A MULTIVARIATE GENERALIZATION OF
Hoeffding’s Inequality

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Summary: We prove a multivariate version of Hoeffding’s inequality about the
distribution of homogeneous polynomials of Rademacher functions. The proof
is based on such an estimate about the moments of homogeneous polynomials
of Rademacher functions which can be considered as an improvement of Borell’s
inequality in a most important special case.

1. Introduction. Formulation of the main results.

Hoeffding’s inequality states the following result. (see e.g. [2], Proposition 1.3.5.)

Theorem A. (Hoeffding’s inequality). Let \( \varepsilon_1, \ldots, \varepsilon_n \) be independent random variables, \( P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = \frac{1}{2}, 1 \leq j \leq n \), and let \( a_1, \ldots, a_n \) be arbitrary real numbers. Put \( Z = \sum_{j=1}^{n} a_j \varepsilon_j \) and \( V^2 = \sum_{j=1}^{n} a_j^2 \). Then

\[
P(Z > u) \leq \exp \left\{ -\frac{u^2}{2V^2} \right\} \quad \text{for all } u > 0. \tag{1.1}
\]

In the study of \( U \)-statistics we need a multivariate version of this result. The goal of this paper is to present such an inequality. To formulate it first we have to introduce some notations.

Let us fix a positive integer \( k \) and some real numbers \( a(j_1, \ldots, j_k) \) for all sets of arguments \( \{j_1, \ldots, j_k\} \) such that \( 1 \leq j_l \leq n, 1 \leq l \leq k \), and \( j_l \neq j_{l'} \) if \( l \neq l' \), in such a way that the numbers \( a(j_1, \ldots, j_k) \) are symmetric functions of their arguments, i.e. \( a(j_1, \ldots, j_k) = a(j_{\pi(1)}, \ldots, j_{\pi(k)}) \) for all permutations \( \pi \in \Pi_k \) of the set \( \{1, \ldots, k\} \).

Let us define with the help of the above real numbers and a sequence of independent random variables \( \varepsilon_1, \ldots, \varepsilon_n \), \( P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = \frac{1}{2}, 1 \leq j \leq n \), the random variable

\[
Z = \sum_{(j_1, \ldots, j_k) : 1 \leq j_l \leq n \text{ for all } 1 \leq l \leq k} a(j_1, \ldots, j_k) \varepsilon_{j_1} \cdots \varepsilon_{j_k} \tag{1.2}
\]

and the number

\[
V^2 = \sum_{(j_1, \ldots, j_k) : 1 \leq j_l \leq n \text{ for all } 1 \leq l \leq k} a^2(j_1, \ldots, j_k). \tag{1.3}
\]

Now we formulate the following result.
Theorem 1. (The multivariate version of Hoeffding’s inequality). The random variable $Z$ defined in formula (1.2) satisfies the inequality

$$P(|Z| > u) \leq A \exp\left\{ -\frac{1}{2} \left( \frac{u}{V} \right)^{2/k} \right\} \quad \text{for all } u \geq 0$$

(1.4)

with the constant $V$ defined in (1.3) and some constants $A > 0$ depending only on the parameter $k$ in the expression $Z$.

Let us remark that the condition that the coefficients $a(j_1, \ldots, j_k)$ are symmetric functions of their variables does not mean a real restriction, since by replacing all coefficients $a(j_1, \ldots, j_k)$ by $a_{\text{Sym}}(j_1, \ldots, j_k) = \frac{1}{k!} \sum_{\pi \in \Pi_k} a(j_{\pi(1)}, \ldots, j_{\pi(k)})$ in formula (1.2), where $\Pi_k$ denotes the set of all permutations of the set $\{1, \ldots, k\}$ we do not change the random variable $Z$. The identities $EZ = 0$, $EZ^2 = k!V^2$ hold. A comparison of Theorem A and Theorem 1 shows that Theorem 1 yields a slightly weaker estimate in the special case $k = 1$ because of the pre-exponential coefficient $A$ in the estimate (1.4). But the expressions in the exponent agree in formula (1.1) and in formula (1.4) in the special case $k = 1$.

