MULTIVARIATE INTERPOLATION IN UNISOLVENT NODES
LIFTING THE CURSE OF DIMENSIONALITY

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Abstract. We extend Newton and Lagrange interpolation to arbitrary di-
mensions. The core contribution that enables this is a generalized notion of
non-tensorial unisolvent nodes, i.e., nodes on which the multivariate polyno-
mial interpolant of a function is unique. By validation, we reach the optimal
exponential Trefethen rates for a class of analytic functions, we term Trefethen
functions. The number of interpolation nodes required for computing the op-
timal interpolant depends sub-exponentially on the dimension, hence resisting
the curse of dimensionality. Based on these results, we propose an algorithm
to efficiently and numerically stably solve arbitrary-dimensional interpolation
problems, with at most quadratic runtime and linear memory requirement.

1. Introduction

Polynomial interpolation goes back to Newton, Lagrange, and others [65], and
its fundamental importance for mathematics and computing is undisputed. Inter-
polation is based on the fact that, in 1D, one and only one polynomial $Q_{f,n}$
of degree $n$ can interpolate a function $f : \mathbb{R} \rightarrow \mathbb{R}$ on $n + 1$ distinct unisolvent
interpolation nodes $P_n \subseteq \mathbb{R}$, i.e., $Q_{f,n}(p_i) = f(p_i)$ for all $p_i \in P_n$, $0 \leq i \leq n$.
This makes interpolation fundamentally different from approximation. For the
latter, the famous Weierstrass Theorem [87] states that any continuous function
$f \in C^0(\Omega, \mathbb{R})$, $\Omega = [-1, 1]$, can be uniformly approximated by polynomials in prin-
ciple [30] [46] [87] [88]. However, the Weierstrass Theorem does not require the poly-
nomials to coincide with $f$ at all, i.e., it is possible that $Q_{\text{Weierstrass},f,n}(x) \neq f(x)$
for all $x \in \Omega$, but still
\begin{equation}
Q_{\text{Weierstrass},f,n} \xrightarrow{n \to \infty} f \quad \text{uniformly on } \Omega.
\end{equation}

Even though the constructive version of the Weierstrass Theorem given by Serge
Bernstein [6] provides a recipe for computing such approximations it only delivers a
linear convergence rate. In 1D, however, interpolation on Chebyshev and Legendre
nodes is known to avoid Runge’s phenomenon for a generic class of functions and
to yield exponential approximation rates [84], which is much faster than what has
been shown possible by Weierstrass-type approximations [6]. There has therefore
been much research into faster multi-dimensional (mD) interpolation schemes that
extend Newton or Lagrange interpolation schemes to arbitrary dimensions. Any

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approach that addresses this problem has to resolve Runge’s phenomenon and resist the curse of dimensionality, thus, has to answer the following question:

**Question 1.** How to construct interpolation nodes $P_{A_{m,n}} \subseteq \Omega = [-1,1]^m$, $m, n \in \mathbb{N}$, $A_{m,n} \subseteq \mathbb{N}^m$ and an efficient and numerically stable interpolation algorithm such that a generic class of functions $f : \Omega \rightarrow \mathbb{R}$ can be uniformly approximated

\[ Q_{f,A_{m,n}} \xrightarrow{n \rightarrow \infty} f, \quad Q_{f,A_{m,n}}(q) = f(q), \quad \forall q \in P_{A_{m,n}}, \]

with a fast (ideally exponential) convergence rate while keeping the number of interpolation nodes $|P_{A_{m,n}}|$ required small (ideally, sub-exponential)?

Question 1 implicitly requires answering the underlying algebraic question of unisolvent:

**Question 2.** Given a polynomial space $\Pi = \text{span}\{q_\alpha\}_{\alpha \in A}$, $A \subseteq \mathbb{N}^m$ generated by some polynomial basis. How to construct unsisolvent interpolation nodes $P_A \subseteq \Omega$, such that the polynomial interpolant $Q_{f,A}$, with $Q_{f,A}(p) = f(p)$, $\forall p \in P_A$, of any function $f : \Omega \rightarrow \mathbb{R}$ is uniquely determined in $\Pi$?

While many approaches exist to extending polynomial interpolation to higher dimensions, providing partial answers to Questions 1 and 2, none of them fully answers these questions.

As far as we recognize, tensorial Chebyshev interpolation best answers these questions among all state-of-the-art approaches. However, Chebyshev interpolation is done on a (full) tensorial grid, which suffers from the curse of dimensionality requiring $|P_{A_{m,n}}| \in \mathcal{O}(n^m)$ interpolation nodes. Therefore, practical implementations are usually limited to dimensions $m \leq 3$.

Here, we therefore provide a generalized notion of unisolvent interpolation nodes, which allows for non-tensorial grids on which we reach exponential convergence rates for the Runge function, as a prominent example of a Trefethen function. We show that the number of nodes required scales sub-exponentially with space dimension. We therefore believe that the present generalization of unisolvent nodes to non-tensorial grids is key to lifting the curse of dimensionality. Our results also directly inspire an efficient algorithm to practically solve high-dimensional interpolation problems. We therefore provide a numerically robust and computationally efficient algorithm and its software implementation, and we use it to empirically verify our theoretical predictions. Combining sub-exponential node numbers with exponential approximation rates, non-tensorial unisolvent nodes are thus able to lift the curse of dimensionality for multivariate interpolation tasks.

In the following extensive introduction we discuss subjects across these fields and summarize the state of the art from previous works.

### 1.1. Runge’s phenomenon – Approximation theory

Already in 1D, it is long known that for any sequence of interpolation nodes $P_n \subseteq \Omega$, $n \in \mathbb{N}$ there exists at least one continuous function $f : \Omega \rightarrow \mathbb{R}$ that can not be approximated by interpolation on $P_n$, i.e.,

\[ Q_{f,n} \xrightarrow{n \rightarrow \infty} f, \quad \text{where } Q_{f,n}(q) = f(q), \quad \forall q \in P_n. \]

Instead, the approximation quality of an interpolation polynomial is sensitive to the choice of the interpolation nodes $P_n \subseteq \Omega$. In other words: Interpolating $f$ with a polynomial $Q_{f,n}$ of increasing degree $n \in \mathbb{N}$ does not guarantee that the interpolant
that any continuous function \( f \in C^0(\Omega, \mathbb{R}) \) can be approximated by polynomials in principle \([60][1][88]\), asking whether such polynomials can be determined by interpolation on specified nodes \( P_A \) is in regard of Runge’s phenomenon, Eq. (1.2), a different question.

Whatsoever, any answer to Questions 2 that is to be of practical relevance must provide a recipe to construct interpolation nodes \( P_A \) that allow efficient approximation while resisting the curse of dimensionality in terms of Question 1.

1.2. Lifting the curse of dimensionality. Recently, Lloyd N. Trefethen \([83]\) proposed a way of delivering a potential solution to the problem: For continuous functions \( f : \Omega \rightarrow \mathbb{R} \) that are analytic in the unbounded Trefethen domain (a generalization of a Bernstein ellipse) \( N_{m,\rho} \subseteq \Omega = [-1,1]^m \), of radius \( \rho > 1 \), an upper bound on the convergence rate applies:

\[
\|f - Q_{f,A_{m,n,p}}\|_{C^0(\Omega)} \in \begin{cases} 
\mathcal{O}_\varepsilon(\rho^{-n/\sqrt{m}}), & p = 1 \\
\mathcal{O}_\varepsilon(\rho^{-n}), & p = 2 \\
\mathcal{O}_\varepsilon(\rho^{-n}), & p = \infty
\end{cases}
\]

where \( g \in \mathcal{O}_\varepsilon(\rho^{-n}) \) iff \( g \in \mathcal{O}((\rho - \varepsilon)^{-n}), \forall \varepsilon > 0 \).

The multi-index sets \( A_{m,n,p} = \{ \alpha \in \mathbb{N}^m \mid \|\alpha\|_p \leq n \} \subseteq \mathbb{N}^m \) generalize the notion of polynomial degree to multi-dimensional \( l_p \)-degree. This suggests that interpolating a function with respect to the polynomial space \( \Pi_{m,n,2} = \text{span}\{x^\alpha\}_{\alpha \in A_{m,n,2}} \) spanned by all \( l_2 \)-monomials can reach the same convergence rate as interpolating with respect to the \( l_\infty \)-degree \( A_{m,n,\infty} \), while \( l_1 \)-degree can not reach such a fast rate. Indeed, this expectation is validated and argued to be genuine in \([83]\).

The number of coefficients \( |A_{m,n,1}| = \binom{m+n}{n} \in \mathcal{O}(m^n) \) required is of polynomial cardinality for \( p = 1 \), whereas \( |A_{m,n,2}| \in o(n^m) \) is of sub-exponential size for \( p = 2 \), but \( |A_{m,n,\infty}| = (n+1)^m \) scales exponentially with dimension \( m \in \mathbb{N} \) for \( p = \infty \). Thus, combining sub-exponential node numbers with exponential approximation rates, interpolation with respect to \( l_2 \)-degree polynomials might yield a way of lifting the curse of dimensionality and answering Question 1.

However, in \([83]\) the interpolants \( Q_{f,A_{m,n,1}}, Q_{f,A_{m,n,2}} \) were computed by regression over a full Chebyshev grid \( P_{A_{m,n,\infty}} \), which requires evaluating \( f \) on the exponentially many \( |A_{m,n,\infty}| = (n+1)^m \) nodes \( P_{A_{m,n,\infty}} \). In other words: There is no numerically stable and efficient algorithm known that can compute \( Q_{f,A_{m,n,2}} \) by evaluating \( f \) only on \( |A_{m,n,2}| \) unisolvent nodes while reaching the optimal Trefethen approximation rates.

1.3. Interpolation with optimal Trefethen rates – The core contribution of this article. Here, we answer Questions 1, 2. To do so, we generalize the notion of unisolvent nodes \( P_A \), \( A \subseteq \mathbb{N}^m \) to non-tensorial grids. This allows us to extend Newton (NI) and Lagrange (LI) interpolation to arbitrary-dimensional spaces such that:

- P1) \( P_A \) are given by non–tensorial, non–symmetric grids, \( A = A_{m,n,p} \), scaling sub-exponentially with dimension \( m \) if \( p < \infty \).
P2) Computing the uniquely determined multivariate interpolant $Q_{f,A} \in \Pi_A$, requires $O(|A|^2)$ or $O(|A|)$ runtime for NI and LI, respectively, and $O(|A|)$ memory.

P3) Evaluating the interpolant $Q_{f,A}(x)$ at any argument $x \in \mathbb{R}^m$ requires $O(|A|)$ or $O(|A|^2)$ runtime for NI and LI, respectively, and $O(|A|)$ memory.

P4) Numerical experiments validate that the Runge function $f(x) = 1/(1 + 10\|x\|^2)$ can be interpolated on $P_{A,m,n,2}$ to machine precision up to dimension 5 while reaching the optimal Trefethen rate $O(\rho^{-n})$ stated in Eq. (1.3) with the theoretically predicted Trefethen radius $\rho$.

We want to add the following remarks:

Remark 1.1. If one only requires the nodes $P_A$ to be unisolvent, then they do not have to be given by a (sub) grid at all. The nodes used for the present $mD$ Newton interpolation are given by a sub-grid, but that sub-grid is neither symmetric nor tensorial. Nevertheless, we reach the optimal exponential Trefethen rates on these grids. If we would add nodes to make the grid symmetric or tensorial, then the number of nodes of the resulting (sparse) tensorial grid would scale exponentially $O(n^m)$ with space dimension $m \in \mathbb{N}$. In contrast, our proposed interpolation nodes scale sub-exponentially $o(n^m)$ and thus lift the curse of dimensionality.

Remark 1.2. We complement the established notion of unisolvent nodes by the dual notion of unisolvence. That is: For given arbitrary nodes $P$, determine the polynomial space $\Pi$ such that $P$ is unisolvent with respect to $\Pi$. In doing so, we revisit earlier results by Carl de Boor and Amon Ros [28, 29] and answer their question from our perspective.

Remark 1.3. Combining our multivariate Lagrange interpolation with the dual notion of unisolvence allows establishing a polynomial regression scheme for scattered data on planar and curved manifolds. In particular, we demonstrate that the level set function $L$ of the torus $T^2_{R,r} = L^{-1}(0)$, its gradient $\nabla L$, and the Runge function $f$ restricted to $T^2_{R,r}$ can be approximated to high (machine) precision with interpolation nodes $P \subseteq T^2$ sampled uniformly at random on the torus.

In summary: We answer Questions 1–2 by establishing an efficient $mD$ interpolation scheme that can approximate a generic class of functions and, at least empirically, reaches the proposed exponential approximation rate for strongly varying Trefethen functions, such as the Runge function $f(x) = 1/(1 + 10\|x\|^2)$, requiring only a sub-exponential number of non-tensorial interpolation nodes. Combining sub-exponential node numbers with exponential approximation rates, non-tensorial unisolvent nodes are thus able to lift the curse of dimensionality for multivariate interpolation tasks.

