LENGTH OF SETS UNDER RESTRICTED FAMILIES OF PROJECTIONS ONTO LINES

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Abstract. Let $\gamma : I \to S^2$ be a $C^2$ curve with $\det(\gamma, \gamma', \gamma'')$ nonvanishing, and for each $\theta \in I$ let $\rho_\theta$ be orthogonal projection onto the span of $\gamma(\theta)$. It is shown that if $A \subseteq \mathbb{R}^3$ is a Borel set of Hausdorff dimension strictly greater than 1, then $\rho_\theta(A)$ has positive length for a.e. $\theta \in I$. This answers a question raised by Käenmäki, Orponen and Venieri.

1. Introduction

Let $S^2$ be the unit sphere in $\mathbb{R}^3$, let $\gamma : I \to S^2$ be a $C^2$ curve with $\det(\gamma, \gamma', \gamma'')$ nonvanishing on an interval $I$, and let $\rho_\theta$ be orthogonal projection onto the span of $\gamma(\theta)$, given by $\rho_\theta(x) = \frac{\langle x, \gamma(\theta) \rangle \gamma(\theta)}{\|\gamma(\theta)\|}$, $x \in \mathbb{R}^3$.

In [2, Conjecture 1.6], Fässler and Orponen conjectured that for any analytic set $A \subseteq \mathbb{R}^3$ $\dim \rho_\theta(A) = \min\{\dim B, 1\}$, a.e. $\theta \in I$, where $\dim B$ means the Hausdorff dimension of $B$. For general $C^2$ curves this was resolved earlier by Käenmäki, Orponen and Venieri [7], who asked [7, p. 4] whether $\dim A > 1$ implies that $\rho_\theta(A)$ has positive length for a.e. $\theta \in I$. This had been shown previously by Fässler and Orponen [2, Theorem 1.9] in the special case where $A$ is a self-similar set without rotations. The following theorem resolves the general case.

Theorem 1.1. If $A \subseteq \mathbb{R}^3$ is an analytic set with $\dim A > 1$, then $\dim \{\theta \in I : \mathcal{H}^1(\rho_\theta(A)) = 0\} \leq 4 - \frac{\dim A}{3}$.

Throughout, the symbol $\mathcal{H}^s$ will be used for the $s$-dimensional Hausdorff measure on Euclidean space. By Frostman’s lemma, Theorem 1.1 will follow from Theorem 1.2 below. For the statement, some notation will be defined first. Given a Borel measure $\mu$ on $\mathbb{R}^3$ and $\alpha \geq 0$, define $c_\alpha(\mu) = \sup_{x \in \mathbb{R}^3, r > 0} \frac{\mu(B(x, r))}{r^\alpha}$. For each $\theta \in I$, the pushforward measure $\rho_\theta#\mu$ is defined by $(\rho_\theta#\mu)(E) = \mu(\rho_\theta^{-1}(E))$, for any Borel set $E$.

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Theorem 1.2. If $\mu$ is a Borel measure on $\mathbb{R}^3$ with $c_{\alpha}(\mu) < \infty$ for some $\alpha > 1$, then
\[
\dim \{ \theta \in \mathcal{I} : \rho_{\theta} \# \mu \not\ll \mathcal{H}^1 \} \leq \frac{4 - \alpha}{3}.
\]

The notation $\rho_{\theta} \# \mu \not\ll \mathcal{H}^1$ means that $\rho_{\theta} \# \mu$ is not absolutely continuous with respect to $\mathcal{H}^1$. The proof of Theorem 1.2 is similar to the proof of Theorem 8 in [4], which solved the analogous problem of projections onto planes. It uses a variant of the decomposition of a measure into “good” and “bad” parts, which originated in [6].

Given $s \geq 0$, a set $A \subseteq \mathbb{R}^3$ is called an $s$-set if $A$ is $\mathcal{H}^s$-measurable with $0 < \mathcal{H}^s(A) < \infty$. Theorem 1.2 implies Theorem 1.1, but it also implies the following slightly stronger version for $s$-sets.

Theorem 1.3. Suppose that $s > 1$, and that $A \subseteq \mathbb{R}^3$ is an $s$-set. Let $\mu$ be the Borel measure defined by $\mu(F) = \mathcal{H}^s(F \cap A)$ for any Borel set $F \subseteq \mathbb{R}^3$, and let
\[
E = \{ \theta \in \mathcal{I} : \rho_{\theta} \# \mu \not\ll \mathcal{H}^1 \}.
\]
Then
\[
\dim E \leq \frac{4 - s}{3},
\]
and
\[
\mathcal{H}^1(\rho_{\theta}(B)) > 0, \quad \text{for all } \theta \in \mathcal{I} \setminus E.
\]
for any $\mathcal{H}^s$-measurable set $B \subseteq A$ with $\mathcal{H}^s(B) > 0$.

Theorem 1.3 is related to a lemma of Marstrand (see [9, Lemma 13]), which states that for $s > 1$, for any $s$-set $A$ in the plane, there is a measure zero set of exceptional directions, such that any $s$-set $B \subseteq A$ projects onto a set of positive Lebesgue measure outside of this set of exceptional directions.

Theorem 1.3 also implies Theorem 1.1 since any analytic set of infinite $\mathcal{H}^s$ measure contains a closed set of positive finite $\mathcal{H}^s$ measure [1].

2. Proof of Theorem 1.2 and Theorem 1.3

Throughout this section, $\gamma : I \to S^2$ will be a fixed $C^2$ unit speed curve with $\det(\gamma, \gamma', \gamma'')$ nonvanishing on $I$, where $I$ is a compact interval. For $A \subseteq \mathbb{R}^3$, $m(A)$ will denote the Lebesgue measure of $A$.

Definition 2.1. Let
\[
\Lambda = \bigcup_{j \geq 1} \Lambda_j,
\]
where each $\Lambda_j$ is a collection of boxes $\tau$ of dimensions $1 \times 2^{j/2} \times 2^j$, forming a finitely overlapping cover of the $\sim 1$-neighbourhood of the truncated light cone $\Gamma_j$ in the standard way, where
\[
\Gamma_j = \{ t\gamma(\theta) : 2^{j-1} \leq |t| \leq 2^j, \quad \theta \in I \}.
\]
Each $\tau \in \Lambda_j$ has an angle $\theta_{\tau}$ such that the long axis of $\tau$ is parallel to $\gamma(\theta_{\tau})$, the medium axis of $\tau$ is parallel to $\gamma'(\theta_{\tau})$, and the short axis of $\tau$ is parallel to $(\gamma \times \gamma')(\theta_{\tau})$. Let $\{ \psi_{\tau} \}_{\tau \in \Lambda}$ be a smooth partition of unity subordinate to the cover
Lemma 2.2. There exists an $r > 0$, depending only on $\gamma$, such that the following holds. Let $j \geq 1$ and let $\tau \in \Lambda_j$. If $\theta \in I$ is such that $2^j(\delta - 1/2) \leq |\theta_\tau - \theta| \leq r$, then for any $T \in T_\tau$, for any positive integer $N$ and for any $f \in L^1(\mathbb{R}^3)$,

\begin{equation}
\|\pi_{\theta\#} M_T f\|_{L^1(\mathbb{R}^3)} \leq C 2^{-j \delta N} \|f\|_{L^1(\mathbb{R}^3)},
\end{equation}

where $C = C \left( N, \gamma, \delta \right)$.

