OPTIMAL FORWARD AND REVERSE ESTIMATES OF MORAWETZ AND KATO–YAJIMA TYPE WITH ANGULAR SMOOTHING INDEX

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Abstract. For the solution of the free Schrödinger equation, we obtain the optimal constants and characterise extremisers for forward and reverse smoothing estimates which are global in space and time, contain a homogeneous and radial weight in the space variable, and incorporate a certain angular regularity. This will follow from a more general result which permits analogous sharp forward and reverse smoothing estimates and a characterisation of extremisers for the solution of the free Klein–Gordon and wave equations. The nature of extremisers is shown to be sensitive to both the dimension and the size of the smoothing index relative to the dimension. Furthermore, in four spatial dimensions and certain special values of the smoothing index, we obtain an exact identity for each of these evolution equations.

1. Introduction

For \( d \geq 2 \) and \( s \in (-\frac{1}{2}, \frac{d}{2} - 1) \) the solution of the free Schrödinger equation \( i\partial_t u + \frac{1}{2}\Delta u = 0 \) satisfies the smoothing estimate

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(x,t)|^2 \frac{dxdt}{|x|^{2(1+s)}} \leq C \|u(0)\|^2_{\dot{H}^s(\mathbb{R}^d)},
\]

where \( \dot{H}^s(\mathbb{R}^d) \) is the usual homogeneous Sobolev space of order \( s \). This estimate was established by Kato and Yajima \[9\] for \( s \in (-\frac{1}{2}, 0] \) whenever \( d \geq 3 \), and \( s \in (-\frac{1}{2}, 0) \) for \( d = 2 \) (see also \[2\] for an alternative approach, and \[18\], \[22\] and \[24\] for the full range \( s \in (-\frac{1}{2}, \frac{d}{2} - 1) \)). Estimates like (1.1) are often referred to as Kato–Yajima smoothing estimates, or Morawetz estimates, since similar estimates for the Klein–Gordon equation were established in the earlier work \[10\].

The focus of this paper are certain angular refinements of (1.1). Hoshino \[8\] proved that whenever \( d \geq 3 \) and \( s \in (-\frac{1}{2}, 0] \), there is a finite constant \( C \) such that

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} |(1 - \Lambda)^{\frac{1+2s}{2}} u(x,t)|^2 \frac{dxdt}{|x|^{2(1+s)}} \leq C \|u(0)\|^2_{\dot{H}^s(\mathbb{R}^d)},
\]

where \(-\Lambda\) is the Laplace–Beltrami operator on the unit sphere \( S^{d-1} \) homogeneously extended to \( \mathbb{R}^d \). In fact, (1.2) is also valid in the range \( s \in (-\frac{d}{2}, \frac{d}{2} - 1) \) for any \( d \geq 2 \) (see \[18\] for the full range). Interestingly, it was recently observed by Fang and Wang \[7\] that a reverse form of (1.2) exists; that is, for the same \((d,s)\) there exists a strictly positive constant \( c \) such that

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} |(1 - \Lambda)^{\frac{1+2s}{2}} u(x,t)|^2 \frac{dxdt}{|x|^{2(1+s)}} \geq c \|u(0)\|^2_{\dot{H}^s(\mathbb{R}^d)},
\]

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In the critical case $s = -\frac{1}{2}$ the estimate (1.1) fails, and the full gain of a half-derivative does not materialise in this way. One may interpret (1.2) as a replacement for this false estimate since, formally, $(1 - \Lambda)^s$ behaves like $|x|^{2s} |\nabla|^{2s}$ in the sense of the order of the derivative and the decay. A different replacement for the failure of (1.1) when $s = -\frac{1}{2}$ is the local smoothing estimate

\begin{equation}
\sup_{R > 0} \frac{1}{R} \int_R \int_{|x| \leq R} |\nabla u(x, t)|^2 \, dx \, dt \leq C \|u(0)\|_{H^\frac{1}{2} (\mathbb{R}^d)}^2
\end{equation}

established in [4, 16, 20]. We remark that it was recently observed by Vega and Visciglia [21] that (1.4) also enjoys a reverse form; in fact, they prove

\begin{equation}
\sup_{R > 0} \frac{1}{R} \int_R \int_{|x| \leq R} |\nabla u(x, t)|^2 \, dx \, dt \geq 2\pi \|u(0)\|_{H^\frac{1}{2} (\mathbb{R}^d)}^2.
\end{equation}

The critical case was also considered in [19] and [12], in the context of more general elliptic operators, and applied to time global existence of solutions to certain derivative nonlinear equations in [13].

One of our main results in this paper is to compute the optimal constants and characterise extremisers for the forward and reverse estimates in (1.2) and (1.3). These optimal estimates will follow from a replacement for the failure of (1.1) when $\Lambda$ is the propagator, and we consider forward and reverse estimates of the form

\begin{equation}
c \|u(0)\|_{L^2(\mathbb{R}^d)}^2 \leq \int_\mathbb{R} \int_{\mathbb{R}^d} |\psi(|\nabla|) \theta(-\Lambda) u(t, x)|^2 \frac{dx \, dt}{|t|^\tau} \leq C \|u(0)\|_{L^2(\mathbb{R}^d)}^2
\end{equation}

for solutions of $i\partial_t u + \phi(|\nabla|) u = 0$, where $\tau \in (1, d)$, and $\phi$ and $\psi$ are such that

\begin{equation}
\psi(\rho)^2 = |\phi'(\rho)| \rho^{1-\tau}.
\end{equation}

Here, we are assuming that the dispersion relation $\phi$ is injective and differentiable.

**Theorem 1.1.** Let $d \geq 2$ and $\tau \in (1, d)$. Suppose

\[ \beta_k = \pi 2^{2-\frac{\tau}{2}} \frac{\Gamma(\tau - 1) \Gamma(k + \frac{\tau}{2})}{\Gamma(\frac{\tau}{2}) \Gamma(k + \frac{d + 2 - \tau}{2} - 1)} |\theta(k(k + d - 2))|^2, \]

and

\[ k = \{ k \in \mathbb{N}_0 : \inf_{\ell \in \mathbb{N}_0} \beta_\ell = \beta_k \} \quad \text{and} \quad K = \{ k \in \mathbb{N}_0 : \sup_{\ell \in \mathbb{N}_0} \beta_\ell = \beta_k \}. \]

If $i\partial_t u + \phi(|\nabla|) u = 0$ and $\psi(\rho)^2 = |\phi'(\rho)| \rho^{1-\tau}$ then

\[ \inf_{k \in \mathbb{N}_0} \beta_k \|u(0)\|_{L^2(\mathbb{R}^d)}^2 \leq \int_\mathbb{R} \int_{\mathbb{R}^d} |\psi(|\nabla|) \theta(-\Lambda) u(t, x)|^2 \frac{dx \, dt}{|t|^\tau} \leq \sup_{k \in \mathbb{N}_0} \beta_k \|u(0)\|_{L^2(\mathbb{R}^d)}^2 \]

and the constants are optimal. Furthermore, nonzero initial data $u(0)$ is an extremiser for the lower bound if and only if $u(0)$ belongs to $\bigoplus_{k \in K} \mathcal{H}_k$, and an extremiser for the upper bound if and only if $u(0)$ belongs to $\bigoplus_{k \in K} \mathcal{H}_k$.

Here, we are using the notation $\mathbb{N}_0$ for $\{0, 1, 2, \ldots\}$, and $\mathcal{H}_k$ for the space of all linear combinations of functions

\[ \xi \mapsto P(\xi) f_0(|\xi|) |\xi|^{-d/2-k+1/2} \]

where $P$ is a homogeneous harmonic polynomial of order $k$ and $f_0 \in L^2(0, \infty)$. Also, $\theta(-\Lambda)$ is the homogeneous extension of the operator $\theta(-\Lambda)$ on the sphere $S^{d-1}$; an explicit definition will be given later in Section 3.
We remark that $H_0$ is the space of square-integrable radially symmetric functions. If the index set $k$ is empty then there are no extremisers for the lower bound, and similarly for $K$ and the upper bound.

The statement of Theorem 1.1 is rather general and as a consequence of the minimal assumptions on $\theta$, the theorem does not guarantee the strict positivity of $\inf_{k \in \mathbb{N}_0} \beta_k$ or the finiteness of $\sup_{k \in \mathbb{N}_0} \beta_k$. The case of primary interest in this paper is

$$\theta(\rho) = (1 + \rho)^{\frac{d-\tau}{2}}.$$ 

For such $\theta$, it is true that $\inf_{k \in \mathbb{N}_0} \beta_k$ is strictly positive and $\sup_{k \in \mathbb{N}_0} \beta_k$ is finite, and we will obtain the optimal constants in (1.2) and (1.3) by taking $\phi(\rho) = \frac{1}{2} \rho^2$ and $\tau = 2(1 + s)$ (so that $\psi(\rho) = \rho^{-s}$).

