ON SYNTACTICALLY SIMILAR LOGIC PROGRAMS
AND
SEQUENTIAL DECOMPOSITIONS

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ABSTRACT. Rule-based reasoning is an essential part of human intelligence prominently formalized in artificial intelligence research via logic programs. Describing complex objects as the composition of elementary ones is a common strategy in computer science and science in general. The author has recently introduced the sequential composition of logic programs in the context of logic-based analogical reasoning and learning in logic programming. Motivated by these applications, in this paper we construct a qualitative and algebraic notion of syntactic logic program similarity from sequential decompositions of programs. We then show how similarity can be used to answer queries across different domains via a one-step reduction. In a broader sense, this paper is a further step towards an algebraic theory of logic programming.

1. INTRODUCTION

Rule-based reasoning is an essential part of human intelligence prominently formalized in artificial intelligence research via logic programs with important applications to expert systems, database theory, and knowledge representation and reasoning (cf. Apt, 1990; Lloyd, 1987; Baral, 2003). Describing complex objects as the composition of elementary ones is a common strategy in computer science and science in general. Antić (2023) has recently introduced the sequential composition of logic programs in the context of logic-based analogical reasoning and learning in logic programming. The sequential composition operation has been studied by Antić (2020) in the propositional case and by Antić (2021) in the non-monotonic case of answer set programming. O’Keefe (1985) is the first to study the composition of logic programs and in Antić (2020, §Related work) the author has compared O’Keefe’s composition and the related composition of Bugliesi, Lamma, and Mello (1994) and Brogi, Lamma, and Mello (1992) to sequential composition. Other notable works dealing with composition are, for example, Dong and Ginsburg (1990, 1995), Plambeck (1990, 1990), and Ioannidis and Wong (1991).

Motivated by the aforementioned applications to analogical reasoning, in this paper we study the sequential decomposition of logic programs and show that it gives rise to a qualitative and algebraic notion of syntactic logic program similarity. More precisely, we say that a program $P$ can be one-step reduced to a program $R$—in symbols, $P \preceq R$—iff $P = (Q \circ R) \circ S$ for some programs $Q$ (prefix) and $S$ (suffix). Now given a query $q$ to $P$, we can answer $q$ by translating every SLD-resolution step of $P$ via the prefix $Q$ into an SLD-resolution step of $R$ and translating the result back again with the suffix $S$ (cf. Example 4). We then say that $P$ and $R$ are syntactically similar iff $P$ can be one-step reduced to $R$ and vice versa, that is, iff $P \preceq R$ and $R \preceq P$. This definition has appealing mathematical properties. For example, we can show that the programs Plus for the addition of numerals and the program
Append for list concatenation are syntactically similar according to our definition (cf. Example 4). Interestingly, when translating queries to Append into queries of the seemingly simpler program Plus, we obtain ‘entangled’ terms of the form \( s([1]) \) and \( s([b,c]) \) which are neither numerals nor lists, and we believe that ‘entangled’ syntactic objects like these are essential for cross-domain analogical reasoning (see the discussion in Section 5). Syntactic similarity is related to the number of bound variables per rule of a program and we deduce from this fact that, for example, the program Member for checking list membership, containing maximally two bound variables per rule, can be one-step reduced to the program Append containing maximally three bound variables per rule, but not vice versa: Member < Append (cf. Examples 5 and 7). This is interesting as the prefix \( Q \) witnessing Member = \( Q \circ \text{Append} \) resembles the mapping between Member and Append of Tausend and Bell (1991) introduced for analogical reasoning.

In a broader sense, this paper is a further step towards an algebraic theory of logic programming.

2. Logic Programs

We recall the syntax and semantics of logic programs by mainly following the lines of Apt (1990).

2.1. Syntax. An (unranked first-order) language \( L \) is defined as usual from predicate, function, and constant symbols and variables. Terms and atoms over \( L \) are defined in the usual way. We denote the set of all ground \( L \)-atoms by \( HBL \) or simply by \( HB \) called the Herbrand base over \( L \), and we denote the set of all \( L \)-atoms not containing function or constant symbols by \( XHBL \). Substitutions and (most general) unifiers of terms and (sets of) atoms are defined as usual. We call any bijective substitution mapping variables to variables a renaming.

