POINTWISE DECAY FOR THE ENERGY-CRITICAL NONLINEAR WAVE EQUATION

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Abstract. This second article in a two-part series (following [arXiv:2105.02865], listed here as [26]) proves optimal pointwise decay rates for the quintic defocusing wave equation with large initial data on nonstationary spacetimes, and both the quintic defocusing and quintic focusing wave equations with small initial data on nonstationary spacetimes. We prove a weighted local energy decay estimate, and use local energy decay and Strichartz estimates on these variable-coefficient backgrounds. By using an iteration scheme, we obtain the optimal pointwise bounds. In addition, we explain how the iteration scheme reaches analogous pointwise bounds for other integral power nonlinearities that are either higher or lower than the quintic power, given the assumption of global existence for those powers (and in the case of the lower powers, given certain initial decay rates).

1. Introduction

We study the energy-critical nonlinear wave equation in three spatial dimensions on a variety of spacetimes, which we call nonstationary, that are allowed to depend on both $t$ and $x$. For clarity, we emphasise that they are allowed to depend only on $x$ as well. The goal is to obtain the optimal pointwise decay rate, stated in Theorem 1.1; this is achieved by an iteration scheme that is outlined in Section 4.1. Along the way, we prove an $r$-weighted integrated local energy decay estimate (Theorem 4.10). The results can be viewed as extensions of the results for the linear problem studied in [26], albeit under stronger assumptions on the coefficients of the wave operator $P$ than in [26]. See also Remark 1.6 for how we reach analogous pointwise decay rates for both focusing and defocusing power nonlinearities that are cubic or higher order; the present article focuses on the quintic order case.

We consider the operator

$$ P := \partial_\alpha g^{\alpha \beta}(t, x) \partial_\beta + g^{\omega}(t, x) \Delta_\omega + B^\alpha(t, x) \partial_\alpha + V(t, x) \quad \text{on } \mathbb{R}^{1+3} \quad (1.1) $$

where the coefficients are allowed to depend on $t$ and we use the summation convention. Here $\Delta_\omega$ denotes the Laplace operator on the unit sphere, and $\alpha, \beta$ range across $0, \ldots, 3$. The main assumptions on $P$ are that it is hyperbolic and a small asymptotically flat perturbation of the d'Alembertian $\Box = -\partial_t^2 + \Delta$; see Section 1 for the precise assumptions on $P$. The precise conditions on the potential $V$, the coefficients $B, g^{\omega}$ and the Lorentzian metric $g$ are given in the main result, Theorem 1.1. We also remark on the iteration for obtaining pointwise decay rates for other integer powers, both lower and higher than the quintic, in Remark 1.6.
We study the nonlinear Cauchy problem

\[
\begin{aligned}
P\phi(t, x) &= \mu\phi(t, x) + \phi(t, x) \in (0, \infty) \times \mathbb{R}^3, \quad \mu \in \{-1, 0, 1\}; \\
(\phi(0, x), \partial_t \phi(0, x)) &= (\phi_0, \phi_1)
\end{aligned}
\]

the convention we adopt is that \(\mu = 1\) corresponds to the defocusing sign.

Our main theorem (Theorem 1.1) states, informally, that if the coefficients of \(P - \square\) are small and asymptotically flat, then the solution to (1.2), as well as its vector fields, obey the global pointwise decay rates of \((t - r)^{-1 - \min(c(P), 2)}(t + r)^{-1}\); here \(c(P)\) is a constant depending on the coefficients in (1.1). The rate of decay coincides with the one obtained by [16] in the case \(P = \square\), but in contrast to [16] we obtain it, on these general backgrounds \(P\), for the solution everywhere in spacetime for initial data in the wider class (1.6), rather than merely in forward light cones \(|x| \leq \lambda t, \lambda < 1\) for the Minkowski background and for the narrower class of compactly supported initial data. We believe this pointwise rate of decay to be sharp. An overview of the proof is contained in Section 1.1, where both an outline of the proof and the main novelties of the proof are explained.

**History of the problem.** The theory of global existence, uniqueness and scattering for the semilinear wave equation on Minkowski spacetime, in three spatial dimensions, for \(\square \phi = \pm \phi^{p+1}\), \(\phi(0, x) = \phi_0(x), \quad \partial_t \phi(0, x) = \phi_1(x)\) was studied extensively; for instance, in the articles [6, 16, 19, 43, 47, 50]. For small initial data, there is a unique global solution if \(p > \sqrt{2}\); see [15, 18, 56]. Work has also been done for the pointwise decay of solutions; see [42, 50, 61]. In the case of compactly supported smooth data, decay rates were proved in [54] (for small data) and in [7, 16] (for large data).

We now briefly remark on other spacetimes. Much work has been done for solutions to the initial value problem

\[
\square g \phi = |g|^{-1/2} \partial_\mu (|g|^{1/2} g^{\mu \nu} \partial_\nu \phi) = 0, \quad \phi(0, x) = \phi_0(x), \quad \partial_t \phi(0, x) = \phi_1(x)
\]

for various Lorentzian metrics \(g\). For the Schwarzschild metric, the solution to the wave equation was conjectured to decay at the rate of \(t^{-3}\) on a compact region in [44]. This rate of decay was shown to hold for the Schwarzschild spacetime, the subextremal Kerr spacetime with \(|a| < M\), and other spacetimes; see [3, 14, 20, 37, 57]. We continue this discussion in Section 1.2, in which we elaborate upon the history and utility of local energy decay estimates, and then conclude with a discussion of pointwise estimates and asymptotic behaviour of solutions on both Minkowski backgrounds and perturbations thereof.

**Statement of the main theorem.** We state some notation that we use throughout the paper. We write \(X \lesssim Y\) to denote \(|X| \leq CY\) for an implicit constant \(C\) which may vary by line. Similarly, \(X \ll Y\) will denote \(|X| \leq cY\) for a sufficiently small constant \(c > 0\). In \(\mathbb{R}^{1+3}\), we consider

\[
\partial := (\partial_t, \partial_1, \partial_2, \partial_3), \quad \Omega := (x^i \partial_j - x^j \partial_i)_{i,j}, \quad S := t \partial_t + \sum_{i=1}^3 x^i \partial_i,
\]
which are, respectively, the generators of translations, rotations and scaling. We denote
the angular derivatives by \( \partial \). We set
\[
Z := (\partial, \Omega, S)
\]
and we define the function class
\[
S^Z(f)
\]
to be the collection of real-valued functions \( g \) such that \( |Z^J g(t, x)| \lesssim_J |f| \) whenever \( J \) is a multiindex. We will frequently use \( f = \langle r \rangle^\alpha \) for some real \( \alpha \leq 0 \), where \( \langle r \rangle := (1 + |r|^2)^{1/2} \). We also define \( S^Z_{\text{radial}}(f) := \{ g \in S^Z(f) : g \) is spherically symmetric.} We denote
\[
\phi_J := Z^J \phi := \partial^i \Omega^j S^k u, \quad \text{if} \; J = (i,j,k)
\]
(1.3)

**Assumptions on \( P \).** Let \( h = g - m \), where \( m \) denotes the Minkowski metric. Let \( \sigma \in (0, \infty) \) be real. We make the following assumptions on the coefficients of \( P \):
\[
\begin{align*}
h^{\alpha\beta}, B^\alpha &\in S^Z((\langle r \rangle^{-1-\sigma}) \\
\partial_t B^\alpha, V &\in S^Z((\langle r \rangle^{-2-\sigma}) \\
g^\alpha &\in S^Z_{\text{radial}}((\langle r \rangle^{-2-\sigma})
\end{align*}
\]
(1.4)

In addition, suppose that for a sufficiently small \( \epsilon > 0 \) we have, for \( A_j := \{2^j \leq |x| \leq 2^{j+1}\} \) and an arbitrary interval \( I \subset \mathbb{R}_+ \),
\[
\sum_{j \geq 0} \sup_{I \times A_j} \langle x \rangle^2 |\partial^2 h^{\alpha\beta}| + \langle x \rangle |\partial h^{\alpha\beta}| + |h^{\alpha\beta}| \leq \epsilon
\]
\[
\sum_{j \geq 0} \sup_{I \times A_j} \langle x \rangle^2 |\partial B^\alpha| + \langle x \rangle |B^\alpha| \leq \epsilon
\]
(1.5)

\[
\sum_{j \geq 0} \sup_{I \times A_j} \langle x \rangle^2 |V| \leq \epsilon, \quad \sup_{I \times \mathbb{R}^3} |x|^2 |V| \leq \epsilon.
\]

We use the assumptions (1.5) in order to apply Strichartz estimates on such variable-coefficient backgrounds.

**Theorem 1.1 (Main theorem).** Let
\[
\kappa := \min(\sigma, 2)
\]
and let \( \phi \) solve (1.2) with the assumptions (1.4) and (1.5). Fix \( m \in \mathbb{N} \). We assume that for a fixed \( N \gg m \),
\[
\phi_0 \in L^2(\mathbb{R}^3), \quad \|\langle r \rangle^{1/2 + \kappa} \partial \phi_{\leq m}(0)\|_{L^2(\mathbb{R}^3)} < \infty.
\]
(1.6)

(1) Suppose in addition that
\[
\|\phi_0\|_{H^{N+1}(\mathbb{R}^3)} + \|\phi_1\|_{H^N(\mathbb{R}^3)} \ll 1.
\]
Then for any \( \mu \in \{-1, 0, 1\} \), we have for all times \( t > 0 \)
\[
\sum_{|J| = 0}^m |\phi_J(t, x)| \lesssim \frac{1}{(t + |x|)(t - |x|)^{1+\kappa}},
\]
(1.7)
(2) If \( \mu \in \{0,1\} \), we have for all times \( t > 0 \)

\[
\sum_{|\ell| = 0}^{m} |\phi_j(t,x)| \lesssim \frac{1}{(t + |x|)(t - |x|)^{1+\kappa}}.
\]

(1.8)

where the implicit constant is allowed to depend on \( \|\phi\|_{L^5(\mathbb{R}^+;L^{10}(\mathbb{R}^3))} \).

Remark 1.2 (Commentary on the main theorem). Thus, the main theorem states that for \( \sigma \) close to 0, the solution decays at the rate \( |\phi| \lesssim (t + r)^{-1}(t - r)^{-(1+\sigma)} \). For large \( \sigma \), the solution decays at the rate \( (t + r)^{-1}(t - r)^{-2} \). Moreover, we have these rates for vector fields of \( \phi \). The parameter \( \kappa \) can be of any size (in particular, arbitrarily small positive) and appears in, for instance, the assumptions for proving the \( r \)-weighted local energy decay estimate of Theorem 4.10.

Remark 1.3 (The linear problem: \( \mu = 0 \) in (1.2)). When \( \mu = 0 \), Theorem 1.1 states that the solution to

\[
P\phi = 0, \quad \phi(0,x) = \phi_0(x), \quad \partial_t \phi(0,x) = \phi_1(x)
\]

obeys the bound

\[
\phi \lesssim \langle v \rangle^{-1} \langle u \rangle^{-1-\sigma}, \quad v := t + r, u := t - r;
\]

(1.9)

this result is part of the main theorem in the article [26]. In [26], all coefficients were allowed to be large perturbations of the Minkowski metric (thus \( \epsilon \in (0,\infty) \) in that setting), and the assumptions (1.5) were not needed; (1.5) were brought in to apply Strichartz estimates for the variable-coefficient backgrounds encoded in \( P \), and are unneeded for \( \mu = 0 \) due to the absence of the nonlinearity. For instance, in [26] no assumptions were made on the second derivatives of \( h^{\alpha\beta} \) in order to obtain the final pointwise bound stated in Theorem 1.1.