In the subsequent section we give a very precise description of these optimal constants; we delay our presentation of this result because it is necessary to first establish some technical notation. At this stage we emphasise that the case $(d, \tau) = (4, 2)$ is particularly special. Here, the sharp form of (1.2) and (1.3) is in fact an exact identity. By taking appropriate choices of $\phi$ and $\tau$, we also obtain the analogous identities for solutions of the free wave and Klein–Gordon equations. We collect these in the following.

**Theorem 1.2.** On $\mathbb{R}^{4+1}$, all solutions of the Schrödinger equation $i\partial_t u + \frac{1}{2} \Delta u = 0$ satisfy

$$\int_{\mathbb{R}} \int_{\mathbb{R}^4} |(1 - \Lambda)^{\frac{1}{2}} u(x, t)|^2 \frac{dx dt}{|x|^2} = \pi \|u(0)\|^2_{L^2(\mathbb{R}^4)},$$

all solutions of the wave equation $\partial_{tt} u - \Delta u = 0$ satisfy

$$2 \int_{\mathbb{R}} \int_{\mathbb{R}^4} |(1 - \Lambda)^{\frac{1}{2}} u(x, t)|^2 \frac{dx dt}{|x|^2} = \pi \left( \|u(0)\|^2_{H^\frac{1}{2}(\mathbb{R}^4)} + \|\partial_t u(0)\|^2_{H^{-\frac{1}{2}}(\mathbb{R}^4)} \right),$$

and all solutions of the Klein–Gordon equation $\partial_{tt} u - \Delta u + u = 0$ satisfy

$$2 \int_{\mathbb{R}} \int_{\mathbb{R}^4} |(1 - \Lambda)^{\frac{1}{2}} (1 - \Lambda)^{\frac{1}{2}} u(x, t)|^2 \frac{dx dt}{|x|^2} = \pi \left( \|u(0)\|^2_{L^2(\mathbb{R}^4)} + \|\nabla u(0)\|^2_{L^2(\mathbb{R}^4)} + \|\partial_t u(0)\|^2_{L^2(\mathbb{R}^4)} \right).$$

In the related case where $\theta(\rho) = \rho^{\frac{d-\tau}{2}}$, the situation is different because $\inf_{k \in \mathbb{N}_0} \beta_k = 0$ and there is no reverse estimate. We do, however, provide an explicit description of the upper bound $\sup_{k \in \mathbb{N}_0} \beta_k$ in Section 3. We remark in passing that $\sup_{k \in \mathbb{N}_0} \beta_k$ is always finite whenever $\theta(\rho) = O(\rho^{\frac{d-1}{2}})$; this is true because

$$\frac{\Gamma(k + \frac{d-\tau}{2})}{\Gamma(k + \frac{d+\tau}{2} - 1)} = O(k^{1-\tau}) \quad \text{as } k \to \infty,$$

which easily follows from Stirling’s formula.

The case where $\theta$ is identically equal to one with no smoothing along the sphere corresponds to (1.1). The optimal constant for the forward estimate in (1.1) has appeared in a number of earlier works, including [5], [11], [24] and our own [3] in the general case, and [15] for the case $s = 0$ (see also [23]). In [3], we proceed using spectral considerations; in this work, we build on [3] and the proof of Theorem 1.1 is also based on spectral considerations. In [11], sharp angular refinements of (1.1) of a different nature to those considered in this paper are established, in the forward direction, by a different approach through the sharp Hardy–Littlewood–Sobolev inequality on the sphere. We also remark that when $\theta$ is identically equal to one, the sequence $(\beta_k)_{k \in \mathbb{N}_0}$ is decreasing and tends to zero as $k$ tends to infinity (see [3]). Hence there is no reverse inequality in this case.
Overview. The upper bound in Theorem 1.1 is stated in the case of the Schrödinger propagator in [4], and the substantially more complete results of this paper were partially announced in [4].

In Section 2 we consider the important case \( \theta(\rho) = (1 + \rho)^{\frac{\tau-1}{4}} \), where we provide a comprehensive description of the optimal constants \( \inf_{k \in \mathbb{N}_0} \beta_k \) and \( \sup_{k \in \mathbb{N}_0} \beta_k \) and when these extrema are attained. The proofs of these results are contained in Section 4. As applications, we provide the optimal constants and characterise the extremisers for (1.2) and (1.3), along with analogous results for the wave and Klein–Gordon equations. In Section 3 we prove Theorem 1.1.

Finally, in Section 5, we provide several further results, including an analysis of the case \( \theta(\rho) = \rho^\frac{\tau-1}{4} \). We also include some further generalisations of Theorem 1.1 to allow weights which are not homogeneous, and dispersion relations and smoothing functions \( \phi \) and \( \psi \) which are not required to satisfy (1.6). The disadvantage of working in such generality is that a completely explicit description of optimal constants and extremisers is not possible. The main focus of this paper is to establish such information and this is the reason that we have presented the results in the Introduction in the case where the weight is homogeneous, and \( \phi \) and \( \psi \) satisfy (1.6).

2. The case \( \theta(\rho) = (1 + \rho)^{\frac{\tau-1}{4}} \)

Theorem 1.1 makes it clear that to obtain explicit expressions for the optimal constants in estimates of the form (1.5), and to characterise the space of extremisers, we must compute

\[
\mathbf{b}(d, \tau; \theta) = \inf_{k \in \mathbb{N}_0} \beta_k \quad \text{and} \quad \mathbf{B}(d, \tau; \theta) = \sup_{k \in \mathbb{N}_0} \beta_k,
\]

and understand the index sets

\[
k(d, \tau; \theta) = \{ k \in \mathbb{N}_0 : \beta_k = \mathbf{b}(d, \tau; \theta) \} \quad \text{and} \quad K(d, \tau; \theta) = \{ k \in \mathbb{N}_0 : \beta_k = \mathbf{B}(d, \tau; \theta) \},
\]

where \( \beta_k \) is given by

\[
\beta_k = \beta_k(d, \tau; \theta) = \pi 2^{2-\tau} \frac{\Gamma(\tau-1)\Gamma(d+\frac{d\tau}{2})}{\Gamma(\frac{d}{2})\Gamma(k+\frac{d\tau}{2}-1)} |\theta(k(d-2))|^2.
\]

The main result in this section is to do this in the case

\( \theta(\rho) = (1 + \rho)^{\frac{\tau-1}{4}} \).

In order to state our result here, it is necessary to introduce a little notation. For \( d \geq 5 \), we introduce two parameters \( \tau_* \in (1, d) \) and \( \tau^* \in (1, d) \), depending only on \( d \), as the unique solution of the equations

\[
d^\frac{\tau-1}{4} \left( \frac{d - \tau_*}{2} \right) = \frac{d + \tau_*}{2} - 1
\]

and

\[
\Gamma \left( \frac{d - \tau^*}{2} \right) = \Gamma \left( \frac{d + \tau^*}{2} - 1 \right)
\]

respectively. It is not immediately clear that \( \tau_* \) and \( \tau^* \) are well-defined so we provide a proof of this at the end of this section, in order not to delay the presentation of the main results of this section.

Remark. We will show in the course of the proof of the following theorem that \( \tau_* \leq \tau^* \).
For $d \geq 5$ and $\tau \in [\tau_s, d)$ we let $k(\tau)$ be the unique non-negative real number such that

$$
(2.1) \quad \frac{2k(\tau) + d - \tau}{2k(\tau) + d + \tau - 2} \left( \frac{1 + (k(\tau) + 1)(k(\tau) + d - 1)}{1 + k(\tau)(k(\tau) + d - 2)} \right)^{\frac{\tau - 1}{d - 2}} = 1
$$

and let $k^*(\tau)$ denote the smallest integer greater than or equal to $k(\tau)$. We show that $k(\tau)$ is well-defined during the proof of the following (in Section 4).

