General warped solutions in 5D dilaton gravity

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Abstract

We present explicit analytic form of general warped solutions of the string inspired dilaton gravity system with bulk cosmological constant in 5 dimensions. The general solution allows for either nonvanishing effective 4–dimensional cosmological constant or the nontrivial 4–dimensional dilaton but not both.

1 Introduction

String theory is the leading candidate for the generalization of gravity. Indeed one of the massless string modes is the graviton and the effective action involves the Einstein–Hilbert action. In the low energy string effective action there is necessarily also another field called the dilaton that couples to (almost) all other fields. In this paper we consider the string-inspired dilaton gravity system in 5 dimensions. We seek solutions of the warped form [1, 2] for the metric and as sum of \( x^\mu \) and \( y \) dependent parts for the dilaton (as usually we denote by \( y \) the last coordinate). Surprisingly the general solutions of the bulk equations of motion can be found and we give their explicit analytic form in

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the string frame. These solutions are the building blocks which, when supplemented by appropriate branes, can be used to construct global warped solutions.

The transition from the string frame to the usually used Einstein frame is immediate but the results in general cannot be given in the explicit analytic form – we present in the paper such a transition for the cases where it can be explicitly done. Some special solutions for this system in the Einstein frame were obtained previously in [3, 4, 5, 6, 7, 8] (we do not consider here theories with other bulk scalar fields, often also called dilaton or radion fields, which may have different form of interactions).

2 5–dimensional dilaton gravity

The 5–dimensional gravity–dilaton action we consider is the tree level string effective action and reads in the string frame:

\[ S = \frac{1}{2\kappa^2_5} \int d^4x dy \sqrt{-g} e^{-\varphi} \left( -2\Lambda + R^{(5)} + \partial_{\mu} \varphi \partial^\mu \varphi \right). \]  

The equations of motion for the action (1) read

\[ 0 = R^{(5)}_{\mu\nu} + \nabla_{\mu} \partial_{\nu} \varphi, \]  

\[ 0 = 2\Lambda + \partial_{\mu} \varphi \partial^\mu \varphi - \Box^{(5)} \varphi. \]  

We are looking for solutions for which the metric has the warped form

\[ ds^2 = e^{2A(y)} g_{\alpha\beta}(x) dx^\alpha dx^\beta + dy^2 \]  

and the dilaton field separates

\[ \varphi(x^\alpha, y) = \sigma(x^\alpha) + \phi(y), \]  

where the indices \( \alpha, \beta = 0, \ldots, 3 \) while \( \mu, \nu = 0, \ldots, 4 \).

With such an ansatz the equations of motion (2–3) can be written as

\[ 0 = R^{(4)}_{\alpha\beta} + \nabla_{\alpha} \partial_{\beta} \sigma - g_{\alpha\beta} \left[ e^{2A} \left( A'' + 4(A')^2 - A' \phi' \right) \right], \]  

\[ 0 = -4A'' - 4(A')^2 + \phi'', \]  

\[ 0 = 2\Lambda - \phi'' - 4A' \phi' + (\phi')^2 + e^{-2A} \left[ \partial^\alpha \sigma \partial_{\alpha} \sigma - \nabla^\alpha \partial_{\alpha} \sigma \right], \]
where prime denotes the derivative with respect to $y$.

For the trivial case $A(y) = \text{const}$, $\phi'(y) = \text{const}$ the eqs. (6–8) just reduce to eqs. (2–3) with fields depending only on $x$ (with possibly different value of $\Lambda$).

We now concentrate on the more interesting situation when $A(y) \neq \text{const}$ and the metric (4) is really a warped one. In such a case the coordinates $x$ and $y$ must separate in eq. (6) and we obtain two conditions:

$$e^{2A} \left( A'' + 4(A')^2 - A' \phi' \right) = \gamma,$$  
$$R^{(4)}_{\alpha \beta} - \gamma g_{\alpha \beta} + \nabla_\alpha \partial_\beta \sigma = 0.$$  

Similarly the $x$–dependent part of eq. (8) reads

$$\partial^\alpha \sigma \partial_\alpha \sigma - \nabla^\alpha \partial_\alpha \sigma = -2\lambda.$$  

However equations (7), (8) and (9) are compatible with (11) only for vanishing constant $\lambda$. The 4–dimensional metric and dilaton must therefore satisfy eqs. (10) and (11) with $\lambda = 0$. Acting on eq. (10) with $\nabla_\alpha$ and using eq. (11) gives the condition

$$\gamma \partial_\beta \sigma = 0.$$  

Thus either $\sigma = \text{const}$ or $\gamma = 0$ and the $x$–dependent parts of the warped solutions must satisfy one of the two following sets of equations:

Case I

$$R^{(4)}_{\alpha \beta} = \frac{R}{4} g_{\alpha \beta}, \quad R = \text{const}, \quad \sigma = \text{const}.$$  

Case II

$$R^{(4)}_{\alpha \beta} + \nabla_\alpha \partial_\beta \sigma = 0, \quad \partial^\alpha \sigma \partial_\alpha \sigma = \nabla^\alpha \partial_\alpha \sigma, \quad \sigma \neq \text{const}.$$  

The case I corresponds to the 4–dimensional equations of motion derived from the 4–dimensional action

$$S^I = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(-2\Lambda + R^{(4)}\right),$$  

and the case II corresponds to the action

$$S^{II} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} e^{-\sigma} \left(R^{(4)} + \partial_\alpha \sigma \partial^\alpha \sigma\right).$$
It is interesting to note that none of these effective actions is the 4–dimensional version of the general dilaton gravity action given in eq. (1). From the effective 4–dimensional point of view the warped solutions cannot simultaneously have the 4–dimensional cosmological constant and the nontrivial dilaton. This feature points once again to the far–reaching differences between effective theories obtained by dimensional reduction (where there may be additional nontrivial constraints on the parameters) and theories defined ab initio in 4 dimensions (where such constraints are absent).

3 Exact solutions in the string frame

It turns out that for both cases I and II eqs. (6–8) can be solved for $A(y)$ and $\phi(y)$ in full generality.

Let us first discuss the case I with $R \neq 0$ which from the 4–dimensional point of view has nonvanishing effective cosmological constant and trivial 4–dimensional dilaton. The $y$-dependent part of eqs. (6–8) reads then

$$R = 4e^{2A} \left( A'' + 4(A')^2 - A' \phi' \right),$$  \hspace{1cm} (17)

$$0 = -4A'' - 4(A')^2 + \phi'',$$  \hspace{1cm} (18)

$$0 = 2\Lambda - \phi'' - 4A' \phi' + (\phi')^2.$$  \hspace{1cm} (19)

These equations are dependent – the derivative of (17) combined with (18) and its derivative gives the derivative of (19).

The character of the solutions depends on the sign of the bulk cosmological constant $\Lambda$. For $\Lambda > 0$ we choose the integration constants ($\alpha$, $y_1$, $\phi_0$) in such a way that the most general solution can be written in the following form

$$e^{2A(y)} = \frac{R}{2\Lambda} + \frac{R}{2\Lambda} \cot \left( \sqrt{\Lambda/2}(y - y_1) \right) \left( \pi \alpha - \sqrt{\Lambda/2}(y - y_1) \right),$$  \hspace{1cm} (20)

$$\phi(y) = \phi_0 + \ln \left[ \frac{R}{2\Lambda} + \frac{R}{2\Lambda} \cot \left( \sqrt{\Lambda/2}(y - y_1) \right) \left( \pi \alpha - \sqrt{\Lambda/2}(y - y_1) \right) \right]$$

$$- \ln \left( \sin^2 \left( \sqrt{\Lambda/2}(y - y_1) \right) \right),$$  \hspace{1cm} (21)

where the allowed values of the constant $\alpha$ are $\alpha > 0$ or $\alpha \leq -1$. Formally the above solutions have infinitely many branches separated by singularities of $A(y)$ and $\phi(y)$.  

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But it is enough to consider only one branch for a given set of the integration constants. That special branch is defined as the one that starts at $y_1$ and for which $A(y) \to +\infty$ when $y \to y_1$. All other branches can be obtained, up to a shift in $y$, by some other choice of the integration constants.

The special branch connected to $y_1$ has also a second singularity of $A(y)$. For $0 < \alpha < 1$ the warp function $A(y)$ goes at that singularity to $+\infty$ while for $|\alpha| \geq 1$ it goes to $-\infty$. The solutions with $A(y) \to +\infty$ at both singularities exist only when the effective 4-dimensional curvature scalar $R$ is positive. In such a case we take the branch $y_1 < y < y_1 + \pi \sqrt{2}/\Lambda$. The solutions with $A(y) \to -\infty$ at one of the singularities can be obtained for both signs of $R$. The range of $y$ in such a case is between $y_1$ and the next singularity. We take the singularity to the right (left) from $y_1$ when the product $\alpha R$ is positive (negative). The position of that singularity can not be found in a closed form. It is a value of $y$ for which the right hand side of eq. (20) vanishes.

