The string tension in massive $QCD_2$

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Abstract

We compute the string tension in massive $QCD_2$. It is shown that the string tension vanishes when the mass of the dynamical quark is zero, with no dependence on the representations of the dynamical or of the external charges. When a small mass ($m \ll e$) is added, a tension appears and we calculate its value as a function of the representations.

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In a recent paper by Gross et al.\cite{1} it was argued that two dimensional QCD exhibits a screening nature when the dynamical quarks have no mass. Confinement appears when mass is given to the quarks. Solutions to the equations of motion and a large $N_f$ analysis which support this statement were given in refs.\cite{2}.

Similar phenomena occur in two dimensional QED. It is well known\cite{3} that integer charges can screen fractional charges when the dynamical electrons are massless. The naive picture of confinement is restored when a mass is given to the dynamical electrons and when the external charge cannot be composed of the dynamical charge. The expression for the string tension in the abelian case is

$$\sigma = m \mu \left(1 - \cos\left(\frac{2\pi q_{\text{ext}}}{q_{\text{dyn}}}\right)\right),$$

(1)

where $m$ is the electron mass, $\mu = e^\gamma \frac{e^{\pi/2}}{4\pi}$ ($e$ is the coupling and $\gamma$ is the Euler number) and $q_{\text{ext}}, q_{\text{dyn}}$ are the external and dynamical charges respectively.

In this note we generalize the proof of \cite{3} to the non-abelian case. We show that when the dynamical quarks are massless, the external source can be extracted from the action by a redefinition of the (bosonized) matter field and therefore there is no string tension. When a mass term does exist, we use this redefinition to calculate the string tension. The proof is given in the interesting case of dynamical and external charges in the fundamental/adjoint representations. Additional remarks concerning the symmetric and anti-symmetric cases are also given.

The action of bosonized QCD$_2$ with massive quarks in the fundamental representation of $SU(N)$ is the following \cite{4}

$$S_{\text{fundamental}} = \frac{1}{8\pi} \int \Sigma d^2x \left(\partial_\mu g \partial^\mu g^\dagger + \frac{1}{12\pi} \int_B d^3y e^{ijk} \text{tr}(g^\dagger \partial_i g)(g^\dagger \partial_j g)(g^\dagger \partial_k g) + \frac{1}{2} m_{\mu} \text{fund} \int d^2x \left(\text{tr}(g + g^\dagger) - \int d^2x \frac{1}{4e^2} F_{\mu\nu}^a F^{a\mu\nu} + \right. \right.$$  

$$\left. - \frac{1}{2\pi} \int d^2x \left(ig^\dagger \partial_+ g A_- + ig \partial_- g^\dagger A_+ + A_+ g A_- - A_+ A_- \right)\right),$$

where $e$ is the gauge coupling, $m$ is the quark mass, $\mu \sim e$, $g$ is an $N \times N$ unitary matrix, $A_\mu$ is the gauge field and the trace is over $U(N)$ indices. Note, however, that only the $SU(N)$ part of the matter field $g$ is gauged.
When the quarks transform in the adjoint representation, the expression for the action is\[^{[3]}\]

\[ S_{\text{adjoint}} = \frac{1}{16\pi} \int_{\Sigma} d^{2}x \; tr(\partial_{\mu}g\partial^{\mu}g) + \]

\[ \frac{1}{24\pi} \int_{B} d^{3}y \epsilon^{ijk} tr(g^{\dag}\partial_{i}g)(g^{\dag}\partial_{j}g)(g^{\dag}\partial_{k}g) + \]

\[ \frac{1}{2} m_{\mu \text{adj}} \int d^{2}x \; tr(g + g^{\dag}) - \int d^{2}x \frac{1}{4e^{2}} F_{\mu \nu}^{a} F^{a \mu \nu} + \]

\[ -\frac{1}{4\pi} \int d^{2}x \; tr(i g^{\dag}\partial_{+}g A_{-} + ig\partial_{-}g^{\dag}A_{+} + A_{+}g A_{-}g^{\dag} - A_{+}A_{-}) \]

