MOVING AND AMPLE CONES OF HOLOMORPHIC SYMPLECTIC FOURFOLDS

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Abstract. We analyze the ample and moving cones of holomorphic symplectic manifolds, in light of recent advances in the minimal model program. As an application, we establish a numerical criterion for ampleness of divisors on fourfolds deformation-equivalent to punctual Hilbert schemes of K3 surfaces.

1. Introduction

Let $F$ be an irreducible holomorphic symplectic variety. The integral cohomology group $H^2(F, \mathbb{Z})$ is equipped with a natural integral quadratic form $(,)$, known as the Beauville-Bogomolov form; the complex structure on $F$ induces a Hodge structure on the complex cohomology group $H^2(F, \mathbb{C})$. The birational geometry of $F$ is tightly coupled to these cohomological invariants. For example, the ample cone of a polarized K3 surface $(F, g)$ is explicitly determined by these structures on $H^2(F, \mathbb{Z})$ and the class $[g]$ [18, §2].

In higher-dimensions, qualitative descriptions of the ample cone have been obtained by Huybrechts [13] and Boucksom [3]. However, these fall short of the precise picture we have for K3 surfaces. The birational geometry of higher-dimensional varieties is much richer than surfaces, and this is reflected in the numerous invariants we can assign to these, e.g., the moving cone parametrizing effective divisors without fixed components. Again, we have a qualitative description of this due to Huybrechts [13] (and Theorem 7 below), but this is not sufficient to determine whether a given divisor is moving.

The last ten years have seen great advances in the birational geometry of holomorphic symplectic varieties and the classification of their birational contractions, especially in the four-dimensional case [23, 22, 19, 4, 5]. This is therefore a natural testing-ground for conjectures on

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the shape of the ample and moving cones. For fourfolds deformation-equivalent to the punctual Hilbert scheme of a K3 surface, we formulated in 1999 precise conjectures characterizing the ample and moving cones [9].

The recent progress in the log minimal model program [2] lends new impetus to the efforts to resolve these problems. Our principal result is Theorem [22], which gives one implication of our conjecture, namely, that the divisors claimed to be ample are in fact ample. This entails a numerical classification of extremal rays, given in Theorem [21]. We also prove general results about the structure of ample and moving cones on irreducible holomorphic symplectic varieties of arbitrary dimension.

The paper is organized as follows: in Section 2 we recall basic notation, constructions and conjectures relating to holomorphic symplectic fourfolds. Section 3 outlines applications of the minimal model program to our situation. In Section 4 we specialize to the four-dimensional example mentioned above and recall the conjectures of [9]. Section 5 offers an analysis ‘from first principles’ of cohomology classes of extremal rays. Finally, Section 6 contains the proof of the main theorem.

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2. Generalities on ample cones of holomorphic symplectic manifolds

Let $F$ be a projective irreducible holomorphic variety, $N_1(F,\mathbb{Z})$ the group of one-cycles (up to numerical equivalence), $N^1(F,\mathbb{Z})$ the group of divisor-classes, $NE_1(F) \subset N_1(F,\mathbb{R})$ the cone of effective curves, and $NE_1(F)$ its closure. The dual to $NE_1(F)$ in $N^1(F,\mathbb{R})$ is the nef cone of $F$. Recall that $\mathbb{R}_{\geq 0} \subset NE_1(F)$ is an extremal ray if whenever $\varrho = c_1C_1 + c_2C_2$ for $C_1, C_2 \in NE_1(F)$ and $c_1, c_2 > 0$ then $C_1, C_2 \in \mathbb{R}_{\geq 0}\varrho$.

Let $(,)$ denote the Beauville-Bogomolov form on $H^2(F,\mathbb{Z})$, normalized so that it is integral but not divisible. The induced $\mathbb{Q}$-valued form on $H^2(F,\mathbb{Z})$ is also denoted $(,)$. Let $C_F$ denote the connected component of the positive cone of $F$

$$\{ \alpha \in H^2(F,\mathbb{R}) \cap H^{1,1}(F,\mathbb{C}) : (\alpha, \alpha) > 0 \}$$
containing the polarization.

We recall some general facts:

• The form $(,)$ has signature $(3, \dim H^2(F, \mathbb{R}) - 3)$ on $H^2(F, \mathbb{R})$ and signature $(1, \dim H^2(F, \mathbb{R}) - 3)$ on $H^2(F, \mathbb{R}) \cap H^{1,1}(F, \mathbb{C})$ [11, 1.9].

• Fujiki [7, 11, 1.11] showed there exists a positive constant $c_0$ such that for each $\alpha \in H^2(F, \mathbb{R})$

$$\alpha^{\dim(F)} = c_0 (\alpha, \alpha)^{(\dim(F)/2).}$$

More generally, for each Chern class $c_i(F)$ there exists a constant $c_i$ such that

$$c_i(F)\alpha^{\dim(F) - i} = c_i (\alpha, \alpha)^{(\dim(F) - i)/2).$$

• Each divisor class $D$ with $(D, D) > 0$ is big [11, 3.10] [12].

