On the asymptotic expansion of the correlators in the XX spin chain.

A.A.Ovchinnikov

Institute for Nuclear Research, RAS, 117312, Moscow

Abstract

Using the XX- spin chain as an example we show that in general the asymptotic expansion of the correlator at large distances is not given by the sum over the low-lying particle-hole intermediate states. Only the first two terms in the expansion agree with the predictions of the Luttinger liquid theory. The other terms are in general given by the intermediate states with the particles and holes with the quantum numbers far away from the Fermi- points (at the distances of order of the length of the chain). We argue that the whole expansion cannot be described by the Luttinger liquid theory.

1. Introduction

Calculation of correlation functions in the 1D quantum liquids or spin systems remains important problem both from theoretical and experimental points of view. Although the values of the critical exponents corresponding to the power-law decay at large distances obtained with the help of the mapping to the Luttinger model (bosonization) [1],[2] or conformal field theory [3], [4] are known for a long time, the calculation of the constants before the asymptotics (prefactors) remains an open problem. Recently some progress in this field was achieved in relating of the prefactors to the certain (lowest) formfactors of local operators by means of of the conformal field theory [5]. Using the Luttinger liquid theory (bosonization) the same results where obtained in [6]. Moreover the universal form of the particle-hole formfactors of local operators for the low-lying states was obtained [5]. The results [5] are confirmed by an explicit calculation of the leading order term of the correlator for the XXZ- spin chain in the magnetic field [7] and an explicit calculations of the formfactors for the low-lying particle-hole excitations [8] for the XXZ- spin chain in the magnetic field.

In the present paper we study different formfactors for the XX- spin chain. Our results are based on the expression for the general formfactor for the XX- spin chain obtained in Ref.[9] in the form of the product depending on the momenta of the eigenstates similar to the Cauchy determinant. We calculate both the lowest formfactors corresponding to the terms of an arbitrary order in the asymptotics of the correlator
and the particle-hole formfactors for the low-lying states corresponding to an arbitrary lowest formfactor. In each order we find the particle-hole formfactors in agreement with the predictions of the Luttinger liquid theory [5], [6]. Comparing the asymptotics of the correlator predicted by the Luttinger liquid theory with the exact results we argue that the intermediate states with the particles and holes far from the Fermi-points also give the power-law terms in the asymptotics of the correlator. Only the first two terms in the asymptotics are given by the low-energy excited states and can be described by the Luttinger liquid theory.

In Section 2 we briefly review the theory of the Luttinger liquid and present the derivation of the results [5] on the universal relations for different formfactors in the XXZ-spin chain. In Section 3 we calculate the lowest formfactors corresponding to the terms of an arbitrary order in the asymptotics of the correlator in the XX-spin chain. The expression for the correlator obtained in this section is equivalent to the contribution of the low-lying intermediate excited states described by the Luttinger liquid theory. In Section 4 we calculate the particle-hole formfactors for different terms in agreement with the predictions of the Luttinger liquid theory. Finally in Section 5 we compare the predictions of the Luttinger liquid theory for the correlator with the exact results obtained by the rigorous method.

2. Universal relations for the prefactors of the correlator.

To explain the method of the calculations let us briefly review the derivation of the scaling relations for the lowest formfactors and the expressions for the particle-hole formfactors in various 1D models [5], [6]. For definiteness as an example we will take the general XXZ-spin chain in the critical region with the Hamiltonian:

$$H = \sum_{i=1}^{L} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right),$$

where the sites $L+1$ and $1$ are coincide. It is well known (for example see [2]) that the low-energy effective theory for this Hamiltonian is given by the Luttinger liquid theory. In this model one usually defines the fields $\tilde{N}_{1,2}(x)$ corresponding to the excitations around the right and the left Fermi-points:

$$\tilde{N}_{1,2}(x) = \frac{i}{L} \sum_{p \neq 0} \frac{\rho_{1,2}(p)}{p} e^{-ipx} e^{-\alpha|p|/2}, \quad \rho_1(p) = \sum_k a_k^+ a_k, \quad \rho_2(p) = \sum_k c_k^+ c_k,$$

where $a_k(c_k)$ are the quasiparticle fermionic operators corresponding to the right(left) branch of the Luttinger model and $\alpha \to 0$ is the regularization parameter. The critical exponents of the correlators and other universal relations are determined by a single parameter $\xi$ which is defined through the ground state energy in the sector with the total number of particles and the momentum $\Delta N = \Delta N_1 + \Delta N_2, \quad \Delta Q = \Delta N_1 - \Delta N_2,$
where $\Delta N_{1,2}$ are the numbers of additional particles at the two Fermi points, according to the relation

$$\Delta E = \frac{\pi}{2L} v \left[ \xi (\Delta N)^2 + (1/\xi)(\Delta Q)^2 \right],$$

where $v$ is the speed of sound. For the XXZ-spin chain (1) the parameter $\xi$ equals $\xi = 2(\pi - \eta)/\pi$, where $\Delta = \cos(\eta)$ [10]. The general operator equation for the operator $\sigma_x^-$ of the original model have the following form:

$$\sigma_x^- = \sum_m C'_m K_1 K_2^m e^{-2p_F m x} e^{i\pi \sqrt{\xi} (N_1(x) - N_2(x))} e^{-i\pi m (1/\sqrt{\xi})(N_1(x) + N_2(x))},$$