Let \( L \) be a language. A (Horn logic) program over \( L \) is a set of rules of the form

\[
A_0 \leftarrow A_1, \ldots, A_k, \quad k \geq 0,
\]

where \( A_0, \ldots, A_k \) are \( L \)-atoms. It will be convenient to define, for a rule of the form (1), \( h(r) := \{A_0\} \) and \( b(r) := \{A_1, \ldots, A_k\} \), extended to programs by \( h(P) := \bigcup_{r \in P} h(r) \) and \( b(P) := \bigcup_{r \in P} b(r) \). In this case, the size of \( r \) is \( k \) denoted by \( sz(r) \). A fact is a rule with empty body and a proper rule is a rule which is not a fact. We denote the facts and proper rules in \( P \) by \( facts(P) \) and \( proper(P) \), respectively. A program \( P \) is ground if it contains no variables and we denote the grounding of \( P \) which contains all ground instances of the rules in \( P \) by \( gnd(P) \). We call a ground program propositional if it contains only propositional atoms with no arguments. The set of all variants of \( P \) is defined by \( variants(P) := \bigcup_{\theta \text{ renaming}} P[\theta] \). The variants of a program will be needed to avoid clashes of variables in the definition of composition. A variable is bound in a rule if it appears in the head and body. Define the dual of \( P \) by

\[
P^d := facts(P) \cup \{A \leftarrow h(r) \mid r \in proper(P) : A \in b(r)\}.
\]

Roughly, we obtain the dual of a theory by reversing all the arrows of its proper rules.

2.2. Semantics. An interpretation is any set of ground atoms. We define the entailment relation, for every interpretation \( I \), inductively as follows: (i) for an atom \( a \), \( I \models a \) if \( a \in I \); (ii) for a set of atoms \( B \), \( I \models B \) if \( B \subseteq I \); (iii) for a rule \( r \) of the form (1), \( I \models r \) if \( I \models h(r) \) implies \( I \models b(r) \); and, finally, (iv) for a program \( P \), \( I \models P \) if \( I \models r \) holds for each rule \( r \in P \). In case \( I \models P \), we call \( I \) a model of \( P \). The set of all models of \( P \) has a least element with respect to set inclusion called the least model of \( P \). We say that \( P \) and \( R \) are logically equivalent if their least models coincide. Define the van Emden-Kowalski operator of \( P \), for every interpretation \( I \), by

\[
T_P(I) := \{h(r) \mid r \in gnd(P) : I \models b(r)\}.
\]
It is well-known that an interpretation $I$ is a model of $P$ iff $I$ is a prefixed point of $T_P$ and the least model of $P$ coincides with the least fixed point of $T_P$.

We define the left and right reduct of $P$, with respect to some interpretation $I$, respectively by

$$\overset{\text{l}}{I}P := \{ r \in P \mid I \models h(r) \} \quad \text{and} \quad \overset{\text{r}}{I}P := \{ r \in P \mid I \models b(r) \}.$$  

2.3. SLD-Resolution. Logic programs compute via a restricted form of resolution, called SLD-resolution, as follows. For simplicity, we consider here only the ground case. Let $q$ be a ground query $\leftarrow A_1, \ldots, A_k$, $k \geq 1$, and suppose that for some $i$, $1 \leq i \leq k$ and $m \geq 0$, $r = A_i \leftarrow A_i', \ldots, A_m'$ is a rule from $\text{gnd}(P)$. Then $q'$ given by

$$\leftarrow A_1, \ldots, A_{i-1}, A_i', \ldots, A_m', A_{i+1}, \ldots, A_k$$

is called a resolvent of $q$ and $r$, and $A_i$ is called the selected atom of $q$. By iterating this process we obtain a sequence of resolvents which is called an SLD-derivation. A derivation can be finite or infinite. If its last query is empty then we speak of an SLD-refutation of the original query $q$. In this case we have derived an SLD-proof of $A_1, \ldots, A_k$. A failed SLD-derivation is a finite derivation which is not a refutation. In case $A$ is a ground atom with an SLD-proof from $P$, we say that $A$ is an SLD-consequence of $P$ and write $P \models A$. For a rule $r$ of the form $(\varphi)$, we write $P \models r$ in case $P \models A_0$ whenever $P \models A_i$ holds for every $1 \leq i \leq k$. We denote the empty query by $\Box$.

