ABSTRACT

We discuss some recent results on black hole thermodynamics within the context of effective gravitational actions including higher-curvature interactions. Wald’s derivation of the First Law demonstrates that black hole entropy can always be expressed as a local geometric density integrated over a space-like cross-section of the horizon. In certain cases, it can also be shown that these entropy expressions satisfy a Second Law. One such simple example is considered from the class of higher curvature theories where the Lagrangian consists of a polynomial in the Ricci scalar.

1. Introduction

Thermodynamics and statistical mechanics describe systems with a large number of degrees of freedom by following the evolution of a few macroscopic parameters, rather than trying to understand all of the details of the microphysics. One such parameter is entropy. Within the context of thermodynamics, entropy is a measure of the degradation of energy in processes. For a thermal system undergoing an infinitesimal change, the First Law of thermodynamics states

\[ T \delta S = \delta U + \delta W \]  

where \( T, S, U \), and \( W \) are the temperature, the entropy, the internal energy, and the work done, respectively. On the left hand side, \( T \delta S \) indicates the amount of energy which becomes unavailable for work in future processes, \( i.e., \) the heat loss. In the context of statistical mechanics, entropy has quite a different significance. Here, it is a measure of one’s lack of detailed information about the microphysical state of a system. The First Law then specifies how this imprecision changes in some process. In fact, the Second Law of thermodynamics dictates that in any process the total

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entropy will never decrease, \( i.e., \delta S_{\text{tot}} \geq 0 \). This law is then a statement that one’s uncertainty about the microscopic physics increases as systems evolve. It is perhaps a surprise that consistency of the Second Law on very large scales requires recourse to general relativity\[1\]. Regarding the entire universe as a single closed system, one would expect that it should have relaxed into equilibrium in the maximum entropy configuration, in contradiction with observations. Such an equilibrium has not been reached because of the cosmological expansion of the universe, and hence there is no inconsistency when this effect of general relativity is taken into account.

Another remarkable connection between thermodynamics and gravity arises in black hole physics — namely, black holes carry an intrinsic entropy. This result relies on the fundamental property that a black hole is a region of spacetime which is inaccessible to observations, and an essential role is played by the event horizon, the boundary between the regions observable and unobservable from infinity. Consider a box carrying some thermal system travelling through spacetime. If the box interacts with other external systems, one may expect that its internal state will be taken out of equilibrium. According to the Second Law, the subsequent evolution would then be characterized by a continued increase of the entropy as the system returns to equilibrium. If the box were to fall into a black hole, it would move out of the region of spacetime in which measurements can be observed from infinity, and there would no longer be any evidence of the entropy carried by the box. The entropy in the observable spacetime would thus appear to have decreased, yielding an apparent violation of the Second Law. To restore the validity of the Second Law, one can assign an extra entropy to the boundary of the recorded spatial region (at each instant of time). This boundary is, of course, a spatial cross-section of the event horizon. Thus, by these very simple considerations, one is led naturally to the concept of black hole (or more generally horizon) entropy.

Similar reasoning led Bekenstein\[2\] to make the bold conjecture some twenty years ago that, within general relativity, black holes carry an intrinsic entropy given by the surface area of the horizon measured in Planck units multiplied by a dimensionless number of order one. This conjecture was also suggested by Hawking’s area theorem\[3\] which had shown that, like entropy, the horizon area can never decrease in classical general relativity. Bekenstein offered arguments for the proportionality of entropy and area, which relied on information theory, as well as the properties of charged rotating black holes in general relativity\[2\].

The next crucial insight came from Hawking while investigating quantum fields in a black hole spacetime. He found that external observers detect the emission of thermal radiation from a black hole with a temperature proportional to its surface gravity, \( \kappa \)\[4\]:

\[
k_B T = \frac{\hbar \kappa}{2\pi c}.
\]

The surface gravity may be thought of as the redshifted acceleration of a fiducial observer moving just outside the horizon\[3\]. In the simplest (\( i.e., \) spherically symmetric) case, \( \kappa = c^4/(4GM) \) where \( M \) is the mass of the black hole.

Previously, extensive studies of Einstein’s equations had culminated in the formulation of four laws of black hole physics\[4\]. Hawking’s discovery of the thermal
radiance of black holes was the key to realizing that these results were the laws of thermodynamics applied to black holes. In the present discussion, our primary interest will be in the First and Second Laws. The First Law of black hole mechanics takes the form

\[ \frac{c^2}{8\pi G} \delta A = c^2 \delta M - \Omega \delta J \]  

(3)

where \( A, M, \Omega \) and \( J \) are the horizon area, the mass, the angular velocity (of the horizon), and angular momentum of the black hole[7]. Here, \( M c^2 \) is naturally identified with the internal energy of the black hole, and \( -\Omega \delta J \) appears to be a work term. Thus given the relation of the surface gravity to the black hole temperature in Eq. (2), the identification of Eq. (3) with the usual thermodynamic First Law (1) is completed by recognizing that the black hole entropy is

\[ S = \frac{k_B c^3 A}{\hbar G} \]  

(4)

This formula applies for any black hole solution of Einstein’s equations[8]. The Second Law of black hole thermodynamics is then established by Hawking’s area theorem, which states that in any classical process involving black holes and positive energy matter, provided that naked singularities do not develop, the total surface area of the event horizon will never decrease[3]. Thus one arrives at a consistent framework for black hole thermodynamics by drawing upon results from thermodynamics, quantum field theory and general relativity. Much of the subsequent interest was and is motivated by the hope that it may provide some insight into the nature of quantum gravity.

Now we would like to motivate studying the thermodynamic properties of black holes within higher curvature gravity theories. Whatever framework physicists eventually uncover to describe quantum gravity, there should be a low energy effective action which describes the dynamics of a “background metric field” for sufficiently weak curvatures at sufficiently long distances. On general grounds, one expects that this effective gravity action will consist of the classical Einstein action plus a series of covariant, higher-dimension interactions (i.e., higher curvature terms, and also higher derivative terms involving all of the physical fields) induced by quantum effects. While such effective actions are typically pathological when considered as fundamental, they may also be used to define mild perturbations for Einstein gravity coupled to conventional matter fields. It is within this latter context of Einstein gravity “corrected” by higher dimension operators that we wish to consider modifications of black hole thermodynamics.

