BEHAVIOR AS $t \to \infty$ OF SOLUTIONS OF A PROBLEM IN MATHEMATICAL PHYSICS

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Abstract. A class of solutions, decaying as $t \to \infty$, of a two-dimensional model problem on the oscillations of an ideal rotating fluid in some domains with angular points is constructed explicitly. The existence of solutions whose $L_2$-norms decrease more rapidly than any negative power of $t$, is established.

INTRODUCTION

In the paper, the first initial-boundary problem for the Poincaré–Sobolev equation is considered,

$$
\frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) + \frac{\partial^2 p}{\partial y^2} = 0, \quad (x, y; t) \in D \times (0, \infty), \quad D \subset \mathbb{R}^2,
$$

$$
p|_{\partial D \times (0, \infty)} = 0,
$$

$$
p|_{t=0} = p_0(x, y), \quad p_t|_{t=0} = p_1(x, y), \quad (p_0|_{\partial D} = 0, p_1|_{\partial D} = 0),
$$

where $D$ is a bounded domain with piecewise smooth boundary and $D$ satisfies the cone condition. Let functions $p_0$ and $p_1$ belong to the Sobolev space $\tilde{W}^{1,2}(D)$, which is a completion of the set $C_0^\infty(D)$ of infinitely differentiable functions whose supports are contained in $D$ with respect to the norm generated by the inner product

$$
(f, g)_1 = \iint_D (\nabla f, \nabla g) \, dx \, dy.
$$

By a generalized solution of the problem (in the sense of the theory of distributions, or the so-called generalized functions) we mean the functions $p(x, y; t)$ with values in $\tilde{W}^{1,2}(D)$ that are twice continuously differentiable with respect to $t$ and satisfy condition (3) and the relation

$$
\int_D (p_{txx} \varphi_x + p_{txy} \varphi_y + p_{yy} \varphi_y) \, dD = 0 \quad \forall t \in (0, \infty)
$$

keeps for any function $\varphi \in C_0^\infty(D)$. If the function $p$ is sufficiently smooth here, then this generalized solution is a classical solution.

This problem arises in hydrodynamics when describing small oscillations of a rotating ideal fluid in the following model two-dimensional case, i.e., under the assumption that the components of velocity and the pressure of the fluid depend only on time and on...
two spatial variables, and the domain (the vessel) filled by the fluid is the cylinder $Q = \{(x, y, z) | (x, y) \in D, z \in \mathbb{R}\}$. A linearized system of equations describing the dynamics of a rotating fluid was first considered by H. Poincaré in [1]. S. L. Sobolev in his well-known paper [2] initiated the investigation of qualitative properties of solutions of this system, and also of related equations of type (1) in diverse domains. To study problem (1)–(3), it is natural to introduce the following operator $A$ acting on the Sobolev space $\overset{\circ}{W}^{\frac{1}{2}}(D)$. The operator $A$ is defined on any smooth function $h \in C^\infty_0(D)$ as the solution of the problem

\[ \Delta(Ah) = \frac{\partial^2 h}{\partial y^2}, \quad Ah \in \overset{\circ}{W}^{\frac{1}{2}}(D), \]

and then is extended by continuity to a bounded operator on $\overset{\circ}{W}^{\frac{1}{2}}(D)$ which is selfadjoint with respect to the inner product (4). Using the operator $A$, we can represent problem (1)–(3) as an abstract Cauchy problem:

\[ \Psi'' = -A\Psi, \quad \Psi(0) = \Psi_0, \quad \Psi'(0) = \Psi_1, \]

where $t \mapsto \Psi(x, y, t)$ stands for a function with values in $\overset{\circ}{W}^{\frac{1}{2}}(D)$, and $\Psi_0, \Psi_1 \in \overset{\circ}{W}^{\frac{1}{2}}(D)$ (see, for example, [3]). One of the main problems arising in the study of problem (7) is that on the behavior of the solutions of (7) as $t \to \infty$, which is closely related to the structure of the spectrum of the operator $A$, as is well known.

As is also well known, for any domain $D$, the spectrum of the operator $A$ is the closed interval $[0, 1]$; however, the qualitative structure of the spectrum substantially depends on the shape of the domain. The investigation of the spectral properties of the operator $A$ was initiated in [3] and continued by many other authors. A rich bibliography concerning the problem and its surrounding can be found in [4]–[10]. The structure of the spectrum of the operator $A$ is known completely in two cases only, namely, if $\partial D$ is an ellipse or if it is a rectangle, provided that these figures are symmetric with respect to the axis $Oy$. In these cases the spectrum of the operator $A$ is purely point, i.e. $A$ admits a complete system of eigenfunctions. Therefore, in this domains all solutions of problem (7) are almost periodic in time. At the same time (see [6]), arbitrary small modifications of the boundary $\partial D$ can result in the occurrence of a continuous spectrum for the operator $A$, whereas, for some deformations of the boundary of the domain, a singular component of the spectrum can appear (see [10]).

The present paper is devoted to the study of the problem (7) for the case in which $D$ is a triangle $\square$

\[ D := \{(x, y) \mid 0 < x < \frac{1}{\alpha}, 0 < y < \alpha x\}, \quad (0 < \alpha < +\infty). \]

It follows from the papers [11, 12] of the author that the spectrum of the operator $A$ is purely continuous in this case. Therefore, the forthcoming investigation of the behavior of solutions of problem (7), which are no almost periodic functions of $t$, and, in particular, the existence problem for solutions decaying in time, is closely related to the properties of differential solutions of the spectral equation for the operator $A$. In Section 4 of the present paper we explicitly construct a class of differential solutions of this kind, and also

\footnote{The results of the paper can be extended (with the corresponding modifications) to a rather large class of domains with angular points (see the remark at the end of the paper). However, in the main text, we deliberately avoid any generalizations to maximally simplify the presentation of the material.}
prove that the spectrum of the operator $A$ is absolutely continuous on the subspace which is the closure of the linear span of these differential solutions. This enables us to explicitly construct a class of solutions of problem (7) whose $L^2$-norm decays as $t \to \infty$ (see Section 2). Moreover, it turns out that some solutions of this class tend to zero more rapidly than any negative power of $t$, and, in the course of time, the entire “energy” of these solutions turns out to be concentrated in an arbitrarily small neighborhood of one of the vertices of the triangle $D$. On one hand, this result agrees with the possible behavior of solutions of the problem under consideration in a neighborhood of the angular points, in the form predicted in [13], whereas, on the other hand, it suggests the idea that, in the investigation of the motions of an actual rotating fluid in a vessel with such a boundary, for large values of $t$, one should consider the corresponding nonlinear systems.

Some results of the present paper were announced in the papers [14], [15].

1. Absolutely continuous spectrum of the operator $A$ for the domain $D$

1. Let $D$ be of the form (8). Consider the spectral problem for the operator $A$,

$$ Au = \lambda u. $$

It can readily be seen that $\lambda \in (0, 1)$ is an eigenvalue of the operator $A$ if and only if the hyperbolic equation

$$ \frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 u}{\partial y^2} = 0, \quad a^2 = \frac{\lambda}{1-\lambda}, $$

has at least one nontrivial generalized solution $u \in \overset{\circ}{W}_2^2(D)$. It follows from the results of [11, 12] that there are no solutions of this kind for the domain $D$ under consideration, and the spectrum of the operator $A$ is purely continuous, $\sigma(A) = \sigma_c(A)$. This means that, if $E_\lambda$ is the spectral function of the operator $A$ and $h$ is an arbitrary element of the space $\overset{\circ}{W}_2^2(D)$, then the function $(E_\lambda h, h)_1$ is continuous on $[0, 1]$. In this section we explicitly construct a function $E_\lambda h$ for the elements $h$ in some $A$-invariant subspace and prove that the corresponding functions $(E_\lambda h, h)_1$ are absolutely continuous.

As is well known, for any $h$ and any interval $[\lambda_1, \lambda_2] \subset [0, 1]$, the function $U(\lambda) := E_\lambda h$ satisfies the equation

$$ A (U(\lambda_2) - U(\lambda_1)) = \int_{\lambda_1}^{\lambda_2} \lambda dU(\lambda), $$

and, conversely, every function $U(\lambda) : [0, 1] \to \overset{\circ}{W}_2^2(D)$, continuously depending on the parameter $\lambda$, and satisfying equation (11) for any $\lambda_1, \lambda_2 \in [0, 1]$ and the condition $U(0) = 0$, where 0 stands for the zero element of the space $\overset{\circ}{W}_2^2(D)$, is necessarily of the form $U(\lambda) = E_\lambda h$, $h \in \overset{\circ}{W}_2^2(D)$ (see, for example, [16]). Every function $U(\lambda)$ of this kind is referred to as a differential solution of equation (7). In the next subsection, we construct a certain class of differential solutions of the spectral equation for the operator $A$.

2. Consider the hyperbolic equation (10) in the domain $D$. We refer to the characteristics of the form $x = a y + c'$ of (10) as the characteristics of the first family and those of the form $x = -a y + c''$ as the characteristics of the second family. Choose an
arbitrary value $\lambda \in (0, (1 + \alpha^2)^{-1})$. Introduce the following polygonal lines by introducing
the following reflection law of the rays of the characteristic directions at the boundary. Suppose that a ray issuing from the point $B(1/\alpha, 1)$ in the direction of the first family of characteristics inside the domain $D$ meets the boundary $\partial D$ and goes inside $D$, after
the reflection at the boundary, already in the direction of the characteristics of the other
family, after which, this ray meets the boundary $\partial D$ again, makes a reflection at $\partial D$, and
goes along a characteristics of the first family, and so on. We also assume that the ray
issued from the point $A(1/\alpha, 0)$ in the direction of the second family of characteristics
inside the domain $D$ admits a similar behavior under the reflection at $\partial D$. Note that,
for the values of $\lambda \in (0, (1 + \alpha^2)^{-1})$ under consideration, both the rays “hide” into the
the angle with the vertex $O(0, 0)$. The polygonal lines that are trajectories of these rays
divide $D$ into infinitely many triangles $T_\lambda^{1}$ and parallelograms $P_\lambda^{i}$. Denote the vertices of
these polygonal lines belonging to the segment $[OB]$ by $B_\lambda^{i}$, those on the segment $[OA]$ by $A_\lambda^{i}$, and the interior points of their intersections by $C_\lambda^{i}$, $i = 1, 2, \ldots$, and index them in
the ascending order in the direction from the segment $[AB]$ to the point $O(0, 0)$.

Let $\theta_1 \in L_2(0, 1)$ be an arbitrary function.

