RIGIDITY OF MARGINALLY OUTER TRAPPED 
(HYPER)SURFACES WITH NEGATIVE $\sigma$-CONSTANT

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Abstract. In this paper we generalize the main result of [13] in two different situations: in the first case for MOTSs of genus greater than one and, in the second case, for MOTSs of high dimension with negative $\sigma$-constant. In both cases we obtain a splitting result for the ambient manifold when it contains a stable closed MOTS which saturates a lower bound for the area (in dimension 2) or for the volume (in dimension $\geq 3$). These results are extensions of [21, Theorem 3] and [20, Theorem 3] to general (non-time-symmetric) initial data sets.

1. Introduction

Let $M^3 = (M^3, g)$ be a Riemannian 3-manifold. It is a very interesting question to know how the topology of a minimal surface $\Sigma^2 \subset M^3$ can influence the geometry of $M^3$, and vice-versa.

Using the Gauss-Bonnet theorem, the Gauss equation, and the stability inequality for minimal surfaces, Schoen and Yau observed that if $M^3$ is oriented and has positive scalar curvature, then $M^3$ does not admit orientable closed stable minimal surfaces of positive genus. Furthermore, they proved the following result (see [24]).

**Theorem 1.1** (Schoen-Yau). Let $M^3 = (M^3, g)$ be an oriented closed Riemannian 3-manifold with nonnegative scalar curvature. If the fundamental group of $M^3$ admits a subgroup abstractly isomorphic to the fundamental group of the 2-torus, then $M^3$ is flat.

In order to prove this result, Schoen and Yau first proved that the hypothesis on the fundamental group of $M^3$ ensures the existence of a stable minimal 2-torus. After, they used that if $M$ admits a metric of nonnegative scalar curvature which is not flat, then $M$ also admits a metric of positive scalar curvature (see [17]). Then, the result follows.

As observed by Fischer-Colbrie and Schoen [10], if $\Sigma^2$ is an orientable closed stable minimal surface of genus $g(\Sigma) \geq 1$ in an oriented Riemannian 3-manifold $M^3$ with scalar curvature $R \geq 0$, then $\Sigma^2$ is a totally geodesic flat 2-torus in $M^3$, and $R = 0$ on $\Sigma^2$. In the same work (see [10, Remark 4]), Fischer-Colbrie and Schoen posed the problem of establishing a stronger (more global) rigidity statement if, say, the 2-torus is suitably area minimizing.
Partially motivated by some issues concerning the topology of back holes, Cai and Galloway [8] solved the problem posed by Fischer-Colbrie and Schoen, in a paper which has inspired a great deal of subsequent works in the subject (e.g. [1, 6, 19, 21]). They proved:

**Theorem 1.2 (Cai-Galloway).** Let $M^3$ be a Riemannian 3-manifold with non-negative scalar curvature. If $\Sigma^2 \subset M^3$ is a two-sided embedded 2-torus which is locally area minimizing, then a neighborhood of $\Sigma$ in $M$ is isometric to the product $((−\varepsilon,\varepsilon) \times \Sigma, dt^2 + g_0)$, where $g_0$, the metric on $\Sigma$ induced from $M$, is flat. Moreover, if $M$ is complete and $\Sigma$ has least area in its isotopy class, $M$ is globally flat.

As observed by Cai and Galloway, in the first part of the theorem above, $M$ need not be globally flat. Also, the second part remains true if $M$ is a manifold with boundary $\partial M \neq \emptyset$, assuming that its boundary is mean convex. A higher dimensional version of Theorem 1.2 was obtained in [7]; see also [12] for a simplified proof.

Two similar results to Theorem 1.2 were obtained by Bray, Brendle, and Neves [6] and by Nunes [21] under different hypotheses on the scalar curvature of $M^3$ and the topology of $\Sigma^2$, which we paraphrase as follow.

**Theorem 1.3 (Bray-Brendle-Neves).** Let $M^3$ be a Riemannian 3-manifold with scalar curvature bounded from below by $2c$, for some constant $c > 0$. If $\Sigma^2$ is an embedded 2-sphere which is locally area minimizing, then the area of $\Sigma$ satisfies

$$A(\Sigma) \leq \frac{4\pi}{c}.$$  

Furthermore, if equality holds, a neighborhood of $\Sigma$ in $M$ is isometric to the product $((−\varepsilon,\varepsilon) \times \Sigma, dt^2 + g_0)$, where $(\Sigma^2, g_0)$ is the round 2-sphere of Gaussian curvature $\kappa = c$. In this case, if further $M$ is complete and $\Sigma$ has least area in its isotopy class, the universal cover of $M$ is isometric to the product $(\mathbb{R} \times \Sigma, dt^2 + g_0)$.

**Theorem 1.4 (Nunes).** Let $M^3$ be a Riemannian 3-manifold with scalar curvature bounded from below by $-2c$, for some constant $c > 0$. If $\Sigma^2 \subset M^3$ is a two-sided embedded closed Riemann surface of genus $g(\Sigma) \geq 2$ which is locally area minimizing, then the area of $\Sigma$ satisfies

$$A(\Sigma) \geq \frac{4\pi(g(\Sigma) - 1)}{c}.$$  

Furthermore, if equality holds, a neighborhood of $\Sigma$ in $M$ is isometric to the product $((−\varepsilon,\varepsilon) \times \Sigma, dt^2 + g_0)$, where $(\Sigma^2, g_0)$ has constant Gaussian curvature $\kappa = -c$. In this case, if further $M$ is complete and $\Sigma$ has least area in its isotopy class, the universal cover of $M$ is isometric to the product $(\mathbb{R} \times \Sigma, dt^2 + g_0)$.

The original proofs of Theorems 1.2, 1.3, and 1.4 are very different. However, Micallef and Moraru [19] presented a unified proof for them. From the point of view of relativity, Theorems 1.2, 1.3, and 1.4 may be viewed as statements about time-symmetric (totally geodesic) initial data sets. In [13], Galloway and the author generalized Theorem 1.3 to general (non-time-symmetric) initial data sets. In that more general situation, minimal surfaces were replaced by *marginally outer trapped surfaces* (MOTSs). We proved (see Section 2 for definitions):
Theorem 1.5 (Galloway-Mendes). Let $M^3 = (M^3, g, K)$ be a 3-dimensional initial data set in a spacetime $\bar{M}^4 = (\bar{M}^4, \bar{g})$. Let $\Sigma^2$ be a spherical MOTS in $M^3$ which is weakly outermost and outer area minimizing. Suppose that $\mu - |J| \geq c$ on $M_+$ for some constant $c > 0$. Then, the area of $\Sigma$ satisfies

$$A(\Sigma) \leq \frac{4\pi}{c}.$$ 

Furthermore, if equality holds, we have:

1. An outer neighborhood $U \approx [0, \varepsilon) \times \Sigma$ of $\Sigma$ in $M$ is isometric to

$$(0, \varepsilon) \times \Sigma, dt^2 + g_0),$$

where $(\Sigma, g_0)$ is the round 2-sphere of Gaussian curvature $\kappa = c$.

2. Each slice $\Sigma_t \approx \{t\} \times \Sigma$ is totally geodesic as a submanifold of spacetime. Equivalently, $\chi_+(t) = \chi_-(t) = 0$, where $\chi_{\pm}$ are the null second fundamental forms of $\Sigma_t$ in $\bar{M}^4$.

3. $K(\cdot, \cdot)|_{\Sigma_t, \Sigma_t} = 0$ and $K(\nu_t, \cdot)|_{\Sigma_t, \Sigma_t} = 0$ for each $x \in \Sigma_t$, where $\nu_t$ is the outer unit normal to $\Sigma_t$, and $J = 0$ on $U$.

Above, $\mu$ and $J$ are defined in terms of the Einstein tensor of $\bar{M}^4$, $G = \bar{Ric} - \frac{\bar{R}}{2}\bar{g}$, by $\mu = G(u, u)$ and $J(\cdot) = G(u, \cdot)|_{\tau_p, M}$, $p \in M$, where $u$ is the future directed timelike unit normal to $M$.

