Minimal length, maximal momentum and thermodynamics of black body radiation

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Abstract

In this paper we study thermodynamics of black body radiation in the presence of quantum gravitational effects through a Generalized Uncertainty Principle (GUP) that admits both a minimal measurable length and a maximal momentum. We focus on quantum gravity induced modification of thermodynamical quantities in this framework. Some important issues such as the generalized Planck distribution, Generalized Wien’s law and Generalized Dulong-Petit law are studied with details.

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1 Introduction

Various approaches to quantum gravity such as String Theory and Doubly Special Relativity predict the existence of a minimal measurable length or a maximum observable momentum. These theories argue that near the Planck scale, the Heisenberg Uncertainty Principle should be replaced by the so called Generalized Uncertainty Principle (GUP) (see [1-5] and references therein). Modification of the standard uncertainty relation of ordinary quantum mechanics by incorporation of quantum gravity effects requires a reformulation of several important physical laws such as thermodynamical laws of black body radiation. The importance of black body radiation lies in the fact that cosmic microwave background radiation (CMB) is shown to have the same spectrum as the black body radiation. Since CMB physics today has obtained a very appreciable place in modern cosmology, any quantum gravitational correction imposed on this issue will provide a better framework to understand the real world. On the other hand, these type of study may provide direct clue to test quantum gravity ideas in the lab. Planck’s early analysis of black body radiation was based on statistical arguments of black-body radiation within a cavity. When we consider quantum gravitational modifications, some important laws that related to the black body radiation should be corrected. In this framework, the spectral energy density of black body radiation gets modified and this modification causes modifications to other thermodynamical quantities. Although these modifications are important only in very high energy scales, but understanding phenomenological aspects of these results will shed light both in the formulation of the ultimate quantum gravity proposal and possible detection of these quantum gravity effects in experiments. Thermodynamics of black body radiation in the presence of minimal measurable length has been studied by some authors [6,7]. However, these studies are not complete since they ignore the fact that a minimal measurable length essentially requires the existence of a maximal momentum encoded in the duality of position-momentum spaces or uncertainty principle. In fact, based on Doubly Special Relativity theories a test particles momentum has upper limit of the order of Planck momentum [8-10]. Existence of a maximal measurable momentum for a test particle modifies the results of the mentioned studies considerably. This is because the existence of a natural cutoff on momentum restricts the number of physically accessible modes considerably. The purpose of this paper is mainly to address this issue. We introduce a GUP that admits both a minimal length and a maximal momentum and we reformulate black body radiation in this framework. Some important issues such as generalization of Planck distribution, equipartition theorem, Stefan-Boltzmann law, Wien’s law and Dulong-Petit law will be studied with details and some important thermodynamical quantities such as the entropy and specific heat of black body radiation are calculated in the presence of quantum gravity effects. We also compare our results with those results that are obtained by ignoring the role of maximal momentum. For simplicity, in which follows we set $\hbar = k_B = c = 1$. 

2
2 Quantum Gravity Effects and Black Body Radiation

2.1 GUP with minimal length and maximal momentum

In the context of the Doubly Special Relativity (DSR) (see [11] for review), one can show that a test particle’s momentum cannot be arbitrarily imprecise. In fact, there is an upper bound for momentum fluctuations [8,9]. As a nontrivial assumption, this may lead to a maximal measurable momentum for a test particle [10]. In this framework, the GUP that predicts both a minimal observable length and a maximal momentum can be written as follows [10]

\[ \Delta x \Delta p \geq \frac{1}{2} \left[ 1 + \left( \frac{\alpha L_p}{\langle p^2 \rangle} + 4\alpha^2 L_p^2 \right)(\Delta p)^2 + 4\alpha^2 L_p^2 \langle p \rangle^2 - 2\alpha L_p \sqrt{\langle p^2 \rangle} \right]. \]  

(1)

Since \( (\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 \), by setting \( \langle p \rangle = 0 \) for simplicity, we find

\[ \Delta x \Delta p \geq \frac{1}{2} \left[ 1 - 2\alpha L_p (\Delta p) + 4\alpha^2 L_p^2 (\Delta p)^2 \right]. \]  

(2)

It is easy to show how this setup leads to a maximal momentum. To show this end, we note that the absolute minimal measurable length in our setup is given by \( \Delta x_{\text{min}}(\langle p \rangle = 0) \equiv \Delta x_0 = \alpha L_p \). Due to duality of position and momentum operators, it is reasonable to assume \( \Delta x_{\text{min}} \propto \Delta p_{\text{max}} \).

Now, saturating the inequality in relation (2), we find

\[ 2(\Delta x \Delta p) = \left( 1 - 2\alpha L_p (\Delta p) + 4\alpha^2 L_p^2 (\Delta p)^2 \right). \]  

(3)

This results in

\[ (\Delta p)^2 - \frac{(2\Delta x + 2\alpha L_p)}{4\alpha^2 L_p^2} (\Delta p) + \frac{1}{4\alpha^2 L_p^2} = 0. \]  

(4)

So, we find

\[ (\Delta p_{\text{max}})^2 - \frac{(2\Delta x_{\text{min}} + 2\alpha L_p)}{4\alpha^2 L_p^2} (\Delta p_{\text{max}}) + \frac{1}{4\alpha^2 L_p^2} = 0. \]  

(5)

