Abstract. We analyze the steady fluid flow in a porous medium containing a network of thin fissures i.e. width $O(\epsilon)$, where all the cracks are generated by the rigid translation of a continuous piecewise $C^1$ functions in a fixed direction. The phenomenon is modeled in mixed variational formulation, using the stationary Darcy’s law and setting coefficients of low resistance $O(\epsilon)$ on the network. The singularities are removed performing asymptotic analysis as $\epsilon \to 0$ which yields an analogous system hosting only tangential flow in the fissures. Finally the fissures are collapsed into two dimensional manifolds.

1. Introduction

Groundwater and oil reservoirs are frequently fissured or layered i.e. the bed rock contains fissures of characteristic dimensions considerably higher than those of the average pore size of the rock. The modeling of saturated flow through geological structures such as these, gives rise to singular problems of partial differential equations [19]. On one hand the singularities are due to the drastic change of permeability from the rock matrix to the fissures. On the other hand a geometric singularity is introduced due to the thinness of the fractures. The presence of singularities in the model has non-desirable effects in their numerical implementation; some of these are ill-condition matrices, high computational costs, numerical stability, etc. This subject is a very active research field, see [2, 5, 9, 11, 12] for numerical analysis aspects, [8, 10] for modeling discussion and [1, 3, 4, 13, 14] for rigorous mathematical treatment of the phenomenon. Homogenization and asymptotic analysis techniques are a common approach for the analytical point of view. However, the remarkable achievements in the field require very restricted hypotheses for the description of the geometry such as uniformly distributed, regular geometric shapes or periodic arrayed structures [7, 16]. In general the variational methods for partial differential equations can formulate successfully a wide class of geometric domains, the limited treatment of the geometry comes from the notorious difficulties it introduces in the asymptotic analysis of the problem.

In the present work, the geometric possibilities of the medium are broaden to an unprecedented setting: free from the aforementioned hypotheses. We use the mixed mixed formulation and the scaling for the flow resistance coefficients presented in

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then a careful choice of directions or “stream lines”, consistent with the natural scaling of the problem permits a successful asymptotic analysis of the model. This leads to a system coupled though multiple two dimensional manifolds representing the fissures in the upscaled model. Additionally, the formulation allows remarkable generality in the fluid exchange balance conditions between the rock matrix and the channels, substantial efficiency for handling the system of equations as well as the information (coefficients, matrices, etc) describing the geometry of the fractures, mostly due to the fact that it does not demand coupling constraints on the underlying spaces of functions. The main goal of the paper is to emphasize on the geometry, consequently the study is limited to the steady case. We describe flow with Darcy’s law
\[ a(\cdot)u + \nabla p + g = 0, \]
(1.1a)

\[ \nabla \cdot u = F. \]
(1.1b)

Drained and non-flux boundary conditions on different parts of the domain boundary will be specified to set a boundary value problem. The fluid exchange across the interface separating the regions are given by
\[ p^1 - p^2 = \alpha u^1 \quad \text{and} \]
\[ u^1 \cdot \hat{n} - u^2 \cdot \hat{n} = f_{\Gamma} \quad \text{on } \Gamma. \]
(1.1c)

Here, the coefficient \( a(\cdot) \) is the flow resistance i.e. the fluid viscosity times the inverse of the permeability of the medium, to be scaled consistently with the fast and slow flow regions of the medium. Finally, the coefficient \( \alpha \) indicates the fluid entry resistance of the rock matrix.

In the following section we define the geometric setting, formulate the problem in mixed variational formulation and establish its well-posedness. In section three the problem is referred to a common geometric setting in order make possible the asymptotic analysis, the existence of a-priori estimates and the structure of the limiting solution are also shown. Section four studies the formulation and well-posedness of the limiting problem and finds its strong form, particularly important for boundary and interface conditions and proves the strong convergence of the solutions. Section five sets the limiting problem as a coupled system with two dimensional interfaces and section six discusses the possibilities and limitations of the technique as well as related future work.

2. Formulation and Geometric Setting

Vectors are denoted by boldface letters as are vector-valued functions and corresponding function spaces. We use \( \bar{x} \) to indicate a vector in \( \mathbb{R}^2 \); if \( x \in \mathbb{R}^3 \) then the \( \mathbb{R}^2 \times \{0\} \) projection is identified with \( \bar{x} = (x_1, x_2) \) so that \( x = (\bar{x}, x_3) \). The symbol \( \nabla \) represents the gradient in the first first two directions: \( \hat{i}, \hat{j} \). Given a function \( f : \mathbb{R}^2 \to \mathbb{R} \) then \( \int_{\mathcal{M}} f \, dS \) is the notation for its surface integral on the \( \mathbb{R}^2 \) manifold \( \mathcal{M} \subseteq \mathbb{R}^3 \). \( \int_A f \, dx \) stands for the volume integral in the set \( A \subseteq \mathbb{R}^2 \); whenever the context is clear we simply write \( \int_A f \). In the same fashion, whenever there is no confusion \( \sum_i, \prod_i \) indicate \( \sum_{i=1}^I \) and \( \prod_{i=1}^I \) respectively.

\[ A + t \stackrel{\text{def}}{=} \{ x + t \hat{k} : x \in A \} \]
(2.1)
The symbol $\hat{\nu}$ denotes the outwards normal vector on the boundary of a given domain $\mathcal{O}$ and $\hat{n}$ denotes the normal upwards vector to a given surface i.e. $\hat{n} \cdot \hat{k} \geq 0$.

For any $A \subseteq \mathbb{R}^3$ and $t \in \mathbb{R}$ we define its $t$-vertical shift by

2.1. General Geometric Setting. The present work will be limited to the study of fractured media where each fissure can be described in a specific way.

**Definition 2.1.** Let $G \subseteq \mathbb{R}^2$ be open a bounded open simply connected set and $\zeta \in C(G)$ be a piecewise $C^1$. Define the surface

$$\Gamma \overset{\text{def}}{=} \{[\tilde{x}, \zeta(\tilde{x})] : \tilde{x} \in G\}. \quad (2.2)$$

We say $\Gamma$ is a surface eligible for vertical translation fissure generation if $\text{ess inf} \{\hat{n}(s) \cdot \hat{k} : s \in \Gamma\} > 0$. Given vertical height $h > 0$ define the fissure of height $h$ generated by a rigid vertical translation of $\Gamma$ by the domain

$$\Omega(h, \Gamma) \overset{\text{def}}{=} \{(\tilde{x}, y) : \zeta(\tilde{x}) < y < \zeta(\tilde{x}) + h\}. \quad (2.3)$$

**Remark 2.1.** Notice that in the definition of $\Omega(h, \Gamma)$ we mention $h$ as the height and not as the width of the crack. Figure 1 shows that, depending on the gradient of the surface the height $h$ can become significantly different from the actual width.

The analysis will be limited to the type of geological system shown in figure 1. It depicts a region $\Omega \subseteq \mathbb{R}^3$ containing a network of fissures generated by vertical rigid translation continuous piecewise $C^1$ surfaces. Such a region is completely characterized in the following definition

**Definition 2.2.** We say a totally fractured medium of vertical translation generated fissures is a finite collection of surface functions

$$\{\zeta_i \in C(G_i) : G_i \subseteq \mathbb{R}^2 \text{ open bounded simply connected region}; \zeta_i \text{ piecewise } C^1 \text{ functions such that } \text{ess inf} \, \hat{n}_i \cdot \hat{k} > 0, 1 \leq i \leq I\}. \quad (2.4a)$$
vertical heights
\[ \{ h_i > 0 : 1 \leq i \leq I \}, \quad (2.4b) \]
and rock-matrix regions
\[ \{ \Omega_i \subseteq \mathbb{R}^3 : \Omega_i \neq \emptyset \text{ open bounded simply connected region, } 0 \leq i \leq I \}. \quad (2.4c) \]

Verifying the following properties:

Non-overlapping condition and indexed ordered
\[ \sup \{ \zeta_i(\bar{x}) + h_i : \bar{x} \in G_i \} < \inf \{ \zeta_{i+1}(\bar{x}) : \bar{x} \in G_{i+1} \} \forall 1 \leq i \leq I - 1. \quad (2.5a) \]

The interface-domain condition
\[
\begin{align*}
\partial \Omega_i \cap \partial \Omega(h_{i+1}, \Gamma_{i+1}) &= \Gamma_{i+1} \quad \forall 0 \leq i \leq I - 1, \\
\partial \Omega_i \cap \partial \Omega(h_i, \Gamma_i) &= \Gamma_i + h_i \quad \forall 1 \leq i \leq I.
\end{align*} \quad (2.5b)
\]

And the condition of connectivity only through fissures
\[ cl(\Omega_\ell) \cap cl(\Omega_k) = \emptyset, \quad \text{whenever } \ell \neq k. \quad (2.5c) \]

For convenience of notation define \( \Gamma_0 \overset{\text{def}}{=} \partial \Omega_0 - \Gamma_1 \) and \( h_0 \overset{\text{def}}{=} 0 \). The fissured system described above will be denoted \( \{(\Gamma_i, h_i, \Omega_i) : 0 \leq i \leq I \} \). The sets \( \Omega_1, \Omega_2 \) are the rock matrix and the fissures regions respectively i.e.
\[
\begin{align*}
\Omega_1 &\overset{\text{def}}{=} \bigcup_{i=0}^{I} \Omega_i, \quad \Omega_2 \overset{\text{def}}{=} \bigcup_{i=1}^{I} \Omega(h_i, \Gamma_i) \\
\Omega &\overset{\text{def}}{=} \Omega_1 \cup \Omega_2.
\end{align*} \quad (2.6)
\]

The global bottom and top interfaces are defined by
\[
\Gamma_t \overset{\text{def}}{=} \bigcup_{i=1}^{I} \Gamma_i, \quad \Gamma_b \overset{\text{def}}{=} \bigcup_{i=1}^{I} \Gamma_i + h_i \\
\Gamma &\overset{\text{def}}{=} \Gamma_b \cup \Gamma_t \quad (2.7)
\]

Finally, \( \tilde{n}^{(i)} \) indicates the upwards normal vector to the surface \( \Gamma_i \) i.e.
\[ \tilde{n}^{(i)} \overset{\text{def}}{=} \frac{(-\nabla \zeta_i, 1)}{|(-\nabla \zeta_i, 1)|} \quad (2.8) \]

When there is no confusion \( \tilde{n} \) denotes the normal vector with respect to the surface of the crack.

**Remark 2.2.** The condition \( (2.5c) \) of connectivity only through fissures is not required for modeling the problem in mixed formulation as it is presented in section \( (2.4) \); however it is necessary for the asymptotic analysis of the system. The same holds for the requirement of simply connected domains.

