SRB-SYMMETRIC DIFFUSIONS ON HYPERBOLIC 
ATTRACTORS

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ABSTRACT. We prove the existence of symmetric diffusions and self-adjoint 
Laplacians on (uniformly) hyperbolic attractors, endowed with SRB-measures. 
The proof is based on Dirichlet form theory. We observe some features of such 
diffusions, for instance, a quasi-invariance property of energy densities in the 
$u$-conformal case and the existence of non-constant harmonic functions of zero 
energy in the ergodic case.

CONTENTS

1. Introduction 1
Acknowledgements 4
2. Preliminaries on uniformly hyperbolic attractors 4
3. Preliminaries on measures and absolute continuity 6
4. Restriction of smooth functions 8
5. Self-adjoint leafwise Laplacians 9
6. Symmetric Markov semigroups 11
7. Regularity and symmetric leafwise diffusions 11
8. Quasi-invariance in the $u$-conformal case 11
9. Functions of zero energy and measurable partitions 19
10. Dirichlet forms and semigroups on domains 22
11. Examples 23
Appendix A. Disintegration and Rokhlin’s theorem 24
Appendix B. The geometric construction of SRB-measures 24
Appendix C. Dirichlet integrals on weighted manifolds 27
Appendix D. Superposition of closable forms 30
References 31

1. INTRODUCTION

In the present paper we consider (uniformly) hyperbolic attractors, $^{13,14,20}$ $^{28,59,64}$, endowed with SRB-measures, $^{9,27,28,53,59,81}$. We assume that the 
densities of their conditional measures are bounded and bounded away from zero, 
which is always the case for the standard constructions by taking limits of forward 
averages, see e.g. $^{60}$. We prove the existence of diffusions and Laplacians that move 
respectively act leafwise in the unstable directions and are symmetric respectively 
self-adjoint with respect to the given SRB-measure.

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On smooth manifolds, graphs and certain groups and metric measure spaces symmetric diffusions and self-adjoint Laplacians are well-understood. Their study provides additional insight into structural features of the space, reflected in the spectrum of the Laplacian, in the behaviour of related partial differential equations and in particular, in properties of the associated heat semigroup. A self-adjoint Laplacian is also the basis for a corresponding elliptic approach to differential forms. In terms of geometric complexity, attractors of dynamical systems may be more involved than manifolds, but they still display more features of smoothness than say, fractals. Analysis on manifolds and on certain classes of fractals is well-developed (see for instance [30,38,49] and the references cited there), but a similar analysis on hyperbolic attractors is yet to be explored. On the other hand hyperbolic attractors have natural tangential structures, unambiguously determined by the stable manifold theorem, [14,20,64,68], and they carry distinguished volume measures that capture the dynamics of the system in an optimal way, namely SRB-measures, [18,19,25,27,52,53,60,67,69,70,81]. In this respect the situation is even better than for sub-Riemannian manifolds, for which the choice of a suitable volume measure is a delicate matter.

There are classical, [40,41], and new, [4,5,72], results on analysis on inverse limit spaces, and in particular, there is a well-developed analysis on abstract solenoids from a group theoretic point of view, cf. (10.12) Definition and (10.13) Theorem. At least expanding hyperbolic attractors in Williams’ sense, [80], are, roughly speaking, topologically conjugate to such abstract solenoids, see Remark 2.2 below. A related theory of metric spaces with similar structure had been established in [62]. The existence results for diffusions and Laplacians we provide here are new, and they are of a different type: Since they are based on the stable manifold theorem and the concept of SRB-measures, they are results in the differentiable category rather than in the topological. A related study of symmetric diffusions on pattern spaces of aperiodic Delone sets can be found in [6]. A detailed analysis of symmetric diffusions on fractals invariant under Kleinian groups, which in some aspects is also connected to our topic, is provided in the recent articles [46,47]. Symmetric diffusions and self-adjoint Laplacians on certain repellers (Julia sets) had been constructed in [65].

We use a Dirichlet form approach, [8,17,30,33], which seems rather well-adapted to the situation: The minimal smoothness requirement for both the stable manifold theorem, Theorem 2.1, and the existence result for SRB-measures, Theorem 3.1, is that the given diffeomorphism is $C^{1+\alpha}$. In this case the local unstable manifolds are $C^1$, so that one cannot talk about $C^2$-functions in leafwise direction. Moreover, even if the diffeomorphism is $C^{r+\alpha}$ with $r \geq 2$, the densities of the conditional measures on the local unstable manifolds can only be expected to be Hölder (or maximally Lipschitz), cf. Theorem 3.1 and Appendix B, so that one is confronted with weighted Riemannian manifolds with densities of too low regularity to introduce classical Laplacians. On the other hand, $C^1$-regularity in leafwise direction is available, and this is sufficient to introduce quadratic forms, Section 5 and Appendix C. The circumstances remind of the situation encountered with second order differential operators in divergence form and with nonsmooth coefficients, and these are typically studied using Dirichlet forms.

There is also a rich theory on analysis on foliated spaces, [21,23,45,56], and the abstract solenoids conjugate to expanding attractors fit into this class, [22, p.284].
Although our goals and methods are quite different, a brief comparison of some aspects might help to put our results into context. In [35] diffusion semigroups on foliated manifolds were introduced using leafwise Laplacians and heat semigroups on the leaves, and in [23] and [21] this was generalized to foliated spaces. A strong incentive for these studies was that invariant measures for the semigroups, referred to as harmonic measures, provide structurally interesting volume measures on the space. These measures are naturally created by and optimally adapted to the dynamics of the heat semigroup. Hyperbolic attractors originate from a given dynamical system, and SRB-measures are already the naturally created and optimally adapted volumes. The semigroups constructed here act on $L^2$-spaces, and by construction they are symmetric with respect to the given SRB-measure, but we do not claim they are Feller. (In probabilistic terms this symmetry of the semigroup means that the associated diffusion process is reversible, [8], with respect to the reference measure.)

In principle our method is straightforward: Restricting the Riemannian metric of the ambient manifold to the unstable manifolds, one can introduce a natural gradient on functions that are leafwise $C^1$ in the unstable directions. Such functions are dense in $L^2$-spaces, as can be seen by restricting $C^1$-functions on the manifold to the attractor, and one can then define an analog of the classical Dirichlet integral on a dense a priori domain, cf. formulas (5.2) and (5.3) in Section 5. Also bump functions and partitions of unity can be imported by restriction, Section 4. This allows an easy localization to rectangles, in which the SRB-measures disintegrate with conditional measures on the local unstable manifolds that are equivalent to their Riemannian measures, cf. formula (3.2) in Section 3. We can interpret the local unstable manifolds as weighted manifolds (with a low-regularity volume weight that is bounded and bounded away from zero), and Dirichlet integrals on such weighted manifolds are closable, Corollary C.1. Using well-known results on the closability of superpositions of closable forms, cf. Proposition D.1 we can eventually conclude that the Dirichlet integral on the attractor is closable and that its closure is a Dirichlet form. By general Dirichlet form theory, [17,30,33], this implies the existence of a self-adjoint Laplacian and symmetric Markov semigroup, Theorems 5.1 and 6.1 and also the existence of a symmetric diffusion follows, Theorem 7.1. We complement these existence results by some first statements on properties of these objects. One observation is that if the diffeomorphism is conformal in the unstable directions ($\psi$-conformal), then the constructed Dirichlet form displays a kind of quasi-invariance property under iterates of the diffeomorphism, Theorem 8.1.
and Corollary 8.1. Another observation is that in the ergodic case there may be non-
constant harmonic functions of zero energy, Corollary 9.1. We finally provide a short
discussion of the relationship between Dirichlet forms and semigroups restricted to
domains in rectangles on the one hand and corresponding objects on the local
unstable manifolds on the other hand, Proposition 10.1 and Corollary 10.2.

Extensions to more general hyperbolic attractors with SRB-measures, [16, 25, 27,
59, 60], and possibly to other equilibrium states, [28], spectral properties (motivated
by [9, 13, 36, 57] on the one hand and [51, 73, 76] on the other), and geometric analysis
questions (related to [56] and to [6, 44]), will be considered in follow up articles.

In Sections 2 and 3 we collect well-known facts on uniformly hyperbolic attrac-
tors and SRB-measures, and in Section 4 some preliminaries on the restriction of
functions from the ambient manifold to the attractor. The Dirichlet integral is
defined in Section 5 and its closability is proved, this also implies the existence of
associated self-adjoint Laplacians. Section 6 concludes the existence of symmetric
semigroups and Section 7 that of symmetric diffusions. Quasi-invariance properties
in the $u$-conformal case are discussed in Section 8, the structure of functions of
zero energy is studied in Section 9. Forms, Laplacians and semigroups restricted
to domains are commented on in Section 10. To make the article as self-contained
as possible, we provide basic facts on Rokhlin’s theorem in Appendix A, a brief
sketch of the geometric approach to SRB-measures in Appendix B, a discussion
on weighted manifolds of low regularity in Appendix C and a sketch of a known
superposition argument for quadratic forms in Appendix D.

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2. Preliminaries on uniformly hyperbolic attractors

We collect preliminaries on topological attractors and uniformly hyperbolic dy-
namical systems.

Let $X$ be a compact topological space and $f : X \to X$ be a continuous map. A
compact subset $\Lambda \subset X$ is called a (topological) attractor for $f$ if there is an open
neighborhood $U$ of $\Lambda$ (referred to as trapping region) such that $f(U) \subset U$ and

\begin{equation}
\Lambda = \bigcap_{n \geq 0} f^n(U).
\end{equation}

Clearly $\Lambda$ is an $f$-invariant subset of $X$, $f(\Lambda) = \Lambda$. If $\Lambda$ is an attractor, then $f|_\Lambda$
is called topologically transitive if for any nonempty open sets $V, W \subset \Lambda$ there is
some $n \geq 0$ such that $f^n(V) \cap W \neq \emptyset$.

Let $M$ be a compact smooth Riemannian manifold, let $r \geq 1$ be an integer or
$r = +\infty$ and $\alpha \in (0, 1]$ and suppose that $f : M \to M$ is a $C^{r+\alpha}$ diffeomorphism. A
compact $f$-invariant subset $\Lambda \subset M$ is called a uniformly hyperbolic set if for each $p \in \Lambda$
there is a $df$-invariant splitting of the tangent space $T_p M = E^s(p) \oplus E^u(p)$
and there are constants $C > 0$, $\lambda \in (0, 1)$ such that for each $p \in \Lambda$ we have

\[ \|d_pf^n v\|_{T_p f^n(M)} \leq C \lambda^n \|v\|_{T_p M} \quad \text{for } v \in E^s(p) \text{ and } n \geq 0; \]

\[ \|d_pf^{-n} v\|_{T_p f^{-n}(M)} \leq C \lambda^n \|v\|_{T_p M} \quad \text{for } v \in E^u(p) \text{ and } n \geq 0. \]
The subspaces $E^s(p)$ and $E^u(p)$ are called stable and unstable subspaces at $p$, respectively. A topological attractor $\Lambda \subseteq M$ is called a uniformly hyperbolic attractor if it is a uniformly hyperbolic set. See for instance [20, Section 5.2].

One of the central results in the theory of hyperbolic dynamical systems is the Stable Manifold Theorem, which we quote to fix notation and as a reference. See [14, Chapter 7] or [20, Section 5.6] for a proof and further details. By $d$ we denote the geodesic distance on $M$.

**Theorem 2.1.** Let $f : M \to M$ be a $C^{\alpha+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold $M$, and $\Lambda \subseteq M$ be a uniformly hyperbolic set with constants $C > 0$ and $\lambda \in (0, 1)$. Then

(i) For every $p \in \Lambda$ and any sufficiently small $\varepsilon > 0$, the sets

$$W^s_\varepsilon(p) := \{ q \in M \mid d(f^n(p), f^n(q)) < \varepsilon, \ \forall n \geq 0 \},$$

$$W^u_\varepsilon(p) := \{ q \in M \mid d(f^{-n}(p), f^{-n}(q)) < \varepsilon, \ \forall n \geq 0 \}$$

are $C^\alpha$-embedded submanifolds of $M$. Their dimensions $d_s = \dim W^s_\varepsilon(p)$ respectively $d_u = \dim W^u_\varepsilon(p)$ are constant on $\Lambda$ and satisfy $d_s + d_u = \dim M$.

(ii) For every $p \in \Lambda$ and any sufficiently small $\varepsilon > 0$ we have

$$T_p W^s_\varepsilon(p) = E^s(p) \quad \text{and} \quad T_p W^u_\varepsilon(p) = E^u(p).$$

(iii) For every $p \in \Lambda$ and any sufficiently small $\varepsilon > 0$ we have

$$f(W^s_\varepsilon(p)) \subset W^s_\varepsilon(f(p)) \quad \text{and} \quad f^{-1}(W^u_\varepsilon(p)) \subset W^u_\varepsilon(f^{-1}(p)).$$

(iv) We have

$$d(f^n(p), f^n(q)) \leq C\lambda^n d(p, q), \quad q \in W^s_\varepsilon(p),$$

and

$$d(f^{-n}(p), f^{-n}(q)) \leq C\lambda^n d(p, q), \quad q \in W^u_\varepsilon(p).$$

To $W^s_\varepsilon(p)$ and $W^u_\varepsilon(p)$ one refers as the local stable and local unstable manifolds of $p$, respectively.

Given a point $p \in \Lambda$, the global stable and global unstable manifolds of $p$ are defined as

$$W^s(p) = \bigcup_{n \geq 0} f^{-n}(W^s_\varepsilon(f^n(p))) \quad \text{and} \quad W^u(p) = \bigcup_{n \geq 0} f^n(W^u_\varepsilon(f^{-n}(p))).$$

They are $C^\alpha$-injectively immersed submanifolds of $M$, and can be topologically characterized by

$$W^s(p) = \{ q \in M \mid d(f^n(p), f^n(q)) \to 0, \ \text{as} \ n \to \infty \}$$

and

$$W^u(p) = \{ q \in M \mid d(f^{-n}(p), f^{-n}(q)) \to 0, \ \text{as} \ n \to \infty \}.$$
Remark 2.1. If \( \Lambda = M \) then \( f \) is called an Anosov diffeomorphism. In this case the set \( \mathcal{F}^u \) forms a (Hölder-) continuous foliation on \( M \) with \( C^r \) leaves in the sense of \([20]\) Section 5.13, see also \([14]\) Section 1.1, p.7] or \([48]\) remarks after Definition A.3.5). In general this foliation is not smooth, and we cannot claim that \((M, \mathcal{F}^u)\) is a foliated manifold in the sense of \([22]\) Definition 1.1.18). We have \( \dim M \geq d_u \), the inequality being strict unless \( f \) is purely expanding.

Remark 2.2. If the topological dimension \( \dim \Lambda \) of the compact space \( \Lambda \) equals the dimension \( d_u \) of the unstable leaves, \( \dim \Lambda = d_u \), then \( \Lambda \) is called an expanding attractor; \([80]\) Definition (4.2)]. An \( n \)-solenoid \( \Sigma \) in the sense of \([80]\) Definition (3.1)] is a foliated space in the sense of \([22]\) Definition 11.2.12], and one can find a manifold \( M \) and a \( C^r \)-diffeomorphism \( f : \tilde{M} \to M \) having an expanding attractor \( \Lambda \) such that \( f|\Lambda \) is conjugate to the shift map \( h : \Sigma \to \Sigma \) of \( \Sigma \). This is proved in \([80]\) Theorems B and C], see also \([22]\) p. 284]. Under additional regularity assumptions one can also find a conjugate \( n \)-solenoid for a given expanding attractor. \([80]\) Theorem A]. (These results may be seen as far reaching generalizations of the elementary fact that the dyadic solenoid is conjugate to an inverse limit space. \([64]\) Section 8.7.1.)

