Extreme Mass-Ratio Binary Black Hole Merger: Universal Characteristics of the Test-Particle Limit

Barak Rom and Re'em Sari
Racah Institute of Physics, The Hebrew University of Jerusalem, Israel
(Dated: April 26, 2022)

We study binary black hole mergers in the extreme mass-ratio limit. We determine the energy, angular momentum, and linear momentum of the post-merger, remnant black hole. Unlike previous works, we perform our analysis directly in the test-particle limit, by solving the Regge-Wheeler-Zerilli wave equation with a source that moves along a geodesic. We build on the fact that towards the merger, small mass-ratio binary systems follow a Geodesic Universal Infall (GUI) trajectory. This formalism captures the final pre-merger stages of small mass-ratio binaries and thus provides a straightforward universal description in a region inaccessible to numerical relativity simulations. We present a general waveform template that may be used in the search for gravitational wave bursts from small and intermediate mass-ratio binary systems. Finally, this formalism gives a formal proof that the recoil velocity is quadratic in the symmetric mass-ratio \( \nu \). Specifically, the velocity is given by \( V/c \approx 0.0467\nu^2 \). This result is about 4% larger than previously estimated. Most of this difference stems from the inclusion of higher multipoles in our calculation.

I. INTRODUCTION

The recent detection of Gravitational Waves (GW) [1], transformed Einstein’s theoretical prediction to an empirically measured phenomenon. One of the key landmarks in the theoretical study of GW is the pioneering work of Regge, Wheeler, and Zerilli [2, 3], who formulated a simple wave equation from the field equations of general relativity, given the Schwarzschild geometry. Teukolsky [4], using Newman-Penrose formalism, generalized this to Kerr geometry as well. The Regge-Wheeler-Zerilli (RWZ) equation, as presented below, is a wave equation with a potential, induced by the spacetime curvature, and a source term, derived from the stress-energy tensor:

\[
\partial_t^2 \psi_{\ell m}^{(\lambda)} - \partial_r^2 \psi_{\ell m}^{(\lambda)} + \frac{\ell}{r} \partial_r \psi_{\ell m}^{(\lambda)} = S_{\ell m}^{(\lambda)}
\]  

(1)

Where \( (\ell, m) \) are the multipolar indices, \( \lambda \in \{ e, o \} \) is the parity, \( r^* = r + 2M \log \left( \frac{r}{2M} - 1 \right) \) is the tortoise coordinate, \( V_{\ell}^{(\lambda)} \) is the curvature potential and \( S_{\ell m}^{(\lambda)} \) is the source term; for the explicit expressions see Appendix A.

The RWZ equation opened the door for quantitative numerical studies of GW emission from binary systems. To mention a few, from the early rival infall studies [5, 6], through circular orbits [7–10], to full mergers [11–13]. We investigate the merger of a non-spinning binary Black Hole (BH) system, initially on quasi-circular orbit, in the extreme mass-ratio limit; \( \mu \ll M \), where \( M \) is the total mass and \( \mu \) is the reduced mass of the system. The merger scenario can be qualitatively divided into three stages: (i) Quasi-circular inspiral, during which the orbit evolves by the emission of GW. (ii) Universal plunge, where the infall path tends to the Geodesic Universal Infall (GUI) trajectory. (iii) Quasi-normal modes (QNM) ringdown.

A binary system, initially at large separation, emits GW which leads to a fast circularization of the orbit followed by a slow decrease of its semi-major axis [14], with a radial velocity that is much smaller than the angular one. This is the quasi-circular inspiral stage, which continues until the secondary BH crosses the Innermost Stable Circular Orbit (ISCO), at \( R_{\text{ISCO}} = 6M \). External to the ISCO there are stable circular orbits, so the secondary BH slowly descends from one to another, but after the ISCO there are no further stable circular orbits. Therefore, the secondary BH motion smoothly shifts from the GW-driven inspiral to a geodesic free-fall. Finally, after the secondary BH crosses the peak of the curvature potential, at \( R \sim 3M \), the GW signal is dominated by the QNM ringing, the intrinsic vibration modes of the remnant BH [12, 15].

The initial motivation for this work lies in the insight that the plunge path is universal in the sense that it is insensitive to the initial separation and eccentricity nor the exact mass-ratio. Although the mass-ratio determines the number of orbits internal to the ISCO, for any small mass-ratio, the infall path tends to the GUI trajectory; namely, coincides with the free-fall trajectory of a test-particle that is initially at the ISCO. Therefore, phenomena that strongly depend on the final pre-merger orbits, like the recoil velocity, are universal.

