Permanency and bifurcations of bounded solutions near homoclinics with symmetric eigenvalues

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Abstract We consider a system with a homoclinic orbit. We decompose the corresponding variational equation on the space of solutions and provide sufficient conditions for the permanency of the homoclinic in the space of $C^1$ vector fields. We also provide new sufficient conditions for the persistence and multiple bifurcations of the bounded solutions nearby. Our results can be verified numerically and do not meet the limitations of classic methods (like Melnikov’s integrals and Poincare’s map).

Keywords Homoclinic bifurcation · Lyapunov-Schmidt reduction · Saddle-node bifurcation · Transcritical bifurcation

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1 Introduction

In the theory of differential equations, the homoclinics are usually known because of the structural unstability often imposing on systems. Some of the most studied chaotic motions arise from homoclinic bifurcations and many classic methods of the theory (like Melnikov’s integrals and Poincare’s maps, [18, 9]) are developed for studying such behaviors. And today, homoclinics are still receiving attention from researchers of both the sciences and engineering areas (see for example [1, 2, 3, 6, 11, 19, 20, 21]).
In the last decades, non-classic methods have been continuously developed by the researchers for studying the behavior of systems around a perturbed homoclinic. In the years after that, some studies have been concerned about providing sufficient conditions for the persistence of regular behaviors (like periodic orbits, bounded solutions and etc.) near a perturbed homoclinic. In an interesting study, [23], Zhu and Zhang considered the $n$-dimensional differential equation $\dot{x} = f(x) + g(x, t)$ with $f, g \in C^3$. They provided sufficient conditions for the existence of an invariant manifold which is spanned by homoclinic orbits. They called this invariant manifold as homoclinic finger-ring and showed that solutions on this manifold remain bounded.

In this paper, we will provide a sufficient condition for the permanency of bounded solutions near a perturbed homoclinic and will explain how it can include the permanency of the homoclinic itself. We will also provide new sufficient conditions for the bifurcations of solutions nearby. Consider the system

$$\dot{x} = f(x) + \epsilon g(x, t), \quad x \in \mathbb{R}^2 \tag{1}$$

where $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are respectively $C^2$ and $C^1$ w.r.t. $x$ and $g$ is bounded (but not necessarily periodic) w.r.t. $t$. We assume that the unperturbed system $\dot{x} = f(x)$ has a homoclinic orbit $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ based on the equilibrium $x_0 = 0$ with the corresponding eigenvalues $\pm \omega \neq 0$. We will impose a decomposition on the corresponding variational equation and will develop results and methods to find bounded solutions and their bifurcations near $\gamma(t)$. Our method can be verified numerically and does not meet the limitations of classic methods (like Melnikov's integrals and Poincare's map). Apparently, this makes the results more appropriate for real-world applications in compare with former results. The main results are summarized below.

We introduce the conditions:

$C_1$: \[ F_1 = \int_{\mathbb{R}} \frac{1}{2\pi} f(\gamma(s)) \wedge Df(\gamma(s)) \gamma(s) ds \neq 0. \]

$C_1'$: \[ F_1' = \int_{\mathbb{R}} \frac{1}{2\pi} f(\gamma(s)) \wedge g(\gamma(s), s) ds \neq 0. \]

$C_2$: The map $\gamma \wedge f(\gamma) \in C_0^1(\mathbb{R}, \mathbb{R}^2)$ is not identically zero. Here the wedge product of two vectors $u = (a, b)$ and $v = (a', b')$ is defined as $u \wedge v = ab' - ba'$.

Note that, the assumption $C_2$ implies that $\gamma$ and $\gamma'$ are linearly independent functions.

**Theorem 1** Consider system (1).

(i) If $C_1$ holds then for $0 \leq |\epsilon| < 1$, (1) has a bounded solution $x(t)$ near $\gamma(t)$. Furthermore, $\|x(t) - \gamma(t)\| \to 0$ as $\epsilon \to 0$.

(ii) If $C_1'$ and $C_2$ hold then for $0 \leq |\epsilon| < 1$, the system has a bounded solution $x(t)$ near $\gamma(t)$. Furthermore, $\|x - \gamma\| \to 0$ as $\epsilon \to 0$.

