Dynamic feedback $H_{\infty}$ optimal control for singular systems

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Abstract. Time delay and packet loss often occur in the process of information transmission in descriptor systems, and $H_{\infty}$ optimal control is adopted to deal with this phenomenon. Firstly, the stability of singular systems is proved by using Lyapunov function, secondly, the existence of $H_{\infty}$ controller and-suboptimal control are proved, and finally, the necessary and sufficient condition of $H_{\infty}$-optimal controller is proved. The validity and feasibility of the proposed method are verified by numerical simulation.

1. Introduction

In recent years, the optimal control problem of singular systems has been widely applied in the fields of industrial production, enterprise management, aerospace, finance and insurance, etc., as a result, some uncertain factors often appear in the control system, which may cause disturbance and influence to the system. For the complexity of the singular system itself, the complexity and difficulty of its theoretical analysis are increased. Delay and packet loss are common in singular networked control systems (NCS), which seriously affect the stability of NCS, therefore, it is of great significance to guarantee the stability of the system in the case of delay and packet loss. In recent years, the researches on the delay and packet loss of networked control systems (ncs) have attracted the attention of scholars and some achievements have been obtained[1-6].

When the controlled object of networked control system is a singular system, normal networked control system is called the control problem of singular networked control system. Reference [7] for the output feedback control problem of a class of linear switched descriptor systems, a sufficient condition for the stabilization of the system is proposed, but the problem of state feedback control is considered. In this paper, Reference [8] the design of dynamic output feedback robust $H_{\infty}$ control for a class of switched singular systems with special uncertainties is presented. Reference [9] design method of dynamic output feedback controller for singular networked control systems with both measurement and control data loss. The design of the general controller is not optimized. Aiming at the internal stability of uncertain continuous-time singular systems, the optimality equations corresponding to the optimal control problem of uncertain continuous-time singular systems are given in reference [10]. But it is only the general optimal control method. The problem of state feedback $H_{\infty}$ optimal control for singular systems with time delay and packet dropout is studied in reference [11]. However, there are few reports on $H_{\infty}$optimal control for singular systems with dynamic feedback. In this paper, the time delay, packet loss and external input disturbance are considered in the generalized networked control systems (NCS) driven by clock, controller and actuator events, in this paper, the problem of $H_{\infty}$-optimal state feedback
control for singular networked control systems with dynamic feedback is studied, the validity and feasibility of the research results are demonstrated by a system simulation example.

2. Problem description

In the generalized networked Control Systems, data communication and network cross-nodes are often used to complete the system operation. But because of the complexity of the system, there are some problems in the transmission of structural data such as sensor, controller and actuator, such as incomplete data, data error, data packet not reaching the target node in a certain time, etc., that is, packet loss. The existence of this kind of situation makes the system data unreliable, the system control stability reduces, seriously affects the system performance. Therefore, it is the key to guarantee the stable operation of the system to deal with the problem of time delay and data packet loss. In this paper, the stability optimal control of singular systems with time delay and packet loss is analyzed.

In the generalized networked control system shown in Fig. 1, the controlled object is a generalized system with impulsive characteristics, \( w(t) \) is the input external disturbance, \( x(t) \) is the measured state, \( u(t) \) is the control input, \( y(t) \) is the system output. The system output of the controlled object is sampled according to a constant period, and the measured data is transmitted to the controller node through the network. When the network induced delay of the whole closed loop does not exceed one sampling period \( t \), the control goal of the system is to ensure the stable operation of the system. For the normal system, the real-time data is accepted normally, but the data collected by the system is still the value of the previous time because of the delay caused by the packet loss. Based on the above considerations, the structure of the singular networked control system is represented as shown in figure 1.

![Fig 1 System Structure Diagram with packet loss](image)

When the switch \( k_1 \) points to \( s_1 \), it means that the system is normal and there is no loss of data packets, the \( K \) period control quantity is \( u(k) = u_c(k) \), when there is the loss of data packets, the \( K \) period control quantity is \( u(k) = u(k-1) \). The generalized controlled object model is described as

\[
\begin{align*}
E\dot{x}(t) &= Ax(t) + Bu(t - \tau) + H_0 w(t) \\
y(t) &= C_1 x(t) + H_1 w(t)
\end{align*}
\]  

(1)

Here \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \) and \( y(t), z(t) \in \mathbb{R}^l \) represent the state, control input and expected output of the generalized controlled object; \( E, A \in \mathbb{R}^{nxn}, B \in \mathbb{R}^{nxm} \) and \( C_1, C_2 \in \mathbb{R}^{lxn} \) are stationary Matrix, \( E \) is a singular matrix, and \( \text{rank}(E) = q < n \), \( w(t) \) is external disturbance, \( H_0, H_1, H_2 \) are constant matrices, \( \tau \) is the network-induced delay of the entire closed loop.

