On spinfoam models in large spin regime

Muxin Han

Centre de Physique Théorique, CNRS UMR7332, Aix-Marseille Université and Université de Toulon, F-13288 Marseille, France

E-mail: Muxin.Han@cpt.univ-mrs.fr

Received 10 May 2013, in final form 10 October 2013
Published 14 November 2013

Abstract
We study the semiclassical behavior of Lorentzian Engle–Pereira–Rovelli–Livine (EPRL) spinfoam model, by taking into account the sum over spins in the large spin regime. We also employ the method of stationary phase analysis with parameters and the so-called, almost analytic machinery, in order to find the asymptotic behavior of the contributions from all possible large spin configurations in the spinfoam model. The spins contributing the sum are written as $J_f = \lambda f$, where $\lambda$ is a large parameter resulting in an asymptotic expansion via stationary phase approximation. The analysis shows that at least for the simplicial Lorentzian geometries (as spinfoam critical configurations), they contribute the leading order approximation of spinfoam amplitude only when their deficit angles satisfy $\gamma \Theta_f \leq \lambda^{-1/2} \mod 4\pi \mathbb{Z}$. Our analysis results in a curvature expansion of the semiclassical low energy effective action from the spinfoam model, where the UV modifications of Einstein gravity appear as subleading high-curvature corrections.

Keywords: covariant loop quantum gravity, lattice models of gravity, models of quantum gravity
PACS number: 04.60.Pp

1. Programme of semiclassical spinfoam state-sum

Loop quantum gravity (LQG) is an attempt to make a background independent, non-perturbative quantization of four-dimensional general relativity (GR)—for reviews, see [1, 2]. Currently, the covariant formulation of LQG is understood in terms of the Spinfoam Model, which is a sum-over-history of quantum geometries [3].

Here, in this paper, we study the semiclassical behavior of Lorentzian Engle–Pereira–Rovelli–Livine (EPRL) spinfoam model [4, 5] by considering the sum over spins in the large spin regime. The analysis is a natural continuation of the previous studies of large spin asymptotics [6–9], which do not take into account the sum over spins. The result of the present

1 Unité mixte de recherche (UMR 6207) du CNRS et des Universités de Provence (Aix-Marseille I), de la Méditerranée (Aix-Marseille II) et du Sud (Toulon-Var); laboratoire affilié à la FRUMAM (FR 2291).
analysis also connects with the recent argument about the ‘flatness problem’ proposed in [10] where summing over spins is taken into account.

It is proposed in [9] that the EPRL spinfoam state-sum model [4, 5] on a simplicial complex \( K \) has the following path integral representation:

\[
A(K) = \sum_{f \in J_f} \int_{\text{SL}(2, \mathbb{C})} \prod_{(e,f)} \int_{\mathbb{C}^3} \prod_{v \in \partial f} \mathrm{d}z_{ef} e^{S[J_f; g_{ve}, z_{ef}]/2},
\]

where \( f \) labels a triangle in the simplicial complex \( K \) or a dual face in the dual complex \( K^* \), \( e \) labels a tetrahedron in \( K \) or an edge in \( K^* \), and \( v \) labels a four-simplex in \( K \) or a dual vertex in \( K^* \). \( J_f \) labels the \( \text{SU}(2) \) irreps associated to each edge. \( g_{ve} \) is a \( \text{SL}(2, \mathbb{C}) \) group variable associated with each dual half-edge. \( z_{ef} \) is a two-component spinor. The integrand written into an exponential form \( e^S \) with the spinfoam action \( S \) is written as

\[
S[J_f, g_{ve}, z_{ef}] = \sum_{(e,f)} \left[ \ln \frac{(Z_{ef}, Z_{ef})^2}{(Z_{ef}, Z_{ef})/(Z_{ef}, Z_{ef})} + i\gamma \ln \frac{(Z_{ef}, Z_{ef})}{(Z_{ef}, Z_{ef})} \right],
\]

where \( Z_{ef} = g^*_{ve} z_{ef} \). The spinfoam action \( S \) has the following discrete gauge symmetry: Flipping the sign of individual group variable \( g_{ve} \Rightarrow -g_{ve} \) leaves \( S \) invariant. Thus, the space of group variable is essentially the restricted Lorentz group \( \text{SO}^+(1, 3) \) rather than its double-cover \( \text{SL}(2, \mathbb{C}) \). \( S \) also has the following continuous gauge degree of freedom:

- Rescaling of each \( z_{ef}^2 \):
  \[ z_{ef} \mapsto \lambda z_{ef}, \quad \lambda \in \mathbb{C} \setminus \{0\}. \]

- \( \text{SL}(2, \mathbb{C}) \) gauge transformation at each vertex \( v \):
  \[ g_{ve} \mapsto x^{-1}_{ve} g_{ve}, \quad z_{ef} \mapsto x^1_{ve} z_{ef}, \quad x_v \in \text{SL}(2, \mathbb{C}). \]

- \( \text{SU}(2) \) gauge transformation on each edge \( e \):
  \[ g_{ve} \mapsto g_{ve} h^{-1}_e, \quad h_e \in \text{SU}(2). \]

We can make the following parametrization of the two-spinor \( z_{ef}^a \) and the group variables \( g_{ve} \in \text{SL}(2, \mathbb{C}) \) in the spinfoam action \( S \):

\[
z_{ef}^a = \begin{pmatrix} 1 \\ 3_{ef} \end{pmatrix} \quad \text{and} \quad g_{ve} = \begin{pmatrix} a_{ve} & c_{ve} \\ b_{ve} & d_{ve} \end{pmatrix}, \quad a_{ve} d_{ve} - c_{ve} b_{ve} = 1,
\]

where \( 3_{ef}, a_{ve}, b_{ve}, c_{ve}, d_{ve} \in \mathbb{C} \). The arguments of the logarithmics in equation (1.2) are two ratios of the polynomials in the variables \( 3_{ef}, a_{ve}, b_{ve}, c_{ve}, d_{ve} \) and their complex conjugates. If we write the \( \text{SL}(2, \mathbb{C}) \) and \( \text{SU}(2) \) gauge transformations by

\[
x^+ = \begin{pmatrix} \alpha_v \\ \gamma_v \\ \delta_v \end{pmatrix}, \quad h = \begin{pmatrix} \mu_v \\ \nu_v \\ \mu_v \end{pmatrix}
\]

then \( 3_{ef}, a_{ve}, b_{ve}, c_{ve}, d_{ve} \) transform in the following way:

\[
3_{ef} \mapsto \gamma_v + \delta_v 3_{ef}, \quad \text{and} \quad \begin{pmatrix} a_{ve} \\ b_{ve} \\ d_{ve} \end{pmatrix} \mapsto \begin{pmatrix} \alpha_v & 0 & 0 \\ 0 & \beta_v & 0 \\ 0 & 0 & \delta_v \end{pmatrix} \begin{pmatrix} a_{ve} \\ b_{ve} \\ d_{ve} \end{pmatrix} \begin{pmatrix} \mu_v \\ \nu_v \\ \mu_v \end{pmatrix}.
\]

For the convenience of the discussion, we define the notion of the partial amplitude \( A_{J_f}(K) \) by collecting all the integrations

\[
A_{J_f}(K) := \int \mathrm{d}g_{ve} \int \mathrm{d}z_{ef} e^{S[J_f; g_{ve}, z_{ef}]/2}
\]

2. The measure \( \mathrm{dz}_{ef} \) is a scaling invariant measure on \( \mathbb{C}^3 \).
So that the spinfoam state-sum model is given by a sum of partial amplitude over all the spin configurations \( \{ J_f \} \) on the simplicial complex \( K \):

\[
A(K) = \sum_{J_f} d_{J_f} A_{J_f}(K).
\]

Note that the infinite spin-sum in \( A(K) \) may result in a divergent result. A way to regularizing the spin-sum is to replace \( SL(2, \mathbb{C}) \) in the definition by the quantum group \( SL_q(2, \mathbb{C}) \) [11], which also relates to the cosmological constant term in spinfoam formulation [12].

So far, the semiclassical properties of the spinfoam model are mostly understood via the asymptotic analysis of the partial amplitude \( A_{J_f}(K) \) as all the spins \( J_f = \lambda j_f \) are uniformly large, where \( \lambda \) is a large parameter to scale the spins uniformly and \( j_f \sim o(1) \). In such a context, the analysis in [8, 9] shows that given a Regge-like spin configuration \( \{ J_f \} \) on the simplicial complex \( K \). A critical configuration \( (j_f, g_{ve}, z_{vf}) \) satisfying \( \mathcal{H}(S) = \delta S = \delta S = 0 \) and a nondegeneracy condition\(^6\) has a canonical geometric interpretation as a classical simplicial Lorentzian geometry on \( K \). More precisely, there is an equivalence between spinfoam critical data and the geometrical data (see [9] for details):

\[
\text{Critical configuration } (j_f, g_{ve}, z_{vf}) \text{ with nondegeneracy } \iff (\pm, E_{\ell}(v), \varepsilon). \tag{1.12}
\]

\( E_{\ell}(v) \) is a set of edge-vectors, or namely, a discrete cotetrad, assigned to the simplicial complex \( K \), up to global sign ambiguity \( \pm \), at each \( v \). \( E_{\ell}(v) \) specifies the Lorentzian simplicial geometry on \( K \), and results in nonvanishing four-volume \( V_4(v) \) of all geometrical four-simplices. \( \varepsilon = \pm 1 \) is a global sign ambiguity on the simplicial complex, which can be fixed by the boundary data if there is a boundary of the spinfoam.

The set of critical configurations contributes to the leading asymptotic behavior of the partial amplitude \( A_{J_f}(K) \) as the spins \( J_f = \lambda j_f \) become largely uniform for all \( f \). If we consider the (physical) critical configurations which corresponds geometrically to the globally oriented and time-oriented spacetimes\(^4\), these critical configurations give the Regge action of discrete GR at the leading order of the \( 1/\lambda \) asymptotic expansion.

In the present analysis, we study the spinfoam state-sum \( A(K) \) perturbatively by using the background field method, in the regime that \( J_f = \lambda j_f \) is uniformly large, where \( \lambda \) is a fixed large parameter (smaller than the cut-off \( \Lambda \)). The perturbative analysis is performed in a neighborhood of a given critical configuration \( (\jmath_f, g_{ve}, z_{vf}) \) of \( S \) as the background/vacuum, which corresponds to the globally oriented and time-oriented spacetime. The background spins are \( J_f = \lambda j_f \) with \( j_f \sim o(1) \). Sometimes, we also refer \( \lambda \) to be the (averaged) background spin.

Given the critical configuration \( (j_f, g_{ve}, z_{vf}) \) as the vacuum, we consider the fluctuations of

\(^3\) See also [7] concerning Euclidean spinfoam models.

\(^4\) A Regge-like spin configuration \( \{ j_f \} \) means that \( j_f \) for each \( f \) can be interpreted as the area of the triangle dual to \( f \), i.e. there exists a set of edge-lengths \( \{ s \} \) for the edges of the simplicial complex, such that \( \{ j_f \} \) and \( \{ s \} \) satisfy the relation between areas and edge-lengths for all triangles.

\(^5\) \( J_f \) is parameter in the analysis of partial amplitude \( A_{J_f}(K) \) since it hasn’t been summed yet.

\(^6\) The nondegeneracy condition is an assumption for the five group variables \( g_{ve} \) at each \( v \):

\[
\prod_{e_1, e_2, e_3, e_4 = 1} \det(N_{e_1}(v), N_{e_2}(v), N_{e_3}(v), N_{e_4}(v)) \neq 0 \quad \text{where} \quad N_{e}(v) = g_{ve} \triangleright (1, 0, 0, 0)^t. \tag{1.11}
\]

\(^7\) The notion of oriented and time-oriented spacetimes for spinfoam critical data is defined in [9] by the equivalence equation (1.12). A discrete oriented spacetime requires that the oriented four-volume \( V_4(v) \) from cotetrad is uniformly positive/negative. A time-oriented spacetime requires that the discrete spin connection along a loop belongs to \( SO^+(1, 3) \). These notions are independent of the \( \pm \varepsilon \) ambiguity.
all the variables in a neighborhood\(^8\) at \((\hat{J}_f, \hat{g}_{ve}, \hat{z}_{ef})\) in the space \(\mathcal{M}_{j,g,z}\) of spinfoam variables \((j_f, g_{ve}, z_{ef})\).