Uniformly hyperbolic attractors \( \Lambda \) possess a local product structure: There exist \( \varepsilon > 0 \) and \( \delta > 0 \) such that

(i) for every \( p, q \in \Lambda \), \( W^s_\varepsilon(p) \cap W^u_\varepsilon(q) \) consists of at most one point which belongs to \( \Lambda \),

(ii) for every \( p, q \in \Lambda \) with \( d(p, q) < \delta \), \( W^s_\varepsilon(p) \cap W^u_\varepsilon(q) \) consists of exactly one point of \( \Lambda \) denoted by \( [p, q] \),

see for instance \([20]\) Proposition 5.9.3] for further details. For any \( p \in \Lambda \) the map

\[
\Pi_\varepsilon : B(p, \delta) \cap \Lambda \to W^s_\varepsilon(p), \quad q \mapsto \Pi_\varepsilon(q) := [p, q],
\]

is continuous, and there exists \( 0 < r < \delta/2 \) such that for all \( p \in \Lambda \) the bracket map \([\cdot, \cdot] \) is a homeomorphism from \( (W^s_\varepsilon(p) \cap B(p, r) \cap \Lambda) \times (W^u_\varepsilon(p) \cap B(p, r)) \) onto an open neighborhood of \( p \) in \( \Lambda \), see \([68]\) Proposition 10.10 and its proof].

Remark 2.3. In general one has \( \dim(W^s_\varepsilon(p) \cap \Lambda) = \dim \Lambda - d_u \) for each \( p \in \Lambda \). For expanding attractors this is zero, and \( W^s_\varepsilon(p) \cap \Lambda \) is homeomorphic to a Cantor set, \([80]\) Lemma 4.3, formula (4.3.1) and Theorems A and C].

A subset \( \mathcal{R} \subset \Lambda \) of the attractor \( \Lambda \) with diameter smaller than \( \delta \) is said to be a rectangle if \( [p, q] \in \mathcal{R} \) whenever \( p, q \in \mathcal{R} \), \([68]\) Definition 10.11].

For an arbitrary point \( x \in \Lambda \) we can find a rectangle \( \mathcal{R}_x \) (which we may additionally choose to be open or closed in \( \Lambda \)) and a neighborhood \( U_x \subset M \) of \( x \), open in \( M \), such that \( \overline{U_x} \cap \Lambda \subset \mathcal{R}_x \). Moreover, since \( \Lambda \) is a compact subset of \( M \), we can find \( x_1, \ldots, x_N \in \Lambda \) so that

\[
\Lambda \subset \bigcup_{i=1}^{N} U_{x_i} \subset \bigcup_{i=1}^{N} \mathcal{R}_{x_i}.
\]

3. Preliminaries on measures and absolute continuity

We recall some basics on measures on the attractor \( \Lambda \) and their disintegration on rectangles, and in particular, we recall the definition of SRB-measures.

Let \( \mathcal{R} \subset \Lambda \) be a rectangle. For \( q \in \mathcal{R} \) we define

\[
W^u_\mathcal{R}(q) = W^u_\varepsilon(q) \cap \mathcal{R} \quad \text{and} \quad W^s_\mathcal{R}(q) = W^s_\varepsilon(q) \cap \mathcal{R},
\]
Recall that by the local product structure for each \((3.2)\) a finite Borel measure \(\hat{\mu}\) we can write this partition as 

\[
P_R \text{ if and only if } \mu \text{ is absolutely continuous w.r.t. } m.
\]

By Theorem A.1 there are a finite Borel measure \(\hat{\mu}_R\) on \(P_R\) and a family \(\{\mu_{W_R^s(z)}\}_{z \in W_R^s(p)}\) of finite Borel measures \(\mu_{W_R^s(z)}\) on the sets \(W_R^s(z)\), respectively, into which \(\mu\) disintegrates. It is convenient to interpret \(\hat{\mu}_R\) as a finite Borel measure on \(W_R^s(p)\), so that by the disintegration formula (A.1) we have

\[
\mu(E) = \int_{W_R^s(p)} \int_{E \cap W_R^s(z)} \mu_{W_R^s(z)}(dy) \hat{\mu}_R(dz) \quad \text{for any } E \subset R \text{ Borel.}
\]

Recall that by the local product structure for each \(q \in R\) there is a unique point \([p,q] \in W_R^s(p)\) such that \(W_R^s([p,q]) = W_R^s(q)\).

For each \(q \in \Lambda\) let now \(m_{W_R^s(q)}\) denote the Riemannian volume on the local unstable manifold \(W_R^u(q)\), one also refers to it as the leaf volume on \(W_R^u(q)\). We say that a finite Borel measure \(\mu\) on \(\Lambda\) satisfies the (AC)-property if for any closed rectangle \(R\) and \(\mu\)-a.e. \(q \in R\) the measures \(\mu_{W_R^s(q)}\) on the sets \(W_R^s(q)\) are absolutely continuous with respect to \(m_{W_R^s(q)}\). \(\hat{\mu}_R\) is known as a measure satisfying the (AC)-property, then it follows automatically that the measures \(\mu_{W_R^s(q)}\) are actually equivalent to \(m_{W_R^s(q)}\), respectively. If \(\mu\) satisfies the (AC)-property and in addition for each measure \(\mu_{W_R^s(q)} = \rho_{W_R^s(q)} \cdot m_{W_R^s(q)}\) the Radon-Nikodym density \(\rho_{W_R^s(q)}\) is bounded and bounded away from zero on \(W_R^s(q)\), then we say that \(\mu\) has uniformly bounded densities.

Remark 3.1. For a given rectangle \(R\) with \(\mu(R) > 0\) the measures \(\mu_{W_R^s(q)}\) are absolutely continuous w.r.t. \(m_{W_R^s(q)}\) for \(\mu\)-a.e. \(q \in R\) if and only if the same absolute continuity holds for \(\hat{\mu}_R\)-a.e. \(z \in W_R^s(p)\): If \(S \subset W_R^s(p)\) is such that \(\hat{\mu}_R(S) > 0\) and absolute continuity fails for all \(z \in S\) then by (3.2) we would obtain

\[
\mu(S) = \int_{W_R^s(p)} \mu_{W_R^s(z)}(dy) \hat{\mu}_R(dz) = \hat{\mu}_R(S) > 0,
\]

for the set \(\Pi^1_s(S) = \{q \in R \mid [p,q] \subset S\}\), here \(\Pi_s\) is the (restriction to \(R\) of the) map from (2.4), and by its continuity \(\Pi^1_s(S)\) is a Borel subset of \(R\). The other implication is trivial.

An \(f\)-invariant Borel probability measure \(\mu\) on \(\Lambda\) which satisfies the (AC) property is called an SRB- (Sinai-Ruelle-Bowen-) measure. We quote the following well-known result, which was first proved using methods from statistical physics, [18, 19, 67, 69, 70], and later using a geometric approach, [52, 60]. Detailed accounts with precise attributions can be for instance found in [28, 81].

**Theorem 3.1.** Assume that \(f\) is a \(C^{1+\alpha}\) diffeomorphism and \(\Lambda\) is a uniformly hyperbolic attractor. Then there is an SRB-measure \(\mu\) on \(\Lambda\) with uniformly bounded H"older continuous densities. If \(f\)|\(\Lambda\) is topologically transitive, then this measure is the only SRB-measure on \(\Lambda\), supp \(\mu = \Lambda\) and \(\mu\) is ergodic.

**Remark 3.2.** Recall that a measure on \(M\) is called smooth if it has a continuous density with respect to the Riemannian volume on \(M\). For Anosov diffeomorphisms the existence of an ergodic SRB-measure is classical, [7], see [14 Theorem 1.1]: Any \(C^{1+\alpha}\)-Anosov diffeomorphism of a smooth compact Riemannian manifold \(M\)
keeping a smooth measure invariant is ergodic with respect to this measure, and this measure is an SRB-measure on $M$ with uniformly bounded Hölder continuous densities.

4. Restriction of smooth functions

Let $M$ be a compact smooth Riemannian manifold and let $\Lambda$ be the uniformly hyperbolic attractor associated with a given $C^{r+\alpha}$-diffeomorphism $f : M \to M$, where $r \geq 1$ is an integer or $+\infty$ and $\alpha \in (0, 1]$.

Recall that $\Lambda$ is a compact subset of $M$. As usual we denote the space of continuous functions on $\Lambda$ (with respect to the subspace topology) by $C(\Lambda)$. We write $C^k(\Lambda)|_\Lambda := \{g|_\Lambda : g \in C^k(M)\}$, $k \leq r$, to denote the space of pointwise restrictions $g|_\Lambda$ to $\Lambda$ of $C^k(M)$-respectively $C^k(\Lambda)$-functions $g$. Note that we have $C^0(\Lambda)|_\Lambda = C(\Lambda)$ by Tietze’s extension theorem.

Given a nonnegative integer $k \leq r$ we say that a Borel function $\varphi : \Lambda \to \mathbb{R}$ is of class $C^k$ in the leafwise sense if for any $x \in \Lambda$ the function $\varphi|_{W^u(x)}$ is in $C^k(W^u(x))$. We write $C^{u,k}(\Lambda)$ for the vector space of leafwise $C^k$-functions on $\Lambda$ and use the shortcut notation $C^u(\Lambda) := C^{u,0}(\Lambda)$ for the space of leafwise continuous functions on $\Lambda$. For any $x \in \Lambda$ the restriction $\varphi|_{W^u(x)}$ of a function $\varphi \in C^{u,k}(\Lambda)$ to the global unstable manifold $W^u(x)$ is an element of $C^k(W^u(x))$.

**Proposition 4.1.** For any $k \leq r$ we have $C^k(\Lambda)|_\Lambda \subset C^{u,k}(\Lambda)$, in particular, $C(\Lambda) \subset C^{u}(\Lambda)$.

**Proof.** For any $x \in \Lambda$ the pointwise restriction $g|_\Lambda$ of a function $g \in C^k(\Lambda)$ is in $C^k(W^u(x))$ because $W^u(x)$ is a $C^r$-embedded submanifold, [55, Theorem 5.27].

**Remark 4.1.** Not every function in $C(W^u(x))$ is necessarily a restriction of a function from $C(\Lambda)$, and not every function in $C^{u}(\Lambda)$ is in $C(\Lambda)$.

Clearly $C^{k+1}(\Lambda)|_\Lambda \subset C^{k}(\Lambda)|_\Lambda$ and $C^{u,k+1}(\Lambda) \subset C^{u,k}(\Lambda)$, and the spaces form algebras under pointwise multiplication.

As usual, we endow each embedded submanifold $W^u_\varepsilon(x)$ with the restriction of the original Riemannian metric on $M$, making it a Riemannian manifold. By $\nabla_M$ and $\nabla_{W^u_\varepsilon(x)}$ we denote the gradient operators on $M$ and on $W^u_\varepsilon(x)$, respectively.

**Proposition 4.2.** For any $g \in C^1(M)$ and any $x \in \Lambda$ we have

$$
\|\nabla_{W^u_\varepsilon(x)}g|_\Lambda(x)\|_{T_xW^u_\varepsilon(x)} \leq \|\nabla_Mg(x)\|_{T_xM}.
$$

**Proof.** We abbreviate $W^u_\varepsilon(x)$ by $W$ and write $P_{T_xW}$ to denote the orthogonal projection in $T_xM$ onto $T_xW$. By definition the gradient $\nabla_Wg|_\Lambda(x)$ on $W$ of $g|_\Lambda$ at $x$ is the unique element of $T_xW$ such that

$$
v(g|_\Lambda) = \langle \nabla_Wg|_\Lambda(x), v \rangle_{T_xW}
$$

for all $v \in T_xW$. For any such $v$ we can find an open interval $I \subset \mathbb{R}$ around zero and a $C^1$-curve $\gamma : I \to W \subset M$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$, and we have

$$
v(g|_\Lambda) = \frac{d}{dt}g(\gamma(t))|_{t=0} = v(g).
$$

For the gradient $\nabla_Mg(x)$ on $M$ of $g$ at $x$ we have, again by definition,

$$
v(g) = \langle \nabla_Mg(x), v \rangle_{T_xM} = \langle P_{T_xW}(\nabla_Mg(x)), v \rangle_{T_xW} = \langle P_{T_xW}(\nabla_Mg(x)), v \rangle_{T_xW},
$$

where $\nabla_Mg$ is the gradient of $g$ on $M$. This completes the proof.
because $v$ is an element of $T_x W$. Since $v$ was arbitrary, this implies the equality $\nabla_W g|_\Lambda(x) = P_{T_x W}(\nabla_M g(x))$ in $T_x W$, from which the statement follows.

**Remark 4.2.** For any $x \in \Lambda$ we have $T_x W^u(x) = T_x W^u_\epsilon(x)$, and for a function $\varphi \in C^{u,1}(\Lambda)$ also $\nabla W^u(\varphi)(x) = \nabla W^u_\epsilon(\varphi)(x)$.

We endow the space $C^u(\Lambda)$ with the supremum norm, $\|\varphi\|_{\sup,\Lambda} := \sup_{x \in \Lambda} |\varphi(x)|$, and the space $C^{u,1}(\Lambda)$ with the norm

$$\|\varphi\|_{C^{u,1}(\Lambda)} := \|\varphi\|_{\sup,\Lambda} + \sup_{x \in \Lambda} \|\nabla W^u(\varphi)(x)\|_{T_x W^u(x)}.$$

For the subspaces $C^k(M)|_\Lambda$ as in Proposition 4.1 one can observe the following.

**Corollary 4.1.** For any $1 \leq r \leq k$ the space $C^k(M)|_\Lambda$ is dense in $C(\Lambda)$ and for all $1 \leq k \leq r$ also in $C^1(M)|_\Lambda$.

**Proof.** For any $g \in C^1(M)$ we can find a sequence $(g_k)_k \subset C^k(M)$ with $\lim_k g_k = g$ in $C^1(M)$, and by Proposition 4.2 we have

$$\limsup_{k} \|\nabla W^u(x)(g(x) - g_k(\Lambda))(x)\|_{T_x W^u(x)} \leq \limsup_{k} \|\nabla M(g(x) - g_k(x))\|_{T_x M} = 0.$$

The remaining statements are trivial by pointwise restriction.

Proposition 4.2 allows to import finite partitions of unity from $M$. Let $\{\chi_i\}_{i=1}^N$ be a $C^k$-partition of unity subordinate to a given finite open cover $\{U_i\}_{i=1}^N$ of the compact set $\Lambda$, that is, each $\chi_i$ is a nonnegative $C^k$-function on $M$ with compact support contained in $U_i$, we have $\sum_{i=1}^N \chi_i \equiv 1$ on a neighborhood of $\Lambda$ and $\sum_{i=1}^N \chi_i \leq 1$ everywhere on $M$. Then the family $\{\chi_i^\Lambda\}_{i=1}^N$ of functions $\chi_i^\Lambda := \chi_i|_\Lambda$ is a partition of unity on $\Lambda$ subordinate to the cover $\{U_i \cap \Lambda\}_{i=1}^N$ of $\Lambda$, and Proposition 4.2 ensures that their gradients are bounded:

**Corollary 4.2.** Let $\{U_i\}_{i=1}^N$ and $\{\chi_i\}_{i=1}^N$ as above.

(i) Each $\chi_i^\Lambda$ is a nonnegative element of $C^k(\Lambda)|_\Lambda$ compactly supported in $U_i \cap \Lambda$ and such that $\sum_{i=1}^N \chi_i^\Lambda \equiv 1$ on $\Lambda$.

(ii) If $1 \leq k \leq r$ then we also have

$$\max_{1 \leq i \leq N} \sup_{x \in \Lambda} \|\nabla W^u(x)(\chi_i^\Lambda(x))\|_{T_x W^u(x)} < \max_{1 \leq i \leq N} \|\chi_i\|_{C^1(M)}.$$

5. **Self-adjoint leafwise Laplacians**

Let $f : M \to M$ be a $C^{r+\alpha}$-diffeomorphism with $r \geq 1$ and $\Lambda$ a uniformly hyperbolic attractor for $f$. Since by Theorem 2.1 the local and global unstable manifolds $W^u_\epsilon(p)$ and $W^u(p)$ are of class $C^1$, we can introduce a ‘classical’ leafwise gradient operator $\nabla$ on $C^{u,1}(\Lambda)$ by

$$\nabla \varphi(x) := \nabla W^u_\epsilon(\varphi)(x), \quad x \in \Lambda, \quad \varphi \in C^{u,1}(\Lambda).$$

In the following we will generally suppress the pointwise restrictions $\varphi \mapsto \varphi|_{W^u(x)}$ from notation, unless there is a specific reason to point it out, and for instance simply write $\nabla W^u(\varphi)(x)$ for the right hand side of (2.1).

