On the Rate of Convergence of Weak Euler Approximation for Nondegenerate Itô Diffusion and Jump Processes

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Abstract
The paper studies the rate of convergence of the weak Euler approximation for Markov processes with Hölder-continuous generators. The main part of the jump intensity measure has a nondegenerate density with respect to the Lévy measure of a spherically-symmetric stable process. It covers a variety of stochastic processes including the nondegenerate diffusions and a class of SDEs driven by spherically-symmetric stable processes. To estimate the rate of convergence of the weak Euler approximation, the existence of a unique solution to the corresponding backward Kolmogorov equation in Hölder space is first proved. It then shows that the Euler scheme yields positive weak order of convergence.

Keywords: weak Euler approximation, rate of convergence, Itô processes, stochastic differential equations

1. Introduction
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with a filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,T]}\) of \(\sigma\)-algebras satisfying the usual conditions. For a fixed \(\alpha \in (0, 2]\), consider an \(\mathcal{F}\)-adapted \(d\)-dimensional stochastic process \(X = \{X_t\}_{t \in [0,T]}\), which solves the equation

\[
X_t = X_0 + \int_0^t \int_0^t y p^X(ds, dy), \quad \text{if } \alpha \in (0, 1),
\]

\[
X_t = X_0 + \int_0^t a_\alpha(X_s)ds + 1_{\{\alpha=2\}} \int_0^t b(X_s)dW_s + \int_0^t \int_{|y|>1} y p^X(ds, dy) + \int_0^t \int_{|y|\leq 1} y q^X(ds, dy), \quad \text{if } \alpha \in [1, 2],
\]

where \(W = \{W_t\}_{t \in [0,T]}\) is a \(d\)-dimensional \(\mathcal{F}\)-adapted standard Wiener process, \(p^X(dt, dy)\) is the jump measure of \(X\) with \(p^X([0, t], \Gamma) = \sum_{s \leq t} 1_\Gamma(X_s - X_t)\).
\( X_{s-} \), \( \Gamma \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \), and \( q^X(dt, dy) = p^X(dt, dy) - \pi^{(\alpha)}(X_t, dy)dt \) is the corresponding martingale measure. The coefficient functions \( a_\alpha = (a^i_\alpha)_{1 \leq i \leq d} \) and \( b = (b^{ij})_{1 \leq i,j \leq d} \) are measurable and bounded, \( \pi^{(\alpha)}(x, dy), x \in \mathbb{R}^d \) is a measurable family of non-negative measures on \( \mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\} \).

For \( \alpha \in (0, 2) \), it is assumed that the principal part of \( \pi^{(\alpha)} \) has a non-degenerate density with respect to the Lévy measure of the spherically-symmetric \( \alpha \)-stable process defined in (16) and the other part has a density with respect to a lower order Lévy measure. That is,

\[
\pi^{(\alpha)}(x, dy) = m^{(\alpha)}(x, y) \frac{dy}{|y|^d + \alpha} + \rho^{(\alpha)}(x, y) \nu^{(\alpha)}(dy),
\]

where \( \nu^{(\alpha)} \) is a non-negative measure on \( \mathbb{R}^d_0 \) and \( m^{(\alpha)}, \rho^{(\alpha)} \) are non-negative measurable functions such that \( m^{(\alpha)} \) and \( \int_{\mathbb{R}^d_0} (|y|^\alpha \wedge 1) \rho^{(\alpha)}(x, y) \nu^{(\alpha)}(dy), x \in \mathbb{R}^d \) are bounded. If \( \alpha = 2 \), there is a non-degenerate diffusion part in \( X_t \) and \( \pi^{(2)}(x, dy) = \rho^{(2)}(x, y) \nu^{(2)}(dy) \). In particular, if \( \pi^{(2)} = 0 \), then \( X_t \) is a diffusion. A large class of strong Markov processes satisfying (1) has been constructed \([1, 7, 10, 14, 22]\) (see references therein as well). These processes are characterized by their Lévy jump measure \( \pi^{(\alpha)}(x, dy) \) (conditional intensity of jumps), drift coefficient \( a_\alpha \), and diffusion coefficient \( b \), or equivalently, by their generators (see Remark 9 and (15)).

The process defined in (1) is used as a mathematical model for random dynamic phenomena in applications from fields such as finance and insurance, to capture continuous and discontinuous uncertainty. It naturally arises in stochastic differential equations driven by Lévy processes.

For many applications, the practical computation of functionals of the type \( F = \mathbb{E}[g(X_T)] \) plays an important role. For instance in finance, derivative prices can be expressed by such functionals. One possibility to numerically approximate \( F \) is given by the discrete time Monte-Carlo simulation of the Itô process \( X_t \). The simplest discrete time approximation of \( X_t \) that can be used for such Monte-Carlo methods is the weak Euler approximation.

Let \( \mathcal{F} = \{ \mathcal{F}_t \}_{t \in [0,T]} \) be a filtration of \( \sigma \)-algebras in \((\Omega, \mathcal{F}, \mathbb{P})\) and let the time discretization \( \{ \tau_i, i = 0, \ldots, n_T \} \) of the interval \([0,T]\) with maximum step size \( \delta \in (0,1) \) be a partition of \([0,T]\) such that \( 0 = \tau_0 < \tau_1 < \cdots < \tau_{n_T} = T \) and \( \max_i (\tau_i - \tau_{i-1}) \leq \delta \). The weak Euler approximation of \( X_t \) is an \( \mathcal{F} \)-adapted stochastic process \( Y = \{ Y_t \}_{t \in [0,T]} \) with \( \mathbb{P}(Y_0 \in A) = \mathbb{P}(X_0 \in A) \).
A), ∀A ∈ ℬ(R^d), defined by the stochastic equation

\[ Y_t = Y_0 + \int_0^t \int yp^Y(ds,dy), \]  
\[ Y_t = Y_0 + \int_0^t a_{\alpha}(Y_{\tau_i})ds + 1_{\{\alpha=2\}} \int_0^t b(Y_{\tau_i})d\tilde{W}_s \]

\[ + \int_0^t \int_{|y|>1} yp^Y(ds,dy) + \int_0^t \int_{|y|\leq 1} yq^Y(ds,dy), \]  
if \( \alpha \in (0,1) \),

\[ Y_t = Y_0 + \int_0^t a_{\alpha}(Y_{\tau_i})ds + 1_{\{\alpha=2\}} \int_0^t b(Y_{\tau_i})d\tilde{W}_s \]

\[ + \int_0^t \int_{|y|>1} yp^Y(ds,dy) + \int_0^t \int_{|y|\leq 1} yq^Y(ds,dy), \]  
if \( \alpha \in [1,2] \),

where \( \tau_{i_s} = \tau_i \) if \( s \in [\tau_i,\tau_{i+1}) \), \( \tilde{W} = \{\tilde{W}_t\}_{t \in [0,T]} \) is a \( d \)-dimensional \( \tilde{F} \)-adapted standard Wiener process, \( p^Y(dt,dy) \) is the jump measure of \( Y_t \), and \( q^Y(dt,dy) = p^Y(dt,dy) - \pi^{(\alpha)}(Y_{\tau_t},dy)dt \) is the corresponding \( \tilde{F} \)-adapted martingale measure on \([0,T] \times \mathbb{R}^d \). Contrary to those in (1), the coefficients in (3) are piecewise constants in each time interval \([\tau_i,\tau_{i+1}) \). The approximation \( Y \) and \( X \) could be defined on different probability spaces but taking their product allows to reduce everything to a single probability space.

An Euler approximation defined by (3) always exists (see Appendix). The weak Euler approximation \( Y \) is said to converge with order \( \kappa > 0 \) if for each bounded smooth function \( g \) with bounded derivatives, there exists a constant \( C \), depending only on \( g \), such that

\[ |Eg(Y_T) - Eg(X_T)| \leq C\delta^{\kappa}, \]

where \( \delta > 0 \) is the maximum step size of the time discretization. Since \( Eg(X_T) \) is the value of the solution to the backward Kolmogorov equation (17), the estimate provides the rate of convergence of a probabilistic approximation \( Eg(Y_T) \) to \( Eg(X_T) \). The backward Kolmogorov equation is a parabolic integro-differential equation of order \( \alpha \leq 2 \).

A very simple example of a Markov process defined by (1) is the solution to the following stochastic differential equation driven by a spherically-symmetric stable process.

**Example 1.** Let \( Z = \{Z_t\}_{t \in [0,T]} \) be a standard \( d \)-dimensional spherically-symmetric \( \alpha \)-stable process defined in (16) and \( U = (U^1,\ldots,U^d) \) be independent standard one-dimensional symmetric \( \alpha_i \)-stable processes independent of \( Z \). Consider for \( t \in [0,T] \),

\[ X_t = X_0 + 1_{\{\alpha \in (1,2)\}} \int_0^t a_{\alpha}(X_s)ds + \int_0^t c(X_s-)dZ_s + \int_0^t \text{diag}(k(X_s-))dU_s, \]

(5)
where \( a_i(x) = (a_i^j(x))_{1 \leq i \leq d}, c(x) = (c^{ij}(x))_{1 \leq i,j \leq d}, \) \( k(x) = (k^i(x))_{1 \leq i \leq d}, \) \( x \in \mathbb{R}^d \) are \( \beta \)-Hölder continuous and bounded functions with \( \beta > 0, \beta \notin \mathbb{N}, \) and \( \text{diag}(k) \) is the diagonal \( d \times d \)-matrix with \( k^1, \ldots, k^d \) on the diagonal. It is assumed that \( c \) is non-degenerate with \( \inf_x \det |c(x)| > 0 \) and \( \alpha_i < \alpha \leq 2. \) In this case (see Corollary 7), the equality (2) for \( \pi^{(\alpha)} \) holds with

\[
m^{(\alpha)}(x,y) = \frac{|y|^{d+\alpha}}{|\det c(x)| |c(x)|^{-1} y|^{d+\alpha}},
\]

\[
\rho^{(\alpha)}(x,y) = \sum_{i=1}^d |k^i(x)|^{\alpha_1} 1_{\{y = y_i e_i\}},
\]

\[
\nu^{(\alpha)}(dy) = \sum_{i=1}^d 1_{\{y = y_i e_i\}} \frac{dy_i}{|y_i|^{1+\alpha}},
\]

where \( \{e_i, i = 1, \ldots, d\} \) is the canonical basis of \( \mathbb{R}^d. \)

For \( t \in [0,T], \) the weak Euler approximation is defined as

\[
Y_t = X_0 + \mathbb{1}_{\{\alpha \in (1,2]\}} \int_0^t a_\alpha(Y_{\tau_s}) ds + \int_0^t c(Y_{\tau_s}) dZ_s + \int_0^t \text{diag}(k(Y_{\tau_s})) dU_s.
\]

As shown in Corollary 7 (4) holds with \( \kappa = \left( \frac{\beta}{\alpha} \right) \wedge 1 \) if \( g \in C^{\alpha+\beta}(\mathbb{R}^d), \) the space of \((\alpha+\beta)\)-Hölder continuous functions.

The case of smooth coefficients, especially for diffusion processes, has been considered by many authors. Milstein was one of the first to study the order of weak convergence for diffusion processes and derived \( \kappa = 1 \) [16, 17]. Talay investigated a class of the second order approximations for diffusion processes [23, 24]. For Itô processes with jump components, Mikulevičius & Platen showed the first-order convergence in the case that the coefficient functions possess fourth-order continuous derivatives [11]. Platen and Kloeden studied not only Euler but also higher order approximations [6, 18]. Protter and Talay considered the weak Euler approximation for

\[
X_t = X_0 + \int_0^t c(X_{s-}) dZ_s, t \in [0,T],
\]

where \( Z_t = (Z_t^1, \ldots, Z_t^m) \) is a Lévy process and \( c = (c^{ij})_{1 \leq i \leq d, 1 \leq j \leq m} \) is a measurable and bounded function [19]. They showed the order of convergence \( \kappa = 1, \) provided that \( c \) and \( g \) are smooth and the Lévy measure of \( Z \) has finite moments of sufficiently high order. Because of this, the main theorems in [19] do not apply even to (5). On the other hand, (11) with (2) and non-degenerate \( m^{(\alpha)} \) does not cover (7), which can degenerate completely.
In general, the coefficients and the test function \( g \) do not always have the smoothness properties assumed in the papers cited above. Mikulevičius & Platen proved that there is still some order of convergence of the weak Euler approximation for non-degenerate diffusion processes under Hölder conditions on the coefficients and \( g \) \([12]\). Kubilius & Platen generalized the result to non-degenerate diffusion processes with a finite number of jumps in finite time intervals \([9]\).

In this paper, as in \([12]\), we employ the idea of Talay (see \([23]\)) and derive the rate of convergence for \((1)\) under \( \beta \)-Hölder conditions on the coefficients, \( b, a, m^{(\alpha)}, \rho^{(\alpha)} \), by using the solution to the backward Kolmogorov equation associated with \( X_t \). In the following Section 2, the main result is stated and the proof is outlined. In Section 3, the necessary and essential technical results are presented. The main theorem is proved in Section 4.

2. Main Result and Outline of the Proof

2.1. Notation

Denote \( H = [0, T] \times \mathbb{R}^d \), \( N = \{0, 1, 2, \ldots\} \), \( \mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\} \). For \( x, y \in \mathbb{R}^d \), write \( (x, y) = \sum_{i=1}^d x_i y_i \), \( |x| = \sqrt{(x, x)} \) and \( |B| = \sum_{i=1}^d |B^i| \), \( B \in \mathbb{R}^{d \times d} \).