It differs from (2) by a factor of one half in front of the WZW and interaction terms because \(g\) is real and represents Majorana fermions. Another difference is that \(g\) now is an \((N^{2} - 1) \times (N^{2} - 1)\) orthogonal matrix. The two actions (2) and (3) can be schematically represented by the following action

\[ S = S_{0} + \frac{1}{2} m_{R} \int d^{2}x \; tr(g + g^{\dag}) - \frac{i k_{\text{dyn}}}{4\pi} \int d^{2}x \; (g\partial_{-}g^{\dag})^{a} A_{+}^{a}. \]

where \(A_{-} = 0\) gauge was used, \(S_{0}\) stands for the WZW action and the kinetic action of the gauge field. \(k_{\text{dyn}}\) is the level (the chiral anomaly) of the dynamical charges (\(k = 1\) for the fundamental representation of \(SU(N)\) and \(k = N\) for the adjoint representation).

Let us add an external charge to the action. We choose static (with respect to the light-cone coordinate \(x^{+}\)) charge and therefore we can omit its kinetic term from the action. Thus an external charge coupled to the gauge field would be represented by

\[ -\frac{i k_{\text{ext}}}{4\pi} \int d^{2}x \; (u\partial_{-}u^{\dag})^{a} A_{+}^{a} \]

Suppose that we want to put a quark and an anti-quark at a very large separation. A convenient choice of the charges would be a direction in the algebra in which the generator has a diagonal form. The simplest choice is a generator of an \(SU(2)\) subalgebra. As an example we write down the

\[^{4}\text{Since we do not consider instantons, the remark of ref.}\[^{[3]}\]\text{about the bosonized form of gauged adjoint matter is not crucial to our discussion.}\]
generator in the case of fundamental and adjoint representations.

\[ T_{\text{fundamental}}^3 = \text{diag}(\frac{1}{2}, -\frac{1}{2}, 0, 0, \ldots, 0) \]  
\( N-2 \) doublets \hfill (6) 

\[ T_{\text{adjoint}}^3 = \text{diag}(1, 0, -1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, \ldots, 0) \]  
\( 2(N-2) \) doublets \hfill (N-2)^2 \hfill (7) 

Generally \( T^3 \) can be written as

\[ T^3 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_i, 0, 0, \ldots) \]  \hfill (8) 

where \( \{\lambda_i\} \) are the ‘isospin’ components of the representation under the \( SU(2) \) subgroup.

We take the \( SU(N) \) part of \( u \) as (see Appendix)\(^5\)

\[ u = \exp(-i4\pi \left( \theta(x^- + L) - \theta(x^- - L) \right) T_{\text{ext}}^3) \]  \hfill (9) 

where \( T_{\text{ext}}^3 \) represents the ‘3’ generator of the external charge and \( u \) is static with respect to the light-cone time coordinate \( x^+ \). The theta function is used as a limit of a smooth function which interpolates between 0 and 1 in a very short distance. In that limit \( u = 1 \) everywhere except at isolated points, where it is not well defined.

The form of the action (4) in the presence of an external source is

\[ S = S_0 + \frac{1}{2} m \mu_R \int d^2 x \left\{ \text{tr}(g + g^\dagger) + \right\} \]  
\[ -\frac{i k_{\text{dyn}}}{4\pi} (g \partial_- g^\dagger)^a + k_{\text{ext}} \delta^a_3 (\delta(x^- + L) - \delta(x^- - L)) A_+^a \right\} \]  \hfill (10) 

The external charge can be eliminated from the action by a transformation of the matter field. A new field \( \tilde{g} \) can be defined as follows

\[ -\frac{i k_{\text{dyn}}}{4\pi} (\tilde{g} \partial_- \tilde{g}^\dagger)^a = \]  \hfill (11) 

\[ -\frac{i k_{\text{dyn}}}{4\pi} (g \partial_- g^\dagger)^a + k_{\text{ext}} \delta^a_3 \left( \delta(x^- + L) - \delta(x^- - L) \right) \] 

\(^5\)In the case of \( SU(2) \) the choice is \( u = \exp(-i2\pi \left( \theta(x^- + L) - \theta(x^- - L) \right) T_{\text{ext}}^3) \)
This definition leads to the following equation for \( \bar{g}^\dagger \)

\[
\partial_- \bar{g}^\dagger = \bar{g}^\dagger \left( g \partial_- g^\dagger + i4\pi \frac{k_{\text{ext}}}{k_{\text{dyn}}} (\delta(x^- + L) - \delta(x^- - L)) T_{\text{dyn}}^3 \right) \tag{12}
\]