**Theorem 2.1.** Let $d \geq 2$, $\tau \in (1, d)$ and $\theta(\rho) = (1 + \rho)^{\frac{\tau - 1}{d - 2}}$. Then the constants $b(d, \tau; \theta)$ and $B(d, \tau; \theta)$ are given by

$$
\begin{array}{|c|c|c|}
\hline
(d, \tau) & b(d, \tau; \theta) & B(d, \tau; \theta) \\
\hline
(d, \tau) & lim_{k \to \infty} \beta_k & \beta_0 \\
\hline
2 \leq d \leq 3 & \beta_0 & \lim_{k \to \infty} \beta_k \\
\hline
d = 4, \tau \in (1, 2) & \beta_0 & \lim_{k \to \infty} \beta_k \\
\hline
d = 4, \tau = 2 & \pi & \pi \\
\hline
d = 4, \tau \in (2, 4) & \lim_{k \to \infty} \beta_k & \beta_0 \\
\hline
d = 5, \tau \in (1, \tau_s) & \beta_0 & \lim_{k \to \infty} \beta_k \\
\hline
d = 5, \tau \in (\tau_s, \tau^*) & \beta_{k^*(\tau)} & \lim_{k \to \infty} \beta_k \\
\hline
d = 5, \tau \in [\tau^*, 5) & \beta_{k^*(\tau)} & \beta_0 \\
\hline
\hline
\end{array}
$$

and the index sets $k(d, \tau; \theta)$ and $K(d, \tau; \theta)$ are given by

$$
\begin{array}{|c|c|c|}
\hline
(d, \tau) & k(d, \tau; \theta) & K(d, \tau; \theta) \\
\hline
(d, \tau) & \emptyset & \{0\} \\
\hline
\hline
(d, \tau) & \emptyset & \emptyset \\
\hline
(d, \tau) & \emptyset & \emptyset \\
\hline
(d, \tau) & \{0\} & \emptyset \\
\hline
(d, \tau) & \{0\} & \{0\} \\
\hline
(d, \tau) & \{0\} & \emptyset \\
\hline
(d, \tau) & \{0\} & \emptyset \\
\hline
(d, \tau) & \{0\} & \emptyset \\
\hline
(d, \tau) & \{0\} & \emptyset \\
\hline
(d, \tau) & \{0\} & \emptyset \\
\hline
(d, \tau) & \{1\} & \emptyset \\
\hline
(d, \tau) & \{1\} & \{0\} \\
\hline
\end{array}
$$

We can combine Theorems 1.1 and 2.1 and give a precise description of the optimal constants and extremisers in (1.2) and (1.3).

**Notation.** For $d \geq 2$ and $s \in \left(-\frac{1}{2}, \frac{d - 1}{2}\right)$, define constants $c(d, s)$ and $C(d, s)$ by

$$
c(d, s) = \inf_{k \in \mathbb{N}_0} \pi^{2s} \frac{\Gamma(1 + 2s)\Gamma(k + \frac{d - 2}{2} - s)}{\Gamma(1 + s)^2\Gamma(k + \frac{d - 2}{2} + s)} \frac{(1 + k(k + d - 2))^{s + \frac{d}{2}}}{(1 + k(k + d - 2))^{s + \frac{d}{2}}}
$$
and
\[
\mathbf{C}(d, s) = \sup_{k \in \mathbb{N}} \pi^{2-2s} \frac{\Gamma(1+2s) \Gamma(k+\frac{d-2}{2} - s)}{\Gamma(1+s)^2 \Gamma(k+\frac{d}{2} + s)} (1+k(k+d-2))^{s+\frac{1}{2}}.
\]

Observe that
\begin{equation}
\mathbf{c}(d, s) = \mathbf{b}(d, 2(1+s); \theta)
\end{equation}
and
\begin{equation}
\mathbf{C}(d, s) = \mathbf{B}(d, 2(1+s); \theta)
\end{equation}
where \( \theta \) is given by
\[
\theta(\rho) = (1 + \rho)^{\frac{1+2s}{2}}.
\]

**Corollary 2.2.** Let \( d \geq 2, s \in (-\frac{1}{2}, \frac{d}{2} - 1) \) and suppose that \( i\partial_t u + \frac{\rho}{2} \Delta u = 0 \) on \( \mathbb{R}^{d+1} \). Then
\[
\mathbf{c}(d, s)\|u(0)\|^2_{H^s(\mathbb{R}^d)} \leq \int \int_{\mathbb{R}^d} |(1-\Lambda)u(x,t)|^2 \frac{dxdt}{|x|^{1+2s}} \leq \mathbf{C}(d, s)\|u(0)\|^2_{H^s(\mathbb{R}^d)},
\]
and the constants are optimal.

Likewise, for the wave and Klein–Gordon equations we have the following.

**Corollary 2.3.** Let \( d \geq 2, s \in (0, \frac{d-1}{2}) \) and suppose that \( \partial_t u - \Delta u = 0 \) on \( \mathbb{R}^{d+1} \). Then
\[
\mathbf{c}(d, s - \frac{1}{2})\|(u(0), \partial_t u(0))\|^2 \leq 2 \int \int_{\mathbb{R}^d} |(1-\Lambda)u(x,t)|^2 \frac{dxdt}{|x|^{1+2s}} \leq \mathbf{C}(d, s - \frac{1}{2})\|(u(0), \partial_t u(0))\|^2
\]
and the constants are optimal. Here, the norm on the initial data is given by
\[
\|(u(0), \partial_t u(0))\|^2_{H^s(\mathbb{R}^d)} + \|\partial_t u(0)\|^2_{H^{s-1}(\mathbb{R}^d)}.
\]

**Corollary 2.4.** Let \( d \geq 2, s \in (-\frac{1}{2}, \frac{d}{2} - 1) \) and suppose that \( \partial_t u - \Delta u + u = 0 \) on \( \mathbb{R}^{d+1} \). Then
\[
\mathbf{c}(d, s)\|(u(0), \partial_t u(0))\|^2 \leq 2 \int \int_{\mathbb{R}^d} |(1-\Lambda)\frac{1}{2}\chi u(x,t)|^2 \frac{dxdt}{|x|^{1+2s}} \leq \mathbf{C}(d, s)\|(u(0), \partial_t u(0))\|^2
\]
and the constants are optimal. Here, the norm on the initial data is given by
\[
\|(u(0), \partial_t u(0))\|^2 = \|u(0)\|^2_{H^s(\mathbb{R}^d)} + \|u(0)\|^2_{H^{s+1}(\mathbb{R}^d)} + \|\partial_t u(0)\|^2_{H^{s}(\mathbb{R}^d)}.
\]

We remark that Theorem 1.1 is a straightforward consequence of Corollaries 2.2, 2.3, and Theorem 2.1 to obtain \( C(4, 0) = c(4, 0) = \pi \).

**Proof of Corollaries 2.2, 2.3 and 2.4.** Corollary 2.2 follows immediately from Theorem 1.1 by taking \( \phi(\rho) = \frac{1}{2} \rho^2, \psi(\rho) = \rho^{-s} \) and \( \tau = 2(1+s) \), clearly satisfying (1.4). For Corollary 2.3 we write the solution of the wave equation \( u \) as \( u_+ + u_- \), where
\[
\begin{align*}
  u_\pm(t) &= \exp(\pm it|\nabla|) f_\pm \\
  u(0) &= f_+ + f_- \\
  \partial_t u(0) &= i|\nabla|(f_+ - f_-).
\end{align*}
\]
Then
\[
\int \int_{\mathbb{R}^d} |(1-\Lambda)^{\frac{1}{2}} u(x,t)|^2 \frac{dxdt}{|x|^{1+2s}} = \int \int_{\mathbb{R}^d} |(1-\Lambda)^{\frac{1}{2}} u_+(x,t)|^2 \frac{dxdt}{|x|^{1+2s}} + \int \int_{\mathbb{R}^d} |(1-\Lambda)^{\frac{1}{2}} u_-(x,t)|^2 \frac{dxdt}{|x|^{1+2s}}
\]
by Plancherel’s theorem and the fact that the Fourier transforms in time of $u_+$ and $u_-$ are disjoint. Corollary 2.3 now follows from two applications of Theorem 1.1 with $\phi(\rho) = \pm \rho$, $\psi(\rho) = \rho^{-s}$ and $\tau = 1 + 2s$, and the parallelogram law. The proof of Corollary 2.4 is similar, using $\phi(\rho) = \pm (1 + \rho^2)^{1/2}$, $\psi(\rho) = (1 + \rho^2)^{-s}$ and $\tau = 2(1 + s)$, and we omit the details. □

Of course, Theorem 2.1 provides a precise description of the optimal constants $c(d, s)$ and $C(d, s)$ appearing in Corollaries 2.2, 2.3 and 2.4 (through (2.2) and (2.3)). The constants $\beta_0$, $\beta_1$, and $\lim_{k \to \infty} \beta_k$ appearing in Theorem 2.1 are given explicitly in terms of $d$ and $\tau$ as follows

\begin{align}
\beta_0 &= \pi 2^{d-1} \frac{\Gamma(\tau - 1)\Gamma\left(\frac{d+\tau}{2}\right)}{\Gamma\left(\frac{d-\tau}{2}\right)^2}\frac{\Gamma\left(\frac{d-\tau}{2}\right)^2}{\Gamma\left(\frac{d-\tau}{2}\right) - 1} \\
\beta_1 &= \pi 2^{d-\tau} \frac{\Gamma\left(\frac{d+\tau}{2}\right)}{(d+\tau)^2}\frac{\Gamma\left(\frac{d+\tau}{2}\right)}{\Gamma(\tau - 1)}\frac{\Gamma\left(\frac{d-\tau}{2}\right)^2}{\Gamma\left(\frac{d-\tau}{2}\right) - 1} \\
\lim_{k \to \infty} \beta_k &= \pi 2^{d-\tau} \frac{\Gamma(\tau - 1)}{\Gamma\left(\frac{d-\tau}{2}\right)^2},
\end{align}

where (2.6) follows easily from Stirling’s formula. In the exceptional case $d = 5$ and $\tau \in (\tau_*, 5)$ the lower bound $b(5, \tau; \theta)$ is given in terms of $k(\tau)$ which is implicitly defined. We are, at least, able to provide the following bounds on $k(\tau)$.