• There is an integral formula for the Beauville-Bogomolov form [11, §1.9] [11]. Choose $\sigma \neq 0 \in \Gamma(F, \Omega^2_F)$, normalized such that

$$\int_F (\sigma \bar{\sigma})^{\dim(F)} = 1.$$

Then there exists a positive real constant $c$ such that

$$(\alpha, \beta) = c \int_F \alpha \beta (\sigma \bar{\sigma})^{\dim(F) - 1}$$

for all $\alpha, \beta \in H^{1,1}(F, \mathbb{C})$.

• Let $D$ be a nef and big divisor class. By Kawamata-Viehweg vanishing [6, §8], $D$ has no higher cohomology. By basepoint-freeness [6, §9], $ND$ is globally generated for some $N \gg 0$.

Let

$$\mathcal{K}_F \subset \mathcal{C}_F, \quad \overline{\mathcal{K}_F} \subset \overline{\mathcal{C}_F}$$

denote the Kähler cone of $F$ and its closure. The intersection $\mathcal{K}_F \cap H^2(F, \mathbb{Z})$ (resp. $\overline{\mathcal{K}_F} \cap H^2(F, \mathbb{Z})$) is the set of ample (resp. nef) divisors on $F$.

We recall the following result [11, §5]:

**Theorem 1.** Choose $\alpha \in \mathcal{C}_F$ ‘very general’, e.g., not orthogonal to any integral class, cf. [11, 5.9]. Then there exists an irreducible holomorphic variety $F'$ and correspondence $\Gamma \subset F \times F'$ inducing a birational map $\phi : F' \dasharrow F$ such that

• $\Gamma : H^2(F, \mathbb{Z}) \to H^2(F', \mathbb{Z})$ is an isomorphism respecting the Beauville-Bogomolov forms;

• $\Gamma_\ast \alpha \in \mathcal{K}_{F'}$. 

The correspondence $\Gamma$ is the specialization of the graph of an isomorphism $F'_t \sim F_t$, where $F'_t$ and $F_t$ are fibers of small deformations $\mathcal{F}'$, $\mathcal{F} \to \{ t : |t| < 1 \}$ of $F'$ and $F$ respectively.

**Example 2.** The simplest nontrivial example is the Atiyah flop. Let $F$ be a K3 surface containing a $(-2)$-curve $E$ and

$\mathcal{F} \to \{ t : |t| < 1 \}$

a general deformation of $F$, so the class $[E]$ does not remain algebraic. Let

$\mathcal{F}' \to \{ t : |t| < 1 \}$

denote the flop of $E$; the fiber $F'$ over $t = 0$ contains a $(-2)$-curve $E'$. Note that in this case $\phi : F' \sim F'$ but

$\Gamma = \text{Graph}(\phi) + E \times E' \subset F \times F'$.

**Remark 3.** From our example, it is evident that

$\phi^*\alpha \neq \Gamma^*\alpha$

in general. Equality holds iff

$\Gamma = \text{Graph}(\phi) + \sum Z_i$

where each $Z_i$ maps to a codimension $\geq 2$ subvariety in each factor.

Let

$\overline{\mathcal{BK}}_F \subset \mathcal{C}_F \subset H^2(F, \mathbb{R}) \cap H^{1,1}(F, \mathbb{C})$

denote the closure of the birational Kähler cone of $F$, i.e., the union $\mathcal{BK}_F$ of the Kähler cones of all holomorphic symplectic varieties birational to $F$. This has the following numerical interpretation:

**Proposition 4.** [13, 4.2] A class $\alpha \in \mathcal{C}_F$ lies in $\overline{\mathcal{BK}}_F$ if and only if $(\alpha, D) \geq 0$ for each uniruled divisor $D \subset F$.

**Definition 5.** An effective divisor $M$ on $F$ is moving if there exists an integer $N > 0$ such that $NM$ has no fixed components. The moving cone of $F$ is the cone generated by moving divisors.

**Remark 6.** It is clear from the definition that elements of the birational Kähler cone $\mathcal{BK}_F$ are contained in the moving cone.

**Theorem 7** (Symplectic interpretation of moving divisors). Each moving divisor is contained in the closure of the birational Kähler cone $\overline{\mathcal{BK}}_F$. Thus $\overline{\mathcal{BK}}_F$ equals the closure of the moving cone of $F$. 
Remark 8. Corollary 15 below is a partial converse to this result.

Proof. We are grateful to Professor D. Huybrechts for his help with this argument.

Suppose that $M$ is moving. To show that $M \in \overline{\text{BK}}_F$, it suffices that
$(M, D) \geq 0$ for each irreducible uniruled divisor $D \subset F$. We write
$n = \dim(F)$.

Replacing $M$ by a suitable multiple if necessary, we may assume that $M$ has no fixed
components, i.e., its base locus has codimension at least two.  There exists a diagram
\[
\begin{array}{ccc}
Z & \xrightarrow{p} & F' \\
q \downarrow & & \\
F \\
\end{array}
\]
where $Z \to F$ is a smooth projective resolution of the base locus of $|M|$ and $p$ is the resulting
morphism. Thus there exists an ample line divisor $H$ on $F'$ such that
\[q^* M = \sum_i c_i E_i + p^* H\]
where each $c_i \geq 0$ and $E_i$ is a $q$-exceptional divisor in $Z$.

