Non-linear Schrödinger Dynamics of Matrix D-branes

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Abstract

We formulate an effective Schrödinger wave equation describing the quantum dynamics of a system of D0-branes by applying the Wilson renormalization group equation to the worldsheet partition function of a deformed \( \sigma \)-model describing the system, which includes the quantum recoil due to the exchange of string states between the individual D-particles. We arrive at an effective Fokker-Planck equation for the probability density with diffusion coefficient determined by the total kinetic energy of the recoiling system. We use Galilean invariance of the system to show that there are three possible solutions of the associated non-linear Schrödinger equation depending on the strength of the open string interactions among the D-particles. When the open string energies are small compared to the total kinetic energy of the system, the solutions are governed by freely-propagating solitary waves. When the string coupling constant reaches a dynamically determined critical value, the system is described by minimal uncertainty wavepackets which describe the smearing of the D-particle coordinates due to the distortion of the surrounding spacetime from the string interactions. For strong string interactions, bound state solutions exist with effective mass determined by an energy-dependent shift of the static BPS mass of the D0-branes.
I. INTRODUCTION AND SUMMARY

Despite the enormous amount of activity over the past few years towards understanding the dynamics of Dirichlet $p$-branes [1], the problem of demonstrating that a system of $N$ moving D-particles can form a bound state is still unresolved. It is relevant to the Matrix Theory conjecture [2] in which these particles are interpreted as Kaluza-Klein modes of 11-dimensional M-Theory compactified on a circle and are described by the supersymmetric $N \times N$ matrix quantum mechanics that is obtained from dimensional reduction of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in ten dimensions. The existence of such a tower of states is equivalent to the statement that the quantum mechanics admits exactly one bound state for each $N$. The original threshold bound state problem was addressed in [3] and recent progress has been made in [4]. However, beyond the proof of existence in the case $N = 2$, the general case $N > 2$ remains in large part an open problem. In this paper we shall address the bound state problem for a system of $N$ non-relativistic, recoiling D0-branes by studying their moduli space dynamics in certain limits.

The effective worldvolume dynamics of a single D$p$-brane coupled to a worldvolume gauge field and to background supergravity fields is described by the action [1]

$$I_{Dp} = \mathcal{T}_p \int d^{p+1}\sigma \ e^{-\phi} \sqrt{-\det \left[ G_{\alpha\beta} + B_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta} \right]} + \mathcal{T}_p \int d^{p+1}\sigma \left[ C \wedge e^{2\pi \alpha' F + B} \wedge \mathcal{G}_{p+1} \right]$$

(1.1)

The first term in (1.1) is the Dirac-Born-Infeld action with $\mathcal{T}_p$ the $p$-brane tension, $\alpha'$ the string Regge slope, $\phi$ the dilaton field, $F = dA$ the worldvolume field strength, and $G$ and $B$ the pull-backs of the target space metric and Neveu-Schwarz two-form fields, respectively, to the D$p$-brane worldvolume. It is a generalization of the geometric volume of the brane trajectory. The second term is the Wess-Zumino action (restricted to its $p+1$-form component) with $C$ the pullback of the sum over all electric and magnetic Ramond-Ramond (RR) form potentials and $\mathcal{G}$ a geometrical factor accounting for the possible non-trivial curvature of the tangent and normal bundles to the $p$-brane worldvolume. It describes the coupling of the D$p$-brane to the supergravity RR $p+1$-form fields as well as to the topological charge of the worldvolume gauge field and to the worldvolume gravitational connections. The fermionic completion of the action (1.1), compatible with spacetime supersymmetry and worldvolume $\kappa$-symmetry, has been described in [5]. For a recent review of the Born-Infeld action and its various extensions in superstring theory, see [6].

While the generalization of the Wess-Zumino Lagrangian to multiple D$p$-branes is obvious (one simply traces over the worldvolume gauge group in the fundamental representation), the complete form of the non-abelian Born-Infeld action is not known. In [7] it was proposed
that the background independent terms can be derived using T-duality from a 9-brane action obtained from the corresponding abelian version by symmetrizing all gauge group traces in the vector representation \[6\]. A direct calculation of the leading terms in a weak supergravity background has been calculated using Matrix Theory methods in \[8\]. Based on the Type I formulation, i.e. by viewing a D-particle in the Neumann picture and imposing T-duality as a functional canonical transformation in the string path integral \[1\], the effective moduli space Lagrangian was derived in \[10\] and shown to coincide (to leading orders in a velocity expansion) with the non-abelian Born-Infeld action of \[7\]. In the following we will use this moduli space approach to D-brane dynamics to describe some properties of the multiple D-brane wavefunction.

The novel aspect of the approach of \[10\] is that the moduli space dynamics induces an effective target space geometry for the D-branes which contains information about the short-distance spacetime structure probed by multiple D-particles. Based on this feature, string-modified spacetime and phase space uncertainty relations can be derived and thereby represent a proper quantization of the noncommutative spacetime seen by low-energy D-particle probes \[11\]. The crucial property of the derivation is the incorporation of proper recoil operators for the D-branes and the short open string excitations connecting them. The smearing of the spacetime coordinates \(y_a^i\) (in general \(i = 1, \ldots, 9 - p\) label the transverse coordinates of the \(D_p\)-brane and \(a = 1, \ldots, N\) the component branes of the multiple D-brane configuration) of a given D-particle as a result from its open string interactions with other branes can be seen directly from the formula for the variance

\[
(\Delta y_a^i)^2 \equiv \left( Y_i - Y_i^{aa} I_N \right)^2 \equiv \sum_{b \neq a} |Y_i^{ab}|^2
\]

where \(Y_i^{ab}\) are the \(u(N)\)-valued positions of the D-particles \((a = b)\) and of the open strings connecting branes \(a\) and \(b\) \((a \neq b)\), and \(I_N\) is the \(N \times N\) identity matrix. The recoil operators give a relevant deformation of the conformal field theory describing free open strings, and thus lead to non-trivial renormalization group flows on the moduli space of coupling constants. The moduli space dynamics is thereby governed by the Zamolodchikov metric and the associated \(C\)-theorem. Physically, the recoil operators describe the appropriate change of quantum state of the D-brane background after the emission or absorption of open or closed strings. They are a necessary ingredient in the description of multiple D-brane dynamics, in which coincident branes interact with each other via the exchange of open string states. The quantum uncertainties derived in \[10,11\] were found to exhibit quantum decoherence effects through their dependence on the recoil energies of the system of D-particles. This suggests that the appropriate quantum dynamics of D0-branes should be described by some sort of stochastic string field theory involving a Fokker-Planck Hamiltonian.
As in [6,7], the derivation in [10] assumes constant background supergravity fields. However, another important ingredient missing in the moduli space description are the appropriate residual fermionic terms from the supersymmetry of the initial static D-brane configuration. While the recoil of the D-branes breaks supersymmetry, it is necessary to include these terms to have a complete description of the stability of the D-particle bound state. As shown in [12], the energy of the bound states of D-branes and strings is determined by the central charge of the corresponding spacetime supersymmetry algebra. Nonetheless, the bosonic formalism that we display below can be exploited to a large extent to describe at least heuristically the quantum phase structure of the multiple D-particle system and, in particular, determine the mass and stability conditions of the candidate bound state. One reason that this approach is expected to yield reliable results is that we view the system of D-branes and strings as a quantum mechanical system (rather than a quantum field theoretical system as might be the case from the fact that $T$-duality is used to effectively integrate over the transverse coordinates of the branes), with the D-brane recoil constituting an excitation of this system. The recoiling system of D-branes and strings can be viewed as an excited state of a supersymmetric (static) vacuum configuration. The breaking of target space supersymmetry by the excited state of the system may thereby constitute a symmetry obstruction situation in the spirit of [13]. According to the symmetry obstruction hypothesis, the ground state of a system of (static) strings and D-branes is a BPS state, but the excited (recoiling) states do not respect the supersymmetry due to quantum diffusion and other effects. Phenomenologically, the supersymmetry breaking induced by the excited system of recoiling D-particles will distort the spacetime surrounding them and may result in a decohering spacetime foam, on which low energy (point-like) excitations live. This motivates the study of non-supersymmetric D-branes recoiling under the exchange of strings. Such quantum mechanical systems exhibit diffusion and may be viewed as non-equilibrium (open) quantum systems, with the non-equilibrium state being related naturally to the picture of viewing the recoiling D-brane system as an excited state of some (non-perturbative) supersymmetric D-brane vacuum configuration.

The main relationship we shall exploit in obtaining the quantum dynamics of multiple D-particle systems is that between the Dirichlet partition function in the background of Type II string fields and the semi-classical (Euclidean) wavefunctional $\Psi[Y^i]$ of a $Dp$-brane. This relation is usually expressed as [14,15]

$$Z = \int DY^i \Psi[Y^i]$$

(1.3)

The wavefunction $\Psi[Y^i]$ is expressed in terms of the generating functional which sums up all one-particle irreducible connected worldsheet diagrams whose boundaries are mapped onto the D-brane worldvolume. Integration over the worldvolume gauge field is implicit in $\Psi$ to
ensure Type II winding number conservation. Dirichlet string perturbation theory yields

$$
\Psi[Y_{i}] = \exp \sum_{h=1}^{\infty} e^{(h-2)\phi} S_{h}[Y_{i}] 
$$

(1.4)

where $S_{h}$ denotes the amplitude with $h$ holes, in which an implicit sum over handles is assumed. However, as we will discuss in the following, the identification (1.3) is not the only one consistent with the approach to D-brane dynamics advocated in [10], and one may instead identify the worldsheet Dirichlet partition function, summed over all genera, with the probability distribution corresponding to the wavefunction $\Psi$. Using this identification, the Wilson renormalization group equation has been proposed as a defining principle for obtaining string field equations of motion, including the appropriate Fischler-Susskind mechanism for the contributions from higher genera [14]. When applied to Dirichlet string theory, we shall find that the consistent D-brane equation of motion follows from the renormalization group equation.

More precisely, within the framework of a perturbative logarithmic conformal field theory approach to multiple D-brane dynamics [10], we will show that the intricate quantum dynamics of a system of interacting D-particles is described by a non-linear Schrödinger wave equation. The corresponding probability density is of the Fokker-Planck type, with quantum diffusion coefficient $D$ given by the square of the modulus of the recoil velocity matrix of the bound state system of D-particles and strings:

$$
D = c_{G} \sqrt{\alpha'} \sum_{i=1}^{9} \text{tr}|\bar{U}_{i}|^{2} 
$$

(1.5)

where $c_{G}$ is a numerical constant and $\bar{U}_{i}$ is the (renormalized) constant velocity matrix of a system of $N$ D-particles arising due to the D-particle recoil from the scattering of string states. This phenomenon is in fact characteristic of Liouville string theory, on which the above approach is based. Since the D-particle interactions distort their surrounding spacetime, these non-linear structures may be thought of as describing short-distance quantum gravitational properties of the D-brane spacetime. Non-linear equations of motion for string field theories have been derived in other contexts in [16]. From this nonlinear Schrödinger dynamics we shall describe a multitude of classes of solutions, using Galilean invariance of the D-brane dynamics which is a consequence of the corresponding logarithmic conformal algebra. We will show that bound state solutions do indeed exist for string couplings $g_{s}$ larger than a dynamically determined critical value. The effective bound state mass is likewise determined as an energetically induced shift of the static, BPS mass of the D0-branes. In fact, we shall find that there are essentially three different phases of the quantum dynamics in string coupling constant space. Below the critical string coupling the multiple D-brane
wavefunction is described by solitary waves, in agreement with the description of free D-branes as string theoretic solitons, while at the critical coupling the quantum dynamics is described by coherent Gaussian wavepackets which determine the appropriate quantum smearing of the multiple D-particle spacetime. These results are shown to be in agreement with the previous results concerning the structure of quantum spacetime [10,11].

We close this section by summarizing some of the generic guidelines that we shall use in this paper for constructing a wavefunctional for the system of D-branes. We will use a field theoretic approach by identifying the Hartle-Hawking wavefunction

$$\Psi_0 \simeq e^{-S_E}$$

where $S_E$ is the effective Euclidean action. We shall discuss the extension to string theory and highlight the advantages and disadvantages of using this identification. We shall also identify the probability density with the genus expansion of an appropriate worldsheet $\sigma$-model:

$$P = \Psi_0^\dagger \Psi_0 = \sum_{\text{genera}} \int Dx \ e^{-S_\sigma[x]}$$

The arguments in favour of this identification will be reality, and the occurrence of statistical probability distribution factors which appear in the wormhole parameters after resummation of (1.7) over pinched genera. The Wilson-Polchinski worldsheet renormalization group flow, coming from the sum over genera as in (1.7), yields a Fokker-Planck diffusion equation

$$\partial_t P = D \nabla^2 P - \nabla \cdot J$$

where $D$ is the diffusion operator defined in (1.5) in terms of (renormalized) recoil velocity matrices, and $J$ is the associated probability current density. The equation (1.8) will follow from the gradient flow property of the $\sigma$-model $\beta$-functions, which is also necessary for the Helmholtz conditions or equivalently for canonical quantization of the string moduli space.

The knowledge of the Fokker-Planck equation (1.8) alone does not lead to an unambiguous construction of the wavefunction $\Psi$. There are ambiguities associated with non-linear $\Psi$-dependent phase transformations of the wavefunction:

$$\Psi \mapsto e^{iN_{\gamma,\lambda}(\Psi)} \Psi$$

$$N_{\gamma,\lambda}(\Psi) = \gamma \log |\Psi| + \lambda \arg \Psi + \theta(\{Y^{ab}_i\}, t)$$

where $t$ is the Liouville zero mode. Furthermore, $\Psi$ is then necessarily determined by a non-linear wave equation if a diffusion coefficient $D$ is present, as will be the case in what follows. The non-linear Schrödinger equation has the form
\[ i\hbar \partial_t \Psi = \mathcal{H}_0 \Psi + \frac{i\hbar}{2} \mathcal{D} \frac{\nabla^2 \mathcal{P}}{\mathcal{P}} \Psi \]  

(1.10)

where \( \mathcal{P} = \Psi^\dagger \Psi \) is the probability density. This is a Galilean-invariant but time-reversal violating equation, exactly as expected from previous considerations of non-relativistic D-brane dynamics and Liouville string theory. Eq. (1.10) will be the proposal in the following for the non-linear quantum dynamics of matrix D-branes (this was noted in passing in [17]).