In this paper, we determine the post-merger energy, angular momentum, and linear momentum of the remnant BH and present the merger waveform, at the test-particle limit. The energy and angular momentum, up to first order in the mass-ratio, are derived from their values at the ISCO while the recoil velocity is numerically calculated by solving the RWZ equation for a source that moves along the GUI trajectory. This method allows calculating directly at the test-particle limit, without introducing any finite mass-ratio and extrapolating to the \( \mu \to 0 \) limit. For a small, finite mass-ratio, there will be higher orders corrections beyond the first order value, that we calculate in this paper. Finally, we sum the high
multipoles contribution and thus evaluate the full recoil coefficient, by extrapolating the results to higher $\ell$ than we numerically calculate.

Throughout the paper we use geometric units, $G = c = 1$.

II. RELATIVISTIC DYNAMICS

The equations of motion of a test-particle that moves around a non-spinning BH, as derived from the Schwarzschild metric, are:

$$\frac{d\phi}{dt} = \left(1 - \frac{2M}{R}\right) \frac{L}{ER^2} \quad (2a)$$

$$\frac{dR}{dt} = \pm \left(1 - \frac{2M}{E}\right) \sqrt{E^2 \left(1 - \frac{2M}{R}\right) \left(1 + \frac{L^2}{R^2}\right)} \quad (2b)$$

Where $E$ and $L$ are the energy and angular momentum, per unit mass, respectively.

A. The GUI Trajectory

We examine the geodesic infall of a test-particle that is initially at the ISCO. Since it is a marginally stable orbit, the test-particle will fall, after an infinitely long time, to the BH. The path towards the merger is schematically composed of infinite quasi-circular orbits in the vicinity of the ISCO, followed by a rapid fall, during which the test-particle passes most of the radial distance in $O(1)$ cycles, as demonstrated in Fig. (1).

This orbit is a geodesic. It is universal in the sense that towards the merger any small mass-ratio binary system tends to this trajectory and therefore we identify it as the Geodesic Universal Infall (GUI) trajectory.

Given the values of energy and angular momentum at the ISCO, $E_{ISCO} = \sqrt{\frac{2}{3}}$, $L_{ISCO} = \sqrt{12}M$, the GUI trajectory can be analytically calculated, yielding the implicit relation $t = g(R_0) - g(R)$, with $R_0 = R(t = 0)$ and:

$$g(R) = 8\sqrt{2}\sin^{-1}\left(\sqrt{\frac{6 - R}{6}}\right) - 4 \tanh^{-1}\left(\sqrt{\frac{6 - R}{2R}}\right) + \frac{48\sqrt{3}}{\sqrt{6 - R}} \left[ \frac{R}{24} + 2F_1\left(\frac{3}{2}, \frac{1}{2}; \frac{1}{6}; \frac{6 - R}{6}\right) \right]$$

Where $2F_1$ is the Gaussian hypergeometric function.

B. Energy & Linear Momentum Fluxes

The physical quantities associated with the GW can be determined from the RWZ function, $\psi_{\ell m}^{(\lambda)}$. Specifically, the energy and linear momentum fluxes are given by [16, 17]:

$$\dot{E} = \frac{1}{8\pi} \sum_{\ell \geq 2, 0 \leq m \leq \ell} \delta_m \left(\frac{\ell + 2}{(\ell - 2)!}\right)^2 |\psi_{\ell m}^{(\lambda)}|^2 \quad (4)$$

Where we sum only over $m \geq 0$, using the symmetry $\psi_{\ell, -m}^{(\lambda)} = (-1)^m \psi_{\ell m}^{(\lambda)*}$ and denoting $\delta_m = \begin{cases} 1/2 & m = 0 \\ 1 & \text{otherwise} \end{cases}$.

$$\dot{P}_x + i\dot{P}_y = \frac{1}{8\pi} \sum_{\ell \geq 2, 0 \leq m \leq \ell} \delta_m \left[ia_{\ell m}\psi_{\ell m}^{(c)\star} \psi_{\ell+1 m+1}^{(a)} + ib_{\ell m}\psi_{\ell m}^{(\lambda)\star} \psi_{\ell+1 m+1}^{(\lambda)\star} - (ia_{\ell m}\psi_{\ell m}^{(c)} \psi_{\ell-1 m-1}^{(a)\star} + b_{\ell m}\psi_{\ell m}^{(\lambda)\star} \psi_{\ell-1 m-1}^{(\lambda)\star}) \right]$$

Where:

$$a_{\ell m} = 2(\ell - 1)(\ell + 2)\sqrt{(\ell - m)(\ell + m + 1)}$$

$$b_{\ell m} = \frac{(\ell + 3)!}{(\ell + 1)(\ell - 2)!} \sqrt{\frac{(\ell + m + 1)(\ell + m + 2)}{(2\ell + 1)(2\ell + 3)}}$$

III. NUMERICAL METHOD

Following [12], we numerically solve the RWZ equation using a second-order Lax-Wendoff scheme, with Sommerfeld absorbing boundary conditions [18, 19]:

$$\lim_{r \to \pm \infty} \left( \frac{\partial r}{\partial t} \psi_{\ell m}^{(\lambda)} \pm \frac{\partial_r}{\partial r} \psi_{\ell m}^{(\lambda)} \right) = 0 \quad (6)$$

For the initial conditions, adopting [12] pragmatic approach, we set $\psi_{\ell m}^{(\lambda)}(r^*, t = 0) = \psi_{\ell m}^{(\lambda)}(r^*, t = 0) = 0$. 

