ANALYSIS OF STOCHASTIC VECTOR-HOST EPIDEMIC MODEL WITH DIRECT TRANSMISSION

YANZHao CAO AND DAWIT DENU
Department of Mathematics and Statistics
Auburn University, Auburn, AL 36849, USA

Abstract. In this paper, we consider the stochastic vector-host epidemic model with direct transmission. First, we study the existence of a positive global solution and stochastic boundedness of the system of stochastic differential equations which describes the model. Then we introduce the basic reproductive number $R_0^s$ in the stochastic model, which reflects the deterministic counterpart, and investigate the dynamics of the stochastic epidemic model when $R_0^s < 1$ and $R_0^s > 1$. In particular, we show that random effects may lead to extinction in the stochastic case while the deterministic model predicts persistence. Additionally, we provide conditions for the extinction of the infection and stochastic stability of the solution. Finally, numerical simulations are presented to illustrate some of the theoretical results.

1. Introduction. Infectious diseases are the leading cause of deaths of children and adolescents especially in the developing countries [14,15,16,31]. The burden of infectious diseases is manifested through death as well as socioeconomic impacts.

Vector-borne disease is transmitted by the bite of infected arthropods such as mosquitoes, ticks, mites, rats and blackflies. It is one of the most common infections accounting for more than 17% of infectious diseases. Many vector-borne diseases, such as malaria, dengue fever, West Nile virus, and recently Zika virus, are transmitted to the human population by insects (vectors) such as mosquitoes. Some vector-borne diseases may also be transmitted directly through blood transfusions, organ transplantation, exposure in a laboratory setting, or from mother to baby during pregnancy, delivery and breast feeding [16,17,29]. Direct transmission has an impact on the dynamics of many vector-borne diseases.

One way to improve control and ultimately eradicate infectious diseases is to choose an appropriate model that best describes the demography and epidemiology of the population being modeled. Indeed, significant improvement has been made in mathematical research of epidemic problems.

Epidemic models can be deterministic, in which the output of the model is fully determined by the parametric and initial values, or stochastic, where the model possesses some inherent randomness. Deterministic vector-host epidemic models have been studied by several authors [1,2,6,8,20,21,25,26,32,37]. In [8], Cai and Li analyzed a simple vector-host epidemic model with direct transmission. Yang and...
Wei discussed the global stability of an epidemic model for vector-borne disease and they showed that the global dynamics are completely determined by the basic reproductive number $R_0$ [39]. Belayneh, et al. provided a cost effective control effort for treatment of hosts and prevention of vector-host contacts for a non-autonomous model, while they establish global stability conditions for the autonomous case [6]. Particular vector-borne diseases such as malaria [21,24,25,26,32], dengue fever [1,2,34] and West Nile virus [5,38] have been modeled and studied by a number of authors.

The deterministic vector-host epidemic models with direct transmission have been studied by Cai and Li [8]. They showed that the stability of the equilibria can be controlled by the basic reproductive number. Similar studies can also be found in [6,32,37]. On the other hand, Jovanovic and Krstic [18] discussed the corresponding stochastic model and studied the stability conditions using Lyapunov functions.

Even though deterministic models are crucial in understanding more about the dynamics of the disease, they do not incorporate the effect of environmental fluctuations. In reality, the system is exposed to influences that are not completely understood and the spread of the disease is inherently random. Thus, for a better analysis of the model, we will include such stochastic influences [35].

There are several ways to include these fluctuations in the deterministic model. For example, [9,11,33] introduced parametric perturbations, since the parameters in the model are always altered due to continuous environmental fluctuations. Another approach, pioneered in the works of May and Beddington [4], assumes that the environmental noise is generated by an $m$-dimensional standard Brownian motion. Some other authors used this idea to study the properties of stochastic epidemic models in order to find a more efficient way to reduce infections [7,9,30,36,40,41].

The aim of this paper is to study the well-posedness of the stochastic vector born disease model and examine how the introduction of stochastic noise affects the dynamics of the vector-host epidemic model with direct transmission. We shall show that the solution of the stochastic model is ultimately bounded in probability. We also seek the sufficient conditions for the solution to be stochastically permanent. Furthermore, we will investigate how the basic reproductive number is affected by the introduction of the stochastic noises.

The organization of the paper is as follows: In section 2, we formulate the stochastic epidemic model from the corresponding deterministic one. In section 3, we analyze the stochastic system using different Lyapunov functions and Ito formula. In the last section we provide numerical simulation using the Milstein method to support the theoretical results in the previous sections.

2. Stochastic model and well-posedness.

2.1. Deterministic model. Let $S_H, I_H$ respectively represent the number of susceptible and infected hosts and $S_V, I_V$ respectively represent the number of susceptible and infected vectors. Assume that susceptible hosts can be infected both directly through contact with an infected host, such as blood transfusion, and indirectly by a bite from an infected vector, such as a mosquito. Similarly, we assume that if a susceptible vector bites an infected host, it will acquire the disease. The model does not assume disease-induced deaths in both species, that is, no one has died from the disease in the given time. However, it can be easily modified to include such assumptions [23,27,32]. The graph below depicts the transmission cycle of the vector-host epidemic model. Here $\mu$ and $\eta$ are the mortality rates of the host
and vector, $\phi$ is the recovery rate of infected hosts, $\beta_1$ is the direct transmission rate from an infected host to susceptible host, and $\beta_2$ is the indirect transmission rate from an infected vector to a susceptible host, and $\beta$ represents the transition rate from infected host to susceptible vector. To begin with, let $b_1$ and $b_2$ be respectively the recruitment rates of the host and the vector. Based on the above discussion, we have the following system of non-linear differential equations [8,18].

$$\begin{align*}
\frac{dS_H}{dt} &= b_1 - \mu S_H - \beta_2 S_H I_V - \beta_1 S_H I_H + \phi I_H \\
\frac{dI_H}{dt} &= \beta_1 S_H I_H + \beta_2 S_H I_V - (\mu + \phi) I_H \\
\frac{dS_V}{dt} &= b_2 - \eta S_V - \beta S_V I_H \\
\frac{dI_V}{dt} &= \beta S_V I_H - \eta I_V.
\end{align*}$$

(1)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Vector-host epidemic model with direct transmission.}
\end{figure}

Let
\[ \Gamma := \{(S_H, I_H, S_V, I_V) \in \mathbb{R}^4 : S_H, I_H, S_V, I_V > 0, S_H + I_H = \frac{b_1}{\mu}, S_V + I_V = \frac{b_2}{\eta} \}. \]

Then it is easy to see that $\Gamma$ is positively invariant under system (1). Also, since the vector field associated with system (1) is bounded and continuous, and satisfies the Lipchitz condition on $\Gamma$, system (1) has a unique solution in $\Gamma$.

Next we introduce the basic reproductive number $R_0$ which is defined as the expected number of secondary cases produced by a single infection in a completely susceptible population. Using the next-generation matrix introduced by Diekmann [12], $R_0$ is given by the expression:

$$R_0 = \frac{\beta_1 b_1}{\mu (\mu + \phi)} + \frac{\beta \beta_2 b_1 b_2}{\eta^2 (\phi + \mu)}. \quad (2)$$

Now setting
\[ \frac{dS_H}{dt} = \frac{dI_H}{dt} = \frac{dS_V}{dt} = \frac{dI_V}{dt} = 0, \]
we obtain the disease-free equilibrium $E_0 = (S_H, I_H, S_V, I_V) = (\frac{b_1}{\mu}, 0, \frac{b_2}{\eta}, 0)$ and the endemic-equilibrium point $E_1 = (S_H^*, I_H^*, S_V^*, I_V^*)$, where
\[ S_V^* = \frac{b_2}{\eta + \beta I_H^*}, \quad I_V^* = \frac{\beta b_2}{\eta} \frac{I_H^*}{\eta + \beta I_H^*}, \quad S_H^* = \frac{\eta (\mu + \phi)(\eta + \beta I_H^*)}{\beta_1 \eta (\eta + \beta I_H^*) + \beta \beta_2 b_2}, \quad (3)\]
and $I_H^*$ is the positive solution of the equation,
\[ k_2 (I_H^*)^2 + k_1 I_H^* + k_0 = 0, \quad (4)\]
where
\[ k_0 = -\mu \eta^2 (\mu + \phi)(R_0 - 1), \quad k_2 = \beta_1 \beta \mu. \]
\[ k_1 = \phi \beta \eta \mu + \eta^2 \beta_1 \mu + \beta b_2 \beta_2 \mu + \beta \eta^2 - \beta b_1 \eta \beta_1. \]

The local and global stabilities of the equilibrium points can be summarized as follows [8].

