ON COMPUTABILITY OF JULIA SETS: ANSWERS TO QUESTIONS OF MILNOR AND SHUB

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Abstract. In this note we give answers to questions posed to us by J. Milnor and M. Shub, which shed further light on the structure of non-computable Julia sets.

1. Introduction

Computability of real sets. The reader is directed to [BY] for a more detailed discussion of the notion of computability of subsets of \( \mathbb{R}^n \) as applied, in particular, to Julia sets. We recall the principal definitions here. The exposition below uses the concept of a Turing Machine. This is a standard model for a computer program employed by computer scientists. Readers unfamiliar with this concept should think instead of an algorithm written in their favorite programming language. These concepts are known to be equivalent.

Denote by \( \mathbb{D} \) the set of the dyadic rationals, that is, rationals of the form \( \frac{p}{2^m} \). We say that \( \phi : \mathbb{N} \to \mathbb{D} \) is an oracle for a real number \( x \), if \( |x - \phi(n)| < 2^{-n} \) for all \( n \in \mathbb{N} \). In other words, \( \phi \) provides a good dyadic approximation for \( x \). We say that a Turing Machine (further abbreviated as TM) \( M^\phi \) is an oracle machine, if at every step of the computation \( M \) is allowed to query the value \( \phi(n) \) for any \( n \).

Let \( K \subset \mathbb{R}^k \) be a compact set. We say that a TM \( M \) computes the set \( K \) if it approximates \( K \) in the Hausdorff metric. Recall that the Hausdorff metric is a metric on compact subsets of \( \mathbb{R}^n \) defined by

\[
d_H(X, Y) = \inf\{\epsilon > 0 \mid X \subset U_\epsilon(Y) \text{ and } Y \subset U_\epsilon(X)\},
\]

where \( U_\epsilon(S) \) is defined as the union of the set of \( \epsilon \)-balls with centers in \( S \).

We introduce a class \( C \) of sets which is dense in metric \( d_H \) among the compact sets and which has a natural correspondence to binary strings. Namely \( C \) is the set of finite unions...
of dyadic balls:
\[ \mathcal{C} = \left\{ \bigcup_{i=1}^{n} B(d_i, r_i) \mid \text{where } d_i, r_i \in \mathbb{D} \right\}. \]

Members of \( \mathcal{C} \) can be encoded as binary strings in a natural way.

We now define the notion of computability of subsets of \( \mathbb{R}^n \) (see [Wei], and also [RW]).

**Definition 1.2.** We say that a compact set \( K \subset \mathbb{R}^k \) is computable, if there exists a TM \( M(d, n) \), where \( d \in \mathbb{D}, \ n \in \mathbb{N} \) which outputs a value 1 if \( \text{dist}(d, K) < 2^{-n} \), the value 0 if \( \text{dist}(d, K) > 2 \cdot 2^{-n} \), and in the “in-between” case it halts and outputs either 0 or 1.

In other words, it computes, in the classical sense, a function from the family \( F_K \) of functions of the form

\[
(2) \quad f(d, n) = \begin{cases} 
0, & \text{if } \text{dist}(d, K) > 2 \cdot 2^{-n} \\
1, & \text{if } \text{dist}(d, K) < 2^{-n} \\
0 \text{ or } 1, & \text{otherwise}
\end{cases}
\]

**Theorem 1.1.** For a compact \( K \subset \mathbb{R}^k \) the following are equivalent:

1. \( K \) is computable as per definition 1.2.
2. there exists a TM \( M(m) \), such that on input \( m \), \( M(m) \) outputs an encoding of \( C_m \in \mathcal{C} \) such that \( d_H(K, C_m) < 2^{-m} \) (global computability),
3. the distance function \( d_K(x) = \inf\{|x - y| \mid y \in K\} \) is computable as per definition 1.1.

Note that in the case \( k = 2 \) computability means that \( K \) can be drawn on a computer screen with arbitrarily good precision (if we imagine the screen as a lattice of pixels).