Now using the value of \( \Delta x_{\text{min}} \), we find

\[ (\Delta p_{\text{max}})^2 - \frac{1}{\alpha L_p} (\Delta p_{\text{max}}) + \frac{1}{4\alpha^2 L_p^2} = 0. \]  

(6)

The solution of this equation is

\[ \Delta p_{\text{max}} = \frac{1}{2\alpha L_p}. \]  

(7)

So, there is an upper bound on particle’s momentum uncertainty. As a nontrivial assumption, we assume that this maximal uncertainty in particle’s momentum is indeed the maximal measurable momentum. This is of the order of Planck momentum. We note that neglecting a factor of \( \frac{1}{2} \) for simplicity in our forthcoming arguments, the GUP formulated as (2) gives the following generalized commutation relation

\[ [x, p] = i \left( 1 - \alpha L_p p + 2\alpha^2 L_p^2 p^2 \right). \]  

(8)
This is equivalent to set \( p \rightarrow p\left(1 - \alpha L_p p + 2\alpha^2 L_p^2 p^2\right) \). Note that the term \(-\alpha L_p p\) that was absent in previous analysis of the black body radiation [6,7], is related to the existence of maximal momentum and provides the basic difference of our analysis with previous works.

In the presence of a minimal measurable length and maximal particle’s momentum as natural cutoffs, the spectrum of black body radiation should be modified. Because of these modifications, the de Broglie relation is modified too

\[
\lambda \simeq \frac{1}{p}(1 - \alpha L_p p + 2\alpha^2 L_p^2 p^2) \tag{9}
\]

and therefore,

\[
E \simeq \nu(1 - \alpha L_p \nu + 2\alpha^2 L_p^2 \nu^2), \tag{10}
\]

where we have supposed \( \Delta p \simeq p \) and \( \Delta x \simeq \lambda \).

Now we consider photons in a cubic box with the length of \( L \) and volume of \( V = L^3 \). According to the boundary condition, the photons’ wavelengths are equal to \( \frac{1}{\lambda} = \frac{n}{2L} \), where \( n \) is a positive integer. By considering the above conditions, we assume that the de Broglie relation is left unchanged. Therefore the photons have momenta given by

\[
p = \frac{n}{2L}.
\]

So the momentum space is divided into cells of volume \( V_p = \left(\frac{1}{2L}\right)^3 = \frac{1}{8V} \). Now it follows that the number of modes with momentum in the interval \([p, p + dp]\) is given by

\[
g(p)dp = 8\pi V p^2 dp.
\]

Similarly for oscillators in a box the number of modes in an infinitesimal frequency interval\([\nu, \nu + d\nu]\) would be written by the following standard formula

\[
g(\nu)d\nu = 8\pi V \nu^2 d\nu \tag{11}
\]

According to (10), the average energy of each oscillator would be given by

\[
E = \left(\frac{\nu}{e^{\frac{\nu}{T}} - 1}\right)\left[1 - \alpha L_p \nu \left(1 - \frac{\nu}{T} e^{-\nu/T}\right) + 2\alpha^2 L_p^2 \nu^2 \left(1 - \frac{\nu}{T} e^{-\nu/T}\right)\right] \tag{12}
\]

This is the generalization of the equipartition theorem in the presence of minimal length and maximal momentum. We see that in the limit of \( L_p \rightarrow 0 \) we find the ordinary quantum mechanics result.

Now we want to find the modified energy density of the black body radiation at temperature \( T \) and frequency interval \([\nu, \nu + d\nu]\). In general the energy density is given by

\[
u(\nu)d\nu = \frac{\bar{E} g(\nu)d\nu}{V}. \tag{13}
\]
Now according to the modifications induced by the existence of minimal length and maximal momentum, the energy density appears in the following form

\[ u_\nu(T)d\nu = 8\pi\left(\frac{\nu^3d\nu}{e^\frac{\nu}{T} - 1}\right)\left[1 - \alpha L_p\nu\left(1 - \frac{\nu}{T}\right) + 2\alpha^2 L_p^2\nu^2\left(1 - \frac{\nu}{T}\right)^2\right] \tag{14} \]

This is actually the generalized Planck distribution for black body radiation in the presence of the quantum gravity effects encoded in the GUP with minimal length and maximal momentum. In our forthcoming arguments we use this relation as our primary input.

2.2 Energy density

By integrating of Eq. (14) on frequency, we can calculate the energy density of black body. This integral gives

