Spectrum of Radiation of a Classical Electron Moving in the de Sitter Spacetime

L.I. Tsaregorodtsev and N.N. Medvedev

March 24, 2022

Abstract

The radiation spectrum of a classical charged particle (electron) moving in the de Sitter universe, has been calculated. The de Sitter metric is taken in the quasi-Euclidean Robertson-Walker form. It is shown that in the de Sitter spacetime an electron radiates as if it moved in a constant and homogeneous electric field with the strength vector \( \mathbf{E} \) collinear to the electron momentum \( \mathbf{p} \).

1 Introduction

In the studies of quantum effects of field interaction in curved spacetimes, the problem of the choice of the initial and final vacuum states of the quantum fields is known to arise. The choice of a vacuum to a considerable extent depends on the choice of the frame of reference since this choice leads to a natural criterion for the choice of a quantum field vacuum.

For the de Sitter spacetime both static and nonstatic frames of reference can be constructed. There are nonstatic frames where the line element for the de Sitter space \( ds^2 \) has the Robertson-Walker form where spatial sections of the de Sitter manifold can represent open hyperboloids, three-spheres or three-planes. In the latter case the metric is called quasi-Euclidean. A review of properties of coordinate systems for the de Sitter spacetime can be found in [1].

In the calculations of quantum effects of field interaction in the Robertson-Walker spacetimes the quasi-Euclidean form of the metric is often used. This choice is explained by the opportunity to apply the well developed \( S \)-matrix formalism of quantum electrodynamics in an external gravitational field [2]. Besides, the results of the calculations can be interpreted in terms of particles moving in flat spacetime in the presence of an external gravitational field.

The quasi-Euclidean, explicitly conformally flat form of the metric the de Sitter spacetime is

\[
\begin{align*}
ds^2 &= a^2(\eta)ds^2, \\
\tag{1}
ds^2 &= (d\eta)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \\
-\infty < \eta < \infty.
\end{align*}
\]
Here $ds^2$ is the Minkowski metric and $a(\eta)$ is the expansion law of the Universe,

$$a(\eta) = \frac{\alpha}{\eta}, \quad \alpha = \text{const.} \quad (2)$$

A natural criterion for the choice of a vacuum state of a quantum massive field in the de Sitter space with the metric (1)-(2) is the requirement for the vacuum to be adiabatic [1]. However, in the case under consideration the metric has a coordinate singularity at $\eta = 0$ and the requirement for the vacuum to be adiabatic does not determine the choice of the initial and final vacua in a unique manner. The ambiguity of vacuum choice in the de Sitter space with the metric (1)-(2) makes it expedient to study the effects of interaction of a charged particle with an electromagnetic field in the framework of classical electrodynamics in curved spacetime. The results obtained can serve as an additional criterion for the vacuum choice in quantum theory. Namely, when the curvature of the de Sitter spacetime is small and the particle momenta are large, the results of calculations of quantum processes having classical analogues should coincide with the results obtained in the framework of classical theory.

In this paper we consider the bremsstrahlung process for a classical charged particle moving in the de Sitter universe with the metric (1)-(2). The plan of the paper is as follows. In Sec. 2 we consider the motion law of a free charged particle in the de Sitter spacetime with the metric (1)-(2). Sec. 3 is devoted to a search for the field created by this particle at large distances from it. In Sec. 4 we calculate the spectral distribution of the energy radiated from the charged particle and analyze the dependence of the spectrum on the spacetime curvature. Finally, in Sec. 5 we discuss our results.

