Abstract

This paper investigates the local times and modulus of non-differentiability of the spherical Gaussian random fields. We extend the methods for studying the local times of Gaussian to the spherical setting. The new main ingredient is the property of strong local nondeterminism established recently in Lan et al (2018).

Key words: Spherical Gaussian Fields, Local Times, Modulus of non-differentiability.

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1 Introduction and statement of main results

In this paper, we shall be concerned with the local time of an isotropic Gaussian random field $T = \{T(x), x \in S^2\}$ with values in $\mathbb{R}^d$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$T(x) = (T_1(x), \ldots, T_d(x)), \quad x \in S^2,$$

where $T_1, \ldots, T_d$ are independent copies of $T_0 = \{T_0(x), x \in S^2\}$, which is a zero-mean Gaussian random field that satisfies

$$E[T_0(x)T_0(y)] = E[T_0(gx)T_0(gy)]$$ (2)
for all $g \in SO(3)$ and all $x, y \in \mathbb{S}^2$. We call $T_0$ the Gaussian field associated with $T$.

It follows from (2) that the Gaussian field $T_0$ is both 2-weakly and strongly isotropic (cf. [S] p.121). It is well-known (cf. [S], p.123) that $T_0$ has the following spectral representation:

$$T_0(x) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x),$$  

(3)

where

$$a_{\ell m} = \int_{\mathbb{S}^2} T_0(x) \overline{Y}_{\ell m}(x) \, d\nu(x),$$

and $\nu$ is the canonical area measure on the sphere. In the spherical coordinates $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi)$, $\nu(dx) = \sin \theta d\theta d\varphi$. Note that the equality in (3) holds in both the $L^2(\mathbb{S}^2 \times \Omega, d\nu(x) \otimes P)$ sense and the $L^2(P)$ sense for every fixed $x \in \mathbb{S}^2$. The set of homogenous polynomials $\{Y_{\ell m} : \ell \geq 0, m = -\ell, \ldots, \ell\}$ are the spherical harmonic functions on $\mathbb{S}^2$, representing an orthonormal basis for the space $L^2(\mathbb{S}^2, \nu)$. In this setting, the random coefficients $\{a_{\ell m}, \ell \geq 0, m = -\ell, \ldots, \ell\}$ are Gaussian complex random variables such that

$$E[a_{\ell m}] = 0;$$

$$E[a_{\ell m} \overline{a}_{\ell_1 m_1}] = \delta_{\ell \ell_1} \delta_{m m_1} C_\ell,$$

where the sequence $\{C_\ell\}$ of nonnegative numbers is called the angular power spectrum of $T_0$, which fully characterizes the dependence structures of $T_0$. A celebrated theorem of Schoenberg [14] provides the following expansion for the covariance function:

$$E[T_0(x) T_0(y)] = \sum_{\ell=0}^{+\infty} C_\ell \frac{2\ell + 1}{4\pi} P_\ell(\langle x, y \rangle),$$

where for $\ell = 0, 1, 2, \ldots$, $P_\ell : [-1, 1] \to \mathbb{R}$ denotes the Legendre polynomial satisfying the normalization condition $P_1(1) = 1$. Thus for every $x \in \mathbb{S}^2$,

$$E[T_0(x) T_0(y)] = \sum_{\ell=0}^{+\infty} C_\ell \frac{2\ell + 1}{4\pi} := K.$$

For simplicity, we assume in this paper that $K = 1$. Otherwise, we consider the random field $\{K^{-1/2} T_0(x), x \in \mathbb{S}^2\}$, which does not cause any essential loss of generality.

As shown by Lang and Schwab [6], Lan et al. [5] that the degree of smoothness of the sample paths of $T_0$ can be precisely characterized by the asymptotic behavior of the angular power spectrum $\{C_\ell\}$ at high multipoles $\ell >> 0$. In this paper, we further illustrate this point by investigating regularity properties of the local times of the vector-valued random field $T$ in [11]. For this purpose, we recall the following condition from Lan et al. [5]:

$$E[T_0(x) T_0(y)] = \sum_{\ell=0}^{+\infty} C_\ell \frac{2\ell + 1}{4\pi} := K.$$

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$$E[T_0(x) T_0(y)] = \sum_{\ell=0}^{+\infty} C_\ell \frac{2\ell + 1}{4\pi} := K.$$
Condition (A) The random field $T_0$ is a zero-mean, Gaussian and isotropic random field indexed by $\mathbb{S}^2$, with unit variance and angular power spectrum satisfying:

$$C_\ell = C_\ell(G, \alpha) = G(\ell) \ell^{-\alpha} > 0,$$

where $\alpha > 2$ and there exists a constant $K_0 \geq 1$ such that for all $\ell = 1, 2, \ldots$

$$K_0^{-1} \leq G(\ell) \leq K_0.$$

The regularity of the trajectories of the Gaussian field $T_0$ is governed by $\alpha$. It has been shown in [4, 5, 6] that the sample function of $T_0$ is almost surely $k$-times continuous differentiable if and only if $\alpha > 2 + 2k$. Hence, the field $T_0$ is twice differentiable almost surely (as needed for the Kac-Rice argument in [1], Theorem 12.1.1) if and only if $\alpha > 6$.

In this paper we focus on the non-smooth regime $2 < \alpha < 4$. In this case, the uniform and local moduli of continuity have recently been proved by Lan et al. [5]. We remark that the regime of $2 < \alpha < 4$ is indeed the most relevant for many areas of applications; in particular, a major driving force for the analysis of spherical random fields has been provided over the last decade by cosmological applications, for instance in connection to the analysis of Cosmic Microwave Background radiation data (CMB). Data analysis on CMB maps is currently an active research area, and the geometry of CMB maps has been deeply investigated (see [9, 10, 11]). In this framework there are strong theoretical and empirical evidence to support the belief that the values of $\alpha$ belong to the non-smooth region $2 < \alpha < 4$.

Our objective of the present paper is to establish the joint continuity, and the uniform and local moduli of continuity for the local times of the vector-valued random field $T$. Based on these results, we show that the sample functions of $T$ and $T_0$ are a.s. nowhere differentiable, and we determine the exact modulus of non-differentiability of $T$.

In order to state our main results, we need some notations. In spherical coordinates $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi)$, every point $x \in \mathbb{S}^2$ can be written as $x = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$. In this paper, we always identify the Cartesian and spherical coordinates of $x \in \mathbb{S}^2$. For any $x \in \mathbb{S}^2$, $0 < r < \pi$, let $D(x, r) = \{y \in \mathbb{S}^2 : d_{\mathbb{S}^2}(x, y) < r\}$ be an open disk on $\mathbb{S}^2$, where $d_{\mathbb{S}^2}(x, y) = \arccos \langle x, y \rangle$ denotes the standard geodesic distance on the sphere, and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^3$. Given $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi)$, denote by $V(\vartheta, \varphi) = \{(\theta, \phi) : 0 \leq \theta \leq \vartheta, 0 \leq \phi \leq \varphi\}$ the angular section originated from the North Pole. It is easy to see that, respectively, the spherical area of $D(x, r)$ and $V(\vartheta, \varphi)$ are

$$\nu(D(x, r)) = 2\pi(1 - \cos r) \quad \text{and} \quad \nu(V(\vartheta, \varphi)) = \varphi(1 - \cos \vartheta). \quad (4)$$

Our first theorem is concerned with the joint continuity of local times of the $T$, see Section 3 below for the definition of local times and more information.

**Theorem 1.1** Let $T = \{T(x), x \in \mathbb{S}^2\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined in [1]. Assume that the associated isotropic random field $T_0$
satisfies Condition (A) with $2 < \alpha < 4$ and $\beta = 4 - (\alpha - 2) d > 0$. Then for any open set $D \subseteq \mathbb{S}^2$ with $\nu(D) > 0$, $\mathbf{T}$ has local times $L(t, D)$ which is in $\mathbb{R}^d \times \mathbb{S}^2$ almost surely. Moreover, there is a modification of $L(t, D)$ such that it is jointly continuous in the following sense:

(i) The local time $L(t, D(x, r))$ is continuous in $(t, x, r)$. Moreover, for any two open disks $D_i = D(x_i, r_i) \subseteq \mathbb{S}^2$ with $x_i \in \mathbb{S}^2$, $r_i \in (0, \delta)$, $i = 1, 2$, and any $\gamma \in (0, \gamma_0)$ with $\gamma_0 = \min \left\{ \frac{\beta}{2(\alpha - 2)}, 1 \right\}$, it satisfies

$$|L(t, D_1) - L(s, D_2)| \leq K_1 \left[ \|t - s\|^\gamma + d_{\mathbb{S}^2}(x, y)^{\beta/4} r^{\gamma_0} \right] r_\eta \ a.s. \tag{5}$$

where $\|\cdot\|$ is the Euclidean distance on $\mathbb{R}^d$ and

$$\eta = \frac{\beta}{2} - (\alpha - 2) \gamma > 0; \tag{5}$$

(ii) The local time $L(t, V(\vartheta, \varphi))$ is continuous in $(t, \vartheta, \varphi)$. Moreover, for any two angular sections $V_i = V(\vartheta_i, \varphi_i) \subseteq \mathbb{S}^2$ with $(\vartheta_i, \varphi_i) \in [0, \pi] \times [0, 2\pi)$, $i = 1, 2$, and any $\gamma \in (0, \gamma_0)$, it satisfies

$$|L(t, V_1) - L(s, V_2)| \leq K_2 \left[ \vartheta_1 - \vartheta_2 \min \left\{ \varphi_1^2, \varphi_2^2 \right\} + \min \left\{ \varphi_1^2, \varphi_2^2 \right\} \right]^{\beta/4} + K_2 \|t - s\|^{\gamma} \varphi_1^\eta/2 \varphi_2^\eta, \ a.s.$$ 

where the constant $\delta \in (0, 1)$ depending only on $\alpha$, $K_0$, and $K_1, K_2$ are positive depending only on $\alpha$, $d$, $K_0$ and $\gamma$.

