Entropy production in 2D $\lambda\phi^4$ theory
in the Kadanoff-Baym approach

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Abstract

We study non-equilibrium quantum dynamics of the single-component scalar field theory in 1+1 space-time dimensions on the basis of the Kadanoff-Baym equation including the next-to-leading-order (NLO) skeleton diagrams. As an extension of the non-relativistic case, we derive relativistic kinetic entropy at the first order in the gradient expansion of the Kadanoff-Baym equations. The derived entropy satisfies the H theorem. Next we perform numerical simulations in spatially homogeneous configurations to investigate thermalization properties of the system by evaluating the system entropy. We find that at later times the kinetic entropy increases approaching the equilibrium value, although the limited time interval in the early stage invalidates the use of it.

1 Introduction

Non-equilibrium quantum field theories provide a suitable framework to investigate a large variety of topical problems in high energy particle physics, astrophysics, cosmology, as well as condensed matter physics[1, 2]. In the context of heavy ion collision physics, the early time evolution of the colliding system toward the quark-gluon plasma (QGP) state has attracted a lot of theoretical interests for recent years. Success of ideal hydrodynamic models for describing bulk properties of the matter created at Brookhaven’s Relativistic Heavy Ion Collider (RHIC) seems suggesting that the produced system is strongly interacting and nearly thermalized within a short time[3] compared with perturbative analysis[4]. There are various theoretical studies on the possibility for this short time thermalization, some of which rely on the instabilities in the plasma[5, 6, 7, 8, 9, 10, 11], and some others include the 2-to-3 processes in parton cascade simulations[12].

In the earliest stage of the high-energy nuclear collisions, the system will be so dense that it would be more suitable to describe the system in terms of the quantum field degrees of freedom than in the particle basis. As a first step of this approach toward the early time dynamics of the nuclear collisions, we study here the non-equilibrium $\lambda\phi^4$ scalar field theory in 1+1 dimensions on the basis of the Kadanoff-Baym (KB) equations.

As early as 1960s, based on a functional formulation of Luttinger and Ward [13], Baym and Kadanoff studied the Dyson-Schwinger equation for the two-point function $G(x, y)$ [14]. Then Baym reformulated it in terms of variational principal, introducing the so-called $\Phi$-derivable approximation [15, 16]. The functional $\Phi[G]$ is given by a truncated set of closed two-particle irreducible (2PI) diagrams, and generates the driving terms of the equations of motion. The main virtue of this approximation is that the resulting KB equations conserve the energy and momentum of the system. This approach was extended to relativistic systems and formulated using the path integral by Cornwall, Jackiw and Tomboulis in [17]. It can be extended further to more general non-equilibrium many-body systems based on the Schwinger-Keldysh real-time path integral method[18, 19].

In the last several years, the real-time field dynamics has been newly investigated by several authors. A seminal work was carried out by Danielewicz[20], who for the first time studied the
full KB equations in the context of the heavy ion collisions at non-relativistic energies. He used a spatially homogeneous initial condition with the non-spherical Fermi distribution for the nucleon momentum. Thermalization problem in the relativistic $\lambda \phi^4$ scalar field theory in 1+1 [21,22], 2+1 [23] and 3+1 [24,25] dimensions, has been investigated with keeping the NLO skeleton diagrams in $\Phi$. Extension to the $O(N)$ theory at the next-to-leading order in $1/N$ expansion can be found in Ref. [26]. Importantly, all these analyses indicate that thermalization is achieved in course of the time evolution of the system independently of the initial conditions. The number distribution functions of the quasi-particles are found to approach the Bose distribution.

The approach to the equilibrium state will be quantitatively characterized if system entropy can be introduced properly. In fact it is an open question how to choose the gross variables and define the corresponding entropy of the system in general non-equilibrium situations. There is no entropy production in fully microscopic calculations. We use the variable $G(x,y)$ in the KB approach. In the non-relativistic case, the kinetic entropy is introduced at the first order of the gradient expansion in Refs. [27] and [28]. To our knowledge, the entropy production has not ever been estimated in the relativistic KB dynamics. Here we shall extend the entropy to the relativistic case in the first order of the gradient expansion. This will provide us, for example, of a criterion how much each microscopic process contributes to thermalization of the system.

The KB equations deal with the evolution of the field $\langle \phi \rangle$ and the two-point function $G(x,y)$, and effectively contain particle number changing processes such as $1 \leftrightarrow 3$ even in the NLO, if interpreted in the particle basis. In contrast, the Boltzmann equation includes only the $2 \leftrightarrow 2$ scattering processes to this order, which preserves the total particle number. This difference should be reflected in the behavior of the system evolution, especially in the entropy production. We expect that this aspect of the KB equations is important to understand the possibility of the rapid thermalization.

For demonstration, we shall numerically solve the non-equilibrium dynamics of $\lambda \phi^4$ theory on the basis of the KB equations. In order to reduce the numerical cost, we restrict our simulations to the spatially uniform case without the mean field $\langle \phi \rangle = 0$ in 1+1 dimensions. We start the simulations with the non-thermal initial conditions, and show the time evolution of the system through the particle number distribution functions, the energy content, the entropy production and so on.

This paper is organized as follows. In Sec. 2 we briefly review the formulation of the KB equation for the relativistic scalar field theory, using the Schwinger-Keldysh formalism and 2PI effective action in the NLO. Next, we present the derivation of the entropy for the relativistic KB equations in the first order in the gradient expansion in Sec. 3, which is the main theoretical part of this work. The expression for the entropy is found to be a natural extension of the non-relativistic one given in [27,28,29,30] in the local approximation. This entropy satisfies the H-theorem. In Sec. 4 we show the numerical simulations of the KB equations. The particle number distribution and the entropy of the system are calculated in terms of the numerical solutions of the KB equations. Finally Secs. 5 and 6 are devoted to discussions and summary of this study.

2 Kadanoff Baym equation

We briefly review the derivation of Kadanoff-Baym equation and fix our notations[1]. For the scalar field theory $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4$, the 2PI effective action with vanishing mean field $\langle \phi \rangle = 0$ (unbroken phase) is written as

$$\Gamma[G] = \frac{i}{2} \text{Tr} \ln (G)^{-1} + \frac{i}{2} G_0^{-1} G + \frac{1}{2} \Phi[G].$$  \hspace{1cm} (1)

Here $iG_0^{-1}(x,y) = -(\partial^2 + m^2)\delta_C(x-y)$ is the free Green’s function and $G$ is the full Green’s function, both of which are defined on the closed time path $C$. The functional $\Phi[G]$ in (1) is

\[\text{(1)}\]

\[\text{[1]}\]

However, their expressions are different from each other in the higher order terms of the skeleton expansion.
generally a sum of all possible 2PI graphs written in terms of $G$. A graph is called 2PI when it remains connected upon cutting two Green’s function lines.

The stationary condition for the effective action \( \delta \Gamma / \delta G = 0 \) gives rise to the Schwinger-Dyson equation for the Green’s function $G(x, y)$

$$G^{-1}(x, y) = G_0^{-1}(x, y) - \Sigma(x, y)$$

with the proper self-energy defined as $\Sigma = i \delta \Phi [G] / \delta G$. The self-energy is divided into the local and the non-local part $\Sigma = \Sigma_{\text{loc}} + \Sigma_{\text{nonl}}$. The $\Sigma_{\text{loc}}$ contributes to the effective mass while the $\Sigma_{\text{nonl}}$ induces the mode-coupling between the different wavenumbers. The 2PI effective action should be invariant under the symmetry transformations of the system. Although we need to approximate the functional $\Phi [G]$ in practical applications, any truncation of $\Phi [G]$ which preserves the symmetry property gives the equations of motion consistent with the corresponding conservation laws\[14, 15\].