**Proposition 2.5.** Let $d = 5$ and $\tau \in (\tau_*, 5)$. The unique positive real number $k(\tau)$ for which (2.1) holds satisfies the following bounds

\[
\frac{C_1}{(5 - \tau)^{1/4}} \leq k(\tau) \leq \frac{C_2}{(5 - \tau)^{1/2}}
\]

for some positive constants $C_1$ and $C_2$.

We have not attempted to sharpen these bounds by bringing the exponents $\frac{1}{4}$ and $\frac{1}{2}$ closer together, although this would be an interesting problem to solve.

Additionally, Theorem 2.1 allows one to characterise the space of extremising initial data in Corollaries 2.2, 2.3 and 2.4. For example, in spatial dimensions $d = 2, 3$, and any $\tau \in (1, d)$, we know that the lower bounds do not possess extremising initial data, and the upper bounds are realised if and only if the initial data is radially symmetric.

Clearly, the case of five spatial dimensions is the most subtle in Theorem 2.1. Although we cannot provide a concrete explanation for this, it is conceivable this is related to the amusing fact that the volume of the unit sphere as a function of the dimension has a global maximum in five dimensions.

As promised, we end this section with a justification that the parameters $\tau_*$ and $\tau^*$ are well-defined.

**Proof that $\tau_*$ is well-defined.** Recall that we are assuming $d \geq 5$. Observe that

\[
\Phi(\tau) \coloneqq \frac{\partial}{\partial \tau} \left(\frac{d-\tau}{2-\tau} \frac{\tau}{2-\tau} - 1\right) = \frac{d-\tau}{2(d-2+\tau)^2} ((\log d)(d-\tau)(d-2+\tau) - 4(d-1))
\]
has at most two roots. These roots are given by
\[ \tau = 1 \pm \sqrt{1 + d(d - 2) - \frac{4(d - 1)}{\log d}} \]
and therefore at most one of these roots lies in \((1, d)\). Furthermore
\[ \Phi(1) = \frac{1}{2(d - 1)}((\log d)(d - 1) - 4) > 0 \]
for \(d \geq 5\), and
\[ \Phi(d) = -\frac{d^{d-1}}{2(d - 1)} < 0 \]
and it follows that there is precisely one root of \(\Phi\) in the interval \((1, d)\). Clearly
\[ d\tau - \frac{1}{2}d - \tau^2 + \frac{1}{2}d - 1 = \begin{cases} 
1 & \text{if } \tau = 1 \\
0 & \text{if } \tau = d
\end{cases} \]
and it follows that \(\tau^*\) exists and is unique.

\[ \square \]

Proof that \(\tau^*\) is well-defined. Again, here we are only considering \(d \geq 5\). Let
\[ \Upsilon(t) = \frac{\Gamma(t)}{\Gamma(d - 1 - t)} \]
for \(t \in (0, \frac{d-1}{2})\). Then, of course,
\[ \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-1}{2} - 1)} = \Upsilon(t) \]
if \(t = \frac{d-1}{2}\) (and note that \(t \in (0, \frac{d-1}{2})\) if and only if \(\tau \in (1, d)\)). So, it suffices to show that there exists a unique \(t^* \in (0, \frac{d-1}{2})\) such that \(\Upsilon(t^*) = 1\).

To this end, we observe that \(\Upsilon\) is log-convex on \((0, \frac{d-1}{2})\) because
\[ (\log \Upsilon)'(t) = \psi(t) + \psi(d - 1 - t) \]
where \(\psi := (\log \Gamma)'\) is the digamma function. We note that
\[ \psi(t) = -\gamma - \frac{1}{t} + t \sum_{j=1}^{\infty} \frac{1}{j(t + j)} \quad \text{and} \quad \psi'(t) = \sum_{j=0}^{\infty} \frac{1}{(t + j)^2}, \]
where
\[ \gamma = \lim_{m \to \infty} \left\{ \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m} - \log m \right\} = 0.5772157 \ldots \]
(see Whittaker–Watson [25, Section 12.16]) and hence \(\psi'\) is a decreasing function on \((0, \infty)\). Then
\[ (\log \Upsilon)'(t) = \psi(t) - \psi'(d - 1 - t) > 0 \]
because we have \(t < \frac{d-1}{2}\), giving the claimed log-convexity of \(\Upsilon\).

So, in particular, \(\Upsilon\) must be convex on \((0, \frac{d-1}{2})\). We have \(\lim_{t \to 0^+} \Upsilon(t) = +\infty\) and at the other endpoint, we have \(\Upsilon(\frac{d-1}{2}) = 1\). Also,
\[ \Upsilon'(t) = \Upsilon(t)(\psi(t) + \psi(d - 1 - t)). \]
It can be shown from (2.7) that \(\psi(\frac{d-1}{2}) > 0\) for \(d \geq 5\) and therefore \(\Upsilon'(\frac{d-1}{2}) = 2\Upsilon(\frac{d-1}{2})\psi(\frac{d-1}{2}) > 0\) for \(d \geq 5\). From this we know that \(\Upsilon(t)\) is increasing for \(t\) sufficiently close to \(\frac{d-1}{2}\). By the Intermediate
Value Theorem, there exists \( t^* \in (0, \frac{d-1}{2}) \) such that \( \Upsilon(t^*) = 1 \). This must be unique since \( \Upsilon \) is convex on \( (0, \frac{d-1}{2}) \).

\[ \square \]

3. Proof of Theorem 1.1

First we will need to provide a brief discussion of spherical harmonics. Essentially, the arguments in this section are already present in [4]. We include the details here for self-containment, and to clarify a small technical point in the expression of the projection \( H_k \) from \( L^2(S^{d-1}) \) to \( \mathcal{H}_k \); here we use Legendre polynomials instead of Gegenbauer polynomials to include the case \( d = 2 \) in a more transparent way.

Let \( A_k \) be the space of solid spherical harmonics of degree \( k \) (these are harmonic polynomials on \( \mathbb{R}^d \) which are homogeneous of degree \( k \)), and let \( \mathcal{H}_k \) be the space of spherical harmonics of degree \( k \) (these are restrictions of functions in \( A_k \) to the sphere \( S^{d-1} \)). Then the eigenvalues of the Laplace–Beltrami operator \(-\Lambda\) on the sphere \( S^{d-1} \) are

\[ \mu_k = k(k + d - 2) \]

and the corresponding eigenspaces are \( \mathcal{H}_k \). The projection \( H_k \) from \( L^2(S^{d-1}) \) to \( \mathcal{H}_k \) can be written

\[ H_k f(\omega) = \frac{N_{k,d}}{|S^{d-1}|} \int_{S^{d-1}} P_{k,d}(\omega \cdot \tilde{\omega}) f(|\tilde{\omega}|) d\tilde{\omega}, \]

where

\[ N_{k,d} = \frac{(2k + d - 2)(k + d - 3)!}{k!(d - 2)!}, \]

\(|S^{d-1}|\) is the surface area of the sphere and \( P_{k,d} \) is the Legendre polynomial of degree \( k \) (see [1]).

Recall that we use the notation \( \mathcal{H}_k \) for the space of all linear combinations of functions

\[ \xi \mapsto P(\xi)f_0(|\xi|)|\xi|^{-d/2 - k + 1/2} \]

where \( P \in A_k \) and \( f_0 \in L^2(0, \infty) \). These spaces allow us to decompose \( L^2(\mathbb{R}^d) \) as

\[ L^2(\mathbb{R}^d) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k, \]

where this is a complete orthogonal direct sum decomposition in the sense that the closed subspaces \( \mathcal{H}_k \) are mutually orthogonal in \( L^2(\mathbb{R}^d) \) for \( k \in \mathbb{N}_0 \), and every \( f \in L^2(\mathbb{R}^d) \) can be written \( f = \sum_{k=0}^{\infty} f_k \) for some \( f_k \in \mathcal{H}_k \). We refer the reader to [13] and [17] for further details.

The operator \( H_k \) on \( L^2(S^{d-1}) \) can be homogeneously extended to \( L^2(\mathbb{R}^d) \) in a natural way by

\[ H_k f(x) = \frac{N_{k,d}}{|S^{d-1}|} \int_{S^{d-1}} P_{k,d}(x' \cdot \tilde{\omega}) f(|x|\tilde{\omega}) d\tilde{\omega}, \]

where \( x' = |x|^{-1} x \), and we shall use the same notation \( H_k \) as long as there is no confusion. In this way, the Laplace–Beltrami operator \(-\Lambda\) can be also regarded as an operator on \( L^2(\mathbb{R}^d) \) by using the spectral decomposition

\[ -\Lambda = \sum_{k=0}^{\infty} \mu_k H_k. \]
It is easy to see that the eigenvalues of this operator are again \( \{ \mu_k \}_{k=0}^\infty \), and \( H_k \) is the projection to the eigenspace \( \mathcal{H}_k \) of \( \mu_k \) for each \( k \in \mathbb{N}_0 \). For any functions \( \theta(\rho) \) of \( \rho \in [0, \infty) \), we can also define \( \theta(-\Lambda) \) as an operator on \( L^2(\mathbb{R}^d) \) by

\[
\theta(-\Lambda) = \sum_{k=0}^{\infty} \theta(\mu_k) H_k.
\]

**Proposition 3.1.** For each \( k \in \mathbb{N}_0 \), the operator \( H_k \) commutes with the Fourier transform and the inverse Fourier transform. In particular, each subspace \( \mathcal{H}_k \) is invariant under the action of these operators.

