SINGULAR DIVISORS AND SYZYGIES OF POLARIZED ABELIAN THREEFOLDS

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Abstract. In this paper we provide a geometric condition for polarized abelian threefolds \((X, L)\) to have simple syzygies, that is, to satisfy property \((N_p)\) and the vanishing of the Koszul cohomology groups \(K_{p, 1}\) for a given natural number \(p \geq 0\). As in [KL15], we use the reduction method of [LPP11] with Newton–Okounkov body techniques, this time augmented by inversion of adjunction techniques from birational geometry and those related to Fujita’s conjecture, together with the idea of differentiation of sections developed in [ELN94]. As a by-product, we are able to construct effective divisors on \(X\) of an ample class with high self-intersection whose singularities are all concentrated on an abelian subvariety. This can be thought as the dual picture of the classical problem about singularities of theta divisors from [EL97].

1. Introduction

The deep connection between the geometric data of a subvariety of a projective space and the defining equations of the embedding in terms of its syzygies has been known since classical times. But, aside from sporadic efforts, it was not until the work of M. Green [G84] that a coherent general picture started to emerge.

Green’s way of encoding syzygies was to reinterpret them in terms of certain cohomology groups. More precisely, he associated the so-called Koszul cohomology groups \(K_{p, q}(X, L, B)\) (for all \(p, q \in \mathbb{N}\)) to the data \((X, L, B)\), where \(X\) is a projective variety, \(L\) a very ample line bundle on \(X\) defining an embedding \(X \subseteq \mathbb{P}(H^0(X, L)) = \mathbb{P}^N\)

and \(B\) an arbitrary line bundle on \(X\).

Out of Green’s work two syzygetic properties have grown in prominence due to their connection to the geometry of the triple \((X, L, B)\). First, property \((N_p)\), introduced by Green–Lazarsfeld [GL86], amounts to asking that the section ring \(R(X, L)\) as a module over \(\text{Sym}^d H^0(X, L)\) has linear syzygies up to step \(p\). Second, there seems to emerge a partially conjectural picture on the asymptotic vanishing of the Koszul cohomology groups \(K_{p, 1}(X, L; dL) = 0\) for \(d \gg 0\) (for a nice overview about these ideas see the recent survey of Ein–Lazarfeld [EL16] or the more standard reference by now [PAG, Section 1.8]).

Both questions have been reasonably well understood in the case of curves. For property \((N_p)\) some milestones are the results of Green [G84], Green–Lazarsfeld [GL86], and Voisin [V02]. For the vanishing of \(K_{p, 1}\) the main theorem is the recent work of Ein–Lazarfeld [EL15]. In both cases these syzygetic properties can be translated into very geometric properties that in essence are of Riemann-Roch type.

In higher dimensions the picture is very murky and it’s not clear what are the geometric properties that have an impact on the syzygetic picture. For the vanishing of Koszul groups \(K_{p, 1}(X, L, dL)\) for \(d \gg 0\) recent work of Ein–Lazarfeld–Yang [ELY15] and Agostini [A17] seem to suggest some conjectural picture.

One class of projective varieties where property \((N_p)\) has been successfully studied are abelian varieties. Extrapolating from elliptic curves, Lazarsfeld conjectured that if \(L\) is an ample line bundle on a complex abelian variety, then \(L^\otimes m\) should have property \((N_p)\) whenever \(m \geq p + 3\). This was proven in due course
by Pareschi [P00] and generalized by Pareschi-Popa in [PP03]. Note that this kind of statement seems somewhat incomplete. It cannot account for those line bundles that are primitive, but have for instance large intersection numbers. More importantly, it doesn’t really connect the geometric data to the syzygetic one, for example by revealing those subvarieties having the strongest impact on the syzygetic side of the ambient space.

In recent years there has been some progress in obtaining more precise estimates for property $(N_p)$ of abelian varieties. First, motivated by work of Hwang–To [HT11], Lazarsfeld–Pareschi–Popa [LPP11] reduced verifying $(N_p)$ to constructing singular divisors and then go on linking it to the Seshadri constant.

Building on [LPP11], the author with Küronya [KL15] use the theory of infinitesimal Newton–Okounkov bodies, as developed in [KL14, KL17], and provide a geometric characterization of $(N_p)$-property on abelian surfaces in terms of non-existence of elliptic curves of low degree. Very recently, Ito [It17] streamlined the proof of [KL15] by making use of properties of log canonical centers instead. We point out that in and of themselves both proofs, [KL15] and [It17], are specific to surfaces.

Parallel to the above, [AKL17] obtained analogous (and numerically better) results for K3 surfaces. The methods employed here however are very different. They rely on the work of Aprodu–Farkas [AF11] and Voisin [V02] on Green’s conjecture for curves lying on a K3 surface. Interestingly, a unified proof for all surfaces with numerically trivial canonical bundle is still missing.

Here we continue the line of research initiated in [KL15]. The goal of this paper is to show a similar result as in [KL15] in the case of abelian threefolds, both for property $(N_p)$ and the vanishing of the Koszul cohomology groups $K_{p,1}(X,L;dL)$ for any $d \geq 2$. In addition to the reduction procedure of [LPP11] and Newton–Okounkov bodies theory, we rely on inversion of adjunction techniques, inspired by [EL93, AS95, Ka97, H97], aided by differentiation techniques pioneered in algebraic geometry by Ein–Lazarsfeld–Nakamaye [ELN94] and asymptotic theory of linear series as developed in [ELMNP06].

Our main result goes as follows.

**Theorem 1.1.** Let $(X,L)$ be a complex polarized abelian threefold such that $(L^3) > 59 \cdot (p+2)^3$ for some integer $p \geq 0$. Assume the following conditions:

1. For any abelian surface $S \subseteq X$ one has $(L^2 \cdot S) > 4 \cdot (p+2)^2$;
2. For any elliptic curve $C \subseteq X$ one has $(L \cdot C) > 2 \cdot (p+2)$.

Then the pair $(X,L)$ satisfies property $(N_p)$ and $K_{p,1}(X,L;dL) = 0$ for any $d \geq 2$.

Property $(N_0)$, or, as it is classically known, projective normality, implies automatically that $L$ is very ample. The strongest numerical result on very ampleness of line bundles on abelian threefolds to date is [BMS16, Corollary 1.5] due to Bayer–Macrì–Stellari, and it is obtained as a consequence of their work on Bridgeland stability conditions on abelian threefolds. As the sequence of properties $(N_p)$ is best considered as increasingly stronger algebraic versions of positivity of line bundles starting with base point freeness and very ampleness, our main result can be seen as a natural generalization of [BMS16, Corollary 1.5].

On the other hand the vanishing of the Koszul cohomology groups $K_{p,1}$ seems to becoming slowly better understood. With the work of Ein–Lazarsfeld–Yang [ELY15] it became clear that local positivity properties do have an effect upon the asymptotic vanishing of these groups in all dimensions. Our statement can be seen, at least for abelian threefolds, as an improvement or a step forward of this circle of ideas. On a philosophical level we are able to translate information about the local positivity of a line bundle into easily computable global conditions in terms of intersection numbers and special abelian subvarieties.
Abelian subvarieties play a manifestly important role in the geometry of abelian varieties. A surprising by-product of our work yields the existence of effective \(\mathbb{Q}\)-divisors in an ample class whose singularities are all concentrated on an abelian subvariety of the ambient space. More concretely:

**Corollary 1.2.** Let \(X\) be an abelian threefold and \(B\) an ample \(\mathbb{Q}\)-divisor on \(X\) with \((B^3) > 40\). Then there exists an effective \(\mathbb{Q}\)-divisor

\[ D \equiv_{\text{num}} B, \]

whose singularities are all concentrated on an abelian subvariety of \(X\). More concretely, the cosupport of the multiplier ideal \(J(X,D)\) as a subset of \(X\) is either an abelian surface, an elliptic curve or the origin.

To put it in a different light, this result is related to the classical problem of singularities of theta divisors. Generalizing an earlier idea of Kollár, Ein–Lazarsfeld [EL97] proved that any effective \(\mathbb{Q}\)-divisor numerically equivalent to the theta divisor of an abelian variety is log canonical. Since theta divisors possess the smallest self-intersection number among all ample classes of an abelian variety, Corollary 1.2 turns the picture upside down in that it looks at ample classes with large self-intersection on an abelian threefold.

Next, we will outline the strategy of the proof of both results. The basic idea follows the path laid down in [KL15]: after a series of reduction steps coming from [LPP11] and the recent work of Ito [It17], we are left finding effective \(\mathbb{Q}\)-divisors whose log-canonical center is zero-dimensional.

To this end, consider the blow-up \(\pi : X' \to X\) of \(X\) at the origin \(0 \in X\) with exceptional divisors \(E\). We will simultaneously work on \(X\) and \(X'\). In order to be able to find our singular divisors, we study the function

\[ t \in \mathbb{R}_+ \to B(\pi^* L - tE) \subseteq X'. \]

Our goal is to study the variation of the stable base locus \(B(\pi^* L - tE)\) and of the multiplicities along its irreducible components as a function of \(t\).

Inspired by work of Faltings in diophantine geometry, Ein–Lazarsfeld–Nakamaye [ELN94] have pioneered the idea of differentiating sections in algebraic geometry. Based on [N96], we first show that the multiplicity function of any irreducible component of \(B(\pi^* (L) - tE)\) has slope at least one, when varying "\(t\)". Furthermore, if there is a non-degenerate subvariety contained in a base locus, the fact that the multiplicity grows fast forces \(B(\pi^* (L) - tE)\) to start containing pretty quickly also sums of this subvariety until finally \(\pi^* (L) - tE\) is not anymore pseudo-effective. In terms of infinitesimal Newton–Okounkov bodies the behavior of the multiplicity function and the existence of a non-degenerate subvariety in the base locus translate into strong conditions on its vertical slices and horizontal width. In particular, the volume of these convex sets is bounded from above under these conditions. But since, these convex sets in \(\mathbb{R}^3\) encode how all the sections of all the powers of \(L\) vanish along a flag of subspaces of tangency directions at the origin \(0 \in X\), their Euclidean volume is equal to \(L^3/6\), and thus quite big, by our assumptions. This provides us with a contradiction.

However, the method above cannot be used for example when our base loci consists only of abelian subvarieties. Furthermore the intersection numbers it provides are quite big. For this reason inversion of adjunction type techniques come well in hand. First, they force the base loci to contain non-trivial components with high multiplicity for not so large "\(t\)". Second, if the intersection numbers of \(L\) with respect to these loci are quite large, then by changing divisors we can cut these loci and get to a zero-dimensional one. Third, once "\(t\)" grows inversion of adjunction forces the Seshadri constant \(\varepsilon(L; 0)\) to become smaller and smaller. So, by [KL17] this imposes again strong conditions on the shape of infinitesimal Newton–Okounkov bodies.
In order to prove our main statements, we use simultaneously differentiation and inversion of adjunction techniques together with interesting phenomena about infinitesimal base loci $B(\pi^*L - tE)$ on abelian manifolds. This is enhanced by seeing all this data through the lenses of Newton-Okounkov body theory, both local and infinitesimal. The main reason for using these convex sets is that it allows not only to understand better intersection numbers, but they also can be viewed as a powerful computational tool.

Much of the notation and definitions in this article follow closely the standard textbooks \cite{Har} and \cite{PAG}.

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2. Preliminary results

In this section we present in detail the preliminary results that will be used in the proof of our main results. Some parts of the discussion below might be known to the experts, but for the benefit of the reader we will try to be as concise and as complete as possible. We don’t include here the definitions and main properties of (asymptotic) multiplier ideals, since their theory is well presented in \cite{PAG}.

2.1. Asymptotic multiplicity, multiplier ideals and base loci. The theory of asymptotic multiplier ideals for big effective $\mathbb{Q}$-divisors will play an important role in our paper. In \cite{PAG} this theory was developed only for line bundles. Even though there is not much difference between the two, we will state the main properties of the asymptotic multiplier ideals for rational classes that will be necessary for later use. Furthermore, we will show how they can be deduced from the same theory for line bundles.

Let $X$ be a smooth complex projective variety, $D$ an effective big $\mathbb{Q}$-Cartier divisor on $X$. Let $m > 0$ be a positive integer such that $mD$ is an integral Cartier divisor. The asymptotic multiplier ideal of $D$

$$J(X, \|D\|) \overset{\text{def}}{=} \mathcal{J}(X; \frac{1}{m} \cdot \|mD\|),$$

where the right-hand side is the appropriate asymptotic multiplier ideal of the line bundle $\mathcal{O}_X(mD)$.

**Lemma 2.1.** Let $X$ be a smooth projective variety and $D$ an effective big $\mathbb{Q}$-Cartier divisor on $X$.

(1) The ideal sheaf $\mathcal{J}(X, \|D\|)$ is a numerical invariant and does not depend on the choice of $m > 0$.

(2) For any $D' \equiv D$ effective $\mathbb{Q}$-divisor, we have the inclusion of ideals

$$\mathcal{J}(X, D') \subseteq \mathcal{J}(X; \|D\|).$$

Let $D'' \in |mD|$ be a general element for some $m \in \mathbb{N}$ sufficiently large and divisible. Then for $D' = \frac{1}{m}D''$ the inclusion turns out to be an equality.

**Proof.** (1) For those $m_1, m_2 > 0$ that make sense, Theorem 11.1.8 from \cite{PAG} yields the equalities

$$\mathcal{J}(X; \frac{1}{m_1} \cdot \|m_1D\|) = \mathcal{J}(X; \frac{1}{m_1m_2} \cdot \|m_2m_1D\|) = \mathcal{J}(X; \frac{1}{m_2} \cdot \|m_2D\|),$$

Together with Example 11.3.12 from \cite{PAG}, these imply the statement.
Taking \( m \gg 0 \) and divisible enough, we have the following inclusions
\[
\mathcal{J}(X; D') \subseteq \mathcal{J}(X; \frac{1}{m} \cdot |mD'|) = \mathcal{J}(X; \frac{1}{m} \cdot |mD'|) = \mathcal{J}(X; \frac{1}{m} \cdot |mD|) = \mathcal{J}(X; |D|) .
\]
The first inclusion is implied by Proposition 9.2.32 from \[\text{PAG}\]. The equalities follow in order from Proposition 11.1.4 and Example 11.3.12 from \[\text{PAG}\] and finally from the definition. The inclusion becomes an equality exactly as in our statement is due to Proposition 9.2.26 from \[\text{PAG}\]. \( \square \)

### 2.2. Base loci of numerical classes.

On abelian manifolds our syzygetic properties can be translated, by using \[\text{LPP11}\], to the problem of finding effective divisors with special singularities. On the other hand, these singularities are located in some base loci of the numerical class. So, based on \[\text{ELMNP06}\], we give a brief introduction to these base loci and their basic properties.

Let \( D \) be a big \( \mathbb{Q} \)-divisor on a complex smooth projective variety \( X \). The stable base locus of \( D \) is the Zariski-closed set
\[
B(D) \overset{\text{def}}{=} \bigcap_{m \gg 0} Bs(mD) ,
\]
where the intersection is taken over all \( m \gg 0 \) such that \( mD \) is a Cartier divisor and the set \( Bs(mD) \) denotes the base locus of the linear system \( |mD| \). It turns out that this locus is not a numerical invariant.

With this idea in mind, the authors of \[\text{ELMNP06}\] introduced approximations of this base locus that are now numerical invariants and encode much of the bad loci of the numerical class of \( D \). First, the augmented base locus of \( D \) is the closed set
\[
B_+(D) \overset{\text{def}}{=} B(D - \frac{1}{m}A) ,
\]
where \( A \) is an ample class and \( m \gg 0 \). Second, the restricted base locus of \( D \) is given by
\[
B_-(D) \overset{\text{def}}{=} \bigcup_{m > 0} B(D + \frac{1}{m}A) .
\]
Note that we have the inclusions \( B_-(D) \subseteq B(D) \subseteq B_+(D) \). Furthermore, \( B_+ \) and \( B_- \) don’t depend on the choice of the ample class \( A \) and are numerical invariants of \( D \). Finally, \( B_-(D) \) might not be Zariski closed. Lesieutre in \[\text{Le14}\] constructs divisors \( D \), based on the Cremona transformation of the projective plane, where \( B_-(D) \) is not closed but is a countable union of subvarieties. In our setup we will see that all these base loci can be assumed to be equal. So, any strange phenomenon does not interfere with our computations.

