Biracks and Switch Braid Quivers

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Abstract

We consider birack and switch colorings of braids. We define a switch structure on the set of permutation representations of the braid group and consider when such a representation is a switch automorphism. We define quiver-valued invariants of braids using finite switches and biracks and use these to categorify the birack 2-cocycle invariant for braids. We obtain new polynomial invariants of braids via decategorification of these quivers.

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1 Introduction

Initially defined in [6], biracks and related structures known as switches have been studied using several different notational conventions in works such as [9, 5, 4]. Biracks have the property that colorings of oriented framed knot, link or braid diagrams with birack elements satisfying a condition at the crossings are preserved by Reidemeister II and III moves, making the number of colorings by a finite birack an easily-computable integer-valued invariant. Switches give colorings to semi-arcs in braid diagrams which are preserved by the braid Reidemeister moves, i.e. the Reidemeister III and direct Reidemeister II moves. These switch colorings give rise to a large family of permutation representations of the braid group, as discussed in [5]. Moreover, the preservation of colorings under moves means that any invariant of birack- or switch-colored knots or braids can be used to define generally stronger invariants known as enhancements.

In this paper we introduce a new family of braid group representations derived from switch colorings, called switch braid representations. These are precisely the permutation representations from [5] which restrict to representations in the category of switches. We show that they are equivalently the representations arising from medial switches, and that they generalize the classical Burau representations.

We also introduce a finite quiver-valued invariant of braids using finite biracks and switches. Since finite quivers are small categories, this construction categorifies the birack and switch counting invariants for braids. The quivers we obtain, like those in [1], decategorify to give us new polynomial invariants of braids. Moreover, the set of these invariant quivers has an algebraic structure of its own which may prove to be of future interest. Weighting these quivers with 2-cocycles in the birack case yields a quiver categorification of the birack 2-cocycle invariant for braids. We are then obtain an infinite family of new two-variable polynomial invariants of braids by decategorifying these weighted quivers.

The paper is organized as follows. In Section 2 we recall the basics of biracks and birack colorings. In Section 3 we recall the basics of switches and introduce our new switch braid representations, characterizing them as the representations induced by medial switches and showing that they specialize to the Burau representations. In Section 4 we define our new switch braid quivers and a new polynomial invariant of braids, the switch braid quiver polynomial, associated to each finite switch. In Section 5 we enhance the switch braid quiver invariants with birack cocycles and obtain two new families of 2-variable polynomial invariants of braids as decategorifications of the birack braid cocycle quiver. We illustrate the new invariants with examples. We conclude in Section 6 with some questions for future research.

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2 Biracks and Braids

We begin this section with a definition. See [4] for more.

**Definition 1.** Let $X$ be a set. A **birack structure** on $X$ is a pair of binary operations $\triangleright, \triangleright : X \times X \to X \times X$ satisfying the following conditions:

(i) For all $y \in X$ the maps $\alpha_y, \beta_y : X \to X$ defined by

$$\alpha_y(x) = x \triangleright y \quad \text{and} \quad \beta_y(x) = x \triangleright y$$

are bijective.

(ii) The map $S : X \times X \to X \times X$ defined by

$$S(x, y) = (y \triangleright x, x \triangleright y)$$

is bijective.

(iii) For all $x, y, z \in X$ the exchange laws are satisfied:

$$\begin{align*}
(x \triangleright y) \triangleright (z \triangleright y) &= (x \triangleright z) \triangleright (y \triangleright z) \\
(x \triangleright y) \triangleright (z \triangleright y) &= (x \triangleright z) \triangleright (y \triangleright z) \\
(x \triangleright y) \triangleright (z \triangleright y) &= (x \triangleright z) \triangleright (y \triangleright z).
\end{align*}$$

A birack structure additionally satisfying $x \triangleright x = x \triangleright x$ for all $x \in X$ is called a **biquandle structure**. We refer to the triple $(X, \triangleright, \triangleright)$ (or just the set $X$) as a **birack** or **biquandle** if the operations $\triangleright, \triangleright$ define a birack structure or biquandle structure, respectively.

**Example 1.** Standard examples of biracks include

- **Constant action biracks.** For any set $X$ and commuting bijections $\sigma, \tau : X \to X$, the operations $x \triangleright y = \sigma(x)$ and $x \triangleright y = \tau(x)$ define a birack structure on $X$. This is called a constant action birack because, for a given element $x$, the actions $y \mapsto x \triangleright y : X \to X$ and $y \mapsto x \triangleright y : X \to X$ are constant. If $\sigma = \tau$, then the birack is a biquandle; if $\sigma = \tau = \text{Id}$ is the identity map, then the birack is called trivial.

- **Alexander biquandles.** For any module $X$ over the ring $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$ of two-variable Laurent polynomials, the operations $x \triangleright y = tx + (s - t)y$ and $x \triangleright y = sx$ define a biquandle structure known as an Alexander biquandle. The special case $s = 1$ yields the classical Alexander module.

- **Skew brace.** Let $X$ have two (not necessarily abelian) group operations $\ast$ and $\circ$ satisfying the modified distributivity condition

$$x \circ (y \ast z) = (x \circ y) \ast x^\ast \circ (x \circ z)$$

where $x^\ast$ is the inverse of $x$ with respect to the $\ast$ operation. Then $X$ is a skew brace, with birack operations given by $x \triangleright y = y^\circ \circ (x \ast y)$ and $x \triangleright y = y \circ (y^\circ \ast x)$ where $y^\circ$ is the inverse of $y$ with respect to the $\circ$ operation.

