INTEGRAL HOMOLOGY GROUPS OF DOUBLE COVERINGS AND RANK ONE $\mathbb{Z}$-LOCAL SYSTEMS FOR A MINIMAL CW COMPLEX

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Abstract. Given a finite CW complex $X$, a nonzero cohomology class $\omega \in H^1(X, \mathbb{Z}_2)$ determines a double covering $X^\omega$ and a rank one $\mathbb{Z}$-local system $L_\omega$. We investigate the relations between the homology groups $H_\ast(X^\omega, \mathbb{Z})$ and $H_\ast(X, L_\omega)$, when $X$ is homotopy equivalent to a minimal CW complex. In particular, this settles a conjecture recently proposed by Ishibashi, Sugawara and Yoshinaga for a hyperplane arrangement complement [ISY22, Conjecture 3.3].

1. Introduction

Finite coverings of CW complexes are classical subjects in algebraic topology. Recently, double coverings attract a lot of attentions for the topology of hyperplane arrangement complement [Yos20, Suc22, ISY22].

A finite collection $\mathcal{A}$ of hyperplanes in $\mathbb{C}^n$ (or $\mathbb{C}P^n$) is called a complex affine (resp. projective) hyperplane arrangement. The topology of the hyperplane arrangement complement is very interesting. For instance, Dimca and Papadima [DP03] and Randell [Ran02] independently showed that the complement of a hyperplane arrangement is homotopy equivalent to a minimal CW complex. A fundamental problem in the theory of hyperplane arrangements is to decide whether various topological invariants of the complement of $\mathcal{A}$ are determined by the combinatorial structure of $\mathcal{A}$. It is well known that the Betti numbers and the cohomology ring of the hyperplane arrangement complements are combinatorially determined (e.g., see [OT92]). However, it is still an open question whether the Betti numbers of a finite abelian cover of a hyperplane arrangement complement are combinatorially determined. This includes the Milnor fiber of a central hyperplane arrangement. See [PS17] for recent progress in this direction and also [Suc01] for an overview of the theory.

Yoshinaga studied the mod-2 Betti numbers of double coverings for a hyperplane arrangement complement and showed that they are combinatorially determined [Yos20, Theorem 3.7]. As an application, he showed that the first integral homology group of the Milnor fiber of the icosidodecahedral arrangement has 2-torsion [Yos20, Theorem 1.2]. Ishibashi, Sugawara and Yoshinaga [ISY22] further studied the 2-torsion part of the homology groups of the double coverings and gave a refinement of Papadima and Suciu’s work [PS10]. Based on computations, they proposed a conjecture [ISY22, Conjecture 3.3] regarding the first integral homology group of a double covering and the first homology group of a rank one...
Z-local system. In this note, we settle this conjecture in all degrees with the more general setting: the CW complex is homotopy equivalent to a minimal CW complex. As an application, we show that the integral homology groups of double coverings of hyperplane arrangement complement are combinatorially determined under certain conditions.

Let $X$ be a finite connected CW complex. Fix a nonzero element $\omega \in H^1(X, \mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2$. Then $\omega$ determines a surjective map $\pi_1(X) \to \mathbb{Z}/2 \cong \{\pm 1\}$. This gives a double covering $X^\omega \to X$. On the other hand, the group homomorphism $\pi_1(X) \to \{\pm 1\} = \mathbb{Z}/2$ also gives a rank one $\mathbb{Z}/2$-local system which we denote by $\mathcal{L}_\omega$.

What is the relation between $H_*(X^\omega, \mathbb{Z})$ and $H_*(X, \mathcal{L}_\omega)$? It is easy to see that

\[ H_i(X^\omega, \mathbb{C}) \cong H_i(X, \mathbb{C}) \oplus H_i(X, \mathcal{L}_\omega \otimes \mathbb{C}). \]

So the difficult part is about the torsions of the homology groups. We give a complete answer to this question when $X$ is homotopy equivalent to a minimal CW complex.