An important generalization of Theorem 1.5 is due to Moraru [20], who extended Nunes’ result to closed hypersurfaces $\Sigma^n$ of dimension $n \geq 3$. He proved:

Theorem 1.6 (Moraru). Let $M^{n+1}$ be a Riemannian manifold of dimension $n + 1$, $n \geq 3$, with scalar curvature bounded from below by $-2c$, for some constant $c > 0$. If $\Sigma^n \subset M^{n+1}$ is a two-sided embedded closed hypersurface with $\sigma(\Sigma) < 0$ which is locally volume minimizing, then the volume of $\Sigma$ satisfies

$$\text{Vol}(\Sigma^n) \geq \left(\frac{\sigma(\Sigma^n)}{2c}\right)^{\frac{n}{2}}.$$ 

Furthermore, if equality holds, a neighborhood of $\Sigma$ in $M$ is isometric to the product $((-\varepsilon, \varepsilon) \times \Sigma, dt^2 + g_\Sigma)$, where $g_\Sigma$, the metric on $\Sigma$ induced from $M$, is Einstein with constant scalar curvature $R_\Sigma = -2c$.

Above, $\sigma(\Sigma)$ is the topological invariant introduced by Schoen [23], called the $\sigma$-constant of $\Sigma$ (see also Kobayashi [18]).

In the same spirit of Theorem 1.5, in the present paper we generalize Theorems 1.4 and 1.6 to non-time-symmetric initial data sets.

Our first result is the following.

Theorem 1.7 (Theorem 1.8). Let $M^3 = (M^3, g, K)$ be a 3-dimensional initial data set. Let $\Sigma^2$ be an orientable weakly outermost closed MOTS in $M^3$ of genus $g(\Sigma) \geq 2$. Suppose that $\mu - |J| \geq -c$ for some constant $c > 0$ and that $K$ is 2-convex, both on $M_+$. Then, the area of $\Sigma$ satisfies

$$A(\Sigma) \geq \frac{4\pi(g(\Sigma) - 1)}{c}.$$ 

Furthermore, if equality holds, we have:

1. An outer neighborhood $U \approx [0, \varepsilon) \times \Sigma$ of $\Sigma$ in $M$ is isometric to

$$(0, \varepsilon) \times \Sigma, dt^2 + g_0),$$

where $(\Sigma, g_0)$ has constant Gaussian curvature $\kappa = -c$. 

$$F$$
(2) Each slice \( \Sigma_t \approx \{ t \} \times \Sigma \) is totally geodesic as a submanifold of spacetime. Equivalently, \( \chi_+(t) = \chi_-(t) = 0 \), where \( \chi_\pm(t) \) are the null second fundamental forms of \( \Sigma_t \).

(3) \( K(\cdot,\cdot)|_{T_x \Sigma_t} = 0 \) and \( K(\nu_t,\cdot)|_{T_x \Sigma_t} = 0 \) for each \( x \in \Sigma_t \), where \( \nu_t \) is the outer unit normal to \( \Sigma_t \), and \( J = 0 \) on \( \Sigma_t \).

We point out that the area estimate (1.1) was proved by Gibbons [16] in the time-symmetric case and by Woolgar [26] in the general case. Our second result is the following.

**Theorem 1.8** (Theorem [1.4]). Let \( M^{n+1} = (M^{n+1}, g, K) \) be a \( (n+1) \)-dimensional initial data set, \( n \geq 3 \). Let \( \Sigma^n \) be a weakly outermost closed MOTS in \( M^{n+1} \) with \( \sigma \)-constant \( \sigma(\Sigma) < 0 \). Suppose that \( \mu - |J| \geq -c \) for some constant \( c > 0 \) and that \( K \) is \( n \)-convex, both on \( M_+ \). Then, the volume of \( \Sigma \) satisfies

\[
\text{Vol}(\Sigma^n) \geq \left( \frac{\sigma(\Sigma^n)}{2c} \right)^{\frac{2}{n+1}}.
\]

Furthermore, if equality holds, we have:

1. An outer neighborhood \( U \approx [0,\varepsilon) \times \Sigma \) of \( \Sigma \) in \( M \) is isometric to \((\{0,\varepsilon\} \times \Sigma, dt^2 + g_\Sigma)\), and \((\Sigma, g_\Sigma)\) is Einstein with constant scalar curvature \( R_\Sigma = -2c \).
2. Each slice \( \Sigma_t \approx \{ t \} \times \Sigma \) is totally geodesic as a submanifold of spacetime. Equivalently, \( \chi_+(t) = \chi_-(t) = 0 \), where \( \chi_\pm(t) \) are the null second fundamental forms of \( \Sigma_t \).
3. \( K(\cdot,\cdot)|_{T_x \Sigma_t} = 0 \) and \( K(\nu_t,\cdot)|_{T_x \Sigma_t} = 0 \) for each \( x \in \Sigma_t \), where \( \nu_t \) is the outer unit normal to \( \Sigma_t \), and \( J = 0 \) on \( \Sigma_t \).

The volume estimate (1.2) was obtained by Cai and Galloway [9] in the time-symmetric case and by Galloway and Murchadha [14] in the general case. Although we are assuming a convexity hypothesis on the initial data set \( M \) in Theorems [1.7] and [1.8] we are not assuming any minimizing hypothesis on the area (in dimension 2) nor the volume (in dimension \( \geq 3 \)) of \( \Sigma \).

Since in time-symmetric initial data sets \( \mu - |J| \) reduces to the half of the scalar curvature, the hypothesis \( \mu - |J| \geq -c \) in general initial data sets is the analogue of \( R \geq -2c \) in time-symmetric ones.

2. **Preliminaries**

Let \( \bar{M}^{n+2} = (\bar{M}^{n+2}, \bar{g}, \bar{\nabla}) \) be a spacetime of dimension \( n+2 \), i.e., a connected, oriented and time-oriented Lorentzian \((n+2)\)-manifold, where \( \bar{g} \) and \( \bar{\nabla} \) denote a Lorentzian metric and its Levi-Civita connection, respectively. Let \( M^{n+1} \) be a spacelike hypersurface embedded into \( \bar{M}^{n+2} \) and let \( u \) denote the future directed timelike unit normal to \( M \). In this case, we can see \( M^{n+1} \) as an initial data set \( M^{n+1} = (M^{n+1}, g, K) \), where \( g \) is the Riemannian metric on \( M \) induced from \( \bar{M} \) and \( K \) is the second fundamental form of \( M \) in \( \bar{M} \). More precisely,

\[
K(V, W) = \bar{g}(\bar{\nabla}_{\nu} u, W), \text{ for all } V, W \in T_p M \text{ and } p \in M.
\]

Now, let \( \Sigma^n \) be a two-sided connected closed (compact with no boundary) hypersurface embedded into \( M^{n+1} \). Fix a unit normal to \( \Sigma \) in \( M \), say \( \nu \), that is,
\( \nu(x) \in T_x \Sigma^\perp \cap T_x M \) for all \( x \in \Sigma \). By convention, we say that \( \nu \) is outward pointing and, in turn, \(-\nu\) is inward pointing. Then, we can define two future directed null normal vector fields along \( \Sigma \), \( l_+ = u + \nu \) and \( l_- = u - \nu \), which are outward and inward pointing, respectively. The null second fundamental forms of \( \Sigma^\alpha \) in \( \bar{M}^{n+2} \), \( \chi^\pm \), and \( \theta^\pm \), are defined by

\[
\chi^\pm(Y,Z) = K(Y,Z) \pm A(Y,Z), \quad \text{for all } Y,Z \in T_x \Sigma \text{ and } x \in \Sigma,
\]

where \( A \) is the second fundamental form of \( \Sigma \) in \( \bar{M} \). On the other hand, the null mean curvatures of \( \Sigma \) in \( \bar{M} \), \( \theta_+ \) and \( \theta_- \), are defined by taking the traces of \( \chi_+ \) and \( \chi_- \) with respect to \( \Sigma \),

\[
\theta^\pm = \text{tr}_\Sigma \chi^\pm.
\]

Observe that

\[
\theta_\pm = \text{div}_\Sigma(l_\pm) = \text{tr}_\Sigma K \pm H,
\]

where \( H = \text{tr} A \) is the mean curvature of \( \Sigma \) in \( M \).