In addition, for \( \mu = 0 \), the main theorem in [26] shows that the final decay rate (1.9) still holds as long as a weak local energy decay estimate is assumed to hold. As its name implies, assuming such an estimate is a more general assumption than assuming a local energy decay estimate holds. Weak local energy decay estimates are satisfied in a large variety of situations.

Remark 1.4 (Differing increments). The argument shown in this paper straightforwardly yields a proof of a more general version of Theorem 1.1 when the decay increments \( \sigma \) differ for the coefficients. (In (1.4), the increments are all assumed to be equal to \( \sigma \).) See also the main theorem in the companion article [26].

Remark 1.5 (Black hole spacetimes). All the arguments in this paper can be adapted to the exterior of a ball and hence the proofs in this paper can be applied in the case of black hole spacetimes.

1.1. Outline of the paper and strategy of the proof. Here we overview the proof of Theorem 1.1, first by presenting an outline of the paper and then stating the main novelties of the proof.

- In Section 2 we define notation that is used throughout the article.
- In Section 3 we connect pointwise bounds to \( L^2 \) estimates and norms, thereby connecting local energy decay (see Section 1.2) to pointwise bounds. We show how the derivative of \( \phi \) decays at a certain better rate than \( \phi \) and its vector fields; this improvement depends on the distance from the light cone \( \{r = t\} \) at which one is evaluating the pointwise value of the solution.
• In Section 4 we rewrite the equation in a way amenable to our pointwise decay iteration scheme. We state and prove lemmas that are used in the scheme to improve the pointwise decay rates of the solution. Finally, we prove an $r$-weighted local energy decay estimate in Theorem 4.10. To do so, we use Strichartz estimates on such variable-coefficient backgrounds that satisfy the assumptions in Section 1 (more precisely, only (1.5) is assumed\footnote{The Strichartz estimate is written as Theorem 2 in the published version of [33], or Theorem 6 in the current arXiv version of [33].} in order to use Strichartz estimates). This result of Theorem 4.10 is then used to improve the decay rate of $\phi$ (and its vector fields)—see Proposition 4.12.

• In Section 5 we prove the final decay rate for $\phi$ and its vector fields in the region exterior to the light cone $\{r = t\}$, that is, in the region $\{r \geq t\}$.

• In Section 6 we prove the final decay rate for $\phi$ and its vector fields in the region inside of the light cone, that is, $\{r \leq t\}$.

In Remark 6.4 we explain how the iteration applies to other power nonlinearities, reaching pointwise bounds analogous to the one stated in Theorem 1.1.

Main novelties of the proof. Compared to the linear problem $P\phi = 0$ (see [26]), the present article additionally employs Strichartz estimates and proves an $r$-weighted local energy decay estimate (see Theorem 4.10). We show that we are able to use these tools for $\phi$ and vector fields of $\phi$; see Corollary 4.7, Proposition 4.8, and Theorem 4.10.

In contrast to the linear problem $P\phi = 0$ or to power nonlinearities that are sextic or higher order, the initial global decay rate (3.4) (which holds for any function that is finite in the $LE^1$ local energy norm) alone is not sufficient for reaching the final decay rates (1.7) and (1.8). The purpose of proving Theorem 4.10 is precisely to obtain a slightly better initial decay rate (for both the solution and its vector fields) than (3.4).

We would like to control the local energy norm and the Strichartz norm (i.e. $L^5L^{10}$ norm) of not only the solution $\phi$, but also its vector fields $\phi_{\leq m}$. Prior to proving the $r$-weighted estimate, we must first (see Proposition 4.8) control the nonlinearity and its vector fields on the right-hand side of the Strichartz estimate. We partition the time interval $\mathbb{R}_+$ into finitely many sub-intervals $I_k$ so that the nonlinearity norm is small on each $I_k$. This smallness enables us to treat the nonlinearity norm perturbatively (that is, we absorb it to the left-hand side of the Strichartz estimate). However, this perturbative argument comes with the cost of the implicit constant in the Strichartz estimate for $\phi_{\leq m}$ now being dependent on the Strichartz norm of the solution $\phi$; this explains the appended remark in Item 2 (the final bound for the large data problem) in Theorem 1.1.

In proving the $r$-weighted estimate (Theorem 4.10), for the small data problem the use of (3.4) suffices to control the nonlinearity and finish the proof. This is because the bound (3.4) then comes with a small factor, and we can immediately treat the nonlinearity perturbatively by bounding four of the five functions in the nonlinearity using (3.4).\footnote{The reader is encouraged to compare this with the analysis in Remark 1.6 on the sextic and higher powers and to see why this approach fails for the lower power nonlinearities (cubic and quartic).} In contrast, for the large data problem this factor can be large; nonetheless the goal will still be to treat the nonlinearity perturbatively. To achieve this, we make use of an inductive argument that takes advantage of the defocusing nature of the nonlinearity. More precisely, even though the defocusing structure is lost upon application of one or
more vector fields to the equation, we are able to make use of the zeroth-order \( r \)-weighted estimate (wherein no vector fields have been applied to (1.2)) to prove higher order \( r \)-weighted estimates. Compared to the higher order case, the zeroth-order case controls an additional type of term, namely the nonlinearity term: see (4.19). The estimate (4.19) is then used to prove higher order estimates. The idea is that control of some lower order norms allows one to treat the higher order norm perturbatively.

Once these tools are in hand, we commence the iteration and prove the final decay rate in relatively short order: see Sections 5 and 6. The iteration scheme used in the present article is not dissimilar to that used in [26]; this scheme is outlined in Section 4.1 and the complete details of the scheme are located in Sections 5 and 6.

1.2. Local energy decay (LED) estimates. Before stating what an LED estimate is, we define the LED norms. We consider a partition of \( \mathbb{R}^3 \) into the dyadic sets \( A_R = \{ R \leq \langle r \rangle \leq 2R \} \) for \( R > 1 \) and \( A_{R=1} = \{ r \leq 1 \} \). We define

\[
\| \phi \|_{LE} = \sup_R \| \langle r \rangle^{-\frac{1}{2}} \phi \|_{L^2((0,\infty) \times A_R)}
\]

its \( H^1 \) counterpart

\[
\| \phi \|_{LE^1} = \| \partial \phi \|_{LE} + \| \langle r \rangle^{-1} \phi \|_{LE},
\]

as well as the dual norm

\[
\| f \|_{LE^*} = \sum_{R \geq 1} \| \langle r \rangle^{\frac{1}{2}} f \|_{L^2((0,\infty) \times A_R)}
\]

\[
\| f \|_{LE^*[t_0,t_1]} = \sum_{R \geq 1} \| \langle r \rangle^{\frac{1}{2}} f \|_{L^2([t_0,t_1] \times A_R)}.
\]

We have the following scale-invariant estimate on Minkowski backgrounds:

\[
\| \partial \phi \|_{L^2_tL^2_x} + \| \phi \|_{LE^1} \lesssim \| \partial \phi(0) \|_{L^2} + \| \Box \phi \|_{LE^*+[L^1_tL^2_x]}
\] (1.13)

and a similar estimate involving the \( LE^1[t_0, t_1] \) and \( LE^*[t_0, t_1] \) norms. This is called a local energy decay estimate, or integrated local energy decay estimate. Morawetz obtained a local energy decay estimate for the Klein-Gordon equation in [31]. Some other work on local energy decay estimates and their applications can be found in, for instance, [4, 21, 22, 33, 35, 48]. For local energy decay estimates for small and time-dependent long range perturbations of the Minkowski space-time, see for instance [4, 34, 35] for time dependent perturbations, and for example [10, 11, 49] for stationary and nontrapping perturbations. There is a related family of local energy decay estimates for the Schrödinger equation.

Moreover, even for large perturbations of the Minkowski metric, if one assumes the absence of trapping then local energy decay estimates can still hold; see for instance [10, 36]. In addition, even in the presence of sufficiently weak trapping, then estimates similar to these local energy estimates—albeit with a loss of regularity—have been established; see for instance [8, 12, 40, 59]. However, in the presence of sufficiently strong trapping, local energy estimates fail; see [45, 46].
We now remark on how local energy decay relates to two types of asymptotic behaviour, namely pointwise decay rates and scattering. Local energy decay in a compact region on an asymptotically flat region implies pointwise decay rates that are related to how rapidly the metric coefficients decay to the Minkowski metric; see, for example, the works [1, 2, 17, 25, 26, 32, 37–39, 41, 57]. Local energy decay is also involved in proving scattering (another type of asymptotic behaviour) on variable-coefficient backgrounds. In particular, they imply Strichartz estimates on certain variable-coefficient backgrounds, see [33]. [28] used local energy decay to prove scattering for the version of the defocusing problem considered in this paper but with only perturbations to the metric, but the argument extends easily to the version of the problem that includes the lower-order terms and angular terms defined in (1.1).

Pointwise estimates and asymptotic behaviour. We begin with works studying the Minkowski background. In [53], pointwise decay estimates were proven for linear wave equations with a source term using the comparison theorem (positivity of the fundamental solution) in 1+3 dimensions. In [9], numerics were shown for the asymptotic behaviour of small spherically symmetric solutions of nonlinear wave equations with a potential which showed that the dominant tail results from a competition between linear and nonlinear effects.

Still on the Minkowski background, in [61], pointwise decay estimates in $\mathbb{R}^{1+3}$ for various ranges of $p$ in the defocusing nonlinearity $|\phi|^p \phi$ were shown given data in a weighted energy space; more precisely, the solution is shown to decay as rapidly as the linear case for $p+1 > (1+\sqrt{17})/2$. The paper [60] investigated similar questions for 1+d dimensions where $d \geq 3$. While prior investigations along these lines of questioning used the time decay of $t \mapsto \int_{\mathbb{R}^{1+d}} |\phi(t,x)|^{p+2} \, dx$ for $1 < p < 5$ to study pointwise decay estimates and scattering, [61] uses the method introduced in [13] to obtain pointwise decay estimates, via a weighted spacetime energy estimate for $2 < p + 1 < 5$.

We now comment on various works that study other backgrounds, or general backgrounds that include the Minkowski spacetime as a special case. The work [58] considers power type nonlinearities with small initial data on Kerr backgrounds. The work [29] considers the null condition on nonstationary spacetimes similar to those in the present article; it proves global existence and sharp pointwise decay, assuming a local energy decay estimate holds. On these nonstationary spacetimes, the work [27] proves global existence for wave equations with the null condition $P\phi = Q(\partial \phi, \partial \phi)$ assuming a weak LED estimate and for wave equations with cubic and higher order nonlinearities $P\phi = N_{\geq \text{cubic}}(\partial^2 \phi, \partial \phi, \phi)$, given small initial data. Under the assumption of global existence (which holds when, for instance, the initial data is small) this work also proves pointwise decay rates for these families of wave equations and for a family $\mathcal{F}$ of quasilinear wave equations. This family $\mathcal{F}$ contains, as a special case, the quasilinear wave equations close to Schwarzschild and Kerr spacetimes whose global existence was proved in [24] and [25] respectively. The upcoming work [30] obtains sharp pointwise asymptotics, given certain assumptions, for a variety of nonlinearities.

