Special Classes of CR Singularities II

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Abstract. It is constructed a Formal Normal Form for a Special Class of Real-Formal Submanifolds in $\mathbb{C}^{2N}$.

1. Introduction and Main Result

In this note, we construct a normal form (see [3, 15]) following advices from Zaitsev [11, 12, 13] for a Special Class of Real-Formal Submanifolds in Complex Spaces. Provided the coordinates $(w, z) = (w_1, w_2, \ldots, w_N; z_1, z_2, \ldots, z_N)$ in $\mathbb{C}^{2N}$, let $M \subset \mathbb{C}^{2N}$ be the Real-Smooth Submanifold defined near $p = 0$ by

$$M : w_1 = Q_1(z) + \overline{Q_1(z)} + O(3), \quad w_2 = Q_2(z) + \overline{Q_2(z)} + O(3), \ldots, \quad w_N = Q_N(z) + \overline{Q_N(z)} + O(3).$$

(1.1)

where $Q_1(z), Q_2(z), \ldots, Q_N(z)$ are polynomials of degree 2 in $z$ written as

$$Q_m(z) = \sum_{n=1}^{N} \alpha_{mn} z_n^2, \quad \text{for all } m = 1, \ldots, N, \text{ such that } \det(\alpha_{mn})_{1 \leq m, n \leq N} \neq 0.$$

(1.2)

Recalling the strategy from [3], we define

$$\text{tr}_m = \sum_{n=1}^{N} \left( \alpha_{mn} \frac{\partial^2}{\partial z_n^2} + \alpha_{mn} \frac{\partial^2}{\partial \overline{z_n}^2} \right), \quad \text{for all } m = 1, \ldots, N.$$

(1.3)

The main result is the following:

Theorem 1.1. Let $M \subset \mathbb{C}^{2N}$ be the Real-Smooth Submanifold defined near $p = 0$ by the equation

$$w_1 = Q_1(z) + \overline{Q_1(z)} + \sum_{k \geq 3} \varphi^{(1)}_k(z, \overline{z}),$$

$$w_2 = Q_2(z) + \overline{Q_2(z)} + \sum_{k \geq 3} \varphi^{(2)}_k(z, \overline{z}),$$

$$\vdots$$

$$w_N = Q_N(z) + \overline{Q_N(z)} + \sum_{k \geq 3} \varphi^{(N)}_k(z, \overline{z}),$$

(1.4)

where $\varphi^{(1)}_k(z, \overline{z}), \ldots, \varphi^{(N)}_k(z, \overline{z})$ are polynomials of degree $k$ in $(z, \overline{z})$, for all $k \geq 3$.

Then there exists a unique formal map

$$(w'_1, \ldots, w'_N; z'_1, \ldots, z'_N) = (G(z, w), F(z, w)) = (w_1 + O(2), \ldots, w_N + O(2); z_1 + O(2), \ldots, z_N + O(2)),$$

(1.5)

that transforms $M$ into the normal form

$$w'_1 = Q_1(z') + \overline{Q_1(z')} + \sum_{k \geq 3} \varphi^{(1)}_k(z', \overline{z'}),$$

$$w'_2 = Q_2(z') + \overline{Q_2(z')} + \sum_{k \geq 3} \varphi^{(2)}_k(z', \overline{z'}),$$

$$\vdots$$

$$w'_N = Q_N(z') + \overline{Q_N(z')} + \sum_{k \geq 3} \varphi^{(N)}_k(z', \overline{z'}),$$

(1.6)

where $\varphi^{(1)}_k(z', \overline{z'}), \ldots, \varphi^{(N)}_k(z', \overline{z'})$ are polynomials of degree $k$ in $(z', \overline{z'})$, for all $k \geq 3$, satisfying

$$\varphi^{(l)}_k(z', \overline{z'}) \in \ker \left( \text{tr}_1^{n_1} \text{tr}_2^{n_2} \ldots \text{tr}_N^{n_N} \right), \quad \text{for all } k \geq 3 \text{ and } l = 1, \ldots, N.$$ (1.7)

These resulted homogeneous polynomials are used in order to apply the generalized version of the Fischer Decomposition [10] by separating the real parts and the imaginary parts of the local defining equations at each degree level.

