On the Maximal Subgroups of $E_7(q)$ and Related Almost Simple Groups

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Contents

1 Introduction 2

2 Preliminary results 4
  2.1 Basic notation and definitions 4
  2.2 Results about algebraic groups 6
  2.3 Stabilizing subspaces 7
  2.4 Action of outer automorphisms 8
  2.5 Alternating forms on vector spaces 10

3 Two algorithms for determining conjugacy classes 11
  3.1 Lie product method 11
  3.2 Subalgebra method 12

4 Reducing to almost simple groups 14

5 Determination of the almost simple groups 16
  5.1 Alternating groups 18
    5.1.1 Alt(7) 19
  5.2 Sporadic groups 21
    5.2.1 HS 21
    5.2.2 Ru 21
    5.2.3 M_{12} 22
    5.2.4 M_{22} 22
    5.2.5 J_2 23
  5.3 Lie type not PSL_2(r) 23
    5.3.1 PSL_3(4) 23
    5.3.2 PSU_3(3) 24
    5.3.3 PSU_3(8) 26
    5.3.4 Ω^+(2) 27
    5.3.5 {2B_2(8} 28
  5.4 Lie type PSL_2(r) 28
1 Introduction

To apply the classification of finite simple groups requires more than just labels for them, but often a significant amount of structural information. One of the major open questions in this area is to compute the maximal subgroups of the finite simple groups (and their almost simple versions). For classical groups one may expect lists in small dimensions and general results for arbitrary dimensions; for exceptional groups of Lie type, complete tables can in theory be produced. This was done decades ago for the small-rank cases of Suzuki and Ree groups, $G_2(q)$ and $3D_4(q)$ (see, for example, [40]). The cases of $F_4(q)$, $E_6(q)$ and $2E_6(q)$ were very recently completed by the author [12]. This leaves $E_7(q)$ and $E_8(q)$.

This paper is a major contribution to tackling the case of $E_7(q)$, leaving only a small handful of ambiguities (Table 1.2). As a result of a number of papers by various authors (see Section 4 for details), all maximal subgroups $M$ of an almost simple group with socle $E_7(q)$ are known except for $M$ almost simple.

**Theorem 1.1.** Let $\bar{G}$ be an almost simple group with socle $G = E_7(q)$, for $q$ a power of a prime $p$. Let $M$ be a maximal subgroup of $\bar{G}$, and assume that $M$ does not contain $G$. One of the following holds:

(i) $M = N_{\bar{G}}(M_0)$ is the normalizer of the intersection $M_0$ of $G$ with a positive-dimensional subgroup of an adjoint algebraic group of type $E_7$, and is given in Table 4.1;

(ii) $M$ is an almost simple group and appears in Table 1.1;

(iii) $M$ is the normalizer of one of the groups $\text{PSL}_2(r)$ for $r = 7, 8, 9, 13$, with possibilities for $p$ given in Table 1.2.

For $\text{PSL}_2(8)$, we are able to eliminate possible maximal subgroups of that type for $E_7(2)$ (already done in [2], but our proofs here are independent of their paper), and $q = 4$, (which is new, see Proposition 6.2 for the case of $\text{PSL}_2(8)$ and $q = 2, 4$).

**Proposition 1.2.** The maximal subgroups of $E_7(4)$ are known, and appear in Tables 1.1 and 4.1.

The principal contribution to Theorem 1.1 in this paper is the determination of the groups in the second part of the list, i.e., Table 1.1. Results of Borovik and Liebeck–Seitz (see Section 4) show that a maximal
| Group         | $p$ | $q$ | No. classes | Stabilizer |
|--------------|-----|-----|-------------|------------|
| $M_{12,2}$   | 5   | 5   | 1           | $(\delta)$|
| $HS$         | 5   | 5   | 2           | 1          |
| $Ru$         | 5   | 5   | 2           | 1          |
| PSU$_3(3)$   | $p \equiv \pm 1 \text{ mod } 8$ | $p$ | 2           | 1          |
| PSU$_3(3)$   | $p \neq 3, p \equiv \pm 3 \text{ mod } 8$ | $p^2$ | 2 | $(\phi)$ |
| PSU$_3(8).6$ | $p \equiv \pm 1 \text{ mod } 8$ | $p$ | 2 | 1 |
| PSU$_3(8).3_1$ | $p \equiv \pm 3 \text{ mod } 8$ | $p$ | 1 | $(\delta)$ |
| PSL$_2(37)$  | $p \neq 37 \text{ a square modulo } 37$ | $p$ | 2 | 1 |
| PSL$_2(37)$  | $p$ a non-square modulo $37$ | $p^2$ | $d$ | $(\phi)$ |
| PSL$_2(29)$  | $p \equiv \pm 1 \text{ mod } 5, p \neq 29 \text{ a square modulo } 29$ | $p$ | 4 | 1 |
| PSL$_2(29)$  | $p \equiv \pm 1 \text{ mod } 5, p$ a non-square modulo $29$ | $p^2$ | 4 | $(\phi)$ |
| PSL$_2(29)$  | $p \equiv \pm 2 \text{ mod } 5$ | $p^2$ | $2d$ | 1 |
| PSL$_2(27).3$ | 13  | 13  | 2           | 1          |
| PSL$_2(27)$  | $p \equiv \pm 1 \text{ mod } 13$ | $p$ | 4 | 1 |
| PSL$_2(27)$  | $p \equiv \pm 5 \text{ mod } 13$ | $p^2$ | 4 | 1 |
| PSL$_2(27)$  | $p \equiv \pm 3, \pm 4 \text{ mod } 13$ | $p^3$ | 4 | $(\phi)$ |
| PSL$_2(27)$  | $p \equiv \pm 2, \pm 6 \text{ mod } 13$ | $p^6$ | $2d$ | $(\phi^2)$ |
| PSL$_2(19)$  | 5   | 5   | 2           | 1          |
| PSL$_2(19)$  | $p \neq 19, p \equiv \pm 1 \text{ mod } 5, p_{19} \equiv p_3 \text{ mod } 2$ | $p$ | 4 | 1 |
| PSL$_2(19)$  | $p \equiv \pm 1 \text{ mod } 5, p_{19} \neq p_3 \text{ mod } 2$ | $p^2$ | 4 | $(\phi)$ |
| PGL$_2(19)$  | $p \equiv \pm 2 \text{ mod } 5$ | $p^2$ | $2d$ | 1 |
| PSL$_2(19)$  | 5   | 5   | 2           | 1          |
| PGL$_2(19)$  | $p \equiv \pm 1, \pm 9 \text{ mod } 40$ | $p$ | 4 | 1 |
| PSL$_2(19)$  | $p \neq 19, p \equiv \pm 11, \pm 19 \text{ mod } 40$ | $p$ | 2 | $(\delta)$ |
| PGL$_2(19)$  | $p \neq 2, p \equiv \pm 2 \text{ mod } 5$ | $p^2$ | 4 | 1 |
| $M_{22,2}$   | 5   | 5   | 1           | $(\delta)$ |