Naive dimensional analysis suggests that the coefficients of all higher dimension terms in such an effective lagrangian should be dimensionless numbers of order unity times the appropriate power of the Planck length. Thus one might worry the effect of the higher dimension terms would be the same order as those of quantum fluctuations, and so there would seem to be little point in studying modifications to classical black hole thermodynamics from higher dimension terms. One motivation for studying the classical problem is that it is of course possible that the coefficients of some higher dimension terms are larger than what would be expected from simple dimensional
We would like to know whether or not consistency with classical black hole thermodynamics places any new restrictions on these coefficients. Moreover, it is interesting to explore black hole thermodynamics in generalized gravity theories in order to see whether the thermodynamic “analogy” is just a peculiar accident of Einstein gravity, it is a robust feature of all generally covariant theories of gravity, or it is something in between.

Certainly, Bekenstein’s original considerations require only the existence of an event horizon, but make no reference to the details of the dynamics of the gravity theory which determines the precise spacetime geometry. Hence these observations would be equally applicable in higher-curvature gravity theories. Similarly, the emission of Hawking radiation with a temperature as in Eq. (2) is a result of quantum field theory for a spacetime containing a horizon[4, 9]. Again, this result is independent of the details of any particular theory of gravity.

Thus black holes in higher-curvature theories will also emit thermal radiation with a temperature proportional to the surface gravity.

From Euclidean path integral methods[10], it is clear that a version of the First Law of black hole thermodynamics still applies in any higher-curvature theory. Applying these techniques to various specific theories and specific black hole solutions, though, showed that Eq. (1) which equates the black hole entropy with the surface area of the horizon no longer holds in general[12]. Recently, the entropy was shown to be given always by a local expression evaluated at the horizon[13, 14, 15, 16].

Wald[14] established the latter result very generally for any diffeomorphism invariant theory via a new (Minkowski signature) derivation of the First Law of black hole mechanics. For variations around a stationary black hole background, he derived a modified First Law

$$\frac{\kappa}{2\pi c} \delta S = c^2 \delta M - \Omega \delta J$$

where $S$ is expressed as a local geometric density integrated over a space-like cross-section of the horizon. Since the black hole temperature is always given by Eq. (2) independent of the details of the dynamics of the gravity theory, the black hole entropy is naturally identified as $S = (k_B/h)S$. If these expressions are truly to play the role of an entropy, they should also satisfy the Second Law of thermodynamics as a black hole evolves — i.e., $S$ should never decrease in any dynamical processes. We have been able to establish this result at least within a certain class of theories in which the curvature only enters the action as powers of the Ricci scalar[17].

In the following, we begin with a brief review of Wald’s derivation of the First Law in section 2. In section 3, we demonstrate that the Second Law holds for arbitrary dynamical processes in a theory where the gravitational Lagrangian is $R + \alpha R^2$. In section 4, another proof of the Second Law is constructed which follows closely Hawking’s proof of the area theorem[3]. In fact, both of these proofs of the Second Law may be generalized to include a larger class of theories where the gravitational action is a polynomial of the Ricci scalar$^{17}$. Section 5 presents a brief discussion of our results.

$^1$Alternatively, this identification follows from a semiclassical evaluation of the partition function or the density of states in quantum gravity with any action, as has been shown explicitly in the case of Einstein gravity[10, 11].
In the following, we adopt the standard convention of setting $\hbar = c = k_B = 1$. Also, we employ the conventions of [5] throughout, and we will only consider asymptotically flat (four-dimensional) spacetimes.

2. Black Hole Entropy as Noether Charge

Wald [14] constructed a new derivation of the First Law of black hole mechanics (5) for any theory which is invariant under diffeomorphisms (i.e., coordinate transformations). In this construction, the black hole entropy $S = S$ is related to the Noether charge of diffeomorphisms under the Killing vector field which generates the event horizon in the stationary black hole background. Further, $S$ can always be expressed as a local geometric density integrated over a space-like cross-section of the horizon \[^{14, 18}\]. Thus the general result, Eq. (5), has in common with the original First Law, Eq. (3), the rather remarkable feature that it relates variations in properties of the black hole as measured at asymptotic infinity to a variation of a geometric property of the horizon. In the following, we will provide a brief introduction to Wald’s techniques. The interested reader is referred to Refs. [14, 16, 18] for complete details.

An essential element of Wald’s approach is the Noether current associated with diffeomorphisms \[^{19}\]. Let $L$ be a Lagrangian built out of some set of dynamical fields, including the metric, collectively denoted as $\psi$. Under a general field variation $\delta \psi$, the Lagrangian varies as

$$\delta (\sqrt{-g} L) = \sqrt{-g} \mathbf{E} \cdot \delta \psi + \sqrt{-g} \nabla_\alpha \theta^\alpha (\delta \psi) \ ,$$

(6)

where \(\cdot\) denotes a summation over the dynamical fields including contractions of tensor indices, and $\mathbf{E} = 0$ are the equations of motion. With symmetry variations for which $\delta (\sqrt{-g} L) = 0$, $\theta^\alpha$ is the Noether current which is conserved when the equations of motion are satisfied — i.e., $\nabla_\alpha \theta^\alpha (\delta \psi) = 0$ when $E = 0$. Rather than vanishing for diffeomorphisms, $\delta \psi = L_\xi \psi$ \[^{14}\] the variation of a covariant Lagrangian is a total derivative, $\delta (\sqrt{-g} L) = L_\xi (\sqrt{-g} L) = \sqrt{-g} \nabla_\alpha (\xi^\alpha L)$. Thus one constructs a new Noether current,

$$J^\alpha = \theta^\alpha (L_\xi \psi) - \xi^\alpha L \ ,$$

which satisfies $\nabla_\alpha J^\alpha = 0$ when $E = 0$.