1.4. Related Work. The importance of the present problem is manifested in the large number of attempts that were made to solve this central problem of applied mathematics. Consequently, an exhaustive overview of multivariate interpolation schemes can not be given here. In the following, we restrict ourselves to mention the most relevant state-of-the-art approaches.

1.4.1. Interpolation in tensorial (sub) grids. There are approaches, e.g. [33, 48, 59, 75], that interpolate polynomials on (sparse) tensorial grids, which are know to be
unisolvent for the corresponding tensorial polynomial spaces. This, for instance, allows to theoretically derive formulas for Lagrange polynomials on these grids [75]. However, this does not mean that efficient algorithms to evaluate the resulting interpolants to machine precision are known. Furthermore, so far none of these approaches is known to reach the optimal Trefethen approximation rates when requiring the number of nodes of the underlying tensorial grids to scale sub-exponential with space dimension. As the numerical experiments in Section 5 suggest, we believe that only non-tensorial grids are able to lift the curse of dimensionality, which requires non–tensorial interpolation schemes such as those presented here.

1.4.2. Weierstrass-type Approximation. Several approaches [2, 14, 17, 18, 19, 67] are available to realize mD Weierstrass-type approximations by computing the $L^2$–projections onto a-priori specified orthogonal polynomials. However, the linear convergence rate of the Bernstein approximation [6] is reflected in the circumstance that these approaches are prevented from approximating a generic class of functions, but are limited to well-behaving bounded analytical or holomorphic functions occurring, for instance, as solutions of elliptic PDEs. In these scenarios, reasonable uniform approximations of the function $f$ can be reached by sparse samples that avoid the curse of dimensionality in high dimensions $m \in \mathbb{N}, m \leq 16$. However, when asking such approaches to deliver approximations to machine precision, or to leave the tight class of well-behaving functions, their resistance to the curse of dimensionality disappears already for low dimensions.

1.4.3. Multivariate splines. A prominent and well established alternative to the above approaches is the multivariate spline interpolation by Carl de Boor et al. [23, 24, 25, 26]. The approximation result [27] for bivariate splines due to de Boor is given as:

**Theorem 1.1** (Carl de Boor). Let $f : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a $(n+1)$-times continuously differentiable bivariate function, $\Delta$ be a triangulation, and $S_{f,n,\Delta} = \{g \in C^0(\Omega, \mathbb{R}) \mid g_{\delta} \in \Pi_{m,n,1}, \forall \delta \in \Delta\}$, $n > 3\rho+1$ be the space of piecewise polynomial functions of degree $n$. Then there exists $c(\Delta) > 0$ such that

$$\text{dist}(f, S_{f,n,\Delta}) \leq c(\Delta)\|D^{n+1}f\|_{C^0(\Omega)}|\Delta|^{n+1},$$

where $|\Delta| = \max_{T \in \Delta} |T|$ is the mesh size.

This result states that any sufficiently smooth function $f$ can be approximated by piecewise polynomial functions, which allows to approximate $f$ by Hermite or spline interpolation. Generalizations of this result rely on this fact and are formulated in a similar manner [23, 24, 26].

Despite its fundamental importance in linking interpolation and approximation, the above result and the resulting interpolation algorithms have some weak points:

A) The strong regularity assumption, i.e., the $(n+1)$-fold differentiability of $f$;

B) The error bound in Eq. (1.4) only guarantees a polynomial convergence rate, but no exponential convergence;

C) The approach is sensitive to the curse of dimensionality, i.e., the number of polynomial coefficients $C_{m,n}$ scales exponentially with dimension: $|C_{m,n}| \in \mathcal{O}(n^m)$. 
Especially for Trefethen functions, such as the Runge function \( f(x) = 1/(1 + 10|x|^2) \), (B) prevents spline interpolation. Several improvements have been presented, including Floatman–Hormann interpolation \([16, 38]\), that reach better approximation quality than splines. However, all of them share the above weaknesses (A,B,C), as we demonstrate in the numerical experiments of Section 5. Though, approximations of lower accuracy might be reached faster then by polynomial interpolation, this makes these approaches incapable for answering Question 1 when higher-precision approximations are required. The multivariate polynomial interpolation method presented here reaches this goal.

1.5. Notation. Let \( m, n \in \mathbb{N} \), \( p \geq 1 \). We denote by \( e_1 = (1, 0, \ldots, 0) \), \( \ldots \), \( e_m = (0, \ldots, 0, 1) \in \mathbb{R}^m \) the standard basis, by \( \| \cdot \| \) the euclidean norm on \( \mathbb{R}^m \), and by \( \| M \|_p \) the \( l_p \)-norm of a matrix \( M \in \mathbb{R}^{m \times m} \). Further, \( A_{m,n,p} \subseteq \mathbb{N}^m \) denotes all multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \) with \( \| \alpha \|_p = (\alpha_1^p + \cdots + \alpha_m^p)^{1/p} \leq n \). We order a finite set \( A \subseteq \mathbb{N}^m \), \( m \in \mathbb{N} \) of multi-indices with respect to the lexicographical order \( \leq_L \) on \( \mathbb{N}^m \) starting from \( x_m \) to \( x_1 \), e.g. \( (5,3,1) \leq_L (1,0,3) \leq_L (1,1,3) \). Then, \( \alpha_{\min}, \alpha_{\max} \) shall denote the minimum and maximum of \( A = \{ \alpha_{\min}, \ldots, \alpha_{\max} \} \) with respect to \( \leq_L \). We call \( A \) downward closed or complete iff there is no \( \beta = (b_1, \ldots, b_m) \in \mathbb{N}^m \setminus A \) with \( b_i \leq a_i \), \( \forall i = 1, \ldots, m \) for some \( \alpha = (a_1, \ldots, a_m) \in A \). In \([19]\) the terminology “downward closed set” is used. Note that \( A_{m,n,p} \) is complete for all \( m, n \in \mathbb{N} \), \( p \geq 1 \). Given \( A \subseteq \mathbb{N}^m \) complete and a matrix \( R_A \in \mathbb{R}^{|A| \times |A|} \) we slightly abuse notation by denoting

\[
R_A = (r_{\alpha,\beta})_{\alpha,\beta \in A} = (r_{i,j})_{1 \leq i,j \leq |A|},
\]

with \( \alpha, \beta \) being the \( i \)-th, \( j \)-th entry of \( A \) ordered by \( \leq_L \), respectively.

We consider the real polynomial ring \( \mathbb{R}[x_1, \ldots, x_m] \) in \( m \) variables and denote by \( \Pi_m \) the \( \mathbb{R} \)-vector space of all real polynomials in \( m \) variables. Further, \( \Pi_A \subseteq \Pi_m \) denotes the polynomial subspace induced by \( A \) and generated by the canonical basis given by the monomials \( x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \) with \( \alpha \in A \). For \( A = A_{m,n,p} \) we write \( \Pi_{m,n,p} \) to mean \( \Pi_{A_{m,n,p}} \). Given a polynomial \( Q(x) = \sum_{\alpha \in A} c_\alpha x^\alpha \), \( A \subseteq \mathbb{N}^m \), we call \( \max_{\alpha \in A, \alpha \neq 0} \| \alpha \|_p \) the \( l_p \)-degree of \( Q \). As it will turn out, the notion of \( l_p \)-degree plays a crucial role for the approximation quality of polynomial interpolation. While \( A_{1,n,p} = \{ 0, \ldots, n \} \), considering \( A \subseteq \mathbb{N}^m \) for \( m > 1 \) generalizes the concept of polynomial degree to multi-dimensions.

Throughout this article \( \Omega = [-1,1]^m \) denotes the \( m \)-dimensional standard hypercube and \( C^0(\Omega, \mathbb{R}) \) the Banach space of continuous functions \( f : \Omega \rightarrow \mathbb{R} \) with norm \( \| f \|_{C^0(\Omega)} = \sup_{x \in \Omega} |f(x)| \).

2. Unisolvence nodes

While the pioneering works by Kuntzmann, Guenther, and Roetman \([48, 59]\) and extensions in \([15]\) proposed a partial answer to Question 2 we provide a generalized notion of unisolvence nodes for multivariate polynomial interpolation with respect to arbitrary finite-dimensional polynomial spaces \( \Pi \subseteq \Pi_m \), i.e., even in the case where \( \Pi \neq \Pi_A \) is not induced by a complete or downward closed set \( A \subseteq \mathbb{N}^m \). We start by reviewing the usual notion of unisolvence.

2.1. The notion of unisolvence. We consider a (not necessarily downward closed or complete) finite set \( A \subseteq \mathbb{N}^m \) of multi-indices, a set of interpolation nodes \( P_A = \{ p_{\alpha_{\min}}, \ldots, p_{\alpha_{\max}} \} \subseteq \mathbb{R}^m \), a set \( B_A = \{ q_{\alpha_{\min}}, \ldots, q_{\alpha_{\max}} \} \) of multivariate polynomials
e.g. $q_\alpha(x) = x^\alpha$, the polynomial space $\Pi_A = \text{span}(B_A)$ generated by the $q_\alpha$, and a function $f : \mathbb{R}^m \rightarrow \mathbb{R}$. The multivariate Vandermonde matrix is given by

$$V(P) = (q_\beta(p_\alpha))_{\alpha, \beta \in A}. \tag{2.1}$$

For $q_\alpha(x) = x^\alpha$ this results in the classic notion $V(P) = (p_\alpha^\beta)_{\alpha, \beta \in A}$. If $V(P)$ is (numerically) invertible then one can interpolate $f$ by solving the linear system of equations

$$V(P)C = F, \quad C = (c_{\alpha_{\text{min}}}, \ldots, c_{\alpha_{\text{max}}}), \quad F = (f(p_{\alpha_{\text{min}}}), \ldots, f(p_{\alpha_{\text{max}}}))$$

using $O(|A|^3)$ operations. Indeed,

$$Q_{f, A}(x) = \sum_{\alpha \in A} c_\alpha q_\alpha \in \Pi_A \tag{2.2}$$

yields the unique interpolant of $f$ on $P_A$, i.e., $Q_{f, A}(p) = f(p)$ for all $p \in P_A$. We call a set of nodes $P_A$ unisolvent with respect to $\Pi_A$ if and only if $V(P)$ is invertible, i.e., if and only if $\ker V(P) = 0$. The condition $\ker V(P) = 0$ is equivalent to requiring that there exists no hypersurface $H = Q^{-1}(0)$ generated by a polynomial $0 \neq Q \in \Pi_A$ with $P \subseteq H$. Indeed, the coefficients $C$ of such a polynomial would be a non-trivial solution of $V(P)C = 0$.

However, even if $P$ is unisolvent, as is well known and shown in our previous work [51], the inversion of the matrix $V$ becomes numerically ill-conditioned when represented in the canonical basis $q_\alpha(x) = x^\alpha$, $\alpha \in A$. Therefore, alternative interpolation schemes with better numerical condition and lower computational complexity are desirable. While previous approaches to addressing this problem relied on tensorial interpolation schemes [33] [48] [59] [60], we here propose a different approach.

### 2.2. Construction of unisolvent nodes

In the following, we develop our concept of unisolvent nodes that generalizes over previous works [48] [59] [61] [52] [63]. We start by stating the definitions on which our concept rests:

**Definition 2.1** (Transformations). An affine transformation $\tau : \mathbb{R}^m \rightarrow \mathbb{R}^m$, $m \in \mathbb{N}$, is a map $\tau(x) = Bx + b$, where $B \in \mathbb{R}^{m \times m}$ is an invertible matrix and $b \in \mathbb{R}^m$. An affine translation is an affine transformation with $B = I$ the identity matrix. A linear transformation is an affine transformation with $b = 0$.

**Lemma 2.2.** Any affine transformation $\tau : \mathbb{R}^m \rightarrow \mathbb{R}^m$, $m \in \mathbb{N}$, induces a ring isomorphism $\tau^* : \mathbb{R}[x_1, \ldots, x_m] \rightarrow \mathbb{R}[x_1, \ldots, x_m]$, i.e., the induced transformation

$$\tau^* : \Pi_m \rightarrow \Pi_m \quad \text{given by} \quad \tau^*(Q)(x) = Q(\tau(x)) \quad \forall x \in \mathbb{R}^m$$

is a linear transformation, such that

i) $\tau^*(\lambda Q_1 + \mu Q_2) = \lambda \tau^*(Q_1) + \mu \tau^*(Q_2)$ for all $Q_1, Q_2 \in \Pi_m$, and $\lambda, \mu \in \mathbb{R}$.

ii) $\tau^*(Q_1, Q_2) = \tau^*(Q_1) \tau^*(Q_2)$ for all $Q_1, Q_2 \in \Pi_m$, and $\tau^*(1) = 1$.

iii) $\tau^*(Q_1/Q_2) = \tau^*(Q_1) / \tau^*(Q_2)$ for all $Q_1, Q_2 \in \Pi_m$, and $\tau^*(1) = 1$.