The next result provides optimal upper and lower bounds for the exact moduli of continuity for the maximum of local time $L^*(D) = \sup_{t \in \mathbb{R}^d} L(t, D)$ w.r.t to the variable $r$ in the set $D = D(x, r)$.

**Theorem 1.2** Under conditions of Theorem 1.1, there exist positive constant $K_3, K_4$ depending only on $\alpha$, $d$ and $K_0$, such that for any $x \in \mathbb{S}^2$,

$$K_3^{-1} \leq \liminf_{r \to 0} \frac{L^*(D(x, r))}{\phi(r)} \leq \limsup_{r \to 0} \frac{L^*(D(x, r))}{\phi(r)} \leq K_3, \ a.s. \tag{6}$$

where the functions $\phi$ is defined by

$$\phi(r) = r^2 \left[ \rho_\alpha(r/\sqrt{\log(\log r)}) \right]^d, \tag{7}$$

with $\rho_\alpha(r) = r^{\frac{\alpha}{d} - 1}$ for $r \geq 0$. 



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Remark 1.3 The result above further confirms that the sample functions of $T_0$ (and $T$) are nowhere differentiable. Actually, in Section 5 we establish the Chung’s law of iterated logarithm, which is essential to the proof of Theorem 1.2 as well.

The rest of this paper is as follows: Section 2 reviews some auxiliary tools for the arguments to follow. Section 3 and 4 present the proofs of Theorems 1.1 and 1.2, respectively. The Chung’s law of iterated logarithm of $T_0$ is established in Section 5. Our method for establishing joint continuity and upper bound of Hölder conditions for the local times relies on moment estimates on the local times, where the property of strong local nondeterminism of $T_0$ proved in Lan et al. [5], plays an essential role. We also make use of a chaining argument that is similar to those in [3, 16].

Throughout this paper, we use $C, K$ to denote a constant whose value may change in each appearance, and $C_{i,j}, K_{i,j}$ to denote the $j$th more specific positive finite constant in Section $i$.

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2 Preliminaries

In this section, we collect a few technical results which will be instrumental for most of the proofs to follow. We recall first the following lemma from [5], which characterizes the variogram and the property of strong local nondeterminism of $T_0$.

Lemma 2.1 Under Condition (A) with $2 < \alpha < 4$, there exist positive constants $K_{2,1} \geq 1, 0 < \delta < 1$ depending only on $\alpha$ and $K_0$, such that for any $x, y \in \mathbb{S}^2$, if $d_{\mathbb{S}^2}(x, y) < \delta$, we have

$$K_{2,1}^{-1} \rho^2_\alpha \langle d_{\mathbb{S}^2}(x, y) \rangle \leq \mathbb{E} \left[ (T_0(x) - T_0(y))^2 \right] \leq K_{2,1} \rho^2_\alpha (d_{\mathbb{S}^2}(x, y)).$$

Moreover, there exists a constant $K_{2,2} > 0$ depending on $\alpha$ and $K_0$ only, such that for all integers $n \geq 1$ and all $x, x_1, ..., x_n \in \mathbb{S}^2$, we have

$$\text{Var} \left( T_0(x) | T_0(x_1), ..., T_0(x_n) \right) \geq K_{2,2} \min_{1 \leq k \leq n} \rho^2_\alpha (d_{\mathbb{S}^2}(x, x_k)).$$

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For any fixed point $x_0 \in \mathbb{S}^2$, consider the spherical random field $Z_0(x) = T_0(x) - T_0(x_0), x \in \mathbb{S}^2$. By taking the same argument as in the proof of Theorem 1 in [5], we obtain the following consequence of (4) in Lemma 2.1.

Corollary 2.2 Under Condition (A) with $2 < \alpha < 4$, there exists a constant $K'_{2,2} > 0$, such that for all integers $n \geq 1$ and all $x, x_1, ..., x_n \in \mathbb{S}^2$,

$$\text{Var} \left( Z_0(x) | Z_0(x_1), ..., Z_0(x_n) \right) \geq K'_{2,2} \min_{0 \leq k \leq n} \left\{ \rho^2_\alpha (d_{\mathbb{S}^2}(x, x_k)) \right\}.$$
The next auxiliary tool that we will use to prove Theorem 1.1 is the following lemma, where $(\cdot)^T$ denotes the transpose of a matrix or a vector.

**Lemma 2.3** Let $X = (X_1, \ldots, X_n)^T \sim N(\mu, \Theta)$ with a positive definite covariance matrix $\Theta$ and mean vector $\mu \in \mathbb{R}^d$ ($d \geq 1$), then for all vectors $t \in \mathbb{R}^d$, we have

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i t^T \xi} \mathbb{E} \left[ e^{i \xi^T X} \right] d\xi \leq \left[ \text{Var} (X_1) \prod_{j=2}^{n} \text{Var} (X_j | X_1, \ldots, X_{j-1}) \right]^{-1/2};$$

Moreover,

$$\text{Var} (X_1) \prod_{j=2}^{n} \text{Var} (X_j | X_1, \ldots, X_{j-1}) = \det \{ \Theta \}.$$

In order to prove the lower bounds in Theorems 1.2, we will also need to exploit the following two lemmas from Talagrand [15]. Let $\{f(x), x \in M\}$ be a centered Gaussian field indexed by $M$ and let $d_f(x, y) = \sqrt{\mathbb{E}[(f(x) - f(y))^2]}$ be its canonical metric on $M$. For a $d_f$-compact manifold $M$, denote by $N_{d_f}(M, \varepsilon)$ the smallest number of balls of radius $\varepsilon$ in metric $d_f$ that are needed to cover $M$.

**Lemma 2.4** If $N_{d_f}(M, \varepsilon) \leq \Psi(\varepsilon)$ for all $\varepsilon > 0$ and the function $\Psi$ satisfies

$$\frac{1}{K_{2,3}} \Psi(\varepsilon) \leq \Psi \left( \frac{\varepsilon}{2} \right) \leq K_{2,3} \Psi(\varepsilon), \quad \forall \varepsilon > 0,$$

where $K_{2,3} > 0$ is a finite constant. Then

$$\mathbb{P} \left\{ \sup_{s, t \in M} |f(s) - f(t)| \leq u \right\} \geq \exp \left( -K_{2,4} \Psi(u) \right),$$

where $K_{2,4} > 0$ is a constant depending only on $K_{2,3}$.

**Lemma 2.5** Let $\{f(x), x \in M\}$ be a centered Gaussian field a.s. bounded on a $d_f$-compact set $M$. There exists a universal constant $K_{2,5} > 0$ such that for any $u > 0$, we have

$$\mathbb{P} \left\{ \sup_{x \in M} f(x) \geq K_{2,5} \left( u + \int_0^\infty \sqrt{\log N_{d_f}(M, \varepsilon)} d\varepsilon \right) \right\} \leq \exp \left( -\frac{u^2}{d_f^2} \right),$$

where $d_f = \sup \{d_f(x, y) : x, y \in M\}$ is the diameter of $M$ in the metric $d_f$.

