STABILITY OF EQUILIBRIUM SHAPES IN SOME FREE BOUNDARY PROBLEMS INVOLVING FLUIDS

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Abstract. In this chapter the motion of two-phase, incompressible, viscous fluids with surface tension is investigated. Three cases are considered: (1) the case of heat-conducting fluids, (2) the case of isothermal fluids, and (3) the case of Stokes flows. In all three situations, the equilibrium states in the absence of outer forces are characterized and their stability properties are analyzed. It is shown that the equilibrium states correspond to the critical points of a natural physical or geometric functional (entropy, available energy, surface area) constrained by the pertinent conserved quantities (total energy, phase volumes). Moreover, it is shown that solutions which do not develop singularities exist globally and converge to an equilibrium state.

1. Introduction

In this chapter, the motion of two heat-conducting, incompressible, viscous Newtonian fluids in $\mathbb{R}^n$, $n \geq 2$, that are separated by a free interface is considered. The position of the separating interface is unknown and has to be determined as part of the problem.

More precisely, the fluids are assumed to fill a bounded region $\Omega \subset \mathbb{R}^n$. Let $\Gamma_0 \subset \Omega$ be a given surface which bounds the region $\Omega_1(0)$ occupied by an incompressible viscous fluid, fluid$_1$, called the dispersed phase, and let $\Omega_2(0)$ be the complement of the closure of $\Omega_1(0)$ in $\Omega$, corresponding to the region occupied by a second incompressible viscous fluid, fluid$_2$, called the continuous phase. The two fluids are assumed to be immiscible, and the dispersed phase is assumed to not being in contact with the boundary $\partial \Omega$ of $\Omega$.

Let $\Gamma(t)$ denote the position of $\Gamma_0$ at time $t$. Thus, $\Gamma(t)$ is a sharp interface which separates the fluids occupying the regions $\Omega_1(t)$ and $\Omega_2(t)$, respectively. The unit normal field on $\Gamma(t)$, pointing from $\Omega_1(t)$ into $\Omega_2(t)$, is denoted by $\nu_\Gamma(t, \cdot)$. Moreover, $V_\Gamma(t, \cdot)$ is the normal velocity and $H_\Gamma(t, \cdot) := -\text{div}_\Gamma \nu_\Gamma$ the $(n-1)$-fold mean curvature (that is, the sum of the principal curvatures) of $\Gamma(t)$ with respect to $\nu_\Gamma(t, \cdot)$, respectively. Here, $H_\Gamma$ is negative when $\Omega_1(t)$ is convex in a neighborhood of $x \in \Gamma(t)$. With this convention, $H_\Gamma = -(n-1)/R$ if $\Gamma$ is a sphere of radius $R$ in $\mathbb{R}^n$.

The following notation for the physical quantities involved will be employed throughout this chapter:

The research of G.S. was partially supported by the NSF Grant DMS-1265579.
The densities \( \rho_i > 0 \) as well as the surface tension \( \sigma > 0 \) are taken to be constant. Let \( \psi_i(\theta) \) denote the Helmholtz free energy of fluid \( i \). It is noted that the free energy is a constitutive quantity that depends on the physical properties of the respective fluids. Several physical and thermodynamic quantities are derived from \( \psi_i \) as follows:

\[
\begin{align*}
\epsilon_i(\theta) &= \psi_i(\theta) + \theta \eta_i(\theta) \quad \text{the (mass specific) internal energy of fluid } i, \\
\eta_i(\theta) &= -\psi_i'(\theta) \quad \text{the (mass specific) entropy of fluid } i, \\
\kappa_i(\theta) &= e_i'(\theta) = -\theta \psi_i''(\theta) \quad \text{the heat capacity of fluid } i.
\end{align*}
\]

In the sequel, the index \( i \) is occasionally dropped, but it should be noted that the quantities \( \psi, \epsilon, \eta, \kappa, \mu, \) and \( d \) may have a jump across the interface \( \Gamma \).

It is assumed throughout that the velocity \( u \) and the temperature \( \theta \) be continuous across \( \Gamma \). The motion of two heat-conducting, incompressible, viscous Newtonian fluids is then described by the following coupled system

\[
\begin{align*}
\rho_i (\partial_t u + (u|\nabla)u) - \text{div} \; T &= 0 \quad \text{in } \Omega \setminus \Gamma(t), \\
\text{div} \; u &= 0 \quad \text{in } \Omega \setminus \Gamma(t), \\
u &= 0 \quad \text{on } \partial\Omega, \\
[u] &= 0 \quad \text{on } \Gamma(t), \\
-T[\nu_T] &= \sigma H T \nu_T \quad \text{on } \Gamma(t), \\
u(0) &= u_0 \quad \text{in } \Omega_0,
\end{align*}
\]

\( \rho \kappa(\partial_t \theta + (u|\nabla)\theta) - \text{div}(d(\theta)\nabla \theta) = 2\mu(\theta)|D(u)|^2 \)

in \( \Omega \setminus \Gamma(t), \)

\[
\begin{align*}
\partial_{\nu} \theta &= 0 \quad \text{on } \partial\Omega, \\
[\theta] &= 0 \quad \text{on } \Gamma(t), \\
[d(\theta)\partial_{\nu} \theta] &= 0 \quad \text{on } \Gamma(t), \\
\theta(0) &= \theta_0 \quad \text{in } \Omega_0.
\end{align*}
\]
Here, $(z|w) = \sum_{j=1}^{n} z_j \bar{w}_j$ denotes the inner product in $\mathbb{C}^n$ for $z, w \in \mathbb{C}^n$.

\[
\|\phi\|(t, x) = \lim_{h \to 0^+} \left( \phi(t, x + h\nu_T(x)) - \phi(t, x - h\nu_T(x)) \right), \quad x \in \Gamma(t),
\]
denotes the jump of the quantity $\phi$, defined on the respective domains $\Omega_i(t)$, across the interface $\Gamma(t)$, and $|D(u)|_2 := (\text{trace}\ |D(u)|^2)^{1/2}$ is the Hilbert-Schmidt norm of the (symmetric) matrix $D(u)$.

For $\Omega$ and the quantities $\psi_i, d_i, \mu_i$ the following regularity and positivity conditions are assumed:

\[
\partial \Omega \in C^3, \quad \psi_i \in C^3(0, \infty), \quad d_i, \mu_i \in C^2(0, \infty), \quad \psi_i''(s), d_i(s), \mu_i(s) > 0, \quad s > 0.
\]

Given is the initial position $\Gamma_0$, the initial velocity $u_0 : \Omega_0 \to \mathbb{R}^n$ and the initial temperature $\theta_0 : \Omega_0 \to \mathbb{R}$, where $\Omega_0 := \Omega \setminus \Gamma_0$.

The unknowns $\Gamma(t)$, the velocity field $u(t, \cdot) : \Omega \setminus \Gamma(t) \to \mathbb{R}^n$, the pressure $\pi(t, \cdot) : \Omega \setminus \Gamma(t) \to \mathbb{R}$, and the temperature $\theta(t, \cdot) : \Omega \setminus \Gamma(t) \to \mathbb{R}$, where $n \geq 2$.

In case $\mu$ is constant, the Navier-Stokes system (1.1) decouples from the advection - diffusion equation (1.2).

In the isothermal case, that is, in case that $\theta$ is constant, system (1.1)-(1.3) reduces to

\[
\rho(\partial_t u + (u|\nabla)u) - \mu \Delta u + \nabla \pi = 0 \quad \text{in} \quad \Omega \setminus \Gamma(t),
\]
\[
\text{div} u = 0 \quad \text{in} \quad \Omega \setminus \Gamma(t),
\]
\[
u = 0 \quad \text{on} \quad \partial \Omega,
\]
\[
[D(u)] = 0 \quad \text{on} \quad \Gamma(t),\]
\[
-T\nu_T \cdot \sigma H_T \nu_T \quad \text{on} \quad \Gamma(t),
\]
\[
V_T = (u|\nu_T) \quad \text{on} \quad \Gamma(t),
\]
\[
u(0) = u_0 \quad \text{in} \quad \Omega_0,
\]
\[
\Gamma(0) = \Gamma_0.
\]
resulting in the isothermal Navier-Stokes problem with surface tension. The corresponding one-phase problem is obtained by setting $\rho_2 = \mu_2 = 0$ and discarding $\Omega_2$.

If $\theta$ is constant and inertia (i.e., the term $\rho(\partial_t u + (u|\nabla)u)$) is ignored, one is left with a quasi-stationary problem, the two-phase Stokes problem with surface tension, which generates the two-phase Stokes flow with surface tension. More
precisely, this problem reads

\[-\mu \Delta u + \nabla \pi = 0 \quad \text{in} \quad \Omega \setminus \Gamma(t),\]
\[\text{div} \ u = 0 \quad \text{in} \quad \Omega \setminus \Gamma(t),\]
\[u = 0 \quad \text{on} \quad \partial \Omega,\]
\[[u] = 0 \quad \text{on} \quad \Gamma(t),\]
\[-[\Gamma \nu_T] = \sigma H_\Gamma \nu_T \quad \text{on} \quad \Gamma(t),\]
\[V_T = (u|\nu_T) \quad \text{on} \quad \Gamma(t),\]
\[\Gamma(0) = \Gamma_0.\]  

(1.6)

The isothermal one-phase Navier-Stokes problem has received wide attention in the last three decades or so. Existence and uniqueness of solutions for \(\sigma = 0\), as well as \(\sigma > 0\), in case that \(\Omega_0\) is bounded has been extensively studied in a series of papers by Solonnikov, see for instance [41, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52] and Mogilevskii and Solonnikov [18]. Results were established in anisotropic Sobolev-Slobodetskii as well as in Hölder spaces. Moreover, it was shown in [43] that if \(\Omega_0\) is sufficiently close to a ball and the initial velocity \(u_0\) is sufficiently small, then the solution exists globally and converges to a uniform rigid rotation of the liquid about a certain axis which is moving uniformly with a constant speed, see also Padula and Solonnikov [20].

More recently, local and global existence and uniqueness results (in case that \(\Omega_0\) is a bounded domain, a perturbed infinite layer, or a perturbed half-space) in anisotropic Sobolev spaces \(W^{2,1}_{q,p}\) have been established by Shibata and Shimizu [37, 38, 39] for \(\sigma = 0\) as well as \(\sigma > 0\). Additional existence results can be found in Mucha and Zajączkowski [19] and Abels [1].

The motion of a layer of an incompressible, viscous fluid in an ocean of infinite extent, bounded below by a solid surface and above by a free surface which includes the effects of surface tension and gravity, was considered by Allain [2], Beale [3], Beale and Nishida [4], Tani [55], and by Tani and Tanaka [56]. If the initial state and the initial velocity are close to equilibrium, global existence of solutions was proved in [3] for \(\sigma > 0\), and in [56] for \(\sigma \geq 0\), and the asymptotic decay rate for \(t \to \infty\) was studied in [4].

Existence and uniqueness of local strong solutions for the isothermal two-phase problem (1.5) was first studied by Denisova [7, 8] and Denisova and Solonnikov [12, 13], while global existence results were established in [10, 11, 14, 53]. Shimizu [40] obtained existence and uniqueness results in anisotropic Sobolev \(W^{2,1}_{q,p}\)-spaces.

Prüss and Simonett [26, 27, 28] considered the two-phase Navier-Stokes equations with \(\sigma > 0\) in a situation where the free boundary \(\Gamma\) is given as the graph of a function over a hyperplane, and gravity is acting on the fluids [26, 28]. It was shown in [27, 28] that solutions regularize and immediately become real analytic in space and time. It is well-known that the situation where gravity acts on two superposed immiscible fluids - with the heavier fluid lying above a fluid of lesser density - can lead to an instability, the famous Rayleigh-Taylor instability, see
Prüss and Simonett [27], Wang and Tice [57], and Wilke [58] for results in this direction.

Köhne, Prüss, and Wilke [17] obtained existence and uniqueness of strong solutions with maximal regularity for (1.5). It was shown that the equilibrium states are given by zero velocities, constant pressures in the phase components, while \( \Omega_1 \) consists of a collection of balls (of possibly different) radii. Moreover, nonlinear stability of equilibria and convergence was established. A similar result was also obtained in [14, 53] by a different approach.

Tanaka [54] and Denisova [9] considered the two-phase Navier-Stokes equations with thermo-capillary convection (1.1)-(1.3) and obtained existence and uniqueness of strong local solutions.

Existence and uniqueness of solutions for the quasi-stationary one-phase Stokes flow was first obtained by Günther, Prokert [16, 21], see also Solonnikov [47]. It was shown in [16] that in case the initial domain of a fluid drop is close to a ball, the solution exits globally and converges to a ball at an exponential rate. This result was rederived by Friedman and Reitich [15] by a different method.

A common approach employed by many authors to analyze free boundary problems in fluid flows relies on the use of Lagrangian coordinates.

Here, a different approach is used, namely the direct mapping method based on the Hanzawa transform, which has proven itself to be particularly useful in the study of phase transitions, and in the study of fluid flows in the presence of phase transitions, see for instance [22, 23, 24, 25, 32, 35, 36], and the monograph [30]. This approach was also essential in establishing regularity properties of solutions for incompressible fluid flows, see [26, 27, 28] and [30]. The direct mapping method has also been employed by Solonnikov in the recent works [48, 50, 51, 52, 53].

The reader will find a systematic exposition of the Hanzawa transform and its application to free boundary problems in Chapter 7.1 of this handbook.

The main results of this chapter concern the stability analysis of equilibria and the qualitative behavior of global solutions. It turns out that the equilibrium states for the problems introduced above are not isolated, but instead give rise to a finite-dimensional smooth manifold \( \mathcal{E} \). Consequently, the kernel of the linearization \( L \) (to be discussed below) at an equilibrium is nontrivial and has at least the dimension of the manifold \( \mathcal{E} \).

It will be shown that for any equilibrium \( e_* \in \mathcal{E} \), the kernel \( N(L) \) is in fact isomorphic to the tangent space of \( \mathcal{E} \) at \( e_* \), the eigenvalue 0 of \( L \) is semi-simple, and the remaining spectral part of the linearization \(-L\) is stable. Hence, every equilibrium \( e_* \) is normally stable.

It will then be shown that every solution that starts out close to an equilibrium exists globally and converges to a (possibly different) equilibrium at an exponential rate. In a simpler context, this result has been termed the generalized principle of linearized stability, see [33, 34]. For problem (1.1)-(1.3), as well as problem (1.5), a significant challenge arises as solutions live on a nonlinear manifold, the state manifold \( \mathcal{SM} \), caused by nonlinear compatibility conditions.
Nonlinear stability for problem (1.1)-(1.3) and problem (1.5) will be obtained by an application of the implicit function theorem in combination with degree theory.

It will be shown that each of the problems introduced above possesses a natural (strict) Lyapunov functional. Consequently, the limit sets of solutions are contained in the set of equilibria $E$. Combining this with compactness of solutions and the local stability of equilibria one shows that any solution which does not develop singularities converges to an equilibrium in the topology of the state manifold $SM$.

Stability of equilibria for the two-phase isothermal Navier-Stokes problem (1.5) was established by Köhne, Prüss, and Wilke [17], and Denisova, Solonnikov [14, 53], while the corresponding results for the two-phase Navier-Stokes equations with heat conduction (1.1)-(1.3) as well as the two-phase Stokes flow (1.6) are contained in the monograph by Prüss and Simonett [30].

It is interesting to note that similar stability results also hold in the more complex situation of fluid flows with phase transitions, with the notable difference that multiple spheres turn out to be unstable in the presence of phase transitions. The reader is once more referred to the monograph [30] for a comprehensive discussion of fluid flows with phase transitions.