For \((t, x) \in H\), multiindex \( \gamma \in \mathbb{N}^d \), and \( i, j = 1, \ldots, d \), denote \( \partial_t u(t, x) = \frac{\partial}{\partial t} u(t, x) \), \( \partial_\gamma^x u(t, x) = \frac{\partial |\gamma|}{\partial \gamma_1 x_1 \ldots \partial \gamma_d x_d} u(t, x) \), \( \partial_i u(t, x) = \frac{\partial}{\partial x_i} u(t, x) \), \( \partial_{ij}^x u(t, x) = \frac{\partial^2}{\partial x_i x_j} u(t, x) \), \( \partial_i^x u(t, x) = \nabla_i u(t, x) = (\partial_1 u(t, x), \ldots, \partial_d u(t, x)) \), \( \partial_i^2 u(t, x) = \Delta u(t, x) = \sum_{i=1}^d \partial_i^2 u(t, x) \).

For \( \alpha \in (0, 2) \), write \( \partial^\alpha v(x) = \mathcal{F}^{-1}[|\xi|^\alpha \mathcal{F} v(\xi)](x) \), where \( \mathcal{F} \) is the Fourier transform with respect to \( x \in \mathbb{R}^d \) and \( \mathcal{F}^{-1} \) is the inverse Fourier transform: \( \mathcal{F} v(\xi) = \int_{\mathbb{R}^d} e^{-i(\xi, x)} u(x) dx \) and \( \mathcal{F}^{-1} v(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\xi, x)} v(\xi) d\xi \).

\( C_0^\infty(H) \) is the set of all functions \( u \) on \( H \) such that for all \( t \in [0, T] \) the function \( u(t, x) \) is infinitely differentiable in \( x \) and for every multiindex \( \gamma \in \mathbb{N}^d \), \( \sup_{(t,x) \in H} |\partial_\gamma^x u(t, x)| < \infty \). \( C_0^\infty(G) \) is the set of all infinitely differentiable functions on an open set \( G \subseteq \mathbb{R}^d \) with compact support.

\( C = C(\cdot, \ldots, \cdot) \) denotes constants depending only on quantities appearing in parentheses. In a given context the same letter is generally used to denote different constants depending on the same set of arguments.

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For $\beta = [\beta^-] + \{\beta\}^+ > 0$, where $[\beta^-] \in \mathbb{N}$ and $\{\beta\}^+ \in (0, 1]$, let $C^\beta(H)$ denote the space of measurable functions $u$ on $H$ such that the norm

$$
|u|_\beta = \sum_{|\gamma| \leq [\beta^-]} \sup_{t,x} |\partial^\gamma_x u(t, x)| + \sum_{|\gamma| = \{\beta\}^+} \sup_{t,x} \frac{|\partial^\gamma_x u(t, x + h) - \partial^\gamma_x u(t, x)|}{|h|^{\{\beta\}^+}} + \sum_{|\gamma| = 1} \sup_{t,x,h \neq 0} \frac{|\partial^\gamma_x u(t, x + h) - 2\partial^\gamma_x u(t, x) + \partial^\gamma_x u(t, x - h)|}{|h|^{\{\beta\}^+}}
$$

is finite. Accordingly, $C^\beta(\mathbb{R}^d)$ denotes the corresponding space of functions on $\mathbb{R}^d$. The classes $C^\beta$ are Hölder-Zygmund spaces and coincide with Hölder spaces if $\beta \notin \mathbb{N}$ (see 1.2.2 in [26]).

2.2. Assumptions and Main Result

Assume $m^{(\alpha)}(x, y)$ and its partial derivatives $\partial^\gamma_x m^{(\alpha)}(x, y), |\gamma| \leq d_0 = [\frac{d}{2}] + 1$ are continuous in $(x, y)$. Moreover, $m^{(\alpha)}(x, y)$ is homogeneous in $y$ with index zero, and

$$
\int_{S^{d-1}} ym^{(1)}(\cdot, y)\mu_{d-1}(dy) = 0, \quad m^{(2)} \equiv 0,
$$

where $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and $\mu_{d-1}$ is the Lebesgue measure.

For $\beta = [\beta^-] + \{\beta\} > 0$ with $[\beta^-] \in \mathbb{N}$ and $\{\beta\} \in (0, 1)$, define

$$
M^{(\alpha)}_{\beta} = 1_{\{\alpha = 1\}}|a_1|_\beta + 1_{\{\alpha = 2\}}|B|_\beta + 1_{\{\alpha \in (0, 2)\}} \sup_{|\gamma| \leq d_0, |\gamma| = 1} |\partial^\gamma_x m^{(\alpha)}(\cdot, y)|_\beta,
$$

$$
N^{(\alpha)}_{\beta} = 1_{\{\alpha \in (1, 2]\}}|a_\alpha|_\beta + \sup_{|\gamma| = [\beta^-], x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (|y|^{\alpha} \wedge 1) \left[ |\partial^\gamma_x \rho^{(\alpha)}(x, y)| + |\partial^\gamma_x \rho^{(\alpha)}(x, y)| \right] \mu^{(\alpha)}(dy) + \sup_{x \neq \bar{x}} \frac{1}{|x - \bar{x}|^{\beta - [\beta^-]}} \int_{\mathbb{R}^d} (|y|^{\alpha} \wedge 1) \left| \partial^\gamma_x \rho^{(\alpha)}(x, y) - \partial^\gamma_x \rho^{(\alpha)}(\bar{x}, y) \right| \mu^{(\alpha)}(dy).
$$

The following assumptions hold.

A1 (i) There exists a constant $\mu > 0$ such that for all $x \in \mathbb{R}^d$ and $|\xi| = 1$,

$$
(B(x)\xi, \xi) \geq \mu, \text{ if } \alpha = 2,
$$

$$
\int_{S^{d-1}} |(w, \xi)|^{\alpha} m^{(\alpha)}(x, w) d\xi \geq \mu, \text{ if } \alpha \in (0, 2),
$$

where $B(x) = b(x)^* b(x), x \in \mathbb{R}^d$.
\[(ii) \lim_{\delta \to 0} \sup_{x \in \mathcal{R}^d} \int_{|y| \leq \delta} |y|^{\alpha} \rho^{(\alpha)}(x, y) \nu^{(\alpha)}(dy) = 0.\]

\[A2(\beta) \quad M^{(\alpha)}_\beta + N^{(\alpha)}_\beta < \infty.\]

The main result of this paper is the following statement.

**Theorem 2.** Let \(\alpha \in (0, 2)\) and \(\beta > 0, \beta \notin \mathbb{N}\). Assume \(A1\) and \(A2(\beta)\) hold. Then there exists a constant \(C\) such that for all \(g \in C^{\alpha+\beta}(\mathcal{R}^d)\),

\[|E g(Y_T) - E g(X_T)| \leq C|g|_{\alpha+\beta} \delta^{\kappa(\alpha, \beta)}, \quad (12)\]

where

\[\kappa(\alpha, \beta) = \begin{cases} \frac{\beta}{\alpha}, & \beta < \alpha, \\ 1, & \beta > \alpha. \end{cases}\]

Below are some comments on the assumptions and the main result.

**Remark 3.** The assumptions \(A1\) and \(A2(\beta)\) guarantee that the solution to the backward Kolmogorov equation associated with \(X_t\) is \((\alpha + \beta)\)-Hölder. They are in direct correspondence to the standard classical assumptions when the operator is differential. The regularity of the solution determines the rate of convergence of the weak Euler approximation.

**Remark 4.** Since \(E[g(X_T)]\) is the value of the solution to the backward Kolmogorov equation (see (56)), the estimate provides the rate of convergence of a probabilistic approximation \(E[g(Y_T)]\) to \(E g(X_T)\).

**Remark 5.** (i) The second condition of \(A1(i)\) holds with some constant \(\mu > 0\) if, for example, there is a Borel set \(\Gamma \subseteq S^{d-1}\) such that \(\mu \mathcal{H}^{d-1}(\Gamma) > 0\) and \(\inf_{x \in \mathcal{R}^d, w \in \Gamma} m^{(\alpha)}(x, w) > 0\).

The assumption \(A1(ii)\) holds if there is a measurable function \(\rho^{(\alpha)}(y)\) such that \(\rho^{(\alpha)}(x, y) \leq \rho^{(\alpha)}(y)\) and \(\int_{\mathcal{R}^d} (|y|^{\alpha} \wedge 1) \rho^{(\alpha)}(y) \nu^{(\alpha)}(dy) < \infty\).

(ii) For \(A2(\beta)\), let \(\beta \notin \mathbb{N}\). \(M^{(\alpha)}_\beta + N^{(\alpha)}_\beta < \infty\) if and only if

\[1_{\{\alpha \in [1, 2]\}} |a_{\alpha}|_{\beta} + 1_{\{\alpha = 2\}} |B|_{\beta} < \infty,\]

and there exists a constant \(C\) such that for all multiindices \(|\gamma| \leq |\beta|, |\gamma'| \leq d_0\) and \(x \in \mathcal{R}^d, \int_{\mathcal{R}^d} (|y|^{\alpha} \wedge 1) |\partial^{\gamma} \rho^{(\alpha)}(x, y)| \nu^{(\alpha)}(dy) + |\partial^\gamma \partial_{y'} m^{(\alpha)}(x, y)| \leq C,\]
and for all \( x, \tilde{x} \in \mathbb{R}^d, w \in S^{d-1} \), and multiindices \( |\gamma| = [\beta], |\gamma'| \leq d_0 \),

\[
|\partial_x^\gamma \partial_{w'}^\gamma m^{(\alpha)}(x, w) - \partial_x^\gamma \partial_{w'}^\gamma m^{(\alpha)}(\tilde{x}, w)| \leq C|x - \tilde{x}|^{\beta - |\beta'|},
\]

\[
\int_{\mathbb{R}_0^d} (|y|^{\alpha} \wedge 1)|\partial_x^\gamma \rho^{(\alpha)}(x, y) - \partial_x^\gamma \rho^{(\alpha)}(\tilde{x}, y)|\nu^{(\alpha)}(dy) \leq C|x - \tilde{x}|^{\beta - |\beta'|}.
\]

The very last inequality holds if, for example, for all \( x, \tilde{x} \in \mathbb{R}^d, y \in \mathbb{R}_0^d \),

\[
|\partial_x^\gamma \rho^{(\alpha)}(x, y) - \partial_x^\gamma \rho^{(\alpha)}(\tilde{x}, y)| \leq C|x - \tilde{x}|^{\beta - |\beta'|} \quad \text{and} \quad \int_{\mathbb{R}_0^d} (|y|^{\alpha} \wedge 1)\nu^{(\alpha)}(dy) < \infty.
\]

**Remark 6.** For the process defined in (5), the assumptions A1 and A2(\( \beta \)) are satisfied if \( c \) is non-degenerate with \( \inf_x \det |c(x)| > 0 \) and for \( i, j = 1, \ldots, d, c^{ij} \in C^{\beta}({\mathbb{R}}^d), |k^i|^{\alpha_i} \in C^{\beta}({\mathbb{R}}^d) \).

By applying Theorem 2 to (5), we have the following statement.

**Corollary 7.** Let \( X = \{X_t\}_{t \in [0, T]} \) satisfy (5) and the assumptions of Example 7 hold. Then (12) holds.

**Proof.** Indeed, by applying Theorem 14.80 in [3] and changing the variables of integration, for any \( f \in C_0^\infty(\mathbb{R}_0^d) \), the compensator of

\[
\int_0^t \int f(y)p^X(ds, dy) = \int_0^t \int f(c(X_{s-})y)p^Z(ds, dy) + \sum_{i=1}^d \int_0^t \int f(k^i(X_{s-})y_i e_i)p^{U_i}(ds, dy_i)
\]

is

\[
\int_0^t \int f(c(X_{s-})y)\frac{dy ds}{|y|^{d+\alpha}} + \sum_{i=1}^d \int_0^t \int f(k^i(X_{s-})y_i e_i)\frac{dy_i ds}{|y_i|^{1+\alpha_i}}
\]

\[
= \int_0^t \int f(y)\frac{m^{(\alpha)}(X_s, y)dy ds}{|y|^{d+\alpha}} + \int_0^t \int f(y)\rho^{(\alpha)}(X_s, y)\nu^{(\alpha)}(dy)ds.
\]

Thus,

\[
g^X(dt, dy) = p^X(dt, dy) - m^{(\alpha)}(X_t, y)\frac{dy}{|y|^{d+\alpha}} - \rho^{(\alpha)}(X_t, y)\nu^{(\alpha)}(dy)
\]
is a martingale measure with \(m^{(\alpha)}, \rho^{(\alpha)},\) and \(\nu^{(\alpha)}\) defined by (13). Obviously, (11) holds for \(\alpha \in (0, 1)\). Since \(m^{(\alpha)}(x, y)\) and \(\rho^{(\alpha)}(x, y)\) are symmetric in \(y\), we have for \(\alpha \in [1, 2)\),

\[
X_t = X_0 + \int_0^t \int_{|y|>1} yp_X(ds, dy) + \int_0^t \int_{|y|\leq 1} yq_X(ds, dy).
\]

Hence, the statement follows by Theorem (2). \(\square\)

**Remark 8.** Under the assumption of Corollary (7) with \(\alpha = 2\), it was derived in [12] that the convergence rate of diffusion processes is of the order \(\frac{1}{3-\beta} < \kappa(2, \beta) = \frac{3}{2}\) if \(\beta \in (1, 2)\). Corollary (7) improves that rate of convergence.

