DIRECT METHOD OF MOVING PLANES FOR LOGARITHMIC LAPLACIAN SYSTEM IN BOUNDED DOMAINS

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Abstract. Chen, Li and Li [Adv. Math., 308(2017), pp. 404-437] developed a direct method of moving planes for the fractional Laplacian. In this paper, we extend their method to the logarithmic Laplacian. We consider both the logarithmic equation and the system. To carry out the method, we establish two kinds of narrow region principle for the equation and the system separately. Then using these narrow region principles, we give the radial symmetry results for the solutions to semi-linear logarithmic Laplacian equations and systems on the ball.

1. Introduction. We are concerned in this paper with the following nonlocal semi-linear Dirichlet problem on the ball

\[
\begin{cases}
(-\Delta)^{L} u(x) = f(u,v), & x \in \Omega = B_R(0) \subset \mathbb{R}^n, \\
(-\Delta)^{L} v(x) = g(u,v), & x \in \Omega = B_R(0) \subset \mathbb{R}^n, \\
u(x) = 0, v(x) = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

(1)

Here, \((-\Delta)^{L}\) is the logarithmic Laplacian operator \([10]\) assuming the form

\[
(-\Delta)^{L} u(x) = C_n P.V. \int_{\mathbb{R}^n} \frac{u(x)1_{B_1(x)} - u(y)}{|x-y|^n} dy
\]

(2)

\[
= C_n P.V. \int_{B_1(x)} \frac{u(x) - u(y)}{|x-y|^n} dy - C_n \int_{\mathbb{R}^n \setminus B_1(x)} \frac{u(y)}{|x-y|^n} dy.
\]

\(C_n\) is a normalization positive constant. Let

\[
L_0 = \left\{ u : \mathbb{R}^n \to \mathbb{R} \left| \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^n} dx < +\infty \right. \right\}.
\]

For \(u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_0\), the integral on the right side of (2) is well defined.

In recent years, motivated by impressive applications in obstacle problem \([27, 25]\), optimization \([13]\), anomalous diffusion phenomena \([28, 24]\), finance \([12]\) and et.al.,
there has been tremendous interest in developing equations with nonlocal operators. In particular, the fractional Laplacian operator $(-\Delta)^{\alpha/2}$ can be defined by

$$(-\Delta)^{\alpha/2} u(x) = C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_{\epsilon}(x)} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy, \quad \alpha \in (0, 2). \quad (3)$$

And the prototype of fractional Laplacian equation is

$$\begin{cases}
(-\Delta)^{\alpha/2} u(x) = f(u), & x \in \Omega, \\
u(x) = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases} \quad (4)$$

Nonlocal equations, such as (1) and (4), do not act by point-wise differentiation but a global integral with respect to a singular kernel, that causes the main difficulty in studying problems involving it. There are several ways to deal with the nonlocal difficulty. By constructing a Dirichlet to Neumann operator of a degenerate elliptic equation, Caffarelli and Silvestre [4] introduced an extension method to localize equations involving fractional Laplacian. This extension method has been applied to a wide kinds of fractional Laplacian equations [11, 26, 14, 3, 2]. Another way to deal with the nonlocal difficulty is considering the corresponding equivalent integral equation, tremendous results including symmetry properties for the fractional Laplacian equation have been obtained [9, 8, 7, 20, 21, 23, 19, 29].

In the literature, many new kinds of maximum principles have been developed [18, 15, 16, 30, 6]. Those maximum principles together with the moving plane methods [17, 1, 5] make it possible to work directly on the nonlocal equation. By showing radially symmetry for solution of (4) with the direct method of moving plane, Chen et al.[6] proved that problem (4) with $\Omega = \mathbb{R}^n$ has no positive solution in $L_\alpha \cap C^{1,1}_{loc}(\mathbb{R}^n)$ ($L_\alpha := \{ u : \mathbb{R}^n \to \mathbb{R} | \int_{\mathbb{R}^n} \frac{|u(y)|}{1+|x|^{n+\alpha}} \, dy < +\infty \}$) in the subcritical case and the solution is radially symmetric in the critical case for all $\alpha \in (0, 2)$. In our previous work [22], we have generalized the direct method of moving planes to the system of fractional Laplacian.

Logarithmic Laplacian (2) could be seen as an extremal of the fractional Laplacian. In fact, $\mathcal{F}((-\Delta)^L)(\zeta) = 2 \ln |\zeta|$ and

$$\lim_{\alpha \to 0^+} \partial_\alpha((-\Delta)^\alpha u(x)) = (-\Delta)^L u(x), \quad \forall u \in C^2(\mathbb{R}^n)$$

(see [10] for more information). Different from the fractional Laplacian, the kernel of the logarithmic Laplacian is $|x|^{-n}$, which is non-integrable at the origin and also at infinity.

