Stochastic Nested Variance Reduction for Nonconvex Optimization

Dongruo Zhou∗ and Pan Xu† and Quanquan Gu‡

Abstract

We study finite-sum nonconvex optimization problems, where the objective function is an average of \( n \) nonconvex functions. We propose a new stochastic gradient descent algorithm based on nested variance reduction. Compared with conventional stochastic variance reduced gradient (SVRG) algorithm that uses two reference points to construct a semi-stochastic gradient with diminishing variance in each iteration, our algorithm uses \( K + 1 \) nested reference points to build a semi-stochastic gradient to further reduce its variance in each iteration. For smooth nonconvex functions, the proposed algorithm converges to an \( \epsilon \)-approximate first-order stationary point (i.e., \( \|\nabla F(x)\|_2 \leq \epsilon \)) within \( \tilde{O}(n \wedge \epsilon^{-2} \wedge n^{1/4} \epsilon^{-2}) \) number of stochastic gradient evaluations. This improves the best known gradient complexity of SVRG \( O(n + n^{2/3} \epsilon^{-2}) \) and that of SCSG \( O(n \wedge \epsilon^{-2} + \epsilon^{-10/3} \wedge n^{2/3} \epsilon^{-2}) \). For gradient dominated functions, our algorithm also achieves a better gradient complexity than the state-of-the-art algorithms.

1 Introduction

We study the following nonconvex finite-sum problem

\[
\min_{x \in \mathbb{R}^d} F(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x),
\]

where each component function \( f_i : \mathbb{R}^d \to \mathbb{R} \) has \( L \)-Lipschitz continuous gradient but may be nonconvex. A lot of machine learning problems fall into (1.1) such as empirical risk minimization (ERM) with nonconvex loss. Since finding the global minimum of (1.1) is general NP-hard (Hillar and Lim, 2013), we instead aim at finding an \( \epsilon \)-approximate stationary point \( x \), which satisfies \( \|\nabla F(x)\|_2 \leq \epsilon \), where \( \nabla F(x) \) is the gradient of \( F(x) \) at \( x \), and \( \epsilon > 0 \) is the accuracy parameter.

In this work, we mainly focus on first-order algorithms, which only need the function value and gradient evaluations. We use gradient complexity, the number of stochastic gradient evaluations,
to measure the convergence of different first-order algorithms.\footnote{While we use gradient complexity as in Lei et al. (2017) to present our result, it is basically the same if we use incremental first-order oracle (IFO) complexity used by Reddi et al. (2016a). In other words, these are directly comparable.} For nonconvex optimization, it is well-known that Gradient Descent (GD) can converge to an $\epsilon$-approximate stationary point with $O(n \cdot \epsilon^{-2})$ (Nesterov, 2014) number of stochastic gradient evaluations. It can be seen that GD needs to calculate the full gradient at each iteration, which is a heavy load when $n \gg 1$. Stochastic Gradient Descent (SGD) has $O(\epsilon^{-4})$ gradient complexity to an $\epsilon$-approximate stationary point under the assumption that the stochastic gradient has a bounded variance (Ghadimi and Lan, 2016). While SGD only needs to calculate a mini-batch of stochastic gradients in each iteration, due to the noise brought by stochastic gradients, its gradient complexity has a worse dependency on $\epsilon$.

In order to improve the dependence of the gradient complexity of SGD on $n$ and $\epsilon$ for nonconvex optimization, variance reduction technique was firstly proposed in Roux et al. (2012); Johnson and Zhang (2013); Xiao and Zhang (2014); Defazio et al. (2014a); Mairal (2015); Bietti and Mairal (2017) for convex finite-sum optimization. Representative algorithms include Stochastic Average Gradient (SAG) (Roux et al., 2012), Stochastic Variance Reduced Gradient (SVRG) (Johnson and Zhang, 2013), SAGA (Defazio et al., 2014a), Stochastic Dual Coordinate Ascent (SDCA) (Shalev-Shwartz and Zhang, 2013) and Finito (Defazio et al., 2014b), to mention a few. The key idea behind variance reduction is that the gradient complexity can be saved if the algorithm use history information as reference. For instance, one representative variance reduction method is SVRG, which is based on a semi-stochastic gradient that is defined by two reference points. Since the the variance of this semi-stochastic gradient will diminish when the iterate gets closer to the minimizer, it therefore accelerates the convergence of stochastic gradient method. The convergence of SVRG under nonconvex setting was first analyzed in Garber and Hazan (2015); Shalev-Shwartz (2016), where $F$ is still convex but each component function $f_i$ can be nonconvex. The analysis for the general nonconvex function $F$ was done by Reddi et al. (2016a); Allen-Zhu and Hazan (2016), which shows that SVRG can converge to an $\epsilon$-approximate stationary point with $O(n^{2/3} \cdot \epsilon^{-2})$ number of stochastic gradient evaluations. This result is strictly better than that of GD. Recently, Lei et al. (2017) proposed a Stochastically Controlled Stochastic Gradient (SCSG) based on variance reduction, which further reduces the gradient complexity of SVRG to $O(n \wedge \epsilon^{-2} + \epsilon^{-10/3} \wedge (n^{2/3} \epsilon^{-2}))$. This result outperforms both GD and SGD strictly. To the best of our knowledge, this is the state-of-art gradient complexity under the smoothness (i.e., gradient lipschitz) and bounded stochastic gradient variance assumptions. A natural and long standing question is:

*Is there still room for improvement in nonconvex finite-sum optimization without making additional assumptions beyond smoothness and bounded stochastic gradient variance?*

In this paper, we provide an affirmative answer to the above question, by showing that the dependence on $n$ in the gradient complexity of SVRG (Reddi et al., 2016a; Allen-Zhu and Hazan, 2016) and SCSG (Lei et al., 2017) can be further reduced. We propose a novel algorithm namely Stochastic Nested Variance-Reduced Gradient descent (SNVRG). Similar to SVRG and SCSG, our proposed algorithm works in a multi-epoch way. Nevertheless, the technique we developed is highly nontrivial. At the core of our algorithm is the multiple reference points-based variance reduction technique in each iteration. In detail, inspired by SVRG and SCSG, which uses two reference points to construct a semi-stochastic gradient with diminishing variance, our algorithm uses $K + 1$ reference points to
construct a semi-stochastic gradient, whose variance decays faster than that of the semi-stochastic gradient used in SVRG and SCSG.

1.1 Our Contributions

Our major contributions are summarized as follows:

- We propose a stochastic nested variance reduction technique for stochastic gradient method, which reduces the dependence of the gradient complexity on $n$ compared with SVRG and SCSG.

- We show that our proposed algorithm is able to achieve an $\epsilon$-approximate stationary point with $\widetilde{O}(n^{1/2} + n^{1/2}/\epsilon^2)$ stochastic gradient evaluations, which outperforms all existing first-order algorithms such as GD, SGD, SVRG and SCSG.

- As a by-product, when $F$ is a $\tau$-gradient dominated function, a variant of our algorithm can achieve an $\epsilon$-accurate solution (i.e., $F(x) - \min_x F(x) \leq \epsilon$) within $\widetilde{O}(n^{1/2} + \tau(n^{1/2}/\epsilon^2)^{1/2})$ stochastic gradient evaluations, which also outperforms the state-of-the-art.

1.2 Additional Related Work

Since it is hardly possible to review the huge body of literature on convex and nonconvex optimization due to space limit, here we review some additional most related work on accelerating nonconvex (finite-sum) optimization.

**Acceleration by high-order smoothness assumption** With only Lipschitz continuous gradient assumption, Carmon et al. (2017b) showed that the lower bound for both deterministic and stochastic algorithms to achieve an $\epsilon$-approximate stationary point is $\Omega(\epsilon^{-2})$. With high-order smoothness assumptions, i.e., Hessian Lipschitzness, Hessian smoothness etc., a series of work have shown the existence of acceleration. For instance, Agarwal et al. (2017) gave an algorithm based on Fast-PCA which can achieve an $\epsilon$-approximate stationary point with gradient complexity $\widetilde{O}(n^{3/4}/\epsilon + n^{3/4}/\epsilon^{7/4})$. Carmon et al. (2016, 2017a) showed two algorithms based on finding exact or inexact negative curvature which can achieve an $\epsilon$-approximate stationary point with gradient complexity $\widetilde{O}(n^{3/4}/\epsilon^{7/4})$. In this work, we only consider gradient Lipschitz without assuming Hessian Lipschitz or Hessian smooth. Therefore, our result is not directly comparable to the methods in this category.

**Acceleration by momentum** The fact that using momentum is able to accelerate algorithms has been shown both in theory and practice in convex optimization (Polyak, 1964; Nesterov, 2005; Hu et al., 2009; Lan, 2012; Ghadimi and Lan, 2012; Nesterov, 2014; Lin et al., 2015; Allen-Zhu, 2017a). However, there is no evidence that such acceleration exists in nonconvex optimization with only Lipschitz continuous gradient assumption (Ghadimi and Lan, 2016; Li and Lin, 2015; Paquette et al., 2017; Li et al., 2017; Lan and Zhou, 2017). If $F$ satisfies $\lambda$-strongly nonconvex, i.e., $\nabla^2 F \succeq -\lambda I$, Allen-Zhu (2017b) proved that Natasha 1, an algorithm based on nonconvex momentum, is able to find an $\epsilon$-approximate stationary point in $\widetilde{O}(n^{2/3}L^{2/3}\lambda^{1/3}\epsilon^{-2})$. Later, Allen-Zhu (2017b) further showed that Natasha 2, an online version of Natasha 1, is able to achieve an $\epsilon$-approximate stationary point within $\widetilde{O}(\epsilon^{-3.25})$ stochastic gradient evaluations.

---

3In fact, Natasha 2 is guaranteed to converge to an $(\epsilon, \sqrt{\epsilon})$-approximate second-order stationary point with $\widetilde{O}(\epsilon^{-3.25})$ gradient complexity, which implies the convergence to an $\epsilon$-approximate stationary point.
To give a thorough comparison of our proposed algorithm with existing first-order algorithms for nonconvex finite-sum optimization, we summarize the gradient complexity of the most relevant algorithms in Table 1. We also plot the gradient complexities of different algorithms in Figure 1 for nonconvex smooth functions. Note that GD and SGD are always worse than SVRG and SCSG according to Table 1. In addition, GNC-AGD and Natasha2 needs additional Hessian Lipschitz condition. Therefore, we only plot the gradient complexity of SVRG, SCSG and our proposed SNVRG in Figure 1.

Table 1: Comparisons on gradient complexity of different algorithms. The second column shows the gradient complexity for a nonconvex and smooth function to achieve an $\epsilon$-approximate stationary point (i.e., $\|\nabla F(x)\|_2 \leq \epsilon$). The third column presents the gradient complexity for a gradient dominant function to achieve an $\epsilon$-accurate solution (i.e., $F(x) - \min_x F(x) \leq \epsilon$). The last column presents the space complexity of all algorithms.

| Algorithm         | nonconvex      | gradient dominant | Hessian Lipschitz |
|-------------------|----------------|-------------------|-------------------|
| GD                | $O\left(\frac{n}{\epsilon^2}\right)$ | $\tilde{O}(\tau n)$ | No                |
| SGD               | $O\left(\frac{1}{\epsilon}\right)$ | $O\left(\frac{1}{\epsilon^2}\right)$ | No                |
| SVRG (Reddi et al., 2016a) | $O\left(\frac{n^{2/3}}{\epsilon^{2/3}}\right)$ | $\tilde{O}(n + \tau n^{2/3})$ | No                |
| SCSG (Lei et al., 2017) | $O\left(\frac{n}{\epsilon^{2/3}}\right)$ | $\tilde{O}\left(n \wedge \frac{n}{\epsilon^2} + \tau n \wedge \frac{n^{2/3}}{\epsilon^{2/3}}\right)$ | No |
| GNC-AGD (Carmon et al., 2017a) | $\tilde{O}\left(\frac{n}{\epsilon^{2/3}}\right)$ | N/A | Needed |
| Natasha 2 (Allen-Zhu, 2017b) | $\tilde{O}\left(\frac{1}{\epsilon^{5/6}}\right)$ | N/A | Needed |
| SNVRG (this paper) | $\tilde{O}\left(\frac{n}{\epsilon^2} \wedge \frac{n^{2/3}}{\epsilon^{2/3}}\right)$ | $\tilde{O}\left(n \wedge \frac{n}{\epsilon^2} + \tau n \wedge \frac{n^{2/3}}{\epsilon^{2/3}}\right)$ | No |

Figure 1: Comparison of gradient complexities among SNVRG, SVRG and SCSG.