Table 1.1: The known maximal subgroups of the simple group $G = E_7(q)$, for $q$ a power of $p$, other than those that are fixed points of a positive-dimensional subgroup of the algebraic group. $M_{22,2}$ is a novelty maximal that occurs in the group $E_7(5).2$. Here, $d = \gcd(2, p - 1)$, $p_n$ is the order of $p$ modulo $n$, $\phi$ is a generator for the group of field automorphisms and $\delta$ is a generator for the group of outer diagonal automorphisms of $G$.

| Prime $p$ | Group $G$ |
|-----------|-----------|
| $p \nmid M_0$ | $\text{Alt}(6), \text{PSL}_2(q), q = 7, 13$ |
| $p = 2$ | $\text{PSL}_2(8)$ |
| $p = 3$ | $\text{PSL}_2(13)$ |
| $p = 5$ | $\text{Alt}(6)$ |
| $p = 7$ | $\text{PSL}_2(q), q = 7, 13$ |

Table 1.2: Potential simple subgroups $M_0$ of $G = E_7(q)$ ($q$ a power of $p$) with $N_G(M_0)$ maximal that have not been eliminated in this document.
subgroup $M$ of $\tilde{G}$ either comes from a positive-dimensional subgroup of the algebraic group, or is an almost simple group and does not arise in that way, the list of which we call $\mathcal{S}$ in analogy with classical groups.

We start with a few sections of preliminary results, and then in Section 3 discuss the two computational methods that will be implemented in determining most of the subgroups in Table 1.1. The determination of the subgroups in Table 1.1 is known from previous papers (described in Section 4), except for the technical question of whether the outer diagonal automorphism fuses two classes of maximal subgroups or normalizes a single class. This is also known for some of the groups already, and for others we compute the answer here. For one set of groups, $\text{PGL}_3(q)$ and $\text{PGU}_3(q)$ for $q$ an odd power of a prime, we cannot determine here whether this has index 1 or 2 in its normalizer in $E_7(q)$, (i.e., whether the diagonal automorphism normalizes these classes) and we leave this case open. We do this analysis for the other groups in Section 4.

The determination of the groups in $\mathcal{S}$ (except for $\text{PSL}_2(r)$, $r = 7, 8, 9, 13$, which we do not do here) takes place in Section 5, starting from the list in Table 5.1. Information about the four remaining groups is given in Section 4. Finally, Section 7 proves that the subgroups in Table 1.1 are genuinely maximal by checking there are no proper subgroups of $E_7(q)$ containing them.

The author would like to thank Alex Ryba profusely, who while at the Isaac Newton Institute in January 2020 explained his ideas, from [16], which form the subalgebra method from Section 3.2. He would also like to thank Friedrich Knop for his proof of Proposition 2.3. Finally, he would like to thank Ben Grossmann for discussions on the StackExchange website that led to producing a short algorithm for determining ranks of linear combinations of two matrices, which is used in several of the computer programs for this article.

2 Preliminary results

Readers of previous papers in this series [9, 10, 12] will be familiar with this notation and setup. Many of the preliminary results in this section will also have been come across before. There are some simplifications in this paper versus previous ones because we no longer have to consider Steinberg endomorphisms that are not Frobenius, and will not need to consider the action of graph automorphisms on maximal subgroups. Chief among the new actors in this paper is the alternating form on the 56-dimensional minimal module for $E_7$, which has not been needed before.

2.1 Basic notation and definitions

Let $p$ be a prime or 0 and let $k$ be an algebraically closed field of characteristic $p$. Let $G$ be a simple, simply connected algebraic group defined over $k$, and let $\sigma$ be a Frobenius endomorphism of $G$. Write $G$ for the fixed points $G^\sigma$. Write $F_p$ for a generator of the semigroup of ‘standard’ Frobenius endomorphisms, i.e., with trivial action on the Weyl group. Write $E_7(q)$ for the simple group of this type when $G$ has type $E_7$. (Thus if $\sigma = F_p$ for $p$ odd then $G$ is the simply connected group $2 \cdot E_7(p)$.) Let $\tilde{G}$ be an almost simple group with socle $G/Z(G)$. Write $G_{ad}$ for the preimage in the simply connected version of $G$ of the fixed points under a Frobenius morphism of the adjoint version of $G$ compatible with $\sigma$. Thus $G_{ad}$ has $G$ as a normal subgroup and the quotient by $G$ is the group of outer diagonal automorphisms.

Let $M(G)$ denote a non-trivial Weyl module of smallest dimension, and $L(G)$ denote the adjoint module of $G$. We will endow $L(G)$ with a Lie bracket (this is crucial in this article) that turns it into a Lie algebra. If $L(G)$ has a single trivial composition factor and a single non-trivial one (so the important one for us is $G$ of type $E_7$ and $p = 2$) then write $L(G)^\circ$ for the non-trivial composition factor.
If $k$ has characteristic $p \neq 0$ and $u \in \text{GL}_n(k)$ has $p$-power order then $u$ is a unipotent element, and acts (up to conjugacy) as a sum of Jordan blocks of various sizes. We write $n_1^{a_1}, \ldots, n_r^{a_r}$ to describe the sizes, for example $5^3, 4, 1$ for an element of $\text{GL}_{15}(k)$, as in [20]. We also use the labelling of unipotent classes from that paper.

Following [21] (and in [9, 10, 12]), write $\text{Aut}^+(G)$ for the set of inner, graph, and $p$-power field automorphisms of $G$. Since we do not consider $G_2$ for $p = 3$ and $F_4$ for $p = 2$, we do not need to consider the very twisted groups. For $G$ of type $E_7$, $\text{Aut}^+(G)$ induces the whole of $\text{Aut}(G)$ on $G = G^\sigma$. A subgroup $H$ of $G$ is Lie primitive if there is no closed, proper, positive-dimensional subgroup of $G$ containing $H$, and Lie imprimitive otherwise. While Lie primitive subgroups often yield maximal subgroups of the finite groups $G$, there may be others. A subgroup $H$ of $G$ is strongly imprimitive if it is contained in a closed, proper, positive-dimensional subgroup $X$, and in addition $X$ may be chosen to be $N_{\text{Aut}^+(G)}(H)$-stable. In this case, the normalizer of $H \cdot Z(G)/Z(G)$ in $\tilde{G}$ is always contained in the normalizer of $X^\sigma \cdot Z(G)/Z(G)$, and so the only strongly imprimitive maximal subgroups are (normalizers of) fixed points of Frobenius endomorphisms of positive-dimensional subgroups of $G$, which are known (see Section 4). Write $\mathcal{X}^\sigma$ for the set of proper, closed, positive-dimensional subgroups of $G$.

