LOCAL STRUCTURE OF SINGULAR HYPERKÄHLER QUOTIENTS

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Abstract. When a compact Lie group acts freely and in a Hamiltonian way on a symplectic manifold, the Marsden–Weinstein theorem says that the reduced space is a smooth symplectic manifold. If we drop the freeness assumption, the reduced space might be singular, but Sjamaar–Lerman (1991) showed that it can still be partitioned into smooth symplectic manifolds which “fit together nicely” in the sense that they form a stratification. In this paper, we prove a hyperkähler analogue of this statement, using the hyperkähler quotient construction. We also show that singular hyperkähler quotients are complex spaces which are locally biholomorphic to affine complex-symplectic GIT quotients with biholomorphisms that are compatible with natural holomorphic Poisson brackets on both sides.

1. Introduction

Let \( K \) be a compact Lie group acting on a symplectic manifold \( M \) in a Hamiltonian way with moment map \( \mu : M \to \mathfrak{t}^* \). Recall that the Marsden–Weinstein theorem [27] says that if the action is free, the quotient

\[
M//_\mu K := \mu^{-1}(0)/K
\]

is a smooth symplectic manifold, called the symplectic reduction of \( M \) by \( K \) with respect to \( \mu \). If the action is not necessarily free, then \( M//_\mu K \) is usually singular, but Sjamaar–Lerman [33] showed that it can still be partitioned into smooth symplectic manifolds (using the partition by orbit types). Moreover, these manifolds fit together nicely in the sense that they form a stratification of \( M//_\mu K \). This means, in particular, that for each stratum \( S \subseteq M//_\mu K \), the closure of \( S \) is a union of strata, and the way in which \( S \) embeds in \( M//_\mu K \) is topologically constant along \( S \) (see §2.1 for a precise definition). Also, the symplectic structures on these strata are compatible with a Poisson bracket on the subalgebra of continuous functions on \( M//_\mu K \) which descend from smooth \( K \)-invariant functions on \( M \). Moreover, every point of \( M//_\mu K \) has a neighbourhood homeomorphic to a linear symplectic reduction (i.e. the reduction of a symplectic vector space by a linear action) with a homeomorphism respecting the natural stratifications and Poisson brackets on both sides. Thus, linear symplectic reductions are universal local models for all symplectic reductions.

In hyperkähler geometry, there is an analogue of symplectic reduction due to Hitchin–Karlhede–Lindström–Roček [20] which has been a very important tool for constructing new examples of these special manifolds. The goal of this paper is to get analogues of Sjamaar–Lerman’s results in this setting. It is already known [5] that hyperkähler quotients by non-free actions of compact Lie groups are partitioned into smooth symplectic manifolds. The main contribution of this paper is to show that this partition is a stratification and obtain a holomorphic version of the above local model.

More precisely, recall that a hyperkähler manifold is a Riemannian manifold \((M,g)\) with three complex structures \( I,J,K \) that are Kähler with respect to \( g \) and satisfy \( IJ = K \). This implies that for all \( a,b,c \in \mathbb{R} \) such that \( a^2 + b^2 + c^2 = 1 \), the endomorphism \( aI + bJ + cK \) is another complex structure which is Kähler with respect to \( g \). Thus, \( M \) has a two-sphere of complex structures. Let \( \omega_1, \omega_J, \omega_K \) be the Kähler forms of \( I,J,K \), respectively. If \( K \) is a compact Lie group acting on \( M \) by preserving the hyperkähler structure, a hyperkähler moment map is a map \( \mu = (\mu_I, \mu_J, \mu_K) : M \to \mathfrak{t}^* \times \mathfrak{t}^* \times \mathfrak{t}^* \), where \( \mathfrak{t} := \text{Lie}(K) \) and \( \mu_I, \mu_J, \mu_K \) are moment maps for \( \omega_1, \omega_J, \omega_K \), respectively. If such a map \( \mu \) exists, we say that the \( K \)-action is tri-Hamiltonian and call the triple \((M,K,\mu)\) a tri-Hamiltonian hyperkähler manifold. The group \( K \) in such a triple will always be assumed to be compact. The hyperkähler quotient of \( M \) by \( K \) with respect to \( \mu \) is the quotient space

\[
M//_\mu K := \mu^{-1}(0)/K.
\]

This construction was introduced in [20, §3(D)], where it is shown that if \( K \) acts freely on \( \mu^{-1}(0) \), then \( M//_\mu K \) and \( M//_\mu K \) are smooth manifolds and \( M//_\mu K \) has a canonical hyperkähler structure descending from \( M \). If the \( K \)-action is not necessarily free, then \( M//_\mu K \) can be partitioned by orbit types as in the

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symplectic case. That is, we partition $M///_\mu K$ into the connected components of the spaces $\mu^{-1}(0)_{(H)}/K$ for all subgroups $H \subseteq K$, where $\mu^{-1}(0)_{(H)}$ is the set of points $p \in \mu^{-1}(0)$ whose stabilizer $K_p$ is conjugate to $H$ in $K$. We call this the orbit type partition of $M///_\mu K$. By adapting Sjamaar–Lerman’s arguments in [33, Theorem 3.5], Dancer–Swann [5, §2] showed each piece in the orbit type partition is a hyperkähler manifold. We state this result in the following form (see §2.5 for details).

Theorem 1.1. Let $((M,g,I,J,K),\mu)$ be a tri-Hamiltonian hyperkähler manifold, let $\pi : \mu^{-1}(0) \to M///_\mu K$ be the quotient map, and let $S \subseteq M///_\mu K$ be a piece of the orbit type partition. Then, $S$ is a topological manifold, $\pi^{-1}(S)$ is a smooth submanifold of $M$, there is a unique smooth structure on $S$ such that $\pi^{-1}(S) \to S$ is a smooth submersion, and there is a unique hyperkähler structure $(g_S,\omega_S,K_S)$ on $S$ such that the pullbacks of the Kähler forms $\omega_{\beta_S},\omega_{J_S},\omega_{K_S}$ to $\pi^{-1}(S)$ are the restrictions of $\omega_1,\omega_J,\omega_K$.

However, the question of whether the orbit type partition of $M///_\mu K$ is a stratification as in the symplectic case was left open in Dancer–Swann’s work. The main issue is that the arguments used by Sjamaar–Lerman [33] to show that the orbit type partition of a symplectic reduction is a stratification is based on the local normal form for the moment map [9, 26], but there is no hyperkähler equivalent. In this paper, we show that if the $K$-action extends to a holomorphic action of the complexification $K_\mathbb{C}$, then we do get a stratification:

Theorem 1.2. Let $(M,K,\mu)$ be a tri-Hamiltonian hyperkähler manifold whose $K$-action extends to an action of $K_\mathbb{C}$ which is holomorphic with respect to some element in the two-sphere of complex structures. Then, the orbit type partition of $M///_\mu K$ is a stratification.

Here, we recall that $K_\mathbb{C}$ is a complex Lie group containing $K$ as a maximal compact subgroup and such that $\text{Lie}(K_\mathbb{C}) = \text{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C}$. The assumption on the $K$-action holds, for example, if $M$ is compact or if $M$ is a complex affine variety and the action map $K \times M \to M$ is real algebraic.

The reason for introducing this assumption is that it implies that $M///_\mu K$ is isomorphic to a symplectic reduction in the category of complex spaces and then we can adapt Sjamaar–Lerman’s arguments to the holomorphic setting. More precisely, let $G := K_\mathbb{C}$ and suppose, without loss of generality, that the action of $G$ on $M$ is holomorphic with respect to the complex structure $I$. Let $\mu_{\mathbb{R}} := \mu_1 : M \to \mathfrak{t}^*$ and $\mu_{\mathbb{C}} := \mu_1 + i\mu_K : M \to \mathfrak{g}^*$,

where $g := \text{Lie}(G)$. Then, $\mu_{\mathbb{C}}$ is $\mathbb{C}$-holomorphic and is a complex moment map for the $G$-action on $M$ with respect to the $\mathbb{C}$-holomorphic complex-symplectic form $\omega_{\mathbb{C}} := \omega_I + i\omega_K$.

Moreover, by letting

$M^{\mu_{\mathbb{R}}-\text{ss}} := \{ p \in M : G \cdot p \cap \mu_{\mathbb{R}}^{-1}(0) \neq \emptyset \}$ and $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}-\text{ss}} := \mu_{\mathbb{C}}^{-1}(0) \cap M^{\mu_{\mathbb{R}}-\text{ss}}$,

we have $\mu_{\mathbb{R}}^{-1}(0) \subseteq \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}-\text{ss}}$ and, by a result of Heinzner–Loose [16], this inclusion descends to a homeomorphism $M///_\mu K \cong \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}-\text{ss}}//G$, where $///$ is a categorical quotient in the category of complex spaces (we will review Heinzner–Loose’s work in §2.4). Thus, it suffices to get a local normal form for the complex part $\mu_{\mathbb{C}}$ of the moment map, and this is one of the main technical results of this paper.

To state this normal form, let $p \in \mu^{-1}(0)$ and let

$V = (T_p(G \cdot p))^{\text{ss}}/T_p(G \cdot p),$

where $(\cdot)^{\text{ss}}$ is the complex-symplectic complement with respect to $\omega_{\mathbb{C}}$. Then, $V$ is a complex-symplectic vector space on which the stabilizer $H := G_p$ acts linearly. Roughly speaking, the local normal form says that the complex-Hamiltonian manifold $(M,1,\omega_{\mathbb{C}},G,\mu_1)$ is completely determined in a neighbourhood of $p$ by the representation of $H$ on $V$. More precisely, let $E$ be the complex-symplectic reduction of $T^*G \times V$ by $H$, where $H$ acts by translations on $T^*G$ and linearly on $V$. Then, $E$ is a complex-Hamiltonian $G$-manifold (see §3 for details). As a complex $G$-manifold, $E$ can be identified with the associated vector bundle $G \times_H (\mathfrak{h}^* \times V)$, where $\mathfrak{h}^* \subseteq \mathfrak{g}^*$ is the annihilator of $\mathfrak{h} := \text{Lie}(H)$ and $G$ acts by left multiplication on the $G$-factor. Moreover, there is an explicit expression for the moment map (see (3.5)). We will show:

Theorem 1.3. Let $(M,K,\mu)$ be a tri-Hamiltonian hyperkähler manifold whose $K$-action extends to an $\mathbb{R}$-holomorphic action of $G := K_\mathbb{C}$. Let $p \in \mu^{-1}(0)$, $H = G_p$, and $V = (T_p(G \cdot p))^{\text{ss}}/T_p(G \cdot p)$. Then, there is a $G$-saturated neighbourhood of $p$ in $M^{\mu_{\mathbb{R}}-\text{ss}}$ which is isomorphic as a complex-Hamiltonian $G$-manifold to a $G$-saturated neighbourhood of $[1,0,0]$ in $G \times_H (\mathfrak{h}^* \times V)$.

1Indeed, the local normal form implies the Darboux theorem, so we would have a canonical form describing all three symplectic forms simultaneously and hence they could not carry any local information. But the symplectic forms on a hyperkähler manifold determine the Riemannian metric which does carry local information into the curvature.
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Stratified spaces.

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This result enables us to study the local complex-symplectic structure of a singular hyperkähler quotient.

In particular, Theorem 1.2 follows from Part (ii) below.

Theorem 1.4 (Local Structure of Singular Hyperkähler Quotients). Let $((M, g, I, J, K), K, µ)$ be a tri-

(i) Complex Structure. The inclusion $µ^{-1}(0) ⊆ µ_C^{-1}(0)_{R^{ss}}$ descends to a homeomorphism $M\sslash\mu 

K ∼= µ_C^{-1}(0)_{R^{ss}}//G$ and hence $M\sslash\mu_K$ inherits the structure $O_I$ of a complex space. For each

S ⊆ M\sslash\mu_K in the orbit type partition, we have:

- $S$ is a non-singular complex submanifold of $(M\sslash\mu_K, O_I)$.
- Let $(g_S, I_S, J_S, K_S)$ be the hyperkähler structure of $S$ as in Theorem 1.1. Then, the inclusion

$S ↪ M\sslash\mu_K$ is holomorphic with respect to $I_S$ and $O_I$.

(ii) Stratification Structure. The orbit type partition of $M\sslash\mu_K$ is a complex Whitney stratification

with respect to $O_I$ (see §2.1 for definitions).

(iii) Poisson Structure. There is a unique Poisson bracket on $O_I$ such that for each $S$ in the orbit

type partition, the inclusion $S ↪ M\sslash\mu_K$ is a Poisson map with respect to the $I_S$-holomorphic

complex-symplectic form $ω_{I_S} + iω_{K_S}$ on $S$.

(iv) Local Model. Let $x ∈ M\sslash\mu_K$. Take a point $p ∈ µ^{-1}(0)$ above $x$, let $H = G_p$, let $V = (T_p(G \cdot p))^{\ast}/T_p(G \cdot p)$, and let $Φ_V : V → h^{\ast}$ be the canonical complex-symplectic moment map for the action of $H$ on $V$, i.e. $Φ_V(v)(X) = \frac{1}{2}ω_C(Xv, v)$. Then, $H$ is a complex reductive group and $x$ has a neighbourhood biholomorphic with respect to $O_I$ to a neighbourhood of 0 in the affine

GIT quotient $Φ_V^{-1}(0)//H = Spec C[Φ_V^{-1}(0)]^H$. Moreover, this biholomorphism respects the natural

partitions and holomorphic Poisson brackets on both sides.

Remark 1.5. Using the Kempf-Ness theorem, there are many situations where $M\sslash\mu_K$ is isomorphic

to a GIT quotient $µ_C^{-1}(0)//L G$ for some linearisation $L$, i.e. when $µ_C^{-1}(0)_{R^{ss}}$ coincides with the set of

$L$-semistable points. In that case, the sheaf $O_I$ is simply the underlying complex analytic structure (see Examples 2.11).

Remark 1.6. In [29], the author has studied in detail a specific family of singular hyperkähler quotients

whose orbit type partitions can be described explicitly. In this case, we have shown directly that the orbit
type partitions are stratifications (but in a weaker sense). Thus, Theorem 1.2 generalizes some of the results

of [29].

The paper is organized as follows. In §2 we introduce the necessary background on stratified spaces

and on the links between symplectic reduction and quotients of complex spaces. In §3 we prove the local

normal form Theorem 1.3 and in §4 we prove Theorem 1.4 about the local complex-symplectic structure

of singular hyperkähler quotients.

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2. Preliminaries

This section gives background material on stratified spaces, symplectic reduction, quotients of complex

analytic spaces, and the links between these notions. We start with a review of the theory of stratified

spaces and explain the work of Sjamaar–Lerman [33] on the stratification of singular symplectic reductions.

We then discuss links with complex geometry and also recall the construction of the hyperkähler structures

on the orbit type pieces of a singular hyperkähler quotient.

2.1. Stratified spaces. The idea behind stratified spaces is to describe singular topological spaces by

decomposing them into manifolds which “fit together nicely”. The underlying object for this theory is

thus the following:

Definition 2.1. A partitioned space is a pair $(X, P)$ where $X$ is a topological space and $P$ a partition

of $X$, i.e. a collection of non-empty disjoint subsets of $X$ whose union is $X$. The elements of $P$ are called

the pieces. An isomorphism between two partitioned spaces $(X, P)$ and $(Y, Q)$ is a homeomorphism

$f : X → Y$ which maps each piece of $X$ bijectively to a piece of $Y$.