### 2.3. Outline of the Proof

First, let us define the operators in the Kolmogorov equation. For \(u \in C^{\alpha+\beta}(H)\), denote

\[
A^{(\alpha)}_y u(t, x) = u(t, x + y) - u(t, x) - \chi^{(\alpha)}(y)(\nabla_x u(t, x), y),
\]

\[
B^{(\alpha)}_y u(t, x) = u(t, x + y) - u(t, x) - 1_{\{|y|\leq 1\}}1_{\{|\alpha|\leq 1\}}(\nabla_x u(t, x), y),
\]

where \(\chi^{(\alpha)}(y) = 1_{\{|y|\leq 1\}}1_{\{|\alpha|\leq 1\}} + 1_{\{|\alpha|\leq 1\}}\). Let

\[
A^{(\alpha)}_z u(t, x) = 1_{\{|\alpha|\leq 1\}}(a_1(z), \nabla_x u(t, x)) + \frac{1}{2}1_{\{|\alpha|\leq 2\}} \sum_{i,j=1}^d B^{ij}(z)\partial^{2}_{ij} u(t, x)
\]

\[+ \int_{\mathbb{R}^d_0} A^{(\alpha)}_y u(t, x)m^{(\alpha)}(z, y)\frac{dy}{|y|^{d+\alpha}}, x, z \in \mathbb{R}^d, \tag{13}\]

\[
A^{(\alpha)} u(t, x) = A^{(\alpha)}_x u(t, x) = A^{(\alpha)}_z u(t, x)|_{z=x}, x \in \mathbb{R}^d,
\]

and

\[
B^{(\alpha)}_z u(t, x) = 1_{\{|\alpha|\leq 1\}}(a_1(z) + \int_{\{|y|>1\}} ym^{(\alpha)}(z, y)\frac{dy}{|y|^{d+\alpha}}, \nabla_x u(t, x))
\]

\[+ \int_{\mathbb{R}^d_0} B^{(\alpha)}_y u(t, x)\rho^{(\alpha)}(z, y)\nu^{(\alpha)}(dy), x, z \in \mathbb{R}^d, \tag{14}\]

\[
B^{(\alpha)}_x u(t, x) = B^{(\alpha)}_z u(t, x) = B^{(\alpha)}_z u(t, x)|_{z=x}, x \in \mathbb{R}^d.
\]

**Remark 9.** Under assumptions A1 and A2(\(\beta\)), for any \(\beta > 0\), there exists a unique weak solution to equation (11) and for every \(u \in C^{\alpha+\beta}(\mathbb{R}^d)\), the stochastic process

\[
u(X_t) - \int_0^t (A^{(\alpha)} + B^{(\alpha)})u(X_s)ds \tag{15}\]
is a martingale \(^{14}\). The operator \(L^{(\alpha)} = A^{(\alpha)} + B^{(\alpha)}\) is the generator of \(X_t\) defined in (11); \(A^{(\alpha)}\) is the principal part of \(L^{(\alpha)}\) and \(B^{(\alpha)}\) is the lower order or subordinated part of \(L^{(\alpha)}\).

If \(v(t, x), (t, x) \in H\) satisfies the backward Kolmogorov equation
\[
(\partial_t + A^{(\alpha)} x + B^{(\alpha)} x) v(t, x) = 0, \quad 0 \leq t \leq T,
\]
\[v(T, x) = g(x),\]
then as interpreted in Section 4, by Itô’s formula
\[
E[g(Y_T)] - E[g(X_T)] = E[v(T, Y_T) - v(0, Y_0)] = E[\int_0^T (\partial_t + L^{(\alpha)}_{Y_T}) v(s, Y_s) ds],
\]
and the regularity of \(v\) determines the one-step estimate and the rate of convergence of the approximation. For \(\beta \in (0, 1)\), the results for the Kolmogorov equation in Hölder classes are available \([13, 15]\). The results can be extended to the case \(\beta > 1\) in a standard analytic way. Due to the lack of regularity, probabilistic techniques like stochastic flows cannot be applied and Fourier multipliers are used to estimate precisely the principal part of the operator in Hölder spaces (see Lemma 15 and Corollary 16) (We do not know any other way to do it). The main difficulty is to derive the one-step estimates (see Lemma 21).

Remark 10. If \(m^{(\alpha)} = 1\), \((B^{ij}) = I\) (d×d-identity matrix), \(a_1(z) = 0\), then \(A^{(\alpha)}\) is the generator of a standard spherically-symmetric \(\alpha\)-stable process
\[
Z_t = \int_0^t \int yp^{Z}(ds, dy), \alpha \in (0, 1),
\]
\[
Z_t = \int_0^t \int_{|y| \leq 1} yq^{Z}(ds, dy) + \int_0^t \int_{|y| > 1} yp^{Z}(ds, dy), \alpha = 1,
\]
\[
Z_t = \int_0^t \int yq^{Z}(ds, dy), \alpha \in (1, 2),
\]
where \(p^{Z}(ds, dy)\) is the jump measure and \(q^{Z}(ds, dy) = p^{Z}(ds, dy) - \frac{dy}{|y|^{d+\alpha}} ds\) is the martingale measure. \(Z_t\) is the standard Wiener process if \(\alpha = 2\).

3. Backward Kolmogorov Equation

In Hölder-Zygmund spaces, consider the backward Kolmogorov equation associated with \(X_t\):
\[
(\partial_t + A^{(\alpha)} x + B^{(\alpha)} x) v(t, x) = f(t, x),
\]
\[v(T, x) = 0.\]
The regularity of its solution is essential for the one step estimate which determines the rate of convergence.

It has been shown (see Theorem 5 in [15]) that for \( \beta \in (0, 1) \) and any \( f \in C^{\beta}(H) \) there is a unique \( v \in C^{\alpha+\beta}(H) \) solving (17). Moreover, \( |v|_{\alpha+\beta} \leq C|f|_{\beta} \). In this section the result is extended to the case \( \beta > 1 \) using a standard induction method, by considering the differences defining the derivatives, interpreting them as solutions to (17), using uniform estimates, and passing to the limit. The main result of this section is Theorem 11.

**Theorem 11.** Let \( \alpha \in (0, 2], \beta > 0, \beta \notin \mathbb{N} \), and \( f \in C^{\beta}(H) \). Assume \( A_1 \) and \( A_2(\beta) \) hold. Then there exists a unique solution \( v \in C^{\alpha+\beta}(H) \) to (17). Moreover, there is a constant \( C \) independent of \( f \) such that \( |v|_{\alpha+\beta} \leq C|f|_{\beta} \).

An immediate consequence of Theorem 11 is the following statement.

**Corollary 12.** Let \( \alpha \in (0, 2] \) and \( \beta > 0, \beta \notin \mathbb{N} \). Assume \( A_1 \) and \( A_2(\beta) \) hold, \( f \in C^{\beta}(H) \), and \( g \in C^{\alpha+\beta}(\mathbb{R}^d) \). Then there exists a unique solution \( v \in C^{\alpha+\beta}(H) \) to the Cauchy problem

\[
\begin{align*}
(\partial_t + A^{(\alpha)}_x + B^{(\alpha)}_x)v(t,x) &= f(t,x), \\
v(T,x) &= g(x),
\end{align*}
\]

and \( |v|_{\alpha+\beta} \leq C(|f|_{\beta} + |g|_{\alpha+\beta}) \) with a constant \( C \) independent of \( f \) and \( g \).

To prove Theorem 11 and Corollary 12 the Hölder norm estimates of \( A^{(\alpha)}f \) and \( B^{(\alpha)}f \) are first derived for \( f \in C^{\alpha+\beta}(\mathbb{R}^d), \beta > 0 \). An auxiliary lemma about uniform convergence of Hölder functions is proved as well.

**3.1. Estimates of Operators and Uniform Convergence of Hölder Functions**

For \( f \in C^{\alpha+\beta}, A^{(\alpha)}f \) and \( B^{(\alpha)}f \) are estimated separately.

**3.1.1. Estimates of \( A^{(\alpha)}f \)**

It is well known (see [7, 13] for example) that for \( z \in \mathbb{R}^d, f \in C^{\alpha+\beta}(\mathbb{R}^d), \alpha \in (0, 2], \beta > 0 \),

\[
A^{(\alpha)}_z f(x) = F^{-1}[\psi^{(\alpha)}(z,\xi)F f(\xi)](x), x \in \mathbb{R}^d,
\]

where

\[
\psi^{(\alpha)}(z,\xi) = -K\int_{S^{d-1}} |(w,\xi)|^{\alpha}[1 - i(1_{\{\alpha\neq 1\}} \tan \frac{\alpha \pi}{2} \text{sgn}(w,\xi))
- \frac{2}{\pi} 1_{\{\alpha=1\}} \text{sgn}(w,\xi) \ln |(w,\xi)|]m^{(\alpha)}(z, w)\mu_{d-1}(dw)
- i1_{\{\alpha=1\}}(a_1(z),\xi) - \frac{1}{2} 1_{\{\alpha=2\}}(B(z)\xi,\xi), z, \xi \in \mathbb{R}^d,
\]
and $K = K(\alpha)$ is a constant depending on $\alpha$. Also, for a fixed $z \in \mathbb{R}^d$, $\exp\{t\psi^{(\alpha)}(z, \xi)\}$ is the characteristic function of a stable process and $A^{(\alpha)}_z$ is its generator (see Theorem 2.3.1 in [20]). For $\alpha \in (0, 2)$, the generator has the property

$$
\int [f(x + y) - f(x) - \chi^{(\alpha)}(y)(\nabla f(x), y)] \frac{dy}{|y|^{d+\alpha}} = c(\alpha, d)\partial^\alpha f \quad (21)
$$

where $\chi^{(\alpha)}(y) = 1_{\{|y| \leq 1\}}1_{\{\alpha = 1\}} + 1_{\{\alpha \in (1, 2)\}}$ and $c(\alpha, d)$ is a constant.

For the derivation of the estimates related to the principal part $A^{(\alpha)}$, Fourier multipliers in $C^{\alpha+\beta}(\mathbb{R}^d)$ is used (see [25]). Since for $\beta > 0$, the Hölder-Zygmund space $C^\beta(\mathbb{R}^d)$ coincides with the Besov space $B^{\beta}_{\infty, \infty}$ (see 2.5.7 in [25]), the theory of multipliers in Besov spaces can be applied, by considering the equivalent norms in $C^\beta(\mathbb{R}^d)$.

Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be a non-negative function such that supp$\phi = \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ and $\sum_{j=-\infty}^{\infty} \phi(2^{-j} \xi) = 1, \forall \xi \neq 0$. Define $\varphi_k \in \mathcal{S}(\mathbb{R}^d)$, $k = 0, \pm 1, \ldots$ by

$$
\mathcal{F}\varphi_k = \phi(2^{-k} \xi) \quad (22)
$$

and $\psi \in \mathcal{S}(\mathbb{R}^d)$ by

$$
\mathcal{F}\psi = 1 - \sum_{k \geq 1} \mathcal{F}\varphi_k(\xi), \quad (23)
$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly-decaying smooth functions on $\mathbb{R}^d$ (see Lemma 6.1.7 in [2] for example).

**Lemma 13.** (see 2.3.8 and 2.3.1 in [25] for (i) and Lemma 12 in [15] for the case $\beta \in (0, 1)$ of (ii)) For $\alpha \in (0, 2]$, $\beta > 0$, and $\gamma = \alpha + \beta$,

(i) $|u|_\beta \sim \sup_x |\psi * u(x)| + \sup_{k \geq 1} 2^{\beta k} \sup_x |\varphi_k * u(x)|$;

(ii) $|u|_{\alpha, \beta} = |u|_\alpha + |\partial^\alpha u|_\beta \sim |u - \partial^\alpha u|_\beta \sim |u|_\gamma$.

**Proof.** Define the family of operators $J^s : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d), s > 0$ by

$$
J^s u = \mathcal{F}^{-1}((1 + |\cdot|^2)^{\frac{s}{2}}\mathcal{F} u) \quad \text{and} \quad I^s u = \mathcal{F}^{-1}((1 + |\cdot|^s)\mathcal{F} u),
$$

where $\mathcal{S}'(\mathbb{R}^d)$ is the Schwartz space of generalized functions. By Theorem 2.3.8 in [25], $J^s : C^{\beta + \epsilon}(\mathbb{R}^d) \to C^\beta(\mathbb{R}^d), \beta > 0$ is an isomorphism. By
Lemma 2 of Section V.3.2 in [21] and Proposition 2 of Section V.3.1 in [21],

\[ F^{-1}\left[\frac{1+|\xi|^s}{1+|\xi|^2}\mathcal{F}u\right] \text{ and } F^{-1}\left[\frac{(1+|\xi|^2)^{\frac{s}{2}}}{1+|\xi|^s}\mathcal{F}u\right], \ s > 0 \]

map \( L_\infty(\mathbb{R}^d) \) onto \( L_\infty(\mathbb{R}^d) \). Therefore \( I^s : C^{\beta+s}(\mathbb{R}^d) \to C^{\beta}(\mathbb{R}^d), s > 0, \beta > 0 \) is an isomorphism as well. Part (ii) then follows.

