_N^p \).

**Lemma 21** For every node \( n_j \) in \( \pi \), the rectified values of all variables/array elements used at \( n_j \) during the execution of \( \partial P_N \) are identical to the values of the same variables/array elements at the corresponding node \( n_j' \) during the execution of \( P_N^p \).

**Proof** If \( vA \) is not identified as an affected variable/array by function COMPUTEFFECTED, the result follows from the proof of Lemma 17 and the no-overwriting property of renaming.

If \( vA \) is identified as an affected variable, we induct on the length of \( \pi \). The only difference in this case is that we also need to consider the assignment statements modified by function NODEDIFF at lines 20 and 24 of function PROGRAMDIFF. Lemmas 18, 19, and 20 guarantee the correctness of the rectified value of \( vA \) computed by these additional statements, given the unrectified value of \( vA \) and rectified and unrectified values of all variables and array elements used in the right hand side of the assignment. By the inductive hypothesis, the rectified values of the latter set of variables and array elements are indeed available. By the no-overwriting property, the unrectified values of \( vA \) and all other variables and arrays used in the right hand side of the assignment are also available. Therefore, the correct rectified value of \( vA \) is computed at each node in \( \pi \).

Finally, note that once a rectified value is generated at a non-glue node in the difference program, renaming ensures that it is not re-defined by subsequent statements in the difference program. Therefore, rectified values, once computed in the difference program, are available for use at subsequent nodes in the execution path. Putting the above parts together completes the proof.

**Theorem 2** \( \partial P_N \) generated by PROGRAMDIFF is such that, for all \( N > 1 \), \( (\psi(N)) P_{N-1} ; \partial P_N \ (\psi(N)) \) holds iff \( (\psi(N)) P_N \ (\psi(N)) \) holds.

**Proof** Lemma 6 guarantees that \( (\psi(N)) P_N \ (\psi(N)) \) holds iff \( (\psi(N)) P_N^p \ (\psi(N)) \) holds. Furthermore, Lemma 21 ensures that for every state \( \sigma \) satisfying \( \psi(N) \), if we execute \( P_N^p \) and \( P_{N-1} ; \partial P_N \) starting from \( \sigma \), then if the final state after termination of \( P_N^p \) satisfies \( \psi(N) \), so does the final state after termination of \( P_{N-1} ; \partial P_N \). This proves the theorem.

**Example 13** We illustrate the difference computation performed by the routine PROGRAMDIFF in Algorithm 4 on our running example. Figure 23 shows the difference program obtained after executing the algorithm on the program in Fig. 17. Notice that while some program statements in Fig. 23 are syntactically similar to corresponding statements in Fig. 17, the additional statements (e.g., at lines 3, 6, and 8) in Fig. 23 have no syntactic counterpart in Fig. 17. For the first loop, since variable \( S \) is not affected, only the peeled iteration is retained. Since \( A1 \) and \( S1 \) both are affected, the statements in the second and the third loop are replaced with statements that rectify their values, along with inserting the peeled statements for both loops. Initialization of variable \( S1 \) is also replaced with the statement that rectifies its value since it depends on the value of \( S \) computed in a peel.

### 4.6 Simplifying the difference program

While we have described a simple strategy to generate a difference program \( \partial P_N \) above, this may lead to unoptimized as well as redundant statements in the naively generated difference program. Our implementation aggressively optimizes \( \partial P_N \) and removes redundant code, renaming variables/arrays as needed. The routine SIMPLIFYDIFF in Algorithm 5 simplifies program statements that compute rectified values, removes redundant loops from the difference program and substitutes loops with the summarized statements computed using acceleration. This helps in \( \partial P_N \) having fewer and simpler loops in a lot of cases. Below, we describe these optimizations and illustrate them using examples.

Since the generation of statements that compute rectified values is not fully optimized, these statements may have
Algorithm 5 \textsc{SimplifyDiff}((Locs, C, E, μ): difference program \(\partial P_N\))

1: \(\partial P_N := (\text{Locs}', C', E', \mu')\), where \(\text{Locs}' := \text{Locs}, C' := C, \text{and } \mu' := \mu; \)
2: for each loop \(L \in \text{LOOPS}(\partial P_N)\) do
3: for each node \(n \in \text{NODES}(L)\) do
4: \(\mu'(n) := \text{Simplify}(\mu'(n)); \quad \triangleright \text{Simplify the statement} \)
5: end for
6: \(\text{if body of } L \text{ is of the form } w_N := w_N \oplus \text{expr}, \text{wherein } w_N \text{ is a scalar variable} \quad \triangleright c \text{ is the label of the edge from } n_1 \text{ to } n \)
7: \(n_{\text{acc}} = \text{FreshNode}(); \quad \triangleright \text{Accelerated statement} \)
8: \(\text{end if} \)
9: \(CE' := CE' \cup (\{n_{\text{acc}}, n, c\}, (n_{\text{acc}}, n_2, \emptyset)) \backslash \{(n_1, n, c), (n, n_2, \text{ff})\} \); \(\triangleright \text{Remove redundant loops} \)
10: \(\text{end if} \)
11: \(CE' := CE' \cup (\{n_{\text{acc}}\} \backslash \text{NODES}(L)); \quad \triangleright \text{Accelerated statement} \)
12: \(\text{end if} \)
13: \(\text{end if} \)
14: \(\text{end if} \)
15: \(\text{return } \partial P_N'; \)

expressions that can be further simplified using the values computed in other statements in the generated difference program \(\partial P_N\). The function \textsc{Simplify} performs this optimization aggressively and simplifies the statements in the difference program (lines 3–4 in Algorithm 5). Let us take an example to illustrate the effect of the \textsc{Simplify} function.

Example 14 Suppose the difference program \(\partial P_N\) has statements of the form \(B_N[i] = B_{N-1}[i] + \text{expr}; \quad \text{and } v_N = \text{expr2} v_{N-1}; \) if \(\text{expr1} \) and \(\text{expr2} \) are constants or functions of \(N \) and loop counters, then expressions such as \(B_N[i] - B_{N-1}[i] \) and \(v_N/v_{N-1} \) can often be simplified from the statements in the difference program. The expression \(\text{expr1} \) is substituted for \(B_N[i] - B_{N-1}[i] \) and \(\text{expr2} \) for \(v_N/v_{N-1} \), respectively.

The difference program \(\partial P_N\) may contain loops that perform redundant computation, for example, copying values across versions of an array corresponding to \(P_N \) and \(P_{N-1} \), due to the simplification of the statements that compute its rectified value. We remove such loops from \(\partial P_N\) in lines 20–23 of Algorithm 5. Let us illustrate this with an example.

Example 15 Suppose the difference program \(\partial P_N\) has the loop \(\text{for } (i=0; i<N-1; i++) \{ \text{sum} = \text{sum} + 1; \} \) where \(\text{sum} \) can be computed from the version of array \(A \) in \(P_{N-1} \) to its version in \(P_N \), and hence, are redundant.

The difference program \(\partial P_N\) may also contain loops that compute values of variables that can be accelerated. We perform this optimization in lines 8–18 of \textsc{SimplifyDiff} in Algorithm 5. We first check if the body of a loop \(L \) is in the specific form eligible for this optimization in line 8. If so, we create a fresh node in line 9 to replace \(L \). Lines 10–16 of Algorithm 5 label the fresh node with the accelerated statement. If we encounter operators that are not supported, then we report a failure of our technique using the \texttt{throw} statement in line 15. Next, we replace the loop with the fresh node in lines 17–18. We demonstrate this optimization with the following example.

Example 16 Suppose the difference program has the loop \(\text{for } (i=0; i<N-1; i++) \{ \text{sum} = \text{sum} + (N-1); \} \). The semantics of the loop can be summarized using the accelerated statement \(\text{sum} = \text{sum} + (N-1) \). \textsc{SimplifyDiff} removes this loop from the program and introduces the accelerated statement instead.

In the following lemma, we use \(\partial P_N'\) to denote the program generated by \textsc{SimplifyDiff}.