Compute the Beauville-Bogomolov form by pulling back to $Z$:
\[
(M, D) = c \int_F [M][D](\sigma \bar{\sigma})^{n-1} = c \int_Z q^*[M]q^*[D]q^*((\sigma \bar{\sigma})^{n-1}) = c \int_Z (\sum_i c_i [E_i] + p^*[H])(q^*[D])q^*((\sigma \bar{\sigma})^{n-1}).
\]
First, note that
\[
\int_Z [E_i]q^*[D]q^*((\sigma \bar{\sigma})^{n-1}) = 0.
\]
Indeed, any degree-$(4n - 2)$ form pulled back from $F$ integrates to zero along $E_i$ because codim$_n q(E_i) \geq 2$. To evaluate the second term
\[
\int_Z p^*[H]q^*[D]q^*((\sigma \bar{\sigma})^{n-1}),
\]
observe that the intersection $p^*[H] \cap q^*[D]$ involves a semiample divisor and an effective divisor. In particular, we can express
\[
p^*[H] \cap q^*[D] = \sum_j n_j W_j, \quad n_j > 0,
\]
where each $W_j$ is a $(2n - 2)$-dimensional subvariety of $Z$. Thus we have
\[
\int_Z p^*[H]q^*[D]q^*((\sigma \bar{\sigma})^{n-1}) = \sum_j n_j \int_{W_j} q^*((\sigma \bar{\sigma})^{n-1}).
\]
Let $\tilde{W}_j \to W_j$ denote a resolution of singularities and $r : \tilde{W}_j \to F$ the induced morphism. We have
\[
\int_{\tilde{W}_j} q^*((\sigma\bar{\sigma})^{n-1}) = \int_{\tilde{W}_j} (r^*\sigma r^*\bar{\sigma})^{n-1} \geq 0
\]
because the integrand is a nonnegative multiple of the volume form on $\tilde{W}_j$.

We have the following result of Boucksom [3] and Huybrechts [13, §3]:

**Theorem 9.** A class $\alpha \in C_F$ (resp. $\overline{C}_F$) is in $K_F$ (resp. $\overline{K}_F$) if and only if $\alpha.C > 0$ (resp. $\alpha.C \geq 0$) for each rational curve $C \subset F$.

However, this does not imply, a priori, that these classes determine a locally-finite rational polyhedral cone, nor does it provide a geometric interpretation of these rational curves.

The Cone Theorem does shed some light on this.

**Proposition 10 (Cone Theorem for varieties with trivial canonical class).** [17, 3.7] Let $Y$ be a smooth projective variety with $K_Y = 0$ and $\Delta$ an effective $\mathbb{Q}$-divisor on $Y$ such that $(Y, \Delta)$ has Kawamata log terminal singularities (see [15, 2.13] for the definition.) Then the closed cone of effective curves $\overline{NE}_1(Y)$ can be expressed
\[
\overline{NE}_1(Y) = \overline{NE}_1(Y)_{\Delta.C \geq 0} + \sum_j \mathbb{R}_{\geq 0}[C_j], \quad \Delta.C_j < 0
\]
where the $C_j$ are extremal and represent rational curves collapsed by contractions of $Y$. This is locally finite in the following sense: For an ample divisor $A$ and $\epsilon > 0$, there are a finite number of $C_j$ with $C_j.(\Delta + \epsilon A) < 0$.

Which parts of $\overline{NE}_1(F)$ can be analyzed using this fact?

**Proposition 11.** Let $M$ be a divisor class contained in $\overline{K}_F$ and in the interior of $\overline{BK}_F$. Then $\overline{K}_F$ is locally-finite rational polyhedral in a neighborhood of $M$.

**Proof.** There is nothing to prove unless $M$ lies on the boundary of $\overline{K}_F$.

We translate the statement using the duality between curves and divisors: Suppose that $R$ is an extremal ray of $\overline{NE}_1(F)$ such that $R^\perp$ meets the interior of $\overline{BK}_F$. We claim that $\overline{NE}_1(F)$ is finite rational polyhedral in a neighborhood of $R$.

Each $M$ in the interior of $\overline{BK}_F$ is contained in $C_F$, i.e., $(M, M) > 0$. Since $R$ is orthogonal to $M$ we have $(R, R) < 0$. 

By Theorem 7, there exists an \( \alpha \in \mathcal{BK}_F \) with \( \alpha R < 0 \). We may assume \( \alpha \) is ‘general’ in the sense of Theorem 1. Let \( F' \) be the hyperkähler manifold birational to \( F \) with Kähler class \( \alpha \). Suppose that \( A' \) is a very ample divisor of \( F' \) and let \( A \) be its proper transform in \( F \). We have \( AR < 0 \) as well.

Choose \( \epsilon > 0 \in \mathbb{Q} \) such that \( \epsilon A \) is Kawamata log terminal. Indeed, since \( F \) is smooth if we choose \( \epsilon \) such that \( \frac{1}{\epsilon} > \max_{x \in F} \{ \text{mult}_x(A) \} \) then the singularities are Kawamata log terminal by [16, 8.10]. Then the Cone Theorem implies that the effective cone is finite polyhedral in some neighborhood of \( R \).

\[ \Box \]

3. Application of the log minimal model program

We will use the following consequence of the log minimal model program:

**Theorem 12.** Let \( Y \) be a smooth projective variety with \( K_Y \) trivial. Suppose that \( D_1, \ldots, D_r \) are big divisors on \( Y \). Then the ring

\[ \bigoplus_{(n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r} \Gamma(F, \mathcal{O}_F(n_1D_1 + \ldots + n_rD_r)) \]

is finitely generated.

