Vacuum Curves, Classical Integrable Systems in Discrete Space-Time and Statistical Physics

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Abstract

A dynamical system with discrete time is studied by means of algebraic geometry. The system admits a reduction that is interpreted as a classical field theory in 2+1-dimensional wholly discrete space-time. The integrals of motion of a particular case of the reduced system are shown to coincide, in essence, with the statistical sum of the well-known (inhomogeneous) 2-dimensional dimer model (the statistical sum is here a function of two parameters). Possible generalizations of the system are examined.

Vacuum curves and vacuum vectors are algebro-geometrical objects that have arisen in the theory of the quantum Yang—Baxter equation. They seem to have their origin in R. Baxter's works [1, 2], and their general definition was formulated by I. M. Krichever [3]. Baxter, and also Takhtajan and Faddeev [4] used the vacuum vectors to obtain a generalization of the Bethe ansatz for the XYZ spin model—an integrable one-dimensional quantum field theory (and also for the eight-vertex model of two-dimensional statistical physics). Krichever has applied the vacuum curves to classification of the solutions of the quantum Yang—Baxter equation in the tensor product of 2-dimensional vector spaces. Then, the author of this paper has found some further applications of the vacuum curves and vacuum vectors. In the paper [8] (see also [9]), new solutions of the quantum Yang—Baxter equation were constructed for the first time. They correspond to what is now known as Chiral Potts model. In [10, 11, 12], the degeneracies of the spectrum of the XXZ quantum chain hamiltonian were examined by means of the vacuum curves, and in [12] the studying of vacuum vector bundles has resulted in the construction of solutions to the tetrahedron equation with commuting spin variables on the links.

Here, I try to demonstrate that the vacuum curves may be useful for studying the classical (not quantum) field theory models as well. A difference equation on the 2+1-dimensional cubic lattice is presented, for which the solution to the Cauchy problem is constructed, at least in principle, through a rather simple scheme. The evolution is of hyperbolic nature, i.e. the “perturbations” propagate not faster than fixed speed. The interesting feature is that, in a particular “scalar” case, the model reveals a quite natural connection with the well-known

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dimer model of statistical physics. The statistical sum of this latter model, which depends here on two parameters (and the model itself is, of course, inhomogeneous), is the integral of motion for any values of these parameters.

This field theory comes as a “reduction” of some very simply described “non-local” dynamical system. On the other hand, generalizations of this latter system are constructed in this paper, and I use a discrete analog of Lax pair for this purpose.

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1 Definition of the dynamical system. Gauge invariance

Let

\[ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]

be a block matrix, \( A, \ldots, D \) being \( n \times n \) matrices consisting of complex numbers. Consider the following two operations: construction of the inverse matrix

\[ L \to L^{-1} \]

and the block transposing

\[ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to L^t = \begin{pmatrix} A & C \\ B & D \end{pmatrix}. \]

Now let a (birational) mapping \( f \) be a composition of these two operations:

\[ f(L) = (L^{-1})^t. \] (1)

Let us introduce the discrete integer-valued time \( \tau \), and let the matrix \( L \) depend on \( \tau \) so that

\[ L(\tau + 1) = f(L(\tau)). \] (2)

This “dynamical system” has been already mentioned in literature [13]. In the present paper, the integrability of this system is demonstrated, assuming that the “motion” is considered up to a “gauge transformation” (see below).

Let \( G \) and \( H \) be non-degenerate \( n \times n \) matrices. The gauge transformation of the matrix \( L \) is the following transformation of its blocks:

\[ A \to GAH, \ldots, D \to GDH. \] (3)

Two matrices \( L \) and \( L' \) connected by the transformation (3) will be called gauge equivalent. It is clear that if \( L(\tau) \) and \( L'(\tau) \) belong to the same class of gauge equivalence, the matrices \( L(\tau + 1) \) and \( L'(\tau + 1) \) also do so. Thus, dynamics (2) induces a dynamics on the set of classes of gauge invariance.
2 Vacuum curves and vacuum vectors

It turns out that the dynamics (2) preserves the so-called vacuum curve $\Gamma$ of the operator $L$ (the bases being fixed, we make no difference between a linear operator and its matrix). To be exact, $\Gamma$ remains unchanged under the transformation $f \circ f$, and undergoes a simple transformation under $f$. The curve $\Gamma$ together with the class of linear equivalence of the pole divisor of the vacuum vectors (see below) determines the matrix $L$ up to a gauge transformation. The set of those classes of linear equivalence is isomorphic to a complex torus—the Jacobian of the curve $\Gamma$. The dynamics (2) linearizes on the Jacobian, i.e. the transformation $f$ corresponds to a constant shift on the torus. Now, let us discuss these facts in detail.