We will specify finite birack structures by listing their operation tables. For example, the smallest nontrivial birack has underlying set $X = \{1, 2\}$ and operation table

|  | 1 | 2 |
|---|---|---|
| 1 | 2 | 1 |
| 2 | 1 | 2 |

|  | 1 | 2 |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 1 |
The birack axioms are chosen so that the number of birack colorings of an oriented knot, link or braid diagram is invariant under the Reidemeister II and III moves. A birack coloring of a diagram $D$ by a birack $X$ (also called an $X$-coloring of $D$) is an assignment of an element of $X$ to each of the semi-arcs of $D$ (i.e., sections of the diagram between crossing points) such that at every crossing, we have the following relationship:

\[
\begin{align*}
  x & \quad y \quad x \\
  y & \quad x \quad y
\end{align*}
\]

It is then a standard exercise to check that the birack axioms [(i), (ii), (iii) in Definition 1] are precisely the conditions required by the Reidemeister II and III moves to guarantee that for any birack-colored oriented knot (or link or braid) diagram before a move, there is a unique birack-colored knot (or link or braid) diagram after the move which agrees with the initial diagram outside the neighborhood of the move. In particular,

- axiom (i) corresponds to the direct Reidemeister II moves,
- axiom (ii) corresponds to the reverse Reidemeister II moves,
- axiom (iii) corresponds to the all-positive Reidemeister III move.

Combined with the direct Reidemeister II moves, this gives us invariance under all of the Reidemeister III moves.

Let us write $C(D, X)$ for the set of all $X$-colorings of $D$. Biracks have traditionally been applied to study knots and links in the following manner:

**Definition 2.** Given a knot or link diagram $K$ and a finite birack $X$, the birack counting invariant $\Phi^X_Z(K)$ is defined as the cardinality of the set $C(K, X)$.

It is a standard exercise to check that if two knot or link diagrams $K, K'$ are related by Reidemeister II and III moves, then

\[
\Phi^X_Z(K) = |C(K, X)| = |C(K', X)| = \Phi^X_Z(K').
\]

Consequently, we have:

**Corollary 1.** The birack counting invariant is an invariant of knots and links.

We could define a similar counting invariant for braids. However, taken naively, the number of birack colorings of a braid diagram is a trivial invariant. Indeed, at any crossing, the colors of the top two semi-arcs determine the colors of the bottom two semi-arcs.
so it follows that the vector of colors \((x_1, \ldots, x_n)\) at the top of the braid diagram uniquely determines the rest of the coloring. Moreover, any vector of colors \((x_1, \ldots, x_n) \in X^n\) at the top will yield a valid coloring. Hence, the number colorings of an n-strand braid diagram \(B\) by a finite birack \(X\) is simply \(|X|^n\), and thus is completely determined by the size of \(X\) and the number of strands of \(B\). Thus we will need a different approach to obtain fruitful braid invariants from biracks.

### 3 Switch Braid Representations

When dealing with braids, the only relevant Reidemeister moves are the Reidemeister III moves and the direct Reidemeister II moves, which we will call the braid Reidemeister moves. The reverse Reidemeister II moves are unnecessary because braid strands are always oriented in the same direction. Hence, we may allow colorings by a slightly more general class of structures than biracks, which are only required to respect the Reidemeister III moves and direct Reidemeister II moves. Such structures are defined as follows.

**Definition 3.** A switch structure on a set \(X\) is an invertible map \(\rho : X \times X \to X \times X\) satisfying the set-theoretic Yang-Baxter equation:

\[
(\rho \times \text{Id}_X)(\text{Id}_X \times \rho)(\rho \times \text{Id}_X) = (\text{Id}_X \times \rho)(\rho \times \text{Id}_X)(\text{Id}_X \times \rho).
\]

We refer to the pair \((X, \rho)\) (or just the set \(X\)) as a switch.

**Remark 1.** Any birack \((X, \cdot, \triangleright)\) can be realized as a switch \((X, \rho)\) by defining

\[
\rho(x, y) = (\alpha_x^{-1}(y), x \triangleright \alpha_x^{-1}(y)).
\]

A switch coloring of a braid diagram \(B\) by a switch \((X, \rho)\), or an \(X\)-coloring of \(B\), is an assignment of an element of \(X\) to each semiarc of \(B\) such that, at every crossing, we have the following relationship:

\[
\begin{align*}
{x, y} & \implies \rho_1(x, y) \\
{x, y} & \implies \rho_2(x, y) \\
{x, y} & \implies \rho_1^{-1}(x, y) \\
{x, y} & \implies \rho_2^{-1}(x, y)
\end{align*}
\]

where \(\rho_i^{\pm1}(x, y)\) is the \(i\)th coordinate of \(\rho^{\pm1}(x, y)\). Note that switch colorings are top-down, as opposed to the left-right birack colorings described in Section 2. Just as with biracks, any vector of colors \((x_1, \ldots, x_n) \in X^n\) at the top of the braid diagram determines a unique \(X\)-coloring.

Given a color vector \(\pi = (x_1, \ldots, x_n) \in X^n\), let us write \(\pi \cdot B\) for the color vector \((y_1, \ldots, y_n) \in X^n\) induced at the bottom of \(B\) upon coloring the top of \(B\) with \(\pi\). This gives us a map

\[
(\pi, B) \to \pi \cdot B : X^n \times \{\text{diagrams of n-strand braids}\} \to X^n.
\]

It is a standard exercise to check that for any \(\pi \in X^n\) and any braid diagrams \(B, B'\) related by braid Reidemeister moves, \(\pi \cdot B = \pi \cdot B'\). Hence, we have:

**Theorem 2.** The function

\[
\varphi : \{\text{diagrams of n-strand braids}\} \to (X^n)^n
\]

sending \(B \mapsto \varphi_B\), in which \(\varphi_B : X^n \to X^n\) is defined by \(\varphi_B(\pi) = \pi \cdot B\), is an invariant of braids.
Remark 2. Note that we can treat $\varphi$ as a function $B_n \to (X^n)^X$ (with domain the $n$-strand braid group). We will do this going forward.

