Strictly increasing and decreasing sequences in subintervals of words and a conjecture of Guo and Poznanović

Jonathan S. Bloom
Dan Saracino

June 15, 2022

Abstract

We prove a conjecture of Guo and Poznanović concerning chains in certain 01-fillings of moon polyominoes. A key ingredient of our proof is a correspondence between words $w$ and pairs $(W(w), M(w))$ of increasing tableaux such that $M(w)$ determines the lengths of the longest strictly increasing and strictly decreasing sequences in every subinterval of $w$. We define this correspondence by using Thomas and Yong’s $K$-infusion operator and then use it to obtain the bijections that prove the conjecture of Guo and Poznanović. In constructing our bijections we introduce new variants of the RSK correspondence and Knuth equivalence.

1 Introduction

The motivation for the work presented in this paper was to prove a conjecture of Guo and Poznanović ([7], Conjecture 15) about chains in certain 01-fillings of moon polyominoes. We state the conjecture after recalling some definitions and related results, and we prove the conjecture in Section 5.

Let $S$ be the set of square boxes in the $xy$-plane determined by the points of $\mathbb{Z} \times \mathbb{Z}$. A polyomino is a finite subset of $S$. A polyomino $M$ is a moon polyomino if it contains every box situated between any two of its boxes that are in the same row or column, and for any two of its rows (or columns), one is a subset of the other, up to translation. Adopting the terminology of Guo and Poznanović, we call a moon polyomino a stack polyomino if its rows are left justified. (We note that some authors use the phrase stack polyomino to refer to a moon polyomino whose columns are bottom justified.) A Ferrers board is a stack polyomino in which the lengths of the rows are weakly decreasing from top to bottom.

By a 01-filling of a moon polyomino we mean an assignment of either a 0 or a 1 to each box of the polyomino. We sometimes refer to the boxes containing 0’s as empty. A sequence of 1’s in a filling of $M$ is a chain in $M$ if the smallest rectangle containing all the 1’s in the
sequence is completely contained in $M$. A chain is called an \textit{ne chain} if each 1 in the chain is strictly above and strictly to the right of all the preceding 1’s in the chain. In an analogous way, we define \textit{se chains}. The \textit{length} of a chain is the number of 1’s in the chain.

A number of authors have contributed results about the enumeration of fillings of polyominoes with restrictions on the maximum lengths of chains of various types. Specializing to 01-fillings with at most one 1 in each row and at most one 1 in each column (which can be dealt with by considering permutations), Krattenthaler \cite{Krattenthaler10} proved the following: For any Ferrers board $F$ and any positive integers $n, s, t$, the number of such fillings of $F$ with exactly $n$ 1’s and longest ne and se chains of lengths $s$ and $t$, respectively, equals the number of such fillings of $F$ with exactly $n$ 1’s and longest ne and se chains of lengths $t$ and $s$, respectively. He then showed that this result implies the main results of \cite{JonssonWelker9} about crossings and nestings in matchings and set partitions, and went on to obtain extensions of his result for general 01-fillings and fillings with nonnegative integer entries. His motivation for proving these extensions was the result of Jonsson \cite{Jonsson8} and Jonsson and Welker \cite{JonssonWelker9}, which states that the total number of 01-fillings of a stack polyomino $M$ with exactly $n$ 1’s and longest ne chains of length $s$ depends only on the multiset of heights of the columns of $M$. While Jonsson used a complicated induction argument, and Jonsson and Welker used commutative algebra, Krattenthaler sought to use the growth diagrams of Fomin and variants of the RSK correspondence to obtain their result. Although he did not completely succeed in doing so, he paved the way for the work of Rubey \cite{Rubey11}.

By supplementing Krattenthaler’s approach with the new idea of promotion of fillings of rectangular shapes, Rubey was able to reprove the result of Jonsson and Welker, and indeed to prove its generalization to moon polyominoes, which had been conjectured by Jonsson. Rubey was also able to prove many additional results involving weakly ne and se chains.

In \cite{Krattenthaler10}, Problem 2, Krattenthaler asked whether his result about interchanging the lengths of the longest ne and se chains would hold in the context of all 01-fillings, instead of just those fillings having at most one 1 in each row and column. A counterexample was provided by de Mier in \cite{DeMier4}, and it follows that, in the context of all 01-fillings, the result of Jonsson and Welker for stack polyominoes cannot be strengthened so as to fix the lengths of the longest ne chains \textit{and} the lengths of the longest se chains.

In \cite{GuoPoznanovic7}, however, Guo and Poznanović proved that the strengthened version of Jonsson and Welker’s result for stack polyominoes does hold in the context of 01-fillings with at most one 1 in each column (which can be dealt with by considering words). They conjectured (\cite{GuoPoznanovic7}, Conjecture 15) that their theorem would hold for all moon polyominoes.

We prove the Guo and Poznanović conjecture in this paper. To state our result precisely, we adopt the following notation of Guo and Poznanović. For any moon polyomino $M$ and positive integers $n, u, v$, we let $N(M; n; ne = u, se = v)$ denote the number of 01-fillings of $M$ with exactly $n$ 1’s such that each column contains at most one 1, and the lengths of the longest ne and se chains in $M$ are $u$ and $v$, respectively.

\textbf{Theorem 1.1.} For every moon polyomino $M$ and all positive integers $n, u, v$ the quantity $N(M; n; ne = u, se = v)$ depends only on the multiset of lengths of the rows of $M$.

Note that two moon polyominoes have the same multiset of lengths of rows if and only
if they have the same multiset of heights of columns. Guo and Poznanović phrased their results in terms of rows, while Jonsson and Welker phrased theirs in terms of columns.

We need some terminology to describe our approach to the proof of Theorem 1.1. Let \( \mathcal{W}_n \) be the set of all words of length \( n \) whose letters are positive integers. For any word \( u \) define \( \text{lis}(u) \) (respectively, \( \text{lds}(u) \)) to be the length of a longest strictly increasing (respectively, decreasing) subsequence in \( u \). We say two words \( u, w \in \mathcal{W}_n \) have the same \textit{CID data} and write \( u \sim_{\text{cid}} w \) provided that for any choice of \( i \leq j \leq n \) we have

\[
\text{lis}(u_i \cdots u_j) = \text{lis}(w_i \cdots w_j) \quad \text{and} \quad \text{lds}(u_i \cdots u_j) = \text{lds}(w_i \cdots w_j).
\]

(We note that “CID” stands for Column Increasing Decreasing, and “Column” refers to the columns in the representation of \( w \in \mathcal{W}_n \) as a 01-filling of a rectangular polyomino with one 1 in each column.) If we only know the first (respectively, second) equality above holds we write \( u \sim_{\text{ci}} w \) (respectively, \( u \sim_{\text{cd}} w \)). We say two words \( u, w \in \mathcal{W}_n \) have the same \textit{RID data} and write \( u \sim_{\text{rid}} w \) if, for every integer interval \( [a, b] \), we have \( \text{lis}(u') = \text{lis}(w') \) and \( \text{lds}(u') = \text{lds}(w') \), where \( u' \) and \( w' \) are the words obtained from \( u \) and \( w \) by deleting all integers not in \( [a, b] \).

Our proof of Theorem 1.1, which is bijective, rests on developing a correspondence between words and pairs of certain increasing tableaux, such that if \( w \mapsto (\mathcal{W}(w), \mathcal{M}(w)) \) then \( \mathcal{W}(u) = \mathcal{W}(w) \) implies \( u \sim_{\text{rid}} w \) and \( \mathcal{M}(u) = \mathcal{M}(w) \) implies \( u \sim_{\text{cid}} w \).

This may be of interest in its own right, since it may be viewed as an extension of known results. It is known that if permutations \( u, w \) have the same Robinson-Schensted (RS) insertion tableau \( P \) then they have the same RID data. (This follows from Lemmas 2, 3, and Exercise 1 in Chapter 3 of [5].) By considering the symmetry property of RS, it follows that if two permutations have the same RS recording tableau then they have the same CID data.

When one tries to understand CID or RID data in the more general context of words, one can try to appeal to the Robinson-Schensted-Knuth (RSK) correspondence, but this provides only partial information. From the same results of [5] in the previous paragraph, it follows that if two words \( u, w \) have the same RSK insertion tableau then \( u \sim_{\text{rd}} w \). By considering the symmetry property of RSK for 2-line arrays, it follows that if words \( u, w \) have the same RSK recording tableau then \( u \sim_{\text{cd}} w \). (This weakened conclusion results from the fact that words, unlike permutations, can have repeated letters.) In fact, the RSK tableaux do not determine strictly increasing data. For example, if \( u = 32311, w = 32211 \) then their shared recording tableau is

\[
\begin{array}{ccc}
1 & 3 \\
2 & 5 \\
4 \\
\end{array}
\]

but we see that \( u \not\sim_{\text{ci}} w \) as the 2nd and 3rd columns of \( u \) have longest increasing sequence length 2, while in \( w \) the same columns have longest increasing sequence length 1.

In proving their result about stack polyominoes, Guo and Poznanović used a variant of the RSK algorithm, called Hecke insertion, which was developed in [1]. It was shown in [2, 6] that if two words have the same Hecke insertion tableau then they do have the same RID data. But it is not true that the Hecke recording tableau (which is a set-valued tableau)
determines CID data. For example, take \( u = 13221, w = 23231 \) which both have the Hecke recording tableau

\[
\begin{array}{cc}
1 & 2, 4 \\
3 & 5
\end{array}
\]

and note that \( u \circ_{ci} w \) as the 3rd and 4th columns of \( u \) have longest increasing sequence length 1, while in \( w \) the same columns have longest increasing sequence length 2.

We will obtain our new correspondence \( w \mapsto (W(w), M(w)) \) that captures both RID and CID data by using the K-infusion operator introduced by Thomas and Yong in \[15\], along with new variants of the RSK algorithm and Knuth equivalence.

The outline of our paper is as follows. In Section 2 we review the ideas of K-Knuth equivalence and K-infusion. In Section 3 we establish the two bijections that we use to prove the conjecture of Guo and Poznanović, modulo the proof that if words \( u \) and \( w \) are such that \( M(u) = M(w) \) then \( u \sim_{cid} w \). The proof of this fact occupies Section 4 and relies on our new variants of the RSK algorithm and Knuth equivalence, which are introduced in that section. Finally, in Section 5 we prove the conjecture of Guo and Poznanović.

2 K-theoretic tools

2.1 K-Knuth equivalence

Definition 2.1. We say two words \( u \) and \( w \) are \( K\)-Knuth equivalent and write \( u \equiv_K w \) provided that one can be obtained from the other by a finite number of the following moves:

\[
\begin{align*}
\cdots bca\cdots & \leftrightarrow \cdots bac\cdots \quad (a < b < c) \\
\cdots acb\cdots & \leftrightarrow \cdots cab\cdots \quad (a < b < c) \\
\cdots a\cdots & \leftrightarrow \cdots aa\cdots \\
\cdots aba\cdots & \leftrightarrow \cdots bab\cdots.
\end{align*}
\]

We will need the following two results about K-Knuth equivalence.

Fact 2.2. ([2], Lemma 5.5) If \( u \) and \( w \) are \( K\)-Knuth equivalent words and \([a, b]\) is an integer interval, then the words obtained from \( u \) and \( v \) by deleting all integers not in \([a, b]\) are \( K\)-Knuth equivalent.

Fact 2.3. ([2], Proposition 37) If two words are \( K\)-Knuth equivalent, then the lengths of the longest strictly decreasing subsequences in these words are the same, and the lengths of the longest strictly increasing subsequences are the same.
2.2 K-Infusion

Throughout this subsection let $\lambda \subseteq \mu$ be partitions, which we view as Ferrers boards. We define an *inner corner* of the skew shape $\mu/\lambda$ to be a maximally southeast box in $\lambda$. We shall also need the notion of an outer corner. To define this, let $\nu$ be any rectangular partition such that $\nu_1 > \mu_1$ and the length of $\nu$ is greater than the length of $\mu$. Then an *outer corner* of $\mu/\lambda$ is a maximally northwest box in $\nu/\mu$.

A filling $T$ of the skew shape $\mu/\lambda$ with positive integers is called a *semistandard Young tableau* if the entries are weakly increasing along rows and strictly increasing along columns, from top to bottom. A filling $T$ is called an *increasing tableau* if the entries are strictly increasing along both rows and columns. We write $\text{sh}(T) = \mu/\lambda$ and refer to this as the *shape* of $T$. In the case $\lambda = \emptyset$ we say $T$ has *straight* shape. We denote by $\text{Inc}(\mu/\lambda)$ the set of all increasing tableaux of shape $\mu/\lambda$. We denote by $\text{Inc}^{st}(\mu/\lambda)$ the set of elements of $\text{Inc}(\mu/\lambda)$ whose set of entries constitute an integer interval with left endpoint 1.

