ON THE BROWN-PETE RSON COHOMOLOGY OF $BPU_n$ IN LOWER DIMENSIONS AND THE THOM MAP

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ABSTRACT. For an odd prime $p$, we determined the Brown-Peterson cohomology of $BPU_n$ in dimensions $-(2p - 2) \leq i \leq 2p + 2$, where $BPU_n$ is the classifying space of the projective unitary group $PU_n$. We construct a family of $p$-torsion classes $\eta_{p,k} \in BP^{2p^k + 2}(BPU_n)$ for $p|n$ and $k \geq 0$ and identify their images under the Thom map with well understood cohomology classes in $H^*(BPU_n; \mathbb{Z}(p))$.

1. INTRODUCTION

Let $p$ be an odd prime number, and let BP be the corresponding Brown-Peterson spectrum. The Brown-Peterson cohomology $BP^*(BG)$ of the classifying space of a compact Lie group or a finite group $G$ is the subject of various works such as Kameko and Yagita [8], Kono and Yagita [11], Leary and Yagita [12], and Yan [21].

One case that $BP^*(BG)$ is particularly interesting is when $G$ is homotopy equivalent to a complex algebraic group via a group homomorphism. In this case the Chow ring of $BG$, $CH^*(BG)$ is defined by Totaro [16], and one has the cycle class map

$$\text{cl} : CH^*(BG) \to H^{even}(BG; \mathbb{Z})$$

which is a ring homomorphism from the Chow ring to the subring of $H^*(BG)$ of even dimensional classes. Although for complex algebraic varieties, Chow rings are in general much more complicated than ordinary cohomology, it is shown in many cases that $CH^*(BG)$ is simpler than $H^*(BG; \mathbb{Z})$. On the other hand, Totaro [16] shows that the cycle class map (1.1) factors as

$$CH^*(BG) \xrightarrow{\tilde{\text{cl}}} MU^{even}(BG) \otimes_{MU^*} \mathbb{Z} \xrightarrow{T} H^{even}(BG; \mathbb{Z})$$

where MU denotes the complex cobordism theory, and the second map $T$ is the Thom map. The first map $\tilde{\text{cl}}$ is called the refined cycle class map. Therefore, the BP theory, being a $p$-local approximation of the MU theory, plays the role of a bridge between the Chow ring and the ordinary cohomology of $BG$. Indeed, it is an interesting problem to find out for which $G$ is the refined cycle class map

$$\tilde{\text{cl}} : CH^*(BG) \xrightarrow{\tilde{\text{cl}}} MU^*(BG) \otimes_{MU^*} \mathbb{Z}$$

an isomorphism. For this to hold, it is necessary that $BP^*(BG)$ concentrates in even dimensions. This property is studied for various $G$ by Kono and Yagita [11].

Key words and phrases. the Brown-Peterson cohomology, the classifying spaces of the projective unitary groups.

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In this paper we focus on $BP^*(BU_n)$, where $PU_n$ is the $n$th projective unitary group. The algebraic invariants of $BU_n$ are much less known compared to $BG$ for most of the other compact Lie groups $G$. The Chow ring of $BP_U$ is determined, up to one relation, by Vezzosi [18]. The additive structure of $CH^*(BU_3)$ is independently determined by Kameko and Yagita [8]. Vistoli [19] improved Vezzosi’s method and determined the additive structures of the Chow ring and ordinary cohomology with integral coefficients of $BU_p$ for an odd prime $p$ and much of the ring structures, and completed Vezzosi’s study of the Chow ring of $BU_3$. The ordinary mod $p$ cohomology and Brown-Peterson cohomology of $BU_p$ are studied by Kameko and Yagita [8]. Kono and Yagita [14], and Vavpetič and Viruel [17]. The mod 2 ordinary cohomology ring of $BU_n$ for $n = 2$ (mod 4) is determined by Kono and Mimura [10] and Toda [15].

For a general positive integer $n$, the cohomology groups $H^k(BU_n;\mathbb{Z})$ for $k \leq 3$ are trivial and are briefly discussed in Section 3. The group $H^k(BU_n;\mathbb{Z})$ is determined by Woodward [20] and $H^k(BU_n;\mathbb{Z})$ by Antieau and Williams [2]. The ring structure of $H^*(BU_n;\mathbb{Z})$ in dimensions less than or equal to 10 is determined by the author [5]. The author [6] also studies some $p$-torsion classes of $CH^*(BU_n)$ for $n$ with $p$-adic valuation 1, i.e., $p|n$ but $p^2 \nmid n$. To the author’s best knowledge, $BP^*(BU_n)$ for $n$ not a prime number has not been studied in any published work before.