It is very useful to decompose the two-point function $G(x, y)$ into two real functions, the statistical function $F(x, y)$ and the spectral function $\rho(x, y)$ defined, respectively, as

$$F(x, y) = \frac{1}{2} \langle \{ \phi(x), \phi(y) \} \rangle = \frac{1}{2} [G^{21}(x, y) + G^{12}(x, y)]$$

and

$$\rho(x, y) = i \langle [\phi(x), \phi(y)] \rangle = i [G^{21}(x, y) - G^{12}(x, y)] ,$$

where $\langle \cdots \rangle$ represents the expectation value taken over a certain initial density matrix. The indices 1 and 2 specify the branch of the contour $C$ in the Schwinger-Keldysh formalism. The function $F$ is called the statistical function because it turns out to be the Bose distribution function in the equilibrium state. The Schwinger-Dyson equation (3) can be equivalently rewritten in terms of $F(x, y)$ and $\rho(x, y)$ as coupled integro-differential equations

\[
(\partial^2 + m^2 + \Sigma_{\text{loc}}(x))F(x, y) = \int_{t_0}^{y_0} dz \Sigma_F(x, z) \rho(z, y) - \int_{t_0}^{x_0} dz \Sigma_{\rho}(x, z) F(z, y) ,
\]

\[
(\partial^2 + m^2 + \Sigma_{\text{loc}}(x))\rho(x, y) = -\int_{y_0}^{x_0} dz \Sigma_{\rho}(x, z) \rho(z, y) ,
\]

where $t_0$ is the initial time. Note that the non-local self-energy has been re-expressed similarly as

$$\Sigma_{\rho}(x, y) = \frac{1}{2} [\Sigma^{21}_{\text{nonl}}(x, y) + \Sigma^{12}_{\text{nonl}}(x, y)] ,$$

$$\Sigma_F(x, y) = i [\Sigma^{21}_{\text{nonl}}(x, y) - \Sigma^{12}_{\text{nonl}}(x, y)] .$$

The set of equations (6) and (7) is called the Kadanoff-Baym equation, which is the two-time formalism and describes the time evolution of the system from a certain initial configuration for $F$ and $\rho$. Note that at each time step the spectral function $\rho$ must satisfy the conditions following from the commutation relations:

$$\rho(x, y)|_{z^0 \to y^0} = 0 ,$$

$$\partial_{z^0} \rho(x, y)|_{z^0 \to y^0} = \delta^d(x - y) ,$$

$$\partial_{z^0} \partial_{y^0} \rho(x, y)|_{z^0 \to y^0} = 0 .$$

Importantly, Eqs. (6) and (7) are non-local in time due to the so-called memory integrals appearing on the RHS. In other words, the evolution is non-Markovian depending on the evolution
history in the past. In many stable systems, however, the integrand of the memory integral dies away exponentially and the macroscopic time scale is separated from the microscopic one. It is instructive to consider the case of a uniform equilibrium state with a small value for the self-energy $\Sigma(\rho_0, p)$. Then we find that the spectral function $\rho$ turns out to be the Breit-Wigner form (See Sec. 3),

$$\rho(p_0, p) = \frac{-\Sigma(\rho_0)}{[(p_0)^2 - \Omega_p^2] - \Sigma^2/4} \rightarrow 2\pi i\epsilon(p_0)\delta((p_0)^2 - \Omega_p^2),$$

(11)

where $\Omega_p^2 = p^2 + m^2 + \text{Re}\Sigma_R$ is the single particle energy including the mean-field effect. The arrow denotes the quasi-particle limit $\Sigma \rightarrow 0$. In this limit the $\rho$ becomes a delta-function and for thermal equilibrium the statistical function $F$ reduces to the Bose distribution

$$F(p_0, p) = 2\pi\delta((p_0)^2 - \Omega_p^2) \left(\frac{1}{2} + \frac{1}{e^{\beta p_0} - 1}\right).$$

(12)

In this paper we restrict ourselves to the spatially homogeneous situation. From the translational invariance, the statistical function $F(x, y) = F(x^0, y^0, x - y)$ and the spectral function $\rho(x, y) = \rho(x^0, y^0, x - y)$ can be Fourier transformed to $F(x^0, y^0; p)$ and $\rho(x^0, y^0; p)$. Then KB equations are simplified in the momentum space as

$$(\partial_0^2 + p^2 + m^2 + \Sigma_{\text{tad}}(x^0))F(x^0, y^0; p) = \int_{t_0}^{t_0'} dz^0 \Sigma_F(x^0, z^0; p)\rho(z^0, y^0; p)$$

$$- \int_{t_0}^{t_0'} dz^0 \Sigma_\rho(x^0, z^0; p)F(z^0, y^0; p),$$

(13a)

$$(\partial_0^2 + p^2 + m^2 + \Sigma_{\text{tad}}(x^0))\rho(x^0, y^0; p) = - \int_{y^0}^{x^0} dz^0 \Sigma_\rho(x^0, z^0; p)\rho(z^0, y^0; p).$$

(13b)

Regarding the functional $\Phi[G]$, we approximate it with the skeleton diagrams obtained at the next-leading order in $\lambda$. The self-energy $\Sigma$ then becomes the sum of the local tadpole diagram and the nonlocal sunset diagram (Fig. 1):

$$\Sigma_{\text{loc}}(x) = \Sigma_{\text{tad}}(x) = \frac{\lambda}{2}F(x, x),$$

$$\Sigma_{\text{sun}}^{ab}(x, y) = - \frac{\lambda^2}{6}G^{ab}(x, y)^3$$

(14)

(15)

where indices $a, b$ denote the branch 1 and 2 of Schwinger-Keldysh contour $C$. Furthermore the

Figure 1: Tadpole and sunset diagrams.
nonlocal part is divided into $\Sigma_F$ and $\Sigma_P$ and written explicitly in terms of $F$ and $\rho$ as

$$
\Sigma_F(x^0, z^0; p) = -\frac{\lambda^2}{6} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} F(x^0, z^0; p - k - q)
$$

$$
\times \left[ F(x^0, z^0; k) F(x^0, z^0; q) - \frac{3}{4} \rho(x^0, z^0; k) \rho(x^0, z^0; q) \right],
$$

(16)

$$
\Sigma_P(x^0, z^0; p) = -\frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \rho(x^0, z^0; p - k - q)
$$

$$
\times \left[ F(x^0, z^0; k) F(x^0, z^0; q) - \frac{1}{12} \rho(x^0, z^0; k) \rho(x^0, z^0; q) \right].
$$

(17)

We solve these KB equations [13] with the self-energy functions [14], [16] and [17] numerically in Sec. 4.

We need to specify the initial condition for $\rho$ and $F$ at $x^0 = y^0 = t_0$ in order to solve this evolution equations. For the spectral function $\rho$ it is fixed by the commutation relation as given in Eqs. (10). For the statistical function $F$, we choose to set the initial conditions of the following functional form

$$
F(x^0, y^0; p)|_{x^0 = y^0 = t_0} = \frac{1}{\omega(p)} \left( n_p + \frac{1}{2} \right),
$$

(18)

$$
\partial_{x^0} F(x^0, y^0; p)|_{x^0 = y^0 = t_0} = 0,
$$

(19)

$$
\partial_{x^0} \partial_{y^0} F(x^0, y^0; p)|_{x^0 = y^0 = t_0} = \omega(p) \left( n_p + \frac{1}{2} \right),
$$

(20)

where $\omega(p)^2 = p^2 + m^2$ and $n_p$ is a function we can freely specify. This form is assumed in analogy with the equilibrium solution in the quasi-particle limit.

At later times in course of the evolution, we define the particle number distribution $n_p(X^0)$ and the frequency $\omega_p(X^0)$ in terms of $F(x^0, y^0; p)$ [21] [23] [24] [25] [1]

$$
n_p(X^0) + \frac{1}{2} = \left[ \partial_{x^0} \partial_{y^0} F(x^0, y^0; p)|_{x^0 = y^0 = X^0} \right. \left. F(X^0, X^0; p) - \left( \partial_{x^0} F(x^0, y^0; p)|_{x^0 = y^0 = x^0} \right) \right]^{1/2}
$$

(21)

$$
\omega_p(X^0) = \sqrt{\frac{\left[ \partial_{x^0} \partial_{y^0} F(x^0, y^0; p)|_{x^0 = y^0 = X^0} \right]}{F(X^0, X^0; p)}}
$$

(22)

Strictly speaking, these definitions [21] and [22] are valid only when the quasi-particle picture works well. Nevertheless, we expect that these quantities are good estimators to characterize the behavior of the system evolution. The system is expected to have a quasi-particle spectrum for a sufficiently small coupling $\lambda$ as shown in 1 + 1 [21] [22], 2 + 1 [24] and 3 + 1 [24] [25] dimensions.

Before proceeding to the next section let us compare the KB equations with the Boltzmann equation in 1+1 dimensions. In a homogeneous system the Boltzmann equation becomes

$$
\frac{\partial n_p(t)}{\partial t} = \frac{\lambda^2}{4} \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{d^d p_3}{(2\pi)^d} \left[ \frac{1}{8 \Omega_{p_1} \Omega_{p_2} \Omega_{p_3}} \times \left[ (1 + n_{p_3})(1 + n_{p_1} n_{p_2} - n_{p_3} n_{p_1}(1 + n_{p_2})) \right] \times (2\pi)^{d+1} \delta^d(p_1 + p_2 - p_3) \delta(\Omega_{p_1} + \Omega_{p_2} - \Omega_{p_3} - \Omega_p) \right],
$$

(23)

where $\Omega_p = \sqrt{p^2 + \mu^2(t)}$ and the mass $\mu^2(t)$ is the self-consistent solution of

$$
\mu(t)^2 = m^2 + \frac{\lambda}{2} \int \frac{d^d k}{(2\pi)^d} \frac{n_k(t)}{\sqrt{\mu(t)^2 + k^2}}.
$$

(24)
In fact, this Boltzmann equation can be derived from the KB equations at the leading order in the gradient expansion and with the Markov and quasi-particle approximations \cite{31}. We remark here that in 1+1 dimensions the Boltzmann equation cannot lead to thermalization because the particle momenta must be unchanged in each 2-to-2 collision in order to satisfy the energy and momentum conservations.