3. Composition

In this section, we recall the sequential composition of logic programs as defined by Antić (2023) and studied in the propositional case by Antić (2020).

Notation 1. In the rest of the paper, $P$ and $R$ denote logic programs over some joint language $L$.

The rule-like structure of logic programs induces naturally a compositional structure as follows (Antić, 2023, Definition 4).

We define the (sequential) composition of $P$ and $R$ by $\overset{[1]}{\circ}$

$$P \circ R := \begin{cases} h(\varnothing) \leftarrow b(S \varnothing) & r \in P \\ h(S \varnothing) = b(\varnothing) & S \subseteq S(\varnothing) \text{ variants}(R) \\ \varnothing = \text{mgu}(b(\varnothing), h(S)) & \end{cases}.$$  

Roughly, we obtain the composition of $P$ and $R$ by resolving all body atoms in $P$ with the ‘matching’ rule heads of $R$. This is illustrated in the next example, where we construct the even from the natural numbers via composition.

Example 2. Consider the program

$$\text{Nat} := \begin{cases} \text{nat}(0) \\ \text{nat}(s(x)) \leftarrow \text{nat}(x) \end{cases}$$

generating the natural numbers. By composing the only proper rule in $\text{Nat}$ with itself, we obtain

$$\{\text{nat}(s(x)) \leftarrow \text{nat}(x)\} \circ \{\text{nat}(s(x)) \leftarrow \text{nat}(x)\} = \{\text{nat}(s(s(x))) \leftarrow \text{nat}(x)\}.$$  

Notice that this program, together with the single fact in $\text{Nat}$, generates the even numbers.

\[1\text{We write } X \subseteq_k Y \text{ in case } X \text{ is a subset of } Y \text{ consisting of } k \text{ elements.}\]
Notice that we can reformulate sequential composition as
\[ P \circ R = \bigcup_{r \in P} (\{r\} \circ R), \]
which directly implies right-distributivity of composition, that is,
\[ (P \cup Q) \circ R = (P \circ R) \cup (Q \circ R) \]
holds for all propositional Horn theories \(P, Q, R\).

However, the following counter-example shows that left-distributivity fails in general:
\[ \{a \leftarrow b, c\} \circ (\{b\} \cup \{c\}) = \{a\} \quad \text{and} \quad ((a \leftarrow b, c) \circ \{b\}) \cup ((a \leftarrow b, c) \circ \{c\}) = \emptyset. \]

Define the unit program by the Krom program\(^2\)
\[ 1_L := \{A \leftarrow A \mid A \in XHB_L\}. \]

In the sequel, we will omit the reference to \(L\).

The space of all logic programs over some fixed language is closed under sequential composition with the neutral element given by the unit program and the empty program serves as a left zero, that is, we have
\[ P \circ 1 = 1 \circ P = 1 \quad \text{and} \quad \emptyset \circ P = \emptyset. \]

We can simulate the van Emden-Kowalski operator on a syntactic level without any explicit reference to operators via sequential composition, that is, for any interpretation \(I\), we have
\[ T_P(I) = \text{gnd}(P) \circ I. \]

As facts are preserved by composition and since we cannot add body atoms to facts via composition on the right, we have
\[ I \circ P = I. \]

3.1. Ground Programs. Ground programs are possibly infinite propositional programs where ground atoms can have a fixed inner structure, which means that we can import here the results of Antić (2020) which do not depend on finiteness.

**Notation 3.** In the rest of this subsection, \(P\) and \(R\) denote ground programs.

For ground programs, sequential composition simplifies to
\[ P \circ R = \{h(r) \leftarrow b(S) \mid r \in P, S \subseteq \text{sz}(r) R : h(S) = b(r)\}. \]

Our first observation is that we can compute the heads and bodies of a ground program \(P\) via
\[ h(P) = P \circ HB \quad \text{and} \quad b(P) = \text{proper}(P)^d \circ HB. \]
Moreover, we have
\[ h(PR) \subseteq h(P) \quad \text{and} \quad b(PR) \subseteq b(R). \]

