Fractional Edge Reconstruction in Integer Quantum Hall Phases

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(Dated: July 29, 2020)

Protected edge modes are the cornerstone of topological states of matter. The simplest example is provided by the integer quantum Hall state at Landau level filling unity, which should feature a single chiral mode carrying electronic excitations. In the presence of a smooth confining potential it was hitherto believed that this picture may only be partially modified by the appearance of additional counter-propagating integer-charge modes. Here we demonstrate the breakdown of this paradigm: the system favors the formation of edge modes supporting fractional excitations. This accounts for previously unexplained observations, and leads to new predictions amenable to experimental tests.

Introduction.— Edge modes are responsible for many of the exciting properties of quantum Hall (QH) states [1]: While the bulk of a QH state is gapped, the edge supports one-dimensional gapless chiral modes [2]. Although several transport properties of these modes are universal and determined by the topological invariants characterizing the bulk state, their detailed structure depends on the interplay between the edge confining potential, electron-electron interaction, and disorder-induced backscattering. As the confining potential is made less steep, the chiral edges of integer [3–8] and fractional [9–15] QH phases and the helical edges of time-reversal-invariant topological insulators [16] may undergo a quantum phase transition (or “edge reconstruction”), while the bulk state remains untouched. Edge reconstruction may be driven by charging or exchange effects and leads to a change in the position, ordering, number and/or nature of the edge modes.

Arguably the simplest example is provided by the edge of the ν = 1 QH state. When confined by a sharp potential, this state supports a single gapless chiral integer mode with charge e∗ = 1; the electronic density steeply falls from its bulk value to zero at the edge. Smoothening the confining potential and accounting for the incompressibility of QH states leads to the formation of an outer, finite density reconstructed strip. Employing a self-consistent Hartree-Fock (HF) scheme, Chamon and Wen [3] found that this additional strip can be described as a ν = 1 QH state [Fig. 1(a)]. Such a state allows the local density to assume an integer value, leading to a smooth variation of the coarse-grained density from its bulk value to zero. Reconstruction introduces an additional pair of counter-propagating gapless chiral modes at the edge. The HF approximation is limited to Slater-determinant states, entailing these to be integer modes (e∗ = 1). Exact diagonalization confirms this picture [4] but is limited to very small systems, rendering it hard to confirm the precise filling factor of the side-strip or the nature of edge modes.

Recent transport experiments on the ν = 1 state [17–19] have led to some surprising observations regarding the edge structure. Exciting the ν = 1 edge at a quantum point contact (QPC), Ref. [17] observed a flow of energy but not charge upstream from the QPC, possibly indicating the presence of upstream neutral modes. Ref. [18] has studied the interference of the edge modes in an electronic Mach-Zehnder interferometer. As the bulk filling factor is reduced from 2 to less than 1, reduction in the visibility of the interference pattern has been observed, with full suppression for ν ≤ 1. This is another indication of the presence of upstream neutral modes [20]. However, it is inconsistent with Chamon and Wen’s picture of only integer charge modes, which can lead to upstream charge propagation, but not to upstream neutral modes. Ref. [18] also found a fractional conductance plateau with g = 1/3 × e2/h by partially pinching off a QPC in the ν = 1 bulk state. This too is incompatible with the edge structure of Fig. 1(a). To cap it all, the conductance plateau observed was accompanied by shot noise with a quantized Fano factor 1, which seems to suggest the edge modes do possess an integer charge. Ref. [19] also reported evidence of fractional modes at the ν = 1 edge.

Here we propose a novel picture of the reconstructed edge of the ν = 1 phase, and show that it accounts for all these seemingly contradictory observations. We establish that reconstruction may introduce a new type of counter-propagating modes, namely fractionally charged (e∗ = 1/3) modes. This is the case when the strip of electrons separated at the edge forms a ν = 1/3 Laughlin

FIG. 1. Schematic representation of two possible configurations at the reconstructed edge of ν = 1 state. Letting the confining potential become smoother, N3 electrons may separate from the bulk by L∗ guiding centers, forming a strip of (a) a ν = 1 state [3] or (b) a ν = 1/3 Laughlin state.
The energy of the unreconstructed state has been subtracted to make comparison easier. The blue (red) dots correspond to states with fractional charges. (c-d) depict the electronic densities of the reconstructed edge supports counterpropagating modes \( \nu = 1 \) and \( \nu = 2/3 \), may appear without an accompanying upstream charge flow. Such neutral modes have been observed in hole conjugate QH states \[17, 23–24\]. Below we show that this could account for the aforementioned Fano factor \[1\]. Moreover, neutral modes may also lead to suppression of interference in Mach-Zehnder interferometers \[20\], in line with existing experiments.