Remark 1.6 (High and low power nonlinearities). In this remark we explain how the methods of the present article automatically give either partial or complete (if certain known results are assumed) proofs of pointwise decay rates (1.14) for various other power nonlinearities. Beyond the present remark, we provide more specifics about this in Remark 6.4, after the iteration has been presented.
We shall distinguish between the small and large data cases:

(1) For small initial data, we consider the Cauchy problems

\[ P \phi = \pm \phi^{p+1}, \quad p \in \mathbb{Z}_{\geq 2} \]

with smooth and compactly supported initial data that is small in a \( H^{n+1} \times H^n \) norm. If \( p + 1 > 1 + \sqrt{2} \) then for sufficiently small and smooth initial data, there exist smooth global solutions. Then the techniques in the present article prove the decay rate

\[ |\phi(t,x)| \lesssim \frac{1}{(t + |x|)(t - |x|)^{1+\min(\sigma,p-2)}}. \quad (1.14) \]

Here we simply mention that

(a) for \( p \geq 5 \) the estimate found in Theorem 4.10 is unnecessary; instead, the bound (3.4) and Lemmas 4.3 and 4.4 alone suffice to reach (1.14). Heuristically, this is because if the nonlinearity contains enough decay, then the initial global decay rate from local energy decay alone (see (3.4)) suffices to bootstrap the solution toward the sharp decay rate stated in (1.14).

(b) The case \( p = 4 \) is the subject of the present article (wherein both large and small data are considered).

(c) For \( p = 2, 3 \), if one had the initial decay rate\(^3\)

\[
\phi|_{r \leq t/2} \lesssim \langle r \rangle^{-1-\delta}, \quad \phi|_{r = t} \lesssim \langle r \rangle^{-1+\delta} \langle u \rangle^{-2\delta}, \quad p = 2 \quad (1.15) \\
\phi|_{r \leq t/2} \lesssim \langle r \rangle^{-3/4-\delta}, \quad \phi|_{r = t} \lesssim \langle r \rangle^{-3/4+\delta} \langle u \rangle^{-2\delta}, \quad p = 3 \quad (1.16)
\]

then the iteration also follows Sections 5 and 6 nearly verbatim and one reaches the rate (1.14) from the method in this paper. See [58] for a proof of the small data problem on the Kerr background. In the small data case, additional lemmas become available.

(2) For large initial data, we consider the (defocusing) Cauchy problems

\[ P \phi = |\phi|^p \phi, \quad p \in 2\mathbb{Z}_{\geq 1} \]

where we avoid the odd integer values of \( p \) because of the issue regarding the smoothness of the modulus of \( \phi \) close to the zero set of \( \phi \). (The method presented in this article applies vector fields to the nonlinearity.)

The value \( p = 4 \) is covered in the present article. For higher \( p \) values, we remark that global existence has not been explicitly established in the literature. If global existence is assumed for the values \( p \geq 5 \), then the iteration presented in Sections 5 and 6 automatically give (1.14), with the same remark in Item 1a above applying here. For \( p = 2 \), if global existence is assumed, then the remark in Item 1c above applies here as well.

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\(^3\)for some sufficiently small \( \delta \)—more precisely, for \( p = 2 \), we need \( 2 - 3\delta > 1 \), while for \( p = 3 \) we need \( 2 - 4\delta > 1 \). This is because of an integration in the radial variable \( \rho \) in the backward light cone \( D_{tr} \) (see Definition 2.1), when bounding \( \int_{D_{tr}} \rho |\phi|^{p+1} dA \). More precisely, we are taking the exponent \( 1/x > 1 \) in \( 1/\langle \rho \rangle^{ex} \). (1.15) and (1.16) follow if Proposition 4.12 holds with \( \gamma > 1, \gamma > 1/2 \) respectively, and see also Remark 4.13.
2. Notation

We begin by defining dyadic numbers and dyadic conical subregions. We work only with dyadic numbers that are at least 1. We denote dyadic numbers by capital letters for that variable; for instance, dyadic numbers that form the ranges for radial (resp. temporal and distance from the cone \( \{|x|=t\} \)) variables will be denoted by \( R \) (resp. \( T \) and \( U \)); thus

\[ R, T, U \geq 1. \]

We choose dyadic integers for \( T \) and a power \( a \) for \( R, U \)—thus \( R = a^k \) for \( k \geq 1 \)—different from 2 but not much larger than 2, for instance in the interval \((2, 5]\), such that for every \( j \in \mathbb{N} \), there exists \( j' \in \mathbb{N} \) with \( a^{j'} = \frac{3}{2} \).

Dyadic decomposition of spacetime. We decompose the region \( \{r \leq t\} \) based on either distance from the cone \( \{|r|=t\} \) or distance from the origin \( \{r=0\} \). We fix a dyadic number \( T \). The regions \( C^T, C^R_T \) and \( C^U_T \) are where we shall apply Sobolev embedding.

\[ C^T := \left\{ (t, x) \in [0, \infty) \times \mathbb{R}^3 : T \leq t \leq 2T, \ r \leq t \right\}, \quad T > 1 \]
\[ C^R_T := \begin{cases} C^T \cap \{R < r < 2R\} & R > 1 \\ C^T \cap \{0 < r < 2\} & R = 1 \end{cases} \]
\[ C^U_T := \begin{cases} (t, x) \in [0, \infty) \times \mathbb{R}^3 : T \leq t \leq 2T \cap \{U < |t-r| < 2U\} & U > 1 \\ (t, x) \in [0, \infty) \times \mathbb{R}^3 : T \leq t \leq 2T \cap \{0 < |t-r| < 2\} & U = 1 \end{cases} \]

As a subregion inside the forward light cone, we define

\[ C^{<3T/4}_T := \bigcup_{R < 3T/8} C^{R}_T. \]

Now letting \( R > T \), we define

\[ C^T_R := \{(t, x) \in [0, \infty) \times \mathbb{R}^3 : r \geq t, \ T \leq t \leq 2T, R \leq r \leq 2R, R \leq |r-t| \leq 2R\} \]

2.0.1. The symbols \( n \) and \( N \). Throughout the paper the integer \( N \) will denote a fixed and sufficiently large positive number, signifying the highest total number of vector fields that will ever be applied to the solution \( \phi \) to (1.2) in the paper.

We use the convention that the value of \( n \) may vary by line.

Tildes atop sets. If \( \Sigma \) is a dyadic set as defined above, we shall use \( \tilde{\Sigma} \) to indicate a slight enlargement of \( \Sigma \) on \( \Sigma \)'s scale. We only perform a finite number of slight enlargements in this paper to dyadic sets. The symbol \( \tilde{\Sigma} \) may vary by line.

The variables \( r, u, v \). If \( x = (x^1, x^2, x^3) \in \mathbb{R}^3 \), we write

\[ r := |x| = \left( \sum_{i=1}^{3} (x^i)^2 \right)^{1/2}, \quad u := t-r, \quad v := t+r. \]
**Summation of norms.** Recall the subscript notation (1.3) for vector fields. Let \( \| \cdot \| \) be any norm used in this paper. Given any nonnegative integer \( N \geq 0 \), we write \( \| g_{\leq N} \| \) to denote \( \sum_{|J| \leq N} \| g_{J} \| \). For instance, taking the absolute value as an example of the norm, the notation \( | \phi_{\leq m}(t,x) | \) means
\[
| \phi_{\leq m}(t,x) | = \sum_{J : |J| \leq m} | \phi_{J}(t,x) |.
\]

**Definition 2.1.** Let
\[
R_{1} := \{ R : R < u/8 \}, \quad R_{2} := \{ R : u/8 < R < v \}, \quad u > 0.
\]

Let \( \mathbb{R}_{+} := [0,\infty) \).

- Let \( D_{tr} \) denote the backward light cone with apex \((r,t)\)
\[
D_{tr} := \{ (\rho,s) \in \mathbb{R}_{+}^{2} : -(t+r) \leq s - \rho \leq t - r, \ |t - r| \leq s + \rho \leq t + r \}.
\]

When we work with \( D_{tr} \) we shall use \((\rho,s)\) as variables, and \( D_{tr}^{R} \) is short for \( D_{tr}^{\rho < R} \).

- For \( R > 1 \), let
\[
D_{tr}^{R} := D_{tr} \cap \{ (\rho,s) : R < \rho < 2R \}
\]

and let
\[
D_{tr}^{R=1} := D_{tr} \cap \{ (\rho,s) : \rho < 2 \}.
\]

### 3. From local energy decay to pointwise bounds

In this section we will show that local energy decay bounds imply certain slow decay rates for the solution, its vector fields, and its derivatives—see Propositions 3.3 and 3.5.

We start with the following pointwise estimate for the second derivative. We shall use it, for instance, when applying Lemma 3.2 to the functions \( w = \partial \phi_{\leq m} \) (that is, when we bound the first-order derivatives pointwise); this will be done in Proposition 3.5.

**Lemma 3.1.** Assume \( \phi \) is sufficiently regular. Then for any point \((t,x)\)
\[
| \partial^{2} \phi_{J}(t,x) | \lesssim \left( \frac{1}{\langle r \rangle} + \frac{1}{\langle u \rangle} \right) | \partial \phi_{|J|+1} | + \left( 1 + \frac{t}{\langle u \rangle} \right) \langle r \rangle^{-2} | \phi_{|J|+2} | + \left( 1 + \frac{t}{\langle u \rangle} \right) | (P\phi)_{\leq |J|} |.
\]

A proof of Lemma 3.1 can be found in [29], which in fact proves Lemma 3.1 under more relaxed assumptions than those made on \( P \) in this article.

By (3.4), (3.1) immediately implies for solutions to (1.2):
\[
| \partial^{2} \phi_{J} | \lesssim \left( \frac{1}{\langle r \rangle} + \frac{1}{\langle u \rangle} \right) | \partial \phi_{|J|+1} | + \left( 1 + \frac{t}{\langle u \rangle} \right) \langle r \rangle^{-2} | \phi_{|J|+2} | \tag{3.2}
\]

The primary estimates that let us pass from local energy decay to pointwise bounds are contained in the following lemma.
Lemma 3.2. Let \( w \in C^4 \), \( Z_{ij} := S^i \Omega^j \), \( \mu := \langle \min(r, |t - r|) \rangle \), and \( \mathcal{R} \in \{ C^R_T, C^V_T, C^T_R \} \). Then we have
\[
\|w\|_{L^\infty(\mathcal{R})} \lesssim \sum_{i \leq j \leq 2} \frac{1}{|\mathcal{R}|^{1/2}} \left( \|Z_{ij}w\|_{L^2(\mathcal{R})} + \|\mu \partial Z_{ij}w\|_{L^2(\mathcal{R})} \right). \tag{3.3}
\]
where we assume \( U \leq \frac{3}{8}T \), \( R \leq \frac{3}{8}T \) and \( R > T \) in the cases \( C^V_T, C^R_T, C^T_R \) respectively, and \( |\mathcal{R}| \) denotes the measure of \( \mathcal{R} \).

The proof of the \( \mathcal{R} \in \{ C^V_T, C^R_T \} \) case is contained in [37] or [26], while the proof for \( C^T_R \) can be found in [26]. We only sketch the proof here. One uses exponential coordinates, which results in \( \mathcal{R} \) being transformed into a region of size \( O(1) \) in all directions. The result is then proved using the fundamental theorem of calculus for the \( s, \rho \) variables and Sobolev embedding for the angular variables.

The next proposition yields an initial global pointwise decay rate for \( \phi_J \) under the assumption that the local energy decay norms are finite. We shall improve this rate of decay in future sections (see Sections 5 and 6) for solutions to (1.2), culminating ultimately in the final pointwise decay rate stated in the main theorem.