Just like manifolds are topological spaces satisfying additional conditions (second countable, Hausdorff,

and locally Euclidean), stratified spaces are partitioned spaces with additional conditions imposed. The

first step is the following notion.
Definition 2.2 ([7, §1.1]). A decomposed space is a partitioned space \((X, P)\) such that \(X\) is a second countable Hausdorff space and the following conditions hold:

- **Manifold condition.** Each element of \(P\) is a topological manifold in the subspace topology.
- **Local condition.** \(P\) is locally finite and its elements are locally closed.
- **Frontier condition.** For all \(S, T \in P\) we have \(S \cap T \neq \emptyset \implies S \subseteq T\).

In that case, we say that \(P\) is a decomposition of \(X\).

Remark 2.3. If \((X, P)\) is a decomposed space, then there is a natural relation on \(P\) given by \(S \leq T\) if \(S \subseteq T\). It follows from the local closedness of the strata that this relation is a partial order. Moreover, the frontier condition is equivalent to

\[
\overline{S} = \bigcup_{T \leq S} T, \quad \text{for all } S \in P.
\]

This notion is sometimes incorporated in the definition of decomposed space, namely we fix a poset \(I\) and say that an \(I\)-decomposed space is a stratified space \((X, P)\) with an isomorphism \(P \cong I\) of posets.

This definition captures the intuitive idea of a space decomposed into manifolds, but it does not tell us how the pieces fit together. For example, the topologist’s sine curve

with two strata (the vertical segment on the left and the curve on the right) is a perfectly valid decomposed space. Roughly speaking, stratified spaces avoid such pathologies by requiring that every point has a neighbourhood which retracts continuously onto it. We also impose that this neighbourhood is compatible with the partition in some sense. To make this precise, we need a few extra notions. First, the dimension of a decomposed space \((X, P)\) is

\[
\dim(X, P) := \sup \{ \dim S : S \in P \}.
\]

Given two partitioned spaces \((X, P)\) and \((Y, Q)\), their cartesian product is the partitioned space \((X \times Y, P \times Q)\) where \(P \times Q = \{ S \times T : S \in P, T \in Q \}\). If \((X, P)\) and \((Y, Q)\) are decomposed spaces, then so is \((X \times Y, P \times Q)\), and \(\dim(X \times Y, P \times Q) = \dim(X, P) + \dim(Y, Q)\). Next, the cone over a partitioned space \((X, P)\) is the partitioned space \((CX, CP)\) where \(CX\) is the open cone over \(X\), i.e.

\[
CX := (X \times [0, \infty))/(\{(p, 0) \sim (q, 0)\}, \text{for all } p, q \in X)
\]

and \(CP\) is the natural partition of \(CX\) given by

\[
CP := \{ S \times (0, \infty) : S \in P \} \cup \{ \text{vertex} \}.
\]

The cone over a decomposed space \((X, P)\) is itself a decomposed space and has dimension \(\dim(CX, CP) = \dim(X, P) + 1\). A stratified space is defined inductively as a decomposed space \((X, P)\) which is locally isomorphic to \(\mathbb{R}^n\) times a cone over a lower-dimensional stratified space:

**Definition 2.4 ([7, 33]).** A zero-dimensional stratified space is any countable set of points with the discrete topology and with any partition. A stratified space is a finite-dimensional decomposed space \((X, P)\) such that every point \(p \in X\) has a neighbourhood isomorphic as a partitioned space to \(\mathbb{R}^n \times CL\) for some \(n \geq 0\) and some compact stratified space \(L\), by a map sending \(p \mapsto \{0\} \times \{\text{vertex}\}\). In that case, we say that \(P\) is a stratification of \(X\).

For example, one-dimensional stratified spaces are locally modelled on cones over finite sets of points, which means that they are the same thing as graphs:

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 one-dimensional local models
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Then, two-dimensional stratified spaces are locally modelled on cones over graphs, etc. Also, all manifolds with corners are stratified spaces.

The compact stratified space \(L\) associated to a point \(p\) in Definition 2.4 is called the link at \(p\) and is unique up to homeomorphisms. Moreover, for a connected stratum \(S \in P\), every point of \(S\) has the same link, so we may speak of the link of the stratum. This is the closest notion of “locally Euclidean” that
we can get for partitioned spaces, namely, the local structure along a stratum is constant. Note that for any link \( L \), the space \( \mathbb{R}^n \times CL \) is contractible. In particular, the topologist’s sine curve above is not a stratified space.

A typical way of proving that a decomposed space \( (X, \mathcal{P}) \) is a stratified space is by the Whitney conditions [36].

**Definition 2.5.** Let \( S \) and \( T \) be two disjoint smooth submanifolds of \( \mathbb{R}^n \). We say that \( S \) is regular over \( T \) if the following two conditions hold for all \( y \in S \cap T \):

- **Whitney Condition A.** If \( x_i \in S \) is a sequence converging to \( y \) and the sequence of subspaces \( T_{x_i}S \subseteq \mathbb{R}^n \) converges (in the Grassmannian) to some \( V \subseteq \mathbb{R}^n \), then \( T_yT \subseteq V \).
- **Whitney Condition B.** If \( x_i \in S \) and \( y_i \in T \) are two sequences converging to \( y \) in such a way that the sequence of lines \( \mathbb{R}(x_i - y_i) \subseteq \mathbb{R}^n \) converges to some \( l \in \mathbb{R}^{n-1} \) and the subspaces \( T_{x_i}S \) to some \( V \subseteq \mathbb{R}^n \), then \( l \subseteq V \).

A Whitney stratification of a subset \( X \) of \( \mathbb{R}^n \) is a decomposition \( \mathcal{P} \) of \( X \) into smooth submanifolds of \( \mathbb{R}^n \) such that \( S \) is regular over \( T \) for all \( S, T \in \mathcal{P} \).

We have (see e.g. Goresky–MacPherson [7, Ch. 1, §1.4] or Mather [28]):

**Proposition 2.6.** Whitney stratifications are stratifications in the sense of Definition 2.4. \( \square \)

Although Whitney stratifications are initially defined in \( \mathbb{R}^n \), the definition is purely local and is invariant under diffeomorphisms [28, §2]. In particular, it makes sense for complex spaces:

**Definition 2.7.** A complex Whitney stratified space is a complex space \((X, \mathcal{O}_X)\) together with a decomposition \( \mathcal{P} \) of \( X \) into complex submanifolds satisfying Whitney conditions A and B.

In particular, complex Whitney stratified spaces are also stratified spaces as in Definition 2.4.

### 2.2. Smooth manifold quotients.

Let \( K \) be a Lie group acting smoothly and properly on a smooth manifold \( M \). Then, the quotient space \( M/K \) is a stratified space with respect to a natural partition by orbit types. To define this partition, for each subgroup \( H \subseteq K \), let \((H)\) be the conjugacy class of \( H \) in \( K \). We say that \( p \in M \) has orbit type \((H)\) if its stabilizer subgroup \( K_p \) is in \((H)\). Denote the set of points of orbit type \((H)\) by

\[
M(H) := \{ p \in M : K_p \in (H) \}.
\]

Then, the connected components of the sets \( M(H)/K \) for \( H \subseteq K \) form a stratification of \( M/K \). The proof is an application of the slice theorem for proper group actions which gives a local model for the \( K \)-manifold \( M \) near a point \( p \in M \) in terms of \( K, K_p \) and \( T_pM/T_pK \cdot p \) (see e.g. [6, Theorem 2.7.4]).

### 2.3. Stratified symplectic spaces.

Another important source of stratified spaces is given by symplectic reduction, as shown by Sjamaar–Lerman [33]. We say that a Hamiltonian manifold is a triple \((M, K, \mu)\) where \( M \) is a symplectic manifold, \( K \) a compact Lie group acting on \( M \) by symplectomorphisms, and \( \mu : M \to \frak{t}^* \) a \((K\text{-equivariant})\) moment map. Sjamaar–Lerman generalized the Marsden–Weinstein theorem [27] by showing that the symplectic reduction \( M//\mu K := \mu^{-1}(0)/K \) has a natural stratification into symplectic manifolds. The strata are the connected components of the spaces \( \mu^{-1}(0)/(\mu(H)/K) \) for closed subgroups \( H \subseteq K \).

The symplectic forms on the strata can be seen as follows (see [33, Theorem 3.5]). For a closed subgroup \( H \subseteq K \), let \( M_H \) be the set of points \( p \in M \) whose stabilizer is precisely \( H \). Then, the connected components of \( M_H \) are smooth symplectic submanifolds of \( M \) (of possibly different dimensions) and the group \( L := N_K(H)/H \) (where \( N_K(H) \) is the normalizer of \( H \) in \( K \)) is compact and acts freely on \( M_H \) by preserving the symplectic forms. Now, \( \frak{t}^* := \text{Lie}(L)^* \) can be identified with a subspace of \( \frak{t}^* \), namely, \( \frak{h}^* \cap (\frak{t}^*)^H \), where \( \frak{h}^* \) is the annihilator of \( \frak{h} := \text{Lie}(H) \) and \( (\frak{t}^*)^H \) is the set of points fixed by \( H \). Moreover, if \( M^H_H \) denotes the union of the connected components of \( M_H \) which intersect \( \mu^{-1}(0) \), then \( \mu \) restricts to a moment map \( \mu_H : M^H_H \to \frak{t}^* \) for the action of \( L \) on \( M^H_H \). Since this action is free, each connected component of \( M^H_H//_{\mu_H} L = \mu_H^{-1}(0)/L \) is a smooth symplectic manifold by the standard Marsden–Weinstein theorem. Then, the inclusion \( \mu_H^{-1}(0) \subseteq \mu^{-1}(0)/(\mu(H)/K) \) descends to a homeomorphism \( M^H_H//_{\mu_H} L \cong \mu^{-1}(0)/(\mu(H)/K) \), and this endows each connected component of \( \mu^{-1}(0)/(\mu(H)/K) \) with a symplectic structure. Furthermore, the pullback of each symplectic form to the corresponding connected component of \( \mu^{-1}(0)/(\mu(H)/K) \) (which is a smooth submanifold of \( M \)) is the restriction of the symplectic form of \( M \).

The symplectic structures on the strata of \( M//\mu K \) can also be viewed more globally as a Poisson structure on \( M//\mu K \). Let \( C^\infty(M//\mu K) \) be \( \mathbb{R} \)-algebra of continuous functions on \( M//\mu K \) which descend from smooth \( K \)-invariant functions on \( M \). Then, there is a natural Poisson bracket on \( C^\infty(M//\mu K) \) such
that the inclusion of each stratum into $M//_\mu K$ is a Poisson map. This motivated Sjamaar–Lerman to make the following definition.

**Definition 2.8.** A stratified symplectic space is a stratified space $(X,\mathcal{P})$ with a smooth symplectic structure on each stratum, a subalgebra $C^\infty(X)$ of the $\mathbb{R}$-algebra of continuous functions on $X$, and a Poisson bracket on $C^\infty(X)$ such that for each stratum $S \in \mathcal{P}$ the embedding $S \to X$ is a Poisson map, i.e. for all $f,g \in C^\infty(X)$ the restrictions $f|_S, g|_S$ are smooth and $\{f|_S, g|_S\} = \{f, g\}|_S$.

**Theorem 2.9** (Sjamaar–Lerman [33]). For every Hamiltonian manifold $(M, K, \mu)$, the quotient $M//_\mu K$ is a stratified symplectic space.

In fact, they showed the stronger statement that $M//_\mu K$ has an embedding in $\mathbb{R}^n$ such that the orbit type partition is a Whitney stratification and used Proposition 2.6 to deduce that $M//_\mu K$ is a stratified space.

Just as for quotients of smooth manifolds (§2.2), the proof is obtained by an appropriate local model. This time, it is the local normal form for the moment map of Guillemin–Sternberg [9] and Marle [26], which is a generalization of the Darboux theorem to Hamiltonian manifolds. In the next chapter, we will adapt Sjamaar–Lerman’s argument to the hyperkähler setting by proving a holomorphic version of this normal form.

Recall that the Darboux theorem can be interpreted as saying that every point $p$ in a symplectic manifold $(M,\omega)$ has a neighbourhood symplectomorphic to a neighbourhood of $0$ in the symplectic vector space $V = T_p M$, i.e. symplectic forms can be linearised and $V$ is the local model. Similarly, the local normal form for the moment map says that a Hamiltonian manifold $(M, K, \mu)$ is completely determined in a neighbourhood of a point $p \in \mu^{-1}(0)$ by the representation of $H = K_p$ on the symplectic slice $V := (T_p(K \cdot p))^{\omega}/\Pi_p(K \cdot p)$ (where $\Pi^\omega$ the symplectic complement). In this case, the local model is the associated vector bundle $K \times_H (\mathfrak{h}^0 \times V)$ over $K/H = K \cdot p$.

This space is homeomorphic to a symplectic reduction of $T^*K \times V$ by $H$ and hence has a canonical symplectic form. Moreover, the left $K$-action $k \cdot [g,\xi,v] = [kg,\xi,v]$ is Hamiltonian and there is an explicit expression for the moment map. One shows that a neighbourhood of $K \cdot p$ in $M$ is isomorphic as a Hamiltonian $K$-manifold to a neighbourhood of the zero section in $K \times_H (\mathfrak{h}^0 \times V)$. Setting $K = 1$ recovers the Darboux theorem. Sjamaar–Lerman used this to prove Theorem 2.9 by reducing to the case of the Hamiltonian manifold $K \times_H (\mathfrak{h}^0 \times V)$ near the zero section.

Our approach for the hyperkähler case will be similar, using a version of the local normal form which describes the underlying complex-Hamiltonian structure of a tri-Hamiltonian hyperkähler manifold.

### 2.4. Kähler quotients

A Hamiltonian Kähler manifold is a Hamiltonian manifold $(M, K, \mu)$ with a $K$-invariant Kähler structure compatible with its symplectic form. If the $K$-action is free, it is a standard result that $M//_\mu K$ has a Kähler structure compatible with the reduced symplectic form (e.g. [20, Theorem 3.1]). More generally, when the action is not necessarily free, each symplectic stratum in Sjamaar–Lerman’s stratification is Kähler. To see this, it suffices to note that for each closed subgroup $H \subseteq K$, the space $M_H$ of points with stabilizer $H$ is now a complex submanifold of $M$ and hence is Kähler.

Thus, the connected components of $M//_\mu L$ (where $\mu_H$ and $L$ are as in §2.3) are Kähler manifolds, and the homeomorphism $M//_\mu L \cong \mu^{-1}(0)(/H)/K$ gives the desired Kähler structures.

But we can say much more about the holomorphic aspect of $M//_\mu K$ if we assume that the action of $K$ extends to a holomorphic action of the complexification $G := K\mathbb{C}$. In that case, we say that the action is integrable and call $(M, K, \mu)$ an **integrable Hamiltonian Kähler manifold**. This terminology comes from the fact the action is integrable if and only if for all $X \in \text{Lie}(K)$, the vector field $IX$ is complete, where $I$ is the complex structure on $M$ and $X^\#$ is the vector field generated by $X$. This holds, for example, if $M$ is compact (since all vector fields are complete). Also, it holds if $M$ is a smooth complex affine variety whose underlying complex structure on $M$ and $X^\#$ the vector field generated by $X$. This holds, for example, if $M$ is a compact complex manifold.