Given a finite Borel measure $\mu$ on $\Lambda$, we define a quadratic form $(\mathcal{E}(\mu), \mathcal{D}_0(\mathcal{E}(\mu)))$ by

$$\mathcal{D}_0(\mathcal{E}(\mu)) := \left\{ \varphi \in C(\Lambda) \cap C^{u,1}(\Lambda) : x \mapsto \|\nabla \varphi(x)\|_{T_x W^u(x)} \text{ is in } L^2(\Lambda, \mu) \right\}.$$
and
\begin{equation}
E(\mu)(\varphi) := \int_{\Lambda} \|\nabla \varphi(x)\|_{T_{x,W^2}(x)}^2 \mu(dx), \quad \varphi \in D_0(E(\mu)).
\end{equation}

By Proposition 4.2 we have \(C^1(M) \subset D_0(E(\mu))\). The form \((E(\mu), D_0(E(\mu)))\) is an analog of the classical Dirichlet integral. We extend it to a standard functional analysis setup, \[63\], Section VIII.6, and begin with a density result for \(L^2\)-spaces.

**Proposition 5.1.** For any finite Borel measure \(\mu\) on \(\Lambda\) and any nonnegative integer \(k \leq r\) the space \(C^k(M) \subset D(\mu)\) is a dense subspace of \(L^2(\Lambda, \mu)\).

For convenience we provide the standard argument.

**Proof.** Let \(\varphi \in L^2(\Lambda, \mu)\). Choosing a Borel version of \(\varphi\) we may, in view of approximability by bounded functions, assume that \(\varphi\) is a bounded Borel function on \(\Lambda\). Given \(\delta > 0\) we can, by Lusin’s theorem, find a subset \(\Lambda_\delta \subset \Lambda\), compact w.r.t. the subspace topology inherited from \(M\), such that \(\mu(\Lambda \setminus \Lambda_\delta) < \delta (2\|\varphi\|_{\sup, \Lambda} + 1)^{-2}\) and \(\varphi|_{\Lambda_\delta}\) is continuous on \(\Lambda_\delta\). Since \(\Lambda_\delta\) is a compact subset of \(M\), Tietze’s extension theorem shows that there is some \(g \in C(M)\) such that \(\varphi|_{\Lambda_\delta} = g|_{\Lambda_\delta}\) and \(\|g\|_{\sup} = \sup_{x \in \Lambda_\delta} |\varphi(x)|\). Let \((g_j)_{j=1}^\infty\) be a sequence of \(C^k(M)\)-functions \(g_j\) such that \(\lim_j \|g - g_j\|_{\sup} = 0\). Then for any large enough \(j\) we have

\[
\int_{\Lambda} |\varphi - g_j|^2 d\mu \leq \delta + \int_{\Lambda_\delta} |\varphi - g_j|^2 d\mu \leq \delta + \|g - g_j\|_{\sup}^2 \mu(\Lambda),
\]

and letting \(j \to \infty\) implies the result. \(\square\)

Since from a quadratic form \(u \mapsto \mathcal{E}(u)\) one obtains a uniquely defined symmetric bilinear form \((u, v) \mapsto \mathcal{E}(u, v)\) by polarization, we use both notions in parallel, and we use the shortcut notation \(E(u) := \mathcal{E}(u, u)\).

A pair \((\mathcal{E}, D(\mathcal{E}))\) is said to be a *densely defined nonnegative definite and symmetric bilinear form* on the Hilbert space \(L^2(\Lambda, \mu)\) if \(D(\mathcal{E})\) is a dense subspace of \(L^2(\Lambda, \mu)\) and \(\mathcal{E}\) is a nonnegative definite and symmetric bilinear form on \(D(\mathcal{E})\). In addition \(D(\mathcal{E})\), endowed with the scalar product \((\varphi, \psi) \mapsto \mathcal{E}(\varphi, \psi) + (\varphi, \psi)_{L^2(\Lambda, \mu)}\), is a Hilbert space, then \((\mathcal{E}, D(\mathcal{E}))\) is said to be a *closed form*, \[63\] Chapter VIII. If \(D_0(\mathcal{E})\) is a dense subspace of \(L^2(\Lambda, \mu)\) and \(\mathcal{E}\) is a nonnegative definite and symmetric bilinear form on \(D_0(\mathcal{E})\), then \((\mathcal{E}, D_0(\mathcal{E}))\) is said to be *closable* if there is a closed form \((\mathcal{E}', D(\mathcal{E}'))\) such that \(D_0(\mathcal{E}) \subseteq D(\mathcal{E}')\) and \(\mathcal{E}' = \mathcal{E}\) on \(D_0(\mathcal{E})\). This is the case if for any sequence \((\varphi_j) \subset D_0(\mathcal{E})\) that is Cauchy with respect to the seminorm \(E^{1/2}\) and such that \(\lim_j \varphi_j = 0\) in \(L^2(\Lambda, \mu)\) we have \(\lim_j \mathcal{E}(\varphi_j) = 0\).

A *Dirichlet form* \((\mathcal{E}, D(\mathcal{E}))\) on \(L^2(\Lambda, \mu)\) is a densely defined nonnegative definite and symmetric bilinear closed form such that

\begin{equation}
\tag{5.4}
u \in D(\mathcal{E}) \text{ implies } u \wedge 1 \in D(\mathcal{E}) \text{ and } \mathcal{E}(u \wedge 1) \leq \mathcal{E}(u),
\end{equation}

see for instance \[17\] Chapter I, 1.1.1 and 3.3.1 or \[8, 30, 33\]. A Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) is called *conservative* if \(1 \in D(\mathcal{E})\) and \(\mathcal{E}(1) = 0\). \[33\] p. 49. If \(1 \in D(\mathcal{E})\) then the Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) is said to be *local* if for any \(F, G \in C^\infty(\mathbb{R})\) with disjoint supports and any \(u \in D(\mathcal{E})\) we have \(\mathcal{E}(F(u), G(u)) = 0\). \[17\] Chapter I, Corollary 5.1.4.

The following theorem is our first main result.

**Theorem 5.1.** Let \(\mu\) be a finite Borel measure on \(\Lambda\) which satisfies the \((AC)\)-property with uniformly bounded densities.
(i) The form \((\mathcal{E}(\mu), D_0(\mathcal{E}(\mu)))\) is closable on \(L^2(\Lambda, \mu)\). Its closure \((\mathcal{E}(\mu), D(\mathcal{E}(\mu)))\) is a local conservative Dirichlet form.

(ii) There is a unique non-positive definite self-adjoint operator \((\mathcal{L}(\mu), D(\mathcal{L}(\mu)))\) on \(L^2(\Lambda, \mu)\) such that

\[
\langle \mathcal{L}(\mu)u, \varphi \rangle_{L^2(\Lambda, \mu)} = -\mathcal{E}(\mu)(u, \varphi)
\]

for all \(u \in D(\mathcal{L}(\mu))\) and \(\varphi \in D(\mathcal{E}(\mu))\). In particular,

\[
\int_{\Lambda} \mathcal{L}(\mu)u \, d\mu = 0, \quad u \in D(\mathcal{L}(\mu)),
\]

and the bottom of the spectrum of \((-\mathcal{L}(\mu), D(\mathcal{L}(\mu)))\) is zero.

Theorem 5.1 holds in particular if \(\mu\) is an SRB-measure with uniformly bounded densities. In this case \(\mathcal{L}(\mu)\) may be viewed as a natural self-adjoint Laplacian on \(\Lambda\).

**Remark 5.1.**

(i) If \(r = 1\) then the (local) unstable manifolds are of class \(C^1\) only, so that no classical leafwise Laplacian can be defined, but self-adjoint Laplacians exist by Theorem 5.1. See the related Remark C.1. Although for \(r \geq 2\) the (local) unstable manifolds are \(C^2\), but nevertheless it does not seem clear how to define any kind of related classical leafwise Laplacian if, as in Theorem 5.1, the densities are maximally Hölder.

(ii) The Laplacians on foliated spaces in [23, Sections 2.1 and 2.2] are constructed from classical Laplacians on global leaves \(W\) of form \(\text{div}_{m_W} \circ \nabla_W\). Here \(\text{div}_{m_W}\) denotes the divergence operator on \(W\) with respect to the Riemannian measure \(m_W\). The Laplacians in Theorem 5.1 are very different, they correspond to the composition \(\text{div}_{\mu_{W^y(z)}} \circ \nabla_{W^y(z)}\) on, roughly speaking, the local unstable leaves, and the divergence operators \(\text{div}_{\mu_{W^u(z)}}\) are defined with respect to weighted measures \(\mu_{W^u(z)}\), see Corollary C.1. If \(\mu\) is an SRB measure constructed following the geometric approach, the (local) weights emerge from a (local) renormalization procedure, cf. Appendix B.

The key fact to prove Theorem 5.1 is the following consequence of (3.2).

**Lemma 5.1.** Let \(\mu\) be a finite Borel measure on \(\Lambda\) which satisfies the \((AC)\)-property with uniformly bounded densities. Then the form \((\mathcal{E}(\mu), D_0(\mathcal{E}(\mu)))\) is closable on \(L^2(\Lambda, \mu)\).

A proof of Lemma 5.1 is given at the end of this section. We prove Theorem 5.1.

**Proof.** As \((\mathcal{E}(\mu), D_0(\mathcal{E}(\mu)))\) is densely defined by Proposition 5.1 and closable by Lemma 5.1 its closure is a densely defined nonnegative definite symmetric bilinear closed form \((\mathcal{E}(\mu), D(\mathcal{E}(\mu)))\) on \(L^2(\Lambda, \mu)\). The chain rule for the gradients \(\nabla_{W^u(z)}\) and definition (5.3) imply that for any \(\varphi \in D_0(\mathcal{E}(\mu))\) and any \(F \in C^1(\mathbb{R})\) with \(F(0) = 0\) we have \(F(\varphi) \in D_0(\mathcal{E}(\mu))\) and \(\mathcal{E}(\mu)(F(\varphi)) \leq \|F\|_{\sup}^2 \mathcal{E}(\mu)(\varphi)\). This property extends to all \(\varphi \in D(\mathcal{E}(\mu))\), [33, Theorem 3.1.1], and it also implies (5.4), [33, Section 1.1]. Conservativity and locality follow from (5.3). The first statement in (ii) with formula (5.5) is clear by general theory, see [33, Theorem 1.3.1] or [17, Chapter I, Propositions 1.2.2 or 3.2.1]. The fact that the bottom of the spectrum of \(-\mathcal{L}(\mu)\) is zero follows from conservativeness, and together with (5.5) also (5.6) follows. \(\square\)
It is not difficult to see that the collection \((T_xW^u(x))_{x \in \Lambda}\) is a measurable field of Hilbert spaces, see [31, Part II, Chapter 1], [75, Chapter IV, Section 8] or [44, Section 2], and we write \(L^2(\Lambda, (T_xW^u(x))_{x \in \Lambda}, \mu)\) for its direct integral with respect to \(\mu\). By \(\Gamma(\mu)\) we denote the carré du champ operator of \((\mathcal{E}(\mu), D(\mathcal{E}(\mu)))\), that is, the unique positive definite continuous symmetric bilinear \(L^1(\Lambda, \mu)\)-valued map on \(D(\mathcal{E}(\mu))\) such that \(\mathcal{E}(\mu)(uw, v) + \mathcal{E}(\mu)(vw, u) - \mathcal{E}(\mu)(w, uv) = \int_\Lambda w \Gamma(\mu)(u, v) \,\mu(dx)\) for all \(u, v, w \in D(\mathcal{E}(\mu)) \cap L^\infty(\Lambda, \mu)\), [17, Chapter I, Section 4.1]

Theorem 5.1 implies that the leafwise gradient \((\mathcal{L}(\mu), D(\mathcal{L}(\mu)))\) extends from \(D_0(\mathcal{E}(\mu))\) to the \(L^2\)-context, that this extension can be used to express the carré du champ, and that an associated divergence operator can be defined as minus its adjoint.

**Corollary 5.1.** Let \(\mu\) be a finite Borel measure on \(\Lambda\) which satisfies the \((AC)\)-property with uniformly bounded densities.

(i) The operator \((\nabla, D_0(\mathcal{E}(\mu)))\) extends to a densely defined closed unbounded linear operator

\[
\nabla : L^2(\Lambda, \mu) \to L^2(\Lambda, (T_xW^u(x))_{x \in \Lambda}, \mu)
\]

with domain \(D(\mathcal{E}(\mu))\), and [5.3] generalizes to

\[
\mathcal{E}(\mu)(\varphi) := \int_\Lambda \|\nabla \varphi(x)\|^2_{T_xW^u(x)} \,\mu(dx), \quad \varphi \in D(\mathcal{E}(\mu)).
\]

In particular, the carré du champ operator \(\Gamma(\mu)\) of \(\mathcal{E}(\mu)\) satisfies

\[
\Gamma(\mu)(\varphi)(x) = \|\nabla \varphi(x)\|^2_{T_xW^u(x)} \text{ for } \mu\text{-a.e. } x \in \Lambda
\]

and all \(\varphi \in D(\mathcal{E}(\mu))\).

(ii) The adjoint \((-\text{div}_\mu, D(\text{div}_\mu))\) of \((\nabla, D(\mathcal{E}(\mu)))\) is a densely defined closed unbounded operator

\[
-\text{div}_\mu : L^2(\Lambda, (T_xW^u(x))_{x \in \Lambda}, \mu) \to L^2(\Lambda, \mu).
\]

(iii) A function \(\varphi \in D(\mathcal{E}(\mu))\) is an element of \(D(\mathcal{L}(\mu))\) if and only if \(\nabla \varphi \in D(\text{div}_\mu)\), and in this case we have

\[
\mathcal{L}(\mu)\varphi = \text{div}_\mu(\nabla \varphi).
\]

**Proof.** Statement (i) is immediate from Theorem 5.1 (ii) follows by general theory, [63, Theorem VIII.1], (iii) is clear by (5.5) and (ii). □

We prove Lemma 5.1.

**Proof.** Let \(\{U_i\}_{i=1}^N\) be a finite cover of \(\Lambda\) by open subsets \(U_i = U_{z_i}\) as in [2.5] and let \(\{\chi_i\}_{i=1}^N \subset C^1(M)\) be a partition of unity subordinate to it as in Corollary 4.2 and \(\{\chi_i^A\}_{i=1}^N\) its restriction to \(\Lambda\). Suppose that \((\varphi_j)_{j=1}^\infty \subset D_0(\mathcal{E}(\mu))\) is Cauchy w.r.t. the seminorm \((\mathcal{E}(\mu))^{1/2}\) and that \(\lim_{j \to \infty} \|\varphi_j\|_{L^2(\Lambda, \mu)} = 0\). By the product rule for \(\nabla W^u(x)\) and Corollary 4.2 we have

\[
\int_{U_i \cap \Lambda} \|\nabla W^u(x)(\varphi_j \chi_i^A - \varphi_k \chi_i^A)(x)\|^2_{T_xW^u(x)} \,\mu(dx)
\]

\[
\leq 2 \int_{U_i \cap \Lambda} \|\nabla W^u(x)(\varphi_j - \varphi_k)(x)\|^2_{T_xW^u(x)} \,\mu(dx)
\]

\[
+ 2C \int_{U_i \cap \Lambda} (\varphi_j - \varphi_k)^2(x) \,\mu(dx)
\]
for any $i = 1, \ldots, N$, here $C$ denotes the square of the quantity on the left hand side of (4.2). This becomes arbitrarily small if $j$ and $k$ are chosen large enough. Clearly also

$$\lim_{j \to \infty} \int_{U_i \cap \Lambda} (\varphi_j \chi_i^A)^2(x) \mu(dx) = 0. \tag{5.11}$$

We will show that

$$\lim_{j \to \infty} \int_{U_i \cap \Lambda} \left\| \nabla W_{\varphi_j}(x) (\varphi_j \chi_i^A)(x) \right\|^2_{T_x W_{\varphi_j}(x)} \mu(dx) = 0, \quad i = 1, \ldots, N. \tag{5.12}$$

Then, by the triangle inequality for the seminorm $(5.12)$,

$$\lim_{j \to \infty} (5.12)^{1/2} \leq \sum_{i=1}^N \lim_{j \to \infty} \left( \int_{U_i \cap \Lambda} \left\| \nabla W_{\varphi_j}(x) (\varphi_j \chi_i^A)(x) \right\|^2_{T_x W_{\varphi_j}(x)} \mu(dx) \right)^{1/2} = 0$$

for all $i = 1, \ldots, N$. In other words, the form $(5.12)$ is closable.