A specific increasing tableau of interest is the *minimal tableau of shape* $\lambda$, which is the increasing tableau of shape $\lambda$ where each box is filled with the smallest positive integer possible. For each $n > 0$ we define the *staircase shape* to be the partition $\sigma_n = (n, n - 1, \ldots, 1)$ and define $M_n$ to be the minimal tableau of shape $\sigma_n$. For example, $M_4$ is the increasing tableau

```
1 2 3 4
2 3 4
3 4
4
```

Next we recall the idea of $K$-theoretic jeu de taquin, which was first introduced and studied in [15]. For any increasing tableau $T$ of shape $\mu/\lambda$ let $B$ be a set of inner corners. We then define the increasing tableau $K\text{jdt}_B(T)$ to be the tableau given by the following inductive procedure. First place a $\bullet$ in each of the inner corners in $B$. Now, identify all the $\bullet$’s that are directly above or to the left of a box containing a 1. Replace these dots with 1’s and replace those 1’s, that were below or to the right of a dot, with a dot. Next, repeat this process for all dots directly above or to the left of a 2, then for all dots directly above or to the left of a 3, and so on, until no dot has an entry below or to its right. Then delete the dots. For example, consider the following increasing tableau

```
    2 3
 1 2 3 6
2 4 5
```

where the selected inner corners are denoted with $\bullet$’s. Then the steps of the $K$-theoretic jeu
de taquin procedure are

and \( K_{\lambda}(T) \) is the increasing tableau on the right, minus the dots.

Naturally, we also have the notion of reverse \( K \)-theoretic jeu de taquin. Again we start with an increasing tableau \( S \) of shape \( \mu / \lambda \) with maximal entry \( k \). This time we take \( C \) to be a set of outer corners for \( S \) and define \( Krjdt_C(S) \) as follows. First, place a dot \( \bullet \) in each outer corner from \( C \) and find all the dots that are directly below or to the right of a box containing a \( k \). Replace these dots with \( k \)'s and replace those \( k \)'s with dots. Repeat this procedure for the numbers \( k-1, k-2, \ldots, 1 \), and then delete the dots. For example, let \( S \) be the increasing tableau on the far right in the above example and let \( C \) be the set of outer corners marked with a dot. The construction of \( Krjdt_C(S) \) is now given by reading the sequence of tableaux in the above example from right to left, so that \( Krjdt_C(S) \) is the increasing tableau with which we began.

It is not difficult to show from the definitions that both \( K \)-theoretic jeu de taquin and its reverse yield increasing tableaux. Moreover, if we start with an increasing tableau \( T \) and a set of inner corners \( B \) then

\[
T = Krjdt_C(S)
\]

where \( S = K_{\lambda}(T) \) and \( C \) is the set of outer corners given by the final location of the dots in the construction of \( S \).

To prepare for the definition of the K-infusion operator, introduced in [15], we define \( K_{\lambda}(U) \), where \( \lambda \in \mu, T \in \text{Inc}(\lambda) \), and \( U \in \text{Inc}(\mu / \lambda) \). Let \( k \) be the largest entry in \( T \) and let \( B_i \) be the set of boxes in \( T \) that contain the entry \( i \). Setting \( U_k := U \) we recursively define

\[
U_{i-1} := K_{B_i}(U_i)
\]

for \( i = k, k-1, \ldots, 1 \). (Note that \( B_i \) is a collection of inner corners for \( U_i \).) Finally, we set

\[
K_{\lambda}(U) := U_0.
\]

For example, take the increasing tableaux

\[
T = \begin{array}{ccc}
1 & 2 & 4 \\
2 & 3 \\
4
\end{array} \quad U = \begin{array}{ccc}
1 & 2 \\
1 & 3 & 4 \\
2 & 3
\end{array}.
\]

Then \( U_3 \) is constructed as

\[
\begin{array}{ccc}
\bullet & 1 & 2 \\
1 & 3 & 4 \\
\bullet & 2 & 3
\end{array} \quad \begin{array}{ccc}
1 & \bullet & 2 \\
\bullet & 3 & 4 \\
2 & 3
\end{array} \quad \begin{array}{ccc}
1 & 2 & \bullet \\
\bullet & 3 & 4 \\
2 & 3 & \bullet
\end{array} \quad \begin{array}{ccc}
1 & 2 & \bullet \\
3 & 4 \\
\bullet & 3 & 4
\end{array} \quad \begin{array}{ccc}
1 & 2 & 4 \\
3 & 4 \\
\bullet & 3 & 4
\end{array}.
\]
and $U_2$ is constructed as

\[
\begin{array}{ccc}
1 & 2 & 4 \\
\bullet & 3 & 4 \\
2 & 3 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 4 \\
3 & \bullet & 4 \\
2 & \bullet & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 4 & \bullet \\
2 & \bullet & \\
\end{array}
\]

and $U_1$ is constructed as

\[
\begin{array}{ccc}
\bullet & 1 & 2 & 4 \\
3 & \bullet & 4 \\
2 & 2 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & \bullet & 2 & 4 \\
3 & 4 & \\
2 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & \bullet & 4 \\
2 & 3 & 4 \\
\bullet & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 4 \\
2 & 3 & 4 \\
\bullet & \\
\end{array}
\]

and then an easy calculation shows that $\text{Kjdt}_T(U) = U_0$ is

\[
\begin{array}{ccc}
1 & 2 & 4 \\
2 & 3 & \\
\end{array}
\]

As illustrated in this example, the fact that $T$ is a straight shape implies that $\text{Kjdt}_T(U)$ will also be a straight shape. Using classical terminology, we call $\text{Kjdt}_T(U)$ a rectification of $U$ by $T$. In the classical setting involving standard Young tableaux, rectification is invariant under the choice of a standard Young tableau $T$ with shape $\lambda$. In the setting of increasing tableaux this is no longer the case. In \cite{6}, Example 22, the authors give an example of a $U$ such that $\text{Kjdt}_T(U) \neq \text{Kjdt}_S(U)$ for two different choices of $T, S \in \text{Inc}(\lambda)$.

On the other hand, there are special cases where rectifications are unique.

**Definition 2.4.** An increasing tableau $T$ with straight shape is called a unique rectification target (URT) if whenever $T$ is a rectification of a skew tableau $U$, then $T$ is the only rectification of $U$. In this case we define $\text{Rect}_T(\mu/\lambda)$ to be the set of all tableaux with shape $\mu/\lambda$ that rectify to $T$.

The phenomenon of URT’s was first isolated in \cite{2}. For our purposes we shall only need the following instance of this phenomenon, which was proved in \cite{2}, Corollary 4.7.

**Fact 2.5.** Every minimal tableau is a unique rectification target.

As expected, we can also define $\text{Krjdt}_T(U)$. To do so, let $k$ be the largest entry of $U$ and let $C_i$ be the set of boxes in $U$ that contain the entry $i$. Set $T_1 := T$, and recursively define

\[
T_{i+1} = \text{Krjdt}_{C_i}(T_i)
\]

for $i = 1, 2, \ldots, k$. Set $\text{Krjdt}_T(U) := T_{k+1}$.

For example, if we take $T$ and $U$ as above then similar calculations show that $T_2, T_3, T_4, T_5$ are, respectively,

\[
\begin{array}{ccc}
1 & 2 & 4 \\
2 & 3 & 4 \\
4 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 4 & \\
2 & 4 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 4 & \\
2 & 3 & 4 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 \\
3 & 4 & \\
2 & 3 & 4 \\
\end{array}
\]

From $\text{Kjdt}_T(U)$ and $\text{Krjdt}_T(U)$ we obtain the $K$-infusion operator, which is defined as follows.
Definition 2.6. For tableaux $T \in \text{Inc}(\lambda)$ and $U \in \text{Inc}(\mu/\lambda)$ we define
\[ \text{fus}(T, U) := (K_{\text{jd}_T}(U), K_{\text{jd}_U}(T)). \]

Two fundamental properties of K-infusion, which we shall need, are the following.

Fact 2.7. Let $T \in \text{Inc}(\lambda)$ and $U \in \text{Inc}(\mu/\lambda)$ and set $(U', T') = \text{fus}(T, U)$. Then there is some shape $\rho \subseteq \mu$ so that $U' \in \text{Inc}(\rho)$ and $T' \in \text{Inc}(\mu/\rho)$.

Fact 2.8 ([15, Theorem 3.1]). Letting $T$ and $U$ be as above, we have
\[ \text{fus}(\text{fus}(T, U)) = (T, U), \]
\[ \text{i.e., the K-infusion operator is an involution.} \]

Lemma 2.9. Fix shapes $\tau \subseteq \nu$ and let $T \in \text{Inc}(\tau)$ be a URT. Then the function
\[ \text{fus}(T, \cdot) : \text{Inc}(\nu/\tau) \rightarrow \bigcup \left( \text{Inc}(\lambda) \times \text{Rect}_T(\nu/\lambda) \right), \]
\[ \text{where the union is over all shapes } \lambda \subseteq \nu, \text{ is a bijection.} \]

Proof. It follows from the fact that K-infusion is an involution (Fact 2.8) that our mapping is injective. To see that it is also surjective, consider some $(U, R)$ in the codomain and set
\[ \text{fus}(U, R) = (T, U'). \]
As $U \cup R$ has shape $\nu$, it follows from Fact 2.7 that $T \cup U'$ also has shape $\nu$. As $T$ has shape $\tau$, we see that $U'$ must have shape $\nu/\tau$. Fact 2.8 now implies that our mapping is surjective.

Remark 2.10. We will later use the fact that Lemma 2.9 remains valid if we replace $\text{Inc}(\mu)$ throughout by $\text{Inc}^{\text{st}}(\mu)$.

The special case of Lemma 2.9 where $\tau = \sigma_{n-1}, \nu = \sigma_n$ and $T = M_{n-1}$ will be used extensively in what follows. In this case, we identify $\mathcal{M}_n$ with $\text{Inc}(\sigma_n/\sigma_{n-1})$ by mapping the $i$th integer in a word to box $(n + 1 - i, i)$, using matrix notation, and our mapping becomes
\[ \text{fus}(M_{n-1}, \cdot) : \mathcal{M}_n \rightarrow \bigcup \left( \text{Inc}(\lambda) \times \text{Rect}_{M_{n-1}}(\sigma_n/\lambda) \right). \]

Definition 2.11. We define $\mathcal{W}(w)$ and $\mathcal{M}(w)$ by writing $\text{fus}(M_{n-1}, w) = (\mathcal{W}(w), \mathcal{M}(w))$.

As discussed in the introduction, one of the central points of this paper is to capture the RID and CID data of a word $w$ in a unified way. We shall show that $\mathcal{W}(w)$ captures RID data (see Theorem 2.13) and that $\mathcal{M}(w)$ captures CID data (see Theorem 2.14). Our result for $\mathcal{W}(w)$ will be easy to obtain, once we have a connection between K-jeu de taquin and K-Knuth equivalence.
Two increasing tableaux \( T \) and \( T' \) are said to be \( K \)-Knuth equivalent if their reading words are \( K \)-Knuth equivalent. Recall that the reading word of a tableau \( T \) is the word \( \text{Read}(T) \) obtained by reading \( T \) from bottom to top and left to right. For example, if

\[
T = \begin{array}{cccc}
2 & 3 & 4 & 5 \\
3 & 4 & 4 & 7 \\
6 & 6 & 7
\end{array}
\]

then \( \text{Read}(T) = 667344722345 \). On the other hand, \( T \) and \( T' \) are said to be \( K \)-jeu de taquin equivalent if each can be obtained from the other by applying \( \text{Kjdt}_B \) and \( \text{Kjdt}_C \) for some finite number of choices of \( B \) and \( C \).

**Fact 2.12.** ([2], Theorem 6.2) Increasing tableaux \( T \) and \( T' \) are \( K \)-Knuth equivalent if and only if they are \( K \)-jeu de taquin equivalent.

**Theorem 2.13.** If \( \mathcal{W}(u) = \mathcal{W}(w) \) then \( u \sim_{\text{rid}} w \).

*Proof.* Since \( \mathcal{W}(u) \) and \( \mathcal{W}(w) \) are obtained from the tableaux representing \( u \) and \( w \) by applying \( \text{Kjdt} \), it follows from Fact 2.12 that \( u \equiv_K \text{Read}(\mathcal{W}(u)) \) and \( w \equiv_K \text{Read}(\mathcal{W}(w)) \). Since \( \mathcal{W}(u) = \mathcal{W}(w) \), we obtain \( u \equiv_K w \).

By Fact 2.2 this implies that for any integer interval \( [a, b] \), the words obtained from \( u \) and \( w \) by deleting all entries not in \( [a, b] \) are \( K \)-Knuth equivalent. By Fact 2.3 this implies that the lengths of longest decreasing subsequences in these reduced words are the same, and the lengths of longest increasing subsequences are the same. So \( u \sim_{\text{rid}} w \). \( \square \)

**Theorem 2.14.** If \( \mathcal{M}(u) = \mathcal{M}(w) \) then \( u \sim_{\text{cid}} w \).

The proof of Theorem 2.14 requires new ideas and a significant amount of effort. As mentioned in the introduction, its proof occupies the entirety of Section 4. We close this section with the following remark.

