Rotundity and monotonicity properties of selected Cesàro function spaces

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Abstract We study rotundity, strict monotonicity, lower local uniform monotonicity and upper local uniform monotonicity in some classes of Cesàro function spaces. We present full criteria of these properties in the Cesàro–Orlicz function spaces $Ces_\phi$ (under some mild assumptions on the Orlicz function $\phi$). Next, we prove a characterization of strict monotonicity, lower local uniform monotonicity and upper local uniform monotonicity in the Cesàro–Lorentz function spaces $CA_\phi$. We also show that the space $CA_\phi$ is never rotund. Finally, we will prove that Cesàro–Marcinkiewicz function space $CM_\phi^{(\ast)}$ is neither strictly monotone nor order continuous for any quasi-concave function $\phi$.

Keywords Cesàro function space · Cesàro–Orlicz function space · Cesàro–Lorentz function space · Cesàro–Marcinkiewicz function space · Rotundity · Monotonicity properties · Order continuity

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1 Introduction

The structure of different types of spaces defined by a Cesàro operator has been intensively investigated during the last decades from the isomorphic as well as isometric point of view. The classical Cesàro sequence $ces_p$ and function $Ces_p$ spaces have been studied by many authors (see [1,2] also for further references). It is worth to mention that some properties are fulfilled in the sequence case and are not in function case. Furthermore, sometimes the cases $Ces_p[0, 1]$ and $Ces_p[0, \infty)$ are essentially different (see an isomorphic description of the Köthe dual of Cesàro spaces in [1,34], see also [23] for the respective isometric description). The spaces generated by the Cesàro operator (including abstract Cesàro spaces) have been considered by Curbera, Delgado, Soria, Ricker, Leśnik and Maligranda in several papers (see [13–16,34–36]).

The Cesàro–Orlicz sequence spaces denoted by $ces_\varphi$ are generalization of the Cesàro sequence spaces $ces_p$. The spaces $ces_\varphi$ have been studied intensively (see [10–12,22]). We are going to discuss several properties of some Cesàro function spaces $CX$. The Cesàro–Orlicz function spaces $Ces_\varphi$ have been studied in [26,27] where the authors, among others, prove the criteria for the existence of order isomorphic (order isometrically isometric) copy of $l^\infty$ in these spaces. We prove a characterization of strict monotonicity, lower local uniform monotonicity, upper local uniform monotonicity and rotundity in the spaces $Ces_\varphi$. We admit the largest possible class of Orlicz functions giving the maximal generality of spaces under consideration. Finally, we study rotundity, monotonicity properties and order continuity of Cesàro–Lorentz function spaces $C_{\Lambda_\varphi}$ and Cesàro–Marcinkiewicz function spaces $CM_{\varphi}^{(*)}$.

2 Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ be the sets of real, nonnegative real and natural numbers, respectively. Denote by $\mu$ the Lebesgue measure on $I$ and by $L^0 = L^0(I)$ the space of all classes of real-valued Lebesgue measurable functions defined on $I$, where $I = [0, 1]$ or $I = [0, \infty)$.

A Banach lattice $X = (X, \| \cdot \|)$ is said to be a Banach ideal space on $I$ if $X$ is a linear subspace of $L^0(I)$ which satisfies the ideal property: if $g \in X$, $f \in L^0$ and $|f| \leq |g|$ a.e. on $I$ then $f \in X$ and $\|f\| \leq \|g\|$.

Unless it is stated otherwise then we understand that a Banach ideal space has the so-called weak unit that is there is an element $f \in X$ that is positive on whole $I$. Sometimes we write $\|\cdot\|_X$ to be sure in which space the norm has been taken. By $X_+$ we denote the positive cone of $X$, that is, $X_+ = \{x \in X : x \geq 0\}$.

For two Banach ideal spaces $X$ and $Y$ on $I$ the symbol $X \hookrightarrow Y$ means that the embedding $X \subset Y$ is continuous, i.e., there exists constant a $C > 0$ such that $\|x\|_Y \leq C \|x\|_X$ for all $x \in X$. Moreover, $X = Y$ means that the spaces are the same as the sets and the norms are equivalent.

A Banach ideal space $X$ is called order continuous ($X \in (OC)$ shortly) if every element of $X$ is order continuous, that is, for each $\bar{f} \in X$ and for each sequence $(f_n) \subset X$ satisfying $0 \leq f_n \leq |f|$ and $f_n \to 0$ a.e. on $I$, we have $\|f_n\| \to 0$. By $X_\alpha$
we denote the subspace of all order continuous elements of $X$. It is worth to notice that $x \in X_a$ if and only if $\|x_{X_{An}}\| \to 0$ for any decreasing sequence of Lebesgue measurable sets $A_n \subset I$ with empty intersection (see [4, Proposition 3.5, p. 15]).

A Banach lattice $X$ is strictly monotone ($X \in (SM)$ for short) if for any $x, y \in X_+$ such that $x \leq y$ and $y \neq x$, we have $\|x\| < \|y\|$, see [5,21]. Moreover, we say that $X$ is upper (lower) locally uniformly monotone, writing shortly $X \in (ULUM)$ ($X \in (LLUM)$), if $\|x_n - x\| \to 0$ for any $x \in X_+$ and any sequence $(x_n) \subset X$ such that $x \leq x_n$ ($0 \leq x_n \leq x$) and $\|x_n\| \to \|x\|$.

We say a normed space $X$ is rotund ($X \in (R)$ for short) if $\|x + y\| < 2$ whenever $x$ and $y$ are distinct points on the unit sphere of $X$. It is well known that rotundity (strict monotonicity) is a useful tool in the theory of Banach spaces (Banach lattices), e.g. in the best approximation problems in Banach spaces (in the best dominated approximation problems in Banach lattices), see [6] for a local approach and further references. Moreover, rotundity and strict monotonicity are closely related in Banach lattices (see [21]). Note that a Banach space (Banach lattice) is rotund (strictly monotone) if and only if the unit sphere contains no nontrivial line segment (nontrivial order interval).

The continuous Cesàro operator $C : L^0(I) \to L^0(I)$ is defined by

$$ Cf(x) = \frac{1}{x} \int_0^x f(t) dt, $$

for $0 < x \in I$. For a Banach ideal space $X$ on $I$ we define an abstract Cesàro space $CX = CX(I)$ by

$$ CX = \{ f \in L^0(I) : C|f| \in X \} $$

with the norm $\|f\|_{CX} = \|C|f|\|_X$ (see [34–36]). We always assume that $CX \neq \{0\}$. If $C : X \to X$ is bounded and $X \neq \{0\}$ then $CX \neq \{0\}$. The reader is referred to [32] for more informations about the boundedness of the operator $C$.

**Remark 1** Let $X$ be a Banach ideal space on $I$. If $X \in (ULUM)$ then $CX \in (ULUM)$. The same implication is true if we replace the property $ULUM$ by $LLUM$ (or by $SM$ property). The proof follows just from the definition (see also Fact in [26]).

### 3 The Cesàro–Orlicz function spaces $Ces_\varphi$

A function $\varphi : [0, \infty) \to [0, \infty]$ is called an Orlicz function if:

(i) $\varphi$ is convex,
(ii) $\varphi(0) = 0$,
(iii) $\varphi$ is neither identically equal to zero nor infinity on $(0, \infty)$,
(iv) $\varphi$ is left continuous on $(0, \infty)$, i.e., $\lim_{u \to b^-_\varphi} \varphi(u) = \varphi(b_\varphi)$ if $b_\varphi < \infty$, where

$$ b_\varphi = \sup\{u > 0 : \varphi(u) < \infty\}. $$

If we denote

$$ a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\}, $$

then $a_\varphi$ and $b_\varphi$ are finite.

If $\varphi$ satisfies the growth conditions $\varphi(u) = O(u^p)$ and $\varphi(u) = O(\log(u))$ for $u \to \infty$ then $a_\varphi = 0$. If $\varphi$ satisfies the growth conditions $\varphi(u) = O(u^p)$ and $\varphi(u) = O(u^{p-1})$ for $u \to \infty$ then $b_\varphi = \infty$.

**Remark 2** Let $\varphi : [0, \infty) \to [0, \infty]$ be an Orlicz function. Then $\varphi$ is convex, $a_\varphi \leq b_\varphi$, $\varphi(0) = 0$, and $\varphi$ is left continuous on $(0, \infty)$.

**Proposition 3** Let $\varphi : [0, \infty) \to [0, \infty]$ be an Orlicz function and $X = CX$ be the abstract Cesàro space defined above. Then $X$ is a Banach space if and only if $\varphi(a_\varphi) / \varphi(b_\varphi) \leq 1$.

**Proof**

If $\varphi(a_\varphi) / \varphi(b_\varphi) \leq 1$ then $X$ is a Banach space by the Orlicz–Mazurkiewicz–Steinhaus theorem. Conversely, suppose that $X$ is a Banach space. Then there is a constant $C > 0$ such that $\|C|f|\|_X \leq C \|f\|_X$ for all $f \in L^0(I)$.