2. Equilibria, energy, and entropy

In this section it will be shown that the total energy for problem (1.1)-(1.3) is preserved, while the total entropy is nondecreasing, implying that the model is thermodynamically consistent. In addition, it will be shown that equilibrium states correspond to zero velocities, constant pressures in the components of the phases, constant temperature, while the dispersed phase consists of a union of disjoint balls.

2.1. Local existence. The basic well-posedness result for problem (1.1)-(1.3) reads as follows, where $\mathcal{P}_T = I - \nu_T \otimes \nu_T$ denotes the orthogonal projection onto the tangent space of $\Gamma$.

**Theorem 2.1.** Let $p > n + 2$ and suppose that condition (1.4) holds. Assume the regularity conditions

\[(u_0, \theta_0) \in W^{2-2/p}_p(\Omega \setminus \Gamma_0)^n \cap W^{3-2/p}_p,
\]

the compatibility conditions

\[
\begin{align*}
div u_0 &= 0 \text{ in } \Omega \setminus \Gamma_0, \quad u_0 = 0 \quad \text{and} \quad \partial_t \theta_0 = 0 \quad \text{on } \partial \Omega, \\
[u_0] &= 0, \quad [\mathcal{P}_{\Gamma_0} \mu(\theta_0) D(u_0) \nu_{\Gamma_0}] = 0 \quad \text{on } \Gamma_0, \\
[\theta_0] &= 0, \quad [d(\theta_0) \partial_\nu \theta_0] = 0 \quad \text{on } \Gamma_0,
\end{align*}
\]

and the well-posedness condition $\theta_0 > 0$ in $\tilde{\Omega}$.

Then there exists a number $a = a(u_0, \theta_0, \Gamma_0)$ and a unique classical solution
Proof. For a proof the reader is referred to [30, Sections 9.2 and 9.4]. □

2.2. The state manifold. It can be shown that the closed $\mathcal{C}^2$-hypersurfaces contained in $\Omega$ which bound a region $\Omega_1 \subset \subset \Omega$ form a $\mathcal{C}^2$-manifold, denoted by $\mathcal{MH}^2(\Omega)$, see for instance [29], or [30, Chapter 2].

The state manifold $\mathcal{S}M$ for problem (1.1)-(1.3) is defined by

$$\mathcal{S}M := \{(u, \theta, \Gamma) \in \mathcal{C}(\bar{\Omega})^n+1 \times \mathcal{MH}^2(\Omega) :$$
$$\begin{align*}
(u, \theta) &\in W^{2-\frac{2}{p}}(\Omega \setminus \Gamma)^n+1, \quad \Gamma \in W^{3-\frac{2}{p}}_p, \\
\text{div } u &= 0 \text{ in } \Omega \setminus \Gamma, \quad \theta > 0 \text{ in } \bar{\Omega}, \quad u = 0, \quad \partial_t \theta = 0 \text{ on } \partial \Omega, \\
P_{\Gamma}[\mu(\theta)D(u)\nu_{\Gamma}] &= 0, \quad [d(\theta)\partial_{\nu}\theta] = 0 \text{ on } \Gamma.
\end{align*}$$

Charts for this manifold are obtained by the charts induced by $\mathcal{MH}^2(\Omega)$, followed by a Hanzawa transformation.

Applying Theorem 2.1 and re-parameterizing the interface repeatedly, one shows that (1.1)-(1.3) yields a local semiflow on $\mathcal{S}M$.

It is noticeable that the pressure $\pi$ does not explicitly occur as a variable in the definition of the state manifold $\mathcal{S}M$. In fact, $\pi$ is determined at each time $t$ from $(u, \theta, \Gamma)$ by means of the weak transmission problem

$$(\varrho^{-1} \nabla \pi, \nabla \phi)_{L^2(\Omega)} = (2\varrho^{-1} \text{div}(\mu(\theta)D(u)) - (u|\nabla)u|\nabla \phi)_{L^2(\Omega)}, \quad \phi \in H^1_{\nu}(\Omega),$$

$$[\pi] = \sigma H_{\Gamma} + ([2\mu(\theta)D(u)\nu_{\Gamma}]|\nu_{\Gamma}) \text{ on } \Gamma.$$ 

Concerning such transmission problems the reader is referred to [17, Theorem 8.5] or [30, Proposition 8.6.2].

2.3. Conservation of phase volumes. Suppose that the dispersed phase $\Omega_1$ consists of $m$ disjoint connected components, $\Omega_1 = \bigcup_{k=1}^m \Omega_{1,k}$. Let $\Gamma_k := \partial \Omega_{1,k}$, $\Gamma = \bigcup_{k=1}^m \Gamma_k$, and let

$$M_k := |\Omega_{1,k}|, \quad k = 1, \cdots, m,$$

denote the volume of $\Omega_{1,k}$. Then for any (sufficiently smooth) solution $(u, \pi, \theta, \Gamma)$ of problem (1.1)-(1.3) one obtains, see for instance [30, Chapter 2],

$$\frac{d}{dt}|\Omega_{1,k}(t)| = \int_{\Gamma_k} V_{\Gamma} d\Gamma_k = \int_{\Gamma_k} (u|\nu_{\Gamma}) d\Gamma_k = \int_{\Omega_{1,k}} \text{div } u \, dx = 0$$

for $k = 1, \cdots, m$. This shows that problem (1.1)-(1.3) preserves the volume of each individual phase component.
2.4. Conservation of energy. The total energy for problem (1.1)-(1.3) is defined by
\[ E := E(u, \theta, \Gamma) := \frac{1}{2} \int_{\Omega \setminus \Gamma} \rho |u|^2 \, dx + \int_{\Omega} \rho \epsilon(\theta) \, dx + \sigma |\Gamma|, \]
where |\Gamma| denotes the surface area of \Gamma. For the time derivative of \( E \) one obtains
\[ \frac{d}{dt} E = \int_{\Omega} \{ \rho(\partial_t u|u) + \rho \partial_t \epsilon(\theta) \} \, dx - \int_{\Gamma} \{ \frac{\rho}{2} |u|^2 + \rho \epsilon(\theta) \} \, d\Gamma \]
for (1.1)-(1.3) is defined by
\[ \frac{d}{dt} \Phi = \int_{\Omega \setminus \Gamma} \rho \partial_t \eta(\theta) \, dx - \int_{\Gamma} [\partial_t \rho \eta(\theta)] \, d\Gamma \]
with the relation \( \epsilon'(\theta) = \theta \eta'(\theta) \) one obtains
\[ \frac{d}{dt} \Phi(t) = \int_{\Omega} \rho \partial_t \eta(\theta) \, dx - \int_{\Gamma} [\partial_t \rho \eta(\theta)] \, d\Gamma \]
\[ = \int_{\Omega} \rho \partial_t \eta(\theta) \, dx - \int_{\Gamma} [\partial_t \rho \eta(\theta)] \, d\Gamma \]
\[ = \int_{\Omega} \left( \frac{1}{\theta} \left\{ 2 \mu(\theta)|D(u)|^2 + \text{div} (d(\theta) \nabla \theta) \right\} - \rho |u| \nabla \eta(\theta) \right) \, dx - \int_{\Gamma} [\partial_t \rho \eta(\theta)] \, d\Gamma \]
\[ = \int_{\Omega} \left( \frac{2 \mu(\theta)|D(u)|^2}{\theta} + \frac{d(\theta)|\nabla \theta|^2}{\theta^2} \right) \, dx + \text{div} \left( \frac{d(\theta)}{\theta} \nabla \theta \right) - \rho |u| \nabla \eta(\theta) \right) \, dx \]
\[ - \int_{\Gamma} [\partial_t \rho \eta(\theta)] \, d\Gamma \]
\[ = \int_{\Omega} \left\{ 2 \mu(\theta)|D(u)|^2 + \frac{d(\theta)|\nabla \theta|^2}{\theta^2} \right\} \, dx. \]
Hence the total entropy is nondecreasing, and this shows that the model is thermodynamically consistent.

It follows that the negative entropy is a Lyapunov functional for system (1.1)-(1.3).

2.5. Entropy and equilibria. The total entropy for (1.1)-(1.3) is defined by
\[ \Phi(\theta, \Gamma) = \int_{\Omega \setminus \Gamma} \rho \eta(\theta) \, dx. \]
and as $\theta$ is continuous, it follows that $\theta = \theta_*$ on $\Omega$, with $\theta_*$ a constant. Next, by $[u] = 0$ and Korn’s inequality we have $\nabla u = 0$, and then $u = 0$ by the no-slip condition on $\partial \Omega$. This implies further $(\partial_t \theta, \partial_t u) = 0$ and $V_T = 0$, i.e., the system is at equilibrium. Furthermore, $\nabla \pi = 0$, and consequently the pressure is constant in the components of the bulk phases. Therefore, $\sigma H = [\pi]$ is constant on each component $\Gamma_k$ of the interface $\Gamma$. This implies that the dispersed phase $\Omega_1$ is a ball if it is connected, or a collection $\bigcup B(x_k, R_k)$ of balls, with the radii $R_k$ of the balls related to the pressures by the Young-Laplace law

$$[\pi]_{|\Gamma_k} = \sigma H_{\Gamma_k} = \frac{\sigma (n-1)}{R_k}. \quad (2.2)$$

**Remark 2.2.** (i) It is noted that at equilibrium the dispersed phase consists of at most countably many disjoint balls $B(x_k, R_k)$. If there are infinitely many of them, then $R_k \to 0$ as $k \to \infty$, hence the corresponding curvatures $H_{\Gamma_k} = -(n-1)/R_k$ tend to infinity, and so do the pressures inside these balls. This is due to the model assumption that there are no phase transitions. On the other hand, phase transitions will occur at very high pressure levels. Therefore, although thermodynamically consistent, the model $\Pi_1$ is physically not very realistic.

To avoid this contradiction, in the sequel only equilibria in which the dispersed phase consists of finitely many balls are considered. Note also that the free boundary will not be of class $C^2$ if $\Omega_1$ has infinitely many components.

(ii) There is another pathological case which will be excluded in the sequel, namely the one where the dispersed phase contains balls touching each other. This can only happen if the radii of these balls are equal, as the pressure jump would otherwise not be constant on $\Gamma$. Physically one would expect such an equilibrium to be unstable. Observe that also in this situation the free boundary $\Gamma$ is not a manifold of class $C^2$.

2.6. The manifold of equilibria. As shown above the equilibrium states of system $\Pi_1$ are zero velocities $u_*$, constant pressures $\pi_*$ in the phases, constant temperature $\theta_*$, and the dispersed phase $\Omega_1$ consists of a collection of balls. An equilibrium is called non-degenerate if

(i) $\Omega_1$ consists of a finite collection of balls;

(ii) the balls do not touch the outer boundary $\partial \Omega$ and do not touch each other.

Suppose then that $\Omega_1 = \bigcup_{k=1}^n B(x_k, R_k)$, with $B(x_k, R_k)$ a ball with center $x_k$ and radius $R_k$. Then

$$S := \left\{ \Sigma = \bigcup_{k=1}^m \Sigma_k : \Sigma_k = \partial B(x_k, R_k), \quad B(x_k, R_k) \subset \Omega, \right\}$$

$$B(x_k, R_k) \cap B(x_j, R_j) = \emptyset \quad \text{for } j \neq k$$

forms a smooth (in fact analytic) manifold of dimension $m(n+1)$. To verify this, it will be shown how a neighboring sphere can be parameterized over a given one.
Let us assume that $\Sigma = \partial B(0, R)$ is centered at the origin of $\mathbb{R}^n$. Suppose $S \subset \Omega$ is a sphere that is sufficiently close to $\Sigma$ and denote by $(y_1, \ldots, y_n)$ the coordinates of its center and let $y_0$ be such that $R + y_0$ corresponds to its radius. One verifies that $(R + y_0)^2 = \sum_{j=1}^{n} ((R + \delta)Y^j - y_j)^2$, where $\delta$ denotes the signed distance function with respect to $\Sigma$, and $Y^j$ are the spherical harmonics of degree one on $\Sigma$. Using the relation $\sum_{j=1}^{n} (Y^j)^2 = 1$ and solving the quadratic equation for $R + \delta$ shows that $S$ can be parameterized over $\Sigma$ (in normal direction) by

$$
\delta(y_0, y_1, \ldots, y_n) = \sum_{j=1}^{n} y_j Y^j - R + \sqrt{\left(\sum_{j=1}^{n} y_j Y^j \right)^2 + (R + y_0)^2 - \sum_{j=1}^{n} y_j^2}.
$$

Clearly, the mapping $z \mapsto \delta(z)$ provides a real analytic parameterization. The case of $m$ spheres can be treated analogously, yielding $m(n + 1)$ degrees of freedom.

Finally,

$$
E = \{ (0, \theta, \Sigma) : \theta > 0 \text{ is constant, } \Sigma \in S \}
$$

(2.4)
denotes the set of all non-degenerate equilibria of (1.1)-(1.3). It follows that $E$ is a real analytic manifold of dimension $m(n+1)+1$. Here we note that the (constant) pressures at equilibrium can be determined by the Young-Laplace law (2.2).

2.7. Equilibria as critical points of the entropy. Suppose that $\Omega_1$ is the union of $m$ disjoint components $\Omega_{1,k}$ and $\Gamma = \bigcup_{k=1}^{m} \Gamma_k$ with $\Gamma_k = \partial \Omega_{1,k}$.

The goal of this subsection is to determine the critical points of the total entropy $\Phi$ under the constraints of given total energy $E = E_0$ and given phase volumes $M_k = M_{0,k}$, say on $C(\bar{\Omega})^{n+1} \times \mathcal{M}\mathcal{H}^2(\Omega)$. See Section 2.2 for the definition of $\mathcal{M}\mathcal{H}^2(\Omega)$.

With $\Phi(\theta, \Gamma) = \int_{\Omega \setminus \Gamma} \varrho \eta(\theta) \, dx$ and

$$
M_k(\Gamma) = |\Omega_{1,k}|, \quad E(u, \theta, \Gamma) = \int_{\Omega \setminus \Gamma} \{ (\varrho/2)|u|^2 + \varrho \epsilon(\theta) \} \, dx + \sigma |\Gamma|,
$$

the method of Lagrange multipliers yields

$$
\Phi'(\theta, \Gamma) + \sum_{k=1}^{m} \lambda_k M_k'(\Gamma) + \mu E'(u, \theta, \Gamma) = 0
$$

at a critical point $e = (u, \theta, \Gamma)$. We will now compute the derivatives of these functionals in the direction of

$$
z = (v, \vartheta, h_1, \ldots, h_m),
$$
where each component \( \Gamma_k \) is varied in the direction of the normal vector field \( h_k \nu_{\Gamma_k} \), with \( h_k : \Gamma_k \to \mathbb{R} \) a (sufficiently smooth) function. This yields

\[
\langle \Phi'(\theta, \Gamma) | z \rangle = \int_{\Omega} \eta'(\theta) \eta d\Omega - \sum_{k=1}^{m} \int_{\Gamma_k} \|\eta(\theta)\| h_k d\Gamma_k,
\]

\[
\langle M'_k(\Gamma) | z \rangle = \int_{\Gamma_k} h_k d\Gamma_k, \quad k = 1, \ldots, m,
\]

\[
\langle E'(u, \theta, \Gamma) | z \rangle = \int_{\Omega} \{ \rho(u|v) + \rho\epsilon'(\theta) \} dx - \sum_{k=1}^{m} \int_{\Gamma_k} \{ \| u \|^2 + \rho\epsilon(\theta) \} + \sigma H_{\Gamma_k} \} h_k d\Gamma_k.
\]

Varying \( \theta \) first, while setting all other variables of \( z \) equal to zero, yields

\[
\rho\epsilon'(\theta) + \mu \rho\epsilon'(\theta) = 0,
\]

hence \( \epsilon'(\theta) = \theta \rho'(\theta) = \kappa(\theta) > 0 \) implies \( \theta = -1/\mu > 0 \) constant. Next we vary \( v \) to obtain \( u = 0 \) as \( \mu \neq 0 \). Finally, each component \( h_k \) is varied separately, to the result

\[
-\| \rho\eta(\theta) \| + \lambda_k + (\| \rho\epsilon(\theta) \| + \sigma H_{\Gamma_k}) / \theta = 0
\]
on \( \Gamma_k \), which by the definition of \( \epsilon \) yields

\[
\| \rho\psi(\theta) \| + \sigma H_{\Gamma_k} = -\lambda_k \theta \text{ on } \Gamma_k. \tag{2.5}
\]

This implies that \( H_{\Gamma_k} \) is constant for each \( k \in \{1, \ldots, m\} \), hence \( \Omega_1 \) consists of a finite collection of balls.