Based on Lemmas 2.4 and 2.5 we obtain the following result:

**Lemma 2.6** Under Condition (A) with $2 < \alpha < 4$, there exists positive constants $K_{2,6}, K_{2,7}$ depending only on $\alpha$ and $K_0$ such that for any $D(z, r) \subset S^2$ and $0 < r < \delta$, we have for any $u > K_{2,6} r^{(\alpha - 2)/2}$,

$$\mathbb{P} \left\{ \sup_{x, y \in D(z, r)} |T_0(x) - T_0(y)| \geq K_{2,7} u \right\} \geq \exp \left( -\frac{u^2}{|\rho_\alpha(2r)|^2} \right), \quad (10)$$
Proof. Recall Lemma 2.1 we have
\[ \sqrt{K_{2,1}^{-1}} \rho_\alpha (x, y) \leq d_{T_0} (x, y) \leq \sqrt{K_{2,1}} \rho_\alpha (x, y). \]
It follows immediately that, for any \( D(z, r) \subset S^2 \), and any \( \epsilon \in (0, r) \),
\[ N_{d_{T_0}} (D(z, r), \epsilon) \leq \frac{2 \pi r^2}{\pi (\epsilon / \sqrt{K_{2,1}})^{4/(\alpha-2)}} \leq 2 (K_{2,1})^{2/(\alpha-2)} \frac{r^2}{\epsilon^{4/(\alpha-2)}}, \]
and
\[ d = \sup \{ d_{T_0}(x, y) : x, y \in D(z, r) \} \leq \sqrt{K_{2,1}} \rho_\alpha (2r), \]
whence similar to the argument in [?], we have
\[ \int_0^d \sqrt{\log N_{d_{T_0}} (D(z, r), \epsilon)} d\epsilon \leq C_2 \pi r^{\alpha/2-1}, \]
for some positive constant \( C_{2,1} \) depending on \( K_{2,1} \). Let the constants \( K_{2,6} = C_{2,1} \) and \( K_{2,7} = 2K_{2,5}C_{2,1} \), then by exploiting Lemma 2.4 with \( K_{4,1} = K_{2,4}C_{2,2} \).

Finally, we prove the following approximation for small ball probability, which is analogy to the argument in [17], Theorem 5.1.

Lemma 2.7 Under Condition (A) with \( 2 < \alpha < 4 \), there exists a positive constant \( K_{2,8} \) depending only on \( \alpha \) and \( K_0 \), such that for any \( \epsilon > 0 \) and \( D(z, r) \subset S^2 \) with \( 0 < r < \delta \), we have
\[ P \left\{ \sup_{x,y \in D(z,r)} |T_0(x) - T_0(y)| \leq \epsilon \right\} \geq \exp \left( -K_{2,8} \frac{r^2}{\epsilon^{4/(\alpha-2)}} \right). \]

Proof. We first prove the lower bound of the small ball probability. The canonical metric \( d_{T_0} \) on \( S^2 \) is defined by
\[ d_{T_0}(x, y) = \sqrt{\mathbb{E} |T_0(x) - T_0(y)|^2}. \]
By (8) in Lemma 2.1 we have for any \( x, y \in S^2 \) with \( d_{S^2}(x, y) < \delta \),
\[ d_{T_0}(x, y) \leq \sqrt{K_{2,1}} \rho_\alpha (x, y). \]
In the meantime, recall the metric entropy, it follows immediately that
\[ N_{d_{T_0}} (D(z, r), \epsilon) \leq \frac{2 \pi r^2}{\pi (\epsilon / \sqrt{K_{2,1}})^{4/(\alpha-2)}} \leq C_{2,2} \frac{r^2}{\epsilon^{4/(\alpha-2)}}, \]
where \( C_{2,2} \) is a positive constant depending only on \( K_{2,1} \) and \( \alpha \). Hence, Lemma 2.7 is derived by exploiting Lemma 2.4 with \( K_{4,1} = K_{2,4}C_{2,2} \).
Local times and their joint continuity

Let us first recall that, for any Borel set \( D \subseteq S^2 \) and \( \omega \in \Omega \), the occupation measure of \( T \) on \( D \) is defined by

\[
\mu_D(I, \omega) := \nu \{ x \in D : T(x, \omega) \in I \}
\]

for all Borel sets \( I \subset \mathbb{R}^d \), where \( \nu \) is the canonical area measure on the unit sphere. If \( \mu_D(\cdot, \omega) \) is absolutely continuous w.r.t the Lebesgue measure \( \lambda_d \) on \( \mathbb{R}^d \), then we say that \( T(\cdot, \omega) \) has local times on \( D \), and define a local time \( L(t, D, \omega) \) as the Radon–Nikodým derivative of \( \mu_D \) with respect to \( \lambda_d \), i.e.,

\[
L(t, D, \omega) = \frac{d\mu_D(\cdot, \omega)}{d\lambda_d}(t), \quad \forall t \in \mathbb{R}^d.
\]  

(11)

As usual, we will from now on omit \( \omega \) from the notation for the local times.

We refer to Geman and Horowitz [3] for a comprehensive survey on local times of random fields. In particular, Theorems 6.3 and 6.4 in [3] show that \( L(t, D) \) satisfies the following occupation density identity: For every Borel set \( D \subseteq S^2 \) and for every measurable function \( f : \mathbb{R}^d \rightarrow \mathbb{R}^+ \),

\[
\int_D f(T(x)) d\nu(x) = \int_{\mathbb{R}^d} f(t)L(t, D) dt.
\]

(12)

We split the proof of Theorem 1.1 into two parts.

**Proof of Theorem 1.1**. Existence. The Fourier transform of the occupation measure \( \mu_D \) is

\[
\hat{\mu}_D(\xi) = \int_D e^{i\xi^T T(x)} d\nu(x), \quad \forall \xi \in \mathbb{R}^d.
\]

Now applying Fubini’s theorem and the independence of components of \( T \), we derive

\[
\mathbb{E} \left[ \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 d\xi \right] = \int_D d\nu(x) \int_{\mathbb{R}^d} d\nu(y) \int_{\mathbb{R}^d} \mathbb{E} \left[ e^{i\xi_x^T (T(x) - T(y))} \right] d\xi
\]

\[
= \int D \times D \mathbb{E} \left[ \left| T_0(x) - T_0(y) \right|^2 \right]^{-d/2} d\nu(x) d\nu(y)
\]

\[
\leq K^{d/2}_{2,1} \int D \times D [d_{S^2}(x, y)]^{(1-\alpha/2)d} d\nu(x) d\nu(y)
\]

\[
\leq K^{d/2}_{2,1} 2\pi \int_0^{2\pi} \int_0^\pi \theta^{(1-\alpha/2)d} \sin \theta d\theta d\phi < \infty.
\]

In the above, the first inequality follows from \( \mathcal{N} \) in Lemma 2.4 and the last inequality follows from the condition that \( \alpha \in (2, 4) \) and \( (\alpha - 2)d < 4 \). Hence, by the Plancherel theorem (see i.e., [12], Ch. 9), we see that \( T \) a.s. has local times which can be represented in the \( L^2(\mathbb{R}^d) \)-sense as

\[
L(t, D) = \frac{1}{2\pi} \int_{\mathbb{R}^d} e^{-i\xi^T t} \int_D e^{i\xi^T T(x)} d\nu(x) d\xi.
\]

(13)
The proof of existence is thus completed. ■

An immediate consequence of the above proof of the existence of local times is the following:

**Corollary 3.1** Under the same conditions as in Theorem 1.1, almost surely the range \( T(S^2) \) has positive Lebesgue measure.

In order to prove the joint continuity of local times, we first prove the moment estimates in Lemmas 3.2 and 3.3 by extending the argument in Xiao [16] to the spherical random field \( T \). The new ingredient for the proofs is the property of strong local nondeterminism in Lemma 2.1.

**Lemma 3.2** Under conditions of Theorem 1.1, there exists a positive constant \( K_{3,1} \) depending only on \( \alpha, d \) and \( K_0 \), such that for any open set \( D \subset S^2 \) with \( \nu(D) > 0 \), \( t \in \mathbb{R}^d \), and integers \( n \geq 1 \), it holds that

\[
E[L(t, D)^n] \leq K_{3,1}^{n} ((n - 1)!)^{d(\alpha - 2)/4} \nu(D)^{(n-1)\beta/4 + 1}.
\]

where \( \beta \) is defined in Theorem 1.1.