**II. QUANTUM MECHANICS ON MODULI SPACE**

In [10] it was shown how a description of non-abelian D-particle dynamics, based on canonical quantization of a \( \sigma \)-model moduli space induced by the worldsheet genus expansion (i.e. the quantum string theory), yields quantum fluctuations of the string soliton collective coordinates and hence a microscopic derivation of spacetime uncertainty relations, as seen by short distance D-particle probes. In the following we will proceed to construct a wavefunction for the system of D0-branes which encodes the pertinent quantum dynamics. To start, in this section we shall clarify certain facts about wavefunctionals in non-critical string theories in general, completing the discussion put forward in [15].

**A. Liouville-dressed Renormalization Group Flows**

Consider quite generally a non-critical string \( \sigma \)-model, defined as a deformation of a conformal field theory \( S_* \) with coupling constants \( \{g^I\} \). The worldsheet action is

\[ S_\sigma[x; \{g^I\}] = S_*[x] + \int \Sigma d^2z \ g^I V_I[x] \]  

(2.1)

where \( V_I \) are the deformation vertex operators and an implicit sum over repeated upper and lower indices is always understood. We assume that the deformation is relevant, so that the worldsheet theory must be dressed by two-dimensional quantum gravity in order to restore conformal invariance in the quantum string theory. The corresponding Liouville-dressed renormalized couplings \( \{\lambda^I\} \) satisfy the renormalization group equations

\[ \ddot{\lambda}^I + Q \dot{\lambda}^I = -\beta^I(\lambda) \]  

(2.2)

where the dots denote differentiation with respect to the worldsheet zero mode of the Liouville field. Here \( Q \) is the square root of the running central charge deficit on moduli space and

\[ \beta^I(\lambda) = h^I \lambda^I + c^I_{JK} \lambda^J \lambda^K + \ldots \]  

(2.3)
are the flat worldsheet $\beta$-functions, expressed in terms of Liouville-dressed coupling constants. In (2.3), $h^I$ are the conformal dimensions and $c_{IJK}$ the operator product expansion coefficients of the vertex operators $V_I$. The minus sign in (2.2) owes to the fact that we confine our attention here to the case of central charge $c > 25$ (corresponding to supercritical bosonic or fermionic strings).

Upon interpreting the Liouville zero mode as the target space time evolution parameter, eq. (2.2) is reminiscent of the equation of motion for the inflaton field $\phi$ in inflationary cosmological models [18,19]. In the present case of course one has a collection of fields $\{g^I\}$, but the analogy is nevertheless precise. The role of the Hubble constant $H$ is played by the central charge deficit $Q$. The precise correspondence actually follows from the gradient flow property of the string $\sigma$-model $\beta$-functions for flat worldsheets:

$$\beta^I = G^{IJ} \frac{\partial}{\partial g^J} C$$ (2.4)

where $C = Q^2$ is the Zamolodchikov $C$-function which is associated with the generating functional for one-particle irreducible correlation functions [20], and $G^{IJ}$ is the matrix inverse of the Zamolodchikov metric

$$G_{IJ} = 2|z|^4 \langle V_I(z, \bar{z}) V_J(0,0) \rangle$$ (2.5)

on the moduli space $\mathcal{M}(\{g^I\})$ of $\sigma$-model couplings $\{g^I\}$. Then the right-hand side of (2.2) also corresponds to the gradient of the potential $V$ in inflationary models:

$$\ddot{\phi} + 3H \dot{\phi} = -\frac{dV}{d\phi}$$ (2.6)

where $\phi$ is the inflaton field in a sufficiently homogeneous domain of the universe.

B. The Hartle-Hawking Wavefunction

In [10,15] it was shown, through the energy dependence of quantum uncertainties, that some sort of stochasticity characterizes non-critical Liouville string dynamics, implying that the analogy of eq. (2.2) with the equations of motion in inflationary models should be made with those involving chaotic inflation [19]. Let us now briefly review the properties of these latter models. In such cases, the ground state wavefunction of the universe may be identified as [21]:

$$\psi_0(a, \phi) = \exp -S_E(a, \phi)$$ (2.7)

where $S_E$ is the Euclidean action for the scalar field $a(\tau)$ and the inflaton scalar field $\phi(\tau)$ which satisfy the boundary conditions:
\[
a(0) = a, \quad \phi(0) = \phi
\]

(2.8)

and \( \tau \) is the Euclidean time.

To understand how eq. (2.7) comes about, we appeal to the Hartle-Hawking interpretation [21]. Consider the Green’s function \( \langle x, t | 0, t' \rangle \) of a particle which propagates from the spacetime point \((0, t')\) to \((x, t)\):

\[
\langle x, t | 0, t' \rangle = \sum_n \psi_n^\dagger(x) \psi_n(0) e^{iE_n(t-t')} = \int Dx \ e^{iS(x,|t-t'|)}
\]

(2.9)

where \( \{ \psi_n \} \) is the complete set of energy eigenstates with energy eigenvalues \( E_n \geq 0 \) (the sum in (2.9) should be replaced by an appropriate integration in the case of a continuous spectrum). To obtain an expression for the ground state wavefunction, we make a Wick rotation \( t = -i\tau \), and take the limit \( \tau \to -\infty \) to recover the initial state. Then in the summation over energy eigenvalues in (2.9), only the ground state \((n = 0)\) term survives if \( E_0 = 0 \). The corresponding path integral representation becomes \( \int Dx \ e^{-S_E(x)} \), and one obtains eq. (2.7) in the semi-classical approximation.

For inflationary models which are based on the de Sitter spaces \( dS_4 \) with

\[
a(\tau) = \kappa^{-1}(\phi) \cos \kappa(\phi) \tau
\]

(2.10)

one has

\[
S_E(a, \phi) = -\frac{3}{16V(\phi)}
\]

(2.11)

and hence

\[
\psi_0(a, \phi) = \exp \left( \frac{3}{16V(\phi)} \right)
\]

(2.12)

Thus the probability density for finding the universe in a state with \( \phi = \text{const.}, \ a = \kappa^{-1}(\phi) = \sqrt{\frac{3}{8\pi V(\phi)}} \) is

\[
\mathcal{P} = |\psi_0|^2 = e^{3/8V(\phi)}
\]

(2.13)

The distribution function (2.13) has a sharp maximum as \( V(\phi) \to 0 \). For inflationary models this is a bad feature, because it diminishes the possibility of finding the universe in a state with a large \( \phi \) field and thereby having a long stage for inflation. However, from the point of view of Liouville string theory, the result (2.13), if indeed valid, implies that the critical string theory (since \( V \propto Q^2 \) there) is a favourable situation statistically, and hence any consideration (such as those in [10]) made in the neighbourhood of a fixed point of the renormalization group flow on the moduli space of running coupling constants is justified.
C. Moduli Space Wavefunctionals

Let us now proceed to discuss the possibility of finding a Schrödinger wave equation for the D-particle wavefunction. The identification (2.7) in the inflationary case needs some careful verification in the case of the topological expansion of the worldsheet $\sigma$-model (2.1). In Liouville string theory, the genus expansion of the partition function may be identified \[15\] with the wavefunctional of non-critical string theory in the moduli space of coupling constants $\{g^I\}$:

$$\Psi(\{g^I\}) = \sum_{\text{genera}}\int Dx\ e^{-S_\sigma[x;\{g^I\}]} \equiv e^{-\mathcal{F}[\{g^I\}]}$$  \hspace{1cm} (2.14)

where

$$\mathcal{F}[\{g^I\}] = \sum_{h=0}^{\infty} (g_s)^{h-2} \mathcal{F}_h(\{g^I\})$$  \hspace{1cm} (2.15)

is the effective target space action functional of the non-critical string theory. The sum on the right-hand side of (2.15) is over all worldsheet genera, which sums up the one-particle irreducible connected worldsheet amplitudes $\mathcal{F}_h$ with $h$ handles. The gradient flow property (2.4) of the $\beta$-functions ensures \[10,15\] that the Helmholtz conditions for canonical quantization are satisfied, which is consistent with the existence of an off-shell action $\mathcal{F}[\{g^I\}]$.

In that case, the effective Lagrangian on moduli space whose equations of motion coincide with the renormalization group equations (2.2) is given by \[10\]

$$\mathcal{L}_M(t) = -\beta^I G_{IJ} \beta^J$$  \hspace{1cm} (2.16)

and it coincides with the Zamolodchikov $C$-function. The semi-classical wavefunction determined by (2.14) is thereby determined by the action $C[\lambda]$ regarded as an effective action on the space of two-dimensional renormalizable field theories. Thus the probability density is $\mathcal{P}[\{g^I\}] = e^{-2\mathcal{F}[\{g^I\}]}$, which implies that the minimization of $\mathcal{F}[\{g^I\}]$ yields a maximization of $\mathcal{P}[\{g^I\}]$, provided that the effective action is positive-definite. This is an ideal situation, since then the minimization of $\mathcal{F}[\{g^I\}]$, in the sense of solutions of the equations $\delta\mathcal{F}/\delta g^I = 0$, corresponds to the conformally-invariant fixed point of the $\sigma$-model moduli space, thereby justifying the analysis in a neighbourhood of a fixed point.

However, the identification (2.14) is not the only possibility in non-critical string theory, as will be discussed below, in particular in connection with the Schrödinger dynamics of D0-branes. The main point is that upon taking the topological expansion in Liouville string theory, the couplings $g^I$ become quantized in such a way so that

$$\sum_{\text{genera}}\int Dx\ e^{-S_\sigma[x;\{g^I\}]} = \int_{\mathcal{M}(\{g^I\})} D\alpha^I e^{-\frac{1}{2\pi^2} G_{IJ} \alpha^J} \int Dx\ e^{-S_s^{(0)}[x;\{g^I+\alpha^I\}]}$$  \hspace{1cm} (2.17)
where the prime on the sum means that the genus expansion is truncated to a sum over pinched annuli of infinitesimal strip size, $S^{(0)}_\sigma[x; \{g^I\}]$ is the tree-level (disc or sphere) action for the $\sigma$-model, and $\alpha^I$ are worldsheet wormhole parameters on the moduli space $M(\{g^I\})$ of the two-dimensional quantum field theory. The Gaussian spread in the $\alpha^I$ in (2.17) can be interpreted as a probability distribution characterizing the statistical fluctuations of the coupling constants $g^I$. The width $\Gamma$ is proportional to the logarithmic modular divergences on the pinched annuli, which may be identified with the short-distance infinities $\log \Lambda$ at tree-level [10] ($\Lambda$ is the worldsheet ultraviolet cutoff scale). The result (2.17) suggests that one may directly identify the genus expansion of the worldsheet partition function as the probability density

$$|\Psi(\{g^I\}, t)|^2 \equiv P(\{g^I\}, t) \quad (2.18)$$

for finding non-critical strings in the moduli space configuration $\{g^I\}$ at Liouville time $t$ (the worldsheet zero mode of the Liouville field). In this way one has a natural explanation for the reality of eq. (2.17) on Euclidean worldsheets. If the identification of the genera summed partition function with the probability density holds, i.e. with the square of the wavefunction $\Psi(\{g^I\}, t)$ rather than the wavefunctional itself, then one may obtain a temporal evolution equation for (2.18) using the Wilson-Polchinski renormalization group equation on the string worldsheet [14]. This will be described in section IV.

One may argue formally in favour of the above identification in the case of Liouville strings, within a world-sheet formalism, by noting [22] that the conventional interpretation of the Liouville (world-sheet) correlators as target-space $S$-matrix elements breaks down upon the interpretation of the Liouville zero-mode as target time. Instead, the only well-defined concept in such a case is the non-factorizable $S$-matrix, which acts on target-space density matrices rather than state vectors. This in turn implies that the corresponding world-sheet partition function, summed over topologies, which in the case of critical strings would be the generating functional of such $S$-matrix elements in target space, should be identified with the probability density in the moduli space of the non-critical strings (2.18). In the Appendix we review this approach [22] by focusing on those aspects of the formalism that are most relevant to our purposes here. As we shall discuss there, the above identification follows from specific properties of the Liouville string formalism.

Notice that if one interprets the topological expansion of the worldsheet partition function as the probability density for the non-critical string configuration $\{g^I\}$, then the simple argument leading to eq. (2.7) is not valid here. In such a situation the action in eq. (2.9), which refers to the string moduli space, is not the same as the effective target space action $\mathcal{F}(\{g^I\})$, but rather something different, corresponding to the phase of the wavefunctional $\Psi(\{g^I\}, t)$ whose probability density (2.18) corresponds to the genera summed worldsheet
partition function. This is not necessarily a bad feature, as we shall see, although in most treatments the target space effective action $\mathcal{F}[\{g^I\}]$ is identified with the moduli space action upon identification of the Liouville zero mode (i.e. the local worldsheet renormalization group scale) with target time. For this, we observe that the statistical interpretation of the resummed worldsheet partition function is compatible with the interpretation in [10] of the Gaussian wormhole parameter distribution function in eq. (2.17) as being responsible for the quantum uncertainties of D-branes. This follows trivially from the fact that

$$|\Psi(\{g^I\}, t)|^2 = e^{-2\mathcal{F}(\{g^I\}, t)}$$

(2.19)

Then, any correlation function may be written as

$$\langle V_1 \cdots V_n \rangle = \int_{\mathcal{M}(\{g^I\})} Dg^I \left|\Psi(\{g^I\}, t)\right|^2 V_1 \cdots V_n$$

$$= \int_{\mathcal{M}(\{g^I\})} Dg^I \int_{\mathcal{M}(\{g^I\})} Da^I e^{-\frac{1}{2\pi^2} a^I G_{IJ} a^J} \int Dx e^{-S^0_{\sigma}(x; \{g^I + a^I\})} V_1 \cdots V_n$$

(2.20)

which using eq. (2.19) gives the connection between the two probability distributions.

