Analysis.

An homotopy of isometries related to a probability density.

Roland Groux

Lycée Polyvalent Rouvière, rue Sainte Claire Deville,
BP 1205. 83070 Toulon Cedex. France.

Email: roland.groux@orange.fr

Abstract.

We are studying here a family of probability density functions indexed by a real parameter, and constructed from homographic relations between associated Stieltjes transforms. From the analysis of orthogonal polynomials we deduce a family of isometries in relation to the classical operators creating secondary polynomials and we give an application to the explicit resolution of specific integral equations.

1. Introduction and notations.

Let \( \rho \) a probability density function on an interval \( I \) bounded with \( a \) and \( b \). We note \( c_n = \int_a^b x^n \rho(x) dx \) the moment of order \( n \).

The Stieltjes transform of the measure of density \( \rho \) is defined on \( C - I \) by the formula:

\[
z \mapsto S_\rho(z) = \int_a^b \frac{\rho(t)dt}{z - t}.
\]

We note \( n \mapsto P_n \) a Hilbert base of normalized polynomials for the classic inner product:

\[
(f, g) \mapsto \langle f, g \rangle = \int_a^b f(t)g(t)\rho(t)dt
\]

on the associated Hilbert' space \( L^2(I, \rho) \).

\( T_\rho \) denotes the operator \( f(x) \mapsto g(x) = \int_a^b \frac{f(t) - f(x)}{t - x} \rho(t)dt \) creating secondary polynomials \( Q_n = T_\rho(P_n) \).

Let us recall the results below: (See [3])

Let \( \mu \) be a positive measure on \( I \) associated to a density function \( \mu \), also allowing moments of any order and having Stieltjes’s transformation linked to these of \( \rho \) by the equality:

\[
S_\mu(z) = z - c_1 - \frac{1}{S_\rho(z)}.
\]

We can then conclude:

- Secondary polynomials \( A_n = Q_{n+1} \) relative with \( \rho \) then form an orthonormal family for the inner product induced by \( \mu \).

- Secondary polynomials \( B_n = T_\mu(A_n) \) relative with \( \mu \) are defined by the formula:

\[
B_n(x) = (x - c_1)Q_{n+1}(x) - P_{n+1}(x).
\]
If the density \( \rho \) is continuous and provided the existence of 
\[
\int_a^b \left( \frac{\rho(x)}{(x-t)^2 + \varepsilon^2} \right) dt,
\]
we can make \( \mu \) explicit by the formula:
\[
\mu(x) = \frac{\varphi(x)}{\frac{\varphi^2(x)}{4} + \pi^2 \rho^2(x)}.
\]

The operator \( f(x) \mapsto g(x) = \int_a^b \frac{f(t) - f(x)}{t-x} \rho(t) dt \) creating secondary polynomials extends
to a continuous linear map linking the space \( L^2(I, \rho) \) to the Hilbert'space \( L^2(I, \mu) \), whose
restriction to the hyperplane \( H_\rho \) of the functions orthogonal for \( \rho \) with \( P_0 = 1 \) constitutes an
isometric function for both norms respectively.

Under the mentioned assumptions, the function \( \varphi \) presented above will be call the **reducer** of
\( \rho \), the measure of density \( \mu \) will be call **secondary measure** associated with \( \rho \). Its moment of
order 0 is equal to \( d_0 = c_2 - (c_1)^2 \). If we normalize \( \mu \), we introduce \( \mu_0 = \frac{\mu}{d_0} \) a probability
density function called the ‘**normalized secondary measure**’ of \( \rho \).

By definition we said in what follows that two measures are equi-normal if they lead to the
same normalized secondary measure.

2. A family of equi-normal measures.

In this section we suppose that \( I \) is a compact interval.

We consider in this section a real parameter \( t > 0 \) and a density of probability \( \rho_t \) whose
secondary measure density is \( t\mu_0 \). We also requires that the moment of order 1 of the two
densities \( \rho \) and \( \rho_t \) are the same. Under these assumptions, the coupling of Stieltjes
tansforms results in:

\[
S_\mu(z) = z - c_1 - \frac{1}{S_\rho(z)} \quad \text{and} \quad S_{\mu_t}(z) = tS_\mu(z) = z - c_1 - \frac{1}{S_{\rho_t}(z)}.
\]

We deduce the Stieltjes tansform of \( \rho_t \):
\[
S_{\rho_t}(z) = \frac{S_\rho(z)}{t + (1-t)(z-c_1)}.
\]

We recall the formula of Stieltjes-Perron making explicit the density from its Stieltjes
transform:
\[
\rho(x) = \lim_{\varepsilon \to 0^+} \frac{S_\rho(x+i\varepsilon) - S_\rho(x-i\varepsilon)}{2i\pi} \quad \text{and} \quad \varphi(x) = \lim_{\varepsilon \to 0^+} S_\rho(x+i\varepsilon) + S_\rho(x-i\varepsilon).
\]

We easily deduce:
\[
\lim_{\varepsilon \to 0^+} S_\rho(x+i\varepsilon)S_\rho(x+i\varepsilon) = \frac{\varphi^2(x)}{4} + \pi^2 \rho^2(x).
\]
By applying Stieltjes-Perron to \( S_\rho \), we obtain after simplifying:

\[
\rho_t(x) = \frac{t \varphi(x)}{[(t-1)(x-c_1) - \frac{\varphi(x)}{2} - t]^2 + \pi^2 \rho^2(x)(t-1)^2(x-c_1)^2} 
\]

However, as shown by the following examples, the function defined by the above formula does not always its Stieltjes transform equals to \( H(z) = \frac{S_\rho(z)}{t + (1-t)(z-c_1)S_\rho(z)} \).