**Definition 1.1.** Let $X$ be a connected finite CW complex. We say that the CW-structure on $X$ is minimal if the number of $i$-cells of $X$ coincides with the (rational) Betti number $b_i(X)$, for every $i \geq 0$. Equivalently, the boundary maps in the cellular chain complex $C_*(X, \mathbb{Z})$ are all the zero maps.

**Theorem 1.2.** Let $X$ be a connected finite CW complex, which is homotopy equivalent to a minimal CW complex. Fix a nonzero element $\omega \in H^1(X, \mathbb{Z}/2\mathbb{Z})$. Then there exists a bounded complex $(E_*, \partial_*)$ of finitely generated free abelian groups

\[ \cdots \to E_{i+1} \xrightarrow{\alpha_i} E_i \xrightarrow{\alpha_i^{-1}} E_{i-1} \to \cdots \]

such that $H_i(E_*, \alpha_*) \cong H_i(X, \mathcal{L}_\omega)$, $\mathrm{rank} E_i = b_i(X)$ and every entry in the boundary map $\alpha_i$ is divisible by 2. Then the complex $(E_*, \frac{1}{2} \alpha_*)$ is well defined. Moreover, we have

\[ H_i(X^\omega, \mathbb{Z}) \cong H_i(X, \mathbb{Z}) \oplus H_i(E_*, \frac{1}{2} \alpha_*). \]

**Remark 1.3.** The minimal CW complex assumption is crucial for the above theorem. For example, consider the real projective space $\mathbb{R}P^n$ with $n > 1$. Its double covering space is the sphere $S^n$. For any $0 < i < n$, we have that $H_i(S^n, \mathbb{Z}) = 0$, yet

\[ H_i(\mathbb{R}P^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}_2, & \text{for } i \text{ odd}, \\ 0, & \text{for } i \text{ even}. \end{cases} \]

**Corollary 1.4.** With the same assumptions and notations as in Theorem 1.2, we have that

\[ H_1(X^\omega, \mathbb{Z}) \cong \mathbb{Z}^{b_1(X)+r} \oplus \mathbb{Z}/d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k \mathbb{Z} \]

with $1 < d_1 | \cdots | d_k$, if and only if

\[ H_1(X, \mathcal{L}_\omega) \cong \mathbb{Z}^r \oplus \mathbb{Z}/2d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2d_k \mathbb{Z} \oplus (\mathbb{Z}/2)^{b_1(X) - r - k - 1}. \]

Since the complement $M(\mathcal{A})$ of a complex hyperplane arrangement $\mathcal{A}$ is homotopy equivalent to a minimal CW complex (see [DP03] or [Ran02]), this corollary gives a positive answer to the following conjecture proposed by Ishibashi, Sugawara and Yoshinaga recently.
Conjecture 1.5. [ISY22, Conjecture 3.3] Let $M(A)$ be the complement of a complex hyperplane arrangement $A$ in $\mathbb{C}^n$. For a nonzero $\omega \in H^1(M(A), \mathbb{Z}_2)$, the integral homology $H_1(M(A)^\omega, \mathbb{Z})$ has a non-trivial 2-torsion if and only if the local system homology $H_1(M(A), L_\omega)$ has a non-trivial 4-torsion.

In particular, Corollary 1.4 is compatible with the computations in [ISY22, Example 3.1, Example 3.2].

Corollary 1.6. With the same assumptions and notations as in Theorem 1.2, the torsion part of $H_i(X, L_\omega)$ is either 0 or a finite direct sum of $\mathbb{Z}_2$ for all $i$ if and only if $H_i(X^\omega, \mathbb{Z})$ is torsion-free for all $i$.

Corollary 1.6 follows directly from Theorem 1.2 and the Universal Coefficient Theorem. Using Corollary 1.6 and Sugawara’s recent work on real hyperplane arrangements [Sug22], we give a partial answer to a question asked by Yoshinaga [Yos20, Remark 3.8].

Corollary 1.7. Let $A$ be a real hyperplane arrangement in $\mathbb{R}^n$ and $X$ be the complexified complement of $A$ in $\mathbb{C}^n$. Fix a nonzero element $\omega \in H^1(X, \mathbb{Z}_2)$. If $L_\omega$ satisfies the CDO-condition (see [Sug22, Definition 2.1]), then $H_i(X^\omega, \mathbb{Z})$ is torsion-free for all $i$. Moreover, $H_i(X^\omega, \mathbb{Z})$ is combinatorially determined.