As introduced by Penrose, \( \Sigma \) is said to be a trapped surface if both \( \theta_+ \) and \( \theta_- \) are negative everywhere. Also, \( \Sigma \) is an outer trapped surface if \( \theta_+ \) is negative (with no assumption on \( \theta_- \)). Finally, we say that \( \Sigma \) is a marginally outer trapped surface (MOTS) if \( \theta_+ \) vanishes identically (with no assumption on \( \theta_- \)). For simplicity, we drop the plus sign of \( l_+ \), \( \chi_+ \), and \( \theta_+ \).

Remark 2.1. Observe that in the time-symmetric case, i.e., when \( K \equiv 0 \), \( \chi \) and \( \theta \) reduce to \( A \) and \( H \), respectively. Then, in this case, a MOTS is just a minimal hypersurface.

Before presenting the notion of stability for MOTSs introduced by Andersson, Mars, and Simon \([2, 3]\), we are going to fix some notations:

- Let \( G \) be the Einstein tensor of \( \bar{M} = (\bar{M}, \bar{g}, \bar{\nabla}) \) given by
  \[
  G = \bar{\text{Ric}} - \frac{\bar{R}}{2} \bar{g},
  \]
  where \( \bar{\text{Ric}} \) and \( \bar{R} \) are the Ricci tensor and the scalar curvature of \( \bar{M} \), respectively;
- Denote \( G(u, u) \) by \( \mu \) on \( M \) and \( G(u, \cdot) \) by \( J(\cdot) \) on \( T_p M \) for each \( p \in M \);
- Define
  \[
  Q = \frac{1}{2} R_\Sigma - (\mu + J(\nu)) - \frac{1}{2} |\chi|^2,
  \]
  where \( R_\Sigma \) is the scalar curvature of \( \Sigma \) with respect to the Riemannian metric \( \langle \cdot, \cdot \rangle \) induced from \( M = (M, g) \);
- Denote by \( X \in \Gamma(T\Sigma) \) the vector field on \( \Sigma \) dual to \( K(\nu(x), \cdot)|_{T_x \Sigma} \) for each \( x \in \Sigma \);
- Denote by \( \tau \) the mean curvature of \( M \) in \( \bar{M} \), i.e., \( \tau = \text{tr} K \).

Now, consider \( t \mapsto \Sigma_t \subset M, \ t \in (-\varepsilon, \varepsilon) \), a variation of \( \Sigma_0 = \Sigma \) in \( M \) with variation vector field \( \frac{\partial}{\partial t} \big|_{t=0} = \phi \nu, \ \phi \in C^\infty(\Sigma) \). Denote by \( \theta(t) \) the null mean curvature of \( \Sigma_t \) in \( \bar{M} \) with respect to \( l_t = u + \nu_t \), that is, \( \theta(t) = \text{div}_{\Sigma_t}(l_t) \), where \( \nu_t \) is the unit normal to \( \Sigma_t \) in \( M \) such that \( \nu_0 = \nu \). A similar calculation to that one in \([3]\) gives,

\[
\frac{\partial \theta}{\partial t} \bigg|_{t=0} = L\phi + \left( \theta(0) \tau - \frac{1}{2} \theta(0)^2 \right) \phi,
\]

\[ \tag{2.2} \]
where $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is the operator given by

\begin{equation}
L \phi = -\Delta \phi + 2 \langle X, \nabla \phi \rangle + (Q - |X|^2 + \text{div} \; X) \phi.
\end{equation}

Here, $\Delta$ and $\nabla$ denote the Laplacian and the gradient operators on $\Sigma = (\Sigma, \langle \cdot, \cdot \rangle)$, respectively. The operator $L$ is known as the MOTSs stability operator.

**Remark 2.2.** Using the Gauss-Codazzi equations, we can express $\mu$ and $J$ solely in terms of the initial data set $M = (\Sigma, \mu, J)$:

\begin{align*}
\mu &= \frac{1}{2} (R + \tau^2 - |K|^2), \\
J &= \text{div} \; K - d\tau,
\end{align*}

where $R$ is the scalar curvature of $M$. Then, using the Gauss equation,

\begin{equation}
\text{Ric}(\nu, \nu) + |A|^2 = \frac{1}{2} (R - R_{\Sigma} + |A|^2 + H^2),
\end{equation}

where $\text{Ric}$ is the Ricci tensor of $M$, we can see that in the time-symmetric case the operator $L$ reduces to the classical stability operator for minimal hypersurfaces,

\begin{equation}
-\Delta - (\text{Ric}(\nu, \nu) + |A|^2),
\end{equation}

when $\Sigma$ is a MOTS (in this case, a minimal hypersurface).

The operator $L$ is not self-adjoint in general. But, it has the following properties (see [3] and the references therein).

**Lemma 2.3.** The following holds for the operator $L$.

1. There is a real eigenvalue $\lambda_1 = \lambda_1(L)$, called the principal eigenvalue of $L$, such that for any other eigenvalue $\mu$, $\text{Re}(\mu) \geq \lambda_1$. The associated eigenfunction $\phi$, $L \phi = \lambda_1 \phi$, is unique up to a multiplicative constant, and can be chosen to be strictly positive.

2. $\lambda_1 \geq 0$ (resp., $\lambda_1 > 0$) if and only if there exists $\psi \in C^\infty(\Sigma)$, $\psi > 0$, such that $L \psi \geq 0$ (resp., $L \psi > 0$).

The lemma above is true for any operator of the form $\phi \mapsto -\Delta \phi + \langle Y, \nabla \phi \rangle + q \phi$, where $Y \in \Gamma(T\Sigma)$ and $q \in C^\infty(\Sigma)$.

A closed MOTS $\Sigma$ is said to be stable if $\lambda_1(L) \geq 0$. It follows from the lemma above that $\Sigma$ is stable if and only if there exists $\psi \in C^\infty(\Sigma)$, $\psi > 0$, such that $L \psi \geq 0$. Observe that in the time-symmetric case, $\Sigma$ is stable as a MOTS if and only if $\Sigma$ is stable as a minimal hypersurface, since, in this case, $\lambda_1(L)$ is the first eigenvalue of the Jacobi operator $L$.

Now, consider the “symmetrized” operator $L_0 : C^\infty(\Sigma) \to C^\infty(\Sigma)$,

\begin{equation}
L_0 \phi = -\Delta \phi + Q \phi,
\end{equation}

obtained formally from (2.3) by taking $X = 0$. As first observed by Galloway [11], using the Rayleigh’s formula for the first eigenvalue of $L_0$ (which coincides with the principal eigenvalue of $L_0$),

\begin{equation}
\lambda_1(L_0) = \inf_{f \in C^\infty(\Sigma) \setminus \{0\}} \frac{\int_{\Sigma} (|\nabla f|^2 + Q f^2) dA}{\int_{\Sigma} f^2 dA},
\end{equation}

and the main argument presented in [15], we have the following result.
Lemma 2.4. \( \lambda_1(L_0) \geq \lambda_1(L) \). In particular, if \( \lambda_1(L) \geq 0 \),

\[
(2.5) \quad \int_{\Sigma} \left( |\nabla f|^2 + Qf^2 \right) dA \geq 0
\]

for all \( f \in C^\infty(\Sigma) \).

The next result is a very useful consequence of the proof of the main theorem proved in [11] (see also [13]).