Since we are handling explicit constants, we clarify that \( \hat{f} \) is the Fourier transform of \( f \) given by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \exp(-ix \cdot \xi) \, dx.
\]

**Proof of Proposition 3.1.** Using polar coordinates, we have

\[
\int_{\mathbb{S}^{d-1}} \int_0^\infty \exp(-ir \omega \cdot x) P_{k,d}(\omega \cdot \tilde{\omega}) f(r \tilde{\omega}) r^{d-1} \, d\omega d\tilde{\omega} dr
\]

and

\[
\int_{\mathbb{S}^{d-1}} \int_0^\infty \exp(-i|\tilde{\omega} \cdot r \omega|) P_{k,d}(\tilde{\omega} \cdot x') f(r \omega) r^{d-1} \, d\omega d\tilde{\omega} dr
\]

so it suffices to check that

\[
\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \exp(-i r \omega \cdot x) P_{k,d}(\omega \cdot \tilde{\omega}) f(r \tilde{\omega}) \, d\omega d\tilde{\omega} = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \exp(-i|\tilde{\omega} \cdot r \omega|) P_{k,d}(\tilde{\omega} \cdot x') f(r \omega) \, d\omega d\tilde{\omega}
\]

for each \( x \in \mathbb{R}^d \) and \( r > 0 \). By switching the \( \omega \) and \( \tilde{\omega} \) variables on the left-hand side, it now suffices to show

\[
\int_{\mathbb{S}^{d-1}} \exp(-ir |\tilde{\omega} \cdot x'|) P_{k,d}(\tilde{\omega} \cdot \omega) d\tilde{\omega} = \int_{\mathbb{S}^{d-1}} \exp(-ir |\tilde{\omega} \cdot x'|) P_{k,d}(\tilde{\omega} \cdot \omega) d\tilde{\omega}
\]

for each \( x \in \mathbb{R}^d \), \( r > 0 \) and \( \omega \in \mathbb{S}^{d-1} \). However, \( \tilde{\omega} \mapsto P_{k,d}(\tilde{\omega} \cdot \omega) \) and \( \tilde{\omega} \mapsto P_{k,d}(\tilde{\omega} \cdot x') \) are spherical harmonics of degree \( k \) and we may apply the Funk–Hecke theorem (see, for example, [1]) to see that both sides of (3.2) are equal to

\[
|\mathbb{S}^{d-2}| P_{k,d}(x' \cdot \omega) \int_{-1}^{1} P_{k,d}(s) \exp(-irs|x|)(1 - s^2)^{\frac{d-3}{2}} \, ds,
\]

which gives the desired claim. The proof for the inverse Fourier transform is almost identical and we omit the details. \( \square \)

It follows from Proposition 3.1 that

\[
\langle H_k \Psi(|\nabla|) f \rangle(\xi) = H_k \langle \Psi(|\nabla|) f \rangle(\xi) = \Psi(|\xi|) H_k \hat{f}(\xi) = \langle \Psi(|\nabla|) H_k f \rangle(\xi)
\]

and therefore \( H_k \Psi(|\nabla|) = \Psi(|\nabla|) H_k \). From this, we also know that \( \theta(-\Lambda) \) also commutes with the Fourier transform, its inverse, and \( \Psi(|\nabla|) \). We use this observation in order to prove Theorem 1.1.
Proof of Theorem 1.1. Let \( S_\theta : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{d+1}) \) be the linear operator given by
\[
S_\theta f(x,t) = |x|^{-\frac{d}{2}} \theta(-\Lambda) \int_{\mathbb{R}^d} \exp(i(x \cdot \xi + t\phi(|\xi|)) \psi(|\xi|) f(\xi) d\xi
\]
for Schwartz functions \( f : \mathbb{R}^d \to \mathbb{C}, (x,t) \in \mathbb{R}^d \times \mathbb{R} \) and where \( \theta(-\Lambda) \) is an operation in the \( x \)-variable. The relevance of the operator \( S_\theta \) is seen through the expression
\[
|x|^{-\frac{d}{2}} \theta(-\Lambda) \psi(|\nabla|) \exp(it\phi(|\nabla|)) f(x) = (2\pi)^{-d} S_\theta \hat{f}(x,t).
\]

Proposition 3.2. Let \( \tau \in (1,d) \). Then the operator \( S_\theta^* S_\theta \) has the spectral decomposition
\[
S_\theta^* S_\theta = \sum_{k=0}^{\infty} \lambda_k |\theta(\mu_k)|^2 H_k,
\]
where, for each \( k \in \mathbb{N}_0 \),
\[
\lambda_k = (2\pi)^{d+1} 2^{1-\tau} \Gamma(\tau-1) \Gamma(k + \frac{d+\tau}{2}) \quad \frac{\Gamma(\frac{\tau}{2})^2 \Gamma(k + \frac{d+\tau}{2} - 1)}{\Gamma(k + \frac{d+\tau}{2})}
\]
and \( \mu_k = k(k + d - 2) \).

Proof. When \( \theta \) is identically equal to one, this follows from [3] (see Theorem 1.5). The general case follows from
\[
S_\theta = S_1 \circ \theta(-\Lambda).
\]
To see (3.6), we simply use our observation that \( \theta(-\Lambda) \) commutes with the inverse Fourier transform and, for each fixed \( t \), commutes with the operator \( \psi(|\nabla|) \exp(it\phi(|\nabla|)) \). \[\Box\]

Clearly, from Proposition 3.2 we have
\[
\|S_\theta f\|_{L^2(\mathbb{R}^{d+1})}^2 = \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \lambda_k |\theta(\mu_k)|^2 (H_k f, H_k f)_{L^2(\mathbb{R}^d)} = \sum_{k=0}^{\infty} \lambda_k |\theta(\mu_k)|^2 \|H_k f\|_{L^2(\mathbb{R}^d)}^2
\]
and therefore
\[
\inf_{k \in \mathbb{N}_0} \lambda_k |\theta(\mu_k)|^2 \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \|S_\theta f\|_{L^2(\mathbb{R}^{d+1})}^2 \leq \sup_{k \in \mathbb{N}_0} \lambda_k |\theta(\mu_k)|^2 \|f\|_{L^2(\mathbb{R}^d)}^2.
\]
Using (3.1) and Plancherel’s theorem \( \|\hat{f}\|_{L^2(\mathbb{R}^d)}^2 = (2\pi)^d \|f\|_{L^2(\mathbb{R}^d)}^2 \) we obtain
\[
\inf_{k \in \mathbb{N}_0} \beta_k \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \int_\mathbb{R} \int_{\mathbb{R}^d} |\psi(|\nabla|) \theta(-\Lambda) \exp(it\phi(|\nabla|)) f(x)|^2 \frac{dxdy}{|x|^7} \leq \sup_{k \in \mathbb{N}_0} \beta_k \|f\|_{L^2(\mathbb{R}^d)}^2,
\]
where the \( \beta_k \) are as given in the statement of Theorem 1.1. The optimality of the constants and the remaining claims concerning extremisers follow in a straightforward way using the fact that
\[
\|S_\theta f\|_{L^2(\mathbb{R}^{d+1})}^2 = \lambda_k |\theta(\mu_k)|^2 \|f\|_{L^2(\mathbb{R}^d)}^2
\]
for any \( f \in \mathcal{D}_k \setminus \{0\} \) and any \( k \in \mathbb{N}_0 \), orthogonality arguments and Proposition 3.1. \[\Box\]
4. Proofs of Theorem 2.1 and Proposition 2.5

Recall that $b(d, \tau; \theta) = \inf_{k \in \mathbb{N}_0} \beta_k$ and $B(d, \tau; \theta) = \sup_{k \in \mathbb{N}_0} \beta_k$ where
\[
\beta_k = \beta_k(d, \tau; \theta) = 2^{2-d} \frac{\Gamma(\tau-1)\Gamma(k+\frac{d-\tau}{2})}{\Gamma(\frac{d}{2})^2 \Gamma(k+\frac{d-\tau}{2}+1)} (1+k(k+d-2))^{\frac{\tau-1}{2}}
\]
for $\theta(\rho) = (1+\rho)\frac{\tau-1}{2}$.

