Attitude Tracking for Rigid Bodies Using Vector and Biased Gyro Measurements

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Abstract—In this article, rigid-body attitude tracking using vector measurements and gyro rate is studied. A gyro-bias observer and an attitude tracking controller are first devised, which in combination ensure the exponential stability of the desired equilibrium of the overall system. To address inertia-matrix uncertainties and external disturbances, an adaptive controller relying on a modified gyro-bias observer and a disturbance estimator are then developed. Simulations are included to illustrate the proposed adaptive controller under realistic conditions.

Index Terms—Attitude tracking, gyro bias estimation, vector measurements.

I. INTRODUCTION

The attitude control for rigid bodies has been studied for decades under the assumption that attitude measurements are available [1], [2], [3]. However, in practice, the attitude is commonly estimated from vector measurements provided by inertial measurement units (see, e.g., [4], [5], [6], [7]). Therefore, the combined observer–controller must be analyzed together to ensure overall stability. Instead of attitude measurements, vector measurements of some known reference vectors in the inertia frame have been used for attitude control since the pioneering work [8]. Attitude control using vector measurements can be categorized as a direct and indirect approach. In the direct approach, no attitude representation is needed, and therefore, no attitude estimation of the attitude is present, but vector measurements are used directly in the control synthesis [8], [9], [10]. In the indirect approach, on the other hand, the attitude is first estimated using the vector measurements and then the controller is designed based on the estimated attitude [11], [12], [13], [14]. A combined approach was presented in [15]. The direct approach has the advantage of a simpler stability analysis, and most importantly, the undesired equilibria introduced by the attitude observer in the closed loop are eliminated. For the attitude stabilization problem, vector measurements are sufficient [8], [13]. When attitude tracking is considered, gyro-rate measurements are also necessary for feedback [9], [15], and the consequent gyro bias must be corrected [12], [14]. The problem becomes even more complicated if, in addition, the inertia matrix is unknown; thus, the gyro bias and the inertia matrix must be estimated simultaneously, arising a nonlinear parameterization issue. To overcome nonlinear parameterization, overparameterization is often employed (see, e.g., [12], [14] and the references cited therein).

Furthermore, external torque disturbances must also be addressed to improve attitude tracking precision [15]. In the aforementioned references, some but not all of these issues were addressed together. The main contribution of this technical note is to provide an integrated solution to address these issues by developing an adaptive controller using vector measurements directly in the presence of uncertainties in the gyro bias, inertia matrix, and external disturbances to track a desired attitude trajectory.

The rest of this article is organized as follows. Section II presents the preliminaries. A gyro-bias observer is designed in Section III, and global exponential convergence to its true value is established (see Theorem 3.2). In Section IV, the attitude tracking problem is first formulated as an alignment problem between the vector measurements and their desired values, with the alignment errors defined by the inner product and the cross-product between them. Therefore, neither an attitude representation nor its estimation is necessary, and thus, critical points created by attitude observers are not present. The closed-loop system contains, in addition to a set of undesired equilibria, the desired attitude as its equilibrium. By exploiting the relationship between these error variables near the undesired equilibria, it allows proposing a strict Lyapunov function to show the exponential stability of the desired equilibrium in a closed loop with an attitude controller when the inertia matrix is known (see Theorem 4.2). A separation property in the combined observer–controller is also proven. In the presence of uncertainties in the inertia matrix and external disturbances, an adaptive attitude controller is developed based on a modified gyro-bias observer plus a disturbance estimator to establish the almost global asymptotic stability (AGAS) of the overall system (see Theorem 4.8). The modified gyro-bias observer ensures the asymptotic convergence of the gyro-bias estimate to its true value and the boundedness independent of the convergence of the inertia-parameter estimate, which avoids the adaptive controller from using overparameterization. In Section V, the proposed adaptive controller is simulated under realistic conditions. Finally, Section VI concludes this article. Appendix A gives proof of some technical results.

II. PRELIMINARIES

A. Notations

The vector norm is denoted by $\|x\| = (x^T x)^{1/2}$ for $x \in \mathbb{R}^n$, while for a matrix $A \in \mathbb{R}^{n \times n}$, its norm is $\|A\| = \lambda_{\text{max}}(A^T A)$. If, in addition, $A$ is symmetric, then $A = A^T > 0$ indicates a positive definite matrix with $A^T$ denoting its transpose, and $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ are its maximum and minimum eigenvalue, respectively. $I_n$ denotes the identity matrix of $n \times n$, and $0_{n \times m}$ denotes a zero matrix of $n \times m$. $S(\cdot) \in \mathbb{R}^{n \times 3}$ is the cross-product operator $S(u)v = u \times v$, which satisfies $S(u)v = -S(v)u$, $S(u)S(v) = S(v)S(u) = S(S(u)v)$, and $\|S(u)v\| \leq \|u\|\|v\|$ $\forall u, v \in \mathbb{R}^n$. A unit sphere of dimension $n-1$ embedded in $\mathbb{R}^n$ is denoted by $S^{n-1} = \{x \in \mathbb{R}^n | x^T x = 1 \}$. A closed unit ball is expressed as $B^{n-1} = \{x \in \mathbb{R}^n | x^T x \leq 1 \}$.

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B. Rotational Kinematics and Dynamics

The attitude of a rigid body is represented by a rotation matrix \( R \in SO(3) \) that transforms the body-fixed frame into an inertial frame. Unit quaternions \( q = [q_0, q_1, q_2, q_3]^T \in S^3 \) are commonly used to represent the attitude, where \( q_0 \in [-1, 1] \) and \( q \in \mathbb{B}^2 \) are the scalar part and the vector part of the quaternion, respectively, and \( S^3 \) denotes the quaternion group with \( I = [1, 0, 0, 0]^T \) as its identity. A rotation matrix \( R \) is related to a unit quaternion \( q \) through the Rodriguez formula

\[
R(q) = I_3 + 2q_0S(q) + 2S^2(q) .
\]

The rotational kinematics and dynamics of a rigid body are

\[
\dot{\omega} = S(M\omega)\omega + \tau + \delta,
\]

\[
M\dot{\omega} = \mathcal{K}(M\omega)\omega + \tau + \delta.
\]

where \( \omega, \tau, \delta \in \mathbb{R}^3 \) are angular velocity, applied torque, and constant external disturbances, respectively. \( M = [m_{ij}]_{j=1}^{3} = MT > 0 \) is the inertia matrix with lower and upper bounds \( mI_k \leq M \leq mI_k \), where \( \mathcal{M} := \lambda_{\text{max}}(M) \), and \( m := \lambda_{\text{min}}(M) \). All are expressed in the body frame.