FIG. 1. The Geodesic Universal Infall (GUI) trajectory of a test-particle from the ISCO, $R = 6M$, to the BH's Horizon (black dashed line).
This convenient choice leads to an unphysical initial burst that propagates outwards, as can be seen in Fig. (4). In our numerical scheme, we model the delta function in the source term as a narrow Gaussian, with a standard deviation of 4 grid cells.

1. Extrapolation to $I^+$

A conceptual limitation of the numerical calculation stems from the extraction of the GW at a finite distance. This obstacle is commonly overcome by extracting the GW at several different radii and extrapolating to null infinity, $I^+$, by expanding it as a series in $1/r$ [20,22]. There are other approaches for extracting the signal directly at $I^+$, for example, compactification of the spatial domain [23] or characteristic extraction [24]. We present a different method of frequency-domain extrapolation.

Given a time-domain signal, $\psi_R^*$, numerically extracted at some finite distance $R^* \gg M$, its propagation to $I^+$ can be analytically calculated. At this limit, the curvature potential is approximately $V(r^*) \sim \frac{\Lambda}{r^*}$, and the RWZ equation can be solved in the frequency-domain [25]

\[
\frac{\partial^2}{\partial \xi^2} \psi + \left(1 - \frac{\Lambda}{\xi^2}\right) \psi = 0 \tag{7a}
\]

\[
\psi = A_\omega e^{-i\xi} \sqrt{\xi} J_\nu(\ell + \frac{1}{2}; \xi) \tag{7b}
\]

Where $\tilde{\psi}(\omega, r^*)$ is the Fourier transform of $\psi(t, r^*)$, $\xi = \omega r^*$, $J_\nu(\xi - i\nu) \equiv J_\nu(\xi) + iY_\nu(\xi)$ is a combination of the Bessel functions of the first and second kind, and $A_\omega$ is a $\omega$-dependant coefficient.

The GW at $I^+$ can be evaluated by taking the limit $\xi \to \infty$:

\[
\tilde{\psi}_\infty = \tilde{\psi}_{R^*}/\tilde{\chi}(\xi_{R^*}) \tag{8}
\]

Where $\xi_{R^*} = \omega R^*$, $\tilde{\psi}_{R^*}$ is the Fourier transform of the numerically extracted $\psi_{R^*}$, and $\tilde{\chi}(\xi_R)$ is a correction function, defined as:

\[
\tilde{\chi}(\xi_R) = \sqrt{\frac{\pi e^{-i[\xi_{R^*} - \frac{\xi}{2}(\ell + 1)]}}{\xi_{R^*} J_{\nu}(\ell + \frac{1}{2}; \xi_{R^*})}} \tag{9}
\]

Thus, the GW at $I^+$ can be determined based on a numerical calculation of the RWZ function at a single finite distance. We examine this method by extracting the RWZ function at different radii, $R = 250M$, $750M$, $1500M$, and independently extrapolating them to $I^+$. As can be seen in Fig. (2), there is a clear mismatch, in amplitude and phase, between the extrapolated signals (dashed lines), which improves significantly after the extrapolation (solid lines). Quantitatively, the relative differences in the maximum absolute magnitude and phase are: $\delta A/A \sim 2.5 \times 10^{-3}$, $\delta \phi \sim 0.069$ between $R = 1500M$ and $R = 250M$, which after extrapolation improves by more than an order of magnitude, $\delta A/A \sim 7 \times 10^{-5}$, $\delta \phi |_{I^+} \sim 0.003$, and between $R = 1500M$ and $R = 750M$, $\delta A/A \sim 5 \times 10^{-4}$, $\delta \phi \sim 0.067$, and after extrapolation $\delta A/A |_{I^+} \sim 6 \times 10^{-6}$, $\delta \phi |_{I^+} \sim 5 \times 10^{-4}$. We see the same consistent alignment of the extrapolated signals in higher multipoles as well.

IV. RESULTS

We implement our numerical method to calculate the GW and emitted fluxes from a source that moves along a geodesic in the Schwarzschild geometry. We begin by reconstructing known results for circular orbits, as a validation test, and then continue to estimate the emission from a source that moves along the GUI trajectory.