\[
\begin{align*}
 u(T) &= \frac{8}{15} \pi^5 T^4 + \frac{640}{63} \pi^7 T^6 \alpha^2 L_p^2 + 128\pi^2 \alpha L_p m_p^3 L_2(e\frac{m_p}{e}) + 32\pi T \alpha L_p m_p^4 \ln(1 - e\frac{m_p}{e}) \\
 &\quad -384\pi^3 \alpha L_p m_p^2 L_3(e\frac{m_p}{e}) + 768\pi^4 \alpha L_p m_p L_4(e\frac{m_p}{e}) - 80\pi T \alpha^2 L_p^2 m_p^5 \ln(1 - e\frac{m_p}{e}) \\
 &\quad -400\pi^2 \alpha^2 L_p^2 m_p^3 L_2(e\frac{m_p}{e}) + 1600\pi^3 \alpha^2 L_p^2 m_p^3 L_3(e\frac{m_p}{e}) - 4800\pi^4 \alpha^2 L_p^2 m_p^3 L_4(e\frac{m_p}{e}) \\
 &\quad + 9600\pi^5 \alpha^2 L_p^2 m_p L_5(e\frac{m_p}{e}) - 2\pi m_p^4 + 768\pi T^5 \alpha L_p \xi(5) - 768\pi T^5 \alpha L_p L_5(e\frac{m_p}{e}) - 9600\pi T^6 \alpha^2 m_p^2 L_2(e\frac{m_p}{e}) \\
 &\quad + 48\pi T^4 L_4(e\frac{m_p}{e}) + 8\pi T m_p^3 \ln(1 - e\frac{m_p}{e}) + 24\pi T^2 m_p^2 L_2(e\frac{m_p}{e}) \\
 &\quad - 48\pi T^3 m_p L_3(e\frac{m_p}{e}) - \frac{224}{35} \pi m_p^5 \alpha L_p \frac{e\frac{m_p}{e}}{e^\frac{m_p}{e} - 1} + \left(\frac{840}{63} m_p^6 \alpha L_p \pi \alpha^2 m_p^6 L_2\right) \tag{15}
\end{align*}
\]

where \( L_i(z) \) is the Polylogarithm function defined as

\[ L_i(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^i} \]

This equation shows the modified Stefan-Boltzmann law and \( m_p \) is the Planck mass. We can use the expansion of the Polylogarithm function to obtain the following result

\[
\begin{align*}
 u(T) &= \frac{8}{15} \pi^5 T^4 + \frac{640}{63} \pi^7 T^6 \alpha^2 L_p^2 + \pi^2 \alpha L_p m_p^3 (128e\frac{m_p}{e} + 32e\frac{2m_p}{e}) + 32\pi T \alpha L_p m_p^4 \ln(1 - e\frac{m_p}{e}) \\
 &\quad - \pi^3 \alpha L_p m_p^2 (384e\frac{m_p}{e} + 48e\frac{2m_p}{e}) + \pi^4 \alpha L_p m_p (768e\frac{m_p}{e} + 48e\frac{2m_p}{e}) - 80\pi T \alpha^2 L_p^2 m_p^5 \ln(1 - e\frac{m_p}{e}) \\
 &\quad - \pi^2 \alpha^2 L_p^2 m_p^4 (400e\frac{m_p}{e} + 100e\frac{2m_p}{e}) + \pi^3 \alpha^2 L_p^2 m_p^3 (1600e\frac{m_p}{e} + 200e\frac{2m_p}{e}) - \pi^4 \alpha^2 L_p^2 m_p^4 (4800e\frac{m_p}{e} + 300e\frac{2m_p}{e}) + \pi^5 \alpha^2 L_p^2 m_p (9600e\frac{m_p}{e} + 300e\frac{2m_p}{e}) - 2\pi m_p^4 + 768\pi T^5 \alpha L_p \xi(5) - \pi^5 \alpha L_p (768e\frac{m_p}{e} + 24e\frac{2m_p}{e})
\end{align*}
\]
\[ -\pi T^6 \alpha^2 m_p L_p^2 (9600 e^{\frac{m_p}{T}} + 150 e^{\frac{2m_p}{T}}) + \pi T^4 (48 e^{\frac{m_p}{T}} + 3 e^{\frac{2m_p}{T}}) + 8\pi T m_p^3 \ln(1 - e^{\frac{m_p}{T}}) + \pi T^2 m_p^2 (24 e^{\frac{m_p}{T}} + 6 e^{\frac{2m_p}{T}}) - \pi T^3 m_p (48 e^{\frac{m_p}{T}} + 3 e^{\frac{2m_p}{T}}) - \frac{224}{35} \pi m_p^5 L_p e^{\frac{m_p}{T}} \frac{e^{\frac{m_p}{T}}}{e^{\frac{m_p}{T}} - 1} + \frac{840}{63} e^{\frac{m_p}{T}} + \frac{168}{63} \pi \alpha^2 m_p^6 L_p^2 \] (16)

It should be mentioned that we considered up to second order of correction terms in each expansion.

### 2.3 Modified Wien’s law

In ordinary quantum mechanics we have the following relation between the wavelength at which the energy density distribution maximizes and corresponding temperature of black body radiation

\[ \lambda_{\text{max}} = \frac{C}{T}, \] (17)

where C is Wien’s constant. In quantum gravity era this relation should be modified. Since \( u_{\lambda}(T) = \nu^2 u_{\nu}(T) \) and \( \nu = \frac{1}{\lambda} \), we rewrite Eq. (14) in terms of wavelength to find

\[ u_{\lambda}(T) = \frac{8\pi}{\lambda^5 (e^{\frac{1}{\lambda T}} - 1)} - \frac{8\pi \alpha L_p}{\lambda^6 (e^{\frac{1}{\lambda T}} - 1)} \left( 1 - \frac{1}{\lambda T} \frac{1}{1 - e^{\frac{1}{\lambda T}}} \right) + \frac{16\pi \alpha^2 L_p^2}{\lambda^7 (e^{\frac{1}{\lambda T}} - 1)} \left( 1 - \frac{1}{\lambda T} \frac{1}{1 - e^{\frac{1}{\lambda T}}} \right) \] (18)