III. MATRIX D-BRANE DYNAMICS

In this section we shall briefly review the worldsheet description of [10] for matrix D0-brane dynamics. The partition function is given by [11]

$$Z[A_0, Y] = \int D\mu(x, \bar{\xi}, \xi) \exp \left(-\frac{1}{4\pi\alpha'} \int d^2z \eta_{\mu\nu} \partial x^\mu \bar{\partial} x^\nu + \frac{1}{2\pi\alpha'} \oint d\tau x_i(\tau) \partial_\sigma x^i(\tau) \right)$$

$$\times \mathcal{W}[x, \bar{\xi}, \xi]$$

(3.1)

where

$$\mathcal{W}[x, \bar{\xi}, \xi] = \exp ig_a \oint d\Sigma \left( \bar{\xi}_a(\tau) A_0^a \xi_b(\tau) \partial_\tau x^0(\tau) + \frac{i}{2\pi\alpha'} \bar{\xi}_a(\tau) Y_i^a(x^0) \xi_b(\tau) \partial_\sigma x^i(\tau) \right)$$

(3.2)

is the deformation action of the free $\sigma$-model in [11]. Here the indices $\mu = 0, 1, \ldots, 9$ and $i = 1, \ldots, 9$ label spacetime and spatial directions of the target space, which we assume has a flat metric $\eta_{\mu\nu}$. The functional integration measure in (3.1) is given by

$$D\mu(x, \bar{\xi}, \xi) = Dx^\mu D\bar{\xi} D\xi \exp \left[-\sum_{a=1}^N \left( \oint d\Sigma \bar{\xi}_a(\tau) \partial_\tau \xi_a(\tau) + \bar{\xi}_a(0) \xi_a(0) \right) \right] \sum_{a=1}^N \bar{\xi}_a(0) \xi_a(1)$$

(3.3)
The complex auxiliary fields $\bar{\xi}_a(\tau)$ and $\xi_a(\tau)$, $a = 1, \ldots, N$, transform in the fundamental representation of the brane gauge group, and they live on the boundary of the worldsheet $\Sigma$ which at tree-level is a disc whose boundary is a circle $\partial \Sigma$ with periodic longitudinal coordinate $\tau \in [0, 1]$ and normal coordinate $\sigma \in \mathbb{R}$. They have the propagator $\langle \bar{\xi}_a(\tau) \xi_b(\tau') \rangle = \delta_{ab} \Theta(\tau' - \tau)$, where $\Theta$ denotes the usual step function. The integration over the auxiliary fields with the measure (3.3) therefore turns (3.2) into a path-ordered exponential functional of the fields $x$ which is the $T$-dual of the usual Wilson loop operator for the ten-dimensional gauge field $(A^0, -\frac{1}{2\pi \alpha'} Y^i)$ dimensionally reduced to the D-particle worldlines. In this picture, $A^0$ is thought of as a gauge field living on the brane worldline, while $Y_{aa}^i$, $a = 1, \ldots, N$, are the transverse coordinates of the $N$ D-particles and $Y_{ab}^i$, $a \neq b$, of the short open string excitations connecting them. We shall subtract out the center of mass motion of the assembly of $N$ D-branes and assume that $Y_i^a \in su(N)$. We shall also use $SU(N)$-invariance of the theory (3.1) to select the temporal gauge $A^0 = 0$.

The action in (3.1) may be formally identified with the deformed conformal field theory (2.1) by taking the couplings $g^I \sim Y_{ab}^i$ and introducing the one-parameter family of bare matrix-valued vertex operators

$$V^i_{ab}(x; \tau) = \frac{g_s}{2\pi \alpha'} \partial_\sigma x^i(\tau) \bar{\xi}_a(\tau) \xi_b(\tau)$$  \hspace{1cm} (3.4)

This means that there is a one-parameter family of Dirichlet boundary conditions for the fundamental string fields $x^i$ on $\partial \Sigma$, labelled by $\tau \in [0, 1]$ and the configuration fields

$$y_i(x^0, \tau) = \bar{\xi}_a(\tau) Y_{ab}^i(x^0(\tau)) \xi_b(\tau)$$  \hspace{1cm} (3.5)

Instead of being forced to sit on a unique hypersurface as in the case of a single D-brane, in the non-abelian case there is an infinite set of hypersurfaces on which the string endpoints are situated. In this sense the coordinates (3.5) may be thought of as an “abelianization” of the non-abelian D-particle coordinate fields $Y_{ab}^i$.

To describe the non-relativistic dynamics of heavy D-particles, the natural choice is to take the couplings to correspond to the Galilean boosted configurations $Y_{ab}^i(x^0) = Y_{ab}^i + U_i x^0$, where $U_i$ is the non-relativistic velocity matrix. However, logarithmic modular divergences appear in matter field amplitudes at higher genera when the string propagator $L_0$ is computed with Dirichlet boundary conditions. These modular divergences are cancelled by adding logarithmic recoil operators [10,23] to the matrix $\sigma$-model action in (3.1). From a physical point of view, if one is to use low-energy probes to observe short-distance spacetime structure, such as a generalized Heisenberg microscope, then one needs to consider the scattering of string matter off the assembly of D-particles. For the Galilean-boosted multiple D-particle system, the recoil is described by taking the deformation of the $\sigma$-model action in (3.1) to be of the form [10]
\[ Y^{ab}_{i}(x^0) = \sqrt{\alpha'} Y^{ab}_{i} C_c(x^0) + U^{ab}_{i} D_c(x^0) = \left( \sqrt{\alpha'} \epsilon Y^{ab}_{i} + U^{ab}_{i} x^0 \right) \Theta_{\epsilon}(x^0) \]  

(3.6)

where

\[ C_c(x^0) = \epsilon \Theta_{\epsilon}(x^0) , \quad D_c(x^0) = x^0 \Theta_{\epsilon}(x^0) \]  

(3.7)

and

\[ \Theta_{\epsilon}(x^0) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dq}{q - i\epsilon} e^{iqx^0} \]  

(3.8)

is the regulated step function whose \( \epsilon \rightarrow 0^+ \) limit is the usual step function. The operators \( (3.7) \) have non-vanishing matrix elements between different string states and therefore describe the appropriate change of quantum state of the D-brane background. They can be thought of as describing the recoil of the assembly of D-particles in an impulse approximation, in which it starts moving as a whole only at time \( x^0 = 0 \). The collection of constant matrices \( \{Y^{ij}_{ab},U^{ij}_{cd}\} \) now form the set of coupling constants \( \{g'\} \) for the worldsheet \( \sigma \)-model \( (3.1) \).

The recoil operators \( (3.7) \) possess a very important property. They lead to a deformation of the free \( \sigma \)-model action in \( (3.1) \) which is not conformally-invariant, but rather defines a logarithmic conformal field theory [24]. Such a quantum field theory contains logarithmic scaling violations in its correlation functions on the worldsheet, which can be seen in the present case by computing the pair correlators of the fields \( (3.7) \)

\[ \langle C_\epsilon(z) C_\epsilon(0) \rangle = 0 \]

\[ \langle C_\epsilon(z) D_\epsilon(0) \rangle = \frac{b}{z^{h_\epsilon}} \]

\[ \langle D_\epsilon(z) D_\epsilon(0) \rangle = \frac{b \alpha'}{z^{h_\epsilon}} \log z \]  

(3.9)

where

\[ h_\epsilon = -\frac{|\epsilon|^2 \alpha'}{2} \]  

(3.10)

is the conformal dimension of the recoil operators. The constant \( b \) is fixed by the leading logarithmic divergence of the conformal blocks of the theory. Note that \( (3.10) \) vanishes as \( \epsilon \rightarrow 0 \), so that the logarithmic worldsheet divergences in \( (3.9) \) cancel the modular annulus divergences mentioned above. An essential ingredient for this cancellation is the identification

\[ \frac{1}{\epsilon^2} = -2\alpha' \log \Lambda \]  

(3.11)
which relates the target space regularization parameter $\epsilon$ to the worldsheet ultraviolet cutoff scale $\Lambda$.

Logarithmic conformal field theories are characterized by the fact that their Virasoro generator $L_0$ is not diagonalizable, but rather admits a Jordan cell structure. Here the operators (3.7) form the basis of a $2 \times 2$ Jordan block and they appear in the spectrum of the two-dimensional quantum field theory as a consequence of the zero modes that arise from the breaking of the target space translation symmetry by the topological defects. The mixing between $C$ and $D$ under a conformal transformation of the worldsheet can be seen explicitly by considering a scale transformation

$$\Lambda \to \Lambda' = \Lambda e^{-t/\sqrt{\alpha'}}$$  \hspace{1cm} (3.12)

Using (3.11) it follows that the operators (3.7) are changed according to $D'_\epsilon = D_\epsilon + t\sqrt{\alpha'} C_\epsilon$ and $C'_\epsilon = C_\epsilon$. Thus in order to maintain scale-invariance of the theory (3.1) the coupling constants must transform under (3.12) as $Y'_i = Y_i + U_i t$ and $U'_i = U_i$, which are just the Galilean transformation laws for the positions $Y^i$ and velocities $U^i$. Thus a scale transformation of the worldsheet is equivalent to a Galilean transformation of the moduli space of $\sigma$-model couplings, with the parameter $\epsilon^{-2}$ identified with the time evolution parameter $t = -\sqrt{\alpha'} \log \Lambda$. The corresponding $\beta$-functions for the worldsheet renormalization group flow are

$$\beta_{Y_i} \equiv \frac{dY_i}{dt} = h_\epsilon Y_i + \sqrt{\alpha'} U_i$$

$$\beta_{U_i} \equiv \frac{dU_i}{dt} = h_\epsilon U_i$$

and they generate the Galilean group $G(9)^{N^2}$ in nine-dimensions.

The associated Zamolodchikov metric

$$G_{ab,cd}^{ij} = 2N\Lambda^2 \left\langle V_{ab}^i(x;0) V_{cd}^j(x;0) \right\rangle$$

(3.14)

can be evaluated to leading orders in $\sigma$-model perturbation theory using the logarithmic conformal algebra (3.9) and the propagator of the auxiliary fields to give [10]

$$G_{ab,cd}^{ij} = \frac{4\tilde{g}_s^2}{\alpha'} \left[ \eta^{ij} I_N \otimes I_N + \frac{\tilde{g}_s^2}{36} \left\{ I_N \otimes \left( U^i U^j + U^j U^i \right) \right. \right. \right.$$

$$+ \left. \left. U^i \otimes U^j + U^j \otimes U^i + \left( U^i U^j + U^j U^i \right) \otimes I_N \right\} \right]_{ab,cd} + O\left( \tilde{g}_s^6 \right)$$

(3.15)

where $I_N$ is the identity operator of $SU(N)$ and we have introduced the renormalized coupling constants

$$\tilde{g}_s = g_s/\sqrt{\alpha'\epsilon} \quad , \quad \tilde{U}^i = U^i/\sqrt{\alpha'\epsilon}$$

(3.16)
From the renormalization group equations (3.13) it follows that the renormalized velocity operator in target space is truly marginal, 

$$\frac{d\bar{U}^i}{dt} = 0 \quad (3.17)$$

which ensures uniform motion of the D-branes. It can also be shown that the renormalized string coupling $\bar{g}_s$ is time-independent [10]. If we further define the position renormalization

$$\bar{Y}^i = Y^i / \sqrt{\alpha'} \epsilon \quad (3.18)$$

then the $\beta$-function equations (3.13) coincide with the Galilean equations of motion of the D-particles, i.e.

$$\frac{d\bar{Y}^i}{dt} = \bar{U}^i \quad (3.19)$$

Note that the Zamolodchikov metric (3.15) is a complicated function of the D-brane dynamical parameters, and as such it represents the appropriate effective target space geometry of the D-particles. The moduli space Lagrangian (2.16) is then readily seen to coincide with the expansion to $O(\bar{g}_s^4)$ of the symmetrized form of the non-abelian Born-Infeld action for the D-brane dynamics [7],

$$L_{\text{NBI}} = \frac{1}{2 \sqrt{2 \pi \alpha'}} \text{tr} \text{Sym} \left[ \text{det} \left[ \eta_{\mu\nu} I_N + 2 \pi \alpha' \bar{g}_s^2 F_{\mu\nu} \right] \right] \quad (3.20)$$

where tr denotes the trace in the fundamental representation of $SU(N)$,

$$\text{Sym}(M_1, \ldots, M_n) = \frac{1}{n!} \sum_{\pi \in S_n} M_{\pi_1} \cdots M_{\pi_n} \quad (3.21)$$

is the symmetrized matrix product and the components of the dimensionally reduced field strength tensor are given by

$$F_{0i} = \frac{1}{2 \pi \alpha'} \frac{d\bar{Y}_i}{dt}, \quad F_{ij} = \frac{\bar{g}_s}{(2 \pi \alpha')^2} [\bar{Y}_i, \bar{Y}_j] \quad (3.22)$$

IV. EVOLUTION EQUATION FOR THE PROBABILITY DISTRIBUTION

In this section we will derive the temporal evolution equation for the probability density $P(\{g^I\}, t)$ following the identification of time with a worldsheet renormalization group scale (i.e. the Liouville zero mode). The basic identity is the Wilson-Polchinski equation for the case of the worldsheet action (2.1) which reads [14]
0 = \frac{\partial Z}{\partial \log \Lambda} \\
= \int Dx^\mu e^{-S_\sigma[x;\{g^i\}]} \left\{ \frac{\partial S_{\text{int}}}{\partial \log \Lambda} \right. \\
- \int d^2z \int d^2w \left( \frac{\partial}{\partial \log \Lambda} G(z - w) \right) \left[ \frac{\delta^2 S_{\text{int}}}{\delta x^\mu(z) \delta x^\mu(w)} + \frac{\delta S_{\text{int}}}{\delta x^\mu(z) \delta x^\mu(w)} \right] \right\} \tag{4.1}

and it is the requirement of conformal invariance of the quantum string theory. Here $S_{\text{int}} = S_\sigma - S_\star$, $Z$ is the partition function of the $\sigma$-model, and

$$G(z - w) = \left\langle \circ x^\mu(z) x^\mu(w) \circ \right\rangle_s$$ \tag{4.2}

is the two-point function computed with respect to the conformal field theory action $S_\star[x]$. The basic assumption in arriving at eq. (4.1) is that the ultra-violet cut-off $\Lambda$ on the string worldsheet appears explicitly only in the propagator $G(z - w)$, as can always be arranged by an appropriate regularization [14].