If $\varphi(a_\varphi) / \varphi(b_\varphi) > 1$ then $\varphi(a_\varphi) / \varphi(b_\varphi) \leq 1$.

**Remark 4** Let $\varphi : [0, \infty) \to [0, \infty]$ be an Orlicz function. Then $\varphi$ is convex, $a_\varphi \leq b_\varphi$, $\varphi(0) = 0$, and $\varphi$ is left continuous on $(0, \infty)$.

**Proposition 5** Let $\varphi : [0, \infty) \to [0, \infty]$ be an Orlicz function and $X = CX$ be the abstract Cesàro space defined above. Then $X$ is a Banach space if and only if $\varphi(a_\varphi) / \varphi(b_\varphi) \leq 1$.

**Proof**

If $\varphi(a_\varphi) / \varphi(b_\varphi) \leq 1$ then $X$ is a Banach space by the Orlicz–Mazurkiewicz–Steinhaus theorem. Conversely, suppose that $X$ is a Banach space. Then there is a constant $C > 0$ such that $\|C|f|\|_X \leq C \|f\|_X$ for all $f \in L^0(I)$.

If $\varphi(a_\varphi) / \varphi(b_\varphi) > 1$ then $\varphi(a_\varphi) / \varphi(b_\varphi) \leq 1$.

**Remark 6** Let $\varphi : [0, \infty) \to [0, \infty]$ be an Orlicz function. Then $\varphi$ is convex, $a_\varphi \leq b_\varphi$, $\varphi(0) = 0$, and $\varphi$ is left continuous on $(0, \infty)$.

**Proposition 7** Let $\varphi : [0, \infty) \to [0, \infty]$ be an Orlicz function and $X = CX$ be the abstract Cesàro space defined above. Then $X$ is a Banach space if and only if $\varphi(a_\varphi) / \varphi(b_\varphi) \leq 1$.

**Proof**

If $\varphi(a_\varphi) / \varphi(b_\varphi) \leq 1$ then $X$ is a Banach space by the Orlicz–Mazurkiewicz–Steinhaus theorem. Conversely, suppose that $X$ is a Banach space. Then there is a constant $C > 0$ such that $\|C|f|\|_X \leq C \|f\|_X$ for all $f \in L^0(I)$.

If $\varphi(a_\varphi) / \varphi(b_\varphi) > 1$ then $\varphi(a_\varphi) / \varphi(b_\varphi) \leq 1$.
then \(0 \leq a_\varphi \leq b_\varphi \leq \infty\). Moreover, \(a_\varphi < \infty\) and \(b_\varphi > 0\), since an Orlicz function is neither identically equal to zero nor infinity on \((0, \infty)\). The function \(\varphi\) is continuous and non-decreasing on \([0, b_\varphi)\) and is strictly increasing on \([a_\varphi, b_\varphi)\). We use notations \(\varphi > 0\), \(\varphi < \infty\) when \(a_\varphi = 0\), \(b_\varphi = \infty\), respectively.

We say an Orlicz function \(\varphi\) satisfies the condition \(\Delta_2\) for large arguments \((\varphi \in \Delta_2(\infty)\) for short\) if there exists \(K > 0\) and \(u_0 > 0\) such that \(\varphi(u_0) < \infty\) and

\[
\varphi(2u) \leq K \varphi(u)
\]

for all \(u \in [u_0, \infty)\). If \(\varphi(2u) \leq K \varphi(u)\) for all \(u \geq 0\) then we say that \(\varphi\) satisfies the condition \(\Delta_2\) for all arguments \((\varphi \in \Delta_2(\mathbb{R}_+))\). These conditions play a crucial role in the theory of Orlicz spaces, see \([8, 31, 38]\) and \([41]\). We will write \(\varphi > 0\), neither identically equal to zero nor infinity on

\[
\varphi > 0,
\]

neither identically equal to zero nor infinity on \((\mathbb{R}_+, \varnothing)\) for large arguments \((\varphi \in \Delta_2(\mathbb{R}_+))\) if \(I = [0, \infty)\).

Recall that an Orlicz function \(\varphi\) is strictly convex whenever \(\varphi \left(\frac{u+v}{2}\right) < \frac{1}{2} [\varphi(u) + \varphi(v)]\) for all \(u \neq v\) and \(u, v \geq 0\).

The Orlicz function space \(L^\varphi = L^\varphi(I)\) generated by an Orlicz function \(\varphi\) is defined by

\[
L^\varphi = \{ f \in L^0(I) : I_\varphi(f/\lambda) < \infty \text{ for some } \lambda = \lambda(f) > 0 \},
\]

where \(I_\varphi(f) = \int_I \varphi(|f(t)|)dt\) is a convex modular (for the theory of Orlicz spaces and modular spaces see \([38, 41]\)). The space \(L^\varphi\) is a Banach ideal space with the Luxemburg–Nakano norm

\[
\|f\|_\varphi = \inf \{ \lambda > 0 : I_\varphi(f/\lambda) \leq 1 \}.
\]

It is well known that \(\|f\|_\varphi \leq 1\) if and only if \(I_\varphi(f) \leq 1\).

The Cesàro–Orlicz function space \(Ces_\varphi = Ces_\varphi(I)\) is defined by \(Ces_\varphi(I) = CL^\varphi(I)\). Consequently, the norm in the space \(Ces_\varphi\) is given by the formula

\[
\|f\|_{Ces_\varphi} = \inf \{ \lambda > 0 : \rho_\varphi(f/\lambda) \leq 1 \},
\]

where \(\rho_\varphi(f) = I_\varphi(C|f|)\) is a convex modular.

Let discuss in details when \(Ces_\varphi \neq \{0\}\). If \(I = [0, \infty)\) then \(Ces_\varphi[0, \infty) \neq \{0\}\) if and only if for each \(\lambda > 0\) there exist \(\gamma_0 \in [0, \infty)\) with \(\int_{\gamma_0}^\infty \varphi(\lambda/t)dt < \infty\) (see Proposition 3 from \([26]\), cf. Theorem 1 (a) in \([34]\)). However, \(Ces_\varphi[0, 1] \neq \{0\}\) for any Orlicz function \(\varphi\). Indeed, \(L^\varphi[0, 1]\) is symmetric and \(Ces_\varphi[0, 1] \neq \{0\}\) if and only if \(\chi_{[a, 1]} \in L^\varphi[0, 1]\) for some \(0 < a < 1\) (see Theorem 1 (b) in \([34]\), cf. Remark 12).

Note that if \(0 < a_\varphi = b_\varphi\) then \(L^\varphi = L^\infty\) and \(\|x\|_\varphi = \frac{1}{b_\varphi}\|x\|_\infty\), see Example 1 in \([38, \text{p.} \ 98]\). Consequently, \(Ces_\varphi = Ces_\infty\) in that case (see \([1, 34]\)). Therefore we can assume that if \(b_\varphi < \infty\), then \(a_\varphi < b_\varphi\).

The following Lemma is formulated for any Banach ideal space. We will apply it later in the case \(X = L^\varphi\).

**Lemma 2** Let \(X\) be a Banach ideal space on \(I\). If \(X\) is rotund then \(CX\) is rotund.
Proof Suppose $X$ is rotund. It is well known that $X \in (R)$ is equivalent to $X_+ \in (R)$ (see [21, Theorem 2]). Take $f, g \in CX$, $f, g \geq 0$, $\|f\|_{CX} = \|g\|_{CX} = 1$ and $\|(f + g)/2\|_{CX} = 1$. We will show that $f = g$. Indeed,

$$\frac{\|f + g\|}{2}_{CX} = \left\| \frac{1}{2} C_f + \frac{1}{2} C_g \right\|_{X} = 1,$$

therefore $Cf = Cg$ by our assumption and $(f - g)$ is identically 0 on $I$, i.e.,

$$\int_0^x (f - g)(t) \, dt = 0, \quad (3.1)$$

for all $0 < x \in I$. It is well known (by the Fundamental Theorem of Calculus) that if $\lambda \in L^1[a, b]$, $r \in \mathbb{R}$ and we define the function $\Lambda : [a, b] \to \mathbb{R}$ as $\Lambda(x) = r + \int_a^x \lambda(t) \, dt$ then $\frac{d}{dx} \Lambda = \lambda$ a.e. on $[a, b]$. Therefore $f - g = 0$ a.e. and so $CX \in (R)$. However, we can also use a direct and easy argument. Indeed, put $h = f - g$ and suppose for the contrary that $h \neq 0$. We can assume that $h(t) > 0$ for all $0 < t \in A \subset I$ where $m(A) > 0$. There is a closed set $B \subset A$ with $m(B) > 0$. We have

$$\int_I h \, d\mu = \int_B h \, d\mu + \int_{I \setminus B} h \, d\mu = 0,$$

and

$$\int_{I \setminus B} h \, d\mu = - \int_B h \, d\mu < 0.$$

Since the set $I \setminus B$ is open, so $I \setminus B = \bigcup_{n=1}^{\infty} (a_n, b_n)$, where $(a_n, b_n) \cap (a_m, b_m) = \emptyset$ for $n \neq m$. We have

$$\int_{I \setminus B} h \, d\mu = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} h \, d\mu < 0,$$

whence there exists $n_0 \in \mathbb{N}$ with $\int_{a_{n_0}}^{b_{n_0}} h \, d\mu \neq 0$. But

$$\int_0^{b_{n_0}} h \, d\mu - \int_0^{a_{n_0}} h \, d\mu = \int_{a_{n_0}}^{b_{n_0}} h \, d\mu < 0.$$

Then either $\int_0^{b_{n_0}} h \, d\mu \neq 0$ or $\int_0^{a_{n_0}} h \, d\mu \neq 0$ and this contradicts (3.1). Therefore $f = g$ and the proof is finished.

\[\square\]

Theorem 3 Suppose $\varphi \in \Delta_2$. Then the Cesàro–Orlicz function space $Ces_\varphi$ is rotund if and only if $\varphi$ is strictly convex.