In our context, \(j_f\) is a perturbation of the background \(\hat{j}_f\). The gap of two neighboring spins is \(\Delta j_f = \frac{1}{\lambda}\) while \(\Delta J_f = \frac{1}{2}\). The spinfoam action \(S[j_f, g_{ve}, z_{ef}]\) is written as \(\lambda S[j_f, g_{ve}, z_{ef}]\) since \(S\) is linear to \(J_f\). The formula of \(S_f\) is identical to equation (1.2) with \(J_f\) replaced by \(j_f\).

In order to analyze the semiclassical behavior of the spinfoam state-sum, we recall the spinfoam state-sum written in the following form

\[ A(K) \equiv \sum_j d_{j_f} \int d\mu(x) \ e^{i S[j_f]} \quad \text{with} \quad x = (g_{ve}, z_{ef}). \]

The semiclassical behavior of the above state-sum is analyzed in the following by assuming the background spin \(\lambda\) is a fixed large but finite parameter, and expand the summand \(\int d\mu(x) \ e^{i S[j_f]}\) perturbatively into an asymptotic series of \(1/\lambda\) for all possible spin configurations. Such an expansion gives us an effective action \(W_k[J_f]\) for the spin variables as an asymptotic series.

Then, the sum of the expansion \(W_k[J_f]\) over all spins gives us the information about the semiclassical behavior of the state-sum.

It is clear that the \(1/\lambda\) asymptotic analysis of the partial amplitude \(A_{j_f}(K)\) depends on the spin parameters \(j_f\), which are not integrated in \(A_{j_f}(K)\). The parameter \(j_f\) controls whether the critical equations \(\mathfrak{R}(S) = \delta_S S = 0\) have solutions or not. Given a neighborhood \(\Omega\)\(^{10}\) at \((\hat{J}_f, \hat{g}_{ve}, \hat{z}_{ef})\), some values of \(j_f\)'s do not result in solutions of the critical equations \(S' = 0\) and \(\mathfrak{R}(S) = 0\) \([7–9]\). Here, we call these values of \(j_f\) Non-Regge-like. A Regge-like spin configuration \(j_f\) admits the critical configuration \((j_f, g_{ve}, z_{ef})\) which can be interpreted as nondegenerate geometry, assuming this neighborhood \(\Omega\) does not touch any critical configuration of degenerate geometry\(^{11}\). The non-Regge-like spin configurations in \(\Omega\) do not result in any critical point.

According to the general result of generalized stationary phase approximation (theorems 7.7.5 and 7.7.1 in \([13]\)), in case there is no critical point in the region of integral \(\int_k e^{i S(x)} d\mu\),

\[ \left| \int_k e^{i S(x)} d\mu(x) \right| \leq C \left( \frac{1}{\lambda} \right)^k \ \sup_{k} \frac{1}{(\|S'\|^2 + \mathfrak{R}(S))^k} \]  

the integral decays faster than \((1/\lambda)^k\) for all \(k \in \mathbb{Z}_+\), provided that \(\sup(||S'||^2 + \mathfrak{R}(S))^{-k}\) is finite (i.e. it does not cancel the \((1/\lambda)^k\) behavior in front).

For the non-Regge-like spins \(j_f\), however, it is not necessary that the corresponding \(A_{j_f}(K)\) decays faster than \((1/\lambda)^k\) for all \(k \in \mathbb{Z}_+\). As an example, for a non-Regge-like \(j_f\) very close to a Regge-like \(j_f\) with the minimal gap \(\Delta j_f = \frac{1}{\lambda^2}\), \(\sup(||S'||^2 + \mathfrak{R}(S))^{-k}\) is likely to be large and cancel the \((1/\lambda)^k\) behavior (such a cancellation is described more precisely in section 4). Therefore, if we consider the semiclassical behavior of the state-sum \(A(K) = \sum_j d_j A_{j_f}(K)\) in the large spin regime, we should in general not ignore these ‘almost Regge-like’ contributions. A general way is presented in sections 3 and 4 to obtain an \(1/\lambda\) asymptotic expansion, taking into account the contributions from all the spins, both Regge-like and non-Regge-like.

---

\(^8\) The full non-perturbative spinfoam model can be understood as a sum of the perturbation theories over different vacuums.

\(^9\) We shall see that we have to consider the complexified space \(\mathcal{M}_{j,g,z}^\mathbb{C}\) of the spinfoam variables in order to study the sum over \(j_f\).

\(^{10}\) \(\Omega\) is a compact neighborhood in the space of spinfoam configurations \(\mathcal{M}_{j,g,z} = \mathbb{R}_+^j \times \text{SL}(2,\mathbb{C})^{A_{inv}} \times (\mathbb{C}P^1)^{\mathcal{A}_{inv}}\) modulo gauge transformations.

\(^{11}\) The degeneracy restricts of the group variables \(g_{ve}\) into a lower-dimensional submanifold. Thus, such a neighborhood always exists.
2. Large spin effective theory

Given that the spinfoam state-sum is a finite sum of the partial amplitude:
\[ A(K) = \sum_{J_f} d_{J_f} A_{J_f}(K). \quad (2.1) \]

If we define an ‘Spin Effection Action’ \( \mathcal{W}_K[J_f] \) by
\[ A_{J_f}(K) = \exp \mathcal{W}_K[J_f]. \quad (2.2) \]

Then, formally the spinfoam state-sum is effectively a partition function of a spin system
\[ A(K) = \sum_{J_f} d_{J_f} \exp \mathcal{W}_K[J_f]. \quad (2.3) \]

Here, we study the spin effective action \( \mathcal{W}_K[J_f] \) as an asymptotic expansion in terms of \( 1/\lambda \).
Thus, the effective theory of spinfoam state-sum is studied in the large spin regime.

In the large spin regime, by the integral expression of the partial amplitude, the spin effective action can be written as
\[ \mathcal{W}_K[\lambda J_f] = \ln \int d_{\mathcal{G}^{\mathcal{G}}_0} \int dz_{\mathcal{G}} f e^{\lambda \mathcal{S}[J_f; \mathcal{G}^{\mathcal{G}} \cup \mathcal{E}]} \quad (2.4) \]

We would like to compute \( \mathcal{W}_K[\lambda J_f] \) perturbatively at the background spin configuration \( J_f = \bar{J}_f \). So, \( \mathcal{W}_K[\lambda J_f] \) can be written as a power series of the perturbations \( s_f := j_f - \bar{j}_f \):
\[ \mathcal{W}_K[\lambda J_f] = \lambda \left[ \mathcal{W}_0 + \sum_{f} \mathcal{W}_1^f s_f + \sum_{f, f'} \mathcal{W}_2^{f, f'} s_f s_{f'} + \sum_{f, f', f''} \mathcal{W}_3^{f, f', f''} s_f s_{f'} s_{f''} + o(s_f^3) \right] \quad (2.5) \]

where the coefficients are given by
\[ \lambda \mathcal{W}_n^{f_1, \ldots, f_n} = \frac{\partial^n}{\partial j_{f_1} \cdots \partial j_{f_n}} \left[ \ln \int d_{\mathcal{G}^{\mathcal{G}}_0} \int dz_{\mathcal{G}} f e^{\lambda \mathcal{S}[J_f; \mathcal{G}^{\mathcal{G}} \cup \mathcal{E}]} \right]_{j_f = \bar{j}_f}. \quad (2.6) \]

Each of the coefficients \( \mathcal{W}_n \) can be computed as an \( 1/\lambda \) asymptotic series, by using the generalized stationary phase analysis guided by the following general result (theorem 7.7.5 in [13]):

**Theorem 2.1.** Let \( K \) be a compact subset in \( \mathbb{R}^N \), \( X \) an open neighborhood of \( K \), and \( k \) a positive integer. If (1) the complex functions \( u \in C_0^k(K) \), \( S \in C^{k+1}(X) \) and \( \Im(S) \leq 0 \) in \( X \); (2) there is a unique point \( x_0 \in K \) satisfying \( \Im(S)(x_0) = 0 \), \( S'(x_0) = 0 \), and \( \det S''(x_0) \neq 0 \). \( S' \neq 0 \) in \( K \setminus \{x_0\} \), then we have the following estimation:
\[ \left| \int_K u(x) e^{iS(x)} \, dx - e^{iS(x_0)} \left( \frac{2\pi}{\lambda} \right)^{\frac{N}{2}} \frac{e^{i\text{Ind}(S')(x_0)}}{\sqrt{\det(S')(x_0)}} \sum_{j=0}^{k-1} \left( \frac{1}{\lambda} \right)^j L_j u(x_0) \right| \leq C \left( \frac{1}{\lambda} \right)^{k+\frac{N}{2}} \sum_{|\alpha| \leq 2k} \sup_{|x|} |D^\alpha u| \quad (2.7) \]

Here, the constant \( C \) is bounded when \( f \) stays in a bounded set in \( C^{k+1}(X) \). We have used the standard multi-index notation \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and
\[ D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \text{where} \quad |\alpha| = \sum_{i=1}^{n} \alpha_i. \quad (2.8) \]

\( L_j u(x_0) \) denotes the following operation on \( u \):
\[ L_j u(x_0) = i^{-j} \sum_{l=-m}^{m} \sum_{2|m| \geq 3m} \frac{2^{-l}}{l! m!} \left[ \sum_{a, b=1}^{N} H_{ab}^{jm}(x_0) \frac{\partial^2}{\partial x_a \partial x_b} \right]^j (g_{x_0}^m u)(x_0). \quad (2.9) \]
where \( H(x) = S''(x) \) denotes the Hessian matrix and the function \( g_{u_0}(x) \) is given by
\[
g_{u_0}(x) = S(x) - S(x_0) - \frac{1}{2} H^{ab}(x_0)(x - x_0)_a(x - x_0)_b
\]
such that \( g_{u_0}(x_0) = g'_{u_0}(x_0) = 0 \).

Note that in equation (2.9), the expression of \( L_s u(x_0) \) only sums a finite number of terms for all \( s \). For each \( s \), \( L_s \) is a differential operator of order \( 2s \) acting on \( u(x) \). For example, we list the possible types of terms in the sums corresponding to \( s = 1 \) and \( s = 2 \):

- In the case \( s = 1 \), the possible \((m, l)\) are \((0, 1), (1, 2), (2, 3)\) to satisfy \( 2l \geq 3m \). The corresponding terms are of the types
  \[
  (m, l) = (0, 1) : \partial^3 u(x_0)
  
  (m, l) = (1, 2) : \partial^3 g_{u_0}(x_0) \partial u(x_0), \quad \partial^3 g_{u_0}(x_0) u(x_0)
  
  (m, l) = (2, 3) : \partial^3 g_{u_0}(x_0) \partial^3 g_{u_0}(x_0) u(x_0),
  \]
  where the indices of \( \partial \) are contracted with the Hessian matrix \( H(x_0) \).

- In the case \( s = 2 \), the possible \((m, l)\) are \((0, 2), (1, 3), (2, 4), (3, 5), (4, 6)\) to satisfy \( 2l \geq 3m \). The corresponding terms are of the types
  \[
  (m, l) = (0, 2) : \partial^5 u(x_0)
  
  (m, l) = (1, 3) : \partial^5 g_{u_0}(x_0) \partial^p u(x_0), \quad (p \geq 3, \ p + q = 6)
  
  (m, l) = (2, 4) : \partial^5 g_{u_0}(x_0) \partial^p g_{u_0}(x_0) \partial^q u(x_0)
  
  (p_1, p_2 \geq 3; p_1 + p_2 + q = 8)
  
  (m, l) = (3, 5) : \partial^5 g_{u_0}(x_0) \partial^p g_{u_0}(x_0) \partial^q g_{u_0}(x_0) \partial^r u(x_0)
  
  (p_1, p_2, p_3 \geq 3; p_1 + p_2 + p_3 + q = 10)
  
  (m, l) = (4, 6) : \partial^5 g_{u_0}(x_0) \partial^l g_{u_0}(x_0) \partial^3 g_{u_0}(x_0) \partial^3 g_{u_0}(x_0) u(x_0),
  \]
  where the indices of \( \partial \) are contracted with the Hessian matrix \( H(x_0) \).