Before stating the main conclusions of this paper, we fix some notations. For a spectrum $A$, we denote by $A_+$ its homotopy groups, or the group of coefficients of the homology theory $A$, considered as a graded abelian group. Denote by $A^*$ the group of coefficients of the cohomology theory associated to $A$. Then $A^*$ and $A_+$ are isomorphic, but the gradings are opposite to each other. For instance, we have $BP_+ \cong \mathbb{Z}_p[v_1, v_2 \cdots]$ where $\dim v_k = 2p^k - 2$ and $BP^* \cong \mathbb{Z}_p[v_1, v_2 \cdots]$ where $\dim v_k = -(2p^k - 2)$.

**Theorem 1.1.** Assume $p|n$. There is a homomorphism of $\mathbb{Z}_p$-algebras

$$Z_p(v_1) \otimes R^* \otimes \mathbb{Z}_p[\eta_{p, 0}]/(p\eta_{p, 0}) \to BP^*(BU_n) \quad (1.3)$$

which is an isomorphism in dimensions $-(2p - 2) \leq i \leq 2p + 2$. Here we have $\dim \eta_{p, 0} = 2p + 2$, $\dim v_1 = -(2p - 2)$ and $R^*$ a graded $\mathbb{Z}_p$-algebra such that

(a) the Thom map $T : BP^*(BU_n) \to H^*(BU_n; \mathbb{Z}_p)$ takes $\eta_{p, 0}$ to $y_{p, 0}$, and

(b) $R$ is free as a graded $\mathbb{Z}_p$-module, and

(c) we have an isomorphism

$$R^* \otimes \mathbb{Q} \cong \mathbb{Q}[e_2, e_3, \cdots, e_p + 1]$$

with $\dim e_i = 2i$, and

(d) the following homomorphism induced from (1.3)

$$R^* \otimes \mathbb{Z}_p[\eta_{p, 0}]/(p\eta_{p, 0}) \to BP^*(BU_n) \otimes_{BP^*} \mathbb{Z}_p \quad (1.3)$$

is an isomorphism in dimensions $0 \leq i \leq 2p + 2$.

In particular, in dimensions $-(2p - 2) \leq i \leq 2p + 2$, $BP^*(BU_n)$ concentrates in even dimensions.

For the next conclusion, we note that there are $p$-torsion classes

$$y_{p, k} \in H^{2p^k+1+2}(BU_n; \mathbb{Z}_p)$$

which are studied in [5] and discussed in more details in Section 3.
Theorem 1.2. For $k \geq 0$ and $p \mid n$, there are $p$-torsion classes

$$\eta_{p,k} \in \text{BP}^{2p^{k+1}+2}(\text{BP} U_n)$$

such that $T(\eta_{p,k}) = y_{p,k}$, where $T$ is the Thom map.

The class $\eta_{p,0}$ is already given in Theorem 1.1.

Remark 1.2.1. Localizing at $p$ the homotopy fiber sequence $B\mathbb{Z}_n \to \text{BSU}_n \to \text{BP} U_n$, we obtain a $p$-local homotopy equivalence $\text{BSU}_n \simeq \text{BP} U_n$ in the case $p \nmid n$. Therefore this case is not very interesting.

Remark 1.2.2. In [6], the author constructs $p$-torsion classes $\rho_{p,k}$, $k \geq 0$ in the Chow ring of $\text{BP} U_n$ when the $p$-adic valuation of $n$ is 1, satisfying $\text{cl}(\rho_{p,k}) = y_{p,k}$. This suggests that the classes $\rho_{p,k}$ should exist for $n$ with $p$-adic valuation greater than 1 and that the refined cycle class map ([16]) should take $\rho_{p,0}$ to $\eta_{p,0}$.

This paper is organized as follows. In Section 2 we present some preliminaries on the the cohomology theories $P(n)$ defined by Johnson and Wilson [7]. In Section 3 we give some results on the ordinary cohomology of $\text{BP} U_n$, most of which are proved in [5] and [6], and presented in slightly different forms. In Section 4 and Section 5, we prove Theorem 1.1 and Theorem 1.2, respectively.