It is natural to seek the properties of the equation involving the logarithmic Laplacian operator,

$$\begin{cases}
(-\Delta)^L u(x) = f(u), & x \in \Omega = B_R(0), \\
u(x) = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases} \quad (5)$$

Chen and Weth [10] has considered (5) and obtained the following Theorem 1.1 by the moving planes method with a maximum principle for narrow domain based on the Aleksandrov-Bakelman-Pucci (ABP) estimate.

**Theorem 1.1.** Let $u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_0$ be a positive solution of equation (5) with $f(\cdot)$ being locally Lipschitz continuous on $[0, +\infty)$. Then $u$ must be radially symmetric and monotone decreasing about the origin.

The motivation of this present paper is to generalize the direct method of moving planes [6] to the logarithmic Laplacian equation and system. We work directly on
the equation and give a new narrow region principle for logarithmic Laplacian (see Proposition 1). Then as an application, we give a simple proof of Theorem 1.1.

Our main concern is the system case. A narrow region principle for problem (1) is given in Proposition 2, by using which we obtain the following symmetry result.

**Theorem 1.2.** Let \((u, v) \in \left( L_0 \cap C_{loc}^{1,1}(\mathbb{R}^n) \right)^2\) be a positive solution of system (1) with \(f(\cdot, \cdot), g(\cdot, \cdot)\) being locally Lipschitz continuous and satisfying
\[
\begin{align*}
  f(u, v_1) &< f(u, v_2), \quad \forall u \geq 0, 0 \leq v_1 < v_2; \\
  g(u_1, v) &< g(u_2, v), \quad \forall v \geq 0, 0 \leq u_1 < u_2.
\end{align*}
\]
Then \(u, v\) must be radially symmetric and monotone decreasing about the origin.

The paper is organized as follows. We devote Section 2 to the scalar equation (5), including the important narrow region principle for the logarithmic Laplacian equation. We prove Theorem 1.2 in Section 3. Note that in the following, \(c\) and \(C\) will be constants which can be different from line to line.

2. Symmetry result for the logarithmic Laplacian equation. In what follows, we shall use the method of moving planes. Choose any direction to be the \(x_1\) direction. For each \(\lambda \in \mathbb{R}\), we write \(x = (x_1, x')\) with \(x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}\) and define \(\Sigma_\lambda := \{ x \in \mathbb{R}^n \mid x_1 < \lambda \}\), \(T_\lambda := \partial \Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 = \lambda \}\). For each point \(x = (x_1, x') \in \Sigma_\lambda\), let \(x^\lambda = (2\lambda - x_1, x')\) be the reflected point with respect to the hyperplane \(T_\lambda\). Define the reflected functions by \(u_\lambda(x) = u(x^\lambda)\) and introduce function
\[
U_\lambda(x) = u_\lambda(x) - u(x).
\]
For any \(A \subset \Sigma_\lambda\), let \(\tilde{A}\) be the reflection of \(A\) with respect to \(T_\lambda\). According to the fact that for all \(f \in C_{loc}^{1,1}(\mathbb{R}^n) \cap L_0\),
\[
((-\Delta)^l f_\lambda)(x) = ((-\Delta)^l f)(x^\lambda),
\]
we know that for all \(x \in \Sigma_\lambda \cap \Omega\),
\[
(-\Delta)^l U_\lambda(x) = ((-\Delta)^l u)(x^\lambda) - ((-\Delta)^l u)(x) = f(u_\lambda(x)) - f(u(x)) = c_\lambda(x) U_\lambda(x),
\]
where
\[
c_\lambda(x) = \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)}.
\]
Without the classical maximum principle, we shall introduce the following narrow region principle, which will play an important role in the proof of Theorem 1.1.

**Proposition 1** (Narrow Region Principle for Equation). Let \(\Omega_{\lambda,l} = \{ x \in \Omega \mid \lambda - l < x_1 < \lambda \}\). Assume that \(U_\lambda(x) \in C_{loc}^{1,1}(\mathbb{R}^n) \cap L_0\) satisfies
\[
\begin{align*}
  &\begin{cases}
    (-\Delta)^l U_\lambda(x) + c(x) U_\lambda(x) \geq 0, & x \in \Omega_{\lambda,l}, \\
    U_\lambda(x) \geq 0, & x \in \Sigma_\lambda \setminus \Omega_{\lambda,l},
  \end{cases}
\end{align*}
\]
with \(c(\cdot)\) being bounded from below. Then there exists \(l_0 > 0\) such that for all \(0 < l \leq l_0\) and \(\lambda \leq 0\), we have
\[
U_\lambda(x) \geq 0, \quad x \in \Omega_{\lambda,l}.
\]
Proof. Assume for contradiction that there is \( x_0 \in \Omega_{\lambda,l} \) such that \( U_\lambda(x_0) < 0 \). Without loss of generality, we assume

\[
U_\lambda(x_0) = \min_{x \in \Omega_{\lambda,l}} U_\lambda(x) < 0.
\]