Proof ($\Leftarrow$). If $\varphi$ is strictly convex and $\varphi \in \Delta_2$ then $L^\varphi$ is rotund (see [8]) and $Ces_\varphi$ is rotund by Lemma 2.
We have to consider two cases.

I. Suppose \( I = [0, 1] \), \( \varphi \in \Delta_2(\infty) \) and \( \varphi \) is not strictly convex. Then there is an interval \((a, b), a, b \in [0, \infty)\) on which \( \varphi \) is affine, i.e., \( \varphi(t) = \alpha t + \beta \) for some \( \alpha, \beta \in \mathbb{R} \) and all \( t \in (a, b) \). Note that the case \( a_\varphi \geq 0 \) is included in the below proof and is even much simpler. Take \( \tilde{a}, \tilde{b} \in (a, b) \), \( \tilde{a} < \tilde{b} \) such that

\[
\frac{\tilde{a} + \tilde{b}}{2} = \frac{a + b}{2} \quad \text{and} \quad \tilde{b} - \tilde{a} \leq \frac{2(a + b)}{1 + 4\pi} \tag{3.2}
\]

Let us define functions

\[
F_n(t) = \frac{\tilde{b} - \tilde{a}}{4} \sin(4\pi nt) + \frac{a + b}{2},
\]

\[
G_n(t) = \frac{\tilde{b} - \tilde{a}}{4} \sin(8\pi nt) + \frac{a + b}{2},
\]

for \( n \in \mathbb{N} \) and \( t \in [0, 1] \). Since \( F_n \) and \( G_n \) are absolutely continuous functions for all \( n \in \mathbb{N} \) and product of absolutely continuous functions is also absolutely continuous function, we can find elements \( f_n, g_n \in L^1[0, 1] \) with

\[
tF_n(t) = \int_0^t f_n(s)ds \quad \text{and} \quad tG_n(t) = \int_0^t g_n(s)ds. \tag{3.3}
\]

For all \( n \in \mathbb{N} \) we have:

(i) \( C(f_n)(x) = F_n(x) \) and \( C(g_n)(x) = G_n(x) \) for all \( 0 < x < I \),

(ii) \( F_n \left( \frac{1}{2n} \right) = G_n \left( \frac{1}{2n} \right) = \frac{a + b}{2} \),

(iii) \( \rho_\varphi(f_n\chi(0,1/2n)) = \rho_\varphi(g_n\chi(0,1/2n)) \).

Properties (i) and (ii) follow from the definition of functions \( F_n \) and \( G_n \) and from equalities (3.3).

Now we prove (iii). We have

\[
C \left( f_n\chi \left( 0, \frac{1}{2n} \right) \right) (t) = F_n(t)\chi \left( 0, \frac{1}{2n} \right)(t) + \frac{1}{t}\chi \left( \frac{1}{2n}, 1 \right)(t) \int_0^{1/2n} f_n(t)dt,
\]

\[
C \left( g_n\chi \left( 0, \frac{1}{2n} \right) \right) (t) = G_n(t)\chi \left( 0, \frac{1}{2n} \right)(t) + \frac{1}{t}\chi \left( \frac{1}{2n}, 1 \right)(t) \int_0^{1/2n} g_n(t)dt.
\]

We claim that functions \( tF_n(t) \) and \( tG_n(t) \) are nondecreasing on the interval \( \left( 0, \frac{1}{2n} \right) \).

We have

\[
\frac{d}{dt}[tG_n(t)] = \frac{\tilde{b} - \tilde{a}}{4} \sin(8\pi nt) + \frac{a + b}{2} + \frac{\tilde{b} - \tilde{a}}{4} 8\pi n \cos(8\pi nt) \\
\geq \frac{\tilde{b} - \tilde{a}}{4} (-1) + \frac{a + b}{2} + \frac{\tilde{b} - \tilde{a}}{2n} 8\pi n (-1) \geq 0,
\]
where the last inequality follows from condition (3.2). The case of the function \( t F_n (t) \) is analogous. This proves the claim and consequently functions \( f_n \) and \( g_n \) are nonnegative on the interval \((0, \frac{1}{2n})\). Thus

\[
\rho_\varphi \left( f_n \chi (0, 1/2n) \right) = I \varphi \left( C \left( \left| f_n \chi (0, \frac{1}{2n}) \right| \right) \right) = I \varphi \left( C \left( f_n \chi (0, \frac{1}{2n}) \right) \right) \\
= \int_0^{1/2n} \varphi \left( F_n (t) \right) dt + \int_{1/2n}^1 \varphi \left( \frac{1}{t} \int_0^{1/2n} f_n (s) ds \right) dt,
\]

\[
\rho_\varphi \left( g_n \chi (0, 1/2n) \right) = I \varphi \left( C \left( \left| g_n \chi (0, \frac{1}{2n}) \right| \right) \right) = I \varphi \left( \left| g_n \chi (0, \frac{1}{2n}) \right| \right) \\
= \int_0^{1/2n} \varphi \left( G_n (t) \right) dt + \int_{1/2n}^1 \varphi \left( \frac{1}{t} \int_0^{1/2n} g_n (s) ds \right) dt. \tag{3.4}
\]

Since \( F_n (t) \subset (a, b) \) and \( G_n (t) \subset (a, b) \), we have

\[
\int_0^{1/2n} \varphi \left( F_n (t) \right) dt = \alpha \int_0^{1/2n} F_n (t) dt + \frac{\beta}{2n}, \\
\int_0^{1/2n} \varphi \left( G_n (t) \right) dt = \alpha \int_0^{1/2n} G_n (t) dt + \frac{\beta}{2n}. \tag{3.5}
\]

Applying (ii), (3.4) and (3.5) it is enough to prove that

\[
\int_0^{1/2n} F_n (t) dt = \int_0^{1/2n} G_n (t) dt.
\]

Observe that

\[
2 \int_0^{1/4n} G_n (t) dt = \int_0^{1/2n} G_n (t/2) dt = \int_0^{1/2n} F_n (t) dt.
\]

Consequently,

\[
\int_0^{1/2n} G_n (t) dt = \int_0^{1/4n} G_n (t) dt + \int_0^{1/2n} G_n (t) dt \\
= \int_0^{1/4n} G_n (t) dt + \int_0^{1/4n} G_n (t) dt = \int_0^{1/2n} F_n (t) dt
\]

and the proof of condition (iii) is finished.