It will be easily seen by noting that the denominator of \( H(z) \) may vanish outside \( I \) if the parameter \( t \) becomes too high.

But if effectively this function \( H \) is the Stieltjes transform of \( \rho_t \), then we deduce from

\[
S_\mu(z) = z - c_1 - \frac{1}{S_\rho(z)} \quad \text{the formula:} \quad S_{\rho_t}(z) = \frac{1}{z - c_1 - tS_\mu(z)}.
\]

So the moment of order 0 of \( \rho_t \) is obtained by:

\[
\lim_{|z| \to \infty} zS_{\rho_t}(z) = \lim_{|z| \to \infty} \frac{1}{1 - \frac{c_1}{z} - tS_\mu(z)} = 1.
\]

Thus, \( \rho_t \) is a probability density function and its moment of order 1 is \( c_1 \) because:

\[
\lim_{|z| \to \infty} z^2(S_{\rho_t}(z) - \frac{1}{z}) = \lim_{|z| \to \infty} \frac{c_1 - tS_\mu(z)}{1 - \frac{c_1}{z} - tS_\mu(z)} = c_1.
\]

We can finally conclude thanks to \( tS_\mu(z) = z - c_1 - \frac{1}{S_\rho(z)} \), that the secondary measure of the measure of density \( \rho_t \) is \( t\mu \). So \( \rho_t \) is effectively equi-normal with \( \rho \).

Now consider some examples. We will study particularly the function

\[
t \mapsto f(t) = t \int\frac{\rho(x)dx}{[(t-1)(x-c_1) - \frac{\varphi(x)}{2} - t]^2 + \pi^2 \rho^2(x)(t-1)^2(x-c_1)^2} \quad \text{making explicit the moment of order 0 for } \rho_t.
\]

- The Tchebychev measure of the second kind over \([-1,1]\]

\[
\rho(x) = \frac{2}{\pi \sqrt{1-x^2}}, \quad \text{and we have} \quad c_1 = 0; \varphi(x) = 4x \quad \text{and } \mu(x) = \frac{\rho(x)}{4}
\]

The function \( f \) is made explicit here by:

\[
t \mapsto f(t) = \frac{2t}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{t^2 + 4(1-t)x^2} dx
\]
A quick viewing using MAPLE shows that \( f \) keeps the constant value 1 all over the interval \([0,2]\).

\[
\begin{align*}
> & f:=\text{proc}(t) \\
& \text{local} \ x,g; \\
& g:=\text{evalf}((2*t/Pi)\times\text{int}(\sqrt{1-x^2}/(t^2+4*(1-t)*x^2),x=-1..1));\text{end}: \\
> & \text{plot}(f);
\end{align*}
\]

This can be directly verified by direct calculation, as well as the following formulas:

\[
\rho_t(x) = \frac{2t\sqrt{1-x^2}}{\pi[t^2 + 4(1-t)x^2]}, \text{ with } t \text{ in } [0,2].
\]

The secondary measure of the density \( \rho_t \) is \( \mu_t = \frac{t}{4} \left( \frac{2\sqrt{1-x^2}}{\pi} \right) \).

The reducer of \( \rho_t \) is defined by: \( \Phi_t(x) = \frac{2(4-2t)x}{t^2 + 4(1-t)x^2} \).

Stieltjes transform is given by: \( S_t(z) = \frac{2}{(2-t)z + t\sqrt{z^2 - 1}} \).

For \( t=1 \) we find of course the Tchebychev measure of the second kind.

For \( t=2 \) we obtain the Tchebychev measure of the first kind.

For \( t = \frac{4}{3} \) we have \( \rho(x) = \frac{6\sqrt{1-x^2}}{\pi(4-3x^2)} \) and \( \Phi(x) = \frac{6x}{4-3x^2} \).
The uniform Lebesgue’s measure over [0,1].

Uniform density \( \rho(x) = 1 \) with the reducer \( \varphi(x) = 2 \ln \left( \frac{x}{1-x} \right) \) and \( c_i = \frac{1}{2} \).

Here we have: \( t \mapsto f(t) = \int_0^1 \frac{dx}{[(t-1)(x-1/2)\ln(x/(1-x)) - t]^2 + \pi^2(t-1)^2(x-1/2)^2} \)

> \textbf{f:=proc}(t)
> local x, g;
> g:=evalf(t*int(1/(((t-1)*(x-1/2)*ln(x/(1-x))-t)^2+Pi*Pi*(t-1)^2*(x-1/2)^2),x=0..1));end :
> \textbf{f}(1.3);

So it seems that, for any \( t \) in \( [0,1] \), the secondary measure of the density

\[ \rho_t(x) = \frac{t}{[(t-1)(x-1/2)\ln(x/(1-x)) - t]^2 + \pi^2(t-1)^2(x-1/2)^2} \]

is : \( \mu_t(x) = \frac{t}{[\ln^2 \left( \frac{x}{1-x} \right) + \pi^2]} \).