**Remark 1** In part (i) of theorem 1 the condition $C_1$ is independent of $g$. This means that, if $C_1$ holds, then for any appropriate $C^1$ map $g(x, t)$ and $0 \leq |\epsilon| < 1$, (1) has a bounded solution near $\gamma(t)$. Especially, if $g = g(x)$ is
independent of \( t \), then this bounded solution is homoclinic. This means that the homoclinic of the unperturbed system in (1) is permanence in the space of \( C^1 \) vector fields.

Now let \( \zeta(t) = (\zeta_1(t), \zeta_2(t)) \) be the unbounded solution of \( \dot{x} = Df(\gamma(t))x \) and \( \Delta(t) = \gamma_1(t)\zeta_2(t) - \gamma_2(t)\zeta_1(t) \). We introduce the following conditions applied in the next theorems.

**C3:**

\[
\mathcal{F}_3 = \int_{\mathbb{R}_2} \frac{f_2(\gamma(s))f_1(\gamma(t))}{\Delta(t)\Delta(s)} \left[ G(s, t) \wedge \mathcal{F}(s, t) \right] ds dt \neq 0
\]

Here \( \mathcal{F}(s, t) = \left( \langle \nabla f_1(\gamma(s)), \gamma(s) \rangle - f_1(\gamma(s)) \right) \), \( G(s, t) = \begin{pmatrix} g_1(\gamma(s)) \\ g_2(\gamma(t)) \end{pmatrix} \).

**C4:**

\[
\begin{align*}
\mathcal{F}_{4,1} &:= \int_{\mathbb{R}} \frac{1}{\Delta(s)} f_2(\gamma(s))g_1(\gamma(s), s - \beta) ds \neq 0, \\
\mathcal{F}_{4,2} &:= \int_{\mathbb{R}^2} \frac{f_1(\gamma(t))f_2(\gamma(s))}{\Delta(t)\Delta(s)} \left[ F_2(t) \wedge \mathcal{F}_1(s) \right] dt ds \neq 0, \\
\mathcal{F}_{4,3} &:= \int_{\mathbb{R}} \frac{f_2(\gamma(s))}{\Delta(s)} \left( \langle \nabla f_1(\gamma(s)), \gamma(s) \rangle - f_1(\gamma(s)) \right) ds \neq 0.
\end{align*}
\]

**C4’:**

\[
\begin{align*}
\mathcal{F}_{4,1}' &:= \int_{\mathbb{R}} \frac{1}{\Delta(s)} f_1(\gamma(s))g_2(\gamma(s), s - \beta) ds \neq 0, \\
\mathcal{F}_{4,2}' &:= \int_{\mathbb{R}^2} \frac{f_2(\gamma(t))f_1(\gamma(s))}{\Delta(t)\Delta(s)} \left[ F_1(t) \wedge \mathcal{F}_2(s) \right] dt ds \neq 0, \\
\mathcal{F}_{4,3}' &:= \int_{\mathbb{R}} \frac{f_1(\gamma(s))}{\Delta(s)} \left( \langle \nabla f_2(\gamma(s)), \gamma(s) \rangle - f_2(\gamma(s)) \right) ds \neq 0.
\end{align*}
\]

**C5:**

\[
\mathcal{F}_5 := \int_{\mathbb{R}^2} \frac{f_1(\gamma(t))f_2(\gamma(s))}{\Delta(t)\Delta(s)} \left[ \mathcal{F}_2(t) \wedge \mathcal{F}_1(s) \right] dt ds \neq 0.
\]

Here \( \tilde{F}_k(s) = \begin{pmatrix} \sum_{i,j=1}^2 D^2_{x_i x_j} f_k(\gamma(s)) \\ \langle \nabla f_k(\gamma(s)), \gamma(s) \rangle - f_k(\gamma(s)) \end{pmatrix} \), \( k = 1, 2, \).

\[
\begin{align*}
\tilde{F}_1(s) &= \begin{pmatrix} \nabla g_1(\gamma(s), s - \beta) + \sum_{i,j=1}^2 D^2_{x_i x_j} f_1(\gamma(s)) \left( \kappa_1 \gamma_j(s) + \kappa_2 j + \kappa_3 j \right) \\
\langle \nabla f_1(\gamma(s)), \gamma(s) \rangle - f_1(\gamma(s)) \end{pmatrix}, \\
\tilde{F}_2(t) &= \begin{pmatrix} \nabla g_2(\gamma(s), s - \beta) + \sum_{i,j=1}^2 D^2_{x_i x_j} f_2(\gamma(t)) \left( \kappa_1 \gamma_j(t) + \kappa_2 j + \kappa_3 j \right) \\
\langle \nabla f_2(\gamma(t)), \gamma(t) \rangle - f_2(\gamma(t)) \end{pmatrix}.
\end{align*}
\]
\[ \kappa_1 = -\int_{\mathbb{R}} \frac{f_2(\gamma(s))g_1(\gamma(s), t - \beta)/\Delta(s)ds}{\int_{\mathbb{R}} f_2(\gamma(s))\left(\langle \nabla f_1(\gamma(s)), \gamma(s)\rangle - f_1(\gamma(s))\right)/\Delta(s)ds}, \]