Furthermore, the map $(\pi, B) \to \pi \cdot B : X^n \times B_n \to X^n$ defines a right group action of $B_n$ on $X^n$. Indeed, the identity braid $1_n \in B_n$ satisfies $\pi \cdot 1_n = \pi$ for every $\pi \in X^n$. And, since the group operation in $B_n$ is vertical stacking of braids, we have $\pi \cdot (B_1 B_2) = (\pi \cdot B_1) \cdot B_2$ for all $\pi \in X^n$ and $B_1, B_2 \in B_n$. Consequently, each $\varphi_B$ is in $\text{Sym}(X^n)$, the automorphism group of the set $X^n$, and moreover:

Theorem 3. $\varphi : B_n \to \text{Sym}(X^n)$ is a contravariant functor.

The functor $\varphi$ gives us a representation of $B_n$ in the category of bijections (set-automorphisms) of $X^n$. This much is known and well-documented in the literature. See, for example, [5].

However, $X$ is not merely a set: it is endowed with the additional structure of a switch. It is natural to ask whether the switch structure on $X$ induces a switch structure on $X^n$ and, if so, whether $\varphi$ maps into the category of switch automorphisms of $X^n$. We will show that the answer to the first question is “yes!” and the answer to the second is “sometimes.”

3.1 Switch Structure on $X^n$

To make our lives easier, we will first move to a universal-algebra definition of switch. Given a switch structure $\rho$ on a set $X$, we can define binary operations $\vartriangleright, \triangleleft, \vartriangleright^{-1}, \triangledown^{-1} : X \times X \to X$ by

$$x \vartriangleright y = \rho_2(x, y), \quad x \triangleleft y = \rho_1(y, x), \quad x \vartriangleright^{-1} y = \rho_1^{-1}(y, x), \quad x \triangledown^{-1} y = \rho_2^{-1}(x, y).$$

Elsewhere in the literature, $x \vartriangleright y$, $x \triangleleft y$, $x \vartriangleright^{-1} y$, $x \triangledown^{-1} y$ are written $x^y$, $x^\vartriangledown$, $x^\triangleleft$, $x^\triangledown$, respectively (see [5]).

The fact that $\rho$ is a switch structure ensures that

(i) for all $x, y, z \in X$,

$$(z \vartriangleright (x \vartriangleright y)) \vartriangleright (y \vartriangleright x) = (z \vartriangleright y) \vartriangleright x$$

$$(y \vartriangleright x) \vartriangleright (z \vartriangleright (x \vartriangleright y)) = (y \vartriangleright z) \vartriangleright (x \vartriangleright (z \vartriangleright y))$$

$$(x \vartriangleright y) \vartriangleright z = (x \vartriangleright (z \vartriangleright y)) \vartriangleright (y \vartriangleright z)$$

(ii) for all $x, y \in X$,

$$(x \vartriangleright y) \vartriangleright^{-1} (y \vartriangleright x) = x = (x \vartriangledown^{-1} y) \vartriangleright (y \vartriangleright^{-1} x)$$

$$(y \vartriangleright x) \vartriangledown^{-1} (x \vartriangleright y) = y = (y \vartriangleright^{-1} x) \vartriangledown (x \vartriangledown^{-1} y).$$

Conversely, given a 4-tuple $(\vartriangleright, \triangleleft, \vartriangleright^{-1}, \triangledown^{-1})$ of binary operations on $X$ satisfying (i) and (ii), we can define functions $\rho, \rho' : X \times X \to X \times X$ by

$$\rho(x, y) = (y \vartriangleright x, x \vartriangleright y) \quad \text{and} \quad \rho'(x, y) = (y \vartriangledown^{-1} x, x \vartriangledown^{-1} y).$$

Then (i) ensures that $\rho$ satisfies the Yang-Baxter equation and (ii) ensures that $\rho \rho' = \rho' \rho = \text{Id}_{X \times X}$. Hence, $\rho$ defines a switch structure on $X$.

We have now described two maps:

$$\{\text{switch structures on } X\} \leftarrow \{\text{tuples } (\vartriangleright, \triangleleft, \vartriangleright^{-1}, \triangledown^{-1}) \text{ satisfying (i) and (ii)}\}.$$

One can see that these maps are inverses of each other, thus proving the following theorem:

Theorem 4. Let $X$ be a set. Then there is a bijective correspondence

$$\{\text{switch structures on } X\} \leftrightarrow \{\text{tuples } (\vartriangleright, \triangleleft, \vartriangleright^{-1}, \triangledown^{-1}) \text{ satisfying (i) and (ii)}\}.$$
Moreover, switch colorings are the same as colorings in which the following relationships hold at crossings:

\[
\begin{align*}
\text{x} & \quad \text{y} \\
\text{y} \lor \text{x} & \quad \text{x} \lor \text{y} \\
\text{x} \lor^{-1} \text{y} & \quad \text{y} \lor^{-1} \text{x}
\end{align*}
\]

Henceforth, we will use the term switch to refer to either of the two equivalent structures discussed in Theorem 4. Now we are ready to define switch products. Given a family of switches \( \{(X_i, \lor_i, \lor_i^{-1}, \lor_i^{-1})\}_{i \in I} \), we define binary operations \( \lor, \lor \lor^{-1}, \lor^{-1} \) on \( \prod_{i \in I} X_i \) as follows:

\[
\begin{align*}
\forall \forall y = (x_i \lor_i y_i)_{i \in I} \\
\forall \lor y = (x_i \lor_i y_i)_{i \in I} \\
\forall \lor^{-1} y = (x_i \lor_i^{-1} y_i)_{i \in I} \\
\forall \lor^{-1} y = (x_i \lor_i^{-1} y_i)_{i \in I}
\end{align*}
\]

Since the operations are defined pointwise, it follows immediately that:

**Theorem 5.** \( \prod_{i \in I} (X_i, \lor, \lor^{-1}, \lor^{-1}) \) is a switch.

In particular, any switch structure on \( X \) induces a natural switch structure on \( X^n \). Hence, for every \( B \in B_n \), the map \( \varphi_B \) is a bijection of a switch. But when is \( \varphi_B \) a switch automorphism?