It remains to verify (5.12). For each $i$ let $\mathcal{R}_i = \mathcal{R}_x$, be a closed rectangle containing $U_i$ as in (2.5) and let $\{ W_{\varphi_j}^z(\varphi) : z \in W_{\mathcal{R}_i}(p_i) \}$ with notation as in (3.1) be a measurable partition of $\mathcal{R}_i$ with $p_i \in \mathcal{R}_i$. For fixed $i$ let $\tilde{\mu}_{\mathcal{R}_i}$ and $\mu_{W_{\varphi_j}^z}(\varphi)$ be as in the local product formula (3.2). By Remark 4.1 we can find a $\tilde{\mu}_{\mathcal{R}_i}$-null set $\mathcal{N} \subset W_{\mathcal{R}_i}(p_i)$ such that for each $z \in W_{\mathcal{R}_i}(p_i) \setminus \mathcal{N}$ the measure $\mu_{W_{\varphi_j}^z}(\varphi)$ has a uniformly bounded density $\rho_{W_{\varphi_j}^z}(\varphi)$ with respect to $m_{W_{\varphi_j}^z}(\varphi)$. For any such $z$ let $\rho_{W_{\varphi_j}^z}(\varphi) := \rho_{W_{\varphi_j}^z}(\varphi)$ on $W_{\mathcal{R}_i}(p_i)$ and $\rho_{W_{\varphi_j}^z}(\varphi) := 1$ on $W_{\mathcal{R}_i}(p_i) \setminus W_{\mathcal{R}_i}(p_i)$ and consider the measure

$$\mu_{W_{\varphi_j}^z}(\varphi) := \rho_{W_{\varphi_j}^z}(\varphi) \cdot m_{W_{\varphi_j}^z}(\varphi).$$

Since $\text{supp} \chi_i^A \subset U_i \cap \Lambda \subset \mathcal{R}_i$, limit relation (5.11) yields

$$\lim_{j \to \infty} \int_{W_{\mathcal{R}_i}(p_i)} \left\| \varphi_j \chi_i^A \right\|^2_{L^2(W_{\varphi_j}^z(\varphi), \mu_{W_{\varphi_j}^z}(\varphi))} \tilde{\mu}_{\mathcal{R}_i}(dz)$$

$$= \lim_{j \to \infty} \int_{W_{\mathcal{R}_i}(p_i)} \left\| \varphi_j \chi_i^A \right\|^2_{L^2(W_{\mathcal{R}_i}(p_i), \mu_{W_{\varphi_j}^z}(\varphi))} \tilde{\mu}_{\mathcal{R}_i}(dz)$$

$$= 0.$$

For each $z \in W_{\mathcal{R}_i}(p_i) \setminus \mathcal{N}$ the measure $\mu_{W_{\varphi_j}^z}(\varphi)$ on the Riemannian manifold $W_{\varphi_j}^z(\varphi)$ satisfies (C.4) in Appendix C so that by Corollary C.1 the Dirichlet integral

$$D(\mu_{W_{\varphi_j}^z}(\varphi))(\psi) = \int_{W_{\varphi_j}^z(\varphi)} \left\| \nabla W_{\varphi_j}^z(\psi)(y) \right\|^2_{T_y W_{\varphi_j}^z(\varphi)} \mu_{W_{\varphi_j}^z}(\varphi)(dy), \quad \psi \in C_c^1(W_{\varphi_j}^z(\varphi));$$

on the manifold $W_{\varphi_j}^z(\varphi)$ with respect to the measure $\mu_{W_{\varphi_j}^z}(\varphi)$, as defined in (C.3), is a closable quadratic form on $L^2(W_{\varphi_j}^z(\varphi), \mu_{W_{\varphi_j}^z}(\varphi))$. By Proposition D.1 then also the quadratic form

$$\varphi \mapsto \int_{W_{\mathcal{R}_i}(p_i)} D(\mu_{W_{\varphi_j}^z}(\varphi))(\varphi) \tilde{\mu}_{\mathcal{R}_i}(dz), \quad \varphi \in C_c(U_i \cap \Lambda) \cap D_0(\mathcal{E}(\mu)),$$

is closable on $L^2(U_i \cap \Lambda, \mu)$. (Note that by Fubini $z \mapsto D(\mu_{W_{\varphi_j}^z}(\varphi))(\varphi)$ is measurable.)
By (3.2) and (5.10)

\[
\int_{W_{R_1}(p_i)} D(\mu_{W^+_n(z)}) (\varphi_j \chi^A_i - \varphi_k \chi^A_i) \hat{\mu}_{R_i}(dz)
\]

\[
= \int_{W_{R_1}(p_i)} \int_{W_{R_1}(z)} \| \nabla W^+(z) (\varphi_j \chi^A_i - \varphi_k \chi^A_i) (y) \|^2 \mu_{W^+_n(z)} W_{R_1}(z) (dy) \hat{\mu}_{R_i}(dz)
\]

can be made arbitrarily small if \( j, k \) are chosen large enough. Since (5.13) is closable, it follows that

\[
\lim_j \int_{W_{R_1}(p_i)} \int_{W_{R_1}(z)} \| \nabla W^+(z) (\varphi_j \chi^A_i) (y) \|^2 \mu_{W^+_n(z)} W_{R_1}(z) (dy) \hat{\mu}_{R_i}(dz)
\]

\[
= 0,
\]

what shows (5.12). \( \square \)

**Remark 5.2.** Alternatively, one could replace \( D_0(\mathcal{E}(\mu)) \) in (5.2), (5.3) and Theorem 5.1 by \( D_0(\mathcal{E}(\mu)) := C^r(M) | \Lambda \) or by the space \( D_0(\mathcal{E}(\mu)) \) of all \( \varphi \in L^2(\Lambda, \mu) \cap C^r(M) \) such that \( x \mapsto \| \nabla \varphi(x) \|_{T_x W^+(x)} \) is in \( L^2(\Lambda, \mu) \). Obviously \( D_0(\mathcal{E}(\mu)) \subset D_0(\mathcal{E}(\mu)) \subset D_0(\mathcal{E}(\mu)) \), these inclusions are strict. All three of these a priori domains lead to closable forms, and their closures inherit the inclusions. By Corollary 4.1 the choice \( D_0(\mathcal{E}(\mu)) \) would readily guarantee that all \( C^k(M) | \Lambda \), \( k \leq r \), are dense subspaces of the domain of the closure. The choice \( D_0(\mathcal{E}(\mu)) \) would be more adequate in the sense that Theorem 5.1 does not need membership in \( C(\Lambda) \). The domain \( D_0(\mathcal{E}(\mu)) \)

we decided to focus on is relatively large but still well-connected to the topology of \( \Lambda \) (see Section 7 below).

6. Symmetric Markov Semigroups

We continue working under the hypotheses stated at the beginning of the preceding section and now consider the related Markov semigroups.

Recall that a strongly continuous semigroup \((P_t)_{t \geq 0}\) on \( L^2(\Lambda, \mu) \) is said to be Markov if for any \( t > 0 \) and any \( u \in L^2(\Lambda, \mu) \) such that \( 0 \leq u \leq 1 \) \( \mu \)-a.e. we have \( 0 \leq P_t u \leq 1 \) \( \mu \)-a.e. A Markov semigroup \((P_t)_{t \geq 0}\) on \( L^2(\Lambda, \mu) \) is called symmetric if

\[
(P_t u, v)_{L^2(\Lambda, \mu)} = (u, P_t v)_{L^2(\Lambda, \mu)}, \quad u, v \in L^2(\Lambda, \mu), \quad t > 0.
\]

A symmetric Markov semigroup on \( L^2(\Lambda, \mu) \) extends to a semigroup of contractive operators on each \( L^p(\Lambda, \mu), 1 \leq p \leq +\infty \), see [30, Theorem 1.4.1]. A symmetric Markov semigroup \((P_t)_{t \geq 0}\) on \( L^2(\Lambda, \mu) \) is said to be conservative if \( P_t 1 = 1 \) for all \( t > 0 \) and recurrent if for any nonnegative \( u \in L^1(\Lambda, \mu) \) we have \( \int_0^\infty P_t u \, dt = 0 \) or \(+\infty\) \( \mu \)-a.e. If it is recurrent, it is also conservative, see [33, Section 1.6, in particular Lemma 1.6.5]. The finite Borel measure \( \mu \) is said to be invariant for \((P_t)_{t \geq 0}\) if \( \int A P_t u \, d\mu = \int_A u \, d\mu \) for all \( u \in L^2(\Lambda, \mu) \).

**Theorem 6.1.** Let \( \mu \) be a finite Borel measure on \( \Lambda \) which satisfies the \( (AC) \) property with uniformly bounded densities. The operator \((\mathcal{L}(\mu), D(\mathcal{L}(\mu))) \) as in Theorem 5.1 generates a strongly continuous Markov semigroup \((P_t)_{t \geq 0}\) on \( L^2(\Lambda, \mu) \) which is symmetric, recurrent and in particular, conservative. The measure \( \mu \) is an invariant measure for \((P_t)_{t \geq 0}\).
Proof. The existence of $(P_t)_{t>0}$ as stated is clear by general theory, see [17] Chapter I, Proposition 3.2.1 or [33] Theorem 1.4.1. That $(P_t)_{t>0}$ is recurrent follows from [33] Theorem 1.6.3. The invariance of $\mu$ is a reformulation of (5.6).

Remark 6.1. Of course also $(P_t)_{t>0}$ depends heavily on the choice of the measure $\mu$, and it would be more appropriate to denote it by $(P_t^{(\mu)})_{t>0}$. We suppress $\mu$ only to have more compact notation in the main statements.

The semigroup $(P_t)_{t>0}$ can also be characterized as the unique symmetric Markov semigroup on $L^2(\Lambda, \mu)$ such that

$$D(\mathcal{E}^{(\mu)}) = \{ \varphi \in L^2(\Lambda, \mu) : \sup_{t>0} t^{-1} \langle \varphi - P_t \varphi, \varphi \rangle_{L^2(\Lambda, \mu)} < +\infty \}$$

and

$$\mathcal{E}^{(\mu)}(\varphi) = \sup_{t>0} t^{-1} \langle \varphi - P_t \varphi, \varphi \rangle_{L^2(\Lambda, \mu)}; \quad \varphi \in D(\mathcal{E}^{(\mu)}),$$

see for instance [17] Sections I.1.2 and I.3.2 or [33] Sections 1.3 and 1.4.

Remark 6.2. In [21] Section 4 and [23] Section 2.2 it was proved that the leafwise Laplacians as mentioned in Remark 5.1 (ii) generate Feller semigroups on the spaces of continuous functions on compact foliated spaces (cf. Remark 2.2). Their invariant measures were characterized by a disintegration formula, [23] Proposition 2.4.10. In that situation it is not clear whether the semigroup is symmetric with respect to any such measure. In the present setup the symmetry (6.1) of $(P_t)_{t>0}$ is immediate from Theorem 5.1.

Remark 6.3. We briefly mention the integrated Gaussian upper estimates, [34], and short-time asymptotics, [78], shown in [74] and [43]. Let $D_1$ be the collection of all $\varphi \in D(\mathcal{E}^{(\mu)}) \cap L^\infty(X, \mu)$ such that $\|\nabla \varphi(x)\|_{T_x W^\infty(x)} \leq 1$ for $\mu$-a.e. $x \in \Lambda$, and for any $A, B \subset \Lambda$ Borel, define

$$d^{(\mu)}_{\mathcal{E}}(A, B) := \sup_{\varphi \in D_1} \varphi(A, B),$$

where $\varphi(A, B) := \text{ess inf}_{x \in B} \varphi(x) - \text{ess sup}_{y \in A} \varphi(y),$

the ess inf and ess sup are taken with respect to $\mu$ and $\sup \emptyset := -\infty$ and $\inf \emptyset := +\infty$.

By [43] Theorem 2.8 resp. [74] Theorem 1.8 we have

$$\int_A P_t \mathbf{1}_B \, d\mu \leq \sqrt{\mu(A)} \sqrt{\mu(B)} \exp \left(-\frac{1}{2} d^{(\mu)}_{\mathcal{E}}(A, B)^2 \right)$$

for any $A, B \subset \Lambda$ Borel, and by [43] Theorem 1.1

$$\lim_{t \to 0} t \log \int_A P_t \mathbf{1}_B \, d\mu = -\frac{1}{2} d^{(\mu)}_{\mathcal{E}}(A, B)^2.$$

7. REGULARITY AND SYMMETRIC LEAFWISE DIFFUSIONS

If the support of $\mu$ is all of $\Lambda$ then the theory in [33] applies without modifications. Recall that in this case a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\Lambda, \mu)$ is called regular if $D(\mathcal{E}) \cap C(\Lambda)$ is dense in $D(\mathcal{E})$ and uniformly dense in $C(\Lambda)$. If it is regular it is said to be strongly local if $\mathcal{E}(u, v) = 0$ whenever $u, v \in D(\mathcal{E}) \cap C(\Lambda)$ are such that $v$ is constant on supp $u$. If a Dirichlet form is regular and strongly local, then there is an associated symmetric diffusion process. Since there are various technical notions involved, we first state the following theorem and then briefly comment on these notions in Remark 7.1 below. By $d|_{\Lambda \times \Lambda}$ we denote the restriction of the geodesic distance $d$ on $M$ to $\Lambda$. 


**Theorem 7.1.** Let \( \mu \) be a finite Borel measure on \( \Lambda \) which satisfies the \((AC)\)-property with uniformly bounded densities. Suppose in addition that \( \text{supp} \mu = \Lambda \). Then \( (E(\mu), D(E(\mu))) \) is a strongly local regular Dirichlet form on \((\Lambda, d_{\Lambda \times \Lambda})\). There is a symmetric Hunt diffusion process \( ((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \Lambda \setminus \mathcal{N}}) \) with starting point \( x \) outside some properly exceptional Borel set \( \mathcal{N} \subset \Lambda \), such that for any bounded Borel function \( u \) on \( \Lambda \) and any \( t > 0 \) the function \( \mathcal{P}_t u \), defined by

\[
(7.1) \quad \mathcal{P}_t u(x) := \begin{cases} \mathbb{E}^x[u(X_t)] & \text{for } x \in \Lambda \setminus \mathcal{N} \\ 0 & \text{for } x \in \Lambda, \end{cases}
\]

satisfies

\[
(7.2) \quad \mathcal{P}_t u(x) = u(x) \quad \text{for } \mu\text{-a.e. } x \in \Lambda.
\]

This process is unique up to equivalence. It has infinite life time. Moreover, for each bounded Borel \( u \) and \( t > 0 \) the function \( \mathcal{P}_t u \) is an \((E(\mu), D(E(\mu)))\)-quasi-continuous version of \( P_t u \).

**Remark 7.1.** A diffusion process is a strong Markov process with almost surely continuous paths, [33, Section 4.5]. A Hunt process is a particularly regular type of strong Markov process, see [15, Section I.9] or [33, Appendix A.2]. A Borel set \( \mathcal{N} \subset \Lambda \) is called properly exceptional if it is a \( \mu \)-null set and \( \mathbb{P}_x(X_t \in \mathcal{N} \text{ for some } t \geq 0) = 0 \), \( x \in \Lambda \setminus \mathcal{N} \), see [33, p. 134 and Theorem 4.1.1]. The process \( ((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \Lambda \setminus \mathcal{N}}) \) is said to be symmetric (with respect to \( \mu \)) if for any \( t > 0 \) and any bounded Borel functions \( u, v \) on \( \Lambda \) we have \( \langle \mathcal{P}_t u, v \rangle_{L^2(\Lambda, \mu)} = \langle u, \mathcal{P}_t v \rangle_{L^2(\Lambda, \mu)}, \) [33 Lemma 4.1.3]. Two symmetric Hunt processes are said to be equivalent if they have a common properly exceptional set \( \mathcal{N} \) such that \( \mathcal{P}_t^{(1)} u(x) = \mathcal{P}_t^{(2)} u(x), x \in \Lambda \setminus \mathcal{N}, \) for all \( t > 0 \) and bounded \( \mu \)-measurable \( u \), see [33, p. 147]; here \( \mathcal{P}_t^{(1)} u \) and \( \mathcal{P}_t^{(2)} u \) are defined for the respective process as in the theorem. Quasi-continuous versions in the context of regular Dirichlet forms are discussed in [33, Section 2.1].