If $M$ is a $kG$-module for $G$ of type $E_7$, and $H$ is a finite subgroup of $G$, then $H$ is a blueprint for $M$ if there is a closed, positive-dimensional subgroup of $G$ stabilizing exactly the same subspaces of $M$ stabilized by $H$. If $H$ is a blueprint for a module $M$, then either $H$ is strongly imprimitive or it stabilizes the same subspaces of $M$ as $G$ itself [10, Corollary 3.3].

Our notation for modules and their structures is as in previous papers in this series [9, 10, 12]. Let $H$ be a finite group. We will generally label simple $kH$-modules by their dimension, for example 10 for a 10-dimensional module. If there are two 10-dimensional modules that are dual to one another, write 10 and $10^*$ for the two modules, and if they are self-dual write $10_1$ and $10_2$. If $M$ is a $kH$-module, write $P(M)$ for its projective cover. We use ‘/’ to distinguish between socle layers of a module, so that a module $1, 5/10$ has socle 1 and second socle $1 \oplus 5$. If $L$ is a subgroup of $H$, write $M|_L$ for the restriction of the $kH$-module $M$ to $L$. Write $S^i(M)$ and $A^i(M)$ for the $i$th symmetric and exterior power of $M$ respectively.

For algebraic groups we use the ‘standard’ notation for simple highest-weight modules $L(\lambda)$ for $\lambda$ a dominant weight. The labelling of such modules follows [4], and is the one used by the references in our paper, and those in previous papers in this series. We often abbreviate $L(\lambda)$ to just $\lambda$, particularly if we have a product of simple groups, so $(10, \lambda_1)$ is the label for the product of the natural modules of $A_2A_5$, for example. The composition factors of maximal reductive subgroups and Levi subgroups of exceptional algebraic groups on $M(G)$ and $L(G)$ can be found in [38], for instance.

The possible sets of composition factors for a simple group $H$, not of Lie type in characteristic $p$, on the modules $M(E_7)$ and $L(E_7)$, are collated in [29, Section 6.3]. One of the main results of [29] states that, unless a row of a table is labelled with ‘P’ (the first is in Table 6.114), every subgroup with those composition factors must be strongly imprimitive. This yields a significant reduction in the number of cases that need to be considered in this paper.

If $H$ is a subgroup of $G$ such that $N_G(H)$ is not maximal, then it could be that $N_G(H)$ is maximal in $\tilde{G}$, where $\tilde{H}$ is the image of $H$ in $\tilde{G}$; such maximal subgroups are called novelty maximal subgroups. Denoting by $A$ the set of outer automorphisms induced by $\tilde{G}$ on $G$, $\tilde{H}$ is a novelty maximal subgroup if $H$ is $A$-stable but no intermediate group $H < X < G$ is $A$-stable. A thorough description of such maximal subgroups is given in [12, Section 3.6], but we do not need this as, apart from $M_{22}$, all subgroups are either strongly imprimitive or produce ‘standard’ maximal subgroups.
Proposition 2.3. Let use the result for $e_A$ subscheme

Proof. Dimension of an abelian subalgebra is given in Table 2.1. Suppose that $p$ is not an embedding with the reduction modulo $H$ is an almost simple subgroup purely to avoid complications with normal primitive subgroups becoming Lie imprimitive. But by a result of Serre [35, Proposition 8] at least there is of classes stays the same; there are examples of two Lie primitive classes becoming one, and also of Lie primitive subgroups isomorphic to $H$ with two results: the first, by Larsen [15, Theorem A.12], establishes that the number of conjugacy classes is independent of the characteristic $p$, and also that $H$ is an almost simple subgroup purely to avoid complications with normal $p$-subgroups, and it is no loss for us as all of our subgroups are almost simple.

The next result is a generalization of an old result of Mal’cev [30] to all primes due to Friedrich Knop.

We only need the case of $e_7$ here, but it is of independent and general interest; for example, the author will use the result for $e_8$ in a future paper.

Proposition 2.2. Let $G_0$ be the algebraic group defined over $C$ of the same type as $G$. If $H$ is a subgroup of $G_0$ such that $H \cdot Z(G_0)/Z(G_0)$ is almost simple, embedding with character $\chi_1, \ldots, \chi_n$ on $p$-restricted highest weight modules $L_1, \ldots, L_n$ respectively, then there exists a subgroup of $G$ isomorphic to $H$, and with Brauer character the reduction modulo $p$ of $\chi_i$, for each corresponding $L_i$.

The next result is a generalization of an old result of Mal’cev [30] to all primes due to Friedrich Knop.

Proposition 2.3. Let $G$ be a simple exceptional algebraic group in characteristic $p$, and let $g$ be its Lie algebra. Suppose that $p \neq 2$ for $G$ of types $B_n$, $C_n$ and $F_4$, and $p \neq 2, 3$ for $G$ of type $G_2$. The maximal dimension of an abelian subalgebra is given in Table 2.1.

Proof. In the Grassmannian of $g$, the condition of being abelian is a closed condition. Thus we have a closed subscheme $A_d \subset Gr_d(g)$ classifying all abelian subalgebras of a given dimension $d$, and $G$ acts on $A_d$ by conjugation. Let $d$ be such that $A_d$ is non-empty. As $A_d$ is a projective variety, any Borel subgroup $B$ of $G$ has a fixed point on $A_d$. Thus, if there exists an abelian subalgebra of $g$ of dimension $d$ then there exists a

This was posted in response to a question asked by the author on the MathOverflow website.

| Lie algebra | Dimension | Lie algebra | Dimension |
|-------------|-----------|-------------|-----------|
| $a_n$ ($n \geq 1$) | $((n+1)^2/4) + \epsilon_n$ | $\mathfrak{g}_2$ | 3 |
| $b_n$ ($n \geq 4$) | $n(n-1)/2 + 1$ | $\mathfrak{f}_4$ | 9 |
| $b_3$ | 5 | $\mathfrak{e}_6$ | $16 + \delta_{p,3}$ |
| $c_n$ ($n \geq 2$) | $n(n+1)/2$ | $\mathfrak{e}_7$ | $27 + \delta_{p,2}$ |
| $d_n$ ($n \geq 4$) | $n(n-1)/2 + 2\delta_{p,2}$ | $\mathfrak{e}_8$ | 36 |

Table 2.1: Maximal dimensions of abelian subalgebras of simple Lie algebras in characteristic $p$. Here $\epsilon_n = 0$ if $p \mid (n+1)$ and $\epsilon_n = 1$ if $p \mid (n+1)$. Also, $p \neq 2$ for types $b_n$, $c_n$ and $f_4$, and $p \neq 2, 3$ for $g_2$.

