Efficient Contraction of Large Tensor Networks for Weighted Model Counting through Graph Decompositions

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Abstract. Constrained counting is a fundamental problem in artificial intelligence. A promising new algebraic approach to constrained counting makes use of tensor networks, following a reduction from constrained counting to the problem of tensor-network contraction. Contracting a tensor network efficiently requires determining an efficient order to contract the tensors inside the network, which is itself a difficult problem. In this work, we apply graph decompositions to find contraction orders for tensor networks. We prove that finding an efficient contraction order for a tensor network is equivalent to the well-known problem of finding an optimal carving decomposition. Thus memory-optimal contraction orders for planar tensor networks can be found in cubic time. We show that tree decompositions can be used both to find carving decompositions and to factor tensor networks with high-rank, structured tensors. We implement these algorithms on top of state-of-the-art solvers for tree decompositions and show empirically that the resulting weighted model counter is quite effective and useful as part of a portfolio of counters.

Keywords: Weighted Model Counting · Tensor Network Contraction · Tree Decomposition · Carving Decomposition

1 Introduction

Constrained counting is a fundamental problem in artificial intelligence, with applications in probabilistic reasoning, planning, inexact computing, engineering reliability, and statistical physics [3,16,24]. In constrained counting (also called weighted model counting) the task is to count the total weight, subject to a given weight function, of the set of solutions of input constraints. Even when the weight function is a constant function, constrained counting is #P-Complete [56]. Nevertheless, the development of tools that can successfully compute the total weight on large industrial formulas is an area of active research [43,55].

Constrained counting can be reduced to the problem of tensor-network contraction [7]. Tensor networks are a tool used across quantum physics and computer science for describing and reasoning about quantum systems, big-data processing, and more [5,11,42]. A tensor network describes a complex tensor as
a computation on many simpler tensors, and the problem of tensor-network contraction is to perform this computation. Although tensor networks can be seen as a variant of factor graphs [34], working directly with tensor networks allows us to leverage massive practical work in machine learning and high-performance computing on tensor contraction [12, 20, 32, 57] (which also includes GPU support [31, 41]) to perform constrained counting. Since tensor networks are relatively unknown in the artificial intelligence community, we give an introduction of relevant material on tensors and tensor networks in Section 3.

Contracting a tensor network requires determining an order to contract the tensors inside the network, and so efficient contraction requires finding a contraction order that minimizes computational cost. Since the number of possible contraction orders grows exponentially in the number of tensors, exhaustive algorithms, e.g. [44], cannot scale to handle the large tensor networks required for the reduction from constrained counting. Instead, recent work [33] gave heuristics that can sometimes find a “good-enough” contraction order. Finding efficient contraction orders for tensor networks remains an area of active research [45].

The primary contribution of this work is the application of heuristic graph-decomposition techniques to find efficient contraction orders for tensor networks that perform weighted model counting. Algorithms based on graph decompositions have been successful across computer science [26, 40], and their success in practice relies on finding good decompositions of arbitrary graphs. This, along with several recent competitions [15], has spurred the development of a variety of heuristics and tools for efficiently finding graph decompositions [2, 28, 54]. While we do not establish new parameterized complexity results for model counting (as fixed-parameter algorithms for model counting are well-known for a variety of parameters [23, 48]), we combine these theoretical results with high-performance tensor network libraries and with existing heuristic graph-decomposition tools to produce a competitive tool for weighted model counting.

We first discuss the Line-Graph method (LG) for finding efficient contraction orders. First applied to tensor networks by Markov and Shi [39], we contribute a new analysis that more closely matches the memory usage of existing tensor libraries. Our analysis provides two novel theoretical insights: (1) memory-efficient contraction orders are equivalent to low-width carving decompositions, and (2) tree decompositions can be used to find carving decompositions. Moreover, our analysis implies that memory-optimal contraction orders for planar tensor networks can be found in cubic time.

Although LG is a general-purpose method for finding contraction orders, LG cannot handle high-rank tensors and so cannot solve many existing counting benchmarks. We therefore contribute a novel method for finding efficient contraction orders, tailored for constrained counting: the Factor-Tree method (FT). FT factors high-rank, highly-structured tensors as a preprocessing step, leveraging prior work on Hierarchical Tucker representations [25].

In order to compare our approaches against other model counters (cachet [50], miniC2D [43], d4 [37], dynQBF [9], dynasp [21] and SharpSAT [55]) and other tensor-based methods, we implement LG and FT using three state-of-the-
art heuristic tree-decomposition solvers in TensorOrder, a new weighted model counter. LG outperforms other model counters and tensor-based methods on a set of unweighted benchmarks, while FT improves the virtual best solver on 21\% of a standard set of weighted model counting benchmarks. TensorOrder is thus useful as part of a portfolio of weighted model counters. Full proofs are provided in Section B of the appendix. All code, benchmarks, and detailed data of benchmark runs are available at [https://github.com/vardigroup/TensorOrder](https://github.com/vardigroup/TensorOrder).

# 2 Preliminaries: Graph Notations

A graph $G$ has a nonempty set of vertices $\mathcal{V}(G)$, a set of (undirected) edges $\mathcal{E}(G)$, a function $\delta_G : \mathcal{V}(G) \to 2^{\mathcal{E}(G)}$ that gives the set of edges incident to each vertex, and a function $\epsilon_G : \mathcal{E}(G) \to 2^{\mathcal{V}(G)}$ that gives the set of vertices incident to each edge. Each edge must be incident to exactly two vertices, but multiple edges can exist between two vertices.

A tree is a simple, connected, and acyclic graph. A leaf of a tree $T$ is a vertex of degree one, and we use $L(T)$ to denote the set of leaves of $T$. A rooted binary tree is a tree $T$ where either $T$ consists of a single vertex (called the root), or every vertex of $T$ has degree one or three except a single vertex of degree two (called the root). If $|\mathcal{V}(T)| > 1$, the immediate subtrees of $T$ are the two rooted binary trees that are the connected components of $T$ after the root is removed.

In this work, we use two decompositions of a graph as a tree: carving decompositions [52] and tree decompositions [46]. Both decompose the graph into an unrooted binary tree, which is a tree where every vertex has degree one or three. First, we describe carving decompositions [52]:

**Definition 1 (Carving Decomposition).** Let $G$ be a graph. A carving decomposition for $G$ is an unrooted binary tree $T$ whose leaves are the vertices of $G$, i.e. $L(T) = \mathcal{V}(G)$.

For every edge $a$ of $T$, deleting $a$ from $T$ yields exactly two trees, whose leaves define a partition of the vertices of $G$. Let $C_a \subseteq \mathcal{V}(G)$ be an arbitrary element of this partition. The width of $T$, denoted $\text{width}_c(T)$, is the maximum number of edges in $G$ between $C_a$ and $\mathcal{V}(G) \setminus C_a$ for all $a \in \mathcal{E}(T)$, i.e.,

$$\text{width}_c(T) = \max_{a \in \mathcal{E}(T)} \left| \left( \bigcup_{v \in C_a} \delta_G(v) \right) \cap \left( \bigcup_{v \in \mathcal{V}(G) \setminus C_a} \delta_G(v) \right) \right|.$$

The width of a carving decomposition $T$ with no edges is 0.

The carving width of a graph $G$ is the minimum width across all carving decompositions for $G$. Next, we define tree decompositions [46]:

**Definition 2 (Tree Decomposition).** Let $G$ be a graph. A tree decomposition for $G$ is an unrooted binary tree $T$ together with a labeling function $\chi : \mathcal{V}(T) \to 2^{\mathcal{V}(G)}$ that satisfies the following three properties:
1. Every vertex of $G$ is contained in the label of some vertex of $T$. That is, $V(G) = \bigcup_{n \in V(T)} \chi(n)$.

2. For every edge $e \in E(G)$, there is a vertex $n \in V(T)$ whose label is a superset of $\epsilon_G(e)$, i.e. $\epsilon_G(e) \subseteq \chi(n)$.

3. If $n$ and $o$ are vertices in $T$, and $p$ is a vertex on the path from $n$ to $o$, then $\chi(n) \cap \chi(o) \subseteq \chi(p)$.

The width of a tree decomposition, denoted $\text{width}_t(T, \chi)$, is the maximum size (minus 1) of the label of every vertex, i.e.,

$$\text{width}_t(T, \chi) = \max_{n \in V(T)} |\chi(n)| - 1.$$ 

The treewidth of a graph $G$ is the minimum width across all tree decompositions for $G$. The treewidth of a tree is 1. Treewidth is bounded by thrice the carving width \[51\]. Carving decompositions are the dual of branch decompositions \[46\].

3 An Introduction to Tensors and Tensor Networks

In this section, we introduce tensors and tensor networks and discuss prior work on the optimization of tensor-network contraction. To aid in exposition, along the way we build an analogy between the language of databases \[52\], the language of factor graphs \[34,13\], and the language of tensors: see Table 1.

3.1 Tensors

Tensors are a generalization of vectors and matrices to higher dimensions– a tensor with $r$ dimensions is a table of values each labeled by $r$ indices. An index is analogous to a variable in constraint satisfaction or an attribute in database theory.

