Grounding Occam’s Razor in a Formal Theory of Simplicity

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Abstract

It is proposed that the Occam’s Razor heuristic – when in doubt, choose the simplest hypothesis – requires a theory of simplicity in order to become fully rigorous and meaningful. With this in mind, a simple formal theory of simplicity is introduced. The relationship between simplicity and computation is explored, with simplicity shown reducible to a variant of computational complexity under certain special assumptions. The theory of simplicity is then used as a foundation for a theory of pattern, and a theory of hierarchy and heterarchy in systems of patterns. The end result is to re-envision Occam’s Razor as something like the following: In order for a mind to effectively understand the world, it should interpret itself and the world in the context of some simplicity measure obeying certain basic criteria. Doing so enables it to build up hierarchical and heterarchical pattern structures that help it interpret the world in a subjectively meaningful and useful way.

Simplicity was so complicated
I couldn’t understand it
– Mariela Ivanova

1 Introduction

Everybody loves Occam’s Razor, the heuristic often phrased as ”When in doubt, choose the simplest option,” and elegantly expressed by Albert Einstein via his maxim that theories should be ”As simple as possible, but no simpler.” This sort of advice sounds intuitively sensible, but without some precise understanding of what ”simplicity” means, it’s not particularly crisp guidance.

My own interest in Occam’s Razor arises largely from my work in artificial intelligence. A host of theorists have argued for Occam’s central role in AI – going back to Ray Solomonoff in the late 1960s, whose theory of ”Solomonoff induction” involves, essentially, AIs that understand the world via choosing the hypothesis represented by the shortest computer program [Sol64]. Marcus Hutter [Hut05] has built a rigorous theory of general intelligence under infinite or near-infinite computing resources, founded on this idea; and Eric Baum has argued the merits of similar ideas from a broad conceptual perspective [Bau04].

Occam’s Razor has also been considered foundational in the philosophy of science, by many different thinkers [Gau83]. There is a lot of power in the idea that complex hypotheses, like the Ptolemaic epicycles, have been systematically cast aside in favor of simpler, more compact hypotheses like the Copernican model.

However, all these applications of Occam’s Razor either rely on very specialized formalizations of the ”simplicity” concept (e.g. shortest program length), or neglect to define simplicity at all.

In my own prior work I have been guilty of a similar laxity in specifying the meaning of ”simplicity.” My work on the theory of complex and cognitive systems has relied substantially on the formal theory of pattern [Goe06], which relies on an assumed quantitative measure of simplicity – but I’ve never said much about this measure, except to give some specific examples of potential measures (such as some involving algorithmic information).
In this brief paper I take the bull by the horns and – after some conceptual exploration of the interpretation of the concept of simplicity in the context of the Occam’s Razor heuristic – give a fairly simple formal theory of what “simplicity” means. There is a close conceptual connection between the ideas given here and algorithmic information theory, which is briefly sketched, but the two theories have differences beyond emphasis.

The approach taken is not to specify one particular ”correct measure of simplicity,” but rather to indicate a set of formal criteria, along with the proposal that any reasonable measure of simplicity must obey these. It is shown that, under certain assumptions, a combination of program length and runtime fulfills the criteria to qualify as a simplicity measure. Conversely, it is shown that if a certain special ”finitude criterion” is assumed in addition to the basic simplicity criteria, then any simplicity measure has an interpretation as a measure of program length and runtime on some formal computer.

The notion of simplicity thus formalized is then used to give slightly revised definitions of other concepts previously defined in [Goe06]: pattern, subpattern and emergence. It is shown that a partial order may be obtained via interpreting \( a < b \) to mean, roughly: \( b \) is a part of a pattern in \( a \). Given a set of entities, some of which are patterns in each other, one may then obtain a natural hierarchical structure for these patterns, in which elements higher in the hierarchy are patterns in elements lower in the hierarchy. This hierarchy is then shown to naturally spawn a metric structure, which may be interpreted as a heterarchy to complement the hierarchy, in the spirit of the “dual network” concept discussed in [Goe06].