**Proof.** There exists a positive \( \epsilon \in \mathbb{Q} \) such that each \( \epsilon D_i \) has divisorial log terminal singularities (see [15, 2.13] for the definition.) Indeed, if we choose \( \epsilon \) such that

\[ 1/\epsilon > \max_{y \in Y, i=1,\ldots,r} \{ \text{mult}_y(D_i, y) \} \]

then [16, 8.10] guarantees the singularities have the desired property. It follows from [2, 1.1.9] that the graded ring

\[ \bigoplus_{(m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r} \Gamma(F, \mathcal{O}_F(\lfloor \sum_i m_i \epsilon D_i \rfloor)) \]

is finitely generated. It remains finitely generated when we restrict to the multidegrees such that each \( m_i \epsilon \in \mathbb{Z} \).

\[ \Box \]

**Proposition 13.** Let \( F \) be a projective irreducible holomorphic symplectic manifold. Consider the chamber decomposition

\[ \bigcup_{F'} \mathcal{K}_{F'} \subset \overline{\mathcal{BK}}_F \]

where the union is taken over holomorphic symplectic birational models of \( F \). This is locally finite polyhedral near divisors \( M \in \mathcal{C}_F \), i.e., given a small neighborhood \( U \ni M \), the cone \( \overline{\mathcal{BK}}_F \cap U \) is defined in \( U \) by a
finite number of rational linear inequalities, with chambers equal to the complement to a finite number of rational hyperplanes.

In particular, \( \overline{K}_F \cap C_F \) is locally-finite rational polyhedral at each divisor \( M \in \overline{K}_F \cap C_F \).

**Proof.** Recall that divisors \( M \) with \( (M, M) > 0 \) are big. Since being big is an open condition in \( N^1(F, \mathbb{R}) \), we can express \( M \) as an element of the convex hull of a collection of big divisors \( D_1, \ldots, D_r \) which freely generate \( N^1(F, \mathbb{Z}) \). Consider the intersection of \( \mathcal{B}K_F \) with the cone \( \langle D_1, \ldots, D_r \rangle \);

it suffices to verify that this is locally finite polyhedral.

The graded ring associated to these divisors

\[
R(D_1, \ldots, D_r) := \bigoplus_{(n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r} \Gamma(F, \mathcal{O}_F(n_1D_1 + \ldots + n_rD_r))
\]

is finitely generated by Theorem 12. As discussed in [10, 2.9], this finite generation has implications for the birational geometry of \( F \):

- the subcone \( \overline{K}_F \cap \langle D_1, \ldots, D_r \rangle \subset \langle D_1, \ldots, D_r \rangle \)
  is determined by a finite number of linear rational inequalities;
- the moving divisors in \( \langle D_1, \ldots, D_r \rangle \) are the union of the rational polyhedral cones

\[
\bigcup_{F'} \overline{K}_{F'} \cap \langle D_1, \ldots, D_r \rangle
\]

where each \( F \to F' \) is a small birational modification.

Indeed, the chamber decompositions of \( \langle D_1, \ldots, D_r \rangle \) are governed by the various Geometric Invariant Theory quotients of \( R(D_1, \ldots, D_r) \) under the \( \mathbb{G}_m^r \)-action associated with the multigrading. We consider linearizations of the action corresponding to positive characters of \( \mathbb{G}_m^r \).

Theorem 7 implies each \( F' \) is also a projective irreducible holomorphic symplectic manifold, which completes the proof. \( \square \)

**Corollary 14.** Let \( F \) be a projective irreducible holomorphic symplectic manifold. Then the intersection

\[
\overline{NE}_1(F) \cap \{R \in H_2(F, \mathbb{R}) : (R, R) < 0\}
\]

is locally-finite rational polyhedral.

**Proof.** As in the proof of Proposition 11, supporting hyperplanes to \( \overline{NE}_1(F) \) in the region

\[
\{R : (R, R) < 0\}
\]

correspond to divisor classes \( M \) with \( (M, M) > 0 \), and Proposition 13 applies. \( \square \)
Corollary 15. Let $F$ be a projective irreducible holomorphic symplectic manifold. Each divisor $M \in \overline{\mathcal{B}_F} \cap \mathcal{C}_F$ is moving.

Proof. Proposition 13 implies that $M$ corresponds to a nef and big divisor $M'$ on some small birational modification $F \sim F'$, where $F'$ is a projective irreducible holomorphic symplectic manifold. Thus basepoint-freeness implies that some multiple of $M'$ is basepoint free. Since $F$ and $F'$ are isomorphic in codimension one, we conclude that $M$ is moving on $F$. □

Remark 16. This analysis only applies to divisor classes with positive Beauville-Bogomolov form. The case where the form is zero is remains open (cf. Conjecture 19).

4. Conjectures for Four-Dimensional Manifolds

In this section, we recall a conjecture on the ample cones of polarized varieties $(F, g)$ deformation equivalent to $S^{[2]}$, the Hilbert scheme of length-two subschemes on a K3 surface $S$. This was first formulated in [9].

