Using of unitarity equations for the calculation of fermion interaction amplitudes in the superstring theory

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Abstract

The unitarity equations for the boson interaction amplitudes in the superstring theory are used to calculate the interaction amplitudes including the Ramond states, which are 10-fermion and Ramond bosons. The n-loop, 4-point amplitude with two massless Neveu-Schwarz bosons and two massless Ramond states is obtained explicitly. It is shown that, in addition, the unitarity equations require some integral relations for local functions determining the amplitude. For the tree amplitude the validness of the above integral relations is verified.

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1 Introduction

A calculation of interaction amplitudes with Ramond state legs (they are 10-dimensional fermions and Ramond bosons) for the Ramond-Neveu-Schwarz superstring \([1]\) looks \([1, 2, 3]\) as a very complicated task. We propose to calculate the desired amplitudes from the unitarity equations for amplitudes of the interaction of the Neveu-Schwarz boson states. In this case the desired \(n\)-loop amplitude is given by integral of an expression, which is represented through measures and vacuum correlators on the genus-\(n\) supermanifold where certain handles are degenerated. Now the measures and vacuum correlators are known \([4, 5, 6, 7]\) for any genus and for all spin structures, the Ramond sector being included.

To obtain the desired representation of the Ramond state interaction amplitude, we calculate the fermion loop discontinuity in the center mass energy (is more exact, in a relevant by-linear 10-invariant) of the boson state amplitude. The discontinuity is represented by an integral of by-linear product of the amplitudes with fermions in the intermediate state. As far as Ramond states are constructed using Majorano-Weyl spinors \([1]\), the amplitude contains only odd number of Dirac matrices sandwiched by spinors. A number of the above spinor structures is less, than number of conditions, which is possible to satisfy locally when the discontinuity is represented in the form of the trace of Dirac matrix products. Thus, the unitarity equations require, in addition, certain integral relations to be valid.

The above calculation use some general properties of integration measures and vacuum correlators when certain handles are degenerated. Details of expressions given in \([1, 2, 6, 7]\) are not cruel for the calculation.

In the present paper we calculate the \(n\)-loop, 4-point amplitude of two massless NS bosons and two massless Ramond states. The \(n\)-loop amplitude is represented by the integral of local functions determining the integrand for the \((n + 1)\)-loop massless boson scattering amplitude in the region where one of the handles is degenerated.

The paper is organized as follows. In section 2 expression for the boson emission amplitude is given along with some formulas necessary for the following consideration. In section 3 the two-particle unitarity equations are derived. In section 4 the unitarity equations due the boson loop are considered. In section 5 the fermion loop unitarity equations are discussed. The 4-point amplitude with two NS massless bosons ant two massless Ramond states is calculated for any number of loops. Additional integration relations are obtained, which are necessary to provide the unitarity equations. The tree approximated relations are verified.

2 Expressions for amplitudes

The superstring amplitude with number of loops \(n\) is given by integral of the sum over super-spin structures. The above super-spin structures are defined on genus-\(n\) complex \((1|1)\) supermanifold \([\text{8}]\) by the super-Schottky groups \([1, 3, 8]\). For \(2\pi\)-twist around \(B_s\)-cycle the corresponding transformation is determined by the Schottky multiplier \(k_s\) along with two fixed points \(U_s = (u_s|\mu_s)\) and \(V_s = (v_s|\nu_s)\) on the supermanifold. Here \(\mu_s\) and
nu are Grassmann partners of local points us and respectively, vs. For the boson handle (the NS case), to 2π-tweak around As-cycle the identical transformation corresponds. For the Ramond type handle 2π-tweak around As-cycle over (ks, us, vs, μs, νs) and their complex conjugated, and over interaction vertex coordinates tj = (zj|θj) on the supermanifold. In doing so the {Ns} set of any (3|2) complex parameters is fixed due to SL(2) symmetry that leads to an additional factor |H({Ns})|^2 in the integrand. The above factor is given in [4]. For each superspin structure in amplitude of interaction m bosons, the integrand is the product F^{(n)} of the integration measure Z^{(n)}_{L,L'}({q, q}) over (ks, us, vs, μs, νs) and their complex conjugated, and over interaction vertex coordinates tj = (zj|θj) is the coordinate of the j-th boson vertex and {q} is the set the super-Schottky group parameters. Further, pj and ϵ(j) is the momentum and, respectively, the polarization tensor of the j-th boson. Furthermore, L (L') is the superspin structure for the holomorphic (anti-holomorphic) movers. Like usual spin structure [10], the super-spin one is given by theta-function characteristics l1s and l2s assigned to the given handle s. In doing so l1s and l2s can be restricted by zero or 1/2. For the boson loop l1s = 0. For the fermion one l1s = 1/2 (l2s = 0 and l1s = 1/2 corresponds to the loop of even spin structure and, respectively, of the odd spin one). The n-loop amplitude A^{(n)} of (pj, ϵ(j)) for interaction of m bosons is given by

$$A^{(n)}(\{p_j\}, \{\epsilon(j)\}) = \frac{g^{2n+2}}{2^n n!} \int |H(\{N_0\})|^2 \sum L, L' Z^{(n)}_{L,L'}({q, q}) \times F^{(n)}_{m}({t_j, \bar{t}_j}, \{p_j\}, \{\epsilon(j)\}, \{q, \bar{q}\}; L; L')(dq d\bar{q} dt d\bar{t}) \tag{1}$$

where g is a coupling constant, and other definitions have been explained above. The integration is performed over (q, q) and over the interaction vertex coordinates (t, \bar{t}) with t = (z|θ). Furthermore, (3|2) complex variables are fixed due to SL(2)-symmetry. For the 4-point massless boson amplitude only even super-spin structures present (including those which contain even number of the fermion loops of the odd spin structure). The multiplier 1/2n is due to the symmetry under the Us = Vs interchange of fixed points Us and Vs for the given handle, and 1/n! appears due to the symmetry under the interchange of the handles. For any boson variable x we define dxd\bar{x} = d(Re x)d(Im x)/(4\pi). For any Grassmann variable η we define \int d\eta = 1. The tension of a string is equal 1/π. The normalizations are as in [4].

Due to factorization in left, right movers [11], the integration measure is given by [4]:

$$Z^{(n)}_{L,L'}({q, \bar{q}}) = (8\pi)^{5n} [\det \Omega^{(n)}_{L,L'}({q, \bar{q}})]^{-5} Z^{(n)}_{L}({q}) Z^{(n)}_{L'}({q}) \tag{2}$$

where Z^{(n)}_{L}({q}) is a holomorphic function of {q}, and Ω^{(n)}_{L,L'}({q, \bar{q}}) is expressed through the

\^1 The overline denotes the complex conjugation.
period matrix $\omega^{(n)}(\{q\}, L)$:

$$
\Omega^{(n)}_{L; L'}(\{q, \overline{q}\}) = 2\pi i[\omega^{(n)}(\{q\}, L') - \omega^{(n)}(\{q\}, L)].
$$

(3)

Because of the boson-fermion mixing the period matrix depends $[12, 4]$ on $L$. The holomorphic function in (3) is given by

$$
Z_L^{(n)}(\{q\}) = \tilde{Z}^{(n)}(\{q\}, L) \prod_{s=1}^{n} \frac{Z(1)(k_s; l_{1s}, l_{2s})}{k_s^{3-2l_{1s}}/2 (u_s - v_s - \mu_s v_s)}
$$

(4)

where the theta-characteristics $(l_{1s}, l_{2s})$ are equal 0 or 1/2, function $\tilde{Z}^{(n)}(\{q\}, L)$ is given in $[4]$, and

$$
Z(1)(k; l_1, l_2) = (-1)^{2l_1+2l_2} 16^{2l_1} \prod_{p=1}^{\infty} \frac{[1 + (-1)^2 k^2 k_p (2l_1-1)/2]^8}{[1 - k^2]^8}.
$$

(5)