**Theorem 2.1.** If \( R_0 < 1 \), then the disease-free equilibrium \( E_0 \) is both locally and globally asymptotically stable.

**Theorem 2.2.** If \( R_0 > 1 \), then the endemic equilibrium \( E_1 \) is both locally and globally asymptotically stable.

The above theorems indicate that \( R_0 \) is a sharp threshold to determine if the disease will die out or stay endemic (see Figure 2).

![Figure 2](image-url)

**Figure 2.** Trajectories of solution of the deterministic model (1), when \( R_0 = 0.71 \) and \( R_0 = 34.20 \) respectively.

### 2.2. Stochastic model

In this subsection we derive the stochastic epidemic model from the corresponding deterministic one by including a random effect in the deterministic case. Throughout the rest of this paper, let \((\Omega, \mathcal{F}, P)\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions, that is, it is right continuous and increasing with \(\mathcal{F}_0\) containing all \(P\)-null sets. Also, we define the differential operator \(\mathcal{L}\) associated with the stochastic differential equation

\[
\frac{dX(t)}{dt} = f(X(t), t) \, dt + g(X(t), t) \, dB
\]

by

\[
\mathcal{L} := \partial_t + f \frac{\partial}{\partial x} + g^2 \frac{\partial^2}{2 \partial x^2}.
\]

Let \( \Delta t > 0 \) be fixed and let \( X^{(\Delta t)}(t) = \left(S_{(\Delta t)}^H(t), I_{(\Delta t)}^H(t), S_{(\Delta t)}^V(t), I_{(\Delta t)}^V(t)\right) \) be a discrete time Markov chain (DTMC) for \( t \in \{0, \Delta t, 2\Delta t, \ldots\} \), such that \( X^{(\Delta t)}(0) \in \mathbb{R}_+^4 \). Also let

\[
\left\{R_{(\Delta t)}^{S_{(\Delta t)}}(k)\right\}_{k=0}^{\infty}, \left\{R_{(\Delta t)}^{I_{(\Delta t)}}(k)\right\}_{k=0}^{\infty}, \left\{R_{(\Delta t)}^{S_{(\Delta t)}}(k)\right\}_{k=0}^{\infty}, \left\{R_{(\Delta t)}^{I_{(\Delta t)}}(k)\right\}_{k=0}^{\infty}
\]

be sequences of random variables defined on \((\Omega, \mathcal{F}, P)\) which are jointly independent to each other and each sequence is identically distributed such that, for any \( k \in \{0, 1, 2, \ldots\} \)

\[
E[R_{(\Delta t)}^{S_{(\Delta t)}}(k)] = E[R_{(\Delta t)}^{I_{(\Delta t)}}(k)] = E[R_{(\Delta t)}^{S_{(\Delta t)}}(k)] = E[R_{(\Delta t)}^{I_{(\Delta t)}}(k)] = 0,
\]

and

\[
E \left[R_{(\Delta t)}^{S_{(\Delta t)}}(k)\right]^2 = \sigma_{S_{(\Delta t)}}^2 \Delta t, \quad E \left[R_{(\Delta t)}^{I_{(\Delta t)}}(k)\right]^2 = \sigma_{I_{(\Delta t)}}^2 \Delta t,
\]

\[
E \left[R_{(\Delta t)}^{S_{(\Delta t)}}(k)\right]^2 = \sigma_{S_{(\Delta t)}}^2 \Delta t, \quad E \left[R_{(\Delta t)}^{I_{(\Delta t)}}(k)\right]^2 = \sigma_{I_{(\Delta t)}}^2 \Delta t.
\]
where $E$ is the expectation and $\sigma_{S_H}, \sigma_{I_H}, \sigma_{S_V}, \sigma_{I_V}$ are some non-negative constants which show the intensity of the fluctuations.

Each sequence of random variables measures the effects of random influence on each compartment during $[k\Delta t, (k+1)\Delta t]$ for $k \in \{0, 1, 2, \ldots \}$. Thus, during $[k\Delta t, (k+1)\Delta t]$, each compartment changes according to the deterministic equation (1) and by a random amount, that is, for $k \in \{0, 1, 2, \ldots \},$

\[
S_H^{(\Delta t)}((k+1)\Delta t) = S_H^{(\Delta t)}(k\Delta t) + \Delta t(b_1 - \mu S_H - \beta_S I_V - \beta_I I_H + \phi I_H) + R_{S_H}^{(\Delta t)}(k)S_H^{(\Delta t)}(k\Delta t),
\]

\[
I_H^{(\Delta t)}((k+1)\Delta t) = I_H^{(\Delta t)}(k\Delta t) + \Delta t(\beta_S S_H I_V + \beta_I S_H I_V - (\mu + \phi)I_H) + R_{I_H}^{(\Delta t)}(k)I_H^{(\Delta t)}(k\Delta t),
\]

\[
S_V^{(\Delta t)}((k+1)\Delta t) = S_V^{(\Delta t)}(k\Delta t) + \Delta t(\beta_H S_H - \eta S_V - \beta_S V_H) + R_{S_V}^{(\Delta t)}(k)S_V^{(\Delta t)}(k\Delta t),
\]

\[
I_V^{(\Delta t)}((k+1)\Delta t) = I_V^{(\Delta t)}(k\Delta t) + \Delta t(\beta_S S_H - \eta I_V) + R_{I_V}^{(\Delta t)}(k)I_V^{(\Delta t)}(k\Delta t).
\]

Following a standard procedure [3,10,13,19] we can show that, as $\Delta t \to \infty$, $X^{\Delta t}(t)$ converges to a diffusion process $X(t) = (S_H(t), I_H(t), S_V(t), I_V(t))$ which satisfies the following system of stochastic differential equations.

\[
dS_H = (b_1 - \mu S_H - \beta_S S_H I_V - \beta_I S_H I_H + \phi I_H)dt + \sigma_{S_H} S_H dB_{S_H}(t),
\]

\[
dI_H = (\beta_I S_H I_H + \beta_S S_H I_V - (\mu + \phi)I_H)dt + \sigma_{I_H} I_H dB_{I_H}(t),
\]

\[
dS_V = (b_2 - \eta S_V - \beta_S V_H)dt + \sigma_{S_V} S_V dB_{S_V}(t),
\]

\[
dI_V = (\beta_S S_H - \eta I_V)dt + \sigma_{I_V} I_V dB_{I_V}(t).
\]

2.3. **Existence of a global solution.** Clearly, there exists a unique local solution $X(t)$ on an interval $[0, \tau]$ where $\tau$ is the explosion time [33,Theorem 3.1]. Next, we show that $\tau = \infty$, that is, this solution is in fact global.

Let $k_0 > 0$ such that $X(0) \in \left(\frac{1}{k_0}, k_0\right]^4$. Now, for any $k \in \mathbb{N}$ such that $k > k_0$ define

\[
\tau_k = \inf \left\{ t \in [0, \tau] : (S_H(t), I_H(t), S_V(t), I_V(t)) \notin \left(\frac{1}{k}, k\right)^4 \right\}
\]

then, $\{\tau_k\}_k$ is an increasing sequence. Denote $\tau = \lim_{k \to \infty} \tau_k$. Clearly $\tau \leq \tau_e$.

Next, we show that $\tau = \infty$. This implies that the explosion time is infinity and thus we conclude the model in (5) has a unique solution and will remain in $\Gamma$ with probability 1.

