Stokes’ formulae on classical symbol valued forms and applications

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Abstract

The Wodzicki residue and the cut-off integral extend to classical symbol valued forms. We show that they obey a Stokes’ type property and that the extended Wodzicki residue can be interpreted as a complex residue like the ordinary one. In the case of cut-off integrals, Stokes’ property (i.e. vanishing on exact forms) only holds for non integer order symbol valued forms and leads to an integration by parts formula and translation invariance for cut-off integrals on non integer order classical symbols. The extended Wodzicki residue yields an even residue cycle on classical symbols and an odd cochain (the cosphere cochain) which measures an obstruction to Stokes’ property of the cut-off integral on integer order symbol-valued forms.

Résumé

Le résidu de Wodzicki et l’intégrale régularisée par troncature s’étendent aux formes à coefficients symboles classiques. Nous montrons que que l’un et l’autre possèdent une propriété de Stokes et que le résidu de Wodzicki des formes s’interprète comme un résidu complexe, de la même manière que le résidu de Wodzicki ordinaire. Dans le cas de l’intégrale régularisée par troncature, la propriété de Stokes (i.e. l’annulation sur les formes exactes) n’est vérifiée que pour les formes d’ordre non-entier. Elle implique une formule d’intégration par parties et une invariance par translation pour l’intégrale régularisée des symboles d’ordre non-entier. Le résidu de Wodzicki étendu induit quant à lui un cycle de dimension paire sur l’algèbre engendrée par les symboles classiques, ainsi qu’une cochaîne de degré un de moins (la cochaîne cosphère) qui mesure l’obstruction à la propriété de Stokes pour les formes d’ordre entier.
Introduction

We discuss generalisations of Stokes’ property \( \int_U d\alpha = 0 \) for ordinary integrals of forms \( \alpha \) with compact support (or tending to zero rapidly enough at infinity) in an open subset \( U \) of \( \mathbb{R}^n \) to regularised integrals of classical symbol valued forms on an open subset of \( \mathbb{R}^n \). Although consequences of such a formula such as integration by parts and translation invariance for regularised integrals are commonly used in the physics literature to compute Feynman graphs, the only explicit reference we could find in the literature to Stokes’ formula for regularised integrals is in [E]. Etingof considers dimensional regularisation which he applies to a class of functions relevant for physics, namely functions of Feynman type, proving Stokes’ formula for corresponding regularised integrals of top degree forms.

Here, we consider general regularisation procedures and all classical symbol valued forms, proving Stokes’ formula with pseudodifferential theoretic tools; an essential obstacle to Stokes’ formula turns out to be the Wodzicki residue extended to forms, to which we devote a large part of the paper.

The Wodzicki residue extended to classical symbol valued forms is the topic of the first part of the paper. It satisfies Stokes’ property and therefore defines a \( 2n \)-cycle on the algebra of classical symbols with compact support on an open subset \( U \subset \mathbb{R}^n \) equipped with the left product of symbols \( \ast \) (Theorem 2). Its associated residue character is a cyclic \( \ast \)-Hochschild cocycle:

\[
(\sigma_0, \cdots, \sigma_{2n}) \mapsto \text{res} (\sigma_0 \ast d\sigma_1 \ast \cdots \ast d\sigma_{2n})
\]

where \( \text{res} \) is the extended residue and where \( \wedge_\ast \) is the product on the graded differential algebra of classical symbol valued forms induced by the left product on symbols and \( d \) the exterior differentiation on \( T^*U \).

On classical pseudodifferential operators of order 0, this \( \ast \)-Hochschild cocycle reduces to a cyclic Hochschild cocycle for the ordinary product for we have:

\[
\text{res} (\sigma_0 \ast d\sigma_1 \ast \cdots \ast d\sigma_{2n}) = \text{res} (\sigma_0 \, d\sigma_1 \wedge \cdots \wedge d\sigma_{2n}).
\]

It coincides up to a multiplicative constant with the analog in the context of classical symbols of the antisymmetrised \( 2n \)-cocycle introduced in [CFS] and further investigated in [H] in the context of star-deformed algebras:

\[
\text{res} (\sigma_0 \ast d\sigma_1 \ast \cdots \ast d\sigma_{2n}) = \frac{(-i)^n}{n!} A \text{res} (\sigma_0 \ast \theta(\sigma_1, \sigma_2) \ast \cdots \ast \theta(\sigma_{2n-1}, \sigma_{2n}))
\]

where we have set \( \theta(\sigma_i, \sigma_j) = \sigma_1 \ast \sigma_j - \sigma_i \cdot \sigma_j \) as in [H], [CFS]. Here \( A \) is the antisymmetrisation over all but the first variable.

The second part of the paper is devoted to cut-off integrals which we also extend to classical symbol valued forms. We show they obey Stokes’ property when restricted to non integer order symbols with compact support (see Theorem 3). As
a result, we get an integration by parts formula for cut-off integrals on non integer order symbols and show translation invariance for cut-off integrals on non integer order symbols.

Stokes’ property does not hold anymore on integer order symbol valued forms with compact support; as a result, one does not expect to define a cycle on the algebra of classical symbols using cut-off integrals. Rather, we express the obstruction to the cyclicity of a $2n$-cochain defined in terms of cut-off integrals of symbols $\int_{T^* U}^{}$:}

$$(\sigma_0, \cdots, \sigma_{2n}) \mapsto \int_{T^* U} (\sigma_0 \ast d \sigma_1 \wedge \cdots \wedge d \sigma_{2n})_0$$

where the subscript 0 stands for the 0-order part of the symbol valued form, in terms of the cosphere $2n - 1$-cochain defined in a similar way to the residue character (Proposition 5).

Finally, in the third part of the paper, we show that the relation between complex residues and the Wodzicki residue extends to symbols valued forms (Theorem 5):

$$\text{Res}_{\omega(z_0)} \int \omega(z) = -\frac{1}{\alpha'(z_0)} \text{res}(\text{res}(\omega(z_0))),$$

where $\omega(z)$ is a holomorphic family of classical symbol valued forms of order $\alpha(z)$ and $\text{res}(\omega(z_0))$ the Wodzicki residue of $\omega(z_0)$.

We also extend Stokes’ formula to cut-off integrals of holomorphic families of symbol valued forms $\omega(z)$ obtained from a symbol valued form $\omega$ via a regularisation procedure (see Theorem 6):

$$-\int d(\omega(z)) = 0.$$

In the case of dimensional regularisation and when applied to forms built from Feynman type functions, this corresponds to a result already proven in [1].

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However, our emphasis is on symbols rather than on operators so that we are led to considering cocycles that mix the star and the ordinary function product on symbols which do not arise in their work.
1 General scheme

Take \( X \) an open subset of \( \mathbb{R}^m \), and \( \mathcal{A} \subset C^\infty(X) \). Any associative (not necessarily commutative) product \(*\) on \( \mathcal{A} \) induces a product \( \wedge_\ast \) on the set \( \Omega \mathcal{A} \) of forms \( \alpha \) on \( X \) which are of the type:

\[
\alpha(x) = \sum_I \alpha_I(x) dx_I, \quad \alpha_I \in \mathcal{A}
\]

as follows:

\[
(\alpha_I(x) dx_{i_1} \cdots dx_{i_p}) \wedge_\ast (\beta_J(x) dx_{j_1} \cdots dx_{j_q}) = (\alpha_I \ast \beta_J)(x) dx_{i_1} \cdots dx_{i_p} dx_{j_1} \cdots dx_{j_q}
\]

which makes it a \( \mathbb{N} \)-graded algebra. If \( \mathcal{A} \) is stable under partial derivations, then the exterior differential \( d \) acts on \( \Omega \mathcal{A} \) increasing the degree by 1:

\[
d (\alpha_I(x) dx_{i_1} \cdots dx_{i_p}) = \sum_{j=1}^m \partial_j \alpha_I(x) dx_j dx_{i_1} \cdots dx_{i_p}.
\]

Clearly, equality \( d^2 = 0 \) comes from the odd parity of the \( dx_i \) which implies \( dx_i dx_j = -dx_j dx_i \).