\[ \kappa_2 = \zeta_j(t) \int_{-\infty}^{t} \frac{1}{\Delta(s)} f(\gamma(s)) \wedge Df(\gamma(s))\gamma(s)ds \]

\[ + f_j(\gamma(t)) \int_{0}^{t} \frac{1}{\Delta(s)} [Df(\gamma(s))\gamma(s) - f(\gamma(s))] \wedge \zeta(s)ds, \quad (j = 1, 2) \]

\[ \kappa_3 = \zeta_j(t) \int_{-\infty}^{t} \frac{1}{\Delta(s)} f(\gamma(s)) \wedge g(\gamma(s), s - \beta)ds \]

\[ + f_j(\gamma(t)) \int_{0}^{t} \frac{1}{\Delta(s)} g(\gamma(s), s - \beta) \wedge \zeta(s)ds, \quad (j = 1, 2) \]

**C6:** \( g(x, t) \) is not \( t \)-constant in a neighborhood of \( \gamma(t) \), that is, there exists an open subset \( U \subseteq \mathbb{R}^2 \) containing \( \gamma(t) \) such that for \( (x, t_1), (x, t_2) \in U \times \mathbb{R}, t_1 \neq t_2 \) implies \( g(x, t_1) \neq g(x, t_2) \).

**Theorem 2** If the conditions C3, (C4 or C4'), C5 and C6 hold then depending on the sign of \( 0 \leq |\epsilon| < 1 \), (1) has two bounded solutions \( x_1(t) \) and \( x_2(t) \) near \( \gamma(t) \). Furthermore, \( \|x_{1,2} - \gamma\| \to 0 \) as \( \epsilon \to 0 \).

**Theorem 3** If the condition C3 fails and the conditions (C4 or C4'), C5 and C6 hold then independent of the sign of \( 0 \leq |\epsilon| < 1 \), (1) has two bounded solutions \( x_1(t) \) and \( x_2(t) \) near \( \gamma(t) \) such that \( \|x_{1,2} - \gamma\| \to 0 \) as \( \epsilon \to 0 \).

## 2 Lyapunov-Schmidt reduction

Here we explain the Lyapunov-Schmidt reduction method in brief. Consider the sufficiently differentiable map

\[ \varphi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n, \quad \varphi : (x, \epsilon) \longrightarrow \varphi(x, \epsilon). \]

We denote the first order derivative of \( \varphi \) w.r.t. \( x \) by \( D_x \varphi \). Assume that the unperturbed map \( x \mapsto \varphi(x, 0) \) has a zero \( x = x_0 \), i.e. \( \varphi(x_0, 0) = 0 \). The question is whether the perturbed map has a zero too? One of the main tools for answering this question is the implicit function theorem; but, if the conditions of the implicit function theorem do not hold then the Lyapunov-Schmidt reduction is an effective tool.

Let \( N(L) \) be the kernel of \( L := D_x \varphi(x_0, 0) \) with \( \text{dim} N(L) = k > 0 \) and a complementary subspace \( N^\perp(L) \). Also suppose that \( p : \mathbb{R}^n \to R(L) \) is a projection with complement \( I - p : \mathbb{R}^n \to R^\perp(L) \). Here \( R(L) \) and \( R^\perp(L) \) are respectively the range of \( L \) and a complementary subspace of \( R(L) \). Since \( \mathbb{R}^n = N(L) \oplus N^\perp(L) \) so we can decompose \( x \in \mathbb{R}^n \) to its components \( \xi \in \mathbb{R}^n \).
\( N(L) \approx \mathbb{R}^k \) and \( \eta \in N^\perp(L) \approx \mathbb{R}^{n-k} \), hence we have \( x = \xi + \eta \). Now consider the map

\[
v : \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{R} \to R(L), \quad v : (\xi, \eta, \epsilon) \mapsto p\varphi(\xi + \eta + x_0, \epsilon).
\]