Suppose on the contrary that $\tau < \infty$. Then there exists $T > 0$ such that $P(\tau \leq T) > \epsilon$ for all $\epsilon \in (0, 1)$. This implies that there exists $k_1 > k_0$ such that $P(\tau_k \leq T) \geq \epsilon$ for all $k \geq k_1$. Define $V : \mathbb{R}_+^4 \to \mathbb{R}_+$ by

\[
V(X(t)) = (S_H - 1 - \ln S_H) + (I_H - 1 - \ln I_H) + (S_V - 1 - \ln S_V) + (I_V - 1 - \ln I_V).
\]

Using Itô’s formula we have

\[
dV = \mathcal{L}V dt + \sigma_{S_H}(S_H - 1)dB_{S_H} + \sigma_{I_H}(I_H - 1)dB_{I_H} + \sigma_{S_V}(S_V - 1)dB_{S_V} + \sigma_{I_V}(I_V - 1)dB_{I_V},
\]

where $\mathcal{L}$ is the infinitesimal generator of the diffusion.
Thus, \( (5) \) is always positive and remains in \( \Gamma \) and noting that for any \( \int \). Integrating both sides of the above inequality on \( (0, T) \), taking the expectation, and noting that for any \( G \in L^2(0, T), E \left( \int_0^T GdB \right) = 0 \), we obtain

\[
EV(X(\tau_k \wedge T)) \leq V(X(0)) + C_1 T.
\]

For \( k \in \mathbb{N} \) such that \( k \geq k_0 \), let \( A_k = \{ \tau_k \leq T \} \). Then, \( P(A_k) \geq \epsilon \). If \( t \in A_k \), then at least one of the following will hold true:

\[
S_H(t) \notin \left( \frac{1}{k}, k \right), \quad I_H(t) \notin \left( \frac{1}{k}, k \right), \quad S_V(t) \notin \left( \frac{1}{k}, k \right), \quad I_V(t) \notin \left( \frac{1}{k}, k \right).
\]

Generally, since \( f(x) = x - 1 - \ln x \) is increasing on \((1, \infty)\) and decreasing on \((0, 1)\) it follows

\[
V(X(\tau_k \wedge t)) \geq (k - 1 - \frac{1}{k}) \land (\frac{1}{k} - 1 - \ln(\frac{1}{k})).
\]

Now we have

\[
V(X(0)) + C_1 T \geq EV(X(\tau_k \wedge T)) \geq \epsilon \left( (k - 1 - \ln k) \land (\frac{1}{k} - 1 - \ln \frac{1}{k}) \right).
\]

Finally, letting \( k \to \infty \) we have

\[
\infty > V(X(0)) + C_1 T = \infty
\]

which is a contradiction. Thus we conclude \( \tau = \infty \).

We summarize the above result in the following theorem.

**Theorem 2.3.** For any initial value \( X(0) \in \Gamma \), system \( (5) \) has a unique global solution on \( t \geq 0 \) and the solution will remain in \( \Gamma \) with probability 1.

3. **Stochastic boundedness and permanence.** Define

\[
\|X(t)\| = (S_H(t)^2 + I_H(t)^2 + S_V(t)^2 + I_V(t)^2)^{1/2}.
\]

Theorem 2.3 shows that for any initial condition \( X(0) \in \Gamma \) the solution of system \( (5) \) is always positive and remains in \( \Gamma \). Next we exam how \( X(t) \) varies in \( \Gamma \). First, we give the definition of a stochastically ultimately bounded solution.

**Definition 3.1.** [22, Page 395] The solution \( X(t) \) of system \( (5) \) is called stochastically ultimately bounded or ultimately bounded in probability if for any \( \epsilon \in (0, 1) \) there is a constant \( \chi = \chi(\epsilon) > 0 \) such that for any initial solution \( X(0) \in \Gamma \), the solution \( X(t) \) of system \( (5) \) has the property that

\[
\limsup_{t \to \infty} P\{\|X(t)\| > \chi\} \leq \epsilon.
\]
Lemma 3.2. For any given initial value $X(0) \in \Gamma$ and $\theta > 1$, there exists $\kappa = \kappa(\theta) > 0$ such that the solution of system (5) satisfies

$$\limsup_{t \to \infty} E\{\|X(t)\|^\theta\} < \kappa(\theta).$$

Proof. Define $V_1(S_H, I_H) = S_H^{\theta} + I_H^{\theta}$. Then, we have

$$\mathcal{L}V_1 = \theta S_H^{\theta-1} dS_H + \frac{1}{2} \theta(\theta - 1) S_H^{\theta-2} \sigma_H^2 dS_H + \theta I_H^{\theta-1} dI_H + \frac{1}{2} \theta(\theta - 1) I_H^{\theta-2} \sigma_I^2 dI_H$$

$$\quad + \theta I_H^{\theta-1} (b_1 - \mu S_H - \beta_2 S_H I_V - \beta_1 S_H I_H + \phi I_H) + \frac{1}{2} \theta(\theta - 1) S_H^{\theta} \sigma_H^2 dS_H$$

$$\quad + \beta_1 S_H I_H - \beta_2 S_H I_V - \beta_1 S_H I_H + \phi I_H + \beta_2 S_H I_V$$

$$\leq \theta \left( \frac{b_1}{\mu} \right)^{\theta-1} (b_1 - \mu S_H - \mu I_H) + \frac{1}{2} \theta(\theta - 1) \left( S_H^{\theta} \sigma_H^2 + I_H^{\theta} \sigma_I^2 \right)$$

$$\quad + \beta_1 S_H I_H - \beta_2 S_H I_V - \beta_1 S_H I_H + \phi I_H + \beta_2 S_H I_V$$

$$= \theta \left( \frac{b_1}{\mu} \right)^{\theta-1} b_1 - \mu S_H + \frac{1}{2} \theta(\theta - 1) \left( \frac{b_1}{\mu} \right)^{\theta} \left( S_H^{\theta} \sigma_H^2 + I_H^{\theta} \sigma_I^2 \right)$$

$$\leq \theta \left( \frac{b_1}{\mu} \right)^{\theta-1} \frac{1}{2} \theta(\theta - 1) \left( \frac{b_1}{\mu} \right)^{\theta} \left( S_H^{\theta} \sigma_H^2 + I_H^{\theta} \sigma_I^2 \right)$$

$$= \left( \frac{b_1}{\mu} \right)^{\theta-1} \left( \theta b_1 + \frac{b_1}{\mu} \theta(\theta - 1) \left( \frac{b_1}{\mu} \right)^{\theta} \left( S_H^{\theta} \sigma_H^2 + I_H^{\theta} \sigma_I^2 \right) \right).$$

Similarly, letting $V_2(S_V, I_V) = S_V^{\theta} + I_V^{\theta}$ we have $\mathcal{L}V_2 \leq M_2$, where

$$M_2 = \left( \frac{b_2}{\eta} \right)^{\theta-1} \left( \theta b_2 + \frac{b_2 \theta(\theta - 1)(S_H^{\theta} \sigma_H^2 + I_H^{\theta} \sigma_I^2)}{2\eta} \right).$$

Let $V = V_1 + V_2$. Then $V \leq M_3 = \left( \left( \frac{b_1}{\mu} \right)^{\theta} + \left( \frac{b_2}{\eta} \right)^{\theta} \right) < \infty$, and by Itô’s formula we have

$$dV = (\mathcal{L}V_1 + \mathcal{L}V_2) dt + \theta (S_H^{\theta} S_H dS_H + \sigma_H^2 dB_H + S_V^{\theta} dS_V + \sigma_V^2 dB_V)$$

$$\leq (M_1 + M_2) dt + \theta (S_H^{\theta} S_H dS_H + \sigma_H^2 dB_H + S_V^{\theta} dS_V + \sigma_V^2 dB_V),$$

and

$$d(e^tV) = e^t(V + \mathcal{L}V) dt + e^t \theta (S_H^{\theta} S_H dS_H + \sigma_H^2 dB_H + S_V^{\theta} dS_V + \sigma_V^2 dB_V)$$

$$+ \sigma_V^2 \sigma_H^2 dWX$$

$$\leq M e^t dt + e^t \theta (S_H^{\theta} S_H dS_H + \sigma_H^2 dB_H + S_V^{\theta} dS_V + \sigma_V^2 dB_V) + \sigma_V^2 \sigma_H^2 dWX,$$

where $M = M_1 + M_2 + M_3$.

