AX-SCHANUEL WITH DERIVATIVES FOR MIXED PERIOD MAPPINGS

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ABSTRACT. We prove the Ax-Schanuel property of the derivatives of mixed period mappings. We also prove the jet space analogue of this result. The proofs use the Ax-Schanuel result for foliated principal bundles obtained by Blázquez-Sanz, Casale, Freitag, and Nagloo.

1. INTRODUCTION

1.1. Motivation. In 1971, Ax proved the function field analogue \cite{Ax} of the Schanuel conjecture for exponentials. This result was extended to other functions in variational Hodge theory, e.g. the j-function by Pila-Tsimerman \cite{PilaTsimerman}, uniformizations of Shimura varieties by Mok-Pila-Tsimerman \cite{MokPilaTsimerman}, period mappings by Bakker-Tsimerman \cite{BakkerTsimerman}, and mixed period mappings by Gao-Klingler \cite{GaoKlingler} and the author \cite{Chiu} independently. In Pila-Tsimerman \cite{PilaTsimerman} and Mok-Pila-Tsimerman \cite{MokPilaTsimerman}, Ax Schanuel theorems for derivatives of respectively the j-function and uniformizations of Shimura varieties were also included. In this paper, we generalize the above results by proving the Ax-Schanuel property of the derivatives of mixed period mappings.

These Ax-Schanuel results have applications on existential closedness problems. For instance, Ax-Schanuel for derivatives of j-functions by Pila-Tsimerman \cite{PilaTsimerman} was applied in the work of Aslanyan, Eterović, and Kirby \cite{Aslanyan} on the existential closedness problem for the j-function; while Ax-Schanuel for derivatives of uniformizations of Shimura varieties \cite{MokPilaTsimerman} is applied in the work of Eterović and Zhao \cite{EterovicZhao} on the same problem for uniformizations of Shimura varieties. One expects our result will have applications on existential closedness problems for mixed period mappings.

Moreover, Ax-Schanuel results have applications in Diophantine geometry. For instance, Ax-Schanuel for period mappings \cite{Baker} was used to prove Shafarevich conjectures for hypersurfaces \cite{Shafarevich1, Shafarevich2}; while Ax-Schanuel for mixed period mappings \cite{Chiu} \cite{GaoKlingler} was used in higher dimensional Chabauty-Kim method \cite{ChabautyKim}. Gao used the Ax-Schanuel theorem for mixed Shimura varieties of Kuga type \cite{Gao} to study the generic rank of Betti map \cite{DimitrovGao}, which was then used by Dimitrov-Gao-Habegger \cite{DimitrovGaoHabegger} to prove a uniform bound for the number of rational points on curves. Ax-Schanuel also have applications on the geometric aspects of

2020 Mathematics Subject Classification. 11G18 and 14G35.
the Zilber-Pink conjecture [8, 10, 13, 34] on unlikely intersections via the Pila-Zannier method [36]. One expects our results will also have applications in the study of rational points and on Zilber-Pink conjectures with derivatives for variations of mixed Hodge structures. A detailed account of the Zilber-Pink conjecture can be found in Pila’s recent book [33].

Recently, Bakker and Tsimerman [7] prove a geometric version of André’s generalization of the Grothendieck period conjecture. This generalizes many Ax-Schanuel theorems.

We will use the notions of Mumford-Tate groups, variations of mixed \( \mathbb{Z} \)-Hodge structures, and mixed period mappings. A reference is [24].

1.2. Statement of results. Let \( X \) be a smooth irreducible quasiprojective complex algebraic variety equipped with an admissible graded-polarized variation of mixed \( \mathbb{Z} \)-Hodge structures (VMHS). Let \( \eta \) be a Hodge generic point of \( X \). Let \( \Gamma \) be the monodromy group of the underlying local system of the variation. Let \( G \) be the identity component of the \( \mathbb{Q} \)-Zariski closure of \( \Gamma \) in the automorphism group of the underlying \( \mathbb{Q} \)-vector space of the generic Hodge structure at \( \eta \). Let \( G_u \) be the unipotent radical of \( G \). Let \( G(\mathbb{R})^+ \) be the identity component of \( G(\mathbb{R}) \). Let \( H_0 \) be the graded-polarized mixed Hodge structure at \( \eta \). Let \( D \) be the \( G(\mathbb{R})^+ G_u(\mathbb{C}) \)-orbit of \( H_0 \) in the classifying space \( \mathcal{M} \) of mixed Hodge structures with the same graded-polarization and Hodge numbers as \( H_0 \). Let \( \tilde{D} \) be the \( G(\mathbb{C})^- \)-orbit of \( H_0 \) in the compact dual \( \tilde{\mathcal{M}} \) of \( \mathcal{M} \). The precise definitions of \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) are in [4, p. 6].

Let \( G(\mathbb{Z}) := G(\mathbb{Q}) \cap GL(H_{\mathbb{Z}, \eta}) \). First assume \( \Gamma \subset G(\mathbb{Z}) \cap G(\mathbb{R})^+ =: G(\mathbb{Z})^+ \). This assumption is assumed everywhere outside Corollary [1, 3] Corollary [1, 4] and their proofs. Let \( \psi : X \to \Gamma \backslash D \) be the period mapping attached to the variation. Let \( \phi \) be the composition of \( \psi \) with \( \Gamma \backslash D \to G(\mathbb{Z})^+ \backslash D \). Let \( q : D \to G(\mathbb{Z})^+ \backslash D \) and \( q' : D \to \Gamma \backslash D \) be the quotient maps.