A. Circular Orbits

We test the numerical calculation scheme in the well-studied circular orbit case [9,10,23,25]. For comparison with the literature, we examine the GW emission from a circular orbit at distance $R = 7.9456M$. We calculate $\psi_{\ell m}^{(4)}$ up to $\ell = 8$, and determine the corresponding energy flux, using Eq. (4). In addition, we develop a

---

1 For a given $(\ell, m)$ and parity $\lambda$, which are omitted from the derivation to simplify the notation.
semi-analytical method to calculate the GW from circular orbits, as discussed in the following section. We get very good agreement between the results of the numerical calculation, the semi-analytical method, and known results in the literature [23], as summarized in Table (I) of Appendix B.

1. Comparison to Semi-Analytic Solution

The RWZ source term has in the circular orbit case a simple time dependence, \( S \propto e^{-i\Omega t} \), where \( \Omega = \sqrt{M/R^3} \) is the orbital frequency. Substituting the ansatz \( \psi(r,t) = f(r)e^{-i\Omega t} \), gives an homogeneous ODE for \( f(r) \):

\[
\frac{d^2 f}{dr^2} + \left(m^2\Omega^2 - V\right)f = 0 \tag{10}
\]

Thus, the original problem reduces to solving Eq. (10) in two separate regimes, \( r^* < R^* \) and \( r^* > R^* \). A unique solution is obtained by imposing outgoing wave boundary conditions and specific finite discontinuity conditions at \( r^* = R^* \):

\[
\begin{align*}
\Delta f|_{R^*} &= \tilde{D}(R)F(R) \\
\Delta f'|_{R^*} &= \tilde{D}(R)G(R)
\end{align*}
\]

Where \( \tilde{D}, F, \) and \( G \) are given in Appendix A. The ODE solution is in a good agreement with the full numerical one, as can be seen in Fig. 3.

Asymptotically, for \( r \to \infty \), \( \psi \to \beta e^{-i\Omega(t-r^*)} \), where \( \beta \) is a complex coefficient determined by the discontinuity conditions. By substituting this expression into Eq. (4) and (5), we get the energy and the linear momentum fluxes as functions of \( \beta \). Thus, for example, the energy flux can be written as:

\[
\dot{E}_{\ell m} = \frac{1}{8\pi} \frac{(\ell + 2)\ell m^2 R^3}{(\ell - 2)!} |\beta_{\ell m}|^2 \tag{11}
\]

In appendix B we provide an analytic solution for \( \beta \) at the Newtonian limit, \( R \gg M \), which allows reconstructing the known quadrupole radiation formula [26], \( \dot{E}_{22} = \frac{32}{5} \frac{M^3 \mu^2}{R^5} \). Moreover, we show that asymptotically, the contribution of the high multipoles to the emitted fluxes decays exponentially, at a constant, radius-dependent, rate:

\[
\frac{\dot{E}_{\ell+1}}{\dot{E}_{\ell}} \sim \frac{\dot{P}_{\ell+1}}{\dot{P}_{\ell}} \sim e^{\frac{2}{4\ell}} \tag{12}
\]

for \( \ell \gg 1 \).

B. Inspiral & Merger

We now move to calculate numerically the GW emitted by a particle moving along the GUI trajectory. Qualitatively, after the initial induced burst, the signal oscillates, with approximately constant amplitude and frequency, corresponding to the quasi-circular orbits in the vicinity of the ISCO. Then, it sharply increases, in amplitude and frequency, due to the particle’s rapid infall at the last few orbits, and quickly decays. Thus, for example, Fig. 4 presents the dominant, quadrupole RWZ function. As can be seen, the amplitude increases by about 45% at its peak, compared to its initial magnitude during the quasi-circular orbits.

As the angular momentum flux can be calculated in a similar manner and the known relation \( \frac{dE}{dt} = \Omega \frac{dJ}{dt} \) can be easily derived.

\[
\frac{dE}{dt} = \Omega \frac{dJ}{dt} \tag{11}
\]
1. The GUI Waveform

The universality of the GUI trajectory entails that the test-particle GUI waveform, as presented in Fig. 4, can be used as a general template for small mass-ratio binary mergers. The GUI waveform captures the GW emission pattern from the merger back to the ISCO crossing; since the secondary BH’s deviation from the GUI trajectory, due to the GW emission, does not yield a significant phase difference between the GUI waveform and the small mass-ratio waveform during the infall from the ISCO to the horizon.

The two signals, the GUI and the small mass-ratio one, go out of phase on a time scale that corresponds to the orbital frequency change during the GW-driven inspiral: $\psi t^2 \sim R t^2 \sim 1 \rightarrow t \propto \nu^{-3/10}$. Where we used that the radial velocity at the vicinity of the ISCO scales as $R \propto \nu^{3/5}$ [27, 28].

As a preliminary proof of concept, we compare the GUI waveform to a waveform from a Numerical Relativity (NR) simulation of a binary merger with mass-ratio 1:10 [29], as presented in Fig. 5. We get a remarkably well agreement between the maximum amplitudes of the two signals, with a relative difference of $0.3\%$, and they stay in phase for about 3 cycles.