Proof. For any $x \in (\gamma \times \gamma')(\theta)^{\perp}$,

\begin{equation}
(\pi_{\theta\#} M_T f)(x) = \int_{\mathbb{R}^3} \int_{\mathbb{R}} f(y) \psi_T(\xi)e^{-2\pi i \langle \xi, x - y \rangle} \left[ \int_{\mathbb{R}} \eta_T(x + t(\gamma \times \gamma')(\theta))e^{-2\pi i t \langle \xi, (\gamma \times \gamma')(\theta) \rangle} dt \right] d\xi dy.
\end{equation}

Let $\xi \in \tau$. By integrating by parts $n$ times, and using (2.2),

\begin{equation}
\int_{\mathbb{R}} \eta_T(x + t(\gamma \times \gamma')(\theta))e^{-2\pi i t \langle \xi, (\gamma \times \gamma')(\theta) \rangle} dt \leq \int_{\mathbb{R}} \chi_T(x + t(\gamma \times \gamma')(\theta)) dt \times \frac{2^{j \delta}}{|((\gamma \times \gamma')(\theta), \xi)|^n} \left( 2^{j(1 - \delta)} |(\gamma \times \gamma')(\theta), (\gamma(\theta_\tau))| + 2^{j(1/2 - \delta)} |(\gamma \times \gamma')(\theta), (\gamma'(\theta_\tau))| 
+ 2^{-j \delta} |(\gamma \times \gamma')(\theta), (\gamma \times \gamma')(\theta_\tau)| \right)^n.
\end{equation}

For each $\theta \in I$, let $\pi_\theta$ be orthogonal projection onto the orthogonal complement of $(\gamma \times \gamma')(\theta)$. The following lemma is essentially a special case of Lemma 5 from [4], but the proof will be included here for completeness.

Lemma 2.2. There exists an $r > 0$, depending only on $\gamma$, such that the following holds. Let $j \geq 1$ and let $\tau \in \Lambda_j$. If $\theta \in I$ is such that $2^j(\delta - 1/2) \leq |\theta_\tau - \theta| \leq r$, then for any $T \in T_\tau$, for any positive integer $N$ and for any $f \in L^1(\mathbb{R}^3)$,
By the definition of $\tau$,
\[
\langle (\gamma \times \gamma')(\theta), \xi \rangle = \\
\xi_1 \langle (\gamma \times \gamma')(\theta), \gamma(\theta) \rangle + \xi_2 \langle (\gamma \times \gamma')(\theta), \gamma'(\theta) \rangle + \xi_3 \langle (\gamma \times \gamma')(\theta), (\gamma \times \gamma')(\theta) \rangle,
\]
where $2^{-2} \leq |\xi_1| \leq 2^{j+2}$, $|\xi_2| \leq 2^{j/2}$ and $|\xi_3| \leq 1$. Let $\varepsilon := |\theta - \hat{\theta}|$. By a second order Taylor approximation (using that $\gamma$ is $C^2$) and the scalar triple product formula,
\[
|\xi_1 \langle (\gamma \times \gamma')(\theta), \gamma(\theta) \rangle| \sim \varepsilon^2 2^j,
\]
provided $r$ is small enough. Moreover
\[
|\xi_2 \langle (\gamma \times \gamma')(\theta), \gamma'(\theta) \rangle| \sim 2^{j/2} \varepsilon, \quad |\xi_3 \langle (\gamma \times \gamma')(\theta), (\gamma \times \gamma')(\theta) \rangle| \sim \varepsilon.
\]
Since $\varepsilon \geq 2^{(j-1)/2}$, it follows that
\[
|\langle (\gamma \times \gamma')(\theta), \xi \rangle| \sim \varepsilon^2 2^j,
\]
provided that $r$ is sufficiently small and $j$ is sufficiently large. Using second order Taylor approximation in a similar way gives that
\[
2^{(1-\delta)} \left| \langle (\gamma \times \gamma')(\theta), \gamma(\theta) \rangle \right| \\
+ 2^{(1/2-\delta)} \left| \langle (\gamma \times \gamma')(\theta), \gamma'(\theta) \rangle \right| + 2^{-\delta} \left| \langle (\gamma \times \gamma')(\theta), (\gamma \times \gamma')(\theta) \rangle \right| \leq 2^{(1-\delta)} \varepsilon^2.
\]
It follows that
\[
2^{3j} \lesssim 2^{-j} 2^{(n-1)} \int R \chi_T(x + t(\gamma \times \gamma')(\theta)) \, dt.
\]
Substituting this into (2.1) gives that
\[
|\langle \pi_{\theta \#} M_T f(x) \rangle| \lesssim 2^{-j} 2^{(n-1)} \| f \|_{L^1(\mathbb{R}^3)} m(\tau) \int R \chi_T(x + t(\gamma \times \gamma')(\theta)) \, dt,
\]
for any $x \in (\gamma \times \gamma')(\theta)^\perp$. Integrating over $t \in \mathbb{R}$ and $x \in (\gamma \times \gamma')(\theta)^\perp$ gives (2.2).

The following lemma is essentially the same as Lemma 2 from [4], but again the proof is included for completeness.

**Lemma 2.3.** Let $j \geq 1$ and let $\tau \in \Lambda_j$. For any finite compactly supported Borel measure $\mu$,
\[
\| M_T \mu \|_{L^1(\mathbb{R}^3)} \leq 2^{3j} \mu(2T) + C_N 2^{-j} \mu(\mathbb{R}^3),
\]
for any positive integer $N$.

**Proof.** By definition,
\[
\| M_T \mu \|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left| \int_{2T} \tilde{\psi}_\tau(x-y) \, d\mu(y) \right| \, dx \\
\leq \int_{2T} \int_{\mathbb{R}^3} \left| \tilde{\psi}_\tau(x-y) \right| \, d\mu(y) \, dx \\
+ \int_{2T} \int_{\mathbb{R}^3 \setminus 2T} \left| \tilde{\psi}_\tau(x-y) \right| \, d\mu(y) \, dx.
\]
(2.6)
The first integral satisfies
\[
\int_{2T} \int_{\mathbb{R}^3 \setminus 2T} \left| \tilde{\psi}_\tau(x-y) \right| \, d\mu(y) \, dx \leq 2^{3j} \mu(2T).
\]
(2.7)
The second integral satisfies
\[ \int_T \int_{\mathbb{R}^3 \setminus 2T} |\tilde{\psi}_\tau(x-y)| \, d\mu(y) \, dx \leq \mu(\mathbb{R}^3) \int_{\mathbb{R}^3 \setminus T_0} |\tilde{\psi}_\tau|, \]
where \( T_0 \) is the translate of the plank \( T \) to the origin, parallel to \( T \). Integrating by parts, and using (2.4), gives that for any \( k \geq 0 \), for any \( x \in \mathbb{R}^3 \setminus 2^k T_0 \),
\[ |\tilde{\psi}_\tau(x)| \lesssim_N 2^{-kN - \beta N} m(\tau). \]
Summing a geometric series over \( k \geq 0 \) gives (with relabelled \( N \))
\[ \int_T \int_{\mathbb{R}^3 \setminus 2T} |\tilde{\psi}_\tau(x-y)| \, d\mu(y) \, dx \leq C_N \mu(\mathbb{R}^3) 2^{-\beta N}. \]
Putting (2.7) and (2.8) into (2.6) finishes the proof. \( \square \)

For a function \( f : X \to [0, +\infty] \) on a measure space \((X, \mathcal{A}, \mu)\), let \( \int f \, d\mu \) denote the lower integral of \( f \), defined by
\[ \int f \, d\mu = \sup \left\{ \int g \, d\mu : g \text{ is simple and } \mathcal{A}\text{-measurable with } 0 \leq g \leq f \right\}, \]
where “simple” means that \( g \) takes finitely many values. In the application of Lemma 2.4 below, the integrand will be measurable, so the use of the lower integral is not important and is only a technical convenience to avoid measurability issues. The definition of the lower integral is standard; see e.g. [10, p. 13].

