The Split Anti Fuzzy Domination in Anti Fuzzy Graphs

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Abstract. We will discuss the concept of a split anti-fuzzy dominating set (SAFD) in anti fuzzy graph (GAF) and investigate the relationship of γ_a_f(GAF) (split anti fuzzy domination number) with other known parameters of anti-fuzzy graph. Some bounds and interesting results for this parameter are obtained. The split anti-fuzzy domination on some standard anti-fuzzy graph has been discussed with some suitable graphs.

Keywords: anti fuzzy graph (GAF), Anti fuzzy dominating set (AFD) and Split anti fuzzy Domination number.

1. Introduction
The fuzzy set introduced by L.A. Zadeh [1] to explain vagueness mathematically and tried to resolve problems by giving a particular grade of membership to every member of a given set, which laid the basis of set theory. In (1975) the fuzzy Graph introduced by A. Rosenfeld [2]. The basic idea of fuzzy graph introduced by Kaufmann [3], and fuzzy relation represents the relationship between the objects of the given set. Domination in fuzzy graphs has been introduced by A.Somasundaram and S. Somasundaram [4] and they defined by effective edge. Domination in fuzzy graphs by strong edge it was discussed by A. Nagoorgani and V. T. Chandrasekaran [5] Anti fuzzy structures on graphs has been introduced by Muhammad Akram [6] and discussed the concepts of self-centroid anti fuzzy graphs and constant anti fuzzy graphs and other concepts. on anti fuzzy graph and domination on anti fuzzy graph has been introduced by R. Muthuraj and A. Sasireka [7, 8] Antipodal anti fuzzy graph has been discussed by Seethalakshmi, R.B. Gnanajothi [9]. Split domination in Fuzzy graph has been introduced by Q. M. Mahioub and N.D Soner [10]. In this paper, we introduce the concept of Split anti fuzzy domination on Anti Fuzzy Graph. Some theorems are discussed and suitable examples are given.

2. Basic Definitions:
2.1. Definition [6]: Let η: V→ [0, 1] and ρ: V× V→ [0, 1], then GAF = (η, ρ) is known as anti fuzzy Graph if ρ (u_1, u_2) ≥ η (u_1) ∨ η (u_2) ∀ u_1, u_2 ∈ V and is denoted by GAF = (η, ρ) and ∨ refer to maximum.

2.2. Definition [6]: G_A* = (η*, ρ*) is known as underlying crisp graph of GAF = (η, ρ)
Where \( \eta^* = \{ w \in V / \eta (w) > 0 \} \) and \( \rho^* = \{ (u, w) \in V \times V / \rho (u, w) > 0 \} \).

Note: \( \rho \) is taken into account as reflexive and symmetric. For each example, \( \eta \) is selected suitably. i.e., only undirected \( G_{AF} \) are studied.

2.3. Definition [7]: The size \( S \) and order \( P \) of \( G_{AF} = (\eta, \rho) \) are defined to be
\[
S = \sum_{u \in V} \rho (u, v) \\
P = \sum_{v \in V} \eta (v),
\]
Denoted by \( S(G_{AF}) \) and \( O(G_{AF}) \) respectively.

2.4. Definition [8]: \( G_{AF} \) is complete if \( \rho (u, w) = \max \{ \eta (u), \eta (w) \}, \forall u, w \in \eta^* \) and it is denoted by \( K_{\eta} \).

2.5. Definition [9]: The complement of \( G_{AF} = (\eta, \rho) \) is an anti-fuzzy graph such that: \( \eta = \overline{\eta} \), and
\[
\rho(x, y) = 1 - \rho (u, w) + \max \{ \eta (u), \eta (w) \} \text{ for all } \rho (u, w) \in E.
\]

2.6. Definition [8]: The effective edge \( e = (u, w) \) in \( G_{AF} \) is defined as if \( \rho (u, w) = \max \{ \eta (u), \eta (w) \} \).

2.7. Definition [8]: Let \( w \) be a vertex in \( G_{AF} \), \( N (w) = \{ u : (w, u) \text{ is an effective edge} \} \) is known as The Neighbourhood of \( w \), \( N[w] \cup \{ w \} \) is known as the closed neighbourhood of \( w \).

2.8. Definition [6]: The \( G_{AF} = (\eta, \rho) \) is connected if there exist a fuzzy path between any two vertices of \( G_{AF} \).

2.9. Definition [12]: The \( G_{AF} = (\eta, \rho) \) is a strong anti fuzzy graph if \( \rho (u, w) = \max \{ \eta (u), \eta (w) \}, \forall \rho (u, w) \in \rho^* \).