The spin effective action \( W_K \) is evaluated at a compact neighborhood of \( K \) a critical \((g_{\text{ve}}, z_{\text{ve}})\), i.e., we impose a smooth function \( u(g_{\text{ve}}, z_{\text{ve}}) \) of compact support \( K \) such that
\[
e^{W_{\text{K}}(\mu)\mu} u(g_{\text{ve}}, z_{\text{ve}}) = \int dg_{\text{ve}} dz_{\text{ve}} e^{S[\gamma; g_{\text{ve}}, z_{\text{ve}}]} u(g_{\text{ve}}, z_{\text{ve}}).
\]
The compact support function \( u(g, z) \) comes from a partition of unity: i.e., a collection of smooth functions \( 0 \leq u_{\mu}(g, z) \leq 1 \) of compact support, such that (1) \( \sum_{\mu} u_{\mu}(g, z) = 1 \) (2) the support \( K_\mu \) of each of them containing at most one critical configuration\(^\text{12}\). So
\[
A_{\dot{\gamma}}(K) = \sum_{\mu} \int dg_{\text{ve}} dz_{\text{ve}} e^{S[\gamma; g_{\text{ve}}, z_{\text{ve}}]} u_{\mu}(g_{\text{ve}}, z_{\text{ve}}).
\]
Employing theorem 2.1, we can evaluate each coefficient \( W_\mu \) in equation (2.5) perturbatively at the large spin regime. Here, we denote the background data by \( \dot{x} = (\dot{f}, \dot{g}_{\text{ve}}, \dot{z}_{\text{ve}}) \leftrightarrow (\pm \dot{E}_\varepsilon(v), \varepsilon = -1) \) (without loss of generality, we set the global sign \( \varepsilon = -1 \) for convenience).

\( 0\text{th order.} \) If we denote by \( N_{\dot{g}_{\text{ve}}} \) the number of degrees of freedom for \( g_{\text{ve}}, z_{\text{ve}} \) modulo gauge, and denote by \( \mu(x) \) the Jacobian between \( dg_{\text{ve}}, dz_{\text{ve}} \) and the Lebesgue measure \( d^N x \), then we have
\[
\lambda \mathcal{W}_0 = \lambda S[\dot{x}] - \frac{N_{\dot{g}_{\text{ve}}}}{2} \ln \lambda + \ln \left( 2\pi \right)^{\frac{N_{\dot{g}_{\text{ve}}}}{2}} \frac{e^{\text{Ind}(S)[\dot{x}]} }{\sqrt{\text{det}(\nabla^2 S)[\dot{x}]} } \mu(\dot{x}) + O(\lambda^{-1}),
\]
\(^\text{12}\) In the present paper, we only consider the sector of geometric critical configurations, which clearly contains a finite number of isolated critical points (modulo the gauge obits) \([9]\).
where the leading order contribution $\mathcal{S}[\hat{x}]$ is the Lorentzian Regge action on the simplicial complex $\mathcal{K}$ with the triangle area given by $\gamma_{f}$:

$$\mathcal{S}[\hat{x}] = i \sum_f \gamma_{f} \hat{\Theta}_f$$  \hspace{1cm} (2.16)

which is resulting from the asymptotic analysis in [8, 9]. $\hat{\Theta}_f$ denotes the deficit/dihedral angle in the bulk/boundary at the triangle $f$ from the discrete background geometry $\hat{E}_b(\nu)$.

1st order. If we write the action $\mathcal{S}[f, g_{e\nu}, z_{ef}] = \sum_f \gamma_{f} \hat{F}_f[g_{e\nu}, z_{ef}]$, then

$$\lambda \mathcal{W}_1^{f} = \frac{\partial}{\partial \gamma_{f}} \ln \int \, dg_{e\nu} \int \, dz_{ef} \, e^{\lambda \mathcal{S}[f, g_{e\nu}, z_{ef}]},$$

$$= \lambda \left[ \int \, dg_{e\nu} \int \, dz_{ef} \, \hat{F}_f[g_{e\nu}, z_{ef}] \right] e^{\lambda \mathcal{S}[f, g_{e\nu}, z_{ef}]},$$

$$= i \lambda \gamma \hat{\Theta}_f - i \nu \hat{\Theta}_f L_1(u_{\mu} \hat{\delta}(\hat{x})) u_{\mu} \hat{\delta}(\hat{x}) + L_1(u_{\mu} \hat{F}_f \hat{\delta})(\hat{x}) + o(\lambda^{-1}),$$  \hspace{1cm} (2.17)

where the critical value $\hat{F}_f[g_{e\nu}, z_{ef}]$ is given by $i \gamma \hat{\Theta}_f$.

2nd order. $\lambda \mathcal{W}_2^{f, f'}$ can be expressed by a connected correlation function of $\hat{F}_f[g_{e\nu}, z_{ef}]$

$$\lambda \mathcal{W}_2^{f, f'} = \lambda^2 \left[ \frac{\hat{F}_f \hat{F}_{f'} e^{\lambda \mathcal{S}[f, g_{e\nu}, z_{ef}]}}{e^{\lambda \mathcal{S}[f, g_{e\nu}, z_{ef}]}} - \lambda^2 \left[ \frac{\hat{F}_f e^{\lambda \mathcal{S}[f, g_{e\nu}, z_{ef}]}}{e^{\lambda \mathcal{S}[f, g_{e\nu}, z_{ef}]}} \right] \frac{\hat{F}_{f'} e^{\lambda \mathcal{S}[f, g_{e\nu}, z_{ef}]}}{e^{\lambda \mathcal{S}[f, g_{e\nu}, z_{ef}]}} \right]$$

$$= \lambda \left[ 2 \sum_{a, b} \mathcal{H}_{ab}^{-1} \hat{\partial}_a \hat{F}_f \hat{\partial}_b \hat{F}_{f'}(\hat{x}) \right] + o(1).$$  \hspace{1cm} (2.18)

where $H$ is the Hessian matrix of the action $S$. The computation of $\mathcal{W}_2^{f, f'}$ is similar to the formalism of graviton propagator computation in LQG, see e.g. [14]. Notice that $\partial_{e\nu} \hat{F}_f(\hat{x}) = 0$ which is nothing but the equation of motion satisfied by the critical configurations. Thus,

$$\sum_{a, b} \mathcal{H}_{ab}^{-1} \hat{\partial}_a \hat{F}_f \hat{\partial}_b \hat{F}_{f'}(\hat{x}) = \sum_{g, g'} \mathcal{H}_{g, g'}^{-1} \hat{\partial}_g \hat{F}_f \hat{\partial}_{g'} \hat{F}_{f'}(\hat{x}),$$  \hspace{1cm} (2.19)

where $g, g'$ denotes the group variables $g_{e\nu}, g_{e'\nu'}$. We compute $\mathcal{W}_2^{f, f'}$ by using the expression of Hessian matrix in [8, 14]. As a result, $\mathcal{W}_2^{f, f'}$ is local in the sense that it vanishes unless the triangles $f, f'$ belong to the same tetrahedron $e$. The nonvanishing $\mathcal{W}_2^{f, f'}$ reads

$$\mathcal{W}_2^{f, f'} = \frac{2(1 + 2i \gamma - 4 \gamma^2 - 2i \gamma)}{5 + 2i \gamma} \hat{n}_e \hat{X}_e \hat{n}_{ef},$$  \hspace{1cm} (2.20)

where the matrix $\hat{X}_e$ is given by

$$\hat{X}_e = \sum_f \hat{j}_f (g^{ij} + \hat{n}_{ef} \hat{n}_{ef} + i e^{ij} \hat{n}_{ef}).$$  \hspace{1cm} (2.21)

Here, the unit three-vector $\hat{n}_{ef}$ determined by $(\hat{j}_f, g_{e\nu}, \hat{z}_{ef})$ is the normal vector of the triangle $f$ in the frame of the tetrahedron $e$ (see [9]).

3rd order. We write $\lambda \mathcal{W}_3^{f_1, f_2, f_3}$ in terms of a connected correlation function $(\hat{F}_f, \hat{F}_{f_2}, \hat{F}_{f_3})$ connect:

$$\lambda \mathcal{W}_3^{f_1, f_2, f_3} = \lambda^3 \left[ (\hat{F}_{f_1} \hat{F}_{f_2} \hat{F}_{f_3}) - (\hat{F}_{f_1} \hat{F}_{f_2})(\hat{F}_{f_3}) - (\hat{F}_{f_1} \hat{F}_{f_2})(\hat{F}_{f_3}) - (\hat{F}_{f_2} \hat{F}_{f_3})(\hat{F}_{f_1}) + 2(\hat{F}_{f_1})(\hat{F}_{f_2})(\hat{F}_{f_3}) \right]$$  \hspace{1cm} (2.22)
where e.g., \( \langle \mathcal{F}_{f_1} \mathcal{F}_{f_2} \mathcal{F}_{f_3} \rangle = \int \mathcal{F}_{f_1} \mathcal{F}_{f_2} \mathcal{F}_{f_3} e^{S(\mathcal{F}_{f_1} \mathcal{F}_{f_2} \mathcal{F}_{f_3})} \int e^{S(\mathcal{F}_{f_1} \mathcal{F}_{f_2} \mathcal{F}_{f_3})}. \) It is shown in general in the LQG three-point function computation [15] that the leading contribution of \( \langle \mathcal{F}_{f_1} \mathcal{F}_{f_2} \mathcal{F}_{f_3} \rangle \) is of order \( o(\lambda^{-2}) \). Thus, the leading contribution of \( \lambda \mathcal{W}_3(\mathcal{F}_{f_1} \mathcal{F}_{f_2} \mathcal{F}_{f_3}) \) is again of order \( \lambda \), i.e.

\[
\lambda \mathcal{W}_3(\lambda \mathcal{F}_{f_1} \mathcal{F}_{f_2} \mathcal{F}_{f_3}) = \lambda \left[ - g^{abc} \mathcal{F}_{f_1} \mathcal{F}_{f_2} \mathcal{F}_{f_3} H_{ab}^{-1} H_{bc}^{-1} + \mathcal{F}_{f_1} \mathcal{F}_{f_2} \mathcal{F}_{f_3} + \mathcal{F}_{f_1} \mathcal{F}_{f_2} \mathcal{F}_{f_3} H_{ab}^{-1} H_{bc}^{-1} \right] (\hat{\phi}) + o(1),
\]

(2.23)

where \( g^{abc} = \partial_a \partial_b \partial_c, \mathcal{F}^{\alpha}_{\beta} \equiv \partial_\alpha \mathcal{F}_{\beta}, \) and \( \mathcal{F}^{\alpha}_{\beta} \equiv \partial_\alpha \partial_\beta \mathcal{F}_{J} \).

The above computation suggests that the coefficients \( \mathcal{W}_n \) may be written as an asymptotic series in terms of \( \lambda^{-1} \):

\[
\mathcal{W}_n = W_n^{(0)} + \left( \frac{1}{\lambda} \right) W_n^{(1)} + \left( \frac{1}{\lambda^2} \right) W_n^{(2)} + o(\lambda^{-3}),
\]

(2.24)

where \( W_n^{(0)} \sim o(1) \). Such an expansion is illustrated by the above computation up to \( \mathcal{W}_3 \), and will be shown to arbitrary \( \mathcal{W}_n \) (\( n > 0 \)) by the application of almost analytic machinery in section 4. The only exception \( \mathcal{W}_0 \) contains a term proportional to \( \ln \lambda \) in its asymptotic expansion. Thus, we write \( W_0^{(0)} = i \sum_{J_f} \tau(\hat{\phi}_f - \frac{N_f}{2\lambda} \ln \lambda) \) so that equation (2.24) is valid for all \( n \).