### 3.2 When \( \varphi_B \) is a Switch Automorphism

We begin with a definition.

**Definition 4.** A switch homomorphism from \( (X, \rho) \) to \( (Y, \tau) \) is a map \( f : X \to Y \) satisfying

\[
\tau(f(x), f(y)) = (f(\rho_1(x, y)), f(\rho_2(x, y)))
\]

for all \( x, y \in X \).

Equivalently, a switch homomorphism from \( (X, \lor_1, \lor_1^{-1}, \lor_1^{-1}) \) to \( (X, \lor_2, \lor_2^{-1}, \lor_2^{-1}) \) is a map \( f : X \to Y \) satisfying

\[
\begin{align*}
f(x \lor_1 y) &= f(x) \lor_2 f(y) \\
f(x \lor_1^{-1} y) &= f(x) \lor_2^{-1} f(y)
\end{align*}
\]

for all \( x, y \in X \). One can readily check that this definition yields a category whose objects are switches and whose morphisms are switch homomorphisms. Furthermore, one can check that the isomorphisms in this category are precisely the morphisms whose underlying maps are bijective (in other words, the inverse of a bijective switch homomorphism is also a switch homomorphism). Thus, \( \varphi_B : X^n \to X^n \) is a switch automorphism if and only if it is a switch homomorphism.

So it suffices to characterize when \( \varphi_B \) is a switch homomorphism. This is the main result of the section:

**Theorem 6.** \( \varphi_B : X^n \to X^n \) is a switch homomorphism for every \( B \in B_n \) if and only if \( X \) is medial.

The definition of medial switch is a special case of the more general concept of entropic variety from universal algebra (see [3] for more). It generalizes the concepts of medial quandle and medial biquandle studied in works such as [8, 2, 7].
**Definition 5.** A *medial switch* (also called an *entropic switch* or *abelian switch*) is a switch $X$ such that

\[
(x \sqcap y) \sqcap (w \sqcap z) = (x \sqcap w) \sqcap (y \sqcap z)
\]

\[
(x \sqcap y) \sqcap (w \sqcap z) = (x \sqcap w) \sqcap (y \sqcap z)
\]

\[
(x \sqcap y) \sqcap (w \sqcap z) = (x \sqcap w) \sqcap (y \sqcap z)
\]

for all $x, y, w, z \in X$.

**Lemma 7.** A switch $X$ is medial if and only if $\varphi_{\sigma_i} : X^n \to X^n$ is a switch homomorphism for every generator $\sigma_i$ of $B_n$.

As usual, $\sigma_1, \ldots, \sigma_{n-1}$ denote the generators in the Artin presentation of $B_n$.

**Proof.** The lemma is most easily understood by studying the following diagrams:

\[
\begin{align*}
(a_{i+1} \sqcap b_{i+1}) \sqcap (a_i \sqcap b_i) &\text{ vs. } (a_{i+1} \sqcap b_{i+1}) \sqcap (a_i \sqcap b_i) \\
(a_i \sqcap b_i) \sqcap (a_{i+1} \sqcap b_{i+1}) &\text{ vs. } (a_i \sqcap b_i) \sqcap (a_{i+1} \sqcap b_{i+1}) \\
(a_{i+1} \sqcap b_{i+1}) \sqcap (a_i \sqcap b_i) &\text{ vs. } (a_{i+1} \sqcap b_{i+1}) \sqcap (a_i \sqcap b_i) \\
(a_i \sqcap b_i) \sqcap (a_{i+1} \sqcap b_{i+1}) &\text{ vs. } (a_i \sqcap b_i) \sqcap (a_{i+1} \sqcap b_{i+1})
\end{align*}
\]

In more detail, we argue as follows:

$(\Rightarrow)$: Suppose $X$ is medial. Let $i \in \{1, \ldots, n-1\}$ and $\vec{a}, \vec{b} \in X^n$. Then for all $k \in \{1, \ldots, n\}$, we have

\[
[\varphi_{\sigma_i}(\vec{a} \sqcap \vec{b})]_k = \begin{cases} 
(a_{i+1} \sqcap b_{i+1}) \sqcap (a_i \sqcap b_i) & \text{if } k = i \\
(a_i \sqcap b_i) \sqcap (a_{i+1} \sqcap b_{i+1}) & \text{if } k = i + 1 \\
ak \sqcap b_k & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
(a_{i+1} \sqcap a_i) \sqcap (b_{i+1} \sqcap b_i) & \text{if } k = i \\
(a_i \sqcap a_{i+1}) \sqcap (b_i \sqcap b_{i+1}) & \text{if } k = i + 1 \\
ak \sqcap b_k & \text{otherwise}
\end{cases}
\]

\[
= [\varphi_{\sigma_i}(\vec{a}) \sqcap \varphi_{\sigma_i}(\vec{b})]_k
\]
and

\[
[\varphi_{\sigma_i}(\overline{a} \triangledown \overline{b})]_k = \begin{cases} 
(a_{i+1} \triangledown b_{i+1}) \triangledown (a_i \triangledown b_i) & \text{if } k = i \\
(a_i \triangledown b_i) \triangledown (a_{i+1} \triangledown b_{i+1}) & \text{if } k = i + 1 \\
ak \triangledown bk & \text{otherwise}
\end{cases}
\]

Hence,

\[
\varphi_{\sigma_i}(\overline{a} \triangledown \overline{b}) = \varphi_{\sigma_i}(\overline{a}) \triangledown \varphi_{\sigma_i}(\overline{b}) \quad \text{and} \quad \varphi_{\sigma_i}(\overline{a} \triangledown \overline{b}) = \varphi_{\sigma_i}(\overline{a}) \triangledown \varphi_{\sigma_i}(\overline{b})
\]

Therefore, \( \varphi_{\sigma_i} \) is a switch homomorphism for every braid group generator \( \sigma_i \).