Note that \( \rho_\varphi \left( f_n \chi (0, 1/2n) \right) \to 0 \) as \( n \to \infty \). Indeed, from condition (ii) and the proof of part (iii) we have
\[ \rho_{\varphi}(f_n \chi(0,1/2n)) = \alpha \int_0^{1/2n} F_n(t) dt + \frac{\beta}{2n} + \int_0^{1/2n} \varphi \left( \frac{1}{t} \int_0^{1/2n} f_n(s) ds \right) dt \]
\[ = \alpha \int_0^{1/2n} F_n(t) dt + \frac{\beta}{2n} + \int_0^{1/2n} \varphi \left( \frac{1}{t} F_n(1/2n) \right) \frac{1}{1/2n} dt \]
\[ \leq \frac{\alpha b}{2n} + \frac{\beta}{2n} + \int_0^{1/2n} \varphi \left( \frac{a+b}{4nt} \right) dt, \]
and
\[ \lim_{n \to \infty} \varphi \left( \frac{a+b}{4nt} \right) dt = \varphi \left( \lim_{n \to \infty} \frac{a+b}{4nt} \right) = 0, \]
for all \( t \in [1/2n, 1] \). Denote \( z_n(t) = \varphi \left( \frac{a+b}{4nt} \right) \) for \( t \in [1/2n, 1] \) and \( z_n(t) = 0 \) otherwise. Then \( z_n(t) \leq \varphi \left( \frac{a+b}{2} \right) \in L^1[0,1] \) and \( z_n \to 0 \) pointwisely. Therefore, by the Lebesgue Dominated Convergence Theorem, \( \rho_{\varphi}(f_n \chi(0,1/2n)) \to 0 \) as \( n \to \infty \). Thus there exists \( n_0 \in \mathbb{N} \) such that
\[ \rho_{\varphi}(f_n \chi(0,1/2n_0)) < 1. \]

Now we put
\[ x_c(t) = f_n \chi(0, \frac{1}{2n_0})(t) + c \chi_{[\frac{1}{2}, 1]}(t), \quad t \in I, \]
\[ y_c(t) = g_n \chi(0, \frac{1}{2n_0})(t) + c \chi_{[\frac{1}{2}, 1]}(t), \quad t \in I, \]
for \( n \in \mathbb{N} \) and \( c \in \mathbb{R} \). We define the function
\[ h(c) = \rho_{\varphi}(x_c) \]
for \( c \in [0, \infty) \). Note that \( h(0) < 1 \) and \( h(c) \leq \varphi(\max\{b,c\}) < \infty \), which means that \( h \) takes finite values because \( \varphi < \infty \). Moreover, for \( 0 < \lambda < 1 \) and \( c_1, c_2 \in [0, \infty) \) we have
\[ h(\lambda c_1 + (1 - \lambda)c_2) = \rho_{\varphi} \left( f_n \chi(0, \frac{1}{2n_0}) + (\lambda c_1 + (1 - \lambda)c_2) \chi_{[\frac{1}{2}, 1]} \right) \]
\[ = \rho_{\varphi} \left( \lambda f_n \chi(0, \frac{1}{2n_0}) + \lambda c_1 \chi_{[\frac{1}{2}, 1]} + (1 - \lambda) f_n \chi(0, \frac{1}{2n_0}) + (1 - \lambda)c_2 \chi_{[\frac{1}{2}, 1]} \right) \]
\[ \leq \lambda h(c_1) + (1 - \lambda) h(c_2), \]
because \( \rho_{\varphi} \) is a convex modular. This means that \( h \) is convex and therefore continuous function on \([0, \infty)\). Applying the Darboux property we find a number \( c_0 \) satisfying \( h(c_0) = 1 \) because \( h(c) \to \infty \) as \( c \to \infty \). Put \( x_{c_0} = x \) and \( y_{c_0} = y \). By the condition (ii) we have \( C x(t) = C y(t) \) for \( t \in [1/(2n_0), 1] \). Then applying the equality (iii) we obtain \( \rho_{\varphi}(x) = \rho_{\varphi}(y) = 1 \). Consequently,
\[
\rho_\varphi \left( \frac{x + y}{2} \right) = \int_0^1 \varphi \left( \frac{1}{2} C(x(t)) + \frac{1}{2} C(y(t)) \right) dt
\]

\[
= \int_0^{1/2n_0} \varphi \left( \frac{1}{2} C(x(t)) + \frac{1}{2} C(y(t)) \right) dt + \int_{1/2n_0}^1 \varphi \left( \frac{1}{2} C(x(t)) + \frac{1}{2} C(y(t)) \right) dt
\]

\[
= \int_0^{1/2n_0} \left[ \alpha \left( \frac{1}{2} C(x(t)) + \frac{1}{2} C(y(t)) \right) + \beta \right] dt + \int_{1/2n_0}^1 \varphi(C(x(t)))dt
\]

\[
= \frac{1}{2} \int_0^{1/2n_0} \varphi(C(x(t)))dt + \frac{1}{2} \int_{1/2n_0}^1 \varphi(C(y(t)))dt + \frac{1}{2} \int_{1/2n_0}^1 \varphi(C(x(t)))dt
\]

\[
= \frac{1}{2} \int_0^1 \varphi(C(x(t)))dt + \frac{1}{2} \rho_\varphi(x) + \frac{1}{2} \rho_\varphi(y) = 1.
\]

Thus

\[
\left\| \frac{x + y}{2} \right\|_{\text{Ces}(\varphi)} = 1,
\]

and \( \|x\|_{\text{Ces}(\varphi)} = \|y\|_{\text{Ces}(\varphi)} = 1 \). Clearly, \( x \neq y \), which means that \( \text{Ces}_\varphi[0, 1] \) is not rotund.

II. Assume that \( I = [0, \infty) \). The proof is the same. We need only to notice that all constructed elements are well defined (the respective modulars are finite) by Proposition 3 from [26].

\[\square\]

**Remark 4** In the proof of Theorem 3 above, the assumption \( \varphi \in \Delta_2 \) has been used only in the proof of implication: if \( \varphi \) is strictly convex then \( \text{Ces}_\varphi \in (R). \) In the proof of reverse implication we use only the assumption \( \varphi < \infty. \) Obviously, if \( \varphi \in \Delta_2(\infty) \) or \( \varphi \in \Delta_2(\mathbb{R}_+) \), then \( \varphi < \infty. \)

Now we present criteria for several properties in the spaces \( \text{Ces}_\varphi. \) The following well known notion will be useful.

Suppose \( \psi, \gamma : [0, \infty) \rightarrow [0, \infty) \). We say \( \psi \) and \( \gamma \) are equivalent (weak equivalent) for "all arguments" if there are constants \( A, B > 0 \) such that \( A \gamma(t) \leq \psi(t) \leq B \gamma(t) \) for all \( t > 0 \) (\( \gamma(At) \leq \psi(t) \leq \gamma(Bt) \) for all \( t > 0 \)). We will write \( \psi \sim_a \gamma \) (\( \psi \sim_w \gamma \) for simplicity. If the above inequalities are satisfied for \( u \in [u_0, \infty) \) with \( u_0 > 0 \) such that \( \psi(u_0) > 0 \) then we say that \( \psi, \gamma \) are equivalent (weak equivalent) for "large arguments" and we write shortly \( \psi \sim_l \gamma \) (\( \psi \sim_w \gamma \)). The proof of the next fact is just an easy exercise - it is enough to apply equivalent formulation for condition \( \Delta_2 \), see [8].

**Remark 5** Suppose \( \gamma \) is a convex function. If \( \psi \sim_a \gamma \) then \( \psi \sim_w \gamma \). Moreover, the converse implication is true if \( \gamma \in \Delta_2(\mathbb{R}_+) \). Similarly, if \( \psi \sim_l \gamma \) then \( \psi \sim_w \gamma \). Moreover, the converse implication is true if \( \gamma \in \Delta_2(\infty) \).

Let \( \varphi \) be an Orlicz function with \( b_\varphi < \infty \) and \( \varphi(b_\varphi) = \infty, \gamma \) be a convex function and \( p > 1 \). We will write \( \varphi^{1/p} \sim_a \gamma \) (\( \varphi^{1/p} \sim_l \gamma \)) provided there are constants
A, B > 0 (there are constants A, B, u₀ > 0 such that ϕ(u₀) > 0) such that \( Aγ(u) \leq ϕ(u)^{1/p} \leq Bγ(u) \) for all \( u \in [0, b_ϕ) \) (for all \( u \in [u₀, b_ϕ) \)).

Recall that we have the following implications

\[ R \Rightarrow SM, \quad ULUM \Rightarrow SM \quad and \quad LLUM \Rightarrow SM. \]

Easy proof follows, in fact, from the definitions of these properties, cf. [21].

**Corollary 6** Let \( ϕ \) be an Orlicz function.

(1) If \( b_ϕ < ∞ \) and \( ϕ(b_ϕ) < ∞ \) then \( Ces_ϕ \notin (SM) \).

(2) Suppose \( b_ϕ < ∞, \, ϕ(b_ϕ) = ∞ \) and there is a number \( p > 1 \) and a convex function \( γ \) such that \( ϕ^{1/p} \sim_1 γ \) if \( I = [0, 1] \) or \( ϕ^{1/p} \sim_γ γ \) if \( I = [0, ∞) \) (as we mean above). Then \( Ces_ϕ \notin (SM) \).

(3) Suppose \( ϕ < ∞ \) and

\( (+) \) there is a number \( p > 1 \) and a convex function \( γ \) such that \( ϕ^{1/p} \sim_1 γ \) if \( I = [0, 1] \) or \( ϕ^{1/p} \sim_γ γ \) if \( I = [0, ∞) \) (as we mean above).

Then the following statements hold:

(i) \( Ces_ϕ \in (R) \) if and only if \( ϕ \in Δ₂ \) and \( ϕ \) is strictly convex.

(ii) Assume additionally that \( \lim_{u \to ∞} \frac{ϕ(u)}{u} = ∞ \) in the case \( I = [0, 1] \). Then the following conditions are equivalent:

(a) \( Ces_ϕ \in (LLUM) \),

(b) \( Ces_ϕ \in (ULUM) \),

(c) \( Ces_ϕ \in (SM) \),

(d) \( ϕ \in Δ₂ \) and \( ϕ > 0 \).