**Proof.** While i), ii) are straightforward to prove iii) follows by ii) using the identity $\tau^*(Q_1) = \tau^*(1 \cdot Q_1) = \tau^*((Q_2/Q_2)Q_1) = \tau^*(Q_2)\tau^*(Q_1/Q_2)$. \hfill $\square$

\footnote{While algorithms of lower complexity exist, such as the Strassen algorithms or the Coppersmith-Winograd algorithm, their break-even points are reached only for problems so large that memory becomes the limiting factor.}
Definition 2.3. If $\Pi \subseteq \Pi_m$, $m \in \mathbb{N}$, is a polynomial subspace, then we call $\tau : \mathbb{R}^m \rightarrow \mathbb{R}^m$ a canonical transformation with respect to $\Pi$ if and only if $\tau$ is an affine transformation such that the induced transformation $\tau^* : \Pi \rightarrow \Pi_m$ maps onto $\Pi$, i.e., $\tau^*(\Pi) \subseteq \Pi$.

Remark 2.4. Note that any affine translation and any affine transformation with $B$ being a diagonal matrix is a canonical transformation with respect to $\Pi_{m,n,p}$, $m, n \in \mathbb{N}$, $1 \leq p \in \mathbb{R} \cup \{\infty\}$.

Using these definitions, we formalize the concept of unisolvent nodes as:

Definition 2.5 (Unisolvent nodes). Let $m \in \mathbb{N}$ and $\Pi \subseteq \Pi_m$ be a polynomial subspace. We call a finite non-empty set $\emptyset \neq P \subseteq \mathbb{R}^m$ unisolvent with respect to $\Pi$ if and only if there exists no non-zero polynomial $Q \in \Pi \setminus \{0\}$ with $Q(p) = 0$, $\forall p \in P$.

Let further $H \subseteq \mathbb{R}^m$ be a hyperplane defined by a linear polynomial $Q_H \in \Pi_{m,1,1} \setminus \{0\}$, i.e., $H = Q_H^{-1}(0)$ such that any affine transformation $\tau_H : \mathbb{R}^m \rightarrow \mathbb{R}^m$ with $\tau_H(H) = \mathbb{R}^{m-1} \times \{0\}$ is canonical with respect to $\Pi$. We consider

\begin{align*}
\Pi|_H &= \{Q \in \Pi \mid \tau_H^*(Q) \in \Pi \cap (\Pi_{m-1} \times \{0\})\} \\
\Pi^*_H &= \{Q \in \Pi_m \mid Q_H Q \in \Pi\}.
\end{align*}

We call $P$ unisolvent with respect to $(\Pi, H)$ if and only if

i) There is no polynomial $Q \in \Pi|_H$ with $\tau^*_H(Q) \neq 0$ and $Q(P \cap H) = 0$.

ii) There is no polynomial $Q \in \Pi^*_H \setminus \{0\}$ with $Q(P \setminus H) = 0$.

Theorem 2.6. Let $m \in \mathbb{N}$, $\Pi \subseteq \Pi_m$ a polynomial subspace, $P \subseteq \mathbb{R}^m$ a finite set, and $H = Q_H^{-1}(0)$ be a hyperplane of co-dimension 1 defined by a polynomial $Q_H \in \Pi_{m,1,1} \setminus \{0\}$ such that:

i) The affine transformation $\tau_H : \mathbb{R}^m \rightarrow \mathbb{R}^m$ with $\tau_H(H) = \mathbb{R}^{m-1} \times \{0\}$ induces a canonical transformation $\tau^*_H : \Pi \rightarrow \Pi$.

ii) $P$ is unisolvent with respect to $(\Pi, H)$.
Then $P$ is unisolvent with respect to $\Pi$.

Proof. Let $Q \in \Pi$ with $Q(P) = 0$. We consider the affine transformation $\tau_H : \mathbb{R}^m \rightarrow \mathbb{R}^m$ with $\tau_H(H) = \mathbb{R}^{m-1} \times \{0\}$ and the projection $\pi_{m-1} : \Pi_m \rightarrow \Pi_{m-1} \times \{0\}$. We consider

$$Q_1 = \tau_H^{-1}\pi_{m-1}\tau_H(Q) \in \Pi_H, \quad \text{and} \quad Q_2 = (Q - Q_1)/Q_H.$$

**Step 1:** We show that $Q_2 \in \Pi_H^\perp$. Certainly, $Q_2$ is well defined on $\mathbb{R}^m \setminus H$. Furthermore, we note that $\tau_H(Q(P)) = \lambda x_m$, $\lambda \in \mathbb{R} \setminus \{0\}$. W.l.o.g. we assume $\lambda = 1$ and use Lemma 2.2(ii) to reformulate Eq. (2) as

$$Q_2 = \tau_H^{-1}(\tau_H(Q) - \pi_{m-1}\tau_H(Q))/\pi_{m-1}(\tau_H(Q_H))$$

$$= \tau_H^{-1}(\tau_H(Q) - \pi_{m-1}\tau_H(Q))/x_m.$$ 

Since $Q_0 := \tau_H(Q) - \pi_{m-1}\tau_H(Q)$ is a polynomial consisting of monomials all sharing the variable $x_m$, the quotient $(\tau_H(Q) - \pi_{m-1}\tau_H(Q))/x_m \in \Pi$ and therefore $Q_2 \in \Pi$. Further, by Lemma 2.2(ii) we obtain

$$Q_HQ_2 = \tau_H^{-1}(x_m)\tau_H^{-1}(Q_0/x_m) = \tau_H^{-1}(Q_0) \in \Pi.$$ 

Hence, $Q_2 \in \Pi_H^\perp$ as claimed.

**Step 2:** We show that $Q = 0$. Indeed, $Q(p) = Q_1(p) = 0$ for all $p \in P \cap H$ implies that $Q_1 = 0$ in regard of assumption i). Consequently, $Q_HQ_2(p) = 0$ for all $p \in P \setminus H$. Since $Q_H(p) \neq 0$ for all $p \in P \setminus H$ we get $\tau_H^{-1}(Q_0)(p) = 0$, $\forall p \in P \setminus H$, which, due to assumption ii) yields that $\tau_H^{-1}(Q_0) = Q_2 = 0$. Thus, $Q = 0$ is the zero polynomial and therefore $P$ is unisolvent with respect to $\Pi$. 

In Fig. 1 we show examples of unisolvent nodes in 2D, generated by recursively applying Theorem 2.6. This illustrates how the notion of unisolvence presented here extends beyond notions resting on specified (sparse) grids.

**Theorem 2.7.** Let the assumptions of Theorem 2.6 be fulfilled and $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a function. Assume that there are polynomials $Q_1 \in \Pi_H, Q_2 \in \Pi_H^\perp$ with $\Pi_{H\perp}, \Pi_{H\parallel}$ from Eq. (2.3), such that:

i) $Q_1(p) = f(p), \forall p \in P \cap H$
ii) \( Q_2(p) = (f(p) - Q_1(p))/Q_H(p), \forall p \in P \setminus H. \)

Then \( Q = Q_1 + Q_HQ_2 \in \Pi \) is the unique polynomial in \( \Pi \) that interpolates \( f \) on \( P \), i.e., \( Q(p) = f(p), \forall p \in P \).

**Proof.** Indeed, \( Q_H \neq 0 \) on \( \mathbb{R}^m \setminus H \) implies that \( Q(p) = f(p), \forall p \in P \). Thus, \( Q \) interpolates \( f \) on \( P \). To show the uniqueness of \( Q \) let \( Q' \in \Pi \) interpolate \( f \) on \( P \). Then \( Q - Q' \in \Pi \) and \( (Q - Q')(p) = 0, \forall p \in P \). Due to Theorem 2.6 we have that \( P \) is unisolvent with respect to \( \Pi \). Thus, \( Q' - Q \) has to be the zero polynomial, proving that \( Q \) is uniquely determined in \( \Pi \).

By recursion, Theorem 2.7 yields a general divided difference scheme for polynomial spaces \( \Pi_A \) that are not induced by downward closed sets \( A \subseteq \mathbb{N}^m \). The recursion is illustrated in Fig. 2 and rephrases our earlier results [51]:

**Corollary 2.8** (Generalized Divided Differences). Let \( \Pi \subseteq \Pi_m, m \in \mathbb{N}, \) be a finite-dimensional polynomial space with \( \dim \Pi = N \in \mathbb{N} \). Assume that there are unisolvent nodes \( P \),

\[
P = (P_0 \cup P_1) = (P_{0,0} \cup P_{0,1}) \cup (P_{1,0} \cup P_{1,1}) = \cdots = \bigcup_{\alpha \in A} P_\alpha, \ A \subseteq \mathbb{N}^m,
\]
generated by recursively applying Theorem 2.6 with respect to hyperplanes

\[
H_{0,1} \subseteq \mathbb{R}^m, \ H_{0,0,0,1} \subseteq \mathbb{R}^{m-1}, \ldots, H_{\alpha,\beta} \subseteq \mathbb{R}^2, \alpha, \beta \in A,
\]
such that \( P_1 \cap H_{0,1} = P_{0,1} \cap H_{0,0,0,1} = P_{1,1} \cap H_{1,0,1,1} = \cdots = P_\alpha \cap H_{\alpha,\beta} = \emptyset \) as illustrated in Fig. 2 (left). Then the unique interpolant \( Q_f \in \Pi \) of any function \( f: \mathbb{R}^m \rightarrow \mathbb{R} \) can be computed in \( \mathcal{O}(N^2) \).

**Proof.** The proof follows by using Theorem 2.7 as an induction argument on \( N \), observing that polynomials \( Q \in \Pi \) can be evaluated in \( \mathcal{O}(N^2) \) and classic 1D Newton interpolation requires \( \mathcal{O}(n^2) \) runtime [65].

While Corollary 2.8 proves the existence of a quadratic-runtime interpolation algorithm even in the general case of irregular unisolvent nodes (Fig. 1 left), the next statement allows us to derive a more suitable approach for implementing such an algorithm in practice:

**Corollary 2.9.** Let \( m \in \mathbb{N}, A \subseteq \mathbb{N}^m \) be a complete set of multi-indices, and \( \Pi_A \subseteq \Pi_m \) by the polynomial sub-space induced by \( A \). We consider the generating nodes given by the grid

\[
(2.4) \quad GP = \bigoplus_{i=1}^m P_i, \quad P_i = \{p_{0,i}, \ldots, p_{n_i,i}\} \subseteq \mathbb{R}, \ n_i = \max_{\alpha \in A}(\alpha_i),
\]

where the \( P_i \) are arbitrary finite sets. Then, the node set

\[
P_A = \{(p_{\alpha,1}, \ldots, p_{\alpha,m}) \mid \alpha \in A\}
\]
is unisolvent with respect to \( \Pi_A \).

**Proof.** We argue by induction on \( m \) and \( |A| \). If \( m = 1 \) then the claim follows from the fact that \( \dim \Pi_A = |A| \) and no polynomial \( Q \in \Pi_A \) can vanish on \( |A| \) distinct nodes \( P_A = \{p_{\alpha(1,1)} \mid \alpha \in A\} \). The claim becomes trivial for \( |A| = 1 \). Now assume that \( m > 1 \) and \( |A| > 1 \). We consider \( A_1 = \{\alpha \in A \mid \alpha_m = 0\}, A_2 = A \setminus A_1 \). By decreasing \( m \) if necessary and w.l.o.g., we can assume that \( A_2 \neq \emptyset \). Consider the hyperplane \( H = \{(x_1, \ldots, x_{m-1}, p_{0,m}) \mid (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}\} \) and \( Q_H \in \Pi_{m,1,1} \)
with $Q_H(x) = x_m - p_{0,m}$. By induction we have that $P_A$ is unisolvent with respect to $(\Pi_A, H)$. Thus, we finish the proof by Theorem 2.6 and induction.

**Definition 2.10** (Essential assumptions). We say that the essential assumptions hold with respect to $A \subseteq \mathbb{N}^m$ and $P_A \subseteq \mathbb{R}^m$, where $m \in \mathbb{N}$ and $A$ is a complete (i.e., downward closed) set of multi-indices, if and only if there exist generating nodes (2.5) $GP = \oplus_{i=1}^m P_i$, \quad $P_i = \{p_{0,i}, \ldots, p_{n_i,i}\} \subseteq \mathbb{R}$, \quad $n_i = \max_{\alpha \in A}(\alpha_i)$, and the unisolvent nodes $P_A$ are given by

$$P_A = \{(p_{\alpha_1,1}, \ldots, p_{\alpha_m,m}) \mid \alpha \in A\}.$$ 

Unless further specified, the generating nodes GP are arbitrary.