For the estimates of the Hölder differences, Lemma 14 is needed.

**Lemma 14.** (see 2.6.1 in [23]) Let \( \beta > 0, h \in C^{d_0}(\mathbb{R}^d), d_0 = \left[\frac{d}{2}\right] + 1, \) and \( K_0 \) be a constant such that \( |\partial^{\gamma} h(\xi)| \leq K_0(1 + |\xi|)^{-|\gamma|} \) for any \( \xi \in \mathbb{R}^d \) and every multiindex \( \gamma \) with \( |\gamma| \leq d_0. \) Then there exists a constant \( C \) such that

\[ |F^{-1}(h\mathcal{F}f)|_\beta \leq CK_0|f|_\beta, \forall f \in C^{\beta}(\mathbb{R}^d). \]

In order to have the solution to (17) \( u \in C^{\alpha+\beta}(H) \) with \( \beta > 1, \) it is necessary to consider \( A^{(\alpha)} \) whose coefficients are differentiated. This requires estimating \( A^{(\alpha)} \) with coefficients not satisfying A1(i). Let the functions \( \bar{a}_1 = (\bar{a}_1^j)_{1 \leq j \leq d} = \bar{a}_1(x), \bar{b} = (\bar{b}^j)_{1 \leq j \leq d} = \bar{b}(x), \bar{m}^{(\alpha)} = \bar{m}^{(\alpha)}(x,y), \) and \( \bar{\rho}^{(\alpha)} = \bar{\rho}^{(\alpha)}(x,y), x \in \mathbb{R}^d, y \in \mathbb{R}_0^d, \) be measurable. In addition, assume that \( \bar{m}^{(\alpha)}(x,y) \) and its partial derivatives \( \partial_y \bar{m}^{(\alpha)}(x,y), |\gamma| \leq d_0 = \left[\frac{d}{2}\right] + 1 \) are continuous in \((x,y), \bar{m}^{(\alpha)}(x,y)\) is homogeneous in \( y \) with index zero, and

\[ \int_{S^{d-1}} y\bar{m}^{(\alpha)}(1,y)\mu_{d-1}(dy) = 0. \]

Define \( \bar{M}^{(\alpha)}_{\beta}, \beta > 0, \bar{A}^{(\alpha)}, \) and \( \bar{\psi}^{(\alpha)} \) by (19), (13), and (20), respectively, with \( m^{(\alpha)}, b, \) and \( a_1 \) replaced by \( \bar{m}^{(\alpha)}, \bar{b}, \) and \( \bar{a}_1, \) respectively. For \( A^{(\alpha)} \), the equality (19) holds with \( \bar{\psi}^{(\alpha)} \) as well.

Let \( \bar{\phi}(z,\xi) = \bar{\psi}^{(\alpha)}(z,\xi)(1 + |\xi|^{\alpha}-1, z, \xi \in \mathbb{R}^d. \) By Remark 10 in [15], it is readily checked that for every multiindex \( \gamma \) with \( |\gamma| \leq d_0 = \left[\frac{d}{2}\right] + 1, \) any \( \beta > 0, \) and any \( \xi \in \mathbb{R}^d, \) the following inequalities hold,

\[ \left| \partial_{\xi}^{\gamma}\bar{\phi}(\cdot,\xi) \right|_\beta \leq C\bar{M}^{(\alpha)}_{\beta}|\xi|^{-|\gamma|}, \left| \partial_{\xi}^{\gamma}\bar{\psi}(\cdot,\xi) \right| \leq C\bar{M}^{(\alpha)}|\xi|^{-|\gamma|}, \]

(24)

where

\[ \bar{M}^{(\alpha)} = \mathbf{1}_{\{\alpha=1\}} \sup_x [1_{\{\alpha=1\}}|\bar{a}_1(x)| + 1_{\{\alpha=2\}}|\bar{b}(x)| + 1_{\{\alpha=0,2\}}] \sup_{|\gamma| \leq d_0, |y| = 1} |\partial_y^{\gamma}\bar{m}^{(\alpha)}(x,y)|. \]

Denote \( \bar{\Phi}f(z,x) = F_x^{-1}[\bar{\phi}(z,\xi)\mathcal{F}f(\xi)](x) \) and \( \bar{\Phi}f(x) = \bar{\Phi}f(x,x), z, x \in \mathbb{R}^d. \) The estimates for \( A^{(\alpha)} \) are derived from Lemma 15.
Lemma 15. Let $\beta > 0$ and $\bar{\beta} \in (0, \beta]$. Assume $\bar{M}^{(\alpha)}_\beta < \infty$. Then there exists a constant $C$ such that for all $f \in C^\beta(R^d)$,

$$|	ilde{\Phi} f(z, \cdot)|_\beta \leq C \bar{M}^{(\alpha)} f|_\beta, z \in R^d,$$

$$|\Phi f(\cdot, x)|_\beta \leq C \bar{M}^{(\alpha)} f|_\beta, x \in R^d,$$

$$|\Phi f|_\beta \leq C \left( \bar{M}^{(\alpha)} f|_\beta + \bar{M}^{(\alpha)}_\beta |f|_\beta \right).$$

Proof. Let $\zeta_1 \in C^\infty_0(R^d)$ and $\zeta_2 = 1 - \zeta_1$ with $\zeta_1 \in [0, 1]$ and $\zeta_1(x) = 1$ if $|x| \leq 1$. Then $\tilde{\Phi} f(z, x) = \Phi_1 f(z, x) + \Phi_2 f(z, x)$, where $\Phi_k f(z, x) = \mathcal{F}^{-1} \left[ \hat{\phi}(z, \xi) \zeta_k(\xi) \mathcal{F} f(\xi) \right](x), k = 1, 2$.

Obviously, $\Phi_k f(z, x) = \eta_k(z, x) * \tilde{f}$, with

$$\tilde{f} = \mathcal{F}^{-1} \left[(1 + |\xi|^\alpha)^{-1} \mathcal{F} f\right] \quad \text{and} \quad \eta_k(z, x) = \mathcal{F}^{-1} \left[\tilde{\psi}^{(\alpha)}(z, \xi) \zeta_k(\xi)\right](x).$$

Let $\tilde{u} = \mathcal{F}^{-1} \zeta_1$, then $\tilde{u} \in S(R^d)$ and

$$\eta_1(z, x) = \mathcal{F}^{-1} \left[\tilde{\psi}^{(\alpha)}(z, \xi) \zeta_1(\xi)\right](x) = \mathcal{F}^{-1} \left[\tilde{\psi}^{(\alpha)}(z, \xi) \mathcal{F} \tilde{u}(\xi)\right](x) = 1_{\{\alpha=1\}}(\tilde{u}(z), \nabla \tilde{u}(x)) + \frac{1}{2} 1_{\{\alpha=2\}} \sum_{i,j=1}^d \mathcal{B}^{ij}(z) \partial^2_{ij} \tilde{u}(x) + \int A^{(\alpha)}(x)(z, y) \frac{dy}{|y|^{d+\alpha}}.$$

Thus,

$$\int |\eta_1(\cdot, x)|_\beta dx \leq C \bar{M}^{(\alpha)}_\beta \quad \text{and} \quad \int |\eta_1(z, x)| dx \leq C \bar{M}^{(\alpha)}.$$

Since by Lemma $|\tilde{f}|_{\alpha + \beta} \leq C |f|_\beta < \infty$, it follows that for any $x, z \in R^d$, $\bar{\beta} \leq \alpha + \beta$,

$$|\int \eta_1(x, y) \tilde{f}(x - y) dy|_{\beta} \leq |\tilde{f}|_{\infty} \int |\eta_1(x, y)|_{\beta} dy \leq C \bar{M}^{(\alpha)} f|_\beta,$$

$$|\int \eta_1(z, y) \tilde{f}(z - y) dy|_{\beta} \leq \tilde{f}_{\beta} \int |\eta_1(z, y)| dy \leq C \bar{M}^{(\alpha)} f|_\beta.$$

Hence,

$$|\Phi_1 f(\cdot, x)|_\beta \leq C \bar{M}^{(\alpha)} f|_\beta \quad \text{and} \quad |\Phi_1 f(z, \cdot)|_\beta \leq C \bar{M}^{(\alpha)} f|_\beta.$$  (25)
A straight application of (24) and Lemma 14 implies that for any \( \beta \in (0, \beta] \),
\[
|\tilde{\Phi}_2(f, \cdot)|_\beta \leq C\tilde{M}^{(\alpha)}(A) |f|_\beta \quad \text{and} \quad |\tilde{\Phi}_2(f, \cdot)|_\beta \leq CM^{(\alpha)}(A) |f|_\beta.
\] (26)

For any multiindex \( |\gamma| \leq [\beta]^-, \)
\[
\partial^\gamma[\tilde{\Phi}_k f(x, x)] = \sum_{\mu+\nu=\gamma} \partial_x^\mu \partial_y^\nu \tilde{\Phi}_k f(z, x)|_{z=x}
= \sum_{\mu+\nu=\gamma} \partial_y^\nu \tilde{\Phi}_k (\partial^\mu f)(z, x)|_{z=x}, k = 1, 2.
\] (27)

By (25)-(27), it follows that \( |\tilde{\Phi}_k f|_\beta \leq C(M^{(\alpha)}(A)|f|_\beta + M^{(\alpha)}(A)|f|_{\beta})|f|_\beta, k = 1, 2, \)
where \( \tilde{\Phi}_k f(x) = \tilde{\Phi}_k f(x, x), \forall x \in \mathbb{R}^d. \)

**Corollary 16.** Let \( \beta > 0, \beta \in (0, \beta], \) and \( M^{(\alpha)}(A) < \infty. \) Assume \( f \in C^{\alpha+\beta}(\mathbb{R}^d), \alpha \in (0, 2]. \) Then there is a constant \( C \) such that
\[
|A^{(\alpha)}(f)(x)|_\beta \leq C M^{(\alpha)}(A)|f|_{\alpha+\beta}, x \in \mathbb{R}^d,
|\bar{A}^{(\alpha)}(f)(z)|_\beta \leq C M^{(\alpha)}(A)|f|_{\alpha+\beta}, z \in \mathbb{R}^d,
|\bar{\bar{A}}^{(\alpha)}(f)|_\beta \leq C \left( M^{(\alpha)}(A)|f|_{\alpha+\beta} + M^{(\alpha)}(A)|f|_{\alpha+\beta} \right).
\]

**Proof.** Let \( \tilde{f} = \mathcal{F}^{-1}[1 + |\xi|^{\alpha} \mathcal{F} f]. \) Then by Lemma 13 \( \tilde{f} \in C^\beta(\mathbb{R}^d), |\tilde{f}|_\beta \leq C|f|_{\alpha+\beta}, \) and \( \tilde{A}^{(\alpha)}(f)(x) = \mathcal{F}^{-1}[\tilde{\phi}^{(\alpha)}(z, \xi)(1 + |\xi|^{\alpha})^{-1} \mathcal{F} f] = \mathcal{F}^{-1}[\tilde{\phi}(z, \xi) \mathcal{F} f]. \)
The statement follows by Lemma 15. \( \square \)

### 3.1.2. Estimates of \( B^{(\alpha)}(f) \)

As in the case of \( A^{(\alpha)}(f) \), it needs to consider \( B^{(\alpha)}(f) \) whose coefficients are differentiated. Let the functions \( \tilde{a}_\alpha = (\tilde{a}^{(\alpha)}_\alpha)_{1 \leq i \leq d} = \tilde{a}_\alpha(x) \) and \( \tilde{\rho}^{(\alpha)} = \tilde{\rho}^{(\alpha)}(x, y), x \in \mathbb{R}^d, y \in \mathbb{R}_0^d \) be measurable. Define \( \tilde{N}_\beta^{(\alpha)} \) and \( \bar{N}^{(\alpha)} \) by (10) and (14), respectively, with \( \tilde{\rho}^{(\alpha)} \) and \( a_\alpha \) replaced by \( \tilde{\rho}^{(\alpha)} \) and \( \tilde{a}_\alpha \).

The following equality for the estimates of \( B^{(\alpha)}(f) \) are used.

**Lemma 17.** (Lemma 2.1 in [1]) For \( \delta \in (0, 1) \) and \( u \in C_0^\infty(\mathbb{R}^d), \)
\[
u(x + y) - u(x) = K \int k^{(\delta)}(y, z) \partial^\delta u(x - z)dz,
\] (28)
where \( K = K(\delta, d) \) is a constant, \( k^{(\delta)}(y, z) = |z + y|^{-d + \delta} - |z|^{-d + \delta}, \) and there exists a constant \( C \) such that \( \int |k^{(\delta)}(y, z)|dz \leq C|y|^d, \forall y \in \mathbb{R}^d. \)
Let $\mathcal{N}(\alpha) = 1_{\{\alpha \in (1,2]\}} \sup_{x} |\bar{a}_{\alpha}(x)|_{\beta} + \sup_{x \in \mathbb{R}^{d}} \int_{\mathbb{R}^{d}} (|y|^{\alpha} \wedge 1) |\bar{\rho}^{(\alpha)}(x, y)| \nu^{(\alpha)}(dy)$.