2.10. Definition [12]: The \( v \)-nodal in \( G_{AF} \) is defined as every vertex has equal fuzzy values. i.e \( \eta (x) = k, \forall x \in V(G_{AF}) \).

2.11. Definition [12]: The \( e \)-nodal in \( G_{AF} \) is defined as every edge has equal fuzzy values. i.e. \( \rho (x, y) = k \forall (x, y) \in E(G_{AF}) \).

2.12. Definition [12]: The uninodal in \( G_{AF} \) is defined as for every vertices and edges in \( G_{AF} \) have equal fuzzy values i.e. \( \eta (x) = k = \rho (x, y) \).

2.13. Definition [13]: Let \( A \subseteq V(G_{AF}) \) is known as an anti-fuzzy vertex cover of \( G_{AF} \) if for each effective edge \( e = (u, w) \), at least (one) of \( u \), \( w \) is in \( A \). The maximum anti-fuzzy cardinality of anti-fuzzy vertex cover is known as anti-fuzzy vertex covering number of \( G_{AF} \) and is represented by \( \alpha_0(G_{AF}) \).

Note: If \( e = (v, w) \) is an effective edge in an anti fuzzy graph \( G_{AF} \), then we say that \( v \) and \( e \) cover each other.

2.14. Definition: A vertex \( w \) is known as an isolated vertex if \( \rho (w, u) > \eta (w) \lor \eta (u) \forall u \in V-\{w\} \).

2.15. Definition: Let \( S \subseteq V(G_{AF}) \) is known as the independent anti-fuzzy set if
\[
\begin{align*}
\rho (w, u) &= 0 \quad \forall u, w \in S \text{ such that } \rho (w, u) \not\in E(G_{AF}) \\
\rho (w, u) &> \eta (w) \lor \eta (u) \forall u, w \in S \text{ such that } \rho (w, u) \in E(G_{AF})
\end{align*}
\]

2.16. Definition: An independent anti – fuzzy set \( S \) of \( G_{AF} \) is called the maximal independent anti-fuzzy set if there is no independent anti- fuzzy set \( S^* \) of \( G_{AF} \) such that \( |S^*| > |S| \).
2.17. Definition: The maximum fuzzy cardinality over all maximal independent anti fuzzy set of $G_{AF}$ is known as the independence number of $G_{AF}$ and is denoted by $\beta_0 (G_{AF})$.

2.18. Definition: Two vertices $u_1$ and $u_2$ of $G_{AF}$ dominate each other if $\rho (u_1, u_2) = \max \{\eta (u_1), \eta (u_2)\}$.

2.19. Definition: A vertex subset $D$ of $V(G_{AF})$ is known as anti-fuzzy dominating (AFD) set of $G_{AF}$ if for each vertex $u_1 \in V - D$ there exists a vertex $u_2 \in D$ such that $u_2$ dominates $u_1$. The AFD set $D$ of $G_{AF}$ is called minimal AFD set of $G_{AF}$ if no proper subset $D'$ of $D$ is AFD of $G_{AF}$.

2.20. Definition: The maximum fuzzy cardinality among all minimal AFD set of $G_{AF}$ is called the anti fuzzy domination number and is denoted by $\gamma (G_{AF})$.

3. Split anti fuzzy Domination of $G_{AF}$.

In this section the SAFD set and split anti fuzzy domination number on $G_{AF}$ are defined, uninodal anti fuzzy graph is discussed, and these concepts are studied on some kinds of simple $G_{AF}$.

3.1. Definition: AFD set $D$ of $G_{AF}$ is known as SAFD set of $G_{AF}$ if the induced anti fuzzy subgraph $< V - D >$ is disconnected.

3.2. Definition: The SAFD set $D$ of $G_{AF}$ is known as minimal SAFD set of $G_{AF}$ if no proper subset $D'$ of $D$ is SAFD set of $G_{AF}$.

3.3. Definition: The maximum fuzzy cardinality among all minimal SAFD set of $G_{AF}$ is known as the split anti fuzzy domination number of $G_{AF}$ and is denoted by $\gamma (G_{AF})$.

3.4. Example: Consider $G_{AF}$ in Figure 1. Such that $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$ and $\rho (u, v) = \eta (u) \vee \eta (v) \forall (u, v) \in E (G_{AF})$

We see that the vertex subset $D_1 = \{v_2, v_3, v_4, v_5\}, D_2 = \{v_1, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$, $D_3 = \{v_4, v_5, v_{10}, v_{11}, v_{12}, v_{13}\}$ and $D_4 = \{v_2, v_3, v_6, v_7, v_8, v_9\}$ are minimal SAFD Set of $G_{AF}$ and hence, $\gamma (G_{AF}) = \max \{|D_1|, |D_2|, |D_3|, |D_4|\} = \max \{1, 4.3, 2.8, 2.3\} = 4.3$

Observation 3.1: A minimal SAFD set of $G_{AF}$ with $|D| = \gamma (G_{AF})$ is denoted by $\gamma^{-1} (G_{AF})$.