Henceforth we shall concentrate on the specific case of interest of a system of $N$ interacting D-particles. Then, upon summing up over pinched genera, there are extra logarithmic divergences in the Green’s function (4.2) coming from pinched annulus diagrams, which may be removed by the introduction of logarithmic recoil operators, as explained in the previous section. Using primes to denote the result of resumming the topological expansion over pinched genera, we then have that

$$\frac{\partial}{\partial \log \Lambda} G(z - w)' = \frac{\partial}{\partial \log \Lambda} \sum_{\text{genera}}' \left\langle \circ x^\mu(z) x^\mu(w) \circ \right\rangle = \frac{\partial}{\partial \log \Lambda} \left\langle \circ x^\mu(z) x^\mu(w) \circ \right\rangle_{\text{int}} \tag{4.3}

where the correlator $\left\langle \cdot \right\rangle_{\text{int}}$ includes the disc and recoil interaction contributions. Subtracting the disc $\Lambda$-dependence in normal ordering, the remaining dependence on the worldsheet cutoff comes from the two-point functions of the logarithmic recoil operators, giving terms of the form

$$\frac{\partial}{\partial \log \Lambda} \left\langle \circ x^\mu(z) x^\mu(w) \circ \right\rangle \left( a_{CC} C_\epsilon(z) C_\epsilon(w) + a_{CD} C_\epsilon(z) D_\epsilon(w) + a_{DD} D_\epsilon(z) D_\epsilon(w) \right) \right\rangle_s \tag{4.4}

The leading divergence comes from the correlation function $\langle D_\epsilon(z) D_\epsilon(w) \rangle_s \sim \log \Lambda$, which follows upon the identification (3.11). Thus we may write

$$\frac{\partial}{\partial \log \Lambda} G(z - w) \simeq c_G (\alpha')^2 \log |z - w| \sum_{i=1}^9 \sum_{a,b=1}^N |U^i_{ab}|^2$$ \tag{4.5}

where $c_G > 0$ is a numerical coefficient whose precise value is not important, and we have used the fact that $U^i \in su(N)$.  

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Next, we observe that in the case of D-particles the second term in eq. \((4.1)\) becomes

\[
\int D\mu(x, \bar{\xi}, \xi) \ e^{-S_{\text{e}}} \int d\tau \int d\tau' (\alpha')^2 c_G \sum_{i=1}^{9} \sum_{a,b=1}^{N} |U_{ab}^i|^2 \log[2 - 2\cos(\tau - \tau')]
\]

\[
\times \left[ \frac{\delta^2 S_{\text{int}}}{\delta x^\mu(\tau) \delta x^\mu(\tau')} + \frac{\delta S_{\text{int}}}{\delta x^\mu(\tau)} \frac{\delta S_{\text{int}}}{\delta x^\mu(\tau')} \right] \tag{4.6}
\]

where the interaction Lagrangian is given by

\[
S_{\text{int}} = \frac{g_s}{2\pi \alpha'} \int d\tau \ \partial_\tau x^i(\tau) \bar{\xi}_a(\tau) Y_i^{ab}(x^0) \xi_b(\tau) \tag{4.7}
\]

In the case of a system of recoiling D0-branes, the \(\sigma\)-model couplings in eq. \((4.7)\) are given by \((3.8)\) with the abelianized couplings \((3.5)\) of \(Y_i^{ab}\) viewed as the boundary values for the open string embedding fields \(x^i(\tau)\) on the D-brane. This means that the fields \(x^i(\tau)\) are simply identified with \(\bar{\xi}_a(\tau)Y_i^{ab}\xi_b(\tau)\). All the non-trivial dependence comes from the \(x^0\) field which obeys Neumann boundary conditions and is not constant on the boundary of \(\Sigma\). Then we may write

\[
\frac{\delta^2 S_{\text{int}}}{\delta x^\mu(\tau) \delta x^\mu(\tau')} + \frac{\delta S_{\text{int}}}{\delta x^\mu(\tau)} \frac{\delta S_{\text{int}}}{\delta x^\mu(\tau')}
\]

\[
= \nabla^2 y_i S_{\text{int}} + (\nabla y_i S_{\text{int}})^2 + \left( \frac{g_s}{2\pi \alpha'} \right)^2 U_{ij}^{ab} U_{ij}^{cd} \bar{\xi}_a(\tau) \xi_b(\tau) \xi_c(\tau') \xi_d(\tau')
\]

\[
\times \partial_\tau x^i(\tau) \partial_\tau x^j(\tau') \Theta_\tau(x^0(\tau)) \Theta_\tau(x^0(\tau')) \tag{4.8}
\]

where \(y_i\) denotes the constant abelianized zero modes of \(x^i(\tau)\) on \(\partial\Sigma\). Here we have used the fact that terms of the form \(x^0(\delta(x^0))\) and \(\Theta(x^0)\delta(x^0)\) vanish with the regularization \((3.8)\). The terms involving \(\partial_\tau x^i(\tau) \partial_\tau x^j(\tau')\) will average out to yield terms of the form

\[
|U_{ab}^i|^2 \left( \Theta_\tau(x^0(\tau')) \Theta_\tau(x^0(\tau')) \right) \sim O(\epsilon^2)
\]

where we have used the logarithmic conformal algebra. At leading orders, these terms vanish, but we shall see the importance of such sub-leading terms later on.

Using the Dirichlet correlator

\[
\left\langle \partial_\tau x^i(\tau) \partial_\tau x_i(\tau') \right\rangle_{\ast} = -\frac{36\pi^2 \alpha'}{1 - \cos(\tau - \tau')}
\]

\[
\int \int \frac{\log[2 - 2\cos(\tau - \tau')]}{1 - \cos(\tau - \tau')} \sim \log \Lambda \tag{4.11}
\]

which has the effect of renormalizing the velocity matrix \(U_{ab}^i \rightarrow \bar{U}_{ab}^i\). Thus, ignoring the \(O(\epsilon^2)\) terms for the moment, we find that the remaining terms in the Wilson-Polchinski renormalization group equation \((4.1)\) yield a diffusion term for the probability density.
\[ \partial_t \mathcal{P}[Y, U; t] = c_G \sqrt{\alpha'} \sum_{j=1}^{9} \sum_{a,b=1}^{N} |\bar{U}_{ab}^j|^2 \nabla_{y_i}^2 \mathcal{P}[Y, U; t] + \mathcal{O}(\epsilon^2) \] (4.12)

This equation is of the Fokker-Planck type, with diffusion coefficient

\[ \mathcal{D} = c_G \sqrt{\alpha'} \sum_{i=1}^{9} \sum_{a,b=1}^{N} |\bar{U}_{ab}^i|^2 \] (4.13)

coming from the quantum recoil of the assembly of D-particles. The diffusion disappears when there is no recoil. Note that (4.13) naturally incorporates the short-distance quantum gravitational smearings for the open string interactions (compare with eq. (1.2)), and it arises as an abelianized velocity for the constant auxiliary field configuration \(\bar{\xi}_a(\tau) = \xi_a(\tau) = 1 \; \forall a = 1, \ldots, N\).

The evolution equation (4.12) should be thought of as a modification of the usual continuity equation for the probability density. Indeed, as we will now show, the \(\mathcal{O}(\epsilon^2)\) terms in eq. (4.12) coming from (4.9) are of the form \(-\nabla_{y_i} J_i\), where

\[ J_i = \frac{\hbar_\text{M}}{2im} \left( \Psi^\dagger \nabla_{y_i} \Psi - \Psi \nabla_{y_i} \Psi^\dagger \right) \] (4.14)

is the probability current density. Here

\[ m = \frac{1}{\sqrt{\alpha' g_s}}, \quad \hbar_\text{M} = 4g_s \] (4.15)

are, respectively, the BPS mass of the D-particles and the moduli space “Planck constant”. \[\]

For this, we note first of all that such terms should generically come in the form

\[ -\nabla_{y_i} J_i = -\frac{\hbar_\text{M}}{m} \left( \nabla_{y_i}^2 \arg \Psi \right) \mathcal{P} - \frac{\hbar_\text{M}}{m} \left( \nabla_{y_i} \arg \Psi \right) \nabla_{y_i} \mathcal{P} \] (4.16)

\[\]

\[1\] The identification (4.15) of Planck’s constant in the D-particle quantum mechanics on moduli space with the string coupling constant is actually not unique in the present context of considering only the exchange of strings between D-particles. As discussed in [10], the most general relation, compatible with the logarithmic conformal algebra, involves an arbitrary exponent \(\chi\) through \(\hbar_\text{M} = 4(\tilde{g}_s)^{1+\chi/2}\). The exponent \(\chi\) arises from specific mechanisms for the cancellation of modular divergences on pinched annular surfaces by appropriate world-sheet short-distance infinities at lower genera. The only restriction imposed on \(\chi\) is that it be positive definite. As shown in [10], the standard kinematical properties of D-particles are reproduced by the choice \(\chi = \frac{2}{3}\). A choice of \(\chi \neq 0\) seems more natural from the point of view that modular divergences should be suppressed for weakly interacting strings. However, in the present case, we assume for simplicity the value \(\chi = 0\), which yields the standard string smearing \(\sqrt{\alpha'}\) for the minimum length uncertainty. The incorporation of an arbitrary \(\chi \geq 0\) in the formalism is straightforward and would not affect the qualitative properties of the following results.
The second term in eq. (4.16), upon identification of the probability density $P$ with the genera resummed partition function on the string worldsheet, is proportional to the worldsheet renormalization group $\beta$-function, given the gradient flow property (2.4) of the string effective action [20], so that

$$\nabla_y J_i = -2P G_{ij} \beta^j \quad (4.17)$$

which is to be understood in terms of abelianized quantities. In the present case the renormalization group equations are given by (3.17) and (3.19) and, since the couplings $\bar{U}_{ab}$ are truly marginal, we are left in (4.17) with only a Zamolodchikov metric contribution

$$G_{CC} = 2NA^4 \langle C_c(\tau)C_c(\tau) \rangle \quad \text{(Note that here one should use suitably normalized correlators $\langle \cdot \rangle$ which yield the behaviour (4.17))}$$

From the logarithmic conformal algebra it therefore follows that a term with the structure of the second piece in eq. (4.16) is hidden in the contributions (4.9) which were dropped as being subleading in $\epsilon$. Furthermore, from (3.15), (4.16), (4.5), (4.8) and (4.9) it follows that to leading orders

$$\nabla^2 y_i \arg \Psi = 0 \quad (4.18)$$

where $\bar{u}_i = d\bar{y}_i/dt$ is the worldsheet zero mode of the abelianized, renormalized velocity operator. It then follows that to leading orders we have $\nabla^2 y_i \arg \Psi = 0$.

Thus, keeping the subleading terms in the target space regularization parameter $\epsilon$ leads to the complete Fokker-Planck equation for the probability density $P = \Psi^* \Psi$:

$$\partial_t P[Y, U; t] = -\nabla_y J_i [Y, U; t] + D \nabla^2 y_i P[Y, U; t] \quad (4.19)$$

where $J_i$ is the probability current density (4.14), with $\Psi$ the wavefunctional for the system of D-branes:

$$\Psi[Y, U; t] = \prod_{i=1}^{9} \exp \left[ -\frac{i c_G}{\sqrt{\alpha'}} \frac{y_i}{\bar{u}_i} \left( \sum_{j=1}^{9} \sum_{a,b=1}^{N} |\bar{U}_{ab}^j|^2 \right)^2 \right] |\Psi[Y, U; t]| \quad (4.20)$$

$^2$Noncommutative position dependent terms arising from commutators $[Y_i, Y_j]$ appear only at two-loop order in $\sigma$-model perturbation theory [10]. An interesting extension of the present analysis would be to generalize the results to include these higher-order terms into the quantum dynamics. However, given that the pertinent equations involve only the abelianized coordinates (3.5), we do not expect the inclusion of such terms to affect the ensuing qualitative conclusions. The effect of the noncommutativity is to render the quantum wave equation for the system of D-particles non-linear, through the recoil-induced diffusion from the multi-brane interactions, as we discuss in the subsequent sections (for a single brane one would obtain a free wave equation governing the quantum dynamics).
Such quantum diffusion is characteristic of all Liouville string theories \cite{13,17,20}. The resulting quantum dynamics, including the quantum diffusion which arises from the D-brane recoil, is described by the Schrödinger wave equation which corresponds to this Fokker-Planck equation. This equation is analysed in detail in the next section.

V. NON-LINEAR SCHröDINGER WAVE EQUATIONS

Given the Fokker-Planck equation (4.19), there is no unique solution for the wavefunction $\Psi$, as we discuss below, and the resulting Schrödinger wave equation is necessarily non-linear, due to the diffusion term \cite{27,28}. Consider the quantum mechanical system with diffusion which is described by the Fokker-Planck equation (4.19) for the probability density $P = \Psi^{\dagger}\Psi$. In \cite{27} it was shown that, by imposing diffeomorphism invariance in the space $\vec{y} \in M$ and representing the symmetry through the infinite-dimensional kinematical symmetry algebra $C^\infty(M) \supset \text{Vect}(M)$, one may arrive at the following non-linear Schrödinger wave equation:

$$i\hbar M \frac{\partial \Psi}{\partial t} = \mathcal{H}_0 \Psi + iI(\Psi)\Psi \quad (5.1)$$

where $\mathcal{H}_0$ is the linear Hamiltonian operator

$$\mathcal{H}_0 = -\frac{\hbar^2}{2m} \nabla^2_{y_i} + V_M(\vec{y}, \vec{u}; t) \quad (5.2)$$

and

$$I(\Psi) = \frac{1}{2} \hbar M \mathcal{D} \frac{\nabla^2_{y_i}(\Psi^{\dagger}\Psi)}{\Psi^{\dagger}\Psi} \quad (5.3)$$

Here $V_M(\vec{y}, \vec{u}; t)$ is the interaction potential on moduli space and the real continuous quantum number $\mathcal{D}$ in (4.13) is the classification parameter of the unitarily inequivalent diffeomorphism group representations. Other models which have more than one type of diffusion coefficient can be found in \cite{27,28}.

A crucial point \cite{28} is that there exist non-linear phase transformations of the wavefunction $\Psi$ (known as quantum mechanical “gauge transformations”) which leave invariant appropriate families of non-linear Schrödinger equations, and also the probability density $P$. Such transformations do not affect any physical observables of the system. This implies that the choice of $\Psi$ is ambiguous, once a density $P$ is found as a solution of eq. (4.19) on the collective coordinate space \{Y_i^{ab}\} of the D-branes. An important ingredient in finding such transformations is the assumption \cite{28,29} that all measurements of quantum mechanical systems can be made so as to reduce eventually to position and time measurements. Because of this possibility, a theory formulated in terms of position measurements is complete enough
in principle to describe all quantum phenomena. This point of view is certainly met by
the D-brane moduli space, whereby the wavefunctional depends only on the couplings \( \{ g' \} \)
and not on the conjugate momenta \( p_I = -i\hbar M \partial / \partial g' I \). The group of non-linear gauge trans-
formations acts on each leaf in a foliation of a family of non-linear Schrödinger equations,
such that the two-dimensional leaves of the foliation consist of sets of equivalent quantum
mechanical evolution equations.