The vacuum curve of the operator $L$ is an algebraic curve in the space $C^2$ of two variables $u, v$. Here are two equivalent definitions of it.

**Definition 1** Consider the relation

$$L(U \otimes X) = V \otimes Y,$$

wherein

$$U = \begin{pmatrix} u \\ 1 \end{pmatrix}, \quad V = \begin{pmatrix} v \\ 1 \end{pmatrix}$$

are two-dimensional vectors, $X$ and $Y$ are $n$-dimensional vectors. For a generic matrix $L$, the non-zero solutions $(U, V, X, Y)$ of the relation (4) are parametrized, up to a scalar factor in $X$ and $Y$, by points of an algebraic curve $\Gamma$ of genus $g = (n - 1)^2$ given by an equation of the form

$$P(u, v) = 0,$$

$P(u, v)$ being a polynomial of degree $n$ in each variable, i.e.

$$P(u, v) = \sum_{j,k=1}^{n} a_{jk} u^j v^k. \quad (6)$$

$\Gamma$ is called the vacuum curve of the operator $L$.

**Definition 2** The vacuum curve of the operator $L$ is the curve $\Gamma$ in $C^2$ given by the equation

$$P(u, v) = \det(V^\perp LU) = \det(uA + B - uvC - vD) = 0,$$

where

$$V^\perp = (1, -v).$$

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Let us denote the points of the vacuum curve by the letter \( z = (u, v) \in \Gamma \). Then \( U = U(z) \) and \( V = V(z) \) are meromorphic vectors on \( \Gamma \) with the pole divisors \( D_U \) and \( D_V \) of degree \( n \), while \( X = X(z) \) and \( Y = Y(z) \), if normalized by, e.g., the condition that their \( n \)th coordinates equal unity, become meromorphic vectors with pole divisors \( D_X \) and \( D_Y \) of degree \( n^2 - n \). Under this normalization, a meromorphic scalar factor \( h(z) \) must be added into (4):

\[
L(U(z) \otimes X(z)) = h(z)V(z) \otimes Y(z). \tag{8}
\]

The linear equivalence of divisors

\[
D_U + D_X \sim D_V + D_Y
\]

holds and is provided by the function \( h(z) \) in the sense that \( h(z) \) has its poles in the points of \( D_U + D_X \) and zeros in the points of \( D_V + D_Y \).

As is shown in the paper \(^3\), the vacuum curve equation \( P(u, v) = 0 \) and the class of linear equivalence of divisor \( D_X \) or \( D_Y \) determine a generic matrix \( L \) to within a gauge transformation, and vice versa, the gauge transformations do not change the vacuum curve and the classes of linear equivalence of divisors. In other words, the correspondence

\[
\text{(class of gauge equivalence of } L) \leftrightarrow (\Gamma, \text{ the class of } D_X)
\]

is a birational isomorphism.

We will call \( X(z) \) the vacuum vector and \( Y(z) \) the covacuum vector in the point \( z \) of the curve \( \Gamma \). \( X(z) = X(u, v) \) generates the (one-dimensional) kernel of the matrix

\[
u A + B - uvC - vD. \tag{9}
\]

The Definition \(^4\) allows one to trace what happens with the vacuum curve and vacuum vectors under the transformation \( L \rightarrow L^{-1} \), while the Definition \(^2\) allows one to trace what happens under the transformation \( L \rightarrow L^t \). Namely, it is seen from the relation

\[
L^{-1}(V(z) \otimes Y(z)) = h(z)^{-1}U(z) \otimes X(z)
\]

that the vacuum curve equation for the matrix \( L^{-1} \) is

\[
P(v, u) = 0,
\]

while its vacuum vector in the point \((v, u)\) coincides with the covacuum vector of the initial matrix \( L \). As for the block transposing, the vacuum curve equation for the matrix \( L^t \)

\[
\det(uA + C - uvB - vD) = 0
\]

may be rewritten as

\[
u^n v^n \det(v^{-1}A - B + u^{-1}v^{-1}C - u^{-1}D) = 0,
\]
i.e.

\[ u^n v^m P(-v^{-1}, -u^{-1}) = 0. \]

The vacuum vector of the matrix \( L^t \) in the point \((-v^{-1}, -u^{-1})\) of its vacuum curve coincides with the vacuum vector \( X(u, v) \) of the matrix \( L \).

Combining these considerations, one finds out that the vacuum curve \( \tilde{\Gamma} \) of the matrix \((L^{-1})^t\) is given by equation

\[ u^n v^m P(-u^{-1}, -v^{-1}) = 0, \]

while the vacuum vector \( \tilde{X}(-u^{-1}, -v^{-1}) \) coincides with the vector \( Y(u, v) \) of the matrix \( L \).