Now for finding modified Wien’s law we should calculate the extremum value of Eq. (18). In our calculations we use the approximation \( e^{\frac{1}{\lambda T}} = 1 + \frac{1}{\lambda T} \). So we have the following relation as the generalized Wien’s law in our GUP framework

\[ \lambda = \frac{1}{12T} + \frac{\left( -18\alpha L_p T + 432\alpha^2 L_p^2 T^2 + 1 + 6\sqrt{3} \alpha L_p T \sqrt{-128\alpha L_p T + 7 + 1728\alpha^2 L_p^2 T^2} \right)^{\frac{1}{2}}}{12T} \]

\[ -\frac{12\alpha L_p T - 1}{12T \left( -18\alpha L_p T + 432\alpha^2 L_p^2 T^2 + 1 + 6\sqrt{3} \alpha L_p T \sqrt{-128\alpha L_p T + 7 + 1728\alpha^2 L_p^2 T^2} \right)^{\frac{1}{2}}} \] (19)

The first term on the right hand side is the standard Wien’s law. The other terms are corrections imposed from quantum gravity considerations. We note that there are also two imaginary values for \( \lambda \) that are not acceptable on physical ground. According to this formula we see that when we consider minimal length and maximal momentum, the standard Wien’s law attains some correction terms that are temperature dependent. These corrections are important only in quantum gravity regime. It is important to note that due to quantum gravitational effects, the \( \lambda_{\text{max}} \) will attains a small shift and this shift itself is temperature dependent. This small wavelength shift can be attributed to the nature of spacetime manifold at the Planck scale [5].
3 Some other thermodynamical properties

Now we can use modified energy density (Eq.(16)) to derive specific heat capacity and the entropy of black body radiation. So we need the total energy of the system which is defined as

\[ U(T) = Vu(T) \] (20)

and is given by the following relation

\[
U(T) = \frac{8}{15}V\pi^5T^4 + \frac{640}{63}V\pi^7T^6\alpha^2L_p^2 + \pi T^2V\alpha L_p m_p^3(128e^{\frac{mp}{T}} + 32e^{\frac{2mp}{T}}) + 32V\pi T\alpha L_p m_p^4\ln(1 - e^{\frac{mp}{T}}) \\
-\pi T^3\alpha L_p m_p^2(384e^{\frac{mp}{T}} + 48e^{\frac{2mp}{T}}) + \pi T^4V\alpha L_p m_p(768e^{\frac{mp}{T}} + 48e^{\frac{2mp}{T}}) - 80\pi VT\alpha^2L_p^2m_p^5\ln(1 - e^{\frac{mp}{T}}) \\
-\pi T^2V^2L_p^2m_p^4(400e^{\frac{mp}{T}} + 100e^{\frac{2mp}{T}}) + \pi T^3V^2L_p^2m_p^3(1600e^{\frac{mp}{T}} + 200e^{\frac{2mp}{T}}) - \pi VT^4\alpha^2L_p^2m_p^2(4800e^{\frac{mp}{T}} + 300e^{\frac{2mp}{T}}) + \pi TV^2L_p^2m_p(9600e^{\frac{mp}{T}} + 300e^{\frac{2mp}{T}}) - 2\pi m_p^4 + 768\pi VT^5\alpha L_p(768e^{\frac{mp}{T}} + 24e^{\frac{2mp}{T}}) - \pi T^6V^2L_p^2m_p^2(9600e^{\frac{mp}{T}} + 150e^{\frac{2mp}{T}}) + \pi TV^4(48e^{\frac{mp}{T}} + 3e^{\frac{2mp}{T}}) + 8\pi VTm_p^3\ln(1 - e^{\frac{mp}{T}}) + \pi VT^2m_p^2(24e^{\frac{mp}{T}} + 6e^{\frac{2mp}{T}}) - \pi VT^3m_p(48e^{\frac{mp}{T}} + 6e^{\frac{2mp}{T}}) - \frac{224}{39}\pi Vm_p^5\alpha L_p\frac{e^{\frac{mp}{T}}}{e^{\frac{mp}{T}} - 1} + \left(\frac{840}{63}e^{\frac{mp}{T}} + \frac{168}{63}\right)\frac{48e^{\frac{mp}{T}} + 2mp}{T}\right)
\]

The specific heat of the material is defined as

\[ C_V = \left(\frac{\partial U}{\partial T}\right)_{V = \text{cte}} = T\left(\frac{\partial S}{\partial T}\right)_{V = \text{cte}} \] (22)

So, we have

\[
\frac{\partial S}{\partial T} = \frac{C_V}{T} \] (23)

Therefore the specific heat capacity of black body radiation in the presence of a minimal length and a maximal momentum is given by

\[
C_V = \frac{32}{15}\pi^5VT^3 + \frac{1280}{21}\pi^7VT^5\alpha^2L_p^2 + \frac{32\pi Vm_p^5\alpha L_p e^{\frac{mp}{T}}}{T(1 - e^{\frac{mp}{T}})} - \frac{80V\pi L_p^2\alpha^2m_p^6e^{\frac{mp}{T}}}{T(1 - e^{\frac{mp}{T}})} \\
+ 2\pi TV\alpha L_p m_p^3(128e^{\frac{mp}{T}} + 32e^{\frac{2mp}{T}}) - 3\pi T^2V\alpha L_p m_p^2(384e^{\frac{mp}{T}} + 48e^{\frac{2mp}{T}}) \\
+ 4\pi T^3V\alpha L_p m_p(768e^{\frac{mp}{T}} + 48e^{\frac{2mp}{T}}) - 2\pi TV\alpha^2L_p^2m_p^4(400e^{\frac{mp}{T}} + 100e^{\frac{2mp}{T}}) \\
+ 3\pi T^2\alpha^2L_p^2m_p^3(1600e^{\frac{mp}{T}} + 200e^{\frac{2mp}{T}}) - 4\pi T^3\alpha^2L_p^2m_p^2(4800e^{\frac{mp}{T}} + 300e^{\frac{2mp}{T}})
\]
Now we can use Eq. (23) to obtain entropy of black body radiation. We have