Lemma 2.5. Let \( \Sigma^n \) be a closed MOTS in an initial data set \( M^{n+1} = (M^{n+1}, g, K) \). If \( \lambda_1(L) = 0 \), then there exists a variation \( t \mapsto \Sigma_t \) of \( \Sigma_0 = \Sigma \) in \( M \), \( t \in (-\varepsilon, \varepsilon) \), such that \( \Sigma_t \) is a hypersurface with constant null mean curvature \( \theta = \theta(t) \) for each \( t \in (-\varepsilon, \varepsilon) \). Furthermore, \( \{ \Sigma_t \}_{t \in (-\varepsilon, \varepsilon)} \) is a foliation of a neighborhood of \( \Sigma \) in \( M \) with variation vector field \( \frac{\partial}{\partial t} = \phi_t \nu_t \), where \( \nu_t \) is the outer unit normal to \( \Sigma_t \) and \( \phi_t : \Sigma_t \approx \{ t \} \times \Sigma \to \mathbb{R} \) is a positive function.

Before passing to the next section, let us fix some notation and terminologies.

If \( \Sigma^n \) is a separating MOTS in \( M^n = (M^n, g, K) \), let \( M_+ \) be the region consisting of \( \Sigma \) and the region outside of \( \Sigma \). We say that \( \Sigma \) is outermost if there are no outer trapped or marginally outer trapped surfaces in \( M_+ \) homologous to \( \Sigma \). We say that \( \Sigma \) is weakly outermost if there are no outer trapped surfaces in \( M_+ \) homologous to \( \Sigma \). Also, \( \Sigma \) is said to be outer area minimizing if its area is less than or equal to the area of any surface in \( M_+ \) homologous to \( \Sigma \).

Remark 2.6. It follows from (2.2) that if \( \Sigma \) is a weakly outermost MOTS, then it is stable. Otherwise, taking a variation \( t \mapsto \Sigma_t \) such that \( \frac{\partial}{\partial t}|_{t=0} = \phi \nu \), where \( \phi > 0 \) is the eigenfunction associated to \( \lambda_1 = \lambda_1(L) \), we have that \( \Sigma_t \) is an outer trapped surface for \( t > 0 \) small enough, which is homologous to \( \Sigma \) in \( M_+ \).

3. The Case of High Genus

In this section we generalize the main result obtained in [13] to surfaces \( \Sigma \) of genus \( \text{genus}(\Sigma) \geq 2 \).

Inequality (3.1) below is a well known area estimate (see [16] for the time-symmetric case and [20] for the general case). But, for the sake of completeness, we will present its proof and some of the consequences when equality occurs.

Proposition 3.1. Let \( \Sigma^2 \) be an orientable stable closed MOTS of genus \( \text{genus}(\Sigma) \geq 2 \) in a 3-dimensional initial data set \( M^3 = (M^3, g, K) \). Suppose that \( \mu + J(\nu) \geq -c \) on \( \Sigma \) for some constant \( c > 0 \). Then, the area of \( \Sigma \) satisfies

\[
(3.1) \quad A(\Sigma) \geq \frac{4\pi(\text{genus}(\Sigma) - 1)}{c}.
\]

Furthermore, if equality holds, \( \Sigma \) has constant Gaussian curvature \( \kappa = -c \), the null second fundamental form \( \chi = \chi_+ \) of \( \Sigma \) vanishes identically, \( \mu + J(\nu) = -c \) on \( \Sigma \), and \( \lambda_1(L_0) = \lambda_1(L) = 0 \).
Proof. By hypothesis, $\lambda_1(L) \geq 0$, where $L$ is the MOTSs stability operator. Then, using the Gauss-Bonnet theorem and inequality (2.5) for $f = 1$, we have

$$0 \leq \int_{\Sigma} QdA = \int_{\Sigma} \left( \kappa - (\mu + J(\nu)) - \frac{1}{2} |\chi|^2 \right) dA$$

$$\leq \int_{\Sigma} \left( \kappa + c - \frac{1}{2} |\chi|^2 \right) dA \leq \int_{\Sigma} (\kappa + c) dA$$

$$= 4\pi (1 - g(\Sigma)) + cA(\Sigma),$$

which proves (3.1).

Now, if equality in (3.1) holds, all inequalities above must be equalities. Then, $\chi = 0$, $\mu + J(\nu) = -c$ on $\Sigma$, and $\int_{\Sigma} QdA = 0$. Observe that $Q = \kappa + c$. Using (2.5) again, for all $\alpha \in \mathbb{R}$ and $\psi \in C^\infty(\Sigma)$, we have

$$0 \leq \int_{\Sigma} (|\nabla(\alpha + \psi)|^2 + Q(\alpha + \psi)^2) dA$$

$$= \int_{\Sigma} (|\nabla\psi|^2 + Q\psi^2) dA + 2\alpha \int_{\Sigma} Q\psi dA.$$

This implies $\int_{\Sigma} Q\psi dA = 0$ for all $\psi \in C^\infty(\Sigma)$, hence $Q = \kappa + c = 0$. Finally, $\lambda_1(L_0) = \lambda_1(L) = 0$ follows from Lemma 2.4 and Rayleigh's formula (2.4), remembering that $\lambda_1(L) \geq 0$. □

Before passing to the main result of this section, we are going to prove a calculus lemma and to present the definition of $n$-convexity. The proof of the following lemma is based on the techniques presented in [19].

**Lemma 3.2.** Let $f \in C^1([0, \varepsilon))$ and $\eta, \xi, \rho \in C^0([0, \varepsilon))$ be functions such that max{$f, \rho$} $\geq 0$, $\xi \geq 0$, $\eta > 0$, $f(0) = 0$, and

$$f'(t)\eta(t) \leq \int_0^t f(s)\xi(s)ds + f(t)\rho(t), \ \forall t \in [0, \varepsilon).$$

Then, $f \leq 0$ everywhere. In particular, if $f \geq 0$, it must be identically zero.

**Proof.** Define

$$I = \{ \delta \in (0, \varepsilon); f \leq 0 \text{ in } [0, \delta] \}.$$

**Claim 1.** $I \neq \emptyset$.

**Proof of Claim 1.** By continuity, there exists a constant $C > 0$ satisfying

$$\max_{t \in [0, \varepsilon/2]} \left\{ \frac{1}{\eta(t)} \int_0^t \xi(s)ds, \frac{\rho(t)}{\eta(t)} \right\} \leq C.$$

Choose $\delta \in (0, \varepsilon/2)$ such that $0 < 1 - 2C\delta$. We claim that $\delta \in I$. Otherwise, fix $t_0 \in (0, \delta]$ with $f(t_0) > 0$ and define

$$t_1 = \inf\{t \in [0, t_0]; f(t) \geq f(t_0) \}.$$

By continuity, $f(t_1) = f(t_0) > 0$. Moreover, by definition, $f(t) \leq f(t_1)$ for all $t \in [0, t_1]$ (observe that $t_1 > 0$, since $f(0) = 0$). Then, by the mean value theorem,
there exists \( t^* \in (0, t_1) \) such that \( f(t_1) = f'(t^*)t_1 \). Therefore,
\[
\frac{f(t_1)}{t_1} = f'(t^*) \leq \frac{1}{\eta(t^*)} \int_0^{t^*} f(s)\xi(s)ds + f(t^*)\frac{\rho(t^*)}{\eta(t^*)}
\]
\[
\leq \frac{f(t_1)}{\eta(t^*)} \int_0^{t^*} \xi(s)ds + f(t^*)\frac{\rho(t^*)}{\eta(t^*)}
\]
\[
\leq Cf(t_1) + f(t^*)\frac{\rho(t^*)}{\eta(t^*)}.
\]
Now, observe that \( f(t^*)\rho(t^*)/\eta(t^*) \leq Cf(t_1) \). In fact, if \( f(t^*)\rho(t^*) \leq 0 \), we have done, since \( f(t_1) > 0 \). If \( f(t^*)\rho(t^*) > 0 \), then \( f(t^*) > 0 \) and \( \rho(t^*) > 0 \), because max\{\( f(t^*), \rho(t^*) \} \geq 0 \) by hypothesis, which implies that
\[
f(t^*)(\rho(t^*)/\eta(t^*)) \leq f(t^*)C \leq f(t_1)C.
\]
In any event,
\[
\frac{f(t_1)}{t_1} \leq 2Cf(t_1),
\]
hence \( 1 \leq 2Ct_3 \leq 2C\delta \), which is a contradiction, because \( 0 < 1 - 2C\delta \). This finishes the proof of Claim 1.