By the definition of logarithmic Laplacian, we write

\[
(-\Delta)^L U_\lambda(x_0) = C_n(I + II),
\]

where

\[
I = P.V. \int_{B_1(x_0)} \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^n} dy, \quad II = - \int_{\mathbb{R}^n \setminus B_1(x_0)} \frac{U_\lambda(y)}{|x_0 - y|^n} dy.
\]

Let \( A = B_1(x_0^n) \cap \Sigma_\lambda \), \( B = (B_1(x_0^n) \cap \Sigma_\lambda) \setminus A \). Then \( \tilde{A} = B_1(x_0^n) \setminus \Sigma_\lambda \) and \( B_1(x_0^n) = A \cup B \cup \tilde{A} \). Consequently,

\[
I = P.V. \int_A \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^n} dy + P.V. \int_B \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^n} dy
+ \int_{\tilde{A}} \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^n} dy
= P.V. \int_A \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^n} dy + P.V. \int_B \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^n} dy
+ \int_A \frac{U_\lambda(x_0) + U_\lambda(y)}{|x_0 - y|^n} dy.
\]

Due to \(|x_0 - y| < |x_0 - y^\lambda|\), \( \forall x_0, y \in \Sigma_\lambda \) and \( x_0 \) is the minimal point of \( U_\lambda(x) \) in \( \Sigma_\lambda \), it follows that

\[
I \leq \int_A \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^n} dy + P.V. \int_B \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^n} dy
+ \int_A \frac{U_\lambda(x_0) + U_\lambda(y)}{|x_0 - y|^n} dy
= 2U_\lambda(x_0) \int_A \frac{1}{|x_0 - y|^n} dy + P.V. \int_B \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^n} dy
\leq 2U_\lambda(x_0) \int_A \frac{1}{|x_0 - y|^n} dy + \int_B \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^n} dy.
\]

Let \( C = \Sigma_\lambda \setminus B_1(x_0^n) \) and \( \tilde{B}, \tilde{C} \) be the reflection of \( B, C \) with respect to \( T_\lambda \). A direct computation shows

\[
II = - \int_{\mathbb{R}^n \setminus B_1(x_0^n)} \frac{U_\lambda(y)}{|x_0 - y|^n} dy
= - \int_C \frac{U_\lambda(y)}{|x_0 - y|^n} dy - \int_B \frac{U_\lambda(y)}{|x_0 - y|^n} dy - \int_\tilde{C} \frac{U_\lambda(y)}{|x_0 - y|^n} dy
= - \int_C \frac{U_\lambda(y)}{|x_0 - y|^n} dy + \int_B \frac{U_\lambda(y)}{|x_0 - y|^n} dy + \int_C \frac{U_\lambda(y)}{|x_0 - y|^n} dy
= - \int_C \frac{U_\lambda(y)}{|x_0 - y|^n} dy + \int_B \frac{U_\lambda(y)}{|x_0 - y|^\lambda|^n} dy + \int_C \frac{U_\lambda(y)}{|x_0 - y|^\lambda|^n} dy
= - \int_C \frac{U_\lambda(y)}{|x_0 - y|^n} dy + \int_B \frac{U_\lambda(y)}{|x_0 - y|^\lambda|^n} dy + \int_C \frac{U_\lambda(y)}{|x_0 - y|^\lambda|^n} dy.
\]

Thus, we have

\[
(-\Delta)^L U_\lambda(x_0) = C_n(I + II) < 0
\]

by the minimality of \( x_0 \) in \( \Sigma_\lambda \).
By the fact that $U_\lambda(y) \geq 0$, for $y \in C \setminus \Omega \subset \Sigma_\lambda \setminus \Omega_\lambda$, and $|x_0 - y| < |x_0 - y^\lambda|$, for $x_0, y \in \Sigma_\lambda$, we derive

$$II \leq -\int_{C \setminus \Omega} U_\lambda(y) \left( \frac{1}{|x_0 - y|^n} - \frac{1}{|x_0 - y^\lambda|^n} \right) dy + \int_B U_\lambda(y) \left( \frac{1}{|x_0 - y|^n} - \frac{1}{|x_0 - y^\lambda|^n} \right) dy \quad (12)$$

$$\leq \ -U_\lambda(x_0) \int_{C \setminus \Omega} \left( \frac{1}{|x_0 - y|^n} - \frac{1}{|x_0 - y^\lambda|^n} \right) dy + \int_B U_\lambda(y) \left( \frac{1}{|x_0 - y|^n} - \frac{1}{|x_0 - y^\lambda|^n} \right) dy$$