\[
\begin{align*}
+5\pi T^4 V \alpha^2 L_p^2 m_p (9600 e^{mp \frac{mp}{T}} + 300 e^{2mp \frac{mp}{T}}) - 6\pi T^5 V \alpha^2 L_p^2 m_p (9600 e^{mp \frac{mp}{T}} + 150 e^{2mp \frac{mp}{T}}) \\
+ \frac{8\pi V m_p^4 e^{mp \frac{mp}{T}}}{T^2(1 - e^{mp \frac{mp}{T}})} + \frac{32\pi V m_p^6 \alpha L_p e^{mp \frac{mp}{T}}}{5T^2(e^{mp \frac{mp}{T}} - 1)} - \frac{32\pi V m_p^6 \alpha L_p e^{2mp \frac{mp}{T}}}{5T^2(e^{mp \frac{mp}{T}} - 1)^2} + 3840\pi T^4 V \alpha L_p \zeta(5) \\
+ \pi T^3 V \alpha L_p m_p (768 e^{mp \frac{mp}{T}} + 48 e^{2mp \frac{mp}{T}}) + 2\pi TV m_p^2 (24 e^{mp \frac{mp}{T}} + 6 e^{2mp \frac{mp}{T}}) - 3\pi VT^2 m_p (48 e^{mp \frac{mp}{T}} \\
+ 6e^{2mp \frac{mp}{T}}) + 32\pi \alpha VL_p m_p^4 Ln(1 - e^{mp \frac{mp}{T}}) + \pi \alpha VL_p m_p^4 (-128 e^{mp \frac{mp}{T}} - 64 e^{2mp \frac{mp}{T}}) \\
+ \pi TV \alpha L_p m_p^3 (384 e^{mp \frac{mp}{T}} + 96 e^{2mp \frac{mp}{T}}) + \pi T^2 \alpha L_p m_p^2 (-768 e^{mp \frac{mp}{T}} - 96 e^{2mp \frac{mp}{T}}) \\
- 80\pi V \alpha^2 L_p^2 m_p^5 Ln(1 - e^{mp \frac{mp}{T}}) + \pi \alpha^2 L_p^2 m_p^5 (400 e^{mp \frac{mp}{T}} + 200 e^{2mp \frac{mp}{T}}) - \pi TV \alpha^2 L_p^2 m_p^4 (1600 e^{mp \frac{mp}{T}} + 400 e^{2mp \frac{mp}{T}}) + \pi VT^2 \alpha^2 L_p^2 m_p^4 (4800 e^{mp \frac{mp}{T}} + 600 e^{2mp \frac{mp}{T}}) - \pi VT^3 \alpha^2 L_p^2 m_p^4 (9600 e^{mp \frac{mp}{T}} \\
+ 600 e^{2mp \frac{mp}{T}}) - 5\pi VT^4 \alpha L_p (768 e^{mp \frac{mp}{T}} + 24 e^{2mp \frac{mp}{T}}) + \pi VT^4 \alpha L_p^2 m_p^2 (9600 e^{mp \frac{mp}{T}} + 300 e^{2mp \frac{mp}{T}}) \\
- \frac{40m_p^7 V e^{mp \frac{mp}{T}} \alpha^2 L_p^2}{3T^2} - \pi V m_p^2 T^2 (48 e^{mp \frac{mp}{T}} + 6 e^{2mp \frac{mp}{T}}) + 4\pi VT^3 (48 e^{mp \frac{mp}{T}} + 3 e^{2mp \frac{mp}{T}}) \\
+ 8\pi V m_p^3 Ln(1 - e^{mp \frac{mp}{T}}) - \pi V m_p^3 (24 e^{mp \frac{mp}{T}} + 12 e^{2mp \frac{mp}{T}}) + \pi V m_p^2 T (48 e^{mp \frac{mp}{T}} + 12 e^{2mp \frac{mp}{T}})
\end{align*}
\]

\[(24)\]

\[
C_V = \frac{32}{15} \pi^5 V T^2 + \frac{1280}{21} \pi^7 V T^4 \alpha^2 L_p^2 + \frac{32\pi V m_p^5 \alpha L_p e^{mp \frac{mp}{T}}}{T^2(1 - e^{mp \frac{mp}{T}})} - \frac{80\pi V L_p^2 \alpha^2 m_p^6 e^{mp \frac{mp}{T}}}{T^2(1 - e^{mp \frac{mp}{T}})}
\]