3 Entropy of the relativistic Kadanoff-Baym equations

In this section we derive the expression for the relativistic entropy in terms of the two-point functions $G(x, y)$ for the $\lambda\phi^4$ theory, as an extension from the non-relativistic entropy current given in \cite{27} and \cite{28}.

We start with the Schwinger-Dyson equation \cite{32}. Multiplying $G$ from the right and left hand sides of Eq. (3), respectively, we obtain

$$-\left[\partial^2_t + m^2 - \frac{\lambda}{2} G^{aa}(x, x)\right] G^{ab}(x, y) - i \int dz \Sigma^{ac}_{\text{nonl}}(x, z) c^{cd} G^{db}(z, y) = i e^{ab}\delta(x - y), \quad (25)$$

$$-\left[\partial^2_t + m^2 - \frac{\lambda}{2} G^{bb}(y, y)\right] G^{ab}(x, y) - i \int dz G(x, z)^{ac} c^{cd} \Sigma^{db}_{\text{nonl}}(z, y) = i e^{ab}\delta(x - y), \quad (26)$$

where $a$ and $b$ assign the branch 1 and 2 of the Schwinger-Keldysh contour $\mathcal{C}$ and $e^{ab} = \text{diag}(1, -1)$. We introduce the "center-of-mass" coordinate $X = (x + y)/2$ and the relative coordinate $x - y$. Then making the difference of these equations \cite{29} and \cite{25} and performing the Fourier transform with respect to the relative coordinate $x - y$, we find

$$\left[2ip \cdot \frac{\partial}{\partial X} - \frac{i}{2} \frac{\lambda}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left( \frac{\partial G^{aa}(X, k)}{\partial X} + \frac{\partial G^{bb}(X, k)}{\partial X} \right) \right] G^{ab}$$

$$= i \int d(x - y)e^{ip(x - y)} \int dz (\Sigma^{ac}_{\text{nonl}}(x, z) c^{cd} G^{db}(z, y) - G^{ac}(x, z)c^{cd} \Sigma^{db}_{\text{nonl}}(z, y)), \quad (27)$$

where $p$ and $k$ are the momentum conjugate to $x - y$. When we make the sum of them and perform the Fourier transform, we get the expression

$$\left[p^2 - m^2 - \frac{\lambda}{4} \left( \int \frac{d^{d+1}k}{(2\pi)^{d+1}} (G^{aa}(X, p) + G^{bb}(X, p)) \right) \right] G^{ab}(X, p)$$

$$= i e^{ab} + \frac{i}{2} \int d(x - y)e^{-ip(x - y)} \int dz (\Sigma^{ac}_{\text{nonl}}(x, z)c^{cd} G^{db}(z, y) + G^{ac}(x, z)c^{cd} \Sigma^{db}_{\text{nonl}}(z, y)), \quad (28)$$

Starting the evolution at $x^0 = y^0 = 0$, we have $G(x, y)$ only in a finite region of $x^0$ and $y^0$. It is therefore important to note that the interval of $x^0 - y^0$ is inevitably limited within $\pm X^0$ in the Fourier transformation.

The gradient expansion with respect to the center-of-mass coordinate $X$ is adequate when the $X$-dependence of the system is smooth enough (See for example \cite{33, 27}). We keep just the first order terms in the gradient expansion of Green’s functions and the self energies here. For the expansion of the right hand side of Eqs. (27) and (28), we use the formula for two point functions $K(x, y)$ and $L(x, y)$:

$$\int d(x - y)e^{ip(x - y)} \int dz K(x, z)L(z, y) = \tilde{K}(X, p) \tilde{L}(X, p)$$

$$+ \frac{i}{2} \left( \frac{\partial \tilde{K}}{\partial p^\rho} \frac{\partial \tilde{L}}{\partial X_\mu} - \frac{\partial \tilde{K}}{\partial X_\rho} \frac{\partial \tilde{L}}{\partial p^\mu} \right) + O \left( \frac{\partial^2}{\partial X^2} \right), \quad (29)$$

where $\tilde{K}(X, p)$ and $\tilde{L}(X, p)$ are the Fourier-transforms in $x - y$. We remark here the scale separation between $X^0$ and $x^0 - y^0$. We implicitly assume that the time dependence on the former is smooth.
and mild while the time correlation in the latter is much shorter. At the very early time of the
evolution, this separation cannot be expected and the gradient expansion should be invalid.

The derivation of the entropy current is most simplified in terms of the retarded propagator
\( G_R = i(G^{11} - G^{12}) \). To the first order in the gradient expansion, Eqs. (27) and (28) reduce to the
equations for \( G_R \):

\[
\left[ \frac{\partial}{\partial \rho^\mu} - \frac{\partial}{\partial X^\mu} \right] G_R(X, p) = 0 ,
\]

(30)

\[
(M - \frac{i}{2} \Sigma) G_R(X, p) = -1 ,
\]

(31)

where \( M \) denotes\(^2\)

\[
M = p^2 - m^2 - \text{Re} \Sigma_R , \quad \Sigma_R = \Sigma^{11} - \Sigma^{12} .
\]

(32)

In deriving Eqs. (30) and (31) we have used the well-known relations \( \Sigma^{11} + \Sigma^{22} = \Sigma^{12} + \Sigma^{21} \),
\( G^{11} + G^{22} = G^{12} + G^{21} \) and \( 2i \text{Im} \Sigma_R = \Sigma_{\rho} \). The formal solution of the above simultaneous
equations (30) and (31) is written as (33)

\[
G_R(X, p) = \frac{-1}{M - \frac{i}{2} \Sigma} .
\]

(33)

One should note here that \( M (\Sigma_{\rho}) \) is real (imaginary). Therefore, the real and imaginary parts of
the retarded propagator \( G_R \) are given as

\[
\text{Re} G_R(X, p) = - \frac{M}{M^2 - \frac{i}{2} \Sigma} ,
\]

(34)

\[
\rho(X, p) = 2i \text{Im} G_R(X, p) = - \frac{\Sigma}{M^2 - \frac{i}{2} \Sigma} .
\]

(35)

We see that the spectral function \( \rho(X, p) \) has the Breit-Wigner form (11) in the first order approxima-
tion of the gradient expansion.

Now we are ready for writing down the entropy current. The “derivation” goes somewhat in a
heuristic way. We make the difference of Eq. (27) for \((a, b) = (1, 2)\) multiplied by \( \ln(i G^{12}/\rho) \) and
Eq. (27) for \((a, b) = (2, 1)\) multiplied by \( \ln(i G^{21}/\rho) \). Then we integrate the resultant expression
over \( d^{d+1}p/(2\pi)^{d+1} \) to arrive at the following equation:

\[
\partial_\mu s^\mu = \frac{1}{2} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \ln \frac{G^{12}}{G^{21}} C(X, p) .
\]

(36)

Here the term \( C \),

\[
C(X, p) = i (\Sigma_{\rho}(X, p) F(X, p) - \Sigma_F(X, p) \rho(X, p) ) ,
\]

(37)

may be identified as the collision term in the Boltzmann limit. With Eq. (36), we define the entropy
current \( s^\mu(X) \) as

\[
s^\mu = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \left[ \left( \rho^\mu - \frac{1}{2} \partial \text{Re} \Sigma_R \right) \left( -G^{12} \ln \frac{i G^{12}}{\rho} + G^{21} \ln \frac{i G^{21}}{\rho} \right) \right]

- \frac{1}{2} \text{Re} G_R \left( \frac{\partial}{\partial \rho^\mu} \left( \Sigma_{\rho} \frac{i G^{12}}{\rho} \right) \ln \frac{i G^{12}}{\rho} + \frac{\partial}{\partial \rho^\mu} \left( \Sigma_{\rho} \frac{i G^{21}}{\rho} \right) \ln \frac{i G^{21}}{\rho} \right) ,
\]

(38)

where we have used the relations \( i(\Sigma^{11} - \Sigma^{22}) = 2\text{Re} \Sigma_R \) and \( i(G^{11} - G^{22}) = 2\text{Re} G_R \). We have also
applied the approximations \( \Sigma^{12} \simeq \Sigma_{\rho} G_{12}^{21} \) and \( \Sigma^{21} \simeq \Sigma_{\rho} G_{21}^{21} \) in the first order gradient expansion
\[27\left[27\right] .