In the present paper we are interested in questions concerning the computability of the Julia set \( J_c = J(f_c) = J(z^2 + c) \). Since there are uncountably many possible parameter values for \( c \), we cannot expect for each \( c \) to have a machine \( M \) such that \( M \) computes \( J_c \) (recall that there are countably many TMs). On the other hand, it is reasonable to want \( M \) to compute \( J_c \) with an oracle access to \( c \). Define the function \( J : \mathbb{C} \to K^* \) (\( K^* \) is the set of all compact subsets of \( \mathbb{C} \)) by \( J(c) = J(f_c) \). In a complete analogy to Definition 1.1 we can define

**Definition 1.3.** We say that a function \( \kappa : S \to K^* \) for some bounded set \( S \) is computable, if there exits an oracle TM \( M^\phi(d, n) \), where \( \phi \) is an oracle for \( x \in S \), which computes a function (2) of the family \( \mathcal{F}_\kappa(x) \).

Equivalently, there exists an oracle TM \( M^\phi(m) \) with \( \phi \) again representing \( x \in S \) such that on input \( m \), \( M^\phi \) outputs a \( C \in \mathcal{C} \) such that \( d_H(C, \kappa(x)) < 2^{-m} \).

In the case of Julia sets:

**Definition 1.4.** We say that \( J_c \) is computable if the function \( J : d \mapsto J_d \) is computable on the set \{\( c \)\}.

We have the following (see [BBY2]):

**Theorem 1.2.** Suppose that a TM \( M^\phi \) computes the function \( J \) on a set \( S \subset \mathbb{C} \). Then \( J \) is continuous on \( S \) in Hausdorff sense.
Previous results. We have demonstrated in [BY]:

**Theorem 1.3.** There exists a parameter value $c \in \mathbb{C}$ such that the Julia set of the quadratic polynomial $f_c(z) = z^2 + c$ is not computable.

The quadratic polynomials in Theorem 1.3 possess Siegel disks. It was further shown by I. Binder and the authors of the present paper in [BBY1] that the absence of rotation domains, that is either Siegel disks or Herman rings, guarantees computability of the rational Julia set. This implies, in particular, that all Cremer quadratic Julia sets are computable – this despite the fact that no informative high resolution images of such sets have ever been produced. One expects, however, that such “bad” but still computable examples have high algorithmic complexity, which makes the computational cost of producing such a picture prohibitively high.

**Two questions on computability of Julia sets.** J. Milnor has asked us the following natural question:

*Is the filled Julia set of a quadratic polynomial always computable?*

In this paper we answer in the affirmative:

**Theorem 1.4.** For any polynomial $p(z)$ there is an oracle Turing Machine $M^\phi(n)$ that given an oracle access to the coefficients of $p(z)$ and $n$, outputs a $2^{-n}$-approximation of the filled Julia set $K_{p(z)}$.

Moreover, in the case when $p(z) = z^2 + c$ is quadratic, only two machines suffice to compute all non-parabolic Julia sets: one for $c \in M$, and one for $c \notin M$.

This may come as a surprise, given the negative result for Julia sets. To gain some insight into how non-computability can be destroyed by filling in, consider the following toy example.

Let $A : \mathbb{N} \to \{0, 1\}$ be any uncomputable predicate. Consider the set

$$
\Omega_t = \begin{cases} 
S^1 \bigcup_{k \in \mathbb{N}, d_k = 1} \{re^{2\pi i/k} | r \in [1 - \frac{1}{k}, 1]\} & \text{for } t = (0.d_1d_2d_3\ldots)_2 \in [0, 1) \\
S^1 \bigcup_{k \in \mathbb{N}, A(k) = 1} \{re^{2\pi i/k} | r \in [1 - \frac{1}{k}, 1]\} & \text{for } t = 1
\end{cases}
$$

To avoid ambiguity, we always take the finite expansion for dyadic $t$’s. An example of a set $\Omega_t$ is depicted on Figure 1. Firstly, note that if $t \in (0, 1)$ is not a computable real, then the set $\Omega_t$ is non-computable by a TM without an oracle for $t$. Moreover, even for a TM $M^\phi$ equipped with an oracle input for $t$, the set $\Omega_t$ is clearly non-computable. However, when filled, every $\Omega_t$ becomes a computable set – the unit disk.

The question of M. Shub again has to do with fragility of non-computability. This time, instead of filling in a non-computable Julia set, we will make a “fuzzy” picture of it by letting the parameter $c$ vary in a neighborhood.

To formalize this, consider the following definition. Let $J$ be the subset of $\mathbb{C} \times \mathbb{C}$ given by

$$
J = \{(z, c) : z \in J_c\}.
$$

Shub has asked us:

*Is the set $J$ computable?*

The answer again is “yes”: 3
Theorem 1.5. Let \( d > 0 \) be any computable real. Then the set
\[
J \cap \mathbb{C} \times \overline{B(0, d)}
\]
is a computable subset of \( \mathbb{C} \times \mathbb{C} \).

