Weighted Remez- and Nikolskii-Type Inequalities on a Quasismooth Curve

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Abstract

We establish sharp $L_p$, $1 \leq p < \infty$ weighted Remez- and Nikolskii-type inequalities for algebraic polynomials considered on a quasismooth (in the sense of Lavrentiev) curve in the complex plane.

Keywords. Polynomial, quasismooth curve, Remez inequality, Nikolskii inequality.

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1. Introduction

From the numerous generalizations of the classical Remez inequality (see, for example, [20, 5, 8, 10]), we mention three results which are the starting point of our analysis.

Let $|S|$ be the linear measure (length) of a Borel set $S$ in the complex plane $C$. By $P_n$ we denote the set of all complex polynomials of degree at most $n \in \mathbb{N} := \{1, 2, \ldots\}$. The first result is due to Erdélyi [7]. Assume that for $p_n \in P_n$ and $T := \{z : |z| = 1\}$ we have

$$\{|\{z \in T : |p_n(z)| > 1\}| \leq s, \quad 0 < s \leq \frac{\pi}{2}. \quad (1.1)$$

Then, $|p_n(e^{it})|^2$ is a trigonometric polynomial of degree at most $n$ and, by the Remez-type inequality on the size of trigonometric polynomials (cf. [7, Theorem 2] or [5, p. 230]), we obtain

$$||p_n||_{C(T)} \leq e^{2sn}, \quad 0 < s \leq \frac{\pi}{2}. \quad (1.2)$$

Here $|| \cdot ||_{C(S)}$ means the uniform norm over $S \subset C$.

The second result is due to Mastroianni and Totik [17]. Let $T_n$ be a trigonometric polynomial of degree $n \in \mathbb{N}$, $1 \leq p < \infty$, and $W : [0, 2\pi] \rightarrow \{x \geq 0\}$ be
an $A_{\infty}$ weight function. Then, according to [17, (5.2) and Theorem 5.2], there are positive constants $c_1$ and $c_2$ depending only on the $A_{\infty}$ constant of $W$ and $p$, such that for a measurable set $E \subset [0, 2\pi]$ with $|E| \leq s$, $0 < s \leq 1$, we have

\begin{equation}
\int_{[0,2\pi]} |T_n|^p W \leq c_1 \exp(c_2 sn) \int_{[0,2\pi]\setminus E} |T_n|^p W.
\end{equation}

The third result, which is due to Andrievskii and Ruscheweyh [4], extends (1.1)-(1.2) to the case of algebraic polynomials considered on a Jordan curve $\Gamma \subset \mathbb{C}$ instead of the unit circle $T$. In the present paper, we always assume that $\Gamma$ is quasismooth (in the sense of Lavrentiev), see [19], i.e., for every $z_1, z_2 \in \Gamma$,

\begin{equation}
|\Gamma(z_1, z_2)| \leq \Lambda_{\Gamma} |z_1 - z_2|,
\end{equation}

where $\Gamma(z_1, z_2)$ is the shorter arc of $\Gamma$ between $z_1$ and $z_2$ (including the endpoints) and $\Lambda_{\Gamma} \geq 1$ is a constant.

Let $\Omega$ be the unbounded component of $\overline{\mathbb{C}} \setminus \Gamma$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Denote by $\Phi$ the conformal mapping of $\Omega$ onto $D^* := \{z : |z| > 1\}$ with the normalization $\Phi(\infty) = \infty$, $\Phi'(\infty) := \lim_{z \to \infty} \frac{\Phi(z)}{z} > 0$.

For $\delta > 0$ and $A, B \subset \mathbb{C}$, we set

\begin{equation}
d(A, B) = \text{dist}(A, B) := \inf_{z \in A, \zeta \in B} |z - \zeta|,
\end{equation}

\begin{equation}
\Gamma_\delta := \{\zeta \in \Omega : |\Phi(\zeta)| = 1 + \delta\}.
\end{equation}

Let the function $\delta(t) = \delta(t, \Gamma), t > 0$ be defined by the equation $d(\Gamma, \Gamma_\delta(t)) = t$ and let $\text{diam} S$ be the diameter of a set $S \subset \mathbb{C}$.

According to [4, Theorem 2], if for $p_n \in \mathbb{P}_n$,

\begin{equation}
|\{z \in \Gamma : |p_n(z)| > 1\}| \leq s < \frac{1}{2} \text{diam } \Gamma,
\end{equation}

then

\begin{equation}
\|p_n\|_{C(\Gamma)} \leq \exp(c_3 \delta(s)n)
\end{equation}

holds with a positive constant $c_3 = c_3(\Gamma)$.

Our objective is to provide the weighted $L_p$ analogue of (1.5)-(1.6) which extends (1.3) to the case of complex polynomials considered on $\Gamma$. Some of our proofs and constructions are modifications of arguments from [17, 1, 2, 9]. For the sake of completeness, we describe them in detail.