Lemma 18. Let $\beta > 0$, $\bar{\beta} \in (0, \beta]$, and $\mathcal{N}_{\bar{\beta}}^{(\alpha)} < \infty$. Then there exists a constant $C$ such that for all $f \in C^{\alpha+\bar{\beta}}(\mathbb{R}^{d})$,

$$|\mathcal{B}^{(\alpha)}_{\bar{\beta}} f(z)|_{\beta} \leq C \mathcal{N}^{(\alpha)} \|f\|_{\alpha+\bar{\beta}}, z \in \mathbb{R}^{d},$$

$$|\mathcal{B}^{(\alpha)}_{\bar{\beta}} f(x)|_{\beta} \leq C \mathcal{N}_{\bar{\beta}}^{(\alpha)} \|f\|_{\alpha+\bar{\beta}}, x \in \mathbb{R}^{d},$$

$$|\mathcal{B}^{(\alpha)}_{\bar{\beta}} f|_{\beta} \leq C \mathcal{N}_{\bar{\beta}}^{(\alpha)} \|f\|_{\alpha+\bar{\beta}}.$$  

Proof. Rewrite $\mathcal{B}^{(\alpha)}_{\bar{\beta}} f(x) = \mathcal{B}^{(\alpha),1}_{\bar{\beta}} f(x) + \mathcal{B}^{(\alpha),2}_{\bar{\beta}} f(x)$, where

$$\mathcal{B}^{(\alpha),1}_{\bar{\beta}} f(x) = \int_{|y| > 1} [f(x + y) - f(x)] \bar{\rho}^{(\alpha)}(z, y) \nu^{(\alpha)}(dy) + 1_{\{\alpha \in (1,2]\}} \{\bar{a}_{\alpha}(z), \nabla f(x)\},$$

$$\mathcal{B}^{(\alpha),2}_{\bar{\beta}} f(x) = 1_{\{\alpha \in (0,1]\}} \int_{|y| \leq 1} [f(x + y) - f(x)] \bar{\rho}^{(\alpha)}(z, y) \nu^{(\alpha)}(dy)$$

$$+ 1_{\{\alpha \in (1,2]\}} \int_{|y| \leq 1} \int_{0}^{1} (\nabla f(x + sy) - \nabla f(x, y)) \bar{\rho}^{(\alpha)}(z, y) ds \nu^{(\alpha)}(dy).$$

By Lemma 17,

$$\mathcal{B}^{(\alpha),2}_{\bar{\beta}} f(x) = 1_{\{\alpha \in (0,1]\}} K \int_{|y| \leq 1} k^{(\alpha)}(y, y') \partial^{\alpha} f(x - y') dy' \bar{\rho}^{(\alpha)}(z, y) \nu^{(\alpha)}(dy)$$

$$+ 1_{\{\alpha = 1\}} \int_{|y| \leq 1} (\nabla f(x + sy), y) \bar{\rho}^{(\alpha)}(z, y) \nu^{(\alpha)}(dy)$$

$$+ 1_{\{\alpha \in (1,2]\}} K \int_{|y| \leq 1} \int_{0}^{1} \left( \int k^{(\alpha-1)}(sy, y') \partial^{\alpha-1} \nabla f(x - y') dy' \right)$$

$$\times \bar{\rho}^{(\alpha)}(z, y) ds \nu^{(\alpha)}(dy)$$

$$+ 1_{\{\alpha = 2\}} \int_{|y| \leq 1} \int_{0}^{1} \left( 1 - s \right) \partial^{2}_{y} f(x + sy, y) ds \bar{\rho}^{(2)}(z, y) \nu^{(2)}(dy).$$ (31)

For $\alpha > 1$, according to Lemma 17 for any $\beta > 0$,

$$|\partial^{\alpha-1} \nabla f|_{\beta} \leq |(1 - \partial^{\alpha-1}) \nabla f|_{\beta} + |\nabla f|_{\beta} \leq C(\|\nabla f\|_{\alpha+\beta-1} + |\nabla f|_{\beta}) \leq C \|f\|_{\alpha+\beta}. \quad (32)$$

By Lemma 17 (31), (32), the first two inequalities are obtained. For exam-
ple, if $\alpha \in (1, 2), \beta \notin \mathbb{N}$, by Lemma 17

$$\left| \int_{|y| \leq 1} \int_{0}^{1} \left( \int k^{(\alpha - 1)}(sy, y') \partial^{\alpha - 1} \nabla f(x - y')dy', y)\bar{\rho}^{(\alpha)}(\cdot, y)ds\nu^{(\alpha)}(dy) \right) \right|_{\beta} \leq C \sup_{x} |\partial^{\alpha - 1} \nabla f(x)| \sup_{|\gamma| \leq |\beta|} \int_{|y| \leq 1} |y|^{|\alpha|} \left[ |\partial^{\gamma} \bar{\rho}^{(\alpha)}(z, y)| \right. \left. + \frac{|\partial^{\gamma} \bar{\rho}^{(\alpha)}(z, y)| - |\partial^{\gamma} \bar{\rho}^{(\alpha)}(z, y)|}{|z - z'|^{\beta - |\beta|}} \right] \nu^{(\alpha)}(dy),$$

and

$$\left| \int_{|y| \leq 1} \int_{0}^{1} \left( \int k^{(\alpha - 1)}(sy, y') \partial^{\alpha - 1} \nabla f(x - y')dy', y)\bar{\rho}^{(\alpha)}(z, y)ds\nu^{(\alpha)}(dy) \right) \right|_{\beta} \leq C |\partial^{\alpha - 1} \nabla f|_{\beta} \int_{|y| \leq 1} |y|^{|\alpha|} \bar{\rho}^{(\alpha)}(z, y) \nu^{(\alpha)}(dy).$$

Also, for $|\gamma| \leq |\beta|$, $\partial^{\gamma}(\bar{B}^{(\alpha)} f) = \sum_{k+\mu=\gamma} \partial^{k} \bar{B}^{(\alpha)} \partial^{\mu} f(x) |_{z=x}, k = 1, 2.$

Hence, by (29) and (30), the third inequality of the statement follows. [17]

3.1.3. Uniform Limits of Hölder Continuous Functions

To prove Theorem 11 by induction and passing to the limit, the following statement is needed.

**Lemma 19.** Assume $u_{n} \in C^{\beta}(\mathbb{R}^{d}), n \in \mathbb{N}$ with $\sup_{n} |u_{n}|_{\beta} < \infty$ and $u_{n} \to u$ uniformly on compact subsets. Then $u \in C^{\beta}, |u|_{\beta} \leq \sup_{n} |u_{n}|_{\beta}$ and $\partial^{\beta} \partial^{\beta} u_{n} \to \partial^{\beta} \partial^{\beta} u, |\gamma| \leq |\beta|^{-} \text{ uniformly on compact subsets as } n \to \infty \text{ for any } \delta \in [0, 1) \text{ such that } |\beta|^{-} + \delta < \beta.$

**Proof.** Let $\psi, \varphi_{k}$ be the functions defined by (22), (23), respectively. If $u_{n} \to u$ uniformly on compact sets, then

$$|\psi \ast u(x)| = \lim_{n} |\psi \ast u_{n}(x)| \leq \sup_{n} \sup_{y} |\psi \ast u_{n}(y)|, \forall x \in \mathbb{R}^{d}$$

and

$$2^{\beta k} |\varphi_{k} \ast u(x)| = 2^{\beta k} \lim_{n} |\varphi_{k} \ast u_{n}(x)| \leq \sup_{n} \sup_{k} 2^{\beta k} |\varphi_{k} \ast u_{n}(y)|, \forall x \in \mathbb{R}^{d}.$$

Hence by Lemma 13(i), $|u|_{\beta} \leq \sup_{n} |u_{n}|_{\beta} < \infty$. By the Arzelà–Ascoli theorem, there exist continuous functions $v_{\gamma}(x), x \in \mathbb{R}^{d}, |\gamma| \leq |\beta|^{-}$, and a
subsequence \( u_{n_k} \) such that \( \partial^\gamma u_{n_k} \to v \), uniformly on compact subsets of \( \mathbb{R}^d \) as \( k \to \infty \). Therefore, by Theorem 3.6.1 in \( \text{[3]} \), \( v_\gamma = \partial^\gamma v_0 \) (the limit of \( u_{n_k} \) is continuously differentiable up to \( |\beta^-| \)). According to (21), for \( \delta \in (0,1) \) such that \( |\beta^- + \delta| < |\beta^-| \), \( v_\gamma = K \int [\partial^\mu u_n(x + y) - \partial^\mu u_n(x)] \frac{dy}{|y|^{d+\delta}} \). Passing to the limit yields that \( \partial^\delta \partial^\mu u_n \to \partial^\delta \partial^\mu u \) uniformly on compact subsets as \( n \to \infty \). ■

3.2. Proof of Theorem 11 and Corollary 12

3.2.1. Proof of Theorem 11

The statement is proved by induction. For \( \alpha \in (0,2], \beta \in (0,1) \), given \( f \in C^\beta(H) \), there exists a unique solution \( u \in C^{\alpha+\beta}(H) \) to the Kolmogorov equation (17) and \( |u|_{\alpha+\beta} \leq C|f|_\beta \).

Assume the result holds for \( \beta \in \bigcup_{l=0}^{n-1} (l,l+1), n \in \mathbb{N} \). Let \( \beta \in (n,n+1), \tilde{\beta} = \beta - 1 \), and \( f \in C^{\tilde{\beta}} \). Then \( \tilde{\beta} \in (n-1,n) \), \( f \in C^{\tilde{\beta}}(H) \) as well, and there exists a unique solution \( v \in C^{\alpha+\tilde{\beta}}(H), \alpha \in (0,2] \) to the Cauchy problem (17) and \( |v|_{\alpha+\tilde{\beta}} \leq C|f|_{\tilde{\beta}} \).

For \( h \in \mathbb{R} \) and \( k = 1,\ldots,d \), denote

\[
v^h_k(t,x) = \frac{v(t,x + he_k) - v(t,x)}{h},
\]

where \( \{e_k, k = 1,\ldots,d\} \) is the canonical basis in \( \mathbb{R}^d \). Let

\[
A^{(0)}_{x^{(0)}}^{k,h} v(t,x) = \frac{1}{h} (A^{(0)}_{x^{(0)}}^{k} - A^{(0)}_{z^{(0)}}^{k} v(t,x),
\]

and

\[
B^{(0)}_{x^{(0)}}^{k,h} v(t,x) = \frac{1}{h} (B^{(0)}_{x^{(0)}}^{k} - B^{(0)}_{z^{(0)}}^{k} v(t,x).
\]

Obviously,

\[
(\partial_t + A^{(0)}_{x^{+h e_k}} + B^{(0)}_{x^{+h e_k}}) v(t,x + he_k) = f(t,x + he_k),
\]

\[
v(T,x + he_k) = 0, \quad k = 1,\ldots,d. \quad (33)
\]

Subtracting (17) from (33) and dividing the difference by \( h \) yields

\[
(\partial_t + A^{(0)}_{x^{(0)}} + B^{(0)}_{x^{(0)}}) v^h_k(t,x) = f^h_k(t,x) - A^{(0)}_{x^{(0)}}^{k,h} v(t,x + he_k) - B^{(0)}_{x^{(0)}}^{k,h} v(t,x + he_k),
\]

\[
v^h_k(T,x) = 0, \quad k = 1,\ldots,d. \quad (34)
\]
Since $f \in C^\beta(H)$ and
\[
f_h^k(t, x) = \frac{f(t, x + he_k) - f(t, x)}{h} = \int_0^1 \partial_k f(t, x + he_k s) ds, \quad \forall h \neq 0,
\]
then
\[
|f_h^k|_\beta \leq C|\nabla f|_{\beta - 1} \leq C|f|_\beta
\]
with a constant $C$ independent of $h$. Since $v \in C^{\alpha + \tilde{\beta}}(H)$, then $v_h^k \in C^{\alpha + \tilde{\beta}}(H)$. Let
\[
\bar{a}_{1;h,k}(x) = \frac{1}{h}(a_1(x + he_k) - a_1(x)) = \int_0^1 \partial_k a_1(x + he_k s) ds
\]
\[
\bar{B}_{h,k}^{ij}(x) = \int_0^1 \partial_k B^{ij}(x + she_k) ds,
\]
\[
\bar{m}_{h,k}^{(\alpha)}(x, y) = \int_0^1 \partial_k m^{(\alpha)}(x + he_k s, y) ds, x \in \mathbb{R}^d.
\]
Then for $(t, x) \in H$,
\[
A^{(\alpha),k,h} v(t, x + he_k) = \frac{1}{h} \left( A_x^{(\alpha)} v(t, x + he_k) - A_x^{(\alpha)} v(t, x + he_k) \right)
\]
\[
= 1_{\{\alpha = 1\}} \left( \bar{a}_{1;h,k}(x), \nabla v(t, x + he_k) \right)
\]
\[
+ \frac{1}{2} 1_{\{\alpha = 2\}} \sum_{i,j=1}^d \bar{B}_{h,k}^{ij}(x) \partial_{ij}^2 v(t, x + he_k)
\]
\[
+ \int A_y^{(\alpha)} v(t, x + he_k) \bar{m}_{h,k}^{(\alpha)}(x, y) \frac{dy}{|y|^{d+\alpha}}.
\]
Similarly, for $(t, x) \in H$,
\[
B_x^{(\alpha),k,h} v(t, x + he_k) = \frac{1}{h} \left( B_x^{(\alpha)} v(t, x + he_k) - B_x^{(\alpha)} v(t, x + he_k) \right)
\]
\[
= 1_{\{\alpha \in (1,2]\}} \left( \bar{a}_{\alpha;h,k}(x), \nabla v(t, x + he_k) \right)
\]
\[
+ \int_{|y| > 1} \nabla_y^1 v(t, x + he_k) \bar{p}_{h,k}^{(\alpha)}(x, y) \nu^{(\alpha)}(dy)
\]
\[
+ 1_{\{\alpha \in (1,2]\}} \int_{|y| \leq 1} \nabla_y^2 v(t, x + he_k) \bar{p}_{h,k}^{(\alpha)}(x, y) \nu^{(\alpha)}(dy)
\]
\[
+ 1_{\{\alpha \in (0,1]\}} \int_{|y| \leq 1} \nabla_y^1 v(t, x + he_k) \bar{p}_{h,k}^{(\alpha)}(x, y) \nu^{(\alpha)}(dy),
\]

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with

\[ \nabla_1^1 v(t, x) = v(t, x + y) - v(t, x), \]
\[ \nabla_2^2 v(t, x) = v(t, x + y) - v(t, x) - (\nabla v(t, x), y) \]

and

\[ \bar{a}_{\alpha, h, k}(x) = \int_0^1 \partial_k a_{\alpha}(x + he_k s) ds, x \in \mathbb{R}^d, \]
\[ \bar{\rho}_{h, k}^{(\alpha)}(x, y) = \int_0^1 \partial_k \rho^{(\alpha)}(x + she_k y) ds, x \in \mathbb{R}^d, y \in \mathbb{R}^d. \]

By applying Corollary 16 with \( \bar{a} = \bar{a}_{h, k}, \bar{B} = \bar{B}_{h, k}, \bar{m}^{(\alpha)} = \bar{m}^{(\alpha)}_{h, k}, \beta = \bar{\beta}, \)
and by the induction assumption, it follows that

\[ |A^{(\alpha), k, h} v|_{\bar{\beta}} \leq CM_\beta^{(\alpha)} |v|_{\alpha + \bar{\beta}} \leq CM_\beta^{(\alpha)} |f|_{\bar{\beta}}, k = 1, \ldots, d, \]

with a constant \( C \) independent of \( h, f, \).