We define the perturbative spinfoam state-sum in the large spin regime with the spin effect action given by equation (2.5) \( (N_f \) denotes the number of triangles in \( K \))

\[
A_{\tau}(\mathcal{K}) = (2\lambda)^{N_f} e^{i \mathcal{W}_0} \prod_{J_f = -\infty}^{\infty} \left( \frac{\partial^J_f}{\partial J_f^{\lambda}} \right) e^{\lambda \sum_f \mathcal{W}_1(\hat{\phi}_f + \frac{\lambda}{2})} \exp[\mathcal{W}_0(\lambda \hat{\phi}_f + o(\lambda^{-1}))\tau (\hat{\phi}_f)],
\]

(2.25)

where \( \tau(\hat{\phi}_f) \) is a compact support smooth function on \( \mathbb{R}^{N_f} \) which comes from a partition of unity [13] \( \sum_{J_f} \tau(\hat{\phi}_f) = 1 \) on the space of \( J_f \). So, the spinfoam state-sum in the large spin regime can be written as

\[
A(\mathcal{K}) = \sum_f \sum_{J_f = -\infty}^{\infty} d_{J_f} \tau(J_f) \exp[\mathcal{W}_0(\lambda J_f)].
\]

(2.26)

If we define the spin fluctuation \( \hat{\phi}_f = \lambda J_f - \frac{\lambda}{2} \) with \( \Delta \hat{\phi}_f = \frac{\lambda}{2} \), then

\[
A_{\tau}(\mathcal{K}) = e^{i \mathcal{W}_0} \prod_{J_f = -\infty}^{\infty} \left( 2\hat{\phi}_f + 2J_f + 1 \right) e^{\lambda \sum_f \mathcal{W}_1(\hat{\phi}_f + \frac{\lambda}{2})} \tau(\hat{\phi}_f) \exp[\lambda^{-1} \mathcal{W}_0(\hat{\phi}_f)]
\]

(2.27)

Therefore as an effective theory, the spinfoam state-sum may be viewed analogously as a partition function of a (classical) spin system with the (complex) Hamiltonian

\[
\sum_{J_f} \mathcal{W}_1(\hat{\phi}_f) + \frac{1}{\lambda^2} \sum_{J_f, J_{f'}} \mathcal{W}_1(\hat{\phi}_f, \hat{\phi}_{f'}) + \frac{1}{\lambda^4} \sum_{J_f, J_{f'}, J_{f''}} \mathcal{W}_1(\hat{\phi}_f, \hat{\phi}_{f'}, \hat{\phi}_{f''}) + o(\lambda^{-4} \mathcal{W}_0^4),
\]

(2.28)

where \( \mathcal{W}_0 = i \lambda \phi_f + o(\lambda^{-1}) \) is an external (complex) source in the system, and the complex coefficients \( \mathcal{W}_1(\hat{\phi}_f) \) defines the interactions between the spin fluctuations \( \hat{\phi}_f \) located at each triangle \( f \) of the simplicial complex \( \mathcal{K} \).

Although the quadratic term \( \mathcal{W}_1(\hat{\phi}_f, \hat{\phi}_{f'}) \) has been shown to be local (\( \mathcal{W}_1(\hat{\phi}_f, \hat{\phi}_{f'}) \) vanishes unless the triangles \( f, f' \) belong to the same tetrahedron \( e \)), the interactions \( \mathcal{W}_1(\hat{\phi}_f, \hat{\phi}_{f'}) \) is in general nonlocal, i.e. the spin \( \hat{\phi}_f \) located at the triangle \( f \) has the interactions not only with neighboring spins, but also with many other spins located at the triangles far away from \( f \). It is because
the Hessian matrix $H$ with respect to the variables $g_{ee}, z_{ef}$ is not block-diagonal (see [8] for a computation of Hessian matrix), the inverse $H^{-1}$ may have nonzero entries for all the matrix elements\(^{13}\).

### 3. (Almost) analytic machinery

In order to understand the semiclassical behavior of the spinfoam state-sum, we should perform the spin-sum for $A(K)$ in the spin partition function equation (2.27). It relies on a more detailed understanding of the functional properties of spin effective action $\mathcal{W}_K[J_f]$. In particular, equation (2.24) should be shown for general $n$.

Let us come back to the asymptotic analysis of the partial amplitude $A_{\lambda,j_f}(K) = \exp \mathcal{W}_K[J_f]$ within a neighborhood of the large background spin $\bar{f}_f = \lambda \bar{f}_f$. The partial amplitude $A_{\lambda,j_f}(K)$ equation (1.9) is of the form\(^{14}\)

$$
\int e^{S[j_f, g_{ee}, z_{ef}]} d\mu[g_{ee}, z_{ef}],
$$

(3.1)

where we see that the spins $\bar{f}_f$ enters as free parameters and the asymptotics of $A_{\lambda,j_f}(K)$ clearly depends on $\bar{f}_f$. The analysis in [7–9] applies the generalized stationary phase approximation (theorem 2.1) to the above type of integral (the method generalizes the standard stationary phase approximation to the case of complex action). It is shown that the leading order asymptotics of the integral equation (3.1) are contributed by the critical configurations $(j_f, g_{ee}, z_{ef})$ which solves the critical equations $\delta_{g,e}S = 0$ and $\Re(S) = 0$. However, there is a difficulty in the generalized stationary phase analysis of the complex action $S(j_f, g_{ee}, z_{ef})$ with $j_f$ as parameter. Not all values of $j_f$ results in solutions of the critical equations $\delta_{g,e}S = 0$ and $\Re(S) = 0$ in a neighborhood of $(\bar{f}_f, \bar{g}_{ee}, \bar{z}_{ef})$. The values of $j_f$ which result in solutions are Regge-like spin configurations, which can be interpreted geometrically as triangle areas. For the non-Regge-like spins $j_f$ which do not result in critical solutions, the integral equation (3.1) decays as $\lambda$ becomes large in the following way:

$$
|A_{\lambda,j_f}(K)| \leq \left( \frac{1}{\lambda} \right)^N \frac{C}{\min(\delta_{g,e}S^2 - \Re(S))^N}, \quad \forall N \in \mathbb{Z}^+,
$$

(3.2)

where $C$ is a constant. Given that there is a finite distance in $\mathbb{R}^n$ between the non-Regge-like spin configuration $j_f$ and a Regge-like spin configuration, the above estimate implies the partial amplitude only gives a small contribution bounded by $o(\lambda^{-N})$ as $\lambda$ becomes large. However, when the non-Regge-like spin configuration is close to a Regge-like one (in $\mathbb{R}^n$ distance), the above estimate may not imply $A_{\lambda,j_f}(K)$ to be small, because $\min(\delta_{g,e}S^2 - \Re(S))$ may be close to zero as the $j_f$ is close to be Regge-like, so that $\min(\delta_{g,e}S^2 - \Re(S))^{-N}$ becomes

\(^{13}\) In the spinfoam action $S$, the $(j_f, g_{ee}, z_{ef})$-variables located far away from each other on $K$ do not correlate. The nonzero matrix elements in the Hessian matrix $H$ are close to diagonal, although $H$ is not block-diagonal. As an example, we consider a tridiagonal matrix with its inverse

$$
T = \begin{pmatrix}
  a_1 & b_1 & 0 & \cdots & 0 \\
  c_1 & a_2 & b_2 & \cdots & 0 \\
  0 & c_2 & a_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & c_{n-1} & a_n
\end{pmatrix}
$$

and

$$
(T^{-1})_{ij} = \begin{cases}
  (1)^{i+j}b_i \cdots b_{i-j-1} \frac{\theta_{i+j+1}}{\theta_0} & \text{if } i \leq j \\
  (1)^{i+j}c_j \cdots c_{j-i-1} \frac{\phi_{i+j+1}}{\phi_0} & \text{if } i > j
\end{cases}
$$

(2.29)

where $\{\theta_i\}$, satisfy the recurrence relation $\theta_i = a_i \theta_{i-1} - b_i \phi_{i-1} \theta_{i-2}$ for $i = 2, 3, \ldots, n$ with initial conditions $\theta_0 = 1$, $\theta_1 = a_1$, $\{\phi_i\}$ satisfy $\phi_i = a_i \phi_{i-1} - c_i \phi_{i-2}$ for $i = n - 1, \ldots, 1$ with the initial conditions $\phi_{n+1} = 1$ and $\phi_n = a_n$ [16].

\(^{14}\) The integral is convergent once we gauge-fix the $\text{SL}(2, \mathbb{C})$ gauge freedom at each vertex, see [17] for details.
large and may cancel the decaying factor $(1/\lambda)^N$. For example, since the spin-gap $\Delta f_j = \frac{1}{\lambda}$, as a reasonable guess, $\min(|\delta z_j S|^2 - \Re(S))$ for a non-Regge-like neighbor of a Regge-like spin may be of the order $1/\lambda$, which cancels the decaying behavior given by $(1/\lambda)^N$. Therefore, the generalized stationary phase analysis does not clarify how much a partial amplitude $A_{j,f}(K)$ with a generic non-Regge-like spin configuration contributes to the spinfoam state-sum. But in order to analyze the spinfoam state-sum, we have to understand the contributions from all the partial amplitudes with both Regge-like and non-Regge-like spins, at least inside a neighborhood of $j_f$.

The above difficulty is from the fact that the non-Regge-like spins do not result in any critical point, which prevents us to write a closed formula to estimate the integral equation (3.1) for all possible spins. The mathematical reason of the difficulty may be described as the follows: The integral equation (3.1) are defined on an $n$-dim real space but with a complex action. The equations of motion are obtained by the variation of the complex action with $n$ real variables $x$, which results in $n$ complex equations, i.e. $2n$ real equations, for $n$ real variables. Then, although the Hessian matrix $\delta^2 S|_x$ is assumed to be nondegenerate, the implicit function theorem does not imply that there is always a solution for the $2n$ equations of motion, since the system is over-constrained. Furthermore, there are even some additional constraints coming from $\Re(S) = 0$.

However, the problem of over-constrained equations may be overcome if one can complexify the space and analytic-continuing the action $S$ to the complexified space. Then, the equations of motion $\delta S$ give $n$ complex equations with $n$ complex variables. The nondegenerate Hessian $\delta^2 S|_x$ at a given critical point always implies a solution in the neighborhood by the complex version of implicit function theorem, at least when $S$ is an analytic function.

Such an idea indeed works and well studied in the mathematical literatures, e.g. [18], see also [13]. The machinery can even apply to the case that $S$ is $C^\infty$ functions, which admits (nonunique) an almost analytic extension\(^{15}\). The result can be summarized in the following (theorem 2.3 in [18], see also theorem 7.7.12 in [13])

**Theorem 3.1.** Let $S(j,x)$, $j \in \mathbb{R}^4$, $x \in \mathbb{R}^N$, be a smooth function in a neighborhood of $(j,\dot{x})$. We suppose that $\Re[S(j,x)] \leq 0$, $\Re[S(j,\dot{x})] = 0$, $\delta S(j,\dot{x}) = 0$, and $\delta^2 S|_x$ is nondegenerate. We denote by $S(j,z)$, $j \in \mathbb{C}^4$, $z = x + iy \in \mathbb{C}^n$ an (nonunique) almost analytic extension of $S(j,x)$ to a complex neighborhood of $(j,\dot{x})$. The equations of motion $\delta S = 0$ define an almost analytic manifold $M$ in a neighborhood of $(j,\dot{x})$, which is of the form $z = Z(j)$. On $M$ and inside the neighborhood, there is a positive constant $C$ such that for $j \in \mathbb{R}^4$

$$-\Re[S(j,z)] \geq C\Im(z)^2,$$  \quad $z = Z(j).$  \hfill (3.3)

We have the following asymptotic expansion for the integral

$$\int e^{iS(j,z)} u(x) \, dx \sim e^{iS(j,Z(j))} \left(\frac{1}{\lambda}\right)^{\frac{n}{2}} \sqrt{\det\left(\frac{2\pi i}{S''|_j[Z(j)]}\right)} \sum_{t=0}^{\infty} \left(\frac{1}{\lambda}\right)^{t} [L_t \tilde{u}](Z(j)), \label{(3.4)}$$

where $u(x) \in C^\infty_0(K)$ is a compact support function on $K$ inside the domain of integration. $N$ is the number of independent variables, the same as the number of holomorphic $z$-variables. The differential operator $L_t$ is defined in the same way as in theorem 2.1 but operates on an almost analytic extension $\tilde{u}(z)$ of $u(x)$ and evaluating the result at $z = Z(j)$. The branch of the

\(^{15}\) An almost analytic extension $\tilde{f}$ of $f \in C^\infty(\mathbb{R})$ in a neighborhood $\omega$ satisfies (1) $\tilde{f} = f$ in $\omega \cap \mathbb{R}$, (2) $|\tilde{f}| \leq C|\Im(z)|^N$ for all $\omega \in \mathbb{Z}_+$, i.e. $\tilde{f}$ vanishes to infinite order on the real axis.
square-root is defined by requiring \( \sqrt{\det(2\pi i/S[j, Z(j)])} \) to deform continuously to 1 under the homotopy:

\[
(1-s) \frac{2\pi i}{S[j, Z(j)]} + sI \in \text{GL}(n, \mathbb{C}), \quad s \in [0, 1].
\]

(3.5)

Note that the asymptotic expansions from two different almost analytic extensions of the pair \( S(j, x) \), \( u(x) \) are different only by an contribution bounded by \( C_K \lambda^{-K} \) for all \( K \in \mathbb{Z}_+ \).