Denote by

$$N^1_+(F, g) = \{ v \in N^1(F, \mathbb{Z}) \mid (v, g) > 0 \}$$

the positive halfspace (with respect to $g$ and the Beauville-Bogomolov form). Let $E$ be the set of classes $\rho \in N^1_+(F, g)$ satisfying one of the following:

1. $(\rho, \rho) = -2$ and $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z},$
2. $(\rho, \rho) = -2$ and $(\rho, H^2(F, \mathbb{Z})) = \mathbb{Z},$
3. $(\rho, \rho) = -10$ and $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z},$

Let $E^*$ be the corresponding classes $R \in H^2_+(F, \mathbb{Z})$, i.e., for some $\rho \in E$ we have

$$(v, \rho) = \begin{cases} R.v & \text{where } (\rho, H^2(F, \mathbb{Z})) = \mathbb{Z} \\ 2R.v & \text{where } (\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z} \end{cases}$$

for each $v \in H^2(F, \mathbb{Z})$. In particular, $R$ satisfies one of the following

1. $(R, R) = -\frac{1}{2},$
2. $(R, R) = -2,$
3. $(R, R) = -\frac{5}{2}.$

Let $N_E(F, g) \subset H^2(F, \mathbb{R})$ be the smallest real cone containing $E^*$ and the elements $R \in N_1(F, \mathbb{Z})$ such that $R.g > 0$ and the corresponding $\rho$ has nonnegative square.

Conjecture 17 (Effective curves conjecture).

$$\text{NE}_1(F) = N_E(F, g).$$
The classes in $E^*$ that are extremal in (the closure of) $N_E(F,g)$ will be called *nodal classes* (cf. \[18, 1.4\]). The nodal classes are denoted $E^*_{\text{nod}}$ and the corresponding classes in $E$ are denoted $E_{\text{nod}}$.

**Conjecture 18** (Nodal classes conjecture). Each nodal class $R \in E^*_{\text{nod}}$ represents a rational curve contracted by a birational morphism $\beta$ given by sections of $O_F(m\lambda)$, $m \gg 0$, where $\lambda$ is any nef and big divisor class with $R.\lambda = 0$.

1. If $(R,R) = -\frac{1}{2}, -2$ (i.e., the corresponding $\rho$ is a $(-2)$-class) then $\rho$ is represented by a family of rational curves parametrized by a K3 surface, which blow down to rational double points.
2. If $(R,R) = -\frac{5}{2}$ (i.e., the corresponding $\rho$ is a $(-10)$-class) then $\rho$ is represented by a family of lines contained in a $\mathbb{P}^2$ contracted to a point.

The remaining generators of the cone of curves are given by:

**Conjecture 19.** \[9, 3.8\] Let $\lambda$ be a primitive class on the boundary of the nef cone with $(\lambda,\lambda) = 0$. Then the corresponding line bundle $O_F(\lambda)$ has no higher cohomology and its sections yield a morphism

$$F \to \mathbb{P}^2$$

whose generic fiber is an abelian surface.

This was subsequently generalized to higher dimensions by Huybrechts \[8\].

5. **Deriving $(-2)$ and $(-10)$-classes from first principles**

In this section, we give a conceptual explanation for the occurrence of $(-2)$ and $(-10)$-classes in our conjectural description of the ample cone. This description is crucial for the proof of our main theorem in Section 6.

We recall a definition due to O’Grady \[20\]. All products of cohomology classes are to be taken in the cohomology ring unless otherwise specified:

**Definition 20.** A *numerical K3*\(^2\) is an irreducible holomorphic symplectic fourfold $F$ such that there exists an isomorphism

$$\psi : H^2(F,\mathbb{Z}) \cong H^2(S^{[2]},\mathbb{Z})$$

with $\psi(\alpha)^4 = \alpha^4$ for each $\alpha \in H^2(F,\mathbb{Z})$. Here $S$ is a K3 surface and $S^{[2]}$ its Hilbert scheme of length-two subschemes.
We recall one key property of these manifolds proved by O’Grady [20, §2]: $H^2(F,\mathbb{Z})$ admits a canonical integral primitive quadratic form $(, )$, the Beauville-Bogomolov form, such that
\begin{equation}
H^2(F,\mathbb{Z})_{(,)} \simeq U^{\oplus 3} \oplus_\perp (-E_8)^{\oplus 2} \oplus_\perp (-2)
\end{equation}
where $U$ is the hyperbolic plane and $E_8$ the positive-definite integral lattice associated to the corresponding root system. Moreover, we have
\[\alpha^4 = 3 (\alpha, \alpha)^2\]
for each $\alpha \in H^2(F,\mathbb{Z})$. This form induces $\frac{1}{2}\mathbb{Z}$-valued quadratic form on $H^2(F,\mathbb{Z})$ by duality:
\[H^2(F,\mathbb{Z})_{(,)} \simeq U^{\oplus 3} \oplus_\perp (-E_8)^{\oplus 2} \oplus_\perp (-1/2).
\]

We recall additional properties of numerical K3’s due to O’Grady [20, §2]:

- The intersection product induces an isomorphism
\[\text{Sym}^2 H^2(F,\mathbb{Q}) \xrightarrow{\sim} H^4(F,\mathbb{Q})\]
and the intersection form on the middle cohomology is given by the formula
\[\alpha_1 \alpha_2, \alpha_3 \alpha_4 = (\alpha_1, \alpha_2)(\alpha_3, \alpha_4) + (\alpha_1, \alpha_3)(\alpha_2, \alpha_4) + (\alpha_1, \alpha_4)(\alpha_2, \alpha_3)\]
for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H^2(F,\mathbb{Z})$.