$$\leq \ -U_\lambda(x_0) \int_{C \setminus \Omega} \frac{1}{|x_0 - y|^n} dy + \int_B U_\lambda(y) \left( \frac{1}{|x_0 - y|^n} - \frac{1}{|x_0 - y^\lambda|^n} \right) dy$$

$$\leq \ -cU_\lambda(x_0) + \int_B \frac{U_\lambda(y)}{y - y^\lambda} dy.$$

Here $c > 0$ is a constant, and we have used the fact $\int_{C \setminus \Omega} \frac{1}{|x_0 - y|^n} dy < |C \setminus \Omega| \leq |\Omega|$, since $|x_0 - y| > 1$, for all $y \in C$.

Combining (11) and (12), we get

$$I + II \leq 2U_\lambda(x_0) \int_A \frac{1}{|x_0 - y|^n} dy + \int_B U_\lambda(y) \left( \frac{1}{|x_0 - y|^n} - \frac{1}{|x_0 - y^\lambda|^n} \right) dy - cU_\lambda(x_0)$$

$$< \left( \int_{A \cup B} \frac{1}{|x_0 - y|^n} dy - c \right) U_\lambda(x_0).$$

Plugging the above inequality into (10), we obtain

$$(-\Delta)^L U_\lambda(x_0) \leq C_n \left( \int_{A \cup B} \frac{1}{|x_0 - y|^n} dy - c \right) U_\lambda(x_0). \quad (13)$$

Using the assumption that $c(\cdot)$ is bounded from below, i.e. there is a constant $c_1$, such that $c(x) \geq c_1$. Now we have

$$(-\Delta)^L U_\lambda(x_0) + c(x_0)U_\lambda(x_0) \leq \left( C_n \int_{A \cup B} \frac{1}{|x_0 - y|^n} dy - cC_n + c_1 \right) U_\lambda(x_0). \quad (14)$$

Let $D = \{ y \mid (x_0)_1 + l < y_1 < (x_0)_1 + \frac{1}{2}, |y' - (x_0)'| < \frac{1}{4} \} \subset (A \cup B) \subset \Sigma_\lambda$ and $\omega_{n-2} = |\partial B_1(0)|$ in $\mathbb{R}^{n-1}$. It is not difficult to verify that

$$\int_{A \cup B} \frac{1}{|x_0 - y|^n} dy \geq \int_D \frac{1}{|x_0 - y|^n} dy \quad (15)$$

$$= \int_l^{\frac{1}{4}} \int_0^{\frac{1}{4}} \frac{\omega_{n-2} r^{n-2}}{(\rho^2 + \tau^2)^{\frac{n}{2}}} d\rho d\tau$$

$$= \int_l^{\frac{1}{4}} \int_0^{\frac{1}{4}} \frac{\omega_{n-2} \rho^{n-2} \tau^{n-2}}{(\rho^2 + \tau^2)^{\frac{n}{2}}} \rho d\rho$$

$$\geq \int_l^{\frac{1}{4}} \int_0^{\frac{1}{4}} \frac{\omega_{n-2} \rho^{n-2}}{(1 + \tau^2)^{\frac{n}{2}}} d\rho$$

$$= c' \ln \frac{1}{2l} \to +\infty, \text{ as } l \to 0^+.$$

Therefore, there is $l_0 > 0$, such that for all $l \in (0, l_0]$, we have $C_n \int_{A \cup B} \frac{1}{|x_0 - y|^n} dy > cC_n - c_1$, and

$$(-\Delta)^L U_\lambda(x_0) + c(x_0)U_\lambda(x_0) < \left( C_n \int_{A \cup B} \frac{1}{|x_0 - y|^n} dy - cC_n + c_1 \right) U_\lambda(x_0) < 0, \quad (16)$$
which leads to a contradiction with (9).

Now we prove Theorem 1.1 by using the method of moving planes directly on the equation. We divide our proof into two steps.

**Step 1.** There exists $\lambda^* \in (-R, 0)$, such that
\[ U_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda, \]  
(17)
for all $\lambda \in (-R, \lambda^*)$.

*Proof of Step 1.* Actually, we can choose $\lambda^* = -R + \lambda_0$, in which $\lambda_0 > 0$ is given by Proposition 1.