Let $k_0 > 0$, such that $X(0) \in \left( \frac{1}{k_0}, k_0 \right)^4$ and for $k > k_0$ let

$$\tau_k = \inf \left\{ t > 0 : (S_H(t), I_H(t), S_V(t), I_V(t)) \notin \left( \frac{1}{k}, k \right)^4 \right\}.$$

Integrating (6) from 0 to $t \wedge \tau_k$ and taking the expectation on both sides, we get

$$E(e^{t \wedge \tau_k} V(X(t \wedge \tau_k))) \leq ME \left( \int_0^{t \wedge \tau_k} e^s ds \right) + V(X(0))$$

$$= ME(e^{t \wedge \tau_k} - 1) + V(X(0)).$$
Letting $k \to \infty$ we obtain

$$EV(X(t)) \leq e^{-t}V(X(0)) + M(1 - e^{-t}).$$

Note that $\|X(t)\|^\theta \leq 2^\theta V(X(t))$. Thus

$$E\|X(t)\|^\theta \leq 2^\theta EV(X(t)) \leq 2^\theta (e^{-t}V(X(0)) + M(1 - e^{-t})),$$

and it follows that

$$\limsup_{t \to \infty} E\|X(t)\|^\theta < \kappa(\theta),$$

where $\kappa(\theta) = 2^\theta M$.

Using the above lemma we show that the solution of system (5) is stochastically ultimately bounded.

**Theorem 3.3.** The solution of system (5) is stochastically ultimately bounded.

**Proof.** By above lemma, there exist a positive constant $\zeta > 0$ such that

$$\limsup_{t \to \infty} E\|X(t)\|^{\frac{2}{\nu}} < \zeta.$$

For any $\epsilon > 0$ put $\chi(\epsilon) = \frac{\epsilon^2}{2}\zeta$. Then, by Chebyshev’s inequality we get

$$P\{\|X(t)\| > \chi\} \leq \frac{E(\|X\|^{\frac{2}{\nu}})}{\chi^2}.$$

This concludes $\limsup_{t \to \infty} P\{\|X(t)\| > \chi\} \leq \frac{\epsilon}{\chi^2} = \epsilon$. □

**Lemma 3.4.** Let $k := \min\{\mu, \eta\}$, $\sigma^2 := \max\{\sigma_{S_H}^2, \sigma_{I_H}^2, \sigma_{S_V}^2, \sigma_{I_V}^2\}$ and assume that $b_1 + b_2 - k > 0$. Then, for any initial value $X(0) \in \Gamma$ the solution $X = X(t)$ of system (5) satisfies

$$\limsup_{t \to \infty} E\left(\frac{1}{\|X(t)\|^\nu}\right) \leq M,$$

where

$$M = \frac{4^\nu a_2^2 + 4a_1\theta}{4a_1} \max \left\{1, \left(1 + \frac{a_2 + \sqrt{a_2^2 + 4a_1\theta}}{2a_1}\right)^{\nu - 2}\right\},$$

$$a_1 = -\theta + \nu(b_1 + b_2 - k - \frac{\nu + 1}{2}\sigma^2),$$

$$a_2 = 2\theta + \nu k + \nu\sigma^2.$$

Here $\nu > 0$ and $\theta > 0$ are any constants satisfying the following conditions:

$$k + \frac{\nu + 1}{2}\sigma^2 - b_1 - b_2 < 0, \text{ and } \theta < \nu(b_1 + b_2 - k - \frac{\nu + 1}{2}\sigma^2)$$

**Proof.** Let $U(S_H, I_H, S_V, I_V) = \frac{1}{S_H + I_H + S_V + I_V}$. Then, by Itô’s formula

$$dU = LUdt - U^2(\sigma_{S_H}S_HdB_{S_H} + \sigma_{I_H}I_HdB_{I_H} + \sigma_{S_V}S_VdB_{S_V} + \sigma_{I_V}I_VdB_{I_V}),$$

where

$$LU = -U^2(b_1 + b_2 - \mu(S_H + I_H) - \eta(S_V + I_V)) + U^3(\sigma_{S_H}^2S_H^2 + \sigma_{I_H}^2I_H^2 + \sigma_{S_V}^2S_V^2 + \sigma_{I_V}^2I_V^2).$$
Let $\nu$ be as in the assumption, then

$$
\mathcal{L}((1+U)^{\nu}) = \nu(1+U)^{\nu-1}\mathcal{L}U + \frac{\nu(\nu-1)}{2}(1+U)^{\nu-2}U^4(\sigma_{S_H}^2S_H^2 + \sigma_{I_H}^2I_H^2

+ \sigma_{S_V}^2S_V^2 + \sigma_{I_V}^2I_V^2))

= \nu(1+U)^{\nu-2}\psi,
$$

where

$$
\psi = (1+U)\mathcal{L}U + \frac{\nu-1}{2}U^4(\sigma_{S_H}^2S_H^2 + \sigma_{I_H}^2I_H^2 + \sigma_{S_V}^2S_V^2 + \sigma_{I_V}^2I_V^2)

\leq -U^3(b_1 + b_2 - \frac{k}{U}) - U^2(b_1 + b_2 - \frac{k}{U}) + (U^3 + \frac{\nu+1}{2}U^4)(\sigma_{S_H}^2S_H^2 + \sigma_{I_H}^2I_H^2

+ \sigma_{S_V}^2S_V^2 + \sigma_{I_V}^2I_V^2)

\leq -U^2(b_1 + b_2 - k) + kU + (U^3 + \frac{\nu+1}{2}U^4)(\sigma_{S_H}^2S_H^2 + \sigma_{I_H}^2I_H^2 + \sigma_{S_V}^2S_V^2 + \sigma_{I_V}^2I_V^2)

\leq -U^2(b_1 + b_2 - k - \frac{\nu+1}{2}\sigma^2) + U(k + \sigma^2).
$$

The last inequality follows from the fact that

$$
U^3(\sigma_{S_H}^2S_H^2 + \sigma_{I_H}^2I_H^2 + \sigma_{S_V}^2S_V^2 + \sigma_{I_V}^2I_V^2) \leq U\sigma^2,
$$

and

$$
U^4(\sigma_{S_H}^2S_H^2 + \sigma_{I_H}^2I_H^2 + \sigma_{S_V}^2S_V^2 + \sigma_{I_V}^2I_V^2) \leq U^2\sigma^2.
$$

Let $\theta$ satisfy the assumption of the lemma. Then,

$$
\mathcal{L}(e^{\theta t}(1+U)^{\nu}) = e^{\theta t}(1+U)^{\nu} + \nu\mathcal{L}(1+U)^{\nu}

= e^{\theta t}(1+U)^{\nu-2}(\theta(1+U)^2 + \nu\psi).
$$

Now,

$$
\theta(1+U)^2 + \nu\psi \leq \theta U^2 + 2\theta U + \theta + \nu \left(-U^2(b_1 + b_2 - k - \frac{\nu+1}{2}\sigma^2) + U(k + \sigma^2)\right)

= U^2 \left(\theta - \nu(b_1 + b_2 - k - \frac{\nu+1}{2})\right) + U(2\theta + \nu k + \nu\sigma^2) + \theta.
$$

Thus,

$$
\mathcal{L}(e^{\theta t}(1+U)^{\nu}) \leq e^{\theta t}(1+U)^{\nu-2}(-a_1U^2 + a_2U + \theta),
$$

where $a_1 = \nu(b_1 + b_2 - k - \frac{\nu+1}{2}) - \theta$ and $a_2 = 2\theta + \nu k + \nu\sigma^2$.