**Proof.** It follows from (13) (see also [3, 17]) that for all \( t \in \mathbb{R}^d \) and integers \( n \geq 1 \),

\[
E[L(t, D)^n] = \frac{1}{(2\pi)^{d/2}} \int_{D^n} e^{-i\sum_{j=1}^{n} t^T \xi_j} \mathbb{E} \left[ e^{i\sum_{j=1}^{n} \xi_j^T T(x_j)} \right] \prod_{j=1}^{n} (d\nu(x_j), d\xi_j),
\]

where \( \xi_j \in \mathbb{R}^d \) for \( j = 1, ..., n \). By the positive definiteness of covariance matrix of \( T_0(x_1), ..., T_0(x_n) \), we have

\[
\int_{\mathbb{R}^d} \mathbb{E} \left[ e^{i\sum_{j=1}^{n} \xi_j^T [T(x_j)-t]} \right] \prod_{j=1}^{n} d\xi_j
\]

\[
= \prod_{k=1}^{d} \int_{\mathbb{R}^n} \mathbb{E} \left[ e^{i\sum_{j=1}^{n} \xi_{j,k} [T_k(x_j)-t_k]} \right] \prod_{j=1}^{n} d\xi_{j,k}
\]

\[
\leq \frac{(2\pi)^{nd/2}}{[\text{det} (\text{Cov}(T_0(x_1), ..., T_0(x_n)))]}^{d/2}
\]

\[
\leq (2\pi)^{nd/2} \left[ \text{Var}(T_0(x_1)) \prod_{j=2}^{n} \text{Var}(T_0(x_j)|T_0(x_1), ..., T_0(x_{j-1})) \right]^{-d/2}.
\]

in view of Lemma 2.3. Recall the assumption of unit variance of \( T_0 \) and the inequality (9) in Lemma 2.1, we obtain that

\[
E[L(t, D)^n] \leq C_{3,1}^{n-1} \left[ \int_{D^n} \prod_{j=2}^{n} \frac{1}{\min_{1 \leq i \leq j-1} \rho_0^d(x_i, x_j)} d\nu \right]^{n-1}.
\]
where the constant $C_{3,1} > 0$ depends only on $K_{2,2}$, and $d\nu = d\nu(x_1) \cdots d\nu(x_n)$. Now let $j \in \{2, \ldots, n\}$ fixed, and define the following sets that are disjoint except on the boundaries,

$$
\Gamma_i = \{ x \in D : d_{g_2}(x, x_i) = \min \{ d_{g_2}(x, x_{i'}), i' = 1, \ldots, j - 1 \} \}.
$$

(15)

Observing $D = \bigcup_{i=1}^{j-1} \Gamma_i$, we have

$$
\begin{align*}
\int_D \frac{1}{\min_{1 \leq i \leq j - 1} \rho_\alpha^d(x_i, x_j)} d\nu(x_j) \\
&= \sum_{i=1}^{j-1} \int_{\Gamma_i} \frac{1}{\rho_\alpha^d(x_i, x_j)} d\nu(x) \\
&\leq \sum_{i=1}^{j-1} \int_0^{2\pi} \int_0^{\tau_i(\phi)} \frac{\theta}{\rho_\alpha^d(\theta)} d\theta d\phi \\
&\leq \frac{K_{3,2}}{(j-1)} \left( \frac{1}{(j-1)} \sum_{i=1}^{j-1} \nu(\Gamma_i) \right)^{\beta/4} \\
&\leq K_{3,2} \left( \frac{\nu(D)}{(j-1)} \right)^{\beta/4},
\end{align*}
$$

(16) (17)

where we have used Jensen’s inequality above and the constant $C_{3,2} > 0$ depends on $\alpha$ and $d$. Moreover, the inequality in (16) holds if and only if $\beta = 4-(\alpha-2)d > 0$. It is readily seen that

$$
\mathbb{E} [L(t, D)^n] \leq C_{3,1}^{n-1} \int_D \int_{D^{n-1}} \prod_{j=2}^n \frac{d\nu(x_1) \cdots d\nu(x_{n-1})}{\min_{1 \leq i \leq j - 1} \rho_\alpha^d(x_i, x_j)} d\nu(x_n)
$$

$$
\leq K_{3,1}^{n-1} \nu(D)^{(n-1)\beta/4+1},
$$

in view of the two results (14) and (17), where the constant $K_{3,1} > 0$ depends on $\alpha$ and $d$. Hence, the lemma is proved. \(\blacksquare\)

Recall $\eta = \beta/2 - (\alpha - 2)\gamma$ defined in (13). Obviously, $\eta < \beta/2 < 2$ for any $\gamma \in (0, 1)$. Now we have the following moment estimation:

**Lemma 3.3** Under conditions of Theorem 1.1, there exists a positive constant $K_{3,2}$ depending on $\alpha$, $d$ $K_0$ and $\gamma$, such that for any open set $D \subset \mathbb{S}^2$ with $\nu(D) > 0$, and any $\mathbf{s}, \mathbf{t} \in \mathbb{R}^d$, all even integers $n \geq 2$, $0 < \gamma < 1$ satisfying $\eta > 0$, we have

$$
\mathbb{E} \{ (\mathbf{L}(\mathbf{t}, D) - \mathbf{L}(\mathbf{s}, D))^n \} \leq K_{3,2}^{n} (n!)^{2-\eta/2} \| \mathbf{t} - \mathbf{s} \|^n \eta^{(n-2)/2} \nu(D)^{(n-1)\eta/2+1}. 
$$

**Proof.** Recall (13), we have

$$
\mathbb{E} \{ (\mathbf{L}(\mathbf{t}, D) - \mathbf{L}(\mathbf{s}, D))^n \} = \frac{1}{(2\pi)^n} \int_{D^n} \int_{\mathbb{R}^{nd}} \mathbb{E} \left[ e^{i\sum_{j=1}^n \xi_j^T T(x_j)} \prod_{j=1}^n \left( e^{-i\mathbf{t}^T \xi_j} - e^{-i\mathbf{s}^T \xi_j} \right) \right] d\nu(x_j) d\xi_j.
$$
By the fact that, for any $\gamma \in (0, 1)$, we have
\[ |e^{-it^T\xi_j} - e^{-is^T\xi_j}| = |e^{-i(t-s)^T\xi_j} - 1| \leq 2^{1-\gamma} \|t - s\|^\gamma \|\xi_j\|\gamma, \]
with $j = 1, \ldots, n$, and hence,
\[ \mathbb{E}\{|L(t, D) - L(s, D)|^n\} \leq \frac{2^{n(1-\gamma)}}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \text{Var}(\sum_{j=1}^n \xi_j^T T(x_j))} \prod_{j=1}^n \|\xi_j\|\gamma \, d\xi_j \, d\nu(x). \]
Since $|a + b| \leq |a| + |b|$ for any real numbers $a, b$ and $0 < \gamma < 1$, we have $\|\xi_j\|\gamma \leq \sum_{k=1}^d \|\xi_{j,k}\|\gamma$, which leads to
\[ \int_{\mathbb{R}^n} e^{-\frac{1}{2} \text{Var}(\sum_{j=1}^n \xi_j^T T(x_j))} \prod_{j=1}^n \|\xi_j\|\gamma \, d\xi_j \]
\[ \leq \sum_{k \in \{1, \ldots, d\}^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \text{Var}(\sum_{j=1}^n \xi_j^T T(x_j))} \prod_{j=1}^n \|\xi_{j,k}\|\gamma \, d\xi_j, \]
with $k = (k_1, \ldots, k_n)$ and $k_j \in \{1, \ldots, d\}$ for $j = 1, \ldots, n$. That is
\[ |\mathbb{E}\{L(t, D) - L(s, D)|^n\} \leq (2\gamma \pi)^{-n} \|t - s\|^{n\gamma} \sum_{k \in \{1, \ldots, d\}^n} \int_{\mathbb{R}^n} J_k(x) \, d\nu \]  
(18)
where $d\nu = d\nu_1 \ldots d\nu_n$ and $J_k$ is the integral
\[ J_k (x) = \int_{\mathbb{R}^n} e^{-\frac{1}{2} \text{Var}(\sum_{j=1}^n \xi_j^T T(x_j))} \prod_{j=1}^n \|\xi_{j,k}\|\gamma \, d\xi_j , \]
for each fixed point $k$ in the discrete space $\{1, \ldots, d\}^n$. By a generalized Hölder’s inequality and Lemma 2.4 in [16] (see also [2]), we see that $J_k (x)$ is bounded by
\[ \prod_{j=1}^n \left( \int_{\mathbb{R}^n} e^{-\frac{1}{2} \text{Var}(\sum_{j=1}^n \xi_j^T T_k(x_j))} \left| \xi_{j,k} \right|^{n\gamma} \, d\xi_j \right)^{1/n} \]
\[ = \frac{(2\pi)^{-n} \int_{\mathbb{R}^n} |v|^{n\gamma} \exp \left\{ -\frac{1}{2} v^2 \right\} \, dv \prod_{j=1}^n \left( \sigma_j^2 \right)^{-\gamma/2}}{[\det \text{Cov}(T_k(x_j), 1 \leq k \leq d, 1 \leq j \leq n)]^{1/2}} \]
\[ \leq \frac{(2\pi)^{-n} \left( \sum_{j=1}^d 2^{\frac{n\gamma-1}{2}} \Gamma \left( \frac{n\gamma+1}{2} \right) \prod_{j=1}^n \left( \sigma_j^2 \right)^{-\gamma/2} \right)}{[\det \text{Cov}(T_0(x_1), \ldots, T_0(x_n))]^{d/2}}, \]  
(19)
where $\sigma_j^2$ is the conditional variance of $T_{k_j}(x_j)$ given $T_l(x_i)$ ($l \neq k_j$ or $l = k_j$ but $i \neq j$) and $\Gamma (\cdot)$ is the Gamma function. Now we define a permutation $\pi$ of $\{1, \ldots, n\}$ such that $\pi (1) = 1$, and
\[ d_{\text{SS}}(x_{\pi(j)}, x_{\pi(j-1)}) = \min \left\{ d_{\text{SS}}(x_i, x_{\pi(j-1)}), i \in \{1, \ldots, n\} \setminus \{\pi(1), \ldots, \pi(j-1)\} \right\} , \]
Then by (19) in Lemma 3.1 we see that $K_{2,2}^{γ_n/2}$ is bounded by

