EXPLICIT COMPUTATIONS WITH CUBIC FOURFOLDS, GUSHEL–MUKAI FOURFOLDS, AND THEIR ASSOCIATED K3 SURFACES

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Abstract. We present some applications of the Macaulay2 software package SpecialFanoFourfolds, a package for working with Hodge-special cubic fourfolds and Hodge-special Gushel–Mukai fourfolds. In particular, we show how to construct new examples of such fourfolds, some of which turn out to be rational. We also describe how to calculate K3 surfaces associated with cubic or Gushel-Mukai fourfolds, which relies on an explicit unirationality of some moduli spaces of K3 surfaces.

Introduction

One of the aims of this note is to show how some computations with cubic fourfolds and Gushel–Mukai fourfolds, which may appear very abstract, can be done in an explicit and completely automatic way. For this purpose, we will use the package SpecialFanoFourfolds [Sta22], which is included with Macaulay2 [GS21] (1). This package implements several functions to construct and analyze such fourfolds, revolving around questions of rationality. Most of the ideas on which these functions are based came from the papers [RS19, RS21b]; other references are [BRS19, HS20, HS21a, ABS21, Sta21b]. For the general theory on cubic and Gushel–Mukai fourfolds, we mainly refer to [Has99, Has00, DIM15].

Recall that cubic fourfolds and Gushel–Mukai fourfolds are prime Fano fourfolds of index 3 and 2, respectively. By the work of Kobayashi and Ochiai [KO73], Iskovskikh and Fujita [Fuj90], and Mukai [Muk89], prime Fano fourfolds of index ≥ 2 are completely classified. We summarize in Table 1 this classification. For most of the known types of prime Fano fourfolds, there are classical constructions to produce rational parameterizations of the general fourfold. It is also classically known that some special cubic fourfolds and some special Gushel–Mukai fourfolds are rational (see [Mor40, Fan43, Rot49]). However, the question whether the general cubic fourfold (resp., the general Gushel–Mukai fourfold) is rational or not is still unsolved. It is expected that inside the moduli space of cubic fourfolds (resp., Gushel–Mukai fourfolds), the rational fourfolds should belong to the so-called Noether–Lefschetz locus, which is a countable union of hypersurfaces; in particular, the very general fourfold should be irrational. Roughly speaking, the Noether–Lefschetz locus parametrizes fourfolds containing a surface whose cohomology class does not come from the ambient space (\(\mathbb{P}^5\) for cubic fourfolds, and \(\mathbb{G}(1, 4) \subset \mathbb{P}^9\) for Gushel–Mukai fourfolds).

1A more updated version of the package is available at https://github.com/Macaulay2/M2/blob/development/M2/Macaulay2/packages/SpecialFanoFourfolds.m2.
| Fourfold | Index | Irrationality of the very general | Description of the rational ones | Ref. |
|----------|-------|----------------------------------|---------------------------------|------|
| Projective space $\mathbb{P}^4$ | 5 | no | all | trivial |
| Quadric hypersurface in $\mathbb{P}^5$ | 4 | no | all | trivial |
| Cubic hypersurface in $\mathbb{P}^5$ (cubic fourfold) | 3 | not known | some examples are known but the general picture is not clear | [Fan43] |
| Complete intersection of two quadrics in $\mathbb{P}^6$ | 3 | no | all | trivial |
| Linear section in $\mathbb{P}^{17}$ of $G(1,4) \subset \mathbb{P}^9$ | 3 | no | all | [Rot49] |
| Quartic hypersurface in $\mathbb{P}^6$ | 2 | yes | no known examples | [Tot15] |
| Complete intersection of a quadric and a cubic in $\mathbb{P}^6$ | 2 | yes | no known examples | [NO22] |
| Complete intersection of three quadrics in $\mathbb{P}^7$ | 2 | yes | a countable union of closed subsets in the corresp. moduli space | [HPT17] |
| Gushel–Mukai fourfold | 2 | not known | some examples are known but the general picture is not clear | [Rot49] |
| Linear section in $\mathbb{P}^{18}$ of the spinorial $S^{10} \subset \mathbb{P}^{15}$ | 2 | no | all | [Rot49] |
| Linear section in $\mathbb{P}^{19}$ of $G(1,5) \subset \mathbb{P}^{14}$ | 2 | no | all | [Rot49] |
| Linear section in $\mathbb{P}^{21}$ of $LG(3,6) \subset \mathbb{P}^{13}$ | 2 | no | all | [Rot49] |
| Fourfold of degree 18 and genus 10 in $\mathbb{P}^{12}$ | 2 | no | all | [Rot49] |

*Table 1. Prime Fano fourfolds of index $\geq 2$. The classical constructions of rationality in [Rot49] are implemented in the function `parametrize`.*

One of the useful features of the package is the ability to find somewhat random examples of fourfolds in the Noether–Lefschetz locus and determine to which component they belong. In some likely cases, from an automatic count of parameters performed on the example, it is possible to get information about the Kodaira dimension of the component. This has already been used in [Sta21b] to deduce that the first ten components of the Noether–Lefschetz locus in the moduli space of Gushel–Mukai fourfolds have negative Kodaira dimension. In Theorem 1.7 we will extend this result by including the eleventh component. See also Theorem 1.2 for known analogous results in the case of cubic fourfolds.

A second feature of the package is the ability to establish the rationality of many fourfolds and calculate rational parameterizations. This is based on ideas introduced in [RS19]. We will use this feature to show new examples of rational Gushel–Mukai fourfolds, thus continuing the research initiated in [Rot49] and continued in [DIM15, RS21a, HS20, Sta21c]. See Tables 8 and 9 for a summary of new and old examples of rational Gushel–Mukai fourfolds; see also Table 7 for the case of cubic fourfolds.
The last and most complex feature of the package is the possibility of calculating K3 surfaces “associated” with cubic or Gushel–Mukai fourfolds (see Subsections 1.7). This is based on ideas developed in [RS21b] and then applied also in [HS21a]. As a consequence of the construction, one deduces the (explicit) unirationality of some moduli space $\mathcal{F}_g$ of K3 surfaces of genus $g$ (see Theorem 2.7). Just to give an example of application, the package can give us the explicit equations of general K3 surfaces of genus 22 (see Code Example 2.14) and, if we had enough computing power, we could produce an explicit dominant rational map from a projective space to the moduli space $\mathcal{F}_{22}$. The unirationality of $\mathcal{F}_{22}$ has recently also been proved by Farkas and Verra [FV21], but their construction seems too abstract to be translated into codes.

The paper is organized as follows. In section 1, we recall general facts on the Noether–Lefschetz loci in the moduli spaces of cubic and Gushel–Mukai fourfolds. We also recall the known results about the Kodaira dimension of the irreducible components of these loci. Next we show how to construct new fourfolds. As an application, we deduce Theorem 1.7, extending the main result obtained in [Sta21b]. Moreover, we construct new examples of rational Gushel–Mukai fourfolds. In Section 2, we first recall the known results about the Kodaira dimension of the moduli spaces $\mathcal{F}_g$ of K3 surfaces of genus $g$. Then we recall the constructions from [RS21b, HS21a] of the explicit unirationality of $\mathcal{F}_g$, for $g = 11, 14, 20, 22$, and show how to get equations for general K3 surfaces of these genera.

1. Noether–Lefschetz loci in the moduli spaces of cubic fourfolds and Gushel–Mukai fourfolds

1.1. Notation and definitions. A cubic fourfold is a smooth cubic hypersurface in $\mathbb{P}^5$. Cubic fourfolds are parametrized by a moduli space $C$ of dimension 20.

A Gushel–Mukai fourfold (GM fourfold for short) is a smooth quadric hypersurface in a 5-dimensional linear section $\mathcal{Y}^5 \subset \mathbb{P}^8$ of the cone $\widetilde{G}(1, 4) \subset \mathbb{P}^{10}$ over the Grassmannian $G(1, 4) \subset \mathbb{P}^9$. GM fourfolds are parametrized by a moduli space $\mathcal{GM}$ of dimension 24. The GM fourfolds $X \subset \mathcal{Y}^5$ for which $\mathcal{Y}^5$ contains the vertex of the cone $\widetilde{G}(1, 4)$ are called of Gushel type and describe an irreducible codimension 2 closed subset of $\mathcal{GM}$. The GM fourfolds which are not of Gushel type are called ordinary. These can be realized as smooth quadric hypersurfaces in a smooth Del Pezzo fivefold $\mathbb{P}^8 \cap G(1, 4) \subset \mathbb{P}^{10}$.

The Noether–Lefschetz locus in the moduli space of cubic fourfolds has been studied by Hassett (see [Has99], see also [Has00, Has16]). It is a countable union of irreducible hypersurfaces

$$\bigcup_{d \in \{8, 12, 14, 18, 20, 24, 26, 30, 32, 36, 38, 42, 44, \ldots\}} C_d \subset C,$$

where the discriminant $d$ runs over all integers $d > 6$ with $d \equiv 0, 2 \mod 6$.

The Noether–Lefschetz locus in the moduli space of GM fourfolds has been studied by Debarre, Iliev, and Manivel (see [DIM15], see also [DK18, DK19, DK20]). It is a countable union of hypersurfaces

$$\bigcup_{d \in \{10, 12, 16, 18, 20, 24, 26, 28, 32, 34, 36, 40, \ldots\}} \mathcal{GM}_d \subset \mathcal{GM},$$

where the discriminant $d$ runs over all integers $d > 6$ with $d \equiv 0, 2 \mod 6$.
where the discriminant \( d \) runs over all integers \( d \geq 8 \) with \( d \equiv 0, 2, 4 \mod 8 \). If \( d \equiv 2 \mod 8 \) then \( \mathcal{GM}_d \) is the union of two irreducible components \( \mathcal{GM}'_d \cup \mathcal{GM}_d'' \), otherwise it is irreducible. Cubic and GM fourfolds belonging to the Noether–Lefschetz locus are called Hodge-special.

1.2. Formulas for the discriminant. We provide some details on the calculation of the discriminant of a Hodge-special fourfold; see also [Has00, Section 4] and [DIM15, Section 7].