**Lemma 2.4.** Let \( \beta \in [0, 1] \), let \( \alpha = 4 - 3\beta \), and let \( \lambda \) be a Borel measure supported on \( I \) with \( c_\beta(\lambda) \leq 1 \). Then for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[ \int (\rho_{\theta\#}\mu) \left( \bigcup_{D \in \mathcal{D}_\theta} D \right) \, d\lambda(\theta) \leq C(\lambda, \delta, \epsilon, \gamma) R^{-\delta} \mu(\mathbb{R}^3), \]
for any \( R \geq 1 \), for any Borel measure \( \mu \) on \( B_3(0, 1) \) with \( c_\alpha(\mu) \leq 1 \), and for any family of sets \( \{\mathcal{D}_\theta\} \), where each \( \mathcal{D}_\theta \) is a disjoint set of intervals of diameter \( 2R^{-1} \)
in the span of \( \gamma(\theta) \), each with cardinality \( |\mathcal{D}_\theta| \leq R^{1-\epsilon} \mu(\mathbb{R}^3) \).

**Remark 1.** Lemma 2.4 roughly says that the pushforward \( \rho_{\theta\#}\mu \) of a \((4 - 3\beta)\)-dimensional measure \( \mu \) satisfies a 1-dimensional Frostman condition on average. By taking \( \beta = 1 \) (for example), this is enough to conclude that projections of 1-dimensional sets are a.e. 1-dimensional, but the main application of Lemma 2.4 here will be to bound the \( L^1 \) norm of the “bad” part of the measure in the proof of Theorem 1.12. The “bad” part of the measure corresponds to intervals of large \( \rho_{\theta\#}\mu \)-mass, which will automatically satisfy the cardinality assumption of Lemma 2.4.

**Proof of Lemma 2.4.** It may be assumed that \( \alpha \leq 3 \), since otherwise the lemma is trivial. Since the constant is allowed to depend on \( \gamma \), it can be assumed that \( \gamma \) is localised to a smaller interval on which Lemma 2.2 holds.

Let \( \phi_R \) be a non-negative bump function supported in \( B(0, R^{-1}) \) which integrates to 1, defined by
\[ \phi_R(x) = R^4 \phi(Rx), \]
for some fixed non-negative bump function \( \phi \) with support in \( B(0, 1) \) such that \( \int \phi = 1 \). For any \( \theta \) and any \( D \in \mathcal{D}_\theta \), the 1-Lipschitz property of orthogonal projections implies that
\[ (\rho_{\theta\#}(\mu \ast \phi_R))(2D) \geq (\rho_{\theta\#}\mu)(D), \]
where $2D$ is the interval with the same centre as $D$, but twice the radius. Moreover,

$$c_\alpha(\mu * \phi_R) \lesssim c_\alpha(\mu),$$

so it suffices to prove (2.9) with $\mu * \phi_R$ in place of $\mu$. To simplify notation the new measure will not be relabelled, but it will be assumed throughout that $\mu$ is a non-negative Schwartz function, and that

$$|\tilde{\mu}(\xi)| \leq C_N(R/\xi)^N, \quad \xi \in \mathbb{R}^3,$$

for any positive integer $N$, where $C_N$ is a constant depending only on $N$.

Let $\epsilon_0$ be any positive real number which is strictly larger than the infimum over all positive $\epsilon$ for which the conclusion of the lemma is true. It suffices to prove that the lemma holds for any $\epsilon > (2\epsilon_0)/3$, so let such an $\epsilon$ be given. Let $R \geq 1$ and choose a non-negative integer $J$ such that $2^J \sim R^{\epsilon/1000}$. Let $\epsilon > 0$ be such that $\epsilon \ll \epsilon - 2^{\epsilon_0}$. Choose $\tilde{\delta} > 0$ such that $\tilde{\delta} \ll \min\{\delta_\epsilon, \epsilon\}$, where $\delta_\epsilon$ is a $\delta$ corresponding to $\epsilon_0$ that satisfies (2.9).

Define the “bad” part of $\mu$ by

$$\mu_b = \sum_{j \geq J} \sum_{\tau \in \Lambda_j} \sum_{T \in T_{\tau,b}} M_T \mu,$$

where, for each $\tau \in \Lambda_j$, the set of “bad” planks corresponding to $\tau$ is defined by

$$T_{\tau,b} = \left\{ T \in T_\tau : \mu(4T) \geq 2^{j(\epsilon_0 - 1)} \right\},$$

where $4T$ is a plank with the same centre as $T$, but scaled by a factor of 4. Define the “good” part of $\mu$ by

$$\mu_g = \mu - \mu_b.$$ 

The Schwartz decay of $\mu$ implies that the sum in (2.11) converges in the Schwartz space $S(\mathbb{R}^3)$. This implies that $\mu_b$ and $\mu_g$ are Schwartz functions, and in particular they are finite complex measures. Pushforwards of complex measures are defined just as for positive measures. By Cauchy-Schwarz,

$$\int \left( \rho_\theta \| \mu \|_1 \right) \left( \bigcup_{D \in \mathbb{D}_\theta} D \right) \ d\lambda(\theta) \leq \int \| \rho_\theta \| \| \mu_b \|_1 \ d\lambda(\theta),$$

where $\mathbb{D}_\theta$ is a $\delta$ corresponding to $\epsilon_0$ that satisfies (2.9).