Moreover, estimate (1.4), disregarding the pre-exponential coefficient $A$ in it, is sharp for all parameters $k \geq 1$. To see this let us consider the random variable $Z = Z_n$ defined in (1.2) with the special choice

$$a(j_1, \ldots, j_k) = a_n(j_1, \ldots, j_k) = \frac{V}{\sqrt{n(n-1) \cdots (n-k+1)}}.$$ 

It is known (see e.g. [3]) that the random variables $Z_n$ converge, as $n \to \infty$, in distribution to a random variable which can be expressed by means of a $k$-fold Wiener–Itô integral. Moreover, it can be expressed in a more explicit form as the distribution of $V \cdot H_k(\eta)$, where $\eta$ is a random variable with standard normal distribution, and $H_k(\cdot)$ is the $k$-th Hermite polynomial with leading coefficient 1. Beside this, the tail behaviour of $H_k(\eta)$ is similar to that of $\eta^k$ in a neighbourhood of the infinity. Hence the above example shows that if we have no additional restriction about the coefficients $a(j_1, \ldots, j_k)$ of the random variable $Z$, then the estimate (1.4) is essentially sharp. We cannot write a better expression in the exponent of its right-hand side. This problem is discussed in more detail in a more general context in Example 2 of paper [5].

Theorem 1 can be interpreted in such a way that the distribution of $Z$ satisfies an inequality similar to the distribution of $V \eta^k$, where $\eta$ is a standard normal random variable. We shall prove it as a relatively simple consequence of the following result, which formulates a similar statement about the moments of the random variable $Z$.

Theorem 2. The random variable $Z$ defined in formula (1.2) satisfies the inequality

$$EZ^{2M} \leq 1 \cdot 3 \cdot 5 \cdots (2kM - 1)V^{2M} \quad \text{for all } M = 1, 2, \ldots$$

(1.5)

with the constant $V$ defined in formula (1.3).
We shall prove Theorem 2 with the help of two lemmas. Before their formulation we introduce the following notation:

\[ \bar{Z} = \sum_{(j_1, \ldots, j_k): 1 \leq j_i \leq n \text{ for all } 1 \leq l \leq k} |a(j_1, \ldots, j_k)| \eta_{j_1} \cdots \eta_{j_k}, \] (1.6)

where \( \eta_1, \ldots, \eta_n \) are iid. random variables with standard normal distribution, and the numbers \( a(j_1, \ldots, j_k) \) agree with those in formula (1.2). Now we state

Lemma 1.

\[ EZ^{2M} \leq E\bar{Z}^{2M} \quad \text{for all } M = 1, 2, \ldots, \] (1.7)

and

Lemma 2. \emph{The random variable \( \bar{Z} \) defined in formula (1.6) satisfies the inequality}

\[ EZ^{2M} \leq 1 \cdot 3 \cdot 5 \cdots (2kM - 1)V^{2M} \quad \text{for all } M = 1, 2, \ldots \] (1.8)

\emph{with the constant \( V \) defined in formula (1.3).}

Theorem 2 states an estimate about the moments of homogeneous polynomials of the independent random variables \( \varepsilon_1, \ldots, \varepsilon_n \) which are sometimes called Rademacher functions in the literature. We finish the Introduction by recalling Borell’s inequality (see e.g. [1]) which gives a similar estimate. The proof of the results will be given in Section 2. Then we compare Borell’s inequality with our results and make some comments in Section 3.

**Theorem B. (Borell’s inequality).** \emph{The moments of the random variable} \( Z \) \emph{defined in formula (1.2) satisfy the inequality}

\[ E|Z|^p \leq \left( \frac{p - 1}{q - 1} \right)^{kp/2} (E|Z|^q)^{p/q} \quad \text{if} \quad 1 < q < p < \infty. \] (1.9)
2. Proof of the results.

