The massive Dirac field on a rotating black hole spacetime: angular solutions

Sam R Dolan\textsuperscript{1} and Jonathan R Gair\textsuperscript{2}

\textsuperscript{1} School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland
\textsuperscript{2} Institute of Astronomy, University of Cambridge, Madingley Road, Cambridge, CB3 0HA, UK

E-mail: sam.dolan@ucd.ie and jgair@ast.cam.ac.uk

Received 28 May 2009, in final form 1 August 2009
Published 21 August 2009
Online at stacks.iop.org/CQG/26/175020

Abstract
The massive Dirac equation on a Kerr–Newman background may be solved by the method of separation of variables. The radial and angular equations are coupled via an angular eigenvalue, which is determined from the Chandrasekhar–Page (CP) equation. Obtaining accurate angular eigenvalues is a key step in studying scattering, absorption and emission of the fermionic field. Here we introduce a new method for finding solutions of the CP equation. First, we introduce a novel representation for the spin-half spherical harmonics. Next, we decompose the angular solutions of the CP equation (the mass-dependent spin-half spheroidal harmonics) in the spherical basis. The method yields a three-term recurrence relation which may be solved numerically via continued-fraction methods, or perturbatively to obtain a series expansion for the eigenvalues. In the case $\mu = \pm \omega$ (where $\omega$ and $\mu$ are the frequency and mass of the fermion) we obtain eigenvalues and eigenfunctions in a closed form. We study the eigenvalue spectrum and the zeros of the maximally co-rotating mode. We compare our results with previous studies, and uncover and correct some errors in the literature. We provide series expansions, tables of eigenvalues and numerical fits across a wide parameter range and present plots of a selection of eigenfunctions. It is hoped that this study will be a useful resource for all researchers interested in the Dirac equation on a rotating black hole background.

PACS numbers: 04.70.−s, 04.62.+v

(Some figures in this article are in colour only in the electronic version)
1. Introduction

The interaction of a fermionic (Dirac) field with a rotating charged (Kerr–Newman) black hole is of deep theoretical interest. The interaction depends on three couplings: coupling between quantum-mechanical spin and black hole (BH) rotation; coupling between field mass and BH mass; and coupling between field charge and BH charge. Desire for a deeper understanding of these couplings has motivated a range of studies over the last three decades.

Over 30 years ago, Chandrasekhar [1] and Page [2] showed that separation of variables is possible for the massive Dirac field on the Kerr–Newman spacetime. In other words, the partial differential equations (PDEs) governing the evolution of the Dirac field may be reduced to a set of coupled ordinary differential equations (ODEs), and the solution can be expressed as a sum over modes and integral over frequency. The radial and angular ODEs are coupled through an angular eigenvalue $\lambda$. The eigenvalue is found by solving the so-called Chandrasekhar–Page (CP) equation (see equation (4)). The eigenfunctions of the CP equation—the so-called mass-dependent spin-half spheroidal harmonics (MDSHs)—are required for a full reconstruction of the field. In this paper, we present a new method for computing both the eigenvalues and eigenfunctions. Our aim is to show that finding MDSHs is no more difficult than finding the spin-weighted spheroidal harmonics for massless fields [3]. Along the way, we review alternative methods and highlight some inaccuracies in the literature.

Motivation for this work arose from two separate studies, conducted independently by the present authors. The Kerr–Newman solution takes a very beautiful form in the limit where the electromagnetic field is assumed to dominate over gravity, which is obtained by letting $G \to 0$ in the metric. The result is flat space, in oblate-spheroidal coordinates, with an electromagnetic field $E + i B = -\nabla \left( \frac{1}{\sqrt{(r - ia) \cdot (r - ia)}} \right)$ (1), i.e., a Coulomb field centred at the imaginary point $r = i a = i(0, 0, a)$ [4–6]. This field can be used to model a rapidly rotating but effectively massless nucleus [7] and in his thesis [8] Gair examined the possibility that the hyperfine splitting observed in muonium could be reproduced by the coupling between electron spin and the frame dragging caused by nuclear rotation in this model [7]. Dolan studied the spectrum of fermionic quasi-bound states in the vicinity of a small non-extremal black hole with $a < M$, extending recent work in this area [9, 10]. Both studies required the determination of accurate eigenvalues for the angular eigen equation, and both studies were impeded by a lack of reliable numerical results in the literature. These studies will be presented in full elsewhere.

The massive Dirac–Kerr–Newman system has been studied in many other contexts. For example, authors have investigated the absence of fermionic superradiance [11, 12]; scattering and absorption [13–17]; and Hawking radiation emission [18, 19]. Our intention here is to provide a solid foundation for possible further work. Let us give three examples of future applications. First, it has been shown [20–24] that a rotating BH is stable to fermionic perturbations. Nevertheless, the quantitative effect of field mass [25] on the fermionic quasi-normal mode spectrum [26] has not been studied. Second, field mass is usually neglected when the Hawking emission process is considered [2]. The emission spectrum will change significantly if the Hawking temperature approaches the mass of a particle species of the standard model. Emission of massive scalars by a Kerr–Newman black hole was considered in [27], but a corresponding study for massive fermions is still lacking. Finally, in recent years there has been much interest in theories with extra dimensions which lead to the possibility of BH creation in hadron colliders [28]. The radial equations for fermionic fields on the brane are closely related to their 4D counterparts [29], and the angular equations are unchanged. The
effect of field mass on emission of fermions from non-rotating holes was recently considered [30]. The method outlined here will be of use to study the emission of massive fermions from rapidly rotating higher-dimensional black holes.

This paper is organized as follows. In section 2, we introduce the basic equations. In section 3, we examine the non-rotating limit and introduce a novel form for the spin-half spherical harmonics. In section 4, we make explicit the symmetries of the eigenspectrum. In section 5, we derive exact expressions for the eigenvalues and eigenfunctions for the special cases $a \mu = \pm a o$. In section 6, we derive a spectral decomposition method for tackling the general case, which leads to a three-term recurrence relation. In section 7, we use the recurrence relation to investigate the asymptotic behaviour of the eigenvalue spectrum in the slow-rotation regime. In section 8, we describe some alternative methods used to find eigenvalue solutions and highlight some errors in previous work. Numerical results are presented in section 9. We examine the dependence of the eigenvalues on $a \omega$ and $a \mu$, investigate the zeros of the eigenvalue spectrum and present a gallery of eigenfunctions. We conclude with a brief discussion in section 10. Tables of eigenvalues, plus simple but accurate numerical fits to the data, are given in Appendix B.

2. Basics

In 1976, Chandrasekhar [1] demonstrated how to separate variables for the massive Dirac field on the Kerr spacetime and shortly afterwards, Page [2] extended the analysis to the Kerr–Newman spacetime. More recently, extensions to non-asymptotically flat [31] and other related spacetimes [32] have been considered. In such analyses, the $t$ and $\phi$ dependence may be factorized by using the ansatz $\Psi(t, r, \theta, \phi) = e^{-i \omega t + im \phi} \Psi_{a \mu}(r, \theta)$, and the Newman–Penrose method is applied [33]. The four components of the wavefunction $\Psi_{a \mu}$ can be expressed as products of two radial and two angular functions $\{R_s(r), R_\omega(r), S_+(\theta), S_-(\theta)\}$ that obey coupled ODEs. The angular equations are

$$\frac{dS_+}{d\theta} + \left( \frac{1}{2} \cot \theta - a \omega \sin \theta + m \csc \theta \right) S_+ = - (\lambda - a \mu \cos \theta) S_-, \tag{2}$$

$$\frac{dS_-}{d\theta} + \left( \frac{1}{2} \cot \theta + a \omega \sin \theta - m \csc \theta \right) S_- = + (\lambda + a \mu \cos \theta) S_. \tag{3}$$

Here, $a = J/M$ is the black-hole angular momentum parameter, and $\omega$ and $\mu$ are the frequency and mass of the state under consideration (Note: the symbol $\sigma \equiv -\omega$ is also used for the frequency in the literature [34]). $S_+(\theta)$ and $S_-(\theta)$ are known as mass-dependent spheroidal harmonics of spin one-half, and $\lambda$ is the eigenvalue. The eigenvalue can be regarded as the square root of a generalized squared total angular momentum [35]. The solutions depend on two continuous parameters, $a \omega$ and $a \mu$. For a given $a \omega$, $a \mu$, the eigenstates $\{S_+(\theta), S_-(\theta), \lambda\}$ may be labelled by three discrete numbers: the angular momentum $j = 1/2, 3/2, \ldots$, the azimuthal component of the angular momentum $m = -j, -j + 1, \ldots, j$ and the parity $\mathcal{P} = \pm 1$. When this dependence is to be made explicit, we will write $S_{\pm} = S_{j=\pm 1/2}^{(a \omega, a \mu)}$ and $\lambda = \lambda_{j m \mathcal{P}}^{(a \omega, a \mu)}$. Equations (2) and (3), and hence the eigenvalue spectrum and eigenfunctions, exhibit a number of symmetries (see section 4). Knowledge of the spectrum in the quadrant $a \omega > 0, a \mu > 0$ is sufficient to determine the full spectrum.

The first-order equations (2) and (3) may be combined to obtain a second-order equation,

$$\left[ \frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} - \left( \frac{m - \frac{1}{2} \cos \theta}{\sin^2 \theta} \right) - \frac{1}{2} \right] S_- + \left[ (a^2 \omega^2 - a^2 \mu^2) \cos^2 \theta + a \omega \cos \theta \right] S_-$$

$$+ a \mu \sin \theta S_+ = - [\lambda^2 - a^2 \omega^2 + 2a \omega m] S_-, \tag{4}$$
known as the Chandrasekhar–Page angular equation. In the limit \( \mu = 0 \) this reduces to the well-known equation for the spin-weighted spheroidal harmonics (see for example equation (2.1) in [3], making the identifications \( s = -1/2 \) and \( \mathcal{A}_{lm} = \lambda^2 + 2a \omega_m - a^2 \omega^2 \)). We find, in the case \( a \mu \neq 0 \), it is actually easier to analyse the coupled first-order equations directly.

There have been a number of studies of equations (2)–(4) over the years [13, 34, 36–39]. In section 8, we discuss alternative approaches in some detail, to validate our method and to identify some errors in the literature.