2.2 Results about algebraic groups

This section contains various results about algebraic groups that will be of use in what follows. We start with two results: the first, by Larsen [15, Theorem A.12], establishes that the number of conjugacy classes of Lie primitive subgroups isomorphic to $H$ of an algebraic group $G$ is independent of the characteristic $p$ of $G$, at least so long as $p \geq |H|$. (This result is often referred to as ‘Larsen’s (0, $p$)-correspondence’.)

Theorem 2.1. Let $H$ be a finite group such that $p \mid |H|$. If $G$ is a semisimple split group scheme, and $k$ and $k'$ are algebraically closed fields of characteristic $p$ and 0 respectively, the number of $G(k)$- and $G(k')$-conjugacy classes of subgroups isomorphic to $H$ are the same finite quantity.

The second extends this in one direction to all $p$, including $p \mid |H|$. It is no longer true that the number of classes stays the same; there are examples of two Lie primitive classes becoming one, and also of Lie primitive subgroups becoming Lie imprimitive. But by a result of Serre [35, Proposition 8] at least there is always an embedding with the reduction modulo $p$ of the character from characteristic 0. We require that $H$ is an almost simple subgroup purely to avoid complications with normal $p$-subgroups, and it is no loss for us as all of our subgroups are almost simple.

Proposition 2.2. Let $G_0$ be the algebraic group defined over $C$ of the same type as $G$. If $H$ is a subgroup of $G_0$ such that $H \cdot Z(G_0)/Z(G_0)$ is almost simple, embedding with character $\chi_1, \ldots, \chi_n$ on $p$-restricted highest weight modules $L_1, \ldots, L_n$ respectively, then there exists a subgroup of $G$ isomorphic to $H$, and with Brauer character the reduction modulo $p$ of $\chi_i$, for each corresponding $L_i$.

The next result is a generalization of an old result of Mal’cev [30] to all primes due to Friedrich Knop. We only need the case of $e_7$ here, but it is of independent and general interest; for example, the author will use the result for $e_8$ in a future paper.

Proposition 2.3. Let $G$ be a simple exceptional algebraic group in characteristic $p$, and let $g$ be its Lie algebra. Suppose that $p \neq 2$ for $G$ of types $B_n$, $C_n$ and $F_4$, and $p \neq 2, 3$ for $G$ of type $G_2$. The maximal dimension of an abelian subalgebra is given in Table 2.1.
B-stable such subalgebra \( a \). If \( T \) is a maximal torus inside \( B \), then \( a \) is also \( T \)-stable. Thus \( a \) has a root space decomposition

\[
a = a_0 \oplus \bigoplus_{\alpha \in R} g_{\alpha},
\]

where \( R \) is a set of roots and \( a_0 \) is a subalgebra of the corresponding maximal torus \( t \).

Since \( a_0 \) is the set of semisimple elements in \( a \), it is normalized by \( B \), so consists of \( B \)-fixed, and thus \( G \)-fixed, points. Thus \( a_0 \) is central, and is the contribution of \( \delta_p \), and so on to the entries in Table 2.1.

Note that, for all \( \alpha, \beta \in R \), we have that \([g_\alpha, g_\beta] = 0\), since \( a \) is abelian. If \( \alpha + \beta \) is a root then, for Chevalley generators \( e_\alpha, e_\beta \), we have that

\[
[e_\alpha, e_\beta] = \pm N_{\alpha \beta} e_{\alpha + \beta},
\]

and this is non-zero because of the restrictions on \( p \). Thus \( \alpha + \beta \) cannot be a root for all \( \alpha, \beta \in R \). If \( \alpha + \beta = 0 \) then for \( p > 2 \) we have that \([e_\alpha, e_{-\alpha}] = h_\alpha \neq 0\), another contradiction. Thus for \( p \) odd, \( d \) is the sum of a maximal set of roots the sum of any two of which is neither a root nor zero, which is clearly independent of \( p \).

If \( p = 2 \) then our condition on \( G \) means that the Dynkin diagram is simply laced, and therefore the root spaces have dimension exactly 2 (spanned by \( \alpha \) and ‘\(-\alpha\)’, which is of course equal to \( \alpha \) in characteristic 2). We cannot have that \( g_\alpha \) has dimension 2 though, since the 2-dimensional root space is not abelian (it commutates into a Cartan subalgebra), and thus \( g_\alpha \) is 1-dimensional in this case as well, spanned by a root. Hence \(|R|\) is bounded by the same value as for \( p \neq 2 \), and the proof is complete.

The next result comes from [11, Lemma 1.12], and is an improvement on previous results in the literature. One should be able to remove \( \text{PSU}_3(3) \) from this as a result of this paper, but we do not need to here.

**Lemma 2.4.** Let \( G \) be an exceptional algebraic group, and let \( H \) be a subgroup of \( G \) such that \( H \cdot Z(G)/Z(G) \) is simple. Either \( H \) is Lie primitive, \( H \) is contained in a proper, closed, connected, positive-dimensional subgroup of \( G \), or \( G = E_7 \), and either \( H \cong \text{PSU}_3(3) \), or \( H \cong \text{PSL}_2(8) \) and \( p = 2 \).

### 2.3 Stabilizing subspaces

We begin with [34] (1.3).

**Lemma 2.5.** Let \( H \) be a subgroup of \( G \) and suppose that \( H \) stabilizes a line on \( L(E_7)^{\circ} \). Then \( H \) is contained in a parabolic or maximal-rank subgroup.

This does not tell us exactly what we want to know though, because an almost simple group need not stabilize a line if the simple subgroup does. This lemma is [9] Propositions 4.5 and 4.6, specialized to the case \( G = E_7 \).

**Lemma 2.6.** Suppose that \( H \) is a finite subgroup of \( G \) possessing no subgroups of index 2. If \( H \) centralizes a line on either \( M(E_7) \) or \( L(E_7)^{\circ} \), then \( H \) is strongly imprimitive.
(Of course, if $H$ is perfect then $H$ has no subgroups of index 2, and any line stabilized by $H$ is centralized by it.) The next result is a little complicated in full generality, because it has to include both graph automorphisms and Steinberg endomorphisms, not just Frobenius endomorphisms. For $E_7$ it is much simpler. This may be found in [25, Proposition 1.12], and in various guises in [9, 10, 12, 29].

Proposition 2.7. Let $H$ be a subgroup of $G$. Suppose that $W$ is a subspace of $M(E_7)$ or $L(E_7)^\circ$ stabilized by $H$. If the intersection of the stabilizers of all subspaces in the $N_{\text{Aut}^+G}(H)$-orbit of $W$ is positive dimensional, then $H$ is strongly imprimitive.

Lemma 2.8 (See, for example, [31, Theorem 21.11]). Let $G$ be a connected linear algebraic group with a Steinberg endomorphism $\sigma$, and write $G = G^\sigma$. Let $X$ be a non-empty set such that $G$ and $\sigma$ act on $X$, and such that $(xg\sigma) = (x\sigma)(g\sigma)$ for all $x \in X$ and $g \in G$. Then $X^\sigma = \emptyset$, i.e., there are $\sigma$-fixed points in $X$.