(3)

where $C'_m$ are some non-universal constants and $K_1^\pm$ are the Klein factors - the operators which commute with the operators $\rho_{1,2}(p)$ and create the single particle at the right (left) Fermi-point if they act on the ground state. The general operator equation (3) have the following sense. The matrix elements of the operator $\sigma_x^-$ for the low-lying states in the original model (1) coincide with the matrix elements of the right-hand side of the equation (3) for the corresponding eigenstates in the Luttinger model. If at large distances the correlator $G(x) = \langle \sigma_x^+ \sigma_0^- \rangle$ is determined by the low-energy intermediate states, it is exactly equal to the correlator of the operators at the right-hand side of eq.(3) in the Luttinger liquid theory. In particular, according to this prescription, let us consider the formfactor of the operator $\sigma_x^-$ between the ground state $|t\rangle$ (the eigenstate with $M = L/2$ particles (up-spins)) and the eigenstate $|\lambda(m)\rangle$ (the eigenstate which is obtained from the ground state with $M - 1$ particles $|\lambda\rangle$ by moving $m$ particles from the left to the right Fermi-point). The matrix element at the right-hand side of eq.(3) for the Luttinger model can be easily calculated and we obtain:

$$C_m' = \langle \lambda(m) | \sigma_0^- | t \rangle (L/2\pi \alpha)^{\xi/4 + m^2/\xi}. \tag{4}$$

Calculating the correlator $G(x)$ with the help of the equation (3) and taking into account the equation (4) we get the general expression for the correlator

$$G(x) = \langle \sigma_x^+ \sigma_0^- \rangle = \sum_{m \geq 0} C_m \frac{\cos(2p_F m x)}{(L \sin(\pi x/L))^{\xi/2 + m^2/2(\xi)}}, \tag{5}$$

where the Fermi-momentum $p_F = \pi/2$ and the prefactors $C_m$ satisfy the following scaling relations for the lowest formfactors (see [5], [6]):

$$|\langle \lambda(m) | \sigma_0^- | t \rangle|^2 = \frac{(-1)^m C_m}{2 - \delta_{0,m}} \left( \frac{2}{L} \right)^{\xi/2 + m^2/2(\xi)} \tag{6}$$

where $|\lambda(m)\rangle$ is the eigenstate obtained from the ground state $|\lambda\rangle$ by creating $m$ extra particles at the right Fermi-point and removing $m$ particles from the left Fermi-point. The scaling relations (6) are used to obtain the prefactors of the correlator for the XX-spin chain in Section 3. We refer to the equations (3), (5) as to the predictions of the
Luttinger liquid theory. In fact as we shall see later the equations (3)-(6) are equivalent to taking into account the low-energy excited intermediate states in the correlator. One can argue that the contribution of the irrelevant operators to the equation (3) which also give the power-law terms in the correlator is equivalent to the contribution of the intermediate states with the particles and holes far away from the Fermi-points (high energy excited states). We show that the expression (5) does not agree with the exact result. Only the first two terms in this expansion agree with the the first two terms in the exact expansion of the correlator. That means that in general the low-energy excited states do not give the whole asymptotic expansion of the correlator. The main result of the present paper is that only the first two terms in the expansion correspond to the low-lying particle-hole excitations i.e. can be described by the Luttinger liquid theory.

Now let us present the derivation of the expressions for the particle-hole formfactors i.e. for the formfactors corresponding to the particle-hole intermediate states [5], [6]. Below we present the calculations for the particle-hole formfactors corresponding to the leading term in the asymptotics of the correlator. The particle-hole formfactors corresponding to the other terms in eq.(5) can be obtained in the same way starting from the equation (3). Let us denote by \(\langle \lambda(p_i, q_i) | \sigma_0^- | t \rangle\) the eigenstate obtained from the ground state \(\langle \lambda |\) by creating the holes with the momenta \(q_i\) and the particles with the momenta \(p_i\) with respect to the Fermi-point (i = 1,...,n) located in the vicinity of the right Fermi-point. Calculating the matrix element at the right-hand side of eq.(3) we obtain:

\[
\langle \lambda(p_i, q_i) | \sigma_0^- | t \rangle = C \langle p_i, q_i | e^{\frac{2\pi}{L} \sum_{p > 0} \frac{a_{p}^{(p)}}{p} } | 0 \rangle, \quad a = -\sqrt{\xi/2},
\]

where \(C\) is the value of the lowest formfactor \(C = \langle \lambda | \sigma_0^- | t \rangle\) and \(\langle p_i, q_i |\) is the eigenstate of the Luttinger model with \(n\) particles with the momenta \(p_i\) with respect to the Fermi-point and \(n\) holes with the momenta \(q_i\) with respect to the Fermi-point at the right branch of the Luttinger model. The matrix element at the right-hand side of the equation (7) was calculated in Ref.[11] (see Appendix):