\(^2\)Please don’t confuse this \( M \) with a mass function. Both \( M \) and \( \Sigma \) have mass-dimension 2.
When we write the two-point functions in the form of the Kadanoff-Baym Ansatz $G^{12} = -i\rho f$ and $G^{21} = -i\rho(1 + f)$ with a real function $f$, the above expression for the entropy current becomes

$$s^\mu = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \left[ \frac{\rho}{i} \left( p^\mu - \frac{1}{2} \frac{\partial \text{Re} \Sigma_R}{\partial p_\mu} \right) (-f \ln f + (1 + f) \ln(1 + f)) \right. \left. - \frac{1}{2} \text{Re} G_R \left[ - \frac{\partial}{\partial p_\mu} \left( \frac{\Sigma_\rho}{i} f \right) \ln f + \frac{\partial}{\partial p_\mu} \left( \frac{\Sigma_\rho}{i} (1 + f) \right) \ln(1 + f) \right] \right]. \quad (39)$$

After integration by parts over $p^\mu$ in the second line, we obtain a simple expression:

$$s^\mu = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \left[ \frac{\rho}{i} \left( p^\mu - \frac{1}{2} \frac{\partial \text{Re} \Sigma_R}{\partial p_\mu} \right) + \frac{\Sigma_\rho}{i} \frac{1}{2} \frac{\partial \text{Re} G_R}{\partial p_\mu} \right] \sigma, \quad (40)$$

where we introduced the notation

$$\sigma(X, p) = -f \ln f + (1 + f) \ln(1 + f). \quad (41)$$

One must distinguish this “occupation number” function $f$ in the Kadanoff-Baym Ansatz from the distribution function $n_p$ defined in (21).

Substituting the solution (34) for $G_R$, we can write the entropy current more explicitly as

$$s^\mu = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \left[ \frac{\rho}{i} \left( 1 + \frac{M^2 - \Sigma_\rho^2}{M^2 - \frac{\Sigma_\sigma^2}{4}} \right) \left( p^\mu - \frac{1}{2} \frac{\partial \text{Re} \Sigma_R}{\partial p_\mu} \right) + \frac{\rho \Sigma_\rho}{4i} \frac{\partial \Sigma_\rho}{\partial p_\mu} \right] \sigma. \quad (42)$$

This expression further simplifies with use of (35) to

$$s^\mu = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \rho^2 \Sigma_\rho \left[ \left( p^\mu - \frac{1}{2} \frac{\partial \text{Re} \Sigma_R}{\partial p_\mu} \right) - \frac{1}{2} M \frac{\partial \Sigma_\rho}{\partial p_\mu} \right] \sigma. \quad (43)$$

This is one of the main results of this work. This expression of the entropy current is a natural extension to the relativistic case. The only difference between non-relativistic (27, 28) and our relativistic case is the factor $\frac{1}{2}$ in front of the momentum derivative of the self-energy. We remark here that there is a discussion about the memory correction terms to the kinetic entropy in the non-relativistic case in Refs. (27 and 28) when we deal with the skeleton diagrams $\Sigma_{\text{nonl}}$ beyond the NLO in $\lambda$.

In the quasi-particle limit, $\Sigma_{\text{nonl}} \to 0$, we know that $G^{12} = -i\rho f = 2\pi\delta((p^0)^2 - \Omega_p^2)(\theta(-p^0) + n_p)$ and $G^{21} = -i\rho(1 + f) = 2\pi\delta((p^0)^2 - \Omega_p^2)(\theta(p^0) + n_p)$. In this limit the expression of the entropy current for $\mu = 0$ reduces to the well-known form of the entropy density for bosons

$$s^0 = \int \frac{d^d p}{(2\pi)^d} \left[ -n_p \ln n_p + (1 + n_p) \ln(1 + n_p) \right], \quad (44)$$

as it should be. In general cases, however, the spectral function $\rho(X, p)$ is defined as the Fourier transform of Eq. (3) in $x - y$, and the occupation number function $f(X, p)$ is then obtained with $G^{12}(X, p) = -i\rho(X, p) f(X, p)$. Although $\rho^0 \rho(X, p) \geq 0$ in equilibrium, we have only a finite support in $x^0 - y^0$ in the initial value problem and the resultant Fourier transform $\rho^0 \rho(X, p)$ may oscillate in $p^0$, as shown in the next section. Accordingly the function $f(X, p)$ can become negative, which brings a difficulty in evaluating the entropy density $s^0$ obtained at the leading order of the gradient expansion.

Finally we show the fact that this entropy current obtained in the NLO in $\lambda$ formally satisfies the H-theorem. Namely, the RHS of Eq. (35) is positive semi-definite. This can be verified by

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3 The tadpole part should be the renormalized one in this expression in 1+1 dimensions.
Figure 2: Evolution of the distribution function \( n_p(\tilde{\omega}_p/m) \) from the tsunami initial condition \((\lambda/m^2 = 4)\).

Figure 3: Evolution of the distribution function \( n_p(\tilde{\omega}_p/m) \) from the WS initial condition \((\lambda/m^2 = 4)\).

substituting the expressions for \( \Sigma_F \) and \( \Sigma_\rho \) into the RHS of (36). As a result we obtain the relation

\[
\partial_\mu s^\mu (X) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2} \ln \frac{G^{12}}{G^{21}} C
\]

\[
= \frac{\lambda^2}{8} \cdot \frac{1}{3!} \int \frac{d^{d+1}q}{(2\pi)^{d+1}} \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{d^{d+1}r}{(2\pi)^{d+1}} (2\pi)^{d+1} (p+q-k-r)
\]

\[
\times \left[ G^{12}(p,X)G^{12}(q,X)G^{21}(k,X)G^{21}(r,X) - G^{21}(p,X)G^{21}(q,X)G^{12}(k,X)G^{12}(r,X) \right]
\]

\[
\times \ln \frac{G^{12}(p,X)G^{12}(q,X)G^{21}(k,X)G^{21}(r,X)}{G^{21}(p,X)G^{21}(q,X)G^{12}(k,X)G^{12}(r,X)} \geq 0 ,
\]

where we have used \( G^{12}(-k,X) = G^{21}(k,X) \). The last inequality holds since \((x-y)\ln \frac{x}{y} \geq 0\). Thus we proved that the H-theorem is fulfilled in the NLO in \( \lambda\phi^4 \) theory. However, at higher orders in the coupling constant \( \lambda \), the definition of the entropy current and the proof of the H-theorem are open problems.

4 Numerical simulation

In this section we show numerical results of time evolutions of the KB equations in the \( \lambda\phi^4 \) theory in 1+1 dimensions. We discretize the space \( L = 2Na_s \) into 2N grid points \( x_n = na_s \) \((n = -N, -N+1, \ldots, N-1, N)\) with \( a_s \) the lattice spacing and apply the periodic boundary condition. Accordingly the momentum has discrete value as \( p_n = \frac{2\pi n}{L} \). The space derivative \(-\partial_x^2\) is replaced with \( \hat{p}^2 = \frac{1}{a_s^2} \sin^2 (\frac{2\pi p_n}{L}) \), which removes much of the lattice artifacts[34]. We set \( N = 40 \), which is sufficient to study the momentum dependence. We also performed the simulation with \( N = 80 \) and found no appreciable differences in the numerical results. We solve the evolution with the time step, \( a_t/a_s = 0.1 \).

We set the mass \( ma_s = 0.3 \) and varied the coupling \( \lambda/m^2 = 4, 2 \) and 1. We prepared the two different types of the initial conditions, “tsunami” distribution and the Woods-Saxon (WS) distribution, respectively,

\[
n_T^p = \frac{1}{N_T} \exp \left[ -\frac{(|p_x| - p_T)^2}{2\sigma^2} \right]
\]

(46)
Figure 4: Evolution of the energy content in units of \( m \) from the tsunami initial condition with \( \lambda/m^2 = 4 \); Kinetic \( \varepsilon_{\text{kin}}/m \) (solid), tadpole \( \varepsilon_{\text{tad}}/m \) (dashed), sunset \( \varepsilon_{\text{sun}}/m \) (dotted) and the total energy \( \varepsilon_{\text{tot}}/m \) (bold solid). See the text for the details.

with \( \sigma^2/m^2 = 4.4 \times (2\pi)^2 \), \( p_T = 7 \cdot 2\pi/L \) and \( N_T = 0.25 \), and

\[
n_p^{\text{WS}} = \frac{1}{N_{\text{WS}}} \frac{1}{e^{(\sqrt{p^2 + m^2 - p_{WS}})/\kappa} + 1}
\]

with \( p_{WS}/m = 2.936 \), \( \kappa/m = 0.35 \) and \( N_{WS} = 0.5 \). The “tsunami” initial condition has two peaks at \( \pm p_T \) (“tsunami”) and may be regarded as a toy model of the nuclear collisions. The WS initial condition is used to check the sensitivity of the evolution to the initial condition. These parameters in the WS case are tuned so that both the initial conditions give the same energy for \( \lambda/m^2 = 4 \).

Later in this section, we change the coupling constant \( \lambda \) with other parameters fixed, in order to see how the evolution depends on the coupling strength. We monitor the energy conservation in each time step in order to estimate the numerical accuracy of our simulation.