**Notation:** Let $A = [A_{ij}] \in \mathbb{R}^{d \times d}$ be a matrix and $x = (x_1, ..., x_d)^\top \in \mathbb{R}^d$ be a vector. $I$ denotes an identity matrix. We use $\|v\|_2$ to denote the 2-norm of vector $v \in \mathbb{R}^d$. We use $\langle \cdot, \cdot \rangle$ to represent the inner product of two vectors. Given two sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n = O(b_n)$ if there exists a constant $0 < C < +\infty$ such that $a_n \leq C b_n$. We write $a_n = \Omega(b_n)$ if there exists a constant
0 < C < +∞, such that \( a_n \geq C b_n \). We use notation \( \widetilde{O}(\cdot) \) to hide logarithmic factors. We also make use of the notation \( f_n \lesssim g_n \) (\( f_n \gtrsim g_n \)) if \( f_n \) is less than (larger than) \( g_n \) up to a constant. We use productive symbol \( \prod_{i=a}^{b} c_i \) to denote \( c_a c_{a+1} \ldots c_b \). Moreover, if \( a > b \), we take the product as 1. We use \( \lfloor \cdot \rfloor \) as the floor function. We use \( \log(x) \) to represent the logarithm of \( x \) to base 2. \( a \wedge b \) is a shorthand notation for \( \min(a, b) \).

2 Preliminaries

In this section, we present some definitions that will be used throughout our analysis.

**Definition 2.1.** A function \( f \) is \( L \)-smooth, if for any \( x, y \in \mathbb{R}^d \), we have
\[
\|\nabla f(x) - \nabla f(y)\|_2 \leq L \|x - y\|_2.
\] (2.1)

An equivalent definition of \( L \)-smoothness is that for any \( x, h \in \mathbb{R}^d \), we have
\[
f(x + h) \leq f(x) + \langle \nabla f(x), h \rangle + \frac{L}{2} \|h\|_2^2.
\] (2.2)

**Definition 2.2.** A function \( f \) is \( \lambda \)-strongly convex, if for any \( x, y \in \mathbb{R}^d \), we have
\[
f(x + h) \geq f(x) + \langle \nabla f(x), h \rangle + \frac{\lambda}{2} \|h\|_2^2.
\] (2.3)

**Definition 2.3.** A function \( F \) with finite-sum structure in (1.1) is said to have stochastic gradients with bounded variance \( \sigma^2 \), if for any \( x \), we have
\[
\mathbb{E}_i \|\nabla f_i(x) - \nabla F(x)\|_2^2 \leq \sigma^2,
\] (2.4)
where \( i \) a random index uniformly chosen from \([n]\) and \( \mathbb{E}_i \) denotes the expectation over such \( i \).

\( \sigma^2 \) is called the upper bound on the variance of stochastic gradients (Lei et al., 2017).

**Definition 2.4.** We say a function \( f \) is lower-bounded by \( f^* \) if for any \( x \in \mathbb{R}^d \), \( f(x) \geq f^* \).

We also consider a class of functions namely gradient dominated functions (Polyak, 1963), which is formally defined as follows:

**Definition 2.5.** We say function \( f \) is \( \tau \)-gradient dominated if for any \( x \in \mathbb{R}^d \), we have
\[
f(x) - f(x^*) \leq \tau \cdot \|\nabla f(x)\|_2^2,
\] (2.5)
where \( x^* \in \mathbb{R}^d \) is the global minimum of \( f \).

Note that gradient dominated condition is also known as the Polyak-Lojasiewicz (P-L) condition (Polyak, 1963), and is not necessarily convex. It is weaker than strong convexity as well as other popular conditions that appear in the optimization literature (Karimi et al., 2016).
3 The Proposed Algorithm

In this section, we present our nested stochastic variance reduction algorithm, namely, SNVRG.

Algorithm 1 One-epoch-SNVRG($x_0, F, K, M, \{T_l\}, \{B_l\}, B$)

1: **Input**: initial point $x_0$, function $F$, loop number $K$, step size parameter $M$, loop parameters $T_l, l \in [K]$, batch parameters $B_l, l \in [K]$, base batch size $B > 0$.
2: $x_0^{(l)} \leftarrow x_0, g_0^{(l)} \leftarrow 0, 0 \leq l \leq K$
3: Uniformly generate index set $I \subset [n]$ without replacement, $|I| = B$
4: $g_0^{(0)} \leftarrow 1/B \sum_{i \in I} \nabla f_i(x_0)$
5: $v_0 \leftarrow \sum_{l=0}^K g_0^{(l)}$
6: $x_1 = x_0 - 1/(10M) \cdot v_0$
7: for $t = 1, \ldots, \prod_{l=1}^K T_l - 1$ do
   8: $r = \min\{j : 0 = (t \mod \prod_{l=j+1}^K T_l), 0 \leq j \leq K\}$
   9: $\{x_t^{(l)}\} \leftarrow \text{Update reference points(}$\{x_{t-1}^{(l)}\}, x_t, r\}$, $0 \leq l \leq K$.
10: $\{g_t^{(l)}\} \leftarrow \text{Update reference gradients(}$\{g_{t-1}^{(l)}\}, \{x_t^{(l)}\}, r\}$, $0 \leq l \leq K$.
11: $v_t \leftarrow \sum_{l=0}^K g_t^{(l)}$
12: $x_{t+1} \leftarrow x_t - 1/(10M) \cdot v_t$
end for
14: $x_{\text{out}} \leftarrow$ uniformly random choice from $\{x_t\}$, where $0 \leq t < \prod_{l=1}^K T_l$
15: $x_{\text{end}} \leftarrow x_{\prod_{l=1}^K T_l}$
16: **Output**: $[x_{\text{out}}, x_{\text{end}}]$

**Function**: Update reference points($\{x_{old}^{(l)}\}, x, r$)
17: $x_{\text{new}}^{(l)} \leftarrow x_{old}^{(l)}, 0 \leq l \leq r-1$; $x_{\text{new}}^{(l)} \leftarrow x, r \leq l \leq K$
19: return $\{x_{\text{new}}^{(l)}\}$

**Function**: Update reference gradients($\{g_{old}^{(l)}\}, \{x_{\text{new}}^{(l)}\}, r$)
20: $g_{\text{new}}^{(l)} \leftarrow g_{old}^{(l)}, 0 \leq l < r$
21: for $r \leq l \leq K$ do
22: Uniformly generate index set $I \subset [n]$ without replacement, $|I| = B_l$
23: $g_{\text{new}}^{(l)} \leftarrow 1/B_l \sum_{i \in I} [\nabla f_i(x_{\text{new}}) - \nabla f_i(x_{\text{new}}^{(l-1)})]$
25: end for
26: return $\{g_{\text{new}}^{(l)}\}$.

One-epoch-SNVRG: We first present the key component of our main algorithm, One-epoch-SNVRG, which is displayed in Algorithm 1. The most innovative part of Algorithm 1 attributes to the $K + 1$ reference points and $K + 1$ reference gradients. Note that when $K = 1$, Algorithm 1 reduces to one epoch of SVRG algorithm (Johnson and Zhang, 2013; Reddi et al., 2016a; Allen-Zhu and Hazan, 2016). To better understand our One-epoch SNVRG algorithm, it would be helpful to revisit the original SVRG which is a special case of our algorithm. For the finite-sum optimization
problem in (1.1), the original SVRG takes the following updating formula

\[ x_{t+1} = x_t - \eta (g_t^{(0)} + g_t^{(1)}) \]

where \( \eta > 0 \) is the step size, \( i_t \) is a random index uniformly chosen from \([n]\) and \( \tilde{x} \) is a snapshot for \( x_t \) after every \( T_1 \) iterations. There are two reference points in the update formula at \( x_t \): \( x_t^{(0)} = \tilde{x} \) and \( x_t^{(1)} = x_t \). Note that \( \tilde{x} \) is updated every \( T_1 \) iterations, namely, \( \tilde{x} \) is set to be \( x_t \) only when \( (t \mod T_1) = 0 \). Moreover, in the semi-stochastic gradient \( v_t \), there are also two reference gradients and we denote them by \( g_t^{(0)} = \nabla F(\tilde{x}) \) and \( g_t^{(1)} = \nabla f_{i_t}(x_t) - \nabla f_{i_t}(\tilde{x}) = \nabla f_{i_t}(x_t^{(1)}) - \nabla f_{i_t}(x_t^{(0)}) \). The references points and gradients used in SVRG are illustrated in Figure 2(a) with \( K = 1 \).

Back to our One-epoch-SNVRG, we can define similar reference points and reference gradients as that in the special case of SVRG. Specifically, for \( t = 0, \ldots, \prod_{l=1}^{K} T_l - 1 \), each point \( x_t \) has \( K + 1 \) reference points \( \{x_t^{(l)}\}, l = 0, \ldots, K \), which is set to be \( x_t^{(l)} = x_{t'} \) with index \( t' \) defined as

\[ t' = \left\lfloor \frac{t}{\prod_{k=l+1}^{K} T_k} \right\rfloor \cdot \prod_{k=l+1}^{K} T_k. \]  (3.1)

Specially, note that we have \( x_t^{(0)} = x_0 \) and \( x_t^{(K)} = x_t \) for all \( t = 0, \ldots, \prod_{l=1}^{K} T_l - 1 \). Similarly, \( x_t \)
also has \( K + 1 \) reference gradients \( \{g_t^{(l)}\} \), which can be defined based on the reference points \( \{x_t^{(l)}\} \):

\[
g_t^{(0)} = \frac{1}{B} \sum_{i \in I} \nabla f_i(x_0),
\]

\[
g_t^{(l)} = \frac{1}{B_t} \sum_{i \in I_t} [\nabla f_i(x_t^{(l)}) - \nabla f_i(x_t^{(l-1)})], \quad l = 1, \ldots, K,
\]

where \( I, I_t \) are random index sets with \(|I| = B, |I_t| = B_t\) and are uniformly generated from \([n]\) without replacement. Based on the reference points and reference gradients, we then update \( x_{t+1} = x_t - 1/(10M) \cdot v_t \), where \( v_t = \sum_{l=0}^K g_t^{(l)} \) and \( M \) is the step size parameter. The illustration of reference points and gradients of SNVRG is displayed in Figure 2(b).

We remark that it would be a huge waste for us to re-evaluate \( g_t^{(l)} \) at each iteration. Fortunately, due to the fact that each reference point is only updated after a long period, we can maintain \( g_t^{(l)} = g_{t-1}^{(l)} \) and only need to update \( g_t^{(l)} \) when \( x_t^{(l)} \) has been updated as suggested by Line 24 in Algorithm 1.

**SNVRG:** Using One-epoch-SNVRG (Algorithm 1) as a building block, we now present our main algorithm: Algorithm 2 for nonconvex finite-sum optimization to find an \( \epsilon \)-approximate stationary point. At each iteration of Algorithm 2, it executes One-epoch-SNVRG (Algorithm 1) which takes \( z_{s-1} \) as its input and outputs \([y_s, z_s]\). We choose \( y_{out} \) as the output of Algorithm 2 uniformly from \( \{y_s\} \), for \( s = 1, \ldots, S \).

**SNVRG-PL:** In addition, when function \( F \) in (1.1) is gradient dominated as defined in Definition 2.5 (P-L condition), it has been proved that the global minimum can be found by SGD (Karimi et al., 2016), SVRG (Reddi et al., 2016a) and SCSG (Lei et al., 2017) very efficiently. Following a similar trick used in Reddi et al. (2016a), we present Algorithm 3 on top of Algorithm 2, to find the global minimum in this setting. We call Algorithm 3 SNVRG-PL, because gradient dominated condition is also known as Polyak-Lojasiewicz (PL) condition (Polyak, 1963).

