Josephson current transport through a Quantum Dot in an Aharonov-Bohm Ring

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The Josephson current through an Aharonov-Bohm (AB) interferometer, in which a quantum dot (QD) is situated on one arm and a magnetic flux $\Phi$ threads through the ring, has been investigated. With the existence of the magnetic flux, the relation of the Josephson current and the superconductor phase is complex, and the system can be adjusted to $\pi$ junction by either modulating the magnetic flux or the QD’s energy level $\varepsilon_d$. Due to the electron-hole symmetry, the Josephson current $I$ has the property $I(\varepsilon_d, \Phi) = I(-\varepsilon_d, \Phi + \pi)$. The Josephson current exhibits a jump when a pair of Andreev bound states aligns with the Fermi energy. The condition for the current jump is given. In particular, we find that the position of the current jump and the position of the maximum value of the critical current $I_c$ are identical. Due to the interference between the two paths, the critical current $I_c$ versus the QD’s level $\varepsilon_d$ shows a typical Fano shape, which is similar to the Fano effect in the corresponding normal device. But they also show some differences. For example, the critical current never reaches zero for any parameters, while the current in the normal device can reach zero at the destruction point.

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I. INTRODUCTION

Mesoscopic electron transport through an Aharonov-Bohm (AB) interferometer has attracted considerable attention recently because of its applications in nanotechnology. It reveals information about intrinsic quantum states by detecting interference of electrons in different paths. Interference between a continuum energy spectrum and a discrete energy state gives transmission probability $T(E)$ asymmetric line shape of typical Fano resonance.\textsuperscript{1} Proposed first by Fano, it has been broadly studied and observed in recent experiments.\textsuperscript{2,3} By setting a quantum dot (QD) in one arm of the AB interferometer, the Fano resonance and Kondo-Fano resonance are found.\textsuperscript{3} Interference between direct transmission and QD makes the transport phase through QD observable.\textsuperscript{2} In addition, the extra phase $\Phi$ due to either the magnetic flux or the spin-orbital interaction can modulate the interference.\textsuperscript{2,3} The transmission probability $T(E)$ shows periodical function of the extra phase. In such an AB-Fano system, the $T(E)$ versus QD level shows a Breit-Wigner resonance when the phase $\Phi$ is $\pi/2$ and a typical Fano resonance when the phase is $0$ or $\pi$. The $T(E)$ also shows a Breit-Wigner resonance when the direct tunneling is broken and a Fano type when direct tunneling is increased. Both the phase parameter and direct tunneling can be included in a Fano parameter $g$ with the transmission probability $T(E) \propto \left(\frac{1}{\Gamma\sqrt{\cos^2 \theta + \cos^2 \pi}}\right)^2$, in which $\epsilon = \frac{E - \varepsilon_d}{\Gamma}$, and $E$ is the energy of the incident electron, $\varepsilon_d$ is the QD’s level and $\Gamma$ is the coupling between QD and the lead. The Fano parameter $g$, generally a complex number, decides the line shape of the resonance.

Previous work on the AB-Fano interferometer focused mainly on the normal device, in which the two external leads are normal. By attaching the interferometer to two superconductor leads instead of two normal leads, the Cooper pair transport and Andreev tunneling occur, then Josephson current emerges even at the zero bias. The purpose of this paper is to study how the Josephson current is affected by the interference of the two paths, and whether the Josephson current also has the Fano characters as in normal systems. Recently, the Josephson current through a mesoscopic system has been extensively investigated because of scientific interest and possible applications. The Josephson current through a clean thin superconductor-normal-superconductor (S-N-S) junction has a discontinuous jump at superconductor phase difference $\theta = \pi$ under proper conditions.\textsuperscript{6} This current jump arises from the discontinuity of supercurrent contribution of Andreev bound-states driven by phase difference $\theta$ of coupled leads.\textsuperscript{2} In this situation, the current jump can be broken by impurities,\textsuperscript{10} the finite temperature, the normal lead attachment,\textsuperscript{11} or the electron-electron interaction.\textsuperscript{12} Another interesting characteristic of the Josephson current is the $\pi$ junction, where the sign of Josephson current can be reversed from $I_c \sin(\theta)$ to $I_c \sin(\theta + \pi)$.\textsuperscript{13,14} The $\pi$ junction is broadly researched in S-ferromagnetic-S (S-F-S) junctions,\textsuperscript{14,15} S-F rings,\textsuperscript{16} S-QD-S junctions,\textsuperscript{17,18} S-AB-S junctions,\textsuperscript{19,20} and so on.\textsuperscript{21}