For the convenience of the analysis, it is reasonable to assume that:

1) all states of the singular controlled object can be measured, and the system is regular by adjusting the local structure configuration of the system, which satisfies one of the following conditions:

(1) \( \deg \det(sE - A) = \text{rank}(E), \deg \det(.) \) is the number of determinants, \( \text{rank}(.) \) is the rank.

(2) \( \text{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \text{rank}(E) \);
(3) For invertible matrices \( \tilde{P}, \tilde{Q} \), restricted equivalent transformation is \( \tilde{P} \tilde{E} \tilde{Q} = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix} \), and \( N = 0 \).

2) Sensor clock driven, controller and actuator event driven.
3) The network induced delay of the whole closed loop is less than or equal to one sampling period.
4) Network information is transmitted in a single packet, and there is no time sequence disorder, and the packet loss rate is certain.
5) The external disturbance of the system is finite energy, the closed loop transfer function \( T(z) \) which is from \( w(t) \) to \( z(k) \), and \( \|T(z)\|_\infty < \overline{\beta} \), \( \overline{\beta} \) is scalar quantity.

According to the hypothesis (1)-(4), the discretization of the controlled system is formulated as (2) when the network-induced delay \( \tau \leq T \).

\[
\begin{align*}
\dot{x}(k+1) &= A_x x_1(k) + B_{11}(\tau) u(k-1) + B_{10}(\tau) u(k) + W_0 w(t) \\
\dot{x}_c(k) &= -B_{2c} u(k-1) - W_{2c} w(k) \\
y(k) &= C_{1c} x_1(k) + C_{2c} x_2(k) + H_1 w(k)
\end{align*}
\]

The dynamic feedback control is adopted and the controller model is expressed as a function (3)

\[
\begin{align*}
\dot{x}_c(k+1) &= A_x x_1(k) + B_{c} y(k) \\
\dot{u}_c(k) &= C_{c} x_1(k)
\end{align*}
\]

3. Design of \( \gamma \)-optimal dynamic feedback controller

Here are two theoretical basis for \( \gamma \)-optimal control:

The external disturbance is not considered firstly, if the time delay \( \tau \leq T \) and packet loss rate \( 0 < \beta < 1 \), then the closed-loop model of generalized networked control systems (2) exponential stability.

For a given constant \( \gamma > 0 \), if there is a symmetric positive-definite matrix \( \hat{P}, \hat{Q}, \hat{S} \), scalar \( \beta_i > 0 \), \( i = 1,2 \) then the system can realize the \( \gamma \)-suboptimal dynamic feedback control and the disturbance attenuation degree is \( \gamma \).

Based on the above conditions, the optimal controller is further designed:

**Theorem 1** For the singular networked control systems (2) and (3), if there is a symmetric positive-definite matrix \( \hat{P}, \hat{Q}, \hat{S} \), matrices \( Y_1, Y_2, Y_3 \), scalars \( \epsilon_0 > 0 \), \( \epsilon_1 > 0 \), \( \beta_1 > 0 \), \( \beta_2 > 0 \), and a unit Matrix \( I \) of the consistent dimension, the matrix inequalities of (4)-(6) hold:

\[
\beta_i^{-1} \beta_i' > 1
\]

\[
\begin{bmatrix}
-\beta_i^{-1} \hat{P} & * & * & * & * & * & * & * \\
0 & -\beta_i^{-1} \hat{Q} & * & * & * & * & * & * \\
0 & 0 & -\beta_i^{-1} \hat{S} & * & * & * & * & * \\
0 & 0 & 0 & -\epsilon_{\tau i} & * & * & * & * \\
A_x \hat{P} & B_{1c} Y_1 & B_{2c} \hat{S} & W_c & -\hat{P} & * & * & * \\
0 & Y_1 & 0 & 0 & 0 & -\hat{Q} & * & * \\
0 & Y_1 & 0 & 0 & 0 & 0 & -\hat{S} & * & * \\
c_{10} \hat{P} & 0 & -c_{2c} B_{2c} \hat{S} & H_1 \hat{c}_{w1} & 0 & 0 & 0 & -I & * \\
c_{10} \hat{P} & 0 & -c_{2c} B_{2c} \hat{S} & H_1 \hat{c}_{w1} & 0 & 0 & 0 & 0 & -\epsilon \epsilon_{\tau i} \\
0 & 0 & 0 & 0 & Y_1 & 0 & 0 & 0 & -\epsilon_{\tau i}
\end{bmatrix}
\]