In Fig. 2 we show the time evolution of the number distribution \( n(\tilde{\omega}_p) \) defined in Eqs. (21) and (22) with the tsunami initial condition (46). From this figure we confirm that our simulation reproduces the results of Ref. [21]. The peak of Gaussian distribution disappears rapidly and the values at high and low edge regions grow up with time. As a result, the particle number distribution approaches the Bose distribution function \( n_p = 1/(e^{\beta_p/T} - 1) \), with temperature \( T/m \sim 2.5 \) and zero chemical potential. Similarly in Fig. 3 we show the time evolution of number distribution \( n(\tilde{\omega}_p) \) with the WS initial condition (47). We see that \( n(\tilde{\omega}_p) \) converges to the same thermal distribution as the one in the case of “tsunami” initial condition.

Next we study the energy content of the system. The explicit expression of the energy is given in Appendix A. We plot the kinetic (62a), tadpole (62c), sunset (62d) and total energy (61a) as a function of time \( mX^0 \) in Fig. 4. As for the total and kinetic energies we plot their differences measured from the initial value of the total energy \( \varepsilon_{\text{tot}}/m \sim 260 \) in our discretization. We find that the growth of the kinetic energy is canceled by the tadpole and the sunset energy to have a constant total energy. The energy is conserved within 0.5 percent in Fig. 4.

### 4.1 Kinetic entropy

Let us study the kinetic entropy (40) derived with the gradient expansion of the KB equation. To this end we first examine the shape of the spectral function \( \rho(X, p) \), which appears in the expression (40) and is needed to compute the occupation number function \( f(X, p) \) in \( \sigma \). The spectral function \( \rho(X, p) \) itself is physically important. In Figs. 5 and 6 we show \( \rho(X, p) \) for \( p_x = 2\pi n/L \) with \( n = 0, \)
Figure 5: Spectral function $\rho(X,p^0,p_x)$ with $p_x = 0$ at $mX^0 = 1.5, 3$ and 4.5.

Figure 6: Spectral function $\rho(X,p^0,p_x)$ with $p_x = 20\pi/L$ at $mX^0 = 1.5, \cdots, 10.5$.

Figure 7: Time evolution of kinetic entropy (40) denoted in + and its quasi-particle approximation (44) shown in a curve for the “tsunami” initial condition with coupling $\lambda/m^2 = 4$ and 10, respectively, at several values of time $X^0$, with the “tsunami” initial condition (46). We clearly see peak structures near $p^0/m \sim \sqrt{m^2 + p_x^2/m}$ at later times in both figures.

At early times, however, the spectral function $\rho(X,p)$ shows oscillatory behavior. This can be understood as the uncertainty relation between the energy and the time. In the observation within a finite time interval $|x^0 - y^0| < X^0$, one can resolve the $p^0$ dependence of $\rho(X,p)$ on the scale larger than $1/X^0$, because we have an oscillating factor due to $\int_{-X_0}^{X_0} dt \exp(-ip^0 t) = 2 \sin(p^0 X^0)/p^0$. We numerically confirmed that the oscillation frequency is indeed proportional to $X^0$. This means that any finer structure of $\rho(X,p)$ than a scale $1/T$ is resolved only after the evolution time of $X^0 > T$. The sharper the peak structure is, the longer time it needs to be resolved, which is seen by comparing the cases with $p_x = 0$ and $20\pi/L$ as shown in Figs. 5 and 6, respectively.

The applicable range of the kinetic entropy (27, 28) has not yet been examined so far with using the numerical solution of the KB equation before this study. Given the oscillating $\rho(X,p)$ near the initial time, we also have an oscillation for the occupation number function $f(X,p)$ accordingly. We thus encounter a problem in evaluating $\sigma$ as it contains the logarithm of $f(X,p)$. Here we come to recognize that the form of the kinetic entropy (40) obtained in the leading-order gradient expansion cannot be applied in the early stage of the initial value problem, although the gradient expansion itself becomes unjustifiable there.

In Fig. 7 we try evaluating the kinetic entropy (40) as a function of time $X^0$. Crude as it is,
we simply neglect the contributions from the region of \( p^0 \) where \( \rho(X, p) \) has negative values as an exploratory estimate. We obtain the entropy that decreases until \( mX^0 \sim 10 \). This peculiar behavior is presumably stemming from omitting the negative \( \rho(X, p) \) contribution, which seems likely to overestimate the entropy. In the later stage, say \( mX^0 > 20 \), the kinetic entropy increases monotonically as time proceeds, which is expected from the fact that the kinetic entropy (40) satisfies the H-theorem.

### 4.2 Entropy in quasi-particle approximation

As an alternative estimate for entropy of the system, we use the the quasi-particle (QP) approximation (44) with the number distribution \( n_p(X^0) \) defined in Eq. (21). The quasi-particle approximation (44) may be reasonable because the spectral function is nicely peaked near \( \omega^2 \sim m^2 + p^2 \) as seen in Figs. 5 and 6. Since \( n_p(X^0) \) is defined locally at time \( x^0 = y^0 \) without the Fourier transformation, we have no computational difficulty even at the very early stage, in contrast to the kinetic entropy (40). We should remark, however, that \( n_p(X^0) \) is here obtained using the solution of the full KB equation via Eqs. (21) and (22).

Although the QP entropy (44) yields somewhat a smaller value as compared with the kinetic entropy (40), the evolution profiles of these entropies are quite similar to each other, except at the early times. Smaller value for the QP entropy (44) may be related to the fact that it neglects the finite width of the spectral distribution over \( p^0 \). With the QP approximation we see in Fig. 7 that the entropy production is concentrated at early times \( mX^0 \lesssim 20 \) and slows down at later times \( mX^0 \gtrsim 20 \), approaching an equilibrium value.

In Fig. 8 we show the ratio between kinetic entropy (40) and its quasiparticle approximation (44) for tsunami and WS initial conditions. Until \( mX^0 \sim 10 \) the ratio decreases because of the numerical artifact, explained in the above. In the range of \( 20 < mX^0 < 80 \) the ratio becomes constant for both initial conditions. In the late time region \( mX^0 > 80 \) the ratio starts to increase. This tendency can be explained from the behavior of time evolution of number density. In Fig. 9 we show the time evolution of number density for tsunami and WS initial conditions with \( \lambda = 4 \). At the late time \( mX^0 > 80 \) with \( \lambda = 4 \) the number density decreases continually. In addition the distribution function shown in Figs. 2 and 3 has the small change for \( mX^0 > 80 \). Due to the decrease of number density and small change for \( n_p \) the entropy with quasiparticle approximation (44) increases weakly compared with the kinetic entropy (40).

We compare the evolutions of the QP entropy for three values of the coupling, \( \lambda/m^2 = 4, 2 \) and 1 with the tsunami and WS initial conditions, respectively, in Figs. 10 and 11. In these figures, we see...
that the larger is the coupling the faster is the entropy produced and saturated to the equilibrium value. In order to quantify the approach to the equilibrium value, we fit the entropy evolution with a simple functional form
\[ s(X^0) = s_{\text{max}} - Ae^{-\gamma(mX^0)}, \]
where \( s_{\text{max}}, A \) and \( \gamma \) are parameters. We chose to fit the evolution in the regions \( 100 \leq mX^0 \leq 150, 300 \leq mX^0 \leq 600 \) and \( 600 \leq mX^0 \leq 900 \) for \( \lambda = 4, 2 \) and 1, because the approach to equilibrium is slower for smaller \( \lambda \). The resultant values for the parameters are summarized in Table 1. The parameters \( \gamma \) and \( s_{\text{max}} \) take the same values independent of the initial conditions both for the coupling constants \( \lambda = 4 \) and 2. The \( \lambda \) dependence of the parameter \( \gamma \) seems non-trivial, contrary to the \( \lambda^2 \) dependence naively inferred from (45). For \( \lambda = 1 \), our fit seems still sensitive to the initial conditions. At the later stage \( \gamma \) might coincide for both initial conditions, but in this range \( mX^0 > 900 \) we have energy errors more than 0.5%, so that we stop our simulation.