**Proof** (1) and (2). In both cases we conclude that \( Ces_ϕ \) contains an order isomorphically isometric copy of \( l^∞ \) (see Corollary 13 from [27] and the proof of Theorems 3 and 4 from [27]). The conclusion follows because \( l^∞ \notin (SM) \).

(3) (i) In view of Theorem 3 we need only to prove the implication: if \( Ces_ϕ \in (R) \) then \( ϕ \in Δ₂ \). Suppose \( ϕ \notin Δ₂ \). Then \( Ces_ϕ \) contains an order isomorphically isometric copy of \( l^∞ \) by Corollary 13 from [27]. Of course, \( l^∞ \notin (R) \) and consequently \( Ces_ϕ \notin (R) \).

(ii) The implications \( (a) \Rightarrow (c) \) and \( (b) \Rightarrow (c) \) are clear.

The implication \( (c) \Rightarrow (d) \) . The condition \( ϕ \in Δ₂ \) follows as in case (i) above because \( l^∞ \notin (SM) \). Moreover, applying Theorem 10 from [26], we conclude that \( ϕ > 0 \).

The implications \( (d) \Rightarrow (a) \) and \( (d) \Rightarrow (b) \). If \( ϕ \in Δ₂ \) and \( ϕ > 0 \), then the Orlicz function space \( L^ϕ \) is even uniformly monotone (see Theorem 7 in [21] with \( E = L^1 \)). Thus \( L^ϕ \in (LLUM) \) and \( L^ϕ \in (ULUM) \) . Consequently, \( Ces_ϕ \in (LLUM) \) and \( Ces_ϕ \in (ULUM) \), by Remark 1. \( \square \)

**Remark 7** The criteria of uniform monotonicity and property LLUM have been proved in [26, Theorem 11] using the theorem about the existence of isomorphic copy of \( l^∞ \). If we consider properties SM and ULUM this argument is not enough and we need [27, Th. 3 and Th. 4].
Let \( \varphi < \infty \). Denote by \( \alpha_\varphi \) the lower Matuszewska–Orlicz index (see [26,27,37]). Recall that \( \alpha_\varphi > 1 \) iff \( \varphi^* \in \Delta_2 \), where \( \varphi^* \) is the function complementary to \( \varphi \) in the sense of Young (see [37]). It is natural to ask whether the condition \((+)\) from the Corollary 6 (3) can be replaced by \( \alpha_\varphi > 1 \), which is, of course, easier to apply. To discuss this question we need the following.

**Proposition 8** Suppose \( \varphi < \infty \) and \( I = [0, \infty) \). Consider the following conditions:

(i) there is a number \( p > 1 \) and a convex function \( \gamma \) such that \( \varphi^{1/p} \sim_a \gamma \),

(ii) \( \alpha_\varphi > 1 \),

(iii) there is a number \( p > 1 \) and a convex function \( \gamma \) such that \( \varphi^{1/p} \sim_a \gamma \).

Then (i) \( \Rightarrow \) (ii). If \( \varphi > 0 \) then (ii) \( \Rightarrow \) (iii). If additionally \( \varphi \in \Delta_2 (\mathbb{R}_+) \) then (iii) \( \Rightarrow \) (i).

**Proof** The implication (i) \( \Rightarrow \) (ii) has been proved in [27, Proposition 8]. Recall only that \( C : L^\varphi \to L^\varphi \) if and only if \( \alpha_\varphi > 1 \) (see [26] for further references).

(ii) \( \Rightarrow \) (iii). By Theorem 1.1 from [18], there exists an Orlicz function \( \widetilde{\varphi} \sim_a \varphi \) such that \( \widetilde{\varphi} > 0 \), \( \widetilde{\varphi}(1) = 1 \) and \( p_\mathcal{S}(\widetilde{\varphi}) > 1 \), where \( p_\mathcal{S}(\psi) = \inf_{t > 0} \frac{t \psi(t)}{\psi'(t)} \) is the lower Simonenko index. Take \( 1 < p < p_\mathcal{S}(\widetilde{\varphi}) \). Consequently, by Lemma 2.3 from [19], the function \( \tilde{\varphi}(t) = \varphi(t) \) is non-decreasing (the condition \( \widetilde{\varphi} \in \Delta_2 (\mathbb{R}_+) \) used in the proof of Lemma 2.3 in [19] is not needed in this part). Thus \( \tilde{\varphi}^{1/(p)}(t) \) is non-decreasing. Denote \( \gamma(u) = \int_u^\infty \frac{\widetilde{\varphi}^{1/(p)}(t)}{t} dt \). Then \( \gamma \) is convex. Moreover, it is easy to see that \( \gamma \sim_a \tilde{\varphi}^{1/p} \).

Hence \( \gamma \sim_a \varphi^{1/p} \).

(iii) \( \Rightarrow \) (i). It is enough to apply the condition \( \Delta_2 \) in equivalent form (see [8]). \( \square \)

**Remark 9** (1) One can prove the above proposition in the case “for large arguments” and \( I = [0, 1] \).

(2) We have the equivalence (i) \( \Leftrightarrow \) (ii) under the additional assumption that \( \varphi \in \Delta_2 \). Consequently, as far as we know, it is not possible to replace the condition \((+)\) in the Corollary 6 (3) by the condition \( \alpha_\varphi > 1 \), because we use the condition \((+)\) exactly in proving the necessity of \( \varphi \in \Delta_2 \).

(3) Clearly, if \( \alpha_\varphi > 1 \) then we have only the weak equivalence \( \varphi^{1/p} \sim_a \gamma \). Unfortunately, in the proofs of results concerning the existence of isometric copy of \( L^\infty \) it is not enough (see [27]).

(4) The implication (ii) \( \Rightarrow \) (i) under additional assumption that \( \varphi \in \Delta_2 \) has been proved in [24, Theorem 1.7].

**Question 10** Is the condition \( \varphi \in \Delta_2 \) “really” needed to demonstrate the equivalence (ii) \( \Rightarrow \) (i) in Proposition 8?

**Question 11** Suppose \( X = L^\varphi \) and \( \varphi^{1/p} \) is equivalent to a convex function for some \( p > 1 \). Then \( CX \) is rotund if and only if \( X \) is rotund. Indeed, \( L^\varphi \) is rotund if and only if \( \varphi \in \Delta_2 \) and \( \varphi \) is strictly convex (see [8] ). The natural question is the following: Whether \( CX \) is rotund if and only if \( X \) is rotund for any symmetric function space \( X \)? Note that that the answer is no if we consider the sequence case (see [10, Remark 2.1]).
4 CX construction for (quasi-)symmetric spaces X

We also present some results concerning the spaces CX in the case of symmetric spaces X different than $L^p$.

Given a vector space $X$ the functional $x \mapsto \|x\|$ is called a quasi-norm if the following three conditions are satisfied:

(i) $\|x\| = 0$ if $x = 0$;
(ii) $\|ax\| = |a|\|x\|$, $x \in X$, $a \in \mathbb{R}$;
(iii) there exists $C = CX \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$.

We call $\| \cdot \|$ a $p$-norm where $0 < p \leq 1$ if, in addition, it is $p$-subadditive, that is, $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$.

Recall the important Aoki–Rolewicz theorem (cf. [25, Theorem 1.3 on p. 7], [39, p. 86], [40, pp.6–8]): if $0 < p \leq 1$ is given by $C = 2^{1/p - 1}$, then there exists a $p$-norm $\| \cdot \|_1$ equivalent to the quasi-norm $\| \cdot \|$ so that

$$\|x + y\|_1^p \leq \|x\|_1^p + \|y\|_1^p \quad \text{and} \quad \|x\|_1 \leq \|x\| \leq 2C\|x\|_1$$

(4.1)

for all $x, y \in X$. The quasi-norm $\| \cdot \|$ induces a metric topology on $X$: in fact a metric can be defined by $d(x, y) = \|x - y\|_1^p$, when the quasi-norm $\| \cdot \|_1$ is $p$-subadditive.

We say that $X = (X, \| \cdot \|)$ is a quasi-Banach space if it is complete for this metric.

To find out more about quasi-normed spaces (and about even more general $\Delta$-normed spaces) the reader is referred to [25].