Let \( \mathcal{MT} := \mathcal{MT}_0 \) be the Mumford-Tate group of \( H_0 \). Let \( D^+_{\mathcal{MT}} \) be the connected mixed Mumford-Tate domain, i.e. the \( \mathcal{MT}(\mathbb{R})^+ \mathcal{MT}_u(\mathbb{C}) \)-orbit of \( H_0 \) in \( \mathcal{M} \).

**Definition 1.1.** Let \( H \) be any mixed \( \mathbb{R} \)-Hodge structure in \( D^+_{\mathcal{MT}} \). Let \( M \) be a normal algebraic \( \mathbb{R} \)-subgroup of the Mumford-Tate group \( \mathcal{MT}_H \) of \( H \). Let \( M_u \) be its unipotent radical. Let \( M(\mathbb{R})^+ \) be the identity component of \( M(\mathbb{R}) \). The \( M(\mathbb{R})^+ M_u(\mathbb{C}) \)-orbit \( D(M) \) of \( H \) is called a weak Mumford-Tate domain. For any such \( D(M) \subset D \), we say \( \phi^{-1} q(D(M)) \) is a weakly special subvariety of \( X \).

\(^1\)By [1, Corollary 6.7], weakly special subvarieties are indeed algebraic.

Let \( k \) be a non-negative integer. Let \( J_k(X, \tilde{D}) \) be the set of all \( k \)-jets of germs of holomorphic mappings between open subsets of \( X \) and \( \tilde{D} \). Since \( X \) and \( \tilde{D} \) are algebraic, \( J_k(X, \tilde{D}) \) can be given an algebraic structure. Let \( \pi_X : J_k(X, \tilde{D}) \to X \) be the mapping defined by projecting the \( k \)-jet of a germ to the center of the
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Let $W_k$ (resp. $W_{k,\Gamma}$) be the analytic subset of $J_k(X, \tilde{D})$ consists of all $k$-jets of germs of local liftings of the period mapping $\phi$ (resp. $\psi$). For any irreducible analytic subset $U$ of $W_k$, denote by $U^{\text{Zar}}$ its Zariski closure in $J_k(X, \tilde{D})$.

**Theorem 1.2.** Let $U$ be an irreducible analytic subset of $W_k$. If
\[
\dim U^{\text{Zar}} - \dim U < \dim W_k^{\text{Zar}} - \dim W_k,
\]
then $\pi_X(U)$ is contained in a proper weakly special subvariety of $X$.

This theorem will be used to deduce the following corollary without the assumption that $\Gamma \subseteq G(\mathbb{Z}) \cap G(\mathbb{R})^+$.

**Corollary 1.3.** Let $U$ be an irreducible analytic subset of $W_{k,\Gamma}$. If
\[
\dim U^{\text{Zar}} - \dim U < \dim W_{k,\Gamma}^{\text{Zar}} - \dim W_{k,\Gamma},
\]
then $\pi_X(U)$ is contained in a proper weakly special subvariety of $X$.

Let $\Delta$ be the open unit disk. We have the following version of mixed Ax-Schanuel in terms of transcendence degree and derivatives.

**Corollary 1.4.** Let $\tilde{\phi}$ be a local lifting of the period mapping $\phi$ on an open subset $B$. Let $v : \Delta^\dim \tilde{D} \to \tilde{D}$ and $u : \Delta^\dim X \to B$ be open embeddings such that $(\tilde{\phi} \circ u)(\Delta^\dim X) \subseteq v(\Delta^\dim \tilde{D})$. Let $f : \Delta^m \to B$ be a holomorphic mapping such that $f(\Delta^m) \subset u(\Delta^\dim X)$. Write $z = (z_1, \ldots, z_m)$, where $z_i$ are the coordinates of $\Delta^m$. If
\[
\text{tr. deg.}_C \mathbb{C}((u^{-1} \circ f)(z), \partial^\alpha (v^{-1} \circ \tilde{\phi} \circ f)(z) : |\alpha| \leq k) < \text{rank}(f) + \dim W_{k,\Gamma}^{\text{Zar}} - \dim W_{k,\Gamma},
\]
then $f(\Delta^m)$ is contained in a proper weakly special subvariety of $X$.

### 1.3. Some recent literature on Ax-Schanuel for other functions.

Recently, there are several works proving Ax-Schanuel theorems for other functions. Baldi and Ullmo [9] prove the Ax-Schanuel theorem for certain non-arithmetic ball quotients. They use Simpson’s theory in addition to o-minimality and monodromy (André-Deligne). Blázquez-Sanz, Casale, Freitag, and Nagloo [11] prove the Ax-Schanuel theorem with derivatives for uniformizers of any Fuchsian group of the first kind and any genus. Their proof uses Ax’s arguments, foliated principal bundles, the Maurer-Cartan structure equation, and the model theory of differentially closed fields. Huang and Ng [22] prove the Ax-Schanuel theorem for certain meromorphic functions using Nevanlinna theory. Papas [30] proves the Ax-Schanuel theorem for the exponential functions for general linear groups.