In this paper, we will investigate the Josephson current $I$ through an AB-Fano interferometer consisting of a QD in one arm and a magnetic flux $\Phi$ through the ring. By using the non-equilibrium Green’s function method, the Josephson current expression is obtained. Due to the electron-hole symmetry, the Josephson current has the property $I(\varepsilon_d, \Phi) = I(-\varepsilon_d, \Phi + \pi)$. Without the magnetic flux, the current-superconducting phase ($I-\theta$) relation usually shows a sinusoidal line shape. But when the magnetic flux exists, the curve of $I-\theta$ is quite complex and can be modulated to $\pi$ junction in appropriate parameters. The system has two pairs of Andreev bound states due to the direct arm and the QD. When one pair of Andreev bound states is in a line with the Fermi energy.
$E_F = 0$, the Josephson current jumps. The conditions for this current jump are given. In particular, we find that the position of the current jump is identical to the position of the maximum value of the critical current $I_c$. The critical current $I_c$ versus the QD level $\varepsilon_d$ shows a typical Fano shape due to the interference between the two paths. The positions for the constructive and destructive interferences are same with the corresponding normal device. But the critical current can not reach zero at the destructing position, which is different from the normal device in which the current can be zero at the destructing position when the magnetic flux $\Phi = \pi n$ (integer $n$). In addition, the critical current is a periodic function of the magnetic flux with the period $\pi$ at the QD level $\varepsilon_d = 0$ and period $2\pi$ while $\varepsilon_d \neq 0$.

The rest of the paper is organized as follows. In Sec. II, the Hamiltonian is present and the Josephson current expression is derived. Main numerical results are given in Sec. III, in which we investigate the Josephson current-superconducting phase relation, the condition of the jump of the Josephson current, and the characters of the critical current. A brief conclusion is given in Sec. IV.

II. MODEL AND FORMULATION

The system we considered is the AB-Fano interferometer consisting of a QD and a reference arm connected to two BCS superconductor leads. The Hamiltonian of the superconductor leads is $H_\alpha = \sum_{k,\sigma} \varepsilon_k C_{k,\sigma}^\dagger C_{k,\sigma}$, $\sum_{k}(\Delta_n C_{k,\alpha} C_{-k,\alpha} + \Delta_\alpha^* C_{-k,\alpha}^\dagger C_{k,\alpha}^\dagger)$, where $\alpha = L, R$ represent the left and right lead, and $\Delta_n = \Delta e^{i\varphi_n}$ is the complex superconducting order parameter, with the superconductor gap $\Delta$ and the superconductor phase $\varphi_n$. Coupling Hamiltonian between leads and QD is $H_T = \sum_{k,\sigma,\alpha}(t_{LR} C_{R,k,\sigma} C_{L,k,\sigma} + t_{RL} C_{R,k,\sigma}^\dagger C_{L,k,\sigma})$. Parameter $t_{LR}$ and $t_{RL}$ are the Fermi-Dirac distribution. The current expression becomes

$$I = -e\langle \dot{N} \rangle = \frac{4e}{\hbar} \text{Re} \int \frac{dE}{2\pi} \langle t_{LR} e^{i\varphi} G_{RL,11}^\le(E) + t_e^{\le} C_{RL,11}^\le(E) \rangle$$

Here the Nambu representation has been used. The Green’s function $G_{RL}^{\le}(E)$ and $G_{RL}^{\ge}(E)$ are the Fourier transformation of $G_{RL}(t-t')$ and $G_{RL}^\le(t-t')$, which are defined as:

$$G_{RL}^{\le}(t-t') = i \sum_{k,k'} \begin{pmatrix} \langle C_{k,1,L}^\dagger(t')d_{k'}(t) \rangle & \langle C_{k,1,L}^\dagger(t')d_{k'}^\dagger(t) \rangle \\ \langle C_{-k,1,L}(t)\dot{d}_{k'}(t) \rangle & \langle C_{-k,1,L}(t)\dot{d}_{k'}^\dagger(t) \rangle \end{pmatrix}$$

Notice here that all of the Green’s functions are functions of time difference $t-t'$, because that we investigate the dc Josephson current in the zero bias case.