4. Analytic extension of spinfoam action

When we apply the almostanalytic machinery to the spinfoam action equation (1.2), we find that \( S[j_f, g_{ve}, z_{ef}] \) is actually an analytic function on a neighborhood of \( (j_f, \tilde{g}_{ve}, \tilde{z}_{ef}) \) (without touching a singularity), thus admitting an unique analytic extension. Indeed, as it was mentioned previously, the arguments of the logarithmics are the ratios of polynomials in the variables \( g_{ve}, a_{ve}, b_{ve}, c_{ve}, d_{ve} \) and their complex conjugates, thus are ratios of the polynomials of their real and imaginary parts. Thus, we can analytically extend the spinfoam action \( S \) from the space of spinfoam configurations \( \mathcal{M}_{j_f, g_{ve}} \) to its complexification \( \mathcal{M}_{j_f, g_{ve}}^C \) by replacing all the real variables (the real and imaginary part of spinfoam variables) by complex variables, or equivalently treating \( g_{ve}, a_{ve}, b_{ve}, c_{ve}, d_{ve} \) and their complex conjugates as independent complex variables. The resulting spinfoam action \( S \) is an analytic function in a complex neighborhood of \( (j_f, \tilde{g}_{ve}, \tilde{z}_{ef}) \) in \( \mathcal{M}_{j_f, g_{ve}}^C \).

We first make a precise definition of the complexified space \( \mathcal{M}_{j_f, g_{ve}}^C \). We define two types of group variables \( (g_{ve}, \tilde{g}_{ve}) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \simeq SO(4, \mathbb{C}) \) parameterized by

\[
g_{ve} = \begin{pmatrix} a_{ve} & b_{ve} \\ c_{ve} & d_{ve} \end{pmatrix}, \quad \tilde{g}_{ve} = \begin{pmatrix} \tilde{a}_{ve} & \tilde{b}_{ve} \\ \tilde{c}_{ve} & \tilde{d}_{ve} \end{pmatrix}
\]

(4.1)
as the complexifications of the \( SL(2, \mathbb{C}) \) group variables. They give the complexified Lorentz transformations on complexified Minkowski space. The spinorial representation of the complexified \( SL(2, \mathbb{C}) \) consists of both dotted and undotted spinors, as the complexification of the spinorial representation of \( SL(2, \mathbb{C}) \). Therefore, we define two types of \( \mathbb{C}P^1 \) spinor \( (z_{ef}, \tilde{z}_{ef}) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \) parameterized by

\[
z_{ef}^\alpha = \left( \frac{1}{\bar{\zeta}_{ef}} \right), \quad \tilde{z}_{ef}^\alpha = \left( \frac{1}{\tilde{\zeta}_{ef}} \right).
\]

(4.2)
The complexified space of spinfoam configurations \( \mathcal{M}_{j_f, g_{ve}}^C \) is a collection of the data \( (j_f, \tilde{g}_{ve}, \tilde{z}_{ef}, \tilde{\zeta}_{ef}) \), where \( j_f \) may be also taken as a complex variable.

The analytic extended action is given by

\[
S = \sum_{(j_f)} j_f \ln \left( \frac{(g_{ve}z_{ef} \cdot \tilde{g}_{ve}z_{ef})^2}{(g_{ve}z_{ef} \cdot \tilde{g}_{ve}z_{ef})} \right) + i\gamma \ln \left( \frac{(g_{ve}z_{ef} \cdot \tilde{g}_{ve}z_{ef})}{(g_{ve}z_{ef} \cdot \tilde{g}_{ve}z_{ef})} \right),
\]

(4.3)

where \( Z_1 \cdot \tilde{Z}_2 \equiv \delta_{a\bar{a}}Z_1^{\bar{a}}\tilde{Z}_2^a \) which appears to be not manifestly Lorentz invariant because of the time-gauge imposed in the spinfoam model\(^{16}\). The analytic extended spinfoam action \( S[j_f, g_{ve}, \tilde{g}_{ve}, \tilde{z}_{ef}, \tilde{\zeta}_{ef}] \) is a holomorphic function on the complexified space \( \mathcal{M}_{j_f, g_{ve}}^C \). When we impose the reality condition that

\[
\tilde{g}_{ve} = \tilde{g}_{ve}, \quad \tilde{z}_{ef} = \tilde{z}_{ef}.
\]

(4.4)

\(^{16}\) See [19] for a discussion about the Lorentz covariance of the EPRL spinfoam model.
The analytic extended action reduces the usual spinfoam action in equation (1.2) by identifying (the left-hand side is the variable in equation (4.3) and the right-hand side is the variable in equation (1.2))

$$g_{ve} \sim g_{ve}, \quad \tilde{z}_{ef} \sim z_{ef}. \quad (4.5)$$

The continuous gauge freedom of the analytic extended action is described in the following: First of all at each vertex $v$, there is a $SO(4, \mathbb{C})$ gauge freedom. Given $(x_v, \tilde{x}_v) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, the action equation (4.3) is invariant under the following transformation

$$g_{ve} \mapsto x_v^{-1} g_{ve}, \quad \tilde{z}_{ef} \mapsto x_v^T \tilde{z}_{ef}$$

$$g_{ve} \mapsto \tilde{x}_v^{-1} \tilde{g}_{ve}, \quad \tilde{z}_{ef} \mapsto \tilde{x}_v \tilde{z}_{ef} \quad (4.6)$$

since the transformation leaves the combinations $g_{ve}^T \tilde{z}_{ef}, \tilde{g}_{ve} \tilde{z}_{ef}$ invariant. Secondly, on each edge $e$, there is a $SL(2, \mathbb{C})$ gauge freedom. Given $h_e \in SL(2, \mathbb{C})$, we define $\hat{h}_e = (h_e^{-1})^T$ and parameterize $h_e, \hat{h}_e$ by

$$h_e = \left( \begin{array}{cc} \mu_e & -\tilde{v}_e \\ v_e & \mu_e \end{array} \right), \quad \hat{h}_e = \left( \begin{array}{cc} \tilde{\mu}_e & -\tilde{v}_e \\ \tilde{v}_e & \mu_e \end{array} \right) \quad (4.7)$$

with $\mu_e \tilde{\mu}_e + v_e \tilde{v}_e = 1$. Here, we emphasize that $\mu_e, \tilde{\mu}_e$ are two independent complex variables and the same for $v_e, \tilde{v}_e$, so that both $h_e, \hat{h}_e \in SL(2, \mathbb{C})$. The analytically extended action $S$ is invariant under the following gauge transformation on the edge:

$$g_{ve} \mapsto h_e g_{ve}, \quad \tilde{g}_{ve} \mapsto \hat{h}_e \tilde{g}_{ve} \quad (4.8)$$

Inside each inner product, $h_e, \hat{h}_e$ cancel each other by $h_e^T \hat{h}_e = 1$. Thirdly, the following $\mathbb{C} \times \mathbb{C}$ complex scaling for each pair $(z_{ef}, \tilde{z}_{ef})$

$$z_{ef} \mapsto \lambda z_{ef}, \quad \tilde{z}_{ef} \mapsto \tilde{\lambda} \tilde{z}_{ef}, \quad \forall (\lambda, \tilde{\lambda}) \in \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\}. \quad (4.9)$$

The equations of motion $\delta_{g, \tilde{g}, z} S = 0$ can be computed straightforwardly. Unfortunately, the resulting equations are complicated in the sense that it is unclear to us so far about the geometrical/physical interpretation of these equations. However, it is clear to us that the equations of motion are a set of algebraic equations, which can be solved in principle at least in a neighborhood $\Omega^C \in \mathcal{M}_{j_f, \mu}^C$ of $(j_f, \tilde{g}_{ve}, \tilde{z}_{ef})$ modulo gauge freedom, provided that the Hessian matrix is nondegenerate. The resulting solution defines an analytic manifold $Z(j)$ in $\Omega^C$, where $Z(j)$ stands for $(g_{ve}(j), \tilde{g}_{ve}(j), z_{ef}(j), \tilde{z}_{ef}(j))$ modulo gauge transformations. The background data is certainly a point on the manifold $Z(j)$ with $\tilde{Z} \equiv Z(j) = (g_{ve}(j), \tilde{g}_{ve}(j), \tilde{z}_{ef}(j), z_{ef}(j))$.

Formally, we insert the solution $Z(j)$ into the asymptotic expansion equation (3.4). We see that $\lambda S[j, Z(j)] \equiv \lambda \sum_j j_f \tilde{F}_j[j] Z[j]$ is the leading order contribution of the spin effective action $W_{\mathcal{K}}[\lambda j_f]$. The spin effective action can be written in the following way:

$$W_{\mathcal{K}}[\lambda j_f] = \lambda S[j, Z(j)] - \frac{N_g}{2} \ln \lambda + \frac{1}{2} \ln \det \left( \frac{2\pi i}{S'[j, Z(j)]} \right) + \ln \mu[Z(j)] + o(\lambda^{-1}) \quad (4.10)$$

although we do not derive an explicit expression of $S[j, Z(j)]$ as a function of spins $j_f$. We do know that $S[j, Z(j)]$ is an analytic function of $j_f$ in a neighborhood of $j_f$, since $S[j, \tilde{g}_{ve}, \tilde{z}_{ef}]$ is a holomorphic function and $Z(j)$ is analytic in $\Omega^C$. By an analytic power-series expansion of $S[j, Z(j)]$, we obtain that

$$S[j, Z(j)] = S[j, Z(\hat{\lambda})] + \sum_f s_f \frac{\partial S[j, Z]}{\partial j_f} \bigg|_{j, \hat{\lambda}} + \sum_{f < f'} s_f \frac{\partial S[j, Z]}{\partial Z} \bigg|_{j, \hat{\lambda}} \frac{\partial Z(j)}{\partial j_f} + o(s_f^2), \quad (4.11)$$

The equations of motion are a set of algebraic equations, which can be solved in principle at least in a neighborhood $\Omega^C \in \mathcal{M}_{j_f, \mu}^C$ of $(j_f, \tilde{g}_{ve}, \tilde{z}_{ef})$ modulo gauge freedom, provided that the Hessian matrix is nondegenerate. The resulting solution defines an analytic manifold $Z(j)$ in $\Omega^C$, where $Z(j)$ stands for $(g_{ve}(j), \tilde{g}_{ve}(j), z_{ef}(j), \tilde{z}_{ef}(j))$ modulo gauge transformations. The background data is certainly a point on the manifold $Z(j)$ with $\tilde{Z} \equiv Z(j) = (g_{ve}(j), \tilde{g}_{ve}(j), \tilde{z}_{ef}(j), z_{ef}(j))$.