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1The particular Cauchy-Schwarz technique used here is from [8].
The contribution from the “bad” part will be bounded first. By the triangle inequality,
\[
\int \|\rho_\theta \# \mu_b\|_{L^1(\mathcal{H}_1)} \, d\lambda(\theta) \leq \sum_{j \geq J} \int \sum_{\tau \in \Lambda_j} \sum_{T \in T_{\tau,b}} \|\rho_\theta \# M_T \mu\|_{L^1(\mathcal{H}_1)} \, d\lambda(\theta)
\]
(2.13)
\[
= \sum_{j \geq J} \int \sum_{\tau \in \Lambda_j: |\theta - \theta| \leq 2^{(\delta - 1/2)} \tau} \sum_{T \in T_{\tau,b}} \|\rho_\theta \# M_T \mu\|_{L^1(\mathcal{H}_1)} \, d\lambda(\theta)
\]
(2.14)
\[+ \sum_{j \geq J} \int \sum_{\tau \in \Lambda_j: |\theta - \theta| > 2^{(\delta - 1/2)} \tau} \sum_{T \in T_{\tau,b}} \|\rho_\theta \# M_T \mu\|_{L^1(\mathcal{H}_1)} \, d\lambda(\theta).
\]
Let \(\pi_\theta\) be orthogonal projection onto \((\gamma \times \gamma')(\theta)\). By the inequality
\[
\|\rho_\theta \# f\|_{L^1(\mathcal{H}_1)} \leq \|\pi_\theta \# f\|_{L^1(\mathcal{H}_1)}
\]
followed by Lemma 2.2,
\[
(2.14) \lesssim 2^{-J} \mu(\mathbb{R}^3) \sim R^{-1000} \mu(\mathbb{R}^3).
\]
By the inequality
\[
\|\rho_\theta \# f\|_{L^1(\mathcal{H}_1)} \leq \|f\|_{L^1(\mathbb{R}^3)},
\]
followed by Lemma 2.3,
\[
(2.13) \lesssim \sum_{j \geq J} \int \sum_{\tau \in \Lambda_j: |\theta - \theta| \leq 2^{(\delta - 1/2)} \tau} \sum_{T \in T_{\tau,b}} \|M_T \mu\|_{L^1(\mathbb{R}^3)} \, d\lambda(\theta)
\]
\[\lesssim 2^{-J} \mu(\mathbb{R}^3) + \sum_{j \geq J} 2^{3\delta J} \int \sum_{\tau \in \Lambda_j: |\theta - \theta| \leq 2^{(\delta - 1/2)} \tau} \sum_{T \in T_{\tau,b}} \mu(2T) \, d\lambda(\theta),
\]
The non-tail term satisfies
\[
(2.15) \sum_{j \geq J} 2^{3\delta J} \int \sum_{\tau \in \Lambda_j: |\theta - \theta| \leq 2^{(\delta - 1/2)} \tau} \sum_{T \in T_{\tau,b}} \mu(2T) \, d\lambda(\theta) \lesssim \sum_{j \geq J} 2^{10\delta J} \int \mu(B_j(\theta)) \, d\lambda(\theta),
\]
where, for each \(\theta \in I\) and each \(j\),
\[
B_j(\theta) = \bigcup_{\tau \in \Lambda_j: |\theta - \theta| \leq 2^{(\delta - 1/2)} \tau} \bigcup_{T \in T_{\tau,b}} 2T.
\]
The inequality (2.15) used that for each \(j\) and each \(\theta \in I\), there are \(\lesssim 2^{\delta J}\) sets \(\tau \in \Lambda_j\) with the property that \(|\theta - \theta| \leq 2^{(\delta - 1/2)} \tau\), which means each of the planks \(2T\) in the union defining \(B_j(\theta)\) intersects \(\lesssim 2^{\delta J}\) of the others. For each \(T\) in the union defining \(B_j(\theta)\), the set \((4T) \cap B(0,1)\) is contained in a plank \(T_\theta\) of dimensions
\[
\sim 2^{j(\delta - 1)} \times 2^{(\delta - 1/2)} \times 1,
\]
with short direction parallel to \(\gamma(\theta)\), medium direction parallel to \(\gamma'(\theta)\), and long direction parallel to \((\gamma \times \gamma')(\theta)\), where the implicit constant depends only on \(\gamma\); this
follows from the second order Taylor approximation for \( \gamma \). Therefore, the intervals in the set
\[
\left\{ \rho_\theta(T_\theta) : T \in \mathcal{T}_{\tau,b}, \quad \tau \in \Lambda_j, \quad |\theta_\tau - \theta| \leq 2^{(3\delta - 1/2)} \right\},
\]
all have length \( \sim 2^{j(\delta - 1)} \), and form a cover of \( \rho_\theta(B_j(\theta) \cap B(0,1)) \). By the Vitali covering lemma, there is a disjoint subcollection
\[
\{ \rho_\theta(T_\theta) : T \in \mathcal{B}_\theta \},
\]
indexed by some set \( \mathcal{B}_\theta \), such that
\[
\{ \rho_\theta(T_\theta) : T \in \mathcal{B}_\theta \},
\]
is a cover of \( \rho_\theta(B_j(\theta) \cap B(0,1)) \). The set \( \mathcal{B}_\theta \) has cardinality \( |\mathcal{B}_\theta| \leq \mu(\mathbb{R}^3)^{2^{j(1/2) - \epsilon_0}} \); by disjointness and the definition of the “bad” planks (see (2.12)). Since the conclusion of the lemma holds for \( \epsilon_0 \), it follows that for each \( j \geq J \),
\[
\int \mu(B_j(\theta)) \, d\lambda(\theta) \leq \int \left( (\rho_\theta \# \mu) \left( \bigcup_{T \in \mathcal{B}_\theta} 3\rho_\theta(T_\theta) \right) \right) \, d\lambda(\theta) \lesssim 2^{2j(-\delta_\epsilon_0/2 + 10\tilde{\delta})} \mu(\mathbb{R}^3).
\]
The set \( B_j(\theta) \) is piecewise constant in \( \theta \) over a partition of \( I \) into Borel sets, so it may be assumed that the integrands above are Borel measurable. Since \( \tilde{\delta} \ll \delta_\epsilon_0 \), the inequality above yields
\[
\text{(2.13)} \lesssim \mu(\mathbb{R}^3)2^{-(J\delta_\epsilon_0)/100} \sim \mu(\mathbb{R}^3)R^{-(\epsilon_\delta_0)/100}.
\]
It remains to bound the contribution from \( \mu_g(\rho_\theta(\mathcal{B}_\theta)) \). By the assumptions in the lemma,
\[
\sup_{\theta \in I} \mathcal{H}^1 \left( \bigcup_{D \in \mathcal{B}_\theta} D \right) \lesssim R^{-\epsilon} \mu(\mathbb{R}^3).
\]
Since \( \epsilon \ll \epsilon - (2\epsilon_0)/3 \), it suffices to prove that
\[
\int \| \rho_\theta \# \mu_\theta \|^2_{L^2(\mathcal{H}^1)} \, d\lambda(\theta) \lesssim \max \left\{ R^{2\epsilon_0/3 + 100\epsilon}, R^{\epsilon/2} \right\} \mu(\mathbb{R}^3),
\]
By Plancherel’s theorem in 1 dimension,
\[
\int \| \rho_\theta \# \mu_\theta \|^2_{L^2(\mathcal{H}^1)} \, d\lambda(\theta) = \int \int_{\mathbb{R}} |\hat{\mu}_\theta(t\gamma)\rangle|^2 \, dt \, d\lambda(\theta).
\]
To formally prove this identity, one approach is to rotate \( \gamma(\theta) \) to \( (1,0,0) \) (using that \( \mathcal{H}^1 \) is a rotation invariant measure on \( \mathbb{R}^3 \)), and apply Plancherel’s theorem on \( \mathbb{R} \). By symmetry, by summing a geometric series, and by the rapid decay of \( \hat{\mu} \) outside \( B(0,R) \) (see (2.10)), it will suffice to bound
\[
\int \int_{2^{j-1}}^{2^j} |\hat{\mu}_g(t\gamma(\theta))\rangle|^2 \, dt \, d\lambda(\theta),
\]
for any \( j \geq 2J \) with \( 2^j \leq R^{1+\tilde{\delta}} \), the contribution from the small frequencies can be bounded trivially by the definition of \( J \). For each \( \tau \in \Lambda \), define the set of “good” planks corresponding to \( \tau \) by
\[
\mathcal{T}_{\tau,g} = \mathcal{T}_\tau \setminus \mathcal{T}_{\tau,b}.
\]
Then
\[ \int \int_{2^{j-1}}^{2^j} |\hat{\mu}_g(t\gamma(\theta))|^2 \, dt \, d\lambda(\theta) \leq \int \int_{2^{j-1}}^{2^j} \left| \sum_{\tau \in \bigcup_{|j'-j| \leq 2} \Lambda_{j'}} \sum_{T \in \mathcal{T}_{\tau, g}} \hat{M}_T \mu(t\gamma(\theta)) \right|^2 \, dt \, d\lambda(\theta) + 2^{-j} \mu(\mathbb{R}^3). \]