Also, because of the zero bias case, the system is in equilibrium and the fluctuation-dissipation theorem holds, we now have $G_{RL}^{\le} = -f(E)(G_{RL}^{\ge} - G_{RL}^{\le} - G_{RL}'(E))$ and $G_{RL}^{\ge}(E) = -f(E)(G_{RL}'(E) - G_{RL}^{\le}(E))$, where $f(E) = 1/(\exp(E-E_F)/k_B T) + 1$ is the Fermi-Dirac distribution. The current expression becomes

$$I = \frac{-4e}{\hbar} \int \frac{dE}{2\pi} f(E)\text{Re}[\{G_{RL}^{\le} - G_{RL}^{\ge}\} t_{LR} + t_e^{\le}(G_{RL}' - G_{RL}^{\le})]_{11}.$$
normal density of states, $\eta$ is an infinitesimal real number, $\beta_0(E) = \beta \Delta / E$, and $\beta(E) = E / \sqrt{\Delta^2 - E^2}$ while $|E| < \Delta$ and $\beta(E) = iE / \sqrt{E^2 - \Delta^2}$ while $|E| > \Delta$. Tunneling coefficients in the Nambu representation can also be expressed in $2 \times 2$ matrix with: $t_{\alpha} = t_{\alpha}(e^{i\theta_{\alpha}/2} 0 - e^{-i\theta_{\alpha}/2})$ and $t_{LR} = t(e^{i\varphi/2} 0 - e^{-i\varphi/2})$.

In the following calculation, we take the symmetric barriers with $t_L = t_R$ for convenience. By using Dyson’s equation, the Green’s function $\tilde{g}$ of the system decoupling with the QD (i.e. $t_L = t_R = 0$) can be deduced as: $\tilde{g}_L^0 = \left( g_L^{-1} - t_{LR} g_R(t_{LR})^{-1} \right)^{-1}$, $\tilde{g}_{RR}^0 = \left( g_R^{-1} - t_{LR} g_L(t_{LR})^{-1} \right)^{-1}$, and $\tilde{g}_{LR}^0 = \tilde{g}_L^0 t_{LR} \tilde{g}_R^0$. Then the retarded Green’s function of the whole AB-Fano interferometer device are solved as: $G_{r_{rL}}^0 = G_{r_{rL}}^0(t_{rL} \tilde{g}_L^0 + t_{rR} \tilde{g}_{RL}^0)$, $G_{rR}^0 = \tilde{g}_L^0 + \tilde{g}_L^0 t_{LR} + \tilde{g}_{RL}^0 t_{LR} G_{d}^0(t_{LR}^0 \tilde{g}_L^0 + t_{LR}^0 \tilde{g}_{RL}^0)$, and $G_{d}^0 = (g_1^0 - \Sigma^r)^{-1}$, where the retarded self energy $\Sigma^r$ is: $\Sigma^r = t_{rL}^0 \tilde{g}_L^0 + t_{rR}^0 \tilde{g}_{RL}^0 + t_{L}^0 \tilde{g}_{LR}^0 t_{LR}^0 + t_{R}^0 \tilde{g}_{RL}^0 t_{LR}^0$. After solving the retarded Green’s functions $G^r$, the expression of the Josephson current $I$ through AB-Fano interferometer can be reduced as:

$$\begin{align*}
I &= \frac{-4e}{\hbar} \int dE f(E) \text{Im} \left[ \frac{2\beta^2 \sin \varphi}{D} \right] \\
&= -\frac{\Gamma \beta}{2}(Q_{11}A_{11} + Q_{12}A_{12}) \\
&+ \frac{\Gamma \sqrt{\beta \rho e^{\frac{i\varphi}{\beta}}}}{\Xi} \left[ A_{11}^2 Q_{11} + A_{12}^2 Q_{22} + 2A_{11}A_{12}Q_{12} \right]
\end{align*}$$

where $D = 1 + x^2 + 2x(E^2 - \Delta^2 \cos \varphi)/(E^2 - \Delta^2)$ and

$$
\Xi = (ED + \Gamma \beta (1 + x))^2 - \varepsilon_d D - \Gamma \sqrt{x}(x - \beta^2)\cos \frac{\theta + \varphi}{2} - \Gamma \sqrt{x}\beta_0 \cos \frac{\varphi - \theta}{2} - \Gamma \beta_0 \left( \cos \frac{\theta}{2} + x \cos \frac{\theta}{2} + \varphi \right)^2.
$$