Furthermore, if $G$ is transitive on $X$ and for some $x \in X$ the stabilizer $G_x$ of $x$ is closed, the number of $G$-orbits on the $\sigma$-fixed points of $X$ are in one-to-one correspondence with $\sigma$-conjugacy classes on $G_x/G_x^\sigma$. In particular, if $G_x$ is connected then all $\sigma$-fixed points of $X$ are $G$-conjugate.

The next lemma can be found in, for example, [36, Lemma 3.2.15].

Lemma 2.9. Let $G$ be a connected linear algebraic group, let $P$ be a maximal parabolic subgroup of $G$. Let $U$ be the unipotent radical of $P$ and let $LT$ be a Levi complement to $U$ in $P$, where $[L, L] = L$ and $T$ is toral. Suppose that $U$ forms a $kL$-module.

Let $H$ be a subgroup of $L$. If

$$\dim_k(H^1(H, U)) = 1$$

then there are exactly two $T$-classes of complements to $U$ in $U.H$, one of which lies in $L$.

Specializing to the case where $G$ has type $C_n$ and $L$ has type $A_{n-1}$ yields the following corollary.

Corollary 2.10. Let $G = \text{Sp}_{2n}(k)$, and let $H$ be a perfect subgroup of $G$. Suppose that $H$ stabilizes a subspace $W$ of dimension $n$ but no complement to $W$. If the $kH$-module $S^2(W)$ has 1-dimensional 1-cohomology then any other subgroup $L$ of $G$ that is isomorphic to $H$, and such that $M(G)_L$ and $M(G)_H$ are isomorphic, is $G$-conjugate to $H$. In other words, $H$ is unique up to $G$-conjugacy.

Proof. Since $H$ does not stabilize a complement to $W$, $W$ cannot be non-singular (as then $W^H$ intersects $W$ trivially and also has dimension $n$). Thus $H$ (and the group $L$) lies in an $A_{n-1}$-parabolic subgroup $X$. As all such subspaces are in the same orbit, we may assume that $L$ also stabilizes $W$ (and acts in the same way on $W$).

To apply Lemma 2.9, we need that the $A_{n-1}$ acts on the unipotent radical of $X$ as the symmetric square of $W$. Since $L(C_n)$ is the symmetric square of $M(C_n)$, clearly $L(C_n)_{A_{n-1}}$ has factors $S^2(M(A_{n-1}))$ and its dual, and the tensor product $M(A_{n-1}) \otimes M(A_{n-1})^*$, the Lie algebra of $A_{n-1}T_1$. Thus the unipotent radical has $kH$-module structure either $S^2(M(A_{n-1}))$ or its dual.

To see which one, we can either perform a calculation, or can simply use the computation from Proposition 5.11 which shows that it is the submodule we need to take 1-cohomology of, rather than the quotient module. This completes the proof.

Lemma 2.11 (See, for example, [31, Corollary 21.8]). Let $\sigma$ be a Steinberg endomorphism of a connected group $G$. If $g \in G$ then $\sigma$ and $g\sigma$ are $G$-conjugate. In other words, all outer automorphisms of $G$ corresponding to the same element of $\text{Out}(G)$ as $\sigma$ are conjugate. Consequently, if $\sigma$ acts as an inner automorphism of the subgroup $H$ then there exists a $G$-conjugate of $H$ centralized by $\sigma$. 

8
The following is well known, and may be found, for example, in either \[22, \text{Lemma 4.3}\] or \[8\].

**Proposition 2.12.** Let \(G\) be simply connected \(E_7\), and let \(W\) be a 1-space in \(M(E_7)\). The stabilizer of \(W\) is one of the following:

(i) an \(E_6T_1\)-parabolic subgroup;

(ii) the normalizer of an \(E_6\)-Levi subgroup, \(E_6T_1, 2\);

(iii) a subgroup \(U_{26} \cdot F_4T_1\), where \(U_{26}\) is a unipotent subgroup of dimension 26;

(iv) a subgroup \(U_{33} \cdot B_3T_1\), where \(U_{33}\) is a unipotent subgroup of dimension 33.

### 2.4 Action of outer automorphisms

In this section we will give some general results on how to understand the action of outer automorphisms of the simple groups \(G/Z(G)\) on subgroups \(H \cdot Z(G)/Z(G)\). Because \(G\) has type \(E_7\), we are interested in field automorphisms \(F_p\) and diagonal automorphisms for \(p\) odd.

Let \(\sigma = F_p\) be the generator for the semigroup of field endomorphisms. Suppose that \(H\) is a simple subgroup of \(G\) modulo \(Z(G)\). First, \(\sigma\) might stabilize the \(G\)-conjugacy class containing \(H\) or it might map it to another class.

In all of our cases, if \(H_1\) has the same actions on \(M(E_7)\) and \(L(E_7)\) as \(H\), then \(H\) and \(H_1\) are \(G\)-conjugate. As this is the case, one may determine whether \(\sigma\) stabilizes the class of \(H\) simply by examining the character of \(M(E_7)\) and \(L(E_7)\). As \(F_p\) raises each eigenvalue to the \(p\)th power, this determines the character of \(H^\sigma\), and therefore the class to which it belongs. If there is more than one class with the same character, it would be much harder to determine whether \(\sigma\) stabilizes the class of \(H\).

By replacing \(\sigma\) by a power if necessary, we may now assume that \(\sigma\) stabilizes the \(G\)-class containing \(H\). Replacing \(H\) by a \(G\)-conjugate if necessary, we may assume that \(\sigma\) normalizes \(H\), via Lemma \[2.11\]. We now must determine the action of \(\sigma\) on \(H\), and in particular if it acts as an element of \(N_G(H)\). If this is the case, then \(\sigma\) acts trivially on some other conjugate of \(N_G(H)\) by Lemma \[2.11\] and therefore a conjugate of \(H\) lies in \(G\).

Generally the outer automorphism groups of our subgroups \(H\) are fairly small, since \(H\) is simple and of fairly low order, so this is not too difficult. If the characters of \(H\) on \(M(E_7)\) and \(L(E_7)\) are not stable under an automorphism \(\phi\) of \(H\), then in order for \(\sigma\) to induce \(\phi\) on \(H\), raising eigenvalues to that power must have the effect of \(\phi\) on the characters, just as in the analysis above on swapping \(G\)-classes.

If \(\sigma\) has no effect on the characters of \(M(E_7)\) and \(L(E_7)\), and neither does \(\phi\), then we will need a different method to determine the action of \(\sigma\) on \(H\), and that will have to be ad hoc.

Diagonal automorphisms are generally much more difficult to understand. The next lemma gives us some control in many cases. The remaining issue is when \(p\) is odd and \(N_G(H)/H \cdot Z(G)\) contains an involution, which could be the diagonal automorphism of \(G\).