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2. Preliminaries on the cohomology theories $P(n)$

In [7], Johnson and Wilson consider variations of the BP cohomology theory $P(n)$, $n \geq 0$ defined inductively in terms of the following cofiber sequences of spectra

$$P(n) \xrightarrow{x_n} P(n) \to P(n+1)$$

where $P(0) = \text{BP}$ and $v_0 = p$. In particular, we have the cofiber sequence

$$\text{BP} \xrightarrow{x_p} \text{BP} \to P(1).$$

We immediately deduce

$$P(n)^* = \mathbb{Z}_p[v_n, \cdots].$$

In particular, in the case $n = 0$ and $n = 1$, we have

$$P(0)^* = \text{BP}^* = \mathbb{Z}_p(v_1, v_2, \cdots, v_n, \cdots)$$

and

$$P(1)^* = \mathbb{Z}_p[v_1, \cdots].$$

We consider the Atiyah-Hirzebruch spectral sequence of the cohomology theories $P(1)$ and BP. Brown and Peterson [3] studied the Postnikov tower of the spectrum BP, of which the 0th level is the Eilenberg-Mac Lane spectrum $H\mathbb{Z}_{(p)}$. To obtain the 1st level, we consider the map

$$H\mathbb{Z}_{(p)} \xrightarrow{\bigvee_{i \geq 1} \delta_{p,i}} \bigvee_{i \geq 1} \Sigma^{2p^{i-1}} H\mathbb{Z}_{(p)}.$$
where $\bigvee_{i \geq 1} \Sigma^{2p^i-1}H\mathbb{Z}_{(p)}$ is the wedge sum of the spectra $\Sigma^{2p^i-1}H\mathbb{Z}_{(p)}$. By convention, $\mathcal{P}^k$ denotes the $k$th Steenrod reduced power operation and $\delta$ denotes the connecting homomorphism associated to the short exact sequence $\mathbb{Z}_{(p)} \xrightarrow{x_p} \mathbb{Z}_{(p)} \to \mathbb{Z}_p$. The 1st level of the Postnikov tower is the homotopy fiber of this map. Therefore we have the following lemma that gives the first nontrivial differential of the Atiyah-Hirzebruch spectral sequence of BP in the universal case:

**Lemma 2.1.** The first nontrivial differential in the Atiyah-Hirzebruch spectral sequence of $BP^*(H\mathbb{Z}_{(p)})$ is $d_{2p-1} = v_1 \otimes \delta \mathcal{P}^1$.

The Postnikov tower of $P(1)$ is constructed in a similar way, with the 0th level $H\mathbb{Z}_p$ and the 1st level the homotopy fiber of the map

$$H\mathbb{Z}_p \xrightarrow{\bigvee_{i \geq 1} \beta \mathcal{P}^i} \bigvee_{i \geq 1} \Sigma^{2p^i-1}H\mathbb{Z}_p$$

where $\beta$ is the connecting homomorphism associated to the short exact sequence $\mathbb{Z}_p \xrightarrow{x_p} \mathbb{Z}_p \to \mathbb{Z}_p$. Hence we have the following

**Lemma 2.2.** The first nontrivial differential in the Atiyah-Hirzebruch spectral sequence of $P(1)^*(H\mathbb{Z}_p)$ is $d_{2p-1} = v_1 \otimes \beta \mathcal{P}^1$. If $x$ is a class of the $E_2$ page which is the mod $p$ reduction of an integral cohomology class, we have $d_{2p-1}(x) = v_1 \otimes Q_1(x)$ where $Q_1$ is the Milnor’s operation defined in [13].

Finally we take notes on a comparison between the cohomology operations for BP theory and the mod $p$ ordinary cohomology theory due to Kane [9]. Let $E = (e_1, e_2, \cdots)$ be a sequence of non-negative integers with only finitely many nonzero terms, and let $\mathcal{P}^E$ be the operation defined by Milnor [13]. Then we have a BP cohomology operation $r^E_E \in BP^*BP$ and the following diagram (display (1.1) of [9]):

$$BP^*(X) \xrightarrow{r^E} BP^*(X) \xrightarrow{T} H^*(X; \mathbb{Z}_p)$$

$$H^*(X; \mathbb{Z}_p) \xrightarrow{c(\mathcal{P}^E)} H^*(X; \mathbb{Z}_p)$$

where $c : \mathcal{A}^* \to \mathcal{A}^*$ is the canonical anti-automorphism of the mod $p$ Steenrod algebra $\mathcal{A}$, and the vertical arrows labelled by $T$ are the reduced Thom isomorphisms, i.e., the composition of the Thom map and the mod $p$ reduction.

Since $r^E_E$ is a map of BP-modules, which in particular commutes with the multiplication $BP \xrightarrow{x_p} BP$, it reduces to an operation $r^E_P \in P(1)^*P(1)$, and yields a commutative diagram

$$P(1)^*(X) \xrightarrow{r^E_P} P(1)^*(X) \xrightarrow{T} H^*(X; \mathbb{Z}_p)$$

$$H^*(X; \mathbb{Z}_p) \xrightarrow{c(\mathcal{P}^E)} H^*(X; \mathbb{Z}_p)$$

### 3. ON THE ORDINARY COHOMOLOGY OF $BP\mathbb{U}_n$

In this section we consider the ordinary cohomology of $BP\mathbb{U}_n$. For the most part of this section, we reformulate some results in [5]. By definition we have a short
Then we have
\[ x \text{ defines a generator} \]
which defines the Lie group \( PU_n \).