**Remark 2.15.** Let \( S_{n-1} \) denote the superstandard tableau of shape \( \sigma_{n-1} \), so that the first row of \( S_{n-1} \) has entries 1, \ldots, \( n-1 \), the second row has entries \( n, \ldots, 2n-3 \), and so on. In [16], Theorem 4.2, Thomas and Yong consider (in our notation) \( \text{fus}(S_{n-1}, w) \). They show that the first entry of \( \text{fus}(S_{n-1}, w) \) is the Hecke insertion tableau of \( w \). As can be seen from the proof of Theorem 2.13 that theorem would still hold with \( M_{n-1} \) replaced by \( S_{n-1} \); in fact it holds when \( M_{n-1} \) is replaced by any increasing tableau of the same shape. On the other hand, Theorem 2.14 fails if we replace \( M_{n-1} \) by \( S_{n-1} \). To illustrate, consider the words \( u = 12113 \) and \( w = 13123 \) which clearly do not have the same CID data (as can be seen from their last three letters). Yet, using \( S_4 \) in place of \( M_4 \) we find that \( \mathcal{M}(u) = \mathcal{M}(w) \) is the tableau

\[
\begin{array}{cc}
3 & 4 \\
2 & 6 & 7 \\
1 & 5 & 9 \\
5 & 8 \\
10
\end{array}
\]
3 Row shifting and column shifting

To motivate this section, we recall the definition of Schützenberger’s promotion operator in the context of standard Young tableaux [13]. (A semistandard Young tableau of straight shape is called standard if its set of entries is an integer interval \([1, n]\) and each entry occurs exactly once.) For such a tableau \(T\), its promotion \(\partial T\) is the tableau obtained by deleting the entry 1 from \(T\) and decrementing the remaining entries by 1, then rectifying the resulting skew tableau (using jeu de taquin), and finally placing the largest entry of \(T\) in the vacated box.

We mention without proof two results that follow from well-known properties of the Robinson-Schensted (RS) algorithm and Knuth equivalence. (We will prove an analogue of the second of these results in Theorem 3.1 below.) The first result states that if \(\pi\) is a permutation of length \(n\) and \(\text{RS}(\pi) = (P, Q)\) and \(\pi' = \text{RS}^{-1}(P, \partial Q)\) then

\[
\pi \sim_{\text{rid}} \pi' \quad \text{and} \quad \pi_2 \cdots \pi_n \sim_{\text{cid}} \pi'_1 \cdots \pi'_{n-1}.
\]  

(1)

We think of the mapping from \(\pi\) to \(\pi'\) as fixing RID data but “shifting left” CID data.

Similarly, the second result is obtained by promoting the insertion tableau rather than the recording tableau. To state it, we need some notation. For any word \(w\) and letter \(a\), we let \(w \setminus a\) be the word obtained by deleting all occurrences of \(a\) from \(w\). We let \(w^-\) denote the word obtained from \(w\) by decrementing all the letters by 1. With this notation, if we set \(\pi' = \text{RS}^{-1}(\partial P, Q)\) then

\[
\pi \sim_{\text{cid}} \pi' \quad \text{and} \quad (\pi \setminus 1)^- \sim_{\text{rid}} \pi' \setminus n.
\]

(2)

We think of this mapping from \(\pi\) to \(\pi'\) as fixing CID data but “shifting down” RID data.

Similar constructs have been employed in other contexts. In particular, Rubey [11] used a map similar to (1), in the context of semistandard Young tableaux, variants of the RSK algorithm, and growth diagrams, to study the distribution of strictly decreasing and weakly increasing (or vice versa) subsequences of words, with applications to moon polyominoes. Likewise, Guo and Poznanović [7] used a map similar to that in (2), in the context of increasing tableaux and Hecke insertion, to prove their Conjecture 15 for the special case of stack polyominoes. Their methods did not provide a map with the properties of the map in (1), however. From our point of view, this is why they did not obtain a proof of the full conjecture.

The goal of this section is to use the mapping \(w \mapsto (W(w), M(w))\) to obtain analogues of the maps in (1) and (2) in the context of words. We denote these analogues by \(\mathcal{C}\) and \(\mathcal{R}\), respectively. Our analogue \(\mathcal{R}\) of the map in (2) will be obtained by using a map like that in (2), although we will not use Hecke insertion or Hecke growth diagrams as Guo and Poznanović did. This map yields the special case of Conjecture 15 that Guo and Poznanović proved. Our second map, \(\mathcal{C}\), will be obtained by using new, and considerably more involved, methods. This map will be the key to obtaining a full proof of Conjecture 15.
3.1 Row shifting

To define our row-shifting operator we first describe the analogue of the classical promotion operator $\partial$ for elements of $\text{Inc}^\text{st}(\lambda)$, where $\lambda$ is a straight shape. To start let $W \in \text{Inc}^\text{st}(\lambda)$ and note that, by definition, the smallest entry in $W$ is 1 and occurs in box $(1, 1)$, as $W$ is a straight shape. Let $W^-$ be the tableau obtained by deleting 1 from $W$ and decrementing all the remaining values by 1. Finally, let $\partial W$ be the result of rectifying $W^-$, i.e., computing $\text{Kjdt}_{((1, 1))} W^-$, and filling the vacated boxes with the largest letter in $w$. It is not hard to see that this gives a bijection

$$\partial : \text{Inc}^\text{st}(\lambda) \rightarrow \text{Inc}^\text{st}(\lambda).$$

To simplify notation below, we establish the convention that for a pair of increasing tableaux $(A, B)$, we set $\partial(A, B) := (\partial A, B)$.

For each $n$ we let $\mathbb{M}^\text{st}_n$ denote the set consisting of all elements of $\mathbb{M}_n$ whose entries constitute an integer interval with left endpoint 1. It follows from Lemma 2.9 and the remark following it that for each $n \geq 2$ we have a bijection

$$f : \mathbb{M}^\text{st}_n \rightarrow \bigcup_\lambda \left( \text{Inc}^\text{st}(\lambda) \times \text{Rect}_{n-1}(\sigma_n/\lambda) \right)$$

given by $w \mapsto \text{fus}(M_{n-1}, w)$. As a first step in defining our row-shift operator $\mathcal{R} : \mathbb{M}_n \rightarrow \mathbb{M}_n$, we define

$$\mathcal{R} : \mathbb{M}^\text{st}_n \rightarrow \mathbb{M}^\text{st}_n$$

by letting $\mathcal{R}(w) = (f^{-1} \circ \partial \circ f)(w)$. When $n = 1$, we define $\mathcal{R}$ to be the identity map. Note that, since $\partial$ is bijective on $\text{Inc}^\text{st}(\lambda)$, it follows that $\mathcal{R}$ is bijective on $\mathbb{M}^\text{st}_n$.

As an illustration, let $w = 4231142$. The left-hand figure below depicts the pair $(M_6, w)$. The next three figures then give $\text{fus}(M_6, w) = (\mathcal{W}(w), \mathcal{M}(w))$, $(\partial \mathcal{W}(w), \mathcal{M}(w))$, and, finally, $\text{fus}^{-1}(\partial \mathcal{W}(w), \mathcal{M}(w))$. Hence $\mathcal{R}(w) = 3121143$.

```
\begin{verbatim}
1 2 3 4 5 6 2 3 4 1 0 4
3 4 5 3 0 4 2 4 5 0 4 1
4 5 6 1 3 0 1 3 4 6 1
5 6 2 4 3 0 2 4 5 1
6 2 4 3 0 2 5 4 3 2
\end{verbatim}
```

To extend the definition of $\mathcal{R}$ to $\mathbb{M}_n$, let $w \in \mathbb{M}_n$ and suppose the entries of $w$ are $v_1 < v_2 < \cdots < v_k$. Let $\text{st}(w)$ be the word obtained from $w$ by replacing each $v_i$ by $i$, and define $\mathcal{R}(w)$ to be the word obtained by taking $\mathcal{R}(\text{st}(w))$ and replacing each $i$ by $v_i$.

Our next theorem asserts that $\mathcal{R}$ has the desired properties.

**Theorem 3.1.** For all $n$, the mapping $\mathcal{R} : \mathbb{M}_n \rightarrow \mathbb{M}_n$ is a bijection such that $w \sim_{\text{cid}} \mathcal{R}(w)$ and

$$(w \setminus s)^- \sim_{\text{rid}} \mathcal{R}(w) \setminus m,$$
where $s$ and $m$ are the smallest and largest letters in $w$, respectively. Here, if $s < v_2 < \cdots < v_k$ are the entries in $w$, then $(w \setminus s)^-$ denotes the word obtained from $w \setminus s$ by replacing $v_i$ by $v_{i-1}$ for each $2 \leq i \leq k$.

Finally, the set of integers occurring in $R(w)$ is the same as the set of integers occurring in $w$.

**Proof.** We need only consider $n \geq 2$. By the definition of $R$ on $W_n$, it suffices to deal with the restriction of $R$ to $W_n^{st}$.

We have already shown that this restricted map is a bijection. Set $u = R(w)$. By our definition of $R$ we know $M(w) = M(u)$ and hence an application of Theorem 2.14 tells us that $w \sim_{cid} u$.

We now turn our attention to the remaining claim, which, in the case under consideration, asserts that

$$(w \setminus 1)^- \sim_{rid} R(w) \setminus m,$$

where $m$ is the largest letter in $w$. We have

$$
\text{fus}(M_{n-1}, w) = (W(w), M(w)) \quad \text{and} \quad \text{fus}(M_{n-1}, u) = (\partial W(w), M(w)).
$$

By Fact 2.12 we have

$$w \equiv_K \text{Read}(W(w)) \quad \text{and} \quad u \equiv_K \text{Read}(\partial W(w)).$$

From the definition of $\partial$ and Fact 2.12 it follows that

$$(\text{Read}(W(w)) \setminus 1)^- \equiv_K \text{Read}(\partial W(w) \setminus m).$$

Combining these equivalences and applying Fact 2.2 gives

$$(w \setminus 1)^- \equiv_K (\text{Read}(W(w)) \setminus 1)^- \equiv_K \text{Read}(\partial W(w) \setminus m) \equiv_K u \setminus m.$$

An application of Facts 2.2 and 2.3 now shows that $(w \setminus 1)^- \sim_{rid} R(w) \setminus m$.

It is clear from our definition of $R$ that the set of integers in $R(w)$ is the same as the set of integers in $w$. \hfill \Box

### 3.2 Column shifting

**Theorem 3.2.** For every $n$, there exists an explicit bijection $C : W_n \to W_n$ such that if $u = C(w)$ then $w \sim_{rid} u$,

$$w_2 \cdots w_n \sim_{cid} u_1 \cdots u_{n-1},$$

and the set of integers in $u$ is the same as the set of integers in $w$.

When $n = 1, 2$, we take $C$ to be the identity map. For the remainder of this subsection assume $n \geq 3$.

To define our map $C$ we first make a few definitions. Let $I_n \in \text{Inc}(1^n)$ be the increasing tableau with entries 1, $\ldots$, $n$. Note that $I_n$ is a URT by Fact 2.5. Next, define $N_n \in \text{Inc}(\sigma_n)$...
to be the increasing tableau obtained by concatenating $I_n$ with $(M_{n-1} + n)$, the result of incrementing all the entries of $M_{n-1}$ by $n$. For example, $N_4$ is

$$
\begin{array}{cccc}
1 & 5 & 0 & 7 \\
2 & 6 & 7 & 0 \\
3 & 2 & & \\
4 & & & \\
\end{array}
$$

We remark that since $I_n$ and $M_n$ are both URTs so too is $N_n$.

To motivate our construction of $C$ we illustrate its calculation for the word $w = 4231142 \in \mathcal{M}_7$. First we have

where the first diagram shows the pair $(N_6, w)$ and the second shows fus$(N_6, w)$.

Now we delete the “vertical strip” consisting of all the yellow boxes and slide the green boxes down and left one unit. Doing so will create a “horizontal strip” of empty boxes that we color yellow and label from 1 to 6 as shown below on the left.

If we let $X$ denote the tableau of yellow and green boxes and $W$ denote the tableau of red boxes, then the last figure shows fus$(W, X)$. Our desired word $C(w)$ is given by the red entries on the diagonal and so $C(w) = 2311432$.

Motivated by the map from Figure 1 to Figure 2 above, we have the following lemma.

**Lemma 3.3.** For $n \geq 3$, we have an explicit bijection

$$
G_n : \mathcal{M}_n \rightarrow \bigcup_{\mu, \lambda} \left( \text{Inc} (\lambda) \times \text{Rect}_{I_{n-1}} (\mu/\lambda) \times \text{Rect}_{M_{n-2}} (\sigma_n/\mu) \right)
$$

where the union is over all partitions $\lambda \subseteq \mu \subseteq \sigma_n$ such that $\mu$ has length $n$. Moreover, if $w \mapsto (W, V, M)$ then $w \equiv_K \text{Read}(W)$ and $M = \mathcal{M}(w_2 \cdots w_n)$.