Proposition 3.3. Let \( T \) be fixed and \( \phi \) be any sufficiently regular function. There is a fixed positive integer \( k \), such that for any multi-index \( J \) with \( |J| \leq N - k \), we have:
\[
|\phi_J| \lesssim \tilde{C}_{|J|} \|\phi_{\leq |J| + k}\|_{L^E(\mathcal{R})} (u)^{1/2} (v)^{-1}. \tag{3.4}
\]

We only sketch the proof here; full details are provided in [26]. One uses Lemma 3.2, which proves (3.4) except in the wave zone. For the wave zone, an extra Hardy-like inequality (3.5) is used to finish the proof. Lemma 3.4 is proven by multiplying by a cutoff function localised to the wave zone; [26] contains a full proof.

Lemma 3.4. If \( f \in C^1 \), then
\[
\int_{t/2}^{3t/2} (t-r)^{-2} f(t, x)^2 dx \lesssim \int_{t/4}^{7t/4} |\partial_r f(t, x)|^2 dx + \frac{1}{t^2} \left( \int_{t/4}^{t/2} f(t, x)^2 dx + \int_{3t/2}^{7t/4} f(t, x)^2 dx \right). \tag{3.5}
\]

3.1. Derivative bounds. The next proposition shows that the first-order derivative (of solutions to (1.2)) decays pointwise faster by a rate of \( \min(\langle r \rangle, \langle t - r \rangle) \). It utilises the initial global decay rate (3.4). The estimates in its proof involve the nonlinearity, but for the quintic nonlinearity as in (1.2) it turns out that the global bounds (3.4) alone already suffice to make the pointwise decay of the first-order derivative similar to the linear case, which is the content of Proposition 3.5. The reader can find details for the linear problem in [26]. Proposition 3.5 will be used in the pointwise decay iteration (see later sections, Sections 5 and 6)—more precisely, for iterating upon the linear components of the equation, namely those having to do with the operator \( P - \Box \). By contrast, the nonlinearity in (1.2) does not involve any derivatives, so Proposition 3.5 will not be involved in the iteration for the nonlinearity.

Proposition 3.5. Let \( \phi \) solve (1.2), and assume that
\[
\phi_{\leq m+n} \lesssim \langle r \rangle^{-\alpha} \langle t \rangle^{-\beta} \langle u \rangle^{-\eta}. \tag{11}
\]
for some sufficiently large $n$. We then have
\begin{equation}
\partial \phi \leq m \lesssim \langle r \rangle^{-\alpha} \langle t \rangle^{-\beta} (u)^{-\eta} \mu^{-1}, \quad \mu := \langle \min(r, |t - r|) \rangle. \tag{3.6}
\end{equation}

**Proof.** Let $\mathcal{R} \in \{C^R, C^R_P, C^R_T\}$. Given a function $w$, we have
\begin{equation}
\| \partial w \leq m \|_{L^2(\mathcal{R})} \lesssim \| \frac{w \leq m + n}{\mu} \|_{L^2(\tilde{\mathcal{R}})} + \| (r)(Pw) \leq m \|_{L^2(\tilde{\mathcal{R}})}. \tag{3.7}
\end{equation}

(See [26] or [37] for a proof of (3.7).) By the initial global pointwise estimate (3.4), (3.7) with $w = \phi$ yields
\begin{equation}
\| \partial \phi \leq m \|_{L^2(\mathcal{R})} \lesssim \| \frac{\phi \leq m + n}{\mu} \|_{L^2(\tilde{\mathcal{R}})}. \tag{3.8}
\end{equation}

Recalling Lemma 3.2, we have
\begin{align*}
\| \partial \phi \leq m \|_{L^\infty(\mathcal{R})} & \lesssim |\mathcal{R}|^{-\frac{1}{2}} \sum Z \| Z \partial \phi \leq m \|_{L^2(\mathcal{R})} + \| \mu \partial Z \partial \phi \leq m \|_{L^2(\mathcal{R})} \\
& \lesssim |\mathcal{R}|^{-\frac{1}{2}} \left( \| \partial \phi \leq m + n \|_{L^2(\mathcal{R})} + \| \mu \partial \phi \leq m + n \|_{L^2(\mathcal{R})} \right) \\
& \lesssim |\mathcal{R}|^{-\frac{1}{2}} \left( \| \mu^{-1} \phi \leq m + n \|_{L^2(\mathcal{R})} + \| \mu \partial \phi \leq m + n \|_{L^2(\mathcal{R})} \right) \\
& \lesssim |\mathcal{R}|^{-\frac{1}{2}} \left( \| \mu^{-1} \phi \leq m + n \|_{L^2(\mathcal{R})} + \| \mu (1 + \frac{t}{\langle u \rangle}) \langle r \rangle^{-2} \phi \leq m + n \|_{L^2(\mathcal{R})} \right) \\
& \lesssim |\mathcal{R}|^{-\frac{1}{2}} \| \mu^{-1} \phi \leq m + n \|_{L^2(\mathcal{R})}
\end{align*}
which follows by (3.2) and (3.8). The final line follows because $\mu^2 (1 + t/\langle u \rangle) \lesssim \langle r \rangle^2$. Finally, the claim (3.6) follows because $\| \mu^{-1} \phi \leq m + n \|_{L^2(\mathcal{R})} \lesssim |\mathcal{R}|^{\frac{1}{2}} \| \mu^{-1} \phi \leq m + n \|_{L^\infty(\tilde{\mathcal{R}})}. \quad \square

## 4. Preliminaries for the iteration, including the $r^\kappa$ estimate

**Remark 4.1** (The initial data). Let $w := S(t, 0)\phi[0]$ denote the solution to the free wave equation with initial data $\phi[0]$ at time 0. Then for any $|J| = O_N(1)$,
\begin{equation}
w_J(t, x) = \frac{1}{|\partial B(x, t)|} \int_{\partial B(x, t)} (\phi_0)J(y) + \nabla y(\phi_0)J(y) \cdot (y - x) + t(\phi_1)J(y) dS(y). \tag{4.1}
\end{equation}

By (4.1) and the assumptions $\phi_0 \in L^2(\mathbb{R}^3)$,
\[\| \langle r \rangle^{1/2+\kappa} \partial \phi \leq N(0) \|_{L^2} < \infty, \quad \kappa = \min(\sigma, 2)\]
we have
\[w_J \lesssim \langle v \rangle^{-1} \langle u \rangle^{-1-\kappa}.

### 4.1. Summary of the iteration

By Remark 4.1, we may assume zero initial data in the following iteration. Second, note that it suffices to prove bounds in $|u| \geq 1$, because the desired final decay rate in $|u| < 1$ already holds by (3.4). Third, we distinguish the nonlinearity and the coefficients of $P - \Box$, and for both of these, we apply the fundamental solution. We iterate these two components in lockstep with one another.
Due to the domain of dependence properties of the wave equation, we shall first complete the iteration in \( \{ u < -1 \} \). For the iteration in \( \{ u > 1 \} \), the decay rates obtained from the fundamental solution are insufficient in the region \( \{ r < t/2 \} \). To remedy this, we prove Proposition 6.2. With the new decay rates obtained from Proposition 6.2, we are then able to obtain new decay rates for the solution and its vector fields. At every step of the iteration, Lemma 4.3 is used to turn the decay gained at previous steps into new decay rates.

Remark 4.2. To simplify the iteration, we shall reduce the value of \( \sigma \) if necessary to be equal to some positive irrational number less than the original value of \( \sigma \). We do this to avoid the appearance of logarithms in the iterations for \( \phi_1 \) and \( \phi_2 \) (see the decomposition (4.5) below). We take \( 0 < \sigma \ll 1 \). In the sections spelling out the details of the iteration, namely Sections 5 and 6, we explain how we reach the final decay rate in Theorem 1.1 (wherein the original value of \( \sigma \) is included in the final decay rate).

4.2. Setting up the problem. We rewrite (1.2) as

\[
\Box \phi = (\Box - P)\phi + F = -\partial_\alpha (h^{\alpha\beta} \partial_\beta \phi + B^\alpha \phi) - g^{\alpha\beta} \Delta_\phi \phi - (V - \partial_\alpha B^\alpha) \phi + F, \quad F := \phi^5
\]

Using the assumptions (1.4), we can write this as

\[
\Box \phi \in \partial (S^Z(r^{-1-\sigma})\phi_{\leq 1}) + S^Z(r^{-2-\sigma})\phi_{\leq 2} + F
\]

Pick any multiindex \( |J| \leq N_1 - 2 \). We have after commuting

\[
\Box \phi_J \in \partial (S^Z(r^{-1-\sigma})\phi_{\leq m+1}) + S^Z(r^{-2-\sigma})\phi_{\leq m+2} + F_{\leq m}
\]

Due to the derivative gaining only \( \langle u \rangle^{-1} \) in the wave zone (see Proposition 3.5), we shall perform an additional decomposition as follows. First, we note that, for any function \( w \),

\[
\partial w \in S^Z(r^{-1})w_{\leq 1} + S^Z(1)\partial_t w, \quad r \geq t/2
\]

which is clear for \( \partial_t \) and \( \partial_\omega \), while for \( \partial_r \) we write

\[
\partial_r = \frac{S}{r} - \frac{t}{r} \partial_t.
\]

Let \( \chi_{cone} \) be a cutoff adapted to the region \( t/2 \leq r \leq 3t/2 \). We now rewrite (4.2) as

\[
\Box \phi_J \in S^Z(r^{-2-\sigma})\phi_{\leq m+2} + (1 - \chi_{cone}) (S^Z(r^{-1-\sigma})\partial \phi_{m+1}) + \partial_t \left( \chi_{cone} S^Z(r^{-1-\sigma})\phi_{m+1} \right) + F_{\leq m}
\]

We now write \( \phi_J = \sum_{j=1}^3 \phi_j \) where

\[
\Box \phi_1 = G_1, \quad G_1 \in S^Z(r^{-2-\sigma})\phi_{\leq m+2} + (1 - \chi_{cone}) (S^Z(r^{-1-\sigma})\partial \phi_{m+1})
\]

\[
\Box \phi_2 = \partial_t G_2, \quad G_2 \in \chi_{cone} S^Z(r^{-1-\sigma})\phi_{m+1}
\]

\[
\Box \phi_3 = F_{\leq m} = G_3
\]

Henceforth the convention in Section 2.0.1 will apply to the symbol \( n \).
4.3. Estimates for the fundamental solution. We have the following result, which is similar to previous classical results, see for instance [18], [5], [52], [55].

**Lemma 4.3.** Let \( m \geq 0 \) be an integer and suppose that \( \psi : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R} \) solves

\[
\Box \psi(t, x) = g(t, x), \quad \psi(0) = 0, \quad \partial_t \psi(0) = 0.
\]

Define

\[
h(t, r) = \sum_{i=0}^{2} \|\Omega^i g(t, r\omega)\|_{L^2(S^2)} \tag{4.6}
\]

Assume that

\[
h(t, r) \lesssim \frac{1}{\langle r \rangle^{\alpha} \langle u \rangle^{\beta} \langle u' \rangle^{\eta}}, \quad \alpha \in (2, 3) \cup (3, \infty), \quad \beta \geq 0, \quad \eta \geq -1/2.
\]

Define

\[
\tilde{\eta} = \begin{cases} 
\eta - 2, & \eta < 1 \\
-1, & \eta > 1
\end{cases}.
\]

We then have in both \( \{ u > 1 \} \) and \( \{ u < -1 \} \) in the case \( \alpha + \beta + \eta > 3 \):

\[
\psi(t, x) \lesssim \frac{1}{\langle r \rangle^{\alpha + \beta + \tilde{\eta}}} \tag{4.7}
\]

On the other hand, if \( \alpha + \beta + \eta < 3 \) and \( u < -1 \), we have

\[
\psi(t, x) \lesssim r^{2 - (\alpha + \beta + \eta)}. \tag{4.8}
\]

**Proof.** A detailed proof of (4.7) can be found in Lemma 5.5 of [26] (see also Lemma 6.1 in [58]). The idea is to use Sobolev embedding and the positivity of the fundamental solution of \( \Box \) to show that

\[
r \psi \lesssim \int_{D_{tr}} \rho h(s, \rho) ds d\rho,
\]

where \( D_{tr} \) is the backwards light cone with vertex \( (r, t) \), and use (4.6).