**Remark 2.11.** It is important to note that the nodes $P_A$ are not sampled from a grid, but generate a sub-grid. Consequently, even though the index sets $A$ are assumed to be downward closed, the flexibility in ordering the $P_i$ results in unisolvent nodes $P_A$ that may induce non-tensorial or non-symmetric grids. This can be seen in Fig. 1 (right), Fig. 3 (right), and Fig. 4 where there are nodes with $p = (p_x, p_y) \in P_A$, but $(p_y, p_x) \notin P_A$. In Fig. 3 (left, middle) examples of a tensorial and symmetric grids are shown.

The nodes shown in Fig. 1 result in high approximation power of the interpolation, as proven in Section 7. They are generated by choosing $GP = \oplus_{i=1}^m (-1)^i \text{Cheb}_n^0$, where the Chebyshev extremes $\text{Cheb}_n^0$ defined in Eq. (7.1) are Leja ordered [61]. Since these $P_A$ form a non-tensorial grid, previous interpolation approaches [33, 75] cannot be used. We therefore establish efficient and numerically robust interpolation schemes for such non-tensorial unisolvent nodes in the next section.

3. **Multivariate Newton Interpolation**

We use the above generalized concept of unisolvence on non-tensorial nodes to provide a natural extension of the classic Newton interpolation scheme to arbitrary
dimensions. The extension presented here relies on recursively applying Theorem 2.7 and Corollary 2.9. We start by defining:

**Definition 3.1** (Multivariate Newton polynomials). Let the essential assumptions (Definition 2.10) be fulfilled with respect to $A \subseteq \mathbb{N}^m$ and $P_A \subseteq \mathbb{R}^m$. Then, we define the multivariate Newton polynomials by

$$N_\alpha(x) = \prod_{i=1}^{m} \prod_{j=0}^{\alpha_i-1} (x_i - p_{j,i}), \quad \alpha \in A.$$  

Indeed, in dimension $m = 1$ this reduces to the classic definition of Newton polynomials [45, 80, 84].

**Definition 3.2** (Multivariate divided differences). Let the essential assumptions (Definition 2.10) be fulfilled with respect to $A \subseteq \mathbb{N}^m$ and $P_A \subseteq \mathbb{R}^m$. Further let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a function. Then, we recursively define the multivariate divided differences:

$$p_{\alpha,0,j} = p_\alpha, \quad p_{\alpha,i,j} = p_\beta, \quad \beta_j = i, \beta_k = \alpha_k, \forall k \neq j,$$

$$F_{\alpha,0,m} = f(p_\alpha), \quad F_{\alpha,0,j} = F_{\alpha,\alpha_{j+1},j+1}, \quad \text{for } 1 \leq j < m,$$

and

$$F_{\alpha,i,j} = \frac{F_{\alpha,i-1,j}(p_\alpha) - F_{\alpha,i-1,j}(p_{\alpha_{i-1},j})}{(p_\alpha - p_{\alpha_{i-1},j})}. \quad \text{for } i \leq \alpha_j.$$

We call $F_{\alpha,0,0} = F_{\alpha,\alpha_{1,1}}$ the Newton coefficients of $Q_{f,A} \in \Pi_A$.

In dimension $m = 1$, this definition recovers the classic divided difference scheme for 1D Newton Interpolation [45, 80, 84].

Using these definitions we state the main result of this section, generalizing Newton interpolation to mD:

**Theorem 3.3.** Let the essential assumptions (Definition 2.10) be fulfilled with respect to $A \subseteq \mathbb{N}^m$ and $P_A \subseteq \mathbb{R}^m$, and let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a function. Then, the
unique polynomial \( Q_{f,A} \in \Pi_A \) interpolating \( f \) on \( P_A \), i.e., \( Q_f(p) = f(p), \forall p \in P_A \), can be determined in \( \mathcal{O}(|A|^2) \) operations requiring \( \mathcal{O}(|A|) \) storage and is given by

\[
Q_{f,A}(x) = \sum_{\alpha \in A} c_{\alpha} N_{\alpha}(x),
\]

where \( c_{\alpha} = F_{\alpha,0,0} \) are the Newton coefficients of \( Q_{f,A} \in \Pi_A \).

Earlier versions of this statement were limited to the case where \( P_A \) is given by a (sparse) tensorial grid \([33]\). In contrast, the present Theorem 3.3 also holds for our generalized notion of non-tensorial unisolvent nodes. In addition, Eq. 3.2 enables a practical implementation of a recursive algorithm. We prove Theorem 3.3

**Proof.** We argue by induction on \( |A| \). If \( |A| = 1 \) then the claim follows immediately. For \( |A| > 1 \) we consider \( A_1 = \{ \alpha \in A \mid \alpha_m = 0 \} \), \( A_2 = A \setminus A_1 \).

By decreasing \( m \) if necessary, and w.l.o.g., we can assume that \( A_2 \neq \emptyset \). Let \( H = \{ (x_1, \ldots, x_{m-1}, p_{0,m}) \mid (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1} \} \) and \( \tau_H : \mathbb{R}^m \rightarrow \mathbb{R}^m \) with \( \tau_H(x) = x_m - p_{0,m} \), such that \( \tau_H(H) = \mathbb{R}^m \times \{ 0 \} \). By assumption, we have that \( \tau \) is a canonical transformation with respect to \( \Pi_A \). Let \( \pi_m-1 : \mathbb{R}^m \rightarrow \mathbb{R}^{m-1} \), \( \pi_m-1(x_1, \ldots, x_m) = (x_1, \ldots, x_{m-1}) \) be the natural projection.

**Step 1:** We reduce the interpolation to \( H \). We set \( P_1 = \pi_m-1(\tau_H(P_A \cap H)) \) and consider \( f_0 : \mathbb{R}^m \rightarrow \mathbb{R} \) with

\[
f_0(x_1, \ldots, x_{m-1}) = f(\tau_H^{-1}(x_1, \ldots, x_{m-1}, 0)) = f(x_1, \ldots, x_{m-1}, p_{0,m}).
\]

Let \( M_{\alpha}(x), \alpha \in A_1 \), be the Newton polynomials with respect to \( A_1, P_1 \). Then induction yields that the coefficients \( d_{\alpha} \in \mathbb{R} \) of the unique polynomial

\[
Q_{f_0,A_1}(x_1, \ldots, x_{m-1}) = \sum_{\alpha \in A_1} d_{\alpha} M_{\alpha}(x_1, \ldots, x_{m-1})
\]

interpolating \( f_0 \) on \( P_1 \) can be determined in less than \( D_0|A_1|^2 \) operations, \( D_0 \in \mathbb{R}^+ \), while requiring a linear amount of storage. Consider the natural embedding \( i_{m-1}^* : \Pi_m \rightarrow \Pi_A^m \).

\[
\tilde{M}_{\alpha}(x_1, \ldots, x_m) = i_{m-1}^*(M_{\alpha})(x_1, \ldots, x_{m-1}) = M_{\alpha}(x_1, \ldots, x_{m-1})
\]

and \( i_{m-1}^*(Q_{f_0,A_1})(x_1, \ldots, x_{m-1}) = Q_{f_0,A_1}(x_1, \ldots, x_{m-1}) \), yields \( \tilde{M}_{\alpha}(x) = N_{\alpha}(x) \), for all \( \alpha \in A_1 \), \( x \in \mathbb{R}^m \). Further, \( Q_1 \in \Pi_A \) is given by

\[
Q_1(x_1, \ldots, x_m) = \sum_{\alpha \in A_1} d_{\alpha} N_{\alpha}(x_1, \ldots, x_m)
\]

and satisfies \( Q_1(p) = f(p) \) for all \( p \in P_A \cap H \).

**Step 2:** We interpolate on \( \mathbb{R}^m \setminus H \). Observe that \( Q_1 \) is constant in direction \( x_m \), i.e., \( Q_1(x_1, \ldots, x_{m-1}, x_m) = Q_{f_0,A_1}(x_1, \ldots, x_{m-1}) \). Thus, by Eq. (3.3),

\[
Q_1(q_1, \ldots, q_{m-1}, q_m) = f(q_1, \ldots, q_{m-1}, p_{0,m}) \text{ for all } (q_1, \ldots, q_{m-1}, q_m) \in P_A.
\]

In light of this fact, and for \( f_1(x) = (f(x) - Q_1(x))/Q_H(x) \), it requires \( D_1|A_2| \), \( D_1 \in \mathbb{R}^+ \), operations to compute

\[
F_{\alpha,1,m} = \frac{f(p_{\alpha,1,m}) - f(p_{\alpha,0,m})}{p_{\alpha} - p_{\alpha,0,m}} = \frac{f(p_{\alpha}) - Q_1(p_{\alpha})}{Q_H(p_{\alpha})} = f_1(p_{\alpha})
\]

for all \( \alpha \in A_2 \).
for all \( p_\alpha \in P_2 = P_A \setminus H \), \( \alpha \in A_2 \). Denote by \( K_\alpha(x) \) the Newton polynomial with respect to \( \Pi_{\alpha} \), \( P_2 \) then induction yields that the coefficients \( e_\alpha \in \mathbb{R} \), \( \alpha \in A_2 \), of the unique polynomial

\[
Q_{f_1, A_2}(x_1, \ldots, x_m) = \sum_{\alpha \in A_2} e_\alpha K_\alpha(x_1, \ldots, x_m).
\]

interpolating \( f_1 \) on \( P_2 \) can be determined in less than \( D_2|A_2|^2 \), \( D_2 \in \mathbb{R}^+ \), operations while requiring linear storage. Due to Eq. (3.1) we observe that \( Q_{H}(x)K_\alpha(x) = N_\alpha(x) \) for all \( \alpha \in A_2 \). By Corollary 2.9 we have that \( P_A \) is unisolvent and therefore Theorem 2.7 implies that the unique polynomial \( Q \in \Pi_A \) interpolating \( f \) on \( P_A \) is given by:

\[
Q_{f,A}(x) = Q_1(x) + Q_{H}(x)Q_2(x) = \sum_{\alpha \in A_1} d_\alpha N_\alpha(x) + \sum_{\alpha \in A_2} e_\alpha N_\alpha(x).
\]

Following Definition 3.2 and using Eq. (3.4), one readily observes that \( d_\alpha = c_\alpha \), \( \forall \alpha \in A_1 \), and that \( e_\alpha = c_\alpha \), \( \forall \alpha \in A_2 \). Thus, we have proven Eq. (3.2). In total, the computation can be done in less than \( D_0|A_1|^2 + D_1|A_2| + D_2|A_2|^2 < \max\{D_0, D_1/2, D_2\}|(A_1| + |A_2|)|^2 \in O(|A|^2) \) operations and \( O(|A_1| + |A_2|) = O(|A|) \) amount of storage.

As an immediate consequence, we deduce that even for non-tensorial grids \( P_A \) the Newton polynomials are a basis of \( \Pi_A \).

**Corollary 3.4 (Newton basis).** Let the essential assumptions (Definition 2.10) be fulfilled with respect to \( A \subseteq \mathbb{N}^m \) and \( P_A \subseteq \mathbb{R}^m \). Then the Newton polynomials

\[
\{N_\alpha\}_{\alpha \in A} \subseteq \Pi_A
\]

are a basis of \( \Pi_A \).

**Proof.** Due Theorem 3.3 every polynomial \( Q \in \Pi_A \) can be uniquely expanded as \( Q = \sum_{\alpha \in A} c_\alpha N_\alpha \), proofing the statement. \( \square \)

**Corollary 3.5 (Evaluation in Newton form).** Let the essential assumptions (Theorem 2.10) be fulfilled with respect to \( A \subseteq \mathbb{N}^m \) and \( P_A \subseteq \mathbb{R}^m \). Further let \( Q(x) = \sum_{\alpha} c_\alpha N_\alpha \), \( c_\alpha \in \mathbb{R} \), be a polynomial in Newton form. Then, it requires \( O(|A|) \) operations to evaluate \( Q \) at some \( x_0 \in \mathbb{R}^m \).