The formal axioms and manipulations presented here are relatively elementary, however, they represent an attempt to find an elegant formulation for some of the most foundational concepts underlying the study of complex and intelligent systems: simplicity, pattern, hierarchy and heterarchy.

These ideas allow us to re-envision Occam’s Razor as something like the following: *In order for a mind to effectively understand the world, it should interpret itself and the world in the context of some simplicity measure obeying certain basic criteria. Doing so enables it to build up hierarchical and heterarchical pattern structures that help it interpret the world in a subjectively meaningful and useful way.* Further, there are some tantalizing potential connections between this subjectivist view of Occam’s Razor and Zurek’s view of the relation between Occam’s Razor and entropy in physics [Zur89]; it may end up being sensible to interpret physical entropy as a sort of simplicity measure, according to the framework presented here.

2 The Interpretation of Simplicity in the Context of Occam’s Razor

”Occam’s Razor“ – the principle that, when in doubt, one should choose the simpler hypothesis – is a powerful heuristic, useful in many contexts. However, in order to escape vacuity, Occam’s Razor must be specifically grounded in some other principle guiding simplicity measurement. At least three options exist in this regard:

- Grounding via physics, so that the Razor basically says ”choose the hypothesis that can be created/computed using the least energy” (or some other physical quantity)

- Grounding via cognitive theory, i.e. via stating that an intelligent system should try to model the world in terms of some sort of suitable measure of simplicity (I call this the Meta-Razor below)

- Grounding via computing theory, i.e. giving a formal computational theory of simplicity and then asserting foundational status for the view of the universe as computational

These are not the only possible approaches, but they are the ones we will consider here. Hutter [Hut05] and Schmidhuber [YGSS10] appear basically sympathetic to the computationalist perspective, whereas my own tendency is to view the cognitive perspective as more fundamental, and to seek interpretations of the other two perspectives in a cognitive light. In any case, the interrelationships between the three perspectives are fascinating and well worth exploring. I will focus here mainly on the latter two approaches, exploring their connections carefully, and also noting some possible connections between them and the physics view.
2.1 The Difficulty of Defining Simplicity

The difficulty of defining simplicity an an "objective" way has been observed frequently by philosophers of science. The notion that scientists tend to prefer simpler theories is complicated by the fact that different scientific paradigms will tend to assess simplicity differently. Historically, there has been no standard, agreed-upon formalization of what makes one theory simpler than another. For instance, there is no uniform language in terms of which all scientists will agree to express their theories, agreeing to judge simplicity via length of expression in that language. But merely saying that "scientists will tend to choose the theories that they personally, psychologically find simpler" is a conclusion of rather limited power – because there are obviously factors other than simplicity that drive scientists' individual choices, and because scientists are diverse human beings with diverse psychologies, and may have all manner of irrational or idiosyncratic dynamics guiding their judgments of simplicity.

Mathematical formulations of Occam’s Razor, as are presented in the formal theory of AI (Solomonoff, Hutter, Schmidhuber, etc.), do not show us any way out of this dilemma – though they do present the dilemma in a clearer and more obvious form.

Suppose one agrees to restrict attention to hypotheses that are expressible as computer programs – so that Occam’s Razor comes down to choosing the simplest computer program that is capable of producing some given set of data. In this case, still one has the question – what does "simpler" really mean?

For starters, one is confronted with a bewildering variety of possible measures, including program length \[ \text{Chai03} \], program runtime \[ \text{Benn90} \], various weighted combinations of length and runtime \[ \text{YGSS10} \], "sophistication" \[ \text{Kop87} \], and so forth. And worse of all, once one has chosen one of these measures, one still has to choose a "reference Universal Turing Machine." That is, if one is measuring simplicity as program length (a la classic Solomonoff induction), one still has to answer the question "Program length in what programming language?", or equivalently "Program length in machine language on what computer?"