Let us now prove (4.8). In this case \( D_{tr} \subset \{ r - t \leq u' \leq r + t, \quad r - t \leq \rho \leq r + t \} \) and we obtain, using that \( \langle u' \rangle \lesssim \rho \) and \( \rho > t \) in \( D_{tr} \):

\[
r \psi \lesssim \int_{r-t}^{r+t} \int_{r-t}^{r+t} \rho^{1 - \alpha - \beta} \langle u' \rangle^{-\eta} \rho d\rho d\rho' \lesssim \int_{r-t}^{r+t} \langle u' \rangle^{2 - (\alpha + \beta + \eta)} d\rho' \lesssim v^{3 - (\alpha + \beta + \eta)}
\]

which finishes the proof. \( \square \)

For the function \( \phi_2 \) we will use the following result for an inhomogeneity of the form \( \partial_t g \) supported near the cone. The result is similar to Lemma 4.3, except that we gain an extra factor of \( \langle u \rangle \) in the estimate.

**Lemma 4.4.** Let \( \psi \) solve

\[
\Box \psi = \partial_t g, \quad \psi(0) = 0, \quad \partial_t \psi(0) = 0, \tag{4.9}
\]

where \( g \) is supported in \( \{ \frac{t}{2} \leq \langle u \rangle \leq \frac{3}{2} \} \). Let \( h \) be as in (4.6), and assume that

\[
|h| + |Sh| + |\Omega h| + \langle t - r \rangle |\partial h| \lesssim \frac{1}{\langle r \rangle^{\alpha} \langle u \rangle^\eta}, \quad 2 < \alpha < 3, \quad \eta \geq -1/2.
\]
Then in \( \{ u > 1 \} \), and \( \{ u < -1 \} \) when \( \alpha + \eta > 3 \)

\[
\psi(t, x) \lesssim \frac{1}{(r'(u))^{\alpha+\eta}}. \tag{4.10}
\]

**Proof.** Let \( \tilde{\psi} \) solve \( \Box \tilde{\psi} = g, \tilde{\psi}(0) = 0, \partial_t \tilde{\psi}(0) = 0 \). In the support of \( g \) we have

\[
(t\partial_i + x_i\partial_t)h \lesssim |Sh| + |\Omega h| + (t-r)|\partial_th|.
\]

By Lemma 4.3 (with \( \beta = 0 \)) applied to \( \nabla \tilde{\psi}, \Omega \tilde{\psi}, S\tilde{\psi} \), and the fact

\[
\langle u \rangle \partial_t \tilde{\psi} \lesssim |\nabla \tilde{\psi}| + |S\tilde{\psi}| + |\Omega \tilde{\psi}| + \sum_i |(t\partial_i + x_i\partial_t)\tilde{\psi}|
\]

the claim follows. \( \Box \)

4.4. \( r^7 \) decay.

**Lemma 4.5** (Preliminary LED and Strichartz). Let \( I \in \{ [T_0, T_1], [T_0, \infty) \} \) be an interval, where \( T_1 \geq T_0 \geq 0 \) are real numbers. If for a sufficiently small \( \epsilon > 0 \), (1.5) holds, and \( \psi \) is a function, then

\[
\| \psi \|_{(L^1_t \cap L^5_x)(I \times \mathbb{R}^3)} + \| \partial \psi \|_{L^2_t L^2_x(I \times \mathbb{R}^3)} \lesssim \| \partial \psi(T_0) \|_{L^2} + \| P \psi \|_{(L^1_t L^2_x + L^2_t) (I \times \mathbb{R}^3)}. \tag{4.11}
\]

**Remark 4.6.** We will assume but not prove Lemma 4.5. The statement of Lemma 4.5 is obtained by combining Theorem 3 in [33] and Proposition 8 in [34]. The assumptions on the operator \( P \) in this paper in (1.5) satisfy the assumptions in both results.

**Corollary 4.7** (Lemma 4.5 with vector fields). For any \( m \geq 0 \)

\[
\| \phi \|_{(L^1_t \cap L^5_x)(I \times \mathbb{R}^3)} + \| \partial \phi \|_{L^2_t L^2_x(I \times \mathbb{R}^3)} \lesssim \| \partial \phi(T_0) \|_{L^2} + \| (P \phi) \|_{(L^1_t L^2_x)(I \times \mathbb{R}^3)}. \]

**Proof.** Since \( P\phi_J = (P\phi)_J + [P, Z']\phi \), by (4.11) we have on any fixed \( I \) with left endpoint \( T_0 \) the following estimate

\[
\| \phi_J \|_{(L^1_t \cap L^5_x) \cap L^5_t L^5_x(I \times \mathbb{R}^3)} + \| \partial \phi_J \|_{L^2_t L^2_x(I \times \mathbb{R}^3)} \lesssim \| \partial \phi_J(T_0) \|_{L^2} + \| (P\phi)_J \|_{L^1_t L^2_x(I \times \mathbb{R}^3)} + \| [P, Z']\phi \|_{L^2_t L^2_x(I \times \mathbb{R}^3)}.
\]

The second line follows from assumptions on \( P \). The claim follows for small \( \epsilon \). \( \Box \)

**Proposition 4.8** (Corollary 4.7 with no nonlinearity). Let \( \phi \) solve (1.2). Assume the hypotheses on \( P \) from (1.5). For any interval \( I \in \{ [T_0, T_1], [T_0, \infty) \} \), the following estimate holds:

\[
\| \phi \|_{(L^1_t \cap L^5_x)(I \times \mathbb{R}^3)} + \| \partial \phi \|_{L^2_t L^2_x(I \times \mathbb{R}^3)} \leq C(\| \phi \|_{L^5_t L^{10}_x(I \times \mathbb{R}^3)}, m) \| \partial \phi(T_0) \|_{L^2(\mathbb{R}^3)} \tag{4.12}
\]

Since \( \| \phi \|_{L^5_t L^{10}_x} \) is bounded by the initial data up to an implicit constant, if the initial data are sufficiently small in the energy norm, then the bound (4.12) holds without any specific dependence on the size of \( \| \phi \|_{L^5_t L^{10}_x} \).

**Proof.** We first remark that

\[
\| \phi \|_{L^5(\mathbb{R}^3)} < \infty
\]

which holds by the main theorem in [28]. For \( m \geq 1 \), the bounds

\[
\| \phi \|_{L^5(\mathbb{R}^3)} < \infty
\]
can be proven by induction, which we now proceed with. Suppose that for some integer $m \geq 0$, $\|\phi_m\|_{L^5([t_n; t^{10}])} < \infty$; we shall show that (4.12) holds.

There are intervals $I_0, \ldots, I_n$ “almost-partitioning” $[0, \infty)$ with $t_0 = 0 \in I_0$, and for $0 \leq j \leq n - 1$, $I_j = [t_j, t_{j+1}]$, while $I_n = [t_n, \infty)$, such that for all $j$, the following norm obeys the bound

$$
\|\phi_m\|_{L^5(I_j; L^{10}([0, \infty) \times \mathbb{R}^3))} \leq 1/(4C_{\text{Stri}})^{1/4}, \quad I_j := [t_j, t_{j+1}]
$$

(4.13)

- By Lemma 4.5, we obtain

$$
\|\phi_{m+1}\|_{LE^1(I_j)} + \|\phi_{m+1}\|_{L^5(I_j; L^{10}([0, \infty) \times \mathbb{R}^3))} + \|\partial \phi_{m+1}\|_{L^\infty(I_j; L^{10}([0, \infty) \times \mathbb{R}^3))} 
\leq C_{\text{Stri}} \left( \|\partial \phi_{m+1}(t_j)\|_{L^2} + \|(P\phi)_{m+1}\|_{L^1(I_j; L^2([0, \infty) \times \mathbb{R}^3))} + \epsilon \|\phi_{m+1}\|_{LE^1(I_j)} \right).
$$

(Here, we bounded the commutator in $LE^s(I_j)$. Here $C_{\text{Stri}}$ is the constant from the Strichartz estimate from Lemma 4.5, and $\epsilon$ from Section 1 is chosen sufficiently small.) This estimate implies the estimate

$$
\|\phi_{m+1}\|_{LE^1(I_j)} + \|\phi_{m+1}\|_{L^5(I_j; L^{10}([0, \infty) \times \mathbb{R}^3))} + \|\partial \phi_{m+1}\|_{L^\infty(I_j; L^{10}([0, \infty) \times \mathbb{R}^3))} 
\leq 2C_{\text{Stri}} \left( \|\partial \phi_{m+1}(t_j)\|_{L^2} + \|\phi_m\|_{L^5(I_j; L^{10}([0, \infty) \times \mathbb{R}^3))} \|\phi_{m+1}\|_{L^5(I_j; L^{10}([0, \infty) \times \mathbb{R}^3))} \right).
$$

(4.14)

- By the hypothesis that $\|\phi_m\|_{L^5([0, \infty) \times \mathbb{R}^3)}$ is finite, we apply (4.13). Thus

$$
\|\phi_{m+1}\|_{LE^1([t_j, t_{j+1}])} + \|\phi_{m+1}\|_{L^5([t_j, t_{j+1}]; L^{10}([0, \infty) \times \mathbb{R}^3))} + \|\partial \phi_{m+1}\|_{L^\infty([t_j, t_{j+1}]; L^{10}([0, \infty) \times \mathbb{R}^3))} 
\leq C \|\partial \phi_{m+1}(t_j)\|_{L^2}, \quad C = C(\|\phi\|_{L^5([0, \infty) \times \mathbb{R}^3)}, m)
$$

$$
\leq C \|\partial \phi_{m+1}\|_{L^\infty([t_{j-1}, t_j]; L^2([0, \infty) \times \mathbb{R}^3))} \leq C \cdot C \|\partial \phi_{m+1}(t_{j-1})\|_{L^2} \leq C_{j+1} \|\partial \phi_{m+1}(0)\|_{L^2}
$$

(4.14)

Here when $j = n$, the interval is understood to read $[t_n, \infty)$.

- By adding these estimates together, we obtain

$$
\|\phi_{m+1}\|_{LE^1([0, \infty) \times \mathbb{R}^3)} + \|\phi_{m+1}\|_{L^5([0, \infty) \times \mathbb{R}^3)} + \|\partial \phi_{m+1}\|_{L^\infty([0, \infty) \times \mathbb{R}^3)} \leq C_{j+1} \|\partial \phi_{m+1}(0)\|_{L^2}
$$

which holds because

$$
\|\partial \phi_{m+1}\|_{L^\infty([0, \infty) \times \mathbb{R}^3)} \leq \sum_{j=0}^n \|\partial \phi_{m+1}\|_{L^\infty(I_j; L^{10}([0, \infty) \times \mathbb{R}^3))}.
$$

\[\square\]

Remark 4.9 (Constants can henceforth depend on the $L^5L^{10}$ norm of $\phi$). Henceforth, we always allow the implicit constant in estimates to depend on $\|\phi\|_{L^5L^{10}([0, \infty) \times \mathbb{R}^3)}$. As a consequence of Proposition 4.8, we have

$$
\|\phi_m\|_{LE^1([0, \infty) \times \mathbb{R}^3)} \lesssim \|\partial \phi_m\|_{L^2([0, \infty) \times \mathbb{R}^3)}.
$$

(4.15)

In the following theorem and its subsequent application (Proposition 4.12) to the pointwise decay problem at hand, we have in mind only an arbitrarily small $\gamma > 0$.