Since \( v(0, 0, 0) = 0 \) and \( pL = D_\eta v(0, 0, 0) : N^\perp(L) \to R(L) \) is an isomorphism, so the implicit function theorem implies that there are sufficiently small neighborhoods \( A \) of \( (\xi, \epsilon) = (0, 0) \), \( B \) of \( \eta = 0 \) and a \( C^2 \) map \( \eta : A \to B \) such that \( \eta(0, 0) = 0 \) and \( \eta = \eta(\xi, \epsilon) \) is the unique solution of

\[
v(\xi, \eta, \epsilon) = p\varphi(\xi + \eta + x_0, \epsilon) = 0.
\]

Therefore, the Lyapunov-Schmidt reduction method reduces the problem of finding zeros for \( \varphi \) to finding zeros of the function \( \tau : \mathbb{R}^k \times \mathbb{R} \to R^2(L), \quad \tau : (\xi, \epsilon) \mapsto (I - p)\varphi(\xi + \eta(\xi, \epsilon) + x_0, \epsilon) \).

Note that \( \tau(0, 0) = 0 \); hence, if there exists a function \( \xi(\epsilon) \) such that \( \xi(0) = 0 \) and \( \tau(\xi(\epsilon), \epsilon) = 0 \) then \( x(\epsilon) = \xi(\epsilon) + \eta(\xi(\epsilon), \epsilon) + x_0 \) is a solution for \( \varphi(x, \epsilon) = 0 \). The function \( \tau \) is called a bifurcation function for \( \varphi(x, \epsilon) = 0 \).

Note that \( x = x(\epsilon) \) could not be obtained directly by the implicit function theorem, because \( L := D_x \varphi(x_0, 0) \) has a nontrivial kernel (see [4] for more details and [10] for development of the theory over Banach spaces).

3 Variational equation

In this section, we consider the variational equation of [10] along \( \gamma(t) \), i.e. \( \dot{z} = A(t)z \) with \( A(t) = Df(\gamma(t)) \). We will look for bounded solutions of the equation

\[
\dot{z} - A(t)z = F(t), \quad F = (F_1, F_2) \in C^0_b(\mathbb{R}, \mathbb{R}^2).
\]

(2)

Here \( C^0_b(\mathbb{R}, \mathbb{R}^2) \) shows the Banach space of bounded continuous functions from \( \mathbb{R} \) to \( \mathbb{R}^2 \) with \( \| F \| = \sup_{t \in \mathbb{R}} |F(t)| \).

**Lemma 4** Let \( X(t) \) be a Fundamental matrix of the variational equation \( \dot{z} = A(t)z \) and \( \Delta(t) = \det X(t) \). Then \( \Delta(t) \) is bounded with \( \Delta(+\infty) = \Delta(-\infty) \).

**Proof** It is enough to show that \( \dot{z} = A(t)z \) has a fundamental matrix with bounded determinate and \( \Delta(+\infty) = \Delta(-\infty) \). From [7] lemma 1, there exist a fundamental matrix \( X(t) \) and an invertible constant matrix \( C \) such that

\[
\lim_{t \to +\infty} X(t) \begin{pmatrix} e^{\omega t} & 0 \\ 0 & e^{-\omega t} \end{pmatrix} = C.
\]

(3)

We may complete the steps of the proof of the lemma to verify that

\[
\lim_{t \to -\infty} X(t) \begin{pmatrix} e^{-\omega t} & 0 \\ 0 & e^{\omega t} \end{pmatrix} = C.
\]

(4)
Thus we have
\[ \Delta(\infty) = \lim_{t \to +\infty} \det X(t) = \det C = \lim_{t \to -\infty} \det X(t) = \Delta(-\infty). \]

Since \( \gamma'(t) \) is a solution of \( \dot{z} = A(t)z \) so we can find a fundamental matrix \( X(t) \) as below
\[ X(t) = \begin{pmatrix} \gamma'_1(t) & \zeta_1(t) \\ \gamma'_2(t) & \zeta_2(t) \end{pmatrix} \]  \hspace{1cm} (5)
with \( X(0) = Id \). Here \( \zeta(t) = (\zeta_1(t), \zeta_2(t)) \) is any solution independent of \( \gamma'(t) \); hence both \( \zeta_1(t) \) and \( \zeta_2(t) \) are unbounded as \( t \to \pm \infty \). Thus, from (3) and (4), when \( t \to \pm \infty \), we have
\[ \gamma'(t)e^{+\omega|t|} \to \text{constant}, \quad \zeta(t)e^{-\omega|t|} \to \text{constant}. \]  \hspace{1cm} (6)