### 2.2. A Local System of Coordinates.

Some aspects of the flow through the fissures are handled more conveniently when the velocities are expressed in a coordinate system consistent with the geometry of the surface that generates the crack. Let \( \Gamma \) be a surface as defined in \( (2.1) \) and \( \tilde{n} \) the upwards normal to the surface \( \Gamma \) i.e. \( \tilde{n} = \tilde{n}(s) = \tilde{n}(\bar{x}) \). Now, for each point \( \bar{x} \) we choose a local orthonormal basis in the following way
\[ \mathcal{B}(\bar{x}) \overset{\text{def}}{=} \{ \tilde{e}_1(\bar{x}), \tilde{e}_2(\bar{x}), \tilde{n}(\bar{x}) \} \quad (2.9) \]
Let $M = M(\bar{x})$ be the orthogonal matrix relating the global canonical basis with the local one i.e.

$$M(\bar{x}) \hat{i} = \hat{e}_1(\bar{x})$$

$$M(\bar{x}) \hat{j} = \hat{e}_2(\bar{x})$$

$$M(\bar{x}) \hat{k} = \hat{n}(\bar{x})$$

(2.10a, 2.10b, 2.10c)

The block matrix notation for this local matrix will be

$$M(\bar{x}) \overset{\text{def}}{=} \begin{pmatrix} M^{T, \tau} & M^{T, \hat{n}} \\ M^{\hat{k}, \tau} & M^{\hat{k}, \hat{n}} \end{pmatrix}(\bar{x})$$

(2.11)

Here the index $T$ stands for the first two components in the directions $\hat{i}, \hat{j}$ while the index $\tau$ stands for the expression of the velocity orthogonal to the component in the direction $\hat{n}$. Then $w = [w_\tau, w_{\hat{n}}](\bar{x})$ with the following relations

$$w_{\hat{n}} \overset{\text{def}}{=} w \cdot \hat{n}(\bar{x})$$

$$w_\tau \overset{\text{def}}{=} (w \cdot \hat{e}_1(\bar{x}), w \cdot \hat{e}_2(\bar{x}))$$

(2.12a, 2.12b)

Clearly, the relationship between velocities is given by

$$w(\bar{x}, x_3) = \begin{pmatrix} \bar{w} \\ w \cdot \hat{k} \end{pmatrix}(\bar{x}, x_3) = M(\bar{x}) \begin{pmatrix} w_\tau \\ w \cdot \hat{n} \end{pmatrix}(\bar{x}, x_3)$$

$$= \begin{pmatrix} M^{T, \tau}(\bar{x}) & M^{T, \hat{n}}(\bar{x}) \\ M^{\hat{k}, \tau}(\bar{x}) & M^{\hat{k}, \hat{n}}(\bar{x}) \end{pmatrix} \begin{pmatrix} w_\tau \\ w \cdot \hat{n} \end{pmatrix}(\bar{x}, x_3)$$

(2.13)

**Proposition 2.3.** Let $h > 0$, $\Gamma, \Omega(h, \Gamma)$ be as in definition [2.1]; $\hat{n}$ be the upwards normal to the surface $\Gamma$ and $M$ be the matrix defined by [2.10]. Then

(i) The map $w \mapsto M(\bar{x})w$ is an isometry in $L^2(\Omega(h, \Gamma))$. In particular if $w_\tau, w \cdot \hat{n}$ are defined as in [2.12], then $w \in L^2(\Omega(h, \Gamma))$ if and only if $w_\tau \in L^2(\Omega(h, \Gamma)) \times L^2(\Omega(h, \Gamma))$ and $w \cdot \hat{n} \in L^2(\Omega(h, \Gamma))$.

(ii) If $w \in L^2(\Omega(h, \Gamma))$ is such that $\partial_z w \in L^2(\Omega(h, \Gamma))$ then

$$\partial_z w(\bar{x}, z) = M(\bar{x}) \begin{pmatrix} \partial_z w_\tau \\ (\partial_z w) \cdot \hat{n} \end{pmatrix}(\bar{x}, z)$$

(2.14)

**Proof.** (i) For $\bar{x}$ fixed the matrix $M(\bar{x})$ is orthogonal i.e. for arbitrary functions $v, w \in L^2(\Omega(h, \Gamma))$ and $x \in \Omega(h, \Gamma)$ holds $v(x) \cdot w(x) = M(\bar{x})v(\cdot) \cdot M(\bar{x})w(x)$. Hence

$$\int_{\Omega(h, \Gamma)} v(x) \cdot w(x) \, dx = \int_{\Omega(h, \Gamma)} M(\bar{x})v(\cdot) \cdot M(\bar{x})w(x) \, dx$$

$$= \int_{\Omega(h, \Gamma)} v_\tau(x) \cdot w_\tau(x) \, dx + \int_{\Omega(h, \Gamma)} (v \cdot \hat{n})(\cdot)(w \cdot \hat{n})(x) \, dx$$

The equality of the second line shows the necessity and sufficiency of the tangential and normal components been square integrable in the domain $\Omega(h, \Gamma)$.

(ii) It follows from a direct calculation of distributions with $\varphi \in C^0_0(\Omega(h, \Gamma))]^3$ arbitrary and the fact that $\partial_z M = 0$. 

□
2.3. The Problem and its Formulation. In this section we define the problem in a rigorous way and give a variational formulation in which it is well-posed. Let \(\{\Gamma_i, h_i, \Omega_i\} : 1 \leq i \leq I\) be a totally fractured domain of vertical translation generated fissures. We denote \(v^1, p^1\) the velocity and pressure in the rock matrix region \(\Omega_1\). In the same fashion \(v^2, p^2\) denote the velocity and pressure in the fissures region \(\Omega_2\). Consider the problem

\[
\begin{align*}
\quad a_1 v^1 + \nabla p^1 + g &= 0 \quad \text{and} \quad (2.15a) \\
\nabla \cdot v^1 &= F \quad \text{in } \Omega_1, \quad (2.15b) \\
p^1 &= 0 \quad \text{on } \partial \Omega_1 - \Gamma. \quad (2.15c) \\
p^1 - p^2 &= \alpha \left( u^1 \cdot \hat{n} \right)_{\Gamma_b} - \alpha \left( u^1 \cdot \hat{n} \right)_{\Gamma_t} \quad \text{and} \quad (2.15d) \\
\left( u^1 - u^2 \right) \cdot \hat{n} |_{\Gamma_t} - \left( u^1 - u^2 \right) \cdot \hat{n} |_{\Gamma_b} &= f_1 \quad \text{on } \Gamma. \quad (2.15e) \\
a_2 u^2 + \nabla p^2 + g &= 0 \quad \text{and} \quad (2.15f) \\
\nabla \cdot u^2 &= F \quad \text{in } \Omega_2, \quad (2.15g) \\
u^2 \cdot \hat{n} &= 0 \quad \text{on } \partial \Omega_2 - \Gamma. \quad (2.15h)
\end{align*}
\]

The flow resistance coefficients \(a_1, a_2\) and the fluid entry resistance coefficient \(\alpha\) are assumed to be positively bounded from below and above, see [15]. In equations (2.15d)–(2.15e) the split of cases is made in order to be consistent with the sign of the upwards normal vector \(\hat{n}\).

2.4. Mixed Formulation of the Problem. We start defining the spaces of velocities and pressures

\[
\begin{align*}
V &\overset{\text{def}}{=} \{ v \in L^2(\Omega) : \nabla \cdot v^1 \in L^2(\Omega_1), v^1 \cdot \hat{n} |_{\Gamma} \in L^2(\Gamma) \}. \quad (2.16a) \\
Q &\overset{\text{def}}{=} \{ q \in L^2(\Omega) : \nabla q^2 \in L^2(\Omega_2) \}. \quad (2.16b)
\end{align*}
\]

Endowed with their natural norms

\[
\| v \|_V \overset{\text{def}}{=} \left\{ \| v \|_{L^2(\Omega)}^2 + \| \nabla \cdot v^1 \|_{L^2(\Omega_1)}^2 + \| v^1 \cdot \hat{n} \|_{L^2(\Gamma)}^2 \right\}^{1/2} \quad (2.16c) \\
\| q \|_Q \overset{\text{def}}{=} \left\{ \| q \|_{L^2(\Omega)}^2 + \| \nabla q^2 \|_{L^2(\Omega_2)}^2 \right\}^{1/2} \quad (2.16d)
\]

**Remark 2.3.** In the spaces above it is understood that

\[
\| v \cdot \hat{n} \|_{L^2(\Gamma)}^2 = \| v \cdot \hat{n} \|_{L^2(G_h)}^2 + \| v \cdot \hat{n} \|_{L^2(G_t)}^2
\]

\[
= \sum_{i=1}^{I} \| v \cdot \hat{n}^i \|_{L^2(G_t)}^2 + \sum_{i=1}^{I} \| v \cdot \hat{n}^i \|_{L^2(G_h)}^2 \quad (2.17)
\]

Consider the problem

\[
\begin{align*}
\text{Find } \ p &\in Q, \ u \in V \\
\int_{\Omega_1} a_1 u \cdot v + \int_{\Omega_2} a_2 u \cdot v - \int_{\Omega_1} p \nabla \cdot v + \int_{\Omega_2} \nabla p \cdot v \\
+ \alpha \int_{\Gamma} (u^1 \cdot \hat{n}) (v^1 \cdot \hat{n}) \ dS - \int_{\Gamma_t} p^1 (v^1 \cdot \hat{n}) \ dS + \int_{\Gamma_b} p^2 (v^1 \cdot \hat{n}) \ dS = - \int_{\Omega} g \cdot v \\
\int_{\Omega_1} \nabla \cdot u q - \int_{\Omega_2} u \cdot \nabla q
\end{align*}
\]

(2.18a)
Remark 2.4. In the formulation above the non-symmetric interface terms are split in two pieces in order to express everything in terms of the upwards normal vector \( \hat{n} \). In the case of the symmetric term \( \int_{\Omega} (\mathbf{u}^1 \cdot \hat{n}) (\mathbf{v}^1 \cdot \hat{n}) dS \) in (2.18a) such split becomes unnecessary since the sign of the normal vector changes in both factors canceling each other.

Define the bilinear forms \( A : \mathbf{V} \rightarrow \mathbf{V}' \), \( B : \mathbf{V} \rightarrow Q' \), \( C : Q \rightarrow Q' \) by

\[
A\mathbf{v}(\mathbf{w}) \overset{\text{def}}{=} \int_{\Omega_1} a_1 \mathbf{v} \cdot \mathbf{w} + \int_{\Omega_2} a_2 \mathbf{v} \cdot \mathbf{w} + \alpha \int_{\Gamma} (\mathbf{v}^1 \cdot \hat{n}) (\mathbf{w}^1 \cdot \hat{n}) dS \quad (2.19a)
\]

\[
B\mathbf{v}(q) \overset{\text{def}}{=} -\int_{\Omega_1} \nabla \cdot \mathbf{v} q + \int_{\Omega_2} \mathbf{v} \cdot \nabla q - \int_{\Gamma} (\mathbf{v}^1 \cdot \hat{n}) q^2 dS + \int_{\Gamma} (\mathbf{v}^1 \cdot \hat{n}) q^2 dS \quad (2.19b)
\]

Then, the system (2.18) is a mixed formulation for the problem (2.15) with the abstract form

\[
\mathbf{u} \in \mathbf{V}, \ p \in Q : \quad A\mathbf{u} + B\mathbf{p} = -\mathbf{g} \text{ in } \mathbf{V}',
\]

\[
-\mathbf{B}\mathbf{u} = \mathbf{f} \quad \text{in } Q'. \quad (2.20)
\]

For the sake of completeness recall some well known results

Theorem 2.4. Let \( \mathbf{V}, Q \) be Hilbert spaces and \( \| \cdot \|_\mathbf{V}, \| \cdot \|_Q \) be their respective norms. Let \( A : \mathbf{V} \rightarrow \mathbf{V}' \), \( B : \mathbf{V} \rightarrow Q' \) be continuous linear operators such that

(i) \( A \) is non-negative and \( \mathbf{V} \)-coercive on \( \ker B \).

(ii) The operator \( B \) satisfies the inf-sup condition

\[
\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{\| B\mathbf{v}(q) \|}{\| \mathbf{v} \|_\mathbf{V} \| q \|_Q} > 0. \quad (2.21)
\]

Then, for each \( \mathbf{g} \in \mathbf{V}' \) and \( \mathbf{f} \in Q' \) there exists a unique solution \( [\mathbf{u}, p] \in \mathbf{V} \times Q \) to the problem (2.20). Moreover, it satisfies the estimate

\[
\| \mathbf{u} \|_\mathbf{V} + \| p \|_Q \leq K ( \| \mathbf{g} \|_{\mathbf{V}'} + \| \mathbf{f} \|_{Q'} ) \quad (2.22)
\]

Proof. See [6]. \( \square \)

Lemma 2.5. Let \( \mathcal{O} \) be an open connected bounded set in \( \mathbb{R}^N \) and \( \mathcal{G} \subseteq \partial \mathcal{O} \) with non-null \( \mathbb{R}^{N-1} \)-Lebesgue measure, then there exists \( \kappa = \kappa(\mathcal{O}) > 0 \) such that

\[
\| \nabla \eta \|_{L^2(\mathcal{O})} + \| \eta \|_\mathcal{O} \geq \kappa \| \eta \|_{H^1(\mathcal{O})} \quad (2.23)
\]

for all \( \eta \in H^1(\mathcal{O}) \).