Here $x \equiv t^2 - \rho \alpha^2$ dictates the tunneling through the direct arm and $\Gamma \equiv 2\pi \rho \alpha^2$ is the coupling strength of QD to leads. In the wide-band approximation, $x$ and $\Gamma$ are independent with the energy $E$. The factors $A$ and $Q$ in equation (4) are:

$$
\begin{align*}
A_{11} &= (1 + x) + \frac{\sqrt{x}}{\beta}_0 (x + \beta^2)e^{i\frac{\varphi + \theta}{\beta}} - \frac{\sqrt{x}}{\beta}_0 e^{i\frac{\varphi - \theta}{\beta}} \\
A_{12} &= -\frac{\sqrt{x}}{\beta}_0 e^{i\theta}(1 + x)e^{i\varphi} + 2i \sin \frac{\beta}{2} \beta_0 e^{i\varphi} \\
A_{21} &= \frac{\sqrt{x}}{\beta}_0 e^{-i\theta}(1 + x)e^{-i\varphi} - 2i \sin \frac{\beta}{2} \beta_0 e^{-i\varphi} \\
A_{22} &= -(1 + x) + \frac{\sqrt{x}}{\beta}_0 (x + \beta^2)e^{-i\frac{\varphi + \theta}{\beta}} - \frac{\sqrt{x}}{\beta}_0 e^{-i\frac{\varphi - \theta}{\beta}}
\end{align*}
$$

and

$$
\begin{align*}
Q_{11} &= ED + \varepsilon_d D - \Gamma \beta \text{Re}(A_{22}) \\
Q_{12} &= \Gamma \beta \text{Re}(A_{12}) \\
Q_{21} &= \Gamma \beta \text{Re}(A_{21}) \\
Q_{22} &= ED - \varepsilon_d D + \Gamma \beta \text{Re}(A_{11})
\end{align*}
$$

The first term in equation (4) describes the direct tunneling contribution, the second and third terms are transport through QD and interference between direct arm and QD. The Josephson current in the equation (4) can be split into two parts, the continuous part $I_{con}$, contributed from continuous spectrum while the energy $E$ outside the superconducting gap $\Delta$, and the discrete part $I_{dis}$ contributed by Andreev bound states while $E$ within the gap. In numerical calculation $I_{con}$ is obtained by integral in equation (4) and $I_{dis}$ is approached by dealing with the delta functions due to the infinitesimal imaginary part $i\eta$. The discrete part is usually much larger than the continuous part. The factor $\Xi$ has two pairs of poles at $E_{1,2}^\pm (E_{1,2}^+ = -E_{1,2}^-)$ within the gap, which count for Andreev bound states. These Andreev bound states arise from the hybridization of bound states of QD $E_{QD}^\pm$ and the direct arm $E_0^\pm$. Here $E_0^\pm = \pm\Delta \sqrt{1 - \frac{4\varepsilon_d}{(1 + \varepsilon_d)} \sin^2(\varphi/2)}$ are the Andreev bound states in the direct arm $E_0^\pm$ and $E_{QD}^\pm$ are also the poles of the factor $D$. $E_{QD}^\pm (E_{QD}^+ = -E_{QD}^-)$ are the Andreev bound states in the QD for the S-QD-S device.

### III. Numerical Results

In this section, we present our numerical investigations on the relation of the Josephson current versus the superconducting phase, the condition of the jump of the current, the Fano resonant characters of critical current, and the dependence of critical current on the magnetic flux.
A. Josephson current-superconducting phase relations