It follows that then one can perform the following local, two-parameter projective gauge
transformation of the wavefunction \[28\] :

\[
\Psi' = N_{\gamma, \lambda}(\Psi) = |\Psi| \exp(i \gamma \log |\Psi| + i \lambda \text{ arg } \Psi)
\]  

(5.4)

under which the probability density is invariant, but the probability current transforms as

\[
\mathcal{J}'_i = \lambda \mathcal{J}_i + \frac{\gamma}{2} \nabla \text{arg } \mathcal{P}
\]

(5.5)

Here \( \gamma(t) \) and \( \lambda(t) \neq 0 \) are some real-valued time-dependent functions. The collection of all
non-linear transformations \( N_{\gamma, \lambda} \) obeys the multiplication law of the one-dimensional affine
Lie group \( Aff(1) \). Under (5.4) there are families of non-linear Schrödinger equations that are
\textit{closed} (in the sense of “gauge closure”). A generic form of such a family, to which the
non-linear Schrödinger equation (5.1) belongs, is

\[
i \frac{\partial \Psi}{\partial t} = \frac{1}{\hbar M} \mathcal{H}_0 \Psi + i \nu_2 R_2[\Psi] \Psi + \mu_1 R_1[\Psi] \Psi + \left( \mu_2 - \frac{1}{2} \nu_1 \right) R_2[\Psi] \Psi
\]

\[+ (\mu_3 + \nu_1) R_3[\Psi] \Psi + \mu_4 R_4[\Psi] \Psi + \left( \mu_5 + \frac{1}{4} \nu_1 \right) R_5[\Psi] \Psi
\]

\[= i \sum_{i=1,2} \nu_i R_i[\Psi] \Psi + \sum_{j=1}^{5} \mu_j R_j[\Psi] \Psi + \frac{1}{\hbar M} V_M(\vec{q}, \vec{u}; t) \Psi
\]

(5.6)

where \( \nu_i, \mu_j \) are real-valued coefficients which are related to diffusion coefficients \( \mathcal{D} \) and \( \mathcal{D}' \)
by

\[
\nu_1 = \frac{\hbar M}{2m}
\]

\[
\nu_2 = \frac{1}{2} \mathcal{D}
\]

\[
\mu_1 = c_1 \mathcal{D}'
\]

\[
\mu_2 = \frac{\hbar M}{4m} + c_2 \mathcal{D}'
\]

\[
\mu_3 = \frac{\hbar M}{2m} + c_3 \mathcal{D}'
\]

\[
\mu_4 = c_4 \mathcal{D}'
\]

\[
\mu_5 = \frac{\hbar M}{8m} + c_5 \mathcal{D}'
\]

(5.7)

and \( R_j[\Psi] \) are non-linear homogeneous functionals of degree 0 which are defined by
\[ R_1 = \frac{m}{\hbar_M} \nabla_y J_i \frac{\mathcal{P}}{\mathcal{P}} \]
\[ R_2 = \frac{\nabla_y^2 \mathcal{P}}{\mathcal{P}} \]
\[ R_3 = \frac{m^2 J_i^2}{\hbar_M^2 \mathcal{P}^2} \]
\[ R_4 = \frac{m}{\hbar_M} J_i \nabla_y \mathcal{P} \frac{1}{\mathcal{P}^2} \]
\[ R_5 = \frac{(\nabla_y^1 \mathcal{P})^2}{\mathcal{P}^2} \] (5.8)

In eq. (5.7) the \( c_j \) are constants, while in eq. (5.8) the probability current density is given by (4.14) with \( \mathcal{P} = \Psi^\dagger \Psi \).

The gauge group \( Aff(1) \) acts on the parameter space of the family (5.6). Some members of this family are thereby linearizable to an ordinary Schrödinger wave equation under the action of (5.4). These are the members for which there exists a specific relation between \( D \) and \( D' \) [23], and for which Ehrenfest’s theorem of quantum mechanics receives no dissipative corrections. The quantum mechanics of D-particles is not of this type, given that there is definite diffusion, dissipation and thus time irreversability. However, as discussed in [10, 23, 25], one needs to also maintain Galilean invariance, which is a property originating from the logarithmic conformal algebra of the recoil operators. As described in [28], there is a class of non-linear Schrödinger wave equations which is Galilean invariant but which violates time-reversal symmetry. For this, it is useful to first construct a parameter set of equations of the form (5.7) which remain \textit{invariant} under the gauge transformations (5.4). We may describe the parameter family of equations (5.6) in terms of orbits of \( Aff(1) \) by regarding \( \gamma = 2m\mu_1 \) and \( \lambda = 2m\nu_1 \) as the group parameters of an \( Aff(1) \) gauge transformation (5.4). Then the remaining five parameters in (5.7) are taken to be the functionally-independent parameters \( \eta_j, j = 1, \ldots, 5 \), which are invariant under \( Aff(1) \) and are defined by

\[ \eta_1 = \nu_2 - \frac{1}{2} \mu_1 \]
\[ \eta_2 = \nu_1 \mu_2 - \nu_2 \mu_1 \]
\[ \eta_3 = \frac{\mu_3}{\nu_1} \]
\[ \eta_4 = \mu_4 - \mu_1 \frac{\mu_3}{\nu_1} \]
\[ \eta_5 = \nu_1 \mu_5 - \nu_2 \mu_4 + (\nu_2)^2 \frac{\mu_3}{\nu_1} \]

(5.9)

A detailed discussion of the corresponding physical observables is given in [28]. For our purposes, we simply select the following relevant property of the non-linear Schrödinger equation based on the parameter set (5.9).
Consider the effect of time-reversal on the non-linear Schrödinger wave equation. Setting \( t \rightarrow -t \) is equivalent to introducing the following new set of coefficients:

\[
(\nu_i)^T = -\nu_i \quad i = 1, 2 \\
(\mu_j)^T = -\mu_j \quad j = 1, \ldots, 5 \\
(V_M)^T = -V_M
\]  

(5.10)

where the superscript \( T \) denotes the time-reversal transformation. It is straightforward to show \([28]\) that, in terms of the \( \eta_j \)'s, there is time-reversal invariance in the non-linear Schrödinger equation if the two parameters \( \eta_1 \) and \( \eta_4 \) are both non-vanishing. On the other hand, a straightforward calculation also shows \([28]\) that Galilean invariance sets \( \eta_4 = 0 \), thereby implying that a family of non-linear Schrödinger wave equations which is invariant under \( G(9) \) but not time-reversal invariant indeed exists. For a single diffusion coefficient \( \mathcal{D} \neq 0 \), as in the case \((4.13)\) of recoiling D-branes, one may set \( \mathcal{D}'c_j = 0 \) (corresponding to the \( \text{Aff} (1) \) gauge choice \( \mu_1 = 0 \)) and thereby obtain the set of gauge invariant parameters:

\[
\eta_1 = \frac{1}{2} \mathcal{D} \\
\eta_2 = 2\alpha'\bar{g}_s^4 \\
\eta_3 = -1 \\
\eta_4 = 0 \\
\eta_5 = -\alpha'\bar{g}_s^4 - \frac{1}{4} \mathcal{D}^2
\]  

(5.11)

The parameter set \((5.11)\) breaks time-reversal invariance, as expected from the non-trivial entropy production and decoherence characterizing the worldsheet renormalization group approach to target space time involving Liouville string theory \([10, 15, 30]\). But it does preserve Galilean invariance, as is required by conformal invariance of the non-relativistic, recoiling system of D-particles.

One may therefore propose that the Fokker-Planck equation for the probability density \( \mathcal{P} \) on the moduli space of collective coordinates of a system of interacting D-branes implies a Schrödinger wave equation for the pertinent wavefunctional which is non-linear, Galilean-invariant and has a time arrow, corresponding to entropy production, and hence explicitly broken time-reversal invariance. The existence of a dissipation \( \mathcal{D} \propto \text{tr} |\bar{U}|^2 \), due to the quantum recoil of the D-branes, implies that the Ehrenfest relations acquire extra dissipative terms for this family of non-linear Schrödinger equations. For example, one can immediately obtain the relations \([27]\)

\[
\frac{d}{dt} \left< \hat{p}_i \right> = -\left< \nabla_{y_i} V_M \right> - m \int_{\mathcal{M}} dy \Psi^\dagger \left( J_i^{(D=0)} - \frac{\mathcal{D} \nabla_{y_i}^2 \mathcal{P}}{\mathcal{P}} \right) \Psi
\]
where \( \mathcal{J}_i^{(D=0)} \) is the undissipative current density (4.14). Note that the fundamental renormalization group equations (3.13) receive no corrections due to the dissipation. The existence of extra dissipation terms in (5.12) in the Ehrenfest relation for the momentum operator \( \hat{p}_i = -i\hbar \nabla \psi_i \), which are proportional to \( \text{tr} |\tilde{U}|^2 \), may now be compared to the generalized Heisenberg uncertainty relations that were derived in [10]. These extra terms are determined by the total kinetic energy of the D-branes and their open string excitations, and they show how the recoil of the D-brane background produces quantum fluctuations of the classical spacetime dynamics. The relationship with quantum uncertainty relations will be discussed in the next section.

**VI. SOLUTIONS OF THE MATRIX D-BRANE WAVE EQUATION**

In this section we shall discuss the situation governing some of the solutions [27,31] of the class of non-linear Schrödinger equations (5.6) which will be of interest to us in the context of the quantum states of a system of many D-particles. We may parametrize the generic wavefunctional \( \Psi \) by an amplitude \( \theta_1(\vec{y},t) \) and a phase \( \theta_2(\vec{y},t) \) as

\[
\Psi = \exp (\theta_1 + i\theta_2) \tag{6.1}
\]

which, on account of (5.4), satisfy the following coupled system of non-linear partial differential equations [31]:

\[
0 = \partial_t \theta_1 - 2\nu_2 \nabla^2 \psi_i \theta_1 - \nu_1 \nabla^2 \psi_i \theta_2 - 4\nu_2 (\nabla \psi_i \theta_1)^2 - 2\nu_1 \nabla \psi_i \theta_1 \nabla \psi_i \theta_2 \\
0 = \partial_t \theta_2 + 2\mu_2 \nabla^2 \psi_i \theta_1 + \mu_1 \nabla^2 \psi_i \theta_2 + 4(\mu_2 + \mu_3) (\nabla \psi_i \theta_1)^2 \\
+ 2(\mu_1 + \mu_2) \nabla \psi_i \theta_1 \nabla \psi_i \theta_2 + \mu_3 (\nabla \psi_i \theta_2)^2 + \frac{1}{\hbar \mathcal{M}} V_M(\vec{y}, \vec{u};t) \tag{6.2}
\]

This system can be equivalently expressed in terms of the gauge invariant set of parameters \( \eta_j \) defined in (5.9) as

\[
\partial_t \theta_1 = 2\eta_1 \nabla^2 \psi_i \theta_1 + \frac{1}{2m} \nabla^2 \psi_i \theta_2 + 4\eta_2 (\nabla \psi_i \theta_1)^2 + \frac{1}{m} \nabla \psi_i \theta_1 \nabla \psi_i \theta_2 \\
\partial_t \theta_2 = -4m\eta_2 \nabla^2 \psi_i \theta_1 - 8m \left( \eta_2 + \eta_5 + \eta_4 \eta_1 - (\eta_1)^2 \eta_3 \right) (\nabla \psi_i \theta_1)^2 \\
- 2\eta_4 \nabla \psi_i \theta_1 \nabla \psi_i \theta_2 - \frac{\eta_3}{2m} (\nabla \psi_i \theta_2)^2 + V_M(\vec{y}, \vec{u};t) \tag{6.3}
\]
where we have selected the gauge $\mu_1 = 0$. Notice that there are ambiguities in the phase function $\theta_2$ in (6.1), which imply that the non-linear Schrödinger equation (5.6) and the amplitude and phase equations (6.2) are not completely equivalent. However, any solution of the phase and amplitude equation yields a solution of the non-linear Schrödinger equation (5.6). This is the commonly accepted point of view [31], and the one which we adopt in this paper.

A. Mass Superselection Rules

One of the key properties of the matrix D-brane system that we have exploited extensively thus far is its Galilean invariance (or equivalently that the matrix $\sigma$-model deformation defines a logarithmic conformal field theory). It is worthwhile to first mention some basic facts concerning Galilean invariant quantum field theories in the non-dissipative case ($\mathcal{D} = 0$) [32]. Consider two non-relativistic spinless fields $\phi_1$ and $\phi_2$ of masses $m_1$ and $m_2$ with a quartic $\phi_1\phi_2$ interaction mediated by a two-body potential $V(\vec{x} - \vec{y})$. The corresponding Haag expansions of their asymptotic fields are determined by Haag amplitudes $f_I(\vec{x} - \vec{x}_a, t - t_a; \vec{x} - \vec{x}_I, t - t_I)$, where $a = 1, 2$ labels the two fields and $I$ labels the different bound states of the system. Then the function $F_I(\vec{x}_1 - \vec{x}_2)$ defined through the equal time relation

$$f_I(\vec{x}_1 - \vec{x}_2; \vec{x}_1 - \vec{x}_I) = \delta \left( \vec{x}_I - \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2} \right) F_I(\vec{x}_1 - \vec{x}_2) \quad (6.4)$$

satisfies the usual Schrödinger wave equation [32]

$$\left[ -\frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \nabla^2_{\vec{x}_1 - \vec{x}_2} + V(\vec{x}_1 - \vec{x}_2) \right] F_I = -\varepsilon_I F_I \quad (6.5)$$

for the stationary bound state problem. This shows that the modified Haag amplitude is exactly the Schrödinger wavefunction of the bound state, with the reduced mass of the two particle system.

The important property used in arriving at the result (6.5) is the Galilean invariance of the system. The unitary projective representations of the Galilean group that arise in non-relativistic quantum mechanics cannot be reduced to vector representations, in contrast to the situation for most other physically relevant groups, such as the Poincaré and Lorentz groups. Explicit mass parameters appear in the phase factors of these representations which lead to the Bargmann superselection rule [33] that the sum of the masses appearing in the kinetic terms of the Hamiltonian must be conserved in every physical process. This implies, in particular, that the mass of any bound state must simply be the sum of the masses of its constituents. Note that since we are dealing with a non-relativistic theory, the energy
is not equal to the mass and indeed the energy of a two-particle bound state differs from the sum \( m_1 + m_2 \) by the binding energy. Since in the case of a composite system of \( N \) heavy D-particles, the non-linear Schrödinger equation also stems from the very specific Galilean invariance of the matrix D-brane dynamics, one might expect this superselection rule to carry over to the present diffusive situation. This would then imply that the mass appearing in the effective kinetic term should be \( Nm = N\bar{g}_s^{-1}/\sqrt{\alpha'} \), and the number of colours \( N \) can be absorbed into the normalization of the mass, i.e. the \( N \)-body D0-brane system behaves just like a single D-brane. However, this is not what we shall find below when we derive the extended Haag decomposition which takes into account the diffusion terms, i.e. of the quantum recoil of the D-particles. The massive Dirichlet string exchange which generates contact interactions among D-branes and gives rise to a non-linear quantum mechanical evolution equation must correctly be incorporated in order to distinguish a single brane from the non-abelian matrix D-branes. In the following we shall derive the modified superselection criteria that arise in the present context.