Given an interpretation \(I\), we define
\[ I^\ominus := 1^{HB-I} \cup I \quad \text{and} \quad I^\ominus := \{A \leftarrow ((A) \cup I) \mid A \in HB\}. \]
It is not difficult to show that \(P I^\ominus\) is the program \(P\) where all occurrences of the ground atoms in \(I\) are removed from the rule bodies in \(P\), that is, we have
\[ P I^\ominus = \{h(r) \leftarrow (b(r) - I) \mid r \in P\}. \]

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\(^2\)Recall from Section\([\text{Section Number}]\) that \(XHB_L\) consists of all \(L\)-atoms not containing function or constant symbols.
Similarly, \( \Pi = \text{facts}(P) \cup \{ h(r) \leftarrow (b(r) \cup I) \mid r \in \text{proper}(P) \} \).

The left and right reducts can be represented via composition and the unit program by

\[
1^I P = 1^I \circ P \quad \text{and} \quad P^I = P \circ 1^I.
\]

Interestingly enough, we can represent the grounding of a (non-ground) program \( P \) via composition with the grounding of the unit program by

\[
gnd(P) = (gnd(1) \circ P) \circ gnd(1).
\]

4. Decomposition and Similarity

Let us start this section with a definition of a qualitative and algebraic notion of syntactic logic program similarity in terms of sequential decomposition.

Given two logic programs \( P \) and \( R \), we define \( P \preceq R \) if there exist programs \( Q \) and \( S \) such that \( P = (QR)S \).

In that case, we say that \( P \) can be one-step reduced to \( R \), and we call \( Q \) a prefix and \( S \) a suffix of \( P \). In case \( P \preceq R \) and \( R \preceq P \), we say that \( P \) and \( R \) are (syntactically) similar denoted by \( P \approx R \).

The computational interpretation of \( P = (QR)S \) is that we can answer a query to \( P \) by translating it via \( Q \) into a query of \( R \), computing one SLD-resolution step with \( R \), and translating it back to \( P \) via \( S \) as is demonstrated in the next example.

**Example 4.** Consider the programs

\[
\text{Plus} := \left\{ \begin{array}{l}
\text{plus}(0, y, y) \\
\text{plus}(s(x), y, s(z)) \leftarrow \\
\text{plus}(x, y, z)
\end{array} \right\} \quad \text{and} \quad \text{Append} := \left\{ \begin{array}{l}
\text{append}(\{\}, y, y) \\
\text{append}([u \mid x], y, [u \mid z]) \leftarrow \\
\text{plus}(x, y, z)
\end{array} \right\}
\]

implementing the addition of positive numbers (represented as numerals) and the concatenation of lists, respectively. To show that \( \text{Plus} \) and \( \text{Append} \) are syntactically similar, we define the programs

\[
Q := \left\{ \begin{array}{l}
\text{append}(\{\}, y, y) \leftarrow \text{plus}(0, y, y) \\
\text{append}([u \mid x], y, [u \mid z]) \leftarrow \text{plus}(s(x), y, s(z))
\end{array} \right\}
\]

and

\[
S := \{ \text{plus}(x, y, z) \leftarrow \text{append}(x, y, z) \},
\]

and compute

\[
\text{Append} = (Q \circ \text{Plus}) \circ S \quad \text{and} \quad \text{Plus} = (Q^d \circ \text{Append}) \circ S^d.
\]

This shows:

\[
\text{Plus} \approx \text{Append}.
\]
The following SLD-derivation demonstrates how we can use the prefix $Q$ and suffix $S$ to append two lists via the seemingly simpler program for the addition of numerals:

\[
\begin{align*}
&\leftarrow ? \text{append}([a], [b, c], [a, b, c]) \\
&\leftarrow Q \text{plus}(s([ ]), [b, c], s([b, c])) \\
&\overset{\text{Plus}}{\leftarrow} Q \text{plus}([ ], [b, c], [b, c]) \\
&\leftarrow S \text{append}([ ], [b, c], [b, c]) \\
&\leftarrow Q \text{plus}(0, [b, c], [b, c]) \\
&\overset{\text{Plus}}{\leftarrow} \square.
\end{align*}
\]

This shows, via a translation to numerals,

\[\text{Append} \vdash \text{append}([a], [b, c], [a, b, c]).\]

Interestingly, the SLD-derivation above contains the ‘entangled’ terms $s([ ])$ and $s([a, b])$ which are neither numerals nor lists, and we believe that such ‘entangled’ syntactic objects—which under the conventional doctrine of programming are not well-typed—are characteristic for reasoning across different domains and deserve special attention (this is discussed in Section 6).