Let us furthermore assume that partial derivations \( \partial_l \) on \( \mathcal{A} \) obey the Leibniz rule, i.e.

\[
\partial_l (a \ast b) = \partial_l a \ast b + a \ast \partial_l b \quad \forall \quad l = 1, \ldots, m, \forall a, b \in \mathcal{A}.
\]

Then \( d \) is a graded derivation on \( \Omega \mathcal{A} \); indeed, for any set of indices \( I = \{i_1, \ldots, i_p\} \) and \( J = \{i_{p+1}, \ldots, i_{p+q}\} \) we have

\[
d (\alpha_I(x) dx_{i_1} \cdots dx_{i_p} dx_{i_{p+1}} \cdots dx_{i_{p+q}})
\]

\[
d ((\alpha_I(x) \ast \beta_J(x)) dx_{i_1} \cdots dx_{i_p} dx_{i_{p+1}} \cdots dx_{i_{p+q}})
\]

\[
= \sum_{l=1}^{p+q} \partial_l (\alpha_I(x) \ast \beta_J(x)) dx_l dx_{i_1} \cdots dx_{i_p} dx_{i_{p+1}} \cdots dx_{i_{p+q}}
\]

\[
= \sum_{l=1}^{p} \partial_l \alpha_I(x) \ast \beta_J(x) dx_l dx_{i_1} \cdots dx_{i_p} dx_{i_{p+1}} \cdots dx_{i_{p+q}}
\]

\[
\quad + (-1)^p \sum_{l=1}^{q} \alpha_I(x) \ast \partial_l \beta_J(x) dx_{i_1} \cdots dx_{i_p} dx_l dx_{i_{p+1}} \cdots dx_{i_{p+q}}
\]

A linear map: \( \tau : \mathcal{A} \to \mathbb{C} \) induces a linear map \( \bar{\tau} : \Omega \mathcal{A} \to \mathbb{C} \) defined by:

\[
\bar{\tau} (\alpha_I(x) dx_{i_1} \cdots dx_{i_p}) = \tau (\alpha_I(x) dx_{i_1} \cdots dx_{i_p})
\]

\[
\bar{\tau} (\beta_J(x) dx_{i_{p+1}} \cdots dx_{i_{p+q}}) = \tau (\beta_J(x) dx_{i_{p+1}} \cdots dx_{i_{p+q}})
\]

\[
\quad + (-1)^p \tau (\alpha_I(x) dx_{i_1} \cdots dx_{i_p})
\]
**Definition 1** Let $\alpha \in \Omega A$.

\[ \bar{\tau}(\alpha_I(x)dx_I) = \tau(\alpha_I), \quad \text{if} \quad |I| = m; \quad \bar{\tau}(\alpha_I(x)dx_I) = 0 \quad \text{otherwise.} \]

We set:

\[ \chi^\tau_k(a_0, \ldots, a_k) := \bar{\tau}(a_0 \star da_1 \wedge \ldots \wedge a_k). \]

**Lemma 1** If

\[ \tau([a, b]_\star) = 0 \quad \forall a, b \in A, \quad \text{and} \quad \bar{\tau} \circ d = 0 \]

then $(\Omega A, d, \bar{\tau})$ defines an $m$-dimensional cycle with character $(a_0, \ldots, a_m) \mapsto \chi^\tau_m(a_0, \ldots, a_m)$ which yields a cyclic Hochschild cocycle.

**Proof:** Since $\tau([a, b]_\star) = 0$ we have that

\[ \tau(\alpha \wedge_\star \beta) = (-1)^{|\alpha||\beta|} \cdot \tau(\beta \wedge_\star \alpha), \]

which combined with $\bar{\tau} \circ d = 0$ provides an $m$-cycle. \hfill \Box

**Proposition 1** Let $\rho : A \to \mathbb{C}$ be a linear map, let $\bar{\rho} : \Omega A \to \mathbb{C}$ induced from $\rho$ as above, and let $\bar{\tau} := \bar{\rho} \circ d$.

1. Then for any $a_1, \ldots, a_k \in A$

\[ B_0 \chi^\rho_k(a_1, a_2, \ldots, a_k) = \bar{\tau}(a_1 \star da_2 \wedge_\star \ldots \wedge_\star da_k). \]

2. If moreover there is a trace $\tau$ on $A$ (i.e. $\tau([a, b]_\star) = 0 \quad \forall a, b \in A$) such that $\bar{\tau}$ coincides with the linear form on $\Omega A$ associated with $\tau$ as in definition 1 above, then $(\Omega A, d, \bar{\tau})$ defines an $m$-dimensional cycle with character

\[ \chi^\tau_m(a_0, \ldots, a_m) := \bar{\tau}(a_0 \star da_1 \wedge_\star \ldots \wedge_\star da_m), \]

which yields a cyclic Hochschild cocycle.

**Proof**

1. Since $\bar{\tau} = \bar{\rho} \circ d$

\[ B_0 \chi^\rho_k(a_1, \ldots, a_k) = \chi^\rho_k(1, a_1, \ldots, a_k) \]

\[ = \bar{\rho}(da_1 \wedge_\star \ldots \wedge_\star da_k) \]

\[ = \bar{\tau}(a_1 \star da_2 \wedge_\star \ldots \wedge_\star da_k). \]

2. This follows from the above lemma since $\bar{\tau} \circ d = \bar{\rho} \circ d^2 = 0$. 

5
Let $\mathcal{A}$ now be equipped with two (associative) products, the pointwise commutative one $\cdot$ and a non commutative one $\star$. Following [H] and [CFS] we set
\[
\theta(a, b) := a \star b - a.b.
\]

**Proposition 2** Let $\tau : \mathcal{A} \to \mathbb{C}$ be a trace with respect to the non-commutative product $\star$. Then
\[
\phi^\tau_{2k} (a_0, a_1, \ldots, a_{2k}) := \tau (a_0 \star (a_1 \star (a_2 \star \cdots \star (a_{2k-1} \star a_{2k}))))
\]
defines a $b + B$-cocycle, namely
\[b \phi_{2k} + \frac{1}{k+1} B \phi_{2k+2} = 0.\]

**Proof:** The proof of [H] and [CFS] adapts to this general set up in a straightforward manner. The assumption there that the star product be closed corresponds here to the cyclicity of $\tau$. One first shows that $\bar{b} \phi_{2k} = 0$ with
\[
\bar{b} \chi (a_0, \ldots, a_n) = \chi(a_0 \star a_1, \ldots, a_j, a_{j+1}, \ldots, a_n) + \sum_{j=1}^{n-1} (-1)^j \chi(a_0, \ldots, a_j \cdot a_{j+1}, \ldots, a_n)
\]
\[
+ \ (-1)^{n+1} \chi(a_0, \ldots, a_{n-1} \star a_n).
\]
The result then follows comparing $b \phi_{2k}$ and $\bar{b} \phi_{2k}$, which yields
\[
b \phi_{2k}(a_0, \ldots, a_{2k+1}) = -\tau (\theta(a_0, a_1) \star \theta(a_2, a_3) \cdots) + \tau (\theta(a_{2k+1}, a_0) \star \theta(a_1, a_2) \cdots)
\]
\[
= -\frac{1}{k+1} B \phi_{2k+2}.
\]

In what follows we apply these constructions to the algebra of classical symbols with compact support on an open subset of $\mathbb{R}^n$ letting $\tau$ be the Wodzicki residue on and $\rho$ the cut-off integral of symbols

## 2 Classical symbols valued forms

Let us first set some notations.

Let $U$ be an open subset of $\mathbb{R}^n$. Let $S^m(U) \subset C^\infty(T^*U)$ denote the set of scalar valued symbols on $U$ of order $m \in \mathbb{R}$, $S(U) := \bigcup_{m \in \mathbb{R}} S^m(U) \subset C^\infty(T^*U)$ the algebra of all scalar valued symbols on $U$, $S^{-\infty}(U) := \bigcap_{m \in \mathbb{R}} S^m(U)$ the algebra of scalar smoothing symbols. We fix a norm on $\mathbb{R}^n$. Let $\chi$ be a smooth function on $T^*U$ such that $\chi(x, \xi) = 0$ for $|\xi| \leq 1/2$ and $\chi(x, \xi) = 1$ for $|\xi| \geq 1$. 6
Definition 2 \( \sigma \in S^m(U) \) is a classical symbol if for any positive integer \( N \) we can write:

\[
\sigma = \sum_{i=0}^{N} \chi \sigma_{m-i} + \sigma(N).
\]

where \( \sigma_{m-i} \) is positively homogeneous of order \( m - i \) (i.e.

\( \sigma_{m-i}(x, t\xi) = t^{m-i} \sigma_{m-i}(x, \xi) \)

for any \( t > 0 \) and any \( (x, \xi) \in T^*_xU - \{0\} \), and where \( \sigma(N) \) is a symbol of order \( m - N - 1 \). We write for short

\[
\sigma \sim \sum_{i=0}^{\infty} \chi \sigma_{m-i}.
\]

Let \( CS^m(U) \) denote the class of scalar classical symbols of order \( m \) and \( CS(U) = (\bigcup_{m \in \mathbb{C}} CS^m(U)) \) the algebra generated by scalar classical symbols of all orders. Similarly, let \( CS^m_{com}(U) \) denote the subsets of classical symbols of order \( m \) with compact support in \( U \) and \( CS_{com}(U) = (\bigcup_{m \in \mathbb{C}} CS^m_{com}(U)) \). \( CS^\mathbb{Z}(U) := \bigcup_{m \in \mathbb{Z}} CS^m(U) \) (resp. \( CS^\mathbb{Z}_{com}(U) := \bigcup_{m \in \mathbb{Z}} CS^m_{com}(U) \)) forms an algebra called the algebra of integer order symbols. We shall also consider its complement, namely the class \( CS^{\mathbb{Z}}(U) := CS(U) - CS^{\mathbb{Z}}(U) \) (resp. \( CS^{\mathbb{Z}}_{com}(U) := CS_{com}(U) - CS^{\mathbb{Z}}_{com}(U) \)) of non integer order symbols.

Let us equip \( CS_{com}(U) \) with the left product of symbols, also called the star product, which admits the following asymptotic development:

\[
(\sigma * \sigma') \sim \sum_{k \geq 0} (-i)^k \sum_{|\alpha| = k} \frac{1}{\alpha!} \partial^\alpha_x \sigma \cdot \partial^\alpha_x \sigma'.
\]

(See for instance [Sh] for details).

Symbol valued forms on \( T^*U \) where \( U \) is an open subset of \( \mathbb{R}^n \) are defined as follows.