**Proof.** The statements on regularity and strong locality are immediate. The existence of a symmetric Hunt diffusion process is guaranteed by [33, Theorems 7.2.1 and 7.2.2], its uniqueness up to equivalence by [33, Theorem 4.2.7]. The quasi-continuity of the functions \( \mathcal{P}_t u \) and \( \mathcal{P}_t v \) are verified in [33, Theorem 4.2.3]. That \( ((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \Lambda \setminus \mathcal{N}}) \) has infinite life time follows from the conservativeness of \((E(\mu), D(E(\mu)))\), [33 Problem 4.5.1]. \( \square \)

If \( \mu \) is an SRB-measure on \( \Lambda \), then the process in Theorem [7.1] can be regarded as a natural symmetric leafwise diffusion on \( \Lambda \).

**Remark 7.2.** In the situation of Theorem [7.1] the index of \((E(\mu), D(E(\mu)))\) in the sense of [42] is well-defined, and it is easily seen to equal \( d_{\mathcal{U}} \). By construction the measure \( \mu \) is a minimal energy dominant measure for \((E(\mu), D(E(\mu)))\), [42].

8. QUASI-INVARiance IN THE u-CONFORMAL CASE

Let \( M, f \) and \( \Lambda \) be as in Section 4 and let now \( \mu \) be an SRB-measure on \( \Lambda \) with uniformly bounded densities. Under these hypotheses it seems natural to ask to what extent also the constructed objects remain invariant under \( f \).

We additionally assume that \( f \) is \( u \)-conformal, [58, p. 230], i.e., that there is a function \( a \) on \( \Lambda \) such that for any \( x \in \Lambda \) the map \( d_{\mathcal{U}} f |_{\mathcal{U}_x} \) is \( a(x) \) times an
isometry $I_x : T_x W^u(x) \to T_{f(x)} W^u(x)$,

\begin{equation}
\label{8.1}
d_x f|_{T_x W^u(x)} = a(x) I_x, \quad x \in \Lambda.
\end{equation}

This last condition is true in particular for all diffeomorphisms $f$ that are conformal on $\Lambda$ in the sense of \cite{13} Definition 4.3.1. It is well-known that the unstable distribution (and also the stable) varies Hölder continuously in $x \in \Lambda$, see for instance \cite{14} Theorem 4.11 and Exercise 4.15. As a consequence, $a$ is Hölder continuous on $\Lambda$, cf. \cite{58}. The iterates of $f$ and $f^{-1}$ satisfy

\begin{equation}
\label{8.2}
d_x f^n|_{T_x W^u(x)} = \left[ \prod_{k=0}^{n-1} a(f^k(x)) \right] I_{f^{n-1}(x)} \circ I_{f^{n-2}(x)} \circ \cdots \circ I_x
\end{equation}

and

\begin{equation}
\label{8.3}
d_x f^{-n}|_{T_x W^u(x)} = \left[ \prod_{k=1}^{n} a(f^{-k}(x)) \right]^{-1} I_{f^{-n}(x)} \circ I_{f^{-n+1}(x)} \circ \cdots \circ I_{f^{-1}(x)}
\end{equation}

for any $n \geq 1$ and $x \in \Lambda$, as can be seen using the chain rule. Taking into account uniform hyperbolicity it follows that

\begin{equation}
\label{8.4}
g \leq |a| \leq \overline{\sigma}
\end{equation}

on $\Lambda$ with constants $1 < g \leq \overline{\sigma} < +\infty$. It seems convenient to express quantities in terms of the (Hölder continuous) potential

$$
\Phi(x) := -\log |a(x)|, \quad x \in \Lambda.
$$

We observe the following consequences of the $f$-invariance of $\mu$ and the $u$-conformality of $f$ for the Dirichlet forms $\mathcal{E}^{(\mu)}$ from Section 5.

**Theorem 8.1.** Suppose that $f$ is $u$-conformal and that $\mu$ is an SRB-measure on $\Lambda$ with uniformly bounded densities and such that $\text{supp} \mu = \Lambda$.

(i) For each $n \in \mathbb{Z}$ and $\varphi \in D(\mathcal{E}^{(\mu)})$ we have $\varphi \circ f^n \in D(\mathcal{E}^{(\mu)})$, and

$$
\mathcal{E}^{(\mu,n)}(\varphi) := \mathcal{E}^{(\mu)}(\varphi \circ f^n)
$$

defines a strongly local regular Dirichlet form $(\mathcal{E}^{(\mu,n)}, D(\mathcal{E}^{(\mu)}))$ on $L^2(\Lambda, \mu)$.

(ii) For each $n \in \mathbb{N}$ and $\varphi \in D(\mathcal{E}^{(\mu)})$ we have

\begin{equation}
\label{8.5}
\mathcal{E}^{(\mu,n)}(\varphi) = \int_{\Lambda} \|\nabla \varphi(x)\|^2_{T_x W^u(x)} e^{-2 \sum_{k=1}^{n} \Phi(f^{-k}(x))} \mu(dx)
\end{equation}

and

\begin{equation}
\label{8.6}
\mathcal{E}^{(\mu,-n)}(\varphi) = \int_{\Lambda} \|\nabla \varphi(x)\|^2_{T_x W^u(x)} e^{2 \sum_{k=0}^{n-1} \Phi(f^k(x))} \mu(dx).
\end{equation}

(iii) For each $n \in \mathbb{N}$ and $\varphi \in D(\mathcal{E}^{(\mu)})$ we have

\begin{equation}
\label{8.7}
g^{2n} \mathcal{E}^{(\mu)}(\varphi) \leq \mathcal{E}^{(\mu,n)}(\varphi) \leq \overline{\sigma}^{2n} \mathcal{E}^{(\mu)}(\varphi),
\end{equation}

with $g$ and $\overline{\sigma}$ from \cite{8.4}, and similarly for $\mathcal{E}^{(\mu,-n)}$ with $\overline{\sigma}^{-2n}$ and $\underline{\overline{\sigma}}^{-2n}$. 
Proof. Suppose that \( \varphi \in \mathcal{D}_0(\mathcal{E}^{(\mu)}) \). By the chain rule \( d_x (\varphi \circ f^n) = d_{f^n(x)} \varphi \cdot d_x f^n \), and by (8.2),
\[
||d_x (\varphi \circ f^n)||_{T^n \mathcal{W}^\mu} = \left| \left| d_{f^n(x)} \varphi \right| \right|_{T^n \mathcal{W}^\mu} e^{-\sum_{k=0}^{n-1} \Phi(f^k(x))}, \quad x \in \Lambda.
\]
Clearly \( ||\nabla \varphi(x)||_{T^n \mathcal{W}^\mu} = ||d_x \varphi||_{T^n \mathcal{W}^\mu} \) by duality, and using (8.4) it follows that \( \varphi \circ f^n \in \mathcal{D}_0(\mathcal{E}^{(\mu)}) \). By (5.3) and the \( f \)-invariance of \( \Lambda \), \( \mu \) and the global unstable manifolds, the change of variable \( y = f^n(x) \) yields
\[
\mathcal{E}^{(\mu)}(\varphi \circ f^n) = \int_{\Lambda} ||d_{f^n(x)} \varphi||_{T^n \mathcal{W}^\mu}^2 e^{-2 \sum_{k=0}^{n-1} \Phi(f^k(x))} \mu(dx)
\]
and a similar identity holds for the carré du champ \( \Gamma^{(\mu,-n)} \) of \( \mathcal{E}^{(\mu,-n)} \). Theorem 8.1 may therefore be regarded as a quasi-invariance property of the carré du champ under \( f \), the functions \( e^{-2 \sum_{k=0}^{n-1} \Phi(f^k(x))} \) resp. \( e^{2 \sum_{k=0}^{n-1} \Phi(f^k(x))} \) may be seen as conformal factors. It follows directly from the proof of Theorem 8.1 (or can be concluded from (8.7)) that for any \( \varphi \in \mathcal{D}(\mathcal{E}^{(\mu)}) \) we have
\[
\mathcal{E}^{2n} \Gamma^{(\mu)}(\varphi) \leq \Gamma^{(\mu,n)}(\varphi) \leq \mathcal{E}^{2n-2} \Gamma^{(\mu)}(\varphi) \quad \mu\text{-a.e. on } \Lambda,
\]
and similarly for \( \Gamma^{(\mu,-n)} \).

Remark 8.1.

(i) For the standard Dirichlet integral on Euclidean spaces both the Lebesgue measure and the carré du champ \( \Gamma(\varphi) = |\nabla \varphi|^2 \) are invariant under translations. For the standard Dirichlet form on an abstract Wiener space the carré du champ is invariant under translations in Cameron-Martin directions, and the Gaussian measure is quasi-invariant, see [17, Chapter II, Section 2, in particular Theorem 2.4.4]. In the situation of Theorem 8.1 the measure \( \mu \) is invariant under \( f \), and the carré du champ is quasi-invariant.

(ii) By (8.8) the map \( f \) may also be viewed as a quasi-isometry in a Dirichlet form sense (cf. [10] Definition 2.17 and Lemma 2.18, [12] p.6).

(iii) For \( n \in \mathbb{N} \) the generator \( \mathcal{L}^{(\mu,n)} \) of \( (\mathcal{E}^{(\mu,n)}, \mathcal{D}(\mathcal{E}^{(\mu,n)})) \) can informally be written as
\[
\mathcal{L}^{(\mu,n)} \varphi = \text{div}_\mu(x e^{-2 \sum_{k=1}^{n-1} \Phi(f^{-k} \nabla \varphi)}),
\]
similarly for the generator of \( \mathcal{E}^{(\mu,-n)} \).
If, as in some examples, the function \( a \) in \((8.1)\) is constant, then the preceding reduces to a simple rescaling.

**Corollary 8.1.** Suppose that \( f \) is \( u \)-conformal and such that \( a \) is constant. Suppose further that \( \mu \) is an SRB-measure on \( \Lambda \) with uniformly bounded densities and \( \supp \mu = \Lambda \).

(i) For any \( \varphi \in D(\mathcal{E}(\mu)) \) and \( n \in \mathbb{Z} \) we have
\[
\mathcal{E}(\mu,n)(\varphi) = a^{2n}\mathcal{E}(\mu)(\varphi) \quad \text{and} \quad \Gamma(\mu,n)(\varphi) = a^{2n}\Gamma(\mu)(\varphi).
\]

(ii) The generator \( \mathcal{L}(\mu,n) \) of \( \mathcal{E}(\mu,n) \) has domain \( D(\mathcal{L}(\mu)) \). For any \( \varphi \in D(\mathcal{L}(\mu)) \) and \( n \in \mathbb{Z} \) we have \( \varphi \circ f^n \in D(\mathcal{L}(\mu)) \) and
\[
\mathcal{L}(\mu,n)\varphi = a^{2n}\mathcal{L}(\mu)\varphi.
\]

(iii) For any \( n \in \mathbb{Z} \) the symmetric Markov semigroup \( (P^{(n)}_t)_{t \geq 0} \) generated by \( \mathcal{L}(\mu,n) \) is given by
\[
P^{(n)}_t = P_{a^{2n}t}, \quad t > 0.
\]

(iv) For any \( n \in \mathbb{Z} \) the symmetric Hunt diffusion process uniquely associated with \( (\mathcal{E}(\mu,n), D(\mathcal{E}(\mu))) \) in the sense of Theorem 7.7 is
\[
((X_{a^{2n}t})_{t \geq 0}, (\mathbb{P}^x)_{x \in \Lambda \setminus \Lambda'})
\]

Since \( a > 1 \) it follows that, roughly speaking, for \( n > 0 \) the expansion effect of \( f^n \) makes the process \( (X_{a^{2n}t})_{t \geq 0} \) run faster than \( (X_t)_{t \geq 0} \), while for \( n < 0 \) it slows the process down.

**Proof.** Statement (i) is clear from Theorem 8.1 and it immediately implies (ii). Statement (iii) follows from (ii) and the spectral representation, and (iv) is seen using (iii) and (7.2).

As a consequence of (8.9) harmonicity properties are preserved under \( f \), this is similar to the situation in the (complex) plane, [1], Section 4.1, Problem 4. To formulate this we recall notions of harmonicity in the Dirichlet form sense, [11, 24]. Given a Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) on \( L^2(\Lambda, \mu) \) and \( O \subset \Lambda \) open, a function \( \varphi \in D(\mathcal{E}) \) is called **superharmonic in** \( O \) (in Dirichlet form sense) if \( \mathcal{E}(\varphi, \psi) \geq 0 \) for all nonnegative \( \psi \in D(\mathcal{E}) \cap C_c(O) \). It is called **subharmonic in** \( O \) if \( -\varphi \) is superharmonic in \( O \), and **harmonic in** \( O \) if it is sub- and superharmonic in \( O \).

**Corollary 8.2.** Let the hypotheses of Corollary 8.1 be in force. If \( n \in \mathbb{Z} \) and \( \varphi \) is superharmonic in an open set \( O \subset \Lambda \) then \( \varphi \circ f^n \) is superharmonic on \( f^{-n}(O) \).

**Proof.** For any nonnegative \( \psi \in D(\mathcal{E}) \cap C_c(f^{-n}(O)) \) the function \( \psi \circ f^{-n} \) is nonnegative and in \( D(\mathcal{E}) \cap C_c(O) \), so that
\[
\mathcal{E}(\mu)(\varphi \circ f^{-n}, \psi) = \mathcal{E}(\mu,n)(\varphi, \psi \circ f^{-n}) = a^{2n}\mathcal{E}(\mu)(\varphi, \psi \circ f^{-n}) \geq 0.
\]

\[\square\]

9. **Functions of zero energy and measurable partitions**

Suppose that \( \mu \) is an SRB-measure with uniformly bounded densities. Let us agree to say that a given Borel function \( \varphi : \Lambda \to \mathbb{R} \) has **zero energy** if for any open rectangle \( R \subset \Lambda \) we have
\[
\int_R \left\| \nabla W^\mu(x)\varphi(x) \right\|_{T_xW^\mu(x)}^2 \mu(dx) = 0.
\]

\[(9.1)\]
The left hand side is always well-defined: For $\mu_R$-a.e. $W^u_R \in \mathcal{P}_R$, the conditional measure $\mu_{W^u_R}$ is equivalent to $m_{W^u_R}|_{W^u_R}$, where $W^u_R$ is a local unstable manifold containing $W^u_R$, and the density $\nu_{W^u_R}$ is uniformly bounded. Extending $\mu_{W^u_R}$ to a measure $\mu_{W^u_R}$ on all of $W^u_R$ by adding a portion of $m_{W^u_R}$ as in the proof of Lemma 9.1, it follows that the Dirichlet integral $\int \phi dx$ is as stated, then disintegration and the uniform bounds on the densities imply $\mu$-measure of $W^u_R$ containing $\mu_{W^u_R}$. The following observation is what one would expect, given the fact that each point lies in a local manifold.

**Proposition 9.1.** Let $\mu$ be an SRB-measure on $\Lambda$ with uniformly bounded densities. If $\varphi : \Lambda \to \mathbb{R}$ is a bounded Borel function that has zero energy, then for $\mu$-a.e. $p \in \Lambda$ there is a connected open subset $U_p$ of $W^u(p)$ on which $\varphi$ is constant.