### 1.4. Strategy.

Blázquez-Sanz, Casale, Freitag, and Nagloo established in [11] the Ax-Schanuel theorem for analytically foliated complex algebraic principal bundles. They proved that if the algebraic group acting on the bundle is sparse (a notion introduced in their paper concerning the analytic subgroups), and if the dimension of an algebraic subvariety of the bundle does not drop too much after intersection with a leaf, then the projection of the intersection under the
bundle map is contained in a $\nabla$-special subvariety, which was also introduced in their paper.

To use their result, we prove in Section 2 that when $k \gg 0$, the set $P := G(\mathbb{C}) \cdot W_k$ is an algebraic principal bundle over $X$, and that there is a foliation on $P$ where each leaf is of the form $g \cdot W_{k,\Gamma}$ for some $g \in G(\mathbb{C})$. In particular, the algebraicity is proved in Lemma 2.2 using the definable Chow theorem of Peterzil-Starchenko \cite{32} and the definable fundamental set for the action of $G(\mathbb{Z})^+$ on $D$ constructed in \cite{13}. The freeness of the group action on the fibers is proved in Lemma 2.4 using the Griffiths conjecture proved by Bakker-Brunebarbe-Tsimerman \cite{5}, the Ax-Schanuel (without derivatives) for $G$-unipotent Levi decomposition of $G(\mathbb{C})$ in \cite{11} to prove that $G(\mathbb{C})$ is sparse. Then in Section 3, we prove our main theorems for all $k \geq 0$ by applying the aforementioned Ax-Schanuel theorem for principal bundles \cite{11} followed by projection to lower order jet spaces.

1.5. Acknowledgements. I would like to thank my advisor Jacob Tsimerman for helpful discussions and comments.

2. Foliated jet bundle attached to the mixed period mapping

A subset of $\mathbb{R}^n$ is said to be definable if it is definable in the o-minimal structure $\mathbb{R}_{an,exp}$ \cite{16}. We refer to \cite{23} Section 2 for an introduction to o-minimality.

Let $K$ be the kernel of homomorphism $G_C \to \text{Aut}(\tilde{D})$ induced by the $G_C$-action on $\tilde{D}$. The group $G_C$ acts on $J_k(X,\tilde{D})$ by postcomposition.

By \cite{13} Section 5.1, there exists a definable open fundamental set $F$ for the action of $G(\mathbb{Z})^+$ on $D$. By \cite{4} Prop. 2.3, $q|_F$ is definable. Let $W_{k,F}$ be the analytic set of all $k$-jets of germs of local liftings into $F$ of the period mapping $\phi$. By \cite{4} Prop. 5.2, $W_{k,F}$ is definable.

Let $X$ and $Y$ be smooth complex manifolds. Let $J_k(X,Y)$ be the set of all $k$-jets of germs of holomorphic mappings between open subsets of $X$ and $Y$. For example, $J_0(X,Y) = X \times Y$ and $J_1(X,Y) = p_X^*\Omega_X \otimes p_Y^*TY$, where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections. An atlas of $J_k(X,Y)$ can be constructed by imposing a chart to be in the atlas if and only if it is of the form $(J_k(B,B'),\kappa)$, where $B \to \Delta^{\dim X}$ and $B' \to \Delta^{\dim Y}$ are charts of $X$ and $Y$ respectively, while $\kappa$ is defined by identifying $J_k(B,B')$ with $J_k(\Delta^{\dim X},\Delta^{\dim Y})$ and sending an element $j \in J_k(\Delta^{\dim X},\Delta^{\dim Y})$ to the tuple $(t,(\partial^*j(t))_{|t| \leq k})$, where $t \in \Delta^{\dim X}$ is the center of the jet $j$. The maximal atlas compatible with this atlas is referred to as the complex analytic structure of $J_k(X,Y)$. When $X$ and $Y$ are algebraic, this structure is the analytification of the complex algebraic variety $J_k(X,Y)$.
A subset of an analytic variety is \textit{analytically constructible} if it is in the Boolean algebra generated by closed, complex analytic subvarieties.

**Lemma 2.1.** The set $W_{k,Γ}$ is an analytically constructible subset of $J_k(X, D)$.

\textit{Proof.} Let $X → J_k(X, Γ\setminus D)$ be the analytic map defined by sending $x ∈ X$ to the $k$-jet of the period mapping centered at $x$. Let $J_k(X, D) → J_k(X, Γ\setminus D)$ be the injective analytic map defined by pushing-forward a jet through $D → Γ\setminus D$. It is injective because $D → Γ\setminus D$ is étale. Denote $J_k(X, Γ\setminus D)$ by $Z$. The canonical projection map $X × Z J_k(X, D) → J_k(X, D)$ then maps onto $W_{k,Γ}$. Hence, $W_{k,Γ}$ is the projection to $J_k(X, D)$ of the preimage of the diagonal under $X × J_k(X, D) → Z × Z$. Since $Z$ is Hausdorff as a complex analytic space, the diagonal is closed in $Z$. The preimage of the diagonal is then analytically constructible in the projective compactification of $X × J_k(X, D)$, so $W_{k,Γ}$ is analytically constructible by the Chevalley-Remmert theorem [28, p. 291].