Keywords: C.-R. Geometry, Equivalence Problem, Normal Norm, Real-Submanifold, Formal Power Series, C.-R. Singularity.

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3. Settings

Let a Formal Equivalence between $M$ and $M'$, denoted like

\[(F(z, w); G(z, w)) = (F_1(z, w), F_2(z, w), \ldots, F_{N}(z, w); G_1(z, w), G_2(z, w), \ldots, G_{N}(z, w)).\]

Provided $w = (w_1, w_2, \ldots, w_{N})$ defined by (3.3), we obtain

\[(3.2) \quad G_{(l)}(z, w) = Q_l(F(z, w), \overline{F}(z, w)) + \sum_{k \geq 3} \phi_{k}^{(l)}(F(z, w), \overline{F}(z, w)), \quad \text{for all } l = 1, \ldots, N.\]

In order to understand better the interactions of terms in (3.2), we write (3.1) as follows

\[(3.3) \quad (F(z, w); G(z, w)) = \left( \sum_{m, n_{1}, \ldots, n_{N} \in \mathbb{N}} F_{m, n_{1}, \ldots, n_{N}}^{(1)}(z) w_{n_{1}} \cdots w_{n_{N}}, \ldots, \sum_{m, n_{1}, \ldots, n_{N} \in \mathbb{N}} F_{m, n_{1}, \ldots, n_{N}}^{(N)}(z) w_{n_{1}} \cdots w_{n_{N}} \right),\]

where $F_{m, n_{1}, \ldots, n_{N}}^{(1)}(z), \ldots, F_{m, n_{1}, \ldots, n_{N}}^{(N)}(z)$ are homogeneous polynomials of degree $m$ in $z$, for all $m, n_{1}, \ldots, n_{N} \in \mathbb{N}$. Because the Formal Mapping (3.1) does not possess constant terms, we obtain $G_{0,0,\ldots,0}(z) = 0$ and $F_{0,0,\ldots,0}(z) = 0$.

We replace (3.3) in (3.2). We obtain

\[(3.4) \quad Q_l \left( \sum_{m, n_{1}, \ldots, n_{N} \in \mathbb{N}} F_{m, n_{1}, \ldots, n_{N}}^{(1)}(z) \left( Q_{1}(z) + Q_{1}(\overline{z}) + \sum_{k \geq 3} \phi_{k}^{(1)}(z, \overline{z}) \right) \right)_{n_{1}} \cdots \left( Q_{N}(z) + Q_{N}(\overline{z}) + \sum_{k \geq 3} \phi_{k}^{(N)}(z, \overline{z}) \right)_{n_{N}} =
\]

\[+ \sum_{k \geq 3} \phi_{k}^{(l)} \left( \sum_{m, n_{1}, \ldots, n_{N} \in \mathbb{N}} F_{m, n_{1}, \ldots, n_{N}}(z) \left( Q_{1}(z) + Q_{1}(\overline{z}) + \sum_{k \geq 3} \phi_{k}^{(1)}(z, \overline{z}) \right) \right)_{n_{1}} \cdots \left( Q_{N}(z) + Q_{N}(\overline{z}) + \sum_{k \geq 3} \phi_{k}^{(N)}(z, \overline{z}) \right)_{n_{N}},\]

for all $l = 1, \ldots, N$.

In particular, we write

\[(3.5) \quad \begin{pmatrix}
\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N} \\
\gamma_{21} & \gamma_{22} & \cdots & \gamma_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{NN}
\end{pmatrix} =
\begin{pmatrix}
\frac{\sum_{i=1}^{N} \gamma_{i1} z_{i}}{\sum_{i=1}^{N} \gamma_{i2} z_{i}} \\
\frac{\sum_{i=1}^{N} \gamma_{i2} z_{i}}{\sum_{i=1}^{N} \gamma_{i1} z_{i}} \\
\vdots \\
\frac{\sum_{i=1}^{N} \gamma_{iN} z_{i}}{\sum_{i=1}^{N} \gamma_{i1} z_{i}}
\end{pmatrix}, \quad \text{where } (\gamma_{ij})_{1 \leq i, j \leq N} \in \mathbb{M}_{N \times N}(\mathbb{C}) \text{ is not degenerate.}
\]