Let $\phi$ be a quasi-concave function on $I$, that is, $\phi(0) = 0$, $\phi$ is positive, non-decreasing and $\phi(t)/t$ is non-increasing for $t \in (0, m(I))$. Then the Marcinkiewicz space $M_{\phi}^{(s)}$ is defined as

$$M_{\phi}^{(s)} = M_{\phi}^{(s)}(I) = \{x \in L^0(I) : \|x\|_{M_{\phi}^{(s)}} = \sup_{t \in I} \phi(t)x^*(t) < \infty\}.$$

Moreover, if $\phi$ is a concave function on $I$, $\phi(0) = 0$, $\phi$ is positive and non-decreasing then the Lorentz function space $\Lambda_{\phi}$ is given by the norm

$$\|x\|_{\Lambda_{\phi}} = \int_I x^*(t)d\phi(t) = \phi(0+)\|x\|_{L^\infty(I)} + \int_I x^*(t)\phi'(t)dt.$$

Recall that Marcinkiewicz spaces $M_{\phi}^{(s)}$ and Lorentz spaces $\Lambda_{\phi}$ are symmetric quasi-Banach function spaces, symmetric Banach function spaces on $I$, respectively (see [4,33]). The spaces $M_{\phi}^{(s)}$ and $\Lambda_{\phi}$ are examples symmetrizations $X^{(s)}$ of some Banach ideal spaces $X$ (see [29] for some properties and more references). For more details about symmetric (quasi-)Banach function spaces see [4,30,33,39,40].

Any non-trivial symmetric normed function space $X$ on $I$ ($X$ is non-trivial if $X \neq \{0\}$) is intermediate space between the spaces $L^1(I)$ and $L^\infty(I)$. More precisely,

$$L^1(I) \cap L^\infty(I) \overset{C_1}{\hookrightarrow} X \overset{C_2}{\hookrightarrow} L^1(I) + L^\infty(I),$$

(4.2)
where \( C_1 = 2f_X(1) \), \( C_2 = 1/f_X(1) \) and \( f_X \) is the fundamental function of \( X \), i.e., \( f_X(t) := \|\chi_{[0,t]}\|_X \) for \( t \in I \) (see [33], Theorem 4.1). In particular, \( \text{supp}(X) = I \).

The situation is different if \((X, \|\cdot\|)\) is non-trivial symmetric quasi-Banach function space. Suppose the constant \( C \) comes from the triangle inequality for quasi-norm \( \|\cdot\| \) and the number \( p \) satisfies the equality \( C = 2^{1/p - 1} \) (see (4.1)). Applying Theorem 1 and 2 from [3] we conclude that

\[
L^p(I) \cap L^\infty(I) \xrightarrow{C_1} X \xrightarrow{C_2} L^{p,\infty}(I) + L^\infty(I),
\]

where \((X, \|\cdot\|_1), C_1 = 2^{1/p} \|\chi_{[0,1]}\|_1, C_2 = \frac{4^{1/p}}{\|\chi_{[0,1]}\|_1} \) and \( L^{p,\infty} = M^{(s)}_{\phi} \) with \( \phi(t) = t^{1/p} \).

**Remark 12**

(i) If \( X \) is a quasi-Banach ideal space on \( I \) then the space \( CX \) (defined as in Sect. 2) is a quasi-Banach ideal space on \( I \). Note that \( CX \) need not have a weak unit even if \( X \) has (see Example 2 in [34]). However, if \( C : X \to X \) is bounded and \( X \) has a weak unit then \( CX \) has it also.

(ii) Suppose \( X \) is a symmetric (quasi-)Banach ideal space on \( I = [0,1] \). Then \( CX \neq [0,1] \). Indeed, we apply the conditions (4.2) and (4.3) for symmetric Banach function space and for symmetric quasi-Banach function space, respectively. Note also that if \( X \hookrightarrow Y \) then \( CX \hookrightarrow CY \) (see Remark after Corollary 1 in [26]). Clearly, \( CX \neq [0,1] \) if \( X = L^p(I) \cap L^\infty(I) \) on \( I = [0,1] \), \( 0 < p \leq 1 \).

### 4.1 The Cesàro–Lorentz spaces \( CA_\phi \)

**Remark 13** The Cesàro–Lorentz function space \( CA_\phi[0,1] \) is always non-trivial. Moreover, \( CA_\phi[0,\infty] \neq [0,1] \) if and only if \( \int_0^\infty \frac{\phi(t)}{t+\lambda} \, dt < \infty \) for some \( \lambda > 0 \).

**Proof** The first fact follows from Remark 12 (ii). In the case of \( I = [0,\infty) \) we will use the direct computations. Set \( f(t) = \frac{1}{t} \chi_{[\lambda,\infty)}(t) \). We have

\[
\|f\|_{A_\phi} = \phi(0+) \|f\|_L^\infty + \int_0^\infty f^*(t)\phi'(t) \, dt = \frac{\phi(0+) \lambda}{\lambda} + \int_0^\infty \frac{\phi'(t)}{t+\lambda} \, dt,
\]

and using [34, Theorem 1 (a)] we finish the proof. \( \square \)

**Theorem 14** The following conditions are equivalent:

(a) The Cesàro–Lorentz function space \( CA_\phi \) is strictly monotone.
(b) The inequality \( \phi'(t) > 0 \) holds for all \( t \in (0, \mu(I)) \) and \( \phi(\infty) = \infty \) when \( I = (0, \infty) \).

**Proof** The implication (a) \( \Rightarrow \) (b). Suppose condition (b) does not hold. Since \( \phi' \) is non-increasing, there is \( a \in (0, \mu(I)) \) such that \( \phi'(t) = 0 \) for all \( t \in (a, \mu(I)) \). Let 

\[
x = \chi_{(0,a)} \quad \text{and} \quad y = \chi_{(0,a+\delta)}
\]

where \( \delta > 0 \) is small enough that \( \chi_{(0,a+\delta)} \) is non-trivial. Then \( x \in CA_\phi \) and \( y \in CA_\phi \). However, \( y \not\in CA_\phi \), contradiction. \( \square \)
for some \( \delta \in (0, \mu (I) - a) \). Then \( 0 \leq x \leq y \) and \( x \neq y \). On the other hand,

\[
\phi (0+) + \int_0^a \phi' (t) \, dt = \phi (0+) \|Cx\|_{L^\infty(I)} + \int_0^a \phi' (t) \, dt = \|Cx\|_{\Lambda_\phi}
\]

\[
= \|x\|_{C\Lambda_\phi} \leq \|y\|_{C\Lambda_\phi} = \|Cy\|_{\Lambda_\phi} = \phi (0+) + \int_0^a \phi' (t) \, dt.
\]

Thus \( C\Lambda_\phi \) is not strictly monotone.

Assume that \( I = (0, \infty) \) and \( \phi (\infty) < \infty \). Set

\[
x = \chi_{(0, \infty)} \text{ and } y = \chi_{(1, \infty)}.
\]

Then \( 0 \leq y \leq x \) and \( x \neq y \). Clearly, \( Cx = x \). Moreover,

\[
Cy (t) = \begin{cases} 
0 & \text{if } 0 < t \leq 1, \\
1 - 1/t & \text{if } t > 1.
\end{cases}
\]

Then \( (Cy)^* = (Cx)^* = \chi_{(0, \infty)} \) which implies that \( \|x\|_{C\Lambda_\phi} = \|y\|_{C\Lambda_\phi} \). It means that \( C\Lambda_\phi \) is not strictly monotone.

The implications \( (b) \Rightarrow (a) \). By the assumptions we conclude that \( X = \Lambda_\phi \)

is strictly monotone (see Lemma 3.1 in [28]). Although we use a little different

than in [28] definition of the Lorentz space \( \Lambda_\phi \) (they coincide if \( \phi(0+) = 0 \)),

the proof of required implication is the same. Since \( X \in (SM) \) so \( CX \in (SM) \) (see Remark 1), whence the proof is finished. \( \square \)

**Proposition 15** The Cesàro–Lorentz function space \( C\Lambda_\phi \) is order continuous if and only if \( \phi (0+) = 0 \) and \( \phi (\infty) = \infty \) when \( I = [0, \infty) \).

**Proof Necessity.** We divide the proof into two parts:

(i) Suppose \( \phi (0+) > 0 \). Set \( f := \chi_{[0,1]} \). Since

\[
(Cf)^* (x) = Cf (x) = \chi_{[0,1]} (x) + \frac{1}{x} \chi_{(1, \infty)} (x),
\]

in the view of [34, Theorem 1 (a)], we conclude that \( f \in \Lambda_\phi \) and \( f \in C\Lambda_\phi \). Let

\( A_n = [0, \frac{1}{n}] \) for \( n \in \mathbb{N} \). We have

\[
\left\| f \chi_{A_n} \right\|_{C\Lambda_\phi} = \left\| C (f \chi_{A_n}) \right\|_{\Lambda_\phi} = \phi (0+) \left\| C (f \chi_{A_n}) \right\|_{L^\infty} + \int_0^{\mu (I)} (Cf)^* (t) \phi' (t) \, dt \\
\geq \phi (0+) \left\| C (f \chi_{A_n}) \right\|_{L^\infty} = \phi (0+) \left\| \chi_{[0, \frac{1}{n}]} \right\|_{L^\infty} = \phi (0+) > 0,
\]

which means that \( C\Lambda_\phi \) is not order continuous.