**Space complexity:** Here we briefly compare the space complexity between our algorithms and other variance reduction based algorithms. SVRG and SCSG needs \( O(d) \) space complexity to store one reference gradient, SAGA (Defazio et al., 2014a) needs to store reference gradients for each component functions, and its space complexity is \( O(nd) \) without using any trick. For our algorithm SNVRG, we need to store \( K \) reference gradients, thus its space complexity is \( O(Kd) \). Later in our theory, we will show that \( K = O(\log \log n) \). Therefore, the space complexity of our algorithm is actually \( \tilde{O}(d) \), which is almost comparable to that of SVRG and SCSG.

---

**Algorithm 2** SNVRG\((z_0, F, K, M, \{T_l\}, \{B_l\}, B, S)\)

1: **Input:** initial point \( z_0 \), function \( F, K, M, \{T_l\}, \{B_l\} \), batch \( B, S \).
2: **for** \( s = 1, \ldots, S \) **do**
3: \( \{y_s, z_s\} \leftarrow \text{One-epoch-SNVRG}(z_{s-1}, F, K, M, \{T_l\}, \{B_l\}, B) \) **▷ Algorithm 1**
4: **end for**
5: **Output:** Uniformly choose \( y_{out} \) from \( \{y_s\}, 1 \leq s \leq S \).
Algorithm 3 SNVRG-PL($z_0, F, K, M, \{T_l\}, \{B_l\}, B, S, U$)

1: **Input:** initial point $z_0$, function $F$, $K$, $M$, $\{T_l\}$, $\{B_l\}$, batch $B$, $S$, $U$.
2: for $u = 1, \ldots, U$ do
3: \hspace{1em} $z_u = \text{SNVRG}(z_{u-1}, F, K, M, \{T_l\}, \{B_l\}, B, S)$ \hspace{1em} ⨂ Algorithm 2
4: end for
5: **Output:** $z_{\text{out}} = z_U$.

4 Main Theory

In this section, we provide the convergence analysis of SNVRG.

4.1 Convergence of SNVRG

We first analyze One-epoch-SNVRG (Algorithm 1) and provide a particular choice of parameters.

**Lemma 4.1.** Suppose that each $f_i$ is $L$-smooth, in Algorithm 1, suppose $B \geq 2$ and let the number of nested loops be $K = \log \log B$. Choose the step size parameter as $M = 6L$. For the loop and batch parameters, let $T_1 = 2, B_1 = 6^K \cdot B$ and

\[
T_l = 2^{2^{l-2}}, \quad B_l = 6^{K-l+1} \cdot B/2^{2^{l-1}},
\]

for all $2 \leq l \leq K$. Then the output of Algorithm 1 $[x_{\text{out}}, x_{\text{end}}]$ satisfies

\[
\mathbb{E}\|\nabla F(x_{\text{out}})\|_2^2 \leq C \left( \frac{L}{B^{1/2}} \cdot \mathbb{E}[F(x_0) - F(x_{\text{end}})] + \frac{\sigma^2}{B} \cdot 1(B < n) \right)
\]

within $1 \lor (7B \log^3 B)$ stochastic gradient computations, where $C = 600$ is a constant and $1(\cdot)$ is the indicator function.

The following theorem shows the gradient complexity for Algorithm 2 to find an $\epsilon$-approximate stationary point with a constant base batch size $B$.

**Theorem 4.2.** Suppose that each $f_i$ is $L$-smooth and $F$ has bounded variance with $\sigma^2$. In Algorithm 2, let $B = n \land (2Cn^2/\epsilon^2)$, $S = 1 \lor (2C L \Delta_F/(B^{1/2} \epsilon^2))$ and $C = 600$. The rest parameters $(K, M, \{B_l\}, \{T_l\})$ are chosen the same as in Lemma 4.1. Then the output $y_{\text{out}}$ of Algorithm 2 satisfies $\mathbb{E}\|\nabla F(y_{\text{out}})\|_2^2 \leq \epsilon^2$ with less than

\[
O \left( \log^3 \left( \frac{\sigma^2}{\epsilon^2} \land n \right) \left[ \frac{\sigma^2}{\epsilon^2} \land n + \frac{L \Delta_F}{\epsilon^2} \left[ \frac{\sigma^2}{\epsilon^2} \land n \right]^{1/2} \right] \right)
\]

stochastic gradient computations, where $\Delta_F = F(z_0) - F^*$.

**Remark 4.3.** If we treat $\sigma^2, L$ and $\Delta_F$ as constants, and assume $\epsilon \ll 1$, then (4.2) can be simplified to $\tilde{O}(\epsilon^{-3} \land n^{1/3} \epsilon^{-2})$. This gradient complexity is strictly better than $O(\epsilon^{-10/3} \land n^{2/3} \epsilon^{-2})$, which is achieved by SCSG (Lei et al., 2017). Specifically, when $n \lesssim 1/\epsilon^2$, our proposed SNVRG is faster than SCSG by a factor of $n^{1/6}$; when $n \gtrsim 1/\epsilon^2$, SNVRG is faster than SCSG by a factor of $\epsilon^{-1/3}$. Moreover, SNVRG also outperforms Natasha 2 (Allen-Zhu, 2017b) which attains $\tilde{O}(\epsilon^{-3.25})$ gradient complexity and needs the additional Hessian Lipschitz condition.
4.2 Convergence of SNVRG-PL

We now consider the case when $F$ is a $\tau$-gradient dominated function. In general, we are able to find an $\epsilon$-accurate solution of $F$ instead of only an $\epsilon$-approximate stationary point. Algorithm 3 uses Algorithm 2 as a component.

**Theorem 4.4.** Suppose that each $f_i$ is $L$-smooth, $F$ has bounded variance with $\sigma^2$ and $F$ is a $\tau$-gradient dominated function. In Algorithm 3, let the base batch size $B = n \wedge \left(4C_1 \tau \sigma^2 / \epsilon\right)$, the number of epochs for SNVRG $S = 1 \vee \left(2C_1 \tau L / B^{1/2}\right)$ and the number of epochs $U = \log(2\Delta_F / \epsilon)$. The rest parameters $(K, M, \{B_i\}, \{T_i\})$ are chosen as the same in Lemma 4.1. Then the output $z_{\text{out}}$ of Algorithm 3 satisfies

$$E[F(z_{\text{out}}) - F^*] \leq \epsilon$$

within

$$O\left(\log^3 \left( n \wedge \frac{\tau \sigma^2}{\epsilon} \right) \log \frac{\Delta_F}{\epsilon} \left[ n \wedge \frac{\tau \sigma^2}{\epsilon} + \tau L \left[ n \wedge \frac{\tau \sigma^2}{\epsilon} \right]^{1/2} \right] \right)$$

stochastic gradient computations, where $\Delta_F = F(z_0) - F^*$.

**Remark 4.5.** If we treat $\sigma^2$, $L$ and $\Delta_F$ as constants, then the gradient complexity in (4.3) turns into $\tilde{O}(n \wedge \tau \epsilon^{-1} + \tau(n \wedge \tau \epsilon^{-1})^{1/2})$. Compared with nonconvex SVRG (Reddi et al., 2016b) which achieves $\tilde{O}(n + \tau n^{2/3})$ gradient complexity, our proposed algorithm SNVRG-PL is strictly better than SVRG in terms of the first summand and is faster than SVRG at least by a factor of $n^{1/6}$ in terms of the second summand. Compared with a more general variant of SVRG, namely, the SCSG algorithm (Lei et al., 2017), which attains $\tilde{O}(n \wedge \tau \epsilon^{-1} + \tau(n \wedge \tau \epsilon^{-1})^{2/3})$ gradient complexity, SNVRG-PL also outperforms it by a factor of $(n \wedge \tau \epsilon^{-1})^{1/6}$.

If we further assume that $F$ is $\lambda$-strongly convex, then it is easy to verify that $F$ is also $1/(2\lambda)$-gradient dominated. As a direct consequence, we have the following corollary:

**Corollary 4.6.** Under the same conditions and parameter choices as Theorem 4.4. If we additionally assume that $F$ is $\lambda$-strongly convex, then Algorithm 3 will outputs an $\epsilon$-accurate solution within

$$\tilde{O}\left( n \wedge \frac{\lambda \sigma^2}{\epsilon} + \kappa \cdot \left[ n \wedge \frac{\lambda \sigma^2}{\epsilon} \right]^{1/2} \right)$$

stochastic gradient computations, where $\kappa = L/\lambda$ is the condition number of $F$.

**Remark 4.7.** Corollary 4.6 suggests that when we regard $\lambda$ and $\sigma^2$ as constants and set $\epsilon \ll 1$, Algorithm 3 is able to find an $\epsilon$-accurate solution within $\tilde{O}(n + n^{1/2} \kappa)$ stochastic gradient computations, which matches SVRG-lep in Katyusha X (Allen-Zhu, 2018). Using catalyst techniques (Lin et al., 2015) or Katyusha momentum (Allen-Zhu, 2017a), it can be further accelerated to $\tilde{O}(n + n^{3/4} \sqrt{\kappa})$, which matches the best-known convergence rate (Shalev-Shwartz, 2015; Allen-Zhu, 2018).

5 Conclusions and Future Work

In this paper, we proposed a stochastic nested variance reduced gradient method for finite-sum nonconvex optimization. It achieves substantially better gradient complexity than existing first-order
algorithms. This partially resolves a long standing question that whether the dependence of gradient complexity on \( n \) for nonconvex SVRG and SCSG can be further improved. There is still an open question: whether \( \tilde{O}(n \wedge \epsilon^{-2} + \epsilon^{-3} \wedge n^{1/2} \epsilon^{-2}) \) is the optimal gradient complexity? For finite sum convex optimization, the lower bound has been studied in a sequence of work (Agarwal and Bottou, 2014; Lan and Zhou, 2017; Arjevani and Shamir, 2016; Woodworth and Srebro, 2016). However, for finite-sum nonconvex optimization, the lower bound is still unknown. We plan to derive such a lower bound in our future work. On the other hand, our algorithm can also be extended to deal with nonconvex nonsmooth finite-sum optimization using proximal gradient (Reddi et al., 2016c).

A Proof of the Main Theoretical Results

In this section, we provide the proofs of our main theories in Section 4.

A.1 Proof of Lemma 4.1

We first prove our key lemma on One-epoch-SNVRG. In order to prove Lemma 4.1, we need the following supporting Lemma:

**Lemma A.1.** Let \( T = \prod_{l=1}^{K} T_l \). If the step size and batch size parameters in Algorithm 1 satisfy \( M \geq 6L \) and \( B_l \geq 6^{K-l+1}(\prod_{s=l}^{K} T_s)^2 \), then the output of Algorithm 1 satisfies

\[
\mathbb{E}\|\nabla F(x_{\text{out}})\|^2 \leq C \left( \frac{M}{T} \cdot \mathbb{E}[F(x_0) - F(x_{\text{end}})] + \frac{2\sigma^2}{B} \cdot 1(B < n) \right),
\]

where \( C = 100 \) is a constant.

**Proof of Lemma 4.1.** Note that \( B = 2^K \), we can easily check that the choice of \( M, \{T_l\}, \{B_l\} \) in Lemma 4.1 satisfies the assumption of Lemma A.1. Moreover, we have

\[
T = \prod_{l=1}^{K} T_l = B^{1/2}.
\]

We now submit (A.2) into (A.1), which immediately implies (4.1).