Formally, we insert the solution $Z(j)$ into the asymptotic expansion equation (3.4). We see that $\lambda S[j, Z(j)] \equiv \lambda \sum_j j_f \tilde{F}_j[j] Z[j]$ is the leading order contribution of the spin effective action $W_{\mathcal{K}}[\lambda j_f]$. The spin effective action can be written in the following way:

$$W_{\mathcal{K}}[\lambda j_f] = \lambda S[j, Z(j)] - \frac{N_g}{2} \ln \lambda + \frac{1}{2} \ln \det \left( \frac{2\pi i}{S'[j, Z(j)]} \right) + \ln \mu[Z(j)] + o(\lambda^{-1}) \quad (4.10)$$

although we do not derive an explicit expression of $S[j, Z(j)]$ as a function of spins $j_f$. We do know that $S[j, Z(j)]$ is an analytic function of $j_f$ in a neighborhood of $j_f$, since $S[j, \tilde{g}_{ve}, \tilde{z}_{ef}]$ is a holomorphic function and $Z(j)$ is analytic in $\Omega^C$. By an analytic power-series expansion of $S[j, Z(j)]$, we obtain that

$$S[j, Z(j)] = S[j, Z(\hat{\lambda})] + \sum_f s_f \frac{\partial S[j, Z]}{\partial j_f} \bigg|_{j, \hat{\lambda}} + \sum_{f < f'} s_f \frac{\partial S[j, Z]}{\partial Z} \bigg|_{j, \hat{\lambda}} \frac{\partial Z(j)}{\partial j_f} + o(s_f^2), \quad (4.11)$$
implies that \( \sum \lambda \) decays as
\[
\sum \lambda \sim e^{\ln \lambda - \lambda} = i \gamma \Theta_f, \quad \frac{\partial S[j, Z]}{\partial Z} = 0
\]
(4.12)

which reproduces the result of coefficients \( \mathcal{W}_0, \mathcal{W}'_0 \) in equation (2.5) at their leading orders [see equations (2.15) and (2.17)]. The \( \ln \lambda \)-term and \( o(1) \)-term of \( \mathcal{W}_0 \) in equation (2.15) is also easily recovered in equation (4.10), since \( S''[j, Z(j)] = S''(\lambda) \) and \( \mu[Z(j)] = \mu(\lambda) \) by the (almost) analytic extension. More importantly, by comparing the two expressions of \( \mathcal{W}_k[\lambda j_f] \) in equations (2.5) and (4.10), we find that the leading contributions of all the coefficients \( \mathcal{W}_n \) in equation (2.5) are of \( o(1) \). As a result, all the coefficients \( \mathcal{W}_k[\lambda j_f] \) in equation (2.5) have the following asymptotic expansion:
\[
\mathcal{W}_n = W_n^{(0)} + \left( \frac{1}{\lambda} \right) W_n^{(1)} + \left( \frac{1}{\lambda} \right)^2 W_n^{(2)} + o(\lambda^{-3})
\]
(4.13)

where for \( n = 0 \), we write \( W_n^{(0)} = i \sum_j \gamma_j \Theta_f - \frac{N_c}{2 \pi} \ln \lambda \). It shows that our expectation in the discussion toward equation (2.24) is indeed correct. Moreover, the analyticity of \( S[j, Z(j)] \) implies that \( \sum_{n=0}^{\infty \lambda} W_n^{(0)} j_f + \ldots + j_f \) is convergent in a neighborhood of \( j_f \).

There is another piece of information that we can extract from the almost analytic machinery in theorem 3.1. We find that the action \( S[j, Z(j)] \) is complex, whose real part is constantly non-positive in the neighborhood under consideration, i.e. there is a positive constant \( C \) such that
\[
\Re (S[j, Z(j)]) \leq -C |\Im (Z(j))|^2.
\]
(4.14)

It implies that there is an ‘exponentially decaying’ factor in equation (3.4) except the Regge-like spin configurations:
\[
e^{\Re (S[j, Z(j)])} \leq e^{-C \lambda |\Im (Z(j))|^2}.
\]
(4.15)

If we make an power-series expansion of the analytic function \( Z(j) \):
\[
Z(j) = Z(\hat{j}) + \sum_j s_j \partial_j Z(\hat{j}) + \frac{1}{2} \sum_j s_j s_j \partial_j \partial_j Z(\hat{j}) + o(s_j^2),
\]
(4.16)

where \( Z(\hat{j}) \) is real since it recovers the background spinfoam variables certainly without complexification. Therefore, we obtain the following expression of the decaying factor:
\[
e^{\Re (S[j, Z(j)])} \leq e^{-C \lambda |\sum_j s_j \partial_j Z(\hat{j}) + \frac{1}{2} \sum_j s_j s_j \partial_j \partial_j Z(\hat{j}) + o(s_j^2)|^2}.
\]
(4.17)

If the fluctuation \( s_f = j_f - \hat{j}_f \sim o(1) \), the factor \( e^{\Re (S[j, Z(j)])} \) is exponentially decaying, which results that the corresponding partial amplitude \( A_{\lambda j_f}(K) \) is bounded by \( o(\lambda^{-N}) \) for all \( N \in \mathbb{Z}_+ \). However, if the fluctuation \( s_f = j_f - \hat{j}_f \sim o(\lambda^{-1/2}) \), i.e. the non-Regge-like \( j_f \) is close to a Regge-like spin configuration \( j_f \), one cannot conclude that the factor \( e^{\Re (S[j, Z(j)])} \) decays as \( \lambda \) large, because the \( e^{-2} \) behavior may be cancelled by \( s_f \sim o(\lambda^{-1/2}) \). It is in general not correct if one ignores the contribution in the state-sum from the partial amplitude with \( s_f = j_f - \hat{j}_f \sim o(\lambda^{-1/2}) \) in the large spin regime.
5. Implementation of spin-sum

5.1. $\lambda^{-1}$-expansion and flatness

In this section, we implement the sum over spins in the large spin regime. By the above asymptotic expansion of the spin effective action $W_{\mathcal{K}}[\lambda, j_f]$, we can write the spin-sum by\textsuperscript{17}

$$A(\mathcal{K}) = \sum_{j_f = -\infty}^{\infty} d_j \tau(j_f) \exp W_{\mathcal{K}}[\lambda, j_f] = \left( \frac{1}{\lambda} \right)^{\frac{N_f}{2}} \sum_{j_f \in \mathbb{Z}/2} d_j e^\sum_j \eta_j \bar{F}_j[Z(j_f)] + o(1) \tau(j_f / \lambda), \quad (5.1)$$

where $0 \leq \tau(j_f) \leq 1$ is a smooth function of compact support located in $j_f \geq o(1) > 0$. Since the summand is a compact support function on $\mathbb{R}^{N_f}$, we apply the Poisson resummation formula to the spin-sum:

$$A(\mathcal{K}) = \left( \frac{1}{\lambda} \right)^{\sum_{j_f \in \mathbb{Z}} \int d_j \sum_{j_f \in \mathbb{Z}/2} J_f \bar{F}_j[Z(j_f)] + o(1) \tau(j_f / \lambda)$$

$$= \left( \frac{1}{\lambda} \right)^{\sum_{j_f \in \mathbb{Z}/2} (\sum_{j_f \in \mathbb{Z}/2} J_f \bar{F}_j[Z(j_f)] + o(1) \tau(j_f) \right). \quad (5.2)$$

For each branch $k_f$, we can study the integral using the stationary phase approximation, and we obtain the equation of motion:

$$\bar{F}_j[Z(j_f)] = 4\pi i k_f, \quad k_f \in \mathbb{Z}. \quad (5.3)$$

where we have used $\frac{\delta S[Z(j_f)]}{\delta Z} = 0$ because of the equations of motion from analytically extended spin foam action. On the other hand, by equation (4.14), \( \Im(S) = 0 \) implies $\Im(Z(j)) = 0$, i.e. $Z(j)$ reduces back to the critical data $g_{\nu\nu}, z_{\nu f}$. Then, the equation of motion reduces to

$$\gamma \tilde{g}_f = 4\pi k_f, \quad k_f \in \mathbb{Z}. \quad (5.4)$$

Therefore, all the critical points of $j_f$ are given by the critical configurations with $\gamma \tilde{g}_f = 0$ mod $4\pi \mathbb{Z}$, when we focus on the globally oriented and time-oriented critical configurations of Lorentzian geometry. This reproduces the flatness result given in [10]. Such a flatness as the leading contribution in $\lambda^{-1}$ expansion is exactly resulting from the non-Regge-like fluctuation of $j_f$. Suppose we restrict the spin-sum only in the subset of Regge-like spin, as the leading contribution of the state-sum in $\lambda^{-1}$, we would obtain a sum of $e^{\delta S_{\text{Regge}}}$ over the Regge-like spin configurations, instead of obtaining contributions only from flat (mod $4\pi \mathbb{Z}$) simplicial geometries. As a result, the spin foam state-sum in the large spin regime can be written as a sum over critical configurations $(j_f, g_{\nu\nu}, z_{\nu f})$ with $\gamma \tilde{g}_f = 4\pi \mathbb{Z}$. (we set $\tau(j_f) = 1$ at these critical values of $j_f$):

$$A(\mathcal{K}) = \left( \frac{1}{\lambda} \right)^{\sum_{j_f \in \mathbb{Z}/2} (\sum_{j_f \in \mathbb{Z}/2} J_f \bar{F}_j[Z(j_f)] + o(1) \tau(j_f / \lambda) \right), \quad (5.5)$$

where the perturbative effective action $I(j_f, g_{\nu\nu}, z_{\nu f}; \lambda)$ is an asymptotic power-series in $\lambda^{-1}$, with the leading contribution of $o(1)$, i.e. the one-loop $\ln \det(\cdots)$ contribution. $N_f$ denotes the number of triangles in $\mathcal{K}$, and $N_g, N_{\nu f}$ denotes the number of degrees of freedom in $g_{\nu\nu}, z_{\nu f}$.\textsuperscript{17} The spin foam amplitude $A(\mathcal{K})$ under consideration here is actually $A_r$ in the previous context, since we are only interested in the perturbative corrections at different critical points. The non-perturbative effects are taken into account by summing over critical points. $\tau$ is a cut-off function which makes $A(\mathcal{K})$ finite, and let us to zoom into the neighborhood of the background $(j_f, g_{\nu\nu}, z_{\nu f})$ to perform the perturbation expansion rigorously. It does not have any physical consequence to our analysis.
5.2. Nontrivial curvature from re-expansion

It is obvious that the flatness result \( y \hat{\Theta}_J = 0 \mod 4\pi \mathbb{Z} \) relies on the setting that \( \lambda^{-1} \) is the only expansion parameter. However, such a result may be modified by modifying the setting of expansion parameter. It is shown in the following that it is natural to have another expansion parameter \( y \hat{\Theta} \) (background curvature) in addition to \( \lambda \) (background spin). The two-parameter expansion includes the non-flat curvatures into the effective degrees of freedom, while the expansion with respect to \( y \hat{\Theta} \) is interpreted to be a low-energy expansion as being a curvature expansion. See section 6 and [23] for more physical motivations about the new expansion.

Let us write the spinfoam state-sum perturbatively at \( (j_f, \hat{g}_e, \hat{z}_e) \)

\[
A_r(\mathcal{K}) = \prod_f d_{j_f} e^{iW_0 Z} 
\]

where

\[
Z = \sum_{s,m=\infty}^{\infty} \prod_f \left( 1 + \frac{2\hat{g}_f}{d_{j_f}} \right) e^{i \Sigma_{s,l} W^{(l)}_{2} w_{j_f} + \bar{w}_{j_f} + \alpha \bar{a}^2} \tau(s_f), 
\]

or

\[
Z = \sum_{\hat{g}_e=\infty}^{\infty} \prod_f \left( 1 + \frac{2\hat{g}_f}{d_{j_f}} \right) e^{i \Sigma_{s,l} W^{(l)}_{2} w_{j_f} + \bar{w}_{j_f} + \alpha \bar{a}^2} \tau(\hat{g}_f), 
\]

where the leading contributions of all coefficients \( W_n \) are of \( o(1) \), and recall that \( \tau(s) \) is a smooth function of compact support from a partition of unity in the space of \( j_f \). \( W_0 \) contains the Regge action of GR as the leading order contribution, \( W_1 \) contains the Regge deficit angle as the leading order contribution

\[
W_0 = i \sum_j y j_f \hat{\Theta}_j - \frac{N_{\xi,\lambda}}{2\lambda} \ln \lambda + o(\lambda^{-1}), \quad W_1 = iy \hat{\Theta}_j + o(\lambda^{-1}).
\]

Then, the spin-sum \( Z \) can be written in the following form by using the Poisson resummation formula:

\[
Z = 2^{N_f} \sum_{k_f \in \mathbb{Z}} \int_{\mathbb{R}^{N_f}} [d\bar{s}_f] \prod_f \left( 1 + \frac{2\hat{g}_f}{2\lambda j_f + 1} \right) e^{i \Sigma_{s,l} (y \hat{\Theta}_j - 4\pi k_f) \bar{s}_{j_f} + \frac{1}{2} \Sigma_{l,l'} W^{(l)}_{2} w_{j_f} + \alpha \bar{a}^2} \tau(s_f) 
\]

\[
= (2\lambda)^{N_f} \sum_{k_f \in \mathbb{Z}} \int_{\mathbb{R}^{N_f}} [d\bar{s}_f] \prod_f \left( 1 + \frac{2\hat{g}_f}{2\lambda j_f + 1} \right) e^{i \Sigma_{s,l} (y \hat{\Theta}_j - 4\pi k_f) \bar{s}_{j_f} + \frac{1}{2} \Sigma_{l,l'} W^{(l)}_{2} w_{j_f} + \alpha \bar{a}^2} \tau(s_f)
\]

when we ignore the \( o(\lambda^{-1}) \)-terms in \( W_1 \).