\section*{B. Stationary Solutions}

The first important class of solutions of interest to us here are the stationary solutions which are defined as usual \[27] by \( \partial_t \mathcal{P} = 0 \). This is equivalent to assuming \( \partial_t \theta_1 = 0 \). Consider the free case whereby the moduli space interaction potential is ignored, \( V_{\mathcal{M}} = 0 \). Then, setting \( \theta_1(\vec{y}, t) = \text{const.} \), one obtains from the amplitude equation in (6.3) the plane wave solutions \( \Psi(\vec{y}, t) = \Psi_0 e^{i(\vec{k} \cdot \vec{y} - \omega(\vec{k}) t)} \), with a modified dispersion relation obtained from the resulting phase equation in (6.3):

\[
\partial_t \theta_2(\vec{y}, t) \equiv \omega(\vec{k}) = \left( \frac{\hbar M}{2m} + c_3 \mathcal{D}' \right) \vec{k}^2 
\]

(6.6)

Notice that for the Galilean invariant but time-reversal violating case (5.11), with \( \mathcal{D}' = 0 \), the dispersion relation (6.6) is that of a non-relativistic massive free particle.

The fact that such plane wave solutions exist is an important feature of the present approach to the description of the effective dynamics of a system of many D-particles. As discussed in sections IV and V, the pertinent wave equation of a system of \( N \) D-particles, interacting via the exchange of strings, is written down in terms of the abelianized positions \( y_i \) in (3.5), rather than an \( N \)-body quantum mechanics that one would expect from the effective Yang-Mills reduction arising in the standard low-energy description of multiple D-particle dynamics \[3\]. This seems to tie in nicely with the heuristic “fat brane” picture of the multiple D-brane system advocated in \[34\], according to which one treats the assembly of D-particles as a single one with string interactions now encoded in the diffusion term (4.13).
Had one instead started with the Yang-Mills matrix quantum mechanics from the onset, then one would have obtained a linear Schrödinger equation with interactions taken care of by the Yang-Mills potential. This is what had been studied in the literature so far \[1\], with no definite, general conclusions about the bound state problem for strings and D-particles (see \[1\] for recent developments). In the present approach, we effectively reduce the quantum mechanics to a single body problem (the dynamics of the “fat brane” \[34\]), at the cost of obtaining a non-linear wave equation. This explains quite naturally the dispersion relation \( (6.6) \) in the case where string interactions between the D-branes are ignored. Note also that the energy is proportional to \( \bar{g}_s^2 \) and thereby depends on the open string excitations between the constituent D-particles.

A non-trivial issue concerns the nature of the effective moduli space potential \( V_M(\vec{y}, \vec{u}; t) \) among the constituent D-particles of the “fat D-brane”. In the context of the \( \sigma \)-model approach to the system of recoiling D-particles \[10\], the effective potential \( V_M \) among the D-particles is determined by the Zamolodchikov \( C \)-function for the deformed \( \sigma \)-model at hand, which according to the discussion in \[11\] and in section III, is given by the Born-Infeld Lagrangian \( (3.20) \). In the context of the (abelianized) “fat brane” picture, therefore, one would expect that the relevant Born-Infeld effective action can be reduced to the one containing only velocity terms of the form \( \text{tr} \sqrt{1 - |\bar{U}|^2} \), given that the coordinate-dependent commutator terms in \( (3.22) \) may be neglected to a first-order approximation, i.e. to leading orders in \( \sigma \)-model perturbation theory \[10\]. Such a coordinate independent potential may then be shifted to zero, and hence the free particle case \( V_M = 0 \) discussed above corresponds to the Galilean invariant multiple D-particle system in a first-order approximation. This is similar in spirit (but not identical) to a discussion of the ground-state properties of the multiple D-particle system, given that the absence of the non-commutative commutator terms in the Born-Infeld action \( (3.20) \) also guarantees unbroken supersymmetry in the standard effective Yang-Mills target space description of multiple D-particle systems \[35\]. This commutative limit is a saddle-point solution of the full \( U(N) \) quantum gauge theory.

However, the free case is not a satisfactory approximation if one wishes to include quantum fluctuations about such ground states, i.e. to properly incorporate string interactions between the D-particles which may enable them to form a bound state. Such a case would correspond to off-shell non-conformal backgrounds in a \( \sigma \)-model context which naturally appear when one considers a quantum field theory of D-particles \[17\]\[29\]. Unfortunately, the precise form of the effective moduli space potential in that case is still not known in a closed form (see \[11\] for the first few terms in a velocity expansion of the \( C \)-function). For instance, it is known \[3,33\] that when a string stretches between two D-particles one obtains a linear (confining) potential, but the inclusion of an arbitrary number of strings complicates the
situation. Moreover, as discussed in section V, there is no unique way of associating the
non-linear Schrödinger equation (5.6) to the effective Fokker-Planck equation for the proba-
bility distribution (4.19) which was derived by world-sheet renormalization group methods.
In particular, the precise form of the effective potential in the linear part of the associated
Schrödinger equation is irrelevant for yielding the Fokker-Planck equation (4.19). This latter
equation is essentially that which governs the quantum mechanics on moduli space. As we
now show, the present approach enables one to make some non-trivial (and generic) state-
ments about the existence of bound state solutions in the multiple D-particle system under
consideration, without precise knowledge of the effective moduli space potential \( V_M(\vec{y}, \vec{u}; t) \).

To this end we discuss a second class of stationary solutions of the non-linear Schrödinger
equation (5.6), which are valid for a generic (but appropriate) potential \( V_M \). Such a case
incorporates the complex dynamics that binds strings and D-particles in “fat D-brane”
configurations. Let us make the ansatz

\[
\theta_2(\vec{y}, t) = -m\eta_1 \theta_1(\vec{y}) - \omega t \tag{6.7}
\]

In the Galilean invariant case (5.11) of relevance to us here, the function (6.7) also solves
(6.3) and the resulting phase equation for \( \theta_1 \) is

\[
2\eta_2 \nabla^2_{y_i} \theta_1 + 4(\eta_2 + \eta_5) (\nabla_{y_i} \theta_1)^2 - \frac{1}{2m} V_M = \frac{\omega}{2m} \tag{6.8}
\]

Then the function

\[
\varphi(\vec{y}) = \exp \beta \theta_1(\vec{y}) \tag{6.9}
\]

where

\[
\beta = 2 \left( 1 + \frac{\eta_5}{\eta_2} \right) = 1 - \frac{D^2}{4\alpha' g_s^4} \tag{6.10}
\]

with \( D \) the diffusion coefficient (4.13), satisfies the affiliated linear Schrödinger equation
[27,31]

\[
- \frac{2m(\eta_2)^2}{\eta_2 + \eta_5} \nabla^2_{y_i} \varphi + V_M \varphi = -\omega \varphi \tag{6.11}
\]

Eq. (6.11) corresponds to an ordinary linear Schrödinger wave equation of a particle in the
potential \( V_M \), but with a shifted effective mass

\[
m^* = \beta m \tag{6.12}
\]

The stationary solutions of the non-linear equation (5.6) in the case (5.11) are therefore
given by
\[ \Psi(\vec{y}, t) = \varphi(\vec{y}) \frac{4\alpha' g_s^4}{\Gamma(1 + 4\alpha' g_s^4 - D^2)} \exp\left( -\frac{8i\sqrt{\alpha' g_s^3} D}{4\alpha' g_s^4 - D^2} \log \varphi(\vec{y})^2 - i\omega t \right) \]

The wavefunctions (6.13) are square-integrable, \( \Psi \in L^2(M) \), if \( \varphi(\vec{y})^{1/\beta} \) are.

Depending on the form of the potential \( V_M \), such solutions may correspond to bound states of the affiliated linear equation (6.11), with the same potential \( V_M \) but with a shifted effective mass (6.12). It is desirable that the effective mass be positive, or otherwise the interaction potential changes its sign. This requires \( \beta > 0 \) in (6.10). For a fixed string coupling constant, from (4.13) we see that the requirement of positivity of the effective bound state mass \( m^* \) translates into a limiting velocity for the D-brane dynamics. In the abelian case, this limiting velocity is just the speed of light \([36]\), as follows from the form of the abelian Born-Infeld action which coincides with the standard relativistic free particle action in this case. Here we find the bound on the velocities of the multiple D-particle system appropriate to the non-abelian Born-Infeld action and the formation of bound state composites. Alternatively, for a given order of magnitude of the diffusion coefficient (4.13) set by the recoil velocities of the D-particles, we find that bound state solutions exist for string couplings stronger than a critical value \( \bar{g}_s^* \) given by

\[ \bar{g}_s^* = \sqrt{\frac{c_G}{2} \sum_{i=1}^{9} \text{tr} |\bar{U}_{ij}|^2} \]  

(6.14)

The solutions of the non-linear Schrödinger equation differ from the bound state solutions of the affiliated linear equation by a spatially dependent phase proportional to the diffusion coefficient \( D \). In the framework of the “fat brane” picture of \([34]\), such bound states may be thought of as corresponding to bound state solutions of a system of D-particles interacting via the exchange of strings. The existence of a critical coupling (6.14) is then physically appealing, because it implies the formation of bound state condensates of D-particles and strings for strong enough string interactions, in similar spirit to the conventional quantum field theoretic case. This is in agreement with the fact \([35]\) that for weak string interactions between the constituent D-branes, the effective spacetime is commutative and no bound states can form. When the string interactions become strong, bound states form and render the effective target space geometry noncommutative. Note that in the present case, the critical coupling (6.14) depends non-trivially on the (kinetic) energy scale of the recoiling D-particles, indicating that a certain minimum amount of energy is required to form the bound state. This velocity dependence is also quite natural, since it means that no bound states can form if the D-particles move too quickly relative to one another.

Let us note also that (6.13) is the analog of the Haag decomposition in the non-diffusive case, since it is obtained from a linear Schrödinger equation via a non-linear, quantum mechanical gauge transformation. This non-linearity implies that the bound state solutions
of the affiliated linear wave equation with shifted mass $m^*$ and potential $V_M(y, \vec{u})$ acquire a phase that depends on $\vec{y}$ through the Haag amplitude $\varphi(\vec{y})$. The transition from the non-linear equation to a related linear one is “mediated” in part by this $\vec{y}$-dependent phase. The effective bound state mass (6.12) is then the analog of the Bargmann superselection criterion in the present case, which we see is modified by a non-linear transformation of the original BPS mass $m$ and also the quantum recoil diffusion.

C. Gaussian Wave Solutions

The non-linear Schrödinger equations admit additional non-trivial solutions of a Gaussian wave type. Such configurations acquire particular importance in the “fat D-particle” context, given that among them there are solitary waves, and one would expect their appearance in view of the role of D-branes as solitons in string theory. The Gaussian wave ansatz for the wave-functional $\Psi$ is described by

$$
\theta_1(y, t) = -\frac{(y - s(t))^2}{8m\sigma(t)^2} + \frac{1}{2}\log\sigma(t)
$$

$$
\theta_2(y, t) = -\frac{1}{2m} \left( A(t)y^2 + B(t)y + C(t) \right)
$$

(6.15)

where we have restricted ourselves, for simplicity, to one spatial dimension $y$, given that the Gaussian cases are separable [31]. Here $s, \sigma, A, B, C$ are real-valued functions of $t$ which are determined from the amplitude and phase equations (6.3) by equating the different powers of $y$ that appear. This leads, in general, to a system of coupled non-linear ordinary differential equations for these functions of $t$ [31,37]. This set of equations can be reduced to two ordinary second order differential equations for $\sigma$ and $s$, and $A, B, C$ may be expressed directly in terms of the solutions of these equations.

For our purposes we note that one may explicitly construct Gaussian wave solutions in the free case $V_M = 0$. Among them are also Gaussian solitary wave solutions with time-independent width $\sigma(t) = \sigma_0 = \text{const}$. For the free case $V_M = 0$, the amplitude and phase equations (6.3) imply that a necessary condition for the existence of such solutions is [31]

$$
\eta_2 + \eta_5 = \alpha' \bar{g}_s^4 - \frac{D^2}{4} = 0
$$

(6.16)

which, on account of (1.13) and (1.15), would imply extreme fine tuning between the magnitude of the velocities of the recoiling D-particles and the string coupling $\bar{g}_s$. Thus it is unlikely to be met in a generic situation, so that the class of Gaussian solitary waves do not seem to describe the D-brane dynamics (see however the discussion below for a different interpretation).
Nonetheless, the Gaussian ansatz (6.15) allows one to carry out some explicit calculations and thereby see directly the effect of the quantum recoil through the non-linearity of the wave equation. This also enables one to make some non-trivial consistency checks of the present formalism. As an illustration, consider the free case $V_M = 0$, for which the differential equation for the function $s(t)$ is that of the classical motion of a free particle, $\ddot{s} = 0$. Galilean invariance may then be used to set $s(t) = 0$ for all $t$, because a general solution $s(t) = u_0 t + s_0$ can be obtained from the $s = 0$ one by a $G(1)$ transformation. The remaining differential equations can be easily integrated by quadratures to give the solutions [37]

$$
\sigma(t) = \sigma_0 \sqrt{\left(\frac{2\bar{g}^2 t}{\sqrt{\alpha'}\sigma_0^2} + f_0\right)^2 + \beta}
$$

$$
A(t) = \frac{16}{(\alpha')^{3/2} \bar{g}_s^3 \sigma_0^2} \frac{2\bar{g}^2 t}{\sqrt{\alpha'}\sigma_0^2} + f_0 - \frac{D}{2\sqrt{\alpha'\bar{g}_s^2}} \sqrt{\left(\frac{2\bar{g}^2 t}{\sqrt{\alpha'}\sigma_0^2} + f_0\right)^2 + \beta}
$$

$$
B(t) = 0
$$

$$
C(t) = -\frac{4}{\sqrt{\alpha'\beta}} \arctan \left(\frac{2\bar{g}^2 t}{\sqrt{\alpha'}\sigma_0^2} + f_0\right) + C_0 (6.17)
$$

where $\sigma_0$, $f_0$ and $C_0$ are appropriate integration constants and $\beta$ is defined in (6.10). The complete solution for the wavefunction $\Psi$ is then obtained by substituting (6.17) into (6.15) and (6.1) with $s = 0$. From this solution it is straightforward to compute expectation values of operators in terms of Gaussian integrals, and in particular arrive at the variance relation [37]

$$
\Delta y \Delta p = 2\bar{g}_s \sqrt{1 + \left(\frac{2\bar{g}^2 t}{\sqrt{\alpha'}\sigma_0^2} + f_0 - \frac{D}{2\sqrt{\alpha'\bar{g}_s^2}}\right)^2} (6.18)
$$