We can obtain by MAPLE verification of this proportionality by examining the moments of order two.

Recall the formulas: \( S_\mu(z) = z - c_i - \frac{1}{S_\rho(z)} \) et \( tS_\mu(z) = z - c_i - \frac{1}{S_\rho(z)} \).

Thus: \( c'_2 - (c'_1)^2 = t(c_2 - (c_1)^2) \).

Or in this case: \( c_1 = c'_1 = \frac{1}{2} \) and \( c_2 = \frac{1}{3} \). So we get: \( c'_2 = \frac{t+3}{12} \), confirmed by MAPLE:
\[ f_2 := \text{proc}(t) \]
\[ \text{local } x, g; \]
\[ g := \text{evalf}(t \times \int x^2/\left((t-1) \times (x-1/2) \times \ln(x/(1-x)) - t\right)^2 + \pi \times \pi \times (t-1)^2 \times (x-1/2)^2, x=0..1)); \]end:
\[ \]
\[ \text{plot}(f_2, 0..1); \]

- For \( \rho(x) = 2x \) over \([0, 1]\), the reducer is \( \varphi(x) = -4x \ln\left(\frac{1-x}{x}\right) - 4 \) and \( c_1 = \frac{2}{3} \).

\[ \text{ro} := x \rightarrow 2 \times x; \]
\[ \text{phi} := x \rightarrow -4 \times (x \times \ln((1-x)/x) + 1); \]
\[ \text{c1} := \text{evalf}(\int x \times \text{ro}(x), x=0..1)); \]
\[ \text{f} := \text{proc}(t) \]
\[ \text{local } x, g; \]
\[ g := \text{evalf}(t \times \int \text{ro}(x)/\left((t-1) \times (x-c_1) \times \text{phi}(x)/2-t\right)^2 + \pi^2 \times (t-1)^2 \times \text{ro}(x)^2 \times (x-c_1)^2, x=0..1)); \]end:
\[ \]
\[ \text{f}(0.45); \text{plot}(f); \]
• For $\rho(x) = \frac{3\sqrt{x}}{2} \text{ over } [0, 1]$, we have $\varphi(x) = 3\text{LerchPhi}(x, 1, -\frac{1}{2})$ and $c_1 = \frac{3}{5}$.

\[
\rho(x) = \frac{3\sqrt{x}}{2}, \quad \varphi(x) = 3\text{LerchPhi}(x, 1, -\frac{1}{2})
\]

\[
r : x \rightarrow 3*\sqrt{x}/2; \quad \rho := x \rightarrow \frac{3}{2}\sqrt{x}
\]

\[
c_1 := \int (x\rho(x), x=0..1); \quad c_1 = \frac{3}{5}
\]

\[
\phi := x \rightarrow 3\text{LerchPhi}(x, 1, -1/2); \quad \phi := x \rightarrow 3\text{LerchPhi}(x, 1, -\frac{1}{2})
\]

\[
f := \text{proc}(t)
\]

local x, g;

\[
g := \text{evalf}(\int (x\rho(x)/((t-1)*(x-c_1)\phi(x)/2-t)^2+\pi^2\rho(x)^2((t-1)^2*(x-c_1)^2), x=0..1)); \text{evalf}(t*g)\end{proc}
\]

\[
f(0.6); 1.000000000
\]

\[
f(0.45628); 0.9999999999
\]

\[
f(0.2157); 1.000000000
\]

\[
f(2); 0.7496041742
\]

\[
f(1.24); 0.9911159300
\]

Now we’ll try to explain why we get a good probability density equi-normal with $\rho$ when the parameter $t$ remains in the interval $]0, 1]$.

Consider first the following lemma: (For an easy writing we note in what follows $I = [-1, 1]$)

### 2.3. Lemma.

The equation $\int_{-1}^1 [(z-c_1)S_\rho(z) = 0]$ has no solutions belonging to the open set $O=C-I$ when the parameter $t$ is element of $]0, 1]$.

**Proof.**

Noting $m = \frac{t}{t-1}$ and $z = x + iy$, the equation becomes:

\[
\int_{-1}^1 \frac{(x-c_1+iy)\rho(u)du}{x-u+iy} = m
\]

Separating the real and imaginary we get:

\[
\int_{-1}^1 \frac{x^2+y^2-u(x-c_1)-c_1x}{(x-u)^2+y^2}\rho(u)du = m \quad \text{and} \quad \int_{-1}^1 \frac{c_1-u}{(x-u)^2+y^2}\rho(u)du = 0
\]
Let us consider first the solutions out of the real axis.

If \( y \neq 0 \) we have from imaginary part above:

\[
\int \frac{u}{(x-u)^2 + y^2} \rho(u) du = \int \frac{c_1}{(x-u)^2 + y^2} \rho(u) du.
\]

And for real part:

\[
\int \frac{x^2 + y^2 + c_1^2 - 2c_1x}{(x-u)^2 + y^2} \rho(u) du = m \int \frac{x^2 + y^2 + u^2 - 2c_1x}{(x-u)^2 + y^2} \rho(u) du.
\]

After reductions:

\[
\int \frac{(1-m)((x-c_1)^2 + y^2) + m(c_1^2 - u^2)}{(x-u)^2 + y^2} \rho(u) du = 0
\]

And with the variable \( t \):

\[
\int \frac{((x-c_1)^2 + y^2) + t(u^2 - c_1^2)}{(x-u)^2 + y^2} \rho(u) du = 0
\]

Thus, if \( c_1 = 0 \), there are no non-real solutions when \( t > 0 \).