Theoretical computer science tells us that, in principle, the choice of reference machine doesn’t matter – for sufficiently complex entities, the simplicity will come out roughly the same no matter what reference machine is chosen. However, the definition of "sufficiently complex" depends on the reference machine in question: the theorem is that, for any pair of reference machines \( X \) and \( Y \), there is some constant \( C \) so that simplicity measured via \( X \) and simplicity measured via \( Y \) will be identical within amount \( C \). In practical terms this kind of equivalence is not necessarily all that helpful, and it can make a big difference what reference machine is chosen. Because real-world intelligence is largely about computational efficiency – about making choices in real, bounded situations using bounded space and time resources.

In the context of real-world AI, for Occam’s Razor based on program space to be meaningful, one needs to restrict attention to programs within some specified region of program space (defined so that, for any two programs in the region, the simulation constant \( C \) is not that large). But then, the question is faced: how does one determine this region of program space? If one does so via Occam’s Razor – "choose the simplest reference machine", i.e. "choose the simplest measure of simplicity"! – then one just arrives at a regress. If one does so by some other principle, then this fact certainly needs to be highlighted, as it’s at least equally important to Occam’s Razor in the process of hypothesis choice.

2.2 Grounding Occam’s Razor in Physics

One way to sidestep this Razor regress is to ground simplicity in the physical universe – arguing that in many contexts the appropriate reference machine is, in essence, the laws of physics as currently understood, which happen to assign a fundamental role to energy minimization. For instance, one could say that the simplest program is the one that can be represented via the physical system utilizing the least amount of energy.

Or, taking a cue from Bennett’s theory of logical depth, one could look at the system whose construction requires the least amount of energy. Or some sort of combination of the energy required for construction and the energy required for operation.

The recourse to physics is perfectly reasonable, in some contexts. However, it doesn’t quite resolve the matter, in spite of pointing in some interesting directions.

Firstly, this avenue doesn’t leave any room for positing Occam’s Razor as a fundamental principle for understanding the universe. If one argues that simplicity bottoms out in physics, then does one also want to argue that physics is discovered by choosing the simplest theory? Obviously this is just another Razor regress.
If one is willing to assume a certain theory of physics as true and correct, at least provisionally, then relative to this physics theory, one can formulate objective versions of Occam’s Razor. This approach may indeed be useful. Read Montague [Mon06] and others have argued that aspects of brain function can be understood via thinking of the brain as a solution evolution has found for the problem of achieving effective organism control using minimum energy. In practical computing, minimizing space and time resource utilization of algorithms can be viewed as bottoming out in minimizing the energy required to build and operate the physical computers running the algorithms.

However, I believe additional insight may be obtained by pursuing an alternate perspective, in which intelligence rather than physics is primary.

2.3 Grounding Occam’s Razor in Intelligence

Suppose one takes the view of an intelligent system modeling a stream of data, with constructing a “physics” of the world underlying that data considered as one of the tasks the system confronts. In that general setting, not all “physics” theories the system might construct, would necessarily give rise to Occam’s Razor type principles. Our current standard theories of physics, involve a notion of energy minimization, but not all possible physics theories, broadly speaking, need involve minimization of some quantity in an equally straightforward way.

One might then postulate a principle that: A good theory of the world is one that describes real-world systems as minimizing some quantity, so that this quantity can be taken as the “simplicity” measure grounding Occam’s Razor.

That is, taking the perspective of an intelligent system choosing between different hypotheses about the world around it, one principle this system might follow is: Choose an hypothesis that leads to a world-model possessing an intuitively comprehensible Occam’s Razor principle.

But, on what grounds would said intelligent system consider such a choice “good”? Suppose the system has certain goals that are important to it. Then, presumably, it would have to judge that choosing an Occam-friendly world-model is a good way for it to achieve its goals.

Obviously, not all systems, environments and goals will lead to this conclusion. Some system/environment/goal triples will be more “Occam-friendly” than others.