2. Proofs

Proof of Theorem 1.2. Since the homology groups of local systems are homotopy invariants, without loss of generality, we assume that $X$ is a minimal CW complex from now on. Note that $H_1(X, \mathbb{Z})$ is torsion free in this case. Then there exists a surjective group homomorphism $\nu : \pi_1(X) \to \mathbb{Z}$ with the following commutative diagram

\[
\begin{array}{ccc}
\pi_1(X) & \xrightarrow{\omega} & \mathbb{Z}_2 \\
\downarrow{\nu} & & \downarrow{\nu} \\
\mathbb{Z} & & \mathbb{Z}
\end{array}
\]

where $\mathbb{Z} \to \mathbb{Z}_2$ is the natural quotient map. Hence the representation of the rank one $\mathbb{Z}$-local system $L_\omega$ factors through $\nu$ and it sends the generator $1_\mathbb{Z}$ of $\mathbb{Z}$ to $-1 \in \mathbb{Z}^\times$.

Let $X^\nu$ be the covering space of $X$ associated to $\nu$. The group of covering transformations of $X^\nu$ is isomorphic to $\mathbb{Z}$ and acts on it. Consider the minimal CW-structure of $X$. By choosing lifts of these cells of $X$ to $X^\nu$, we obtain a free basis for the cellular chain complex $C_*(X^\nu, \mathbb{Z})$ of $X^\nu$ as $\mathbb{Z}[t^\pm]$-modules. So $C_*(X^\nu, \mathbb{Z})$ is a bounded complex of finitely generated free $\mathbb{Z}[t^\pm]$-modules:

\[
\cdots \to C_{i+1}(X^\nu, \mathbb{Z}) \xrightarrow{\partial_i} C_i(X^\nu, \mathbb{Z}) \xrightarrow{\partial_{i-1}} C_{i-1}(X^\nu, \mathbb{Z}) \xrightarrow{\partial_{i-2}} \cdots \xrightarrow{\partial_0} C_0(X^\nu, \mathbb{Z}) \to 0.
\]

With the above free basis for $C_*(X^\nu, \mathbb{Z})$, $\partial_i$ can be written down as a matrix, say a $(m \times n)$-matrix $(f_{kj})_{m \times n}$ with $f_{kj} \in \mathbb{Z}[t^\pm]$. We can choose $f_{kj}$ to be an integral valued polynomial simultaneously. Assume that

\[
f_{kj} \equiv a_{kj} t + b_{kj} \mod (t^2 - 1) \text{ with } a_{kj}, b_{kj} \in \mathbb{Z}.
\]
Since $X$ is a minimal CW complex, every entry in $\partial_i$ is divisible by $t - 1$. Hence $f_{kj}$ takes the value of 0 at $t = 1$, which implies $b_{kj} = -a_{kj}$.

Due to the commutative diagram (2), $X^\nu$ is a covering space of $X^\omega$. In particular, $H_*(X^\nu, \mathbb{Z})$ can be computed by the chain complex

$$C_*(X^\nu, \mathbb{Z}) \otimes_{\mathbb{Z}[t^\pm]} \mathbb{Z}[t^\pm]/(t^2 - 1),$$

and $H_*(X, \mathcal{L}_\omega)$ can be computed by the following chain complex

$$(E_*, \alpha_*) := C_*(X^\nu, \mathbb{Z}) \otimes_{\mathbb{Z}[t^\pm]} \mathbb{Z}[t^\pm]/(t + 1),$$

both viewed as complexes of finitely generated free abelian groups. Note that $f_{kj}$ takes the value $-2a_{kj}$ at $t = -1$. Hence the boundary map $\alpha_i$ can be written down as a matrix $(-2a_{kj})_{m \times n}$.