As it was already mentioned, $F^{(n)}_{m}(\{t_j\}, \{p_j\}, \{\epsilon^{(j)}\}, \{q, \overline{q}\}; L; L')$ in $[4]$ is the vacuum expectation of the interaction vertex product. The above vertex for the massless boson with a 10-momentum $p = \{p_M\}$ and polarizations $\zeta = \{\zeta_M\}$ and $\zeta' = \{\zeta'_M\}$ for left and right movers $(M = 0, \ldots, 9)$, is given (with our normalization) by $[4]$

$$
V(t, \overline{t}; p; \zeta) = 4[\zeta D(t) X(t, \overline{t})][\overline{\zeta'} D(t) X(t, \overline{t})] \exp[ipX(t, \overline{t})]
$$

(6)

where for two 10-vectors $a$ and $b$ the scalar product $ab = a_M b^M$ is defined using the ” mainly + ” metrics. Through $X^N(t, \overline{t})$ the scalar superfield of the string is denoted. Further, $p\zeta = p\zeta' = 0$, $p^2 = 0$, and the spinor derivative $D(t)$ is

$$
D(t) = \partial \partial_{\bar{z}} + \partial_{\bar{\eta}}
$$

(7)

Function $F^{(n)}_{m}(\{t_j, \overline{t}_j\}, \{p_j\}, \{\epsilon^{(j)}\}, \{q, \overline{q}\}; L; L') \equiv F^{(n)}_{m}(\{t_j, \overline{t}_j\}, \{p_j\}, \{\epsilon^{(j)}\}; \{q, \overline{q}\}; L; L')$ in $[4]$ is given $[4]$ through the scalar superfield vacuum correlator $\hat{X}_{L; L'}^N(t, \overline{t}_j; \overline{t}; \{q\}) \equiv \hat{X}(j, l)$ by the integral over Grassmann variables $\{\eta_j; \overline{\eta}_j\}$ assigned to the $j$-th boson

$$
F^{(n)}_{m}(\{t_j, \overline{t}_j\}, \{p_j\}, \{\epsilon^{(j)}\}) = \int (d\eta d\overline{\eta}) \exp \left[ -\frac{1}{2} \sum_{j,l} (\hat{\kappa}_j + ip_j)(\hat{\kappa}_l + ip_l)\hat{X}(j, l) \right]
$$

(8)

where the operator $\hat{\kappa}_j$ is $\hat{\kappa}_j = \kappa_j + \overline{\kappa}_j$ with $\kappa_j = 2\eta_j \zeta(j) D(t_j)$ and $\kappa'_j = 2\eta_j \zeta'(j) D(t_j)$. Here $\zeta(j)$ and $\overline{\zeta(j)}$ determine the polarization of the $j$-boson. So $\zeta(j)M \overline{\zeta(j)N} = \epsilon^{(j)}_{M N}$. All particle momenta are implied to be ”entering” to diagram.

The vacuum correlator is given through Green function $R^{(n)}_{L}(t, t'; \{q\})$ along with the super-scalar functions $J^{(n)}_{s}(t; \{q\}; L)$, having periods $(s = 1, \ldots, n)$:

$$
4\hat{X}_{L; L'}^N(t, \overline{t}; t', \overline{t}'; \{q\}) = R^{(n)}_{L}(t, t'; \{q\}) + \overline{R^{(n)}_{L}(t, t'; \{q\})} + I^{(n)}_{L L'}(t, \overline{t}; t', \overline{t}'; \{q\})
$$

(9)
\[ I^{(n)}_{LL'}(t, \bar{t}; t', \bar{t}'; \{q, \bar{q}\}) = [J^{(n)}_{s}(t; \{q\}; L) + \overline{J^{(n)}_{s}(t; \{q\}; L')}] [\Omega^{(n)}_{LL'}(\{q, \bar{q}\})]^{-1} \]
\[ \times [J^{(n)}_{r}(t'; \{q\}; L) + \overline{J^{(n)}_{r}(t'; \{q\}; L')}] \] (10)

Green’s function is normalized by
\[ R^{(n)}_{L}(t, t'; \{q\}) = \ln(z - z' - \partial \partial') + \tilde{R}^{(n)}_{L}(t, t'; \{q\}) \] (11)
where \( \tilde{R}^{(n)}_{L}(t, t'; \{q\}) \) has no a singularity at \( z = z' \). At \( t = t' \), as is usual, \( R^{(n)}_{L}(t, t'; \{q\}) \) is \( \tilde{R}^{(n)}_{L}(t, t; \{q\}) \). The correlator (11) at the same points is calculated accordingly to this agreement.

3 Two-particle unitarity equations

The domain where certain Schottky multipliers \( |k_j| \to 1 \) can be excluded (4) from the integration region in (4). Then the unitarity equations are due (4) to the region where certain \( k_j \to 0 \). Two-particle cut is due to the region where only Schottky multiplier \( k \to 0 \). In that case in (4) it is convenient to fix local coordinate of one of the interaction vertices, for example, \( z_4 \), along with limiting points \( U = (u|\mu) \) and \( V = (v|\nu) \) of the degenerated transformation discussed. Moreover, we take \( \mu = \nu = 0 \). Then \( |H(\{N_0\})|^2 \) in (4) is given by (4):
\[ |H(\{N_0\})|^2 = |(z_4 - u)(z_4 - v)|^2 . \] (12)

We calculate the discontinuities in \( s = -(p_1 + p_2)^2 = -(p_3 + p_4)^2 \). They arise from the region where both \( z_1 \) and \( z_2 \) being outside the Schottky circles, go to \( u \) or to \( v \). The contribution to integral from the \( z_2 \to v \) region is the same as from the \( z_2 \to u \) one. So, we consider \( z_2 \to u \), the result being multiplied by factor 2. Besides, the result needs to be multiplied by factor \( n \) since each of \( n \) handles can be the degenerated handle discussed.

If \( z_2 \to u \), then \( z_1 \to u \) or \( z_1 \to v \). The second case is reduced to the first one by the appropriate Schottky transformation of \( z_1 \). Thus \( z_1 \) appears inside the circle containing \( u \). Therefore we consider \( z_1 \to u \) and \( z_2 \to u \) for \( z_1 \) can be both outside the Schottky circle, and inside it, and \( z_2 \) lays outside of the Schottky one. In that case
\[ |z_2 - u| \geq \sqrt{k}|u - v| . \] (13)

This formula follows from an expression (4, 4) of the Schottky transformation through \( u, v \) and \( k \) at \( k \to 0 \). For simplicity we calculate that part of the discontinuity, which contains the tree amplitude by the \((n - 1)\)-loop one. It is determined by a configuration where limiting points of remaining basic Schottky group transformations are not closed to \( u \). We restrict ourselves by the discussion of the massless states.

The singularity at \( k \to 0 \) of the integration measure (2) arises due to factor \( 1/k^{(3-2l_1)/2} \) in (3). The leading singularity \( \sim 1/k^{3/2} \) for the NS handle (then \( l_1 = 0 \), see section 2) is
cancelled for the sum over two terms \( l_2 = 0 \) and \( l_2 = 1/2 \) since (11) is even function of \( \sqrt{k} \). So the leading singularity in integral always is \( \sim 1/|k|^2 \). The factor at \( 1/|k|^2 \) depends \( \ln |k| \) by means of the period matrix (see the Appendix). Next order in degrees of \( k \) terms do not contribute to the massless state cuts considered. With the required accuracy we can present the contribution \( W \) to (11) from discussed region as the following integral over the loop momentum \( \tilde{p}_1 \) of the sum of expressions factorized in initial and final states

\[
W = \int \frac{d^{10} \tilde{p}_1}{\tilde{p}_1^1 \tilde{p}_2^2} \sum_{(\lambda,\lambda')} A_{(0)}(\{p, \epsilon\}_i, \tilde{p}_1 \lambda, \lambda') \tilde{A}_{(n-1)}(\lambda, \lambda'; \tilde{p}_1, \{p, \epsilon\}_f),
\]