![Fig. 5. Comparison between the GUI waveform (blue line) and NR simulation waveform, from the SXS catalog [29], with mass-ratio 1 : 10 (red dashed line). Both of the signals are shifted so that their real part reaches its maximum value at $t - r^* = 0$. The vertical dashed line corresponds to the time when the test-particle is at $R \sim 5.5M$ and the relative distance in the NR simulation is about $6.15M$.](image)

2. Linear Momentum - Recoil Velocity

Using Eq. 5, we calculate the emitted linear momentum flux. The recoil velocity is then obtained by integration:

$$V_z + iV_y = V_0 - \frac{1}{M} \int_{-\infty}^{t} (\dot{P}_x + i\dot{P}_y) \, dt$$  (13)

First, we point out the scaling. As can be seen in appendix A, the source term in the RWZ equation is linear in $\mu$. Therefore, $\psi \propto \mu$ and its time derivative scales as $\psi \propto \frac{d\psi}{dt} \propto \nu$. Given this scaling, Eq. 4 and 5 imply that the radiated fluxes scale as $\nu^2$, and so does the recoil velocity.

The integration constant, $V_0$, is determined by the requirement that the initially oscillating velocity, corresponding to the GW emission along the quasi-circular orbits, will have zero mean, as discussed in [25]. For comparison with the literature, we note that up to $\ell = 7$ the recoil magnitude is $V^{(7)}/\nu^2 = 0.0455$, which is about $1 - 2\%$ larger than the results of [23, 25, 30], which were calculated for a small, finite, mass-ratios. Our results are in a closer agreement to that of [31]. Further analysis needs to be done to establish if this slight deviation stems from numerical inaccuracies or from the fact that our result is calculated along the exact geodesic trajectory.

The higher multipoles have a decreasing, yet significant, contribution to the total recoil velocity, as can be seen in Fig. 6 and in appendix C, where we present the detailed results for each multipole separately. Therefore, we wish to evaluate the contribution of the infinite “tail” of high multipoles.

$$V = V^{(L)} + \sum_{\ell=L+1}^{\infty} \delta V^{(\ell)}$$  (14)

Where $L$ is the highest multipole that was numerically calculated; for this work $L = 10$. Based on the results for circular orbits as a heuristic guideline, and reinforced by the results of [22] regarding the QNM energy flux exponential decay, we assume that the contribution of the higher multipoles along the merger scenario decreases exponentially as well:

$$\delta V^{(\ell)} = a \times C^{\ell}$$  (15)

We get a good correspondence between the numerical results to an exponential decay trend, as can be seen in Fig. 6 and 7. Using the numerical fit we determine the values of the coefficients in Eq. 15 and evaluate the total recoil velocity: $V/\nu^2 = 0.0467$. This value is about $4\%$ larger than previous results in the literature, which estimated $V/\nu^2 \approx 0.044$. This difference originates mostly due to the summation by extrapolation of the high multipoles contribution, which were previously neglected.

3. Energy and Angular Momentum

We present a brief derivation of the post-merger energy and angular momentum of the remnant BH, in
FIG. 6. The final recoil velocity, up to a given multipole $\ell$. The total velocity converges to $V/\nu^2 \approx 0.0467$ (black dash-dotted line) and correspond to an exponential decay trend (red dashed line). The detailed numerical results are presented in Table (II) of Appendix B.

FIG. 7. An exponential decay fit, in a semi-logarithmic scale, to the relative contribution of each multipole to the total recoil velocity.

leading order of the mass-ratio. Up to the ISCO the emitted energy and angular momentum can be determined directly:

$$\Delta E = (1 - E_{ISCO}) \nu = \left(1 - \sqrt{\frac{8}{9}}\right) \nu,$$
$$\Delta J = -J_{ISCO} M \nu = -\sqrt{12} M^2 \nu.$$

The time from the ISCO crossing to the merger scales as $t \propto \nu^{-1/5}$ [27, 28], and as mentioned above, $\dot{E} \propto \nu^2$. Therefore, the total emitted energy after the ISCO crossing scales as $\nu^{9/5}$, and so does the emitted angular momentum. Hence, the GW emission after the ISCO crossing contributes only to the next order in the mass-ratio. In summary, we get that the final energy and spin of the remnant BH are:

$$M_f/M = 1 - \left(1 - \sqrt{\frac{8}{9}}\right) \nu + O(\nu^{9/5}) \quad (16a)$$
$$a \equiv J/M^2 = \sqrt{12} \nu + O(\nu^{9/5}) \quad (16b)$$

The first order terms are in accordance with known results in the literature [32, 33]. However, the scaling of the next order, $O(\nu^{9/5})$, which was already derived by [27, 28], differs from the broadly used $O(\nu^2)$.