Proof of Lemma 1. We can write, by carrying out the multiplications in the expressions $EZ^{2M}$ and $EZ^{2M}$, by exploiting the additive and multiplicative properties of the expectation for sums and products of independent random variables together with the identities $E\varepsilon_j^{2k+1} = 0$ and $E\eta_j^{2k+1} = 0$ for all $k = 0, 1, \ldots$ that

$$EZ^{2M} = \sum_{j_1, \ldots, j_l, m_1, \ldots, m_l, 1 \leq j_s \leq n, j_s \geq 1, 1 \leq s \leq l, m_1 + \cdots + m_l = M} A(j_1, \ldots, j_l, m_1, \ldots, m_l) E\varepsilon_{j_1}^{2m_1} \cdots E\varepsilon_{j_l}^{2m_l} \quad (2.1)$$

and

$$EZ\tilde{2M} = \sum_{j_1, \ldots, j_l, m_1, \ldots, m_l, 1 \leq j_s \leq n, j_s \geq 1, 1 \leq s \leq l, m_1 + \cdots + m_l = M} B(j_1, \ldots, j_l, m_1, \ldots, m_l) E\eta_{j_1}^{2m_1} \cdots E\eta_{j_l}^{2m_l} \quad (2.2)$$

with some coefficients $A(j_1, \ldots, j_l, m_1, \ldots, m_l)$ and $B(j_1, \ldots, j_l, m_1, \ldots, m_l)$ such that

$$|A(j_1, \ldots, j_l, m_1, \ldots, m_l)| \leq B(j_1, \ldots, j_l, m_1, \ldots, m_l). \quad (2.3)$$

We could express the coefficients $A(\cdot, \cdot, \cdot)$ and $B(\cdot, \cdot, \cdot)$ in an explicit form, but we do not have to do this. What is important for us is that $A(\cdot, \cdot, \cdot)$ can be expressed as the sum of certain terms, and $B(\cdot, \cdot, \cdot)$ as the sum of the absolute value of the same terms, hence relation (2.3) holds. Since $E\varepsilon_j^{2m} \leq E\eta_j^{2m}$ for all parameters $j$ and $m$ formulas (2.1), (2.2) and (2.3) imply Lemma 1.

Proof of Lemma 2. Let us consider a white noise $W(\cdot)$ on the unit interval $[0, 1]$, i.e. let us take a set of Gaussian random variables $W(A)$ indexed by the measurable sets $A \subset [0, 1]$ such that $EW(A) = 0$, $EW(A)W(B) = \lambda(A \cap B)$ with the Lebesgue measure $\lambda$ for all measurable subsets of the interval $[0, 1]$. (We also need the relation $W(A \cup B) = W(A) + W(B)$ with probability 1 if $A \cap B = \emptyset$, but this relation is the consequence of the previous ones. Indeed, they yield that $E(W(A \cup B) - W(A) - W(B))^2 = 0$ if $A \cap B = \emptyset$, and this implies the desired identity.) Let us introduce the random variables $\eta_j = n^{1/2}W(\lfloor j/n \rfloor)$, $1 \leq j \leq n$, together with the function $f(t_1, \ldots, t_k)$, with arguments $0 \leq t_s < 1$ for all indices $1 \leq s \leq k$, defined as

$$f(t_1, \ldots, t_k) = \begin{cases} n^{k/2}|a(j_1, \ldots, j_k)| & \text{if } t_s \in \left[ \frac{j_s - 1}{n}, \frac{j_s}{n} \right), \\ 0 & \text{if } t_s \in \left[ \frac{j_s - 1}{n}, \frac{j_s}{n} \right], \text{and } j_s = j_{s'} \text{ for some } s \neq s', \end{cases} \quad 1 \leq j_s \leq n, 1 \leq s \leq k \quad (2.4)$$