Theorem 4.10 (The $r^\gamma$ estimate). Let $\phi$ solve (1.2). Let $\gamma < 2\sigma, \gamma < 1$ and let the potential $V$ satisfy $V \in S^2(\epsilon/r^2)$. Let $T_2 > T_1 \geq 0$. 


For any integer \( m \geq 0 \), we have

\[
A_{\gamma,m} + E_{\phi \leq m}(T_2) \lesssim \|\phi\|_{L^6(L^{10})} E_{\phi \leq m}(T_1) + \|\partial\phi_{\leq m}\|^2_{L^2(E(T_1,T_2))} + \|\partial^2\phi_{\leq m}\|^2_{L^2(E(T_1,T_2))}
\]

(4.16)

where the \( A, E \) norms are:

\[
A_{\gamma,m} := \int_{T_1}^{T_2} \int_{\mathbb{R}^3} (\phi_{\leq m})^2 r^{\gamma-3} + |\partial\phi_{\leq m}|^2 r^{\gamma-1} \, dx \, dt
\]

\[
E_{\phi \leq m}(T_1) := \|r^{\gamma/2}(\phi_{\leq m}, (\partial_r + \frac{1}{2r})\phi_{\leq m}, \phi_{\leq m})(T_1)\|^2_{L^2(\mathbb{R}^3)}, \|r^\alpha(f_1, \ldots, f_n)\| := \sum_{j=1}^n \|r^\alpha f_j\|.
\]

**Proof.** Fix \( m \geq 0 \). Let \(|J| \leq m\). Fix \( 0 \leq T_1 < T_2 \). Let

\[
A_{\gamma,J} := \int_{T_1}^{T_2} \int_{\mathbb{R}^3} \phi_J^2 r^{\gamma-3} + |\partial\phi_J|^2 r^{\gamma-1} \, dx \, dt.
\]

Integrating by parts,

\[
\int_{[T_1,T_2] \times \mathbb{R}^3} \Box_J (r^\gamma \partial_r \phi_J + r^{\gamma-1} \phi_J) \, dx \, dt = \int_{[T_1,T_2] \times \mathbb{R}^3} \frac{-\gamma r^{\gamma-1}}{2} (\partial_r \phi_J)^2 - \frac{1}{2} (2 - \gamma) r^{\gamma-1} |\phi_J|^2
\]

\[
- \frac{\gamma(1 - \gamma) r^{\gamma-3}}{2} \phi_J^2 \, dx \, dt + \int_{\mathbb{R}^3} \phi_J \left[ \frac{1}{2} |\partial_r \phi_J|^2 + \partial_r \phi_J \partial_r \phi_J + \frac{1}{2r} \partial_r \phi_J \right] \, dx
\]

(4.17)

- We now manipulate the boundary terms to obtain positive definite terms: We have

\[
\int_{\mathbb{R}^3} -r^\gamma \frac{1}{2r} \partial_r \phi_J \, dx = \int -r^\gamma \frac{1}{2r} (\partial_r - \partial_r) \phi_J \, dx
\]

\[
= \int -r^\gamma \frac{1}{2r} \partial_r \phi_J \, dx + \int_{S^2} \int_0^\infty \frac{1}{2r} \partial_r \phi_J \, r^2 \, dr \, d\omega
\]

\[
= \int -r^\gamma \frac{1}{2r} \partial_r \phi_J \, dx + \int_{S^2} \int_0^\infty \frac{1}{2r} \partial_r \phi_J \, r^2 \, dr \, d\omega
\]

\[
= \int -r^\gamma \frac{1}{2r} \partial_r \phi_J \, dx + \int_{S^2} \int_0^\infty \frac{1}{4} \partial_r \phi_J^2 \, r^2 \, dr \, d\omega
\]

\[
= \int -r^\gamma \frac{1}{2r} \partial_r \phi_J \, dx - \int_{S^2} \int_0^\infty \frac{\gamma + 1}{4} r^\gamma \phi_J^2 \, dr \, d\omega
\]

\[
= \int -r^\gamma \frac{1}{2r} \partial_r \phi_J \, dx + \frac{\gamma + 1}{4} \int_{\mathbb{R}^3} r^\gamma \phi_J^2 \, dx
\]
Thus,

\[
\begin{align*}
- \int_{\mathbb{R}^3} r^\gamma & \left( \frac{1}{2} |\phi_J|^2 + \frac{1}{2} (\partial_c \phi_J)^2 + \frac{1}{4} \phi_J^2 + \frac{1}{2} \frac{\phi_J}{r} \partial_c \phi_J \right) T_2 \\
&- \int_{\mathbb{R}^3} r^\gamma \left( \frac{1}{2} |\phi_J|^2 + \frac{1}{2} (\partial_c \phi_J)^2 + \frac{1}{8} \frac{\phi_J^2}{r^2} + \frac{1}{2} \frac{\phi_J}{r} \partial_c \phi_J \right) T_1 \\
&= - \int_{\mathbb{R}^3} r^\gamma \left( \frac{1}{2} |\phi_J|^2 + \frac{1}{2} \left[ \partial_c \phi_J + \frac{\phi_J}{2r} \right]^2 + \left( \frac{\gamma}{4} + \frac{1}{8} \frac{\phi_J^2}{r^2} \right) \right) T_2 \\
&= - \int_{\mathbb{R}^3} r^\gamma \left( \frac{1}{2} |\phi_J|^2 + \frac{1}{2} \left[ \partial_c \phi_J + \frac{\phi_J}{2r} \right]^2 + \left( \frac{\gamma}{4} + \frac{1}{8} \frac{\phi_J^2}{r^2} \right) \right) T_1
\end{align*}
\]

(4.18)

- We shall now prove by induction the claim that

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}^3} (\phi \leq m)^2 r^{\gamma - 3} dx dt \leq C
\]

where \( C \) is a constant depending on the initial data. We make use of the defocusing sign of the nonlinearity. We have

\[
\int \int \phi^5 (r^\gamma \partial_c \phi + r^{\gamma - 1} \phi) dx dt = \int r^\gamma \phi^6 \left[ \frac{T_1}{0} \right] dx + \int \int \left( \frac{2}{3} - \frac{\gamma}{6} \right) r^{\gamma - 1} \phi^6 dx dt
\]

Note that both terms are nonnegative for our range of small \( \gamma \) (indeed any \( \gamma < 4 \)). This implies (for our range of small \( \gamma \))

\[
\int_0^\infty \int_{\mathbb{R}^3} r^{\gamma - 1} \phi^6 dx dt < \infty \quad (4.19)
\]

(because \( T_1 \) and \( T_2 \) were arbitrary); and it also establishes the claim for the base case value \( m = 0 \).

Suppose that for some \( m \geq 0 \),

\[
\int_0^\infty \int_{\mathbb{R}^3} (\phi \leq m)^2 r^{\gamma - 3} dx dt < \infty.
\]

Then, letting \( M \phi_J = r^\gamma (\partial_c \phi_J + r^{-1} \phi_J) \) and \( |J| = m + 1 \),

\[
\int_{T_1}^{T_2} \left( \phi^5 \right)_J M \phi_J dx dt = \int r^{(\gamma - 1)/2} M \phi_J \cdot r^{(\gamma + 1)/2} (\phi^5)_J dx dt
\]

\[
\lesssim \epsilon' \int r^{\gamma - 1} \left( (\partial_c \phi_J)^2 + (r^{-1} \phi_J)^2 \right) dx dt + \frac{1}{\epsilon'} \int r^{\gamma + 1} (\phi^5)_J dx dt
\]

(4.20)

(4.21)

for some small \( \epsilon' > 0 \). We treat that term perturbatively and absorb it to the left hand side. Then we note that:

- In the case when there is no single factor in the nonlinearity that has \( m + 1 \) vector fields falling on it, we have by (3.4)

\[
\int r^{\gamma + 1} (\phi^5)_J dx dt \lesssim \int r^{\gamma - 3} A^B (\phi \leq m)^2 dx dt, \quad A := \| \phi \|_{L^B(\mathbb{R}^+)}
\]

which is bounded by a constant depending on the initial data by the induction hypothesis.
– If there is a factor with \( m + 1 \) fields falling on it, we return to (4.20) and use (4.19).

\[
\int \phi^4 r^{\gamma - 1} \phi^6 dx dt \leq \epsilon' \int r^{\gamma - 2} \phi_J^4 + \frac{1}{\epsilon'} \int r^{\gamma} \phi_1^3 + \frac{1}{\epsilon'} \int r^{\gamma - 3} A^2 \phi_J^2 + \frac{1}{\epsilon'} A^2 \int r^{\gamma - 1} \phi_6^2
\]

for some small \( \epsilon' > 0 \). We again used (3.4). We absorb the small term to the left hand side, and the other term is bounded by a constant depending on the initial data, from (4.19).

• (1) Here, in dealing with the potential \( V \), we assume only that \( V \in S^2(\mathbb{R}^2) \). We have:

\[
\int_{T_1}^{T_2} \int_{\mathbb{R}^3} |V_{\leq m} \phi_{\leq m} r^\gamma (\partial_\nu \phi_J + \frac{\phi_J}{r})| \, dx dt \lesssim \epsilon \int \frac{1}{r^{2 - \gamma}} |\phi_{\leq m}| (|\partial_\nu \phi_J| + |\frac{\phi_J}{r}|) \, dx dt \\
\lesssim \epsilon \int \frac{(\phi_{\leq m})^2 + \phi_J^2}{r^{3 - \gamma}} + \frac{|\partial_\nu \phi_J|^2}{r^{1 - \gamma}} \, dx dt \lesssim \epsilon A_{\gamma,m}
\]

If \( B \in S^2(\mathbb{R}^{1-\sigma_B}) \) and \( 2\sigma_B > \gamma \):

\[
\int_{T_1}^{T_2} \int_{\mathbb{R}^3} |B_{\leq m} \partial_\nu \phi_{\leq m} r^\gamma - 1 \phi_J| \, dx dt \lesssim \int \frac{1}{\langle r \rangle^{1+\sigma_B}} |\partial_\nu \phi_{\leq m} r^{\gamma - 1} \phi_J|
\]

\[
\lesssim \frac{1}{\epsilon} \int \frac{|\partial_\nu \phi_{\leq m}|^2}{\langle r \rangle^{1+2\sigma_B - \gamma}} + \epsilon \int r^{\gamma - 3} \phi_J^2
\]

\[
\lesssim \frac{1}{\epsilon} \|\phi_{\leq m}\|_{LE^1(T_1,T_2)}^2 + \epsilon A_{\gamma,J}
\]

The bound on \( \int |B_{\leq m} \partial_\nu \phi_{\leq m} r^\gamma| \cdot |\partial_\nu \phi_J| \, dx dt \) is similar.