One could formalize this notion in the context of reinforcement learning and statistical decision theory (among other approaches). Consider an agent receiving percepts and rewards from, and enacting actions in, some environment. Will it benefit the agent to model its environment in a way that causes it to choose actions according to some measure of “simplicity”? Marcus Hutter’s results regarding AIXI give a positive answer – in the case that the agent has very large computing resources, and a very large amount of time to interact with the world, and the sequence of rewards is determined by some computable function. But given realistic computing resources and interaction time, it’s not so clear. Hutter has shown that the large-resources, long-time scenario is Occam-friendly under some reasonable assumptions – but still, that’s not the scenario in which we live.

However, we humans do constitute another piece of data – we have limited resources, and have had limited time to interact with the world, and we have also come to model the world in an Occam-friendly manner, via our current theories of physics.

The idea that an intelligent system should try to model the world in such a way that, for some comprehensible simplicity measure, Occam’s Razor applies, might be labeled the “Meta-Razor.” Unlike Occam’s Razor, which relies on some additional assumption about simplicity, the Meta-Razor stands independently of other assumptions. But the Meta-Razor lacks the power of Occam’s Razor, because it leaves the door open for different intelligent systems to construct world-models involving different simplicity measures, and thus to rate hypotheses in different, incommensurate ways.

2.4 Relations Between Physical and Computational Measures of Simplicity

There are interesting relationships between the various approaches to understanding simplicity. In the following, we will give a formal theory of simplicity motivated largely by the notion of the Meta-Razor, and then explore its relationship to computation theory, showing a close connection. And, though we will not
dwell on them extensively here, there are also connections between the computational and physical notions of complexity, that are worthy of future exploration.

For instance, there are mathematical relationships between algorithmic information (the length of the shortest program for computing something) and Shannon entropy, which indicate that the algorithmic information view of Occam’s Razor and the Maximum Entropy prior (resulting from thermodynamics) are equivalent in some cases; e.g. argues that “Algorithmic randomness is typically very difficult to calculate exactly but relatively easy to estimate. In large systems, probabilistic ensemble definitions of entropy (e.g., coarse-grained entropy of Gibbs and Boltzmann’s entropy $H = \ln(W)$, as well as Shannon’s information-theoretic entropy) provide accurate estimates of the algorithmic entropy of an individual system or its average value for an ensemble.”

Zurek goes further, and argues that “Physical entropy is a sum of (i) the missing information measured by Shannon’s formula and (ii) the algorithmic information content – algorithmic randomness – present in the available data about the system.” This is particularly provocative in the context of the axiomatic approach to simplicity given below, which views simplicity as the sum of a term corresponding conceptually to program length (“algorithmic information content”) and the sum of a term corresponding conceptually to computation runtime. If one were to use missing Shannon information in place of computation runtime in fulfilling the abstract simplicity criteria, one would reach the conclusion that physical entropy as interpreted by Zurek is a valid simplicity measure. This is a speculative direction that we will not explore fully here, but is interesting as a possible guide for future investigation.

2.5 The Need for Grounding Simplicity

Occam’s Razor is a powerful heuristic – but it doesn’t stand on its own. It requires some external grounding, because the notion of “simplicity” must come from somewhere. One can ground simplicity “objectively”, via assuming a physical world involving a simplicity measure such as energy, or via assuming a universe governed by a particular computational model; or one can ground it “subjectively” via the Meta-Razor, as a choice that an intelligent agent makes, to model the world in such a way that a comprehensible simplicity measure emerges from that world. Either way the specific grounding of simplicity is a highly significant matter, without which the Razor is vacuous.

The remainder of this paper pursues a formalization of the concept of simplicity, inspired largely by the subjectivist approach to specifying Occam’s Razor. A set of criteria is presented, specifying what constitutes a sensible measure of simplicity. This fits naturally with the hypothesis that, to understand the world and itself, a mind should construct a perspective centered around some sensible measure of simplicity, fulfilling the criteria. This notion of simplicity is then connected with computation theory, showing that computation-theoretic measures of simplicity are in essence a special case of the broader axiomatic notion presented. The remaining sections of the paper then explore how a notion of simplicity may be used by an intelligent system to build up a fuller mathematical framework for understanding the world – e.g. a hierarchy and a metric space of patterns recognized in experience.