Now we assume the value of \( y \hat{\Theta}_j - 4\pi k_f \equiv \alpha \hat{X}_f \) is nonzero but very small for some certain \( k_f \), where we make a new scaling parameter \( \alpha \ll 1 \) and \( \hat{X}_f \sim o(1) \). Then, we have

\[
Z = (2\lambda)^{N_f} \sum_{k_f \in \mathbb{Z}} \int_{\mathbb{R}^{N_f}} [d\bar{s}_f] \prod_f \left( 1 + \frac{2\hat{g}_f}{2\lambda j_f + 1} \right) e^{i \Sigma_{s,l} \beta \bar{X}_{j_f} + \alpha \bar{a}^2} \tau(s_f),
\]

where we have defined \( \beta \equiv \lambda \alpha \) and treat \( \beta \) and \( \lambda \) as two independent scaling parameters, as a reparametrization of the space of parameters \( \alpha, \lambda \). \( A_r(\mathcal{K}) \) is written as

\[
A_r(\mathcal{K}) = \prod_f d_{j_f} e^{i\beta \Sigma_{s,l} \bar{X}_{j_f} + o(1)} Z.
\]
We make the stationary phase approximation of the integrals in different branch $k_f$ with respect to the $\lambda$-scaling. The critical equations $\Re(\sum_{f,f'} W^f_{s,f} s_f s_{f'} + \alpha(s^3)) = 0$ and $\delta x_s (\sum_{f,f'} W^f_{s,f} s_f s_{f'} + \alpha(s^3)) = 0$ are solved by $s_f = 0$. Recall theorem 2.1, the $\lambda^{-1}$ corrections are given by

$$
\left( \frac{1}{\lambda} \right)^s L_s[\exp \sum_j X_j s_j \mu(s_j)]_{s_j=0} \quad \text{where} \quad \mu(s_f) = \prod_f \left( 1 + \frac{2\lambda s_f}{2\lambda j_f + 1} \right) \tau(s_f)
$$

(5.12)

Recall that $L_s$ is a differential operator of order $2s$, so we obtain the following power-counting

$$
\left( \frac{1}{\lambda} \right)^s L_s[\exp \sum_j X_j s_j \mu(s_j)]_{s_j=0} = \sum_{\beta} \frac{2^s}{\lambda^s} f_{s,\beta} = \sum_{\beta} \alpha \lambda^{-s-1} f_{s,\beta},
$$

(5.13)

where we find that such an expansion is valid only when $\beta \ll \lambda$, i.e.

$$\alpha \ll \lambda^{-1/2}.
$$

(5.14)

When such a perturbative expansion is valid, the critical configurations with nonzero $\gamma/\Theta_f - 4\pi k_f \equiv \alpha X_f \ll o(\lambda^{-1/2})$ can contribute to the leading order approximation of $A(\lambda)$:

$$
A(\lambda) = \left( \frac{1}{\lambda} \right)^{2s-2n_f} \sum_{(j_f, g_{s,\alpha} Z_f)} j_f \exp \sum_j X_j + \cdots
$$

(5.15)

where $(j_f, g_{s,\alpha} Z_f)$ denote the critical configurations with $\gamma/\Theta_f - 4\pi k_f \equiv \alpha X_f \ll o(\lambda^{-1/2}),$ and '···' stands for the contributions of $o(1)$ and $o(\beta^\alpha \lambda^{-s})$. On the exponential $i\beta \sum_j j_f X_j$ is an analogue of Regge action if we identify $\gamma j_f$ to be the area of the triangle $f$

$$
\exp \left( i\beta \sum_j j_f X_f \right) = \exp \left( i\beta \sum_j j_f / \Theta_f \right)
$$

(5.16)

but here $\gamma/\Theta_f$ is close to $2 \pi \mathbb{Z}$ up to a difference much smaller than $o(\lambda^{-1/2})$.

### 5.3. Semiclassical low energy effective action

Let us try to understand the above expansion in a more detailed way. We focus on the integral with $k_f = 0$ in $A(\lambda)$. The integrals with nonzero $k_f$ can be analyzed in the same way. We apply the technique in theorem 3.1 to the action

$$
\lambda S[s_f] = i\lambda \sum_j j_f / \Theta_f + \lambda \left[ \sum_f i\gamma / \Theta_f s_f + \sum_{f,f'} W^f_{s,f} s_f s_{f'} + o(s^3) \right]
$$

(5.17)

with $\Re(S) \leq 0$ by equation (3.3).

If we consider the action $S[s_f] \equiv S[\gamma / \Theta_f, s_f]$, where $\gamma / \Theta_f$ is treated as a parameter here, we find

$$
\gamma / \Theta_f = 0 \quad \text{and} \quad s_f = 0 \quad \Rightarrow \quad \Re(S) = 0 \quad \text{and} \quad \delta s_s S = 0.
$$

(5.18)

The critical point $\gamma / \Theta_f = 0$, $s_f = 0$ for the action $S[\gamma / \Theta_f, s_f]$ fulfills the assumption of the almost analytic machinery in theorem 3.1. We apply the almost analytic machinery to $S[\gamma / \Theta_f, s_f]$ by analytic continuing $s_f$ to the complex variables, $S$ is an analytic function of $s_f$, then the equation of motion $\delta_s S = 0$ gives an analytic manifold $s_f = Z_f(\gamma / \Theta_f)$ at least locally. The integral in the branch $k_f = 0$ in $S$ is expressed as an asymptotic expansion

$$
\exp \sum_j j_f / \Theta_f, Z_f = 0 = e^{iS[\gamma / \Theta_f, X_f(\gamma / \Theta_f)]} \left( \frac{1}{\lambda} \right)^{\frac{s_f}{\gamma}} \sqrt{\det \left( \frac{2\pi i}{\delta^2_{s,s'} S[\gamma / \Theta_f, Z_f(\gamma / \Theta_f)]} + \alpha \left( \frac{1}{\lambda} \right) \right)}
$$

(5.19)

$$
\lambda S[s_f] = i\lambda \sum_j j_f / \Theta_f + \lambda \left[ \sum_f i\gamma / \Theta_f s_f + \sum_{f,f'} W^f_{s,f} s_f s_{f'} + o(s^3) \right]
$$

(5.17)
The leading effective action $S[\gamma \hat{\Theta}_f, Z_f(\gamma \hat{\Theta}_f)]$ has the property following from equation (3.3) that
\[ \Re(S[\gamma \hat{\Theta}_f, Z_f(\gamma \hat{\Theta}_f)]) \leq -C|\Im(Z_f(\gamma \hat{\Theta}_f))|^2. \] (5.20)
We can compute more concretely the expression of effective action $S[\gamma \hat{\Theta}_f, Z_f(\gamma \hat{\Theta}_f)]$ as a power series of $\gamma \hat{\Theta}_f$. We expand the action $S[\gamma \hat{\Theta}_f, s_f]$ at the first order solution (in $\gamma \hat{\Theta}_f$) from equation of motion $\delta_{s_f}S = 0$:
\[ s_f = -\frac{i}{2} \sum_{f, f'} (W^{-1})_{f, f'} \gamma \hat{\Theta}_f. \] (5.21)
If we define a short-hand notation: $y_f \equiv s_f + \frac{1}{2} \sum_{f'} (W^{-1})_{f, f'} \gamma \hat{\Theta}_f$, then the expansion of the action reads
\[ \lambda S[\gamma \hat{\Theta}_f, y_f] = i \lambda \sum_{f} \gamma f \hat{\Theta}_f + \lambda \left[ \frac{1}{4} \sum_{f, f'} (W^{-1})_{f, f'} \gamma \hat{\Theta}_f^2 \gamma \hat{\Theta}_f + o((\gamma \hat{\Theta}_f)^3) \right] \]
\[ + \lambda \left[ \sum_{f} o((\gamma \hat{\Theta}_f)^2) y_f + \sum_{f, f'} [2W_{f, f'} + o(\gamma \hat{\Theta}_f)] y_f y_{f'} + o(y_f^2) \right]. \] (5.22)
The equation of motion $\delta_{y_f}S[\gamma \hat{\Theta}_f, y_f]$ with respective to $y_f$ gives an $o((\gamma \hat{\Theta}_f)^2)$ correction to the original approximating solution $y_f = 0$, i.e.
\[ y_f = o((\gamma \hat{\Theta}_f)^2) \quad \text{or} \quad s_f = -\frac{i}{2} \sum_{f'} (W^{-1})_{f, f'} \gamma \hat{\Theta}_f + o((\gamma \hat{\Theta}_f)^2). \] (5.23)
If we expand the action $S[\gamma \hat{\Theta}_f, s_f]$ at the new approximating solution and iterating the above procedure, we approximate the exact solution of $\delta_{s_f}S[\gamma \hat{\Theta}_f, s_f] = 0$ better and better and obtain the exact solution $s_f$ as a power-series of $\gamma \hat{\Theta}_f$:
\[ s_f = Z_f(\gamma \hat{\Theta}_f) = \sum_{n=0}^{\infty} \alpha_{f, f_1, \ldots, f_n} \gamma \hat{\Theta}_{f_1} \cdots \gamma \hat{\Theta}_{f_n}, \] (5.24)
where the series has a finite convergence radius since we know that $Z_f(\gamma \hat{\Theta}_f)$ is analytic.
Evaluating the action $S[\gamma \hat{\Theta}_f, s_f]$ at this exact solution gives
\[ \lambda S[\gamma \hat{\Theta}_f, Z_f(\gamma \hat{\Theta}_f)] = i \lambda \sum_{f} \gamma f \hat{\Theta}_f + \lambda \left[ \sum_{n=2}^{\infty} \beta_{f_1, \ldots, f_n}^{(n)} \gamma \hat{\Theta}_{f_1} \cdots \gamma \hat{\Theta}_{f_n} \right], \] (5.25)
where the quadratic order coefficient is given by
\[ \beta_{f_1, f_2}^{(2)} = \frac{1}{4} (W^{-1})_{f_1, f_2}. \] (5.26)
Therefore, as $\gamma \hat{\Theta}_f$ is small,
\[ S[\gamma \hat{\Theta}_f, Z_f(\gamma \hat{\Theta}_f)] = i \sum_{f} \gamma f \hat{\Theta}_f + \frac{1}{4} \sum_{f, f'} (W^{-1})_{f, f'} \gamma \hat{\Theta}_f^2 \gamma \hat{\Theta}_f + o((\gamma \hat{\Theta}_f)^3). \] (5.27)
Following the same procedure for the $k_f \neq 0$ branches in equation (5.9), we obtain in general
\[ S_0[\gamma \hat{\Theta}_f - 4\pi k_f, Z_f(\gamma \hat{\Theta}_f - 4\pi k_f)] = \sum_{f} \gamma f \hat{\Theta}_f \]
\[ + \frac{1}{4} \sum_{f, f'} (W^{-1})_{f, f'} (\gamma \hat{\Theta}_f - 4\pi k_f) (\gamma \hat{\Theta}_f - 4\pi k_f) + o((\gamma \hat{\Theta}_f - 4\pi k_f)^3). \] (5.28)
In general, $S_b$ has a negative real part coming from the terms of quadratic and higher order in $(\gamma \hat{\Theta}_f - 4\pi k_f)$. By equation (5.20), the exponential $e^{S_b}$ in the asymptotic expansion equation (5.19) for generic $k$ decays exponentially unless $\gamma \hat{\Theta}_f$ from the background data $(\hat{j}_f, \hat{g}_{\mu\nu}, \hat{z}_{\kappa\nu})$ is close to one of $\{4\pi k_f : k_f \in \mathbb{Z}\}$. The non-decaying $e^{S_b}$ requires that the non-negative real part $\lambda S_b$ does not scale to be large by $\lambda \gg 1$, which gives the nontrivial restriction to $\gamma \hat{\Theta}_f$, i.e. for a constant $C \sim o(1)$

$$|\gamma \hat{\Theta}_f| \ll C\lambda^{-1/2} \mod{4\pi \mathbb{Z}}. \quad (5.29)$$

which improves the bound equation (5.14).