The solution of (12) is

\[
\bar{g}^\dagger = P \exp \int d x^- \left( g \partial_- g^\dagger + i4\pi \frac{k_{\text{ext}}}{k_{\text{dyn}}} (\delta(x^- + L) - \delta(x^- - L)) T_{\text{dyn}}^3 \right) = e^{i4\pi \frac{k_{\text{ext}}}{k_{\text{dyn}}} \theta(x^- + L) T_{\text{dyn}}^3} \bar{g}^\dagger e^{-i4\pi \frac{k_{\text{ext}}}{k_{\text{dyn}}} \theta(x^- - L) T_{\text{dyn}}^3},
\]

where \( P \) denotes path ordering and we assumed that \( T_{\text{dyn}}^3 \) commutes with \( g \partial_- g^\dagger \) for \( x^- \geq L \) and with \( g^\dagger \) for \( x^- = -L \) (as we shall see, this assumption is self consistent with the vacuum configuration).

Let us take the limit \( L \to \infty \). For \(-L < x^- < L\), the above relation simply means that

\[
g = \bar{g} e^{i4\pi \frac{k_{\text{ext}}}{k_{\text{dyn}}} \pi^3 T_{\text{dyn}}^3} \tag{14}
\]

Since the Haar measure is invariant (and finite, unlike the fermionic case) with respect to unitary transformations, the form of the action in terms of the new variable \( \bar{g} \) reads

\[
S = S_{WZW}(\bar{g}) + S_{\text{kinetic}}(A_\mu) - \frac{i k_{\text{dyn}}}{4\pi} \int d^2 x \ (\bar{g} \partial_- g^\dagger)^a A_+^a \tag{15}
\]

\[
+ \frac{1}{2} m_{\mu R} \int d^2 x \ tr(\bar{g} e^{i4\pi \frac{k_{\text{ext}}}{k_{\text{dyn}}} T_{\text{dyn}}^3} + e^{-i4\pi \frac{k_{\text{ext}}}{k_{\text{dyn}}} T_{\text{dyn}}^3} \bar{g}^\dagger) = QCD_2 \text{ with a chirally rotated mass term.}
\]

The string tension can be calculated easily from (15). It is simply the vacuum expectation value of the Hamiltonian density, relative to the v.e.v. of the Hamiltonian density of the theory without an external source,

\[
\sigma = \langle H \rangle - \langle H_0 \rangle \tag{16}
\]

The vacuum of the theory is given by \( \bar{g} = 1 \).

\[
\langle H \rangle = - \frac{1}{2} m_{\mu R} tr(e^{i4\pi \frac{k_{\text{ext}}}{k_{\text{dyn}}} T_{\text{dyn}}^3} + e^{-i4\pi \frac{k_{\text{ext}}}{k_{\text{dyn}}} T_{\text{dyn}}^3}) = \tag{17}
\]

\[
- m_{\mu R} \sum_i \cos(4\pi \lambda_i \frac{k_{\text{ext}}}{k_{\text{dyn}}})
\]

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Therefore the string tension is

\[ \sigma = m \mu_R \sum_i \left( 1 - \cos(4\pi \lambda_i \frac{k_{\text{ext}}}{k_{\text{dyn}}}) \right) \]  

(18)

A few remarks should be made:

(i) The string tension (18) reduces to the abelian string tension (1) when abelian charges are considered. It seems that the non-abelian generalization is realized by replacing the charge \( q \) with \( k \).

(ii) The string tension was calculated in the tree level of the bosonized action. Perturbation theory (with \( m \) as the coupling) may change eq.(18), but we believe that it would not change its general character.

(iii) When no dynamical mass is present, the theory exhibits screening. This is simply because non-abelian charges at the end of the world interval can be eliminated from the action by a chiral transformation of the matter field.

(iv) When the test charges are in the adjoint representation \( k_{\text{ext}} = N \), equation (18) predicts screening by the fundamental charges (with \( k_{\text{dyn}} = 1 \)).