\textit{□}

**Lemma 2.2.** The set $P := G(ℂ) · W_k = G(ℂ) · W_{k,Γ}$ is an algebraically constructible subvariety of $J_k(X, ̃D)$.

\textit{Proof.} Define the algebraic morphism $Ψ : G(ℂ) × J_k(X, ̃D) → J_k(X, ̃D)$ by postcomposition. There exist projective compactifications $G(ℂ)'$ and $J_k(X, ̃D)'$ of $G(ℂ)$ and $J_k(X, ̃D)$ respectively, such that $Ψ$ extends to a rational map $Ψ' : G(ℂ)' × J_k(X, ̃D)' → J_k(X, ̃D)'$. By the Chevalley-Remmert theorem [28, p. 291], the set $P$, which is the image under $Ψ'$ of an analytically constructible set (Lemma 2.1), is analytically constructible. Moreover, $P := G(ℂ) · W_k = G(ℂ) · W_{k,Γ}$ is definable. By the definable Chow theorem of Peterzil-Starchenko [32], $P$ is algebraically constructible.

\textit{□}

**Lemma 2.3.** We have $P = W^Z_{k,Γ} = W^Z_k$.

\textit{Proof.} Since $Γ ⊂ G(ℤ)^+ ⊂ G(ℤ)$ and since $G$ is the identity component of the $Q$-Zariski closure of $Γ$, the $Q$-closure of $Γ$ is $G$. By [13, Lemma A.1], the $C$-Zariski closure of $Γ$ is $G_{C}$. Then since $Γ · W_{k,Γ} = W_{k,Γ}$, we have $G_{C} · W^Z_{k,Γ} = W^Z_{k,Γ}$. By Lemma 2.2, $G(ℂ) · W_{k,Γ} = G(ℂ) · W_k =: P$ is algebraic, so $P = W^Z_{k,Γ} = W^Z_k$.

\textit{□}

We explain the idea of the proof of the following lemma. We first use the Griffiths conjecture proved by Bakker-Brunebarbe-Tsimerman [5] to reduce to the case where the liftings are submersions onto its image. Hence if $g ∈ G(ℂ)$ stabilizes the germ of a local lifting, then $g$ fixes the image of the lifting. By Ax-Schanuel for mixed period mappings [13, 20], this will imply that $g ∈ K$. By Noether’s chain condition, similar statement holds when the germ is truncated at some finite order. We then make this order independent of the local lifting using the definable Chow theorem [32] and the chain condition the second time.

**Lemma 2.4.** There exists an integer $k_0 > 0$ such that $K$ is the $G_{C}$-stabilizer of any jet in $W_k$ for any $k ≥ k_0$. 

Proof. Since $K$ is normal in $G_C$, it suffices to show that there exists an integer $k_0 > 0$ such that $K$ is the $G_C$-stabilizer of any jet in $W_{k,F}$ for any $k \geq k_0$.

Let $j$ be the germ of a local lifting into $F$ of the period mapping $\phi : X \to G(\mathbb{Z})^+ \setminus D$. Let $j_k$ be the $k$-jet of $j$. Let $S_{j,k}$ be the $G_C$-stabilizer of $j_k$. By the Griffiths conjecture for mixed period mappings proved in [5], there exists an algebraic variety $Y$ such that the period mapping is the composition of a dominant algebraic morphism $f : X \to Y$ and a closed immersion $\iota : Y \to G(\mathbb{Z})^+ \setminus D$. Let $X^o$ be the Zariski open subset of $X$ on which $f$ is smooth. By [25, Prop. 2.10], the inclusion $X^o \hookrightarrow X$ induces a surjection $\pi_1(X^o) \to \pi_1(X)$, so the VMHS has the same monodromy group after pulling back to $X^o$, and thus by Lemma [23] we can assume $X = X^o$ when proving Theorem [12]. Consider the analytic fiber product

$$
\begin{array}{ccc}
W_{0,\Gamma} & \longrightarrow & D \\
\downarrow & & \downarrow q \\
X & \psi \longrightarrow & \Gamma \setminus D.
\end{array}
$$

Let $K'$ be the pointwise $G(\mathbb{C})$-stabilizer of the image of $W_{0,\Gamma} \to D$. The germ $j$ is fixed by a conjugate $K''$ of $K'$ in $G(\mathbb{C})$. Since $f$ is smooth, it is surjective on tangent spaces, so $K'' = \bigcap_{k \geq 0} S_{j,k}$ by the identity theorem.

Let $a \in K'$. Let $D_a$ be the subset of elements in $D$ that are fixed by $a$. Let $V_{0,\Gamma} := X \times D_a$. We have $W_{0,\Gamma} \subset V_{0,\Gamma}$. The projection of $W_{0,\Gamma}$ to $X$ is equal to $X$. By the Ax-Schanuel for variations of mixed Hodge structures [13, 20], $\dim V_{0,\Gamma} \geq \dim W_{0,\Gamma} + \dim \bar{D}$, so $D_a = D$. Therefore, $K' = K$, so $K'' = K$ by normality of $K$.