Note that $f(U) = -a_1U^2 + a_2U + \theta$ has a maximum value of $f \left( \frac{a_2}{2a_1} \right) = \frac{a_2^2 + 4a_1\theta}{4a_1}$

and also

$$
(1+U)^{\nu-2} \leq \max \left\{ 1, \left( 1 + \frac{a_2 + \sqrt{a_2^2 + 4a_1\theta}}{2a_1} \right)^{\nu-2} \right\}.
$$

In conclusion, we have

$$
\mathcal{L}(e^{\theta t}(1+U)^{\nu}) \leq M_1 e^{\theta t},
$$

where

$$
M_1 = \frac{a_2^2 + 4a_1\theta}{4a_1} \max \left\{ 1, \left( 1 + \frac{a_2 + \sqrt{a_2^2 + 4a_1\theta}}{2a_1} \right)^{\nu-2} \right\}.
$$
Finally, by Itô’s formula,
\[ d(e^{\theta t}(1 + U)\nu) = e^{\theta t}(1 + U)\nu \, d(1 + U)\nu + (1 + U)\nu^{-1}e^{\theta t}(\sigma_S\nu S_HdB_{S_H}(t) + \sigma_I\nu I_HdB_{I_H}(t) + \sigma_S\nu S_VdB_{S_V}(t) + \sigma_I\nu I_VdB_{I_V}(t)). \]

Integrating both sides and taking the expectation will result in
\[ E(e^{\theta t}(1 + U)\nu - (1 + U(0))\nu) \leq M_1 E\left(\frac{e^{\theta t}}{\theta} - 1\right). \]

Simplifying it further, we get
\[ E((1 + U)\nu) \leq e^{-\theta t}(1 + U(0))\nu + M_1\left(1 - e^{-\theta t}\right). \]

Letting \( t \to \infty \), we conclude
\[ \limsup_{t \to \infty} E(U)\nu \leq \limsup_{t \to \infty} E(1 + U)\nu \leq \frac{M_1}{\theta}. \]

Now, for any \((S_H, I_H, S_V, I_V) \in \mathbb{R}_+^4\), we have
\[ (S_H + I_H + S_V + I_V)^\nu \leq 4\nu(S_H^2 + I_H^2 + S_V^2 + I_V^2)^{\frac{\nu}{2}} \leq 4\nu\|X(t)\|^\nu. \]

Thus, it follows that
\[ \limsup_{t \to \infty} E\left(\frac{1}{\|X(t)\|^\nu}\right) \leq 4\nu \limsup_{t \to \infty} E(U)\nu \leq M, \]
where \( M = \frac{4\nu M_1}{\theta}. \)

In the study of epidemic models, an important property is the so-called stochastic permanence, which indicates how the total population in the model changes in the long run. First, we give its definition and then we show that under some conditions, the system in (5) is stochastically permanent.

**Definition 3.5.** [7] System (5) is said to be stochastically permanent, if for any \( \epsilon \in (0, 1) \), there exist positive constants \( \lambda_1 = \lambda_1(\epsilon) \) and \( \lambda_2 = \lambda_2(\epsilon) \) such that for any initial value \( X(0) \in \Gamma \), the solution \( X(t) \) satisfies the following conditions:

\[ \liminf_{t \to \infty} P\{\|X(t)\| \leq \lambda_1\} \geq 1 - \epsilon, \quad \liminf_{t \to \infty} P\{\|X(t)\| \geq \lambda_2\} \geq 1 - \epsilon. \]

Now by Theorem 3.3 for any \( \epsilon \in (0, 1) \), there exists \( \lambda_1 > 0 \) such that
\[ P\{\|X\| > \lambda_1\} \leq \epsilon, \] which is equivalent to \( P\{\|X\| < \lambda_1\} > 1 - \epsilon. \] Thus, it follows that
\[ \liminf_{t \to \infty} P\{\|X\| \leq \lambda_1\} \geq 1 - \epsilon. \]

Also, assume that all the hypotheses in Lemma (3.4) hold. Then, we have
\[ \limsup_{t \to \infty} E\left(\frac{1}{\|X(t)\|^\nu}\right) \leq M. \]

For any \( \epsilon \in (0, 1) \), let \( \lambda_2 = \frac{\epsilon}{\nu}. \) Then,
\[ P\{\|X\| < \lambda_2\} = P\left\{\frac{1}{\|X\|} > \frac{1}{\lambda_2}\right\} \leq E\left(\frac{1}{\|X\|^\nu}\right) = \lambda_2^\frac{1}{\nu} E\left(\frac{1}{\|X\|^\nu}\right) = \lambda_2^\frac{1}{\nu}. \]

Taking the limit, we get
\[ \limsup_{t \to \infty} P\{\|X\| < \lambda_2\} \leq \lambda_2^\frac{1}{\nu} M = \epsilon. \]
Therefore, we conclude that
\[
\liminf_{t \to \infty} P\{\|X\| \geq \lambda_2 \} \geq 1 - \epsilon.
\]
The following theorem summarizes the above discussion.

**Theorem 3.6.** Under the assumptions of Lemma (3.4), the solution of system (5) is stochastically permanent for any initial value \(X(0) \in \Gamma\).

4. **Extinction and stochastic stability.** In section 2 we defined the basic reproductive number \(R_0\). According to [8], for the deterministic model the number of infected hosts \(I_H\) and vectors \(I_V\) will tend to zero in the long run provided that \(R_0 < 1\). In this section we derive a similar condition for the stochastic model so that the number of infected hosts and vectors will decrease exponentially to zero almost surely in the long run. The following theorem provides a condition for the extinction of the infected hosts and vectors.

**Theorem 4.1.** Let \(R_0^* := R_0 - \sigma^2_{1H} / \sigma_{1H}^2\). If \(R_0^* < 1\) and \(\eta < \beta\), then for any initial value \(X(0) \in \Gamma\), \(I_H(t)\) will tend to zero exponentially almost surely. That is, \(\limsup_{t \to \infty} \ln I_H(t) < 0 \ a.s.\)

**Proof.** From system (5), and using \(S_H = b_H / \mu - I_H\) we get
\[
dI_H = \left(\beta_1 \left(\frac{b_1}{\mu} - I_H\right)I_H + \beta_2 \left(\frac{b_1}{\mu} - I_H\right)I_V - (\mu + \phi)I_H\right)dt + \sigma_{1H} I_H dB_{1H}(t).
\]

Now, by Itô’s formula we have
\[
d(\ln(I_H)) = \left\{ \frac{1}{I_H} \left( \beta_1 \left(\frac{b_1}{\mu} - I_H\right)I_H + \beta_2 \left(\frac{b_1}{\mu} - I_H\right)I_V - (\mu + \phi)I_H \right) + \frac{-1}{2I_H^2} \sigma^2_{1H} I_H \right\} dt
\]
\[
+ \frac{1}{I_H} \sigma_{1H} I_H dB_{1H}(t)
\]
\[
= \left( \beta_1 \left(\frac{b_1}{\mu} - I_H\right) + \beta_2 \left(\frac{b_1}{\mu} - I_H\right) I_V - \mu - \phi - \frac{1}{2} \sigma^2_{1H} \right) dt + \sigma_{1H} I_H dB_{1H}(t)
\]
\[
\leq \left( \beta_1 \frac{b_1}{\mu} + \beta_2 \frac{b_1 b_2}{\eta \mu} - \mu - \phi - \frac{1}{2} \sigma^2_{1H} \right) dt + \sigma_{1H} I_H dB_{1H}(t).
\]

Integrating both sides on \([0, t]\), we get
\[
\frac{\ln(I_H(t)) - \ln(I_H(0))}{t} \leq \frac{\beta_1 b_1}{\mu} + \frac{\beta_2 b_1 b_2}{\mu \eta} - \mu - \phi - \frac{1}{2} \sigma^2_{1H} + \frac{1}{t} \int_0^t \sigma_{1H} I_H dB_{1H}(s).
\]

Let \(M(t) = \int_0^t \sigma_{1H} I_H dB_{1H}(s)\). Then, \(M\) is a martingale [22, Theorem 5.14], with a quadratic variation given by
\[
\langle M, M \rangle_t = \int_0^t \sigma^2_{1H} ds = \sigma_{1H}^2 t.
\]

Since \(\limsup_{t \to \infty} \frac{\langle M, M \rangle_t}{t} = \sigma^2_{1H} < \infty\), by the strong law of large numbers, we have \(\limsup_{t \to \infty} \frac{M(t)}{t} = 0\).

Thus,
\[
\limsup_{t \to \infty} \frac{\ln I_H(t)}{t} \leq \frac{\beta_1 b_1}{\mu} + \frac{\beta_2 b_1 b_2}{\mu \eta} - \mu - \phi - \frac{1}{2} \sigma^2_{1H}.
\]
Now if $R_0^0 < 1$ and $\eta < \beta$ we have
\[
\frac{\beta_1 b_1}{\mu} + \frac{\beta_2 b_1 b_2}{\mu \eta} - \mu - \phi - \frac{1}{2} \sigma^2_{I_h} < 0.
\]
Thus,
\[
\limsup_{t \to \infty} \frac{\ln I_H(t)}{t} \leq \frac{\beta_1 b_1}{\mu} + \frac{\beta_2 b_1 b_2}{\mu \eta} - \mu - \phi - \frac{1}{2} \sigma^2_{I_h} < 0 \text{ a.s.}
\]
\[\square\]

Using the above theorem and the following lemma [22, Theorem 3.3], we conclude that the number of infected vectors $I_V$ will also tend to zero exponentially almost surely.