(ii) Let \( \phi (\infty) < \infty \). Put \( f = \chi_{[0, \infty)} \) and \( A_n = [n, \infty) \) for \( n \in \mathbb{N} \). Is easy to see that \( Cf (i) = (1 - \frac{i}{n}) \chi_{[n, \infty)} (i) \) and \( (Cf)^* (i) = \chi_{[0, \infty)} (i) = f (i) \). Thus \( f \in C\Lambda_\phi \). Moreover, we have
\[ \| f \chi A_n \|_{C \Lambda \phi} = \| C(f \chi A_n) \|_{\Lambda \phi} = \phi(0+) \| C(f \chi A_n) \|_{L^\infty} + \int_0^\infty (Cf)^n(t) \phi'(t) \, dt \]
\[ = \phi(0+) + \int_0^\infty f(t) \phi'(t) \, dt = \phi(0+) + \int_0^\infty \phi'(t) \, dt = \phi(\infty) > 0, \]
whence \( C \Lambda \phi \) is not order continuous.

**Sufficiency.** Recall that Lorentz space \( \Lambda \phi \) is order continuous if and only if \( \phi(0+) = 0 \) and \( \phi(\infty) = \infty \) when \( I = [0, \infty) \), see [33, Lemma 5.1, p. 110]. Now the implication follows easily from [26, Fact]. \( \square \)

**Remark 16** Note that the above result for \( I = [0, 1] \) follows from Proposition 2 in [26] under the assumption that \( C : \Lambda \phi \rightarrow \Lambda \phi \). We present the direct proof because we need it for \( I = [0, \infty] \) and it does not require the assumption \( C : \Lambda \phi \rightarrow \Lambda \phi \). Note also that we may reformulate theorem above as: \( C \Lambda \phi \) is order continuous iff \( \Lambda \phi \) is order continuous.

Recall that a Banach ideal space \( X \) has the Kadec–Klee property with respect to global convergence in measure, we write \( X \in (H_g) \) whenever for any sequence \( (x_n) \subset X \) such that \( x_n \rightharpoonup x \) globally in measure and \( \| x_n \|_X \rightarrow \| x \|_X \), we have \( \| x_n - x \|_X \rightarrow 0 \).

**Theorem 17** The following conditions are equivalent:

(a) The Cesàro–Lorentz function space \( C \Lambda \phi \) is lower locally uniformly monotone.

(b) The Cesàro–Lorentz function space \( C \Lambda \phi \) is upper locally uniformly monotone.

(c) The inequality \( \phi'(t) > 0 \) holds for all \( t \in (0, \mu(I)) \), \( \phi(0+) = 0 \) and \( \phi(\infty) = \infty \) when \( I = (0, \infty) \).

**Proof** The implication \((a) \Rightarrow (c)\). Since \( LLUM \Rightarrow SM \), applying Theorem 14, it is enough to prove that \( \phi(0+) = 0 \). Recall also that \( LLUM \Rightarrow OC \) (see Proposition 2.1 in [17]). It is enough to apply Proposition 15.

The implication \((c) \Rightarrow (a)\). By the assumptions we conclude that \( X = \Lambda \phi \) is lower locally uniformly monotone (see Proposition 3 from [20]). Although we use a little different definition of the Lorentz space \( \Lambda \phi \) than in [20], they coincide if \( \phi(0+) = 0 \). By Remark 1, we finish the proof.

The implication \((b) \Rightarrow (c)\). Since \( ULUM \Rightarrow SM \), applying Theorem 14, it is enough to prove that \( \phi(0+) = 0 \). Suppose \( \phi(0+) > 0 \). Let

\[ x = \chi(0,1/2) \text{ and } x_n = \chi(0,1/2+1/n) \text{ for } n > 1. \]

Then \( 0 \leq x \leq x_n \) and \( x_n \rightharpoonup x \) globally in measure. Note also that \( x_n \in C \Lambda \phi \) (see inequality (4.4) below). Moreover, we claim that

\[ \| x_n \|_{C \Lambda \phi} \rightarrow \| x \|_{C \Lambda \phi}. \]
Indeed, we have

\[
\|x\|_{C_{\Lambda^\phi}} = \phi(0+) \|Cx\|_{L^\infty} + \int_0^\infty (Cx)^\ast(t)\phi'(t)dt \\
= \phi(0+) \|x\|_{L^\infty} + \int_0^\infty Cx(t)\phi'(t)dt \\
= \phi(0+) + \int_0^\infty \left(\chi(0,1/2)(t) + \frac{1}{2t} \chi(1/2,\infty)(t)\right)\phi'(t)dt \\
= \phi(0+) + \int_0^\infty \chi(0,1/2)(t)\phi'(t)dt + \int_0^\infty \frac{1}{2t} \chi(1/2,\infty)(t)\phi'(t)dt
\]

and

\[
\|x_n\|_{C_{\Lambda^\phi}} = \phi(0+) \|Cx_n\|_{L^\infty} + \int_0^\infty (Cx_n)^\ast(t)\phi'(t)dt \\
= \phi(0+) \|x\|_{L^\infty} + \int_0^\infty \left(\chi(0,1/2+1/n)(t) + \frac{1}{2} + \frac{1}{n} \chi(1/2+1/n,\infty)(t)\right)\phi'(t)dt \\
= \phi(0+) + \int_0^\infty \chi(0,1/2+1/n)(t)\phi'(t)dt \\
+ \left(\frac{1}{2} + \frac{1}{n}\right) \int_0^\infty \frac{1}{t} \chi(1/2+1/n,\infty)(t)\phi'(t)dt.
\]

Consequently,

\[
\|x_n\|_{C_{\Lambda^\phi}} - \|x\|_{C_{\Lambda^\phi}} \\
= \int_0^{1/2+1/n} \phi'(t)dt + \frac{1}{2} \int_0^{1/2+1/n} \frac{1}{t} \phi'(t)dt + \frac{1}{n} \int_0^\infty \frac{1}{t} \phi'(t)dt.
\]

From the Lebesgue dominated convergence theorem, the first two integrals tends to zero as \(n \to \infty\). Moreover, for all \(\lambda > 0\) and \(t_0 > 0\) there is constant \(C > 0\) with \(\frac{1}{t} \leq \frac{C}{t_0 + \lambda\tau}\) for all \(t \geq t_0\) (it is enough to take \(C = \frac{t_0 + \lambda}{t_0}\)). Take \(t_0 = 1/2\) and the number \(\lambda\) from Remark 13. Then, with \(C = 1 + 2\lambda\), we have

\[
\int_0^\infty \frac{1}{t} \phi'(t)dt \leq \int_0^{1/2} \frac{1}{t} \phi'(t)dt + \frac{1}{1/2} \int_0^\infty \frac{1}{t} \phi'(t)dt < \infty,
\]

(4.4)

by Remark 13. Consequently, \(\frac{1}{n} \int_0^{1/2+1/n} \frac{1}{t} \phi'(t)dt \to 0\). Thus the claim \(\|x_n\|_{C_{\Lambda^\phi}} \to \|x\|_{C_{\Lambda^\phi}}\) is proved. On the other hand,

\[
\|x_n - x\|_{C_{\Lambda^\phi}} = \|C|x_n - x|\|_{\Lambda^\phi} = \|(C|x_n - x|)^\ast\|_{\Lambda^\phi} \\
= \|\chi(0,1/n)\|_{\Lambda^\phi} \geq \phi(0+) > 0.
\]

Thus \(C_{\Lambda^\phi} \notin (ULUM)\).
The Cesàro–Lorentz function space $C\Lambda_{\phi}$ is not rotund for any fundamental function $\phi$.

**Proof** If there is $a \in (0, \mu (I))$ such that $\phi' (t) = 0$ for all $t \in (a, \mu (I))$, by Theorem 14, $C\Lambda_{\phi}$ is not strictly monotone, whence it is not rotund. We go similarly if $\phi(\infty) < \infty$.

Suppose $\phi(\infty) = \infty$ and $\phi'(t) > 0$ for all $t \in (0, \mu (I))$. For $0 < \alpha_0 < \alpha_1 < \alpha_2 < \mu (I)$ and $a > v_0 > u_0$ set

$$x = a\chi(0,\alpha_0) + v_0\chi(\alpha_0,\alpha_1) + u_0\chi(\alpha_1,\alpha_2).$$

Then $x = x^*$, whence $Cx$ is also nonincreasing, that is $Cx = (Cx)^*$. Suppose $I = [0, 1]$. Since $L^{\infty}(I) \hookrightarrow \Lambda_{\phi}(I)$, so $Cx \in \Lambda_{\phi}(I)$. Taking $\tilde{x} = \frac{x}{\|Cx\|_{\Lambda_{\phi}}}$ we get $\|C\tilde{x}\|_{\Lambda_{\phi}} = 1$. Moreover, for $I = [0, \infty)$, in order to prove that $Cx \in \Lambda_{\phi}(I)$ we should follow as in the proof of Theorem 17, see inequalities (4.4). Hence, in both cases, we may assume that $\|Cx\|_{\Lambda_{\phi}} = 1$.