3. Exact solutions for \( a = 0 \)

In the non-rotating limit, \( a = 0 \), the equations reduce to

\[
\frac{dS_+}{d\theta} + \left( \frac{1}{2} \cot \theta + m \csc \theta \right) S_+ = -\lambda S_-
\]

\[
\frac{dS_-}{d\theta} + \left( \frac{1}{2} \cot \theta - m \csc \theta \right) S_- = +\lambda S_+.
\]

Making the substitution

\[
\begin{bmatrix}
S_+ \\
S_-
\end{bmatrix} = \begin{bmatrix}
\cos(\theta/2) & \sin(\theta/2) \\
-\sin(\theta/2) & \cos(\theta/2)
\end{bmatrix}
\begin{bmatrix}
T_+ \\
T_-
\end{bmatrix}
\]

leads to coupled equations

\[
\frac{d}{d\theta} \begin{bmatrix}
T_+ \\
T_-
\end{bmatrix} + \begin{bmatrix}
(m + 1/2) \cot \theta & (m + \lambda + 1/2) \\
(m - 1/2 - \lambda) & -(m - 1/2) \cot \theta
\end{bmatrix}
\begin{bmatrix}
T_+ \\
T_-
\end{bmatrix} = 0.
\]

These can be written in the second-order form,

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dT_+}{d\theta} \right) - \frac{(m \pm 1/2)^2}{\sin^2 \theta} T_+ + \lambda(\lambda + 1) T_+ = 0,
\]

which is simply the general Legendre equation. The eigenvalues are

\[
\lambda_{jm}(0,0) = \mathcal{P}(j + 1/2)
\]

where \( \mathcal{P} = \pm 1 \) and \( j \) is a half-integer. The eigenfunctions are \( T_{\pm}(\theta) \propto P_{L}^{m \pm 1/2}(\cos \theta) \), where \( P_{L}^{m \pm 1/2} \) is an associated Legendre polynomial obeying the Condon–Shortley phase convention, and \( L = j + \mathcal{P}/2 \) is a non-negative integer. Let us define a set of solutions

\[
\begin{bmatrix}
Y_{jm}^{+}(\theta) \\
Y_{jm}^{-}(\theta)
\end{bmatrix} = \begin{bmatrix}
1 & (j - m)! \\
2\pi & (j + m)!
\end{bmatrix}^{1/2}
\begin{bmatrix}
\cos(\theta/2) & \sin(\theta/2) \\
-\sin(\theta/2) & \cos(\theta/2)
\end{bmatrix}
\begin{bmatrix}
P_{L}^{m+1/2}(\cos \theta) \\
P_{L}^{m-1/2}(\cos \theta)
\end{bmatrix}
\]

where \( x = \cos \theta, c_{jm}^{m} = \mathcal{P}(L + 1/2) - m \) and \( L = j + \mathcal{P}/2 \).

It is straightforward to verify that these solutions are normalized so that

\[
2\pi \int_{0}^{\pi} d\theta \sin \theta s Y_{jm}^{m}(\theta) s Y_{jm}^{m}(\theta) = \delta_{jj}.
\]

and that they exhibit the symmetries

\[
s Y_{jm}^{m}(\theta) = (-1)^{j-1/2} s Y_{jm}^{m}(\theta),
\]

\[
= \mathcal{P}(-1)^{m-1/2} s Y_{j-m}^{m}(\theta),
\]

\[
= \mathcal{P}(-1)^{j+m} s Y_{jm}^{m}(\pi - \theta).
\]
Note that by combining these symmetries, we may obtain a further four expressions of similar form.

The solutions (11) are closely related to the spin-weighted spherical harmonics of spin-weight half, \( s_{\pm 1/2} Y_{jm}^{(NP)}(\theta) \), first introduced by Newman and Penrose [40]. We make the identification

\[
Y_{jm}^{(NP)}(\theta) = (-1)^{m+s} Y_{jm, P=\pm 1}(\theta)
\]

(16)

\[
= \left[ \frac{2j + 1}{4\pi} \frac{(j + m)!}{(j + s)! \frac{(j - s)!}{(j - m)!}} \right]^{1/2} [\sin(\theta/2)]^2 j^2 
\times \sum_n \left( \frac{j - s}{n} \right) \left( \frac{j + s}{n + s - m} \right) (-1)^{j-s-n} [\cot(\theta/2)]^{2n+s-m}.
\]

(17)

The representation (17) was introduced by Goldberg et al (see [41], equation (3.1)). The representation of \( s_{\pm 1/2} Y_{jm}^{(NP)}(\theta) \) in terms of Legendre polynomials (equations (11) and (16)) does not seem to be well known in the literature. These functions arise in the guise of spherical monogenics in geometric algebra [42,43], and we find that the representation (11) has many advantages.

4. Symmetries

The eigenvalue spectrum has the following symmetries:

\[
\lambda(-a_\omega, a_\mu)_{j,m,P} = -\lambda(a_\omega, -a_\mu)_{j,-m,-P} = -\lambda(-a_\omega, -a_\mu)_{j,-m,-P} = \lambda(j,-m,-P),
\]

(18)

Hence knowledge of the spectrum in the quadrant \( a_\omega > 0, a_\mu > 0 \) is sufficient to determine the full spectrum. In close correspondence with equations (13) and (14), the eigenfunctions have the following symmetries:

\[
s_{\pm 1/2} S^{(a_\omega,a_\mu)}_{jm,P}(\theta) = (-1)^{j-1/2} s_{\mp 1/2} S^{(-a_\omega,-a_\mu)}_{jm,-P}(\theta),
\]

(19)

\[
P = (-1)^{m-1/2} s_{\pm 1/2} S^{(-a_\omega,a_\mu)}_{jm,P}(\pi - \theta),
\]

(20)

\[
= (-1)^{j+m} s_{\mp 1/2} S^{(a_\omega,a_\mu)}_{jm, -P}(\pi - \theta),
\]

(21)

and combinations thereof. For example, combining (19) and (20) yields

\[
s_{\pm 1/2} S^{(a_\omega,a_\mu)}_{jm,P}(\theta) = P(-1)^{j+m} s_{\pm 1/2} S^{(-a_\omega,-a_\mu)}_{jm, -P}(\theta).
\]

5. Exact solutions for \( a_\omega = \pm a_\mu \)

In the special case that \( a_\omega = a_\mu \) we find that transformation (7) yields the coupled equations

\[
\frac{d}{d\theta} \begin{bmatrix} T_+ \\ T_- \end{bmatrix} + \begin{bmatrix} (m + 1/2) \cot \theta & (\lambda + m + 1/2 - a_\omega) \\ (m - 1/2 - \lambda - a_\omega) & -(m - 1/2) \cot \theta \end{bmatrix} \begin{bmatrix} T_+ \\ T_- \end{bmatrix} = 0.
\]

(22)

Again, the solutions are associated Legendre polynomials, \( T_{\pm} \propto P_{L}^{m\pm 1/2} \), where \( L \) is a non-negative integer. The normalized solutions are

\[
\begin{bmatrix} +1/2 S_{jm,P}^{(a_\omega,a_\mu)}(\theta) \\ -1/2 S_{jm,P}^{(a_\omega,a_\mu)}(\theta) \end{bmatrix} = A \begin{bmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \begin{bmatrix} P_{L}^{m+1/2}(x) \\ b_{jm,P} P_{L}^{m-1/2}(x) \end{bmatrix}
\]

(23)
where $L = j + \mathcal{P}/2$ and $b_{jm^p} = [(L + 1/2)^2 - m^2]/(\lambda + m + 1/2 - a\omega)$ and the normalization factor is
\[
A^2 = \frac{2L + 1}{4\pi} \frac{(L - m - 1/2)!}{(L + m + 1/2)!} \left( 1 + \frac{\mathcal{P}(m - a\omega)}{\sqrt{(L + 1/2)^2 - 2ma\omega + a^2\omega^2}} \right). \tag{24}
\]
The corresponding eigenvalue is
\[
\lambda_{jm^p}^{(a\omega,a\mu)} = -\frac{1}{2} + \mathcal{P}\sqrt{(L + 1/2)^2 - 2ma\omega + a^2\omega^2}. \tag{25}
\]
In the special case $m = -j$, $\mathcal{P} = -1$ we find $b_{jm^p,m,-1} = 1$ and $\lambda_{jm^p,m,-1} = -1/2 - j - a\omega$.

The solution for $a\omega = -a\mu$ can be found by considering the symmetry of equations (2) and (3) under the simultaneous transformations
\[
a\mu \to -a\mu, \quad \lambda \to -\lambda, \quad Y_{jm^p}(\theta) \to (-1)^j a^{-1/2} Y_{jm^p}(\theta). \tag{26}
\]
Hence the solutions are
\[
\begin{bmatrix}
+1/2 S^{(a\omega,a\omega)}_{jm^p}(\theta) \\
-1/2 S^{(a\omega,-a\omega)}_{jm^p}(\theta)
\end{bmatrix}
= B \begin{bmatrix}
\cos(\theta/2) & -\sin(\theta/2) \\
\sin(\theta/2) & \cos(\theta/2)
\end{bmatrix}
\begin{bmatrix}
\mathcal{P}_{L}^{m+1/2}(x) \\
\mathcal{P}_{L}^{m-1/2}(x)
\end{bmatrix}, \tag{27}
\]
where now $L = j - \mathcal{P}/2$ and $b_{jm^p}^{(a\omega,-a\omega)} = -[(L + 1/2)^2 - m^2]/(-\lambda_{jm^p}^{(a\omega,-a\omega)} + m + 1/2 - a\omega)$ and the normalization factor is
\[
B^2 = \frac{2L + 1}{4\pi} \frac{(L - m - 1/2)!}{(L + m + 1/2)!} \left( 1 + \frac{\mathcal{P}(m - a\omega)}{\sqrt{(L + 1/2)^2 - 2ma\omega + a^2\omega^2}} \right). \tag{28}
\]
The corresponding eigenvalue is
\[
\lambda_{jm^p}^{(a\omega,-a\omega)} = \frac{1}{2} + \mathcal{P}\sqrt{(L + 1/2)^2 - 2ma\omega + a^2\omega^2}. \tag{29}
\]

\section{A spectral decomposition method}

An obvious next step is to seek an expansion for the spheroidal harmonics in the basis of spherical harmonics (11). This was the approach followed by Chakrabarti [37], who applied the method to the second-order CP equation (4) to find a five-term recurrence relation. Here we apply the method directly to the first-order equations themselves, to recover a three-term recurrence relation. The key advantage of a three-term relation is that it can be solved using robust continued-fraction methods.