Indeed, the $K$-orbit of every function in $\mathbb{C}[M]$ is contained in a finite-dimensional vector space, so we can embed $M$ as a $K$-invariant subvariety of a finite-dimensional complex representation of $K$ and then the extension to a $K\mathbb{C}$-action follows from the universality property of complexifications (see e.g. [14, p. 226]).

We will recall below how this integrability assumption implies that $M//_\mu K$ is homeomorphic to a categorical quotient of complex spaces $M^\mu_{ss}/K\mathbb{C}$ where $M^\mu_{ss}$ is an open subset of $M$. This quotient is more precisely an analytic Hilbert quotients, which is the complex analogue of Geometric Invariant Theory (GIT) quotients in algebraic geometry. Good expositions can be found in Heinzner–Huckleberry [13, 14] or Greb [8, §2–3]; we summarize the main points in this section. See also [12, 17, 15, 16].

#### 2.4.1. Analytic Hilbert quotients
Definition 2.10. Let $(X, \mathcal{O}_X)$ be a complex space and $G$ a complex reductive group acting holomorphically on $X$. An analytic Hilbert quotient of $X$ by $G$ is a complex space $(Y, \mathcal{O}_Y)$ together with a $G$-invariant surjective holomorphic map $\pi: X \to Y$ such that:

(i) the map $\pi: X \to Y$ is locally Stein, i.e. $Y$ has a cover by Stein open sets whose preimages are Stein;

(ii) $\mathcal{O}_Y = (\pi_*\mathcal{O}_X)^G$.

An important consequence of this definition is that, if it exists, an analytic Hilbert quotient is a categorical quotient for complex spaces. In particular, it is unique up to biholomorphisms. We denote it $X//G :=$ the analytic Hilbert quotient of $X$ by $G$ (if it exists).

Topologically, $X//G$ is the quotient of $X$ by the equivalence relation $x \sim y$ if $G \cdot x \cap G \cdot y \neq \emptyset$ and $\pi: X \to X//G$ is the corresponding quotient map. The space $X//G$ can also be viewed as the set of closed $G$-orbits, i.e. by defining the set of polystable points

$$X^{ps} := \{ x \in X : \text{the orbit } G \cdot x \text{ is closed in } X \}$$

the inclusion $X^{ps} \subseteq X$ descends to a bijection $X^{ps}/G \to X//G$. In particular, for every $p \in X//G$, there is a unique closed $G$-orbit in the fibre $\pi^{-1}(p) \subseteq X$.

Example 2.11. An important class of examples of analytic Hilbert quotients are the GIT quotients. Let $X$ be a complex affine variety, $G$ a complex reductive group acting algebraically on $X$, and consider the affine GIT quotient $X//G := \text{Spec } \mathbb{C}[X]^G$ together with the morphism $X \to X//G$ induced by the inclusion $\mathbb{C}[X]^G \to \mathbb{C}[X]$. Then, the analytification of $X \to X//G$ is an analytic Hilbert quotient [11, §6.4]. More generally, since complex affine varieties are Stein spaces, this shows that the analytification of any GIT quotient is an analytic Hilbert quotient.

Two other properties of analytic Hilbert quotients that we will use later are as follows.

Proposition 2.12. Let $\pi: X \to X//G$ be an analytic Hilbert quotient.

(i) An open set $U \subseteq X$ is $G$-saturated if and only if it is saturated with respect to $\pi$. In that case, $U//G := \pi(U)$ is open in $X//G$ and the restriction $U \to U//G$ is an analytic Hilbert quotient.

(ii) If $Y \subseteq X$ is a $G$-invariant closed complex subspace, then $Y//G := \pi(Y)$ is a closed complex subspace of $X//G$ and the restriction $Y \to Y//G$ is an analytic Hilbert quotient. \[\square\]

For (i) see [17, §2 Remark and §1 Corollary] and for (ii) see [17, §1(iii)].

2.4.2. The Heinzner–Loose theorem. Just as for GIT quotients, the question of existence of analytic Hilbert quotients is a subtle one. In complete analogy with GIT, for an action of a complex reductive group $G$ on a complex space $X$, there does not always exist an analytic Hilbert quotient, but in good cases, one can find a large open subset of $X$ on which the quotient exists. For GIT, this set depends on a choice of a linearisation, and for analytic Hilbert quotients, it depends on a choice of a moment map for the action of a maximal compact subgroup $K \subseteq G$, as we now explain.

Let $(M, K, \mu)$ be an integrable Hamiltonian Kähler manifold and let $G = K_C$. Define the set of $\mu$-semistable points by

$$M^{\mu-ss} := \{ p \in M : \overline{G \cdot p \cap \mu^{-1}(0)} \neq \emptyset \}$$

and the set of $\mu$-polystable points by

$$M^{\mu-ps} := (M^{\mu-ss})^{ps} = \{ p \in M : G \cdot p \text{ is closed in } M^{\mu-ss} \}.$$

Theorem 2.13 (Heinzner–Loose [16]). The set $M^{\mu-ss}$ is open in $M$ and the analytic Hilbert quotient $M^{\mu-ss}//G$ exists. We have

$$p \in M^{\mu-ps} \iff G \cdot p \cap \mu^{-1}(0) \neq \emptyset.$$  

Moreover, the inclusion $\mu^{-1}(0) \hookrightarrow M^{\mu-ss}$ descends to a homeomorphism $M//\mu K \to M^{\mu-ss}//G$. Also, for every $p \in M^{\mu-ps}$ we have $G_p = (K_p)_C$, so $G_p$ is a complex reductive group. \[\square\]

Remark 2.14.

(1) Special cases of Theorem 2.13 were known long before [16]. See, for example, Guillemin–Sternberg [10, §4] and Kirwan [22, §7.5]. It was also obtained independently by Sjamaar [32] under an additional assumption on the moment map. This result can be thought of as an “analytic” version of the Kempf–Ness theorem.
Heinzner–Loose [16] do not mention analytic Hilbert quotients directly, but the above theorem can be deduced from their proofs. The reformulation which we gave can be found in Heinzner–Huckleberry [12, §0]. To translate from [16] and [12, §0] to Theorem 2.13, note the following: the statement that the analytic Hilbert quotient $M^\mu_{ss}//G$ exists is [12, §0(i)]; the equivalence (2.1) is in [16, (1.2)(a) and (1.3)]; the homeomorphism $M//\mu K \to M^\mu_{ss}//G$ is [12, §0(iv)] or [16, (1.3)]; by [12, §0(iii)] or [16, (2.2) and (2.7)] we have that $G_p = (K_p)_\mu$ for all $p \in \mu^{-1}(0)$ and hence also for all $p \in M^\mu_{ps}$ since $M^\mu_{ps} = G \cdot \mu^{-1}(0)$ by (2.1).

The main ingredient in the proof of Heinzner–Loose’s theorem is the Holomorphic Slice Theorem. We briefly review it here, since we will use it later. If $H$ is a complex Lie subgroup of a complex Lie group $G$ and $S$ is a complex $H$-manifold, we denote by $G \times_H S$ the quotient of $G \times S$ by the $H$-action $h \cdot (g,x) = (gh^{-1}, h \cdot x)$. Since the $H$-action is free and proper, there is a unique complex manifold structure on $G \times_H S$ such that $G \times S \to G \times_H S$ is a holomorphic submersion.

Definition 2.15. Let $G$ be a complex reductive group acting holomorphically on a complex manifold $M$. A slice at a point $p$ in $M$ is a $G_p$-invariant complex submanifold $S \subseteq M$ containing $p$ such that $G \cdot S$ is open in $M$ and the map

$$G \times_{G_p} S \to G \cdot S, \quad [g,x] \mapsto g \cdot x$$

is a $G$-equivariant biholomorphism.

Theorem 2.16 (Holomorphic Slice Theorem [16, §2.7] [32, Theorem 1.12]). Let $(M, K, \mu)$ be an integrable Hamiltonian Kähler manifold. Then, there exists a slice at every point of $M^\mu_{ps}$. \hfill \Box

Remark 2.17. In [16], this is stated only for points $p \in M$ such that $\mu(p)$ is fixed by the coadjoint action, but since $M^\mu_{ps} = G \cdot \mu^{-1}(0)$ we deduce the above version.

2.4.3. Stratification of analytic Hilbert quotients. Let $\pi : X \to X//G$ be an analytic Hilbert quotient (e.g. a GIT quotient). Then, as in §2.2, the orbit space $X^G//G$ has a natural partition by $G$-orbit types, i.e. the pieces are the connected components of the sets $(X^G)^H//G$ for $H \subseteq G$. Then, the bijection $X^G//G \to X//G$ defines a natural partition on $X//G$ which we call the $G$-orbit type partition. Equivalently, the orbit type of a point $p \in X//G$ is defined to be the orbit type of the unique closed orbit in $\pi^{-1}(p)$.

If $(M, K, \mu)$ is a Hamiltonian Kähler manifold, then $M^\mu_{//} K \cong M^\mu_{ss}//G$ is an analytic Hilbert quotient and hence has a $G$-orbit type partition. But it also has the $K$-orbit type partition of Sjamaar–Lerman. Moreover, each stratum in the $K$-orbit type partition is a Kähler manifold, and hence has a complex structure. The next result shows that these partitions and complex structures are the same.

Theorem 2.18 (Sjamaar [32, Theorem 2.10]).

(i) The homeomorphism $M^\mu_{//} K \to M^\mu_{ss}//G$ is an isomorphism of partitioned spaces.

(ii) The $G$-orbit type strata of $M^\mu_{ss}//G$ are complex submanifolds.

(iii) Let $S$ be a $K$-orbit type stratum in $M^\mu_{//} K$ and $S'$ the corresponding $G$-orbit type stratum in $M^\mu_{ss}//G$. Then, the restriction $S \to S'$ is a biholomorphism with respect to Kähler structure on $S$ and the complex structure on $S'$ obtained from (ii). \hfill \Box

Remark 2.19. Point (iii) is not stated in this way in [32, Theorem 2.10], but is nonetheless part of the proof.

2.5. Hyperkähler quotients. Let $(M, K, \mu)$ be a tri-Hamiltonian hyperkähler manifold (using the terminology of the introduction). We recall, following Dancer–Swann [5], the construction of a hyperkähler structure on each piece of the orbit type partition of $M^\mu_{//} K$. We also explain how to get the refinement stated in Theorem 1.1 which characterizes these structures uniquely.

The proof is very similar to the construction of the Kähler structures on the orbit type strata of a Kähler quotient as explained in §2.4. For the purpose of this section, it will be convenient to slightly relax the definition of a manifold so that different connected components can have different dimensions. Also, a smooth submanifold will always mean a smooth embedded submanifold.

Let $S \subseteq M^\mu_{//} K$ be an orbit type piece. Then, $S$ is a connected component of a set of the form $\mu^{-1}(0)^H//K$ for some closed subgroup $H \subseteq K$. The set $M_H$ of points with stabilizer $H$ is now a hyperkähler submanifold of $M$ and $\mu$ restricts to a hyperkähler moment map $\mu_H : M'_{H} \to \mathfrak{t}^* \otimes \mathbb{R}^3 \subseteq \mathfrak{t}^* \otimes \mathbb{R}^3$ for the free action of $L := N_K(H)/H$ on the union $M_H$ of the connected components of $M_H$ intersecting $\mu^{-1}(0)$. Hence, the connected components of $M^\mu_{//} K$ are hyperkähler manifolds by the usual hyperkähler quotient construction [20, Theorem 3.2]. Moreover, the inclusion $\mu_H^{-1}(0) \subseteq \mu^{-1}(0)^H//K$ descends to a homeomorphism $M^\mu_{//} K \to M^\mu_{ss}//G$ and hence endows each connected component
of $\mu^{-1}(0)/K$ with a hyperkähler structure. To show that this map is indeed a homeomorphism and also to characterize the hyperkähler structure as in Theorem 1.1, we will need the following lemma. This result is implicit in Sjamaar-Lerman [33], but we give a short proof for completeness.

**Lemma 2.20.** Let $K$ be a compact Lie group acting smoothly on a smooth manifold $M$, let $H$ be a closed subgroup of $K$, and let $L = N_K(H)/H$. Then, $M_H$ and $M_{(H)/K}$ are smooth submanifolds of $M$ and the quotients $M_H/L$ and $M_{(H)/K}$ are topological manifolds with unique smooth structures such that the quotients maps $M_H \to M_{H}/L$ and $M_{(H)/K} \to M_{(H)/K}$ are smooth submersions. Moreover, the inclusion $M_H \hookrightarrow M_{(H)/K}$ descends to a diffeomorphism $M_H/L \to M_{(H)/K}$.

**Proof.** This follows easily from the slice theorem for proper group actions. The map $M_H/L \to M_{(H)/K}$ is clearly bijective, so everything reduces to local statements and hence we may assume (by the slice theorem) that $M = K \times_H W$ for some representation $W$ of $H$. Then, $M_H = L \times W_H$, $M_{(H)/K} = K \times_H W_H$, and $W_H$ is a linear subspace of $W$, so $M_H$ and $M_{(H)/K}$ are smooth submanifolds of $M$. Moreover, $M_H/L = W_H$ and the quotient map $M_H \to M_{H}/L$ is the projection $L \times W_H \to W_H$ and hence is a smooth submersion. Similarly, the quotient map $M_{(H)/K} \to M_{(H)/K}$ is the projection $K/H \times W_H \to W_H$. Under these identifications, the map $M_H/L \to M_{(H)/K}$ is the identity map $W_H \to W_H$. \hfill $\square$

Let

$$\pi : \mu^{-1}(0) \longrightarrow M/\mu K$$

be the quotient map and let $S \subseteq M/\mu K$ be an orbit type piece as above.

**Proposition 2.21.** The space $S$ is a topological manifold, $\pi^{-1}(S)$ is a smooth submanifold of $M$ (of pure dimension), there is a unique smooth structure on $S$ such that $\pi^{-1}(S) \to S$ is a smooth submersion, and there is a unique hyperkähler structure $(g_S, I_S, J_S, K_S)$ on $S$ such that the pullbacks of the Kähler forms $\omega_S, \omega_{J_S}, \omega_{K_S}$ to $\pi^{-1}(S)$ are the restrictions of $\omega, \omega_I, \omega_J, \omega_K$.

**Proof.** Let $Z = \mu^{-1}(0)$ so that $S$ is a connected component of $Z_{(H)}/K$ for some $H \subseteq K$. As explained above, $Z_H$ is a smooth submanifold of $M_H$ and $Z_H/L$ is a hyperkähler manifold where $L = N_K(H)/K$.