As is well known (see [11]), every generalized solution $u_0(x, y, \lambda)$ of (10) on the characteristic triangle $ABC_\lambda^{1}$ such that $u_0(x, y, \lambda) \in W_2^1(ABC_\lambda^{1})$ coincides on this triangle almost
everywhere with a function continuous on $ABC_\lambda^{1}$ and such that the generalized derivative
of this function with respect to the variable $x$ (treated in the sense of distribution theory)
has a trace on the segment $[AB]$, and this trace belongs to the space $L_2(0, 1)$. Therefore,
one can pose the following problem: among all solutions of (10), find solutions that satisfy
the conditions

$$u_0|_{AB} = 0, \quad \frac{\partial u_0}{\partial x}|_{AB} = \theta_1.$$

It follows from [17] that such a solution $u_0$ exists and is unique in the triangle $ABC_\lambda^{1}$.
Moreover, the trace of this solution on the segment $[BC_\lambda^{1}]$ belongs to $W_2^1(BC_\lambda^{1})$ and satisfies the following bound:

$$(12) \quad \left\|u_0|_{BC_\lambda^{1}}\right\|_{W_2^1(BC_\lambda^{1})} \leq C_{0, \varepsilon}\|\theta_1\|_{L_2},$$

where the constant $C_{0, \varepsilon}$ does not depend on $\theta_1$.

Further, using the function $u_0$, we can construct a generalized solution $u_1(x, y, \lambda) \in W_2^1(BC_\lambda^{1}B_\lambda^{1})$ of (10) in the triangle $BC_\lambda^{1}B_\lambda^{1}$ such that

$$u_1|_{BC_\lambda^{1}} = u_0|_{BC_\lambda^{1}}, \quad u_1|_{BB_\lambda^{1}} = 0.$$

The problem of finding such a solution is a generalization of the classical Darboux problem
in which a solution of a hyperbolic equation is uniquely determined by the values of this solution
on two curves issuing from a given point. In the present case, one of the curves is
a characteristic of the equation. In [12], this problem is studied both for the regular and
for the generalized solutions. It follows from the results of [12] that the desired function
$u_1$ exists and is unique and, moreover, the trace of $u_1$ on the segment $[C_\lambda^{1}B_\lambda^{1}]$ belongs to the space $W_2^1(C_\lambda^{1}B_\lambda^{1})$ and satisfies a bound similar to (12).

After this, using the functions $u_0$ and $u_1$, we construct a generalized solution $u_2(x, y, \lambda) \in W_2^1(AB_\lambda^{1}A_\lambda^{1})$ of (10) on the triangle $AB_\lambda^{1}A_\lambda^{1}$ such that

$$u_2|_{AC_\lambda^{1}} = u_0|_{AC_\lambda^{1}}, \quad u_2|_{C_\lambda^{1}B_\lambda^{1}} = u_1|_{C_\lambda^{1}B_\lambda^{1}}, \quad u_2|_{AA_\lambda^{1}} = 0.$$
This process can be continued. Let

\[
\begin{align*}
    u(x, y; \theta_1; \lambda) :=
    \begin{cases}
        u_0(x, y, \lambda) & \text{for } (x, y) \in \Delta A B C_1^1, \\
        u_1(x, y, \lambda) & \text{for } (x, y) \in \Delta B C_1^1 B_1^1, \\
        u_2(x, y, \lambda) & \text{for } (x, y) \in \Delta A B_1^1 A_1^1, \\
        \ldots & \ldots 
    \end{cases}
\end{align*}
\]

(13)

It can readily be proved that, for any \( \theta \in L_2(0, 1) \) and for any \( \lambda \in (0, (1 + \alpha^2)^{-1}) \), the function \( u(x, y; \theta; \lambda) \) thus constructed has the following properties:

a) the function \( u(x, y; \theta; \lambda) \) belongs to \( L_2(D) \) and is a generalized solution of (10);

b) for any \( 0 < \varepsilon < 1 \), the function \( u(x, y; \theta_1; \lambda) \) belongs to the space \( W_{2, \alpha}^1(D \cap \{ x > \varepsilon \}) \);

c) for any smooth curve \( \tilde{r}(t) := \{ x(t), y(t), \lambda(t) \} \), \( t \in [t_0, t_1] \) lying in the domain \( D \times (0, (1 + \alpha^2)^{-1}) \), the function \( u(x(t), y(t); \lambda_1; \lambda_2(t)) \) is absolutely continuous on \([t_0, t_1]; \)

d) for almost all \( x \in (0, 1/\alpha) \), the derivatives \( u_0'(x, \alpha x), u_1'(x, \alpha x), u_2'(x, 0) \), and \( u_0'(x, 0) \) are well defined and, for any \( 0 < \varepsilon < \frac{1}{\alpha} \), there is a constant \( C_{\varepsilon} \) independent of \( \theta_1 \) and such that

\[
\|u_0'(x, \alpha x)\|_{L_2(x, \alpha x)} \leq C_{\varepsilon}\|\theta_1\|_{L_2}, \quad \|u_0'(x, \alpha x)\|_{L_2(x, \alpha x)} \leq C_{\varepsilon}\|\theta_1\|_{L_2},
\]

\[
\|u_2'(x, 0)\|_{L_2(x, 0)} \leq C_{\varepsilon}\|\theta_1\|_{L_2}, \quad \|u_2'(x, 0)\|_{L_2(x, 0)} \leq C_{\varepsilon}\|\theta_1\|_{L_2}.
\]

Let \( 0 < \lambda_1 < \lambda_2 < (1 + \alpha^2)^{-1} \), and let \( \sigma(\mu) \in C^1[\lambda_1, \lambda_2] \) be a function. Introduce the function:

\[
\begin{align*}
    U(x, y; \theta_1; \sigma; \lambda) :=
    \begin{cases}
        0, & \text{for } \lambda \leq \lambda_1, \\
        \int_{\lambda_1}^{\lambda} \sigma(\mu) u(x, y; \theta_1; \mu) d\mu, & \text{for } \lambda_1 < \lambda \leq \lambda_2, \\
        \int_{\lambda_1}^{\lambda} \sigma(\mu) u(x, y; \theta_1; \mu) d\mu, & \text{for } \lambda_2 < \lambda \leq \lambda_2, \\
        \int_{\lambda_1}^{\lambda} \sigma(\mu) u(x, y; \theta_1; \mu) d\mu, & \text{for } \lambda_2 < \lambda,
    \end{cases}
\end{align*}
\]

(14)

where 0 stands for the zero element of the space \( W_{1/2}(D) \).

**Theorem 1.1.** The function \( U(x, y; \theta_1; \sigma; \lambda) \) is a differential solution of (9).

We present the proof of the theorem under several headings.

**Lemma 1.2.** For any \( \lambda_1, \lambda_2 \) such that \( \lambda_1 \leq \lambda_2 \leq \lambda_2 \), the function

\[
V(x, y; \theta_1; \sigma) := \int_{\lambda_1}^{\lambda_2} \sigma(\mu) u(x, y; \theta_1; \mu) d\mu
\]

(15)

belongs to the space \( W_{1/2}(D) \) and satisfies the bound

\[
\|V(x, y; \theta_1; \sigma)\|_{L_2} \leq M (\lambda_2 - \lambda_1) \|\theta_1\|_{L_2},
\]

where \( M \) does not depend on \( \lambda_1, \lambda_2, \theta_1 \).

**Proof of the lemma.** Let us prove first that \( V(x, y; \theta_1; \sigma) \in W_{1/2}(D) \). By property b), to this end, it suffices to show that

\[
\|V_x'\|_{L_2(P_2)} < \infty, \quad \|V_y'\|_{L_2(P_2)} < \infty,
\]

where
where $D_2 := D \cap \{a_2 y > x + a_2 - \frac{1}{\alpha}\}$ and $a_2 = \sqrt{\frac{\lambda_2}{1-\lambda_2}}$. For example, let us prove the first inequality (the other can be proved in a similar way). For $(x, y) \in D_2$, for all $\mu \in [\lambda_1, \lambda_2]$, by the Riemann formula (see [11]) we have

$$u(x, y; \theta_1; \mu) = \frac{\sqrt{\mu}}{2\sqrt{1-\mu}} \int_{P(x,y;\mu)}^{Q(x,y;\mu)} \varphi(x', \mu) dx', \tag{17}$$

where

$$\varphi(x, \mu) := \left(\frac{\alpha w_x' + \frac{1-\mu}{\mu} u_y'}{y=\alpha x}\right), \tag{18}$$

and $P(x, y; \mu)$ and $Q(x, y; \mu)$ are abscissae of the right and left angles of the characteristic triangle with the vertex at the point $(x, y)$ (this triangle corresponds to the value $\lambda = \mu$ and leans on $OB$). It can readily be established that

$$P(x, y; \mu) = \frac{\alpha x - y}{2\alpha}l + \frac{\alpha x + y}{2\alpha}, \quad Q(x, y; \mu) = \frac{\alpha x + y}{2\alpha} + \frac{\alpha x - y}{2\alpha l}, \tag{19}$$

where

$$l = l(\mu) := \frac{\sqrt{1-\mu} + \alpha \sqrt{\mu}}{\sqrt{1-\mu} - \alpha \sqrt{\mu}} \tag{20}$$

is a function strictly monotone increasing on the interval $(0, (1 + \alpha^2)^{-1})$ and taking the values in the interval $(1, +\infty)$, and therefore

$$l(\mu) > 1 \quad \text{for any } \mu \in [\lambda_1, \lambda_2]. \tag{21}$$

Thus, if $(x, y) \in D_2$, then

$$V(x, y; \theta_1; \sigma) = \frac{\lambda_2}{\lambda_1} \int_{\lambda_1}^{\lambda_2} \frac{\sigma(\mu) \sqrt{\mu}}{2\sqrt{1-\mu}} \int_{P(x,y;\mu)}^{Q(x,y;\mu)} \varphi(\tilde{x}, \mu) d\tilde{x} d\mu \tag{22}$$

and therefore

$$\|V'_x\|^2_{L_2(D_2)} = \int_{D_2} \int_{\lambda_1}^{\lambda_2} \left(\frac{\sigma(\mu) \sqrt{\mu}}{2\sqrt{1-\mu}} \left\{Q'_x \varphi(Q, \mu) - P'_x \varphi(P, \mu)\right\}\right)^2 d\mu dxdy \leq$$

$$\leq 2 \int_{D_2} \int_{\lambda_1}^{\lambda_2} \left(\frac{\sigma(\mu) \sqrt{\mu}}{2\sqrt{1-\mu}} P'_x \varphi(P, \mu)\right)^2 d\mu dxdy + 2 \int_{D_2} \int_{\lambda_1}^{\lambda_2} \left(\frac{\sigma(\mu) \sqrt{\mu}}{2\sqrt{1-\mu}} Q'_x \varphi(Q, \mu)\right)^2 d\mu dxdy. \tag{23}$$

Thus, our problem is reduced to the existence problem for these integrals, for any function $\theta_1 \in L_2(0, 1)$ and any $\sigma(\mu) \in C^1[\lambda_* \lambda_*]$. Note that the set of all piecewise constant functions defined on the interval $(0, 1)$ and such that the ends of the intervals on which the functions are constant are rationals is
dense in $L_2(0, 1)$, and therefore, for $\theta_1(x)$, we take the function

$$\theta_{1,n}(x) = \begin{cases} 
    \cdots, & i-\frac{1}{n} < x < \frac{i}{n}, \\
    \cdots & \cdots
\end{cases}$$

where $c_i \in \mathbb{C}$ are some numbers, $i = 1, 2, \ldots, n$. In this case, the function $\varphi$ can be presented explicitly. Namely, let

$$\mu = \frac{(l-1)^2}{\alpha^2(l+1)^2 + (l-1)^2},$$

(it can readily be seen that $\mu$ is the function inverse to $l(\mu)$, i.e. $\mu(l(t)) \equiv t$). Denote

$$\tilde{\varphi}(x, l) := \varphi(x, \mu(l)).$$

Introduce the functions

$$x_{k,j} := \frac{1}{\alpha l^k} \cdot \frac{n+j}{2n} + \frac{1}{\alpha l^k+1} \cdot \frac{n-j}{2n}, \quad k = 0, 1, 2, \ldots, j = 0, \pm 1, \ldots, \pm (n-1).$$

Then

$$\Phi(x, l) := \tilde{\varphi}(x, l) \frac{l-1}{2\alpha l} = \begin{cases} 
    \cdots, & \cdots \\
    c_i l^k, & x_{k,i-1}(l) < x < x_{k,i}(l) \\
    \cdots, & \cdots \\
    c_i l^k, & x_{k,0}(l) < x < x_{k,1}(l) \\
    -c_i l^k, & x_{k,-1}(l) < x < x_{k,0}(l) \\
    -c_i l^k, & x_{k,-i}(l) < x < x_{k,1-i}(l) \\
    \cdots, & \cdots \\
    -c_n l^k, & x_{k,-n}(l) < x < x_{k,1-n}(l) \\
    \cdots & \cdots
\end{cases}$$

where $k = 0, 1, 2, \ldots, i = 1, 2, \ldots, n$.