**Proof.** By Lemma 2.9 we have the bijection

$$
\text{fus}(N_{n-1}, \cdot) : \mathcal{M}_n \rightarrow \bigcup_{\lambda} \left( \text{Inc} (\lambda) \times \text{Rect}_{N_{n-1}} (\sigma_n/\lambda) \right),
$$

13
where the union is over all \( \lambda \subseteq \sigma_n \). Observe that any \( T \in \text{Rect}_{N_{n-1}}(\sigma_n/\lambda) \) corresponds to a pair \((V,M)\), where the entries \( 1, \ldots, n-1 \) in \( T \) determine \( V \) and the remaining entries in \( T \), decremented by \( n \), determine \( M \). From the definitions involved we see that \( V \in \text{Rect}_{I_{n-1}}(\mu/\lambda) \) and \( M \in \text{Rect}_{M_{n-1}}(\sigma_n/\mu) \) for some \( \mu \geq \lambda \). This mapping gives us the correspondence

\[
\text{Rect}_{N_{n-1}}(\sigma_n/\lambda) \to \bigcup_{\mu} \left( \text{Rect}_{I_{n-1}}(\mu/\lambda) \times \text{Rect}_{M_{n-2}}(\sigma_n/\mu) \right)
\]

where the union is over all \( \mu \subseteq \sigma_n \) of length \( n \) containing \( \lambda \). Note that \( \mu \) must have length \( n \) because, in the calculation of \( \text{fus}(N_{n-1},w) \), no entry from \( M_{n-2} + n - 1 \) can move to the bottom box of \( \sigma_n \), so this box must be filled by an entry from either \( w \) or \( I_{n-1} \), i.e., this box is in either \( \lambda \) or \( V = \mu/\lambda \), so it is in \( \mu \).

To see that \( G_v \) has the desired properties, note that, by our construction, the diagonal tableau representing \( w \) is K-jdt equivalent to \( W \), so by Fact 2.12 \( w \equiv_K \text{Read}(W) \). Furthermore, our construction implies that if \( \text{fus}(M_{n-2},w_2\ldots w_n) = (W',M') \) then \( M' = M \) and so the last claim follows. \( \square \)

We also have the following analogous result, where \( * \) denotes conjugation.

**Lemma 3.4.** For \( n \geq 3 \), we have an explicit bijection

\[
G_h : \mathfrak{W}_n \to \bigcup_{\mu, \lambda} \left( \text{Inc}(\lambda) \times \text{Rect}_{I_{n-1}}(\mu/\lambda) \times \text{Rect}_{M_{n-2}}(\sigma_n/\mu) \right)
\]

where the union is over all partitions \( \lambda \subseteq \mu \subseteq \sigma_n \) such that \( \mu \) has width \( n \). Moreover, if \( w \mapsto (W,H,M) \) then \( M = \mathcal{M}(w_1\ldots w_{n-1}) \), and \( w \) and \( W \) are K-jdt equivalent.

**Proof.** First observe that conjugation gives a bijection between the codomain of \( G_v \) and that of \( G_h \). Using this we define

\[
G_h(w) = (W^*,V^*,M^*)
\]

where \( G_v(w^*) = (W,V,M) \) and \( w^r \) is the reverse of \( w \). It follows that \( G_h \) is bijective and \( \mu \) has width \( n \).

We know \( w^r \) and \( W \) are K-jdt equivalent by Lemma 3.3 and so \( w \) and \( W^* \) are K-jdt equivalent too. Additionally, since \( (w_2^r\ldots w_n^r)^r = w_1\ldots w_{n-1} \) we see that

\[
\mathcal{M}(w_1\ldots w_{n-1}) = \mathcal{M}((w_2^r\ldots w_n^r)^r) = M^*.
\]

\( \square \)

Our next definitions are inspired by the yellow boxes in Figures 2 and 3 in our motivating example.

**Definitions 3.5.** For partitions \( \lambda \subseteq \mu \subseteq \sigma_n \) we say the skew shape \( \mu/\lambda \) is a *vertical n-strip* provided that \( \mu \) has length \( n \), \( |\mu/\lambda| \) is \( n \) or \( n-1 \), and \( \mu_i - 1 \leq \lambda_i \) for all \( i \). For such a vertical \( n \)-strip we define \( \text{Vert}_n(\mu/\lambda) \) to be the set of increasing tableaux of shape \( \mu/\lambda \) with entries 1 through \( n-1 \) that weakly increase from top to bottom.
We say the skew shape $\mu/\lambda$ is a horizontal $n$-strip provided that $\mu_1 = n$, $|\mu/\lambda|$ is $n$ or $n-1$, and $\mu_{i+1} \leq \lambda_i$ for all $i$. (The last condition says that no column can contain two boxes of $\mu/\lambda$.) For such a horizontal $n$-strip, we define $\text{Horiz}_n(\mu/\lambda)$ to be the set of increasing tableaux of shape $\mu/\lambda$ with entries 1 through $n-1$ that weakly increase from left to right.

For example, if $\lambda = (3^2, 2^2)$ and $\mu = (4^2, 3^2, 1^3)$ then $\text{Vert}_n(\mu/\lambda)$ consists precisely of the two skew tableaux (shown in yellow):

![Skew Tableaux]

On the other hand, if $\lambda = (3^2, 2)$ and $\mu = (4^2, 2, 1^3)$ then $\text{Vert}_n(\mu/\lambda)$ consists of exactly one skew tableau shown below:

![Skew Tableau]

Observe that our motivating example illustrated a vertical $n$-strip with no repeated numbers. In general, at most one repetition can occur.

**Lemma 3.6.** Fix $\lambda \subseteq \mu \subseteq \sigma_n$. If $\mu$ has length $n$ we have

$$\text{Rect}_{n-1}(\mu/\lambda) = \text{Vert}_n(\mu/\lambda)$$

and if $\mu$ has width $n$ we have

$$\text{Rect}_{n-1}(\mu/\lambda) = \text{Horiz}_n(\mu/\lambda).$$

**Proof.** Let $T \in \text{Vert}_n(\mu/\lambda)$ and denote by

$$T_0, \ldots, T_m$$

the sequence of tableaux in a rectification of $T$ so that $T_0 = T$. (At this point we do not know $T_m = I_{n-1}$.) We will show by induction that each $T_i$ has at most one box in each row and has entries 1, $\ldots$, $n-1$ that are weakly increasing from top to bottom. This certainly holds for $T_0$. Now consider adjacent boxes in $T_k$ like

![Adjacent Boxes]

By our inductive hypothesis $a \leq b$ and so in $T_{k+1}$ these boxes become either

![Inductive Cases]

15
depending on whether \( a < b \) or \( a = b \). As the set of entries in each \( T_i \) is preserved it follows that \( T_m \) has at most one box in each row and contains the entries 1, \ldots, n - 1. As \( T_m \) is a straight shape, it follows that \( T_m = I_{n-1} \). So \( \text{Vert}_n(\mu/\lambda) \subseteq \text{Rect}_{I_{n-1}}(\mu/\lambda) \).

Next, fix \( R \in \text{Rect}_{I_{n-1}}(\mu/\lambda) \) and let

\[
R_0, \ldots, R_m
\]

be the sequence of tableaux in a rectification of \( R \) so that \( R = R_0 \) and \( I_{n-1} = R_m \). Since each step is invertible we can think of starting with \( I_{n-1} = R_m \) and running this process backward. An argument similar to the one above then shows that \( R = R_0 \) has at most one box in each row, has entries that weakly increase from top to bottom, and contains the numbers 1, \ldots, n - 1. It then follows that \( R \in \text{Vert}_n(\mu/\lambda) \) as needed.

The second assertion of the lemma follows from the first by taking conjugates. \( \square \)

We now introduce a bijection between vertical \( n \)-strips and horizontal \( n \)-strips that preserves \( \lambda \). For any shape \( \mu \subseteq \sigma_n \) of length \( n \) we define

\[
\tilde{\mu} = (n, \mu_1 - 1, \ldots, \mu_n - 1).
\]

For such \( \mu \) it follows that \( \mu_n = 1 \) and therefore that \( \tilde{\mu} \) has length at most \( n \).

To get a feeling for how \( \mu \) and \( \tilde{\mu} \) relate consider the example where \( n = 7 \), \( \lambda = (3^2, 2^2) \), and \( \mu = (4^2, 3^2, 1^3) \) so that \( \lambda \subseteq \mu \subseteq \sigma_7 \). Coloring the boxes of \( \lambda \) red, those of \( \mu/\lambda \) yellow, and those of \( \sigma_7/\mu \) green gives

As in the motivating example, if we shift the green boxes down one unit and left one unit and then color all empty boxes yellow, we have the configuration

Note that \( \tilde{\mu} \) is the shape of the yellow and red boxes.

**Lemma 3.7.** If \( \mu/\lambda \) is a vertical \( n \)-strip, then \( \tilde{\mu}/\lambda \) is a horizontal \( n \)-strip so that \( |\mu/\lambda| = |\tilde{\mu}/\lambda| \). Moreover, the skew shape \( \sigma_n/\tilde{\mu} \) is obtained by shifting the shape \( \sigma_n/\mu \) down one unit and left one unit.

**Proof.** Since \( \mu_i - 1 \leq \lambda_i \) for all \( 1 \leq i \leq n \), it follows from our definition of \( \tilde{\mu} \) that \( \tilde{\mu}_{i+1} \leq \lambda_i \) for all \( i \) and that \( \tilde{\mu}_1 = n \). To conclude the proof, it suffices to show that \( \lambda \subseteq \tilde{\mu} \), for then since
\[ |\tilde{\mu}| = |\mu| \] we will have \(|\tilde{\mu}/\lambda| = |\mu/\lambda|\), which is \(n\) or \(n - 1\). And so we aim to show that \(\tilde{\mu}_i \geq \lambda_i\) for all \(1 \leq i \leq n\).

For \(i = 1\) this is clear, since \(\tilde{\mu}_1 = n\) and \(\lambda \subseteq \sigma_n\). Now suppose \(2 \leq i \leq n\) so that \(\tilde{\mu}_i = \mu_{i-1} - 1\). If \(\mu_{i-1} > \lambda_{i-1}\) then we have
\[
\tilde{\mu}_i = \mu_{i-1} - 1 \geq \lambda_{i-1} \geq \lambda_i.
\]
Now consider the case when \(\mu_{i-1} = \lambda_{i-1}\). As \(\mu/\lambda\) is a vertical \(n\)-strip with \(|\mu/\lambda| \geq n - 1\) we must have \(\mu_j = \lambda_j + 1\) for all \(j \neq i - 1\). In particular,
\[
\lambda_i + 1 = \mu_i \leq \mu_{i-1} = \lambda_{i-1}
\]
and so
\[
\tilde{\mu}_i = \mu_{i-1} - 1 = \lambda_{i-1} - 1 \geq \lambda_i.
\]

To address the last claim, note that after shifting \(\sigma_n/\mu\) down and to the left one unit its first row is empty and its \(i\)th row for \(i > 1\) extends, left to right, from column \(\mu_{i-1}\) to column \(n + 1 - i\). On the other hand, the first row of \(\sigma_n/\tilde{\mu}\) is empty (as \(\tilde{\mu}_1 = n\) and its \(i\)th row for \(i > 1\) extends, left to right, from column \(\tilde{\mu}_i + 1 = \mu_{i-1}\) to column \(n + 1 - i\). (If \(\mu_{i-1} > n + 1 - i\) then the \(i\)th row is empty in both cases.)

We now get the following correspondence.

**Lemma 3.8.** We have an explicit bijection \(\varphi : \text{Vert}_n(\mu/\lambda) \to \text{Horiz}_n(\tilde{\mu}/\lambda)\).

*Proof.* We consider two cases depending on \(|\mu/\lambda|\).

Case 1: \(|\mu/\lambda| = n - 1\)

In this case the numbers 1 through \(n - 1\) each appear exactly once in each \(V \in \text{Vert}_n(\mu/\lambda)\). This together with the fact that the entries of \(V\) are weakly increasing from top to bottom implies that
\[
|\text{Vert}_n(\mu/\lambda)| = 1.
\]
By the second claim in Lemma [3.7] we see that \(|\tilde{\mu}/\lambda| = n - 1\). By an analogous argument \(|\text{Vert}_n(\tilde{\mu}/\lambda)| = 1\), making our bijection \(\varphi\) a triviality in this case.