Identifying the curves \( \Gamma \) and \( \tilde{\Gamma} \) by means of the isomorphism

\[(u, v) \leftrightarrow (-u^{-1}, -v^{-1}),\]

one sees that

\[ D_{\tilde{X}} \sim D_Y \sim D_X + D_U - D_V, \]

which means that, in essence, the transformation \([\square]\) results in adding a fixed element of the Picard group, namely the equivalence class of the divisor \( D_U - D_V \), to the pole divisor \( D_X \) of the vacuum vectors. It is clear also that after two transformations one returns to the initial curve:

\[ \tilde{\Gamma} = \Gamma. \]

3 Reduction to evolution equation in the 2+1-dimensional space-time

The dynamical system of the previous section admits an interesting reduction, i.e. some special choice of the matrices \( A, \ldots D \) that is in agreement with the evolution. In this section, it will be convenient to treat the matrices \( A, \ldots D \) as linear operators acting from the linear space \( \mathcal{H}_1 \) into the linear space \( \mathcal{H}_2 \) (of the same finite dimension). This being the situation at the moment \( \tau \), the operators act, of course, from \( \mathcal{H}_2 \) into \( \mathcal{H}_1 \) at the moment \( \tau + 1 \), and so on.

Let each of the spaces \( \mathcal{H}_1, \mathcal{H}_2 \) be a direct sum of \( lm/2 \) identical subspaces of dimension \( d \), where \( l, m \) are even numbers. Let us imagine these subspaces as situated at the vertices of the square lattice on the torus of the sizes \( l \times m \) (which will mean the periodic boundary conditions in both discrete space variables).

Let the subspaces be arranged in checkerboard fashion, as in Fig. 3, where the empty circles correspond to subspaces of the space \( \mathcal{H}_1 \), while the filled circles correspond to those of the space \( \mathcal{H}_2 \).

Let then the operators \( A, \ldots D \) be such that the image of each of the mentioned \( d \)-dimensional subspaces with respect to, say, operator \( A \) lies in the \( d \)-dimensional subspace of \( \mathcal{H}_2 \) at which points the arrow marked “\( A \)” that links
Figure 1: Integrable dynamical system in the 2+1-dimensional space-time
these two subspaces (Fig. 1). Analogously, the restrictions on $B, C, D$ are depicted in Fig. 1 (see also formula (20) for non-degenerate $A, \ldots D$). Thus, to each link of the lattice a $d \times d$ matrix is attached that is a block of one of the “large” matrices $A, \ldots D$. Let us shade half of the squares of the lattice in a checkerboard way, as in Fig. 1. One can verify that the evolution of the system may be described as follows.

At the first step, each of the four $d \times d$ matrices that correspond to the arrows surrounding each shaded square is transformed into a matrix expressed through just these four matrices. This goes according to the following formulae, in which the $d \times d$ blocks are somewhat freely denoted by the same letters $A, \ldots D$ as the “large” matrices:

\begin{align*}
A &\rightarrow (A - BD^{-1}C)^{-1}, \\
B &\rightarrow (B - AC^{-1}D)^{-1}, \\
C &\rightarrow (C - DB^{-1}A)^{-1}, \\
D &\rightarrow (D - CA^{-1}B)^{-1}.
\end{align*}

However, the formulae (10–13) apply equally to the “large” matrices.

After the transformation (10–13), all the arrows reverse, and at the second step the non-shaded squares are engaged in the same way according to the same formulae (10–13). Then everything is repeated. Thus, the evolution is of hyperbolic nature: each local perturbation spreads not faster than one unit of length per unit of time.

Let us clarify the symmetries of vacuum curves and divisors $D_X$ in this “reduced” model. Let us introduce two integer-valued coordinates $\xi, \eta$ for the vertices of the lattice, so that $\xi$ increases by 1 in passing from a vertex one step to the right, and $\eta$ increases by 1 in passing one step upwards. $\xi$ and $\eta$ are defined modulo $l$ and $m$ respectively. A $d$-dimensional subspace of $H_1$ or $H_2$ will be denoted $H_{\xi\eta}$ if it corresponds to a vertex with coordinates $\xi, \eta$. Consider a linear transformation in spaces $H_1$ and $H_2$ consisting in multiplying the vectors of each subspace $H_{\xi\eta}$ by $\omega_{1,2}^\ell$, $\omega_{1,2}$ being a fixed primitive root of the $l$-th degree of unity:

\[ \omega_1^\ell = 1. \]