- There is a distinguished class $q^\vee \in H^4(F,\mathbb{Q}) \cap H^{2,2}(F,\mathbb{C})$ such that
\[q^\vee.\alpha_1.\alpha_2 = 25 (\alpha_1, \alpha_2)\]
for all $\alpha_1, \alpha_2 \in H^2(F,\mathbb{Z})$. This is a rational multiple of the dual Beauville-Bogomolov form induced on $H^2(F,\mathbb{Z})$ via Poincaré duality. The class $q^\vee$ is the unique Hodge class (up to scalar multiplication) in the middle cohomology of a general numerical K3[2].

- We have the formulas
\[c_2(F) = \frac{6}{5} q^\vee, \quad q^\vee.q^\vee = 23 \cdot 25.
\]

**Theorem 21.** Let $F$ be a numerical K3[2]. Suppose $R \in N_1(F,\mathbb{Z})$ is an extremal ray such that there exists a Kawamata log terminal effective divisor $B \subset F$ with $B.R < 0$. If $(R, R) < 0$ then we have
\[(R, R) = -1/2, -2, -5/2.
\]
Moreover, $N^1(F,\mathbb{Z})$ contains an element $\rho$ satisfying one of the following:

- $(\rho, \rho) = -2$ and $(\rho, H^2(F,\mathbb{Z})) = \mathbb{Z}$;
• $(\rho, \rho) = -2$ and $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z};$
• $(\rho, \rho) = -10$ and $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z}.$

Proof. The Cone Theorem [17, 3.7] implies that there exists an extremal contraction $\beta : F \to F'$ with $\beta_* R = 0$ and $\text{Pic}(F/F') \simeq \mathbb{Z}$. Since $(R, R) < 0$ the line bundle $L$ contracting $\rho$ has $(L, L) > 0$; indeed, this is a consequence of the fact that $(\cdot, \cdot)$ has signature $(1, \text{rank}(\text{N}^1(F, \mathbb{Z}))) - 1$ on the Néron-Severi group.

We use the partial description of extremal contractions [23, 1.1], [19, 1.4,1.11], [5]. The morphism $\beta' : F \to F'$ satisfies one of the following alternatives:

• $\beta'$ is a divisorial contraction taking the exceptional divisor to a surface $T \subset F'$. At each smooth point of $T$, $\beta'$ is locally a contraction to a two-dimensional rational double point.
• $\beta'$ is a small contraction, taking a smooth Lagrangian $\mathbb{P}^2 \subset F$ to an isolated singularity of $F'$.

In the divisorial case, the smooth locus of $T$ has codimension $\geq 2$ complement and admits a holomorphic symplectic form.

Consider first the divisorial case. Suppose that $D$ is the exceptional divisor of $\beta$; the generic fiber of $\beta|D : D \to T$ is an ADE-configuration of $\mathbb{P}^1$'s. Since $\beta$ is extremal, the fundamental group of $T^{sm}$ acts transitively on the components of $\beta^{-1}(t)$ for $t \in T$ generic. An analysis of intersection numbers implies that only $A_1$ and $A_2$ configurations may occur (see [22, 5.1]).

Let $\tilde{D}$ denote the normalization of $D$ and
$$\tilde{D} \xrightarrow{\gamma} \tilde{T} \xrightarrow{T}$$
the Stein factorization of $\beta|\tilde{D}$. Then the generic fiber $C = \gamma^{-1}(t)$ is isomorphic to $\mathbb{P}^1$. However, the classification of rational double points yields
$$\mathcal{N}_{\tilde{D}/F}|C \simeq \mathcal{O}_{\mathbb{P}^1}(-2),$$
hence
$$D.C = -1, -2.$$ This analysis does not require $F$ to be a numerical K3[2], only an irreducible holomorphic symplectic fourfold.

We will now use integrality properties of the Beauville-Bogomolov form. Let $\rho \in \text{N}^1(F, \mathbb{Z})$ denote the primitive class identified with a positive multiple of the extremal ray $R$ via the Beauville-Bogomolov form. Precisely, for each $A \in H^2(F, \mathbb{Z})$ we have
$$A.R = r(A, \rho)$$
with \( r = 1, 1/2 \) depending on whether \(( R, H_2(F, \mathbb{Z})) = \mathbb{Z}, \frac{1}{2}\mathbb{Z}\). Clearly \( C = mR \) and \( D = n\rho \) for \( m, n \in \mathbb{N} \), and we have
\[
D.C = mnR, \rho = mnr (\rho, \rho).
\]

The following cases may occur:

(I) \( D.C = -1 \):
   (a) \( r = 1 \): Here \( m = n = 1 \) and \( R, \rho = -1 \), hence \((\rho, \rho) = -1\) which is impossible because \((, )\) is even-valued.
   (b) \( r = 1/2 \): Here \( mn (\rho, \rho) = -2 \) and thus \((\rho, \rho) = -2 \). We conclude that \((R, R) = -1/2 \).