Let us begin with the ansatz
\[
s^{(a\omega,a\mu)}_{jm^p}(\theta) = \sum_{k=[m]}^\infty s^{(a\omega,a\mu)}_{k,m^p}(j) Y_{k,m^p}(\theta) \tag{30}
\]
where $s^{(a\omega,a\mu)}_{k,m^p}(j)$ are expansion coefficients to be determined, and $Y_{k,m^p}(\theta)$ were defined in (11). The symmetries (15) and (21) imply that the coefficients are related by
\[
s^{(a\omega,a\mu)}_{k,m^p}(j) = (-1)^j k^j s^{(a\omega,a\mu)}_{k,m^p}(j) \tag{31}
\]
For clarity, we neglect the least-relevant indices, writing $\pm1/2 s^{(a\omega,a\mu)}_{k,m^p}(j) \equiv c^{(\pm)}_{j}(k)$ and $\pm1/2 Y_{k,m^p} \equiv Y_{k^j}^{(\pm)}$. Substituting ansatz (30) into equation (2) and using (31) leads to
\[
\sum_{k=[m]}^\infty \left( -\mathcal{P}(k' + 1/2) Y_{k'}^{(-)} - a\omega \sin \theta Y_{k'}^{(+)} + (-1)^j k^j (\lambda - a\mu \cos \theta) Y_{k'}^{(-)} \right) c_{k,j}^{(\pm)} = 0. \tag{32}
\]

If we now multiply by $Y_k^{(-)}$ and integrate we obtain the matrix eigenvalue equation
\[ \sum_{k'=|m|}^{\infty} A_{kk'} b_{k'} = \lambda b_k \] (33)
where
\[ A_{kk'} = (-1)^{j-k} (k + 1/2) P \delta_{kk'} + (-1)^{j-k'} a \omega D_{kk'}^{(1)} + a \mu C_{kk'}^{(1)}, \] (34)
\[ b_k = (-1)^{j-k} c_{k(j)}^{(-)} = c_{k(j)}^{(-)}, \] (35)
and
\[ C_{kk'}^{(1)} = 2 \pi \int_0^\pi d\theta \sin \theta Y_k^{(-)}(\theta) \cos \theta Y_{k'}^{(-)}(\theta), \] (36)
\[ D_{kk'}^{(1)} = 2 \pi \int_0^\pi d\theta \sin \theta Y_k^{(-)}(\theta) \sin \theta Y_{k'}^{(+)}(\theta), \] (37)
\[ = P \left( \frac{2k'+1}{2k+1} \right)^{1/2} \langle k'1m0|km\rangle \langle k'1\frac{1}{2}0|k\frac{1}{2} \rangle, \] (38)
\[ = P \left( \frac{2k'+1}{2k+1} \right)^{1/2} \langle k'1m0|km\rangle \langle k'1-\frac{1}{2}1|k\frac{1}{2} \rangle. \] (39)

The Clebsch–Gordan coefficients are only non-zero for $k' - 1 \leq k \leq k' + 1$. Hence $A_{kk'}$ is a tridiagonal matrix, and the eigenvalues $\lambda$ and vectors $b_k$ can be found via linear algebra routines. Alternatively, we can obtain a three-term recurrence relation for the expansion coefficients $c_{k(j)}^{(+)}$. Using expressions for the Clebsch–Gordan coefficients given in Appendix A it is straightforward to show that
\[ \alpha_k b_{k+1} + \beta_k b_k + \gamma_k b_{k-1} = 0, \quad k = |m|, |m| + 1, \ldots \] (40)
where
\[ \alpha_k = (a \mu + \epsilon_k a \omega) \frac{\sqrt{(k+1)^2 - m^2}}{2(k+1)}, \] (41)
\[ \beta_k = \epsilon_k (k + 1/2) \left( 1 - \frac{a \omega m}{k(k+1)} + \frac{a \mu m}{2k(k+1)} - \lambda \right), \] (42)
\[ \gamma_k = (a \mu - \epsilon_k a \omega) \frac{\sqrt{k^2 - m^2}}{2k}. \] (43)
and $\epsilon_k = (-1)^{j-k} P$. Note that $j, k$ and $m$ are half-integers. The advantage of a three-term recurrence relation is that it can be solved via continued-fraction methods. To obtain numerical results, we followed the approach outlined in [44] and used a rescaling algorithm described in [45]. The basic idea is to separate out the eigenvalue by writing $\hat{\beta}_k = \tilde{\beta}_k - \lambda$, and write down an expression for the ratio of consecutive terms
\[ \frac{b_k}{b_{k-1}} = - \frac{\gamma_k}{\tilde{\beta}_k - \lambda + \alpha_k \left( \frac{b_{k+1}}{b_k} \right)}. \] (44)
The series should converge and this condition gives the possible values for the eigenvalues. In practice, we write the eigenvalue as a continued fraction
\[ \lambda = \beta_1 - \frac{\alpha_1 \gamma_1}{\beta_2 - \lambda - \frac{\alpha_2 \gamma_2}{\beta_3 - \lambda - \frac{\alpha_3 \gamma_3}{\beta_4 - \lambda - \cdots}}}. \] (45)
This fraction can be continued to a certain level, and then we use the fact that the ratio $b_{k+1}/b_k \to 0$ as $k \to \infty$ to ignore subsequent terms. This is an equation of the form $\lambda = g(\lambda)$ which can be solved iteratively, by starting with a guess for $\lambda$, evaluating the right-hand side of (45), using this as a new estimate of $\lambda$ and repeating.
7. Asymptotics

Below we show that the new three-term recurrence relation (40) may be used to study the eigenspectrum in the small-\(a_\omega, a_\mu\) limit. To understand the opposite limit, \(a_\omega, a_\mu \to \infty\), we recall some key results in the literature.

7.1. Small \(a_\omega\) and \(a_\mu\)

For small values of \(a_\omega\) and \(a_\mu\) one may express the separation constant \(\lambda\) as a power series.

We start with the continued fraction equation in the form

\[
\beta_j - \alpha_j - \frac{1}{\gamma_j} \beta_{j-1} - \cdots \left( \frac{\alpha_m Y_{m+1}}{\beta_m} \right) = \frac{\alpha_{j+1} Y_{j+1}}{\beta_{j+1}} - \cdots \beta_{j+2} - \cdots \tag{46}
\]

and expand the separation constant as a Taylor series,

\[
\lambda_{j,m} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \Lambda_{pq} (a_\omega)^p (a_\mu)^q. \tag{47}
\]

By grouping together like powers of \(a_\omega\) and \(a_\mu\) we obtain expansion coefficients,

\[
\Lambda_{00} = \mathcal{P}(j + 1/2),
\]

\[
\Lambda_{10} = -\frac{1}{2} \mathcal{P} m K_0^0(j),
\]

\[
\Lambda_{01} = \frac{1}{2} m K_0^0(j),
\]

\[
\Lambda_{20} = \mathcal{P} [H(j) + H(j + 1)],
\]

\[
\Lambda_{02} = \Lambda_{20},
\]

\[
\Lambda_{30} = \frac{1}{2} \mathcal{P} m [K_1^0(j) H(j) + K_1^0(j + 1) H(j + 1)],
\]

\[
\Lambda_{03} = \frac{1}{2} m [K_1^0(j) H(j) - K_1^0(j + 1) H(j + 1)],
\]

\[
\Lambda_{11} = 2[H(j + 1) - H(j)],
\]

\[
\Lambda_{21} = \frac{1}{2} m [(2K_1^0(j + 1) - K_1^0(j + 1)) H(j + 1) - (2K_1^0(j) - K_1^0(j)) H(j)],
\]

\[
\Lambda_{12} = \frac{1}{2} \mathcal{P} [(K_1^0(j + 1) - 2K_1^0(j + 1)) H(j + 1) + (K_1^0(j) - 2K_1^0(j)) H(j)],
\]

where (for compactness) we have defined the functions

\[
H(k) = (k^2 - m^2)/(8k^3),
\]

\[
K_0^\pm(k) = 1/k \pm 1/(k + 1),
\]

\[
K_1^\pm(k) = 1/[(k + 1)(k - 1)] \pm 1/k^2.
\]

If desired, the series may be continued to higher orders with the aid of a symbolic algebra package. It is straightforward to confirm that the expansion coefficients are consistent with the exact eigenvalues for \(a_\omega = \pm a_\mu\) presented in section 5 and expansions for the massless case \((a_\mu = 0)\) given in, e.g. [3]. In section 8, we validate against an alternative series expansion in powers of \((a_\omega \pm a_\mu)\) obtained in [34, 36].
7.2. Large \( a \omega \), massless \( a \mu = 0 \)

In the massless case (\( a \mu = 0 \)), asymptotic results for the eigenvalue in the large-\( a \omega \) limit were obtained by Breuer et al. In our notation,

\[
\lambda^2 = 2(q - m) a \omega + A_0 + A_1/(a \omega) + A_2/(a \omega)^2 + A_3/(a \omega)^3 + \mathcal{O}((a \omega)^{-4}),
\]

where

\[
A_0 = -\frac{1}{4} [q^2 - m^2 + 2s + 1],
\]

\[
A_1 = -\frac{1}{8} [q^3 - m^2 q + q - s^2(q + m)],
\]

\[
A_2 = -\frac{1}{64} (5q^4 - (6m^2 - 10)q^2 + m^4 - 2m^2 - 4s^2(q^2 - m^2 - 1) + 1)
\]

and \( s = -1/2 \) and \( A_3 \) is given in [46]. Breuer et al left the parameter \( q \) undetermined. Casals and Ottewill [47] showed that \( q \) may be obtained by counting the number of zeros of the solution. We obtain the best match between (61) and our numerical results when we take \( q = j + z \), where \( z = 0 \) if \(|j - m|\) is even and \( z = 1 \) otherwise (see figure 2). However, we are unable to find a suitable asymptote for the \( m = j \) mode.

7.3. Large \( a \omega \) and \( a \mu \)

The solutions with \( a \mu = 0 \) are a special case of the asymptotic behaviour. This is most easily seen from the CP equation, equation (4). When \( a \mu = 0 \), the terms quadratic in \( a \omega \) can be combined into a single term \(-a^2 \omega^2 \sin^2 \theta S\). As a result, if the eigensolutions are confined to a region near \( \theta = 0 \) of size \( \Delta \theta \lesssim 1/\sqrt{a \omega} \), this term is actually \( O(a \omega) \) and hence we expect \( \lambda^2 \sim a \omega \). It is clear from the results of Breuer et al. [46], given in equation (61), that this is indeed the case for the asymptotic \( a \mu = 0 \) solutions.

If instead we consider the limit \( a \omega \rightarrow \infty \), with the ratio \( r = \mu/\omega \neq 0 \) fixed, there is an additional quadratic term, \(-a^2 \omega^2 r^2 \cos^2 \theta S\), in equation (4). It is not possible for both this term and the \(-a^2 \omega^2 \sin^2 \theta S\) term to be small simultaneously, and so we expect \( \lambda^2 \sim a^2 \omega^2 \) in this case. This is borne out by our numerical results. Writing \( \lambda = a \omega \tilde{\lambda}(r) + \cdots \), the asymptotic solutions \( \tilde{\lambda}(r) \) may be found by taking the limit \( a \omega \rightarrow \infty \) of the recurrence relation, equations (40)–(43)

\[
\tilde{a}_k b_{k+1} + \tilde{b}_k b_k + \tilde{y}_k b_{k-1} = 0, \quad k = |m|, |m| + 1, \ldots ,
\]

where

\[
\tilde{a}_k = (r + \epsilon_k) \sqrt{(k + 1)^2 - m^2} / (2k + 1),
\]

\[
\tilde{b}_k = -\epsilon_k (k + 1/2) m / k(k + 1) + m / 2k(k + 1) - \tilde{\lambda}(r),
\]

\[
\tilde{y}_k = (r - \epsilon_k) \sqrt{k^2 - m^2} / 2k
\]

and \( \epsilon_k = (-1)^{k-1} \mathcal{P} \) as before. We can solve this recurrence using the same method outlined in section 6. The results are shown in figure 1 for \( 0 < r \lesssim 1 \). We note that as \( r \rightarrow 1 \), all the positive parity modes converge at \( \tilde{\lambda} = 1 \) and the negative parity modes converge at \( \tilde{\lambda} = -1 \), which is consistent with the \( a \omega \rightarrow \infty \) limit of the exact solution for \( r = 1 \), equation (25). We have verified numerically that the eigenvalues of the full problem do indeed have a linear slope asymptotically, and that this slope is correctly predicted by the asymptotic solutions shown in figure 1.
Figure 1. Solutions to the asymptotic problem. Lowest modes of the solution to the asymptotic recurrence relation, equations (65)–(68), for $m = 1/2$ (left panel) and $m = -1/2$ (right panel), as a function of the fixed mass-to-frequency ratio $r = \mu/\omega$. Modes with $\tilde{\lambda}(r) > 0$ have parity $P = 1$ and those with $\tilde{\lambda}(r) < 0$ have parity $P = -1$.