Next we compute how many stochastic gradient computations we need in total after we run One-epoch-SNVRG once. According to the update of reference gradients in Algorithm 1, we only update \( g_t^{(0)} \) once at the beginning of Algorithm 1 (Line 4), which needs \( B \) stochastic gradient computations. For \( g_t^{(l)} \), we only need to update it when \( 0 = (t \mod \prod_{j=l+1}^{K} T_j) \), and thus we need to sample \( g_t^{(l)} \) for \( T/ \prod_{j=l+1}^{K} T_j = \prod_{j=1}^{l} T_j \) times. We need \( 2B_l \) stochastic gradient computations for each sampling procedure (Line 24 in Algorithm 1). We use \( T \) to represent the total number of stochastic gradient computations, then based on above arguments we have

\[
\mathcal{T} = B + 2 \sum_{l=1}^{K} B_l \cdot \prod_{j=1}^{l} T_j.
\]
Now we calculate $\mathcal{T}$ under the parameter choice of Lemma 4.1. Note that we can easily verify the following results:

$$\prod_{j=1}^{l} T_j = 2^{2^{l-1}} = B^2^{2^{l-1}}, \quad B_1 \cdot \prod_{j=1}^{l} T_j = 2 \times 6^K B, \quad B_l \cdot \prod_{j=1}^{l} T_j = 6^{K-l+1} B. \quad (A.4)$$

Submit (A.4) into (A.3) yields the following results:

$$\mathcal{T} = B + 2 \left( 2 \times 6^K B + \sum_{l=2}^{K} 6^{K-l+1} B \right) < B + 6 \times 6^K B = B + 6 \times 6^{\log \log B} B < B + 6B \log^3 B. \quad (A.5)$$

Therefore, the total gradient complexity $\mathcal{T}$ is bounded as follows.

$$\mathcal{T} \leq B + 6B \log^3 B \leq 7B \log^3 B. \quad (A.6)$$

### A.2 Proof of Theorem 4.2

Now we prove our main theorem which spells out the gradient complexity of SNVRG.

**Proof of Theorem 4.2.** By (4.1) we have

$$\mathbb{E}\|\nabla F(y_s)\|_2^2 \leq C \left( \frac{L}{B^{1/2}} \cdot \mathbb{E}[F(z_{s-1}) - F(z_s)] + \frac{\sigma^2}{B} \cdot 1(B < n) \right), \quad (A.7)$$

where $C = 600$. Taking summation for (A.7) over $s$ from 1 to $S$, we have

$$\sum_{s=1}^{S} \mathbb{E}\|\nabla F(y_s)\|_2^2 \leq C \left( \frac{L}{B^{1/2}} \cdot \mathbb{E}[F(z_0) - F(z_S)] + \frac{\sigma^2}{B} \cdot 1(B < n) \cdot S \right). \quad (A.8)$$

Dividing both sides of (A.8) by $S$, we immediately obtain

$$\mathbb{E}\|\nabla F(y_{\text{out}})\|_2^2 \leq C \left( \frac{L\mathbb{E}[F(z_0) - F^*]}{SB^{1/2}} + \frac{\sigma^2}{B} \cdot 1(B < n) \right), \quad (A.9)$$

$$= C \left( \frac{L\Delta_F}{SB^{1/2}} + \frac{\sigma^2}{B} \cdot 1(B < n) \right), \quad (A.10)$$

where (A.9) holds because $F(z_S) \geq F^*$ and by the definition $\Delta_F = F(z_0) - F^*$. By the choice of parameters in Theorem 4.2, we have $B = n \land (2C\sigma^2/\epsilon^2), S = 1 \lor (2CL\Delta_F/(B^{1/2}\epsilon^2))$, which implies

$$1(B < n) \cdot \sigma^2/B \leq \epsilon^2/(2C), \quad \text{and} \quad L\Delta_F/(SB^{1/2}) \leq \epsilon^2/(2C). \quad (A.11)$$
Submitting (A.11) into (A.10), we have $\mathbb{E}\|\nabla F(y_{\text{out}})\|^2 \leq 2C\epsilon^2/(2C) = \epsilon^2$. By Lemma 4.1, we have that each One-epoch-SNVRG takes less than $7B\log^3 B$ stochastic gradient computations. Since we have total $S$ epochs, so the total gradient complexity of Algorithm 2 is less than

$$S \cdot 7B\log^3 B \leq 7B\log^3 B + \frac{L\Delta F}{\epsilon^2} \cdot 7B^{1/2}\log^3 B = O\left(\log^3 \left(\frac{\sigma^2}{\epsilon} \land n\right) \left[\frac{\sigma^2}{\epsilon^2} \land n + \frac{L\Delta F}{\epsilon^2} \left[\frac{\sigma^2}{\epsilon^2} \land n \right]^{1/2}\right]\right),$$

(A.12)

which leads to the conclusion. $\square$

### A.3 Proof of Theorem 4.4

We then prove the main theorem on gradient complexity of SNVRG under gradient dominance condition (Algorithm 3).

**Proof of Theorem 4.4.** Following the proof of Theorem 4.2, we obtain a similar inequality with (A.9):

$$\mathbb{E}\|\nabla F(z_{u+1})\|^2 \leq C\left(\frac{L\mathbb{E}[F(z_u) - F^*]}{SB^{1/2}} + \frac{\sigma^2}{B} \cdot 1(B < n)\right).$$

(A.13)

Since $F$ is a $\tau$-gradient dominated function, we have $\mathbb{E}\|\nabla F(z_{u+1})\|^2 \geq 1/\tau \cdot \mathbb{E}[F(z_{u+1}) - F^*]$ by Definition 2.5. Plugging this inequality into (A.13) yields

$$\mathbb{E}[F(z_{u+1}) - F^*] \leq \frac{C\tau L}{SB^{1/2}} \cdot \mathbb{E}[F(z_u) - F^*] + \frac{C\tau \sigma^2}{B} \cdot 1(B < n) \leq \frac{1}{2}\mathbb{E}[F(z_u) - F^*] + \frac{\epsilon}{4},$$

(A.14)

where the second inequality holds due to the choice of parameters $B = n \land (4C_1\tau \sigma^2/\epsilon)$ and $S = 1 \lor (2C_1\tau L/B^{1/2})$ for Algorithm 3 in Theorem 4.4. By (A.14) we can derive

$$\mathbb{E}[F(z_{u+1}) - F^*] - \frac{\epsilon}{2} \leq \frac{1}{2}\left(\mathbb{E}[F(z_u) - F^*] - \frac{\epsilon}{2}\right),$$

which immediately implies

$$\mathbb{E}[F(z_U) - F^*] - \frac{\epsilon}{2} \leq \frac{1}{2U}\left(\Delta_F - \frac{\epsilon}{2}\right) \leq \frac{\Delta_F}{2U}. \quad (A.15)$$

Plugging the number of epochs $U = \log(2\Delta_F/\epsilon)$ into (A.15), we obtain $\mathbb{E}[F(z_U) - F^*] \leq \epsilon$. Note that each epoch of Algorithm 3 needs at most $S \cdot 7B\log^3 B$ stochastic gradient computations by Theorem 4.2 and Algorithm 3 has $U$ epochs, which implies the total stochastic gradient complexity

$$U \cdot S \cdot 7B\log^3 B = O\left(\log^3 \left(n \land \frac{\tau \sigma^2}{\epsilon}\right) \log \frac{\Delta_F}{\epsilon} \left[n \land \frac{\tau \sigma^2}{\epsilon} + \tau L \left[n \land \frac{\tau \sigma^2}{\epsilon}\right]^{1/2}\right]\right). \quad (A.16)$$
B Proof of Key Lemma A.1

In this section, we focus on proving Lemma A.1 which plays a pivotal role in proving our main theorems. Let $M, \{T_i\}, \{B_i\}, B$ be the parameters as defined in Algorithm 1. We denote $T = \prod_{i=1}^{K} T_i$.

We define filtration $\mathcal{F}_t = \sigma(x_0, \ldots, x_t)$. Let $\{x_t^{(l)}\}, \{g_t^{(l)}\}$ be the reference points and reference gradients in Algorithm 1. We define $v_t^{(l)}$ as

$$v_t^{(l)} := \sum_{j=0}^{l} g_t^{(j)}, \quad \text{for } 0 \leq l \leq K. \quad (B.1)$$

We first present the following definition and two technical lemmas for the purpose of our analysis.

**Definition B.1.** We define constant series $\{c_j^{(s)}\}$ as the following. For each $s$, we define $c_T^{(s)}$ as

$$c_T^{(s)} = \frac{M}{6^{K-s+1} \prod_{l=s}^{K} T_l}. \quad (B.2)$$

When $0 \leq j < T$, we define $c_j^{(s)}$ by induction:

$$c_j^{(s)} = \left(1 + \frac{1}{T} c_j^{(s+1)} + \frac{3L^2}{M} \cdot \prod_{l=s+1}^{K} T_l B_s. \quad (B.3)$$

**Lemma B.2.** For any $p, s$, where $1 \leq s \leq K$ and $0 \leq p \prod_{j=s}^{K} T_j < (p+1) \prod_{j=s}^{K} T_j \leq \prod_{j=1}^{K} T_j$, we define

$$\text{start} = p \cdot \prod_{j=s}^{K} T_j, \quad \text{end} = \text{start} + \prod_{j=s}^{K} T_j,$$

for simplification. Then we have the following results:

$$E \left[ \sum_{j=\text{start}}^{\text{end}-1} \frac{\|\nabla F(x_j)\|_2^2}{100M} + F(x_{\text{end}}) + c_T^{(s)} \cdot \|x_{\text{end}} - x_{\text{start}}\|_2^2 \right] \mathcal{F}_{\text{start}}$$

$$\leq F(x_{\text{start}}) + \frac{2}{M} \cdot E[\|\nabla F(x_{\text{start}}) - v_{\text{start}}\|_2^2] \mathcal{F}_{\text{start}} \cdot \prod_{j=s}^{K} T_j.$$
Proof of Lemma A.1. We have
\[
\frac{T-1}{100M} \sum_{j=0}^{T-1} \mathbb{E} \left\| \nabla F(x_j) \right\|^2_2 + \mathbb{E} [F(x_T)] \leq \frac{T-1}{100M} \sum_{j=0}^{T-1} \mathbb{E} \left\| \nabla F(x_j) \right\|^2_2 + \mathbb{E} [F(x_T) + c^{(1)}_{T}] \cdot \|x_T - x_0\|^2_2
\]
\[
\leq \mathbb{E} [F(x_0)] + \frac{2}{M} \cdot \mathbb{E} \left\| \nabla F(x_0) - g_0 \right\|^2_2 \cdot T, \tag{B.4}
\]
where the second inequality comes from Lemma B.2 with we take \( s = 1, p = 0 \). Moreover we have
\[
\mathbb{E} \left\| \nabla F(x_0) - g_0 \right\|^2_2 = \mathbb{E} \left\| \frac{1}{B} \sum_{i \in I} \left[ \nabla f_i(x_0) - \nabla F(x_0) \right] \right\|^2_2
\]
\[
\leq 1(B < n) \cdot \frac{1}{B} \cdot \sum_{i=1}^{n} \left\| \nabla f_i(x_0) - \nabla F(x_0) \right\|^2_2 \leq \mathbb{E} \left\| \nabla F(x_0) - g_0 \right\|^2_2 \leq 1(B < n) \cdot \frac{\sigma^2}{B}, \tag{B.5}
\]
\[
\mathbb{E} \left\| \nabla F(x_0) - g_0 \right\|^2_2 = \mathbb{E} \left\| \frac{1}{B} \sum_{i \in I} \left[ \nabla f_i(x_0) - \nabla F(x_0) \right] \right\|^2_2
\]
\[
\leq 1(B < n) \cdot \frac{1}{B} \cdot \sum_{i=1}^{n} \left\| \nabla f_i(x_0) - \nabla F(x_0) \right\|^2_2 \leq \mathbb{E} \left\| \nabla F(x_0) - g_0 \right\|^2_2 \leq 1(B < n) \cdot \frac{\sigma^2}{B}, \tag{B.6}
\]
where (B.5) holds because of Lemma B.3. Plug (B.6) into (B.4) and note that we have \( M = 6L \), and then we obtain
\[
\sum_{j=0}^{T-1} \mathbb{E} \left\| \nabla F(x_j) \right\|^2_2 \leq C \left( M \mathbb{E} [F(x_0) - F(x_T)] + \frac{2T \sigma^2}{B} \cdot 1(B < n) \right), \tag{B.7}
\]
where \( C = 100 \). Divide both sides of (B.7) by \( T \), then Lemma A.1 holds trivially. \( \square \)

C Proof of Technical Lemmas

In this section, we provide the proofs of technical lemmas used in Appendix B.