**Proof.** By following the proof of Theorem 3.3 and using an induction argument on the number of coefficients, Eq. (3.5) yields that \( Q_1, Q_2 \) can be evaluated in linear time. Since the evaluation of \( Q_H \) requires constant time, the claim follows. \( \square \)

**Remark 3.6.** Recursively applying the splitting \( Q = Q_1 + Q_H Q_2 \) recovers the classic Aitken-Neville evaluation algorithm [11, 70, 69] for dimension \( m = 1 \). Evaluation is efficiently done by applying a multivariate version of the Horner scheme, see for instance [45, 80, 60].
4. Multivariate Lagrange Interpolation

Even though the present unisolvent nodes do not have to be tensorial, and therefore the results of [75, 76] do not apply, we can still generalize the concept of Lagrange interpolation to multi-dimensions. For this, we define:

**Definition 4.1** (Lagrange polynomials). Let $m \in \mathbb{N}$, $A \subseteq \mathbb{N}^m$ be a complete set of multi-indices, and $P_A = \{ p_\alpha \}_{\alpha \in A}$ be an unisolvent set of nodes with respect to a polynomial subspace $\Pi_{P_A} \subseteq \Pi_m$. Then, we define the multivariate Lagrange polynomials

$$L_\alpha \in \Pi_{P_A} \quad \text{with} \quad L_\alpha(p_\beta) = \delta_\alpha, \beta, \ \alpha, \beta \in A,$$

where $\delta_{\cdot, \cdot}$ is the Kronecker delta.

For $A = A_{m,n,\infty}$ and the grid $P_A$ becomes tensorial and the above definition recovers the known notion of tensorial $mD$ Lagrange interpolation [41, 43], where

$$L_\alpha(x) = \prod_{i=1}^m l_{\alpha_i}(x), \quad l_{\alpha_i}(x) = \prod_{j=0, j \neq \alpha_i}^n \frac{x_i - p_{j,i}}{p_{\alpha_i,j} - p_{j,i}}, \quad \alpha \in A.$$

The following theorem then generalizes the classic facts known for 1D Lagrange interpolation [7, 84] and for tensorial $mD$ Lagrange interpolation [7, 84] to the non-tensorial multi-dimensional case:

**Corollary 4.2** (Lagrange basis). Let the assumptions of Definition 4.1 be fulfilled. Then:

i) The Lagrange polynomials $L_\alpha \in \Pi_A$ are a basis of $\Pi_A$.

ii) The polynomial $Q_{f,A}(x) = \sum_{\alpha \in A} f(p_\alpha) L_\alpha(x) \in \Pi_A$ is the unique polynomial interpolating $f$ on $P_A$ and can be determined in $O(|A|)$ operations.

**Proof.** To show i), observe that there are $|A|$ Lagrange polynomials. Due to Corollary 3.4, we deduce $\dim \Pi_A = |A|$. Given $c_\alpha \in \mathbb{R}, \ \alpha \in A$ such that $\sum_{\alpha \in A} c_\alpha L_\alpha = 0$, the unisolvence of $P_A$ implies that the polynomial $Q(x) = \sum_{\alpha \in A} c_\alpha L_\alpha$ vanishes on $P_A$ and, therefore, has to be the zero polynomial. Hence, $c_\alpha = 0$ for all $\alpha \in A$, implying that the $L_\alpha \in \Pi_A$ are linear independent and thus yield a basis of $\Pi_A$. The claimed uniqueness in ii) follows from i), and the remaining statement holds for trivial reasons.

**Remark 4.3.** The complexity of the interpolation depends on the choice of $A$. As introduced, the cases $A_{m,n,1}$, $A_{m,n,2}$, and $A_{m,n,\infty}$ are of special interest. While

$$|A_{m,n,1}| = \binom{m+n}{n} = \binom{m+n}{m} \in O(m^n) \cap O(n^m), \quad |A_{m,n,\infty}| = (n+1)^m,$$

explicit such formulas for $|A_{m,n,p}|$, $1 < p < \infty$, are unknown. An approximation for $p = 2$ is given in [83] as

$$|A_{m,n,2}| \approx (n+1)^m \text{vol}(B_{l_2}^m) = (n+1)^m \frac{(\pi/4)^{m/2}}{(m/2)!} \approx \frac{(n+1)^m}{\sqrt{\pi m}} \left( \frac{\pi e}{2m} \right)^{m/2},$$

where $B_{l_2}^m$ denotes the $l_2$-ball in dimension $m$. Thus, for fixed degree $n \in \mathbb{N}$ Lagrange interpolation is of polynomial complexity $O(m^n)$ for $p = 1$, of sub-exponential complexity $o(n^m)$ for $p = 2$, and of exponential complexity $O(n^m)$ for $p = \infty$. 
5. Regression on scattered data

We address the question of how to interpolate a function \( f \in C^0(\Omega, \mathbb{R}) \) if the interpolation nodes \( P \) can not be chosen according to Definition 2.10 but are arbitrarily given and fixed. We start by noting that if the locations of the nodes are distributed uniformly at random, then the nodes are unisolvent with probability 1 \(^{52}\). Based on this observation, we state:

**Corollary 5.1** (Regression on given nodes). Let the essential assumptions (Definition 2.10) be fulfilled with respect to \( A \subseteq \mathbb{N}^m \) and \( P_A \subseteq \mathbb{R}^m \), let \( f : \mathbb{R}^m \rightarrow \mathbb{R} \) be a function, and let \( \mathcal{P} \supseteq \mathcal{P}_A \) be any set of nodes containing at least one unisolvent node set \( \mathcal{P}_A = \{\tilde{p}_\alpha\}_{\alpha \in A} \subseteq \mathbb{R}^m \) with respect to \( \Pi_A \). Denote by \( \mathcal{F} = (f(\tilde{p}))_{\tilde{p} \in \mathcal{P}} \in \mathbb{R}^{|\mathcal{P}|} \). Then the following is true:

i) The Lagrange coefficients \( C_{\text{Lag}} = (c_{\alpha_{\min}}, \ldots, c_{\alpha_{\max}}) \) of \( Q_{f,A} \) are uniquely determined by solving

\[
R_A C_{\text{Lag}} = \mathcal{F}, \text{ where } R_A = (r_{i,\alpha}) \in \mathbb{R}^{\mathcal{P}|A|} \quad \text{with} \quad r_{i,\alpha} = L_{\alpha}(\tilde{p}_i) , \tilde{p}_i \in \mathcal{P}
\]

using standard least-squares regression: \( C_{\text{Lag}} = \arg\min_{x \in \mathbb{R}^{|A|} \mid x} ||R_A x - \mathcal{F}||_2^2 \).

ii) If \( f \in \Pi_A \) is a polynomial, then \( Q_{f,A}(p) = f(p) \) for all \( p \in \mathcal{P}_A \).

**Proof.** Since \( \mathcal{P}_A \) is assumed to be unisolvent, the matrix \( R_A \) is of full rank. Thus, i) follows from the existence of a unique solution for the convex least-squares optimization problem. Then ii) follows by the unisolventness and by the fact that \( Q_{f,A}(p) = f(p) \) for all \( p \in \mathcal{P}_A \). \( \Box \)

We validate the practical feasibility of this regression method in Section 8.2. In particular, we empirically observe the matrix \( R_A \) to be well conditioned even for complex regression problems, reflecting the proper choice of unisolvent nodes.

6. The dual notion of unisolvent

Rather than constructing unisolvent nodes de novo, Carl de Boor and Amon Ros [28, 29] considered the dual problem of unisolvent: for a given polynomial space \( \Pi \) and a set of nodes \( P \subseteq \mathbb{R}^m \) that is not unisolvent with respect to \( \Pi \), find a maximum subset \( P_0 \subseteq P \) and a polynomial subspace \( \Pi_{P_0} \subseteq \Pi \) such that \( P_0 \) is unisolvent with respect to \( \Pi_{P_0} \) [28, 29]. We suggest to rephrase the problem in more abstract mathematical terms by considering the map

\[
\Gamma_k : \mathcal{P}_k \rightarrow \text{Gr}(k, X), \quad X = \Pi_{m,k,\infty},
\]

where \( \mathcal{P}_k = \{P \subseteq \mathbb{R}^m \mid |P| = k\} \) is the set of all finite subsets of \( \mathbb{R}^m \) of cardinality \( k \), and \( \text{Gr}(k, X) \) is the Grassmann manifold, i.e., the smooth manifold that consists of all \( k \)-dimensional subspaces of the vector space \( X \) [68]. In particular, \( \text{Gr}(1, \mathbb{R}^m) = \mathbb{R}P_m^{m-1} \) and \( \text{Gr}(1, \mathbb{C}^m) = \mathbb{C}P_m^{m-1} \) are the real and complex projective spaces, respectively [51, 50].

We start by stating:

**Theorem 6.1.** Let \( m, k \in \mathbb{N}, \mathcal{P}_k = \{P \subseteq \mathbb{R}^m \mid |P| = k\} \) be the set of all subsets of \( \mathbb{R}^m \) with cardinality \( k \), and \( X = \Pi_{m,k,\infty} \) be the space of all polynomials with \( \ell_\infty \)-degree at most \( k \). Then, there is one and only one polynomial subspace \( \Pi_P \subseteq X \) such that \( P \) is unisolvent with respect to \( \Pi_P \). In particular, the map

\[
\Gamma_k : \mathcal{P}_k \rightarrow \text{Gr}(k, X), \quad \Gamma_k(P) = \Pi_P,
\]
with $\text{Gr}(k, X)$ denoting the Grassmann manifold \[65\], is well defined and smooth.

An extended and detailed version of this statement is meanwhile given in our contribution \[86\].

**Remark 6.2.** Note that for $m = 1$ we have $\Pi_{m,k,\infty} = \Pi_{1,k,1}$ and $\text{Gr}(k, X) = X = \Pi_{1,n,1}$. Since $n + 1$ nodes are unisolvent in dimension 1, Theorem 6.1 becomes trivial, and $\Gamma_k(P) = X$ is constant in that case.

**Proof.** Let $P_{A_{m,k,\infty}}$ be a set of unisolvent nodes with respect to $X = \Pi_{m,k,\infty}$, generated according to the essential assumptions in Definition 2.10. Denote by $L_\alpha \in X$, $\alpha \in A_{m,k,\infty}$ the corresponding Lagrange polynomials. Fix an ordering $P = \{p_0, \ldots, p_k\}$ and consider the matrix $R = (r_{i,\alpha}) \in \mathbb{R}^{k \times K}$, $K = |A_{m,k,\infty}| = (k + 1)^m$, defined by

$$r_{i,\alpha} = L_\alpha(p_i), \quad i = 1, \ldots, k, \quad \alpha_{\min} \leq \alpha \leq \alpha_{\max}, \quad \alpha \in A_{m,k,\infty}.$$

Let $\mu = \text{rank}(R)$ be the rank of $R$ and $D = \text{diag}(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^{k \times k}$ the diagonal matrix with the first $\mu$ entries equal to 1 and all others equal to 0. Let further $C \in \mathbb{R}^{K \times k}$ be a solution of $RC = D$ and $b_i = (c_{\alpha_{\min}, i}, \ldots, c_{\alpha_{\max}, i})$ be the rows of $C$. Then

$$B_i(x) = \sum_{\alpha \in A_{m,k,\infty}} c_{\alpha,i}L_\alpha(x) \in X, \quad i = 1, \ldots, k$$

is the maximal set of linearly independent polynomials with $B_i(p_j) = \delta_{i,j}, 1 \leq j \leq \mu$, where $\delta_{i,j}$ denotes the Kronecker delta. The $B_i$ are the interpolants of the delta-distribution-functions $d_i : \mathbb{R}^m \to \mathbb{R}$, $d_i(x) = \delta_{x,p_i}$. Since $P_{A_{m,k,\infty}}$ is unisolvent, the $B_i$ are uniquely determined and linearly independent. Hence, $\mu = k$, and the $B_i$ are a basis of the uniquely determined polynomial subspace $\Pi_P = \text{span}(B_i)_{i=1,\ldots,k}$ for which $P$ becomes unisolvent. Since $\text{Gr}(k, X)$ is a smooth manifold \[68\], and the $B_i$ depend smoothly on $P$, this shows that $\Gamma_k$ is a well-defined smooth map.

\[\square\]

**Remark 6.3.** We revisit the work of Carl de Boor and Amon Ros \[28\] \[29\] by considering the subspace

$$\tilde{Y}(P) = \Gamma_k(P) \cap \Pi.$$

The set $P_0 \subseteq P$ and a basis of $\tilde{Y}(P)$ can be derived using Gaussian elimination \[85\] on the associated Vandermonde matrix $V(P)$. We note that properties of the map $P \mapsto \tilde{Y}(P)$ discussed in \[28\] \[29\], such as continuity and differentiability, can be deduced by considering $\Gamma_k$ from Eq. \(6.1\).

However, the fact that $\text{dim}(X) = (k + 1)^m$ implies that Theorem 6.1 is of mostly theoretical interest. Only if $\Gamma_k(P) \approx Y$ is known to be located near a relatively low-dimensional subspace $Y \subseteq X$, e.g. $Y = \Pi_{m,n,2}$ with small $m, n$, then Theorem 6.1 might be of practical relevance, as the following consequence states:

**Theorem 6.4.** Let the essential assumptions (Definition 2.10) be fulfilled with respect to $A \subseteq \mathbb{N}^m$ and $P_A \subseteq \mathbb{R}^m$. Let further $P \subseteq \mathbb{R}^m$ be any set of nodes that is not unisolvent with respect to $\Pi_A$, $\Gamma_k$ be as in Theorem 6.1, and $f : \mathbb{R}^m \to \mathbb{R}$ be a function. Then:

i) There is a set $P_0 \subseteq P$ of maximal cardinality $k = |P_0|$ that can be determined in $O(|A|^3)$ operations such that $\Gamma_k(P_0) \subseteq \Pi_A$. 