where \( \tilde{p}_1 + \tilde{p}_2 = P = p_1 + p_2 = -(p_3 + p_4) \) and the sum goes over polarizations \( (\lambda, \lambda') \) of the intermediate state for holomorphic \( (\lambda) \) and anti-holomorphic \( (\lambda') \) movers. The first multiplier under the sum depends on momenta and polarizations \( \{p, \epsilon\}_i = (p_1, p_2, \epsilon^{(1)}, \epsilon^{(2)}) \) of the initial state and of the intermediate state, as well. The second one depends on parameters of the intermediate state and on parameters \( \{p, \epsilon\}_f = (p_3, p_4, \epsilon^{(3)}, \epsilon^{(4)}) \) of the final state. Both multipliers are calculated for \( \tilde{p}_1^2 = \tilde{p}_2^2 = 0 \). The cut due to the nullification of a denominator \( (\tilde{p}_1^2 \tilde{p}_2^2) \) is calculated without an effort. From the comparison of eq.(14) with the unitarity equation it follows that \( \tilde{A}_{(0)}(\{p, \epsilon\}_i, \tilde{p}_1 \lambda, \lambda') \) and \( \tilde{A}_{(n-1)}(\lambda, \lambda'; \tilde{p}_1, \{p, \epsilon\}_f) \) are the tree and \( (n-1) \)-loop amplitudes. To derive (14) the considered contribution \( W \) to (11) is first represented by

\[
W = \int d^{10} \tilde{p}_1 \int \frac{d^2 z_1}{4\pi} \frac{d^2 z_2}{4\pi} dy \frac{1}{2} \exp \left[ \frac{p_1 p_2}{2} \ln |z_1 - z_2| - \frac{p_1 \tilde{p}_1}{2} \ln |z_1 - u| \right]
\]

\[
- \frac{p_2 \tilde{p}_1}{2} \ln |z_2 - u| + \frac{y \tilde{p}_1^2}{4} \sum_r \hat{O}_r(\{p, \epsilon, z, \tilde{z}\}_i, u, \tilde{u}, \tilde{p}_1) O_r(\{p, \epsilon\}_f, \tilde{p}_1),
\]

where all of 10-vectors are Euclidian, and \( y = \ln |k| \). Function \( \hat{O}_r(\{p, \epsilon, t, \tilde{t}\}_i, u, \tilde{u}, \tilde{p}_1) \) has the translation symmetry. In addition, by dimensional reasons, it receives multiplier \( 1/|\ell|^2 \) if any of \( z_1 \) and \( z_2 \) is multiplied by \( \ell \). Therefore, if, instead of \( z_1 \), one defines a new variable \( z = (z_1 - u)/(z_2 - u) \), then the dependence on \( z_2 \) and on \( y \) is extracted in (13) in the form of the factor \( |z_2 - u|^{-2} \exp[x(\tilde{p}_1^2 - \tilde{p}_2^2)/4 + y\tilde{p}_1^2/4] \) where \( x = \ln |z_2 - u| \). In this case \(-\infty < y < 0 \) and, due to (13), \( 0 < x < y/2 \). The integral over \( z_2 \) and \( y \) being calculated, is \( 4/|\tilde{p}_1^2(\tilde{p}_1^2 + \tilde{p}_2^2)| \). Since this expression is further multiplied by the function, which is symmetrical under \( \tilde{p}_1 \leftrightarrow \tilde{p}_2 \), it can be replaced by \( 2(\tilde{p}_1^2 \tilde{p}_2^2)^{-1} \). Then the required (14) appears. The tree amplitude arises in the form of integral where interaction vertex coordinate for intermediate particles with the \( \tilde{p}_1 \) momentum is fixed to be \( (0|0) \), and for the remaining intermediate state it is \( (\infty|0) \). In integral for the \( (n-1) \)-loop amplitude, the vertex coordinates corresponding to the above states are fixed to be \( (v|0) \) and \( (u|0) \). Local coordinate \( z_4 \) is fixed, too. To derive (15), the vacuum expectation (11) is represented by the integral (8) over \( (d\eta d\bar{\eta}) \). Further, at \( k \to 0 \), every term \( \tilde{W} \) of the integrand is represented by the Gauss integral over the loop momentum as it follows
\[ W = \int d^{10}\tilde{p}_1 \exp \left[ \tilde{G}_0 + \tilde{G}_1 \tilde{p}_1^2 + \tilde{B}_1 \tilde{p}_1 \right] \exp \left[ G_1 \tilde{p}_1^2 + G_12 \tilde{p}_1 \tilde{p}_2 + G_2 \tilde{p}_2^2 + B_i \tilde{p}_1 \right. \\
\left. + B_2 \tilde{p}_2 + G_0 \right] \mathcal{O}_L((p, \zeta, \tilde{p}_1, \{t\}, \{q\})\mathcal{O}_{L'}((p, \zeta', \tilde{p}_1, \{t\}, \{q\})) \\
\times \tilde{Z}^{(n-1)}_{L, L'}((q, \tilde{q}))(z_4 - u)(z_4 - v)^2, \]  

where the integrand depends on the type of the degenerated loop. Factors \( \tilde{G}_0, \tilde{G}_1 \) and \( \tilde{B}_1 \) both depend on the initial state, on \( y = \ln |k| \) and on \( (u, \bar{u}) \). The second exponent depend on the final state and on \( \{q\} \) (with \( k = 0 \) for the degenerated handle). It depends also on super-spin structures \((\tilde{L}, \tilde{L}')\) associated with all handles, except for the degenerated handle. In \( \mathcal{O}_L \) the multiplier \( \mathcal{O}_{L'} \) is given explicitly. Further, \( \{p\}, \{\zeta\}, \{\zeta'\} \) and \( \{t, \bar{t}\} \) denote the sets of momenta, polarizations and vertex coordinates for the bosons in the initial and final states. Factors \( \mathcal{O}_L((p, \zeta, \tilde{p}_1, \{t\}, \{q\}) \) and \( \mathcal{O}_{L'}((p, \zeta', \tilde{p}_1, \{t\}, \{q\}) \) arise due to an expansion in small \( k \), \( (z_1 - u) \) and \( (z_2 - u) \) of the holomorphic or, respectively, anti-holomorphic function in \( \mathcal{O}_L \) and in \( \mathcal{O}_{L'} \). Because of the non-holomorphic factor in \( \mathcal{O}_L \) and due to the last term in \( \mathcal{O}_{L'} \), \( \mathcal{O}_L((p, \zeta, \tilde{p}_1, \{t\}, \{q\}) \) for \( n > 1 \) depends also on \( \{\bar{t}\}, \{\bar{q}\} \) and \( L' \). Correspondingly, the second of the discussed factors depends on \( \{\bar{t}\}, \{\bar{q}\} \) and \( L' \). Both the discussed multipliers may be polynomial in \( \tilde{p}_1 \). Each of multipliers is the sum of expressions factorized in the initial and final states. Factor \( \tilde{Z}^{(n-1)}_{L, L'}((q, \tilde{q}) \) appears due to the integration measure. Below we obtain expressions for the functions discussed. In deriving \( \mathcal{O}_L \) we shall use following formulas for a determinant \( n \)-dimensional matrix \( \Omega \) and for the quadratic form \( J \Omega^{-1} J \) of \( n \) functions \( J_s \) (where \( s = 1, \ldots, n \)):  

\[ J \Omega^{-1} J = [J_1(1 - \Omega_{i1j}\Omega^{-1}_{j1i})J_{i1}]^2[\Omega_{i1} - \Omega_{i1}\Omega^{-1}_{j1i}\Omega_{j1}]^{-1} + J_{i1}[\Omega^{-1}]_{j1i}, \]

\[ \det \Omega = \det(\Omega_{j1i})[\Omega_{i1} - \Omega_{i1}\Omega^{-1}_{j1i}\Omega_{j1}]^{-1}, \]  

where the summation on twice repeating indexes is implied, and alphabetic indexes in \( \mathcal{O}_L \) is run from 2 up to \( n \). The coefficient at \( J_s J_s \) (where \( p, s = 1, \ldots, n \) in \( \mathcal{O}_L \) are non other than the appropriate element \( (\Omega^{-1})_{ps} \) given through elements of \( \Omega \). The second formula follows from equation \( \ln \det \Omega = \text{trace} \ln \Omega \) once \( \text{trace} \ln \Omega \) is calculated in terms of quantities in \( \mathcal{O}_L \).