V. CONCLUSIONS

We present a thorough investigation of the leading order effects of a binary BH merger in the extreme mass-ratio limit. We develop a new approach that allows us to perform the calculation directly in the test-particle limit, without introducing any finite mass-ratio. This can be done due to the universal characteristics of the plunge, specifically its tendency towards the Geodesic Universal Infall - GUI - trajectory. In addition, this straightforward approach allows us to construct a universal waveform that describes well the peak GW emission at the final stages of small mass-ratio binary mergers. This GUI waveform may be used as a computationally inexpensive template in the ongoing search for GW from intermediate mass-ratio binary systems in earth-based detectors.

At last, using this formalism, we show that the recoil velocity has a quadratic dependence on the mass-ratio and is given by $V/c \approx 0.0467 \nu^2$. This result is larger than the known value in the literature by about 4%, mostly as a result of the high multipole contributions, which were neglected in previous studies. As for the final energy and spin of the remnant BH, we derive analytically the first order, linear in the mass-ratio, terms and point out the scaling of the next order, $O(\nu^{9/5})$, which differs from the broadly used value in the literature. Our calculation can be generalized to Spinning BH.

Appendix A: Source Term and Curvature Potential in the RWZ Equation

The explicit form of the curvature potential and the source term, for a test-particle, in the RWZ equation are as follows [16]:

$$S^{(\lambda)} = D^{(\lambda)}(R, \Theta, \Phi) \left[ G^{(\lambda)}(R) \delta(r^* - R^*) + F^{(\lambda)}(R) \partial_{r^*} \delta(r^* - R^*) \right] \quad (A1)$$

Where $\left( R(t), \Theta(t), \Phi(t) \right)$ are the test-particle’s coordinates. Without loss of generality we can assume that the motion is equatorial and so $\Theta(t) = \frac{\pi}{2}$. In the following equations, $r$, $t$ and $L$ are measured in units of the primary BH mass, $M$. 
1. Even Parity Perturbations

\[ V^{(e)}_\ell = \left(1 - \frac{2}{r}\right) \left(\Lambda^2 - \frac{2}{r^2}\right) \frac{(\Lambda - 2)^2 (\Lambda r + 6) + 36(\Lambda - 2)r + 72}{M^2 r^3 [((\Lambda - 2)r + 6)]^2} \]  

Where \( \Lambda = \ell (\ell + 1) \).

\[ D^{(e)} = \mu \left(1 - \frac{2}{r}\right) \frac{8\pi Y^*_{\ell m} (\Theta, \Phi)}{E \xi R \left[(\Lambda - 2) R + 6\right]} \]  

\[ G^{(e)} = \frac{1}{M \left[(\Lambda - 2) R + 6\right]} \left[-12 + 8 \left(1 - 3E^2 + \Lambda\right) R \right. \right. 

\left. \left. + (\Lambda - 2) \Lambda R^2 - 4i m L \left[(\Lambda - 2) R + 6\right]\right] V_r \right. 

\left. + \frac{2L^2}{R^2 (\Lambda - 2)} \left[\Lambda^3 R^2 - \Lambda^2 R \left[-12 + (5 + m^2) R\right] \right. \right. 

\left. \left. - 4 \left[3 - 6 \Lambda + m^2 (R - 3)^2 - 3R + R^2\right] \right. \right. 

\left. \left. - 2 \Lambda [3(5 + m^2) R - 2(2 + m^2) R^2] \right\} \right) \]  

(Appendix B: Circular Orbits)

We present in Table 4 a comparison between the radiated energy fluxes, up to \( \ell = 8 \), as calculated using the semi-analytical method, the numerical PDE scheme, extrapolated to null infinity and extracted at finite radii, and known results in the literature.\(^\text{23}\). We note that as \( \ell \) increases, the amplitude of the \( m \ll \ell \) moments becomes increasingly smaller and so more distorted by numerical noise and rescattering of the initial junk radiation. However, their contribution to the total emitted fluxes is negligible.

1. Expansion at the Newtonian limit

Eq. (10) can be analytically solved in the Newtonian limit, where it becomes equivalent to Eq. (7a), with a specific value of \( \omega = m \Omega \). Therefore, the solution for \( f(r) \) is a combination of Bessel functions of the first and second kind, as in Eq. (7b). Imposing outgoing wave boundary conditions and using the known limits of the Bessel functions yield:

\[ f(r > R) = \beta \left(1\right) \frac{\Gamma (\ell + \frac{1}{2})}{\sqrt{\pi}} \left(\frac{2i}{m \Omega r}\right)^\ell r \sim R \]  

(B1)

Where \( \Gamma \) is the gamma function. \( \beta \) is uniquely determined by imposing the discontinuity conditions at \( r = R \):

\[
\begin{align*}
\Delta f^{(\lambda)} |_{r = R} & = \lambda^{(\lambda)} \\
\Delta f^{(\ell)} |_{r = R} & = -\frac{\delta^{(\ell)} \gamma^{(\lambda)}}{r}
\end{align*}
\]  