To measure how bad a numerical class \( D \) vanishes along a subvariety \( V \subseteq X \) we use the multiplicity. So, the asymptotic multiplicity of \( D \) along \( V \) is defined to be
\[
\text{mult}_V(|D|) = \lim_{m \to \infty} \frac{\text{mult}_V(|mD|)}{m} .
\]
By \[\text{ELMNP06}, \text{Proposition 2.8}\], it turns out that \( \text{mult}_V(|D|) > 0 \) if and only if \( V \subseteq B_-(D) \).

Throughout the paper we will be using quite often implicitly the following lemma.

**Lemma 2.2.** Let \( \varepsilon > 0 \) be some real number. Then the following two conditions are equivalent:

1. \( \text{mult}_V(|D|) \geq \varepsilon \).
2. \( \text{mult}_V(D') \geq m \varepsilon \) for all \( m \gg 0 \) and divisible enough and any generic choice of \( D' \in |mD| \).

**Proof.** By Lemma 3.3 from \[\text{ELMNP06}\], we have the following equality
\[
\text{mult}_V(|D|) = \inf_{D'} \text{mult}_V(D') ,
\]
where the minimum is over all effective \( \mathbb{Q} \)-divisors \( D' \equiv D \). Furthermore, by definition \( \text{mult}_V(|mD|) = \text{mult}_V(D') \) for a generic \( D' \in |mD| \). Applying these two ideas and the definition of asymptotic multiplicity, we can deduce easily the statement. \( \square \)

2.3. **Base loci and singular divisors on the blow-up.** The problem of obtaining singular divisors with special properties at a point is linked to the variation of base loci on the blow-up of this point.

Let \( X \) be a smooth projective variety and \( x \in X \) a point. Denote by \( \pi : X' \to X \) the blow-up of \( X \) at \( x \) with exceptional divisor \( E \). Let \( B \) be an ample \( \mathbb{Q} \)-class on \( X \). We denote by

\[
B_t \overset{\text{def}}{=} \pi^*(B) - tE, \quad \text{for any } t \geq 0.
\]

To this data we can associate two invariants. First, the **Seshadri constant**

\[
\varepsilon(B;x) = \inf_{x \in C} \frac{(B \cdot C)}{\text{mult}_x(C)} = \sup \{ t > 0 \mid B_t \text{ is nef (ample)} \} > 0.
\]

The second invariant, called the **infinitesimal width** of \( B \) at \( x \), is given by the formula

\[
\mu(B;x) \overset{\text{def}}{=} \sup \{ t > 0 \mid B_t \text{ is } \mathbb{Q} \text{-effective} \} < \infty.
\]

Notice that \( \mu(B;x) \geq \varepsilon(B;x) \).

It is worth pointing out here, that it is useful for us to define these invariants for singular ambient spaces. We will do so in Section 3.2 and this will play an important part in the proof of our main results.

Based on the previous subsection it is then natural to ask about the behavior of the following functions

\[
t \in (0, \mu(B;x)] \rightarrow B_-(B_t), B_+(B_t) \subseteq X'.
\]

Since ample divisors don’t have any base loci, then both functions are null on the segment \((0, \varepsilon(B;x))\). It remains to study the problem in the rest of the interval. One interesting lemma is that in this infinitesimal setup all these three loci agree for almost all \( t \).

**Lemma 2.3. (Behavior of infinitesimal base loci).**

(a) The function \( t \to B_-(B_t) \) or \( B_+(B_t) \) is increasing with respect to inclusion of the image.

(b) Let \( t_0 \geq 0 \) be some positive real number. Then there exists \( 0 < \delta \ll 1 \) such that

\[
B_-(B_t) = B(B_t) = B_+(B_t)
\]

is constant for any \( t \in (t_0, t_0 + \delta) \).

**Remark 2.4.** This statement shows that there exists at most countably many points where these base loci can jump and where their behavior can be quite complicated. Presently, we don’t know if the points where the jumping happens has infinitely many accumulation points, finitely many or they accumulate only at \( \mu(B;x) \). In dimension 2 by [KLM12] there are at most finitely many of these points due to Zariski-Fujita decomposition for pseudo-effective divisors.

**Proof.** (a) Since \( E \) is effective, the definition of the stable base locus implies that the function \( t \to B(B_t) \) is increasing. Again by the definition, the same can be said about the other two base loci.

(b) Without loss of generality assume \( t_0 \geq \varepsilon(L;x) \). Let \( r \in (0, \varepsilon(B;x)) \) be some rational number. For any \( 0 < \lambda \ll 1 \) we have the following equalities of sets

\[
B_+(B_{t_0}) = B(B_{t_0} - \lambda B_r) = B_-(B_{t_0} - \lambda B_r) = B(B_{t_0} - \frac{\lambda x}{1 - \lambda}).
\]
Since $B_r$ is ample, then the definition of augmented base locus implies the first equality. The function $t \rightarrow B(B_r)$ is increasing, so the definition of the restricted base yields the second one. Finally, our asymptotic base loci don’t change by taking powers of the class, so we also get the third one.

Since $t_0 - \lambda \cdot r > (1 - \lambda)t_0$, we have the equalities

$$B_+ (B_{t_0} ) = B(B_{t_0} + \delta) = B_+ (B_{t_0} + \delta) ,$$

for any $0 < \delta \ll 1$. So, we finish the proof. \hfill $\square$

Based on these properties about the infinitesimal variation of base loci, we can now try to understand the behavior of the asymptotic multiplicity function along subvarieties on the blow-up.

Before doing so, let’s introduce some notations. For any subvariety $\overline{Y} \subseteq X'$ let $m_{\overline{Y}} : [0, \mu(B;x)] \rightarrow \mathbb{R}_+$ be the multiplicity function of the class $B$ at $x$ along $\overline{Y}$ given by the formula

$$m_{\overline{Y}}(t) \overset{\text{def}}{=} \text{mult}_{\overline{Y} } (||\pi^*(B) - tE||) .$$

It is also important to encode the point $t$ when this function start being non-zero. For this we define

$$t_{\overline{Y}} \overset{\text{def}}{=} \min \{ t > 0 \mid \overline{Y} \subseteq B_+ (B_{t}) \} .$$

Without any sort of confusion, for a subvariety $Y \subseteq X$ we denote by $t_Y$ and $m_Y$ the above quantities computed for the proper transform $\overline{Y}$ of $Y$ on the blow-up $X'$.

With this notation in hand, in the following lemma we show that the multiplicity function is convex.

**Lemma 2.5. (Convexity of infinitesimal multiplicity function).** Let $\overline{Y} \subseteq X'$ be a subvariety. Then the function $m_{\overline{Y}} : [0, \mu(B;x)] \rightarrow \mathbb{R}_+$ is continuous, increasing and convex.

**Proof.** The increasing nature of the multiplicity function is due to the fact that $E$ is effective.

For the convexity property let $0 < t_0 \leq t_1 \leq \mu(B;x)$ and $s \in (0, 1)$. Then we have the following inequalities

$$(1 - s)m_{\overline{Y}}(t_0) + s \cdot m_{\overline{Y}} (t_1) \geq \text{mult}_{\overline{Y} } (||((1 - s)B_{t_0} + sB_{t_1})||) = m_{\overline{Y} }((1 - s)t_0 + st_1) .$$

The first inequality is due to the rescaling and convexity property known for the function

$$\xi \in N^1(X')_{\mathbb{R}} \rightarrow \text{mult}_{\overline{Y} } (||\xi||) \in \mathbb{R} ,$$

from Remark 2.3 and Proposition 2.4 in [ELMNP06]. So, we finish the proof. \hfill $\square$

### 2.4. Inversion of adjunction.

An important ingredient in our proofs is inversion of adjunction. They form a powerful tool in algebraic geometry due to their ability to transport certain geometric properties held on a subvariety to the whole ambient space. Most of the material here is inspired by [PAG] and [E97]. For the benefit of the reader we present here the main results with complete proofs.

As usual, let $X$ be a smooth projective variety and $D$ be an effective $\mathbb{Q}$-divisor on $X$. The log-canonical threshold of $D$ is

$$\text{lct}(D) = \inf \{ c > 0 \mid \mathcal{J}(X;c \cdot D) \neq \emptyset_X \} .$$

The divisor $D$ is called log-canonical if $\text{lct}(D) = 1$. The locus of log-canonical singularities of $D$ is

$$\text{LC}(D) \overset{\text{def}}{=} \text{Zeroes} (\mathcal{J}(X;cD)) ,$$

where $c = \text{lct}(D)$. It is often useful to be able to assume that the LC-locus of a divisor is irreducible. The next lemma asserts that this is the case after arbitrarily small perturbation of the divisor.
Lemma 2.6. (Tie-breaking trick). Let $D$ be an effective $\mathbb{Q}$-divisor on $X$ and fix $c = \lct(D)$. There exists an effective divisor $E \equiv B$, where $B$ is an ample class, such that for any $0 < \varepsilon \ll 1$ there are rational numbers $0 < c_\varepsilon < c$ and $t_\varepsilon > 0$ so the lc-locus of the divisor $D_\varepsilon \overset{\text{def}}{=} c_\varepsilon D + t_\varepsilon E$ is an irreducible normal subvariety of $\text{LC}(D)$. Moreover, $c_\varepsilon \to c$ and $t_\varepsilon \to 0$ whenever $\varepsilon \to 0$.

Proof. This statement is a global analogue of Lemma 10.4.8 from [PAG] and the proof is exactly the same, so we don’t reproduce it here. Normality instead follows from Theorem 1.6 from [Ka97].

In the following we say that a subvariety $Z \subseteq X$ is a critical variety of $D$ if $Z$ is the lc-locus of some divisor $D_\varepsilon$ as in Lemma 2.6. By the proof of Lemma 10.4.8 from [PAG], $Z$ can be chosen not to depend on the choice of $0 < \varepsilon \ll 1$. Although, it might not be unique.

Ito in [It17, Remark 2.1] remarked the following consequence when the critical variety is zero-dimensional.

Lemma 2.7. Let $B$ be an ample $\mathbb{Q}$-divisor on $X$. Let $D \equiv B$ with $c \overset{\text{def}}{=} \lct(D) < 1$ so that
\[
\dim(\text{LC}(D)) = 0.
\]
Then there exists an effective $\mathbb{Q}$-divisor $D' \equiv c' B$ with $0 < c' < 1$ and $\mathcal{J}(X, D') = \mathfrak{m}_x$ for some $x \in X$.

Proof. Without loss of generality we can assume that $D$ is log-canonical. Construct $D'$ out of $D$ as in Lemma 2.6. Let $x \in X$ be the unique point in the co-support of $\mathcal{J}(X, D')$. Let $\pi' : Y \to X$ be a log-resolution of $D'$, obtained by blowing up smooth centers on $X$. Due to its properties any irreducible exceptional divisor $F$ on $Y$ that counts in the construction of $\mathcal{J}(X, D')$ is contracted to $x$ by $\pi'$ and has coefficient $-1$. In particular, $\pi'$ factors out through the blow-up $\pi : X' \to X$ at $x$ and exceptional divisor $E$. Furthermore, any such $F$ maps to $E$. So, taking any $f \in \mathfrak{m}_x$, then $\mu_F(\pi'(f)) \geq 1$ as we have automatically $\mu_F(\pi'(f)) \geq 1$. In particular, $\mathfrak{m}_x \subseteq \mathcal{J}(X, D')$ and since the latter is not-trivial it is actually an equality.

An important property of LC-loci is that by changing the divisor and under certain numerical conditions we can cut down the sets. This is the basic idea of inversion of adjunction and this kind of arguments are tremendously important for inductive arguments and more.

In the following we explain the idea in the case when the LC-locus is of small dimension. Some of this might work more generally, but this will be enough for our means. These statements might be clear to the seasoned expert, but for completeness we include here the proofs.

Proposition 2.8 (Cutting down the LC locus I). Let $B$ be an ample class on $X$ and $D \equiv cB$ be a log-canonical effective $\mathbb{Q}$-divisor for some $0 < c < 1$ whose LC-locus $V$ is an irreducible smooth curve. If
\[
(1 - c) \cdot (B \cdot V) > 1,
\]
then there exists an effective $\mathbb{Q}$-divisor $D' \equiv (1 - c) B$ satisfying the property that
\[
\dim(\text{LC}((1 - \delta)D + D')) = 0
\]
for any $0 < \delta \ll 1$.

Proof. By Riemann-Roch on curves there exists for any $0 < \xi \ll 1$ an effective divisor $D_\xi \equiv B|_V$ with
\[
\mu_\xi(D_\xi) \geq (B \cdot V) - \xi
\]
at a (very general) point $x \in V$. 
Choose now a natural number $m \gg 0$ and divisible enough. Then, we can assume that $mD_V$ is a Cartier divisor, $mB$ defines a very ample line bundle with $mD_V^\xi \in |mB|_V$ and the restriction map on global sections

$$H^0(X, \mathcal{O}_X(mB)) \to H^0(V, \mathcal{O}_V(mB))$$

is surjective. We can assume further that the twisted ideal sheaf $\mathcal{I}_{V,X}(mB)$ is globally generated. This implies that the linear series of divisors

$$\{D_m^\xi \in |mB| \text{ where } D_m^\xi|_V = mD_V^\xi\}$$

has no base points on $X \setminus V$. Let $D^\xi_m \overset{\text{def}}{=} \frac{1 - \epsilon}{m} D_m^\xi$ for some general choice of $D_m^\xi \in |mB|$ with $D_m^\xi|_V = m D_V^\xi$. Then we have the equality of ideal sheaves

$$\mathcal{I}(X, (1 - \delta)D + D^\xi)|_{X \setminus V} = \mathcal{I}(X, (1 - \delta)D)|_{X \setminus V} = \mathcal{O}_{X \setminus V},$$

for any $0 < \delta \ll 1$, by applying Example 9.2.29 from [PAG] and $D$ is globally log-canonical. By the same reasoning, the above equalities take place also at those points $y \in V$ that are not contained in $\text{Supp}(D_V^\xi)$.

It remains to show that $\mathcal{I}(X, (1 - \delta)D + D^\xi)$ is non-trivial at $x$. As $V$ is a smooth curve, then $\text{mult}_x(D^\xi) > 1$ whenever $\xi \ll 1$. The statement is then implied by Proposition 10.4.10 from [PAG].

Another interesting way of cutting down the log-canonical locus on abelian three-folds is by making use of the Seshadri constant, which we know are quite big for line bundles on these ambient spaces by [N96].

**Proposition 2.9 (Cutting down the LC locus II).** Let $B$ be an ample class on $X$ and $D \equiv cB$ be a log-canonical effective $\mathbb{Q}$-divisor for some $0 < c < 1$. Suppose for some point $x \in X$ we have

$$\text{mult}_x(D) + (1 - c)\epsilon(B; x) > \dim(X).$$

Then there is an effective divisor $D' \equiv (1 - c)B$ with $\text{LC}((1 - \delta)D + D') = \{x\}$ for any $0 < \delta \ll 1$.

**Proof.** The inequality in the statement is still valid if we tweak it a little bit, say as follows

$$(1 - \delta)\text{mult}_x(D) + (1 - c)(\epsilon(B; x) - \delta) > \dim(X),$$

for some $0 < \delta \ll 1$. Take $m \gg 1$ and divisible enough. Then the linear system

$$\{D_m \in |mB| \mid \text{mult}_x(D_m) \geq m(\epsilon(B; x) - \delta)\}$$

is base point free on $X \setminus \{x\}$. Setting $D' = \frac{1 - \epsilon}{m} D_m$, then we have

$$\mathcal{I}(X, (1 - \delta)D + D')|_{X \setminus \{x\}} = \mathcal{I}(X, (1 - \delta)D)|_{X \setminus \{x\}} = \mathcal{O}_{X \setminus \{x\}},$$

for any $0 < \delta \ll 1$, by Example 9.2.29 from [PAG]. The latter equality follows as $D$ is log-canonical. So, all this implies that $\mathcal{I}(X, (1 - \delta)D + D')$ vanishes at most at $x$. But the first inequality in the proof together with the choice of the divisor $D'$ yield

$$\text{mult}_x((1 - \delta)D + D') > \dim(X),$$

for any $0 < \delta \ll 1$.

By Proposition 9.3.2 from [PAG], then $\text{LC}((1 - \delta)D + D') = \{x\}$. □
3. Convex geometry and the infinitesimal picture of threefolds

The goal of this section is to introduce infinitesimal Newton–Okounkov bodies on threefolds and discuss how base loci affect the shape of these convex sets. Finally, we discuss infinitesimal data on singular surfaces.