We now turn to the situation when the bulk cosmological constant $\Lambda$ is negative. For this case we introduce three integration constants $(y_0, y_1, \phi_0)$ and the additional discrete parameter $\kappa = \pm 1$. The solutions for $\Lambda < 0$ read

\[ e^{2A(y)} = \frac{R}{2\Lambda} - \frac{R}{2\Lambda} \coth^\kappa \left( \sqrt{-\Lambda/2} (y - y_1) \right) \cdot \left( \sqrt{-\Lambda/2} (y - y_0) + \tanh^\kappa \left( \sqrt{-\Lambda/2} (y_0 - y_1) \right) \right), \]

\[ \phi(y) = \phi_0 + \ln \left[ \frac{R}{2\Lambda} - \frac{R}{2\Lambda} \coth^\kappa \left( \sqrt{-\Lambda/2} (y - y_1) \right) \cdot \left( \sqrt{-\Lambda/2} (y - y_0) + \tanh^\kappa \left( \sqrt{-\Lambda/2} (y_0 - y_1) \right) \right) \right] - \ln \left( \frac{1 - \kappa}{2} + \sinh^2 \left( \sqrt{-\Lambda/2} (y - y_1) \right) \right). \]

(22)

In this case there is no problem of multiple branches. For $\kappa = +1$ the real solutions exist for $y$ inside the interval bounded by $y_0$ and $y_1$ for $R < 0$ and outside this interval for $R > 0$. At each of the points, $y_0$ and $y_1$, there is a singularity of $A(y)$ (and also of $\phi(y)$) but the nature of these singularities is different: $A(y) \to -\infty$ for $y \to y_0$ while $A(y) \to +\infty$ for $y \to y_1$.

The solutions for $\kappa = -1$ are defined for $y$ inside the interval bounded by $y_0$ and $y_2$ for $R < 0$ and outside this interval for $R > 0$. The boundary position $y_2$ is defined as
the second (besides $y_0$) zero of the right hand side of eq. (22). In this case it is different from $y_1$ and cannot be written in a simple form. At both points, $y_0$ and $y_2$, the warp function $A(y)$ approaches $-\infty$.

The solution for $\Lambda = 0$ can be obtained as a limit of (20) and (21):

$$e^{2A(y)} = \frac{R (y - y_1)^3 - (y_0 - y_1)^3}{12 (y - y_1)},$$  \hspace{1cm} (24)

$$\phi(y) = \ln \left[ \frac{C_\phi R (y - y_1)^3 - (y_0 - y_1)^3}{12 (y - y_1)^3} \right],$$  \hspace{1cm} (25)

where we changed the integration constants in order to simplify the result. The above solutions have singularities at $y_0$ and $y_1$ such that $A(y) \to -\infty$ for $y \to y_0$ and $A(y) \to +\infty$ for $y \to y_1$. For $R < 0$ ($R > 0$) the solution is well defined for $y$ inside (outside) the interval bounded by $y_0$ and $y_1$.

Let us now discuss case I with $R = 0$ and case II (they lead to the same $y$-dependent equations). The equations we have to solve are given by (17–19) with $R = 0$.

The solutions for $\Lambda > 0$ read:

$$e^{2A(y)} = c_A \left| \tan \left( \sqrt{\Lambda/2} (y - y_0) \right) \right|,$$  \hspace{1cm} (26)

$$\phi(y) = \phi_0 + 2 \ln \left| \tan \left( \sqrt{\Lambda/2} (y - y_0) \right) \right| - \ln \left| \sin \left( \sqrt{2\Lambda} (y - y_0) \right) \right|.$$  \hspace{1cm} (27)

The range of $y$ is either $y_0 < y < y_0 + \pi \sqrt{2/\Lambda}$ or $y_0 - \pi \sqrt{2/\Lambda} < y < y_0$. The warp function $A(y)$ goes to $-\infty$ at $y = y_0$ and to $+\infty$ at the second singularity.

In the case $\Lambda < 0$ we again introduce additional parameter $\kappa = \pm 1$ and write the solutions in the form:

$$e^{2A(y)} = c_A \left| \tanh \left( \sqrt{-\Lambda/2} (y - y_0) \right) \right|^\kappa,$$  \hspace{1cm} (28)

$$\phi(y) = \phi_0 + 2\kappa \ln \left| \tanh \left( \sqrt{-\Lambda/2} (y - y_0) \right) \right| - \ln \left| \sinh \left( \sqrt{-2\Lambda} (y - y_0) \right) \right|,$$  \hspace{1cm} (29)

where $y > y_0$ or $y < y_0$.