**Proof.** It suffices to restrict attention to a single (arbitrary) open rectangle $R$. If $\varphi$ is as stated, then disintegration and the uniform bounds on the densities imply that

$$\int_{W^u_R} \left\| \nabla_{W^u_R}(x) \varphi(x) \right\|_{T_{x}W^u_R}^2 m_{W^u_R}(dx) = 0$$

for $\mu_R$-a.e. $W^u_R \in \mathcal{P}_R$. Since $W^u_R$ is a connected open subset of $W^u_R$, a standard argument shows that $\varphi|_{W^u_R}$ must be constant: For any $p \in W^u_R$ we can find a chart neighborhood $U_p$ of $p$ in $W^u_R$ and local coordinates $x = (x^1, \ldots, x^{d_u})$ mapping $U_p$ onto a connected open set $V \subset \mathbb{R}^{d_u}$. Since $x$ itself is a $C^r$-diffeomorphism (see [53, p. 60]), we can choose $U_p$ and $V = x(U_p)$ slightly smaller to guarantee that the differential $dx^{-1}$ of $x^{-1}$ is bounded on $V$. Since also the metric is locally bounded, it follows that

$$0 = \int_{U_p} \left\| \nabla_{W^u_R}(x) \varphi(x) \right\|_{T_{x}W^u_R}^2 m_{W^u_R}(dx),$$

$$= \int_{V} g_{ij}(x^{-1}(y)) \frac{\partial \varphi}{\partial x^i}(x^{-1}(y)) \frac{\partial \varphi}{\partial x^j}(x^{-1}(y)) \sqrt{\det g_{W^u}(x^{-1}(y))} \ dy,$$

$$\geq c \int_{V} \left\| \nabla_{W^u_R}(\varphi \circ x^{-1})(y) \right\|^2 \ dy.$$

This implies that $\varphi \circ x^{-1}$ is constant on $V$, hence $\varphi$ is constant on $U_p$.

We quote the following result from [60, Theorem 2] on the existence of a particularly nice measurable partition of $\Lambda$.

**Theorem 9.1.** Let $\mu$ be an ergodic SRB-measure on $\Lambda$ with uniformly bounded densities. Then there is an $f$-invariant open subset $N$ of $\Lambda$ with $\mu(N') = 1$ and a measurable partition $\mathcal{P}_N$ of $N$ such that for each $x \in N$ the partition element $C(x) \in \mathcal{P}_N$, containing $x$ is a connected open subset of the global unstable manifold $W^u(x)$.

One consequence of Theorem 9.1 is the following complement to Proposition 9.1.

**Corollary 9.1.** Let $\mu$ be an ergodic SRB-measure on $\Lambda$ with uniformly bounded densities. There is a bounded Borel function $\varphi$ on $\Lambda$ that is not $\mu$-a.e. constant on $\Lambda$ but satisfies

$$\int_{\Lambda} \left\| \nabla_{W^u_R}(x) \varphi(x) \right\|_{T_{x}W^u_R}^2 \mu(dx) = 0.$$

The fact that (9.2) makes sense is considered part of the statement.
Remark 9.1. Corollary 9.1 may be interpreted as an existence statement for non-constant bounded harmonic functions on $\Lambda$, it shows that no $L^p$-Liouville theorem ($1 \leq p \leq +\infty$) and no Liouville theorem for bounded energy finite functions can hold, cf. [37, 38, p. 319/320].

Proof. Let $\mu_{A'}$ be the quotient measure on the partition $P_{A'}$. There is a subset $A \subset P_{A'}$ with $0 < \mu_{A'}(A) < 1$. If not, $\mu_{A'}$ would have an atom $C \in P_{A'}$ with $\mu_{A'}(C) = 1$, but this would imply $\mu(W^u) \geq \mu(C) = 1$ for the global unstable manifold $W^u$ containing $C$. Then by ergodicity the partition of $\Lambda$ into global unstable manifolds would be measurable, and this would imply that the measure theoretic entropy is zero, [54], see [26, p. 28] for an exposition of the argument. However, this would contradict the fact that SRB-measures have positive entropy, [60, Theorem 1], see also [52, 53, 81]. Accordingly, the set $B = \bigcup_{C \in A} C$ has measure $0 < \mu(B) < 1$. The function $\varphi(x) := 1_B$, $x \in \Lambda'$, is Borel measurable, and it is constant on each $C \in P_{A'}$. Since each such $C$ is an open subset of some global unstable manifold, it follows that

$$\nabla W^u(x)\varphi(x) = \nabla C \varphi(x) = 0, \quad x \in C.$$

This implies that $\int_{P_{A'}} \|\nabla W^u(x)\varphi(x)\|_{T_x W^u(x)}^2 \mu(dx) = 0$, and since $\mu(\Lambda \setminus \Lambda') = 0$, (9.2) follows. \hfill \Box

As another consequence of Theorem 9.1 we can observe the fact that, roughly speaking, heat is not transported transversally.

Corollary 9.2. Let $\mu$ be an ergodic SRB-measure on $\Lambda$ with uniformly bounded densities. If $A$ and $B$ are unions of sets from $P_{A'}$, and no global unstable manifold hits both $A$ and $B$ then $\int_A P_t 1_B \ d\mu = 0$ for any $t > 0$.

Proof. Let $A' \supset A$ denote the union of all $C \in P_{A'}$ contained in one of the global unstable manifolds that hits $A$, and let $B' \supset B$ be defined similarly. Since each global unstable manifold $W^u$ is $f$-invariant and $A' \cap W^u$ is. This implies that also $A'$ and $B'$ are $f$-invariant, and being disjoint, one of them must have measure zero by ergodicity. Since $(P_t)_{t \geq 0}$ is Markovian, $\int_A P_t 1_B \ d\mu \leq \int_{A'} P_t 1_{B'} \ d\mu \leq \sqrt{\mu(A')}\sqrt{\mu(B')} = 0$ for any $t > 0$. \hfill \Box

Theorem 9.1 and Theorem A.1 also yield an additional formula for $\mathcal{E}(\mu)$.

Corollary 9.3. Let $\mu$ be an ergodic SRB-measure on $\Lambda$ with uniformly bounded densities and let $\nu$, $P_{A'}$ and $\mu_{A'}$ be as before. Then there is a family $\{\mu_C\}_{C \in P_{A'}}$ of probability measures $\mu_C$ on the partition elements $C$ of $P_{A'}$ such that

$$\mathcal{E}(\mu)(\varphi) = \int_{P_{A'}} D_{\mu_{A'}}(\varphi) \ \mu_{A'}(dC), \quad \varphi \in \mathcal{D}_0(\mathcal{E}(\mu)).$$

Here $D_{\mu_{A'}}(\varphi)$ denotes the Dirichlet integral on $C \in P_{A'}$ with respect to the measure $\mu_C$ as defined in (C.3).

Proof. For any $\varphi \in \mathcal{D}_0(\mathcal{E}(\mu))$ definitions (5.1) and (5.3), together with Theorem 9.1 and (C.3), yield

$$\mathcal{E}(\mu)(\varphi) = \int_{P_{A'}} \|\nabla W^u(x)\varphi(x)\|_{T_x W^u(x)}^2 \mu(dx) = \int_{P_{A'}} \int_{C} \|\nabla_C \varphi(x)\|_{T_x C}^2 \mu_C(dx) \mu_{A'}(dC).$$
10. Dirichlet forms and semigroups on domains

We revisit the basic idea in the proof of Lemma 5.1 and comment on Dirichlet forms and semigroups on domains inside rectangles, which then can easily be compared to corresponding objects on local leaves. As in Section 5 we assume that \( \mu \) is a finite Borel measure on \( \Lambda \) which satisfies the (AC)-property with uniformly bounded densities.

Let \( O \subset \Lambda \) be an open set such that \( \mathcal{O} \) is contained in an open rectangle \( \mathcal{R} \). As before let \( \mu_{\mathcal{R}} \) be the quotient measure on the partition \( \mathcal{P}_{\mathcal{R}} = \{W^\mathcal{R}_k\} \) of \( \mathcal{R} \). We can fix a \( \mu_{\mathcal{R}} \)-null set \( \mathcal{N}_{\mathcal{R}} \subset \mathcal{P}_{\mathcal{R}} \) so that on all partition elements \( W^\mathcal{R}_k \) outside this null set the (conditional) measures \( \mu_{W^\mathcal{R}_k} \) are absolutely continuous with bounded densities bounded away from zero. The family \( (L^2(O \cap W^\mathcal{R}_k, \mu_{W^\mathcal{R}_k}))_{W^\mathcal{R}_k \in \mathcal{P}_{\mathcal{R}} \setminus \mathcal{N}_{\mathcal{R}}} \) is a measurable field of Hilbert spaces, and the space \( L^2(O, \mu) \) is isometrically isomorphic to the direct integral

\[
L^2(\mathcal{P}_{\mathcal{R}}, (L^2(O \cap W^\mathcal{R}_k, \mu_{W^\mathcal{R}_k}))_{W^\mathcal{R}_k \in \mathcal{P}_{\mathcal{R}} \setminus \mathcal{N}_{\mathcal{R}}} \mu_{\mathcal{R}})
\]

over this measurable field. Therefore any element \( \varphi \) of \( L^2(O, \mu) \) can be interpreted as a class of \( \mu_{\mathcal{R}} \)-square integrable sections

\[
(10.1) \quad \varphi = (\varphi_{W^\mathcal{R}_k})_{W^\mathcal{R}_k \in \mathcal{P}_{\mathcal{R}} \setminus \mathcal{N}_{\mathcal{R}}} \text{ with elements } \varphi_{W^\mathcal{R}_k} \in L^2(W^\mathcal{R}_k, \mu_{W^\mathcal{R}_k}).
\]

One perspective upon localized forms and semigroups on \( O \) is the usual restriction to \( O \), seen as an open subset of \( \Lambda \). By \( \mathcal{D}(\mathcal{E}(\mu, O)) \) we denote the closure of \( \mathcal{D}(\mathcal{E}(\mu)) \cap \mathcal{C}(O) \) in \( \mathcal{D}(\mathcal{E}(\mu)) \) and set

\[
\mathcal{E}(\mu, O)(\varphi) := \mathcal{E}(\mu)(\varphi), \quad \varphi \in \mathcal{D}(\mathcal{E}(\mu, O)).
\]

This corresponds to the choice of Dirichlet boundary conditions for the open set \( O \).

**Proposition 10.1.** Let \( \mu \) be a finite Borel measure on \( \Lambda \) which satisfies the (AC)-property with uniformly bounded densities and \( \text{supp} \mu = \Lambda \). Let the open set \( O \) and the rectangle \( \mathcal{R} \) be as above.

(i) The form \( (\mathcal{E}(\mu, O), \mathcal{D}(\mathcal{E}(\mu, O))) \) is a strongly local regular Dirichlet form on \( L^2(O, \mu) \).

(ii) The space \( \mathcal{D}_0(\mathcal{E}(\mu, O)) \) of all \( \varphi \in \mathcal{C}(O) \cap C^{n,1}(\Lambda) \) for which the function \( x \mapsto ||\nabla \varphi(x)||_{W^{n,1}(x)} \) is in \( L^2(O, \mu) \) is a dense subspace of \( \mathcal{D}(\mathcal{E}(\mu, O)) \).

(iii) For any \( \varphi \in \mathcal{D}_0(\mathcal{E}(\mu, O)) \) we have

\[
(10.2) \quad \mathcal{E}(\mu, O)(\varphi) = \int_{\mathcal{P}_{\mathcal{R}}} \mathcal{D}^{W_{\mathcal{R}}}(\varphi_{W^\mathcal{R}_k}) \mu_{\mathcal{R}}(dW^\mathcal{R}_k),
\]

with notation as in \( (10.1) \).

**Proof.** Statement (i) is immediate and statement (iii) follows by similar arguments as used for Corollary 9.3. To see (ii) suppose that \( \varphi \in \mathcal{D}(\mathcal{E}(\mu, O)) \) and let \( (\varphi_k)_k \subset \mathcal{C}(O) \cap \mathcal{D}(\mathcal{E}(\mu)) \) be a sequence approximating \( \varphi \) in \( \mathcal{D}(\mathcal{E}(\mu)) \). For each fixed \( k \) let \( \chi^k \in C^1(M) \) be such that \( \text{supp} \chi^k \subset \Lambda \) and \( \chi^k = 1 \) on \( \text{supp} \varphi_k \). By the construction of \( (\mathcal{E}(\mu), \mathcal{D}(\mathcal{E}(\mu))) \) there is a sequence \( (\psi_j)_j \subset C(\Lambda) \cap C^{n,1}(\Lambda) \) that approximates \( \varphi_k \) in \( \mathcal{D}(\mathcal{E}(\mu)) \). Then the sequence \( (\chi^k \psi_j)_j \) is in \( \mathcal{D}_0(\mathcal{E}(\mu, O)) \) and approximates \( \varphi_k \) in \( \mathcal{D}(\mathcal{E}(\mu)) \). Statement (ii) now follows easily. \( \square \)
By \((P_{t}^{O})_{t>0}\) we denote the symmetric Markov semigroup on \(L^{2}(O,\mu)\) uniquely associated with \((\mathcal{E}(\mu,O),\mathcal{D}(\mathcal{E}(\mu,O)))\).

A second perspective is to consider restrictions of Dirichlet integrals and Markov semigroups to open subsets of the individual local leaves. As in the proof of Lemma 5.1 and in the preceding section we can continue the measures \(\mu_{W_{R}}^{\psi}\) to measures on the local unstable manifolds \(W_{R}^{u}\) containing \(W_{R}^{u}\) (as in \((4.1)\)) in a way that the resulting measures \(\mu_{W_{R}}^{\psi}\) have uniformly bounded densities as in Corollary C.1. Given \(W_{R}^{u} \in \mathcal{P}_{R} \setminus \mathcal{N}_{R}\) the set \(W_{R}^{u} \cap O\) is an open subset of \(W_{R}^{u}\). Accepting a slight abuse of notation to keep it short, we define quadratic forms by

\[
D^{(\mu_{W_{R}^{u}};O)}(\psi) := D^{(\mu_{W_{R}^{u}})}(\psi), \quad \psi \in C_{c}^{1}(W_{R}^{u} \cap O).
\]

This corresponds to the choice of Dirichlet boundary conditions on \(W_{R}^{u} \cap O\), seen as an open subset of \(W_{R}^{u}\). The following is a direct consequence of Corollary C.1.

**Corollary 10.1.** Let \(\mu\) be a finite Borel measure on \(\Lambda\) which satisfies the \((AC)\)-property with uniformly bounded densities and \(\text{supp } \mu = \Lambda\). Let \(O, R\) and \(\mathcal{N}_{R}\) be as above. For each \(W_{R}^{u} \in \mathcal{P}_{R} \setminus \mathcal{N}_{R}\) the quadratic form \((D^{(\mu_{W_{R}^{u}};O)}, C_{c}^{1}(W_{R}^{u} \cap O))\) is closable on \(L^{2}(W_{R}^{u} \cap O, \mu_{W_{R}^{u}})\) and its closure \((D^{(\mu_{W_{R}^{u}};O)}, D(D^{(\mu_{W_{R}^{u}};O)}))\) is a strongly local regular Dirichlet form.

For each \(W_{R}^{u} \in \mathcal{P}_{R} \setminus \mathcal{N}_{R}\) let \((P_{t}^{(\mu_{W_{R}^{u}};O)})_{t>0}\) denote the symmetric Markov semigroup on \(L^{2}(W_{R}^{u} \cap O, \mu_{W_{R}^{u}})\) uniquely associated with \((D^{(\mu_{W_{R}^{u}};O)}, D(D^{(\mu_{W_{R}^{u}};O)}))\). Fusing these semigroups by integration with respect to \(\hat{\mu}_{R}\), we obtain a Markov semigroup on \(L^{2}(O,\mu)\) which dominates \((P_{t}^{O})_{t>0}\).