This corresponds to the following transformation of the operators $A, \ldots D$ (from now on we speak of each of these operators “as a whole”, not of their blocks):

\[ A \rightarrow \omega_1 A, \ B \rightarrow B, \ C \rightarrow C, \ D \rightarrow \omega_1^{-1}D. \]  

(14)

Consider also another linear transformation in $H_1$ and $H_2$, consisting in multiplying the vectors of each subspace $H_{\xi\eta}$ by $\omega_2^m$, $\omega_2$ being a fixed primitive root of the $m$-th degree of unity:

\[ \omega_2^m = 1. \]

This corresponds to the following transformation:

\[ A \rightarrow A, \ B \rightarrow \omega_2 B, \ C \rightarrow \omega_2^{-1}C, \ D \rightarrow D. \]  

(15)
The vacuum curve of the operator \( L \), which is given by equation (7)
\[
P(u, v) = \det(uA + B - uvC - vD) = 0,
\]
must be invariant under the transformations (14), (15). This leads to the invari-
ance of the polynomial \( P(u, v) \) with respect to the following transformations \( g_1 \) and \( g_2 \):
\[
g_1(u, v) = (\omega_1 u, \omega_1^{-1} v).
\]
\[
g_2(u, v) = (\omega_2^{-1} u, \omega_2^{-1} v).
\]
This invariance, then, leads to the following statement: only those coefficients \( a_{jk} \) are non-zero in the vacuum curve equation (see (3), (6)) for the “reduced”
model, for which
\[
j - k \equiv 0 \pmod{l},
\]
\[
j + k \equiv 0 \pmod{m}.
\]
As for the divisor \( D_X \), let us recall that it consists of such points in the curve
\( \Gamma \) in which vanishes the last coordinate of the vector \( X \) (see \( \mathbf{\text{3}} \)), the latter being
an eigenvector of the matrix (9) with zero eigenvalue:
\[
(uA + B - uvC - vD)X(u, v) = 0.
\]
This immediately leads to the conclusion: the divisor \( D_X \) is invariant with
respect to the transformations (16, 17).
Under some additional condition, the inverse statement also holds: if the
curve \( \Gamma \) and divisor \( D_X \) are invariant under the transformations (16), (17),
then the corresponding \( L \)-operator comes from a “reduced” model described in this
section. For instance, this is true if \( l/2 \) and \( m/2 \) are relatively prime numbers.
If these numbers are not relatively prime, some conditions are to be imposed
on the divisor \( D_X \). To avoid going into details of this latter case, let us not
consider it here.
Thus, let an operator \( L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) be given, \( A, \ldots D \) being \( n \times n \) matrices,
\( n = (lm/2)d \), \( l \) and \( m \) even, and \( l/2 \) and \( m/2 \) being relatively prime. Let the vacuum curve \( \Gamma \) of the operator \( L \) and the divisor \( D_X \) be invariant under the action of the group \( G \) generated by its elements \( g_1, g_2 \) (16, 17), \( \omega_1 \) and \( \omega_2 \) being primitive roots of degrees \( l \) and \( m \) of unity. Then the linear space in which operators \( A, \ldots D \) act decomposes into a direct sum of \( lm/2 \ d \)-dimensional
subspaces \( H_{\xi \eta} \), \( \xi \) and \( \eta \) being integers modulo \( l \) and \( m \) respectively and such
that \( \xi + \eta \) is an even number, and the following equalities between the images
of these subspaces hold (in a “generic” case of non-degenerate \( A, \ldots D \)):
\[
A H_{\xi - 1, \eta + 1} = B H_{\xi \eta} = C H_{\xi, \eta + 2} = D H_{\xi + 1, \eta + 1}.
\]
The equalities (20) mean exactly that one is in the situation of Fig. 1.
Let us prove the above statements. First, the natural projection from the curve $\Gamma$ to its factor $\Gamma / G$ has no branch points (here the fact that $l/2$ and $m/2$ are relatively prime is used to demonstrate that ramification does not occur when $u$ or $v$ equals zero or infinity). Thus, the $n$-dimensional linear space of meromorphic functions $x(z) = x(u, v)$ whose pole divisor is $D_X$ decomposes into a direct sum of subspaces of equal dimensions corresponding to the characters of (commutative) group $G$. Each of these subspaces consists of functions $x(z)$ satisfying relations

$$x(gz) = \chi_{\xi, \eta}(g)x(z),$$

the character $\chi_{\xi, \eta}$ being a scalar factor

$$\chi_{\xi, \eta}(g) = \omega_1^{g_1 a} \omega_2^{g_2 b},$$

where

$$g = g_1^{a} g_2^{b}.$$  

The equality $g_1^{l/2}g_2^{m/2} = 1$ means that $\xi + \eta$ must be an even number.