(B2)

Where \( \gamma^{(\lambda)} = \left\{ \begin{array}{ll} 
16\pi & \text{even} \\
\frac{\lambda}{\sqrt{3}} & \text{odd}
\end{array} \right. \)

and \( \delta^{(\lambda)} = \left\{ \begin{array}{ll} 
\Lambda/2 & \text{even} \\
1 & \text{odd}
\end{array} \right. \)

The radiated fluxes can be calculated using Eq. (4) and (5). For the energy flux we get:

\[ \dot{E}_{\ell m} = \frac{2m^{(\ell + 1)} (\ell + 1) (\ell + 2)}{R^{(\ell + 3 \ell + 1)(2 \ell + 1)(\ell + 2) + 1}} \]  

\[ \times \left\{ \frac{[(-m)!][(-m)!]}{[(-m)!][(-m)!]} \right\} (\ell + m) \text{ even} \]  

\[ \times \left\{ \frac{[(-m)!][(-m)!]}{[(-m)!][(-m)!]} \right\} (\ell + m) \text{ odd} \]  

(B3)

The result for the even energy flux is in accordance with a previous calculation of [7]. The total flux, for a given multipole \( \ell \), can be evaluated up to sub-leading order as \( \dot{E}_\ell = \sum_m \dot{E}_{\ell m} \approx \dot{E}_\ell + \dot{E}_{\ell - 2} \), where the contribution of the odd multipoles is negligible as \( \dot{E}_\ell^{(o)}/\dot{E}_\ell^{(e)} \approx O \left( \frac{1}{\sqrt{r}} \right) \).

\(^3\) At the region \( r < R \), all we need for this calculation is the asymptotic behavior \( f|_{r \to R^-} \propto r^{\ell + 1} \).
The linear momentum flux is given by:

\[ P = \frac{\ell + 2}{\ell \ell + 1} \frac{2 \ell + 1}{2 \ell + 3} \frac{\ell + 1}{\ell} (h_{\ell m}^+ + h_{\ell m}^-) \times \frac{(\ell - m)!}{(\ell - m)! + (\ell + m)!} \frac{2 \ell + 1}{2 \ell + 1} \frac{\ell}{\ell} \]

Table I. Energy flux for a circular orbit at radius \( R = 7.9456 M \) around a non-rotating BH. We compare the results from the semi-analytical method (subscript \( \text{ODE} \)), the full numerical scheme, extrapolated to null infinity (subscript \( PDE_{\ell+} \)) and extracted at finite radii, \( R = 1500M \) and \( R = 250M \) (subscripts \( PDE_{1500} \) and \( PDE_{250} \)), and the results of \( \text{BE} \) (subscript \( \text{Bernuzzi + 11} \)), which were calculated at null infinity. The relative differences, with respect to the semi-analytical result, appear in square brackets.

From Eq. (B3) we get that asymptotically for high multipoles:

\[ \dot{E}_\ell \approx \frac{1}{2 R^3} \sqrt{\frac{\ell}{\pi}} \left( \frac{e^2/4}{R} \right)^\ell B_E(\ell) \]

\[ B_E(\ell) \sim 1.01 + \frac{3.42}{\ell} + \frac{4.08}{\ell^2} \]

for \( \ell \gg 1 \).

The linear momentum flux is given by:

\[ |\dot{P}_{\ell m}| = \frac{2}{R^{2 + \ell/2} (\ell + 1)} \frac{\ell + 1}{\ell + 1} \frac{2 \ell + 1}{2 \ell + 3} \frac{\ell + 1}{\ell} (h_{\ell m}^+ + h_{\ell m}^-) \times \frac{(\ell - m)!}{(\ell - m)! + (\ell + m)!} \frac{2 \ell + 1}{2 \ell + 1} \frac{\ell}{\ell} \]

Odd multipoles is negligible, and we denote:

\[ h_{\ell m}^{(\pm)} = \left( 1 \pm \frac{1}{m} \right)^{\ell + 1} (\ell + m + 1) \times \left[ \frac{(m + 1)(\ell + 3)}{2 \ell + 3} + 4(\ell + m) \frac{2 \ell + 1}{\ell (\ell - 1)} \right] \]

For \( \ell \gg 1 \):

\[ |\dot{P}_\ell| \approx \frac{e}{4 R^3} \sqrt{\frac{\ell}{\pi R}} \left( \frac{e^2/4}{R} \right)^\ell B_P(\ell) \]

\[ B_P(\ell) \sim 1.001 + \frac{3.515}{\ell} + \frac{2.695}{\ell^2} \]

We note that \( \dot{P}_{\ell m} \propto e^{2R(t-r_m)} \), and therefore it vanishes when averaging over one period of the motion. Thus, as can be inferred from the symmetry of this case, there is no accumulation of recoil velocity in circular orbits.
Appendix C: Multipolar decomposition