Basic Setup. — We consider a \( \nu = 1 \) state on a disk. In the symmetric gauge, \( \vec{A}/\hbar = (-y/2\ell^2, x/2\ell^2) \), the wavefunction of single-particle states in the lowest Landau level is \( \phi_m(\vec{r}) = (r/\ell)^m e^{-im\theta} e^{-\left(\pi\ell/\sqrt{2}\right)} e^{-2\pi m^2 + \pi m l^2} \), \( l \) is the magnetic length. Assuming spin polarized electrons and neglecting higher Landau levels, the Hamiltonian is \( H = H_{ee} + H_e \), \( H_{ee} \) is the interaction part while \( H_e \) is a circularly symmetric one-body confining potential. Denoting \( E_e = e^2/\epsilon_0 \ell, H_{ee} = (E_e/2) \sum_{m_1,m_2,n} V_{m_1,m_2,n}^e c_{m_1}^\dagger c_{m_2} c_{m_1+n} c_{m_2+n} \) and \( H_e = E_e \sum_m V_{m_1}^e c_{m_1}^\dagger c_{m_1} \), where \( V_{m_1}^e \) is the two-body Coulomb matrix element and \( V^e \) is the matrix element of the confining potential. The total angular momentum \( L \) is a good quantum number. The energy confining potential reads

\[
V_e(r) = \begin{cases} 
\frac{E_e}{2} (r-r_0+w_\ell), & r < r_0 - \frac{w_\ell}{2} \\
\frac{w_\ell}{2}, & r_0 - \frac{w_\ell}{2} < r < r_0 + \frac{w_\ell}{2} \\
0, & r > r_0 + \frac{w_\ell}{2}
\end{cases}
\]

where \( r_0 \) is the radius of a compact \( \nu = 1 \) state. The dimensionless parameter \( s \) sets the overall height of the potential, which we henceforth fix to \( s = 7 \). The steepness of the potential is controlled by the dimensionless width \( w \).

We consider two classes of variational states (shown in Fig. 1), corresponding to an integer [Chamon-Wen \[3\], Fig. 1(a)] and a fractional [Fig. 1(b)] reconstructed edge. Both are controlled by two parameters: The total occupancy \( N_S \) of the reconstructed edge strip, and the num-

![FIG. 2. Variational analysis for \( N_S + N_B = 100 \) and \( s = 7 \). (a-b) The energy of the two variational states as a function of the total angular momentum at (a) \( w = 6.0 \) and (b) \( w = 10.2 \). The energy of the unreconstructed state has been subtracted to make comparison easier. The blue (red) dots correspond to states with \( \nu = 1 \) \((\nu = 1/2)\) reconstruction at the edge. For sharp edges \((w < 10)\) the ground state is the one with minimum angular momentum, implying that \( L_S = 0 \), hence no edge reconstruction. In this case, we expect a single downstream edge mode supporting \( e^* = 1 \) quasiparticles. For smooth edges \((w > 10)\) the ground state shifts to a higher angular momentum sector implying that the electronic disk expands and the edge undergoes reconstruction. (b) shows that a fractional reconstruction is energetically favorable to an integer reconstruction. This is true for all \( w > 10 \). Thus the reconstructed edge supports counterpropagating modes with fractional charges. (c-d) depict the electronic densities of the ground state at (c) \( w = 6.0 \) and (d) \( w = 10.2 \). The non-monotonic variation of density at the edge is another signature of the presence of additional emergent modes.

state [Fig. 1(b)] instead of the commonly-assumed \( \nu = 1 \) state. To go beyond the constraints of the HF approximation [which imply an integer (0 or 1) occupation of each single particle state], we treat the two edge configurations depicted in Fig. 1 as variational states \[12\], and compare their respective energies for different strip size \((N_S)\) and separation \((L_S)\) as a function of the slope of the confining potential. We find that for smooth slopes the fractionally reconstructed edge [Fig. 1(b)] is energetically favorable. Our analysis then demonstrates that fractional edge reconstruction may be much more robust than integer reconstruction.

The intricate edge structure involving a downstream \( e^* = 1 \) mode along with a pair of counter-propagating \( e^* = 1/3 \) modes has several experimental consequences.
ber $L_S$ of empty orbitals separating it from the bulk. The latter contains $N_B$ electrons, such that the total number of electrons $N_S + N_B$ is fixed (to be 100). The Chamon-Wen family of states includes the compact edge configuration ($N_S = 0 = L_S$) which is the ground state for sharp confining potentials. For smoother confining potentials, the lowest energy state is expected to be at non-zero $N_S$ and $L_S$. In this case, a comparison of the energies of the states in the two classes determines whether fractionally charged modes could appear at the edge of the $\nu = 1$ phase.

**Variational ansatz: Integer edges.**— Fig. 1(a) represents a Slater determinant state of $N_S + N_B$ electrons. It can be written as $|N_B, 0\rangle \otimes |N_S, N_B + L_S\rangle$ where,

$$|N, L\rangle = c_{L+N-1}^{\dagger} c_{L+N-2}^{\dagger} \cdots c_L^{\dagger} c_0\rangle,$$  \hspace{1cm} (2)

The energy and angular momentum of each state in the integer class of reconstructions can be found easily once the Coulomb matrix elements are known [31].

**Variational ansatz: Fractional edges.**— Fig. 1(b) represents the product state of a Slater determinant ($|N_B, 0\rangle$) with an annulus of the $\nu = 1/3$ Laughlin state, containing $N_S$ electrons starting at the guiding center $m = N_B + L_S$. The (unnormalized) wavefunction corresponding to the annulus is,

$$\prod_{i=1}^{N_S} \left[z_i^{N_B+L_S}\right] \left[\prod_{i<j} (z_i - z_j)\right]^3 e^{-\frac{1}{2} \sum_i |z_i|^2},$$  \hspace{1cm} (3)

where $z_n = x_n - iy_n$ is the coordinate of the $n$th particle. The energy and angular momentum of states in this class involve the Coulomb energy and average occupations of the Laughlin state [Eq. [3]]. We evaluate these using standard classical Monte-Carlo techniques [31].