We denote by $\alpha, c, \varepsilon, \alpha_1, c_1, \varepsilon_1, \ldots$ positive constants (different in different sections) that are either absolute or they depend on parameters inessential for the argument; otherwise, such dependence will be explicitly stated. For nonnegative functions $f$ and $g$ we use the expression $f \preceq g$ (order inequality) if $f \leq cg$. The expression $f \asymp g$ means that $f \preceq g$ and $g \preceq f$ simultaneously.
2. Main Results

We say that a finite Borel measure $\nu$ supported on $\Gamma$ is an $A_\infty$ measure ($\nu \in A_\infty(\Gamma)$ for short) if there exists a constant $\lambda_\nu \geq 1$ such that for any arc $J \subset \Gamma$ and a Borel set $S \subset J$ satisfying $|J| \leq 2|S|$ we have

\[ \nu(J) \leq \lambda_\nu \nu(S), \tag{2.1} \]

see for instance [6, 12]. The measure defined by the arclength on $\Gamma$ is automatically the $A_\infty$ measure. Another interesting example is the equilibrium measure $\mu_\Gamma$ on $\Gamma$ (see for example [21]). By virtue of [13] $\mu_\Gamma \in A_\infty(\Gamma)$.

**Theorem 1** Let $\nu \in A_\infty(\Gamma)$, $1 \leq p < \infty$, and let $E \subset \Gamma$ be a Borel set. Then for $p_n \in P_n, n \in \mathbb{N}$, we have

\[ \int_{\Gamma} |p_n|^p d\nu \leq c_1 \exp(c_2\delta(s)n) \int_{\Gamma \setminus E} |p_n|^p d\nu \]

provided that $0 < |E| \leq s < (\text{diam}\, \Gamma)/12$, where the constants $c_1$ and $c_2$ depend only on $\Gamma, \lambda_\nu, p$.

Let $\Gamma = T$. Starting with the trigonometric polynomial

\[ T_n(t) = \sum_{k=0}^{n} (a_k \sin kt + b_k \cos kt), \]

consider the algebraic polynomial

\[ p_{2n}(z) := z^n \sum_{k=0}^{n} \left( \frac{a_k}{2i} (z^k - z^{-k}) + \frac{b_k}{2} (z^k + z^{-k}) \right). \]

Then (2.2) implies (1.3) (up to the upper bound on a parameter $s$).

The sharpness of Theorem 1 is established by our next theorem. Let $ds = |dz|, z \in \Gamma$ be the arclength measure on $\Gamma$.

**Theorem 2** Let $0 < s < \text{diam} \, \Gamma$ and $1 \leq p < \infty$. Then there exist an arc $E_s \subset \Gamma$ with $|E_s| = s$ as well as constants $\varepsilon_1 = \varepsilon_1(\Gamma)$ and $n_0 = n_0(s, \Gamma, p) \in \mathbb{N}$ such that for any $n > n_0$, there is a polynomial $p_{n,s} \in P_n$ satisfying

\[ \int_{\Gamma} |p_{n,s}|^p ds \geq \exp(\varepsilon_1\delta(s)n) \int_{\Gamma \setminus E_s} |p_{n,s}|^p ds. \tag{2.3} \]
If, in the definition of the $A_\infty$ measure, we assume that $S$ is also an arc, then $\nu$ is called a doubling measure. In [17, Section 5] one can find an example showing that the weighted Remez-type inequality may not be true in the case of doubling measures.

A straightforward consequence of Theorem 1 is the following Nikolskii-type inequality which partially overlaps with [24, Corollary 3.10] where the analogous inequality is proved in another way. For more details on the classical Nikolskii inequality, its generalizations, and further references see, for example [11, 5, 8, 16].

**Theorem 3** Let $\nu \in A_\infty(\Gamma)$ satisfy $d\nu = wd\sigma$, $w : \Gamma \to \{x \geq 0\}$, and let $1 \leq p < q < \infty$. Then, for $p_n \in P_n$, $n > n_1$,

$$
(\int_\Gamma |p_n|^q wds)^{1/q} \leq c_3 d(\Gamma, \Gamma_1/n)^{1/q-1/p} \left(\int_\Gamma |p_n|^p w^{p/q}ds\right)^{1/p}
$$

holds with constants $c_3 = c_3(\Gamma, p, q, \lambda_\nu)$ and $n_1 = n_1(\Gamma)$.

For $\Gamma = T$ (2.4) yields [17, Theorem 5.5]. The estimate (2.4) is sharp in the following sense.

**Theorem 4** For $n \in \mathbb{N}$, there exists a polynomial $p_n^* \in P_n$, such that for $1 \leq p < q < \infty$,

$$
(\int_\Gamma |p_n^*|^q ds)^{1/q} \geq \varepsilon_2 d(\Gamma, \Gamma_1/n)^{1/q-1/p} \left(\int_\Gamma |p_n^*|^p ds\right)^{1/p}
$$

holds with $\varepsilon_2 = \varepsilon_2(\Gamma, p, q)$.

Note that $\delta(s)$ and $d(\Gamma, \Gamma_\delta)$ can be further estimated. We mention three well known results. For a more complete theory see, for example [25, 18, 15, 19].

The Ahlfors criterion [14, p. 100] implies that $\Gamma$ is quasiconformal. Therefore, $\Phi$ can be extended to a quasiconformal homeomorphism $\Phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Taking into account Lemma 1 below and distortion properties of conformal mappings with quasiconformal extension (cf. [18, pp. 289, 347]) we have

$$
\delta(s) \lesssim s^{1/\alpha}, \quad 0 < s < \text{diam } \Gamma,
$$

$$
d(\Gamma, \Gamma_\delta) \geq \delta^\alpha, \quad 0 < \delta < 1,
$$

with some $\alpha = \alpha(\Gamma)$ such that $1 \leq \alpha < 2$.