Furthermore, there exists \( k > 0 \) such that
\[ \begin{align*}
&\text{for } t \geq s \geq 0, \quad |\gamma'_i(t)\zeta_j(s)| < k \exp \left( -\omega(t-s) \right), \quad i,j = 1,2, \\
&\text{for } t \leq s \leq 0, \quad |\gamma'_i(t)\zeta_j(s)| < k \exp \left( \omega(t-s) \right), \quad i,j = 1,2.
\end{align*} \]  \hspace{1cm} (7, 8)

**Lemma 5** The system (2) has a bounded solution Iff
\[ \int_{\mathbb{R}} \frac{1}{\Delta(s)} f(\gamma(s)) \wedge F(s) ds = 0. \]  \hspace{1cm} (9)

**Proof**: The proof is under construction and will be appear after publishing.

A direct conclusion from the above lemma is that bounded solutions of (2) is obtained by
\[ \begin{align*}
z_1(t) &= \gamma'_1(t) \left( x_2 - \int_0^t \frac{\zeta(s)}{\Delta(s)} \wedge F(s) ds \right) + \zeta_1(t) \int_{-\infty}^t \frac{\gamma'(s)}{\Delta(s)} \wedge F(s) ds, \\
z_2(t) &= \gamma'_2(t) \left( x_2 - \int_0^t \frac{\zeta(s)}{\Delta(s)} \wedge F(s) ds \right) + \zeta_2(t) \int_{-\infty}^t \frac{\gamma'(s)}{\Delta(s)} \wedge F(s) ds.
\end{align*} \]  \hspace{1cm} (10, 11)

**4 Bifurcation map and homoclinic bifurcations**

In this section, we consider the equation (11) and look for its bounded solutions near \( \gamma(t) \). We will find a bifurcation map for the system and will investigate its homoclinic bifurcations. To this end, let \( x(t) \) be a solution of (11) of the form
\[ x(t-\beta) = \alpha \gamma(t) + z(t), \]  \hspace{1cm} (12)
with \( \alpha \approx 1 \) and \( 0 \leq |\beta| \ll 1 \). By replacing \( x(t-\beta) \) in (11) we get
\[ \dot{z} - A(t)z = F(t,z,\alpha,\beta,\epsilon), \]  \hspace{1cm} (13)
where \( A(t) = Df(\gamma(t)) \) and
\[
F(t, z, \alpha, \beta, \epsilon) = f(\alpha \gamma(t)) + Df(\alpha \gamma(t))z - \alpha f(\gamma(t)) - A(t)z \\
+ \epsilon g(\alpha \gamma(t) + z, t - \beta) + O(z^2, \epsilon z).
\]

It is easy to see that \( x(t) \) is bounded Iff \( z(t) \) is bounded; furthermore, \( \|x - \gamma\| = \|z\| \). Thus, if \( z(t) \) is a nontrivial bounded solution of (13) near zero then \( x(t) \) is a nontrivial bounded solution of (1) near \( \gamma \). Because of the exponential dichotomy of the variational equation \( \dot{z} = A(t)z \), the existence of a nontrivial bounded solution for (13) which is enough near to zero, equals to the existence of a homoclinic bifurcation for (1).

On the other hand, from Lemma 5 the solution \( z(t) \) of (13) is bounded Iff
\[
\int_{\mathbb{R}} \frac{f(\gamma(t))}{\Delta(t)} \land F(t, z(t), \alpha, \beta, \epsilon) dt = 0.
\]

Thus, in order to have \( x(t) \) bounded and near \( \gamma(t) \), we must investigate solutions \( z(t) \) of (13) near zero such that the above integral equality holds. For this purpose, we define the linear projection \( p \) as below
\[
p : C^0_{\text{b}}(\mathbb{R}, \mathbb{R}^2) \to C^0_{\text{b}}(\mathbb{R}, \mathbb{R}^2), \quad p : F(t) \mapsto \frac{\Delta^2(t)}{\|\gamma\|_2^2} J(t) \int_{\mathbb{R}} J(s) F(s) ds, \quad (14)
\]
where
\[
J(s) = \frac{1}{\Delta(s)} \begin{pmatrix} -\gamma_2'(s) \\ \gamma_1'(s) \end{pmatrix}.
\]