Fix a set $\text{Ind}$ and define an index to be an element of $\text{Ind}$. For each index $i$ fix a finite set $[i]$ called the domain of $i$. An assignment to a set of indices $I \subseteq \text{Ind}$ is a function $\tau$ that maps each index $i \in I$ to an element of $[i]$. Let $[I]$ denote the set of assignments to $I$, i.e.,

$$[I] = \{ \tau : I \to \bigcup_{i \in I} [i] \text{ s.t. } \tau(i) \in [i] \text{ for all } i \in I \}.$$ 

We now define tensors as multidimensional arrays of values, indexed by assignments to a set of indices.\[4\]

**Definition 3 (Tensor).** A tensor $A$ over a finite set of indices (denoted $\mathcal{I}(A)$) is a function $A : [\mathcal{I}(A)] \to \mathbb{C}$ (where $\mathbb{C}$ is the set of complex numbers).

\[4\] In some works, a tensor is defined as a multilinear map and Definition \[3\] would be its representation in a fixed basis.
The rank of a tensor $A$ is the cardinality of $\mathcal{I}(A)$. The memory to store a tensor (in a dense way) is exponential in the rank. For example, a scalar is a rank 0 tensor, a vector is a rank 1 tensor, and a matrix is a rank 2 tensor. An example of a higher-rank tensor is the copy tensor on a set of indices $I$, which is the tensor $\text{COPY}_I : [I] \rightarrow \mathbb{C}$ such that, for all $\tau \in [I]$, $\text{COPY}_I(\tau) \equiv 1$ if $\tau$ is a constant function on $I$ and $\text{COPY}_I(\tau) \equiv 0$ otherwise [6].

It is common to consider sets of tensors closed under contraction (see Section 3.2), e.g. tensors with entries in $\mathbb{R}$ as in Section 7. Database tables under bag-semantics [10], i.e., multirelations, are tensors with entries in $\mathbb{N}$. Probabilistic database tables [8] are tensors with entries in $[0,1]$ that sum to 1.

Many tools exist (e.g. numpy\(^2\)) to efficiently manipulate tensors. In Section 7 we use these tools to implement tensor-network contraction, defined next.

### 3.2 Tensor Networks

A tensor network defines a complex tensor by combining a set of simpler tensors in a principled way. This is analogous to how a database query defines a resulting table in terms of a computation across many tables.

**Definition 4 (Tensor Network).** A tensor network $N$ is a nonempty set of tensors across which no index appears more than twice.

Free indices of $N$ are indices that appear once, while bond indices of $N$ are indices that appear twice. We denote the set of free indices of $N$ by $\mathcal{F}(N)$ and the set of bond indices of $N$ by $\mathcal{B}(N)$. The bond dimension of $N$ is the maximum size of $[i]$ for all bond indices $i$ of $N$.

The problem of tensor-network contraction, given an input tensor network $N$, is to compute the contraction of $N$ by marginalizing all bond indices:

**Definition 5 (Tensor Network Contraction).** The contraction of a tensor network $N$ is a tensor $\mathcal{T}(N)$ with indices $\mathcal{F}(N)$ (the set of free indices of $N$), i.e. a function $\mathcal{T}(N) : [\mathcal{F}(N)] \rightarrow \mathbb{C}$, that is defined for all $\tau \in [\mathcal{F}(N)]$ by

$$
\mathcal{T}(N)(\tau) \equiv \sum_{\rho \in [\mathcal{B}(N)]} \prod_{A \in \mathcal{N}} A((\rho \cup \tau)|_{\mathcal{I}(A)}),
$$

\(^2\) http://www.numpy.org/
Algorithm 1 Recursively contracting a tensor network

Input: A tensor network $N$ and a rooted binary tree $T$ whose leaves are the tensors of $N$, i.e. $L(T) = N$.
Output: $T(N)$, the contraction of $N$.

1: procedure $\text{Contract}(N, T)$
2: if $|N| = 1$ then
3: return the tensor contained in $N$
4: else
5: $T_1, T_2 \leftarrow$ immediate subtrees of $T$
6: $A_1 \leftarrow \text{Contract}(L(T_1), T_1)$
7: $A_2 \leftarrow \text{Contract}(L(T_2), T_2)$
8: return $A_1 \cdot A_2$

For example, the contraction of the tensor network $\{\text{COPY}_I, \text{COPY}_J\}$ is the tensor $\text{COPY}_{I \oplus J}$ (where $I \oplus J$ is the symmetric difference of $I$ and $J$). Notice that if a tensor network has no free indices then its contraction is a rank 0 tensor, i.e. a scalar. We write $A \cdot B$ to mean the contraction of the tensor network containing the two tensors $A$ and $B$.

Following our analogy, given a tensor network containing database tables (under bag-semantics) as tensors, its contraction is the join of those tables followed by the projection of all shared attributes. Thus a tensor network is analogous to a project-join query. A tensor network can also be seen as a variant of a factor graph \cite{34} with the additional practical restriction that no variable appears more than twice. The contraction of a tensor network corresponds to the marginalization of a factor graph \cite{47} and can similarly be seen as a special case of the FAQ problem \cite{1}. The restriction on the appearance is heavily exploited in tools for tensor-network contraction, since it allows tensor contraction to be implemented as matrix multiplication and thus leverage significant work in high-performance computing on matrix multiplication, both on the CPU \cite{38} and the GPU \cite{20}.

We focus in this work on tensor networks with relatively few (or no) free indices and hundreds or thousands of bond indices. Such tensor networks are obtained in a variety of applications \cite{11,17}, including the reduction from model counting to tensor network contraction \cite{7}. Although the rank of the contraction $T(N)$ is small in this case, computing entries by directly following Equation \ref{eq:contract} requires performing a summation over an exponential number of terms—one for each assignment in $|B(N)|$—and is therefore infeasible. $T(N)$ can instead be computed by recursively decomposing the tensor network, as in Algorithm \ref{alg:contract} \cite{19}. The choice of rooted binary tree $T$ does not affect the output of Algorithm \ref{alg:contract} but may have a dramatic impact on the running-time and memory usage. We explore this in more detail in the following section.

3.3 Contracting Tensor Networks

The rooted binary trees used by Algorithm \ref{alg:contract} are called contraction trees \cite{19}:
Definition 6 (Contraction Tree). Let $N$ be a tensor network. A contraction tree for $N$ is a rooted binary tree $T$ whose leaves are the tensors of $N$.

In our analogy, a contraction tree for a tensor network representing a project-join query is a join tree of that query (with projections done as early as possible).

The problem of tensor-network-contraction optimization, which we tackle in this paper, is given a tensor network $N$ to find a contraction tree that minimizes the computational cost of Algorithm 1. Several cost-based approaches aim for minimizing the total number of floating point operations to perform Algorithm 1: in step 8, e.g. [44] and the einsum package in numpy. In this work, we instead focus on structure-based approaches to tensor-network-contraction optimization, which analyze the rank of intermediate tensors that appear during Algorithm 1. These ranks indicate the amount of memory and computation required at each recursive stage. Moreover, these ranks are more amenable to analysis.

One line of work [39,18] uses graph decompositions to analyze the contraction complexity of a contraction tree: the maximum over all recursive calls of the sum (minus 1) of the rank of the two tensors contracted in step 8 of Algorithm 1. Contraction complexity measures the memory required when step 8 is computed by summing over each shared index sequentially. However, modern tensor packages (e.g. numpy) instead sum over all shared indices simultaneously, which requires the same number of floating point operations but often requires significantly less intermediate memory. Thus contraction complexity overestimates the memory requirements of many contraction trees.

Instead, another line of structure-based optimization analyzes the maximum rank over all recursive calls of the result of step 8 (and step 3). We call this the max rank of a contraction tree. Max rank measures the memory required when step 8 is computed by summing over all shared indices simultaneously. Thus max rank estimates the memory usage of modern tensor packages. Recent work [33] introduced three methods for heuristically minimizing the max rank: a greedy approach (called greedy), an approach using graph partitioning (called metis), and an approach using community-structure detection (called GN).

In this work, we improve on these methods by using graph decompositions to find contraction trees with small max rank.

4 From Weighted Model Counting to Tensor Networks

In this section, we introduce a framework for solving the problem of weighted model counting with tensor networks. The task in weighted model counting is to count the total weight, subject to a given (literal) weight function, of the set of solutions of input constraints. Formally:

Definition 7 (Weighted Model Count). Let $\varphi$ be a formula over Boolean variables $X$ and let $W : X \times \{0,1\} \rightarrow \mathbb{R}$ be a function (called the weight function). The weighted model count of $\varphi$ w.r.t. $W$ is

$$W(\varphi) = \sum_{\tau \in [X]} \varphi(\tau) \cdot \prod_{x \in X} W(x, \tau(x)).$$
Fig. 1. The tensor network (left) produced by Theorem 1 on $\varphi = (w \lor x \lor \neg y) \land (w \lor y \lor z) \land (\neg x \lor \neg y) \land (\neg y \lor \neg z)$, consisting of 8 tensors and 10 indices. Vertices in this diagram are tensors, while edges indicate that the tensors share an index. The weight function affects the entries of the tensors for $w$, $x$, $y$, and $z$. This tensor network has a contraction tree (right) of max rank 4, but no contraction trees of smaller max rank.

Note that $[X]$ is the set of all functions $\tau$ from $X$ to $\{0, 1\}$. Existing reductions from model counting to tensor-network contraction [7, 33] focus on the unweighted case (i.e., when $W$ is constant). Since we are interested in weighted model counting, we prove that the reduction can be easily extended:

**Theorem 1.** Let $\varphi$ be a CNF formula and let $W$ be a weight function. One can construct in polynomial time a tensor network $N_{\varphi}$ such that $F(N_{\varphi}) = \emptyset$ and the contraction of $N_{\varphi}$ is $W(\varphi)$.

**Sketch of Construction.** Include in $N_{\varphi}$ a tensor for each variable and for each clause of $\varphi$. Each variable tensor and clause tensor share an index if and only if the corresponding variable appears in the corresponding clause.

See Figure 1 for an example of the reduction. This reduction is closely related to the formulation of model counting as the marginalization of a factor graph representing the constraints. Unlike the reduction to factor-graph marginalization, which only assigns factors to clauses, we must also assign a tensor to each variable $x$. For example, if $x$ has weights $W(x, 0) = W(x, 1) = 1$ then the tensor assigned to $x$ is a copy tensor. This reduction can also be extended beyond OR clauses to other types of constraints (e.g., parity or cardinality constraints).

Theorem 1 suggests that the weighted model count of $\varphi$ can be computed by constructing and contracting $N_{\varphi}$. We present this framework as Algorithm 2.