A fact which is not well-appreciated is that for any local symmetry, there exists a globally-defined Noether potential $Q^{ab}$, satisfying $J^a = \nabla_b Q^{ab}$ where $Q^{ab} = -Q^{ba}$ \[^{20}\]. $Q^{ab}$ is a local function of the dynamical fields and a linear function of the symmetry parameter (i.e., $\xi^\alpha$ in the present case). Of course, this equation for $J^a$ is valid up to terms which vanish when the equations of motion are satisfied. Given this expression for $J^a$, it follows that the Noether charge contained in a spatial volume $\Sigma$ can be expressed as a boundary integral $\oint_{\partial \Sigma} d^2x \sqrt{h} \epsilon_{ab} Q^{ab}$, where $h_{ab}$ and $\epsilon_{ab}$ are the induced metric and binormal form on the boundary $\partial \Sigma$.

\[^{1}\] $L_\xi \psi$ denotes the Lie derivative of the field $\psi$ along the vector field $\xi^\alpha$ \[^{3}\] — e.g., $L_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$. 

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Wald’s derivation of the First Law requires that the black hole possess a bifurcate Killing horizon, which is defined as follows: First, a Killing vector is a vector field generating an invariance for a particular solution — i.e., $\mathcal{L}_\xi \psi = 0$ for all fields. A Killing horizon is then a null hypersurface whose null generators are orbits of a Killing vector field. If the horizon generators are geodesically complete to the past (and if the surface gravity is nonvanishing), then the Killing horizon contains a space-like cross-section $B$, the bifurcation surface, on which the Killing field $\chi^a$ vanishes. Such a bifurcation surface is a fixed point of the Killing flow, and lies at the intersection of the two null hypersurfaces that comprise the full Killing horizon. For example, in the maximally extended Schwarzschild black hole spacetime, the bifurcation surface is the two-sphere of area $16\pi M^2$ at the origin of Kruskal $U-V$ coordinates. The existence of bifurcate Killing horizons in general, and its relation to the Zeroth Law (constancy of the surface gravity), will be discussed in section 5.

The key to Wald’s derivation of the First Law is the identity

$$\delta H = \delta \int_\Sigma dV_a J^a - \int_\Sigma dV_a \nabla_b (\xi^a \theta^b - \xi^b \theta^a),$$

where $H$ is the Hamiltonian generating evolution along the vector field $\xi^a$, and $\Sigma$ is a spatial hypersurface with volume element $dV_a$. This identity is satisfied for arbitrary variations of the fields away from any background solution. If the variation is to another solution, then one can replace $J^a$ by $\nabla_b Q^{ab}$, so the variation of the Hamiltonian is given by surface integrals over the boundary $\partial \Sigma$. If, moreover, $\xi^a$ is a Killing vector of the background solution, then $\delta H = 0$. In this case one obtains an identity relating the various surface integrals over $\partial \Sigma$.

Suppose now that the background solution is chosen to be a stationary black hole with horizon-generating Killing field $\chi^a \partial_a = \partial_t + \Omega \partial_\phi$, and the hypersurface $\Sigma$ is chosen to extend from asymptotic infinity down to the bifurcation surface where $\chi^a$ vanishes. The surface integrals at infinity then yield precisely the mass and angular momentum variations, $\delta M - \Omega \delta J$, appearing in Eq. (5), while the surface integral at the bifurcation surface reduces to $\delta \oint_B d^2x \sqrt{h} \epsilon_{ab} Q^{ab}(\tilde{\chi})$. Finally, it can be shown that the latter surface integral always has the form $(\kappa/2\pi)\delta S$, where $\kappa$ is the surface gravity of the background black hole, and $S = 2\pi \oint_B d^2x \sqrt{h} \epsilon_{ab} Q^{ab}(\tilde{\chi})$, with $\tilde{\chi}^a$, the Killing vector scaled to have unit surface gravity.

By construction $Q^{ab}$ involves the Killing field $\tilde{\chi}^a$ and its derivatives. However, this dependence can be eliminated as follows. Using Killing vector identities, $Q^{ab}$ becomes a function of only $\tilde{\chi}^a$ and the first derivative, $\nabla_a \tilde{\chi}_b$. At the bifurcation surface, though, $\tilde{\chi}^a$ vanishes and $\nabla_a \tilde{\chi}_b = \epsilon_{ab}$, where $\epsilon_{ab}$ is the binormal to the bifurcation surface. Thus, eliminating the term linear in $\tilde{\chi}^a$ and replacing $\nabla_a \tilde{\chi}_b$ by $\epsilon_{ab}$ yields a completely geometric functional of the metric and the matter fields, which may be denoted $\tilde{Q}^{ab}$. One can show that the resulting expression,

$$S = 2\pi \oint d^2x \sqrt{h} \epsilon_{ab} \tilde{Q}^{ab},$$

yields the correct value for $S$ when evaluated not only at the bifurcation surface, but in fact on an arbitrary cross-section of the Killing horizon. Thus this latter
expression is a natural candidate for the entropy of a general non-stationary black hole.