The sequence $\{S_{j,k}\}_{k \geq 0}$ of subgroups of $G_C$ is decreasing. Since $G_C$ is Noetherian, there exists $k_j > 0$ such that $K = S_{j,k}$ for all $k \geq k_j$.

For any $k \geq 0$, let $X_k$ be the definable analytic subset of points in $X$ for which the $k$-jets of the germs of local liftings, centered at these points, into $F$ have $G_C$-stabilizer equal to $K$. Let $\alpha : G(\mathbb{C}) \times J_k(X, \bar{D}) \to J_k(X, \bar{D}) \times J_k(X, \bar{D})$

be defined by $(g, j) \mapsto (j, g \cdot j)$. Let $\Delta$ be the diagonal in $J_k(X, \bar{D}) \times J_k(X, \bar{D})$. Let $A_k := (G(\mathbb{C}) \setminus K \times W_{k,\Gamma}) \cap \alpha^{-1}(\Delta)$. The complement of the projection of $A_k$ in $X$ is equal to $X_k$. By Lemma [21], $W_{k,\Gamma}$ is analytically constructible, so $A_k$ and thus $X_k$ are analytically constructible. By definable Chow theorem [32] (see also [31, Cor. 2.3]), $X_k$ is algebraically constructible. From above, $X = \bigcup_{k \geq 0} X_k$. The sequence $\{X_k\}$ is increasing. Hence, there exists $k_0 > 0$ such that $X = X_k$ for all $k \geq k_0$. The claim follows. □

Theorem 2.5. Let $k \geq k_0$. The map $\pi_X|_P$ makes $P$ a principal $G_C / K$-bundle over $X$. There is a foliation on $P$ where each leaf is of the form $g \cdot W_{k,\Gamma}$ for some $g \in G(\mathbb{C})$, and vice versa. The leaves are transverse to the fibers of the bundle.
Proof. Let $\lambda : B \to D$ be a local lifting of $\phi : X \to \Gamma \backslash D$, such that $B$ is an open subset and that $\lambda(B)$ does not intersect any other $G(\mathbb{Z})^+$-translate of it. Let $W_{k,\lambda}$ be the analytic set of all $k$-jets of germs of $\lambda$. For any $x \in B$, let $J_{k,x,\lambda}$ be the $k$-jet of the germ of $\lambda$ at $x$. By Lemma 2.4, the map

$$\kappa : G(\mathbb{C})/K \times B \to (G(\mathbb{C})/K) \cdot W_{k,\lambda}$$

defined by $(gK, x) \mapsto g \cdot J_{k,x,\lambda}$ is a biholomorphism. We then have

$$\pi_X|_P^{-1}(B) = (G(\mathbb{C})/K) \cdot W_{k,\lambda} \simeq (G(\mathbb{C})/K) \times B.$$ 

Moreover, $G_C/K$ acts transitively and freely (by Lemma 2.4) on the fibers of $\pi_X|_P$.

Suppose $\lambda_1 : B_1 \to D$ is another local lifting on an open subset $B_1$ which overlaps with $B$. Similarly, we have $\pi_X|_P^{-1}(B_1) \simeq (G(\mathbb{C})/K) \times B_1$. By restricting $\lambda$ and $\lambda_1$ to $B \cap B_1$, we thus have an automorphism on $(G(\mathbb{C})/K) \times (B \cap B_1)$, which is a product of an automorphism of $G(\mathbb{C})/K$ and the identity on $B \cap B_1$.

Let $\mathcal{L}$ be the set of all local liftings $\lambda$ satisfying the condition as above. Let

$$\mathcal{S} := \{g \cdot W_{k,\lambda} : g \in G(\mathbb{C}), \lambda \in \mathcal{L}\}.$$ 

Define an equivalence relation $\sim$ on $\mathcal{S}$ as follows: $g_0 \cdot W_{k,\lambda_0} \sim g_\ell \cdot W_{k,\lambda_\ell}$ in $\mathcal{S}$ if and only if there exist $g_i \cdot W_{k,\lambda_i} \in \mathcal{S}$ for each $0 < i < \ell$, such that $g_{i-1} \cdot W_{k,\lambda_{i-1}} \cap g_i \cdot W_{k,\lambda_i} \neq \emptyset$ for all $1 \leq i \leq \ell$. Then we have a foliation on $P$ where each leaf has the same dimension as $X$ and is of the form

$$\bigcup_{g \cdot W_{k,\lambda} \sim g_0 \cdot W_{k,\lambda_0}} g_0 \cdot W_{k,\lambda_0} \in \mathcal{S},$$

and vice versa. Hence each leaf is of the form $g_0 \cdot W_{k,\Gamma}$ for some $g_0 \in G(\mathbb{C})$, and vice versa. The transversality follows from that $\kappa$ is a biholomorphism. 

\[\square\]

3. AX-SCHANUEL FOR FOLIATED PRINCIPAL BUNDLES

We recall the definitions of $\nabla$-special subvarieties and sparse groups, and the Ax-Schanuel theorem for foliated principal bundles proved by Blázquez-Sanz, Casale, Freitag, and Nagloo. Then we prove that any $\nabla$-special subvariety of $X$ is contained in a proper weakly special subvariety, and that $G(\mathbb{C})/K$ is sparse.