**Lemma 2.13.** Let \(H\) be a subgroup of \(G\) such that \(C_G(H) = Z(G)\), and with \(H \cdot Z(G)/Z(G)\) simple.

(i) If \(H \leq G\) then \(N_G(H)\) is contained in \(N_G(G) = G_{ad} \cdot Z(G)\).

(ii) The number of \(G\)-classes in the \(G_{ad}\)-class of \(H\) is \(|G_{ad} : G|/|N_G(H) : N_G(H)|\).

In particular, if \(N_G(H) \leq G \cdot Z(G)\) (for example, if \(|N_G(H) : H \cdot Z(G)|\) is prime to \(|G_{ad} : G|\)) then a diagonal automorphism of order \(d\) of \(G\) fused \(d\) distinct \(G\)-classes of subgroups \(H\).
Proof. Since the image of $H$ in the adjoint group is simple, and by a result of Burnside the automorphism group of a simple group is complete, we see that any automorphism of $N_G(H)/Z(G)$ that centralizes the image of $H$—for example, $F_q$—must centralize all of $N_G(H)/Z(G)$. Thus in the adjoint group, $N_G(H)$ is centralized by $F_q$, and so $N_G(H) \leq G_{ad} : Z(G)$, proving the first part.

The second part and the consequence are clear. □

In the remaining case, where $N_G(H)/H \cdot Z(G)$ contains an involution, $N_G(H)$ might lie in $G \cdot Z(G)$ or it might only lie in $G_{ad}$. If the action of $N_G(H)$ on $M(E_7)$ is not definable over $F_q$ then $N_G(H)$ cannot embed in $2 \cdot E_7(q)$, of course, and this is often enough to conclude that the normalizer does not lie inside $G$ (if it does not). Other situations must be dealt with on a case-by-case basis (or in the case of $PGL_3(q)$ not dealt with at all).

2.5 Alternating forms on vector spaces

Fix a non-degenerate, alternating bilinear form $f(-, -)$ on a vector space $V$, and let $X = \text{Sp}(V)$ with respect to this form. Let $H$ be a subgroup of $X$. If $f'(-, -)$ is another $H$-invariant symplectic form on $V$, then there exists $g \in \text{GL}(V)$ such that, if $Y$ is the symmetry group of $f'$, then $Y^g = X$, i.e., $(f')^g = f$. Furthermore, since $X$ is irreducible on $V$, if $g'$ is another such element then there exists $n \in X \cdot Z(\text{GL}(V))$ such that $g' = gn$. Hence, the subgroup $H^g$ of $X$ is determined up to $X$-conjugacy. Thus to every (non-degenerate) $H$-invariant symplectic form on $V$ we may associate a subgroup of $\text{Sp}(V)$, unique up to conjugacy. We see that the number of $X$-conjugacy classes in the image of this function is equal to the number of $\text{Sp}(V)$-conjugacy classes of subgroups isomorphic to $H$ and that lie in the same $\text{GL}(V)$-conjugacy class. In particular, $H$ is ‘unique up to $\text{Sp}(V)$-conjugacy’ (given an action on $V$) if, given $f'$, we may choose $g$ to lie in $C_{\text{GL}(V)}(H)$.

In attempting to understand the $\text{Sp}(V)$-conjugacy classes of a subgroup $H$, we start with a proposition that uses the above observations to split $V$ up into various $H$-invariant summands.

Proposition 2.14. Let $H$ be a finite group and let $V \cong V_1 \oplus V_2$ be a faithful $kH$-module. If $\phi_i$ is an alternating bilinear form on $V_i$ with radical zero, then all subgroups $H$ of $\text{Sp}(V)$ corresponding to the forms $a\phi_1 + b\phi_2$ ($a, b \in k$) are conjugate.

Proof. The subgroup $H$ lies in the direct product of $\text{Sp}(V_1)$ and $\text{Sp}(V_2)$. The scalar matrices on $V_1$ and $V_2$ act transitively on the set of forms $a\phi_1 + b\phi_2$, and these scalars centralize $H$. Thus the corresponding subgroups of $\text{Sp}(V)$ are conjugate. □

Thus if an $H$-invariant alternating form on $V = V_1 \oplus V_2$ (each $V_i$ is $H$-invariant) can always be written as a sum of two forms, one on each $V_i$, then this proposition applies. For example, if the $V_i$ have no composition factors in common as $kH$-modules then this must hold.

This allows us to produce the following proposition, which is often enough for us.

Proposition 2.15. Let $H$ be a finite group and let $V$ be a faithful, semisimple $kH$-module. Suppose that, for all composition factors $W$ of $V$,

(i) if $W$ is not self-dual then both $W$ and $W^*$ have multiplicity exactly 1 in $V$;

(ii) if $W$ is self-dual and supports a symmetric form then $W$ appears with multiplicity exactly 2 in $V$; and

(iii) if $W$ is self-dual and supports an alternating form then $W$ appears with multiplicity 1 in $V$. 

10
Then up to conjugacy we may assume that $H \leq \text{Sp}(V)$, and if $H_1 \leq \text{Sp}(V)$ is $\text{GL}(V)$-conjugate to $H$ then it is $\text{Sp}(V)$-conjugate to $H$. In other words, there is a unique conjugacy class of $H$ in the symplectic group acting on the natural module as $V$.

**Proof.** First, we note that $H$ may be conjugated into $\text{Sp}(V)$, because each homogeneous component (where we group a module and its dual into a component) supports a unique symplectic form with zero radical.

To prove uniqueness of $H$, by the previous paragraph the result holds if $V$ is homogeneous, so we may use induction and assume that $V = V_1 \oplus V_2$ and each $V_i$ has a unique symplectic form $f_i$ with zero radical (up to $\text{Sp}(V_i)$-conjugacy). Then we apply Proposition 2.14.

3 Two algorithms for determining conjugacy classes

The two main algorithmic methods in this article both involve the Lie product on $L(G)$, and one involves the minimal module as well. We start with the easiest.

### 3.1 Lie product method

Let $H$ be a finite group, let $M$ be a $kH$-module of dimension $\dim(L(G))$, and suppose that we wish to determine whether $H$ embeds in $G$, acting on $L(G)$ as the module $M$, and if so, how many $G$-classes of subgroups have such an action.

The Lie product is a non-zero element of the space $\text{Hom}_{kG}(\Lambda^2(L(G)), L(G))$, so we construct the space

$$R^H(M) = \text{Hom}_{kH}(\Lambda^2(M), M),$$

which is $n$-dimensional. We then take a basis $\phi_1, \ldots, \phi_n$ for this space, and let

$$\phi = \sum_{i=1}^n a_i \phi_i,$$

where the $a_i$ are indeterminates from $k$. The element $\phi$ is a generic alternating product on $M$.