Using Wald’s technique, the formula for black hole entropy has been found for a general Lagrangian of the following form:

\[ L = L(\psi, \nabla_\alpha \psi, g_{\alpha\beta}, R_{\alpha\beta\gamma\delta}, \nabla_{e_1} \nabla_{e_2} R_{\alpha\beta\gamma\delta}, \ldots) \]

that is involving only first derivatives of the matter fields (denoted by \( \psi \)), and the Riemann tensor and symmetric derivatives of \( R_{\alpha\beta\gamma\delta} \) up to some finite order \( n \). \( S \) may then be written \[15, 18, 16\]

\[ S = -2\pi \int d^2x \sqrt{h} \sum_{m=0}^{n} (-1)^m \nabla_{(e_1} \cdots \nabla_{e_m)} Z^{e_1 \cdots e_m;abcd} \epsilon_{ab} \epsilon_{cd} \] (8)

where the \( Z \)-tensors are defined by

\[ Z^{e_1 \cdots e_m;abcd} \equiv \frac{\partial L}{\partial \nabla_{(e_1} \cdots \nabla_{e_m)} R_{abcd}}. \]

As a more explicit example, consider the action

\[ I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R + \alpha R^2 + \beta R_{ab} R^{ab} \right) \]

for which one finds

\[ S = \frac{1}{4G} \int d^2x \sqrt{h} \left( 1 + 2\alpha R + \beta(R - h^{ab} R_{ab}) \right) \]. (9)

Here, the first term yields the expected contribution for Einstein gravity, namely \( A/(4G) \). Thus just as the Einstein term in the action is corrected by higher-curvature terms, the Einstein contribution to the black hole entropy receives higher-curvature corrections.

Note that the present description of Wald’s construction has neglected certain important details. For example, the reader is directed to Ref. \[16\] for an explanation of why the elimination of explicit dependence of \( S \) on the Killing field (discussed in the paragraph containing Eq. (7)) is valid even for when variations to non-stationary solutions are allowed. Further, a number of ambiguities arise in the construction of \( \tilde{Q}^{ab} \[18\], so Eqs. (8) and (9) should be understood as the result of making certain (natural) choices in the calculation. None of these ambiguities have any effect when the charge is evaluated on a stationary horizon \[18\], but they will become significant for non-stationary horizons. In this case, a choice which yields an entropy that satisfies the second law would be a preferred definition \[17\].

3. The Second Law in one example

Given that the identification of the surface gravity with the Hawking temperature in Eq. (4) is universal, Wald’s generalized First Law has the natural interpretation
as the First Law of thermodynamics for higher-curvature theories with the black hole entropy $S = S$. If the latter expressions are truly to play the role of an entropy, they should also satisfy the second law of thermodynamics as a black hole evolves — i.e., $S$ should never decrease in any dynamical processes.

For general relativity with $S = S = A/(4G)$, the Second Law is established by Hawking’s area theorem. An essential ingredient in the proof of this theorem is the assumption that the null convergence condition $R_{ab}k^ak^b \geq 0$ holds for all null vectors $k^a$. This condition is implied by the Einstein field equation

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab}$$

(10)

together with the null energy condition for the matter stress-energy, i.e.,

$$T_{ab}k^ak^b \geq 0 \quad \text{for any null vector } k^a .$$

(11)

Another essential ingredient is cosmic censorship — i.e., it is assumed that naked singularities do not develop in the processes of interest.

In theories where higher curvature interactions are included along with the Einstein Lagrangian, the equations of motion may still be written in the form of Eq. (10) if the contributions from the higher curvature interactions are included in the stress-energy tensor. However, these contributions typically spoil the energy condition required to prove the area theorem. Hence, in these theories, one can not establish an area increase theorem, but this result is not relevant since the entropy is no longer identified with the area in such a case. The relevant question is whether or not the quantity $S$ appearing in the First Law satisfies an increase theorem. If so, one would have a Second Law of black hole thermodynamics for these theories, further validating the interpretation of $S$ as the black hole entropy.

In this section we establish the Second Law for arbitrary dynamical processes involving black holes in the theory given by a higher curvature action of the form

$$I_0 = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (R + \alpha R^2) + L_m(\psi, g) \right]$$

(12)

where $L_m$ denotes a conventional Lagrangian for some collection of matter fields, denoted $\psi$. The latter Lagrangian will also contain the metric, but we assume that it contains no derivatives of the metric. As in Eqs. (8) or (9), Wald’s black hole entropy can be written

$$S = \frac{1}{4G} \int_\mathcal{H} d^2x \sqrt{h} (1 + 2\alpha R)$$

(13)

where the integral is taken over a space-like cross-section of the horizon, $\mathcal{H}$.

The gravitational field equations arising from the action (12) are

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T^m_{ab}(\psi, g) + 2\alpha \left( \nabla_a \nabla_b R - g_{ab} \nabla^2 R \right. - R R_{ab} + \frac{1}{4}g_{ab} R^2 ) .$$

(14)
We will assume that matter stress-energy tensor, \( T^m_{\alpha \beta} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^\alpha \beta} \), does satisfy the null energy condition \( (\text{I}) \). However, if one treats the entire expression on the right hand side of this equation as the stress-energy tensor of Einstein’s equations \( (\text{II}) \), it is clear that this total stress-energy does not satisfy the null energy condition \( (\text{I}) \) because of the higher curvature contributions \( (\text{i.e.}, \text{the terms proportional to } \alpha) \). Thus, as discussed above, Hawking’s proof of the area theorem does not apply here. What is desired, though, is to establish an increase theorem for \( S \) in Eq. \( (\text{III}) \).

Our approach will be the following. First, we show that the present higher curvature theory is equivalent to Einstein gravity for a conformally related metric coupled to an auxiliary scalar field, as well as to the original matter fields. Second, we argue that the black hole entropy in the higher curvature theory is identical to that in the conformally related theory. Finally, since Hawking’s area theorem holds in the Einstein-plus-scalar theory, we conclude that the entropy never decreases in the original theory \( (\text{II}) \).

