Bilateral Trade: A Regret Minimization Perspective*

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Abstract

Bilateral trade, a fundamental topic in economics, models the problem of intermediating between two strategic agents, a seller and a buyer, willing to trade a good for which they hold private valuations. In this paper, we cast the bilateral trade problem in a regret minimization framework over \( T \) rounds of seller/buyer interactions, with no prior knowledge on their private valuations. Our main contribution is a complete characterization of the regret regimes for fixed-price mechanisms with different feedback models and private valuations, using as a benchmark the best fixed-price in hindsight. More precisely, we prove the following tight bounds on the regret:

- \( \Theta(\sqrt{T}) \) for full-feedback (i.e., direct revelation mechanisms).
- \( \Theta(T^{2/3}) \) for realistic feedback (i.e., posted-price mechanisms) and independent seller/buyer valuations with bounded densities.
- \( \Theta(T) \) for realistic feedback and seller/buyer valuations with bounded densities.
- \( \Theta(T) \) for realistic feedback and independent seller/buyer valuations.
- \( \Theta(T) \) for the adversarial setting.

1 Introduction

In the bilateral trade problem, two strategic agents —a seller and a buyer— wish to trade some good. They both privately hold a personal valuation for it and strive to maximize their respective quasi-linear utility. The burden of designing a mechanism to reach an agreement is usually delegated to a third party. This scenario arises naturally in many internet applications, such as ridesharing systems like Uber or Lyft, where trades between sellers (drivers) and buyers (riders) are managed by a mechanism designed by the platform.

In general, an ideal mechanism for the bilateral trade problem would optimize the efficiency, i.e., the social welfare resulting by trading the item, while enforcing incentive compatibility (IC) and individual rationality (IR). The assumption that makes a two-sided mechanism design more complex than its one-sided counterpart is budget balance (BB): the mechanism cannot subsidize or make a profit from the market. Unfortunately, as Vickrey observed in his seminal work [Vickrey, 1961], the optimal incentive compatible mechanism maximizing social welfare for bilateral trade may not be budget balanced.

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A more general result due to Myerson and Satterthwaite [Myerson and Satterthwaite, 1983] shows that there are some problem instances where a fully efficient mechanism for bilateral trade that satisfies IC, IR, and BB does not exist. This impossibility result holds even if prior information on the buyer and seller’s valuations is available, the truthful notion is relaxed to Bayesian incentive compatibility (BIC), and the budget balance constraint is loosened to weak budget balance (WBB). To circumvent this obstacle, a long line of research has focused on the design of approximating mechanisms that satisfy the above requirements while being nearly efficient.

These approximation results build on a Bayesian assumption: seller and buyer valuations are drawn from two distributions, which are both known to the mechanism designer. Although in some sense necessary—which any information on the priors there is no way to extract any meaningful approximation result [Dütting et al., 2021]—this assumption is unrealistic. Following a recent line of research [Cesa-Bianchi et al., 2015, Lykouris et al., 2016, Daskalakis and Syrgkanis, 2016], in this work we study this basic mechanism design problem in a regret minimization setting. Our goal is bounding the total loss in efficiency experienced by the mechanism in the long period by learning the salient features of the prior distributions.

At each time $t$, a new seller/buyer pair arrives. The seller has a private valuation $S_t \in [0,1]$ representing the smallest price she is willing to accept in order to trade. Similarly, the buyer has a private value $B_t \in [0,1]$ representing the highest price that she will pay for the item. The mechanism sets a price $P_t \in [0,1]$ which results in a trade if and only if $S_t \leq P_t \leq B_t$.

There are two common utility functions that reflect the performance of the mechanism at each time step: the social welfare, which sums the utilities of the two players after the trade (and remains equal to the seller’s valuation if no trade occurs), and the gain from trade, consisting in the net gain in the utilities. In formulae, for each price $p \in [0,1]$,

- **Social Welfare**: $\text{SW}_t(p) := \text{SW}(p, S_t, B_t) := S_t + (B_t - S_t)I(S_t \leq p \leq B_t)$.
- **Gain from Trade**: $\text{GFT}_t(p) := \text{GFT}(p, S_t, B_t) := (B_t - S_t)I(S_t \leq p \leq B_t)$.

We begin by investigating the standard assumption in which $(S_1, B_1), (S_2, B_2), \ldots$ are i.i.d. random vectors, supported in $[0,1]^2$, representing the valuations of seller and buyer respectively (stochastic i.i.d. setting). We also consider the case where $(S_1, B_1), (S_2, B_2), \ldots$ is an arbitrary deterministic process $(s_1, b_1), (s_2, b_2), \ldots$ (adversarial setting).

In our online learning framework, we aim at minimizing the regret of the mechanism over a time horizon $T$:

$$\max_{p \in [0,1]} E \left[ \frac{1}{T} \sum_{t=1}^{T} \text{GFT}(p) - \frac{1}{T} \sum_{t=1}^{T} \text{GFT}(P_t) \right].$$

Note that since $\text{GFT}(p_t) = \text{SW}(p_t) - S_t$ and $S_t$ does not depend on the choice of $p$, gain from trade and social welfare lead to the same notion of regret.

Hence, the regret is the difference between the expected total performance of our algorithm, which can only sequentially learn the distribution, and our reference benchmark, corresponding to the best fixed-price strategy assuming full knowledge of the distribution. Our main goal is to design strategies with asymptotically vanishing time-averaged regret with respect to the best fixed-price strategy or, equivalently, regret sublinear in the time horizon $T$.

The class of fixed-price mechanisms is of particular importance in bilateral trade as they are simple to implement, clearly truthful, individually rational, budget balanced, and enjoy the desirable property of asking the agents very little information. Moreover, it can be shown that fixed prices are the only direct revelation mechanisms which enjoy budget balance, dominant strategy incentive compatibility, and ex-post individual rationality [Colini-Baldeschi et al., 2016].

To complete the description of the problem, we need to specify the feedback obtained by the mechanism after each sequential round. We propose two main feedback models:

- **Full feedback.** In the full-feedback model, the pair $(S_t, B_t)$ is revealed to the mechanism after the $t$-th trading round. The information collected by this feedback model corresponds to direct revelation.
1.1 Overview of our Results

In Sections 4 to 6, we investigate the stochastic setting (under various assumptions), the adversarial setting, and how regret bounds change depending on the quality of the received feedback. In all cases, we provide matching upper and lower bounds. In particular, our positive results are constructive: explicit algorithms are given in each case. More precisely, we present (see Table 1 for a summary):

- Algorithm 2 (Follow the Best Price) for the full-feedback model achieving a $O(T^{1/2})$ regret in the stochastic (iid) setting (Theorem 1); this rate cannot be improved, not even under some additional natural assumptions (Theorem 2).

- Algorithm 3 (Scouting Bandits) for the harder realistic-feedback model achieving a $O(T^{2/3})$ regret in a stochastic (iid) setting in which the valuations of the seller and the buyer are independent of each other (iv) and have bounded densities (bd) (Theorem 3); this rate cannot be improved (Theorem 4).

- Impossibility results:
  - For the realistic-feedback model, if either the (iv) or the (bd) assumptions are dropped from the previous stochastic setting, no strategy can achieve sublinear worst-case regret (Theorems 5 and 6).
  - In an adversarial setting, no strategy can achieve sublinear worst-case regret, not even in the simple full-feedback model (Theorem 7).

In Section 7, we depart from the budget balance setting in which the learner posts the same price to both the seller and the buyer. We consider a weak budget balance (WBB) setting in which (possibly) distinct prices $p \leq p'$ can be posted, $p$ to the seller, and $p'$ to the buyer. We design Algorithm 4 (Scouting Blindits) which, leveraging the higher amount of information available in the WBB setting, can break the linear lower bound of Theorem 5 (Theorem 8).

Finally, in Section 8 we investigate the case in which the feedback is limited to one single bit (i.e., whether or not a trade occurred), showing a striking difference between the BB and WBB cases.

### Table 1: Our main results for fixed-price mechanisms. The rates are both upper and lower bounds. The slots without references are immediate consequences of the others.

|               | Stochastic (iid) | Adversarial |
|---------------|------------------|-------------|
|               | iid              | +iv         | +bd         | iv+bd       | adv         |
| Full          | $T^{1/2}$ (Thm 1) | $T^{1/2}$   | $T^{1/2}$   | $T^{1/2}$ (Thm 2) | $T$ (Thm 7) |
| Real          | $T$              | $T$ (Thm 6) | $T$ (Thm 5) | $T^{2/3}$ (Thms 3+4) | $T$         |

*mechanisms*, where the agents publicly declare their valuations in each round, and the price proposed by the mechanism at time $t$ only depends on past bids.

- Realistic feedback. In the harder realistic feedback model, only the relative order between $S_t$ and $P_t$ and between $B_t$ and $P_t$ are revealed after the $t$-th round. This model corresponds to posted price *mechanisms*, where seller and buyer separately accept or refuse the posted price. The price computed at time $t$ only depends on past bids, and the values $S_t$ and $B_t$ are never revealed to the mechanisms.

### 1.2 Technical Challenges

In this section, we sum up the technical challenges for various instances of our problem.
Full feedback. The full-feedback model fits nicely in the learning with expert advice framework [Cesa-Bianchi and Lugosi, 2006]. Each price \( p \in [0,1] \) can be viewed as an expert, and the revelation of \( S_t \) and \( B_t \) allows the mechanism to compute \( GFT_t(p) \) for all \( p \), including the mechanism’s own reward \( GFT_t(P_t) \). A common approach to reduce the cardinality of a continuous expert space is to assume some regularity (e.g., Lipschitzness) of the reward function, so that a finite grid of representative prices can be used. This approach yields a \( O(\sqrt{T}) \) bound under density boundedness assumptions on the joint distribution of the seller and the buyer. By exploiting the structure of the reward function \( \mathbb{E}[GFT_t(\cdot)] \), we obtain a better regret bound (by a log factor) without any assumptions on the distribution (other than iid). In Eq. (2), we show how to decompose the expression of the expected gain from trade in pieces that can be quickly learned via sampling. The full feedback received in each new round is used to refine the estimate of the actual gain from trade as a function of the price, while the posted prices are chosen so to maximize it. A Follow the Leader strategy is shown to achieve a \( O(\sqrt{T}) \) bound in the stochastic (iid) setting (Theorem 1). This holds for arbitrary joint distributions of the seller and the buyer. In particular, even when the buyer and seller have a correlated behavior. The main issue for the lower bound is that the (expected) gain from trade cannot be chosen arbitrarily: we can only control its shape indirectly as a function of the distribution of the seller/buyer pair. By designing a suitable family of such distributions, we build a reduction showing that the full-feedback bilateral trade problem is harder than a corresponding 2-action partial monitoring game with a known \( \Omega(\sqrt{T}) \) lower bound (Theorem 2).

Realistic Feedback. Here, at each time \( t \), only \( I(S_t \leq P_t) \) and \( I(P_t \leq B_t) \) are revealed to the learner. In contrast to the full-feedback model, this is not enough to reconstruct the gain from trade \( GFT_t \) at time \( t \): if the trade does not occur, it is unclear which prices would have resulted in a trade. Moreover, in contrast to bandit problems [Cesa-Bianchi and Lugosi, 2006], this feedback is not even enough to determine \( GFT_t(P_t) \): if the trade occurs, there is no way to infer the difference \( B_t - S_t \) directly. Thus, we cannot directly rely on known bandits tools to tackle the two competing goals of estimating the underlying distributions (exploration) while optimizing the estimated gain from trade (exploitation). Instead, we show how to decompose the expected gain from trade at price \( p \) into a global part that can be quickly estimated by uniform sampling on the \([0,1]\) interval, and a local part that can be learned by posting \( p \). Theorem 3 shows that our Algorithm 3 (Scouting Bandits) can take advantage of this decomposition by relying on any bandit algorithm to learn the local part of the expected gain from trade. We derive a sublinear regret of \( O(T^{2/3}) \) in a stochastic (iid) setting in which the valuations of the seller and the buyer are independent of each other (iv) and have bounded densities (bd). The lower bound presents challenges similar to those of the full-feedback model, with additional hurdles due to the specific nature of the realistic feedback that lead to a harder \( \Omega(T^{2/3}) \) rate (Theorem 4). Dropping the (iv) assumption leads to a pathological lack of observability phenomenon, in which it is impossible to distinguish between two scenarios with significantly different optimal prices (Theorem 5). Dropping the (bd) assumption leads to a needle in a haystack, a different pathological phenomenon in which all prices but one suffer a high regret, and it is essentially impossible to find this optimal price among a continuum of suboptimal prices (Theorem 6).

Adversarial setting. Here, the valuations of the buyer and the seller form an arbitrary deterministic process generated by an oblivious adversary. In this setting, learning is impossible. Indeed, using a construction inspired by the Cantor ternary set, we show that even under a full-feedback model, no strategy can lead to a sublinear worst-case regret (Theorem 7).

Lower Bound Techniques. Due to their technical nature, the proofs of the lower bounds (Theorems 2 and 4 to 7) are only sketched in the main text. Detailed versions of all of them are provided in the Appendix where, inspired by partial monitoring, we develop a general setting for sequential games that subsumes, in particular, all instances of our bilateral trade problem (Appendix C). Within this setting, we then build reductions by mapping instances of our problem to other known partial monitoring games. These reductions rely on two key lemmas, introduced in Appendix D: our Embedding and Simulation lemmas (Lemmas 3
The study of the bilateral trade problem dates back to the already mentioned seminal works of Vickrey [Vickrey, 1961] and Myerson and Satterthwaite [Myerson and Satterthwaite, 1983]. A more recent line of research focuses on Bayesian mechanisms that achieve the IC, BB, and IR requirements while approximating the optimal social welfare or the gain from trade. Blumrosen and Dobzinski [Blumrosen and Dobzinski, 2014] proposed the median mechanism that sets a posted price equal to the median of the seller distribution and shows that this mechanism obtains an approximation factor of 2 to the optimal social welfare. Subsequent work by the same authors [Blumrosen and Dobzinski, 2016] improved the approximation guarantee to $\epsilon/(\epsilon-1)$ through a randomized mechanism whose prices depend on the seller distribution in a more intricate way. In [Colini-Baldeschi et al., 2016] it is demonstrated that all DSIC mechanisms that are BB and IR must post a fixed price to the buyer and to the seller. The same result has been previously proven under stronger assumptions in [Hagerty and Rogerson, 1987].

In a different research direction aimed to characterize the information theoretical requirements of two-sided markets mechanisms, [Dütting et al., 2021] show that setting the price equal to a single sample from the seller distribution gives a 2-approximation to the optimal social welfare. In a parallel line of work, the harder objective of approximating the gain from trade has been considered. An asymptotically tight fixed-price $O(\log \frac{1}{\epsilon})$ approximation bound is also achieved in [Colini-Baldeschi et al., 2017], with $r$ being the probability that a trade happens (i.e., the value of the buyer is higher than the value of the seller). A BIC 2-approximation of the second best with a simple mechanism is obtained in [Brustle et al., 2017].

In the following, we discuss the relationship between the approximation results mentioned above and the regret analysis we develop in this work that compares online learning mechanisms against the best ex-ante fixed-price mechanism. First of all, in the realistic feedback setting, the approximation mechanisms for bilateral trade cannot be easily implemented. For example, the single sample 2-approximation to the optimal social welfare [Dütting et al., 2021] requires multiple rounds of interaction in order to obtain, approximately, a random sample from the distribution. The median mechanism of [Blumrosen and Dobzinski, 2014] requires an even larger number of rounds in order to estimate the median of the seller distribution. Furthermore, here we note that these two more demanding approaches may yield worst performances than the best ex-ante fixed price. This implies that there are instances where our online learning approach converges to a mechanism that is strictly better than the median or sample mechanisms, even assuming they have full knowledge of the underlying distributions.

There is a vast body of literature on regret analysis in (one-sided) dynamic pricing and online posted price auctions—see, e.g., the excellent survey published by [den Boer, 2015] and the tutorial slides by [Slivkins and Zeevi, 2015]. In their seminal paper, Kleinberg and Leighton prove a $O(T^{2/3})$ upper bound (ignoring logarithmic factors) on the regret in the adversarial setting [Kleinberg and Leighton, 2003]. Later works show simultaneous multiplicative and additive bounds on the regret when prices have range $[1, h]$ [Blum et al., 2004, Blum and Hartline, 2005]. These bounds have the form $\epsilon G^*_T + O((h\ln h)/\epsilon^2)$ ignoring $\ln \ln h$ factors, where $G^*_T$ is the total revenue of the optimal price $p^*$. Recent improvements on these results prove that the additive term can be made $O(p^* (\ln h)/\epsilon^2)$, where the linear scaling is now with respect to the optimal price rather than the maximum price $h$ [Bubeck et al., 2017]. Other variants consider settings in which the number of copies of the item to sell is limited [Agrawal and Devanur, 2014, Babaioff et al., 2015, Badanidiyuru et al., 2018], buyers act strategically in order to maximize their utility in future rounds [Amin et al., 2013, Devanur et al., 2019, Mohri and Medina, 2014, Drutsa, 2018], or there are features associated with the goods on sale [Cohen et al., 2020]. In the stochastic setting, previous works typically assume parametric [Broder and Rusmevichientong, 2012], locally smooth [Kleinberg and Leighton, 2003], or piecewise constant demand curves [Cesa-Bianchi et al., 2019, den Boer and Keskin, 2020].

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1 Consider a seller with value $\epsilon > 0$ or 0 with equal probability and a buyer with value 1. The best fixed price has welfare of 1. For small $\epsilon$, the median and the sample mechanism, respectively, obtains a welfare close to 1/2 and 3/4.
2 The Bilateral Trade learning protocol

In this section, we present the learning protocol for the sequential problem of bilateral trade (see Learning Protocol 1). We recall that the reward collected from a trade is the gain from trade, defined for all \( p, s, b \in [0,1] \), by

\[
GFT(p, s, b) := (b - s) \mathbb{I}\{s \leq b\}.
\]

**Learning Protocol 1: Bilateral Trade**

```plaintext
for time \( t = 1, 2, \ldots \) do
  a new seller/buyer pair arrives with (hidden) valuations \( (S_t, B_t) \in [0,1]^2 \);
  the learner posts a price \( P_t \in [0,1] \);
  the learner receives a (hidden) reward \( GFT_t(P_t) \in [0,1] \), where \( GFT_t(\cdot) := GFT(\cdot, S_t, B_t) \);
  feedback \( Z_t \) is revealed;
end
```

At each time step \( t \), a seller and a buyer arrive with privately held valuations: \( S_t \in [0,1] \) for the seller and \( B_t \in [0,1] \) for the buyer. The learner then posts a price \( P_t \in [0,1] \) and a trade occurs if and only if \( S_t \leq P_t \leq B_t \). When this happens, the learner gains a reward \( GFT(P_t, S_t, B_t) \) but, instead of observing this reward, the learner only observes the feedback \( Z_t \). The nature of the sequence of valuations \( (S_1, B_1), (S_2, B_2), \ldots \) and feedback \( Z_1, Z_2, \ldots \) depends on the specific instance of the problem and is described below.

The goal of the learner is to determine a strategy \( \alpha \) generating the prices \( P_1, P_2, \ldots \) (as in Learning Protocol 1) achieving sublinear regret

\[
R_T(\alpha) := \max_{p \in [0,1]} \mathbb{E} \left[ \sum_{t=1}^T GFT(p, S_t, B_t) - \sum_{t=1}^T GFT(P_t, S_t, B_t) \right],
\]

where the expectation is taken with respect to the sequence of buyer and seller valuations, and (possibly) the internal randomization of \( \alpha \). To lighten the notation, we denote by \( p^* \) (one of) the \( p \in [0,1] \) maximizing the previous expectation. Such a \( p^* \) always exists (for a proof of this fact, see Appendix B).

We now introduce several instances of bilateral trade, depending on the type of the received feedback and the nature of the environment.

2.1 Feedback

**Full feedback:** the feedback \( Z_t \) received at time \( t \) is the entire seller/buyer pair \( (S_t, B_t) \); in this setting, the seller and the buyer reveal their valuations at the end of a trade.

**Realistic feedback:** the feedback \( Z_t \) received at time \( t \) is just the pair \( (\mathbb{I}\{S_t \leq P_t\}, \mathbb{I}\{P_t \leq B_t\}) \); in this setting, the seller and the buyer only reveal whether or not they accept the trade at price \( P_t \).

2.2 Environment

**Stochastic (iid):** \( (S_1, B_1), (S_2, B_2), \ldots \) is an i.i.d. sequence of seller/buyer pairs, while \( S_t \) and \( B_t \) could be (arbitrarily) correlated. We will also investigate the (iid) setting under the following further assumptions.

**Independent valuations (iv):** \( S_t \) and \( B_t \) are independent of each other.

**Bounded density (bd):** \( (S_1, B_1) \) admits a joint density bounded by some constant \( M \).

**Adversarial (adv):** \( (S_t, B_t)_{t \in \mathbb{N}} \) is an arbitrary deterministic sequence \( (s_t, b_t)_{t \in \mathbb{N}} \subset [0,1]^2 \).
3 The Decomposition Lemma

In this section, we present a key lemma whose purpose is to decompose the gain from trade into terms that depend only on the outcome of yes/no questions. This result allows leveraging DKW inequalities in the proofs of our upper bounds. Moreover, it shows how to use the limited feedback available to reconstruct the expected gain from trade in the realistic feedback settings. Furthermore, it leads to an easy proof of the existence of the maximum of the expected gain from trade, under no assumptions on the seller and buyer distributions. We defer the proofs of the results of this section to Appendix A.