We impose the requirement that $\phi$ is a Lie product, i.e., it satisfies the Jacobi identity. This requirement yields a collection of quadratic equations in the $a_i$,

$$((u \wedge v)\phi \wedge w)\phi + ((v \wedge w)\phi \wedge u)\phi + ((w \wedge u)\phi \wedge v)\phi = 0,$$

for $u, v, w$ basis elements of $M$.

Solving these equations yields the collection $\mathcal{L}$ of all Lie products on $M$. There may well be many elements of $\mathcal{L}$ that are not simple Lie algebras, for example the zero product, or more generally any with a kernel, or whose image is not the whole of $M$. Removing these, and for example any that have an abelian subalgebra of too large a dimension (see Proposition 2.3), we end up with a collection of candidates for being a simple Lie algebra of type $L(G)$. Finally, we need to take orbits under the action of the normalizer in $\text{GL}(M)$ of $H$. For each of these orbits, we simply test on Magma whether the Lie algebra is simple, and then that it is $L(G)$.

This method was used successfully in [12] to determine the number of conjugacy classes of various Lie primitive subgroups for $E_6$, $E_7$ and $E_8$, and we use it here, with much larger dimensions for the space of
alternating products, to deal with some of the larger subgroups. It has the benefit that one does not need to worry about whether $H$ is a non-split extension by the centre of $G$.

If $R^H(M)$ has dimension more than about 100, the number of variables becomes too large to practically determine the subset of Lie products, and so we use a different method.

### 3.2 Subalgebra method

This method is more powerful than the previous one, but requires putative actions $M$ and $L$ on both the minimal and adjoint modules. It also requires an alternating bilinear form on $M$ that yields an embedding of $H$ into $\text{Sp}_{56}(k)$. For this reason, the previous method is preferred in cases where the space of alternating forms is small, because the setup is less complicated. This method, due to Alexander Ryba, appeared in [16], where it was used to compute conjugacy classes of $\text{PSL}_2(27)$ in $E_7(k)$ for $p \nmid |\text{PSL}_2(27)|$. We use it with a few modifications compared with [16] in order to increase its utility in cases where the prime divides the order of the subgroup.

For all primes $p$, the group $G$ of type $E_7$ is contained in the algebraic group $X$ of type $C_{28}$, i.e., $\text{Sp}_{56}(k)$ contains $E_7(k)_\text{sc}$. Thus $L(E_7)$ is a Lie subalgebra of $L(C_{28})$. Since $L(C_{28})$ is the symmetric square of $M(C_{28})$, we place a Lie algebra structure directly on the symmetric square of $M(C_{28})$. For practical purposes when computing we place the Lie algebra structure on $M(C_{28}) \otimes M(C_{28})$, given by

$$[u \otimes v, x \otimes y] = (u, y)x \otimes v - (x, v)u \otimes y$$

(where $u, v, x, y \in M(C_{28})$ and $(, )$ is the bilinear form) and then consider the symmetric part of the tensor square.

Let $H$ be a putative subgroup of $G$, and assume that we are given 56- and 133-dimensional $kH$-modules $M$ and $L$. We will count the number of $G$-conjugacy classes of subgroups $H$ embedding with actions $M$ and $L$ on $M(E_7)$ and $L(E_7)$ respectively.

Suppose first that $M$ supports a unique (up to $\text{Sp}_{56}$-conjugacy) non-degenerate symplectic form. This yields a unique embedding of $H$ into $C_{28}$, and thus $H$ acts on the Lie algebra $L(C_{28}) = S^2(M)$. Let $U$ denote the submodule of $S^2(M)$ that is the sum of the images of all maps in Hom$_{kH}(L, S^2(M))$. If $H \leq G \leq C_{28}$ then the Lie algebra of $G$ lies in the subspace $U$ of $S^2(M)$.

Let $W$ be a simple submodule of $L$, let $\phi_1, \ldots, \phi_n$ denote a basis for Hom$_{kH}(W, U)$, and let $W_i$ denote the image of $\phi_i$. If $w \in W$, and $a_1, \ldots, a_n$ are parameters with values in $k$, write

$$\bar{w} : W \to U, \quad \bar{w} = \sum_{i=1}^n a_i(w\phi_i) \in U.$$  

If $\bar{u}$ and $\bar{w}$ lie in $L(G)$ for $u, w \in W$ then, in particular, $[\bar{u}, \bar{v}] \in U$. By extending a basis for $U$ to a basis for $S^2(M)$, the coefficients of $[\bar{u}, \bar{v}]$ on basis elements outside of $U$, which must be 0, yield quadratic equations in the $a_i$. Of course, choosing three points $u, v, w \in W$, one may obtain cubic equations via $[[\bar{u}, \bar{v}], \bar{w}]$, and so on. These restrictions yield a smaller set of potential subalgebras, and (hopefully) allow us to determine which copies of $W$ in $S^2(M)$ lie in $L(G)$ precisely. We then take the subalgebra generated by each option for $W$, and if $H$ does not preserve any proper subalgebra of $L(G)$ then the subalgebra so generated must be $L(G)$ itself, and we can test this via computer. Running through all possibilities for the image of $W$ up to $N_{C_{28}}(H)$-conjugacy yields all $G$-conjugacy classes of subgroups $H$.

This is the basic idea, but there are many modifications that can be made. First, $W$ might not be enough, and we can allow more than one simple submodule to be chosen simultaneously. If there is a 1-parameter of
symplectic forms then one may still compute \([\bar{u}, \bar{w}]\), but it will come with an extra parameter that must be solved for. (This is the case for \(\text{PSL}_2(19)\), \(p = 2\).) If \(p = 2\) then \(S^2(M)\) is reducible, and we may consider \(\Lambda^2(M)\) instead of \(S^2(M)\), thought of as a submodule of \(M \otimes M^*\), and hence inheriting its Lie product. If even one parameter for the symplectic form is not enough, we might perform the same analysis in the Lie algebra of \(A_{55}\) instead, so work with \(M \otimes M\) rather than \(S^2(M)\). (We demonstrate this method in the supplementary materials for \(\text{PSL}_2(19)\) and \(p = 2\), providing two proofs for that case.)

Having applied the method, probably to multiple submodules, one obtains a number of different possibilities for submodules \(\bar{W}\) of \(L(G)\), up to the action of \(N_{C_{28}}(H)\). For each of these we take the subalgebra generated by \(\bar{W}\). If it has dimension greater than 133, then \(\bar{W}\) cannot lie in \(L(G)\), by Proposition 2.3. One also knows all of the actions of nilpotent classes of the Lie algebra of \(G\) on \(M [37, Tables 2 and 3]\) (they differ slightly from those from [20] of unipotent classes for \(E_7\) and \(p \leq 5\)), and hence on \(S^2(M)\), so if the subalgebra contains a nilpotent element of \(L(C_{28})\) that has the wrong Jordan normal form on \(S^2(M)\), then \(\bar{W}\) cannot lie in \(L(G)\). (We use this for \(\text{Alt}(7)\), \(p = 5\), \(\text{PSL}_2(11)\), \(p = 3\), and \(\text{PSL}_2(13)\), \(p = 2\).) If the subalgebra generated is not a subalgebra of \(\tau_7\), then \(\bar{W}\) cannot lie in \(L(G)\).