Since the \( \tau \)'s are finitely overlapping,
\[ (2.16) \int \int_{2^{j-1}}^{2^j} \left| \sum_{\tau \in \bigcup_{|j'-j| \leq 2} \Lambda_{j'}} \sum_{T \in \mathcal{T}_{\tau, g}} \hat{M}_T \mu(t\gamma(\theta)) \right|^2 \, dt \, d\lambda(\theta) \leq 2^{-j} \mu(\mathbb{R}^3) \]

The uncertainty principle implies that each of the integrals in the right-hand side of (2.16) is bounded by \( 2^{-j\beta} \) times the integral of the same function over \( \mathbb{R}^3 \). More precisely, for each \( \tau \in \bigcup_{|j'-j| \leq 2} \Lambda_{j'} \), the contribution from the planks in the above sum with \( T \cap B\left(0, 2^{10^j\beta}\right) = \emptyset \) is negligible since \( \mu \) is supported in \( B(0, 1) \). The remaining sum
\[ g_\tau = \sum_{T \in \mathcal{T}_{\tau, g} : T \cap B(0, 2^{10^j\beta}) \neq \emptyset} M_T \mu, \]

is equal to \( g_\tau \varphi \) where \( \varphi \) is a smooth bump function on \( B\left(0, 2^{1+10^j\beta}\right) \) obtained by rescaling a bump function on the unit ball, and therefore by the Cauchy-Schwarz inequality,
\[ |g_\tau|^2 \lesssim |g_\tau|^2 \ast \zeta, \]
where
\[ \zeta(\xi) = \frac{2^{30^j\beta}}{1 + |2^{10^j\beta} \xi|^{100}}, \quad \xi \in \mathbb{R}^3. \]

The function in the right-hand side of (2.17) is essentially constant on balls of radius \( 2^{-10^j\beta} \) (as it inherits this property from \( \zeta \)), so by discretising the integral in (2.16) into balls of radius \( 2^{-10^j\beta} \) and using the condition \( c_\beta(\lambda) \leq 1 \),
\[ \int \int_{2^{j-1}}^{2^j} \left| \sum_{T \in \mathcal{T}_{\tau, g}} \hat{M}_T \mu(t\gamma(\theta)) \right|^2 \, dt \, d\lambda(\theta) \lesssim 2^{j(10^3 - 10^j\beta)} \int_{\mathbb{R}^3} \left| \sum_{T \in \mathcal{T}_{\tau, g} : T \cap B(0, 2^{10^j\beta}) \neq \emptyset} \hat{M}_T \mu \right|^2 \, d\xi + 2^{-j} \mu(\mathbb{R}^3), \]
for each $\tau \in \bigcup_{j' - j \leq 2} \Lambda_{j'}$. Let

$$T_{j,g} = \bigcup_{\tau \in \bigcup_{j' - j \leq 2} \Lambda_{j'}} \left\{ T \in T_{\tau,g} : T \cap B \left( 0, 2^{10j} \right) \neq \emptyset \right\}.$$  

By Plancherel’s theorem in $\mathbb{R}^3$ and the finite overlapping property of the $T$’s, it suffices to prove that for any $j \geq 2J$,

$$\sum_{T \in T_{j,g}} \int_{\mathbb{R}^3} |M_T \mu|^2 \, dx \lesssim 2^j (\beta + 2^{j_0} + 100 \varepsilon) \mu(\mathbb{R}^3). \tag{2.18}$$

From the definition $M_T \mu = \eta_T \left( \mu \ast \overline{\psi_{\tau(T)}} \right)$ and by Fubini, the left-hand side of the above is equal to

$$\int \sum_{T \in T_{j,g}} [\eta_T M_T \mu] \ast \overline{\psi_{\tau(T)}} \, d\mu.$$  

If $f_T := [\eta_T M_T \mu] \ast \overline{\psi_{\tau(T)}}$, then by the Cauchy-Schwarz inequality with respect to the measure $\mu$, the square of the above is bounded by

$$\int \left| \sum_{T \in T_{j,g}} f_T \right|^2 \, d\mu \cdot \mu(\mathbb{R}^3).$$

By the uncertainty principle,

$$\int \left| \sum_{T \in T_{j,g}} f_T \right|^2 \, d\mu \lesssim \int \sum_{T \in T_{j,g}} f_T^2 \, d\mu_j,$$

where $\mu_j = \mu \ast \phi_j$ and $\phi_j(x) = \frac{2^{3j}}{1 + \frac{x^2}{2^{2j}N}}$, where $N \sim 1000/\delta^2$. By dyadic pigeonholing, there exists a collection $\mathcal{W}$ of planks $T \in T_{j,g}$ with $\|f_T\|_p$ constant over $T \in \mathcal{W}$ up to a factor of $2$, and a union $Y$ of disjoint $2^{-j}$-balls $Q$ such that each $Q$ intersects $\sim M$ planks $2T \in \mathcal{W}$ for some dyadic number $M$, and such that

$$\int \left| \sum_{T \in T_{j,g}} f_T \right|^2 \, d\mu_j \lesssim j^{10} \int_Y \left| \sum_{T \in \mathcal{W}} f_T \right|^2 \, d\mu_j + 2^{-j} \mu(\mathbb{R}^3)^2.$$  

Let $p = 6$. By Hölder’s inequality with respect to the Lebesgue measure,

$$\int_Y \left| \sum_{T \in \mathcal{W}} f_T \right|^2 \, d\mu_j \leq \left\| \sum_{T \in \mathcal{W}} f_T \right\|_{L^p(Y)}^2 \left( \int_Y \mu_j(x)^{\frac{p-2}{p}} \right)^{\frac{p-2}{p}}, \tag{2.19}$$
By the dimension condition \( c_\alpha(\mu) \leq 1 \) on \( \mu \), the definition of \( Y \), and the definition of the “good” planks,

\[
\left(2.20\right) \quad \int_Y \mu_j^\frac{n}{p-2} \lesssim 2^{\frac{2(n-\alpha)}{p-2}} \int_Y \mu_j
\]

\[
\leq 2^{\frac{2(n-\alpha)}{p-2}} \sum_{Q \subseteq Y} \int_Q \mu_j
\]

\[
\lesssim \left( \frac{1}{M} \right) \cdot 2^{\frac{2(n-\alpha)}{p-2}} \sum_{Q \subseteq Y} \sum_{T \in \mathbb{W}} \int_{Q \cap 3T} \mu_j
\]

\[
\lesssim \left( \frac{1}{M} \right) \cdot 2^{\frac{2(n-\alpha)}{p-2}} \sum_{T \in \mathbb{W}} \int_{3T} \mu_j
\]

\[
\lesssim \left( \frac{1}{M} \right) \cdot 2^{\frac{2(n-\alpha)}{p-2}} \sum_{T \in \mathbb{W}} \mu(4T) + 2^{-100j}
\]

\[
\lesssim 2^{\frac{2(n-\alpha)}{p-2} + j(\epsilon_0 - 1)} \left( \frac{\|W\|}{M} \right).
\]

This bounds the second factor in (2.19), so it remains to bound the first factor.