It is apparent from $u(x) = 0$, if $x \in \mathbb{R}^n \setminus \Omega$ and $u(x) > 0$, if $x \in \Omega$, that $U_\lambda(x) \geq 0$ when $x \in \Sigma_\lambda \setminus \Omega$. From (8), and the fact that $u(x)$, $u_\lambda(x)$ are bounded, we know that $c_\lambda(x)$ is bounded. Now that $\Omega_{\lambda, l} = \{x \in \Omega | -R < x_1 < \lambda\} = \Sigma_\lambda \cap \Omega$ is a narrow region, by using Proposition 1, we obtain $U_\lambda(x) \geq 0$, $x \in \Sigma_\lambda \cap \Omega$.

Consequently, there holds (17).

We now move the plane $T_\lambda$ to the right as long as (17) holds to its limiting position. Define
\[ \lambda_0 = \sup \{\lambda \leq 0 \mid U_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda\}. \]

**Step 2.** $\lambda_0 = 0$.

*Proof of Step 2.* Suppose $\lambda_0 < 0$, we show that the plane $T_\lambda$ can be moved further right.

By continuity, $U_{\lambda_0}(x) \geq 0$, for all $x \in \Sigma_{\lambda_0}$. Moreover, we have
\[ U_{\lambda_0}(x) > 0, \quad \forall x \in \Sigma_{\lambda_0} \cap \Omega. \]  
(18)

If (18) is not true, there exists some point $x_0 \in \Sigma_{\lambda_0} \cap \Omega$ such that
\[ U_{\lambda_0}(x_0) = 0. \]

On the one hand, equation (7) tells us that
\[ (-\Delta)^L U_{\lambda_0}(x_0) = f(u_{\lambda_0}(x_0)) - f(u(x_0)) = 0. \]  
(19)
One the other hand, by direct computation and using the fact that $|x_0 - y_\lambda| > |x_0 - y|$, $\forall y \in \Sigma_{\lambda_0}$ and $U_{\lambda_0} \geq (\neq) 0$ in $\Sigma_{\lambda_0}$, we know that
\[ (-\Delta)^L U_{\lambda_0}(x_0) = C_n P.V. \int_{\mathbb{R}^n} \frac{-U_{\lambda_0}(y)}{|x_0 - y|^n} dy \]
(20)
\[ = C_n P.V. \left( \int_{\Sigma_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|x_0 - y|^n} dy + \int_{\Sigma_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|x_0 - y|^n} dy \right) \]
\[ = C_n P.V. \left( \int_{\Sigma_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|x_0 - y|^n} dy + \int_{\Sigma_{\lambda_0}} \frac{U_{\lambda_0}(y)}{|x_0 - y_\lambda|^n} dy \right) \]
\[ = C_n P.V. \int_{\Sigma_{\lambda_0}} U_{\lambda_0}(y) \left( \frac{1}{|x_0 - y_\lambda|^n} - \frac{1}{|x_0 - y|^n} \right) dy \]
\[ < 0, \]
which contradicts with (19). This proves (18).
It follows from (18) and the continuity that for all given \( \delta \in (0, l_0) \) and \( \delta < R + \lambda_0 \), in which \( l_0 \) is given by Proposition 1, there is \( c_0 > 0 \) such that
\[
U_{\lambda_0}(x) \geq c_0 > 0, \quad x \in \Sigma_{\lambda_0 - \delta} \cap \Omega.
\]

Since \( U_\lambda(x) \) depends on \( \lambda \) continuously, there exists \( \epsilon > 0 \) (\( \epsilon < \min\{l_0 - \delta, -\lambda_0\} \)), such that \( \forall \lambda \in (\lambda_0, \lambda_0 + \epsilon) \),
\[
U_\lambda(x) \geq 0, \quad x \in \Sigma_{\lambda_0 - \delta} \cap \Omega.
\]

While for all \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \), \( \Omega_{\lambda, \lambda - \lambda_0 + \delta} \subset \Omega_{\lambda, \delta + \epsilon} \) is a narrow region. By using Proposition 1, we have
\[
U_\lambda(x) \geq 0, \quad x \in \Omega_{\lambda, \lambda - \lambda_0 + \delta}.
\]

Note that \( \Sigma_\lambda = \Sigma_\lambda \setminus \Omega \cup \Sigma_{\lambda_0 - \delta} \cap \Omega \cup \Omega_{\lambda, \lambda - \lambda_0 + \delta} \) and \( \lambda_0 + \epsilon < 0, U_\lambda(x) \geq 0 \), for \( x \in \Sigma_\lambda \setminus \Omega \). Taking (21) (22) into account, we have for all \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \),
\[
U_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda,
\]
which contradicts with the definition of \( \lambda_0 \).

Therefore, \( \lambda_0 = 0 \).

Moreover, from (18), we have \( u(x_1, x') < u(y_1, x') \), for all \(-R < x_1 < y_1 \leq 0, (x_1, x'), (y_1, y') \in B_R(0) \).