Before doing so, we fix some notation used throughout this section. Let $X$ be a complex projective smooth threefold, $B$ an ample $\mathbb{Q}$-class on $X$ and $x \in X$ be a point. Let $\pi : X' \rightarrow X$ be the blow-up of $X$ at $x$ with the exceptional divisor $E \cong \mathbb{P}^2$. The infinitesimal width of the Seshadri constant of $B$ at $x$ we denote by

$$\mu \overset{\text{def}}{=} \mu(B;x) \quad \text{and} \quad \varepsilon \overset{\text{def}}{=} \varepsilon(B;x).$$

With this in hand, we can proceed further.

3.1. Infinitesimal Newton–Okounkov bodies. By now the theory of Newton–Okounkov bodies is somewhat standard, so we refrain from a detailed exposition. Instead, we refer the interested reader to the original sources [KK12, LM09] and the survey articles [B, KL]. The connection between Newton–Okounkov bodies and local positivity of line bundles is treated in detail in [CHPW15, KL14, KL15a, KL17, KL, Ro16].

First, we will give a short exposition to this theory for the infinitesimal picture in the case of threefolds. With the notation as above fix an infinitesimal flag

$$Y_* : Y_0 = X' \supseteq Y_1 = E \supseteq Y_2 = l \supseteq Y_3 = \{Q\},$$

where $l \subseteq E \cong \mathbb{P}^2$ is a line and $Q \in l$ is a point. We say that the flag $Y_*$ is generic if $l$ is a generic line in $\mathbb{P}^2$ and $Q \in l$ is a generic point on the line.

To this data we can associate the infinitesimal Newton–Okounkov body

$$\tilde{\Delta}_{Y_*}(B) = \text{convex hull}\{v_{Y_*}(D) \mid \text{effective } D \equiv B\} \subseteq \mathbb{R}^3.$$

The valuation $v_{Y_*}(D) = (v_1, v_2, v_3)$ encodes how the divisor $\pi^*(D)$ vanishes along the flag $Y_*$. In more details, it is defined inductively. First, set $v_1 = \text{mult}_x(D)$. Then, automatically we have

$$E \not\subseteq \text{Supp}(\pi^*(D) - v_1 E).$$

In particular, the restriction $D_1 = (\pi^*(D) - v_1 E)|_E$ makes sense as an effective divisor. So, finally set

$$v_2 = \text{mult}_l(D_1) \quad \text{and} \quad v_2 = \text{mult}_Q ((D_1 - v_2 l)|_l).$$

In particular, our convex set encodes how all the effective divisors in the numerical class of $B$ vanish along certain tangency directions.

In the following the inverted simplex of length $\sigma$ is the convex set described as

$$\Delta_{\sigma}^{-1} \overset{\text{def}}{=} \{(t, u, v) \in \mathbb{R}^+_3 \mid 0 \leq t \leq \sigma, 0 \leq u + v \leq t\} \subseteq \mathbb{R}^+_3,$$

In order to make use of the power of the theory of these convex sets we need to discuss first its properties.

**Theorem 3.1. (Properties of infinitesimal Newton–Okounkov bodies).** With notation as above one has

1. The Euclidean volume $\text{vol}_{\mathbb{R}^3}(\Delta_{Y_*}(B)) = B^3/6$.
2. The inclusion $\tilde{\Delta}_{Y_*}(B) \subseteq \Delta_{\mu}^{-1}$ and the intersection $\tilde{\Delta}_{Y_*}(B) \cap \{\mu\} \times \mathbb{R}^2 \neq \varnothing$.
3. The Seshadri constant $\varepsilon(B;x) = \max\{|\sigma| \Delta_{\sigma}^{-1} \subseteq \tilde{\Delta}_{Y_*}(B)\}$ and does not depend on the choice of the flag $Y_*$. 
4. When the flag $Y_*$ is taken to be generic, then $[0, \mu] \times (0, 0) \subseteq \tilde{\Delta}_{Y_*}(B)$. 
The first statement is the main result of [LM09]. The second and third statement follows directly from Proposition 2.6 and Theorem C from [KL15a]. Applying simultaneously Lemma 2.3 together with Theorem 2.1 from [KL15a] we deduce the last statement.

Based on Theorem 3.1, it’s worth pointing out that we have a full understanding of the convex set $\tilde{\Delta}_Y$ in the region $[0, \varepsilon] \times \mathbb{R}^2$. More mysterious is its behavior in the region $(\varepsilon, \mu] \times \mathbb{R}^2$.

We discuss this in the following proposition on three-folds.

**Proposition 3.2. (Conditions on the infinitesimal Newton–Okounkov bodies).** Let $x \in V \subseteq X$ be a subvariety with $m_Y(t) = m_t > 0$ for some $t \in (\varepsilon, \mu)$. Fix $Y$, a generic infinitesimal flag.

1. If $V$ is curve, then
   \[ \tilde{\Delta}_Y(B) \cap \{ t \} \times \mathbb{R}^2 \subseteq \text{convex hull}\{(t,0,0), (t,0,t), (t-t_m,m_t), (t,m_t,0)\} \, . \]

2. If $V$ is a surface with $m = \text{mult}_Y(Y) \geq 1$, then
   \[ \tilde{\Delta}_Y(B) \cap \{ t \} \times \mathbb{R}^2 \subseteq \text{convex hull}\{(t,0,0), (t,0,t-m \cdot m_t), (t,t-m \cdot m_t,0)\} \, . \]

**Proof:** We start first by making some remarks. Since the vertical slices $\tilde{\Delta}_Y(B) \cap \{ t \} \times \mathbb{R}^2$ change continuously with $t$, we can assume that all the base loci of $B_t$ agree, by using Lemma 2.3. In particular, $B_-(B_t)$ is a closed proper subvariety. Furthermore, since the line $l$ and the point $Q$ were chosen to be generic then they will not be contained in this base locus. Finally, let $\overline{V}$ be the proper transform of $V$ by the blow-up morphism.

1. Fix a point $P \in \overline{V} \cap E \neq \emptyset$, which will be automatically distinct from $Q$ and not contained in $l$. The definition of multiplicity along a subvariety yields
   \[ \text{mult}_P(||B_t||) \geq m_t \, . \]

Take now any effective $\mathbb{Q}$-divisor $D \equiv B$ with $m_Y(D) = t$. Denote by $\overline{D}$ the proper transform of $D$. Note, first that $\overline{D} \equiv B_t$, and the valuation vector $v_Y(D) = (t,a,b)$ for some $a,b \in \mathbb{Q}$.

By the proof of Lemma 2.2, the inequality above implies $\text{mult}_P(\overline{D}) \geq m_t$. Restricting it to the exceptional divisor this yields $\text{mult}_P(\overline{D}|_E) \geq m_t$. As $P \notin l$, then it is not hard to see that

\[ a \leq t - m_t \, . \]

Before proving this, note that, by the definition of infinitesimal Newton–Okounkov bodies, this inequality would imply the inclusion in the statement.

So, it remains to prove (3.2.1). We assume the opposite that this does not happen, meaning that we have the inequality $a > t - m_t$. In this case $R \equiv \overline{D}|_E$ is an effective $\mathbb{Q}$-divisor of degree $t$ on $\mathbb{P}^2$. Taking into account the definition of the valuation $v_*$, we can then write

\[ R = al + R', \text{ where } l \not\subset \text{Supp}(R') \]

The properties of $R$ and the fact that $P \notin l$ yield

\[ \text{mult}_P(R') \geq m_t \text{ and } \deg(R') = t - a < m_t \, . \]

So, by Bézout theorem on $\mathbb{P}^2$, we realize that such an effective divisor $R$ cannot exist in the projective plane. Thus, we get our contradiction and prove our statement in this case.

2. Let $D \equiv B$ be an effective $\mathbb{Q}$-divisor with $m_Y(D) = t$. It defines then a valuation vector
   \[ v_{E_*}(\pi^*(D)) = (t,a,b) \, , \]
where \( a, b \in \mathbb{Q} \). Denote by \( \overline{D} \) the proper transform of \( D \) on \( X' \). By Lemma 2.2, we can then write
\[
\overline{D} = D' + (m_t + x)\overline{\nu}, \quad \text{where} \quad \overline{\nu} \notin \text{Supp}(D'),
\]
for some \( x \geq 0 \). Denote by \( C = \overline{\nu} \cap E \) and we consider it with its scheme structure. Thus \( C \) is a curve of degree \( m \) in the plane \( E \cong \mathbb{P}^2 \). The decomposition of the divisor \( D \) above yields then
\[
\mult_C(\overline{D}|_E) \geq m_t + x.
\]
The flag \( Y_\ast \) was chosen to be generic, so \( l \) and \( P \) are not contained in \( C \). Now, to compute \( a \) and \( b \) we use the restricted divisor \( R' = D'|_E \). With this in hand, the above inequality implies
\[
\deg(R') \leq t - m(m_t + x).
\]
Finally, using Bézout theorem and the above inequality we then get
\[
a + b \leq t - m(m_t + x).
\]
Varying now \( D \) in its class and using the definition of infinitesimal Newton–Okounkov bodies the above inequality implies the inclusion in the statement. This finishes the proof. \( \square \)

3.2. **Infinitesimal data on singular surfaces.** Later on it will become very important to use the geometry of those subvarieties appearing in some infinitesimal base locus. Unfortunately, they might not be smooth and the existence of many invariants can break down. However, considering infinitesimal data at a smooth point, much of our invariants transfer well from the subvariety to its desingularization.

We explain these ideas for surfaces. Let \( S \) be an irreducible projective surface, \( A \) an ample line bundle on \( S \) and \( x \in S \) a smooth point. Let \( \pi : S' \to S \) be the blow-up at \( x \) with \( E \cong \mathbb{P}^1 \) the exceptional divisor, which is Cartier on \( S' \). In particular, the definitions of \( \varepsilon(L;x) \) and \( \mu(L;x) \) make sense when \( S \) is just irreducible.

In practice we will need to be able to transfer the infinitesimal data from \( S \) to a desingularization \( \overline{S} \), since things work better in the smooth case. So, consider the following diagram:

\[
\begin{array}{ccc}
\overline{S'} & \xrightarrow{\pi} & \overline{S} \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{\pi} & S
\end{array}
\]

where \( f : \overline{S} \to S \) is a resolution of singularities of \( S \) that is an isomorphism in a neighborhood of \( x \). The rest of the diagram is the fibred product. Note that \( \overline{\pi} : \overline{S'} \to \overline{S} \) is the blow-up of the same point \( x \in \overline{S} \).

With this in hand we have the following lemma:

**Lemma 3.3.** Under the assumptions above, we have \( \mu(A;x) = \mu(f^*(A);x) \) and \( \varepsilon(A;x) = \varepsilon(f^*(A);x) \).

**Proof.** The equality between the Seshadri constants follows from the fact that in their definitions it does not matter whether the class \( A_t \overset{\text{def}}{=} \pi^*(A) - tE \) is ample or nef.

For the infinitesimal width, let \( Y_\ast : E \ni z \) be an infinitesimal flag on \( S' \). Denote also by \( Y_\ast \) its corresponding flag on \( \overline{S'} \). The first goal is to show that
\[
\Delta_{Y_\ast}(\pi^*(A)) = \Delta_{Y_\ast}((f' \circ \pi)^*(A))
\]
as sets in \( \mathbb{R}^2 \), whose existence make sense since all the elements in the flag are smooth at \( z \).
As effective Cartier divisors pull-back to effective Cartier divisors and \( f' \) is an isomorphism in a neighborhood of \( E \), then the direct inclusion follows. By Theorem 2.3 from [LM09] both of these sets have the same area equal to \( (A^2) = (f^*(A)^2) \), where the latter uses projection formula. Thus these bodies coincide.

It remains to show that the horizontal width of each convex set is equal to the infinitesimal width of the corresponding line bundle. For \( f^*(A) \) this is clear, as \( S \) is smooth. For \( A \), take an effective \( \mathbb{Q} \)-Cartier divisor \( D \equiv A_t \). Then we have

\[
(D \cdot E) = (A_t \cdot E) = ((\pi^*(A) - tE) \cdot E) = t > 0.
\]

The formula in Theorem 2.8 from Chapter VI of [Ko96], yields that \( \text{Sup}(D) \) intersects \( E \) non-trivially. So, let \( F \overset{\text{def}}{=} \pi_*(D) \), the appropriate linear combination of the images of the irreducible components of \( D \) through \( \pi \). Then \( F \equiv A \) is a \( \mathbb{Q} \)-Cartier divisor, since \( D \) is and \( x \in S \) is a smooth point. The same reasoning implies that \( D = \pi^*(F) - tE \) where the equality is of effective Cartier divisors. In particular, the infinitesimal width can be described only by those divisors coming from downstairs and this ends the proof. \( \square \)

4. Infinitesimal Picture of Abelian Threefolds

In this section we study the infinitesimal picture on abelian threefolds. In the following \( X \) is an abelian threefold, \( 0 \in X \) its origin and \( B \) a \( \mathbb{Q} \)-ample class on \( X \). The aim is to study the behavior of the function \( t \rightarrow \mathbf{B}_+(B_t) \) on the blow-up \( X' \) of the origin, together with the multiplicity functions of the irreducible components of \( \mathbf{B}_+(B_t) \) and their effect on the shape of infinitesimal Newton–Okounkov bodies.

There are a few things that seem to work in favor of abelian manifolds. First, the origin \( 0 \in X \) for many positivity questions behaves like a very general point. Second, since everything can be moved around the Seshadri constant and the infinitesimal width are constant when varying the base point. Third, the tangent bundle of such an ambient space is trivial, allowing us to use differentiating techniques to positivity properties. The latter tells us that the multiplicity function of an irreducible component of \( \mathbf{B}_+(B_t) \) grows fast. Finally, by [D94], any curve of small degree with respect to \( B \) tends to be degenerate, whenever \( B^3 \) is big.

4.1. Small remarks on abelian threefolds. We start first with an easy consequence of Lemma 2.2 about the continuity and convexity property of the multiplicity function.

**Lemma 4.1.** Let \( X \) be an abelian three-fold, \( B \) an ample \( \mathbb{Q} \)-class on \( X \) and \( 0 \neq x \in X \). Then

\[
m_x(t) = \text{mult}_x(||B_t||) \leq t, \text{ for any } t \geq 0,
\]

where we also denote by \( x \) the proper transform of the point by the blow-up morphism.

**Proof.** Due to the definition of Seshadri constants, it is enough to consider the function \( m_x \) only on the interval \( [\varepsilon, \mu] \). So, assume the opposite, i.e. that there exists \( t_1 \in (\varepsilon; \mu) \) with \( m_x(t_1) > t_1 \). In this case we know that the point \( (0, \varepsilon) \) sits on the graph of the function \( m_x \), but the point \( (t_1, t_1) \) does not.

By Lemma 2.5, \( m_x \) is continuous and convex. Thus for \( t \geq t_1 \) the graph of \( m_x \) sits above the line connecting the points \( (0, \varepsilon) \) and \( (t_1, t_1) \). Furthermore, by the same token, \( m_x \) is differentiable everywhere besides at most countably many points. These last two facts imply that there is a point \( t_0 \in (\varepsilon, t_1) \) where the slope of the function \( m_x \) is strictly bigger than 1. Since the graph is convex then the slopes at any points \( (t, m_x(t)) \) for any \( t \geq t_0 \) (or the left and right slope at those where differentiability is a concern) also have this property.

This heuristic explanation forces \( m_x(\mu) > \mu \). So, if \( t \) is very close to \( \mu \), then continuity implies

\[
\text{mult}_x(||B_t||) > \mu(B; 0).
\]
Thus, there exists an effective divisor $D \equiv B$ with mult$_t(D) > \mu$ and so $\mu(B;0) < \mu(B;x)$. Since $X$ is abelian, the function $x \in X \mapsto \mu(B;x) \in \mathbb{R}_+$ is constant. This contradicts our assumption and finishes the proof. □

An important problem in the proofs of our main results is that our log-canonical centers might not pass through the origin. When this is one-dimensional, we cannot simply assume that. Instead in the two-dimensional case, the following lemma will be very useful.

**Lemma 4.2.** Suppose for some $t \in (0, \mu)$ there exists a surface $\overline{S} \subseteq B_+ (B_t)$. Then the origin $0 \in S \overset{\text{def}}{=} \pi(\overline{S})$. Furthermore, under these circumstances either $S$ is an abelian surface or mult$_0(S) \geq 2$.