Proof. See proposition 5.2 of [18] or lemma 1.2 in [15]. \( \square \)

Corollary 2.6. There exists a constant \( \kappa > 0 \) such that

\[
\| \nabla q \|_{L^2(\Omega_2)}^2 + \| q \|_{L^2(\Gamma)}^2 \geq \kappa \| q \|_{L^2(\Omega_1)}^2. \quad (2.24)
\]

For all \( q \in H^1(\Omega_2) \)

Proof. Apply lemma (2.5) on each connected component \( \Omega(h_i, \Gamma_i) \) and choose \( \kappa \) as the minimum constant associated to each domain. \( \square \)

Lemma 2.7. The operator \( B \) satisfies the inf-sup condition (2.21).
Proof. We use the same strategy presented lemma 1.3 in [15] with a slight modification in the construction of the particular test function. Fix \( q \in Q \) and denote \( \xi_j \) the unique solution of the problem

\[
-\nabla \cdot \nabla \xi_j = q^1 \text{ in } \Omega_j,
\]

\[
\nabla \xi_j \cdot \hat{n} = q^2 \text{ on } \Gamma_j, \quad \nabla \xi_j \cdot \hat{n} = -q^2 \text{ on } \Gamma_j + h_j,
\]

\( \xi_j = 0 \) on \( \partial \Omega_j - \Gamma_j - (\Gamma_j + h_j) \).

Define \( v^1 \defeq \sum_{j=0}^{I} \nabla \xi_j \mathbb{1}_{\Omega_j} \). Thus, \( -\nabla \cdot v^1 = \sum_{j=0}^{I} q^1 \mathbb{1}_{\Omega_j} \)

Due to the Poincaré inequality \( c_1 \| v^1 \|_{H^1(\Omega)} \leq \| q^1 \|_{L^2(\Omega)} + \| q^2 \|_{L^2(\Gamma)} \). Hence, setting \( v^2 \defeq \nabla q^2 \)

\[
\mathcal{B}v(q) = -\int_{\Omega_1} \nabla \cdot v^1 q^1 + \int_{\Omega_2} \nabla \cdot v^2 q^2 - \int_{\Gamma_1} (v^1 \cdot \hat{n}) q^2 dS + \int_{\Gamma_2} (v^1 \cdot \hat{n}) q^2 dS
\]

\[
= \int_{\Omega_1} |q^1|^2 + \int_{\Omega_2} |\nabla q^2|^2 + \int_{\Gamma_1} |q^2|^2 dS + \int_{\Gamma_2} |q^2|^2 dS
\]

\[
\geq \int_{\Omega_1} |q^1|^2 + \frac{k}{2} \int_{\Omega_2} |q^2|^2 + \frac{1}{2} \left( \int_{\Gamma_1} |q^2|^2 + \int_{\Gamma_2} |q^2|^2 dS \right) \geq c \| v \|_V \| q \|_Q \quad (2.26)
\]

For \( c \defeq \min\{c_1, \frac{1}{2}, \frac{a^*}{2}\} \), which gives the inf-sup condition of the operator \( \mathcal{B} \). \( \square \)

**Theorem 2.5.** Suppose that \( 0 \leq \alpha, a_i(\cdot) \in L^\infty(\Omega) \) and

\[
a^* \defeq \min \text{ ess inf } \{a_i(x) : x \in \Omega_i\} \quad (2.27)
\]

If \( a^* \) is positive then, the mixed variational formulation \( (2.20) \) (or equivalently, the system \( (2.18) \)) is well-posed.

**Proof.** Clearly \( A \) is non-negative and \( \mathbf{V} \)-coercive on \( \text{ker } \mathcal{B} \). The operator \( \mathcal{B} \) satisfies the inf-sup condition as seen in the preceding lemma. Due to theorem \( (2.4) \) the result follows. \( \square \)

### 3. Scaling the Problem and Convergence Statements

In order to perform the asymptotic analysis for a the problem \( (2.18) \) in a medium of thin fractures, the heights and resistance coefficients have to be scaled. We have the following definition (see figure \( 2 \)).

**Definition 3.1.** Let \( \{(\zeta_i, h_i, \Omega_i) : 1 \leq i \leq I\} \) be a fractured medium of vertical translation generated fissures. For \( \epsilon \in (0,1) \) we define its associated \( \epsilon \)-scaled fissured system \( \{(\zeta_i^\epsilon, h_i, \Omega_i^\epsilon) : 1 \leq i \leq I\} \) by

\[
\zeta_i^\epsilon = \zeta_i - (1 - \epsilon) \sum_{\ell=0}^{i-1} h_\ell, \quad 1 \leq i \leq I.
\]

\[
\{\epsilon h_i > 0 : 1 \leq i \leq I\}. \quad (3.1b)
\]

\[
\Omega_j^\epsilon \defeq \Omega_j - (1 - \epsilon) \sum_{\ell=0}^{j} h_\ell, \quad 0 \leq j \leq I. \quad (3.1c)
\]
The domains $\Omega_1^\epsilon, \Omega_2^\epsilon, \Omega^\epsilon$ and the surfaces $\Gamma_1^\epsilon, \Gamma_b^\epsilon, \Gamma^\epsilon$ are defined as in (2.4c), (2.6) respectively.

Remark 3.1. Clearly the systems $\{(\Gamma_i^\epsilon, \epsilon h_i, \Omega_i^\epsilon) : 1 \leq i \leq I\}$ satisfies the conditions of definition (2.2).

3.1. Isomorphisms of Spaces and Formulation. Let $\Omega_1^\epsilon, \Omega_2^\epsilon, \Omega^\epsilon$ and $\Gamma_1^\epsilon, \Gamma_b^\epsilon, \Gamma^\epsilon$ be the domains and surfaces associated to the family $\{(\zeta_i^\epsilon, \epsilon h_i, \Omega_i^\epsilon) : 1 \leq i \leq I\}$ as in definition (3.1). Define the spaces

$$V^\epsilon \overset{\text{def}}{=} \{ v \in L^2(\Omega^\epsilon) : \nabla \cdot v^1 \in L^2(\Omega_1^\epsilon), v^1 \cdot \hat{n} |_{V^\epsilon} \in L^2(\Gamma^\epsilon) \},$$

$$Q^\epsilon \overset{\text{def}}{=} \{ q \in L^2(\Omega^\epsilon) : \nabla q^2 \in L^2(\Omega_2^\epsilon) \}.\quad (3.2a)$$

We endow the spaces with the norms coming from the natural inner product

$$\| v \|_{V^\epsilon} = \left\{ \| v \|_{L^2(\Omega_1^\epsilon)}^2 + \| \nabla \cdot v^1 \|_{L^2(\Omega_1^\epsilon)}^2 + \| v^1 \cdot \hat{n} \|_{L^2(\Gamma^\epsilon)}^2 \right\}^{1/2} \quad (3.2c)$$

$$\| q \|_{Q^\epsilon} = \left\{ \| q \|_{L^2(\Omega_2^\epsilon)}^2 + \| \nabla q^2 \|_{L^2(\Omega_2^\epsilon)}^2 \right\}^{1/2} \quad (3.2d)$$

Consider the scaled problem

Find $p^\epsilon \in Q^\epsilon, \ u^\epsilon \in V^\epsilon$,

$$\int_{\Omega_1^\epsilon} a_1 u^\epsilon \cdot v \, dy + \epsilon \int_{\Omega_2^\epsilon} a_2 u^\epsilon \cdot v \, dy - \int_{\Omega_1^\epsilon} p^\epsilon \cdot \nabla v \, dy + \int_{\Omega_2^\epsilon} \nabla p^\epsilon \cdot v \, dy$$

$$+ \alpha \int_{\Gamma^\epsilon} (u^\epsilon \cdot \hat{n}) (v^1 \cdot \hat{n}) dS - \int_{\Gamma_1^\epsilon} p^\epsilon (v^1 \cdot \hat{n}) dS - \int_{\Gamma_b^\epsilon} p^\epsilon (v^1 \cdot \hat{n}) dS = - \int_{\Omega^\epsilon} g^\epsilon \cdot v \, dy \quad (3.3a)$$

$$\int_{\Omega_1^\epsilon} \nabla \cdot u^\epsilon q \, dy - \int_{\Omega_2^\epsilon} u^\epsilon \cdot \nabla q \, dy$$

$$\int_{\Omega_1^\epsilon} a_1 u^\epsilon \cdot v \, dy + \epsilon \int_{\Omega_2^\epsilon} a_2 u^\epsilon \cdot v \, dy - \int_{\Omega_1^\epsilon} p^\epsilon \cdot \nabla v \, dy + \int_{\Omega_2^\epsilon} \nabla p^\epsilon \cdot v \, dy$$

$$+ \alpha \int_{\Gamma^\epsilon} (u^\epsilon \cdot \hat{n}) (v^1 \cdot \hat{n}) dS - \int_{\Gamma_1^\epsilon} p^\epsilon (v^1 \cdot \hat{n}) dS - \int_{\Gamma_b^\epsilon} p^\epsilon (v^1 \cdot \hat{n}) dS = - \int_{\Omega^\epsilon} g^\epsilon \cdot v \, dy$$

$$\int_{\Omega_1^\epsilon} \nabla \cdot u^\epsilon q \, dy - \int_{\Omega_2^\epsilon} u^\epsilon \cdot \nabla q \, dy$$
\[
+ \int_{\Gamma_t^r} (u^{x,1} \cdot \hat{n}) q^2 dS - \int_{\Gamma_b^r} (u^{x,1} \cdot \hat{n}) q^2 dS = \int_{\Omega^r} F^e q \, dy + \int_{\Gamma_r^e} q^2 dS \quad (3.3b)
\]

for all \( q \in Q^r, \, \mathbf{v} \in V^e \).

Clearly, the problem (3.3) is well-posed since it verifies all the hypothesis of theorem (2.5). In order to analyze the asymptotic behavior of the solution \((u^r, p^r)\) as \( \epsilon \downarrow 0 \) the geometry of the \( \epsilon \)-domains must be mapped to a common domain of reference.

### 3.2. The \( \epsilon \)-Problems in a Reference Domain.

We introduce the change of variable (see figure (2)) \( \varphi : \Omega^e \rightarrow \Omega \) defined by

\[
\varphi(y) = \sum_{j=0}^l \left( \bar{y}, y_3 + (1 - \epsilon) \sum_{\ell = 0}^j h_\ell \right) \mathbb{I}_{\Omega_t}(y)
\]

\[
+ \sum_{i=0}^l \left( \bar{y}, \frac{1}{\epsilon} (y_3 - \zeta_i^e(\bar{y})) + \zeta_i^e(\bar{y}) + (1 - \epsilon) \sum_{\ell = 0}^{i-1} h_\ell \right) \mathbb{I}_{\Omega_I}(\epsilon h_i, \Gamma_i^e)(y) \quad (3.4)
\]

Defining \((\bar{x}, z) \equiv \varphi(y)\) the gradients are related as follows

\[
\nabla_y = \left\{ \begin{array}{c}
\nabla_{\bar{x}} \\
\partial_z
\end{array} \right\} \mathbb{I}_{\Omega_t} + \sum_i \left[ \begin{array}{cc}
I & (1 - \frac{1}{\epsilon}) \nabla_{\bar{x}} \zeta_i(\bar{x}) \\
0 & \frac{1}{\epsilon}
\end{array} \right] \left\{ \begin{array}{c}
\nabla_{\bar{x}} \\
\partial_z
\end{array} \right\} \mathbb{I}_{\Omega_I}(\epsilon h_i, \Gamma_i^e) \quad (3.5)
\]

Here, it is understood that \( I \) is the identity matrix in \( \in \mathbb{R}^{2 \times 2} \). We write \( \zeta_i \) instead of \( \zeta_i^e \) for the sake of simplicity recalling that both surfaces differ only by a constant of vertical translation.

**Theorem 3.2.** Let \( \varphi : \Omega^e \rightarrow \Omega \) be the change of variable defined in equation (3.4). Then, the maps defined \( \Phi_1 : V \rightarrow V^e, \, \Phi_2 : Q \rightarrow Q^e \) defined respectively by \( (\Phi_1 \mathbf{v})(y) \equiv \mathbf{v}(\varphi(y)) \) and \( (\Phi_2 q)(y) \equiv q(\varphi(y)) \) are isomorphisms.