We first discuss the current-phase \((I-\phi)\) relation of AB-Fano interferometer. Fig.1 shows Josephson current \(I\) as a function of the superconducting phase difference \(\phi\) of the left and right leads. When \(x = 0\), the device reduces into a QD coupled to two superconductor leads. The \(I-\phi\) relation is sinusous line shape when the level \(\epsilon_d\) is far off the Fermi level \(E_F = 0\), and the current shows a discontinuous jump at \(\phi = \pi\) when \(\epsilon_d = 0\) (shown Fig.1a), in which the Andreev bound states \(E^+_{QD} = E^-_{QD} = 0.42\). On the other hand, while \(\Gamma = 0\), the system reduces into an S-I-S device. The \(I-\phi\) relation is sinusous line shape when \(x \) is much smaller or larger than 1, and the current \(I\) has a discontinuous jump while \(x = 1\) (see Fig.1b), in which the Andreev bound states \(E^+_0 = E^-_0 = 0.10\). When both \(x\) and \(\Gamma\) are non-zero (in other words, the two pathes are opened), the AB-Fano interferometer is formed and the transport can be adjusted by magnetic flux \(\Phi\) through it. Two pairs of Andreev bound states \(E^\pm_{1,2}\) which belong to the direct arm and QD, cause the hybridization to form new Andreev bound states \(E^\pm_{1,2}\), which enable the interference construction or destruction of the Josephson current. Then the current-phase \(I-\phi\) relation is usually not a sinusoidal-like function (except for the special magnetic flux values \(\Phi = 0\) and \(\pi\)). While \(\Phi = 0\) and \(\pi\), the \(I-\phi\) relation is still a sinus line shape, and the current \(I\) is zero at \(\phi = 0\) and \(\pi\), as shown in Fig.1c and 1d. In some specific parameters the discontinuous jump of the current can still occur, which we will detail in the next sub-section. Here, we notice that the current has the relation: \(I(\epsilon_d, \Phi) = I(-\epsilon_d, \Phi + \pi)\). In other words, while the level changes from \(\epsilon_d\) to \(-\epsilon_d\) and the magnetic flux from \(\Phi\) to \(\Phi + \pi\), the current \(I\) does not vary regardless of any other parameters. The relation of \(I(\epsilon_d, \Phi) = I(-\epsilon_d, \Phi + \pi)\) comes from the electron-hole symmetry, i.e. taking the transform \((d_\alpha, d^\dagger_{\alpha,\sigma})\) to \((d^\dagger_\alpha, d_{\alpha,\sigma})\) and simultaneously setting the parameters \((\epsilon_d, \Phi)\) to \((-\epsilon_d, \Phi + \pi)\), the Hamiltonian \(H\) is invariant.

When the magnetic flux \(\Phi\) is not equal to 0 or \(\pi\), the current-phase relation is usually not a sinusoidal-like function, and the current \(I\) has non-zero values at \(\phi = 0\). In some special parameters, the current \(I\) is negative while \(\theta \in [0, \pi]\), which is a \(\pi\) junction. For example, by proper selection of parameters, \(\epsilon_d = 0.5\), \(\Gamma = 0.45\), \(\Phi = 0.6\pi\), and \(x = 0.4 \sim 0.9\), the current \(I\) is negative when the phase \(\phi \in [0, \pi]\) as shown in Fig.2a. But it is not a strict \(\pi\) junction, and the current is not positive in all regions \(\theta \in [\pi, 2\pi]\). The realization of quasi-\(\pi\) junction is because of the introduction of the magnetic flux phase \(\Phi\) which changes the interference of two pathes. Another example, as shown in Fig.2b with the magnetic flux \(\Phi = \pi/2\), the current is negative (positive) in most parts of region \(\theta \in [0, \pi]\) ([\(\pi, 2\pi\)]. Here we notice that the \(\theta\) region for the negative current \(I\) is not equal to that of the positive current. In all curves in Fig.2a and some curves in Fig.2b, the negative-current region is obviously wider than the positive-current region. However,

![FIG. 2: Current I vs. the phase \(\theta\) with the parameters in (a) \(\epsilon_d=-0.5, \Phi=0.45, x\) from 0.4 to 0.9 with space 0.1 in arrow direction, (b) \(x=0.5, \Gamma = 0.1, \Phi=0, \text{ and } \epsilon_d = -0.1\) (solid curve), -0.05 (dashed curve), 0 (dotted curve), 0.05 (dash-dotted curve), and 0.1 (dash-dot-dot curve).]