Lemma 22 \(\psi(N) \land P_{N-1}; \partial P_N' \{\psi(N)\} \text{ holds iff } \{\psi(N)\} \land P_{N-1}; \partial P_N \{\psi(N)\} \text{ holds}. \)

Proof Follows trivially from the fact that \textsc{SimplifyDiff} in Algorithm 5 optimizes program statements, removes only redundant statements/loops, and replaces loops using semantically equivalent accelerated statements. \qed

Example 17 We illustrate the application of the simplification routine \textsc{SimplifyDiff} from Algorithm 5 on our running


4.7 Generating the difference pre-condition $\partial \varphi(N)$

We now present a syntactic routine, called $\text{SYNTACTICDIFF}$, in Algorithm 6 for generation of the difference pre-condition $\partial \varphi(N)$. Although this suffices for all our experiments, for the sake of completeness, we present later a more sophisticated algorithm for generating $\partial \varphi(N)$ simultaneously with $\text{Pre}(N)$ in Sect. 5.3.

Formally, given $\varphi(N)$, the function $\text{SYNTACTICDIFF}$ from Algorithm 6 generates a formula $\partial \varphi(N)$ such that $\varphi(N) \Rightarrow (\varphi(N - 1) \circ \partial \varphi(N))$, where $\circ$ is $\land$ when $\varphi(N)$ is a universally quantified formula and is $\lor$ when $\varphi(N)$ is an existentially quantified formula. Observe that such a $\partial \varphi(N)$ exists for universally quantified formulas $\varphi(N)$, then $\varphi(N) \Rightarrow \varphi(N - 1)$ must hold. Similarly, if such a $\partial \varphi(N)$ exists for existentially quantified formulas $\varphi(N)$, then $\varphi(N - 1) \Rightarrow \varphi(N)$ must hold. Therefore, we can use the validity of $\varphi(N) \Rightarrow \varphi(N - 1)$ and $\varphi(N - 1) \Rightarrow \varphi(N)$, as a test to decide the existence of $\partial \varphi(N)$ for universally and existentially quantified formulas, respectively.

Algorithm 6 incorporates the scenarios described above and Boolean combinations thereof. When $\varphi(N)$ is of the syntactic form $\forall i \in [0 \ldots N] \bigcirc\varphi(i)$, we first check the validity of $\varphi(N) \Rightarrow \varphi(N - 1)$ in line 2. If this test fails, we report failure using the throw statement in line 3. Otherwise, $\partial \varphi(N)$ is set to $\bigcirc\varphi(N)$ in line 5. Similarly, when $\varphi(N)$ is of the syntactic form $\exists i \in [0 \ldots N] \bigcirc\varphi(i)$, then $\partial \varphi(N)$ is set to $\bigcirc\varphi(N)$ in line 10, after checking the validity of $\varphi(N - 1) \Rightarrow \varphi(N)$ (line 7). If the test in line 7 fails, again we report failure using the throw statement in line 8. When $\varphi(N)$ is of the syntactic form $\varphi^1(N) \land \cdots \land \varphi^k(N)$, $\partial \varphi(N)$ is computed by taking the difference of each individual conjunct and disjuncting them as $\partial \varphi^1(N) \lor \cdots \lor \partial \varphi^k(N)$ (line 12).

Note that this operation results in an over-approximation of the difference pre-condition. When $\varphi(N)$ is of the form $\varphi^1(N) \lor \cdots \lor \varphi^k(N)$, $\partial \varphi(N)$ is computed by taking the difference of each individual disjunct as $\partial \varphi^1(N) \lor \cdots \lor \partial \varphi^k(N)$ (line 14). Finally, if $\varphi(N)$ does not belong to any of these syntactic forms (line 15) or if condition 2(a) of Theorem 1 is violated by the $\partial \varphi(N)$ computed in this manner (line 18), then we over-approximate $\partial \varphi(N)$ by True in lines 16 and 19.

Lemma 23 The difference pre-condition $\partial \varphi(N)$ computed by $\text{SYNTACTICDIFF}$ is such that (i) $\varphi(N) \Rightarrow (\varphi(N - 1) \circ \partial \varphi(N))$, where $\circ \in \{\land, \lor, \land\}$. (ii) $P_{N-1}$ does not modify variables/arrays in $\partial \varphi(N)$.

Proof Condition (i) follows from the checks implemented in lines 2 and 7 of function $\text{SYNTACTICDIFF}$. The check in line 18 ensures condition (ii). This concludes the proof. □

Example 18 Consider the pre-condition $\varphi(N) := \forall i \in [0, N] A[i] = 1$ from our running example in Fig. 24. The difference pre-condition computed by function $\text{SYNTACTICDIFF}$ in Algorithm 6 is $\partial \varphi(N) := A[N - 1] = 1$ shown by the assume statement in Fig. 25.

Example 19 Consider the pre-condition $\varphi(N) := \forall i \in [0, N] A[i] = 1 \lor \forall i \in [0, N] A[i] = 2$. $\text{SYNTACTICDIFF}$ in Algorithm 6 enters the recursive case in line 14. The recursive invocations with inputs $\varphi_1(N) := \forall i \in [0, N] A[i] = 1$ and $\varphi_2(N) := \forall i \in [0, N] A[i] = 2$ compute the difference pre-conditions $\partial \varphi_1(N) := A[N - 1] = 1$ and $\partial \varphi_2(N) := A[N - 1] = 2$, respectively. On returning from the recursive case, the algorithm stores the formula $A[N - 1] = 1 \lor A[N - 1] = 2 \in \partial \varphi(N)$.

Example 20 Consider the pre-condition $\varphi(N) := \exists i \in [0, N] A[i] \geq 100 \land \exists j \in [0, N] A[j] \leq 10$. The difference pre-condition computed by function $\text{SYNTACTICDIFF}$ in Algorithm 6 is $\partial \varphi(N) := A[N - 1] \geq 100 \lor A[N - 1] \leq 10$. Notice that the computed difference pre-condition is an over-approximation. Had we computed the difference pre-condition as $\partial \varphi(N) := A[N - 1] \geq 100 \land A[N - 1] \leq 10$, then it would have resulted in a contradiction.

Example 21 For pre-condition formulas $\varphi(N) := \forall i \in [0, N] A[i] = N$ and $\varphi(N) := \exists i \in [0, N] A[i] = N$ the validity checks at lines 2 and 7, respectively, in Algorithm 6 fail. Hence, the algorithm terminates without being able to compute an appropriate pre-condition.

5 Algorithms for full-program induction

In this section, we discuss the algorithms for full-program induction. The algorithm primarily focuses on generation of
Algorithm 6 SYNTACTICDIFF(\(\varphi(N)\)): pre-condition

1: if \(\varphi(N)\) is of the form \(\forall i \in [0 \ldots N] \varphi(i)\) then
2:   if \(\varphi(N) \rightarrow \varphi(N - 1)\) is invalid then
3:     throw “Unable to compute the difference pre-condition”;  
4:   end if
5:   \(\partial \varphi(N) := \hat{\varphi}(N)\);
6: else if \(\varphi(N)\) is of the form \(\exists i \in [0 \ldots N] \varphi(i)\) then
7:   if \(\varphi(N - 1) \rightarrow \varphi(N)\) is invalid then
8:     throw “Unable to compute the difference pre-condition”;  
9:   end if
10:  \(\partial \varphi(N) := \hat{\varphi}(N)\);
11: else if \(\varphi(N)\) is of the form \(\varphi^1(N) \wedge \cdots \wedge \varphi^k(N)\) then
12:  \(\partial \varphi(N) := \text{SYNTACTICDIFF}(\varphi^1(N)) \vee \cdots \vee \text{SYNTACTICDIFF}(\varphi^k(N))\);
13: else if \(\varphi(N)\) is of the form \(\varphi^1(N) \vee \cdots \vee \varphi^k(N)\) then
14:  \(\partial \varphi(N) := \text{SYNTACTICDIFF}(\varphi^1(N)) \vee \cdots \vee \text{SYNTACTICDIFF}(\varphi^k(N))\);
15: else
16:  \(\partial \varphi(N) := \text{True}\);  
17: end if
18: if \(P_{N-1}\) updates scalars or array elements in \(\partial \varphi(N)\) then
19:  \(\partial \varphi(N) := \text{True}\);  
20: end if
21: return \(\partial \varphi(N)\);

the three crucial components: difference program \(\partial P_N\), difference pre-condition \(\partial \varphi(N)\), and the formula \(\text{Pre}(N)\) for strengthening pre- and post-conditions. We have already seen the computation of the difference program and the difference pre-condition in Sect. 4. Before describing the algorithm for full-program induction, however, we present the strategy for computing the formula \(\text{Pre}(N)\).