For example, consider the first integral on the right-hand side of (22). It is clear that the existence of the integral is equivalent to the convergence of the series

$$\sum_{k=0}^{+\infty} \int_{R_k} \left| \int_{\lambda_1}^{\lambda_2} \frac{\sigma(\mu)\sqrt{\mu}}{2\sqrt{1-\mu}} P_x' \varphi(P, \mu) d\mu \right|^2 dx dy,$$

where

$$R_k := \left\{ (x, y) \in D_2 \left| \frac{1}{\alpha l^k+1} < x < \frac{1}{\alpha l^k} \right\}, \quad l_1 := l(\lambda_1) > 1.$$

Introduce the following notation:

$$\tilde{\sigma}(l) := \frac{\sigma(\mu(l))\sqrt{\mu(l)}}{2\sqrt{1-\mu(l)}} \mu'(l) \frac{2\alpha l}{l-1}, \quad \tilde{P}_x(x, y; l) := P_x'(x, y; \mu(l));$$
In this case,
\[
\int_{\lambda_1}^{\lambda_2} \frac{\sigma(\mu)\sqrt{\lambda}}{2\sqrt{1-\mu}} P_x(x, y; \mu) \varphi(P(x, y; \mu), \mu) d\mu =
\]

\[(28)\]
\[
= \int_{l_1}^{l_2} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) \Phi(\tilde{P}(x, y; l), l) dl, \quad l_i = l(\lambda_i), \ i = 1, 2.
\]

For any chosen point \((x, y) \in R_k\), denote by \(l_{k,k',j}\) the values of \(l\) corresponding to the intersection points of the curve \(x = P(x, y; l)\) with the curves \(x_{k',j}(l)\) in the plane \(Olx\). Let \(N(x, y; k)\) be the number of the values \(l_{k,k',j}\) belonging to the interval \([l_1, l_2]\). Assume that \(k'\) takes here the values \(\{k_m', k_m' + 1, ..., k_M'\}\). It can readily be seen that, for sufficiently large values of \(k\), there is a constant \(C_0\) (independent of \((x, y)\), \(k\), \(l_1\), \(l_2\) and depending on \(\lambda_s, \lambda_\infty\) only) such that

\[k \log l_2 l_1 - C_0 < k_m' \leq k' \leq k_M' \leq k + 1.
\]

Therefore, for sufficiently large values of \(k\),

\[N(x, y; k) \leq k + 1 - k \log l_2 l_1 + C_0 (< \gamma k),
\]

where the number \(0 < \gamma < 1\) does not depend on \((x, y), k, l_1, l_2\).

In this case, for any point \((x, y) \in R_k\), we have

\[
\int_{l_1}^{l_2} \mathcal{P} dl = \int_{l_1}^{l_{k,k',j,n}} \mathcal{P} dl + \int_{l_{k,k',M,n}}^{l_{k,k',M-1,n}} \mathcal{P} dl + ... + \int_{l_{k,k',m,n}}^{l_2} \mathcal{P} dl,
\]

where \(\mathcal{P} = \mathcal{P}(x, y, l)\) stands for the integrand in (28). Here, for any \(k' = k_m' + 1, ..., k_M'\),

\[
\int_{l_{k,k',n}}^{l_{k,k'-1,n}} \mathcal{P} dl = \sum_{i=1}^{n} c_i \left( \int_{l_{k,k'-1,i-1}}^{l_{k,k'-1,i}} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) l_{k'}^{k'-1} dl - \int_{l_{k,k'-1,i}}^{l_{k,k',1,i-1}} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) l_{k'}^{k'-1} dl \right),
\]

and we can change the order of summation, because all sums contain finitely many summands for any chosen \(k\) and \(n\), and therefore

\[
\int_{l_1}^{l_2} \mathcal{P} dl = \int_{l_1}^{l_{k,k',j,n}} \mathcal{P} dl + \int_{l_{k,k',n}}^{l_2} \mathcal{P} dl +
\]

\[(29)\]
\[
+ \sum_{i=1}^{n} c_i \sum_{k'=k_m'+1}^{k'_M} \left( \int_{l_{k,k'-1,i-1}}^{l_{k,k'-1,i}} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) l_{k'}^{k'-1} dl - \int_{l_{k,k'-1,i}}^{l_{k,k',1,i-1}} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) l_{k'}^{k'-1} dl \right).
\]
Thus, for any point \((x, y) \in R_k\),

\[
\left| \int_1^{l_2} \mathcal{P} \, dl \right|^2 \leq 3 \left| \int_1^{l_2} \mathcal{P} \, dl \right|^2 + 3 \left| \int_1^{l_2} \mathcal{P} \, dl \right|^2 +
\]

\[
+ 3 \left[ \sum_{i=1}^n \sum_{k=k_i}^{k_i+1} \left( \int_{l_k}^{l_{k',1,i}} \sigma(l) \tilde{P}_x(x, y; l) l^{k'-1} \, dl - \int_{l_k}^{l_{k',1,i}} \sigma(l) \tilde{P}_x(x, y; l) l^{k'-1} \, dl \right)^2 \right].
\]

Let us estimate the last summand. Denote \(t(x, y; l) := \tilde{\sigma}(l) \tilde{P}_x(x, y; l)\). Since \(t(x, y; l) \in C^1[l_*, l**]\) for any point \((x, y) \in R_k\), where \(l_* := l(\lambda_*)\), \(l** := l(\lambda**\))

it follows that

\[
\int_{l_k}^{l_{k',1,i-1}} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) l^{k'-1} \, dl - \int_{l_k}^{l_{k',1,i-1}} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) l^{k'-1} \, dl =
\]

\[
= \frac{1}{k'} \left( (x, y; l) l^{k'} t(x, y; l) l^{k'} \right)_{l_k}^{l_{k',1,i-1}} - t(x, y; l) l^{k'} \left( l_{k',1,i-1} \right)
\]

\[
- \int_{l_k}^{l_{k',1,i-1}} \frac{k'}{k'_{l'}} t(x, y; l) \, dl + \int_{l_k}^{l_{k',1,i-1}} \frac{k'}{k'_{l'}} t(x, y; l) \, dl.
\]

Note that, by the choice of \(l_{k,k',j}\),

\[
l_{k',1,i-1} \, t(x, y; l) l_{k',1,i-1} = \frac{i+n}{2n} l_{k',1,i} + \frac{n-1}{2n} t(x, y; l) l_{k',1,i}.
\]

Denote

\[
G_i(z) := \frac{i+n}{2n} + \frac{n-1}{2n} t(x, y; l) l_{k',1,i}, \quad i = 0, \pm 1, \pm 2, \ldots \pm n.
\]

Then

\[
t(x, y; l) l^{k'} \bigg|_{l_k}^{l_{k',1,i-1}} = G_i(l_{k',1,i}) - G_i(l_{k',1,i-1}) + \frac{1}{2n} (1 - l_{k',1,i-1}) t(x, y; l) l_{k',1,i}.
\]

Similarly,

\[
t(x, y; l) l^{k'} \bigg|_{l_k}^{l_{k',1,i-1}} = G_{i-1}(l_{k',1,i-1}) - G_{i-1}(l_{k',1,i}) + \frac{1}{2n} (1 - l_{k',1,i-1}) t(x, y; l) l_{k',1,i}.
\]

Therefore, writing

\[
K(z) := \frac{z-1}{P(x, y; z)} t(x, y; z),
\]

we obtain

\[
t(x, y; l) l^{k'} \bigg|_{l_k}^{l_{k',1,i-1}} - t(x, y; l) l^{k'} \bigg|_{l_k}^{l_{k',1,i-1}} =
\]

\[
= (G_i(l_{k',1,i}) - G_i(l_{k',1,i-1})) - (G_{i-1}(l_{k',1,i-1}) - G_{i-1}(l_{k',1,i})) +
\]

\[
\]
It can readily be proved that there are constants $C_2, C_3$ independent of $l_1, l_2, n, k$ such that

$$|G'(z)| \leq C_2 l_1^k, \quad |K'(z)| \leq C_3 l_1^k$$

for any point $(x, y) \in R_k$, for any $i = 0, \pm 1, \pm 2, \ldots \pm n$, and for any $z \in [l_*, l_{**}]$. Hence,

$$\frac{1}{k'} \left| t(x, y; l) l^k \left| l_{k,k'-1,i-1} \right| - t(x, y; l) l^k \left| l_{k,k'-1,i-1} \right| \right| \leq \frac{l_1^k}{k'} \left( C_2 (l_{k,k'-1,i} - l_{k,k'-1,i-1}) + C_2 (l_{k,k'-1,i-1} - l_{k,k'-1,i}) + \frac{1}{2n} C_3 (l_{k,k'-1,i-1} - l_{k,k'-1,i}) \right).$$

Further, by the relation

$$\frac{x_{k'-1,1}(l_{k,k'-1,i-1}) - x_{k'-1,1}(l_{k,k'-1,i-1})}{x_{k'-1,1}(l_{k,k'-1,i-1}) - x_{k'-1,1}(l_{k,k'-1,i-1})} = \frac{1}{2n},$$

there exists a constant $C_4$ independent of $n, k, (x, y), k', i$ and such that

$$(l_{k,k'-1,i} - l_{k,k'-1,i-1}) \leq C_4 \left( l_{k,k'-1,n} - l_{k,k'-1,-n} \right),$$

and therefore

$$\frac{1}{k'} \left| t(x, y; l) l^k \left| l_{k,k'-1,i} \right| - t(x, y; l) l^k \left| l_{k,k'-1,i-1} \right| \right| \leq \frac{C_5 l_1^k}{2nk'} \left( l_{k,k'-1,n} - l_{k,k'-1,-n} \right),$$

where $C_5$ does not depend on $l_1, l_2, n, k, (x, y), k', i$.