$$\prod_{j=1}^{n} \min \{ \left[ \frac{1}{2} \rho_\alpha \left( d_{\mathbb{L}}(x_{\pi(j)}, x_i) \right) \right]^2 ; i \neq \pi(j) \}$$

$$\leq \prod_{j=1}^{n} \left\{ \frac{1}{2} \rho_\alpha \left( d_{\mathbb{L}}(x_{\pi(j)}, x_{\pi(j-1)}) \right), \rho_\alpha \left( d_{\mathbb{L}}(x_{\pi(j)}, x_{\pi(j+1)}) \right) \right\}^2$$

$$\leq \prod_{j=1}^{n} \left( \frac{1}{2} \rho_\alpha \left( d_{\mathbb{L}}(x_{\pi(j)}, x_{\pi(j-1)}) \right) \right)^2 \leq \prod_{j=1}^{n} \left[ \min_{1 \leq i \leq j-1} \rho_\alpha \left( d_{\mathbb{L}}(x_{\pi(i)}, x_{\pi(i)}) \right) \right]^{2γ}$$

For the sake of notation’s simplicity, we denote by $\pi(j) = j$ for each $j = 1, \ldots, n$. By (19) and Lemma 2.3 we derive that

$$J_k (x) \leq \frac{C_{3,2}^n (n-1)!}{\Pi_{j=2}^{n} \left[ \min_{1 \leq i \leq j-1} \rho_\alpha \left( d_{\mathbb{L}}(x_j, x_i) \right) \right]^{d+2γ}}$$

(20)

where $C_{3,2}$ is a positive constant depending on $K_{2,2}$ and $d$. Therefore, by (15) and (20) above, we have

$$\mathbb{E} \{ [L(t, D) - L(s, D)]^n \} \leq \left( \frac{dC_{3,2}}{2^{7π}} \right)^n \| t - s \|^n n! \sum_{k \in \{1, \ldots, d\}^n} I_k,$$

(21)

where

$$I_k =: \int_{D^n} \prod_{j=2}^{n} \left[ \min_{1 \leq i \leq j-1} \rho_\alpha \left( d_{\mathbb{L}}(x_j, x_i) \right) \right]^{-(d+2γ)} d\nu.$$  

(22)

Similar to the argument in the proof of Lemma 3.2 for any fixed $j \in \{2, \ldots, n\}$, we define the domains $\Gamma_i$, $i = 1, \ldots, j-1$, same as in (15), then we can obtain

$$A_j := \int_{D} \frac{1}{\min_{1 \leq i \leq j-1} \rho_\alpha \left( d_{\mathbb{L}}(x_j, x_i) \right)^{d+2γ}} d\nu (x_j)$$

$$= \sum_{i=1}^{j-1} \int_{\Gamma_i} \frac{1}{\min_{1 \leq i \leq j-1} \rho_\alpha \left( d_{\mathbb{L}}(x_j, x_i) \right)^{d+2γ}} d\nu (x_j)$$

$$\leq \sum_{i=1}^{j-1} \int_{0}^{2π} r_i (\phi) \frac{1}{\rho_\alpha (\theta)^{d+2γ}} d\theta d\phi = \sum_{i=1}^{j-1} \eta^{-1} \int_{0}^{2π} \left[ r_i (\phi) \right]^n d\phi$$

$$\leq C_{3,3} \sum_{i=1}^{j-1} \left\{ \int_{0}^{2π} \frac{1}{2} \left[ r_i (\phi) \right]^2 d\phi \right\}^{n/2} = C_{3,3} \left( \nu (\Gamma_i) \right)^{n/2}$$

$$\leq C_{3,3} (j-1) \left( \frac{1}{(j-1)} \sum_{i=1}^{j-1} \nu (\Gamma_i) \right)^{n/2} \leq C_{3,3} (j-1) \left( \frac{\nu (D)}{(j-1)} \right)^{n/2}$$

(23)

(24)

where constant $η$ is the one defined in (5) and $C_{3,3} = \eta^{-1} (2π)^{1-n/2}$. The inequality in (24) holds if and only if $0 < η < 2$, which is always true for all
\( \alpha \in (2, 4), \gamma \in (0, 1), \ d \in \mathbb{N}^+ \) satisfying \((\alpha - 2) (d + 2 \gamma) < 4\). Hence, by the definition of \( I_k \) in (22) and the inequality (24) above, we have

\[
I_k = \int_D \left[ \int_{D^{n-1}} \Pi_{j=2}^n A_j \, dv(x_n) \cdot \cdots \cdot dv(x_2) \right] \, dv(x_1)
\]

\[
\leq (2\pi)^{n/2} C_{3,3}^n \|(n-1)\|^{1-\gamma/2} \nu(D)\|(n-1)\|^{1/2+1}.
\]

Thus, by (21) and (25) above, we immediately obtain

\[
\mathbb{E} \left\{ [L(t, D) - L(s, D)]^n \right\} \leq (n-1)! \|t-s\|^{n\gamma} \sum_{k \in \{1, \ldots, p\}} C_{3,4}^n r^{(n-1)\eta+2}
\]

\[
\leq K_{3,2}^n \|(n-1)\|^{2-\eta/2} \|t-s\|^{n\gamma} \nu(D)\|(n-1)\|^{1/2+1}
\]

where the constants \( C_{3,4} = C_{3,2} (2\pi)^{\eta/2} C_{3,3} \) and \( K_{3,2} = dC_{3,4} \). The proof is then completed. ■

**Proof of Theorem 1.1 Joint Continuity.**

It follows immediately from Kolmogorov’s continuity theorem and Lemmas 3.2 and 3.3 that, for any two sets \( D_1, D_2 \subset \mathbb{S}^2 \) with \( \nu(D_i) > 0, i = 1, 2 \), and any \( s, t \in \mathbb{R}^d \), we have

\[
|L(t, D_1) - L(s, D_1)| \leq C_{3,5} \|t-s\| \nu(D_1)^{\eta/2},
\]

and

\[
|L(s, D_1) - L(s, D_2)| \leq C_{3,6} \nu(\Delta D)^{[1-(\alpha-2)d/4]},
\]

where \( C_{3,5}, C_{3,6} \) are positive constants depending only on \( \alpha, A, d, \) and \( C_{3,5} \) depends on \( \gamma \) as well. Moreover, the set \( \Delta D = D_1 \cup D_2 \setminus (D_1 \cap D_2) \), that is, the union of \( D_1 \) and \( D_2 \), excluding their intersection. Therefore, by (26), (27) together with the following inequality

\[
|L(t, D_1) - L(s, D_2)| \leq |L(t, D_1) - L(s, D_1)| + |L(s, D_2) - L(s, D_2)|,
\]

we have

(i) If replacing \( D \) with \( D(x, r) \), then for any \( r_1, r_2 \in (0, \delta), x \in \mathbb{S}^2 \),

\[
|L(t, D(x, r_1)) - L(s, D(x, r_2))| \leq \pi C_{3,5} \|t-s\| r_1^{\eta} + \pi C_{3,6} r_1^2 - r_2^2 |1-(\alpha-2)d/4|
\]

and for any \( r \in (0, \delta), x, y \in \mathbb{S}^2 \) with \( d_{S^2}(x, y) < 2r \),

\[
|L(t, D(x, r)) - L(s, D(y, r))| \leq \pi C_{3,5} \|t-s\| r^\eta + 6C_{3,6} [d_{S^2}(x, y)r]^{[1-(\alpha-2)d/4]},
\]

where we have used the fact that

\[
\nu(D_1 \cap D_2) = 4 \left( \frac{\pi}{2} - \arcsin \frac{d_{S^2}(x, y)}{r} \right) (1 - \cos r)
\]

\[ - d_{S^2}(x, y) \sqrt{r^2 - d_{S^2}(x, y)^2/4} + o(d_{S^2}(x, y) r) \]
and
\[ \nu(\Delta D) = 2\pi(1 - \cos r) - \nu(D_1 \cap D_2) \leq 6d_2(x, y)r \]

Here we have denoted by \( o(\cdot) \) the higher order terms.

(ii) If replacing \( D \) with the angular section \( V(\theta, \varphi) = \{(\theta, \phi) : 0 \leq \theta \leq \vartheta, 0 \leq \phi \leq \varphi \} \) with \( (\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi] \), then for any
\[
|L(t, V(\vartheta_1, \varphi_1)) - L(s, V(\vartheta_2, \varphi_2))| \\
\leq \pi C_{3.5} \|t - s\|^{1/2} \vartheta_1^{\eta/2} \vartheta_2^{\eta/2} \\
+ \pi C_{3.6} \min \{\vartheta_1 - \varphi_2, \varphi_2 - \varphi_1\} \min \{\vartheta_1, \vartheta_2\} \left[\vartheta_1^2 - \vartheta_2^2\right]^{\beta/4}
\]

The joint continuity of local times \( L(t, D) \) w.r.t. \( t, D \) in different cases is then obtained and hence the proof of Theorem 1.1 is completed.