5.1 Generating the formula \(\text{Pre}(N - 1)\)

We use Dijkstra’s weakest pre-condition computation to obtain \(\text{Pre}(N - 1)\) after the difference pre-condition \(\partial \varphi(N)\) and the difference program \(\partial P_N\) have been generated. The weakest pre-condition can always be computed using quantifier elimination engines in state-of-the-art SMT solvers like Z3 if \(\partial \varphi\) is loop free. In such cases, we use a set of heuristics to simplify the calculation of the weakest pre-condition before harnessing the power of the quantifier elimination engine. If \(\partial \varphi\) contains a loop, it may still be possible to obtain the weakest pre-condition if the loop doesn’t affect the post-condition. Otherwise, we compute as much of the weakest pre-condition as can be computed from the non-loopy parts of \(\partial \varphi\), and then try to recursively solve the problem by invoking full-program induction on \(\partial \varphi\) with appropriate pre- and post-conditions.

Example 22 We apply Dijkstra’s weakest pre-condition computation on the Hoare triple from our running example in Fig. 24. The Hoare triple in Fig. 25 shows the difference pre-condition \(\partial \varphi(N)\), post-condition \(\psi(N)\) and the formula \(\psi(N - 1)\) from the induction hypothesis as well as the strengthened pre- and post-condition formulas. The first application of weakest pre-condition computation generates the pre-condition \(A_1[N-1] = N+1\) on array \(A_1\). This is lifted to the quantified form \(\forall i \in [0, N] A_1[i] = N + 1\) in a natural way and is used to strengthen the post-condition. We substitute \(N\) with \(N - 1\) and rename the array to get the formula \(\forall i \in [0, N] A_1[Nm1][i] = N\), which is used to strengthen the pre-condition. Re-applying weakest pre-condition computation generates the predicates on \(S\) and \(S_Nm1\) that further strengthen the pre- and post-condition as shown in Fig. 25.

5.2 Verification by full-program induction

The basic version of full-program induction algorithm is presented as routine FPIVERIFY in Algorithm 7. The important
Algorithm 7 FPIVerify($P_N$: program, $\varphi(N)$: pre-condition, $\psi(N)$: post-condition)

1: if Base case check \{ $\varphi(1)$ \} $P_1$ [$\varphi(1)$] fails then
2:  print "Counterexample found!"
3:  return False;
4: end if

5: ($P_N$, $\varphi(N)$, $\psi(N)$, GlueNodes) := RENAME($P_N$, $\varphi(N)$, $\psi(N)$); ➤ Renaming as described in Sect. 4.1
6: $\partial \varphi(N)$ := SYNTACTICDIFF($\varphi(N)$);
7: $\partial P_N$ := PROGRAMDIFF($P_N$, GlueNodes);
8: $\partial P_N$ := SIMPLIFYDIFF($\partial P_N$); ➤ Simplify and Accelerate loops
9: $i := 0;
10: \text{Pre}_i(N) := \psi(N);
11: \text{c}_{\text{Pre}_i}(N) := \text{True}; ➤ Cumulative conjoined pre-condition
12: do
13:  if \{ $\text{c}_{\text{Pre}_i}(N - 1) \wedge \psi(N - 1) \wedge \partial \varphi(N)$ \} $\partial P_N$ \{ $\text{c}_{\text{Pre}_i}(N) \wedge \psi(N)$ \} then
14:    return True;
15:  end if
16:  $i := i + 1;
17:  \text{Pre}_i(N - 1) := \text{LOOPFREEWP}(\text{Pre}_{i - 1}(N), \partial P_N)$; ➤ Dijkstra’s WP sans WP-for-loops
18:  if no new $\text{Pre}_i(N - 1)$ obtained then
19:    if CHECKPROGRESS($\partial P_N$, $\partial P_N$) then
20:      return FPIVerify($\partial P_N$, $\text{c}_{\text{Pre}_{i - 1}}(N - 1) \wedge \psi(N - 1) \wedge \partial \varphi(N)$, $\text{c}_{\text{Pre}_{i - 1}}(N) \wedge \psi(N)$); ➤ Failed to prove by full-program induction
21:    else
22:      return False;
23:    end if
24:  end if
25:  $\text{c}_{\text{Pre}_i}(N) := \text{c}_{\text{Pre}_{i - 1}}(N) \wedge \text{Pre}_i(N)$;
26: end if
27: while Base case check \{ $\varphi(1)$ \} $P_1$ [$\varphi(1)$] passes;
28: return False; ➤ Failed to prove by full-program induction

steps of Algorithm 7 include checking conditions 3(a), 3(b), and 3(c) of Theorem 1 (lines 1, 27, and 13 resp.), calculating the weakest pre-condition of the relevant part of the post-condition (line 17), recursively invoking our routine FPIVerify with the strengthened pre- and post-conditions (line 20), and accumulating the weakest pre-condition predicates thus calculated for strengthening the pre- and post-conditions (line 25). We now discuss the algorithm in detail.

We first check the base case of the analysis in line 1. The base case of our induction reduces to checking the validity of a Hoare triple of a loop-free program. This is achieved by compiling the pre-condition, program and post-condition into a first-order logic formula. The validity of the formula can be checked with an off-the-shelf back-end SMT solver like Z3. If the check fails, we have found a valid counterexample that is reported to the user in line 2, and the algorithm terminates in line 3.

Next, we rename the variables and arrays in the program $P_N$ as well as the pre- and post-conditions (as described in Sect. 4.1) and collect the set of glue nodes (line 5). Then, in line 6, we compute the difference pre-condition $\partial \varphi(N)$ using function SYNTACTICDIFF (described in Sect. 4.7). We then compute the difference program $\partial P_N$, in line 7, using function PROGRAMDIFF from Sect. 4.5. Note that this function can compute the difference program when the scalar variables/arrays of interest are identified as affected. In line 8, we simplify the statements in the computed difference program, remove redundant statements and try to accelerate loops, if any, using function SIMPLIFYDIFF from Algorithm 5.

The do-while loop in lines 12–27 iteratively checks if the given assertion can be proved. Once the base case succeeds, we check the inductive step in line 13. If the loop terminates via the return statement in line 14, then the inductive claim has been successfully proved. Otherwise, in line 17, we compute Dijkstra’s weakest pre-condition using the formula $\text{Pre}_i(N)$, over the difference program. The formula $\text{Pre}_i(N)$ is initialized to $\psi(N)$ in line 10. We denote the computed weakest pre-condition as $\text{Pre}_i(N - 1)$. Note that, the formula $\text{Pre}_i(N - 1)$ strengthens the pre-condition and the same formula $\text{Pre}_i(N)$, but with $N$ substituted for $N - 1$, strengthens the post-condition. The variable $\text{c}_{\text{Pre}_i}(N - 1)$, initialized to True in line 11, accumulates weakest pre-condition formulas from each loop iteration (line 25).

In case no further weakest pre-conditions can be generated, checked in line 18, we recursively invoke FPIVerify on $\partial P_N$ in line 20. Prior to the recursive invocation, we check if it will be beneficial using function CHECKPROGRESS, in line 19. Discussion about CHECKPROGRESS is deferred to Sect. 6. The recursive invocation helps in situations where the computed difference program $\partial P_N$ has loops. To present an example of this scenario, we modify the program in Fig. 1 by having the statement $C[t3] = N; \text{(instead of $C[t3] = 0; \text{)} in line 10. In this case, $\partial P_N$ retains a loop that rectifies the value of $C[t3]$ corresponding to its computation in the
third loop in Fig. 1. The recursive invocation of full-program induction on ∂P_N as input for the example described here will result in a loop-free difference program. If the check in line 19 reports that further application of full-program induction will not yield any benefits then we report the failure of our technique in line 22.