Let us now obtain similar bounds for the integrals in the right-hand side of $(30)$. Consider the first integral. Since the functions in the integrand are continuous, it follows that

$$\int_{l_{k,k'-1,i-1}}^{l_{k,k'-1,i}} l_{k,k'-1,i} l_{k,k'-1,i-1} dl = \frac{t'(x, y; \eta_{k,k',i}) \eta_{k,k',i}}{k'} \int_{l_{k,k'-1,i}}^{l_{k,k'-1,i-1}} l_{k,k'-1,i} dl =$$

$$= \frac{t'(x, y; \eta_{k,k',i}) \eta_{k,k',i}}{(k')^2} \left( l_{k,k'-1,i} - l_{k,k'-1,i-1} \right),$$

where $\eta_{k,k',i} \in [l_{k,k'-1,i-1}, l_{k,k'-1,i}]$. We have (see $(31)$):

$$l_{k,k'-1,i} - l_{k,k'-1,i-1} = F_i(l_{k,k'-1,i}) - F_i(l_{k,k'-1,i-1}) + \frac{1}{2n} \frac{(1 - l_{k,k'-1,i-1})}{P(x, y; l_{k,k'-1,i-1})},$$

where

$$F_i(z) := \frac{i + n z + \frac{n-i}{2n}}{P(x, y; l_{k,k'-1,i-1})}, \quad i = 0, \pm 1, \pm 2, \ldots \pm n.$$
Here there are constants $C_6, C_7$ independent of $l_1, l_2, n, k, k', i$ and such that the following bounds hold for any points $(x, y) \in R_k$, for any $i = 0, \pm 1, \pm 2, \ldots \pm n$, and for any $z \in [l_*, l_*]$: 

$$|F'_i(z)| \leq C_6 l_i^k, \quad \left| \frac{(1 - l_{k,k'-1,i-1})}{P(x, y; l_{k,k'-1,i-1})} \right| \leq C_7 l_i^k,$$

and therefore

$$\left| \int_{l_{k,k'-1,i-1}}^{l_{k,k'-1,i}} \frac{k'}{k} t_i^k(x, y; l) \, dl \right| \leq \frac{C_8 l_i^k}{(k')^2} (l_{k,k'-1,i} - l_{k,k'-1,i-1}) + \frac{C_9 l_i^k}{2n (k')^2} \leq$$

$$\leq \frac{C_{10} l_i^k}{2n (k')^2} (l_{k,k'-1,n} - l_{k,k'-1,n-1}) + \frac{C_9 l_i^k}{2n (k')^2},$$

where $C_8, C_9, C_{10}$ are constants independent of $l_1, l_2, n, k, (x, y), k', i$. It is clear that the second integral summand in (30) can be estimated in a similar way, and therefore

$$\left| \int_{l_{k,k'-1,i-1}}^{l_{k,k'-1,i}} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) l^{k'-1} \, dl - \int_{l_{k,k'-1,i}}^{l_{k,k'-1,i-1}} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) l^{k'-1} \, dl \right| \leq$$

$$\leq \frac{C_{11} l_i^k}{n k'} (l_{k,k'-1,n} - l_{k,k'-1,n-1}) + \frac{C_{12} l_i^k}{2n (k')^2},$$

where the constants $C_{11}, C_{12}$ do not depend on $l_1, l_2, n, k, (x, y), k', i$.

Let us now estimate the sum

$$\sum_{k'=k_m+1}^{k_M} \left| \int_{l_{k,k'-1,i-1}}^{l_{k,k'-1,i}} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) l^{k'-1} \, dl - \int_{l_{k,k'-1,i}}^{l_{k,k'-1,i-1}} \tilde{\sigma}(l) \tilde{P}_x(x, y; l) l^{k'-1} \, dl \right| \leq$$

$$\leq \sum_{k'=k_m+1}^{k_M} \left| \frac{C_{11} l_i^k}{n k'} (l_{k,k'-1,n} - l_{k,k'-1,n-1}) \right| + \sum_{k'=k_m+1}^{k_M} \frac{C_{12} l_i^k}{2n (k')^2} \leq \sum_{k'=k_m+1}^{k_M} \frac{1}{k'} \leq \frac{C_{13} l_i^k}{n k'} (l_2 - l_1),$$

(32) 

$$\leq \frac{C_{14} l_i^k}{k (1 - \gamma)} \sum_{k'=k_m+1}^{k_M} (l_{k,k'-1,n} - l_{k,k'-1,n-1}) + \sum_{k'=k_m+1}^{k_M} \frac{C_{12} l_i^k}{2n k (1 - \gamma)} \sum_{k'=k_m+1}^{k_M} \frac{1}{k'} \leq \frac{C_{13} l_i^k}{n k'} (l_2 - l_1),$$

where $C_{13}$ does not depend on $l_1, l_2, (x, y), k, i$. Let us clarify the bound for the second summand. For sufficiently large $k$, we have $C_0 + 1 \leq k \left(1 - \log_{l_2} l_1\right)$, and thus

$$\sum_{k'=k_m+1}^{k_M} \frac{1}{k'} \leq \frac{N(x, y; k)}{k_{m+1}} \leq \frac{k \left(1 - \log_{l_2} l_1\right) + C_0 + 1}{k (1 - \gamma)} \leq$$

$$\leq \frac{2k \left(\log_{l_2} l_2 - \log_{l_2} l_1\right)}{k (1 - \gamma)} \leq C_{14} (l_2 - l_1),$$
where $C_{14}$ depends on $l_*$, $l_{**}$ only, which proves the bound (32). Then we have

$$\left| \sum_{i=1}^{n} c_i \sum_{k'=k_0+1}^{k_1} \left( \int_{l_{k',1-i}}^{l_{k,1-i}} \tilde{\sigma}(l) \, \tilde{P}_x(x,y;l) \, l^{k'-1} \, dl - \int_{l_{k',1-i}}^{l_{k,1-i}} \tilde{\sigma}(l) \, \tilde{P}_x(x,y;l) \, l^{k-1} \, dl \right) \right|^2 \leq \sum_{i=1}^{n} \left| c_i \frac{C_{13} \ell k_1}{nk} (l_2 - l_1) \right|^2 \leq \frac{C_{15} \ell k_1}{nk^2} (l_2 - l_1)^2 \left( \sum_{i=1}^{n} |c_i|^2 \right).$$

It is clear that the first two summands in (29) can be estimated in a similar way, and therefore the following inequality holds for any point $(x, y) \in R_k$: 

$$\left| \int_{l_1}^{l_2} \mathcal{P} \, dl \right|^2 \leq \frac{C_{15} \ell k_1}{nk^2} (l_2 - l_1)^2 \left( \sum_{i=1}^{n} |c_i|^2 \right),$$

where $C_{15}$ depends on $l_*$, $l_{**}$ only. Then

$$\left\| \iint_{\mathcal{D}_2} \left| \int_{\lambda_1}^{\lambda_2} \frac{\sigma(\mu) \sqrt{\mu} \partial P}{2 \sqrt{1 - \mu} \partial x}(x,y;\mu) \varphi(P(x,y;\mu),\mu) \, d\mu \right|^2 \, dx \, dy \right\| \leq \sum_{k=k_0}^{+\infty} S(R_k) \frac{C_{15} \ell k_1}{nk^2} (l_2 - l_1)^2 \left( \sum_{i=1}^{n} |c_i|^2 \right) \leq \sum_{k=k_0}^{+\infty} \frac{C_{16} \ell k_1}{nk^2} (l_2 - l_1)^2 \left( \sum_{i=1}^{n} |c_i|^2 \right) \leq \frac{C_{16} \ell k_1}{n} (l_2 - l_1)^2 \left( \sum_{i=1}^{n} |c_i|^2 \right) \leq \frac{C_{17} \ell k_1}{n} (l_2 - l_1)^2 \left( \sum_{i=1}^{n} |c_i|^2 \right) = C_{17} (l_2 - l_1)^2 \left\| \theta_1, n(x) \right\|_L^2,$$

where $S(R_k)$ stands for the area of the domain $R_k$ and the constants $C_{16}$, $C_{17}$ depend on $l_*$, $l_{**}$ and $\sigma(\mu)$ only. Since the quantity $C_{17} (l_2 - l_1)^2$ does not depend on $n$, it follows that the inequality thus obtained holds for an arbitrary function $\theta_1(x) \in L_2(0,1)$,

$$\iint_{\mathcal{D}_2} \left| \int_{\lambda_1}^{\lambda_2} \frac{\sigma(\mu) \partial P}{\partial x}(x,y;\mu) \varphi(P(x,y;\mu),\mu) \, d\mu \right|^2 \, dx \, dy \leq C_{17} (l_2 - l_1)^2 \left\| \theta_1(x) \right\|_L^2.$$ 

Obviously, the other integral in (28) can be estimated in a similar way, and therefore

$$\left\| V'_2 \right\|_{L_2(D_2)}^2 \leq C_{18} (l_2 - l_1)^2 \left\| \theta_1(x) \right\|_{L_2}^2 \leq C_{19} (\lambda_2 - \lambda_1)^2 \left\| \theta_1(x) \right\|_{L_2}^2,$$

where $C_{19}$ does not depend on $\theta_1, \lambda_1, \lambda_2$. This proves the first assertion of the lemma.

To prove the other assertion of the lemma, let us estimate the quantity $\left\| V'_2 \right\|_{L_2(D \setminus D_2)}^2$. Let $M(x,y) \in \{ D \setminus D_2 \}$, i.e. let $M(x,y) \in \triangle AA \lambda_2 B$. Let us draw the characteristic of the first family for equation (10) through $M(x,y)$ until this characteristic meets the segment $OA$ at some point $L(x - ay;0)$. After this, draw the characteristic of the other family for (11) through $M(x,y)$ until it meets the segment $OB$ at some point $K \left( \frac{x + ay}{1 + \alpha}, \frac{x + ay}{1 + \alpha} \right)$.
Note that, by properties b), c), and d) of the function \( u(x, y; \theta_1; \lambda) \), the polygon \( T(x, y, \lambda) \) with the vertices at the points \( M, K, B, A, L \) satisfies the following conditions:

\[
\int_{\partial T(x, y; \lambda)} \frac{\partial u}{\partial x} \, dy + \frac{1}{a^2} \frac{\partial u}{\partial y} \, dx = \int_{T(x, y; \lambda)} \left( \frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 u}{\partial y^2} \right) \, dx \, dy = 0.
\]

This implies the relation

\[
(34) \quad u|_M = u(x, y; \theta_1; \lambda) = -\frac{a}{2} \int_{x-ay}^{1/\alpha} \varphi_0(t, \lambda) \, dt + \frac{a}{2} \int_{\lambda}^{B} \theta_1(t) \, dt + \frac{1}{1+\alpha} \int_{x-ay}^{B} \varphi(t, \lambda) \, dt,
\]

where \( \varphi_0(t, \lambda) := \frac{1}{a^2} \frac{\partial u}{\partial y} \bigg|_{y=0} \). Therefore, for almost all \((x, y) \in \{D \setminus \overline{D}_2\}\), we have

\[
V_x' = \int_{\lambda_1}^{\lambda_2} u(x, y; \theta_1; \lambda)' \, d\lambda = \int_{\lambda_1}^{\lambda_2} \frac{a}{2} \left( \varphi_0(x - ay, \lambda) + \frac{1}{1+\alpha} \varphi \left( \frac{x + ay}{1+\alpha}, \lambda \right) \right) \, d\lambda.
\]