In summary, the critical points \( e = (0, \theta, \Gamma) \) of the total entropy with the constraints of prescribed phase volumes and prescribed total energy are precisely the equilibria of the system.

It is also interesting to note that each equilibrium \( e = (0, \theta, \Gamma) \) with \( \Gamma = \bigcup_{k=1}^{m} \Gamma_k \) is a local maximum of the entropy w.r.t. the constraints \( E = E_0 \) and \( M_k = M_{0,k} \) constant. In order to verify this one needs to show that

\[
\mathcal{D}(e) := [\Phi + \sum_{k=1}^{m} \lambda_k M_k + \mu E]''(e)
\]
is negative definite on \( \ker (E'(e)) \bigcap_{k=1}^{m} \ker (M'_k(e)) \), the intersection of the kernels of \( E'(e) \) and \( M'_k(e) \), where \( (\lambda_1, \ldots, \lambda_m, \mu) \) are the fixed Lagrange multipliers found above. The kernel of \( M'_k(e) \) is given by

\[
\int_{\Gamma_k} h_k d\Gamma_k = 0, \quad k = 1, \ldots, m, \tag{2.6}
\]
and that of \( E'(e) \) by

\[
\int_{\Omega} \rho\kappa(\theta) \theta d\Omega = \sum_{k=1}^{m} (\| \rho\epsilon(\theta) \| + \sigma H_{\Gamma_k}) \int_{\Gamma_k} h_k d\Gamma_k. \tag{2.7}
\]
A straightforward calculation yields

\[-\theta \langle D(e)z|z\rangle = \int_{\Omega} \varrho |v|^2 \, dx + \int_{\Omega} (\varrho \kappa (\theta) / \theta) \vartheta^2 \, dx - \sigma \sum_{k=1}^{m} \int_{\Gamma_k} (H'_{\Gamma_k} h_k) h_k \, d\Gamma_k.\]

It is well-known that the linearization of \( H_{\Gamma_k} \) at a sphere \( \Gamma_k \) is given by

\[H'_{\Gamma_k} = (n - 1)/R_k^2 + \Delta_{\Gamma_k},\]

where \( R_k \) is the radius of the sphere and \( \Delta_{\Gamma_k} \) denotes the Laplace-Beltrami operator on \( \Gamma_k \), see for instance [29], or [30, Chapter 2]. As \( \varrho, \kappa \) are positive and \( H'_{\Gamma_k} \) is negative semi-definite for all functions in \( L_2(\Gamma_k) \) with mean value zero, one concludes that the form \( \langle D(e)z|z\rangle \) is indeed negative definite on \( L_2(\Gamma) \).

An equilibrium \( e \) is generically not isolated. If a sphere \( \Gamma_k \) does not touch the outer boundary, it may be moved inside of \( \Omega \) without changing the total entropy. This fact is reflected in \( D(e) \) by choosing \((v, \vartheta) = 0 \) and \( h_k = Y^j_k \), the spherical harmonics for \( \Gamma_k \), which satisfy \( H'_{\Gamma_k} Y^j_k = 0 \).

Summarizing, the following result has been established.

**Theorem 2.3.** The following assertions hold for problem (1.1)-(1.3).

(a) The phase volumes \( |\Omega_{1,k}| \) and the total energy \( E \) are preserved.

(b) The negative total entropy \( -\Phi \) is a strict Lyapunov functional.

(c) The non-degenerate equilibria are zero velocities, constant pressures in the components of the phases, constant temperature, and the interface is a finite union of non-intersecting spheres which do not touch the outer boundary \( \partial \Omega \).

(d) The set \( E \) of non-degenerate equilibria forms a real analytic manifold of dimension \( m(n + 1) + 1 \), where \( m \) denotes the number of connected components of \( \Omega_1 \).

(e) The critical points of the entropy functional for prescribed phase volumes and prescribed total energy are precisely the equilibria of the system.

(f) All critical points of the entropy functional for prescribed phase volumes and prescribed total energy are local maxima.

3. **Linear stability**

By employing the Hanzawa transformation, see for instance [30], one shows that the pertinent linear problem associated to (1.1)-(1.3) at an equilibrium \((0, \theta_*, \Sigma)\), with \( \Sigma = \bigcup_{k=1}^{m} \Sigma_k \) and \( \Sigma_k = \partial B(x_k, R_k) \), is given by
\[ \rho \partial_t u - \mu_* \Delta u + \nabla \pi = \rho f_u \quad \text{in} \quad \Omega \setminus \Sigma, \]
\[ \text{div} u = g_d \quad \text{in} \quad \Omega \setminus \Sigma, \]
\[ u = 0 \quad \text{on} \quad \partial \Omega, \]
\[ [u] = 0 \quad \text{on} \quad \Sigma, \]
\[ -[T_* \nu \Sigma] + \sigma (A_{\Sigma} h) \nu \Sigma = g_u \quad \text{on} \quad \Sigma, \]
\[ u(0) = u_0 \quad \text{in} \quad \Omega, \]
\[ \rho \kappa_* \partial_t \vartheta - d_* \Delta \vartheta = \rho \kappa_* f_\theta \quad \text{in} \quad \Omega \setminus \Sigma, \]
\[ \partial_t \vartheta = 0 \quad \text{on} \quad \partial \Omega, \]
\[ [\vartheta] = 0 \quad \text{on} \quad \Sigma, \]
\[ -[d_* \partial_t \vartheta] = g_\theta \quad \text{on} \quad \Sigma, \]
\[ \vartheta(0) = \vartheta_0 \quad \text{in} \quad \Omega \setminus \Sigma, \]
\[ \partial_t h - (u|\nu \Sigma) = f_h \quad \text{on} \quad \Sigma, \]
\[ h(0) = h_0 \quad \text{on} \quad \Sigma, \]

where \( \vartheta = \theta - \theta_* \) is the relative temperature, \( \mu_* = \mu(\theta_*), \kappa_* = \kappa(\theta_*), d_* = d(\theta_*), \)
\( T_* = 2\mu_* D(u) - \pi I, \) and \( h \) is the height function used to parameterize \( \Gamma(t) \) over the reference manifold \( \Sigma \) by means of \( \Gamma(t) = \{ q + h(t, \eta) \nu \Sigma(p) : q \in \Sigma, \ t \geq 0 \}. \)

Finally, the linear operator \( A_\Sigma \) is given by
\[
A_\Sigma|_{\Sigma_k} = A_{\Sigma_k} = -H'_{\Sigma_k}(0) = -\frac{n-1}{R_k^2} - \Delta_{\Sigma_k}.
\]

It is well-known that \( A_{\Sigma_k} \) is self-adjoint, positive semi-definite on functions with zero mean, and has compact resolvent in \( L_2(\Sigma_k) \). \( \lambda_0 = 0 \) is an eigenvalue with eigenspace of dimension \( n \), spanned by the spherical harmonics of degree one. \( \lambda_{-1} = -(n-1)/R_k^2 \) is also an eigenvalue, its eigenspace is one-dimensional and consists of the constants.

In order to introduce a functional analytic setting for the linear problem (3.1)-(3.3), let
\[
X_0 = L_{p,\sigma}(\Omega) \times L_p(\Omega) \times W_0^{2-1/p}(\Sigma),
\]
where the subscript \( \sigma \) means solenoidal, and define the operator \( L \) by
\[
L(u, \vartheta, h) = \left( -\mu_*/\rho \Delta u + \nabla \pi/\rho, -(d_*/\rho \kappa_*) \Delta \vartheta, -(u|\nu \Sigma) \right).
\]

The domain \( X_1 := D(L) \) of \( L \) is given by
\[
D(L) = \{ (u, \vartheta, h) \in H_0^2(\Omega \setminus \Sigma)^{n+1} \times W_0^{3-1/p}(\Sigma) : \text{div } u = 0 \quad \text{in} \quad \Omega \setminus \Sigma, \]
\[ u, \partial_t \vartheta = 0 \quad \text{on} \quad \partial \Omega, \quad [u], [\vartheta], [\partial_t \vartheta] = 0 \quad \text{on} \quad \Sigma \},
\]
where \( \mathcal{P}_\Sigma \) denotes the orthogonal projection onto the tangent space of \( \Sigma \). Here \( \pi \in H_0^1(\Omega \setminus \Sigma) \) is determined by means of the weak transmission problem
\[
(g^{-1} \nabla \pi|\nabla \phi)_{L_2(\Omega)} = (g^{-1} \mu_* \Delta u|\nabla \phi)_{L_2(\Omega)}, \quad \phi \in H_0^1(\Omega),
\]
\[
[g \pi] = -\sigma A_{\Sigma} h + ([2\mu_* D(u) \nu \Sigma] | \nu \Sigma) \quad \text{on} \quad \Sigma.
\]
Setting \( z = (u, \vartheta, h), f = (f_u, f_\vartheta, f_h), \) and \( g = (g_d, g_u, g_\vartheta) \) one concludes that the linear problem (3.1)-(3.3) can be rewritten as a Cauchy problem
\[
\dot{z} + Lz = f(t), \quad z(0) = z_0,
\]
provided \( g = 0 \). Associated with the operator \( L \) is the eigenvalue problem
\[
\varrho \lambda u - \mu_s \Delta u + \nabla \pi = 0 \quad \text{in} \quad \Omega \setminus \Sigma,
\]
\[
\operatorname{div} u = 0 \quad \text{in} \quad \Omega \setminus \Sigma,
\]
\[
u u = 0 \quad \text{on} \quad \partial \Omega,
\]
\[
[u] = 0 \quad \text{on} \quad \Sigma, \tag{3.4}
\]
and
\[
\varrho \kappa_s \lambda \vartheta - d_s \Delta \vartheta = 0 \quad \text{in} \quad \Omega \setminus \Sigma,
\]
\[
\partial_\nu \vartheta = 0 \quad \text{on} \quad \partial \Omega,
\]
\[
[\vartheta] = 0 \quad \text{on} \quad \Sigma, \tag{3.5}
\]
and
\[
- \lambda h - (u|\nu_\Sigma) = 0 \quad \text{on} \quad \Sigma.
\]
Note that the eigenvalue problems (3.4) and (3.5) decouple.

**Theorem 3.1.** Let \( e_* \in \mathcal{E} \) be an equilibrium. Then \( L \) has the following properties.

(a) \(-L\) generates a compact, analytic \( C_0\)-semigroup in \( X_0 \) which has the property of maximal \( L_p\)-regularity.

(b) The spectrum of \( L \) consists of countably many eigenvalues of finite algebraic multiplicity.

(c) \(-L\) has no eigenvalues \( \lambda \neq 0 \) with nonnegative real part.

(d) \( \lambda = 0 \) is a semi-simple eigenvalue of \( L \) of multiplicity \( m(n+1) + 1 \).

(e) The kernel \( N(L) \) of \( L \) is isomorphic to the tangent space \( T_{e_*} \mathcal{E} \) of \( \mathcal{E} \) at \( e_* \).

Hence, the equilibrium \( e_* \in \mathcal{E} \) is normally stable.

**Proof.** (a) It follows from the results in [30] that \(-L\) generates a compact, analytic, strongly continuous semigroup in \( X_0 \) which has maximal \( L_p\)-regularity.

(b) As \( L \) has compact resolvent, the spectrum of \( L \) consists entirely of eigenvalues of finite algebraic multiplicity.

(c) Suppose that \( \lambda \neq 0 \) with \( \operatorname{Re} \lambda \geq 0 \) is an eigenvalue of \(-L\) with eigenfunction \((u, \vartheta, h)\). Since (3.4)-(3.5) decouple, either \((u, h) \neq (0,0)\) or \( \vartheta \neq 0 \).
Let us first assume that \((u, h) \neq (0, 0)\). Taking the \(L_2\)-inner product of the equation for \(u\) with \(u\) and integrating by parts yields
\[
0 = \lambda|q^{1/2}u|^2_{L_2(\Omega)} - (\text{div} \ T_* u)_{L_2(\Omega)}
\]
\[
= \lambda|q^{1/2}u|^2_{L_2(\Omega)} + 2\int_{\Omega} \mu_t |D(u)|^2 dx - \int_{\Omega} \text{div} (T_* u) dx
\]
\[
= \lambda|q^{1/2}u|^2_{L_2(\Omega)} + 2|\mu|^{1/2} D(u)|^2_{L_2(\Omega)} + \int_{\Sigma} ([T_* \nu_\Sigma]|u) dx
\]
\[
= \lambda|q^{1/2}u|^2_{L_2(\Omega)} + 2|\mu|^{1/2} D(u)|^2_{L_2(\Omega)} + \sigma \bar{\lambda}(A_\Sigma; h|h)_{L_2(\Sigma)},
\]
where the relations \(\|u\| = 0\), \([T_* \nu_\Sigma] = \sigma A_\Sigma h\nu_\Sigma\) and \((u|\nu_\Sigma) = \lambda h\) are employed. Taking the real part yields the identity
\[
0 = \Re \lambda|q^{1/2}u|^2_{L_2(\Omega)} + 2|\mu|^{1/2} D(u)|^2_{L_2(\Omega)} + \sigma \Re \lambda(A_\Sigma; h|h)_{L_2(\Sigma)}.
\]
On the other hand, if \(\Im \lambda \neq 0\), taking the imaginary part results in
\[
\sigma(A_\Sigma; h|h)_{L_2(\Sigma)} = |q^{1/2}u|^2_{L_2(\Omega)}.
\]
Substituting this expression into (3.7) leads to
\[
0 = 2\Re \lambda|q^{1/2}u|^2_{L_2(\Omega)} + 2|\mu|^{1/2} D(u)|^2_{L_2(\Omega)},
\]
which shows that if \(\Re \lambda \geq 0\) is an eigenvalue of \(-L\), then it must be real. In fact, the last identity shows that \(D(u) = 0\) and hence \(\nabla u = 0\) by Korn’s inequality. Therefore, \(u\) is constant on \(\Omega\), and hence \(u \equiv 0\) by the no-slip boundary condition on \(\partial \Omega\). The condition \(\lambda h = (u|\nu_\Sigma)\) then yields \(h = 0\), a contradiction to the assumption \((u, h) \neq (0, 0)\).

In a next step it will be shown that (3.6) does not have eigenvalues with \(\lambda > 0\). Assume to the contrary that (3.6) has a nontrivial solution \((u, h)\) for \(\lambda > 0\). Let \(\Omega_{1,k}\) denote the components of \(\Omega\) and set \(\Sigma_k = \partial \Omega_{1,k}\). By the divergence theorem
\[
0 = \int_{\Omega_{1,k}} \text{div} u \, dx = \int_{\Sigma_k} (u|\nu_\Sigma) \, d\Sigma_k = \lambda \int_{\Sigma_k} h \, d\Sigma_k.
\]
This shows that the mean values of \(h\) vanish for all components of \(\Sigma\). As \(A_\Sigma\) is positive semi-definite on functions which have mean value zero for each component of \(\Sigma\), (3.7) implies \(\lambda = 0\). Hence there are no eigenvalues with nonnegative real part.