On the other hand, consider the direct sum

$$\mathbb{Z}[t^\pm]/(t^2 - 1) \cong \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot t$$

as $\mathbb{Z}$-modules. Then by this choice of basis, $(f_{kj})_{m \times n}$ gives a $(2m \times 2n)$-matrix with entry $f_{kj}$ being replaced by $(-a_{kj}^t a_{kj}, a_{kj}^t - a_{kj})$. By elementary row and column operations, it becomes

$$\begin{pmatrix}
- a_{kj} \\
0
\end{pmatrix}_{m \times n} \quad 0 \quad \end{pmatrix}_{2m \times 2n}.
$$

This matrix has the same invariant divisors as of the linear map $\frac{1}{2} \alpha_i$. It implies

$$\text{tor}(H_i(X^\omega, \mathbb{Z})) \cong \text{tor}(H_i(E_*, \frac{1}{2} \alpha_*)),$$

where $\text{tor}(-)$ denotes the torsion part of the corresponding abelian group. Since $X$ is a minimal CW complex, $H_*(X, \mathbb{Z})$ is torsion free. So the isomorphism (1) holds for the torsion parts.

Let $\pi : X^\omega \to X$ be the double covering map associated to $\omega$. Then we have

$$\pi_* \mathbb{C}_{X^\omega} \cong \mathbb{C}_X \oplus (\mathcal{L}_\omega \otimes \mathbb{C}),$$

hence

$$H_i(X^\omega, \mathbb{C}) \cong H_i(X, \mathbb{C}) \oplus H_i(X, \mathcal{L}_\omega \otimes \mathbb{C}) \cong H_i(X, \mathbb{C}) \oplus H_i(E_*, \mathbb{C}, \frac{1}{2} \alpha_*).$$

Therefore the isomorphism (1) also holds for the free abelian parts. Then the claim follows.

\textit{Proof of Corollary 1.4.} Since the homology groups of local systems are homotopy invariants, without loss of generality, we assume that $X$ is a minimal CW complex. Then $H_1(X, \mathbb{Z})$ is free of rank $b_1(X)$. Theorem 1.2 implies that $H_1(X^\omega, \mathbb{Z})$ has the displayed form if and only if

$$H_1(E_*, \frac{1}{2} \alpha_*) \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k \mathbb{Z}.$$

Since $X$ is connected and $\mathcal{L}_\omega$ is a non-trivial local system, $E_0 \cong \mathbb{Z}$ and $H_0(E_*, \frac{1}{2} \alpha_*) = 0$. Then the invariant divisors of $\frac{1}{2} \alpha_1 : E_2 \to E_1$ are

$$\{1, \ldots, d_1, \ldots, d_k\},$$
where the multiplicity of 1 is $b_1(X) - 1 - k - r$. One readily sees that this is equivalent to that $H_1(X, L_\omega) \cong H_1(E_\star, \alpha_\star)$ has the displayed form. □

Proof of Corollary 1.7. To give the proof, we need to show that the torsion part of $H_i(X, L_\omega)$ is either 0 or a direct sum of $\mathbb{Z}_2$ for all $i$. The cohomology version of this claim is proved in [Sug22, Theorem 1.3]. Then one gets the homology version by the Universal Coefficient Theorem. Corollary 1.6 implies that $H_*(X^\omega, \mathbb{Z})$ is torsion-free for all degrees. Hence $H_*(X^\omega, \mathbb{Z})$ is combinatorially determined due to [ISY22, Theorem 2.4]. □

Example 2.1 (Double star arrangement). Consider the double star arrangement [ISY22, Figure 2] and we use the same notations as in [ISY22, Example 3.2]. Take

$$\omega = e_8 + e_9 + e_{10} + e_{13} + e_{14} + e_{15}.$$  

It is easy to check that the CDO-condition holds for $L_\omega$ on the hyperplane $H_6$. Then we have $H_1(M(\mathcal{A}_{DS}), L_\omega) \cong \mathbb{Z}_2^9$ (see [Sug22, Theorem 1.3]), hence

$$H_1(M(\mathcal{A}_{DS})^\omega, \mathbb{Z}) \cong \mathbb{Z}^{10}.$$  

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