Then

\[
\|V_x'\|_{L_2(D \setminus \overline{D}_2)}^2 = \int_{D \setminus \overline{D}_2} \int_{\lambda_1}^{\lambda_2} \left( \int_{\lambda_1}^{\lambda_2} \frac{a}{2} \left( \varphi_0(x - ay, \lambda) + \frac{1}{1+\alpha} \varphi \left( \frac{x + ay}{1+\alpha}, \lambda \right) \right) \, d\lambda \right)^2 \, dx \, dy \leq \int_{D \setminus \overline{D}_2} \int_{\lambda_1}^{\lambda_2} \frac{a^2}{4} \left( \varphi_0(x - ay, \lambda) + \frac{1}{1+\alpha} \varphi \left( \frac{x + ay}{1+\alpha}, \lambda \right) \right)^2 \, d\lambda \, dx \, dy \leq M_0(\lambda_2 - \lambda_1) \int_{\lambda_1}^{\lambda_2} \left( \int_{D \setminus \overline{D}_2} \left| \varphi_0(x - ay, \lambda) \right|^2 + \left| \varphi \left( \frac{x + ay}{1+\alpha}, \lambda \right) \right|^2 \, dx \, dy \right) \, d\lambda \leq (\lambda_2 - \lambda_1) \int_{D \setminus \overline{D}_2} \int_{\lambda_1}^{\lambda_2} \frac{a^2}{4} \left( \varphi_0(x - ay, \lambda) + \frac{1}{1+\alpha} \varphi \left( \frac{x + ay}{1+\alpha}, \lambda \right) \right)^2 \, d\lambda \, dx \, dy \leq M_0(\lambda_2 - \lambda_1) \int_{\lambda_1}^{\lambda_2} \left( \int_{D \setminus \overline{D}_2} \left| \varphi_0(x - ay, \lambda) \right|^2 + \left| \varphi \left( \frac{x + ay}{1+\alpha}, \lambda \right) \right|^2 \, dx \, dy \right) \, d\lambda \leq M_1(\lambda_2 - \lambda_1) \int_{\lambda_1}^{\lambda_2} \left\{ \|\varphi_0\|^2_{L_2(A^1(\lambda_2)A)} + \|\varphi\|^2_{L_2(B^2(\lambda_2)B)} \right\} \, d\lambda \leq M_2(\lambda_2 - \lambda_1)^2 \|\theta_1\|^2_{L_2},
\]

where the positive constants \( M_i, i = 0, 1, 2 \) do not depend on \( \lambda_1, \lambda_2 \) and \( \theta_1 \). Taking \([33]\) into account, we obtain

\[
\|V_x'\|^2_{L_2(D)} \leq M_3(\lambda_2 - \lambda_1)^2 \|\theta_1\|^2_{L_2},
\]

where the constant \( M_3 \) does not depend on \( \lambda_1, \lambda_2 \) and \( \theta_1 \). Obviously, for \( \|V_y'\|^2_{L_2(D \setminus \overline{D}_2)} \), we have a similar bound. This proves the lemma.

Proof of Theorem 1.1. Lemma 1.2 implies that the function \( \|U(x, y; \theta_1; \sigma; \lambda)\|_1 \) is continuous and even the absolute continuous with respect to \( \lambda \).
Let us now prove that formula (11) holds for our function $U$ for any $\lambda_1, \lambda_2 \in (0, (1 + \alpha^2)^{-1})$. Since

$$U(x, y; \theta_1; \sigma; \lambda_2) - U(x, y; \theta_1; \sigma; \lambda_1) = \int_{\lambda_1}^{\lambda_2} \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda$$

and

$$dU(x, y; \theta_1; \sigma; \lambda) = \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda,$$

it follows that we are to prove the relation

$$A \int_{\lambda_1}^{\lambda_2} \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda = \int_{\lambda_1}^{\lambda_2} \lambda \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda,$$

or, equivalently, after applying the Laplace operator to this relation, to prove the equality

$$\frac{\partial^2}{\partial y^2} \int_{\lambda_1}^{\lambda_2} \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda = \Delta \int_{\lambda_1}^{\lambda_2} \lambda \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda$$

in the space $W^{2,1}_2(D)$. This means that, for an arbitrary $g \in C_0^\infty(D)$, we must prove that

$$\iint_D \left\{ \frac{\partial^2}{\partial y^2} \int_{\lambda_1}^{\lambda_2} \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda \cdot g - \Delta \int_{\lambda_1}^{\lambda_2} \lambda \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda \cdot g \right\} dxdy = 0.$$

Since the function $\sigma(\lambda)u(x, y; \theta_1; \lambda) \in W^{1,1}_2(D \cap \{x > \varepsilon\})$, $\varepsilon > 0$, is a generalized solution of (10) by construction, it follows that

$$\int_D \sigma u \frac{\partial g}{\partial z} dxdy = \lambda \int_D \sigma u \Delta g dxdy$$

for any $g \in C_0^\infty(D)$. Therefore

$$\begin{align*}
\iint_D &\left\{ \frac{\partial^2}{\partial y^2} \int_{\lambda_1}^{\lambda_2} \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda \cdot g - \Delta \int_{\lambda_1}^{\lambda_2} \lambda \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda \cdot g \right\} dxdy = \\
= &\iint_D \left\{ \int_{\lambda_1}^{\lambda_2} \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda \cdot \frac{\partial^2 g}{\partial y^2} - \int_{\lambda_1}^{\lambda_2} \lambda \sigma(\lambda) u(x, y; \theta_1; \lambda) d\lambda \cdot \Delta g \right\} dxdy = \\
= &\iint_D \int_{\lambda_1}^{\lambda_2} \left\{ \sigma(\lambda) u(x, y; \theta_1; \lambda) \cdot \frac{\partial^2 g}{\partial y^2} - \lambda \sigma(\lambda) u(x, y; \theta_1; \lambda) \cdot \Delta g \right\} d\lambda dxdy = \\
= &\int_{\lambda_1}^{\lambda_2} \int_D \left\{ \sigma(\lambda) u(x, y; \theta_1; \lambda) \cdot \frac{\partial^2 g}{\partial y^2} - \lambda \sigma(\lambda) u(x, y; \theta_1; \lambda) \cdot \Delta g \right\} dxdy d\lambda = 0.
\end{align*}$$

This completes the proof of Theorem 1.1.

Let $\theta_2(x) \in L_2(0, 1/\alpha)$ be an arbitrary function. In just the same way in which the function $u(x, y; \theta_1; \lambda)$ was constructed for $\lambda \in (0, (1 + \alpha^2)^{-1})$, we construct the function $v(x, y; \theta_2; \lambda)$ for $\lambda \in ((1 + \alpha^2)^{-1}, 1)$, namely, the function $v$ is the generalized solution of
Corollary 1.3. If \( \sigma(\lambda) \) is an arbitrary function in \( C^1(0,1) \), whose support \( \text{supp} \sigma \) belongs to the union of the intervals \( (0,(1+\alpha^2)^{-1}) \) and \( ((1+\alpha^2)^{-1},1) \), then the function

\[
U(x,y;\lambda) := \int_0^\lambda \sigma(\mu)w(x,y;\mu)d\mu, \quad \lambda \in [0,1],
\]

with values in the space \( W^\frac{1}{2} \) is a differential solution of (10).

Corollary 1.4 (existence theorem for the absolutely continuous spectrum). Let \( H_0 \) be the closure of the linear span of all differential solutions of the form (38) in \( W^\frac{1}{2}(D) \). Theorem 1 proved above implies that the spectrum of the operator \( A \) is absolutely continuous on \( H_0, \sigma(A|_{H_0}) = \sigma_{ac}(A|_{H_0}) \).

Example 1.5. In some works, the class of bounded convex domains \( \Omega \) under consideration is defined by the condition that \( \Gamma = \partial \Omega = \bigcup_{j=1}^m \Gamma_j, \Gamma_j \in C^4 \), and \( \Gamma_j \) is either a line segment or has positive curvature at every point, including the endpoints (the value of curvature at the endpoints is understood as the limit of the curvature at the interior points). Let a set \( \{\alpha_1, \ldots, \alpha_m\} \subset [0,\pi/2] \) consist of angles between the axis \( Ox \) and all one-side tangents at the endpoints \( \Gamma_j, j \in \{1, \ldots, n\} \). Let \( 0 \leq \alpha_1 < \cdots < \alpha_m \leq \pi/2 \). Write

\[
\alpha = \alpha(\lambda) = \arccos(\sqrt{\lambda}) \in [0,\pi/2], \quad \lambda = \cos^2 \alpha,
\]

\[
\xi = \xi(X,\alpha) = x \sin \alpha + y \cos \alpha, \quad \eta = \eta(X,\alpha) = x \sin \alpha - y \cos \alpha, \quad X = (x,y).
\]

Consider domains of this kind for which there are \( \alpha^* \in \{\alpha_1, \ldots, \alpha_m\} \) and \( X^1, X^2 \in \Gamma \) such that

\[
\begin{align*}
\{ \xi(X^1,\alpha^*) = \xi(X^2,\alpha^*) \text{ or } \eta(X^1,\alpha^*) = \eta(X^2,\alpha^*) \}, \\
\{ X \in \mathbb{R}^2 \mid \xi(X,\alpha^*) = \xi(X^j,\alpha^*) \} \cap \Omega = \emptyset, \quad j = 1,2, \\
\{ X \in \mathbb{R}^2 \mid \eta(X,\alpha^*) = \eta(X^j,\alpha^*) \} \cap \Omega = \emptyset, \quad j = 1,2.
\end{align*}
\]

As it was noted above, the number \( \lambda \in (0,1) \) is an eigenvalue of the operator \( A \) corresponding to the domain \( \Omega \) if and only if the hyperbolic equation (10) has a nontrivial generalized solution in \( W^\frac{1}{2}(\Omega) \).

In particular, let \( \Omega \) be the quadrangle with vertices at the points \( O(0,0), A(\frac{1}{3},\frac{1}{3}), B(\frac{1}{2},1), C(0,1) \). Then \( n = m = 4 \) and the above angles are: \( 0, \pi/4, \arctan 4, \pi/2 \). Let \( \alpha^* = 0, X^1 = C, X^2 = B \). Then \( \xi(X^1,\alpha^*) = \xi(X^2,\alpha^*) \) and, as can readily be seen, the four lines \( \xi(X,\alpha^*) = \xi(X^j,\alpha^*), \eta(X,\alpha^*) = \eta(X^j,\alpha^*) \), \( j = 1,2 \), do not intersect the domain \( \Omega \). Thus, the domain in question belongs to the class of domains described above.