Let \([X] \in \mathcal{C}\) be a cubic fourfold containing an irreducible surface \( S \subset \mathbb{P}^5 \) of degree \( \deg(S) \) and sectional genus \( g(S) \), which has smooth normalization and only a finite number of nodes as singularities. Then one computes (see [Ful84, Theorem 9.3]) that the self-intersection of \( S \) in \( X \) is given by

\[
\left( S \right)_X^2 = 3 \deg(S) + 6 g(S) - 12 \chi(\mathcal{O}_S) + 2 K_S^2 + 2 \delta - 6.
\]

Denoting by \( H_X \) the class of the hyperplane section, we have that \([X] \in \mathcal{C}_d\), where \( d \) is the discriminant of the lattice spanned by \( ([H_X^2], [S]) \), that is

\[
d = \text{disc} \begin{bmatrix} H_X^2 \mid [S] \\ \deg(S) \end{bmatrix} = 3(S)_X^2 - \deg(S)^2.
\]

Similarly, let \([X \subset \mathbb{Y}^5] \in \mathcal{GM}\) be a GM fourfold containing an irreducible surface \( S \subset \mathbb{Y}^5 \), which has smooth normalization and only a finite number of nodes as singularities. Let \( a \sigma_{3,1} + b \sigma_{2,2} \) denote the class of \( \gamma_X^*(S) \) in the Chow ring of \( \mathbb{G}(1,4) \), where \( \gamma_X : X \to \mathbb{G}(1,4) \) is the so-called Gushel map, defined as the composition of the embedding of \( X \) into the cone \( \mathbb{G}(1,4) \) followed by the projection from the vertex. Again one computes that the self-intersection of \( S \) in \( X \) is given by

\[
\left( S \right)_X^2 = a + 2b + 4g(S) - 12 \chi(\mathcal{O}_S) + 2 K_S^2 + 2 \delta - 4.
\]

We have that \([X] \in \mathcal{GM}_d\), where \( d \) is the discriminant of the lattice spanned by \( (\gamma_X^*(\sigma_{1,1}), \gamma_X^*(\sigma_2), [S]) \), that is

\[
d = \text{disc} \begin{bmatrix} \gamma_X^*(\sigma_{1,1}) \mid \gamma_X^*(\sigma_2) \mid [S] \\ 2 \mid 2 \mid b \\ 2 \mid 4 \mid a \\ b \mid a \mid (S)_X^2 \end{bmatrix} = 4(S)_X^2 - 2a^2 + 4ab - 4b^2.
\]

If \( d \equiv 2 \mod 8 \), then \([X] \in \mathcal{GM}'_d\) if \( a + b \) is even, and \([X] \in \mathcal{GM}''_d\) if \( b \) is even (see [DIM15, Corollary 6.3]).

1.3. Implementation. In the Macaulay2 package SpecialFanoFourfolds, Hodge-special fourfolds are implemented as subtypes of the class "embedded projective variety", provided by the package MultiprojectiveVarieties [Sta21a]. However, a Hodge-special fourfold is represented internally by a pair \((S, X)\), where \( X \) is the fourfold and \( S \) is a special hidden surface contained in \( X \). The command \texttt{surface(X)} returns this surface \( S \). Hodge-special fourfolds can be created by the two functions: \texttt{specialCubicFourfold} and \texttt{specialGushelMukaiFourfold}, which typically expect the pair \((S, X)\) as input and always return \( X \). It is also possible to give only the surface \( S \) embedded in \( \mathbb{P}^5 \) or in a fivefold \( \mathbb{Y}^5 \), so that Macaulay2 will randomly choose a fourfold \( X \) containing \( S \). In
the case when $X$ is a GM fourfold given as a subvariety of $\mathbb{P}^8$, the Gushel map of $X$ is calculated behind the scenes and can be retrieved with the command $\texttt{toGrass}(X)$. The function $\texttt{discriminant}$, as its name suggests, calculates the discriminant of the fourfold. This is done by applying the formulas (1.1), (1.2), (1.3), and (1.4), where the value of $K_S^2$ is determined by functions from the packages $\texttt{Cremona}$ [Sta18] and $\texttt{CharacteristicClasses}$ [Jos15], and the normalization of $S$ via the package $\texttt{IntegralClosure}$ [EST20]. In some internal calculations it is required to invert complicated birational maps. For this, functions from the package $\texttt{RationalMaps}$ [BHSS22] are also used.

**Code Example 1.1.** In the following code, we create a GM fourfold containing a $\tau$-quadric surface, that is, a two-dimensional linear section of a Schubert variety $\Sigma_{1,1} \simeq G(1,3) \subset G(1,4)$ (first row of Table 8). We input the equations in the ring of polynomials with 9 variables $a,b,\ldots,i$. Note that by default, several checks are done on the input data; these can be relaxed or strengthened using the option $\texttt{InputCheck}$.

```
M2 -q --no-preload
Macaulay2, version 1.19.1
i1 : needsPackage "SpecialFanoFourfolds"; -- version 2.5.1
i2 : QQ[vars(0..8)]; -- coordinate ring of PP^8
i3 : S = projectiveVariety ideal(i, f, c, b, a, e*g-d*h);
o3 : ProjectiveVariety, surface in PP^8
i4 : X = projectiveVariety ideal(e*g-d*h+b*i, e*f-c*h+a*i,
  d*f-c*g-a*i-b*i-c*i-f*i-i^2, b*f-a*g-a*h-b*h-c*h-f*h-h*i,
  b*c-a*d+e*b*e-c*e+c*h+a*i-2*e*i, 2*a*b+b^2+a*b*c+b*c+2*a*d-c*d+e
  *e+2*b*e+2*c*e+2*a*f+2*c*f+2+3*a*g+b*e+2*c*g+f*g+2*a*h+b*h+
  3*c*h+2*f*h+a*i+3*b*i+3*c*i-d*i+3*e*i+3*f*i+1+2*h*i+2*i^2);
o4 : ProjectiveVariety, 4-dimensional subvariety of PP^8
i5 : X = specialGushelMukaiFourfold(S,X);
o5 : ProjectiveVariety, GM fourfold containing a surface
  of degree 2 and sectional genus 0
i6 : surface X
o6 = S
i6 : ProjectiveVariety, surface in PP^8
i7 : describe X
o7 = Special Gushel–Mukai fourfold of discriminant 10(')
  containing a surface in PP^8 of degree 2 and sectional genus 0
  cut out by 6 hypersurfaces of degrees (1,1,1,1,1,2)
  and with class in G(1,4) given by s_(3,1)+s_(2,2)
Type: ordinary
```

1.4. Kodaira dimension of the components. We recall known results about the Kodaira dimension of the components of the Noether–Lefschetz loci in the moduli spaces $C$ and $G_M$. The first result in this direction has been obtained in [Nue15], by showing the following theorem with the exception of $d = 42$.

**Theorem 1.2** ([Nue15, Lai17, FV21]). Each irreducible component $C_d$ of the Noether–Lefschetz locus in $C$ is unirational if the discriminant $d$ is at most 44.

On the other side we have the following.

**Theorem 1.3** ([TVA19]).
• The component $C_d$ is of general type for any $d \geq 114$, with the possible exceptions $d \in \{120, 122, 128, 132, 138, 150, 152, 180, 192\}$.

• The Kodaira dimension $\kappa(C_d)$ is non-negative for any $d \geq 86$, with the possible exceptions $d \in \{90, 92, 96, 108, 120, 132, 180\}$.

In the case of GM fourfolds we have the following weaker results.

**Theorem 1.4 ([Sta21b]).** Each irreducible component of the Noether–Lefschetz locus in $\mathcal{G}M$ is uniruled if the discriminant $d$ is at most 26; moreover, $\mathcal{G}M'_{10}$, $\mathcal{G}M'_{12}$, and $\mathcal{G}M_{20}$ are unirational.

In Theorem 1.7 we will extend Theorem 1.4 by including the case $d = 28$.

**Remark 1.5.** From the main result of [Ma18] (see also [Pet21]), it follows that only a finite number of components of the Noether–Lefschetz locus in $\mathcal{G}M$ are unirational (in such case the discriminant is at most 224). Nevertheless, it does not yet seem clear if the same holds true about the uniruledness.

Theorem 1.4 is proved in [Sta21b] by constructing very particular examples of Hodge-special GM fourfolds; more precisely, examples of pairs $(S, X)$, where $S \subset Y^5$ is a smooth irreducible surface in a smooth Del Pezzo fivefold $Y^5$, and $[X] \in \mathbb{P}(H^0(O_{Y^5}(2))) \simeq \mathbb{P}^{39}$ is a smooth hyperquadric in $Y^5$ containing $S$ (some of the most relevant examples are included in Table 5). Then, by applying Proposition 1.6 below, which can be done automatically with the function `parameterCount`, we deduce that there exists a family $S \subset \text{Hilb}_{Y^5}^{\chi(O_S((1)))}$ of surfaces with $[S] \in S$ such that the closure of the locus of smooth hyperquadrics in $Y^5$ containing some surface of $S$ describes an irreducible component of the Noether–Lefschetz locus in $\mathcal{G}M$. This component is uniruled since it is covered by curves birational to pencils of GM fourfolds through surfaces of $S$ (indeed we have $h^0(I_{S/Y^5}(2)) > 1$ for each $[S] \in S$).

**Proposition 1.6 (Count of parameters).** Let $V$ be either $\mathbb{P}^5$ or a Del Pezzo fivefold $Y^5 = \mathbb{G}(1, 4) \cap \mathbb{P}^8$, and let $r$ be respectively equal to 3 and 2. Let $S \subset V$ be a smooth irreducible surface which is contained in a smooth hypersurface $X \subset V$ of degree $r$. Assume that

1. $h^1(N_{S/V}) = 0$, and
2. $h^1(O_S(r)) = 0$ and $h^0(I_{S/V}(r)) = h^0(O_V(r)) - \chi(O_S(r))$.

Then there is a unique irreducible component $S \subset \text{Hilb}(V)$ of the Hilbert scheme of $V$ that contains $[S]$, and the family $X_S \subset \mathbb{P}(H^0(O_V(r)))$ of the hypersurfaces in $V$ of degree $r$ containing some surface of the family $S$ has codimension at most

$$\dim(\mathbb{P}(H^0(O_V(r)))) - (h^0(N_{S/V}) + h^0(I_{S/V}(r)) - h^0(N_S/X) - 1).$$

Furthermore, if this last value is 1 and by applying the formulas in Subsection 1.2 we get a non-zero value of $d$, then $X_S$ is a hypersurface; after passing to the quotient modulo $\text{Aut}(V)$, this gives rise to an irreducible component of the Noether–Lefschetz locus parameterizing fourfolds of discriminant $d$.