Next, following [19] we call $\Gamma$ Dini-smooth if it is smooth and if the angle $\beta(s)$ of the tangent, considered as a function of the arc length $s$, has the property

$$
|\beta(s_2) - \beta(s_1)| \leq h(s_2 - s_1), \quad 0 < s_2 - s_1 < |\Gamma|/2,
$$
where \( h \) is a function satisfying
\[
\int_0^{\|\Gamma\|/2} \frac{h(x)}{x} \, dx < \infty.
\]

We call a Jordan arc \( Dini\)-smooth if it is a subarc of some \( Dini\)-smooth curve. According to [4, Theorem 4] if \( \Gamma \) is \( Dini\)-smooth, then
\[
\delta(s) \asymp s, \quad 0 < s < \text{diam} \, \Gamma,
\]
\[
d(\Gamma, \Gamma_\delta) \asymp \delta, \quad 0 < \delta < 1.
\]
Moreover, the distortion properties of \( \Phi \) in the case of a piecewise \( Dini\)-smooth \( \Gamma \) (cf. [19, Chapter 3] or [3, pp. 32-36]) imply that if \( \Gamma \) consists of a finite number of \( Dini\)-smooth arcs which meet under the angles \( \alpha_1 \pi, \ldots, \alpha_m \pi \) with respect to \( \Omega \), where \( 0 < \alpha_j < 2 \), then
\[
\delta(s) \asymp s^{1/\alpha}, \quad 0 < s \leq \text{diam} \, \Gamma,
\]
\[
d(\Gamma, \Gamma_\delta) \asymp \delta^\alpha, \quad 0 < \delta < 1,
\]
hold with \( \alpha := \max(1, \alpha_1, \ldots, \alpha_m) \).

### 3. Auxiliary Constructions and Results

In this section, we review some of the properties of conformal mappings \( \Phi \) and \( \Psi := \Phi^{-1} \) whose proofs can be found, for example, in [1, Section 3]. We also prove some new facts about these conformal mappings which are used in the proofs of the main results.

**Lemma 1** Assume that \( z_j \in \overline{\Omega}, \ t_j := \Phi(z_j), \ j = 1, 2, 3. \) Then,

(i) the conditions \( |z_1 - z_2| \leq |z_1 - z_3| \) and \( |t_1 - t_2| \leq |t_1 - t_3| \) are equivalent;

(ii) if \( |z_1 - z_2| \leq |z_1 - z_3| \), then
\[
\frac{|t_1 - t_3|^{1/\alpha}}{|t_1 - t_2|} \leq \frac{|z_1 - z_3|}{|z_1 - z_2|} \leq \frac{|t_1 - t_3|}{|t_1 - t_2|}^\alpha, \quad \alpha = \alpha(\Gamma) \geq 1.
\]

Most of the geometrical facts below can be obtained by a straightforward application of Lemma 1 to specifically chosen triplets of points.

For \( \delta > 0 \) and \( z \in \Gamma \), set
\[
\rho_\delta(z) := d(\{z\}, \Gamma_\delta), \quad \tilde{z}_\delta := \Psi[(1 + \delta)\Phi(z)].
\]
Then
\[ (3.1) \quad \rho_\delta(z) \asymp |z - \tilde{z}_\delta|. \]

Moreover, for \(0 < v < u \leq 1\) and \(z \in \Gamma\), Lemma 1 for the triplet \(z, \tilde{z}_v, \tilde{z}_u\) and (3.1) yield
\[ (3.2) \quad \left( \frac{u}{v} \right)^{1/\alpha} \leq \frac{\rho_\alpha(z)}{\rho_v(z)} \leq \left( \frac{u}{v} \right)^{\alpha} \]

which implies
\[ (3.3) \quad \delta(2s) \leq \delta(s), \quad 0 < s < \text{diam } \Gamma. \]

Indeed, the only nontrivial case is where \(s\) satisfies \(\delta(2s) \leq 1\). Let \(z_{2s} \in \Gamma\) be such that
\[ \rho_\delta(z_{2s}) = d(\Gamma, \Gamma_{\delta(2s)}) = 2s. \]

Since
\[ \frac{\rho_\delta(z_{2s})}{\rho_\delta(s)(z_{2s})} \leq \frac{\rho_\delta(z_{2s})}{d(L, L_{\delta(s)})} = \frac{2s}{s} = 2, \]

by the left-hand side of (3.2) we obtain (3.3).

Furthermore, for \(0 < \delta \leq 1\) and \(z, \zeta \in L\), the following relations hold:
\[ \text{if } |z - \zeta| \leq \rho_\delta(z), \text{ then } \rho_\delta(\zeta) \asymp \rho_\delta(z); \]
\[ \text{if } |z - \zeta| \geq \rho_\delta(z), \text{ then } \]
\[ (3.5) \quad \left( \frac{\rho_\delta(z)}{|z - \zeta|} \right)^{\alpha_1} \leq \rho_\delta(\zeta) \leq \left( \frac{\rho_\delta(z)}{|z - \zeta|} \right)^{1/\alpha_1}. \]

Let \(\delta_0 = \delta_0(\Gamma) > 0\) be fixed such that
\[ \max_{z \in \Gamma} \rho_{\delta_0}(z) < \frac{|\Gamma|}{2}. \]

For \(z \in \Gamma\) and \(0 < \delta < \delta_0\), denote by \(z'_\delta, z''_\delta \in \Gamma\) the two points with the properties
\[ z \in \Gamma(z'_\delta, z''_\delta), \quad |\Gamma(z'_\delta, z)| = |\Gamma(z, z''_\delta)| = \frac{\rho_\delta(z)}{2}. \]