Lemma 1 (Decomposition lemma). Fix any price \( p \in [0,1] \). Then, for any \( s, b \in [0,1] \),

\[
\text{GFT}(p, s, b) = \int_{[p,1]} \mathbb{I}[s \leq p \leq \lambda] \, d\lambda + \int_{[0,p]} \mathbb{I}[s \leq \lambda \leq b] \, d\lambda .
\]  

Furthermore, let \( S \) and \( B \) be two \([0,1]-valued\) random variables:

- Then
  \[
  \mathbb{E}[\text{GFT}(p, S, B)] = \int_{[p,1]} \mathbb{P}[s \leq p \leq \lambda] \, d\lambda + \int_{[0,p]} \mathbb{P}[s \leq \lambda \leq b] \, d\lambda .
  \]  

- If \( U \) is uniform on \([0,1]\) and independent of \((S, B)\), then
  \[
  \mathbb{E}[\text{GFT}(p, S, B)] = \mathbb{P}[S \leq p \leq U \leq B] + \mathbb{P}[S \leq p \leq B] .
  \]  

- If \( U \) is uniform on \([0,1]\) and \( S, B, U \) are independent, then
  \[
  \mathbb{E}[\text{GFT}(p, S, B)] = \mathbb{P}[S \leq p] \mathbb{P}[p \leq U \leq B] + \mathbb{P}[p \leq B] \mathbb{P}[S \leq U \leq p] .
  \]  

- If \( U \) is uniform on \([p,1]\), \( V \) is uniform on \([0,p]\) and \((U, V)\) is independent of \((S, B)\), then
  \[
  \mathbb{E}[\text{GFT}(p, S, B)] = \mathbb{E}[(1-p)\mathbb{I}[S \leq p \leq U \leq B]] + \mathbb{E}[p\mathbb{I}[S \leq V \leq p \leq B]] .
  \]

We now present a corollary of our Decomposition lemma relating the regularity of the distributions (specifically, the boundedness of the densities) to the regularity of the expected gain from trade (i.e., its Lipschitzness).

If \( S \) and \( B \) are \([0,1]-valued\) random variables such that \((S, B)\) admits joint density \( f \) bounded above by some constant \( M \), then \( \mathbb{E}[\text{GFT}(\cdot, S, B)] \) is \( 4M \)-Lipschitz.

4 Full-Feedback Stochastic (iid) Setting

We begin by considering the full-feedback model (corresponding to direct revelation mechanisms) in a stochastic environment, where the seller/buyer pairs \((S_1, B_1), (S_2, B_2), \ldots\) are \([0,1]^2\)-valued i.i.d. random vectors, without any further assumptions on their common distribution (in particular, \( S_1 \) and \( B_1 \) could be arbitrarily correlated). Here, at the end of each round, sellers and buyers declare their actual valuations to the learner. The incentive-compatibility is guaranteed by the fact that the posted prices do not depend on the declared valuations at each specific round, but only on past ones, so that there is no point in misreporting.

In Section 4.1, we show that a Follow the Leader approach, which we call Follow the Best Price (FBP, Algorithm 2), achieves a \( O(\sqrt{T}) \) upper bound. In Section 4.2, we provide a matching \( \Omega(\sqrt{T}) \) lower bound rate.
Algorithm 2: Follow the Best Price (FBP)

\textbf{initialization:} let $P_1 \leftarrow 1/2$;
\begin{algorithmic}
  FOR $t = 1, 2, \ldots$
    \STATE receive feedback $(S_t, B_t)$;
    \STATE pick $P_{t+1} \in \arg\max_{p \in [0,1]} \frac{1}{t} \sum_{i=1}^t \text{GFT}(p, S_t, B_t)$;
  END
\end{algorithmic}

4.1 Follow the Best Price (FBP)

We begin by presenting our Follow the Best Price (FBP) algorithm. It consists in posting the best price with respect to the samples that have been observed so far. Notably, it does not need preliminary knowledge of the time horizon $T$. For each time $t$, given $(S_1, B_1), \ldots, (S_t, B_t)$, one can reconstruct the gain from trade function $\text{GFT}(c, S_t, B_t)$ at each time step $i \leq t$ and compute (one of) the best price(s) $P_{t+1} \in \arg\max_{p \in [0,1]} \frac{1}{t} \sum_{i=1}^t \text{GFT}(p, S_t, B_t)$. Note that $\frac{1}{t} \sum_{i=1}^t \text{GFT}(c, S_t, B_t)$ is a step-wise constant function that attains its maximum at one of the observed sellers’ valuations $S_1, \ldots, S_t$. Hence, even a naive enumeration approach is computationally efficient. On a technical note, prices $P_{t+1}$ should be defined in a measurable way in order for the regret to be well-defined. For example, this can be done by picking $P_{t+1}$ as the $S_i$ with the smallest index among all the $S_i \in \arg\max_{p \in [0,1]} \frac{1}{t} \sum_{i=1}^t \text{GFT}(p, S_i, B_i)$. (For other ideas on how to break ties in a measurable way, see [Cesari and Colombini, 2021, Section 2.4].)

The main idea of the analysis of Algorithm 2 is to show that the approximation of the expected gain from trade with its empirical means is uniform over all possible seller/buyer distributions and prices. A possible way to achieve this result could be through a pseudo-dimension argument (e.g., see [Li et al., 2001, Introduction and Theorem 5]). However, this approach requires subtle measurability considerations. In contrast, we will show that one could get around these measurability issues altogether by leveraging our Decomposition lemma (Lemma 1) and a bivariate DKW inequality (Theorem 15). Our approach also yields constants that —while still fairly high (see discussion after the proof)— are significantly better than those guaranteed by pseudo-dimension results. Finally, this presentation will be helpful to get the reader acquainted with the techniques that appear in the following sections for the realistic setting.

**Theorem 1.** In the full-feedback stochastic (iid) setting, the regret of Follow the Best Price satisfies, for all horizons $T$,

\[ R_T(\text{FBP}) \leq \frac{1}{2} + c\sqrt{T-1}. \]

where $c \in (0, 1.144240)$ is a universal constant.

**Proof.** Without loss of generality, assume that $T \geq 2$. Fix any $t \in [T - 1]$. For any $p \in [0,1]$ define the random variable

\[ H_t(p) := \frac{1}{t} \sum_{i=1}^t \text{GFT}_i(p) - \mathbb{E}[\text{GFT}_1(p)], \]

where we recall that $\text{GFT}_i(p) := \text{GFT}(p, S_i, B_i)$. By definition of $P_{t+1}$ and the independence of $P_{t+1}$ and $(S_{t+1}, B_{t+1})$, we have that

\[ \mathbb{E} [\text{GFT}_{t+1}(p^*)] - \mathbb{E} [\text{GFT}_{t+1}(P_{t+1})] \leq \mathbb{E} \left[ \frac{1}{t} \sum_{i=1}^t \text{GFT}_i(P_{t+1}) \right] - \mathbb{E} \left[ \text{GFT}_{t+1}(P_{t+1}) \right] \]

\[ = \mathbb{E} \left[ \frac{1}{t} \sum_{i=1}^t \text{GFT}_i(P_{t+1}) - \mathbb{E} \left[ \text{GFT}_{t+1}(P_{t+1}) | P_{t+1} \right] P_{t+1} \right] = \mathbb{E} [H_t(P_{t+1})] = (*) . \]

\footnote{By the symmetry of the problem, the maximum is also attained at one of the buyers’ valuations.}
Then, by the Decomposition lemma (1)-(2), we get

\[ H_r(P_{r+1}) \leq \sup_{p \in [0,1]} \left( \frac{1}{r} \sum_{i=1}^{r} \text{GFT}_i(p) - \mathbb{E}[\text{GFT}_1(p)] \right) \]

\[ = \sup_{p \in [0,1]} \left( \frac{1}{r} \sum_{i=1}^{r} \left( \int_{\{p,1\}} \mathbb{I}[S_i \leq p \leq \lambda \leq B_i] \, d\lambda + \int_{[0,p]} \mathbb{I}[S_i \leq \lambda \leq p \leq B_i] \, d\lambda \right) \right. \]

\[ - \left. \left( \int_{\{p,1\}} \mathbb{P}[S_i \leq p \leq \lambda \leq B_i] \, d\lambda + \int_{[0,p]} \mathbb{P}[S_i \leq \lambda \leq p \leq B_i] \, d\lambda \right) \right) \]

\[ = \sup_{p \in [0,1]} \left( \int_{[0,p]} \left( \frac{1}{r} \sum_{i=1}^{r} \mathbb{I}[S_i \leq \lambda, -B_i \leq -p] - \mathbb{P}[S_i \leq \lambda, -B_i \leq -p] \right) \, d\lambda \right) \]

\[ + \int_{[p,1]} \left( \frac{1}{r} \sum_{i=1}^{r} \mathbb{I}[S_i \leq p, -B_i \leq -\lambda] - \mathbb{P}[S_i \leq p, -B_i \leq -\lambda] \right) \, d\lambda \]

\[ \leq 2 \sup_{x,y \in \mathbb{R}} \left| \frac{1}{r} \sum_{i=1}^{r} \mathbb{I}[S_i \leq x, -B_i \leq y] - \mathbb{P}[S_i \leq x, -B_i \leq y] \right|. \]

Letting \( m_0, c_1, c_2 \) as in Theorem 15, \( \epsilon_t := \sqrt{m_0/t} \), taking expectations to the left and right hand side of Eq. (7), and applying the bivariate DKW inequality (Theorem 15), we get

\[ (\ast) \leq \mathbb{E} \left[ 2 \sup_{x,y \in \mathbb{R}} \left| \frac{1}{r} \sum_{i=1}^{r} \mathbb{I}[S_i \leq x, -B_i \leq y] - \mathbb{P}[S_i \leq x, -B_i \leq y] \right| \right] \]

\[ \leq 2 \epsilon_t + 2 \int_{[\epsilon_t,1]} \mathbb{P} \left[ \sup_{x,y \in \mathbb{R}} \left| \frac{1}{r} \sum_{i=1}^{r} \mathbb{I}[S_i \leq x, -B_i \leq y] - \mathbb{P}[S_i \leq x, -B_i \leq y] \right| > \epsilon \right] \, d\epsilon \]

\[ \leq 2 \epsilon_t + 2 \int_{\epsilon_t}^{1} c_1 \exp(-c_2 t^2) \, d\epsilon \leq 2 \epsilon_t + \frac{c_1}{\sqrt{c_2}} \int_{0}^{\infty} e^{-u} u^{-1/2} \, du = \left( 2 \sqrt{m_0} + c_1 \sqrt{\frac{\pi}{c_2}} \right) \frac{1}{\sqrt{t}}. \]

Being \( t \) arbitrary, using the fact that \( \sum_{i=1}^{T-1} t^{-1/2} \leq 2 \sqrt{T-1} \), and letting \( c := 2 \left( 2 \sqrt{m_0} + c_1 \sqrt{\frac{\pi}{c_2}} \right) < 1144265 \), we have that

\[ R_r(\text{FBP}) \leq \frac{1}{2} + \sum_{i=1}^{T-1} \left( \mathbb{E}[\text{GFT}_{r+1}(p^*)] - \mathbb{E}[\text{GFT}_{r+1}(P_{r+1})] \right) \leq \frac{1}{2} + \frac{c}{2} \sum_{r=1}^{T-1} \frac{1}{\sqrt{t}} = \frac{1}{2} + c \sqrt{T-1}, \]

which concludes the proof.

The loose bound on the constant \( c \) appearing in the statement is due to the (likely suboptimal) large constants appearing in Theorem 15; any improvement on the bivariate DKW inequality would result in an improvement of this constant. For example, it is conjectured [Naaman, 2021, Section 5] that the tightest bound for the bivariate DKW inequality is (with the same notation as Theorem 15), for all \( m \in \mathbb{N} \) and \( \epsilon > 0 \),

\[ \mathbb{P}\left[ \sup_{x,y \in \mathbb{R}} \left| \frac{1}{r} \sum_{k=1}^{r} \mathbb{I}[X_k \leq x, Y_k \leq y] - \mathbb{P}[X_1 \leq x, Y_1 \leq y] \right| > \epsilon \right] \leq 4 \exp(-2m \epsilon^2). \]

If this was the case, we could replace Eq. (8) with

\[ (\ast) \leq 2 \int_{[0,1]} \mathbb{P} \left[ \sup_{x,y \in \mathbb{R}} \left| \frac{1}{r} \sum_{i=1}^{r} \mathbb{I}[S_i \leq x, -B_i \leq y] - \mathbb{P}[S_i \leq x, -B_i \leq y] \right| > \epsilon \right] \, d\epsilon \leq 2 \sqrt{2 \pi} \frac{1}{\sqrt{t}}. \]

leading to a significantly smaller constant \( c := 2 \cdot 2 \sqrt{2 \pi} < 11 \).
For the seller, for any regret. Thus, if \( \varepsilon \) is bounded-away from zero, the only way to avoid suffering linear regret is to identify the sign of \( \varepsilon \) and play accordingly.

\[ \Omega(\varepsilon) \]

### 4.2 \( \sqrt{T} \) Lower Bound (iv+bd)

In this section, we show that the upper bound on the minimax regret we proved in Section 4.1 is tight. No strategy can beat the \( O(\sqrt{T}) \) rate when the seller/buyer pair \((S_i, B_i)\) is drawn i.i.d. from an unknown fixed distribution, even under the further assumptions that the valuations of the seller and buyer are independent of each other and have bounded densities. For a full proof of the following theorem, see Appendix E.

**Theorem 2.** In the full-feedback model, for all horizons \( T \), the minimax regret \( R^*_T \) satisfies

\[
R^*_T := \inf_{\alpha} \sup_{(S_i, B_i) \sim D} R_T(\alpha) = \Omega(\sqrt{T}),
\]

where \( c \geq 1/(8\sqrt{2\pi}) \), the infimum is over all of the learner’s strategies \( \alpha \), and the supremum is over all distributions \( D \) of the seller/buyer pair such that:

- (iid) \((S_1, B_1), (S_2, B_2), \ldots \sim D\) is an i.i.d. sequence.
- (iv) \( S_1 \) and \( B_1 \) are independent of each other.
- (bd) \((S_1, B_1)\) admits a joint density bounded by \( M \geq 4 \).

**Proof sketch.** We build a family of distributions \( D_{\varepsilon} \) for the seller/buyer pair parameterized by \( \varepsilon \in [0, 1] \). For the seller, for any \( \varepsilon \in [0, 1] \), we define the density

\[
f_{S_{\varepsilon}} := 2(1 + \varepsilon)I_{[0, \frac{1}{4}]} + 2(1 - \varepsilon)I_{[\frac{3}{4}, 1]}.
\]

(Fig. 1(a), in red/blue)

For the buyer, we define a single density (independently of \( \varepsilon \))

\[
f_B := 2\mathbb{I}_{[\frac{1}{4}, \frac{3}{4}]} I_{[\frac{1}{4}, 1]}.
\]

(Fig. 1(a), in green)

In the \( +\varepsilon \) (resp., \( -\varepsilon \)) case, the optimal price belongs to the region \([0, 1/2]\) (resp., \([1/2, 1]\)) in the \( +\varepsilon \) (resp., \( -\varepsilon \)) case, the learner incurs a \( \Omega(\varepsilon) \) regret. Thus, if \( \varepsilon \) is bounded-away from zero, the only way to avoid suffering linear regret is to identify the sign of \( \pm \varepsilon \) and play accordingly.

This closely resembles the construction for the lower bound of online learning with expert advice. In fact, a technical proof (see Appendix E), shows that our setting is harder (i.e., it has a higher minimax regret) than an instance of an expert problem (with two experts), which has a known lower bound on its minimax regret of \( \frac{1}{8\sqrt{2\pi}} \sqrt{T} \) [Cover, 1965]. \( \square \)
5 Realistic-Feedback Stochastic (iid) Setting

In this section, we tackle the problem in the more challenging realistic-feedback model, again under the assumption that the seller/buyer pairs \((S_1, B_1), (S_2, B_2), \ldots\) are \([0,1]^2\)-valued i.i.d. random variables, all with the same law as some \((S, B)\). We will first study the case in which \(S\) and \(B\) are independent (iv) and have bounded densities (bd), then discuss what happens if either one of the two assumptions is lifted.

We recall that in the realistic-feedback model, the only information collected by the learner at the end of each round \(t\) consists of \(I\{S_t \leq P_t\}\) and \(I\{P_t \leq B_t\}\).

### 5.1 Scouting Bandits: from Realistic Feedback to Multi-Armed Bandits

The main challenge in designing low-regret algorithms with realistic feedback lies in the fact that posting the same price \(\frac{P}{u}\) (in the price \(\tilde{p}\) now only left to solve a bandit problem on the same law as some and used to simultaneously estimate the integral terms on a suitable grid of \(bd\) implies the Lipschitzness of the expected gain from trade (Section 5.1)), which in turns allows to discretize

\[
\begin{align*}
\mathbb{E}[\text{GFT}(q_k, S, B)] & \approx \mathbb{P}[S \leq q_k] \tilde{F}_k + \mathbb{P}[q_k \leq B] \tilde{G}_k = \mathbb{E}[\mathbb{I}\{S \leq q_k\} \tilde{F}_k + \mathbb{I}\{q_k \leq B\} \tilde{G}_k | H] =: \mathbb{E}[Z(k) | H],
\end{align*}
\]

where \(H\) consists of the estimates \(\tilde{F}_j, \tilde{G}_j\) (for all \(j\)) at the the end of the global exploration phase. We are now only left to solve a bandit problem on \(K\) arms with reward function \(Z\): the only quantities to learn are the two local terms \(\mathbb{P}[S \leq q_k]\) and \(\mathbb{P}[q_k \leq B]\), which can be estimated with the available feedback by posting the price \(q_k\).

**Algorithm 3:** Scouting Bandits

input: exploration time \(T_0\), grid size \(K\), and \(K\)-armed bandit algorithm \(\alpha\);

 initialization: \(q_k \leftarrow k/(K+1), \tilde{F}_k \leftarrow 0, \tilde{G}_k \leftarrow 0\), for all \(k \in [K]\);

for \(t = 1, 2, \ldots, T_0\) do // scouting phase

- draw \(U_t\) from \([0,1]\) uniformly at random;
- post price \(U_t\) and observe feedback \(\mathbb{I}\{S_t \leq U_t\}, \mathbb{I}\{U_t \leq B_t\}\);
- let \(\tilde{F}_k \leftarrow \tilde{F}_k + \frac{1}{T_0} \mathbb{I}\{q_k \leq U_t \leq B_t\}\), and \(\tilde{G}_k \leftarrow \tilde{G}_k + \frac{1}{T_0} \mathbb{I}\{S_t \leq U_t \leq q_k\}\), for all \(k \in [K]\);

end

for \(t = T_0 + 1, T_0 + 2, \ldots\) do // bandit phase

- generate the next arm \(I_t\) with \(\alpha\);
- post price \(q_{I_t}\) and observe \(\mathbb{I}\{S_{I_t} \leq q_{I_t}\}, \mathbb{I}\{q_{I_t} \leq B_{I_t}\}\);
- feed \(\alpha\) the reward \(Z(I_t) \leftarrow \mathbb{I}\{S_{I_t} \leq q_{I_t}\} \tilde{F}_{I_t} + \mathbb{I}\{q_{I_t} \leq B_{I_t}\} \tilde{G}_{I_t}\);

end

The independence of \(S\) and \(B\) (iv) is required for applying Eq. (4), while the bounded density assumption (bd) implies the Lipschitzness of the expected gain from trade (Section 3), which in turns allows to discretize
We bound the four terms separately.

Similarly to previous sections, denote for all times $t$ and prices $p$, $\text{GFT}_t(p) := \text{GFT}(p, S_t, B_t)$. Then

$$R_t(\text{SB}) \leq T_0 + \sum_{t=T_0+1}^T \mathbb{E}[\text{GFT}_t(p^*) - \text{GFT}_t(P_t)]$$

$$= T_0 + \sum_{t=T_0+1}^T \left( \mathbb{E}[\text{GFT}_t(p^*)] - \mathbb{E}[\text{GFT}_t(q_{k^*})] \right) + \sum_{t=T_0+1}^T \left( \mathbb{E}[\text{GFT}_t(q_{k^*})] - \mathbb{E}[Z_{H,t}(k^*)] \right)$$

$$+ \mathbb{E}\left[ \sum_{t=T_0+1}^T Z_{H,t}(k^*) - \sum_{t=T_0+1}^T Z_{H,t}(I_{H,t}) \right] + \sum_{t=T_0+1}^T \left( \mathbb{E}[Z_{H,t}(I_{H,t})] - \mathbb{E}[\text{GFT}_t(P_t)] \right)$$

$$=: T_0 + (I) + (II) + (III) + (IV). \tag{9}$$

We bound the four terms separately.

For the term (I), by the $4M$-Lipschitzness of the gain from trade (Section 3) and the fact that the step size of the grid is $1/(K+1)$, we get

$$(I) = \sum_{t=T_0+1}^T \left( \mathbb{E}[\text{GFT}_t(p^*)] - \mathbb{E}[\text{GFT}_t(q_{k^*})] \right) \leq 4M|p^* - q_{k^*}|(T - T_0) \leq \frac{4M}{K+1}(T - T_0).$$

For the term (II), for any $t \geq T_0 + 1$, by the independence of $H$ and $(S_t, B_t)$, we have

$$\mathbb{E}[Z_{H,t}(k^*)] = \mathbb{E}[\{S_t \leq q_{k^*}\} \tilde{F}_{k^*} + \mathbb{I}(q_{k^*} \leq B_t) \tilde{G}_{k^*}]$$

$$= \mathbb{P}[S_t \leq q_{k^*}] \mathbb{P}[q_{k^*} \leq U_t \leq B_t] + \mathbb{P}[q_{k^*} \leq B_t] \mathbb{P}[S_t \leq U_t \leq q_{k^*}] = \mathbb{E}[\text{GFT}_t(q_{k^*})],$$

where the last identity follows from Eq. (4), and in turn implies that (II) = 0.
For the term (III), using the fact that for \( \mathbb{P}_H \)-almost every \( h \in \mathcal{H} \), the sequence \( (Z_{h,t})_{t \geq T_0 + 1} \) is included in \([0,1]\), we obtain

\[
(III) = \mathbb{E} \left[ \mathbb{E} \left[ \sum_{t=T_0+1}^{T} Z_{h,t}(k^*) - \sum_{t=T_0+1}^{T} Z_{h,t}(I_{h,t}) \bigg| H \right] \right] \\
\leq \left( \int_{\mathcal{H}} \mathbb{E} \left[ \sum_{t=T_0+1}^{T} Z_{h,t}(k^*) - \sum_{t=T_0+1}^{T} Z_{h,t}(I_{h,t}) \right] \mathbb{d}\mathbb{P}_H(h) \leq R_{T-T_0}(\alpha) \right)\]

where \((*)\) follows from the independence of \((I_{h,t}, S_t, B_t)\) and \(H\) (for any \( h \in \mathcal{H} \) and all \( t \geq T_0 + 1 \)) and in the last inequality we upper bounded (for \( \mathbb{P}_H \)-almost every \( h \in \mathcal{H} \)) the regret of \( \alpha \) when run on the sequence of rewards \((Z_{h,t})_{t \geq T_0 + 1}\) with \( R_{T-T_0}(\alpha) \).