When weakest pre-condition computation succeeds, we conjoin the computed strengthening predicate Pre_i(N) with the variable c_Pre_{i-1}(N) in line 25. Since the weakest pre-condition (Pre_i(N − 1) in line 17) computed in every iteration of the loop is conjoined to strengthen the inductive pre-condition (c_Pre_{i−1}(N − 1) in line 25), it suffices to compute the weakest pre-condition of Pre_{i−1}(N) (instead of c_Pre_i(N) ∧ ψ(N)) in line 17. Possibly multiple iterations of the loop in lines 12–27 are required to strengthen the pre-and post-conditions. After each iteration, the base case is checked again in line 27 with the strengthened pre- and post-conditions. If the loop terminates due to violation of the base case with the strengthened post-condition (line 27), we report the failure of our method by returning False in line 28.

Lemma 24 Upon successful termination, if function FPIVERIFY returns True, then {ψ_N} P_N {ψ_N} holds for all N ≥ 1.

Proof Verifying the given Hoare triple requires establishing the conditions mentioned in Theorem 1. The functions PROGRAMDIFF invoked in line 7 and SIMPLIFYDIFF invoked in line 8 ensure condition 1 of Theorem 1 (refer Theorem 2 and Lemma 22). The call to SYNTACTICDIFF in line 6 in FPIVERIFY computes the difference pre-conditions that satisfy conditions 2(a) and 2(b) (refer Lemma 23). The conditions 3(a) and 3(b) of Theorem 1 are checked in lines 1 and 27, respectively. The check in line 13 ensures that the return statement in line 14 executes only when condition 3(c) of Theorem 1 is ensured. Similarly, the statement in line 20 returns True only if the recursive call to FPIVERIFY proves all conditions in Theorem 1. Hence, we conclude that {ψ_N} P_N {ψ_N} holds for all N ≥ 1.

5.3 Generalized FPI algorithm

While the algorithm FPIVERIFY suffices for all of our experiments, it may not always be the case. Specifically, even if ∂P_N is loop free, the analysis may exit the loop in lines 12–27 of FPIVERIFY by violating the base-case check in line 27. To handle (at least partly) such cases, we propose the following strategy. Whenever a (weakest) pre-condition Pre_i(N − 1) is generated, instead of using it directly to strengthen the current pre- and post-conditions, we “decompose” it into two formulas Pre_i′(N − 1) and ∂ψ_i′(N) with a twofold intent: (a) potentially weaken Pre_i(N − 1) to Pre_i′(N − 1), and (b) potentially strengthen the difference formula ∂ψ(N) to ∂ψ_i′(N) ∧ ∂ψ_i(N). The checks for these intended usages of Pre_i′(N − 1) and ∂ψ_i′(N) are implemented in lines 3, 4, 5, 13 and 18 of routine FPIDEcomposeVerify, shown as Algorithm 8. This routine is meant to be invoked as FPIDEcomposeVerify(i) after each iteration of the loop in lines 12–27 of routine FPIVERIFY (so that Pre_i(N), c_Pre_i(N), etc. are initialized properly). In general, several “decompositions” of Pre_i(N) may be possible, and some of them may work better than others. FPIDEcomposeVerify permits multiple decompositions to be tried through the use of the functions NEXTDECOMPOSITION and HASNEXTDECOMPOSITION. The meaning of these two routines is intuitive from their names. Lines 21–24 of FPIDEcomposeVerify implement a simple back-tracking strategy, allowing a search of the space of decompositions of Pre_i(N − 1). Observe that when we use FPIDEcomposeVerify, we simultaneously compute a difference formula (∂ψ_i′(N) ∧ ∂ψ(N)) and an inductive pre-condition (c_Pre_{i−1}(N) ∧ Pre_i(N)).

Lemma 25 Upon successful termination, if function FPIDEcomposeVerify returns True, then {ψ_N} P_N {ψ_N} holds for all N ≥ 1.

Proof The conditions mentioned in Theorem 1 are a prerequisite to verifying the given Hoare triple. Condition 1 of Theorem 1 is ensured by difference computation (functions PROGRAMDIFF and SIMPLIFYDIFF) in FPIVERIFY. Conditions 2(a) and 2(b) are established in FPIVERIFY (via the call to function SYNTACTICDIFF) and the checks on lines 3–5 in FPIDEcomposeVerify ensure that these conditions continue to hold. Further, FPIVERIFY also ensures conditions 3(a) and 3(b) before it invokes FPIDEcomposeVerify. Now, the check in line 13 in FPIDEcomposeVerify ensures condition 3(c) of Theorem 1. Similarly, the statement in line 22 in FPIDEcomposeVerify returns True only if the recursive call to FPIDEcomposeVerify proves all the conditions in Theorem 1. Hence, we conclude that {ψ_N} P_N {ψ_N} holds for all N ≥ 1.

6 Progress measures

Recall from Sect. 5.2 that given the parameterized Hoare triple {ψ(N)} P_N {ψ(N)}, our technique recursively computes difference programs until the given post-condition ψ(N) is proved. The difference computation must eventually result in programs ∂P_N that can be easily verified without the need of further applying inductive reasoning or indicate otherwise. In this section, we define a progress measure that can be used to check if the difference computation will eventually simplify the program to the extent that it can be verified using a back-end SMT solver. The measure is based on the characteristics of the difference programs computed by our technique.

Ranking functions have been traditionally used to show program termination [33–36]. We use the notion of rank-
Algorithm 8 FPIDECOMPOSEVERIFY( i : integer )
1:  do
2:      ⟨ Pre′(N − 1), ∆ϕ′(N) ⟩ := NEXTDECOMPOSITION(Pre(N − 1));
3:      Check if (a) ∆ϕ′(N) ∧ Pre′(N − 1) → Pre(N − 1),
4:         (b) ϕ(N) → ϕ(N − 1) ⊓ (∆ϕ′(N) ∧ ϕ(N)),
5:         (c) P_{N−1} does not update any variable or array element in ∆ϕ′(N)
6:      if any check in lines 3 - 5 fails then
7:         if HASNEXTDECOMPOSITION(Pre(N − 1)) then
8:             continue;
9:         else
10:             return False;
11:         end if
12:     end if
13:     if {c_{Pre−1}(N − 1) ∧ ψ(N − 1) ∧ Pre(N − 1) ∧ ∆ϕ(N)} ∆P_{N} {c_{Pre−1}(N) ∧ ψ(N) ∧ Pre′(N)} then
14:         return True;
15:     else
16:         c_{Pre}(N) := c_{Pre−1}(N) ∧ Pre′(N);
17:         Pre(N + 1) := LOOPFREEWP(Pre′(N), ∆P_{N});
18:         if {ψ(1)} P_{1} {c_{Pre}(1) ∧ Pre(1)} holds then
19:             prev∆ϕ(N) := ∆ϕ(N);
20:             δϕ(N) := ∆ϕ′(N) ∧ δϕ(N);
21:             if FPIDECOMPOSEVERIFY(i + 1) then
22:                 return True;
23:             else
24:                 δϕ(N) := prev∆ϕ(N);
25:             end if
26:         end if
27:         i := i + 1;
28:     end if
29: while HASNEXTDECOMPOSITION(Pre(N − 1));
30: return False;

Further, the difference computation can potentially reduce the dependence on the value of N. For programs with expressions that do not directly or indirectly \(^1\) rely on N, the difference program consists of only the peeled iterations of loops. Clearly, when the difference program ∆P_{N} is impervious to the value of N, additional code to rectify the values of variables is no longer required in the subsequent recursive invocations. This indicates that we have made progress. We thus use the presence of variables in the program whose value directly or indirectly depends on the value of N as another criterion to measure progress. As previously stated in Sect. 4.4, if the value of a variable/array depends on N or on a value computed in a peeled statement, then we call such variables/arrays as affected variables/arrays. Our technique computes the set of affected variables during each recursive attempt to verify the post-condition. The difference program must rectify the values of these affected variables/arrays. When the difference program has no affected variables, the verification attempt can be terminated after the next invocation of our technique.