Claim 2. \( \sup I = \varepsilon \).

Proof of Claim 2. Define \( \delta_0 = \sup I \in (0, \varepsilon) \). If \( \delta_0 < \varepsilon \), set \( \tilde{f}(t) = f(t + \delta_0) \), \( \tilde{\eta}(t) = \eta(t + \delta_0) \), \( \tilde{\xi}(t) = \xi(t + \delta_0) \), and \( \tilde{\rho}(t) = \rho(t + \delta_0) \) for \( t \in [0, \varepsilon - \delta_0] \). Observing that, in this case, \( f \leq 0 \) in \([0, \delta_0]\) (i.e., \( \delta_0 \in I \)), we have
\[
\tilde{f}'(t)\tilde{\eta}(t) = f'(t + \delta_0)\eta(t + \delta_0)
\]
\[
\leq \int_0^{t+\delta_0} f(s)\xi(s)ds + f(t + \delta_0)\rho(t + \delta_0)
\]
\[
\leq \int_0^t \tilde{f}(r)\tilde{\xi}(r)dr + \tilde{f}(t)\tilde{\rho}(t),
\]
for all \( t \in [0, \varepsilon - \delta_0] \). From the definition of \( \delta_0 \), \( \tilde{f}(0) = f(\delta_0) = 0 \). Then, using Claim 1 for these new functions, there exists \( \delta_1 \in (0, \varepsilon - \delta_0) \) such that \( \tilde{f} \leq 0 \) in \([0, \delta_1]\), which implies \( f \leq 0 \) in \([0, \delta_0 + \delta_1]\), contradicting the definition of \( \delta_0 \). This finishes the proof of Claim 2.

The result follows from Claim 2. \( \square \)

Let \( M^{n+1} = (M^{n+1}, g, K) \) be an initial data set of dimension \( n + 1 \). We say that \( K \) is \( n \)-convex (on \( M \)) if \( \text{tr}_\pi K \geq 0 \) for all \( \pi \subset T_pM \) and \( p \in M \), where \( \pi \) is a linear subspace of dimension \( n \). Saying that \( K \) is \( n \)-convex is equivalent to say that the sum of the \( n \) smallest eigenvalues of \( K \) is nonnegative.

The main result of this section is the following.

Theorem 3.3. Let \( M^3 = (M^3, g, K) \) be a 3-dimensional initial data set. Let \( \Sigma^2 \) be an orientable weakly outermost closed MOTS in \( M^3 \) of genus \( g(\Sigma) \geq 2 \). Suppose that \( \mu - |J| \geq -\varepsilon \) for some constant \( \varepsilon > 0 \) and that \( K \) is \( 2 \)-convex, both on \( M_+ \). Then, if \( A(\Sigma) = 4\pi(g(\Sigma) - 1)/\varepsilon \), the following hold.

1. An outer neighborhood \( U = [0, \varepsilon) \times \Sigma \) of \( \Sigma \) in \( M \) is isometric to \( ([0, \varepsilon) \times \Sigma, dt^2 + g_0) \),
where \((\Sigma, g_0)\) has constant Gaussian curvature \(\kappa = -c\).

(2) Each slice \(\Sigma_t \approx \{t\} \times \Sigma\) is totally geodesic as a submanifold of spacetime. Equivalently, \(\chi_+(t) = \chi_-(t) = 0\), where \(\chi_\pm(t)\) are the null second fundamental forms of \(\Sigma_t\).

(3) \(K(\cdot, \cdot)\mid_{T_x \Sigma_t} = 0\) and \(K(\nu_t, \cdot)\mid_{T_x \Sigma_t} = 0\) for each \(x \in \Sigma_t\), where \(\nu_t\) is the outer unit normal to \(\Sigma_t\), and \(J = 0\) on \(U\).

**Proof.** First, observe that \(\Sigma\) is stable, because it is weakly outermost. Then, by Proposition 3.1, we have \(\lambda_1(L) = 0\). In this case, let \(\{\Sigma_t\}_{t \in (-\varepsilon, \varepsilon)}\), \(\theta = \theta(t)\), and \(\phi = \phi_t\) be given by Lemma 2.5. It follows from (2.2) that

\[
\frac{d\theta}{dt} = -\Delta \phi + 2\langle X, \nabla \phi \rangle + \left( Q - \frac{1}{2} \theta^2 + \theta \tau \right) \phi,
\]

where \(\Delta = \Delta_t\), \(\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_t\), \(X = X_t\), \(\nabla = \nabla_t\) (gradient operator), \(Q = Q_t\), and \(\text{div} = \text{div}_t\) are the respective entities associated to \(\Sigma_t\), for each \(t \in (-\varepsilon, \varepsilon)\). Thus,

\[
\frac{\theta'}{\phi} = -\frac{\Delta \phi}{\phi} + 2\langle X, \frac{1}{\phi} \nabla \phi \rangle - |X|^2 + \text{div} X + Q - \frac{1}{2} \theta^2 + \theta \tau
\]

\[
\leq \text{div} Y - |Y|^2 + Q - \frac{1}{2} \theta^2 + \theta \tau,
\]

where \(Y = X - \nabla \ln \phi\). Therefore, observing that \(\theta'(t)\) is also constant on \(\Sigma_t\) and using the divergence theorem, for each \(t \in [0, \varepsilon)\), we have

\[
\frac{\theta'(t)}{\phi} \int_{\Sigma_t} \frac{1}{\phi} dA_t - \theta(t) \int_{\Sigma_t} \tau dA_t \leq \int_{\Sigma_t} Q dA_t
\]

\[
= \int_{\Sigma_t} \left( \kappa - (\mu + J_\nu) - \frac{1}{2} |\chi|^2 \right) dA_t
\]

\[
\leq \int_{\Sigma_t} \left( \kappa - (\mu - |J|) - \frac{1}{2} |\chi|^2 \right) dA_t
\]

\[
\leq \int_{\Sigma_t} (\kappa - (\mu - |J|)) dA_t.
\]

Now, using the last inequality above along with the hypothesis that \(\mu - |J| \geq -c\) on \(U\), we get

\[
\frac{\theta'(t)}{\phi} \int_{\Sigma_t} \frac{1}{\phi} dA_t - \theta(t) \int_{\Sigma_t} \tau dA_t \leq \int_{\Sigma_t} (\kappa + c) dA_t
\]

\[
= 4\pi (1 - g(\Sigma)) + cA(\Sigma_t)
\]

\[
= -cA(\Sigma_0) + cA(\Sigma_t)
\]

\[
= c \int_0^t ds \frac{d}{ds} A(\Sigma_s) ds.
\]

Above we have used the Gauss-Bonnet theorem and the fundamental theorem of calculus. On the other side, \(\theta(t) = \text{tr}_{\Sigma_t} K + H_t \geq H_t\) for \(t \in [0, \varepsilon)\), since \(K\) is 2-convex on \(U\). Here, \(H_t\) is the mean curvature of \(\Sigma_t\) in \(M\). Then, by the first
variation of area,
\[ \theta'(t) \int_{\Sigma_t} \phi \frac{1}{2} dA_t - \theta(t) \int_{\Sigma_t} \tau dA_t \leq c \int_0^t \left( \int_{\Sigma_s} H_s \phi dA_s \right) ds \]
\[ \leq c \int_0^t \theta(s) \left( \int_{\Sigma_s} \phi dA_s \right) ds, \]
i.e.,
\[ \theta'(t) \int_{\Sigma_t} \phi \frac{1}{2} dA_t \leq \int_0^t \theta(s) \left( c \int_{\Sigma_s} \phi dA_s \right) ds + \theta(t) \int_{\Sigma_t} \tau dA_t, \quad \forall t \in [0, \varepsilon). \]

It follows from Lemma 3.2 that \( \theta(t) = 0 \), i.e., \( \Sigma_t \) is a MOTS for each \( t \in [0, \varepsilon) \), because \( \theta(t) \geq 0 \) since \( \Sigma \) is weakly outermost.