Observe that the above defined random variables $\eta_1, \ldots, \eta_n$ are independent with standard normal distribution, hence we may assume that they appear in the definition.
of the random variable $\bar{Z}$ in formula (1.6). With such a choice we can represent $\bar{Z}$ in the form of a $k$-fold Wiener–Itô integral (introduced e.g. in [4])

$$\bar{Z} = \int f(t_1, \ldots, t_k)W(\,dt_1)\ldots W(\,dt_k)$$

of the (elementary) function $f$ defined in formula (2.4) with respect to white noise $W(t)$ we have introduced. Beside this, the identity

$$\int f^2(t_1, \ldots, t_k)\,dt_1\ldots dt_k = V^2$$

also holds with the number $V$ defined in formula (1.3). Hence to complete the proof of Lemma 2 it is enough to show that if a function $f$ of $k$ variables and a $\sigma$-finite measure $\mu$ on some measurable space $(\mathcal{X}, \mathcal{X})$ satisfy the inequality

$$\int f^2(x_1, \ldots, x_k)\mu(dx_1)\ldots\mu(dx_k) = \sigma^2 < \infty$$

with some $\sigma^2 > 0$, then the moments of the $k$-fold Wiener–Itô integral (defined e.g. in [4])

$$J_{\mu, k}(f) = \frac{1}{k!} \int f(x_1, \ldots, x_k)\mu_W(dx_1)\ldots\mu_W(dx_k)$$

of the function $f$ with respect to a white-noise $\mu_W$ with counting measure $\mu$ satisfy the inequality $E(\,k!J_{\mu, k}(f)\,)^{2M} \leq 1 \cdot 3 \cdots (2kM - 1)\sigma^{2M}$ for all $M = 1, 2, \ldots$. But this result (which can be got relatively simply from the diagram formula for the product of Wiener–Itô integrals) is proven in Proposition A of paper [5], hence here I omit the proof.\(^1\)

Theorem 2 is a straightforward consequence of Lemmas 1 and 2. Hence it remained to prove Theorem 1 with the help of Theorem 2.

**Proof of Theorem 1.** By the Stirling formula we get from the estimate of Theorem 2 that

$$EZ^{2M} \leq \frac{(2kM)!}{2^{kM}(kM)!}V^{2M} \leq A \left(\frac{2}{e}\right)^{kM} (kM)^{kM} V^{2M}$$

for any $A \geq \sqrt{2}$ if $M \geq M_0(A)$. Hence we can write by the Markov inequality that

$$P(Z > u) \leq \frac{EZ^{2M}}{u^{2M}} \leq A \left(\frac{2kM}{e} \left(\frac{V}{u}\right)^{2/k}\right)^{kM}$$

(2.5)

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1 For the sake of completeness I put the proof of this result together with some definitions needed to understand it to an Appendix of this paper, but probably it will not belong to the final version of this work.
for all $A > \sqrt{2}$ if $M \geq M_0(A)$. Put $kM = k\tilde{M}(u) = \frac{1}{2} \left( \frac{u}{V} \right)^{2/k}$, and $M = M(u) = \lceil \tilde{M} \rceil$, where $[x]$ denotes the integer part of the number $x$. Let us choose a number $u_0$ by the identity $M(u_0) = M_0(A)$. Formula (2.5) can be applied with $M = M(u)$ for $u \geq u_0$, and it yields that

$$P(Z > u) \leq Ae^{-kM} \leq Ae^{k}e^{-kM} = Ae^{k} \exp \left\{ -\frac{1}{2} \left( \frac{u}{V} \right)^{2/k} \right\} \text{ if } u \geq u_0. \quad (2.6)$$

Formula (2.6) means that relation (1.2) holds for $u \geq u_0$ if the constant $A$ is replaced by $Ae^{k}$ in it. By choosing the constant $A$ sufficiently large we can guarantee that relation (1.2) holds for all $u \geq 0$.