**Proof of Theorem 2.1.** We set
\[
h(k, \tau) := \frac{\beta_{k+1}}{\beta_k} = \frac{2k+d-\tau}{2k+d+\tau-2} \left( \frac{1+(k+1)(k+d-1)}{1+k(k+d-2)} \right)^{\frac{\tau-1}{2}}.
\]
Then we have that $h(k, \tau) \to 1$ as $k \to \infty$ and will often use this fact without notification. Also we have
\[
\frac{\partial h}{\partial k}(k, \tau) = -A(d, k, \tau) \{ B_0(d, \tau) + B_1(d, \tau)k + B_2(d, \tau)k^2 \},
\]
where
\[
A(d, k, \tau) = \frac{\tau-1}{2(2k+d+\tau-2)^2 \{1+k(k+d-2)\}^2} \left( \frac{1+(k+1)(k+d-1)}{1+k(k+d-2)} \right)^{\frac{\tau-1}{2}},
\]
\[
B_0(d, \tau) = d((d-3)\tau(2-\tau) + (d-4)(d^2-d+2)),
\]
\[
B_1(d, \tau) = 2(d-1)\{\tau(2-\tau) + 3d(d-4)\} = (d-1)B_2(d, \tau),
\]
\[
B_2(d, \tau) = 2(\tau(2-\tau) + 3d(d-4)).
\]

Regarding the function $A$, the only property that we use in the rest of the proof is that $A > 0$ since $\tau > 1$.

The cases where $d = 2$, $d = 3$ and $d = 4$ with $\tau \in (2, 4)$ are straightforward to handle because, for such $(d, \tau)$, we have $B_j(d, \tau) < 0$ for each $j = 1, 2, 3$. This is clear from the expressions:

| $(d, \tau)$ | $B_0(d, \tau)$ | $B_1(d, \tau)$ | $B_2(d, \tau)$ |
|-------------|----------------|----------------|----------------|
| $d = 2$     | $-2\tau(2-\tau) - 16$ | $2\tau(2-\tau) - 24$ | $2\tau(2-\tau) - 24$ |
| $d = 3$     | $-24$          | $4\tau(2-\tau) - 36$ | $2\tau(2-\tau) - 18$ |
| $d = 4$     | $4\tau(2-\tau)$ | $6\tau(2-\tau)$ | $2\tau(2-\tau)$ |

This means $h(\cdot, \tau)$ is strictly increasing (to 1) so that $(\beta_k)_{k \in \mathbb{N}_0}$ is a strictly decreasing sequence. This implies
\[
\inf_{k \in \mathbb{N}_0} \beta_k = \lim_{k \to \infty} \beta_k \quad \text{and} \quad \sup_{k \in \mathbb{N}_0} \beta_k = \beta_0.
\]

In the special case $(d, \tau) = (4, 2)$, we clearly have $B_j(4, 2) = 0$ for each $j = 1, 2, 3$. This means $h(\cdot, \tau) = 1$ and the sequence $(\beta_k)_{k \in \mathbb{N}_0}$ is constant. Using, for example, (2.4), we have that this constant value is equal to $\pi$. Also, when $d = 4$ and $\tau \in (1, 2)$ it is clear that $B_j(4, \tau) > 0$ for each $j = 1, 2, 3$. In such a case, $h(\cdot, \tau)$ is strictly decreasing, $(\beta_k)_{k \in \mathbb{N}_0}$ is a strictly increasing sequence, and therefore
\[
\inf_{k \in \mathbb{N}_0} \beta_k = \beta_0 \quad \text{and} \quad \sup_{k \in \mathbb{N}_0} \beta_k = \lim_{k \to \infty} \beta_k.
\]
We next consider the case \( d \geq 5 \) and begin with some preliminary observations. Recall that \( \tau_* \) is uniquely defined by \( h(0, \tau_*) = 1 \). It is also true that, for \( \tau \in (1, d) \), we have \( h(0, \tau) < 1 \) if and only if \( \tau \in (\tau_*, d) \).

Since

\[
\beta_0 = \pi^{2 - \tau} \frac{\Gamma(\tau - 1)}{\Gamma(\frac{\tau}{2})^2} \frac{\Gamma(d - \tau)}{\Gamma(d + \tau - 1)} = \frac{\Gamma(d - \tau)}{\Gamma(d + \tau - 1)} \lim_{k \to \infty} \beta_k
\]

we see that \( \beta_0 = \lim_{k \to \infty} \beta_k \) when \( \tau = \tau^* \), and it is also true that, for \( \tau \in (1, d) \), we have

\[
(4.1) \quad \beta_0 > \lim_{k \to \infty} \beta_k \iff \tau \in (\tau^*, d).
\]

We also record the following lemma, which is completely elementary.

**Lemma 4.1.** Let \( d \geq 5 \) and \( \tau \in (1, d) \). Then \( h(k, \tau) \) is strictly decreasing to 1 for sufficiently large \( k \). Furthermore, \( h(\cdot, \tau) \) has at most one stationary point on \([0, \infty)\), and when it exists it is a global maximum on this domain.

**Proof.** Since \( B_1(d, d) = 4d(d - 1)(d - 5) \) and \( B_2(d, d) = 4d(d - 5) \), it is clear that \( B_1(d, \tau), B_2(d, \tau) > 0 \) for all \( \tau \in (1, d) \). This, of course, means that the quadratic function

\[
k \mapsto B_0(d, \tau) + B_1(d, \tau)k + B_2(d, \tau)k^2
\]

has at most one root on \([0, \infty)\). Since \( A > 0 \), it follows that \( h(\cdot, \tau) \) has at most one stationary point on \([0, \infty)\). It is clear that \( h(k, \tau) \) is strictly decreasing for sufficiently large \( k \) and therefore this stationary point must be a global maximum when it exists. \( \square \)

We can use Lemma 4.1 to argue that if \( \tau \in (1, \tau_*) \) then \( h(0, \tau) > 1 \) and therefore \( h(k, \tau) > 1 \) for all \( k \geq 0 \). Consequently, \( (\beta_k)_{k \in \mathbb{N}_0} \) is strictly increasing so that

\[
(4.2) \quad \inf_{k \in \mathbb{N}_0} \beta_k = \beta_0 \quad \text{and} \quad \sup_{k \in \mathbb{N}_0} \beta_k = \lim_{k \to \infty} \beta_k.
\]

**Remark.** We can now deduce that \( \tau_* \) cannot exceed \( \tau^* \). If it were true that \( \tau^* < \tau_* \) then for \( \tau \in (\tau^*, \tau_*) \) we know from (4.2) that \( \lim_{k \to \infty} \beta_k \beta_k > \beta_0 \). However, this contradicts (4.1).

If \( \tau \in [\tau_*, d) \), so that \( h(0, \tau) \leq 1 \), then from Lemma 4.1 it must be true that \( h(\cdot, \tau) \) has a unique global maximum which is strictly bigger than 1. By the Intermediate Value Theorem there exists \( k(\tau) \in [0, \infty) \) such that \( h(k(\tau), \tau) = 1 \), and this justifies the existence of \( k(\tau) \) satisfying (2.1). Since there is only one stationary point of \( h(\cdot, \tau) \) on \([0, \infty)\) it follows that \( k(\tau) \) is unique.

Suppose \( k(\tau) \not\in \mathbb{N}_0 \). In this case, if \( k \geq k^*(\tau) \) then \( h(k, \tau) > 1 \) and \( \beta_k > \beta_k \); that is,

\[
\beta_k < \beta_k < \beta_k < \beta_k + 1 < \beta_k + 2 < \cdots
\]

Similarly, if \( k \leq k^*(\tau) - 1 \) then \( h(k, \tau) < 1 \) and \( \beta_k < \beta_k \); that is,

\[
\beta_k < \beta_k < \beta_k < \beta_k < \cdots < \beta_0.
\]

This means, of course,

\[
\inf_{k \in \mathbb{N}_0} \beta_k = \beta_{k^*(\tau)}
\]
and, using (4.1),
\[
\sup_{k \in \mathbb{N}_0} \beta_k = \max\{\beta_0, \lim_{k \to \infty} \beta_k\} = \left\{ \begin{array}{ll}
\lim_{k \to \infty} \beta_k & \text{if } \tau \in [\tau_*, \tau^*) \\
\beta_0 & \text{if } \tau \in [\tau^*, d) \end{array} \right.
\]

By a similar argument, the same conclusion is true in the case \(k(\tau) \in \mathbb{N}_0\), except the infimum is not uniquely attained because \(h(k^*(\tau), \tau) = 1\) means \(\beta_{k^*(\tau)+1} = \beta_{k^*(\tau)}\).

The proof of Theorem 2.1 will be complete once we verify that \(k(\tau) \in (0, 1)\) and \(k^*(\tau) = 1\) whenever \(d \geq 6\) and \(\tau \in (\tau_*, d)\). Since \(h(0, \tau) < 1\) for such \(\tau\), it suffices to show that \(h(1, \tau) > 1\). For this, we define
\[
\Theta(\tau) := h(1, \tau) = \frac{d + 2 - \tau}{d + \tau} \left( \frac{2d + 1}{d} \right)^{\frac{d-\eta}{d}}.
\]

Necessarily \(B_0(d, \tau) < 0\); otherwise each \(B_j(d, \tau) > 0\) (see the proof of Lemma 4.1) which implies \(h(\cdot, \tau)\) is decreasing on \((0, \infty)\). Since \(h(0, \tau) < 1\) this means \(\lim_{k \to \infty} h(k, \tau) < 1\), which is false.