### 4.3 Particle changing processes

As seen in the behavior of \( n(\tilde{\omega}_p) \) in Figs. 2 and 3, the system evolves toward the equilibrium state. The entropy in the QP approximation is also written in term of \( n(\tilde{\omega}_p) \). Here we study which microscopic process contributes to the change of the distribution function \( n(\tilde{\omega}_p) \) in course of the time evolution. Although the distribution \( n(\tilde{\omega}_p) \) in Figs. 2 and 3 is computed using Eqs. (21) and (22), it seems instructive to evaluate the time derivative \( dn_p/dX^0 \) in the quasi-particle approximation, which is given in Eq. (66) in Appendix B. Within this approximation we can clearly separate out the contributions of 0-to-4, 1-to-3, 2-to-2 and 3-to-1 processes. In Figs. 12 and 13 shown are the contributions of each microscopic process on the RHS of \( dn_p/dX^0 \) Eq. (66) at the momentum

| \( \lambda \) | \( \gamma_0 \) | \( s_{\text{max}} \) | \( A \) | \( \gamma \) | \( \gamma_0 \) | \( s_{\text{max}} \) | \( A \) | \( \gamma \) |
|---|---|---|---|---|---|---|---|---|
| 4 | 0.24 | 2.54 | 0.16 | 0.030 | 0.14 | 2.54 | 0.14 | 0.030 |
| 2 | 0.084 | 2.48 | 0.10 | 0.0031 | 0.036 | 2.44 | 0.13 | 0.0031 |
| 1 | 0.027 | 2.39 | 0.17 | 0.0024 | 0.0085 | 2.39 | 0.19 | 0.0011 |

Table 1: Slope parameter \( \gamma_0 \) near \( X^0 \sim 0 \) and parameters in Eq. (48) for “tsunami” (left) and WS (right) initial conditions.
Figure 12: Microscopic contributions to $dn_p/dX^0$ at $L \cdot p = 7 \cdot 2\pi$ as functions of $mX^0$ in the case of "tsunami" initial condition with $\lambda/m^2 = 4$. By integrating from $mX^0 = 0$ to $mX^0 = 35$ each contribution to $\delta n_p$ is obtained to be 0.51, 0.62, -3.17 and -0.24 for 0-to-4, 1-to-3, 2-to-2 and 3-to-1 processes.

Figure 13: Microscopic contributions to $dn_p/dX^0$ at $L \cdot p = 7 \cdot 2\pi$ as functions of time $mX^0$ in the case of the WS initial condition with $\lambda/m^2 = 4$. By integrating from $mX^0 = 0$ to $mX^0 = 35$ each contribution to $\delta n_p$ is obtained to be 0.54, 0.75, -0.86 and -0.26 for 0-to-4, 1-to-3, 2-to-2 and 3-to-1 processes.

$p_n = 2\pi n/L$ with $n = 7$ for "tsunami" and WS initial conditions, respectively. We set $\lambda/m^2 = 4$. The mode with $p$ is included as one of the three particles in the 1-to-3 process while it is formed from three particles in the 3-to-1 process.

Even in the quasi-particle approximation, the number changing processes are possible because of the finite memory time as was studied in Ref. [37]. We see that at early times the number changing processes $1 \leftrightarrow 3$ contribute as well as $2 \leftrightarrow 2$ scattering processes. One should recall that $dn_p/dX^0 = 0$ in the Boltzmann limit in 1+1 dimensions; even the 2-to-2 process is possible only when it keeps the particle momenta unchanged.

As is seen in Fig. 2, the momentum of $n = 7$ locates near the peak for the tsunami initial condition, and $n(\tilde{\omega}_p)$ decreases rapidly toward the equilibrium value. In the quasi-particle approximation shown in Fig. 12, the 2-to-2 process contributes largely to this decrease, although other number changing processes are also operative at early times, say $mX^0 < 35$. For the WS initial condition, the momentum of $n = 7$ locates near the shoulder position of the distribution and $n(\tilde{\omega}_p)$ decreases with time as seen in Fig. 3. In this case, the 2-to-2 and other processes seem equally contributing to $dn(\tilde{\omega}_p)/dX^0$. However, the 2-to-2 contribution is negative and rather non-oscillatory while others are fast oscillating.

One should bear in mind the limitation of our discussion here. Namely, these observations are in the quasi-particle approximation, while we evaluated $n(\tilde{\omega}_p)$ using the full solution of the KB equation which includes both the effects of the finite memory time and the spectral distribution $\rho(X, p)$. The latter is missed in the estimate with the quasi-particle approximation.

5 Discussion

We have studied the time evolution of the $\lambda\phi^4$ theory in 1+1 dimensions in the framework of the KB equations with the statistical and spectral functions, $F(X, p)$ and $\rho(X, p)$, as basic ingredients. In this framework we can take into account two kinds of the "offshell" effects; one is the finite memory time effect and the other is the non-trivial form of the spectral functions $\rho(X, p)$.

We have seen that the particle number distribution $n(\tilde{\omega}_p)$, which is defined with the numerical solution of the KB equations, converges to the Bose distribution in the time evolution. We also showed that both the kinetic entropy [40] and the QP entropy [44] increase in course of the
evolution. In order to understand the mechanism of the entropy production, we analyzed the microscopic processes within the quasi-particle approximation and found that the 2-to-2 scattering processes as well as the particle number changing processes are operative in the early stage of the evolution. It is important to note here that in the Boltzmann description in 1+1 dimensions, the particle distribution function never evolves, \( dn(\omega_p)/dX^0 = 0 \), as we mentioned previously.

It is not clear, however, which effect is essential to the system thermalization, the finite memory effect or the spectral function \( \rho(X, p) \). It would be interesting to compare the finite memory time simulations with and without the quasi-particle approximation in 1+1 dimensions, in order to estimate the importance of the spectral structure \( \rho(X, p) \) in thermalization. We leave this for our future study.

We have introduced formally the kinetic entropy (40) associated with the relativistic KB equations, to the leading order in the gradient expansion. In general, the gradient expansion applies only when the \( X^0 \) dependence becomes gentle toward equilibrium, otherwise the higher order terms can give substantial effects. Furthermore, we encountered a difficulty in evaluating the kinetic entropy (40), the leading term in the gradient expansion, because of the fact that the spectral function \( \rho(X, p) \) shows an oscillation in \( p^0 \) with the frequency proportional to \( 1/X^0 \). This is understood as the uncertainty relation between the energy and the time. This yields negative values for the occupation number function \( f(X, p) \), and invalidates the expression for the kinetic entropy (40) at early times. This has not been recognized until our attempt for the numerical evaluation. At sufficiently later times when \( \rho(X, p) \) and \( f(X, p) \) become positive definite, the resultant entropy monotonically increases in time. In contrast, the QP entropy (44) is expressed in terms of the number distribution \( n_p \) obtained from the two-point function \( F \) without the Fourier transformation, and it shows a monotonic increase in Fig. 7. Although the QP entropy has no strict foundation in the context of the KB equation, its time evolution seems physically quite suitable as an indicator of the system entropy.

We have also analyzed and compared the behavior of kinetic entropy (40) and its quasiparticle approximation (44). In the middle range of thermalization the ratio of both entropy becomes constant, so that entropy (44) is a useful indicator of the thermalization. However at the asymptotic stage of thermalization due to the small change of \( n_p \) entropy (44) is affected by the behavior of total number density. Therefore entropy (44) has difficulty a little in estimating asymptotic behavior of the kinetic entropy (40) which is based on KB equation, but does not take so large difference.

6 Summary and outlook

We have extended the kinetic entropy current to relativistic case from non-relativistic one on the basis of the KB equation \( \lambda \phi^4 \) theory. The derivation was done in parallel to the non-relativistic case to the leading order of the gradient approximation and to the NLO of the skeleton expansion of \( \Phi[G] \). The derived kinetic entropy satisfies the H theorem.

The numerical simulation of the KB equations in 1+1 dimensions has been performed with the “tsunami” and Woods-Saxon initial conditions. We have seen that the particle number distribution approaches the Bose-Einstein distribution with time, irrespective of the applied initial distributions, which reconfirms the results of Ref. [21]. We noted that the Boltzmann equation with 2-to-2 scatterings in 1+1 dimensions never thermalizes. Therefore, we see that the “offshell” effects included in the KB approach must be very important for thermalization problem.

In evaluating the kinetic entropy (40) numerically, we recognized that the Fourier transformation within the limited time interval for \( mX^0 \lesssim 10 \), makes the spectral function \( \rho(X, p) \) and the occupation number function \( f(X, p) \) oscillating in \( p^0 \), accordingly the expression (40) must become complex-valued. At later times, on the other hand, the kinetic entropy (40) increases in time, indicating that the system approaches the equilibrium state. Therefore we find that the kinetic entropy (40) should be useful when the shape of the spectral function \( \rho(X, p) \) is well resolved, provided that the gradient expansion is also valid.

As an alternative, we have studied the entropy evolution in the quasi-particle approximation
and found that it increases almost monotonically saturating at the equilibrium value. The late time behavior is similar to that of the kinetic entropy. We further studied the dependences of the evolution on the coupling constant and the initial condition.

The KB equations involve the offshell effects of the finite memory time as well as of the non-trivial spectral functions $\rho(X,p)$ and $F(X,p)$. We examined effects of the finite memory time within the quasi-particle approximation, and showed that the 2-to-2 process as well as the number changing processes are operative for producing the entropy at early stage of the evolution. In order to study the importance of the spectral function $\rho(X,p)$, it will be useful to compare the simulations with and without the quasi-particle approximation in $1+1$ dimensions. We leave this for future study.