\[
\langle p_i, q_i | e^{\frac{2\pi}{L} \sum_{p > 0} \frac{a_{p}^{(p)}}{p} } | 0 \rangle = F_a(p_i, q_i) = \det_{ij} \left( \frac{1}{p_i - q_j} \right) \prod_{i=1}^{n} f^+(p_i) \prod_{i=1}^{n} f^-(q_i),
\]

where

\[
f^+(p) = \frac{\Gamma(p + a)}{\Gamma(p)\Gamma(a)}, \quad f^-(q) = \frac{\Gamma(1 - q - a)}{\Gamma(1 - q)\Gamma(1 - a)}.
\]

In the equation (8) \(p_i\) and \(q_i\) are assumed to be integers (corresponding to the momenta \(2\pi p_i/L\) and \(2\pi q_i/L\)) and \(p_i > 0, q_i \leq 0\). Thus we have calculated the particle-hole formfactor in the form \(\langle \lambda(p_i, q_i) | \sigma_0^- | t \rangle = CF_a(p_i, q_i)\), and have shown that the constant \(C\) in front of the function \(F_a(p_i, q_i)\) is the lowest formfactor. To sum up the formfactor expansion for the correlators in the framework of the approach of Ref.[8] (where the
dependence on the particle and hole positions was found for the formfactors of the XXZ-spin chain in the magnetic field) it is important to have the formula for the sum:

$$\sum_n \sum_{p_i>0,q_i\leq0} \left| F_a(p_i, q_i) \right|^2 e^{i(p-q)2\pi x/L} = \frac{1}{(1 - e^{2\pi x/L})^2},$$  \hspace{1cm} (9)

where $p = \sum_{i=1}^n p_i$, $q = \sum_{i=1}^n q_i$ and $p_i$, $q_i$ are integers. In the same way one can consider the particle-hole excitations in the vicinity of the left Fermi- point. Calculating the corresponding matrix element at the right-hand side of eq.(3) we obtain:

$$\langle \lambda(p_i, q_i) | \sigma^-_0 | t \rangle = C \langle p_i, q_i | e^{2\pi \sum_{p<0} \frac{p^2}{p}} | 0 \rangle,$$

where $C$ is again the value of the lowest formfactor $C = \langle \lambda | \sigma^-_0 | t \rangle$. The matrix element at the right-hand side of Eq.(10) is calculated in the Appendix:

$$\langle p_i, q_i | e^{2\pi \sum_{p<0} \frac{p^2}{p}} | 0 \rangle = F_c(p_i, q_i) = \det_{ij} \left( \frac{1}{p_i - q_j} \right) \prod_{i=1}^n f^+(p_i) \prod_{i=1}^n f^-(q_i),$$  \hspace{1cm} (11)

where now

$$f^+(p) = \frac{\Gamma(-p-c)}{\Gamma(-p)\Gamma(1-c)}, \quad f^-(q) = \frac{\Gamma(1+q+c)}{\Gamma(1+q)\Gamma(c)}.$$

In the equation (11) $p_i$ and $q_i$ are assumed to be integers (corresponding to the momenta $2\pi p_i/L$ and $2\pi q_i/L$) and $p_i < 0$, $q_i \geq 0$. To sum up the formfactor expansion for the correlator it is important to have the formula for the sum:

$$\sum_n \sum_{p_i<0,q_i\geq0} \left| F_c(p_i, q_i) \right|^2 e^{i(p-q)2\pi x/L} = \frac{1}{(1 - e^{-2\pi x/L})^2},$$  \hspace{1cm} (12)

where $p = \sum_{i=1}^n p_i$, $q = \sum_{i=1}^n q_i$ and $p_i$, $q_i$ are integers. The total formfactor is a product of the two factors $F_a(p_i, q_i)$ and $F_c(p_i, q_i)$ corresponding to the right and the left Fermi-points. The particle-hole formfactors corresponding to the higher order terms in eq.(5) can be obtained in the same way starting from the equation (3).

### 3. Formfactor for the XX- spin chain and the calculation of the correlators.

Let us consider the particular case of the XX- spin chain ($\Delta = 0$):

$$H = \sum_{i=1}^L \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right),$$  \hspace{1cm} (13)