Now we can use Eq. (23) to obtain entropy of black body radiation. We have
Now by integrating Eq. (25) we can find the entropy of black body radiation as follows

\[
S = \frac{32}{45} \pi^5 V T^3 + \frac{256}{21} \pi^7 V \alpha^2 L_p^2 T^5 + 960 \pi V \alpha L_p T^4 \zeta(5) + 64 \pi V \alpha L_p m_p^3 \left[ 4 m_p Ei(1, -\frac{m_p}{T}) + 4 e \frac{m_p}{T} T_2^2 + 2 m_p Ei(1, -\frac{2 m_p}{T}) + e \frac{2 m_p}{T} T_2^2 + 2 m_p Ei(1, -\frac{3 m_p}{T}) + e \frac{3 m_p}{T} T_2^2 + 8 m_p T e \frac{m_p}{T} \right] + 4 T^4 m_p^2 Ei(1, -\frac{2 m_p}{T}) + e \frac{2 m_p}{T} T_2^2 + 8 m_p T e \frac{m_p}{T} \]

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Now by integrating Eq. (25) we can find the entropy of black body radiation as follows

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S = \frac{32}{45} \pi^5 V T^3 + \frac{256}{21} \pi^7 V \alpha^2 L_p^2 T^5 + 960 \pi V \alpha L_p T^4 \zeta(5) + 64 \pi V \alpha L_p m_p^3 \left[ 4 m_p Ei(1, -\frac{m_p}{T}) + 4 e \frac{m_p}{T} T_2^2 + 2 m_p Ei(1, -\frac{2 m_p}{T}) + e \frac{2 m_p}{T} T_2^2 + 8 m_p T e \frac{m_p}{T} \right] + 4 T^4 m_p^2 Ei(1, -\frac{2 m_p}{T}) + e \frac{2 m_p}{T} T_2^2 + 8 m_p T e \frac{m_p}{T} \]

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\[-64\pi V\alpha L_p m_p^4 \left[ 2Ei(1, -\frac{m_p}{T}) + Ei(1, -\frac{2m_p}{T}) \right] + 96\pi V\alpha L_p m_p^3 \left[ 4m_p Ei(1, -\frac{m_p}{T}) + 4 e^{m_p T} \right] + 2Ei(1, -\frac{2m_p}{T}) m_p + T e^{2m_p} - 96\pi V\alpha L_p m_p^2 \left[ 8m_p Ei(1, -\frac{m_p}{T}) + 8 e^{m_p T} + 2Ei(1, -\frac{2m_p}{T}) m_p + T e^{2m_p} \right] + 200\pi V\alpha^2 L_p^2 m_p \left[ 2Ei(1, -\frac{m_p}{T}) + Ei(1, -\frac{2m_p}{T}) \right] - 400\pi V\alpha^2 L_p^2 m_p^2 \left[ 4m_p Ei(1, -\frac{m_p}{T}) + 4 e^{m_p T} + 2Ei(1, -\frac{2m_p}{T}) m_p + T e^{2m_p} \right] + 300\pi V\alpha^2 L_p^2 m_p^2 \left[ 8m_p Ei(1, -\frac{m_p}{T}) + 8 e^{m_p T} + 8 e^{m_p T} m_p + 4 Ei(1, -\frac{2m_p}{T}) m_p^2 + T^2 e^{2m_p} + 2T e^{2m_p} m_p \right] - 200\pi V\alpha^2 L_p^2 m_p^2 \left[ 8m_p Ei(1, -\frac{m_p}{T}) + 8 e^{m_p T} m_p + 8 e^{m_p T} T^2 m_p + 16 T^3 e^{m_p T} + 4 Ei(1, -\frac{2m_p}{T}) m_p^3 + 2T e^{2m_p} m_p + T^2 e^{2m_p} + T^3 e^{2m_p} m_p + 3 T^4 e^{2m_p} \right] + 25\pi V\alpha^2 L_p^2 m_p^2 \left[ 16 Ei(1, -\frac{m_p}{T}) m_p^4 + 16 e^{m_p T} m_p^3 + 16 e^{m_p T} T^2 m_p^2 + 32 e^{m_p T} T^2 m_p + 96 e^{m_p T} T^4 \right] + 8 Ei(1, -\frac{2m_p}{T}) m_p^4 + 4 T e^{2m_p} m_p^3 + 2 T^2 e^{2m_p} m_p^2 + 2 T^3 e^{2m_p} m_p + 3 T^4 e^{2m_p} \right] + \frac{40}{3} m_p^6 \pi V^2 L_p e^{m_p T} \left[ -3 V m_p \left[ 8 Ei(1, -\frac{m_p}{T}) m_p^2 + 8 e^{m_p T} T^2 + 8 e^{m_p T} T m_p + 4 Ei(1, -\frac{2m_p}{T}) m_p^2 + T^2 e^{2m_p} + 2 T e^{2m_p} m_p \right] + 4 \pi \left[ 8 Ei(1, -\frac{m_p}{T}) m_p^3 + 8 e^{m_p T} T m_p^2 + 8 e^{m_p T} T m_p + 16 e^{m_p T} T^3 + 4 Ei(1, -\frac{2m_p}{T}) m_p^3 + 2 T e^{2m_p} m_p^2 + T^2 e^{2m_p} m_p + T^3 e^{2m_p} \right] - 12 m_p^3 \pi V \left[ 2 Ei(1, -\frac{m_p}{T}) + Ei(1, -\frac{2m_p}{T}) \right] + 16 m_p^2 \pi V \left[ 4 Ei(1, -\frac{m_p}{T}) m_p + 4 T e^{m_p T} + 2 Ei(1, -\frac{2m_p}{T}) + T e^{2m_p} \right] + \ldots \right]

(26)

Where \( Ei(a, z) \) is the exponential integral defined as

\[
Ei(a, z) = z^{a-1} \Gamma(1 - a, z)
\]

We know from [5,6] that by considering only a minimal length GUP, the corrections of entropy and specific heat capacity of black body radiation contain only the odd power of \( T \). But here with a GUP that admits both minimal length and maximal momentum, we see that both odd and even powers of \( T \) are present in entropy and specific heat relations. In the standard limit we recover the results of ordinary quantum mechanics.