Then by moving plane from \( x_1 = R \) to the left, we derive that \( u \) is symmetric with respect to \( x_1 = 0 \). Because of the fact that any direction can be chosen as \( x_1 \) direction, \( u(x) \) must be radially symmetric and monotonic decreasing about the origin.

\[ \square \]

3. Symmetry result for the logarithmic Laplacian system. To deal with the logarithmic Laplacian system (1), we shall use the same spirit of direct method of moving planes as in Section 2, while we need a new version of the narrow region principle to treat the coupling of the system.

We use \( \Sigma_\lambda, T_\lambda \) and \( x^\lambda \) in the same way as in Section 2. Define the reflected functions by \( u_\lambda(x) = u(x^\lambda), v_\lambda(x) = v(x^\lambda) \) and introduce functions
\[
U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v_\lambda(x) - v(x).
\]

It follows that for all \(-R < \lambda \leq 0 \) and for all \(x \in \Sigma_\lambda \cap \Omega, \)
\[
(-\Delta)^L U_\lambda(x) = ((-\Delta)^L u)(x^\lambda) - ((-\Delta)^L u)(x)
\]
\[
= f(u(x^\lambda), v(x^\lambda)) - f(u(x), v(x))
\]
\[
= \xi_1(x) U_\lambda(x) + \eta_1(x) V_\lambda(x),
\]
in which
\[
\xi_1(x) = \frac{f(u_\lambda(x), v(x)) - f(u(x), v(x))}{u_\lambda(x) - u(x)}, \quad \eta_1(x) = \frac{f(u_\lambda(x), v_\lambda(x)) - f(u_\lambda(x), v(x))}{v_\lambda(x) - v(x)}.
\]

Similarly, there holds
\[
(-\Delta)^L V_\lambda(x) = \xi_2(x) U_\lambda(x) + \eta_2(x) V_\lambda(x), \quad (24)
\]
where
\[
\xi_2(x) = \frac{g(u_\lambda(x), v_\lambda(x)) - g(u(x), v_\lambda(x))}{u_\lambda(x) - u(x)}, \quad \eta_2(x) = \frac{g(u(x), v_\lambda(x)) - f(u(x), v(x))}{v_\lambda(x) - v(x)}.
\]
Proposition 2 (Narrow Region Principle for System). Let $\Omega_{\lambda,l} := \{ x \in \Sigma_{l} \cap \Omega | \lambda - l < x_{1} < \lambda \}$. Assume that $(U_{\lambda}(x), V_{\lambda}(x)) \in \left( C^{1,1}_{loc}(\mathbb{R}^{n}) \cap L_{0} \right)^{2}$ is a solution of system

\[
\begin{cases}
(-\Delta)^{l} U_{\lambda}(x) = \xi_{1}(x) U_{\lambda}(x) + \eta_{1}(x) V_{\lambda}(x), & x \in \Omega_{\lambda,l}, \\
(-\Delta)^{l} V_{\lambda}(x) = \xi_{2}(x) U_{\lambda}(x) + \eta_{2}(x) V_{\lambda}(x), & x \in \Omega_{\lambda,l}, \\
U_{\lambda}(x) \geq 0, V_{\lambda}(x) \geq 0, & x \in \Sigma_{l} \cap \Omega_{\lambda,l}
\end{cases}
\]  

(25)

with $\xi_{i}(x)$ and $\eta_{i}(x)$ ($i = 1, 2$) are bounded from above and satisfying

\[
\xi_{i}(x) > 0, \quad \eta_{1}(x) > 0, \quad x \in \Omega_{\lambda,l}.
\]  

(26)

Then there exists $l_{0} > 0$ such that for all $0 < l \leq l_{0}$ and all system (25) with $\lambda \leq 0$, 

(a) if there is $x^{*} \in \Omega_{\lambda,l}$ satisfying $U_{\lambda}(x^{*}) = \min_{x \in \Sigma_{l}} U_{\lambda}(x) < 0$, then

\[
V_{\lambda}(x^{*}) < 2U_{\lambda}(x^{*}) < 0;
\]

(b) if there is $y^{*} \in \Omega_{\lambda,l}$ satisfying $V_{\lambda}(y^{*}) = \min_{y \in \Sigma_{l}} V_{\lambda}(y) < 0$, then

\[
U_{\lambda}(y^{*}) < 2V_{\lambda}(y^{*}) < 0.
\]

Proof. Let $x^{*} \in \Omega_{\lambda,l}$ and $U_{\lambda}(x^{*}) = \min_{x \in \Sigma_{l}} U_{\lambda}(x^{*}) < 0$, for some $l > 0$ and $\lambda \leq 0$.