**Proof.** First notice for \( \mathbf{v} \in V \) and \( q \in Q \) the functions \( \Phi_1 \mathbf{v} \) and \( \Phi_2 q \) are defined on \( \Omega^e \). Moreover, for \( \ell = 1, 2 \) the restriction of the change of variable is a bijection i.e. \( \varphi : \Omega_1^e \rightarrow \Omega_\ell \) is a bijection. Therefore \( \mathbf{v}(\cdot) \in L^2(\Omega_\ell) \) if and only if \( \mathbf{v}(\varphi(\cdot)) \in L^2(\Omega_1^e) \) and \( q(\cdot) \in L^2(\Omega_\ell) \) if and only if \( q(\varphi(\cdot)) \in L^2(\Omega_1^e) \). Even more, \( \varphi : \Gamma_1^e \rightarrow \Gamma_\ell \) and \( \varphi : \Gamma_1^e + h_i \rightarrow \Gamma_\ell + h_i \) are bijective rigid translations. Therefore, the isomorphisms \( L^2(\Gamma_1^e) \simeq L^2(\Gamma_\ell), \, L^2(\Gamma_1^e + h_i) \simeq L^2(\Gamma_\ell + h_i) \) follow for all \( 1 \leq i \leq I \).

For the isomorphism \( \Phi_1 \), take \( \mathbf{v} \in V \) which is equivalent to \( \mathbf{v}(y) \in L^2(\Omega) \) and \( \nabla_y \cdot \mathbf{v}(y) \in L^2(\Omega_1) \). Due to the previous discussion these two conditions are equivalent to \( \mathbf{v}(\varphi(y)) \in L^2(\Omega^e) \) and \( \nabla_y \cdot \mathbf{v}(\varphi(y)) = \nabla_y \cdot \mathbf{v}(x) \in L^2(\Omega_1^e) \). However, equation (3.4) yields \( \nabla_y \cdot \mathbf{v}(\varphi(y)) = \nabla_y \cdot \mathbf{v}(x) = \nabla_x \cdot \mathbf{v}(x) \) whenever \( x \in \Omega_1 \); i.e. \( \nabla_y \cdot \mathbf{v}(y) \in L^2(\Omega_1^e) \).

For the map \( \Phi_2 \), the \( L^2 \)-integrability condition between spaces \( Q \) and \( Q^e \) is shown using the same arguments of the first paragraph. It remains to show the \( L^2 \)-integrability condition on the gradient. First observe that the last row in the matrix equation (3.5) implies that \( \frac{\partial}{\partial y} q(y) \in L^2(\Omega_2^e) \) if and only if \( \frac{\partial}{\partial z} q(x) \in L^2(\Omega_2) \). Second, for the derivatives in the first two directions equation (3.5) yields

\[
\frac{\partial}{\partial y} q(y) = \frac{\partial}{\partial x} q(x) + \left( 1 - \frac{1}{\epsilon} \right) \frac{\partial}{\partial x} \zeta_i(x) \frac{\partial}{\partial z} q(x), \quad \ell = 1, 2.
\]
Recalling the gradient of $\zeta_i$ is bounded we conclude $\frac{\partial}{\partial y} q(y) \in L^2(\Omega_2^i)$ if an only if $\frac{\partial}{\partial x_i} q(x) \in L^2(\Omega_2)$ for $\ell = 1, 2$. Since $\frac{\partial}{\partial z} q(x) \in L^2(\Omega_2)$ is immediate the proof is complete.

We are to apply the change of variable $\varphi : \Omega^F \to \Omega$ in the problem 3.3, to this end, it is more convenient to write the system in terms of the quantities and directions which yield estimates agreeable with the asymptotic analysis. Hence, recalling the definition of the upwards normal vector (2.8) the following relationships hold

$$ ||(-\nabla \zeta_i, 1)| \cdot \nabla \zeta_i + v_3, \quad (3.6a) $$

$$ (-\nabla \zeta_i, 1) \cdot \nabla \zeta_i + v_3 = 0 \quad \text{in} \quad \Omega(h_i, \Gamma_i). \quad (3.6b) $$

Applying the change of variable (3.4) to the problem (3.3) and combining with the relation 3.6a we get the following variational statement:

**Find** $p^f \in Q$, $\mathbf{u}^f \in \mathbf{V}$:

$$ \int_{\Omega_1} a_1 \mathbf{u}^f \cdot \mathbf{v} + \epsilon^2 \int_{\Omega_2} a_2 \mathbf{u}^f \cdot \mathbf{v} - \int_{\Omega_1} p^f \nabla \cdot \mathbf{v} $$

$$ + \sum_i \int_{\Omega(h_i, \Gamma_i)} \epsilon \left( \nabla p^f + \frac{\partial}{\partial z} p^f \nabla \zeta_i \right) \cdot \nabla \mathbf{v} + \nabla (-\nabla \zeta_i, 1) \partial_z p^f (\mathbf{v} \cdot \nabla \zeta_i) $$

$$ - \int_{\Gamma_t} p^\epsilon \cdot \mathbf{v} + \int_{\Gamma_h} \mathbf{v} \cdot \hat{n} \mathbf{v} \cdot \hat{n} $$

For all $\mathbf{q} \in Q$, $\mathbf{v} \in \mathbf{V}$

Finally, due to the theorem (3.2) on isomorphisms of function spaces we conclude that the problem (3.7) and (3.3) are equivalent.

### 3.2.1. The Strong Rescaled Problem

The solution of the problem (3.7) is the weak solution of the following system of equations

$$ a_1 \mathbf{u}^r \cdot \mathbf{v} + \nabla p^r + \mathbf{g} = 0 \quad \text{and} \quad (3.8a) $$

$$ \nabla \cdot \mathbf{u}^r = f^r \quad \text{in} \quad \Omega_1. \quad (3.8b) $$

$$ p^r = 0 \quad \text{on} \quad \partial \Omega_1 - \Gamma. \quad (3.8c) $$

$$ \mathbf{u}^r \cdot \hat{n} \mathbf{I}_{\Gamma_h} - \mathbf{u} \cdot \hat{n} \mathbf{I}_{\Gamma_h} \quad \text{and} \quad (3.8d) $$

$$ \mathbf{u}^r \cdot \hat{n} \mathbf{I}_{\Gamma_t} - \mathbf{u}^r \cdot \hat{n} \mathbf{I}_{\Gamma_t} = f_1^r \quad \text{on} \quad \Gamma. \quad (3.8e) $$

$$ \sum_i \left[ \epsilon a_2 \mathbf{u}^r \cdot \hat{n} \mathbf{I}_{\Gamma_h} + \nabla p^r \cdot \hat{n} \mathbf{I}_{\Gamma_h} + (1 - \frac{1}{\epsilon}) \partial_z p^r \mathbf{v} \cdot \hat{n} \mathbf{I}_{\Gamma_h} + \mathbf{g} \mathbf{v} \cdot \hat{n} \mathbf{I}_{\Gamma_h} \right] = 0, \quad (3.8f) $$

$$ \epsilon^2 a_2 \mathbf{u}^r \cdot \hat{n} \mathbf{I}_{\Gamma_t} = \partial_z p^r + \epsilon \mathbf{g}^r = 0 \quad \text{and} \quad (3.8g) $$
\[
\sum_i \nabla \left( \epsilon \tilde{u}^{e,2} \cdot \epsilon \tilde{u}^{e,2} \cdot \tilde{\nabla} \xi_i + |(-\tilde{\nabla} \xi_i, 1)| \right) \mathbb{1}_{\Omega(h_i, \Gamma_i)} = \epsilon F^{e,2} \text{ in } \Omega_2. \tag{3.8h}
\]
\[
\tilde{u}^{e,2} \cdot \tilde{\nu}^{(i)} = 0 \text{ on } \partial \Omega(h_i, \Gamma_i) - \Gamma \text{ for all } 1 \leq i \leq I. \tag{3.8i}
\]
As before equations \([3.8d], [3.8e]\) have the separation of cases \(\mathbb{1}_{\Gamma_h}, \mathbb{1}_{\Gamma_e}\) in order to be consistent with the upwards normal vector \(\hat{n}\). However, the equations \([3.8e]\) and \([3.8i]\) need further clarification. We start fixing an index \(i \in \{1, \ldots, I\}\) of the sum in the equation \((3.7b)\); reordering and integrating by parts yield
\[
- \int_{\Omega(h_i, \Gamma_i)} \epsilon \tilde{u}^{e,2} \cdot (\nabla q + \partial_z q \nabla \xi_i) + |(-\tilde{\nabla} \xi_i, 1)|(u^{e,2} \cdot \hat{n}^{(i)}) \partial_z q
\]
\[
= - \int_{\Omega(h_i, \Gamma_i)} \left( \epsilon \tilde{u}^{e,2} \cdot \epsilon \tilde{u}^{e,2} \cdot \tilde{\nabla} \xi_i + |(-\tilde{\nabla} \xi_i, 1)|(u^{e,2} \cdot \hat{n}^{(i)}) \right) \cdot \nabla q
\]
\[
= \int_{\Omega(h_i, \Gamma_i)} \nabla \cdot \left( \epsilon \tilde{u}^{e,2} \cdot \epsilon \tilde{u}^{e,2} \cdot \tilde{\nabla} \xi_i + |(-\tilde{\nabla} \xi_i, 1)|(u^{e,2} \cdot \hat{n}^{(i)}) \right) q
\]
\[
- \int_{\partial \Omega(h_i, \Gamma_i)} q \left( \epsilon \tilde{u}^{e,2} \cdot \epsilon \tilde{u}^{e,2} \cdot \tilde{\nabla} \xi_i + |(-\tilde{\nabla} \xi_i, 1)|(u^{e,2} \cdot \hat{n}^{(i)}) \right) \cdot \tilde{\nu}^{(i)} dS.\]

Where \(\tilde{\nu}^{(i)}\) is the outwards pointing unit normal field of the boundary \(\partial \Omega(h_i, \Gamma_i)\).

We focus on the boundary term
\[
\int_{\partial \Omega(h_i, \Gamma_i)} q \left( \epsilon \tilde{u}^{e,2} \cdot \epsilon \tilde{u}^{e,2} \cdot \tilde{\nabla} \xi_i + |(-\tilde{\nabla} \xi_i, 1)|(u^{e,2} \cdot \hat{n}^{(i)}) \right) \cdot \tilde{\nu}^{(i)} dS
\]
\[
= \int_{\partial \Omega(h_i, \Gamma_i) - (\Gamma_h \cup \Gamma_e)} q \left( \epsilon \tilde{u}^{e,2} \cdot \epsilon \tilde{u}^{e,2} \cdot \tilde{\nabla} \xi_i + |(-\tilde{\nabla} \xi_i, 1)|(u^{e,2} \cdot \hat{n}^{(i)}) \right) \cdot \tilde{\nu}^{(i)} dS
\]
\[
+ \sum_{\ell=0,1} \int_{\ell h_i + \Gamma_i} q \left( \epsilon \tilde{u}^{e,2} \cdot \epsilon \tilde{u}^{e,2} \cdot \tilde{\nabla} \xi_i + |(-\tilde{\nabla} \xi_i, 1)|(u^{e,2} \cdot \hat{n}^{(i)}) \right) \cdot \tilde{\nu}^{(i)} dS.\]

The equality \(\tilde{\nu}^{(i)} \cdot \hat{k} = 0\) holds on the portion of the vertical wall \(\partial \Omega(h_i, \Gamma_i) - (\Gamma_h \cup \Gamma_i)\) i.e. the equation \([3.8d]\) follows. For the remaining pieces of the boundary recall \(\tilde{n}^{(i)} = \tilde{\nu}^{(i)}\) on \(h_i + \Gamma_i\) and \(\tilde{n}^{(i)} = -\tilde{\nu}^{(i)}\) on \(\Gamma_i\); together with the equation \([2.8]\), we get
\[
- \int_{\ell h_i + \Gamma_i} q \left( \epsilon \tilde{u}^{e,2} \cdot \epsilon \tilde{u}^{e,2} \cdot \tilde{\nabla} \xi_i + |(-\tilde{\nabla} \xi_i, 1)|(u^{e,2} \cdot \hat{n}^{(i)}) \right) \cdot \tilde{\nu}^{(i)} dS
\]
\[
= (-1)^{\ell} \int_{\ell h_i + \Gamma_i} q \left( \epsilon \tilde{u}^{e,2} \cdot \epsilon \tilde{u}^{e,2} \cdot \tilde{\nabla} \xi_i + |(-\tilde{\nabla} \xi_i, 1)|(u^{e,2} \cdot \hat{n}^{(i)}) \right) \cdot \frac{(-\tilde{\nabla} \xi_i, 1)}{|(-\tilde{\nabla} \xi_i, 1)|} dS
\]
\[
= (-1)^{\ell} \int_{\ell h_i + \Gamma_i} q (u^{e,2} \cdot \hat{n}^{(i)}) dS \text{ for } \ell = 0, 1.\]

Combining this last identity with the interface terms in equation \((3.7b)\), the strong normal flux balance condition \([3.8d]\) follows.