Consider the short exact sequence of Lie groups
\[ 1 \to S^1 \to U_n \to PU_n \to 1 \]
which defines the Lie group \( PU_n \). Here \( S^1 \) is the complex unit circle, and \( U_n \) is the \( n \)th unitary group. Taking classifying spaces, we obtain a homotopy fiber sequence
\[ BS^1 \to BU_n \to BPU_n. \]

Notice that \( BS^1 \) is of the homotopy type of the Eilenberg-Mac Lane space \( K(\mathbb{Z}, 2) \). Delooping \( BS^1 \), we obtain another homotopy fiber sequence
\[ (3.1) \quad BU_n \to BPU_n \xrightarrow{\chi} K(\mathbb{Z}, 3). \]

Here the map
\[ (3.2) \quad \chi : BPU_n \to K(\mathbb{Z}, 3) \]
defines a generator \( x_1 \) of \( H^3(BPU_n; \mathbb{Z}) \cong \mathbb{Z}_n \) (or \( H^3(BPU_n; \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{e p} \), where \( e \) is the \( p \)adic valuation of \( n \)), which we call the canonical Brauer class of \( BPU_n \).

The homotopy fiber sequence \( \chi \) induces a Serre spectral sequence \( ^U E^{s,t}_2 \) converging to \( H^*(BPU_n; \mathbb{Z}_{(p)}) \):
\[ ^U E^{s,t}_2 \cong H^s(K(\mathbb{Z}, 3); H^t(BU_n; \mathbb{Z}_{(p)})) \Rightarrow H^{s+t}(BPU_n; \mathbb{Z}_{(p)}). \]

Then an element in \( ^U E^{s,t}_2 \) is the sum of elements of the form \( \vartheta \otimes \xi \) where \( \vartheta \in H^s(BPU_n; \mathbb{Z}_{(p)}) \) and \( \xi \in H^t(K(\mathbb{Z}, 3); \mathbb{Z}_{(p)}) \). The cohomology of \( BU_n \) is a polynomial ring in the universal Chern classes:
\[ H^*(BU_n; \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[c_1, c_2, \cdots]. \]

As in [5], we consider the linear operator
\[ \nabla : H^*(BU_n; \mathbb{Z}_{(p)}) \to H^{*+2}(BU_n; \mathbb{Z}_{(p)}) \]
determined by \( \nabla(c_i) = (n - i + 1)c_{i-1} \) and the Leibniz formula \( \nabla(ab) = a\nabla(b) + \nabla(a)b \). The differential \( ^U d_3 \) is completely determined in [5]:

**Lemma 3.1** (Corollary 3.10, [5]). Let \( \vartheta = \vartheta[c_1, \cdots, c_n] \) be a class in \( H^*(BPU_n; \mathbb{Z}) \), and let \( \xi \in H^*(K(\mathbb{Z}, 3); \mathbb{Z}) \). Let \( x_1 \in H^3(K(\mathbb{Z}, 3); \mathbb{Z}) \) be the canonical Brauer class.

Then we have
\[ ^U d_3(\vartheta \otimes \xi) = \nabla(\vartheta) \otimes x_1 \xi. \]

Let \( TU_n \) and \( TPU_n \) be the respective standard maximal tori of \( U_n \) and \( PU_n \). Then we have
\[ H^*(TU_n; \mathbb{Z}) \cong \mathbb{Z}[t_1, t_2, \cdots, t_n] \]
where each \( t_i \) is of dimension 2. It follows from an elementary calculation that the canonical projection \( TU_n \to TPU_n \) identifies \( H^*(BTU_n; \mathbb{Z}) \) as the subring of
$H^*(BTU_2;\mathbb{Z})$ generated by $t_i - t_j$ for all $i \neq j$. Let $W$ be the Weyl group of $TPU_n$. Then it is well known that the pullback
\[ H^*(BU_n;\mathbb{Z}) \rightarrow H^*(BTU_2;\mathbb{Z})^W \]
is an isomorphism, where $H^*(BTU_2;\mathbb{Z})^W$ is the invariant subring of the Weyl group action. Furthermore, an elementary calculation yields the following

**Proposition 3.2.** The pullback
\[ H^*(BU_n;\mathbb{Z}) \rightarrow H^*(BTU_2;\mathbb{Z})^W \]
sends $\text{Ker } \nabla$ isomorphically onto $H^*(BTPU_n;\mathbb{Z})^W$.

**Remark 3.3.** In general $H^*(BTU_2;\mathbb{Z})^W$ is not necessarily a polynomial ring. As shown by Vezzosi [18], there is a class $y_{12} \in H^{12}(BTU_2;\mathbb{Z})^W$ which is not a nontrivial product, but $3y_{12}$ is.