We also consider the complexity of expressions in the program and use it as a criteria for measuring progress. For every expression appearing in assignment statements, its expres-

\(^1\) By indirect dependence, we mean the dependence via another value computed in a peeled or non-peeled statements in the program.
expression complexity can be defined in many possible ways. Once this complexity is defined for expressions, we can take the maximum complexity of all the expressions as the expression complexity of the entire program. For programs with polynomial expressions, we can use the highest degree of the affected variables/arrays in the expression as the expression complexity. Similarly, several other criteria can be used to define the expression complexity. These include nesting levels of array indices, size/weight of the expression trees in $\partial P_N$ vis-a-vis $P_N$, number of variables, operators and constants in the expressions and so on. It is worth pointing out that such notions have been previously studied in term rewriting systems \[37,38\].

Note that each criterion discussed so far, including the number of loops, the number of affected variables, and the expression complexity, is well founded. Hence, the domain of values represented by their Cartesian product is also well-founded and represents a lexicographic ordering on the difference programs computed by our method. Progress is guaranteed if each recursive invocation of our technique in the cycle reduces this measure assigned by such a ranking function. We argue that the cycle of recursive invocations to our technique must eventually terminate, as there are no infinite descending chains of elements in the well-founded domain. We present an algorithm that can compute values from this domain on the fly and return the result of the comparison between the computed quantities. Note that no user intervention is required for checking progress.

\begin{algorithm}
\caption{CHECKPROGRESS($P_N$: program, $\partial P_N$: difference program)}
\begin{algorithmic}[1]
\STATE LoopList $\leftarrow$ LOOPS($P_N$);
\STATE LoopList' $\leftarrow$ LOOPS($\partial P_N$);
\IF {$\#\text{LoopList} < \#\text{LoopList}$}
\RETURN True;
\ENDIF
\STATE AffectedVars $\leftarrow$ COMPUTEFFECTED($P_N$);
\STATE AffectedVars' $\leftarrow$ COMPUTEFFECTED($\partial P_N$);
\IF {$\#\text{AffectedVars} < \#\text{AffectedVars}$}
\RETURN True;
\ENDIF
\STATE EC $\leftarrow$ EXPRESSIONCOMPLEXITY($P_N$);
\STATE EC' $\leftarrow$ EXPRESSIONCOMPLEXITY($\partial P_N$);
\IF {EC < EC'}
\RETURN True;
\ENDIF
\RETURN False;
\end{algorithmic}
\end{algorithm}

The routine CHECKPROGRESS in Algorithm 9 is used for checking progress after the difference program is computed. The algorithm is based on the change in the number of loops, number of affected variables, and the expression complexity of the given program $P_N$ vis-a-vis the difference program $\partial P_N$. First, we compute the number of loops in programs $P_N$ and $\partial P_N$. We compare the number of loops in $P_N$ and $\partial P_N$ in line 3. If the difference program has fewer loops than $P_N$, then we return True concluding that the $\partial P_N$ is simpler to verify than the given program. Note that, we do not consider the glue loops in the difference program that were introduced during the renaming step to copy values across versions. If the number of loops does not decrease in an invocation of our technique, we check if the number of affected variables has decreased. We compute the set of affected variables in $P_N$ and $\partial P_N$ using the routine COMPUTEFFECTED from Algorithm 3. In line 8, we compare the number of affected variables in both the programs. The algorithm returns True if the difference program has fewer affected variables than $P_N$. Subsequently, we check if the expressions in the difference program $\partial P_N$ are “simpler,” and easier to reason with, than $P_N$. We assume the availability of a routine EXPRESSIONCOMPLEXITY that can compute this complexity measure for programs $P_N$ and $\partial P_N$. In line 13 we check if the expression complexity of the difference program $\partial P_N$ is less than that of the given program $P_N$, in which case the algorithm returns True. If none of these criteria are met, then the algorithm returns False.

**Lemma 26** If CHECKPROGRESS in Algorithm 9 returns True, then the difference program $\partial P_N$ is “simpler” to verify (using the full-program induction technique) as compared to the given program $P_N$.

**Proof** The difference program $\partial P_N$ has strictly less loops than $P_N$ when the check in line 3 is satisfied. In this case, verifying $\partial P_N$ is simpler than verifying $P_N$. Further, reduction in the number of affected variables/arrays means less code is retained to rectify their values. Hence, when the check on line 8 is satisfied, $\partial P_N$ is simpler than $P_N$. By Lemma 17, when none of the variables/arrays of interest are identified as affected, only the peeled iterations of loops (referred as Peel($P_N$)) suffice as the difference program $\partial P_N$. This also makes verifying $\partial P_N$ simpler as compared to $P_N$. Similarly, the last condition ensures that the expressions in the difference program are easier to reason with than the given program $P_N$. Further, these characteristics of the program and the ordering among them as specified by CHECKPROGRESS forms a lexicographic ranking function \[36\]. Hence, these quantities are bound to reduce with each application of our technique, making the difference program simpler to verify each time. This concludes the lemma.

**Lemma 27** The routine FPIVERIFY in Algorithm 7 eventual terminates.

**Proof** Function FPIVERIFY presented in Algorithm 7 can execute in infinite recursion only when the invocation of CHECKPROGRESS in line 19 returns True infinitely often. From difference program computation, we know that the
number of loops and affected variables/arrays in the difference program $\partial P_N$ never increase beyond their counts in the given program $P_N$, they either decrease or remain the same. Further, if the expression complexity of all the statements that update an affected variable/array does not decrease then our method returns False, and consequently we report failure. Thus, these three characteristics with the specified ordering among them form a lexicographic ranking function [36]. Since the value of the lexicographic ranking function strictly decreases in each recursive application of our method, it ensures that function FPIVERIFY eventually terminates. □

### 7 Full-program induction in generalized settings

For brevity and ease of explanation, we have presented our technique in simple settings. We have so far considered Hoare triples that have a single parameter $N$. In this section, we show how our technique can be adapted to Hoare triples with multiple parameters as well as peeling loops in different directions for our inductive reasoning. We also state the limitations of our technique.

Based on the ideas previously described, our technique can already verify several interesting scenarios in programs. Our technique can verify programs that manipulate arrays of different sizes as well as loops with non-uniform termination conditions that are a linear function of $N$. It does so by computing a (possibly different) peel count for each loop that manipulates different arrays. For the ease of presentation, our algorithm computes the rectified values of variables/arrays in statements with a single operator. When the program statements have two or more operators, such statements can be split into multiple statements, by introducing temporary variables such that each statement has a single operator, and then computing the difference program using our algorithm.

**Multiple independent program parameters.** Consider proving Hoare triples with multiple parameters $N_1, N_2, \ldots, N_k$. Suppose that the values of these parameters are independent of each other. Verifying Hoare triples for all values of these parameters can be done by inducting on one program parameter at a time while keeping the other parameters fixed. We explain this with the help of a simple example with two parameters. To prove that the Hoare Triple $\{\varphi(N_1, N_2)\} P_{N_1, N_2} \{\psi(N_1, N_2)\}$ for all $N_1 \geq a \land N_2 \geq b$, we prove the following three sub-goals. First, in the base case we prove that the triple $\{\varphi(a, b)\} P_{a,b} \{\psi(a, b)\}$ holds. Second, induction over the parameter $N_1$, where we assume the Hoare Triple $\{\varphi(k, N_2)\} P_{k, N_2} \{\psi(k, N_2)\}$ holds with $k \geq a \land N_2 \geq b$, and prove the Hoare Triple $\{\varphi(k + 1, N_2)\} P_{k+1, N_2} \{\psi(k + 1, N_2)\}$, treating $N_2$ as a symbolic parameter unchanged during the induction. Third, induction over the parameter $N_2$, where we assume that the Hoare Triple $\{\varphi(N_1, l)\} P_{N_1,l} \{\psi(N_1, l)\}$ holds where $N_1 \geq a \land l \geq b$, and prove the Hoare Triple $\{\varphi(N_1, l + 1)\} P_{N_1,l+1} \{\psi(N_1, l + 1)\}$, treating $N_1$ as a symbolic parameter unchanged in the induction. This can be easily extended to Hoare triples with more than two parameters. For programs that manipulate arrays of different independent sizes, we treat each variable representing the symbolic size of arrays as a parameter. As described above, our technique verifies such programs by inducting on each parameter one at a time.