Finally, we upper bound the last term (IV). If the \( K \)-armed bandit algorithm \( \alpha \) is randomized, let \( V_t \) be its internal randomization of at each time step \( t \geq T_0 + 1 \); otherwise, omit all references to \((V_t)_{t \geq T_0 + 1}\). Define, for each time step \( t \geq T_0 + 1 \), \( L_t := (H, V_{t_0+1}, S_{t_0+1}, B_{t_0+1}, \ldots, V_{t-1}, S_{t-1}, B_{t-1}, V_t), \) \( \mathbb{P}_T := \mathbb{P}[\cdot | L_t] \), and take a uniform random variable \( U_t \) on \([0,1]\) independent of \((L_t, B_t, S_t)\). Now, for all \( t \geq T_0 + 1 \), leveraging the measurability of \( q_{h,t}, \hat{F}_{h,t}, \hat{G}_{h,t} \) with respect to \( (L_t) \), the independence of \( L_t \) and \((S_t, B_t)\), and the Decomposition lemma (4), we get

\[
\mathbb{E}[Z_{h,t}(I_{h,t})] - \mathbb{E}[\text{GFT}_t(P_t)] = \mathbb{E}[\mathbb{E}[\{I[S_t \leq q_{h,t}]\hat{F}_{h,t} + I[q_{h,t} \leq B_t] \hat{G}_{h,t} - \text{GFT}(q_{h,t}, S_t, B_t) | L_t] | L_t] \\
= \mathbb{E}[\mathbb{P}[S_t \leq q_{h,t}]\{\hat{F}_{h,t} - \mathbb{P}[q_t \leq U_t \leq B_t]\} + \mathbb{P}[q_{h,t} \leq B_t] \{\hat{G}_{h,t} + \mathbb{P}[S_t \leq U_t \leq q_{h,t}]\}] \\
\leq \mathbb{E}[\max_{k \in [K]} |\hat{F}_k - \mathbb{P}[q_k \leq U_t \leq B_t]|] + \mathbb{E}[\max_{k \in [K]} |\hat{G}_k - \mathbb{P}[S_t \leq U_t \leq q_{k}]|] =: (V) + (VI) .
\]

For the first addend, applying the univariate DKW inequality (Theorem 14), we have

\[
(V) = \int_{[0,1]} \mathbb{P}[\max_{k \in [K]} |\hat{F}_k - \mathbb{P}[q_k \leq U_t \leq B_t]| > \varepsilon] \, \mathbb{d}\varepsilon \\
= \int_{[0,1]} \mathbb{P}[\max_{k \in [K]} \left| \frac{1}{T_0} \sum_{i=1}^{T_0} \mathbb{I}\{-U_i \leq B_t \} - \mathbb{P}[q_k \leq U_t \leq B_t]\right| > \varepsilon] \, \mathbb{d}\varepsilon \\
\leq \int_{[0,1]} \mathbb{P}[\sup_{x \in \mathbb{R}} \left| \frac{1}{T_0} \sum_{i=1}^{T_0} \mathbb{I}\{-U_i \leq B_t \} - \mathbb{P}[q_k \leq U_t \leq B_t]\right| > \varepsilon] \, \mathbb{d}\varepsilon \\
\leq \int_{0}^{1} 2 \exp(-2T_0 \varepsilon^2) \, \mathbb{d}\varepsilon \leq \frac{1}{\sqrt{2T_0}} \int_{0}^{\infty} e^{-u} u^{-1/2} \, \mathbb{d}u = \sqrt{\frac{\pi}{2 \sqrt{T_0}}} .
\]

Similarly, one can show that \((VI) \leq \sqrt{\frac{\pi}{2 \sqrt{T_0}}} \) which in turn yields (IV) \( \leq \sqrt{\frac{\pi}{T_0}} (T - T_0) \).

Putting the bounds on (I)-(IV) together in (9) gives the first part of the result. Substituting the stated choice of the parameters yields the second.

Note that to achieve a regret of order \( O(MT^{2/3}) \) we tuned the parameters \( T_0 \) and \( K \) of Scouting Bandits as a function of \( T \). If the time horizon is unknown, we can obtain the same order of regret with a standard doubling trick [Cesa-Bianchi and Lugosi, 2006]. Also, note that if we allow tuning the parameters as a function of the Lipschitz constant \( M \) (which is however unknown in general), the regret rate would improve to order \( O(M^{1/3}T^{2/3}) \). This can be achieved by taking \( T_0 := \lceil T^{2/3} \rceil \) and \( K := \lceil M^{2/3} T^{1/3} \rceil \).

### 5.2 \( T^{2/3} \) Lower Bound Under Realistic Feedback (iv+bd)

In this section, we show that the upper bound on the minimax regret we proved in Section 5.1 is tight. No strategy can beat the \( O(T^{2/3}) \) rate when the seller/buyer pair \((S_t, B_t)\) is drawn i.i.d. from an unknown fixed
The seller, for any game, observes feedback according to the sign of $\pm\epsilon$. Clearly, the feedback received from the buyer gives no information on its minimax regret of $\Omega(\epsilon)$, but incur $\Omega(\epsilon)$ regret if it is the blue one ($-\epsilon$); the converse happens in $a_3$.

**Theorem 4.** In the realistic-feedback model, for all horizons $T$, the minimax regret $R_T^*$ satisfies

$$R_T^* := \inf_{\alpha} \sup_{(S_t,B_t) \sim \mathcal{D}} R_T(\alpha) \geq c T^{2/3},$$

where $c \geq 11/672$, the infimum is over all learner’s strategies $\alpha$, and the supremum is over all distributions $\mathcal{D}$ of the seller/buyer pair that satisfy:

1. $(S_t, B_t), (S_2, B_2), \ldots \sim \mathcal{D}$ is an i.i.d. sequence.
2. $S_1$ and $B_1$ are independent of each other.
3. $(S_1, B_1)$ admits a joint density bounded by $M \geq 24$.

**Proof sketch.** We build a family of distributions $\mathcal{D}_{a\pm\epsilon}$ of the seller/buyer pair parameterized by $\epsilon \in [0, 1]$. For the seller, for any $\epsilon \in [0, 1]$, we define the density

$$f_{S,\pm\epsilon} := \frac{1}{4\delta} \left( (1 \pm \epsilon) I_{[0,\delta]} + (1 \mp \epsilon) I_{[\frac{1}{2}, \frac{1}{2}+\delta]} + I_{[\frac{1}{2}, \frac{1}{2}+\delta]} + I_{[\frac{1}{2}, \frac{1}{2}+\delta]} \right),$$

where $\delta := 1/4$ is a normalization constant. For the buyer, we define a single density (independently of $\epsilon$)

$$f_B := \frac{1}{4\delta} \left( I_{[\frac{1}{2}, \frac{1}{2}+\delta]} + I_{[\frac{1}{2}, \frac{1}{2}+\delta]} + I_{[\frac{1}{2}, \frac{1}{2}+\delta]} + I_{[\frac{1}{2}, \frac{1}{2}+\delta]} \right).$$

In the $+\epsilon$ (resp., $-\epsilon$) case, the optimal price belongs to a region $a_2$ (resp., $a_3$, see Fig. 2(b)). By posting prices in the wrong region $a_4$ (resp., $a_3$) in the $+\epsilon$ (resp., $-\epsilon$) case, the learner incurs $\Omega(\epsilon)$ regret. Thus, if $\epsilon$ is bounded away from zero, the only way to avoid suffering linear regret is to identify the sign of $\pm\epsilon$ and play accordingly. Clearly, the feedback received from the buyer gives no information on $\pm\epsilon$. Since the feedback received from the seller at time $t$ by posting a price $p$ is $I(S_t \leq p)$, one can obtain information about (the sign of) $\pm\epsilon$ only by posting prices in the costly ($\Omega(1)$-regret) sub-optimal region $a_1$.

This closely resembles the learning dilemma present in the so-called revealing action partial monitoring game [Cesa-Bianchi and Lugosi, 2006]. In fact, a technical proof (see Appendix F), shows that our setting is harder (i.e., it has a higher minimax regret) than an instance of a revealing action problem, which has a known lower bound on its minimax regret of $\Omega(T^{2/3})$ [Cesa-Bianchi et al., 2006]. □
Therefore, given that the optimal price in the (bd) case, the optimal price belongs to the region $[0, 1/2]$ (resp., $(1/2, 1]$, see Fig. 3(b)). By posting prices in the wrong region $(1/2, 1)$ (resp., $[0, 1/2]$) in the $f$ (resp., $g$) case, the learner incurs at least $1/3 - 1/4 = 1/12$ regret. Thus, the only way to avoid suffering linear regret is to determine if the valuations of the seller and buyer are generated by $f$ or $g$. For each price $p \in [0, 1]$, consider the four rectangles with opposite vertices $(p, p)$ and $(u_i, u_i)$, where $(u_i, u_i)_{i=1,\ldots,4}$ are the four vertices of the unit square. Note that the only information on the distribution of $(S, B)$ that the learner can gather from the realistic feedback $(I[S_i \leq p], I[p \leq B_i])$ received after posting a price $p$ is (an estimate of) the area of the portion of the distribution of the support of the distribution included in each of these four rectangles. However, these areas coincide in the cases $f$ and $g$. Hence, under realistic feedback, $f$ and $g$ are completely indistinguishable. Therefore, given that the optimal price in the $f$ (resp., $g$) case is $3/8$ (resp., $5/8$), the best that the learner can do is to sample prices uniformly at random in the set $\{3/8, 5/8\}$, incurring a regret of $1/24$. For a formalization of this argument leveraging the techniques we described in the introduction, see Appendix G.
Figure 4: All prices but $x$ have high regret. However, under realistic feedback, finding $x$ in finite time is impossible.

5.4 Linear Lower Bound Under Realistic Feedback (iv)

In this section, we prove that in the realistic-feedback case, no strategy can achieve sublinear regret without any limitations on how concentrated the distributions of the valuations of the seller and buyer are, not even if they are independent of each other (iv).

At a high level, if the two distributions of the seller and the buyer are very concentrated in a small region, finding an optimal price is like finding a needle in a haystack. For a full proof of the following theorem, see Appendix H.

**Theorem 6.** In the realistic-feedback model, for all horizons $T$, the minimax regret $R^*_T$ satisfies

$$R^*_T := \inf_{\alpha} \sup_{(S_1, B_1) \sim \mathcal{D}} R_T(\alpha) \geq c T,$$

where $c \geq 1/8$, the infimum is over all learner’s strategies $\alpha$, and the supremum is over all distributions $\mathcal{D}$ of the seller/buyer pair such that:

(iid) $(S_1, B_1), (S_2, B_2), \ldots \sim \mathcal{D}$ is an i.i.d. sequence.

(iv) $S_1$ and $B_1$ are independent of each other.

Proof sketch. Consider a family of seller/buyer distributions $(S^*, B^*)$, parameterized by $x \in I$, where $I$ is a small interval centered in $1/2$, $S^*$ and $B^*$ are independent of each other, and they satisfy

$$S^* = \begin{cases} x & \text{with probability } \frac{1}{2}, \\ 0 & \text{with probability } \frac{1}{2}, \end{cases} \quad B^* = \begin{cases} x & \text{with probability } \frac{1}{2}, \\ 1 & \text{with probability } \frac{1}{2}. \end{cases}$$

The distributions and the corresponding gain from trade are represented in Fig. 4(a) and Fig. 4(b), respectively. A direct verification shows that the best fixed price with respect to $(S^*, B^*)$ is $p = x$. Furthermore, by posting any other prices, the learner incurs a regret of approximately $1/2$ with probability $1/4$. It is intuitively clear that no strategy can locate (exactly!) each possible $x \in I$ in a finite number of steps. This results, for any strategy, in regret of at least (approximately) $T/8$. See Appendix H for a more detailed analysis.

6 Adversarial Setting: Linear Lower Bound Under Full Feedback

In this section, we prove that even in the simpler full-feedback case, no strategy can achieve worst-case sublinear regret in an adversarial setting. Lower bounds for the adversarial setting have a slightly different
structure that the stochastic ones. The idea of the proof is to build, for any strategy, a hard sequence of sellers and buyers’ valuations \((s_1, b_1), (s_2, b_2), \ldots\) which causes the algorithm to suffer linear regret for any horizon \(T\).

**Theorem 7.** In the full-feedback adversarial (adv) setting, for all horizons \(T\), the minimax regret \(R^*_T\) satisfies

\[
R^*_T := \inf_{\alpha} \sup_{(s_1, b_1), (s_2, b_2), \ldots} R_T(\alpha) \geq cT,
\]

where \(c \geq 1/4\), the infimum is over all of the learner’s strategies \(\alpha\), and the supremum is over all deterministic sequences \((s_1, b_1), (s_2, b_2), \ldots \in [0, 1]^2\) of the seller and buyer’s valuations.

**Proof.** We begin by fixing any strategy \(\alpha\) of the learner. This is a sequence of functions \(\alpha_t\), mapping the past feedback \((s_1, b_1), \ldots, (s_{t-1}, b_{t-1})\), together with some internal randomization, to the price \(P_t\) to be posted by the learner at time \(t\). In other words, the strategy maintains a distribution \(v_t\) over the prices that is updated after observing each new pair \((s_t, b_t)\) and used to draw each new price \(P_t\). We will show how to constructively determine a sequence of seller/buyer valuations that is hard for \(\alpha\) to learn. This sequence is oblivious to the prices \(P_1, P_2, \ldots\), posted by \(\alpha\), in the sense it does not have access to the realizations of its internal randomization. The idea is, at any time \(t\), to determine a seller/buyer pair \((s_t, b_t)\) either of the form \((c_t, 1)\) or \((0, d_t)\), with \(c_t \approx \frac{1}{2} \approx d_t\), such that the probability \(v_t\) that the strategy picks a price \(P_t \in [s_t, b_t]\) (i.e., that there is a trade) is at most \(\frac{1}{2}\) and, at the same time, there is common price \(p^*\) which belongs to \([s_t, b_t]\) for all times \(t\). This way, since \(b_t - s_t \approx \frac{1}{2}\) for all \(t\), the regret of \(\alpha\) with respect to \((s_1, b_1), (s_2, b_2), \ldots\) is at least (approximately) greater than or equal to \(T/4\).

The formal construction proceeds inductively as follows. Let \(\epsilon \in (0, \frac{1}{10\epsilon})\). Let

\[
\begin{cases}
  c_1 := \frac{1}{2} - \frac{1}{4}\epsilon, \quad d_1 := \frac{1}{2} + \frac{1}{4}\epsilon, \quad s_1 := 0, \quad b_1 := d_1, & \text{if } v_1\left[[0, \frac{1}{2} - \frac{1}{4}\epsilon]\right] \leq \frac{1}{2}, \\
  c_1 := \frac{1}{2} + \frac{1}{4}\epsilon, \quad d_1 := \frac{1}{2} - \frac{1}{4}\epsilon, \quad s_1 := c_1, \quad b_1 := 1, & \text{otherwise}.
\end{cases}
\]

Then, for any time \(t\), given that \(c_i, d_i, s_i, b_i\) are defined for all \(i \leq t\) and recalling that \(v_{t+1}\) is the distribution over the prices at time \(t+1\) (of the strategy \(\alpha\) after observing the feedback \((s_1, b_1), \ldots, (s_t, b_t)\)), let

\[
\begin{cases}
  c_{t+1} := c_t, \quad d_{t+1} := d_t - \frac{2\epsilon}{3}, \quad s_{t+1} := 0, \quad b_{t+1} := d_{t+1}, & \text{if } v_{t+1}\left[[0, c_t + \frac{2\epsilon}{3}]\right] \leq \frac{1}{2}, \\
  c_{t+1} := c_t + \frac{2\epsilon}{3}, \quad d_{t+1} := d_t, \quad s_{t+1} := c_{t+1}, \quad b_{t+1} := 1, & \text{otherwise}.
\end{cases}
\]

Then the sequence of seller/buyer valuations \((s_1, b_1), (s_2, b_2), \ldots\) defined above by induction satisfies:

- \(v_t\left[[s_t, b_t]\right] \leq \frac{1}{2}\), for each time \(t\).
- There exists \(p^* \in [0, 1]\) such that \(p^* \in [s_t, b_t]\), for each time \(t\) (e.g. \(p^* := \lim_{t \to \infty} c_t\)).
- \(b_t - s_t \geq \frac{1-3\epsilon}{2}\), for each time \(t\).

This implies, for any horizon \(T\),

\[
R_T(\alpha) = \sum_{t=1}^{T} \text{GFT}(p^*, s_t, b_t) - \sum_{t=1}^{T} \mathbb{E} [\text{GFT}(P_t, s_t, b_t)] \geq \sum_{t=1}^{T} (b_t - s_t) (1 - v_t[, s_t, b_t]) \geq \frac{1-3\epsilon}{4}T.
\]

Since \(\epsilon\) and \(\alpha\) are arbitrary, this yields immediately \(R^*_T \geq T/4\). \(\square\)

**7 Breaking Linear Lower Bounds: Weakly Budget Balanced Results**

In this section, we show how to break the linear lower bound of Section 5.3 without requiring the independence of the valuations of the seller and the buyer. To do so, we move from a budget balance to a weak budget
balance mechanism. In this setting, rather than posting a single price, the learner can post two (possibly distinct) prices \(0 \leq p \leq p' \leq 1\) to the seller, and \(p'\) to the buyer. This condition allows the platform to extract money from the trade but not to subsidize it. Naturally, this changes the benchmark: if the learner posts a pair \((p, p') \in [0, 1]^2\) and the valuations of the seller and the buyer are \((s, b) \in [0, 1]^2\), the net gain of the seller is \(p - s\) while that of the buyer is \(b - p'\). Thus, the gain from trade in this setting becomes

\[
\text{GFT}: [0, 1]^2 \times [0, 1]^2 \to [0, 1], \quad (p, p', s, b) \mapsto \begin{cases} b - p' + p - s & \text{if } s \leq p \leq p' \leq b \end{cases}.
\]

Note that posting the same price \(p = p'\) to both the seller and the buyer leads to the old definition of gain for trade \((b - s)\{s \leq p \leq b\}\), which we denoted by \(\text{GFT}(p, s, b)\) in previous sections. For this reason and to keep the notation lighter, we will denote \(\text{GFT}(p, p, s, b)\) simply by \(\text{GFT}(p, s, b)\) here.

We design a explore-then-exploit algorithm, that we call Scouting Blindits (Algorithm 4). In the exploration phase (scouting phase), the high-level idea is to leverage the Decomposition lemma (5) to build an accurate estimate of the gain from trade at each point \(q_k\) of a suitably fine grid. By the bounded density assumption, which implies the Lipschitzness of the expected gain from trade, this is sufficient to approximate \(E[\text{GFT}(\cdot, S_1, B_1)]\) uniformly, so that the price \(\hat{P}^*\) that maximizes the estimates \(\hat{F}_k + \hat{G}_k\) is an approximate maximizer of gain from trade. In the exploitation phase (blind phase) the algorithm posts \(\hat{P}^*\) blindly to both the seller and the buyer, ignoring all the feedback it receives. This implies that Scouting Blindits is actually budget balanced during the whole blind phase (which, after tuning, constitutes the majority of time).

**Algorithm 4: Scouting Blindits (SBl)**

**input:** exploration time \(T_0\), grid size \(K\);

**initialization:** \(q_i \leftarrow i/(K + 1), \hat{F}_i \leftarrow 0, \hat{G}_i \leftarrow 0\), for all \(i \in [K]\), \(k \leftarrow 1\);

**for** \(t = 1, \ldots, 2KT_0\) **do** // scouting phase

\[\text{if } t \text{ is odd then} \]
\[\text{draw } U_i \text{ from } [q_k, 1] \text{ uniformly at random;}
\[\text{post the prices } (q_k, U_i) \text{ and observe feedback } \mathbb{I}(S_i \leq q_k \leq U_i \leq B_i);
\[\text{let } \hat{F}_k \leftarrow \hat{F}_k + \frac{1}{T_0}(1 - q_k)\mathbb{I}(S_i \leq q_k \leq U_i \leq B_i);
\[\text{else}
\[\text{draw } V_i \text{ from } [0, q_k] \text{ uniformly at random;}
\[\text{post the prices } (V_i, q_k) \text{ and observe feedback } \mathbb{I}(S_i \leq V_i \leq q_k \leq B_i);
\[\text{let } \hat{G}_k \leftarrow \hat{G}_k + \frac{1}{T_0}q_k\mathbb{I}(S_i \leq V_i \leq q_k \leq B_i);
\[\text{end}
\[\text{if } t \geq 2KT_0 \text{ then}
\[\text{let } k \leftarrow k + 1;
\[\text{end}
\]
compute \(\hat{P}^* \in \text{argmax}_{k \in [K]}(\hat{F}_k + \hat{G}_k)\) and let \(\hat{P}^* \leftarrow q_{j^*};

**for** \(t = 2KT_0 + 1, \ldots\) **do** // blind phase

\[\text{post the price } \hat{P}^* \text{ to both the seller and the buyer;}
\]

We will now show that the regret of suitable tuning of Algorithm 4 is at most \(\tilde{O}(T^{3/4})\).