The equivalence of the higher curvature theory \( (\text{II}) \) to Einstein gravity coupled to an auxiliary scalar field has been discussed previously by many authors \[22\]. The first step is to introduce a new scalar field \( \phi \), and a new action, which is linear in \( R \),

\[
I_1 = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi G} \left[ (1 + 2\alpha \phi)R - \alpha \phi^2 \right] + L_m(\psi,g) \right\} . \tag{15}
\]

The \( \phi \) equation of motion is simply \( \phi = R \), and one recovers the original action upon substituting this equation into Eq. \( (\text{I}) \) — \( \text{i.e.}, I_1(\phi = R) = I_\circ \). In the form of Eq. \( (15) \), the action contains no terms that are more than quadratic in derivatives. However, this action does contain an unconventional interaction, \( \phi R \). Hence in the metric equations of motion,

\[
R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R = 8\pi G T_{\alpha \beta}^m(\psi,g) + 2\alpha \left( \nabla_\alpha \nabla_\beta \phi - g_{\alpha \beta} \nabla^2 \phi \right)
- \phi R_{\alpha \beta} + \frac{1}{2} g_{\alpha \beta} \phi R - \frac{1}{4} g_{\alpha \beta} \phi^2 ,
\]

the total stress-energy tensor appearing on the right hand side still contains some problematic contributions \( (\text{e.g., } \nabla_\alpha \nabla_\beta \phi) \), which prevent the null energy condition from being satisfied.

The \( \phi R \) interaction can be removed by performing the following conformal transformation

\[
g_{\alpha \beta} = (1 + 2\alpha \phi)^{-1} \bar{g}_{\alpha \beta} . \tag{16}
\]

In terms of \( \bar{g}_{\alpha \beta} \), the action \( (\text{II}) \) becomes

\[
I_2 = \int d^4x \sqrt{-\bar{g}} \left\{ \frac{1}{16\pi G} \left[ \bar{R} - \frac{3}{2} \left( \frac{2\alpha}{1 + 2\alpha \phi} \right)^2 \nabla_\alpha \phi \nabla^\alpha \phi - \frac{\alpha \phi^2}{(1 + 2\alpha \phi)^2} \right] \right. 
+ \left. (1 + 2\alpha \phi)^{-2} L_m(\psi, (1 + 2\alpha \phi)^{-1} \bar{g}) \right\} , \tag{17}
\]
which includes the standard Einstein-Hilbert action for $\bar{g}_{ab}$ and the auxiliary scalar $\phi$ with less conventional couplings — see below. The $\bar{g}_{ab}$ equations of motion are now

$$\bar{R}_{ab} - \frac{1}{2} \bar{g}_{ab} \bar{R} = \frac{8\pi G}{1 + 2\alpha \phi} T^m_{ab}(\psi, (1 + 2\alpha \phi)^{-1} \bar{g}) + \frac{3}{2} \left( \frac{2\alpha}{1 + 2\alpha \phi} \right)^2 \bar{\nabla}_a \phi \bar{\nabla}_b \phi$$

$$- \frac{1}{2} \bar{g}_{ab} \left[ \frac{3}{2} \left( \frac{2\alpha}{1 + 2\alpha \phi} \right)^2 \bar{\nabla}_c \phi \bar{\nabla}^c \phi + \frac{\alpha \phi^2}{(1 + 2\alpha \phi)^2} \right].$$

The most important feature of this final theory for our purposes is that, assuming $1 + 2\alpha \phi > 0$, the total stress-energy tensor appearing on the right hand side above satisfies the null energy condition (11).

Given the absence of higher derivative or unconventional gravity couplings, the black hole entropy for $I_2$ is given by $\bar{S} = \bar{A} / (4G)$, just as for Einstein gravity. Since the equations of motion (18) are Einstein’s equations, and the null energy condition is satisfied by the stress-energy tensor, Hawking’s proof of the area theorem is valid for the $I_2$ theory with the assumptions that cosmic censorship holds for $\bar{g}_{ab}$ and that $1 + 2\alpha \phi > 0$. (The latter assumption will be further discussed below.) Hence there is a classical entropy increase theorem for the theory defined by $I_2$ in Eq. (17).

Now Eq. (16) along with $\phi = R$ provides a mapping between the solutions for the Einstein-plus-scalar theory defined by $I_2$, and the original higher curvature theory defined by $I_0$, in which the metrics are related by a conformal transformation

$$\bar{g}_{ab} = (1 + 2\alpha R) g_{ab}.$$  (19)

The conformal transformation (19) preserves the causal structure of the solutions and, if $g_{ab}$ is asymptotically flat, then so is $\bar{g}_{ab}$. Thus, if $g_{ab}$ is an asymptotically flat black hole, then so is $\bar{g}_{ab}$, and they have the same horizon and surface gravities [23]. Furthermore, since the asymptotic forms of $g_{ab}$ and $\bar{g}_{ab}$ agree, the mass and angular momenta of the two spacetimes agree. Further, the angular velocities agree on stationary horizons, because the horizon generating null combination of the time translation and rotation Killing fields is preserved under the conformal transformation.

In short, we have shown all of the ingredients, other than the entropy, in the First Law (1) agree. Thus, for all variations, the changes in the entropies must also agree. Therefore the entropies themselves are equal to within a constant in each connected class of black hole solutions. Since the area increase theorem for the Einstein-plus-scalar theory gives $\delta \bar{S} \geq 0$ in any dynamical process, we conclude that $\delta S \geq 0$ for the corresponding process in the higher curvature theory. We have thus established a classical Second Law in the higher curvature theory defined by the action $I_0$ in Eq. (12).

A number of remarks on the above analysis will now be made. Using the conformal relation (19) between the two metrics, the “barred” entropy can be expressed directly in terms of $g_{\mu\nu}$:

$$\bar{S}(\bar{g}) = \frac{1}{4G} \int_{\mathcal{H}} d^2 x \sqrt{\bar{h}} = \frac{1}{4G} \int_{\mathcal{H}} d^2 x \sqrt{h} (1 + 2\alpha R).$$  (20)
For stationary black holes this agrees with the entropy in the higher curvature theory as determined directly from the First Law in that theory (it was already argued above that $\mathcal{H}$ corresponds to a cross-section of the event horizon for the metric $g_{ab}$ as well). This agreement is explained by the reasoning given in the preceding paragraph. It should be emphasized however that the First Law, which applies to variations away from a stationary black hole background, does not determine the form of the entropy during arbitrary dynamical processes [14, 18].