Definition 3 Let \( k \) be a non negative integer, \( m \) a complex number. We let

\[
\Omega^k CS^m(U) = \{ \alpha \in \Omega^k(T^*U), \text{ with } \alpha_{I,J}(x,\xi) dx_I \wedge d\xi_J \}
\]

with \( \alpha_{I,J} \in CS^{m-|J|}(U) \}




denote the set of order \( m \)-classical symbol valued forms.
The left product of symbols \( * \) extends to symbol valued forms: given

\[
\alpha = \sum_{I,J,|I|+|J|=p} \alpha_{I,J}(x,\xi) \, dx_I \wedge d\xi_J \in \Omega^p CS^m(U)
\]

and

\[
\beta = \sum_{K,L,|K|+|L|=q} \alpha_{K,L}(x,\xi) \, dx_K \wedge d\xi_L \in \Omega^q CS^n(U),
\]

we set

\[
\alpha \wedge \beta := \sum_{I,J,|I|+|J|=k} \sum_{K,L,|K|+|L|=q} \alpha_{I,J}(x,\xi) \ast \alpha_{K,L}(x,\xi) \, dx_I \wedge d\xi_J \wedge dx_K \wedge d\xi_L
\]

which lies in \( \Omega^{p+q} CS^{m+n}(U) \).

Let \( \Omega^k CS(U) := (\bigcup_{m \in \mathbb{Z}} \Omega^k CS^m(U)) \) (resp. \( \Omega^k CS_{com}(U) := (\bigcup_{m \in \mathbb{Z}} \Omega^k CS^m_{com}(U)) \)) be the algebra generated by classical symbol (resp. with compact support) valued \( k \)-forms of all orders. The sets \( \Omega^k CS^Z(U) := \bigcup_{m \in \mathbb{Z}} \Omega^k CS^m(U) \), \( \Omega^k CS^Z_{com}(U) := \bigcup_{m \in \mathbb{Z}} \Omega^k CS^m_{com}(U) \) form algebras. We shall also consider the sets \( \Omega^k CS^{Z}(U) := \bigcup_{m \in \mathbb{Z}} \Omega^k CS^m(U) \) (resp. \( \Omega^k CS^{Z}_{com}(U) := \bigcup_{m \in \mathbb{Z}} \Omega^k CS^m_{com}(U) \)).

**Remark 1**

- With these conventions, \( d\xi_j \) is of order 1. Also, a \( k \)-form of order 0 reads \( \alpha = \sum_{|I|+|J|=k} \alpha_{I,J}(x,\xi) \, dx_I \wedge d\xi_J \) with \( \alpha_{I,J} \) of order \(-|J|\).

- The order of a zero degree symbol valued form \( \sigma \in \Omega^0 CS^m(U) \) coincides with the order of the corresponding classical symbol \( \sigma \).

- More generally, any zero order symbol valued \( k \)-form on \( U \) is of the type

\[
\alpha = \sum_{|I|+|J|=k} \alpha_{I,J} \, dx_I \wedge d\xi_J
\]

with \( \alpha_{I,J} \) of order \(-|J|\). In particular, given any \( \sigma \in CS(U) \), the top form \( \sigma_{-n}(x,\xi) \, dx_1 \wedge \cdots \wedge dx_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \) provides an example of positively homogeneous zero order symbol valued \( n \)-form.

**Lemma 2** A classical symbol valued form \( \alpha \in \Omega^k CS^m(U) \) of order \( m \) has an asymptotic expansion of the following form. For any non negative integer \( N \), there is a symbol valued form \( \alpha_{(N)} \) of order \( m - N - 1 \) such that

\[
\alpha = \sum_{i=0}^{N} \alpha_{m-i} + \alpha_{(N)}
\]

with \( \alpha_{m-i} := \sum_{|I|+|J|=k} \alpha_{I,J,m-|J|-i} \, dx_I \wedge d\xi_J \) is positively homogeneous of order \( m - i \), with \( \alpha_{I,J,m-|J|-i} \) positively homogeneous of order \( m - |J| - i \).

Furthermore, the exterior differentiation \( d \) sends \( \Omega^k CS^m(U) \) to \( \Omega^{k+1} CS^m(U) \) and for any integer \( j \leq m \), we have

\[
(d\alpha)_j = d\alpha_j.
\]
**Proof:** The first part of the statement follows trivially from the description of $\alpha$ combined with the properties of ordinary classical symbols. As for the second part of the statement we write

$$d\alpha = \sum_{|I|+|J|=k} \alpha_{I,J} \ dx_I \wedge d\xi_J$$

$$= \sum_{l=1}^n \sum_{|I|+|J|=k} \frac{\partial}{\partial x_I} \alpha_{I,J} \ dx_I \wedge dx_I \wedge d\xi_J$$

$$+ \sum_{l=1}^n \sum_{|I|+|J|=k} \frac{\partial}{\partial \xi_m} \alpha_{I,J} \ dx_I \wedge dx_I \wedge d\xi_J$$

which lies in $\Omega^k CS(U)$ since the order of $\frac{\partial}{\partial x_m} \alpha_{I,J} \ dx_I \wedge d\xi_J$ coincides with that of $\alpha_{I,J}$. The computation above also shows that if $\alpha$ is positively homogeneous of order $m$, so is $d\alpha$, which ends the proof of the lemma. $\square$

**Remark 2** In particular, for $\alpha \in \Omega CS(U)$ we have:

$$(d\alpha)_0 = d\alpha_0.$$ 

### 3 The Wodzicki residue character on classical symbols

#### 3.1 The Wodzicki residue extended to classical symbol valued valued forms

Let us first briefly recall the notion of Wodzicki residue on classical symbols [W, K].

**Definition 4** Let $U$ be an open subset in $\mathbb{R}^n$ and $x$ a point in $U$. The (local) Wodzicki residue density of a classical symbol $\sigma \in CS(U)$ at point $x$ is given by

$$\text{res}_x(\sigma) = \int_{|\xi|=1} \sigma_{-n}(x,\xi) \ dS\xi,$$

where $dS\xi = \sum_{i=1}^n (-1)^{i+1} \xi_i \ d\xi_1 \wedge \cdots \wedge d\hat{\xi}_i \wedge \cdots \wedge d\xi_n$ and $|\xi| = (\sum_{i=1}^n \xi_i^2)^{1/2}$ is the canonical norm in $\mathbb{R}^n$.

For any $\sigma \in CS(U)$ with compact support the Wodzicki residue of $\sigma$ is then defined as:

$$\text{res}(\sigma) := \int_U \text{res}_x(\sigma) \ dx.$$

**Remark 3** For any $t > 0$ we have $dS(t\xi) = t^n dS\xi$ and $\sigma_{-n}(x, t\xi) = t^{-n} \sigma_{-n}(x, \xi)$ so that the form $\sigma_{-n}(x, \xi) \ dx \wedge dS\xi$ is positively homogeneous of degree 0.
The Wodzicki residue extends from $CS(U)$ to $\Omega CS(U)$ in a straight forward manner.

**Definition 5** For any $\alpha = \sum_{I,J} \alpha_{IJ} \, dx_I \wedge d\xi_J \in \Omega CS_{com}(U)$, for any $x \in U$ we set

$$\text{res}_x \left( \sum_{I,J} \alpha_{IJ} \, dx_I \wedge d\xi_J \right) = \sum_{I,J} \text{res}_x(\alpha_{IJ}) \, dx_I = \sum_{J} \int_{|\xi|=1} (\alpha_{IJ})_{-n}(x,\xi) \, dS_\xi, \quad \text{if } |J| = n$$

and $\text{res}_x \left( \sum_{I,J} \alpha_{IJ} \, dx_I \wedge d\xi_J \right) = 0$ whenever $|J| \neq n$. Similarly, we set:

$$\text{res} \left( \sum_{I,J} \alpha_{IJ} \, dx_I \wedge d\xi_J \right) = \text{res}(\alpha_{IJ}) = \int_{|\xi|=1} (\alpha_{IJ})_{-n}(x,\xi) \, dx \wedge dS_\xi, \quad \text{if } |I| = |J| = n$$

and $\text{res} \left( \sum_{I,J} \alpha_{IJ} \, dx_I \wedge d\xi_J \right) = 0$ whenever $|I| \neq n$ or $|J| \neq n$.

It is useful to give an alternative more intrinsic formulation of this extended Wodzicki residue. The form $dS_\xi$ on $T^*_x U$ can be seen as the interior product $i_X(\Omega_x)$ of the volume form $\Omega_x := d\xi_1 \wedge \cdots \wedge d\xi_n$ on $T^*_x U$ with the Liouville (or radial) field

$$X(x,\xi) = \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i}.$$ 

This Liouville field can also be seen as the generator

$$X(x,\xi) := \frac{d}{dt|_{t=0}} f_t(x,\xi)$$

of the one parameter semigroup of transformations of $T^* U$:

$$\mathbb{R} \times T^* U \to T^* U$$

$$(t, (x,\xi)) \mapsto f_t(x,\xi) := (x, e^t \xi).$$

Let $\rho : T^* U - \{0\} \to S^* U$ denote the radial projection $\rho(x,\xi) = (x, \frac{\xi}{|\xi|})$, and let $j : S^* U \to T^* U - \{0\}$ denote the canonical fibre bundle injection. Clearly $\rho \circ j = Id$. We have the following lemma.