**Proof.** Assume the opposite that $\overline{S} \cap E = \emptyset$. Tweaking $t$ slightly, we can assume by Lemma 2.3 that

$$B_- (B_t) = B (B_t) = B_+(B_t).$$

In particular, our base loci are all numerical invariants of the class $B_t$. Now, take an effective divisor $\overline{D} \equiv B_t$, which can be written as follows

$$\overline{D} = F + s\overline{S},$$

where $\overline{S} \not\subseteq \text{Supp}(F)$ and some $s > 0$.

Since $0 \not\in S$, then for a general point $x \in X$ we can assume that $x \not\in \pi(B(B_t))$ and $0 \not\in S_0 = S - x$. With this data in hand, the new divisor $\overline{D}' = F + s\pi^* (S_0)$ satisfies the properties

$$\overline{D}' \equiv B_t \text{ and } \overline{S} \not\subseteq \text{Supp}(\overline{D}').$$

The latter condition yields, by the definition of the stable base locus, that $\overline{S} \not\subseteq B(B_t)$, which we assumed to be one equal to the augmented base locus. So, we get our contradiction in this case.

For the second part, assume that mult$_0(S) = 1$. The goal is to show that since the proper transform of this surface appears in some infinitesimal base locus, then this surface must be abelian. Take $x \in S$ be another smooth point and denote by $\overline{S}_t = \pi^* (S_x) - E$ the proper transform of $S_x$. It is easy to see that both $\overline{S}_t$ and $\overline{S}$ are lying in the same numerical class on the blow-up $X'$. So, the same argument as above implies that once $\overline{S} \subseteq B_+ (B_t)$ then automatically $\overline{S}_t \subseteq B_+ (B_t)$, whenever $t < \mu$. If $S$ is not abelian then the closure of all the surfaces $S - x$ for any smooth point $x \in S$ becomes the entire $X$. In particular, this contradicts the fact that $B_t$ is a big divisor, as we have chosen $t < \mu$. Hence $S$ must be abelian, whenever $0 \in S$ is a smooth point. □

A classical result of Debarre says that small degree curves on abelian threefolds are degenerate. Based on this we prove the following proposition:

**Proposition 4.3.** Let $B$ be an ample $\mathbb{Q}$-divisor class on $X$ with $(B^3) > 2$. If $C \subseteq X$ is a curve with $(B \cdot C) \leq 2$ then either $C$ is elliptic or $S \overset{\text{def}}{=} C + C \subseteq X$ is an abelian surface with $(B^2 \cdot S) \leq 2$.

**Proof.** Debarre shows in [D94] that whenever $C + C \subseteq X$, we have

$$(B \cdot C) \geq 3 \cdot \sqrt[3]{\frac{B^3}{6}}.$$

This will not happen if $(B \cdot C) \leq 2$ and $B^3 > 2$, therefore $C$ must be degenerate.

If $C$ is elliptic, then we are done. So, assume that $S \overset{\text{def}}{=} C + C \subseteq X$ is an abelian surface, and restrict the question to $S$. As $C \subseteq S$ is not elliptic, then $(C^2) \geq 2$. The Hodge index theorem then implies

$$(B^2 \cdot S) \cdot (C^2) \leq (B \cdot C)^2.$$

Consequently, $(B^2 \cdot S) \leq 2$, as required. □
4.2. **Differentials and infinitesimal base loci on abelian three-folds.** In this subsection we study the behavior of the base loci and the multiplicity function along irreducible components of the classes $B_t$. This part is mainly based on ideas [ELN94], [EKL95], [N96], and [N05].

Taking into account the notations from Section 2, for any subvariety $\mathcal{Y} \subseteq X'$ let $t_\mathcal{Y}$ be the point $t$ where $\mathcal{Y}$ starts appearing in the base locus $B_+(B_t)$ and let $m_{t_\mathcal{Y}} : [0, \mu] \to \mathbb{R}_+$ be the multiplicity function. As before, if $Y \subseteq X$, then $t_Y$ and $m_Y$ is the same data computed on the proper transform $\mathcal{Y}$ of $Y$ on the blow-up $X'$.

Nakamaye, [N05, Lemma 3.2] for the general case or [N96, Lemma 3.4] for abelian ambient spaces, showed that for any subvariety $Y \subseteq X$ the function $m_Y$ has slope at least one. Based on this we show that the same phenomenon happens for any subvariety $\mathcal{Y} \subseteq X'$ not only those coming from downstairs:

**Proposition 4.4.** *(Differentiation techniques).* With the above notation let $t_1 \in [t_\mathcal{Y}, \mu)$. Then

$$m_{t_\mathcal{Y}}(t) - m_{t_\mathcal{Y}}(t_1) \geq t - t_1 \quad \forall t \geq t_1.$$  

**Proof.** We follow here the ideas developed in [ELN94]. Let's explain a bit the general setup. Let $A$ be an ample line bundle on an abelian manifold $Z$ of dimension $n$. Let $\mathcal{D}_A^k$ be the sheaf of differential operators of order $\leq k$ with respect to $A$. Since $Z$ is abelian, this sheaf has two important properties.

First, applying Lemma 2.5 from [ELN94] to abelian manifolds, yields that the sheaf $\mathcal{D}_A^k \otimes \mathcal{O}_Z((3n+3)A)$ is globally generated for any line bundle $L$ on $Z$ and any $k \geq 1$.

Second, for any effective divisor $R \in |L|$, there is a natural homomorphism of vector bundles $\mathcal{D}_A^k \to \mathcal{O}_Z(L)$, which locally is defined as follows: it takes any differential operator $D$ of order $\leq k$ to $D(f)$ where $f$ is the local function of the divisor $R$.

Based on these properties, for any $m, k > 0$ and any section $R \in |mA|$ we can build a subspace of sections

$$(4.4.2) \quad 0 \neq V_{k,m} \subseteq H^0(Z, \mathcal{O}_Z((m+12)A)),$$

where each section is obtained by differentiating $R$ by a differential operator of order at most $k$.

Going back to our setup we take an asymptotic view on (4.4.2). It implies that for any effective $\mathbb{Q}$-divisor $D \equiv B$, any positive rational numbers $0 < \epsilon \ll \rho$, there exists a new divisor effective $\mathbb{Q}$-divisor

$$\partial^\rho (D) = (1+\epsilon)B$$

constructed by applying to $mD$, for some $m \gg 0$ and divisible enough, a differential operator of order $\leq mp$, and then dividing this new effective divisor by $m$. It is very important to note here, that the choice of $\epsilon$ and $\rho$ are independent. So, by fixing $\rho$ and taking $\epsilon \to 0$, we can consider for our computations that the divisor $\partial^\rho (D) \equiv B$, even if that not be true per se.

Furthermore, since multiplicity behaves linearly with respect to differentiation, we also have that if $Y \subseteq X$ some subvariety then we have the inequality

$$\text{mult}_Y(D) - \text{mult}_Y(\partial^\rho(D)) \geq \rho$$

Putting all these ideas together and considering also the properties of asymptotic multiplicity then this shows the statement in case $Y \subseteq X$ is subvariety distinct from the origin.

For infinitesimal subvarieties $\mathcal{Y} \subseteq E$ we need a slightly more careful analysis. Let $\{u_1, u_2, u_3\}$ be a local system of parameters at $0 \in X$. An effective Cartier divisor $R \geq 0$ can be written locally as a Taylor series

$$P_s(u_1, u_2, u_3) + P_{s+1}(u_1, u_2, u_3) + \ldots$$
where each \( P_i \in \mathbb{C}[u_1, u_2, u_3] \) are homogeneous polynomials of degree \( i \) and \( s = \text{mult}_0(R) \). If \( R \) is the proper transform of the divisor to the blow-up \( X' \) of the origin then we have

\[
R|_E = \text{Zeroes}(P_i) \subseteq E \simeq \mathbb{P}^2.
\]

In this local setup, any differential operator of order at most \( k \) is a homogeneous polynomial of degree at most \( k \) in the following order one differential operators

\[
\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3}.
\]

So, it is not hard to see then that for any differential operator \( \partial^k \) of order at most \( k \), then the proper transform

\[
\frac{\partial^k(R)|_E}{\partial^k(P_i)}, \text{if } s > k.
\]

These ideas and those developed in the first part of the proof yield the same inequality for multiplicity function of infinitesimal subvarieties \( Y \subseteq E \).

4.3. Sums of subvarieties and infinitesimal base loci. An important property of abelian manifolds are that they have a group structure. Based on the previous subsection, especially on Proposition 4.4, we are able to manage well when new sums of subvarieties can show up in the base loci \( B_+(B_t) \).

Let \( Y_1, Y_2 \subseteq X \) be two subvarieties. For our purposes we will assume

\[
y_1 \leqslant y_2 < \mu(B; 0).
\]

We consider now the addition map

\[
(y_1, y_2) \in Y_1 \times Y_2 \rightarrow y_1 + y_2 \in X.
\]

The image, denoted by \( Y_{12} \), will then be an irreducible subvariety of \( X \). The following example studies a few cases of sums of subvarieties that will be helpful later.

**Example 4.5 (Sums of subvarieties of abelian threefolds).**

1. Suppose \( Y_1 \subseteq X \) is a non-degenerate curve, i.e. \( Y_1 + Y_1 + Y_1 = X \). Then Example 2.6 and Corollary 2.7 from Chapitre VIII in [D] implies that \( Y_{12} \subseteq X \) is a surface and \( Y_1 + Y_1 + Y_2 = X \) for any curve \( Y_2 \subseteq X \).

2. Suppose \( Y_1, Y_2 \subseteq X \) are curves and the sum \( Y_{12} \) also remains a curve. Then there exists an elliptic curve \( C \subseteq X \) and two points \( x_1, x_2 \in X \) so that \( x_1 + C = Y_1 \) and \( x_2 + C = Y_2 \). We get this result, by translating both curves to pass through the origin and using that the sum variety \( Y_{12} \) is irreducible.

3. Suppose \( Y \subseteq X \) is a non-abelian surface containing the origin. As \( Y + Y \) and \( Y - Y \) are irreducible and contain \( Y \), then \( Y + Y = Y - Y = X \).

The following lemma gives an interesting criterion when the proper transform of \( Y_{12} \) shows up in the base loci \( B_-(B_t) \), based on the behavior of the multiplicity function of \( Y_1 \).

**Lemma 4.6.** With the above notation, suppose the origin \( 0 \in Y_1 \). Then we have the inequality

\[
t_{Y_{12}} \leqslant t_{12} \overset{\text{def}}{=} \min \{ t > 0 \mid m_{Y_{12}}(t) \geq t_{Y_1} \}.
\]

**Proof.** Fix a rational number \( t' > 0 \) so that \( m_{Y_2}(t') > t_{Y_1} \). For any point \( p \neq 0 \in Y_2 \) we denote by \( p \), without confusion, its inverse image on \( Y_2 \). Note that it’s enough to show that for any \( p \neq 0 \in Y_2 \) the proper transform \( Y''_1 \) of \( Y''_1 = Y_1 + p \), satisfies the property \( Y''_1 \subseteq B_-(B_{t'}) \). Furthermore, \( p \in Y''_1 \), since \( 0 \in Y_1 \).
First, by the proof of Lemma 2.2, the condition \( m_{\Delta}(t) > t_{Y_1} \) implies that for any effective \( \mathbb{Q} \)-divisor \( \overline{D} \equiv B_{t'} \) we have \( \text{mult}_{\overline{T}}(\overline{D}) > t_{Y_1} \). So, using the definition of the multiplicity along a subvariety we then have

\[
\text{mult}_p(\overline{D}) > t_{Y_1} \quad \text{for all } \overline{D} \equiv B_{t'}.
\]

Now, let \( \pi_p : X_p \to X \) be the blow-up of \( X \) at \( p \), with \( E_p \) the exceptional divisor and \( B^p_t \equiv \pi^*_p(B) - tE_p \) for any \( t \geq 0 \). Since \( X \) is abelian then any effective divisor can be moved to any point. So, any effective divisor in the class \( B^p_t \) is the pull-back by the map \( T : X \to X \), where \( T(x) = x - p \), of an effective divisor from the class \( B_t \). In particular, we know that the proper transform of \( Y^p_t \) by \( \pi_p \) is contained in \( B_{t_0}^p \), whenever \( t \geq t_{Y_1} \).

As \( \text{mult}_p(\pi_*(\overline{D})) > t_{Y_1} \), then the proper transform of \( \pi_*(\overline{D}) \) by the map \( \pi_p \) lies in the class \( B_{t_0}^p \), where \( t_0 = \text{mult}_p(\overline{D}) \). Hence, we have

\[
Y^p_1 \subseteq \text{Supp}(\pi_*(\overline{D})) \quad \text{and} \quad \text{mult}_p(\pi_*(\overline{D})) > 0.
\]

Finally, this containment happens for any effective \( \overline{D} \equiv B_t \). Thus, \( \overline{Y}_1 \equiv B(B_{t'} \). Tweaking slightly \( t' \), Lemma 2.3 then implies \( \overline{Y}_1 \equiv B_{B_t} \) for any \( p \in Y_2 \). And we finish the proof. \( \square \)

Based on the above Lemma 4.6 and Proposition 4.4, we can deduce easily the following corollary.

**Corollary 4.7 (Bounds on infinitesimal width).** _Under the above assumptions we have:

1. Let \( 0 \in Y \subseteq X \) be a non-degenerate curve. Then \( t_{Y+Y} \leq 2t_Y \) and \( \mu(B; 0) \leq 3 \cdot t_Y \).
2. If \( Y_1, Y_2 \subseteq X \) two subvarieties as in Lemma 4.6 so that \( Y_1 + Y_2 = X \), then \( \mu(B; 0) \leq \min\{t > 0 \mid m_{\Delta}(t) \geq t_{Y_1}\} \).

*Proof.* Note (2) follows from Lemma 4.6 and (1) from combining Lemma 4.6 with Proposition 4.4. \( \square \)

### 4.4. Infinitesimal Newton–Okounkov bodies on abelian threefolds.

In subsection 3.1 we have showed how to bound the vertical slices of infinitesimal Newton–Okounkov bodies based on knowing the dimension of the base loci \( B_+(B_t) \). Together with Proposition 4.4 this gives indeed very strong conditions on these shapes on abelian threefolds, whenever the flag is taken to be generic.

In some cases, like when \( B_+(B_t) \) consists of curves, whose image have high multiplicity at the origin, the estimations in subsection 3.1 aren’t so strong. Using instead intersection theory and the ideas of differentiation we developed in Proposition 4.4, we are able to find much stronger estimations on abelian three-folds.

We will start first with the following proposition:

**Proposition 4.8.** Let \( X \) be an abelian threefold and \( B \) an \( \mathbb{Q} \)-ample class on \( X \) and \( \Delta \) be a generic infinitesimal flag. Suppose that for some \( t_0 < \mu \) the intersection \( B_+(B_t) \cap E \) contains a curve. Then the function

\[
t \mapsto \text{vol}_{\mathbb{R}^2}(\tilde{\Delta}_+(B) \cap \{t\} \times \mathbb{R}^2)
\]

is decreasing on the interval \((t_0, \mu)\).

*Proof.* Fix a rational number \( t \in (t_0, \mu) \) and denote by \( \tilde{\Delta}_+(B)_t \) the intersection of \( \tilde{\Delta}_+(B) \) with \( \{t\} \times \mathbb{R}^2 \). Applying Proposition 3.2 and Proposition 4.4, for any \( 0 < r < \mu - t_0 \) we then have the inclusion of sets

\[
\tilde{\Delta}_+(B)_t - (r, 0, 0) \subseteq \text{Int}(\Delta^{-1}_t),
\]

where \( \text{Int}(\Delta) \) is the topological interior of a convex set \( \Delta \subseteq \mathbb{R}^3 \).
So, to prove that the function in the statement is decreasing, it suffices then to show the implication
\[ \forall (t, a, b) \in \text{Int}(\hat{\Delta}_r(B)) \implies (t, a, b) - (r, 0, 0) \in \text{Int}(\hat{\Delta}_r(B)). \]
Furthermore, the valutative points of $\hat{\Delta}_r(B)$ form a dense subset. Thus it is enough to show the above implication when $t, a, b, r \in \mathbb{Q}$ and there exists a $\mathbb{Q}$-effective divisor $D \equiv B$ such that $\nu_r(\pi^*(D)) = (t, a, b).

The idea is to show that there exists a differential operator $\partial^r$ of order $r$, such that the divisor $\partial^r(D)$, with both $\partial^r(D)$ and $\partial^r(D)$ defined as in the proof of Proposition 4.4, satisfy the property that
\[ (4.8.3) \quad \nu_r(\pi^*(\partial^r(D))) = (t - r, a, b). \]

Even though $\partial^r(D)$ is not in the numerical class of $B$, this suffices. The reason is that we can take a limiting process, as explained in Proposition 4.4, which will imply $(t - r, a, b) \in \hat{\Delta}_r(B)$ and we would be done.