8. Alternative methods

As discussed briefly in section 2, alternative methods for finding eigensolutions exist in the literature, although some mistakes are also present. Here we seek to review and clarify the present situation.

Suffern, Fackerell and Cosgrove (hereafter SFC) [34] showed that the eigenfunctions could be expressed as an infinite series of hypergeometric $_{2}F_{1}$ functions and derived a three-term recurrence relation for the series coefficients ([34], equation (16)). They showed that the series terminates for the special case $a\omega = -a\mu$, and derived an exact result compatible with (29) in this case. Their three-term relation can be solved via the continued-fraction method to determine $\lambda$. We have verified that the eigenvalues calculated via SFC’s recurrence relation agree with the eigenvalues calculated via (33). A key result presented in SFC was a series expansion of the form

$$\lambda = \sum \sum C_{rs} (a\sigma - a\mu)^{r} (a\sigma + a\mu)^{s}.$$  

(69)

where $\sigma = -\omega$. Unfortunately, the expansion coefficients presented in table 1 of [34] are not all correct (for example, diagonal elements $C_{nn}$ should be zero [36]). In the second and fourth tables of table 2 of [34], the eigenvalues are only accurate to one decimal place for $m = \pm 1/2$ and $j = 3/2$ and $j = 1/2$.

A five-term recurrence relation was calculated by Chakrabarti [37]; however, we believe the result should be treated with caution because the numerical eigenvalues presented are incorrect. For example, in tables 1, 2a, 2b and 3 in [37], the eigenvalues are wrong, except for the special case $a\mu = 0$. There seems to be a tacit assumption made that the spectrum is symmetric about zero (i.e. that if $\lambda$ is an eigenvalue then so is $-\lambda$), which is not the case for $a\mu \neq 0$. The correct symmetries of the spectrum are given in equation (18).

Kalnins and Miller [13] also obtained a three-term recurrence relation, and a series expansion in powers of $a$, namely $\lambda = \sum_{r=0}^{\infty} \lambda_{r} a^{r}$. Ranganathan [48] presented exact solutions for the special case $\omega = \pm \mu$, $m = \pm j$, which are compatible with those given in section 5 and in SFC [34].
Batic, Schmid and Winklmeier [36] (hereafter BSW) recently showed that the eigenvalues of (4) satisfy a first-order partial differential equation,

\[(\tilde{\mu} - 2\tilde{\nu}\lambda) \frac{\partial \tilde{\mu}}{\partial \tilde{\mu}} + (\tilde{\nu} - 2\tilde{\mu}\lambda) \frac{\partial \tilde{\nu}}{\partial \tilde{\nu}} + 2m\tilde{\mu} + 2\tilde{\mu}\tilde{\nu} = 0,\]

(70)

where \(\tilde{\mu} = a\mu\) and \(\tilde{\nu} = -a\omega\). Using this PDE, BSW found a method for calculating the series coefficients \(C_{rs}\) in SFCs expansion (69). BSW point out that the method applied by SFC to determine the series coefficients is plagued by divide-by-zero problems at certain orders \((r, s)\). BSW described a method for avoiding the problem, allowing the series expansion to be taken to arbitrary order. We have checked that the expansion coefficients given in [36] are in full agreement with our expansion, equations (48)–(57). The nonlinear PDE (70) may be solved by the method of characteristics, and BSW showed that the characteristic equations can be reduced to a Painlevé III equation [36].

Recently, Batic and Nowakowski [49] (hereafter BN) derived an ordinary differential equation for the eigenvalues at fixed mass-to-frequency ratio \(r = \mu/\omega\), using analytic perturbation theory. This equation was

\[\frac{d\lambda}{da} = \frac{2(\alpha_\omega - m_j)(2\lambda\alpha_\omega - a\mu)}{a(4\lambda^2 - 1)}.\]

(71)

Unfortunately, this equation is not correct (see discussion below). In the corrected calculation, the eigenvalue is given once again by a partial differential equation, so a solution is not so easily obtained.

We have found that it is possible to adapt the approach of BN in order to derive an alternative recurrence relation for the angular eigenvalues. Following the same notation as BN, we write \(S_+ = \tilde{g}_1/\sqrt{\sin \theta}, S_- = \tilde{g}_2/\sqrt{\sin \theta}, k = -m, m_e = \mu\) and \(M = a\). In deriving the ODE (71), BN introduced functions \(U = \tilde{g}_1^2 + \tilde{g}_2^2, V = \tilde{g}_2^2 - \tilde{g}_1^2\) and \(W = 2\tilde{g}_1\tilde{g}_2\), which obey the differential equations

\[U'(\theta) = -2 f(\theta)V(\theta) + 2Mm_e \cos \theta W(\theta),\]

\[V'(\theta) = -2 f(\theta)U(\theta) + 2\lambda W(\theta),\]

\[W'(\theta) = 2Mm_e \cos \theta U(\theta) - 2\lambda V(\theta).\]

These are equations (3.5)–(3.7) of BN, but we have corrected a factor of 2 in equation (72). BN stated correctly that \(d\lambda/da = m_l I_1 + \omega I_2\), where

\[I_1 = \int_0^\pi \cos \theta V(\theta) d\theta, \quad I_2 = \int_0^\pi \sin \theta W(\theta) d\theta.\]

(75)

BN went on to derive three equations for \(I_1, I_2\) and a third integral, \(I_3 = \int_0^\pi \cos^2 \theta U(\theta) d\theta\). However, the sign of \(\omega\) on the left-hand side of their third equation (number 3.15) was wrong, with the consequence that they appeared to obtain three independent equations, whereas in reality the third equation was a linear combination of the first two. The fact that their original solution was not correct was indicated by the consequence that \(I_3 \equiv 0\), while \(I_3\) is the integral of the square of a real function and must therefore be greater than zero.

It is possible to adapt and extend their approach as follows. We introduce two families of integrals, which generalize the integrals used in BN

\[J_n = 2 \int_0^\pi \cos^n \theta \tilde{g}_1^2(\theta) d\theta, \quad L_n = 2 \int_0^\pi \sin \theta \cos^{2n} \theta \tilde{g}_1(\theta)\tilde{g}_2(\theta) d\theta.\]

(76)
Due to the symmetry of the eigenfunctions $\tilde{g}_1(\pi - \theta) = \tilde{g}_2(\theta)$, and the fact that they are normalized such that $\int_0^\pi \tilde{g}_i^2 \, d\theta = 1$, we see that $J_0 = 2$, $J_1 = I_1$, $L_1 = I_2$ and $J_2 = I_3$. Multiplying equation (72) by $\sin \theta \cos^{2n+1} \theta$ and integrating we find
\[(2n - 1)J_{2n-2} - 2nJ_{2n} = 2Mm_e L_n + 2M\omega J_{2n+1} - 2(M\omega + k)J_{2n-1}. \tag{77}\]

Multiplying equation (73) by $\sin \theta \cos^{2n} \theta$ and integrating gives
\[2nJ_{2n-1} - (2n + 1)J_{2n+1} = 2M\omega J_{2n+2} - 2(M\omega + k)J_{2n} + 2\lambda L_n. \tag{78}\]

Finally, multiplying equation (74) by $\cos^{2n+1} \theta$ and integrating gives
\[(2n + 1)L_n = 2Mm_e J_{2n+2} - 2\lambda J_{2n+1}. \tag{79}\]

These three relations together can be solved recursively to obtain all of the integrals. For eigenvalues of the system, $J_n, L_n \to 0$ as $n \to \infty$, and this can be used to solve the recurrences. Combining the three equations gives a five-term recurrence for $\lambda$
\[(4n^2 - 1)J_{2n-2} + 2(2n + 1)(M\omega + k)J_{2n-1} - 2n(2n + 1)J_{2n} + (2(2n + 1)\lambda - 2M\omega)J_{2n+1} - 4M^2 m_e^2 J_{2n+2} = 0 \tag{80}\]
or a four-term recurrence that now depends on $\lambda^2$
\[2n(2n + 1)J_{2n-1} + 2(2n + 1)(M\omega + k)J_{2n} - ((2n + 1)^2 - 4\lambda^2)J_{2n+1} - (2(2n + 1)M\omega + 4Mm_e\lambda)J_{2n+2} = 0 \tag{81}\]

It is clear from the above that it is possible to use the approach of BN to derive the eigenvalues, without resorting to a spectral decomposition. However, the result is a recurrence relation and the nice analytic solution for the eigenvalue derived in the original version of their paper is lost. While it may be possible to modify their approach further to obtain the eigenvalue exactly, we have so far been unable to do this. The recurrences in equations (80) and (81) are more difficult to work with than the three-term recurrence in equation (40) and so we have used expression (40) to compute all the results presented in section 9.

9. Results

In this section, we present some numerical results for the eigenvalues and eigenfunctions of equations (2) and (3). These were computed using the three-term recurrence relation, equations (40)–(43), and the series expansion in spin-half spherical harmonics (30). We have cross-checked against results from the other techniques (section 8). As before we separate variables using the ansatz $\Psi(t, r, \theta, \phi) = e^{-i\omega t} e^{i\mu \phi} \Psi_{\text{om}}(r, \theta)$.

9.1. Eigenvalues

Figure 2 shows eigenvalues for the massless case, $\mu = 0$, for the modes $j = 1/2, \ldots, 11/2$ and parity $\mathcal{P} = +1$. The eigenvalues are split on azimuthal number $m$. For positive $a\omega$, the lowest-lying eigenvalue is the $m = j$ mode, which tends to zero as $a\omega \to \infty$. For negative $a\omega$, the symmetries (18) mean that the spectrum looks the same but the ordering of eigenvalues in $m$ is reversed (i.e. the $m = -j$ mode is lowest-lying). It also follows from (18) that, in the massless case $\mu = 0$, the eigenvalues of negative parity $\mathcal{P} = -1$ are found by inversion, $\lambda \to -\lambda$.