If \(\delta \geq \delta_0\), we set \(\Gamma(z'_\delta, z''_\delta) := \Gamma\). Let
\[ l_n(z) := \Gamma(z'_{1/n}, z''_{1/n}), \quad z \in \Gamma, n \in \mathbb{N}. \]

Hence, we have
\[ (3.6) \quad |l_n(z)| \asymp \rho_{1/n}(z). \]
Let $\nu \in A_{\infty}(\Gamma)$. Consider the function

$$w_n(z) := \frac{\nu(l_n(z))}{\rho_{1/n}(z)}, \quad z \in \Gamma, n \in \mathbb{N}. \quad (3.7)$$

Since $\nu$ is also a doubling measure on $\Gamma$, for any arcs $J_1$ and $J_2$ with $J_1 \subset J_2 \subset \Gamma$,

$$\frac{\nu(J_2)}{\nu(J_1)} \leq c_1 \left( \frac{|J_2|}{|J_1|} \right)^{\alpha_2}, \quad c_1 = c_1(\Gamma, \lambda_\nu), \alpha_2 = \alpha_2(\Gamma, \lambda_\nu). \quad (3.8)$$

The proof of (3.8) follows along the same lines as the proof of [2, (4.1)] (cf. [17, Lemma 2.1]).

Next, according to [1, Lemma 4] for $z, \zeta \in \Gamma$ and $n \in \mathbb{N}$,

$$\frac{1}{c_2} \left( 1 + \frac{|\zeta - z|}{\rho_{1/n}(z)} \right)^{-\alpha_3} \leq \frac{w_n(\zeta)}{w_n(z)} \leq c_2 \left( 1 + \frac{|\zeta - z|}{\rho_{1/n}(z)} \right)^{\alpha_3}, \quad (3.9)$$

where $c_2 = c_2(\Gamma, \lambda_\nu), \alpha_3 = \alpha_3(\Gamma, \lambda_\nu)$.

We follow a technique of [1, (3.12)] and consider for $n, m \in \mathbb{N}$ and $z, \zeta \in \Gamma$ the polynomial (in $z$)

$$q_{n,m}(\zeta, z) = \sum_{j=0}^{N} a_j(\zeta) z^j, \quad N = (10n - 11)m, \quad (3.10)$$

which satisfies the following properties:

if $|\zeta - z| \leq \rho_{1/n}(z) \approx \rho_{1/n}(\zeta)$, then

$$\frac{1}{c_3} \leq |q_{n,m}(\zeta, z)| \leq c_3, \quad c_3 = c_3(\Gamma, m); \quad (3.11)$$

if $|\zeta - z| > \rho_{1/n}(z)$, then by virtue of (3.5),

$$|q_{n,m}(\zeta, z)| \leq c_4 \left( \frac{\rho_{1/n}(\zeta)}{|\zeta - z|} \right)^m \leq c_5 \left( \frac{\rho_{1/n}(z)}{|\zeta - z|} \right)^{m/\alpha_1}, \quad (3.12)$$

where $c_j = c_j(\Gamma, m), j = 4, 5$.

Let for $z \in \Gamma$ and $n, m \in \mathbb{N}$,

$$I_{n,m}(z) := \int_{\Gamma} |q_{n,m}(\zeta, z)| \frac{w_n(\zeta)}{w_n(z) \rho_{1/n}(\zeta)} |d\zeta| \frac{|d\zeta|}{\rho_{1/n}(\zeta)} \quad \frac{1}{w_n(z)} \int_{\Gamma} |q_{n,m}(\cdot, z)| \frac{w_n(\zeta)}{\rho_{1/n}(\zeta)} ds. \quad (3.12)$$

We use the following notation: for $z \in \mathbb{C}$ and $\delta > 0$,

$$D(z, \delta) := \{ \zeta : |\zeta - z| < \delta \}, \quad D^*(z, \delta) := \mathbb{C} \setminus \overline{D(z, \delta)}. \quad (3.12)$$
Lemma 2  There exist sufficiently large $m = m(\Gamma, \lambda_\nu) \in \mathbb{N}$ and $c_6 = c_6(\Gamma, \lambda_\nu)$ such that
\begin{equation}
\frac{1}{c_6} \leq I_{n,m}(z) \leq c_6, \quad z \in \Gamma.
\end{equation}

**Proof.** According to the inequalities (1.4), (3.4), (3.9), and (3.10) we obtain
\begin{align*}
I_{n,m}(z) &\geq \int_{\Gamma \cap D(z, \rho_{1/n}(z))} |q_{n,m}(\zeta, z)| w_n(\zeta) \frac{|d\zeta|}{w_n(z) \rho_{1/n}(\zeta)} \geq 1,
\end{align*}
which yields the left-hand side of (3.13).

Next, by (1.4), (3.4), (3.5), (3.9)-(3.11), and [1, (3.20)] we have
\begin{align*}
I_{n,m}(z) &\leq \int_{\Gamma \cap D(z, \rho_{1/n}(z))} |q_{n,m}(\zeta, z)| w_n(\zeta) \frac{|d\zeta|}{w_n(z) \rho_{1/n}(\zeta)} \\
&\quad + \int_{\Gamma \cap D^*(z, \rho_{1/n}(z))} |q_{n,m}(\zeta, z)| w_n(\zeta) \frac{|\zeta - z|}{w_n(z) \rho_{1/n}(\zeta)} \frac{|d\zeta|}{|\zeta - z|} \\
&\leq 1 + \int_{\Gamma \cap D^*(z, \rho_{1/n}(z))} \left( \frac{|\zeta - z|}{\rho_{1/n}(z)} \right)^{\alpha_3 - m/\alpha_1 + \alpha_1} \frac{|d\zeta|}{|\zeta - z|} \leq 1
\end{align*}
if $m$ is any (fixed) number with $\alpha_3 - m/\alpha_1 + \alpha_1 < 0$.