A same computation as in the proof of Proposition 1 (see inequality (13)) gives us that

\[
(-\Delta)^{l} U_{\lambda}(x^{*}) < C_{n} \left( \int_{A \cup B} \frac{1}{|x^{*} - y|^{n}} dy - c \right) U_{\lambda}(x^{*}),
\]  

(27)

in which constants $C_{n}, c$ are independent on $\lambda$. Combining the above estimate with equation (25), we get

\[
\eta_{1}(x^{*}) V_{\lambda}(x^{*}) \leq b_{1}(x^{*}, \lambda) U_{\lambda}(x^{*}),
\]  

(28)

where

\[
b_{1}(x^{*}, \lambda) = C_{n} \left( \int_{A \cup B} \frac{1}{|x^{*} - y|^{n}} dy - c \right) - \xi_{1}(x^{*}).
\]

Thanks to (26), it follows that

\[
V_{\lambda}(x^{*}) \leq \frac{b_{1}(x^{*}, \lambda)}{\eta_{1}(x^{*})} U_{\lambda}(x^{*}).
\]  

(29)

Due to the assumptions that $\xi_{1}(x)$ is bounded from above, $\eta_{1}(x) > 0$ is bounded and (15), it follows that there is $l_{0} > 0$ such that for all $l \in (0, l_{0}]$, $\frac{b_{1}(x^{*}, \lambda)}{\eta_{1}(x^{*})} > 2$, which completes the proof of conclusion (a).

The proof of conclusion (b) is similar and we shall omit it. \qed

With the help of the narrow region principle for system, we give the proof of Theorem 1.2. The proof is divided in two steps.

Step 1. For $\lambda \in (-R, -R + l_{0})$ and $\lambda \leq 0$, we have

\[
U_{\lambda}(x) \geq 0 \text{ and } V_{\lambda}(x) \geq 0, \quad \forall x \in \Sigma_{l}.
\]  

(30)

Here $l_{0}$ is given by Proposition 2.

Proof of Step 1. Assume for contradiction that there is a $\lambda \in (-R, -R + l_{0})$ ($\lambda \leq 0$) and a point $x^{*} \in \Sigma_{l}$ such that $U_{\lambda}(x^{*}) < 0$. Since, $U_{\lambda}(x) \geq 0$, for each $\lambda \leq 0$ and all $x \in \Sigma_{l} \cap \Omega$, we could assume $x^{*} \in \Sigma_{l} \cap \Omega$,

\[
U_{\lambda}(x^{*}) = \min_{x \in \Sigma_{l}} U_{\lambda}(x) < 0.
\]
We remark that system (23) and (24) satisfies all the assumptions in Proposition 2. Note that \( u(x) \) and \( v(x) \) are bounded on \( \mathbb{R}^n \). Together with the assumption that \( f, g \) are locally Lipschitz continuous, we know that \( \xi_t(x) \) and \( \eta_t(x) \) are uniformly bounded with respect to \( \lambda \). Moreover, it follows from (6) that \( \xi_t(x^*) > 0 \) and \( \eta_t(x^*) > 0 \). In this case, \( \Sigma_\lambda \cap \Omega \subset \Omega_{\lambda, 0} \) is a narrow region. An immediate consequence of Proposition 2 is

\[
V_\lambda(x^*) < 2U_\lambda(x^*) < 0. \tag{31}
\]

Since \( V_\lambda(x) \geq 0 \), for all \( \lambda \leq 0 \) and all \( x \in \Sigma_\lambda \setminus \Omega \), there exists a point \( y^* \in \Sigma_\lambda \cap \Omega \) such that

\[
V_\lambda(y^*) = \min_{y \in \Sigma_\lambda} V_\lambda(y) < 0.
\]

Using Proposition 2 again, we find that

\[
U_\lambda(y^*) < 2V_\lambda(y^*) < 0. \tag{32}
\]

Combining (31) and (32), we obtain

\[
V_\lambda(x^*) < 2U_\lambda(x^*) \leq 2U_\lambda(y^*) < 4V_\lambda(y^*) \leq 4V_\lambda(x^*).
\]

Noticing that \( V_\lambda(x^*) < 0 \), we get \( 1 > 4 \), which is a contradiction. \( \square \)

We now move the plane \( T_\lambda \) to the right as long as (30) holds to its limiting position. Define

\[
\lambda_0 = \sup\{\lambda \leq 0 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \in (-R, \lambda]\}.
\]

**Step 2.** \( \lambda_0 = 0. \)

**Proof.** Assume for contradiction that \( \lambda_0 < 0 \).

By continuity, there hold

\[
U_{\lambda_0}(x) \geq 0, V_{\lambda_0}(x) \geq 0, \forall x \in \Sigma_{\lambda_0}. \tag{33}
\]