We obtain a system of equations written in its matrix form like

\[(3.6) \quad \begin{pmatrix}
Q_{1}(z, \overline{z}) & \cdots & Q_{N}(z, \overline{z}) \\
Q_{2}(z, \overline{z}) & \cdots & Q_{N}(z, \overline{z}) \\
\vdots & \vdots & \vdots \\
Q_{N}(z, \overline{z}) & \cdots & Q_{N}(z, \overline{z})
\end{pmatrix} =
\begin{pmatrix}
\left( \sum_{i=1}^{N} \gamma_{i1} z_{i} \right)^{2} + \left( \sum_{i=1}^{N} \gamma_{i1} z_{i} \right)^{2} \\
\left( \sum_{i=1}^{N} \gamma_{i2} z_{i} \right)^{2} + \left( \sum_{i=1}^{N} \gamma_{i2} z_{i} \right)^{2} \\
\vdots \\
\left( \sum_{i=1}^{N} \gamma_{iN} z_{i} \right)^{2} + \left( \sum_{i=1}^{N} \gamma_{iN} z_{i} \right)^{2}
\end{pmatrix}.
\]
Moreover, we can assume
\[
\det \begin{pmatrix}
G^{(1)}_{0,0,0, \ldots, 0}(z) & G^{(1)}_{0,0,1, \ldots, 0}(z) & \cdots & G^{(1)}_{0,0,1, \ldots, 0}(z) \\
G^{(2)}_{0,0,0, \ldots, 0}(z) & G^{(2)}_{0,0,1, \ldots, 0}(z) & \cdots & G^{(2)}_{0,0,1, \ldots, 0}(z) \\
\vdots & \vdots & \ddots & \vdots \\
G^{(N)}_{0,0,0, \ldots, 0}(z) & G^{(N)}_{0,0,1, \ldots, 0}(z) & \cdots & G^{(N)}_{0,0,1, \ldots, 0}(z)
\end{pmatrix} \neq 0.
\]

Now, we consider the following change of coordinates
\[
\begin{pmatrix}
w_1' \\
w_2' \\
\vdots \\
w_N'
\end{pmatrix} = \begin{pmatrix}
G^{(1)}_{0,0,0, \ldots, 0}(z) & G^{(1)}_{0,0,1, \ldots, 0}(z) & \cdots & G^{(1)}_{0,0,1, \ldots, 0}(z) \\
G^{(2)}_{0,0,0, \ldots, 0}(z) & G^{(2)}_{0,0,1, \ldots, 0}(z) & \cdots & G^{(2)}_{0,0,1, \ldots, 0}(z) \\
\vdots & \vdots & \ddots & \vdots \\
G^{(N)}_{0,0,0, \ldots, 0}(z) & G^{(N)}_{0,0,1, \ldots, 0}(z) & \cdots & G^{(N)}_{0,0,1, \ldots, 0}(z)
\end{pmatrix}^{-1}
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_N
\end{pmatrix},
\]

(3.8)

In particular, we assume
\[
G(z, w) = (w_1 + O(2), w_2 + O(2), \ldots, w_N + O(2)) \quad \text{and} \quad F(z, w) = (z_1 + O(2), z_2 + O(2), \ldots, z_N + O(2)).
\]

In particular, we write
\[
z^I = z^{i_1} \cdots z^{i_N}, \quad \text{where} \quad I = (i_1, \ldots, i_N) \in \mathbb{N}^N \quad \text{such that} \quad |I| = i_1 + \cdots + i_N \geq 3,
\]

(3.10)

\[
z^J = z^{j_1} \cdots z^{j_N}, \quad \text{where} \quad J = (j_1, \ldots, j_N) \in \mathbb{N}^N \quad \text{such that} \quad |J| = j_1 + \cdots + j_N \geq 2.
\]

We consider the following Generalized Fischer Decomposition
\[
z^I = \sum_{l=1}^{N} A_l(z, \overline{z}) q_l(z, \overline{z}) + C(z, \overline{z}), \quad \sum_{l=1}^{N} \text{tr} \left( C(z, \overline{z}) \right) = 0,
\]

(3.11)

where \(\tilde{I} = (i_1, i_2, \ldots, i_N) \in \mathbb{N}^N \) and \(i_1 + \tilde{i}_2 + \cdots + \tilde{i}_N = p\).

Last details are just some computations to be added later.

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