C.1 Proof of Lemma B.2

Let \( M, \{T_i\}, \{B_i\}, B \) be the parameters defined in Algorithm 1 and \( \{x^{(l)}_i\}, \{g^{(l)}_i\} \) be the reference points and reference gradients defined in Algorithm 1. Let \( v^{(l)}_i, \mathcal{F}_i \) be the variables and filtration defined in Appendix B and let \( c^{(s)}_j \) be the constant series defined in Definition B.1.

In order to prove Lemma B.2, we will need the following supporting proportions and lemmas. We first state the proposition about the relationship among \( x^{(s)}_i, g^{(s)}_i \) and \( v^{(s)}_i \):

**Proposition C.1.** Let \( v^{(l)}_i \) be defined as in (B.1). Let \( p, s \) satisfy \( 0 \leq p \cdot \prod_{j=s+1}^{K} T_j < (p + 1) \cdot \prod_{j=s+1}^{K} T_j < T \). For any \( t, t' \) satisfying \( p \cdot \prod_{j=s+1}^{K} T_j \leq t < t' < (p + 1) \cdot \prod_{j=s+1}^{K} T_j \), it holds that
\[
x^{(s)}_i = x^{(s)}_{i'}, \tag{C.1}
g^{(s')}_i = g^{(s')}_{i'}, \tag{C.2}
v^{(s)}_i = v^{(s)}_{i'}, \tag{C.3}
\]
for any \( s' \) that satisfies \( 0 \leq s' \leq s \).
The following lemma spells out the relationship between $c^{(s-1)}_j$ and $c^{(s)}_T$. In a word, $c^{(s-1)}_j$ is about $1 + T_{s-1}$ times less than $c^{(s)}_T$:

**Lemma C.2.** If $B_s \geq 6^{K-s+1}(\prod_{i=s}^{K} T_i)^2$, $T_i \geq 1$ and $M \geq 6L$, then it holds that

$$c^{(s-1)}_j \cdot (1 + T_{s-1}) < c^{(s)}_T, \quad \text{for } 2 \leq s \leq K, 0 \leq j \leq T_{s-1},$$

(C.4)

and

$$c^{(K)}_j \cdot (1 + T_K) < M, \quad \text{for } 0 \leq j \leq T_K.$$  

(C.5)

Next lemma is a special case of Lemma B.2 with $s = K$:

**Lemma C.3.** Suppose $p$ satisfies $0 \leq pT_K < (p+1)T_K \leq \prod_{i=1}^{K} T_i$. If $M > L$, then we have

$$\mathbb{E}\left[ F(x_{(p+1),T_K}) + c^{(K)}_T \cdot \|x_{(p+1),T_K} - x_{p,T_K}\|^2 + \sum_{j=0}^{T_K-1} \left\| \nabla F(x_{p,T_K+j}) \right\|^2_{2} / 100M \right] \leq \mathbb{E}\left[ \|\nabla F(x_{p,T_K}) - v_{p,T_K}\|^2_{2} \right] \cdot T_K.$$  

(C.6)

The following lemma provides an upper bound of $\mathbb{E}\left[ \|\nabla F(x^{(l)}_t) - v^{(l)}_t\|^2_{2} \right]$, which plays an important role in our proof of Lemma B.2.

**Lemma C.4.** Let $t^l$ be as defined in (3.1), then we have $x^{(l)}_t = x^l_t$, and

$$\mathbb{E}\left[ \|\nabla F(x^{(l)}_t) - v^{(l)}_t\|^2_{2} \right] \leq \frac{L^2}{B_t} \left\| x^{(l)}_t - x^{(l-1)}_t \right\|^2_2 + \left\| \nabla F(x^{(l-1)}_t) - v^{(l-1)}_t \right\|^2_2.$$  

(C.7)

**Proof of Lemma B.2.** We use mathematical induction to prove that Lemma B.2 holds for any $1 \leq s \leq K$. When $s = K$, the statement holds because of Lemma C.3. Suppose that for $s+1$, Lemma B.2 holds for any $p'$ which satisfies $0 \leq p' \prod_{j=s+1}^{K} T_j < (p'+1) \prod_{j=s+1}^{K} T_j \leq \prod_{j=1}^{K} T_j$. We need to prove Lemma B.2 still holds for $s$ and $p$, where $p$ satisfies $0 \leq p \prod_{j=s}^{K} T_j < (p+1) \prod_{j=s}^{K} T_j \leq \prod_{j=1}^{K} T_j$. We first define $m = \prod_{j=s+1}^{K} T_j$ for simplification, then we choose $p' = pT_s + u$, and we set indices start$_u$ and end$_u$ as

$$start_u = p' \prod_{j=s+1}^{K} T_j, \quad end_u = start_u + \prod_{j=s+1}^{K} T_j.$$  

(C.8)

It can be easily verified that the following relationship also holds:

$$start_u = start + um, \quad end_u = start + (u+1)m.$$  

(C.8)

Based on (C.8), we have

$$\mathbb{E}\left[ \sum_{j=start_u}^{end_u-1} \left\| \nabla F(x_j) \right\|^2_{2} / 100M + F(x_{start+(u+1)m}) + c^{(s+1)}_{T_{s+1}} \cdot \left\| x_{start+(u+1)m} - x_{start+um} \right\|^2_{2} \right].$$  

16
where the last inequality holds because of the induction hypothesis that Lemma B.2 holds for $s+1$ and $p'$. Note that we have $x_{\text{start}} = x_{\text{start} + u \cdot m} = x_{(s)}_{\text{start}}$ from Proposition C.1, which implies

$$
\mathbb{E}\left[\|\nabla F(x_{\text{start}}) - v_{\text{start}}\|^2_{\mathcal{F}_{\text{start}}}\right] = \mathbb{E}\left[\left\|\nabla F(x_{(s)}_{\text{start}}) - v_{(s)}_{\text{start}}\right\|^2_{\mathcal{F}_{\text{start}}}\right]
$$

$$
\leq \frac{L^2}{B^2_s}\|x_{(s)}_{\text{start}} - x_{(s-1)}_{\text{start}}\|^2 + \|\nabla F(x_{(s-1)}_{\text{start}}) - v_{(s-1)}_{\text{start}}\|^2 \quad \text{(C.10)}
$$

$$
= \frac{L^2}{B^2_s}\|x_{\text{start} + u \cdot m} - x_{\text{start}}\|^2_2 + \|\nabla F(x_{\text{start}}) - v_{\text{start}}\|^2_2 \quad \text{(C.11)}
$$

where (C.10) holds because of Lemma C.4 and (C.11) holds due to Proposition C.1. Plugging (C.11) into (C.9) and taking expectation $\mathbb{E}[-\mathcal{F}_{\text{start}}]$ for (C.9), we have

$$
\mathbb{E}\left[\sum_{j=\text{start}_u}^{\text{end}_u-1} \frac{\|\nabla F(x_j)\|^2}{100M} + F(x_{\text{end}_u}) + c_{T_{s+1}}^{(s)} \left\|x_{\text{end}_u} - x_{\text{start}_u}\right\|^2_{\mathcal{F}_{\text{start}}}\right]
$$

$$
\leq F(x_{\text{start}_u}) + \frac{2}{M} \mathbb{E}\left[\|\nabla F(x_{\text{start}_u}) - v_{\text{start}_u}\|^2_{\mathcal{F}_{\text{start}_u}}\right] \cdot \prod_{j=s+1}^{K} T_j, \quad \text{(C.9)}
$$

Next we bound $\|x_{\text{start} + (u+1) \cdot m} - x_{\text{start}}\|^2_2$ as the following:

$$
\|x_{\text{start} + (u+1) \cdot m} - x_{\text{start}}\|^2_2 = \|x_{\text{start} + u \cdot m} - x_{\text{start}}\|^2_2 + \|x_{\text{start} + (u+1) \cdot m} - x_{\text{start} + u \cdot m}\|^2_2
$$

$$
+ 2\langle x_{\text{start} + (u+1) \cdot m} - x_{\text{start} + u \cdot m}, x_{\text{start} + u \cdot m} - x_{\text{start}} \rangle
$$

$$
\leq \|x_{\text{start} + u \cdot m} - x_{\text{start}}\|^2_2 + \|x_{\text{start} + (u+1) \cdot m} - x_{\text{start} + u \cdot m}\|^2_2 + \frac{1}{T_s} \cdot \|x_{\text{start} + u \cdot m} - x_{\text{start}}\|^2_2 + T_s \cdot \|x_{\text{start} + (u+1) \cdot m} - x_{\text{start} + u \cdot m}\|^2_2 \quad \text{(C.13)}
$$

$$
= \left(1 + \frac{1}{T_s}\right) \cdot \|x_{\text{start} + u \cdot m} - x_{\text{start}}\|^2_2 + (1 + T_s) \cdot \|x_{\text{start} + (u+1) \cdot m} - x_{\text{start} + u \cdot m}\|^2_2 \quad \text{(C.14)}
$$

where (C.13) holds because of Young’s inequality. Taking expectation $\mathbb{E}[\cdot\mathcal{F}_{\text{start}}]$ over (C.14) and multiplying $c_{u+1}^{(s)}$ on both sides, we obtain

$$
c_{u+1}^{(s)} \mathbb{E}\left[\|x_{\text{start} + (u+1) \cdot m} - x_{\text{start}}\|^2_{\mathcal{F}_{\text{start}}}ight] \leq c_{u+1}^{(s)} \left(1 + \frac{1}{T_s}\right) \mathbb{E}\left[\|x_{\text{start} + u \cdot m} - x_{\text{start}}\|^2_{\mathcal{F}_{\text{start}}}ight] + c_{u+1}^{(s)} (1 + T_s) \mathbb{E}\left[\|x_{\text{start} + (u+1) \cdot m} - x_{\text{start} + u \cdot m}\|^2_{\mathcal{F}_{\text{start}}}ight]. \quad \text{(C.15)}
$$
Adding up inequalities (C.15) and (C.12) together yields

\[
E \left[ \sum_{j=\text{start}_u}^{\text{end}_u-1} \frac{\|\nabla F(x_j)\|^2}{100M} + F(x_{\text{start}+(u+1)m}) + c_u^{(s)} \|x_{\text{start}+(u+1)m} - x_{\text{start}}\|^2 \right] \\
+ c_{T_{s+1}}^{(s+1)} \|x_{\text{start}+(u+1)m} - x_{\text{start}+um}\|^2 F_{\text{start}} \leq E \left[ F(x_{\text{start}+um}) + \|x_{\text{start}+um} - x_{\text{start}}\|^2 \left[ c_u^{(s)} \left( 1 + \frac{1}{T_s} \right) + \frac{3L^2}{B_s M} \prod_{j=s+1}^{K} T_j \right] F_{\text{start}} \right] \\
+ \frac{2}{M} E[\|\nabla F(x_{\text{start}}) - v_{\text{start}}\|^2 | F_{\text{start}}] \prod_{j=s+1}^{K} T_j \\
+ c_{T_{s+1}}^{(s+1)} E[\|x_{\text{start}+(u+1)m} - x_{\text{start}+um}\|^2 | F_{\text{start}}], \tag{C.16}
\]

where the last inequality holds due to the fact that \(c_u^{(s)} = c_u^{(s)} (1 + 1/T_s) + 3L^2/(B_s M) \cdot \prod_{j=s+1}^{K} T_j\) by Definition B.1 and \(c_{u+1}^{(s)} \cdot (1 + T_s) < c_{T_{s+1}}^{(s+1)}\) by Lemma C.2. Cancelling out the term \(c_{T_{s+1}}^{(s+1)} \cdot E[\|x_{\text{start}+(u+1)m} - x_{\text{start}+um}\|^2 | F_{\text{start}}]\) from both sides of (C.16), we get

\[
E \left[ \sum_{j=\text{start}_u}^{\text{end}_u-1} \frac{\|\nabla F(x_j)\|^2}{100M} + F(x_{\text{start}+(u+1)m}) + c_u^{(s)} \|x_{\text{start}+(u+1)m} - x_{\text{start}}\|^2 \right] F_{\text{start}} \\
\leq E \left[ F(x_{\text{start}+um}) + c_u^{(s)} \|x_{\text{start}+um} - x_{\text{start}}\|^2 \right] F_{\text{start}} \\
+ \frac{2}{M} E[\|\nabla F(x_{\text{start}}) - v_{\text{start}}\|^2 | F_{\text{start}}] \prod_{j=s+1}^{K} T_j.
\]