Let $G$ be a complex algebraic group. Let $\nabla$ be a flat principal $G$-bundle on a principal $G$-bundle $P$ over a complex algebraic variety $X$. The Galois group $\text{Gal}(\nabla)$ of $\nabla$ is the algebraic group $\{g \in G : g \cdot M = M\}$ for any minimal $\nabla$-invariant subvariety $M$ of $P$. A subvariety $Z$ of $X$ is $\nabla$-special if for each irreducible component $Z_i$ with smooth locus $Z_i^*$, the group $\text{Gal}(\nabla|_{Z_i^*})$ is a proper subgroup of $G$.

A Lie subalgebra of the Lie algebra $\mathfrak{g}$ of $G$ is said to be algebraic if it is the Lie algebra of an algebraic subgroup of $G$. The algebraic envelop $\overline{\mathfrak{h}}$ of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the smallest algebraic Lie subalgebra containing $\mathfrak{h}$. An algebraic group
$G$ is said to be **sparse** if for any proper Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the algebraic envelop $\overline{\mathfrak{h}}$ is a proper Lie subalgebra of $\mathfrak{g}$.

**Theorem 3.1** ([11]). Let $G$ be a sparse complex algebraic group. Let $\nabla$ be a flat principal $G$-connection on the principal $G$-bundle $P$ over a complex algebraic variety $X$. Assume that the Galois group $\text{Gal}(\nabla) = G$. Let $V$ be an algebraic subvariety of $P$ and $L$ an horizontal leaf. If $\dim V < \dim(V \cap L) + \dim G$, then the projection of $V \cap L$ in $X$ is contained in a $\nabla$-special subvariety.

**Theorem 3.2.** Let $P$ be the foliated principal $G\mathbb{C}/K$-bundle over $X$ in Theorem 2.3. Let $\nabla$ be the flat principal $G\mathbb{C}/K$-connection on $P$ induced by the foliation. If $Z$ is an irreducible $\nabla$-special subvariety in $X$, then $Z$ is contained in a proper weakly special subvariety. Moreover, $G\mathbb{C}/K = \text{Gal}(\nabla)$.

**Proof.** Let $\ell$ be a horizontal leaf over the smooth locus $Z^*$ of $Z$. By Theorem 2.3, each leaf in $P$ is of the form $g \cdot W_{k,\Gamma}$ for some $g \in G(\mathbb{C})$. Let $\Gamma_1$ be the monodromy group of $Z^*$. The group $g\Gamma_1g^{-1}$ stabilizes $\ell$. Therefore, the algebraic group $g\overline{\Gamma_1}g^{-1}$ stabilizes the Zariski closure $\overline{\ell}$ of $\ell$. By definition and [11] Lemma 2.2, $\text{Gal}(\nabla|_{Z^*}) = \{gK \in G\mathbb{C}/K : g \cdot \overline{\ell} = \overline{\ell}\}$, so $(g\overline{\Gamma_1}g^{-1})K/K \subset \text{Gal}(\nabla|_{Z^*})$. Then since $Z$ is $\nabla$-special, $(g\overline{\Gamma_1}g^{-1})K/K$ is a proper subgroup of $G\mathbb{C}/K$, so $\overline{\Gamma_1}$ is a proper subgroup of $G\mathbb{C}$. By André-Deligne [11] and the algebraicity [4, Corollary 6.7] of weakly special subvariety, $Z$ is contained in a proper weakly special subvariety. Similarly, $\overline{\Gamma}/K \subset \text{Gal}(\nabla)$. Since $\Gamma \subset G(\mathbb{Z})^+$, the $\mathbb{Q}$-closure of $\Gamma$ is $G$. By [13] Lemma A.4, the $\mathbb{C}$-Zariski closure of $\Gamma$ is $G\mathbb{C}$. Therefore,

$$G\mathbb{C}/K = \overline{\Gamma}/K \subset \text{Gal}(\nabla),$$

so $G\mathbb{C}/K = \text{Gal}(\nabla)$.

**Lemma 3.3.** If $G$ is an algebraic group whose quotient by its unipotent radical $G_u$ is semisimple, then $G$ is sparse.