Case 2: \(|\mu/\lambda| = n\)

We first claim that there is a natural correspondence between the outer corners of \(\lambda\) and \(\text{Vert}_n(\mu/\lambda)\). To see this observe that \(\mu/\lambda\) has exactly one box on each of the \(n\) rows, filled with the numbers 1 through \(n - 1\) so that they are weakly increasing from top to bottom. Hence there must be exactly one repeated value and every \(V \in \text{Vert}_n(\mu/\lambda)\) is characterized by this repeated value. On the other hand, for any outer corner \((i, j)\) of \(\lambda\) we can construct \(V\) by labeling its first \(i\) boxes 1 to \(i\) and then labeling its remaining \(n - i\) boxes \(i\) through \(n - 1\). As \((i, j)\) is an outer corner of \(\lambda\) the boxes labeled \(i\) are not in the same column and so \(V \in \text{Vert}_n(\mu/\lambda)\). A straightforward check shows that this gives our desired correspondence.

Again by Lemma [3.7] we know that \(|\tilde{\mu}/\lambda| = n\). By appealing to conjugates, we also obtain a correspondence between the outer corners of \(\lambda\) and \(\text{Horiz}_n(\tilde{\mu}/\lambda)\). Consequently, we obtain a bijection \(\varphi : \text{Vert}_n(\mu/\lambda) \to \text{Horiz}_n(\tilde{\mu}/\lambda)\).
We are now ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** For \( n \geq 3 \), we define our column-shift map \( C : \mathcal{W}_n \to \mathcal{W}_n \) as follows. For any \( w \in \mathcal{W}_n \) let \( G_v(w) = (W, V, M) \) so that \( V \in \text{Vert}_{I_n-1}((\mu/\lambda)} \) for some \( \lambda \subseteq \mu \) by Lemma 3.6. The same lemma together with Lemma 3.8 shows that \( \varphi(V) \in \text{Horiz}(\tilde{\mu}/\lambda) = \text{Rect}_{I_n-1}(\tilde{\mu}/\lambda) \).

Letting \( M' \) be the tableau obtained by shifting \( M \) down and left one unit, it follows from Lemma 3.7 that \( M' \) has shape \( \sigma_n/\lambda \). We now see that \( (W, \varphi(V), M') \) is in the codomain of \( G_h \) and define \( C(w) = G_h^{-1}(W, \varphi(V), M') \).

As \( G_v, G_h, \) and \( \varphi \) are bijections, so is \( C \).

To check that \( C \) has the desired properties set \( u = C(w) \). By our construction, both \( w \) and \( u \) are K-Knuth equivalent to Read\((W)\), so \( w \equiv_K u \) and by Fact 2.3 we conclude that \( w \sim_{\text{cid}} u \). Additionally, \( M(w_2 \cdots w_n) \) and \( M(u_1 \cdots u_{n-1}) \) are the same tableau (modulo a translation) and so Theorem 2.14 implies that \( w_2 \cdots w_n \sim_{\text{cid}} u_1 \cdots u_{n-1} \).

The set of integers in \( u \) is the same as the set of integers in \( w \) because each of these sets is the set of integers in \( W \).

4. \( \mathcal{M}(u) = \mathcal{M}(w) \) implies \( u \sim_{\text{cid}} w \)

In this section we prove Theorem 2.14 thereby establishing that \( \mathcal{M}(w) \) determines CID data for words. To carry out this proof, we first establish an RSK-like algorithm, which we call iRSK. Like classical RSK, iRSK gives a mapping from words to certain pairs of semistandard Young tableaux

\[ w \mapsto (P, Q) \]

with the same shape. This mapping is introduced in Subsection 4.1. In Subsection 4.2 we prove that if two words \( u, w \) have the same \( \mathcal{M} \) then they have the same \( \mathcal{Q} \). This requires a careful analysis of the rectification process of a word as defined in Subsection 2.2. Finally in Subsection 4.3 we prove that \( \mathcal{Q} \) generalizes the classical recording tableau in that \( \mathcal{Q}(w) = \mathcal{Q}(w) \) implies \( u \sim_{\text{cid}} w \). Combining these results, together with an appeal to word reversal, gives us a proof that if \( \mathcal{M}(u) = \mathcal{M}(w) \) then \( u \sim_{\text{cid}} w \).

4.1 An inflated RSK algorithm

In this subsection we develop an insertion algorithm for words, called iRSK, that is similar to classical RSK. Throughout this subsection we make use of the RSK algorithm and the surrounding language without giving a full review. A treatment of these ideas can be found in [5] [12] [14].

Recall that the classical RSK algorithm consists of two parts: “insertion” and “bumping”. Our first definition isolates the former.
Definition 4.1. For any semistandard Young tableau $p$ consisting of a single row and any positive integer $a$ let $$p \leftarrow a$$ be the semistandard tableau $p'$ obtained as follows. If $p$ is empty or $a$ is greater than or equal to all the entries of $p$ then $p'$ is the result of appending $a$ to the (right) end of $p$. Otherwise, $p'$ is the result of replacing the leftmost entry of $p$ that is larger than $a$ by $a$.

If $w \in \mathcal{W}_n$ then we define $p \leftarrow w$ as $$((p \leftarrow w_1) \leftarrow w_2) \cdots) \leftarrow w_n.$$ For example,

$$\begin{align*}
\begin{array}{cccccc}
2 & 2 & 4 & 5 & 6 & \leftarrow 351
\end{array}
\end{align*}
\begin{align*}
\begin{array}{cccccc}
1 & 2 & 3 & 5 & 5 & 5
\end{array}
\end{align*}
.$$ To motivate our variant of the insertion process, which we will call i-insertion, consider the tableau $$P' = \begin{array}{ccccccc}
2 & 2 & 4 & 5 & 5 & 6 \\
4 & 4 & 6
\end{array}.$$ To i-insert a 5 into this tableau we first replace $P'$ by the skew tableau

$$\begin{array}{ccccccc}
2 & 2 & 4 & 5 & 5 & 6 \\
4 & 4 & 6 & 5
\end{array}$$
and denote the rows by $p'_0, p'_1, p'_3$. We inflate $w = 5$ to $w^+ = 55$ so that the rightmost occurrence of 5 in

$$p_1 := (p'_1 \leftarrow 55) = \begin{array}{ccccccc}
2 & 2 & 4 & 5 & 5 & 5 & 5
\end{array}$$

is in the same column as the leftmost occurrence of 5 in row $p'_0$. Note that the set of elements bumped from $p'_1$ forms the word $\alpha = 6$. We inflate this word to 666 so that in

$$p_2 := (p'_2 \leftarrow 666) = \begin{array}{ccccccc}
4 & 4 & 6 & 6 & 6 & 6
\end{array}$$
the rightmost 6 is in the same column as the leftmost 6 in $p'_1$. In this step nothing was bumped so we stop. The resulting tableau from this operation is then

$$\begin{array}{ccccccc}
2 & 2 & 4 & 5 & 5 & 5 & 5 \\
4 & 4 & 6 & 6 & 6 & 6
\end{array}.$$ We formalize this insertion algorithm in the following definition.

Definition 4.2. Given a semistandard Young tableau $P'$, denote its $i$th row by $p'_i$. For any positive integer $a$ we define $P' \leftarrow a$ to be the semistandard Young tableau $P$ with rows $p_i$ determined as follows.

First redefine $P'$ as $P' \oplus a$, index the rows from 0, and set $S_0 = \{a\}$. We now proceed recursively. Given the set $S_i = \{a_1 < \cdots < a_k\}$, whose values are a subset of $p'_i$, let $\alpha^+_i$ be the inflation of the word $\alpha = a_1 \cdots a_k$ so that if

$$p_{i+1} := p'_{i+1} \leftarrow \alpha^+_i$$

19
then the rightmost occurrence of $a_j$ in $p_{i+1}$ is in the same column as the leftmost occurrence of $a_j$ in $p'_i$. Finally, let $S_{i+1}$ be the set of elements bumped during this insertion process.

For any word $w$ we define its $i$-insertion tableau $\mathcal{P}(w)$ to be the tableau

$$((∅ ↹ w_1) ↹ w_2) \cdots) ↹ w_n.$$  

We define the $i$-recording tableau $\mathcal{Q}(w)$ as follows. For $j \geq 1$, let $\lambda^{(j)}$ be the shape of $\mathcal{P}(w_1 \cdots w_j)$ and set $\lambda^{(0)} = ∅$ so that 

$$\lambda^{(0)} \leq \cdots \leq \lambda^{(n)}.$$  

Let $\mathcal{Q}(w)$ be the tableau with shape $\lambda^{(n)}$ such that its entries in $\lambda^{(j)}/\lambda^{(j-1)}$ are $j$ for $1 \leq j \leq n$. Finally we write

$$iRSK(w) = (\mathcal{P}(w), \mathcal{Q}(w)).$$  

Let us (again) illustrate $i$-insertion but this time using the notation from our definition.

To start take $a = 1$ and 

$$P' = \begin{array}{cccccc}
2 & 2 & 5 & 5 & 5 & 6 \\
3 & 4 & 7 & 7 & 8 & 8 \\
\end{array}.$$  

Then to compute $P' ↹ 1$ we first redefine $P'$ as

$$P' = \begin{array}{cccccc}
1 & & & & & \\
2 & 2 & 5 & 5 & 5 & 6 \\
3 & 4 & 7 & 7 & 8 & 8 \\
\end{array}.$$  

and set $S_0 = \{1\}$. Now $\alpha_0^+ = 1^8$ so that

$$p_1 = \begin{pmatrix}
2 & 2 & 5 & 5 & 5 & 6 \\
\_ & \_ & \_ & \_ & \_ & \_ \\
\end{pmatrix} ↹ \alpha_0^+ = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
\_ & \_ & \_ & \_ & \_ & \_ \\
\end{pmatrix}$$  

and $S_1 = \{2, 5, 6\}$. Now $\alpha_1^+ = 25666$ so that

$$p_2 = \begin{pmatrix}
3 & 4 & 7 & 7 & 8 & 8 \\
\_ & \_ & \_ & \_ & \_ & \_ \\
\end{pmatrix} ↹ \alpha_1^+ = \begin{pmatrix}
2 & 4 & 5 & 6 & 6 & 6 \\
\_ & \_ & \_ & \_ & \_ & \_ \\
\end{pmatrix}$$  

and $S_2 = \{3, 7, 8\}$. Now $\alpha_2^+ = 37788$ so that

$$p_3 = (∅ ↹ \alpha_2^+) = \begin{pmatrix}
3 & 7 & 7 & 8 & 8 \\
\_ & \_ & \_ & \_ & \_ \\
\end{pmatrix}$$  

and $S_3 = ∅$. Consequently,

$$P' ↹ 1 = \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 4 & 5 & 6 & 6 & 6 \\
3 & 7 & 7 & 8 & 8 & 8 \\
\end{array}.$$  

20
To illustrate the entire iRSK algorithm, take $w = 4163276$. Then we have

| $w_1\cdots w_j$ | $P(w_1\cdots w_j)$ | $Q(w_1\cdots w_j)$ |
|------------------|---------------------|---------------------|
| 4                | 4                   | 1                   |
| 41               | 1 1                 | 1 2                 |
|                  | 4                   | 2                   |
| 416              | 1 1 6               | 1 2 3               |
|                  | 4                   | 2                   |
| 4163             | 1 1 3 3             | 1 2 3 4             |
|                  | 4 6 0               | 2 4 4               |
| 41632            | 1 1 2 2 2           | 1 2 3 4 5           |
|                  | 3 3 3               | 2 4 4               |
|                  | 4 6                 | 5 5                 |
| 416327           | 1 1 2 2 2 2 7       | 1 2 3 4 5 6         |
|                  | 3 3 3               | 2 4 4               |
|                  | 4 6                 | 5 5                 |
| 4163276          | 1 1 2 2 2 2 6 6     | 1 2 3 4 5 6 7       |
|                  | 3 3 3 7 7           | 2 4 4 7 7           |
|                  | 4 6                 | 5 5                 |

We leave it to the reader to show that in general $P(w)$ and $Q(w)$ are semistandard Young tableaux.

We can of course extend the definition of iRSK from words to 2-line arrays, just as in RSK (see Subsection 4.3). We shall not need this extension in the present paper.

### 4.2 $\mathcal{M}(u) = \mathcal{M}(w)$ implies $Q(u) = Q(w)$

The goal of this subsection is to prove that if two words $u, w$ have the same $\mathcal{M}$ then they have the same i-recording tableau $Q$. To carry this out we first introduce the following notation.

**Notation 4.3.** Let $\mathcal{W}^{(0)}(w)$ be the diagonal skew-tableau that represents the word $w$. (See the paragraph following Remark 2.10.) For $j \geq 1$, let $\mathcal{W}^{(j)}(w)$ be the tableau obtained from $\mathcal{W}^{(0)}(w)$ by carrying out the first $j$ steps in the calculation of $\mathcal{W}(w) = \text{Kjdt}_{M_{n-1}}(\mathcal{W}^{(0)}(w))$, i.e.,

$$\text{Kjdt}_{B_{n-j}} \circ \cdots \circ \text{Kjdt}_{B_{n-1}}(\mathcal{W}^{(0)}(w)),$$

where $B_i$ is the set of boxes in $M_{n-1}$ containing $i$. 
Let $\mathcal{W}^{(j,0)}(w)$ be the tableau consisting of $\mathcal{W}^{(j)}(w)$ together with the placement of dots used to compute $\mathcal{W}^{(j+1)}(w)$, and for $k \geq 1$ let $\mathcal{W}^{(j,k)}(w)$ be the tableau (including boxes with dots) that results from interchanging the dots with 1, then 2, \ldots, up through $k$.