3. A discussion about the results.

Let us look what kind of estimate yields Borell’s inequality for the expression $Z$ defined in (1.2). It is natural to apply it with the choice $q = 2$. Since $EZ^2 = k!V^2$, Borell’s inequality yields with such a choice the estimate $E|Z|^{2p} \leq (2p - 1)^{kp} (k!V)^p$ for all real numbers $p \geq 1$. Let us compare this inequality for the moments $EZ^2M$ with large integers $M$ with the estimate of Theorem 2. If we disregard some constant factors not depending on $M$ we get that this estimate is of order $(2M)^{kM}V^{2M} \cdot (k!)^M$, while Theorem 2 yields an estimate of order $(2M)^{kM}V^{2M} \cdot \left( \frac{k}{e} \right)^{kM}$. It can be seen that $k! > \left( \frac{k}{e} \right)^{kM}$ for all $k \geq 1$. This means that Borell’s inequality shows that $EZ^2M \leq C^M(kM)^{kM}V^{2M}$ for large $M$ with a universal constant $C$ depending only on the parameter $k$ in formula (1.2), but it does not give the optimal choice for the parameter $C$. As a consequence, it implies a weakened version $P(|Z| > u) \leq A \exp \left\{ -B \left( \frac{u}{V} \right)^{2/k} \right\}$ of the inequality of Theorem 1 with some universal constants $A$ and $B$, but it cannot yield the optimal choice for the number $B$. In short, Theorem 2 is weaker than Borell’s inequality in that respect that it compares only the second and $2M$-th moment of the random variable $Z$, but it yields a sharper bound. Hence it can be more useful in certain applications.

Let us finally remark that actually we have proved a sharper result than Theorems 1 and 2. In those results we have defined the random variable $Z$ with the help of independent random variables $\varepsilon_j$ with distribution $P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = \frac{1}{2}$. But the proof of Theorems 1 and 2 also works without any change in the case of random variables with other distributions. Let us formulate this result. First I introduce the following notion.

Definition of sub-Gaussian distributions. Let us call a random variable $\xi$ or its distribution sub-Gaussian, if its moments satisfy the relations $E\xi^{2M-1} = 0$ and $E\xi^{2M} \leq E\eta^{2M}$ for all $M = 1, 2, \ldots$, where $\eta$ is a random variable with standard normal distribution.

It is clear that a random variable with distribution $P(\varepsilon = 1) = P(\varepsilon = -1) = \frac{1}{2}$ is sub-Gaussian. Because of some symmetrization arguments applied in probability theory this seems to be the most important example of sub-Gaussian random variables, but the following result holds for all of them.
Theorem 3. Let \( \varepsilon_1, \ldots, \varepsilon_n \) be independent sub-Gaussian random variables (with possibly different distributions). Let us define the random variable \( Z \) by formula (1.2) by the replacement of the original random variables \( \varepsilon_1, \ldots, \varepsilon_n \) with these new random variables \( \varepsilon_1, \ldots, \varepsilon_n \). This new random variable \( Z \) also satisfies the estimate (1.4) of Theorem 1 and the estimate (1.5) of Theorem 2.

Theorem 3 means that the distribution and moments of homogeneous polynomials of independent sub-Gaussian random variables satisfy such estimates as the distribution and moments of homogeneous polynomials of Gaussian random variables. Here the sub-Gaussian property plays a most essential role. In the case of homogeneous polynomials of independent, but not necessarily sub-Gaussian random variables the situation is much more complex. But this problem will not be discussed here.

Appendix

To prove the inequality formulated at the end of Lemma 2 we need a result which expresses the expected value of the product of multiple Wiener–Itô integrals in an appropriate way. To formulate this result which is the simple consequence of a basic result of the theory of Wiener–Itô integrals, the so-called diagram formula, first I have to introduce some notations. Let me recall that given a \( \sigma \)-finite measure \( \mu \) on some measurable space \((X, \mathcal{X})\) we call a white noise with counting measure \( \mu \) such a Gaussian field \( \mu_W(A), A \in \mathcal{X} \), indexed by the measurable sets of \( X \) which satisfies the relations \( \mathbb{E}\mu_W(A) = 0 \) and \( \mathbb{E}\mu_W(A)\mu_W(B) = \mu(A \cap B) \) for all \( A, B \in \mathcal{X} \).