### 4 Unitarity for the boson loop

In a case of the boson loop at \( k \to 0 \) the scalar function \( J^{(n)}_1(t; \{q\}; L) \equiv J_1(t) \), as well as elements \( \omega^{(n)}((q), L)_{11} \equiv \omega_{11} \) and \( \omega^{(n)}((q), L)_{11} \equiv \omega_{11} \) matrices of the periods, are expressed through holomorphic Green’s function \( \mathcal{F}_L^{(n-1)}(t, t'; \{q\}) \equiv R(t, t') \) and scalar functions \( J^{(n-1)}_1(t; \{q\}; \tilde{L}) \equiv J_1(t) \) on the genus-(\( n - 1 \)) supermanifold formed by other handles.
as follows:

\[ 2\pi i \omega_{11} = \ln k + R(U, U) + R(V, V) - R(U, V) - R(V, U), \]
\[ 2\pi i \omega_{1l} = J_l(U) - J_l(V), \quad J_l(t) = R(t, U) - R(t, V), \]

where \( U = (u|0) \) and \( V = (v|0) \). For the boson string and in the NS sector of the superstring [13, 14, 12, 13, 16] these relations directly follow from a representation of the discussed functions by Poincare series. In particular, for the boson string the mentioned series are given in [8] (in its Appendix B). For the NS sector the discussed expressions are obtained by the replacement of an interval \((z_1 - z_2)\) by a superinterval \((z_1 - z_2 - \vartheta_1 \vartheta_2)\). For zero Grassmann moduli the relations are evidently true for the Ramond sector, too. Eqs. (18) are necessary for the unitarity equations could be to take place. So it is naturally expect that the relations is true even though the Grassmann moduli are arbitrary, and it is really so in the case (see the Appendix).

For simplicity we assume the boson loop for both movers: \( l_1 = l_1' = 0 \). Then exponents in (18) are found to be:

\[
B_1 = -i \sum_{j=3}^{4} (\hat{k}_j + ip_j)\hat{X}(t_j, \bar{t}_j; V, \bar{V}) ,
\]
\[
B_2 = -i \sum_{j=3}^{4} (\hat{k}_j + ip_j)\hat{X}(t_j, \bar{t}_j; U, \bar{U}) ,
\]
\[
G_0 = -\frac{1}{2} \sum_{j,l=3}^{4} (\hat{k}_j + ip_j)(\hat{k}_l + ip_l)\hat{X}(t_j, \bar{t}_j; \bar{t}_4, \bar{t}_4) ,
\]
\[
G_{12} = \hat{X}(U, \bar{U}; V, \bar{V}) ,
\]
\[
G_1 = \hat{X}(V, \bar{V}; V, \bar{V}) + \ln |u - v| ,
\]
\[
G_2 = \hat{X}(U, \bar{U}; U, \bar{U}) ,
\]
\[
\tilde{G}_0 = -\frac{1}{2} \sum_{j=3}^{4} (\hat{k}_j + ip_j)\hat{X}(t_j, \bar{t}_j; \bar{t}_4, \bar{t}_4) ,
\]
\[
\tilde{G}_1 = \frac{1}{2} \ln |k| ,
\]
\[
\tilde{B}_1 = i \sum_{j=1}^{2} (\hat{k}_j + ip_j)\hat{X}(t_j, \bar{t}_j; \bar{t}_4, \bar{t}_4) .
\]

where \( U = (u|0) \), \( V = (v|0) \), \( \hat{k}_j \) is the same as in [8], and \( \hat{X}(t, \bar{t}; t', \bar{t}') \) is the correlator [8] on the genus-(\( n - 1 \)) supermanifold. Further, \( \tilde{G}_0 \) and every term in \( \tilde{B}_1 \) represents the genus-0 correlator for the given points. To calculate the pre-exponent in (18), one leaves terms \( \geq 1 \), the estimation \( |z_1 - u| \sim |z_2 - u| \sim \sqrt{|k|} \) and \( \vartheta_1 \sim \vartheta_2 \sim |k|^{1/4} \) being used. In the considered case the pre-exponent does not depend on \( \tilde{p}_1 \) and \( \tilde{p}_2 \). To check (19), one substitutes (19) in (16) and calculates the integral, the result being compared with (17). In doing so eqs. (18) are used where the index ”1” concerns to the degenerated handle.

As an example, we consider one-loop amplitude \( (n = 1) \). Then the integration measure in (16) is equal to 1, and \( \hat{X}(t, \bar{t}; t', \bar{t}') \) is \( (\ln |z_1 - z_2 \vartheta_1 \vartheta_2|)/2 \). Product of the two first factors in the pre-exponent is \( O(b)(\{\zeta\})O(b)(\{\zeta'\}) \) where the holomorphic function \( O(b)(\{\zeta\}) \) can be written down as the sum over 10-vectors \( \tilde{\zeta}_1 \) and \( \tilde{\zeta}_2 \) to be polarizations of bosons carrying
momtum \( \tilde{p}_1 \) and, respectively, \( \tilde{p}_2 \):

\[
O_{(0)} = \sum_{\tilde{z}_1, \tilde{z}_2} \left[ -\tilde{z}_1 \tilde{z}_2 + \sum_{j=1}^{2} \tilde{z}_1 (\kappa_j + ip_j) \partial_j \right] \sum_{l=1}^{2} \left( \frac{1}{2} (\kappa_j + ip_j) \partial_j \right)
\]

\[
\times \left[ -\tilde{z}_1 \tilde{z}_2 + \sum_{j=1}^{2} \tilde{z}_1 (\kappa_j + ip_j) \partial_j \right] \sum_{l=1}^{2} \left( \frac{1}{2} (\kappa_j + ip_j) \partial_j \right) - \frac{2}{u-v}. \tag{20}
\]

Here \( \kappa \) is the same as in (8). In (20) summation is performed over all independent polarizations. The term \(-2/(u-v)\) appears because from (5) factor \(8/(u-v)\) appears, but not \(10/(u-v)\). This term being the Faddeev-Popov ghost contribution to the unitarity equations, just cancels the non-physical polarizations to (14) (the proof of this statement will be given in other place).

For the \( n \)-loop amplitude, factor \( \hat{Z}_{L,L'}^{(n-1)}(\{q, \bar{q}\}) \) is equal the genus-(\( n-1 \)) measure on the supermanifold formed by all handles, except for the degenerated handle. As for \( n = 1 \), the right part of (20) is reduced to (14) once the integration over \( y, t_1, t_2 \) and \( t_3 = t \) to be performed. One can verify all the unitarity equations for boson loops, at least, for massless states (such check will be given in another place).

### 5 Fermion loop

We derive now the two-particle unitarity equations for a loop with fermion states in the holomorphic sector \( (l_1 = 1/2) \). Then at \( k \to 0 \) the holomorphic Green function preserves the singularity at \( z = u \) and \( z = v \). Thus, the function and, therefore, other functions in (14), differ from corresponding genus-(\( n-1 \)) functions. In particular, Green function \( R_{(0)}(t_1, t_2) \) being the limit of the appropriate genus-1 one (for \( l_2 = 0 \) and for \( 2l_2 = 1 \)), is given by

\[
R_{(0)}(t, t') = \ln(z - z') - \frac{\partial \theta'}{2(z - z')} \left\{ \frac{(z - u)(z' - v)}{z - v(z' - u)} + \frac{(z - v)(z' - u)}{z - u(z' - v)} \right\} \tag{21}
\]