(2) We consider now all of the terms that involve the metric \( h^{\alpha \beta} \). We may schematically write this as \( \int (|\partial h_{\leq m} \phi_{\leq m}| + |h_{\leq m} \partial^2 \phi_{\leq m}|) r^\gamma (\frac{\phi_J}{r} + |\partial_\nu \phi_J|) \, dx dt \), where \( |J| = m \).

\[
\int_{T_1}^{T_2} \int_{\mathbb{R}^3} (|\partial h_{\leq m} \phi_{\leq m}| + |h_{\leq m} \partial^2 \phi_{\leq m}|) r^\gamma (\frac{\phi_J}{r} + |\partial_\nu \phi_J|) \, dx dt
\]

\[
\lesssim \epsilon \int \frac{1}{\langle r \rangle^{1+\sigma}} r^\gamma (|\partial h_{\leq m}| + |\partial^2 h_{\leq m}|) (|r^{-1} \phi_J| + |\partial_\nu \phi_J|) \, dx dt
\]

\[
\lesssim \epsilon \int \frac{r^{-\gamma}}{\langle r \rangle^{1+\sigma}} (|\partial h_{\leq m}| + |\partial^2 h_{\leq m}|) \cdot r^{\gamma - 1} (|r^{-1} \phi_J| + |\partial_\nu \phi_J|) \, dx dt
\]

\[
\lesssim \epsilon \|\partial h_{\leq m}\|_{LE(T_1,T_2)}^2 + \epsilon \|\partial_\nu \phi_{\leq m}\|_{LE(T_1,T_2)}^2 + \epsilon A_{\gamma,J} \text{ if } 2\sigma > \gamma
\]

which we can control using (4.15) if we assume \( 2\sigma > \gamma \).

Taking the sum of (4.17), (4.18) and (4.22) to (4.24) over all \( |J| \leq m \), i.e. \( \sum_{|J| \leq m} \) ((4.17), (4.18) and (4.22) to (4.24)), and taking into account our argument for the non-linear terms as well, we get

\[
A_{\gamma,m} + E_{\phi_{\leq m}}^\gamma (T) \lesssim E_{\phi_{\leq m}}^\gamma (0) + \|\phi_{\leq m}\|_{LE^1(T_1,T_2)}^2 + \|\partial^2 \phi_{\leq m}\|_{LE(T_1,T_2)}^2.
\]

\[\square\]
Thus we used Section 1’s assumptions to obtain finiteness of the local energy norms $\|\phi_{\leq m}\|_{L^E(R_+ \times \mathbb{R}^3)}$, and now we have, in particular, showed that

$$\int_0^\infty \int_{\mathbb{R}^3} (\phi_{\leq m})^2 r^{\gamma-3} dx dt \lesssim E_{\phi_{\leq m}}^\gamma(0) + \|\partial\phi_{\leq m+1}(0)\|_{L^2}^2 \leq C_0$$

where $C_0$ is a constant depending only on initial data.

**Lemma 4.11.** Recall Definition 2.1. Let $v_+ := \langle s + \rho \rangle$ where $(\rho, s) \in D_{tr}$. Then

$$\|v_+^{-1}\|_{L^2(D_{tr}^R \mathbb{R}^3_1)} \lesssim 1,$$  \hfill (4.25)  

$$\|v_+^{-1}\|_{L^2(D_{tr}^R \mathbb{R}^3_2)} \lesssim \left(\frac{\langle u \rangle}{R}\right)^{\frac{1}{2}}.$$  \hfill (4.26)

**Proof.** (4.25) and (4.26) follow from Lemma 4.3. \hfill \Box

**Proposition 4.12** (Application of the $r^\gamma$ estimate). Let $\phi$ solve $P\phi = \phi^5$. Assume the hypotheses on $\gamma$ in Theorem 4.10 and also (1.4).

$$\langle r \rangle (\phi_3)_{\leq m} \lesssim \langle u \rangle^{1/2-\gamma/2}, \quad u > 1$$  \hfill (4.27)  

$$\|\phi_3\|_{\leq m} \lesssim r^{-\frac{1}{2} - \frac{\gamma}{2}}, \quad u < -1$$  \hfill (4.28)

**Remark 4.13.** If $\gamma \geq 1$, then for $u < -1$ this theorem would instead conclude

$$(\phi_3)_{\leq m} \lesssim r^{-1} \langle u \rangle^{-\frac{1}{2}(\gamma-1)}.$$

**Proof.**

- Let $u > 1$. We now show

$$\int_{D_{tr}} \rho H_3 dA \lesssim \langle u \rangle^{1/2-\gamma/2}, \quad H_3(t, r) := \sum_{k=0}^2 \|\Omega^k(\phi_{\leq m}^5(t, r\omega))\|_{L^3(S^2)}.$$

We have

$$\int_{D_{tr}^R} \rho H_3 dA \lesssim \int_{D_{tr}^R} \rho \langle s + \rho \rangle^2 \|\phi_{\leq m+n}\|_{L^2(S^2)} dA$$

$$\lesssim \int_{D_{tr}^R} \frac{1}{v_+} \|\phi_{\leq m+n}\|_{L^2(S^2)} dA$$

$$\lesssim \|v_+^{-1}\|_{L^2(D_{tr}^R)} \|\phi_{\leq m+n}\|_{L^2_{\rho, s, \omega}}$$

$$\lesssim \|v_+^{-1}\|_{L^2(D_{tr}^R)} \frac{1}{R} \frac{1}{R^{(\gamma-3)/2}} C_0$$

where $C_0$ is a constant depending on the initial data. The first line follows by (3.4) and the last line follows by Theorem 4.10.

1. Let RHS denote “the right-hand side of.” By Lemma 4.11,

$$\sum_{R \in \mathcal{R}_1} \text{RHS}(4.29) \lesssim \langle u \rangle^{1/2-\gamma/2}, \quad \gamma \in (0, 1).$$

2. Fix $R \in \mathcal{R}_2$. By Lemma 4.11 we have

$$\text{RHS}(4.29) \lesssim \left(\frac{\langle u \rangle}{R}\right)^{\frac{1}{2}} \frac{1}{R} \frac{1}{R^{(\gamma-3)/2}} = \langle u \rangle^{1/2} R^{-\gamma/2}.$$
Then we have
\[
\sum_{R \in \mathcal{R}_2} \text{RHS}(4.29) \lesssim \langle u \rangle^{1/2 - \gamma/2}, \quad \text{valid for } \gamma \in (0, \infty).
\]

This finishes the proof of (4.27).

\* Let \( u < -1 \).

\[
\int_{D_{tr}} \rho H_3 \, dA \lesssim \int_{D_{tr}} \rho^{\frac{3-\gamma}{2}} \| \phi_{\leq m+n} \|_{L^2} \cdot \rho^{\frac{3-\gamma}{2}} \| \phi_{\leq m+n} \|_{L^2}^4 \, dA
\]
\[
\lesssim \left( \int \rho^{3-\gamma} \| \phi_{\leq m+n} \|_{L^2}^8 \, dA \right)^{\frac{1}{2}}
\]
\[
\lesssim (r^{1-\gamma})^{\frac{1}{2}}.
\]

The second line follows from Theorem 4.10. The third line follows from (3.4), Lemma 4.3 and the assumption \( \gamma < 1 \). \( \square \)

5. The iteration in \( \{ u < -1 \} \)

**Theorem 5.1.** If \( u < -1 \), then
\[
\phi_{\leq m} \lesssim \langle r \rangle^{-1} \langle u \rangle^{-1 - \min(\sigma, 2)},
\]
Here \( \sigma \) denotes the original value of \( \sigma \) taken from Theorem 1.1.

**Proof.** We begin with the bounds in (3.4) and Propositions 3.5 and 4.12, which in the outside region translate to
\[
\phi_{\leq m} \lesssim \langle u \rangle^{1/2}, \quad \partial_t \phi_{\leq m} \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1/2}}, \quad \phi_{\leq m+n} \lesssim \frac{\langle u \rangle^{1/2 - \gamma/2}}{\langle r \rangle}.
\]
(5.1)

For simplicity, we shall use the first (far left) \( \phi_{\leq m+n} \) bound for \( \phi_1 \) and the other (far right) \( \phi_{\leq m+n} \) bound for \( \phi_3 \). Since \( \langle u \rangle \leq \langle r \rangle \), (5.1) can be weakened to
\[
\phi_{\leq m+n} \lesssim \frac{1}{\langle r \rangle^{1/2}}, \quad \partial_t \phi_{\leq m+n} \lesssim \frac{1}{\langle r \rangle^{1/2} \langle u \rangle}, \quad \phi_{\leq m+n} \lesssim \frac{1}{\langle r \rangle^{1/2 + \gamma/2}}.
\]
(5.2)

Recall the decomposition (4.5), and let
\[
H_i = \sum_{k=0}^{2} \| \Omega^k (G_{i})_{\leq n} (t, r \omega) \|_{L^2(\mathbb{S}^2)}.
\]

Let \( \sigma \) denote the reduced, irrational number mentioned in Remark 4.2 until stated otherwise. We thus have, using (5.2):
\[
H_1 \lesssim \frac{1}{\langle r \rangle^{5/2 + \sigma}}, \quad \partial_t H_2 \lesssim \frac{1}{\langle r \rangle^{3/2 + \sigma} \langle u \rangle}, \quad H_3 \lesssim \frac{1}{\langle r \rangle^{5/2 + 5/2}}.
\]

By (4.8) with \( \alpha = 5/2 + \sigma, \beta = 0, \) and \( \eta = 0 \), we obtain
\[
(\phi_1)_{\leq m+n} \lesssim r^{-1/2 - \sigma}
\]

which gains a factor of $\langle r \rangle^{-\sigma}$ compared to (5.2). Similarly (4.8) with $\alpha = 3/2 + \sigma$, $\beta = 0$, and $\eta = 1$ yields

$$(\phi_2)_{m+n} \lesssim r^{-1/2-\sigma}$$

Finally, (4.8) with $\alpha = 2 + 2\gamma, \beta = 0, \eta = 1/2$ yields

$$(\phi_3)_{m+n} \lesssim r^{-1/2-2\gamma}.$$  

The three inequalities above, combined with Proposition 3.5, give the following improved bounds (by a factor of $\langle r \rangle^{-\sigma'}$ where $\sigma' := \min(2\gamma, \sigma)$).

$$\phi_{m+n} \lesssim \frac{1}{\langle r \rangle^{1/2+\sigma'}}, \quad \partial \phi_{m+n} \lesssim \frac{1}{\langle r \rangle^{1/2+\sigma'} \langle u \rangle}.$$  \hspace{1cm} (5.3)

We now repeat the iteration, replacing $\alpha$ by $\alpha + \sigma'$ and applying (4.8). The process stops after $\lfloor \frac{1}{2\sigma'} \rfloor$ steps, when (4.8), combined with Proposition 3.5 yield

$$\phi_{m+n} \lesssim \frac{1}{\langle r \rangle^{3+\sigma}}, \quad \partial \phi_{m+n} \lesssim \frac{1}{\langle r \rangle^{3+\sigma} \langle u \rangle}.$$  \hspace{1cm} (5.4)

We now switch to using (4.7) for $\phi_1$ and $\phi_3$, and (4.10) for $\phi_2$. Note that (5.4) implies

$$H_1 \lesssim \frac{1}{\langle r \rangle^{3+\sigma}}, \quad H_2 \lesssim \frac{1}{\langle r \rangle^{2+\sigma}}, \quad H_3 \lesssim \frac{1}{\langle r \rangle^{5}}.$$  