**Sketch of the proof.** See also [Nue15] and [Sta21b, Subsection 1.5]. By the condition (1), we deduce that the surface $S$ corresponds to a smooth point $[S]$ of the Hilbert
scheme $\text{Hilb}(\mathcal{V})$. Therefore there exists a unique irreducible component $\mathcal{S}$ of $\text{Hilb}(\mathcal{V})$ that contains $[S]$. Let

$$\mathcal{X}_S = \{(S',[X']): S' \subset X' \subset S, \pi(S') = \pi(S)\}$$

be the incidence correspondence. The fiber at a point $[S'] \in S$ of the first projection $\pi_1: \mathcal{X}_S \to S$ is isomorphic to the linear space $\mathbb{P}(\mathcal{I}_{S'/\mathcal{V}}(r))$. By the semicontinuity theorem, we have that the dimension of the fiber of $\pi_1$ achieves its minimum value on an open set of $S$, and by the condition (2) it follows that the point $[S]$ belongs to this open set. Therefore there exists a unique irreducible component $\mathcal{Z}$ of $\mathcal{X}_S$ that dominates $S$, and its dimension is equal to

$$\dim(\mathcal{Z}) = \dim(S) + h^0(\mathcal{I}_{S/\mathcal{V}}(r)) - 1.$$

Now the fiber at the point $[X] \in \mathbb{P}(\mathcal{I}_{S'/\mathcal{V}}(r))$ of the restriction to $\mathcal{Z}$ of the second projection $\pi_2|_{\mathcal{Z}}: \mathcal{Z} \subset \mathcal{X}_S \to \mathbb{P}(H^0(\mathcal{O}_\mathcal{V}(r)))$ contains the point $[S,X]$, and we have

$$\dim_{\mathbb{P}(\mathcal{I}_{S'/\mathcal{V}}(r))}(\tau_2|_{\mathcal{Z}}([X])) \leq \dim_{\mathbb{P}(\mathcal{I}_{S'/\mathcal{V}}(r))}(\mathbb{P}(\mathcal{I}_{S'/\mathcal{V}}(r)) - 1).$$

By semicontinuity, we deduce that the generic fiber of $\tau_2|_{\mathcal{Z}}$ has dimension at most $h^0(\mathcal{N}_{S/X})$. Thus we have

$$\dim(\mathcal{Z}) = \dim(S) + h^0(\mathcal{I}_{S/\mathcal{V}}(r)) - 1 = h^0(N_{S/\mathcal{V}}) + h^0(\mathcal{I}_{S/\mathcal{V}}(r)) - 1.$$

Finally, if from the formulas in Subsection 1.2 we get a non-zero value of $d$, then we deduce that $\tau_2(\mathcal{X}_S)$ does not fill the whole space since its image in the corresponding moduli space is contained in the Noether–Lefschetz locus. 

1.5. Finding new examples of Hodge-special fourfolds. We present a slight simplification of a construction used in [Sta21b] that allows us to find new examples of Hodge-special GM fourfolds and extend the main result obtained there. We start by fixing a birational transformation $\psi: \mathbb{P}^5 \to G(1,4) \subset \mathbb{P}^9$, which we choose to be the inverse of the projection from a $\sigma_{2,2}$-plane, that is, the transformation defined by the linear system of quadrics through a Segre threefold $\Sigma_3 \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 \subset \mathbb{P}^6$. Now we want to take some good surface $S \subset G(1,4)$ (possibly smooth and cut out by quadrics), obtained as the image via $\psi$ of some surface $T \subset \mathbb{P}^6$, and get a GM fourfold as the transversal intersection of $G(1,4)$ with a hyperplane and a hyperquadric through $S$. If $T$ does not cut the base locus $\Sigma_3$ of $\psi$, we typically obtain uninteresting surfaces in $G(1,4)$ which are neither contained in a hyperplane of $\mathbb{P}^9$ nor in a hyperquadric of $G(1,4)$. Therefore we take surfaces $T \subset \mathbb{P}^6$ together with an automorphism $\sigma$ of $\mathbb{P}^6$ sending a curve $C$ on $T$ to another curve on $\Sigma_3$. By replacing $T$ by $\sigma(T)$, we obtain a surface cutting $\Sigma_3$ at least along a curve, and we can hope the image $S = \psi(T)$ is good enough for our purposes. In practice, this can be achieved automatically only if $T$ is some simple rational surface such that we are able to find curves $C \subset T$ of low degree and low genus. So we restrict ourselves to a simpler case. Let $T = T(a; i, j, k, \ldots) \subset \mathbb{P}^6$ be the rational surface obtained as the image of the plane via the linear system of curves of degree $a$ having $i$ general base points $p_1^i, \ldots, p_j^i$ of multiplicity 1, $j$ general base points $p_1^j, \ldots, p_j^j$ of multiplicity 2, $k$ general base points $p_1^k, \ldots, p_k^k$ of multiplicity 3, and so on. Let $C = C(e; l, m, n, \ldots) \subset T$ be the curve which is represented in the plane by a general curve of degree $e$ passing through $l$ of the $i$ points $p_1^1, \ldots, p_1^1$, $m$ of the $j$ points $p_2^2, \ldots, p_j^2$, $n$ of the $k$ points $p_3^3, \ldots, p_k^3$, and so on. Suppose we are able to get an automorphism $\sigma$ of $\mathbb{P}^6$ which sends $C$ into a curve $\sigma(C) \subset \Sigma_3$, and that the image $S = \psi(\sigma(T)) \subset G(1,4)$
is a surface contained in a (smooth) GM fourfold. We denote by \( G^{a,i,j,k,...}_{e,l,m,n,...} \) the GM fourfold obtained as the intersection of \( G(1,4) \) with a general hyperplane and a general hyperquadric through \( S \) (we leave out the dependence on \( \sigma \)).

1.5.1. Running the construction. Using the package \texttt{SpecialFanoFourfolds} we can perform the above construction of the GM fourfold \( G^{a,i,j,k,...}_{e,l,m,n,...} \) by giving just one command:

\[
\texttt{specialGushelMukaiFourfold([a,i,j,k,...],[e,l,m,n,...])}
\]

Optionally, we can specify the coefficient ring as second argument (a large finite field is used by default). Everything is done automatically, including the searching of the automorphism \( \sigma \) and the needed checks on the surface \( S = \psi(\sigma(T)) \). If no error message occurs, then we can conclude that the GM fourfold \( G^{a,i,j,k,...}_{e,l,m,n,...} \) exists and, most importantly, has been successfully created. We refer to the online documentation for more details. See Tables 5, 6, 8, and 9 for some examples where this procedure works well. These examples have been found by another function provided by the package (available only in debug mode) that automatically scans many combinations of pairs of lists of integers \( ((a,i,j,k,...),(e,l,m,n,...)) \).

1.5.2. Explicit cubic fourfolds of high discriminant. As an application, we can construct Hodge-special cubic fourfolds of high discriminant. This is just one of the examples. The projection of the surface \( S \subset Y^5 \) as in the second row of Table 9 from a general plane in \( Y^5 \) of type \( \sigma_{2,2} \) yields a surface \( R \subset P^5 \) of degree 14 and sectional genus 7, cut out by 3 cubics, 6 quartics, and 2 quintics, having 15 nodes as the only singularities, and \( S \) as its normalization. A general cubic fourfold \( Z \subset P^5 \) containing \( R \) is smooth, and by (1.1) and (1.2) we deduce that \([Z] \in \mathcal{C}_{86}\). This works in other cases. For instance, starting from the surface \( S \subset Y^5 \) as in the fifth row of Table 8, we get a cubic fourfold in \( \mathcal{C}_{62} \).

1.6. Geometric description of the component \( \mathcal{G}M_{28} \). We point out that the existence of examples of GM fourfolds as in Table 6 extends Theorem 1.4 to another case. More precisely, we have the following.

\textbf{Theorem 1.7.} The component \( \mathcal{G}M_{28} \) is uniruled. In particular, each irreducible component of the Noether–Lefschetz locus in \( \mathcal{G}M \) has negative Kodaira dimension if the discriminant \( d \) is at most 28.

The proof of this result follows from the computation performed in Code Example 1.8 below (see Proposition 1.6). Indeed it tells us that there exists a generically smooth, 27-dimensional, irreducible component \( S \) of the Hilbert scheme of a smooth Del Pezzo fivefold \( Y^5 = P^8 \cap G(1,4) \), whose general point corresponds to a smooth rational surface of degree 11 and sectional genus 4 and with class \( 6\sigma_{3,1} + 5\sigma_{2,2} \) in \( G(1,4) \). The family of hyperquadrics in \( Y^5 \) through some surface of \( S \) is a hypersurface in \( P(H^0(O_{Y^5}(2))) \simeq P^{39} \), which gives rise to the component \( \mathcal{G}M_{28} \) by the formulas (1.3) and (1.4). Moreover, a general fourfold in \( \mathcal{G}M_{28} \) contains only a finite number of surfaces of \( S \), and the family of hyperquadrics through a general \([S] \in S \) is a projective space of dimension 11. Note, however, that we are unable to compute the generic members of \( S \) and of \( \mathcal{G}M_{28} \).
**Code Example 1.8** (Geometric description of \(\mathcal{G}_M_{28}\)). In the code we construct a Hodge-special GM fourfold \((S, X)\) as described in the last row of Table 6. The function `parameterCount` gives the three invariants \(h^0(I_{S/Y}(2)), h^0(N_{S/Y}), h^0(N_{S/X})\) reported in the table.

```plaintext
i8 : time X = specialGushelMukaiFourfold([7,0,6,2],[2,0,5,0]);
   -- used 10.3554 seconds
o8 : ProjectiveVariety, GM fourfold containing a surface of degree 11 and sectional genus 4
i9 : time describe X
   -- used 2.22773 seconds
o9 = Special Gushel-Mukai fourfold of discriminant 28 containing a surface in PP^8 of degree 11 and sectional genus 4 cut out by 17 hypersurfaces of degree 2 and with class in \(\mathcal{G}(1,4)\) given by \(6*\sigma_3(1)+5*\sigma_2(2)\)
Type: ordinary
i10 : time X = parameterCount(X,Verbose=>true)
   -- \(h^1(N_{S,Y}) = 0\)
   -- \(h^0(N_{S,Y}) = 27\)
   -- \(h^1(O_S(2)) = 0\), and \(h^0(I_{S,Y}(2)) = 12 = h^0(O_Y(2)) - \chi(O_S(2))\);
   -- in particular, \(h^0(I_{S,Y}(2))\) is minimal
   -- \(h^0(N_{S,X}) = 0\)
   -- \(\text{codim}(\{X\} : S \subset X \subset Y) \leq 1\)
   -- used 1164.59 seconds
o10 = (1, (12, 27, 0))
```