2 The particle motion law

Let us consider the Lagrange function of a particle moving in the curved spacetime (1)–(2):

$$L = -m \frac{d\tilde{s}}{d\eta} = -ma(\eta)\sqrt{1 - \left(\frac{dr}{d\eta}\right)^2}.$$ 

We would like to use in the manifold to be examined, instead of the metric $d\tilde{s}^2$, the conformally related metric of the Minkowski spacetime $ds^2$. Then $v = dr/d\eta = \dot{r}$ is the particle velocity in an external gravitational field relative to the flat metric. Differentiating the Lagrange function with respect to $v$, we shall find the generalized momentum $p$ of the particle in the de Sitter spacetime. Since the Lagrange function does not depend on the radius vector of the particle $r$, the generalized momentum is conserved:

$$p = \frac{ma\dot{v}}{\sqrt{1 - v^2}} = \text{const.} \quad (3)$$

Solving the expression (3) for $v$, we get the time dependence of the particle velocity:

$$v = \pm \frac{p}{\sqrt{m^2a^2(\eta) + p^2}}. \quad (4)$$
By substituting into (4) the function \( a(\eta) \) and performing integration, we find the particle motion law in the de Sitter space:

\[
r - r_0 = \frac{p}{p^2} \left( \sqrt{m^2\alpha^2 + p^2\eta^2 - m\alpha} \right). \tag{5}
\]

Calculating the squared 4-vector of the particle acceleration (relative to the flat metric), we obtain

\[
w^i w_i = -\ddot{v}^2 - \left[ v, \dot{v} \right]^2 = -\frac{p^2}{m^2\alpha^2} = -w^2 \tag{6}
\]

where

\[
w = \frac{p}{m\alpha} = \text{const}
\]

is the particle acceleration vector in the intrinsic frame of reference. It follows from (6) that the particle motion in the de Sitter universe, considered relative to the conformally related Minkowski metric, is the relativistic uniformly accelerated motion \[2\].

In the massless limit Eq. (5) takes the form

\[
r = r_0 + n|\eta|, \quad n = p/p. \tag{7}
\]

Thus, generally speaking, in the massless limit the world line of the particle differs from a null geodesic since the direction of the particle velocity vector at the moment of time \( \eta = 0 \) instantly changes into the opposite.

## 3 The field of a charged particle

Now we will find the electric and magnetic fields created by a charge moving in the de Sitter universe. For this purpose we consider the Maxwell equations in a Riemannian spacetime \[3\]

\[
F_{ik} = J_i, \quad F_{ik} = A_{i;k} - A_{k;i} \tag{8}
\]

with a nonzero right-hand side. Here \( J^i \) is the 4-vector of current density,

\[
J^i = \frac{e}{\sqrt{-g}} \delta(r - r_e) \frac{\partial x^i}{\partial \eta}.
\tag{9}
\]

\( r_e \) is the radius vector of the charge \( e \) and the electromagnetic field potential \( A_i \) satisfies the Lorentz condition

\[
A_{i;\ i} = 0 \tag{10}
\]

(in Eqs. (8) and further rational units of charge are used).

Let us perform in (8)–(10) the conformal transformation \( g_{ik} = a^2(\eta)\eta_{ik} \), where \( \eta_{ik} \) is the Minkowski metric. The Maxwell equations are invariant under this transformation, and the Lorentz condition, as a result of the transformation, takes the form

\[
\eta^{ik} \frac{\partial A_k}{\partial x^i} = -2HA_0. \tag{11}
\]
Here \( H = \dot{a}/a \) is the Hubble parameter and the dot denotes time differentiation. Substituting \( F_{ik} \) expressed in terms of the potentials into the transformed Maxwell equations and taking into account the Lorentz condition (11), we obtain the equations determining the electromagnetic field potential in the Robertson-Walker spacetime
\[
\eta^{ij} \frac{\partial^2 A_k}{\partial x^i \partial x^j} + 2 \frac{\partial}{\partial x^k} \left( HA_0 \right) = j_k
\]
(12)
where
\[
j_k = \eta_{kl} \frac{d x^l}{d \eta}, \quad \rho = e \delta(r - r_e).
\]
(13)