**Lemma 4.2.** Given a stochastic differential equation
\[
dX(t) = f(X(t), t)dt + g(X(t), t)dB(t).
\]
Assume that there exists a function $V \in C^{2,1}(R^d \times [t_0, \infty); R_+)$, and constants $p > 0, c_1 > 0, c_2, c_3 \in R, c_3 \geq 0$, such that for all $X \neq 0$ and $t \geq t_0$,
1. $c_1 \|X\|^p \leq V(X, t),$
2. $LV(X, t) \leq c_2 V(X, t),$
3. $\|V_X(X, t)g(X, t)\|^2 \geq c_3 V^2(X, t)$.

Then,
\[
\limsup_{t \to \infty} \frac{\ln \|X(t; t_0, X_0)\|}{t} \leq \frac{-c_3 - 2c_2}{2p} \text{ a.s.}
\]

**Corollary 4.3.** If $R_0^* < 1$, then for any initial value $X(0) \in \Gamma$, we have
\[
\limsup_{t \to \infty} \frac{\ln I_V(t)}{t} < 0 \text{ a.s.}
\]

**Proof.** By Theorem 4.1, if $R_0^* < 1$, then $I_H(t)$ will tend to zero exponentially almost surely and since exponential stability implies asymptotic stability, we have that
\[
\lim_{t \to \infty} I_H(t) = 0 \text{ a.s.}
\]

Thus, for any $\epsilon > 0$, there exists $k_1 > 0$ and a set $\bar{\Gamma} \subset \Gamma$ such that
\[
P(\bar{\Gamma}) > 1 - \epsilon, \quad 0 \leq I_H(t) < \frac{\eta}{\beta b_2} \epsilon \text{ for all } t > k_1.
\]

Noting that $(-\epsilon - \eta I_V) dt + \sigma I_V dB_{iV} \leq dI_V \leq (\epsilon - \eta I_V) dt + \sigma I_V dB_{iV}$, and $\epsilon$ is arbitrary, we have
\[
dI_V(t) = -\eta I_V dt + \sigma I_V dB_{iV}.
\]

Let $V(I_V(t)) = I_V^2$, then $LV = (2\eta + \sigma^2_{I_V})V$. Also, we have $\|V_{I_V} g\|^2 = 4\sigma^2_{I_V} I_V^4$, thus by lemma (4.2) it follows
\[
\limsup_{t \to \infty} \frac{\ln I_V(t)}{t} \leq -\eta - \frac{\sigma^2_{I_V}}{2} < 0 \text{ a.s.}
\]
\[\square\]

**Remark.** Note that in the deterministic case, if $R_0 < 1$ then the system has a disease-free equilibrium. In this case, $R_0^* < 1$ and by Theorem 4.1 and corollary 4.3, in the stochastic model, the number of infected hosts and vectors will go to zero exponentially almost surely. On the contrary, we might have cases where $R_0^* < 1,$
but $R_0 > 1$. That is, a large environmental fluctuation can suppress the number of infected hosts (see Figures 3, 4 and 5 below).

Figure 3. Trajectories of solution of the stochastic model (5) and frequency histograms of $I_H$ and $I_V$ when $R^*_0 = 0.7797$.

Figure 4. Extinction in the deterministic model implies extinction in the stochastic case, $R_0 = 0.71$ and $R^*_0 = 0.64$ respectively.

Figure 5. Existence of Endemic equilibrium in the deterministic model ($R_0 = 1.13$), while extinction in the stochastic case ($R^*_0 = 0.81$).

Definition 4.4. [22, Definition 2.1] Given the stochastic differential equation

$$dX(t) = f(X(t), t) dt + g(X(t), t) dB(t) \text{ on } t \geq t_0.$$  

Assume that $f(0, t) = 0$, $g(0, t) = 0$ for all $t > 0$ such that $X(t) = 0$ is a trivial solution. Then, the trivial solution is called
1. Stochastically stable if for all \( \epsilon \in (0, 1) \) and \( r > 0 \) there exist \( \delta = \delta(\epsilon, r, t_0) > 0 \) such that

\[
P\{\|X(t; t_0, x_0)\| < r \text{ for all } t \geq t_0\} \geq 1 - \epsilon,
\]

for \( X_0 \in \mathbb{R}^d \) such that \( \|X_0\| < \delta \).

2. Stochastically asymptotically stable if it is stochastically stable and for every \( \epsilon \in (0, 1) \), there exists a \( \delta_0 = \delta_0(\epsilon, t_0) > 0 \) such that,

\[
P\{\lim_{t \to \infty} X(t; t_0, x_0) = 0\} \geq 1 - \epsilon
\]

whenever \( \|X_0\| < \delta_0 \).

The standard method of studying stability is through a Lyapunov function. However, it is difficult to construct such a function for a nonlinear system of stochastic differential equations. In this paper we analyze the stability of the corresponding linear system, and if the coefficients of the nonlinear system satisfy the condition given in the following theorem, then we conclude that it is asymptotically stable almost surely [28, Theorem 7.1].

**Theorem 4.5.** If the linear system

\[
dX(t) = MX(t)\,dt + \sigma X(t)dB(t)
\]

with constant coefficients is asymptotically stable almost surely, and the coefficients of the nonlinear system

\[
dX(t) = b(X, t)\,dt + \sigma(X, t)dB(t)
\]

satisfy an inequality

\[
\|b(x, t) - M x\| + \|\sigma(x, t) - \sigma x\| < \gamma\|x\|,
\]

in a sufficiently small neighborhood of the point \( x = 0 \), and with a sufficiently small constant \( \gamma \), then the solution \( X = 0 \) of the nonlinear system is asymptotically stable almost surely.

For \( S_H = \frac{b_1}{\mu} - I_H \) and \( S_V = \frac{b_2}{\eta} - I_V \), system (5) reduces to the following

\[
dI_H = (\beta H - \beta_1 I_H + \frac{\beta H}{\mu} I_V - \beta_2 I_H I_V - (\mu + \phi)I_H)dt + \sigma I_H dB_{I_H}(t),
\]

\[
dI_V = (\frac{\beta b_2}{\eta} I_H - \beta I_H I_V - \eta I_V)dt + \sigma I_V dB_{I_V}(t).
\]

The corresponding linearized system is given by

\[
dI_H = \left(\frac{\beta_1 b_1}{\mu} - \phi\right) I_H + \frac{\beta_2 b_1}{\mu} I_V\right)dt + \sigma I_H dB_{I_H}(t),
\]

\[
dI_V = \left(\frac{\beta b_2}{\eta} I_H - \eta I_V\right) dt + \sigma I_V dB_{I_V}(t).
\]

Finally, we state the condition for the asymptotic stability of the system (7) as follows.

**Theorem 4.6.** Let \( X(t) = (I_H(t), I_V(t)) \) be the solution of (7) and suppose

\[
\Theta = \min\{\mu + \phi - \frac{\beta_1 b_1}{\mu} - \frac{c_2 b_2}{\eta} - \frac{1}{2}\sigma^2_I, \eta - \frac{1}{2}\sigma^2_I, \frac{1}{2}\sigma^2_I - \frac{c_2 b_1}{\mu}\} > 0
\]
Thus, the trivial solution \( X \) where \( E \) endem given equilibrium point \( l \) of \( (7) \) captures the idea of non-extinction of the system is frequently used. In the analysis of stochastic epidemic models the idea of persistence in mean, which persistence of a population or a collection of interacting populations. Hence, in

5. Persistence in mean. One of the most fundamental issues in epidemiology and population biology is about the necessary conditions to ensure the long-term persistence of a population or a collection of interacting populations. Hence, in

the analysis of stochastic epidemic models the idea of persistence in mean, which captures the idea of non-extinction of the system is frequently used.