Hence, the right-hand side of (3.13) is also proved.

\[\square\]

Lemma 3  For $r \geq 1$,
\begin{equation}
\frac{\rho_{1/n}(\zeta)}{c_7} \leq \int_\Gamma |q_{n,2}(\zeta, z)|^r |dz| \leq c_7 \rho_{1/n}(\zeta), \quad \zeta \in \Gamma,
\end{equation}
where $c_7 = c_7(\Gamma, r)$.

**Proof.** The left-hand side inequality follows from (1.4) and (3.10):
\begin{align*}
\int_\Gamma |q_{n,2}(\zeta, z)|^r |dz| &\geq \int_{\Gamma \cap D(\zeta, \rho_{1/n}(\zeta))} |q_{n,2}(\zeta, z)|^r |dz| \geq \rho_{1/n}(\zeta).
\end{align*}
Furthermore, according to (1.4), (3.10), (3.11), and [1, (3.20)] we have
\begin{align*}
\int_\Gamma |q_{n,2}(\zeta, z)|^r |dz| &\leq \int_{\Gamma \cap D(\zeta, \rho_{1/n}(\zeta))} |dz| + \int_{\Gamma \cap D^*(\zeta, \rho_{1/n}(\zeta))} \left( \frac{\rho_{1/n}(\zeta)}{|\zeta - z|} \right)^{2r} |d\zeta| \\
&\leq \rho_{1/n}(\zeta),
\end{align*}
which proves the right-hand side of (3.14).

\[ \square \]

4. Proofs of Theorems

We start with some preliminaries. Let

\[ q_r(z) := c \prod_{j=1}^{m} |z - z_j|^{\beta_j}, \quad z \in \mathbb{C}, \]

where \( z_j \in \mathbb{C}, c > 0, \beta_j > 0 \) be a generalized polynomial of degree \( r := \beta_1 + \ldots + \beta_m \) and let

\[ E(q_r) := \{ z \in \Gamma : q_r(z) > 1 \}. \]

By [4, Theorem 2], the condition

\[ |E(q_r)| \leq \frac{1}{2} \text{diam } \Gamma \]

yields

\[ \|q_r\|_{C(\Gamma)} \leq \exp(c_1 \delta(s)r), \quad c_1 = c_1(\Gamma). \]

Consider the set

\[ F_s = F_s(q_r) := \{ z \in \Gamma : q_r(z) > \exp(-c_1 \delta(s)r)\|q_r\|_{C(\Gamma)} \} \]

and the generalized polynomial

\[ f_{r,s}(z) := \frac{q_r(z) \exp(c_1 \delta(s)r)}{\|q_r\|_{C(\Gamma)}} \]

so that \( E(f_{r,s}) = F_s \).

We have

\[ |F_s| \geq s, \quad 0 < s < \frac{1}{2} \text{diam } \Gamma. \]

Indeed, the case \( |F_s| \geq (\text{diam } \Gamma)/2 \) is trivial. If \( |F_s| < (\text{diam } \Gamma)/2 \), then by (4.1)-(4.2), applied to \( f_{r,s} \) and \( |F_s| \) instead of \( q_r \) and \( s \), we obtain

\[ \exp(c_1 \delta(s)r) = \|f_{r,s}\|_{C(\Gamma)} \leq \exp \left( c_1 \delta(|F_s|)r \right), \]

that is, \( \delta(|F_s|) \geq \delta(s) \) which implies (4.3).

Let, as before, \( 1 \leq p < \infty \). We claim that if a Borel set \( A \subset \Gamma \) satisfies

\[ |A| \geq |\Gamma| - s, \quad 0 < s < \frac{1}{4} \text{diam } \Gamma, \]
\[ \int_{\Gamma} (q_r)^p ds \leq (1 + \exp(c_2 \delta(s)pr)) \int_{A} (q_r)^p ds, \quad c_2 = c_2(\Gamma). \]

Indeed, by virtue of (4.3) for \(0 < s < (\text{diam } \Gamma)/4\), we have \(|F_{2s}| \geq 2s\) which yields \(|A \cap F_{2s}| \geq s\). Therefore, according to (3.3),

\[
\int_{\Gamma \setminus A} (q_r)^p ds \leq s ||q_r||_{C(\Gamma)}^p \leq \int_{A \cap F_{2s}} ||q_r||_{C(\Gamma)}^p ds \\
\leq \exp (c_1 \delta (2s) pr) \int_{A \cap F_{2s}} (q_r)^p ds \\
\leq \exp (c_2 \delta (s) pr) \int_{A} (q_r)^p ds,
\]

which proves (4.5).

Let \(w_n, n \in \mathbb{N}\) be defined by (3.7).

**Lemma 4** For a Borel set \(A \subset \Gamma\) satisfying (4.4), \(1 \leq p < \infty\), and \(p_n \in P_n, n \in \mathbb{N}\),

\[
\int_{\Gamma} |p_n|^p w_n ds \leq c_3 \exp(c_4 \delta(s)n) \int_{A} |p_n|^p w_n ds,
\]

where \(c_j = c_j(\Gamma, p, \lambda_\nu), j = 3, 4\).