**Example 5.** Tausend and Bell (1991) derive a mapping between the program `Append` from Example 4 above and the program

\[
\begin{align*}
\text{Member} := \left\{ \begin{array}{l}
\text{member}(u, [u \mid x]), \\
\text{member}(u, [v \mid x]) \leftarrow \text{append}([v \mid x], u, [v \mid x]) \\
\text{member}(u, x) \end{array} \right\},
\end{align*}
\]

which computes list membership. We can now ask—by analogy to Tausend and Bell (1991)—whether `Member` can be one-step reduced to `Append` (and vice versa) according to our definition. Define the programs

\[
\begin{align*}
Q := \left\{ \begin{array}{l}
\text{member}(u, [u \mid x]), \\
\text{member}(u, [v \mid x]) \leftarrow \text{append}([v \mid x], u, [v \mid x])
\end{array} \right\}
\end{align*}
\]

and

\[
S := \{ \text{append}(x, y, z) \leftarrow \text{member}(y, x) \}.
\]

It is not hard to compute

\[
\text{Member} = (Q \circ \text{Append}) \circ S,
\]

which shows that membership can indeed be one-step reduced to the program for appending lists:

\[
\text{Member} \preceq \text{Append}.
\]

Interestingly enough, the programs $Q$ and $S$, which in combination permute the first two arguments of `Append` and ‘forget’ about the last one, resemble the mapping computed in Tausend and Bell (1991) by other means. The converse fails, roughly, since `Member` does not contain enough ‘syntactic structure’ to represent `Append`. The intuitive reason is that `Member` has only two bound variables, whereas `Append` has three (see Example 7).
5. Properties of Similarity

Given ground atoms \( A_0, \ldots, A_k \), the identity
\[
\{A_0\} = \{A_0 \leftarrow A_1\}\{A_1\}
\]
shows
\[
\{A_0\} \preceq \{A_0 \leftarrow A_1\}.
\]
Since we cannot add body atoms to facts via composition on the right, we have
\[
\{A_0 \leftarrow A_1\} \not\preceq \{A_0\}.
\]
Hence,
\[
\{A_0\} < \{A_0 \leftarrow A_1\}.
\]
On the other hand, we have
\[
\{A_0 \leftarrow A_1\} = \{A_0 \leftarrow A_1, \ldots, A_k\}\{A_2, \ldots, A_k\}^\circ,
\]
\[
\{A_0 \leftarrow A_1, \ldots, A_k\} = \{A_0 \leftarrow A_1\}\{A_2, \ldots, A_k\}^\circ,
\]
which shows
\[
\{A_0 \leftarrow A_1\} \approx \{A_0 \leftarrow A_1, \ldots, A_k\}.
\]
In the non-ground case, we have, for example:
\[
\{p \leftarrow p\} < \{p(x_1) \leftarrow p(x_1)\} < \{p(x_1, x_2) \leftarrow p(x_1, x_2)\} < \ldots.
\]
This motivates the following definition.

Definition 6. The width of a rule is given by the number of its bound variables, extended to programs via 
\[
\text{width}(P) := \max_{r \in P} \text{width}(r).
\]
The number of bound variables cannot increase via composition—for example:
\[
\{p \leftarrow p\} \circ \{p(x_1) \leftarrow p(x_1)\} \circ \{p(x) \leftarrow p(x, y)\} = \{p(x, y) \leftarrow p(x, z)\}.
\]
This entails
\[
\text{width}(PR) \leq \text{width}(P) \quad \text{and} \quad \text{width}(PR) \leq \text{width}(R).
\]
Hence, we have
\[
\text{width}(P) \leq \text{width}(R).
\]

Example 7. Reconsider the programs Append and Member of Example\(^6\) We have
\[
\text{width}(Member) = 2 \quad \text{whereas} \quad \text{width(Append)} = 3.
\]
By (10) we thus have Append \not\preceq Member.

The following propositions summarize some facts about syntactic similarity.