If \( R_0^* > 1 \), then \( R_0^* > 1 \). By Theorem 2.2, the deterministic model (1) has endemic equilibrium point \( E_1 = (S^*_H, I^*_H, S^*_V, I^*_V) \), where

\[
\frac{\beta_1 b_1}{\mu} I^*_H - \beta_1 I^*_H \frac{2}{\mu} + \frac{\beta_2 b_1}{\mu} I^*_V - \beta_2 I^*_H I^*_V - (\mu + \phi) I^*_H = 0, \\
\frac{\beta_2 b_2}{\eta} I^*_H - \beta_2 I^*_H I^*_V - \eta I^*_V = 0.
\]

Theorem 5.1. Let \( R_0^* > 1 \) and assume that

\[
\frac{\sigma^2_H}{2} < l_1 \quad \text{and} \quad \frac{\sigma^2_V}{2} < l_2,
\]

where \( l_1 = \frac{\beta_1 b_1}{\mu} + \frac{\beta_2 b_1 I^*_V}{\mu I^*_V} \) and \( l_2 = \frac{\beta_2 b_2 I^*_V}{\eta I^*_V} \). Then, for any \( X(0) \in \Gamma \), the solution of \( (7) \) satisfies

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t [(I_H(s) - (1 + \frac{\sigma^2_H}{2 l_1})I_H^2 + (I_V(s) - (1 + \frac{\sigma^2_V}{2 l_2})I_V^2)ds \leq \frac{k_2}{k_1},
\]

where \( k_1 = \min \{l_1, l_2\} \) and \( k_2 = \frac{1}{2} \left( 1 + \frac{\sigma^2_H}{2 l_1} \right) \sigma^2_H I_H \eta^2 + \frac{1}{2} \left( 1 + \frac{\sigma^2_V}{2 l_2} \right) \sigma^2_V I_V \eta^2. \)
Proof. Let $V_1 = \frac{1}{2}(I_H - I_H^*)^2$ and $V_2 = \frac{1}{2}(I_V - I_V^*)^2$. Then,

$$\mathcal{L}V_1 = (I_H - I_H^*)\left(\frac{\beta_1 b_1}{\mu} I_H - \beta_1 I_H^2 + \frac{\beta_2 b_1}{\mu} I_V - \beta_2 I_H I_V - (\mu + \phi) I_H \right) + \frac{1}{2} \sigma_{I_H}^2 I_H^2.$$ 

Using (9) we get

$$\mathcal{L}V_1 \leq (I_H - I_H^*)\left(\frac{\beta_1 b_1}{\mu} (I_H - I_H^*) + \frac{\beta_2 b_1 I_V^*}{\mu I_H^*} (I_H^* - I_H) \right) + \frac{1}{2} \sigma_{I_H}^2 I_H^2$$

$$= -l_I \left( I_H - (1 + \frac{\sigma_{I_H}^2}{2l_1}) I_H^* \right)^2 + \frac{1}{2} (1 + \frac{\sigma_{I_H}^2}{2l_1}) \sigma_{I_H}^2 I_H^*.$$ 

Similarly, we have $\mathcal{L}V_2 \leq -l_2 \left( I_V - (1 + \frac{\sigma_{I_V}^2}{2l_2}) I_V^* \right)^2 + \frac{1}{2} (1 + \frac{\sigma_{I_V}^2}{2l_2}) \sigma_{I_V}^2 I_V^*.$

Now letting $V = V_1 + V_2$, we have

$$\mathcal{L}V = \mathcal{L}V_1 + \mathcal{L}V_2 \leq -l_I \left( I_H - (1 + \frac{\sigma_{I_H}^2}{2l_1}) I_H^* \right)^2 - l_2 \left( I_V - (1 + \frac{\sigma_{I_V}^2}{2l_2}) I_V^* \right)^2 + k_2$$

$$\leq -k_I \left( I_H - (1 + \frac{\sigma_{I_H}^2}{2l_1}) I_H^* \right)^2 + \left( I_V - (1 + \frac{\sigma_{I_V}^2}{2l_2}) I_V^* \right)^2 + k_2.$$ 

By Itô’s formula,

$$dV = \mathcal{L}V dt + \sigma_{I_H} I_H dB_{I_H} + \sigma_{I_V} I_V dB_{I_V}.$$ 

Integrating both sides, we get

$$k_I \int_0^t \left( \left( I_H(s) - (1 + \frac{\sigma_{I_H}^2}{2l_1}) I_H^* \right)^2 + \left( I_V(s) - (1 + \frac{\sigma_{I_V}^2}{2l_2}) I_V^* \right)^2 \right) ds$$

$$\leq k_2 t - V(I_H(t), I_V(t)) + V(I_H(0), I_V(0)).$$

Thus, taking the limit we get

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t [(I_H(s) - (1 + \frac{\sigma_{I_H}^2}{2l_1}) I_H^*)^2 + (I_V(s) - (1 + \frac{\sigma_{I_V}^2}{2l_2}) I_V^*)^2] ds \leq \frac{k_2}{k_1}.$$

\[\square\]

**Definition 5.2.** [11] The system given by (7) is called persistence in mean if

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t I_H(s) ds > 0 \quad \text{and} \quad \liminf_{t \to \infty} \frac{1}{t} \int_0^t I_V(s) ds > 0.$$ 

**Corollary 5.3.** Assume that the conditions on Theorem 5.1 hold, and suppose that

$$(1 + \frac{\sigma_{I_H}^2}{2l_1}) I_H^* > \sqrt{\frac{k_2}{k_1}} \quad \text{and} \quad (1 + \frac{\sigma_{I_V}^2}{2l_2}) I_V^* > \sqrt{\frac{k_2}{k_1}}.$$ 

Then, the system in (7) satisfies

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t I_H(s) ds \geq \frac{1}{2} (1 + \frac{\sigma_{I_H}^2}{2l_1}) I_H^* - \frac{l_1}{I_H^*(2l_1 + \sigma_{I_H}^2)} k_2 > 0,$$ 

and

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t I_V(s) ds \geq \frac{1}{2} (1 + \frac{\sigma_{I_V}^2}{2l_2}) I_V^* - \frac{l_2}{I_V^*(2l_2 + \sigma_{I_V}^2)} k_2 > 0.$$
That is the system is persistent in the mean.

Proof. From Theorem 5.1, we have

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \left( I_H(s) - (1 + \frac{\sigma_H^2}{2l_1} I_H^*) \right)^2 \, ds \leq \frac{k_2}{k_1}
\]

and

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \left( I_V(s) - (1 + \frac{\sigma_V^2}{2l_2} I_V^*) \right)^2 \, ds \leq \frac{k_2}{k_1}.
\]

For any constant \( \xi > 0 \), we have

\[
I_H \geq \frac{1}{2} I_H^* \xi - \frac{1}{2I_H^* \xi} (I_H - I_H^* \xi)^2.
\]

Thus,

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t I_H(s) \, ds \geq \frac{1}{2} I_H^* \xi - \frac{1}{2I_H^* \xi} \limsup_{t \to \infty} \frac{1}{t} \int_0^t (I_H - I_H^* \xi)^2 \, ds.
\]

Let \( \xi = 1 + \frac{\sigma_H^2}{2l_1} \). Then

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t I_H(s) \, ds \geq \frac{1}{2} I_H^*(1 + \frac{\sigma_H^2}{2l_1}) - \frac{l_1}{I_H^*(2l_1 + \sigma_H^2)} \frac{k_2}{k_1}.
\]

The second result can be obtained following a similar procedure.

Finally, we conduct numerical simulations to validate our theoretical results. To this end, we choose the parameters in the system as follows.

\[
\beta_1 = 0.01, \quad \beta_2 = 0.3, \quad \eta = 0.3, \quad \beta = 0.06, \quad \phi = 0.35, \quad \mu = 0.83, \quad b_1 = 5, \quad b_2 = 4, \quad \sigma_S = 0.1, \quad \sigma_H = 0.1, \quad \sigma_V = 0.1, \quad \sigma_I = 0.1
\]

Then \( R_0 = 4.1309 \), \( R_0 = 4.1352 \) and the deterministic model has a unique endemic equilibrium \( E_1 = (2.47, 3.55, 7.79, 5.54) \), which is globally stable. Thus, by Theorem 5.1, the solution of system (7) oscillates about \( E_1 \). Now, if we fix all the values of the parameters but increase the noise intensities to \( \sigma_H = 0.5, \quad \sigma_H = 0.5, \quad \sigma_V = 0.5, \quad \sigma_H = 0.5 \) then, the new solution will still oscillate about the same endemic equilibrium with larger amplitude as can be seen in Figure 6.