Now, $Z_H/L$ is a smooth submanifold of $M_H/L$ and its image under the diffeomorphism $M_H/L \to M_{(H)/K}$ is $Z_{(H)}/K$, so the latter is also a smooth submanifold. Recall that if $f : X \to Y$ is a smooth submersion between smooth manifolds and $Y' \subseteq Y$ is a smooth submanifold, then $f^{-1}(Y')$ is a smooth submanifold and the restriction $f^{-1}(Y') \to Y'$ is a smooth submersion (this follows easily from the rank theorem). Thus, $Z_{(H)}$ is a smooth submanifold of $M_{(H)}$ and $Z_{(H)} \to Z_{(H)}/K$ is a smooth submersion. Note that $\pi^{-1}(S)$ is open in $Z_{(H)}$, so it is also a smooth submanifold and the restriction $\pi^{-1}(S) \to S$ is a smooth submersion. Moreover, $\pi^{-1}(S)$ has pure dimension since $S$ is connected and all fibres are diffeomorphic to $K/H$.

To prove the claim about the hyperkähler structure, let $\eta, \eta_I, \eta_J, \eta_K$ be the Kähler forms on $Z_{(H)}/K$ induced by the diffeomorphism $Z_H/L \to Z_{(H)}/K$ and consider the commutative diagram

$$\begin{array}{ccc}
Z_H & \xrightarrow{i} & Z_{(H)} \\
\rho \downarrow & & \downarrow j \\
Z_H/L & \xrightarrow{\varphi} & Z_{(H)}/K.
\end{array}$$

We want to show that $\pi^*\eta = j^*\omega_I$ and similarly for $J$ and $K$. By construction of the hyperkähler structure on $Z_H/L$ we have $\rho^*\varphi^*\eta = (ji)^*\omega_I$ and hence $i^*(\pi^*\eta) = i^*(j^*\omega_I)$. Hence, $\pi^*\eta$ and $j^*\omega_I$ agree on $T_pZ_H$ for all $p \in Z_H$. Note that since $d\rho_p$ and $d\eta_p$ are surjective we have $T_pZ_{(H)} = T_pZ_H + \ker d\eta_p$. Thus, to prove that $\pi^*\eta$ and $j^*\omega_I$ agree on $T_pZ_{(H)}$ it suffices to show that if $u \in \ker d\eta_p$ and $v \in T_pZ_{(H)}$ then $\pi^*\eta(u, v) = j^*\omega_I(u, v)$. Clearly, $\pi^*\eta(u, v) = 0$ since $d\eta_p(u) = 0$. To show that also $j^*\omega_I(u, v) = 0$, note that $d\eta_p = T_p(K \cdot)$ so $u = X^\#$ for some $X \in k$ and hence $\eta_p(u, v) = X^\# \cdot \omega(u, v) = d\rho_p(u) = d\rho_p(v) = 0$ since $v \in T_pZ_{(H)} \subseteq \ker (d\rho_p)|_p$. Hence, $\pi^*\eta$ and $j^*\omega_I$ agree on $T_pZ_{(H)}$ for all $p \in Z_H$ and since they are $K$-invariant and $K : Z_H = Z_{(H)}$ we conclude that $\pi^*\eta = j^*\omega_I$. The same argument also shows that $\pi^*\eta_J = j^*\omega_J$ and $\pi^*\eta_K = j^*\omega_K$. Since a hyperkähler structure is completely determined by its three symplectic forms (e.g. $I = \omega_I^{-1}\omega_K$; see [19, bottom of p. 63]) this proves the proposition. \hfill $\square$

**3. A local normal form for the underlying complex-Hamiltonian manifold**

The goal of this section is to establish a local normal form for the underlying complex-Hamiltonian manifold of a tri-Hamiltonian hyperkähler manifold analogous to the local normal form of Guillemin–Sternberg [9] outlined in §2.3. It will be used in §4 to show that singular hyperkähler quotients are stratified spaces, in a proof similar to Sjamaar–Lerman’s one for symplectic reductions.
3.1. Statement of result. We first introduce some terminology. A complex-symplectic manifold is a complex manifold \((M,I)\) together with a non-degenerate holomorphic closed 2-form \(\omega_C\). A complex-Hamiltonian manifold is a complex-symplectic manifold \((M,\omega_C,\mu_C)\) together with a holomorphic action of a complex Lie group \(C\) preserving \(\omega_C\) and with a complex moment map, i.e. a \(G\)-equivariant holomorphic map \(\mu_C : M \to \mathfrak{g}^* = \operatorname{Lie}(C)^*\) such that \(d(\mu_C, X) = X^\# \omega_C\) for all \(X \in \mathfrak{g}\).

Now, let \((M,K,\mu)\) be a tri-Hamiltonian hyperkähler manifold with complex structures \(I,J,K\) and suppose that the \(K\)-action is \(I\)-integrable. Let \(G := K_C\) and let \(\omega_I,\omega_J,\omega_K\) be the three Kähler forms. Then, \(\omega_C := \omega_I + i\omega_K\) is a \(G\)-invariant complex-symplectic form on \((M,I)\) and \(\mu_C := \mu_I + i\mu_K : M \to \mathfrak{g}^*\) is a complex moment map for the action of \((M,I,\omega_C)\) (see [20, §3(D)]). Thus, \((M,I,\omega_C,G,\mu_C)\) is a complex-Hamiltonian manifold which we call the underlying complex-Hamiltonian manifold of \((M,K,\mu)\). Let \(\mu_B := \mu_I\) so that \((M,K,\mu_B)\) is a Hamiltonian Kähler manifold as in §2.4. In particular, we have the sets \(\mu_B^{-1}(0)\) and \(\mu_B^{-1}(0)\) as in §2.4.2. In what follows, we will use the notations

\[
\mu_C^{-1}(0)^{\mu_B-ss} := \mu_C^{-1}(0) \cap M^{\mu_B-ss} \quad \text{and} \quad \mu_C^{-1}(0)^{\mu_B-ps} := \mu_C^{-1}(0) \cap M^{\mu_B-ps}.
\]

In analogy with the local normal form in symplectic geometry, the idea is to show that in a neighbourhood of a point \(p \in \mu_C^{-1}(0)\), the underlying complex-Hamiltonian manifold \((M,I,\omega_C,G,\mu_C)\) is completely determined by the representation of \(H := G_p\) on the complex-symplectic slice

\[
V := (T_p(G \cdot p))^{\omega_C} / T_p(G \cdot p).
\]

By the Holomorphic Slice Theorem, the orbit \(G \cdot p\) is embedded in \(M\), so \(T_p(G \cdot p)\) is well-defined. Just as in the real case, the definition of a moment map implies that \(T_p(G \cdot p) \subseteq \ker(d\mu_C) = (T_p(G \cdot p))^{\omega_C}\) and hence \(V\) is a well-defined complex-symplectic vector space. We have \(H = G_p = (K_p)_C\) (by Theorem 2.13) so \(H\) is a complex reductive group acting linearly on \(V\) and preserving its complex-symplectic form. The goal is to construct a complex-Hamiltonian manifold \(E\) from \(G, H\) and \(V\) which is isomorphic to a neighbourhood of \(p\) in \((M,I,\omega_C,G,\mu_C)\). The construction of \(E\) is the same as the one used by Guillemin–Sternberg [9], but in a complex-symplectic setting; see also [33, §2].

We now build the local model \(E\). Let \(G, H\) and \(V\) be as above, so that \(G\) is a complex reductive group, \(H\) a reductive subgroup of \(G\), and \(V\) a complex-symplectic representation of \(H\). Since \(G\) is a complex manifold, the cotangent bundle \(T^*G\) has a canonical complex-symplectic form \(−d\alpha\), where \(\alpha\) is the tautological 1-form. We identify \(T^*G\) with \(G \times \mathfrak{g}^*\) via left translation, i.e. via the biholomorphism

\[
G \times \mathfrak{g}^* \to T^*G, \quad (g, \xi) \mapsto (dL_{g^{-1}})^*(\xi),
\]

where \(L_{g^{-1}} : G \to G\) is left multiplication by \(g^{-1}\). Recall that a Lie group action on any manifold lifts to a Hamiltonian action on the cotangent bundle. By considering the action of \(G \times G\) on \(V\) by left and right multiplications (i.e. \((a,b) \cdot g := agb^{-1}\)) its lift to \(T^*G = G \times \mathfrak{g}^*\) is

\[
(a,b) \cdot (g,\xi) = (agb^{-1}, Ad_g^\mathfrak{g} \xi),
\]

and the moment map is

\[
\mu : T^*G \to \mathfrak{g}^* \times \mathfrak{g}^*, \quad \mu(g,\xi) = (Ad_g^\mathfrak{g} \xi, -\xi)
\]

(see e.g. [1, §4.4]). The vector space \(V\) with its complex-symplectic form \(\omega_C : V \times V \to \mathbb{C}\) can also be viewed as a complex-Hamiltonian \(H\)-manifold with complex moment map

\[
\Phi_V : V \to \mathfrak{h}^*, \quad \Phi_V(v)(X) = \frac{1}{2} \omega_C(X,v,\cdot) \quad (v \in V, X \in \mathfrak{h}).
\]

Thus, there is a Hamiltonian action of \(H\) on \(T^*G \times V\), where \(H\) acts on \(T^*G\) as a subgroup of the right factor of \(G \times G\) and on \(V\) via the given representation. Let \(E\) be the complex-symplectic reduction of \(T^*G \times V\) by \(H\). Since the action of \(H\) on \(T^*G \times V\) is free and proper, \(E\) is a complex-symplectic manifold. Moreover, the Hamiltonian action of the left factor of \(G \times G\) on \(T^*G\) descends to a Hamiltonian action of \(G\) on \(E\), making \(E\) into a complex-Hamiltonian \(G\)-manifold.

We can also rewrite \(E\) in a more convenient form where the complex moment map for the \(G\)-action is explicit. First, note that the complex moment map for the \(H\)-action on \(T^*G \times V\) is

\[
\lambda : T^*G \times V \to \mathfrak{h}^*, \quad \lambda(g,\xi,v) = \Phi_V(v) - \xi|_\mathfrak{h}.
\]

Take a Hermitian inner-product on \(\mathfrak{g}\) invariant under the maximal compact subgroup \(K \subseteq G\) and let \(\mathfrak{m}\) be the orthogonal complement to \(\mathfrak{h}\) in \(\mathfrak{g}\). This defines an \(H\)-equivariant isomorphism \(\mathfrak{h}^* \cong \mathfrak{m}^0 \subseteq \mathfrak{g}^*\) so we can view \(\Phi_V\) as taking values in \(\mathfrak{g}^*\). Then, the map

\[
G \times \mathfrak{g}^* \times V \to \lambda^{-1}(0), \quad (g,\xi,v) \mapsto (g,\xi + \Phi_V(v), v)
\]
is a biholomorphism. The $H$-action on $\lambda^{-1}(0)$ corresponds to the $H$-action on $G \times \mathfrak{h}^0 \times V$ given by $h \cdot (g, \xi, v) = (gh^{-1}, \text{Ad}_h^* \xi, h \cdot v)$, so $E$ is the holomorphic vector bundle
\begin{equation} \label{eqn:moment_map} E = G \times_H (\mathfrak{h}^0 \times V) \end{equation}
onumber
over $G/H$. In this setup, the Hamiltonian $G$-action is
\begin{equation} \label{eqn:Hamiltonian_action} G \times E \to E, \quad a \cdot [g, \xi, v] = [ag, \xi, v] \end{equation}
and the complex moment map is
\begin{equation} \label{eqn:moment_map_complex} \nu_C : G \times_H (\mathfrak{h}^0 \times V) \to \mathfrak{g}^*, \quad [g, \xi, v] \mapsto \text{Ad}_h^* \xi + \Phi_V(v). \end{equation}

We summarize this discussion in the following proposition.

**Proposition 3.1.** Let $G$ be a complex reductive group, $H$ a reductive subgroup of $G$, and $V$ a complex-symplectic representation of $H$. Then, the complex-symplectic manifold 
\begin{equation} \label{eqn:moment_map} \text{of \ref{eqn:moment_map}} \end{equation}
and the complex moment map 
\begin{equation} \label{eqn:moment_map_complex} \text{is \ref{eqn:moment_map_complex}} \end{equation}
is a complex-Hamiltonian manifold.

**Remark 3.2.** Dancer–Swann [4] showed that $E$ is a tri-Hamiltonian hyperkähler manifold whose underlying complex-Hamiltonian manifold is the one described above.

The goal of this section is to prove the following result.

**Theorem 3.3.** Let $(M, K, \mu)$ be a tri-Hamiltonian hyperkähler manifold with complex structures $I, J, K$ and symplectic forms $\omega_I, \omega_J, \omega_K$. Suppose that the $K$-action is $1$-integrable and let $G = K_C$. Let $\mu_R = \mu_I$ and $\mu_C = \mu_J + i \mu_K$. For all $p \in \mu_C^{-1}(0)$-ss, there is a $G$-saturated neighbourhood $U$ of $G \cdot p$ in $M^{\mu_R-ss}$, a $G$-saturated neighbourhood $U'$ of the zero section in $E = G \times_H (\mathfrak{h}^0 \times V)$, and a $G$-equivariant isomorphism $f : U \to U'$ of complex-symplectic manifolds such that $f(p) = [1, 0, 0]$ and $\nu_C \circ f = \mu_C$.

**Remark 3.4.** We will see in the course of the proof that $U'$ can be chosen to be of the form $G \times_H (H \cdot B)$ where $B$ is an open ball around zero in $\mathfrak{h}^0 \times V$.

The structure of the proof is as follows. We first use the Holomorphic Slice Theorem to show that a neighbourhood of $p$ is biholomorphic to a neighbourhood of the zero section of $E$. We then use some basic results of complex-symplectic representations to construct a biholomorphism $E \to E$ which will make the complex-symplectic form $\omega_C$ from the hyperkähler structure match the canonical one $\eta_C$ on the zero section of $E$. Then, we use a holomorphic version of the Darboux–Weinstein theorem (which we prove in the next subsection) to deform $E$ further so that $\omega_C$ match with $\eta_C$ on a full neighbourhood of the zero section.

### 3.2. Holomorphic Darboux–Weinstein theorem

The Darboux–Weinstein theorem [35] is a standard result in symplectic geometry which says that if two symplectic forms $\omega_0$ and $\omega_1$ on a manifold $M$ agree on a submanifold $N \subseteq M$ then we can find a diffeomorphism $f$ on a neighbourhood of $N$ such that $f^* \omega_1 = \omega_0$. There is also an equivariant version of the theorem, where if $\omega_0, \omega_1$ and $N$ are invariant under the action of a compact Lie group, then $f$ can be taken to be equivariant. By the tubular neighbourhood theorem, it suffices to prove the result when $M$ is a vector bundle and $N$ the zero section, and this is indeed how Weinstein’s original proof [35] goes. In the holomorphic category, there is no tubular neighbourhood theorem, but we can still adapt Weinstein’s proof to formulate a similar result on holomorphic vector bundles.

**Theorem 3.5.** Let $G$ be a group acting on a holomorphic vector bundle $E$ by bundle automorphisms. Let $\omega_0$ and $\omega_1$ be two $G$-invariant complex-symplectic forms on a $G$-invariant neighbourhood $U$ of the zero section $Z \subseteq E$ such that $\omega_0|_Z = \omega_1|_Z$. Then, there are $G$-invariant neighborhoods $U_0$ and $U_1$ of $Z$ in $U$ and a $G$-equivariant biholomorphism $f : U_0 \to U_1$ such that $f^* \omega_1 = \omega_0$ and $f|_Z = \text{Id}_Z$.