We present in Table (II) the detailed numerical results for the accumulated recoil velocity.

\[ \ell \left( \frac{V}{\nu} \right)^2 \]

\[ \ell \left( \frac{V}{\nu} \right)^2 \]

\[ \ell \left( \frac{V}{\nu} \right)^2 \]

\[ \ell \left( \frac{V}{\nu} \right)^2 \]

\[
\begin{array}{ccc}
\ell & \left( \frac{V}{\nu} \right)^2 & \ell \\
2 & 0.0289 & 5 & 0.0436 & 8 & 0.0460 \\
3 & 0.0374 & 6 & 0.0448 & 9 & 0.0463 \\
4 & 0.0414 & 7 & 0.0456 & 10 & 0.0464 \\
\end{array}
\]

TABLE II. The accumulated recoil velocity up to a given multipole \( \ell \).

[1] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 116, 061102 (2016).
[2] T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).
[3] F. J. Zerilli, Phys. Rev. D 2, 2141 (1970).
[4] S. A. Teukolsky, ApJ 185, 635 (1973).
[5] M. Davis, R. Ruffini, and J. Tiomno, Phys. Rev. D 5, 2932 (1972).
[6] M. Davis, R. Ruffini, W. H. Press, and R. H. Price, Phys. Rev. Lett. 27, 1466 (1971).
[7] E. Poisson, Phys. Rev. D 47, 1497 (1993).
[8] T. Tanaka, M. Shibata, M. Sasaki, H. Tagoshi, and T. Nakamura, Progress of Theoretical Physics 90, 65 (1993), https://academic.oup.com/ptp/article-pdf/90/1/65/90-1-65.pdf.
[9] K. Martel, Phys. Rev. D 69, 044025 (2004).
[10] C. F. Sopuerta and P. Laguna, Phys. Rev. D 73, 044028 (2006).
[11] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter, Phys. Rev. D 73, 104002 (2006).
[12] A. Nagar, T. Damour, and A. Tartaglia, Classical and Quantum Gravity 24, S109 (2007).
[13] F. Herrmann, E. Hinder, D. Shoemaker, and P. Laguna, Classical and Quantum Gravity 24, S33 (2007).
[14] P. C. Peters, Phys. Rev. 136, B1224 (1964).
[15] E. Berti, V. Cardoso, and A. O. Starinets, Class. Quant. Grav. 26, 163001 (2009), arXiv:0905.2975 [gr-qc].
[16] A. Nagar and L. Rezzolla, Classical and Quantum Gravity 23, 4297 (2006).
[17] D. Pollney, C. Reisswig, L. Rezzolla, B. Szilágyi, M. Ansorg, B. Deris, P. Diener, E. N. Dorband, M. Koppitz, A. Nagar, and E. Schnetter, Phys. Rev. D 76, 124002 (2007).
[18] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, Numerical recipes in c (Cambridge University Press, Cambridge, USA, 2007) Chap. 20, 3rd ed.
[19] L. Rezzolla (2011).
[20] M. A. Scheel, M. Boyle, T. Chu, L. E. Kidder, K. D. Matthews, and H. P. Pfeiffer, Phys. Rev. D 79, 024003 (2009).
[21] M. Boyle and A. H. Mroué, Phys. Rev. D 80, 124045 (2009).
[22] M. Boyle et al., Classical and Quantum Gravity 36, 195006 (2019), arXiv:1904.04831 [gr-qc].
[23] S. Bernuzzi, A. Nagar, and A. Zenginoğlu, Phys. Rev. D 84, 084026 (2011).
[24] C. Reisswig, N. T. Bishop, D. Pollney, and B. Szilágyi, Phys. Rev. Lett. 103, 221101 (2009).
[25] S. Bernuzzi and A. Nagar, Phys. Rev. D 81, 084056 (2010).
[26] L. Landau and E. Lifshitz, The classical theory of fields (Butterworth-Heinemann, 1975) Chap. 11, pp. 323–325, 3rd ed.
[27] A. Buonanno and T. Damour, Phys. Rev. D 62, 064015 (2000).
[28] A. Ori and K. S. Thorne, Phys. Rev. D 62, 124022 (2000).
[29] S. Collaboration, 10.5281/zenodo.3302023 (2019).
[30] P. A. Sundararajan, G. Khanna, and S. A. Hughes, Phys. Rev. D 81, 104009 (2010).
[31] A. Nagar, E. Harms, S. Bernuzzi, and A. Zenginoğlu, Phys. Rev. D 90, 124086 (2014).
[32] T. Damour, A. Nagar, and L. Villain, Phys. Rev. D 89, 024031 (2014).
[33] C. O. Lousto, M. Campanelli, Y. Zlochower, and H. Nakano, Classical and Quantum Gravity 27, 114006 (2010).