**Proof.** Let $\mathfrak{g}, \mathfrak{g}_s$, and $\mathfrak{g}_u$ be the Lie algebras of $G, G_s$ and $G_u$ respectively, where $G_s$ is a Levi subgroup of $G$. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g} = \mathfrak{g}_s \ltimes \mathfrak{g}_u$. It is a general fact that $\mathfrak{h}$ is an ideal in its algebraic envelop $\overline{\mathfrak{h}}$ [11] Example 3.5]. Suppose $\overline{\mathfrak{h}} = \mathfrak{g}$. By [12] §6, no. 8, Cor. 4], $\mathfrak{h} \cap \mathfrak{g}_u$ is the radical of $\mathfrak{h}$ and $\mathfrak{h} \cap \mathfrak{g}_s$ is a Levi subalgebra of $\mathfrak{h}$, so $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{g}_s) \ltimes (\mathfrak{h} \cap \mathfrak{g}_u)$. The ideal $\mathfrak{h} \cap \mathfrak{g}_s$ of the semisimple Lie algebra $\mathfrak{g}_s$ is semisimple, so there exists an algebraic subgroup $H_1$ of $G_s$ whose Lie algebra is $\mathfrak{h} \cap \mathfrak{g}_s$. Moreover, the exponential map gives an algebraic variety isomorphism [29] Prop. 14.32 between the unipotent group $G_u$ and its Lie algebra, so there exists an algebraic subgroup $H_2$ of $G_u$ whose Lie algebra is $\mathfrak{h} \cap \mathfrak{g}_u$. The Lie algebra of the algebraic subgroup $H_1 \ltimes H_2$ of $G$ is thus $\mathfrak{h}$, so $\mathfrak{h} = \overline{\mathfrak{h}} = \mathfrak{g}$. Therefore, if $\mathfrak{h}$ is proper, then $\overline{\mathfrak{h}}$ is also proper.

**Corollary 3.4.** The algebraic monodromy group $G\mathbb{C}$ and the quotient $G\mathbb{C}/K$ are sparse.
Proof. Formations of the radical and the unipotent radical commute with field extensions in characteristic 0, so by André [11, Corollary 2], we can write $G = G_s \ltimes G_u$, where $G_s$ is a semisimple Levi subgroup, while $G_u$ is the unipotent radical and the radical. By Lemma 3.3 $G_C$ and $G_C/K$ are sparse. □

4. PROOFS OF MAIN THEOREM AND COROLLARIES

We now prove Theorem 1.2, Corollary 1.3, and Corollary 1.4, which are restated as Theorem 4.1, Corollary 4.2, and Corollary 4.3 below.

**Theorem 4.1.** Let $U$ be an irreducible analytic subset of $W_k$. If

$$\dim U^{Zar} - \dim U < \dim W_k^{Zar} - \dim W_k,$$

then $\pi_X(U)$ is contained in a proper weakly special subvariety of $X$.

**Proof.** Let $S$ be the set of all distinct representatives of the cosets in $G(\mathbb{Z})^+ / \Gamma$. We have

$$W_k = \bigcup_{g \in S} W_{k, \Gamma}$$

and

$$\dim W_k = \dim X = \dim W_{k, \Gamma}.$$

By Lemma 2.3, we have $P = W_k^{Zar}$. First assume $k \geq k_0$. Since $U$ is irreducible, $g^{-1}U \subset W_{k, \Gamma}$ for some $g \in S$. By Theorem 2.5 $W_{k, \Gamma}$ is a leaf in $P$ and $\dim P - \dim W_{k, \Gamma} = \dim(G_C/K)$. Then

$$\dim (g^{-1}U)^{Zar} = \dim U^{Zar}$$

$$< \dim U + \dim W_k^{Zar} - \dim W_k$$

$$= \dim g^{-1}U + \dim P - \dim W_{k, \Gamma}$$

$$\leq \dim((g^{-1}U)^{Zar} \cap W_{k, \Gamma}) + \dim(G_C/K).$$

We have $\pi_X(U) = \pi_X(g^{-1}U) \subset \pi_X((g^{-1}U)^{Zar} \cap W_{k, \Gamma})$. By Lemma 2.2 $(g^{-1}U)^{Zar} \subset P$. Then by Corollary 3.4 Theorem 3.1 and Theorem 3.2 $\pi_X(U)$ is contained in a proper weakly special subvariety of $X$.

We now prove the theorem for $1 \leq k < k_0$. Let $W_{k_0}$ be the irreducible analytic subset of $J_{k_0}(X, \overline{D})$ consists of all $k_0$-jets of germs of local liftings of the period mapping $\phi$. Let $P_{k_0} := G(\mathbb{C}) \cdot W_{k_0}$. Let $\rho : P_{k_0} \to P$ be the projection defined by lowering the order of jets. Let $U_{k_0} := W_{k_0} \cap \rho^{-1}(U)$, which implies that $U_{k_0}^{Zar} \subset \rho^{-1}(U^{Zar})$. We have

$$\dim \rho^{-1}(U^{Zar}) - \dim U^{Zar} \leq \dim P_{k_0} - \dim P.$$

Moreover, since $\rho|_{W_{k_0}}$ is equidimensional, we have

$$\dim W_{k_0} - \dim W_k = \dim U_{k_0} - \dim U.$$
Hence,
\[\dim U_{k_0}^{\text{Zar}} - \dim U_{k_0} \leq \dim \rho^{-1}(U_{k_0}^{\text{Zar}}) - \dim U_{k_0}\]
\[\leq \dim P_{k_0} - \dim P + \dim U_{k_0}^{\text{Zar}} - \dim U_{k_0}\]
\[< \dim P_{k_0} + \dim U - \dim W_k - \dim U_{k_0}\]
\[= \dim W_{k_0}^{\text{Zar}} - \dim W_k.\]

By the case for \(k = k_0\), \(\pi_X(U_{k_0})\) is contained in a proper weakly special subvariety of \(X\). We are done since \(\pi_X(U) \subset \pi_X(U_{k_0}).\)

**Corollary 4.2.** Let \(U\) be an irreducible analytic subset of \(W_{k, \Gamma}\). If
\[\dim U_{\text{Zar}}^{\text{Zar}} - \dim U < \dim W_{k, \Gamma}^{\text{Zar}} - \dim W_{k, \Gamma},\]
then \(\pi_X(U)\) is contained in a proper weakly special subvariety of \(X\).