The KB equation is one of the most promising approaches to describe the non-equilibrium processes in the quantum field theories. Although we limited our simulations in $1+1$ dimensions to see the importance of the “offshell” effects, the simulations in high dimensions are certainly needed, where those effects will be likely important. In the field of high-energy heavy-ion collisions, early thermalization is one of the important issues under debate. The KB dynamics may provide a suitable framework to investigate the early time evolution of such energetic nuclear collisions, and the entropy introduced in this paper may be one of the useful quantities to characterize thermalization of the system in course of its time evolution. Toward this end, extension and practical development to the case of gluodynamics are much desired.

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A Energy Momentum Tensor

We derive the expression for the energy-momentum tensor, $\Theta^{\mu\nu}$, of the $\lambda\phi^4$ theory. The 2PI effective action $\Gamma[G]$ Eq. (11) is invariant under the translation $x^\mu \rightarrow x^\mu + \epsilon^\mu$. Following Noether’s procedure, we apply the position dependent translation $x^\mu \rightarrow x^\mu + \epsilon^\mu(x)$ to compute the change of the action $\delta \Gamma = \int_x \partial_\nu \left[ \epsilon_\mu(x) \Theta^{\mu\nu}(x) \right]$. For $\epsilon_\mu$, independent of $x$, we can prove the current conservation $\partial_\mu \Theta^{\mu\nu} = 0$ as a result of the invariance of the action $\delta \Gamma = 0$. The energy-momentum tensor $\Theta^{\mu\nu}$ reads from $\delta \Gamma$ as the coefficient factor of $\partial_\mu \epsilon_\nu(x)$ [13, 29, 24].

Under the position dependent translation, Green’s function changes as

$$G(x,y) \rightarrow G'(x,y) \equiv G(x + \epsilon(x,y), y + \epsilon(y)) = G(x,y) + \epsilon^\lambda(x) \partial^\alpha_x G(x,y) + \epsilon^\lambda(y) \partial^\alpha_y G(x,y).$$

Then the change of each term in the action (11) is calculated as follows: the first term in Eq. (11) leaves no $\epsilon$ term

$$\delta \left[ \frac{i}{2} \text{Tr} \ln G^{-1} \right] = -\frac{i}{2} \text{Tr} \left[ \frac{1}{G} \right] \delta G = -\frac{i}{2} \int_{x,y} G^{-1}(x,y) \left[ \epsilon^\mu(y) \partial^\alpha_p G(y,x) + \epsilon^\mu(x) \partial^\alpha_p G(y,x) \right]$$

$$= -\frac{i}{2} \int_x \epsilon^\mu(x) \partial^\alpha_p \delta(x-x) = 0.$$  \hspace{1cm} (50)

The second term gives rise to

$$\delta \left[ \frac{i}{2} G_0^{-1} G \right] = -\frac{i}{2} \int_{x,y} \left[ \left( \partial^\alpha_x + m^2 \right) \delta(x-y) \right] \delta G(x,y)$$

$$= -\frac{1}{2} \int_x \epsilon^\mu(x) \partial^\alpha_x \left[ \delta(x-y) \partial^\alpha_p \frac{1}{G(y,x) + G(x,y)} \left( G(x,y) + G(y,x) \right) - \delta(x-y) g_{\mu\nu} \partial^\alpha_x G(x,y) + m^2 g_{\mu\nu} \delta(x-y) G(x,y) \right].$$

\hspace{1cm} (51)
where we have used
\[
\int_{x,y} \partial^\mu_\delta [\delta(x - y)G(x, y)] = \int_{x,y} \delta(x - y) \left[ \partial^\mu_\delta G(y, x) + \partial^\mu_\delta G(x, y) \right].
\] (52)

The third term yields
\[
\delta \left[ \frac{1}{2} \Phi[G] \right] = \frac{1}{2} \int_{x,y} \frac{\delta \Phi}{\delta G(x, y)} \left( \epsilon^\mu(x) \partial^\mu_\delta G(x, y) + \epsilon^\mu(y) \partial^\mu_\delta G(x, y) \right).
\] (53)

The third term \( \delta \left[ \frac{1}{2} \Phi[G] \right] \) can be rewritten more conveniently by use of the Jacobian. For example the tadpole part in \( \Phi[G] \) changes under the translation as
\[
\delta \left[ \frac{1}{2} \Phi[G] \right] = 2\lambda \int_x G(x, x)^2 \left( \epsilon^\mu(x) \partial^\mu_\delta G(x, y) + \epsilon^\mu(y) \partial^\mu_\delta G(x, y) \right).
\] (54)

However we observe that the translation can be dealt with the change of variables in the integral as
\[
\lambda \int_x G(x + \epsilon(x), x + \epsilon(x))^2 = \int_{x'} \det \left( 1 + \frac{\partial x^\mu}{\partial x'^\mu} \right) G(x', x')^2
\] (55)

and therefore the change can be recast to
\[
\delta \left[ \frac{1}{2} \Phi[G] \right] = -\lambda \int_x \frac{\partial^\mu_\delta}{\partial x^\mu} G(x', x')^2 = -\lambda \int_x \frac{\partial^\mu_\delta}{\partial x^\mu} G(x, x)^2.
\] (56)

In this way, the total change of \( \Phi[G] \) can be re-expressed by using the Jacobian. Furthermore, the number of integration coincides with the power of the coupling \( \lambda \) in general. This means that the Jacobians under the translation can be absorbed in the change of the coupling \( \lambda \to \lambda \zeta(x) \) and that \( \delta \Phi \) can be rewritten as [15 29 21]
\[
\delta \left[ \frac{1}{2} \Phi[G] \right] = \int_x \partial_\zeta \epsilon(x) \frac{\delta \Phi}{\delta \zeta(x)} \bigg|_{\zeta=1}.
\] (57)

Since \( \delta \Gamma = \int_x \partial_\epsilon \epsilon(x) \Theta^{\mu \nu}(x) \), the energy-momentum tensor is found as
\[
\Theta^{\mu \nu}(x) = \frac{1}{2} \int_y \left[ \delta(x - y) \partial^\mu_\delta \partial^\nu_\delta G(x, y) + G(y, y) \right] - \delta(x - y) g^{\mu \nu} \partial^\lambda_\delta \partial^\lambda_\delta G(x, y)
\]
\[
+ g^{\mu \nu} \delta(x - y) m^2 G(x, y) \right] = \frac{1}{2} g^{\mu \nu} \frac{\delta \Phi}{\delta \zeta(x)} \bigg|_{\zeta=1}
\] (58)

In the case of uniform space, by taking the Fourier transform with respect to the spatial relative coordinate, we obtain the explicit expressions for the energy \( \Theta^{00} \) and the pressure \( \Theta^{11} \) as
\[
\Theta^{00}(X^0) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \left( p^2 + m^2 + \partial_\phi \partial_\phi \right) \left. F(x^0, y^0; p) \right|_{x^0=y^0=X^0} \frac{1}{2} \frac{\delta \Phi}{\delta \zeta} - \text{counter term},
\] (59)
\[
\Theta^{11}(X^0) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \left( p^2 - m^2 + \partial_\phi \partial_\phi \right) \left. F(x^0, y^0; p) \right|_{x^0=y^0=X^0} \frac{1}{2} \frac{\delta \Phi}{\delta \zeta} + \text{counter term}.
\] (60)

Here the counter term cancels out the divergence in \( \frac{\delta \Phi}{\delta \zeta} \). In 1 + 1 dimensions the divergence is only in the tadpole part which can be in the similar manner to that of [21].

In the NLO in \( \lambda \) of the skeleton expansion, the tadpole and sunset diagrams contribute to \( \delta \Phi/\delta \zeta \). Then the total energy and pressure are decomposed as
\[
E_{\text{tot}}(X^0) = E_{\text{kin}}(X^0) + E_{\text{tad}}(X^0) + E_{\text{sun}}(X^0),
\] (61a)
\[
P_{\text{tot}}(X^0) = P_{\text{kin}}(X^0) + P_{\text{tad}}(X^0) + P_{\text{sun}}(X^0),
\] (61b)

\textsuperscript{4}The counter term in energy and pressure is the same.
where

\[
E_{\text{kin}}(X^0) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} (p^2 + m^2 + \partial_x \partial_{p^0}) F(x^0, y^0; p) |_{x^0 = y^0 = X^0},
\]

\[
P_{\text{kin}}(X^0) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} (p^2 - m^2 + \partial_x \partial_{p^0}) F(x^0, y^0; p) |_{x^0 = y^0 = X^0},
\]

\[
E_{\text{tad}}(X^0) = -P_{\text{tad}}(X^0) = \frac{1}{4} \int \frac{d^d p}{(2\pi)^d} \delta m_{\text{tad}}^2(X^0) F(X^0, X^0; p) + \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \delta n_{\text{tad}}^2 F(X^0, X^0; p),
\]

\[
E_{\text{sun}}(X^0) = -P_{\text{sun}}(X^0) = \frac{1}{4} \int \frac{d^d p}{(2\pi)^d} I(X^0, p).
\]

Here \(\delta m_{\text{tad}}^2\) denotes the mass counter term:

\[
\delta m_{\text{tad}}^2 = -\frac{\lambda}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{2\omega_p},
\]

and

\[
I(X^0, p) = \int_0^{X^0} dt' \left[ \Sigma_\rho(X^0, t'; p) F(t', X^0; p) - \Sigma_F(X^0, t'; p) \rho(t', X^0; p) \right]
\]

is finite in 1+1 dimensions.\(^b\)

**B Microscopic processes in the Kadanoff-Baym equation**

The evolution of the particle number distribution can be understood in terms of the microscopic processes in the quasi-particle approximation. See [1, 37] for a similar discussion.