3 A (Fairly) Simple Formalization of Simplicity

We consider a space $X$ of ordered lists, defined as the set of finite lists of elements drawn from some set $Y$, and endowed with

- a binary operation $*$ that maps $X \times X \to X$
- a binary operation $t$ that maps $X \times X \to [0, \infty]$

Conceptually, $*$ describes what happens when you combine two entities in $X$ to produce another one; and $t$ measures the cost of executing this combination.

The space $X$ is assumed to have an identity element $e$ under $*$. No commutativity or associativity properties are assumed; however, we make a special assumption about how $*$ maps over lists, namely

$$x * [w, v] = (x * w) * v$$

The operation $t$ is assumed to have the properties
• \( t(y, e) = t(e, y) = 0 \)
• \( t(y, z) \leq \min_w(t(y, w) + t(w, z)) \)
• \( y * z = x \rightarrow t(w, [y, z]) \leq t(w, x) + t(y, z) \)

Assuming the above assumptions are fulfilled, we will call \( * \) and \( t \) a production operator and a cost operator respectively.

We then define a simplicity measure on \( X \) as any function \( s : X \rightarrow [0, \infty] \) satisfying

• \( s(x) = 0 \iff x = e \) (the identity element)
• \( s(x) = \min_{y, z : y * z = x} h(y, z) \), where \( h(y, z) = s(y) + s(z) + t(y, z) \)
• \( s([a, b]) \leq s(a) + s(b) \)

4 Simplicity and Computational Cost

The paradigm case of a simplicity measure is computational cost. Consider the situation where members of \( X \) may be interpreted as programs on some particular Universal Turing Machine. Then, to put it simply, one may consistently interpret

• \( t(y, z) \) as the runtime required (on this UTM) to execute \( y \) with \( z \) as input
• \( s(x) \) as the minimum over all programs \( P \) that produce \( x \) of length(\( P \)) + runtime(\( P \))

Different weightings between program length and runtime may be incorporated via scaling \( t \) by various non-negative constants.

The question arises whether the converse holds – can one interpret an arbitrary simplicity measure in terms of program length and runtime on some computer? We show below that this is true if one makes an additional "finitude" assumption about the nature of the simplicity measure.

4.1 Deriving Simplicity from Computational Complexity

To frame the relation between simplicity and computation more rigorously, suppose one has a formal computer \( M \) that takes one program as its "operating program" and another program as input, and produces a third program \( s \) output. This can be modeled, for example, as a standard sort of multi-tape Turing machine with independently accessible tapes. Suppose also that the computer has an in-built interpreter for a programming language \( L \) that includes lists and the operator \( * \); and suppose that the language \( L \) obeys the syntactic rule \( x * [w, v] = (x * w) * v \) given above, meaning that (roughly speaking) performing this substitution in a program’s executable code doesn’t affect the result of running the program. Let \( t_M(x, y) \) denote the time cost of executing program \( x \) (in language \( L \)) with program \( y \) as input; and suppose that the cost \( t_M \) of operations on \( M \) is measured in such a way that \( y * z = x \rightarrow t_M(w, [y, z]) \leq t_M(w, x) + t_M(y, z) \) (which will be achieved, for example, if the operations of extracting an element from a list, and piping a result into a process as an input, are both rated as free). Let \( s_M(x) \) denote the minimum of (program length + program execution cost), achieved for any program in \( L \) that, run on \( M \), produces \( x \) as output. Consider the identity \( e \) as the null program, and assume \( s_M(e) = 0 \), and \( t_M(e, x) = t_M(x, c) = 0 \). Then, one may show that

**Theorem 1** \( s_M \) is a simplicity measure on the set of expressions in \( L \)

**Proof.** Suppose \( y * z = x \), and let \( h_M(y, z) = s_M(y) + s_M(z) + t_M(y, z) \).

The proof goes by induction. Suppose the theorem statement holds for all expressions \( z \) in \( L \) with \( s_M(z) < s_M(x) \).