### 1.7. Hodge-associated K3 surfaces.
For some infinitely many values of the discriminant \(d\), a fourfold \([X] \in \mathcal{C}_d\) (resp., \([X] \in \mathcal{G}_M_d\)) has a (Hodge-)associated K3 surface of degree \(d\) and genus \(\frac{d}{2} + 1\). We call such values \(\mathcal{C}\)-admissible (resp., \(\mathcal{G}_M\)-admissible). Table 2 reports the first \(\mathcal{C}\)- and \(\mathcal{G}_M\)-admissible values; see [Has00, Theorem 1.0.2] and [DIM15, Proposition 6.5] for the precise definitions.

| \(\mathcal{C}\) | 8  | 12 | 14 | 18 | 20 | 24 | 26 | 30 | 32 | 36 | 38 | 42 | 44 |
|-----------------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| \(\mathcal{G}_M\) | 10 | 12 | 16 | 18 | 20 | 24 | 26 | 28 | 32 | 34 | 36 | 40 | 42 | 44 |

| \(\mathcal{C}\) | 48 | 50 | 54 | 56 | 60 | 62 | 66 | 68 | 72 | 74 | 78 | 80 | 84 |
|-----------------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| \(\mathcal{G}_M\) | 48 | 50 | 52 | 56 | 58 | 60 | 64 | 66 | 68 | 72 | 74 | 76 | 80 | 82 | 84 |

**Table 2.** \(\mathcal{C}\)- and \(\mathcal{G}_M\)-admissible values < 86.

**Remark 1.9** (Geometric interpretation of associated K3 surfaces). An integer \(d\) of the form \(2(n^2 + n + 1)/a^2\) for some \((a, n) \in \mathbb{Z}^2\) is always \(\mathcal{C}\)-admissible, but the converse is not true. For instance, we have \(14 = \frac{2(2^2+2+1)}{1^2}\), \(26 = \frac{2(3^2+3+1)}{1^2}\), \(38 = \frac{2(30^2+30+1)}{7^2}\), \(42 = \frac{2(6^2+1+1)}{3^2}\), \(62 = \frac{2(5^2+5+1)}{2^2}\), but the next \(\mathcal{C}\)-admissible value 74 is not of this form. Addington [Add16] (see also [Has99] and [Ouc20]) proved that if \([X]\) is a cubic fourfold in \(\mathcal{C}_d\), then the following are equivalent:

- \(d\) is of the form \(2(n^2 + n + 1)/a^2\), for some \((a, n) \in \mathbb{Z}^2\);
the Fano variety $F(X)$ of lines in $X$ is birational to the Hilbert scheme $\text{Hilb}^2(U)$ of two points on some K3 surface $U$ of degree $d$ (necessarily associated with $X$). Moreover, if $a = 1$ and $[X] \in C_d$ is general, we have an isomorphism $F(X) \simeq \text{Hilb}^2(U)$.

The notion of associated K3 surface leads to the following rationality conjecture (see [Kuz10, AT14, Kuz16, Has16, DIM15]).

**Conjecture 1.10 (Kuznetsov’s conjecture).** A fourfold $[X] \in C$ is rational if and only if $[X] \in C_d$ for some $C$-admissible value $d$, that is

$$[X] \in C_{14} \cup C_{26} \cup C_{38} \cup C_{42} \cup C_{62} \cup C_{74} \cup C_{78} \cup C_{86} \cup \cdots$$

A fourfold $[X] \in GM$ is rational if and only if $[X] \in GM_d$ for some $GM$-admissible value $d$, that is,

$$[X] \in GM_{10}' \cup GM_{10}'' \cup GM_{26} \cup GM_{26}' \cup GM_{26}'' \cup GM_{34}' \cup \cdots$$

It is classically known that cubic fourfolds in $C_{14}$ are rational (see [Fan43, Mor40], see also [BRS19]), as well as that GM fourfolds in $GM_{26}'$ are rational (see [Rot49, Enr97], see also [DIM15]). The following result summarizes the current state of the conjecture.

**Theorem 1.11 (“first cases” + [RS19, RS21b, HS20]).** The fourfolds in $C_{14} \cup C_{26} \cup C_{38} \cup C_{42} \cup GM_{10}' \cup GM_{10}'' \cup GM_{26} \cup GM_{26}' \cup GM_{26}'' \cup GM_{34}' \cup \cdots$ are rational.

**Remark 1.12.** Although we are unable to prove that every fourfold in $GM_{26}$ is rational, we can exhibit several examples of ordinary rational fourfolds in $GM_{26}''$. We include two of these examples in the last two rows of Table 8. The last row contains an entirely new example. The penultimate line contains a new example of ordinary fourfold, but examples of non-ordinary fourfolds containing the same surface were already constructed in [Sta21c]. See Subsection 2.6 below for more details on the example of the last row of Table 8. See also Table 9 for some example of rational fourfold in $GM_{34}$.

2. **Moduli spaces of K3 surfaces**

The moduli space $F_g$ of polarized K3 surfaces of genus $g$ parametrizes pairs $(S, H)$, where $S$ is a K3 surface and $H \in \text{Pic}(S)$ is a primitive polarization class with $H^2 = 2g - 2$. The dimension of $F_g$ is 19.

2.1. **Kodaira dimension of $F_g$.** There are many values of $g$ (although a finite number) for which we have no information about the Kodaira dimension of $F_g$. We now recall the known results. The following theorem has been established by Mukai in the cases $g \leq 13$ and $g = 16, 17, 18, 20$, by Nuer in the cases $g = 14, 20$, and by Farkas and Verra in the cases $g = 14, 22$. See Tables 3 and 4 for precise references; see also [HS21a, Section 7].

**Theorem 2.1.** The moduli space $F_g$ is unirational for any $g \leq 22$, with the possible exceptions $g \in \{15, 19, 21\}$.

On the other side we have the following.

**Theorem 2.2 ([GHS07]).**

- The moduli space $F_g$ is of general type for any $g \geq 47$, with the possible exceptions $g \in \{48, 49, 50, 52, 53, 54, 56, 57, 60, 62\}$.
The Kodaira dimension $\kappa(F_g)$ is non-negative for $g \geq 41$, with the possible exceptions $g \in \{42, 45, 46, 48\}$.

2.2. Connections with cubic fourfolds and GM fourfolds. Let $d > 6$ be $C$-admissible and $[X] \in C_d$ general. If $d \equiv 2 \mod 6$ then $X$ admits a unique associated K3 surface, while if $d \equiv 0 \mod 6$ then $X$ admits two associated K3 surfaces. Indeed, we have the following:

**Theorem 2.3 ([Has00]).** Assume $d > 6$ is $C$-admissible. There is a dominant rational map

$$\varphi_d : F_{d+2}^2 \rightarrow C_d, \quad [S] \mapsto [X] : S \text{ is associated with } X$$

which is birational for $d \equiv 2 \mod 6$ and a degree 2 cover for $d \equiv 0 \mod 6$.

The map $\varphi_d$ is not explicit since it is defined at the level of Hodge structures. The author is not able to determine an equation of $[X] = \varphi_d([S])$ from equations of $[S]$.

**Remark 2.4 (Non-explicit unirationality of $F_{14}$ and $F_{20}$).** Nuer in [Nue15] showed that $C_{26}$ and $C_{38}$ are unirational. From this and the birationality of the maps $\varphi_{26}$ and $\varphi_{38}$, he deduced the unirationality of $F_{14}$ and $F_{20}$.

**Remark 2.5 (Non-explicit unirationality of $F_{14}$ and $F_{22}$).** We recall the results of Farkas and Verra on the unirationality of $F_{14}$ and $F_{22}$. Let us consider

- $b^{3,7}_{\text{scr}} = \text{PGL}(6)$-quotient of the Hilbert scheme of 3-nodal septic scrolls,
- $b^{8,9}_{\text{scr}} = \text{PGL}(6)$-quotient of the Hilbert scheme of 8-nodal nonic scrolls,

and the incidence correspondences

$$\mathcal{X}_{26} = \{ [R, X] : [R] \in b^{3,7}_{\text{scr}}, \, [X] \in |H^0(\mathcal{I}_R/\mathbb{P}^5(3))| \} / \text{PGL}(6),$$

$$\mathcal{X}_{42} = \{ [R, X] : [R] \in b^{8,9}_{\text{scr}}, \, [X] \in |H^0(\mathcal{I}_R/\mathbb{P}^5(3))| \} / \text{PGL}(6).$$
The main results in [FV18] and [FV21] state that $X_{26}$ is rational and that $X_{42}$ is unirational, and moreover that we have two commutative diagrams:

$$\begin{align*}
\mathcal{F}_{14,1} & \xrightarrow{\sim} \mathcal{X}_{26}, \\
\mathcal{F}_{14} & \xrightarrow{\varphi_{26}} \mathcal{C}_{26},
\end{align*}$$

$$\begin{align*}
\mathcal{F}_{22,1} & \xrightarrow{\sim} \mathcal{X}_{42}, \\
\mathcal{F}_{22} & \xrightarrow{\varphi_{42}} \mathcal{C}_{42}.
\end{align*}$$

From this the unirationality of $\mathcal{F}_{14}$ and $\mathcal{F}_{22}$ follows.

The following remark is particularly useful for us. It follows from Theorem 2.3 in the case of cubic fourfolds, and from the main result of [BP21] in the case of GM fourfolds.

**Remark 2.6 ([Has00, BP21]).** Let $d$ be a $C$-admissible (resp., GM-admissible) value. Let $X$ be a cubic (resp., GM) fourfold corresponding to a general point in an irreducible component of the Noether–Lefschetz locus of cubic (resp., GM) fourfolds of discriminant $d$. Let $S$ be a K3 surface of genus $g = \frac{d^2}{2} + 1$ which is associated with the fourfold $X$. Then $S$ corresponds to a general point in the moduli space $\mathcal{F}_g$.

### 2.3. Explicit unirationality of $\mathcal{F}_g$

We are interested in the *explicit unirationality* of the moduli space $\mathcal{F}_g$, that is, in finding an explicit dominant rational map from a projective space to $\mathcal{F}_g$. More precisely, with “explicit” we mean that there is a computer-implementable procedure to determine the equations of the general member of $\mathcal{F}_g$ as a function of a number of specific independent variables. The original methods used in the proof of Theorem 2.1 provide the explicit unirationality of $\mathcal{F}_g$ only for $g \leq 12$ and $g \neq 11$ (see also [KP18] for the case $g = 12$).

In the following, we recall a construction from [RS21b] and [HS21a] on the explicit unirationality of $\mathcal{F}_g$ for some values of $g$. Then we show how this can be executed in practice to get the equations of general K3 surfaces. To be more precise we have the following (see also Tables 3 and 4).

**Theorem 2.7 ([RS21b, HS21a]).** The moduli space $\mathcal{F}_g$ is explicitly unirational for $g = 6, 8, 11, 14, 20, 22$.