Figures 1, 2 and 3 show the variation of energy density, entropy and specific heat capacity of black body radiation versus temperature. The departure of quantum gravity results from the standard results are enhanced in the high temperature limit. We note that energy density, entropy and specific heat capacity of black body radiation in the presence of the quantum gravity effects encoded in GUP (2) are generally larger than corresponding quantities in ordinary quantum mechanics. So, quantum gravity corrections on the black body spectrum are generally temperature-dependent and increase the thermodynamical quantities relative to their ordinary quantum mechanical values.
Figure 1: Energy density of black body radiation versus its temperature.

Figure 2: Entropy of black body radiation versus its temperature.
4 Modified Dulong-Petit law

4.1 Standard framework

An statement of the Dulong-Petit law in modern term is that regardless of the nature of the substance or crystal, the specific heat capacity $C$ of a solid substance (measured in Joule per Kelvin per Kilogram) is equal to $\frac{3R}{M}$, where $R$ is the gas constant (measured in Joule per Kelvin per Mole) and $M$ is the molar mass (measured in Kilogram per Mole). Thus the heat capacity per Mole of many solids is $3R$. A system of vibrations in a crystalline solid can be modeled by considering harmonic oscillator potentials along each degree of freedom. Then the free energy of system can be written as

$$F = N\varepsilon_0 + K_\beta T \sum_\alpha \log \left(1 - e^{-\frac{\hbar \omega_\alpha}{k_\beta T}}\right), \quad (27)$$

where the index $\alpha$ sums over all the degrees of freedom. We consider the case where $K_\beta T \gg \hbar \omega_\alpha$. So we have

$$1 - e^{-\frac{\hbar \omega_\alpha}{k_\beta T}} \approx \frac{\hbar \omega_\alpha}{k_\beta T},$$

and therefore

$$F = N\varepsilon_0 + K_\beta T \sum_\alpha \log \left(\frac{\hbar \omega_\alpha}{k_\beta T}\right). \quad (28)$$

Now we define geometric mean frequency by

$$\log \bar{\omega} = \frac{1}{M} \sum_\alpha \log \omega_\alpha.$$
where M measures the total number of degrees of freedom. Thus we have

\[ F = N \varepsilon_0 - M K_\beta T \log k_\beta T + M k_\beta T \log \hbar \omega \]  

(29)

By using the definition

\[ U = F - T \left( \frac{\partial F}{\partial T} \right)_V \]  

(30)

we have

\[ U = N \varepsilon_0 + M K_\beta T \]

This gives specific heat capacity as

\[ C_V = \left( \frac{\partial U}{\partial T} \right)_V = M K_\beta \]  

(31)

which is independent of the temperature .

### 4.2 GUP framework

Now we want to rewrite the above equations in the presence of minimal length and maximal momentum effects encoded in GUP (2). In this situations

\[ F = N \varepsilon_0 + K_\beta T \sum \log \left( 1 - e^{\frac{E(1 - \beta E + 2 \beta E^2)}{K_\beta T}} \right) \]  

(32)

and we have

\[ K_\beta T \gg E(1 - \beta E + 2 \beta E^2). \]

So we can use the approximation

\[ 1 - e^{\frac{E(1 - \beta E + 2 \beta E^2)}{K_\beta T}} \approx \frac{E(1 - \beta E + 2 \beta E^2)}{K_\beta T} \]

Therefore

\[ F = N \varepsilon_0 + K_\beta T \sum \log \left( \frac{E(1 - \beta E + 2 \beta E^2)}{K_\beta T} \right) \]  

(33)

and

\[ \log \bar{E} = \frac{1}{M} \sum \log E(1 - \beta E + 2 \beta E^2) \]  

(34)

Now by using of Eq. (12), we have find

\[ F = N \varepsilon_0 + M K_\beta T \log \left( \frac{\nu}{e^\frac{\nu}{\hbar} - 1} \left( 1 - \alpha L_\nu \nu \left( 1 - \frac{\nu}{\hbar} \right) \right) \right) - M K_\beta T \log K_\beta T \]  