For deriving (21) one may, for example, use genus-1 functions in [4]. The same kind singularity appears in the higher genus functions. When all Grassmann moduli are equal to zero it can be seen from expression [4] for the fermion part of the Green function. Of course, the singularity takes place for non-zero Grassmann modules, too (see the Appendix). If \( z \to u \), but \( |z' - u| \sim 1 \), then Green function \( R_{L}^{(n)}(t, t'; \{q\}) \equiv R(t, t') \) is given by

\[
R(t, t') = R_{(r)}(U, t') - \frac{\partial \phi(t')}{2\sqrt{(z - u)}}, \tag{22}
\]

where \( R_{(r)}(U, t') \) is its regular at \( z \to u \) part, \( U = (u|0) \), and \( \phi(t') \) is a coefficient at the singularity. In this case \( \phi(t') \) has no a singularity at \( z' \to u \) due to the symmetry of the
Green function under the interchange between its arguments. The kindred formula arises, if
\( z \rightarrow \nu \). The discussed singularities are due to the fermion part of the Green function, but
for non-zero Grassmann moduli, the singularities arise also for its boson part and, hence, for
scalar functions \( J^{(n)}_{s}(t; \{q\}; L) \). If \( z \rightarrow z' \rightarrow u \), then
\[
R(t, t') = R(0)(t, t') + R_{(rr)}(U, U) - \frac{\partial\phi(U)}{2\sqrt{(z - u)}} - \frac{\partial'\phi(U)}{2\sqrt{(z' - u)}}
\]
(23)
where \( R_{(rr)}(U, U) \) is the regular at \( z' \rightarrow u \) part. The \( \phi(U) \) is proportional to Grassmann
moduli. When any of arguments of \( R(t, t') \) go to \( U = (u|\mu = 0) \) or to \( V = (v|\nu = 0) \) (or
both they does it) the Green function has no the discussed singularities since \( \theta = 0 \). Then
eqs. (18) are trues in the considered case, too (see the Appendix). From (21) at \( z \rightarrow z' \rightarrow u \)
and (18), \( \tilde{G}_{0} \) in (10) is equal to
\[
\tilde{G}_{0} = -k_{1} + ip_{1})(k_{2} + ip_{2})\left[\frac{1}{2}\ln|z_{1} - z_{2}| - \frac{\vartheta_{1}\vartheta_{2}}{8(z_{1} - z_{2})}\right]
\]
(24)
If the loop is the fermion one for both movers \( (l_{1} = l'_{1} = 1/2) \), then the expression in square
brackets is added by the term, which is a complex conjugated to last term in the above
brackets. Other coefficients for the exponents in (16) are given by (13) where \( \dot{X}(t, \bar{t}; t', \bar{t}') \) it
is the limit at \( k \rightarrow 0 \) of the vacuum correlator (13). Hence at \( z \rightarrow u \), the correlator \( \dot{X}(t, \bar{t}; t', \bar{t}') \)
in (16) is given by
\[
\dot{X}(t, \bar{t}; t', \bar{t}') = \dot{X}_{(v)}(U, \bar{U}; t', \bar{t}') - \frac{\vartheta_{2}(t')}{2\sqrt{(z - u)}}
\]
(25)
At \( z \rightarrow u \) and \( z' \rightarrow u \), it is equal to
\[
\dot{X}(t, \bar{t}; t', \bar{t}') = \dot{X}_{rr}(U, \bar{U}; U, \bar{U}) - \frac{\vartheta_{2}(U)}{2\sqrt{(z - u)}} - \frac{\vartheta'(U)}{2\sqrt{(z' - u)}}
\]
(26)
where \( \vartheta(t) \) has no a singularity at \( z = u \). Further, \( \phi(U) \) is proportional Grassmann moduli.
In so doing \( \vartheta(t) \) depends not only on \( t \), but also on \( \bar{t} \). If \( l'_{1} = 1/2 \) then discussed singular
terms need to be added by their complex conjugated. The discussed expressions for factors
in (16) are checked by the calculation of integral (16) after the substitution of (14), (24) and
of the expressions given below for the pre-exponent.
In particular, \( \ddot{Z}_{L(L)}^{(n-1)}(\{q\}, \{\bar{q}\}) \) in (16) is given by (2) with \( Z_{L}^{(n)}(\{q\}) \) to be replaced by\(^{2}\)\n\( kZ_{L}^{(n)}(\{q\})/16 \) at \( k = 0 \). In addition, \( \omega^{(n)}(\{q\}, L) \) is replaced by \( \ddot{\omega}^{(n-1)}(\{q\}, \bar{L}) \), whose
elements are equal to elements \( [\omega^{(n)}(\{q\}, L)]_{jl} \) for \( j, l = 2, \ldots, n \) at \( k = 0 \). As in (17),
indexes \( j, l = 2, \ldots, n \) are assigned to non-degenerated handles. If \( l'_{1} = 1/2 \), then \( Z_{L}^{(n)}(\{q\}) \)

\(^{2}\) The multiplier 1/16 in the next expression in the text presents since the multiplier 16 in (1) at \( 2l_{1} = 1 \)
arises from the trace of the Dirac matrix product in the fermion loop.
is replaced by $kZ^{(n)}_{L'}(\{q\})/16$ at $k = 0$, and $\omega^{(n)}(\{q\}, L')$ is replaced by $\tilde{\omega}^{(n-1)}(\{q\}, \tilde{L})$, its elements being $[\omega^{(n)}(\{q\}, L')]_{ij}$ at $k = 0$. If $l'_1 = 0$, then $Z^{(n)}_{L'}(\{q\})$ and $\omega^{(n)}(\{q\}, L')$ both are replaced by the corresponding functions on the genus-$(n-1)$ supermanifold. At last, $O_L(\{p, \zeta\}, \tilde{p}_1, \{t\}, \{q\})$ is given by

$$O_L(\{p, \zeta\}, \tilde{p}_1, \{t\}, \{q\}) = \exp \left[ \sum_{j=1}^{2} \frac{(2\eta_j \zeta(j) + ip_j \varphi(j))}{8\sqrt{z_j - u}} \tilde{\Psi} \right],$$

where $\eta_j$ is as in (28), and the 10-vector $\tilde{\Psi}$ depending on the final state parameters, is

$$\tilde{\Psi} = \sum_{j=3}^{4} [2\eta_j \zeta(j) D(t_j) + ip_j \varphi(j)] \varphi(t_j) + i\tilde{p}_1 \varphi(V) + i\tilde{p}_2 \varphi(U),$$

where $D(t)$ is given by (29) and $\varphi(t)$ is defined in (23). Like $\varphi(U)$, the function $\varphi(V)$ is proportional the Grassmann moduli. If $l'_1 = 1/2$, then expression for conjugated factor in (28) is calculated in a similar kind. If $l'_1 = 0$, then it is the same as for the boson loop. As it was already noted, all the expressions above are checked by their substitution in (28) with the following calculation of the integral over $\tilde{p}_1$, the result being compared with (28). The desired $n$-loop amplitude $\tilde{A}^{(n)}$ with two Ramond states is

$$\tilde{A}^{(n)} = \tilde{\psi}(\tilde{p}_2)(\tilde{T}_3^{(n)} + \tilde{T}_1^{(n)}) \psi(\tilde{p}_1),$$

where $\psi(p)$ is the Majorano-Weyl spinor, which satisfies to $\Gamma_{11} \psi(p) = \psi(p)$ and to the Dirac equation $(\Gamma p) \psi(p) = 0$. Here $\Gamma^M$ is the Dirac matrix, and $\Gamma_{11}$ is the product of all of ten matrices above. Furthermore, $\tilde{T}_1^{(n)}$ contains one Dirac matrix, and $\tilde{T}_3^{(n)}$ contains the anti-symmetrized product of three matrices discussed. Hence

$$O_L(\{p, \zeta\}, \tilde{p}_1, \{t\}, \{q\}) = \text{trace}[(T^{(0)}_3 + T^{(0)}_1)(\Gamma \tilde{p}_1)(\tilde{T}_3^{(n-1)} + T^{(n-1)}_1)(\Gamma \tilde{p}_2)] + \ldots$$

where dots denote terms, which are nullified once the integrations in (28) and summation over the spin structures are performed. Further, $T^{(n)}_1$ contains one Dirac matrix, and $T^{(n)}_3$ contains the anti-symmetrized product of three such matrixes. As far as 10-spinor in (32) satisfy to the Weyl condition, the trace of the unity matrix is equal to 16.