The cohomology of $K(\mathbb{Z}, 3)$, in principle, is determined in [4], where the homology of $K(\pi, n)$ for any finitely generated abelian group $\pi$ and any $n > 0$ is described in full. In [5], the author uses an alternative description of the cohomology of $K(\mathbb{Z}, 3)$, which is consistent with the notations in this paper. The following lemma is well known and can be easily deduced from Section 2 of [4].

**Lemma 3.4.** In dimensions $0 < i \leq 2p + 2$, we have
\[ H^i(K(\mathbb{Z}, 3);\mathbb{Z}(p)) \cong \begin{cases} \mathbb{Z}(p), & i = 3, \\ \mathbb{Z}, & i = 2p + 2, \\ 0, & 0 < i \leq 2p + 4, \ i \neq 3, \ 2p + 2. \end{cases} \]

Let $x_1 \in H^3(K(\mathbb{Z}, 3);\mathbb{Z}(p))$ be the canonical Brauer class. Then the group
\[ H^{2p+2}(K(\mathbb{Z}, 3);\mathbb{Z}(p)) \cong \mathbb{Z}_p \]
is generated by a class $y_{p,0} = \delta \mathcal{P}^1(x_1)$, where $\delta$ is the connecting homomorphism and $\mathcal{P}^1$ is the first Steenrod reduced power operation. The mod $p$ reduction of $y_{p,0}$, denoted by $\bar{y}_{p,0}$, is equal to $Q_1(x_1)$, where $Q_1$ is one of the Milnor’s operations defined in [13].

**Proposition 3.5.** In dimension $0 \leq i \leq 2p + 2$ we have
\[ (H^*(BTPU_n;\mathbb{Z})^W \otimes \mathbb{Z}(p)[x_1, y_{p,0}])(nx_1, py_{p,0}) \leq 2p + 2 \cong H^{2p+2}(BPU_n;\mathbb{Z}(p)). \]

**Proof.** It follows from Lemma 3.3 that in the spectral sequence $\text{U}E_{s,t}^*$, the only possibly nontrivial differentials from $\text{U}E_{s,t}^{0,1}$ for $t \leq 2p + 2$ are $\text{U}d_{0,2t}^{0,1}$. Therefore, by Lemma 3.4 we have $\text{U}E_{s,t}^{0,1} \cong \text{Ker } \nabla$, and by Proposition 3.2 we have $\text{U}E_{s,t}^{0,1} \cong H^t(BTPU_n;\mathbb{Z})^W$ for $t \leq 2p + 2$. The class $1 \otimes x_1$ being an $n$-torsion permanent cocycle and $1 \otimes y_{p,0}$ being a permanent cocycle follow from Theorem 1.1 and Theorem 1.2 of [5]. Therefore the groups $\text{U}E_{s,t}^s$ for $s + t \leq 2p + 2$ is determined, and we conclude.

In the cohomology ring $H^*(K(\mathbb{Z}, 3);\mathbb{Z}(p))$ we have $p$-torsion classes
\[ y_{p,k} = \delta \mathcal{P}^p \mathcal{P}^{p-1} \cdots \mathcal{P}^1(x_1), \ k \geq 0 \]
of dimension $2p^{k+1} + 2$, where $x_1 \in H^3(BPU_n;\mathbb{Z})$ is the canonical Brauer class. In [6] the author shows that these classes have nontrivial images under
\[ \chi : H^*(K(\mathbb{Z}, 3);\mathbb{Z}(p)) \rightarrow H^*(BPU_n;\mathbb{Z}(p)). \]
Theorem 3.6 (I) of Theorem 1.1, [6]). In $H^{2p^{k+1}+2}(BPUn;\mathbb{Z}(p))$, we have p-torsion classes $y_{p,k} \neq 0$ for all odd prime divisors $p$ of $n$ and $k \geq 0$.

As pointed out in Section 1, the canonical map $BSU_n \to BP_0$ is a rational homotopy equivalence. This means that in principal we do not need to worry about the non-torsion classes in $H^*(BPUn;\mathbb{Z}(p))$ or $BP^*(BPUn)$. We formulate this argument as the following

Proposition 3.7. We have

$$H^*(BPUn;\mathbb{Q}) \cong \mathbb{Q}[e_2, e_3, \cdots, e_n]$$

where $\dim e_i = 2i$, and

$$BP^*_\mathbb{Q}(BPUn) \cong H^*(BPUn;\mathbb{Q}) \otimes_{\mathbb{Q}} (BP^* \otimes \mathbb{Q}) \cong \mathbb{Q}[e_2, e_3, \cdots, e_n, v_1, v_2, \cdots],$$

where $BP^*_\mathbb{Q}$ denotes the localization of $BP$ with respect to $\mathbb{Q}$.