**Results.**— Fig. 2 shows the total energies and the ground state densities for the two class of variational states at different confining potentials. In Figs. 2(a) and 2(b) the blue dots correspond to integer edges while the red dots correspond to the fractional edge states. For a sharp confining potential $|w < 10$, Fig. 2(a)] the lowest energy state is the one with the minimal angular momentum (in this case 4950$h$). This corresponds to the unreconstructed $\nu = 1$ state with a single chiral edge mode. Fig. 2(c) shows the electronic density in this case, which drops monotonically from $1/2\pi e^2$ to 0.

For smoother potentials $|w > 10$, Fig. 2(b)] the lowest energy state has a much larger angular momentum $(5256h$ for $w = 10.2$) than the compact state. Correspondingly, Fig. 2(d) shows that the density varies non-monotonically at the edge. The states with a fractional edge are found to have a lower energy than the states with an integer edge whenever reconstruction is favored. This is the main result of this work. We now turn to discuss the experimental consequences of such a reconstruction and compare them to the observations reported in literature so far.

**Two-terminal conductance.**— Let us consider the setup shown in Fig. 3, where the edge structure is based on our analysis of a disk geometry. The chiral modes emanating from the source (S) are biased with respect to those emerging from the drain (D). Due to disorder-induced intermode tunnelling, the counterpropagating chiralos at each edge will equilibrate over a typical length $\ell_{eq}$. For a fully equilibrated edge ($L \gg \ell_{eq}$), the two-terminal conductance is $e^2/h$, as expected for the $\nu = 1$ QH state. Note that this would be the case for both sharp and smooth edges and for both integer and fractional reconstructions.

For $L \ll \ell_{eq}$, the detailed structure of the edge underlies the conductance. For a sharp edge transport takes place through a single integer chiral, hence the electric conductance would retain the values $e^2/h$. This is different for smooth edges. The electric conductance is sensitive to the number as well as the nature of the modes; with a pair of counterpropagating fractional chiralos, the electric conductance becomes $5/3 \times e^2/h$ [21] [22]. Such an observation would uniquely identify the edge structure proposed here [Fig. 1(b)] – a smoking gun signature of fractional edge reconstruction [37].

**Neutral modes.**— Consider the fractional reconstruction of Fig. 1(b). Labelling the outermost channel as 1 and the innermost edge as 3 [cf. Fig. 4(a)], the low energy dynamics of the three modes is described by three chiral bosonic fields $\phi_j$ ($j = 1, 2, 3$) satisfying the Kac-Moody algebra, $[\phi_j(x), \phi_{j'}(x')] = i\tau_j [K_{j,j'}, \phi_{j'}(x')]$, where the $K$-matrix is diagonal with $K_{1,1} = 3, K_{2,2} = -3, K_{3,3} = 1$. The inner two modes are counter-propagating charge modes of $\nu = 1$ and $\nu = 1/3$ type. This is precisely the edge structure of the hole-conjugate $\nu = 2/3$ FQH state. In the
The presence of disorder-induced backscattering and interactions the two charge modes can hybridize [Fig. 4(a)], resulting in a downstream charged mode \(\phi_c\) and an upstream neutral mode \(\phi_n\), which are effectively decoupled at low energies [13]. The new \(K\) matrix is diagonal with \(K_{1,1} = 3, K_{c,c} = 1, K_{n,n} = -1\). We note that here the outermost mode \((\phi_1)\) is kept untouched (cf. Fig. 4).

The experimental consequences of this emergent neutral mode are similar to the neutral modes in hole-conjugate states. For instance, it can lead to an upstream thermal current, which was reported in [17], accompanied by an upstream shot noise (see below) [38] [39]. The presence of the neutral mode can also hinder observation of interference effects in Mach-Zehnder setups [20] as reported in [18].

Fractional conductance plateau and noise.— The presence of fractionally charged chiral modes at the edge has clear experimental consequences for transport measurements. Consider for example the single QPC setup of Fig. 4(b). Here the bulk filling factor is \(\nu = 1\) and the current is transmitted from the source (S1) to the drain (D1). When the QPC is fully open then the conductance would be \(e^2/h\), as expected from the bulk topological index. However, due to the edge structure discussed above, it is also possible to pinch off the QPC, so that only the outermost mode \((\phi_1)\) is transmitted while the inner two modes are completely reflected. In this case there would be a fractional conductance plateau at \(1/3 \times e^2/h\) while the bulk filling factor remains 1. Such a plateau was reported in Ref. [18].

Interestingly, although the conductance is quantized, the system could exhibit shot noise on the conductance plateau. Under the assumption of coherent propagation of the neutral mode, and provided certain symmetry conditions are satisfied [28] [41], the Fano factor is quantized. Such a quantized noise at the at the 1/3 conductance plateau has been reported in Ref. [18]. Below we sketch the underlying physics relying on our fractionally reconstructed edge picture.