We now telescope the above inequality for \(u = 0 \text{ to } T_s - 1\), then we have

\[
E \left[ \sum_{u=0}^{T_s-1} \sum_{j=\text{start}_u}^{\text{end}_u-1} \frac{\|\nabla F(x_j)\|^2}{100M} + F(x_{\text{end}}) + c_{T_s}^{(s)} \|x_{\text{end}} - x_{\text{start}}\|^2 \right] F_{\text{start}} \\
\leq F(x_{\text{start}}) + \frac{2T_s}{M} \cdot E[\|\nabla F(x_{\text{start}}) - v_{\text{start}}\|^2 | F_{\text{start}}] \cdot \prod_{j=s+1}^{K} T_j.
\]

Since \(\text{start}_u = \text{end}_{u-1}, \text{start}_0 = \text{start}, \text{and end}_{T_s-1} = \text{end}\), we have

\[
E \left[ \sum_{j=\text{start}}^{\text{end}_u-1} \frac{\|\nabla F(x_j)\|^2}{100M} + F(x_{\text{end}}) + c_{T_s}^{(s)} \|x_{\text{end}} - x_{\text{start}}\|^2 \right] F_{\text{start}}
\]

18
\[ \leq F(x_{\text{start}}) + \frac{2}{M} \cdot \mathbb{E} \left[ \| \nabla F(x_{\text{start}}) - v_{\text{start}} \|_2^2 | F_{\text{start}} \right] \cdot \prod_{j=s}^{K} T_j. \quad (C.17) \]

Therefore, we have proved that Lemma B.2 still holds for \( s \) and \( p \). Then by mathematical induction, we have for all \( 1 \leq s \leq K \) and \( p \) which satisfy \( 0 \leq p \cdot \prod_{j=s}^{K} T_j < (p + 1) \cdot \prod_{j=s}^{K} T_j \leq \prod_{j=1}^{K} T_j \), Lemma B.2 holds. \( \square \)

### C.2 Proof of Lemma B.3

The following proof is adapted from that of Lemma A.1 in Lei et al. (2017). We provide the proof here for the self-containedness of our paper.

**Proof of Lemma B.3.** We only consider the case when \( m < N \). Let \( W_j = \mathbb{1}(j \in J) \), then we have

\[ \mathbb{E} W_j^2 = \mathbb{E} W_j = \frac{m}{N}, \quad \mathbb{E} W_j W_{j'} = \frac{m(m-1)}{N(N-1)}. \quad (C.18) \]

Thus we can rewrite the sample mean as

\[ \frac{1}{m} \sum_{j \in J} a_j = \frac{1}{m} \sum_{i=1}^{N} W_i a_i, \quad (C.19) \]

which immediately implies

\[
\mathbb{E} \left\| \frac{1}{m} \sum_{j \in J} a_j \right\|_2^2 = \frac{1}{m^2} \left( \sum_{j=1}^{N} \mathbb{E} W_j^2 \|a_j\|_2^2 + \sum_{j \neq j'} \mathbb{E} W_j W_{j'} \langle a_j, a_{j'} \rangle \right)
\]

\[
= \frac{1}{m^2} \left( \frac{m}{N} \sum_{j=1}^{N} \|a_j\|_2^2 + \frac{m(m-1)}{N(N-1)} \sum_{j \neq j'} \langle a_j, a_{j'} \rangle \right)
\]

\[
= \frac{1}{m^2} \left( \frac{m}{N} - \frac{m(m-1)}{N(N-1)} \right) \sum_{j=1}^{N} \|a_j\|_2^2 + \frac{m(m-1)}{N(N-1)} \left\| \sum_{j=1}^{N} a_j \right\|_2^2
\]

\[
= \frac{1}{m^2} \left( \frac{m}{N} - \frac{m(m-1)}{N(N-1)} \right) \sum_{j=1}^{N} \|a_j\|_2^2
\]

\[
\leq \frac{1}{m} \cdot \frac{1}{N} \sum_{j=1}^{N} \|a_j\|_2^2.
\]

\( \square \)

### D Proofs of the Auxiliary Lemmas

In this section, we present the additional proofs of supporting lemmas used in Appendix C. Let \( M, \{T_l\}, \{B_l\} \) and \( B \) be the parameters defined in Algorithm 1. Let \( \{x_i^{(l)}\}, \{g_i^{(l)}\} \) be the reference
points and reference gradients used in Algorithm 1. Finally, \( v^{(t)}_t, \mathcal{F}_t \) are the variables and filtration defined in Appendix B and \( c_j^{(s)} \) are the constant series defined in Definition B.1.

### D.1 Proof of Proposition C.1

**Proof of Proposition C.1.** By the definition of reference point \( x_t^{(s)} \) in (3.1), we can easily verify that (C.1) holds trivially.

Next we prove (C.2). Note that by (C.1) we have \( x_t^{(s)} = x_t^{(s')} \). For any \( 0 \leq s' \leq s \), it is also true that \( x_t^{(s')} = x_t^{(s')} \) by (3.1), which means \( x_t \) and \( x_t^{(s')} \) share the same first \( s + 1 \) reference points. Then by the update rule of \( g_t^{(s')} \) in Algorithm 1, we will maintain \( g_t^{(s')} \) unchanged from time step \( t \) to \( t' \). In other words, we have \( g_t^{(s')} = g_t^{(s')} \) for all \( 0 \leq s' \leq s \).

We now prove the last claim (C.3). Based on (B.1) and (C.2), we have \( v_t^{(s)} = \sum_{s''=0}^{s} g_t^{(s'')} = \sum_{s''=0}^{s} g_{p\Pi_{j=s+1}^{K} t_j}^{(s'')} = v_t^{(s)} \). Since for any \( s \leq s'' \leq K \), we have the following equations by the update in Algorithm 1 (Line 18).

\[
\begin{align*}
    x_{p\Pi_{j=s+1}^{K} t_j}^{(s'')} &= x_{p\Pi_{j=s+1}^{K} t_j / \Pi_{j=s''+1}^{K} t_j}^{(s''+1)} \prod_{l=s+1}^{l=s''+1} T_l \\
    &= x_{p\Pi_{j=s+1}^{K} t_j / \Pi_{j=s''+1}^{K} t_j}^{(s''+1)} \\
    &= x_t^{(s)}
\end{align*}
\]

Then for any \( s < s'' \leq K \), we have

\[
\begin{align*}
    g_{p\Pi_{j=s+1}^{K} t_j}^{(s'')} &= \frac{1}{B_{s''}} \sum_{i \in I} \left[ \nabla f_i \left( x_{p\Pi_{j=s+1}^{K} t_j}^{(s'')} \right) - \nabla f_i \left( x_{p\Pi_{j=s+1}^{K} t_j}^{(s''+1)} \right) \right] = 0. \quad (D.1)
\end{align*}
\]

Thus, we have

\[
\begin{align*}
    v_{p\Pi_{j=s+1}^{K} t_j}^{(s''+1)} &= \sum_{s''=0}^{s} g_{p\Pi_{j=s+1}^{K} t_j}^{(s'')} = \sum_{s''=0}^{s} g_{p\Pi_{j=s+1}^{K} t_j}^{(s'')} = \sum_{s''=0}^{s} g_{t}^{(s'')} = v_t^{(s)}, \quad (D.2)
\end{align*}
\]

where the first equality holds because of the definition of \( v_{p\Pi_{j=s+1}^{K} t_j}^{(s''+1)} \), the second equality holds due to (D.1), the third equality holds due to (C.2) and the last equality holds due to (B.1). This completes the proof of (C.3).

\[\Box\]

### D.2 Proof of Lemma C.2

**Proof of Lemma C.2.** For any fixed \( s \), it can be seen that from the definition in (B.3), \( c_j^{(s)} \) is monotonically decreasing with \( j \). In order to prove (C.4), we only need to compare \( (1 + T_{s-1}) \cdot c_j^{(s-1)} \) and \( c_{T_{s-1}}^{(s)} \). Furthermore, by the definition of series \( \{ c_j^{(s)} \} \) in (B.3), it can be inducted that when \( 0 \leq j \leq T_{s-1} \),

\[
    c_j^{(s-1)} = \left( 1 + \frac{1}{T_{s-1}} \right)^{T_{s-1}-j} \cdot c_{T_{s-1}}^{(s-1)} + \left( 1 + \frac{1}{T_{s-1}} \right)^{T_{s-1}-j} - 1 \cdot \frac{3L^2}{M} \cdot \prod_{l=s}^{l=s} T_l \cdot \frac{1}{B_{s-1}}. \quad (D.3)
\]

20
We take $j = 0$ in (D.3) and obtain
\[
c_0^{(s-1)} = \left(1 + \frac{1}{T_{s-1}}\right)^{T_{s-1}} \cdot c_{T_{s-1}}^{(s-1)} + \frac{(1 + 1/T_{s-1})^{T_{s-1}} - 1}{1/T_{s-1}} \cdot \frac{3L^2}{M} \cdot \prod_{l=s}^{K} T_l
\]
\[
< 2.8 \times c_{T_{s-1}}^{(s-1)} + \frac{6L^2}{M} \cdot \prod_{l=s}^{K} T_l
\]
\[
\leq \frac{2.8M + 6L^2/M}{6K-s+2 \cdot \prod_{l=s}^{K} T_l}
\]
\[
< \frac{3M}{6K-s+2 \cdot \prod_{l=s}^{K} T_l}
\]

where (D.4) holds because $(1 + 1/n)^n < 2.8$ for any $n \geq 1$, (D.5) holds due to the definition of $c_{T_{s-1}}^{(s-1)}$ in (B.2) and $B_{s-1} \geq 6K-s+2(\prod_{l=s-1}^{K} T_l)^2$ and (D.6) holds because $M \geq 6L$. Recall that $c_j^{(s)}$ is monotonically decreasing with $j$ and the inequality in (D.6). Thus for all $2 \leq s \leq K$ and $0 \leq j \leq T_{s-1}$, we have
\[
(1 + T_{s-1}) \cdot c_j^{(s-1)} \leq (1 + T_{s-1}) \cdot c_0^{(s-1)} \leq (1 + T_{s-1}) \cdot \frac{3M}{6K-s+2 \cdot \prod_{l=s}^{K} T_l}
\]
\[
< \frac{6M}{6K-s+2 \cdot \prod_{l=s}^{K} T_l}
\]
\[
= c_j^{(s)}
\]

where the third inequality holds because $(1 + T_{s-1})/T_{s-1} \leq 2$ when $T_{s-1} \geq 1$ and the last equation comes from the definition of $c_j^{(s)}$ in (B.2). This completes the proof of (C.4).

Using similar techniques, we can obtain the upper bound for $c_j^K$ which is similar to inequality (D.6) with $s-1$ replaced by $K$. Therefore, we have
\[
(1 + T_K) \cdot c_j^{(K)} \leq (1 + T_K) \cdot c_0^{(K)} \leq \frac{6M}{6K-K+1 \cdot \prod_{l=K}^{K} T_l} \leq M,
\]
which completes the proof of (C.5).

\[\square\]

**D.3 Proof of Lemma C.3**

Now we prove Lemma C.3, which is a special case of Lemma B.2 if we choose $s = K$.