Concerning the eigenvalue problem (3.5) one obtains, after taking the \(L_2\)-inner product of the first line with \(\theta\), integrating by parts, and employing the condition \([d_* \partial_\nu \theta] = 0\), the following relation
\[
0 = \lambda (\rho \kappa q)^{1/2} \theta|^2_{L_2(\Omega)} + |d^{1/2} \nabla \theta|^2_{L_2(\Omega)}.
\]
This readily shows that all eigenvalues of (3.5) are real and non-positive.

(d) Suppose \(\lambda = 0\). Then (3.7) and (3.8) yield
\[
|\mu|^{1/2} D(u)|^2_{L_2(\Omega)} = |d^{1/2} \nabla \theta|^2_{L_2(\Omega)} = 0,
\]
hence $\vartheta$ is constant and $D(u) = 0$, and then $u = 0$ by Korn’s inequality and the no-slip condition $u = 0$ on $\partial\Omega$. This implies further that the pressures are constant in the components of the phases. From the relation $[\pi]_{\Sigma_k} = -\sigma A_{\Sigma_k} h_k$

one concludes that $A_{\Sigma_k} h_k$ is constant for $k = 1, \cdots, m$, where $h_k = h|_{\Sigma_k}$.

The kernel of the linearization $L$ is spanned by $e_{\theta} = (0, 1, 0)$, $e_{jk} = (0, 0, Y_{jk})$, with $Y_{jk}$ the spherical harmonics of degree one for the spheres $\Sigma_k$, $j = 1, \cdots, n$, $k = 1, \cdots, m$, and $e_{0k} = (0, 0, Y_{0k})$, where $Y_{0k}$ equals one on $\Sigma_k$ and zero elsewhere. Hence the dimension of the null space $N(L)$ is $m(n + 1) + 1$.

Next it will be shown that $\lambda = 0$ is semi-simple. So suppose $(u, \vartheta, h)$ is a solution of $L(u, \vartheta, h) = \sum_{j,k} \alpha_{jk} e_{jk} + \sum_k \beta_k e_{0k} + \gamma e_{\theta}$. This means

$$-\mu_* \Delta u + \nabla \pi = 0$$
$$\text{div} \, u = 0$$
$$u = 0$$
$$[u] = 0$$
$$-\sum_{j,k} \alpha_{jk} Y_{jk} + \sum_k \beta_k Y_{0k} = 0$$

and

$$-d_* \Delta \vartheta = \rho \kappa \gamma$$
$$\partial_{\nu} \vartheta = 0$$
$$[\vartheta] = 0$$
$$-\sum_{j,k} \alpha_{jk} Y_{jk} = 0$$

It is to be shown that $(\alpha_{jk}, \beta_k, \gamma) = 0$ for all $j, k$. Integrating the divergence equation for $u$ over $\Omega_{1,k}$ yields

$$0 = \int_{\Omega_{1,k}} \text{div} \, u \, dx = \int_{\Sigma_k} (u|_{\Sigma_k}) \, d\Sigma_k = -\sum_k \beta_k \int_{\Sigma_k} Y_{0k} \, d\Sigma_k = -\beta_k |\Sigma_k|,$$

where the property that the spherical harmonics have mean value zero is employed. Therefore, $\beta_k = 0$ for $k = 1, \cdots, m$.

Taking the $L_2$-inner product of the equation for $u$ with $u$, one obtains as in (3.6)

$$|\mu_*^{1/2} D(u)|^2_{L_2(\Omega)} = 0.$$ 

This implies $D(u) = 0$, hence $u = 0$ by Korn’s inequality and the no-slip boundary condition on $\partial\Omega$. This, in turn, yields

$$0 = -(u|_{\Sigma_k}) = \sum_{j,k} \alpha_{jk} Y_{jk}.$$
Thus $\alpha_{jk} = 0$ for all $j,k$, as the spherical harmonics $Y^j_k$ are linearly independent. Finally, integrating the equation for $\vartheta$ yields

$$
\gamma(\kappa|\vartheta) L_2(\Omega) = - \int_\Omega d_\vartheta \Delta \vartheta \, dx = \int_\Omega [d_\vartheta \vartheta \vartheta] \, d\Sigma - \int_{\partial\Omega} d_\vartheta \vartheta \, d(\vartheta) \Omega = 0,
$$

and hence $\gamma = 0$. Therefore, the eigenvalue $\lambda = 0$ is semi-simple.

(e) The assertion follows from the fact that $T e^* E \subset N(L)$ and the relation

$$
\text{dim} N(L) = \text{dim} T e^* E = m(n+1) + 1.
$$

\[ \square \]

4. Nonlinear stability of equilibria

Suppose $e = (0, \theta, \Sigma) \in E$ is a non-degenerate fixed equilibrium. Choosing $\Sigma$ as reference manifold and employing the Hanzawa transformation to problem (1.1)-(1.3) one obtains the nonlinear system

$$
\begin{align*}
\rho \partial_t u - \mu \Delta u + \nabla \pi &= F_u(u, \vartheta, h, \pi) \quad \text{in } \Omega \setminus \Sigma, \\
\text{div } u &= G_d(u, h) \quad \text{in } \Omega \setminus \Sigma, \\
u = 0 &\quad \text{on } \partial\Omega, \\
[u] &= 0 \quad \text{on } \Sigma, \\
&\text{(4.1)} \\
- \mathcal{P}_\Sigma [2\mu D(u)\nu_{\Sigma}] &= G_r(u, \vartheta, h) \quad \text{on } \Sigma, \\
-(\mathcal{P}_\Sigma [2\mu D(u)\nu_{\Sigma}]) |_{\nu_{\Sigma}} + [\pi] + \sigma A_{\Sigma} h &= G_{\pi}(u, \vartheta, h) \quad \text{on } \Sigma, \\
u(0) &= u_0 \quad \text{in } \Omega, \\
&\text{(4.2)} \\
\rho \kappa_\vartheta \partial_\vartheta \vartheta - d_\vartheta \Delta \vartheta &= F_{\vartheta}(u, \vartheta, h) \quad \text{in } \Omega \setminus \Sigma, \\
\partial_\vartheta \vartheta &= 0 \quad \text{on } \partial\Omega, \\
[u] &= 0 \quad \text{on } \Sigma, \\
&\text{(4.2)} \\
- [d_\vartheta \partial_\vartheta \vartheta] &= G_{\vartheta}(\vartheta, h) \quad \text{on } \Sigma, \\
\vartheta(0) &= \vartheta_0 \quad \text{in } \Omega \setminus \Sigma, \\
&\text{(4.3)}
\end{align*}
$$

and

$$
\begin{align*}
\partial_t h - (u|\nu_{\Sigma}) &= F_h(u, h) \quad \text{on } \Sigma, \\
h(0) &= h_0 \quad \text{on } \Sigma,
\end{align*}
$$

where $\vartheta, \kappa_\vartheta, \mu, d_\vartheta$, and $A_{\Sigma}$ have the same meaning as in Section 3. The precise expressions for the nonlinearities will not be listed here, and the reader is referred to [30, Chapter 9] (and also to [17] for the isothermal case). It suffices to point out that the nonlinearities are $C^1$ in all variables and vanish, together with their first order derivatives, at $(u, \vartheta, h, \pi) = (0, 0, 0, c)$, where $c$ is constant in the phase components.

To obtain an abstract formulation of problem (4.1)-(4.3), we choose as the principal system variable $z = (u, \vartheta, h)$. The regularity space for $z$ is

$$
\mathcal{E}(a) := \{ (u, \vartheta, h) \in \mathcal{E}_u(a) \times \mathcal{E}_\vartheta(a) \times \mathcal{E}_h(a): u, \vartheta, h = 0 \text{ on } \partial\Omega, \ [u], [\vartheta] = 0 \text{ on } \Sigma \},
$$

\[ \square \]
where
\[
E_u(a) = \mathbb{E}_0(a)^n, \quad \mathbb{E}_\theta(a) = H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega \setminus \Sigma)),
\]
\[
E_h(a) = W^{2-1/2p}_p(J; L_p(\Sigma)) \cap H^1_p(J; W^{2-1/p}_p(\Sigma)) \cap L_p(J; W^{3-1/p}_p(\Sigma))
\]
and \( J = (0, a) \). Here the regularity for the height function \( h \) is accounted for as follows. Asserting that \( u \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega \setminus \Sigma)) \) one obtains by trace theory
\[
([2\mu, D(u)\nu \Sigma]) |_{\nu \Sigma} \in W^{1/2-1/2p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{1-1/p}_p(\Sigma)),
\]
\[
(u|_{\nu \Sigma}) \in W^{1/2-1/p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{1-1/p}_p(\Sigma)).
\]
Requiring that the function \( h \) in \([4, 11, 18]\) has the best possible regularity then amounts to
\[
\Delta_\Sigma h \in W^{1/2-1/2p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{1-1/p}_p(\Sigma)),
\]
\[
\partial_\nu h \in W^{1-1/2p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{1-3/p}_p(\Sigma)),
\]
which in turn results in \( h \in E_h(a), \) as \( E_h(a) \) embeds into \( W^{1/2-1/2p}_p(J; H^2_p(\Sigma)) \). By the same reasoning one also has \( [\pi] \in W^{1/2-1/p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{1-1/p}_p(\Sigma)).
\]

The trace space \( X_\gamma \) of \( E(a) \) is given by
\[
X_\gamma = \{ (u, \vartheta, h) \in W^{2-2/p}_p(\Omega \setminus \Sigma)^{n+1} \times W^{3-2/p}_p(\Sigma) : u, \vartheta, \partial_\nu \vartheta = 0 \text{ on } \partial \Omega \}
\]
\[
[u], [\vartheta] = 0 \text{ on } \Sigma \}.
\]
Finally, let
\[
\tilde{E}(a) = \{ w = (z, \pi) : z \in E(a), \pi \in L_p(J; \tilde{H}^1_p(\Omega \setminus \Sigma)) \},
\]
\[
\tilde{X}_\gamma = \{ w = (z, \pi) : z \in X_\gamma, \pi \in \tilde{W}^{1-2/p}_p(\Omega \setminus \Sigma), \pi = W^{1-3/p}_p(\Sigma) \},
\]
where \( \tilde{H}^1_p(\Omega \setminus \Sigma) \) and \( \tilde{W}^{1-2/p}_p(\Omega \setminus \Sigma) \) denote corresponding homogeneous spaces.

4.1. **The tangent space at equilibria.** In this subsection, the structure of the state manifold \( SM \) in a neighborhood of a fixed equilibrium \( e_* = (0, \vartheta_*, \Sigma) \) will be studied. Towards this objective, observe that near \( e_* \), the state manifold is described by
\[
SM_* = \{ (u, \theta, h) \in X_\gamma : \text{div } u = G_\theta(u, h) \text{ in } \Omega \setminus \Sigma,
\]
\[
- \mathcal{P}_\Sigma[2\mu, D(u)\nu] = G_\vartheta(u, \vartheta, h), \quad [d_s \partial_\nu \vartheta] = 0 \text{ on } \Sigma \}.
\]
Associated to \( SM_* \) is the linear subspace
\[
SX_* = \{ (u, \vartheta, h) \in X_\gamma : \text{div } u = 0 \text{ in } \Omega \setminus \Sigma,
\]
\[
\mathcal{P}_\Sigma[\mu, D(u)\nu] = 0, \quad [d_s \partial_\nu \vartheta] = 0 \text{ on } \Sigma \};
\]
the boundary trace space
\[
Y_* = W^{1-2/p}_p(1, \Omega \setminus \Sigma) \times W^{1-3/p}_p(\Sigma; T \Sigma) \times W^{1-3/p}_p(\Sigma),
\]
where
\[ W^{1-2/p}_p(\Omega \setminus \Sigma) := \{ v \in W^{1-2/p}_p(\Omega \setminus \Gamma) : \int_{\Omega \setminus \Sigma} v \, dx = 0 \}, \]
with \( T\Sigma \) the tangent bundle of \( \Sigma \); the linear stationary boundary operator
\[ Bz = (\text{div} u, -P_\Sigma[2\mu D(u)\nu_\Sigma], -[d\nu, \vartheta]) , \]
and the stationary boundary nonlinearity
\[ G(z) = (G_d(u, h, \vartheta), G_r(u, \vartheta, h), G_\theta(\vartheta, h)). \]
With this notation one has
\[ SM_* = \{ z \in X_* : Bz = G(z) \text{ in } Y_\gamma \}, \]
\[ SX_* = \{ z \in X_* : Bz = 0 \text{ in } Y_\gamma \} . \]

This structure will now be employed to parameterize \( SM_* \) over \( SX_* \) near \((0, 0, 0)\).

It will be convenient to enlarge the system variable \( z \) by the pressure, setting
\[ w = (z, \pi), \]
where \( \pi \in \dot{W}^{1-2/p}_p(\Omega \setminus \Sigma) \) and \( [\pi] \in W^{1-3/p}_p(\Sigma) \). Furthermore, including the normal component of the normal stress balance
\[ -(\langle 2\mu, D(u)\nu_\Sigma \rangle |\nu_\Sigma) + [\pi] + \sigma A_{\Sigma} h = G_\nu(u, \vartheta, h) \]
leads to the extended operators \( \hat{B} \) and \( \hat{G} \). It is worthwhile to point out that here the pressure appears only linearly, i.e., it does not appear in \( \hat{G} \).

The differential operator \( A \) is defined by the expression for the operator \( L \) introduced in Section 3. Consequently, system \((4.1)-(4.3)\) can be restated as
\[ \partial_t z + A_0 w = \hat{F}(w) \text{ in } \Omega \setminus \Sigma, \]
\[ \hat{B}_\nu w = \hat{G}(z) \text{ on } \Sigma, \]
\[ z(0) = z_0, \]
where \( \hat{F}(w) := (F_u(w)/\rho, F_\vartheta(z)/\rho \kappa_*, F_h(z)) \). It is important to note that, formally, \((\hat{F}(0), \hat{F}'(0)) = 0\) and \((\hat{G}(0), \hat{G}'(0)) = 0\), with \( \hat{F}' \) and \( \hat{G}' \) the Fréchet derivative of \( \hat{F} \) and \( \hat{G} \), respectively.

4.2. Parameterization of \( SM \). In order to parameterize \( SM_* \) over \( SX_* \) one solves the problem
\[ \omega \tau + A_0 \overline{w} = 0 \quad \text{in } \Omega \setminus \Sigma, \]
\[ \hat{B} \overline{w} = \hat{G}(\tau + \tilde{\tau}) \quad \text{on } \Sigma, \]
where \( \omega > 0 \) is sufficiently large and
\[ A_0 w := (-\mu_* \Delta u + (1/\rho) \nabla \pi, -(d_\vartheta/\rho \kappa_*) \Delta \vartheta, 0), w = (u, \vartheta, h, \pi) \in \hat{X}_\gamma. \]
Given \( \tilde{\tau} \in SX_* \) small, we are looking for a solution \( \overline{w} \in \hat{X}_\gamma \). For this, the implicit function theorem will be employed. Obviously, for \( \tilde{\tau} = 0 \) one has the trivial solution \( \overline{w} = 0 \). Moreover, one notes that the first line in \((4.6)\) yields \( \overline{h} = 0 \). As
\( \hat{G} : X_\gamma \to \hat{Y}_\gamma \) is of class \( C^1 \) with \( (\hat{G}(0), \hat{G}'(0)) = 0, \) where \( \hat{Y}_\gamma = Y_\gamma \times W^{1-3/p}_p(\Sigma) \), it is to be shown that the linear problem

\[
\omega \bar{w} + A_0 w = 0 \quad \text{in} \quad \Omega \setminus \Sigma,
\]

\[
\hat{B} \bar{w} = \hat{g} \quad \text{on} \quad \Sigma,
\]

admits a unique solution, for any given datum \( \hat{g} \in \hat{Y}_\gamma \). In fact, the propositions in the next subsection will do this job, up to lower order perturbations. Therefore, one may apply the implicit function theorem to find a ball \( B_{SX}(0,r) \) and a map

\( \hat{\phi} : B_{SX}(0,r) \to \hat{X}_\gamma \)

of class \( C^1 \) with \( (\hat{\phi}(0), \hat{\phi}'(0)) = 0 \) such that

\[
\bar{w} = \hat{\phi}(\tilde{z})
\]

is the unique solution of (4.6) near zero. The map \( (\text{id} + \phi) : B_{SX}(0,r) \to SM_\ast \) is surjective onto a neighborhood of zero, where \( \phi \) means dropping the pressure \( \pi \) in \( \hat{\phi} \). To see this fix any \( z \in SM_\ast \) and solve the linear problem with (4.7) \( \hat{g} = \hat{G}(z) \) to obtain a unique \( w = (z, \pi) \in \hat{X}_\gamma \). Let then \( \tilde{z} = z - w \). If \( z \) is chosen small enough, \( \tilde{z} \in B_{SX}(0,r) \) and hence \( w = \hat{\phi}(\tilde{z}) \) by uniqueness, yielding the representation \( z = \tilde{z} + \phi(\tilde{z}) \). Consequently, the map \( \Phi : B_{SX}(0,r) \to SM_\ast \) defined by

\[
\Phi(\tilde{z}) = \tilde{z} + \phi(\tilde{z})
\]

yields the desired parameterization. In conclusion, the following result has been obtained.