Then, if \( s_M(y) < s_M(x) \) and \( s_M(z) < s_M(x) \), clearly \( h_M(y, z) > s_M(x) \), because \( h_M(y, z) \) represents the sum of the program length plus execution cost for one particular way of producing \( x \), and \( s_M(x) \) is supposed
to be the minimum achievable (program length + execution cost) for any M-program producing \( x \). On the other hand, if \( s_M(y) > s_M(x) \) or \( s_M(z) > s_M(x) \), clearly \( h_M(y, z) > s_M(x) \) as well.

And, \( s_M(x) \) can’t be smaller than every \( h_M(y, z) \) value, because if nothing else \( s_M(x) = h_M(e, x) \), by assumption.

So, we conclude the theorem statement must hold for \( s_M(x) \). And the theorem follows by induction.

Note that, in order to get the proof of Theorem 1 to go through smoothly, we had to assume a somewhat idiosyncratic programming language \( L \) and measure of computational time cost. However, these idiosyncrasies don’t actually make a big difference – if one were using some more standard language and cost measure, one would find that \( s_M \) is almost a simplicity measure, rather than precisely being one. Things would become a little more ugly, but the basic conceptual story would be the same. A similar conclusion may be drawn regarding Theorems 2 and 3 below.

### 4.2 Deriving Computation from Simplicity

It is natural to inquire as to the extent which the axioms presented above as characterizing simplicity constrain the space of mathematical functions that may serve as measures of simplicity. One thinks of Cox’s Theorem in the foundations of probability [Cox06], where some fairly broad-looking axioms characterizing an uncertainty measure are shown to imply that said uncertainty measure must be monotonically transformable into probability. It’s possible that a similar result might hold here, showing that any functions \( s \) and \( t \) obeying the above rules must be scaled versions of some basic simplicity and cost functions.

One interesting conjecture is that any simplicity measure satisfying the above axioms may be represented as "program length + runtime" on some formal computer. This would provide a conceptual result analogous to Cox’s Axioms: due to the possibility bisimulation between different UTMs, the program length and runtime on any two UTMs are equivalent up to a constant, the caveat being that the constant is potentially very large.

While this sounds at first like a far-reaching conclusion, a limited version of it actually follows straightforwardly from the set-up used in the definition of simplicity.

Define \([y, z]\) to be an elementary operation if there does not exist any \( w \) so that \( t(y, z) = t(y, w) + t(w, z) \). This means, intuitively, that in terms of \( t \) there is no intermediate step involved in producing \( y \) from \( z \) – the lowest-cost approach is to do it all in one step.

Next, define \( x \) to be an elementary entity if there does not exist any pair \( (y, z) \) so that \( s(x) = h(y, z) \) and neither \( y = e \) nor \( z = e \). This means, intuitively, that \( x \) cannot be decomposed into any simpler entities.

For an elementary operation \([y, z]\), define \( C([y, z]) = t(y, z) \); and for an elementary operation \( x \), define \( C(x) = s(x) \).

Given a set of functions \( t \) and \( s \), one may thus derive a set of elementary operations and entities, which generate a set of expressions \( E \) (the free algebra on the elements of \( X \) generated by the operation \( * \)). Define the cost \( C(E) \) of an expression \( E \in E \) as the total cost of all the elementary operations and entities in \( E \).

It is then easy to see that, given an element \( x \in X \), \( s(x) \) is equal to the minimum cost \( C(E) \) over all expressions \( E \) that produce \( x \).