Study now real solutions.

For \( y = 0 \), the equation easily simplifies to:

\[
\int \frac{mu + (1-m)x - c_1}{x-u} \rho(u) du = 0.
\]

And with the variable \( t \):

\[
\int \frac{tu - x + c_1(1-t)}{x-u} \rho(u) du = 0
\]

Here again if \( c_1 = 0 \), there are no real solutions out of \( I \) if \( 0 < t < 1 \).

However there may exist if \( t > 1 \) as shown in the following example for a Tchebychev’s measure and the value \( t = 3 \).

```maple
> ro:=x->sqrt(1-x^2):
> f:=proc(x)
local t,f;
f:=evalf(int((3*t-x)*ro(t)/(x-t),t=-1..1));end;

> plot(f);`
```

```
> f(1.06); 0.005893680455
> f(1.07); -0.07793247767
```
Now let us analyze the general case where \( c_1 \neq 0 \) and \( t \in [0,1] \)

- Resuming the first study of non-real solutions. We got the qualities:

\[
\int \frac{u}{(x-u)^2 + y^2} \rho(u) du = \int \frac{c_1}{(x-u)^2 + y^2} \rho(u) du \quad \text{and} \quad \int \frac{([x-c_1]^2 + y^2) + t(u^2 - c_1^2)}{(x-u)^2 + y^2} \rho(u) du = 0.
\]

Or \([x-c_1]^2 + y^2\] + \(t(u^2 - c_1^2) = x^2 - 2c_1x + y^2 + tu^2 + (1-t)c_1^2\)

The first integral equality leads to:

\[
\int \frac{-2xu}{(x-u)^2 + y^2} \rho(u) du = \int \frac{-2xc_1}{(x-u)^2 + y^2} \rho(u) du
\]

(This by multiply by \(-2x\)). So we deduce the transform of the second integral:

\[
J = \int \frac{([x-c_1]^2 + y^2) + t(u^2 - c_1^2)}{(x-u)^2 + y^2} \rho(u) du = 0 = \int \frac{([x-u]^2 + y^2) + (t-1)(u^2 - c_1^2)}{(x-u)^2 + y^2} \rho(u) du = K
\]

Now write: \(0 = (1-t)J + tK\). After simplifications this results to:

\[
\int \frac{((1-t)x-c_1^2 + y^2 + t(x-u)^2}{(x-u)^2 + y^2} \rho(u) du = 0 \; , \; \text{impossible, because} \; t \in [0,1]
\]

- Now examine real solutions. We saw earlier in this case:

\[
\int \frac{t(x-u) + c_1(1-t)}{x-u} \rho(u) du = 0. \; \text{Or we can write denominator as:} \; -t(x-u) + (1-t)(c_1 - x).
\]

From this we deduce:

\[
\int \frac{(1-t)(c_1 - x)}{x-u} \rho(u) du = t.
\]

If we write \(c_1 - x = c_1 - u + u - x\), we gets: \(t - 1 + \int \frac{(1-t)(c_1 - u)}{x-u} \rho(u) du = t\)

So:

\[
\int \frac{(1-t)(c_1 - u)}{x-u} \rho(u) du = 1
\]

Or if \(x\) is located ‘right on the interval’ \(I\), therefore higher than any element \(u\) of \(I\) and also higher than \(c_1\), we have:

\[
\frac{c_1 - u}{x-u} < \frac{x-u}{x-u} = 1.
\]

So we deduce if \(t < 1\):

\[
1 \leq \int \frac{(1-t)(x-u)}{x-u} \rho(u) du = 1 - t. \; \text{So we conclude} \; t \leq 0, \; \text{that is contrary to the hypothesis.}
\]

(We obtain a similar contradiction if \(x\) is located ‘left on the interval’ \(I\).)

This completes the proof of Lemma (2.3)

Now study the following result:
2.4. Lemma.

Let \( H \) a function of complex variable, holomorphic on \( O = \mathbb{C} - I \).

We note \( \tilde{I} = [-a, a] \), with \( a > 1 \) and we suppose here that:

2.4.1 \( |H(z)| \) has limit 0 when the module of \( z \) tends to infinity.

2.4.2 The two sequences of function defined on \( \tilde{I} \) by \( x \mapsto H(x + \frac{j}{n}) \) and \( x \mapsto H(x - \frac{j}{n}) \) have a limit for the classic norm of \( L^1([-a, a]) \) when the index \( n \) tends to infinity.

Under these hypotheses we can conclude that \( H \) is the Sieltjes transform of \( v \) defined on \( I \) by

\[
v(x) = \lim_{\varepsilon \to 0^+} \frac{H(x - i\varepsilon) - H(x + i\varepsilon)}{2i\pi}.
\]

Proof.