**Remark 3.6.** Here $\omega_i|_Z$ is the restriction of $\omega_i$ to $(\mathcal{A}^k T^* E)|_Z$ (this is not the same as the pullback to $Z$).

The rest of this subsection is devoted to the proof of this theorem, which is an adaptation of Weinstein’s proof [35] to the holomorphic setting. Let us first briefly sketch how we will proceed. The first step is to get a “Poincaré lemma” for the retraction of $U$ onto $Z$, i.e. to construct an explicit homotopy operator $I : \Omega^k(U) \to \Omega^{k-1}(U)$ between the identity map and $\pi^*$, where $\pi : U \to U, v \mapsto 0 \cdot v$. Then, $\alpha = I(\omega_0 - \omega_1)$ is a $1$-form on $U$ and, for $t$ small enough, $\omega_t := \omega_0 + t(\omega_1 - \omega_0)$ is non-degenerate, so we get a time-dependent holomorphic vector field $X_t = \omega_t^{-1}(\alpha)$ on a neighbourhood of $Z$. The proof concludes by showing that the time-dependent flow of $X$ gives a biholomorphism with the desired properties.
Let us now construct the homotopy operator. Let \( \overline{B} \) be the closed unit disc centred at 0 in \( \mathbb{C} \) and let \( U \subseteq B \) as in Theorem 3.5. By shrinking \( U \) if necessary, we may assume that it is preserved by \( \overline{B} \), i.e. \( zu \in U \) for all \( z \in \overline{B} \) and \( u \in U \). Let \( \Omega^k(U) \) the space of holomorphic \( k \)-forms on \( U \) and let

\[
W := \{(z, u) \in \mathbb{C} \times U : zu \in U \}.
\]

Then, \( W \) is open in \( \mathbb{C} \times U \) and we have \( \overline{B} \times U \subseteq W \). Let \( \xi \) be the holomorphic vector field on \( W \) given by \( \xi_{(z, u)} = i_z \xi \) under the identification \( T_{(z, u)}W = T_z \mathbb{C} \times T_u U \). Let

\[
\lambda : W \to U, \quad (z, u) \mapsto zu
\]

be the scaling map, and for each \( z \in \overline{B} \) let

\[
i_z : U \to W, \quad p \mapsto (z, p).
\]

Then, for all \( \omega \in \Omega^k(U) \), we have a homomorphic family of \( k \)-forms

\[
W \to \Lambda^k T^*U, \quad (z, u) \mapsto (i_z^*(\xi \omega))(u).
\]

Hence,

\[
I \omega := \int_0^1 i_z^*(\xi \omega) \, dz
\]

is a homomorphic \((k - 1)\)-form on \( U \). Let \( \pi : U \to B \) be the projection onto the zero section, i.e. \( u \mapsto 0 \cdot u \).

**Proposition 3.7.** The map

\[
I : \Omega^k(U) \to \Omega^{k-1}(U)
\]

is a homotopy operator between the identity map and \( \pi^* \), i.e.

\[
d(I \omega) + I(d \omega) = \omega - \pi^* \omega
\]

for all \( \omega \in \Omega^k(U) \).

**Proof.** We have

\[
d(I \omega) + I(d \omega) = \int_0^1 i_z^*(d(\xi \omega)) + \int_0^1 i_z^*(\xi \omega) \, dz.
\]

Moreover, the flow \( \theta_t \) of the vector field \( \xi \) is \( \theta_t(z, u) = (z + tu, u) \) for all \( t \in [0, 1] \). Thus, \( i_z^*(\xi) = i_z \theta_t^* \xi \) and since \( \theta_t^* \xi = \frac{d}{dt} \theta_t^* \xi \) we get

\[
d(I \omega) + I(d \omega) = \int_0^1 i_0^* \frac{d}{dt} \theta_t^* \xi \, dt = i_0^* \xi - i_0^* \xi = \omega - \pi^* \omega.
\]

We will also need the following easy consequence of the definition of \( I \).

**Lemma 3.8.** Let \( \omega \in \Omega^k(U) \) and let \( p \in Z \). If \( \omega_p = 0 \) then \( (I \omega)_p = 0 \).

**Proof.** We have

\[
[i_0^* (\xi \omega)]_p(v) = [\xi \omega]_p(d_i(v)) = \omega_p(d \lambda_i(v)) = 0
\]

since \( tp = p \) is in the zero section. Thus, \( i_0^* (\xi \omega) = 0 \) for all \( t \in [0, 1] \) and hence \( (I \omega)_p = 0 \).

**Proof of Theorem 3.5.** Let \( \eta = \omega_1 - \omega_0 \) and let \( \alpha = -I \eta \), where \( I \) is the homotopy operator of Proposition 3.7. Then, \( \eta = -d \alpha \). Since \( \eta \) is \( G \)-invariant, it follows easily from the definition of \( I \) that \( \alpha \) is also \( G \)-invariant. Moreover, since \( \eta |_Z = 0 \) we have \( \omega |_Z = 0 \). By Lemma 3.8.

For each \( z \in \mathbb{C} \), define a \( G \)-invariant holomorphic 2-form on \( U \) by \( \omega_z = \omega_0 + z \eta \). We have \( \omega_z |_Z = \omega_0 |_Z \), so in particular, \( \omega_z |_p \) is non-degenerate for all \( (z, p) \in \mathbb{C} \times Z \). Let \( \overline{B} \) be the open disc of radius \( r \) centred at 0 in \( \mathbb{C} \). By compactness of \( \overline{B} \), we can find a neighbourhood \( U' \subseteq U \) of \( Z \) such that \( \omega_z |_p \) is non-degenerate for all \( (z, p) \in \overline{B} \times U' \). Moreover, by \( G \)-invariance of \( \omega_z \), we can take \( U' \) to be \( G \)-invariant. Thus, we may assume that \( \omega_z |_p \) is non-degenerate for all \( (z, p) \in \overline{B} \times U \).

In particular, the maps

\[
\omega_z : TU \to T^*U, \quad v \mapsto \omega_z(v, \cdot)
\]

are vector bundle isomorphisms for all \( z \in \overline{B} \). Define a holomorphic family of vector fields on \( U \) by

\[
X : \Omega^1 \times U \to TU, \quad (z, u) \mapsto (\omega_z)^{-1}(\alpha_p).
\]

Let \( J \) be the open subset of \( J \times M \) such that for all \( (t_0, p) \in J \times M \), the map \( \psi^{(t_0,p)}(t) := \psi(t, t_0, p) \) is the maximally extended integral curve of \( X|_{J \times U} \) starting at \((t_0, p)\). From the general theory of smooth time-dependent flows (e.g. [4, Theorem 9.41]), for all \((t_1, t_0) \in J \times J\) the set

\[
U(t_1, t_0) := \{ p \in U : (t_1, t_0, p) \in \mathcal{E} \}
\]
is open and the map

\[
\psi(t_1, t_0) : U(t_1, t_0) \longrightarrow U(t_0, t_1), \quad p \mapsto \psi(t_1, t_0, p)
\]

is a diffeomorphism. Moreover, since \( X \) is holomorphic, \( \psi(t_1, t_0) \) is a biholomorphism (this follows from the holomorphic dependence of solutions to linear system of ODEs on the initial conditions, see e.g. [3, Ch. 1, §8]). Since \( \alpha_Z = 0 \) we have \( X_{(t_0, p)} = 0 \) for all \((t_0, p) \in J \times Z\), and hence \( \psi(t_1, t_0, p) = p \) for all \((t_1, t_0, p) \in J \times J \times Z\). In particular, \( J \times J \times Z \subseteq E \), so \( U(1, 0) \) and \( U(0, 1) \) contain \( Z \). We claim that the biholomorphism \( \psi_1 : U_{1, 0} \rightarrow U_{0, 1} \) is the one we need. First, since \( \alpha_1 \) and \( \omega_2 \) are \( G \)-invariant, this is a complex-symplectic representation of \( H \). The \( t \)-action is Hamiltonian, this is a complex-symplectic representation of \( H \). Sjamaar’s proof of the Holomorphic Slice Theorem: see the top of p. 101 in [32]. It can also be proved by linearising the action of \( G_p \) on the slice \( S \) at \( p \) [32, Theorem 1.21].

It will be important later to know that the open set \( U \) of the preceding proposition can be taken to be \( G \)-saturated. First, we have:

**Proposition 3.10.** Let \((M, K, \mu)\) be an integrable Hamiltonian \( K \)ähler manifold and let \( p \in M^{\mu-ss} \). Then, every \( G \)-invariant neighbourhood of \( p \) contains a neighbourhood of \( p \) which is \( G \)-saturated in \( M^{\mu-ss} \).

**Proof.** Our argument is similar to [18, Remark 14.24]. As observed in [17, Remark 1.1], the quotient map \( \pi : M^{\mu-ss} \rightarrow M^{\mu-ss}/G \) sends \( G \)-invariant closed subsets to closed subsets. Let \( U \) be a \( G \)-invariant neighbourhood of \( p \) in \( M^{\mu-ss} \). Then, \( C := M^{\mu-ss} - U \) is a \( G \)-invariant closed subset of \( M^{\mu-ss} \). So \( \pi(C) \) is closed in \( M^{\mu-ss}/G \). Moreover, since \( G \cdot p \) is closed in \( M^{\mu-ss} \), we have \( \pi(p) \notin \pi(C) \). Hence, \( \pi^{-1}(M^{\mu-ss}/G - \pi(C)) \) is a \( G \)-saturated neighbourhood of \( p \) contained in \( U \).

The set \( H \cdot B \) in Proposition 3.9 is also \( G \)-saturated [34, Corollary 4.9] and it follows that \( G \times_H (H \cdot B) \) is \( G \)-saturated in \( G \times_H W \). We can then restate the Holomorphic Slice Theorem in the following form:

**Theorem 3.11 (Linearised Holomorphic Slice Theorem).** Let \((M, K, \mu)\) be an integrable Hamiltonian \( K \)ähler manifold. Let \( p \in M^{\mu-ss} \), let \( G = K_C \), let \( H = G_p \), and let \( W = T_p M/T_p (G \cdot p) \). Then, there is a \( G \)-saturated neighbourhood \( U \) of \( p \) in \( M^{\mu-ss} \). A \( G \)-saturated neighbourhood \( U' \) of the zero section of the vector bundle \( G \times_H W \), and a \( G \)-equivariant biholomorphism \( U' \rightarrow U \) which maps \([1, 0] \) to \( p \).

**Proof.** Let \( \varphi : G \times_H HB \rightarrow U \) be the biholomorphism of Proposition 3.9. By Proposition 3.10, there is a \( G \)-saturated neighbourhood \( U' \) of \( p \) contained in \( U \). Let \( B' \subseteq B \) be an open ball sufficiently small so that \( U'' := \varphi(G \times_H HB') \subseteq U' \). Then, \( U'' \) is \( G \)-saturated.

**3.4. Proof of the hyperkähler local normal form.** We now complete the proof of Theorem 3.3. The first step is to have an explicit expression for the complex-symplectic form \( \eta_C \) of the local model \( E = G \times q (h^0 \times V) \) at the point \( q = [1,0,0] \). Note that \( G_q = H \), so \( H \) acts linearly on \( T_q E \). Since the \( G \)-action is Hamiltonian, this is a complex-symplectic representation of \( H \) on \( T_q E \). Recall that \( m \subseteq g \) is the orthogonal complement to \( h \).

**Proposition 3.12.** We have \( T_q E \cong m \times m^* \times V \) as complex-symplectic \( H \)-representations, where \( m \times m^* \) has the canonical complex-symplectic form

\[
(m \times m^*) \times (m \times m^*) \rightarrow \mathbb{C}, \quad ((X, \varphi), (Y, \psi)) \mapsto \psi(X) - \varphi(Y).
\]

Moreover, \( T_q (G \cdot q) \cong m \times 0 \times 0 \) under this isomorphism.
\textbf{Proof.} The canonical symplectic form on $T^*G = G \times \mathfrak{g}^*$ at $T_{(g,\xi)}(T^*G) = \mathfrak{g} \times \mathfrak{g}^*$ is
$$((X, \varphi), (Y, \psi)) \mapsto \psi(X) - \varphi(Y) + \xi([X, Y])$$
(1, §4.4). In particular, if $q := (1, 0, 0) \in T^*G \times V$, the symplectic form on $T^*G \times V$ at $T_q(T^*G \times V) = \mathfrak{g} \times \mathfrak{g}^* \times V$ is
$$((X, \varphi, u), (Y, \psi, v)) \mapsto \psi(X) - \varphi(Y) + \Omega(u, v).$$
Now, we have $d\lambda_q(X, \xi, v) = -\xi|_h$, so the tangent space to $\lambda^{-1}(0)$ at $q$ is $\mathfrak{g} \times \mathfrak{h}^\circ \times V$. Moreover, $T_q\lambda^{-1}(0)/H = T_q\lambda^{-1}(0)/T_q\lambda(H \cdot q) = \mathfrak{g}/\mathfrak{h} \times \mathfrak{h}^\circ \times V$.

Identifying $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{m}$ and $\mathfrak{h}^\circ$ with $\mathfrak{m}^*$ gives the result. □

Now, we need to recall a result from representation theory. Recall that a subspace $U$ in a symplectic vector space $(\mathfrak{r}, \omega)$ is called \textit{symplectic} if $U \cap U^\omega = 0$ (or equivalently $\omega$ restricts to a symplectic form on $U$).

\textbf{Proposition 3.13} (see e.g. [23, §2]). \textit{Let $H$ be a complex reductive group. Every finite-dimensional complex-symplectic representation $H \to \text{Sp}(\mathfrak{r}, \omega)$ is of the form}
$$R = U_1 \oplus \cdots \oplus U_m \oplus (V_1 \oplus W_1) \oplus \cdots \oplus (V_n \oplus W_n)$$
\textit{where:}

(i) $U_i$, $V_i$ and $W_i$ are the irreducible $H$-submodules of $R$;

(ii) $U_i$ and $V_i \oplus W_i$ are symplectic subspaces;

(iii) every skew-symmetric $H$-invariant bilinear form on $V_i$ or $W_i$ is zero;

(iv) the space of skew-symmetric $H$-invariant bilinear forms on $U_i$ is one-dimensional.

Moreover, if two symplectic $H$-representations $(R_1, \omega_1)$ and $(R_2, \omega_2)$ are isomorphic as $H$-modules then they are isomorphic as symplectic $H$-representations, i.e. there is an $H$-equivariant linear isomorphism $\varphi : R_1 \to R_2$ such that $\varphi^*\omega_2 = \omega_1$.

We deduce from this proposition a first linear approximation of the hyperkähler local normal form.

\textbf{Lemma 3.14.} \textit{Let $H$ be a complex reductive group acting linearly on a finite-dimensional complex vector space $\mathfrak{r}$. Suppose that $\omega$ and $\eta$ are two $H$-invariant complex-symplectic forms on $\mathfrak{r}$, and $S \subseteq \mathfrak{r}$ is an $H$-invariant subspace that is isotropic with respect to both $\omega$ and $\eta$. Then, there exists an $H$-equivariant linear isomorphism $\varphi : \mathfrak{r} \to \mathfrak{r}$ that restricts to the identity on $S$ and such that $\varphi^*\eta = \omega$.}

\textbf{Proof.} Let $R = U_1 \oplus \cdots \oplus U_m \oplus (V_1 \oplus W_1) \oplus \cdots \oplus (V_n \oplus W_n)$ be the decomposition of Proposition 3.13 with respect to $\omega$. The space $S$ is $H$-invariant, so it is a direct sum of irreducible $H$-submodules. But $S$ is isotropic, so it contains no symplectic subspace, and hence is a direct sum of $V_i$’s and $W_i$’s, no two of which occur in the same pair. Thus, after relabeling, we may assume that $S = V_1 \oplus \cdots \oplus V_k$ for some $k \leq n$.