**Theorem 4.1.** The state manifold \( SM_\ast \) can be parameterized via the map \( \Phi \) over the space \( SX_\ast \). In particular, the tangent space \( T_e SM \) at the equilibrium \( e_\ast \), and equivalently the tangent space \( T_0 SM_\ast \) at zero, is isomorphic to the space \( SX_\ast \).

Note that an equilibrium \( e_\infty \in E \) close to \( e_\ast \in E \) in \( SM \), respectively \( z_\infty \) close to zero in \( X_\gamma \), decomposes as

\[
z_\infty = \tilde{z}_\infty + \tau_\infty = \ddot{z}_\infty + \phi(\ddot{\tau}_\infty),
\]

with \( \tilde{z}_\infty \in SX_\ast \). This follows as \( Aw_\infty = \hat{F}(w_\infty) = 0 \) at an equilibrium.

**4.3. Auxiliary linear elliptic problems.** For the application of the implicit function theorem in Section 4.2 the following results were employed. The first one concerns an elliptic transmission problem for the temperature, and the second one a two-phase Stokes problem.

**Proposition 4.2.** Let \( \omega > 0 \) be large, \( \varrho, \kappa, d_\ast > 0 \), and \( p > n + 2 \). Then the problem

\[
\varrho \kappa_\ast \omega \vartheta - d_\ast \Delta \vartheta = 0 \quad \text{in} \quad \Omega \setminus \Sigma,
\]

\[
\vartheta = 0 \quad \text{on} \quad \partial \Omega,
\]

\[
[\vartheta] = 0, \quad -[d_\ast \partial_\nu \vartheta] = g \quad \text{on} \quad \Sigma,
\]

has a unique solution \( \vartheta \in W^{2-2/p}_p(\Omega \setminus \Sigma) \) if and only if \( g \in W^{1-3/p}_p(\Sigma) \).

The assertion follows, for instance, from the results in [30] Section 6.5.
Proposition 4.3. Let $\omega > 0$ be large, $\rho, \mu_\ast > 0$, and $p > n + 2$. Then the problem
\[
\begin{align*}
\rho \omega u - \mu_\ast \Delta u + \nabla \pi &= 0 \quad \text{in } \Omega \setminus \Sigma, \\
\operatorname{div} u &= g_d \quad \text{in } \Omega \setminus \Sigma, \\
u = 0 \quad \text{on } \partial \Omega, \\
[u] &= 0, \\
- P\Sigma [2\mu_\ast D(u) \nu_\Sigma] &= g_\tau \quad \text{on } \Sigma, \\
([2\mu_\ast D(u) \nu_\Sigma] \nu_\Sigma) + [\pi] &= g_\nu \quad \text{on } \Sigma,
\end{align*}
\]
has a unique solution
\[
egin{align*}
u &\in W^{2-2/p}_p(\Omega \setminus \Sigma), \\
\pi &\in W^{1-2/p}_p(\Omega \setminus \Sigma), \\
[\pi] &\in W^{1-3/p}_p(\Sigma),
\end{align*}
\]
if and only if
\[
g_d \in W^{1-2/p}_p(\Omega \setminus \Sigma), \quad (g_\tau, g_\nu) \in W^{1-3/p}_p(\Sigma; T\Sigma \times \mathbb{R}).
\]
The assertion follows, for instance, from [30, Chapter 8].

4.4. Nonlinear stability analysis. In order to analyze the stability properties of an equilibrium $e_\ast = (0, \theta_\ast, \Sigma)$, the time-dependent variables are decomposed in the same way as in the previous section into $z(t) = \overline{z}(t) + \tilde{z}(t)$ and $w(t) = \overline{w}(t) + \tilde{w}(t)$. The full problem (4.5) may then be decomposed into two systems, formally one for $\overline{w}$ and one for $\tilde{w}$, according to
\[
\begin{align*}
(\omega + \partial_t)\overline{z} + A\overline{w} &= \hat{F}(\overline{w} + \tilde{w}) \quad \text{in } \Omega \setminus \Sigma, \\
\hat{B}\overline{w} &= \hat{G}(\overline{z} + \tilde{z}) \quad \text{on } \Sigma, \\
\overline{z}(0) &= \phi(\overline{z}_0) \quad \text{in } \Omega,
\end{align*}
\]
and
\[
\begin{align*}
\partial_t \tilde{z} + A\tilde{w} &= \omega \overline{z} \quad \text{in } \Omega \setminus \Sigma, \\
\hat{B}\tilde{w} &= 0 \quad \text{on } \Sigma, \\
\tilde{z}(0) &= \tilde{z}_0 \quad \text{in } \Omega.
\end{align*}
\]
Adding these equations yields problem (4.9).

One should think of this decomposition in the following way. The first part has a fast dynamics due to $\omega > 0$ large and takes care of the stationary boundary conditions, while the second equation lives in the tangent space $\mathcal{S} \mathcal{X}$, and carries the actual dynamics.

There should be a word of caution. While for the initial value $z_0$ as well as for $z_\infty$ the decomposition $z_0 = \overline{z}_0 + \phi(\tilde{z}_0)$ and $z_\infty = \overline{z}_\infty + \phi(\tilde{z}_\infty)$ is employed, it no longer holds in the time-dependent case: in general $\overline{z}(t) \neq \phi(\tilde{z}(t))$!

In order to show stability and exponential convergence of solutions starting close to the equilibrium $e_\ast = (0, \theta_\ast, \Sigma)$, the decomposition $z = \overline{z} + \tilde{z} + z_\infty$ is used, with the idea that $z_\infty$ will be the limit of $z(t)$ as $t$ goes to infinity, and $\overline{z}, \tilde{z}$ are
exponentially decaying. This means that the corresponding equations for \( \tau \) and \( \tilde{z} \) are shifted to

\[
(\omega + \partial_t)\tau + Aw = \tilde{F}(\tau + \tilde{w} + w_\infty) - \tilde{F}(w_\infty) \quad \text{in} \quad \Omega \setminus \Sigma,
\]

\[
\tilde{B}w = \tilde{G}(\tau + \tilde{z} + z_\infty) - \tilde{G}(z_\infty) \quad \text{on} \quad \Sigma, \quad (4.11)
\]

\[
\tau(0) = \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty) \quad \text{in} \quad \Omega,
\]

and

\[
\partial_t \tilde{z} + A\tilde{w} = \omega \tau \quad \text{in} \quad \Omega \setminus \Sigma,
\]

\[
\tilde{B}\tilde{w} = 0 \quad \text{on} \quad \Sigma,
\]

\[
\tilde{z}(0) = \tilde{z}_0 - \tilde{z}_\infty \quad \text{in} \quad \Omega,
\]

where \( z_0 = \tilde{z}_0 + \phi(\tilde{z}_0) \) and \( z_\infty = \tilde{z}_\infty + \phi(\tilde{z}_\infty) \). It is convenient to remove the pressure \( \tilde{p} \) from (4.12) by solving the weak transmission problem

\[
(\varrho^{-1} \nabla \tilde{p} | \nabla \phi)_{L_2(\Omega)} = (\varrho^{-1} \mu_1 \Delta \tilde{u} | \nabla \phi)_{L_2(\Omega)} + \omega(\tau | \nabla \phi)_{L_2(\Omega)}, \quad \phi \in H^1_p(\Omega),
\]

\[
[\tilde{p}] = -\sigma \mathcal{A}_\Sigma \tilde{h} + ([2\mu, D(\tilde{u})]_\Sigma) \text{ on } \Sigma
\]

and insert it into (4.11) for \( \tilde{w} \). The first problem may be written abstractly as

\[
L\tau = N(\tau, \tilde{z}, \tilde{z}_\infty), \quad t > 0, \quad \tau(0) = \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty), \quad (4.13)
\]

and with the Helmholtz projection \( \mathcal{P} \), the second one as the evolution equation

\[
\partial_t \tilde{z} + L\tilde{z} = \omega \mathcal{P}\tau, \quad t > 0, \quad \tilde{z}(0) = \tilde{z}_0 - \tilde{z}_\infty. \quad (4.14)
\]

Here \( L : X_1 \to X_0 \) is the operator defined in Section 3. For further use the space \( \mathcal{F}(a) \) of data \( (f_u, f_b, f_d, g_a, g_b) \),

\[
\mathcal{F}(a) = \mathcal{F}_1(a) \times \mathcal{F}_2(a) \times \mathcal{F}_3(a) \times \mathcal{F}_4(a) \times \mathcal{F}_5(a) \times \mathcal{F}_6(a),
\]

is introduced, with

\[
\mathcal{F}_1(a) = L_p(J; L_p(\Omega))^n,
\]

\[
\mathcal{F}_2(a) = L_p(J; L_p(\Omega)),
\]

\[
\mathcal{F}_3(a) = W^{1-1/2p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{1-1/p}_p(\Sigma)),
\]

\[
\mathcal{F}_4(a) = H^1_p(J; H^{-1}_p(\Omega)) \cap L_p(J; H^1_p(\Omega \setminus \Sigma)),
\]

\[
\mathcal{F}_5(a) = W^{1/2-1/2p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{1-1/p}_p(\Sigma))^n,
\]

\[
\mathcal{F}_6(a) = W^{1/2-1/2p}_p(J; L_p(\Sigma)) \cap L_p(J; W^{1-1/p}_p(\Sigma)),
\]

where \( J = (0, a) \) and \( 0 \hat{H}^{-1}_p(\Omega) = [\hat{H}^1_{p', \partial\Omega}(\Omega)]^* \) is the dual of the homogeneous space

\[
\hat{H}^1_{p', \partial\Omega}(\Omega) := \{ \phi \in L_{1,loc}(\Omega) : \nabla \phi \in L_{p'}(\Omega), \, \phi = 0 \text{ on } \partial\Omega \}.
\]

In addition, the function space

\[
\tilde{\mathcal{E}}(a) := H^1_p(J; X_0) \cap L_p(J; X_1), \quad J = (0, a), \quad (4.15)
\]
will be used, with $X_0$ and $X_1$ as in Section 3. It follows from the results in Chapters 6 and 8 of [30] that

$$(\mathbb{L}_\omega, \text{tr}) \in \text{Isom}(\hat{\mathbb{E}}(\infty, \delta), \mathbb{F}(\infty, \delta) \times X_\gamma),$$

provided $\omega$ is chosen sufficiently large, that is, $(\mathbb{L}_\omega, \text{tr})$ is an isomorphism from $\hat{\mathbb{E}}(\infty, \delta)$ onto $\mathbb{F}(\infty, \delta) \times X_\gamma$. Here the following notation is employed:

$$z \in \mathbb{E}(\infty, \delta) \iff e^{\delta t} z \in \mathbb{E}(\infty),$$

and similarly for $\mathbb{F}(\infty, \delta)$, $\hat{\mathbb{E}}(\infty, \delta)$ and $\hat{\mathbb{E}}(\infty, \delta)$.

According to Theorem 3.1, $-L$ is the generator of an analytic $C_0$-semigroup with maximal $L_p$-regularity in $X_0$. Moreover, it follows from the same theorem that there is a number $\delta_0 > 0$ such that $\Re \sigma(-L) \cap (-\delta_0, 0) = \emptyset$. Let $\delta$ be chosen so that $0 < \delta < \delta_0$. On shows that the function $N$ is of class $C^1$ with respect to the variables $(\overline{\gamma}, \hat{z})$ in the function spaces $\hat{\mathbb{E}}(\infty, \delta) \times \hat{\mathbb{E}}(\infty, \delta)$, provided condition (1.4) holds, but merely continuous in $\hat{z}_\infty$, unless one additional degree of regularity is imposed on the coefficients.

The main theorem of this chapter is the following.

**Theorem 4.4.** Let $p > n + 2$ and suppose that condition (1.4) holds.

Then each equilibrium $e_\ast = (0, \theta_\ast, \Sigma) \in \mathcal{E}$ is non-linearly stable in the state manifold $\mathcal{S}M$. Any solution with initial value close to $e_\ast$ in $\mathcal{S}M$ exists globally and converges in $\mathcal{S}M$ to a possibly different stable equilibrium $e_\infty \in \mathcal{E}$ at an exponential rate.

**Proof.** Based on the spectral properties of $L$ derived in Theorem 3.1 let $P^s$ denote the projection onto the stable subspace $X^s_0 = P^s X_0 = R(L)$ and $P^c$ the complementary projection onto the kernel $N(L) = X^c_0 = P^c X_0$. Moreover, let $L^s$ be the part of $L$ in $X^s_0$.