Let \( Z \) a complex number element of \( O = \mathbb{C} - I \).

The function : \( z \mapsto f(z) = \frac{H(z)}{(z - Z)} \) is holomorphic on \( O - \{Z\} \) an its residue at the point \( Z \) is \( H(Z) \). Let \( \lambda \) a fixed real with \( 0 < \lambda < a \)

We consider \( R_\varepsilon \) a rectangle with sides parallel to the axes, oriented in the direct sense, and whose summits have for affix : \( 1 + \lambda + \varepsilon(\pm i); -1 - \lambda + \varepsilon(\pm i), \) with \( \varepsilon > 0 \).

We note \( C_b \) the circle focused on the origin, oriented in the direct sense and whose radius equal \( b \), with \( b > |Z| \).

The two paths \( R_\varepsilon \) and \( C_b \) are homotopic in \( O \). Thus, according with Cauchy’s theorem:

\[
\int_{R_\varepsilon} f(z)dz = \int_{C_b} f(z)dz - 2i\pi H(Z). \quad (2.4.3)
\]

Now study the evolution of this equality when: \( b \to +\infty \) and \( \varepsilon \to 0^+ \).

- It is clear that for the two segments parallel to the imaginary axis, the contribution of the first integral vanished to 0 with \( \varepsilon \), because \( f \) has a finite limit at \( 1 + \lambda \) and \( -1 - \lambda \).

\[
\lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} f(1 + \lambda + it)idt = 0 = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} f(-1 - \lambda + it)idt = 0.
\]

- For the two parallels to the real axis, we obtain by adding the integrals over reversed paths:

\[
\lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \left( H(t - i\varepsilon) - H(t + i\varepsilon) \right) dt = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \left( \int_{-\varepsilon}^{\varepsilon} \frac{H(t - i\varepsilon) - H(t + i\varepsilon)}{t - Z - i\varepsilon} dt \right) dt
\]

\[
= \lim_{\varepsilon \to 0^+} \left( \int_{-\varepsilon}^{\varepsilon} \frac{(t-Z)(H(t - i\varepsilon) - H(t + i\varepsilon))}{(t-Z-i\varepsilon)(t-Z+i\varepsilon)} dt \right) + J(\varepsilon)
\]
with \( J(\varepsilon) = \mathrm{i} e^{\int_{-1+\lambda}^{1+\lambda} H(t-\varepsilon i) + H(t+\varepsilon i) dt} \). From (2.4.2) we easily deduce: \( \lim_{\varepsilon \to 0^+} J(\varepsilon) = 0 \),

and also: \( \lim_{\varepsilon \to 0^+} \int_{-1+\lambda}^{1+\lambda} \frac{(t-Z)[H(t-\varepsilon i) - H(t+\varepsilon i)] dt}{(t-Z-i\varepsilon)(t-Z+i\varepsilon)} = 2i\pi \int_{-1+\lambda}^{1+\lambda} \frac{v(t) dt}{t-Z} \)



Note that according Jordan’s lemma, and thanks to (2.4.1), the integral on the circle \( C_b \) vanished to 0 when radius \( b \) tends to infinity. Thus, by limit passages, the formula (2.4.3) becomes:

\[
2i\pi \int_{-1+\lambda}^{1+\lambda} \frac{v(t) dt}{t-Z} = -2i\pi H(z) .
\]

When \( \lambda \) vanished to 0 we finally obtain the expected formula:

\[
\int_{-1}^{1} \frac{v(t) dt}{Z-t} = S_{\rho}(Z) = H(Z)
\]

Thanks to these two precedent results we can now explain the importance of placing the parameter \( t \) in the interval \( [0,1] \). So in what follows we suppose \( 0 < t \leq 1 \).

We’ll apply the lemma (2.4) to the function \( z \mapsto H(z) = \frac{S_{\rho}(z)}{t + (1-t)(z-c_1)S_{\rho}(z)} \).

First note that thanks to lemma (2.3), the function \( H \) is holomorphic on \( O = C - I \)

(2.4.1) is provided here directly by \( \lim_{|t| \to \infty} S_{\rho}(z) = 0 \) and \( \lim_{|t| \to \infty} zS_{\rho}(z) = 1 \)

If we suppose that the density \( \rho \) is such that the function \( H \) above satisfies (2.4.2) we can effectively conclude that the Stieltjes transform of \( \rho_t \) is \( H \). In this case we have a family of density of probability equi-normal with the initial density \( \rho \) and indexed with the real parameter \( t \in [0,1] \).

We can notice that when \( t \) tends to 0, the limit of this family within the sense of distributions is the Dirac’s measure \( \delta_{c_1} \) concentrated at the first moment \( c_1 \). Indeed we check easily that for any continuous function \( g \) on the interval \( I \): \( \lim_{t \to 0} \int f(u)\rho_t(u) du = f(c_1) \)

Note also that the reducer \( \varphi_t(x) \) of \( \rho_t(x) \) tends towards the Dirac’s reducer: \( \frac{2}{x - c_1} \).

(The Sieltjes transform of \( \delta_{c_1} \) is \( S_{\delta_{c_1}}(z) = \frac{1}{z - c_1} \).