Let us call the $U_i$-factors the $\omega$-symplectic-$H$-summands (which are symplectic by (ii)) and the $V_i$- and $W_i$-factors the $\omega$-isotropic-$H$-summands (which are isotropic by (iii)). We can also consider the decomposition of $R$ into $H$-invariant symplectic subspaces with respect to $\omega$ if the form $U_1 \oplus \cdots \oplus U_m \oplus (P_1 \oplus Q_1) \oplus \cdots \oplus (P_n \oplus Q_n)$ where the $P_i$’s and $Q_i$’s are the same as the $\omega$-isotropic-$H$-summands. Hence, the $\eta$-symplectic-$H$-summands are the same as the $\omega$-symplectic-$H$-summands. Thus the decomposition of $R$ into $H$-invariant symplectic subspaces with respect to $\eta$ is of the form $U_1 \oplus \cdots \oplus U_m \oplus (P_1 \oplus Q_1) \oplus \cdots \oplus (P_n \oplus Q_n)$ where the $P_i$’s and $Q_i$’s are the same as the $\omega$-isotropic-$H$-summands. Hence, the $\eta$-symplectic-$H$-summands are the same as the $\omega$-symplectic-$H$-summands.

To find an $H$-equivariant isomorphism $\psi : R_1 \to R_2$ which is the identity on the $V_i$’s and such that $\psi^*\eta = \omega$. Indeed, in that case $\mathfrak{r} = \mathfrak{r}/\mathfrak{r} = \mathfrak{r}/\mathfrak{r}$ as $H$-modules so also as symplectic representations (by the last part of Proposition 3.13), and then we have an isomorphism $R_1 \oplus R_2 \to R_1 \oplus R_2$ with the desired properties. To find $\psi : R_1 \to R_1$, note that $\omega$ provides an isomorphism $W_i \to V_i^\circ$ and $\eta$ provides an isomorphism $V_i^\circ \to Q_i$. Let $\gamma_i : W_i \to Q_i$ be the composition and let $\beta_i : V_i \oplus W_i \to V_i \oplus Q_i$ be the product of the identity on $V_i$ with $\gamma_i$. Then, $\beta_i$ is an $H$-invariant isomorphism such that $\beta_i^*\eta|_{V_i \oplus Q_i} = \omega|_{V_i \oplus W_i}$. Putting those $\beta_i$’s together we get an $H$-equivariant isomorphism $\psi : R_1 \to R_1$ which is the identity on the $V_i$’s. Moreover, $\psi^*\eta = \omega$ since the factors $V_i \oplus W_i, \ldots, V_k \oplus W_k$ are $\omega$-orthogonal (since if $A \subseteq R_1$ is an $H$-invariant symplectic subspace then $R_1 = A \oplus A^\omega$ and $A^\omega$ is $H$-invariant so $A^\omega$ is the sum of the irreducible $H$-submodules of $R_1$ complementary to $A$) and similarly the factors $V_1 \oplus Q_1, \ldots, V_k \oplus Q_k$ are $\eta$-orthogonal. □
Lemma 3.15. Let $H \rightarrow \text{Sp}(R, \omega)$ be a complex-symplectic representation and $S \subseteq R$ an $H$-invariant isotropic subspace. Then, $R/S \cong S^* \times S^*/S$ as $H$-modules.

Proof. Let $R \rightarrow S^*$ be the composition of the isomorphism $R \rightarrow R^*$ induced by $\omega$ with the restriction map $R^* \rightarrow S^*$. Let $R \rightarrow S^w$ be the projection along the $H$-invariant complement of $S^w$ in $R$ (by complete reducibility). These maps give an $H$-equivariant surjective map $R \rightarrow S^* \times S^*/S$ with kernel $S^w \cap S$. Since $S$ is isotropic, we have $S^* \cap S = S$. □

Proof of Theorem 3.3. Let $p \in \mu_c^{-1}(0)^{\mu_c-\text{ps}}$ and let $H = G_p$. Then, $T_p(G \cdot p) = g/\mathfrak{h}$ is isotropic in $T_pM$ so, by Lemma 3.15, we have $T_pM/T_p(G \cdot p) = \mathfrak{h}^\circ \times V$, where $V = (T_p(G \cdot p))^{\omega_c}/T_p(G \cdot p)$ is the complex-symplectic slice at $p$. Thus, by the Linearised Holomorphic Slicing Theorem (Theorem 3.11), a $G$-saturated neighbourhood of $p$ in $M^{\mu_c-\text{ps}}$ is $G$-equivariantly biholomorphic to a $G$-saturated neighbourhood of $q = [1,0,0]$ in the local model $E = G \times_H (\mathfrak{h}^\circ \times V)$. Note that by Proposition 3.12, $T_q(G \cdot q)$ is also isotropic with respect to the canonical complex-symplectic form on $E$. Note also that any $G$-invariant neighbourhood of the zero section $0_E = G \cdot q$ of $E$ contains a $G$-saturated neighbourhood, namely $G \times_Q H B$ for a sufficiently small open ball $B$. Hence, it suffices to show that any two $G$-invariant complex-symplectic forms $\omega_c$ and $\eta_c$ on a $G$-invariant neighbourhood of the zero-section $0_E = G \cdot q$ in $E$ such that $T_q0_E$ is isotropic with respect to both can be deformed from one into the other by a $G$-equivariant holomorphic deformation on a possibly smaller neighbourhood of $0_E$. By the holomorphic Darboux–Weinstein theorem, it suffices to first find a deformation that makes them match on $0_E$. This is carried out by the linear algebra developed in Lemma 3.14, as we now explain.

By Lemma 3.14, there exists an $H$-equivariant linear isomorphism $\varphi : T_qE \rightarrow T_qE$ which restricts to the identity on $T_q0_E$ and such that $\varphi^*\eta_c = \omega_c$. We have $T_qE = m \times m^* \times V$ and $T_q0_E = m \times 0 \times 0$, so $\varphi$ is of the form

$$\varphi : m \times m^* \times V \rightarrow m \times m^* \times V,$$

where $A : m^* \times V \rightarrow m$ and $B : m^* \times V \rightarrow m^* \times V$ are some linear maps, with $B$ invertible. Then,

$$\psi : E \rightarrow E, \quad \psi(g, \xi, \nu) = \{ge^{A(\xi, \nu)}, B(\xi, \nu)\}$$

is a $G$-equivariant biholomorphism with $d\psi = \varphi$. In particular, $\varphi^*\eta_c|_{g \cdot q} = \omega_c|_{g \cdot q}$ and since $\omega_c$ and $\eta_c$ are $G$-invariant and $\varphi$ is $G$-equivariant this implies that $\varphi^*\eta_c|_{g \cdot q} = \omega_c|_{g \cdot q}$ for all $g \in G$, i.e. $\varphi^*\eta_c|_{0_E} = \omega_c|_{0_E}$.

We can now apply the holomorphic Darboux–Weinstein theorem. This shows the existence of a $G$-equivariant complex-symplectic isomorphism $\varphi : U \rightarrow U'$ such that $\varphi(p) = q$ where $U$ is $G$-saturated neighbourhood of $p$ in $M^{\mu_c-\text{ps}}$ and $U'$ a $G$-saturated neighbourhood of $q \in E$. It remains to show that $\nu_c \circ \varphi = \mu_c$. Since $(\nu_c \circ \varphi)(p) = 0 = \mu_c(p)$ and since moment maps are unique up to a constant (see e.g. [2, Ch. 26]) it suffices to show that $\nu_c \circ \varphi$ is a moment map for the action $-\text{action}$ on $M^{\mu_c-\text{ps}}$. This follows from the fact that $\varphi$ is a $G$-equivariant complex-symplectic isomorphism. □

4. Local structure of singular hyperkähler quotients

Throughout this section, $(M, K, \mu)$ will be a tri-Hamiltonian hyperkähler manifold whose $K$-action is $I$-integrable, i.e. extends to an $I$-holomorphic action of $G = K_C$. The goal of this section is to use the local normal form of §3 to describe the local complex-symplectic structure of the singular hyperkähler quotient $M\!//\!\mu K$, i.e. we prove Theorem 1.4 in the introduction. In particular, we endow $M\!//\!\mu K$ with the structure of a complex space and show that the orbit type partition is a complex Whitney stratification.

4.1. Complex structure. Let us first explain how the results on analytic Hilbert quotients of §2.4 help us define a complex structure on $M\!//\!\mu K$. We use the notation of §3; in particular, $\mu_c = \mu_1 + i\mu_2$ and $\mu_2 = \mu_1$. First note that $\mu_c^{-1}(0)^{\mu_c-\text{ps}} = \mu_c^{-1}(0) \cap M^{\mu_c-\text{ps}}$ is a $G$-invariant closed complex subspace of $M^{\mu_c-\text{ps}}$. Hence, its image $\mu_c^{-1}(0)^{\mu_c-\text{ps}}//G$ in $M^{\mu_c-\text{ps}}//G$ is a complex closed subspace and the restriction

$$\mu_c^{-1}(0)^{\mu_c-\text{ps}} // G \rightarrow \mu_c^{-1}(0)^{\mu_c-\text{ps}} // G$$

is an analytic Hilbert quotient (Proposition 2.12(ii)). The space $\mu_c^{-1}(0)^{\mu_c-\text{ps}}//G$ has a $G$-orbit type partition as in §2.4.3 and $M\!//\!\mu K$ has a $K$-orbit type partition into hyperkähler manifolds by Theorem 1.1. By Heinzner–Loose’s Theorem 2.13 and Sjamaar’s Theorem 2.18(i), we get:

**Proposition 4.1.** We have $\mu^{-1}(0) \subseteq \mu_c^{-1}(0)^{\mu_c-\text{ps}}$ and this inclusion descends to an isomorphism

$$M\!//\!\mu K \rightarrow \mu_c^{-1}(0)^{\mu_c-\text{ps}} // G$$

of partitioned spaces.

In particular, $M\!//\!\mu K$ has the structure of a complex space. We denote the structure sheaf by $\mathcal{O}_1$. □
4.2. Linear hyperkähler quotients. Let us first consider the case of a linear hyperkähler quotient; this example will be important later. Let \( V \) be a quaternionic vector space, i.e. a real vector space endowed with three endomorphisms \( l, J, K \) such that \( l^2 = J^2 = K^2 = lJK = -1 \). Then, \( V \cong \mathbb{H}^n \) for some \( n \), so we may endow \( V \) with a real inner-product \( \langle \cdot, \cdot \rangle \) such that \( l, J, K \) are skew-symmetric. This makes \( V \) into a hyperkähler manifold, with Kähler forms \( \omega_l(u,v) = (lu,v) \), etc. Let \( K \) be a compact Lie group acting linearly on \( V \) by preserving \( \langle \cdot, \cdot \rangle \) and \( l, J, K \). Then, there is a hyperkähler moment map, namely,

\[
\phi = (\phi_l, \phi_J, \phi_K) : V \longrightarrow \mathfrak{t}^* \times \mathfrak{t}^* \times \mathfrak{t}^*, \quad \phi(v)(X) = \frac{1}{2}\langle \omega_l(Xv,v), \omega_J(Xv,v), \omega_K(Xv,v) \rangle.
\]

Moreover, the \( K \)-action extends to an \( L \)-linear action of \( H := K_C \), and the underlying complex-Hamiltonian manifold is simply \( (V,l,\omega_C,H,\Phi_V) \) where \( \omega_C \) is the complex-symplectic bilinear form \( \omega_C^O : V \times V \to \mathbb{C} \) and \( \Phi_V \) is the canonical complex moment map \( \Phi_V(v)(X) = \frac{1}{2}\omega_C^O(Xv,v) \) as before. By the Kempf–Ness theorem [21], every point in \( V \) is \( \phi_l \)-semistable (see e.g. [30, Proposition 3.9]), so the complex space \( (V//_O K,\mathcal{O}_1) \) is simply the analytification of the affine GIT quotient \( \Phi_V^O(0)//H = \text{Spec} \mathbb{C}[\Phi_V^O(0)]^H \) (Example 2.11).

Conversely, if \( H \) is any complex reductive group and \( H \to \text{Sp}(V,\omega_C) \) is a complex-symplectic representation (e.g. a complex-symplectic slice) then \( V \cong \mathbb{C}^{2n} \cong \mathbb{H}^n \) for some \( n \), so we may endow \( V \) with the structure of a quaternionic vector space invariant under the action of a maximal compact subgroup \( K \) of \( H \) (by averaging). Hence, the GIT quotient \( \Phi_V^O(0)//H \) can always be viewed as a hyperkähler quotient.

4.3. Local holomorphic structure. Let \( x \in M//_\mu K \) and let \( p \in \mu^{-1}(0) \) be a point above \( x \). Let \( H = G_p \) and let \( V = (T_p(G \cdot p))^{\omega_C}/T_p(G \cdot p) \) be the complex-symplectic slice at \( p \). As in §3, let \( \Phi_V : V \to \mathfrak{t}^* \) be the canonical complex moment map (3.2) for the action of \( H \) on \( V \). The first step in proving Whitney conditions will be to use Theorem 3.3 to show that \( x \) has a neighbourhood \( U \) which is isomorphic as a complex and partitioned space to a neighbourhood \( U' \) of \( 0 \) in the GIT quotient \( \Phi_V^O(0)//H \). Since Whitney conditions are local, this will reduce the problem to the spaces \( \Phi_V^O(0)//H \) near \( 0 \). However, note here that the natural partition of \( \Phi_V^O(0)//H \) is by \( H \)-orbit types rather than \( G \)-orbit types. To show that the biholomorphism \( U \to U' \) is an isomorphism of partitioned spaces, we will first need to show that once we refine the partitions into their connected components, the \( G \)-orbit type partition of \( \Phi_V^O(0)//H \) (i.e. using conjugacy classes in \( G \) rather than in \( H \)) is identical to the \( H \)-orbit type partition. This will follow from a result of Palais which says that when a compact Lie group \( K \) acts on a completely regular space \( X \), every \( x \in X \) has a neighbourhood \( V \) such that if \( y \in V \) then \( K_y \) is conjugate to a subgroup of \( K_z \):

**Lemma 4.2.** Let \( (M,K,\mu) \) be a Hamiltonian Kähler manifold and let \( K' \) be a compact Lie group containing \( K \) as a Lie subgroup. Then, the \( K \)- and \( K' \)-orbit type partitions of \( M//_\mu K \) coincide. Moreover, if \( (M,K,\mu) \) is integrable, then the \( K'_{\mathbb{C}} \)- and \( K_{\mathbb{C}} \)-orbit type partitions of \( M//_\mu K \) also coincide.