**Theorem 8.** If \((S_1, B_1), (S_2, B_2), \ldots\) is an i.i.d. sequence and \((S_1, B_1)\) has a density bounded by some constant \(M\), then the regret of Scouting Blindits run with parameters \(T_0\) and \(K\) satisfies, for any time horizon \(T \geq 2KT_0\),

\[
\mathbb{R}_T(\text{SBl}) \leq 2KT_0 + 2\left(\frac{2M}{K} + \inf_{\epsilon > 0}(\epsilon + K \exp(-2\epsilon^2 T_0))\right)(T - 2KT_0).
\]
In particular, tuning the parameters \(K := \lceil T^{1/4} \rceil\) and \(T_0 := \lceil (\sqrt{T \log T})/2 \rceil\) yields
\[
\mathbb{E}_T(SBI) = O((M + \log T) T^{3/4}).
\]

**Proof.** Let \(p^* \in \arg\max_{p \in [0,1]} \mathbb{E}[\text{GFT}(p, S_1, B_1)]\). Fix any \(\varepsilon > 0\) and define the good event \(E\) as
\[
E := \bigcap_{k=1}^{K}\left\{ |\mathbb{E}[\text{GFT}(q_k, S_1, B_1)] - (\bar{F}_k + \bar{G}_k)| \leq \varepsilon \right\}.
\]

By Eq. (5), for each \(k \in [K]\), we have that \(\bar{F}_k + \bar{G}_k\) is the empirical mean of \(T_0\) i.i.d. \([0,1]\)-valued copies of a random variable whose expected value is \(\mathbb{E}[\text{GFT}(q_k, S_1, B_1)]\). Then, by Chernoff-Hoeffding inequality and a union bound, we have that
\[
\mathbb{P}[\mathcal{E}] \leq 2K \exp(-2\varepsilon^2 T_0).
\]

On the other hand, for each \(p \in [0,1]\), define \(k(p) \in [K]\) as the index of a point in the grid \((q_1, \ldots, q_K)\) closest to \(p\). Then, on the good event \(\mathcal{E}\), for all \(t \geq 2KT_0 + 1\), we have that:
\[
\mathbb{E}[\text{GFT}(p^*, S_t, B_t)] - \mathbb{E}[\text{GFT}(\bar{P}^*, S_t, B_t)] = \mathbb{E}[\text{GFT}(p^*, S_t, B_t)] - \mathbb{E}[\text{GFT}(q_{k(p)} S_t, B_t)] + \mathbb{E}[\text{GFT}(q_{k(p)}, S_t, B_t)] - (\bar{F}_{k(p)} + \bar{G}_{k(p)}) - (\bar{F}_{k(p)} + \bar{G}_{k(p)}) - \mathbb{E}[\text{GFT}(\bar{P}^*, S_t, B_t)] - \mathbb{E}[\text{GFT}(\bar{P}^*, S_t, B_t)] \\
\leq 4M/K + \varepsilon + 0 + \varepsilon = 4M/K + 2\varepsilon.
\]

So, if \((P_t, Q_t)_{t \in [T]}\) are the prices posted by Algorithm 4, we have that
\[
\sum_{t=1}^{T} \left( \mathbb{E}[\text{GFT}(p^*, S_t, B_t)] - \mathbb{E}[\text{GFT}(P_t, Q_t, S_t, B_t)] \right) \\
\leq 2KT_0 + \sum_{t=2KT_0 + 1}^{T} \left( \mathbb{E}[\text{GFT}(p^*, S_t, B_t)] - \mathbb{E}[\text{GFT}(\bar{P}^*, S_t, B_t)] \right) \\
= 2KT_0 + \sum_{t=2KT_0 + 1}^{T} \mathbb{E}[\text{GFT}(p^*, S_t, B_t)] - \mathbb{E}[\text{GFT}(\bar{P}^*, S_t, B_t)] \\
\leq 2KT_0 + \mathbb{P}[\mathcal{E}](T - 2KT_0) + \sum_{t=2KT_0 + 1}^{T} \mathbb{E}\left[ \left( \mathbb{E}[\text{GFT}(p^*, S_t, B_t)] - \mathbb{E}[\text{GFT}(\bar{P}^*, S_t, B_t)] \right) 1_{\mathcal{E}} \right] \\
\leq 2KT_0 + 2K \exp(-2\varepsilon^2 T_0)(T - 2KT_0) + \sum_{t=2KT_0 + 1}^{T} \mathbb{E}\left[ \left( \frac{4M}{K} + 2\varepsilon \right) 1_{\mathcal{E}} \right] \\
\leq 2KT_0 + 2\left( \frac{2M}{K} + \varepsilon + K \exp(-2\varepsilon^2 T_0) \right)(T - 2KT_0).
\]

By the arbitrariness of \(\varepsilon\), we have the first part of the result. Substituting the stated choice of the parameters in the last expression (doing the calculation choosing e.g. \(\varepsilon = \lceil T^{-1/4} \rceil\)) yields the second part of the result. \(\square\)

As we noted in Section 5.1, if the time horizon is unknown, we can retain the regret guarantees of the previous result with a standard doubling trick.

It is straightforward to see that the same construction of Theorem 4 applies, giving a lower bound on the regret in the weakly budget balance setting of order \(\Omega(T^{2/3})\). Indeed, there the distribution of the buyer is known. Therefore, it is counterproductive to post two different prices to the seller and the buyer (same quality of feedback but lower gain from trade). We leave the gap between this \(\Omega(T^{2/3})\) lower bound and the \(\tilde{O}(T^{3/4})\) regret of Algorithm 4 open for future research.
8 Learning with One Bit

In this section, we discuss the (im)possibility of learning with less than a realistic feedback. We start by noting that Scouting Blindits requires only one bit of feedback $\mathbb{I}\{S_t \leq P_t \leq P'_t \leq B_t\}$, i.e., whether or not the trade occurred at time $t$ if $P_t$ was posted to the seller and $P'_t$ to the buyer. In this setting, one can therefore achieve sublinear regret without observing the two bits $\mathbb{I}\{S_t \leq P_t\}$ and $\mathbb{I}\{P'_t \leq B_t\}$ provided by realistic feedback. Thus, it is natural to wonder whether the single bit $\mathbb{I}\{S_t \leq P_t \leq P'_t \leq B_t\}$ is sufficient for obtaining sublinear regret bounds also in the budget balanced setting. This is not the case: even under the further assumptions of bounded densities (bd) and independent valuations (iv), a single bit in the budget balance setting does not provide sufficient observability. Indeed, consider a first instance in which the seller $S$ and buyer $B$ have uniform distributions on $[0,1]$, independent of each other. In this case, the only maximizer of the expected gain from trade is $p^* = \frac{1}{2}$. As a second instance, consider two independent distributions of the seller $S'$ and buyer $B'$ with densities (bounded by 2 and even infinite differentiable) $f_{S'}(s) := 4(4 - 2s^3 + s^2)/(s^3 - s^2 + 4)^2$ and $f_{B'}(b) = b(b - \frac{1}{2})(b - 1) + 1$ respectively. Then, for all $p \in [0,1]$, we have $\mathbb{P}[S \leq p \leq B] = \mathbb{P}[S' \leq p \leq B']$. Therefore, the two instances are indistinguishable under the single-bit feedback, but a direct verification shows that in the second instance, $p^* = \frac{1}{2}$ is not a maximizer of the expected gain from trade. Leveraging these facts and the continuity of the gain from trade in the two instances leads to a linear minimax regret, using the same ideas as in Theorem 5.

9 Conclusions

This work initiates the study of the bilateral trade problem in a regret minimization framework. We designed algorithms and proved tight bounds on the regret rates achieved under various feedback and private valuation models.

Our work opens several possibilities for future investigation. One first and natural research direction is related to the more general settings of two-sided markets with multiple buyers and sellers, different prior distributions, and complex valuation functions. A second direction is related to the tight characterization of the regret rates for weak budget balance mechanisms (which we proved are strictly better than the budget balance rates in some cases). Finally, we believe other classes of markets, which assume prior knowledge of the agent’s preferences, could be fruitfully studied in a regret minimization framework.

A Missing Details of Section 3

In this section, we prove the Decomposition lemma and its corollary as stated in Section 3.

Lemma 1 (Decomposition lemma). Fix any price $p \in [0,1]$. Then, for any $s, b \in [0,1]$,

$$GFT(p, s, b) = \int_{[0,1]} \mathbb{I}[s \leq p \leq \lambda \leq b] \, d\lambda + \int_{[0,p]} \mathbb{I}[s \leq \lambda \leq p \leq b] \, d\lambda. \quad (1)$$

Furthermore, let $S$ and $B$ be two $[0,1]$-valued random variables:

- Then

  $$\mathbb{E}[GFT(p, S, B)] = \int_{[0,1]} \mathbb{P}[S \leq p \leq \lambda \leq B] \, d\lambda + \int_{[0,p]} \mathbb{P}[S \leq \lambda \leq p \leq B] \, d\lambda. \quad (2)$$

- If $U$ is uniform on $[0,1]$ and independent of $(S, B)$, then

  $$\mathbb{E}[GFT(p, S, B)] = \mathbb{P}[S \leq p \leq U \leq B] + \mathbb{P}[S \leq U \leq p \leq B]. \quad (3)$$

- If $U$ is uniform on $[0,1]$ and $S, B, U$ are independent, then

  $$\mathbb{E}[GFT(p, S, B)] = \mathbb{P}[S \leq p] \mathbb{P}[p \leq U \leq B] + \mathbb{P}[p \leq B] \mathbb{P}[S \leq U \leq p]. \quad (4)$$
We begin by proving Eq. (5). Thus, Eq. (6) is an immediate consequence of Eq. (1) and Fubini’s theorem. We now prove Eq. (3). Under the assumptions, Eq. (2) implies
\[
\mathbb{P}[S \leq p \leq U \leq B] = \mathbb{P}\left(\{S \leq p\} \cap \{U \leq B\} \cap \{U \in [p, 1]\}\right) = \int_{[p, 1]} \mathbb{P}\left(\{S \leq p\} \cap \{U \leq B\} \mid U = \lambda\right) d\mathbb{P}_U(\lambda) = \int_{[p, 1]} \mathbb{P}[S \leq p \leq \lambda \leq B] d\lambda.
\]
The equality \(\mathbb{P}[S \leq U \leq p \leq B] = \int_{[0, p]} \mathbb{P}[S \leq \lambda \leq p \leq B] d\lambda\) can be shown analogously, proving Eq. (3).

Eq. (4) is an immediate consequence of Eq. (3), leveraging independence.

We now prove Eq. (5). If \(p \in (0, 1)\), the result follows from Eq. (4). Thus, assume \(p \in (0, 1)\). Then
\[
\mathbb{E}[GFT(p, S, B)] = \int_{[p, 1]} \mathbb{P}[S \leq p \leq \lambda \leq B] d\lambda + \int_{[0, p]} \mathbb{P}[S \leq \lambda \leq p \leq B] d\lambda.
\]
For the first addend, we have,
\[
\int_{[p, 1]} \mathbb{P}[S \leq p \leq \lambda \leq B] d\lambda = (1 - p) \int_{[p, 1]} \mathbb{P}[S \leq p \leq \lambda \leq B] d\mathbb{P}_U(\lambda)
\]
\[
= (1 - p) \int_{[p, 1]} \mathbb{P}[S \leq p \leq U \leq B \mid U = \lambda] d\mathbb{P}_U(\lambda) = (1 - p)\mathbb{P}[S \leq p \leq U \leq B] = \mathbb{E}\left[(1 - p)\mathbb{I}\{S \leq p \leq U \leq B\}\right].
\]
Analogously, one shows \(\int_{[0, p]} \mathbb{P}[S \leq \lambda \leq p \leq B] d\lambda = \mathbb{E}\left[p\mathbb{I}\{S \leq V \leq p \leq B\}\right]\), which gives Eq. (5).

We conclude this section by showing that the bounded-density assumption implies the Lipschitzness of the expected gain from trade.

If \(S\) and \(B\) are \([0, 1]\)-valued random variables such that \((S, B)\) admits joint density \(f\) bounded above by some constant \(M\), then \(\mathbb{E}[GFT(\cdot, S, B)]\) is \(4M\)-Lipschitz.

**Proof.** Take any two \(0 \leq p < q \leq 1\). We have that
\[
\left|\int_{[p, 1]} \mathbb{P}[S \leq p \leq \lambda \leq B] d\lambda - \int_{[q, 1]} \mathbb{P}[S \leq q \leq \lambda \leq B] d\lambda\right|
\]
\[
= \left|\int_{[q, 1]} \left(\mathbb{P}[S \leq p \leq \lambda \leq B] - \mathbb{P}[S \leq q \leq \lambda \leq B]\right) d\lambda + \int_{(p, q]} \mathbb{P}[S \leq p \leq \lambda \leq B] d\lambda\right|
\]
\[
\leq \sup_{\lambda \in [0, 1]} \left|\mathbb{P}[S \leq p \leq \lambda \leq B] - \mathbb{P}[S \leq q \leq \lambda \leq B]\right| + |p - q|
\]
\[
\leq \sup_{\lambda \in [0, 1]} \left|\int_{[\lambda, 1]} \int_{(p, q]} f(s, b) \, ds \right| \, db\right| + |p - q| \leq 2M|p - q|.
\]
Analogously, we can prove that
\[
\left|\int_{[0, p]} \mathbb{P}[S \leq \lambda \leq p \leq B] d\lambda - \int_{[0, q]} \mathbb{P}[S \leq \lambda \leq q \leq B] d\lambda\right| \leq 2M|p - q|.
\]
Thus, Eq. (2) yields \(\mathbb{E}[GFT(p, S, B)] - \mathbb{E}[GFT(q, S, B)] \leq 4M|p - q|\).
B Existence of the Best Price

In this section, we show that a price \( p^* \) maximizing the expected regret always exists.

**Lemma 2.** The function \( p \mapsto \mathbb{E}[\text{GFT}(p, S, B)] \) is upper semicontinuous. In particular, there exists a maximizer \( p^* \in [0, 1] \).

**Proof.** Let \( U \) be a random variable that is uniform on \([0, 1]\) and independent of \((S, B)\). By the Decomposition lemma (3), it is sufficient to show that

\[
 f: \mathbb{R} \to [0, 1], \quad p \mapsto \mathbb{P}[S \leq p \leq U \leq B] \quad \text{and} \quad g: \mathbb{R} \to [0, 1], \quad p \mapsto \mathbb{P}[S \leq U \leq p \leq B]
\]

are both upper semicontinuous. We now prove that \( f \) is upper semicontinuous, i.e., that for any \( p \in \mathbb{R} \), we have

\[
 \limsup_{q \to p} f(q) \leq f(p).
\]

To do so, we show that for any \( p \in \mathbb{R} \) and any two sequences \( q_n \uparrow p, r_n \downarrow p \), we have that

\[
 \limsup_{q_n \uparrow p} f(q_n) \leq f(p) \quad \text{and} \quad \limsup_{r_n \downarrow p} f(r_n) \leq f(p).
\]

If \( p \in \mathbb{R} \setminus [0, 1] \), the result is trivially true. Thus, let \( p \in [0, 1] \), \( q_n \uparrow p \) and \( r_n \downarrow p \). Then,

\[
 \mathcal{I}_{\{S \leq q_n \leq U \leq B\}} \to \mathcal{I}_{\{S < p \leq U \leq B\}}, \quad n \to \infty, \\
 \mathcal{I}_{\{S \leq r_n \leq U \leq B\}} \to \mathcal{I}_{\{S \leq p < U \leq B\}}, \quad n \to \infty,
\]

pointwise everywhere. By Lebesgue’s dominated convergence theorem, it follow that, if \( n \to \infty \),

\[
 f(q_n) \to \mathbb{P}[S < p \leq U \leq B] \leq \mathbb{P}[S \leq p \leq U \leq B] = f(p), \\
 f(r_n) \to \mathbb{P}[S \leq p < U \leq B] = \mathbb{P}[S \leq p \leq U \leq B] = f(p).
\]

By the arbitrariness of \( p \), \( (q_n)_{n \in \mathbb{N}} \) and \( (r_n)_{n \in \mathbb{N}} \), \( f \) is therefore upper semicontinuous. Analogously, one can prove that \( g \) is upper semicontinuous. Hence, \( p \mapsto \mathbb{E}[\text{GFT}(p, S, B)] = f(p) + g(p) \) is an upper semicontinuous function defined on the compact set \([0, 1]\), so it attains its maximum at some \( p^* \in [0, 1] \) by the Weierstrass theorem. \(\square\)

C Model and Notation

For all \( T \in \mathbb{N} \), we denote the set of the first \( T \) integers \( \{1, \ldots, T\} \) by \([T]\). If \( \mathbb{P} \) is a probability measure and \( X \) is a random variable, we denote by \( \mathbb{P}_X \) the probability measure defined for any (measurable) set \( E \), by \( \mathbb{P}_X[E] := \mathbb{P}[X \in E] \). We denote the expectation of a random variable \( X \) with respect to the probability measure \( \mathbb{P} \) by \( \mathbb{E}_\mathbb{P}[X] \). If a measure \( \nu \) is absolutely continuous with respect to another measure \( \mu \) with density \( f \), we denote \( \nu \) by \( f\mu \), so that for any (measurable) set \( E \), \( (f\mu)[E] := \nu[E] = \int_E f(x) \, d\mu(x) \). We denote the Lebesgue measure on the interval \([0, 1]\) by \( \mu_\ell \) and the product Lebesgue measure on \([0, 1]^3 \) by \( \mu_3 \). For any set \( E \) and \( x \in E \), we denote the Dirac measure on \( x \) by \( \delta_x \) (the dependence on \( E \) will always be clear from context).

C.1 The Learning Model

In this section, we introduce an abstract notion of sequential games which encompasses all the settings we discussed in the main part of the paper, providing a unified perspective. This will be especially useful when proving lower bounds.

**Definition 1** (Sequential game). A (sequential) game is a tuple \( \mathcal{E} := (X, Y, Z, \rho, \varphi, \mathcal{P}) \), where:
• $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are sets called the player’s action space, adversary’s action space, and feedback space.

• $\rho: \mathcal{X} \times \mathcal{Y} \to [0,1]$ and $\varphi: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ are called the reward and feedback functions$^3$.

• $\mathcal{P}$ is a set of probabilities on the set $\mathcal{Y}^N$ of sequences in $\mathcal{Y}$, called the adversary’s behavior.

This definition generalizes the partial monitoring games of [Lattimore and Szepesvári, 2020, Bartók et al., 2014] to settings with infinitely many arms and is able to model adversarial, i.i.d., and more general stochastic settings all at once. Before proceeding, we introduce another few handy definitions that will be used throughout the paper.

**Definition 2.** If $\mathcal{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \rho, \varphi, \mathcal{P})$ is a game, then we say the following. The sample space is the set $\Omega := \mathcal{Y}^N \times [0,1]^N$. The adversary’s actions $(Y_t)_{t \in \mathbb{N}}$ and the player’s randomization $(U_t)_{t \in \mathbb{N}}$ are sequences of random variables defined, for all $t \in \mathbb{N}$ and $\omega = ((y_n)_{n \in \mathbb{N}}, (u_n)_{n \in \mathbb{N}}) \in \Omega$, by $Y_t(\omega) := y_t$ and $U_t(\omega) := u_t$. The set of scenarios $\mathcal{S}$ is the set of probability measures $\mathcal{P}$ on $\Omega$ of the form $\mathcal{P} = \mu \otimes \mu_\mathcal{L}$, where $\mu \in \mathcal{P}$.

For the sake of conciseness, whenever we fix a game $\mathcal{G}$, we will assume that all the objects (sets, functions, random variables) presented in Definitions 1–2 are fixed and denoted by the same letters without declaring them explicitly each time, unless strictly needed.

Note that this setting models an oblivious adversary since its actions are independent of the player’s past randomization, i.e., for all $t \in \mathbb{N}$, $\mathbb{P}_{Y_t|Y_1,...,Y_t,U_1,...,U_t} = \mathbb{P}_{Y_t|Y_1,...,Y_t}$. Note also that we are assuming that the randomization of the player’s strategy is carried out by drawing numbers in the interval $[0,1]$ independently and uniformly at random. We can restrict ourselves to this case in light of the Skorokhod Representation Theorem [Williams, 1991, Section 17.3] without losing (much) generality. We now introduce formally the strategies of the player, the resulting played actions, and the corresponding feedback.

**Definition 3** (Player’s strategies, actions, and feedback). Given a game $\mathcal{G}$, we define a player’s strategy as a sequence of functions $\alpha = (\alpha_t)_{t \in \mathbb{N}}$ such that, for each $t \in \mathbb{N}$, $\alpha_t: [0,1]^t \times \mathcal{Z}^{t-1} \to \mathcal{X}$. Given a player’s strategy $\alpha$, we define inductively (on $t$) the corresponding sequences of player’s actions $(X_t)_{t \in \mathbb{N}}$ and player’s feedback $(Z_t)_{t \in \mathbb{N}}$ by $X_t := \alpha_t(U_1,...,U_t,Z_1,...,Z_{t-1})$, $Z_t := \varphi(X_t,Y_t)$. In the sequel, we will denote the set of all strategies for a game $\mathcal{G}$ by $\mathcal{A}(\mathcal{G})$.

To lighten the notation, we will write $\mathcal{A}$ instead of $\mathcal{A}(\mathcal{G})$ if it is clear from context. We can now extend the standard notions of regret, worst-case regret, and minimax regret to our general setting.