Theorem 2. The Lagrange coefficients \( c_e \in \mathbb{R}, e \in E \), are uniquely determined up to adding a polynomial \( q \in \Pi_A \) with \( q(p) = 0 \) for all \( p \in P \), i.e., \( Q_{P_0, f} + q \) is a valid interpolant.

Proof. All statements follow from the following observation: We order the nodes \( P = \{p_1, \ldots, p_l\} \) and a basis \( \{\mu_1, \ldots, \mu_{|A| - k}\} \) of the quotient space \( \Pi_A / \Gamma_k(P_0) \) can be computed in \( \mathcal{O}(|A|^3) \) operations.

ii) A basis \( \{\rho_1, \ldots, \rho_k\} \) of \( \Gamma_k(P_0) \) and a basis \( \{\mu_1, \ldots, \mu_{|A| - k}\} \) of the quotient space \( \Pi_A / \Gamma_k(P_0) \) can be computed in \( \mathcal{O}(|A|^3) \) operations.

iii) The Lagrange coefficients \( c_e \in \mathbb{R}, e \in E \), are uniquely determined up to adding a polynomial \( q \in \Pi_A \) with \( q(p) = 0 \) for all \( p \in P \), i.e., \( Q_{P_0, f} + q \) is a valid interpolant.

iv) The interpolant \( Q_{P_0, f} \) is uniquely determined up to adding a polynomial \( q \in \Pi_A \) with \( q(p) = 0 \) for all \( p \in P \), i.e., \( Q_{P_0, f} + q \) is a valid interpolant.

v) If \( \Gamma(P_0) \neq \Pi_A \), then the Lagrange coefficients \( d_\alpha \in \mathbb{R} \) of a polynomial \( \alpha \in \Pi_A \) satisfying \( Q_\alpha(p) = 0 \) for all \( p \in P \) can be computed in \( \mathcal{O}(|A|^3) \) operations.
One of the practical applications of Theorem 6.4 can be stated as follows:

**Corollary 6.5.** Let the essential assumptions (Definition 2.10) be fulfilled with respect to \( A \subseteq \mathbb{N}^m \) and \( P_A \subseteq \mathbb{R}^m \). Let further \( Q_M \in \Pi_A \) be a polynomial and \( M = Q_M^{-1}(0) \) be the affine algebraic variety given by the zero-level set of \( Q_M \).

Denote by

\[
(6.4) \quad \Pi_M = \{Q | Q \in \Pi_A \} \subseteq \Pi_m
\]

the polynomial subspace of all restrictions \( Q_M \) of polynomials \( Q \in \Pi_A \) to \( M \).

i) Assume that there is a finite set \( P_0 \subseteq M \) such that \( \Gamma_k(P_0) \subseteq \Pi_A \), with \( k = |P_0| = |A| - 1 \). Then \( \Gamma_k(P_0) \cong \Pi_M \).

ii) If \( P_0 \) is as in i), then by replacing \( \Gamma_k(P_0) \) with \( \Pi_M \) the statements ii)–v) of Theorem 6.4 apply. In particular, \( Q_M \) can be determined up to a constant factor.

**Proof.** Observe that \( \Pi_M \) can be understood as the quotient of \( \Pi_A \) by all polynomials \( m_M := \{Q \in \Pi_A \colon Q | M \equiv 0\} \) vanishing on \( M \), i.e.,

\[
\Pi_M \cong \Pi_A/m_M, \quad \dim(\Pi_M) = \dim(\Pi_A) - \dim(m_M) = l \in \mathbb{N}.
\]

Since \( Q_M \in \Pi_A \) we have that \( Q_M \in m_M \) and thereby \( \dim(\Pi_M) \leq |A| - 1 \). On the other hand, for \( P_0 \) as in i) we have \( \Gamma_k(P_0) \subseteq \Pi_M \) with \( \dim(\Gamma_k(P_0)) = |A| - 1 \). Thus, i) follows. Statement ii) is then obvious. Indeed, \( m_M = \text{span}(Q_M) \cong \Pi_A/\Gamma_k(P_0) = \text{span}(\mu) \) yields that \( Q_M = c\mu \), \( c \in \mathbb{R} \setminus \{0\} \) with \( \mu \) as in Theorem 6.4 ii). \( \square \)

**Remark 6.6.** Since unisolvence is a generic property \([52, 79]\), and due to Theorem 6.4), randomly sampling \( |A| \) nodes on \( M \) yields a set \( P_0 \) as required in i) with probability 1. Consequently, any restriction \( Q_M \in \Pi_M \) of a polynomial \( Q \in \Pi_A \) to \( M \) can be interpolated as in Theorem 6.4 ii). We demonstrate the numerical stability of such an approach in Section 8.2. But before that, we discuss the approximation power of polynomial interpolation in \( mD \).

### 7. Approximation Theory

We address the fundamental question of how well polynomial interpolation can approximate continuous functions in \( mD \). We derive several statements that enable control over the approximation error

\[
\| f - Q_f \|_{C^0(\Omega)}.
\]

We denote by \( \partial^\alpha f(x) = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_m}^{\alpha_m} f(x) \) the partial derivative of \( f \) with respect to the multi-index \( \alpha \), evaluated at point \( x \in \Omega \). Further, we denote by \( C^0(\Omega, \mathbb{R}) \) the \( \mathbb{R} \)-vector space of continuous functions on \( \Omega \) with norm \( \| f \|_{C^0(\Omega)} = \sup_{x \in \Omega} |f(x)| \).

For a set of multi-indices \( A \subseteq \mathbb{N}^m \), \( m \in \mathbb{N} \), we consider

\[
C_A(\Omega, \mathbb{R}) = \{ f \in C^0(\Omega, \mathbb{R}) \mid \partial^\alpha f \in C^0(\Omega, \mathbb{R}), \forall \alpha \in A \},
\]

\[
\| f \|_{C_A(\Omega)} = \sum_{\alpha \in A} \| \partial^\alpha f \|_{C^0(\Omega)}.
\]

For \( A = A_{m,n,1} \) the normed vector space \( (C_A(\Omega, \mathbb{R}), \| \cdot \|_{C_A(\Omega)}) \) coincides with the classic definition of the Banach space \( (C^0(\Omega, \mathbb{R}), \| \cdot \|_{C^0(\Omega)}) \). In light of this fact, one can easily deduce that \( (C_A(\Omega, \mathbb{R}), \| \cdot \|_{C_A(\Omega)}) \) is a Banach space for all downward closed sets \( A \subseteq \mathbb{N}^m \).
The central challenge we address here is to free interpolation from Runge’s phenomenon, as discussed in Section 1.4.3. Therefore, we introduce the Chebyshev nodes of first kind and the Chebyshev extreme nodes

\[
\text{Cheb}_{\text{1st}}^n = \left\{ \cos \left( \frac{2k-1}{2(n+1)} \pi \right) \mid 1 \leq k \leq n+1 \right\},
\]

\[
\text{Cheb}^0_n = \left\{ \cos \left( \frac{k\pi}{n} \right) \mid 0 \leq k \leq n \right\}.
\]

The nodes \(\text{Cheb}_{\text{1st}}^n\) are minimizers of the product \(M_{P_n}(x) = \prod_{p \in P_n} |x - p|, |P_n| = n+1\), i.e.,

\[
\min_{P_n \subseteq \Omega} \| M_{P_n} \|_{C^0(\Omega)} = \frac{1}{2^n} \quad \text{for} \quad P_n = \text{Cheb}_{\text{1st}}^n.
\]

The nodes \(\text{Cheb}^0_n\) are the extrema of \(M_{\text{Cheb}^\text{1st}_{n+1}}\), with values oscillating between \(M_{\text{Cheb}^\text{1st}_{n+1}}(q) \in \{-1, 1\}\) for \(q \in \text{Cheb}^0_n\).

### 7.1. Combining numerical accuracy with approximation ability.

The following technical detail is of central importance for the remainder of this section:

**Definition 7.1** (2nd essential assumption). We say that the 2nd essential assumption is fulfilled if and only if in addition to the essential assumption from Definition 2.10 the generating nodes

\[
GP = \oplus_{i=1}^m P_i, \quad P_i = \{p_{0,i}, \ldots, p_{n_i,i}\} \subseteq \mathbb{R}, \quad n_i = \max_{\alpha \in A} (\alpha_i),
\]

are given by *Leja-ordered* Chebyshev extreme nodes, i.e.,

\[P_i = \{p_0, \ldots, p_n\} = \pm\text{Cheb}^\text{1st}_n, \pm\text{Cheb}^0_n\]

and the following holds:

\[
|p_0| = \max_{p \in P} |p|, \quad \prod_{i=0}^{j-1} |p_j - p_i| = \max_{1 \leq k \leq m} \prod_{i=0}^{j-1} |p_k - p_i|, \quad 1 \leq j \leq n.
\]

Leja ordering is known to minimize numerical rounding errors in 1D Newton interpolation [10]. In particular, \(\{p_0, p_1\} = \{-1, 1\}\) holds for \(\text{Cheb}^0_n, n \geq 1\). Consequently, as we demonstrate in Section 8, this allows approximating highly varying functions, such as the Runge function, to machine precision.

**Definition 7.2** (Lebesgue constant). Let the assumptions of Definition 4.1 be fulfilled and \(f \in C^0(\Omega, \mathbb{R})\) with \(Q_{f,A}(x) = \sum_{\alpha \in A} f(p_\alpha) L_\alpha(x)\) the interpolant of \(f\) in Lagrange form. Then we define the *Lebesgue constant* as the operator norm of the interpolation operator \(I_{P_A} : C^0(\Omega, \mathbb{R}) \rightarrow C^0(\Omega, \mathbb{R}), f \mapsto Q_{f,A}\) with respect to \(P_A\), i.e.,

\[
\Lambda(P_A) := \| I_{P_A} \| = \sup_{f \in C^0(\Omega, \mathbb{R}), \|f\|_{C^0(\Omega)} \leq 1} \| Q_{f,A} \|_{C^0(\Omega)} = \left\| \sum_{\alpha \in A} |L_\alpha| \right\|_{C^0(\Omega)}.
\]

Since \(P_A\) is unisolvent, \(I_{P_A}\) is linear and exact on \(\Pi_A\), i.e, \(I_A(Q) = Q\) for all \(Q \in \Pi_A\). The Lebesgue constant provides a relative measure of the approximation...
quality of $P_A$ in the following sense: Let $Q^*_f \in P_A$ be an optimal polynomial approximation of $f$, then:

$$
\|f - Q_{f,A}\|_{C^0(\Omega)} \leq \|f - Q^*_f\|_{C^0(\Omega)} + \|Q_{f,A} - Q^*_f\|_{C^0(\Omega)} \\
\leq \|f - Q^*_f\|_{C^0(\Omega)} + \|I_P(f - Q^*_f)\|_{C^0(\Omega)} \\
\leq (1 + \Lambda(P_A))\|f - Q^*_f\|_{C^0(\Omega)}.
$$

(7.5)

In dimension $m = 1$, it is known that for any arbitrary sequence of interpolation nodes $P_n \subseteq \Omega$, $\Lambda(P_n)$ is unbounded. However, by choosing $P_n = \text{Cheb}^1_n$ or $P_n = \text{Cheb}^0_n$, the following estimate applies [12, 13, 35, 55, 56, 64, 73, 72]:

$$
\Lambda(P_n) = \frac{2}{\pi} \left( \log(n) + \gamma + \log(8/\pi) \right) + O(1/n^2),
$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. The estimates of [17] [Eq. (3.19)] show that for $A = A_{m,n,p}$, $p = 1, 2$ and fixed dimension $m \in \mathbb{N}$, we observe algebraic growth

$$
\Lambda(P_A) = O(|A|A(P_n)^m)), \quad |A| = O(n^m).
$$

Due to Eq. (7.5), this implies that all functions $f$ for which the optimal approximation error $\|f - Q^*_n,f\|_{C^0(\Omega)}$ decreases faster than $O(\Lambda(P_A))$ can be approximated by polynomial interpolation in $P_A$.

7.2. Approximation rate in multiple dimensions. In practice, the question of how fast the interpolant $Q_{f,A}$ converges to $f$ is of certain interest. Lloyd N. Trefethen recently used the famous result of Bernstein’s prize-winning memoir from 1914 [6] to derive upper bounds on the convergence rates [83]. He assumed the function $f$ to be analytically over generalized versions of Hooke and Newton ellipses [4]. Numerical experiments suggested that these rates are also lower bounds, with first steps already done to mathematically prove this expectation [9]. Here we revisit these results and adapt them to our problem.