By rescaling by \( 2^j \), applying the refined decoupling inequality (see Theorem A.1 of the appendix), and then rescaling back,

\[
\left\| \sum_{T \in \mathbb{W}} f_T \right\|_{L^p(Y)} \lesssim 2^{j\epsilon} \left( \frac{M}{\|W\|} \right)^{\frac{1}{2} - \frac{1}{p}} \left( \sum_{T \in \mathbb{W}} \|f_T\|_p^2 \right)^{1/2}.
\]

Recall that \( f_T = \widehat{\nu_T \mu} * \psi_{2^j T} \). By applying the Hausdorff-Young inequality, then Hölder’s inequality, and then Plancherel’s theorem,

\[
\|f_T\|_p \lesssim \|\nu_T \mu\|_2 2^{\frac{3p}{2} - \frac{3p}{2}} (\frac{M}{\|W\|})^{\frac{1}{2} - \frac{1}{p}} \left( \sum_{T \in \mathbb{W}} \|\nu_T \mu\|_2^2 \right)^{1/2}.
\]

Hence

\[
\left(2.21\right) \quad \left\| \sum_{T \in \mathbb{W}} f_T \right\|_{L^p(Y)} \lesssim 2^{j\epsilon} \left( \frac{M}{\|W\|} \right)^{\frac{1}{2} - \frac{1}{p}} \left( \sum_{T \in \mathbb{W}} \|\nu_T \mu\|_2^2 \right)^{1/2}.
\]

Putting (2.21) and (2.20) into (2.19) gives

\[
\sum_{T \in \mathbb{W}} \int_{\mathbb{R}^3} |\nu_T \mu|^2 \leq 2^{j \left( \frac{5 - 2\alpha}{2} + \frac{3p}{2} - \frac{3(n-p)}{2p} \right)} \left( \sum_{T \in \mathbb{W}} \int_{\mathbb{R}^3} |\nu_T \mu|^2 \right)^{1/2} \mu(\mathbb{R}^3)^{1/2}.
\]

By cancelling the common factor, this gives

\[
\sum_{T \in \mathbb{W}} \int_{\mathbb{R}^3} |\nu_T \mu|^2 \leq 2^{j \left( \frac{5 - 2\alpha}{2} + \frac{3p}{2} - \frac{3(n-p)}{2p} \right)} \mu(\mathbb{R}^3).
\]

Since \( p = 6 \) and \( \alpha = 4 - 3\beta \), this simplifies to

\[
\sum_{T \in \mathbb{W}} \int_{\mathbb{R}^3} |\nu_T \mu|^2 \leq 2^{j \left( \beta + \frac{2p}{5} + 3\epsilon \right)} \mu(\mathbb{R}^3),
\]

which verifies (2.18), and as explained above, this proves the lemma. \( \square \)
The proof of Theorem 1.2 will be similar to the proof of the lemma.

**Proof of Theorem 1.2.** It will first be shown that the set
\[ \{ \theta \in I : \rho_{\theta \# \mu} \not\ll \mathcal{H}^1 \} \]
is Borel measurable. It may be assumed that \( \mu \) is compactly supported, since if \( \mu_k \) is the restriction of \( \mu \) to \( B(0, k) \), then
\[ \{ \theta \in I : \rho_{\theta \# \mu} \not\ll \mathcal{H}^1 \} = \bigcup_{k=1}^{\infty} \{ \theta \in I : \rho_{\theta \# \mu_k} \not\ll \mathcal{H}^1 \} . \]

For any positive integer \( n \), the function
\[ (\theta, t) \mapsto (\rho_{\theta \# \mu}) (B(t \gamma(\theta), 1/n)), \]
is lower semicontinuous on \( I \times \mathbb{R} \), and is therefore Borel measurable from \( I \times \mathbb{R} \) to \([0, +\infty)\) (here \( B(x, r) \) denotes the open ball or interval of radius \( r \) around \( x \)). It follows that the function
\[ \theta \mapsto \limsup_{n \to \infty} n (\rho_{\theta \# \mu}) (B(t \gamma(\theta), 1/n)) \]
is Borel measurable from \( I \) to \([0, +\infty]\). It follows that the set
\[ \{ \theta \in I : \int_{\mathbb{R}} \limsup_{n \to \infty} n (\rho_{\theta \# \mu}) (B(t \gamma(\theta), 1/n)) \, dt < \mu(R^3) \} , \]
is a Borel measurable subset of \( I \), and this set is equal to \( \{ \theta \in I : \rho_{\theta \# \mu} \not\ll \mathcal{H}^1 \} \) by the Lebesgue differentiation theorem (see e.g. [3, Theorem 3.22]).

As in the proof of the lemma, it may be assumed that \( \gamma \) is localised to a small interval on which Lemma 2.2 holds. It may also be assumed that \( \alpha \leq 3, c_\alpha(\mu) \leq 1 \) and (by countable stability of the Hausdorff dimension) that \( \mu \) has support in the unit ball. Let \( \beta \) be such that \( 0 \leq \beta < (4 - \alpha)/3 \). Let \( \lambda \) be a Borel measure supported on \( I \) with \( c_\beta(\lambda) \leq 1 \). Let \( \epsilon > 0 \) be such that \( \epsilon \ll \frac{4 - \alpha}{3} - \beta \). Choose \( \delta > 0 \) such that \( \delta \ll \min \{ \epsilon, \delta_\varepsilon \} \), where \( \delta_\varepsilon \) is an exponent corresponding to \( \epsilon \) from Lemma 2.4.

Using Definition 2.1 define \( \mu_b \) by
\[ (2.22) \quad \mu_b = \sum_{j \geq 1} \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau, b}} M_T \mu, \]
where, for each \( j \geq 1 \) and \( \tau \in \Lambda_j \), the set of “bad” planks corresponding to \( \tau \) is defined by
\[ \mathbb{T}_{\tau, b} = \left\{ T \in \mathbb{T}_{\tau} : \mu(4T) \geq 2^j (4\epsilon - 1) \right\} . \]

Since the frequencies are no longer localised, the sum in (2.22) only converges a priori in the space of tempered distributions, but the individual functions \( M_T \mu \) are smooth and compactly supported. For \( \lambda \)-a.e. \( \theta \in I \),
\[ (2.23) \quad \rho_{\theta \# \mu_b} := \sum_{j \geq 1} \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau, b}} \rho_{\theta \#} M_T \mu, \]
where, for $\lambda$-a.e. $\theta \in I$, the series will be shown to be absolutely convergent in $L^1(\mathcal{H}^1)$. Since $L^1$ is always a Banach space, any absolutely convergent series of $L^1$ functions is convergent in $L^1$, so the $\lambda$-a.e. absolute convergence of (2.23) in $L^1(\mathcal{H}^1)$ will imply that $\rho_{\theta\#}\mu_b \in L^1(\mathcal{H}^1)$ for $\lambda$-a.e. $\theta \in I$, and will imply that the series is well-defined as an $L^1(\mathcal{H}^1)$ limit. Define

$$\rho_{\theta\#}\mu_g = \rho_{\theta\#}\mu - \rho_{\theta\#}\mu_b,$$

for each $\lambda \in I$ such that the sum defining $\rho_{\theta\#}\mu_b$ converges in $L^1(\mathcal{H}^1)$ (which will include $\lambda$-a.e. $\theta \in I$). It will be shown that $\rho_{\theta\#}\mu_g \in L^2(\mathcal{H}^1)$, for $\lambda$-a.e. $\theta \in I$. Together with $\rho_{\theta\#}\mu_b \in L^1(\mathcal{H}^1)$, this will imply that $\rho_{\theta\#}\mu \in L^1(\mathcal{H}^1)$ (or equivalently $\rho_{\theta\#}\mu \ll \mathcal{H}^1$) for $\lambda$-a.e. $\theta \in I$.