(v) String tension appears when the test charges are in the fundamental representation and the dynamical charges are in the adjoint [7]. The value of the string tension is

\[ \sigma = m \mu_{\text{adj}} \left( 2(1 - \cos \frac{4\pi}{N}) + 4(N - 2)(1 - \cos \frac{2\pi}{N}) \right) \]

(19)

The case of \( SU(2) \) is special. The \( 4\pi \) which appears in eq.(18) is replaced by \( 2\pi \), since the bosonized form of the external \( SU(2) \) fundamental matter differs by a factor of a half with respect to the other \( SU(N) \) cases (see Appendix). Hence, the string tension in this case is \( 4m \mu_{\text{adj}} \).

The generalization of (18) to arbitrary representations is not straightforward. However, we can comment about its nature (without rigorous proof).

Let us focus on the interesting case of the antisymmetric representation. One can show that in a similar manner to [3], the WZW action with \( g \) taken to be \( \frac{1}{2} N(N - 1) \times \frac{1}{2} N(N - 1) \) unitary matrices, is a bosonized version of \( QCD_2 \) with fermions in the antisymmetric representation.

The antisymmetric representation is described in the Young-tableaux notation by two vertical boxes. Its dimension is \( \frac{1}{2} N(N - 1) \) and its diagonal...
The SU(2) generator is

\[ T^3_{\text{anti-symmetric}} = \text{diag}(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, ..., 0, 0, ..., 0), \quad (20) \]

and consequently \( k = N - 2 \). When the dynamical charges are in the fundamental and the external in the antisymmetric the string tension should vanish because tensor product of two fundamentals include the antisymmetric representation. Indeed, (18) predicts this result.

The more interesting case is when the dynamical charges are antisymmetric and the external are fundamentals. In this case the value of the string tension depends on whether \( N \) is odd or even\footnote{7}. When \( N \) is odd the string tension should vanish because the anti-fundamental representation can be built by tensoring the antisymmetric representation with itself \( \frac{1}{2}(N - 1) \) times. When \( N \) is even string tension must exist. Note that (18) predicts

\[ \sigma = 2m\mu_{as}(N - 2)(1 - \cos\left(\frac{2\pi}{N - 2}\right)) \quad (21) \]

which is not zero when \( N \) is odd.

The resolution of the puzzle seems to be the following. Non-Abelian charge can be static with respect to its spatial location. However, its representation may change in time due to emission or absorption of soft gluons (without cost of energy). Our semi-classical description of the external charge as a c-number is insensitive to this scenario. We need an extension of (18) which takes into account the possibilities of all various representations. We propose the following external current

\[ j^a_{\text{ext}} = \delta^a_3 k_{\text{ext}}(1 + lN)(\delta(x^- + L) - \delta(x^- - L)) \quad (22) \]

where \( l \) is an arbitrary positive integer. The value of the external current corresponds to \( 1 + lN \) charges which were multiplied in a symmetric way. The resulting string tension is

\[ \sigma = m\mu_R \sum_i \left(1 - \cos\left(4\pi\lambda_k\frac{k_{\text{ext}}}{k_{\text{dyn}}}(1 + lN)\right)\right), \quad (23) \]

which includes the arbitrary integer \( l \). What is the value of \( l \) that we should pick?
The dynamical charges are attracted to the external charges in such a way that the total energy of the configuration is minimal. Therefore the value of $l$ which is needed, is the one that guarantees minimal string tension.

Thus the extended expression for string tension is the following

$$\sigma = \min_k \left\{ m\mu_R \sum_i \left( 1 - \cos \left( 4\pi \lambda \frac{k_{\text{ext}}}{k_{\text{dyn}}} (1 + lN) \right) \right) \right\}$$  \hspace{1cm} (24)

In the case of dynamical antisymmetric charges and external fundamentals and odd $N$, $l = \frac{1}{2}(N - 3)$ gives zero string tension. When $N$ is even the string tension is given by (21).

The expression (24) yields the right answer in some other cases also.

Another example is the case of dynamical charges in the symmetric representation. The bosonization for this case can be derived in a similar way to that of the antisymmetric representation, and $T^3$ is given by

$$T^3_{\text{symmetric}} = \text{diag}(1, 0, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, ..., -\frac{1}{2}, \frac{1}{2}, 0, 0, ..., 0),$$  \hspace{1cm} (25)

and therefore $k = N + 2$. When the external charges transform in the fundamental representation and $N$ is odd, eq.(24) predicts zero string tension (as it should). When $N$ is even the string tension is given by

$$\sigma = 2m\mu_{\text{symm}} \left( (1 - \cos \frac{4\pi}{N + 2}) + (N - 2)(1 - \cos \frac{2\pi}{N + 2}) \right)$$  \hspace{1cm} (26)

We discussed only the cases of the fundamental, adjoint, anti-symmetric and symmetric representations since we used bosonization techniques which are applicable to a limited class of representations\cite{8}.