It can immediately be proved that the function

\[
u(x,y) := C^* \cdot \begin{cases} x, & \text{for } (x,y) \in \triangle COM; \\ 1 - y, & \text{for } (x,y) \in \triangle CMB; \\ y - x, & \text{for } (x,y) \in \triangle OMA; \\ \frac{1}{2}(y+1) - 2x, & \text{for } (x,y) \in \triangle MAB; \end{cases}
\]
where \( M \) has coordinates \((\frac{1}{3}, \frac{1}{2})\) and \( C^* \neq 0 \) is an arbitrary constant, belongs to the space \( \dot{W}^{\frac{1}{2}}(\Omega) \) and is the generalized solution of the Dirichlet problem for equation (10) for \( \lambda = \cos^2(\arctan 2) = \frac{1}{5} \). Thus, \( \lambda = \frac{1}{5} \) is an eigenvalue of the operator \( A \) corresponding to the domain \( \Omega \).

2. Solutions of the nonstationary Sobolev equation

Using the differential solutions \( U(x, y; \lambda) \) of the form (35), which are constructed above, we can write out some exact solutions of the nonstationary Sobolev equation. Namely, the following assertion is an immediate corollary to Theorem 1.1.

Theorem 2.1. Let

\[
(39) \quad p_0(x, y) := \int_0^1 \sigma_0(\lambda) w_0(x, y; \lambda) d\lambda, \quad p_1(x, y) := \int_0^1 \sigma_1(\lambda) w_1(x, y; \lambda) d\lambda,
\]

where \( w_0, w_1 \) are constructed in the way described above from functions \( \theta_1^{(0)}, \theta_2^{(0)}, \theta_1^{(1)}, \theta_2^{(1)} \), respectively, and the supports \( \text{supp} \sigma_i \) are contained in the union of the intervals \((0, (1 + \alpha^2)^{-1})\) and \(((1 + \alpha^2)^{-1}, 1)\). Then the function

\[
(40) \quad p(x, y; t) := \int_0^1 \cos(\sqrt{\lambda} t) \sigma_0(\lambda) w_0(x, y; \lambda) d\lambda + \int_0^1 \frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}} \sigma_1(\lambda) w_1(x, y; \lambda) d\lambda,
\]

is a solution to problem (11) – (3) and \( \|p(x, y; t)\|_{L_2(D)} \to 0 \) as \( t \to \infty \).

Note that here we have \( p_0(x, y), p_1(x, y) \in H_0 \).

It follows from the results of the papers [11], [17], and [12] that, for any point \((x, y) \in D\), the function \( w(x, y; \lambda) \) is an absolutely continuous function of the variable \( \lambda \) on the closed interval \([0, 1]\) (see the property c) of the function \( u(x, y; \theta_1; \lambda) \), and therefore

\[
p(x, y; t) \to 0 \quad (t \to \infty)
\]

for any point \((x, y) \in D\).

Theorem 2.2. Let \( \theta_1^{(0)}, \theta_1^{(1)} \in C_0^\infty[0, 1], \theta_2^{(0)}, \theta_2^{(1)} \in C_0^\infty[0, 1/\alpha], \sigma_i(\lambda) \in C_0^\infty[0, 1], \) let the supports \( \text{supp} \sigma_i \) are contained in the union of the intervals \((0, (1 + \alpha^2)^{-1})\) and \(((1 + \alpha^2)^{-1}, 1)\), \( i = 0, 1 \). In this case, for any \( n = 1, 2, ..., \) there exist a constant \( K_n \) independent of \( t \) and such that the bound

\[
(41) \quad \|p(x, y; t)\|_{L_2(D)} \leq \frac{K_n}{t^n}
\]

holds for any \( t \in (0, \infty) \), where \( p(x, y; t) \) stands for the solution of problem (11) – (3) of the form (40).

Proof. Obviously, it suffices to consider the case in which \( \theta_1^{(1)} \equiv 0, \theta_2^{(0)} \equiv 0, \theta_2^{(1)} \equiv 0, \sigma_1(\lambda) \equiv 0, \text{supp} \sigma_0 \subset [\lambda_*, \lambda_{**}] \subset (0, (1 + \alpha^2)^{-1}), \) i.e.

\[
(42) \quad p(x, y; t) := \int_0^1 \cos(\sqrt{\lambda} t) \sigma_0(\lambda) u(x, y; \lambda) d\lambda = \int_0^{(1+\alpha^2)^{-1}} \cos(\sqrt{\lambda} t) \sigma_0(\lambda) u(x, y; \lambda) d\lambda,
\]
where \( u(x, y; \lambda) = u(x, y; \hat{\theta}_1^{(0)}; \lambda) \). Under the assumptions formulated in the theorem, \( u(x, y; \lambda) \) is infinitely differentiable with respect to \( x, y, \lambda \) everywhere inside the prisme \( D \times (0, (1 + \alpha^2)^{-1}) \) and can be continuously extended with all its derivatives, to the entire surface of the prisme, except for the line \( x = 0, \ y = 0 \), and the corresponding function \( \tilde{\varphi}(x, l) \) (see (18)) is infinitely differentiable with respect to \( x, l \) everywhere inside the half-strip \((0, 1/\alpha) \times (1, \infty) \). Let \( l_1 := l(\lambda_*), \ l_2 := l(\lambda_{**}) \). Then we obviously have \( 1 < l_1 \leq l_2 < +\infty \).

**Lemma 2.3.** For any \( n = 0, 1, 2, \ldots \), the following bounds hold:

\[
(43) \quad \left| \frac{\partial^n \tilde{\varphi}}{\partial l^n} \right| \leq C(n)k^n l^k, \quad \frac{1}{\alpha l^{k+1}} < x \leq \frac{1}{\alpha l^k}, \quad l_1 \leq l \leq l_2,
\]

where \( k = 1, 2, \ldots \), and the constant \( C(n) \) does not depend on \( k \) and \((x, l)\).

**Proof.** Since \( 1 < l_1 \leq l \leq l_2 < +\infty \), it follows that, obviously, it suffices to prove the above bounds for the function \( \Phi(x, l) := \tilde{\varphi}(x, l) \left| \frac{1}{2\alpha l} \right| \).

Let \(-\frac{1}{2} \leq \tau < \frac{1}{2} \) and let

\[
C_\tau := \Phi \left( \frac{1}{\alpha} \left( \frac{1}{2} - \tau \right) + \frac{1}{\alpha l_*} \left( \frac{1}{2} + \tau \right), l_* \right).
\]

Introduce the functions

\[
\hat{x}_{k, \tau}(l) = \frac{1}{\alpha l^k} \left( \frac{1}{2} - \tau \right) + \frac{1}{\alpha l^{k+1}} \left( \frac{1}{2} + \tau \right), \quad k = 0, 1, 2, \ldots.
\]

Using (25) and (26) it can readily be proved that

\[
1) \quad \Phi \left( \frac{x}{l}, \frac{l}{l} \right) = l \Phi(x, l), \quad (x, l) \in (0, 1/\alpha) \times (1, \infty),
\]

\[
2) \quad \Phi(x, l) \big|_{x=\hat{x}_{k, \tau}(l)} = C_\tau l^k, \quad k = 0, 1, 2, \ldots,
\]

which immediately implies the assertion of the lemma for \( n = 0 \).

Further, it follows from property 1) that

\[
\frac{\partial \Phi}{\partial x} \left( \frac{x}{l}, \frac{l}{l} \right) = l^2 \frac{\partial \Phi}{\partial x}(x, l).
\]

Let

\[
M_1 = \max_{(x, l) \in \Omega_1} \left| \frac{\partial \Phi}{\partial l} \right|,
\]

where \( I_1 := \left\{ (x, l) \left| \frac{1}{\alpha l} < x \leq \frac{1}{\alpha l}, \ l_1 \leq l \leq l_2 \right. \right\} \).

Then the following bound holds in the \( k \)-th strip \( I_k := \left\{ (x, y) \left| \frac{1}{\alpha l^{k+1}} < x \leq \frac{1}{\alpha l^k}, \ l_1 \leq l \leq l_2 \right. \right\} \):

\[
\left| \frac{\partial \Phi}{\partial x}(x, l) \right| \leq M_1 l^{2k}.
\]

Let \((x_0, l_0)\) be an arbitrary point of the \( k \)-th strip, and let \( \tau_0 \) be such that

\[
x_0 = \hat{x}_{k, \tau_0}(l_0).
\]

Then, on one hand,

\[
\frac{d \Phi(\hat{x}_{k, \tau_0}(l), l)}{dl} \bigg|_{l_0} = C_{\tau_0} k^{l_0^{-1}}.
\]
and, on the other hand,
\[
\frac{d\Phi(\tilde{x}_{k, \tau_0}(l), l)}{dl} = \left[ \frac{\partial \Phi}{\partial x} \cdot \frac{d\tilde{x}_{k, \tau_0}}{dl} + \frac{\partial \Phi}{\partial l} \right]_{x=\tilde{x}_{k, \tau_0}(l)},
\]
and therefore
\[
\frac{d\Phi(\tilde{x}_{k, \tau_0}(l), l)}{dl} \bigg|_{l_0} = \frac{\partial \Phi}{\partial x} \bigg|_{(x_0, l_0)} \cdot \left( \frac{-k}{\alpha l_0^{k+1}} \left( \frac{1}{2} - \tau_0 \right) + \frac{-k - 1}{\alpha l_0^{k+2}} \left( \frac{1}{2} + \tau_0 \right) \right) + \frac{\partial \Phi}{\partial l} \bigg|_{(x_0, l_0)}.
\]
Hence, for any point \((x_0, l_0) \in I_k,
\[
\left| \frac{\partial \Phi}{\partial l} \bigg|_{(x_0, l_0)} \right| \leq C_{\tau_0} k l_0^{k-1} + M_1 l_0^{2k} \cdot \left| \frac{k}{\alpha l_0^{k+1}} + \frac{k + 1}{\alpha l_0^{k+2}} \right| \leq C(1) k l_0^k.
\]
which implies the assertion of the lemma for \(n = 1\). To obtain the desired bound for \(n = 2\), it suffices to note that
\[
\frac{\partial^2 \Phi}{\partial x^2} \left( \frac{x}{l}, l \right) = l^3 \frac{\partial^2 \Phi}{\partial x^2}(x, l)
\]
and to consider the second derivative of the function \(\Phi(\tilde{x}_{k, \tau_0}(l), l)\), respectively, and so on.