### 3.3. A-priori Estimates and Convergence Statements

In order to get a-priori estimates on the norm of the solutions the following hypothesis are assumed

\[
\|F^e\|_{L^2(\Omega)} \text{ is bounded and } F^{1,e} \overset{w}{\rightarrow} F^1 \text{ in } L^2(\Omega_1), \tag{3.9a}
\]
Now test equation (3.7a) with $u$ and combined with equation (3.8f) yields bounded (3.14) is covered and the global inequality (3.8c) implies that the left hand side of inequality (3.10) is bounded. 

Next the term $\|p^\epsilon\|_Q$ must be bounded in terms of the flux $u^{\epsilon^{-1}}\mathbf{1}_{\Omega_1} + \epsilon u^{\epsilon^{-2}}\mathbf{1}_{\Omega_2}$ and the forcing terms. Due to the equation (3.8a) we have

$$
\|\frac{1}{\epsilon} \partial_z p^{\epsilon^{-2}}\|_{0,\Omega_2} \leq \epsilon \|a_2\|_{L^\infty(\Omega_2)} \|u^{\epsilon^{-2}}\|_{0,\Omega_2} + \|g_2\|_{0,\Omega_2}.
$$

(3.11a)

Combined with equation (3.8d) yields

$$
\|\nabla p^{\epsilon^{-2}}\|_{0,\Omega_2} \leq C \left( \|a_2\|_{L^\infty(\Omega_2)} \|\epsilon u^{\epsilon^{-2}}\|_{0,\Omega_2} + \|g\|_{0,\Omega_2} \right).
$$

(3.11b)

For $C > 0$ an adequate constant. Thus

$$
\|\nabla p^{\epsilon^{-2}}\|_{0,\Omega_2} \leq C \left( \|a_2\|_{L^\infty(\Omega_2)} \|\epsilon u^{\epsilon^{-2}}\|_{0,\Omega_2} + \|g\|_{0,\Omega_2} \right).
$$

(3.12)

With $C > 0$ a constant independent from $\epsilon > 0$. Additionally, the equation (3.8a) yields

$$
\|\nabla p^{\epsilon^{-2}}\|_{0,\Omega_1} \leq \|a_1\|_{L^\infty(\Omega_1)} \|u^{\epsilon^{-2}}\|_{0,\Omega_1} + \|g\|_{0,\Omega_1}.
$$

(3.13)

The boundary condition (3.8c) together with Poincaré inequality give the control $\|p^{\epsilon^{-1}}\|_{H^{1}(\Omega_1)} \leq C \|\nabla p^{\epsilon^{-2}}\|_{0,\Omega_1}$. On the other hand, the inequality (2.24) implies $\|p^{\epsilon}_{1,\Omega_2} \leq C(\|p^{\epsilon}_{0,\Gamma} + \|p^{\epsilon}_{1,\Omega_2}\|_{1,\Omega_2})$; combined with the normal stress balance conditions (3.8d) we conclude:

$$
\|p^\epsilon\|_Q \leq \|p^\epsilon\|_{1,\Omega} \leq C \|\nabla p^\epsilon\|_{0,\Omega}.
$$

(3.14)

And $C > 0$ is independent from $\epsilon > 0$. Finally, a combination of inequalities (3.14), (3.13) and (3.12) imply that the left hand side of inequality (3.10) is bounded.

**Remark 3.2.** The previous estimate on $\|p^{\epsilon^{-2}}\|_{H^{1}(\Omega_2)}$ could have been attained without requiring the drained condition (3.8c) on the whole matrix rock region external boundary. It was enough to set the drained condition on a subset of positive measure contained in $\partial\Omega_j$ — $\Gamma$ for $j$ fixed to have control on $\|p^{\epsilon^{-1}}\|_{1,\Omega_1}$ by $\|\nabla p^{\epsilon^{-1}}\|_{0,\Omega_1}$. Combining this fact with the normal stress balance conditions (3.8d), an inequality of the type (3.14) can be deduced for the union of adjacent domains $\Omega(h_j, \Gamma_j) \cup \Omega_j \cup \Omega(h_{j+1}, \Gamma_{j+1})$ and continue the process until the whole domain $\Omega$ is covered and the global inequality (3.14) is obtained.

Due to the observations above we conclude that the following sequences are bounded

$$
\|u^{\epsilon^{-1}}\|_{0,\Omega_1}, \|\epsilon u^{\epsilon^{-2}}\|_{0,\Omega_2}, \sqrt{\alpha} \|u^{\epsilon^{-1}} \cdot \mathbf{n}\|_{L^2(\Gamma)}
$$

(3.15a)

and

$$
\|p^{\epsilon^{-1}}\|_{H^{1}(\Omega_1)}, \|p^{\epsilon^{-2}}\|_{H^{1}(\Omega_2)}, \|\frac{1}{\epsilon} \partial_z p^\epsilon\|_{0,\Omega_2}, |\nabla u|^{\epsilon^{-1}}\|_{L^2(\Omega_1)}.
$$

(3.15b)
Remark 3.3. The change of variable $\varphi$ modifies the structure of the divergence on the domains $\Omega(h_i, \Gamma_i)$ for all $1 \leq i \leq I$, therefore it can only be claimed that the linear combination $\epsilon \nabla \cdot \vec{u}^{\epsilon - 2} + \epsilon (1 - \frac{1}{\epsilon}) \partial_z (\nabla \zeta_i \cdot \vec{u}^{\epsilon - 2}) + \partial_z u_3^{\epsilon - 2}$ is bounded in $L^2(\Omega(h_i, \Gamma_i))$.

3.4. Weak Limits. The previous section state bounds independent from $\epsilon > 0$ for $[u^{\epsilon, 1}, \epsilon u^{\epsilon, 2}] \in V$ and $p^\epsilon = [p^{\epsilon, 1}, p^{\epsilon, 2}] \in H^1(\Omega_1) \times H^1(\Omega_2)$, consequently in $Q$. Then, there must exist $u \in V$, $p \in Q$, $\eta \in L^2(\Omega_2)$ and a subsequence, from now on denoted the same, such that

$$p^\epsilon \rightharpoonup p \text{ in } Q \text{ and strongly in } L^2(\Omega),$$

$$u^{\epsilon, 1} \rightharpoonup u^1 \text{ in } L^2(\Omega_1) \text{ and } \nabla \cdot u^{\epsilon, 1} \rightharpoonup \nabla \cdot u^1 \text{ in } L^2(\Omega_1),$$

$$\sqrt{\alpha} u^{\epsilon, 1} \cdot \vec{n} \rightharpoonup \sqrt{\alpha} u^1 \cdot \vec{n} \text{ in } L^2(\Gamma),$$

$$\epsilon u^{\epsilon, 1} \rightharpoonup u^2 \text{ in } L^2(\Omega_2),$$

$$\frac{1}{\epsilon} \partial_z p^{\epsilon, 2} \rightharpoonup \eta \text{ in } L^2(\Omega_2) \text{ and } \partial_z p^{\epsilon, 2} \to 0 \text{ strongly in } L^2(\Omega_2).$$

Choose $\phi \in C_0^\infty(\Omega(h_i, \Gamma_i))$ arbitrary, test the equation (3.7b) with $q \overset{\text{def}}{=} \epsilon \phi$ and let $\epsilon \downarrow 0$. Recalling (3.16d) this gives

$$0 = \lim_{\epsilon \downarrow 0} \int_{\Omega(h_i, \Gamma_i)} |(-\nabla \zeta_i, 1)| (\epsilon u^{\epsilon, 2} \cdot \vec{n}^{(i)}) \partial_z \phi = \int_{\Omega_2} |(-\nabla \zeta_i, 1)| (u^2 \cdot \vec{n}^{(i)}) \partial_z \phi$$

$$= - \langle \partial_z |(-\nabla \zeta_i, 1)| (u^2 \cdot \vec{n}^{(i)}), \phi \rangle_{D'(\Omega(h_i, \Gamma_i)), D(\Omega(h_i, \Gamma_i))}.$$
Now fix \( i \in \{1, \ldots, I \} \) and take a function \( \mathbf{v}_i^2 \in (C_0^\infty(\Omega(h_i, \Gamma_i))^2 \). Recalling (2.13) define \( \mathbf{v} = M_i^{T, \tau} \mathbf{v}_i^2 \) and \( v_3 = M_i^{k, \tau} \mathbf{v}_i^2 \). Then, the function \( \mathbf{v}^2 = \frac{1}{\varepsilon} (\mathbf{v}, v_3) \) has the structure (3.18) or equivalently \( \mathbf{v}^2 \cdot \mathbf{n} = 0 \) inside \( \Omega(h_i, \Gamma_i) \). Define \( \mathbf{v} \) as the trivial extension of \( \mathbf{v}^2 \) to the whole domain \( \Omega \), therefore \( \mathbf{v} \in \mathbf{V} \). Test (3.7a) with \( \mathbf{v} \) and let \( \varepsilon \downarrow 0 \), this gives
\[
\int_{\Omega(h_i, \Gamma_i)} a_2(\mathbf{x}) \mathbf{u}_i^2 \cdot (\mathbf{v}, v_3) + \int_{\Omega(h_i, \Gamma_i)} \nabla p^2 \cdot \mathbf{v} + \int_{\Omega(h_i, \Gamma_i)} \mathbf{g} \cdot (\mathbf{v}, v_3) = 0.
\]
Consequently
\[
\int_{\Omega(h_i, \Gamma_i)} a_2(\mathbf{x}) \mathbf{u}_i^2 \cdot \mathbf{v}_\tau + \int_{\Omega(h_i, \Gamma_i)} \nabla p^2 \cdot M_i^{T, \tau} \mathbf{v}_\tau + \int_{\Omega(h_i, \Gamma_i)} \mathbf{g} \cdot \mathbf{v}_\tau = 0.
\]
The equation above holds for all \( \mathbf{v}_\tau \in (C_0^\infty(\Omega(h_i, \Gamma_i))^2 \) and due to the isomorphism of proposition (2.3) we conclude
\[
a_2(\mathbf{x}) \mathbf{u}_i^2 + \left( M_i^{T, \tau} \right) \nabla p^2 + \mathbf{g} \cdot \mathbf{v}_\tau = 0 \quad \text{in} \quad \Omega(h_i, \Gamma_i), \quad 1 \leq i \leq I. \tag{3.19}
\]
The equation (3.16) implies that \( p^2 \) does not depend on the variable \( z \) on \( \Omega_2 \) i.e. \( p^2 = p^2(\mathbf{x}) \). Therefore assuming
\[
a_2 = a_2(\mathbf{x}), \quad \mathbf{g} = \mathbf{g}(\mathbf{x}) \quad \text{in} \quad \Omega_2 \tag{3.20}
\]
the equation (3.19) gives \( \mathbf{u}_i^2 = \mathbf{u}_{i}(\mathbf{x}) \) i.e. \( \mathbf{u}_i^2 \) is independent from \( z \) in \( \Omega_2 \). Together with the fact \( \mathbf{u}_i^2 \cdot \mathbf{n} = 0 \) in \( \Omega_2 \) we conclude that the whole vector velocity \( \mathbf{u}_i^2 \) is independent from \( z \) in \( \Omega_2 \).