\section{Hölder conditions of local times}

In order to prove Theorem 1.2, we need the following lemma which is readily seen in view of Lemmas 3.2 and 3.3.

\begin{lemma}
Under conditions of Theorem 1.1, there exist positive constants \( K_{4.1}, K_{4.2} \) depending on \( \alpha, K_0, d \) and \( K_{4.2} \) depends on \( \gamma \) as well, such that for any open disk \( D = D(x, r) \subset S^2 \) with \( r \in (0, \delta) \), any \( t \in \mathbb{R}^d, x_0 \in S^2 \) and all even integers \( n \geq 2 \), and \( 0 < \gamma < 1 \),
\[
\mathbb{E} \left( \left| L(t + T(x_0), D) \right|^n \right) \leq K_{4.1}^n n! \left( \pi/2 \left( n! \right)^{1/2} \right)^{\alpha/2} \vartheta_1^{\eta/2},
\]
\[
\mathbb{E} \left( \left| L(t + T(x_0), D) - L(t + T(x_0), D) \right|^n \right) \leq K_{4.2}^n n! \left( \pi/2 \left( n! \right)^{1/2} \right)^{\alpha/2} \left\| t - s \right\|^{\epsilon/2} r^{n\gamma},
\]
where \( \eta \) is the constant defined in [5].
\end{lemma}

\begin{proof}
For any points \( x_1, ..., x_n \in S^2 \), let \( Z(x_j) = T(x_j) - T(x_0), j = 1, ..., n \), we have
\[
\det \text{Cov}[Z_0(x_1), ..., Z_0(x_n)] = 
\begin{align*}
\text{Var}(Z_0(x_1)) \prod_{j=2}^n \text{Var}(Z_0(x_j)) &\text{Var}(Z_0(x_1), ..., Z_0(x_{j-1})) \\
\geq K_{2.2}(K_{2.2}')(n-1)^{\alpha/2} &\rho_\alpha(x_1, x_0) \prod_{j=2}^n \min_{0 \leq i \leq j-1} \rho_\alpha(x_j, x_i).
\end{align*}
\end{proof}
in view of Corollary 2.2. Let \( L_Z(t, D) \) be the local time of \( Z \) at \( t \) in \( D \), and recall (14) and (17), then we can obtain that

\[
\mathbb{E} \{ |L_Z(t, D)|^n \} \leq \left( \frac{K_{1,2}' / K_{1,2}}{\pi} \right)^{-n/2} (2\pi)^{-nd/2} (K_{1,2}')^{-nd/2} \\
\times \int_{D} \left[ \int_{D^{n-1}} \frac{d\nu(x_1) \cdots d\nu(x_{n-1})}{\prod_{j=2}^{n} \min_{0 \leq i < j-1} [\rho_\alpha(d_{S^2}(x_i, x_j))]^d} \right] d\nu(x_1) \\
\leq (C_{4,1})^n n! \left( \frac{\eta^2}{\eta!} \right) ^{n^3/4} ,
\]

where \( C_{4,1} \) is a positive constant depending on \( K_{3,1}, K_{2,2} \) and \( K_{1,2}' \). Likewise, recall (21) and (25), we have

\[
\mathbb{E} \{ |L_Z(t, D) - L_Z(s, D)|^n \} \leq C_{4,2} \| t - s \|^{\eta \gamma} \\
\times \int_{D} \left[ \int_{D^{n-1}} \frac{d\nu(x_n) \cdots d\nu(x_2)}{\prod_{j=2}^{n} \min_{0 \leq i < j-1} [\rho_\alpha(d_{S^2}(x_i, x_j))]^{d+2\gamma}} \right] d\nu(x_1) \\
\leq (C_{4,3})^n (n!)^{2-\eta/2} \| t - s \|^{\eta \gamma} r^m ,
\]

where \( C_{4,2} \) and \( C_{4,3} \) are positive constants depending on \( K_{2,2}, K_{1,2}', K_{3,2} \) and \( \gamma, d \). The results in Lemma 4.1 is then derived by the fact that

\[ L_Z(t, D) = L(t + T(x_0), D). \]

Based on Lemma 4.1, we now follow the similar line as in the proof of Lemma 2.7 in [16], and obtain the results below:

**Lemma 4.2** Assume conditions of Theorem 1.1 hold, there exists a positive constant \( K_{4,3}, K_{4,4} \) depending on \( d, \alpha, K_0 \) and \( K_{4,4} \) depends on \( \gamma \) as well, such that for any open disk \( D = D(x, r) \subset S^2 \) with \( r \in (0, \delta) \), any \( u > 0 \), \( t \in \mathbb{R}^d \), \( x_0 \in S^2 \),

\[
\mathbb{P} \left\{ L(t + T(x_0), D) \geq \frac{K_{4,3} r^2}{[\rho_\alpha(u r)]^d} \right\} \leq e^{-1/u^2}. \tag{28}
\]

and

\[
\mathbb{P} \left\{ |L(t + T(x_0), D) - L(s + T(x_0), D)| \geq \frac{K_{4,4} r^2}{[\rho_\alpha(u r)]^d u^4 \gamma} \right\} \leq e^{-u - 2}. \tag{29}
\]

**Proof.** Let

\[
\Lambda = \frac{L(t + T(x_0), D)}{r^2}, \quad u_n = \frac{1}{\sqrt{n}}
\]

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with \( n \in \mathbb{N}^+ \). Then by Chebyshev’s inequality and Lemma 4.1, we have

\[
\mathbb{P} \left\{ \Lambda \geq \frac{K_{4,3}}{\rho_\alpha (u_r)^d} \right\} \leq \frac{\mathbb{E} [\Lambda^n] \rho_\alpha^n (u_r)^{n(\alpha-2) d/8 \beta}}{(K_{3,4})^n} \leq \left( \frac{K_{4,1}}{K_{4,3}} \right)^n (n!)^{d/8} \left( \frac{u_r}{\ln K_{4,1} K_{4,3}} \right) \leq \exp \left\{ -\frac{1}{8} (\alpha-2) d - \ln \frac{K_{4,1}}{K_{4,3}} \right\} \leq \exp \left\{ -2/u_r^2 \right\},
\]

where the last inequality follows from Stirling’s formula, and the constants satisfy \( K_{4,3} = K_{4,1} \exp \left\{ \frac{1}{8} (\alpha-2) d \right\} \). Now for any \( u > 0 \) small enough, there exists \( n \in \mathbb{N}^+ \), such that \( u_{n+1} \leq u_n \), and

\[
\mathbb{P} \left\{ \Lambda \geq \frac{K_{4,3}}{\rho_\alpha (u_r)^d} \right\} \leq \mathbb{P} \left\{ \Lambda \geq \frac{K_{4,3}}{\rho_\alpha (u_r)^d} \right\} \leq \exp \left\{ -2/u_n^2 \right\} \leq \exp \left\{ -u^{-2} \right\}.
\]

Hence inequality (25) is derived.

The proof of estimation (29) is similar to the argument above and we omit it here. Thus the proof of Lemma 4.2 is completed. □

**Proof of Theorem 1.2**

The proof for the upper bounds in the inequalities (11) is based on a chaining argument and quite similar to the proof of Theorem 1.1 and 1.2 in [16] section 3 by replacing the notations \( X(t), B(r, 2^{-n}) \) with ones of \( T(x), D(x_0, 2^{-n}) \) for some \( x_0 \in S^2 \), and Lemma 2.7, 3.1 in [16] with Lemmas 2.3 and 4.2 in this paper.

For the lower bounds in the inequities (10), we first let \( I \) be the closure of the set \( T(D) = \{ T(x), x \in D \} \) for any disk \( D = D(z, r) \subset S^2 \). Recall the definition of local time (11) or formula (12), we have for any \( \omega \in \Omega \),

\[
\nu(D) = \mu_D(1, \omega) = \int I L(t, D, \omega) \, dt \
\leq L^\ast(D, \omega) \cdot \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \sup_{x,y \in D} \| T(x) - T(y) \|^d.
\]

That is, for any \( z \in S^2 \),

\[
\liminf_{r \to 0} \frac{L^\ast(D(z, r))}{\phi_1(r)} \geq C_{4,4} \cdot \liminf_{r \to 0} \left\{ \frac{\rho_\alpha (r/\sqrt{\log \log r^{-1}})}{\sup_{x,y \in D(z, r)} \| T(x) - T(y) \|^d} \right\}^{d} \geq C_{4,4} \left( \lim_{r \to 0} \sup_{x,y \in D(z, r)} \frac{\| T(x) - T(y) \|}{\rho_\alpha (r/\sqrt{\log \log r^{-1}})} \right)^{-d} = C_{4,4} K_{5,3}^{-d}.
\]

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in view of Proposition 5.2 in Section 5, which derives the lower bounds in the equalities in Theorem 1.2. The positive constant $C_{4,4}$ depends only on $d$. The proof is then completed. ■

5 Modulus of non-differentiability

In this section, we establish local and global Chung’s law of the iterated logarithm, or we say, the modulus of non-differentiability of the random field $T$, which are the essential of proving the lower bound of maximum local time in Theorem 1.2.