**Proof of Lemma C.3.** To simplify notations, we use $\mathbb{E}[]$ to denote the conditional expectation $\mathbb{E} [\cdot | \mathcal{F}_{p:T_K}]$ in the rest of this proof. For $0 \leq j < T_K$, we denote $h_{p:T_K+j} = -(10M)^{-1} \cdot v_{p:T_K+j}$. According to the update in Algorithm 1 (Line 12), we have
\[
x_{p:T_K+j+1} = x_{p:T_K+j} + h_{p:T_K+j},
\]

(D.8)
which immediately implies
\[
F(x_{p,T_K+j+1}) = F(x_{p,T_K+j} + h_{p,T_K+j})
\le F(x_{p,T_K+j}) + \langle \nabla F(x_{p,T_K+j}), h_{p,T_K+j} \rangle + \frac{L}{2} \|h_{p,T_K+j}\|_2^2
\]
(D.9)
\[
= [\langle v_{p,T_K+j}, h_{p,T_K+j} \rangle + 5M \|h_{p,T_K+j}\|_2^2] + F(x_{p,T_K+j})
+ \langle \nabla F(x_{p,T_K+j}) - v_{p,T_K+j}, h_{p,T_K+j} \rangle + \left( \frac{L}{2} - 5M \right) \|h_{p,T_K+j}\|_2^2
\]
\[
\le F(x_{p,T_K+j}) + \langle \nabla F(x_{p,T_K+j}) - v_{p,T_K+j}, h_{p,T_K+j} \rangle + (L - 5M) \|h_{p,T_K+j}\|_2^2,
\] (D.10)
where (D.9) is due to the L-smoothness of F and (D.10) holds because \( \langle v_{p,T_K+j}, h_{p,T_K+j} \rangle + 5M \|h_{p,T_K+j}\|_2^2 = -5M \|h_{p,T_K+j}\|_2^2 \le 0 \). Further by Young’s inequality, we obtain
\[
F(x_{p,T_K+j+1}) \le F(x_{p,T_K+j}) + \frac{1}{2M} \|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|_2^2 + \left( \frac{M}{2} + L - 5M \right) \|h_{p,T_K+j}\|_2^2
\]
\[
\le F(x_{p,T_K+j}) + \frac{1}{M} \|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|_2^2 - 3M \|h_{p,T_K+j}\|_2^2,
\] (D.11)
where the second inequality holds because \( M > L \). Now we bound the term \( c_{j+1}^{(K)} \|x_{p,T_K+j+1} - x_{p,T_K}\|_2^2 \). By (D.8) we have
\[
c_{j+1}^{(K)} \|x_{p,T_K+j+1} - x_{p,T_K}\|_2^2 = c_{j+1}^{(K)} \|x_{p,T_K+j} - x_{p,T_K} + h_{p,T_K+j}\|_2^2
\]
\[
= c_{j+1}^{(K)} \left[ \|x_{p,T_K+j} - x_{p,T_K}\|_2^2 + \|h_{p,T_K+j}\|_2^2 + 2 \langle x_{p,T_K+j} - x_{p,T_K}, h_{p,T_K+j} \rangle \right].
\]
Applying Young’s inequality yields
\[
c_{j+1}^{(K)} \|x_{p,T_K+j+1} - x_{p,T_K}\|_2^2 \le c_{j+1}^{(K)} \left[ \|x_{p,T_K+j} - x_{p,T_K}\|_2^2 + \|h_{p,T_K+j}\|_2^2 
\right.
\]
\[
+ \frac{1}{T_K} \|x_{p,T_K+j} - x_{p,T_K}\|_2^2 + T_K \|h_{p,T_K+j}\|_2^2 \right]
\]
\[
= c_{j+1}^{(K)} \left[ \left( 1 + \frac{1}{T_K} \right) \|x_{p,T_K+j} - x_{p,T_K}\|_2^2 + (1 + T_K) \|h_{p,T_K+j}\|_2^2 \right],
\] (D.12)
Adding up inequalities (D.12) and (D.11), we get
\[
F(x_{p,T_K+j+1}) + c_{j+1}^{(K)} \|x_{p,T_K+j+1} - x_{p,T_K}\|_2^2
\]
\[
\le F(x_{p,T_K+j}) + \frac{1}{M} \|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|_2^2 - \left[ 3M - c_{j+1}^{(K)} (1 + T_K) \right] \|h_{p,T_K+j}\|_2^2
\]
\[
+ c_{j+1}^{(K)} \left( 1 + \frac{1}{T_K} \right) \|x_{p,T_K+j} - x_{p,T_K}\|_2^2
\]
\[
\le F(x_{p,T_K+j}) + \frac{1}{M} \|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|_2^2 - 2M \|h_{p,T_K+j}\|_2^2
\]
\[
+ c_{j+1}^{(K)} \left( 1 + \frac{1}{T_K} \right) \|x_{p,T_K+j} - x_{p,T_K}\|_2^2,
\] (D.13)

22
where the last inequality holds due to the fact that \((K)_{j+1}(1 + T_K) < M\) by Lemma C.2. Next we bound \(\|\nabla F(x_{p,T_K+j})\|^2\) with \(\|h_{p,T_K+j}\|^2\). Note that by (D.8), we have
\[
\|\nabla F(x_{p,T_K+j})\|^2 = \| \nabla F(x_{p,T_K+j}) - v_{p,T_K+j} \|^2 + 10Mh_{p,T_K+j}^2 \\
\leq 2(\|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|^2 + 100M^2\|h_{p,T_K+j}\|^2),
\]
which immediately implies
\[
-2M\|h_{p,T_K+j}\|^2 \leq \frac{2}{100M}(\|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|^2 - \frac{1}{100M}\|\nabla F(x_{p,T_K+j})\|^2).
\]
Plugging (D.14) into (D.13), we have
\[
F(x_{p,T_K+j+1}) + c_{j+1}(K)\|x_{p,T_K+j+1} - x_{p,T_K}\|^2 \\
\leq F(x_{p,T_K+j}) + \frac{1}{M}\|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|^2 + \frac{1}{50M}\|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|^2 \\
- \frac{1}{100M}\|\nabla F(x_{p,T_K+j})\|^2 + c_{j+1}(K)\left(1 + \frac{1}{T_K}\right)\|x_{p,T_K+j} - x_{p,T_K}\|^2 \\
\leq F(x_{p,T_K+j}) + \frac{2}{M}\|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|^2 - \frac{1}{100M}\|\nabla F(x_{p,T_K+j})\|^2 \\
+ c_{j+1}(K)\left(1 + \frac{1}{T_K}\right)\|x_{p,T_K+j} - x_{p,T_K}\|^2. 
\] (D.15)
Next we bound \(\|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|^2\). First, by Lemma C.4 we have
\[
\mathbb{E}\|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|^2 \leq \frac{L^2}{B_K}\mathbb{E}\|x_{p,T_K+j}^{(K)} - x_{p,T_K+j}^{(K-1)}\|^2 + \mathbb{E}\|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|^2. 
\]
Since \(x_{p,T_K+j}^{(K)} = x_{p,T_K+j}, v_{p,T_K+j}^{(K)} = v_{p,T_K+j}, x_{p,T_K+j}^{(K-1)} = x_{p,T_K}\) and \(v_{p,T_K+j}^{(K-1)} = v_{p,T_K}\), we have
\[
\mathbb{E}\|\nabla F(x_{p,T_K+j}) - v_{p,T_K+j}\|^2 \leq \frac{L^2}{B_K}\mathbb{E}\|x_{p,T_K+j} - x_{p,T_K}\|^2 + \mathbb{E}\|\nabla F(x_{p,T_K}) - v_{p,T_K}\|^2. 
\] (D.16)
We now take expectation \(\mathbb{E}[\cdot]\) with (D.15) and plug (D.16) into (D.15). We obtain that
\[
\mathbb{E}\left[F(x_{p,T_K+j+1}) + c_{j+1}(K)\|x_{p,T_K+j+1} - x_{p,T_K}\|^2 + \frac{1}{100M}\|\nabla F(x_{p,T_K+j})\|^2\right] \\
\leq \mathbb{E}\left[F(x_{p,T_K+j}) + \left(c_{j+1}(K)\left(1 + \frac{1}{T_K}\right) + \frac{3L^2}{B_KM}\right)\|x_{p,T_K+j} - x_{p,T_K}\|^2 + \frac{2}{M}\|\nabla F(x_{p,T_K}) - v_{p,T_K}\|^2\right] \\
\leq \mathbb{E}\left[F(x_{p,T_K+j}) + c_{j}(K)\|x_{p,T_K+j} - x_{p,T_K}\|^2 + 2\frac{1}{M}\cdot \|\nabla F(x_{p,T_K}) - v_{p,T_K}\|^2\right], 
\] (D.17)
where (D.17) holds because we have \(c_{j}(K) = c_{j+1}(1 + 1/T_K) + 3L^2/(B_KM)\) by Definition B.1.
Telescoping (D.17) for \( j = 0 \) to \( T_K - 1 \), we have
\[
\mathbb{E} \left[ F(x_{(p+1):T_K}) + c_T^{(K)} \cdot \|x_{(p+1):T_K} - x_{p:T_K}\|^2 \right] + \frac{1}{100M} \sum_{j=0}^{T_K-1} \mathbb{E} \|\nabla F(x_{p:T_K+j})\|^2 \\
\leq F(x_{p:T_K}) + \frac{2T_K}{M} \cdot \mathbb{E} \|\nabla F(x_{p:T_K}) - v_{p:T_K}\|^2,
\]
(D.18)
which completes the proof.

\[ \square \]

D.4 Proof of Lemma C.4

Proof of Lemma C.4. If \( t^l = t^{l-1} \), we have \( x_{t^l} = x_{t^{l-1}}^{(1)} \) and \( v_{t^l} = v_{t^{l-1}}^{(1)} \). In this case the statement in Lemma C.4 holds trivially. Therefore, we assume \( t^l \neq t^{l-1} \) in the following proof. Note that
\[
\mathbb{E} \left[ \|\nabla F(x_{t^l}^{(1)}) - v_{t^l}^{(1)}\|^2 \big| \mathcal{F}_{t^l} \right] \\
= \mathbb{E} \left[ \|\nabla F(x_{t^l}^{(1)}) - v_{t^l}^{(1)} - \mathbb{E}[\nabla F(x_{t^l}^{(1)}) - v_{t^l}^{(1)}]|\mathcal{F}_{t^l}]\|^2 \big| \mathcal{F}_{t^l} \right] + \mathbb{E} \left[\|\nabla F(x_{t^l}^{(1)}) - v_{t^l}^{(1)}|\mathcal{F}_{t^l}\|^2 \right] \\
= \mathbb{E} \left[ \|\nabla F(x_{t^l}^{(1)}) - \sum_{j=0}^{l} g_{t^l}^{(j)} - \mathbb{E}\left[\nabla F(x_{t^l}^{(1)}) - \sum_{j=0}^{l} g_{t^l}^{(j)}\right]\|^2 \big| \mathcal{F}_{t^l} \right] + \mathbb{E} \left[\|\nabla F(x_{t^l}^{(1)}) - \sum_{j=0}^{l} g_{t^l}^{(j)}|\mathcal{F}_{t^l}\|^2 \right],
\]
(D.19)
where in the second equation we used the definition \( v_{t^l}^{(1)} = \sum_{i=0}^{l} g_{t^l}^{(i)} \) in (B.1). We first upper bound term \( J_1 \). According to the update rule in Algorithm 1 (Line 21-25), when \( j < l \), \( g_{t^l}^{(j)} \) will not be updated at the \( t^l \)-th iteration. Thus we have \( \mathbb{E}[g_{t^l}^{(j)}|\mathcal{F}_{t^l}] = g_{t^l}^{(j)} \) for all \( j < l \). In addition, by the definition of \( \mathcal{F}_{t^l} \), we have \( \mathbb{E}[\nabla F(x_{t^l}^{(1)})|\mathcal{F}_{t^l}] = \nabla F(x_{t^l}^{(1)}) \). Then we have the following equation
\[
J_1 = \mathbb{E} \left[ \|g_{t^l}^{(1)} - \mathbb{E}[g_{t^l}^{(1)}|\mathcal{F}_{t^l}]\|^2 \big| \mathcal{F}_{t^l} \right].
\]
(D.20)
We further have
\[
g_{t^l}^{(1)} = \frac{1}{B_t} \sum_{i \in I} \left[ \nabla f_i(x_{t^l}^{(1)}) - \nabla f_i(x_{t^l}^{(l-1)}) \right], \quad \mathbb{E}[g_{t^l}^{(1)}|\mathcal{F}_{t^l}] = \nabla F(x_{t^l}^{(0)}) - \nabla F(x_{t^l}^{(l-1)}).
\]
(D.21)
Therefore, we can apply Lemma B.3 to (D.20) and obtain
\[
J_1 \leq \frac{1}{B_t} \cdot \frac{1}{n} \sum_{i=1}^{n} \left\| \nabla f_i(x_{t^l}^{(1)}) - \nabla f_i(x_{t^l}^{(l-1)}) - \left[ \nabla F(x_{t^l}^{(0)}) - \nabla F(x_{t^l}^{(l-1)}) \right] \right\|^2 \\
\leq \frac{1}{B_t n} \sum_{i=1}^{n} \left\| \nabla f_i(x_{t^l}^{(1)}) - \nabla f_i(x_{t^l}^{(l-1)}) \right\|^2 \\
\leq \frac{L^2}{B_t} \left\| x_{t^l}^{(1)} - x_{t^l}^{(l-1)} \right\|^2,
\]
(D.22)
where the second inequality is due to the fact that $E[\|X - E[X]\|^2] \leq E[\|X\|^2]$ for any random vector $X$ and the last inequality holds due to the $L$-smoothness condition of $F$.