Consider the setup shown in Fig. 4(b). The source S1 on the upper left side of the QPC biases both charge modes emanating from it with the same voltage (say \(V\)). The current in the two modes is \(I_1 = V/3 \times e^2/h\), \(I_c = 2V/3 \times e^2/h\) and the total current is thus \(I = I_1 + I_c = V \times e^2/h\). The current \((I_i, i = 1, c)\) in a given mode is related to the corresponding quasiparticle density \((n_i)\) through \(I_1 = e/3 \times v_1 n_1\) and \(I_c = 2e/3 \times v_c n_c\), where \(v_i\) are the corresponding velocities, implying \(v_1 n_1 = v_c n_c\).

Therefore if \(N\) quasiparticles of charge \(\frac{1}{2}\) emanate from the S1 in time \(\tau\), then \(N\) quasiparticles of charge \(\frac{3}{2}\) also emanate in the same time interval. The total current \((I)\) is \(I = e/3 \times N/\tau + 2e/3 \times N/\tau = eN/\tau\).

Now, on the upper right side of the QPC, the outermost \(e/3\) mode is biased while the inner \(2e/3\) mode is grounded, and therefore the two modes will equilibrate through tunnelling processes, which would also create excitations in the neutral mode. If there were \(N\) quasiparticles in \(\phi_1\), then after equilibration with \(\phi_c\) there would be \(N/3\) quasiparticles left in both charged modes and \(2N/3\) neutral excitations in the upstream neutral mode. These neutral excitations would move to the lower right side of the QPC and decay into quasiparticle-quasihole pairs in the charge modes. This generates stochastic noise in the charged modes because each decay process can randomly generate either a quasiparticle (quasihole) in the outermost (inner) mode or vice versa. This decay process would lead to a stochastic tunneling of \(N/3\) electronic excitations into \(\phi_c\), which eventually reach the drain D1. Similarly, on the lower left side of the QPC, a biased \(2e/3\) mode flows in parallel to an unbiased \(e/3\) mode. Their mutual equilibration would again generate \(2N/3\) neutral excitations. These decay on the upper left side of the QPC and generate \(2N/3\) excitations in the \(\phi_1\) mode entering the drain D1.

As a result of the above, the charge entering the drain in time \(\tau\) is \(Q = e/3 \times N/3 + 2e/3 \times N/3 + e/3 \times \sum_{i=1}^{N/3} a_i + 2e/3 \times \sum_{i=1}^{N/3} b_i\), where \(a_i\) and \(b_i\) are random variables which take values \(\pm 1\) with equal probability, and describe the noise generated in the modes due to the neutral excitations decay described above. This implies that the average current arriving at the drain is \(I_D = \langle Q \rangle / \tau = eN/3 = I/3\) (consistent with a transmission of 1/3). The variance of the charge is \(\delta Q^2 = \langle Q^2 \rangle - \langle Q \rangle^2\).
\((Q^2) - \langle Q \rangle^2 = e^2/9 \times \sum_{i=1}^{2N/3} a_i^2 + 4e^2/9 \times \sum_{i=1}^{N/3} b_i^2 = 2Ne^2/9 = 2e/9 \times I\tau.\) The the effective Fano factor is \(F_{\text{eff}} = \frac{2}{9I\tau} / 1 / \text{et}(1 - t).\) Using \(t = 1/3\) we obtain \(F_{\text{eff}} = 1,\) which coincides with the observation of Ref. [15].

Conclusions. — We have studied edge reconstruction that at the boundary of \(\nu = 1\) integer quantum Hall state. Previously reported Hartree-Fock calculations show that upon smoothening the confining potential a new strip of \(\nu = 1\) QH state is formed at the edge, introducing counterpropagating integer modes [5]. Going beyond the mean-field approximation, we have performed a variational calculation, where we have compared the above ansatz to a new one, in which the electronic strip forms a \(\nu = 1/3\) Laughlin state. We have found that such fractional reconstruction is always energetically favorable, implying that fractional modes can appear at the boundary of integer QH states. We have discussed experimental consequences of such a fractionally reconstructed edge, which nicely square with previous unaccounted for measurements, and provide predictions for future experiments. Our finding sets the stage for a future detailed investigation of coherent as well as incoherent transport in designed geometries, implementing the idea of fractionally reconstructed edges.

We acknowledge useful discussions with M. Heiblum and J. Park. U. K. was supported by the Raymond and Beverly Sackler Faculty of Exact Sciences at Tel Aviv University and by the Raymond and Beverly Sackler Center for Computational Molecular and Material Science. M. G. and Y. G. were supported by the Israel Ministry of Science and Technology (Contract No. 3-12419). M. G. was also supported by the Israel Science Foundation (ISF, Grant No. 227/15) and US-Israel Binaitional Science Foundation (BSF, Grant No. 2016224). Y. G. was also supported by CRC 183 (project C01), the Minerva Foundation, DFG Grant No. RO 2247/8-1, DFG Grant No. MI 658/10-1 and the GIF Grant No. I-1505-303.10/2019.

[1] B. I. Halperin, Quantized hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential, Phys. Rev. B 25, 2185 (1982).