Consider the solution of Eqs. (12) in the de Sitter universe. Putting
\[
A_0 = \eta B_0,
\]
(14)
we find that the function \( B_0(r, \eta) \) is a solution of the inhomogeneous wave equation
\[
\frac{\partial^2 B_0}{\partial \eta^2} - \triangle B_0 = \frac{j_0}{\eta}
\]
and hence can be written down as
\[
B_0(r, \eta) = \frac{1}{4\pi} \int \frac{d\eta' d^3 x' j_0(r', \eta') \delta(\eta' - \eta + |r - r'|)}{\eta' |r - r'|}.
\]
(15)

Having calculated, with the help of (14)–(15), the scalar potential \( A_0 \) of the field of the point charge, we present it as
\[
A_0(r, \eta) = A_0^M(r, \eta) + A_0^S(r, \eta).
\]
(16)

Here \( A_0^M(r, \eta) \) is the scalar potential of the field created by the charge in accelerated motion in flat spacetime (the Liénart-Wiechert potentials)
\[
A_0^M(r, \eta) = \frac{e}{4\pi (R - Rv)}, \quad R = |r - r_e(\eta')|, \quad \eta' + |r - r_e(\eta')| = \eta, \quad v = \dot{r}_e(\eta').
\]
The additional term \( A_0^S(r, \eta) \) is caused by the spacetime curvature:
\[
A_0^S = \frac{R}{\eta'(R - Rv)}.
\]

Now consider the equation for the vector potential \( A \)
\[
\frac{\partial^2 A_\alpha}{\partial \eta^2} - \triangle A_\alpha - 2 \frac{\partial B_0}{\partial x^\alpha} = j_\alpha \quad (\alpha = 1, 2, 3).
\]
(17)
where $\mathbf{A}^M = (-A_1^M, -A_2^M, -A_3^M)$ is the Liénart-Wiechert vector potential

$$\mathbf{A}^M = \frac{e}{4\pi} \frac{\mathbf{v}}{(R - R\mathbf{v})},$$

and the function $\varphi(r, \eta)$ has the form

$$\varphi = \frac{e}{8\pi^2} \int \frac{d^3x'}{|r - r'| |\eta''(R' - R'\mathbf{v}')|}.$$

Here

$$R' = r' - r_e(\eta''), \quad \mathbf{v}' = \dot{r}_e(\eta''),$$

and the moment of time $\eta''$ is determined by the equation

$$\eta'' - \eta + |r - r'| + |r' - r_e(\eta'')| = 0.$$

Having calculated the electromagnetic field potentials, we can find the electric and magnetic field strengths using the formulae

$$\mathbf{H} = \text{rot} \mathbf{A} = \text{rot} \left( \mathbf{A}^M - \text{grad} \varphi \right) = \text{rot} \mathbf{A}^M = \mathbf{H}^M,$$  \hspace{1cm} (18)

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial \eta} - \text{grad} A_0.$$  \hspace{1cm} (19)

It follows from (18) that the magnetic field strength $\mathbf{H}$ created by the point charge moving in the de Sitter universe has precisely the same form as the strength $\mathbf{H}^M$ of the field of the charge moving in the Minkowski spacetime. In turn, Eq. (19) for the electric field strength $\mathbf{E}$ can be presented as

$$\mathbf{E} = \mathbf{E}^M + \mathbf{E}^S,$$  \hspace{1cm} (20)

where $\mathbf{E}^M$ is the field strength of the point charge moving in flat spacetime, and the additional term $\mathbf{E}^S$ has the form

$$\mathbf{E}^S = \text{grad} \left( \frac{\partial \varphi}{\partial \eta} - A_0^S \right).$$

Let us consider the field $\mathbf{E}^S$ at large distances from the charge. Calculations show that for $r \gg r'$, $R \gg r_e(\eta')$ the field $\mathbf{E}^S$ is oriented along the radius vector $\mathbf{r}$ of the observation point:

$$\mathbf{E}^S = \frac{\mathbf{r}}{r^2(\eta - r)} E(\eta - r)$$

and hence does not give any contribution to the total intensity of charge radiation.