The lower plots in figure 2 compare numerically determined eigenvalues against asymptotic results of section 7. In the small-$a\omega$ regime, we use expansion (47), taken to third order. In the large-$a\omega$ regime, we compare against the asymptotics (61). Note that there...
Figure 2. Massless spheroidal eigenvalues \((a\mu = 0)\). The top plot shows the eigenvalues \(\lambda\) for modes \(j = 1/2, \ldots, 11/2, m = -j, \ldots, j\) in the parameter range \(a\omega = 0, \ldots, 8\) and \(a\mu = 0\). For positive \(a\omega\), the \(m = +j\) eigenvalue is smallest; it tends to zero as \(a\omega \to \infty\). The massless spectrum \((a\mu = 0)\) is symmetric under \(\lambda \to -\lambda\). The lower plots compare numerical results (solid) with approximations (dotted) valid in the small-\(a\omega\) (left) and large-\(a\omega\) limits, given by equations (47) and (61).

Figure 3 compares the eigenvalue spectrum for the case \(a\omega = a\mu\) (equation (25)) with the massless \((a\mu = 0)\) spectrum. The \(j = 5/2\) modes are shown here; other modes follow a similar pattern. Observe that the \(m = j, P = -1\) eigenvalue crosses the \(\lambda = 0\) line at high \(a\omega\), but no other eigenvalue changes sign. This can be understood by examining the exact result, equation (25). For \(P = -1, m = -j\) we obtain \(\lambda = -(j + 1/2) + a\omega\) and hence \(\lambda = 0\) when \(a\omega = j + 1/2\). Likewise, the \(m = +j\) eigenvalue is zero when \(a\omega = -(j + 1/2)\). For all other states, the term under the square root in (25) is non-zero for all \(a\omega \geq 0\), and hence the eigenvalue does not change sign.

The zeros of the eigenvalue \(\lambda\) are worth some further consideration. Figure 4 shows that, for each \(j\), the \(m = j, P = -1\) mode has zero eigenvalue along a line in the quadrant \(a\omega > 0, a\mu > 0\). We find no evidence to suggest that the eigenvalue of any other \(m, P\) mode passes through zero in this quadrant. The zeros in the other quadrants follow immediately from the symmetries of the spectrum (equation (18)).

In Appendix B, we present some tables of numerically determined eigenvalues, for the \(j = 1/2\) and \(j = 3/2\) modes in the range \(0 < a\omega < 1, 0 < \mu/\omega < 1\). Our intention is to provide a resource for checking and validating future studies.
Figure 3. Eigenvalues for \( j = 5/2, m = -j, \ldots, j \) and \( P = \pm 1 \). The solid lines show the \( \mu = \omega \) eigenvalues (equation (25)) for positive (top/red) and negative (bottom/blue) parities \( P \). The dotted black lines show the massless eigenvalues \( \mu = 0 \), also shown in figure 2. The eigenvalue of the \( a\mu = a\omega, m = j = 5/2, P = -1 \) mode crosses the axis at \( a\omega = 3 \) (i.e. \( \lambda_{3/2,5/2,-1} = 0 \)).

Figure 4. Contours in the \( a\omega-a\mu \) plane along which \( \lambda = 0 \). The three lines show maximally co- and counter-rotating modes with \( j = 1/2, j = 3/2 \) and \( j = 5/2 \) (dashed red, dotted blue, fine dotted green). The plot illustrates the symmetries of the eigenvalue spectrum, given in equation (18).

9.2. Eigenfunctions

Figures 5 and 6 show the eigenfunctions \( S_+ (\theta) \) (red) and \( S_- (\theta) \) (blue) of the \( j = 1/2 \) and \( j = 3/2 \) modes, for a range of \( a\omega \geq 0 \) and \( a\mu \geq 0 \). Figure 5 shows the massless spheroidal
harmonics \((a\mu = 0)\). The solid line represents the spherical \((a = 0)\) harmonics, and the broken lines represent the spheroidal harmonics with \(a\omega = 1, 2, 3\) and 4. As the rotational coupling increases, the eigenfunctions show a tendency to increase in magnitude near the poles \((\theta \sim 0^\circ, \theta \sim 180^\circ)\), and decrease in magnitude around the equatorial plane \((\theta = 90^\circ)\).

Figure 6 shows \(j = 1/2, j = 3/2\) eigenfunctions for the special case \(\omega = \mu\), in the range \(a\omega = 0 \cdots 4\). In this case, the eigenfunctions are known in closed form (27). The plot makes it clear that the eigenfunctions of the \(P = +1, m = \pm j, a\mu = a\omega\) mode do not depend on \(a\omega\). For the other modes, we again see a general ‘enhancement’ towards the poles.

The eigenfunctions plotted in figures 5 and 6 exhibit the symmetries stated in equations (19) and (20). For example, the reflection symmetry \(S_+ (\theta) = P(-1)^{j+m} S_-(\pi - \theta)\) is obvious.
Figure 6. Angular solutions $S_{+}(\theta)$ (red) and $S_{-}(\theta)$ (blue) for $a\omega = a\mu$. The top four plots show the $j = 1/2$ modes, and the bottom eight plots show the $j = 3/2$ modes, as a function of polar angle $\theta$. The co-rotating modes $m > 0$ are on the left, and the counter-rotating modes $m < 0$ are on the right. Plots come in pairs: one for positive parity ($P = +1$) and negative parity ($P = -1$). The solid line shows the solution for $a\omega = 0$. The broken lines show the solutions for $a\omega = 1.0, 2.0, 3.0$ and $4.0$. (Colour in online version)

In the massless case (figure 5), there is also a clear symmetry under parity change $P = -P$; whereas in the general case, this only holds if we simultaneously flip the sign of the mass ($\mu \rightarrow -\mu$). Solutions for arbitrary $a\omega, a\mu$ can be found by applying the symmetries (19)–(21) to the solutions in the first quadrant.

Figure 7 shows the influence of field mass on the shape of the eigenfunctions. Here, we plot the eigenfunctions of the $j = 1/2, j = 3/2$ modes, for mass-to-frequency ratios $\mu/\omega$ from 0 to 2. In general, it would seem that mass seems to lead to additional structure in the eigensolutions around the equatorial plane. However, we defer any attempt at physical interpretation to a future in-depth study.
10. Conclusion

We have described a technique for the determination of eigenvalues and eigenfunctions of the angular equation for the massive Dirac field on a rotating black-hole background. By carrying out a spectral decomposition of the angular eigenfunctions, we obtained a three-term recurrence relation. Exact eigenvalues can be easily and quickly obtained by numerically solving the three-term relation (40); alternatively, they may be estimated from series expansion (47). We have described two alternative methods to obtain the eigenfunctions, including an alternative set of recurrence relations that were derived in an attempt to correct one of the previous attempts to obtain a solution for the eigenvalues [49]. We have tabulated eigenvalues for a range of values of the spin of the central black hole and the mass of the fermion field (tables 1–6), and derived fits which reproduce the numerical results to high precision (tables 7–12).

The motivation for this work was the need for angular eigenvalues as input for two separate studies—an investigation of bound states in the massless Kerr–Newman background [8] and a study of the spectrum of quasi-bound states in the vicinity of a small black hole [9, 10]. The solution to the radial equation for these two problems will be presented elsewhere, together with an estimate of the error arising from using fitting function equation (B.1) in place of exact angular eigenvalues.

We hope this work will also be of benefit to other studies, for example, investigations into (i) the effect of field mass on fermionic quasi-normal ringing of black holes; (ii) the effect of particle mass on the Hawking radiation emission spectrum for temperatures close to the mass of the particle species; and (iii) the interaction of brane-localized fermions with rotating higher-dimensional black holes.
Table 1. Eigenvalues $\lambda$ for $j = 1/2, m = \pm 1/2$ and $P = +1$. In each row, the top line is the $m = +1/2$ eigenvalue and the lower line is the $m = -1/2$ value. Here, $R$ is the ratio between mass $\mu$ and frequency $\omega$.

| $P = +1$ | $R = \mu/\omega$ | $a\omega$ |
|----------|------------------|-----------|
| 0.0      | 0.934 097        | 0.869 818 |
|          | 1.067 385        | 1.136 116 |
| 0.2      | 0.941 098        | 0.884 521 |
|          | 1.061 036        | 1.124 021 |
| 0.4      | 0.948 159        | 0.899 463 |
|          | 1.054 745        | 1.112 158 |
| 0.6      | 0.955 278        | 0.914 640 |
|          | 1.048 514        | 1.100 528 |
| 0.8      | 0.962 457        | 0.930 048 |
|          | 1.042 342        | 1.089 133 |
| 1.0      | 0.969 694        | 0.945 683 |
|          | 1.036 229        | 1.077 973 |

Table 2. Eigenvalues $\lambda$ for $j = 1/2, m = \pm 1/2$ and $P = -1$. In each row, the top line is the $m = +1/2$ eigenvalue and the lower line is the $m = -1/2$ value. Here, $R$ is the ratio between mass $\mu$ and frequency $\omega$.

| $P = -1$ | $R = \mu/\omega$ | $a\omega$ |
|----------|------------------|-----------|
| 0.0      | -0.934 097       | -0.869 818 |
|          | -1.067 385       | -1.136 116 |
| 0.2      | -0.927 156       | -0.855 357 |
|          | -1.073 792       | -1.148 441 |
| 0.4      | -0.920 276       | -0.841 142 |
|          | -1.080 258       | -1.160 994 |
| 0.6      | -0.913 456       | -0.827 176 |
|          | -1.086 781       | -1.173 773 |
Table 2. (Continued)

\[ P = -1 \]

\[ R = \mu/\omega \]

|   | 0.1  | 0.2  | 0.3  | 0.4  | 0.5  |
|---|------|------|------|------|------|
| 0.8 | – 0.906 697 | – 0.813 461 | – 0.720 298 | – 0.627 215 | – 0.534 219 |
| | – 1.093 362 | – 1.186 776 | – 1.280 236 | – 1.373 736 | – 1.467 268 |
| 1.0 | – 0.900 000 | – 0.800 000 | – 0.700 000 | – 0.600 000 | – 0.500 000 |
| | – 1.100 000 | – 1.200 000 | – 1.300 000 | – 1.400 000 | – 1.500 000 |

\[ \omega = \mu/\omega \]

|   | 0.6  | 0.7  | 0.8  | 0.9  | 1.0  |
|---|------|------|------|------|------|
| 0.0 | – 0.631 876 | – 0.577 928 | – 0.526 476 | – 0.477 646 | – 0.431 544 |
| | – 1.422 021 | – 1.495 674 | – 1.569 992 | – 1.644 879 | – 1.720 244 |
| 0.2 | – 0.580 627 | – 0.515 589 | – 0.452 221 | – 0.390 625 | – 0.330 897 |
| | – 1.453 910 | – 1.531 603 | – 1.609 681 | – 1.688 073 | – 1.766 714 |
| 0.4 | – 0.531 711 | – 0.456 447 | – 0.382 152 | – 0.308 942 | – 0.236 738 |
| | – 1.487 709 | – 1.570 089 | – 1.652 658 | – 1.735 371 | – 1.818 185 |
| 0.6 | – 0.485 243 | – 0.400 704 | – 0.316 603 | – 0.232 973 | – 0.149 846 |
| | – 1.523 370 | – 1.611 058 | – 1.698 816 | – 1.786 622 | – 1.874 453 |
| 0.8 | – 0.441 318 | – 0.348 520 | – 0.255 831 | – 0.163 261 | – 0.070 817 |
| | – 1.560 827 | – 1.654 406 | – 1.747 998 | – 1.841 598 | – 1.935 199 |
| 1.0 | – 0.400 000 | – 0.300 000 | – 0.200 000 | – 0.100 000 | 0.000 000 |
| | – 1.600 000 | – 1.700 000 | – 1.800 000 | – 1.900 000 | – 2.000 000 |