In presenting the result (13) for the black hole entropy as determined by the First Law, we chose the simplest geometric formula which naturally extends to a dynamical horizon. Here we have shown that the Second Law is obeyed, even during a dynamical process, by the entropy given by that particular formula, reproduced in Eq. (20). The relation (19) gives an unambiguous result for the dynamical entropy and so can be used to resolve the ambiguities [14, 18] inherent in the Noether charge construction of [14]. In the present case, the alternate proposal for dynamical entropy of Ref. [14], which uses a boost invariant projection, gives a result for the dynamical entropy that differs from Eq. (20) for non-stationary black holes. Unless there are two different entropy functionals obeying a local increase law, it appears that the proposal of Ref. [14] is inconsistent with the Second Law during the dynamical process in the present theories.

There is one considerable assumption in preceding discussion which we have not yet addressed. For the mapping between the solutions of the two theories (19) to exist, it is necessary that the factor $1 + 2\alpha R$ is positive. (Note that in asymptotically flat regions this factor tends to one.) Thus, given a black hole solution of the higher curvature theory, one must have $R > -\frac{1}{2\alpha}$ (for positive $\alpha$) everywhere outside of the black hole and on the event horizon.

From the point of view of the Einstein-plus-scalar theory, one requires that $\phi > -\frac{1}{2\alpha}$ everywhere outside of the event horizon for the mapping to a solution of the higher curvature theory to exist. Recall that cosmic censorship was assumed in the proof of the area increase theorem for this theory. This assumption rules out dynamical processes in which a black hole begins with a configuration with $\phi > -\frac{1}{2\alpha}$ everywhere initially, and evolves to one with $\phi \leq -\frac{1}{2\alpha}$ in some region outside of the horizon. The reason is that, by the equations of motion (18) when $\phi = -\frac{1}{2\alpha}$, the stress-energy tensor is singular and hence the Einstein tensor, $\bar{R}_{ab} - \frac{1}{2}\bar{g}_{ab}\bar{R}$, is singular [24]. Cosmic censorship for $\bar{g}_{ab}$ would only allow such curvature singularities to develop behind the event horizon, and hence rules out any process in which $\phi = -\frac{1}{2\alpha}$ is reached outside of the horizon.

An alternative argument showing that it is consistent to make the restriction $\phi > -\frac{1}{2\alpha}$ can be given by considering the character of the potential term in the Einstein-plus-scalar theory [23]. The non-standard kinetic term for the scalar field $\phi$ in Eq. (17) can be replaced with an ordinary one by defining a new scalar field $\varphi := \beta^{-1} \ln(1 + 2\alpha \phi)$, where $\beta = \sqrt{16\pi G/3}$. In terms of $\varphi$ the action $I_2$ takes the form

$$I_3 = \int d^4x \sqrt{-\bar{g}} \left[ \frac{1}{16\pi G} \bar{R} - \frac{1}{2} \nabla_a \varphi \nabla^a \varphi - V(\varphi) + e^{-2\beta \varphi} L_m(\psi, e^{-\beta \varphi} \bar{g}) \right],$$

(21)
where \( V(\varphi) = \frac{1}{64 \pi G \alpha} e^{-2\beta \varphi} (e^{\beta \varphi} - 1)^2 \). Now the singular point \( \varphi = -\frac{1}{2\alpha} \) corresponds to \( \varphi \to -\infty \). Provided \( \alpha > 0 \), the potential \( V(\varphi) \) rises exponentially as \( \varphi \to -\infty \). The term involving the matter Lagrangian has the same exponential for a prefactor, and so one may worry that it may undermine the barrier due to \( V(\varphi) \). The kinetic terms for the matter fields will include at least one inverse metric which will bring the rate of exponential growth down by a factor of \( \exp(\beta \varphi) \) for these contributions. We will assume that any matter potential is non-negative — this amounts to extending the null energy condition for \( T_{ab}^m \) to the dominant energy condition — so that these terms can only increase the potential barrier as \( \varphi \to -\infty \). Thus, as long as the metric and matter fields do not become singular, the dynamics of \( \varphi \) as \( \varphi \to -\infty \) will be dominated by the potential barrier so \( \varphi \) will not run off to \( -\infty \). Therefore, initial data satisfying the bound \( \phi > -\frac{1}{2\alpha} \) will evolve within the bound, as long as the other fields remain nonsingular.

Note that the argument just given breaks down if \( \alpha < 0 \) since the potential is then exponentially falling as \( \varphi \to -\infty \). In fact, the theory is unstable for negative \( \alpha \). The previous argument did not seem to require that \( \alpha \) be positive, however it did assume cosmic censorship which, presumably, would be violated in the unstable theory with \( \alpha < 0 \).

Given the above arguments that \( \varphi = -\frac{1}{2\alpha} \) is never reached outside the event horizon for positive \( \alpha \), one may also rule out processes in the higher curvature theory in which a black hole evolves to reach \( R = -\frac{1}{2\alpha} \) somewhere outside of the event horizon. From a superficial examination of the higher curvature equations of motion (14), \( R = -\frac{1}{2\alpha} \) does not appear to be singular. However, there is no obstruction to mapping the initial part of the evolution to the Einstein-plus-scalar theory, where it becomes a process leading up to a naked singularity of \( \bar{g}_{ab} \) at the point where \( \phi = -\frac{1}{2\alpha} \), as discussed above. Such processes were ruled out by the assumption of cosmic censorship for \( g_{ab} \), though, hence we have also ruled out the corresponding evolution in the higher curvature theory.

### 4. Direct proof of Second Law

Our method of establishing the Second Law above used the fact that the higher curvature gravity theory is conformally related to Einstein gravity in which the area theorem holds. This special feature of these theories is not shared by most general higher curvature theories, so it would be most interesting to see how the Second Law could be established directly, without making use of the conformal transformation technique. In the present section we will construct such a direct proof for the curvature squared theory studied in the previous section. The exercise will be instructive for efforts to establish entropy increase theorems for theories that are not susceptible to the conformal transformation “trick”.