(\( \Leftarrow \)) Suppose \( \varphi_{\sigma_i} : X^n \to X^n \) is a switch homomorphism for every generator \( \sigma_i \) of \( B_n \). Let \( x, y, w, z \in X \). Choose \( i \in \{1, \ldots, n-1\} \) and \( \overline{a}, \overline{b} \in X^n \) so that

\[
a_i = x, \quad a_{i+1} = w, \quad b_i = y, \quad b_{i+1} = z.
\]

Then, since \( \varphi_{\sigma_i} \) is a switch homomorphism, we have

\[
\varphi_{\sigma_i}(\overline{a} \triangledown \overline{b}) = \varphi_{\sigma_i}(\overline{a}) \triangledown \varphi_{\sigma_i}(\overline{b}) \quad (1)
\]

and

\[
\varphi_{\sigma_i}(\overline{a} \triangledown \overline{b}) = \varphi_{\sigma_i}(\overline{a}) \triangledown \varphi_{\sigma_i}(\overline{b}). \quad (2)
\]

Taking the \((i+1)\)th components of Equation \([1]\) we obtain

\[
(x \triangledown y) \triangledown (w \triangledown z) = (a_i \triangledown b_i) \triangledown (a_{i+1} \triangledown b_{i+1})
\]

\[
= [\varphi_{\sigma_i}(\overline{a} \triangledown \overline{b})]_{i+1}
\]

\[
= [\varphi_{\sigma_i}(\overline{a}) \triangledown \varphi_{\sigma_i}(\overline{b})]_{i+1}
\]

\[
= (a_i \triangledown a_{i+1}) \triangledown (b_i \triangledown b_{i+1})
\]

\[
= (x \triangledown w) \triangledown (y \triangledown z).
\]

Taking the \(i\)th components of Equation \([1]\) we obtain

\[
(w \triangledown z) \triangledown (x \triangledown y) = (a_{i+1} \triangledown b_{i+1}) \triangledown (a_i \triangledown b_i)
\]

\[
= [\varphi_{\sigma_i}(\overline{a} \triangledown \overline{b})]_i
\]

\[
= [\varphi_{\sigma_i}(\overline{a}) \triangledown \varphi_{\sigma_i}(\overline{b})]_i
\]

\[
= (a_{i+1} \triangledown a_i) \triangledown (b_{i+1} \triangledown b_i)
\]

\[
= (w \triangledown x) \triangledown (z \triangledown y).
\]

Taking the \(i\)th components of Equation \([2]\) we obtain

\[
(w \triangledown z) \triangledown (x \triangledown y) = (a_{i+1} \triangledown b_{i+1}) \triangledown (a_i \triangledown b_i)
\]

\[
= [\varphi_{\sigma_i}(\overline{a} \triangledown \overline{b})]_i
\]

\[
= [\varphi_{\sigma_i}(\overline{a}) \triangledown \varphi_{\sigma_i}(\overline{b})]_i
\]

\[
= (a_{i+1} \triangledown a_i) \triangledown (b_{i+1} \triangledown b_i)
\]

\[
= (w \triangledown x) \triangledown (z \triangledown y).
\]

Therefore, \( X \) is medial.
This proves the lemma.

Proof of Theorem 6. By Lemma 7, it suffices to show that \( \varphi_B : X^n \to X^n \) is a switch homomorphism for every \( B \in \mathcal{B}_n \) if and only if \( \varphi_{\sigma_i} : X^n \to X^n \) is a switch homomorphism for every generator \( \sigma_i \) of \( \mathcal{B}_n \).

(\( \Rightarrow \)): This direction is trivial.

(\( \Leftarrow \)): Suppose \( \varphi_{\sigma_i} : X^n \to X^n \) is a switch homomorphism for every generator \( \sigma_i \) of \( \mathcal{B}_n \). Let \( B \in \mathcal{B}_n \). Then we can write \( B = \sigma_{(1)}^{i(1)} \cdots \sigma_{(k)}^{i(k)} \) for some \( k \) and functions \( \iota : \{1, \ldots, k\} \to \{1, \ldots, n-1\} \) and \( \epsilon : \{1, \ldots, k\} \to \{+, -\} \). Then, using the fact that \( \varphi \) is a contravariant functor, we obtain

\[
\varphi_B = \varphi(B) = \varphi(\sigma_{(1)}^{i(1)} \cdots \sigma_{(k)}^{i(k)}) = \varphi(\sigma_{(1)}^{i(1)}) \cdots \varphi(\sigma_{(k)}^{i(k)}) = \varphi(\sigma_i)^{\epsilon(1)} \cdots \varphi(\sigma_i)^{\epsilon(k)} = \varphi_{\sigma_{(1)}^{i(1)} \cdots \sigma_{(k)}^{i(k)}}.
\]

Since each \( \varphi_{\sigma_{(i)}} \) is a switch homomorphism, it follows that \( \varphi_B \) is a switch homomorphism.

This completes the proof.