B. the condition of the jump of the current

The current \(I_{con}\) from continuous spectrum with \(|E| > \Delta\) is always continuous. The discontinuous current arises from the part \(I_{dis}\) which is from the Andreev bound states. When one of the two pairs of Andreev bound states \(E^\pm_{1,2}\) just aligns with the Fermi level \(E_F = 0\) (i.e. \(E^+_{1,2} = E^-_{1,2} = 0\)), an abrupt jump occurs in the current \(I_{dis}\) so that the current \(I = I_{con} + I_{dis}\). The condition of the jump of the current \(I\) is thus \(\Xi(E) = 0\). With the help of Eq.(5), the condition \(\Xi(E) = 0\) can be reduced to:

\[
\begin{align*}
\epsilon_d D - \Gamma x \sqrt{x} \cos \Phi - \Gamma \sqrt{x} \cos (\theta - \Phi) &= 0 \\
\Gamma^2 [\cos \frac{\Phi}{2} + x \cos (2\Phi - \theta/2)] &= 0
\end{align*}
\]

(6)

When the above two equations are tenable, the current will jump. For example, (i) while \(\Gamma = 0\), the condition in Eq.(6) reduces into \(x = 1\) and \(\theta = \pi\). This is consistent with the jump in Fig.1b. (ii) While \(x = 0\), the condition in Eq.(6) reduces into \(\epsilon_d = 0\) and \(\theta = \pi\), which agrees with the jump in Fig.1a. In general, when both \(x\) and \(\Gamma\) are non-zero with a magnetic flux through the AB-Fano interferometer, the condition in Eq.(6) can be rewritten as:

\[
\theta = -2 \arctan \left[ \frac{1 + x \cos 2\Phi}{x \sin 2\Phi} \right]
\]

(7)
where the probability here, we simply recall the results of the normal AB-Fano interferometer. In the corresponding intersections in Fig. 3a.

\[ \Gamma = 0 \]

Specifically, for \( \Phi = \pi/2 \), we have \( \theta = \pi \) and \( \frac{x}{\sqrt{2}} = 0 \), which is consistent with the jump position in Fig. 2b. Fig. 3a shows the jump-occurrence region in the parameter space of \((x, \frac{x}{\sqrt{2}})\). When the parameters are just at the curves of Fig. 3a, a current jump will occur. Fig. 3b shows the current \( I \) versus the level \( \varepsilon_d/\Gamma \) at \( x = 0.2 \) and \( \Gamma = 0.5 \). It clearly shows that the jump occurs at the corresponding intersections in Fig. 3a.

### C. Fano resonant characters of critical current

In this and the next sub-section, we will focus on the critical current, which is experimentally accessible. In fact, the critical current in the superconducting AB-Fano interferometer behaves similarly to the current in the normal AB-Fano interferometer under the small bias. So here, we simply recall the results of the normal AB-Fano device, which consists of an AB ring attached to two normal leads with a QD in one of its arms. The transmission probability \( T \) of the normal AB-Fano device is [4, 6]

\[
T(E) = \frac{4x}{(1+x)^2} + \frac{4\Gamma(1-x)\sqrt{x}\cos\Phi}{(1+x)^3} \text{Re}G^r(E) - \frac{\Gamma[(1+x)^2 - 4x(1+\cos^2\Phi)]}{(1+x)^3} \text{Im}G^r(E), \tag{9}
\]

where \( G^r^{-1}(E) = E - \varepsilon_d + \Gamma\sqrt{x}\cos\Phi/(1+x) - i\Gamma/(1+x) \) and the meaning of the parameters \((\varepsilon_d, \Gamma, \Phi, \text{and } x)\) is the same as the above superconducting device. From the Eq.(9), we found that the interference construction can be realized at

\[
\varepsilon_d = -\frac{\Gamma\sqrt{x}\cos\Phi}{1-x}, \tag{10}
\]

and interference destruction is at

\[
\varepsilon_d = \frac{(1+x\cos2\Phi)\Gamma}{2\sqrt{x}(1+x)\cos\Phi}. \tag{11}
\]

Here the interference construction position is the same as Eq.(8) of the position of the jump of the superconducting current in the superconducting device. The \( I-\varepsilon_d \) relation of the normal device shows a symmetric line shape when \( x = 0 \) and a typical Fano resonance when at finite \( x \) and zero magnetic flux \( \Phi \). The shape of the curve of \( I-\varepsilon_d \) can be affected by the magnetic flux \( \Phi \) and is symmetric at \( \Phi = \pi/2 \).