Let $y = P^c \hat{z}, x = P^c \hat{x}$ and note that the equilibria over $P^c \mathcal{S}X_s = P^c X_0$ may be parameterized according to

$$\hat{z}_\infty = x_\infty + \psi(x_\infty) + \phi(x_\infty + \psi(x_\infty)), \quad \hat{x}_\infty \in X^c_0,$$

by solving the nonlinear stationary problem

$$L^s y = \omega P^s \mathbb{P} \phi(x + y)$$

by the implicit function theorem. Finally, let $y_\infty = \psi(x_\infty)$. Applying the projection $P^s$ to the equation for $\hat{z}$ one obtains the problem

$$\partial_t y + L^s y = \omega P^s \mathbb{P} \mathbb{P} \frac{\partial y}{\partial t} = 0, \quad t > 0, \quad y(0) = y_0 - y_\infty,$$

and for $x$ analogously

$$\partial_t x = \omega P^s \mathbb{P} \mathbb{P} \mathbb{P} \frac{\partial x}{\partial t} = 0, \quad t > 0, \quad x(0) = x_0 - x_\infty.$$

It is important to observe that

$$(\partial_t + L^s, \text{tr}) \in \text{Isom}(P^s \hat{\mathbb{E}}(\infty, \delta), L_p(\mathbb{R}^+, \delta; X^s_0) \times X^s_0).$$
Finally, the whole problem (4.13)-(4.14) may be rewritten as \( \mathbb{H}(v, (x_\infty, y_0)) = 0 \), where \( v = (w, y, x, x_0) \) and

\[
\begin{pmatrix}
\frac{d}{dx} w - N(v, x_\infty), \pi(0) - \phi(x_0 + y_0) + \phi(x_\infty + y_\infty)

\partial_t y + L^* y - \omega P^* \pi, y(0) - y_0 + y_\infty

x(t) + \omega \int_t^\infty P^* \pi(s) \, ds

x_0 - x_\infty + \omega \int_0^\infty P^* \pi(s) \, ds
\end{pmatrix}.
\]

One shows that the mapping

\[
\mathbb{H} : \mathbb{E}(\infty, \delta) \times P^* \mathbb{E}(\infty, \delta) \times P^* \mathbb{E}(\infty, \delta) \times X_\infty^c \times (X_0^c \times X_\infty^c)
\]

\[
\rightarrow (\mathbb{E}(\infty, \delta) \times X_\gamma) \times (L_p(\mathbb{R}_+; \delta, X_0^c) \times X_\infty^c) \times L_p(\mathbb{R}_+; \delta, X_0^c) \times X_\infty^c
\]

is of class \( C^1 \) w.r.t \( (v, y_0) \), continuous w.r.t \( x_\infty \), and differentiable w.r.t \( x_\infty \) at \( x_\infty = 0 \). One notes that \( X_0^c = X_\infty^c = X_\gamma^c \). The Fréchet derivative \( D_c \mathbb{H}(0, 0) \) w.r.t. the variable \( v \) is given by the operator matrix

\[
D_c \mathbb{H}(0, 0) = \begin{bmatrix}
\frac{d}{dx}, tr & 0 & 0 & 0 & 0 \\
* & (\partial_t + L^*, tr) & 0 & 0 & 0 \\
* & 0 & I & 0 & 0 \\
* & 0 & 0 & I & 0
\end{bmatrix}.
\]

Here the stars indicate bounded linear operators which, due to the triangular structure of the operator matrix, do not need to be computed explicitly, as the diagonal terms of this operator matrix are invertible. Therefore, by the implicit function theorem, see for instance [6] Theorem 15.1, there are balls \( B_{X_0^c}(0, r) \) and \( B_{X_\gamma^c}(0, r) \), and a continuous map

\[
\mathcal{T} : B_{X_0^c}(0, r) \times B_{X_\gamma^c}(0, r) \rightarrow \mathbb{E}(\infty, \delta) \times \mathbb{E}(\infty, \delta) \times X_\infty^c,
\]

\[
\mathcal{T}(x_\infty, y_0) = (w, \hat{z}, x_0),
\]

with \( \mathcal{T}(0, 0) = 0 \). Then \((z, \pi) := (w + \hat{z} + z_\infty, \pi + \hat{\pi} + \pi_\infty)\) yields the unique solution of (4.15) such that

\[
z(t) := x_\infty + \psi(x_\infty) + \phi(x_\infty + \psi(x_\infty)) \text{ in } X_\gamma \quad \text{as } t \rightarrow \infty.
\]

One should observe that \( \hat{z} := x + y \in \mathbb{E}(\infty, \delta) \) seemingly has less regularity than its counterpart \( \pi \in \mathbb{E}(\infty, \delta) \). However, a moment of reflection shows that any solution \( \hat{z} \in \mathbb{E}(\infty, \delta) \) of problem (4.14) inherits the additional regularity \( \hat{z} \in \mathbb{E}(\infty, \delta) \).

It follows from the implicit function theorem that \( \mathcal{T} \) is \( C^1 \) in \( y_0 \), but only continuous in \( x_\infty \), unless more regularity for the parameter functions is required. Nonetheless, \( \mathcal{T} \) is differentiable with respect to \( x_\infty \) at \( x_\infty = 0 \).

The properties of \( \mathcal{T} \) can be summarized as follows: given an equilibrium

\[
z_\infty = x_\infty + \psi(x_\infty) + \phi(x_\infty + \psi(x_\infty)) \in \mathcal{E}
\]

and an initial value \( y_0 \in X_\gamma^c \), one determines with the help of the implicit function theorem a value \( x_0 \) in \( X_0^c \) and a solution \( z \) of (4.14) with initial value \( (x_0, y_0) \) such that \( z(t) \) converges to \( z_\infty \) exponentially fast. Exponential convergence is
obtained by setting up the implicit function theorem in a space of exponentially
decaying functions. Next, the mapping

$$S : B_{X_0}^c(0, r) \times B_{X_0}^c(0, r) \to X_0^c \times X_0^c, \quad (x_\infty, y_0) \mapsto (x_0, y_0),$$

will be analyzed in more detail. It is worthwhile to point out that this mapping
gives rise to the construction of a stable foliation for \( \text{(1.5)} \) (by invariant, locally
stable manifolds) in a neighborhood of the fixed equilibrium \( e_* \in \mathcal{E} \), see [31].

To complete the proof, the question which remains is whether \( S \) is surjective
near \((0, 0)\). Indeed, surjectivity would imply that for any initial value \((x_0, y_0)\) in
a sufficiently small neighborhood of \((0, 0)\) in \( X_0^c \times X_0^c \) there exists \( z_\infty \in \mathcal{E} \) and a
unique solution \( z \) to problem \( \text{(1.5)} \) which converges to \( z_\infty \) at an exponential rate
in the topology of \( SM \).

To prove surjectivity of \( S \), degree theory will be employed. For this purpose,
define a map \( f : B_{X_0}^c(0, r) \times B_{X_0}^c(0, r) \to X_0^c \) by means of \( f(x_\infty, y_0) = x_0(x_\infty, y_0) \).
As has already been established, this map is continuous, and it is close to the
identity. In fact, differentiating the relation

$$\deg H_1(T(x_\infty, y_0), T(x_\infty, y_0)) = 0$$

with respect to \((x_\infty, y_0)\) at \((0, 0)\) one obtains \((D_1 T_1(0, 0), D_2 T_1(0, 0)) = 0\). Here \( H_1 \)
denotes the first line of \( \mathbb{H} \) and \( T_1 \) the first component of \( T \), respectively. This
implies \((D_1 \tau(0, 0), D_2 \tau(0, 0)) = 0\). From the representation

$$f(x_\infty, y_0) = x_\infty - \omega \int_0^\infty P^c t \, ds,$$

one infers that for every \( \varepsilon > 0 \) there is a constant \( \rho > 0 \) such that

$$|f(x_\infty, y_0) - x_\infty|_X^c \leq \omega \int_0^\infty |P^c t|_X^c ds \leq \varepsilon (y_0|x^c|_X^c + |x_\infty|x^c|_X^c),$$

whenever \(|(x_\infty, y_0)| \leq \rho\), with \( \rho \leq r \). In the following, let \( \varepsilon = 1/3 \) be fixed. Here \( y_0 \)
only serves as a parameter, so we are in a finite dimensional setting and may
employ the Brouwer degree, in particular its homotopy invariance. Define the
homotopy \( h(\tau, x, y_0) = \tau f(x, y_0) + (1 - \tau)x \), and consider the degree

$$\deg (h(\tau, \cdot, y_0), B_{X_0}^c(0, \rho/2), \xi), \quad (\xi, y_0) \in B_{X_0}^c(0, \rho/2) \times B_{X_0}^c(0, \rho/2).$$

For \( \tau = 0 \) it is equal to one, hence it is equal to one for all \( \tau \in [0, 1] \), provided
there are no solutions of \( h(\tau, x, y_0) = \xi \) with \( |x|_{X^c_0} = \tau \). To show this, suppose
\( h(\tau, x, y_0) = \xi \), i.e., \( \xi - x = \tau(f(x, y_0) - x) \), and \( |x|_{X^c_0} = \tau \). Then by the above
estimate

$$r = |x|_{X^c_0} \leq |\xi|_{X^c_0} + |x - \xi|_{X^c_0} \leq |\xi|_{X^c_0} + \varepsilon (|y_0|_{X^c_0} + |x|_{X^c_0}) < r,$$

provided \(|\xi|_{X^c_0} < \rho/2\) and \(|y_0|_{X^c_0} < \rho/2\). Hence, \( \deg (f(\cdot, y_0), B_{X_0}^c(0, \rho/2), \xi) \) equals
one as well, showing that the equation \( f(x_\infty, y_0) = \xi \) has at least one solution for
each \((\xi, y_0) \in B_{X_0}^c(0, \rho/2) \times B_{X_0}^c(0, \rho/2), \) i.e., the mapping is surjective near zero.
This completes the proof of the theorem.
Remark 4.5. It should be noted that the proof of surjectivity can be based on
the inverse function theorem (in lieu of employing degree theory), provided the
mappings involved are $C^1$ in all variables. This property can be ensured by asking
for one more degree of regularity for the functions $\psi_i$ and $d_i, \mu_i$ in condition (1.4).

5. Global Existence and Convergence

It has been shown in Section 2 that the negative total entropy is a strict Lyapunov functional for (1.1)-(1.3). Therefore, the $\omega$-limit sets

$$\omega(u, \theta, \Gamma) := \{(u, \theta, \Gamma) \in SM :$$

$$\exists \ t_n \nearrow \infty \text{ s.t. } (u(t_n), \theta(t_n), \Gamma(t_n)) \to (u, \theta, \Gamma) \text{ in } SM \},$$
of solutions $(u, \theta, \Gamma)$ in $SM$ are contained in the manifold $E \subset SM$ of equilibria.

There are several obstructions for global existence:
- **regularity**: the norms of either $u(t), \theta(t), \Gamma(t)$ may become unbounded;
- **geometry**: the topology of the interface may change;
or the interface may touch the boundary of $\Omega$;
or a part of the interface may contract to a point.

Let $\Gamma \subset \Omega$ be a hypersurface. Then $\Gamma$ satisfies the **ball condition** if there is a number $r > 0$ such that for each point $p \in \Gamma$ there are balls $B(x_i, r) \subset \Omega_i$ such that $\Gamma \cap B(x_i, r) = \{p\}$ for $i = 1, 2$. The subset $M\mathcal{H}^2(\Omega, r) \subset M\mathcal{H}^2(\Omega)$ consists, by definition, of all hypersurfaces $\Gamma \in M\mathcal{H}^2(\Omega)$ that satisfy the ball condition for a fixed radius $r > 0$.

Let $(u, \theta, \Gamma)$ be a solution of (1.1)-(1.3) on its maximal existence interval $[0, t_+)$.

Then $\Gamma(t)$ is said to satisfy the **uniform ball condition** if there exists a number
$r > 0 \text{ such that } \Gamma(t) \in M\mathcal{H}^2(\Omega, r) \text{ for all } t \in [0, t_+)$. Note that this condition bounds the curvature of $\Gamma(t)$, prevents parts of $\Gamma(t)$ to shrink to points, to touch the outer boundary $\partial \Omega$, and to undergo topological changes.

With this property, combining the local semiflow for problem (1.1)-(1.3) with the corresponding Lyapunov functional (i.e., the negative total entropy), relative compactness of bounded orbits, and the convergence results from the previous section, one obtains the following global result.

**Theorem 5.1.** Let $p > n + 2$ and suppose that condition (1.4) holds.

Suppose that $(u, \theta, \Gamma)$ is a solution of (1.1)-(1.3) in the state manifold $SM$ on its maximal time interval $[0, t_+)$. Assume there are constants $M, m > 0$ such that the following conditions hold on $[0, t_+)$:

(i) $|u(t)|_{W^{2-2/p}_p, \theta(t)|_{W^{2-2/p}_p, |\Gamma(t)|_{W^{3-2/p}_p} \leq M < \infty,$

(ii) $m \leq \theta(t);$

(iii) $\Gamma(t)$ satisfies the uniform ball condition.

Then $t_+ = \infty$, i.e., the solution exists globally, and the solution converges in $SM$ to an equilibrium $(0, \theta, \Gamma, \infty) \in E$. 

On the contrary, if \((u(t), \theta(t), \Gamma(t))\) is a global solution in \(\mathcal{SM}\) which converges to an equilibrium \((0, \theta_\infty, \Gamma_\infty)\) in \(\mathcal{SM}\) as \(t \to \infty\), then (i)-(iii) hold.

**Proof.** It is well known that each \(\Gamma \in \mathcal{MH}^2(\Omega)\) admits a tubular neighborhood \(U_a := \{x \in \mathbb{R}^n : \text{dist}(x, \Gamma) < a\}\) of width \(a = a(\Gamma) > 0\) such that the signed distance function

\[
d_\Gamma : U_a \to \mathbb{R}, \quad |d_\Gamma(x)| := \text{dist}(x, \Gamma),
\]

is well-defined and \(d_\Gamma \in C^2(U_a, \mathbb{R})\). Here \(d_\Gamma(x) < 0\) iff \(x \in \Omega_1 \cap U_a\) by convention. One then defines a level function \(\varphi_\Gamma\) by means of

\[
\varphi_\Gamma(x) := \begin{cases} d_\Gamma(x) \chi(3d_\Gamma(x)/a) + \text{sgn}(d_\Gamma(x))(1 - \chi(3d_\Gamma(x)/a)), & x \in U_a, \\ \chi_{\text{ex}}(x) - \chi_{\text{in}}(x), & x \notin U_a,
\end{cases}
\]

where \(\Omega_{\text{ex}}\) and \(\Omega_{\text{in}}\) denote the exterior and interior component of \(\mathbb{R}^n \setminus U_a\), respectively, and \(\chi\) is a smooth cut-off function with \(\chi(s) = 1\) if \(|s| < 1\) and \(\chi(s) = 0\) if \(|s| > 2\). The level function \(\varphi_\Gamma\) is then of class \(C^2\), \(\varphi_\Gamma(x) = d_\Gamma(x)\) for \(x \in U_a/3\), and \(\varphi_\Gamma(x) = 0\) iff \(x \in \Gamma\).

Let \(\mathcal{MH}^2(\Omega, r)\) denote the subset of \(\mathcal{MH}^2(\Omega)\) which consists of all \(\Gamma \in \mathcal{MH}^2(\Omega)\) such that \(\Gamma \subset \Omega\) satisfies the ball condition with fixed radius \(r > 0\). This implies in particular that dist(\(\Gamma, \partial\Omega\)) \(\geq 2r\) and all principal curvatures of \(\Gamma \in \mathcal{MH}^2(\Omega, r)\) are bounded by \(1/r\). Furthermore, the level functions \(\varphi_\Gamma\) are well-defined for \(\Gamma \in \mathcal{MH}^2(\Omega, r)\), and form a bounded subset of \(C^2(\bar{\Omega})\). The map

\[
\Phi : \mathcal{MH}^2(\Omega, r) \to C^2(\bar{\Omega}), \quad \Phi(\Gamma) = \varphi_\Gamma,
\]

is a homeomorphism of the metric space \(\mathcal{MH}^2(\Omega, r)\) onto \(\Phi(\mathcal{MH}^2(\Omega, r)) \subset C^2(\bar{\Omega})\), see [30, Section 2.4.2].

Let \(s - (n - 1)/p > 2\). For \(\Gamma \in \mathcal{MH}^2(\Omega, r)\) one defines \(\Gamma \in W^s_p(\Omega, r)\) if \(\varphi_\Gamma \in W^s_p(\Omega)\). In this case the local charts for \(\Gamma\) can be chosen of class \(W^s_p\) as well. A subset \(A \subset W^s_p(\Omega, r)\) is said to be (relatively) compact, if \(\Phi(A) \subset W^s_p(\Omega)\) is (relatively) compact. Finally, one defines dist\(_{W^s_p}\) (\(\Gamma_1, \Gamma_2\) := \(|\varphi_{\Gamma_1} - \varphi_{\Gamma_2}|_{W^s_p(\Omega)}\) for \(\Gamma_1, \Gamma_2 \in \mathcal{MH}^2(\Omega, r)\).