Now \(B_0(d, \tau) < 0\) if and only if \(\tau \in (\tau(d), d)\), where \(\tau(d)\) is the largest root of
\[
\tau(2 - \tau) = \frac{(d - 4)(d^2 - d + 2)}{d - 3};
\]
that is,
\[
\tau(d) = 1 + \sqrt{1 + \frac{d - 4}{d - 3}(d^2 - d + 2)}.
\]
Define \(\eta(d) = d - \tau(d)\) to be the length of the interval for which \(B_0(d, \tau) < 0\). Then it is straightforward to check that
\[
\eta(d) = d - 1 - \sqrt{1 + \frac{d - 4}{d - 3}(d^2 - d + 2)} = \frac{8}{(d - 3) \left( d - 1 + \sqrt{1 + \frac{d - 4}{d - 3}(d^2 - d + 2)} \right)} \leq \eta(6),
\]
where
\[
\eta(6) = \frac{8}{3(5 + \sqrt{67}/3)} = 0.274...
\]
We shall prove that
\[
0 \leq \eta \leq \eta(d) \implies \Theta(d - \eta) > 1.
\]
To establish (4.3) first notice that
\[
\Theta(d - \eta) = \frac{2 + \eta}{2d - \eta} \left( \frac{2d + 1}{d} \right)^{\frac{d-\eta}{d}} \geq \frac{1}{d} 2^{\frac{d-\eta}{d}} \left( 1 + \frac{1}{2d} \right)^{\frac{d-\eta}{d}}
\]
and therefore
\[
\Theta(d - \eta) \geq 2^{\frac{d-\eta}{d}} \left( \frac{5d - \eta - 1}{4d^2} \right) \geq 2^{\frac{d-\eta(6)}{d}} \left( \frac{5d - \eta(6) - 1}{4d^2} \right)
\]
since \(0 \leq \eta \leq \eta(d) \leq \eta(6)\). This means we will have shown (4.3) once we show that
\[
2^{\frac{d-\eta(6)}{d}} > \frac{4d^2}{5d - \eta(6) - 1}.
\]
When \( d = 6 \), we have \( 2^{\frac{\eta(6)}{d}} = 5.144... \) and \( \frac{144}{d-\eta(6)} = 5.012... \) and hence (4.4) is true. We will show that (4.4) is true for larger dimensions using a straightforward induction argument, and so we assume (4.4) is true for some fixed \( d \geq 6 \). Using this assumption,

\[
2^{\frac{d-\eta(6)}{5d-\eta(6)-1}} > \frac{4\sqrt{2d^2}}{5d-\eta(6)-1}
\]

and so it suffices to check that

(4.5) \[
\frac{4\sqrt{2d^2}}{5d-\eta(6)-1} \geq \frac{4(d+1)^2}{5(d+1)-\eta(6)-1}.
\]

It is clear that if \( d \geq \left(2^{1/4} - 1\right)^{-1} \) then \( \sqrt{2d^2} \geq (d + 1)^2 \), and hence (4.5) holds for such \( d \). But \( (2^{1/4} - 1)^{-1} = 5.285... \), therefore (4.5), and hence (4.4), is true for all \( d \geq 6 \).

Bringing everything together, whenever \( d \geq 6 \) and \( \tau \in (\tau_*, d) \), with \( \eta = d - \tau \) then we have \( 0 < \eta < \eta(d) \) and consequently (4.3) implies \( h(k, \tau) = \Theta(\tau) > 1 \).

From the above proof, one can easily extract the claimed expressions for the index sets \( k(d, \tau; \theta) \) and \( K(d, \tau; \theta) \) on account of Theorem 1.1; we omit the details. \( \square \)

**Proof of Proposition 2.5.** To obtain the claimed lower bound, define

\[
\Delta_k(\varepsilon) = \frac{\partial}{\partial \varepsilon}(h(k, 5 - 2\varepsilon))
\]

for \( \varepsilon \geq 0 \). Then

\[
\Delta_k(\varepsilon) = \left[ \frac{2(k + 2)}{(k + 4 - \varepsilon)^2} - \frac{k + \varepsilon}{k + 4 - \varepsilon} \log \left( \frac{k^2 + 5k + 5}{k^2 + 3k + 1} \right) \right] \left( \frac{k^2 + 5k + 5}{k^2 + 3k + 1} \right)^{2-\varepsilon}
\]

\[
\leq \frac{2(k + 2)}{(k + 3)^2} \left( \frac{k^2 + 5k + 5}{k^2 + 3k + 1} \right)^{2}
\]

for \( 0 < \varepsilon < 1 \), and it follows from the Mean Value Theorem that

\[
h(k, 5 - 2\varepsilon) \leq h(k, 5) + \varepsilon \frac{2(k + 2)}{(k + 3)^2} \left( \frac{k^2 + 5k + 5}{k^2 + 3k + 1} \right)^{2}.
\]

Hence

(4.6) \[
0 \leq \varepsilon < \varepsilon(k) \implies h(k, 5 - 2\varepsilon) < 1,
\]

where

\[
\varepsilon(k) := (1 - h(k, 5)) \frac{(k + 3)^2(k^2 + 3k + 1)^2}{2(k + 2)(k^2 + 5k + 5)^2}.
\]

A straightforward calculation shows that

\[
1 - h(k, 5) = 1 - \frac{k(k^2 + 5k + 5)^2}{(k + 4)(k^2 + 3k + 1)^2} = \frac{4}{(k + 4)(k^2 + 3k + 1)^2}
\]

and therefore

\[
\varepsilon(k) = \frac{2(k + 3)^2}{(k + 2)(k + 4)(k^2 + 5k + 5)^2} \geq \frac{C}{(k + 1)^4}
\]

for some absolute constant \( C \) and all \( k \in \mathbb{N}_0 \).
For fixed $\tau \in (\tau_*, 5)$, if we take $k \in \mathbb{N}_0$ such that
\[
k < \left( \frac{2C}{5 - \tau} \right)^{1/4} - 1
\]
then $\frac{1}{2}(5 - \tau) < \varepsilon(k)$ and, by (4.6), we get $h(k, \tau) < 1$. This means that $k(\tau) \in \mathbb{N}_0$ satisfying (2.1) satisfies the lower bound
\[
k(\tau) \geq \left( \frac{2C}{5 - \tau} \right)^{1/4} - 1.
\]

For the upper bound, by Lemma 4.1 we make the observation that $k(\tau)$ cannot exceed the positive value of $k$ at which $\frac{\partial h}{\partial k}(k, \tau)$ is equal to zero, given by
\[
k = -2 + \sqrt{\frac{B_1(5, \tau) - B_0(5, \tau)}{B_2(5, \tau)}};
\]
that is
\[
k(\tau) \leq -2 + \sqrt{\frac{5 - \tau(2 - \tau)}{(5 - \tau)(3 + \tau)}}.
\]
Hence, there exists positive constants $C_1$ and $C_2$ such that, for all $\tau \in (\tau_*, 5)$,
\[
\frac{C_1}{(5 - \tau)^{1/4}} \leq k(\tau) \leq \frac{C_2}{(5 - \tau)^{1/2}},
\]
as claimed. □

5. Further results

We begin by considering the case $\theta(\rho) = \rho^{2-\tau}$, and $\phi$ and $\psi$ satisfying (1.6), with $d \geq 2$ and $\tau \in (1, d)$. Then we have
\[
\beta_k = \pi 2^{2-\tau} \frac{\Gamma(\tau - 1)\Gamma(k + \frac{d-\tau}{2})}{\Gamma(\frac{d}{2})^2\Gamma(k + \frac{d+\tau}{2} - 1)} (k(k + d - 2))^{\frac{\tau-1}{2}}
\]
and it is clear that $b(d, \tau; \theta) = 0$. Also, if
\[
h(k, \tau) = \frac{\beta_{k+1}}{\beta_k} = \frac{2k + d - \tau}{2k + d + \tau - 2} \left( \frac{k+1}{k} \right)^{\frac{\tau-1}{2}}
\]
then
\[
\frac{\partial h}{\partial k}(k, \tau) = -A(d, k, \tau) \{ B_0(d, \tau) + B_1(d, \tau)k + B_2(d, \tau)k^2 \},
\]
where
\[
A(d, k, \tau) = \frac{\tau - 1}{2(2k + d + \tau - 2)^2(k(k + d - 2))^2} \left( \frac{(k+1)(k+d-1)}{k(k+d-2)} \right)^{\frac{\tau-1}{2}},
\]
\[
B_0(d, \tau) = (d - 1)(d - 2)(d - 2 + \tau)(d - \tau),
\]
\[
B_1(d, \tau) = 2(d - 1)(3(d - 2)^2 + (2 - \tau)\tau),
\]
\[
B_2(d, \tau) = 2(3(d - 2)^2 + (2 - \tau)\tau).
\]
For $d \geq 2$ and $\tau \in (1, d)$, obviously we have $A(k, d, \tau) > 0$ and $B_0(d, \tau) \geq 0$. 
Since \((2 - \tau)r\) is strictly decreasing for \(r \in (1, d)\), we have \(3(d - 2)^2 + (2 - \tau)r > 2(d - 2)(d - 3)\) and therefore \(B_1(d, \tau) > 0\) and \(B_2(d, \tau) > 0\). This means \(h(\cdot, \tau)\) is strictly decreasing and tends to 1 from above. It follows that \((\beta_k)_{k \in \mathbb{N}_0}\) is strictly increasing and

\[
B(d, \tau; \theta) = \lim_{k \to \infty} \beta_k = \pi 2^{2-\tau} \frac{\Gamma(\tau - 1)}{\Gamma(\tau)^2}.
\]

Using Theorem 1.1 we may use the above analysis to obtain the following.