Algorithm 2 is a fixed-parameter algorithm for model counting, parameterized by carving-width of the incidence graph. The existence of such algorithms is easily implied by fixed-parameter algorithms for model counting parameterized by treewidth [23, 48] since treewidth is bounded by thrice the carving width [51]. A variety of methods can be used in Step 2 to find a contraction tree to contract $N_{\varphi}$, including the methods LG and FT that we discuss in the following sections.

### 5 The Line-Graph Method for Finding Contraction Trees

The Line-Graph method for finding contraction trees for a tensor network $N$ applies graph-decomposition techniques to a particular graph constructed from
Algorithm 2 Computing the weighted model count with a TN

**Input:** A CNF formula \( \varphi \) and a weight function \( W \)

**Output:** \( W(\varphi) \), the weighted model count of \( \varphi \) w.r.t. \( W \)

1: \( N_\varphi \leftarrow \) tensor network constructed via Theorem 1
2: \( T \leftarrow \text{Find Contraction Tree}(N_\varphi) \) \( \triangleright \) e.g., LG or FG
3: \( \text{return } \text{Contract}(N_\varphi, T) \)

\( N \). Prior work \cite{39} on tensor networks with no free indices constructed a graph from a tensor network where tensors correspond to vertices and indices shared between tensors correspond to edges. In the context of constraint networks \cite{14}, this is analogous to the dual constraint graph (if multiple edges are drawn between constraints with multiple variables in common).

Although tensor networks constructed from weighted model counting instances do not have free indices, we utilize tensor networks with free indices as part of the preprocessing in Section 6 and so we need a more general graph construction that can handle free indices. Other works, e.g. \cite{59}, extend the graph construction of \cite{39} to tensor networks with free indices by treating free indices as “half-edges” (i.e., edges incident to one vertex), but decompositions of such graphs are not well-studied. In order to cleanly extend our decomposition-based analysis to tensor networks with free indices, in this work we instead add an extra vertex incident to all free indices, which we call the *free vertex*. We call this the *structure graph* of a tensor network:

**Definition 8 (Structure Graph).** Let \( N \) be a tensor network. The **structure graph** of \( N \) is the graph \( G \) whose vertices are the tensors of \( N \) and a fresh vertex \( z \) (called the *free vertex*) and whose edges are the indices of \( N \). Each tensor is incident to its indices, and \( z \) is incident to all free indices. That is, \( V(G) = N \sqcup \{ z \} \), \( E(G) = B(N) \cup F(N) \), \( \delta_G(A) = I(A) \) for all \( A \in N \), and \( \delta_G(z) = F(N) \).

Intuitively, the structure graph of a tensor network \( N \) captures how indices are shared by the tensors of \( N \). For example, on a CNF formula \( \varphi \) Theorem 1 produces a tensor network \( N_\varphi \) whose structure graph is exactly the *incidence graph* of \( \varphi \). Note that if \( N \) has no free indices, the free vertex has no incident edges and the remaining graph is exactly the graph analyzed in prior work.

### 5.1 Finding Contraction Trees from Carving Decompositions

Contraction trees are closely connected to decompositions of the structure graph. We prove in Theorem 2 that contraction trees of a tensor network correspond to carving decompositions of its structure graph, where max rank corresponds exactly to carving width:

**Theorem 2.** Let \( N \) be a tensor network with structure graph \( G \) and let \( w \in \mathbb{N} \). Then \( N \) has a contraction tree of max rank \( w \) if and only if \( G \) has a carving decomposition of width \( w \). Moreover, given one of these objects the other can be constructed in \( O(|N|) \) time.
Sketch of Construction. Given a carving decomposition for $G$ of width $w$, remove the leaf (and the incident edge) corresponding to the free vertex of $G$. This is a contraction tree for $N$ that has max rank $w$. Conversely, given a contraction tree for $N$ of max rank $w$, invert this transformation by attaching the free vertex of $G$ as a leaf incident to the root. This carving decomposition for $G$ has width $w$.

Carving decompositions have been studied in several settings. For example, there is an algorithm to find a carving decomposition of minimal width of a planar graph in time cubic in the number of edges [27]. It follows that if the structure graph of a tensor network $N$ is planar, one can construct a contraction tree of $N$ of minimal max rank in time $O(|B(N) \cup F(N)|^3)$.

There is limited work on the heuristic construction of “good” carving decompositions for non-planar graphs. Instead, we leverage the work behind finding tree decompositions to find carving decompositions and subsequently find contraction trees of small max rank.

5.2 Finding Contraction Trees from Tree Decompositions

One technique for join-query optimization [12,40] focuses on analysis of the join graph. The join graph of a project-join query consists of all attributes of a database as vertices and all tables in the join as cliques. In this approach, tree decompositions for the join graph of a query are used to find optimal join trees. The analogous technique on factor graphs analyzes the primal graph of a factor graph, which consists of all variables as vertices and all factors as cliques. Similarly, tree decompositions of the primal graph can be used to find variable elimination orders [30]. The graph analogous to join graphs and primal graphs for tensor networks is the line graph of the structure graph:

**Definition 9 (Line Graph).** The line graph of a graph $G$ is a graph $\text{Line}(G)$ whose vertices are the edges of $G$, and where the number of edges between each $e, f \in \mathcal{E}(G)$ is $|\mathcal{E}_G(e) \cap \mathcal{E}_G(f)|$, the number of endpoints shared between $e$ and $f$.

This technique was applied in the context of tensor networks by Markov and Shi [39], who proved that tree decompositions for $\text{Line}(G)$ (where $G$ is the structure graph of a tensor network $N$) can be transformed into contraction trees for $N$ of small contraction complexity. Specifically, tree decompositions of optimal width $w$ yield contraction trees of contraction complexity $w + 1$.

In the following theorem we analyze the max rank of the resulting contraction trees, which has not previously been studied. We present this result as a new relationship between carving width and treewidth:

**Theorem 3.** Let $G$ be a graph with $\mathcal{E}(G) \neq \emptyset$. Given a tree decomposition $T$ for $\text{Line}(G)$ of width $w \in \mathbb{N}$, one can construct in polynomial time a carving decomposition for $G$ of width no more than $w + 1$.

**Sketch of Construction.** For each $v \in \mathcal{V}(G)$, some vertex $n_v \in \mathcal{V}(T)$ must contain all edges incident to $v$ in its bag. Add vertices to $T$ to ensure that each $n_v$ is a unique leaf of $T$ and label each $n_v$ with $v$. These labels indicate a subtree of $T$ that is the desired carving-decomposition.
Applying Theorem 3 when $G$ is the structure graph of a tensor network (together with Theorem 2) gives us the Line-Graph method, which finds contraction trees by finding tree decompositions of the corresponding line graph. There are several advantages to our new analysis over the analysis of [39]: our analysis holds for tensor networks with free indices, and we analyze the max rank of contraction trees instead of the contraction complexity. Although the contraction complexity (and, for factor graphs, the width of the elimination order) is equal to one plus the width of the used tree decomposition, the max rank is smaller on some graphs; see Section A in the appendix for an experimental analysis of this.

6 The Factor-Tree Method for Finding Contraction Trees

Approaches to tensor-network contraction that do not modify the input tensor network (e.g., LG) are inherently limited by the ranks of the input tensors. If a tensor network has a rank $r$ tensor, then all contraction trees have max rank of at least $r$. This is a problem for tensor networks with high-rank tensors.

One example of tensor networks that may contain high-rank tensors are the networks obtained by the reduction from model counting. The tensor network produced from a formula $\phi$ contains a tensor representing each variable $x$, where the rank of this tensor is the number of appearances of $x$ in $\phi$ (e.g., the rank 4 tensor for $y$ in Figure 1). For many benchmarks, where a single variable might appear tens or even hundreds of times, this reduction will therefore produce tensor networks containing tensors of infeasibly-high rank. Reductions exist from model counting on arbitrary formulas to model counting on formulas where the number of appearances of each variables is small. However, existing reductions do not consider the carving width of the resulting incidence graph and so often do not significantly improve the max-rank of available contraction trees.

We introduce here a novel method Factor-Tree that avoids this barrier by preprocessing the input tensor network. Our insight is that a tree decomposition for the incidence graph of $\phi$ can be used as a guide to introduce new variables in a principled way, so that the resulting tensor network has good contraction trees. In the language of tensors, introducing new variables corresponds to factoring: replacing each high-rank tensor $A$ with a tensor network $N_A$ of low-rank tensors that contracts to $A$. The key idea of FT, then, is to use a tree decomposition for the structure graph to factor high-rank tensors.

We state this new result as Theorem 4. Since not all tensors can be factored in the ways that we require for this theorem and for FT, we first characterize the required property: that every tensor is factorable as an arbitrary tree of tensors:

**Definition 10.** A tensor $A$ is **tree factorable** if, for every tree $T$ whose leaves are $I(A)$ (called a dimension tree of $A$), there is a tensor network $N_A$ and a bijection $g_A: V(T) \rightarrow N_A$ s.t.

1. $A$ is the contraction of $N_A$,
2. \( g_A \) is an isomorphism between \( T \) and the structure graph of \( N_A \) with the free vertex (and incident edges) removed,

3. for every index \( i \) of \( A \), \( i \) is an index of \( g_A(i) \), and

4. for some index \( i \) of \( A \), the bond dimension of \( N_A \) is no bigger than \( \| [i] \| \).

All tensors in the reduction of Theorem 4 from weighted model counting to tensor networks are tree factorable. A tensor network \( N_A \) that satisfies properties 1, 2, and 3 of Definition 10 for some tree is called a Hierarchical Tucker representation of \( A \) [25]. Property 4 ensures the result of Theorem 4 has small bond dimension.

We now state the main result of this section, which allows us to use a tree decomposition for the structure graph of a tensor network (containing only tree factorable tensors) to factor each tensor in the network and find a contraction tree of low max rank for the resulting network:

**Theorem 4.** Let \( N \) be a tensor network of tree-factorable tensors such that \( |F(N)| \leq 3 \) and the structure graph of \( N \) has a tree decomposition \( T \) of width \( w \).