Suppose that the assumptions (i)-(iii) are valid. Then \(\Gamma([0, t_+)) \subset W^{3-2/p}_p(\Omega, r)\) is bounded, hence relatively compact in \(W^{3-2/p}_p(\Omega, r)\). Thus \(\Gamma([0, t_+))\) can be covered by finitely many balls with centers \(\Sigma_k\) such that

\[
\text{dist}_{W^{3-2/p-\epsilon}_p}(\Gamma(t), \Sigma_j) \leq \delta \quad \text{for some } j = j(t), t \in [0, t_+).
\]

Let \(J_k = \{t \in [0, t_+) : j(t) = k\}\). Using for each \(k\) a Hanzawa-transformation \(\Xi_k\), we see that the pull backs \(\{(u(t, \cdot), \theta(t, \cdot)) \circ \Xi_k : t \in J_k\}\) are bounded in \(W^{2-2/p}_p(\Omega \setminus \Sigma_k)^{n+1}\), hence relatively compact in \(W^{2-2/p-\epsilon}_p(\Omega \setminus \Sigma_k)^{n+1}\). Employing now [30, Theorem 9.2.1] one obtains solutions \((u^1, \theta^1, \Gamma^1)\) with initial configurations \((u(t), \theta(t), \Gamma(t))\) in the state manifold \(\mathcal{SM}\) on a common time interval, say \((0, a]\), and by uniqueness one has

\[
(u^1(a), \theta^1(a)\Gamma^1(a)) = (u(t + a), \theta(u + a), \Gamma(t + a)).
\]
Continuous dependence implies then relative compactness of \( \{(u(\cdot), \theta(\cdot), \Gamma(\cdot)) : 0 \leq t < t_+\} \) in \( \mathcal{SM} \); in particular \( t_+ = \infty \) and the orbit \( (u, \theta, \Gamma)(\mathbb{R}_+) \subset \mathcal{SM} \) is relatively compact. The entropy is a strict Lyapunov functional, hence the limit set \( \omega(u, \theta, \Gamma) \) of a solution is contained in the set \( \mathcal{E} \) of equilibria. By compactness, \( \omega(u, \theta, \Gamma) \subset \mathcal{SM} \) is non-empty, hence the solution comes close to \( \mathcal{E} \). Finally, one may apply the convergence result Theorem 4.4 to complete the sufficiency part of the proof. Necessity follows by a compactness argument. □

6. The isothermal problem

In this section the isothermal Navier-Stokes problem with surface tension \([15]\) will be considered. It turns out that the main results for this problem concerning well-posedness and the stability analysis of equilibria parallel those in Sections 2-4 for problem \([11]\). A decisive difference is caused by the fact that, as temperature is neglected, the principles of thermodynamics do no longer apply.

The basic local well-posedness result for problem \([15]\) reads as follows.

**Theorem 6.1.** Let \( p > n + 2 \) and suppose that \( \partial \Omega \in C^3 \). Assume the regularity conditions

\[
\begin{align*}
  u_0 &\in W^{2-2/p}(\Omega \setminus \Gamma_0), \quad \Gamma_0 \in W^{3-2/p}, \\
  \text{div } u_0 &= 0 \text{ in } \Omega \setminus \Gamma_0, \quad u_0 = 0 \text{ on } \partial \Omega, \\
  [u_0] &= 0, \quad \mathcal{P}_\Gamma [\mu D(u)v] = 0 \text{ on } \Gamma_0.
\end{align*}
\]

Then there exists a number \( a = a(\Gamma_0, u_0) \) and a unique classical solution \((u, \pi, \Gamma)\) of \([15]\) on the time interval \((0, a)\). Moreover, \( \mathcal{M} = \bigcup_{t \in (0, a)} \{t\} \times \Gamma(t) \) is real analytic.

**Proof.** The reader is referred to \([17]\), see also \([30\text{, Sections 9.2 and 9.4}]\). □

The state manifold for \([15]\) is defined by

\[
\mathcal{SM} := \left\{ (u, \Gamma) \in C(\bar{\Omega})^n \times MH^2 : u \in W^{2-2/p}(\Omega \setminus \Gamma)^n, \, \Gamma \in W^{3-2/p}, \right. \\
\text{div } u = 0 \text{ in } \Omega \setminus \Gamma, \quad u = 0 \text{ on } \partial \Omega, \quad \mathcal{P}_\Gamma [\mu D(u)v] = 0 \text{ on } \Gamma
\left\}.
\]

Applying Theorem 6.1 and re-parameterizing the interface repeatedly, one shows once more that problem \([15]\) generates a semiflow on \( \mathcal{SM} \). As in problem \([11]\), the pressure \( \pi \) is determined for each time \( t \) from \((u, \Gamma)\) by means of the weak transmission problem

\[
(\varrho^{-1} \nabla \pi \mid \nabla \phi)_{L^2(\Omega)} = (\varrho^{-1} \mu \Delta u - (u \mid \nabla)u \mid \nabla \phi)_{L^2(\Omega)}, \quad \phi \in H^1_{\varrho} (\Omega), \\
[\pi] = \sigma H_{\Gamma} + ([2\mu D(u)v] \mid v) \text{ on } \Gamma.
\]

Suppose that the dispersed phase \( \Omega_1 \) consists of \( m \) connected disjoint components, \( \Omega_1 = \bigcup_{k=1}^m \Omega_{1,k} \). Let \( \Gamma_k := \partial \Omega_{1,k} \), \( \Gamma = \bigcup_{k=1}^m \Gamma_k \), and let \( M_k := |\Omega_{1,k}| \) denote the
volume of $\Omega_{1,k}$. Then one shows exactly as in Section 2.3 that problem (1.5) preserves the volume of each individual phase component.

The available energy for problem (1.5) is defined by

$$\Phi_0(u, \Gamma) := \frac{1}{2} \int_{\Omega \setminus \Gamma} \rho |u|^2 \, dx + \sigma |\Gamma|.$$ 

For the time derivative of $\Phi_0$ one obtains

$$\frac{d}{dt} \Phi_0 = \int_{\Omega} \rho (\partial_t u |u|) \, dx - \int_{\Gamma} \left\{ \left[ \frac{\rho}{2} |u|^2 \right] + \sigma H_\Gamma \right\} V_{\Gamma} \, d\Gamma$$

$$= -\int_{\Omega} \{ \rho((u |\nabla|u|u) - (\text{div} \, T |u|) \} \, dx - \int_{\Gamma} \left\{ \left[ \frac{\rho}{2} |u|^2 \right] + \sigma H_\Gamma \right\} V_{\Gamma} \, d\Gamma$$

$$= -2 \int_{\Omega} \mu |D(u)|^2 \, dx - \int_{\Gamma} \left\{ \left[ (T u |\nu_{\Gamma}) \right] + \sigma H_\Gamma V_{\Gamma} \right\} \, d\Gamma$$

$$= -2 \int_{\Omega} \mu |D(u)|^2 \, dx,$$ 

showing that the available energy is decreasing. Therefore, $\Phi_0$ constitutes a Lyapunov functional for (1.5). As in Section 2.5 one shows that $\Phi_0$ is a strict Lyapunov functional. The same arguments as in Section 2 also imply that the equilibria of (1.5) consist of zero velocities, constant pressures in the phase components, and that the dispersed phase consists of a collection of non-intersecting balls in $\Omega$. Consequently, the set $E$ of non-degenerate equilibria for (1.5) is given by

$$E = \{ (0, \Sigma) : \Sigma \in \mathcal{S} \},$$

where $\mathcal{S}$ is defined in (2.3). $E$ defines a real analytic manifold of dimension $m(n+1)$.

In analogy to Section 2.7 one shows that the critical points of the energy functional $\Phi_0$ under the constraints of $M_k = M_{0,k}$ constant correspond exactly to the equilibria of (1.5), and that all critical points are local minima of the energy functional with the given constraints.

**Theorem 6.2.** The following assertions hold for problem (1.5).

(a) The phase volumes $|\Omega_{1,k}|$ are preserved.

(b) The energy functional $\Phi_0$ is a strict Lyapunov functional.

(c) The non-degenerate equilibria are zero velocities, constant pressures in the components of the phases, and the interface is a finite union of non-intersecting spheres which do not touch the outer boundary $\partial \Omega$.

(d) The set $E$ of non-degenerate equilibria forms a real analytic manifold of dimension $m(n+1)$, where $m$ denotes the number of connected components of $\Omega_1$.

(e) The critical points of the energy functional for prescribed phase volumes are precisely the equilibria of the system.

(f) All critical points of the energy functional for prescribed phase volumes are local minima.
This result was first established in [17, Proposition 5.2], see also [5, Theorem 3.1]. It should be observed that the assertions in Remark 2.3 do also apply to the isothermal case.

Suppose \((0, \Sigma) \in \mathcal{E}\), with \(\Sigma = \bigcup_{k=1}^{m} \Sigma_k\) and \(\Sigma_k = \partial B(x_k, R_k)\) is a fixed equilibrium for problem (1.5). In analogy to Section 3, and using the notation introduced there, one associates with system (1.5) the following linear problem

\[
\begin{align*}
\frac{\partial}{\partial t} u - \mu \Delta u + \nabla \pi &= \varrho f_u \quad \text{in} \quad \Omega \setminus \Sigma, \\
\text{div} \ u &= g_d \quad \text{in} \quad \Omega \setminus \Sigma, \\
u &= 0 \quad \text{on} \quad \partial \Omega, \\
[u] = 0 \quad \text{on} \quad \Sigma, \\
- [T \nu] + \sigma(A_\Sigma h) \nu = g_u \quad \text{on} \quad \Sigma, \\
\partial_t h - (u|\nu) = f_h \quad \text{on} \quad \Sigma, \\
u(0) &= u_0 \quad \text{in} \quad \Omega, \\
h(0) &= h_0 \quad \text{on} \quad \Sigma, \\
\end{align*}
\]

(6.1)

and the linear operator \(L\),

\[
L(u, h) := (- \varrho^{-1}(\mu \Delta u - \nabla \pi), -(u|\nu)),
\]

defined on \(X_0 = L_{p,\sigma}(\Omega) \times W_p^{3-1/p}(\Sigma)\) with domain

\[
X_1 = D(L) = \{(u, h) \in H^{2}_p(\Omega \setminus \Sigma)^n \times W_p^{3-1/p}(\Sigma) : \text{div} \ u = 0 \ \text{in} \ \Omega \setminus \Sigma, \\
u = 0 \ \text{on} \ \partial \Omega, \ [u] = 0 \ \text{on} \ \Sigma, \ \mathcal{P}_\Sigma[\mu D(u)|\nu] = 0 \ \text{on} \ \Sigma\}.
\]

Here \(\pi\) is again determined as the solution of the weak transmission problem

\[
(q^{-1} \nabla \pi|\nabla \phi)_{L_2(\Omega)} = (q^{-1} \mu \Delta u|\nabla \phi)_{L_2(\Omega)}, \quad \phi \in H^{1}_p(\Omega), \\
[\pi] = -\sigma A_\Sigma h + ([2 \mu D(u)|\nu])|\nu| \quad \text{on} \ \Sigma.
\]

The principal variable in (6.1) is \(z = (u, h)\), the dynamic inhomogeneities are \(f = (f_u, f_h)\), and the static ones are \(g = (g_d, g_u)\). The eigenvalue problem associated with \(L\) becomes

\[
\begin{align*}
\varrho \lambda u - \mu \Delta u + \nabla \pi &= 0 \quad \text{in} \quad \Omega \setminus \Sigma, \\
\text{div} \ u &= 0 \quad \text{in} \quad \Omega \setminus \Sigma, \\
u &= 0 \quad \text{on} \quad \partial \Omega, \\
[u] = 0 \quad \text{on} \quad \Sigma, \\
- [T \nu] + \sigma(A_\Sigma h) \nu = 0 \quad \text{on} \quad \Sigma, \\
\lambda h - (u|\nu) &= 0 \quad \text{on} \quad \Sigma.
\end{align*}
\]

(6.2)
Theorem 6.3. Let \( e_* \in \mathcal{E} \) be an equilibrium. Then the operator \( L \) has the following properties.

(a) \(-L\) generates a compact, analytic \( C_0 \)-semigroup in \( X_0 \) which has the property of maximal \( L^p \)-regularity.

(b) The spectrum of \( L \) consists of countably many eigenvalues of finite algebraic multiplicity.

(c) \(-L\) has no eigenvalues \( \lambda \neq 0 \) with nonnegative real part.

(d) \( \lambda = 0 \) is a semi-simple eigenvalue of \( L \) of multiplicity \( m(n+1) \).

(e) The kernel \( \mathcal{N}(L) \) of \( L \) is isomorphic to the tangent space \( T_{e_*} \mathcal{E} \) of the manifold of equilibria \( \mathcal{E} \) at \( e_* \).

Hence, each equilibrium \( e_* \in \mathcal{E} \) is normally stable.

Proof. The proof proceeds in the same way as the corresponding proof of Theorem 3.1, with the only difference that here all quantities and assertions relating to the temperature are dismissed. One verifies, for instance, that the kernel of \( L \) is spanned by the functions \( e_{jk} = (0, Y^j_k) \) with \( Y^j_k \) the spherical harmonics of degree one for the spheres \( \Sigma_k, j = 1, \cdots, n, k = 1, \cdots, m \), and \( e_{0,k} = (0, Y^0_k) \), where \( Y^0_k \) equals one on \( \Sigma_k \) and zero elsewhere. Hence the dimension of the null space \( \mathcal{N}(L) \) is \( m(n+1) \).

The main theorem of this chapter concerning the stability of equilibria reads as follows.

Theorem 6.4. Let \( p > n + 2 \). Then every equilibrium \( e_* = (0, \Sigma) \in \mathcal{E} \) is nonlinearly stable in the state manifold \( \mathcal{S} \mathcal{M} \). Any solution with initial value close to \( e_* \) in \( \mathcal{S} \mathcal{M} \) exists globally and converges in \( \mathcal{S} \mathcal{M} \) to a possibly different stable equilibrium \( e_{\infty} \in \mathcal{E} \) at an exponential rate.

Proof. The proof proceeds in the same way as the corresponding proof for Theorem 4.4, with the obvious modification that all quantities and assertions relating to the temperature are to be disregarded.

Analogous assertions as in Theorem 5.1 hold for solutions \( (u(t), \Gamma(t)) \) of the isothermal problem (1.5), again with the obvious modification that the temperature variable is dropped, see also [17, Theorem 7.1].

7. The two-phase Stokes flow with surface tension

In this section the two-phase quasi-stationary Stokes problem with surface tension (1.6) will be considered. This problem is considerably easier to analyze than problem (1.1)-(1.3), or problem (1.5). In fact, it turns out that in this case, the
only system variable is the unknown hypersurface $\Gamma$. In order to obtain this reduction, the two-phase Stokes problem

$$
-\mu \Delta u + \nabla \pi = 0 \quad \text{in} \quad \Omega \setminus \Gamma,
\text{div} u = 0 \quad \text{in} \quad \Omega \setminus \Gamma,
u u = 0 \quad \text{on} \quad \partial \Omega,
[u] = 0 \quad \text{on} \quad \Gamma,
-[[T \nu] \Gamma] = g \nu \Gamma \quad \text{on} \quad \Gamma
$$

(7.1)

will play an important role. One shows that (7.1) admits for each $g \in W^{1-1/p}_p(\Gamma)$ a unique solution $(u, \pi)$ (up to constants in the pressure) with regularity

$$(u, \pi) \in H^2_p(\Omega \setminus \Gamma) \times \dot{H}^1_p(\Omega \setminus \Gamma),$$

where $1 < p < \infty$, and $\dot{H}^1_p$ denotes the homogeneous Sobolev space of order one.

For a given function $g \in W^{1-1/p}_p(\Gamma)$ let $(u, \pi)$ be the solution of (7.1). Then the Neumann-to-Dirichlet operator $N_\Gamma : W^{1-1/p}_p(\Gamma) \to W^{2-1/p}_p(\Gamma)$ is defined by

$$N_\Gamma g := (u|\nu_\Gamma).$$

The following results hold for $N_\Gamma$.