Eq.(27) is a 4-degree polynomial in the exponent since the exponent is the sum of products of Grassmann variables. Odd degrees disappear once Grassmann integration is performed. Even degrees being represented as (23), determine the integrand for $\tilde{A}^{(0)}$ and $\tilde{A}^{(n-1)}$. For this purpose (28) is re-written as

$$\tilde{\Psi} = \Psi - iP\Phi + i\tilde{p}[\varphi(V) - \varphi(U)],$$

where $P = p_1 + p_2 = \tilde{p}_1 + \tilde{p}_2$, $\tilde{p} = (p_1 - \tilde{p}_2)/2$,

$$\Phi = \frac{1}{2}[(\varphi(t_3) + \varphi(t_4) - \varphi(U) - \varphi(V)],$$

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\begin{equation}
\Psi = 2\eta_3\zeta(3)D(t_3)\varphi(t_3) + 2\eta_4\zeta(4)D(t_4)\varphi(t_4) + i(p_3 - p_4)\frac{1}{2}[\varphi(t_3) - \varphi(t_4)].
\end{equation}

Function \(\varphi(t)\) is defined in \(\text{[23]}\). In that case \(\text{[27]}\) appears to be

\begin{equation}
O_L(\{p, \zeta\}, \tilde{p}_1, \{t\}, \{q\}) = \exp \left[ \sum_{a=1}^{2} \frac{4\eta a \zeta(a)}{16\sqrt{z_r - u}} \left( -2i(p_1 - p_2)\partial_1 \partial_2 + \sum_{j=1}^{2} \eta_j \zeta_j \zeta_j \zeta_2 \zeta_3 \zeta_4 \right) \right].
\end{equation}

The tree amplitude is calculated from the one-loop case where Grassmann moduli is absent. Representing the factor in \(\text{[10]}\) as \(\text{[30]}\), one obtains that

\begin{equation}
4\sqrt{2T_1^{(0)}} = -\left( \Gamma[\zeta(3)\zeta(4)] \frac{1}{2}(p_3 - p_4) - (\zeta(4)p_3)\zeta_3 + (\zeta(3)p_4)\zeta_4 \right) \frac{\eta_3 \delta(3)3\eta_4}{2(z_3 - z_4)}
\end{equation}

\begin{equation}
	imes \left[ \frac{v - u}{(z_3 - u)(z_4 - v)} + \frac{v - u}{(z_3 - v)(z_4 - u)} \right] \left[ \frac{\zeta_3\tilde{p}_3 - \tilde{p}_4\zeta_4}{4(z_3 - u)(z_3 - v)(z_4 - v)} \right] \left[ \zeta_3\tilde{p}_3 - \tilde{p}_4\zeta_4 \right],
\end{equation}

where \([\ldots]_a\) denotes the anti-symmetrized (with 1/6) expression. In \(\text{[33]}\) terms omitted, which are nullified once the Grassmann integrations are performed. As above, \(v\) or \(u\) is the coordinate of the fermion emission vertex with momentum \(\tilde{p}_1\) and, respectively, \(\tilde{p}_2\). For the amplitude with boson momenta to be \(p_1\) and \(p_2\), the appropriate expressions arise as the limit \(v \to u\) \(\to \infty\) of \(\text{[33]}\) with the following replacement \(v \to u\) along with the replacement of indexes 3 \(\to 1\) and 4 \(\to 2\). Being substituted in \(\text{[14]}\), eqs.\(\text{[33]}\) give (up to the normalization) the amplitude obtained in \(\text{[2]}\). To derive \(\text{[35]}\), one integrates by pert those terms in \(\text{[14]}\), which are proportional \(\eta_1\eta_2p_1p_2/[(z_1 - u)(z_1 - z_2)]\) and \(\eta_1\eta_2p_1p_2/[(z_2 - u)(z_1 - z_2)]\). The first term is integrated over \(z_2\), and the second one is integrated over \(z_1\). Due to \(\text{[24]}\) for \(\tilde{G}_0\), the first term turns to \(-\eta_1\eta_2p_1\tilde{p}_1/[(z_1 - u)(z_2 - u)]\), and the second one appears to be \(\eta_1\eta_2p_1\tilde{p}_1/[(z_1 - u)(z_2 - u)]\). Being multiplied by the \(1/(z_3 - z_4)\) factor, expression in square brackets in first of eqs. \(\text{[33]}\) is represented either as

\begin{equation}
\frac{2(v - u)}{(z_4 - u)(z_4 - v)(z_3 - z_4)} = \frac{(v - u)^2}{(z_3 - u)(z_4 - u)(z_4 - v)} + \frac{(v - u)^2}{(z_3 - v)(z_4 - v)(z_4 - u)},
\end{equation}

or that obtained by replacements \(z_3 \leftrightarrow z_4\) and \(v \leftrightarrow u\). When \(\text{[36]}\) is multiplied by \(\eta_3\eta_4p_3p_4\), the first term in \(\text{[34]}\) is integrated by parts over \(z_3\). Then it appears to be proportional to

\begin{equation}
-\frac{2\eta_3\eta_4(v - u)}{(z_4 - u)(z_4 - v)} \left[ \frac{p_3\tilde{p}_1}{z_3 - v} + \frac{p_3\tilde{p}_2}{z_3 - u} \right].
\end{equation}

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It is useful to note that from (21), in the considered $n = 1$ case the function $\varphi(t)$ in (34) is given by

$$\varphi(t) = -\theta \sqrt{\frac{u - v}{(z - u)(z - v)}}. \quad (38)$$

For the arbitrary $n$ (including $n = 0$) the term $T_3^{(n-1)}$ in (30) turns out from the term $\sim \Psi_{M_1}\Psi_{M_2}\Psi_{M_3}\Phi$, which arises in the expansion over the exponent in (27) from product of the third degree of the sum on $s$ in (27) by $\sim$. Here $\Psi_M$ is 10-component of $\Psi$. Being proportional to $(\tilde{p}_1\tilde{p}_2)$, the discussed terms gives $\sim (\tilde{p}_1\tilde{p}_2)$ terms in (30), which are due to $\text{trace}[T_3^{(0)}(\Gamma\tilde{p}_1)T_3^{(n-1)}(\Gamma\tilde{p}_1)]$. As far as spinor structure in $T_3^{(n-1)}$ is independent from the intermediate state variables, $T_3^{(n-1)}$ is calculated in the unique way as

$$T_3^{(n-1)} = \frac{i}{48}(\Gamma\Psi)(\Gamma\Psi)(\Gamma\Psi)\Phi. \quad (39)$$

As far as $T_3^{(n-1)}$ and $T_3^{(0)}$ both are anti-symmetrized in Dirac matrices, in the discussed $\text{trace}$, there is no terms proportional to scalar products of those vectors, which both belong to $T_3^{(0)}$ or to $T_3^{(n-1)}$. For the same reason, there are no terms where each of $\tilde{p}_1$ and $\tilde{p}_2$ in (30) forms the scalar product with the vector belonging to the same $T_3^{(0)}$ or $T_3^{(n-1)}$. Other terms in the $\text{trace}$ may arise only from those terms in the expansion of the exponent (27), which contain a square of the sum over $r$. If $T_3^{(n-1)}$ is given by (30), the quadratic in $(\tilde{p}_1 + \tilde{p}_2)$ part of the $\text{trace}$ is obtained, too. The remaining terms are quadratic in $(\tilde{p}_1 - \tilde{p}_2)$. They do not contained in the expansion of the exponent (27) since they do not include $\sim \Psi\Psi\Phi$. Hence, for the corresponding terms to be in the unitarity equation for the boson amplitude, the following integration equation must be satisfied

$$<\Psi_M\Psi_N \left[ 1 - \frac{1}{16}[\varphi(V) - \varphi(U)](P^2\Phi + i(P\Psi)) \right] - \frac{i}{4}\Psi_M\Psi_N(\Psi\Phi)\Phi >= 0, \quad (40)$$

where $< ... >$ means the integration of the explicit expression in (10) (for details, see eq.(13) and the text below), which is previously multiplied by the following from (16) factors, the summation over $L, L'$ being performed. The integral is calculated over variables assigned to $A^{(n-1)}$. To calculate $T_1^{(n-1)}$ in (30), we consider $\text{trace}[T_3^{(0)}(\Gamma\tilde{p}_1)T_3^{(n-1)}(\Gamma\tilde{p}_1)]$. Extracting it from terms in (27), we obtain that

$$T_1^{(n-1)} = -i(\Gamma\Psi)\Phi + \frac{i}{8}(\Gamma\Psi)(P\Psi)[\varphi(V) - \varphi(U)]\Phi, \quad (41)$$

where notations are the same as in (27). With (11) and (10) to be taken into account, the remaining linear $\Psi$ terms in (27) originate all remaining terms in (30), except that, which is proportional to $\text{trace}[(\Gamma\tilde{p})T_1^{(0)}]$ times $\text{trace}[(\Gamma\tilde{p})T_1^{(n-1)}]$. In addition, it can not be originated by the unity in (27) since it does not contain $\Psi$. Hence the integration relation arises to be

$$<16 + \text{trace}\left[\frac{1}{2}(\Gamma T_1^{(n-1)})(\Gamma\tilde{p})\right] + \sqrt{2}[\varphi(V) - \varphi(U)][P^2\Phi + (\Psi P)] >= 0, \quad (42)$$
where, as in (10), $< ... >$ means integration of the explicit expression in (10) multiplied by all following from (10) factors, the summation over $(\bar{L}, L')$ being performed. Other notations are the same as in (27).