3.5. Preposition: Let anti fuzzy graph $G_{AF} = K_\eta$ then SAFD set does not exist.

3.6. Preposition: Let $G_{AF} = K_{2, \eta}$ a star anti fuzzy graph then $\gamma (K_{1, \eta}) = \eta (v)$, $v$ is a root vertex.

3.7. Preposition: Let $G_{AF} = K_{n1,n2}$ be a complete anti fuzzy bipartite graph where $|V_1| = m$ and $|V_2| = n$ where $m = \sum \eta (v), v \in V_1$ and $n = \sum \eta (v), v \in V_2$ then $\gamma (K_{n1,n2}) = \max \{m, n\}$. 

Figure 1
3.8 Theorem: Let $D$ be a (SAFD) set of $G_{AF}$ be a minimal SAFD set of $G_{AF}$ if and only if for every vertex $u_2 \in D$ one of the next situations holds:

(a) There exists a vertex $u_1 \in V - D$ such that $N(u_1) \cap D = \{u_2\}$;
(b) $u_2$ is an isolated in $D$;
(c) $<V - D>\cup\{u_2\}$ is connected.

Proof: Consider $D$ is a minimal SAFD of GAF and $u_2 \in D$ such that $u_2$ does not satisfy any one of the three situations, Then by (a) and (b) $D^* = D - \{u_2\}$ is AFD set of GAF and by condition (c) $<V - D^*$ > is disconnected. This implies that $D^*$ is a minimal SAFD set of $G_{AF}$; this is a contradiction with minimalist $D$. Therefore, for every vertex $u_2 \in D$ satisfies one of the above conditions.

Conversely, assume that for every vertex $u_2 \in D$ one of the above situations holds. Further, if $D$ is not minimal, then there exists a vertex $u_2 \in D$ such that $D - \{u_2\}$ is SAFD set of $G_{AF}$ and there exists a vertex $u_1 \in D - \{u_2\}$ such that $u_1$ dominates $u_2$. That is $u_1 \in N(u_2)$. Therefore, $u_2$ does not satisfy the conditions (b) and (c), thus it must satisfy the condition (a). Then there exists $u_1 \in V - D$ such that $N(u_1) \cap D = \{u_2\}$. Since $D - \{u_2\}$ is a SAFD set of $G_{AF}$, then there exists $h \in D - \{u_2\}$ such that $h \in N(u_1)$. Therefore, $h \in N(u_1) \cap D$, $h \neq u_2$, is a contradiction with $N(u_1) \cap D = \{u_2\}$. Clearly, $D$ is a minimal SFD set for $G_{AF}$.

3.9. Theorem: The AFD set $D$ of $G_{AF}$ is a (SAFD) set of $G_{AF}$ if and only if there exist $u_1, u_2 \in V - D$ such that every $u_1-u_2$ path contains a vertex of $D$.

Proof: Suppose that $D$ is a minimal SAFD set of $G_{AF}$, then $<V - D>$ is disconnected, take $u_1, u_2 \in V - D$ such that every $u_1-u_2$ path-joining $u_1$ and $u_2$ must contain a vertex of $D$. Conversely, assume that $u_1, u_2 \in V - D$ such that every $u_1-u_2$ path contains a vertex of $D$. Let $D$ be an AFD set of $G_{AF}$, $<V - D>$ either connected or disconnected. $<V - D>$ is connected, then for any two vertices $u_1, u_2 \in V - D$ there is a $u_1-u_2$ path joining $u_1$ and $u_2$ in $<V - D>$ which does not contain a vertex of $D$, this impossible with our assumption. Therefore, $D$ is a SAFD set of $G_{AF}$.

3.10. Preposition: Let $G_{AF} = (\eta, \rho)$ be a strong anti fuzzy graph and $D$ be a $\gamma_{SAF}(G_{AF})$ – set of $G_{AF}$, then $V - D$ is AFD set of $G_{AF}$.

Proof: Assume that $D$ is a minimal SAFD set of $G_{AF}$. If $V - D$ is not AFD set of $G_{AF}$, then there exists $w \in D$ which does not dominate any vertex of $V - D$. Thus $D^* = D - \{w\}$ is a SAFD set of $G_{AF}$, this is a contradiction, therefore $V - D$ is AFD set of $G_{AF}$.