Thus, without loss of generality, there are sequences \( \{\lambda_k\} \) and \( \{x^k\} \) satisfying

\[
\lambda_0 < \lambda_{k+1} < \lambda_k < 0, k = 1, 2, \ldots, \lim_{k \to \infty} \lambda_k = \lambda_0,
\]

\[
x^k \in \Sigma_{\lambda_k}, \quad U_{\lambda_k}(x^k) = \min_{x \in \Sigma_{\lambda_k}} U_{\lambda_k}(x) < 0.
\]

Due to the fact that \( U_{\lambda_k}(x) \geq 0 \) in \( \Sigma_{\lambda_k} \setminus \Omega \), it follows that \( x^k \in \Sigma_{\lambda_k} \cap \Omega \). By the boundedness of \( \Omega = B_R(0) \), there is a subsequence, which we still denote by \( \{x^k\} \), satisfying \( x^k \to x^* \), as \( k \to \infty \), and

\[
U_{\lambda_0}(x^*) = \lim_{k \to \infty} U_{\lambda_k}(x^k) \leq 0, x^* \in \bigcap_{k=1}^{\infty} \Sigma_{\lambda_k} \cap \Omega = \Sigma_{\lambda_0} \cap \Omega.
\]

Using (33), we know that \( U_{\lambda_0}(x^*) = 0 \).

There are two possible cases.

**Case 1.** \( x^* \in \Sigma_{\lambda_0} \cap \Omega \).

In this case, equation (23) becomes

\[
(-\Delta)^L U_{\lambda_0}(x^*) = \eta_1(x^*) V_{\lambda_0}(x^*) \geq 0, \tag{34}
\]

in which we have used (6). According to the fact that \( u \) is positive in \( \Omega \) and zero outside \( \Omega \), it follows \( U_{\lambda_0}(x) \not\equiv 0 \) in \( \Sigma_{\lambda_0} \). By a same computation as in (20), we have

\[
(-\Delta)^L U_{\lambda_0}(x^*) < 0,
\]
which contradicts with (34).

Case 2. $x^* \in T_{\lambda_{0}} \cap \Omega$.

In this case, for sufficiently large $k$, there hold $|x^k - x^*| < \frac{l_0}{2}$ and $|\lambda_k - \lambda_0| < \frac{l_0}{2}$, which means $x^k \in \Omega_{\lambda_k, l_0}$. By using Proposition 2, we obtain

$$V_{\lambda_k}(x^k) < 2U_{\lambda_k}(x^k) < 0.$$  \hfill (35)

Since $V_{\lambda_k}(x) \geq 0$ for $x \in \Sigma_{\lambda_k} \setminus \Omega$, then there is $y^k \in \Sigma_{\lambda_k} \cap \Omega$ such that

$$V_{\lambda_k}(y^k) = \min_{x \in \Sigma_{\lambda_k}} V_{\lambda_k}(x) < 0.$$  

Up to a subsequence, we may assume that $\lim_{k \to \infty} y^k = y^* \in \Sigma_{\lambda_0} \cap \Omega$ and $V_{\lambda_0}(y^*) = 0$.

There are also two possible cases. One is $y^* \in \Sigma_{\lambda_0} \cap \Omega$ and it will lead to a contradiction as in Case 1. The other one is $y^* \in T_{\lambda_{0}} \cap \Omega$. Hence, for sufficiently large $k$, $y^k \in \Omega_{\lambda_k, l_0/2}$ and Proposition 2 implies that

$$U_{\lambda_k}(y^k) < 2V_{\lambda_k}(y^k) < 0.$$  \hfill (36)

From (36) and (35), we derive

$$V_{\lambda_k}(x^k) < 2U_{\lambda_k}(x^k) \leq 2U_{\lambda_k}(y^k) < 4V_{\lambda_k}(y^k) \leq 4V_{\lambda_k}(x^k),$$

for sufficiently large $k$. Noticing that $V_{\lambda_k}(x^k) < 0$, we get $1 > 4$, which is a contradiction.

Therefore, $\lambda_0 = 0$. \hfill \Box

Using the same argument as above and applying the moving planes from $x_1 = R$, we will also find $\lambda_0 = 0$. Hence, $U_0(x) \equiv 0$ and $V_0(x) \equiv 0$, i.e. $u(x_1, x') = u(-x_1, x')$, $v(x_1, x') = v(-x_1, x')$. From the proof we also know that $u(x_1, x') < u(\bar{x}_1, x')$ and $v(x_1, x') < v(\bar{x}_1, x')$, if $|x_1| > |\bar{x}_1|$ and $(x_1, x'), (\bar{x}_1, x') \in \Omega$.

Since any direction can be chosen as $x_1$, we have $u$ and $v$ are radially symmetric and monotone decreasing about the origin. The proof is completed.

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