With this in hand, we need to show (4.8.3), and for this we can assume without loss of generality that all the data is integral: $r, t, a, b \in \mathbb{Z}$ and $D$ is a Cartier divisor. Take $\{u_1, u_2, u_3\}$ a set of local coordinates of the origin $0 \in X$ and we assume that in $E \cong \mathbb{P}^2$ the flag $Y_\bullet$ is defined by $\{u_1 = 0\} \supseteq \{0 : 0 : 1\}$.

In this language, then our effective divisor $D$ can be written locally as a Taylor series
\[ P_1(u_1, u_2, u_3) + P_{t+1}(u_1, u_2, u_3) + \ldots \]
where each polynomial $P_i$ is homogeneous of degree $i$. Since we know that $\nu_r(\pi^*(D)) = (t, a, b)$, then there exists two polynomials $Q \in \mathbb{C}[u_1, u_2, u_3]$ and $R \in \mathbb{C}[u_2, u_3]$, satisfying the properties
\[ P_1 = u_0^t \cdot Q, \quad Q(0, u_2, u_3) = u_1^1 \cdot R(u_2, u_3), \quad u_0^\perp Q_t \quad \text{and} \quad u_1^\perp R_t \]
Taking all these properties into account we can then write
\[ P_1(u_1, u_2, u_3) = u_1^t \cdot u_3^b (c \cdot u_3^{-a-b} + R'_1(u_2, u_3)) + u_1^{a+1} R''(u_1, u_2, u_3), \]
where $c \neq 0$, $R'$ is a degree $t - a - b$ polynomial and $R''$ is of degree $t - a - 1$. Finally, from the above we know that $a + b + r < t$, and thus $\frac{\partial^r(D)}{\partial u_3}(P)$ contains the monomial $u_1^t u_2^r u_3^{-a-b-r}$ with a non-zero coefficient.

With this in hand, we take the differential operator to be $\partial^r = \frac{\partial^r}{\partial u_3}$. Based on the descriptions and properties above, it is not hard to show that the new divisor $\partial^r(D)$ satisfies (4.8.3) and thus we finish the proof. □

As a consequence of the material above, we obtain the following theorem.

**Theorem 4.9.** (Conditions of infinitesimal Newton–Okounkov bodies). Let $X$ be an abelian three-fold, $B$ be an ample $\mathbb{Q}$-divisor on $X$, and $C \subseteq X$ a curve with $\text{mult}_0(C) \geq 2$, so that $\mu(B; 0) > t_C$. Then for a generic choice of infinitesimal flag $Y_\bullet$ on $X'$ and any $t \geq t_C$, we have the inequality
\[ \text{vol}_{\mathbb{R}^2}(\hat{\Delta}_r(B) \cap \{t\} \times \mathbb{R}^2) \leq V_C(t) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{q-1} \frac{t}{t_C}^2, & \text{when } t_C \leq t \leq \frac{q-t_C}{q-1} t_C \\ \frac{t^2}{2}, & \text{when } t \geq \frac{q-t_C}{q-1} t_C \end{cases} \]

**Remark 4.10.** The proof of Theorem 4.9 is inspired by the work of Cascini–Nakayama from [CN14].

**Proof.** Let $t \in (t_C, \mu(B; 0))$ be a rational number, for which Lemma 2.3 holds. Any vector with rational coordinates in $\text{Int}(\hat{\Delta}_r(B))$ is the valuation of a section. Thus, $E \not\subset B(B_t)$. So, Example 4.22 and Theorem 4.24 from [LM09] yield
\[ \text{vol}_{X/E}(B_t) = 2 \cdot \text{vol}_{\mathbb{R}^2}(\hat{\Delta}_r(B) \cap \{t\} \times \mathbb{R}^2), \]
where the left side is the restricted volume, introduced in [ELMNP06].
In [ELMNP06] the authors give a geometric description of the restricted volume given by the formula
\[
\text{vol}_{X'|E}(B_t) = \lim_{m \to \infty} \frac{\#(E \cap D_{m,1} \cap D_{m,2} \setminus B(B_t))}{m^2}
\]
where \(D_{m,1}, D_{m,2} \in |mB_t|\) are two general choice of divisors and the limit is considered for those \(m > 0\) whenever the definitions and data make sense.

First consider the case when \(B_+(B_t)\) contains a curve in the exceptional divisor \(E\) for any \(t \geq t_C\). Then the function \(t \to \text{vol}_{X'|E}(B_t)\) is decreasing for any \(t \geq t_C\) by Proposition 4.8. In particular,
\[
\text{vol}_{X'|E}(B_t) \leq 2V_C(t), \forall t \geq t_C,
\]
as \(V_C(t)\) is an increasing function.

By Lemma 4.2, we are left to consider the case when \(B_+(B_t)\) is one dimensional and has no irreducible components contained in \(E\) for any \(t \in [t_C,t']\) for some \(t' > t_C\). In this setup, then Proposition 4.4 yields
\[
\text{mul}(\mathcal{C}([B_t])) \geq t - t_C \text{ for any } t \in (t_C,t').
\]
Now, for any \(m \gg 0\) and divisible enough choose general divisors \(D_{m,1}, D_{m,2} \in |mB_t|\). The above properties, aided by Bertini’s theorem, implies that the intersection \(D_{1,m} \cap D_{2,m}\) is one dimensional and does not have components contained in \(E\). So, as a cycle we can write
\[
D_{1,m} \cap D_{2,m} = (m^2(t - t_C)^2 + d)\overline{C} + F_m,
\]
for some \(d > 0\), such that \(\overline{C} \notin \text{Supp}(F_m)\). Under these conditions we have the inequality
\[
m^2 \cdot t^2 = (D_{1,m} \cdot D_{2,m} \cdot E) = \left((m^2(t - t_C)^2 + d)\overline{C} + F_m \cdot E\right) \geq q(m^2(t - t_C)^2) + (F_m \cdot E).
\]
Since \(B_+(B_t) = B(B_t)\) by our assumption above, this inequality yields finally
\[
\text{vol}_{X'|E}(B_t) = \lim_{m \to \infty} \frac{\#(E \cap D_{m,1} \cap D_{m,2} \setminus B(B_t))}{m^2} \leq \lim_{m \to \infty} \frac{(F_m \cdot E)}{m^2} \leq V_C(t) = t^2 - q(t - t_C)^2.
\]
If a curve in \(E\) appears also in \(B_+(B_{t'})\) for some \(t'\), then Proposition 4.8 yields that the function \(t \to \text{vol}_{X'|E}(B_t)\) is decreasing for any \(t > t'\). But since the function \(V_C(t)\) is actually increasing, then this function bounds from above the areas of the vertical slices \(\Delta_{V_C}(B_t)\) for any \(t \geq t'\) also. This finishes the proof.

4.5. An inductive approach. Induction plays an important role in our proofs. When \(S \subseteq X\) is an abelian surface, then [KL15] or [It17] develop criteria for existence of singular divisors with certain properties on \(S\). Inversion of adjunction provides ways to extend this divisor to one on \(X\) with the same properties.

**Proposition 4.11. (Inductive process vs. singular divisors).** Let \(X\) be an abelian three-fold, containing an abelian surface \(S \subseteq X\). Suppose \(B\) is an ample \(\mathbb{Q}\)-divisor on \(X\) that satisfies the following two properties:

1. \((B^2 \cdot S) > 4\).
2. For any elliptic curve \(C \subseteq S\) we have \((B \cdot C) > 1\).

If the class \(B - S\) is ample then there exists an effective \(\mathbb{Q}\)-divisor \(D \equiv B + \mathcal{F}(X; D) = m_{x,0}\).

**Proof.** By rigidity lemma any morphism between two abelian varieties is the composition of a translation and a group homomorphism. Thus, there exists a point \(x \in X\) and an abelian surface \(S_0 \subseteq X\), where the embedding preserves the group structure, so that \(S = S_0 + x\).
Furthermore, this also implies that \((S_0 + y) \cap S_0 = \emptyset\) for any \(y \notin S_0\). Since any translation of \(S\) remains in the same numerical class, then these ideas imply
\[(4.11.4) \quad (B \cdot S^2) = (S^3) = (S \cdot C) = 0, \text{ for any curve } C \subseteq S.
\]
In order to understand how to construct \(D\), write the class \(B\) as follows
\[D \equiv \left( \left( (1 - \varepsilon)B - S \right) + (1 - \delta)S \right) + (\varepsilon B + \delta S) = B, \text{ for any } 0 < \delta < \varepsilon < 1.
\]
The goal is to construct out of each term an effective divisor with certain properties.

First, let \(0 < \varepsilon < 1\) be a rational number, so that \((1 - \varepsilon)B\) also satisfies (1) and (2) in the statement and
\[B^\varepsilon \overset{\text{def}}{=} (1 - \varepsilon)B - S \text{ is ample.}
\]

By (4.11.4) the class \(B^\varepsilon\) also satisfies condition (1) and (2). So, applying Proposition 3.1 from [It17] and Lemma 2.6 to the class \(B^\varepsilon|_S\) on \(S\), we deduce the existence of an effective \(\mathbb{Q}\)-divisor \(D_1^S \equiv B^\varepsilon|_S\) with
\[\mathcal{O}(S; (1 - c)D_1^S) = m_{S,0}, \quad \forall \, 0 < c < 1.
\]
Next step is to use the ideas from the proof of Proposition 2.8. Let \(m \gg 0\) and divisible enough so that \(mD_1^S\) on \(S\) and \(mB^\varepsilon\) on \(X\) become Cartier, the divisor \(mB^\varepsilon\) becomes very ample, the restriction map
\[H^0(X, \mathcal{O}_X(mB^\varepsilon)) \rightarrow H^0(S, \mathcal{O}_S(mB^\varepsilon))
\]
is surjective, and the twisted ideal sheaf \(\mathcal{I}_{S,X}(mB^\varepsilon)\) is globally generated. In particular, all these assumptions imply that the non-trivial linear series of divisors
\[\left| V^\varepsilon_m \right| \overset{\text{def}}{=} \{ D^\varepsilon_m \in |mB^\varepsilon|, \text{ where } D^\varepsilon_m|_S = mD_1^S \}
\]
has no base points on \(X \setminus S\).

Let \(D^\varepsilon_1 = \frac{1}{m}D^\varepsilon_m\) for some general \(D^\varepsilon_m \in \left| V^\varepsilon_m \right|\). This choice forces the multiplier ideal \(\mathcal{I}(X, D^\varepsilon_1)\) to be trivial on \(X \setminus S\). Since \(D^\varepsilon_1|_S = D_1^S\), then applying Example 9.5.16 from [PAG] we have the inclusions
\[\mathcal{I}(S; D^\varepsilon_1) \subseteq \mathcal{I}(X; D^\varepsilon_1 + (1 - \delta)S) \subseteq \mathcal{I}(S; (1 - c_1)D^\varepsilon_1)\]
for some \(0 < c_1 < 1\) and any \(0 < \delta < 1\). Thus, \(\text{LC}(D_1 + (1 - \delta)S)\) is zero-dimensional. By Lemma 2.7, changing slightly the divisor we get a new one whose multiplier ideal is the maximal ideal \(m_{X,0}\). Finally, as \(\varepsilon B + \delta S\) is ample, choose a general effective \(\mathbb{Q}\)-divisor \(D_2 \equiv \varepsilon B + \delta S\). By Example 9.2.29 from [PAG], then
\[\mathcal{I}(X; D^\varepsilon_1 + (1 - \delta)S + D_2) = \mathcal{I}(X; D^\varepsilon_1 + (1 - \delta)S) = m_{X,0}.
\]
This finishes the proof of the proposition. \(\square\)

An interesting consequence of the ideas used in the proof of the previous proposition is the following corollary. They play an important role in the proof of Corollary 1.2.

**Corollary 4.12.** If \(S \subseteq X\) is an abelian surface and \(B\) and ample \(\mathbb{Q}\)-divisor on \(X\) such that \(B - S\) is ample, then there is a divisor \(D \equiv B\) with \(\mathcal{I}(X; D) = \mathcal{O}_X(-S)\). If \(B - S\) is not ample then \(3 \cdot (B^2 \cdot S) \geq B^3\).

**Proof.** If we assume that \(B - S\) is ample then the construction of a divisors as above is done as in Example 9.2.29 from [PAG].

In the following let’s assume that \(B - S\) is not ample. Since our ambient space is an abelian manifold then every pseudo-effective divisor is ample. Thus our assumption yields the inequality
\[1 \geq \rho \overset{\text{def}}{=} \max\{t > 0 \mid B - tS \text{ is pseudo-effective}\}.
\]
Based on the theory of Newton–Okounkov bodies from [LM09], we have the identity
\[ \text{vol}_X(B) = 3 \int_0^\rho \text{vol}_{X|S}(B - tS) \, dt. \]
But for any \( t \in [0, \rho] \) the class \( B - tS \) is ample, due to the fact that \( X \) is abelian. So, based on Serre vanishing, and asymptotic Riemann Roch it is not hard to deduce the equalities
\[ \text{vol}_X(B) = B^3 \text{ and } \text{vol}_{X|S}(B - tS) = ((B - tS)^2 \cdot S) \text{ for any } t \in [0, \rho]. \]
As \( S \) is an abelian surface then we can use (4.11.4), and thus deduce that \( \text{vol}_{X|S}(B - tS) = (B^2 \cdot S) \) for any \( t \in [0, \rho] \). Putting together all the ideas above we then deduce easily the inequality in the statement. \( \square \)

Proposition 4.13 is important, whenever we have a singular divisor whose log-canonical locus is an abelian surface. If this locus is a surface \( S \subseteq X \) that is not abelian, we do not expect to have a criteria as above. The simple reason is that the class of \( S \) does not behave numerically as that of an abelian surface.

On the other hand, the infinitesimal width behaves very well in this case, especially when the origin is a smooth point on this surface.

**Proposition 4.13. (Inductive process vs. infinitesimal width.)** Let \( S \subseteq X \) be an irreducible projective non-abelian surface that is smooth at the origin \( 0 \in X \). Then
\[ \mu(B;0) \leq \mu(B|_S;0) \]
for any ample \( \mathbb{Q} \)-class \( B \) on \( X \).

**Proof.** First, fix the ample class \( B \). Since the infinitesimal width is homogeneous of degree one, it suffices to prove the statement when \( B \) is a line bundle on \( X \). In the following, we will suppose that
\[ \mu(B;0) > \mu(B|_S;0) \]
and the goal is to get a contradiction.

Denote by \( S_x \overset{\text{def}}{=} S - x \) for any very general point \( x \in S \). The main idea is to show the inequalities
\[ t_{S_x} \leq \mu(B|_S;x) \leq \mu(B|_S;0). \]
Let’s first see how these inequalities lead to a contradiction. Since \( \mu(B|_S;0) = \mu(B|_S;x) \), they yield
\[ \overline{S}_x \subseteq B_+(B_t) \text{ for any } t > \mu(B|_S;0), \]
where \( \overline{S}_x \) is the proper transform of \( S_x \) by \( \pi \). As \( S \) is not abelian, we then have \( S - S = X \). In particular, the closure of the union of \( S_x \), with \( x \in S \) very general, is \( X \) and consequently \( B_+(B_t) = X \) for any \( t > \mu(B|_S;0) \). This contradicts our initial assumption and the definition of infinitesimal width.

It remains to show the inequalities in (4.13.5). Let’s start with the first one. Choose a rational number \( t_0 > \mu(B|_S;0) \) and let \( D = \pi_x^* (B|_S) - t_0 E_{S_x} \), where \( \pi_x : \overline{S}_x \rightarrow S_x \) is the blow-up of \( S_x \) at 0 and \( E_{S_x} \) is the exceptional divisor. For any \( m \gg 0 \) large and divisible consider the following exact sequence
\[ 0 \rightarrow H^0(X', \mathcal{O}_{X'}(mB_{t_0} - \overline{S}_x)) \rightarrow H^0(X', \mathcal{O}_{X'}(mB_{t_0})) \rightarrow H^0(\overline{S}_x, \mathcal{O}_{\overline{S}_x}(mD)) \]
By the definition of \( \mu(B|_S;0) \) the group on the right vanishes. Thus \( \overline{S}_x \subseteq B(D) \subseteq B_+(D) \) and the left inequality in (4.13.5) follows immediately from the definition of the invariant \( t_{S_x} \).