A Appendix

We give here a detailed derivation of the external quark anti-quark field (9).

For the case of external charges in a real representation the $u$ field can chosen to point in some special direction in the $SU(N)$ algebra which we take to be '3', namely $u = \exp -iT^3\phi$. For external charges in a complex representation one has to dress this ansatz with a baryon number part, namely
\( u = \exp -i\chi \exp -iT^3\phi \). In both cases the gauge field is taken in the same direction \( A_+ = a_+T^3 \). The Lagrangian for the former case takes the form

\[
\mathcal{L} = \frac{k}{8\pi} (\partial_- \phi)(\partial_+ \phi) + \frac{1}{2e^2} (\partial_- a_+)^2 + M^2 \sum_i \cos \lambda_i \phi + \frac{k}{4\pi} \partial_- \phi a_+, \tag{27}
\]

where \( A_- = 0 \) gauge was used, \( k \) is the level and \( \lambda_i \) are the isospin components of the diagonal sub SU(2) generator \( T^3 \).

The equations of motion for the matter and gauge fields are the following

\[
\frac{k}{4\pi} \partial_- \partial_+ \phi + M^2 \sum_i \lambda_i \sin \lambda_i \phi + \frac{k}{4\pi} \partial_- a_+ = 0 \tag{28}
\]
\[
\partial_+^2 a_+ = e^2 \frac{k}{4\pi} \partial_- \phi \tag{29}
\]

Integrating (29) with zero boundary conditions and substituting in (28) we obtain

\[
\frac{k}{4\pi} \partial_- \partial_+ \phi + M^2 \sum_i \lambda_i \sin \lambda_i \phi + e^2 \left( \frac{k}{4\pi} \right)^2 \phi = 0 \tag{30}
\]

Let us assume a solution for \( \phi \) which describes an infinitely heavy light-cone static quark anti-quark system

\[
\phi = \alpha \left( \theta(x^+ - L) - \theta(x^- - L) \right), \tag{31}
\]

where \( \alpha \) is a yet unknown coefficient.

For the region \(-L < x^- < L\) we obtain

\[
M^2 \sum_i \lambda_i \sin \lambda_i \alpha + e^2 \left( \frac{k}{4\pi} \right)^2 \alpha = 0 \tag{32}
\]

When \( M^2 \gg e^2 \) the solution for \( \alpha \) is of the following form

\[
\alpha = 4\pi n + \epsilon, \tag{33}
\]

where \( n \) is integer (we will pick the minimal \( n = 1 \) possibility) and \( \epsilon \) is determined by the substitution in (32)

\[
M^2 \sum_i \lambda_i^2 \epsilon + e^2 \left( \frac{k}{4\pi} \right)^2 4\pi \approx 0 \tag{34}
\]
Thus $\alpha$ is given by

$$\alpha = 4\pi - \frac{e^2}{M^2} \left(\frac{k}{4\pi}\right)^2 \frac{4\pi}{\sum_i \lambda_i^2} + O \left(\frac{e^2}{M^2}\right)^2$$

(35)

In the limit $M^2 \to \infty$, $u$ is

$$u = \exp -i4\pi \left(\theta(x^- + L) - \theta(x^- - L)\right) T^3$$

(36)

When $u$ is in a complex representation, namely $u$ is represented by $u = \exp -i\chi \exp -iT^3 \phi$, we find by repeating the above derivation the following expression

$$u =$$

$$\exp -i2\pi \left(\theta(x^- + L) - \theta(x^- - L)\right) \exp -i4\pi \left(\theta(x^- + L) - \theta(x^- - L)\right) T^3,$$

(37)

for $U(N > 2)$ and

$$u =$$

$$\exp -i\pi \left(\theta(x^- + L) - \theta(x^- - L)\right) \exp -i2\pi \left(\theta(x^- + L) - \theta(x^- - L)\right) T^3,$$

(38)

for $U(2)$. Note that the $SU(2)$ part has a $2\pi$ prefactor.

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