**Corollary 10.2.** Let the hypotheses of Corollary 10.1 be in force. For any \(t > 0\) and \(\varphi = (\varphi_{W_{R}^{u}})_{W_{R}^{u} \in \mathcal{P}_{R} \setminus \mathcal{N}_{R}}\) from \(L^{2}(O,\mu)\), interpreted as in 10.1, we can define an element \(\hat{P}_{t}^{O} \varphi\) by \((\hat{P}_{t}^{O} \varphi)_{W_{R}^{u} \in \mathcal{P}_{R} \setminus \mathcal{N}_{R}}\) of \(L^{2}(O,\mu)\), again interpreted as in 10.1, by setting

\[
(\hat{P}_{t}^{O} \varphi)_{W_{R}^{u}} := P_{t}^{(\mu_{W_{R}^{u}};O)} \varphi_{W_{R}^{u}}, \quad W_{R}^{u} \in \mathcal{P}_{R} \setminus \mathcal{N}_{R}.
\]

The map \(\varphi \mapsto \hat{P}_{t}^{O} \varphi\) is a linear contraction on \(L^{2}(O,\mu)\), and the family \((\hat{P}_{t}^{O})_{t>0}\) is a symmetric Markov semigroup on \(L^{2}(O,\mu)\). For any \(t > 0\) we have \(P_{t}^{O} \leq \hat{P}_{t}^{O}\).

**Proof.** For any \(t > 0\) and \(\varphi \in L^{2}(O,\mu)\) we have

\[
\|\hat{P}_{t}^{O} \varphi\|_{L^{2}(O,\mu)}^{2} = \int_{\mathcal{P}_{R}} \|P_{t}^{(\mu_{W_{R}^{u}};O)} \varphi_{W_{R}^{u}}\|_{L^{2}(W_{R}^{u} \cap O, \mu_{W_{R}^{u}})}^{2} \hat{\mu}_{R}(dW_{R}^{u})
\]

\[
\leq \int_{\mathcal{P}_{R}} \|\varphi_{W_{R}^{u}}\|_{L^{2}(W_{R}^{u} \cap O, \mu_{W_{R}^{u}})}^{2} \hat{\mu}_{R}(dW_{R}^{u})
\]

\[
= \|\varphi\|_{L^{2}(O,\mu)}^{2}
\]

by the contractivity of the operators \(P_{t}^{(\mu_{W_{R}^{u}};O)}\). This implies that \(\hat{P}_{t}^{O}\) is a linear contraction on \(L^{2}(O,\mu)\). The semigroup property \(\hat{P}_{s+t}^{O} = \hat{P}_{s}^{O} \hat{P}_{t}^{O}\), \(s,t > 0\), is immediate from that of the operators \(P_{t}^{(\mu_{W_{R}^{u}};O)}\) on the leaves, and similarly we can observe the Markov property. The symmetry follows from the symmetry of the operators \(P_{t}^{(\mu_{W_{R}^{u}};O)}\) in the spaces \(L^{2}(W_{R}^{u} \cap O, \mu_{W_{R}^{u}})\) and integration, and the strong continuity follows using dominated convergence and \((10.3)\).
Now let \((\tilde{\mathcal{E}}^O, \mathcal{D}(\tilde{\mathcal{E}}^O))\) be the Dirichlet form on \(L^2(O, \mu)\) uniquely associated with \((\tilde{P}_t^O)_{t > 0}\), that is, defined by analogs of (6.2) and (6.3) with \(O\) and \(\tilde{P}_t^O\) in place of \(\Lambda\) and \(P_t\). We have
\[
\begin{align*}
t^{-1} \langle \varphi - P_t^O \varphi, \varphi \rangle_{L^2(O, \mu)} &= \int_{\mathcal{P}_R} t^{-1} \int_{W_R^u} \left( \varphi_{W_R^u}(x) - P_t^{(\mu W^e_R, O)} \varphi_{W_R^u}(x) \right) \varphi_{W_R^u}(x) \mu_{W_R^u}(dx) \hat{\mu}_R(dW_R^u) \\
&\leq \int_{\mathcal{P}_R} P_t^{(\mu W^e_R, O)}(\varphi_{W_R^u}) \hat{\mu}_R(dW_R^u)
\end{align*}
\]
for any \(\varphi \in \mathcal{D}_0(\mathcal{E}(\mu, O))\) and \(t > 0\). Taking the supremum over \(t > 0\) and using monotone convergence, we see that \(\tilde{\mathcal{E}}^O(\varphi) = \mathcal{E}(\mu, O)(\varphi)\) for any such \(\varphi\), and as both forms are closed, this extends to all \(\varphi \in \mathcal{D}(\mathcal{E}(\mu))\). Therefore \(\mathcal{D}(\mathcal{E}(\mu, O)) \subseteq \mathcal{D}(\tilde{\mathcal{E}}^O)\), what implies that for the generators \(\mathcal{L}(\mu, O)\) and \(\tilde{\mathcal{L}}^O\) of \((P_t^O)_{t > 0}\) and \((\tilde{P}_t^O)_{t > 0}\), respectively, we have \(\mathcal{L}(\mu, O) \leq \tilde{\mathcal{L}}^O\) in the sense of non-positive definite self-adjoint operators. By the spectral theorem it follows that \(P_t^O = e^{t\mathcal{L}(\mu, O)} \leq e^{t\tilde{\mathcal{L}}^O} = \tilde{P}_t^O\). □

**Remark 10.1.** The operators \(P_t^{(\mu W^e_R, O)}\) admit integral representations
\[
P_t^{(\mu W^e_R, O)} \psi(x) = \int_{W_R^u \cap O} p_t^{(\mu W^e_R, O)}(x, y) \psi(y) \mu_{W_R^u}(dy),
\]
valid in particular for any \(\psi \in L^2(W_R^u \cap O, \mu_{W_R^u})\). Here \(p_t^{(\mu W^e_R, O)}(x, y)\) denotes the Dirichlet heat kernel with respect to \(\mu_{W^e_R} \mid W_R^u \cap O\) on the open subset \(W_R^u \cap O\) of \(W_R^u\) in the sense of Remark 9.1 (iv).

11. Examples

We comment on some concrete examples of uniformly hyperbolic attractors.

11.1. **Hyperbolic toral automorphisms.** Prototype examples of Anosov diffeomorphisms as mentioned in Remark 2.1 are hyperbolic automorphisms \(f_A : T^n \to T^n\) of the \(n\)-torus \(T^n = \mathbb{R}^n / \mathbb{Z}^n\), defined by \(f_A([x]) = [Ax]\), \(x \in \mathbb{R}^n\), where \([x] = x \mod 1\), interpreted component-wise, denotes the class of \(x \in \mathbb{R}^n\) in \(T^n\), and \(A\) is an integer \((n \times n)\)-matrix with \(|\det A| = 1\) and no eigenvalue of modulus 1. [20 Section 1.7], [64 Section 8.5]. For the special case that \(n = 2\) and
\[
A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
\]
the map \(f_A\) is known as Arnold’s cat map, it is \(u\)-conformal with \(a = \lambda_u\) in (8.1), \(\lambda_u = (3 + \sqrt{5})/2 > 1\) being the larger eigenvalue of \(A\). In general, hyperbolic toral automorphisms do of course not have to be \(u\)-conformal, but typically have various eigenvalues greater one. Any hyperbolic toral automorphism is ergodic with respect to the (normalized) Riemannian volume \(m_{T^n}\) on \(T^n\). [20 Proposition 4.4.3], in particular, topologically transitive, so that \(m_{T^n}\) is the unique SRB-measure for \(f_A\) on \(\Lambda = T^n\) and has uniformly bounded densities.
11.2. Solenoids. Prototype example of expanding attractors, [80], are solenoids, also referred to as Smale-Williams attractors, [20] Section 1.9, [48] Section 17.1, [64] Section 8.7. Let $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the closed unit disk and let $S^1$ be the unit circle, which we parametrize by the angle $\theta \in [0, 2\pi)$. Then the solid torus $T = D^2 \times S^1$ is a smooth compact manifold that admits local coordinates $(x, y, \theta)$ and becomes Riemannian if endowed with the restriction of the Euclidean scalar product in $\mathbb{R}^3$. Given $r \in (0, 1)$ and $\alpha, \beta \in (0, \min \{r, 1-r\})$, we can define a smooth map $f : T \to T$ by

$$f(x, y, \theta) = (\alpha x + r \cos \theta, \beta y + r \sin \theta, 2\theta),$$

and the resulting uniformly hyperbolic attractor $\Lambda$ is called a solenoid. The map $f$ is topologically transitive on $\Lambda$, and it is $u$-conformal with constant $a = 2$ (its Lyapunov exponent for the expanding direction) in [8.1]. There is a unique SRB-measure $\mu$ on $\Lambda$, it is ergodic, and since $\Lambda$ may locally be viewed as the product of an interval (in the expanding direction) and a Cantor set (in the contracting direction), $\mu$ may locally be viewed as the product of a one-dimensional Lebesgue measure and a Cantor measure. Occurrences of solenoids in physical models are discussed in [50], along with striking related findings in experimental physics, [50] Sections 12.2 and 13.1.

11.3. DA attractors. A non-Anosov uniformly hyperbolic attractor can also be derived from an Anosov diffeomorphism by a deformation in a small neighborhood of a fixed point. For instance, let $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ be Arnold’s cat map, let $\lambda_s < 1 < \lambda_u$ be its eigenvalues and $v^s$ and $v^u$ the associated eigenvectors. We use the coordinates $\alpha_1 v^u + \alpha_2 v^s$ in a neighborhood $U \subset \mathbb{T}^2$ of the only fixed point $p_0$ of $f_A$ corresponding to $(0, 0) \in \mathbb{R}^2$. Let $r_0 > 0$ be such that the ball $B(p_0, r_0)$ is contained in $U$ and let $h : [0, +\infty) \to [0, 1]$ be a function such that $\text{supp } h \subset [0, r_0)$ and $h(x) = 1$ for $0 \leq x \leq r_0/2$. Now let $\phi^\tau$ be the flow of the differential equations $\alpha_1 = 0$ and $\alpha_2 = \alpha_2 \cdot h((\alpha_1, \alpha_2))$ and define $F : \mathbb{T}^2 \to \mathbb{T}^2$ as $F(\alpha_1, \alpha_2) := \phi^\tau \circ f_A(\alpha_1, \alpha_2)$ on $U$, where $\tau > 0$ is any fixed number such that $e^{\tau} \lambda_s > 1$, and $F = f_A$ on $\mathbb{T}^2 \setminus U$. The map $F$ is called a DA-diffeomorphism, and the associated attractor $\Lambda$ is uniformly hyperbolic. Details can for instance be found in [48] Section 17.2a] or [64] Section 8.8, graphical visualizations in [29].

11.4. Plykin attractor. If $J : \mathbb{T}^2 \to \mathbb{T}^2$ is defined by $J([x]) = [-x]$ and $F$ is a DA-diffeomorphism on $\mathbb{T}^2$ as constructed above, then $F$ and $J$ commute and one can find periodic point for $F$. It is then possible to construct a map $\tilde{F} : \mathbb{T}^2 \to \mathbb{T}^2$ which also commutes with $J$, has four fixed points and defines a hyperbolic attractor. Projecting this attractor onto a specific sphere with four holes, one obtains the Plykin attractor, [61]. By stereographic projection it can also be regarded as a compact subset $\Lambda$ of a planar region $D$ with three holes, and moreover, there is an equivalent ‘graphic’ way to construct it as the attractor of a suitable map $f : D \to D$, Figure 1. See [48] Section 17.2b] or [64] Section 8.9], and [29], [50] Sections 1.2.4, 2.4 and 2.5 for visualizations and phenomenological explanations. Detailed discussions of Plykin type attractors in physical models can be found in [50].
Appendix A. Disintegration and Rokhlin’s theorem

We recall a classical theorem due to Rokhlin [66], further details can be found in [79, Chapter 5].

Let \((X, \mathcal{A}, \mu)\) be a probability space and \(P\) a partition of \(X\). A subset \(Q\) of \(P\) is called measurable if the preimage of \(Q\) under the projection map \(\pi : X \rightarrow P\) taking each point \(x \in X\) to the element \(P(x) \in P\) containing \(x\) is measurable. The quotient probability measure \(\hat{\mu}\) on \(P\) is defined as the pushforward of \(\mu\) under \(\pi\).

A disintegration of \(\mu\) with respect to \(P\) is a family \(\{\mu_P\}_{P \in P}\) of probability measures on \(X\), called the conditional measures (or conditional probabilities), such that

(i) \(\mu_P(P) = 1\) for \(\hat{\mu}\)-a.e. \(P \in P\),
(ii) for any \(A \in \mathcal{A}\) the function \(P \mapsto \mu_P(A)\) is measurable,
(iii) we have

\[
\mu(A) = \int \mu_P(A) \, d\hat{\mu}(P) \quad \text{for any } A \in \mathcal{A}.
\]

A partition \(P\) of \(X\) is called measurable if there exists a sequence \((P_n)_n\) of countable partitions \(P_n\) such that for \(\mu\)-a.e. \(x \in X\), \(P_{n+1}(x) \subset P_n(x)\) for every \(n \in \mathbb{N}\), and \(P(x) = \bigcap_{n \in \mathbb{N}} P_n(x)\).

Remark A.1. The partition into orbits of an ergodic measure theoretic dynamical system is not measurable, unless one orbit has full measure, [79, Example 5.1.10]. The same is true for the partition of a uniformly hyperbolic attractor into global unstable manifolds as in [2,3], endowed with an ergodic probability measure, see for instance [53, p. 513].

We quote the following from [79, Proposition 5.1.7 and Theorem 5.1.11].

Theorem A.1. Let \(\mu\) be a Borel probability measure on a compact metric space \(X\) and \(P\) a measurable partition of \(X\). There is a disintegration of \(\mu\) with respect to \(P\). If \(\{\mu_P\}_{P \in P}\) and \(\{\mu'_P\}_{P \in P}\) are two disintegrations of \(\mu\) with respect to \(P\) then \(\mu_P = \mu'_P\) for \(\hat{\mu}\)-a.e. \(P \in P\).

Appendix B. The geometric construction of SRB-measures

For the convenience of the reader we recall the ansatz for the well-known geometric construction of SRB-measures, [60, Propositions 2 and 3 and Theorem 4].

For any \(z \in \Lambda\) we write \(J^u_z f\) to denote the Jacobian of \(f\) along \(W^u(z)\) (the modulus of the determinant of \(d_z f|_{T_z W^u} : T_z W^u \rightarrow T_{f(z)} W^u\) with respect to the
Riemannian metric). For any $n \geq 1$, any $x \in \Lambda$ and any $y \in W^u(x)$ we set $g_n(x, y) := \prod_{j=1}^n J^u_{f^{-j}(y)} f \left[ J^u_{f^{-j}(y)} f \right]^{-1}$. Let $\varepsilon > 0$ be fixed. There exists a constant $K_0 \geq 1$ such that for all $n \geq 1$ and for all $x \in \Lambda$ and $y \in W^u_\varepsilon(x)$, we have

\[
K_0^{-1} \leq g_n(x, y) \leq K_0.
\]

This follows using a traditional argument, [16, Lemma 3.3]: Since $K$ and since $R$ rectangle, then the weak convergence, together with disintegration on weak subsequential limit $\varrho$ with a renormalization constant $|1+\varepsilon|^n$, it follows that $\varrho_t$ is $\alpha$-Hölder. Taking into account uniform hyperbolicity,

\[
|\log J^u_{f^{-j}(x)} f - \log J^u_{f^{-j}(y)} f| \leq L d(f^{-j}(x), f^{-j}(y)) \alpha \leq LC \lambda^\alpha d(x, y)^\alpha
\]

with $L, C > 0$ and $\lambda \in (0, 1)$ as in Theorem 2.1 and summing over $j \in \mathbb{N}$, we see that $g_n(x, y) \leq \exp((LC \lambda^\alpha d(x, y)^\alpha)/(1 - \lambda^\alpha))$, what implies (B.1). In a similar way it follows that $|g_n(x, y) - 1| \leq C' d(x, y)^\alpha$ with $C' > 0$ depending only on $\alpha, L, C, \lambda$ and $\varepsilon$. Combined with (B.1) this shows that

\[
|g_n(x, y) - g_n(x, z)| \leq C' K_0 d(y, z)^\alpha,
\]

for all $y, z \in W^u_\varepsilon(x), y \neq z$.

Applying Arzelà-Ascoli on a compact subset $K(x)$ of $W^u_\varepsilon(x)$ containing $x$, we then see that $g(x, y) := \lim_n g_n(x, y)$ with convergence uniform in $x \in \Lambda$ and $y \in K(x)$, [60, Proposition 2]. Clearly $g$ admits the same bounds as $g_n$ in (B.1), and using [B.3] it follows that for any $x$ the function $g(x, \cdot)$ is $\alpha$-Hölder on $K(x)$.