Always within the sense of distributions we note also: \( \lim_{t \to 0^+} \frac{(x - c_1)^2 \rho_t}{t} = \mu \).
3. **Orthogonal polynomials for the density \( \rho \).**

We note here \( n \mapsto P_n \) a classic sequence of orthonormal polynomials related to the density \( \rho \), and \( Q_n = T_\rho (P_n) \) the associated secondary polynomials. In accordance with the notations of the first paragraph, we have: \( P_n = P_n \) and \( Q_n = Q_n \).

Recall the important result mentioned in this introduction section: (3.1)

- Secondary polynomials \( A_n = Q_{n+1} \) relative with \( \rho \) then form an orthonormal family for the inner product induced by \( \mu \).

- Secondary polynomials \( B_n = T_\mu (A_n) \) relative with \( \mu \) are defined by the formula:

\[
B_n(x) = (x - c_1)Q_{n+1}(x) - P_{n+1}(x)
\]

When we normalize \( \mu \) into \( \mu_0 = \frac{\mu}{d_0} \), the set of primary orthonormal polynomials becomes \( \tilde{A}_n = \sqrt{d_0} A_n \) and the associated secondary polynomials are now made explicit by

\[
\tilde{B}_n = \frac{B_n}{\sqrt{d_0}} \quad \text{because} \quad T_{\mu_0} = \frac{T_\mu}{d_0}.
\]

The same normalization from the density equi-normal \( \rho \) leads to the same set of polynomials, because normalize secondary measure of the density \( \rho \) is also \( \mu_0 \).

So, thanks to the formulas (3.1) we get:

\[
\begin{cases}
\sqrt{d_0} Q_{n+1}(x) = \sqrt{d_0} Q'_{n+1}(x) \\
\frac{1}{\sqrt{d_0}}[(x - c_1)Q_{n+1}(x) - P_{n+1}(x)] = \frac{1}{\sqrt{d_0}}[(x - c_1)Q'_{n+1}(x) - P'_{n+1}(x)]
\end{cases}
\]

Then, for \( n \geq 1 \):

\[
(3.2) \quad \begin{cases}
P'_n(x) = \frac{1}{\sqrt{t}}[tP_n(x) + (1-t)(x - c_1)Q_n(x)] \\
Q'_n(x) = \frac{1}{\sqrt{t}} Q_n(x)
\end{cases}
\]

**Associated isometries.**

From \( Q_n = T_\rho (P_n) \) and (3.2), we see that the map \( f(x) \mapsto \frac{1}{\sqrt{t}}[tf(x) + (1-t)(x - c_1)T_\rho (f)(x)] \) transforms \( (P_n)_{n \geq 1} \) orthonormal for the density \( \rho \) into \( (P'_n)_{n \geq 1} \) orthonormal for the density \( \rho \).
According to Cauchy’s theorem, we can extends this application to an isometric map linking the hyperplane $H_\rho$ of the functions of $L^2(I, \rho)$ orthogonal for $\rho$ with $P_0 = 1$ to the hyperplane $H'_\rho$ constituted by the elements of $L^2(I, \rho)$ orthogonal with $P_0 = 1$.

If we introduce $\tilde{\rho}_i = \frac{\rho_i}{t}$, we deduce that $V'_\rho$ defined by:

$$f(x) \mapsto V'_\rho(f(x)) = tf(x) + (1 - t)(x - c_1)T'_\rho(f(x))$$

is an isometric function linking $H_\rho$ equipped with the norm of $L^2(I, \rho)$ to $H'_\rho$ equipped with the norm associated to $\tilde{\rho}_i$. This translates into the following equal:

$$\int f^2(x)\rho(x)dx = \int [tf(x) + (1 - t)(x - c_1)T'_\rho(f(x))]^2 \frac{\rho_i(x)dx}{t} = \int [V'_\rho(f(x))]^2 \tilde{\rho}_i(x)dx$$

For example, for the Tchebychev’s measure of the second kind over $]-1,1[ : \rho(x) = \frac{2}{\pi}\sqrt{1 - x^2}$, we have $\tilde{\rho}_i(x) = \frac{2\sqrt{1 - x^2}}{\pi[t^2 + 4(1 - t)x^2]}$; $c_i = 0$. So (3.3) translates in this case to:

$$\int_{-1}^{1} f^2(x)\sqrt{1 - t^2}dx = \int_{-1}^{1} [tf(x) + (1 - t)xT'_\rho(f(x))]^2 \frac{\sqrt{1 - x^2}dx}{t^2 + 4(1 - t)x^2}$$

under the assumption that $\bar{f} = \frac{2}{\pi} \int_{-1}^{1} f(x)\sqrt{1 - x^2}dx$ equals 0. (The formula (3.3) requires belonging of $f$ to $H_\rho$). In the general case we adjust the formula by changing $f$ into $f - \bar{f}$, as shown in the following MAPLE program.

```maple
> rho:=x->2*sqrt(1-x^2)/Pi;
> rho2:=(x,s)->2*sqrt(1-x^2)/(Pi*(s^2+4*(1-s)*x^2)):
> V:=proc(f,t)
local u,x,g,y,C;
C:=evalf(int(f(u)*rho(u),u=-1..1));
g:=t*(f(x)-C)+(1-t)*x*int((f(u)-f(x))*rho(u)/(u-x),u=-1..1);
y:=evalf(int(g^2*rho2(x,t),x=-1..1));end:
> V(x->x^3-2/(x+5)+1/(x^2+3),1.35);
0.10100202639029552321

> W:=proc(g)
local x,C,y;
C:=evalf(int(g(x)*rho(x),x=-1..1));
y:=evalf(int(g(x)*g(x)*rho(x),x=-1..1)-C^2));end:
> W(x->x^3-2/(x+5)+1/(x^2+3));
0.1010020263902957
```