(35)
To calculate specific heat capacity we should calculate total energy from Eq.(30). We find

$$U = N\varepsilon_0 + MK_\beta T - \frac{MK_\beta T^2}{e^\frac{T}{\tau} - 1} \left[ 1 - \alpha L_p \nu \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) + 2\alpha^2 L_p^2 \nu^2 \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) \right] \times $$

$$\left[ - \frac{\alpha L_p \nu}{T^2(1-e^{-\frac{T}{\tau}})} - \frac{\nu^2 e^{-\frac{T}{\tau}}}{T^3(1-e^{-\frac{T}{\tau}})^2} \right] + 2\alpha^2 L_p^2 \nu^2 \left[ - \frac{\nu}{T^2(1-e^{-\frac{T}{\tau}})} - \frac{\nu^2 e^{-\frac{T}{\tau}}}{T^3(1-e^{-\frac{T}{\tau}})^2} \right] $$

$$MK_\beta \nu^2 \left[ 1 - \alpha L_p \nu \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) + 2\alpha^2 L_p^2 \nu^2 \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) \right] e^{-\frac{T}{\tau}} \frac{e^\frac{T}{\tau} - 1}{e^\frac{T}{\tau} - 1} \right] \right] \right]$$

(36)

Now after finding $U$, we can calculate the specific heat capacity to find

$$C_V = MK_\beta - T \left[ \frac{1}{\nu} \left( 1 - \alpha L_p \nu \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) + 2\alpha^2 L_p^2 \nu^2 \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) \right) \right] \frac{MK_\beta}{e^\frac{T}{\tau} - 1} - \left[ \nu \left[ - \alpha L_p \nu \right] \right]$$

$$- \frac{2\nu}{T^3(1-e^{-\frac{T}{\tau}})} + \frac{4\nu^2 e^{-\frac{T}{\tau}}}{T^4(1-e^{-\frac{T}{\tau}})^2} - \frac{\nu^3 e^{-\frac{T}{\tau}}}{T^5(1-e^{-\frac{T}{\tau}})^2} - \frac{2\nu^3(e^{-\frac{T}{\tau}})^2}{T^5(1-e^{-\frac{T}{\tau}})^3} + 2\alpha^2 L_p^2 \nu^2 \left[ - \frac{2\nu}{T^3(1-e^{-\frac{T}{\tau}})} + \frac{4\nu^2 e^{-\frac{T}{\tau}}}{T^4(1-e^{-\frac{T}{\tau}})^2} - \frac{\nu^3 e^{-\frac{T}{\tau}}}{T^5(1-e^{-\frac{T}{\tau}})^2} - \frac{2\nu^3(e^{-\frac{T}{\tau}})^2}{T^5(1-e^{-\frac{T}{\tau}})^3} \right]$$

$$\left[ \frac{2\nu^2}{T^2(1-e^{-\frac{T}{\tau}})} - \frac{\nu^2 e^{-\frac{T}{\tau}}}{T^3(1-e^{-\frac{T}{\tau}})^2} \right] \right] + \frac{2\nu^2}{(e^\frac{T}{\tau} - 1)^2 T^2}$$

$$\left[ 2\alpha^2 L_p^2 \nu^2 \left[ \frac{\nu}{T^2(1-e^{-\frac{T}{\tau}})} - \frac{\nu^2 e^{-\frac{T}{\tau}}}{T^3(1-e^{-\frac{T}{\tau}})^2} \right] \right] (e^\frac{T}{\tau})$$

$$\nu^3 \left( 1 - \alpha L_p \nu \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) + 2\alpha^2 L_p^2 \nu^2 \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) \right) e^{-\frac{T}{\tau}} \frac{e^\frac{T}{\tau} - 1}{e^\frac{T}{\tau} - 1} \right] \right]$$

$$\frac{2\nu^3 \left( 1 - \alpha L_p \nu \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) + 2\alpha^2 L_p^2 \nu^2 \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) \right) \left( e^\frac{T}{\tau} \right)^2}{(e^\frac{T}{\tau} - 1)^2 T^4}$$

$$2\nu^2 \left( 1 - \alpha L_p \nu \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) + 2\alpha^2 L_p^2 \nu^2 \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) \right) e^{-\frac{T}{\tau}} \frac{e^\frac{T}{\tau} - 1}{e^\frac{T}{\tau} - 1} \right] \right]$$

$$\frac{2\nu^3 \left( 1 - \alpha L_p \nu \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) + 2\alpha^2 L_p^2 \nu^2 \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) \right) \left( e^\frac{T}{\tau} \right)^2}{(e^\frac{T}{\tau} - 1)^3 T^4}$$

$$2\nu^2 \left( 1 - \alpha L_p \nu \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) + 2\alpha^2 L_p^2 \nu^2 \left( 1 - \frac{\nu}{T(1-e^{-\frac{T}{\tau}})} \right) \right) e^{-\frac{T}{\tau}} \frac{e^\frac{T}{\tau} - 1}{e^\frac{T}{\tau} - 1} \right] \right]$$

14
In the absence of quantum gravity this relation reduces to the standard Dulong-Petit law. Note that the GUP-corrected terms are temperature-dependent as a generic feature of quantum gravity effects.
5 Summary and Conclusion

In this paper we have studied the effects of minimal length and maximal momentum, as natural cutoffs encoded in a generalized uncertainty principle, on the thermodynamics of black body radiation. The importance of the black body radiation lies in the fact that cosmic microwave background radiation is shown to have the same spectrum as the black body radiation. We have found the generalization of equipartition theorem, the generalized Planck distribution, Modified Wien's law and the modified Stefan- Boltzmann law. In the next step we have found some thermodynamical properties of black body in the presence of mispecific heat capacity and entropy. And finally we have calculated the modified Dulong-Petit law in this framework. We have shown that quantum gravity corrections are generally temperature-dependent. This is a generic feature and has its origin probably on the very nature of space-time at quantum gravity scales.

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