Let us prove now the estimation (41) for \(n = 1\). We have
\[
p(x, y; t) = \int_0^1 \cos(\sqrt{t} \ t) \sigma_0(\lambda) \ u(x, y; \lambda) \ d\lambda = \int_0^1 \cos(\nu t) \ 2\nu \ \sigma_0(\nu^2) \ u(x, y; \nu^2) \ d\nu =
\]
\[
= -\frac{1}{t} \int_0^1 \sin(\nu t) \ (2\nu \sigma_0(\nu^2))'_{\nu} u(x, y; \nu^2) \ d\nu \ - \frac{1}{t} \int_0^1 \sin(\nu t) \ 2\nu \ \sigma_0(\nu^2) \ u'_{\nu}(x, y; \nu^2) \ d\nu.
\]
We claim that the \(L_2\)-norm of the first integral is bounded,
\[
\left\| \int_D \left\{ \int_0^1 \sin(\nu t) \ (2\nu \sigma_0(\nu^2))'_{\nu} u(x, y; \nu^2) \ d\nu \right\}^2 \right\| dx dy \leq
\]
\[
\leq \int_D \int_0^1 \left\| (2\nu \ \sigma_0(\nu^2))'_{\nu} u(x, y; \nu^2) \right\|^2 d\nu dx dy = \int_D \int_0^1 \left\| (2\nu \ \sigma_0(\nu^2))'_{\nu} u(x, y; \nu^2) \right\|^2 d\nu dy d\nu \leq K_1,
\]
where \(K_1\) does not depend on \(t\). Further, for the second integral in (41), we obtain
\[
\int_D \left\| \int_0^1 \sin(\nu t) \ 2\nu \ \sigma_0(\nu^2) \ u'_{\nu}(x, y; \nu^2) \ d\nu \right\|^2 dx dy =
\]
where $D_*$ is obvious, and therefore we study the first summand by using the representation (17) of the convergence of the series (46) + 4∫D \left| \sigma_0(\mu) u'_\mu(x, y; \mu) 2\sqrt{\mu} \right|^2 d\mu dx dy =

\leq (\lambda_* - \lambda) \int \int_D \left| \sigma_0(\mu) u'_\mu(x, y; \mu) 2\sqrt{\mu} \right|^2 d\mu dx dy +

+ (\lambda_* - \lambda) \int \int_{D \cap [D_*]} \left| \sigma_0(\mu) u'_\mu(x, y; \mu) 2\sqrt{\mu} \right|^2 d\mu dx dy,

where $D_* := D \cap \{a_* y > x + a_* - \frac{1}{\alpha} \}$ for $a_* = \sqrt{\frac{\lambda_*}{1 - \lambda_*}}$. The existence of the last summand is obvious, and therefore we study the first summand by using the representation (17) of the function $u(x, y; \mu)$ in the domain $D_*:

\int \int \int_{D_*} \left| \sigma_0(\mu) u'_\mu(x, y; \mu) 2\sqrt{\mu} \right|^2 d\mu dx dy \leq

\leq 4 \int \int_{D_*} \left| \sigma_0(\mu) 2\sqrt{\mu} \left( \frac{\sqrt{\mu}}{2\sqrt{1 - \mu}} \right)' \int_{P(x,y;\mu)} Q(x,y;\mu) \varphi(x', \mu) dx' \right|^2 d\mu dx dy +

+ 4 \int \int_{D_*} \left| \sigma_0(\mu) \frac{\mu}{\sqrt{1 - \mu}} Q'_\mu(x, y; \mu) \varphi(Q(x, y; \mu), \mu) \right|^2 d\mu dx dy +

+ 4 \int \int_{D_*} \left| \sigma_0(\mu) \frac{\mu}{\sqrt{1 - \mu}} P'_\mu(x, y; \mu) \varphi(P(x, y; \mu), \mu) \right|^2 d\mu dx dy +

+ 4 \int \int_{D_*} \left| \sigma_0(\mu) \frac{\mu}{\sqrt{1 - \mu}} \int_{P(x,y;\mu)} \varphi'_\mu(\bar{x}, \mu) d\bar{x} \right|^2 d\mu dx dy.

(45)

Let us prove the existence of the last integral. To this end, it suffices to prove the convergence of the series

(46)

$$
\sum_{k=k_0}^{+\infty} \int \int_{R_k} \left| Q(x,y;\mu) \int_{P(x,y;\mu)} \varphi'_\mu(\bar{x}, \mu) d\bar{x} \right|^2 d\mu dx dy,
$$
where the integrals are taken over trapezium

\[ R_k := \left\{ (x, y) \mid \frac{1}{l_{k+1}} < x < \frac{1}{l_k}, \ 0 < y < \alpha x \right\} \]

and \( k_0 \) is sufficiently large. Passing to the new variable of integration \( l = l(\mu) \), we obtain

\[
\int_{\lambda_*}^{\lambda_{**}} Q(x, y; \nu) \left( \frac{2}{P(x, y; \nu)} \right)^2 d\mu \leq K_2 \left\{ \int_{l_1}^{l_2} \left( \tilde{P}(x, y, l) - \tilde{Q}(x, y, l) \right) \left( \int_{P(x, y; l)}^{l_{**}} |\tilde{\varphi}'(\tilde{x}, l)|^2 d\tilde{x} \right) dl \right\}.
\]

where \( K_2 \) stands for a positive constant depending on \( \lambda_* \) and \( \lambda_{**} \) only. Let

\[ P_k := \max_{(x, y) \in R_k, l \in [l_1, l_2]} \tilde{P}(x, y; l), \quad Q_k := \min_{(x, y) \in R_k, l \in [l_1, l_2]} \tilde{Q}(x, y; l). \]

It can readily be seen that

\[ P_k = \frac{l_2 + 1}{2l_1^k}, \quad Q_k = \frac{l_2 + 1}{2l_1^{k+1}l_2}, \]

and therefore, for \((x, y) \in R_k\),

\[ \int_{l_1}^{l_2} \left( \tilde{P}(x, y; l) - \tilde{Q}(x, y; l) \right) \left( \int_{P(x, y; l)}^{l_{**}} |\tilde{\varphi}'(\tilde{x}, l)|^2 d\tilde{x} \right) dl \leq K_3 \left\{ \int_{l_1}^{l_2} |\tilde{\varphi}'(\tilde{x}, l)|^2 dl \right\} d\tilde{x}, \]

where \( K_3 \) stands for a positive constant depending on \( \lambda_* \) and \( \lambda_{**} \) only. Moreover, it also follows from (47) that there are positive integers \( m_0 \) and \( r_0 \) independent of \( k \) and such that

\[ l_1^{-(k+m_0)} \leq Q_k < P_k \leq l_1^{-(k-r_0)} \]

for any \( k \). Therefore, denoting the integrand on the right-hand side of (48) by \( \mathcal{F} \), we see that \( \mathcal{F} \) is nonnegative and

\[
\int_{Q_k}^P \mathcal{F} d\tilde{x} \leq \int_{l_1^{-(k+m_0)}}^{l_1^{-(k-m_0-1)}} \mathcal{F} d\tilde{x} + \int_{l_1^{-(k+m_0-1)}}^{l_1^{-(k-m_0-2)}} \mathcal{F} d\tilde{x} + \ldots + \int_{l_1^{-(k-r_0+1)}}^{l_1^{-(k-r_0)}} \mathcal{F} d\tilde{x}.
\]

Then

\[
\sum_{k=k_0}^{+\infty} \int_{R_k}^{\lambda_{**}} Q(x, y; \nu) \left( \int_{P(x, y; \nu)}^{l_{**}} |\varphi'(\tilde{x}, \nu)| d\tilde{x} \right)^2 d\mu dx dy \leq
\]

\[ (m_0 + r_0 + 1)K_3 \sum_{k=k_0}^{+\infty} \int_{l_1^{-(k+m_0)}}^{l_1^{-(k-m_0-1)}} \left\{ \int_{l_1}^{l_2} |\tilde{\varphi}'(\tilde{x}, l)|^2 dl \right\} d\tilde{x}, \]

(49)

\[ \quad \leq \left( m_0 + r_0 + 1 \right) K_3 \sum_{k=k_0}^{+\infty} \frac{S_k}{l_k} \int_{l_1^{-(k+1)}}^{l_1^{-(k-m_0-1)}} \left\{ \int_{l_1}^{l_2} |\tilde{\varphi}'(\tilde{x}, l)|^2 dl \right\} d\tilde{x}, \]
where \( S_k \) stands for the area of \( R_k \).

Let \( \tilde{x} \in [l_1^{-(k+1)}, l_1^{-k}] \). Denote by \( l_{\tilde{x},i} \) the values of \( l \) corresponding to the points of intersection of the line \( x = \tilde{x} \) with the curves \( x = \frac{1}{t_i}, i = 0, 1, 2, \ldots \),

\[
\tilde{x} = \frac{1}{(l_{\tilde{x},0})^{k+1}} = \frac{1}{(l_{\tilde{x},1})^k} = \frac{1}{(l_{\tilde{x},2})^{k-1}} = \cdots = \frac{1}{(l_{\tilde{x},i+1})^{k-i}} = \cdots ,
\]

and by \( N(\tilde{x}, k) \) the number of intersections which correspond to \( l_{\tilde{x},i} \) belonging to the interval \([l_1, l_2]\). It can readily be seen that \( N(\tilde{x}, k) < \gamma k \) for sufficiently large \( k \), where \( 0 < \gamma < 1 \) does not depend on \( k \) and \( \tilde{x} \). Here we obviously have \( l_{\tilde{x},0} \leq l_1 \leq l_{\tilde{x},1} \) and

\[
\int_{l_1}^{l_2} d\tilde{x} \int_{l_1^{-k}}^{l_2^{-k}} |\varphi'_l(\tilde{x}, l)|^2 dl \leq \int_{l_1^{-k}}^{l_2^{-k}} \left\{ \sum_{i=0}^{N(\tilde{x}, k)+1} \left( \int_{l_{\tilde{x},i}}^{l_{\tilde{x},i+1}} |\varphi'_l(\tilde{x}, l)|^2 dl \right) \right\} d\tilde{x}.
\]

Let us now use the bound (43) for \( (n = 1) \). In this case,

\[
\int_{l_1^{-k}}^{l_2^{-k}} \left\{ \sum_{i=0}^{N(\tilde{x}, k)+1} \left( \int_{l_{\tilde{x},i}}^{l_{\tilde{x},i+1}} |\varphi'_l(\tilde{x}, l)|^2 dl \right) \right\} d\tilde{x}
\]

\[
\leq \int_{l_1^{-k}}^{l_2^{-k}} \left\{ \sum_{i=0}^{N(\tilde{x}, k)+1} \int_{l_{\tilde{x},i}}^{l_{\tilde{x},i+1}} C^2(1) (k - i)^2 l^{2(k-i)} dl \right\} d\tilde{x}
\]

\[
\leq k^2 C^2(1) \int_{l_1^{-k}}^{l_2^{-k}} \left\{ \sum_{i=0}^{N(\tilde{x}, k)+1} (l_{\tilde{x},i+1})^{2(k-i)} (l_{\tilde{x},i+1} - l_{\tilde{x},i}) \right\} d\tilde{x}
\]

\[
\leq k^2 C^2(1) \int_{l_1^{-k}}^{l_2^{-k}} \left\{ (N(\tilde{x}, k) + 2) (l_{\tilde{x},0})^{2(k+1)} \right\} d\tilde{x} \leq k^2 C^2(1) \int_{l_1^{-k}}^{l_2^{-k}} \left\{ (\gamma k + 2) l_1^{2(k+1)} \right\} d\tilde{x}
\]

\[
\leq K_4 k^3 l_1^k,
\]

where \( K_4 \) stands for a positive constant independent of \( k \). This obviously implies that the series (49) converges.