Then for each \( A \in N \) there is a tensor network \( N_A \) whose contraction is \( A \) that consists only of rank 3 or smaller tensors. Moreover, the disjoint union of these networks, \( N' = \cup_{A \in N} N_A \), is a tensor network that contracts to \( T(N) \), has the same bond dimension as \( N \), and has a contraction tree of max rank no larger than \( \lfloor 4(w + 1)/3 \rfloor \).

**Sketch of Construction.** For each \( i \in B(N) \cup F(N) \), \( e_G(i) \) must be a subset of the bag of some \( n_i \in V(T) \). Attach a new vertex \( n'_i \), whose bag is the same as \( n_i \), as a leaf to an arbitrary edge incident to \( n_i \). In this way, we can assume each \( n_i \) is a unique leaf of \( T \).

Next, for each \( A \in N \), the smallest connected component of \( T \) containing \( \{n_i : i \in I(A)\} \) is a dimension tree \( T_A \) of \( A \) (where each \( n_i \) indicates the leaf of index \( i \)). Factor \( A \) with \( T_A \) using Definition 10 to get \( N_A \) and \( g_A \).

We now construct the contraction tree for \( M = \cup_{A \in N} N_A \). For each \( n \in V(T) \), let \( M_n = \{B : B \in N_A, g_A(B) = n\} \). At each leaf \( \ell \in L(T) \), attach an arbitrary contraction tree of \( M_\ell \). At each non-leaf \( n \in V(T) \), partition \( M_n \) into three equally-sized sets and attach an arbitrary contraction tree for each to the three edges incident to \( n \). Next, if \( |F(N)| = 3 \), then without loss of generality the vertices of \( T \) whose bag contains the free vertex of \( N \) form a subtree of \( T \) with exactly one vertex \( n' \in V(T) \) of degree 3. If \( |F(N)| < 3 \), let \( n' \) instead be an arbitrary vertex of \( T \) whose bag contains the free vertex of \( N \). Attach the free vertex of \( M \) as a leaf at \( n' \). These attachments create a carving decomposition from \( T \) for the structure graph of \( M \), of width no larger than \( \lfloor 4(w + 1)/3 \rfloor \).

Finally, apply Theorem 2 to construct a contraction tree for \( M \).

A similar technique was applied in the context of constraint satisfaction by Samer and Szeider [19]. Their construction can be translated to tensor networks to produce a factored network \( N' \) with structure graph \( G \) that satisfies all requirements of Theorem 4 except with a bound of \( w \) on the treewidth of \( G \) in place of the bound on max-rank. Since the treewidth of \( \text{Line}(G) \) plus 1 is bounded by the product of the maximum degree of \( G \) (namely 3) and the treewidth of \( G \).
plus 1 [39], we can combine [19] with LG to produce a contraction tree for $N'$ of max rank no larger than $3(w + 1)$. Theorem [4] thus gives an improvement over [19] on max-rank from $3(w + 1)$ to $\lceil 4(w + 1)/3 \rceil$.

The construction of Theorem 4 gives us the Factor-Tree method, which uses tree decompositions of the structure graph to preprocess the tensor network and factor high-rank tensors. See Figure 2 for an example of the preprocessing.

We show in Section 7.3 that FT can significantly improve the quality of the contraction tree on benchmarks with high-rank tensors.

### 7 Implementation and Evaluation

We implement Algorithm 2 in TensorOrder, a new tool for weighted model counting using tensor networks. TensorOrder can be configured to perform Step 2 using one of the methods from Kourtis et al. [33] (greedy, metis, and GN) or one of the methods presented in this paper (LG and FT). Implementation details are given in Section 7.1.

We use TensorOrder to compare tensor-based methods with existing state-of-the-art tools for weighted model counting: cachet [50], miniC2D [10] and d4 [37]. We also compare with dynQBF [9], dynasp [21] and SharpSAT [55] when the benchmarks are unweighted. Note dyneQBF and dynasp are solvers from related domains (that can be used as model counters) that also use tree decompositions.

We compare TensorOrder on two sets of existing benchmarks. First, in Section 7.2 we compare on formulas that count the number of vertex covers of randomly-generated cubic graphs [33]. Second, in Section 7.3 we compare on formulas whose weighted count is exact inference on Bayesian networks [50].

Each experiment was run in a high-performance cluster (Linux kernel 2.6.32) using a single 2.80 GHz core of an Intel Xeon X5660 CPU and 48 GB RAM. Each implementation was run once on each benchmark with a timeout of 1000 seconds. We provide all code, benchmarks, and detailed data of benchmark runs at [https://github.com/vardigroup/TensorOrder](https://github.com/vardigroup/TensorOrder).
7.1 Implementation Details of TensorOrder

TensorOrder is implemented in Python 3.6. All tensor contractions are performed using numpy 1.15 and 64-bit double precision floats. TensorOrder also supports infinite-precision integer arithmetic, but the performance is significantly degraded by limited numpy support. Note that numpy is able to leverage SIMD parallelism for tensor contraction.

Both LG and FT require first finding a tree decomposition. To do this, we leverage three heuristic tree-decomposition solvers: Tamaki [54], FlowCutter [28], and htd [2]. TensorOrder therefore has three implementations of LG (LG+Tamaki, LG+Flow, and LG+htd) and three implementations of FT (FT+Tamaki, FT+Flow, and FT+htd) for different choices of solver.

All the tree-decomposition solvers we consider are online solvers and so each implementation must decide how long to run the solver (this time is included in the measured running time). TensorOrder estimates the time to contract each potential contraction tree (using techniques from the einsum package of numpy) and continues to look for better tree decompositions until it expects to have spent more than half of the running time on finding a tree decomposition. This strikes a balance between improving and using the contraction trees.

7.2 Counting Vertex Covers of Cubic Graphs

We first compare on benchmarks that count the number of vertex covers of randomly-generated cubic graphs [33]. In particular, for each number of vertices \( n \in \{50, 60, 70, \cdots, 250\} \) we randomly sample 100 connected cubic graphs using a Monte Carlo procedure [58]. These benchmarks are monotone 2-CNF formulas in which every variable appears 3 times.

Results on these benchmarks are summarized in Figure 3. For ease of presentation, we display only the best-performing of the LG and FT implementations: LG+Flow. We observe that tensor-based methods are fastest when \( n \geq 110 \). On large graphs our contribution LG+Flow is fastest and able to find the lowest max-rank contraction trees. LG+Flow is the only implementation able to solve at least 50 benchmarks within 1000 seconds when \( n = 220 \).

7.3 Weighted Model Counting: Exact Inference

We next compare on a set of weighted model counting benchmarks from Sang, Beame, and Kautz [50]. These 1091 benchmarks are formulas whose weighted model count corresponds to exact inference on Bayesian networks. We first evaluate numerical accuracy, since our approach uses 64-bit double precision floats: on all benchmarks that miniC2D also finishes, the weighted model count returned by our approaches agrees within \( 10^{-3} \).

Results on these benchmarks are summarized in Figure 4. Our implementations of FT each solve fewer benchmarks than cachet, miniC2D, and d4. Nevertheless, FT+* are together able to solve 231 benchmarks faster than existing counters (FT+Flow is fastest on 175, FT+Tamaki is fastest on 50, and
Fig. 3. Median solving time (top) and max-rank of the computed contraction tree (bottom) of various model counters and tensor-based methods on counting the number of vertex covers of 100 cubic graphs with $n$ vertices. Solving time of datapoints that ran out of time (1000 seconds) or memory (48 GB) are not shown. Our contribution \textbf{LG+Flow} is faster than all other methods when $n \geq 170$. \textbf{LG+Flow} finds contraction trees of lower max-rank than all other tensor-based methods when $n \geq 170$.

\textbf{FT+htd} is fastest on 6), including 62 benchmarks on which \texttt{cachet}, \texttt{miniC2D}, and \texttt{d4} all time out. This significantly improves the virtual best solver (VBS) when \textbf{FT+*} are included.

The tensor-based methods (\textbf{LG}, \textbf{greedy}, \textbf{metis}, and \textbf{GN}) that do not perform factoring were only able to count a single benchmark from this set within 1000 seconds. We observe that most of these benchmarks have a variable that appears many times, which significantly hinders tensor-based methods that do not perform factoring (see Section 6).
We conclude from these experiments that both \textbf{LG} and \textbf{FG} are useful as part of a portfolio of weighted model counters.

8 Conclusions and Future Work

We presented two methods, \textbf{LG} and \textbf{FT}, for using graph decompositions to find contraction trees of small max rank of tensor networks. \textbf{LG} is a general-purpose method for finding contraction orders, while \textbf{FT} is tailored for constrained counting to handle high-rank, highly-structured tensors. We evaluated \textbf{LG} and \textbf{FT} in the context of exact weighted model counting and demonstrated that \texttt{TensorOrder} is useful as part of a portfolio of weighted model counters. It would be interesting in the future to analyze the types of benchmarks amenable to tensor-network methods, e.g. by computing lower bounds on carving width in addition to the upper bounds given by heuristic methods. It would also be interesting to explore the impact of other preprocessing techniques (e.g., PMC \cite{hutter2014pruning} or B+E \cite{fichte2016exact}) on carving width and treewidth.

Although we restricted our experiments to a single core, a variety of libraries exist for efficiently performing tensor contractions on multiple cores or on GPUs \cite{bammes2017efficient,roberts2020comparative}. One direction for future work is to analyze and improve the potential parallelism of tensor-based algorithms. This would allow comparison against other recent GPU-based counters, including Fichte \textit{et al.} \cite{fichte2020parallel} which also uses graph decompositions.

Tensor-based methods can also be used to count other classes of CSPs. For example, all techniques we introduced in this work would have similar performance computing the weighted model count of formulas that mix OR clauses.
with XOR clauses and Exactly-One clauses (as such clauses can also be represented as tree-factorable tensors). More generally, our algorithms for tensor-network contraction can be used to improve many other applications of tensor networks. Evaluating our techniques on a wider collection of tensor networks is an exciting direction for future work.