**Remark 3.4.** Observe that due to the assumptions for the data (3.20) the equation (3.19) is independent from \( z \) becoming a lower-dimensional Darcy-type constitutive law on the stream lines parallel to \( \zeta_i \).

### 4. The Limit Problem

Define the subspaces
\[
\mathbf{V}_0 \overset{\text{def}}{=} \left\{ \mathbf{v} \in \mathbf{V} : \partial_z \mathbf{v} = 0, \, \mathbf{v}_i^2 \cdot \mathbf{n}^{(i)} = 0 \quad \text{in} \quad \Omega(h_i, \Gamma_i) \right\}. \tag{4.1a}
\]
\[
Q_0 \overset{\text{def}}{=} \{ q \in Q : \partial_z q = 0 \quad \text{in} \quad \Omega_2 \}. \tag{4.1b}
\]

**Remark 4.1.** Notice that if \( q \in Q_0 \) the fact that \( \partial_z q = 0 \) in \( \Omega_2 \) implies \( q^2|_{\Gamma_i + h_i} = q^2|_{\Gamma_i + h_i} \) for all \( 1 \leq i \leq I \). Therefore, the spaces
\[
\left\{ \mathbf{v}^1 \in L^2(\Omega_1) : \nabla \cdot \mathbf{v}^1 \in L^2(\Omega_1), \, \mathbf{v}^1 \cdot \mathbf{n}^{(i)} \in L^2(\Gamma_i) \right\} \times \prod_i L^2(\Gamma_i)
\]
\[
L^2(\Omega_1) \times \prod_i H^1(\Omega(h_i, \Gamma_i))
\]
are isomorphic to (4.1a) and (4.1b) respectively.

Due to the structure of the space if \( \mathbf{v} = [\mathbf{v}_1^1, \mathbf{v}_2^2] \in \mathbf{V}_0 \) then the function \([\mathbf{v}_1^1, \frac{\varepsilon}{2} \mathbf{v}_2^2]\) is also in \( \mathbf{V}_0 \). Using the latter to test (3.7a) and \( q \in Q_0 \) for testing (3.7b) we let \( \varepsilon \downarrow 0 \) and conclude that the limits \([\mathbf{u}_1^1, \varepsilon \mathbf{u}_2^2] \rightarrow \mathbf{u} \) and \( p^\varepsilon \rightarrow p \) are a solution of the limit problem

\[
\text{Find} \, p \in Q_0, \, \mathbf{u} \in \mathbf{V}_0.
\]
\[ \int_{\Omega_1} a_1 \mathbf{u} \cdot \mathbf{v} - \int_{\Omega_1} p \nabla \cdot \mathbf{v} + \int_{\Omega_2} a_2 \mathbf{u}^2 \cdot \mathbf{v}^2 + \int_{\Omega_2} \nabla p^\tau \cdot \mathbf{v} \]

\[ - \int_{\Gamma_b} p^2 (\mathbf{v} \cdot \hat{\mathbf{n}}) dS + \int_{\Gamma_b} \hat{\mathbf{n}} \cdot \mathbf{v} dS + \alpha \int_{\Gamma} (\mathbf{u} \cdot \hat{\mathbf{n}})(\mathbf{v} \cdot \hat{\mathbf{n}}) dS = - \int_{\Omega_1} \mathbf{g} \cdot \mathbf{v} - \int_{\Omega_2} \mathbf{g} \cdot \mathbf{v}^2 \quad (4.2a) \]

\[ \int_{\Omega_1} \nabla \cdot \mathbf{u} \mathbf{v} - \int_{\Omega_2} \mathbf{u}^2 \cdot \nabla q + \int_{\Gamma_b} (\mathbf{u} \cdot \hat{\mathbf{n}}) q^2 dS - \int_{\Gamma_t} (\mathbf{u} \cdot \hat{\mathbf{n}}) q^2 dS = \int_{\Omega_1} F^1 q + \int_{\Gamma} f r^2 dS \quad (4.2b) \]

**4.1. Well-Posedness of the Limit Problem.** The problem (4.2) is a mixed formulation of the type (2.20) with the operators \( A^0 : V_0 \to V_0 \) and \( B^0 : V_0 \to Q_0 \) defined by

\[ A^0 \mathbf{v}(\mathbf{w}) \overset{\text{def}}{=} \int_{\Omega_1} a_1 \mathbf{v} \cdot \mathbf{w} + \int_{\Omega_2} a_2 \mathbf{v}^2 \cdot \mathbf{w}^2 + \alpha \int_{\Gamma} (\mathbf{v} \cdot \hat{\mathbf{n}})(\mathbf{w} \cdot \hat{\mathbf{n}}) dS \quad (4.3a) \]

\[ B^0 q(\mathbf{v}) \overset{\text{def}}{=} - \int_{\Omega_1} \nabla \cdot \mathbf{v} q + \int_{\Omega_2} \mathbf{v} \cdot \nabla q - \int_{\Gamma_t} (\mathbf{v} \cdot \hat{\mathbf{n}}) q^2 + \int_{\Gamma_b} (\mathbf{v} \cdot \hat{\mathbf{n}}) q^2 \quad (4.3b) \]

**Theorem 4.1.** The operator \( B^0 \) satisfies the inf-sup condition.

**Proof.** The proof has the same structure as lemma (2.7), there is only one detail to be examined in the construction of the test functions. Fix \( q = [q^1, q^2] \in Q_0 \), construct \( \mathbf{v}^1 \) in the same way it is built in problem (2.25). On the other hand since \( q^2 \in H^1(\Omega_2) \), \( \partial_z q^2 = 0 \), define

\[ \mathbf{v}^2 \overset{\text{def}}{=} \sum_i (\nabla q^2_i \cdot \nabla \zeta_i) \mathbf{I}_{\Omega(h_i, \Gamma_i)} \]

Then \( \mathbf{v}^2 \cdot \hat{\mathbf{n}} = 0 \) and \( \partial_z \mathbf{v}^2 = 0 \) in \( \Omega(h_i, \Gamma_i) \) for all \( 1 \leq i \leq I \), i.e. \( \mathbf{v}^2 \in V_0 \) and \( \| \mathbf{v}^2 \|_{0, \Omega_2} \leq C \| q^2 \|_{1, \Omega_2} \) as desired. Repeating the inequalities presented in (2.26) the proof is complete.

Since the inf-sup condition holds the theorem (2.5) applies to the operators (4.3) on the spaces \( V_0, Q_0 \) and the limit problem (4.2) is well-posed. Due to the uniqueness of the solution of the limit problem it follows that the original sequence converges weakly to the limit \( \mathbf{u} \in V_0 \), \( p \in Q_0 \).

**4.2. The Strong Form.** In order to describe the strong limit problem corresponding to (4.2) two features have to be exploited. First, the structure \( \mathbf{v} \cdot \hat{\mathbf{n}} = 0 \) in \( \Omega_2 \) for all \( \mathbf{v} \in V_0 \) implying \( \nabla \mathbf{v} = M^T \tau \mathbf{v}^2 \), for \( M^T \tau \) the matrix defined in (2.13). Second, the independence of the velocities and pressures with respect to \( z \) in \( \Omega_2 \). This last property allows to write the integrals over \( \Omega(h, \Gamma_i) \) as surface integrals on \( \Gamma_i \). Hence, the system (4.2) transforms in

Find \( p \in Q_0, \mathbf{u} \in V_0 \):

\[ \int_{\Omega_1} a_1 \mathbf{u} \cdot \mathbf{v} - \int_{\Omega_1} p \nabla \cdot \mathbf{v} + \sum_i h_i (\hat{\mathbf{n}}, \hat{\mathbf{k}})(a_2 \mathbf{u}^2 + (M^T \tau)^i \nabla p^\tau + \mathbf{g} \cdot \nabla \cdot \mathbf{v}^2) dS \]
\[- \int_{\Gamma_t} p^2 (\mathbf{v}^1 \cdot \mathbf{n}) \, dS + \int_{\Gamma_b} p^2 (\mathbf{v}^1 \cdot \mathbf{n}) \, dS + \alpha \int (\mathbf{u}^1 \cdot \mathbf{n}) (\mathbf{v}^1 \cdot \mathbf{n}) \, dS = - \int_{\Omega_1} \mathbf{g} \cdot \mathbf{v} \tag{4.4a} \]

\[
\int_{\Omega_1} \nabla \cdot \mathbf{u} \, q - \sum_i h_i \int_{\Gamma_i} (\mathbf{n}^{(i)} \cdot \mathbf{k}) M_i^{T,\tau} \mathbf{u}^2 \cdot \mathbf{\tilde{u}} \, dS
\]

\[
+ \int_{\Gamma_t} (\mathbf{u}^1 \cdot \mathbf{n}) q^2 \, dS - \int_{\Gamma_b} (\mathbf{u}^1 \cdot \mathbf{n}) q^2 \, dS = \int_{\Omega_1} F^1 q + \int_{\Gamma} F^2 q^2 \, dS \tag{4.4b} \]

Integrating by parts the above statement we get the strong lower dimensional problem

\[
a_2 \mathbf{u} + \nabla p^1 + \mathbf{g}^1 = 0, \tag{4.5a} \]

\[
\nabla \cdot \mathbf{u} = F^1 \text{ in } \Omega_1. \tag{4.5b} \]

\[
p^1 = 0 \text{ on } \partial \Omega_1 - \Gamma. \tag{4.5c} \]

\[
\mathbf{u}^2 \cdot \mathbf{n} = 0, \quad \partial_n p^2 = 0, \tag{4.5d} \]

\[
\sum_i \left[ a_2(s) \mathbf{u}^2 + (M_i^{T,\tau}) \mathbf{\tilde{u}} p^2 + g_i^2(s) \right] \mathbf{1}_{\Gamma_i} = 0, \tag{4.5e} \]

\[
\sum_i \left[ (\mathbf{u}^1 \cdot \mathbf{n}^{(i)})_{|\Gamma_i + h_i} - (\mathbf{u}^1 \cdot \mathbf{n}^{(i)})_{|\Gamma_i} \right] \mathbf{1}_{\Gamma_i}
\]

\[
+ \sum_i h_i (\mathbf{n}^{(i)} \cdot \mathbf{k}) \mathbf{\tilde{u}} \cdot (M_i^{T,\tau} \mathbf{u}^2) \mathbf{1}_{\Gamma_i} = f_i \text{ in } \Gamma. \tag{4.5f} \]

\[
p^1 - p^2 = \alpha \mathbf{u}^1 \cdot \mathbf{n} \mathbf{1}_{\Gamma_b} - \alpha \mathbf{u}^1 \cdot \mathbf{n} \mathbf{1}_{\Gamma_i}. \tag{4.5g} \]

\[
\mathbf{u}^2 \cdot \mathbf{n}^{(i)} = 0 \text{ on } \partial G_i \text{ for all } 1 \leq i \leq I. \tag{4.5h} \]

The statement of equation (4.5e) was already shown in (3.19), however the statements (4.5f) and (4.5h) need further discussion.

4.3. The Interface Integrals Setting.

**Definition 4.2.** Let \( G, \zeta \) and \( \Gamma \) be as in definition (2.1), define the spaces

\[
L^2(\Gamma) \overset{\text{def}}{=} \{ h : \Gamma \to \mathbb{R} : \int_{\Gamma} h^2(s) \, dS < +\infty \} \tag{4.6a} \]

\[
H^1(\Gamma) \overset{\text{def}}{=} \{ h \in L^2(\Gamma) : \mathbf{\tilde{\nabla}} h \in L^2(\Gamma) \times L^2(\Gamma) \} \tag{4.6b} \]

\[
H^1_0(\Gamma) \overset{\text{def}}{=} \{ h \in H^1(\Gamma) : h|_{\partial G} = 0 \} \tag{4.6c} \]

Here \( \mathbf{\tilde{\nabla}} \) indicates the gradient with respect to the variables \((x_1, x_2)\) contained in \( G \).