Note that cases $g = 6$ and $g = 8$ are very elementary: general K3 surfaces of genus 6 and 8 can be respectively realized as linear sections of quadratic sections of $\mathbb{G}(1, 4) \subset \mathbb{P}^9$ and as linear sections of $\mathbb{G}(1, 5) \subset \mathbb{P}^{14}$. However the new method works in the same way in all cases. The main ingredient used is a *good description* of a unirational component of the Noether–Lefschetz locus in the moduli space of cubic fourfolds or GM fourfolds such that the fourfolds in this component have a Hodge-associated K3 surface of genus $g$. So, taking into account Table 2 and Theorems 1.3 and 2.2, we do not exclude that the same method might be also applied to other values of $g$ in the set

$$\{6, 8, 11, 14, 18, 20, 22, 26, 27, 30, 32, 35, 38, 40\}.$$  

In the next subsection we provide more details on the procedure, which is strongly connected with the proof of Theorem 1.11.
2.4. An overview of the construction to obtain Theorem 2.7. For more details on this subsection, we refer to [RS21b, Section 4] and [HS21a, Section 6]. Let $V$ be either $\mathbb{P}^5$ or a Del Pezzo fivefold $\mathbb{V}^5$, and let $r$ be respectively equal to 3 and 2, the degree of the hypersurfaces we consider in $V$. For each of the components $D$ considered in Theorem 1.11 parameterizing fourfolds of discriminant $d$ (with the exception of $\mathcal{G}_{10}''$), we are able to exhibit an explicit unirational family $S \subset \text{Hilb}(V)$ of surfaces in $V$ such that the following hold (actually, we have more than one example; see Tables 7 and 8):

1. The closure of the locus of smooth hypersurfaces $X \subset V$ of degree $r$ containing some surface $S$ of the family $S$ describes a hypersurface in $\mathbb{P}(H^0(\mathcal{O}_V(r)))$, which gives rise to the component $D$ of the Noether–Lefschetz locus in the corresponding moduli space $C$ or $\mathcal{G}_{10}$.

2. For some $e \geq 1$, the general member $[S] \in S$ admits a congruence of $(re - 1)$-secant curves of degree $e$, parametrized by a variety $H$. Thus we have a universal family $X = \{(\lbrack C \rbrack, p) : \lbrack C \rbrack \in H, \ p \in C \subset V \} \subset H \times V$ and two projections

(2.1)

$$
\begin{diagram}
\node{\mathcal{X}} 
\arrow{e, \pi} 
\node{\mathcal{H}} 
\arrow{se, V} 
\node{V,}
\end{diagram}
$$

where $\pi$ is birational and its general fiber $\pi^{-1}(p)$ corresponds to the unique curve $\lbrack C \rbrack \in H$ passing through $p$ and cutting $S$ at $re - 1$ points.

3. The linear system $|H^0(\mathcal{I}_{S/V}(re - 1))|$ of hypersurfaces in $V$ of degree $re - 1$ with points of multiplicity $e$ along $S$ defines a dominant rational map

$$
\mu : V \dashrightarrow W
$$
ono
onto a prime Fano fourfold $W$ of index $s = i(W)$. In particular, we can realize the general curve of the congruence as the general fiber of the map $\mu$.

4. By degree reasons, the restriction of $\mu$ to a general hypersurface $[X] \in |H^0(\mathcal{I}_{S/V}(r))|$ of degree $r$ that contains $S$ gives a birational map $\mu|_X : X \dashrightarrow W$. We have that the inverse map of $\mu|_X$ is defined by the linear system of hypersurfaces in $W$ of degree $se - 1$ having points of multiplicity $e$ along an irreducible surface $U \subset W$. These maps fit in the following commutative diagram:

(2.2)

$$
\begin{diagram}
\node{Z \cap \mathbb{P}^{N-1}} 
\arrow{se, \varphi_{|X}} 
\node{|H^0(\mathcal{I}_{S}(r))|} 
\arrow{e, \mu} 
\node{|H^0(\mathcal{I}_{U}(s))|} 
\node{V} 
\arrow{se, \mu_{|V}} 
\node{|H^0(\mathcal{I}_{S}(re-1))|} 
\node{X} 
\arrow{e, \mu} 
\node{W} 
\node{U} 
\end{diagram}
$$
where $Z \subset \mathbb{P}^N$ is the closure of the image of the map $\varphi : V \longrightarrow \mathbb{P}^N$ defined by the linear system of hypersurfaces of degree $r$ through $S$.

(5) Finally, the surface $U \subset W$ can be obtained as a linear projection of a minimal K3 surface $\tilde{U} \subset \mathbb{P}^{\frac{d}{2}+1}$ of degree $d$ and genus $\frac{d}{2} + 1$, which is associated with the fourfold $X$.

In this framework, we define the fundamental locus $E \subset V$ of the congruence as the base locus of the inverse of the projection $\pi$ in (2.1), that is, $E = \{ p \in V : \dim(\pi^{-1}(p)) > 0 \}$.

Then notice that there are two kinds of points on $U$:

- the points $p \in W$ such that $\mu^{-1}(p)$ is a curve of the congruence but contained in the hypersurface $X \subset V$;
- and the points $p \in \mu(E) \subset W$ in the image via $\mu$ of $E$; so that $\dim \mu^{-1}(p) \geq 2$.

A key remark here is that the points of the second kind do not depend on the hypersurface $X \subset V$ through $S$. So that if $X' \subset V$ is another general hypersurface of degree $r$ through $S$, and $U' \subset W$ is the corresponding surface defining the inverse of the restriction of $\mu$ to $X'$, we must have $\overline{\mu(E)} \subseteq U \cup U'$. Actually, in all cases we have analyzed, it turns out that $\overline{\mu(E)}$ is the union of the exceptional curves on $U$ and on $U'$, which are recoverable as the top-dimensional components of $U \cap U'$. In particular, we can determine their equations from the equations of $S$ in $V$. Once we have computed the surface $U$ and its exceptional curves $C_1, \ldots, C_t$, we get the minimal associated K3 surface $\tilde{U} \subset \mathbb{P}^{\frac{d}{2}+1}$ as the closure of the image of the rational map $U \longrightarrow \mathbb{P}^{\frac{d}{2}+1}$ defined by $|H + \sum_{i=1}^t \deg(C_i)C_i|$ (in some cases a normalization of $U$ is also needed). We summarize with the following diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
X' & \leftarrow & |H^0(T_{U'}(se-1))| \\
| & \mu & | \\
S & \leftarrow & V \\
| & \mu & | \\
X & \leftarrow & |H^0(T_U(se-1))|
\end{array}
\end{array}
\]

By Remark 2.6 we deduce that if the pair $(S, X)$ is general enough in the incidence correspondent $\mathcal{X}_S = \{(S, X) : [S] \in \mathcal{S}, \ [X] \in |H^0(L_{S/S'}(r))|\}$, then the K3 surface $\tilde{U}$ will be general enough in the moduli space $\mathcal{F}_g$ with $g = \frac{d}{2} + 1$. If the coordinates of the generic point of $\mathcal{X}_S$ belong to a pure transcendental extension of $\mathbb{C}$ (which is always possible if $S$ is unirational), then the same holds for the coordinates of the generic point of $\mathcal{F}_g$, thus obtaining the unirationality of $\mathcal{F}_g$.

2.5. **Implementation of the construction.** Everything said in the previous subsection is automatically detected and handled by the function

\[\text{associatedK3surface}\]
This function takes as input a Hodge-special fourfold represented by a pair \((S,X)\), with \(S \subset X \subset \mathbb{V}\) as described above. Then, if nothing goes wrong, it returns four objects:

- the dominant rational map \(\mu : \mathbb{V} \to W\) defined by the linear system of hypersurfaces of degree \(re - 1\) having points of multiplicity \(e\) along \(S\);
- the surface \(U \subset W\) determining the inverse map of the restriction of \(\mu\) to \(X\);
- the list \(C = \{C_1, C_2, \ldots\}\) of the exceptional curves on the surface \(U\) (more precisely, \(C_1\) is the union of the exceptional lines, \(C_2\) of the conics, and so on);
- the birational map \(f : U \to \tilde{U} \subset \mathbb{P}^{\frac{d}{2}+1}\) defined by \(|H + \sum_{i=1}^4 \deg(C_i)C_i|\).

Therefore, by taking the image of the last map we get an associated K3 surface \(\tilde{U} \subset \mathbb{P}^{\frac{d}{2}+1}\) with the fourfold \(X\).

Here are a few more details on the computation behind the scenes. The first task of the procedure is to try to detect the congruence of curves. This is done by considering the map \(\varphi : \mathbb{V} \to Z \subset \mathbb{P}^N\) that appears in (2.2) defined by the linear system of hypersurfaces of degree \(r\) through \(S\), checking that it is birational, and then analyzing the lines on \(Z\) passing through its general point. Indeed, if \(\varphi\) is birational, it induces a 1–1 correspondence:

\[
\bigcup_{e \geq 1} \left\{ \text{curves of degree } e \text{ in } \mathbb{V} \text{ passing through a general point } p \in \mathbb{V} \text{ and that are } (re - 1)\text{-secant to } S \right\} \xrightarrow{\sim} \left\{ \text{lines contained in } Z \subset \mathbb{P}^N \text{ and passing through } \varphi(p) \right\}.
\]

Once the congruence is detected, the next task is to compute the map \(\mu : \mathbb{V} \to W\) and the surface \(U \subset W\) corresponding to \(X\). There are several ways to determine the equations of \(U\). The strategy often adopted by Macaulay2 is based on the fact that we have a 1–1 correspondence between the two linear systems \(|H^0(I_{S/X}(r))|\) and \(|H^0(I_{U/W}(s))|\), which follows from the commutativity of the diagram (2.2). In most of the known cases, \(S\) is cut out by hypersurfaces of degree \(r\) and \(U\) is cut out by hypersurfaces of degree \(s\).

### 2.6. Running the construction with a example.

In the following, we run the procedure with an example of fourfold constructed as described in the last row of Table 8. This is a very special GM fourfold \(X \subset \mathbb{Y}_5 = \mathbb{G}(1,4) \cap \mathbb{P}^8\) which contains a smooth rational surface \(S \subset \mathbb{P}^8\) of degree 9 and sectional genus 2 with class \(5\sigma_{3,1} + 4\sigma_2\) in \(\mathbb{G}(1,4)\). Thus \([X] \in \mathbb{G}M_{26}''\) by (1.3) and (1.4), although not all fourfolds in \(\mathbb{G}M_{26}''\) are of this type (see Code Example 2.10). It seems relevant to notice that this surface \(S\), as a subvariety of \(\mathbb{P}^8\), is the same surface that appears in the fourth row of Table 8 and in Code Example 2.11, that is the surface considered in [HS20] to describe the component \(\mathbb{G}M_{20}\) (indeed, both can be realized as the image of the plane via the linear system of quartic curves through three simple points and one double point in general position).