Before giving the main results, we first introduce the following band-limited random field. For any two integers $1 \leq L < U \leq \infty$, we define a random field as follows:

$$T_{0}^{L,U}(x) = \sum_{\ell=L}^{U} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \quad x \in S^2.$$ 

Observe that $T_{0}^{L,U}(x)$ and $T_{0}^{L',U'}(x)$ are independent for $L < U < L' < U'$ in view of the orthogonality properties of the Fourier components of the field $T_{0}(x)$ and the assumption of Gaussianity.

Meantime, let $T_{0}^{\Delta}$ be the random field defined by

$$T_{0}^{\Delta} = T_{0} - T_{0}^{L,U}.$$ 

For any $r$ small, take $L = \lfloor r^{-1}(B(r))^{-\kappa_{1}} \rfloor$ and $U = \lfloor r^{-1}(B(r))^{1-\kappa_{1}} \rfloor$, where $B(r)$ is a function such as $(\log \log r)^{\kappa_{2}}$ or $(\log r)^{\kappa_{2}}$ and the constants $\kappa_{1}, \kappa_{2}$ are to be determined. Here $[\cdot]$ denotes integer part as usual. Then we have the following approximation for $T_{0}^{\Delta}$:

**Lemma 5.1** Under the same condition as in Theorem 1.1, there exist positive constants $K_{5,1}$ and $K_{5,2}$ depending only on $\alpha$ and $K_{0}$, such that for any $0 < r < \delta$, $0 < \kappa_{1} \leq \frac{\alpha}{2} - 1$ and $u > K_{5,1}(B(r))^{-\kappa_{1}(2-\frac{\alpha}{2})} \sqrt{\log B(r)} r^{\frac{\alpha}{2} - 1}$, we have

$$\mathbb{P}\left\{ \sup_{x,y \in D(z,r)} |T_{0}^{\Delta}(x) - T_{0}^{\Delta}(y)| \geq u \right\} \leq \exp\left( -\frac{1}{K_{5,2}} \frac{(B(r))^{\kappa_{1}(4-\alpha)} u^2}{r^{\alpha-2}} \right).$$

**Proof.** Like in many other arguments in this paper, we start by introducing a suitable Gaussian metric $d_{T_{0}^{\Delta}}$ defined on $D(z,r) \subset S^2$ by

$$d_{T_{0}^{\Delta}}(x,y) := \left[ \mathbb{E} \left| T_{0}^{\Delta}(x) - T_{0}^{\Delta}(y) \right|^2 \right]^{1/2}.$$ 

Once again, due to the fact that $d_{T_{0}^{\Delta}}(x,y) \leq \sqrt{K_{2,1} \beta_{\alpha}(x,y)}$, a simple metric entropy argument yields

$$N_{d_{T_{0}^{\Delta}}}(D(z,r), \varepsilon) \leq C_{5,1} \frac{r^{2} \varepsilon^{4/(\alpha-2)}}{\varepsilon^{2/(\alpha-2)}},$$

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with \( C_{5,1} \) depending on \( K_{2,1} \). More precisely, recall

\[
d^2_{\Delta_0} (x,y) = \left( \sum_{\ell=0}^{L-1} + \sum_{\ell=U+1}^{\infty} \right) \frac{2\ell + 1}{4\pi} C_\ell \{ 1 - P_t(\cos \theta) \}.
\]

where \( \theta = d_{S^2} (x,y) \). Let \( 0 < \theta < r \), by Lemma 6.1 in the Appendix, we obtain

\[
d^2_{\Delta_0} (x,y) \leq K_A (L^{1-\alpha} \theta^2 + U^{2-\alpha}) \leq K_A \left[ B(r)^{-\kappa_1 (4-\alpha)} + B(r)^{-(1-\kappa_1) (\alpha-2)} \right] \leq K_A r^{\alpha-2} \leq \theta^2 r^{\alpha-2},
\]

where

\[
h(r) =: \sqrt{K_A} B(r)^{-\frac{\alpha}{(4-\alpha)}}.
\]

Hence, if we let

\[
\overline{d} := \sup \{ d_{\Delta_0} (x,y) : x, y \in D(z,r) \},
\]

which obviously have \( \overline{d} \leq h(r) r^{\frac{\alpha}{2} - 1} \), then

\[
\int_0^{\overline{d}} \sqrt{\log N_{\Delta_0} (D(z,r), \epsilon)} \, d\epsilon \leq \int_0^{h(r) r^{\frac{\alpha}{2} - 1}} \sqrt{\log C_{4,1}} \frac{r^2}{\epsilon^{(2-\alpha)/2}} \, d\epsilon
\]

\[
\leq \frac{2}{\sqrt{\alpha - 2}} \sqrt{K_{2,1}} r^{\frac{\alpha}{2} - 1} \int_0^{\infty} \frac{u}{\log \frac{\sqrt{K_{2,1}}}{h(r)}} \, u^{-\frac{1}{2}} \exp \left( -u \right)
\]

\[
\leq \frac{4h(r)}{\sqrt{\alpha - 2}} \sqrt{K_{2,1}} r^{\frac{\alpha}{2} - 1}.
\]

By exploiting Lemma 2.5, we immediately derive that, for any

\[
u > C_{5,2} B(r)^{-\kappa_1 (2-\alpha/2)} \sqrt{\log B(r)} r^{\alpha/2 - 1}
\]

with \( C_{5,2} > 0 \) depending on \( K_{2,5}, K_A, \alpha, \kappa_1 \), it holds that

\[
P \left\{ \sup_{x,z \in \mathbb{D}(z,r)} | T_{\Delta_0} (x) - T_{\Delta_0} (y) | \geq \nu \right\} \leq \exp \left( - \frac{u^2}{4K_{2,5}^2 |h(r)|^{2r^{\alpha-2}}} \right)
\]

Letting \( K_{5,1} = 2K_{2,5}C_{5,2} \) and \( K_{5,2} = 4K_{2,5}^2 K_A \), the proof is then completed.

Now let us focus on the Chung’s law of the iterated logarithm.

**Proposition 5.2** Under the conditions of Theorem 1.1 there exists positive constants \( K_{4,2} \) such that for any \( z \in S^2 \) with

\[
\lim_{r \to 0} \sup_{x,y \in \mathbb{D}(z,r)} \frac{||T(x) - T(y)||}{\rho_\alpha \left( \frac{r}{\sqrt{\log \log r^{-1}}} \right)} = K_{5,3}.
\]
**Proof.** Due to Lemma 7.1.1 in Marcus and Rosen [7] and the fact that \( \| T(x) - T(y) \| = \sqrt{d} | T_0(x) - T_0(y) | \), we only need to prove the upper and lower bounds of the following form: there exist positive and finite constants \( C_{5,3} \) and \( C_{5,4} \) such that

\[
\lim_{r \to 0} \sup_{x, y \in B(z, r)} \frac{|T_0(x) - T_0(y)|}{\rho_\alpha \left( \frac{r}{\sqrt{\log \log r^{-1}}} \right)} \geq C_{5,3}, \quad \text{a.s.} \tag{32}
\]

and

\[
\lim_{r \to 0} \sup_{x, y \in B(z, r)} \frac{|T_0(x) - T_0(y)|}{\rho_\alpha \left( \frac{r}{\sqrt{\log \log r^{-1}}} \right)} \leq C_{5,4}, \quad \text{a.s.} \tag{33}
\]

which implies \( \lim_{r \to 0} \sup_{x, y \in B(z, r)} \frac{|T_0(x) - T_0(y)|}{\rho_\alpha \left( \frac{r}{\sqrt{\log \log r^{-1}}} \right)} \leq C_{5}, \quad \text{a.s.} \) with \( K_{5,3} \in [C_{5,3}, C_{5,4}] \).