Proposition 8. For any program \( P \) and interpretations \( I \) and \( J \), we have
\[
\begin{align*}
\text{gnd}(P) \preceq P & \quad \text{and} \quad T_P(I) \preceq \text{gnd}(P), \\
I \preceq P & \quad \text{and} \quad P \cup I \preceq P, \\
I \approx J & \quad \text{and} \quad P \approx I \quad \text{if} \quad P \text{ is an interpretation}.
\end{align*}
\]
Proof. The relations in the first line are immediate consequences of (9) and (4), respectively. The relation $I \preceq P$ follows from (5). The computation

$$P \cup I \equiv P \cup IP \equiv (1 \cup I)P$$

shows $P \cup I \preceq P$. The similarity $I \approx J$ follows from $I \preceq P$. The last similarity follows from (13) together with

$$P \preceq I \equiv P \cup (QI S) \equiv (QI S) T_{Q}(I), \text{ for some } Q, S \Rightarrow P \text{ is an interpretation.}$$

□

Proposition 9. For any ground program $P$ and interpretation $I$, we have

(14) $h(P) \preceq P$ and $b(P) \preceq \text{proper}(P)^d$,

(15) $\text{facts}(P) \preceq P$ and $P \preceq 1 \cup \text{facts}(P)$,

(16) $P f^\circ \approx P$ and $P f^\circ \preceq P$,

(17) $^1P \preceq P$ and $P ^1 \preceq P$.

Moreover, we have

(18) $P \approx P f^\circ \iff \text{facts}(P) = \text{facts}(P f^\circ)$,

(19) $P \approx \text{facts}(P) \iff P = \text{facts}(P)$

Proof. The relations in the first line are immediate consequences of (6). The relation $\text{facts}(P) \preceq P$ follows from $\text{facts}(P) = P \circ \emptyset$, and the relation $P \preceq 1 \cup \text{facts}(P)$ follows from

$$(1 \cup \text{facts}(P))\text{proper}(P) \equiv \text{proper}(P) \cup \text{facts}(P)\text{proper}(P) \equiv \text{proper}(P) \cup \text{facts}(P) = P.$$ The relation $P f^\circ \preceq P$ holds trivially; $P \preceq P f^\circ$ follows from $P = (P f^\circ)f^\circ$. Similarly, the relation $P f^\circ \preceq P$ holds trivially; for $P \preceq P f^\circ$ see (18). The relations in (17) are immediate consequences of (8).

Next, we prove (18): $\text{facts}(P) = \text{facts}(P f^\circ)$ means that by removing the ground atoms in $I$ from all rule bodies in $P$ we do not obtain novel facts—hence, we can add the ground atoms from $I$ back to the rule bodies of $P$ via $P = (P f^\circ)f^\circ$ which shows $P \preceq P f^\circ$ and see (16) (recall that we cannot add body atoms to facts via composition); the other direction is analogous.

Finally, the equivalence in (19) is shown as follows: we have $P \preceq \text{facts}(P)$ iff

(20) $P = (Q \circ \text{facts}(P)) \circ S = Q \circ \text{facts}(P)$, \text{ for some } Q \text{ and } S$

since

$$Q \circ \text{facts}(P) \equiv T_{Q}(\text{facts}(P))$$

is an interpretation and

$$T_{Q}(\text{facts}(P)) \circ S \equiv T_{Q}(\text{facts}(P)).$$

The identity (20) holds iff $P = T_{Q}(\text{facts}(P))$ which is equivalent to $P = \text{facts}(P)$ since $T_{Q}(\text{facts}(P))$ yields an interpretation. □
The following characterization of syntactic similarity follows immediately from the definitions.

**Proposition 10.** For any programs $P$ and $R$, we have $P \preceq R$ iff for each rule $r \in P$ there is a rule $s_r$, a subset $R_r$ of $R$ with $\text{width}(r) \leq \text{width}(R_r)$, and a program $S_r$ such that

$$\{r\} = ((s_r)R_r)S_r \quad \text{and} \quad ((s_r)R)S \subseteq P,$$

where $S := \bigcup_{r \in P} S_r$.

In this case, we have $P = (QR)S$ with $Q := \bigcup_{r \in P} \{s_r\}$.