Applying Lemma 18 with \( a_{\alpha} = \bar{a}_{\alpha, h, k}, \bar{\rho}^{(\alpha)} = \bar{\rho}_{h, k}^{(\alpha)}, \beta = \bar{\beta}, f = v(t, \cdot + he_k) \)
together with the induction assumption yields

\[ |B^{(\alpha), k, h} v|_{\bar{\beta}} \leq CN_\beta^{(\alpha)} |v|_{\alpha + \bar{\beta}} \leq CN_\beta^{(\alpha)} |f|_{\bar{\beta}}, k = 1, \ldots, d, \quad (36) \]

with a constant \( C \) independent of \( h, f, \).

Hence, \( f_k^h(t, x) - A^{(\alpha), k, h} v(t, x + he_k) - B^{(\alpha), k, h} v(t, x + he_k) \in \mathcal{C}^{\bar{\beta}}(H) \)
and \( v_k^h \in \mathcal{C}^{\alpha + \bar{\beta}}(H) \) satisfies (34). Thus, by the induction assumption and

\[ |v_k^h|_{\alpha + \bar{\beta}} \leq C |f_k^h - A^{(\alpha), k, h} v - B^{(\alpha), k, h} v|_{\bar{\beta}} \leq C |f|_{\bar{\beta}}, k = 1, \ldots, d, \quad (37) \]

where \( C \) is a constant independent of \( h, f, \). Also by Corollary 16 and Lemma 18

\[ |B^{(\alpha)} v_k^h|_{\bar{\beta}} \leq C |v_k^h|_{\alpha + \bar{\beta}} \leq C |f|_{\bar{\beta}} \quad \text{and} \quad |A^{(\alpha)} v_k^h|_{\bar{\beta}} \leq C |v_k^h|_{\alpha + \bar{\beta}} \leq C |f|_{\bar{\beta}}. \quad (38) \]

Therefore by (34), for any \( (t, x) \in H, \)

\[ v_k^h(t, x) - v_k^h(s, x) = \int_s^t [f_k^h(r, x) - A^{(\alpha), k, h} v(r, x + he_k) - B^{(\alpha), k, h} v(r, x + he_k)] dr \]
\[ - \int_s^t (A^{(\alpha)} + B^{(\alpha)}) v_k^h(r, x) dr, 0 \leq s < t \leq T, k = 1, l \ldots, d, \]

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and by (35)-(38),

\[ |v_k^h(t,x) - v_k^h(s,x)| \leq (|f_k^h - A_x^{(a)}v - B_x^{(a)}v|_\beta + |(A_x^{(a)} + B_x^{(a)})v_k^h|_\beta)|t-s| \leq C|t-s|, \]

with a constant C independent of h, k. Hence (see (37) as well), \( v_k^h(t,x), k = 1, \ldots, d \) are equicontinuous in \((t,x)\) and by the Arzelà–Ascoli theorem, for each \( h_n \to 0 \), there exist a subsequence \( \{h_{n_j}\} \) and continuous functions \( v_k(t,x), (t,x) \in H, k = 1, \ldots, d \), such that \( v_{k_{n_j}}^h(t,x) \to v_k(t,x) \) uniformly on compact subsets of \( H \) as \( j \to \infty \). By Lemma 19, \( v_k \in C^{\alpha+\beta} \) and \( |v_k|_{\alpha+\beta} \leq C|f|_{\beta} \), \( k = 1, \ldots, d \).

It then follows from passing to the limit in (34) and the dominated convergence theorem that \( u_k \) is the unique solution to

\[ (\partial_t + A_x^{(a)} + B_x^{(a)})v_k(t,x) = \partial_t f(t,x) - (\partial_k A_x^{(a)})v(t,x) - (\partial_k B_x^{(a)})v(t,x), \]

\[ v_k(T,x) = 0, k = 1, \ldots, d \]

and so \( v_{k_{n_j}}^h(t,x) \to v_k(t,x), \forall h_n \to 0 \). Hence,

\[ v_k(t,x) = \lim_{h \to 0} v_k^h(t,x) = \lim_{h \to 0} \frac{v(t,x + he_k) - v(t,x)}{h} = \partial_k v(t,x), \]

\( \partial_k v \in C^{\alpha+\beta}(H), k = 1, \ldots, d, \) and \( |\nabla v|_{\alpha+\beta} \leq C|f|_{\beta} \). Therefore, \( v \in C^{\alpha+\beta}(H) \) and the statement of Theorem 11 follows.

### 3.2.2. Proof of Corollary 12

By Lemmas 15 and 18 for \( g \in C^{\alpha+\beta}(\mathbb{R}^d), |A^{(a)}g|_{\beta} \leq C|g|_{\alpha+\beta} \) and \( |B^{(a)}g|_{\beta} \leq C|g|_{\alpha+\beta} \) with a constant \( C \) independent of \( f \) and \( g \). It then follows from (17) that there exists a unique solution \( \tilde{v} \in C^{\alpha+\beta}(H) \) to the Cauchy problem

\[ (\partial_t + A_x^{(a)} + B_x^{(a)})\tilde{v}(t,x) = f(t,x) - A_x^{(a)}g(x) - B_x^{(a)}g(x), \]

\[ \tilde{v}(T,x) = 0 \quad (39) \]

and \( |\tilde{v}|_{\alpha+\beta} \leq C(|g|_{\alpha+\beta} + |f|_{\beta}) \) with \( C \) independent of \( f \) and \( g \). Let \( v(t,x) = \tilde{v}(t,x) + g(x) \), where \( \tilde{v} \) is the solution to problem (39). Then \( v \) is the unique solution to the Cauchy problem (18) and \( |v|_{\alpha+\beta} \leq C(|g|_{\alpha+\beta} + |f|_{\beta}). \)

**Remark 20.** If the assumptions of Corollary 12 hold and \( v \in C^{\alpha+\beta}(H) \) is the solution to (18), then \( \partial_t v = f - A_x^{(a)}v - B_x^{(a)}v \) and according to Corollary 16 and Lemma 18, \( |\partial_t v|_{\beta} \leq C(|g|_{\alpha+\beta} + |f|_{\beta}). \)
4. One Step Estimate and Proof of the Main Result

The following Lemma provides a one-step estimate of the conditional expectation of an increment of the Euler approximation.

**Lemma 21.** Let $\alpha \in (0, 2]$, $\beta > 0$, $\beta \notin \mathbb{N}$, and $\delta > 0$. Assume A1 and A2($\beta$) hold. Then there exists a constant $C$ such that for all $f \in C^\beta(\mathbb{R}^d)$,

$$|E[f(Y_s) - f(Y_{\tau_is})]| \leq C|f|\beta\delta^{\kappa(\alpha, \beta)}, \forall s \in [0, T],$$

where $i_s = i$ if $\tau_i \leq s < \tau_{i+1}$ and $\kappa(\alpha, \beta)$ is as defined in Theorem 2.

The proof of Lemma 21 is based on applying Itô’s formula to $f(Y_s) - f(Y_{\tau_is})$, $f \in C^\beta(\mathbb{R}^d)$. If $\beta > \alpha$, by Remark 9 and Itô’s formula, the inequality holds. If $\beta < \alpha$, $f$ is first smoothed by using $w \in C^\infty_0(\mathbb{R}^d)$, a nonnegative smooth function with support on $\{|x| \leq 1\}$ such that $w(x) = w(\|x\|)$, $x \in \mathbb{R}^d$, and $\int w(x)dx = 1$ (see (8.1) in [4]). Note that, because of the symmetry,

$$\int_{\mathbb{R}^d} x^i w(x)dx = 0, i = 1, \ldots, d. \quad (40)$$

For $x \in \mathbb{R}^d$ and $\varepsilon \in (0, 1)$, define $w^\varepsilon(x) = \varepsilon^{-d}w(\frac{x}{\varepsilon})$ and the convolution

$$f^\varepsilon(x) = \int f(y)w^\varepsilon(x-y)dy = \int f(x-y)w^\varepsilon(y)dy, x \in \mathbb{R}^d. \quad (41)$$

4.1. Auxiliary Estimates

For the estimates of $A_z^{(\alpha)} f^\varepsilon$, the following integral estimates are needed.

**Lemma 22.** Let $\alpha \in (0, 2)$ and $v \in C^\infty_0(\mathbb{R}^d)$.

(i) For $\chi^{(\alpha)}(y) = 1_{\{|y| \leq 1\}}1_{\{\alpha = 1\}} + 1_{\{\alpha \in (1, 2)\}}$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d_0} \left|v(y + y') - v(y) - \chi^{(\alpha)}(y') (\nabla v(y), y')\right| \frac{dydy'}{\|y'\|^{d+\alpha}} < \infty;$$

(ii) For $\beta \in (0, 1 \wedge \alpha)$, $z \in \mathbb{R}^d$, $\sup_z \int_{\mathbb{R}^d} |(A_z^{(\alpha)} w)(y)| |y|^\beta dy < \infty$;

(iii) For $\beta \in (1, \alpha)$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d_0} \int_0^1 \left|w(y + sy') - w(y)\right| |y|^\beta \frac{dsdydy'}{\|y'\|^{d+\alpha - 1}} < \infty.$$
Proof. (i) Clearly,

\[ |v(y + y') - v(y) - \chi^{(\alpha)}(y')(\nabla v(y), y')| \]

\[ \leq 1_{\{|y'| \leq 1\}} \left\{ \int_0^1 \max_{i,j} |\partial^2_{ij} v(y + sy')| |y'|^2 + 1_{\{\alpha \in (0, 1)\}} |\nabla v(y + sy')||y'||ds \right\} 

+ 1_{\{|y'| > 1\}} \left\{ |v(y + y')| + |v(y)| + 1_{\{\alpha \in (1, 2)\}} |\nabla v(y)| |y'| \right\}, y, y' \in \mathbb{R}^d.

The claim then follows.

(ii) For $\beta \in (0, 1)$, $\beta < \alpha$, $z \in \mathbb{R}^d$,

\[
\int_{\mathbb{R}^d} |(A_z^{(\alpha)} w)(y)||y|^\beta dy \leq \int_{\mathbb{R}^d} |v(y + y')||y|^\beta \frac{dy dy'}{|y'|^{d+\alpha}} 
\]

\[ + \int_{\mathbb{R}^d} |v(y)||y|^\beta \frac{dy dy'}{|y'|^{d+\alpha}} 
\]

\[ + \max_{i,j} \int_{\mathbb{R}^d} \int_{|y'| \leq 1} \int_0^1 |\partial^2_{ij} w(y + sy')| |y'|^2 |y|^\beta \frac{ds dy dy'}{|y'|^{d+\alpha}} \]

and

\[
\int_{\mathbb{R}^d} \int_{|y'| > 1} |w(y + y')||y|^\beta \frac{dy dy'}{|y'|^{d+\alpha}} \leq C \left[ \int_{\mathbb{R}^d} \int_{|y'| > 1} |w(y + y')||y + y'|^\beta \frac{dy dy'}{|y'|^{d+\alpha}} 
\]

\[ + \int_{\mathbb{R}^d} \int_{|y'| > 1} |y'|^\beta \frac{dy dy'}{|y'|^{d+\alpha}} \right].

Part (ii) follows.