**Lemma 3** A form $\alpha$ on $T^* U - \{0\}$ is positively homogeneous of order zero if and only if it satisfies one of the two equivalent conditions:
1. the form can be written
\[ \alpha = \rho^* \beta + \frac{dr}{r} \wedge \rho^* \gamma \] (4)
with \( \beta, \gamma \in \Omega(S^*U) \), and more precisely:
\[ \beta = j^* \alpha, \quad \gamma = j^*(\iota_X \alpha). \]

2. \( L_X(\alpha) = 0 \) where \( L_X \) is the Lie derivative in direction \( X \).

**Proof:** The second condition is equivalent to \( \alpha(x, e^t \xi) = \alpha(x, \xi) \quad \forall t > 0 \) and hence to positive homogeneity of order zero since:
\[ L_X \alpha = \left. \frac{d}{dt} f_t^* \alpha \right|_{t=0} = \frac{d}{dt} |_{t=0} \alpha(x, e^t \xi). \]

For any \( \beta \in \Omega(S^*U) \) the differential form \( \rho^* \beta \) is invariant by dilations, hence positively homogeneous of order zero. The first condition then clearly implies that \( \alpha \) is positively homogeneous of order zero, as \( \frac{dr}{r} \) obviously is, hence (1) \( \Rightarrow \) (2).

Suppose now that (2) is verified, and seek for \( \beta \) and \( \gamma \) such that (4) holds. As \( j^*(\frac{dr}{r}) = 0 \) and \( \rho \circ j = Id \) we clearly have:
\[ j^* \alpha = j^*(\rho^* \beta + \frac{dr}{r} \wedge \gamma) = \beta. \]

Now \( \iota_X \alpha = \iota_X \frac{dr}{r} \wedge \rho^* \gamma = \rho^* \gamma \), hence \( \gamma = j^* \rho^* \gamma = j^*(\iota_X \alpha) \). We have then proved the uniqueness of \( \beta \) and \( \gamma \). To prove the existence, notice that the difference:
\[ \delta = \alpha - (\rho^* \iota_X \alpha) \]
verifies \( j^* \delta = \iota_X \delta = 0 \), hence it easily follows that \( \delta = 0 \). So (2) \( \Rightarrow \) (1).

**Example 1** Given any \( \sigma \in CS(U) \), the top form
\[ \alpha_\sigma(x, r \cdot \omega) := \sigma_{-n}(x, r \cdot \omega) \, dx_1 \wedge \cdots \wedge dx_n \wedge \frac{dr}{r} \wedge dS \omega \]
\[ = \sigma_{-n}(x, \xi) \, dx_1 \wedge \cdots \wedge dx_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \]
is a positively homogeneous zero order symbol valued \( n \)-form and we have:
\[ \iota_X \alpha_\sigma = \sigma_{-n} dx_1 \wedge \cdots \wedge dx_n \wedge dS \xi. \]

The following elementary result provides a more intrinsic formulation of the Wodzicki residue extended to forms.
Proposition 3 Let $U$ be an open subset of $\mathbb{R}^n$. Denote by $j$ (resp. $j_x$ for any $x \in U$) the injection of $S^*U$ (resp. $S_x^*U$) inside the cotangent bundle $T^*U$ (resp. inside $T_x^*U$). Given $\alpha \in \Omega CS(U)$, for any $x \in U$:

$$\text{res}_x(\alpha) := \int_{S^*U} j_x^* (\alpha)$$

where $\Lambda$ stands for the volume element $n! \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$, and:

$$\text{res}(\alpha) := \int_U \text{res}_x(\alpha) \, dx_1 \cdots dx_n = \int_{S^*U} j^* (\alpha).$$

### 3.2 Stokes’ formula for the Wodzicki residue

**Theorem 1** For any $\beta \in \Omega CS(U)$ with compact support we have

$$\text{res} (d\beta) = 0.$$

**Proof:** Using Cartan’s formula, this follows from Stokes’ property for ordinary integrals, since $(d\beta)_0 = d\beta_0$ implies $\mathcal{L}_\mathbf{X} d\beta_0 = 0$, hence:

$$\text{res} (d\beta) = \int_{S^*U} j^* (\mathbf{X} d\beta_0) = - \int_{S^*U} j^* (d\mathbf{X} \beta_0) = - \int_{S^*U} d \left( j^* (\mathbf{X} \beta_0) \right) = 0$$

since $S^*U$ is boundaryless. \qed

We recover this way a known integration by parts formula for the Wodzicki residue which underlies the traciality property of the Wodzicki residue on classical pseudodifferential operators.

**Corollary 1** For any $\sigma \in CS(U)$ with compact support,

$$\text{res} \left( \frac{\partial}{\partial \xi_i} \sigma \sigma' \right) = - \text{res} \left( \sigma \frac{\partial}{\partial \xi_i} \sigma' \right) \quad \forall i \in \{1, \ldots, n\}$$

and

$$\text{res} \left( \frac{\partial}{\partial x_i} \sigma \sigma' \right) = - \text{res} \left( \sigma \frac{\partial}{\partial x_i} \sigma' \right) \quad \forall i \in \{1, \ldots, n\}.$$

**Proof:** Let $\tau \in CS(U)$ with compact support. Applying Theorem 1 to $\beta^i := \tau_{-n+1} (x, \xi) \, dx_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n$, we get

$$\text{res} \left( \frac{\partial}{\partial \xi_i} \tau (x, \xi) \right) = \text{res} \left( \frac{\partial}{\partial \xi_i} \tau (x, \xi) \right) \, dx_1 \wedge \cdots \wedge dx_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n$$

$$= \text{res} \left( \frac{\partial}{\partial \xi_i} \tau_{-n+1} (x, \xi) \, dx_1 \wedge \cdots \wedge dx_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \right)$$

$$= (-1)^{i-1} \text{res} \left( d \left( \tau_{-n+1} (x, \xi) \, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n \right) \right)$$

$$= 0.$$
Applying this to $\tau \equiv \sigma \sigma'$ yields the first part of the corollary. A similar proof replacing $\frac{\partial}{\partial \xi_i}$ by $\frac{\partial}{\partial x_i}$ using Stokes’ formula applied to $\beta_i := \tau_n(x, \xi) d\xi_1 \wedge \cdots \wedge d\xi_i \wedge d\xi_1 \wedge \cdots \wedge d\xi_n$ gives the second equality of the corollary.

Corollary 2 The Wodzicki residue defines a trace on the subalgebra $CS_{\text{com}}(U) \in CS(U)$ of symbols with compact support in $x$

$$\text{res}([\sigma, \sigma'] \ast) = 0 \quad \forall \sigma, \sigma' \in C\ell_{\text{comp}}(U, \mathbb{C})$$

where we have set $[\sigma, \sigma'] \ast := \sigma \ast \sigma' - \sigma \ast \sigma'$.

Proof: We use the asymptotic development of the left product of symbols. There exists a positive integer $N$ such that :

$$\text{res}(\sigma \ast \sigma') = \sum_{k \leq N} (-i)^k \sum_{|\alpha| = k} \frac{1}{\alpha!} \text{res}(\partial^{\alpha}_x \sigma \partial^{\alpha}_x \sigma').$$

Indeed, the remainder term will be of order $<-n$ for sufficiently big $N$, and then will have vanishing residue. By the above lemma, we have for $\sigma, \sigma' \in CS(U)$ with compact support in $U$

$$\text{res}(\sigma \ast \sigma') = \sum_{|\alpha| \leq N} i^{|\alpha|} \frac{1}{\alpha!} \text{res}(\partial^{\alpha}_x \sigma \partial^{\alpha}_x \sigma')$$

$$= \sum_{|\alpha| \leq N} (-i)^{|\alpha|} \frac{1}{\alpha!} \text{res}(\partial^{\alpha}_\xi \partial^{\alpha}_x \sigma \cdot \sigma')$$

$$= \sum_{|\alpha| \leq N} i^{|\alpha|} \frac{1}{\alpha!} \text{res}(\partial^{\alpha}_x \sigma \cdot \partial^{\alpha}_\xi \sigma')$$

$$= \text{res}(\sigma' \ast \sigma).$$

\[\square\]

3.3 A Wodzicki residue cycle on zero order classical symbols

The exterior differential:

$$d : \Omega^k CS(U) \to \Omega^{k+1} CS(U)$$

obeys the usual “Leibniz rule”:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{k} \alpha \wedge d\beta \quad \forall \alpha \in \Omega^k CS(U), \beta \in \Omega^* CS(U)$$

as can easily be checked from [2] and [3] so that $(\Omega CS_{\text{com}}(U), d)$ is a graded differential algebra with $CS_{\text{com}}(U)$ equipped with the left product of symbols.
Theorem 2 Let $\text{CS}_{\text{com}}(U)$ be equipped with the left product of symbols. The triple $(\Omega \text{CS}_{\text{com}}(U), d, \text{res})$ yields an $2n$-cycle which we refer to as the Wodzicki residue cycle.

Proof: As previously observed, the Wodzicki residue vanishes on $\Omega^k \text{CS}_{\text{com}}(U)$ for $k < 2n$. It is closed by the Stokes’ formula since $\text{res}(d\beta) = 0$ for any $\beta \in \Omega^{2n-1}\text{CS}_{\text{com}}(U)$.