**Example 11.** Consider the propositional programs

$$P = \begin{cases} c & a \leftarrow b, c \\ b & a \leftarrow a, c \end{cases} \quad \text{and} \quad \pi_{(a,b)} = \begin{cases} a & \leftarrow b \\ b & \leftarrow a \end{cases}.$$

We construct the programs $Q$ and $S$ such that $P = (Q\pi_{(a,b)})S$ according to Proposition 10. Define

$$r_1 := c \quad \Rightarrow \quad s_{r_1} := c \quad \text{and} \quad S_{r_1} := \emptyset,$$

$$r_2 := a \leftarrow b, c \quad \Rightarrow \quad s_{r_2} := a \leftarrow a \quad \text{and} \quad S_{r_2} := \{b \leftarrow b, c\},$$

$$r_3 := b \leftarrow a, c \quad \Rightarrow \quad s_{r_3} := b \leftarrow b \quad \text{and} \quad S_{r_3} := \{a \leftarrow a, c\},$$

and

$$Q := \{s_{r_1}, s_{r_2}, s_{r_3}\} = 1^{[a,b]} \cup \{c\} \quad \text{and} \quad S := S_{r_1} \cup S_{r_2} \cup S_{r_3} = \{c\}^{\oplus} - 1^{[c]}.$$

This yields

$$P = ((1^{[a,b]} \cup \{c\})\pi_{(a,b)})\{c\}^{\oplus} - 1^{[c]}).$$

Similar computations yield

$$\pi_{(a,b)} = (1^{[a,b]} P)\{c\}^{\oplus}.$$

This shows

$$P \approx \pi_{(a,b)}.$$

**Example 12.** Consider the propositional programs

$$\pi_{(a,b)} := \begin{cases} a & \leftarrow b \\ b & \leftarrow a \end{cases} \quad \text{and} \quad R := \begin{cases} a & \leftarrow b \\ b & \leftarrow b \end{cases}.$$

We have

$$R = \pi_{(a,b)}R \quad \Rightarrow \quad R \preceq \pi_{(a,b)}.$$

On the other hand, there can be no programs $Q$ and $S$ such that $\pi_{(a,b)} = (QR)S$ since we cannot rewrite the rule body $b$ of $R$ into $a$ and $b$ simultaneously via composition on the right. This shows

$$R < \pi_{(a,b)}.$$

The following simple example shows that syntactic similarity and logical equivalence are ‘orthogonal’ concepts.

**Example 13.** The empty program is logically equivalent with respect to the least model semantics to the propositional program $a \leftarrow a$ consisting of a single rule. Since we cannot obtain the rule $a \leftarrow a$
from the empty program via composition, logical equivalence does not imply syntactic similarity. For
the other direction, the computations
\[
\left\{ a \leftarrow b \rightarrow a \right\} = \left\{ a \leftarrow b, a \right\} \{b\}^\oplus \quad \text{and} \quad \left\{ a \leftarrow b, a \right\} = \left\{ a \leftarrow b \right\} \{b\}^\oplus
\]
show that the programs \( P = \{a, b \leftarrow a\} \) and \( R = \{a, b \leftarrow a, b\} \) are syntactically similar; however, \( P \) and \( R \) are not logically equivalent.

6. Conclusion

This paper showed how a qualitative and algebraic notion of syntactic logic program similarity can be constructed from the sequential decomposition of programs. We derived some elementary properties of syntactic similarity and, more importantly, demonstrated how it can be used to answer queries across different domains by giving some illustrative examples. Interestingly, in Example 4, in the process of answering queries in the list domain by translating it to the seemingly simpler domain of numerals, we obtained ‘entangled’ terms of the form \( s([\ ]) \) and \( s([b, c]) \) which are neither numerals nor lists. We believe that ‘entangled’ syntactic objects of this form—which under the conventional doctrine of programming are not well-typed (see, for example, the remarks in (Apt, 1997, §5.8.2)—are characteristic for ‘creative’ analogies across different domains and deserve special attention. More precisely, if we think of (logic) programs as tools for solving problems, then using those tools ‘creatively’ often requires, as in the example mentioned above, a tight coupling between seemingly incompatible objects. However, if those objects are statically typed by the programmer, then this might prevent the creation and exploitation of useful analogical ‘bridges’ for knowledge transfer.

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