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Appendix

A  A Comparison of Treewidth and Carving Width

In this section, we perform an experimental comparison of treewidth and carving width of the incidence graphs of the model counting benchmarks in Section 7.

A1 Experimental Setup

On each incidence graph $G$, we ran each of the three heuristic tree-decomposition solvers integrated with \textsc{TensorOrder–Tamaki}, \textsc{FlowCutter}, and \textsc{htd–} for 1000 seconds on $G$ and $\text{Line}(G)$ and recorded the width of the best tree decomposition found amongst all tree-decomposition solvers. On each tree decomposition for $\text{Line}(G)$ found by the solvers, we used \textsc{LG} to compute the corresponding carving decomposition of $G$ and recorded the smallest width found amongst all decompositions. Similarly, on each tree decomposition for $G$ found by the solvers, we used \textsc{FT} to compute the corresponding carving decomposition of the preprocessed graph and recorded the smallest width found amongst all decompositions.

Unlike the experiments in Section 7, we do not estimate the time to contract each potential contraction tree. Instead, we run each tree-decomposition solver for the full 1000 seconds on each benchmark. This allows us to more fully estimate the treewidth and carving width of these benchmarks and so more fully evaluate the potential of decomposition solvers. Each experiment was run in a high-performance cluster (Linux kernel 2.6.32) using a single 2.80 GHz core of an Intel Xeon X5660 CPU and 48 GB RAM.

A2 Results

We first compare treewidth and carving width using the benchmarks from Section 7.2: randomly-generated cubic graphs [33]. Since \textsc{FT} only factors tensors of order 4 or higher and all vertices in each cubic graph has exactly three incident edges, \textsc{FT} performs no factoring on these graphs. Thus both \textsc{LG} and \textsc{FT} can be used to find carving decompositions.

Results on these benchmarks are summarized in Figure A1. We observe that, for most large graphs, the carving width of $G$ is smaller than the treewidth of $G$, which is smaller that the treewidth of $\text{Line}(G)$. On these benchmarks, the width of the carving decompositions of $G$ found by \textsc{LG} are indeed smaller than the upper bound guaranteed by Theorem 3 of the width of the used tree decomposition plus one.

We next compare treewidth and carving width using the benchmarks from Section 7.3: incidence graphs of probabilistic inference benchmarks [50]. Results on these benchmarks are summarized in Figure A2 and Figure A3.

We observe that the carving width of $G$ found by \textsc{LG} is extremely large on these benchmarks (larger than 50 on 1064 of the 1091 benchmarks), since these graphs have vertices of high degree. Nevertheless, the carving width of $G$
Fig. A1. Median of the best upper bound found for treewidth and carving width of 100 cubic graphs with \( n \) vertices. Each graph-decomposition solver was run for 1000 seconds. For most large graphs, the carving width of \( G \) is smaller than the treewidth of \( G \), which is smaller than the treewidth of \( \text{Line}(G) \).

Fig. A2. A histogram comparing an upper bound on the treewidth of \( \text{Line}(G) \) (the width of the best tree decomposition found across all solvers after 1000 seconds) with the width of the best carving decomposition of \( G \) constructed by \( \text{LG} \), across 1091 probabilistic inference benchmarks. On most of these benchmarks, no solver is able to find a tree decomposition of \( \text{Line}(G) \) of width smaller than 50 and so \( \text{LG} \) is unable to find carving decompositions of \( G \) of width smaller than 50.
Fig. A3. A histogram comparing an upper bound on the treewidth of $G$ (the width of the best tree decomposition found across all solvers after 1000 seconds) with the width of the best carving decomposition of preprocessed constructed by FT, across 1091 probabilistic inference benchmarks. On most benchmarks, the tree decompositions found of $G$ have significantly smaller width than the tree decompositions found of Line($G$). Thus the carving decompositions produced by FT are significantly better than those produced by LG.

found by LG is still smaller than the upper bound guaranteed by Theorem 3 of the treewidth of Line($G$) plus one on most benchmarks. We also observe that FT does significantly reduce the carving width by preprocessing the graph (to smaller than 50 on 1066 of the 1091 benchmarks). Moreover, the carving width found by FT is smaller than the upper bound guaranteed by Theorem 4 on most benchmarks.

B Proofs

In this section we present proofs of the theorems stated in the main body of the paper. Throughout, we often refer to a vertex of a tree as a node and an edge as an arc to avoid confusion, since our proofs will frequently work simultaneously with a graph and an associated tree.

B1 A Proof of Theorem 1

In this section, we present a proof of Theorem 1. We additionally prove the claim (made in Section 6) that every tensor used in Theorem 1 is tree factorable. In order to prove this additional claim, it is convenient to state and prove another characterization of tree-factorable tensors.
Lemma 1. Let $A$ be a tensor such that, for every nonempty $J \subseteq \mathcal{I}(A)$, there is a fresh index $i$ and two tensors $B$ and $C$ such that $A = B \cdot C$ and:

1. $|i| \leq |j|$ for some $j \in J$,
2. $\mathcal{I}(B) = \{i\} \cup J$,
3. $\mathcal{I}(C) = \{i\} \cup \mathcal{I}(A) \setminus J$,
4. $B$ is tree factorable, and
5. $C$ is tree factorable.

Then $A$ is tree factorable.

Proof. Let $T$ be a dimension tree of $A$. If $T$ has two or fewer leaves, then $|\mathcal{I}(A)| \leq 2$ and so $A$ is tree factorable.

Otherwise, there is an arc $a \in \mathcal{E}(T)$ such that deleting $a$ from $T$ partitions $T$ into two trees $T_1$ and $T_2$ whose leaves are sets $J \subseteq \mathcal{I}(A)$ and $\mathcal{I}(A) \setminus J$, both nonempty. Apply the hypothesis to obtain a fresh index $i$ and two tree-factorable tensors $B$ and $C$ such that $A = B \cdot C$ and all four properties above are satisfied. Moreover, adding $i$ as a leaf to $T_1$ at $a$ produces a dimension tree for $B$, and similarly adding $i$ as a leaf to $T_2$ at $b$ produces a dimension tree for $C$.

Apply Definition 10 to $B$ to produce a tensor network $N_B$ and a bijection $g_B : V(T_1) \to N_B$. Apply Definition 10 to $C$ to produce a tensor network $N_C$ and a bijection $g_C : V(T_2) \to N_C$. One can check that $N_A = N_B \cup N_C$ and $g = g_B \cup g_C$ satisfies the properties of Definition 10 and therefore $A$ is tree factorable.

We now prove Theorem 1, which we restate as the following lemma (including the additional claim from Section 6 that every tensor is tree factorable).

Lemma 2. Let $\varphi$ be a CNF formula over Boolean variables $X$ and let $W : X \times \{0, 1\} \to \mathbb{R}$ be a weight function. One can construct in polynomial time a tensor network $N_\varphi$ such that $F(N_\varphi) = \emptyset$, the contraction of $N_\varphi$ is $W(\varphi)$, and every tensor in $N_\varphi$ is tree factorable.

Proof. For each clause $C$ in $\varphi$, let $\sup(C)$ be the set of Boolean variables that appear in $C$. Define $I = \{(x, C) : C \in \varphi, x \in \sup(C)\}$ to be a set of indices, each with domain $\{0, 1\}$. That is, $I$ has an index for each appearance of each variable in $\varphi$. For each $x \in X$, define $A_x : [I \cap \{x\} \times \varphi] \to \mathbb{R}$ to be the following tensor:

$$\tau \mapsto \begin{cases} W(x, 1) & \text{if } \tau \text{ is the constant function } i \mapsto 1 \\ W(x, 0) & \text{if } \tau \text{ is the constant function } i \mapsto 0 \\ 0 & \text{otherwise.} \end{cases}$$

Next, for each $C \in \varphi$, define $B_C : [I \cap (\{x\} \times \varphi)] \to \mathbb{R}$ to be the following tensor:

$$\tau \mapsto \begin{cases} 1 & \text{if } \{x : x \in \sup(C) \text{ and } \tau((x, C)) = 1\} \text{ satisfies } C \\ 0 & \text{otherwise.} \end{cases}$$
Let \( N_\varphi = \{ A_x : x \in X \} \cup \{ B_C : C \in \varphi \} \). Observe that each index \((x, C)\) appears in \(A_x\) and \(B_C\), so \(N_\varphi\) is indeed a tensor network with \(\mathcal{F}(N_\varphi) = \emptyset\).

Observe that \(\{ A_x : x \in X \}\) contracts to a tensor \(A : [I] \rightarrow \mathbb{R}\) such that \(A(\tau) = 0\) whenever \(\tau(x, C) \neq \tau(x, D)\) for some \(C, D \in \varphi\), and otherwise \(A(\tau) = \prod_{x \in X} W(x, \tau(x, C_x))\), where each \(C_x \in \varphi\) is an arbitrary clause that contains \(x\). Thus \(A\) captures the weights of the weighted model counting computation.

Similarly, observe that \(\{ B_C : C \in \varphi \}\) contracts to a tensor \(B : [I] \rightarrow \{0, 1\}\) such that \(B(\tau) = 0\) whenever some clause \(C\) is not satisfied by truth assignment indicated by \(\tau\), and 1 whenever all clauses are satisfied. Thus \(B\) captures the tensor \(\varphi\). It follows that the contraction of \(A\) and \(B\) is indeed \(W(\varphi)\).