The fact that the ordinary Wodzicki residue defines a trace on $\text{CS}_{\text{com}}(U)$ immediately implies: 

$$\text{res}(\alpha \wedge_* \beta) = (-1)^{|\alpha| \cdot |\beta|} \text{res}(\beta \wedge_* \alpha)$$

so that $(\Omega \text{CS}_{\text{com}}(U), d, \text{res})$ defines a cycle. We call residue character the associated $2n$-character (see Appendix A).

Definition 6 Let the residue $k$-cochain denote the $k+1$-linear form on $\text{CS}_{\text{com}}(U)$

$$\chi^\text{res}_k(\sigma_0, \ldots, \sigma_k) = \text{res}(\sigma_0 \ast d\sigma_1 \wedge \cdots \wedge d\sigma_k)$$

for all $\sigma_0, \ldots, \sigma_k \in \text{CS}_{\text{com}}(U)$.

Residue $k$-cochains vanish for $k < 2n$ and the residue character is the $2n$-residue cochain $\chi^\text{res}_{2n}$. It satisfies the following properties (with the notations of Appendix A):

- $B_0 \chi^\text{res}_{2n} = 0$ and $B \chi^\text{res}_{2n} = 0$,
- $b_* \chi^\text{res}_{2n} = 0$ where $b_*$ is the Hochschild coboundary operator associated with the left product on symbols.

Restricting to zero order symbols we get:

Theorem 3 For any symbols $\sigma_0, \ldots, \sigma_{2n} \in \text{CS}_{\text{com}}^0(U)$,

$$\chi^\text{res}_{2n}(\sigma_0, \ldots, \sigma_{2n}) = \text{res}(\sigma_0 \cdot d\sigma_1 \wedge \cdots \wedge d\sigma_{2n})$$

$$= \int_{S^*U} j^* \chi(\sigma^L_0, d\sigma^L_1 \wedge \cdots \wedge d\sigma^L_{2n})$$

$$= \frac{(-1)^n}{n!} A[\text{res}(\sigma_0 \theta(\sigma_1, \sigma_2) \cdots \theta(\sigma_{2n-1}, \sigma_{2n}))].$$

Here $\sigma^L_i$ stands for the leading symbol of $\sigma_i$ and where we have set $\theta(\sigma_i, \sigma_j) = \sigma_i \ast \sigma_j - \sigma_i \cdot \sigma_j$ as in section 1. $A$ denotes the antisymmetrisation over all but the first variable.

Proof: The difference $\sigma_0 \ast d\sigma_1 \wedge \cdots \wedge d\sigma_{2n} - \sigma_0 d\sigma_1 \wedge \cdots \wedge d\sigma_{2n}$ has clearly vanishing residue as top form of order $\leq -1$, hence the first equality. The second equality then follows since the top order term $(\sigma_0 d\sigma_1 \wedge \cdots \wedge d\sigma_{2n})_0$ is precisely
\[ \sigma_0^L \sigma_1^L \wedge \cdots \wedge \sigma_{2n}^L. \]

As for the last equality, we have \( \theta(\sigma_1, \sigma_j) \sim \sum_{|\alpha| \neq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha \sigma_1 \partial^2 \sigma_j \) so that

\[
A \left[ \text{res} \left( \sigma_0 \theta(\sigma_1, \sigma_2) \cdots \theta(\sigma_{2n-1}, \sigma_{2n}) \right) \right]
= A \left[ \sum_{|\alpha_1| \neq 0, \cdots, |\alpha_n| \neq 0} \frac{(-i)^{|\alpha|}}{\alpha_1! \cdots \alpha_n!} \text{res} \left( \sigma_0 \partial_{\xi_1}^{\alpha_1} \sigma_1 \partial_{\xi_2}^{\alpha_2} \sigma_2 \cdots \partial_{\xi_n}^{\alpha_n} \sigma_{2n-1} \partial_{\xi_n}^{\alpha_n} \sigma_{2n} \right) \right]
= A \left[ \sum_{i_1, \ldots, i_n \in \{1, \ldots, n\}} (-i)^n \text{res} \left( \sigma_0 \partial_{\xi_{i_1}} \sigma_1 \partial_{x_{i_1}} \sigma_2 \cdots \partial_{\xi_{i_n}} \sigma_{2n-1} \partial_{x_{i_n}} \sigma_{2n} \right) \right]
= i^n n! \text{res} \left( \sigma_0 \partial_1 \wedge \partial_2 \wedge \cdots \wedge \partial_{2n-1} \wedge \partial_{2n} \right).
\]

4 Cut-off integrals of symbol valued forms and cosphere cochain

4.1 Cut-off integrals extended to classical symbols valued forms

Defining cut-off integrals amounts to extracting finite parts from otherwise divergent integrals, a procedure which we recall here (without proofs) in the case of ordinary classical symbols \( \mathbb{H}, \mathbb{G}, \mathbb{W}, \mathbb{KV} \).

**Proposition 4** Let \( U \) be an open subset of \( \mathbb{R}^n \) and let \( x \in U \). Given \( \sigma \sim \sum_{i=0}^{\infty} \chi_{m-i} \in CS^m(U) \), the expression \( \int_{B^*_r(0,R)} \sigma(x, \xi) \, d\xi \) has an asymptotic expansion

\[
\int_{B^*_r(0,R)} \sigma(x, \xi) \, d\xi = c(x) + \sum_{i=0, m-i+n \neq 0} a_i(x) \frac{R^{m-i+n}}{m-i+n} + b(x) \log R
\]

where \( c(x), a_i(x), b(x) \in \mathbb{C} \). The finite part called the cut-off integral of \( \sigma(x, \cdot) \)
which is given by the constant \( c(x) \) reads:

\[
\int_{T^*_x U} \sigma(x, \xi) d\xi := \text{fp}_{R \to \infty} \int_{B^*_x(0, R)} \sigma(x, \xi) d\xi
\]

\[
= \int_{B^*_x(0, 1)} \sigma(x, \xi) d\xi
\]

\[
+ \int_{T^*_x U - B^*_x(0, 1)} \sigma_N(x, \xi) d\xi
\]

\[
- \sum_{i=0, m-i+n \neq 0}^{K_N} \frac{1}{m-i+n} \int_{|\xi|=1} \sigma_{m-i}(x, \xi) dS_\xi.
\]

If \( \sigma \in CS_{\text{com}}(U) \) we set

\[
\int_{T^*_x U} \sigma(x, \xi) := \int_U dx \int_{T^*_x U} \sigma(x, \xi).
\]

The constant \( b(x) \) coincides with the local Wodzicki residue density \( \text{res}_x(\sigma) \). When it vanishes, the finite part \( \text{fp}_{R \to \infty} \int_{B^*_x(0, R)} \sigma(x, \xi) d\xi \) is independent of the rescaling \( R \to \lambda R \). Specifically, this holds for non integer order symbols.

**Remark 4** This cut-off integral extends the ordinary integral in the following sense: if \( \sigma \) has order smaller than \(-n\) then \( \int_{B^*_x(0, R)} \sigma(x, \xi) d\xi \) converges when \( R \to \infty \) and \( \int_{T^*_x U} \sigma(x, \xi) dx = \int_{T^*_x U} \sigma(x, \xi) d\xi \).

**Definition 7** The cut-off integral on \( T^*_x U \) of a form \( \alpha = \sum_{I,J} \alpha_{I,J} dx_I \wedge d\xi_J \in \Omega CS(U) \) with compact support in \( x \) is defined by:

\[
\int_{T^*_x U} \alpha := \int_{T^*_x U} \alpha_{I,J}(x, \xi) dx_1 \wedge \cdots \wedge dx_n d\xi_1 \wedge d\xi_n \text{ if } |I| = |J| = n
\]

and which vanishes otherwise.

As in the case of ordinary integrals, we recover the cut-off integral on symbol valued functions \( \sigma \in CS^{\mathbb{Z}}(U) \) via the integral on forms by integrating the top form \( \sigma(x, \xi) dx \wedge d\xi \) setting:

\[
\int_{T^*_x U} \sigma(x, \xi) := \int_{T^*_x U} \sigma(x, \xi) dx \wedge d\xi
\]

where the right hand side is now seen as a cut-off integral on a symbol valued form.

Similarly to ordinary integrals, cut-off integrals on forms satisfy Stokes’ property (compare with Lemma 5.5 in [LP]).
Theorem 4 Let $U$ be an open subset of $\mathbb{R}^n$ and let $\beta \in \Omega^{2n-1} CS_{com}(U)$ be a symbol valued form. Then

$$\int_{T^*U} d\beta = \sum_{I,J} \int_{S^*(0,1)} \beta_{I,J,-n+1}(x,\xi) \, dx_I \wedge d\xi_J$$

so that Stokes’ formula:

$$\int_{T^*U} d\beta = 0$$

holds whenever $\beta \in \Omega^{2n-1} CS_{com}^Z(U)$.

Here $\beta(x,\xi) = \sum_{I,J \subset \{1,\ldots,n\}, |I|+|J|=2n-1} \beta_{I,J}(x,\xi) \, dx_I \wedge d\xi_J$ with $\beta_{I,J} \in CS(U)$.