For example if $w = 4351243$ then

\[
\mathcal{W}^{(0)}(w) = \begin{array}{ccccccc}
\text{ } & 3 & \\
4 & 2 & 1 & \\
3 & 5 & \\
\text{ } & 4
\end{array} \quad \text{and then} \quad \mathcal{W}^{(1,0)}(w) = \begin{array}{ccccccc}
\text{ } & 3 & \\
\bullet & 4 & 1 & \\
\bullet & 3 & 5 & \\
\text{ } & 4
\end{array} \quad \text{and then} \quad \mathcal{W}^{(1,1)}(w) = \begin{array}{ccccccc}
\text{ } & 3 & \\
\bullet & 4 & 1 & \\
\text{ } & 1 & 3 & 5 & \\
\text{ } & 4
\end{array} \quad \text{and then} \quad \mathcal{W}^{(1,2)}(w) = \begin{array}{ccccccc}
\text{ } & 3 & \\
\bullet & 3 & 5 & \\
\bullet & 4 & 1 & \\
\text{ } & 1
\end{array}.
\]

Using this notation we now give an outline of this section. Recall that $\mathcal{Q}(w)$ is completely determined by the shapes of $\mathcal{P}(w_1 \cdots w_j)$ for all $j$. Our strategy is to show that, for all $j$, $\mathcal{M}(w)$ determines (part of) the shape of $\mathcal{W}^{(j)}$ (Lemma 4.8) which in turn determines the shape of $\mathcal{P}(w_1 \cdots w_j)$ (Lemma 4.7).

To facilitate our arguments, observe first that for any word $w$ the shape of $\mathcal{W}^{(j)}(w)$ is such that the first elements in each of the first $n - j$ columns are along the diagonal given by the boxes $(n - j, 1), (n - j - 1, 2), \ldots, (1, n - j)$. It turns out that in what follows, we need only concern ourselves with these $n - j$ initial columns. Since they start on successive rows we introduce the following (unorthodox) coordinate system to reference boxes among these $n - j$ initial columns. In $\mathcal{W}^{(j)}(w)$ we write $\langle k, i \rangle$ to indicate the $k$th box (from the top) in column $i \leq n - j$. For convenience we further define $\langle 0, i \rangle$ when $i < n - j$ to be the box that is directly above $\langle 1, i \rangle$.

For example, consider the tableau

\[
\mathcal{W}^{(4)}(4351243) = \begin{array}{ccccccc}
\text{ } & 1 & 2 & 3 \\
1 & 4 & \\
3 & 5 & \\
\text{ } & 4
\end{array}.
\]

The first $7 - 4 = 3$ columns have topmost boxes that are along a diagonal. Under our new coordinate system, $\langle 2, 3 \rangle$ references the box containing the 4 in the third column. Likewise, $\langle 2, 1 \rangle$ references the box containing the 3 in the first column.
It is important to note that under this coordinate system $⟨k, i⟩$ in $W^{(j)}(w)$ references the same box as $⟨k + 1, i⟩$ in $W^{(j+1)}(w)$ whenever $i + j < n$. Furthermore, when working with the skew tableaux $W^{(j,k)}(w)$ we shall use the coordinate system for $W^{(j)}(w)$. For example, in $W^{(4,2)}(4351243) = \begin{array}{ccc} 1 & 2 & \bullet \ 1 & \bullet & 3 \\ \bullet & 4 & \bullet \ 3 & 5 & 4 \end{array}$, the 1’s are in positions $⟨0,1⟩$ and $⟨0,2⟩$ while $⟨1,3⟩$ references the box containing the 2.

To motivate our first few lemmas, consider the intermediate tableau $x$ in the construction of $P(w)$ where $w = 4351243$: 

Likewise consider the tableaux $W^{(0)}(w), \ldots, W^{(6)}(w)$ in the construction of $W(w)$:

**Notation 4.4.** For any skew tableau $T$ we denote its $i$th column by $T_i$.

Observe that the first column of $P(w)$ is the same as the first column in $W^{(6)}(w)$, and the second column of $P(w)$ is the same as the second column in $W^{(5)}(w)$, and in general

$$W^{(7-i)}_i(w) = P_i(w_1 \cdots w_7).$$

Additionally, we see that the first column of $P(w_1 \cdots w_6)$ is the same as the first column in $W^{(5)}(w)$, and the second column is the same as the second column in $W^{(4)}(w)$, etc. In general we have

$$W^{(6-i)}_i(w) = P_i(w_1 \cdots w_6).$$

We first work to prove that this pattern holds in general.

**Lemma 4.5.** For any word $w$ of length $n$ and $m \leq n$ we have $W_i^{(j)}(w) = W_i^{(j)}(w_1 \cdots w_m)$ for all $i + j \leq m$. 

23
Proof. We begin with a general observation. Assume \( u \) and \( w \) are two words such that for some \( j \) and some \( k \) we have
\[
W_i^{(j-1)}(w) = W_i^{(j-1)}(u),
\]
for all \( i \leq k + 1 \). By definition, it follows that
\[
W_i^{(j)}(w) = W_i^{(j)}(u),
\]
for all \( i \leq k \).

Now observe that if \( j = 0 \) then
\[
W_i^{(0)}(w) = w_i = W_i^{(0)}(w_1 \cdots w_m),
\]
for all \( i \leq m \). By the above observation, it follows that
\[
W_i^{(1)}(w) = w_i = W_i^{(1)}(w_1 \cdots w_m),
\]
for all \( i + 1 \leq m \). Repeating this argument proves our claim. \( \square \)

**Lemma 4.6.** Let \( w \) be a word of length \( n \) and fix \( i \geq 1 \). If
\[
\begin{align*}
W_i^{(n-i)}(w) &= \mathcal{P}_{i-1}(w_1 \cdots w_{n-1}) \quad (3) \\
W_i^{(n-i-1)}(w) &= \mathcal{P}_i(w_1 \cdots w_{n-1}) \quad (4) \\
W_i^{(n-i-1)}(w) &= \mathcal{P}_{i+1}(w) \quad (5)
\end{align*}
\]
then \( W_i^{(n-i)}(w) = \mathcal{P}_i(w) \).

Proof. To see that the first entries of \( \mathcal{P}_i(w) \) and \( W_i^{(n-i)}(w) \) are the same, define \( s_i = \min\{w_i, \ldots, w_n\} \) for \( 1 \leq i \leq n \). By the definition of \( W^{(n-i)}(w) \), the first entry of \( W_i^{(n-i)}(w) \) is \( s_i \).

Now for \( 1 \leq i \leq n \) define \( t_i \) as follows. Let \( a_1 \) be the smallest entry in \( w \) and let the rightmost occurrence of \( a_1 \) be in position \( r_1 \). For \( 1 \leq i \leq r_1 \), let \( t_i = a_1 \). Then let \( a_2 \) be the smallest entry occurring to the right of position \( r_1 \) in \( w \) and let the rightmost occurrence of \( a_2 \) in \( w \) be in position \( r_2 \). For \( r_1 + 1 \leq i \leq r_2 \), let \( t_i = a_2 \). Continue in this way. By the definition of \( i \)-insertion, the first entry of \( \mathcal{P}_i(w) \) is \( t_i \). It is easy to see, by thinking about the way the \( s_i \) are chosen, that \( s_i = t_i \) for \( 1 \leq i \leq n \).

Having shown that the first entries of \( \mathcal{P}_i(w) \) and \( W_i^{(n-i)}(w) \) are equal, we now assume that their \((k - 1)\)st entries are equal with the aim of proving that their \( k \)th entries also coincide. To this end we consider two cases depending on whether there is ever a dot in position \( (k - 1, i) \) during the transformation from \( W^{(n-i-1)}(w) \) to \( W^{(n-i)}(w) \).

**Case 1:** The position \((k - 1, i)\) never contains a dot.

Assume for the moment that \( i \geq 2 \) and let the following diagrams depict \( W^{(n-i-1)}(w) \) and \( W^{(n-i)}(w) \), respectively, where \( a \) is the \((k - 1)\)st element in the \( i \)th column on the left. By the assumption of Case 1, \( a \) is the \( k \)th element in the \( i \)th column on the right. (Here each letter denotes a positive integer or \( \infty \), indicating an empty box. It is convenient here to use \( \infty \) rather than 0.)

24
Note that (3) implies that \( b \) is in position \( (k - 1, i - 1) \) in \( \mathcal{P}(w_1, \ldots, w_{n-1}) \). Additionally, (4) implies that \( a \) is in position \( (k - 1, i) \) in \( \mathcal{P}(w_1 \cdots w_{n-1}) \). Finally, by our inductive hypothesis we know that \( c \) is in position \( (k - 1, i) \) in \( \mathcal{P}(w) \).

Now assume \( a < \infty \). As \( c, b < a \) it follows by the definition of i-insertion that \( a \) must be in position \( (k, i) \) in \( \mathcal{P}(w) \), as required.

If \( a = \infty \) then the above argument implies, by the definition of i-insertion, that the position \( (k, i) \) in \( \mathcal{P}(w) \) must be empty.

Finally, if \( i = 1 \) then \( a \) is in the first column of \( \mathcal{P}(w_1 \cdots w_{n-1}) \) and the same argument, in the absence of \( b \), yields the desired result.

**Case 2:** The position \( (k - 1, i) \) eventually contains a dot.

Assume for the moment that \( i \geq 2 \) and let the following diagrams depict \( \mathcal{W}^{(n-i-1)}(w) \) and \( \mathcal{W}^{(n-i)}(w) \) respectively, where \( a \) is the \( k \)th element in the \( i \)th column on the left and \( b' \) is the \( k \)th element in the \( i \)th column on the right.
It follows from our assumptions that in $\mathcal{P}(w_1 \cdots w_{n-1})$ and $\mathcal{P}(w)$ the $i-1, i, i+1$ columns and $k-1, k$ rows give, respectively, the rectangles

$$
\begin{array}{|c|c|c|}
\hline
  d & b & * \\
\hline
  * & a & * \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
  * & e & * \\
\hline
  * & a' & c \\
\hline
\end{array}
$$

where (3) yields the placement of $d$, (4) yields the placement of $a, b$, (5) yields the placement of $c$, and finally, our inductive hypothesis yields the placement of $e$. It now suffices to show that $a' = b'$.

Since position $\langle k-1, i \rangle$ in $\mathcal{W}^{(n-i-1)}(w)$ eventually contains a dot it follows that $b' = \min(a, c)$. Further, one of positions $\langle k-2, i-1 \rangle$ or $\langle k-2, i \rangle$ in $\mathcal{W}^{(n-i-1)}(w)$ must eventually contain a dot during this transformation and hence $d = b$ or $e = b$. (Note when $k = 2$ the transformation from $\mathcal{W}^{(n-i-1)}(w)$ to $\mathcal{W}^{(n-i)}(w)$ begins by placing dots at the top of columns $1, \ldots, i$.)

By definition of iRSK we have $a' \leq a, c$ and so $a' \leq \min(a, c)$. For a contradiction assume $a' < \min(a, c)$, where one or both of $a$ or $c$ might be $\infty$. This means that $a'$ bumps $a$ and that this is the rightmost occurrence of $a'$ in the $k$th row of $\mathcal{P}(w)$. Consequently, $b = a'$ and $d < b$. As $\mathcal{P}(w)$ is increasing along columns we have $e < a' = b$. This contradicts the fact that either $b = d$ or $b = e$.

If $i = 1$ then $d$ is absent yet an argument analogous to that in the previous paragraph proves that $a' = b'$ as needed.

Lemma 4.7. If $w \in \mathfrak{W}_n$ then

$$
\mathcal{W}_i^{(n-i)}(w) = \mathcal{P}_i(w)
$$

for all $i \leq n$.