This surface also admits a congruence of 3-secant conics inside \(\mathbb{Y}_5\), and the linear system of cubic hypersurfaces in \(\mathbb{Y}_5\) with double points along \(S\) gives a dominant rational map \(\mu : \mathbb{Y}_5 \to W \subset \mathbb{P}^5\) onto a smooth quadric hypersurface \(W \subset \mathbb{P}^5\). The restriction of \(\mu\) to \(X\) is a birational map \(X \to W\) whose inverse is defined by the linear system of hypersurfaces of degree 7 in \(W\) with double points along a smooth irreducible surface \(U \subset W\).
of degree 13 and sectional genus 11. This surface $U$ has four exceptional lines $L_1, \ldots, L_4$ and one exceptional twisted cubic $C$. The linear system $|H + L_1 + \cdots + L_4 + 3C|$ on $U$ gives a birational map onto a minimal K3 surface $\tilde{U} \subset \mathbb{P}^{14}$ of genus 14. As mentioned earlier, everything is done by Macaulay2 as in the following.

**Code Example 2.8.** We construct a fourfold $X$ as described in the last row of Table 8. Then we call the function `associatedK3surface` (by turning on the “Verbose” option).

```
i11 : time X = specialGushelMukaiFourfold([4,5,1],[2,3,0]);
-- used 9.5484 seconds
o11 : ProjectiveVariety, GM fourfold containing a surface
of degree 9 and sectional genus 2
i12 : describe X
o12 = Special Gushel-Mukai fourfold of discriminant 26('')
containing a smooth surface in \mathbb{P}^8 of degree 9 and
sectional genus 2 cut out by 19 hypersurfaces of degree 2
and with class in G(1,4) given by 5*s_(3,1)+4*s_(2,2)
Type: ordinary
i13 : time (mu,U,C,f) = associatedK3surface(X,Verbose=>true);
-- computing the Fano map mu from the fivefold in \mathbb{P}^8 to \mathbb{P}^5 defined
by the hypersurfaces of degree 3 with points of multiplicity 2
along the surface S of degree 9 and genus 2
-- computing the surface U corresponding to the fourfold X
-- computing the surface U' corresponding to another fourfold X'
-- computing the 4 exceptional line(s) in U \cap U'
-- computing the top components of U \cap U'
-- computing the map f from U to the minimal K3 surface of degree 26
-- computing the image of f
-- used 58.1472 seconds
i14 : f;
o14 : RationalMap (rational map from U to \mathbb{P}^{14})
i15 : ? ideal image f
o15 = surface of degree 26 and sectional genus 14 in \mathbb{P}^{14}
cut out by 66 hypersurfaces of degree 2
```

The congruence of 3-secant conics to $S$ can be detected by considering the map $\varphi : \mathbb{P}^5 \rightarrow Z \subset \mathbb{P}^{13}$ defined by the quadrics through $S$. Indeed, we have that through the general point of $Z$ there pass 6 lines, which come from five 1-secant lines to $S$ and one single 3-secant conic to $S$. This is performed by the following code.

**Code Example 2.9.** We run the function `detectCongruence` for the fourfold $X$ constructed in Code Example 2.8 (last row of Table 8).

```
i16 : time c = detectCongruence(X,Verbose=>true);
number lines contained in the image and passing through a general point: 6
number 1-secant lines = 5
number 3-secant conics = 1
-- used 6.18911 seconds
```

```
i16 : Congruence of 3-secant conics to surface in a Del Pezzo fivefold
```
At this point, Macaulay2 knows that the fourfold $X$ is rational. A command such as $\text{parametrize}(X)$ gives us a rational parameterization of $X$.

We conclude with a count of parameters (see Proposition 1.6) which shows that the family of fourfolds $X$ as in the last row of Table 8 describes a locus of codimension 2 in $\mathcal{G}M$, hence of codimension 1 in $\mathcal{G}M_{26}$.

**Code Example 2.10.** We run the function $\text{parameterCount}$ for the fourfold $X$ constructed in Code Example 2.8 (last row of Table 8).

```
i17 : time parameterCount(X,Verbose=>true)
-- h^1(N_{S,Y}) = 0
-- h^0(N_{S,Y}) = 24
-- h^1(O_S(2)) = 0, and $h^0(I_{S,Y}(2)) = 14 = h^0(O_Y(2)) - \chi(O_S(2))$;
-- in particular, $h^0(I_{S,Y}(2))$ is minimal
-- $h^0(N_{S,X}) = 0$
-- codim$\{X : S \subset X \subset Y\} \leq 2$
-- used 337.969 seconds
```

2.7. Running the construction with other examples.

**Code Example 2.11** (Explicit unirationality of $\mathcal{F}_{11}$). Here we construct a random GM fourfold $X \subset Y^5 = G(1,4) \cap P^8$ in the component $\mathcal{G}M_{20}$ which contains a smooth rational surface $S \subset Y^5$ of degree 9, sectional genus 2, and class $6\sigma_{3,1} + 3\sigma_{2,2}$ in $G(1,4)$, as described in [HS20] and [RS21b, Remark 5.8] (see also the fourth row of Table 8). Therefore, the corresponding associated K3 surface is a random K3 surface of genus 11 in $\mathbb{P}^{11}$. In the notation of the diagrams (2.2) and (2.3), we have $r = 2$, $e = 2$, $s = 5$, $W = P^4$, $U \subset W$ is a non-normal surface of degree 10 and sectional genus 8, and the exceptional curves on (the normalization of) $U$ are one line and one twisted cubic curve. See also [HS21a, Section 6] for more details on this calculation.

```
i18 : X = specialGushelMukaiFourfold "general GM 4-fold of discriminant 20";
o18 : ProjectiveVariety, GM fourfold containing a surface of degree 9 and sectional genus 2
i19 : describe X
o19 = Special Gushel-Mukai fourfold of discriminant 20
  containing a smooth surface in PP^8 of degree 9 and sectional genus 2 cut out by 19 hypersurfaces of degree 2
  and with class in G(1,4) given by 6*s_{3,1}+3*s_{2,2}
i20 : time (mu,U,C,f) = associatedK3surface(X,Verbose=>true);
-- computing the Fano map $\mu$ from the fivefold in PP^8 to PP^4 defined
-- by the hypersurfaces of degree 3 with points of multiplicity 2
-- along the surface $S$ of degree 9 and genus 2
-- computing the surface $U$ corresponding to the fourfold $X$
-- computing the 1 exceptional line(s) in $U \cap U'$
-- computing the top components of $U \cap U'$
-- computing desingularization of $U$
-- computing the map $f$ from $U$ to the minimal K3 surface of degree 20
-- computing the image of $f$
-- used 114.893 seconds
RationalMap (rational map from U to PP^{11})

? ideal image f

= surface of degree 20 and sectional genus 11 in PP^{11}
cut out by 36 hypersurfaces of degree 2

Remark 2.12. The command K3(11) provided by the package K3Surfaces [HS21b] executes the procedure in Code Example 2.11 and returns a random K3 surface of genus 11. The same command can be called with other values of g, for which an explicit unirationality construction is known (see Tables 3 and 4).

Code Example 2.13 (Explicit unirationality of \( F_{20} \)). Here we construct a random cubic fourfold in the component \( C_{38} \) which contains a smooth rational surface of degree 10 and sectional genus 6 as described in [Nue15] and [RS21b, Section 3] (see also the fifth row of Table 7). Therefore, the corresponding associated K3 surface is a random K3 surface of genus 20 in \( \mathbb{P}^{20} \). In the notation of the diagrams (2.2) and (2.3), we have \( r = 3, \ e = 2, \ s = 5, \ W = \mathbb{P}^{4}, \ U \subset W \) is a smooth surface of degree 12 and sectional genus 14, and the exceptional curves on \( U \) are 10 lines and one rational normal quartic curve. See also [RS21b, Section 6.1] for more details on this calculation.

Code Example 2.14 (Explicit unirationality of \( F_{22} \)). Here we construct a random cubic fourfold in the component \( C_{42} \) which contains a 5-nodal rational surface of degree 9 and sectional genus 2 as described in [RS21b, Section 4] (see also the last row of
Table 7). Therefore, the corresponding associated K3 surface is a random K3 surface of genus 22 in $\mathbb{P}^{22}$. In the notation of the diagrams (2.2) and (2.3), we have $r = 3$, $e = 3$, $s = 3$, $W = G(1, 4) \cap \mathbb{P}^7$ is a Del Pezzo fourfold, $U \subset W$ is a smooth surface of degree 21 and sectional genus 18, and the exceptional curves on $U$ are 5 lines and 4 conics. See also [RS21b, Section 6.2] for more details on this calculation.