Proof: The $2n-1$ form reads $\beta(x,\xi) = \sum_{I,J \subset \{1,\ldots,n\}, |I|+|J|=2n-1} \beta_{I,J}(x,\xi) \, dx_I \wedge d\xi_J$ with $\beta_{I,J} \in CS^Z(U)$ so that, letting $B^*(0,R)$, resp. $S^*(0,R)$ be respectively the ball in the cotangent bundle of radius $R$ centered at the origin, and the sphere
in the cotangent bundle of radius $R$ centered at the origin, we have

$$\int_{T^*U} d\beta = \sum_{I,J} \int_{T^*U} d(\beta_{I,J}(x, \xi) \, dx \wedge d\xi)$$

$$= \sum_{I,J} \lim_{R \to \infty} \int_{B^*(0,R)} d(\beta_{I,J}(x, \xi) \, dx \wedge d\xi)$$

$$= \sum_{I,J} \lim_{R \to \infty} \int_{S^*(0,R)} \beta_{I,J}(x, \xi) \, dx \wedge d\xi$$

using Stokes' property for ordinary integrals

$$= \sum_{I,J} \lim_{R \to \infty} \int_{S^*(0,R)} \beta_{I,J}(x, \xi) \, dx \wedge d\xi$$

$$= \sum_{I,J} \lim_{R \to \infty} \int_{S^*(0,R)} \chi(\xi) \beta_{I,J}(x, \xi) \, dx \wedge d\xi$$

since $\lim_{|\xi| \to \infty} |\xi|^{n-1} \beta_{I,J}(N)(x, \xi) = 0$ and $\chi = 1$ outside $B^*(0,1)$

$$= \sum_{I,J} \lim_{R \to \infty} \int_{S^*(0,R)} \beta_{I,J}(x, \xi) \, dx \wedge d\xi$$

$$= \sum_{I,J} \lim_{R \to \infty} \int_{S^*(0,R)} \beta_{I,J}(x, \xi) \, dx \wedge d\xi$$

$$= \sum_{I,J} \lim_{R \to \infty} \int_{S^*(0,R)} \beta_{I,J}(x, \xi) \, dx \wedge d\xi$$

$$= \sum_{I,J} \lim_{R \to \infty} \int_{S^*(0,R)} \beta_{I,J}(x, \xi) \, dx \wedge d\xi$$

$$= \sum_{I,J} \lim_{R \to \infty} \int_{S^*(0,R)} \beta_{I,J}(x, \xi) \, dx \wedge d\xi$$

where $m_{I,J} \notin \mathbb{Z}$ is the order of $\beta_{I,J}$.

As a consequence, cut-off integrals on non-integer order symbols satisfy an integration by parts formula:

**Corollary 3** For any $\sigma \in CS_{\text{com}}(U)$ then

$$\int_{T^*U} \frac{\partial}{\partial \xi_i} \sigma(x, \xi) \, d\xi \, dx = (-1)^{i-1} \int_{S^*(0,1)} \sigma_{-n+1}(x, \xi) \, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n \wedge dx_1 \wedge \cdots \wedge dx_n.$$

In particular, if $\sigma \in CS^0_{\text{com}}(U)$ then

$$\int_{T^*U} \frac{\partial}{\partial \xi_i} \sigma(x, \xi) \, d\xi \, dx = 0 \quad \forall i \in \{1, \cdots, n\}.$$
\textbf{Proof:} Applying Stokes’ formula to $\beta := \sigma(x, \xi) \, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n \wedge dx_1 \wedge \cdots \wedge dx_n$ we have:

\begin{align*}
\int_{T^* U} \frac{\partial}{\partial \xi_i} \sigma(x, \xi) \, d\xi &= (-1)^{i-1} \int_{T^* U} d \left( \sigma(x, \xi) \, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n \wedge dx_1 \wedge \cdots \wedge dx_n \right) \\
&= (-1)^{i-1} \int_{S^*(0,1)} \sigma_{n+1}(x, \xi) \, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n \wedge dx_1 \wedge \cdots \wedge dx_n.
\end{align*}

This last term vanishes whenever $\sigma$ has non integer order. \hfill \Box

The integration by parts formula yields translation invariance of cut-off integrals on non integer order symbols.

\textbf{Corollary 4} For any $\sigma \in CS_{\text{com}}(U)$

\begin{align*}
\int_{T^* U} \sigma(x, \xi + \eta) \, dx \, d\xi = \int_{T^* U} \sigma(x, \xi) \, dx \, d\xi \quad \forall \eta \in C^\infty(U, T^* U).
\end{align*}

If $\sigma \in CS_{\text{com}}^\underline{\mathbb{Z}}(U)$ then

\begin{align*}
\int_{T^* U} \sigma(x, \xi + \eta) \, dx \, d\xi = \int_{T^* U} \sigma(x, \xi) \, dx \, d\xi \quad \forall \eta \in C^\infty(U, T^* U).
\end{align*}

\textbf{Proof:} A Taylor expansion $\eta \mapsto \sigma(\xi + \eta)$ in $\eta$ at 0 yields, for any $x \in U$, the existence of some $\theta \in ]0,1[$ such that:

\begin{align*}
\int_{T^* U} \sigma(x, \xi + \eta) \, d\xi &= \sum_{|\alpha| \leq K} \int_{T^* U} d\xi \left( \frac{D^\alpha \sigma(x, \xi)}{\alpha!} \eta^\alpha + \sum_{|\alpha| = K} \int_{T^* U} d\xi \left( \frac{D^\alpha \sigma(x, \xi + \theta \eta)}{\alpha!} \eta^\alpha \right) \right).
\end{align*}

Since $\sigma$ has non integer order symbol, neither has $D^\alpha \sigma$ an integer order. After integrating over $U$, the terms corresponding to $|\alpha| \neq 0$ vanish by the integration by parts formula, as a result of which we are left with the $|\alpha| = 0$ term and

\begin{align*}
\int_{T^* U} \sigma(x, \xi + \eta) \, dx \, d\xi = \int_{T^* U} \sigma(x, \xi) \, dx \, d\xi.
\end{align*}

\hfill \Box

\subsection{4.2 The cosphere cochain as a $B_0$-coboundary}

\textbf{Definition 8} Let the cosphere $k$-cochain denote the $(k+1)$-linear form on $CS_{\text{com}}(U)$

\begin{align*}
\psi_k(\sigma_0, \cdots, \sigma_k) = \int_{S^* U} j^* (\sigma_0 \ast d\sigma_1 \wedge \cdots \wedge d\sigma_k)_0
\end{align*}

for all $\sigma_0, \cdots, \sigma_k \in CS_{\text{com}}(U)$.
Since $\psi_k$ vanishes for $k < 2n - 1$, we shall focus on $\psi_{2n-1}$.

We introduce a cochain on $\text{CS}_{\text{com}}(U)$ built from cut-off integrals of classical symbol valued forms:

**Definition 9** For any $\sigma_0, \ldots, \sigma_k \in \text{CS}_{\text{com}}(U)$ we set

$$\chi_k^{\text{cut-off}}(\sigma_0, \ldots, \sigma_k) = \int_{T^* U} (\sigma_0 \ast d \sigma_1 \wedge \ldots \wedge d \sigma_k)_0.$$ 

**Remark 5** $\chi_k^{\text{cut-off}}$ vanishes for $k < 2n$ so that we focus on the $2n$-cochain $\chi_{2n}^{\text{cut-off}}$.

By Stokes’ formula for cut-off integrals on non integer order symbol valued forms, we have (with $B_0$ as in Appendix A):

$$B_0 \chi_{2n}^{\text{cut-off}}(\sigma_0, \ldots, \sigma_{2n-1}) = \chi_{2n}^{\text{cut-off}}(1, \sigma_0, \ldots, \sigma_{2n-1})$$

$$= \int_{T^* U} (d \sigma_0 \wedge \ldots \wedge d \sigma_{2n-1})_0$$

$$= \int_{T^* U} d (\sigma_0 \wedge \ldots \wedge d \sigma_{2n-1})_0$$

$$= 0$$

whenever the sum of the orders of the $\sigma_i$’s is non integer.

However, $\chi_{2n}^{\text{cut-off}}$ is not cyclic in general; the obstruction to its cyclicity is measured by the cosphere cochain.

**Proposition 5**

$$B_0 \chi_{2n}^{\text{cut-off}}(\sigma_0, \ldots, \sigma_{2n-1}) = \chi_{2n}^{\text{cut-off}}(1, \sigma_0, \ldots, \sigma_{2n-1})$$

$$= \psi_{2n-1}(\sigma_0, \ldots, \sigma_{2n-1})$$

for any $\sigma_0, \ldots, \sigma_{2n-1} \in \text{CS}_{\text{com}}(U)$.

It vanishes whenever the $\sigma_i$’s have orders which sum up to a non integer.

**Proof:**

$$B_0 \chi_{2n}^{\text{cut-off}}(\sigma_0, \ldots, \sigma_{2n-1}) = \chi_{2n}^{\text{cut-off}}(1, \sigma_0, \ldots, \sigma_{2n-1})$$

$$= \int_{T^* U} (d \sigma_0 \wedge \ldots \wedge d \sigma_{2n-1})_0$$

$$= \mathfrak{p}\int_{B^*(0, R)} d (\sigma_0 \ast d \sigma_1 \wedge \ldots \wedge d \sigma_{2n-1})_0$$

$$= \mathfrak{p}\int_{S^*(0, R)} (\sigma_0 \ast d \sigma_1 \wedge \ldots \wedge d \sigma_{2n-1})_0$$

$$= \int_{S^*(0, 1)} (\sigma_0 \ast d \sigma_1 \wedge \ldots \wedge d \sigma_{2n-1})_0$$

$$= \psi_{2n-1}(\sigma_0, \sigma_1, \ldots, \sigma_{2n-1}).$$
5 The Wodzicki residue extended to forms as a complex residue

We first recall how the ordinary residue density on symbols can be interpreted as a complex residue via cut-off integrals of symbols.