4. ON THE P(1) AND BP COHOMOLOGY OF BPUn

In this section we prove Theorem [11] by studying an Atiyah-Hirzebruch spectral sequence, and the Thom map for the Eilenberg-Mac Lane space $K(\mathbb{Z}, 3)$. Indeed, the Thom map for Eilenberg-Mac Lane spaces are studied in Tamanoi [14], and we make use of one of his main conclusions. In what follows, we denote by $A^*$ the mod $p$ Steenrod algebra, and we use the notations for the stable cohomology operations in Milnor [13].

Theorem 4.1 (Tamanoi, (I) of Theorem A, [14]). Let $p$ be a prime, and let $n \geq 1$. The image of the Thom map

$$T: BP^*(K(\mathbb{Z}, n + 2)) \to H^*(K(\mathbb{Z}, n + 2); \mathbb{Z}_p)$$

is an $A^*$-invariant polynomial subalgebra with infinitely many generators:

$$\text{Im} T = \mathbb{Z}_p[Q_{s_n}Q_{s_{n-1}} \cdots Q_{s_1}(\tau_{n+2}) | 0 < s_1 < \cdots < s_n],$$

where $\tau_{n+2} \in H^{n+2}(K(\mathbb{Z}, n + 2); \mathbb{Z}_p)$ is the fundamental class.

Consider the Atiyah-Hirzebruch spectral sequence $(E(1)^*, d_*)$ converging to $P(1)^*(BPUn)$:

$$E(1)^*_{2^t} = H^*(BPUn; P(1)^t) \Rightarrow P(1)^{s+t}(BPUn).$$

Lemma 4.2.

$$P(1)^3(BPUn) = 0.$$

Proof. Consider the Atiyah-Hirzebruch spectral sequence converging to $P(1)^*(BPUn)$ with

$$E(1)^*_{2^t} \cong H^*(BPUn; P(1)^t).$$

We use overhead bars to indicate mod $p$ reductions of classes in $H^*(BPUn; \mathbb{Z}(p))$. The only nontrivial entry $E(1)^*_{2^t}$ with $s + t = 3$ is $E(1)^{3,0}_2$, which is generated by $1 \otimes x_1$. It follows from Lemma 2.2 that in the spectral sequence $E(1)^*_*$, we have

$$d_{2p-1}(1 \otimes x_1) = v_1 \otimes Q_1(x_1) = v_1 \otimes y_{p,0} \neq 0,$$

and we conclude.

□

Lemma 4.3.

$$BP^3(BPUn) = 0.$$
Proof. Let $E_{s,t}^*$ be the Atiyah-Hirzebruch spectral sequence for $BP^*(BPU_n)$:
\[ E_{s,t}^2 \cong H^s(BPU_n; BP^t) \Rightarrow BP^{s+t}(BPU_n). \]
Consider the cofiber sequence (2.2)\[ \text{BP} \xrightarrow{x_p} \text{BP} \to P(1), \]
which induces a long exact sequence of $BP^*$-modules\[ (4.2) \]
\[ \cdots \to BP^i(BPU_n) \xrightarrow{x_p} BP^i(BPU_n) \to P(1)^i(BPU_n) \to BP^{i+1}(BPU_n) \to \cdots. \]
By Proposition 3.5, we have\[ P(1) \cong \mathbb{Z} \]
where $e$ is the $p$-adic valuation of $n$, i.e., we have $n = p^e n'$ where $p \nmid n'$. Therefore,\[ BP^3(BPU_n) \]
is a subring of $\mathbb{Z}/p^e$. Let $i = 3$ in the long exact sequence (4.2). Then we have a monomorphism\[ BP^3(BPU_n)/p \hookrightarrow P(1)^3(BPU_n). \]
By Lemma 4.2 we have $P(1)^3(BPU_n) = 0$. Hence we have $BP^3(BPU_n) = 0$. \hfill \Box

Lemma 4.4. The class $1 \otimes y_{p,0} \in E_2^{2p+2,0}$ is a permanent cocycle.

Thus we denote by $\eta_{p,0}$ the class represented by $1 \otimes y_{p,0}$. The Thom map therefore takes $\eta_{p,0}$ to $y_{p,0}$.