The components of the vector $X(u, v)$ are exactly the functions $x(z)$. In an appropriate basis, $d$ components correspond to each character $\chi_{\xi, \eta}$. Let us denote $H_{\xi, \eta}$ the set of vectors with other components equal to zero. Now, the equalities (20) are to be proved to end this section.

Consider the decomposition of vector $X(u, v)$ into a sum

$$X(u, v) = \sum_{\xi, \eta} X_{\xi, \eta}(u, v),$$

where $X_{\xi, \eta} \in H_{\xi, \eta}$. Then

$$X_{\xi, \eta}(g(u, v)) = \chi_{\xi, \eta}(g)X_{\xi, \eta}(u, v).$$

Consider the sum

$$\sum_{g \in G} \chi_{\xi, \eta}(g^{-1})g\{(uA + B - uvC - vD)X(u, v)\} = 0 \quad (21)$$

(which is equal to zero because of (19)). The action of $g$ upon the braces in (21) means that each $u$ and $v$ in the braces is transformed according to (16), (17), i.e., $u$ changes into $\chi_{1, -1}(g)u$, and $v$ changes into $\chi_{-1, -1}(g)v$. The equality (21) gives thus

$$uAX_{\xi-1, \eta+1}(u, v) + BX_{\xi, \eta}(u, v) - uvCX_{\xi, \eta+2}(u, v) - vX_{\xi+1, \eta+1}(u, v)D = 0. \quad (22)$$

Let us set $u = 0$ in (22). Then $v$ can take $n$ different values $v_j$ satisfying relation $P(0, v_j) = 0$. To these values $v_j$ correspond $d$ linearly independent vectors $X_{\xi, \eta}(0, v_j)$, and also $d$ vectors $X_{\xi+1, \eta+1}(0, v_j)$. Thus, the equalities

$$BX_{\xi, \eta}(0, v_j) = v_j DX_{\xi+1, \eta+1}(0, v_j)$$
that result from (22) give

\[ BH_{\xi,\eta} = DH_{\xi+1,\eta+1}. \]

Analogously, one can as well obtain the rest of equalities (20).

4 Connection to dimer model

As has been demonstrated, the integrals of motion of the dynamical system of Section 1 and its reductions (if the even degrees of the transformation (1) are considered) are the coefficients \( a_{jk} \) of the vacuum curve (6). These coefficients are determined up to a common factor, so they may be divided by \( a_{00} \). As one can see, the resulting coefficients are those of the polynomial

\[ \det(1 + uAB^{-1} - vDB^{-1} - uvCB^{-1}). \] (23)

In other words, the determinant (23) is an integral of motion for any \( u, v \).

Let us turn now to the model from section 3, that is to the model in 2+1-dimensional discrete space-time with periodic boundary conditions, and let the dimension \( d \) of the linear space corresponding to each vertex be equal to 1. Each of the “small” matrices \( A, B, C, D \) corresponding to the links will then be a single (depending on the link) number \( a, b, c \) or \( d \). It is well known that the determinant of any \( N \times N \) matrix is a sum of its matrix elements products corresponding in a certain way to the permutations of \( N \) objects, while each permutation decomposes into a product of the cyclic ones. In our situation, the cyclic permutations correspond to the non-selfintersecting closed paths (contours) going along the arrows of the following diagram (Fig. 2) (thus, general permutations correspond to the sets of non-intersecting paths). To each closed path corresponds the product of the weights \( ua, -uvc, -vd, b^{-1} \) on its links, and, to get right signs for the terms of which the determinant (23) is made up, one should add a minus sign to each such product containing an even number of the factors \( b^{-1} \).

**Remark 1** Another way to obtain right signs is: to multiply each \( b \) by \(-1\) and then multiply each product corresponding to a closed path (and containing any number of \( b \)’s) by \(-1\).

It turns out that the determinant (23) is connected with the statistical sum of the well known dimer model [6]. Let us define the correspondence between the sets of paths and the dimer configurations as follows. Let the empty set of paths correspond to the “standard” dimer configuration, the dimers being placed on the “B-links” (Fig. 3). For a non-empty set of paths, let us change the standard configuration along all the paths, replacing each dimer by a free link and vice versa. One can verify that this is a bijective correspondence.
The statistical sum being considered, let the weights \(-b\) (not \(b^{-1}\)) correspond to the “B-links”, while to the other links correspond the unchanged weights \(ua, -vd, -uvc\). Then one can see that the statistical sum, if multiplied by \(\prod_{\text{over all links}}(-b^{-1})\) (let us call the result the normalized statistical sum), consists of the same terms as the determinant \(\det\), up to different signs of some of them. Let us emphasize that the dimer model is, of course, inhomogeneous: the weights \(a, b, c, d\) are different for different links.