**Proof.** Let \(q_{n,m}\) be the polynomial defined in Section 3. By (4.5) applied to the generalized polynomial \(q_r := |p_n||q_{n,m}(\cdot, \cdot)|^{1/p}\), where \(\zeta \in \Gamma, m = m(\Gamma)\) is from Lemma 2 and \(r \asymp n\), we have

\[
\int_{\Gamma} |p_n|^p |q_{n,m}(\zeta, \cdot)| ds \leq (1 + \exp(c_4 \delta(s)n)) \int_{A} |p_n|^p |q_{n,m}(\zeta, \cdot)| ds.
\]

Multiplying the both sides of this inequality by \(w_n(\zeta)/\rho_{1/n}(\zeta)\), integrating by \(\zeta\) over \(\Gamma\), and applying the Fubini theorem we obtain

\[
\int_{\Gamma} |p_n|^p w_n I_{n,m} ds \\
= \int_{\Gamma} \int_{\Gamma} |p_n(z)|^p |q_{n,m}(\zeta, z)| \frac{w_n(\zeta)}{\rho_{1/n}(\zeta)} d\zeta |dz| \\
\leq (1 + \exp(c_4 \delta(s)n)) \int_{A} \int_{\Gamma} |p_n(z)|^p |q_{n,m}(\zeta, z)| \frac{w_n(\zeta)}{\rho_{1/n}(\zeta)} d\zeta |dz| \\
= (1 + \exp(c_4 \delta(s)n)) \int_{A} |p_n|^p w_n I_{n,m} ds,
\]
where $I_{n,m}$ is defined by (3.12), which, together with (3.13), yields (4.6).

\[ \square \]

**Proof of Theorem 1.** The construction below is partially adapted from the proof of [17, Theorem 3.1] and the proof of [2, Theorem 4]. Let $m = m(n, s, \Gamma) \in \mathbb{N}$ be a sufficiently large number to be chosen later and let

$$
\begin{align*}
\theta_k &= \frac{2\pi k}{N}, \\
\xi_k &= \Psi(e^{i\theta_k}), \\
J'_k &= \{ e^{i\theta} : \theta_{k-1} \leq \theta < \theta_k \}, \\
J_k &= \Psi(J'_k), \\
J_{k+1} &= \{ e^{i\theta} : \theta_k \leq \theta < \theta_{k+1} \},
\end{align*}
$$

By virtue of Lemma 1, (1.4), and (3.1) for $k = 1, \ldots, N$, we have

$$
|J_k| \asymp |\xi_k - \xi_{k-1}| \asymp |\xi_k - (\xi_k)/N| \asymp \rho_{1/N}(\xi_k).
$$

(4.7)

Let $K := \{ k : |E \cap J_k| < |J_k|/2 \}$. Then

$$
\sum_{k \in K} |J_k| \leq 2 \sum_{k \notin K} |E \cap J_k| \leq 2|E| < 2s,
$$

which for

$$
A := \Gamma \setminus E \text{ and } A^* := \bigcup_{k \in K} (A \cap J_k) \subset A
$$

implies

$$
|\Gamma| = \bigcup_{k \notin K} |A \cap J_k| + |A^*| + |E| \leq \bigcup_{k \notin K} |J_k| + |A^*| + |E| < |A^*| + 3s,
$$

that is,

$$
(4.8) \quad |A^*| > |\Gamma| - 3s.
$$

Let

$$
B_k := \sup_{\xi \in J_k} w_n(\xi),
$$

and let $\eta_k, \nu_k \in J_k$ be such that

$$
|p_n(\eta_k)| = \min_{\xi \in J_k} |p_n(\xi)|, \quad |p_n(\nu_k)| = \max_{\xi \in J_k} |p_n(\xi)| = ||p_n||_{C(J_k)}.
$$

Consider

$$
R = R(p_n, p, m, n) := \sum_{k \in K} |p_n(\nu_k)|^p B_k |J_k \cap A|
$$
and
\[ V = V(p_n, p, m, n) := R - \sum_{k \in K} |p_n(\eta_k)|^p B_k |J_k \cap A| \]

which satisfy
\[
V = \sum_{k \in K} (|p_n(v_k)|^p - |p_n(\eta_k)|^p) B_k |J_k \cap A| \leq p \sum_{k \in K} |p_n(v_k) - p_n(\eta_k)||p_n(v_k)|^{p-1} B_k |J_k \cap A|. 
\]

If \( p > 1 \) and \( q > 1 \) satisfy \( 1/p + 1/q = 1 \), Hölder’s inequality implies
\[
V \lesssim \left( \sum_{k \in K} |p_n(v_k) - p_n(\eta_k)|^p B_k |J_k \cap A| \right)^{1/p} R^{1/q} 
\]
\[
\leq \left( \sum_{k \in K} \left( \int_{J_k} |p_n'|^p ds \right)^{p} B_k |J_k \cap A| \right)^{1/p} R^{1/q}. 
\]

If \( p = 1 \), setting \( R^{1/q} := 1 \), we have the same estimate for \( V \).

Note that by (3.9) and (4.7) \( B_k \simeq A_k := \inf_{\xi \in J_k} w_n(\xi) \).