(iii) For $1 < \beta < \alpha < 2$,

\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |w(y + sy') - w(y)||y|^\beta - 1 \frac{dy dy' ds}{|y'|^{d+\alpha - 1}} \]

\[ \leq \int_{\mathbb{R}^d} \int_{|y'| > 1} \int_0^1 |w(y + sy')||y|^\beta - 1 \frac{dy dy' ds}{|y'|^{d+\alpha - 1}} 
\]

\[ + \int_{\mathbb{R}^d} \int_{|y'| > 1} \int_0^1 |w(y)||y|^\beta - 1 \frac{dy dy' ds}{|y'|^{d+\alpha - 1}} 
\]

\[ + \int_{\mathbb{R}^d} \int_{|y'| \leq 1} \int_0^1 \int_0^1 |\nabla w(y + sy')||y|^\beta - 1 \frac{ds d\tau dy dy'}{|y'|^{d+\alpha - 2}}. \]
Lemma 23. Let $\beta < \alpha, \beta \neq 1, \alpha \in (0, 2), \text{and } \varepsilon \in (0, 1)$. Then

(i) there exists a constant $C$ such that for all $f \in C^\beta(\mathbb{R}^d), x \in \mathbb{R}^d$,

$$|f^\varepsilon(x) - f(x)| \leq C\varepsilon^\beta |f|_{\beta};$$

(ii) there exists a constant $C$ such that for all $z, x \in \mathbb{R}^d$,

$$|A^\varepsilon_2 f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta} |f|_{\beta} \quad (42)$$

and in particular, for all $f \in C^\beta(\mathbb{R}^d), z, x \in \mathbb{R}^d$,

$$|\partial^\alpha f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta} |f|_{\beta}; \quad (43)$$

(iii) there exist constants $C$s such that for $k, l = 1, \ldots, d, x \in \mathbb{R}^d$,

$$|\partial_k f^\varepsilon(x)| \leq C\varepsilon^{-1+\beta} |f|_{\beta}, \text{if } \beta < 1, \quad (44)$$

$$|f^\varepsilon|_1 \leq C|f|_1,$$

$$|\partial^2_{kl} f^\varepsilon(x)| \leq C\varepsilon^{-2+\beta} |f|_{\beta}, \text{if } \beta < 2, \quad (45)$$

and

$$|f^\varepsilon|_{\alpha} \leq C\varepsilon^{-\alpha+\beta} |f|_{\beta}, \text{if } \beta \in (0, 1], \alpha \in (1, 2), \quad (46)$$

$$|\partial^{\alpha-1} \nabla f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta} |f|_{\beta}, \text{if } \beta \in (1, \alpha), \alpha \in (1, 2). \quad (47)$$

Since

$$\int_{\mathbb{R}^d} \int_{|y'| > 1} \int_0^1 |w(y + sy')||y|^{\beta-1} \frac{dy'ds}{|y'|^{d+\alpha-1}}$$

$$\leq C \left[ \int_{\mathbb{R}^d} \int_{|y'| > 1} \int_0^1 |w(y + sy')||y + sy'|^{\beta-1} \frac{dy'ds}{|y'|^{d+\alpha-1}} \right]$$

$$+ \int_{\mathbb{R}^d} \int_{|y'| > 1} \int_0^1 |w(y + sy')||y'|^{\beta-1} \frac{dy'ds}{|y'|^{d+\alpha-1}}$$

and similarly,

$$\int_{\mathbb{R}^d} \int_{|y'| \leq 1} \int_0^1 \int_0^1 |\nabla w(y + s\tau y')||y|^{\beta-1} \frac{dsd\tau dy'dy}{|y'|^{d+\alpha-2}}$$

$$\leq C \left[ \int_{\mathbb{R}^d} \int_{|y'| \leq 1} \int_0^1 \int_0^1 |\nabla w(y + s\tau y')||y + s\tau y'|^{\beta-1} \frac{dsd\tau dy'dy}{|y'|^{d+\alpha-2}} \right]$$

$$+ \int_{\mathbb{R}^d} \int_{|y'| \leq 1} \int_0^1 \int_0^1 |\nabla w(y + s\tau y')||y'|^{\beta-1} \frac{dsd\tau dy'dy}{|y'|^{d+\alpha-2}}$$

are finite, part (iii) follows. □

In the following some estimates for $A^\varepsilon f^\varepsilon$ and $B^\varepsilon f^\varepsilon$ are proved.

Lemma 23. Let $\beta < \alpha, \beta \neq 1, \alpha \in (0, 2), \text{and } \varepsilon \in (0, 1)$. Then
Proof. (i) For $\beta \in (1, 2)$, by (40),
\[
f^\varepsilon(x) - f(x) = \int [f(x - y) - f(x)] w^\varepsilon(y) dy
\]
\[
= \int [f(x + y) - f(x) - (\nabla f(x), y)] w^\varepsilon(y) dy
\]
\[
= \int \int^1_0 (\nabla f(x + sy) - \nabla f(x), y) dsw^\varepsilon(y) dy
\]
and $|f^\varepsilon(x) - f(x)| \leq C|\nabla f|_{\beta-1} \int |y|^{1+\beta-1} w^\varepsilon(y) dy \leq C|f|_{\beta^2}$.

For $\beta \in (0, 1]$,
\[
f^\varepsilon(x) - f(x) = \int [f(x - y) - f(x)] w^\varepsilon(y) dy = \int [f(x + y) - f(x)] w^\varepsilon(y) dy
\]
and $f^\varepsilon(x) - f(x) = \frac{1}{2} \int [f(x + y) + f(x - y) - 2f(x)] w^\varepsilon(y) dy$.

Hence, for $\beta \in (0, 1]$, $|f^\varepsilon(x) - f(x)| \leq C|f|_{\beta^2}$.

(ii) For $z, x \in \mathbb{R}^d$, by changing the variable of integration with $\bar{y} = \frac{y}{\varepsilon}$ and using (8) for $\alpha = 1$, it yields that for $\chi^{(\alpha)}(y) = 1_{\{|y| \leq 1\}} 1_{\{\alpha = 1\}} + 1_{\{\alpha \in (1, 2)\}}$,
\[
A^{(\alpha)}_z w^\varepsilon(x) = 1_{\{\alpha = 1\}}(a_1(z), \nabla w^\varepsilon(x))
\]
\[
+ \int \left[w^\varepsilon(x + y) - w^\varepsilon(x) - \chi^{(\alpha)}(y)(\nabla w^\varepsilon(x), y)\right] m^{(\alpha)}(z, y) \frac{dy}{|y|^{d+\alpha}}
\]
\[
= \varepsilon^{-\alpha} \varepsilon^{-d} \left(A^{(\alpha)}_z w\right)(\frac{x}{\varepsilon}), \quad (48)
\]
It then follows from Lemma (22.1), the Fubini theorem, (48), and changing the variable of integration with $\bar{y} = \frac{y}{\varepsilon}$ as well, that
\[
A^{(\alpha)}_z f^\varepsilon(x) = \int_{\mathbb{R}^d} \varepsilon^{-\alpha} \varepsilon^{-d} \left(A^{(\alpha)}_z w\right)(\frac{x - y}{\varepsilon}) f(y) dy \quad (49)
\]
\[
= \int_{\mathbb{R}^d} \varepsilon^{-\alpha} \varepsilon^{-d} \left(A^{(\alpha)}_z w\right)(\frac{y}{\varepsilon}) f(x - y) dy
\]
\[
= \int \varepsilon^{-\alpha} \left(A^{(\alpha)}_z w\right)(y) f(x - \varepsilon y) dy, x, z \in \mathbb{R}^d.
\]

By Lemma (22(i)) and the Fubini theorem, $\int_{\mathbb{R}^d} A^{(\alpha)}_z w(y) dy = 0$. Hence, if $\beta \in (0, 1], \beta < \alpha$, then
\[
A^{(\alpha)}_z f^\varepsilon(x) = \int \varepsilon^{-\alpha} \left(A^{(\alpha)}_z w\right)(y) f(x - \varepsilon y) dy
\]
\[
= \int \varepsilon^{-\alpha} \left(A^{(\alpha)}_z w\right)(y) [f(x - \varepsilon y) - f(x)] dy
\]
and $|A_2^{(\alpha)}f^\varepsilon(x)| \leq C\varepsilon^{-\alpha + \beta} |f|_\beta \int_{\mathbb{R}^d} |(A_2^{(\alpha)}w)(y)| |y|^{\beta} dy $.

By Lemma 22(ii), (42) follows in this case.

Assume $1 < \beta < \alpha < 2$. By Theorem 2.27 in [4], differentiation and integration can be switched:

$$A_2^{(\alpha)}w(y) = \int \left[ w(y + y') - w(y) - \langle \nabla w(y), y' \rangle \right] m^{(\alpha)}(z, y') \frac{dy'}{|y'|^{d+\alpha}}$$

$$= \int \int_0^1 \langle \nabla_y w(y + s y') - \nabla_y w(y), y' \rangle ds m^{(\alpha)}(z, y') \frac{dy'}{|y'|^{d+\alpha}}$$

$$= \sum_{i=1}^d \frac{\partial}{\partial y_i} \int \int_0^1 \left[ w(y + s y') - w(y) \right] y'_i ds m^{(\alpha)}(z, y') \frac{dy'}{|y'|^{d+\alpha}}.$$

Integrating by parts, it follows that

$$A_2^{(\alpha)}f^\varepsilon(x) = \int \varepsilon^{-\alpha} A_2^{(\alpha)}w(y) f(x - \varepsilon y) dy$$

$$= \varepsilon^{-\alpha + 1} \int \int_0^1 \left[ w(y + s y') - w(y) \right] \times (\nabla f(x - \varepsilon y), y') m^{(\alpha)}(z, y') \frac{dsdy'y'}{|y'|^{d+\alpha}}, x \in \mathbb{R}^d.$$

Since $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |w(y + s y') - w(y)| \frac{dsdy'y'}{|y'|^{d+\alpha}} < \infty$, the Fubini theorem applies and $\int[w(y + sy') - w(y)]dy = 0$. Hence, (50) can be rewritten as

$$A_2^{(\alpha)}f^\varepsilon(x) = \varepsilon^{-\alpha + 1} \int \int_{\mathbb{R}^d} \int_0^1 \left[ w(y + s y') - w(y) \right] \times (\nabla f(x - \varepsilon y) - \nabla f(x), y') m^{(\alpha)}(z, y') \frac{dsdy'y'}{|y'|^{d+\alpha}}, x, z \in \mathbb{R}^d.$$

Thus,

$$|A_2^{(\alpha)}f^\varepsilon(x)| \leq C\varepsilon^{-\alpha + \beta - 1} |\nabla f|_{\beta - 1} \int \int_0^1 |w(y + s y') - w(y)| |y|^{\beta - 1} \frac{dydy'}{|y'|^{d+\alpha - 1}}$$

$$\leq C\varepsilon^{-\alpha + \beta} |f|_{\beta}, x, z \in \mathbb{R}^d,$$

and according to Lemma 22(iii), (42) is proved. By taking $m^{(\alpha)} = 1$, (43) is obtained. Lastly, the case of $\alpha \in (1, 2), \beta = 1$ is solved by interpolation. Fix
\( \alpha \in (1, 2), z \in \mathbb{R}^d \). Let \( B \) be the Banach space of continuous bounded functions on \( \mathbb{R}^d \) with supremum norm. Consider the operator \( T(f) = A_\varepsilon^{(\alpha)} f^\varepsilon(x) \). \( T : C^{1+\frac{\alpha}{2}}(\mathbb{R}^d) \to B \) is proved to be bounded:

\[
|T(f)| = \sup_{x} |A_\varepsilon^{(\alpha)} f^\varepsilon(x)| \leq C \varepsilon^{-\alpha+\frac{1}{2}} |f|_{\frac{3}{2+\alpha}} = C \varepsilon^{\frac{3}{2}(1-\alpha)} |f|_{\frac{3}{2+\alpha}}, f \in C^{\frac{3}{2+\alpha}}(\mathbb{R}^d)
\]

Therefore, by interpolation, \( T : C^1(\mathbb{R}^d) \to B \) is bounded and

\[
|T(f)| = \sup_{x} |A_\varepsilon^{(\alpha)} f^\varepsilon(x)| \leq C \varepsilon^{-\alpha+1} |f|_1, f \in C^1(\mathbb{R}^d).
\]

(iii) If \( \beta < 1 \), by changing the variable of integration,

\[
\partial_k f^\varepsilon(x) = \varepsilon^{-1} \int_{\mathbb{R}^d} \varepsilon^{-d} \partial_k w(\frac{x-y}{\varepsilon}) f(y) dy
= \varepsilon^{-1} \int_{\mathbb{R}^d} \varepsilon^{-d} \partial_k w(\frac{y}{\varepsilon}) f(x-y) dy
= \varepsilon^{-1} \int_{\mathbb{R}^d} \partial_k w(y) [f(x-\varepsilon y) - f(x)] dy.
\]

If \( \beta = 1 \), then

\[
f^\varepsilon(x + h) + f^\varepsilon(x - h) - 2f^\varepsilon(x)
= \frac{1}{2} \int w_\varepsilon(y) [f(x-y+h) + f(x-y-h) - 2f(x-y)] dy
\]

and \( |f^\varepsilon|_1 \leq |f|_1 \). Also, since \( \partial_{kl}^2 w(y) = \partial_{kl}^2 w(-y), k, l = 1, \ldots, d, y \in \mathbb{R}^d \),

\[
\partial_{kl}^2 f^\varepsilon(x) = \varepsilon^{-2} \int_{\mathbb{R}^d} \varepsilon^{-d} \partial_{kl}^2 w(\frac{x-y}{\varepsilon}) f(y) dy
= \varepsilon^{-2} \int_{\mathbb{R}^d} \varepsilon^{-d} \partial_{kl}^2 w(\frac{y}{\varepsilon}) f(x-y) dy
= \varepsilon^{-2} \int_{\mathbb{R}^d} \partial_{kl}^2 w(y) [f(x-\varepsilon y) - f(x)] dy
= \frac{1}{2} \varepsilon^{-2} \int_{\mathbb{R}^d} \partial_{kl}^2 w(y) [f(x+\varepsilon y) + f(x-\varepsilon y) - 2f(x)] dy.
\]

Thus

\[
|\partial_k f^\varepsilon(x)| \leq C \varepsilon^{-1+\beta} |f|_\beta \text{ if } \beta \in (0, 1),
|\partial_{kl}^2 f^\varepsilon(x)| \leq C \varepsilon^{-2+\beta} |f|_\beta \text{ if } \beta \in (0, 1),
\]

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By Theorem 6.4.5 in [2],
\[ T^\beta \text{ and (47) follows.} \]
for all \( x \in \mathbb{R}^d \). Similarly, if \( 1 < \beta < 2 \),
\[ \partial_k f^\varepsilon(x) = \int \varepsilon^{-d} w\left(\frac{y}{\varepsilon}\right) \partial_k f(x - y)dy = \int \varepsilon^{-d} w\left(\frac{x - y}{\varepsilon}\right) \partial_k f(y)dy \]
and
\[ \partial^2_{kl} f^\varepsilon(x) = \varepsilon^{-1} \int \partial_k w\left(\frac{y}{\varepsilon}\right) \partial_l f(x - y)dy \]
\[ = \varepsilon^{-1} \int \partial_k w(y)\partial_l f(x - \varepsilon y) - \partial_k f(x)dy. \]

Hence, \( |\partial^2_{kl} f^\varepsilon(x)| \leq C\varepsilon^{-1-\beta}|f|_\beta \).