**Different direction of peeling loops.** Recall that the difference program $\partial P_N$ is sequentially composed with $P_{N-1}$. Earlier, we have been peeling the last iterations of the loops in $P_N$ so that $P_{N-1}$ and $P_N$ have the same number of iterations in each loop. However, there are programs where peeling the last iterations of loops may not be possible such that our technique can compute a difference program. In such cases, we may need to peel the initial iterations of the loops in the program. As an example, consider a loop where the value of the loop counter decreases in each iteration. A possible way is to rotate these loops to fit the template of loops defined in our grammar and then apply our technique. However, not all loops are such that they can be rotated easily using the standard loop transformation techniques. For such loops, we may need to peel it at the beginning. We peel the initial iterations of these loops and add the code that rectifies values of variables computed in the loop after the peeled iterations such that $P_{N-1}; \partial P_N$ is semantically equivalent to $P_N$. Thus, by peeling initial iterations of loops when computing the difference program, our technique can be easily adapted to programs with such loops in a sound way.

### 7.1 Limitations

There are several scenarios under which the full-program induction technique may not produce a conclusive result.

Program computation with side-effects may make it difficult to compute the difference program such that there is a clear separation between the program $P_{N-1}$ and the rest of the computation that can make up $P_N$. Computation that results in side-effects includes I/O operations, allocation, de-allocation and modification of heap memory and other operations that modify the environment which is not local to the given program. When the given program is not free of such side-effects, our technique may not be able to decompose it into $P_{N-1}$ and $\partial P_N$. Note that we only disallow the computation that impacts the post-condition to be proved. In our experience, a large class of array manipulating programs are naturally free of side-effects. In particular, the programs discussed in this paper (including Fig. 1) and those used for experimentation are free of side-effects.
Currently, our technique is unable to verify programs with branch conditions that are dependent on the parameter \( N \). Computing the difference program becomes cumbersome in such cases. This stems from the fact that the branch condition may evaluate to different outcomes in \( P_N \) and \( P_{N-1} \), for the same value of \( N \), and hence, may require us to compute the difference of two arbitrary pieces of code blocks. We identify such cases while computing the difference program in Algorithm 4 and suspend our verification attempt on line 7 of the routine NODEDIF. Note that this does not include the loop conditions, which are handled by peeling the loop. To illustrate this case, consider the Hoare triple shown in Fig. 26. The first loop in the program initializes array \( A \) and the second loop updates array \( A \) within a branch statement with the conditional expression \( N \% 2 == 0 \). It is easy to see that this branch condition will evaluate to different outcomes in \( P_N \) and \( P_{N-1} \). As a result, it is difficult for our technique to compute a difference program. Invariant generation techniques may be better suited for verifying this example. The weakest loop invariants needed to prove the post-condition are: \( \forall j \in [0,i) (A[j] = 0) \) for the first loop and \( \forall k \in [0,i) (A[k] \% 2 = N \% 2) \) for the second loop.

The difference program includes all peeled iterations of \( P_N \) that are missed in \( P_{N-1} \). Hence, our technique needs to know the symbolic upper bound on the value of the loop counter to be able to compute the number of iterations to be peeled from the program. Further, when programs have loops with nonlinear termination conditions, the construction of the difference program becomes challenging. The number of peeled iterations itself may be a function of \( N \) and possibly result in a loop in the difference program. For example, consider a loop in \( P_N \) with the counter \( i \) initialized to 0 and the loop termination condition \( "i < N^{2}." \) The corresponding loop in \( P_{N-1} \) has the same initialization but the termination condition is \( "i < (N - 1)^{2}.\) "2 \times N + 1" iterations must be peeled from this loop. For such loop conditions, an entire loop appears as the peel in the difference program. Since the number of iterations to be peeled is not a constant number, currently while computing this peel (in line 7 of Algorithm 1), our technique reports a failure to handle such programs. Further, our grammar restricts the shape of loops that can be verified using our technique. Most notably, we analyze programs with only non-nested loops. We have designed a variant of the full-program induction technique [39] that can verify a class of programs with nested loops. The technique greatly simplifies the computation of difference programs. It infers and uses relations between two slightly different versions of the program during the inductive step. We refer the interested reader to [39].

The inductive reasoning may remain inconclusive when the rank of the difference programs, as defined in Sect. 6, does not reduce during the successive invocations to verify the post-condition using our technique. Continuous reduction in the rank/progress measure is crucial to the success of full-program induction. When no progress is observed, we suspend the verification attempt in line 22 of the routine FPIVERIFY in Algorithm 7. Though the ranking functions can be defined in many possible ways, there are programs that pose a challenge in computing the difference program in a way that the rank of the computed difference program does not reduce. However, such programs are rarely seen in practice.

Our technique may fail to verify a correct program if the heuristics used for weakest pre-condition either fail or return a pre-condition that causes violation of the base case checked on line 27 of the routine FPIVERIFY in Algorithm 7.

Apart from the conceptual limitations mentioned above, our prototype implementation has a few limitations. We currently support expressions in assignment statements with only \( \{+, -, \times , /\} \) operators. In the implementation, we support a single program parameter and peel only the last iterations of loops. Despite all these limitations, our experiments show that full-program induction performs remarkably well on a large suite of benchmarks.

### 8 Experimental results

In this section, we present an extensive experimental evaluation of the full-program induction technique on a large set of array manipulating benchmarks.

#### 8.1 Implementation

We have implemented our technique in a prototype tool called \textsc{Vajra}. Our tool and the benchmarks used in the experiments are publicly available at [40]. \textsc{Vajra} takes a C program in SV-COMP format as input. The tool, written in C++, is built.
on top of the LLVM/CLANG [23] 6.0.0 compiler infrastructure. We use CLANG front-end to obtain LLVM bitcode. Several normalization passes such as constant propagation, dead code elimination, static single assignment (SSA) generation for renaming variables and arrays, loop normalization for running loop-dependent passes that identify program constructs such as loop counter, lower bound and upper bound expressions, branch conditions and so on, are performed on the bitcode. We use Z3 [12] v4.8.7 as the SMT solver to prove the validity of the parametric Hoare triples for loop-free programs and to compute weakest pre-conditions. We have also implemented a Gaussian elimination-based procedure that propagates array equalities and simplifies select store nests in the generated SMT formula to compute weakest pre-conditions.

8.2 Benchmarks

We have evaluated VAJRA on a test-suite of 231 benchmarks inspired from different algebraic functions that compute polynomials as well as standard array operations such as copy, min, max, and compare. Of these, there are 121 safe benchmarks and 110 unsafe benchmarks. All our programs take a symbolic parameter N which specifies the size of each array as well as the number of times each loop executes. Several benchmarks in the test-suite follow different types of templates wherein either the number of loops in the program increases or they use potentially different data values. Program from the first kind of templates allows us to gauge the scalability aspect of our technique as the number of loops in the program increases. Programs from the latter templates allow for checking the robustness of our technique to the content of arrays and scalars.

Assertions in the benchmarks are either universally quantified or quantifier-free safety properties. The predicates in these assertions are (in-)equalities over scalar variables, array elements, and possibly nonlinear polynomial terms over N. Although our technique can handle some classes of existentially quantified assertions as discussed in Sect. 4.7, all the examples considered for our experiments have universally quantified or quantifier-free assertions. The approach described in the paper is naturally applicable to programs with such assertions, given that the underlying SMT solver can discharge the verification conditions containing formulas with existential quantification and quantifier alternation when a loop-free difference program is automatically computed. Handling post-conditions with existential quantification and quantifier alternation are part of future work.

8.3 Setup

All experiments were performed on an Ubuntu 18.04 machine with 16 GB RAM and running at 2.5 GHz. We have compared our tool VAJRA against the verifiers for array programs VIAP (v1.1) [5], VERIABS (v1.3.10) [7], BOOSTER (v0.2) [10], VAPHOR (v1.2) [9], and FREQHORN (v.0.5) [4]. C programs were manually converted to mini-Java as required by VAPHOR and CHC formulae as required by FREQHORN. Since FREQHORN does not automatically find counterexamples, so we used the supplementary tool EXPL from its repository on unsafe benchmarks as recommended by them. We have used the same version of VERIABS that was used to perform the experiments in [24], since the later version of VERIABS invokes our tool VAJRA in its pipeline for verifying array programs (refer [7]). A timeout of 100 s was set for these experiments.