5.1 The Wodzicki residue density on symbols as a complex residue

Recall that given an open subset $U \subset \mathbb{R}^n$ (resp. an $n$-dimensional manifold $M$), for any real number $m$ the class $C_{\text{com}}^m(U)$ of classical symbols of order $m$ with compact support on $U$ (resp. of classical symbols of order $m$) can be equipped with a natural Fréchet topology so that $\bigcup_{m \in \mathbb{R}} C_{\text{com}}^m(U)$ comes equipped with an inductive limit Fréchet topology. We first recall the notion of holomorphic regularisation (see e.g. [P] for a review of various regularisations):

**Definition 10** A holomorphic regularisation procedure on $C_{\text{com}}^m(U)$ is a map

$$\mathcal{R} : C_{\text{com}}^m(U) \to \text{Hol}(C_{\text{com}}^m(U))$$

$$\sigma \mapsto \sigma(z)$$

where $\text{Hol}(C_{\text{com}}^m(U))$ is the algebra of holomorphic maps with values in $C_{\text{com}}^m(U)$, such that

1. $\sigma(0) = \sigma$,
2. $\sigma(z)$ has holomorphic order $\alpha(z)$ (in particular, $\alpha(0)$ is equal to the order of $\sigma$) such that $\alpha'(0) \neq 0$.

By holomorphic map we mean that each positively homogeneous component $\sigma_{\alpha(z) - j}(z)$ is holomorphic and that for any integer $N \geq 1$ the remainder

$$\sigma_{(N)}(z)(x, \xi) := \sigma(z)(x, \xi) - \sum_{j=0}^{N-1} \sigma_{\alpha(z) - j}(z)(x, \xi)$$

is holomorphic in $z$ as an element of $C^\infty(U \times \mathbb{R}^n)$ with $k^{th}$ $z$-derivative

$$\sigma_{(N)}^{(k)}(z)(x, \xi) := \partial_z^k(\sigma_{(N)}(z)(x, \xi)) \in S^{\alpha(z) - N + \epsilon}(U, V)$$

for any $\epsilon > 0$. 

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A first example of holomorphic regularisation is the well known Riesz regularisation, which sends a classical symbol $\sigma$ of order $m$ to

$$\sigma(z)(x, \xi) := \sum_{j=0}^{N} \chi(\xi) \sigma_{m-j}(x, \xi) \cdot |\xi|^{-z} + \sigma_{(N)}(x, \xi)$$

with the notations of (1) and where $N$ is chosen large enough so that $m - N < -n$. Generalisations of the type $\sigma \mapsto \sigma(z)(x, \xi) := H(z) \sum_{j=0}^{N} \chi(\xi) \sigma_{\alpha-j}(x, \xi) \cdot |\xi|^{-z} + \sigma_{(N)}(x, \xi)$ where $H$ is a holomorphic function such that $H(0) = 1$ include dimensional regularisation which arises in physics (see [P]).

**Remark 6**

**Proposition 6** [G], [KV], [L] Given a holomorphic regularisation procedure $\mathcal{R}$ : $\sigma \mapsto \sigma(z)$ on $CS_{\text{com}}(U)$ and any symbol $\sigma \in CS_{\text{com}}(U)$, for any $x \in U$ the map $z \mapsto \int_{T^*x} \sigma(z)(x, \xi) d\xi$ (resp. $z \mapsto \int_{T^*U} dx d\xi \sigma(z)$) is meromorphic with simple poles at points in $\alpha^{-1}([-n, +\infty[ \cap \mathbb{Z})$ where $\alpha$ is the order of $\sigma(z)$. Moreover for any $x \in U$

$$\text{Res}_{z=0} \int_{T^*x} \sigma(z)(x, \xi) d\xi = -\frac{1}{\alpha'(0)} \text{Res}_x(\sigma(0)),$$

respectively

$$\text{Res}_{z=0} \int_{T^*U} \sigma(z)(x, \xi) d\xi = -\frac{1}{\alpha'(0)} \text{Res}(\sigma(0)).$$

On the grounds of this proposition we set:

**Definition 11** The $\mathcal{R}$-regularised integral of $\sigma \in CS_{\text{com}}(U)$ is defined by:

$$\int_{T^*x}^{\mathcal{R}} \sigma(x, \xi) d\xi := \lim_{z \to 0} \left( \int_{T^*x} \sigma(z)(x, \xi) d\xi - \frac{1}{z} \text{Res}_{z=0} \int_{T^*U} d\xi \sigma(z)(x, \xi) \right)$$
Proof of the proposition: We identify $T^*_x U$ with $\mathbb{R}^n$ using a coordinate chart. From equation (5) we have

$$\int_{\mathbb{R}^n} \sigma(z)(x, \xi) \, d\xi = \int_{B(0,1)} \sigma(z)(x, \xi) \, d\xi$$

$$- \sum_{i=0, \alpha(z) - i + n \neq 0}^N \frac{1}{\alpha(z) - i + n} \int_{S(0,1)} \sigma_{\alpha(z) - i}(z)(x, \xi) \, d\xi$$

$$\int_{\mathbb{R}^n} \sigma_{(N)}(z)(\xi) \, d\xi$$

$$= \int_{B(0,1)} \sigma(z)(x, \xi) \, d\xi$$

$$- \sum_{i=0, \alpha(z) - i + n \neq 0}^N \frac{1}{\alpha(0) - i + n + \alpha'(0) z + o(z)} \int_{S(0,1)} \sigma_{\alpha(z) - i}(z)(x, \xi) \, d\xi$$

$$\int_{\mathbb{R}^n} \sigma_{(N)}(z)(x, \xi) \, d\xi,$$

where we have written $\alpha(z) = \alpha(0) + \alpha'(0).z + o(z)$. As a consequence, we have that:

$$\text{Res}_{z=0} \int_{\mathbb{R}^n} \sigma(z)(x, \xi) \, d\xi$$

$$= \text{Res}_{z=0} \int_{B(0,1)} \sigma(z)(x, \xi) \, d\xi$$

$$- \text{Res}_{z=0} \sum_{i=0}^K \frac{1}{\alpha(0) - i + n + \alpha'(0) z + o(z)} \int_{S(0,1)} \sigma_{\alpha(z) - i}(z)(x, \xi) \, d\xi$$

$$+ \text{Res}_{z=0} \int_{\mathbb{R}^n} \sigma_{(N)}(z)(x, \xi) \, d\xi$$

$$= -\frac{1}{\alpha'(0)} \int_{S(0,1)} \sigma_{-n}(0)(x, \xi) \, d\xi$$

$$= -\frac{1}{\alpha'(0)} \text{res}_x(\sigma(0)).$$

This result extends to classical symbol valued forms.
5.2 Cut-off integrals of holomorphic families of symbol valued forms

**Definition 12** A holomorphic regularisation procedure on $\Omega CS(U)$ is a map

$$R : \Omega C_{\text{com}}(U) \rightarrow \Omega \text{Hol}(CS_{\text{com}}(U))$$

$$\omega \mapsto \omega(z)$$

where

$$\Omega \text{Hol}(CS_{\text{com}}(U)) := \{ z \mapsto \omega(z) = \sum_{I,J} \omega_{IJ} dx_I \wedge d\xi_J \in \Omega CS_{\text{com}}(U), \ z \mapsto \omega_{IJ}(z) \text{ lies in Hol} CS(U) \ \text{for all multi-indices } I,J \}$$

and

1. $\omega(0) = \omega$,
2. $\omega(z)$ has holomorphic order $\alpha(z)$ (in particular, $\alpha(0)$ is equal to the order of $\omega$) such that $\alpha'(0) \neq 0$.