For the second inequality, by Lemma 3.3, we can assume that \( S \) is smooth everywhere. Then, let
\[ \pi_S : Y = \text{Bl}_{S'} \rightarrow S \times S \]
be the blow-up of the diagonal $\Delta_S \subseteq S \times S$ with $E_\Delta \subseteq Y$ the exceptional divisor. Let $\pi_1, \pi_2 : Y \to S$ be the projection morphisms to each factor.

We study the flat family $\pi_1 : Y \to X$, where for any $x \in S$ the fiber $\pi_1^{-1}(x) = \text{Bl}_x(S)$ is the blow-up of $S$ at $x$. Let $\mathcal{B} = \pi_1^*(B)$ and note that $\mathcal{B}|_{\pi_1^{-1}(x)} = \pi_x^*(B)$, where $\pi_x = \pi_1|_{\pi_1^{-1}(x)} : \text{Bl}_x(S) \to S$ is the blow-up of $S$ at $x$ with the exceptional divisor $E_x = E_S \cap \pi_1^{-1}(x)$. Consider on $Y$ the following $\mathbb{Q}$-Cartier divisor

$$\mathcal{B} - t \cdot E_S, \text{ for any } t \in \mathbb{Q}_+.$$  

Fix $t \in \mathbb{Q}_+$ and suppose the restriction $(\mathcal{B} - t \cdot E_S)|_{\pi_1^{-1}(x')}$ is $\mathbb{Q}$-linear equivalent to an effective divisor for some $x' \in S$. By Theorem III.12.8 from [Har], the same property takes place either for all $x \in S$ or those lying on a countable union of curves and points of $S$ including $x'$. Together with the definition of infinitesimal width this implies the second inequality in (4.13.5). \hfill $\Box$

By Section 4.2 we know that the shape of the Seshadri constant on an abelian threefold affects the magnitude of the infinitesimal width. Geometry will tell us that this connection is much richer.

Let $B$ be an ample class on $X$ and $C \subseteq X$ an irreducible with $q \overset{\text{def}}{=} \text{mult}_0(C) \geq 2$ and denote by $S = C + C \subseteq X$. Assume that $\varepsilon(B; 0) = \frac{(B \cdot C)}{q}$. In the literature these are so called Seshadri exceptional curve. Inspired by Lemma 4.2 from [CN14], we have the following proposition:

**Proposition 4.14.** Under the above notation, assume $S$ is not an abelian surface. Then at least one of the two conditions is satisfied:

1. $\mu(B; 0) \leq \frac{q^2}{q^2 - q + 2} \varepsilon(B; 0)$.
2. $\text{mult}_C(S) \geq 2$.

**Remark 4.15.** It is worth pointing out that in the above two statements we have assumed that $S$ is not abelian. So, it is natural to ask what happens when it is.

Taking a closer look at the proofs of Proposition 4.13 and Proposition 4.14, then whenever $C$ is a Seshadri exceptional curve for $B$ on $X$ with $q \geq 2$, i.e. $C$ is at least not elliptic, then

$$t_S \leq \frac{q^2}{q^2 - q + 2} \varepsilon(B; 0).$$

This is due to two facts. First, the proof of Proposition 4.13 implies that $t_S \leq \mu(B|_S; 0)$ instead of the stronger one with the width. Second, the proof of Proposition 4.14 uses the inequality $C^2 \geq q^2 - q$ from [EL93] applied on any surface. If $S$ is abelian one can use the stronger version $C^2 \geq q^2 - q + 2$ from [KSS09].

**Proof.** Suppose that for a general point $x \in C$ the surface $S$ is smooth at $x$. By Proposition 4.13, in order to prove the first outcome in this case, it is enough to show the inequality

$$\mu(B|_S; x) \leq \frac{q}{q - 1} \varepsilon(B; 0).$$

On $S$ the curve $C_x \overset{\text{def}}{=} C + x$ has the same data at $x$ as the curve $C$ at the origin 0. Since the Seshadri constant increases when restricting to subvarieties, then $\varepsilon(B; x) = \varepsilon(B|_S; x)$. So, it suffices to show the inequality

$$\mu(B|_S; x) \leq \frac{q}{q - 1} \varepsilon(B|_S; x).$$

Taking a resolution of singularities of $S$ that is an isomorphism around $x$, it suffices, By Lemma 3.3, to prove this inequality when $S$ is smooth, the class $A = B|_S$ is big and nef and there is no curve $F \subseteq S$ containing the
point $x$ with $(A \cdot F) = 0$. But in this setup, we have the following inequality
\[ \varepsilon(A; x) \cdot \mu(A; x) \leq (A^2) . \]
Let’s first see how this inequality implies our statement. So, as $q \geq 2$ then $C_x$ moves in a non-trivial family on $S$, even after an appropriate desingularization on $S$. Corollary 1.2 from [EL93] then yields $(C_x^2) \geq q^2 - q$.
Finally, the above inequality and Hodge index imply the following sequence of inequalities
\[ \mu(A; x) \leq \frac{L^2}{\varepsilon(A; x)} \leq \frac{(L \cdot C)^2}{C^2 \cdot \varepsilon(A; x)} \leq \frac{(L \cdot C)^2}{(q^2 - q)\varepsilon(A; x)} \leq \frac{q}{q - 1} \varepsilon(A; x) . \]
So, this would finish the proof.

The above inequality can be seen best from the theory of infinitesimal Newton–Okounkov polygons. Basically, Theorem D from [KL14] shows that any infinitesimal Newton–Okounkov polygon contains the triangle with the vertices $(0,0), (\varepsilon(A; x), 0)$ and $(\varepsilon(A; x), \varepsilon(A; x))$. Since $\mu(A; x)$ is the horizontal width of any of these sets, whose areas is $(A^2)/2$ by Theorem A from [LM09], this implies the inequality. $\square$

5. SYZYGIES VS. SINGULAR DIVISORS ON ABELIAN MANIFOLDS

In this section we discuss the connection between syzygies and singular divisors on abelian manifolds. The $N_p$-property was considered in [LPP11]. For the vanishing of the groups $K_{p,1}(X; L; dL) = 0$ for any $d \geq 1$, we use the ideas from [ELY15] together with those from [LPP11], to show that on any abelian manifold the vanishing of $K_{p,1}$ is also related to singular divisors in the same fashion as in the case of $N_p$-property.

**Theorem 5.1 (Syzygies vs. singular divisors).** Let $X$ be an abelian variety, $L$ an ample divisor on $X$, and $p \geq 0$ an integer. Suppose there exists an effective $\mathbb{Q}$-divisor $F$ on $X$ satisfying the following two properties:

1. $F \equiv \frac{1-c}{p+2} \cdot L$ for some $0 < c < 1$,
2. The multiplier ideal $\mathcal{J}(X, F)$ has zero-dimensional support.

Then $(X, L)$ satisfies property $N_p$ and $K_{p,1}(X; L, B) = 0$, for any line bundle $B$, as long as $B - L$ is ample.

**Remark 5.2.** Using Remark 1.4 and Remark 1.8 from [ELY15], and using the proof of Theorem 5.1, we can prove that under the same above conditions, $L$ is also $p$-jet very ample.

**Proof of Theorem 5.1.** First, we tackle $N_p$-property. By the work of [LPP11] we need to find a divisor
\[ F' \equiv \frac{1-c}{p+2} \cdot L \] with $\mathcal{J}(X; F') = \mathcal{M}_o$.

Take $F$ as in the statement. Then Lemma 2.7 produces another divisor $F''$ with $\mathcal{J}(X; F'') = \mathcal{M}_{X,x}$ for some $x \in X$. Setting $F' = F'' - x$, then $F'$ satisfies the conditions in [LPP11] and we have $N_p$-property.

Moving forward to the other syzygetic property, we can assume already the existence of an effective divisor $F'$ satisfying (5.2.6). The goal is to use this divisor and prove the vanishing of the $K_{p,1}$ group.

Before doing so, we would like to make some notation. Let $Y = X \times X \times (p+1)$ and $\pi_{ij} : Y \to X \times X$ the projection of $Y$ to the $i$ and $j$ factor. Let $\Delta_i = \pi_{ij}^*(\Delta)$, where $\Delta \subseteq X \times X$ is the diagonal. Now, define by
\[ Z = \Delta_0 \cup \ldots \cup \Delta_{0,p+1} \subseteq X \times X \times (p+1) . \]
Moving forward, note that $H^1(X, B) = 0$ by Kodaira’s vanishing, as $B$ is ample and $K_X = 0$. Apply now Lemma 1.2 from [ELY15], and deduce that $K_{p,1}(X; L, B) = 0$ if the restriction map
\[ H^0(Y, L \boxtimes B^{\boxtimes (p+1)}) \longrightarrow H^0(Y, (L \boxtimes B^{\boxtimes (p+1)})|_Z) \] is surjective,
where \( M_1 \boxtimes M \boxtimes \ldots \boxtimes M_{p+2} \) is the tensor product of the pull-backs of \( M_i \) from \( X \) to \( Y \) by the projection morphism to the \( i \)-th factor. So, the goal is to obtain the latter statement as a consequence of Nadel’s vanishing for multiplier ideals (see [PAG, Theorem 9.4.8]).

For this we need to construct an effective \( \mathbb{Q} \)-divisor \( E \) on \( Y \) so that \( \mathcal{J}(Y; E) = \mathcal{J}_{Z/Y} \) and the class \( (L \boxtimes B^{\otimes(p+1)}) - E \) is ample. Based on the ideas from [LPP11], we construct \( E \) from the initial divisor \( F' \).

Let \( \pi_i : X^{\times(p+1)} \to X \) be the projection to the \( i \)-th factor and \( \delta : X \times X^{\times(p+1)} \to X^{\times(p+1)} \) given by the formula

\[
\delta(x_0, \ldots, x_{p+1}) = (x_0 - x_1, \ldots, x_0 - x_{p+1}).
\]

Then we define our divisor \( E \) as follows

\[
E \stackrel{\text{def}}{=} \delta^* \left( \sum_{i=1}^{p+1} \pi_i^*(F') \right).
\]

Going further we then have the following equalities of ideals

\[
\mathcal{J}(Y, E) = \delta^* \left( \mathcal{J} \left( X^{\times(p+1)}, \sum_{i=1}^{p+1} \pi_i^*(F') \right) \right) = \delta^* \left( \prod_{i=1}^{p+1} \pi_i^* \left( \mathcal{J}(X, F') \right) \right) = \delta^* \left( \prod_{i=1}^{p+1} \pi_i^* \left( m_{X, 0} \right) \right) = \mathcal{J}_{Z/X}.
\]

The first one is due to the fact that \( \delta \) is a smooth morphism and multiplier ideals commute under such maps, as proven in Example 9.5.45 from [PAG]. The second one follows by making use of Proposition 9.5.22 from [PAG] and the last one by doing computations on a local level.

It remains to show that \( (L \boxtimes B^{\otimes(p+1)}) - E \) is ample. From [LPP11, Proposition 1.3] we have the equality

\[
E \equiv_{\text{num}} \frac{1 - c}{p+2} \delta^* \left( L^{\otimes(p+1)} \right) = (1 - c)L \boxtimes L^{\otimes(p+1)} - \frac{1 - c}{p+2} N,
\]

where \( N \) is some nef class on \( Y \). So, then I can write

\[
L \boxtimes B^{\otimes(p+1)} - E \equiv_{\text{num}} cL \boxtimes (B - L)^{\otimes(p+1)} + \frac{1 - c}{p+2} N.
\]

Since \( B - L \) was chosen to be ample, then the first summand of the expression on the right is ample. As we know that \( N \) is a nef class, then the whole expression is ample. So, we finish the proof. \( \square \)

6. Proofs of the main results

This section is devoted to the proof of Theorem 1.1 and Corollary 1.2. Before delving into the proofs, we fix first some notation that will be used throughout this section.

Let \((X, L)\) be a polarized abelian three-fold such that \(L^3 > 59(p+2)^3\). We denote by

\[
B \stackrel{\text{def}}{=} \frac{1}{p+2} L.
\]

As usual, let \( \pi : X' \to X \) be the blow-up of the origin \( 0 \in X \) and denote by \( B_t \stackrel{\text{def}}{=} \pi^*(B) - tE \), for any \( t \geq 0 \). We denote the infinitesimal width and the Seshadri constant of the class \( B \) at the origin by

\[
\mu \stackrel{\text{def}}{=} \mu(B; 0) \text{ and } \epsilon \stackrel{\text{def}}{=} \epsilon(B; 0).
\]

At this point it is worth pointing out that \( B^3 \geq 59 \). So, in particular we have \( \mu > 3 \).
For an effective \( \mathbb{Q} \)-divisor \( \overline{D} \equiv B \), call the divisor \( D \overset{\text{def}}{=} \pi_*(\overline{D}) \), hopefully without confusion, the push-forward of \( \overline{D} \). It is defined as the linear span of the images of all irreducible components of the divisor \( \overline{D} \) that are distinct from \( E \) and taken with the appropriate coefficients. In particular, we have the formula

\[
\overline{D} = \pi^*(D) - \text{mult}_0(D)E
\]

as effective divisors.

With this notation in hand, we can proceed first with the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Theorem 5.1, if we can find an effective \( \mathbb{Q} \)-divisor \( D \equiv B \) so that \( \lct(D) < 1 \) and \( \LC(D) \cdot D \) is zero-dimensional then the pair \((X, L)\) satisfies our syzygetic properties.

If \( \overline{D}_t \equiv B_t \) is an effective divisor for some \( t > 3 \), then its push-forward \( D_t = \pi_*(D_t) \) has \( \lct(D_t) < 1 \), by Proposition 9.3.2 from [PAG]. As \( \mu > 3 \), then this discussion implies that it remains to tackle the case when for any effective divisor \( \overline{D}_t \equiv B_t \) and any \( t > 3 \), its proper push-forward \( D_t \) has all the critical varieties, obtained as in Lemma 2.6, to be at least one-dimensional. This latter condition, by Proposition 9.5.13 from [PAG] and Lemma 2.2, implies that \( B_3 \) is not nef. In particular, this yields that \( \epsilon(B; 0) < 3 \).

With this in hand, we divide the proof in a few steps. The first two use inversion of adjunction techniques. Here we will get that either the syzygetic properties are satisfied or we obtain upper bounds on the Seshadri constant \( \epsilon(B; 0) \). In the last step we use this upper bounds and heavy computational techniques on infinitesimal Newton–Okounkov bodies to get to a contradiction.

**Step: 1** Fix \( t \in (3, \mu) \) and suppose that for some divisor \( \overline{D}_t \equiv B_t \) the proper push-forward \( D_t \) has a smooth curve as a critical variety. Then either our syzygetic properties are satisfied or \( \epsilon(B; 0) \leq 6 - t \).

Denote by \( C_t \) this smooth curve and by \( \overline{C}_t \) its proper transform on \( X' \). By the construction of critical varieties from Lemma 2.6, there exists an effective divisor \( D'_t \equiv c'_tB \) with

\[
\text{Zeroes}(\mathcal{J}(X; D'_t)) = C_t \quad \text{and} \quad \mathcal{J}(X; (1 - \delta)D'_t) = 0_X \quad \text{for any } 0 < \delta < 1.
\]

The divisor \( D'_t \) can be chosen as close as one wants to the divisor \( \lct(D_t) \cdot D_t \), i.e. the multiplicity and the coefficient \( c'_t \) of \( D'_t \) can be chosen to be arbitrarily close to the multiplicity of \( \lct(D_t)D_t \) and \( \lct(D_t) \).

We now can use inversion of adjunction techniques. Assume first the following inequality

\[
t \cdot c'_t + \epsilon \cdot (1 - c'_t) > 3.
\]

As \( \text{mult}_0(D_t) \geq t \) and \( \text{mult}_0(D'_t) \) can be chosen arbitrarily close to \( c_t \cdot \text{mult}_0(D_t) \), then by (6.0.7) all the conditions of Proposition 2.9 are satisfied by \( D'_t \). So, we can construct another effective \( \mathbb{Q} \)-divisor \( D''_t \equiv B \) with \( \lct(D''_t) < 1 \) and zero-dimensional LC center. By Theorem 5.1, this would then imply that our pair \((X, L)\) satisfies both syzygetic properties.