```
\begin{verbatim}
i29 : X = specialCubicFourfold "general cubic 4-fold of discriminant 42";
o29 : ProjectiveVariety, cubic fourfold containing a surface of degree 9 and sectional genus 2
i30 : describe X
  o30 = Special cubic fourfold of discriminant 42 containing a 5-nodal surface of degree 9 and sectional genus 2 cut out by 9 hypersurfaces of degree 3
i31 : time (mu,U,C,f) = associatedK3surface(X,Verbose=>true);
  -- computing the Fano map mu from PP^5 to PP^7 defined by the hypersurfaces of degree 8 with points of multiplicity 3 along the surface S of degree 9 and genus 2
  -- computing the surface U corresponding to the fourfold X
  -- computing the surface U' corresponding to another fourfold X'
  -- computing the 5 exceptional line(s) in U \cap U'
  -- computing the top components of U \cap U'
  -- computing the map f from U to the minimal K3 surface of degree 42
  -- computing the image of f
  -- used 692.743 seconds
i32 : f;
o32 : RationalMap (rational map from U to PP^22)
i33 : ? ideal image f
  o33 = surface of degree 42 and sectional genus 22 in PP^22 cut out by 190 hypersurfaces of degree 2
\end{verbatim}
```
3. Summary tables of examples

| Surface $S$ in $\mathbb{Y}^5 = G(1,4) \cap \mathbb{P}^8$ | $K_S^2$ | Class in $G(1,4)$ | Codim in $\mathcal{G}M$ | Locus in $\mathcal{G}M$ | $h^0(\mathcal{I}_S/\mathbb{Y}^5(2))$ | $h^0(N_{S/\mathbb{Y}^5})$ | $h^0(N_{S/X})$ | Curves of degree $e$ in $\mathbb{Y}^5$ passing though a general point of $\mathbb{Y}^5$ and that are $(2e-1)$-secant to $S$ for $e \leq 5$. |
|------------------------------------------------|--------|-------------------|------------------|------------------|-----------------|-----------------|----------------|-------------------------------------------------|
| Quadric surface [DIM15] | 8 | $\sigma_{3,1} + \sigma_{2,2}$ | 1 | $\mathcal{G}M_{10}$ | 31 | 8 | 0 | 1, 0, 0, 0, 0 |
| M2-command: specialGushelMukaiFourfold "tau-quadric" (The same case as in the first row of Table 8) |
| Quintic Del Pezzo surface [Rot49] | 5 | $3\sigma_{3,1} + 2\sigma_{2,2}$ | 1 | $\mathcal{G}M_{10}$" | 24 | 18 | 3 | 3, 0, 0, 0, 0 |
| M2-command: specialGushelMukaiFourfold "quintic del Pezzo surface" |
| Cubic scroll [DIM15] | 8 | $2\sigma_{3,1} + \sigma_{2,2}$ | 1 | $\mathcal{G}M_{12}$ | 28 | 11 | 0 | 2, 0, 0, 0, 0 |
| M2-command: specialGushelMukaiFourfold "cubic scroll" |
| Rational surface of degree 10 and genus 4 | 0 | $6\sigma_{3,1} + 4\sigma_{2,2}$ | 1 | $\mathcal{G}M_{16}$" | 15 | 29 | 5 | 6, 0, 0, 0, 0 |
| M2-command: specialGushelMukaiFourfold([3,6],[2,2],"quartic scroll") |
| Rational surface of degree 12 and genus 5 | $-1$ | $7\sigma_{3,1} + 5\sigma_{2,2}$ | 1 | $\mathcal{G}M'_{18}$ | 11 | 32 | 4 | 7, 5, 0, 0, 0 |
| M2-command: specialGushelMukaiFourfold([5,9,0,1],[2,3,0,1]) |
| Rational surface of degree 9 and genus 3 | 2 | $5\sigma_{3,1} + 4\sigma_{2,2}$ | 1 | $\mathcal{G}M'_{18}$" | 16 | 26 | 3 | 5, 0, 0, 0, 0 |
| M2-command: specialGushelMukaiFourfold([4,6,1],[2,2,1]) |
| Rational surface of degree 9 and genus 2 | | | | $\mathcal{G}M_{20}$ | 14 | 25 | 0 | 6, 1, 0, 0, 0 |
| M2-command: specialGushelMukaiFourfold "general GM 4-fold of discriminant 20" (The same case as in the fourth row of Table 8) |
| Rational surface of degree 10 and genus 3 | 3 | $6\sigma_{3,1} + 4\sigma_{2,2}$ | 1 | $\mathcal{G}M_{24}$ | 13 | 27 | 1 | 6, 2, 0, 0, 0 |
| M2-command: specialGushelMukaiFourfold([4,5,1],[2,1,1]) |
| Rational surface of degree 12 and genus 5 | 0 | $7\sigma_{3,1} + 5\sigma_{2,2}$ | 1 | $\mathcal{G}M'_{26}$" | 11 | 30 | 2 | 7, 4, 0, 0, 0 |
| M2-command: specialGushelMukaiFourfold([4,9],[2,4]) |

Table 5. Selected codimension-one families of GM fourfolds described as the closure of the locus of smooth quadric hypersurfaces in a del Pezzo fivefold $\mathbb{Y}^5$ containing some smooth surface $S \subset \mathbb{Y}^5$ cut out by quadrics and varying in an irreducible component of $\text{Hilb}_{\mathbb{Y}^5}^{(S)}$. See [Sta21b, Table 1] for more examples.
Explicit computations with cubic fourfolds and GM fourfolds

Surface \( S \) in \( Y^5 = G(1,4) \cap \mathbb{P}^8 \)

| Surface S in \( Y^5 = G(1,4) \cap \mathbb{P}^8 \) | \( K_S^2 \) | Class in \( G(1,4) \) in \( \mathcal{G}M \) | Codim | Locus in \( \mathcal{G}M \) | \( h^0(\mathcal{I}_S/Y^5(2)) \) | \( h^0(N_{S/Y^5}) \) \( h^0(N_{S/X}) \) | Curves of degree \( e \) in \( Y^5 \) passing through a general point of \( Y^5 \) and that are \( (2e-1) \)-secant to \( S \) for \( e \leq 5 \). |
|-----------------|--------|-----------------|-------|-----------------|-----------------|-----------------|-----------------|
| Rational surface of degree 13 and genus 6 | \(-18\sigma_{3,1} + 5\sigma_{2,2}\) | 1 | \( \mathcal{G}M_{28} \) | 10 | 31 | 2 | 8, 6, 0, 0, 0 |
| Construction of an example: take the isomorphic image via \( \psi_{\Sigma_3} : \mathbb{P}^6 \longrightarrow G(1,4) \) of a rational surface \( T \subset \mathbb{P}^5 \subset \mathbb{P}^6 \) of degree 8 and genus 4, obtained as the image of the plane via the linear system of sextic curves with four general simple base points and six general double points, which cuts \( \Sigma_3 \) along a quintic elliptic curve obtained as the image of a general cubic passing through three of the four simple points and five of the six double points. |
| M2-command: \( \text{specialGushelMukaiFourfold([6,4,6],[3,3,5])} \) |
| Rational surface of degree 11 and genus 4 | \( 26\sigma_{3,1} + 5\sigma_{2,2} \) | 1 | \( \mathcal{G}M_{28} \) | 12 | 27 | 0 | 6, 2, 0, 0, 0 |
| Construction of an example: take the image via \( \psi_{\Sigma_3} : \mathbb{P}^6 \longrightarrow G(1,4) \) of a rational surface \( T \subset \mathbb{P}^5 \subset \mathbb{P}^6 \) of degree 7 and genus 3, obtained as the image of the plane via the linear system of septic curves with six general double base points and two triple points, which cuts \( \Sigma_3 \) along a rational normal quartic curve obtained as the image of a conic passing through five of the six double points. |
| M2-command: \( \text{specialGushelMukaiFourfold([7,0,6,2],[2,0,5,0])} \) |

Table 6. Continuation of Table 5. New families of GM fourfolds.
| Surface $S$ in $\mathbb{P}^5$ | $K_S^2$ | Nodes | Codim in $\mathcal{C}$ | Locus in $\mathcal{C}$ | $h^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$ | $h^0(N_{S/\mathbb{P}^5})$ | $h^0(N_{S/X})$ | Curves of degree $e$ passing through a general point of $\mathbb{P}^5$ and that are $(3e-1)$-secant to $S$ for $e \leq 5$. |
|-------------------------------|---------|-------|----------------------|-------------------|-----------------|-----------------|-----------------|---------------------------------|
| Del Pezzo surface of degree 5 | 5       | 0     | 1                    | $\mathcal{C}_{14}$| 25              | 35              | 5               | 1, 0, 0, 0, 0          |
| $M2$-command (1): specialCubicFourfold surface $\{3,4\}$ | | | | | | | | |
| $M2$-command (2): specialCubicFourfold "quintic del Pezzo surface" | | | | | | | | |
| Rational scroll of degree 4 | 8       | 0     | 1                    | $\mathcal{C}_{14}$| 28              | 29              | 2               | 1, 0, 0, 0, 0          |
| $M2$-command: specialCubicFourfold(PP$[2,2]$) | | | | | | | | |
| Del Pezzo surface of degree 7 | 7       | 1     | 1                    | $\mathcal{C}_{26}$| 14              | 42              | 1               | 5, 1, 0, 0, 0          |
| $M2$-command: specialCubicFourfold "one-nodal septic del Pezzo surface" | | | | | | | | |
| Rational scroll of degree 7 | 8       | 3     | 1                    | $\mathcal{C}_{26}$| 13              | 44              | 2               | 7, 1, 0, 0, 0          |
| $M2$-command: specialCubicFourfold "3-nodal septic scroll" | | | | | | | | |
| Rational surface of degree 10 and genus 6 | $-1$    | 0     | 1                    | $\mathcal{C}_{38}$| 10              | 47              | 2               | 7, 1, 0, 0, 0          |
| $M2$-command (1): specialCubicFourfold surface $\{10,0,0,10\}$ | | | | | | | | |
| $M2$-command (2): specialCubicFourfold "general cubic 4-fold of discriminant 38" | | | | | | | | |
| Rational scroll of degree 8 | 8       | 6     | 1                    | $\mathcal{C}_{38}$| 10              | 47              | 2               | 9, 4, 1, 0, 0          |
| $M2$-command: specialCubicFourfold "6-nodal octic scroll" | | | | | | | | |
| Rational surface of degree 9 and genus 2 | 5       | 5     | 1                    | $\mathcal{C}_{42}$| 9               | 48              | 2               | 9, 7, 1, 0, 0          |
| $M2$-command: specialCubicFourfold "general cubic 4-fold of discriminant 42" | | | | | | | | |

**Table 7.** Selected descriptions of families of rational cubic fourfolds for which we can explicitly calculate associated K3 surfaces. See [RS21b, Table 1] for more examples.
Quadric surface \([\text{DIM15]}\) 8, \(\sigma_3, 1 + \sigma_2, 2\) 1 \(\mathcal{G}M_{10}\) 31 8 0 1, 0, 0, 0, 0

\textit{M2\text{-command}}: \texttt{specialGushelMukaiFourfold "tau-quadric"

K3 surface of degree 14 and genus 8 \([\text{RS21a]}\)