Table 3. Eigenvalues for \( j = 3/2, m = \pm 3/2 \) and \( P = +1 \).

\[ P = +1 \]

\[ R = \mu/\omega \]

|   | 0.1  | 0.2  | 0.3  | 0.4  | 0.5  |
|---|------|------|------|------|------|
| 0.0 | 1.920 329 | 1.841 350 | 1.763 119 | 1.685 695 | 1.609 143 |
| | 2.080 312 | 2.161 215 | 2.242 663 | 2.324 614 | 2.407 027 |
| 0.2 | 1.924 473 | 1.849 942 | 1.776 486 | 1.704 190 | 1.633 144 |
| | 2.076 449 | 2.153 751 | 2.231 842 | 2.310 661 | 2.390 152 |
| 0.4 | 1.928 643 | 1.858 641 | 1.790 098 | 1.723 128 | 1.657 851 |
| | 2.072 612 | 2.146 386 | 2.221 237 | 2.297 085 | 2.373 856 |
| 0.6 | 1.932 840 | 1.867 446 | 1.803 952 | 1.742 503 | 1.683 251 |
| | 2.068 799 | 2.139 119 | 2.210 849 | 2.283 888 | 2.358 139 |
| 0.8 | 1.937 063 | 1.876 356 | 1.818 048 | 1.762 311 | 1.709 329 |
| | 2.065 012 | 2.131 951 | 2.200 679 | 2.271 067 | 2.342 997 |
| 1.0 | 1.941 311 | 1.885 372 | 1.832 381 | 1.782 542 | 1.736 068 |
| | 2.061 250 | 2.124 881 | 2.190 725 | 2.258 623 | 2.328 427 |
| 0.6 | 1.533 529 | 1.458 925 | 1.385 410 | 1.313 063 | 1.241 970 |
| | 2.489 862 | 2.573 085 | 2.656 660 | 2.740 555 | 2.824 737 |
| 0.2 | 1.563 445 | 1.495 195 | 1.428 503 | 1.363 483 | 1.300 256 |
| | 2.470 262 | 2.550 942 | 2.632 146 | 2.713 831 | 2.795 954 |
Table 3. (Continued)

\[
P = +1
\]

| \( R = \mu/\omega \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|----------------------|-----|-----|-----|-----|-----|
| 0.4                  | 1.594 395 | 1.532 895 | 1.473 492 | 1.416 335 | 1.361 576 |
|                      | 2.451 480 | 2.529 894 | 2.609 036 | 2.688 852 | 2.769 289 |
| 0.6                  | 1.626 354 | 1.571 979 | 1.520 298 | 1.471 488 | 1.425 727 |
|                      | 2.433 514 | 2.509 932 | 2.587 316 | 2.665 597 | 2.744 709 |
| 0.8                  | 1.659 291 | 1.614 391 | 1.568 825 | 1.528 789 | 1.492 475 |
|                      | 2.416 357 | 2.491 044 | 2.566 964 | 2.644 029 | 2.722 158 |
| 1.0                  | 1.693 171 | 1.654 066 | 1.618 962 | 1.588 061 | 1.561 553 |
|                      | 2.400 000 | 2.473 214 | 2.547 950 | 2.624 100 | 2.701 562 |

Table 4. Eigenvalues for \( j = 3/2, m = \pm 3/2 \) and \( P = -1 \).

\[
P = -1
\]

| \( R = \mu/\omega \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|----------------------|-----|-----|-----|-----|-----|
| 0.0                  | -1.920 329 | -1.841 350 | -1.763 119 | -1.685 695 | -1.609 143 |
|                      | -2.080 312 | -2.161 215 | -2.242 663 | -2.324 614 | -2.407 027 |
| 0.2                  | -1.916 210 | -1.832 864 | -1.749 998 | -1.667 649 | -1.585 858 |
|                      | -2.084 199 | -2.168 777 | -2.253 701 | -2.338 945 | -2.424 480 |
| 0.4                  | -1.912 118 | -1.824 486 | -1.737 125 | -1.650 054 | -1.563 297 |
|                      | -2.088 112 | -2.176 436 | -2.264 955 | -2.353 651 | -2.442 509 |
| 0.6                  | -1.908 053 | -1.816 216 | -1.724 500 | -1.632 913 | -1.541 465 |
|                      | -2.092 050 | -2.184 194 | -2.276 423 | -2.368 731 | -2.461 110 |
| 0.8                  | -1.904 013 | -1.808 054 | -1.712 125 | -1.616 228 | -1.520 366 |
|                      | -2.096 012 | -2.192 048 | -2.288 106 | -2.384 182 | -2.480 276 |
| 1.0                  | -1.900 000 | -1.800 000 | -1.700 000 | -1.600 000 | -1.500 000 |
|                      | -2.100 000 | -2.200 000 | -2.300 000 | -2.400 000 | -2.500 000 |

| \( R = \mu/\omega \) | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|----------------------|-----|-----|-----|-----|-----|
| 0.0                  | -1.533 529 | -1.458 925 | -1.385 410 | -1.313 063 | -1.241 970 |
|                      | -2.489 862 | -2.573 085 | -2.656 660 | -2.740 555 | -2.824 737 |
| 0.2                  | -1.504 668 | -1.424 125 | -1.344 277 | -1.265 176 | -1.186 878 |
|                      | -2.510 280 | -2.596 321 | -2.682 579 | -2.769 030 | -2.855 652 |
| 0.4                  | -1.476 877 | -1.390 819 | -1.305 151 | -1.219 898 | -1.135 092 |
|                      | -2.531 512 | -2.620 646 | -2.709 896 | -2.799 247 | -2.888 686 |
| 0.6                  | -1.450 166 | -1.359 027 | -1.268 059 | -1.177 273 | -1.086 681 |
|                      | -2.553 551 | -2.646 048 | -2.738 593 | -2.831 179 | -2.923 800 |
| 0.8                  | -1.424 541 | -1.328 755 | -1.233 011 | -1.137 312 | -1.041 660 |
|                      | -2.576 385 | -2.672 507 | -2.768 640 | -2.864 783 | -2.960 933 |
| 1.0                  | -1.400 000 | -1.300 000 | -1.200 000 | -1.100 000 | -1.000 000 |
|                      | -2.600 000 | -2.700 000 | -2.800 000 | -2.900 000 | -3.000 000 |
Table 5. Eigenvalues for $j = 3/2$, $m = \pm 1/2$ and $P = +1$.

| $P = +1$ | $a \omega$ |
|---------|-------------|
| $R = \mu/\omega$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| 0.0 | 1.974 582 | 1.951 771 | 1.931 737 | 1.914 653 | 1.900 690 |
| 2.027 860 | 2.058 002 | 2.090 270 | 2.124 517 | 2.160 601 |
| 0.20 | 1.975 854 | 1.954 168 | 1.935 077 | 1.918 712 | 1.905 206 |
| 2.026 477 | 2.055 157 | 2.085 916 | 2.118 632 | 2.153 187 |
| 0.40 | 1.977 225 | 1.956 967 | 1.939 328 | 1.924 407 | 1.912 300 |
| 2.025 190 | 2.052 692 | 2.082 402 | 2.114 216 | 2.148 033 |
| 0.60 | 1.978 695 | 1.960 169 | 1.944 501 | 1.931 764 | 1.922 023 |
| 2.023 999 | 2.050 604 | 2.079 720 | 2.111 253 | 2.145 187 |
| 0.80 | 1.980 265 | 1.963 777 | 1.950 603 | 1.937 641 | 1.925 641 |
| 2.021 904 | 2.047 548 | 2.076 820 | 2.109 598 | 2.145 751 |
| 1.00 | 1.981 935 | 1.967 793 | 1.957 641 | 1.949 490 |
| 2.020 478 | 2.047 548 | 2.076 820 | 2.109 598 | 2.145 751 |

Table 6. Eigenvalues for $j = 3/2$, $m = \pm 1/2$ and $P = -1$.

| $P = -1$ | $a \omega$ |
|---------|-------------|
| $R = \mu/\omega$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| 0.0 | -1.974 582 | -1.951 771 | -1.931 737 | -1.914 653 | -1.900 690 |
| -2.027 860 | -2.058 002 | -2.090 270 | -2.124 517 | -2.160 601 |
| 0.2 | -1.973 408 | -1.949 771 | -1.929 299 | -1.912 203 | -1.898 688 |
| -2.029 340 | -2.061 228 | -2.095 471 | -2.131 886 | -2.170 299 |
| 0.4 | -1.972 333 | -1.948 166 | -1.927 750 | -1.911 326 | -1.899 125 |
| -2.030 917 | -2.064 837 | -2.101 522 | -2.140 746 | -2.182 294 |
| 0.6 | -1.971 356 | -1.946 953 | -1.927 075 | -1.911 986 | -1.901 920 |
| -2.032 590 | -2.068 830 | -2.108 428 | -2.151 104 | -2.196 590 |
| 0.8 | -1.970 476 | -1.946 126 | -1.927 259 | -1.914 140 | -1.906 981 |
| -2.034 361 | -2.073 209 | -2.116 190 | -2.162 958 | -2.213 182 |
| 1.0 | -1.969 694 | -1.945 683 | -1.928 286 | -1.917 745 | -1.914 214 |
| -2.036 229 | -2.077 973 | -2.124 808 | -2.176 305 | -2.232 051 |
Table 6. (Continued)

| $P = -1$ | $a_0$ |
|----------|-------|
| $R \equiv \mu/\omega$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| 0.0 | – 1.890 016 | – 1.882 792 | – 1.879 170 | – 1.879 284 | – 1.883 249 |
| 0.2 | – 2.198 388 | – 2.237 750 | – 2.278 567 | – 2.320 724 | – 2.364 112 |
| 0.4 | – 2.210 546 | – 2.252 476 | – 2.295 943 | – 2.340 814 | – 2.386 963 |
| 0.6 | – 1.891 353 | – 1.888 185 | – 1.889 758 | – 1.896 163 | – 1.907 439 |
| 0.8 | – 2.225 963 | – 2.271 567 | – 2.318 931 | – 2.367 895 | – 2.418 308 |
| 1.0 | – 1.897 068 | – 1.897 568 | – 1.903 499 | – 1.914 877 | – 1.931 650 |
| 2.244 637 | – 2.347 498 | – 2.401 899 | – 2.458 034 |
| 0.8 | – 1.905 924 | – 1.911 032 | – 1.922 289 | – 1.939 595 | – 1.962 780 |
| 0.6 | – 2.266 546 | – 2.322 758 | – 2.381 549 | – 2.442 671 | – 2.505 900 |
| 1.0 | – 1.917 745 | – 1.928 286 | – 1.945 683 | – 1.969 694 | – 2.000 000 |
| 2.291 647 | – 2.354 724 | – 2.420 937 | – 2.489 975 | – 2.561 553 |

Table 7. Fitting coefficients for $j = 1/2, m = \pm 1/2$ and $P = +1$. The upper line in each row is for $m = 1/2$ and the lower line for $m = -1/2$.