13
We will see now that under an hypothesis of density related to $\rho$, we have the composition pattern: $V'_{\rho} = (T'_{\rho})^{-1} \circ T_{\rho}$, with the following usual notations:

- $T_{\rho}$ is the isometric function linking $H_{\rho}$ with the norm of $L^2(I,\rho)$ to $L^2(I,\mu)$.
- $T'_{\rho}$ linking $H'_{\rho}$ with the norm of $L^2(I,\rho')$ to $L^2(I,\mu)$.

The existence of $T'_{\rho}$ requires the density of the space of polynomials in $L^2(I,\rho)$, which is verified for instance if $I$ is a compact interval. Under this hypothesis, and thanks to the density of polynomials in $H_{\rho}$, it is sufficient to study the transform of $P_n$.

By definitions: $V'_{\rho}(P_n) = tP_n + (1-t)(x-c_1)Q_n$

Thanks (3.2) and linearity of $T'_{\rho}$, we have: $T_{\rho} \circ V'_{\rho}(P_n) = \sqrt{t}T_{\rho}(P_n') = \sqrt{t}Q'_n = Q_n$

So we get for every $n$: $V'_{\rho}(P_n) = (T_{\rho})^{-1} \circ T_{\rho}(P_n)$, that validates the composition formula.

(3.4) $V'_{\rho} = (T_{\rho})^{-1} \circ T_{\rho}$

**Inversion of the operator $V'_{\rho}$**.

From (3.4) we can write: $(V'_{\rho})^{-1} = (T_{\rho})^{-1} \circ T'_{\rho}$

Recall now that the definition of equi-normal measure is symmetric, and that the secondary measure associated to the density $\rho$ is $\mu$. So, by changing the roles, $\rho \leftrightarrow \rho'$, the primary density $\rho$ becomes equi-normal with $\rho'$ and this with an inverse proportionality coefficient.

So, with the change $t \leftrightarrow \frac{1}{t}$ we get: (3.5) $(V'_{\rho})^{-1}(f(x)) = \frac{1}{t} f(x) + (1 - \frac{1}{t})(x-c_1)T'_{\rho}(f(x))$

Note : the new parameter $\frac{1}{t}$ no longer belongs to the interval $[0,1]$. This is not necessary here because $\rho$ deduced from $\rho'$ with this coefficient is by primary hypothesis a density of probability.

Application to solving specific integral equations.

We consider here the equation: $(E_{\alpha}): f(x) + \lambda(x-c_1)\int_{I} \frac{f(u)-f(x)}{u-x} \rho(u)du = g(x)$

with $\lambda > 0$, $g$ a given function and $f$ unknown function in the hyperplane $H_{\rho}$.
If we note: \( t = \frac{1}{1 + \lambda} \), the equation above easily translates into:

\[
 tf(x) + (1-t)(x-c_i)T_\rho'(f)(x) = tg(x), \text{ or else: } \ V'_\rho(f(x)) = tg(x), \text{ with } t \in [0,1].
\]

According to the precedent results, it can be resolved if \( g \) is an element of \( H_\rho' \), using the formula:

\[
 f(x) = g(x) - \frac{\lambda}{1+\lambda}(x-c_i)T_\rho'(g)(x)
\]

In integral terms, this solution is made explicit by:

\[
 (3.6) \quad f(x) = g(x) - \frac{\lambda}{1+\lambda}(x-c_i)\int_1^{1+\lambda} \frac{g(u)-g(x)}{u-x} \rho_{\lambda}(u) \, du
\]

Note that the condition \( \lambda > 0 \) is not necessary, it is sufficient that \( \rho_{\lambda} \) is a probability density function. Note also that the equation \( (E_{\lambda}) \) is realized when \( g \) is a constant function \( g(x) = C \) with the evident solution \( f(x) = g(x) = C \) . So, thanks linearity, the formula (3.6) can be used with the simple hypothesis: \( g \) is element of \( L^2(I,\rho_{\lambda}) \).

For example if we chooses the Tchebychev measure of the second kind over \([-1,1]\]

\[
 \rho(x) = \frac{2}{\pi} \sqrt{1-x^2}, \text{ and } \lambda = -\frac{1}{2} \text{ we have } c_i = 0, t = 2 \text{ and so } \rho_2(x) = \frac{1}{\pi\sqrt{1-x^2}} \text{ is the Tchebychev density of the first kind.}
\]

Then in this case the equation \( (E_{\lambda}) \): \( f(x) - \frac{x}{\pi} \int_{-1}^1 \frac{f(t)-f(x)}{t-x} \sqrt{1-t^2} \, dt = g(x) \) has for solution:

\[
 f(x) = g(x) + \frac{x}{\pi} \int_{-1}^1 \frac{g(t)-g(x)}{t-x} \times \frac{dt}{\sqrt{1-t^2}}
\]