**Definition 7.3.** Let $E_{2, h_z}$ be the Newton ellipse with foci 0 and 1 and leftmost point $-h$. For $m \in \mathbb{N}$ and $h \in [0, 1]$ we call $\rho = h + \sqrt{1 + h^2}$ the Trefethen radius and the open region

$$
N_{m, \rho} = \left\{ (x_1, \ldots, x_m) \in \mathbb{C}^m \mid (x_1^2 + \cdots + x_m^2)/m \in E_{2, h_z} \right\}
$$

the Trefethen domain [83].

We restate Lloyd N. Trefethen’s statement [83] here in adapted form to match our notation. We call a continuous function $f : \Omega \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$ analytical in the Trefethen domain $N_{m, \rho} \subseteq \mathbb{C}^m$ if and only if $f$ can be continuously extended to a function $\overline{f} : \Omega \subseteq \mathbb{C}^m \rightarrow \mathbb{R}$ that possesses an absolutely convergent Taylor series in $N_{m, \rho}$.

**Theorem 7.4** (Lloyd N. Trefethen). Let the 2nd essential assumption be fulfilled with respect to $A_{m,n,p}$. Further, assume that $f \in C^0(\Omega, \mathbb{R})$ is analytical in the Trefethen domain $N_{m, \rho}$. Then

$$
\|f - Q_{f,A_{m,n,p}}\|_{C^0(\Omega)} \in \left\{ O(e^{-n/\sqrt{m}}), \quad p = 1, \right.
\left. O(e^{-n}), \quad p = 2, \right. \quad O(e^{-n}), \quad p = \infty \right\}.
$$
Therefore, \( g(n) \in O(\rho^{-n}) \) if and only if \( g(n) \in O((\rho-\varepsilon)^{-n}) \) for all \( \varepsilon > 0 \). Thus, the above are upper bounds on the approximation rates.

**Remark 7.5.** The above statement indicates that the \( l_2 \)-degree suffices for the approximation to be as good as when using the entire \( P_{A,m,\infty} \) grid. In light of Remark 4.3, the interpolation can therefore reach any approximation accuracy with sub-exponential complexity \( O(|A_{m,n,2}|) \), \( |A_{m,n,2}| \in o(n^m) \), in any dimension \( m \) whenever the optimal rate for \( p = 2 \) applies. This can be observed in the numerical experiments presented in the next section.

We complete this section by extending classic approximation error estimates to arbitrary dimensions.

### 7.3. Error bound in multiple dimensions.

We generalize the classic 1D approximation error bound for polynomial interpolation to arbitrary dimensions \( m \in N \). For a given complete set of multi-indices \( A \subseteq N^m \), we denote by

\[
\partial A = \{ \alpha \in A \mid \alpha + e_i \not\in A \text{ for all } 1 \leq i \leq m \}
\]

\[
\overline{A} = \{ \beta \in N^m \mid \beta = \alpha + e_i \text{ for some } \alpha \in \partial A, 1 \leq i \leq m \}
\]

the discrete inner and outer boundaries of \( A \) and define \( \overline{A} = A \cup \overline{A} \) to be the closure of \( A \) in this sense. If \( m = 1 \) then for any \( n \in N \), \( A = \{0, \ldots, n\}, \partial A = \{n\}, \overline{A} = \{n + 1\} \), and \( \overline{A} = \{0, \ldots, n + 1\} \). With this we can state:

**Theorem 7.6 (Approximation error).** Let the essential assumptions (Definition 2.16) be fulfilled with respect to \( A \subseteq N^m \) and \( P_A \subseteq \mathbb{R}^m \). Further, let \( f \in C_A(\Omega, \mathbb{R}) \). Then, for any \( x \in \Omega \), there are \( \xi_{x,\beta} \in \Omega, \beta \in \overline{A} \), such that:

\[
f(x) - Q_{f, A}(x) = \sum_{\beta \in \overline{A}} \frac{\partial^\beta f(\xi_{x,\beta})}{\beta!} N_{\beta}(x), \quad \beta! := \prod_{i=1}^m \beta_i!.
\]

**Proof.** We argue by induction on \( |A| \). For \( |A| = 1 \) we have \( A = \{0\} \) and \( Q_{f, A}(x) = f(p_0) \) for all \( x \in \Omega \). Thus, by the Mean Value Theorem, we have

\[
f(x) - Q_{f, A}(x) = f(x) - f(p_0) = \sum_{i=1}^m \partial_{x_i} f(\xi_{x,i})(x_i - p_{0,i})
\]

for some \( \xi_{x,i} \in \Omega \), yielding Eq. (7.8) in this case. For \( |A| > 1 \), we consider the subsets \( A_1 = \{ \alpha \in A \mid \alpha_m = 0 \}, A_2 = A \setminus A_1 \), and the hyperplane \( H = \{(x_1, \ldots, x_{m-1}, p_{0,m}) \mid (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}\} \) defined by the polynomial \( Q_H = (x_m - p_{0,m}) \in \Pi_{m,1,1} \). As in Eq. (3.5), we use the splitting \( Q_{f, A}(x) = Q_1(x) + Q_H(x)Q_2(x), x \in \mathbb{R}^m \). We denote by \( x_H = (x_1, \ldots, x_{m-1}, p_{0,m}) \) the projection of \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \) onto \( H \) and recall that Theorem 2.7 guarantees \( Q_1(x) = Q_1(x_H) \). By decreasing \( m \) if necessary and w.l.o.g., we can assume that \( A_2 \neq \emptyset \). We denote by \( K_\beta(x) \) the multivariate Newton polynomials with respect to
and recursively compute
\begin{equation}
\begin{aligned}
A_2 \quad & \text{and recursively compute} \\
(7.9) \quad & f(x) - Q(x) = f(x_H) - Q_1(x_H) + Q_H(x) \left( \frac{f(x) - f(x_H)}{x_m - p_{0,m}} - Q_2(x) \right) \\
& = \sum_{\beta \in \mathcal{D}A_1} \frac{\partial^\beta f(\xi_{x,H,\beta})}{\beta!} N_\beta(x_H) + \left( \partial_{x_m} f(\eta_x) - Q_2(x) \right) Q_H(x) \\
& = \sum_{\beta \in \mathcal{D}A_1} \frac{\partial^\beta f(\xi_{x,H,\beta})}{\beta!} N_\beta(x) + \sum_{\beta \in \mathcal{D}A_2} \frac{\partial^\beta \partial_{x_m} f(\xi_{x,\beta})}{\beta!} Q_H(x) K_\beta(x) \\
& = \sum_{\beta \in \mathcal{D}A} \frac{\partial^\beta f(\xi_{x,\beta})}{\beta!} N_\beta(x),
\end{aligned}
\end{equation}

where we used the Mean Value Theorem for the second term in Eq. (7.9) and the fact that $Q_H(x)K_\beta(x) = N_\beta(x)$ for all $\beta \in \mathcal{D}A_2$ to yield Eq. (7.10). □

Remark 7.7. Note that for $m = 1$, Eq. (7.8) reduces to the classic 1D result
\begin{equation}
\begin{aligned}
f(x) - Q_{f,A}(x) = f^{(n+1)}(\xi_x) \prod_{i=0}^n (x - p_i)
\end{aligned}
\end{equation}

with $P_A = \{p_0, \ldots, p_n\}$, $|A| = n + 1$. This yields the known approximation error bound in 1D [45]:
\begin{equation}
\begin{aligned}
|f(x) - Q_{f,A}(x)| \leq \frac{|f^{(n+1)}(\xi_x)|}{2^n(n+1)!} \leq \frac{\|f^{(n+1)}\|_{C^0(\Omega)}}{2^n(n+1)!}, \quad P_A = \text{Cheb}_n^{1st}.
\end{aligned}
\end{equation}

Our result in Eq. (7.8) provides a similar bound on the approximation error in $m$D whenever the $k$-th derivatives of $f$ are known or bounded. However, usually these bounds are unknown. By validating the proposed Trefethen approximation rates in the next section, we even though provide a potential control of the approximation error being applicable in practice.

8. Numerical experiments

We implement a prototype of our multivariate interpolation solver, named MIP, in MATLAB. The code implements the multivariate divided difference scheme of Definition 3.2 for interpolation nodes $P_A$, $A = A_{m,n,p}$, generated by Leja-ordered Chebyshev extreme nodes, i.e., $GP = \oplus_{i=1}^m \text{Cheb}_n^0$ according to the 2nd essential assumption in Definition 7.1. We compare our solver with the following alternative methods:

1. Chebfun from the corresponding MATLAB package [32];
2. Cubic splines and 5th-order splines from the MATLAB Curve Fitting Toolbox;
3. Floater-Hormann interpolation [38] from the R package chebpol [42];
4. Multi-linear (piecewise linear) interpolation from chebpol [42];
5. Chebyshev interpolation of 1st kind from chebpol [42];
6. Uniform (grid) interpolation by Chebyshev polynomials from chebpol [42];
7. Vandermonde interpolation on $P_{A_{m,n,p}}$ in MATLAB.

Note that apart from MIP and Vandermonde, all other schemes use regular grids $P_{A_{m,n,\infty}}$ as interpolation nodes. Therefore, Chebfun and Chebyshev only deliver $l_{\infty}$-degree interpolations.
All implementations were benchmarked using MATLAB version R2019b, Chebfun package version 5.7.0, and R versions 3.2.3/Linux and 3.6.2/macOS with chebpol package version 2.1.2 on a standard personal computer (Intel(R) Xeon(R) CPU E5-2660 v3 @2.60GHz, 128GB RAM).

The code and all benchmark datasets are freely available from: https://git.mpicbg.de/mosaic/polyapprox. The implementation of MIP is provided as a prototype, which can be used to reproduce the results presented here. Currently, we optimize and re-implement the code from a software engineering perspective into an open source Python package including further algorithmic improvements, such as the multivariate barycentric Lagrange interpolation [7] discussed in Section 9.2.

8.1. Approximation on the hypercube. In the first set of experiments, we illustrate the statements of Theorem 7.4. Therefore, we consider the Runge function

$$f_R(x) = \frac{1}{1 + 10\|x\|^2}.$$ 

Observe that $f_R$ has poles in $z = \pm i/\sqrt{10} \in \mathbb{C}$. In light of this fact, one can show that $f_R$ is analytic in the Trefethen domain $N_{m,\rho}$ with $h = 1/\sqrt{10} \approx 0.316$ and with Trefethen radius $\rho = h + \sqrt{1 + h^2} \approx 1.365$, thus fulfilling the requirements of Theorem 7.4.

**Experiment 1.** We measure the approximation errors of the interpolants computed by the mentioned methods. To do so, we sample 100 randomly nodes $P \subseteq \Omega$, $|P| = 100$, independently generated for each degree, but identical for all methods and determine $\max_{q \in P} |f(q) - Q_f(q)| \approx \|f - Q_f\|_{C^0(\Omega)}$.

Figure 5 shows the results of this experiment in dimension $m = 2$. We observe that Chebyshev, Chebfun, and MIP are the only methods that converge down to machine precision (32-bit double-precision arithmetics). The convergence rate is as stated in Theorem 7.4 and reproduces earlier results by Lloyd N. Trefethen [83] as introduced in Section 1. However, we only use the $P_A$, $A = A_{m,n,p}$, $p = 1, 2$, unsisolvent nodes to determine the interpolants, whereas Trefethen computed the rates for the $l_1$- and $l_2$-degree approximations by regression over the whole $l_{\infty}$-grid. This detail might be the reason for the slight advantage of MIP over Chebfun and Chebyshev for high degrees. Further, we recognize that the Vandermonde approach is inaccurate and even becomes numerically unstable (rising errors) for higher degrees. It is therefore inappropriate for approximating strongly varying functions, such as the Runge function. As expected, (Chebyshev) polynomial interpolation on uniform grids (uniform) and multi-linear interpolation also do not converge.

Finally, we observe that Floater-Hormann interpolation performs better than multivariate cubic splines. It is comparable to 5th-order splines, but reaches an accuracy of $10^{-7}$ faster than any other approach.

Figure 6 shows the results of the same experiment in dimension $m = 3$, leaving out the infeasible methods. The observations made in 2D remain valid. However, Floater-Hormann becomes indistinguishable from 5th-order splines. Further, when considering the amount of coefficients/nodes required to determine the interpolant, plotted in the right panel (with logarithmic scales on both axes). The polynomial convergence rates of Floater-Hormann and all spline-type approaches become visible. MIP requires $122^3/899028 \approx 2$-times less coefficients/nodes than Chebyshev or Chebfun to approximate $f$ to machine precision for $n = 121$. 
Figure 5. Approximation errors for the benchmarked methods interpolating the Runge function in dimension $m = 2$.