**Definition 4** (Regret). Given a game $\mathcal{G}$ and a horizon $T \in \mathbb{N}$, we define the regret (of $\alpha \in \mathcal{A}$ in a scenario $\mathcal{P} \in \mathcal{S}$), the worst-case regret (of $\alpha \in \mathcal{A}$), and the minimax regret (of $\mathcal{G}$), respectively, by

$$R_T^\mathcal{P}(\alpha) := \sup_{X \in \mathcal{X}} \mathbb{E}_\mathcal{P} \left[ \sum_{t=1}^{T} \rho(x,Y_t) - \sum_{t=1}^{T} \rho(X_t,Y_t) \right], \quad R_T^\mathcal{S}(\alpha) := \sup_{\mathcal{P} \in \mathcal{S}} R_T^\mathcal{P}(\alpha), \quad R_T^\mathcal{A}(\mathcal{G}) := \inf_{\alpha \in \mathcal{A}(\mathcal{G})} R_T^\mathcal{S}(\alpha).$$

If $\mathcal{G}$ and $\mathcal{\tilde{G}}$ are two games and $R_T^\mathcal{A}(\mathcal{G}) \geq R_T^\mathcal{A}(\mathcal{\tilde{G}})$, we say that $\mathcal{\tilde{G}}$ is easier than $\mathcal{G}$ (or equivalently, that $\mathcal{G}$ is harder than $\mathcal{\tilde{G}}$). When it is clear from the context, we will omit the dependence on $\mathcal{G}$ in $R_T^\mathcal{A}(\mathcal{G})$.

### C.2 Bilateral Trade as a Game

We now formally cast the various instances of bilateral trade we introduced in Section 2 into our sequential game setting.$^5$ In this context, we think of the learner as the player and the environment as the adversary.

---

$^3$More precisely, we need $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ to be non-empty measurable spaces and $\rho, \varphi$ to be measurable functions. To avoid clutter, in the following we will never mention explicitly these types of standard measurability assumptions unless strictly needed.

$^4$When $t = 1$, $[0,1]^t \times \mathcal{Z}^{t-1} = [0,1]$. In the following, we will always adopt this type of convention without mention it.

$^5$Straightforwardly, the same can be done for the weak budget balance setting we studied in Section 7.
C.2.1 Player’s Actions, Adversary’s Actions, and Reward

The player’s action space $X$ is the unit interval $[0, 1]$. This corresponds to the player posting the same price to both the seller and the buyer (budget balance). The adversary’s action space $Y$ is $[0, 1]^2$. They are the pairs of valuations of the seller and buyer. The reward function $\rho$ is the gain from trade $GFT:[0,1] \times [0,1]^2 \rightarrow [0,1]$, $(p, (s,b)) \rightarrow (b - s)I\{s \leq p \leq b\}$.

C.2.2 Available Feedback

**Full:** the feedback space $\mathcal{Z}$ is the unit square $[0, 1]^2$ and the feedback function is $\varphi:[0,1] \times [0,1]^2 \rightarrow [0,1]^2$, $(p, (s,b)) \rightarrow (s,b)$. This corresponds to the seller and the buyer revealing their valuations at the end of a trade.

**Realistic:** the feedback space $\mathcal{Z}$ is the boolean square $\{0,1\}^2$ and the feedback function is $\varphi:[0,1] \times [0,1]^2 \rightarrow \{0,1\}^2$, $(p, (s,b)) \rightarrow (I\{s \leq p\}, I\{p \leq b\})$. This corresponds to the seller and the buyer accepting or rejecting a trade at a price $p$.

C.2.3 Adversary’s Behavior

**Stochastic (iid):** the adversary’s behavior $\mathcal{P} = \mathcal{P}_{iid}$ consists of products of a single probability on $Y = [0,1]^2$, i.e., $\mu \in \mathcal{P}_{iid}$ if and only if there exists a probability measure $\mu$ on $[0,1]^2$ such that $\mu = \otimes_{t \in \mathbb{N}} \mu$. This corresponds to a stochastic i.i.d. environment, where however the valuations of the seller and the buyer could be correlated.

We will also investigate the following stronger assumptions.

**Independent valuations (iv):** the adversary’s behavior $\mathcal{P} = \mathcal{P}_{iv}$ is the subset of $\mathcal{P}_{iid}$ in which the valuations of the seller and the buyer are independent, i.e., $\mu \in \mathcal{P}_{iv}$ if and only if there exist two probability measures $\mu_S, \mu_B$ on $[0,1]$ such that $\mu = \otimes_{t \in \mathbb{N}} (\mu_S \otimes \mu_B)$.

**Bounded density (bd):** for a fixed $M \geq 1$, the adversary’s behavior $\mathcal{P} = \mathcal{P}_{bd}^M$ is the subset of $\mathcal{P}_{iid}$ in which the joint distribution of the valuations of buyer and seller has a density bounded by $M$, i.e., $\mu \in \mathcal{P}_{bd}^M$ if and only if there exists a density $f:[0,1]^2 \rightarrow [0,1]$ such that $\mu = \otimes_{t \in \mathbb{N}} (f \mu)$, where $\mu = \mu_S \otimes \mu_B$.

**Independent valuations with bounded density (iv+bd):** for a fixed $M \geq 1$, the adversary’s behavior $\mathcal{P} = \mathcal{P}_{iv+bd}^M$ is the subset $\mathcal{P}_{iv} \cap \mathcal{P}_{bd}^M$ of $\mathcal{P}_{iid}$.

**Adversarial (adv):** the adversary’s behavior $\mathcal{P} = \mathcal{P}_{adv}$ consists of products of Dirac measures on $Y = [0,1]^2$, i.e., $\mu \in \mathcal{P}_{adv}$ if and only if there exists a sequence $(s_t, b_t)_{t \in \mathbb{N}} \subset [0,1]^2$ such that $\mu = \otimes_{t \in \mathbb{N}} \delta(s_t, b_t)$. This corresponds to a deterministic, oblivious, and adversarial environment.

D Two Key Lemmas on Simplifying Sequential Games

In this section we introduce some useful techniques that could be of independent interest for proving lower bounds in sequential games. The idea is to give sufficient conditions for a given game to be harder than another, where the second one has a known lower bound on its minimax regret.

At a high level, the first lemma shows that if the adversary’s actions are independent of each other, a game $\mathcal{G}$ is easier than game $\mathcal{F}$ if $\mathcal{F}$ can be embedded in $\mathcal{G}$ in such a way that:

1. The optimal player’s actions of $\mathcal{F}$ are no better than the ones in $\mathcal{G}$.
2. The suboptimal player’s actions of $\mathcal{F}$ no worse than the ones in $\mathcal{G}$.
3. At distributional level, the quality of the feedback in $\mathcal{F}$ is no worse than that in $\mathcal{G}$.

The proof is deferred to Appendix D.1.
Lemma 3 (Embedding). Let $\mathcal{G} := (X, Y, Z, \rho, \varphi, \mathcal{P})$ and $\mathcal{G} := (\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{\rho}, \tilde{\varphi}, \tilde{\mathcal{P}})$ be two games, $\delta, \tilde{\delta}$ their respective sets of scenarios, $(Y_t)_{t \in \mathbb{N}}, (\tilde{Y}_t)_{t \in \mathbb{N}}$ their adversaries’ actions, and $T \in \mathbb{N}$ a horizon. Assume that $Y_1, \ldots, Y_T$ are $P$-independent for any scenario $P \in \delta$, $\tilde{Y}_1, \ldots, \tilde{Y}_T$ are $\tilde{P}$-independent for any scenario $\tilde{P} \in \tilde{\delta}$, and that there exist $\tilde{f} : X \to \tilde{X}$, $g : \tilde{Z} \to Z$, and $h : \tilde{\delta} \to \delta$ satisfying:

1. $\sup_{x \in X} \sum_{t=1}^{T-1} \mathbb{E}_{P}[\tilde{\rho}(\tilde{x}, \tilde{y}_t)] \leq \sup_{x \in X} \sum_{t=1}^{T-1} \mathbb{E}_{\tilde{P}}[\rho(x, Y_t)]$ for any scenario $\tilde{P} \in \tilde{\delta}$.

2. $\mathbb{E}_{\tilde{P}}[\tilde{\rho}(\tilde{x}, \tilde{y}_t)] \geq \mathbb{E}_{\tilde{\varphi}(\tilde{P})}[\rho(x, Y_t)]$ for any time $t \in [T]$, scenario $\tilde{P} \in \tilde{\delta}$, and action $x \in X$.

3. $\tilde{P} \mathbb{E}_{\tilde{P}}[\tilde{\rho}(\tilde{x}(x), \tilde{y}_t)] = (h(\tilde{P}))_{\varphi(x, Y_t)}$ for any time $t \in [T]$, scenario $\tilde{P} \in \tilde{\delta}$, and action $x \in X$.

Then $R^*_T(\mathcal{G}) \geq R^*_T(\mathcal{G})$.

The second lemma addresses feedback with uninformative (i.e., scenario-independent) components. At a high level, if the feedback of some of the player’s actions has one or more uninformative components, the game can be simplified by getting rid of them. The player can achieve this by simulating the uninformative parts of the feedback using her randomization. The proof is deferred to Appendix D.1.

Lemma 4 (Simulation). Let $\mathcal{V}, \mathcal{W}$ be two sets, $\mathcal{G} := (X, Y, Z, \rho, \varphi, \mathcal{P})$ a game with $Z = \mathcal{V} \times \mathcal{W}$, $\delta$ its set of scenarios, $(Y_t)_{t \in \mathbb{N}}$ its adversary’s actions, $\pi : \tilde{Z} \to \mathcal{V}$ the projection on $\mathcal{V}$, and $T \in \mathbb{N}$ a horizon. Assume that $Y_1, \ldots, Y_T$ are $P$-independent for any scenario $P \in \delta$ and that there exist disjoint sets $I, U \subset X$ such that $I \cup U = X$ and

1. For any time $t \in [T]$ and action $x \in I$ there exists $\psi_{tx} : [0, 1] \to \mathcal{W}$ such that, for all $P \in \delta$,

   $$P_{\varphi(x, Y_t)} = P_{\pi(\varphi(x, Y_t))} \otimes (\mu_L)_{\psi_{tx}}.$$  

2. For any time $t \in [T]$ and action $x \in U$, there exists $\psi_{tx} : [0, 1] \to Z$ such that, for all $P \in \delta$,

   $$P_{\varphi(x, Y_t)} = (\mu_L)_{\psi_{tx}}.$$  

Let $* \in \mathcal{V}$ and define

$$\tilde{\varphi} : X \times Y \to \mathcal{V}, (x, y) \mapsto \begin{cases} \pi(\varphi(x, y)), & \text{if } x \in I, \\ * \in \mathcal{V}, & \text{if } x \in U. \end{cases}$$

Define the game $\mathcal{G} := (X, Y, \mathcal{V}, \rho, \tilde{\varphi}, \tilde{\mathcal{P}})$. Then $R^*_T(\mathcal{G}) \geq R^*_T(\mathcal{G})$.

D.1 Proofs of the lemmas

In this section, we will give a full proof of the two useful Embedding and Simulation lemmas introduces in Appendix D. To lighten the notation, for any $m, n \in \mathbb{N}$, with $m \leq n$ and a family $(\lambda_k)_{k \in \mathbb{N}}$ we let $\lambda_{mn} := (\lambda_m, \lambda_{m+1}, \ldots, \lambda_n)$ and similarly $\lambda_{nm} := (\lambda_n, \lambda_{n-1}, \ldots, \lambda_m)$.

We begin by proving the Embedding lemma, that we restate for ease of reading.

Lemma 3 (Embedding). Let $\mathcal{G} := (X, Y, Z, \rho, \varphi, \mathcal{P})$ and $\mathcal{G} := (\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{\rho}, \tilde{\varphi}, \tilde{\mathcal{P}})$ be two games, $\delta, \tilde{\delta}$ their respective sets of scenarios, $(Y_t)_{t \in \mathbb{N}}, (\tilde{Y}_t)_{t \in \mathbb{N}}$ their adversaries’ actions, and $T \in \mathbb{N}$ a horizon. Assume that $Y_1, \ldots, Y_T$ are $P$-independent for any scenario $P \in \delta$, $\tilde{Y}_1, \ldots, \tilde{Y}_T$ are $\tilde{P}$-independent for any scenario $\tilde{P} \in \tilde{\delta}$, and that there exist $\tilde{f} : X \to \tilde{X}$, $g : \tilde{Z} \to Z$, and $h : \tilde{\delta} \to \delta$ satisfying:

1. $\sup_{x \in X} \sum_{t=1}^{T-1} \mathbb{E}_{P}[\tilde{\rho}(\tilde{x}, \tilde{y}_t)] \leq \sup_{x \in X} \sum_{t=1}^{T-1} \mathbb{E}_{\tilde{P}}[\rho(x, Y_t)]$ for any scenario $\tilde{P} \in \tilde{\delta}$.

2. $\mathbb{E}_{\tilde{P}}[\tilde{\rho}(\tilde{x}, \tilde{y}_t)] \geq \mathbb{E}_{\tilde{\varphi}(\tilde{P})}[\rho(x, Y_t)]$ for any time $t \in [T]$, scenario $\tilde{P} \in \tilde{\delta}$, and action $x \in X$.

3. $\tilde{P} \mathbb{E}_{\tilde{P}}[\tilde{\rho}(\tilde{x}(x), \tilde{y}_t)] = (h(\tilde{P}))_{\varphi(x, Y_t)}$ for any time $t \in [T]$, scenario $\tilde{P} \in \tilde{\delta}$, and action $x \in X$.  

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Then $R^*_t(\mathcal{F}) \geq R^*_t(\tilde{\mathcal{F}})$.

**Proof.** Fix any strategy $\alpha \in \mathcal{A}(\mathcal{F})$. For each time $t \in \mathbb{N}$, define

$$\tilde{a}_t : [0,1]^t \times \tilde{X}^{t-1} \rightarrow \tilde{X}, (u_1,\ldots,u_t,\tilde{x}_1,\ldots,\tilde{x}_{t-1}) \mapsto \tilde{f}(a_t(u_1,\ldots,u_t, g(\tilde{x}_1),\ldots,g(\tilde{x}_{t-1}))).$$

Then $\tilde{a} := (\tilde{a}_t)_{t \in \mathbb{N}} \in \mathcal{A}(\tilde{\mathcal{F}})$. As usual, let $(Y_t)_{t \in \mathbb{N}}$ and $(U_t)_{t \in \mathbb{N}}$ be the adversary’s actions and the player’s randomization in game $\mathcal{F}$ and $(X_t)_{t \in \mathbb{N}}$ and $(Z_t)_{t \in \mathbb{N}}$ be the player’s actions and the feedback according to the strategy $\alpha$. Let $(\tilde{Y}_t)_{t \in \mathbb{N}}, (\tilde{U}_t)_{t \in \mathbb{N}}, (\tilde{X}_t)_{t \in \mathbb{N}}, (\tilde{Z}_t)_{t \in \mathbb{N}}$ be the corresponding objects for the game $\tilde{\mathcal{F}}$ and the strategy $\tilde{a}$. Furthermore, define

$$\tilde{X}_1 = a_1(\tilde{U}_1), \quad \tilde{Z}_1 = g(\tilde{f}(\tilde{X}_1,\tilde{Y}_1)), \quad \tilde{X}_2 = a_2(\tilde{U}_1,\tilde{U}_2,\tilde{Z}_1), \quad \tilde{Z}_2 = g(\tilde{f}(\tilde{X}_2,\tilde{Y}_2)), \ldots \ .$$

Fix $\tilde{\mathcal{P}} \in \tilde{\mathcal{D}}$, where $\tilde{\mathcal{D}}$ is the set of scenarios of the game $\tilde{\mathcal{F}}$. Then $\tilde{\mathcal{P}}_{\tilde{U}_1} = (\tilde{h}(\tilde{\mathcal{P}}))_{\tilde{U}_1}$. Now, since $X_1 = a_1(U_1)$ and $\tilde{X}_1 = a_1(\tilde{U}_1)$, we also have that $\tilde{\mathcal{P}}_{\tilde{X}_1,\tilde{U}_1} = (\tilde{h}(\tilde{\mathcal{P}}))_{\tilde{X}_1,\tilde{U}_1} =: Q_1$. Now, up to a set with $Q_1$-probability zero, if $x_1 \in X$ and $u_1 \in [0,1]$, we get, using Item 3:

$$\tilde{\mathcal{P}}_{\tilde{X}_1,\tilde{U}_1}(\tilde{X}_1=x_1,\tilde{U}_1=u_1) = \tilde{\mathcal{P}}(\tilde{f}(\tilde{X}_1,\tilde{Y}_1))_{\tilde{X}_1=x_1,\tilde{U}_1=u_1} = \tilde{\mathcal{P}}(\tilde{f}(\tilde{X}_1,\tilde{Y}_1))$$

$$= (\tilde{h}(\tilde{\mathcal{P}}))_{\tilde{X}(x_1),Y_1}(\tilde{X}_1=x_1,\tilde{U}_1=u_1) = (\tilde{h}(\tilde{\mathcal{P}}))_{\tilde{Z}(x_1),Y_1}(\tilde{Z}_1,\tilde{X}_1,\tilde{U}_1)(A_1 \times D),$$

from which it follows that $\tilde{\mathcal{P}}_{\tilde{Z}_1,\tilde{X}_1,\tilde{U}_1} = (\tilde{h}(\tilde{\mathcal{P}}))_{\tilde{Z}_1,\tilde{X}_1,\tilde{U}_1}$. By induction, suppose that for time $t \in [T-1]$ we have that

$$\tilde{\mathcal{P}}_{\tilde{Z}_t,\tilde{X}_t,\tilde{U}_t,\ldots,\tilde{X}_1,\tilde{U}_1} = (\tilde{h}(\tilde{\mathcal{P}}))_{\tilde{Z}_t,\tilde{X}_t,\tilde{U}_t,\ldots,\tilde{X}_1,\tilde{U}_1}.$$ 

Then, using independence we have that

$$\tilde{\mathcal{P}}_{\tilde{Z}_t,\tilde{X}_t,\tilde{U}_t,\ldots,\tilde{X}_1,\tilde{U}_1,\tilde{X}_{t+1},\tilde{U}_{t+1},\ldots,\tilde{U}_1} = (\tilde{h}(\tilde{\mathcal{P}}))_{\tilde{Z}_t,\tilde{X}_t,\tilde{U}_t,\ldots,\tilde{X}_1,\tilde{U}_1,\tilde{X}_{t+1},\tilde{U}_{t+1},\ldots,\tilde{U}_1}.$$ 

Furthermore, since $X_{t+1} = a_{t+1}(U_1,\ldots,U_{t+1},Z_1,\ldots,Z_t)$ and $\tilde{X}_{t+1} = a_{t+1}(\tilde{U}_1,\ldots,\tilde{U}_{t+1},\tilde{Z}_1,\ldots,\tilde{Z}_t)$, we have that

$$\tilde{\mathcal{P}}_{\tilde{Z}_t,\tilde{X}_t,\tilde{U}_t,\ldots,\tilde{X}_1,\tilde{U}_1,\tilde{X}_{t+1},\tilde{U}_{t+1},\ldots,\tilde{U}_1} = (\tilde{h}(\tilde{\mathcal{P}}))_{\tilde{Z}_t,\tilde{X}_t,\tilde{U}_t,\ldots,\tilde{X}_1,\tilde{U}_1,\tilde{X}_{t+1},\tilde{U}_{t+1},\ldots,\tilde{U}_1} =: Q_{t+1}.$$ 

Now, up to a set with $Q_{t+1}$-probability zero, if $x_1,\ldots,x_{t+1} \in X$, $u_1,\ldots,u_{t+1} \in [0,1]$, and $z_1,\ldots,z_t \in Z$, by the $\tilde{\mathcal{P}}$-independence of $\tilde{Y}_1,\ldots,\tilde{Y}_{t+1}$, Item 3, and the $\tilde{h}(\tilde{\mathcal{P}})$-independence of $Y_1,\ldots,Y_{t+1}$, we have

$$\tilde{\mathcal{P}}_{\tilde{Z}_{t+1} \mid \tilde{Z}_t=z_1,\ldots,\tilde{Z}_1=x_1,\tilde{X}_t=x_1,\tilde{X}_1=x_1,\tilde{U}_1=u_1,\tilde{U}_1=u_1}$$

$$= \tilde{\mathcal{P}}_{\tilde{Z}_{t+1} \mid \tilde{Z}_t=z_1,\ldots,\tilde{Z}_1=x_1,\tilde{X}_t=x_1,\tilde{X}_1=x_1,\tilde{U}_1=u_1,\tilde{U}_1=u_1}$$

$$= \tilde{\mathcal{P}}_{\tilde{Z}_{t+1} \mid \tilde{Z}_t=z_1,\ldots,\tilde{Z}_1=x_1,\tilde{X}_t=x_1,\tilde{X}_1=x_1,\tilde{U}_1=u_1,\tilde{U}_1=u_1}$$

$$= \tilde{\mathcal{P}}_{\tilde{Z}_{t+1} \mid \tilde{Z}_t=z_1,\ldots,\tilde{Z}_1=x_1,\tilde{X}_t=x_1,\tilde{X}_1=x_1,\tilde{U}_1=u_1,\tilde{U}_1=u_1}.$$
So, if $A_{t+1} \subseteq \mathcal{Z}, D \subseteq \mathcal{Z} \times X_{t+1} \times [0,1]$, we have that

$$
\begin{align*}
\tilde{P}_{Z_{t+1},(\tilde{X}_{t+1:1}, \tilde{U}_{t+1:1})}(A_{t+1} \times D) \\
= \int_D \tilde{P}_{Z_{t+1},(\tilde{Z}_{t+1:1}=\tilde{X}_{t+1:1}, \tilde{U}_{t+1:1}=u_{t+1:1})}(A_{t+1}) \, d\tilde{P}_{Z_{t+1},(\tilde{Z}_{t+1:1}=\tilde{X}_{t+1:1}, \tilde{U}_{t+1:1}=u_{t+1:1})}(\tilde{Z}_{t+1:1}, \tilde{U}_{t+1:1}, t+1) \\
= \int_D (\tilde{h}(\tilde{P}))_{Z_{t+1},(\tilde{Z}_{t+1:1}=\tilde{X}_{t+1:1}, \tilde{U}_{t+1:1}=u_{t+1:1})}(A_{t+1}) \, d(\tilde{h}(\tilde{P}))_{Z_{t+1},(\tilde{Z}_{t+1:1}=\tilde{X}_{t+1:1}, \tilde{U}_{t+1:1}=u_{t+1:1})}(\tilde{Z}_{t+1:1}, \tilde{U}_{t+1:1}) \\
= (\tilde{h}(\tilde{P}))_{Z_{t+1},(\tilde{X}_{t+1:1}, \tilde{U}_{t+1:1})}(A_{t+1} \times D),
\end{align*}
$$

from which follows that $\tilde{P}_{Z_{t+1},(\tilde{X}_{t+1:1}, \tilde{U}_{t+1:1})} = (\tilde{h}(\tilde{P}))_{Z_{t+1},(\tilde{X}_{t+1:1}, \tilde{U}_{t+1:1})}$. In particular, for each $t \in [T]$ we have $\tilde{P}_{X_t} = (\tilde{h}(\tilde{P}))_{X_t}$. Hence, using the $\tilde{h}(\tilde{P})$-independence of $Y_1, \ldots, Y_T$, Item (2), and the $\tilde{P}$-independence of $Y_1, \ldots, Y_T$, we get

$$
\begin{align*}
\sum_{t=1}^T \mathbb{E}_{\tilde{P}}[\tilde{h}(X_t, Y_t)] &= \sum_{t=1}^T \int_X \mathbb{E}_{\tilde{h}(\tilde{P})}[\tilde{h}(x, Y_t)] \, d(\tilde{h}(\tilde{P}))_{X_t}(x) \\
&\leq \sum_{t=1}^T \int_X \mathbb{E}_{\tilde{P}}[\tilde{\rho}(\tilde{X}_t, \tilde{Y}_t)] \, d(\tilde{h}(\tilde{P}))_{X_t}(x) \\
&= \sum_{t=1}^T \int_X \mathbb{E}_{\tilde{P}}[\tilde{\rho}(\tilde{X}_t, \tilde{Y}_t)] \, d\tilde{P}_{X_t}(x) \\
&= \sum_{t=1}^T \mathbb{E}_{\tilde{P}}[\tilde{\rho}(\tilde{X}_t, \tilde{Y}_t)] = \sum_{t=1}^T \mathbb{E}_{\tilde{P}}[\tilde{\rho}(\tilde{X}_t, \tilde{Y}_t)].
\end{align*}
$$