Proof. Recall that the mod $p$ reduction of $y_{p,0}$ is $\bar{y}_{p,0} = Q_1(x_1)$. It follows from Theorem 4.4.1 that $\bar{y}_{p,0}$, as a class in $H^*(K(\mathbb{Z}, 3); \mathbb{Z}_p)$, is in the image of the reduced Thom map\[ (4.3) \quad \text{BP}^{2p+2}(K(\mathbb{Z}, 3)) \xrightarrow{T} H^{2p+2}(K(\mathbb{Z}, 3); \mathbb{Z}_p) \xrightarrow{\eta} H^{2p+2}(K(\mathbb{Z}, 3); \mathbb{Z}_p), \]
where the first arrow is the Thom map and the second one is the mod $p$ reduction. Moreover, the second arrow is an isomorphism, from which it follows that $y_{p,0} \in H^{2p+2}(K(\mathbb{Z}, 3); \mathbb{Z}_p)$ is in the image of $T$. Therefore, the class $y_{p,0} \in H^*(BPU_n; \mathbb{Z}_p)$ is in the image of the reduced Thom map, since we have the following commutative diagram:
\[
\begin{array}{ccc}
\text{BP}^{2p+2}(K(\mathbb{Z}, 3)) & \xrightarrow{\chi} & \text{BP}^{2p+2}(BPU_n) \\
\downarrow T & & \downarrow T \\
H^{2p+2}(K(\mathbb{Z}, 3); \mathbb{Z}_p) & \xrightarrow{\chi} & H^{2p+2}(BPU_n; \mathbb{Z}_p)
\end{array}
\]
where the vertical arrows are the Thom maps, and the horizontal ones are induced by the canonical Brauer class $\chi$ defined in (3.2). Therefore the class $y_{p,0} \in H^{2p+2}(BPU_n; \mathbb{Z}_p)$ is in the image of $T$ and we conclude. \hfill \Box

Next we consider the classes\[ 1 \otimes e_i \in \text{BP}^0 \otimes H^{2i}(BPU_n; \mathbb{Z}_p) \cong E_2^{2i,0} \]
and\[ v_1 \otimes 1 \in E_2^{0,-(2p-2)}. \]
It follows from Proposition 3.7 that there are nonnegative integers \( \lambda_1, \lambda_2 \geq 2 \) and \( \mu \), such that \( p^\lambda v_i \) and \( p^\mu v_1 \) are permanent cocycles. Therefore, the groups \( E_{s,t}^\infty \) for \( 0 \leq s \leq 2p + 2 \) and \(-2p - 2 \leq t \leq 0\) are determined up to nonzero multiples of non-torsion classes, from which Theorem 1.1 follows.

5. THE CLASSES \( \eta_{p,k} \)

In this section we prove Theorem 1.2 by constructing the \( p \)-torsion classes \( \eta_{p,k} \). One might be tempted to consider copying the construction of \( \eta_{p,0} \). Indeed, by Tamanoi’s Theorem 4.1, one can show that the classes

\[
\bar{y}_{p,k} = Q_k(x_1) \in H^*(K(\mathbb{Z},3); \mathbb{Z}_p)
\]

is in the image of the Thom map. However, one may run into trouble trying to produce an analog of the isomorphism in (4.3) in higher dimensions. Hence we take a different approach.

For \( k \geq 0 \), we have the classes

\[
(x_{p,k} = \mathcal{R}^p_k(x_{p,k-1}) \in H^{2p^{k+1}+1}(K(\mathbb{Z},3); \mathbb{Z}_p))
\]

and

\[
(y_{p,k} = \delta(x_{p,k}) \in H^{2p^{k+1}+2}(K(\mathbb{Z},3); \mathbb{Z}_p)).
\]

Let \( \bar{y}_{p,k} \) denote mod \( p \) reduction of \( y_{p,k} \). Then the classes \( x_{p,k} \) and \( \bar{y}_{p,k} \) may be obtained inductively on \( k \) as follows:

**Proposition 5.1.** For \( k \geq 1 \), we have

\[
x_{p,k} = \mathcal{R}^p_k(x_{p,k-1}) \text{ and } \bar{y}_{p,k} = \mathcal{R}^p_k(\bar{y}_{p,k-1}).
\]

**Proof.** Recall that for positive integers \( a, b \) such that \( a \leq pb \), we have the Adem relation (11)

\[
\mathcal{A}^a \beta \mathcal{A}^b = \sum_i (-1)^{ai} \binom{(p-1)(b-i)}{a-p_i} \beta \mathcal{A}^{a-b+i} \mathcal{A}^i + \sum_i (-1)^{ai+b+1} \binom{(p-1)(b-i)-1}{a-p_i-1} \mathcal{A}^{a-b+i} \beta \mathcal{A}^i.
\]

For \( k > 0 \), let \( a = p^k \) and \( b = p^{k-1} \). Then the only choice of \( i \) to offer something nontrivial on the right hand side of (5.3) is \( i = p^{k-1} \), and (5.3) becomes

\[
\mathcal{R}^p \beta \mathcal{R}^{p^{k-1}} = \beta \mathcal{R}^p \mathcal{R}^{p^{k-1}}.
\]