The uncertainty relation (6.18) corresponds to that of a minimum uncertainty wavepacket, with effective Planck constant determined by the string coupling, as is conjectured in [10]. In fact, eq. (6.18) has a remarkably similar form to the modified Heisenberg uncertainty relations that were derived in [10] directly from the string $\sigma$-model genus expansion, in that it contains a diffusion term proportional to the total kinetic energy of the D-particles and their open string excitations which implies that the accuracy with which one can measure position and momentum in a system of D-particles depends on the (recoil) energy that arises in the measurement process. Such an energy dependence is characteristic of string-modified uncertainties and quantum gravitational effects. At time $t = 0$ we can expand the right-hand side of (6.18) for slowly moving branes to get

$$
\Delta y \Delta p \simeq \bar{g}_s \left(2 + f_0^2\right) - \frac{f_0 c_G}{\bar{g}_s} \sum_{i=1}^{g} \text{tr} |\bar{U}^i|^2 (6.19)
$$
To leading orders in $\sigma$-model perturbation theory and for weakly coupled strings, the momentum fluctuations $\Delta p$ and recoil velocities $\bar{U}$ coincide up to the BPS mass of the D-branes $[10]$. Choosing $\text{sgn} \ f_0 = -1$, it follows that (6.19) has the usual form of the string-modified Heisenberg uncertainty relation:

$$\Delta y \Delta p \simeq \bar{g}_s \left( 2 + f_0^2 \right) + \frac{N|f_0|c_G\alpha'}{\bar{g}_s^3} (\Delta p)^2$$

(6.20)

The solution to the equation for the extrema of $\Delta y$ in (6.20) as a function of $\Delta p$ leads to the usual minimum measurable length, and hence to the standard expectations that it is not possible to probe distances smaller than the intrinsic string length or the spacetime Planck length using only D-particle probes (Note that for $f_0 \geq 0$ there is no bound on the measurability of lengths in the spacetime). Notice that one may in fact use the complete form of the phase space uncertainty relations derived in $[10]$, along with the modified Ehrenfest relations (5.12), as a necessary criterion to determine in general if a candidate wavefunction represents the true quantum states of the D-particles and strings. We shall not do so here, but rather carry on with the heuristic analysis and present one final possible description of the quantum dynamics of the system.

**D. Solitary Wave Solutions**

There exist other types of solitary waves which may occur for certain regions of the two-dimensional parameter space spanned by $\eta_2$ and $\eta_5$. The construction of such solutions in the free case $V_M = 0$ may be carried out as follows $[31]$. For the Galilean invariant case under consideration, if $\theta_1$ and $\theta_2$ are solutions of the equations (6.7) and (6.8), then the Galilean boosted amplitude and phase

$$\tilde{\theta}_1(\bar{y},t) = \theta_1(\bar{y} - \bar{u}t)$$

$$\tilde{\theta}_2(\bar{y},t) = \theta_2(\bar{y} - \bar{u}t, t) - m\bar{u} \cdot \bar{y} - \frac{m\bar{u}^2}{2} \eta_3 t$$

(6.21)

with $\bar{u}$ a constant velocity, are non-stationary solutions of (6.3). Such transformations take the original Galilean invariant solution of the non-linear Schrödinger equation (5.6),(5.11), with $\eta_4 = 0, \eta_3 = -1$, to solutions in the more general case

$$-2\eta_1(1 + \eta_3) + \eta_4 = 0$$

(6.22)

The resulting solution for the wave-functional $\Psi$ then reads

$$\Psi(\bar{y},t) = \Psi_0 \exp \left( \theta_1(\bar{y} - \bar{u}t) - i \left[ 4m\eta_1 \theta_1(\bar{y} - \bar{u}t) + m\bar{u} \cdot \bar{y} + \frac{m\bar{u}^2}{2} \eta_3 t + \omega t \right] \right)$$

(6.23)
where $\Psi_0$ is a constant. The associated probability density $\mathcal{P} = \Psi^\dagger \Psi$ is itself a solitary wave, moving with constant speed and without changing its shape:

$$\mathcal{P}(\vec{y}, t) = |\Psi_0|^2 \exp 2\theta_1(\vec{y} - \vec{u}t) \quad (6.24)$$

In the particular case where $\eta_2 (\eta_2 + \eta_5) < 0$, i.e. $\beta < 0$, one obtains square-integrable solitary wave solutions of the form $[31]

$$\Psi(\vec{y}, t) = \Psi_0 \cosh \left( \vec{k} \cdot (\vec{y} - \vec{u}t) \right) \frac{4\alpha' \bar{g}_s^4}{4\alpha' \bar{g}_s^4 - D^2} \exp \left[ -\frac{8i\sqrt{\alpha' \bar{g}_s^2 D}}{4\alpha' \bar{g}_s^4 - D^2} \log \cosh \left( \vec{k} \cdot (\vec{y} - \vec{u}t) \right) \right] - \frac{16\alpha' \bar{g}_s^8}{4\alpha' \bar{g}_s^4 - D^2} \vec{k}^2 t \quad (6.25)$$

Notice that the condition $\beta < 0$ is opposite to that for the existence of positive effective mass bound states in the affiliated linear Schrödinger equation (6.11). This implies that such coherent solitonic states cease to exist for string couplings stronger than the critical value (6.14), for given recoil velocities of the D-particles.

Such solitary waves may play an important role in the underlying dynamics of a system of D-particles. The fact that they arise in the “fat brane” picture indicates that the latter degree of freedom constitutes an effective coarse-grained description of macroscopic quantum coherent states that may characterize (under the specified conditions) a certain phase of the ground state of a system of D-particles interacting among themselves via the exchange of strings. From this point of view, the critical coupling situation (6.10) where $\bar{g}_s = \bar{g}_s^*$, corresponding to Gaussian solitary waves, may be thought of as a critical regime separating the bound state phase from the solitary wave coherent phase. It should be noted, however, that although the wavefunctions (6.13) and (6.25) are non-analytic around $\bar{g}_s = \bar{g}_s^*$, this change of solution should not be regarded as some phase transition in string coupling constant space. Rather, when the string interactions become weak enough to untighten the bound states of D-particles and strings, the quantum dynamics is described by minimum uncertainty wavepackets which produce the appropriate smearing to the quantum spacetime. Outside of this regime the D-particles behave as freely propagating solitary waves lending another interpretation to the notion of D-branes as string theoretic solitons. It should also be stressed that this heuristic picture is complementary to the picture discussed in [10], where the microscopic structure of groups of recoiling D-particles under the exchange of strings has been studied in detail in a $\sigma$-model context. In the present picture one finds an effective quantum mechanics for a “coarsed grained” description of the multiple D-particle system, in which “lumps” of D-particles are described as single bodies obeying a non-linear diffusive Schrödinger dynamics which encodes their interactions. As mentioned before, this analysis is at present preliminary and must thought of as a heuristic picture for the D-brane dynamics, given that it neither incorporates supersymmetry nor the precise form of the effective
moduli space potential $V_M$. The above discussion has only centered around the free case as an illustration, which is correct only to a certain approximation, since the abelianization of the collective coordinates of the D-particles, on which the “free” particle picture is based, is not accurate beyond this mean field approximation.

VII. CONCLUSIONS

The above discussion completes the first analysis concerning the construction of a Schrödinger-type wave equation (here non-linear) for the wavefunctional of a collection of D-particles. This equation is the present proposal for the complicated quantum dynamics of Dirichlet branes, at least within the framework of worldsheet $\sigma$-model perturbation theory. Further analysis should concentrate on the applicability of such studies to interferometric devices, which might probe Planckian physics. The non-linear Schrödinger dynamics that we have described above to characterize generic systems of D-branes should also be compared to the linear modifications of Schrödinger wave equations in open quantum mechanical systems interacting with a stochastic type environment. Such types of interactions are customarily assumed in many approaches to quantum gravity.

There are, in general, arguments against non-linear Schrödinger wave equations, mostly stemming from the fact that non-linearities may lead to superluminal propagation, i.e. motion which goes faster than the speed of light (see the discussion in [28] and references therein). In the D-brane picture advocated above, there is the advantage of expressing the system in a “closed” but non-linear form, i.e. writing a self-consistent quantum equation which comprises only known degrees of freedom, including recoil. In the above analysis we did not use environment operators, which are usually unknown and assumed to be generically of a stochastic form. In the present approach we have demonstrated the existence of quantum diffusion and of stochasticity [10,15] in the sense of Gaussian probability distributions obtained explicitly as a result of summing over all worldsheet genera (i.e. coming from the quantum string theory).

Moreover, as shown in [10], the resulting off-shell target space action underlying the Schrödinger dynamics near a fixed point in moduli space (where the entire approach is valid) is the non-abelian Born-Infeld action, which implies a limiting velocity (equal to the speed of light) for the D-particle dynamics [36]. This feature prevents superluminal propagation. Thus the issues of superluminal propagation due to a non-linear Schrödinger wave equation, which were originally advocated in [38], cannot apply to the D-particle cases. These issues express such conceptual difficulties as the possibility of using long-ranged quantum correlations to send instantaneous signals, and of communicating among...
different quantum mechanical worlds in a “multiple world” interpretation. As noted in [28], since non-linear gauge transformations on the wavefunctional can be found which linearize the non-linear Schrödinger wave equation, such claims cannot be correct in general. The Fokker-Planck equation that we have derived for the probability density may correspond to a linear Schrödinger wave equation, but of an open system, i.e. including interactions with environmental degrees of freedom (in agreement with the analysis of [17,38]). However, the non-linear wave equation involving only functionals of \(\Psi\), and thereby closing the system, is preferred. We do not need to appeal here to the unknown environmental degrees of freedom to describe the recoil-induced diffusion of D-brane dynamics.

Although the arguments for superluminal propagation are overcome by the existence of a Born-Infeld Lagrangian for the D0-brane dynamics, a more direct approach is desired since the Fokker-Planck diffusion equation implies that an external environment is present for the D-brane dynamics. This may be argued through the fact that the non-linear effects that we have considered cannot produce superluminal propagation because the present formalism is self-consistent only for weakly-coupled strings with \(\bar{g}_s \to 0\) (corresponding to heavy D-particles and hence very small velocities). Light string states are then obtained by S-duality under which the resulting Schrödinger wave equation may indeed become linear.

Notice also that the time-reversal violation of the effective wave equation that we have obtained is implied directly (through entropy production) by the fact that decoherence is present in the system of D-particles. However, if \(CPT\) is conserved in a standard string theory, then \(T\) violation should occur if \(CP\) symmetry breaking is present. In the Born-Infeld Lagrangian one may have induced \(CP\) violation in four dimensions, for example, by the introduction of topological instanton terms. In string phenomenology such terms arise quite naturally upon compactification of the target space. The present approach which allows decoherence (due to the energy dependence of the quantum uncertainties) therefore provides a natural explanation of the supression of \(CP\) symmetry in string theory. But generally in decohering quantum mechanical systems \(CPT\) is not conserved, so the above approach also differentiates between standard particles and recoiling D-particles.

We close by mentioning that non-linear quantum dynamics may be desirable from another point of view discussed recently in [39]. The non-linear evolution of correlated systems (i.e. those involving the dynamics of many “particle” states) may be free from the contamination by negative probabilities which afflict the corresponding linear case. The D-brane systems we are considering here always involve many D-branes interacting with each other via the exchange of closed string states. In the present picture [10] single D-particles are only viewed as a limiting case of a multi-brane system where the constituent branes are separated at macroscopic distances compared to the string length \(\sqrt{\alpha'}\). The non-linear dynamics,
therefore, which we proposed in this article, may be essential for guaranteeing complete positivity of the system in the generalized sense discussed in [39]. If one accepts the modern viewpoint of string-inspired phenomenology of the observable world [40], according to which the latter is represented as a D-brane interacting with other branes via the exchange of closed string bulk states only, then the above-mentioned issues may be of crucial importance for a consistent formulation.

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APPENDIX

In this Appendix we present a world-sheet formalism in support of the identification of the world-sheet partition function of a Liouville string with the probability density in moduli space rather than a wavefunctional, as happens in the case of critical strings.

We commence our analysis by considering the correlation functions among vertex operators in a generic Liouville theory, viewing the Liouville field as a local renormalization-group scale on the world sheet [22]. Standard computations [41] yield for an $N$-point correlation function among world-sheet integrated vertex operators $V_i \equiv \int d^2 z V_i(z, \bar{z})$:

$$A_N \equiv \langle V_{i_1} \ldots V_{i_N} \rangle \propto \Gamma(-s) \mu^s < \int d^2 z \sqrt{\gamma} \tilde{e}^{\alpha \phi} \tilde{V}_{i_1} \ldots \tilde{V}_{i_N} >_{\mu=0} \tag{7.1}$$

where the tilde denotes removal of the Liouville field $\phi$ zero mode, which has been path-integrated out in (7.1). The world-sheet scale $\mu$ is associated with cosmological constant terms on the world sheet, which are characteristic of the Liouville theory. The quantity $s$ is the sum of the Liouville anomalous dimensions of the operators $V_i$:

$$s = -\sum_{i=1}^{N} \frac{\alpha_i}{\alpha} - \frac{Q}{\alpha} ; \quad \alpha = -\frac{Q}{2} + \frac{1}{2} \sqrt{Q^2 + 8} \tag{7.2}$$
The $\Gamma$ function can be regularized \cite{12, 22} (for negative-integer values of its argument) by analytic continuation to the complex-area plane using the the Saaschultz contour of Fig. 1. Incidentally, this yields the possibility of an increase of the running central charge due to the induced oscillations of the dynamical world sheet area (related to the Liouville zero mode). This is associated with an oscillatory solution for the Liouville central charge near the fixed point. On the other hand, the bounce interpretation of the infrared fixed points of the flow, given in refs. \cite{12, 22}, provides an alternative picture of the overall monotonic change at a global level in target space-time.