![Figure 6](image-url)

**Figure 6.** The effect of noise intensity on the trajectories of \( I_H(t) \) and \( I_V(t) \). In both cases the solution oscillates about the endemic equilibrium \( E_1 \) with different amplitudes. In (a) the noise intensity is \( \sigma_I = \sigma_V = 0.1 \), and in (b) \( \sigma_I = \sigma_V = 0.5 \).

For the long-term persistence of the disease, we consider the same set of parameters as above. Then \( 1 + \frac{\sigma_H^2}{2l_1} I_H^* = 3.56, \quad (1 + \frac{\sigma_V^2}{2l_2}) I_V^* = 5.59 \) and \( \sqrt{\frac{k_2}{k_1}} = 0.65 \).
It is easy to check that all the hypotheses of Corollary 5.3 are satisfied, and we conclude that \( \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t I_H(s) \, ds = 1.72 > 0 \) and \( \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t I_V(s) \, ds = 2.76 > 0 \). This is verified in Figure 7.

![Figure 7](image)

**Figure 7.** Persistence of the disease and histogram of \( I_H \) and \( I_V \).

**REFERENCES**

[1] M. Aguiar, N. Stollenwerk and B. W. Kooi, *Modeling Infectious Diseases Dynamics: Dengue Fever, a Case Study*, Epidemiology Insights, ISBN: 978-953-51-0565-7, InTech, DOI: 10.5772/31920.

[2] M. Andraud, N. Hens, C. Marais and P. Beutels, *Dynamic epidemiological models for dengue transmission: A systematic review of structural approaches*, PLoS One, 7 (2012), e49085.

[3] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, A Wiley-Interscience Publication, 1971.

[4] J. R. Beddington and R. May, *Harvesting natural populations in a randomly fluctuating environment*, Science, 197 (1977), 463–465.

[5] K. W. Blayneh, A. B. Gumel, S. Lenhart and T. Clayton, *Backward bifurcation and optimal control in transmission dynamics of west nile virus*, Bull. Math. Biol., 72 (2010), 1006–1028.

[6] K. W. Blayneh, Y. Cao and H.-D. Kwon, *Optimal control of vector-borne diseases: Treatment and prevention*, Discrete and Continuous Dynamical Systems, Series B, 11 (2009), 587–611.

[7] Y. Cai, X. Wang, W. Wang and M. Zhao, *Stochastic dynamics of an sirs epidemic model with ratio-dependent incidence rate*, Abstract and Applied Analysis, 2013 (2013), Article ID 172631, 11pp.

[8] L. Cai and Xuezhi Li, *Analysis of a simple vector-host epidemic model with direct transmission*, Discrete Dynamics in Nature and Society, (2010), Article ID 679613, 12pp.

[9] N. Dalal, D. Greenhalgh and X. Mao, *A stochastic model of AIDS and condom use*, J. Math. Anal. Appl., 325 (2007), 36–53.

[10] L. C. Evans, *An Introduction to Stochastic Differential Equations*, University of California, Berkeley, CA, 2013.

[11] A. Gray, D. Greenhalgh, L. Hu, X. Mao and J. Pan, *A stochastic differential equation sis epidemic model*, SIAM J. Appl. Math., 71 (2011), 876–902.

[12] D. O. J. A. P. Heesterbeek and J. A. J. Metz, On the definition and the computation of the basic reproduction ratio \( R_0 \) in models for infectious diseases, Math. Biol., 35 (1990), 503–522.

[13] D. J. Higham, *An algorithmic introduction to numerical simulation of stochastic differential equations*, SIAM Review, 43 (2001), 525–546.

[14] http://www.who.int/mediacentre/factsheets/fs387/en/.

[15] http://www.who.int/whosis/en/.

[16] http://www.cdc.gov/Westnile/transmission/.

[17] http://www.cdc.gov/dengue/epidemiology/.
[18] M. Jovanovic and M. Krstic, Stochastically perturbed vector-borne disease models with direct transmission, *Applied Mathematical Modelling*, 36 (2012), 5214–5228.

[19] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag Berlin Heidelberg, 1992.

[20] X. Ling, *Modeling and Analysis of Vector-borne Diseases on Complex Networks*, PhD Thesis, Kansas State University, 2013.

[21] S. Mandal, R. R. Sarkar and S. Sinha, *Mathematical Models Of Malaria - A Review*, *Malar J.*, 10, 2011, 202.

[22] X. Mao, *Stochastic Differential Equations and Applications*, Woodhead Publishing, second edition, January 13, 2008.

[23] M. Martcheva, *An Introduction to Mathematical Epidemiology*, Texts in Applied Mathematics, 61, Springer, New York, 2015.

[24] F. E. Mckenzie, *Why Model Malaria?*, *Parasitology Today*, 16 (2000), 511–516.

[25] G. A. Ngwa and W. S. Shu, A mathematical model for endemic malaria with variable human and mosquito populations, *Mathematical and Computer Modelling*, 32 (2000), 747–763.

[26] K. Okosun and O. Makinde, Optimal control analysis of malaria in the presence of non-linear incidence rate, *Appl. Comput. Math.*, 12 (2013), 20–32.

[27] Z. Qiu, Dynamical behavior of a vector-host epidemic model with demographic structure, *Computers and Mathematics with Applications*, 56 (2008), 3118–3129.

[28] K. Rafail, *Stochastic Stability of Differential Equations*, Springer-Verlag Berlin Heidelberg, 2012.

[29] R. Rosenberg and C. B. Beard, Vector-borne infections, *CDCEID journal*, 17 (2011), 2pp.

[30] M. Samsuzzoha, M. Singh and D. Lucy, Uncertainty and sensitivity analysis of the basic reproduction number of a vaccinated epidemic model of influenza, *Applied Mathematical Modelling*, 37 (2013), 903–915.

[31] Smallpox: Disease, Prevention, and Intervention. The CDC and the World Health Organization, *History and Epidemiology of Global Smallpox Eradication From the training course*, Slide 16–17.

[32] M. O. Souza, Multiscale analysis for a vector-borne epidemic model, *J. Math. Biol.*, 68 (2014), 1269–1293.

[33] D. R. Stirzaker, A perturbation method for the stochastic recurrent epidemic, *EpidemicIMA J Appl Math.*, 15 (1975), 135–160.

[34] N. Stollenwerk, M. Aguiar, S. Ballesteros, J. Boto, B. Kooi and L. Mateus, *Dynamic Noise, Chaos and Parameter Estimation in Population Biology*, Interface Focus, 2012.

[35] J. E. Truscott and C. A. Gilligan, Response of a deterministic epidemiological system to a stochastically varying environment, *PNAS*, 100 (2003), 9067–9072.

[36] Q. Wei, Z. Xiong and F. Wang, Dynamic of a Stochastic SIR Model Under Regime Switching, *Journal of Information & Computational Science*, 10 (2013), 2727–2734.

[37] H. M. Wei, X. Z. Li and M. Martcheva, An epidemic model of a vector-borne disease with direct transmission and time delay, *Journal of Mathematical Analysis and Applications*, 342 (2008), 895–908.

[38] M. J. Wonham and M. A. Lewis, A Comparative Analysis of Models for West Nile Virus, *Mathematical Epidemiology Lecture Notes in Mathematics*, 1945 (2008), 365–390.

[39] H. Yang, H. Wei and X. Li, Global stability of an epidemic model for vector-borne disease, *J Syst Sci Complex Journal*, 23 (2010), 279–292.

[40] C. Zhu and G. Yin, Asymptotic properties of hybrid diffusion systems, *Control Optim.*, 46 (2007), 1155–1179.

[41] L. Zu, D. Jiang and D. O’Regan, *Stochastic Permanence, Stationary Distribution and Extinction of a Single-Species Nonlinear Diffusion System with Random Perturbation*, Abstract and Applied Analysis, Article ID 320460, 2014.

Received December 2015; revised February 2016.

E-mail address: yzc0009@auburn.edu
E-mail address: dbd0005@auburn.edu