where the sites $L+1$ and 1 are coincide. The solution of the model (13) has the following form [12],[9]. We assume for simplicity $L$- to be even and $M = L/2$ to be odd ($S^z = 0$ for the ground state and $(M-1)$- even, we also assume $L$ to be even so that the ground state is not degenerate). Then each eigenstate in the sector with $M$ particles (up-spins)
is characterized by the set of the momenta \( \{ p \} = \{ p_1, \ldots, p_M \} \) such that \( p_i = 2\pi n_i/L \), \( n_i \in \mathbb{Z} \) and each eigenstate in the sector with \( M - 1 \) particles is characterized by the set of the momenta \( \{ q \} = \{ q_1, \ldots, q_{M-1} \} \), \( q_i = 2\pi (n_i + 1/2)/L \), \( n_i \in \mathbb{Z} \). The ground-state in the sector with \( M \) particles (up-spins) is given by the configuration \( \{ p \} = \{ p_1, \ldots, p_M \} \), \( p_i = 2\pi/L(i - (M + 1)/2) \), (\( M \) is odd), and the ground state in the sector with \( M - 1 \) particles \( \{ q_0 \} = \{ q_1, \ldots, q_{M-1} \} \), \( q_i^{(0)} = 2\pi/L(i - M/2) \). Equivalently one can take the shifted momenta

\[
p_i = (2\pi/L)(i), \quad i = 1, \ldots, M, \quad q_j^{(0)} = (2\pi/L)(j + 1/2), \quad j = 1, \ldots, M - 1.
\]

Thus the ground states in the sector \( S^z = 0 \) (\( M = L/2 \)) corresponds to the momenta \((2\pi/L)n\) while the eigenstates in the sector \( S^z = -1 \) (\( M = L/2 - 1 \)) corresponds to the momenta \((2\pi/L)(n + 1/2)\), where \( n \) is an integer. In terms of the sets of the momenta \( \{ p \} \) and \( \{ q \} \) the formfactor \( \psi(\{ q \}) = \langle \{ q \}|\sigma_0^-|\{ p \} \rangle \) can be represented in the following form [9]:

\[
\psi(\{ q \}) = \frac{1}{\sqrt{L}} \left( \frac{i}{L} \right)^{M-1} \left( e^{i \sum_{i=1}^{M-1} q_i} \right) \frac{\prod_{i<j} \sin((p_i - p_j)/2) \prod_{i<j} \sin((q_i - q_j)/2)}{\prod_{i,j} \sin((p_i - q_j)/2)}.
\]

Let us note that the expression (14) for the formfactor is valid both for the ground state configuration of the momenta \( \{ q \} \) and for an arbitrary excited state characterized by the momenta \( q_1, \ldots, q_{M-1} \). The advantage of the expression (14) for the formfactors in comparison with the determinant expression is that with the help of this expression one can calculate the formfactor as a function of the set of the momenta \( \{ q \} \) for the low-energy particle-hole excited states and then calculate the asymptotics of the correlator.

To calculate the formfactor corresponding to the \( m \)-th term in the expansion of the correlator (5) we should calculate the expression (14) for the set of the momenta \( \{ p \} \) corresponding to the ground state and the set of the momenta \( \{ q \} \) shifted to the right by \( m \) in the units \( 2\pi/L \) with respect to the set \( \{ q \} \) corresponding to the ground state. Clearly, only the denominator gets modified in the process of this shift. Using the fact that only the particles close to the Fermi-points change their quantum numbers \( (m << L) \) and for small arguments the sinususes can be replaced by their arguments the resulting formfactor can be easily calculated:

\[
\psi_m = \psi_0 \prod_{k=1}^{m} \left( (1/2)(3/2) \ldots (k - 1 + 1/2)(\pi/L)^k \right) \prod_{k=2}^{m} \left( (1/2)(3/2) \ldots (k - 2 + 1/2)(\pi/L)^{k-1} \right),
\]

where \( \psi_0 \) is the value of the formfactor corresponding to the ground state (in the notations of Section 3 \( \psi_0 = \langle \lambda|\sigma_0^-|t \rangle, \psi_m = \langle \lambda(m)|\sigma_0^-|t \rangle \)). Then, calculating the products we obtain:

\[
\psi_m = \psi_0 \left( \frac{\pi}{L} \right)^{m^2} \prod_{k=1}^{m-1} \left( \frac{\Gamma(k + 1/2)}{\Gamma(1/2)} \right)^2 \frac{\Gamma(m + 1/2)}{\Gamma(1/2)}.
\]
where $\psi_0$ is again the value of the lowest formfactor. The product in this equation can be expressed through the Barnes G-function defined by the relations $G(1 + z) = \Gamma(z)G(z)$, $G(1) = 1$. The formfactor takes the form:

$$
\psi_m = \psi_0 \left( \frac{\pi}{L} \right)^{m^2} \left( \frac{G(m + 1/2)}{\sqrt{\pi}G(1/2)} \right)^2 \frac{\Gamma(m + 1/2)}{(\pi)^{m-1/2}}. \tag{15}
$$

Using the expression (15) the prefactors $C_m$ (5) for $m > 0$ can be calculated:

$$
C_m = (-1)^m |\psi_0|^2 \sqrt{2L} \left( \frac{\pi}{2} \right)^{2m^2} \left( \frac{G(m + 1/2)}{\sqrt{\pi}G(1/2)} \right)^4 \left( \frac{\Gamma(m + 1/2)}{\pi^{m-1/2}} \right)^2. \tag{16}
$$