C. Measurements

It is assumed that \( n \geq 2 \) reference vectors in the inertial frame \( r_i \in S^2 \), \( i = 1, 2, \ldots, n \), are known and their measurements in the body frame \( v_i \in S^2 \) are accessible for all \( t \geq 0 \). These unit vectors are related through

\[
v_i = R^T r_i, \quad i = 1, 2, \ldots, n .
\]

Furthermore, a biased gyro-rate measurement \( \omega_g \)

\[
\omega_g = \omega + b
\]

is available, where \( b \in \mathbb{R}^3 \) is a unknown constant gyro bias.

The following assumptions are made.

1) Assumption A1: Among the \( n \) inertial reference vectors \( r_i \), there are at least two noncollinear vectors.

2) Assumption A2: Gyro bias is a constant unknown vector with a known bound, i.e., \( \| b \| \leq \theta_b \), with \( \theta_b \) known.

Note that, for each pair of noncollinear vector measurements \( v_1, v_2 \in S^2 \), a third virtual vector measurement \( v_3 = \frac{S(v_1)v_2}{\| S(v_1)v_2 \|} \) can be obtained, which satisfies (4) with \( r_3 = \frac{S(r_1)r_2}{\| S(r_1)r_2 \|} \).

III. GYRO-BIAS OBSERVER

The estimated angular velocity \( \hat{\omega} := \omega_g - \hat{b} \) can be obtained by the gyro-rate measurement \( \omega_g \) and the gyro-bias estimate \( \hat{b} \), which is given by the following observer:

\[
\dot{\hat{b}} = K_F \hat{\omega} + \gamma_f \sum_{i=1}^{n} k_i S(\Lambda_i v_i) (v_i - v_f)_i
\]

\[
\hat{b} = \hat{b} - \sum_{i=1}^{n} k_i S^T(v_f)_i \Lambda_i v_i
\]

where \( 0 < \Lambda_i = \Lambda_i^T \in \mathbb{R}^{3 \times 3} \) is a constant matrix gain, \( k_i > 0 \) is the weight assigned to each vector measurement according to its confidence level, \( K_F := \sum_{i=1}^{n} k_i S^T(v_f)_i \Lambda_i S(v_i) \), and \( v_f \) is the filtered \( v_i \), as follows:

\[
\dot{v}_f = \gamma_f (v_i - v_f), \quad v_f(0) = v_i(0) \quad \forall i = 1, 2, \ldots, n
\]

with \( \gamma_f > 0 \) the filter gain.

The following technical lemma [16] is needed for the design of the bias observer.

Lemma 3.1 (Linear filter): Consider the linear filter (8). Then, \( \forall \epsilon_f > 0 \), there exists a \( \lambda_f := \omega_{\text{max}}(K_b) \) being \( \omega_{\text{max}} < \infty \) the bandwidth of the overall system, such that \( \| v_i - v_f_i \| < \epsilon_f \) for \( i = 1, 2, \ldots, n \) and \( t \geq 0 \) provided that \( \gamma_f > \lambda_f \).

Theorem 3.2 (Globaly exponentially gyro-bias observer): Let \( \Lambda_i \in \mathbb{R}^3 > 0, k_i > 0, \) for \( i = 1, 2, \ldots, n \), and \( \gamma_f > 0 \) be chosen according to Lemma 3.1 such that \( \lambda_o := \lambda_{\text{min}}(K_o) - \epsilon_f \sum_{i=1}^{n} k_i \lambda_{\text{max}}(\Lambda_i) > 0 \), where \( K_o := \sum_{i=1}^{n} k_i S^T(v_f)_i \Lambda_i S(v_i) \). Then, the gyro-bias observer (6), (7) drives \( \hat{b} \to b \) exponentially \( \forall \epsilon(0) \in \mathbb{R}^3 \).

Proof: It follows from (4) and (2) that \( \dot{v}_i = S(v_i)\omega \). Let the bias estimation error be defined as \( \hat{b} := b - \hat{b} \). Its time derivative, by using (6)–(8), and \( K_F \) is \( \hat{b} \to b - \sum_{i=1}^{n} k_i S^T(v_f)_i \Lambda_i v_i = K_F(\omega - \omega) = -K_F \hat{b} \), which can be expressed by adding \( \pm K_o \) as follows,

\[
\dot{\hat{b}} = -K_F \hat{b} = -K_o \hat{b} - \left( \sum_{i=1}^{n} k_i S^T(v_f)_i \Lambda_i S(v_i) \right) \hat{b} .
\]

Note that \( K_o = K_o^T > 0 \) under Assumption A1 (see [13, Lemma 2]). Consider the Lyapunov function candidate \( V_1 = \frac{1}{2} \| \hat{b} \|^2 \). Its time evolution along (9) is

\[
\dot{V}_1 = -\hat{b}^T K_o \hat{b} - \hat{b}^T \left( \sum_{i=1}^{n} k_i S^T(v_f)_i \Lambda_i S(v_i) \right) \hat{b}
\]

\[
\leq -\lambda_{\text{min}}(K_o) \| \hat{b} \|^2
\]

\[
+ \sum_{i=1}^{n} k_i \| S(v_f)_i \| \| \Lambda_i \| \| S(v_i) \| \| \hat{b} \|,
\]

\[
\leq -\lambda_{\text{min}}(K_o) \| \hat{b} \|^2
\]

\[
+ \sum_{i=1}^{n} k_i \| v_f_i \| - v_i \| \| \Lambda_i \| \| \hat{b} \|^2
\]

\[
\leq -\left( \lambda_{\text{min}}(K_o) - \epsilon_f \sum_{i=1}^{n} k_i \lambda_{\text{max}}(\Lambda_i) \right) \| \hat{b} \|^2
\]

\[
= -\lambda_o \| \hat{b} \|^2 = -2\lambda_o V_1
\]

where the property \( \| S(v_f) \| \leq \| v_f \| \forall v_f, v \in \mathbb{R}^3 \), the unit norm of \( \| v_f \| = 1 \), and Lemma 3.1 were used. Therefore, \( \hat{b} \to 0 \) exponentially for any initial condition \( \hat{b}(0) \in \mathbb{R}^3 \) with a decaying rate \( \lambda_o \).
IV. ATTITUDE CONTROL DESIGN

A. Control Objective and Error Variables

Let $R_d \in SO(3)$ be a desired attitude related to a differentiable desired angular velocity $\omega_d \in \mathbb{R}^3$ through $R_d = R_d(\omega_d)$. Define the attitude error as $R_e = R_{d}R_{d}^{T}$, which is associated with its quaternion parameterization $e = [e_0, e_1^T]^T \in S^3$ by the Rodrigues formula $R_e = R(e)$ in (1). The control objective is to design a control law to achieve asymptotic convergence of the attitude tracking error $e \to [\pm 1, 0, 0]^T$ and the angular velocity error $\omega - \omega_d \to 0$.