Proof. We proceed by induction on $n$ and note that the result clearly holds when $n = 1$. Now fix some $n > 1$ and some word $w$ with length $n$. Observe that when $i = n$ we have

$$
\mathcal{W}_n^{(0)}(w) = w_n = \mathcal{P}_n(w).
$$

In light of this we further proceed by (reverse) induction on $i$, the column index. Fix $i < n$ and assume that

$$
\mathcal{W}_{i+1}^{(n-i-1)}(w) = \mathcal{P}_{i+1}(w).
$$

By Lemma 4.5 and our inductive hypothesis on $n$ we have

$$
\mathcal{W}_i^{(n-i-1)}(w) = \mathcal{W}_i^{(n-i-1)}(w_1 \cdots w_{n-1}) = \mathcal{P}_i(w_1 \cdots w_{n-1})
$$
and

\[ W_{i-1}^{(n-i)}(w) = W_{i-1}^{(n-i)}(w_1 \cdots w_{n-1}) = P_{i-1}(w_1 \cdots w_{n-1}). \]

It now follows by Lemma 4.6 that

\[ W_{i-1}^{(n-i)}(w) = P_i(w) \]

and hence the result holds for all words of length \( n \) as needed. \( \square \)

Lemma 4.8. For \( i + j \leq n \) we have

\[ |W_i^{(j)}(w)| = |\{ x \in [n-j, n] | x \notin M_i(w) \}|. \]

Proof. First observe that this statement trivially holds for \( j = 0 \) since \( n \notin M_i(w) \) and \( W_i^{(0)} = w_i \) for all \( i \leq n \). Proceeding inductively, assume the claim holds for some \( j < n - i \) and consider the transformation from \( W^{(j)}(w) \) to \( W^{(j+1)}(w) \). A consequence of \( W^{(j)}(w) \) being an increasing tableau is that during this transformation there is at most one dot in the \( i \)th column at any given step. Further, any time a dot enters this column the number of elements in this column decreases by exactly 1. Likewise, whenever a dot exits this column the number of elements in this column increases by exactly 1.

Now consider two cases. First assume there is no dot in the \( i \)th column in \( W^{(j,n)}(w) \). This means that \( n - (j + 1) \notin M_i(w) \). From the fact that \( i < n - j \) (we assumed \( j < n - i \) above), it follows that there is a dot in column \( i \) in \( W^{(j,0)}(w) \). As we start with a dot in this column and end with no dot, it follows from the first paragraph that \( |W_i^{(j+1)}(w)| = 1 + |W_i^{(j)}(w)| \). By induction we now have

\[ |\{ x \in [n-(j+1), n] | x \notin M_i(w) \}| = 1 + |\{ x \in [n-j, n] | x \notin M_i(w) \}| = |W_i^{(j+1)}(w)|. \]

Next, assume there is a dot in the \( i \)th column in \( W^{(j,n)}(w) \) so that \( n - (j + 1) \notin M_i(w) \).

By reasoning similar to the preceding paragraph, we have \( |W_i^{(j+1)}(w)| = |W_i^{(j)}(w)| \) and hence, by induction,

\[ |\{ x \in [n-(j+1), n] | x \notin M_i(w) \}| = |\{ x \in [n-j, n] | x \notin M_i(w) \}| = |W_i^{(j+1)}(w)|. \]

This completes our proof. \( \square \)

Lemma 4.9. Fix \( u, w \in \mathfrak{M}_n \). If \( M(u) = M(w) \) then \( Q(u) = Q(w) \).

Proof. Let \( \lambda^{(j)} \) be the shape of \( P(w_1 \cdots w_j) \). By our definition we know that \( Q(w) \) is completely determined by the sequence \( \varnothing \leq \lambda^{(1)} \leq \cdots \leq \lambda^{(n)} \). Therefore to prove the lemma it suffices to show that \( M(w) \) completely determines the \( \lambda^{(j)} \)'s.

To this end observe that, by Lemma 4.7, Lemma 4.5 and Lemma 4.8, the height of the \( i \)th column of \( \lambda^{(j)} \), for all \( i \leq j \), is given by

\[ |P_i(w_1 \cdots w_j)| = |W_i^{(j-i)}(w)| = |\{ x \in [n-j+i, n] | x \notin M_i(w) \}|. \]

In other words, \( \lambda^{(j)} \) is determined by the first \( j \) columns of \( M(w) \). This implies our result. \( \square \)
To accomplish the goals of this section, we shall need an alternative description of the iRSK algorithm. Not surprisingly, this description will rest on well-known properties of classical RSK. We begin by recalling RSK and some of these classical properties.

First we recall that a 2-line array is an array of positive integers consisting of two rows of equal length, such that the columns are ordered lexicographically from left to right, with the top entries taking precedence. We call the elements of the top row indices, and those of the bottom row values. The length of such an array is the length of these rows.

For any 2-line array \( A \) we define \( A^\star \) to be the 2-line array obtained by switching the top and bottom rows of \( A \) and then placing the columns in lexicographic order.

We next recall that RSK is a bijection between all 2-line arrays of length \( n \) and all pairs of semistandard Young tableaux of size \( n \) with the same shape. Going forward we write

\[ RSK(A) = (P(A), Q(A)) \]

where \( P(A) \) and \( Q(A) \) are the usual insertion and recording tableaux, respectively. Since any word \( w \) of length \( n \) can be viewed as a 2-line array with values \( w_1, \ldots, w_n \) and indices \( 1, \ldots, n \), we can also write \( P(w) \) and \( Q(w) \). A fundamental connection between RSK and inverses of 2-line arrays is the following. (For a proof see Chapter 4.2 in [5].)

**Fact 4.10.** For any 2-line array \( A \) with \( RSK(A) = (P, Q) \) we have

\[ RSK(A^\star) = (Q, P). \]

There is an important equivalence relation on words that is related to RSK. Recall that two words \( u, w \) are said to be Knuth equivalent, in which case we write \( u \equiv w \), provided that one can be obtained from the other by a finite number of moves of the following two types:

\[
\begin{align*}
\cdotsaba & \leftrightarrow \cdotsaab \quad (a < b \leq c) \\
\cdotsacba & \leftrightarrow \cdotsabac \quad (a \leq b < c).
\end{align*}
\]

Knuth equivalence and RSK are connected by the following classic result, which we shall need momentarily. (For a proof see Theorem 3.4.3 in [12].)

**Fact 4.11.** If \( u, w \) are words, then \( u \equiv w \) if and only if \( P(u) = P(w) \).

With these basics in hand we now introduce a variant of Knuth equivalence that is tailored to our needs.

**Definition 4.12.** We say two words \( u \) and \( w \) are i-Knuth equivalent and write \( u \equiv_i w \) provided that one can be obtained from the other by a finite number of the following moves:

\[
\begin{align*}
\cdotsaba & \leftrightarrow \cdotsaab \quad (a < b \leq c) \\
\cdotsacba & \leftrightarrow \cdotsabac \quad (a \leq b < c).
\end{align*}
\]
We call the first type of move an inflation, and we say that any two words differ in multiplicity provided that they differ by a finite number of inflations.

For our purposes, there are two salient properties of this definition. The first is the obvious fact that if \( u \equiv w \) then \( u \equiv_i w \). The second property is the subject of our next lemma.

**Lemma 4.13.** If \( u \equiv_i w \) then \( u \sim_{rd} w \).

**Proof.** We first show that if \( u \equiv_i w \) then \( \text{lds}(u) = \text{lds}(w) \). This is clear if \( u \) and \( w \) differ by a single inflation. On the other hand, if \( u \equiv w \) then, by Fact 4.11, \( P(u) = P(w) \) so by Exercise 1 in Chapter 3 of [5], \( \text{lds}(u) = \text{lds}(w) \). So \( u \equiv_i w \) implies that \( \text{lds}(u) = \text{lds}(w) \).

It now suffices to show that if \( u \equiv_i w \) and \( u' \) and \( w' \) are obtained from \( u \) and \( w \), respectively, by deleting all occurrences of the largest (respectively, smallest) letter then \( u' \equiv_i w' \). If \( u \) and \( w \) differ by a single inflation then either \( u' = w' \) or \( u' \) and \( w' \) differ by a single inflation, so \( u' \equiv_i w' \). On the other hand, if \( u \) and \( w \) differ by a single Knuth move then by Lemma 3 in Chapter 3 of [5] we have \( u' \equiv w' \) and so \( u' \equiv_i w' \).

In order to bring CID information into the picture we need the following definition.

**Definition 4.14.** For any \( w \in \mathcal{W}_n \) we define \( w^\star \) to be the word obtained as follows. Let \( A \) be the 2-line array representing \( w \), i.e., its values are \( w_1, \ldots, w_n \) and its indices are \( 1, \ldots, n \). Define \( w^\star \) to be the word given by the values of \( A^\star \).

For example, if \( w = 4113252 \), then

\[
A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 1 & 3 & 2 & 5 & 2 \end{pmatrix}
\quad \text{and} \quad
A^\star = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 7 & 4 & 1 & 6 \end{pmatrix},
\]

giving \( w^\star = 2357416 \). Observe that \( w^\star \) is always a permutation and so the map \( w \mapsto w^\star \) is not an involution.

To prove our main theorems we require two lemmas.

**Lemma 4.15.** Let \( u, w \) be words. Then \( u \sim_{cd} w \) if and only if \( u^\star \sim_{rd} w^\star \).

**Proof.** Fix an interval \([i, j]\). Observe that we have a length-preserving correspondence between the decreasing sequences of \( u_i \cdots u_j \) and the decreasing sequences of \( u^\star \) with values in \([i, j]\), given by the map

\[
u_{x_1} > \cdots > u_{x_k} \mapsto x_k > \cdots > x_1.
\]

Note that this correspondence relies on the fact that in any 2-line array a strictly decreasing sequence is, by definition, indexed by a strictly increasing sequence. The lemma is now a consequence of this fact.

**Lemma 4.16.** For every word \( w \),

\[
w \equiv_i \text{Read} \left( \mathcal{P}(w) \right) \quad \text{and} \quad w^\star \equiv_i \text{Read} \left( \mathcal{Q}(w) \right).
\]
We postpone the proof of this result since it is a bit involved and requires new ideas. Instead we turn to the main theorems of the section.

**Theorem 4.17.** Let \( u, w \in \mathcal{W}_n \). If \( Q(u) = Q(w) \) then \( u \sim_{cd} w \).

**Proof.** By Lemma 4.16 we have
\[
 u^* \equiv_i \text{Read}(Q(u)) \quad \text{and} \quad w^* \equiv_i \text{Read}(Q(w)).
\]
Since \( Q(u) = Q(w) \) we have \( u^* \equiv_i w^* \), so by Lemma 4.13 we have \( u^* \sim_{vd} w^* \). Lemma 4.15 now implies \( u \sim_{cd} w \), as desired. \( \square \)

**Remark 4.18.** In light of this theorem, a natural question is whether \( Q \) captures CID data. Unfortunately, it does not, as evidenced by the words 11 and \( u = 12 \) which are not CID equivalent but do have the same \( Q \) tableau.

We are now in position to prove Theorem 2.14.

**Proof of Theorem 2.14.** If \( \mathcal{M}(u) = \mathcal{M}(w) \) then by Lemma 4.9 and Theorem 4.17 we have \( u \sim_{cd} w \).

Now consider the reverse words \( u^r \) and \( w^r \). By taking transposes we have \( \mathcal{M}(u^r) = \mathcal{M}(w^r) \) and so \( u^r \sim_{cd} w^r \) by the above argument. Equivalently, we have \( u \sim_{ci} w \). Combining this with the result of the preceding paragraph gives \( u \sim_{cid} w \) as needed. \( \square \)

**4.3.1 Proof of Lemma 4.16**

To prove this lemma we make use of an alternative description of the iRSK algorithm, in terms of 2-line arrays.

**Definition 4.19.** For any 2-line array \( A \) define the triple
\[
 f(A) = (B, p, q),
\]
where \( p \) and \( q \) are the top rows of \( P(A) \) and \( Q(A) \), respectively, and \( B = \text{RSK}^{-1}(P(A), Q(A)) \) where \( \widehat{T} \) denotes the tableau obtained from a tableau \( T \) by deleting the top row.

**Definition 4.20.** Let \( w \in \mathcal{W}_n \) and consider the construction of the i-insertion tableau \( P(w) \). For each \( i > 0 \) and \( j \leq n \), let \( A_{ij} \) be the 2-line array whose values are given by the sequence of elements bumped from row \( i \) during the insertion
\[
 P(w_1 \cdots w_{j-1}) \leftarrow w_j
\]
and whose indices are \( j \). We then define
\[
 A_i = A_{i1} \cdots A_{in}
\]
and refer to this array as the \( \textit{i} \text{th bumped array for } w \). We define \( A_0 \) to be the 2-line array given by the word \( w \).

We make the same definition with RSK in place of iRSK. Context will make it clear which variant we are referencing.
For example, take the word \( w = 5143642 \) with \( \text{RSK}(w) \) given by

\[
\begin{array}{cccc}
1 & 2 & 4 & 3 \\
3 & 6 & 4 & 5 \\
4 & 5 & 7 & 6 \\
\end{array}
\quad \begin{array}{cccc}
1 & 3 & 5 & 4 \\
2 & 6 & 4 & 7 \\
4 & 7 & 5 & 6 \\
\end{array}
\]

A simple check shows that the bumped arrays in this case are

\[
A_1 = \begin{pmatrix} 2 & 4 & 6 & 7 \\ 5 & 4 & 6 & 3 \end{pmatrix} \quad A_2 = \begin{pmatrix} 4 & 7 \\ 5 & 4 \end{pmatrix} \quad A_3 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}.
\]

Observe that the insertion and recording tableaux for \( A_i \) are precisely the tableaux obtained by deleting the first \( i \) rows of \( P \) and \( Q \), respectively. For example \( \text{RSK}(A_1) \) gives

\[
\begin{array}{cccc}
3 & 6 & 2 & 6 \\
4 & 5 & 4 & 7 \\
\end{array}
\]

To state our next result we shall need the idea of two 2-line arrays \( A \) and \( B \) differing in multiplicity. By this we simply mean that \( A \) can be obtained from \( B \) by a finite number of the following moves:

\[
\begin{array}{cccc}
\cdots & a & \cdots \\
\cdots & b & \cdots
\end{array} \quad \leftrightarrow \quad \begin{array}{cccc}
\cdots & a & a & \cdots \\
\cdots & b & b & \cdots
\end{array}
\]

Note that if \( A \) and \( B \) differ in multiplicity then the words given by their values are i-Knuth equivalent.