3.8. Preposition: For any strong anti fuzzy graph $G_{AF} = (\eta, \rho)$,

$$\gamma_{AF}(G_{AF}) + \gamma_{SAF}(G_{AF}) \leq |P|.$$

Proof: Let $D$ be a $\gamma_{SAF^{-}}$ set of $G_{AF}$, thus from Preposition 3.3, $V - D$ is AFD set of $G_{AF}$. Therefore $\gamma_{AF}(G_{AF}) \leq |V - D| = |P| - \gamma_{SAF}(G_{AF})$. Hence $\gamma_{AF}(G_{AF}) + \gamma_{SAF}(G_{AF}) \leq |P|$. □

3.11. Preposition: Let $D$ be a $\gamma_{SAF^{-}}$ set of $G_{AF}$, if $<D>$ is disconnected anti fuzzy subgraph of $G_{AF}$, then $\gamma_{SAF}(G_{AF}) \leq |P|/2$.

Proof: Let $D$ be a $\gamma_{SAF^{-}}$ set of $G_{AF}$, thus $V - D$ is AFD set of $G_{AF}$, since $<D>$ is disconnected, then $V - D$ is a SAFD set of $G_{AF}$. Therefore, $\gamma_{SAF}(G_{AF}) \leq |V - D| = |P| - \gamma_{SAF}(G_{AF})$. Hence $\gamma_{SAF}(G_{AF}) \leq |P|/2$. □

3.12. Preposition: For any anti fuzzy graph $G_{AF} = (\eta, \rho)$, $\gamma_{AF}(G_{AF}) \leq \gamma_{SAF}(G_{AF})$;
Proof: from definitions of $\gamma_{Af}(G_{Af})$ and $\gamma_{SAf}(G_{Af})$. □

3.13. Preposition: $V - A$ is a SAFD set of strong anti fuzzy graph $G_{AF} = (\eta, \rho)$ If A is maximal Independent anti fuzzy set of $G_{AF}$.

Proof: Since A is maximal independent anti fuzzy set of strong anti-fuzzy graph $G_{AF}$, then $V - A$ is AFD set of $G_{AF}$. Further $< A > = < V - (V - A) >$ is disconnected. This implies $V - A$ is a SAFD set. □

3.14. Theorem: A set $S_{i} \subseteq V(G_{AF})$ is independent anti fuzzy set of $G_{AF}$ if and only if $V(G_{AF}) - S_{i}$ is an anti-vertex covering of $G_{AF}$.

Proof: Let $S_{i}$ be an independent anti-fuzzy set of $G_{AF}$. By the definition of independent anti fuzzy set, there exist no effective edge between any two vertices in $S_{i}$, thus no edges of $G_{AF}$ has at least one end in $S_{i}$ Then $V(G_{AF}) - S_{i}$ contains at least one end for every edge, Hence $V(G_{AF}) - S_{i}$ is an anti-vertex covering of $G_{AF}$. And similarly if $S_{C}$ is anti-vertex covering then it is clear that $V(G_{AF}) - S_{C}$ is independent anti-fuzzy Set. □

3.15 Theorem: If $G_{AF}$ is an anti-fuzzy graph, then $P \leq \alpha_{0} + \beta_{0}$, where $\alpha_{0}, \beta_{0}$ are anti-fuzzy covering number and independence number respectively.

Proof: Let $G_{AF}$ be an anti-fuzzy graph. Let $S_{i}$ be a maximal anti independent set and $S_{C}$ be an anti-vertex covering of $G_{AF}$. By theorem 3.3, we get $V(G_{AF}) - S_{C}$ is an anti-independent set of $G_{AF}$. Hence $|V - S_{C}| \leq |S_{i}| \Rightarrow P - \alpha_{0} \leq \beta_{0} \Rightarrow P \leq \alpha_{0} + \beta_{0}$. □

3.16 Theorem: Let $G_{AF} = (\eta, \rho)$ be a uninodal anti-fuzzy graph then $\gamma_{SAf}(G_{AF}) \leq \alpha_{0}(G_{AF})$, where $\alpha_{0}(G_{AF})$ is a vertex covering number of $G_{AF}$.

Proof: Let $A$ be a maximal independent anti-fuzzy set of $G_{AF}$, then it is contains at least two vertices and for each vertex $u \in A$ there exists $w \in V - A$ such that $\rho(u, w) = \eta(u) \lor \eta(w)$. Thus $V - A$ is a SAFD set of $G_{AF}$. Hence $\gamma_{SAf}(G_{AF}) \leq |V - A| = P - \beta_{0}(G_{AF}) = \alpha_{0}(G_{AF})$. □

3.17. Theorem: Let $G_{AF} = (\eta, \rho)$ be any anti fuzzy graph with end-vertex, $\gamma_{Af}(G_{AF}) = \gamma_{SAf}(G_{AF})$. Furthermore, there exists a SAFD set of $G_{AF}$ containing all vertices adjacent to anti fuzzy end-vertices.