Informally, we may think of projection of \( J \cap \mathbb{C} \times (c - \epsilon, c + \epsilon) \) to the first coordinate as the picture that a computer would produce when \( J \) itself is uncomputable.

To understand how the mechanism of non-computability is destroyed in this case, consider again the set \( \Omega_t \) for \( t \in (0, 1] \) as the toy model. The set \( W = \{(z, t) : z \in \Omega_t, t \in (0, 1]\} \subset \mathbb{C} \times \mathbb{R} \) is computable even though \( \Omega_t \) itself is non-computable for \( t = 1 \). This happens because in the closure of \( W \) the “slice” corresponding to \( t = 1 \) is
\[
S^1 \bigcup_{k \in \mathbb{N}} \{re^{2\pi i/k} | r \in [1 - 1/k, 1]\} \supset \Omega_1.
\]
This set “masks” the computational hardness of \( \Omega_1 \), and makes \( W \) computable.

2. Computability of filled Julia sets

Our goal in this section is to prove Theorem 1.4. For a given polynomial \( p(z) \) we construct a machine computing the corresponding filled Julia set \( K_p \). We will use the following combinatorial information about \( p \) in the construction. Note the this information can be encoded using a finite number of bits.

- Information that would allow us to compute the non-repelling orbits of the polynomial with an arbitrary precision. Note that there are at most \( \deg p - 1 \) of them. Such information could, for example, consist of the list of periods \( k_i \) of such orbits; and for each \( i \) a finite collection of dyadic balls \( \{D_{ij}^k\}_{j=1}^{k_i} \) separating the points of the corresponding orbit from the other solutions of the equation \( p^{k_i}(z) = z \). This allows for an arbitrarily precise approximation of the orbits by using an iterative root-finding algorithm for \( p^{k_i}(z) = z \) in \( D_{ij}^k \).

- In the case of a hyperbolic or a parabolic orbit \( \zeta \), a domain of attraction \( D_\zeta \) such that every orbit converging to \( \zeta \) eventually reaches and stays in \( D_\zeta \). In the hyperbolic case \( D_\zeta \) is just a collection of discs. In the parabolic case, it is a collection of sectors around the points of \( \zeta \), and it can be computed with an arbitrarily high precision.
Recall the Fatou-Sullivan classification of Fatou components of a polynomial mapping of \( \hat{\mathcal{C}} \) (see e.g. [Mil]).

2.1. Computing \( K_p \). We are given a dyadic point \( d \in \mathbb{D} \) and an \( n \in \mathbb{N} \). Our goal is to always terminate and output 1 if \( B(d, 2^{-n}) \cap K_p \neq \emptyset \) and to output 0 if \( B(d, 2 \cdot 2^{-n}) \cap K_p = \emptyset \). We do it by constructing five machines. They are guaranteed to terminate each on a different condition, always with a valid answer. Together they cover all the possible cases.

**Lemma 2.1.** There are five oracle machines \( M_{\text{ext}}, M_{\text{jul}}, M_{\text{hyp}}, M_{\text{par}}, M_{\text{sieg}} \) such that

1. if \( d \) is at distance \( \geq \frac{4}{3} \cdot 2^{-n} \) from \( K_p \), \( M_{\text{ext}}(d,n) \) will halt and output 0. If \( d \) is at distance \( \leq 2^{-n} \) from \( K_p \), \( M_{\text{ext}}(d,n) \) will never halt;
2. if \( d \) is at distance \( \leq \frac{5}{3} \cdot 2^{-n} \) from \( J_p \), \( M_{\text{jul}}(d,n) \) will halt and output 1. If \( d \) is at distance \( \geq 2 \cdot 2^{-n} \) from \( J_p \), \( M_{\text{jul}}(d,n) \) will never halt;
3. \( M_{\text{hyp}}(d,n) \) halts and outputs 1 if and only if \( d \) is inside an attracting basin of a hyperbolic orbit of \( p \);
4. \( M_{\text{par}}(d,n) \) halts and outputs 1 if and only if \( d \) is inside an attracting basin of a parabolic orbit of \( p \);
5. \( M_{\text{sieg}}(d,n) \) halts and outputs 1 if the orbit of \( d \) reaches a Siegel disc, and \( d \) is at distance \( \geq \frac{4}{3} \cdot 2^{-n} \) from \( J_p \). It never halts if \( d \) is at distance \( \geq 2 \cdot 2^{-n} \) from \( K_p \).