Let us have a \( \sigma \)-finite measure \( \mu \) together with a white noise \( \mu_W \) with counting measure \( \mu \) on \((X, \mathcal{X})\). Let us consider \( L \) real valued functions \( f_i(x_1, \ldots, x_{k_l}) \) on \((X^{k_1}, \mathcal{X}_l)\) such that \( \int f_i^2(x_1, \ldots, x_{k_l})\mu(dx_1) \ldots \mu(dx_{k_l}) < \infty, 1 \leq l \leq L \). Let us consider the Wiener–Itô integrals \( k_l!J_{\mu,k_l}(f_l) = \int f_l(x_1, \ldots, x_{k_l})\mu_W(dx_1) \ldots \mu_W(dx_{k_l}), 1 \leq l \leq L \), and let us describe how the expected value \( \mathbb{E}\left(\prod_{l=1}^{L} k_l!J_{\mu,k_l}(f_l)\right) \) can be calculated by means of the diagram formula.

For this goal let us introduce the following notations. Put

\[
F(x_{(l,j)}, 1 \leq l \leq L, 1 \leq j \leq k_l) = \prod_{l=1}^{L} f_l(x_{(l,1)}, \ldots, x_{(l,k_l)}), \tag{A1}
\]

and define a class of diagrams \( \Gamma(k_1, \ldots, k_L) \) in the following way: Each diagram \( \gamma \in \Gamma(k_1, \ldots, k_L) \) is a (complete, undirected) graph with vertices \((l,j), 1 \leq l \leq L, 1 \leq j \leq k_l\), and we shall call the set of vertices \((l,j)\) with a fixed index \( l \) the \( l \)-th row of the graphs \( \gamma \in \Gamma(k_1, \ldots, k_L) \). The graphs \( \gamma \in \Gamma(k_1, \ldots, k_L) \) will have edges with the following properties. Each edge connects vertices \((l,j)\) and \((l',j')\) from different rows, i.e. \( l \neq l' \) for the end-points of an edge. From each vertex there starts exactly one edge. \( \Gamma(k_1, \ldots, k_L) \) contains all graphs \( \gamma \) with such properties. If there is no such graph, then \( \Gamma(k_1, \ldots, k_L) \) is empty.
Put $2N = \sum_{l=1}^{L} k_l$. Then each $\gamma \in \Gamma(k_1, \ldots, k_L)$ contains exactly $N$ edges. If an edge of the diagram $\gamma$ connects some vertex $(l, j)$ with some other vertex $(l', j')$, $l' > l$, then we call $(l', j')$ the lower end-point of this edge, and we denote the set of lower end-points of $\gamma$ by $A_{\gamma}$ which has $N$ elements. Let us also introduce the following function $\alpha_\gamma$ on the vertices of $\gamma$. Put $\alpha_\gamma(l, j) = (l, j)$ if $(l, j)$ is the lower end-point of an edge, and $\alpha_\gamma(l, j) = (l', j')$ if $(l, j)$ is connected with the point $(l', j')$ by an edge of $\gamma$, and $(l', j')$ is the lower end-point of this edge. Then we define the function

$$ F_\gamma(x_{(l,j)}, (l,j) \in A_\gamma) = F(x_{\alpha_\gamma(l,j)}, 1 \leq l \leq L, 1 \leq j \leq k_l) $$

with the function $F$ introduced in (A1), i.e. we replace the argument $x_{(l,j)}$ by $x_{(l',j')}$ in the function $F$ if $(l, j)$ and $(l', j')$ are connected by an edge in $\gamma$, and $l' > l$. Then we enumerate the lower end-points somehow, and define the function $B_\gamma(r), 1 \leq r \leq N$, such that $B_\gamma(r)$ is the $r$-th lower end-point of the diagram $\gamma$. Write

$$ F_\gamma(x_1, \ldots, x_N) = \tilde{F}_\gamma(x_{B_\gamma(r)}), 1 \leq r \leq N $$

and

$$ F_\gamma = \int \cdots \int F_\gamma(x_1, \ldots, x_N) \mu(dx_1) \cdots \mu(dx_N) \quad \text{for all } \gamma \in \Gamma(k_1, \ldots, k_L). $$

Now we formulate the corollary of the diagram formula we need.