Finally for $\Lambda = 0$ we obtain:

$$e^{2A(y)} = c_A \left| y - y_0 \right|^\kappa,$$  \hspace{1cm} (30)

$$\phi(y) = (2\kappa - 1) \ln \left( C_\phi \left| y - y_0 \right| \right),$$  \hspace{1cm} (31)
where again \( y > y_0 \) or \( y < y_0 \). In the last two types of solutions \((28-31)\) we have \( \kappa A(y) \to -\infty \) for \( y \to y_0 \).

Let us now summarize the structure of singularities for all the solutions. We denote by \( S_+ \) (respectively \( S_- \)) singularity at finite \( y \) where \( A(y) \to +\infty \) (respectively \( -\infty \)) and by \( S_{inf} \) the fact that a solution extends to \( y \to \infty \) or \( y \to -\infty \). The character of the solutions depends on the signs of the bulk cosmological constant \( \Lambda \) and the effective 4-dimensional curvature scalar \( R \) and also on the presence of a nontrivial 4-dimensional dilaton (case II). All the cases are summarized in the following table:

| \( \Lambda > 0 \) | case I, \( R < 0 \) and case II | case I, \( R = 0 \) | case I, \( R > 0 \) |
|-------------------|-------------------------------|-----------------|------------------|
| \( S_+ - S_- \)  | \( S_+ - S_- \)               | \( S_+ - S_+ (S_-) \) |                  |
| \( \Lambda = 0 \) | \( S_+ - S_- \)               | \( S_+ (S_-) - S_{inf} \) | \( S_+ (S_-) - S_{inf} \) |
| \( \Lambda < 0 \) | \( S_- - S_+ (S_-) \)        | \( S_+ (S_-) - S_{inf} \) | \( S_+ (S_-) - S_{inf} \) |

One can see that each solution has at least one singularity at finite \( y \). Solutions with non–positive bulk cosmological constant \( \Lambda \) and non–negative effective curvature scalar (case I with \( R \geq 0 \) and case II) start at such singularity and extend to infinity. All other solutions have two singularities at finite values of \( y \). The distance between the singularities is of the order of \( 1/\sqrt{\Lambda} \) for \( \Lambda > 0 \) and is a free parameter for \( \Lambda \leq 0 \).

### 4 Solutions in the Einstein frame

So far we have presented general warped solutions in 5–dimensional dilaton gravity in the string frame. Let us now discuss the form of these solutions in the Einstein frame.

In order to go to the Einstein frame we perform a Weyl transformation for which \( \varphi(x, y) \) is unchanged and the metric transforms as

\[
\tilde{g}_{\mu\nu} = e^{-2\varphi/3} g_{\mu\nu}.
\] (32)

Then the 5–dimensional action reads

\[
S_E = \frac{1}{2\kappa^2} \int d^5x \sqrt{-\tilde{g}} \left( -2\Lambda \epsilon^{\frac{2}{3}\varphi} + \tilde{R}^{(5)} - \frac{1}{3} \partial_{\mu}\varphi \partial^{\mu}\varphi \right) .
\] (33)
Of course we are not going to solve the equations of motion derived from the above action since we can simply translate the solutions obtained before in the string frame. These solutions have the warped form also in the Einstein frame only for $\sigma = \text{const}$ (i.e. case I). Then we have

$$ds_E^2 = e^{2A_E(y)}\tilde{g}_{\alpha\beta}(x)dx^\alpha dx^\beta + e^{-2\phi(y)/3}dy^2, \quad (34)$$

where

$$e^{2A_E(y)} = e^{2A(y)-2\phi(y)/3}, \quad (35)$$

In order to put the metric (34) in the standard form one should redefine the last coordinate by the relation

$$dz = e^{-\phi(y)/3}dy. \quad (36)$$

Using this new coordinate $z$ the line element in the Einstein frame can be written in the usually assumed form

$$ds_E^2 = e^{2\tilde{A}_E(z)}\tilde{g}_{\alpha\beta}(x)dx^\alpha dx^\beta + dz^2. \quad (37)$$

However, in general there is no analytic formula for $\tilde{A}_E(z)$ because for the string–frame general solutions found in the paper it is not possible to integrate and invert in closed form the equation (36). One can transform our general string–frame solutions to the Einstein frame either as explicit solutions for a modified form of the metric (34) using the coordinate $y$ or as non-explicit solutions for a standard form of the metric (37). Although the latter transformation is quite complicated but nevertheless some important general remarks can be made. The most interesting one is related to the character of singularities at finite $z$. It is not very difficult to check that for all the solutions both $S_+$ and $S_-$ singularities in the string frame are translated into $S_-$ singularity (i.e. $\tilde{A}_E(z) \to -\infty$) in the Einstein frame. One can show also that finite distance between singularities in the string frame in the $y$ coordinate is also a finite distance between singularities in the $z$ coordinate.