Next we turn to bound term $J_2$. Note that

$$E[g(l)\mid F_t] = E\left[\frac{1}{B_t} \sum_{i \in I} (\nabla f_i(x_t^{(l)}) - \nabla f_i(x_t^{(l-1)}) \mid F_t)\right] = \nabla F(x_t^{(l)}) - \nabla F(x_t^{(l-1)}),$$

which immediately implies

$$E\left[\nabla F(x_t^{(l)}) - \sum_{j=0}^{l} g_t^{(j)} \mid F_t\right] = E\left[\nabla F(x_t^{(l)}) - \nabla F(x_t^{(l-1)}) + \nabla F(x_t^{(l-1)}) - \sum_{j=0}^{l-1} g_t^{(j)} \mid F_t\right]\right.
\begin{equation}
= E[\nabla F(x_t^{(l-1)}) - v_t^{(l-1)} \mid F_t]
= \nabla F(x_t^{(l-1)}) - v_t^{(l-1)},
\end{equation}

where the last equation is due to the definition of $F_t$. Plugging $J_1$ and $J_2$ into (D.19) yields the following result:

$$E\left[\|\nabla F(x_t^{(l)}) - v_t^{(l)}\|^2 \mid F_t\right] \leq \frac{L^2}{B_t} \|x_t^{(l)} - x_t^{(l-1)}\|^2 + \|\nabla F(x_t^{(l-1)}) - v_t^{(l-1)}\|^2,$$

which completes the proof.

\section*{E An Equivalent Version of Algorithm 1}

Recall the One-epoch-SNVRG algorithm in Algorithm 1. Here we present an equivalent version of Algorithm 1 using nested loops, which is displayed in Algorithm 4 and is more aligned with the illustration in Figure 2(b). Note that the notation used in Algorithm 4 is slightly different from that in Algorithm 1 to avoid confusion.

\section*{References}

AGARWAL, A. and BOTTOU, L. (2014). A lower bound for the optimization of finite sums. arXiv preprint arXiv:1410.0723.

AGARWAL, N., ALLENZHU, Z., BULLINS, B., HAZAN, E. and MA, T. (2017). Finding approximate local minima for nonconvex optimization in linear time.

ALLEN-ZHU, Z. (2017a). Katyusha: The first direct acceleration of stochastic gradient methods. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing. ACM.

ALLEN-ZHU, Z. (2017b). Natasha 2: Faster non-convex optimization than sgd. arXiv preprint arXiv:1708.08694.

ALLEN-ZHU, Z. (2018). Katyusha x: Practical momentum method for stochastic sum-of-nonconvex optimization. arXiv preprint arXiv:1802.03866.
Algorithm 4 One-epoch SNVRG($F, x_0, K, M, \{T_i\}, \{B_i\}, B$)

1: **Input:** Function $F$, starting point $x_0$, loop number $K$, step size parameter $M$, loop parameters $T_i, i \in [K]$, batch parameters $B_i, i \in [K]$, base batch $B > 0$.

2: **Output:** $[x_{\text{out}}, x_{\text{end}}]$ 

3: $T \leftarrow \prod_{i=1}^{K} T_i$

4: Uniformly generate index set $I \subset [n]$ without replacement

5: $g_{[0]}^{(0)} \leftarrow \frac{1}{T} \sum_{i \in I} \nabla f_i(x_0)$

6: $x_{[0]}^{(l)} \leftarrow x_0, \quad 0 \leq l \leq K$,

7: for $t_1 = 0, \ldots, T_1 - 1$ do

8: Uniformly generate index set $I \subset [n]$ without replacement, $|I| = B_1$

9: $g_{[t_1]}^{(1)} \leftarrow \frac{1}{T_1} \sum_{i \in I} \left[ \nabla f_i(x_{[t_1]}^{(1)}) - \nabla f_i(x_{[0]}^{(0)}) \right]$

10: end for

11: Uniformly generate index set $I \subset [n]$ without replacement, $|I| = B_t$

12: $g_{[t]}^{(l)} \leftarrow \frac{1}{T} \sum_{i \in I} \left[ \nabla f_i(x_{[t]}^{(l)}) - \nabla f_i(x_{[t-1]}^{(l)}) \right]$

13: for $t_K = 0, \ldots, T_K - 1$ do

14: Uniformly generate index set $I \subset [n]$ without replacement, $|I| = B_K$

15: $g_{[t_K]}^{(K)} \leftarrow \frac{1}{T_K} \sum_{i \in I} \left[ \nabla f_i(x_{[t_K]}^{(K)}) - \nabla f_i(x_{[t_K-1]}^{(K-1)}) \right]$

16: Denote $t = \sum_{j=1}^{K} t_j \prod_{i=j+1}^{K} T_i$, then let $x_{t+1} \leftarrow x_t - 1/(10M) \cdot \sum_{l=0}^{K} g_{[t]}^{(l)}$

17: end for

18: $x_{[t+1]}^{(K)} \leftarrow x_{[t]}^{(K)}$

19: end for

20: $x_{[t]}^{(l)} \leftarrow x_{[T]}^{(l+1)}$

21: end for

22: $x_{[t+1]}^{(1)} \leftarrow x_{[T]}^{(2)}$

23: end for

24: $x_{\text{out}} \leftarrow$ a uniformly random choice from $\{x_0, \ldots, x_{T-1}\}$

25: $x_{\text{end}} \leftarrow x_T$

26: **return** $[x_{\text{out}}, x_{\text{end}}]$
Carmon, Y., Duchi, J. C., Hinder, O. and Sidford, A. (2017a). “convex until proven guilty”: Dimension-free acceleration of gradient descent on non-convex functions. In International Conference on Machine Learning.

Carmon, Y., Duchi, J. C., Hinder, O. and Sidford, A. (2017b). Lower bounds for finding stationary points of non-convex, smooth high-dimensional functions.

Defazio, A., Bach, F. and Lacoste-Julien, S. (2014a). Saga: A fast incremental gradient method with support for non-strongly convex composite objectives. In Advances in Neural Information Processing Systems.

Defazio, A., Domke, J. et al. (2014b). Finito: A faster, permutable incremental gradient method for big data problems. In International Conference on Machine Learning.

Garber, D. and Hazan, E. (2015). Fast and simple pca via convex optimization. arXiv preprint arXiv:1509.05647.

Ghadimi, S. and Lan, G. (2012). Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization i: A generic algorithmic framework. SIAM Journal on Optimization 22 1469–1492.

Ghadimi, S. and Lan, G. (2016). Accelerated gradient methods for nonconvex nonlinear and stochastic programming. Mathematical Programming 156 59–99.

Hillar, C. J. and Lim, L.-H. (2013). Most tensor problems are np-hard. Journal of the ACM (JACM) 60 45.

Hu, C., Pan, W. and Kwok, J. T. (2009). Accelerated gradient methods for stochastic optimization and online learning. In Advances in Neural Information Processing Systems.

Johnson, R. and Zhang, T. (2013). Accelerating stochastic gradient descent using predictive variance reduction. In Advances in neural information processing systems.

Karimi, H., Nutini, J. and Schmidt, M. (2016). Linear convergence of gradient and proximal-gradient methods under the polyak-lojasiewicz condition. In Joint European Conference on Machine Learning and Knowledge Discovery in Databases. Springer.

Lan, G. (2012). An optimal method for stochastic composite optimization. Mathematical Programming 133 365–397.

Lan, G. and Zhou, Y. (2017). An optimal randomized incremental gradient method. Mathematical programming 1–49.

Lei, L., Ju, C., Chen, J. and Jordan, M. I. (2017). Non-convex finite-sum optimization via scsg methods. In Advances in Neural Information Processing Systems.

Li, H. and Lin, Z. (2015). Accelerated proximal gradient methods for nonconvex programming. In Advances in neural information processing systems.
Li, Q., Zhou, Y., Liang, Y. and Varshney, P. K. (2017). Convergence analysis of proximal gradient with momentum for nonconvex optimization. *arXiv preprint arXiv:1705.04925*.

Lin, H., Mairal, J. and Harchaoui, Z. (2015). A universal catalyst for first-order optimization. In *Advances in Neural Information Processing Systems*.

Mairal, J. (2015). Incremental majorization-minimization optimization with application to large-scale machine learning. *SIAM Journal on Optimization* 25 829–855.

Nesterov, Y. (2005). Smooth minimization of non-smooth functions. *Mathematical programming* 103 127–152.

Nesterov, Y. (2014). *Introductory Lectures on Convex Optimization*. Kluwer Academic Publishers.

Paquette, C., Lin, H., Drusvyatskiy, D., Mairal, J. and Harchaoui, Z. (2017). Catalyst acceleration for gradient-based non-convex optimization. *arXiv preprint arXiv:1703.10993*.

Polyak, B. T. (1963). Gradient methods for minimizing functionals. *Zhurnal Vychislitel’noi Matematiki i Matematicheskoi Fiziki* 3 643–653.

Polyak, B. T. (1964). Some methods of speeding up the convergence of iteration methods. *USSR Computational Mathematics and Mathematical Physics* 4 1–17.

Reddi, S. J., Hefny, A., Sra, S., Poczos, B. and Smola, A. (2016a). Stochastic variance reduction for nonconvex optimization 314–323.

Reddi, S. J., Sra, S., Poczos, B. and Smola, A. (2016b). Fast incremental method for smooth nonconvex optimization. In *Decision and Control (CDC), 2016 IEEE 55th Conference on*. IEEE.

Reddi, S. J., Sra, S., Poczos, B. and Smola, A. J. (2016c). Proximal stochastic methods for nonsmooth nonconvex finite-sum optimization. In *Advances in Neural Information Processing Systems*.

Roux, N. L., Schmidt, M. and Bach, F. R. (2012). A stochastic gradient method with an exponential convergence rate for finite training sets. In *Advances in Neural Information Processing Systems*.

Shalev-Shwartz, S. (2015). Sdca without duality. *arXiv preprint arXiv:1502.06177*.

Shalev-Shwartz, S. (2016). Sdca without duality, regularization, and individual convexity. In *International Conference on Machine Learning*.

Shalev-Shwartz, S. and Zhang, T. (2013). Stochastic dual coordinate ascent methods for regularized loss minimization. *Journal of Machine Learning Research* 14 567–599.

Woodworth, B. E. and Srebro, N. (2016). Tight complexity bounds for optimizing composite objectives. In *Advances in neural information processing systems*.

Xiao, L. and Zhang, T. (2014). A proximal stochastic gradient method with progressive variance reduction. *SIAM Journal on Optimization* 24 2057–2075.