Then, using Item (1), we have

$$
R_T(\tilde{P})(\sigma) = \sup_{x \in X} \left( \sum_{t=1}^T \mathbb{E}_{\tilde{h}(\tilde{P})}[\tilde{h}(x, Y_t)] - \sum_{t=1}^T \mathbb{E}_{\tilde{h}(\tilde{P})}[\tilde{h}(X_t, Y_t)] \right) \\
\geq \sup_{x \in X} \left( \sum_{t=1}^T \mathbb{E}_{\tilde{P}}[\tilde{\rho}(\tilde{X}_t, \tilde{Y}_t)] - \sum_{t=1}^T \mathbb{E}_{\tilde{P}}[\tilde{\rho}(\tilde{X}_t, \tilde{Y}_t)] \right) = R_T(\tilde{P})(\sigma).
$$

Since $\tilde{P}$ was arbitrary, we get

$$
R_T(\tilde{P}) = \inf_{\tilde{P} \in \mathcal{P}} R_T(\tilde{P}) \leq R_T(\tilde{P})(\sigma) = \sup_{\tilde{P} \in \mathcal{P}} R_T(\tilde{P})(\sigma) \geq \sup_{\tilde{P} \in \mathcal{P}} R_T(\tilde{P})(\sigma) = R_T(\sigma),
$$

and since $\sigma$ was arbitrary, we get

$$
R_T(\tilde{P}) \leq \inf_{\sigma \in \mathcal{P}} R_T(\sigma) = R_T(\mathcal{P}).
$$

\[\square\]

We now prove the Simulation lemma we introduced in Appendix D showing how to get rid of uninformative feedback.

**Lemma 4 (Simulation).** Let $\mathcal{V}, \mathcal{W}$ be two sets, $\mathcal{G} := (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \rho, \varphi, \mathcal{P})$ a game with $\mathcal{Z} = \mathcal{V} \times \mathcal{W}$, $\mathcal{S}$ be set of scenarios, $(Y_t)_{t \in \mathbb{N}}$ its adversary’s actions, $\pi: \mathcal{Z} \to \mathcal{V}$ the projection on $\mathcal{V}$, and $T \in \mathbb{N}$ a horizon. Assume that $Y_1, \ldots, Y_T$ are $P$-independent for any scenario $P \in \mathcal{S}$ and that there exist disjoint sets $I, U \subseteq X$ such that $I \cup U = X$ and

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1. For any time $t \in [T]$ and action $x \in I$ there exists $\psi_{t,x} : [0,1] \rightarrow \mathcal{W}$ such that, for all $P \in \mathcal{D}$,

$$P_{\psi(x,y_t)} = P_{\pi(x,y_t)} \otimes (\mu_t)_{\psi_{t,x}}.$$ 

2. For any time $t \in [T]$ and action $x \in \mathcal{U}$, there exists $y_{t,x} : [0,1] \rightarrow \mathcal{Z}$ such that, for all $P \in \mathcal{D}$,

$$P_{\psi(x,y_t)} = (\mu_t)_{y_{t,x}}.$$ 

Let $* \in \mathcal{V}$ and define

$$\bar{\phi} : X \times \mathcal{Y} \rightarrow \mathcal{V}, \ (x,y) \mapsto \begin{cases} \pi(\phi(x,y)) , & \text{if } x \in I, \\ * , & \text{if } x \in \mathcal{U}. \end{cases}$$

Define the game $\mathcal{G} := (X, \mathcal{Y}, \mathcal{V}, \rho, \bar{\phi}, \mathcal{P})$. Then $R^*_t(\mathcal{G}) \geq R^*_t(\mathcal{F})$.

**Proof.** For each number $a \in [0,1]$, fix a binary representation $0.a_1a_2a_3 \ldots$ of $a$ and define $\xi(a) := 0.a_1a_2a_3a_4 \ldots$. Note that the two resulting functions $\xi, \zeta : [0,1] \rightarrow [0,1]$ are $\mu_t$-independent with common (uniform) push-forward distribution $(\mu_t)_{\xi} = (\mu_t)_{\zeta}$.

Let $(Y_t)_{t \in \mathbb{N}}, (U_t)_{t \in \mathbb{N}}$ be the sequences of adversary’s actions and player’s randomization for the sequential game $\mathcal{G}$ and note that they are also the same for the sequential game $\mathcal{F}$. For each $t \in \mathbb{N}$ define $\beta_t : X \times \mathcal{V} \times [0,1] \rightarrow \mathcal{Z}$ via

$$(x,v,u) \mapsto \begin{cases} (v, \psi_{t,x}(u)) , & \text{if } x \in I, \\ \psi_{t,x}(u) , & \text{if } x \in \mathcal{U}, \end{cases}$$

if $t \leq T$, and in an arbitrary manner if $t \geq T + 1$. Fix $a = (a_t)_{t \in \mathbb{N}} \in \mathcal{A}(\mathcal{G})$. Let $(X_t)_{t \in \mathbb{N}}, (Z_t)_{t \in \mathbb{N}}$ be the sequences of player’s actions and feedback associated to the strategy $a$.

Fix $(u_t)_{t \in \mathbb{N}} \in \mathcal{U}[0,1]$ and $(v_t)_{t \in \mathbb{N}} \in \mathcal{V}$. Define by induction (on $t$) the sequences $(x_t)_{t \in \mathbb{N}}$ and $(z_t)_{t \in \mathbb{N}}$ via the relationships

$$x_t = a_t(\xi(u_1), \ldots, \xi(u_t), z_1, \ldots, z_{t-1}), \quad z_t = \beta_t(x_t, v_t, \zeta(u_t)).$$

Note that for each $t \in \mathbb{N}$, we have that $x_t$ depends only on $u_1, \ldots, u_t, v_1, \ldots, v_{t-1}$, so we can define

$$\bar{a}_t(u_1, \ldots, u_t, v_1, \ldots, v_{t-1}) := x_t.$$ 

Being $(u_t)_{t \in \mathbb{N}}$ and $(v_t)_{t \in \mathbb{N}}$ arbitrary, this defines a sequence of functions $(\bar{a}_t)_{t \in \mathbb{N}}$ such that, for all $t \in \mathbb{N}$,

$$\bar{a}_t : [0,1]^t \times \mathcal{V}^{t-1} \rightarrow X$$

i.e., $\bar{a} := (\bar{a}_t)_{t \in \mathbb{N}} \in \mathcal{A}(\mathcal{F})$. Let $(X_t)_{t \in \mathbb{N}}$ and $(\bar{V}_t)_{t \in \mathbb{N}}$ be respectively the sequence of player’s actions and the feedback sequence associated with the strategy $\bar{a}$. For each $t \in \mathbb{N}$, define also $\bar{Z}_t := \beta_t(X_t, \bar{V}_t, \zeta(U_t))$. Note that for each $t \in \mathbb{N}$ it holds that $\bar{X}_t = a_t(\xi(U_1), \ldots, \xi(U_t), \bar{Z}_1, \ldots, \bar{Z}_{t-1})$.

Fix a scenario $P \in \mathcal{S}$. Note first that $P_{\xi(U_t)} = P_{U_t}$, and since $X_t = a_t(U_t)$ and $X_t = \bar{a}_t(U_t) = a_t(\xi(U_t))$, we also have that $P_{X_t,U_t} = P_{X_t,U_t} = Q_1$. Now, up to a set with $Q_1$-probability zero, if $x_1 \in X$ and $u_1 \in [0,1]$, using Items (1) and (2), we have that

$$P_{Z_1|X_1=x_1,\zeta(U_1)=u_1} = P_{\beta_1(X_1,\bar{\phi}(X_1,\xi(U_1)))|X_1=x_1,\xi(U_1)=u_1} = P_{\beta_1(X_1,\bar{\phi}(X_1,\xi(U_1)))}$$

$$= \begin{cases} P_{\beta_1(x_1,\pi(\phi(x_1,y_1),\xi(U_1)))} , & \text{if } x_1 \in I, \\ P_{\beta_1(x_1,\pi(\phi(x_1,y_1)))} , & \text{if } x_1 \in \mathcal{U}, \end{cases}$$

$$= \begin{cases} P_{\pi(\phi(x_1,y_1))} \otimes P_{\xi(U_1)} , & \text{if } x_1 \in I, \\ P_{\pi(\phi(x_1,y_1))} , & \text{if } x_1 \in \mathcal{U}, \end{cases}$$

$$= \begin{cases} P_{\pi(\phi(x_1,y_1))} \otimes (\mu_t)_{\psi_{1,x_1}} , & \text{if } x_1 \in I, \\ (\mu_t)_{\psi_{1,x_1}} , & \text{if } x_1 \in \mathcal{U}, \end{cases}$$

$$= \begin{cases} P_{\phi(x_1,y_1)} \otimes (\mu_t)_{\psi_{1,x_1}} , & \text{if } x_1 \in I, \\ P_{\phi(x_1,y_1)} , & \text{if } x_1 \in \mathcal{U}, \end{cases}$$

$$= P_{\phi(x_1,y_1)} = P_{\phi(x_1,y_1)|X_1=x_1,U_1=u_1} = P_{Z_1|X_1=x_1,U_1=u_1}.$$
So, if \( A_1 \subset \mathcal{Z} \) and \( D \subset X \times [0, 1] \), then

\[
\mathbb{P}_{\bar{Z}_1, (\bar{X}_i, \xi(U_i))}(A_1 \times D) = \int_D \mathbb{P}_{\bar{Z}_1, (\bar{X}_i, \xi(U_i))=u_1}(A_1) \, d\mathbb{P}_{\bar{X}_i, \xi(U_i)}(x_i, u_1)
\]

\[
= \int_D \mathbb{P}_{Z_1, X_i=1, u_1}(A_1) \, d\mathbb{P}_{X_i, U_i}(x_i, u_1) = \mathbb{P}_{Z_1, (X_i, U_i)}(A_1 \times D),
\]

from which it follows that \( \mathbb{P}_{\bar{Z}_1, \bar{X}_i, \xi(U_i)} = \mathbb{P}_{Z_1, X_i, U_i} \). By induction, suppose that for \( t \in [T-1] \) we have that

\[
\mathbb{P}_{\bar{Z}_t, \cdots, \bar{Z}_1, \bar{X}_t, \cdots, \bar{X}_1, \xi(U_t), \cdots, \xi(U_1)} = \mathbb{P}_{Z_t, \cdots, Z_1, X_t, \cdots, X_1, U_t, \cdots, U_1}.
\]

Then, using independence we have that

\[
\mathbb{P}_{\bar{Z}_t, \cdots, \bar{Z}_1, \bar{X}_t, \cdots, \bar{X}_1, \xi(U_t), \cdots, \xi(U_1)} = \mathbb{P}_{Z_t, \cdots, Z_1, X_t, \cdots, X_1, U_t, \cdots, U_1} = Q_{t+1}.
\]

Furthermore, since \( X_{t+1} = a_{t+1}(U_1, \ldots, U_{t+1}, Z_1, \ldots, Z_t) \) and

\[
\bar{X}_{t+1} = \bar{a}_{t+1}(U_1, \ldots, U_{t+1}, \bar{V}_1, \ldots, \bar{V}_t) = a_{t+1}(\xi(U_1), \ldots, \xi(U_{t+1}), \bar{Z}_1, \ldots, \bar{Z}_t)
\]

we have that

\[
\mathbb{P}_{\bar{Z}_{t+1} = z_{t+1}, \ldots, \bar{Z}_1 = z_1, \bar{X}_{t+1} = x_{t+1}, \ldots, \bar{X}_1 = x_1, \xi(U_{t+1}) = u_{t+1}, \ldots, \xi(U_1) = u_1}
\]

\[
= \mathbb{P}_{\bar{a}_{t+1}}(\bar{X}_{t+1}, \bar{a}_{t+1}(X_{t+1}, Y_{t+1}), \xi(U_{t+1})) | \bar{Z}_2 = z_2, \ldots, \bar{Z}_1 = z_1, \bar{X}_{t+1} = x_{t+1}, \ldots, \bar{X}_1 = x_1, \xi(U_{t+1}) = u_{t+1}, \ldots, \xi(U_1) = u_1
\]

\[
= \mathbb{P}_{\bar{a}_{t+1}}(x_{t+1}, \bar{a}_{t+1}(x_{t+1}, Y_{t+1}), \xi(U_{t+1})) = \begin{cases} \mathbb{P}_{\bar{a}_{t+1}}(x_{t+1}, \bar{a}_{t+1}(x_{t+1}, Y_{t+1}), \xi(U_{t+1})) & \text{if } x_{t+1} \in I \\ \mathbb{P}_{\bar{a}_{t+1}}(x_{t+1}, \bar{a}_{t+1}(x_{t+1}, Y_{t+1}), \xi(U_{t+1})) & \text{if } x_{t+1} \in U \end{cases}
\]

\[
= \begin{cases} \mathbb{P}(\pi(x_{t+1}, Y_{t+1})) \bar{X}_{t+1} & \text{if } x_{t+1} \in I \\ \mathbb{P}(\pi(x_{t+1}, Y_{t+1}) \bar{X}_{t+1} & \text{if } x_{t+1} \in U \end{cases}
\]

\[
= \begin{cases} \mathbb{P}(\pi(x_{t+1}, Y_{t+1})) \bar{X}_{t+1} & \text{if } x_{t+1} \in I \\ \mathbb{P}(\pi(x_{t+1}, Y_{t+1}) \bar{X}_{t+1} & \text{if } x_{t+1} \in U \end{cases}
\]

\[
= \begin{cases} \mathbb{P}(\pi(x_{t+1}, Y_{t+1})) \bar{X}_{t+1} & \text{if } x_{t+1} \in I \\ \mathbb{P}(\pi(x_{t+1}, Y_{t+1}) \bar{X}_{t+1} & \text{if } x_{t+1} \in U \end{cases}
\]

\[
= \begin{cases} \mathbb{P}(\pi(x_{t+1}, Y_{t+1})) \bar{X}_{t+1} & \text{if } x_{t+1} \in I \\ \mathbb{P}(\pi(x_{t+1}, Y_{t+1}) \bar{X}_{t+1} & \text{if } x_{t+1} \in U \end{cases}
\]

Now, up to a set with \( Q_{t+1} \)-probability zero, if \( x_1, \ldots, x_{t+1} \in X, u_1, \ldots, u_{t+1} \in [0, 1] \) and \( z_1, \ldots, z_t \in \mathcal{Z} \), using the \( \mathbb{P} \)-independence of \( Y_1, \ldots, Y_{t+1} \) and Items (1)–(2), we have that

\[
\mathbb{P}_{Z_{t+1} | Z_t = z_t, \ldots, Z_1 = z_1, X_{t+1} = x_{t+1}, \ldots, X_1 = x_1, U_{t+1} = u_{t+1}, \ldots, U_1 = u_1}
\]

\[
= \mathbb{P}_{\pi(x_{t+1}, Y_{t+1})} | Z_t = z_t, \ldots, Z_1 = z_1, X_{t+1} = x_{t+1}, \ldots, X_1 = x_1, U_{t+1} = u_{t+1}, \ldots, U_1 = u_1
\]

\[
= \mathbb{P}_{Z_{t+1} | Z_t = z_t, \ldots, Z_1 = z_1, X_{t+1} = x_{t+1}, \ldots, X_1 = x_1, U_{t+1} = u_{t+1}, \ldots, U_1 = u_1}
\]
In conclusion

Theorem under the further assumptions that the valuations of the seller and buyer are independent of each other and have bounded densities.

Proof.

In this section, we prove that in the full-feedback case, no strategy can beat the each $X$.

The idea of the proof is to build a family of scenarios $\tilde{\tilde{Z}}_{t+1:i} = \tilde{Z}_{t+1:i} \in \{0, 1\}^t \times X_{t+1} \times [0, 1]^{t+1}$, we have that

$$\mathbb{P}_{\tilde{Z}_{t+1}}(\tilde{Z}_{t+1}, \tilde{X}_{t+1}, ..., \tilde{X}, \tilde{Z}(U_{t+1}), ..., \tilde{Z}(U_1)) (A_{t+1} \times D)$$

from which it follows that $\mathbb{P}_{z_{t+1:i}}(\tilde{Z}_{t+1:i} = \{0, 1\}^t \times X_{t+1} \times [0, 1]^{t+1})$. In particular, for each $t \in [T]$ we have that $\mathbb{P}_{X_t} = \mathbb{P}_{\tilde{X}_t}$. So, for each $t \in [T]$, using the $\mathbb{P}$-independence of $Y_1, ..., Y_t$, we have that

$$\mathbb{P}_{X_t, Y_t} = \mathbb{P}_{X_t} \otimes \mathbb{P}_{Y_t} = \mathbb{P}_{\tilde{X}_t} \otimes \mathbb{P}_{Y_t} = \mathbb{P}_{\tilde{X}_t, Y_t},$$

and then

$$\mathbb{E}_{\tilde{X}_t, Y_t} [\rho(X_t, Y_t)] = \mathbb{E}_{X_t, Y_t} [\rho] = \mathbb{E}_{\tilde{X}_t, Y_t} [\rho] = \mathbb{E}_{\tilde{X}_t} [\rho(X_t, Y_t)].$$

In conclusion

$$R^p_T(\alpha) = \sup_{x \in X} \mathbb{E}_{\tilde{X}_t} \left[ \sum_{t=1}^{T} \rho(x, Y_t) - \sum_{t=1}^{T} \rho(X_t, Y_t) \right] = \sup_{x \in X} \left[ \sum_{t=1}^{T} \mathbb{E}_{\tilde{X}_t} [\rho(x, Y_t)] - \sum_{t=1}^{T} \mathbb{E}_{\tilde{X}_t} [\rho(X_t, Y_t)] \right]$$

$$= \sup_{x \in X} \left[ \sum_{t=1}^{T} \mathbb{E}_{\tilde{X}_t} [\rho(x, Y_t)] - \sum_{t=1}^{T} \mathbb{E}_{\tilde{X}_t} [\rho(X_t, Y_t)] \right] = \sup_{x \in X} \left[ \sum_{t=1}^{T} \mathbb{E}_{\tilde{X}_t} [\rho(x, Y_t)] - \sum_{t=1}^{T} \mathbb{E}_{\tilde{X}_t} [\rho(X_t, Y_t)] \right] = R^p_T(\tilde{\alpha}).$$

Since $\mathbb{P}$ was arbitrary, it follows that $R^p_T(\alpha) = R^p_T(\tilde{\alpha})$. Since $\alpha$ was arbitrary, it follows that

$$R^*_{T} = \inf_{\alpha \in \mathcal{A}} R^p_T(\alpha) = \inf_{\alpha \in \mathcal{A}} R^p_T(\tilde{\alpha}) \geq \inf_{\alpha \in \mathcal{A}} R^p_T(\alpha') = R^*_{T}(\mathcal{F}).$$

$\square$

E $\sqrt{T}$ Lower Bound Under Full-Feedback (iv+bd)

In this section, we prove that in the full-feedback case, no strategy can beat the $\sqrt{T}$ rate that we proved in Theorem 1 when the seller/buyer pair $(S_t, B_t)$ is drawn i.i.d. from an unknown fixed distribution, not even under the further assumptions that the valuations of the seller and buyer are independent of each other and have bounded densities.

The idea of the proof is to build a family of scenarios $\mathbb{P}^{\pm\varepsilon}$ parameterized by $\varepsilon \in [0, 1]$, like in Fig. 1. The only way to avoid suffering linear regret in a scenario $\mathbb{P}^{\pm\varepsilon}$ is to identify the sign of $\pm\varepsilon$. Leveraging the Embedding and Simulation lemmas (Lemmas 3 and 4), this construction leads to a reduction to a two-action expert problem, which has a know lower bound on the regret of order $\sqrt{T}$.