Let us study these signs in detail. Note that the conditions of non-intersecting and non-self-intersecting impose strong restrictions on the possible path configurations. Every closed path on the torus is homologically equivalent to a linear combination with integer coefficients of two basis cycles \(a\) and \(b\) whose intersection number is 1 (I use the boldface font for cycles, because the letters \(a, b\ldots\) are already in use). If the torus is cut along a closed non-self-intersecting
path $c$ not equivalent to zero, the result will be homeomorphic to the lateral surface of a cylinder (this follows, e.g., from [3], chapter 1, section 3). Then the contour $d$ going along a generatrix of the cylinder in a properly chosen direction has the intersection number 1 with the contour $c$. The intersection number being bilinear and integer-valued, we find that if the contour $c$ is homologically equivalent to a sum $l \mathbf{a} + m \mathbf{b}$, then $l$ and $m$ cannot have common divisors (not equal to $\pm 1$). Thus, the following lemma is valid.

**Lemma 1** Every closed non-selfintersecting path on the torus is homologically equivalent to a linear combination of the basis paths $\mathbf{a}$ and $\mathbf{b}$ with relatively prime integer coefficients.

Now let us pass to the case of several contours on the torus. If they do not intersect, their intersection numbers equal 0 (of course) and thus their homological classes must be proportional to one another. This together with Lemma [3] leads to the following lemma.

**Lemma 2** Several closed non-intersecting and non-selfintersecting paths going along the arrows on the torus, as in Fig. [4], are necessarily all homologically equivalent to one another.

If two paths are homologically equivalent, then the terms of the same degrees in $u$ and $v$ correspond to them (one can see in Fig. [5] that the different ways round an “elementary square” yield the same degrees of $u$ and $v$). Let the basis paths $\mathbf{a}$ and $\mathbf{b}$ yield the terms proportional to $x = u^{\alpha_1}v^{\beta_1}, y = u^{\alpha_2}v^{\beta_2}$ correspondingly (with the factors of proportionality not depending on $u,v$). According to Lemma [1] the determinant (23) and the statistical sum of the dimer model are polynomials in $x, y$. The following lemma sums up this section.

**Lemma 3** Let $f(x,y)$ and $s(x,y)$ be the determinant (23) and the normalized statistical sum of the dimer model considered as functions of $x$ and $y$. Then

\[
s(x,y) = \frac{1}{2} \left( -f(x,y) + f(-x,y) + f(x,-y) + f(-x,-y) \right), \tag{24}
\]

\[
f(x,y) = \frac{1}{2} \left( -s(x,y) + s(-x,y) + s(x,-y) + s(-x,-y) \right). \tag{25}
\]

**Proof.** If the normalized statistical sum consists of the terms

\[
c_{jk}x^jy^k = c_{jk}(u^{\alpha_1}v^{\beta_1})^j(u^{\alpha_2}v^{\beta_2})^k,
\]

then the determinant consists of the same terms multiplied by

\[
(-1)^{\text{number of contours}} = (-1)^{\text{g.c.d.}(j,k)} = (-1)^{j+k+j+k}
\]

(here Remark [1] and Lemmas [1] and [3] are used). This means that the signs of all the terms must be changed except where both numbers $j$ and $k$ are even. This is exactly what the formulae (24, 25) do. The lemma is proved.
5 The discrete analog of Lax pair and a generalization of the dynamical system

Now let us return from the reduction of Section 3 to general matrices $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Let us consider the evolution described in Section 1 from another viewpoint. Denote

$$(L^{-1})^t = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}.$$  

This means that

$$\begin{pmatrix} \tilde{A} & \tilde{C} \\ \tilde{B} & \tilde{D} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1. \quad (26)$$

It follows from the equality (26) that

$$\tilde{A}A + \tilde{C}C = \tilde{B}B + \tilde{D}D, \quad (27)$$

$$\tilde{A}B + \tilde{C}D = 0,$$

$$\tilde{B}A + \tilde{D}C = 0.$$  

These three equations are equivalent to the fact that the following equality holds for any complex $u$:

$$-(\tilde{A} - u\tilde{B})^{-1}(\tilde{C} - u\tilde{D}) = (uA + B)(uC + D)^{-1}. \quad (28)$$

Vice versa, from (28) follows

$$\tilde{L}^tL = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix},$$  

$F$ being equal to both sides of (27), i.e.

$$\tilde{L} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}(L^{-1})^t.$$  

It is clear that with any choice of $F$ the matrix $\tilde{L}$ belongs to the same equivalence class. The formula (28) defines the same evolution in the space of these classes as it was in Section 3 with the agreement that the operators without a tilde correspond to the moment of time $\tau$, while those with a tilde correspond to the moment $\tau + 1$.