To relate the quaternion error $e = [e_0, e_1^T]^T$ with the vector measurements, two error variables are defined: the weighted inner product between $v_i$ in (4) and its desired value $v_{d_i} := R_d^T v_i$

$$e_R = \frac{1}{2}\sum_{i=1}^{n} k_i \|v_i - v_{d_i}\|^2$$

(11)

where $k_i > 0$ are given in the observer (6), (7), and the weighted cross product between $v_i$ and $v_{d_i}$

$$z = \sum_{i=1}^{n} k_i S(v_i)v_{d_i}$$

(12)

Some useful results for the subsequent analysis are given in the following lemma, proved in the Appendix A.

Lemma 4.1 (Properties of error variables $e_R$ and $z$):

1) The error $e_R$ in (11) can be expressed as follows:

$$e_R = \sum_{i=1}^{n} k_i \left(1 - v_i^T v_{d_i}\right) = 2e_1^T We_v$$

(13)

where $W := \sum_{i=1}^{n} k_i S^2(v_i) = W > 0$ under Assumption A1. Furthermore, $0 \leq e_R \leq 2 \sum_{i=1}^{n} k_i e_R = 0$ implies $v_i = v_{d_i}$ and $e = \pm 1$; while $e_R = 2 \sum_{i=1}^{n} k_i$ implies $v_i = -v_{d_i}$. In addition, $e_R = 2\lambda_{w,j}$, when $e = [e_0, e_1^T]^T = [0, v_{w,j}^T]^T$, where $v_{w,j}, j = 1, 2, 3$, are the unit eigenvectors of $W$ associated with the eigenvalues $\lambda_{w,j}$, that is, $Wv_{w,j} = \lambda_{w,j}v_{w,j}$, ordered as $\lambda_{w,1} \geq \lambda_{w,2} \geq \lambda_{w,3} > 0$.

2) The time derivative of $e_R$ is

$$\dot{e}_R = z^T(\omega - \omega_d)$$

(14)

3) The error variable $z$ in (12) can be expressed as follows:

$$z = 2R_d^T(e_0I_3 - S(e_v))Wv_e$$

(15)

Furthermore, $0 \leq \|z\| \leq \sum_{i=1}^{n} k_i$, and $z = 0_{3 \times 1}$ implies that $e = \pm 1$, or $e = [0, v_{w,j}^T]^T$, $j = 1, 2, 3$.

4) The dynamics of $z$ is described by

$$\dot{z} = J(\omega - \omega_d) + S(z)\omega_d$$

(16)

where $J = \sum_{i=1}^{n} k_i S^T(v_{d_i})S(v_i)$, with $\|J\| \leq \sum_{i=1}^{n} k_i$.

5) For any $\alpha_1 > 0$, there exists $\beta > 0$, such that

$$\alpha_1 e_R \leq \frac{\beta}{2} \|z\|^2 \quad \forall t \geq 0$$

(17)

for all $e \in S^3 \setminus \bigcup B_j$, where $\bigcup B_j := B_1 \cup B_2 \cup B_3$, and

$$B_j := \left\{ e \in S^3 | e = -\rho \sqrt{1 - \rho^2} v_{w,j}^T, \rho \in [0, 1] \right\}$$

is a closed ball with center in $e_j := [0, v_{w,j}^T]^T$, and (arbitrarily small) radius $\epsilon > 0$, for $j = 1, 2, 3$.

B. Attitude Tracking Controller

Define the reference velocity $\omega_r := -\lambda_c z + \omega_d$ and the composite tracking error $\sigma := \omega - \omega_r$, where $\kappa_c > 0$ is a design parameter. Note that the control objective is achieved if $\sigma(t) \to 0$ asymptotically. Toward this end, the following control law is proposed:

$$\tau = M\dot{\omega}_r - S(M\omega_r)\sigma - (\alpha_1 I_3 + \alpha_2 J^T)z$$

(18)

where $\dot{\omega}_r = -\lambda_c J(\dot{\omega} - \omega_r) - S(z)\dot{v}_{d} + \dot{\sigma}$, $\sigma = \omega - \omega_r$, $\dot{\omega} = \omega_j - b$, with $b$ given by the gyro-bias observer (7), $K_c \in \mathbb{R}^{3 \times 3}$, $K_c = K_c^T > 0$, and $\alpha_1, \alpha_2 > 0$ are design parameters.

Theorem 4.2 (Exponential stability): Let the design parameters be chosen as follows.

1) Gyro-bias observer: $\lambda_c = \lambda^T_c > 0$, $k_1 > 0$, for $i = 1, 2, \ldots, n$, and $\gamma_f > 0$ according to Theorem 3.2.

2) Controller: $K_c = K_c^T > 0$, $\lambda_c > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$ such that

$$\lambda_a := \alpha_1 - \alpha_2 \sum_{i=1}^{n} k_i > 0$$

(19)

$$\lambda_o := \lambda_{\min}(K_o) - \epsilon \sum_{i=1}^{n} k_i \lambda_{\max}(A_i) > \frac{\|G\|^2}{4\gamma_{\min}(K_c)}$$

(20)

where $G := K_c - S(\omega_r)M + \lambda_c M J$, which is bounded in light of the boundedness of $z$ and $\omega_d$. Then, the attitude controller (18) and the gyro-bias observer (6), (7), in closed loop with the system (2), (3) ignoring the external disturbance $d \in \mathbb{R}^3$, have the following properties for $j = 1, 2, 3$:

i) The set of equilibria for $(e, \sigma, b) \in S^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ is \((\pm 1, 0_{3 \times 1}, 0_{3 \times 1}), (e_0, 0_{3 \times 1}, 0_{3 \times 1})\).

ii) The equilibrium $(\pm 1, 0_{3 \times 1}, 0_{3 \times 1})$ is exponentially stable $\forall (e(0), \sigma(0), b(0)) \in X := \mathbb{S}^3 \mathbb{B}_{\min} \times \mathbb{R}^3 \times \mathbb{R}^3$, where $B_{\min} := \{ e \in S^3 | e \leq 2\lambda_{\omega,d}\}$.

iii) The third undesired equilibrium $(e_j, 0_{3 \times 1}, 0_{3 \times 1})$ are unstable.