These results allow us to divide the solutions into two classes. In one class the solutions end at two points at which the metric vanishes and some components of the curvature tensor have singularities. The distance between these points is finite. The
solutions have such features when $\Lambda > 0$ or $R < 0$. For other parameters, namely when $\Lambda \leq 0$ and $R \geq 0$, a solution has only one singularity and extends infinitely in one of the directions in the coordinate $y$ (the same is true also for coordinate $z$). The metric vanishes at the singularity and grows to infinity for $|y| \to \infty$ ($|z| \to \infty$).

Any of the globally well defined warped solutions in such a theory has to be piecewise (i.e. outside the branes) given by the solutions found in this paper since as we have shown these are the most general warped solutions of the bulk equations of motion. If branes with appropriate properties are added then we can eventually build models which are effectively 4–dimensional.

We now discuss situations when it is possible to find explicitly the solutions in terms of the coordinate $z$. They correspond to case I ($\sigma = \text{const}$) with vanishing 4–dimensional curvature scalar $R$. For $\Lambda = 0$, using eqs. (30), (31) and (36) we get

$$e^{2\tilde{A}_E(z)} = \sqrt{\tilde{C}_A|z - z_0|}, \quad \phi(z) = \frac{3}{2} \ln \left(\tilde{C}_\phi|z - z_0|^\kappa\right). \quad (38)$$

where the new coordinate $z$ is given by $z = z_0 + C_z(y - y_0)^{2\kappa/(2\kappa + 1)}$. These are the solutions discussed in ref. [4] (with different normalization of the dilaton). For nonzero bulk cosmological constant $\Lambda$ we obtain

$$e^{2\tilde{A}_E(z)} = \sqrt{\tilde{C}_A(z - z_0)\left[(z_1 - z_0)^3 - (z - z_0)^3\right]}, \quad (40)$$

$$\phi(z) = \frac{3}{2} \ln \left[\frac{2}{9\Lambda(z_1 - z_0)^3 - (z - z_0)^3}\right]. \quad (41)$$

For $\Lambda > 0 (\Lambda < 0)$ the solution is well defined for $z$ inside (outside) the interval bounded by $z_0$ and $z_1$, at which the metric vanishes. The coordinate $z$ for $\Lambda > 0$ is equal to

$$z = z_0 + C_z \left[\sin \left(\sqrt{\Lambda/2}(y - y_0)\right)\right]^{2/3}, \quad (42)$$

while for $\Lambda < 0$

$$z = z_0 + C_z \left[\sinh \left(\sqrt{\Lambda/2}(y - y_0)\right)\right]^{2/3}, \quad (43)$$

$$z = z_1 + C_z \left[\cosh \left(\sqrt{\Lambda/2}(y - y_0)\right)\right]^{2/3}, \quad (44)$$
for $\kappa = +1$ and $\kappa = -1$ respectively.

The solutions in the Einstein frame (40–41) are new. Let us emphasize that these solutions constitute only a small subset of general solutions found in the paper (the others in general cannot be explicitly transformed to the Einstein frame and only some partial results can be obtained for example for $R \neq 0$ and $\Lambda = 0$ [5]).

5 Conclusions

The general warped solutions of the 5–dimensional dilaton gravity presented in this paper may be useful in the brane–like applications of string inspired models and supplemented by appropriate branes may lead to globally well defined warped solutions. It is worth emphasizing once again that the 5–dimensional theory admits warped solutions only of two 4–dimensional types: Einstein metric with a constant dilaton or nontrivial dilaton without a cosmological constant. This observation may have important consequences for all theories that include 4–dimensional dynamical dilaton.

The solutions described in this paper have curvature singularities (in the 5–dimensional sense). It is therefore necessary to supplement them with appropriate branes to obtain solutions which are well behaved for the whole range of $y$. It is easy to transpose the solutions to the “cosmological case” where the distinguished coordinate is not $y$ but $t$ which is timelike. It is then not necessary to add any branes because the evolution starts (and eventually ends) at singularities which are of the same type as in standard cosmology.

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