**Proof.** Let \( X = \mu^{-1}(0) \) and let \( \pi : X \to X/K \) be the quotient map. Let \( S \subseteq X/K \) be a \( K' \)-orbit type piece, i.e. a connected component of a set of the form \( X(H)_{K'}'/K \) for some closed subgroup \( H \subseteq K \), where \( (H)_{K'} \) is the conjugacy class of \( H \) in \( K' \). We have \( S = \pi(T) \) for some connected component \( T \) of \( X(H)_{K'} \).

Fix \( x = \pi(p) \in S \) with \( p \in T \). We want to show that if \( q = \pi(y) \in S \) for some \( y \in T \) then \( K_x \) and \( K_y \) (which are conjugate in \( K' \)) are in fact conjugate in \( K \). Let

\[
A := \{ y \in T : K_y \text{ is conjugate to } K_z \text{ in } K \}.
\]

It suffices to show that \( A \) is both open and closed in \( T \). **Closed:** Let \( y \in \overline{A} \cap T \) and write \( y = \lim_{n \to \infty} y_n \) with \( y_n \in A \). Then, there exists \( k_n \in K \) such that \( k_nk_xk_n^{-1} = K_{y_n} \) for all \( n \). Since \( K \) is compact, we may assume that \( \lim_{n \to \infty} k_n = k \) for some \( k \in K \). Then, \( kK_xk^{-1} \subseteq K_y \) by continuity of the action. Moreover, \( kK_xk^{-1} \) and \( K_y \) are isomorphic since they are conjugate in \( K' \) and since they have finitely many connected components, the inclusion \( kK_xk^{-1} \subseteq K_y \) implies that \( kK_xk^{-1} = K_y \). Thus, \( A \) is closed. **Open:** Let \( y \in A \). By Palais [31, Corollary 2 on p. 313] there is a neighbourhood \( V \) of \( y \) in \( X \) such that if \( z \in V \) then \( K_z \) is conjugate (in \( K \)) to a subgroup of \( K_y \). Then, \( V \cap T \) is a neighbourhood of \( y \) in \( T \) and \( V \cap T \subseteq A \), so \( A \) is open in \( T \).

The second statement amounts to show that if \( H \) and \( L \) are two closed subgroups of a compact Lie group \( R \), then \( H \) and \( L \) are conjugate in \( R \) if and only if \( H_{\mathbb{C}} \) and \( L_{\mathbb{C}} \) are conjugate in \( R_{\mathbb{C}} \). This follows from Mostow’s decomposition, as explained by Sjamaar [32, Proof of Theorem 2.10, first paragraph].

Now, by picking a quaternionic structure on the complex-symplectic slice \( V \) as explained in §4.2, we can apply this result to \( (V,K_\mu,\phi_\mu) \) and infer that the \( G \)- and \( H \)-orbit type partitions of \( \Phi_V^O(0)//H \) coincide. This will be used for the last part of the following result.

Proposition 4.3. Let \( x \in M_{/\mu} K \). Take a point \( p \in \mu^{-1}(0) \) above \( x \), let \( H = G_p = (K_p)_{/\mu} \), and let \( V = (T_p(G \cdot p))^{cc}/T_p(G \cdot p) \). Then, there is a neighbourhood \( U \) of \( x \) in \( M_{/\mu} K \), an open ball \( B \subseteq V \) around \( 0 \), and a biholomorphism (with respect to \( O_1 \)) from \( U \) to the GIT quotient \( \Phi_V^{-1}(0)/H = \text{Spec}(\mathbb{C}[\Phi_V^{-1}(0)]^H) \) which maps \( x \) to the image of \( 0 \in \Phi_V^{-1}(0) \). Moreover, this biholomorphism is an isomorphism of partitioned spaces.

Proof. Let \( E = G \times_H (h^0 \times V) \). Since \( H \) is reductive and acts freely on \( G \times (h^0 \times V) \), \( E \) is an affine variety. Moreover, \( \nu_C : E \rightarrow g^* \) is a morphism so \( \nu_C^{-1}(0) \) is an affine variety in \( E \) and we can consider the GIT quotient \( \nu_C^{-1}(0)/G = \text{Spec}(\mathbb{C}[\nu_C^{-1}(0)]^G) \). We claim that \( \nu_C^{-1}(0)/G \cong \nu_C^{-1}(0)/H \) as affine varieties. Indeed, we have \( \nu_C^{-1}(0) = G \times_H \Phi_V^{-1}(0) \) so the inclusion \( \Phi_V^{-1}(0) \rightarrow \nu_C^{-1}(0) : v \mapsto [1, v] \) descends to a morphism \( \psi : \Phi_V^{-1}(0)/H \rightarrow \nu_C^{-1}(0)/G \). Also, the projection \( \nu_C^{-1}(0) = G \times_H \Phi_V^{-1}(0) \rightarrow \nu_C^{-1}(0)/H \) onto the second factor descends to a morphism \( \nu_C^{-1}(0)/G \rightarrow \Phi_V^{-1}(0)/H \) which is an inverse of \( \psi \).

Now, for an element \( [g, v] \in G \times_H \Phi_V^{-1}(0) = \nu_C^{-1}(0) \) we have \( G_{[g, v]} = gHg^{-1} \), so \( \psi \) is an isomorphism of partitioned spaces with the \( G \)-orbit type partitions on both sides. As explained above, Lemma 4.2 implies that the \( G \)-orbit type partition on \( \Phi_V^{-1}(0)/H \) coincides with the \( H \)-orbit type partition.

Let \( U \subseteq M^{\mu\text{-ss}}, U' \subseteq E \) and \( f : U \rightarrow U' \) be as in Theorem 3.3. By Remark 3.4, we may assume \( U' = G \times_H (H \cdot B) \) for some open ball \( B \) around zero in \( m^* \times V \). Then, \( W = U \cap \mu_C^{-1}(0)^{\mu\text{-ss}} \) is a \( G \)-saturated open subset of \( \mu_C^{-1}(0)^{\text{as}}, \) and so is \( W' = U' \cap \nu_C^{-1}(0) \in \nu_C^{-1}(0) \). Moreover, by Proposition 2.12(i), the image \( W/G \) of \( W \) in \( \mu_C^{-1}(0)^{\mu\text{-ss}}/G \) is open and \( W' \rightarrow W/G \) is an analytic Hilbert quotient. Similarly, \( W' \rightarrow W'/G \subseteq \nu_C^{-1}(0)/G \) is an analytic Hilbert quotient. Since \( f : U \rightarrow U' \) is a \( G \)-equivariant biholomorphism with \( \nu_C \circ f = \mu_C \), it restricts to a \( G \)-equivariant biholomorphism \( W \rightarrow W' \) and hence to a biholomorphism \( W/G \rightarrow W'/G \) which respects the \( G \)-orbit type partitions. Moreover, under the isomorphism \( \nu_C^{-1}(0)/G \cong \Phi_V^{-1}(0)/H \) above we have an isomorphism \( W'/G \cong (H \cdot B \cap \Phi_V^{-1}(0))/H \) of complex and partitioned spaces.

4.4. The orbit type pieces are complex submanifolds. As a first application of Proposition 4.3, we will show that the pieces in the orbit type partition are complex submanifolds with respect to \( O_1 \); this is one of the requirements in the definition of complex Whitney stratifications.

We shall achieve this by describing the orbit type partition of \( \Phi_V^{-1}(0)/H \), where \( H \) is a complex reductive group, \( H \rightarrow \text{Sp}(V, \omega_C) \) a complex-symplectic representation, and \( \Phi_V \) the moment map (3.2). First note that the set \( V_H \) of fixed points of \( H \) in \( V \) is a complex-symplectic subspace. Let \( W \) be its symplectic complement, so that \( V = W \oplus V_H \). Then, \( W \) is complex-symplectic and \( H \)-invariant so it provides a complex-symplectic representation of \( H \). The complex moment map \( \Phi_W : W \rightarrow h^* \) associated to this representation is simply the restriction of \( \Phi_V \) to \( W \), so we have the decomposition

\[
\Phi_V^{-1}(0)/H = \Phi_W^{-1}(0)/H \times V_H.
\]

For each \( L \subseteq H \), let \( (\Phi_W^{-1}(0)/H)_{(L)} \) be the image of \( \Phi_W^{-1}(0)^{\mu_L}/H \) under the bijection \( \Phi_W^{-1}(0)^{\mu_L}/H \rightarrow \Phi_V^{-1}(0)/H \). Then, the pieces of the orbit type partition of \( \Phi_V^{-1}(0)/H \) are the connected components of the sets of the form \( (\Phi_W^{-1}(0)/H)_{(L)} \times V_H \).

Lemma 4.4. The orbit type piece of \( \Phi_V^{-1}(0)/H \) containing 0 is \([0] \times V_H \).

Proof. Note that \( V_H = V^{H} \) since \( v \in V \) and \( H_v = gHg^{-1} \) for some \( g \in H \), then \( gHg^{-1} \subseteq H \) and since \( gHg^{-1} \) and \( H \) are isomorphic LIE groups with finitely many connected components this implies \( gHg^{-1} = H \) and hence \( H_v = H \). In particular, \( W_H = W \cap V_H = 0 \), so the piece containing 0 is \( (\Phi_W^{-1}(0)/H)_{(H)} \times V_H = [0] \times V_H \).

Proposition 4.5. The pieces of the orbit type partition of \( M_{/\mu} K \) are non-singular complex subspaces with respect to \( O_1 \).

Proof. By Lemma 4.4 and Proposition 4.3, the embedding of a \( K \)-orbit type piece in \( M_{/\mu} K \) is locally biholomorphic to the embedding of \([0] \times V_H \) in \( \Phi_V^{-1}(0)/H \times V_H \).

4.5. Compatibility with the hyperkähler structures. We show that for each orbit type stratum \( S \subseteq M_{/\mu} K \), the sheaf \( O_1 \) is compatible with the complex structure \( I_S \) on \( S \), where \( (g_S, I_S, J_S, K_S) \) is its hyperkähler structure as in Theorem 1.1.

Proposition 4.6. The inclusion \( S \hookrightarrow M_{/\mu} K \) is holomorphic with respect to \( I_S \) and \( O_1 \).

Proof. We want to show that the composition \( S \hookrightarrow M_{/\mu} K \rightarrow \mu_C^{-1}(0)^{\mu\text{-ss}}/G \) is holomorphic, where \( S \) has the complex structure \( I_S \). Since \( \mu_C^{-1}(0)^{\mu\text{-ss}}/G \) is a closed complex subspace of \( M^{\mu\text{-ss}}/G \), it suffices
to show that the composition $S \rightarrow \mu_{-1}(0)\mu_{-1}^*G \rightarrow M_{\mu_{-1}^*G}G$ is holomorphic, which is the same as the composition $S \rightarrow M/\mu_{-1}^*K \rightarrow M_{\mu_{-1}^*K} \rightarrow M_{\mu_{-1}^*G}G$. Let $H \subseteq K$ be as above so that $S$ is a connected component of $\mu_{-1}(0)(H)/K$. Then, $S$ is a subset of a connected component $T$ of $\mu_{-1}(0)(H)/K$. Moreover, $T$ is a stratum in the Kähler quotient $M/\mu_{-1}K$ and, from the definition of the Kähler structure on $T$ given in §2.4 and the definition of $I$ given above, the inclusion $S \hookrightarrow T$ is holomorphic. Hence, it suffices to show that the composition $T \hookrightarrow M/\mu_{-1}K \rightarrow M_{\mu_{-1}^*G}G$ is holomorphic, and this follows from Theorem 2.18(iii). \qed

4.6. The frontier condition. In this section, we prove that the orbit type partition of $M/\mu_{-1}K$ is a decomposition in the sense of Definition 2.2 (this is a requirement in the definition of Whitney stratified spaces). Since $K$ is compact, $\mu_{-1}(0)/K$ satisfies the local condition, so the only thing left to show is the frontier condition. This will be achieved by the local model of Proposition 4.3, so we first need to discuss how the frontier condition can be inferred locally.

Given a partitioned space $(X, P)$ we will denote by $P^\circ$ the refinement of $P$ obtained by decomposing every stratum of $P$ into its connected components. In particular, the orbit type partition of $M/\mu_{-1}K$ which we are considering is the refinement $P^\circ$ of $P := \{\mu_{-1}(0)(K_p) : p \in \mu_{-1}(0)\}$. Also, we will say that a partitioned space $(X, P)$ is conical at a stratum $S \in P$ if $S \subseteq T$ for all $T \in P$.

The following lemma provides a local criterion for partitioned spaces to satisfy the frontier condition.

**Lemma 4.7.** Let $(X, P)$ be a partitioned space. Suppose that every point $x \in X$ has a neighbourhood $U$ such that if $S \in P$ is the stratum containing $x$, then $S \cap U$ is connected and $(P|_U)^\circ$ is conical at $S \cap U$. Then, $P^\circ$ satisfies the frontier condition.

**Proof.** Let $S, T \in P$ and let $S = \bigcup_i S_i$, $T = \bigcup_j T_j$ be their decompositions into connected components. Suppose that $S_{i_0} \cap T_{j_0} \neq \emptyset$ for some $i_0, j_0$. We want to show that $S_{i_0} \subseteq T_{j_0}$. The set $R := S_{i_0} \cap T_{j_0}$ is closed in $S_{i_0}$ so it suffices to show that $R$ is also open in $S_{i_0}$. Let $x \in R$. Take a neighbourhood $U$ of $x$ in $X$ such that $S \cap U$ is connected and $(P|_U)^\circ$ is conical at $S \cap U$. We claim that $S_{i_0} \cap U \subseteq R$, or equivalently, $S_{i_0} \cap U \subseteq T_{j_0}$. If $T \cap U = \bigcup_k C_k$ is the decomposition of $T \cap U$ into connected components, then, since $(P|_U)^\circ$ is conical at $S \cap U$, we have $S_{i_0} \cap U \subseteq C_k$ for all $k$. But the set of connected components of $T \cap U$ is the union of the set of connected components of $T_j \cap U$ for all $j$, so there exists $k_0$ such that $C_{k_0} \subseteq T_{j_0} \cap U$ and hence $S_{i_0} \cap U \subseteq S \cap U \subseteq C_{k_0} \subseteq T_{j_0}$. \qed

**Proposition 4.8.** The orbit type partition of $M/\mu_{-1}K$ satisfies the frontier condition and hence is a decomposition.