Proof: Suppose that $D$ is AFD set of $G_{AF}$ and $v$ be an end vertex of $G_{AF}$, then there exists a cut vertex $u$ adjacent to $v$ and $\rho(u, v) = \eta(u) \lor \eta(v)$. Assume that $u \in D$, then $D$ is a SAFD set of $G_{AF}$, if $u \in V - D$ then $v \in D$ Hence $D - \{v\} \cup \{u\}$ is SAFD set. Repeating this process for all such cut-vertices adjacent to end-vertices, we obtain a SAFD set of $G_{AF}$ containing all cut-vertices adjacent to end-vertices of $G_{AF}$. □

3.18 Theorem: Let $G_{AF} = (\eta, \rho)$ be any anti fuzzy graph, then $\gamma_{SAf}(G_{AF}) = t$, $t \in [0, 1]$, $t = \eta(w)$, $w \in V(G_{AF})$ if and only if GAF has only one cut vertex $w \in V(G_{AF})$ which has $n - 1$ neighbors of vertices.

Proof: Assume that $D = \{w\}$ is a $\gamma_{SAf}$ set of $G_{AF}$, thus $< V - \{w\} >$ is disconnected. Hence $v$ is a cut vertex of $G_{AF}$, so $N(w) = \{V - \{w\}\}$ then $w$ has $n - 1$ neighbors in $G_{AF}$. Assume that there exists another cut vertex say $u$ in $G_{AF}$ which has $n - 1$ neighbors in $G_{AF}$, then $u$ is adjacent to all remaining vertices of $G_{AF}$. In this case $< V - \{w\} >$ is connected, this is a contradiction. Then $w$ is only the cut vertex of $G_{AF}$ has $n - 1$ neighbors in $G_{AF}$.
Conversely, assume that \( w \) is only one cut vertex of \( G_{AF} \) has \( n-1 \) neighbors in \( G_{AF} \), then \( w \) is adjacent to all vertices of \( G_{AF} \). Hence there exists \( u \in V - \{w\} \), \( u \neq w \) which it is not adjacent with other vertex of \( V - \{w\} \), the \( <V - \{w\}> \) is disconnected. Thus \( \{w\} \) is SAFD set of \( G_{AF} \) and hence \( \gamma_{SADF}(G_{AF}) = t, t = \eta(w) \).

\[3.1.9. \text{Theorem:} \] Every SAFD set of \( G_{AF} = (\eta, \rho) \) is a split dominating set in crisp graph \( G_{A^*} = (\eta^*, \rho^*) \).

\[\text{Proof:} \] Let \( D \) be a SAFD set of \( G_{AF} = (\eta, \rho) \) then for each vertex \( u \in V - D \) there exist \( w \in D \) such that \( \rho(u, w) = \eta(u) \vee \eta(w) > 0 \), and \( <V - D> \) is disconnected. Thus \( \rho(u, w) \in \mu^* \), hence each vertex in \( V - D \) is dominated by at least one vertex in \( D \) and \( <V - D> \) is disconnected, thus \( D \) is a split dominating set in \( G_{A^*} = (\eta^*, \rho^*) \).

\[\text{Note:} \] The convers theorem 3.8 is not true.

\[3.1.9.1. \text{Example:} \] Let \( G_{A^*} = (\eta^*, \rho^*) \) and \( G_{AF} = (\eta, \rho) \), be a crisp graph of \( GA \) and anti fuzzy graph are considered in figure (2) and figure (3) respectively.

\[\text{Figure.2 crisp graph (} G_{A^*} \text{)} \]

\[\text{Figure.3 anti fuzzy graph (} G_{AF} \text{)} \]

We see that the split dominating set in crisp graph \( G_{A^*} = (\eta^*, \rho^*) \), \( D = \{x, u\} \) which is not a split anti fuzzy dominating set in anti fuzzy graph \( G_{AF} = (\sigma, \mu) \).

\[4. \text{Conclusion} \]

In this work, we studied (SAFD) set and a split anti fuzzy domination number of an anti-fuzzy graph \( (G_{AF}) \). For some standard an anti-fuzzy graphs, we found the exact value of \( \gamma_{SADF}(G_{AF}) \). In addition, we got some relationships between split anti-fuzzy domination number and for some parameters.

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