The following isomorphism result is necessary

**Theorem 4.3.** Let \( G, \zeta \) and \( \Gamma \) be as in definition (2.1). Consider the natural embedding \( j : \Gamma \to G \) defined by \( j(\mathbf{x}, \zeta(\mathbf{x})) = \mathbf{x} \) and the map

\[
\varphi \mapsto \varphi \circ j \tag{4.7} \]

Then

(i) The embedding (4.7) is an isomorphism between \( L^2(G) \) and \( L^2(\Gamma) \).

(ii) The embedding (4.7) is an isomorphism between \( H^1(G) \) and \( H^1(\Gamma) \).

(iii) The embedding (4.7) is an isomorphism between \( H^1_0(G) \) and \( H^1_0(\Gamma) \).
Proof. By definition \( j : G \to \Gamma \) is linear and bijective, therefore the map (4.7) is bijective between spaces of functions.

(i) Due to the hypothesis \( \zeta \) satisfies \( C_1 = \text{ess inf}\{ \hat{n}(s) \cdot \hat{k} : s \in \Gamma \} > 0 \) then, for any \( \phi \in L^2(\Gamma) \)

\[
\int (\phi \circ j)^2 dS = \int_G (\hat{n}(\bar{x}) \cdot \hat{k})^{-1} \phi^2(\bar{x}) d\bar{x} \leq \frac{1}{C_1} \int_G \phi^2(\bar{x}) d\bar{x}.
\]

The inequality above gives the continuity of the application \( \varphi \mapsto \varphi \circ j \). Due to Banach’s inversion theorem the map is an isomorphism.

(ii) By definition \( \nabla (\phi \circ j) = \nabla \phi(\bar{x}) \) holds for any \( \phi \in H^1(G) \).

(iii) Is immediate from (ii).

\[ \square \]

Fix \( i \in \{1, \ldots, I\} \), choose \( q \in Q_0 \) supported inside \( \Omega(h_i, \Gamma_i) \) and test equation (4.4b); hence

\[
-\int_{\Omega(h_i, \Gamma_i)} \tilde{u}^2 \cdot \nabla q - \int_{\Gamma_i} (u^1 \tilde{n}^{(i)}) \cdot \nabla q \, dS + \int_{\Gamma_i + h_i} (u^1 \tilde{n}^{(i)}) \cdot q^2 \, dS = \int_{\Gamma_i \cup \Gamma_i + h_i} f_i q^2 \, dS.
\]

We focus on the first term of the left hand side. First \( \partial_z q^2 = 0 \) implies \( \tilde{u}^2 \cdot \nabla q^2 = u^2 \cdot \nabla q^2 \), then

\[
-\int_{\Omega(h_i, \Gamma_i)} u^2 \cdot \nabla q = \int_{\Omega(h_i, \Gamma_i)} \nabla \cdot u^2 \, q^2 - \int_{\partial\Omega(h_i, \Gamma_i)} q^2 \cdot \nu \, dS.
\]

The two summands of the left hand side are treated separately. For the first summand the independence from the variable \( z \) implies \( \nabla \cdot u^2 = \nabla \cdot \tilde{u}^2 \), the fact \( u^2 \cdot \tilde{n}^{(i)} = 0 \) in \( \Omega(h_i, \Gamma_i) \) gives \( \tilde{u}^2 = M^{T,R} u_2^2 \). Thus

\[
\int_{\Omega(h_i, \Gamma_i)} \nabla \tilde{u}^2 \, q^2 \, d\bar{x} = h_i \int_{\Gamma_i} \tilde{\nu} \cdot (M^{T,R} u_2^2) \, q^2 \, d\bar{x} = h_i \int_{\Gamma_i} (\tilde{n}^{(i)} \cdot \tilde{k}) \tilde{\nu} \cdot (M^{T,R} u_2^2) \, q^2 \, d\bar{x}.
\]

The boundary term in (4.9) can be written as

\[
-\int_{\Gamma_i \cup \Gamma_i + h_i} q^2 \cdot \nu \, dS - \int_{\partial\Omega(h_i, \Gamma_i) \setminus \{\Gamma_i \cup \Gamma_i + h_i\}} q^2 \, \nu \cdot \nu^{(i)} \, dS.
\]

The first summand vanishes since \( u^2 \cdot \tilde{n}^{(i)} = 0 \) in \( \Omega(h_i, \Gamma_i) \). The boundary piece described in the second summand is a vertical wall, then \( \tilde{\nu}^{(i)} \cdot \tilde{k} = 0 \) and it can be identified with the outwards normal vector to the set \( G_i \subseteq \mathbb{R}^2 \). Moreover, due to the independence of the integrand with respect to the variable \( z \), the surface integral can be collapsed to a line integral over \( \partial G_i \). Combining these observations with (4.10) and (4.9) the equation (4.8) transforms in

\[
h_i \int_{\Gamma_i} (\tilde{n}^{(i)} \cdot \tilde{k}) \tilde{\nu} \cdot (M^{T,R} u_2^2) \, q^2 \, d\bar{x} = -h_i \int_{\Gamma_i} q^2 \, \nu \cdot \nu^{(i)} \, dC - \int_{\Gamma_i} (u^1 \tilde{n}^{(i)}) \cdot q^2 \, dS + \int_{\Gamma_i + h_i} (u^1 \tilde{n}^{(i)}) \cdot q^2 \, dS = \int_{\Gamma_i \cup \Gamma_i + h_i} f_i q^2 \, dS.
\]

Where \( dC \) is the arc-length measure on \( \partial G_i \). The isomorphisms provided by theorem (4.3) imply that the quantifier \( q^2 \mid_{\Gamma_i} \) can hit any function in the space \( H^1_0(\Gamma_i) \). Therefore, the equation (4.5b) follows. Finally, using again theorem (4.3) the trace
Figure 3. System of 2-D Manifold Fissures

of test function \( q^2 \mid \Gamma_i \) can hit any function in the space \( H^1_0(\Gamma_i) \) and combined with equation (4.5f) give (4.5h).

4.4. Strong Convergence of the Solutions.

Theorem 4.4. Under the hypothesis

\[
\| F^{\epsilon,1} - F^1 \|_{0, \Omega_1} \to 0, \quad \| f^{\epsilon}_\Gamma - f_\Gamma \|_{0, \Gamma} \to 0 \quad \text{and} \quad \| g^{\epsilon} - g \|_{0, \Omega} \to 0
\]

(4.11)

The solutions \( u^{\epsilon}, p^{\epsilon} \) satisfy the following strong convergence statements

\[
\| u^{\epsilon,1} - u^1 \|_{0, \Omega_1} \to 0, \quad \| \epsilon u^{\epsilon,2} - u^2 \|_{0, \Omega_2} \to 0,
\]

(4.12)

\[
\| p^{\epsilon,1} - p^1 \|_{1, \Omega_1} \to 0, \quad \| p^{\epsilon,2} - p^2 \|_{1, \Omega_2} \to 0.
\]

(4.13)

Proof. The proof uses exactly the same arguments presented in \[15\], theorem 3.2.

Finally, assume that \( u^{\epsilon,2}_2 \neq 0 \) and consider the quotients:

\[
\frac{\| u^{\epsilon,2}_\tau \|_{0, \Omega_2}}{\| u^{\epsilon,2} \cdot \hat{n} \|_{0, \Omega_2}} = \frac{\epsilon u^{\epsilon,2}_\tau \|_{0, \Omega_2}}{\| u^{\epsilon,2} \cdot \hat{n} \|_{0, \Omega_2}} > \frac{\| u^{\epsilon,2}_\tau \|_{0, \Omega_2} - \delta}{\| \epsilon u^{\epsilon,2}_\tau \cdot \hat{n} \|_{0, \Omega_2}} > 0
\]

(4.14)

The lower bound holds true for \( \epsilon > 0 \) small enough and adequate \( \delta > 0 \) then we conclude that the magnitudes ratio of the flux tangential component over normal component blows-up to infinity, i.e. the flow in the thin channel is predominantly tangential. Finally if \( u^{\epsilon,2}_2 = 0 \), unlike the analysis for flat interfaces presented in \[15\], no conclusions can be obtained due to the complexity introduced by the geometry of the fissures.

5. A Problem with two dimensional Manifolds

In this section, using the independence of the limit functions with respect to \( z \) in \( \Omega_2 \) it will be shown that the limiting problem (4.5) can be formulated as a
system coupling Darcy flow in three dimensions with tangential flow hosted two dimensional manifolds as depicted in figure 3.

5.1. Geometric Setting.

Definition 5.1. We say a totally fractured medium of two dimensional manifold fissures is a finite collection of

Surface functions

\{ \lambda_i \in C(\Gamma_i) : G_i \subseteq \mathbb{R}^2 \text{ open bounded simply connected region}; \}

\lambda_i \text{ piecewise } C^1 \text{ function such that } \text{ess inf } \hat{n}^{(i)} \cdot \hat{n} > 0, 1 \leq i \leq I \}. \quad (5.1a)

And rock-matrix regions

\{ \Theta_j \subseteq \mathbb{R}^3 : \Theta_j \neq \emptyset \text{ open bounded simply connected region}, 0 \leq j \leq I \}. \quad (5.1b)

Verifying the following properties

Non-overlapping condition and indexed ordered

\sup \{ \lambda_i(\vec{x}) : \vec{x} \in G_i \} < \inf \{ \lambda_{i+1}(\vec{x}) : \vec{x} \in G_{i+1} \}, \ \forall 1 \leq i \leq I - 1 \quad (5.2a)

The interface domain condition

\Lambda_i = \partial \Theta_i \cap \partial \Theta_{i-1}, \ \forall 1 \leq i \leq I \quad (5.2b)

for \( \Lambda_i \overset{\text{def}}{=} \{ [\vec{x}, \lambda_i(\vec{x})] : \vec{x} \in G_i \} \). And the connectivity through fissures condition

\begin{align*}
\text{cl}(\Theta_\ell) \cap \text{cl}(\Theta_k) &= \emptyset \text{ whenever } |\ell - k| > 1. \\
(5.2c)
\end{align*}

For convenience of notation define \( \Lambda_0 \overset{\text{def}}{=} \partial \Omega_0 - \Gamma_1 \). We denote this fissured system by \( \{(\Lambda_i, \Theta_i) : 0 \leq i \leq I\} \). The rock matrix and fissures regions are the sets

\begin{align*}
\Theta &\overset{\text{def}}{=} \bigcup_{i=0}^{I} \Theta_i, \\
\Lambda &\overset{\text{def}}{=} \bigcup_{i=0}^{I} \Lambda_i, \\
\Theta_{FR} &\overset{\text{def}}{=} \Theta \cup \Lambda.
\end{align*}

\( \hat{n}^{(i)} \) indicated the upwards normal vector to the surface \( \Lambda_i \). Finally, we introduce the notations \( \Lambda_i^+ \) and \( \Lambda_i^- \) for the upper and lower faces of the manifold \( \Lambda_i \).