Figure 6. Approximation errors for the benchmarked methods interpolating the Runge function in dimension $m = 3$.

Figure 7 shows the results for dimension $m = 4$. Spline interpolation was not able to scale to high degrees due to computer memory requirements. To simulate the behavior for higher degrees we rescale the hypercube to $\Omega = \left[-\sqrt{10}, \sqrt{10}\right]^m$. That is, we approximate the Runge function on two scales: Once for Runge factor $RF = 10$ and once for $RF = 1$. The results for $RF = 10$ show a similar situation as the results in 3D. However, Figure 6 suggests that again degree $n \approx 75$ is the crossover point, where MIP and Chebyshev become the superior approaches. Indeed, for $RF = 1$, only Chebyshev and MIP converge down to machine precision. But MIP reaches that goal earlier ($n = 40/47$) than Chebyshev, and with less interpolation nodes $\frac{|C_{\text{Chebyshev}}|}{|C_{\text{MIP}}|} = \frac{5398416}{858463} \approx 6$.

The same is true in dimension $m = 5$, as Fig. 8 illustrates. Especially when considering the right plot (with logarithmic scales on both axes), we observe that MIP best resists the curse of dimensionality by yielding 2 orders of magnitude better accuracy than Chebyshev for $n = 40$ (3.0 $\cdot$ 10$^{-14}$ vs. 2.1 $\cdot$ 10$^{-12}$) with less interpolation nodes $\frac{|C_{\text{Chebyshev}}|}{|C_{\text{MIP}}|} = \frac{115856201}{18920038} \approx 6$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Approximation errors for the benchmarked methods interpolating the Runge function in dimension $m = 2$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Approximation errors for the benchmarked methods interpolating the Runge function in dimension $m = 3$.}
\end{figure}
Figure 7. Approximation errors for the benchmarked methods interpolating the Runge function in dimension $m = 4$.

Figure 8. Approximation errors for the benchmarked methods interpolating the Runge function in dimension $m = 5$.

Table 1. Fitted convergence rates of MIP.

| dim | fitting range $\rho_{RF}$ | $c_{RF}$ |
|-----|---------------------------|----------|
| 2   | $2 \sim 121$              | 1.35     |
| 3   | $2 \sim 121$              | 1.34     |
| 4   | $2 \sim 80$               | 1.32     |
| 5   | $2 \sim 40$               | 2.33     |

To assess the convergence rates, we fit the data for MIP with the model $y = c\rho^{-n}$ with an $R$-squared of 0.99 or better, indicated by the dashed lines in the corresponding figures. This yields the exponential decays reported in Table 1 for the approximation errors in the corresponding ranges. The Trefethen radius is given by $\rho_{\text{max}} = 1 + \sqrt{2} \approx 2.41$ for $RF = 1$. Thus, the upper bounds in Theorem 7.4 are almost achieved by the MIP method. In contrast, Chebyshev just reaches convergence rates $\rho_{RF=1} \approx 1.9$ in dimension $m = 4, 5$.

In summary, Experiment 1 confirms that MIP converges as expected from Theorem 7.4. Compared to the other methods tested, MIP is efficient in reaching machine precision. MIP also seems to resistant to the curse of dimensionality best,
which becomes increasingly visible in higher dimensions, thus supporting the prediction of Remark 7.5.

8.2. Regression on curved manifolds. We realize the algorithm of Carl de Boor and Amon Ros [28, 29] in terms of Corollary 6.5 in case of the torus $M = \mathbb{T}_{r^2}^2$.

That is, we consider

$$Q_{T_{r^2}^2}(x, y, z) = (x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2)$$

with $R = 0.7$ and $r = 0.3$. $T_{r^2}^2 = Q_{T_{r^2}^2}^{-1}(0)$ is an algebraic hypersurface of degree 4. Given a function $f : \Omega \to \mathbb{R}$, we aim to interpolate the restriction $f_{|T_{r^2}^2}$.

**Experiment 2.** We choose $A = A_{m,n,p}$ with $m = 3$, $p = 2$, and sample $S = [1, 5 \cdot |A|] \in \mathbb{N}$ uniformly random nodes $P_{T_{r^2}^2}$ on the torus $T_{r^2}^2$, as illustrated in Figure 9 (left). Further, we generate $P_A$ with respect to

$$GP = \text{Cheb}_n^0 \oplus \text{Cheb}_n^0 \oplus r \cdot \text{Cheb}_n^0,$$

yielding a feasible grid near the sampled nodes $P_{T_{r^2}^2}$. Since $T_{r^2}^2$ is a hypersurface of degree 4, the nodes $p \in P_{T_{r^2}^2}$ are not unisolvent for $\Pi_{A_{m,n,p}}$ with $n \geq 4$. Thus, due to Corollary 6.3

$$m_M := \{Q \in \Pi_A \mid Q_{|M} \equiv 0 \neq \{0\}, \quad M = T_{r^2}^2 \}$$

However, by Remark 6.4 and Theorem 6.4 the splitting

$$\Pi_A \cong \Pi_A/m_M + m_M = : \Pi_{A,T^2} + \Pi_{A,T^2}^\perp$$

corresponds with probability 1. For $A = A_{4,4,2}$, we have $\dim \Pi_{A,T^2} = 1$ and by computing a basis $\mu \in \Pi_A$ of $\dim \Pi_{A,T^2}$ as in Eq. (6.3) we have determined a level-set function $Q_{T_{r^2}^2} = \mu$, i.e., $T_{r^2}^2 = Q_{T_{r^2}^2}^{-1}(0)$.

Further, considering $R_A = (L_{\alpha}(p_i))_{(i, \alpha) \in S \times A}$, where $L_{\alpha}$, $\alpha \in A$ denote the Lagrange polynomials w.r.t. $P_A$, we find the Lagrange coefficients $C_{\text{lag}}$ of the interpolant $Q_{f,A|T_{r^2}^2}$ of $f_{|T_{r^2}^2}$ for any $f : \Omega \to \mathbb{R}$ by solving

$$R_A C_{\text{lag}} \approx F, \quad F = (f(p_1), \ldots, f(p_S)) \in \mathbb{R}^S, \quad p_i \in P_{T_{r^2}^2}, \quad 1 \leq i \leq S$$

using standard MATLAB least-square regression.

The results of this experiment are shown in Fig. 9. All errors are measured on the $S$ nodes $P_{T_{r^2}^2}$, plus additional 200 uniform at random sampled points $P \subseteq T_{r^2}^2$ for 10 independent repetitions in each case. The inset table in Fig. 9 shows that the approximation errors for the level-set function $Q_{T_{r^2}^2}$ and its gradient $\nabla Q_{T_{r^2}^2}$, corresponding to the surface normal, are within machine precision. The right panel of Fig. 9 shows the mean approximation errors with min–max error bars for interpolating the restriction $f_{|T_{r^2}^2}$ of the Runge function $f(x) = 1/(1 + 10\|x\|^2)$ to the torus, measured analogously.

The results suggest that the notion of the matrix $R_A$ defined with respect to the unisolvent grid $P_A$ is the main reason for the fast approximation rate on the Runge function. Implicitly, we therefore also validate the numerical stability of the pre-computation approach.

We are not aware of any other approach that can handle such an interpolation task without requiring a triangulation or parameterization of the manifold $T_{r^2}^2$. Therefore, these results demonstrate the flexibility of MIP and suggest its use in
However, by Remark 7 and Theorem 6 the splitting
face of degree yielding a feasible grid near the sampled nodes
illustrated in Figure 7 (left). Further, we generate

Experiment 3

function with Corollary 6 in case of the torus

We realize the algorithm of Carl de Boor and Amon Ros \[28,29\] in terms of

8.3 Regression on Curved Manifolds

approximation error for the restriction
approximation errors of the level set function

Fig. 7

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| function | approximation error | MIP |
|----------|---------------------|-----|
| \(Q_{R,r}^2\) | 6.4358e-15 | |
| \(\nabla Q_{R,r}^2\) | 5.3956e-13 | |

Figure 9. Uniformly random nodes on the torus \(T^2_{R,r}\) with \(R = 0.7\) and \(r = 0.3\) (left), approximation errors of the level set function \(Q_{T^2_{R,r}}\) and its gradient \(\nabla Q_{T^2_{R,r}}\) (table) and approximation error for the restriction \(f_{T^2_{R,r}}\) of Runge function to the torus (right).

numerical methods on curved surfaces, including quadrature schemes, ODE and PDE solvers, and optimization algorithms.

9. Conclusion

We have generalized the notion of unisolvent nodes for polynomial interpolation in arbitrary dimensions with respect to a generalized concept of polynomial degree and for non-tensorial node distributions. This allowed us to extend the classic 1D Newton and Lagrange interpolation methods to multivariate schemes in a numerically stable and efficient way, resulting in a practically implemented algorithm with \(O(|A|^2)\) runtime complexity and \(O(|A|)\) memory complexity. We also provided the theory for generalizing the approach to interpolation of scattered data on curved manifolds.

We empirically observed that the resulting algorithm reaches the optimal approximation rate given by Lloyd N. Trefethen’s Theorem \[83\], supporting the conjecture that these rates apply in general \[9\]. In contrast to previous approaches, such as Chebfun \[32\], multivariate splines \[26\], and Floater-Hormann interpolation \[38\], the present MIP algorithm achieves exponential approximation rates for the Runge function using only sub-exponentially many interpolation nodes. This suggests that we have found an efficient approximation scheme that overcomes the curse of dimensionality for a generic class of functions.

In closing, we discuss related concepts and possible further developments.

9.1. Generalizing the notion of unisolvence. We generalized the notion of unisolvence beyond the pioneering work of Kuntzmann, Guenther, and Roetman \[48,59\]. However, an even more general notion should be possible by considering (more) general hypersurfaces \(\tilde{H}\) instead of the linear hyperplanes \(H \subseteq \mathbb{R}^m\) used in Theorem 2.6. This might also enable the native construction of unisolvent nodes on
general level-set manifolds and to derive a Newton interpolation scheme on curved spaces.

9.2. Barycentric Lagrange interpolation. In 1D, barycentric Lagrange interpolation is the most efficient interpolation scheme \([7]\) for fixed nodes. Both determining the interpolant \(Q_{f,n}\) and evaluating \(Q_{f,n}\) at any given \(x_0 \in \mathbb{R}\) require linear time \(O(n)\). This is achieved by precomputing the constant barycentric weights that only depend on the locations of the nodes, but not on the function \(f\).

We have already established preliminary theoretical results towards generalizing this approach to \(mD\) for the case of \(l_1\)-degree \([75]\). This suggests that the current complexity \(O(|A|^2)\) of interpolation and evaluation for the case of multi-indices \(A = A_{m,n,p}, p > 1\), can be further reduced.

9.3. Multivariate polynomial regression. For a given function \(f : \Omega \rightarrow \mathbb{R}\) and a set of nodes \(P \subseteq \Omega = [-1,1]^m, m \in \mathbb{N}\), we consider the graph \(G = f(P) \times P \subseteq \mathbb{R}^{m+1}\). Corollary \([6,5]\) and Experiment \([2]\) allow identifying the hypersurface \(M \supseteq G\) that contains \(G\) as a level set of polynomials \(Q_{M,i} \in \Pi_{m+1}\), \(M = \cap_{i=1}^{n+1-dim M} Q_{M,i}^{-1}(0)\) and to interpolate the restriction \(g|_M\) of any function \(g : \mathbb{R}^{m+1} \rightarrow \mathbb{R}\) to \(M\). We consider both aspects crucial steps towards developing a multivariate polynomial regression scheme, opening up applications in multivariate analysis, statistics, and topological analysis \([3, 34, 39, 63, 62, 77, 89]\).

9.4. Trigonometric interpolation. In 1D, trigonometric Clairaut–Lagrange–Gauss interpolation can be used to compute the discrete Fourier transform (DFT) of a periodic function \(f\) \([54]\). This fact was for example used in the development of the famous Cooley-Tukey algorithm \([20]\), which re-instantiated an algorithm by Carl F. Gauss \([44]\) to yield a modern realization of the Fast Fourier Transform. These concepts are also closely related to the invention of wavelets \([81]\). Revisiting these aspects from the perspective of multivariate Lagrange interpolation might be worthwhile to make progress in open problems, such as Fourier transformation of non-periodic, highly oscillating signals or fast Fourier transforms on scattered nodes \([5, 47, 53]\).

9.5. Numerical integration. Until today, the classic Gauss quadrature formula is the best approach to approximating integrals \(I_{Gauss}(f) \approx \int_{\Omega} f(x) \, dx\) in one variable \([43, 55]\). Many contributions toward extending this approach to higher dimensions have been made \([21, 22, 49, 82]\). This list is by no means exhaustive, and research in this direction is actively ongoing. The present notion of unisolvent nodes might be helpful in this endeavor.

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