(II) \( D.C = -2 \):
   (a) \( r = 1 \): Here we have \((\rho, \rho) = -2/mn \) which forces \( m = n = 1 \) and \((\rho, \rho) = -2 \). We conclude that \((R, R) = -2 \).
   (b) \( r = 1/2 \): Here \((\rho, \rho) = -4/mn \) so \( mn = 1 \) or \( 2 \). However, the lattice \((\mathfrak{2})\) does not admit primitive vectors \( \rho \) of length four with \((\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z} \). Indeed, if we had
\[
\rho = 2v + a\delta, \quad 2 \nmid a
\]
with
\[
v \in U^\oplus 3 \oplus (-E_8)^\oplus 2, (\delta, \delta) = -2, (v, \delta) = 0,
\]
then it would follow that
\[
(\rho, \rho) = 4(v, v) - 2a^2 \equiv -2 \pmod{8}.
\]
We conclude that \( mn = 2 \), \((\rho, \rho) = -2 \), and \((R, R) = -1/2 \).

This completes the proof in the divisorial case.

We turn to the case where \( \beta : F \to F' \) is small contraction of a Lagrangian \( \mathbb{P}^2 \). Some multiple of the extremal ray \( R \) is necessarily the class \( L \) of a line in \( \mathbb{P}^2 \). We shall show that \((L, L) = -5/2 \) which implies that \( R = L \), completing the proof of the theorem.

Suppose that \( \lambda \in H^2(F, \mathbb{Z}) \) is the unique class with
\[
2A.L = (A, \lambda)
\]
for all \( A \in H^2(F, \mathbb{Z}) \). We do not assume a priori that \( \lambda \) is primitive. Consider a deformation \( F_t \) of \( F \) for which \([L] \in H_2(F_t, \mathbb{Z})\) (or equivalently, \( \lambda \) remains a Hodge class. The Lagrangian plane also deforms in \( F_t \) (see [21] and [9]). For a general deformation \( F_t \), the only Hodge classes in \( H^4(F_t, \mathbb{Z}) \) are rational linear combinations of \( q^\vee \) and \( \lambda^2 \). Indeed, the Torelli map is locally an isomorphism and \( q^\vee, \lambda^2 \in H^4(F_t, \mathbb{Q}) \) are the only Hodge classes in \( \text{Sym}^2 H^2(F_t, \mathbb{Z}) \) for generic Hodge structures on \( H^2(F_t, \mathbb{Z}) \) (see [20], §3 for a detailed proof).
We may put
\[(4) \quad [\mathbb{P}^2] = aq^\omega + b\lambda^2.\]
Geometric properties of the Lagrangian plane translate into algebraic conditions on the coefficients \(a, b\); we use the intersection properties of numerical K3\(^2\)'s listed above:

- The normal bundle to any Lagrangian submanifold is equal to its cotangent bundle. Thus we have
  \([\mathbb{P}^2].[\mathbb{P}^2] = c_2(\Omega^1_{\mathbb{P}^2}) = 3\]
  which implies
  \[25 \cdot 23a^2 + 50ab \lambda , \lambda ) + 3b^2 (\lambda, \lambda)^2 = 3.\]
- Using the exact sequence
  \[0 \to T_{\mathbb{P}^2} \to T_F|\mathbb{P}^2 \to N_{\mathbb{P}^2/F} \to 0\]
  we compute that \(c_2(T_F)|\mathbb{P}^2 = -3.\) It follows that
  \[-3 = \frac{6}{5}(25 \cdot 23a + 25b \lambda , \lambda).\]
- We know that \(\lambda|\mathbb{P}^2\) is some multiple of the hyperplane class, i.e., \(\lambda.[\mathbb{P}^2] = (\lambda.L) L.\) We deduce that
  \[\lambda.\lambda.[\mathbb{P}^2] = (\lambda.L)^2 = (\lambda, \lambda)^2 / 4.\]
  Using formula (1) to evaluate \(\lambda.\lambda.[\mathbb{P}^2]\) we obtain
  \[(\lambda, \lambda)^2 / 4 = 25a (\lambda, \lambda) + 3b (\lambda, \lambda)^2.\]
Altogether, we obtain three Diophantine equations in the variables \((\lambda, \lambda), a,\) and \(b.\) Eliminating \(a\) and \(b\) and solving for \((\lambda, \lambda)\) we obtain the quadratic equation
\[23 (\lambda, \lambda)^2 + 20 (\lambda, \lambda) - 2100 = 0\]
with solutions \((\lambda, \lambda) = -10, 210/23.\) Only the first makes sense. We conclude that \((L, L) = -5/2\) and \(\lambda\) is primitive and \((\lambda, H^2(F, \mathbb{Z})) = 2\mathbb{Z}.\) \(\square\)

6. Applications to ample divisors

In this section, we prove one implication of Conjecture [17]

**Theorem 22.** Let \((F, g)\) be a polarized irreducible holomorphic symplectic manifold, deformation equivalent to Hilbert scheme of length-two subschemes of a K3 surface. Then we have
\[\text{NE}_1(F) \subseteq \text{NE}(F, g).\]
Equivalently, the divisors predicted to be ample by our conjectures are indeed ample.

**Proof.** Let $M$ be a divisor such that

- $(M, M) > 0$; and
- $M.R > 0$ for each $R \in E^*$.