For a switch \( X \), let \( \text{Aut}(X^n) \) denote the automorphism group of the switch \( X^n \). We then have the following:

**Corollary 8.** Let \( X \) be a switch. Then

\( \varphi \) defines a functor \( \mathcal{B}_n \to \text{Aut}(X^n) \) if and only if \( X \) is medial.

**Definition 6.** Let \( X \) be a medial switch. Then the contravariant functor \( \varphi : \mathcal{B}_n \to \text{Aut}(X^n) \) is called a switch braid representation.

This defines an infinite family of braid group representations.

### 3.3 Examples

**Example 2.** Let \( X \) be a module over a commutative ring \( R \) and let \( \lambda, \mu \) be invertible elements of \( R \). The map \( \rho : X \times X \to X \times X \) given by

\[
\rho(x, y) = (\lambda y + (1 - \mu \lambda)x, \mu x)
\]

defines a switch structure on \( X \) called the Alexander switch. The \( \triangledown, \bigtriangledown \) operations are given by

\[
x \triangledown y = \mu x \quad \text{and} \quad x \bigtriangledown y = \lambda x + (1 - \mu \lambda)y.
\]

Observe that for all \( x, y, w, z \in X \), we have

\[
(x \triangledown y) \triangledown (w \bigtriangledown z) = (\mu x) \triangledown (w \bigtriangledown z) = \mu^2 x = (\mu x) \triangledown (y \bigtriangledown z) = (x \bigtriangledown w) \bigtriangledown (y \bigtriangledown z)
\]
and

\[(w \boxtimes z) \nabla (x \boxtimes y) = (\mu w) \nabla (\mu x)\]
\[= \lambda \mu w + (1 - \mu \lambda) \mu x\]
\[= \mu (\lambda w + (1 - \mu \lambda) x)\]
\[= (\lambda w + (1 - \mu \lambda) x) \nabla (z \nabla y)\]
\[= (w \nabla x) \nabla (z \nabla y)\]

and

\[(w \nabla z) \nabla (x \nabla y) = (\lambda w + (1 - \mu \lambda) z) \nabla (\lambda x + (1 - \mu \lambda) y)\]
\[= \lambda (\lambda w + (1 - \mu \lambda) z) + (1 - \mu \lambda) (\lambda x + (1 - \mu \lambda) y)\]
\[= \lambda^2 w + \lambda (1 - \mu \lambda) z + (1 - \mu \lambda) x + (1 - \mu \lambda)^2 y\]
\[= \lambda (\lambda w + (1 - \mu \lambda) x) + (1 - \mu \lambda) (\lambda z + (1 - \mu \lambda) y)\]
\[= (\lambda w + (1 - \mu \lambda) x) \nabla (\lambda z + (1 - \mu \lambda) y)\]
\[= (w \nabla x) \nabla (z \nabla y)\]

Therefore, the Alexander switch is medial. Consequently, each Alexander switch gives rise to a switch braid representation of the braid group.

**Example 3.** In this example, we show that the classical Burau representation of the braid group is a special case of the switch braid representation. Let \( \Lambda = \mathbb{Z}[t, t^{-1}] \) be the ring of Laurent polynomials. The contravariant Burau representation of \( B_n \) is the unique group homomorphism \( \psi_n : (B_n)^{op} \to \text{GL}_n(\Lambda) \) sending \( \sigma_i \mapsto U_i = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix} \).

First observe that \( \Lambda \) can be viewed as a module over itself. We can then define an Alexander switch structure on \( \Lambda \) by taking \( \rho : \Lambda \times \Lambda \to \Lambda \times \Lambda \) to be \( \rho(x, y) = (ty + (1-t)x, x) \).

Here we have chosen our invertible elements to be \( \lambda = t \) and \( \mu = 1 \). Then by Example 2, \( \Lambda \) defines a medial switch. Moreover, the switch braid representation \( \varphi : B_n \to \text{Aut}(\Lambda^n) \) arising from \( \Lambda \) satisfies the following: for all generators \( \sigma_i \) of \( B_n \) and all \( \vec{a} \in \Lambda^n \),

\[
\varphi_{\sigma_i}(\vec{a}) = (a_1 \ldots a_{i-1} a_i a_{i+1} a_{i+2} \ldots a_n)^\top
\]
\[= (I_{i-1} \begin{pmatrix} a_1 \\ a_{i+1} \\ a_{i+2} \vdots \\ a_n \end{pmatrix} + t a_{i+1} + (1-t) a_i a_{i+2} \ldots a_n)^\top
\]
\[= U_i \vec{a} = \psi_n(\sigma_i)(\vec{a}).\]

Hence, \( \varphi_{\sigma_i} = \psi_n(\sigma_i) \) for all braid group generators \( \sigma_i \). Consequently, since both \( \varphi \) and \( \psi_n \) are contravariant functors \( B_n \to \text{GL}_n(\Lambda) \), it follows that \( \varphi = \psi_n \).
4 Switch Braid Quivers

We begin this section with our main definition. Given a switch structure on a set $X$, we will assign to each braid diagram $B$ a quiver (i.e., a directed multi-graph) with $|X|^n$ vertices, such that this assignment is invariant under the braid Reidemeister moves.

**Definition 7.** Let $X$ be a switch and $B$ an $n$-strand braid diagram. The switch braid quiver associated to $B$ and $X$, denoted $SQ_X(B)$, has

- vertex set $X^n$, and
- a directed edge from $\mathbf{x} = (x_1, \ldots, x_n)$ to $\mathbf{y} = (y_1, \ldots, y_n)$ if and only if $\varphi_B(\mathbf{x}) = \mathbf{y}$.