Here bellows some checks using MAPLE:

\[
 > \rho := x -> 2 * \sqrt{1-x^2} / \pi;
 > \rho_2 := x -> 1 / (\pi * \sqrt{1-x^2});
 > V := proc(f)
 \text{local } t, x, g;
 g := f(x) - (x/2) * \text{int}((f(t)-f(x)) * \rho(t)/(t-x), t=-1..1);
 g := \text{simplify}(g); g := \text{unapply}(g, x);
 end:
 > W := proc(g)
 \text{local } t, x, f;
 f := g(x) + x * \text{int}((g(t)-g(x)) * \rho_2(t)/(t-x), t=-1..1); f := \text{simplify}(f); f := \text{unapply}(f, x);
 end:
 > V(W(x->2*x^11-7*x^10+8*x^5-3*x+2)); x \rightarrow 2 \, x^{11} - 7 \, x^{10} + 8 \, x^{5} - 3 \, x + 2
\]
4. Formulas of compositions.

Recall that the secondary measure of $\rho_s$ equals $t\mu$. So, if $t$ and $s$ are two elements defining equi-normal densities $\rho_t$ and $\rho_s$, we have the obvious relation $(\rho_t)_s = \rho_{t,s}$.

We also saw in the previous paragraph that the operator $\frac{1}{\sqrt{t}}V^t\rho_t$ transforms $(P_n)_{n \geq 1}$ orthonormal for the density $\rho_t$ into $(P_n^s)_{n \geq 1}$ orthonormal for the density $\rho_s$.

From these two points we deduce easily the relation: $\frac{1}{\sqrt{t} \cdot s}V^{t,s}_\rho = \left(\frac{1}{\sqrt{s}}V^s_{\rho_s}\right) \circ \left(\frac{1}{\sqrt{t}}V^t_{\rho_t}\right)$ and so we get the composition formula: (4.1) $V^{t,s}_\rho = V^t_{\rho_t} \circ V^s_{\rho_s}$.

By changing the initial density into $\rho_u$, the formula above extends to $V^{t,s}_{\rho_u} = V^{s}_{\rho_{u,s}} \circ V^{t}_{\rho_s}$.

Using the definition: $f(x) \mapsto V^t_{\rho_t}(f(x)) = tf(x) + (1-t)(x-c_t)T_{\rho_t}(f(x))$, the development of the formula (4.1) leads after simplifications to:

$$T^t_{\rho_t} \circ T^s_{\rho_s}((x-c_t)f(x)) = \frac{T^t_{\rho_t} - tT^s_{\rho_s}}{1-t}(f(x)) \tag{4.2}$$

By obvious changing we get the more general formula:

$$T^t_{\rho_t} \circ T^s_{\rho_s}((x-c_t)f(x)) = \frac{sT^t_{\rho_t} - tT^s_{\rho_s}}{s-t}(f(x)) \tag{4.3}$$

If we note for simplifying $S$ the operator: $f(x) \mapsto (x-c_t)f(x)$, we get the barycentric formula above:

$$T^t_{\rho_t} \circ T^s_{\rho_s} \circ S = \frac{sT^t_{\rho_t} - tT^s_{\rho_s}}{s-t} \tag{4.4}$$
Here is a check using MAPLE in the case of Tchebychev’s measures mentioned above:

> rho:=x->2*sqrt(1-x^2)/Pi:
> rho2:=x->1/(Pi*sqrt(1-x^2)):

> T:=proc(f)
local t,x,g;
g:=int((f(t)-f(x))*rho(t)/(t-x),t=-1..1);
g:=simplify(g);g:=unapply(g,x);end:

> T2:=proc(f)
local t,x,g;
g:=int((f(t)-f(x))*rho2(t)/(t-x),t=-1..1);
g:=simplify(g);g:=unapply(g,x);end:

> f:=x->7*x^5-4*x^3+x/(x^2+3);

\[
\begin{align*}
\rho &:= x \rightarrow 7x^5 - 4x^3 + \frac{x}{x^2 + 3} \\
\end{align*}
\]

> g:=x->x*f(x):
> a:=T2(f)(x):
> b:=T(f)(x):
> simplify(2*a-b);

\[
\begin{align*}
\frac{41x^2 - 24\sqrt{3} + 81 + 56x^6 + 178x^4}{8(x^2 + 3)}
\end{align*}
\]

Note finally that if we apply the formula (4.3) to a function of type \( x \mapsto f(x) = \frac{1}{x-z} \) who is an eigenvector for any operator \( T^I_\rho \), we obtain the relation linking the Stieltjes transforms

\[
S^I_\rho(z) = \frac{(z-c_i)S^I_\rho(z)S^*_\rho(z)}{t-s} \frac{tS^I_\rho(z) - sS^I_\rho(z)}{t-s}
\]

References.

[1] Christian Berg, Annales de la Faculté des Sciences de Toulouse, Sér 6 Vol. S5 (1996), p.9-32.
[2] G.A Baker, Jr and P. Graves-Morris, ‘Padé approximants’, Cambridge University Press, London 1996.
[3] R. Groux. C.R Acad.Sci. Paris, Ser.I. 2007. Vol 345. pages 373-376.