**Theorem 9** (Theorem 2, restated). In the full-feedback stochastic (iid) setting with independent valuations (iv) and densities bounded by a constant $M = 4$ (bd), for all horizons $T \in \mathbb{N}$, the minimax regret satisfies

$$R^*_{T} = \Omega(\sqrt{T}).$$

**Proof.** Fix any horizon $T \in \mathbb{N}$ and any $M \geq 4$. Recalling Appendix C.2, the full-feedback stochastic (iid) setting with independent valuations (iv) and densities bounded (bd) by $M$ is a game $\mathcal{F} := (X, Y, Z, \rho, \varphi, \mathcal{P})$, where $X = [0, 1]$, $Y = [0, 1]^2$, $Z = [0, 1]^2$, $\rho = \text{GFT}$, $\varphi: (p, (s, b)) \mapsto (s, b)$, and $\mathcal{P} = \mathcal{G}_{iv+bd}^M$. Define, for each $\varepsilon \in [\pm 1]$, the densities $f_{S, \varepsilon} = 2(1 + \varepsilon)[[0, 1]] + 2(1 - \varepsilon)[[0, 1]]$ and $f_{B} = 2[[0, 1]]$. Fix the adversary’s...
which hinges in a non-trivial way that highlights that only its first component is informative. In step four and six, we

behavior \( \mathcal{P}_1 \) as the subset of \( \mathcal{P} \) whose elements have the form \( \mu_i = \otimes_{\epsilon \in \mathbb{N}} (f_{\epsilon, s} \mu_s \otimes f_{\epsilon, 1} \mu_1) \), for some \( \epsilon \in [-1, 1] \). Since \( \mathcal{P}_1 \subset \mathcal{P} \), the game \( \mathcal{G}_1 := (X, Y, Z, \rho, \varphi, \mathcal{P}_1) \) is easier than \( \mathcal{G} \) (i.e., \( R^*_t(\mathcal{G}) \geq R^*_t(\mathcal{G}_1) \)) by the Embedding lemma (Lemma 3) with \( \tilde{f} \) and \( g \) as the identities, and \( \tilde{h} \) as the inclusion. Now, define \( \rho_1 : X \times Y \to [0, 1], (p, (s, b)) \mapsto (b - s)I[s \leq \frac{1}{2} \leq b]I[p \leq \frac{1}{2}] + (s - \frac{1}{2})I[\frac{1}{2} \leq s \leq \frac{3}{4}]I[p > \frac{1}{2}] \) and note that, defining \( \mathcal{G}_2 := (X, Y, Z, \rho_1, \varphi, \mathcal{P}_1) \), by the Embedding lemma with \( \tilde{f}, g, \tilde{h} \) as the identities, we have that the game \( \mathcal{G}_2 \) is easier than the game \( \mathcal{G}_1 \) (i.e., \( R^*_t(\mathcal{G}_1) \geq R^*_t(\mathcal{G}_2) \)). Then, let \( \mathcal{Z}_3 := \{0, 1\} \times [0, \frac{1}{4}] \times [0, 1] \) and \( \varphi_3 : X \times Y \to \mathcal{Z}_3, (p, (s, b)) \mapsto I[s \leq \frac{1}{4}], sI[s \leq \frac{1}{4}] + (s - \frac{1}{2})I[\frac{1}{4} \leq s \leq \frac{3}{4}], b \). Define the game \( \mathcal{G}_3 := (X, Y, Z, \rho_3, \varphi_3, \mathcal{P}_1) \). By the Embedding lemma with \( \tilde{f}, \tilde{h} \) as the identities and \( g : \mathcal{Z}_3 \to \mathcal{Z}_3, (i, s, b) \mapsto (i + 2 + 5)(1 - i), b \), we have that the game \( \mathcal{G}_3 \) is easier than the game \( \mathcal{G}_2 \) (i.e., \( R^*_t(\mathcal{G}_2) \geq R^*_t(\mathcal{G_3}) \)). Next, let \( \varphi_4 : X \times Y \to \mathcal{Z}_3, (p, (s, b)) \mapsto I[s \leq \frac{1}{4}] \), and define the game \( \mathcal{G}_4 := (X, Y, Z, \rho_3, \varphi_4, \mathcal{P}_1) \). Let \( (Y_t)_{t \in \mathbb{N}} \) be the adversary’s actions in \( \mathcal{G}_4 \). A tedious computation verifies that for all \( t \in \mathbb{N}, p \in X, \) and scenarios \( \mathcal{P} \) of game \( \mathcal{G}_1, \mathcal{P}_{\varphi_1(p, Y_t)} \) is easier than \( \mathcal{G}_3 \) (i.e., \( R^*_t(\mathcal{G}_3) \geq R^*_t(\mathcal{G}_4) \)). Finally, consider the game \( \mathcal{G}_5 := \{(1, 2), (1, 2), (0, 1), \rho_5, \varphi_5, \mathcal{P}_3 \} \), where in matrix notation, \( \rho_5 = \left[ \rho_5(i, j) \right]_{i, j \in \{1, 2\}} \) and \( \varphi_5 = \left[ \varphi_5(i, j) \right]_{i, j \in \{1, 2\}} \) are given by

\[
\rho_5 := \begin{bmatrix}
1/2 & 3/8 \\
3/8 & 1/2 \\
\end{bmatrix}, \quad \varphi_5 := \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix},
\]

and \( \mathcal{P}_3 \) is the set of all measures \( \tilde{\mu}_i \) of the form \( \tilde{\mu}_i = \otimes_{\epsilon \in \mathbb{N}} \left( \frac{1 + \epsilon}{2} \delta_1 + \frac{1 - \epsilon}{2} \delta_2 \right) \) for some \( \epsilon \in [-1, 1] \), where \( \delta_i \) is the Dirac measure at \( i \in \{1, 2\} \). Thus, letting \( \delta_4 \) and \( \delta_5 \) be the two sets of scenarios in games \( \mathcal{G}_4 \) and \( \mathcal{G}_5 \) respectively (note that \( \delta_4 \) coincides with the set of scenarios of \( \mathcal{G}_4 \)) and using again the Embedding lemma, this time with \( \tilde{f} : [0, 1] \to \{1, 2\}, p \mapsto I(p \leq \frac{1}{2}) + 2I(p > \frac{1}{2}), g : [0, 1] \to \{0, 1\}, i \mapsto i, \) and \( \tilde{h} : \delta_5 \to \delta_4, \tilde{\mu}_i \otimes \mu_1 \mapsto \mu_i \otimes \mu_1 \), we obtain that \( \mathcal{G}_5 \) is easier than \( \mathcal{G}_4 \) (i.e., \( R^*_t(\mathcal{G}_4) \geq R^*_t(\mathcal{G}_5) \)). This last game \( \mathcal{G}_5 \) is an online learning problem with full information (also known as learning with expert advice), whose minimax regret is known to be lower bounded by \( \frac{1}{\sqrt{2} \pi} \sqrt{T} \) [Cover, 1965]. In conclusion, we proved that \( R^*_t(\mathcal{G}) \geq R^*_t(\mathcal{G}_5) \geq \frac{1}{\sqrt{2} \pi} \sqrt{T} \).

\section{Proof of \( T^{2/3} \) Lower Bound Under Realistic Feedback (iv+bd)}

In this section we give a detailed proof of our \( T^{2/3} \) lower bound of Section 5.2 which hinges in a non-trivial way on our Embedding and Simulation lemmas (Lemmas 3 and 4). We denote Bernoulli distributions with parameter \( \lambda \) by \( \text{Ber}_\lambda \).

\begin{theorem}[Theorem 4, restated] In the realistic-feedback stochastic (iid) setting with independent valuations (iv) and densities bounded by a constant \( M \geq 24 \) (bd), for all horizons \( T \in \mathbb{N} \), the minimax regret satisfies

\[
R^*_T \geq \frac{11}{672} T^{2/3}.
\]
\end{theorem}

\begin{proof}
Fix an arbitrary horizon \( T \in \mathbb{N} \) and any \( M \geq 24 \). Recalling Appendix C.2, the realistic-feedback stochastic (iid) setting with independent valuations (iv) and densities bounded (bd) by \( M \) is a game \( \mathcal{G} := (X, Y, Z, \rho, \varphi, \mathcal{P}) \), where \( X = [0, 1], Y = [0, 1]^2, Z = [0, 1]^2, \rho = \text{GFT}, \varphi : (p, (s, b)) \mapsto (I[p \leq s], I[p < b]) \), and \( \mathcal{P} = \mathcal{P}_M^{\text{bd}} \). The idea of the proof is to build a sequence of games, each one easier than the former, the last of which has a known lower bound on its minimax regret. In the first step we limit the adversary’s behavior to a parametric family which is easily manageable and well-represents the difficulty of the problem (see Fig. 2). In the second step, we increase the reward of suboptimal actions in order to have only three possible expected-reward values in each scenario. In the third and fifth steps we increase the feedback, presenting it in a way that highlights that only its first component is informative. In step four and six, we

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simulate-away the uninformative parts of the feedback. Finally, in step 7 we show that the resulting game is harder than a known partial monitoring game with minimax regret of order at least $T^{2/3}$.

**Step 1.** Let $\mathcal{G} := \frac{1}{4s}$. Define the following densities of the seller and buyer, respectively, by

$$f_s := \frac{1}{4s} \left( (1+\epsilon)I_{[0,\frac{1}{8}]} + (1-\epsilon)I_{[\frac{1}{8},\frac{1}{4}]} + I_{[\frac{1}{4},\frac{1}{2}]} + I_{[\frac{1}{2},\infty]} \right), \forall \epsilon \in [-1,1], \quad \text{(red/blue in Fig. 2)}$$

$$f_B := \frac{1}{4s} \left( I_{[1-\frac{\mathcal{G}}{2},1]} + I_{[\frac{1}{2},1-\frac{\mathcal{G}}{2}]} + I_{[\frac{1}{4},\frac{1}{2}]} + I_{[\frac{1}{8},\frac{1}{4}]} \right). \quad \text{(green in Fig. 2)}$$

Define $\mathcal{P}_1$ as the subset of $\mathcal{P}$ whose elements have the form $\mu_s := \otimes_{t \in \mathbb{N}} (f_s, s) \otimes \delta_B$ for $\epsilon \in [-1,1]$. Since $\mathcal{P}_1 \subset \mathcal{P}$, the game $\mathcal{G}_1 := (\mathcal{X}, \mathcal{T}, \mathcal{Z}, \rho, \phi, \mathcal{P}_1)$ is easier than $\mathcal{G}$ (i.e., $R^*_\mathcal{G}(\mathcal{G}_1) \geq R^*_\mathcal{G}(\mathcal{G})$) by the Embedding lemma (Lemma 3) with $\mathcal{F}$ and $\mathcal{G}$ as the identities, and $\mathcal{H}$ as the inclusion.

**Step 2.** Define $\rho_2 : \mathcal{X} \times \mathcal{Y} \to [0,1]$, $(p, (s, b)) \mapsto \operatorname{GFT}(\frac{1}{2} + \frac{1}{\mathcal{G}}, (s, b)) \mathbb{I}_{\{p < \frac{1}{2}\}} + \operatorname{GFT}(\frac{1}{2} + \frac{1}{\mathcal{G}}, (s, b)) \mathbb{I}_{\{\frac{1}{2} \leq p < \frac{3}{4}\}} + \operatorname{GFT}(\frac{1}{2} + \frac{1}{\mathcal{G}}, (s, b)) \mathbb{I}_{\{\frac{3}{4} \leq p < 1\}}$. By the Embedding lemma with $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ as the identities, we have that the game $\mathcal{G}_2 := (\mathcal{X}, \mathcal{Y}, \rho_2, \phi_2, \mathcal{P}_1)$ is easier than $\mathcal{G}_1$ (i.e., $R^*_\mathcal{G}(\mathcal{G}_1) \geq R^*_\mathcal{G}(\mathcal{G}_2)$).

**Step 3.** Define $\mathcal{G}_3 := \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1\} \times [0, \mathcal{G}] \times \{0, 1\} \times \{0, 1\} \times \mathcal{X}$ and $\varphi_3 : \mathcal{X} \times \mathcal{Y} \to \mathcal{G}_3$,

$$(p, (s, b)) \mapsto \left\{ \begin{array}{ll} (\eta(s), s - \eta(s), 0, \mathbb{I} \{p \leq b\}, p), & \text{if } p < \frac{1}{4}, \\ (0, 0, \mathbb{I} \{s \leq p\}, \mathbb{I} \{p \leq b\}, p), & \text{if } p \geq \frac{1}{4}, \end{array} \right.$$ where $\eta : [0,1] \to \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1\}$, $s \mapsto \frac{1}{6} \mathbb{I} \{\frac{1}{6} \leq s \leq \frac{1}{6} + \mathcal{G}\} + \frac{1}{3} \mathbb{I} \{\frac{1}{3} \leq s \leq \frac{1}{3} + \mathcal{G}\} + \frac{1}{2} \mathbb{I} \{\frac{1}{2} \leq s \leq \frac{3}{4} + \mathcal{G}\ \}$ + $\frac{3}{4} \mathbb{I} \{\frac{3}{4} \leq s \leq \frac{3}{4} + \mathcal{G}\}$ + $\mathbb{I} \{\frac{3}{4} + \mathcal{G} \leq s \leq 1\}$ + $\mathbb{I} \{\frac{1}{2} \leq s \leq 1\}$. Define the game $\mathcal{G}_3 := (\mathcal{X}, \mathcal{Y}, \mathcal{G}_3, \rho_2, \varphi_3, \mathcal{P}_1)$. By the Embedding lemma with $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ as the identities and

$$\varphi : \mathcal{G}_3 \to \mathcal{G}_2, \quad (a, u, i, j) \mapsto \left\{ \begin{array}{ll} \mathbb{I} \{a \leq b\}, j), & \text{if } p < \frac{1}{4}, \\ (i, j), & \text{if } p \geq \frac{1}{4}, \end{array} \right.$$ we have that the game $\mathcal{G}_3$ is easier than $\mathcal{G}_2$ (i.e., $R^*_\mathcal{G}(\mathcal{G}_2) \geq R^*_\mathcal{G}(\mathcal{G}_3)$).

**Step 4.** Let $\mathcal{G}_4 := \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $\varphi_4 : \mathcal{X} \times \mathcal{Y} \to \mathcal{G}_4$, $(p, (s, b)) \mapsto \eta(s) \mathbb{I} \{p < \frac{1}{4}\}$. Define the game $\mathcal{G}_4 := (\mathcal{X}, \mathcal{Y}, \mathcal{G}_4, \varphi_3, \mathcal{P}_1)$. Let $(Y_t)_{t \in \mathbb{N}} = (S_t, B_t)_{t \in \mathbb{N}}$ be the adversary's actions in $\mathcal{G}_4$, $E := \{0, \mathcal{G}\} \cup [\frac{1}{2}, \frac{1}{2} + \mathcal{G}] \cup [\frac{1}{2} + \mathcal{G}, \frac{3}{4} + \mathcal{G}]$ and $F := [\frac{1}{2} - \mathcal{G}, \frac{1}{2}] \cup [\frac{1}{2} - \mathcal{G}, \frac{3}{4}] \cup [\frac{1}{2} - \mathcal{G}, \frac{3}{4} + \mathcal{G}] \cup [1 - \mathcal{G}, 1]$. A long and tedious computation verifies that for all $t \in \mathbb{N}$,

- For each $p \in [0,1/4)$ and any scenario $\mathcal{P}$ of game $\mathcal{G}_3$, $\mathbb{P}_{\mathcal{P}_3(p,Y_t)} = \mathbb{P}_{\mathcal{P}_3(s,Y_t)} \otimes (v \otimes \delta_0 \otimes \text{Ber}_{\lambda_F} \otimes \delta_p)$, where $v$ is the uniform distribution on $[0, \mathcal{G}]$ and $\lambda_F := \frac{1}{4\mathcal{G}} \|\mathcal{E}\|_1 \{[p, 1] \cap F\}$. By the well-known Skorohod representation [Williams, 1991, Section 17.3], there exists $\psi_p : [0,1] \to [0, \mathcal{G}] \times \{0, 1\} \times \{0, 1\} \times \mathcal{X}$ such that $v \otimes \delta_0 \otimes \text{Ber}_{\lambda_F} \otimes \delta_p = (\mu_L)_\psi_p$.

- For each $p \in [1/4, 1]$ and any scenario $\mathcal{P}$ of game $\mathcal{G}_3$, $\mathbb{P}_{\mathcal{P}_3(p,Y_t)} = \delta_0 \otimes \delta_0 \otimes \text{Ber}_{\lambda_E} \otimes \text{Ber}_{\lambda_F} \otimes \delta_p$, where $\lambda_E := \frac{1}{4\mathcal{G}} \|\mathcal{E}\|_1 \{[p, 1] \cap E\}$ and $\lambda_F := \frac{1}{4\mathcal{G}} \|\mathcal{E}\|_1 \{[p, 1] \cap F\}$. By the Skorohod representation, there exists $\psi_p : [0,1] \to \mathcal{G}_3$ such that $\delta_0 \otimes \delta_0 \otimes \text{Ber}_{\lambda_E} \otimes \text{Ber}_{\lambda_F} \otimes \delta_p = (\mu_L)_\psi_p$.

Thus, by the Simulation lemma (Lemma 4) with $\mathcal{I} = [0,1/4)$ and $\mathcal{U} = [1/4, 1]$, the game $\mathcal{G}_4$ is easier than $\mathcal{G}_3$ (i.e., $R^*_\mathcal{G}(\mathcal{G}_3) \geq R^*_\mathcal{G}(\mathcal{G}_4)$).
Step 5. Let $Y_5 := Y^{\|}$, $Z_5 := \{0,1\} \times (\mathbb{N} \cup \{\infty\}) \times \{0,1\} \times X$, $\rho_5 : X \times Y_5 \rightarrow [0,1]$, $(p, (s_k, b_k)_{k \in \mathbb{N}}) \mapsto \rho_2(p, s_1, b_1)$, where $\eta$ is defined in game $\mathcal{B}_3$, $\tau := \inf\{k \in \mathbb{N} \mid \eta(s_k) \in \{0,1/\delta\}\} \in \mathbb{N} \cup \{\infty\}$, and $s_\infty := 0$. Let $\mathcal{P}_5$ be the set of measures on $Y_5^3$ of the form $\mu_\epsilon := \otimes_{k \in \mathbb{N}}(f_{s_k, \mu_L} \otimes f_{b_k, \mu_L})$ for $\epsilon \in [-1,1]$, and define the game $\mathcal{B}_5 := (X, Y_5, Z_5, \rho_5, \varphi_5, \mathcal{P}_5)$. By the Embedding lemma with $\mathcal{F}$ as the identity,

$$g : Z_5 \rightarrow Z_4, \quad (z, k, j, p) \mapsto \frac{1}{6}(1-z)I\{p < \frac{1}{4}, k = 1\} + \frac{1}{4}j + \frac{2}{3}(1-j)I\{p < \frac{1}{4}, k > 1\},$$

and $\mathcal{H} : \mu_L \mapsto \mu_L \otimes \mu_L$, we have that the game $\mathcal{B}_5$ is easier than $\mathcal{B}_4$ (i.e., $R^*_{\mu}(\mathcal{F}_4) \geq R^*_{\mu}(\mathcal{B}_5)$).

Step 6. Now, define $\pi : Z_5 \rightarrow \{0,1\}$ as the projection on the first component $\{0,1\}$ of $Z_5$, $Z_6 := \{0,1\}$, $\varphi_6 := \pi \circ \varphi_5$, and the game $\mathcal{B}_6 := (X, Y_5, Z_6, \rho_5, \varphi_6, \mathcal{P}_5)$. Let $(Y_t)_{t \in \mathbb{N}}$ be the adversary’s actions in $\mathcal{B}_5$. A straightforward verification shows that for all $t \in \mathbb{N}$,

- For each $p \in [0,1/4]$ and any scenario $\mathcal{P}$ of game $\mathcal{B}_5$, $P_{\mathcal{P},(p,\tilde{\nu})} = P_{\mathcal{P},(\varphi(p,\tilde{\nu}))} \otimes (\nu \otimes \delta_p)$, where $\nu$ is the unique distribution on $(\mathbb{N} \cup \{\infty\}) \times \{0,1\}$ such that, for all $k \in \mathbb{N} \cup \{\infty\}$, $j \in \{0,1\}$, $\nu \{((k,j))\} = \frac{1}{2}I\{k = 1, j = 0\} + \frac{1}{2}I\{1 < k < \infty\}$. Using again the Skorokhod representation, there exists $\psi_p : [0,1] \rightarrow (\mathbb{N} \cup \{\infty\}) \times \{0,1\} \times \{0,1\}$ such that $\nu \otimes \delta_p = (\mu_L)_{\psi_p}$.

- For each $p \in [1/4,1]$ and any scenario $\mathcal{P}$ of game $\mathcal{B}_5$, $P_{\mathcal{P},(p,\tilde{\nu})} = \delta_{(0,1,0,p)} = (\mu_L)_{\gamma_p}$, where $\gamma_p : [0,1] \rightarrow Z_5$, $\lambda \mapsto (0,1,0,p)$.

Thus, by the Simulation lemma with $I = [0,1/4]$ and $U = [1/4,1]$, the game $\mathcal{B}_6$ is easier than $\mathcal{B}_5$ (i.e., $R^*_{\mu}(\mathcal{F}_5) \geq R^*_{\mu}(\mathcal{B}_6)$).