**Corollary 5.1.** Let \(d \geq 2\), \(s \in (-\frac{1}{2}, \frac{d}{2} - 1)\) and suppose that \(i\partial_t u + \frac{1}{2} \Delta u = 0\) on \(\mathbb{R}^{d+1}\). Then

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(-\Lambda)^{\frac{1}{2}} u(x, t)|^2 \frac{dx dt}{|x|^{2(1+s)}} \leq \pi 2^{2s} \frac{\Gamma(2s + 1)}{\Gamma(s + 1)^2} \left( \|u(0)\|_{H^s(\mathbb{R}^d)}^2 + \|\partial_t u(0)\|_{H^{s-1}(\mathbb{R}^d)}^2 \right),
\]

the constant is optimal and there are no extremisers.

**Corollary 5.2.** Let \(d \geq 2\), \(s \in (0, \frac{d-1}{2})\) and suppose that \(\partial_t u - \Delta u = 0\) on \(\mathbb{R}^{d+1}\). Then

\[
2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(\Lambda)^{\frac{1}{2}} u(x, t)|^2 \frac{dx dt}{|x|^{2(1+s)}} \leq \pi 2^{2s} \frac{\Gamma(2s + 1)}{\Gamma(s + 1)^2} \left( \|u(0)\|_{H^s(\mathbb{R}^d)}^2 + \|\partial_t u(0)\|_{H^{s-1}(\mathbb{R}^d)}^2 \right),
\]

the constant is optimal and there are no extremisers.

**Corollary 5.3.** Let \(d \geq 2\), \(s \in (-\frac{1}{2}, \frac{d}{2} - 1)\) and suppose that \(\partial_t u - \Delta u + u = 0\) on \(\mathbb{R}^{d+1}\). Then

\[
2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(1 - \Lambda)^{\frac{1}{2}} u(x, t)|^2 \frac{dx dt}{|x|^{2(1+s)}} \leq \pi 2^{2s} \frac{\Gamma(2s + 1)}{\Gamma(s + 1)^2} \left( \|u(0)\|_{H^s(\mathbb{R}^d)}^2 + \|\partial_t u(0)\|_{H^{s+1}(\mathbb{R}^d)}^2 + \|\partial_t u(0)\|_{H^{s+1}(\mathbb{R}^d)}^2 \right),
\]

the constant is optimal and there are no extremisers. Here, the norm on the initial data is given by

\[
\|(u(0), \partial_t u(0))\|_2 = \|u(0)\|_{H^s(\mathbb{R}^d)}^2 + \|u(0)\|_{H^{s+1}(\mathbb{R}^d)}^2 + \|\partial_t u(0)\|_{H^{s+1}(\mathbb{R}^d)}^2.
\]

We finish by stating a more general result than Theorem 1.1 which is not restricted to homogeneous weights, and does not require the dispersion relation and smoothing functions \(\phi\) and \(\psi\) to satisfy (1.6). The cost of this generality is that the optimal constants are less explicit and precise information about the extremisers is less readily available.

**Theorem 5.4.** Suppose \(w : [0, \infty) \to [0, \infty), \psi : [0, \infty) \to [0, \infty)\) and \(\phi : [0, \infty) \to \mathbb{R}\) are such that \(\alpha_k : [0, \infty) \to [0, \infty)\) is continuous for each \(k \in \mathbb{N}_0\), where

\[
\alpha_k(\rho) = \frac{p \psi(\rho)^2}{|\phi'(\rho)|} \int_0^{\infty} J_{\nu(k)}(r \rho)^2 r w(r) dr,
\]

\(\nu(k) = \frac{d}{2} + k - 1\) and \(J_{\nu(k)}\) is the Bessel function of the first kind with order \(\nu(k)\). If \(i\partial_t u + \phi(|\nabla|)u = 0\) then

\[
\inf_{k \in \mathbb{N}_0} \inf_{\rho > 0} \beta_k(\rho) \|u(0)\|_{L^2(\mathbb{R}^d)}^2 \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi(|\nabla|) \theta(-\Lambda) u(t, x)|^2 w(|x|) dx dt \leq \sup_{k \in \mathbb{N}_0} \sup_{\rho > 0} \beta_k(\rho) \|u(0)\|_{L^2(\mathbb{R}^d)}^2,
\]

where

\[
\beta_k(\rho) = 2\pi |\theta(k(k + d - 2))|^2 \alpha_k(\rho)
\]

and the constants are optimal.

In the case where the weight \(w\) is homogeneous, but \(\phi\) and \(\psi\) do not necessarily satisfy (1.6), we may deduce the following.
Corollary 5.5. Let $\tau \in (1, d)$. Suppose $\psi : [0, \infty) \to [0, \infty)$ and $\phi : [0, \infty) \to \mathbb{R}$ are such that $\zeta : [0, \infty) \to [0, \infty)$ is continuous, where

$$\zeta(\rho) = \rho^{\tau-1} \psi(\rho)^2 \left| \frac{\phi'(\rho)}{\phi(\rho)} \right|.$$ 

Let

$$\beta_k = \pi^{2d-\tau} \frac{\Gamma(\tau - 1) \Gamma(k + \frac{d-\tau}{2})}{\Gamma\left(\frac{d}{2} \right)^2 \Gamma(k + \frac{d+\tau}{2} - 1)} |\theta(k(k + d - 2))|^2$$

for $k \in \mathbb{N}_0$. If $i\partial_t u + \phi(|\nabla|) u = 0$ then

$$\inf_{k \in \mathbb{N}_0} \beta_k \inf_{\rho > 0} \zeta(\rho) \|u(0)\|_{L^2(\mathbb{R}^d)}^2 \leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\psi(|\nabla|) \theta(-\Lambda) u(t, x)|^2 \frac{dx dt}{|x|^\tau} \leq \sup_{k \in \mathbb{N}_0} \beta_k \sup_{\rho > 0} \zeta(\rho) \|u(0)\|_{L^2(\mathbb{R}^d)}^2$$

and the constants are optimal.

It is clear that Corollary 5.5 extends the sharp estimates in Theorem 1.1 since $\zeta$ is identically equal to one under the assumption (1.6). However, the situation regarding extremisers is more complicated when (1.6) does not hold. We may use Theorem 1.2 from [3] to see that the existence of extremisers for the upper bound is equivalent to the existence of some $k_0 \in \mathbb{N}_0$ such that

$$\beta_{k_0} = \sup_{k \in \mathbb{N}_0} \beta_k$$

and a subset $S$ of $(0, \infty)$ with positive Lebesgue measure such that

$$\zeta(\rho_0) = \sup_{\rho > 0} \zeta(\rho) \quad \text{for each } \rho_0 \in S.$$ 

An analogous remark is also valid for the lower bound, where each instance of sup is replaced by inf.

Corollary 5.5 follows immediately from Theorem 5.4 and the formula

$$\int_0^\infty J_\nu(kr)^2 \frac{dr}{r^{\tau-1}} = 2^{1-\tau} \frac{\Gamma(\tau - 1) \Gamma(k + \frac{d-\tau}{2})}{\Gamma\left(\frac{d}{2} \right)^2 \Gamma(k + \frac{d+\tau}{2} - 1)} r^{\tau-2}.$$ 

This formula was also used in the proof of Theorem 1.6 in [3], and Corollary 5.5 is in fact a generalisation of Theorem 1.6 in [3] where the case $\theta$ identically equal to one is given.

In a similar manner, Theorem 5.4 is a generalisation of Theorem 4.1(b) of [23] where the case $\theta$ identically equal to one is given. To prove Theorem 5.4 we use our observation that the operator $S_\theta$, introduced in [3.3], satisfies (3.6) and proceed using the same argument in [23]; we omit the details. We remark that this proof via the argument in [23] is also based on a spherical harmonic decomposition and orthogonality arguments, but does not yield precise spectral information as in Proposition 5.2.

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