Recall \( \lim_{r \to 0} \sup_{x, y \in B(z, r)} \frac{|T_0(x) - T_0(y)|}{\rho_\alpha \left( \frac{r}{\sqrt{\log \log r^{-1}}} \right)} \leq C_{5,4} \) and the upper bound of \( \lim_{r \to 0} \sup_{x, y \in B(z, r)} \frac{|T_0(x) - T_0(y)|}{\rho_\alpha \left( \frac{r}{\sqrt{\log \log r^{-1}}} \right)} \leq C_{5,4} \) in Theorem 1.2, we have for any \( z \in S^2 \),

\[
\lim_{r \to 0} \sup_{x, y \in B(z, r)} \frac{|T_0(x) - T_0(y)|}{\rho_\alpha \left( \frac{r}{\sqrt{\log \log r^{-1}}} \right)} \leq C_{5,4} \left[ \lim_{r \to 0} \frac{L^*(D(z, r))}{\phi_1(r)} \right]^{-1} \geq \frac{C_{4,4}}{K_3}.
\]

where \( \phi_1 \) is defined in \( \Theta_4 \). One can verify (cf. Lemma 7.1.6 in \( \Theta_4 \)) that this implies \( \Theta_2 \).

Now let us focus on the Proof of \( \Theta_3 \). Let \( B(r) = (\log \log r^{-1})^{\kappa_2} \) for any \( r \in (0, \delta) \) and \( \kappa_2 > 0 \) to be determined. Now we choose \( r_k = (2 \log k)^{-\kappa_2} \), \( k = 1, 2, \cdots \), and \( L_k = B(r_k)^{-\kappa_1} r_k^{-1} \) as well as \( U_k = B(r_k)^{1-\beta} r_k^{-1} \), where \( \beta \in (0, \frac{\alpha}{2} - 1] \) and \( \lfloor \cdot \rfloor \) denotes the integer part as before. Obviously \( L_{k+1} < U_k < L_k \) for any \( k \in N^+ \). We would like to prove that for some constant \( C_{5,5} > 0 \),

\[
\sum_{k=1}^{\infty} P \left\{ \sup_{x, y \in B(z, r_k)} \left| T_0^{L_k, U_k}(x) - T_0^{L_k, U_k}(y) \right| \leq C_{5,5} \rho_\alpha \left( \frac{r_k}{\sqrt{\log \log r_k^{-1}}} \right) \right\} = \infty. \tag{34}
\]

Thus, due to the independence of \( T_0^{L_k, U_k} \) for different \( k \)'s, we have

\[
\limsup_{k \to \infty} \sup_{x, y \in B(z, r_k)} \left| T_0^{L_k, U_k}(x) - T_0^{L_k, U_k}(y) \right| \leq C_{5,5} \rho_\alpha \left( \frac{r_k}{\sqrt{\log \log r_k^{-1}}} \right), \quad \text{a.s.} \tag{35}
\]

in view of the Borel-Cantelli Lemma. The equation \( \Theta_3 \) can be derived by the fact that

\[
d_{T_0^{L_k, U_k}} = \sqrt{E \left| T_0^{L_k, U_k}(x) - T_0^{L_k, U_k}(y) \right|^2} \leq d_{T_0},
\]

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and thus, similar to the argument as in the proof of Lemma 2.7, we have for some constant $C_{4,5} > 0$,

$$
\mathbb{P} \left\{ \sup_{x,y \in D(z,r_k)} \left| T_0^{L_k} - T_0^{U_k} (x) - T_0^{L_k} (y) \right| \leq C_{5,5} \rho_{\alpha} \left( \frac{r_k}{\log \log r_k^{-1}} \right) \right\} 
\geq \exp \left( \frac{2K_{2,8}}{C_{5,5}} \log k \right) = k^{-1/2},
$$

where we have taken $C_{5,5} = 4 \log e K_{2,8}$.

On the other hand, recall $T_0^{\Delta_k} \equiv T_0^{L_k} - T_0^{U_k}$. Let $B(r_k)^{\kappa_1 (4 - \alpha)} = (\log \log r_k^{-1})^{\alpha/2}$, then it is readily seen that for any $k \in \mathbb{N}^+$,

$$
\rho_{\alpha} \left( \frac{r_k}{\sqrt{\log \log r_k^{-1}}} \right) > (B(r_k))^{\kappa_1 (2 - \alpha/2)} \sqrt{\log B(r_k) r_k^{(\alpha - 2)/2}},
$$

and thus, by Lemma 5.1, we have for any constant $C_{5,6} > 0$,

$$
\mathbb{P} \left\{ \sup_{x,y \in D(z,r_k)} \left| T_0^{\Delta_k} - T_0^{\Delta_k} (x) - T_0^{\Delta_k} (y) \right| > C_{5,6} \rho_{\alpha} \left( \frac{r_k}{\sqrt{\log \log r_k^{-1}}} \right) \right\} 
\leq \exp \left( - \frac{C_{5,6}^2}{K_{5,2}} \frac{(B(r_k))^{\kappa_1 (4 - \alpha)}}{(\log \log r_k^{-1})^{\alpha/2 - 1}} \right) \leq \exp \left( - \frac{C_{5,6}^2}{K_{5,2}} \log \log r_k^{-1} \right) \leq \exp \left( - \frac{C_{5,6}^2}{K_{5,2}} \log \log r_k^{-1} \right) = k^{-2},
$$

where we have chosen $C_{5,6} \log e = 2K_{5,3}$. Therefore, we have

$$
\sum_{k=1}^{\infty} \mathbb{P} \left\{ \sup_{x,y \in D(z,r_k)} \left| T_0^{\Delta_k} - T_0^{\Delta_k} (x) - T_0^{\Delta_k} (y) \right| > C_{5,6} \rho_{\alpha} \left( \frac{r_k}{\sqrt{\log \log r_k^{-1}}} \right) \right\} < \infty,
$$

and again by the Borel-Cantelli Lemma, we have

$$
\limsup_{r \to 0} \sup_{x,y \in D(z,r_k)} \left| T_0^{\Delta_k} - T_0^{\Delta_k} (x) - T_0^{\Delta_k} (y) \right| \leq C_{5,6} \rho_{\alpha} \left( \frac{r_k}{\sqrt{\log \log r_k^{-1}}} \right), \ a.s. \ (36)
$$

The inequality (33) is then derived in view of (35) and (36). □

6 Appendix

Lemma 6.1 There exists a positive constant $K_A$ depending on $K_0$ and $\alpha$, such that for any $\theta > 0$ small, and positive integers $L < \frac{K_A}{\theta}$, we have

$$
\sum_{\ell=1}^{L} \frac{2\ell + 1}{4\pi} C_{\ell} \{ 1 - P_{\ell} (\cos \theta) \} \leq K_A L^{1-\alpha} \theta^2,
$$

20
and, for any $U > 1$,

$$\sum_{\ell=U}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell \{1 - P_\ell(\cos \theta)\} \leq K_A U^{2-\alpha}.$$  

**Proof.** Recall first that, from Condition (A), there exists a positive constant $K_0$ such that for all $\ell = 1, 2, \ldots$

$$K_0^{-1} \ell^{-\alpha+1} \leq \frac{2\ell + 1}{4\pi} C_\ell \leq K_0 \ell^{-\alpha+1}.$$ 

We recall also the following Hilb’s asymptotics results (see [13], page 195, Theorem 8.21.6): for $K_A > 0$, we have uniformly

$$P_\ell(\cos \theta) = \left\{ \frac{\theta}{\sin \theta} \right\}^{1/2} J_0((\ell + \frac{1}{2})\theta) + \delta_\ell(\theta),$$

where

$$\delta_\ell(\theta) \ll \begin{cases} \frac{\theta^2 O(1)}{\theta^{1/2} O(\ell^{-3/2})} & \text{for } 0 < \theta < \frac{K_A}{\ell} \\ \ell^{1/2} O(\ell^{-3/2}) & \text{for } \theta > \frac{K_A}{\ell} \end{cases}.$$ 

Moreover,

$$\lim_{u \to 0} \frac{1 - J_0(K_A u)}{K_A^2 u^2} = \frac{1}{2}.$$ 

Thus, by using the fact that

$$\frac{\theta}{\sin \theta} - 1 = \frac{\theta^2}{6} + O(\theta^3), \text{as } \theta \to 0,$$

we obtain that

$$\sum_{\ell=1}^{L} \frac{2\ell + 1}{4\pi} C_\ell \{1 - P_\ell(\cos \theta)\} \leq K_0 \sum_{\ell=1}^{L} \ell^{1-\alpha} \left( \frac{\ell^2}{K_A^2} - \frac{1}{6} \right) \theta^2 \leq \frac{K_0}{(4 - \alpha) K_A^2} L^{4-\alpha} \theta^2.$$ 

On the other hand, recall that $1 - P_\ell(\cos \theta) \leq 2$ uniformly for all $\theta$, whence we have, for any $U > 1$,

$$\sum_{\ell=U}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell \{1 - P_\ell(\cos \theta)\} \leq K_{1,0} \sum_{\ell=U}^{\infty} \ell^{1-\alpha} \leq \frac{2K_{1,0} U^{2-\alpha}}{\alpha - 2}.$$ 

Let $K_A = \max \left\{ \sqrt[3]{\frac{K_0}{(4-\alpha)^3}} \cdot \frac{2K_0}{\alpha - 2} \right\}$, the proof is then completed. □
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