Proof: The proof starts with calculating the dynamics of the error state $x_1 := [z^T \sigma^T \hat{b}^T]^T$ in closed loop with the control law (18). By (16), the dynamics of $z$ is rewritten as

$$\dot{z} = J - \lambda_c Jz + S(z)\omega_d$$

(21)

The dynamics of $\sigma$ can be calculated by the definition of $\omega_r$, and its time derivative $\dot{\omega}_r = -\lambda_c J(\dot{\omega} - \omega_r) - S(z)\dot{v}_{d} + \dot{\sigma}$, the definition of $\dot{\omega}_r$, the fact $\omega - \omega_r = -\hat{b}$, the property of the skew-symmetric matrix, and the control law (18) as follows:

$$\dot{M}\sigma = M(\dot{\omega}_r - \omega_r) = S(M\omega_r) + \tau - M\dot{\omega}_r$$

$$= S(M\omega_r) + \tau - M\omega_r + S(M\omega_r)$$

$$= S(M\omega_r) - K_c\sigma + G\hat{b} - (\alpha_1 I_3 + \alpha_2 J^T)$$

(22)

The error dynamics of the closed-loop system is completed by the dynamics of the bias-estimation error in (9). Note that $x_{1_d} = 0_{n \times 1}$ is the only equilibrium of the closed-loop system (21), (22), and (9).

Therefore, according to item 3) of Lemma 4.1, the set of equilibria for $(e, \sigma, b) \in S^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ is given in the item i).

To prove item ii), consider the following Lyapunov function candidate:

$$V_2(x_1, t) = \frac{1}{2} \sigma^T M \sigma + \frac{1}{2} \|\hat{b}\|^2 + \alpha_2 \|z\|^2 + \alpha_1 e_R$$

(23)

$$\forall x_1 = [z^T, \sigma^T, \hat{b}^T]^T \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$$

Note that in the region $X$, the equilibrium $x_{1_d} = 0_{n \times 1} \Rightarrow e = \pm 1$ because $2\lambda_{\omega,d}\|e_v\|^2 \leq $
\[ 2e_1^T W e_v = e_R < 2\alpha_{\omega,\beta} \text{ implies } \| e_v \| < 1, \text{ and therefore, the undesired equilibria are excluded from the region } V_2 \text{ is bounded by} \]
\[ \gamma_1 \| x_1 \|^2 \leq V_2(x_1, t) \leq \gamma_2 \| x_1 \|^2 \]
\[ (24) \]

with \( \gamma_1 = \frac{1}{2} \min(\lambda_{\min}(M), 1, \alpha_2 \beta), \text{ and } \gamma_2 = \frac{1}{2} \max(\lambda_{\max}(M), 1, \alpha_2 + \beta) \) for some \( \beta > 0 \) according to item 5 of Lemma 4.1.

The time derivative of (23) along the error dynamics (21), (22), and (9), and the time derivative of \( e_R \) in (14) written as \( e_R = z^T (\omega - \omega_d) = z^T \sigma - \lambda \omega_d z \), is as follows:
\[ V_2 = \sigma^T M \sigma + b^T b + \alpha_2 z^T z + \alpha_1 e_R \]
\[ = -\sigma^T K_s \sigma + \sigma^T Gb - b^T K_j b - \lambda z^T (\alpha_1 I_3 + \alpha_2 J) z \]
\[ = -x_1^T Q_1 x_1 \leq -\lambda_{\min}(Q_1) \| x_1 \|^2 \leq -\frac{\lambda_{\min}(Q_1)}{\gamma_2} V_2 \]
\[ (25) \]
where
\[ Q_1 = \begin{bmatrix} \lambda \, (\alpha_1 I_3 + \alpha_2 J) & 0 & 0 \\ 0 & -K_c & -\frac{1}{2} G \\ 0 & -\frac{1}{2} G & K_j \end{bmatrix} \]
\[ (26) \]
is positive definite under conditions (19) and (20). Therefore, by [17, Th. 4.10] it follows that:
\[ \| x_1(t) \| \leq \left( \frac{\gamma_2}{\gamma_1} \right)^{1/2} \| x_1(t_0) \| e^{-\frac{\lambda_{\min}(Q_1)}{\gamma_2}(t-t_0)}. \]

Then, the equilibrium \( x_1 = 0_{3\times 1} \) is globally exponentially stable.
\[ V(x_1(0)) = [z^T (0), \sigma(0), b^T (0)]^T \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3. \]
This implies that \( (e_0, \sigma, b) = (\pm 1, 0_{3\times 1}, 0_{3\times 1}) \) is (locally) exponentially stable for (10).

To prove item iii), that is, the instability of the undesired equilibrium \( e = e_j, j = 1, 2, 3 \), the Lyapunov function \( V_2 \) near the equilibrium \( e = e_j \) is analyzed. With little notation abuse, denote \( V_2(x_1, t) \) by \( V_2(\omega, \epsilon) \).

Note that the attraction region in item ii) of Theorem 4.2 may be enlarged to cover almost the entire space of \( \mathbb{S}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \), as the following corollary states.

**Definition 4.3** (Almost global asymptotic stability (AGAS) and almost semiglobal exponential stability (ASGES)) [15]: Let the origin be an equilibrium of a dynamic system.

1. AGAS, if it is asymptotically stable and the set of initial states that do not converge to the origin has zero Lebesgue measure.
2. ASGES, if it is asymptotically stable and for almost all initial states, there exist finite controller parameters such that the corresponding trajectory exponentially converges to the origin.

**Corollary 4.4** (ASGES and AGAS): The equilibrium \( (1, 0_{3\times 1}, 0_{3\times 1}) \) is of AGAS \( \forall (e(0), \sigma(0), b(0)) \in \mathbb{S}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \) and ASGES \( \forall (e(0), \sigma(0), b(0)) \in \mathbb{S}^3 \cup B_1 \times \mathbb{R}^3 \times \mathbb{R}^3 \), where \( e(0), \sigma(0), b(0) \) are defined in item 5 of Lemma 4.1.