**Theorem B.** With the above introduced notation

$$ E \left( \prod_{l=1}^{L} k_l! J_{\mu,k_l}(f_l) \right) = \sum_{\gamma \in \Gamma(k_1, \ldots, k_L)} F_\gamma. $$

(If $\Gamma(k_1, \ldots, k_L)$ is empty, then the expected value of the above product of random integrals equals zero.) Beside this

$$ F_\gamma^2 \leq \prod_{l=1}^{L} \int f_l^2(x_1, \ldots, x_{k_l}) \mu(dx_1) \cdots \mu(dx_{k_l}) \quad \text{for all } \gamma \in \Gamma(k_1, \ldots, k_L). $$

Now we turn to the proof of the inequality

$$ E \left( k! J_{\mu,k}(f) \right)^{2M} \leq 1 \cdot 3 \cdot 5 \cdots (2kM - 1) \left( \int f^2(x_1, \ldots, x_{k}) \mu(dx_1) \cdots \mu(dx_{k}) \right)^M. \quad (A2) $$

**Proof of Relation** (A2). Relation (A2) can be simply proved with the help of Theorem B if we apply it with $L = 2M$ and the functions $f_l(x_1, \ldots, x_{k_l}) = f(x_1, \ldots, x_k)$ for all $1 \leq l \leq 2M$. Then Theorem B yields that

$$ E \left( k! J_{\mu,k}(f)^{2M} \right) \leq \left( \int f^2(x_1, \ldots, x_{k}) \mu(dx_1) \cdots \mu(dx_{k}) \right)^M \mid \Gamma_{2M}(k), $$
where \(|\Gamma_{2M}(k)|\) denotes the number of diagrams \(\gamma\) in \(\Gamma(k, \ldots, k)\). Thus to complete the proof of relation (A2) it is enough to show that \(|\Gamma_{2M}(k)| \leq 1 \cdot 3 \cdot 5 \cdots (2kM - 1)\). But this can be seen simply with the help of the following observation. Let \(\bar{\Gamma}_{2M}(k)\) denote the class of all graphs with vertices \((l, j), 1 \leq l \leq 2M, 1 \leq j \leq k,\) such that from all vertices \((l, j)\) exactly one edge starts, all edges connect different vertices, but we also allow edges connecting vertices \((l, j)\) and \((l, j')\) with the same first coordinate \(l\). Let \(|\bar{\Gamma}_{2M}(k)|\) denote the number of graphs in \(\bar{\Gamma}_{2M}(k)\). Then clearly \(|\Gamma_{2M}(k)| \leq |\bar{\Gamma}_{2M}(k)|\).

On the other hand, \(|\bar{\Gamma}_{2M}(k)| = 1 \cdot 3 \cdot 5 \cdots (2kM - 1)\). Indeed, let us list the vertices of the graphs from \(\bar{\Gamma}_{2M}(k)\) in an arbitrary way. Then the first vertex can be paired with another vertex in \(2kM - 1\) way, after this the first vertex from which no edge starts can be paired with \(2kM - 3\) vertices from which no edge starts. By following this procedure the next edge can be chosen \(2kM - 5\) ways, and by continuing this calculation we get the desired relation.

References

1.) Borell, C. (1979) On the integrability of Banach space valued Walsh polynomials. Séminaire de Probabilités XIII, Lecture Notes in Math. 721 1–3. Springer, Berlin.

2.) Dudley, R. M. (1998) Uniform Central Limit Theorems. Cambridge University Press, Cambridge U.K.

3.) Dynkin, E. B. and Mandelbaum, A. (1983) Symmetric statistics, Poisson processes and multiple Wiener integrals. Annals of Statistics 11, 739–745

4.) Major, P. (1981) Multiple Wiener–Itô integrals. Lecture Notes in Mathematics 849, Springer Verlag, Berlin Heidelberg, New York,

5.) Major, P. (2004) On a multivariate version of Bernstein’s inequality. Submitted to Ann. Probab.

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