First, we differentiate the distribution \(n_p\) defined in Eq. (21) with respect to the time \(t\) to find

\[
\left( \frac{1}{2} + n_p \right) \partial_t n_p = \int_0^t \frac{d^d z}{(2\pi)^d} \left\{ \left[ \Sigma_\rho(t, z; p) F(z^0, t; z; p) - \Sigma_F(t, z; p) \rho(z^0, t; p) \right] \partial_t F(t, t'; p) \right\}
\]

\[
- \left[ \Sigma_\rho(t, z; p) \partial_z F(z, t; p) - \Sigma_F(t, z; p) \partial_z \rho(z, t; p) \right] F(t, t; p).
\]

(65)

Meaning of the memory integrals on the RHS of Eq. (65) becomes clear if we take the quasi-particle limit for \(\rho\) and \(F\) as given in Eqs. (11) and (12). We find \(\rho(t, t'; p) = \tilde{\omega}(p)^{-1} \sin[\tilde{\omega}(p)(t - t')]\) and \(F(t, t'; p) = \tilde{\omega}(p)^{-1} \cos[\tilde{\omega}(p)(t - t')] (n + \frac{1}{2})\). We substitute these to the self-energy \(\Sigma_\rho\) and \(\Sigma_F\) at the NLO in Eq. (65) and arrive at the following expression:

\[
\partial_t n_p(t) = \frac{\lambda^2}{3} \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \int_0^t \frac{d^d t'}{2\tilde{\omega}(p)2\tilde{\omega}(q)2\tilde{\omega}(k)2\tilde{\omega}(p - k - q)}
\]

\[
\left\{ \left[ (1 + n_p)(1 + n_q)(1 + n_k)(1 + n_{p-k} - q) - n_p n_q n_k n_{p-k} - (t') \right] \times \cos \left[ (\tilde{\omega}(p) + \tilde{\omega}(q) + \tilde{\omega}(k) + \tilde{\omega}(p - k - q)) (t - t') \right]
\]

\[
+ 3 \left[ (1 + n_p)(1 + n_q)(1 + n_k)(1 + n_{p-k} - q) - n_p n_q n_k (1 + n_{p-k} - q) (t') \right] \times \cos \left[ (\tilde{\omega}(p) + \tilde{\omega}(q) + \tilde{\omega}(k) - \tilde{\omega}(p - k - q)) (t - t') \right]
\]

\[
+ 3 \left[ (1 + n_p)(1 + n_q) n_{k} n_{p-k-q} - n_p n_q (1 + n_k)(1 + n_{p-k-q}) (t') \right] \times \cos \left[ (\tilde{\omega}(p) + \tilde{\omega}(q) - \tilde{\omega}(k) - \tilde{\omega}(p - k - q)) (t - t') \right]
\]

\[
+ 3 \left[ (1 + n_p) n_{q} n_{p-k-q} - n_p n_{q} (1 + n_k)(1 + n_{p-k-q}) (t') \right] \times \cos \left[ (\tilde{\omega}(p) - \tilde{\omega}(q) - \tilde{\omega}(k) - \tilde{\omega}(p - k - q)) (t - t') \right] \right\}.
\]

(66)

\(^b\)In the case of 3 + 1 dimensions, the renormalization of self-consistent theories is required and developed in [39].
We can interpret physically each microscopic process in the bracket \{\cdots\} of Eq. (66). The first term represents creation and annihilation processes of four particles $0 \leftrightarrow 4$. The second term describes the process $1 \to 3$ where $n_p$ is involved as one of the three particles and its reverse. The third term corresponds to $2 \leftrightarrow 2$ scattering process. The last term describes the decay of $n_p$ to 3 particles and its reverse.

The number changing processes are allowed because of the finite memory time $t - t'$; the energy conservation in each microscopic process can be violated \cite{37}. We evaluated these contributions and showed them in Figs. 12 and 13. Oscillatory behaviors seen in these figures obviously come from cosine functions. Removing the memory time effect by taking the infinite time limit $t - t_0 \to \infty$, we recover the strict energy conservation in the microscopic process, \[ \lim_{t - t_0 \to \infty} \int_{t_0}^{t} dt' \cos(\omega(t - t')) = \pi \delta(\omega). \]

In this limit only the $2 \leftrightarrow 2$ scatterings are allowed and then we obtain the Boltzmann equation (23).

References

[1] J. Berges, AIP Conf. Proc. 739, 3 (2005) [hep-ph/0409233].
[2] For example, see “Progress in Nonequilibrium Green’s Functions III,” M. Bonitz and A. Filinov (eds.), J. Phys. Conf. 35, 1 (2006).
[3] U. W. Heinz, AIP Conf. Proc. 739, 163 (2005).
[4] R. Baier, A. H. Mueller, D. Schiff and D. T. Son, Phys. Lett. B502, 51 (2001).
[5] For review, S. Mrowczynski, PoS CPOD2006: 042 (2006) [hep-ph/0611067].
[6] S. Mrowczynski, Phys. Lett. B314, 118 (1993); Phys. Rev. C 49, 2191 (1994); Phys. Lett. B 393, 26 (1997).
[7] P. Arnold, J. Lenaghan and G. D. Moore, JHEP 0308, 002 (2003).
[8] A. Rebhan, P. Romatschke and M. Strickland, Phys. Rev. Lett. 94, 102303 (2005).
[9] A. Dumitru, Y. Nara and M. Strickland, Phys. Rev. D 75, 025016 (2007).
[10] P. Romatschke and R. Venugopalan, Phys. Rev. D 74, 045011 (2006).
[11] A. Iwazaki, Phys. Rev. C 77, 034907 (2008).

H. Fujii and K. Itakura, Nucl. Phys. A 809, 88 (2008).
[12] Z. Xu and C. Greiner, Phys. Rev. C 71, 064901 (2005); Phys. Rev. C 76, 024911 (2007).
[13] J. Luttinger and J. Ward, Phys. Rev. 118, 1417 (1960).
[14] G. Baym and L. Kadanoff, Phys. Rev. 124, 287 (1961).
[15] G. Baym, Phys. Rev. 127, 1391 (1962).
[16] L.P. Kadanoff, G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).
[17] J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).
[18] J. Schwinger, J. Math. Phys. 2 (1961) 407.
[19] L.V. Keldysh, ZHETF 47 1515 (1964) [Sov. Phys. JETP 20, 235 (1965)].
[20] P. Danielewicz, Ann. Phys. (N.Y.) 152, 305 (1984).
[21] G. Aarts and J. Berges, Phys. Rev. D 64, 105010 (2001).
[22] J. Berges and J. Cox, Phys. Lett. B 517, 369 (2001).
[23] S. Juchem, W. Cassing and C. Greiner, Phys. Rev. D 69, 025006 (2004).
[24] A. Arrizabalaga, J. Smit and A. Tranberg, Phys. Rev. D 72, 025014 (2005).
[25] M. Lindner and M. M. Müller; Phys. Rev. D 73 125002 (2006).
[26] J. Berges, Nucl. Phys. A699, 847 (2002).
[27] Y.B. Ivanov, J. Knoll, and D.N. Voskresensky, Nucl. Phys. A672, 313 (2000)
[28] T. Kita, J. Phys. Soc. Jpn. 75, 114005 (2006)
[29] Y.B. Ivanov, J. Knoll, and D.N. Voskresensky, Nucl. Phys. A657, 413 (1999)
[30] Y.B. Ivanov, J. Knoll, and D.N. Voskresensky, Ann. Phys. 293, 126 (2001)
[31] E. Calzetta, B.L. Hu, Phys. Rev. D 37 (1988) 2878.
[32] W. Cassing, nucl-th 08080715
[33] W. Botermans and R. Malfliet, Phys. Rep. 198 (1990) 115
[34] I. Montvay and G. Münster; Quantum Fields on a Lattice, Cambridge University Press (1994)
[35] J. Rau and B. Müller, Phys. Rep. 272 (1996) 1
[36] H. van Hees and J. Knoll, Phys. Rev. D 65, 025010 (2002); Phys. Rev. D 65, 105005 (2002); Phys. Rev. D 66, 025028 (2002)
[37] T. Ikeda, Phys. Rev. D 69, 105018 (2004).