**Proof:** For \( e \in \mathbb{S}^3 \), the inequality (17) no longer holds, nor is the upper bound on the Lyapunov function \( V_2 \) in (23). However, it follows from \( V_2 = -x_1^T Q_1 x_1 \leq 0 \) in the proof of Theorem 4.2 that \( V_2 \) is nonincreasing along the error dynamics (21), (22), and (9).

This implies that \( x_1 = [z^T, \sigma^T, b^T]^T \) and \( x_1 \) are bounded, which in turn results in \( V_2 \) being bounded. It follows from Barbalat’s lemma that \( V_2 \to 0 \), which shows the asymptotic stability of \( x_1 = 0_{3\times 1} \).

This implies the asymptotic stability of \( (e(0), \sigma, b) = (\pm 1, 0_{3\times 1}, 0_{3\times 1}) \) of AGAS [13, 15]. On the other hand, \( \forall \epsilon > 0 \), the proof of item ii) in Theorem 4.2 holds for \( e \in \mathbb{S}^3 \cup B_1 \).

Note that the set \( \mathbb{S}^3 \cup B_1 \) can be arbitrarily enlarged for a given \( \alpha_1 \), and any arbitrarily small \( \epsilon > 0 \) by a sufficiently large \( \beta \) in item 5 of Lemma 3.1 to cover almost all \( \mathbb{S}^3 \), so the equilibrium \( (\pm 1, 0_{3\times 1}, 0_{3\times 1}) \) is of ASGES.

**Remark 4.5** (Comparison with some reported controllers): By exploiting the relationship between the error variables \( z \) and \( e_R \) near the equilibrium \( e = e_j \) in item 5 of Lemma 3.1, it allows proposing a strict Lyapunov function (23) for the desired equilibrium. This in turn enables the control (18) to achieve AGAS and ASGES under the conditions stated in Theorem 4.2, compared with AGAS in [8, 9, 12, and 13].

**Remark 4.6** (Separation property): Note that the bias observer (6), (7) is used in the control loop without modifications. This implies that the gyro-bias observer (6), (7), and the controller (18) can be designed separately, provided that the convergence rate \( \lambda_\omega \) in (20) of the observer is sufficiently large. This condition may also be obtained through finite-time convergence, e.g., [7].

**Remark 4.7** (Controller tuning): The controller and the observer can be tuned as follows to satisfy the conditions of Theorem 4.2.

1) Assign the weights \( k_j \) to the observer (6), (7) according to the confidence level of each sensor.
2) Choose the controller gains \( K_c = K_j > 0, \lambda_\omega > 0, \alpha_1 > 0, \alpha_2 > 0 \) such that condition (19) holds.
3) Choose the filter gain \( \gamma_j \) as in Lemma 3.1 and the matrix gain \( \Lambda_j \) in (6), (7) such that condition (20) holds.

### C. Adaptive Attitude Tracking Controller

In the presence of uncertainties in the inertia matrix and external constant disturbances \( d \), the following adaptive control law with disturbance compensation is proposed:

\[ \tau = Y(\dot{\omega}, h) \dot{\theta} - K_c \dot{\tilde{\sigma}} - (\alpha_1 I_4 + \alpha_2 J^T) z - \dot{d} \]
\[ (28) \]
where the controller gains \( K_c, \alpha_1, \text{ and } \alpha_2 \) are the same as in the control law (18). \( \theta(t) \) is the estimation of the inertia-parameter vector \( \theta := [m_{11}, m_{22}, m_{33}, m_{23}, m_{13}, m_{12}]^T \in \mathbb{R}^6 \), with \( m_{ij} \) denoting the...
entries of the inertia matrix, which is updated according to
\[ \dot{\hat{\omega}} = -\Gamma^T (\hat{\dot{\omega}}, h) \hat{\sigma} \]
(29)
where \( 0 < \Gamma = \Gamma^T \in \mathbb{R}^{6 \times 6} \) is the adaptation gain. \( \dot{Y}(\hat{\omega}, h) \) is the regressor
\[ Y(\hat{\omega}, h):= S(\hat{\omega})F_1(\hat{\omega}) + F_1(h) \]
\[ h := -\lambda_c (J(\hat{\omega} - \omega_d) + S(z)\omega_d) + \hat{\omega}_d + \left( \alpha_1 I_3 + \alpha_2 J^T \right) z \]
(30)
and \( F_1(u) \), for a vector \( u = [u_1, u_2, u_3]^T \), is defined as
\[ F_1(u) := \begin{bmatrix} u_1 & 0 & 0 & 0 & u_3 & u_2 \\ 0 & u_2 & 0 & u_3 & 0 & u_1 \\ 0 & 0 & u_3 & u_2 & u_1 & 0 \end{bmatrix} \]
Likewise, \( \tilde{d} \in \mathbb{R}^3 \) is the estimated disturbance, which is updated by
\[ \dot{\hat{\alpha}} = k_d \hat{\sigma} \]
(31)
being \( k_d > 0 \) a gain parameter.

The estimated composition error \( \hat{\sigma} := \hat{\omega} - \omega_r \), where \( \hat{\omega} = \omega_0 - \hat{b} \)
is obtained by the following modified gyro-bias observer:
\[ \dot{\hat{b}} = \mu_b \text{Tanh}(\tilde{b}) - \sum_{i=1}^{n} k_i \lambda_i S^T(v_{f_i}) \Lambda_i v_i \]
(32)
\[ \dot{\hat{\omega}} = \frac{1}{\mu_b} \text{Cosh}^2(\tilde{b}) \left( K_f \hat{\omega} + \sum_{i=1}^{n} k_i \lambda_i \Lambda_i v_i - (\alpha_1 I_3 + \alpha_2 J^T) z \right) \]
(33)
where \( \mu_b > \theta_b \), with \( \theta_b \) the bound on \( \|b\| \) in Assumption A2. The hyperbolic functions are defined entry wise
\[ \text{Tanh}(b) := [\tanh(b_1), \tanh(b_2), \tanh(b_3)]^T \in \mathbb{R}^3 \]
\[ \text{Cosh}(b) := \text{diag}\{\text{cosh}(b_1), \text{cosh}(b_2), \text{cosh}(b_3)\} \in \mathbb{R}^{3 \times 3} \]
\[ \text{Sech}(b) := \text{diag}\{\text{sech}(b_1), \text{sech}(b_2), \text{sech}(b_3)\} \in \mathbb{R}^{3 \times 3} \]
Note that \( \tilde{b} \) in (32) is bounded by \( \|\tilde{b}\| \leq \sqrt{3} \mu_b + \sum_{i=1}^{n} k_i \lambda_{max}(\Lambda_i) \). The stability of the proposed adaptive controller is summarized in the following theorem.