Since Hölder’s inequality also yields
\[
\left( \int_{J_k} |p_n'|^p ds \right)^{p} \leq |J_k|^{p-1} \int_{J_k} |p_n'|^p ds, 
\]
by [1, Lemma 1], Lemma 1, (3.2)-(3.4), (3.9), and (4.6)-(4.8) for the nonzero polynomial \( p_n \) we further have
\[
VR^{1/q} \leq \left( \sum_{k=1}^{N} \left( \frac{\rho_{1/n}(\xi_k)}{\rho_{1/n}(\xi_k)} \right)^p \int_{J_k} (|p_n'|(\rho_{1/n})^p w_n ds \right)^{1/p} 
\]
\[
\lesssim m^{-\varepsilon} \left( \int_{\Gamma} (|p_n'|(\rho_{1/n})^p w_n ds \right)^{1/p} 
\]
\[
\lesssim m^{-\varepsilon} \left( \int_{\Gamma} |p_n|^p w_n ds \right)^{1/p} 
\]
\[
\lesssim m^{-\varepsilon} \exp(c_4 \delta(3s)n) \left( \int_{A^*} |p_n|^p w_n ds \right)^{1/p} 
\]
\[
\lesssim m^{-\varepsilon} \exp(c_5 \delta(s)n) R^{1/p}, 
\]
Taking $m$ to be the integral part of
$$1 + (2c_6 \exp(c_5 \delta(s)n))^{1/\epsilon}$$
we have $V \leq R/2$ and $m \asymp \exp(c_7 \delta(s)n)$. Therefore,
$$R \leq 2 \sum_{k \in K} |p_n(\eta_k)|^p B_k |J_k \cap A| \asymp \sum_{k \in K} |p_n(\eta_k)|^p A_k |J_k \cap A|.$$ 

Since $\nu \in A_\infty(\Gamma)$ and $|J_k \cap A| \geq |J_k|/2$, $k \in K$,
according to (2.1), (3.6), (3.8), and (4.7) for $\xi \in J_k$ we have
$$w_n(\xi) = \frac{\nu(l_n(\xi))}{\rho_{1/n}(\xi)} \leq \frac{1}{|J_k|} \frac{\nu(l_n(\xi)) \nu(l_N(\xi))}{\nu(l_N(\xi))} \frac{\nu(J_k)}{\nu(J_k \cap A)} \nu(J_k \cap A) \leq m^{\alpha_1} \frac{\nu(J_k \cap A)}{|J_k|}.$$ 

Therefore,
$$R \leq m^{\alpha_1} \sum_{k \in K} |p_n(\eta_k)|^p \nu(J_k \cap A) \leq m^{\alpha_1} \int_{A^*} |p_n|^p d\nu.$$ 

Moreover, by [1, Lemma 2], (3.3), (4.8), and Lemma 4,
$$\int_{\Gamma} |p_n|^p d\nu \leq \int_{\Gamma} |p_n|^p w_n ds \leq \exp(c_4 \delta(3s)n) \int_{A^*} |p_n|^p w_n ds \leq \exp(c_5 \delta(s)n) R \leq \exp((c_5 + c_7 \alpha_1) \delta(s)n) \int_{A^*} |p_n|^p d\nu \leq \exp(c_8 \delta(s)n) \int_A |p_n|^p d\nu, \quad c_8 = c_5 + c_7 \alpha_1,$$
which is the desired conclusion.

Proof of Theorem 2. Let $z_s \in \Gamma$ and $\zeta_s \in \Gamma_{\delta(s)}$ satisfy $|z_s - \zeta_s| = s = \rho_{\delta(s)}(z_s)$. Define points $z^*_s, z^{**}_s \in \Gamma$ such that $z_s \in \Gamma(z^*_s, z^{**}_s) =: E_s$ and
$$|\Gamma(z^*_s, z_s)| = |\Gamma(z_s, z^{**}_s)| = \frac{s}{2},$$

13
Lemma 1 and (1.4) yield

\[ |\Phi(z^*_s) - \Phi(z_s)| \asymp |\Phi(z_s) - \Phi(z^*_s)| \asymp |\Phi(z_s) - \Phi(\zeta_s)| \geq \delta(s). \]

Let \( A_s := \Gamma \setminus E_s \) and let \( \Phi_s \) be the conformal mapping of \( \Omega_s := C \setminus A_s \) onto \( D^* \) normalized by

\[ \Phi_s(\infty) = \infty, \quad \Phi_s'(\infty) > 0. \]

According to [4, Lemma 5], (1.4), and (4.9) we obtain

\[ \log |\Phi_s(z_s)| \geq |\Phi_s(z^*_s) - \Phi_s(z_s^*)| \geq \delta(s), \]

that is,

\[ |\Phi_s(z_s)| \geq \exp(\varepsilon_1 \delta(s)) \geq 1 + \varepsilon_1 \delta(s), \quad \varepsilon_1 = \varepsilon_1(\Gamma). \]

Let \( p_{n,s} \in P_n \) be the \( n \)-th Faber polynomial for \( \Omega_s \) (see [22, Chapter II, §1] or [23, Chapter II]). From a result by Pommerenke [18, p. 85, Theorem 3.11] (see also [23, Chapter IX, §3]) it follows that

\[ ||p_{n,s}||_{C(A_s)} \leq 2 \sqrt{n \log n + 2}. \]

Moreover, according to [22, Chapter II, §1] for \( \xi \in \Omega_s \),

\[ p_{n,s}(\xi) = \Phi_s(\xi)^n + \omega_{n,s}(\xi), \]

where

\[ |\omega_{n,s}(\xi)| \leq \left( n \log \frac{1}{|\Phi_s(\xi)|^2 - 1} \right)^{1/2}. \]

Next, by (1.4) for \( d_s := \text{dist}(z_s, A_s) \) we have \( \varepsilon_2 s \leq d_s \leq s/2 \), where \( \varepsilon_2 = \varepsilon_2(\Gamma) \).