Next, assume that (6.0.7) does not hold. Then this gives an upper bound on \( c'_t \) and lower bound on \( 1 - c'_t \):

\[
c'_t \leq \frac{3 - \epsilon}{1 - \epsilon} \quad \text{and} \quad 1 - c'_t \geq \frac{t - 3}{t - \epsilon}.
\]

With this in hand, we will again be using inversion of adjunction. Assume for this that

\[
t \cdot \frac{t - 3}{t - \epsilon} (B \cdot C_t) > 1.
\]

Combining this with (6.0.8), and use Proposition 2.8, we can construct again a new divisor \( D'''_t \) with \( \lc(D'''_t) < 1 \) and \( \dim(\LC(D'''_t)) = 0 \). So, we would be again done in this case.
It remains to study then the case when the above inequality does not hold. Thus, we can assume
\[
(B \cdot C_t) \leq \frac{t - \varepsilon}{t - 3}.
\]
Finally, if \((B \cdot C_t) \leq 2\) then Proposition 4.3 leads to a contradiction due to the assumptions in our statement. So, assuming that \((B \cdot C_t) > 2\), then the inequality above yields the upper bound \(\varepsilon(B; 0) < 6 - t\).

**Step 2:** Fix \(t \in (3, \mu)\) and suppose that for some divisor \(\overline{D}_t \equiv B_t\) the proper push-forward \(D_t\) has a surface \(S_t \subseteq X\) as a critical variety. Then either we go back to Step 1 or \(\text{mult}_0(S_t) = 2\).

By Step 1, we can assume that for a "general enough" choice of \(\overline{D}_t \equiv B_t\), a surface \(S_t\) appears as the critical variety of its push-forward divisor \(D_t\). Here by "general enough" choice means a divisor that satisfies both Lemma 2.1 and Lemma 2.2. In particular, this implies that \(\overline{S}_t \subseteq B_{\pm}(B_t)\), where \(\overline{S}_t\) is the proper transform of \(S_t\). Furthermore, as \(S_t\) is the log-canonical center of a general choice of \(D_t\) with \(\text{mult}_0(D_t) = t\), then Proposition 9.5.13 from [PAG] yields \(\text{mult}_{\overline{S}_t}(D_t) \geq \frac{t}{3}\). In particular, the general choice of \(D_t\) then implies
\[
(6.0.9) \quad \text{mult}_{\overline{S}_t}(||B_t||) \geq \frac{t}{3} > 1.
\]

Going forward, assume first that \(S_t\) is abelian. Then (6.0.9) implies that \(B - \frac{t}{3}S_t\) is pseudo-effective and so \(B - S_t\) is ample as \(X\) is abelian. Making use now of Proposition 4.11 and Theorem 5.1, then our initial conditions in the main statement imply that our syzygetic properties of the pair \((X, L)\) hold.

Now assume that \(S_t\) is not abelian. By Lemma 4.2 this forces \(\text{mult}_0(S_t) \geq 2\) and so it remains to study the case when \(\text{mult}_0(S_t) \geq 3\). For this apply (6.0.9) together with Proposition 3.2 to the Newton–Okounkov body \(\overline{\Delta}_{Y_{\star}}(B)\), where \(Y_{\star}\) is a general infinitesimal flag. Since \(\text{mult}_0(S) \geq 3\) then the lower bound on the multiplicity yields \(\mu(B; 0) \leq t\). But this contradicts the choice of \(t \in (3, \mu)\) we made.

**Step 3:** The bad cases coming out of Step 1 and 2 contradict the condition \(B^3 > 59\).

In this final step the idea is to translate the "bad" cases that come out of Step 1 and 2 into strong conditions on some infinitesimal Newton–Okounkov body of the class \(B\) with respect to some generic fixed choice of infinitesimal flag \(Y_{\star}\). This will force its volume to be quite small, which will lead to a contradiction that it is equal to \(2B^3/6\), by Theorem 3.1, and thus at least \(59/6\).

Before doing so, let’s try to understand our "bad" cases. Let \(t \in (3, \mu)\) and suppose for some \(\overline{D}_t \equiv B_t\) the LC-locus of \(D_t\) contains a smooth curve. By Step 1 we have a bound on the Seshadri constant.

The second "bad" case is when for some \(t \in (3, \mu)\) there exists a divisor \(D_t\) whose LC-locus contains a surface with multiplicity 2 at the origin. Without loss of generality we can assume that the choice of \(D_t \equiv B_t\) is general, as otherwise we fall back to the first "bad" case. Denote by \(t_1\) the minimum \(t \in (3, \mu]\) for which the second "bad" case happen. Let \(S_1\) be the "bad" surface appearing and \(\overline{S}_1\) its proper transform. The choice of \(D_t\) together with Proposition 9.5.13 from [PAG] and Proposition 4.4 implies then
\[
(6.0.10) \quad \text{mult}_{S_1}(||B_t||) \geq \frac{t_1}{3} + r, \text{ for any } r \geq 0.
\]

The definition of \(t_1\) implies also that for any \(3 < t < t_1\) for a general choice of the divisor \(D_t\) we fell in the first "bad" case. Under these circumstances, Step 1 then yields
\[
(6.0.11) \quad t_1 \leq 6 - \varepsilon(B; 0).
\]

Besides \(S_1\) imposing strong condition on the shape of an infinitesimal Newton–Okounkov body, there is a second surface \(S_2\). If \(\varepsilon(B; 0)\) is defined by a surface, then this is \(S_2\). When it is defined by a curve \(C \subseteq X\)
with \( q \overset{\text{def}}{=} \text{mult}_0(C) \geq 1 \), set \( S_2 \overset{\text{def}}{=} C + C \). By Lemma 4.6, this yields the following inequalities
\[
(6.0.12) \quad t_{S_2} \leq 2\varepsilon(B;0) \text{ and } \mu(B;0) \leq 3\varepsilon(B;0).
\]
It is worth pointing out here that when \( C \) is elliptic then there is no surface \( S_2 \).

The proof is divided in three parts depending on the size of \( \varepsilon(B;0) \). In each case either \( S_1 \) or \( S_2 \) or both give strong conditions on the slices of \( \Delta_{Y_\varepsilon}(B) \). By applying Fubini’s theorem this gives an upper bound on the volume of this convex set, which by Theorem 3.1 is \( B^3/6 \), i.e. at least 59/6. Then this immediately leads to a contradiction as our upper bound will be smaller than 59/6.

**Case 3.1:** \( 0 < \varepsilon(B;0) \leq 1.5 \).

In the following assume that \( \varepsilon(B;0) \) is defined by a curve \( C \subseteq X \). If it is not, then we can still find a curve \( C \) with \( \frac{\langle L_C \rangle}{q} \) as close as it gets to \( \varepsilon(B;0) \) and the argument below would still work.

Our first goal is to show that under the assumptions of our statement \( C \) is a non-degenerate curve and \( q = \text{mult}_0(C) \geq 5 \). Suppose \( C \) is degenerate. If \( C \) is elliptic, then the bounds on \( \varepsilon(B;0) \) imply that \( (B \cdot C) \leq 1.5 \), contradicting our initial assumptions. If \( S_2 = C + C \) is abelian, then [KSS09] yields \( C^2 \geq q^2 - q + 2 \), as \( C \) is not elliptic. Applying Hodge index, we finally have the following sequence of inequalities
\[
(B^2 \cdot S_2) \leq \frac{(B \cdot C)^2}{C^2} \leq \frac{q^2 \cdot \varepsilon(B;0)^2}{q^2 - q + 2} \leq (1.5)^2 \cdot \frac{8}{7} < 4.
\]
This leads again to a contradiction, as we assumed that such a surface does not exist on \( X \).

We know now that \( C \) is non-degenerate. Thus, by Debarre [D94], this yields the inequalities
\[
1.5 > \frac{(B \cdot C)}{q} \geq \frac{3}{q \cdot \sqrt{B^3/6}}.
\]
Since \( B^3 > 59 \) and \( q \) is an integer then this forces \( q \geq 5 \).

With this in hand, let’s describe the upper bounds on vertical slices of the convex set \( \Delta_{Y_\varepsilon}(B) \). For the region \([0, \varepsilon] \times \mathbb{R}^2\), we use the upper bound from Theorem 3.1 for our convex set. For the region \([\varepsilon, 3\varepsilon] \times \mathbb{R}^2\) we apply Theorem 4.9 with \( q = 5 \) and for higher multiplicity the upper bounds will much stronger.

Having these bounds, we use now Fubini’s theorem to give an upper bound of the volume of \( \Delta_{Y_\varepsilon}(B) \), equal to \( B^3/6 \) by Theorem 3.1. The latter is bigger than 58/6, so this yields the following sequence of inequalities
\[
\frac{58}{6} < \text{vol}_{\mathbb{R}^3}(\Delta_{Y_\varepsilon}(B)) \leq \int_0^\varepsilon t^2 \frac{1}{2} dt + \left( \int_{\varepsilon}^{2\varepsilon} t^2 - 5(t - \varepsilon)^2 \right) \frac{1}{2} dt + \int_{\varepsilon}^{3\varepsilon} \frac{5}{8} \varepsilon^3 dt.
\]
The expression on the right is equal to \( \frac{45}{32} \varepsilon^3 \). So, this inequality does not hold for \( \varepsilon \in (0,1.5) \), leading to a contradiction.

**Case 3.2:** \( 1.5 < \varepsilon(B;0) < 2 \).

The same proof, as at the start of Case 3.1, implies that the bounds on \( \varepsilon(B;0) \) yield that either the curve \( C \) is non-degenerate with \( q = \text{mult}_0(C) \geq 4 \), or \( S_2 = C + C \) is actually an abelian surface with
\[
4 \leq (B^2 \cdot S_2) \leq \frac{32}{7}.
\]
Let’s deal first with the latter case, when \( S_2 \) is an abelian surface with the intersection numbers just as above.

First note that \( q \geq 2 \), as otherwise this contradicts, by Hodge index, the conditions \( (B \cdot C) < 2 \), \( \varepsilon(B;0) \leq 2 \), \( (B \cdot S_2) \leq 4 \) and \( C^2 \geq 2 \), as \( C \) is not elliptic. Second, we can further assume the inequalities
\[
t_{S_2} \leq \frac{8}{7} \varepsilon(B;0) \text{ and } \mu(B;0) \leq \frac{8}{7} \varepsilon(B;0) + 1.
\]
The first inequality follows from Remark 4.15. The second holds if \( \mu(B;0) - t_S \leq 1 \). This latter can be assumed as otherwise \( B - S \) is ample and our syzygetic properties hold, by Proposition 4.11 and Theorem 5.1.

Finally, since we have assumed that \( \varepsilon(B;0) < 2 \), the inequalities above imply that \( \mu(B;0) \leq \frac{23}{3} \), contradicting the condition that \( B^3 \geq 59 \).

It remains to tackle the case when \( S \) is not abelian. In particular, \( C \) is non-degenerate and \( q \geq 4 \). By Step 3.2 and Lemma 4.2 both \( S_1 \) and \( S_2 \) are singular at 0. Also, the bounds on \( \varepsilon(B;0) \) imply by Proposition 4.14 that \( S_2 \) is not normal. Since \( S_1 \) is an LC-locus of a divisor and thus normal, then \( S_1 \neq S_2 \).

Going back to the inequalities (6.0.12) and (6.0.11), we get
\[
t_1 \leq 6 - \varepsilon \text{ and } t_{S_2} \leq 2\varepsilon.
\]

On the other hand, by (6.0.10) and Proposition 4.4, we also get the inequalities
\[
\text{mult}_{\tilde{\Sigma}_1}(|B_i|) \geq t - 4 + \frac{2\varepsilon}{3} \text{ and } \text{mult}_{\tilde{\Sigma}_2}(|B_i|) \geq t - 2\varepsilon
\]
for any \( t \geq 6 - \varepsilon \). In this latter range we can apply Proposition 3.2 and use that both surfaces \( S_1 \) and \( S_2 \) are singular at the origin to deduce for any \( t \geq 6 - \varepsilon \) we have the inclusion
\[
\tilde{\Delta}_Y(B) \cap \{t\} \times \mathbb{R}^2 \subseteq \text{convex hull}\{(t,0,0), (t,x_t,0), (t,0,x_t)\},
\]
where \( x_t \leq \frac{24 + 8\varepsilon}{3} - 3t \). By virtue that \( x_t \geq 0 \), then
\[
\mu(B;0) \leq \frac{24 + 8\varepsilon}{9}.
\]

With everything said, we can now find the upper bounds on the vertical slices of \( \tilde{\Delta}_Y(B) \). In \([0,\varepsilon] \times \mathbb{R}^2 \) we use Theorem 3.1 and in \([\varepsilon,6-\varepsilon] \times \mathbb{R}^2 \) we apply Theorem 4.9. Finally, note that the interval \([6-\varepsilon, \frac{24 + 8\varepsilon}{9}] \times \mathbb{R}^2 \) is non-trivial when \( \varepsilon \geq \frac{30}{17} \) and null otherwise. Let \( A_\varepsilon = 1 \), when the latter interval is not trivial and \( A_\varepsilon = 0 \), when it is. Applying as usual Fubini’s theorem, this yields the following inequalities
\[
\frac{58}{6} < \text{vol}_{\mathbb{R}^3}(\tilde{\Delta}_Y(B)) \leq \int_{0}^{\varepsilon} \frac{t^2}{2} dt + \left( \int_{\varepsilon}^{\frac{24 + 8\varepsilon}{9}} \frac{t^2}{2} dt + \int_{\varepsilon}^{6-\varepsilon} \frac{2t^2}{3} dt \right) + A_\varepsilon \cdot \int_{6-\varepsilon}^{\frac{24 + 8\varepsilon}{9}} \frac{2(6-\varepsilon - 3t)^2}{2} dt.
\]

If we count only the first three integrals then we get the following equivalent inequality
\[
4\varepsilon^2 - \frac{64}{54} \varepsilon^3 > \frac{58}{6},
\]
which definitely is not satisfied for \( \varepsilon \in [1.5,2] \). Now, the last integral is maximal when \( \varepsilon = 2 \). In particular, it can be bounded from above by \( \frac{32}{243} \). This then implies that we should have the following inequality
\[
4\varepsilon^2 - \frac{64}{54} \varepsilon^3 + \frac{32}{243} \geq \frac{58}{6},
\]
which actually cannot hold for \( \varepsilon \in [1.5,2] \). So, we get the desired contradiction also in this case.

**Case 3.3:** \( 2 \leq \varepsilon(B;0) < 3 \).

In this case we cannot assume that the Seshadri exceptional curve \( C \) (or surface) is actually non-degenerate. It might be elliptic (abelian), based on the assumptions in the main statement. Due to this we cannot rely on the surface \( S_2 \). But we still have the non-abelian surface \( S_1 \).

As in Case 3.2 we have the following inequalities
\[
t_1 \leq 6 - \varepsilon \text{ and } \mu(B;0) \leq 8 - \frac{4\varepsilon}{3}.
\]
The first one is the same as (6.0.11). The second one follows from (6.0.10) and Proposition 3.2.

Finally, we go back to the set \( \tilde{\Delta}_r(B) \). On \([0, \varepsilon] \times \mathbb{R}^2 \) we use Theorem 3.1 and on \([\varepsilon, 6 - \varepsilon] \) we apply Proposition 3.2, using the curve \( C \). On \([6 - \varepsilon, 8 - \frac{4\varepsilon}{3}] \times \mathbb{R}^2 \) we apply again Proposition 3.2, taking into account that \( \text{mult}_0(S_2) = 2 \) and (6.0.10). All this information together yield the following inequalities

\[
\frac{59}{6} < \text{vol}_{\mathbb{R}^3}(\tilde{\Delta}_r(B)) \leq \int_0^\varepsilon \frac{t^2}{2} dt + \int_\varepsilon^{6-\varepsilon} \frac{t^2 - (t - \varepsilon)^2}{2} dt + \int_{6-\varepsilon}^{8-\frac{4\varepsilon}{3}} \frac{(8 - \frac{4\varepsilon}{3} - t)^2}{2} dt,
\]

which is equivalent

\[
\frac{59}{6} < \left( \frac{(6 - \varepsilon)^3}{6} - \frac{(6 - 2\varepsilon)^3}{6} \right) + \frac{(2 - \frac{4\varepsilon}{3})^3}{6}.
\]

It is not hard to check that both terms are maximal when \( \varepsilon = 2 \), by using the derivative trick. Plugging in \( \varepsilon = 2 \) in the right side leads to a contradiction to this inequality.\(^1\)

\( \square \)

Proof of Corollary 1.2. The proof is similar to the one of Theorem 1.1, so we will be making use of some ideas developed there. As before, the bad case is when for any \( t \geq 3 \) and a general choice of divisor \( D_t \equiv B_t \), \( \text{LC}(D_t) \) is either a non-elliptic curve \( C \) or a non-abelian surface.