5.2. Spaces of Functions and Isomorphisms.

Definition 5.2. We define the following spaces for velocity and pressure

\begin{align*}
V_f &\overset{\text{def}}{=} \{ v \in L^2(\Theta_{FR}) : \nabla \cdot v^1 \in L^2(\Theta_j), 0 \leq j \leq I; \\
v^1 \cdot \hat{n}^{(i)}|_{\Lambda_i^+}, v^1 \cdot \hat{n}^{(i)}|_{\Lambda_i^-} &\in L^2(\Lambda_i), v^2|_{\Lambda_i} \in L^2(\Lambda_i), 1 \leq i \leq I \}, \\
Q_f &\overset{\text{def}}{=} \{ q \in L^2(\Theta_{FR}) : q|_{\Lambda_i} \in H^1(\Lambda_i), 1 \leq i \leq I \}. \\
(5.4a, b)
\end{align*}

Endowed with the norms coming from the natural inner products

\begin{align*}
\|v\|_{V_f} &\overset{\text{def}}{=} \|v\|_{L^2(\Theta_{FR})} + \|\nabla \cdot v\|_{L^2(\Theta_{FR})} \\
&+ \sum_i \|v \cdot \hat{n}^{(i)}|_{\Lambda_i^+}\|_{L^2(\Lambda_i)}^2 + \|v \cdot \hat{n}^{(i)}|_{\Lambda_i^-}\|_{L^2(\Lambda_i)}^2 + \|v\|_{L^2(\Lambda_i)}^2 + \|v\|_{H^1(\Lambda_i)}^2 \\
&\|q\|_{Q_f} &\overset{\text{def}}{=} \|q\|_{L^2(\Theta_{FR})} + \sum_i \|q\|_{H^1(\Lambda_i)}^2. \\
(5.4c, d)
\end{align*}
Remark 5.1. Notice that definition \(5.4a\) demands only \(v \in H_{\text{div}}(\Theta_i)\) i.e. the divergence is square integrable only on these subdomains. Therefore, both normal traces \(v \cdot \hat{n}^{(i)}|_{\Lambda_i^+}\) and \(v \cdot \hat{n}^{(i)}|_{\Lambda_i^-}\) make sense in \(H^{-1/2}(\Gamma_i)\) but we require the extra condition of been in \(L^2(\Lambda_i)\). We do not demand the global condition \(v \in H_{\text{div}}(\Theta_{FR})\) because this would imply the continuity of the normal traces across a surface i.e. \(u^1 \cdot \hat{n}^{(i)}|_{\Lambda_i^+} = u^1 \cdot \hat{n}^{(i)}|_{\Lambda_i^-}\). Such condition can not model jumps across the fissures as the normal stress balance interface \(4.5g\) and the limit equation \(4.5f\).

Next define a change of variable based on piecewise translations

**Definition 5.3.** Let \(x = (\tilde{x}, x_3)\) and define the map \(T : \Omega \to \mathbb{R}^3\)

\[
Tx \overset{\text{def}}{=} \sum_{j=0}^{I} \left( \tilde{x}, x_3 - \sum_{\ell=0}^{j} h_\ell \right) \mathbb{I}_{\Omega_j}(\tilde{x}, x_3) - \sum_{i=1}^{I} \left( \tilde{x}, \zeta_i(\tilde{x}) - \sum_{\ell=0}^{i-1} h_\ell \right) \mathbb{I}_{\Theta(h_i, \Gamma_i)}(\tilde{x}, x_3)
\]

(5.5)

Define \(\Theta_j \overset{\text{def}}{=} T(\Omega_j)\) and \(\lambda_i : G_i \to \mathbb{R}\) by \(\lambda_i \overset{\text{def}}{=} \zeta_i(\tilde{x}) - \sum_{\ell=0}^{i-1} h_\ell\).

Clearly the system \(\{(\Lambda_i, \Theta_i) : 1 \leq i \leq I\}\) satisfies the conditions of definition \(5.1\). With the previous definitions we have the following result

**Theorem 5.4.** (i) The application \(v \mapsto v \circ T\) is an isometric isomorphism from \(\mathbf{V}_0\) to \(\mathbf{V}_f\).

(ii) The application \(q \mapsto q \circ T\) is an isometric isomorphism from \(\mathbf{Q}_0\) to \(\mathbf{Q}_f\).

**Proof.** (i) The proof is a direct application of part (i) in theorem \(4.3\). The only detail that needs further clarification is to observe that

\[
\begin{align*}
v^1 \cdot \hat{n}^{(i)}|_{\Gamma_i} &\mapsto (v^1 \circ T) \cdot \hat{n}^{(i)}|_{\Lambda_i^-} \\
v^1 \cdot \hat{n}^{(i)}|_{\Gamma_i+h_i} &\mapsto (v^1 \circ T) \cdot \hat{n}^{(i)}|_{\Lambda_i^+}
\end{align*}
\]

(ii) It is a direct application of parts (i) and (ii) in theorem \(4.3\).

\(\square\)

5.3. The Lower Dimensional Mixed Problem. Due to the previous theorem the problem \(4.2\) is equivalent to the following mixed problem with two dimensional manifolds

Find \(p \in Q_f, \ u \in V_f:\)

\[
\begin{align*}
\int_{\Theta} a_1 \mathbf{u} \cdot \mathbf{v} - \int_{\Theta} p \mathbf{v} \cdot \mathbf{v} + \sum_i h_i \int_{\Lambda_i} \hat{\alpha} \mathbf{u}^2 + (M_{T,T'} \mathbf{v}) \mathbf{p} + \mathbf{g}_\tau \cdot \mathbf{v}^2 dS \\
+ \sum_i \alpha \int_{\Lambda_i} \left( (u^1 \cdot \hat{n}^{(i)}|_{\Lambda_i^+})(v^1 \cdot \hat{n}^{(i)}|_{\Lambda_i^-}) + (u^1 \cdot \hat{n}^{(i)}|_{\Lambda_i^-})(v^1 \cdot \hat{n}^{(i)}|_{\Lambda_i^+}) \right) dS \\
- \sum_i \int_{\Lambda_i} p^2 \left( (v^1 \cdot \hat{n}^{(i)}|_{\Lambda_i^+}) - (v^1 \cdot \hat{n}^{(i)}|_{\Lambda_i^-}) \right) dS = - \int_{\Theta} \mathbf{g} \cdot \mathbf{v} - \int_{\Lambda} \mathbf{g}_\tau \cdot \mathbf{v}^2 dS \\
\int_{\Theta} \nabla \cdot \mathbf{u} q - \sum_i h_i \int_{\Lambda_i} \hat{\alpha} \mathbf{u}^2 \cdot \nabla q dS \\
+ \sum_i \int_{\Lambda_i} \left( (u^1 \cdot \hat{n}|_{\Lambda_i^+}) - (u^1 \cdot \hat{n}|_{\Lambda_i^-}) \right) q^2 dS = \int_{\Theta} F^1 q + \int_{\Lambda} f_r q^2 dS \tag{5.6a}
\end{align*}
\]

\[
\begin{align*}
\int_{\Theta} \mathbf{u} \cdot \mathbf{q} - \sum_i h_i \int_{\Lambda_i} \hat{\alpha} \mathbf{u}_\tau^2 \cdot \hat{n} \mathbf{q} dS \\
+ \sum_i \int_{\Lambda_i} \left( (u^1 \cdot \hat{n}|_{\Lambda_i^+}) - (u^1 \cdot \hat{n}|_{\Lambda_i^-}) \right) q^2 dS = \int_{\Theta} F^1 q + \int_{\Lambda} f_r q^2 dS \tag{5.6b}
\end{align*}
\]
for all $q \in Q_f, v \in V_f$.

Finally the equivalence of problems (4.2) and (4.4) gives the well-posedness of the system above.

6. Final Discussion and Future Work

(i) The formulation presented in this work can manage large amounts of information in a remarkably efficient way. One of the main reasons is the notation introduced by Showalter in [15] for the description of function spaces.

(ii) The results can be generalized immediately to the $\mathbb{R}^N$-setting using the same arguments presented here. The structure of the problems is analogous.

(iii) The approach based on analytic semigroups theory presented in section [15] can be directly applied here to model the time dependent problem for totally fissured systems with singularities.

(iv) Although the mathematical analysis is solid, the approach used throughout the paper stops been suitable for surfaces with high gradients such as the one depicted in the right hand side of figure (4) where $\hat{n}_2 \cdot \hat{k} \ll \hat{n}_1 \cdot \hat{k}$. In this case the translation in the direction $\hat{k}$ generates a fissure whose cross section areas can be very different from one piece to another i.e. $A_2 \ll A_1$. Such a fissure is not realistic. On the other hand consider a fissure such as the one depicted in the left hand side of figure (4). Here the translation is made in the bisector vector direction

$$\hat{e} = \frac{1}{|\hat{n}_1 + \hat{n}_2|/2} \hat{n}_1 + \hat{n}_2$$

This process generates a more realistic fissure. Additionally, demanding the fissures to be defined by the parallel translation of a surface in a fixed direction although is a step forward with respect to previous achievements, is still restrictive for modeling the phenomenon in natural geological formations. Setting the problem in the mixed variational formulation used here can be easily extended to systems with fissures described by a very general type of geometry. However, the difficulty of the asymptotic analysis increases substantially.

Figure 4. Translation Generated Fissures
Such question will be addressed in future work by the introduction of correction factors obtained comparing the flow energy dissipation in a real fissure and an artificial one e.g. replacing the presence of the fissure in the left hand side of (4) with the one on the right side affected by a correction factor. In the same way, fissures defined by walls which are not rigid translations of the other will be compared to a fissure generated by vertical translation of its “average surface” and having the same “average width”.

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References

[1] Grégorie Allaire, Marc Briane, Robert Brizzi, and Yves Capdeboscq. Two asymptotic models for arrays of underground waste containers. *Applied Analysis*, 88 (no. 10-11):1445–1467, 2009.
[2] Todd Arbogast and Dana Brunson. A computational method for approximating a Darcy-Stokes system governing a vuggy porous medium. *Computational Geosciences*, 11, No 3:207–218, 2007.
[3] Todd Arbogast and Heather Lehr. Homogenization of a darcy-stokes system modeling vuggy porous media. *Computational Geosciences*, 10, No 3:291–302, 2006.
[4] Nan Chen, Max Gunzburger, and Xiaoming Wang. Asymptotic analysis of the differences between the Stokes-Darcy system with different interface conditions and the Stokes-Brinkman system. *Journal of Mathematical Analysis and Applications*, 368 (2):658–676, 2009.
[5] Gabriel N. Gatica, Salim Meddahi, and Ricardo Oyarzúa. A conforming mixed finite-element method for the coupling of fluid flow with porous media flow. *IMA Journal of Numerical Analysis*, 29, 1:86–108, 2009.
[6] V. Girault and P.-A. Raviart. *Finite element approximation of the Navier-Stokes equations*, volume 749 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1979.
[7] Ulrich Hornung, editor. *Homogenization and Porous Media*, Ulrich Hornung editor, volume 6 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 1997.
[8] ZhaoQin Huang, Jun Yao, YaJun Li, ChenChen Wang, and XinRui Lü. Permeability analysis of fractured vuggy porous media based on homogenization theory. *Science China Technological Sciences*, 53 (3):839–847, 2010.
[9] W. J. Layton, F. Scheiweck, and I. Yotov. Coupling fluid flow with porous media flow. *SIAM Journal of Numerical Analysis*, 40 (6):2195–2218, 2003.
[10] Thérèse Lévy. Fluid flow through an array of fixed particles. *International Journal of Engineering Science*, 21:11–23, 1983.
[11] J. San Martín, J.-F. Scheid, and L. Smaranda. A modified lagrange-galerkin method for a fluid-rigid system with discontinuous density. *Numerische Mathematik*, 122 (2):341–382, 2012.
[12] Vincent Martin, Jérôme Jaffré, and Jean E. Roberts. Modeling fractures and barriers as interfaces for flow in porous media. *SIAM J. Sci. Comput.*, 26(5):1667–1691, 2005.
[13] A. Mikelić. A convergence theorem for homogenization of two-phase miscible flow through fractured reservoirs with uniform fracture distribution. *Applicable Anal.*, 33:203–214, 1089.
[14] Fernando Morales and Ralph Showalter. The narrow fracture approximation by channeled flow. *Journal of Mathematical Analysis and Applications*, 365:320–331, 2010.
[15] Fernando Morales and Ralph Showalter. Interface approximation of Darcy flow in a narrow channel. *Mathematical Methods in the Applied Sciences*, 35:182–195, 2012.
[16] Enrique Sánchez-Palencia. *Nonhomogeneous media and vibration theory*, volume 127 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 1980.
[17] R. E. Showalter. *Hilbert space methods for partial differential equations*, volume 1 of *Monographs and Studies in Mathematics*. Pitman, London-San Francisco, CA-Melbourne, 1977.

[18] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*, volume 49 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[19] R. E. Showalter. *Microstructure Models of Porous Media*. In Ulrich Hornung editor *Homogenization and Porous Media*, volume 6 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 1997.

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