According to the formulas of Section 3 the square of the formfactor $|\psi_0|^2$ can be expressed through the prefactor of the leading asymptotics as $|\psi_0|^2 = (2/L)^{1/2}C_0$, $C_0/2\sqrt{\pi} = 0.147088...$ [9]. Taking into account the explicit expression for the leading prefactor $C_0$, $C_0 = 2^{-1/2}\pi^{3/2}(G(1/2))^4$ we find that the general prefactor takes the form:

$$
C_m = (-1)^m \frac{1}{2^{m^2-1}} \pi^{2m^2-2m+1/2} (G(m + 1/2))^4 \left( \frac{\Gamma(m + 1/2)}{\pi^{m-1/2}} \right)^2, \quad m > 0. \tag{17}
$$

The equation (17) is the central result of the present paper. In thermodynamic limit the asymptotic expansion of the correlator takes the following form:

$$
G(x) = \frac{C_0}{\sqrt{\pi}} \left( \frac{1}{x^{1/2}} + \sum_{m \geq 1} y_m \frac{\cos(\pi mx)}{x^{1/2 + 2m^2}} \right), \tag{18}
$$

where the coefficients $y_m$ are equal

$$
y_m = (-1)^m \frac{1}{2^{2m^2-1}} \left( \frac{G(m + 1/2)}{\sqrt{\pi}G(1/2)} \right)^4 \left( \frac{\Gamma(m + 1/2)}{\pi^{2m-1}} \right). \tag{19}
$$

The first coefficients are: $y_1 = -1/8$, $y_2 = 9/2^{15}$. For small values of $m$ $y_m$ decreases very fast, however for large $m$ they begin to increase very fast due to the behaviour of the function $G(m + 1/2)$. We did not have the explanation of this growth. As we will show in Section 4 the results (17)-(19) are equivalent to the contribution of the low-lying intermediate excited states to the correlator which will be compared with the exact results for the first few terms in Section 5. Let us note that the same expression could be obtained with the help of the generalized Fisher-Hartwig conjecture [13], [14].

4. Calculation of the particle-hole formfactors for the XX- spin chain.

Let us calculate the particle-hole formfactors for the state $|\lambda(m)\rangle$ (the set of the momenta $\{q\}$) corresponding to $m$ extra particles at the right Fermi-point and $m$ extra holes at the left Fermi-point. Let us start with the calculation of the formfactor with
one particle and one hole at the right Fermi-point. Let us denote the positions of the
particle and the hole with respect to the first filled level by the positive integers \( p \) and
\( h \) \((p > 0, h \geq 0)\) in the units \( 2\pi/L \). The creation of the particle-hole pair leads to the
modification of both the numerator and the denominator of the equation (14).

Substituting the corresponding set of the momenta \( \{ q \} \) into the expression (14),
taking into account the condition \( p, h \ll L \), using the property \( \sin((\pi/2 + \pi/2)/2) = 1 \),
we obtain after the cancellation of the identical terms in the numerator and the
denominator the expression

\[
\psi_m(p, h) = \psi_m \frac{(1/2)(3/2) \ldots (p + m - 1 - 1/2)(1/2)(3/2) \ldots (h - m + 1/2)}{(p + h)h!(p - 1)!},
\]

where \( \psi_m \) is the value of the lowest formfactor considered in the previous section. Note
that all factors \((\pi/L)\) are cancelled. We considered the case \( h > m \). One can prove that
the case \( h \leq m \) leads to the same expression. Introducing the hole momenta \( q = -h \leq 0 \),
the last formula takes the following form:

\[
\psi_m(p, q) = \psi_m \frac{1}{(p - q)} \frac{\Gamma(p + a)}{\Gamma(p)} \frac{\Gamma(1 - q - a)}{\Gamma(1 - q)\Gamma(1 - a)},
\]

where the parameter \( a \) equals

\[
a = -\frac{1}{2} + m.
\]

In the same way one can easily prove that for \( n \) particles and \( n \) holes the formfactor
takes the form \( \psi_m F_a(p_i, q_i) \) where the function \( F_a(p_i, q_i) \) is given by the equation (8)
with the same parameter \( a = -1/2 + m \). Thus we observe that the expression for the
total formfactor coincides with the prediction of the Luttinger liquid theory presented
in Section 3 according to the operator equation (3). In fact the equation (3) gives
\( a = -\sqrt{\xi}/2 + m/\sqrt{\xi} \) which coincides with our value of \( a \) at \( \xi = 1 \). At \( m = 0 \) the
formfactor corresponds to the leading term in the asymptotics of the correlator.