It remains to show that each \(A_x\) and \(B_C\) are tree factorable. For a set of indices \(J \subseteq I\) and \(a, b \in \mathbb{R}\), define \(g_{\text{copy}}(J, a, b)\) to be the tensor \([J] \rightarrow \mathbb{R}\) defined by

\[
\tau \mapsto \begin{cases} 
    a & \text{if } \tau \text{ is the constant function } 1 \\
    b & \text{if } \tau \text{ is the constant function } 0 \\
    0 & \text{otherwise.}
\end{cases}
\]

Notice that \(A_x = g_{\text{copy}}(I \cap \{(x) \times \varphi\}, W(x, 1), W(x, 0))\). Moreover, for every nonempty \(K \subseteq J\) observe that

\[
g_{\text{copy}}(J, a, b) = g_{\text{copy}}(\{i\} \cup K, a, b) \cdot g_{\text{copy}}(\{i\} \cup J \setminus K, 1, 1)
\]

where \(i\) is a fresh index with domain \(\{0, 1\}\). By induction using Lemma 1 it follows that \(g_{\text{copy}}(J, a, b)\) is tree factorable for every \(J \subseteq I\) and \(a, b \in \mathbb{R}\), and thus \(A_x\) is tree factorable for every \(x \in X\) as well.

Next, for a set of indices \(J \subseteq I\) and a fresh index \(i\) (with domain \(\{0, 1\}\)), define \(\text{OR}(J, i)\) to be the tensor \([J \cup \{i\}] \rightarrow \mathbb{R}\) defined by

\[
\tau \mapsto \begin{cases} 
    \tau(i) & \text{if } \tau(j) = 1 \text{ for some } j \in J \\
    1 - \tau(j) & \text{otherwise.}
\end{cases}
\]

Moreover, for every nonempty \(K \subseteq J\) observe that

\[
\text{OR}(J, i) = \text{OR}(K, k) \cdot \text{OR}(\{k\} \cup J \setminus K, i)
\]

where \(k\) is a fresh index with domain \(\{0, 1\}\). By induction using Lemma 1 it follows that \(\text{OR}(J)\) is tree factorable for every \(J \subseteq I\) and index \(i \notin J\).

Moreover, for an index \(i\) with \([i] = \{0, 1\}\) define \(\text{CAP}(i)\) to be the tensor \([\{i\}] \rightarrow \mathbb{R}\) that maps \(\tau\) to \(\tau(i)\). Notice that, for every \(C \in \varphi\), \(B_C = \text{OR}(I \cap (\sup(\varphi) \times \{C\}), i) \cdot \text{CAP}(i)\) for a fresh index \(i\) (with domain \(\{0, 1\}\)). It follows that \(B_C\) is tree factorable as well.

### B2 A Proof of Theorem 2

In this section we prove Theorem 2, which ultimately follows from Lemma 4 and Lemma 5.
For the proofs of these lemmas, we first observe that the structure graph of a tensor network \( N \) allows you to compute the set of free indices of each arbitrary subset of \( N \). Thus the structure graph contains all information necessary to compute the max order of a contraction tree of \( N \). We formalize this as the following lemma. Note that for a graph \( G \) and \( V \subseteq \mathcal{V}(G) \) we define \( \delta_G(V) \) to be the set of edges of \( G \) incident to some vertex in \( V \).

**Lemma 3.** Let \( N \) be a tensor network with structure graph \( G \). If \( N' \subseteq N \) is nonempty, then \( N' \) is a tensor network and \( F(N') = \delta_G(N') \cap \delta_G(\mathcal{V}(G) \setminus N') \).

**Proof.** \( N' \) is a tensor network since \( N \) is. Let \( z \) be the free vertex of \( G \).

Consider an arbitrary free index \( i \in F(N') \). Since \( i \) is an index of exactly one \( A \in N' \), we have \( i \in \delta_G(N') \). Moreover, \( i \) is either a free or bond index of \( N \). If \( i \) is a bond index of \( N \), \( i \) must also be an index of some \( B \in N \setminus N' \) and so \( i \in \delta_G(N \setminus N') \). If \( i \) is a free index of \( N \), then \( i \in \delta_G(z) \). In either case, \( i \in \delta_G(\mathcal{V}(G) \setminus N') \) and so \( i \in \delta_G(N') \cap \delta_G(\mathcal{V}(G) \setminus N') \).

Conversely, consider an arbitrary \( i \in \delta_G(N') \cap \delta_G(\mathcal{V}(G) \setminus N') \). Since \( i \in \delta_G(N') \), \( i \) is the index of some tensor of \( N' \). Since \( i \in \delta_G(\mathcal{V}(G) \setminus N') \), either \( i \in \delta_G(z) \) (and so \( i \) appears exactly once in \( N \)) or \( i \) is also an index of some tensor in \( N \setminus N' \). In either case, since \( N \) is a tensor network \( i \) cannot appear twice in \( N' \) and so \( i \in F(N') \).

We now prove the first part of Theorem 2, namely that carving decompositions can be constructed from contraction trees. The construction is straightforward: given a contraction tree, attach the free vertex as a leaf connected to the root of the contraction tree. The following lemma shows the width of the resulting carving decomposition is the max order of the input contraction tree.

**Lemma 4.** Let \( N \) be a tensor network with structure graph \( G \). Let \( T \) be a contraction tree of \( N \) of max-width \( w \). Construct \( T' \) from \( T \) by adding the free vertex \( z \in \mathcal{V}(G) \) as a leaf and adding an arc from \( z \) to the root of \( T \). Then \( T' \) is a carving decomposition of \( G \) and \( \text{width}_c(T) = w \).

**Proof.** By construction, \( T' \) is an unrooted binary tree and \( \mathcal{L}(T') = \mathcal{L}(T) \cup \{ z \} = \mathcal{V}(G) \), so \( T' \) is indeed a carving decomposition of \( G \). It remains to check the width of \( T' \). To do this, define \( \text{next} : \mathcal{V}(T) \to \mathcal{E}(T') \) by defining, for each \( n \in \mathcal{V}(T) \), \( \text{next}(n) \) to be the first arc on the path from \( n \) to \( z \) in \( T' \). Since \( \text{next} \) is a bijection, the width of \( T' \) is

\[
\text{width}_c(T') = \max_{a \in \mathcal{E}(T')} |\delta_G(C_a) \cap \delta_G(\mathcal{V}(G) \setminus C_a)| = \max_{n \in \mathcal{V}(T)} |\delta_G(C_{\text{next}(n)}) \cap \delta_G(\mathcal{V}(G) \setminus C_{\text{next}(n)})|.
\]

There is a bijection between the nodes of \( T \) and the recursive calls of Algorithm 1, namely where each \( n \in \mathcal{V}(T) \) is matched with the recursive call where \( n \) is the root. Consider an arbitrary \( n \in \mathcal{V}(T) \) and let \( N_n \) be the block of the partition \( \{ C_{\text{next}(n)}, \mathcal{V}(G) \setminus C_{\text{next}(n)} \} \) that does not contain \( z \). By construction,
\( T(N_n) \) is the tensor returned from the recursive call of Algorithm 1 when \( n \) is the root.

By Lemma 3, \( F(N_n) = \delta_G(N_n) \cap \delta_G(V(G) \setminus N_n) \). Plugging in \( N_n \), we get that \( F(N_n) = \delta_G(C_{\text{next}(n)}) \cap \delta_G(V(G) \setminus C_{\text{next}(n)}) \). Thus the order of \( T(N_n) \) is exactly \( |\delta_G(C_{\text{next}(n)}) \cap \delta_G(V(G) \setminus C_{\text{next}(n)})| \).

It follows that the width of \( T' \) is exactly the max order of \( T \), as desired.

The construction described in Lemma 4 is invertible, since \( T \) can be reobtained from \( T' \) by removing the free vertex of \( G \) (and incident arc). The second part of Theorem 3.3 (that contraction trees can be constructed from carving decompositions) is thus a corollary of Lemma 4.

**Lemma 5.** Let \( N \) be a tensor network with structure graph \( G \). Let \( T \) be a carving decomposition of \( G \). Construct \( T' \) from \( T \) by removing the free vertex \( z \in V(G) \) (and the arc in \( T \) incident to \( z \)) from \( T \). Then \( T' \) is a contraction tree of \( N \) and the max order of \( T' \) is the width of \( T \).

**Proof.** Notice \( \mathcal{L}(T') = \mathcal{L}(T) \setminus \{z\} = N \) and \( T' \) is a rooted binary tree (whose root is the node previously attached by an arc to \( z \)). Thus \( T' \) is a contraction tree of \( N \). Moreover, applying the construction from Lemma 4 to \( T' \) produces \( T \), and so the max order of \( T' \) is \( \text{width}_e(T) \).

### B3 A Proof of Theorem 3

For the proof of Theorem 3, it will be convenient to focus on tree decompositions for which the labels of the internal nodes are unimportant. In particular, we would like to be able to find tree decompositions so that we can check properties 1 and 2 of a tree decomposition without considering the labels of internal nodes. We formalize this in the following lemma, which we use in Theorem 3 and Theorem 4.

**Lemma 6.** Let \( G \) be a graph with no vertices of degree 0, let \( (T, \chi) \) be a tree decomposition of \( G \), and let \( f : A \to 2^{V(G)} \) for some finite set \( A \). Assume that:

1. For every \( e \in E(G) \), there is some \( a \in A \) with \( \epsilon_G(e) \subseteq f(a) \).
2. For every \( a \in A \), there is some \( n \in V(T) \) with \( f(a) \subseteq \chi(n) \).

Then we can construct in polynomial time a tree decomposition \((S, \psi)\) of \( G \) such that \( \text{width}_e(S, \psi) \leq \text{width}_e(T, \chi) \) and there is a bijection \( g : A \to \mathcal{L}(S) \) such that \( \psi \circ g = f \).