**Lemma 4.21.** Fix a word \( w \) of length \( n \). Let \( A_0, \ldots, A_k \) be the bumped arrays from the construction of \( \mathcal{P}(w) \) and let the \( i \)th rows of \( \mathcal{P}(w) \) and \( \mathcal{Q}(w) \) be \( p_i \) and \( q_i \), respectively. Then there exist 2-line arrays \( A_i^+, \ldots, A_k^+ \) such that \( A_i \) and \( A_i^+ \) differ in multiplicity and

\[
f(A_i^+) = (A_{i+1}, p_{i+1}, q_{i+1}).
\]

**Proof.** By definition of iRSK there exist arrays \( A_i^+ \) which differ in multiplicity from \( A_i \) so that if

\[
\text{RSK}(A_i^+) = (P, Q)
\]

then \( p_{i+1} \) and \( q_{i+1} \) are the top rows of \( P \) and \( Q \), respectively, and the first bumped array from the construction of \( P(A_i^+) \) is \( A_{i+1} \). In other words,

\[
\text{RSK}(A_{i+1}) = (\widehat{P}, \widehat{Q})
\]

and so \( f(A_i^+) = (A_{i+1}, p_{i+1}, q_{i+1}) \) as needed.

To prove our next lemma we make use of the following well-known fact. A proof of this can be found in Proposition 1 of Chapter 2 in [5].
Fact 4.22. For any word $w$, we have $w \equiv \text{Read}(P(w))$.

Since Knuth equivalence implies i-Knuth equivalence, it follows immediately that $w \equiv_i \text{Read}(P(w))$. We make use of this fact in this form below.

Lemma 4.23. Let $A$ be a 2-line array and set $f(A) = (B, p, q)$. Denote by $\alpha$ and $\beta$ the words given by the values of $A$ and $B$, respectively. Then we have

$$\alpha \equiv_i \beta \cdot p$$

and

$$f(A^*) = (B^*, q, p).$$

Proof. Let $\text{RSK}(A) = (P, Q)$ so that $P(\alpha) = P$. So $\alpha \equiv_i \text{Read}(P)$. By definition of $B$ we also know that $P(\beta) = \tilde{P}$ and so $\beta \equiv_i \text{Read}(\tilde{P})$. Consequently,

$$\alpha \equiv_i \text{Read}(P) = \text{Read}(\tilde{P}) \cdot p \equiv_i \beta \cdot p,$$

which proves the first claim. The second claim follows immediately by applying Fact 4.10 as this shows that $\text{RSK}(A^*) = (Q, P)$ and $\text{RSK}(B^*) = (\tilde{Q}, \tilde{P})$.

Proof of Lemma 4.23. To prove the first claim let $A_0, \ldots, A_k$ and $A_0^+, \ldots, A_k^+$ be as in Lemma 4.21 and for $0 \leq j \leq k$ let $\alpha_j$ and $\alpha_j^+$ be the words obtained from the values of $A_j$ and $A_j^+$, respectively. So

$$f(A_j^+) = (A_{j+1}, p_{j+1}, q_{j+1}),$$

where $p_j$ and $q_j$ are the $j$th rows of $P(w)$ and $Q(w)$, respectively. Furthermore, $\alpha_j$ and $\alpha_j^+$ differ in multiplicity and so $\alpha_j \equiv_i \alpha_j^+$. By repeated applications of (6) in Lemma 4.23 we have

$$w \equiv_i \alpha_0^+ \equiv_i \alpha_1 p_1 \equiv_i \alpha_1^+ p_1 \equiv_i \alpha_2 p_2 p_1 \equiv_i \cdots \equiv_i p_k \cdots p_1 = \text{Read}(P(w)).$$

To prove the second claim set $C_j = A_j^*$ and $C_j^* = (A_j^+)^*$. By Equation (7) in Lemma 4.23 we have

$$f(C_j^*) = (C_{j+1}, q_{j+1}, p_{j+1}).$$

Since $w^*$ is the word given by the values in $C_0 = A_0^*$, an argument analogous to that given for the first claim proves the second claim.

5. The conjecture of Guo and Poznanović

The purpose of this section is to prove the conjecture of Guo and Poznanović (Theorem 15), concerning the lengths of the longest ne and se chains in certain 01-fillings of moon polyominoes. For the convenience of the reader, we recall that $N(M; n; ne = u, se = v)$ denotes the number of 01-fillings of $M$ with exactly $n$ 1’s such that each column contains at most one 1, and the lengths of the longest ne and se chains in $M$ are $u$ and $v$, respectively.
Proof of Theorem 1.1. We will prove that, starting with any moon polyomino $M$, we can perform a sequence of transformations of $M$ that turn it first into a stack polyomino and then into a Ferrers board, preserving the multiset of row lengths and $N(M; n; ne = u; se = v)$ at each step. Since a Ferrers board is completely determined by its multiset of row lengths, this will prove the theorem.

To begin, let $M$ be a moon polyomino and let $c$ be the leftmost column of $M$. Let $R$ be the largest rectangle in $M$ that contains column $c$ and let $M'$ be the moon polyomino obtained from $M$ by deleting $c$ and adding a new column $c'$ at the right end of $R$. Let $R'$ be the rectangle in $M'$ that has the same size as $R$, so that $R'$ is “$R$ shifted one unit to the right.”

It is clear that $M$ and $M'$ have the same multiset of row lengths. We will show that

$$N(M; n; ne = u, se = v) = N(M'; n; ne = u, se = v). \quad (8)$$

To each 01-filling of $M$ that has at most one 1 in each column we associate a corresponding filling of $M'$ as follows. For boxes of $M$ outside $R$ we retain the given filling. We modify the filling of $R$ to obtain a filling of $R'$ as follows. If $c$ is empty we leave $c'$ empty and retain the given filling of the rest of $R$. If $c$ is not empty, we let $w$ be the word whose matrix representation is given by the nonempty rows and columns of the given filling of $R$ and we take the word $w'$ associated to $w$ by Theorem 3.2, recalling that the values that occur in $w'$ are precisely those that occur in $w$. We place the matrix representation of $w'$ in the columns of $R'$ that were not empty in $R$, and $c'$. This leaves empty the rows of $R$ that were empty in the given filling of $M$. 

33
To prove (8) it suffices to show that for every 01-filling of $M$ with at most one 1 in each column, and any maximal rectangle $X$ of $M$, the length of the longest ne (respectively, se) chains in $X$ is the same as the length of the longest ne (respectively, se) chains in the corresponding maximal rectangle $X'$ of $M'$ under the associated filling of $M'$. For $R$ and $R'$ this is clear, since the bijection of Theorem 3.2 preserves the lengths of the longest increasing and decreasing sequences in a word.

Now suppose $X$ is a maximal rectangle wider than $R$, so that $X$ consists of a rectangle $X_1$ contained in $R$ with another rectangle $X_2$ of the same height adjoined to its righthand side, as shown above.

Any ne chain of maximal length $\ell$ in $X$ consists of an initial segment contained in some interval $I$ of the rows of $R$, and a terminal segment in $X_2$. The initial segment must have maximal length among ne chains in the interval $I$ of rows of $R$, so by Theorem 3.2 the interval $I$ of rows of $R'$ contains a ne chain of the same length. This sequence, adjoined to the terminal segment in $X_2$, gives a ne chain of length $\ell$ in $X'$. A similar argument applies to se chains. If we then repeat the argument, interchanging the roles of $X'$ and $X$, we conclude that the lengths of the longest ne and se chains in $X$ and $X'$ coincide.

If $X$ is a maximal rectangle narrower than $R$, then a ne chain of maximal length $\ell$ in $X$ consists of initial and terminal segments outside $R$ and a middle segment contained in some interval $J$ of the columns of $R$. The middle segment must have maximal length among ne chains in the interval $J$ of columns of $R$, so by Theorem 3.2 the interval $J$ of columns of $R'$ contains a ne chain of the same length. This chain, together with the initial and terminal segments outside $R$, gives us a ne chain of length $\ell$ in $X'$. As in the preceding paragraph, we can now conclude that the lengths of the longest ne and se chains in $X$ and $X'$ coincide.

By iterating the argument contained in the preceding five paragraphs, we can convert $M$ into a stack polyomino $M^\diamond$, preserving the multiset of row lengths and $N(M; n; ne = u, se = v)$. To complete the proof, we can invoke the result of Guo and Poznanović ([7], Theorem 1), or we can convert the stack polyomino into a Ferrers board (in French notation, with longest rows at the bottom) by doing an argument similar to the above, starting with the bottom row $r$ of $M^\diamond$ instead of the left column, taking the largest rectangle in $M^\diamond$ that contains $r$, and adding a row $r'$ to the top of this rectangle and deleting $r$. We then use Theorem 3.1 in place of Theorem 3.2 to complete an argument similar to the above. (In this case, when we place the word $w'$ in the columns of $R'$ that were not empty in $R$, we must be careful to place the entries, from smallest to largest, into the rows of $R'$ that were not empty in $R$, and $r'$.)

The strategy of transforming a stack polyomino to a Ferrers board by repeatedly moving up the bottom row is the same strategy used by Guo and Poznanović (and the same idea, moving columns of a moon polyomino instead of rows, was employed earlier by Rubey in [11], Theorem 5.3). In carrying out this strategy for stack polyominoes, Guo and Poznanović used methods (Hecke insertion and Hecke growth diagrams) that enabled them to deal with longest ne and se chains in the maximal rectangles that occurred. But new methods were needed to deal with the maximal rectangles that occur in transforming a moon polyomino into a stack polyomino.
References

[1] Anders S. Buch, Andrew Kresch, Mark Shimozono, Harry Tamvakis, and Alexander Yong. Stable Grothendieck polynomials and K-theoretic factor sequences. *Mathematische Annalen*, 340:359–382, 2005.

[2] Anders S. Buch and Matthew J. Samuel. K-theory of minuscule varieties. *arXiv: Algebraic Geometry*, 2013.

[3] William Y. C. Chen, Eva Yu-Ping Deng, Rosena R. X. Du, Richard P. Stanley, and Catherine Huafai Yan. Crossings and nestings of matchings and partitions. *Transactions of the American Mathematical Society*, 359:1555–1575, 2005.

[4] Anna de Mier. On the symmetry of the distribution of k-crossings and k-nestings in graphs. *Electr. J. Comb.*, 13, 11 2006.

[5] William Fulton. *Young tableaux: with applications to representation theory and geometry*. London Mathematical Society Student Texts. Cambridge University Press, 1996.

[6] Christian Gaetz, Michelle Mastrianni, Rebecca Patrias, Hailee Peck, Colleen Robichaux, David Schwein, and Ka Yu Tam. K-Knuth equivalence for increasing tableaux. *Electron. J. Comb.*, 23:P1.40, 2016.

[7] Ting Guo and Svetlana Poznanović. Hecke insertion and maximal increasing and decreasing sequences in fillings of stack polyominoes. *Journal of Combinatorial Theory, Series A*, 176:105304, 11 2020.

[8] Jakob Jonsson. Generalized triangulations and diagonal-free subsets of stack polyominoes. *Journal of Combinatorial Theory, Series A*, 112(1):117–142, 2005.

[9] Jakob Jonsson and Volkmar Welker. A spherical initial ideal for Pfaffians. *Illinois Journal of Mathematics*, 51:1397–1407, 2006.

[10] Christian Krattenthaler. Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes. *Adv. Appl. Math.*, 37:404–431, 2006.

[11] Martin Rubey. Increasing and decreasing sequences in fillings of moon polyominoes. *Adv. Appl. Math.*, 47:57–87, 2011.

[12] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.

[13] Marcel Paul Schützenberger. Quelques remarques sur une construction de Schensted. *Mathematica Scandinavica*, 12:117–128, 1963.
[14] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[15] Hugh Thomas and Alexander Yong. A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus. *Algebra & Number Theory*, 3:121–148, 2007.

[16] Hugh Thomas and Alexander Yong. Longest increasing subsequences, Plancherel-type measure and the Hecke insertion algorithm. *Adv. Appl. Math.*, 46:610–642, 2011.