**Theorem 4.8 (Adaptive controller):** Let the design parameters be chosen as follows.
1) **Gyro-bias observer:** \( \Lambda_i = \Lambda_i^T > 0, k_i > 0 \), for \( i = 1, 2, \ldots, n \), and \( \gamma_f > 0, \mu_b > \theta_b \),
2) **Adaptive controller:** \( K_c = K_c^T > 0, \lambda_c > 0, \alpha_1 > 0, \alpha_2 > 0, k_d > 0 \), and \( \Gamma = \Gamma^T > 0 \)
such that condition (19) is satisfied and
\[ \lambda_o = \lambda_{min}(K_o) > 0 \quad \text{and} \quad \lambda_{min}(K_c) > \frac{\|H\|^2}{4\lambda_o} + \|S(\hat{\sigma})M\| \]
(35)
where \( H := M(K_f + \lambda_c J) + S(M\omega_d) - S(\hat{\omega}_d + \omega_r)M \), which is bounded. Then, the adaptive controller (28), the gyro-bias observer (32), and the adaptation law (29)–(31), in closed loop with the system (2), (3), have the following properties, for \( j = 1, 2, 3 \).

i) All variables are bounded, and \( (e, \sigma, \tilde{b}) \rightarrow (\pm 1, 0_{3 \times 1}, 0_{3 \times 1}) \) almost globally asymptotically \( \forall(e(0), \sigma(0), \tilde{b}(0), \tilde{d}(0), \theta(0)) \in \mathcal{X}_e := S^3 \setminus \bigcup B_j \in \mathbb{R}^9 \times \mathbb{R}^6 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^6 \), where \( \theta = \hat{\omega} - \omega \) and \( \tilde{d} = d - d \) are the parameter estimation errors.

ii) The three undesired equilibria \( (e_j, 0_{3 \times 1}, 0_{3 \times 1}, 0_{3 \times 1}, 0_{6 \times 1}) \), \( j = 1, 2, 3 \), are unstable.

**Proof:** Let \( x_2 := [z^T, \sigma^T, \tilde{b}^T, \tilde{d}^T, \tilde{\theta}^T]^T \) denote the state of the closed-loop system with the control law (28). Its dynamics is calculated as follows. The dynamics of the bias estimation error by (32) and (33) is
\[ \dot{\tilde{b}} = \dot{\tilde{b}} - \mu_b \text{Sech}^2(\tilde{b}) \tilde{b} - K_f \tilde{\omega} - \sum_{i=1}^{n} k_i S(\Lambda_i v_i) \hat{v}_{f_i} \]
\[ = -K_f \tilde{b} - (\alpha_1 I_3 + \alpha_2 J^T) z \]
(36)
The dynamics of \( \tilde{\sigma} = \tilde{\omega} - \omega_r \), in view of (5) and (36), is now given by
\[ M \tilde{\sigma} = M \tilde{\omega} - M \omega_r = M \omega - M \hat{\omega}_d - M \tilde{b} \]
\[ = - (S(\omega) F_1(\omega) + F_1(h)) \theta + M (K_f + \lambda_c J) \tilde{b} + d + \tau \]
\[ = -Y(\omega, h) \theta + M (K_f + \lambda_c J) \tilde{b} + d + \tau \]
Adding and subtracting \( Y(\omega, h) \theta \) gives
\[ M \tilde{\sigma} = -Y(\omega, h) \theta + Y(\omega, h) \theta - \tilde{\sigma} \tilde{b} - \tilde{b} \]
\[ = -Y(\omega, h) \theta + M (K_f + \lambda_c J) \tilde{b} + d + \tau \]
which in closed loop with the controller (28) gives
\[ M \tilde{\sigma} = -Y(\omega, h) \theta - \tilde{\sigma} (K_c + S(\tilde{b})M) \tilde{\sigma} - \tilde{b} \]
\[ - (\alpha_1 I_3 + \alpha_2 J^T) z + H \tilde{b} + \tilde{d} \]
(37)
The error state dynamics is completed with the following:
\[ \dot{\tilde{z}} = J \tilde{\sigma} - \lambda_c J z + S(z) \omega_d + J \tilde{b} \]
\[ \dot{\tilde{b}} = -\Gamma^T (\hat{\dot{\omega}}, h) \hat{\sigma} \]
\[ \dot{\tilde{d}} = -k_d \tilde{\sigma} \]
(39)
Note that the state \( x_2 \) of the closed-loop system (36)–(40) has the only equilibrium at \( x_2 = 0_{16 \times 1} \). To determine its stability, consider the following Lyapunov function candidate:
\[ V_3 = \frac{1}{2} \sigma^T M \sigma + \frac{1}{2} \|\tilde{b}\|^2 + \frac{1}{2} \tilde{d}^T \Gamma^{-1} \tilde{d} + \frac{1}{2 k_d} \|d\|^2 + \frac{\alpha_2}{2 \alpha_1} \|z\|^2 \]
\[ + \alpha_1 e_R \]
(41)
\[ \forall x_2 \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^6 \text{ and } e \in S^3 \setminus \bigcup B_j \].

The time evolution of (41) along the closed-loop dynamics, by using the fact that \( e_R = z^T \sigma - \lambda_c z^T \sigma + z^T \tilde{b} \), is given by
\[ \dot{V}_3 = \sigma^T M \dot{\sigma} + \tilde{b}^T \tilde{b} + \tilde{d}^T \tilde{d} + \frac{1}{k_d} \tilde{d}^T \tilde{d} + \alpha_2 z^T \dot{z} + \alpha_1 e_R \]
\[ = -\tilde{\sigma}^T (K_c + S(\tilde{b})M) \tilde{\sigma} + \tilde{b}^T K_f \tilde{b} \]
\[ - \lambda_c z^T (\alpha_1 I_3 + \alpha_2 J^T) z \]
\begin{equation}
\leq -\ddot{x}^T Q_2 \ddot{x}
\end{equation}

where \( \ddot{x} := [||z||, ||\dot{\sigma}||, ||\dot{\theta}||]^T \), and

\begin{equation}
Q_2 = \begin{bmatrix}
\lambda, \lambda \alpha & 0 & 0 \\
0 & \lambda_{\min}(K_e) - ||S(\dot{\theta})M|| - \frac{1}{2} ||H|| \\
0 & -\frac{1}{2} ||H|| \\
\end{bmatrix}
\end{equation}