Recall the Fatou-Sullivan classification of Fatou components of a polynomial mapping of \( \hat{\mathcal{C}} \) (see e.g. [Mil]):

**Theorem 2.2 (Fatou-Sullivan classification).** Every Fatou component of a polynomial mapping of \( \hat{\mathcal{C}} \) of degree at least two is a preimage of a periodic component. Every periodic component is of one of the following types: the immediate basin of an attracting (or a super-attracting) periodic point; a component of the immediate basin of a parabolic periodic point; a Siegel disk.

**Proof of Theorem 2.4, given Lemma 2.7** By Fatou-Sullivan classification it is not hard to see that for each \((d,n)\) at least one of the machines halts. Moreover, by the definition of the machines, they always output a valid answer whenever they halt. Hence running the machines in parallel and returning the output of the first machine to halt gives the algorithm for computing \( K_p \). \( \square \)

It remains to prove Lemma 2.1.

**Proof. (of Lemma 2.1)** We give a simple construction for each of the five machines.

1. \( M_{\text{ext}} \): Take a large ball \( B \) such that \( K_p \subseteq B \). Intuitively, we pull the ball back under \( p \) to get a good approximation of \( K_p \). Let \( B_k \) be a \( 2^{-(n+3)} \)-approximation of the set \( p^{-k}(B) \). Output 0 iff \( B_k \cap B(d, \frac{5}{3} \cdot 2^{-n}) = \emptyset \). It is not hard to see that this algorithm satisfies the conditions on \( M_{\text{ext}} \).
2. \( M_{\text{jul}} \): Enumerate all the repelling periodic orbits of \( p \). Let \( C_k \) be a \( 2^{-(n+3)} \)-approximation of the union of the first \( k \) orbits enumerated. Output 1 iff \( d(d, C_k) < \frac{11}{6} \cdot 2^{-n} \). The repelling periodic orbits are all in \( J_p \) and are dense in this set. Hence the algorithm satisfies the conditions on \( M_{\text{jul}} \).
(3) $M_{hyp}$: Let $z_k$ be a $2^{-k}$-approximation of $p^k(d)$. If $d$ is inside the basin of attraction for a some orbit $\zeta$, then $z_k$ for some $k$ will be inside $D_\zeta$ for some $k$. Output 1 if $z_k$ is at least $2^{-k}$-far from the boundary of $D_\zeta$.

(4) $M_{par}$: Very similar to $M_{hyp}$. The only difference is that now we are checking for convergence to an attracting petal of a parabolic orbit.

(5) $M_{sieg}$: This is the most interesting case. It is not hard to see that for each $k$, we can compute a union $E_k$ of dyadic balls such that

$$\bigcup_{i=0}^{k} p^i \left( B(d, \frac{4}{3} \cdot 2^{-n}) \right) \subset E_k \subset \bigcup_{i=0}^{k} p^i \left( B(d, \frac{5}{3} \cdot 2^{-n}) \right).$$

Let $c$ be the center of the Siegel disc (one of the centers, in case of an orbit), and let $y$ be the given periodic point in the connected component of $c$. We terminate and output 1 if $E_n$ separates $c$ from $y$ in $\mathbb{C}$ (or covers either one of them).

If $d$ is inside the Siegel disc, then the forward images of $B(d, \frac{4}{3} \cdot 2^{-n})$ will cover an annulus in the disc that will separate $c$ from the boundary of the disc, and in particular from $y$. Hence $M_{sieg}$ will terminate and output 1.

If the distance from $d$ to $K_c$ is $\geq 2 \cdot 2^{-n}$, then $E_k \cap K_p = \emptyset$ for all $k$. In particular, $E_k$ cannot separate $c$ from $y$, since they are connected in $K_p$.

□

2.2. The quadratic case. In the quadratic case there is at most one non-repelling orbit. In the case there is a Siegel disc, the Julia set is connected, and any repelling periodic orbit can be taken as the orbit connected to the Siegel disc. In fact, it is not hard to see that if we exclude the parabolic case, one machine suffices to take care of all the connected filled Julia sets. As a corollary we get:

Corollary 2.3. Denote by $M$ the Mandelbrot set, and by $P$ the set of $c$’s for which $J_c$ is parabolic. The function $K : c \mapsto K_{z_2+c}$ is continuous in the Hausdorff metric on the set $M - P$.