It will first be shown that

$$\int \sum_{j \geq 1} \sum_{\tau \in \Lambda_j} \sum_{T \in T_{\tau,b}} \|\rho_{\theta\#} M T \mu\|_{L^1(\mathcal{H}^1)} \ d\lambda(\theta) < \infty. \quad (2.24)$$

The proof of this is similar to the proof of Lemma 2.4, but some of the details will be included for readability. The left-hand side of (2.24) can be written as

$$\sum_{j \geq 1} \int \sum_{\tau \in \Lambda_j} \sum_{T \in T_{\tau,b}} \|\rho_{\theta\#} M T \mu\|_{L^1(\mathcal{H}^1)} \ d\lambda(\theta)$$

(2.25)

$$= \sum_{j \geq 1} \int \sum_{\tau \in \Lambda_j: |\theta - \theta| \leq 2^{j(\delta - 1/2)}} \sum_{T \in T_{\tau,b}} \|\rho_{\theta\#} M T \mu\|_{L^1(\mathcal{H}^1)} \ d\lambda(\theta)$$

(2.26)

$$+ \sum_{j \geq 1} \int \sum_{\tau \in \Lambda_j: |\theta - \theta| > 2^{j(\delta - 1/2)}} \sum_{T \in T_{\tau,b}} \|\rho_{\theta\#} M T \mu\|_{L^1(\mathcal{H}^1)} \ d\lambda(\theta).$$

By Lemma 2.2

$$L \lesssim \mu(\mathbb{R}^3). \quad (2.20)$$

By Lemma 2.3

$$\leq \sum_{j \geq 1} \int \sum_{\tau \in \Lambda_j: |\theta - \theta| \leq 2^{j(\delta - 1/2)}} \sum_{T \in T_{\tau,b}} \|M T \mu\|_{L^1(\mathbb{R}^3)} \ d\lambda(\theta)$$

$$\lesssim \mu(\mathbb{R}^3) + \sum_{j \geq 1} 2^{3j\delta} \int \sum_{\tau \in \Lambda_j: |\theta - \theta| \leq 2^{j(\delta - 1/2)}} \sum_{T \in T_{\tau,b}} \mu(2T) \ d\lambda(\theta).$$

As in the proof of Lemma 2.4 the non-tail term satisfies

$$\sum_{j \geq 1} 2^{3j\delta} \int \sum_{\tau \in \Lambda_j: |\theta - \theta| \leq 2^{j(\delta - 1/2)}} \sum_{T \in T_{\tau,b}} \mu(2T) \ d\lambda(\theta) \lesssim \sum_{j \geq 1} 2^{10j\delta} \int \mu(B_j(\theta)) \ d\lambda(\theta),$$

where, for each $\theta \in I$ and each $j$,

$$B_j(\theta) = \bigcup_{\tau \in \Lambda_j: |\theta - \theta| \leq 2^{j(\delta - 1/2)}} T \cup 2T.$$
For each $T$ in the union defining $B_j(\theta)$, the set $(4T) \cap B(0,1)$ is contained in a plank $T_\theta$ of dimensions
\[
\sim 2^{\left(2^j - 1\right)} \times 2^{\left(\beta - 1/2\right)} \times 1,
\]
with short direction parallel to \(\gamma(\theta)\), medium direction parallel to \(\gamma(\beta)\), and long direction parallel to \((\gamma \times \gamma')(\theta)\). The intervals in the set
\[
\left\{ \rho_\theta(T_\theta) : T \in \mathcal{T}_\tau, \quad \tau \in \Lambda_j, \quad |\theta_\tau - \theta| \leq 2^{\left(\beta - 1/2\right)} \right\},
\]
all have length \(\sim 2^{\left(2^j - 1\right)}\), and cover \(\rho_\theta(B_j(\theta) \cap B(0,1))\). By the Vitali covering lemma, there is a disjoint subcollection
\[
\{ \rho_\theta(T_\theta) : T \in B_\theta \},
\]
indexed by some set $B_\theta$, such that
\[
\{3\rho_\theta(T_\theta) : T \in B_\theta\},
\]
is a cover of \(\rho_\theta(B_j(\theta) \cap B(0,1))\). The set $B_\theta$ has cardinality $|B_\theta| \leq \mu(\mathbb{R}^3)2^j(1 - \epsilon)$; by disjointness and the definition of the “bad” planks. By Lemma \[2.4\] for each $j \geq 1$,
\[
\int \mu(B_j(\theta)) \, d\lambda(\theta) \leq \int (\rho_\theta \mu)( \bigcup_{T_\theta \in B_\theta} 3\rho_\theta(T_\theta) ) \, d\lambda(\theta) \lesssim 2^{\beta(-\epsilon/2 + 10\epsilon)} \mu(\mathbb{R}^3).
\]
Since $\delta \ll \delta_\epsilon$, summing the above inequality over $j$ gives
\[
(2.20) \quad \mu(\mathbb{R}^3).
\]