**Proof.** Let \( a \in A \) and let \( n_a \in V(T) \) be an arbitrary node with \( f(a) \subseteq \chi(n_a) \). Choose an arbitrary arc \( b \in \epsilon_T(n_a) \) and construct \( T' \) from \( T \) by attaching a new leaf \( \ell_a \) at \( b \) (and introducing a new internal node). We can extend \( \chi \) into a labeling \( \chi' : V(T') \to 2^{E(G)} \) by labeling the new internal node with \( \chi(n_a) \) and labeling the new leaf node with \( f(a) \). Note that \((T', \chi')\) is still a tree decomposition of \( G \) of width \( \text{width}_e(T, \chi) \), that all labels of leaves of \((T, \chi)\) are still labels of leaves of \((T', \chi')\), and that the new leaf of \((T', \chi')\), namely \( \ell_a \), is labeled by \( f(a) \).
By repeating this process for every \( a \in A \), we can prove by induction that we produce in polynomial time a tree decomposition \((T', \chi')\) of width \( \text{width}_h(T, \chi) \). Moreover, we can define a function \( g : A \to \mathcal{L}(T') \) given by \( g(a) = \ell_a \) (the new leaf attached at each step). By construction, \( \chi' \circ g = f \) (since each \( \ell_a \) was labeled by \( f(a) \)) and moreover \( f \) is an injection (since a new leaf was introduced at each step). It remains to make \( f \) a bijection by removing leaves of \( T' \).

In particular, notice that properties (1) and (2) of a tree decomposition can be satisfied purely looking at the nodes of \( T' \) in the range of \( g \). Moreover, removing leaves of \( T' \) cannot falsify property (3) of a tree decomposition. Thus we can repeatedly remove leaves of \( T' \) not in the range of \( g \) until we eventually reach a tree decomposition \((S, \psi)\) for \( G \) whose leaves are exactly the range of \( g \). Thus \( g \) is a bijection as a function onto \( \mathcal{L}(S) \) and \( \psi \circ f = \delta_G \). Moreover, since \( \psi(V(S)) \subseteq \chi'(V(T')) \) it follows that \( \text{width}_h(S, \psi) \leq \text{width}_h(T', \chi') = \text{width}_h(T, \chi) \) as desired.

We now prove Theorem 3 which is stated as Lemma 7. The key idea of the proof is to observe that in a tree decomposition \((T, \chi)\) of \( \text{Line}(G) \), \( \delta_G(v) \) must be a subset of one of the labels of \((T, \chi)\) for every \( v \in V(G) \). We use Lemma 8 to assume WLOG that \( \delta_G(v) \) must, in fact, be the label of some leaf of \((T, \chi)\). Thus we can replace, for each \( v \in V(G) \), the leaf labeled by \( \delta_G(v) \) with \( v \) to obtain a carving decomposition \( T' \) of \( G \). Moreover, the width of \( T' \) is bounded by the size of the labels of \((T, \chi)\).

In this lemma we use one additional notation: given a graph \( G \) and a subset of vertices \( V \subseteq V(G) \), we use \( G \cap V \) to denote the largest subgraph of \( G \) whose vertices are \( V \), i.e. the graph \( G \cap V \) such that \( V(G \cap V) = V \), \( E(G \cap V) = \{ e \in E(G) : \epsilon_G(e) \subseteq V \} \), and \( \delta_{G \cap V}(v) = \delta_G(v) \cap E(G \cap V) \) for every \( v \in V \).

**Lemma 7.** Let \( G \) be a graph with \( \mathcal{E}(G) \neq \emptyset \). If \((T, \chi)\) is a tree decomposition of \( \text{Line}(G) \), then we can construct in polynomial time a carving decomposition \( T' \) of \( G \) such that \( \text{width}_h(T') \leq \text{width}_h(T, \chi) + 1 \).

**Proof.** WLOG, we may assume that \( \text{Line}(G) \) has no vertices of degree 0 (vertices in \( \text{Line}(G) \) of degree 0 are an edge in \( G \) between two vertices of degree 1, so we can construct a carving decomposition of each pair of these vertices, use this lemma to find a carving decomposition of the remaining graph, and finally attach each carving decomposition at an arbitrary arc of \( T' \) by introducing two new internal nodes).

We first aim to apply Lemma 6 with \( G = \text{Line}(G) \), \( A = V(G) \), and \( f = \delta_G \). To verify the first required property for Lemma 6 observe for every \( (v, F) \in \text{Line}(G) \) that \( \epsilon_{\text{Line}(G)}((v, F)) = F \subseteq \delta_G(v) \). To verify the second required property, consider an arbitrary vertex \( v \in V(G) \). Define \( \chi_v : V(T) \to 2^{\delta_G(v)} \) by \( \chi_v(n) = \chi(n) \cap \delta_G(v) \) for all \( n \in V(T) \). Notice that \((T, \chi_v)\) is a tree decomposition of \( \text{Line}(G) \cap \delta_G(v) \), a complete graph with \( |\delta_G(v)| \) vertices. Since the treewidth of a complete graph with \( k \) vertices is \( k - 1 \), it follows that \( \text{width}_h(T, \chi_v) \geq |\delta_G(v)| - 1 \). That is, there is some \( n_v \in V(T) \) such that \( |\chi_v(n_v)| \geq |\delta_G(v)| \). It follows that \( \chi_v(n_v) = \delta_G(v) \) and so \( \delta_G(v) \subseteq \chi(n_v) \).

By Lemma 6 there is therefore a tree decomposition \((S, \psi)\) of width no larger than \((T, \chi)\) and a bijection \( g : V(G) \to \mathcal{L}(S) \) such that \( \psi \circ g = \delta_G \).
Construct $T'$ from $S$ by replacing every leaf $\ell \in \mathcal{L}(S)$ with $g^{-1}(\ell)$. Since $g$ is a bijection, $\mathcal{L}(T') = \mathcal{V}(G)$ and so $T'$ is a carving decomposition. It remains to check that $\text{width}_c(T') \leq \text{width}_c(T, \chi) + 1$.

Consider an arbitrary arc $a \in \mathcal{E}(T')$ and let $C_a$ be a block of the partition of $\mathcal{V}(G)$ defined by removing $a$. Consider an arbitrary edge $e \in \delta_G(C_a) \cap \delta_G(\mathcal{L}(T') \setminus C_a)$. It follows that there are vertices $v \in C_a$ and $w \in \mathcal{L}(T') \setminus C_a$ that are both incident to $e$. Thus $e \in \delta_G(v) \cap \delta_G(w)$, which together with $\chi \circ g = \delta_G$ implies that $e \in \chi(g(v)) \cap \chi(g(w))$. Property 3 of tree decompositions implies that $e$ must also be in the label of every node in the path from $g(v)$ to $g(w)$ in $T$; in particular, $e$ is in the label of both endpoints of $a$. Since $e$ was arbitrary, $\delta_G(C_a) \cap \delta_G(\mathcal{L}(T') \setminus C_a)$ is a subset of the label of both endpoints of $a$. Thus $|\delta_G(C_a) \cap \delta_G(\mathcal{L}(T') \setminus C_a)| \leq \text{width}_c(T, \chi) + 1$.

Since $a$ was arbitrary, it follows that $\text{width}_c(T') \leq \text{width}_c(T, \chi) + 1$ as desired.

**B4 A Proof of Theorem 4**

At its core, the proof of Theorem 4 relies on a graph construction. Our goal is to expand each vertex $v$ of the structure graph $G$ of a tensor network into a tree so that the resulting graph $H$ has small carving width. We do this in Lemma 8 below. To prove Theorem 4 we then use the tree (in the lemma below, $H \cap H_v$) for each vertex $v$ as a guide to factor the tensor $v$, so that the resulting structure graph is isomorphic to $H$ and thus the resulting tensor network has small carving width.

The proof of Lemma 8 can be divided into three parts. First, we construct $H$, the subsets $H_v$, and the injection $f$. Second, we construct a carving decomposition of $H$. Third, we compute the width of the carving decomposition.

**Lemma 8.** Let $G$ be a graph with a tree decomposition of width $w$. There is a graph $H$, subsets $H_v \subseteq \mathcal{V}(H)$ for each $v \in \mathcal{V}(G)$, and an injection $f : \mathcal{E}(G) \rightarrow \mathcal{E}(H)$ such that:

1. The sets $H_v$ form a partition of $\mathcal{V}(H)$,
2. For each $v \in \mathcal{V}(G)$, $H \cap H_v$ is a tree with $|\delta_G(v)|$ leaves, and each leaf is incident to exactly one edge in the range of $f$,
3. For all $e \in \mathcal{E}(G)$, we have $e \in \delta_G(v) \cap \delta_G(w)$ if and only if $f(e) \in \delta_H(H_v) \cap \delta_H(H_w)$, and
4. The carving width of $H$ is no larger than $[4(w + 1)/3]$.

**Proof.** **Part 1: Constructing $H$.** WLOG, we may assume that $G$ has no vertices of degree 0 (for each $v \in \mathcal{V}(G)$ of degree 0, add $v$ as a vertex of degree 0 to $H$ and take $H_v = \{v\}$). By Lemma 6 applied to the input tree decomposition (with $f = \epsilon_G$), there is a tree decomposition $(T, \chi)$ of width no more than $w$ so that there is a bijection $g : \mathcal{E}(G) \rightarrow \mathcal{L}(T)$ that satisfies $\epsilon_G = \chi \circ g$.

For each $v \in \mathcal{V}(G)$, let $V_v = \{n \in \mathcal{V}(T) : v \in \chi(n)\}$ be the set of nodes of $T$ whose label contains $v$. By property 3 of tree decompositions, $T \cap V_v$ is a tree. Let $T_v$ be the smallest connected subgraph of $T \cap V_v$ that contains every leaf in
the first element of \( J \) between \( J \) and \( I \) consecutive elements of \( \delta_G(v) \). Then \( T_v \) is a tree and \( g|_{\delta_G(v)} \) is a bijection between the edges incident to \( v \) and the leaves of \( T_v \).