Put

$$y_1 = a\chi(0,\alpha_0) + u_0\chi(\alpha_0,\alpha_2)$$

and

$$y_2 = a\chi(0,\alpha_0) + v_0\chi(\alpha_0,\alpha_2).$$

Similarly as above, $y_1, y_2 \in C\Lambda_{\phi}$. Of course, we have $y_1 \leq x \leq y_2$, $y_1 \neq x$ and $y_2 \neq x$. Thus $\|y_1\|_{C\Lambda_{\phi}} \leq 1 \leq \|y_2\|_{C\Lambda_{\phi}}$. From our assumptions and Theorem 14 we can conclude that the Cesàro–Lorentz space $C\Lambda_{\phi}$ is strictly monotone, so the above inequalities are sharp, i.e., $\|y_1\|_{C\Lambda_{\phi}} < 1 < \|y_2\|_{C\Lambda_{\phi}}$. Let us define a function $f$ as

$$f : [u_0, v_0] \ni \lambda \mapsto \|a\chi(0,\alpha_0) + \lambda\chi(\alpha_0,\alpha_2)\|_{C\Lambda_{\phi}} \in [0, \infty).$$

It follows from the definition that $f(u_0) = \|y_1\|_{C\Lambda_{\phi}}$ and $f(v_0) = \|y_2\|_{C\Lambda_{\phi}}$, so $f(u_0) < 1 < f(v_0)$. In particular, the function $f$ takes finite values. Furthermore, $f$ is a convex function. In fact, for $0 < \omega < 1$ and $u_0 \leq \lambda_1, \lambda_2 \leq v_0$ we get
f(ωλ₁ + (1 − ω)λ₂) = \left\| aχ(0, a₁) + (ωλ₁ + (1 − ω)λ₂)χ(a₁, a₂) \right\|_{C_Aφ} \\
= \left\| ωaχ(0, a₁) + ωλ₁χ(a₁, a₂) + (1 − ω)aχ(0, a₁) + (1 − ω)λ₂χ(a₁, a₂) \right\|_{C_Aφ} \\
≤ \left\| ωaχ(0, a₁) + ωλ₁χ(a₁, a₂) \right\|_{C_Aφ} + \left\| (1 − ω)aχ(0, a₁) + (1 − ω)λ₂χ(a₁, a₂) \right\|_{C_Aφ} \\
= ω \left\| aχ(0, a₁) + λ₁χ(a₁, a₂) \right\|_{C_Aφ} + (1 − ω) \left\| aχ(0, a₁) + λ₂χ(a₁, a₂) \right\|_{C_Aφ} \\
= ωf(λ₁) + (1 − ω)f(λ₂).

Therefore the function f is continuous and then applying the Darboux property we find a number λ₀ ∈ [u₀, v₀] with f(λ₀) = 1. For an element

\[ y = aχ(0, a₁) + λ₀χ(a₁, a₂), \]

we would get y = y*, Cy = (Cy)* and \( \|Cy\|_{Aφ} = 1 \). Consequently, \( (C(x + y))^* = C(x + y) \). Moreover,

\[ \phi (0+) \| C|x + y\|_{L∞} = 2aφ (0+) = \phi (0+) \left( \|Cx\|_{L∞} + \|Cy\|_{L∞} \right). \]

Thus

\[ \|x + y\|_{C_Aφ} = \|C(x + y)\|_{Aφ} = \|Cx\|_{Aφ} + \|Cy\|_{Aφ} = 2, \]

whence \( C_Aφ \) is not rotund.

\[ \square \]

4.2 The Cesàro–Marcinkiewicz spaces \( CM^{(s)}_φ \)

Remark 20 For any fundamental function φ the Cesàro–Marcinkiewicz space \( CM^{(s)}_φ \) is non-trivial.

Proof If \( I = [0, 1] \), then we apply Remark 12 (ii). Suppose \( I = [0, ∞) \). Set \( f(t) = \frac{1}{t}χ_{[a, ∞)}(t) \). We have

\[ \|f\|_{M^{(s)}_φ} = \sup_{0 ≤ t < ∞} f^*(t)φ(t) = \sup_{0 ≤ t < ∞} \frac{1}{t + a}χ_{[0, ∞)}(t)φ(t) = \sup_{0 ≤ t < ∞} \frac{φ(t)}{t + a} < ∞, \]

for any \( a > 0 \), because \( φ(t)/t \) is non-increasing. Therefore, for \( a = 1 \) we have

\[ \frac{1}{t}χ_{[1, ∞)}(t) \in M^{(s)}_φ[0, ∞) \]

and consequently \( CM^{(s)}_φ[0, ∞) \neq \{0\} \), by [34, Theorem 1 (a) and (b)] (note that the proof in [34] does not change in the case of quasi-Banach ideal space instead of Banach ideal space).

\[ \square \]

Proposition 21 The Cesàro–Marcinkiewicz space \( CM^{(s)}_φ \) is never strictly monotone.

Proof The idea of this proof comes from [7, Lemma 2.4]. The elements constructed below are in fact the same but because of slightly different character of the norm in Cesàro–Marcinkiewicz space and different monotonicity property considered in [7] we
present the details. Moreover, we use part of this construction again in Proposition 23. Let
\[ f(t) = \frac{t}{\phi(t)} \]
for \( t \in [0, \mu(I)) \). From [4, Theorem 5.2] we conclude that function \( f \) is a fundamental function of some Köthe dual space, so this function must be quasi-concave. Thus, the function defined as \( \lambda(t) = \frac{d}{dt} f(t) \) is a non-increasing function defined almost everywhere. Put
\[ g(t) = \lambda(t)\chi_{[0, \frac{1}{2})}(t) \quad \text{and} \quad h(t) = \lambda(t)\chi_{[0, \mu(I))}(t). \]
We have
\[ \int_0^t g(s)ds = f(t) = \frac{t}{\phi(t)} \]
for any \( 0 \leq t < 1/2 \), so
\[ Cg(t) = \frac{1}{t} \int_0^t g(s)ds = \frac{1}{\phi(t)} \chi_{[0, \frac{1}{2})}(t) + \frac{1}{2t \phi(\frac{1}{2})} \chi_{[\frac{1}{2}, \mu(I))}(t) \]
and
\[ Ch(t) = \frac{1}{t} \int_0^t h(s)ds = \frac{1}{\phi(t)} \chi_{[0, \mu(I))}(t) \]
for \( t > 0 \).

Of course,

(i) \( g \leq h \) and \( g \neq h \),

(ii) \((Cg)^*(t) = Cg(t)\) and \((Ch)^*(t) = Ch(t)\) for \( t \in I \).

Finally,
\[ \|g\|_{CM(\phi)} = \|Cg\|_{M(\phi)} = \sup_{0 \leq t < \mu(I)} (Cg)^*(t)\phi(t) = \max\{1, \sup_{\frac{1}{2} \leq t < \mu(I)} \frac{\phi(t)}{2t \phi(\frac{1}{2})}\} = 1, \]
because the function \( \phi(t)/t \) is non-increasing, cf. [4, Corollary 5.3 (5.9), p. 67]. Moreover, even easier we get \( \|h\|_{CM(\phi)} = 1 \). This finishes the proof.

\[ \square \]

**Corollary 22** The Cesàro–Marcinkiewicz function space \( CM(\phi) \) is neither lower local uniformly monotone nor upper local uniformly monotone for any quasi-concave function \( \phi \).

**Proposition 23** The Cesàro–Marcinkiewicz function space \( CM(\phi) \) is never order continuous.

**Proof** It is enough to take (we use the notation from the proof of Proposition 21)
\[ u(t) = \lambda(t)\chi_{[0,1]}(t) \]
and $A_n = [0, \frac{1}{n}]$ for $n \in \mathbb{N}$. Indeed, just as in the mentioned proof we get

$$\|u \chi_{A_n}\|_{CM^*(\phi)} = \|C(u \chi_{A_n})\|_{M^*(\phi)}$$

$$= \sup_{0 \leq t < \mu(I)} \left( C \left( \lambda \chi_{[0, \frac{1}{n}]} \right)(t) \phi(t) \right)$$

$$= \sup_{0 \leq t < \mu(I)} \left( \frac{1}{\phi(t)} \chi_{[0, \frac{1}{n}]}(t) + \frac{1}{nt \phi \left( \frac{1}{n} \mu(I) \right)}(t) \phi(t) \right)$$

$$= \max \left\{ 1, \sup_{\frac{1}{n} < t < \mu(I)} \frac{\phi(t)}{nt \phi \left( \frac{1}{n} \right)} \right\} = 1.$$

Therefore, an element $u$ is not order continuous. Since $\phi$ is the arbitrary quasi-concave function we finish the proof. □

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