8.4 Summary of the results

We executed all six tools on the entire set of 231 benchmarks. A table with the summary of obtained results is shown in Table 2. We present the results on safe and unsafe benchmarks separately for a fair representation of each tool on the set of benchmarks.

8.5 Analysis on safe benchmarks

VAJRA verified 110 safe benchmarks, compared to 58 verified by VIAP, 50 by VERIABS, 36 by BOOSTER, 27 by VAPHOR, and 26 by FREQHORN. VAJRA was inconclusive on 11 benchmarks. The reasons for the inability of our tool to generate

| Tool   | Success | CE | Inconclusive | TO |
|--------|---------|----|--------------|----|
| Safe   | VAJRA   | 110| 0            | 11 |
|        | VIAP    | 58 | 0            | 2  |
|        | VERIABS | 50 | 1            | 0  |
|        | BOOSTER | 36 | 27           | 17 |
|        | VAPHOR  | 27 | 9            | 2  |
|        | FREQHORN| 26 | 0            | 19 |

| Unsafe | VAJRA   | 0  | 109          | 1  |
|        | VIAP    | 1  | 108          | 0  |
|        | VERIABS | 0  | 102          | 0  |
|        | BOOSTER | 0  | 84           | 15 |
|        | VAPHOR  | 1  | 106          | 1  |
|        | FREQHORN| 0  | 99           | 0  |
a conclusive result are as follows: (1) the difficulty in computing a difference program due to the presence of a branch condition dependent on \( N \) or complex operations such as modulo, (2) difficulty in computing the required strengthening of the pre- and post-conditions and (3) the back-end SMT solver returning an inconclusive result.

\textsc{Vajra} verified 52 benchmarks on which VIAP diverged, primarily due to the inability of VIAP’s heuristics to get closed-form expressions. VIAP verified 5 benchmarks that could not be verified by the current version of \textsc{Vajra} due to syntactic limitations. \textsc{Vajra}, however, is two orders of magnitude faster than VIAP on programs that were verified by both (refer Fig. 27).

\textsc{Vajra} proved 60 benchmarks on which \textsc{VeriAbs} diverged. \textsc{VeriAbs} ran out of time on programs where loop shrinking and merging abstractions were not strong enough to prove the assertions. \textsc{VeriAbs} reported 1 program as unsafe due to the imprecision of its abstractions and it proved 3 benchmarks that \textsc{Vajra} could not.

\textsc{Vajra} verified 74 benchmarks that \textsc{Booster} could not. \textsc{Booster} reported 27 benchmarks as unsafe due to imprecise abstractions, its fixed-point computation engine reported unknown result on 17 benchmarks and it ended abruptly on 41 benchmarks. \textsc{Booster} also proved 2 benchmarks that could not be handled by the current version of \textsc{Vajra} due to syntactic limitations.

\textsc{Vajra} verified 83 benchmarks that \textsc{Vaphor} could not. The distinguished cell abstraction technique implemented in \textsc{Vaphor} is unable to prove safety of programs, when the value at each array index needs to be tracked. \textsc{Vaphor} reported 9 programs unsafe due to imprecise abstraction, returned unknown on 2 programs and ended abruptly on 83 programs. \textsc{Vaphor} proved 2 benchmarks that \textsc{Vajra} could not.

\textsc{Vajra} verified 84 programs on which \textsc{FreqHorn} diverged, especially when constants and terms that appear in the inductive invariant are not syntactically present in the program. \textsc{FreqHorn} ran out of time on 76 programs, reported unknown result or ended abruptly on 19 benchmarks. \textsc{FreqHorn} verified a benchmark with a single loop that \textsc{Vajra} could not.

All the benchmarks that are uniquely solved by \textsc{Vajra} have multiple sequentially composed loops and/or a form of aggregation/cross-iteration dependence via a scalar variable or an array.

### 8.6 Analysis on unsafe benchmarks

\textsc{Vajra} disproved 109 benchmarks, compared to 108 disproved by VIAP, 102 by \textsc{VeriAbs}, 84 by \textsc{Booster}, 106 by \textsc{Vaphor}, and 99 by \textsc{expl}, the supplementary tool that comes with \textsc{FreqHorn}. \textsc{Vajra} was unable to disprove 1 benchmark.

\textsc{Vajra} disproved 1 benchmark which VIAP could not. VIAP concluded 1 benchmark as safe and timed out on 1 benchmark. Even on unsafe benchmarks, \textsc{Vajra}, is an order of magnitude faster than VIAP (refer Fig. 28). \textsc{Vajra} disproved 7 benchmarks which \textsc{VeriAbs} could not. \textsc{VeriAbs} ran out of time on 8 programs.

\textsc{Vajra} disproved 25 benchmarks that \textsc{Booster} could not. \textsc{Booster} reported unknown result on 15 benchmarks and it timed out on 11 benchmarks. \textsc{Vajra} disproved 3 benchmarks that \textsc{Vaphor} could not. \textsc{Vaphor} proved 1 program as safe, returned unknown on 2 programs and timed out on 2 programs. \textsc{Vajra} disproved 10 programs which \textsc{expl} could not. \textsc{expl} ran out of time on 11 programs.

### 8.7 Comparing the performance

The quantile plots in Figs. 27 and 28 show the performance of each tool on all the safe and unsafe benchmarks, respectively, in terms of time taken to produce the result. \textsc{Vajra} verified/disproved each benchmark within 3 s. In compar-
ison, as seen from the plots, other tools took significantly more time in proving the programs.

As mentioned previously, the test-suite has several benchmarks that are instantiated from different templates. For such instantiated benchmarks that only change the data values in the instances of the templates, we did not see any change in the performance of Vajra. Hence, we do not discuss them further. We now discuss the results for a set of templates where the number of loops in the benchmarks instantiated from them increases. In Table 3, we present the results of executing Vajra on the benchmarks instantiated from the “zerosum” templates. The first column indicates the benchmark instance number, the second column indicates the number of loops in the instantiated benchmark, columns three to six indicate the benchmark template name that is instantiated and give the time (in seconds) taken by Vajra to prove the given assertion in the benchmark instance. It can be seen from the table that as the number of loops increase in the benchmark, our tool requires more time in solving the benchmark. This is primarily attributed to pre- and post-condition strengthening step in our technique that requires our technique to infer and prove auxiliary predicates iteratively during the inductive step.

9 Related work

Earlier work on inductive techniques can be broadly categorized into those that require loop-specific invariants to be provided or automatically generated, and those that work without them. Requiring a “good” inductive invariant for every loop in a program effectively shifts the onus of assertion checking to that of invariant generation. Among techniques that do not require explicit inductive invariants or mid-conditions for each loop, there are some that require loop invariants to be implicitly generated by a constraint solver. These include techniques based on constrained Horn clause solving [2–4,9], acceleration and lazy interpolation for arrays [10] and those that use inductively defined predicates and recurrence solving [5,6,41], among others.

QUIC3 [3], FreqHorn [4], and the technique in [2] infer universally quantified inductive invariants of array programs specified as Constrained Horn Clauses. QUIC3 [3] extends the IC3 framework to a combination of SMT theories and performs lazy quantifier instantiations. FreqHorn [4] infers universally quantified invariants over arrays within its syntax-guided synthesis framework and can reason with complex array index expressions by adopting the tiling [8] ideas.

Vaphor [9] transforms array programs to array-free Horn formulas. Their technique is parameterized by the number of array cells to be tracked resulting in an eager quantifier instantiation.

Booster [10] combines acceleration [42,43] and lazy abstraction with interpolation for arrays [44]. Performing interpolation to infer universally quantified array properties is difficult [20,45]. The technique does not always succeed, especially for programs where simple interpolants are difficult to compute [8].