Since \( \theta(t) = 0 \) for each \( t \in [0, \varepsilon) \), all inequalities above must be equalities. Then, \( Y = X - \nabla \ln \phi = 0 \), \( \chi = 0 \), and \( \mu + J(\nu) = \mu - |J| = -c \) on \( U \approx [0, \varepsilon) \times \Sigma \). Moreover, \( A(\Sigma_t) = A(\Sigma_0) = 4\pi(\gamma(\Sigma_0) - 1)/c \) for all \( t \in [0, \varepsilon) \). Because \( \theta'(t) = 0 \), it follows from (2.2) and Lemma 2.3 that \( \Sigma_t \) is stable. Thus, using Proposition 3.1, \( \Sigma_t \) has constant Gaussian curvature \( \kappa_t = -c \), for each \( t \in [0, \varepsilon) \).

Now, because \( t \mapsto A(\Sigma_t) \) is constant for \( t \in [0, \varepsilon) \),
\[ 0 = \frac{d}{dt} A(\Sigma_t) = \int_{\Sigma_t} H_t \phi dA_t. \]
This implies \( H_t = 0 \) for each \( t \in [0, \varepsilon) \), since \( 0 = \theta(t) \geq H_t \) and \( \phi = \phi_t > 0 \). Then, \( \Sigma_t \) is a minimal MOTS, which implies \( \text{tr}_\Sigma K = 0 \) for each \( t \in [0, \varepsilon) \). In this case, the null mean curvature \( \theta_-(t) = \text{tr}_\Sigma K = H_t \) of \( \Sigma_t \) with respect to \( l_-(t) = u - \nu_t \) also vanishes everywhere. Applying (2.2) for \( \theta_- \) and \( \phi_- = -\phi \) instead of \( \theta = \theta_+ \) and \( \phi \), respectively, we have
\[ 0 = \theta'_- = -\Delta \phi_- + 2(\nabla \phi_- \cdot X_-) + (Q_- - |X_-|^2 + \text{div} X_-) \phi_- , \]
where
\[ Q_- = \kappa - (\mu + J(\nu)) - \frac{1}{2}|\chi_-|^2 \]
\[ = -c - (\mu + |J|) - \frac{1}{2}|\chi_-|^2 \]
\[ = -2|J| - \frac{1}{2}|\chi_-|^2 , \]
\[ X_- = -X = -\frac{1}{\phi} \nabla \phi . \]
Substituting (3.3) and (3.4) into (3.2) (and remembering that \( \phi_- = -\phi \)), we get
\[ \Delta \phi + \frac{\nabla \phi |^2}{\phi} + \left( |J| + \frac{1}{4} |\chi_-|^2 \right) \phi = 0 , \]
which, after integration over \( \Sigma_t \), implies
\[ |\nabla \phi_0 | = |\chi_- | = |J| = 0 \quad \text{on} \ U. \]

Equation 2.1 now implies that \( K |_{\Sigma_t \times T_\Sigma \Sigma_t} = 0 \) and that \( (\Sigma, g_t) \) is totally geodesic in \( M \) for each \( t \in [0, \varepsilon) \), where \( g_t \) is the Riemannian metric on \( \Sigma_t \approx \{ t \} \times \Sigma \) induced from \( (M, g) \). Writing \( g = \phi^2 dt^2 + g_t \) on \( U \approx [0, \varepsilon) \times \Sigma \) and observing that \( \phi = \phi_t \) depends only on \( t \in [0, \varepsilon) \), it follows that \( g_t \) does not depend on \( t \in [0, \varepsilon) \).

Then, after the simple change of variable \( ds = \phi(t) dt \), \( \phi(t) = \phi_t \), we can see that
g has the product structure $ds^2 + g_0$ on $U$, where $(\Sigma, g_0)$ has constant Gaussian curvature $\kappa = -c$. \qed

4. The Case of High Dimension with Negative $\sigma$-Constant

In this section, we extend the main result of last section to the case of high-dimensional MOTSs with negative $\sigma$-constant. But, before stating our result, we are going to present some terminologies.

Let $\Sigma^n$ be a connected closed (compact with no boundary) $n$-manifold, $n \geq 3$. Denote by $\mathcal{M}(\Sigma)$ the set of all Riemannian metrics on $\Sigma$. The Einstein-Hilbert functional $\mathcal{E} : \mathcal{M}(\Sigma) \to \mathbb{R}$ is defined by

$$\mathcal{E}(g) = \frac{\int_{\Sigma} R_g \, dv_g}{\text{Vol}(\Sigma^n, g)^{-\frac{2}{n}}},$$

where $R_g$ is the scalar curvature of $(\Sigma, g)$. Denote by $[g] = \{e^{2f}g; f \in C^\infty(\Sigma)\}$ the conformal class of $g \in \mathcal{M}(\Sigma)$. The Yamabe invariant of $(\Sigma, [g])$ is defined as the following conformal invariant:

$$\mathcal{Y}(\Sigma, [g]) = \inf_{\tilde{g} \in [g]} \mathcal{E}(\tilde{g}).$$

The classical solution of the Yamabe problem by Yamabe [27], Trudinger [25], Aubin [4] (see also [5]), and Schoen [22] says that every conformal class $[g]$ contains metrics $\hat{g}$, called Yamabe metrics, which realize the minimum:

$$\mathcal{E}(\hat{g}) = \mathcal{Y}(\Sigma, [g]).$$

Such metrics have constant scalar curvature given by

$$R_{\hat{g}} = \mathcal{Y}(\Sigma^n, [g]) \text{Vol}(\Sigma^n, \hat{g})^{-\frac{2}{n}}.$$

Furthermore,

$$\mathcal{Y}(\Sigma^n, [g]) \leq \mathcal{Y}(\mathbb{S}^n, [g_{\text{can}}]),$$

and equality holds if and only if $(\Sigma^n, g)$ is conformally diffeomorphic to the Euclidean $n$-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ endowed with the canonical metric $g_{\text{can}}$.

The following topological invariant was introduced by Schoen in lectures given in 1987, which were published two years later [24] (see also Kobayashi [18]),

$$\sigma(\Sigma^n) = \sup_{g \in \mathcal{M}(\Sigma^n)} \mathcal{Y}(\Sigma^n, [g]).$$

This invariant is called the $\sigma$-constant of $\Sigma$.

Remark 4.1. Using the solution of the Yamabe problem, it is not difficult to prove that $\sigma(\Sigma) > 0$ if and only if $\Sigma$ admits a Riemannian metric of positive scalar curvature. On the other hand, observe that if $S^2$ is an orientable closed surface, then its Euler-Poincaré characteristic $\chi(S) = 2(1 - g(S)) > 0$ if and only if $g(S) = 0$, i.e., $S^2$ is topologically $\mathbb{S}^2$, which by Gauss-Bonnet and uniformization theorems is equivalent to say that $S^2$ admits a Riemannian metric of positive Gaussian curvature. Having this in mind, the $\sigma$-constant in dimension $n \geq 3$ shares the same property with the Euler-Poincaré characteristic in dimension 2.