It remains to show that $\rho_{\theta \# \mu_{\gamma}} \in L^2(\mathcal{H}^1)$ for $\lambda$-a.e. $\theta \in I$. To prove this, by Plancherel’s theorem in 1 dimension it suffices to show that
\[
\int \int_{\mathbb{R}} |\hat{\mu}_{\gamma}(t\gamma(\theta))|^2 \, dt \, d\lambda(\theta) < \infty.
\]
By symmetry and by summing a geometric series, it is enough to show that for any $j \geq 1$,
\[
\int \int_{2^{j-1}} |\hat{\mu}_{\gamma}(t\gamma(\theta))|^2 \, dt \, d\lambda(\theta) \lesssim 2^{-j\epsilon}.
\]
By similar reasoning to the proof of Lemma \[2.4\] it suffices to show that
\[
\sum_{T \in \mathcal{T}_{j,g}} \int_{\mathbb{R}^3} |M_T\mu|^2 \lesssim 2^{j(\beta - 10\epsilon)},
\]
where
\[
\mathcal{T}_{j,g} = \bigcup_{\tau \in \bigcup_{|\nu| = j} A_{\tau,\nu}} \left\{ T \in \mathcal{T}_\tau : T \cap B(0,2^{10j}) \neq \emptyset \right\}.
\]
By a similar argument to the proof of Lemma \[2.3\]
\[
\sum_{T \in \mathcal{T}_{j,g}} \int_{\mathbb{R}^3} |M_T\mu|^2 \lesssim 2^{j\left[\frac{\beta - 20}{p} + \frac{1}{2} + 10n\right]}.
\]
Since $\alpha > 4 - 3\beta$, $p = 6$ and $\epsilon \ll \frac{4 - 3\beta}{3} - \beta$, this implies that
\[
\sum_{T \in \mathcal{T}_{j,g}} \int_{\mathbb{R}^3} |M_T\mu|^2 \lesssim 2^{j(\beta - 10\epsilon)},
\]
which finishes the proof of the theorem. \[\square\]
Proof of Theorem 1.3. By the density theorem for Hausdorff measures ([10, Theorem 6.2]),
\[
\limsup_{r \to 0^+} \frac{\mathcal{H}^s(A \cap B(x,r))}{r^s} \leq 2^s \quad \text{H}^s\text{-a.e. } x \in A.
\]
It follows that if, for each positive integer \(n\),
\[
A_n := \left\{ x \in A : \sup_{0 < r < 1/n} \frac{\mathcal{H}^s(A \cap B(x,r))}{r^s} < 2^{s+1} \right\},
\]
and \(\mu_n\) is the Borel measure defined by
\[
\mu_n(F) = \mathcal{H}^s \left( F \cap A_n \setminus \bigcup_{k=1}^{n-1} A_k \right),
\]
for any Borel set \(F\), then
\[
(2.27) \quad \mu = \sum_{n=1}^{\infty} \mu_n,
\]
and for any \(n \geq 1\)
\[
c_s(\mu_n) \leq \max \left\{ 2^{2s+1}, (2n)^s \mathcal{H}^s(A) \right\}.
\]
By Theorem 1.2 for any \(n \geq 1\),
\[
\dim \left\{ \theta \in I : \rho_{\theta \#} \mu_n \ll \mathcal{H}^1 \right\} \leq \frac{4 - s}{3},
\]
By (2.27),
\[
\left\{ \theta \in I : \rho_{\theta \#} \mu \ll \mathcal{H}^1 \right\} = \bigcup_{n=1}^{\infty} \left\{ \theta \in I : \rho_{\theta \#} \mu_n \ll \mathcal{H}^1 \right\}.
\]
By countable stability of the Hausdorff dimension, it follows that
\[
\dim \left\{ \theta \in I : \rho_{\theta \#} \mu \ll \mathcal{H}^1 \right\} \leq \frac{4 - s}{3}.
\]
This proves the first half of the theorem. For the second half, let \(B \subseteq A\) be an \(\mathcal{H}^s\)-measurable set with \(\mathcal{H}^s(B) > 0\). Let \(\nu\) be the Borel measure
\[
\nu(F) = \mathcal{H}^s(F \cap B) \quad (= \mu(F \cap B)),
\]
for any Borel set \(F\). It will be shown that
\[
(2.28) \quad \left\{ \theta \in I : \rho_{\theta \#} \nu \ll \mathcal{H}^1 \right\} \subseteq \left\{ \theta \in I : \mathcal{H}^1(\rho_{\theta}(B)) > 0 \right\}.
\]
Let \(\theta \in I\) be such that \(\rho_{\theta \#} \nu \ll \mathcal{H}^1\). Suppose for a contradiction that \(\mathcal{H}^1(\rho_{\theta}(B)) = 0\). Let \(\delta > 0\) be such that
\[
(\rho_{\theta \#} \nu)(F) < \mathcal{H}^s(B),
\]
for any Borel set \(F\) with \(\mathcal{H}^1(F) < \delta\). Since \(\mathcal{H}^1(\rho_{\theta}(B)) = 0\), there exists a Borel set \(F\) containing \(\rho_{\theta}(B)\) with \(\mathcal{H}^1(F) < \delta\). Hence
\[
\mathcal{H}^s(B) = \mathcal{H}^s(\rho_{\theta}^{-1}(F) \cap B) = (\rho_{\theta \#} \nu)(F) < \mathcal{H}^s(B),
\]
and this contradiction proves (2.28). Thus
\[
\left\{ \theta \in I : \mathcal{H}^1(\rho_{\theta}(B)) = 0 \right\} \subseteq \left\{ \theta \in I : \rho_{\theta \#} \nu \ll \mathcal{H}^1 \right\}
\subseteq \left\{ \theta \in I : \rho_{\theta \#} \mu \ll \mathcal{H}^1 \right\}. \quad \Box
Appendix A. Refined decoupling

The following inequality is Theorem 9 from [4]; it is a refined version of the decoupling theorem for generalised cones.

**Theorem A.1 (\([4, \text{Theorem 9}]\)).** Let \(I\) be a compact interval, and let \(\gamma : I \to S^2\) be a \(C^2\) unit speed curve with \(\det(\gamma, \gamma', \gamma'')\) nonvanishing on \(I\). Then if \(c > 0\) is sufficiently small (depending only on \(\gamma_0\)) such that the following holds for all \(\gamma\), for each \(\Theta_{\gamma}\) be a maximal \(cR^{-1/2}\)-separated subset of \(I\), and for each \(\gamma \in \Theta_{\gamma}\), let

\[
\tau(\gamma) := \left\{ \lambda_1 \gamma(\theta) + \lambda_2 \gamma'(\theta) + \lambda_3 (\gamma \times \gamma')(\theta) : 1/2 \leq \lambda_1 \leq 1, |\lambda_2| \leq R^{-1/2}, |\lambda_3| \leq R^{-1} \right\}.
\]

For each \(\tau = (\tau(\gamma))\), let \(\mathcal{T}_\tau\) be a \(\sim 1\)-overlapping cover of \(\mathbb{R}^3\) by translates of

\[
\left\{ \lambda_1 \gamma(\theta) + \lambda_2 \gamma'(\theta) + \lambda_3 (\gamma \times \gamma')(\theta) : |\lambda_1| \leq R^3, |\lambda_2| \leq R^{1+\delta}, |\lambda_3| \leq R^{1+\delta} \right\}.
\]

If \(2 \leq p \leq 6\), and

\[
\mathbb{W} \subset \bigcup_{\gamma \in \Theta_{\gamma}} \mathcal{T}_{\tau(\gamma)},
\]

and

\[
\sum_{T \in \mathbb{W}} f_T
\]

is such that \(\|f_T\|_p\) is constant over \(T \in \mathbb{W}\) up to a factor of 2, with \(\text{supp} \hat{f}_T \subseteq \tau(T)\) and

\[
\|f_T\|_{L^\infty(B(0,R),T)} \leq AR^{-10000}\|f_T\|_p,
\]

and \(Y\) is a disjoint union of balls in \(B_3(0,R)\) of radius 1, such that each ball \(Q \subseteq Y\) intersects at most \(M\) planks \(2T\) with \(T \in \mathbb{W}\), then

\[
\left\| \sum_{T \in \mathbb{W}} f_T \right\|_{L^p(Y)} \leq C_{A,\gamma,c,\epsilon,\delta} R^c \left( \frac{M}{|\mathbb{W}|} \right)^{\frac{1}{2} - \frac{1}{2p}} \left( \sum_{T \in \mathbb{W}} \|f_T\|_p^2 \right)^{1/2}.
\]

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