The formula (28) suggests the following generalization. Let, from now on, $A(u)$ and $B(u)$ be matrices depending polynomially on $u$:

$$A(u) = A_0 + A_1 + \ldots + A_m u^m,$$  

$$B(u) = B_0 + B_1 + \ldots + B_m u^m.$$  

13
We will look for matrices $\tilde{A}(u), \tilde{B}(u)$—the matrix polynomials of the same degrees $m_A$ and $m_B$ in $u$—that satisfy, for any $u$, the equation

$$\tilde{B}(u)^{-1} \tilde{A}(u) = A(u)B(u)^{-1}. \quad (31)$$

The relation (31) provides what is called a discrete analog of the Lax $L, A$-pair, which means here that the operators $\tilde{A}(u)\tilde{B}(u)^{-1}$ and $A(u)B(u)^{-1}$ (which are playing the role of $L$ of the pair) are “isospectral deformations” of one another:

$$\tilde{A}(u)\tilde{B}(u)^{-1} = \tilde{A}(u)A(u)B(u)^{-1}\tilde{A}(u)^{-1}. \quad (32)$$

Let $v$ be an eigenvalue of both sides of (31). Let $Y(u, v)$ be the corresponding eigenvector normalized, as in Section 2, so that its last coordinate equals unity, and let $X(u, v)$ be the vector proportional to $B(u)^{-1}Y(u, v)$ and normalized in the same way. One can verify that this may be described by the following formula ($h(u, v)$ being a scalar factor):

$$\begin{pmatrix} A(u) \\ B(u) \end{pmatrix} X(u, v) = h(u, v) \begin{pmatrix} v \\ 1 \end{pmatrix} \otimes Y(u, v), \quad (32)$$

which is in obvious analogy to (31). The divisor equivalence is

$$mD_u + DX \sim D_v + DY, \quad (33)$$

$D_u$ and $D_v$ being pole divisors of the functions $u$ and $v$, $m = \max(m_A, m_B)$.

For a given $u$, the eigenvalues $v$ come from the equation

$$P(u, v) = \det (A(u) - vB(u)) = 0. \quad (34)$$

It defines an algebraic curve $\Gamma$—“generalized vacuum curve”. Let us calculate the genus $g$ of the curve $\Gamma$. First, we need to know the number of branch points of the projection

$$(u, v) \longrightarrow u \quad (34)$$

of the curve $\Gamma$ onto the complex plane.

Consider $P(u, v)$ as a polynomial in $v$:

$$P(u, v) = a_0(u) + a_1(u) + \ldots + a_n(u)v^n. \quad (35)$$

One can verify that $a_j(u)$ has a degree

$$\deg a_j(u) = (n - j)m_A + j_B. \quad (36)$$

From this one can deduce that the discriminant of $P(u, v)$ considered as a polynomial in $v$ is a polynomial of degree

$$b = (m_A + m_B)n(n - 1)$$
in $u$. The mapping $\{34\}$ being $n$-sheeted and the number of branch points equalling $b$, one obtains from the Riemann—Hurwitz formula that

$$g = (n - 1) \left( \frac{m_A + m_B}{2} n - 1 \right).$$

(37)

So, the following construction has been described. Given two polynomial matrix functions $A(u)$ and $B(u)$, one considers the meromorphic matrix function $A(u)B(u)^{-1}$ (or else $B(u)^{-1}A(u)$), and from this function the algebro-geometrical objects arise: the generalized vacuum curve $\Gamma$ and the linear equivalence class of the pole divisor $D_Y$ (or, respectively, $D_X$) of the eigenvectors of the mentioned meromorphic matrix function. Instead of the pair $(A(u), B(u))$, it is sufficient to indicate its equivalence class with respect to gauge transformations

$$A(u) \to GA(u)H, \quad B(u) \to GB(u)H;$$

(38)

instead of the function $A(u)B(u)^{-1}$, its equivalence class with respect to transformations

$$A(u)B(u)^{-1} \to GA(u)B(u)^{-1}G^{-1}$$

will suffice. Then it turns out that the correspondence between such equivalence classes (either of the pairs $(A(u), B(u))$ or the functions $A(u)B(u)^{-1}$) and the abovementioned algebro-geometrical objects is a birational isomorphism, the divisors $D_X$ and $D_Y$ being of degree $g + n - 1$, as in Section 2.