In this sense, any simplicity measure produces a "computational model" of sorts. However, there’s a catch. In general, if \( Y \) is infinite, we have no way of knowing that the set of elementary operations or entities is finite. In the case where the sets of elementary operations and entities are both finite (the "finitude condition"), then we can define a formal computer which contains the elementary entities in its memory and the elementary operations as its basic instruction set. And it’s easy to see that, for this computer, if we assume the time required for access to the elementary entities is instantaneous, the minimum cost \( C(E) \) over all expressions \( E \) producing \( x \) is equal to the minimal (program length + runtime) over all programs producing \( x \).

So we have the conclusion

**Theorem 2** In the case that the finitude condition holds (which includes, but is not restricted to, the case that \( Y \) is finite), there exists some formal computer so that the simplicity \( s(x) \) is equal to the minimum of (program length + runtime) over all programs running on the computer and producing \( x \).
Combining this with the existence of bisimulations between Turing machines lets us conclude that: Given two simplicity measures \( s_1 \) and \( s_2 \) both satisfying the finitude condition, and at least one of which corresponds to a formal computer that is a UTM, the limit of \( s_1(x)/s_2(x) \) as \( s_1(x) \to \infty \) or \( s_2(x) \to \infty \) tends to 1.

We have not quite fully reduced simplicity to computation. Rather, we have provided a slight tweak of the conventional view of computation, portraying it as a particular case of simplicity in which the set of elementary entities and operations is finite.

5 Pattern From Simplicity; Hierarchy from Pattern

Using the above notion of simplicity, we may define pattern as follows: the pair \((y, z)\) is a pattern in \(x\) relative to weight \(w\) with degree

\[
I_{y,z;w}(x) = \frac{s(x) - h(y, z)}{s(x)}
\]

Ordinarily one will say that \((y, z)\) is a pattern in \(x\) (relative to \(w\)) if the degree \(I_{y,z;w}(x) > 0\).

5.1 Hierarchy from Pattern

The concept of pattern can be used to build up a natural hierarchical structure on the space \(X\), deriving directly from the notion of simplicity underlying pattern.

We can use the notion of pattern to define a binary operation on \(X\), such that

\[
y \leq x \iff \max_z I_{y,z}(x) \geq 0
\]

If \(y \leq x\), we will say that \(y\) is a compositional subpattern of \(x\). I.e., this means \(y\) can be combined with some other entity \(z\) to form a pattern in \(x\).

It is not hard to show that

**Theorem 3** The relation \(\leq\) defined above is a partial order on \(X\)

**Proof.**

It’s immediate to observe that \(y \leq x\) and \(x \leq y\) can hold simultaneously iff \(x = y\). To have \(y \leq x\), one must have

\[
s(y) + s(z) + t(y, z) \leq s(x)
\]

for some \(z\) so that \(x = y \ast z\); but, to have \(x \leq y\), one must have

\[
s(x) + s(w) + t(x, w) \leq s(y)
\]

for some \(w\) so that \(y = x \ast w\). So, we must have

\[
0 \leq s(w) + t(y, z) \leq s(y) - s(x) \leq -s(z) - t(x, w) \leq 0
\]

meaning \(s(w) = s(z) = 0\) so \(w = z = e\), so \(x = y\).

Next, we show transitivity, i.e. that \(x \leq y\) and \(y \leq z\) implies \(x \leq z\).

Firstly, \(x \leq y\) means that for some \(w\), we have \(x \ast w = y\) and

\[
s(x) + s(w) + t(x, w) \leq s(y)
\]  

(1)

Similarly, \(y \leq z\) means that for some \(v\), we have \(y \ast v = z\) and

\[
s(y) + s(v) + t(y, v) \leq s(z)
\]  

(2)

How then can we show \(x \leq z\)? We must demonstrate some \(u\) so that \(x \ast u = z\) and

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1 This is identical to the prior definition of pattern given in [Goe06], if one sets \(w = 0\) and sets the metric in the prior definition to a trivial metric so that \(d(x, z)\) is 0 for \(x = z\) and \(\infty\) otherwise.
\[ s(x) + s(u) + t(x, u) \leq s(z) \]