3. Computability of the set $\mathbb{J}$

Recall that

$$\mathbb{J} = \{(z, c) : z \in J_c\} \subset \mathbb{C}^2.$$

Theorem 1.5 asserts that $\mathbb{J}$ is computable. We prove it by showing that $\mathbb{J}$ is weakly computable.

Definition 3.1. We say that a set $C$ is weakly computable if there is an oracle Turing Machine $M^\phi(n)$ such that if $\phi$ represents a real number $x$, then the output of $M^\phi(n)$ is

$$M^\phi(n) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } B(x, 2^{-n}) \cap C = \emptyset \\ 0 \text{ or } 1 & \text{otherwise} \end{cases}$$

It has been shown that the weak definition is equivalent to the standard definition. See [Brv2], for example. We will need the following lemma.

Lemma 3.1. For any point $(z, c)$ in the complement of the closure $\mathbb{J}$, $z$ converges to an attracting periodic orbit of $f_c : z \mapsto z^2 + c$. 

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The proof of the lemma occupies §

The following lemma allows us to “cover” all points that belong to \( \mathcal{J} \).

**Lemma 3.2.** There is an algorithm \( A_1(n) \) that on input \( n \) outputs a sequence of dyadic points \( p_1, p_2, \ldots \in \mathbb{C} \times \mathbb{C} \) such that

\[
B(\mathcal{J}, 2^{-(n+3)}) \subset \bigcup_{j=1}^{\infty} B(p_j, 2^{-(n+2)}) \subset B(\mathcal{J}, 2^{-(n+1)}).
\]

**Proof.** It is well known that repelling periodic orbits of \( f_c \) are dense in \( J_c \). Hence, the set

\[
S_{rep} = \{(z, c) : z \text{ is in a repelling periodic orbit of } f_c \}
\]

is dense in \( \mathcal{J} \). \( S_{rep} \) is a union of a countable number of algebraic curves \( S^m_{rep} \) given by the constraints

\[
\begin{cases}
  f_c^m(z) = z \\
  |(f_c^m)'(z)| > 1
\end{cases}
\]

For each \( m \) we can compute a finite number of points \( p^1_{r_1}, \ldots, p^m_{r_m} \) approximating \( S^m_{rep} \) such that

\[
B(S^m_{rep}, 2^{-(n+3)}) \subset \bigcup_{j=1}^{r_m} B(p^m_{j}, 2^{-(n+2)}) \subset B(S^m_{rep}, 2^{-(n+1)}).
\]

We have

\[
\mathcal{J} = \overline{S_{rep}} = \bigcup_{m=1}^{\infty} S^m_{rep}.
\]

Hence the computable sequence \( p^1_1, \ldots, p^1_{r_1}, p^2_{r_1}, \ldots, p^2_{r_2}, \ldots, p^m_{r_1}, \ldots, p^m_{r_m}, \ldots \) satisfies the conditions of the lemma.

**Corollary 3.3.** There is an oracle machine \( M_1^{\phi_1, \phi_2}(n) \), where \( \phi_1 \) is an oracle for \( z \in \mathbb{C} \) and \( \phi_2 \) is an oracle for \( c \in \mathbb{C} \), such that \( M_1^{\phi_1, \phi_2} \) always halts whenever \( d((z, c), \mathcal{J}) < 2^{-(n+4)} \) and never halts if \( d((z, c), \mathcal{J}) \geq 2^{-n} \).

**Proof.** Query the oracles for a point \( p \in \mathbb{C} \times \mathbb{C} \) such that \( d(p, (z, c)) < 2^{-(n+4)} \). Then run the following loop:

\[
i \leftarrow 0
\]

**do**

\[
i \leftarrow i + 1
\]

**generate** \( p_i \) using \( A_1(n) \) from Lemma 3.2

**while** \( d(p, p_i) > 2^{-(n+2)} \)

\[
\text{If } d((z, c), \mathcal{J}) < 2^{-(n+4)}, \text{ then } d(p, \mathcal{J}) < 2^{-(n+3)}, \text{ hence by Lemma 3.2 there is an } i \text{ such that } d(p, p_i) \leq 2^{-(n+2)}, \text{ and the loop terminates. If } d((z, c), \mathcal{J}) > 2^{-n}, \text{ then } d(p, \mathcal{J}) > 2^{-n} - 2^{-(n-4)} > 1.5 \cdot 2^{-(n+1)}. \text{ Hence, by Lemma 3.2, } p \notin B(p_i, 2^{-(n+1)}) \text{ for all } i, \text{ and the loop will never terminate.} \]

The following lemma allows us to exclude points outside \( \mathcal{J} \) from \( \mathcal{J} \).