We can also consider (2) as the operator equation \( (Lz)(t) = F(t) \) where
\[
L : C^1_{\text{b}}(\mathbb{R}, \mathbb{R}^2) \to C^0_{\text{b}}(\mathbb{R}, \mathbb{R}^2), \quad (Lz)(t) = \dot{z}(t) - A(t)z(t). \quad (15)
\]

The lemma 5 implies that the enough and sufficient condition for a map \( F(t) = (F_1(t), F_2(t)) \in C^0_{\text{b}}(\mathbb{R}, \mathbb{R}^2) \) belongs to \( R(L) \) is that \( pF(t) = 0 \), i.e. \( R(L) = N(p) \). Also it is easy to see that
\[
N(L) = \{ \xi \gamma' : \xi \in \mathbb{R} \} \quad \text{and} \quad R(p) = \{ \xi \Delta(t) \begin{pmatrix} -\gamma_2'(t) \\ \gamma_1'(t) \end{pmatrix} : \xi \in \mathbb{R} \}
\]
are one dimensional subspaces, so we can consider \( \xi \in \mathbb{R} \) as an element of \( N(L) \) (or \( R(p) \)). Since \( I - p : C^0_{\text{b}}(\mathbb{R}, \mathbb{R}^2) = N(p) \oplus R^\perp(L) \to N(p) \) so \( k \), the inverse of \( L : N(L) \to N(p) \), is a well defined linear isomorphism as below
\[
k : (I - p)C^0_{\text{b}}(\mathbb{R}, \mathbb{R}^2) \to N(L),
\]
This enables us to decompose \( z \in C^1_{\text{b}}(\mathbb{R}, \mathbb{R}^2) \) as \( z = \xi + \eta \in N(L) \oplus R^\perp(L) \); then by using the Lyapunov-Schmidt reduction, the problem of finding bounded solutions for (13) is equivalent to the solving of the system of equations
\[
0 = pF(t, \xi + \eta, \alpha, \beta, \epsilon), \quad (16)
\]
\[
\eta(t) = k(I - p)F(t, \xi + \eta, \alpha, \beta, \epsilon). \quad (17)
\]
Let
\[ G : C^0_b([0,1] \times \mathbb{R}^2) \times \mathbb{R}^3 \to N^+(L), \]
\[ G(\xi + \eta, \alpha, \beta, \epsilon) := \eta - k(I-p)F(\xi + \eta, \alpha, \beta, \epsilon). \]
Then it is easy to check that \( G(0,1,0,0) = 0 \) and \( D_\eta G(0,1,0,0) = I \). Hence, by applying the implicit function theorem, there exist a neighborhood \( U \) of \((0,1,0,0) \in N(L) \times \mathbb{R}^3 \) and a unique map \( \eta : U \to N^+(L) \) such that \( \eta(0,1,0,0) = 0 \) and \( \eta(\xi, \alpha, \beta, \epsilon) \) is the unique solution of (17) in \( U \). Replacing \( \eta(\xi, \alpha, \beta, \epsilon) \) in (16), finally we obtain the bifurcation function as below
\[ pF(t, \xi + \eta(\xi, \alpha, \beta, \epsilon), \alpha, \beta, \epsilon) = 0 \]
or equivalently
\[ B(\xi, \alpha, \beta, \epsilon) = \int_{\mathbb{R}} J(s)F(s, \xi + \eta(\xi, \alpha, \beta, \epsilon), \alpha, \beta, \epsilon) ds = 0. \] (18)
It is useful to note that, by differentiating from (17) w.r.t. \( \xi \) we find that
\[ D_\xi \eta = [I - k(I-p)\frac{\partial F}{\partial z}]^{-1}k(I-p)\frac{\partial F}{\partial z} \]
which implies that
\[ D_\xi \eta(0,1,0,0) = 0. \] (19)

\textbf{Proof (Proof of theorem 1)} The proof is under construction and will be appear after publishing.

\textbf{Proof (Proof of theorem 2)} The proof is under construction and will be appear after publishing.

\textbf{Proof (Proof of theorem 3)} The proof is under construction and will be appear after publishing.