We construct \( H \) by taking a copy of \( T_v \) for each \( v \in V(G) \) and connecting these copies where indicated by \( g \). Formally, the vertices of \( H \) are \( \{(v, n) : v \in V(G), n \in V(T_v)\} \). Since \( V(T_v) \subseteq V(T) \) for each \( v \in V(G) \), we have \( V(H) \subseteq V(G) \times V(T) \). Thus there are two projections \( \pi_G : V(H) \to V(G) \) and \( \pi_T : V(H) \to V(T) \), indicating respectively the first or second component of a vertex of \( H \). For each \( v \in V(G) \), define \( H_v = \pi^{-1}_G(v) \). Thus the sets \( H_v \) form a partition of \( V(H) \).

For every \( v \in V(G) \) and every arc in \( T \) between \( n, m \in V(T_v) \), we add an edge between \((v, n)\) and \((v, m)\). This ensures that \( H \cap H_v \) is isomorphic to \( T_v \) and so \( H \cap H_v \) is a tree with \(|\delta_G(v)| \) leaves. Moreover, for each \( e \in E(G) \) incident to \( v, w \in V(G) \), we add an edge \( f(e) \) between \((v, g(e))\) and \((w, g(e))\). The map \( f : E(G) \to E(H) \) constructed in this way is an injection and satisfies property 3 above. Moreover, since \( g|_{\delta_G(v)} \) is exactly the leaves of \( T_v \), each leaf \( \ell \in E(H \cap H_v) \) is incident to exactly one edge in the range of \( f \), namely \( f(g^{-1}(\ell)) \).

**Part 2: Constructing a carving decomposition \( S \) of \( H \).** The idea of the carving decomposition \( S \) is to attach the elements of \( \pi^{-1}_T(n) \) as leaves along arcs incident to \( n \in V(T) \) and so transform \( T \) into a carving decomposition.

To do this, we need to choose for each \( x \in V(H) \) some arc \( \alpha(x) \in \delta_T(\pi_T(x)) \). For each \( v \in V(G) \) and each vertex \((v, \ell) \in H_v \) that is a leaf of \( H \cap H_v \), we define \( \alpha((v, \ell)) \) to be the unique arc in \( T \) incident to \( \ell \). For each \( v \in V(G) \) and \((v, n) \in H_v \) of degree 2 in \( H \cap H_v \), let \((v, m) \in H_v \) be an arbitrary neighbor of \((v, n) \) and define \( \alpha((v, n)) \) to be the arc between \( n \) and \( m \).

We will consider all vertices \((v, n)\) of degree 3 corresponding to the same \( n \in T \) as a group. In particular, for each \( n \in T \) of degree 3 in \( T \) we define the set \( C = \{v : n \in T_v, (v, n) \text{ has degree } 3 \text{ in } H \cap H_v\} \). Observe that \(|C| \leq |\pi^{-1}_T(n)| \leq |\chi(n)| \leq w + 1 \). We partition the elements of \( C \) into three sets, denoted \( C_{n,a} \) for each \( a \in \delta_T(n) \), such that \(|C_{n,a}| \leq \lfloor |C|/3 \rfloor \) and so \(|C_{n,a}| \leq \lfloor (w+1)/3 \rfloor \) for each \( a \in \delta_T(n) \). Finally, define \( \alpha((v, n)) = a \) for every \( a \in \delta_T(n) \) and \( v \in C_{n,a} \).

We have now assigned every vertex \( x \in V(H) \) an arc \( \alpha(x) \) incident to \( \pi_T(x) \). We use this to construct a carving decomposition \( S \) from \( T \) by adding each \( x \in V(H) \) as a leaf along the arc \( \alpha(x) \) of \( T \). Formally, for each \( x \in V(H) \) let \( n_x \) denote a fresh vertex. The vertices of \( S \) are the vertices of \( T \), the elements of \( H \), and \( \{n_x : x \in V(H)\} \). We add an arc in \( S \) between \( x \) and \( s_x \) for every \( x \in V(H) \).

For each arc \( a \) in \( T \), let \( \alpha, \beta \in V(T) \) be the nodes incident to \( a \). Construct an arbitrary sequence \( I \) from the elements of \( \{n_x : x \in \alpha^{-1}(a)\} \) and an arbitrary sequence \( J \) from the elements of \( \{n_x : x \in \alpha^{-1}(a)\} \). Add arcs between consecutive elements of \( I \) and consecutive elements of \( J \). If \( I \) and \( J \) are both empty, add an arc \( a' \) between \( a \) and \( p \). Otherwise, if \( I \) is empty add an arc \( a' \) between \( a \) and the first element of \( J \), and an arc between the last element of \( J \) and \( p \) (and similarly if \( J \) is empty). If neither \( I \) nor \( J \) are empty, add an arc between \( a \) and the first element of \( J \), an arc \( a' \) between the last element of \( I \) and the first element of \( J \), and an arc between the last element of \( J \) and \( p \).
Finally, remove the previous leaves of $T$ from $S$ and contract their neighbors of degree 2. The resulting unrooted binary tree $S$ is a carving decomposition of $H$, since we have added all vertices of $H$ as leaves and removed the previous leaves of $T$.

**Part 3: Computing the width of $S$.** Since all vertices of $H$ are degree 3 or lower, all of the arcs between $x \in \mathcal{V}(H)$ and $n_x$ define a partition of width at most $3 \leq [4(w+1)/3]$.

Let us consider the arcs we added to replace each arc $a \in \mathcal{E}(T)$ between $p, o \in \mathcal{V}(T)$. Observe that $a$ defines a partition $\{B_o, B_p\}$ of $\mathcal{V}(T)$, denoted so that $o \in B_o$ and $p \in B_p$. First, consider the arc $a'$, which defines the partition $\{\pi_T^{-1}(B_o), \pi_T^{-1}(B_p)\}$ of $\mathcal{V}(H)$. Since all edges of $H$ follow the tree structure of $T$, all edges in $H$ between $\pi_T^{-1}(B_o)$ and $\pi_T^{-1}(B_p)$ are between $\pi_T^{-1}(o)$ and $\pi_T^{-1}(p)$. Since $\pi_G(\pi_T^{-1}(o)) \subseteq \chi(o)$ and $\pi_G(\pi_T^{-1}(o)) \subseteq \chi(p)$, it follows that the partition defined by $a'$ has width no larger than $|\chi(o) \cap \chi(p)| \leq w + 1$.

Now consider an arc $b$ added between $o$ and $a'$ (or equivalently between $p$ and $a'$). Some elements of $\pi_T^{-1}(o)$ have changed blocks from the partition defined by $a'$. Elements of degree 2 changing blocks does not affect the total number of edges between the partitions, but each element of degree 3 that changes blocks increases the number of shared edges by 1. There are $|C_{o,a}| \leq [(w+1)/3]$ elements of degree 3 added as leaves between $o$ and $a'$, and so the partition defined by $b$ has width at most $w + 1 + [(w+1)/3] = [4(w+1)/3]$.

It follows that the width of $S$ is at most $[4(w+1)/3]$, and so the carving width of $H$ is at most $[4(w+1)/3]$. 

We now prove Theorem 4, which is stated as the following lemma.

**Lemma 9.** Let $N$ be a tensor network of tree-factorable tensors s.t. $|\mathcal{F}(N)| \leq 3$ and the structure graph of $N$ has a tree decomposition of width $w$.

Then for each $A \in N$ there is a tensor network $N_A$ whose contraction is $A$ that consists only of rank 3 or smaller tensors. Moreover, the disjoint union of these networks, $\bigcup_{A \in N} N_A$, is a tensor network that contracts to $\mathcal{T}(N)$, has the same bond dimension as $N$, and has a contraction tree of max order no larger than $[4(w+1)/3]$.

**Proof.** Let $G$ be the structure graph of $N$ with free vertex $z$. We begin by applying Lemma 4 to $G$ to obtain a graph $H$, a partition $\{H_v : v \in \mathcal{V}(G)\}$ of the vertices of $H$, and an injection $h : \mathcal{E}(G) \rightarrow \mathcal{E}(H)$ satisfying the properties of Lemma 4.

Consider each $A \in N$ and observe that by property 2 of Lemma 4 for every leaf $\ell \in \mathcal{L}(H \cap H_A)$ there is exactly one $i_{\ell} \in \mathcal{E}(G)$ s.t. $\ell \in e_H(h(i_{\ell}))$. Moreover, property 3 of Lemma 4 implies that $i_{\ell} \in \delta_G(A) = \mathcal{L}(A)$. Construct a tree by replacing every leaf $\ell \in \mathcal{L}(H \cap H_A)$ by $i_{\ell}$ and apply Definition 10 to obtain a tensor network $N_A$ and a bijection $h_A : H_A \rightarrow N_A$ s.t.:

1. $N_A$ contracts to $A$,
2. $h_A$ is an isomorphism between $H \cap H_A$ and the structure graph of $N_A$ with the free vertex (and all incident edges) removed,
3. for every index $i$ of $A$, $i$ is an index of $h_A(g(i))$, and,
4. The bond dimension of $N_A$ is smaller than the bond dimension of $N$.

It follows from property 1 that $M = \cup_{A \in N} N_A$ is a tensor network that contracts to $T(N)$. It follows from property 4 that $M$ has the same bond dimension as $N$.

Let $G'$ be the structure graph of $M$, with free vertex $z'$. We construct $H'$ from $H$ by contracting all vertices of $H_f$ into a single vertex $z'$. Since $|F(N)| \leq 3$, $H \cap H_f$ is a tree with at most 3 leaves and so $H_f$ contains at most one vertex of degree 3 in $H$. It follows that the carving width of $H'$ is no larger than the carving width of $H$.

Define $h' : V(H') \to V(G')$ by mapping $x$ to $h_A(x)$ for each $x \in H_A$ and mapping $z'$ to $z'$. By construction, $h'$ is an isomorphism between $H'$ and $G'$ (edges within each $H_A$ are maintained since each $h_A$ is an isomorphism, and edges between $H_A$ and $H_B$ are maintained by property 3 of Lemma 8 and property 3 of Definition 10). It follows that the carving width of $G'$ is no larger than the carving width of $H$, and so by Theorem 5 $M$ has a contraction tree of max order no larger than $[4(w+1)/3]$. 