Suppose that the black hole entropy of a gravity theory takes the following form

\[
S = \frac{1}{4G} \int_{\mathcal{H}} d^2 x \sqrt{\mathcal{h}} e^\rho, \tag{22}
\]

where \( e^\rho \) is a scalar function of the local geometry at the horizon. For the theory
considered in the preceding section one has \( e^\rho = 1 + 2\alpha R \). The method to be used here will rely critically on the fact that \( e^\rho \) is necessarily positive, and \( \rho = 0 \) when the curvature vanishes.

We wish to consider the change of this entropy along the null congruence generating the event horizon under any dynamical evolution. Let \( k^a \) be the null tangent vector field of the horizon generators with respect to the affine parameter \( \lambda \). Then one has

\[
S' = \frac{1}{4G} \int d^2x \sqrt{h} e^\rho \tilde{\theta} \tag{23}
\]

with

\[
\tilde{\theta} := \theta + \partial_\lambda \rho, \tag{24}
\]

where \( \theta = d(\ln \sqrt{h})/d\lambda = \nabla_a k^a \) is the expansion of the horizon generators.

Now the question is whether or not there can exist a point along the null geodesics at which \( \tilde{\theta} \) becomes negative. In order to answer this question, we use the Raychaudhuri equation, as in the proof of area theorem, to obtain an expression for \( \partial_\lambda \tilde{\theta} \):

\[
\partial_\lambda \tilde{\theta} = \partial_\lambda \theta + \partial_\lambda^2 \rho = -\frac{1}{2} \theta^2 - \sigma^2 - k^a k^b R_{ab} + k^a k^b \nabla_a \nabla_b \rho, \tag{25}
\]

where \( \sigma^2 \) is the square of the shear.

For the \( R + \alpha R^2 \) theories, it is easy to see that the equations of motion (14) imply that \( k^a k^b (R_{ab} - \nabla_a \nabla_b \rho) = (8\pi G e^{-\sigma} T_{ab}^m + \nabla_a \rho \nabla_b \rho) k^a k^b \), which is non-negative provided the null energy condition holds for the matter fields. Thus in those theories one has

\[
\partial_\lambda \tilde{\theta} \leq -\frac{1}{2} \theta^2, \tag{26}
\]

or

\[
\partial_\lambda [\tilde{\theta}^{-1}] \geq \frac{1}{2} (\theta/\tilde{\theta})^2. \tag{27}
\]

Now we follow Hawking’s proof of the area theorem, with \( \tilde{\theta} \) in place of \( \theta \). Suppose at some point on the horizon we have \( \tilde{\theta} < 0 \). Then in a neighborhood of that point one can deform a space-like slice of the horizon slightly outward to obtain a compact space-like two-surface \( \Sigma \) so that \( \tilde{\theta} < 0 \) everywhere on \( \Sigma \), \( \tilde{\theta} \) being defined along the outgoing null geodesic congruence orthogonal to \( \Sigma \). If cosmic censorship is assumed, then there is necessarily some null geodesic orthogonal to \( \Sigma \) that remains on the boundary of the future of \( \Sigma \) all the way out to \( I^+ \). Asymptotic flatness (where components of the Riemann tensor in an orthonormal frame all fall off at least as \( r^{-3} \)) implies that \( \rho \to 0 \) like \( \lambda^{-3} \) at infinity, whereas \( \theta \) goes like \( \lambda^{-1} \), where \( \lambda \) is the affine parameter along an outgoing null geodesic. Therefore \( \theta/\tilde{\theta} \to 1 + O(\lambda^{-3}) \), so the inequality (27) implies that, as one follows the geodesic outwards from \( \Sigma \), \( \tilde{\theta} \) reaches \( -\infty \) at some finite affine parameter. Since \( \tilde{\theta} = \theta + \partial_\lambda \rho \), this means that either \( \theta \) or \( \partial_\lambda \rho \) goes to \( -\infty \). In the former case we have a contradiction, as in the area theorem, since it implies there is a conjugate point on the geodesic, which cannot happen since the geodesic stays on the boundary of the future of \( \Sigma \) all the way out to \( I^+ \). In the
latter case we have a naked singularity, since $\partial_\lambda \rho = e^{-\rho} \partial_\lambda e^\rho$, and $e^\rho$ was assumed from the beginning in (22) to be a nonvanishing function of the curvature.

For the present theory, we have in particular $\partial_\lambda \rho = k^a \nabla_a \rho = 2\alpha(1+2\alpha R)^{-1} k^a \nabla_a R$. In the preceding section we argued that $1 + 2\alpha R$ never goes to zero outside the horizon in the case of a stable theory, so the divergence of $\partial_\lambda \rho$ implies a divergence of $k^a \nabla_a R$, but this divergence also violates cosmic censorship. Therefore we conclude that cosmic censorship and the null energy condition on the matter imply that the black hole entropy (22) can never decrease for the stable theories. Note that in making this argument we have used the condition $1 + 2\alpha R > 0$ that was established via the conformal transformation trick, so we do not quite have a fully “direct proof” of the Second Law.

The above argument suggests that the “weakest” naked singularity which might create a violation of the Second Law would be a divergence in $k^a \nabla_a R$. It seems that this could happen even if the curvature itself is nonsingular everywhere. However, if one imposes also the equations of motion of the theory, then a divergence in $k^a \nabla_a R$ would necessarily entail also a divergence in either the curvature tensor or the matter stress tensor.

5. Discussion

Present investigations indicate quite strongly that black hole thermodynamics is a robust feature of all generally covariant theories of gravity. In particular, Wald’s construction provides a general First Law of black hole mechanics

$$\frac{\kappa}{2\pi} \delta S = \delta M - \Omega \delta J$$

for any such theory. The principle difference between separate theories is the precise formula for the black hole entropy, $S$. In general, though, $S$ is expressed as a local geometric density integrated over a cross-section of the horizon. One might have expected the latter result following Bekenstein’s original observations.