**Lemma 3.4.** There is an oracle machine \( M_2^{\phi_1, \phi_2} \), where \( \phi_1 \) is an oracle for \( z \in \mathbb{C} \) and \( \phi_2 \) is an oracle for \( c \in \mathbb{C} \), such that \( M_2^{\phi_1, \phi_2} \) halts if and only if \( z \) converges to an attracting periodic orbit (or to \( \infty \)) under \( f_c : z \mapsto z^2 + c \).
Proof. $M_2$ is systematically looking for an attracting cycle of $f_c$. It also iterates $f_c$ on $z$ with increasing precision and for increasingly many steps until we are sure that either one of the two things holds:

1. the orbit of $z$ converges to $\infty$; or
2. we find an attracting orbit of $f_c$ and the orbit of $z$ converges to it.

If the search is done systematically, the machine will eventually halt if one of the possibilities above holds. It obviously won’t halt if neither holds. □

Proof. (of Theorem 1.5) The algorithm is: Run the machines $M_{\phi_1,\phi_2}(n)$ from Corollary 3.3 and $M_{\phi_1,\phi_2}^2$ from Lemma 3.4 in parallel. Output 1 if $M_1$ terminates first and 0 if $M_2$ terminates first.

First we observe that $M_1(n)$ only halts on points that are $2^{-n}$-close to $\mathbb{J}$, in which case 1 is a valid answer according to Definition 3.1. Similarly, $M_2$ only halts on points that are outside $\mathbb{J}$, in which case 0 is a valid answer. Hence if the algorithm terminates, it outputs a valid answer. It remains to see that it does always terminate. Consider two cases.

Case 1: $(z, c) \in \mathbb{J}$. In this case $d((z, c), \mathbb{J}) = 0 < 2^{-(n+4)}$, and the first machine is guaranteed to halt.

Case 2: $(z, c) \notin \mathbb{J}$. By Lemma 3.1, $z$ converges to an attracting periodic orbit of $f_c$ in this case, and hence the second machine is guaranteed to halt. □

4. Proof of Lemma 3.1

Suppose $z \notin J_c$ and the orbit of $z$ does not belong to an attracting basin. By the Fatou-Sullivan classification (see e.g. [Mil]), there exists $k \in \mathbb{N}$ such that $w \equiv f_c^k(z)$ belongs to a Siegel disk or to the immediate basin of a parabolic orbit. Our aim is to show that for an arbitrary small $\delta > 0$, there exists a pair $(\tilde{z}, \tilde{c}) \in \mathbb{C} \times \mathbb{C}$ with $|z - \tilde{z}| < \delta$, $|c - \tilde{c}| < \delta$, and for which $\tilde{z} \in J_{\tilde{c}}$. We will treat the Siegel case first.

4.1. The case when $w$ lies in a Siegel disk. Let us denote $\Delta$ the Siegel disk containing $w$, and let $m \in \mathbb{N}$ be its period, that is, the mapping

$$f_c^m : \Delta \to \Delta$$

is conjugated by a conformal change of coordinates $\phi : \Delta \to \mathbb{D}$ to an irrational rotation of $\mathbb{D}$.

The following statement is elementary (cf. Prop. 7.1 in [Dou]):

**Proposition 4.1.** Denote $\zeta = \phi^{-1}(0) \in \Delta$ the center of the Siegel disk. For each $s > 0$ there exists $\tilde{c} \in B(c, s)$ such that $f_c$ has a parabolic periodic point $\tilde{\zeta}$ of period $m$ in $B(\zeta, s)$. In particular, $J_{\tilde{c}}$ is connected, and $B(\zeta, s) \cap J_{\tilde{c}} \neq \emptyset$.