\[
< 0
\]
In the formula, the symbol "*" means the corresponding symmetric term, the disturbance attenuation degree is $\gamma = \sqrt{\beta}$, and the $\gamma$-suboptimal dynamic feedback controller is

$$
\begin{align*}
  x_c(k+1) &= Y_1 \hat{Q}^{-1} x_c(k) + \frac{Y_2}{\varepsilon_1} y(k) \\
  u_c(k) &= Y_3 \hat{Q}^{-1} x_c(k)
\end{align*}
$$

**Theorem 2** For closed-loop model (2) and (3), base on theoretical basis 2), the if (8) the optimization problem has a feasible solution:

$$
\min_{\beta} \beta
$$

subject to \( \varepsilon > 0, \varepsilon_1 > 0, \beta > 0 \)

The $\gamma$-optimal dynamic feedback controller is (9):

$$
\begin{align*}
  x_c^+(k+1) &= Y_1^+ \hat{Q}^{-1} x_c^+(k) + \frac{Y_2^+}{\varepsilon_1} y(k) \\
  u_c^+(k) &= Y_3^+ \hat{Q}^{-1} x_c^+(k)
\end{align*}
$$

disturbance attenuation degree is $\gamma^* = \sqrt{\beta^*}$.

4. Simulation analysis

Select the generalized controlled object model as

$$
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} \dot{x}(t) = \begin{bmatrix}
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
 -1 & 0 & 0 & 1 \\
  0 & 1 & 1 & 1
\end{bmatrix} x(t) + \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} w(t)
$$

The measurement period is 0.1 s, the network-induced delay is 0.05 and the packet loss rate is 0.05.

Applying theorem 1 and choosing scalars $\beta_1 = 1.0261, \beta_2 = 0.7079$, a set of feasible solutions is obtained:

$$
\begin{bmatrix}
  0.6658 & 0.0006 \\
  0.0006 & 0.9206
\end{bmatrix}, \hat{Q} = 0.7677, \hat{S} = 1.2543, Y_1 = 1, Y_2 = 1, Y_3 = [0.5, 1], \varepsilon_1 = 0.9717, \varepsilon_2 = 0.9720,
$$

$\beta_1 + \beta_2 = 1.0336 > 1, \beta = 0.9715, \gamma = \sqrt{\beta} = 0.9856$

Then the $\gamma$-suboptimal dynamic feedback controller is

$$
\begin{align*}
  x_c(k+1) &= 1.3026 x_c(k) + y(k) \\
  u_c(k) &= \begin{bmatrix}
  0.6513 & 1.3026
\end{bmatrix} x_c(k)
\end{align*}
$$

Applying theorem 2, we can get the optimal solution is
The disturbance attenuation degree is \( \gamma^* = \sqrt{\beta^*} = 0.0098 \).

The \( \gamma \)-optimal dynamic feedback controller is:

\[
\begin{align*}
\dot{x}_c^*(k+1) &= 0.0195 x_c^*(k) + 0.2523 y(k) \\
u_c^*(k) &= [0.1376 \quad 0.0923]x_c^*(k)
\end{align*}
\]

And the simulation results as follow Fig 2:

**Fig 2** The simulation results of \( \gamma \)-optimal dynamic (red curve) and \( \gamma \)-suboptimal dynamic (blue curve)

The simulation results show that the optimal control effect is reliable. Further research on high-order system can be done in the future, which can be applied to more fields and solve more practical problems.

5. **Conclusion**

In this paper, the \( H_{\infty} \) optimal controller is designed and analyzed for a class of generalized networked control systems which are clock-driven by sensors, event-driven by controllers and actuators, and whose time delay is uncertain but less than one sampling period, a sufficient condition for the existence of \( H_{\infty} \) optimal controller observer is given. The effectiveness of the proposed approach is verified by simulation with the LMI toolbox of Matlab. In the future, it can be used to solve more practical problems in the field of generalized networked control systems.

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