Thus the radiation field created by a free charge moving in the de Sitter universe is described by the same formulae as the radiation field of a charge moving with acceleration in Minkowski spacetime.
4 The spectrum of radiation

Now we find the spectrum of radiation of a charge moving in the de Sitter universe. The amount of energy $dI$ propagating per unit of time through the element $df = a^2(\eta)r^2d\Omega$ of a spherical surface centred at the origin is determined by the relation

$$dI = T^{0\alpha}df_{\alpha} \quad (\alpha = 1, 2, 3)$$

where $T^{ik}$ is the energy-momentum tensor of the electromagnetic field

$$T^{ik} = -F^i_mF^{km} + \frac{1}{4}g^{ik}F_{mn}F^{mn}.$$  

Expressing its components $T^{0\alpha}$ in terms of the electric and magnetic field strengths, we find that the radiation intensity in the element of solid angle $d\Omega$ in the de Sitter universe has the form

$$dI = (|\mathbf{E}\mathbf{H}|\mathbf{n})r^2d\Omega.$$  

(21)

As shown above, in a wave zone the field strengths $\mathbf{E}$ and $\mathbf{H}$ coincide with those produced by an accelerated charge in flat spacetime. Hence, when calculating the spectral distribution of radiation energy from a charge in the de Sitter spacetime, we can use the result obtained in the framework of classical electrodynamics in flat spacetime [3]. Namely, the particle radiation energy spectrum is determined by the expression

$$dE_k = \frac{1}{16\pi^3}j_m(k)j^m_*(k)d^3k.$$  

(22)

where $j^m(k)$ is the Fourier component of the 4-vector of current density

$$j^k = e\int dx^k(\eta)\exp\left(ik_0\eta - i\mathbf{k}\mathbf{r}(\eta)\right).$$  

(23)

In (23) the integration is carried out along the particle world line. If one takes into account that the particle motion law has the form (5), then the calculation of the particle radiation spectrum in the de Sitter universe will be reduced to the calculation of radiation spectrum of a uniformly accelerated charge in Minkowski spacetime. The solution of this problem was obtained by A.I. Nikishov and V.I. Ritus as a special case when considering a more general problem of radiation spectrum of a classical charged particle moving in a constant electric field [3]. Expressing in Nikishov’s formula the field strength $\mathbf{E}$ in terms of the acceleration $w$ and putting $w = p/(m\alpha)$, we find the spectral distribution of electron radiation energy in the de Sitter spacetime:

$$dE_k = \frac{e^2}{4\pi^3w^2}K_1^2\left(\frac{k_{\perp}}{w}\right)d^3k.$$  

(24)

Here $k_{\perp}$ is the wave vector component perpendicular to the particle momentum $\mathbf{p}$ and $K_1(a)$ is the Macdonald function [3].
Let us now investigate the dependence of the spectral distribution (24) on the spacetime curvature. If the curvature is small, \( R = 12\alpha^{-2} \ll m^2 \), then \( w \to 0 \) and the radiation spectrum takes the form

\[
dE_k = \frac{e^2 d k \perp d k \parallel d\varphi}{8\pi^2 w} \exp\left(-\frac{2k \perp}{w}\right)
\]

where \( k \parallel \) is the projection of the wave vector on the direction of the momentum \( p \). As it was to be expected, \( dE_k \to 0 \) when \( R \to 0 \).

In the opposite limiting case, when \( m\alpha \to 0 \), we obtain

\[
dE_k = \frac{e^2 d^3 k}{4\pi^3 k \perp^2}.
\]

It follows from (26) that the radiation spectral density does not depend on the curvature \( R \) of the de Sitter universe when \( R \to \infty, k \ll R \). Eq. (26) remains valid for arbitrary values of \( R \neq 0 \) if \( m \to 0 \). Thus, in the massless limit the radiation spectral density does not tend to zero. The behaviour of the spectral density in the massless limit is explained by the particle world line being different from a null geodesic when \( m \to 0 \), the particle experiencing an instantaneous acceleration at the moment \( \eta = 0 \).