Let us proceed with the calculation of the formfactor with one particle and one hole
at the left Fermi-point. Again we denote the positions of the particle and the hole with
respect to the first filled level by \( p \) and \( h \) \((p > 0, h \geq 0)\) in the units \( 2\pi/L \). Substituting
the corresponding set of the momenta \( \{ q \} \) into the expression (14), taking into account
the condition \( p, h \ll L \), we obtain after the cancellation of the identical terms the expression

\[
\psi_m(p, h) = \psi_m \frac{(1/2)(3/2) \ldots (h + m + 1/2)(1/2)(3/2) \ldots (p - m - 2 + 1/2)}{(p + h)h!(p - 1)!},
\]

where \( \psi_m \) is the value of the lowest formfactor. We considered the case \( p > m \). One
can prove that the case \( p \leq m \) leads to the same expression. Introducing the particle
momenta $p \rightarrow -p$, $p < 0$, and the hole momenta $q = h$ the last formula takes the following form:

$$
\psi_m(p, q) = \psi_m \frac{1}{(p - q)} \frac{\Gamma(-p - c)}{\Gamma(-p) \Gamma(1 - c)} \frac{\Gamma(q + 1 + c)}{\Gamma(q + 1) \Gamma(c)},
$$

(21)

where the parameter $c$ equals

$$
c = \frac{1}{2} + m.
$$

In the same way one can easily prove that for $n$ particles and $n$ holes the formfactor takes the form $\psi_m F_c(p_i, q_i)$ where the function $F_c(p_i, q_i)$ is given by the equation (11) with the same parameter $c = 1/2 + m$. Again we observe that the expression for the total formfactor coincides with the prediction of the Luttinger liquid theory according to the operator equation (3). In fact the equation (3) gives $c = \sqrt{\xi}/2 + m/\sqrt{\xi}$ which coincides with our value of $c$ at $\xi = 1$. Let us stress once more that we have calculated the sets of the formfactors corresponding to the $m$-th term in the correlator (5). Thus for the XX- spin chain we obtained the particle-hole formfactors for the low-lying excitations in agreement with the universal formulas predicted by the Luttinger liquid theory. Note that once the particle-hole formfactors for different $m$ are obtained one can calculate the correlator using the formulas (9), (12) without mentioning the bosonization at all. Thus it is clear that the results of Section 3 for the correlator (prediction of the Luttinger liquid theory) is nothing else as the contribution of the complete set of the intermediate states corresponding to the low excitation energy. That means that in general the contributions of order $\sim 1/L$ to the formfactors at low $p_i, q_i \sim 1$ can give the contributions to the correlator which are not suppressed by a power of $L$. Equivalently, in order to obtain the whole asymptotic expansion of the correlator one should take into account the contributions of the high-energy excited states.

5. Exact expansion of the correlator.

In this section we present the method to calculate the exact asymptotic expansion of the correlator up to the terms of arbitrary order. The correlator in the XX- spin chain $G(x)$ which can be represented as a Toeplitz determinant [12]. In fact using the Jordan-Wigner transformation relating spin operators to the Fermi operators $(a_i^+, a_i) \sigma_x^+ = \exp(i\pi \sum_{l<x} n_l) a_i^+$, the correlation function $G(x)$ can be represented as the following average for over the free-fermion ground state:

$$
G(x) = \langle 0 | a_x^+ e^{i\pi N(x)} a_0 | 0 \rangle,
$$

where $N(x) = \sum_{i=1}^{x-1} n_i$. Introducing the operators, which anticommutate at different sites,

$$
A_i = a_i^+ + a_i, \quad B_i = a_i^+ - a_i, \quad A_i B_i = e^{i\pi n_i},
$$
where \( n_i = a_i^+ a_i \) - is the fermion occupation number, with the following correlators with respect to the free-fermion vacuum,

\[
\langle 0 | B_i A_j | 0 \rangle = 2G_0(i - j) - \delta_{ij}, \quad \langle 0 | A_i A_j | 0 \rangle = 0, \quad \langle 0 | B_i B_j | 0 \rangle = 0,
\]

where \( G_0(i - j) \) is the free-fermion Green function, one obtains the following expression for the bosonic correlator:

\[
G(x) = \frac{1}{2} \langle 0 | B_0( A_1 B_1)(A_2 B_2) \cdots (A_{x-1} B_{x-1}) A_x | 0 \rangle.
\]

Clearly in the thermodynamic limit the correlator is given by the determinant of the Toeplitz matrix \( M(i - j) \):

\[
G(x) = \frac{1}{2} \det_{ij} (M(i - j)), \quad i, j = 1 \ldots x,
\]

where the matrix \( M(i - j) \) corresponds to the following generating function: \( f(x) = e^{ix\text{sign}(\pi/2 - |x|)}, -\pi < x < \pi. \) Due to the form of the matrix \( M(i - j) \) (\( M(l) = 0 \) for \( l \)- odd) the correlator \( G(x) \) can be represented in the following form [12]:

\[
G(x) = \frac{1}{2} (R_N)^2, \quad (x = 2N), \quad G(x) = \frac{1}{2} R_N R_{N+1}, \quad (x = 2N + 1), \quad (22)
\]

where \( R_N \) is the determinant of the \( N \times N \) matrix of the following form:

\[
R_N = \det_{ij} \left( (-1)^{i-j}G_0(2i - 2j - 1) \right), \quad i, j = 1 \ldots N.
\]