**Proof.** Let $x \in M/\mu_{-1}K$, let $V$, $H$ and $B \subseteq V$ be as in Proposition 4.3, and let $U = (H \cdot B) \cap \Phi_V^{-1}(0)/H$. We denote by $[v]$ the image of a point $v \in \Phi_V^{-1}(0)$ in the GIT quotient $\Phi_V^{-1}(0)/H$. Then, $x$ has a neighbourhood isomorphic to $U$ as partitioned spaces, with an isomorphism sending $x$ to $0$. Let $P$ be the orbit type partition of $\Phi_V^{-1}(0)/H$ and let $S \in P$ be the piece containing $[v]$. By Lemma 4.7, it suffices to show that $S \cap U$ is connected and $(P|_U)^\circ$ is conical at $S \cap U$. By Lemma 4.4, $S = \{[0]\} \times V^H$ so $S \cap U = \{[0]\} \times (V^H \cap B)$ is connected. To show that $(P|_U)^\circ$ is conical at $S \cap U$, let $T' \in (P|_U)^\circ$. Then, $T'$ is a connected component of $T \cap U$, where $T := \Phi_V^{-1}(0)/H) \times V^H$ for some $L \subseteq H$. We need to show that $S \cap U \subseteq T'$. Let $([0], v) \in S \cap U$, where $v \in V^H \cap B$. Take any point $([w], u)$ of $T'$, where $w \in (\Phi_V^{-1}(0)_{(p)})_{(L)}$, $u \in V^H$, and $w + u \in H \cdot B$. It suffices to find a continuous path $\gamma : (0, 1) \rightarrow T \cap U$ such that $\gamma(0) = ([w], u)$ and $\lim_{t \rightarrow 0} \gamma(t) = ([0], v)$. Let $h \in H$ be such that $w + u \in h^{-1}B$. Then, $hw + u \in B$. We also have $v \in B$, so there exists $t_0 > 0$ small enough so that $t_0 hw + v \in B$ and hence $t_0 w + v \in H \cdot B$. Now, $\Phi_V(hw) = t^2 \Phi_V(w) = 0$ and hence $([t], v) \in T \cap U$ for all $t > 0$ and $([t], v) \rightarrow ([0], v)$ as $t \rightarrow 0$. Moreover, since $B$ is convex, the straight line from $t_0 w + v$ to $w + u$ will stay in $(H \cdot B) \cap ((\Phi_V^{-1}(0)_{(p)})_{(L)} \times V^H)$ and hence $([t_0 w], v)$ and $([w], u)$ are in the same path component $T'$ of $T \cap U$. \qed

4.7. Whitney conditions. We show that the orbit type partition of $M/\mu_{-1}K$ is a complex Whitney stratification with respect to $\mathcal{O}_H$ and hence a stratification in the sense of Definition 2.4. Our proof is very similar to that of Sjamaar–Lerman [33, §6]. Let us first recall the following result of Whitney.

**Lemma 4.9** (Whitney [36, Lemma 19.3]). Let $S$ and $T$ be disjoint complex submanifolds of a complex space $X$ with $S \subseteq \overline{T}$ and $\dim S < \dim T$. There is a complex subspace $A$ of $S$ with $\dim A < \dim S$ such that $T$ is regular over $S - A$. \qed

**Corollary 4.10.** Let $X$ be a complex space and $T \subseteq X$ a complex submanifold with $\dim T > 0$. Then, $T$ is regular over $\{x\}$ for all $x \in T - T$. \qed
Proof. Use Lemma 4.9 with \( S = \{ x \} \). (Remark: Whitney [36] defines a set \( A \) to have \( \dim A < 0 \) if and only if \( A = \emptyset \), see page 500, lines 24–25 in that paper.)

**Lemma 4.11.** Let \( X \) be a complex space and \( S, T \subseteq X \) disjoint complex submanifolds such that \( T \) is regular over \( S \). Then, for all \( n \geq 0 \), \( T \times \mathbb{C}^n \) is regular over \( S \times \mathbb{C}^n \) in \( X \times \mathbb{C}^n \).

**Proof.** This follows directly from the definition since \( T(x) = T_x S \times \mathbb{C}^n \).

**Proposition 4.12.** The orbit type partition of \( M/\mu K \) is a complex Whitney stratification with respect to \( \mathcal{O}_I \). In particular, it is a stratification in the sense of Definition 2.4.

**Proof.** By Proposition 4.3, the problem reduces to checking Whitney conditions for the \( H \)-orbit type partition of \( \Phi^{-1}(0)/H \) at \( [0] \). By \S 4.4, we have \( \Phi^{-1}(0)/H = \Phi^{-1}(0)/H \times V \) and by Lemma 4.11 it suffices to check Whitney condition for \( \Phi^{-1}(0)/H \) at \( [0] \). But the piece containing \( [0] \) is the singleton \( \{ [0] \} \), so this follows from Corollary 4.10.

4.8. Poisson structure. We show that there is a natural Poisson bracket on \( \mathcal{O}_I \) making \( M/\mu K \) a stratified symplectic space in the sense of Sjamaar–Lerman’s work (§2.3) but in a complex analytic sense:

**Definition 4.13.** A stratified complex-symplectic space is a complex space \((X, \mathcal{O}_X)\) with a complex Whitney stratification \( \mathcal{P} \), a complex-symplectic structure on each stratum, and a sheaf of Poisson brackets on \( \mathcal{O}_X \) such that the embeddings \( S \to X \) for \( S \in \mathcal{P} \) are holomorphic Poisson maps.

The definition of the Poisson bracket on \( \mathcal{O}_I \) is as follows. Let \( U \subseteq M/\mu K \) be open, let \( f, g \in \mathcal{O}_I(U) \) and let \( x \in U \). To define \( \{ f, g \}(x) \), let \( S \subseteq M/\mu K \) be the orbit type stratum containing \( x \) and let \( (g_S, l_S, l_S, K_S) \) be its hyperkähler structure. Then, \( \omega_S \in \mathcal{O}_X \) is a complex-symplectic form on \( (S, l_S) \). By Proposition 4.6, the restrictions \( f|_{S \cap U}, g|_{S \cap U} \) are \( l_S \)-holomorphic and hence we can take their Poisson bracket \( \{ f|_{S \cap U}, g|_{S \cap U} \} : S \cap U \to \mathbb{C} \) with respect to \( \omega_S \). Define \( \{ f, g \}(x) := \{ f|_{S \cap U}, g|_{S \cap U} \}(x) \). This defines a function \( \{ f, g \} : U \to \mathbb{C} \), and the goal is to show that it is holomorphic, i.e. \( \{ f, g \} \in \mathcal{O}_I(U) \).

In what follows, we identify \( S \) with a \( G \)-orbit type stratum in \( \mu^{-1}(0)/\mu \mathbb{K} \). Let \( \pi : Z \to \mathbb{C} \) be a complex submanifold of \( M \), the map \( \pi : Z \to S \) is a holomorphic submersion, and \( \pi^*(\omega_S) = i^{*} \omega_Z \) where \( i : Z \hookrightarrow M \).

**Lemma 4.14.** The set \( Z \) is a complex submanifold of \( M \), the map \( \pi : Z \to S \) is a holomorphic submersion, and \( \pi^*(\omega_S) = i^{*} \omega_Z \) where \( i : Z \hookrightarrow M \).

**Proof.** By Theorem 3.3, the embedding of \( Z \) in \( M \) is locally biholomorphic to the embedding of \( G/H \times V^H \) in \( G \times H (h^* \times V) \) and \( \pi \) is locally biholomorphic to the projection \( G/H \times V^H \to V^H \). This proves the first and second assertions. For the third assertion, we first note that since the pull back of the symplectic forms \( \omega_{S, l_S}, \omega_{S, l_S}, \omega_{K_S} \) on \( \mu^{-1}(0)/\mu \mathbb{K} \) are the restriction of the symplectic forms \( \omega, \omega_j, \omega_K \) on \( M \), we have \( j^*(\pi^*(\omega_S)) = j^*(i^{*} \omega_Z) \) where \( j : \mu^{-1}(0)/\mu \mathbb{K} \to Z \). Since \( j \) descends to a diffeomorphism \( \mu^{-1}(0)/\mu \mathbb{K} \to \mu^{-1}(0)/\mu \mathbb{K} \) we get that for all \( p \in \mu^{-1}(0)/\mu \mathbb{K} \), \( T_p Z = T_p \mu^{-1}(0)/\mu \mathbb{K} + T_p (G \cdot p) \). Hence, the result follows by the same argument as in the proof of Proposition 2.21.

**Lemma 4.15.** Let \( f : U \to \mathbb{C} \) be a holomorphic \( G \)-invariant function on an open set \( U \subseteq M \), and let \( \Xi_f \) be the holomorphic vector field on \( U \) dual to \( df \) under \( \omega_C \). Then, \( \Xi_f \) is tangent to \( Z \), i.e. \( \Xi_f(p) \in T_p Z \) for all \( p \in Z \cap U \).

**Proof.** Let \( \mathfrak{m} = h^\perp \) as in §3.1. By the local normal form we may assume that \( M = G \times_H (\mathfrak{m}^* \times V) \), \( p = [1, 0, 0] \) and \( Z = G/H \times V^H \). By Lemma 3.12, \( T_p M = \mathfrak{m} \times \mathfrak{m}^* \times V \), \( Z = \mathfrak{m} \times V^H \), and \( T_p (G \cdot p) = \mathfrak{m} \times 0 \times 0 \). Let \( (X, \xi, v) = \Xi_f(p) \in \mathfrak{m} \times \mathfrak{m}^* \times V \). Then,

\[
df(p)(Y, \eta, w) = \eta(X) - \xi(Y) + \omega_C(v, w)
\]

for all \( (Y, \eta, w) \in \mathfrak{m} \times \mathfrak{m}^* \times V \). Since \( f \) is \( G \)-invariant, we have \( df(p)(\mathfrak{m} \times 0 \times 0) = 0 \), so \( \xi = 0 \). Also, \( G \)-equivariance implies that for all \( v \in V \) and \( h \in H \), we have \( df(p)(0, h \cdot v, w) = df(p)(0, 0, w) \), so \( \omega_C(v, h \cdot w) = \omega_C(v, w) \). Since \( \omega_C \) is \( H \)-invariant, this implies \( \omega_C(h^{-1}v - v, w) = 0 \) for all \( v \in V \) and \( h \in H \), so \( v \in V^H \). Thus, \( \Xi_f(p) = (Y, 0, v) \in \mathfrak{m} \times 0 \times V^H = T_p Z \).

**Lemma 4.16.** For all open set \( U \subseteq M/\mu K \) and \( f, g \in \mathcal{O}_I(U) \), we have \( \{ f, g \} \in \mathcal{O}_I(U) \).
Proof. We identify $M / \mu K$ with $\mu^{-1}(0)^{\mu_\text{ss}} / G$. Let $\Pi : \mu^{-1}(0)^{\mu_\text{ss}} \to \mu^{-1}(0)^{\mu_\text{ss}} / G$ be the quotient map. Then, $\{f,g\} \in \Omega(U)$ if and only if the pullback $\Pi^*\{f,g\} : \Pi^{-1}(U) \to \mathbb{C}$ is holomorphic. This is a local statement, so we may suppose that $\Pi^{-1}(U) = \mu^{-1}(0)^{\mu_\text{ss}} \cap U'$ for some $G$-invariant open set $U' \subseteq M^{\mu_\text{ss}}$ such that $\Pi^*f$ and $\Pi^*g$ extend to holomorphic $G$-invariant functions $f, g : U' \to \mathbb{C}$. Then, it suffices to show that $\Pi^*\{f,g\} = \{f,g\}_{|\Pi^{-1}(U)}$. Since $f, g$ and $\omega_c$ are $G$-invariant, so is $\{f,g\}$. Thus, it suffices to show that $\Pi^*\{f,g\}(x) = \{\hat{f},\hat{g}\}(x)$ for every polystable point $x \in \Pi^{-1}(U) \cap \mu^{-1}(0)^{\mu_\text{ss}}$. We have $p \in Z$ for some $Z$ as above. Let $S = \Pi(Z)$, $\pi = \Pi|_Z : Z \to S$ and $i : S \to M$, as before. Then, we have $d\pi(\Xi_f(p)) = \Xi_{\hat{f}}(\pi(p))$, where $\Xi_f$ is the Hamiltonian vector field of $f$ on $U$ for all $v \in T_p Z$.

$$\omega_c(d\pi(\Xi_f(p)), d\pi(v)) = \omega_c(\Xi_f(p), v) = df_p(v) = d\pi(\{f,\pi\}(d\pi(v))) = (\omega_c)(\Xi_f(p), d\pi(v)).$$

Thus, $\{f,g\}(\Pi(p)) := \eta_c(\Xi_f(\pi(p)), \Xi_g(\pi(p))) = \eta_c(d\pi(\Xi_f(p)), d\pi(\Xi_g(p))) = \omega_c(\Xi_f(p), \Xi_g(p)) = \{\hat{f},\hat{g}\}(p)$.

So $\Pi^*\{f,g\} = \{\hat{f},\hat{g}\}_{|\Pi^{-1}(U)}$ and hence $\{f,g\} \in \Omega(U)$. □

It is clear from its construction that the Poisson bracket is uniquely determined by the property that the inclusions of the strata are Poisson maps. Thus, we have:

**Proposition 4.17.** There is a unique Poisson bracket on $\Omega$ such that for every $S \subseteq M / \mu / K$ in the orbit type partition, the inclusion $S \hookrightarrow M / \mu / K$ is a Poisson map with respect to $(\omega_S|_C = \omega_{\text{aff}} + i\omega_{\text{K}})$. Thus, $(M / \mu / K, \Omega)$ is a stratified complex-symplectic space.

**4.9. Local model.** Let $H$ be a complex reductive group and $H \to \text{Sp}(V, \omega_c)$ a complex-symplectic representation. Then, as explained in §4.2, we can view the affine GIT quotient $V_0 := \Phi^{-1}_V(0)/H$ as a hyperkähler quotient. Hence, if $\mathcal{O}_{V_0}$ denotes the underlying complex analytic structure of $V_0$, then $(V_0, \mathcal{O}_{V_0})$ together with the $H$-orbit type partition is a stratified complex-symplectic space. In particular, there is a canonical Poisson bracket on $\mathcal{O}_{V_0}$ (which does not depend on the choice of quaternionic structure, as can be seen from its construction). Moreover, $\mathcal{O}_{V_0}(V_0)$ contains $\mathbb{C}[\Phi^{-1}_V(0)]^G$ and it is easy to see from the proof of Proposition 4.16 that this Poisson bracket restricts to the usual one on $\mathbb{C}[\Phi^{-1}_V(0)]^G$. Recall from Proposition 4.3 that $\Phi^{-1}_V(0)/H$ provides a local model for the complex structure of $M / \mu / K$. Here we show that $V^{-1}_V(0)/H$ is also a local model for the Poisson structure.

**Proposition 4.18.** Let $x \in M / \mu / K$. Let $p \in \mu^{-1}(0)$ be a point above $x$, let $H = G_p$, let $V$ be the complex symplectic slice at $p$, and let $\Phi_V : V \to \mathfrak{h}^*$ the complex-moment map. Then, $H$ is a complex reductive group and $x$ has a neighbourhood which is isomorphic as a stratified complex-symplectic space to a neighbourhood of $[0]$ in $\Phi^{-1}_V(0)/H$.

**Proof.** By Proposition 4.3, all it remains to check is that the biholomorphism $U \to U'$ respects the Poisson brackets. Let $\eta_c$ be the complex-symplectic form on the local model $E = G \times_H (\mathfrak{h}^* \times V)$ and $\nu_c : E \to \mathfrak{g}^*$ the complex moment map. Since the hyperkähler local normal form for $(M, K, \mu)$ is an isomorphism of complex-symplectic manifolds, we only need to show that the isomorphism $\nu_c^{-1}(0)/G \cong \Phi^{-1}_V(0)/H$ of affine varieties (see proof of Proposition 4.3) respects the Poisson brackets. This follows from the fact that $V$ is a complex-symplectic submanifold of $E$ via the embedding $\iota : V \hookrightarrow G \times_H (\mathfrak{h}^* \times V)$, $v \mapsto [1,0,v]$ and that the isomorphism in question descends from this map. □

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