Then it follows by induction that we have

\[
\mathcal{R}^{p^k} \beta \mathcal{R}^{p^{k-1}} \cdots \mathcal{R}^p \mathcal{R}^1 = \beta \mathcal{R}^{p^k} \cdots \mathcal{R}^p \mathcal{R}^1.
\]

Compare the above with the definitions (5.1) and (5.2), and we conclude. \( \square \)

We abuse notations and let \( x_{p,k} \), \( y_{p,k} \) and \( \bar{y}_{p,k} \) denote the image of themselves under \( \chi^* : H^*(K(\mathbb{Z},3);L) \rightarrow H^*(BP^0;L) \) for \( L = \mathbb{Z}_\ell \) or \( \mathbb{Z}_p \).

**Lemma 5.2.** The class \( x_{p,0} \in H^*(BP^0;\mathbb{Z}_p) \) is in the image of the Thom map

\[
T: \mathcal{P}(1)^*(BP^0_n;\mathbb{Z}_p) \rightarrow H^*(BP^0_n;\mathbb{Z}_p).
\]

We denote by \( \xi_{p,0} \in \mathcal{P}(1)^{2p+1}(BP^0_n) \) a class satisfying \( T(\xi_{p,0}) = x_{p,0} \).
Proof. It follows from the long exact sequence of BP\(^\ast\)-modules (4.2)
\[
\cdots \to \text{BP}\(i\)(BP\(U_n\)) \xrightarrow{\times p} \text{BP}\(i\)(BP\(U_n\)) \xrightarrow{\delta} \text{BP}\(i+1\)(BP\(U_n\)) \xrightarrow{\times p} \cdots
\]
that the \(p\)-torsion class \(\eta_{p,0}\) is in the image of
\[
\delta: \text{BP}\(2p+1\)(BP\(U_n\)) \to \text{BP}\(2p+2\)(BP\(U_n\)).
\]
Let \(\xi_{p,0} \in \text{BP}\(2p+1\)(BP\(U_n\))\) be a class satisfying \(\delta(\xi_{p,0}) = \eta_{p,0}\). The commutative diagram
\[
P(1)^{2p+1}(BP\(U_n\)) \xrightarrow{\delta} \text{BP}\(2p+2\)(BP\(U_n\))
\]
\[
\Downarrow T
\]
\[
H^{2p+1}(BP\(U_n\);Z\(_p\)) \xrightarrow{\delta \cong} H^{2p+2}(BP\(U_n\);Z\(_p\)).
\]
yields \(T(\xi_{p,0}) = x_{p,0}\), and we conclude. \(\square\)

We proceed to construct a collection of classes \(\xi_{p,k}\) for \(k \geq 0\) by induction on \(k\). For \(k = 0\), the class \(\eta_{p,0}\) is given in Theorem 1.1. For the inductive argument, recall Kane’s commutative diagram (2.6):
\[
P(1)^{\ast}(X) \xrightarrow{r^{\ast}_k} P(1)^{\ast}(X)
\]
\[
\Downarrow T
\]
\[
H^{\ast}(X;Z\(_p\)) \xrightarrow{c(\mathscr{P}^E)} H^{\ast}(X;Z\(_p\)).
\]
The canonical anti-automorphism satisfies \(c^2 = \text{id}\), and restricts to an anti-automorphism on the subalgebra of \(\mathscr{A}\) generated by \(\mathscr{P}^E\) for various sequences \(E\). Therefore, for \(k \geq 0\), we have sequences \(E_i\) such that
\[
c(\mathscr{P}^k) = \sum_i \mathscr{P}^{E_i},
\]
or equivalently
\[
\mathscr{P}^k = \sum_i c(\mathscr{P}^{E_i}).
\]
By (5.5) we have \(R_k \in P(1)^{\ast}P(1)\) such that the following diagram commutes:
\[
P(1)^{\ast}(X) \xrightarrow{R_k} P(1)^{\ast}(X)
\]
\[
\Downarrow T
\]
\[
H^{\ast}(X;Z\(_p\)) \xrightarrow{\mathscr{P}^k} H^{\ast}(X;Z\(_p\)).
\]
By induction on \(k\) and Proposition 5.1, if \(x_{p,k}\) is in the image of the Thom map \(T\), then so is \(x_{p,k+1} = \mathscr{P}^k(x_{p,k})\). The induction also gives classes \(\xi_{p,k} = R_k(\xi_{p,k-1})\) such that \(T(\xi_{p,k}) = x_{p,k}\). The desired classes \(\eta_{p,k}\) are thus given by \(\eta_{p,k} = \delta(\xi_{p,k})\).

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