To even state the First Law presupposes the validity of the Zeroth Law, namely, that the surface gravity or temperature is constant across a stationary event horizon. In Einstein gravity the Zeroth Law for a Killing horizon can be proved if the dominant energy condition is assumed[6]. It can be similarly established for the $R + \alpha R^2$ theory using the conformal transformation technique described in section 3. One need only notice that the stress energy tensor for the action $I_2$ (17) or $I_3$ (21) satisfies the dominant energy condition, assuming that $\alpha$ is positive, and that the original matter fields satisfy the dominant energy condition. Hence the surface gravity of any Killing horizon for the $\bar{g}_{ab}$ metric must be constant and, since surface gravity is conformally invariant[23], the same must be true of the Killing horizons for $g_{ab}$. In more general higher curvature theories the validity of the Zeroth Law remains an important open question.

It is worth emphasizing that unless the event horizon is a Killing horizon, the abovementioned proofs of the Zeroth Law are not applicable. In Einstein gravity, Hawking proved that the event horizon of a stationary black hole must be a Killing
horizon[3]. To our knowledge, this proof has not been extended to general higher curvature theories, or even to higher dimensional Einstein gravity. For the higher curvature theories considered in this paper, at least in four dimensions, it seems likely that Hawking’s proof can be imported via the conformal transformation relating the theory to Einstein gravity with matter. For stationary black holes in more general higher curvature theories, whether or not stationary event horizons are necessarily Killing horizons is another important open question.

Recall that Wald’s derivation of the First Law required a Killing horizon with a bifurcation surface. Although not all Killing horizons have bifurcation surfaces, those that satisfy the Zeroth Law do[21]. More precisely, if the surface gravity is constant and nonvanishing on a slice of a Killing horizon, then there exists a stationary extension of the spacetime (at least in a neighborhood of the horizon) to one that includes a bifurcation surface. Moreover given a slice of a Killing horizon where the surface gravity is not constant, then a similar extension exists but there is necessarily a curvature singularity where the bifurcation surface would have been. Conversely, if a Killing horizon has a regular bifurcation surface, then the surface gravity is necessarily constant. The Zeroth Law thus holds for any bifurcate Killing horizon, irrespective of the underlying gravitational dynamics. However, since one has no independent proof that Killing horizons in a given theory necessarily possess a regular bifurcation surface, this does not help one to establish the Zeroth Law in general.

In sections 3 and 4, we presented two proofs of the Second Law for the higher curvature theory in Eq. (12). For this theory,

$$S = \frac{1}{4G} \oint d^2x \sqrt{h} (1 + 2\alpha R)$$

will never decrease in any classical process involving black holes. In fact, both of these proofs (as well as the proof of the Zeroth Law) are easily generalized for higher curvature theories[17] with actions of the form

$$I_0 = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (R + P(R)) + L_m(\psi, g) \right]$$

where $P$ is a polynomial in the Ricci scalar, $P(R) = \sum_{n=2} a_n R^n$. For these theories, the black hole entropy is

$$S = \frac{1}{4G} \oint d^2x \sqrt{h} (1 + P'(R)) \quad (28)$$

where $P'(R) = \sum_{n=2} na_n R^{n-1}$. Similarly to the $R^2$ theory, these proofs of the Second Law require that the matter fields satisfy the null energy condition, and that the coupling constants $a_n$ appearing in $P(R)$ be restricted such that the slope of the entropy density $1 + P'(R)$ (plotted versus $R$) is positive for positive $R$, and also between $R = 0$ and the first negative value of $R$ where $1 + P'(R)$ vanishes.[17]

As for the $R^2$ example discussed in detail, *i.e.*, $P(R) = \alpha R^2$, the expression $1 + P'(R)$ must be positive in order to implement the conformal transformation between the original higher curvature theory and the Einstein-plus-scalar theory, and also to
ensure the null energy condition is satisfied in the latter theory. This positivity is also an essential ingredient for a direct proof, as in section 4. It is interesting that precisely the same expression plays the role of the entropy surface density in Eq. (28). Thus the positivity restriction translates on the horizon to the condition that the local entropy density should be positive everywhere. In particular it leads to the total black hole entropy always being positive. The latter is a minimum requirement that must be fulfilled if this entropy is to have a statistical mechanical origin. The fact that we actually require a local positivity condition on the entropy density is suggestively consistent with the idea that this density may have a statistical interpretation. In any event, these (and other higher curvature) theories may provide a more refined test of the various proposals to explain the statistical origin of black hole entropy.

One would like to extend the Second Law of black hole thermodynamics to more general higher curvature theories. One extension which we have found is to consider general higher curvature theories but restricting the black hole evolution to quasi-stationary processes[17]. For such processes in which a (vacuum) black hole accretes positive energy (neutral) matter — i.e., $T_{ab}^{m} \ell_{a} \ell_{b} \geq 0$ for any null vector $\ell^{a}$ — the Second Law is a direct consequence of the First Law of black hole mechanics, independent of the details of the gravitational action.

The investigations presented in sections 3 and 4[17] have been limited to an intrinsic or classical Second Law — i.e., we have only dealt with the increase of the black hole entropy alone. In general relativity, we know that the effective transfer of negative energy from quantum fields to a black hole can lead to a decrease in the horizon entropy (i.e., horizon area), and the same is true for these higher curvature effective theories since black holes still produce Hawking radiation in these theories. Thus it is important to ask whether a generalized Second Law ($\delta(S_{BH} + S_{outside}) \geq 0$) also holds. In general relativity, there are arguments that the generalized Second Law applies for quasi-stationary processes involving positive energy matter[20]. These arguments seem to carry over to stable higher curvature gravity theories as well, since they do not involve the equations of motion but rather lean on the First Law and the maximum entropy property of thermal radiation. The validity of the generalized Second Law for dynamical processes that are not quasi-stationary remains an important open question.

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