**Proof.** Let \(\rho : \pi_1(X) \to \Gamma\) be the monodromy representation attached to the VMHS. Let \(\Gamma^0 = \Gamma \cap G(\mathbb{Z}) \subset G(\mathbb{R})^+\). Let \(A = \rho^{-1}(\Gamma^0)\). Then \(\pi_1(X)/A \cong \Gamma/\Gamma^0\) is finite since \(\Gamma^{\text{Zar}}/G\) and \(G(\mathbb{R})/G(\mathbb{R})^+\) are finite. Let \(\widetilde{X}\) be the finite covering of \(X\) such that \(\rho^*\pi_1(\widetilde{X}) = A\). The monodromy representation of the pullback of the VMHS to \(\widetilde{X}\) is given by \(\tilde{\rho} : \pi_1(\widetilde{X}) \to \Gamma^0\). Since \(\Gamma/\Gamma^0\) and \(\Gamma^{\text{Zar}}/G\) are finite, we have \(\dim \Gamma^{\text{Zar}}/\Gamma^0 = \dim \Gamma^{\text{Zar}} = \dim G\), so \(\Gamma^{\text{Zar}} = G\) by connectedness of \(G\). Let \(\widetilde{W}_k\) be the analytic subset of \(J_k(\widetilde{X}, \widetilde{D})\) consists of all \(k\)-jets of germs of local liftings of the period mapping \(\tilde{\phi} : \widetilde{X} \to \Gamma^0 \setminus D\). Since \(\widetilde{X} \to X\) is étale, there is a canonical étale map \(\widetilde{W}_k \to W_k\). Let \(\widetilde{U}\) be an irreducible component of the preimage of \(U\) under this map. There is also an étale map \(J_k(\widetilde{X}, D) \to J_k(X, D)\). We have
\[\dim \widetilde{U}_{\text{Zar}}^{\text{Zar}} - \dim \widetilde{U} \leq \dim U_{\text{Zar}}^{\text{Zar}} - \dim U\]
\[< \dim W_{k, \Gamma}^{\text{Zar}} - \dim W_{k, \Gamma}\]
\[\leq \dim W_{k}^{\text{Zar}} - \dim W_k.\]

By Theorem 4.1, the projection \(\pi_X(\widetilde{U})\) is contained in a proper weakly special subvariety of \(X\), so \(\pi_X(U)\) is contained in a proper weakly special subvariety of \(X\).

Let \(\Delta\) be the open unit disk.

**Corollary 4.3.** Let \(\tilde{\phi}\) be a local lifting of the period mapping \(\phi\) on an open subset \(B\). Let \(v : \Delta_{\dim \tilde{D}} \to \tilde{D}\) and \(u : \Delta_{\dim X} \to B\) be open embeddings such that \((\tilde{\phi} \circ u)(\Delta_{\dim X}) \subset v(\Delta_{\dim \tilde{D}})\). Let \(f : \Delta^m \to B\) be a holomorphic mapping such that \(f(\Delta^m) \subset u(\Delta_{\dim X})\). Write \(z = (z_1, \ldots, z_m)\), where \(z_i\) are the coordinates of \(\Delta^m\). If
\[\text{tr.deg.}_\mathbb{C}(u^{-1} \circ f)(z), \partial^\alpha(v^{-1} \circ \tilde{\phi} \circ f)(z) : |\alpha| \leq k < \text{rank}(f) + \dim W_{k, \Gamma}^{\text{Zar}} - \dim W_{k, \Gamma},\]
then \(f(\Delta^m)\) is contained in a proper weakly special subvariety of \(X\).
Proof. We have a map $\sigma : \Delta^n \to W_k$ defined by $\sigma(z) = J_{k,f(z)}\tilde{\phi}$, where $J_{k,f(z)}\tilde{\phi}$ is the $k$-jet of $\tilde{\phi}$ at $f(z)$. Let $U$ be the image of $\sigma$. Using the coordinate charts $u$ and $v$, the map $\sigma$ can be expressed as a tuple of functions, including $(u^{-1} \circ f)(z)$ and $\partial^\alpha (v^{-1} \circ \tilde{\phi} \circ f)(z)$, where $|\alpha| \leq k$. Then $\operatorname{rank}(f) \leq \operatorname{rank}(\sigma) = \dim U$. We also have
\[
\dim U^\text{Zar} = \operatorname{tr} \deg \mathbb{C}((u^{-1} \circ f)(z), \partial^\alpha (v^{-1} \circ \tilde{\phi} \circ f)(z)) : |\alpha| \leq k).
\]
Then by assumption, $\dim U^\text{Zar} < \dim U + \dim W_k^\text{Zar} - \dim W_{k,\Gamma}$, so $f(\Delta^n) \subset \pi_X(U)$ is contained in a proper weakly special subvariety of $X$ by Corollary 4.2. $\square$

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