It is clear that the same arguments prove the convergence of the first integral on the right-hand side of inequality (45), because the function \( \varphi(x, \mu) \) can not increase more rapidly than \( \varphi'_l(x, \mu) \).

Consider now the second integral on the right-hand side of (45). Since

\[
P'_\mu(x, y; \mu) = \frac{\alpha x - y}{2\alpha} l'_\mu,
\]

we obviously have

\[
\int_{D_2} \int_{D_1} \int_{\mathbb{R}^+} \left\{ \sigma_0(\mu) \frac{\mu}{\sqrt{1 - \mu}} P'_\mu(x, y; \mu) \phi(P(x, y; \mu), \mu) \right\}^2 d\mu dx dy
\]
Thus, Theorem 2.2 is proved for $l_1$ does not depend on $\nu$ and $\lambda$. Setting $\lambda_1 = 1. To prove the validity of the bound (41) for $\lambda_1$, we have

$$
\int_{l_1}^{l_2} \left| \tilde{\varphi}(\tilde{P}(x, y; l), l) \right|^2 dl = \int_{l_1}^{l_2} \left| \tilde{\varphi}(\tilde{P}(x, y; l), l) \right|^2 dl + \int_{l_1}^{l_2} \left| \tilde{\varphi}(\tilde{P}(x, y; l), l) \right|^2 dl + \ldots + \int_{l_1}^{l_2} \left| \tilde{\varphi}(\tilde{P}(x, y; l), l) \right|^2 dl \leq \sum_{k=k_0}^{+\infty} \int_{l_1}^{l_2} \left| \tilde{\varphi}(\tilde{P}(x, y; l), l) \right|^2 dl \leq C(0) \left\{ \int_{l_1}^{l_2} l_{k_{M,n}'}^{2k_{M}'} dl + \int_{l_1}^{l_2} l_{k_{M,n}'}^{2k_{M}'} dl + \ldots + \int_{l_1}^{l_2} l_{k_{M,n}'}^{2k_{M}'} dl \right\} \leq C(0)(l_2 - l_1) \left( (l_{k_{M,n}})_{k_{M,n}'}^{2k_{M}'} + (l_{k_{M,n}})_{k_{M,n}'}^{2k_{M}'} + \ldots + (l_{k_{M,n}})_{k_{M,n}'}^{2k_{M}'} \right) = C(0)(l_2 - l_1) \left( P(x, y; l_{k_{M,n}})'^2 + P(x, y; l_{k_{M,n}})'^2 + \ldots + P(x, y; l_{k_{M,n}})'^2 \right) \leq C(0)(l_2 - l_1)(N(x, y; k) + 1) \frac{4\alpha^2}{(\alpha x + y)^2} \leq \frac{K_7 k}{x^2},
$$

where $K_7$ does not depend on $k$, and therefore the series (50) converges indeed.

Thus, Theorem 2.2 is proved for $n = 1$. To prove the validity of the bound (41) for $n = 2$, note that

$$
U(x, y; t) =
$$

$$
= \frac{1}{t} \int_0^1 \sin(\nu t) (2\nu \sigma_0(\nu^2))' \nu u(x, y; \nu^2) d\nu - \frac{1}{t} \int_0^1 \sin(\nu t) 2\nu \sigma_0(\nu^2) u'(x, y; \nu^2) d\nu =
$$

$$
= \frac{1}{t^2} \left( \int_0^1 \cos(\nu t) (2\nu \sigma_0(\nu^2))'' \nu u(x, y; \nu^2) d\nu + 2 \int_0^1 \cos(\nu t) (2\nu \sigma_0(\nu^2))' \nu u'(x, y; \nu^2) d\nu +
$$
The existence of the first two integrals in the last expression was proved above, and, to prove the convergence of the third integral, with regard to the representation of \(u(x, y; \nu^2)\) in the form (17), it is obviously sufficient to prove that the series

\[
+ \int_0^1 \cos(\nu t) 2\nu \sigma_0(\nu^2) u_{\nu\nu}(x, y; \nu^2) d\nu
\]

converges. The proof of the convergence of this series repeats verbatim the proof of the convergence of the series (46) with the only difference that the bound (43) is used for \(n = 2\).

The cases \(n = 3, 4, \ldots\) are treated in a similar way. This completes the proof of Theorem 2.2.

3. DISTRIBUTION OF THE ENERGY OF THE INITIAL STATE OF THE FLUID

It can readily be proved that law of conservation of energy holds for the solutions of problem (1)–(3) (see, for example, [5]):

\[
\mathcal{E}(t, D) := \int_D \left( |p_y|^2 + |p_{xt}|^2 + |p_{yt}|^2 \right) dx dy = \text{const.}
\]

Problem (1)–(3) was studied in [18] in the complement \(\mathbb{R}^2 \setminus \overline{\Omega}\) to some convex bounded domain \(\Omega\). In this case, the energy of the initial perturbation is redistributed as \(t \to \infty\) in such a way that the part of energy concentrated on every compact set \(A \in \mathbb{R}^2 \setminus \overline{\Omega}\) tends to zero, i.e. a scattering of energy occurs. In our case the following assertion holds.

**Theorem 3.1.** Let \(p_i, i = 1, 2\) satisfy the conditions of Theorem 2.2. In this case, for any \(\varepsilon > 0\) and \(\delta > 0\), there is a \(T = T(\varepsilon, \delta)\) such that

\[
\mathcal{E}(t, D_\varepsilon) := \int_{D_\varepsilon} \left( |p_y|^2 + |p_{xt}|^2 + |p_{yt}|^2 \right) dx dy < \delta
\]

for the corresponding solution \(p = p(x, y; t)\) of problem (1)–(3) for all \(t > T\), where \(D_\varepsilon\) stands for the set \(D \cap \{x > \varepsilon\} \cap \{y < 1 - \varepsilon\}\).

**Proof.** As in the proof of Theorem 2.2, assume for simplicity that \(\theta^{(1)}_1 \equiv 0, \theta^{(0)}_2 \equiv 0, \theta^{(1)}_2 \equiv 0, \sigma_1(\lambda) \equiv 0, \text{ and supp } \sigma_0 \subset [\lambda_*, \lambda_{**}] \subset (0, (1+\alpha^2)^{-1})\). In this case the solution of problem (1)–(3) is the function

\[
p(x, y; t) = \int_0^1 \cos(\sqrt{\lambda} t) \sigma_0(\lambda) u(x, y; \lambda) d\lambda =
\]

\[
= -\frac{1}{t} \int_0^1 \sin(\sqrt{\lambda} t) \left(2\sqrt{\lambda} \sigma_0(\lambda)\right)' u(x, y; \lambda) d\lambda - \frac{1}{t} \int_0^1 \sin(\sqrt{\lambda} t) \sigma_0(\lambda) u'(x, y; \lambda) 2\sqrt{\lambda} d\lambda
\]
As was noted above, under the assumptions of the theorem, \( u(x, y; \lambda) \) is infinitely differentiable with respect to \( x, y, \lambda \) everywhere inside the prisme \( D \times (0, (1 + \alpha^2)^{-1}) \) and can be continuously extended together with all its derivatives to the entire surface of the prism
\[
(D \cap \{x > \varepsilon\}) \times (0, (1 + \alpha^2)^{-1}), \quad \varepsilon > 0.
\]

We have
\[
p_{xt} = -\frac{1}{t} \int_0^1 \sqrt{\lambda} \sin(\sqrt{\lambda} t) \left(2\sqrt{\lambda} \sigma_0(\lambda) \right)'_{\lambda} u_x'(x, y; \lambda) d\lambda -
-\frac{1}{t} \int_0^1 \sin(\sqrt{\lambda} t) \sigma_0(\lambda) u''_{x\lambda}(x, y; \lambda) 2\lambda d\lambda,
\]
and therefore
\[
\int \int_{D \cap \{x > \varepsilon\}} |p_{xt}|^2 \, dx \, dy \leq \frac{2}{t^2} \int \int_{D \cap \{x > \varepsilon\}} \left| \int_0^1 \sqrt{\lambda} \sin(\sqrt{\lambda} t) \left(2\sqrt{\lambda} \sigma_0(\lambda) \right)'_{\lambda} u_x'(x, y; \lambda) d\lambda \right|^2 \, dx \, dy +
\]
\[
+ \frac{2}{t^2} \int \int_{D \cap \{x > \varepsilon\}} \left| \int_0^1 \sin(\sqrt{\lambda} t) \sigma_0(\lambda) u''_{x\lambda}(x, y; \lambda) 2\lambda d\lambda \right|^2 \, dx \, dy \leq
\]
\[
\leq \frac{2}{t^2} \int \int_{D \cap \{x > \varepsilon\}} \int_0^1 \sqrt{\lambda} \left(2\sqrt{\lambda} \sigma_0(\lambda) \right)'_{\lambda} u_x'(x, y; \lambda) \, d\lambda \, dx \, dy +
\]
\[
+ \frac{2}{t^2} \int \int_{D \cap \{x > \varepsilon\}} \int_0^1 |\sigma_0(\lambda) u''_{x\lambda}(x, y; \lambda) 2\lambda|^2 d\lambda \, dx \, dy \leq \frac{K}{t^2} \rightarrow 0 \quad (t \rightarrow \infty),
\]
where the positive constant \( K \) does not depend on \( t \). The other summands in (52) can be estimated in a similar way.

It follows from the last theorem that, in the course of time, the total energy of the initial state of the fluid turns out to be almost completely concentrated in arbitrary small neighborhoods of the vertices \( O \) and \( B \) of the domain \( D \). It is clear here that, if we have \( \theta_2(i) \equiv 0, i = 0, 1 \), or \( \text{supp} \sigma_i \in (0, (1 + \alpha^2)^{-1}), i = 0, 1 \), then the energy is accumulated in a neighborhood of the point \( O \) only. Respectively, if \( \theta_1(i) \equiv 0, i = 0, 1 \), or \( \text{supp} \sigma_i \in ((1 + \alpha^2)^{-1}, 1), i = 0, 1 \), then the energy is accumulated in a neighborhood of the point \( B \). It is clear that this picture occurs due to the fact that the Poincaré-Sobolev equation describes the behavior of an ideal fluid, whereas, in the case of a real fluid, one should consider the corresponding nonlinear systems of equations.

In conclusion we note that the approach to the construction of exact solutions of problem (1)-(3) which is suggested in the present paper can be used for a rather wide class of domains with angular points. For example, let \( D \) be a “curvilinear triangle” whose sides \( OA \) and \( OB \) are some smooth curves intersecting at the point \( O \) and forming a nonzero
angle at this point. If, for any $\lambda \in (\lambda', \lambda'')$ the rays of characteristic directions whose reflection low at the boundary is described in Section 11 hide into the angle with the vertex $O$, then, on this interval $(\lambda', \lambda'')$, one can construct differential solutions of the spectral equation for the operator $A$ similarly to the rule used in Section 11 and the solutions of the nonstationary problem (11–33) corresponding to these differential solutions. It is clear that the behavior of these solutions as $t \to \infty$ is similar to the behavior of the solutions described above.

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