Now let $x \in \Lambda$ be fixed. Set $c_0 := 1$ and $c_n := \left[ \prod_{k=0}^{n-1} J^u_{f^k(x)} f \right]^{-1}$ and consider the measures on $f^n(W^u(x))$ defined by $\nu_n(dy) := c_n g^n(x, y) m^{f^n(W^u(x))}(dy)$. Using the change of variable formula and balancing all cancellations, one can see that $\nu_n = f^n_* \nu_0$ for any $n$, [60, Proposition 3 and Section 2.6]. Similarly as in the Krylov-Bogoliubov theorem one can now define averages

\[
\mu_n := \frac{1}{n} \sum_{k=0}^{n-1} \nu_k,
\]

and since $\nu_0$ is a finite measure, Prohorov’s theorem implies that $(\mu_n)_n$ has a weak subsequential limit $\mu$, which obviously is $f$-invariant. If now $\mathcal{R} \subset \Lambda$ is a rectangle, then the weak convergence, together with disintegration on $\mathcal{R}$ and a uniform convergence argument for the densities of the conditional measures show that for $\mu_\mathcal{R}$-a.e. $x$ we have

\[
\mu_{W^\mathcal{R}_{\varepsilon}(x)}(dy) = g(x)^{-1} g(x, y) m_{W^\mathcal{R}_{\varepsilon}(x)}(dy)
\]

with a renormalization constant $g(x) := \int_{W^\mathcal{R}_{\varepsilon}(x)} g(x, y) m_{W^\mathcal{R}_{\varepsilon}(x)}(dy)$, [60, Theorem 4, Lemma 13 and Section 2.7]. In particular, $\mu$ is an SRB-measure with uniformly bounded Hölder continuous densities $g_{W^\mathcal{R}_{\varepsilon}(x)} = g(x)^{-1} g(x, \cdot)$.

**Appendix C. Dirichlet integrals on weighted manifolds**

This section collects some facts on Laplacians on weighted manifolds of low regularity and with uniformly bounded Borel densities.

We begin with a well-known standard fact which we provide for completeness.

**Proposition C.1.** Let $(M, g)$ be a $C^1$-Riemannian manifold, let $\nabla_M$ denote the gradient operator on $M$ and $m_M$ the Riemannian volume. The quadratic form $(D, C^1_0(M))$, defined by the classical Dirichlet integral

\[
\begin{align*}
D(\varphi) &= \int_M \|
abla_M \varphi(x)\|^2_{T_x M} m_M(dx), & \varphi \in C^1_0(M),
\end{align*}
\]

is a closed quadratic form.
is closable on \( L^2(M,m_M) \).

**Proof.** Suppose that \( (\varphi_j)_{j=1}^{\infty} \subset C^1(M) \) is Cauchy w.r.t. \( D^{1/2} \) and such that \( \lim_{j \to \infty} \varphi_j = 0 \) in \( L^2(M,m_M) \). Then \( (\nabla_M \varphi_j)_j \) is Cauchy in the Hilbert space \( L^2(M,TM,m_M) \) of \( L^2 \)-vector fields on \( M \) and therefore has a limit \( w \) in this space. A \( C^1 \)-version of the divergence theorem follows by the traditional arguments, see for instance \([38] \) Theorem 3.5, and it implies that for any compactly supported \( C^1 \)-vector field \( v \in C^1(M,TM) \) we have

\[
\int_M \langle v, w \rangle_{TM} dm_M = \lim_{j \to \infty} \int_M \langle v, \nabla_M \varphi_j \rangle_{TM} dm_M = -\lim_{j \to \infty} \int_M (\text{div}_M v) \varphi_j dm_M = 0,
\]

where \( \text{div}_M \) is the classical divergence operator on \( M \). Since \( v \) was arbitrary, this implies that \( w = 0 \) in \( L^2(M,TM,m_M) \) and consequently \( \lim_{j \to \infty} D(\varphi_j) = 0 \). \( \square \)

Now suppose that \( (M,g) \) is a \( C^1 \)-Riemannian manifold and \( g_M : M \to (0, +\infty) \) is a Borel function. We endow \( M \) with the the measure

\[
(\mu_M = g_M \cdot m_M)
\]

having density \( g_M \) w.r.t. to the Riemannian volume \( m_M \) on \( M \). Then \( (M,g,\mu_M) \) could be seen as a low regularity version of the weighted manifolds in \([38] \) Definition 3.17. We can define a quadratic \( [0, +\infty] \)-valued functional on \( C^1(M) \) by

\[
D^{(\mu_M)}(\varphi) = \int_M \|\nabla_M \varphi(x)\|_{TM}^2 \mu_M(dx), \quad \varphi \in C^1(M).
\]

The following is immediate from Proposition \( C.1 \)

**Corollary C.1.** Let \( (M,g) \) be a \( C^1 \)-Riemannian manifold and let \( \mu_M \) be as above with a Borel density \( g_M \) satisfying

\[
K^{-1} \leq g_M(x) \leq K, \quad x \in M.
\]

with a constant \( K > 1 \). Then the quadratic form \( (D^{(\mu_M)}, C^1(M)) \) is closable on \( L^2(M,\mu_M) \). Its closure \( (D^{(\mu_M)}, D(D^{(\mu_M)})) \) is a strongly local regular Dirichlet form. Its generator \( (\mathcal{L}^{(\mu_M)}, D(\mathcal{L}^{(\mu_M)})) \) is a non-positive definite self-adjoint operator.

The operator \( (\mathcal{L}^{(\mu_M)}, D(\mathcal{L}^{(\mu_M)})) \) is a natural self-adjoint Laplacian on \( (M,g,\mu_M) \), we refer to it as the Dirichlet Laplacian on \( M \).

**Remark C.1.**

(i) If \( g_M \equiv 1 \) and \( (M,g) \) is \( C^1 \) only, one cannot introduce a classical Laplacian on \( C^2 \)-functions. However, as just seen, a self-adjoint Laplacian exists.

(ii) For the special case \( g_M \equiv 1 \) we recover the original Riemannian measure \( \mu_M = m_M \) and the Dirichlet integral \( D^{(m_M)} = D \), which is the closure on \( L^2(M,m_M) \) of \( C.1 \). For this case the additional conclusions made in Corollary \( C.1 \) are standard and complement Proposition \( C.1 \). If \( (M,g) \) is \( C^2 \) then the corresponding Dirichlet Laplacian \( (\mathcal{L}^{(m_M)}, D(\mathcal{L}^{(m_M)})) \) is the Friedrichs extension of the classical Laplacian on \( C^2(M) \).

(iii) In the general case the Dirichlet Laplacian can informally be rewritten as

\[
\mathcal{L}^{(\mu_M)} \varphi = \mathcal{L}^{(m_M)} \varphi + \langle g_M^{-1} \nabla_M g_M, \nabla_M \varphi \rangle_{TM}.
\]
The symmetric Hunt diffusion process on $M$ it generates is a distorted Brownian motion, see for instance \[2,32,45,77\] and the references therein.

(iv) The symmetric Markov semigroup $(P_t^{(\mu_M)})_{t \geq 0}$ on $L^2(M, \mu_M)$ generated by $(\mathcal{L}^{(\mu_M)}, \mathcal{D}(\mathcal{L}^{(\mu_M)}))$ admits heat kernels with respect to $\mu_M$ in the sense of \[39\] Definition 2.3 and the remarks following it. This follows from \[39\] Theorem 2.12: On the $C^1$-Riemannian manifold $(M, g, \mu_M)$ Nash (or Faber-Krahn) inequalities hold locally, as can be seen using local charts. By \[C.4\] this implies the validity of such inequalities locally on $(M, g, \mu_M)$, so that $(P_t^{(\mu_M)})_{t > 0}$ is seen to be locally ultracontractive in the sense of \[39\] Definition 2.11.

We would like to point out that even if the density $\varrho_M$ is only Borel measurable, the hypotheses of Corollary C.1 permit to define a kind of classical divergence operator. Consider the space
\[
\varrho_M^{-1}C^1_c(M, TM) := \{ \varrho_M^{-1}w : w \in C^1_c(M, TM) \},
\]
where $\varrho_M^{-1}$ denotes the reciprocal of $\varrho_M$, and set
\[
(C.5) \quad \dive_{\mu_M} v := \varrho_M^{-1} \dive_M(\varrho_M v), \quad v \in \varrho_M^{-1}C^1_c(M, TM).
\]
Obviously the classical divergence on $C^1$-vector fields corresponds to the special case $\varrho_M \equiv 1$. The following version of the divergence theorem is immediate from a $C^1$-version of the classical one, \[38\] Theorem 3.5. It could be used to give an alternative, more direct proof of Corollary C.1.

**Corollary C.2.** Let $(M, g)$ be a $C^1$-Riemannian manifold and let $\mu_M$ be as before. Then for any $\varphi \in C^1_c(M)$ and $v \in \varrho_M^{-1}C^1_c(M, TM)$ we have
\[
(C.6) \quad \int_M (\dive_{\mu_M} v) \varphi \, d\mu_M = \int_M \langle v, \nabla_M \varphi \rangle_{TM} \, d\mu_M.
\]

Gradient and divergence extend to the $L^2$-setting.

**Corollary C.3.** Let $(M, g)$ be a $C^1$-Riemannian manifold and let $\mu_M$ be as before.

(i) The operator $(\nabla_M, C^1_c(M))$ extends to a densely defined closed unbounded linear operator
\[
\nabla_M : L^2(M, \mu_M) \to L^2(M, TM, \mu_M)
\]
with domain $\mathcal{D}(\mathcal{L}^{(\mu_M)})$.

(ii) The adjoint $(-\dive_{\mu_M}, \mathcal{D}(\dive_{\mu_M}))$ of $(\nabla_M, \mathcal{D}(\mathcal{L}^{(\mu_M)}))$ is a densely defined closed unbounded operator
\[
-\dive_{\mu_M} : L^2(M, TM, \mu_M) \to L^2(M, \mu_M),
\]
and it extends the operator $(-\dive_{\mu_M}, \varrho_M^{-1}C^1(M, TM))$ with $\dive_{\mu_M}$ as in \[C.5\]. Identity \[C.6\] holds for all $v \in \mathcal{D}(\dive_{\mu_M})$ and $\varphi \in \mathcal{D}(\mathcal{L}^{(\mu_M)})$.

(iii) A function $\varphi \in \mathcal{D}(\mathcal{L}^{(\mu_M)})$ is an element of $\mathcal{D}(\mathcal{L}^{(\mu_M)})$ if and only if $\nabla_M \varphi \in \mathcal{D}(\dive_{\mu_M})$, and in this case we have
\[
\mathcal{D}(\mathcal{L}^{(\mu_M)}) \varphi = \dive_{\mu_M}(\nabla_M \varphi).
\]

**Remark C.2.** One could try to define operators $\Delta_{\mu_M} \varphi := \dive_{\mu_M}(\nabla_M \varphi)$ on the space of all $\varphi \in C^1(M)$ such that $\nabla_M \varphi \in \varrho_M^{-1}C^1_c(M, TM)$. If $(M, g)$ is $C^2$ and $\varrho_M$ is $C^1$ it is obvious that this space does not only consist of constant functions (and in particular, that it is not trivial), but in general this may not be clear.
Remark C.3. Similarly as for \( C^\infty \)-weighted manifolds definition (C.5) can be used to define the Dirichlet integral in \( (C.3) \) on a larger domain. For \( \varphi \in L^1_{\text{loc}}(M, \mu_M) \) we can define \( \nabla_M \varphi \) as an element of the dual \( (\varphi^{-1}C^1_c(M,TM))^\ast \) of \( \varphi^{-1}C^1_c(M,TM) \), endowed with the norm \( v \mapsto \|\varphi^{-1}C^1_c(M,TM)\| \), by
\[
\nabla_M \varphi(v) := -\int_M \varphi \text{div}_\mu_M v \, d\mu_M, \quad v \in \varphi^{-1}C^1_c(M,TM).
\]
Then we can consistently extend (C.3) to a quadratic \([0, +\infty]\)-valued functional on \( L^1_{\text{loc}}(M, \mu_M) \) by
\[
D^{(\mu_M)}(\varphi) := \int_M \|\nabla_M \varphi(x)\|^2_{L^2_M} \, \mu_M(dx), \quad \varphi \in L^1_{\text{loc}}(M, \mu_M).
\]
Obviously this is finite if and only if \( \nabla_M \varphi \in L^2(M, TM, \mu_M) \).

Appendix D. Superposition of closable forms

The following is a special case of [3] Theorem 1.2 respectively a small modification of [17] Chapter V, Proposition 3.11 or [33, Section 3.1. (2°)]. For details on measurable fields of Hilbert spaces and their direct integrals see for instance [31, Part II, Chapter 1] or [75, Chapter IV, Section 8].

Proposition D.1. Let \((Y, \mathcal{Y}, \mu)\) be a \( \sigma \)-finite measure space and let \((H_y, \langle \cdot, \cdot \rangle_{H_y})_{y \in \mathcal{Y}}\) be a measurable field of Hilbert spaces \( H_y \) on \( \mathcal{Y} \). Suppose that for all \( y \in \mathcal{Y} \) we are given a dense subspace \( C_y \) of \( H_y \) and a quadratic form \((Q^{(y)}, C_y)\). Consider the subspace
\[
\mathcal{C} := \{ v = (v_y)_{y \in \mathcal{Y}} \in L^2(\mathcal{Y}, (H_y)_{y \in \mathcal{Y}}, \mu) : v_y \in C_y \text{ for all } y \in \mathcal{Y} \text{ and the map } \mu \mapsto Q^{(y)}(v_y) \text{ is } \mu \text{-integrable} \}
\]
of the direct integral \( L^2(\mathcal{Y}, (H_y)_{y \in \mathcal{Y}}, \mu) \). If for \( \mu \text{-a.e. } y \in \mathcal{Y} \) the form \( (Q^{(y)}, C_y) \) is closable on \( H_y \), then the quadratic form \((Q, \mathcal{C})\), defined by
\[
Q(v) = \int_\mathcal{Y} Q^{(y)}(v_y) \, \mu(dy), \quad v \in \mathcal{C},
\]
is closable on \( L^2(\mathcal{Y}, (H_y)_{y \in \mathcal{Y}}, \mu) \).

For convenience we provide the well-known proof, [3, Theorem 1.2].

Proof. Suppose that \( (v_y)_{y=1}^\infty \subset \mathcal{C} \) is Cauchy w.r.t. the seminorm \( Q^{1/2} \) and such that \( \lim_{j \to \infty} \int_\mathcal{Y} \|Q^{(y)}(v_j)\|^2_{H_y} \, \mu(dy) = 0 \). Passing to a subsequence if necessary, we may assume that
\[
\sum_{j=1}^\infty Q(v_{j+1} - v_j) < +\infty \quad \text{and} \quad \sum_{j=1}^\infty \int_\mathcal{Y} \|Q^{(y)}(v_j)\|^2_{H_y} \, \mu(dy) < +\infty.
\]
Then for \( \mu \text{-a.e. } y \in \mathcal{Y} \) we have
\[
\sum_{j=1}^\infty Q^{(y)}(v_{j+1} - v_j) < +\infty \quad \text{and} \quad \sum_{j=1}^\infty \|Q^{(y)}(v_j)\|^2_{H_y} < +\infty,
\]
in particular, \( ((v_j)_y)_{j=1}^\infty \) is Cauchy w.r.t. \( (Q^{(y)})^{1/2} \) and \( \lim_{j \to \infty} \| (v_j)_y \|_{H_y} = 0 \). The closability of the forms \( Q^{(y)} \) implies that \( \lim_{j \to \infty} Q^{(y)}((v_j)_y) = 0 \) for \( \tilde{\mu} \)-a.e. \( y \in Y \), so that by Fatou’s lemma we obtain

\[
Q(v_k) = \int_X \lim_{j \to \infty} Q^{(y)}((v_k)_y - (v_j)_y) \tilde{\mu}(dy) \leq \liminf_{j \to \infty} Q(v_k - v_j),
\]

what becomes arbitrarily small if \( k \) is large enough. Hence \( \lim_{k \to \infty} Q(v_k) = 0 \). □

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