Next, we study the critical current \( I_c \) in the superconducting AB-Fano interferometer. Here we select the maximum of current in a period \( 2\pi \) of superconducting phase difference \( \theta \) as the critical current. Due to the interference between two pathes, the construction or destruction transport occurs, and the critical current \( I_c \) versus the QD’s level \( \varepsilon_d \) usually exhibits a Fano characters. To show the details of Fano characters, we plot \( I_c \) as a function of \( \varepsilon_d \) at zero magnetic flux (\( \Phi = 0 \)) for small \( x \) values and large \( x \) values in Fig. 4(a) and (b), respectively. For small \( x \) values (\( x = 0.1 \), for instance), \( I_c \) shows a Fano resonance near \( \varepsilon_d = 0 \) with the peak at \( \varepsilon_d = -\Gamma\sqrt{x}/(1+x) \) and the valley at approximately \( \varepsilon_d = \Gamma/(2\sqrt{x}) \). The positions of the peak and valley are the same with the normal device [see Eqs.(10) and (11)], though in the normal device the normal current is under the small bias, while in the present superconducting device the critical current \( I_c \) is at zero bias. With the
enhancement of $\Gamma$, the line shape tends to behave more symmetric characters as the Breit-Wigner resonance and the magnitude of the critical current is enhanced at negative $\varepsilon_d$ side but reduced at positive $\varepsilon_d$ side. For large $x$ values (e.g., $x = 1.0$, as shown in Fig.4b), the destructive interference plays the core role, and the curves of $I_c-\varepsilon_d$ are almost symmetric for the small $\Gamma$ and show an obvious Fano valley at $\varepsilon_d = \Gamma/2$. The Fano peak is pushed to infinity at $x = 1.0$ (as shown in Eq.(10)), so it is invisible in Fig.4(b). With $\Gamma$ increasing, the transmission probability through the QD grows and the destruction of two paths occurs, causing the valley to deepen and the line shape to become more asymmetric.

In this paragraph, we investigate the effect of the direct path $x$ on the critical current $I_c$. In Figs. 4(c) and 4(d), we show the critical current $I_c$ versus $\varepsilon_d$ at different $x$ for a fixed $\Gamma = 0.5$ and $\Phi = 0$. While $x = 0$, the direct path is closed, $I_c$ is completely symmetric with the peak at $\varepsilon_d = 0$ and the valley at $\varepsilon_d = \infty$, which are the same with Eqs.(10) and (11). When $x$ increases, the peak position is moved off from the Fermi level $E_F = 0$ but the valley position gradually from infinity to 0, and the curve of $I_c-\varepsilon_d$ shows a Fano resonance. At the middle $x$, the Fano resonance is the most prominent. While $x$ tends to 1, the direct path is completely open and the peak position tends to infinity, leading the $I_c-\varepsilon_d$ curve into a symmetric valley.

In the above $I_c-\varepsilon_d$ relation discussion, the magnetic flux $\Phi$ is fixed at zero. In the normal AB-Fano device, the line shape of the current $I$ versus $\varepsilon_d$ is strongly modified by the magnetic flux $\Phi$. So, in the following, we investigate how the $I_c-\varepsilon_d$ curve in the superconducting AB-Fano device is affected by $\Phi$. Figs.4(e) and (f) show $I_c-\varepsilon_d$ with $\Gamma > x$ and $\Gamma < x$, respectively. These two cases when $\Gamma > x$ and $\Gamma < x$ represent situations where the QD path or direct arm path dominates the transport. While $\Gamma > x$, the $I_c-\varepsilon_d$ curve is strongly affected by the magnetic flux $\Phi$. When $\Phi$ increases from 0, the Fano shape first grows more notable, then the peak is greatly reduced while $\Phi$ near $\pi/2$, and at last the peak is increased and the Fano shape is recovered again at $\Phi = \pi$ (see Fig.4e). On the other hand, while $\Gamma < x$, the $I_c-\varepsilon_d$ curve is only slightly affected by $\Phi$ (see Fig.4f). We add the following three points: (i) The critical current $I_c$ has the relation: $I_c(\varepsilon_d, \Phi) = I_c(-\varepsilon_d, \Phi + \pi)$, because of the electron-hole symmetry and $I(\varepsilon_d, \Phi) = I(-\varepsilon_d, \Phi + \pi)$. Due to the relation $I_c(\varepsilon_d, \Phi) = I_c(-\varepsilon_d, \Phi + \pi)$, the critical current $I_c$ is a periodic function of $\Phi$ with period $\pi$ at $\varepsilon_d = 0$. (ii) At $\Phi = \pi/2$, the $I_c-\varepsilon_d$ curve is still not symmetric, which is different from the current in the normal device. (iii) Though the valley in the normal device can reach zero in some parameter regions, at no parameters does the valley value of the critical current $I_c$ reach zero.