The first condition implies that $M.R > 0$ for $R \neq 0 \in N_1(F, \mathbb{Z})$ with $(R, R) \geq 0$. Indeed, $(, )$ has signature $(1, \dim N_1(F, \mathbb{R}) - 1)$ on $N_1(F, \mathbb{R})$. To prove the Theorem, it suffices to show that $M$ is ample on $F$.

Suppose that $M$ fails to be ample. After a small perturbation of $g$, the line segment

$$tM + (1-t)g, \quad t \in [0, 1]$$

meets the boundary of the ample cone of $F$ in the interior of the facet of the nef cone. Indeed, Proposition 13 shows that $\overline{\mathcal{C}}_F \cap \mathcal{C}_F$ is locally-finite rational polyhedral. Under these assumptions

$$\tau := \sup \{t : tM + (1-t)g \text{ is ample} \}$$

is rational. Let $R$ be the (primitive, integral) generator of the extremal ray corresponding to our facet; we have $(R, R) < 0$. Theorem 21 implies that

$$(R, R) = -1, -2, -5/2$$

whence $R \in N_E(F, g)$. \hfill $\square$

**Remark 23.** The underlying techniques here are reminiscent of those used in the proof that ‘minimal models are connected by flops’ [14] [2, 1.1.3].

**References**

[1] Arnaud Beauville. Variétés kählériennes compactes avec $c_1 = 0$. Astérisque, (126):181–192, 1985. Geometry of K3 surfaces: moduli and periods (Palaiseau, 1981/1982).

[2] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type, 2006.

[3] Sébastien Boucksom. Le cône kählérien d’une variété hyperkählérienne. C. R. Acad. Sci. Paris Sér. I Math., 333(10):935–938, 2001.

[4] Dan Burns, Yi Hu, and Tie Luo. HyperKähler manifolds and birational transformations in dimension 4. In Vector bundles and representation theory (Columbia, MO, 2002), volume 322 of Contemp. Math., pages 141–149. Amer. Math. Soc., Providence, RI, 2003.

[5] Koji Cho, Yoichi Miyaoka, and Nicholas I. Shepherd-Barron. Characterizations of projective space and applications to complex symplectic manifolds. In Higher dimensional birational geometry (Kyoto, 1997), volume 35 of Adv. Stud. Pure Math., pages 1–88. Math. Soc. Japan, Tokyo, 2002.
[6] Herbert Clemens, János Kollár, and Shigefumi Mori. Higher-dimensional complex geometry. Astérisque, (166):144 pp. (1989), 1988.
[7] Akira Fujiki. On the de Rham cohomology group of a compact Kähler symplectic manifold. In Algebraic geometry, Sendai, 1985, volume 10 of Adv. Stud. Pure Math., pages 105–165. North-Holland, Amsterdam, 1987.
[8] Mark Gross, Daniel Huybrechts, and Dominic Joyce. Calabi-Yau manifolds and related geometries. Universitext. Springer-Verlag, Berlin, 2003. Lectures from the Summer School held in Nordfjordeid, June 2001.
[9] Brendan Hassett and Yuri Tschinkel. Rational curves on holomorphic symplectic fourfolds. Geom. Funct. Anal., 11(6):1201–1228, 2001.
[10] Yi Hu and Sean Keel. Mori dream spaces and GIT. Michigan Math. J., 48:331–348, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.
[11] Daniel Huybrechts. Compact hyper-Kähler manifolds: basic results. Invent. Math., 135(1):63–113, 1999.
[12] Daniel Huybrechts. Erratum: “Compact hyper-Kähler manifolds: basic results” [Invent. Math. 135 (1999), no. 1, 63–113; MR1664696 (2000a:32039)]. Invent. Math., 152(1):209–212, 2003.
[13] Daniel Huybrechts. The Kähler cone of a compact hyperkähler manifold. Math. Ann., 326(3):499–513, 2003.
[14] Yujiro Kawamata. Flops connect minimal models, 2007.
[15] János Kollár, editor. Flips and abundance for algebraic threefolds. Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).
[16] János Kollár. Singularities of pairs. In Algebraic geometry—Santa Cruz 1995, volume 62 of Proc. Sympos. Pure Math., pages 221–287. Amer. Math. Soc., Providence, RI, 1997.
[17] János Kollár and Shigefumi Mori. Birational geometry of algebraic varieties, volume 134 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
[18] Eduard Looijenga and Chris Peters. Torelli theorems for Kähler K3 surfaces. Compositio Math., 42(2):145–186, 1980/81.
[19] Yoshinori Namikawa. Deformation theory of singular symplectic n-folds. Math. Ann., 319(3):597–623, 2001.
[20] Kieran G. O’Grady. Irreducible symplectic 4-folds numerically equivalent to Hilb^2(K3), 2005.
[21] Claire Voisin. Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes. In Complex projective geometry (Trieste, 1989/Bergen, 1989), volume 179 of London Math. Soc. Lecture Note Ser., pages 294–303. Cambridge Univ. Press, Cambridge, 1992.
[22] Jan Wierzba. Contractions of symplectic varieties. J. Algebraic Geom., 12(3):507–534, 2003.
[23] Jan Wierzba and Jarosław A. Wiśniewski. Small contractions of symplectic 4-folds. Duke Math. J., 120(1):65–95, 2003.