[2] X. G. Wen, Electrodynamical properties of gapless edge excitations in the fractional quantum hall states, Phys. Rev. Lett. 64, 2290 (1990).

[3] D. B. Chklovskii, B. I. Shklovskii, and L. I. Glazman, Electrostatics of edge channels, Phys. Rev. B 46, 4026 (1992).

[4] J. Dempsey, B. Y. Gelfand, and B. I. Halperin, Electron-electron interactions and spontaneous spin polarization in quantum hall edge states, Phys. Rev. Lett. 70, 3639 (1993).

[5] C. d. C. Chamon and X. G. Wen, Sharp and smooth boundaries of quantum hall liquids, Phys. Rev. B 49, 8227 (1994).

[6] A. Karlhede, S. A. Kivelson, K. Leijnell, and S. L. Sondhi, Textured edges in quantum hall systems, Phys. Rev. Lett. 77, 2061 (1996).

[7] Y. Zhang and K. Yang, Edge spin excitations and reconstructions of integer quantum hall liquids, Phys. Rev. B 87, 125140 (2013).

[8] U. Khanna, G. Murthy, S. Rao, and Y. Gefen, Spin mode switching at the edge of a quantum hall system, Phys. Rev. Lett. 119, 186804 (2017).

[9] A. H. MacDonald, Edge states in the fractional-quantum-hall-effect regime, Phys. Rev. Lett. 64, 220 (1990).

[10] M. D. Johnson and A. H. MacDonald, Composite edges in the \(\nu = 2/3\) fractional quantum hall effect, Phys. Rev. Lett. 67, 2060 (1991).

[11] A. H. MacDonald, E. Yang, and M. D. Johnson, Quantum dots in strong magnetic fields: Stability criteria for the maximum density droplet, Australian Journal of Physics 46, 345 (1993).

[12] Y. Meir, Composite edge states in the \(\nu = 2/3\) fractional quantum hall regime, Phys. Rev. Lett. 72, 2624 (1994).

[13] C. L. Kane, M. P. A. Fisher, and J. Polchinski, Randomness at the edge: Theory of quantum hall transport at filling \(\nu = 2/3\), Phys. Rev. Lett. 72, 4129 (1994).

[14] C. L. Kane and M. P. A. Fisher, Impurity scattering and transport of fractional quantum hall edge states, Phys. Rev. B 51, 13449 (1995).

[15] Y. N. Joglerak, H. K. Nguyen, and G. Murthy, Edge reconstructions in fractional quantum hall systems, Phys. Rev. B 68, 035332 (2003).

[16] J. Wang, Y. Meir, and Y. Gefen, Spontaneous breakdown of topological protection in two dimensions, Phys. Rev. Lett. 118, 046801 (2017).

[17] V. Venkatachalam, S. Hart, L. Pfeiffer, K. West, and A. Yacoby, Local thermometry of neutral modes on the quantum hall edge, Nature Physics 8, 676 (2012).

[18] R. Bhattacharyya, M. Banerjee, M. Heiblum, D. Mahalu, and V. Umansky, Melting of interference in the fractional quantum hall effect: Appearance of neutral modes, Phys. Rev. Lett. 122, 246801 (2019).

[19] T. Maiti, P. Agarwal, S. Purkait, G. J. Sreejith, S. Das, G. Basiol, L. Sorba, and B. Karmakar, Magnetic field dependent equilibration of fractional quantum hall edge modes, arXiv:2006.11621 (2020).

[20] M. Goldstein and Y. Gefen, Suppression of interference in quantum hall mach-zehnder geometry by upstream neutral modes, Phys. Rev. Lett. 117, 276804 (2016).

[21] I. Protopopov, Y. Gefen, and A. Mirlin, Transport in a disordered \(\nu = \frac{2}{3}\) fractional quantum hall junction, Annals of Physics 385, 287 (2017).

[22] C. Nosiglia, J. Park, B. Rosenow, and Y. Gefen, Incoherent transport on the \(\nu = 2/3\) quantum hall edge, Phys. Rev. B 98, 115408 (2018).

[23] A. Bid, N. Ofek, M. Heiblum, V. Umansky, and D. Mahalu, Shot noise and charge at the 2/3 composite fractional quantum hall state, Phys. Rev. Lett. 103, 236802 (2009).

[24] A. Bid, N. Ofek, H. Inoue, M. Heiblum, C. L. Kane, V. Umansky, and D. Mahalu, Observation of neutral modes in the fractional quantum hall regime, Nature 466, 585 (2010).

[25] I. Gurman, R. Sabo, M. Heiblum, V. Umansky, and D. Mahalu, Extracting net current from an upstream neutral mode in the fractional quantum hall regime, Nature Communications 3, 1289 (2012).

[26] Y. Gross, M. Dolev, M. Heiblum, V. Umansky, and
D. Mahalu, Upstream neutral modes in the fractional quantum hall effect regime: Heat waves or coherent dipoles, Phys. Rev. Lett. 108, 226801 (2012).

[27] H. Inoue, A. Grivnin, Y. Ronen, M. Heiblum, V. Umansky, and D. Mahalu, Proliferation of neutral modes in fractional quantum hall states, Nature Commun. 5, 4067 (2014).