Note that braid Reidemeister moves do not change the colors at the boundary (top and bottom rows) of a switch-colored braid diagram. Hence, if $B, B'$ are braid diagrams related by a braid Reidemeister move, then

$$SQ_X(B) = SQ_X(B').$$

Consequently, we have the following result:

**Theorem 9.** For any switch $X$, the function

$$SQ_X : \{\text{diagrams of } n\text{-strand braids}\} \to \{\text{quivers}\}$$

sending $B \mapsto SQ_X(B)$ is an invariant of braids.

**Remark 3.** We can now treat $SQ_X$ as a function $B_n \to \{\text{quivers}\}$.

For each braid $B$, $SQ_X(B)$ is a small category which determines the switch braid counting invariant, and hence is a categorification. Note that the subquiver of loop edges (edges for which the initial and terminal vertices are the same) in $SQ_X(B)$ corresponds to colorings of the braid closure $\overline{B}$.

**Example 4.** Consider the braid

whose closure is the Hopf link and let $X$ be the birack (in fact, it is a biquandle) with operation tables

$$\begin{array}{c|ccc} \triangleright & 1 & 2 & 3 \\ \hline 1 & 1 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 3 \end{array} \quad \begin{array}{c|ccc} \triangleright & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{array}$$

The set $X^2$ has $|X|^2 = 9$ elements; we compute that the pair $(x, y)$ is sent by the braid to the element $(x \triangleright y, y \triangleright (x \triangleright y))$.
and thus obtain switch braid quiver

\[
\begin{array}{ccc}
(1,1) & (1,2) & (1,3) \\
(2,1) & (2,2) & (2,3) \\
(3,1) & (3,2) & (3,3)
\end{array}
\]

Since \( \varphi_B \) is a bijection for every \( B \), we have:

**Theorem 10.** For any braid \( B \) and finite switch \( X \), the switch braid quiver \( SQ_X(B) \) decomposes into disjoint directed cycles.

For each vertex \( v \) in \( SQ_X(B) \), let \( L(v) \) denote the length of the directed cycle containing \( v \). Then we make the following definition:

**Definition 8.** Let \( X \) be a finite switch and \( B \) a braid diagram. We define the switch braid quiver polynomial of \( B \) with respect to \( X \) to be the polynomial obtained by summing over the vertices \( v \) in \( SQ_X(B) \):

\[
\Phi^C_X(B) = \sum_{v \in SQ_X(B)} u^{L(v)}.
\]

Since the switch braid quiver is an invariant of braids, it follows that:

**Corollary 11.** The switch braid quiver polynomial is an invariant of braids.

**Example 5.** The braid in Example 4 has switch braid quiver invariant value \( \Phi^C_X(B) = 6u^3 + 3u \) with respect to the birack in the example.

Since a coloring with switch braid quiver cycle length 1 is a coloring of the braid closure, we obtain:

**Corollary 12.** When \( X \) is a birack, the coefficient of the linear term \( u \) in \( \Phi^C_X(B) \) is the birack counting invariant of the braid closure of \( B \), namely \( \Phi^Z_X(B) \).

This last corollary poses a natural question: What is the meaning of the other coefficients? It is not difficult to see that a coloring with cycle length \( k \) is a coloring of the closure of \( B^k \), the \( k \)th power of \( B \) in the braid group; hence we have the more general result:

**Corollary 13.** When \( X \) is a birack, the coefficient of the term \( u^k \) in \( \Phi^C_X(B) \) is the birack counting invariant of the braid closure of \( B^k \), namely \( \Phi^Z_X(B^k) \).

**Example 6.** The invariant value \( \Phi^C_X(B) = 6u^3 + 3u \) from example 5 says that closure of the braid in Example 4, i.e. the Hopf link, has three colorings by the birack \( X \) in the example, distinguishing it from the two-component unlink which has nine. Moreover, it also says that the closure of the third power of the braid, i.e. the (6,2)-torus link, has six colorings by \( X \).
5 Switch Braid Quiver Cocycle Enhancements

The meanings of the individual terms of the switch braid quiver polynomial all have been identified. However, like the counting invariants from which the polynomial arises, this invariant can be enhanced in various ways. In this section we will consider enhancements via birack 2-cocycles, leaving further enhancements for future work.

Let \( X \) be a finite birack and \( A \) an abelian group. Recall (see [4] for more detail) that the group of birack \( k \)-chains is the \( A \)-module generated by ordered \( k \)-tuples of elements of \( X \), i.e. \( C_k(X; A) = A[X]^k \). The map \( \partial_k : C_k(X; A) \rightarrow C_{k-1}(X; A) \) defined on generators by

\[
\partial_k(x_1, \ldots, x_k) = \sum_{j=1}^{k} (-1)^j [(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k) - (x_1 \geq x_j, \ldots, x_{j-1} \geq x_j, x_{j+1} \wedge x_j, \ldots, x_k \wedge x_j)]
\]

and extended by linearity is known as the birack boundary map. For each \( k \) we have the groups of boundaries \( B_{k+1}(X; A) = \text{Im} \partial_{k+1} \subset C_k(X; A) \) and cycles \( Z_k(X; A) = \text{Ker} \partial_k \subset C_k(X; A) \) with the quotient group giving us the \( k \)th birack homology \( H_k(X; A) = Z_k(X; A)/B_k(X; A) \).

Dualizing yields the corresponding cohomology modules, \( H^k(X; A) = Z^k(X; A)/B^{k-1}(X; A) \).

Given an element of \( Z^2(X; A) \) representing a cohomology class in \( H^2(X; A) \), we collect contributions from the crossings in a birack coloring of \( B \) given by

\[
BW(v) = \sum_{c \in B} (-1)^{\epsilon(c)} \phi(x(c), y(c))
\]

where we sum over crossings \( c \) in \( B \), with \( \epsilon(c) \) denoting the sign of crossing \( c \) and \((x(c), y(c))\) denoting the tuple of colors on the left of crossing \( c \) induced by the color vector \( v \), as in the diagram above.