To see technically why the above formalism leads to a breakdown in the interpretation of the correlator $A_N$ as a target-space string amplitude, which in turn leads to the interpretation of the world-sheet partition function as a probability density rather than a wave-function in target space, one first expands the Liouville field in (normalized) eigenfunctions $\{\phi_n\}$ of the Laplacian $\Delta$ on the world sheet

$$\phi(z, \bar{z}) = \sum_n c_n \phi_n = c_0 \phi_0 + \sum_{n \neq 0} \phi_n \propto A^{-1/2}$$

(7.3)

with $A$ the world-sheet area, and

$$\Delta \phi_n = -\epsilon_n \phi_n \quad n = 0, 1, 2, \ldots, \quad \epsilon_0 = 0 \quad (\phi_n, \phi_m) = \delta_{nm}$$

(7.4)

The result for the correlation functions (without the Liouville zero mode) appearing on the right-hand-side of eq. (7.1) is, then

$$\tilde{A}_N \propto \Pi_{n \neq 0} dc_n e^{x_p(-\frac{1}{8\pi} \sum_{n \neq 0} \epsilon_n c_n^2 - \frac{Q}{8\pi} \sum_{n \neq 0} R_n c_n + \sum_{n \neq 0} \alpha_i \phi_n(z_i) c_n) \left( \int d^2 \xi \sqrt{\hat{\gamma}} e^{\alpha \sum_{n \neq 0} \phi_n c_n} \right)^s}$$

(7.5)

with $R_n = \int d^2 \xi R^{(2)}(\xi) \phi_n$. We can compute \cite{7.3} if we analytically continue \cite{11} $s$ to a positive integer $s \to n \in \mathbb{Z}^+$. Denoting

$$f(x, y) \equiv \sum_{n, m \neq 0} \frac{\phi_n(x) \phi_m(y)}{\epsilon_n}$$

(7.6)

one observes that, as a result of the lack of the zero mode,

$$\Delta f(x, y) = -4\pi \delta^{(2)}(x, y) - \frac{1}{A}$$

(7.7)

We may choose the gauge condition $\int d^2 \xi \sqrt{\hat{\gamma}} \phi = 0$. This determines the conformal properties of the function $f$ as well as its ‘renormalized’ local limit

$$f_R(x, x) = \lim_{x \to y}(f(x, y) + \ln d^2(x, y))$$

(7.8)
where $d^2(x, y)$ is the geodesic distance on the world sheet. Integrating over $c_n$ one obtains

$$\tilde{A}_{n+N} \propto \exp \left( \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j f(z_i, z_j) + \frac{Q^2}{128\pi^2} \int \int R(x)R(y) f(x, y) - \sum_i \frac{Q}{8\pi} \alpha_i \left[ \sqrt{\gamma}R(x) f(x, z_i) \right] \right)$$  \hfill (7.9)

We now consider infinitesimal Weyl shifts of the world-sheet metric, $\gamma(x, y) \rightarrow \gamma(x, y)(1 - \sigma(x, y))$, with $x, y$ denoting world-sheet coordinates. Under these, the correlator $A_N$ transforms as follows [22]

$$\delta \tilde{A}_N \propto \left[ \sum_i h_i \sigma(z_i) + \frac{Q^2}{16\pi} \int d^2 x \sqrt{\gamma}R \sigma(x) + \frac{1}{A} \left\{ Qs \int d^2 x \sqrt{\gamma}R \sigma(x) + (s)^2 \int d^2 x \sqrt{\gamma}R \hat{f}_R(x, x) + Qs \int d^2 x d^2 y \sqrt{\gamma}R(x) \sigma(y) \hat{G}(x, y) - s \sum_i \alpha_i \int d^2 x \sqrt{\gamma}R(x) \hat{G}(x, z_i) - \frac{1}{2} s \sum_i \alpha_i \hat{f}_R(z_i, z_i) \int d^2 x \sqrt{\gamma}R(x) - \frac{Qs}{16\pi} \int d^2 x d^2 y \sqrt{\gamma}(x) \gamma(y) \hat{R}(x) \hat{f}_R(x, x) \sigma(y) \right\} \tilde{A}_N$$ \hfill (7.10)

where the hat notation denotes transformed quantities, and the function $G(x, y)$ is defined as

$$G(z, \omega) \equiv f(z, \omega) - \frac{1}{2} (f_R(z, z) + f_R(\omega, \omega))$$ \hfill (7.11)

and transforms simply under Weyl shifts [22]. We observe from (7.10) that if the sum of the anomalous dimensions $s \neq 0$ (‘off-shell’ effect of non-critical strings), then there are non-covariant terms in (7.10), inversely proportional to the finite-size world-sheet area $A$. Thus the generic correlation function $A_N$ does not have a well-defined limit as $A \rightarrow 0$.

In our approach to string time we identify [22] the target time as $t = \phi_0 = -\log A$, where $\phi_0$ is the world-sheet zero mode of the Liouville field. The normalization follows from a consequence of the canonical form of the kinetic term for the Liouville field $\phi$ in the Liouville $\sigma$ model [43] [22]. The opposite flow of the target time, as compared to that of the Liouville mode, is, on the other hand, a consequence of the ‘bounce’ picture [12] [22] for Liouville flow of Fig. 1. In view of this, the above-mentioned induced time (world-sheet scale $A$-) dependence of the correlation functions $A_N$ implies the breakdown of their interpretation as well-defined $S$-matrix elements, whenever there is a departure from criticality $s \neq 0$.

In general, this is a feature of non-critical strings wherever the Liouville mode is viewed as a local renormalization-group scale of the world sheet [22]. In such a case, the central charge of the theory flows continuously with the world-sheet scale $A$, as a result of the
Zamolodchikov c-theorem [14]. In contrast, the screening operators in conventional strings yield quantized values [13]. Due to the analytic continuation curve illustrated in Fig. 1, we observe that upon interpreting the Liouville field $\phi$ as time [22]: $t \propto \log A$, the contour of Fig. 1 represents evolution in both directions of time between fixed points of the renormalization group: Infrared fixed point $\rightarrow$ Ultraviolet fixed point $\rightarrow$ Infrared fixed point.

When one integrates over the Saalschultz contour in Fig. 1, the integration around the simple pole at $A = 0$ yields an imaginary part [12, 22], associated with the instability of the Liouville vacuum. We note, on the other hand, that the integral around the dashed contour shown in Fig. 1, which does not encircle the pole at $A = 0$, is well defined. This can be interpreted as a well-defined $S$-matrix element, which is not, however, factorisable into a product of $S$- and $S^\dagger$-matrix elements, due to the $t$ dependence acquired after the identification $t = -\log A$.

Note that this formalism is similar to the Closed-Time-Path (CTP) formalism used in non-equilibrium quantum field theories [15]. Such formalisms are characterized by a ‘doubling of degrees of freedom’ (c.f. the two directions of the time (Liouville scale) curve of Fig. 1, in each of which one can define a set of dynamical fields in target space). As we discussed above, this prompts one to identify the corresponding Liouville correlators $A_N$ with $S$-matrix elements rather than $S$-matrix elements in target space. Such elements act on the density matrices $\rho = \text{Tr}_M|\Psi><\Psi|$ rather than wave vectors $|\Psi>$ in target space of the string: $\rho_{\text{out}} = S\rho_{\text{in}}$ (c.f. the analogy with the $S$-matrix, $|\text{out}> = S|\text{in}>$).

This in turn implies that the world-sheet partition function $\tilde{Z}_{\chi,L}$ of a Liouville string at a given world-sheet genus $\chi$, which is connected to the generating functional of the Liouville correlators $A_N$, when defined over the closed Liouville (time) path (CTP) of figure 1, can be associated with the probability density (diagonal element of a density matrix) rather than the wavefunction in the space of couplings. Indeed, one has

$$\tilde{Z}_{\chi,L}[g^I] = \int_{\text{CTP}} d\phi_0 Z_{\chi,L}[\phi_0, g^I]$$

(7.12)

where $\{g^I\}$ denotes the set of couplings of the (non-conformal) deformations, $\phi_0 \sim \ln A$ is the Liouville zero mode, and $A$ is the world-sheet area (renormalization-group scale). If one naively interprets $Z_{\chi,L}[\phi_0, g^I]$ as a wavefunctional in moduli space $\{g^I\}$, $\Psi[\phi_0, g^I]$, then, in view of the double contour of figure 1 over which $\tilde{Z}_{\chi,L}$ is defined, one encounters at each slice of constant $\phi_0$ a product of $\Psi[\phi_0, g^I]\Psi^\dagger[\phi_0, g^I]$, the complex conjugate wavefunctional corresponding to the second branch of the contour of opposite sense to the branch defining $\Psi[\phi_0, g^I]$. This is analogous to the doubling of degrees of freedom in conventional thermal field theories [15]. Such products represent clearly probability densities $\mathcal{P}[t, g^I]$ in moduli space of the non-critical strings upon the identification of the Liouville zero mode $\phi_0$ with the target time $t$ [22].
In the above spirit, one may then consider the (formal) summation over world-sheet topologies $\chi$, and identify the summed-up world-sheet partition function $\sum_{\chi} Z_{\chi,L}[\phi_0, g']$ with the associated probability density in moduli space. In the case of D-particles, discussed in this work, the moduli space coincides with the configuration space (collective) coordinates of the D-particle soliton, and hence the corresponding probability density is associated with the position of the D-particle in target space. We stress once again that the above conclusion is based on the crucial assumption of the definition of the Liouville-string world-sheet partition function over the closed-time-path of figure $[1]$. As we demonstrate in the main text, the specific D–brane example provide us with highly non-trivial consistency checks of this approach.

Before closing this appendix we would like to give an explicit demonstration of the above ideas for the specific (simplified) case of recoiling (Abelian) D-particles. We shall demonstrate below that, upon considering the non-critical $\sigma$-model of a recoiling D-particle at a fixed world-sheet (Liouville) scale $\phi_0 = \ln A$, and identifying the Liouville mode with the target time, the Euclideanized world-sheet partition function can describe a probability density in moduli (collective coordinate) space.

To this end, let us first consider the pertinent $\sigma$ model partition function for a D-particle, at tree level and in a Minkowskian world-sheet $\Sigma$ formalism:

$$Z_{\chi=0,L} = \int (DX^i) \, e^{-i\frac{1}{4\pi\alpha'} \int_{\Sigma} \partial X^i \overline{\partial X^j} n_{ij} + i\frac{1}{4\pi\alpha'} \int_{\partial \Sigma} (\epsilon g^C + g^{D\pi}_t) \partial_n X^i}$$  \hspace{1cm} (7.13)$$

where $\epsilon^{-2} \sim \ln \Lambda^2 = \ln A$ (c.f. (3.11)), on account of the logarithmic algebra [23]. In our approach $\epsilon^{-2}$ is identified with the target time. This is why in (7.13) we have not path-integrated over $X^0$, but we consider an integral only over the spatial collective coordinates $X^i, i = 1, \ldots 9$ of the D-particle. The combination of $\sigma$-model couplings $\epsilon g^C + g^{D\pi}_t$ may be identified with the generalized (Abelian) position $\epsilon Y^i$ of the recoiling D-particle (3.9).

Notice that, since here we have already identified the time with the scale $\epsilon^{-2} > 0$, the step function in the recoil deformations of the $\sigma$-model (3.7) acquires trivial meaning. We shall come back to a discussion on how one can incorporate a world-sheet dependence in the time coordinate later on.

Suppose now that, following the spirit of critical strings [14], one identifies the Minkowskian world-sheet partition function (7.13) with a wavefunctional $\Psi[Y^i, \phi_0 = t]$. The probability density in $Y^i$ space, $P[Y^i, t] = \Psi[Y^i, t] \Psi^*[Y^i, t]$, reads in this case:

$$|Z_{\chi=0,L}[Y^i, t]|^2 = \int DX^i \int DX'' e^{-i\frac{1}{4\pi\alpha'} \int_{\Sigma} \partial X^i \overline{\partial X^j} n_{ij} + i\frac{1}{4\pi\alpha'} \int_{\partial \Sigma} \epsilon Y^i(t) \partial_n (X^i - X^n)} = \left( \int DX^i e^{i\frac{1}{4\pi\alpha'} \int_{\Sigma} \partial X^i \overline{\partial X^j} n_{ij} - i\frac{1}{4\pi\alpha'} \int_{\partial \Sigma} Y^i(t) \partial_n X^i \right)^\infty \left( \int DX^i e^{-i\frac{1}{4\pi\alpha'} \int_{\Sigma} \partial X^i \overline{\partial X^j} n_{ij}} \right),$$  \hspace{1cm} (7.14)$$

where $X^i_{\pm} = X^i \pm X'^i$. Upon passing to a Euclidean world-sheet formalism, and taking into account that the $Y_i$ independent factor can be absorbed in appropriate normalization.
of the \(\sigma\)-model correlators, one then proves our statement that \(\sigma\)-model partition functions in non-critical strings can be identified with moduli space probability densities.

Notice that similar conclusions can be reached even in the case where the time \(X^0\) is included in the analysis as a full fledged world-sheet field and *is only eventually* identified with the Liouville mode. In such a case, by considering the probability density as above, one is confronted with path integration over \(X^0_\pm = X_0 \pm X^0\) \(\sigma\)-model fields, which also appear in the arguments of the step function operators \(\Theta_\epsilon(X^0_\pm)\) in the recoil deformations \(\text{(3.7)}\), that are non trivial in this case. However, upon Liouville dressing and the *requirement* that the Liouville mode be identified with the target time, one is forced to restrict oneself on the hypersurface \(X^- = 0\) in the corresponding path integral \(\int DX^0_+ DX^0_- (\ldots)\). As a consequence, one is then left with a world-sheet partition function integrated only over the Liouville mode \(X_+ = 2\phi\) (c.f. \(\tilde{Z}\) in \(\text{(3.12)}\)), and hence the identification of a Liouville string partition function with a probability density in moduli space is still valid, upon passing onto a Euclideanized world-sheet formalism. It can also be seen, in a straightforward manner, that summing upon higher world-sheet topologies, as in \(\text{[10]}\), will not change this conclusion.
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FIG. 1. Contour of integration in the analytically-continued (regularized) version of $\Gamma(-s)$ for $s \in \mathbb{Z}^+$. The quantity $A$ denotes the (complex) world-sheet area. This is known in the literature as the Saalschutz contour, and has been used in conventional quantum field theory to relate dimensional regularization to the Bogoliubov-Parasiuk-Hepp-Zimmermann renormalization method. Upon the interpretation of the Liouville field with target time, this curve resembles closed-time-paths in non-equilibrium field theories.