**Remark 7** Clearly, any holomorphic regularisation $R$ on $CS_{\text{com}}(U)$ induces one on $\Omega CS_{\text{com}}(U)$ setting:

$$R(\omega) = \sum_{I,J} R(\omega_{IJ}) dx_I \wedge d\xi_J.$$ 

**Theorem 5** Given a holomorphic regularisation procedure $R : \omega \mapsto \omega(z)$ on $\Omega CS_{\text{com}}(U)$ induced by a regularisation $R : \sigma \mapsto \sigma(z)$ on $CS_{\text{com}}(U)$ and any symbol valued form $\omega \in \Omega CS_{\text{com}}(U)$, the map $z \mapsto \int_{T^* x} \omega(z)$ (resp. $z \mapsto \int_{T^* x} \omega(z)$) is meromorphic with simple poles at points in $\alpha^{-1}([-n, +\infty[ \cap \mathbb{Z})$ where $\alpha$ is the order of $\omega(z)$. Moreover for any $x \in U$

$$\text{Res}_{z=0} \int_{T^* x} \omega(z)(x,\xi) = -\frac{1}{\alpha'(0)} \text{res}_x (\omega(0)),$$

respectively

$$\text{Res}_{z=0} \int_{T^* x} \omega(z)(x,\xi) = -\frac{1}{\alpha'(0)} \text{res}(\omega(0)).$$

On the grounds of this theorem, we set the following definition:
**Definition 13** The $\mathcal{R}$-regularised integral of $\omega \in \Omega CS_{com}(U)$ is defined by:
\[
\int_{T^*U}^{\mathcal{R}} \omega(x, \xi) := \lim_{z \to 0} \left( \int_{T^*U} \omega(z)(x, \xi) - \frac{1}{z} \text{Res}_{z=0} \int_{T^*U} \omega(z)(x, \xi) \right).
\]

**Proof of the theorem:** The result follows from applying Proposition 6 to each component $\omega_{IJ}(z)$ of the form $\omega(z) = \sum_{IJ} \omega_{IJ}(z) dx_I \wedge d\xi_J$. The symbol valued form $\omega_{IJ}(z)$ has order $\alpha_{IJ}(z) = \alpha(z) - |J|$ so that $\alpha'_{IJ}(0) = \alpha'(0)$. Since $z \mapsto \int_{T^*U} \omega_{IJ}(z)$ is meromorphic with simple poles so is $z \mapsto \int_{T^*U} \omega(z)$ and we have
\[
\text{Res}_{z=0} \int_{T^*U} \omega(z)(x, \xi) = \sum_{IJ} \text{Res}_{z=0} \int_{T^*U} \omega_{IJ}(z)(x, \xi) dx_I \wedge d\xi_J
\]
\[
= - \sum_{IJ} \frac{1}{\alpha'_{IJ}(0)} \text{Res}_{z}(\omega_{IJ}(0)) dx_I \wedge d\xi_J
\]
\[
= - \frac{1}{\alpha'(0)} \sum_{IJ} \text{Res}_{z}(\omega_{IJ}(0)) dx_I \wedge d\xi_J
\]
\[
= - \frac{1}{\alpha'(0)} \text{Res}_{z}(\omega(0)),
\]

Stokes’ formula holds as an equality of meromorphic functions:

**Theorem 6** Given a holomorphic regularisation procedure $\mathcal{R} : \omega \mapsto \omega(z)$ on $\Omega CS_{com}(U)$ induced by a regularisation $\mathcal{R} : \sigma \mapsto \sigma(z)$ on $CS_{com}(U)$ and any symbol valued form $\omega \in \Omega CS_{com}(U)$, we have the following equality of meromorphic functions:
\[
\int_{T^*U} d(\omega(z)) = 0,
\]

**Proof:** Since $\omega(z)$ has non integer order outside a discrete set of complex numbers, and since by Theorem 4, Stokes’ property holds for non integer order symbols valued forms, the statement holds outside this discrete set of poles. The meromorphicity of the function $z \mapsto \int_{T^*U} d(\omega(z))$ proved in Theorem 5 then yields the expected equality of meromorphic functions.

**Remark 8**
- This statement in the case of dimensional regularisation and transposed to forms built from Feynman type functions as in [E] corresponds to Proposition 12 of [E].
- In general,
\[
\int_{T^*U}^{\mathcal{R}} d\omega \neq 0
\]

since exterior differentiation and regularisation $\mathcal{R}$ do not “commute”.

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Appendix A

We recall here a few definitions borrowed from non commutative geometry see e.g. [C], [GVF]. Let \((A, \star)\) be an associative algebra over some ring \(R\) with unit 1. The space \(C^n(A, R)\) of \(R\)-valued \(n + 1\)-linear forms on \(A\) corresponds to the space of \(n\)-cochains on \(A\). Equivalently, these spaces can be seen as spaces of \(R\)-multilinear \(n\)-forms on \(A\) with values in the \(R\)-algebraic dual \(A^*\), seen as an \(A\)-bimodule, where for \(\chi \in A^*\) we put \(a' \chi(a) a'' = \chi(a'' a')\).

Following [C] we define the operators \(B_0\) and \(B\) acting on cochains:

**Definition 14** Let

\[
B_0 : C^n(A) \to C^{n-1}(A) \\
\chi \mapsto B_0 \chi(a_0, \cdots, a_{n-1}) := \chi(1, a_0, \cdots, a_{n-1}) - (-1)^n \chi(a_0, \cdots, a_{n-1}, 1).
\]

Let \(B := A B_0\) where \(A\) denotes cyclic antisymmetrisation in all variables so that

\[
B \chi(a_0, \cdots, a_{n-1}) = \sum_{i=0}^{n-1} (-1)^i \chi(1, a_i, a_{i+1}, \cdots) - (-1)^n \sum_{i=0}^{n-1} (-1)^i \chi(a_i, a_{i+1}, \cdots, a_{i-1}, 1)
\]

One can check that \(B^2 = 0\) so that \(B\) defines a homology on \(C^\bullet(A)\) [C].

**Definition 15** The Hochschild coboundary for the product \(\star\) of an \(n\)-cochain \(\chi\) is defined by:

\[
b_\star \chi(a_0, \cdots, a_{n+1}) = \sum_{j=0}^{n} (-1)^j \chi(a_0, \cdots, a_j \star a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1} \chi(a_{n+1} \star a_0, \cdots, a_n).
\]

It satisfies the condition \(b_\star^2 = 0\) and hence defines a cohomology called the Hochschild cohomology of \((A, \star)\).

**Definition 16** An \(n\)-dimensional cycle is given by a triple \((\Omega, d, \int)\) where \(\Omega\) is a graded differential algebra on \(C\) equipped with the differential \(d\) such that \(d^2 = 0\) and \(\int : \Omega^n \to C\) is a closed graded trace i.e. \(\int\) is a linear map which, when extended to \(\Omega\) by 0, satisfies

\[
\int \alpha \wedge \beta = (-1)^{|\alpha||\beta|} \cdot \int \beta \wedge \alpha, \quad \int d\beta = 0 \quad \forall \beta \in \Omega^{n-1}(A).
\]

An \(n\)-cycle on an algebra \(A\) on \(C\) is a cycle \((\Omega, d, \int)\) together with a homomorphism \(\rho : A \to \Omega^0\). The character \(\chi_n\) of an \(n\)-cycle is defined by:

\[
\chi_n(a_0, \cdots, a_n) = \int \rho(a_0) d\rho(a_1) \cdots d\rho(a_n) \quad \forall a_i \in A.
\]

Let us also recall that the character of a cycle has the following properties:
1. $\chi_n$ is cyclic i.e.

$$\chi_n(a_0, \cdots, a_n) = (-1)^n \chi_n(a_1, \cdots, a_n, a_0), \quad \forall a_i \in A$$

2. $\chi_n(1, a_1, \cdots, a_n) = 0 \quad \forall a_i \in A$.

3. $b \chi_n = 0$ where $b$ is the Hochschild coboundary associated with the product on $A$.

References

[C] A. Connes, Non commutative Geometry, Academic Press (1994)

[CFS] A. Connes, M. Flato, D. Sternheimer, *Closed star products and cyclic cohomology*, Lett. Math. Phys. 24 1–12 (1992)

[F] B. Fedosov, Deformation quantization and index theory, Akademie Verlag, Mathematical topics 9 (1996)

[G] V. Guillemin, *Residue traces for certain algebras of Fourier integral operators*, Journ. Funct. Anal. 115 (1993) 391–417; *A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues*, Adv. Math. 55 (1985) 131–160

[E] P. Etingof, *Note on dimensional regularization*, in Quantum Fields and Strings: A course for Mathematicians, Vol 1. AMS/IAS 1999

[GVF] J.Gracia-Bondia, J. Varilly, H. Figueroa, Elements of non commutative geometry, Birkhäuser Advanced texts (2000)

[H] G. Halbout, *Calcul d’un invariant de star-produit fermé sur une variété symplectique*, Comm. Math. Phys. 205 53–67 (1999)

[HH] J. Helton, R. Howe, *Traces of commutators of integral operators*, Acta Mathematica 135 271–305 (1975)

[K] Ch. Kassel, *Le résidu non commutatif (d’après M. Wodzicki)*, Séminaire Bourbaki, Astérisque 177-178 199-229 (1989)

[KV] M. Kontsevich, S. Vishik, *Determinants of elliptic pseudo-differential operators*, Max Planck Institut preprint, 1994 (arXiv:hep-th 940 40 46); Geometry of determinants of elliptic operators, Funct. Anal. on the Eve of the 21st. century, Birkhäuser, Progr. Math. 131, 1995, 173–197

[L] M. Lesch, *On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols*, Annals of Global Anal. and Geom. 17 151–187 (1999)
[LP] M. Lesch, M. Pflaum *Traces on algebras of parameter dependent pseudodifferential operators and the eta-invariant*, Trans. Amer. Soc. **352** n.11 4911-4936 (2000)

[P] S. Paycha, *From heat-operators to anomalies; a walk through various regularization techniques in mathematics and physics*, Emmy Nöther Lectures, Göttingen, 2003 [http://www.math.uni-goettingen.de]

[Sh] M. Shubin, *Pseudodifferential operators and spectral theory*, Springer (1987).

[W] M. Wodzicki, *Non commutative residue, Chapter I. Fundamentals, K-theory, Arithmetic and Geometry*, Springer Lecture Notes **1289**, 1987, pp.320-399.