VIAP [5] translates the program to an array-free quantified first-order logic formula in the theory of equality and uninterpreted functions using the scheme proposed in [46]. They use several tactics to simplify the generated formula and apply induction over array indices to prove the property. Unlike our method, it does not have heuristics for finding additional pre-conditions that are required for the induction proof to succeed which our method successfully infers.

[41] uses theorem provers to introduce and prove lemmas that implicitly capture inductive loop invariants at arbitrary points in the program described in trace logic.

Thanks to the impressive capabilities of modern constraint solvers and the effectiveness of carefully tuned heuristics for stringing together multiple solvers, approaches that rely on constraint solving have shown a lot of promise in recent years. However, at a fundamental level, these formulations rely on solving implicitly specified loop invariants garbed as constraint solving problems.

Template-based techniques [17–19] search for inductive invariants by instantiating the parameters of a fixed set of templates within the abstract interpretation framework. They can generate invariants with alternating quantifiers, however, the user must supply invariant templates and the cost of generating invariants is quite high.

A large number of techniques have been proposed in the literature that use induction [47–53] and its pragmatically

| Instance | Loops | Zerosum | Zerosum-const | Zerosum-m | Zerosum-const-m |
|----------|-------|---------|---------------|----------|----------------|
| 1        | 3     | 0.54    | 0.49          | –        | –              |
| 2        | 5     | 0.87    | 0.88          | 0.86     | 0.92           |
| 3        | 7     | 1.28    | 1.27          | 1.26     | 1.21           |
| 4        | 9     | 1.73    | 1.94          | 2.07     | 1.76           |
| 5        | 11    | 2.32    | 2.56          | 2.31     | 2.48           |
| 6        | 13    | –       | 2.94          | –        | 2.95           |
more useful version k-induction [1,54–64]. These techniques generate and use loop invariants, especially when aimed at verifying safety properties of programs. In contrast, our novel technique does not rely on generation or use of loop-specific invariants and differs significantly from these methods in the way in which the inductive step is formulated using the computed difference programs and difference pre-conditions.

There are yet other inductive techniques, such as that in [8,39,65,66], that truly do not depend on loop invariants being generated. In fact, the technique of [65] comes closest to our work in principle. However, [65] imposes severe restrictions on the input programs to move the peel of one loop across the next sequentially composed loop such that the program with the peeled loops composed with the program fragment consisting of only the peeled iterations is semantically equivalent to the input program. They call these restrictions on the input programs as non-interference of loops. In practice, such restrictive conditions and data dependencies are not satisfied by a large class of programs. For instance, the example in Fig. 1 and our running example in Fig. 4 do not meet these restrictions. The technique of [65] is thus applicable only to a small part of the program-assertion space over which our technique works.

The tiling [8] technique for verifying universally quantified properties of array programs reasons one loop at a time and applies only when loops have simple data dependencies across iterations (called non-interference of tiles in [8]). It effectively uses a slice of the post-condition of a loop as an inductive invariant. In the case of sequentially composed loops, it also requires strong enough mid-conditions to be automatically generated or supplied by the user. Our full-program induction technique circumvents all of these requirements.

The method proposed in [66] proves programs correct by induction on a rank, chosen as the size of program states. It constructs a safety proof by automatically synthesizing a squeezing function that can map higher-ranked states to a lower-ranked state, while ensuring that original states are faithfully simulated by their squeezed counterparts. This allows the method to shrink program traces of unbounded length, limiting the reasoning to only minimally ranked states. A guess-and-check approach combined with heuristics for making educated guesses is employed for computing the squeezing functions necessary to prove a given program. Successful synthesis of a squeezing function is equivalent to establishing the inductive step. These functions can be quite useful in practice, for example, to prove programs that may not have a first-order representable loop invariant. In general, squeezing functions are not easy to synthesize and automatically searching for such functions is a non-trivial and an exceedingly time-consuming task. Further, the squeezing functions can only consist of commutative and invertible operations, restricting their applicability. The technique may be used in tandem with the classical loop-invariant-based methods. In comparison, our technique generates and uses difference invariants in an explicit inductive step and it does not rely on generation and use of squeezers to shrink the state space of the program.

The technique presented in [39] also performs induction on the entire program and is a parallel line of work. As stated in [39], full-program induction forms the basis of their technique but the way in which the inductive step is formulated differs significantly from ours. It coins difference invariants that relate two slightly different versions of the given program. They use just the peeled iterations of loops as difference programs and amend the inductive reasoning using difference invariants. The technique supports nested loops as well as branch conditions with value dependent on the program parameter N. The prototype tool Diffy [67] implements the method. We believe that there are programs for which [39] may not be able to successfully infer and use difference invariants, but full-program induction (with its recursive invocation ability) will be able to verify the post-conditions in such programs.

There are several techniques that approximate program computation during verification. [68] has proposed a counterexample-guided abstraction refinement scheme for programs that manipulate arrays. Their idea relies on prophecy variables to refine the abstraction. VERIABS [7] is an abstraction-based verifier to prove properties of programs. It implements a portfolio of abstractions that enable the tool to leverage bounded model checking. These abstractions tend to restrict the array manipulating loops to a fixed number of (possibly initial) iterations. The tool makes a series of attempts to prove the property and uses program features to choose the next abstraction/strategy to be applied. Fluid updates [69] use bracketing constraints, which are over- and under-approximations of indices, to specify the concrete elements being updated in an array without explicit partitioning. While their abstraction is independent of the given property, they assume that only a single index expression updates the array in each loop, severely restricting the technique. Analyses proposed in [13,14] partition the array into symbolic slices and abstracts each slice with a numeric scalar variable. Abstract interpretation-based techniques [15,16] propose an abstract domain which utilizes cell contents to split array cells into groups. In particular, the technique in [15] is useful when array cells with similar properties are non-contiguously present in the array. These approaches require the implementation of abstract transformers for each specialized domain which is not a necessity with our framework. Other techniques for analyzing array manipulating programs include [20,70].

Program differencing [71], program integration [72], and differential static analysis [73] have been studied in the literature for various purposes. Incremental computation of
VeriAbs: verification by abstraction and test generation (competition). SymDiff [78] tool, based on differential static analysis [73], displays semantic differences between different program versions and checks equivalence. However, the method neither supports checking quantified post-conditions nor does it support loops and arrays of potentially unbounded size. Unfortunately, these techniques do not always generate code fragments that are well suited for property verification, especially when the input programs manipulate arrays. To the best of our knowledge, full-program induction is the first technique to successfully employ difference computation customized for verification in an inductive setting.

Full-program induction also offers several other advantages over the existing techniques. For instance, it can reason with different quantifiers over multiple variables, it does not require implementation of specialized abstract domains for handling quantified formulas and it can enable the use of existing tools and techniques for reasoning over arrays. We believe that verification tools need to have an arsenal of techniques to be able to efficiently prove a wide range of challenging problems. Our novel technique, full-program induction, is a suitable fit for such an arsenal and has been adopted by verifiers such as VeriABS in practice. Since the 2020 edition of the international software verification competition (SV-COMP), VeriABS [7] invokes our tool Vajra in its pipeline of tools for verifying programs with arrays from the set of benchmarks in the competition.

10 Conclusion and future work

We presented a novel property-driven verification technique, called full-program induction, that performs induction over the entire program via parameter $N$. Significantly, our analysis obviates the need for loop-specific invariants during verification. The technique automatically computes the difference program and difference pre-condition that enable the inductive step of the reasoning. Our technique is general and can be applied to programs manipulating arrays that store integers, matrices, polynomials, vectors, and so on. This gives our technique the potential of verifying APIs used in machine learning and cryptography libraries. Experiments show that Vajra performs remarkably well vis-a-vis state-of-the-art tools for analyzing array manipulating programs.

Possible directions of future work include investigations into possible ways of incorporating automatically generated and externally supplied invariants during our analysis, especially for computing simpler difference programs and handling programs with nested loops. Automated support for handling assertions with existential quantification and quantifier alternation and for verifying heap-manipulating programs as well as programs that operate on tensors using our technique. Investigations into the use of synthesis-based techniques for automatically computing the difference programs and adapting them to programs from various interesting domains forms another line of work. Improvements to the algorithms for simultaneous strengthening of pre- and post-conditions can be considered.

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