[28] Y. Cohen, Y. Ronen, W. Yang, D. Banitt, J. Park, M. Heiblum, A. D. Mirlin, Y. Gefen, and V. Umansky, Synthesizing a $\nu = 2/3$ fractional quantum hall effect edge state from counter-propagating $\nu = 1$ and $\nu = 1/3$ states, Nature Comm. 10, 1920 (2019).

[29] J. Park, A. D. Mirlin, B. Rosenow, and Y. Gefen, Noise on complex quantum hall edges: Chiral anomaly and heat diffusion, Phys. Rev. B 99, 161302 (2019).

[30] C. Spanslatt, J. Park, Y. Gefen, and A. D. Mirlin, Conductance plateaus and shot noise in fractional quantum hall point contacts, Phys. Rev. B 101, 075308 (2020).

[31] See Supplemental Materials for more details about the variational calculations, which includes Refs. [32–36].

[32] E. V. Tsiper, Analytic coulomb matrix elements in the lowest landau level in disk geometry, J. Math. Phys. 43, 1664 (2002).

[33] J. K. Jain, *Composite Fermions* (Cambridge University Press, Cambridge, 2007).

[34] S. Mitra and A. H. MacDonald, Angular-momentum-state occupation-number distribution function of the Laughlin droplet, Phys. Rev. B 48, 2005 (1993).

[35] R. B. Laughlin, Anomalous quantum hall effect: An incompressible quantum fluid with fractionally charged excitations, Phys. Rev. Lett. 50, 1395 (1983).

[36] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, Equation of state calculations by fast computing machines, The Journal of Chemical Physics 21, 1087 (1953).

[37] The thermal Hall conductance is $3 \times \frac{\pi^2 k_B^2 T}{3h}$ for an unequilibrated edge for both integer and fractional reconstructions since it only depends on the number of chiralies participating in transport. For an equilibrated edge it reduces to $\frac{\pi^2 k_B^2 T}{3h}$, as expected for the $\nu = 1$ QH state.

[38] R. Sabo, I. Gurman, A. Rosenblatt, F. Lafont, D. Banitt, J. Park, M. Heiblum, Y. Gefen, V. Umansky, and D. Mahalu, Edge reconstruction in fractional quantum hall states, Nature Physics 13, 491 (2017).

[39] C. Spanslatt, J. Park, Y. Gefen, and A. D. Mirlin, Topological classification of shot noise on fractional quantum hall edges, Phys. Rev. Lett. 123, 137701 (2019).

[40] J. Wang, Y. Meir, and Y. Gefen, Edge reconstruction in the $\nu = \frac{2}{3}$ fractional quantum hall state, Phys. Rev. Lett. 111, 246803 (2013).

[41] J. Park, B. Rosenow, and Y. Gefen, Symmetry-related transport on a fractional quantum Hall edge, arXiv:2003.13727 (2020).
Supplemental material for “Fractional Edge Reconstruction in Integer Quantum Hall Phases”

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This set of supplemental materials provides additional details about the variational calculation used to find the lowest energy state for integer (Section I) and fractional (Section II) edge reconstructions.

I. INTEGER RECONSTRUCTION

Fig. 1(a) represents a Slater determinant of \( N_S + N_B \) electrons. For convenience, we write it as the product of two Slater determinants, \(|N_B,0 \rangle \otimes |N_S, N_B + L_S \rangle \) where

\[
|N, L \rangle = c_{L+N-1}^{\dagger} c_{L-N}^{\dagger} \cdots c_{L+1}^{\dagger} c_L^{\dagger} |0 \rangle. \tag{S1}
\]

The total angular momentum (in units of \( h \)) of \(|N, L \rangle \) is \( NL + N(N-1)/2 \), and that of the combined state is just the sum of the angular momenta of its two components

\[
N_S L_S + 1/2 (N_B + N_S)(N_B + N_S - 1). \tag{S2}
\]

The second term above is the angular momentum of the compact state (\( L_S = 0 \)). Thus the unreconstructed state has the smallest possible angular momentum for a fixed number of electrons \((N_S + N_B)\) in the lowest Landau level. We have used \( N_S + N_B = 100 \), which corresponds to minimum angular momentum 4950 \((h)\).

The energy of \(|N, L \rangle \) is

\[
\langle N, L | H_{ee} | N, L \rangle + \langle N, L | H_c | N, L \rangle,
\]

where

\[
\sum_{i,j = 0}^{N_L-1} \left( V^{ee}_{ij;0} - V^{ee}_{ij;j-i} \right), \tag{S3}
\]

\[
\sum_{i = 0}^{N_B - 1} \sum_{j = L_S}^{N_B + L_S - 1} V_c^{ee}_{i,j} \tag{S4}
\]

The energy of the full state consists of the sum of the energies of its constituents, as well as their two-body interaction energy,

\[
E_c \sum_{i=0}^{N_B-1} \sum_{j=L_S}^{N_B+L_S-1} \left( V^{ee}_{ij;0} - V^{ee}_{ij;j-i} \right). \tag{S5}
\]

Therefore, the energy and angular momentum of each state in the integer class of reconstructions can be computed easily once the matrix elements are known. In the disk geometry, the Coulomb matrix elements for lowest Landau level states can be found analytically \cite{[1], [2]}. The matrix elements of confining potentials are given by,

\[
V_m^c = \int d^2r \ V_c(r) |\phi_m(r)|^2 \tag{S6}
\]