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| 1 | $-1.33287 \times 10^{-1}$ | $1.66774 \times 10^{-4}$ | $-1.82477 \times 10^{-4}$ | $2.69509 \times 10^{-4}$ |
| 0 | $-1.89002 \times 10^{-5}$ | $7.8727 \times 10^{-10}$ | $4.68693 \times 10^{-5}$ | $-7.76405 \times 10^{-5}$ |
| 2 | $2.90019 \times 10^{-4}$ | $7.32430 \times 10^{-2}$ | $7.62158 \times 10^{-4}$ | $-2.00106 \times 10^{-3}$ |
| 0 | $8.39768 \times 10^{-5}$ | $0.0737528$ | $-7.84339 \times 10^{-4}$ | $8.33219 \times 10^{-4}$ |
| 3 | $-3.56377 \times 10^{-4}$ | $2.97546 \times 10^{-2}$ | $-5.99395 \times 10^{-5}$ | $4.35512 \times 10^{-3}$ |
| 0 | $-1.5588 \times 10^{-6}$ | $-0.0288713$ | $8.79491 \times 10^{-5}$ | $-2.42401 \times 10^{-3}$ |
| 4 | $-1.87177 \times 10^{-5}$ | $1.61106 \times 10^{-3}$ | $4.59520 \times 10^{-3}$ | $-8.23406 \times 10^{-3}$ |
| 0 | $1.33331 \times 10^{-4}$ | $-8.50242 \times 10^{-4}$ | $4.21954 \times 10^{-3}$ | $-1.17962 \times 10^{-3}$ |

Table 8. Fitting coefficients for $j = 1/2, m = \pm 1/2$ and $P = -1$. The upper line in each row is for $m = 1/2$, and the lower line for $m = -1/2$.

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| 1 | $3.33047 \times 10^{-1}$ | $3.31222 \times 10^{-1}$ | $3.98816 \times 10^{-3}$ | $-2.03543 \times 10^{-3}$ | $2.37639 \times 10^{-4}$ |
| 2 | $-3.32884 \times 10^{-1}$ | $-3.34729 \times 10^{-1}$ | $1.82192 \times 10^{-3}$ | $-1.36801 \times 10^{-3}$ | $4.66671 \times 10^{-4}$ |
| 3 | $-2.92723 \times 10^{-1}$ | $3.12502 \times 10^{-1}$ | $-1.08338 \times 10^{-1}$ | $1.89525 \times 10^{-2}$ | $-2.68764 \times 10^{-3}$ |
| 0 | $-3.00825 \times 10^{-1}$ | $3.09671 \times 10^{-1}$ | $-8.99404 \times 10^{-2}$ | $1.03932 \times 10^{-2}$ | $-3.18256 \times 10^{-3}$ |
| 4 | $-8.06550 \times 10^{-2}$ | $6.85237 \times 10^{-3}$ | $9.79197 \times 10^{-2}$ | $-6.14738 \times 10^{-2}$ | $9.59822 \times 10^{-3}$ |
| 0 | $8.14284 \times 10^{-2}$ | $-8.29826 \times 10^{-2}$ | $3.77385 \times 10^{-2}$ | $-2.10873 \times 10^{-2}$ | $7.55711 \times 10^{-3}$ |
| 2 | $4.03322 \times 10^{-2}$ | $-5.06553 \times 10^{-2}$ | $-6.96088 \times 10^{-3}$ | $2.74214 \times 10^{-2}$ | $-8.09157 \times 10^{-3}$ |
| 0 | $-9.26754 \times 10^{-3}$ | $4.99333 \times 10^{-3}$ | $2.46876 \times 10^{-3}$ | $1.95452 \times 10^{-3}$ | $-2.47130 \times 10^{-3}$ |
Table 9. Fitting coefficients for $j = 3/2, m = \pm 3/2$ and $P = +1$. The upper line in each row is for $m = 3/2$, and the lower line for $m = -3/2$.

| $n$ | $k$ | $m = 3/2$ | $m = -3/2$ |
|-----|-----|-----------|-------------|
| 1   | 0   | 1.99945 × 10^{-1} | 7.87532 × 10^{-4} | 6.5803 × 10^{-4} |
| 1   | 1   | -2.0063 × 10^{-1} | 2.84647 × 10^{-4} | 1.8922 × 10^{-4} |
| 2   | 0   | 2.8785 × 10^{-4}  | 2.99573 × 10^{-2} | 4.57896 × 10^{-3} | 2.73991 × 10^{-3} |
| 0   | 3.7295 × 10^{-4} | 3.02612 × 10^{-2} | 2.33862 × 10^{-3} | -1.13402 × 10^{-3} |
| 3   | 0   | -4.8153 × 10^{-4} | 1.25710 × 10^{-2} | -8.94371 × 10^{-3} | 4.94845 × 10^{-3} |
| 0   | -6.35154 × 10^{-4} | -6.12322 × 10^{-3} | -3.03913 × 10^{-3} | 1.83562 × 10^{-3} |
| 4   | 0   | 2.62158 × 10^{-4} | -1.84259 × 10^{-3} | 7.31973 × 10^{-3} | -3.94239 × 10^{-3} |
| 0   | 3.25972 × 10^{-4} | -1.57711 × 10^{-3} | 3.59247 × 10^{-3} | -1.49975 × 10^{-3} |

Table 10. Fitting coefficients for $j = 3/2, m = \pm 3/2$ and $P = -1$. The upper line in each row is for $m = 3/2$, and the lower line for $m = -3/2$.

| $n$ | $k$ | $m = 3/2$ | $m = -3/2$ |
|-----|-----|-----------|-------------|
| 1   | 0   | 5.99797 × 10^{-1} | 2.00423 × 10^{-1} | 3.40585 × 10^{-3} | -4.77152 × 10^{-4} | 2.28612 × 10^{-4} |
| -6.00024 × 10^{-1} | -2.00033 × 10^{-1} | 2.87207 × 10^{-5} | 7.14085 × 10^{-5} | -6.24219 × 10^{-5} |
| 2   | -1.26255 × 10^{-1} | 1.24048 × 10^{-1} | -3.12219 × 10^{-2} | 2.82263 × 10^{-3} | -1.47819 × 10^{-3} |
| -1.27827 × 10^{-1} | 1.28410 × 10^{-1} | -3.22650 × 10^{-2} | -8.40480 × 10^{-4} | 6.80416 × 10^{-4} |
| 3   | -3.53405 × 10^{-2} | 3.91967 × 10^{-2} | -9.37469 × 10^{-3} | -5.64216 × 10^{-3} | 3.06728 × 10^{-3} |
| 3.03746 × 10^{-2} | -2.96270 × 10^{-2} | 6.62180 × 10^{-3} | 2.07091 × 10^{-3} | -1.47298 × 10^{-3} |
| 4   | 2.41821 × 10^{-4} | -1.29284 × 10^{-2} | 1.51332 × 10^{-2} | -3.26087 × 10^{-3} | -9.82596 × 10^{-4} |
| -4.08651 × 10^{-3} | 3.00329 × 10^{-3} | 2.36225 × 10^{-3} | -3.51752 × 10^{-3} | 1.39410 × 10^{-3} |

Table 11. Fitting coefficients for $j = 3/2, m = \pm 1/2$ and $P = +1$. The upper line in each row is for $m = 1/2$, and the lower line for $m = -1/2$.

| $n$ | $k$ | $m = 1/2$ | $m = -1/2$ |
|-----|-----|-----------|-------------|
| 1   | 0   | -3.33047 × 10^{-1} | 6.87003 × 10^{-2} | -3.52559 × 10^{-3} | 1.25094 × 10^{-3} | 1.90819 × 10^{-4} |
| 3.32884 × 10^{-1} | -6.52992 × 10^{-2} | -1.56447 × 10^{-3} | 8.64220 × 10^{-4} | -2.19611 × 10^{-4} |
| 2   | 2.92722 × 10^{-1} | -3.12089 × 10^{-1} | 1.53920 × 10^{-1} | -1.50603 × 10^{-2} | 5.51959 × 10^{-3} |
| 3.00824 × 10^{-1} | -3.09458 × 10^{-1} | 1.36164 × 10^{-1} | -6.81511 × 10^{-3} | 1.38445 × 10^{-3} |
| 3   | 8.06564 × 10^{-2} | -7.56215 × 10^{-3} | -8.92653 × 10^{-3} | 5.52054 × 10^{-2} | -6.37204 × 10^{-4} |
| -8.14264 × 10^{-2} | 8.25932 × 10^{-2} | -3.92363 × 10^{-2} | 1.47876 × 10^{-2} | -4.17156 × 10^{-3} |
| 4   | -4.03332 × 10^{-2} | 5.10444 × 10^{-2} | 4.71906 × 10^{-3} | -2.46746 × 10^{-2} | 6.22122 × 10^{-3} |
| 9.26669 × 10^{-3} | -4.80496 × 10^{-3} | -3.88778 × 10^{-3} | 2.34438 × 10^{-3} | -1.17466 × 10^{-4} |
Table 12. Fitting coefficients for $j = 3/2, m = \pm 1/2$ and $P = -1$. The upper line in each row is for $m = 1/2$, and the lower line for $m = -1/2$.