Thus, the $(n-1)$-loop amplitude $\tilde{A}^{(n-1)}$ for the transition of two massless Ramond states (carrying $\bar{p}_1$ and $\bar{p}_2$) to two massless bosons with momenta and polarizations to be $(p_3, \zeta(3))$ and $(p_4, \zeta(4))$, is given by

$$A^{(n-1)} = \frac{2g^{2n}}{2n(n - 1)!} \int |(z_4 - u)(z_4 - v)|^2 \sum_{\bar{L}, L'} \tilde{Z}^{(n-1)}_{\bar{L}, L'}(\{q, \bar{q}\}) \tilde{F}^{(n-1)}_{\bar{L}, L'} \times \mathcal{O}_L \mathcal{O}_{L'}(dqd\bar{q}dt)',$$  \hspace{1cm} (43)

where $g$ is a coupling constant, and summation is performed over super-spin structures on the genus-$(n-1)$ supermanifold. The sum includes odd superspin structures due to the contribution to the unitarity from the fermion loop with $l_2 = 1/2$. The integrand in (13) is expressed through functions determining the $n$-loop amplitude (11) in the region where $k \to 0$. In doing so $Z^{(n-1)}_{\bar{L}, L'}(\{q, \bar{q}\})$ is calculated through $Z^{(n)}_{\bar{L}, L'}(\{q, \bar{q}\})$ as it was explained next eq.(26). Among other things, it depends on coordinates $v$ and $u$ of the fermion vertices. Further, $\tilde{F}^{(n-1)}_{\bar{L}, L'}$ is calculated through $\tilde{F}^{(n-1)}_{\bar{L}, L'}(\eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2)$, which is the expression in (8) integrated over $(\eta_3, \bar{\eta}_3)$ and $(\eta_4, \bar{\eta}_4)$. In this case the genus-$n$ correlator (1) is replaced by $\tilde{X}(t, \bar{t}; t', \bar{t}')$ to be the above correlator at $k \to 0$. Besides, $(z_1|\vartheta_1) \to (v|0)$, $(z_2|\vartheta_2) \to (u|0)$, $p_1 \to \bar{p}_1$ and $p_2 \to \bar{p}_2$. If $\bar{p}_1$ and $\bar{p}_2$ are carried by the Ramond bosons, then

$$\mathcal{O}_L = \tilde{\psi}(\bar{p}_2)(T_3^{(n-1)} + T_1^{(n-1)})\tilde{\psi}(\bar{p}_1)$$ \hspace{1cm} (44)

where $\psi(p)$ is the spinor, and other definitions are given in (13) and (11). In this case $\mathcal{O}_{L'}$ is calculated in the like manner. In doing so $\tilde{F}^{(n-1)}_{\bar{L}, L'}$ in (13) is equal to $\tilde{F}^{(n-1)}_{\bar{L}, L'}(\eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2)$ at $\eta_1 = \eta_2 = 0$ and $\bar{\eta}_1 = \bar{\eta}_2 = 0$. So

$$\tilde{F}^{(n-1)}_{\bar{L}, L'} = \tilde{F}^{(n-1)}_{\bar{L}, L'}(0, 0, 0, 0).$$ \hspace{1cm} (45)

If $\bar{p}_1$ and $\bar{p}_2$ are carried by the Ramond fermions, then either holomorphic pare, or the anti-holomorphic one of the state is described by boson wave function. If it is the holomorphic part, then $\mathcal{O}_L = 1$, and

$$\tilde{F}^{(n-1)}_{\bar{L}, L'} = \int d\eta_1 d\eta_2 \tilde{F}^{(n-1)}_{\bar{L}, L'}(\eta_1, \eta_2, 0, 0).$$ \hspace{1cm} (46)

Further, $\mathcal{O}_{L'}$ is calculated with (14). The fermion vertex coordinates are fixed to be $(v|0)$ and $(u|0)$, along with the boson local coordinate $z_4$. Since (13) has $SL(2)$-symmetry, one could fix any other (3|2) variables, but we do not develop this topic here.

It can be checked that with (10) and (12), the amplitude (13) for longitudinal boson emission vanishes as it is required by gauge invariance (to be shown otherwise).
Expressions (41) and (42) are given by (13) with replacing $O_L$ (or $\overline{O_L}$, or both them) by functions inside $< ... >$ (or their complex conjugated). At present we check the discussed relations only for the tree approximated amplitude (13). Then the first multiplier under the sum symbol in (13) is equal to unity, and other factors are calculated by (24) and (38). For the checking, it is convenient to fix $u \to \infty$ or $v \to \infty$. If $u \to \infty$, then terms with $\eta_3 \eta_4 p_3 p_4/[(z_4-v)(z_3-z_4)]$ are integrated by parts, like the expressions discussed after eq.(35). Of course, the check of the relation can be performed for any $u$ and $v$. In that case eq.(36) is used with the integration by parts leading to (57). The verifying the discussed relations for loop amplitudes needs an additional study.

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A Green functions

Here we argue relations (13) for the Ramond sector and clarify formulas of Section 5 for the vacuum correlator. For this purpose we use eq.[3] of the genus-$n$ Green function through functions of genera-$n_i$ where $\sum_i n_i = n$. In this case $n$ handles are divided into groups where the $i$-th group consists from the $n_i$ handles. Each group is given by the set of \{q\}, parameters of the super-Schottky group. For simplicity we consider all the super-spins structures $L_i$ to be even ($L = \{L_i\}$). Generally, to construct the Green function, one needs solely to build an expression, which satisfies known relations under transformations $t \to t_s^b$ and $t \to t_s^a$ for $2\pi$-twist about $B_s$ and $A_s$-cycle:

$$R_L^{(n)}(t_s^b, t'; \{q\}) = R_L^{(n)}(t, t'; \{q\}) + J_s^{(n)}(t'; \{q\}; L),$$

$$R_L^{(n)}(t_s^a, t'; \{q\}) = R_L^{(n)(r)}(t, t'; \{q\}),$$

where $R_L^{(n)(s)}(t, t'; \{q\})$ is obtained by $2\pi$-twist about the corresponding Schottky circle (if the circle is assigned to the Ramond type handle, the Green function has the square cut). Simultaneously, in doing so one determines the scalar functions. Furthermore, one determines the period matrix using the relations for these functions under transformations $t \to t_s^b$ and $t \to t_s^a$ for $2\pi$-twist about $B_s$ and $A_s$-cycle. The above relations are

$$J_r^{(n)}(t_s^b; \{q\}; L) = J_r^{(n)}(t; \{q\}; L) + 2\pi i \omega_{sr}(\{q\}, L),$$

$$J_r^{(n)}(t_s^a; \{q\}; L) = J_r^{(n)(s)}(t; \{q\}; L) + 2\pi i \delta_{rs}.$$

(A.2)

In doing so we consider $K_L^{(n)}(t, t'; \{q\})$ defined to be

$$K_L^{(n)}(t, t'; \{q\}) = D(t') R_L^{(n)}(t, t'; \{q\})$$

(A.3)

where the spinor derivative is defined by (1). We build (see [3, 5]) a matrix operator $\hat{K} = \{\hat{K}_{sr}\}$ where $\hat{K}_{sr}$ is an integral operator vanishing at $s = r$. For $s \neq r$, the kernel of $\hat{K}_{sr}$ is
\( \tilde{K}^{(n_s)}(t, t'; \{q\}_s)dt' \). Here \( \tilde{K}^{(n_s)}(t, t'; \{q\}_s) \) is related by (A.3) with \( \tilde{R}^{(n_s)}(t, t'; \{q\}_s) \), which is the non-singular part (11) of the Green function. So

\[
K^{(n_s)}(t, t'; \{q\}_s) = \frac{\vartheta - \vartheta'}{z - z'} + \tilde{K}^{(n_s)}(t, t'; \{q\}_s). \tag{A.4}
\]