Consider now the $f_c^m$-invariant analytic circle

$$S_r = \phi^{-1}\left(\{z = re^{2\pi i \theta}, \theta \in [0, 2\pi]\}\right)$$

which contains $w$. Let $\epsilon > 0$ be such that

$$B(w, \epsilon) \subset f_c^k(B(z, \delta)) \cap \Delta.$$
Set $B \equiv B(w, \epsilon/2)$ and let $n \in \mathbb{N}$ be such that the union
\[ \bigcup_{0 \leq i \leq n} f^{mi}_c(B) \supset S_r. \]
By Proposition 4.1 for all $\delta > 0$ small enough, there exist $\hat{c} \in B(c, \delta)$ for which $J_{\hat{c}}$ is connected and there is a point of $J_{\hat{c}}$ inside the domain bounded by $S_r$. Since repelling periodic orbits of $f_c$ are dense in $\partial \Delta$, again for $\delta$ small enough, there are points of $J_{\hat{c}}$ on the outside of $S_r$ as well, and so there exists a point $\xi \in J_{\hat{c}} \cap S_r$. By construction, there exists $j \in \mathbb{N}$ such that $f_j^\ast(B(z, \delta)) \ni \xi$. By invariance of Julia set, if $\hat{c}$ is close enough to $c$ we have $B(z, \delta) \cap J_{\hat{c}} \neq \emptyset$, and the proof is complete.

4.2. The case when $w$ lies in a parabolic basin. We need to recall the Douady-Lavaurs theory of parabolic implosion \cite{Doul, Lav}. Denote $\zeta$ the parabolic periodic point of $f_c$ whose immediate basin contains $w$, and let $m \in \mathbb{N}$ be its period.

Recall that an attracting petal $P_A$ is a topological disk whose boundary contains $\zeta$, such that $f_c^{mk}(P_A) \subset P_A$ for some $k \in \mathbb{N}$, and such that the quotient Riemann surface
\[ C_A = P_A/f_c^{mk} \simeq \mathbb{C}/\mathbb{Z}. \]
The quotient $C_A$, called an attracting Fatou cylinder, parametrizes the orbits converging under the dynamics of the iterate $f_c^m$ to $\zeta$ inside the periodic cycle of petals $P_A, f_c^m(P_A), \ldots, f_c^{m(k-1)}(P_A)$. Recall (see \cite{Mil}) that a quadratic polynomial $f_c$ has only one cycle of petals at $\zeta$. A repelling petal $P_R$ is an attracting petal for the local inverse $f_c^{-m}$ fixing $\zeta$; the union
\[ \bigcup_{0 \leq i \leq k-1} f^{mi}_c(P_A \cup P_R) \]
forms a neighborhood of $\zeta$. The repelling Fatou cylinder $C_R$ is defined in a similar fashion.

Let $\tau$ be any conformal isomorphism $C_A \to C_R$. After uniformization,
\[ C_A \xrightarrow{\approx} \mathbb{C}/\mathbb{Z}, \quad C_R \xrightarrow{\approx} \mathbb{C}/\mathbb{Z} \]
$\tau(z) \equiv z + q \mod \mathbb{Z}$ for some $q \in \mathbb{C}$. Let $g : P_A \to P_R$ be any lift of $\tau$; it necessarily commutes with $f_c^{mk}$. Consider the semigroup $G$ generated by the dynamics of the pair $(f_c, g)$. The orbit $Gz$ of a point $z \in \mathbb{C}$ is independent of the choice of the lift $g$ and only depends on $\tau$.

Set
\[ J_{(c, \tau)} = \{ z \in \mathbb{C} \text{ such that } Gz \cap J_c \neq \emptyset \}. \]
It can be shown that this set is the boundary of
\[ K_{(c, \tau)} = \{ z \in \mathbb{C} \text{ such that } Gz \text{ is bounded} \}. \]

The Douady-Lavaurs theory postulates:

**Theorem 4.2.** For every $\tau$ as above and every $s > 0$ there exists $\tilde{c} \in B(c, s)$ such that $B(J_{\hat{c}}, s) \supset J_{(c, \tau)}$.

Since $\zeta \in J_c$, and $J_c$ is connected, there exists a point $u \in J_c \cap P_R$. Let $\hat{w} \in C_A$ be the orbit of $w$, and let $\hat{u} \in C_R$ be the orbit of $u$. Choose $\tau : C_A \to C_R$ so that $\tau(\hat{w}) = \hat{u}$. Then $J_{(c, \tau)} \ni z$, and the claim follows by Theorem 4.2.


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