Having executed, with the help of Eq. (2.16.33.2) from [7], the integration in (24) in \( k_0 \), we find the angular distribution of radiation of an electron in the de Sitter spacetime. The result has the form

\[
dE = \frac{3e^2}{128\pi} \frac{w d\Omega}{\sin^2 \theta},
\]

i.e. the energy emitted by the particle into the solid angle \( d\Omega \) for the whole time of the motion is proportional to an intrinsic acceleration of the particle relative to the Minkowski metric.

It follows from (27) that the total radiation energy for infinite time from the whole particle trajectory is infinite. The divergence of the total radiation energy for infinite time of motion is easy to be understood by taking into account that according to the Larmor formula the source radiation intensity is finite.

5 Conclusion

We have carried out the calculation of radiation spectrum of a classical charged particle in de Sitter spacetime. It seems to be of interest to compare Eq. (24) with the results of quantum-mechanical calculations of the electron radiation spectrum in the spacetime with the metric (1)–(2).

The results of quantum mechanical calculations obviously depend on the choice of the vacuum state of the quantum field. If, for example, we use the Bunch-Davies vacuum [1] as the initial and the final vacuum of the spinor field, then we obtain in the classical limit (\( w \to 0, k_0 \ll p, k_0/w = \text{const} \)) the spectral distribution of radiation energy to be

\[
dE_k = \frac{e^2}{16\pi w^2} \left[L_1^2\left(\frac{k \perp}{w}\right)-L_1\left(\frac{k \perp}{w}\right)\right] d^3k.
\]
where $I_1(z)$ is the modified Bessel function of the first kind and $L_{-1}(z)$ is the modified Struve function [6]. As we see, the classical limit of the electron radiation spectrum (28) does not coincide with (24). In our point of view, it means that the vacuum state definition in the de Sitter universe (1)–(2) requires a specification. However, in the present paper we would like to be restricted to ascertaining the fact that the use of the Bunch-Davies vacuum for calculations of quantum effects in the de Sitter universe leads to the results that, in the classical limit, contradict the results of the classical theory. A discussion of the problem of vacuum choice in the spacetime (1)–(2) exceeds the limits of our article. This problem and also the results of calculations of the electron radiation spectrum in the de Sitter universe in the framework of quantum electrodynamics in an external gravitational field will be considered in our next paper.

Acknowledgment

The authors would like to express their thanks to V.G. Bagrov and I.L. Buchbinder for helpful discussions.

References

[1] N.D. Birrell and P.C.W. Davies, “Quantum Fields in Curved Space-Time”, Cambridge Univ. Press, 1982.

[2] I.L. Buchbinder, E.S. Fradkin and D.M. Gitman, Fortschr. Phys. 29, 187 (1981)

[3] L.D. Landau and E.M. Lifshitz, “Field Theory”, Nauka, Moscow 1974 (in Russian).

[4] J.D. Jackson, ”Classical Electrodynamics”, John Wiley & Sons Inc., New-York-London 1962.

[5] A.I. Nikishov and V.I. Ritus, Zh. Eksp. Teor. Fiz 56, 2035 (1969).

[6] H. Bateman, “Higher Transcendental Functions”, McGraw-Hill 1953.

[7] A.P. Prudnikov, U.A. Brychkov and O.I.Marichev, “Integrals and Series. Special Functions”, Nauka, Moscow, 1983 (in Russian).

[8] L.I.Tsaregorodtsev, in: “Quantum Field Theory and Gravity. Second International Conference”, ed. I.L. Buchbinder and K.E. Osetrin, Tomsk 1998 (in Russian).