Now, let us present the first result of this section. The following volume estimate was obtained by Galloway and Murchadha [14] (see also [9] for the time-symmetric case). Our contribution consists in the infinitesimal rigidity obtained when equality occurs, which is based on [13] and [20].
Proposition 4.2. Let $\Sigma^n$ be a stable closed MOTS with $\sigma(\Sigma) < 0$ in an initial data set $M^{n+1} = (M^{n+1}, g, K)$, $n \geq 3$. Suppose that $\mu + J(\nu) \geq -c$ on $\Sigma$ for some constant $c > 0$. Then, the volume of $\Sigma$ satisfies

$$\text{Vol}(\Sigma^n) \geq \left( \frac{\sigma(\Sigma^n)}{2c} \right)^\frac{n-2}{n}.$$  \hspace{1cm} (4.1)

Furthermore, if equality holds, the metric $g_\Sigma$ on $\Sigma$ induced from $M$ is Einstein and has constant scalar curvature $R_\Sigma = -2c$, the null second fundamental form $\chi = \chi_+$ of $\Sigma$ vanishes identically, $\mu + J(\nu) = -c$ on $\Sigma$, and $\lambda_1(L_0) = \lambda_1(L) = 0$.

Proof. Since $\Sigma$ is stable, $\lambda_1(L) \geq 0$, where $L$ is the MOTSs stability operator. Then, by (2.5),

$$0 \leq 2 \int_{\Sigma} (|\nabla u|^2 + Q u^2) \, dv$$

$$= \int_{\Sigma} (2|\nabla u|^2 + (R_\Sigma - 2(\mu + J(\nu) - |\chi|^2)u^2) \, dv$$

$$\leq \int_{\Sigma} (2|\nabla u|^2 + (R_\Sigma + 2c)u^2) \, dv,$$  \hspace{1cm} (4.2)

for all $u \in C^\infty(\Sigma)$, $u > 0$. Using $2 < \frac{4(n-1)}{n-2}$ and Hölder inequality into (4.3), we have

$$0 \leq \int_{\Sigma} \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + R_\Sigma u^2 \right) \, dv + 2c \int_{\Sigma} u^2 \, dv$$

$$\leq \int_{\Sigma} \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + R_\Sigma u^2 \right) \, dv + 2c \text{Vol}(\Sigma)^\frac{n}{2} \left( \int_{\Sigma} u^{\frac{2n}{n-2}} \, dv \right)^{\frac{n-2}{n}}$$  \hspace{1cm} (4.4)

for all $u \in C^\infty(\Sigma)$, $u > 0$. On the other hand, it is well known that for $g = u^{\frac{4}{n-2}} g_\Sigma$, $u > 0$, we have

$$R_g u^{\frac{4}{n-2}} = \frac{4(n-1)}{n-2} \Delta u + R_\Sigma u,$$  \hspace{1cm} (4.5)

which implies

$$\mathcal{E}(u^{\frac{4}{n-2}} g_\Sigma) = \frac{\int_{\Sigma} \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + R_\Sigma u^2 \right) \, dv}{\left( \int_{\Sigma} u^{\frac{2n}{n-2}} \, dv \right)^{\frac{n-2}{n}}}.$$  \hspace{1cm} (4.6)

Therefore, using (4.7) into (4.5),

$$0 \leq \mathcal{E}(u^{\frac{4}{n-2}} g_\Sigma) + 2c \text{Vol}(\Sigma)^\frac{n}{2},$$  \hspace{1cm} (4.8)

for all $u \in C^\infty(\Sigma)$, $u > 0$, which implies

$$0 \leq \inf_{u \in C^\infty(\Sigma), u > 0} \mathcal{E}(u^{\frac{4}{n-2}} g_\Sigma) + 2c \text{Vol}(\Sigma)^\frac{n}{2}$$

$$= \mathcal{Y}(\Sigma, [g_\Sigma]) + 2c \text{Vol}(\Sigma)^\frac{n}{2} \hspace{1cm} (4.9)$$

$$\leq \sigma(\Sigma) + 2c \text{Vol}(\Sigma)^\frac{n}{2}.$$  \hspace{1cm} (4.10)

This finishes the proof of (4.1).

Now, suppose that equality in (4.1) holds. Then, inequalities (4.9) and (4.10) must be equalities. Choose $u_0 \in C^\infty(\Sigma)$, $u_0 > 0$, such that $\hat{g} = u_0^{4/(n-2)} g_\Sigma$ is a Yamabe metric. Then, inequalities (4.2), (4.3), (4.4), (4.5), and (4.8) must be
equalities when \( u = u_0 \). Therefore, \( \chi = 0, \mu + J(\nu) = -c \) on \( \Sigma \), and \( |\nabla u_0| \) vanishes identically, since \( 2 < \frac{4(n-1)}{n-2} \). This implies that \( u_0 \) is constant (in particular, \( g_{\Sigma} \) is also a Yamabe metric) and \( \int_{\Sigma} Qdv = 0 \). Proceeding analogously to the proof of Proposition 3.1, we have \( R_{\Sigma} = -2c \) and \( \lambda_1(L_0) = \lambda_1(L) = 0 \). Equality in \( (4.10) \) implies that \( g_{\Sigma} \) realizes \( \sigma(\Sigma) \), i.e., \( \mathcal{E}(g_{\Sigma}) = \mathcal{V}(\Sigma, [g_{\Sigma}]) = \sigma(\Sigma) \). Then, by [23] pp. 126-127, \( g_{\Sigma} \) is Einstein. □

In the following, we are going to present the main result of this section. But, before doing that, let us make an important remark.

Remark 4.3. Let \( \Sigma^n \) be a closed manifold of dimension \( n \geq 3 \). Suppose that \( g \in \mathcal{M}(\Sigma) \) satisfies \( \mathcal{V}(\Sigma^n, [g]) < 0 \). Then, it is well known that given \( c > 0 \), there exists a unique metric \( \tilde{g} \in [g] \) such that \( R_{\tilde{g}} = -2c \). In fact, the existence follows from the solution of the Yamabe problem. To prove the uniqueness, let \( g_1, g_2 \in [g] \) be such that \( R_{g_1} = -2c = R_{g_2} \). Since \( g_1 \) and \( g_2 \) are conformal, we have \( g_2 = u^{4/(n-2)}g_1 \) for some \( u \in C^\infty(\Sigma), u > 0 \). Then, using \( (4.10) \) for \( g_1 \) and \( g_2 \), it follows that

\[
-2cu_n^{n+2} = \frac{4(n-1)}{n-2} \Delta g_1 u - 2cu,
\]

i.e.,

\[
2cu_n^{n+2} = \frac{4(n-1)}{n-2} \Delta g_1 u + 2cu.
\]

Then, taking \( x_0 \in \Sigma \) such that \( u(x_0) = \min u \),

\[
2c(u(x_0))^{n+2} = \frac{4(n-1)}{n-2} \Delta g_1 u(x_0) + 2cu(x_0) \geq 2cu(x_0),
\]

which implies \( u(x_0) \geq 1 \). In turn, taking \( x_1 \in \Sigma \) such that \( u(x_1) = \max u \), we have \( u(x_1) \leq 1 \). Thus, \( u \equiv 1 \).

The main result of this section is the following. Its proof is based on [13] and [20].

**Theorem 4.4.** Let \( M^{n+1} = (M^{n+1}, g, K) \) be a \((n+1)\)-dimensional initial data set, \( n \geq 3 \). Let \( \Sigma^n \) be a weakly outermost closed MOTS in \( M^{n+1} \) with \( \sigma \)-constant \( \sigma(\Sigma) < 0 \). Suppose that \( \mu - |J| \geq -c \) for some constant \( c > 0 \) and that \( K \) is \( n \)-convex, both on \( M_+ \). Then, if \( \text{Vol}(\Sigma^n) = (|\sigma(\Sigma^n)|/2c)^{n/2} \), the following hold.

1. An outer neighborhood \( U \approx (0, \varepsilon) \times \Sigma \) of \( \Sigma \) in \( M \) is isometric to

\[
([0, \varepsilon] \times \Sigma, dt^2 + g_\Sigma),
\]

and \( (\Sigma, g_\Sigma) \) is Einstein with constant scalar curvature \( R_{\Sigma} = -2c \).