| $n$ | 0   | 1   | 2   | 3   | 4   |
|-----|-----|-----|-----|-----|-----|
| 1   | $1.99974 \times 10^{-1}$ | $6.65908 \times 10^{-2}$ | $2.08259 \times 10^{-4}$ | $-2.82760 \times 10^{-4}$ | $-5.93627 \times 10^{-5}$ |
|     | $-1.99866 \times 10^{-1}$ | $-6.69306 \times 10^{-2}$ | $9.28649 \times 10^{-5}$ | $1.25203 \times 10^{-4}$ | $-8.67219 \times 10^{-5}$ |
| 2   | $-1.91707 \times 10^{-1}$ | $1.92687 \times 10^{-1}$ | $-1.24489 \times 10^{-1}$ | $3.25586 \times 10^{-3}$ | $2.08439 \times 10^{-4}$ |
|     | $-1.93239 \times 10^{-1}$ | $1.94559 \times 10^{-1}$ | $-1.23591 \times 10^{-1}$ | $-8.37842 \times 10^{-5}$ | $-2.56220 \times 10^{-4}$ |
| 3   | $-1.65338 \times 10^{-2}$ | $1.26573 \times 10^{-2}$ | $-2.56257 \times 10^{-2}$ | $-2.31309 \times 10^{-3}$ | $-8.48344 \times 10^{-4}$ |
|     | $1.90423 \times 10^{-2}$ | $-2.18047 \times 10^{-2}$ | $3.85236 \times 10^{-2}$ | $-9.44174 \times 10^{-3}$ | $1.13187 \times 10^{-3}$ |
| 4   | $8.26685 \times 10^{-3}$ | $-1.46583 \times 10^{-2}$ | $7.67572 \times 10^{-3}$ | $-5.46338 \times 10^{-3}$ | $7.20356 \times 10^{-4}$ |
|     | $1.78362 \times 10^{-3}$ | $-4.28852 \times 10^{-3}$ | $2.04973 \times 10^{-3}$ | $-3.65754 \times 10^{-3}$ | $1.31285 \times 10^{-3}$ |

Acknowledgments

SD would like to thank Anthony Lasenby and Chris Doran for helpful discussions and geometric insight, and Adrian Ottewill and Marc Casals, and the Irish Research Council for Science, Engineering and Technology (IRCSET) for financial support. JG would like to thank Donald Lynden-Bell for useful discussions. JG’s work is supported by the Royal Society.

Appendix A. Clebsch–Gordan coefficients

In this section, we give explicit forms for the Clebsch–Gordan coefficients that appear in integrals (37) and (39), and present simplified expressions for $C_{k\tilde{k}}$ and $D_{k\tilde{k}}^{(1)}$. The relevant coefficients are

\[
\langle k - 1, 1m|0km\rangle = \frac{\sqrt{2[k^2 - m^2]}}{(2k)(2k - 1)},
\]

\[
\langle k, 1m|0km\rangle = \frac{m}{\sqrt{k(k + 1)}},
\]

\[
\langle k + 1, 1m|0km\rangle = -\frac{\sqrt{2[(k + 1)^2 - m^2]}}{(2k + 3)(2k + 2)}
\]

and

\[
\langle k - 1, 1\frac{1}{2}|0\frac{1}{2}\rangle = \frac{1}{2}\sqrt{\frac{2k + 1}{k}},
\]

\[
\langle k, 1\frac{1}{2}|0\frac{1}{2}\rangle = \frac{1}{2}\sqrt{\frac{1}{k(k + 1)}},
\]

\[
\langle k + 1, 1\frac{1}{2}|0\frac{1}{2}\rangle = -\frac{1}{2}\sqrt{\frac{2k + 1}{k + 1}}
\]

and

\[
\langle k - 1, 1 - \frac{1}{2}|0\frac{1}{2}\rangle = \sqrt{\frac{2k + 1}{8k}}.
\]
\[ \langle k, 1 - \frac{1}{2} | k \frac{1}{2} \rangle = -\frac{2k + 1}{\sqrt{8k(k + 1)}}, \]  
(A.8)

\[ \langle k + 1, 1 - \frac{1}{2} | k \frac{1}{2} \rangle = \sqrt{\frac{2k + 1}{8(k + 1)}}. \]  
(A.9)

Hence the integrals (37) and (39) are

\[ C_{kk'}^{(1)} = \begin{cases} \sqrt{k^2 - m^2}/(2k) & k' = k - 1 \\ m/[2(k(k + 1)] & k' = k \\ \sqrt{(k + 1)^2 - m^2}/[2(k + 1)] & k' = k + 1 \\ 0 & |k - k'| > 1. \end{cases} \]  
(A.10)

\[ P D_{kk'}^{(1)} = \begin{cases} \sqrt{k^2 - m^2}/(2k) & k' = k - 1 \\ -m(k + 1/2)/[k(k + 1)] & k' = k \\ -\sqrt{(k + 1)^2 - m^2}/[2(k + 1)] & k' = k + 1 \\ 0 & |k - k'| > 1. \end{cases} \]  
(A.11)

Appendix B. Tables of eigenvalues

Numerical eigenvalues for the \( j = 1/2 \) and \( j = 3/2 \) modes are tabulated in tables 1–6. The parameter range is the same as in [37], \( a\sigma = 0, \ldots, 1 \) and \( r = \mu/\omega = 0, \ldots, 1 \). The eigenvalues are accurate to six decimal places and have been checked using the method of SFC [34]. They disagree with the eigenvalues presented in Chakrabarti [37], tables 1, 2a, 2b and 3. It is possible to find simple polynomial fits that reproduce these eigenvalues with high precision. We take an ansatz of the form and fit it in the range \( a\omega \in [-1, 0] \), \( r = \mu/\omega \in [0, 1] \)

\[ \lambda = \lambda_0 + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} f_{nk}(a\omega)^n(1 + \mu/\omega)^k. \]  
(B.1)

The coefficients \( f_{nk} \) that reproduce the data in tables 1–6 are tabulated in tables 7–12.

References

[1] Chandrasekhar S 1976 Proc. R. Soc. Lond. A 349 571
[2] Page D N 1976 Phys. Rev. D 14 1509
[3] Berti E, Cardoso V and Casals M 2006 Phys. Rev. D 73 024013 (arXiv:gr-qc/0511111)
[4] Lynden-Bell D 2003 A Magic Electromagnetic Field (Cambridge: Cambridge University Press) pp 369–75
[5] Lynden-Bell D 2004 Phys. Rev. D 70 104021 (arXiv:gr-qc/0407066)
[6] Lynden-Bell D 2004 Phys. Rev. D 70 105017 (arXiv:gr-qc/0410109)
[7] Pekeris C L and Frankowski K 1989 Phys. Rev. A 39 518
[8] Gair J R 2005 PhD Thesis University of Cambridge
[9] Lasenby A N, Doran C J L, Pritchard J, Caceres A and Dolan S R 2005 Phys. Rev. D 72 105014 (arXiv:gr-qc/0209090)
[10] Dolan S R 2007 Phys. Rev. D 76 084001 (arXiv:0705.2880)
[11] Iyer B R and Kumar A 1978 Phys. Rev. D 18 4799
[12] Wagh S M and Dadhich N 1983 Phys. Rev. D 28 1863
[13] Kalmus E G and Miller W Jr 1992 J. Math. Phys. 33 286
[14] Chakrabarti S K and Mukhopadhyay B 2000 Mon. Not. R. Astron. Soc. 317 979 (arXiv:astro-ph/0007277)
[15] Mukhopadhyay B and Chakrabarti S K 2000a Il Nuovo Cimento B 115 885 (arXiv:astro-ph/0007253)
[16] Mukhopadhyay B and Chakrabarti S K 2000b Nucl. Phys. B 582 627 (arXiv:astro-ph/0007016)
[17] Batic D 2007 J. Math. Phys. 48 022502 (arXiv:gr-qc/060651)
[18] Page D N 1976b Phys. Rev. D 14 3260
[19] Zhou S and Liu W 2008 Phys. Rev. D 77 104021
[20] Finster F, Kamran N, Smoller J and Yau S-T 2002 Commun. Math. Phys. 230 201 (arXiv:gr-qc/0107094)
[21] Finster F, Kamran N, Smoller J and Yau S-T 2003 Adv. Theor. Math. Phys. 7 25 (arXiv:gr-qc/0005088)
[22] He X and Jing J 2006 Chin. Phys. 15 2850
[23] He X and Jing J 2006 Nucl. Phys. B 755 313 (arXiv:gr-qc/0611003)
[24] Finster F, Kamran N, Smoller J and Yau S-T 2007 J.Math. Phys. 48 022502 (arXiv:gr-qc/060651)
[25] Simone L and Will C 1992 Class. Quantum Grav. 9 963
[26] Jing J and Pan Q 2005 Nucl. Phys. B 728 109 (arXiv:gr-qc/0506098)
[27] Grain J and Barrau A 2008 Eur. Phys. J. C 53 641 (arXiv:hep-th/0701265)
[28] Kanti P 2004 Int. J. Mod. Phys. A 19 4899 (arXiv:hep-ph/0402168)
[29] Casals M, Dolan S R, Kanti P and Winstanley E 2007 J. High Energy Phys JHEP03(2007)019 (arXiv:hep-th/0608193)
[30] Rogatko M and Szyplowska A 2009 Phys. Rev. D 79 104005 (arXiv:0904.4544)
[31] Belgiorno F and Cacciatori S L 2008 arXiv:0803.2496
[32] Newman E and Penrose R 1962 J. Math. Phys. 3 566
[33] Suffern K G, Fackerell E D and Cosgrove C M 1983 J. Math. Phys. 24 1350
[34] Batic D and Schmid H 2008 Rev. Colomb. Math. 42 183 (arXiv:gr-qc/0512112)
[35] Batic D, Schmid H and Winklmeier M 2005 J. Math. Phys. 46 012504 (arXiv:math-ph/0402047)
[36] Chakrabarti S K 1984 Proc. R. Soc. Lond. A 391 27
[37] Chandrasekhar S 1983 The Mathematical Theory of Black Holes (Oxford: Oxford University Press)
[38] Winklmeier M 2008 J. Diff. Eqns 245 2145 (arXiv:0806.1866)
[39] Newman E and Penrose R 1966 J. Math. Phys. 3 566
[40] Goldberg J, Macfarlane A, Newman E, Rohrlich F and Sudarshan E 1967 J. Math. Phys. 8 2155
[41] Doran C J L, Lasenby A N, Gull S F, Somaroo S and Challinor A D 1996 Adv. Imag. Elect. Phys. 95 271 (arXiv:quant-ph/0509178)
[42] Doran C and Lasenby A 2003 Geometric Algebra for Physicists (Cambridge: Cambridge University Press)
[43] Leaver E W 1985 Proc. R. Soc. Lond. A 402 285
[44] Press W H, Teukolsky S A, Vetterling W T and Flannery B P 1992 Numerical Recipes in C 2nd edn (Cambridge: Cambridge University Press)
[45] Breuer R A, Ryan M P Jr and Waller S 1977 Proc. R. Soc. Lond. A 358 71
[46] Casals M and Oettewill A C 2005 Phys. Rev. D 71 064025 (arXiv:gr-qc/0409012)
[47] Ranganathan D 2006 arXiv:gr-qc/0604057
[48] Batic D and Nowakowski M 2008 Class. Quantum Grav. 25 225022 (arXiv:0805.4828)