II. FRACTIONAL RECONSTRUCTION

Fig. 1(b) represents the product state of a Slater determinant \((|N_B,0 \rangle \) with an annulus of the \( \nu = 1/3 \) Laughlin state \((|\Psi_3^1 \rangle \), containing \( N_S \) electrons starting at the guiding center \( m = N_B + L_S \). The (unnormalized) wavefunction corresponding to \(|\Psi_3^1 \rangle \) is,

\[
\prod_{i=1}^{N_B} \left[ z_i^{N_B+L_S} \right] \left[ \prod_{i<j} (z_i - z_j)^3 \right] e^{-\frac{1}{3} \sum_i |z_i|^2}, \tag{S7}
\]

where \( z_i = (x_i - iy_i)/\ell \) is the coordinate of the \( i \)th particle.

The angular momentum of the (standard) Laughlin state with \( N_S \) particles is \( \frac{2}{3} N_S (N_S - 1) \). Adding \( N_B + L_S \) holes in the center increases the angular momentum by \( N_B (N_B + L_S) \). Then the combined state has a total angular momentum \( N_S (L_S + N_S - 1) + \frac{2}{3} (N_B + N_S) (N_B + N_S - 1) \). Comparing this expression with that of the corresponding integer-edge state, we note that this is larger by \( N_S (N_S - 1) \). This indicates that the electronic density of the fractionally reconstructed state varies much more smoothly than the corresponding integer reconstructed state.

The energy of the combined state is the sum of the energy of the two components (the \( \nu = 1 \) bulk and the \( \nu = 1/3 \) annulus) and their mutual interaction energy. The energy of \(|\Psi_3^1 \rangle \) is \(|\langle \Psi_3^1 | H_{ee} |\Psi_3^1 \rangle + \langle \Psi_3^1 | H_c |\Psi_3^1 \rangle)/|\langle \Psi_3^1 |\Psi_3^1 \rangle|\) where

\[
\langle \Psi_3^1 |\Psi_3^1 \rangle = \prod_{i} d^2r_i |\Psi_3^1 \rangle^2; \tag{S8}
\]

\[
\langle \Psi_3^1 | H_{ee} |\Psi_3^1 \rangle = \prod_{i} d^2r_i |\Psi_3^1 \rangle^2 \left[ \sum_{i<j} E_c \ell \right], \tag{S9}
\]

\[
\langle \Psi_3^1 | H_c |\Psi_3^1 \rangle = E_c \sum_{m} \langle \Psi_3^1 | c_m^\dagger c_m |\Psi_3^1 \rangle V_m^c, \tag{S10}
\]
and its interaction energy with the bulk $\nu = 1$ state is

$$E_c \sum_{i=0}^{N_B-1} \sum_{j=LS} \frac{\langle \Phi_j | c_j^\dagger c_j | \Psi_3 \rangle}{\langle \Psi_3 | \Psi_3 \rangle} \left( V_{ij}^{ee} - V_{ij}^{ee_{i-1}} \right).$$

(S11)

These expressions involve the Coulomb energy and average occupations of the Laughlin states, which we evaluate using standard classical Monte-Carlo techniques [2] briefly described below.

Coulomb Energy

The Coulomb energy of $|\Psi_3\rangle$ is

$$\frac{1}{\prod_i d^2 r_i | \Psi_3 \rangle^2} \int \prod_i d^2 r_i | \Psi_3 \rangle^2 \left[ \sum_{i<j} E_c \ell (|r_i - r_j|) \right]. \quad (S12)$$

Since $|\Psi_3\rangle^2$ is real and positive, it can be interpreted as a (unnormalized) classical probability distribution [3].

Writing $|\Psi_3\rangle^2$ as a Boltzmann distribution $e^{-B_U}$, we can make this interpretation concrete by recognizing $U$ as the potential for a two-dimensional plasma of charged particles in presence of an impurity of charge $N_B + L_S$ at the origin. The Coulomb energy can then be computed using standard Metropolis sampling [6].

Average Occupation

The average occupation of $m^{th}$ single-particle state in $|\Psi_3\rangle$ is

$$\langle c_m^\dagger c_m \rangle_{1/3} = \frac{\langle \Psi_3 | c_m^\dagger c_m | \Psi_3 \rangle}{\langle \Psi_3 | \Psi_3 \rangle} = \int d^2 r_1 d^2 r_2 \rho_{\Psi_3} (\vec{r}_1, \vec{r}_2) \phi_m(\vec{r}_1) \phi_m(\vec{r}_2), \quad (S13)$$

where $\rho_{\Psi_3}$ is the one-particle density matrix of $|\Psi_3\rangle$,

$$\rho_{\Psi_3} (\vec{r}_1, \vec{r}_2) = \frac{1}{\prod_i d^2 r_i | \Psi_3 \rangle^2} \int \prod_i d^2 r_i | \Psi_3 \rangle^2 \Psi_3^\dagger (\vec{r}_1, \vec{r}_2, \cdots) \Psi_3 (\vec{r}_1, \vec{r}_2, \cdots). \quad (S14)$$