Denote by \( t_1 \) the maximum \( t \in (3, \mu) \) for which \( \text{LC}(D_t) \) contains a non-elliptic curve. This definition implies also \( B_+(B_t) \) contains a surface \( S_1 \) for any \( t \geq t_1 \), which appears as the \( \text{LC} \)-locus of some general choice of divisor \( D_t \). By Lemma 4.2 the surface \( S_1 \) must be singular at the origin 0.

Let \( t \in [3, t_1) \). The first step in the proof is to show that we either have the following inequality

(6.0.13)

\[
(B \cdot C_t) \geq 5,
\]

or there is a divisor whose multiplier ideal is supported on an abelian surface.

Suppose first that there exists an abelian surface \( S \) containing the curve \( C_t \). If the class \( B - S \) is ample then by Corollary 4.12 we are done. Otherwise, the following string of inequalities

\[
(B \cdot C_t)^2 \geq (B \cdot S) \cdot C_t^2 \geq \frac{2B^3}{3} \geq 25
\]

where the first one follows from Hodge index theorem applied on \( S \). The second is due to Corollary 4.12 and the fact that \( C_t \) is not elliptic and thus \( C_t^2 \geq 2 \).

In this setup the inequalities above imply (6.0.13). Conversely, if there is no abelian surface \( S \) as above, then at least we know that \( C_t \) is non-degenerate. Using [D94] as usual, we obtain again (6.0.13).

The second step consists of using the same techniques (inversion of adjunction) of Step 1 from the proof of Theorem 1.1. Based on those ideas, we either find a divisor whose \( \text{LC} \)-locus is the origin and we are done or combining with (6.0.13) we have the inequalities

\[
5 < (B \cdot C_t) \leq \frac{t - \varepsilon}{t - 3}.
\]

Since this happens for any \( t < t_1 \), then it yields a very strong upper bound on the Seshadri constant as in

(6.0.14)

\[
\varepsilon(B; 0) \leq 15 - 4t_1.
\]

In particular this implies that \( t_1 < 3.75 \) as \( \varepsilon(B; 0) > 0 \). In order to obtain a stronger statement we divide the proof in two parts based on the range of \( t_1 \).

\(^1\)Using deeper convex geometry and more complex computations in Step 3.3, it is not hard to see that the condition \( B^3 > 56 \) is strong enough for Theorem 1.1.
**Case 1:** $3.5 \leq t_1 < 3.75$ ($\Rightarrow 0 < \varepsilon(B;0) \leq 1$).

Since $\varepsilon(B;0) \leq 1$, let’s show first that $\varepsilon(B;0)$ is defined by a curve $C \subseteq X$. To see this assume that it is defined by an abelian surface $S$. Then Proposition 4.4 yields $\text{mult}_S(||B_2||) \geq 1$ and by Corollary 4.12 we would be done. If $S$ is not abelian then Corollary 4.7 yields $\mu(B;0) \leq 2$, contradicting $B^3 > 40$.

Let $C_3$ be a curve appearing in the LC-locus of a general choice of divisor $D_3$. We can assume that $C_3$ is not elliptic, as otherwise we would be done. In particular, this implies that the sum $S = C + C_3$ is a surface. Furthermore, we know that

$$\text{mult}_C(B_t) \geq t - 2, \text{ for any } t \geq 3.$$ 

Taking into account Proposition 4.4 for the curve $C$ and $C_3$, together with the bound in (6.0.14) and repeatedly Lemma 4.6 we then get the inequality

$$\text{mult}_S(||B_t||) \geq 4t_1 + t - 17, \text{ for any } 3 \leq t \leq t_1.$$ 

Finally, we want a strong upper bound on $\mu(B;0)$. First, the inequality above yields $\text{mult}_S(||B_4||) \geq 1$. If $S$ is abelian then Corollary 4.12 implies that in the bad case $\mu(B;0) \leq 4$. When $S$ is not abelian, then $\text{mult}_0(S) \geq 2$ by Lemma 4.2. Applying Proposition 3.2 and use that $S + C = X$, we obtain again that $\mu(B;0) \leq 4$.

We can now go back to our usual trick of using the convex set $\hat{\Delta}_Y(B)$. In the region $[0, 15 - 4t_1] \times \mathbb{R}^2$ we use Theorem 3.1. In the region $[15 - 4t_1, 3] \times \mathbb{R}^2$ we use the curve $C$ in $[3, t_1] \times \mathbb{R}^2$ the surface $S$, and in $[t_1, 4] \times \mathbb{R}^2$ the surface $S_1$, and applying in all cases Proposition 3.2. Note here that $\text{mult}_0(S_1) = 2$.

Putting all this information together, we then get the following inequalities

$$\frac{40}{6} < \text{vol}_{\mathbb{R}^3} \left( \hat{\Delta}_Y(B) \right) \leq \int_0^{15 - 4t_1} \frac{t^2}{2} dt + \int_{15 - 4t_1}^{3} \frac{t^2}{2} dt + \int_3^{t_1} \frac{9}{2} dt + \int_{t_1}^{4} \frac{(\frac{9}{2} - t)^2}{2} dt,$$

which is equivalent by rearranging some terms to the following one

$$\frac{40}{6} < \frac{27}{6} t_1^3 + t_1 - 3 + \frac{9}{2} \left( - \frac{64}{3} + \frac{64}{27} \right).$$

To get a contradiction note that we can maximize the right product easily. For the term $\frac{4t_1 - 3}{3}$ it is maximal when $t_1 = 3.5$ and for the second term in the product when $t_1 = 3.75$. Doing so, we get to a contradiction.

**Case 2:** $3 < t_1 < 3.5$ ($\Rightarrow 1 < \varepsilon(B;0) < 3$).

This case is simpler as we use only the surface $S_1$ and not $S$. As in Case 3.2 from the proof of Theorem 1.1, we have the following upper bound

$$\mu(B;0) \leq \frac{4t_1}{3}.$$

Going to the usual suspect $\hat{\Delta}_Y(B)$, we use Theorem 3.1 in the region $[0, 15 - 4t_1] \times \mathbb{R}^2$. In $[15 - 4t_1, t_1] \times \mathbb{R}^2$ we use the curve $C$, $[t_1, \frac{4t_1}{3}] \times \mathbb{R}^2$ the surface $S_1$, and apply in both cases Proposition 3.2. Note that $\text{mult}_0(S_1) = 2$.

Putting all this information together, we then get the following inequalities

$$\frac{40}{6} < \text{vol}_{\mathbb{R}^3} \left( \hat{\Delta}_Y(B) \right) \leq \int_0^{15 - 4t_1} \frac{t^2}{2} dt + \int_{15 - 4t_1}^{t_1} \frac{t^2}{2} dt + \int_{t_1}^{\frac{4t_1}{3}} \frac{(\frac{9}{2} - t)^2}{2} dt,$$

which is equivalent by rearranging some terms to the following one

$$\frac{40}{6} < \frac{t_1^3}{6} - \frac{(5t_1 - 15)^3}{6} + \frac{t_1^3}{27}.$$ 

And this inequality does not hold whenever $t_1 \in (3, 3.5)$. So, this finishes the proof of our corollary. \qed
7. FURTHER DISCUSSIONS AND OPEN PROBLEMS

In this section we propose a conjectural picture concerning this circle of ideas.

A first interesting line of research would be to study the syzygetic properties of any abelian manifold. Taking into account the two dimensional picture as discussed in [It17] and [KL15], and the three dimensional picture here, we suggest the following conjecture (also partly proposed by Ito):

**Conjecture 7.1.** Let $X$ be a $g$-dimensional abelian manifold over the complex number and $L$ an ample line bundle on $X$. Suppose the following inequality holds
\[(V \cdot L^\dim(V)) > (\dim(V))^{\dim(V)}\]
for any abelian submanifold $V \subset X$. Then $(X, L)$ satisfies $N_p$-property and $K_{p,1}(X, L; dL) = 0$ for any $d \geq 2$.

Ito proves this conjecture for $g = 2$. The goal of this paper was to give strong evidence of it for $g = 3$.

In our work there is one important problem making it hard to improve our bounds to the conjectural picture, even for abelian three-folds. Basically, we don’t understand the full effect of the LC-loci of a divisor on the infinitesimal geometry (convex or not) of the classes $\pi^*(L) - tE$. This is especially true when these LC-loci don’t pass through the origin. In this paper, we realized that they affect dramatically the Seshadri constant. But there should be other strong conditions around.

It is tempting here to ask for a statement as Conjecture 7.1 for Calabi-Yau manifolds, where we exchange everywhere the word abelian with CY above. And there is (not so strong) evidence to support this. For property $N_p$ [AKL17] proves a similar statement for $K3$ and Enriques surfaces, but for bi-elliptic surfaces this still remains unknown. To make matter worse, the proofs use different techniques from one type of surfaces to another. For vanishing of $K_{p,1}$ and for small $p$, the recent work [A17] might be a good starting point.

As pointed out by Corollary 1.2, this work exhibits interesting patterns of the singularity loci of effective divisors on an abelian manifold. With this in mind we propose the following conjecture:

**Conjecture 7.2.** Let $X$ be a $g$-dimensional abelian manifold and $B$ an ample $\mathbb{Q}$-divisor on $X$. If $(B^g) > g^g$ then there exists an effective $\mathbb{Q}$-divisor $D \equiv B$ such that the cosupport of $\mathcal{J}(X; D)$ is set-theoretically an abelian subvariety of $X$. Moreover, if this abelian subvariety is at least one-dimensional, then the pair $(X, B)$ splits into polarized abelian submanifolds.

Classically, it was important to study the singularities pluri-theta divisors, i.e. $B = \Theta$ as in [EL97]. In this case they show that $\mathcal{J}(X; D)$ is trivial for any $D \equiv B$ with one exception when the pair $(X; B)$ splits as a product of principally polarized abelian submanifolds.

Our conjecture tackles the dual case when $B^g \geq g^g \gg \Theta^g = g!$. In this range there is always divisors with non-trivial multiplier ideal, but the goal of this conjecture is that there exists divisors whose LC-loci are special subvarieties, which affect the geometry of the whole space. Furthermore, this kind of statement opens the door to induction type arguments when studying the properties of $(X, B)$, be them geometric, algebraic, differential or arithmetic, that can be connected to the singularity theory of the divisors in the class of $B$.

As underlined in this paper, it is very important to understand the infinitesimal picture at the origin of an ample line bundle on an abelian variety. This suggests the following problem:
Problem 7.3. If $X$ is an abelian manifold and $B$ an ample $\mathbb{Q}$-class on $X$, then are there good approximations of the convex set $\tilde{\Delta}_Y(B)$, for some very general infinitesimal flag $Y$, in terms of the geometry of $X$? Furthermore, what are the properties of the base loci $B_+((\pi^*(B) - tE))$?

In this paper we tried to tackle this problem in the three-dimensional case. But even here there seems to be lot of place for improvements. Besides being important for the above conjectures, the shape of the infinitesimal Newton–Okounkov bodies allows us to better understand the Seshadri constant $\varepsilon(B; 0)$ and so generalize the works of Nakamaye [N96] or Debarre [D94]. Knowing more about the properties of the base loci $B_{+}((\pi^*(B) - tE))$ is also important. Some examples we have seen in Lemma 4.2 or Section 4.5. Furthermore, it is not hard to show that they are connected and invariant under the automorphism $-1: X \to X$. For both conjectures these properties (and perhaps more) should play an important role in understanding better the case when some LC-loci of the class $\pi^*(L) - tE$ don’t pass through the origin.

REFERENCES

[A17] Daniele Agostini, *Asymptotic syzygies and spanned line bundles*, online. ↑ 1, 7

[AKL17] Daniele Agostini, Alex Küronya, and Victor Lozovanu, *Higher syzygies on surfaces with numerically trivial canonical bundle*, online. ↑ 1, 7

[AS95] Urban Anghean and Yum Tong Siu, *Effective freeness and point separation for adjoint bundles*, Invent. Math. 122 (1995), no. 2, 291–308. ↑ 1

[AF11] Marian Aprodu and Gavril Farkas, *Green’s conjecture for curves on arbitrary K3 surfaces*, Compos. Math. 147 (2011), no. 3, 839–851. ↑ 1

[BMS16] Arend Bayer, Emanuelle Macri, and Paolo Stellari, *The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds*, Invent. Math. 206 (2016), no. 3, 869–933. ↑ 1

[B] Sébastien Boucksom, *Corps D’Okounkov*, Séminaire Bourbaki 65 (2012), no. 1059, 1–38. ↑ 3.1

[CN14] Paolo Cascini and Michael Nakamaye, *Seshadri constants on smooth threefolds*, Adv. Geom. 14 (2014), no. 1, 59–79. ↑ 4.10, 4.5

[CHPW15] Sung Rak Choi, Yoonsuk Hyun, Jinhyung Park, and Joonyeong Won, *Asymptotic base loci via Okounkov bodies* (2015). ↑ 3.1

[D94] Olivier Debarre, *Degrees of curves in abelian varieties*, Bull. Soc. Math. France 122 (1994), no. 3, 343–361. ↑ 4, 6, 7

[DG] Olivier Debarre, *Tores et variétés abéliennes complexes*, Cours Spécialisés, vol. 6, Société Mathématique de France, Paris; EDP Sciences, Les Ulis, 1999. ↑ 1

[E97] Lawrence Ein, *Multiplier ideals, vanishing theorems and applications*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 203–219. ↑ 2.4

[EKL95] Lawrence Ein, Oliver Küchle, and Robert Lazarsfeld, *Local positivity of ample line bundles*, J. Differential Geom. 42 (1995), no. 2, 193–219. ↑ 4.2

[EL93] Lawrence Ein and Robert Lazarsfeld, *Global generation of pluricanonical and adjoint linear series on smooth projective threefolds*, J. Amer. Math. Soc. 6 (1993), no. 4, 875–903. MR1207013 ↑ 1, 4.15, 4.5

[EL97] Lawrence Ein and Robert Lazarsfeld, *Singularities of theta divisors and the birational geometry of irregular varieties*, J. Amer. Math. Soc. 10 (1997), no. 1, 243–258. ↑ (document), 1, 7

[EL15] Lawrence Ein, *The gonality conjecture on syzygies of algebraic curves of large degree*, Publ. Math. Inst. Hautes Études Sci. 10 (2015), no. 212, 301–313. ↑ 1

[EL16] Lawrence Ein, *Syzygies of projective varieties of large degree: recent progress and open problems*, online. ↑ 1

[ELMNP06] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, *Asymptotics invariants of base loci*, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 6, 1701–1734. ↑ 1, 2.2, 2.2, 2.3, 3.4

[ELN94] Lawrence Ein, Robert Lazarsfeld, and Michael Nakamaye, *Zero-estimates, intersection theory, and a theorem of Demailly*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 183–207. ↑ (document), 1, 1, 4.2, 4.2

[Ely15] Lawrence Ein, Robert Lazarsfeld, and D. Yang, *A vanishing theorem for weight one syzygies*, Algebra and Number Theory 10 (2016), no. 9, 1965–1981. ↑ 1, 1, 5, 5.2, 5
SINGULAR DIVISORS AND SYZYGIES OF POLARIZED ABELIAN THREEFOLDS

[GL86] Mark Green and Robert Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math. 83 (1986), no. 1, 73–118.

[GL91] Mark Green, *Koszul cohomology and the geometry of projective varieties - I*, J. Diff. Geom. 19 (1984), 125–171.

[Har] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977, 45–236.

[Helm] Stefan Helmke, *On Fujita’s conjecture*, Duke Math. J. 88 (1997), no. 2, 201–216. MR1455517

[IIT11] Jun-Muk Hwang and Wing-Keung To, *Bauer-Sarneck invariant and projective normality of abelian varieties*, Complex and differential geometry, Springer Proc. Math., vol. 8, Springer, Heidelberg, 2011, pp. 157–170.

[KK12] Kiumars Kaveh and Askold Khovanskii, *Newton–Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*, Annals of Mathematics 176 (2012), 925–978.

[Ka97] Yujiro Kawamata, *On Fujita’s freeness conjecture for 3-folds and 4-folds*, Math. Ann. 308 (1997), no. 3, 491–505.

[Ko06] Michael Nakamaye, *Seshadri constants on abelian varieties*, Annals of Mathematics 168 (2008), no. 3, 597–635.

[Koll97] Yujiro Kawamata, *On Fujita’s conjecture*, Duke Math. J. 83 (1996), no. 2, 329–341.

[Ko02] Claire Voisin, *Green’s generic syzygy conjecture for curves of even genus lying on a K3 surface*, J. Eur. Math. Soc. 4 (2002), no. 4, 363–404.

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