\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Degree & Class & Locus & \(h^0(I_{S/\mathbb{P}}(2))\) & \(h^0(N_{S/\mathbb{P}})\) & \(h^0(N_{S/S})\) & Curves of degree \(e\) in \(\mathbb{P}^5\) passing through a general point of \(\mathbb{P}^5\) and that are \((2e - 1)\)-secant to \(S\) for \(e \leq 5\) \\
\hline
14 & \(9\sigma_3, 1 + 5\sigma_2, 2\) & \(1\) & \(\mathcal{G}M_{10}\) & 10 & 39 & 10, 8, 1, 0, 0 \\
\hline
\end{tabular}

\textit{M2\text{-command}}: \texttt{specialGushelMukaiFourfold "K3 surface of genus 8 with class (9,5)"

Plane \([\text{Rot49]}\)

\begin{tabular}{|c|c|c|c|c|c|}
\hline
Degree & Class & Locus & \(h^0(I_{S/\mathbb{P}}(2))\) & \(h^0(N_{S/\mathbb{P}})\) & \(h^0(N_{S/S})\) & Curves of degree \(e\) in \(\mathbb{P}^5\) passing through a general point of \(\mathbb{P}^5\) and that are \((2e - 1)\)-secant to \(S\) for \(e \leq 5\) \\
\hline
9 & \(\sigma_3, 1\) & \(2\) & \(\mathcal{G}M_{10}\) & 34 & 4 & 1, 0, 0, 0, 0 \\
\hline
\end{tabular}

\textit{M2\text{-command}} (1): \texttt{specialGushelMukaiFourfold "sigma-plane"

\textit{M2\text{-command}} (2): \texttt{specialGushelMukaiFourfold schubertCycle([3,1],[\mathcal{G}M(1,4)])

Rational surface of degree 9 and genus 2 \([\text{HS20, HS21a]}\)

\begin{tabular}{|c|c|c|c|c|}
\hline
Degree & Class & Locus & \(h^0(I_{S/\mathbb{P}}(2))\) & \(h^0(N_{S/\mathbb{P}})\) & \(h^0(N_{S/S})\) & Curves of degree \(e\) in \(\mathbb{P}^5\) passing through a general point of \(\mathbb{P}^5\) and that are \((2e - 1)\)-secant to \(S\) for \(e \leq 5\) \\
\hline
9 & \(6\sigma_3, 1 + 3\sigma_2, 2\) & \(1\) & \(\mathcal{G}M_{20}\) & 14 & 25 & 0, 6, 1, 0, 0, 0 \\
\hline
\end{tabular}

\textit{Contraction of an example}: take the isomorphic image via \(\psi_{\Sigma_3}: \mathbb{P}^6 \rightarrow \mathbb{G}(1,4)\) of a rational surface \(T \subset \mathbb{P}^5 \subset \mathbb{P}^6\) of degree 8 and genus 4, obtained as the image of the plane via the linear system of quintic curves with nine general simple base points and two general double points, which cuts \(\Sigma_3\) along a quintic elliptic curve obtained as the image of a generic cubic passing through six of the nine simple points and the two double points.

\textit{M2\text{-command}}: \texttt{specialGushelMukaiFourfold([5,9,2],[3,6,2])

Rational normal scroll of degree 7 (see also \[\text{Sta21b]}\)

\begin{tabular}{|c|c|c|c|c|}
\hline
Degree & Class & Locus & \(h^0(I_{S/\mathbb{P}}(2))\) & \(h^0(N_{S/\mathbb{P}})\) & \(h^0(N_{S/S})\) & Curves of degree \(e\) in \(\mathbb{P}^5\) passing through a general point of \(\mathbb{P}^5\) and that are \((2e - 1)\)-secant to \(S\) for \(e \leq 5\) \\
\hline
7 & \(4\sigma_3, 1 + 3\sigma_2, 2\) & \(3\) & \(\mathcal{G}M_{20}\) & 16 & 21 & 0, 4, 1, 0, 0, 0 \\
\hline
\end{tabular}

\textit{Contraction of an example}: take the isomorphic image via \(\psi_{\Sigma_3}: \mathbb{P}^6 \rightarrow \mathbb{G}(1,4)\) of a rational normal quintic scroll \(T \subset \mathbb{P}^6\), obtained as the image of the plane via the linear system of cubic curves with one double base point, which cuts \(\Sigma_3\) along a rational normal quartic curve obtained as the image of a conic passing through the double point.

\textit{M2\text{-command}}: \texttt{specialGushelMukaiFourfold([3,0,1],[2,0,1])

Triple projection of K3 surface of degree 26 and genus 14

\begin{tabular}{|c|c|c|c|c|}
\hline
Degree & Class & Locus & \(h^0(I_{S/\mathbb{P}}(2))\) & \(h^0(N_{S/\mathbb{P}})\) & \(h^0(N_{S/S})\) & Curves of degree \(e\) in \(\mathbb{P}^5\) passing through a general point of \(\mathbb{P}^5\) and that are \((2e - 1)\)-secant to \(S\) for \(e \leq 5\) \\
\hline
9 & \(-11\sigma_3, 1 + 6\sigma_2, 2\) & \(2\) & \(\mathcal{G}M_{26}\) & 7 & 37 & 6, 11, 22, 32, 6, 1 \\
\hline
\end{tabular}

\textit{Contraction of an example}: Let \(X \subset \mathbb{Y}^5\) be an example of GM fourfold containing a surface \(S\) of degree 9 and genus 2 as in last row, and let \(\mu: \mathbb{Y}^5 \rightarrow \mathbb{P}^5\) be the rational map defined by the linear system of cubic hypersurfaces in \(\mathbb{Y}^5\) with double points along \(S\). Then the base locus of \(\mu\) is the union of a threefold \(B\) and a \(\sigma_3, 1\)-plane \(P\) such that \(P \cap B = P \cap S\) is a conic. The projection \(\xi: \mathbb{Y}^5 \rightarrow \mathbb{P}^5\) from \(P\) sends the surface \(S\) isomorphically into a quintic del Pezzo surface \(\xi(S) \subset \mathbb{P}^5\). The inverse image \(\xi^{-1}((S))\) is isomorphic to an internal projection of a Fano threefold \(G(1,5) \cap \mathbb{P}^5 \subset \mathbb{P}^5\) and intersects \(X\) in the union of \(S\) and another smooth surface of degree 17 and genus 11, which is isomorphic to a triple projection of a minimal K3 surface of degree 26 and genus 14 in \(\mathbb{P}^{14}\).

\textit{M2\text{-command}}: \texttt{specialGushelMukaiFourfold("triple projection of K3 surface of degree 26")

Rational surface of degree 9 and genus 2

\begin{tabular}{|c|c|c|c|c|}
\hline
Degree & Class & Locus & \(h^0(I_{S/\mathbb{P}}(2))\) & \(h^0(N_{S/\mathbb{P}})\) & \(h^0(N_{S/S})\) & Curves of degree \(e\) in \(\mathbb{P}^5\) passing through a general point of \(\mathbb{P}^5\) and that are \((2e - 1)\)-secant to \(S\) for \(e \leq 5\) \\
\hline
5 & \(5\sigma_3, 1 + 4\sigma_2, 2\) & \(2\) & \(\mathcal{G}M_{26}\) & 14 & 24 & 0, 5, 1, 0, 0, 0 \\
\hline
\end{tabular}

\textit{Contraction of an example}: take the image via \(\psi_{\Sigma_3}: \mathbb{P}^6 \rightarrow \mathbb{G}(1,4)\) of a rational surface \(T \subset \mathbb{P}^5\) of degree 7 and genus 2, obtained as the image of the plane via the linear system of quartic curves with five general simple base points and one general double point, which cuts \(\Sigma_3\) along a rational normal quintic curve obtained as the image of a general conic passing through three of the five simple points.

\textit{M2\text{-command}}: \texttt{specialGushelMukaiFourfold([4,5,1],[2,3,0])

TABLE 8. New and selected descriptions of families of rational GM fourfolds for which we can explicitly calculate associated K3 surfaces. In all cases, the surface \(S\) is smooth and cut out by quadrics.
Surface $S$ in $\mathcal{Y}^5 = G(1,4) \cap \mathbb{P}^8$

| $K_S^2$ | Class in $G(1,4)$ | Codim in $\mathcal{GM}$ | Locus in $\mathcal{GM}$ | $h^0(\mathcal{I}_{S/\mathcal{Y}^5}(2))$ | $h^0(N_{S/\mathcal{Y}^5})$ | $h^0(N_{S/\mathcal{X}})$ | Curves of degree $e$ in $\mathcal{Y}^5$ passing through a general point of $\mathcal{Y}^5$ and that are $(2e-1)$-secant to $S$ for $e \leq 5$. |
|--------|-----------------|-----------------|----------------|----------------|----------------|----------------|----------------|
| 1-nodal surface of degree 11 and genus 3 (see also [Sta21c]) | 3 | $7\sigma_{3,1} + 4\sigma_{2,2}$ | 2 | $\mathcal{GM}_{26}$ | 11 | 29 | 2 | 7, 4, 1, 0, 0 |

Contraction of an example: Let $X \subset \mathcal{Y}^5$ be an example of GM fourfold containing a surface $S$ of degree 9 and genus 2 as in last row of Table 8, and let $\mu : \mathcal{Y}^5 \rightarrow \mathbb{P}^5$ be the rational map defined by the linear system of cubic hypersurfaces in $\mathcal{Y}^5$ with double points along $S$. The locus $\{ p \in \mu(\mathcal{Y}^5) : \dim \mu^{-1}(p) \geq 2 \}$ consists of a twisted cubic and four lines. If $L$ is one of these four lines, the inverse image $\mu^{-1}(L) \subset \mathcal{Y}^5$ is a threefold cutting $X$ in the union of the surface $S$ and a surface of degree 11 and genus 3 with a node.

M2-command: `specialGushelMukaiFourfold("surface of degree 11 and genus 3 with class (7,4)")`

Rational surface of degree 14 and genus 7

| $K_S^2$ | Class in $G(1,4)$ | Codim in $\mathcal{GM}$ | Locus in $\mathcal{GM}$ | $h^0(\mathcal{I}_{S/\mathcal{Y}^5}(2))$ | $h^0(N_{S/\mathcal{Y}^5})$ | $h^0(N_{S/\mathcal{X}})$ | Curves of degree $e$ in $\mathcal{Y}^5$ passing through a general point of $\mathcal{Y}^5$ and that are $(2e-1)$-secant to $S$ for $e \leq 5$. |
|--------|-----------------|-----------------|----------------|----------------|----------------|----------------|----------------|
| $-1$ | $9\sigma_{3,1} + 5\sigma_{2,2}$ | - | $\mathcal{GM}_{34}$ | 9 | - | - | 9, 8, 1, 0, 0 |

Contraction of an example: take the isomorphic image via $\psi_{\Sigma_3} : \mathbb{P}^6 \rightarrow G(1,4)$ of a rational surface $T \subset \mathcal{Y}^5 \subset \mathbb{P}^6$ of degree 8 and genus 4, obtained as the image of the plane via the linear system of sextic curves with four general simple base points and six general double points, which cuts $\Sigma_3$ along a quintic elliptic curve obtained as the image of a general cubic passing through one of the four simple points and the six double points.

M2-command: `specialGushelMukaiFourfold([6,4,6],[3,1,6])`

Table 9. Continuation of Table 8. Note, however, that in the last row the surface $S \subset \mathcal{Y}^5$ is not cut out by quadrics, and moreover the function `associatedK3surface` currently fails (some requirements are not met).
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