If we assume $\gamma \hat{\Theta}_f$ is small and the bound equation (6.3) is satisfied, the above analysis lets us obtain a perturbative effective action for the spinfoam state-sum $A(\mathcal{K})$:

$$A(\mathcal{K}) = \prod_f d_{j_f} (1 + j_f^{-1}Z_f(\gamma \hat{\Theta}_f))e^{\lambda_{\mathrm{mod}4}(j_f, \hat{g}_{\mu\nu}, \hat{z}_{\kappa\nu})}, \quad (5.30)$$

where the leading order contribution to the effective action $I_{\mathrm{eff}}$ is given by a power-series of $\gamma \hat{\Theta}_f$

$$I_{\mathrm{eff}}(\hat{j}_f, \hat{g}_{\mu\nu}, \hat{z}_{\kappa\nu}) = \frac{1}{d} \sum_f \gamma j_f \hat{\Theta}_f + \frac{1}{4} \sum_f (W_2^{-1})_{f, f'} \gamma \hat{\Theta}_f \gamma \hat{\Theta}_{f'} + o((\gamma \hat{\Theta}_f)^3)
- \frac{N_{\mathrm{eff}}}{2\lambda} \ln \lambda + o\left(\frac{1}{\lambda}\right). \quad (5.31)$$

We have neglected the $k_f \neq 0$ integrals in equation (5.9) since they are exponentially decaying. Recall equation (2.20) for the expression of $W_2^{1/f'}$. Although $W_2^{1/f'}$ is local in $f, f'$, $(W_2^{1})_{f, f'}$ is nonlocal. $\hat{\Theta}_f$ is the deficit angle of the background simplicial geometry from $(\hat{j}_f, \hat{g}_{\mu\nu}, \hat{z}_{\kappa\nu})$. Thus, the above expression of effective action is a curvature expansion whose leading order is the Regge action.

6. Discussion

The above analysis studies the asymptotic behavior of spinfoam state-sum model by taking into account the sum over spins in the large spin regime. We mainly focus on the contributions from the globally oriented and time-oriented critical configurations of Lorentzian geometry. The analysis shows that:

1. If we only consider a single scaling parameter $\lambda$, which is the large spin parameter, the leading contributions to the spinfoam state-sum $A(\mathcal{K})$ are only given by the critical configurations with their deficit angle satisfying

$$\gamma \hat{\Theta}_f = 0 \mod{4\pi \mathbb{Z}}. \quad (6.1)$$

2. However, if we analyze the situation more carefully with $\gamma \hat{\Theta}_f \ll 1 \mod{4\pi \mathbb{Z}}$, a different expansion of $A(\mathcal{K})$ should be employed by using two scaling parameters $\lambda$ and $\gamma \hat{\Theta}_f$. Such an expansion is valid only when $\gamma \hat{\Theta}_f \ll \lambda^{-1/2} \mod{4\pi \mathbb{Z}}$.

3. The new two-parameter expansion gives a curvature expansion of the semiclassical low energy effective action from the spinfoam model. In the expansion of effective action, the leading contribution is the simplicial Einstein gravity (Regge action), while the UV modifications of Einstein gravity appear as subleading high-curvature corrections.

Let us focus on the branch $k_f = 0$, a small $\gamma \hat{\Theta}_f = \alpha \chi_f \ll o(\lambda^{-1/2})$ may come from two possibilities in particular (the combinations of the two are also possible):
- If we require a small Barbero–Immirzi parameter, e.g. $\gamma \ll \lambda^{-1/2}$, then a finite $\Theta_f \sim o(1)$ is admitted, i.e. we thus identify $a = \gamma$ and $\Theta_f = X_f$. Then, the leading contribution equation (5.15) gives the quantum Regge calculus with a discrete functional integration measure on the space of critical configurations (equivalent to the discrete cotetrad). If we make a further stationary phase approximation with respect to $\beta$-scaling, the leading order contributions are given by the discrete cotetrad satisfying the discrete Einstein equation in the Regge calculus.

The requirement of a small $\gamma$ has already been proposed in the context of spinfoam graviton propagator computation [14, 15], see also [21]. A detailed and systematic study of this possibility is presented in [22].

- We assume the Barbero–Immirzi parameter $\gamma \sim o(1)$, and consider a critical configuration $(f_j, g_{ij}, \varsigma_{ij})$ with the deficit angle $\Theta_f \sim o(X_f \leq o(\lambda^{-1/2})$ (the deficit angles are scaled to be small by $\gamma$). The simplicial geometry from the critical data approximates a continuum geometry with a typical curvature radius $\rho$. The (dimensionful) averaged edge-length is $L$. In the continuum limit, the edge-length is much larger than the curvature radius. Then, the Regge deficit angle can be written approximately by [20]

$$\Theta_f = \frac{\text{Area}(f^\ast)f^2}{\rho^2} \left[ 1 + o\left( \frac{L^2}{\rho^2} \right) \right] \sim o\left( \frac{\lambda^2}{\rho^2} \right),$$

where $\text{Area}(f^\ast)$ is the area of the face dual to the triangle $f$. Therefore, if we keep the Barbero–Immirzi parameter $\gamma \sim o(1)$,

$$\Theta_f \sim o(\lambda^{-1/2}) \iff \rho^2 \gtrsim \lambda^{3/2} L^2,$$

i.e. the typical curvature radius of the background geometry should be much larger than the Planck length. If the large background spin parameter $\lambda \sim 10^4$, then the requirement from equation (6.3) for the curvature radius is $\rho \gtrsim 10^3 L^2 \sim 10^{-32} m$. It means that a large range of nontrivial curvature is still admitted as the background geometry, even when $\gamma \sim o(1)$. The expansion parameters contain the scaling $\alpha$ of deficit angle, so the expansion may be understood as a curvature expansion.

In such an approximation, only the critical configurations with small deficit angles contribute in the leading order. One needs a large triangulation in order to approximate a curved geometry. Such a situation perhaps relates to the regime where the model approximates the GR on the continuum.

The analysis here mainly focuses on the critical configurations of globally oriented, time-oriented Lorentzian geometry. However, for the other critical configurations (with $\varepsilon = -1, \text{sgn}(V_4) = 1$), the analysis can be carried out essentially in the same way [22]. Equation (5.2) is a universal formula valid independent of the choice of critical configurations. Here, we list the critical value of $\mathcal{F}_f$ at different type of critical configurations classified in [8, 9]:

| $\mathcal{F}_f$ | 
|--------------------|
| Lorentz time-oriented: $-i \varepsilon \text{sgn}(V_4) \gamma \Theta_f$ |
| Lorentz time-unoriented: $-i \varepsilon [\text{sgn}(V_4) \gamma \Theta_f + \pi]$ |
| Euclidean: $-i \varepsilon [\text{sgn}(V_4) \Theta_f + \pi n_f]$ |
| Vector: $i \Theta_f$ |

18 In the geometrical interpretation of the spinfoam critical data, we assume the area $\gamma j_f$ is measured in the unit $\lambda L_p^2$, which leads to (1) the area spectrum $A_f = \gamma j_f L^2_p$ coincides with the result from canonical LQG in the large spin regime; (2) the Regge action is obtained by $\lambda \sum_f \gamma j_f \Theta_f = \frac{1}{\rho^2} \sum_f A_f \Theta_f$. 

19
where $\Theta^f_i$ is the deficit angle in Euclidean geometry, $n_f = 0, 1$, and $\Phi^f_i$ is the vector geometry angle. The generalization of the previous results to other critical configurations can be done straightforwardly by replacing the quantity $\gamma^i/\Theta^f_i$ in the previous discussion by other critical values of $F^i_f$.

Acknowledgments

The author would like to thank H Haggard, S Speziale, A Riello, C Rovelli, and M Zhang for many helpful discussions. He also would like to thank Y Ma for the invitation to visit the Center for Relativity and Gravitation, Beijing Normal University, China, where a part of this research work is carried out. The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement no 298786.

References

[1] Thiemann T 2007 Modern Canonical Quantum General Relativity (Cambridge: Cambridge University Press)
[2] Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)
[3] Han M, Huang W and Ma Y 2007 Fundamental structure of loop quantum gravity Int. J. Mod. Phys. D 16 1397–474 (arXiv:gr-qc/0509064)
[4] Rovelli C 2011 Zakopane lectures on loop gravity arXiv:1102.3660
[5] Rovelli C 2011 Simple model for quantum general relativity from loop quantum gravity J. Phys.: Conf. Ser. 314 012006
[6] Barrett J W, Dowdall R J, Fairbairn W J, Hellmann F and Pereira R 2010 Lorentzian spin foam amplitudes: graphical calculus and asymptotics Class. Quantum Grav. 27 165009
[7] Conrady F and Freidel L 2008 On the semiclassical limit of 4D spin foam models Phys. Rev. D 78 104023
[8] Han M and Zhang M 2012 Asymptotics of spin foam amplitude on simplicial manifold: Euclidean theory Class. Quantum Grav. 29 165004 (arXiv:1109.0500)
[9] Han M and Zhang M 2011 Asymptotics of spin foam amplitude on simplicial manifold: Lorentzian theory arXiv:1109.0499
[10] Fairbairn W J and Meusburger C 2012 Quantum deformation of two four-dimensional spin foam models J. Math. Phys. 53 022501
[11] Mizoguchi S and Tada T 1992 3-dimensional gravity from the Turaev-viro invariant Phys. Rev. Lett. 68 1795–6
[12] Hörmander L 1990 The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis 2nd edn (Berlin: Springer)
Modesto L and Rovelli C 2005 Particle scattering in loop quantum gravity Phys. Rev. Lett. 95 191301
Bianchi E, Modesto L, Rovelli C and Speziale S 2006 Graviton propagator in loop quantum gravity Class. Quantum Grav. 23 6989–7028

[15] Rovelli C and Zhang M 2011 Euclidean three-point function in loop and perturbative gravity Class. Quantum Grav. 28 175010 (arXiv:1105.0566)
Zhang M Saddle point expansion unpublished note
[16] Usmani R A 1994 Inversion of a tridiagonal jacobi matrix Linear Algebra Appl. 212 413
[17] Engle J and Pereira R 2009 Regularization and finiteness of the Lorentzian LQG vertices Phys. Rev. D 79 084034
[18] Melin A and Sjöstrand J 1975 Fourier integral operators with complex-valued phase functions Lecture Notes Math. 459 120–223
[19] Rovelli C and Speziale S 2011 Lorentz covariance of loop quantum gravity Phys. Rev. D 83 104029
[20] Feinberg G, Friedberg R, Lee T D and Ren H C 1984 Lattice gravity near the continuum limit Nucl. Phys. B 245 343
Feinberg G and Lee T D 1984 Derivation of Regge action Nucl. Phys. B 242 145
[21] Magliaro E and Perini C 2011 Emergence of gravity from spinfoams Europhys. Lett. 95 30007
Magliaro E and Perini C 2011 Regge gravity from spinfoams arXiv:1105.0216
[22] Han M 2013 Semiclassical analysis in spinfoam model with a small Barbero-immirzi parameter Phys. Rev. D 88 044051 (arXiv:1304.5628)
[23] Han M 2013 Covariant loop quantum gravity, low energy perturbation theory, and Einstein gravity arXiv:1308.4063