Under conditions (35) and (34), \( Q_2 > 0 \); therefore, \( \dot{V}_3 \leq 0 \). This implies that the state \( x_2 \) is bounded. Next, it is straightforward to verify that \( \dot{V}_3 \) is bounded, and then \( (\varepsilon, \dot{\sigma}, \dot{\theta}) \rightarrow (0_{3:1}, 0_{3:1}, 0_{3:1}) \) asymptotically by invoking Barbalat’s Lemma. Moreover, \( z \rightarrow 0_{3:1} \) implies that \( (\varepsilon, \dot{\sigma}, \dot{\theta}) \rightarrow (\pm 1, 0_{3:1}, 0_{3:1}) \) almost globally asymptotically, and \( \theta \) and \( d \) remain bounded. Furthermore, the three undesired equilibria \( (\varepsilon_j, 0_{3:1}, 0_{3:1}, 0_{3:1}) \) are unstable by the same arguments as in the proof of Theorem 4.2.

Remark 4.9 (Adaptive controller): Compared to similar adaptive control laws for tracking the attitude of rigid bodies (e.g., [12, 14]), the proposed scheme does not require any representation of the attitude or its estimation—a property derived from the nonadaptive design in the previous section. Note that the estimated gyro bias \( \hat{b} \) in (32) is bounded and converges to its true value, which is independent of the estimated vector of inertia parameters \( \hat{\theta} \). This feature allows updating the inertia-parameter estimate \( \hat{\theta} \) with the standard gradient-type update law (29) without overparameterization, in contrast to [12] and [14], where the overparameterizations were required to obtain a linear parameterization.

When a persistent excitation condition on the regressor \( Y(\omega, h) \) is met, the exponential convergence of \( \hat{\theta} \rightarrow \theta_0 \) can be established as in classical adaptive control designs [18].

Remark 4.10 (Tuning procedure): The design parameters can be tuned to verify conditions (19), (34), and (35) as follows.
1) Assign the weight \( k_i \) according to the confidence of each vector measurement \( v_i \).
2) Choose the observer gains \( \Lambda_i = \Lambda_i^T > 0 \) and \( \gamma_j > 0 \) to verify condition (34). Likewise, the observer bound \( \mu_0 > 0 \) is chosen to be \( \mu_0 \geq \theta_0 \), according to Assumption A2.
3) The controller gains \( \alpha_1 > 0 \) and \( \alpha_2 > 0 \) are selected to fulfill the condition (19).
4) Fix a controller gain \( \lambda_c > 0 \), and choose \( K_c = K_c^T > 0 \) to verify (35).
5) The design parameter \( \Gamma = \Gamma^T > 0 \) in the adaptive law can be set independently of the design parameters of the observer and the controller.

V. SIMULATIONS

The adaptive controller (28) was simulated in the presence of uncertainties in the gyro bias, the inertia matrix, and external disturbances under noisy measurements, and its performance is shown in Fig. 1.

In the simulation, three inertial reference vectors \( r_1 = [0, 0, 1]^T \), \( r_2 = (1/\sqrt{3})[1, 1, 1]^T \), and \( r_3 = r_1 \times r_2/||r_1 \times r_2|| \) were considered. For comparison purposes, the desired trajectory was chosen as in [12], given by \( \omega_d = [\cos(t) + 0.5 \cos(0.2t), 0.75 \sin(2t), \sin(5t e^{0.001t}) + \cos(0.5t)]^T \) (rad/s). The initial conditions were \( q(0) = [-1, 0, 0, 0]^T \), \( \omega(0) = [0, 0, 0]^T \) (rad/s), \( \dot{q}(0) = [0.8, 0.6, 0.0]^T \), \( \dot{\omega}(0) = [1.5, 0, 0]^T \) (rad/s), \( b(0) = [0, 0, 0]^T \) (rad/s), \( d(0) = [0, 0, 0]^T \) (N·m), and \( \theta(0) = 0_{3:1} \) (Kg·m²). The vector of inertia parameters was \( \theta = [0.0360, 0.0869, 0.0935, 0.0004, 0.0015, -0.0007]^T \) (Kg·m²). The vector measurement weights were \( k_i = 0.1 \), and the observer gains were \( \Lambda_i = 10I_3 \) for \( i = 1, 2, 3 \), \( \gamma_j = 1000 \), and \( \mu_0 = 1 \). The adaptive controller gains were \( K_c = 3I_3 \), \( \lambda_c = 1 \), \( \alpha_1 = 0.1 \), \( \alpha_2 = 0.01 \), \( k_d = 1 \), and \( \Gamma = I_3 \).

The external disturbance and the gyro bias were \( d = [0.2, -0.1, -0.05]^T \) (N·m) and \( b = [0.2, 0.1, -0.1]^T \) (rad/s), respectively. The noise was introduced in the vector measurements and gyro measurements as follows. Vector measurements: \( v_{m_i} = (v_i + m_i \nu_i)/||v + m_i \nu_i|| \), \( \nu_i = \nu_i/||\nu_i|| \), where \( \nu_i \in \mathbb{R}^3 \) are zero-mean Gaussian distributions with unitary variance, for \( i = 1, 2, 3 \), and \( m_i \in [0, 0.1] \) is a uniform distribution; gyro-rate measurement: \( \omega_m = \omega + m_w \nu_w + b \), with \( \nu_w \in \mathbb{R}^3 \) a zero-mean Gaussian distributions and unit-variance, and \( m_w \in [0, 0.1] \) a uniform distribution.

It is worth mentioning that the estimates of gyro bias, inertia parameters, and disturbances converged to their true values when the noise
in the measurements was removed. These simulation results were not included due to space limitations.

VI. CONCLUSION

In this note, attitude tracking using vector and gyro-rate measurements was considered. An adaptive controller was devised to deal with uncertainties in the gyro bias and inertia matrix, and external torque disturbances. The limitations of the proposed scheme include the measurement errors caused by accelerometers when the rigid body undergoes an apparent significant acceleration, which is unknown. Possible remedies to this problem may be using other vector measurement sensors (e.g., charge-coupled device (CCD) cameras) or relying on schemes with an estimation of the apparent acceleration.