By (4.7) with $\alpha = 2 + \sigma, \beta = 1, \eta = 0$, we obtain

$$(\phi_1)_{m+n} \lesssim r^{-1}\langle u \rangle^{-\sigma}$$

Similarly (4.10) with $\alpha = 2 + \sigma, \eta = 0$ yields

$$(\phi_2)_{m+n} \lesssim r^{-1}\langle u \rangle^{-\sigma}$$

Finally, (4.7) with $\alpha = 5, \beta = 0, \eta = 0$ yields

$$(\phi_3)_{m+n} \lesssim r^{-1}\langle u \rangle^{-2}$$

We now repeat the iteration. We can continue improving the decay rates of $\phi_1$ and $\phi_2$ all the way to

$$(\phi_1)_{m}, (\phi_2)_{m} \lesssim r^{-1}\langle u \rangle^{-1-\sigma}.$$  \hspace{1cm} (5.5)

For $\phi_3$, we note that after the bounds $$(\phi_1)_{m+n}, (\phi_2)_{m+n} \lesssim r^{-1}\langle u \rangle^{-1/5}$$ are obtained, we have

$$(\phi_3)_{m+n} \lesssim r^{-1}\langle u \rangle^{-3}.$$  \hspace{1cm} (5.6)

By (5.5) and (5.6) we now have, for the original value of $\sigma$ from Theorem 1.1,

$$H_1 \lesssim \frac{1}{\langle r \rangle^{3+\sigma} \langle u \rangle^{1+}}, \quad H_2 \lesssim \frac{1}{\langle r \rangle^{2+\sigma} \langle u \rangle^{1+}}, \quad H_3 \lesssim \frac{1}{\langle r \rangle^{5} \langle u \rangle^{1+}}.$$  

Using (4.7) and (4.10) now completes the proof.  \hspace{1cm} □
6. The iteration in \( \{ u > 1 \} \)

6.1. Converting \( r \) decay to \( t \) decay. The pointwise decay rates for the solution and its vector fields obtained from the estimates for the fundamental solution (see Section 4.3) are by themselves insufficient for completing the iteration. We show below that if the \( \langle r \rangle^{-1} \) decay from the fundamental solution is converted into \( \langle t \rangle^{-1} \), then the iteration does work.

Lemma 6.1.

\[
\| \phi_{\leq m} \|_{L^2_t(C_T^{<3T/4})} \lesssim T^{-1} \| \langle r \rangle \phi_{\leq m+n} \|_{L^2_t(C_T^{<3T/4})} + \| (\phi^5)_{\leq m+n} \|_{L^2_t(C_T^{<3T/4})}.
\]

We only sketch the proof of Lemma 6.1 here, and refer the reader to [26] or [37] for details. We may assume that \( \phi \) is supported in \( C_T^{<3T/4} \). By the local energy decay estimate, we control \( \| \phi_{\leq m} \|_{L^2_t(C_T^{<3T/4})} \) by an energy term plus \( \| (\phi^5)_{\leq m} \|_{L^2_t(C_T^{<3T/4})} \). Then averaging in time is used to bound the energy term by the \( LE^1 \) norm, leading to the desired estimate.

The next proposition uses Lemma 6.1 to obtain better pointwise decay for the solution and its vector fields in the region \( \{ r < t/2 \} \).

Proposition 6.2. Let \( \phi \) solve (1.2). Let \( \delta > 0 \). Assume that

\[
\phi_{\leq M} |_{r \leq 3t/4} \lesssim \langle r \rangle^{-1} \langle u \rangle^{1/2-n\delta}, \quad \phi_{\leq M} |_{r \leq 3t/4} \lesssim \langle t \rangle^{-1} \langle u \rangle^{1/2-(n-1)\delta}, \quad n \geq 1
\]

for an \( M \) that is sufficiently larger than \( m \). Then we have

\[
\phi_{\leq m} |_{C_T^{<3T/4}} \lesssim \langle t \rangle^{-1} \langle u \rangle^{1/2-n\delta}.
\]

Proof. By Proposition 3.5 and (6.1),

\[
T^{-1} \| \langle r \rangle \phi_{\leq m+n} \|_{L^2_t} \lesssim T^{-1} \| \phi_{\leq m+n} \|_{LE} \lesssim T^{-q}, \quad q = n\delta.
\]

Fix \( n \geq 1 \). For \( A_R=1 \), we use the latter bound in (6.1) to obtain

\[
\| (\phi^5)_{\leq m+n} \|_{L^2_t[L^2_t]} \lesssim T^{-2-5(n-1)\delta}.
\]

For \( A_R, R > 1 \), we use the former bound and the latter bound in (6.1) in a three-to-two ratio (respectively). This yields

\[
\| \langle r \rangle^{1/2} (\phi^5)_{\leq m+n} \|_{L^2_t[L^2_t]} \lesssim \left( T^{-10n\delta+6\delta} \right)^{1/2} = T^{-5n\delta+3\delta}, \quad R > 1.
\]

Therefore Lemma 6.1 implies, after the dyadic sum,

\[
\| \phi_{\leq m} \|_{L^2_t(C_T^{<3T/4})} \lesssim T^{-n\delta}
\]

and the conclusion now follows by Lemma 3.2. \( \square \)

6.2. The iteration.

Theorem 6.3. If \( u > 1 \), then

\[
\phi_{\leq m} \lesssim \langle u \rangle^{-1} \langle u \rangle^{-1-\min(\sigma,2)}.
\]

Here \( \sigma \) denotes the original value of \( \sigma \) taken from Theorem 1.1.
Proof. As before, we begin with the bounds in (3.4) and Propositions 3.5 and 4.12, which in the inside region translate to

\[ \phi_{m+n} \leq \frac{\langle u \rangle^{1/2}}{\langle t \rangle}, \quad \partial_t \phi_{m+n} \leq \frac{1}{\langle r \rangle \langle u \rangle^{1/2}}, \quad \phi_{m+n} \leq \frac{\langle u \rangle^{1/2-\gamma/2}}{\langle t \rangle} \]

(6.2)

where again for simplicity we use the far left \( \phi_{m+n} \) bound for \( \phi_1 \) and the far right \( \phi_{m+n} \) bound for \( \phi_3 \).

Let \( \sigma \) denote the reduced, irrational number mentioned in Remark 4.2 until stated otherwise. We thus have, using (6.2):

\[ H_1 \lesssim \frac{\langle u \rangle^{1/2}}{\langle r \rangle^{2+\sigma} \langle t \rangle}, \quad \partial_t H_2 \lesssim \frac{1}{\langle r \rangle^{1+\sigma} \langle t \rangle}, \quad H_3 \lesssim \frac{\langle u \rangle^{5(1/2-\gamma/2)}}{\langle t \rangle^5} \lesssim \frac{\langle u \rangle^{1/2}}{\langle t \rangle^{3+5\gamma/2}}. \]

By (4.7) with \( \alpha = 2 + \sigma, \beta = 1, \) and \( \eta = -1/2, \) we obtain

\[ (\phi_1)_{m+n} \lesssim \langle r \rangle^{-1} \langle u \rangle^{1/2-\sigma} \]

Similarly (4.7) with \( \alpha = 2 + \sigma, \beta = 0, \) and \( \eta = 1/2 \) yields

\[ (\phi_2)_{m+n} \lesssim \langle r \rangle^{-1} \langle u \rangle^{1/2-\sigma} \]

Finally, (4.7) with \( \alpha = 0, \beta = 3 + 5\gamma/2, \) and \( \eta = -1/2 \) yields

\[ (\phi_3)_{m+n} \lesssim \langle r \rangle^{-1} \langle u \rangle^{1/2-5\gamma/2} \]

The three inequalities above give

\[ \phi_{m+n} \lesssim \langle r \rangle^{-1} \langle u \rangle^{1/2-\sigma'}, \quad \sigma' := \min(2\gamma, \sigma). \]

By Proposition 6.2 we obtain the following improved bounds (by a factor of \( \langle u \rangle^{-\sigma'} \)):

\[ \phi_{m+n} \lesssim \frac{\langle u \rangle^{1/2-\sigma'}}{\langle t \rangle}, \quad \partial_t \phi_{m+n} \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1/2+\sigma'}}. \]

(6.3)

We now repeat the iteration, replacing \( \eta \) by \( \eta + \sigma' \), applying (4.8) and then improving decay in \( \{t/r > 2\} \) by Proposition 6.2. The process stops after \( \lceil \frac{1}{2\sigma'} \rceil \) steps, when (4.7) and Proposition 6.2 yield

\[ |\phi_{m+n}| \lesssim \frac{1}{\langle t \rangle}, \quad |\partial_t \phi_{m+n}| \lesssim \frac{1}{\langle r \rangle \langle u \rangle}. \]

(6.4)

At this point we switch to using (4.10) for \( \phi_2 \), and the iteration process follows the same pattern as in Section 5, with the extra use of Proposition 6.2. Like before in Theorem 5.1, we make the final iterate involving the original value of \( \sigma \) from Theorem 1.1. \( \square \)

Remark 6.4 (Other integral powers \( p \in \mathbb{Z}_{\geq 2} \)). For \( \phi \) solving \( P\phi = G_3 \) with

\[ G_3 = |\phi|^p \phi, \quad p \geq 5 \text{ with large initial data} \]

or

\[ G_3 = \pm \phi^{p+1}, \quad p \geq 5 \text{ with small initial data} \]

we remark that if global existence holds for the large data case (as this is clear for the small data case), then just by the initial global decay rate (3.4) alone, the decay rates in Proposition 4.12 are immediately achieved for some \( \gamma \) (and hence there is no need to prove Theorem 4.10), and letting \( \phi_3 \) solve \( \Box \phi_3 = G_3 \) as before in Section 4.2, the iterations in
Sections 5 and 6 follow nearly verbatim, with the natural modification that in the end we reach the final decay rate 

\[(\phi_3)_{\leq m+n} \lesssim r^{-1}\langle u \rangle^{-(p-1)}, \text{ for } u < -1, \quad \phi_3 \leq m + n \lesssim v^{-1}\langle u \rangle^{-(p-1)}, \text{ for } u > 1.\]

Thus

\[\phi \leq m \lesssim \langle v \rangle^{-1}\langle u \rangle^{-\min\{1+\sigma,p-1\}}.\]

For \(\phi\) solving \(P\phi = G_3\) with

\[G_3 = |\phi|^2\phi\]

with large initial data, we remark here that if global existence holds and if Proposition 4.12 holds with \(\gamma > 1\), then the iteration also follows Sections 5 and 6 nearly verbatim, and we reach the final decay rates

\[(\phi_3)_{\leq m+n} \lesssim r^{-1}\langle u \rangle^{-1} \text{ for } u < -1, \quad \phi_3 \leq m + n \lesssim v^{-1}\langle u \rangle^{-1} \text{ for } u > 1\]

and thus

\[\phi \leq m \lesssim \langle v \rangle^{-1}\langle u \rangle^{-1}.\]

Similarly we have global existence for \(P\phi = \pm \phi^3\) with small initial data, and if we had Proposition 4.12 with \(\gamma > 1\) then this remark holds as well.

For \(\phi\) solving \(P\phi = G_3\) with

\[G_3 = \pm \phi^4\]

with small initial data, if we had Proposition 4.12 with \(\gamma > 1/2\) then after setting \(\Box \phi_3 = G_3\) in Section 4.2, the iterations in Sections 5 and 6 also hold nearly verbatim and we reach

\[(\phi_3)_{\leq m+n} \lesssim \langle v \rangle^{-1}\langle u \rangle^{-2}\]

thus

\[\phi \leq m \lesssim \langle v \rangle^{-1}\langle u \rangle^{-\min\{1+\sigma,2\}}.\]

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