We end this section by giving an application of the theorem to the power-law nonlinear oscillatory system (20). Bifurcations of such systems have been widely studied in engineering and sciences (see, for example, [3,15,16,13,12,22]). They mostly concerned on investigating of chaotic motions by using the Melnikov method, however, our purpose here is finding a bounded solution near the perturbed homoclinic. Consider the system
\[ \dot{x} = y, \quad \dot{y} = \nu x - \mu x^{p+1} + \epsilon g(x, y, t) \] (20)
where \( \nu, \mu \) are positive parameters, \( p > 1 \) is the integral power of the strongly nonlinear term and \( g(x, y, t) \) is a self-excited force and damping. Let \( \gamma(t) = (x(t), y(t)) \) be the homoclinic of the unperturbed system, \( (\epsilon = 0) \), with \( \gamma(0) = (x_{\text{max}}, 0), x_{\text{max}} = \sqrt{\frac{(p+2)\nu}{2\mu}} \). The orbit of \( \gamma(t) \) is the graph of the functions
\[ y_{\pm}(x) = \pm x \sqrt{\nu - \frac{2\mu}{p+2} x^p}, \quad 0 \leq x \leq x_{\text{max}}. \]
It must be noted that, since (20) is Hamiltonian so the determinant $\Delta(t)$ is constant; thus it can be omitted from the computations through the conditions.

It is easy to see that $C_2$ holds, thus we calculate $F'_1$ in $C_1'$ for (20) and obtain:

$$F'_1 = \int_{\gamma(t)} g(x, y, t) dx$$

$$= \int_{0}^{x_{\text{max}}} g_2(x, y(x), t(x)) - g_2(x, y_-(x), -t(x)) dx$$

where $t(x) \leq 0$ shows the time $x(t) = x$. Thus, if one of the following assumptions holds then $F'_1 \neq 0$ and theorem 1 implies that (20) has a bounded solution near $\gamma$, for $|\epsilon| \ll 1$.

**A1:** The function $g(x, y, t)$ is increasing w.r.t. $y$ and it is even and bounded w.r.t. $t$.

**A2:** The function $g(x, y, t) \geq 0$ is even w.r.t. $y$ and it is odd and bounded w.r.t. $t$.

**A3:** The function $g(x, y, t) \geq 0$ is odd w.r.t. $y$ and it is even and bounded w.r.t. $t$.

5 Conclusion

Although homoclinic bifurcations are mostly known because of the chaotic behavior they might impose on a system, here we studied them from the bifurcation theory point of view. The chaotic behavior, of course, exists if the phase space can be decomposed to the direct sum of the stable and unstable subspaces of the corresponding variational equation. We left this for further studies in another paper. At this point, our results guarantee the existence and bifurcations of bounded solutions near an unperturbed homoclinic.

In a special case, when the function $g(x, t)$ in (1) is $T$-periodic w.r.t. $t$, the bounded solution implied by theorem 1 is a homoclinic orbit based on the unique hyperbolic $T$-periodic solution near the origin. The same statement is valid for bounded solutions in theorems 2 and 3. In a more general case, if (1) is time-dependent then from (??) and the structure of the proof of theorems 2 and 3 the bounded solutions $x_i(t), i = 1, 2$, are not a time rescale of each other. Thus these solutions are distinct.

If (1) is autonomous (i.e. $g(x, t) = g(x)$ is independent of $t$) then the bounded solution implied by theorem 1 is a unique homoclinic orbit based on a unique hyperbolic fixed point near the origin. In this case the two distinct solutions of theorem 2 and 3 are time rescales of each other.

Among the conditions of this paper, $C_5$ is probably the hardest to verify. It needs usually to be verified numerically due to the unbounded solution $\zeta(t)$ which appears in the formula. What is interesting is that, although the conditions $C_1$, $C_4$ and $C_4'$ are generic, they always fail for Hamiltonian systems because of $F_{4,2} = F'_{4,2} = 0$. Thus, for applying the theorems 2 and 3 for a Hamiltonian system, we have to make a change of variables and bring
the system into a non-Hamiltonian system. Indeed the theorems of this paper are easier to apply for non-Hamiltonian systems.

Finally, It is useful to note that, although the conditions of this paper are formulated by terms of usual integrals on $\mathbb{R}$ or $\mathbb{R}^2$, for a real application, it is more appropriate to consider them as integrals on the curve $\gamma(t)$ (see the example of section 4). The later mentioned forms are considered because they are more suitable for the proofs of theorems.

6 Conflict of interest

The authors declare that they have no conflict of interest.

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