According to [3, p. 23, Lemma 2.3] for \( \xi \in W_s := \Gamma \cap D(z_s, d_s/32) \) we obtain

\[ |\Phi_s(\xi) - \Phi_s(z_s)| \leq \frac{1}{2}(|\Phi_s(z_s)| - 1), \]

and by (4.10)

\[ |\Phi_s(\xi)| \geq 1 + \frac{\varepsilon_1}{2} \delta(s), \]

which, together with (4.13), implies

\[ ||\omega_{n,s}||_{C(W_s)} \leq \left( n \log \frac{1}{1 + \frac{\varepsilon_1}{2} \delta(s)} \right)^{1/2} \leq \left( n \log \left( 1 + \frac{1}{\varepsilon_1 \delta(s)} \right) \right)^{1/2}. \]
Furthermore, (4.12), (4.14), and (4.15) yield
\[ ||p_{n,s}||_{C(W_s)} \geq \left( 1 + \frac{\varepsilon_1}{2} \delta(s) \right)^n - \left( n \log \left( 1 + \frac{1}{\varepsilon_1 \delta(s)} \right) \right)^{1/2}. \]

Let \( n_2 = n_2(\Gamma, s) \in \mathbb{N} \) and \( \varepsilon_3 = \varepsilon_3(\Gamma) \) be such that
\[ ||p_{n,s}||_{C(W_s)} \geq \frac{1}{2} \left( 1 + \frac{\varepsilon_1}{2} \delta(s) \right)^n, \quad n > n_2, \]
and
\[ 1 + \frac{\varepsilon_1}{2} \delta(s) \geq \exp(2\varepsilon_3 \delta(s)), \]
that is,
\[ ||p_{n,s}||_{C(W_s)} \geq \frac{1}{2} \exp(2\varepsilon_3 \delta(s)n), \quad n > n_2. \]
Summarizing, by virtue of (4.11), we have
\[ \exp(\varepsilon_3 \delta(s)n) \int_{A_s} |p_{n,s}|^p ds \leq \frac{\int_{W_s} |p_{n,s}|^p ds}{\int_{W_s} |p_{n,s}|^p ds} \leq \frac{\exp(\varepsilon_3 \delta(s)n)\Gamma[2^p(n \log(n + 2))^{p/2}}{2^{-p} \exp(2\varepsilon_3 \delta(s)np)\varepsilon_2 16^{-1}s} = 4^{p+2}|\Gamma|(n \log(n + 2))^{p/2}s^{-1} \varepsilon_2^{-1} \exp(-\varepsilon_3 \delta(s)n) \to 0 \quad \text{as } n \to \infty. \]

Let \( n_0 = n_0(s, \Gamma, p) > n_2 \) be such that for \( n > n_0 \) the right-hand side of the last inequality is at most 1. Then, the left-hand side is also \( \leq 1 \) from which (2.3) follows.

Proof of Theorem 3. Modifying the reasoning from the proof of [17, Theorem 5.5], we let \( d_n := d(\Gamma, \Gamma_{1/n}) \) and
\[ E_n = E_{n,q} := \left\{ z \in \Gamma : |p_n(z)|^q w(z) \geq d_n^{-1} \int_{\Gamma} |p_n|^q w ds \right\}. \]
Since
\[ \int_{\Gamma} |p_n|^q w ds \geq |E_n| d_n^{-1} \int_{\Gamma} |p_n|^q w ds, \]
we have \( |E_n| \leq d_n. \)

According to (1.4) and Lemma 1, there exists \( n_1 = n_1(\Gamma) \in \mathbb{N} \) such that for \( n > n_1 \) we have \( d_n < (\text{diam } \Gamma)/12. \)
Since $\delta(d_n) = 1/n$, by Theorem 1 for $n > n_1$ we obtain
\[
\int_{\Gamma} |p_n|^q w ds \leq \int_{\Gamma \setminus E_n} |p_n|^q w ds = \int_{\Gamma \setminus E_n} (|p_n|^p w^{p/q})(|p_n|^q w)^{(q-p)/q} ds
\]
\[
\leq \left( d_n^{-1} \int_{\Gamma} |p_n|^q w ds \right)^{(q-p)/q} \int_{\Gamma} |p_n|^p w^{p/q} ds,
\]
that is,
\[
\left( \int_{\Gamma} |p_n|^q w ds \right)^{p/q} \leq d_n^{(p-q)/q} \int_{\Gamma} |p_n|^p w^{p/q} ds,
\]
which establishes (2.4).

\[\blacksquare\]

Proof of Theorem 4. There is no loss of generality in assuming that $n > 100$ (for $n \leq 100$ take $p_n^* \equiv 1$). Let $z_{1/n} \in \Gamma$ satisfy
\[
\rho_{1/n}(z_{1/n}) = \min_{z \in \Gamma} \rho_{1/n}(z) = d(\Gamma, \Gamma_{1/n}).
\]
Consider polynomial
\[
p_n^*(z) := q_{k,2}(z_{1/n}, z),
\]
where $k$ is the integral part of $n/20$ and $q_{k,2}$ is introduced in Section 3. By (3.2) and Lemma 3 for any fixed $r \geq 1$,
\[
\int_{\Gamma} |p_n^*|^r w ds \simeq \rho_{1/k}(z_{1/n}) \simeq d(\Gamma, \Gamma_{1/n}),
\]
which implies (2.5).

\[\blacksquare\]

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