To obtain (46), (44) and the interpolation theorem are applied. Let \( \beta \in (0,1) \). Consider an operator on \( C^\beta \) defined by \( T^\beta(f) = f^\varepsilon \). According to [2], \( T^\beta : C^\beta(\mathbb{R}^d) \rightarrow C^k(\mathbb{R}^d), k = 1,2 \) is bounded,
\[ |T^\varepsilon(f)|_k \leq C\varepsilon^{-k+\beta}|f|_\beta, k = 1,2, f \in C^\beta(\mathbb{R}^d). \]

By Theorem 6.4.5 in [2], \( T^\alpha : C^\alpha(\mathbb{R}^d) \rightarrow C^\beta(\mathbb{R}^d) \) is bounded and
\[ |T^\varepsilon(f)|_\alpha \leq C\varepsilon^{-(1+\beta)(2-\alpha)}\varepsilon^{-(2+\beta)(\alpha-1)}|f|_\beta = C\varepsilon^{-\alpha+\beta}|f|_\beta, f \in C^\beta(\mathbb{R}^d). \]
If \( \beta \in (1,\alpha) \), then \( \partial^{\alpha-1}\nabla f^\varepsilon = \partial^{\alpha-1}(\nabla f)^\varepsilon \) and by (43),
\[ |\partial^{\alpha-1}\nabla f^\varepsilon(x)| = |\partial^{\alpha-1}(\nabla f)^\varepsilon(x)| \leq C\varepsilon^{1-\alpha+(\beta-1)}|\nabla f|_{\beta-1} \leq C\varepsilon^{-\alpha+\beta}|f|_\beta, \]
and (47) follows. ■

**Remark 24.** If \( \Delta_h f = f(x + h) - f(x), \tau_h f(x) = f(x + h) \), then
\[ \Delta_h (fg) = \tau_h f \Delta_h g + g \Delta_h f, \]
\[ \Delta_h^2(fg) = \tau_{2h} f \Delta_h^2 g + \Delta_h g \Delta_h f + \tau_h g \Delta_h^2 f + \Delta_h f \Delta_h g. \]

**Corollary 25.** Assume \( a_\alpha(x) \) and \( \int_{\mathbb{R}^d} (|y|^\alpha + 1) \rho(x) \nu \alpha(x)y\nu \alpha(x)(dy), x \in \mathbb{R}^d \)
are bounded, \( \varepsilon \in (0,1) \). Then there exists a constant \( C \) such that for all \( z, x \in \mathbb{R}^d, f \in C^\beta(\mathbb{R}^d), |B^\alpha_z f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta}|f|_\beta. \)
Proof. If $\beta < \alpha < 1$, by Lemma 17

$$f^\varepsilon(x + y) - f^\varepsilon(x) = \int k^{(a)}(y, y') \partial^\alpha f^\varepsilon(x - y') dy',$$

and by Lemma 23, \[ |f^\varepsilon(x + y) - f^\varepsilon(x)| \leq C \varepsilon^{-\alpha + \beta} |y|^\alpha \wedge 1, \quad x, y \in \mathbb{R}^d. \] (51)

If $\beta = \alpha = 1$, by Lemma 23(ii), \[ |f^\varepsilon(x + y) - f^\varepsilon(x)| \leq C \sup_x |f(x)| + |\nabla f^\varepsilon(x)| (|y| \wedge 1) \leq C \varepsilon^{-1 + \beta} |y|^\alpha \wedge 1, \quad x, y \in \mathbb{R}^d. \] (52)

Assume $\alpha \in (1, 2)$, then for $x, y, y' \in \mathbb{R}^d, |y| \leq 1$,

$$f^\varepsilon(x + y) - f^\varepsilon(x) - (\nabla f^\varepsilon(x, y) = \int_0^1 (\nabla f^\varepsilon(x + sy) - \nabla f^\varepsilon(x, y)ds. \] (53)

If $\beta < \alpha$, by Lemmas 17, 23, \[ |\nabla f^\varepsilon(x + y') - \nabla f^\varepsilon(x)| \leq C \sup_x |\partial^\alpha \nabla f^\varepsilon(x)| |y'|^{\alpha - 1} \leq C \varepsilon^{-\alpha + \beta} |y'|^{\alpha - 1}. \] (54)

If $\beta > 1$, clearly $|\nabla f^\varepsilon(x + y') - \nabla f^\varepsilon(x)| \leq C |f|^\alpha |y'|^{\alpha - 1}$.

If $\beta \in (0, 1], \alpha \in (1, 2)$, by Lemma 23, \[ |\nabla f^\varepsilon(x + y') - \nabla f^\varepsilon(x)| \leq C \varepsilon^{-\alpha + \beta} |y'|^{\alpha - 1} |f|^\beta. \] (55)

By applying (54) to (53), it follows that for $x, y \in \mathbb{R}^d, |y| \leq 1$,

$$|f^\varepsilon(x + y) - f^\varepsilon(x) - (\nabla f^\varepsilon(x, y)| \leq C \varepsilon^{-\alpha + \beta} |y|^\alpha |f|^\beta.$$ (56)

Therefore, the statement follows by the assumptions and Lemma 23. \[\square\]

4.2. Proof of Lemma 21

If $\beta < \alpha$, define $f^\varepsilon$ by (41) for $\varepsilon \in (0, 1)$ and apply Ito’s formula (see Remark 9): for $s \in [0, T]$,

$$E[f^\varepsilon(Y_s) - f^\varepsilon(Y_{t_\varepsilon})|\tilde{\mathcal{F}}_{t_\varepsilon}] = E\left[ \int_{t_\varepsilon}^s (A^{(\alpha)}_{Y_{t_\varepsilon}} f^\varepsilon(Y_r) + B^{(\alpha)}_{Y_{t_\varepsilon}} f^\varepsilon(Y_r))dr|\tilde{\mathcal{F}}_{t_\varepsilon}\right].$$
Hence, by Lemma 23 and Corollary 25 for \( \varepsilon \in (0, 1) \),
\[
|\mathbb{E}[f(Y_s) - f(Y_{\tau_{\varepsilon}})]| \leq |\mathbb{E}[(f - f^\varepsilon)(Y_s) - (f - f^\varepsilon)(Y_{\tau_{\varepsilon}})]| \\
+ |\mathbb{E}[f^\varepsilon(Y_s) - f^\varepsilon(Y_{\tau_{\varepsilon}})]| \\
\leq C(\varepsilon^\beta + \delta \varepsilon^{-\alpha + \beta})|f|_\beta,
\]
with a constant \( C \) independent of \( \varepsilon, f \). Minimizing \( \varepsilon^\beta + \delta \varepsilon^{-\alpha + \beta} \) in \( \varepsilon \in (0, 1) \) gives
\[
|\mathbb{E}[f(Y_s) - f(Y_{\tau_{\varepsilon}})]| \leq C\delta |f|_\beta.
\]

If \( \beta > \alpha \), apply Itô’s formula directly (see Remark 9) and
\[
\mathbb{E}[f(Y_s) - f(Y_{\tau_{\varepsilon}})] = \mathbb{E}\left[\int_{\tau_{\varepsilon}}^s (A^{(\alpha)}_{Y_{\tau_{\varepsilon}}} f(Y_r) + B^{(\alpha)}_{Y_{\tau_{\varepsilon}}} f(Y_r))dr\right].
\]

Hence, by Corollary 16 and Lemma 23,
\[
|\mathbb{E}[f(Y_s) - f(Y_{\tau_{\varepsilon}})]| \leq C\delta |f|_\beta.
\]
The statement of Lemma 21 follows.

4.3. Proof of Theorem 2

Let \( v \in C^{\alpha + \beta}(H) \) be the unique solution to (18) with \( f = 0 \) (see Corollary 12). By Itô’s formula (see Remark 9, (15)) and (18),
\[
\mathbb{E}v(0, X_0) = \mathbb{E}v(T, X_T) = \mathbb{E}g(X_T).
\]  

By Lemma 18, Corollaries 16, 12, and Remark 20,
\[
|A_{z}^{(\alpha)}v(s, \cdot)|_\beta + |B_{z}^{(\alpha)}v(s, \cdot)|_\beta \leq C|v|_\alpha + C|g|_\alpha + \beta,
\]
\[
|\partial_t v(s, \cdot)|_\beta \leq C|g|_\alpha + \beta, s \in [0, T].
\]

Then, by Itô’s formula (Remark 9 (15)) and Corollary 12 with (56) and (57), it follows that
\[
\mathbb{E}g(Y_T) - \mathbb{E}g(X_T) = \mathbb{E}v(T, Y_T) - \mathbb{E}v(0, X_0) \\
= \mathbb{E}\left[\int_0^T [\partial_t v(s, Y_s) + A^{(\alpha)}_{Y_{\tau_{\varepsilon}}} v(s, Y_s) + B^{(\alpha)}_{Y_{\tau_{\varepsilon}}} v(s, Y_s)]ds\right] \\
= \mathbb{E}\left[\int_0^T \{[\partial_t v(s, Y_s) - \partial_t v(s, Y_{\tau_{\varepsilon}})] \\
+ [A^{(\alpha)}_{Y_{\tau_{\varepsilon}}} v(s, Y_s) - A^{(\alpha)}_{Y_{\tau_{\varepsilon}}} v(s, Y_{\tau_{\varepsilon}})] \\
+ [B^{(\alpha)}_{Y_{\tau_{\varepsilon}}} v(s, Y_s) - B^{(\alpha)}_{Y_{\tau_{\varepsilon}}} v(s, Y_{\tau_{\varepsilon}})]\}ds\right].
\]
Hence, by (57) and Lemma 21 there exists a constant $C$ independent of $g$ such that
\[ |E_g(Y_T) - E_g(X_T)| \leq C\delta^{\alpha,\beta}|g|_{\alpha+\beta}. \]
The statement of Theorem 2 follows.

5. Appendix

By Lemma 26, an Euler approximation defined by (3) always exists.

Lemma 26. Given $X_t$ defined by (7), there is an approximation (3).

Proof. According to Lemma 14.50 in [3], there is a measurable function $l^{(\alpha)} : \mathbb{R}^d \times \mathbb{R}_0 \to \mathbb{R}^d$ such that $\pi^{(\alpha)}(x,dy) = \int_{\mathbb{R}_0} 1_{dy} (l^{(\alpha)}(x,z)) \frac{dz}{z^2}, x \in \mathbb{R}^d$. Given $X_t$ satisfying (11), let $p(dt,dz)$ be an independent Poisson point measure on $[0,T] \times \mathbb{R}_0$ with compensator $\frac{dzdt}{z^2}$, the weak Euler approximation for $t \in [0,T]$ is then defined as

\[
Y_t = X_0 + \int_0^t \int l^{(\alpha)}(Y_{\tau_is},z)p(ds,dz), \text{ if } \alpha \in (0,1),
\]

\[
Y_t = X_0 + \int_0^t a_\alpha(Y_{\tau_is})ds + 1_{\{\alpha=2\}} \int_0^t b(Y_{\tau_is})dW_s
\]

\[
+ \int_0^t \int_{|l^{(\alpha)}(Y_{\tau_is},z)|>1} l^{(\alpha)}(Y_{\tau_is},z)p(ds,dz)
\]

\[
+ \int_0^t \int_{|l^{(\alpha)}(Y_{\tau_is},z)|\leq1} l^{(\alpha)}(Y_{\tau_is},z)(p(ds,dz) - \frac{dzds}{z^2}), \text{ if } \alpha \in [1,2].
\]

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