Step 7. Finally, consider the game $\mathcal{B}_7 := \{(1, 2, 3), (1, 2), (0, 1), \rho_7, \varphi_7, \mathcal{P}_7\}$, where in matrix notation, $\rho_7 = \left[\rho(i, j)\right]_{i \in \{1,2,3\}, j \in \{1,2\}}$ and $\varphi_7 = \left[\varphi(i, j)\right]_{i \in \{1,2,3\}, j \in \{1,2\}}$ are given by

$$\rho_7 := \frac{1}{96} \begin{bmatrix} 1 & 2 & 3 \\ 34 & 45 & 37 \\ 38 & 44 & \end{bmatrix}, \quad \varphi_7 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and $\mathcal{P}_7$ is the set of all measures of the form $\otimes_{\epsilon \in \{1,2,3\}}(\epsilon \rho \otimes (1-\epsilon) \varphi)$, for $\epsilon \in [-1,1]$. Thus, using again the Embedding lemma, this time with $\mathcal{F} : [0,1] \rightarrow (1,2,3)$, $p \mapsto I\{p < \frac{1}{4}\} + 2I\{\frac{1}{4} \leq p \leq \frac{1}{2}\} + 3I\{\frac{1}{2} < p\}$, $g : [0,1] \rightarrow \{0,1\}, i \mapsto i$, and $\mathcal{H} : \otimes_{\epsilon \in \{1,2,3\}}(\epsilon \rho \otimes (1-\epsilon) \varphi) \mapsto \mu_L \otimes \mu_L$, we obtain that $\mathcal{B}_7$ is easier than $\mathcal{B}_6$ (i.e., $R^*_{\mu}(\mathcal{F}_6) \geq R^*_{\mu}(\mathcal{B}_7)$). This last game is an instance of the so-called revealing action partial monitoring game, whose minimax regret is known to be lower bounded by $\frac{\Omega}{n} T^{2/3}$ [Cesa-Bianchi et al., 2006]. In conclusion, we proved that $R^*_{\mu}(\mathcal{F}) \geq R^*_{\mu}(\mathcal{B}_7) \geq \frac{\Omega}{n} T^{2/3}$.

G Linear Lower Bound Under Realistic Feedback (bd)

In this section, we prove that in the realistic-feedback case, no strategy can achieve sublinear regret in the worst case if the valuations of the buyer and the seller may be dependent, not even if they have a bounded density.

The idea of the proof is to exploit the lack of observability in this setting, building a family of scenarios $\mathcal{P}^\lambda$ (parameterized by $\lambda \in [0,1]$) as convex combinations of the two measures in Fig. 3. If $\lambda < 1/2$, the optimal action is $3/8$, while if $\lambda > 1/2$, the optimal action becomes $5/8$. This family is built in such a way that the
feedback gives no information on λ, making it impossible to distinguish between the two cases. Leveraging the Embedding and Simulation lemmas (Lemmas 3 and 4), this construction leads to a reduction to an instance of a non-observable partial monitoring game, whose regret is trivially lower bounded by T/24.

**Theorem 11** (Theorem 5, restated). In the realistic-feedback stochastic (iid) setting with joint density bounded by a constant M ≥ 64/3 (bd), for all horizons T ∈ N, the minimax regret satisfies

\[ R^*_T \geq \frac{1}{24} T. \]

*Proof.* Fix any horizon T ∈ N and M ≥ 64/3. Recalling Appendix C.2, the realistic-feedback stochastic (iid) setting with joint density bounded by M (bd) is a game \( \mathcal{G} = (X, Y, Z, \rho, \mathcal{P}, \varphi) \), where \( X = [0,1], Y = [0,1]^2, Z = \{0,1\}^2, \rho = \text{GFT}, \varphi: (p, (s,b)) \mapsto \{(s \leq p), \{(p \leq b)\} \), and \( \mathcal{P} = \mathcal{P}_{bd} \). Define the two joint densities \( f = \sum_{i,j \in \{0,1\}} [\{s \leq \frac{1}{2}\} + (b-s)] \{s \leq \frac{1}{2}\} \{p \leq \frac{1}{2}\} \) and \( g: [0,1]^2 \rightarrow [0,M], (s,b) \mapsto f(1-b,1-s) \) (see Fig. 3, left). Let \( \mathcal{P}_1 \) be the subset of \( \mathcal{P}_{bd} \) whose elements have the form \( \mu_1 := \otimes_{i \in \mathbb{N}} \{((1-\lambda)f + \lambda g)(\mu_L \otimes H)\} \) for \( \lambda \in [0,1] \). Since \( \mathcal{P}_1 \subset \mathcal{P} \) the game \( \mathcal{G}_{1} := (X, Y, Z, \rho, \mathcal{P}_1) \) is easier than \( \mathcal{G} \) (i.e., \( R^*_T(\mathcal{G}) \geq R^*_T(\mathcal{G}_1) \)) by the Embedding lemma (Lemma 3) with \( \bar{I} \) and \( \varrho \) as the identities, and \( \mathcal{B} \) as the inclusion. Define \( \mathcal{Z}_1 := \{0\} \) and \( \varphi_1: X \times Y \rightarrow \mathcal{Z}_1, (p, (s,b)) \mapsto 0. \) Let \( (Y_1)_i \in \mathcal{B} \) be the adversary’s actions in \( \mathcal{G}_1 \). Now, since for all \( t \in \mathbb{N} \) any two scenarios \( \mathcal{P} \) and \( \mathcal{Q} \) of game \( \mathcal{G}_{1} \), and each \( p \in [0,1], \mathcal{P}_{\varphi_1(p,Y_1)} \) then by the well-known Skorokhod representation [Williams, 1991, Section 17.3], for each \( t \in \mathbb{N} \) and each \( p \in [0,1] \) there exists \( \mu_{p,t} : [0,1] \rightarrow [0,1]^2 \) such that for any scenario \( \mathcal{P} \) of game \( \mathcal{G}_{1}, \mathcal{P}_{\varphi_1(p,Y_1)} = \mu_{p,t} \). Thus, the Simulation lemma (Lemma 4) with \( I = \emptyset \) and \( \mathcal{U} = X \) implies that the game \( \mathcal{G}_2 := (X, Y, Z_{2,b}, \varrho_2, \mathcal{P}_1) \) is easier than \( \mathcal{G}_1 \) (i.e., \( R^*_T(\mathcal{G}_1) \geq R^*_T(\mathcal{G}_2) \)). Define \( \varphi_2: X \times Y \rightarrow [0,1], (p, (s,b)) \mapsto (b-s) \{s \leq \frac{1}{2}\} \{p \leq \frac{1}{2}\} \) and \( \mathcal{G}_3 := (X, Y, Z_2, \varrho_2, \mathcal{P}_1) \). By the Embedding lemma with \( \bar{I}, \varrho, \mathcal{B} \) as the identities, we have that the game \( \mathcal{G}_3 \) is easier than the game \( \mathcal{G}_2 \) (i.e., \( R^*_T(\mathcal{G}_2) \geq R^*_T(\mathcal{G}_3) \)). Finally, consider the game \( \mathcal{G}_4 := \{i \in \{1,2\}, \varphi_4, \varphi_4, \mathcal{P}_4\} \), where in matrix notation, \( \varphi_4 = \left[ \begin{array}{cccc} 1/2 & 1/4 & 1/2 & 1/3 \\ 1/3 & 1/4 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \) and \( \mathcal{P}_4 \) is the set of all measures of the form \( (1-\lambda) \delta_1 + \lambda \delta_2 \), for \( \lambda \in [0,1] \). Using again the Embedding lemma, this time with \( \bar{I}: [0,1] \rightarrow \{1,2\}, p \mapsto \{p \leq 1/2\} + 2\{1/2 < p\}, \varphi: \{0\} \rightarrow \{0\}, i \leftrightarrow i, \) and \( \mathcal{B}: \otimes_{i \in \mathbb{N}} ((1-\lambda) \delta_1 + \lambda \delta_2) \otimes \mu_L \mapsto \mu_L \otimes \mu_L, \) we obtain that \( \mathcal{G}_4 \) is easier than \( \mathcal{G}_3 \) (i.e., \( R^*_T(\mathcal{G}_3) \geq R^*_T(\mathcal{G}_4) \)). This last game has (trivially) minimax regret at most \( \left( \frac{1}{3} - \frac{1}{4} \right) \frac{T^2}{2} \). In conclusion, we proved that \( R^*_T(\mathcal{G}) \geq R^*_T(\mathcal{G}_4) \geq \frac{1}{24} T. \)

**H Linear Lower Bound Under Realistic Feedback (iv)**

In this section, we prove that in the realistic-feedback case, no strategy can achieve sublinear regret without any limitations on how concentrated the distributions of the valuations of the seller and buyer are, not even if they are independent of each other (iv).

The idea of the proof is that if the two distributions are very concentrated in a small region, finding an optimal price is like finding a needle in a haystack. Each strategy that (at each time step) receives as feedback only a finite number of bits, as in our realistic setting, can assign positive probability to at most a countable set of points. Thus one could find concentrated distributions of the buyer and seller that have a unique optimal point in which the strategy has zero probability of posting prices at all time steps, and such that all other prices suffer large regret.

**Theorem 12** (Theorem 6, restated). In the realistic-feedback stochastic (iid) setting with independent valuations (iv), for all horizons \( T \in \mathbb{N} \), the minimax regret satisfies

\[ R^*_T \geq \frac{1}{8} T. \]

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Proof. To lighten the notation, for any $n \in \mathbb{N}$ and a family $(\lambda_k)_{k \in \mathbb{N}}$, we let $\lambda_{1:n} := (\lambda_1, \ldots, \lambda_n)$. Fix an arbitrary horizon $T \in \mathbb{N}$. Recalling Appendix C.2, the realistic-feedback stochastic (iid) setting with independent valuations (iv) is a game $\mathcal{G} := (X, \mathcal{Y}, \mathcal{Z}, \rho, \varphi, \mathcal{P})$, where $X = [0,1]$, $\mathcal{Y} = [0,1]^2$, $\mathcal{Z} = [0,1]$, $\rho = \text{GFT}$, $\varphi : (p, (s, b)) \mapsto ((s \leq p), (p \leq b))$, and $\mathcal{P} = \mathcal{P}_\mathcal{Y}$. Let $\mathcal{E}$ be the set of scenarios of $\mathcal{G}$. Fix a strategy $\alpha$ for game $\mathcal{G}$ and let $\varepsilon \in (0,1)$. Define $\alpha_t := \alpha_t$, $v_t = (\mu_t)_{i|t}$, and for each $t \in \mathbb{N}$ and $z_1, \ldots, z_T \in \{0,1\}^T$, $\tilde{\alpha}_{t+1, z_{t+1}} : [0,1]^{t+1} \rightarrow [0,1]$, $u_{1:t+1} \mapsto \tilde{\alpha}_{t+1}(u_{1:t+1}, z_{t+1})$ and $v_{t+1, z_{t+1}} := (\otimes_{s=1}^{t+1} \mu_s)_{\tilde{\alpha}_{t+1, z_{t+1}}}$. Define also the set $A_1 := \{x \in [0,1] \mid v_1([x]) \geq 0\}$ and, for each $t \in \mathbb{N}$, the union $A_{t+1} := \bigcup_{z_t \in \{0,1\}} \{x \in [0,1] \mid v_{t+1, z_{t+1}}([x]) > 0\}$. Note that, for each $t \in \mathbb{N}, A_t$ is countable, being the union of $4^{t-1}$ countable sets. Then $A := \bigcup_{t \in \mathbb{N}} A_t$ is countable. Since $B := [\frac{1}{2}, \frac{1}{2}]$ has the power of continuum, we have that the same holds for $B \backslash A$. In particular, $B \backslash A$ is non-empty. Pick $x^* \in B \backslash A$ and define $\mu_S := \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{x^*}$, $\mu_B := \frac{1}{2}\delta_{-x^*} + \frac{1}{2}\delta_1$, and $\mathbb{P} := (\otimes_{t \in \mathbb{N}} (\mu_S \otimes \mu_B)) \otimes \mu_L \in \mathcal{E}$. Then for each $t \in \mathbb{N}$, we have that

$$E_{\mathbb{P}}[\rho(x^*, Y_t)] = \frac{x^* + (1 - x^*) + 1}{4}.$$ On the other hand, $\mathbb{P}[X_t = x^*] = v_1([x^*]) = 0$ and for each $t \in \mathbb{N}$, we have that

$$\mathbb{P}[X_{t+1} = x^*] = \mathbb{P}[\alpha_{t+1}(U_1, \ldots, U_{t+1}, Z_1, \ldots, Z_t) = x^*] = \sum_{z_1, \ldots, z_t \in \{0,1\}} \mathbb{P}[\alpha_{t+1}(U_1, \ldots, U_{t+1}, Z_1, \ldots, Z_t) = x^* \cap Z_1 = z_1 \cap \cdots \cap Z_t = z_t] \leq \sum_{z_1, \ldots, z_t \in \{0,1\}} \mathbb{P}[\alpha_{t+1}(U_1, \ldots, U_{t+1}, Z_1, \ldots, Z_t) = x^*] = \sum_{z_1, \ldots, z_t \in \{0,1\}} v_{t+1, z_1, \ldots, z_t}([x^*]) = 0,$$

which in turn gives

$$E_{\mathbb{P}}[\rho(X_t, Y_t)] = \frac{x^* \mathbb{P}_{x^*}[[0,x^*]] + (1 - x^*) \mathbb{P}_{x^*}[[x^*,1]] + 1}{4} = \frac{x^* \mathbb{P}_{x^*}[[0,x^*]] + (1 - x^*) \mathbb{P}_{x^*}[[x^*,1]] + 1}{4} \leq \frac{\max(x^*, 1 - x^*) + 1}{4} = \frac{x^* + (1 - x^*) + 1 - \min(x^*, 1 - x^*)}{4}.$$ So, if $T \in \mathbb{N}$ we get

$$R_T^\mathbb{P}(\alpha) = \mathbb{E}_{\mathbb{P}} \left[ \sum_{t=1}^{T} \rho(x^*, Y_t) - \sum_{t=1}^{T} \rho(X_t, Y_t) \right] \geq \frac{\min(x^*, 1 - x^*) T}{4} \geq \frac{1 - \varepsilon}{8} T.$$ Since $\varepsilon$ was arbitrary, we get, for all $T \in \mathbb{N}$, $R_T^\mathbb{P}(\alpha) = \sup_{\mathbb{P} \in \mathcal{E}} R_T^\mathbb{P}(\alpha) \geq \sup_{\varepsilon \in (0,1)} \frac{1 - \varepsilon}{8} T = T/s$. Since $\alpha$ was arbitrary we get, for each $T \in \mathbb{N}$, $R_T^* = \inf_{\alpha \in \mathcal{E}} R_T^\mathbb{P}(\alpha) \geq T/s$. $\square$

I Adversarial Setting: Linear Lower Bound Under Full Feedback

In this section, we give a more detailed proof of Theorem 7 with a notation consistent with our abstract setting of sequential games.

Theorem 13 (Theorem 7, restated). In the full-feedback adversarial (adv) setting, for all horizons $T \in \mathbb{N}$, we have

$$R_T^* \geq \frac{1}{4} T.$$
Proof. Recalling Appendix C.2, the full-feedback adversarial (adv) bilateral trade setting is a game \( \mathcal{G} := (X, \mathcal{Y}, \mathcal{Z}, \rho, \phi, \mathcal{P}) \), where \( X = [0, 1], \mathcal{Y} = [0, 1]^2, \mathcal{Z} = [0, 1]^2, \rho = \text{GFT}, \phi : (p, (s, b)) \mapsto (s, b) \), and \( \mathcal{P} = \mathcal{P}_{\text{adv}} \).

Let \( \mathcal{S} \) be the set of scenarios of \( \mathcal{G} \). Fix a strategy \( \hat{a} \in \mathcal{A} \) and an \( \epsilon \in (0, 1/18) \). Define \( \tilde{a}_1 := a_1, v_1 := (\mu_L)_{\tilde{a}_1}, \) and

\[
\begin{cases}
  c_1 := \frac{1}{2} - \frac{3}{2} \epsilon, & d_1 := \frac{1}{2} - \frac{1}{2} \epsilon, & s_1 := 0, & b_1 := 1, & \text{if } v_1 \left[ \left[ 0, \frac{1}{2} - \frac{1}{2} \epsilon \right] \right] \leq \frac{1}{2}, \\
  c_1 := \frac{1}{2} + \frac{3}{2} \epsilon, & d_1 := \frac{1}{2} + \frac{1}{2} \epsilon, & s_1 := 1, & b_1 := 1, & \text{otherwise}.
\end{cases}
\]

If \( t \in \mathbb{N} \), suppose we defined \( \tilde{a}_t, v_t, c_t, d_t, s_t, b_t \) and let

\[
(a_{t+1}, [0, 1]^t) \rightarrow [0, 1], (u_1, \ldots, u_{t+1}) \rightarrow a_{t+1}(u_1, \ldots, u_{t+1}, (s_1, b_1), \ldots, (s_t, b_t)),
\]

\[
v_{t+1} := (\Phi_{t+1})_{\tilde{a}_{t+1}}, \text{ and}
\]

\[
\begin{cases}
  c_{t+1} := c_t, & d_{t+1} := d_t - \frac{2}{\epsilon}, \quad s_{t+1} := 0, & b_{t+1} := d_{t+1}, & \text{if } v_{t+1} \left[ \left[ 0, c_t + \frac{2}{\epsilon} \right] \right] \leq \frac{1}{2}, \\
  c_{t+1} := c_t + \frac{2}{\epsilon}, & d_{t+1} := d_t, \quad s_{t+1} := c_{t+1}, & b_{t+1} := 1, & \text{otherwise}.
\end{cases}
\]

Then \( (a_t)_{t \in \mathbb{N}}, (v_t)_{t \in \mathbb{N}}, (c_t)_{t \in \mathbb{N}}, (d_t)_{t \in \mathbb{N}}, (s_t, b_t)_{t \in \mathbb{N}} \) are well-defined by induction and satisfy:

- For each \( t \in \mathbb{N} \), \( d_t - c_t = \frac{2^t}{\epsilon} \).
- For each \( t \in \mathbb{N} \), \( c_1 \leq c_2 \leq \cdots \leq c_t \leq d_t \leq \cdots \leq d_3 \leq d_2 \leq d_1 \).
- \( \exists x^* \in \bigcap_{t=1}^{\infty} \{c_t, d_t\} \).
- For each \( t \in \mathbb{N} \), \( \rho(x^*, (s_t, b_t)) = b_t - s_t \geq \frac{1 - 3 \epsilon}{2} \).
- For each \( t \in \mathbb{N} \), \( \mathbb{P}[a_t(U_1, \ldots, U_t, (s_1, b_1), \ldots, (s_t-1, b_{t-1})) \in [s_t, b_t]] \leq \frac{1}{2} \).

Now, define \( \mathbb{P} := (\Phi_{t+1})_{\delta(s_t, b_t)} \otimes \mu_L \in \mathcal{S} \). Then, for each \( t \in \mathbb{N} \),

\[
\mathbb{E}^\mathbb{P}[\rho(X_t, Y_t)] = \mathbb{E}^\mathbb{P}[\rho \left( a_t(U_1, \ldots, U_t, (s_1, b_1), \ldots, (s_{t-1}, b_{t-1})), (s_t, b_t) \right)] 
\leq \left( \frac{1}{2} + \frac{3 \epsilon}{2} \right) \mathbb{P}[a_t(U_1, \ldots, U_t, (s_1, b_1), \ldots, (s_{t-1}, b_{t-1})) \in [s_t, b_t]] \leq \frac{1}{4} + \frac{3 \epsilon}{4}, 
\]

and so, for each \( T \in \mathbb{N} \)

\[
R_T^\mathbb{P}(\alpha) = \mathbb{E}^\mathbb{P} \left[ \sum_{t=1}^{T} \rho(x^*, Y_t) - \sum_{t=1}^{T} \rho(X_t, Y_t) \right] = \sum_{t=1}^{T} \rho(x^*, (s_t, b_t)) - \sum_{t=1}^{T} \mathbb{E}^\mathbb{P}[\rho(X_t, Y_t)] 
\geq \sum_{t=1}^{T} (b_t - s_t) \left( 1 - \mathbb{P}[a_t(U_1, \ldots, U_t, (s_1, b_1), \ldots, (s_{t-1}, b_{t-1})) \in [s_t, b_t]] \right) \geq \frac{1 - 3 \epsilon T}{4}.
\]

Since \( \epsilon \) was arbitrary, we get, for all \( T \in \mathbb{N} \), \( R_T^\mathbb{P}(\alpha) = \sup_{\mathbb{P} \in \mathcal{D}} R_T^\mathbb{P}(\alpha) \geq \sup_{\epsilon \in (0, 1/18)} \frac{1 - 3 \epsilon T}{4} = \frac{T}{4} \). Since \( \alpha \) arbitrariness, we get, for each \( T \in \mathbb{N} \), \( R_T^\alpha = \inf_{\mathbb{P} \in \mathcal{D}} R_T^\mathbb{P}(\alpha) \geq \frac{T}{4} \).

\[\square\]

### J  DKW Inequalities

We begin this section by presenting the univariate DKW inequality as proved in [Massart, 1990].

**Theorem 14.** If \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space and \((X_n)_{n \in \mathbb{N}}\) is a \( \mathbb{P} \)-i.i.d. sequence of random variables, then, for any \( \epsilon > 0 \) and all \( m \in \mathbb{N} \), it holds

\[
\mathbb{P} \left[ \sup_{x \in \mathbb{R}} \frac{1}{m} \sum_{k=1}^{m} \mathbb{I}[X_k \leq x] - \mathbb{P}[X_1 \leq x] \right] > \epsilon \right] \leq 2 \exp \left( -2m \epsilon^2 \right).
\]

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We now present a bivariate DKW inequality which can be proved by applying the VC-type bound of [Anthony and Bartlett, 2009, Theorem 4.9; see also Lemmas 4.4, 4.5, and 4.11 for the explicit constants].

**Theorem 15.** There exist positive constants $m_0 \leq 1200$, $c_1 \leq 13448$, $c_2 \geq 1/576$ such that, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(X_n, Y_n)_{n \in \mathbb{N}}$ is a $\mathbb{P}$-i.i.d. sequence of two-dimensional random vectors, then, for any $\epsilon > 0$ and all $m \in \mathbb{N}$ such that $m \geq m_0/\epsilon^2$, it holds

$$
\mathbb{P}\left[ \sup_{x, y \in \mathbb{R}} \frac{1}{m} \sum_{k=1}^{m} I[X_k \leq x, Y_k \leq y] - \mathbb{P}[X_1 \leq x, Y_1 \leq y] > \epsilon \right] \leq c_1 \exp(-c_2m\epsilon^2).
$$

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