2. Each slice \( \Sigma_t \approx \{t\} \times \Sigma \) is totally geodesic as a submanifold of spacetime. Equivalently, \( \chi_+ (t) = \chi_- (t) = 0 \), where \( \chi_\pm (t) \) are the null second fundamental forms of \( \Sigma_t \).

3. \( K(\cdot, \cdot)|_{T_\Sigma} = 0 \) and \( K(\nu_t, \cdot)|_{T_\Sigma} = 0 \) for each \( x \in \Sigma_t \), where \( \nu_t \) is the outer unit normal to \( \Sigma_t \), and \( J = 0 \) on \( U \).

**Proof.** Since weakly outermost closed MOTSs are stable (Remark 2), by Proposition 1.2 we have \( \lambda(L) = 0 \), where \( L \) is the MOTSs stability operator. Then, we
can take \( \{ \Sigma_t \}_{t \in (-\varepsilon, \varepsilon)} \), \( \theta = \theta(t) \), and \( \phi = \phi_t \) as in Lemma 2.5. By (2.2),

\[
\theta' = -\Delta \phi + 2\langle X, \nabla \phi \rangle + \left( Q - |X|^2 + \text{div} \, X - \frac{1}{2} \theta^2 + \theta \tau \right) \phi \\
= \left( -|Y|^2 + \text{div} \, Y + Q - \frac{1}{2} \theta^2 + \theta \tau \right) \phi \\
\leq \left( -|Y|^2 + \text{div} \, Y + Q + \theta \tau \right) \phi,
\]

(4.11)

where \( Y = X - \nabla \ln \phi \). Above, \( \Delta = \Delta_t \), \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_t \), \( X = X_t \), \( \nabla = \nabla_t \) (gradient operator), \( Q = Q_t \), and \( \text{div} = \text{div}_t \) are the respective entities associated to \( \Sigma_t \), for each \( t \in (-\varepsilon, \varepsilon) \). It follows from (4.11) that

\[
\theta'(t) \frac{u^2}{\phi} - \theta(t) \tau u^2 \\
\leq -u^2|Y|^2 + u^2 \text{div} \, Y + Qu^2 \\
= -u^2|Y|^2 - 2u\langle \nabla u, Y \rangle + \text{div}(u^2 Y) \\
+ \left( \frac{1}{2} R_{\Sigma_t} - (\mu + J(\nu)) - \frac{1}{2} |\chi|^2 \right) u^2 \\
\leq -u^2|Y|^2 + 2|u||\nabla u||Y| + \text{div}(u^2 Y) \\
+ \left( \frac{1}{2} R_{\Sigma_t} - (\mu - |J|) - \frac{1}{2} |\chi|^2 \right) u^2 \\
\leq |\nabla u|^2 + \text{div}(u^2 Y) + \left( \frac{1}{2} R_{\Sigma_t} + c \right) u^2,
\]

for all \( u \in C^\infty(\Sigma_t) \), \( u > 0 \), and \( t \in [0, \varepsilon) \). Above we have used that \( \mu - |J| \geq -c \) on \( M_\varepsilon \). Integrating over \( \Sigma_t \), observing that \( \theta'(t) \) is also constant on \( \Sigma_t \), and using Hölder inequality, we get

\[
2 \left( \theta'(t) \int_{\Sigma_t} \frac{u^2}{\phi} dv_t - \theta(t) \int_{\Sigma_t} \tau u^2 dv_t \right) \\
\leq \int_{\Sigma_t} (2|\nabla u|^2 + R_{\Sigma_t} u^2) dv_t + 2c \int_{\Sigma_t} u^2 dv_t \\
\leq \int_{\Sigma_t} \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + R_{\Sigma_t} u^2 \right) dv_t \\
+ 2c \text{Vol}(\Sigma_t)^{\frac{n}{n-2}} \left( \int_{\Sigma_t} u^{\frac{2n}{n-2}} dv_t \right)^{\frac{n-2}{n}},
\]

for all \( u \in C^\infty(\Sigma_t) \), \( u > 0 \), and \( t \in [0, \varepsilon) \). It follows from Remark 4.3 that for each \( t \in [0, \varepsilon) \), there exists a unique \( u_t \in C^\infty(\Sigma_t) \), \( u_t > 0 \), such that \( \hat{g}_t = u_t^{4/(n-2)} g_{\Sigma_t} \) is a Yamabe metric with constant scalar curvature \( R_{\hat{g}_t} = -2c \), given that

\[
\mathcal{Y}(\Sigma_t, [g_{\Sigma_t}]) \leq \sigma(\Sigma) < 0,
\]
where \( g_{\Sigma_t} \) is the Riemannian metric on \( \Sigma_t \) induced from \( M \). Therefore,

\[
2 \left( \theta'(t) \int_{\Sigma_t} u_t^2 \phi^{-1} dv_t - \theta(t) \int_{\Sigma_t} \tau u_t^2 dv_t \right) \left( \int_{\Sigma_t} u_t^{2m} dv_t \right)^{-\frac{2m}{n-2}} \leq E(u_t^{-\frac{1}{n-2}} g_{\Sigma_t}) + 2c \text{Vol}(\Sigma_t)^{\frac{2}{n}}
\]

\[
= Y(\Sigma_t, [g_{\Sigma_t}]) + 2c \text{Vol}(\Sigma_t)^{\frac{2}{n}}
\]

\[
\leq \sigma(\Sigma) + 2c \text{Vol}(\Sigma_t)^{\frac{2}{n}}
\]

\[
= \left( -2c \text{Vol}(\Sigma)^{\frac{2}{n}} + 2c \text{Vol}(\Sigma_t)^{\frac{2}{n}} \right)
\]

\[
= \frac{4c}{n} \int_0^t \text{Vol}(\Sigma_s)^{\frac{2-n}{n}} \frac{d}{ds} \text{Vol}(\Sigma_s) ds,
\]

for each \( t \in (0, \varepsilon) \). Using that \( \theta(t) = \text{tr}_{\Sigma_t} K + H_t \leq H_t \) for \( t \in (0, \varepsilon) \), since \( K \) is \( n \)-convex on \( M_+ \), together with the first variation of area (volume), we have

\[
\theta'(t) \int_{\Sigma_t} u_t^2 \phi^{-1} dv_t - \theta(t) \int_{\Sigma_t} \tau u_t^2 dv_t \leq \frac{2c}{n} \int_0^t \text{Vol}(\Sigma_s)^{\frac{2-n}{n}} \left( \int_{\Sigma_s} H_s \phi dv_s \right) ds
\]

\[
\leq \frac{2c}{n} \int_0^t \theta(s) \left( \text{Vol}(\Sigma_s)^{\frac{2-n}{n}} \int_{\Sigma_s} \phi dv_s \right) ds.
\]

Now, observing that by the uniqueness of \( \tilde{g}_t \), \( u_t \) is continuous on \( t \in (0, \varepsilon) \), we can use Lemma \ref{lem:continuity} for

\[
f(t) = \theta(t), \quad \eta(t) = \left( \int_{\Sigma_t} u_t^2 \phi^{-1} dv_t \right) \left( \int_{\Sigma_t} u_t^{2m} dv_t \right)^{-\frac{2m}{n-2}},
\]

\[
\rho(t) = \left( \int_{\Sigma_t} \tau u_t^2 dv_t \right) \left( \int_{\Sigma_t} u_t^{2m} dv_t \right)^{-\frac{n-2}{2m}} \text{ and } \xi(t) = \frac{2c}{n} \text{Vol}(\Sigma_t)^{\frac{2-n}{n}} \int_{\Sigma_t} \phi dv_t.
\]

Thus, \( \theta(t) = 0 \) for all \( t \in (0, \varepsilon) \). Then, continuing as in the proof of Theorem \ref{thm:main}, we have the result. \( \square \)

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