Since \( R_N \) is the Cauchy determinant in thermodynamic limit we obtain the following expression:

\[
R_N = \left( \frac{2}{\pi} \right)^N \prod_{k=1}^{N-1} \left( \frac{(2k)^2}{(2k+1)(2k-1)} \right)^{N-k} . \quad (23)
\]

The sum corresponding to the logarithm of the determinant \( R_N \) was calculated in Ref.[15] in the context of the Ising model in the following form:

\[
\ln(R_N) = \ln A - \frac{1}{4} \ln N + \sum_{k=2}^{\infty} \frac{(2^{2k} - 1)B_{2k}}{k(k-1)2^{2k}N^{2(k-1)}}, \quad (24)
\]

where \( A = 2^{1/12}e^{3\zeta(-1)} = 0.6450024... \) and \( B_{2k} \) are the Bernoulli numbers. Using the expression (24) one can obtain from the equation (22) an arbitrary number of terms in the asymptotic expansion of the correlator (for example, see [16]). The first few terms are:

\[
G(x) = \frac{C_0}{\sqrt{\pi}x^{1/2}} (1 - (-1)^x \frac{1}{8} \frac{1}{x^2} + \frac{1}{128}\frac{1}{x^4} + (-1)^x \frac{1}{8}\frac{1}{x^4} - \frac{1}{64}\frac{1}{x^6} - (-1)^x \frac{363}{1024}\frac{1}{x^6} \\
+ \frac{1707}{32768}\frac{1}{x^8} + (-1)^x \frac{1985}{1024}\frac{1}{x^8} + \ldots) . \quad (25)
\]
One can see that only the first two terms in the equation (18) coincide with the exact result (25) i.e. only the first two terms in the expansion of the correlator correspond to the sum over the low-lying intermediate excited states.

**Conclusion.**

Using the expressions for the formfactors of local operators for the XX - quantum spin chain as a Cauchy determinants we obtained the contribution of the low-energy excited intermediate states to the asymptotic expansion of the correlator in the XX spin chain. We obtained the particle-hole formfactors corresponding to the higher order terms in the asymptotics of the correlator in agreement with the predictions of the Luttinger liquid theory. We have shown that in order to obtain the correct asymptotic expansion of the correlator in the XX- spin chain one should take into account the intermediate states with the high (of order of the band width) excitation energy. In the other words the contributions to the original formfactors which are suppressed by the power of $1/L$ at small quantum numbers of the particles and the holes, $p_i, q_i \sim 1$, leads to the power-law contributions to the asymptotics of the correlator, which are not suppressed by a power of $L$. Equivalently, if various irrelevant operators are included into the equation (3), the sum over the corresponding low-energy formfactors (which are suppressed at low energies) can be saturated by the quantum numbers $p_i, q_i$ of order of $L$. Therefore the corresponding contributions cannot be described in the framework of the Luttinger liquid theory. We have seen that only the first two terms are in agreement with predictions of the Luttinger liquid theory. We expect the same results for the general case of the XXZ- spin chain.

**Acknowledgments.**

The author is grateful to Professor J.H.H.Perk for providing the author with the result (25).

**Appendix.**

Let us calculate the matrix elements (8), (11) which correspond to the particle-hole formfactors. Let us begin with the matrix element (8) corresponding to the first branch of the Luttinger model [11]. We start with the matrix element for the operators in the coordinate space $\langle 0|a^+(x)a(y)e^{B_a}|0\rangle$, $B_a = a(2\pi/L)\sum_{p>0} p_1(p)/p$. Using the formula $Ae^B = e^B(\sum_n (1/n!) [A, B]_n)$, one can commute the exponent $e^{B_a}$ to the left. Thus we
obtain the following equation:

\[ \langle 0 | a^+(x) a(y) e^{B_u} | 0 \rangle = \left( \frac{1 - e^{i x'}}{1 - e^{i y'}} \right)^a \langle 0 | a^+(x) a(y) | 0 \rangle = \frac{1}{L} \left( \frac{1 - e^{i x'}}{1 - e^{i y'}} \right)^a \frac{e^{i y'}}{e^{i y'} - e^{i x'}}, \]

where \( x' = 2\pi x / L, \ y' = 2\pi y / L. \) Then for the derivative we obtain

\[ (i \partial_x + i \partial_y) \langle 0 | a^+(x) a(y) e^{B_u} | 0 \rangle = -2\pi \frac{2}{L^2} \left( 1 - e^{i x'} \right)^a e^{i y'} \left( 1 - e^{i y'} \right)^{-(a+1)}. \]

Calculating the Fourier transform for both sides of this equation we obtain for the matrix element \( \langle 0 | a_q^+ a_p e^{B_u} | 0 \rangle \) the expression (8) for \( n = 1 \) with the factors

\[ f^+(p) = a \int_0^{2\pi} \frac{dy}{2\pi} e^{-i(p-1)y} \left( 1 - e^{i y} \right)^{-(a+1)}, \quad p > 0, \]

\[ f^-(q) = \int_0^{2\pi} \frac{dx}{2\pi} e^{i qx} \left( 1 - e^{i x} \right)^{a-1}, \quad q \leq 0, \]

where \( p \) and \( q \) are integers. Calculating the integrals and using the Wick's theorem we obtain the equation (8) with the functions \( f^+(p), \ f^-(q) \) presented in the text.