We claim that it works to set \( u = [w, v] \). First, we have \( x * [w, v] = (x * w) * v \) because this is an assumed property of \(*\). So, what we need to show that \( u \) realizes \( x \leq z \) is that

\[ s(x) + s([w, v]) + t(x, [w, v]) \leq s(z) \]

Recall, we know \( t(x, [w, v]) \leq t(x, w) + t(y, v) \) and \( s([w, v]) \leq s(w) + s(v) \) via assumed properties of \( t \). Thus, it suffices to show that

\[ s(x) + s(w) + s(v) + t(x, w) + t(y, v) \leq s(z) \]

But this follows by inserting Equation 1 into Equation 2.

6 Similarity and Heterarchy from Hierarchy and Pattern

Systems of patterns are generally characterized by both hierarchical and heterarchical structures (as argued extensively in [Goe93]). We have modeled hierarchical structure above using a partial order; heterarchical structure, on the other hand, is effectively mathematically modelable using metric structure. One important form of heterarchy is a network in which patterns are associated with other similar patterns, and similarity is merely a rescaling of distance.

One may also derive a metric structure on \( \mathcal{X} \) from the definition of hierarchy as given above, leveraging the underlying definition of simplicity. This may be done in many ways; one approach is to associate two sets with each \( x \in X \): the pattern intension \( int(x) \) consisting of all \( y \) so that \( x \leq y \); and the pattern extension \( ext(x) \) consisting of all \( y \) so that \( y \leq x \). One may the construct a metric associated with each of these, i.e. where \( d_T \) is the Tanimoto distance [RT60], one can set

- \( d_{int}(x, y) = d_T(int(x), int(y)) \)
- \( d_{ext}(x, y) = d_T(ext(x), ext(y)) \)

One may then define a composite metric as

\[ d(x, y) = \alpha d_{int}(x, y) + (1 - \alpha)d_{ext}(x, y) \]

This gives a metric structure to the space of processes \( \mathcal{X} \) in a way that coincides naturally with the hierarchical structure.

An alternate approach is to look more closely at the quantity used in the definition of the order \( \leq \) given above, which we may name

\[ Q(y, x) = \max_z I_{y,z}(x) \]

This quantity measures the degree to which \( y \) is a compositional subpattern in \( x \). If one normalizes the set of \( Q(\ast, x) \) values over the pattern extension of \( x \), then one obtains a probability distribution of \( Q \) values over the pattern extension of \( x \); and one can measure the distance between the pattern extension distribution associated with \( x \) and the one associated with \( y \) using the Hutchinson metric [Hut81]. One can do similarly for pattern intensions, and then create a composite metric analogous to the above.

The relation between this probabilistically defined (Hutchinson) metric and the previously specified crisply defined (Tanimoto) metric is unclear, though one expects they will often be strongly correlated.

In either case, it seems, one obtains a metric plays nicely with the hierarchical structure defined by the partial order, which is important if one wishes to explore the general relation between hierarchy and heterarchy in systems of patterns.
7 Conclusion

Occam’s Razor is a valuable heuristic, yet vacuous except when paired with a theory of simplicity. Grounding simplicity in physics or computation theory can be valuable for certain purposes, yet ultimately is conceptually unsatisfactory, because one cannot then use one’s grounding of simplicity to evaluate one’s theory of physics or computation without succumbing to circular reasoning. An alternative approach is to place cognitive theory at the center, and re-envision Occam’s Razor as something like the following: In order for a mind to effectively understand the world, it should interpret itself and the world in the context of some simplicity measure obeying certain basic criteria. Doing so enables it to build up hierarchical and heterarchical pattern structures that help it interpret the world in a subjectively meaningful and useful way. In this paper we have put some meat on the bones of this idea, via presenting a formal axiomatic theory of simplicity, and showing that it relates to the computational approach to simplicity, and gives rise to hierarchical and heterarchical cognitive structures with appropriate properties. This provides a more solid foundation for cognitive theories founded on notions of simplicity and pattern, such as those presented previously in [Goe06].

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