**Proposition 7.1.** Suppose $\partial \Omega \in C^3$ and $\Gamma \in \mathcal{M} H^2(\Omega)$ consists of $m$ components, $\Gamma = \bigcup_{k=1}^m \Gamma_k$. Then the operator $N_\Gamma$ has the following properties.

(a) $(N_\Gamma g| h)_{L^2(\Gamma)} = (g|N_\Gamma h)_{L^2(\Gamma)}$, $g, h \in W^{1/2}_2(\Gamma)$.

(b) $(N_\Gamma g| g)_{L^2(\Gamma)} = 2 \int_{\Omega} |\text{D}(u)|^2 \frac{1}{2} \, dx$, where $(u, \pi)$ is the solution of (7.1).

(c) Let $e_k$ be the function which is one on $\Gamma_k$ and zero elsewhere. Then

$$(N_\Gamma g| e_k)_{L^2(\Gamma)} = 0 \quad \text{for each} \quad g \in W^{1/2}_2(\Gamma) \quad \text{and each} \quad 1 \leq k \leq m.$$

In particular, $N_\Gamma e_k = 0$ for each $k$, and $N_\Gamma g$ has mean value zero for each function $g \in W^{1/2}_2(\Gamma)$.

(d) $N(N_\Gamma) = \text{span}\{e_1, \ldots, e_m\}$.

**Proof.** Let $g, h \in W^{1/2}_2(\Sigma)$ be given, and let $(u, \pi)$ denote the solution of (7.1) corresponding to $g$, and $(v, q)$ the solution corresponding to $h$, respectively. Then one obtains

$$(N_\Gamma g| h)_{L^2(\Gamma)} = \int_\Gamma (u|h \nu_\Gamma) \, d\Gamma = -\int_\Gamma [(u|T(v, q)\nu_\Gamma)] \, d\Gamma$$

$$= \int_\Omega \text{div} (T(v, q)u) \, dx = 2 \int_\Omega \mu D(v) : D(u) \, dx$$

and the assertions in (a)-(b) follow at once. Here, $D(u) : D(v) = \text{trace} (D(u)D(v))$ denotes the Frobenius inner product of the (symmetric) matrices $D(u)$ and $D(v)$. 
(c) Let \( g \in W^{1/2}_{2}(\Sigma) \) be given, and let \((u, \pi)\) be the solution of (7.1). By the divergence theorem

\[
(N_{\Gamma}g|e_k)_{L^2(\Sigma)} = \int_{\Gamma_k} (u|\nu_{\Gamma}) \, d\Gamma_k = \int_{\Omega_{1,k}} \text{div} \, u \, dx = 0,
\]

with \( \Omega_{1,k} \) the region enclosed by \( \Sigma_k \). Therefore, by density of \( W^{1/2}_{2}(\Gamma) \) in \( L^2(\Gamma) \),

\[
(N_{\Gamma}g|e_k)_{L^2(\Sigma)} = 0 \quad \text{for all} \quad g \in L^2(\Sigma),
\]

and hence \( N_{\Gamma}e_k = 0 \).

(d) It remains to show that \( N_{\Gamma}N_{\Gamma}e_k \subset \text{span}\{e_1, \cdots, e_m\} \). Suppose \( g \in N(N_{\Gamma}) \).

It then follows from part (b) that \( D(u) = 0 \). Korn’s inequality and the no-slip boundary condition readily imply that \( u = 0 \) on \( \Omega \), and hence \( \pi \) is constant on each boundary component \( \Gamma_k \), and hence \( g|_{\Gamma_k} = \{ \pi \}|_{\Gamma_k} \) is constant on each boundary component \( \Gamma_k \), and hence \( g \in \text{span}\{e_1, \cdots, e_m\} \).

By means of the Neumann-to-Dirichlet operator \( N_{\Gamma} \), problem (1.6) can be reformulated as a geometric evolution equation

\[
V_{\Gamma} = \sigma N_{\Gamma}H_{\Gamma}, \quad \Gamma(0) = \Gamma_0.
\]

In order to study problem (7.2) one may parameterize \( \Gamma \) over an analytic reference manifold \( \Sigma \) which is \( C^2 \) close to \( \Gamma_0 \). Problem (7.2) can then be cast as a quasilinear evolution equation

\[
\partial_t h + A(h)h = F(h), \quad h(0) = h_0,
\]

where \( h(t) \) denotes the height function which parameterizes \( \Gamma(t) \), that is,

\[
\Gamma(t) = \{ q + h(t,q)\nu_{\Sigma}(q) : q \in \Sigma, \ t \geq 0 \}.
\]

The resulting problem (7.3) is amenable to the theory of maximal \( L^p \)-regularity for quasilinear parabolic evolution equations, see for instance [30, Chapter 5] for a comprehensive account of this theory. The following basic well-posedness result holds true.

**Theorem 7.2.** Suppose \( p > n + 2 \). Then for each \( \Gamma_0 \in W^{3-2/p}_{p} \) there is a number \( a = a(\Gamma_0) \) and a unique classical solution \( \Gamma = \{ \Gamma(t) : t \in (0, a) \} \) for (7.2). Moreover, \( \mathcal{M} = \bigcup_{t \in (0, a)} \{ t \} \times \Gamma(t) \) is real analytic.

**Proof.** The reader is referred to [30, Section 12.5] □

Suppose, as in the previous sections, that the dispersed phase \( \Omega_1 \) consists of \( m \) disjoint connected components, \( \Omega_1 = \bigcup_{k=1}^{m} \Omega_{1,k} \). Let \( \Gamma_k := \partial \Omega_{1,k} \), \( \Gamma = \bigcup_{k=1}^{m} \Gamma_k \), and let \( M_k := |\Omega_{1,k}| \) denote the volume of \( \Omega_{1,k} \). Then one shows, as in Section 2.3, that problem problem (7.2), or equivalently problem (1.6), preserves the volume of each individual phase component. Indeed, by Proposition 7.1(c)

\[
\frac{d}{dt}|\Omega_{1,k}(t)| = \int_{\Gamma_k} V_{\Gamma} \, d\Gamma_k = \sigma \int_{\Gamma_k} N_{\Gamma}H_{\Gamma} \, d\Gamma_k = \sigma (N_{\Gamma}H_{\Gamma}|e_k)_{L^2(\Sigma)} = 0.
\]
The time derivative of the surface area $|\Gamma(t)|$ is given by
\[
\frac{d}{dt}|\Gamma(t)| = -\int_{\Gamma} V_\Gamma H_\Gamma d\Gamma = -\sigma \int_{\Gamma} (N_\Gamma H_\Gamma)H_\Gamma d\Gamma = -2\sigma \int_\Omega |D(u)|^2 dx,
\]
where $(u, \pi)$ is the solution of (7.1) with $g = H_\Gamma$. This shows that surface area is decreasing. Therefore, $\Phi_0(\Gamma) = |\Gamma|$ constitutes a Lyapunov functional for (7.2). As in Section 2.5 one shows that $\Phi_0$ is a strict Lyapunov functional. An analogous argument as in Section 2 also implies that the equilibria of (7.6) consist of zero velocities, constant pressures in the phase components, and that the dispersed phase consists of a collection of non-intersecting balls in $\Omega$. Consequently, the set $\mathcal{E}$ of non-degenerate equilibria for (7.2) is given by
\[
\mathcal{E} = \{\Sigma : \Sigma \in \mathcal{S}\},
\]
where $\mathcal{S}$ is defined in (2.3). $\mathcal{E}$ gives rise to a real analytic manifold of dimension $m(n+1)$.

In analogy to Section 2.7 one also shows that the critical points of the area functional $\Phi_0$ under the constraints of $M_k = M_{0,k}$ constant correspond to the equilibria of (7.2), and that all critical points are local minima of the area functional $\Phi_0$ under the given constraints.

**Theorem 7.3.** The following assertions hold for problem (7.2).

(a) The phase volumes $|\Omega_{1,k}|$ are preserved.
(b) The area functional $\Phi_0$ is a strict Lyapunov functional.
(c) Each non-degenerate equilibrium consists of a finite union of non intersecting spheres which do not touch the outer boundary $\partial \Omega$.
(d) The set $\mathcal{E}$ of non-degenerate equilibria forms a real analytic manifold of dimension $m(n+1)$, where $m$ denotes the number of connected components of $\Omega_1$.
(e) The critical points of the area functional for prescribed phase volumes are precisely the equilibria of the system.
(f) All critical points of the area functional for prescribed phase volumes are local minima.

In order to analyze the stability properties of equilibria for the geometric evolution equation (7.2) one may proceed as follows. Suppose $\Sigma = \bigcup_{k=1}^m \Sigma_k \in \mathcal{E}$ is an equilibrium for (7.2). Choosing $\Sigma$ as a reference manifold one shows that problem (7.2), or for that matter also problem (7.3), can be written as
\[
\partial_t h + \sigma N_\Sigma A_\Sigma h = G_\Sigma(h), \quad h(0) = h_0, \tag{7.4}
\]
where $A_\Sigma$ has the same meaning as in Section 4. The nonlinear function $G_\Sigma$ satisfies $(G_\Sigma(0), G'_\Sigma(0)) = 0$. 
Let $X_0 := W_p^{2-1/p}(\Sigma)$, $X_1 := W_p^{3-1/p}(\Sigma)$, and set
$$L : D(L) = X_1 \subset X_0 \rightarrow X_0, \quad L := \sigma N_{\Sigma} A_{\Sigma}.$$  \hfill (7.5)

**Theorem 7.4.** The operator $L$ has the following properties.

(a) $-L$ generates a compact, analytic $C_0$-semigroup in $X_0$ which has the property of maximal $L^p$-regularity.

(b) The spectrum of $L$ consists of countably many real eigenvalues of finite algebraic multiplicity. The spectrum is independent of $p$.

(c) $-L$ has no positive eigenvalues.

(d) $\lambda = 0$ is a semi-simple eigenvalue of $L$ of multiplicity $m(n+1)$.

(e) The kernel $N(L)$ of $L$ is isomorphic to the tangent space $T_\Sigma E$.

Hence, the equilibrium $\Sigma \in E$ is normally stable.

**Proof.** The assertions in (a)-(b) follow from standard arguments.

(c) Suppose that $\lambda \in \mathbb{C}$, $\lambda \neq 0$, is an eigenvalue for $-L$, that is,
$$\lambda h + \sigma N_{\Sigma} A_{\Sigma} h = 0 \quad \text{(7.6)}$$
for some nontrivial function $h \in W_2^{5/2}(\Sigma)$. Taking the inner product of (7.6) with $A_{\Sigma} h$ in $L^2(\Sigma)$ yields
$$\lambda (h | A_{\Sigma} h)_{L^2(\Sigma)} + \sigma (N_{\Sigma} A_{\Sigma} h | A_{\Sigma} h)_{L^2(\Sigma)} = 0 \quad \text{(7.7)}$$
As $N_{\Sigma}$ and $A_{\Sigma}$ are symmetric, this identity implies that $\lambda$ must be real, hence the spectrum of $L$ is real.

Suppose that $\lambda > 0$. By Proposition 7.1(c), $(N_{\Sigma} A_{\Sigma} h | e_k)_{L^2(\Sigma)} = 0$ and consequently, $(h | e_k)_{L^2(\Sigma)} = 0$ as well, which implies $(h | A_{\Sigma} h)_{L^2(\Sigma)} \geq 0$. As $N_{\Sigma}$ is positive semi-definite on $L^2(\Sigma)$, see Proposition 7.1(b), one concludes that $(h | A_{\Sigma} h)_{L^2(\Sigma)} = 0$. This yields $A_{\Sigma} h = 0$, and then $h = 0$ by (7.6) as $\lambda > 0$.

(d) Suppose $h \in N(L)$. Then $A_{\Sigma} h = \sum_{k=1}^m a_k e_k$ by Proposition 7.1(d). This implies
$$h = h_0 - \sum_{k=1}^m (a_k R_k^2/(n-1)) e_k,$$
with $h_0 \in N(A_{\Sigma})$, where $R_k$ denotes the radius of the sphere $\Sigma_k$. As $N(A_{\Sigma})$ is spanned by the spherical harmonics $Y_j^k$ on $\Sigma_k$; we see that $\dim N(L) = m(n+1)$.

Next it will be shown that the eigenvalue 0 is semi-simple. Suppose $L^2 h = 0$. Then
$$N_{\Sigma} A_{\Sigma} h = h_0 + \sum_{k=1}^m a_k e_k,$$
for some $h_0 \in N(A_{\Sigma})$ and $a_k \in \mathbb{C}$.

Multiplying this relation with $e_j$ in $L^2(\Sigma)$ one obtains $a_k = 0$ for all $k$. Taking the $L^2(\Sigma)$ inner product of the relation $N_{\Sigma} A_{\Sigma} h = h_0$ with $A_{\Sigma} h$ yields
$$(N_{\Sigma} A_{\Sigma} h | A_{\Sigma} h)_{L^2(\Sigma)} = (h_0 | A_{\Sigma} h)_{L^2(\Sigma)} = 0$$
as $\mathcal{A}_\Sigma$ is symmetric and $h_0 \in \mathcal{N}(\mathcal{A}_\Sigma)$. Therefore, $\mathcal{N}_\Sigma \mathcal{A}_\Sigma h = 0$, that is, $h \in \mathcal{N}(L)$.

(e) The assertion follows as $\mathcal{N}(L)$ and $T_\Sigma \mathcal{E}$ are of the same dimension. □

**Theorem 7.5.** Let $p > n+2$ and suppose that $\Sigma$ is a (nondegenerate) equilibrium of (7.2).

Then any solution of (7.4) starting close to 0 in $W_p^{3-2/p}(\Sigma)$ exists globally and converges to an equilibrium $h_\infty$ in $W_p^{3-2/p}(\Sigma)$ at an exponential rate. Here, $h_\infty$ corresponds to some $\Gamma_\infty \in \mathcal{E}$.

**Proof.** The proof of this result is based on the generalized principle of linearized stability for quasilinear parabolic equations introduced in [33], see also Chapter 5 in the monograph [30]. □

The state manifold for (7.2) is defined by means of $\mathcal{S}\mathcal{M} = \mathcal{M}\mathcal{H}^2(\Omega)$. The main result of this section reads as follows.

**Theorem 7.6.** Let $p > n + 2$. Suppose that $\Gamma(t)$ is a solution of (7.2), defined on its maximal existence interval $[0,t_+]$. Assume there is a constant $M > 0$ such that the following conditions hold on $[0,t_+]$:

(i) $|\Gamma(t)|_{W_p^{3-2/p}} \leq M < \infty$;

(ii) $\Gamma(t)$ satisfies a uniform ball condition.

Then $t_+ = \infty$, i.e., the solution exists globally, and $\Gamma(t)$ converges in $\mathcal{S}\mathcal{M}$ to an equilibrium $\Gamma_\infty \in \mathcal{E}$ at an exponential rate. The converse is also true: if a global solution converges in $\mathcal{S}\mathcal{M}$ to an equilibrium, then (i) and (ii) are valid.

**Proof.** The proof is similar to that of Theorem 5.1, see also [30, Section 12.5]. □

8. Conclusions

In this chapter, the equilibrium states for the two-phase Navier-Stokes problem with heat-advection and surface tension (1.1)-(1.3), the two-phase isothermal Navier-Stokes problem with surface tension (1.5), and the two-phase Stokes flow with surface tension (1.6) are characterized. It is shown that every equilibrium is normally stable, and that every solution that starts close to an equilibrium exists globally and converges to a (possibly different) equilibrium at an exponential rate. Moreover, it is shown that the negative total entropy for (1.1)-(1.3), the available energy for (1.5), and the surface area for (1.6) constitute strict Lyapunov functionals. This implies that solutions which do not develop singularities converge to an equilibrium in the topology of the state manifold $\mathcal{S}\mathcal{M}$.

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