Definition 9. Let \( B \) be a braid diagram, \( X \) a finite birack and \( A \) an abelian group. For each element \( \phi \in H^2(X; A) \), we define the birack braid cocycle quiver \( BCQ^X(B) \) to be the switch braid quiver \( SQ^X(B) \) with vertices weighted with the Boltzmann weights:

\[
BW(v) = \sum_{c \in B} (-1)^{\epsilon(c)} \phi(x(c), y(c))
\]
By construction, we have:

**Theorem 14.** For any finite birack \( X \), abelian group \( A \) and birack 2-cocycle \( \phi \in H^2(X; A) \), the function

\[
\mathcal{B}CQ_X^\phi : \{ \text{diagrams of } n\text{-strand braids} \} \to \{ \text{vertex-weighted quivers} \}
\]

sending \( B \to \mathcal{B}CQ_X^\phi(B) \) is an invariant of braids.

**Remark 4.** We can now treat \( \mathcal{B}CQ_X^\phi \) as a map \( B_n \to \{ \text{vertex-weighted quivers} \} \).

**Example 7.** Let \( X = \{1, 2, 3\} \) have the constant action birack structure given by \( \sigma = \tau = (13) \) and let \( \phi : X \times X \to \mathbb{Z}_5 \) be given by

\[
\phi(x, y) = \chi_{(1,2)} + 4\chi_{(1,3)} + 3\chi_{(2,1)} + 2\chi_{(2,3)} + \chi_{(3,1)} + 2\chi_{(3,2)}.
\]

The reader can verify that \( \phi \) is a birack 2-cocycle. Then, for example, the \( X \)-coloring of the braid \( B = \sigma_1^2 \in B_2 \) determined by the top row vector \((2, 3)\) shown has Boltzmann weight \( \phi(2, 1) + \phi(1, 2) + \phi(2, 1) = 3 + 1 + 3 = 2 \):

![Braid Diagram](image)

Repeating over the set \( X^2 \), \( B \) has birack braid cocycle quiver

![Braid Diagram](image)

**Definition 10.** For any braid diagram \( B \), finite birack \( X \), abelian group \( A \) and birack 2-cocycle \( \phi \in C^2(X; A) \), make the following definitions.

- The **birack braid cocycle quiver polynomial** \( \Phi_{X}^{\mathcal{B}CQ, \phi}(B) \) is the sum over the vertices \( v \) in \( \mathcal{B}CQ_X^\phi(B) \):

\[
\Phi_{X}^{\mathcal{B}CQ, \phi}(B) = \sum_{v \in \mathcal{B}CQ_X^\phi(B)} u^\mathcal{C}(v) v^B W(v)
\]

where the terms encode the cycle length and Boltzmann weight of each vertex.
• The birack braid cocycle 2-variable polynomial is the sum over the edges $e$ in $\mathcal{BCQ}^\phi_X(B)$:

$$
\Phi^\phi_{X,2}(B) = \sum_{e \in \mathcal{BCQ}^\phi_X(B)} s^{BW(s(e))}t^{BW(t(e))}
$$

where $BW(s(e))$ is the Boltzmann weight of the source vertex of $e$ and $BW(t(e))$ is the Boltzmann weight of the target vertex of $e$.

Since both polynomials are decategorifications of the braid invariant $\mathcal{BCQ}^\phi_X$, we have:

**Corollary 15.** Both polynomials $\Phi^{\mathcal{BCQ},\phi}_X$ and $\Phi^{\phi,2}_X$ are invariants of braids.

**Example 8.** The braid in Example 7 has

$$
\Phi^{\mathcal{BCQ},\phi}_X(B) = u^2v^4 + u^2v^2 + 3u^2v + u^2 + 3
$$

and

$$
\Phi^{\phi,2}_X(B) = s^4t + s^2 + st + 2 + t^2 + 3.
$$

6 Questions

We conclude with some questions for future research.

• What other enhancements of $\Phi^C_X$ are possible?

• Beyond replacing with biracks with biquandles, what strategies are needed to obtain invariants of knots and link from these invariants of braids?

• Is there a natural notion of an inner automorphism of a switch? If so, which braids $B$ make $\varphi_B$ an inner automorphism of $X^n$?

References

[1] K. Cho and S. Nelson. Quandle coloring quivers. *Journal of Knot Theory and Its Ramifications*, 28(01), 2018.

[2] A. S. Crans and S. Nelson. Hom quandles. *Journal of Knot Theory and Its Ramifications*, 23(02), 2014.

[3] B. A. Davey and G. Davis. Tensor products and entropic varieties. *Algebra Universalis*, 21:68--88, 1985.

[4] M. Elhamdadi and S. Nelson. Quandles—An Introduction to the Algebra of Knots, volume 74 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2015.

[5] R. Fenn, M. Jordan-Santana, and L. Kauffman. Biquandles and virtual links. *Topology and its Applications*, 145(1-3):157--175, 2004.

[6] R. Fenn, C. Rourke, and B. Sanderson. Trunks and classifying spaces. *Applied Categorical Structures*, 3(4):321--356, 1995.

[7] E. Horvat and A. S. Crans. From biquandle structures to hom-biquandles. *Journal of Knot Theory and Its Ramifications*, 29(02), 2020.

[8] P. Jedlička, A. Pilitowska, D. Stanovský, and A. Zamojska-Dzienio. The structure of medial quandles. *Journal of Algebra*, 443:300–334, 2015.
[9] L. H. Kauffman and D. Radford. Bi-oriented quantum algebras, and a generalized Alexander polynomial for virtual links. In Diagrammatic Morphisms and Applications, volume 318 of Contemp. Math., pages 113–140. Amer. Math. Soc., Providence, RI, 2003.

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