We define kernels together with the differential \( dt' = dz'd\vartheta'/2\pi i \). The discussed operator performs the integration with \( \tilde{K}^{(n_s)}(t, t'; \{q\}_s) \) over \( t' \) along \( C_r \)-contour, which surrounds the limiting points associated with the considered group \( r \) of the handles and the cuts between limiting points for the Ramond handles. The desired relation for the Green function is derived in Section 5 of [5]. For every case of interest, the expression obtained can be transformed to the relation, which allows to verify (A.1) under the transformations assigned to the \( r \)-th group of the handles (Section 5 of [5]). To give the desired expression, we define \( (1 - \hat{K})^{-1}\hat{K}^{(n_s)}(t, t_1)dt_1 \) to be the kernel of the operator

\[
(1 - \hat{K})^{-1}\hat{K} = \hat{K} + \hat{K}\hat{K} + \ldots \tag{A.5}
\]

The desired Green function is given by

\[
R^{(n)}_L(t, t'; \{q\}) = R^{(n_r)}_L(t, t'; \{q\}_r) + \sum_{s \neq r} \int_{C_s} K^{(r_r)}_L(t, t_1; \{q\}_r) R^{(n_s)}_L(t_1, t; \{q\}_s) dt_1 \\
+ \sum_{p \neq r} \sum_{s} \int_{C_p} K^{(r_r)}_L(t, t_1; \{q\}_r) dt_1 \int_{C_s} [(1 - \hat{K})^{-1}\hat{K}]_{ps}(t_1, t_2)dt_2 R^{(n_s)}_L(t, t'; \{q\}_s) \tag{A.6}
\]

where \( R^{(n_r)}(t, t'; \{q\}_r) \) and \( K^{(r_r)}_L(t, t_1; \{q\}_r) \) are total Green functions including the singular term in (11) and (A.4). The scalar function \( J^{(n)}_j(t; \{q\}; L) \) associated with the \( j \)-th handle of the \( r \)-th supermanifold appears to be [4]

\[
J^{(n)}_j(t; \{q\}; L) = J^{(n_r)}_j(t; \{q\}_r; L_r) + \sum_{s \neq r} \int_{C_s} D(t_1)J^{(n_r)}_j(t_1; \{q\}_r; L_r)dt_1 R^{(n_s)}_L(t_1, t; \{q\}_s) \\
+ \sum_{p \neq r} \sum_{s} \int_{C_p} D(t_1)J^{(n_r)}_j(t_1; \{q\}_r; L_r) dt_1 \int_{C_s} [(1 - \hat{K})^{-1}\hat{K}]_{ps}(t_1, t_2)dt_2 R^{(n_s)}_L(t, t'; \{q\}_s). \tag{A.7}
\]

The period matrix is calculated from (A.7) presented to be

\[
J^{(n)}_j(t; \{q\}; L) - J^{(n)}_j(t_0; \{q\}; L) = J^{(n_r)}_j(t; t_0; \{q\}_r; L_r) + \int_{C_p} D(t_1)J^{(n_r)}_j(t_1; \{q\}_r; L_r)dt_1 \\
\times R^{(n_s)}_L(t_1, t; t_0; \{q\}_s) \\
+ \sum_{p \neq r} \int_{C_p} D(t_1)J^{(n_r)}_j(t_1; \{q\}_r; L_r) dt_1 \int_{C_s} [(1 - \hat{K})^{-1}\hat{K}]_{ps}(t_1, t_2)dt_2 R^{(n_s)}_L(t_2, t; t_0; \{q\}_s). \tag{A.8}
\]

where \( t_0 \) is a fixed parameter. Both \( z \) and \( z_0 \) lie inside the \( C'_{s0} \) contour, and

\[
R^{(n_s)}_L(t_1, t; t_0; \{q\}_s) = R^{(n_s)}_L(t, t; \{q\}_s) - R^{(n_s)}_L(t_1, t; \{q\}_s). \\
J^{(n_r)}_j(t; t_0; \{q\}_r; L_r) = J^{(n_r)}_j(t; \{q\}_r; L_r) - J^{(n_r)}_j(t_0; \{q\}_r; L_r). \tag{A.9}
\]
where $R^{(n)\alpha}_{\alpha}(t_1; \{q\}; s)$ is the total Green function \[1\] including the singular term. To prove \[A.8\], one, using \[A.3\], calculates the contribution from the $\ln[(z_2 - z - \partial_2 \vartheta)/(z_2 - z_0 - \partial_2 \vartheta_0)]$ term due to the singularity of the Green function. The corresponding integral is transformed to the one along the cut between $z_2 = z - \partial_2 \vartheta$ and $z_2 = z_0 - \partial_2 \vartheta_0$. Then it is found to be

$$ \int \frac{D(t_2)f(t_2)[\theta(z_2 - z - \partial_2 \vartheta) - (z_2 - z_0 - \partial_2 \vartheta_0)]dz_2d\vartheta_2 = f(t) - f(t_0) \quad (A.10) $$

where $f(t)$ denotes either the Green function, or $J^{(nr)}_r(t_1; \{q\}_r; L_r)$. As the result, one obtains \[A.7\]. From \[A.8\], the $\omega^{(n)\alpha}_{\alpha j_s}(\{q\}; L)$ element of the period matrix is found to be

$$ 2\pi i \omega^{(n)\alpha}_{\alpha j_s}(\{q\}; L) = \delta_{j_s j_s} \ln k_j + (1 - \delta_{j_s j_s}) \int_{C_s} D(t)J^{(nr)}_r(t; \{q\}; L_r)dt J^{(ns)}_s(t; \{q\}_s; L_s) + \sum \int_{C_p} D(t)J^{(nr)}_r(t; \{q\}_r; L_r)dt \int_{C_s} [(1 - \bar{K})^{-1}\bar{K}]_{pr}(t, t')dt' J^{(ns)}_s(t'; \{q\}_s; L_s) \quad (A.11) $$

Now we use the above expressions in the case of two group of the handles present. The first group consists of solely the degenerated handle, and the second group consists of all the remaining ones. The genus-1 scalar function is given by

$$ J(t) = \ln \frac{z - u}{z - v}. \quad (A.12) $$

To receive the first relation in \[18\] for the boson loop, we substitute eq.\[A.12\] in \[A.11\] for $\omega_{11}$ and calculate \[A.11\] at $k \to 0$. In this case the integration contour over $t_1$ rounds the non-degenerated handles, but it can be transformed to a contour round the poles of $J(z)$. Since for $k \to 0$ the remaining part of the integrand has no singularities at $z = (u, v)$ and $z' = (u, v)$, the integral is the sum over the residues in these points. With \[A.6\], the desired relation appears. To see the second relation in \[18\], one uses \[A.11\] for $\omega_{11}$ where the integration contour over $t$ rounds the non-degenerated handles, \[A.12\] being substituted. At $k \to 0$ the integral \[A.11\] over $t$ is given by the sum over the residues in $z = u$ and $z = v$. With \[A.9\] the desired relation appears. The relation for $J_1$ is obtained by substitution of \[A.12\] in \[A.9\] for $J_1$ and the calculation it at $k \to 0$. The integral is equal to the sum over the residues in $z = u$ and $z = v$. With \[A.10\], the desired relation appears. To receive \[22\], we use \[A.6\] where the integration contour over $t_1$ rounds the degenerated handle. The first factor in \[A.9\] is calculated by \[A.3\] for Green’s function \[21\]. The regular at $z \to u$ part in \[22\] is given by the contribution from the pole at $z_1 = z = u$ originated by the logarithmic term in \[21\]. Eq.\[23\] and formulas for scalar functions are derived in the similar way. As the result, with \[9\], expressions \[23\] and \[26\] appear.

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