APPENDIX

A. Proof of Lemma 4.1

1) Recall the attitude error \( R_e = R R_d^T \), which is associated with its quaternion parameterization \( e = [e_0, e_1, e_2, e_3]^T \in S^3 \) by the Rodriguez formula \( R_e = R(e) \) in (1). Then, it follows from (13), \( v_i = R^T R_e v_i \), and \( v_d = R^T R_0 v_d \) that

\[
e_R = \sum_{i=1}^n k_i (1 - v_i^T v_d) = \frac{1}{2} \sum_{i=1}^n k_i (1 - r_i^T R_0 r_i)
\]

\[
= \sum_{i=1}^n k_i (1 - r_i^T (I_3 + 2e_0 S(e_0) + 2S^2(e_0)) r_i)
\]

\[
= -\sum_{i=1}^n k_i (r_i^T (2S^2(e_0)) r_i)
\]

\[
= 2e_0^T \left( -\sum_{i=1}^n k_i S^2(r_i) \right) e_0 = 2e_0^T W e_0.
\]

Clearly, \( 0 \leq e_R = \sum_{i=1}^n k_i (1 - v_i^T v_d) \leq 2 \sum_{i=1}^n k_i e_0^T S(e_0) + 2S^2(e_0) v_0^T v_0 \). Furthermore, \( e_R = 0 \iff v_i = v_d \iff e = [\pm 1, 0, 0, 0]^T \), \( e_R = 2 \sum_{i=1}^n k_i e_0 \iff v_i = -v_d \). In particular, \( e_R = 2e_0^T W v_0 = 2\lambda_0^2 v_0 = \lambda_0^2 v_0 \), where \( v_0 \) is the unit eigenvector of the eigenvalues \( \lambda_{w,j} \), i.e., \( W v_0 = \lambda_0 v_0 \), ordered as \( \lambda_{w,1} \geq \lambda_{w,2} \geq \lambda_{w,3} > 0 \).

2) By \( \dot{v}_i = S(v_i) \omega \) and \( \dot{v}_d = S(v_d) \omega_d \), the time derivative of \( e_R = \sum_{i=1}^n k_i (1 - v_i^T v_d) \) is as follows:

\[
\dot{e}_R = -\sum_{i=1}^n k_i (v_i^T S(v_d) \omega_d + v_d^T S(v_i) \omega)
\]

\[
= -\sum_{i=1}^n k_i (v_i^T S(v_d) \omega - S(v_d) \omega)
\]

\[
= z^T (\omega - \omega_d).
\]

3) \( \|z\| \leq \sum_{i=1}^n k_i \|v_i \times v_d\| \leq \sum_{i=1}^n k_i e_0 \). The rest of the proof is given by [13, Lemmas 1 and 3].

4) By the property of the skew-symmetric matrix, the time derivative of the error variable \( z \) in (12) is

\[
\dot{z} = \sum_{i=1}^n k_i (S(v_i) \dot{v}_d - S(v_d) \dot{v}_i)
\]

\[
= \sum_{i=1}^n k_i (S(v_i) S(v_d) \omega_d - S(v_d) S(v_i) \omega)
\]

\[
= \sum_{i=1}^n k_i ((S(v_i) S(v_d) - S(v_d) S(v_i)) \omega_d
\]

\[
- (S(v_d) S(v_i) (\omega - \omega_d))
\]

\[
= \sum_{i=1}^n k_i (S(v_i) S(v_d) \omega_d + J(\omega - \omega_d)
\]

\[
= \sum_{i=1}^n k_i (S(v_i) S(v_d) \omega_d) + J(\omega - \omega_d).
\]

5) By (15), the norm \( \|z\|^2 = z^T z \) is given by

\[
\frac{1}{2} \|z\|^2 = \frac{1}{2} \left( 2x_0^T W (e_0 I_3 + S(e_0)) R_d \right)
\]

\[
- (2R_0^T (e_0 I_3 - S(e_0)) W e_0)
\]

\[
= 2x_0^T W e_0 - 2 (e_0^T W e_0)^2.
\]

It is easy to see that the equality of (17) holds when \( e = [e_0, e_1, e_2, e_3] = [\pm 1, 0, 0, 0]^T \), which corresponds to \( R = R_d \).

Now, consider the undesired equilibria \( e_j = [0, v_{w,j}, 1, 0]^T \), being \( v_{w,j} \), \( j = 1, 2, 3 \), the \( j \)th unit-eigenvector of \( W \). Given any \( 1 > \epsilon > 0 \), applying an arbitrarily small rotation \( \delta_j = [(\sqrt{1-\epsilon^2}) v_{w,j}^T e_j \), \( e_j \), \( j \) yields

\[
e_j^* = [e_0, e_1, e_2, e_3] = e_j \otimes \delta_j = -\epsilon, (\sqrt{1-\epsilon^2}) v_{w,j}^T e_j.
\]

Then, the error variable \( z \) in (15) and \( e_R \) in (11) evaluated at \( e_j^* \) and \( e_j^* \), respectively, is

\[
\frac{1}{2} \|z\|^2 = 2x_0^T W e_j^* - 2 (e_0^T W e_j^*)^2
\]

\[
= 2x_0^T e_j^* - 2 (e_0^T W e_j^*)^2
\]

\[
= 2x_0^T e_j^* - 2 (1 - \epsilon^2)
\]

\[
\epsilon^2 \lambda_{w,j} \leq \lambda_{w,2} \leq \lambda_{w,1} \), near the undesired equilibria \( e_j \), it has

\[
\frac{1}{2} \|z\|^2 \leq \min_j \frac{1}{2} \|z\|^2 = 2x_0^T e_j^* - 2 (1 - \epsilon^2)
\]

\[
e_R \leq \max_i \epsilon_i \leq 2 \frac{1}{2} \|z\|^2
\]

\[
\epsilon_{R}^2 \leq \frac{1}{2} \|z\|^2 + 2 \epsilon_{\lambda_{w,j}} \geq 2 \lambda_{w,j} \geq \lambda_{w,1} \), \( \lambda_{w,2} \geq \lambda_{w,3} > 0 \).

Therefore, \( \epsilon_{R}^2 \leq \frac{1}{2} \|z\|^2 + 2 \epsilon_{\lambda_{w,j}} \), \( \lambda_{w,2} \geq \lambda_{w,1} \), \( \lambda_{w,2} \geq \lambda_{w,3} > 0 \).

Therefore, \( \forall \alpha > 0 \), selecting \( \beta \) such that \( \frac{\epsilon_{\lambda_{w,j}}}{\epsilon_{\lambda_{w,j}}} > \alpha \), results in the inequality (17).

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