The derivation of the equation (11) is similar. We consider the matrix element \( \langle 0 | c^+(x) c(y) e^{B_c} | 0 \rangle \) with \( B_c = c(2\pi/L) \sum_{p<0} \rho_2(p) / p. \) Commuting \( e^{B_c} \) to the left we obtain:

\[ \langle 0 | c^+(x) c(y) e^{B_c} | 0 \rangle = \left( \frac{1 - e^{-i y'}}{1 - e^{-i x'}} \right)^c \langle 0 | c^+(x) c(y) | 0 \rangle = \frac{1}{L} \left( \frac{1 - e^{-i y'}}{1 - e^{-i x'}} \right)^c \frac{e^{-i y'}}{e^{-i y'} - e^{-i x'}}. \]

Calculation of the derivatives gives

\[ (i \partial_x + i \partial_y) \langle 0 | c^+(x) c(y) e^{B_c} | 0 \rangle = -c \frac{2\pi}{L^2} \left( 1 - e^{-i y'} \right)^{c-1} \left( 1 - e^{-i x'} \right)^{-c+1}. \]

Calculating the Fourier transform we obtain for the matrix element \( \langle 0 | c_q^+ c_p e^{B_c} | 0 \rangle \) the expression (11) for \( n = 1 \) with the factors

\[ f^+(p) = c \int_0^{2\pi} \frac{dy}{2\pi} e^{-i(p+1)y} \left( 1 - e^{-i y} \right)^{c-1}, \quad p < 0, \]

\[ f^-(q) = c \int_0^{2\pi} \frac{dx}{2\pi} e^{i qx} \left( 1 - e^{-i x} \right)^{-c+1}, \quad q \geq 0. \]

Calculating the integrals we obtain the equation (11) with the functions \( f^+(p), \ f^-(q) \) presented in the text.
References

[1] A.Luther, I.Peschel, Phys.Rev.B 9 (1974) 2911; Phys.Rev.B 12 (1975) 3908.

[2] F.D.M.Haldane, Phys.Rev.Lett. 47 (1981) 1840; J.Phys.C 14 (1981) 2585 ;
Phys.Rev.Lett. 45 (1980) 1358.

[3] J.Cardy, Nucl.Phys. B 270 (1986) 186.

[4] A.Mironov, A.Zabrodin, Int.J.Mod.Phys. A7 (1992) 3885.

[5] A.Shashi, L.I.Glazman, J.S.Caux, A.Imambekov, Phys.Rev.B 84 (2011) 045408.

[6] A.A.Ovchinnikov, Phys.Lett.A 375 (2011) 2694. arXiv:1107.4295 [math-ph].

[7] N.Kitanine, K.K.Kozlowski, J.M.Maillet, N.A.Slavnov, V.Terras,
J.Stat.Mech.:Theory and Experiment 04 (2009) P04003.

[8] N.Kitanine, K.K.Kozlowski, J.M.Maillet, N.A.Slavnov, V.Terras, arXive:1003.4557
[math-ph]; J.Stat.Mech. (2011) P12010.

[9] A.A.Ovchinnikov, J.Phys.:Condens.Matter 16 (2004) 3147.

[10] H.J.de Vega, M.Karowski, Nucl.Phys.B 285 (1987) 619; F.Woynarovich, H.P.Eckle,
J.Phys.A 20 (1987) L97; M.Karowski, Nucl.Phys.B 300 (1988) 473; F.C.Alcaraz,
M.N.Barber, T.M.Batchelor, Phys.Rev.Lett. 58 (1987) 771.

[11] E.Bettelheim, A.G.Abanov, P.Wiegmann, J.Phys.A 40 (2007) F193.

[12] E.Lieb, T.Schultz, D.Mattis, Ann.Phys. 16 (1961) 407.

[13] M.E.Fisher, R.E.Hartwig, Adv.Chem.Phys. 15 (1968) 333.

[14] E.L.Basor, K.E.Morrison, Lin.Alg.Appl. 202 (1994) 129. E.L.Basor, C.A.Tracy,
Physica A 177 (1991) 167. P.Deift, A.Its, I.Krasovsky, Ann. of Math. 174 (2011)
1243.

[15] H.Au-Yang, J.H.H.Perk, Phys.Lett.A 104 (1984) 131.

[16] B.M.McCoy, Phys.Rev. 173 (1968) 531; E.Barouch, B.M.McCoy, Phys.Rev.A 3
(1971) 786; B.M.McCoy, J.H.H.Perk, R.E.Shrock, Nucl.Phys.B 220 (1983) 269;
J.H.H.Perk, H.Au-Yang, J.Stat.Phys. 135 (2009) 599.