D. Critical current-magnetic flux relations

Finally, we investigate the relation of the critical current $I_c$ with the magnetic flux phase $\Phi$. Fig.5(a) shows $I_c$ versus $\Phi$ for the different $x$ values at $\varepsilon_d = 0$. Several characteristics can be noticed: i) $I_c$ is a periodic function of $\Phi$ (or the magnetic flux $\phi$) with the period $2\pi$ (or $e/\hbar$) while $\varepsilon_d \neq 0$, and period $\pi$ (or $e/2\hbar$) at $\varepsilon_d = 0$. ii) For a small $x$ value, $I_c$ is almost a constant, because the direct path is almost closed. For a large $x$ value, the oscillation of $I_c-\Phi$ is strong. The oscillation shape greatly departs from sin $\Phi$ or cos $\Phi$ shape because the higher order tunneling processes are numerous at $\varepsilon_d = 0$. iii) The critical current $I_c$ shows a peak at $\Phi = (2n + 1)\pi/2$ with the integer $n$, which is consistent with equations (7) and (10). Fig.5(b) shows $I_c$ versus $\Phi$ for the QD level $\varepsilon_d$ at $x = 0.2$. When $\varepsilon_d$ is far away from zero, $I_c$ shows a standard sin $\Phi$/$\cos \Phi$ behavior because the higher order tunneling processes are weak at the small $x$ and $|\varepsilon_d| \gg 0$. But $I_c$ for positive $\varepsilon_d$ bears a phase lapse of $\pi$ with that of $-\varepsilon_d$. This phase lapse is from the phase $\Theta_{QD}$ of the transmission coefficient of a QD, with $\Theta_{QD} = \pi/2$ when $\varepsilon_d \gg 0$ and $-\pi/2$ when $\varepsilon_d \ll 0$. As $\varepsilon_d$ approaches zero, the oscillation of $I_c-\Phi$ is enhanced and departs from the sin $\Phi$/$\cos \Phi$ behavior, with peaks emerging at about $\Phi = \pi/2$ and $3\pi/2$.

IV. CONCLUSION

In conclusion, the Josephson current through an Aharonov-Bohm interferometer consisting of a quantum dot and a direct arm with magnetic flux through the ring has been investigated. An equation for the occurrence
of supercurrent discontinuity is given. In particular, we found that the position of the supercurrent discontinuity, the position of the peak of critical current, and the constructive interference of the current in a corresponding normal device are the same. By adjusting the device’s parameters, such as the magnetic flux phase and direct arm coupling, the Josephson junction can vary from a normal junction to a π-junction. Fano characters of the critical current are similar to those of the current in the normal device are the same. By adjusting the device’s parameters, such as the magnetic flux phase and direct arm coupling, the Josephson junction can vary from a normal junction to a π-junction. Fano characters of the critical current are similar to those of the current in the normal device are the same. By adjusting the device’s parameters, such as the magnetic flux phase and direct arm coupling, the Josephson junction can vary from a normal junction to a π-junction. Fano characters of the critical current are similar to those of the current in the normal device are the same. By adjusting the device’s parameters, such as the magnetic flux phase and direct arm coupling, the Josephson junction can vary from a normal junction to a π-junction. Fano characters of the critical current are similar to those of the current in the normal device are the same. By adjusting the device’s parameters, such as the magnetic flux phase and direct arm coupling, the Josephson junction can vary from a normal junction to a π-junction. Fano characters of the critical current are similar to those of the current in the normal device are the same. By adjusting the device’s parameters, such as the magnetic flux phase and direct arm coupling, the Josephson junction can vary from a normal junction to a π-junction. Fano characters of the critical current are similar to those of the current in the normal device are the same. By adjusting the device’s parameters, such as the magnetic flux phase and direct arm coupling, the Josephson junction can vary from a normal junction to a π-junction. Fano characters of the critical current are similar to those of the current in the normal device are the same. By adjusting the device’s parameters, such as the magnetic flux phase and direct arm coupling, the Josephson junction can vary from a normal junction to a π-junction. Fano characters of the critical current are similar to those of the current in the normal device are the same.

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