The easiest way to show this is to start from a given curve $\Gamma$ defined by the equation

$$P(u, v) = \sum_{j=0}^{n} \sum_{k=0}^{(n-j)m_A+jm_B} a_{jk}v^j u^k = 0$$

(compare with $(35, 36)$) and a divisor $D_X$ in it of degree $g + n - 1$. The number of coefficients $a_{jk}$ minus one common factor equals

$$(n + 1) \left( \frac{m_A + m_B}{2} n + 1 \right) - 1.$$  

(39)

The linear equivalence class of divisor $D_X$ is defined, as is known, by $g$ parameters. Adding up the expressions $(39)$ and $(37)$, one gets the total of

$$(m_A + m_B)n^2 + 1$$  

(40)

parameters.

Then, the gauge equivalence class of the pair $(A(u), B(u))$ is constructed out of relation $(32)$. To give more details, one must at first choose a divisor $D_Y$ satisfying the equivalence $(33)$. Then the poles and zeros of the function $h(u, v)$ are determined. For $X(u, v)$ and $Y(u, v)$ one must take columns consisting each of $n$ linearly independent meromorphic functions with corresponding pole
divisors. The arbitrariness in these constructions leads exactly to the fact that $A(u)$ and $B(u)$ are determined up to a transformation (38).

The pair $(A(u), B(u))$, up to a scalar common factor, is determined by $(m_A + m_B + 2)n^2 - 1$ parameters (see (29, 30)). In taking the gauge equivalence class, the number of parameters is reduced by $2(n^2 - 1)$. The result is again (41). This means that, indeed, to a generic pair $(A(u), B(u))$ corresponds a divisor $D_X$ of degree $g + n - 1$ and the correspondence

$$(\text{gauge equivalence class of the pair } (A(u), B(u))) \longleftrightarrow (\Gamma, \text{class of } D_X)$$

is a birational isomorphism.

Now let us recall that $Y(u, v)$ was defined as an eigenvector of the operator $A(u)B(u)^{-1}$, while $X(u, v)$, as is easily seen, is an eigenvector of $B(u)^{-1}A(u)$. The relation (31) means that for the pair $(\tilde{A}(u), B(u))$ its vector $\tilde{X}(u, v)$ is nothing else than $Y(u, v)$, i.e. the equivalence holds

$$D_X \sim D_{\tilde{X}} + (mD_u - D_v). \quad (41)$$

Now, assuming that if a quantity without a tilde corresponds to the moment of time $\tau$ then that with a tilde corresponds to $\tau + 1$, one comes to a conclusion that to the adding of unity to the time corresponds a constant shift (41) in the Jacobian of the curve $\Gamma$. Thus, the dynamics of the system in this section, as well as in Section 2, linearizes.

### 6 Discussion

In this paper I study a dynamical system in discrete time, i.e. a mapping and its iterations, acting on finite sets of $n \times n$ matrices. The system appears in several modifications, on which depends the number of matrices as well as the additional conditions that may be imposed on them. The “law of motion” is formulated in a rather simple way, and a large number of “integrals of motion” turn out to exist and be the coefficients of the “vacuum curve”—the object coming from the theory of the quantum Yang—Baxter equation. If the motion in the system is considered up to a “gauge transformation”, the system is integrable in the sense that there exists a birational isomorphism between the “phase space” and the set of pairs (a vacuum curve, an element of its Picard group), so that in the process of “motion” the vacuum curve doesn’t change, while the element of its Picard group depends on the time linearly. Thus, the Cauchy problem is solved through the following scheme: the initial point in the phase space $\rightarrow$ the vacuum curve and the element of its Picard group at the initial moment of time $\rightarrow$ the same at the moment $\tau$ $\rightarrow$ the element of the phase space at the moment $\tau$.

Connections with statistical physics are exposed in Section 4, where a special reduction of the model is considered. Note that there exists one more dynamical
system in 2+1-dimensional discrete space-time that is also connected with statistical physics and seems to be completely integrable. This system is as follows. Consider the inhomogeneous Ising model on the triangle lattice. Imagine this lattice as consisting of triangles of the form $\triangle$ and perform for each of them the “triangle—star” transformation \[7\]:

$$
\triangle \rightarrow \star
$$

Thus, the hexagonal lattice appears, which now may be imagined as made up of its parts of the form $\bigstar$. So, let us perform the “star—triangle” transformation

$$
\bigstar \rightarrow \bigtriangleup
$$

for each of them. One step of the evolution is over. It would be of interest to reveal possible connections of this model with the model on the square lattice from Section 4.

Another interesting problem still unsolved: to describe the evolution of the system in Section 1 in full, not up to gauge equivalence.

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