Computing $\rho_{\Psi_3}$ for all $\vec{r}_a$ and $\vec{r}_b$ using the above expression is very costly. To simplify the calculation, we note that both $\phi_m$ and $\Psi_3$ are eigenstates of the angular-momentum operator. Therefore the one-particle density matrix also satisfies

$$\rho_{\Psi_3} (\vec{r}_a, \vec{r}_b) = \sum_m \langle c_m^\dagger c_m \rangle_{1/3} \phi_m(\vec{r}_a) \phi_m(\vec{r}_b). \quad (S15)$$

In the special case of $\vec{r}_b = re^{i\theta_b}$ and $\vec{r}_a = re^{i\theta_a + i\theta}$, the above expression reduces to

$$\rho_{\Psi_3} (\vec{r}_b; \theta; \vec{r}_a) = \sum_m \langle c_m^\dagger c_m \rangle_{1/3} |\phi_m(\vec{r}_b)|^2 e^{-i m \theta}. \quad (S16)$$

Since $\langle c_m^\dagger c_m \rangle_{1/3}$ is non-zero over a contiguous, finite and known range of $m$ [namely from $m = N_B + L_S$ to $m = N_B + L_S + 3(N_S - 1)$], the summation over $m$ can be restricted to this range without any error. Then we may interpret the above relation as a discrete Fourier transform from $m$ to its conjugate $\theta$ [4]. Inverting the Fourier transform we get

$$\langle c_m^\dagger c_m \rangle_{1/3} |\phi_m(\vec{r})|^2 = \frac{1}{3(N_S - 1) + 1} \sum_{j=0}^{3(N_S - 1)} e^{i m \theta} \rho_j (\vec{r}; \theta; \vec{r}). \quad (S17)$$

where $\theta_j = 2\pi j / [3(N_S - 1) + 1]$. Note that Eq. (S17) is only true for $N_B + L_S \leq m \leq N_B + L_S + 3(N_S - 1)$. In principle Eq. (S17) is valid for any value of $r$, but in practice the statistical error is minimum when $r \sim \sqrt{2m} \ell$ [4]. Since for large $m$, $|\phi_m|^2$ is very sharply peaked at this value of $r$, in this work we evaluate the occupation by integrating Eq. (S17) over $\vec{r}$ to get,

$$\langle c_m^\dagger c_m \rangle_{1/3} = \frac{1}{3(N_S - 1) + 1} \sum_{j=0}^{3(N_S - 1)} e^{i m \theta} \rho_j, \quad (S18)$$

where $\rho_j = \int d^2 r |\psi_j (\vec{r}; \theta; \vec{r})|^2$. (S19)

Note that $\theta_j$ is not being integrated over in the previous expression. Then the occupation at any $m$ (within the appropriate range) can be found after we evaluate $\rho_j$ for all $j = 0, \cdots, 3(N_S - 1)$. Using Eq. (S14) we have,

$$\rho_j = \frac{1}{\prod_i d^2 r_i | \Psi_3 \rangle^2} \int \prod_{i=1}^{N_S} d^2 r_i \Psi_3 (\vec{r}_1 e^{i \theta_1}, \vec{r}_2, \cdots) \Psi_3^\dagger (\vec{r}_1, \vec{r}_2, \cdots). \quad (S20)$$

From the definition of $\Psi_3$ we obtain

$$\Psi_3 (\vec{r}_1 e^{i \theta_1}, \vec{r}_2, \cdots) = \Psi_3 (\{\vec{r}_1\}) \times Z_1 (\theta_j; \{\vec{r}_1\}), \quad (S21)$$

$$Z_1 (\theta_j; \{\vec{r}_1\}) = e^{-i \theta j (N_B + L_S)} \prod_{j \neq a} \frac{(z_a e^{-i \theta_j} - z_j)^3}{(z_a - z_j)^3}. \quad (S22)$$

Therefore, $\rho_j$ can be expressed as

$$\frac{1}{\prod_i d^2 r_i | \Psi_3 \rangle^2} \int \prod_i d^2 r_i | \Psi_3 \rangle^2 \sum_{a=1}^{N_S} \frac{Z_1 (\theta_j; \{\vec{r}_1\})}{N_S}, \quad (S23)$$
where we have symmetrized $Z$ over all particles to increase the rate of convergence. The above expression has the same form as Eq. (12) and can therefore be evaluated through very similar Metropolis sampling.

[1] E. V. Tsiper, Analytic coulomb matrix elements in the lowest landau level in disk geometry, J. Math. Phys. 43, 1664 (2002).
[2] J. K. Jain, Composite Fermions (Cambridge University Press, Cambridge, 2007).
[3] Y. Meir, Composite edge states in the $\nu=2/3$ fractional quantum hall regime, Phys. Rev. Lett. 72, 2624 (1994).
[4] S. Mitra and A. H. MacDonald, Angular-momentum-state occupation-number distribution function of the Laughlin droplet, Phys. Rev. B 48, 2005 (1993).
[5] R. B. Laughlin, Anomalous quantum hall effect: An incompressible quantum fluid with fractionally charged excitations, Phys. Rev. Lett. 50, 1395 (1983).
[6] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, Equation of state calculations by fast computing machines, The Journal of Chemical Physics 21, 1087 (1953).