For smaller examples \(H\), the process becomes increasingly difficult. One useful tool is a linear matrix pencil problem. Given nilpotent matrices \(A\) and \(B\), such that \(A + xB\) is nilpotent for all \(x \in k\), we want to compute the Jordan normal form of \(A + xB\), both generically and for the finitely many values of \(x\) for which the normal form is different. Assuming these actions are not correct, this allows us to eliminate 1-parameter possibilities for \(\bar{W}\). We do this for, for example, \(\text{PSL}_2(11)\).

These various tests are enough to eliminate almost all possibilities other than submodules \(\bar{W}\) that generate a copy of \(\tau_7\). In some cases, for example \(\text{Alt}(7)\) for \(p = 5\), we always show that the submodule generates a \(\mathfrak{d}_4\) subalgebra. At this stage we may apply a recent result of Premet and Stewart [33], which classifies maximal subalgebras of exceptional Lie algebras in good characteristic. In the few cases where we need to do this we will describe exactly what occurs in the relevant section.

We now give an example using this method, which in Section 5.2.3 is solved using the Lie product method. It will offer another proof of the result, and be small enough to illustrate the technique.

**Example 3.1.** Let \(H = 2 \cdot M_{12}\) and \(p = 5\). Examining [29, Table 6.148], we see there is a possible embedding of \(H\) into \(G\) with composition factors on \(M(E_7)\) and \(L(E_7)\) given by 32, 12 and 78, 55 respectively (the character 55 is the one of the three with Brauer character value 7 for a non-central involution). This is labelled ‘\(P\)’ in that table, and would be Lie primitive if it exists.

The module 12 has no self-extensions, but there is an extension between 12 and 32, yielding a self-dual module 12/32/12. The actions of \(u \in H\) of order 5 on 32 \(\oplus 12^{52}\) and 12/32/12 are 510, 16 and 510, 32. Both of these actions are allowable according to [20, Table 7], they of course correspond to different nilpotent classes. The action on \(L(E_7)\) is clearly fixed though as 78 \(\oplus 55\), and \(u\) acts on this module with Jordan blocks 526, 3. Consulting [20, Tables 7 and 8], we see that \(u\) belongs to class \(A_4 + A_2\) and \(H\) must act uniserially on \(M(E_7)\). A computer check confirms that the module 12/32/12 has a unique symplectic form up to scalar multiplication, and so we are in a good position to apply the subalgebra method. Write \(M\) and \(L\) for the \(kH\)-modules 12/32/12 and 55, \(\oplus 78\) respectively.

We first determine the normalizer of \(H\) in \(\text{Sp}_{56}(k)\). In this case it is quite easy to check that \(H\) is centralized by a 1-dimensional unipotent subgroup \(X\) (and the centre of \(\text{Sp}_{56}(k)\)). In addition, \(H\) is normalized by an element of order 2 acting as an outer automorphism on \(H\). This will either swap two copies of \(L(E_7)\) inside \(S^2(M)\) or stabilize one, and this corresponds to \(H\) and \(H.2\) embedding in \(G\) respectively.
We now consider $S = S^2(M)$. The space $\text{Hom}_{kH}(55, S)$ is 4-dimensional, and the space $\text{Hom}_{kH}(78, S)$ is 2-dimensional. This means that $U$, the sum of the images of all maps from $L$ to $S$, has dimension $55 \cdot 4 + 78 \cdot 2 = 376$, whereas $S$ has dimension 1596. We can illustrate a number of our techniques here. First, $X$ acts on the set of 78-dimensional $kH$-submodules of $U$; there is a regular orbit, with representative $W_1$, and a fixed point, with representative $W_2$. Let $L_i$ denote the subalgebra generated by $W_i$. It turns out that both $L_i$ are isomorphic to $L$ as $kH$-modules; the subalgebra $W_1$ is of type $e_7$, but the second is a nilpotent Lie algebra with a 55-dimensional abelian ideal.

We can simply test whether $L_1$ is an $e_7$-Lie algebra, so it remains to eliminate the other one. We can do this in a number of ways: we can prove that the 553 forms an abelian subalgebra of $L_2$, thus $L_2$ is not $e_7$ by Proposition 2.3. We could also check that the 553 forms an ideal of the Lie algebra. Finally, we could find some nilpotent element of $S$ that lies in $W_2$ and has the wrong Jordan normal form. In the supplementary materials we demonstrate each of these proofs.

We also demonstrate an approach using $W = 553$ and $W' = 78$, using the action of $X$ on the set of possibilities for $W$. Here we therefore have six variables, $a_1, \ldots, a_4$ parametrizing the submodule $W$, and $a_5$ and $a_6$ parametrizing $W'$. The action of $X$ shows that we may assume that $a_2 = 0$, and then that $a_4 = 1$. This yields two solutions, one corresponding to $W_2$ and the other corresponding to $W_1$ (up to the action of $X$).

Since $W_1$ is unique up to the action of the centralizer $X$, we see that $H$ must extend to $H.2$ in $G$.

4 Reducing to almost simple groups

In this section we determine all maximal subgroups that arise from fixed points of a positive-dimensional subgroup. Apart from these, there is a finite list of other possibilities, each of which will be considered in turn in the next section.

We start with a result of Borovik and Liebeck–Seitz, which reduces us to almost simple groups.

Theorem 4.1 ([3, 24]). Let $M$ be a maximal subgroup of an almost simple group $G$ of exceptional type. One of the following holds:

(i) $M = N_G(X')$ for $X$ a member of $\mathcal{X}$, as listed in [24, Theorem 2];

(ii) $M$ is an exotic $r$-local subgroup, as listed in [3];

(iii) $G$ is of type $E_8$, $p \geq 5$, and $M = (\text{Alt}(5) \times \text{Alt}(6)) \rtimes 2^2$;

(iv) $M$ has the same type as $G$, and is the normalizer of the fixed points of a field or field-graph automorphism of $G$;

(v) $M$ is almost simple.

If $G$ is of type $E_7$ then the third option does not apply, and the second option is a subset of the first, as it happens. The fourth is simply (the normalizer of) $E_7(q_0)$ for $q = q_0^r$ with $r$ a prime. Note that if $M$ is almost simple but does not appear in (i)–(iv), then $C_G(M) = Z(G)$. To see this, notice that otherwise $M \leq C_G(C_G(M))$, and this latter group is proper in $G$. It is certainly $\sigma$-stable whenever $M$ is, and is infinite if $M$ is simple by [24, Lemma 1.2(i)] (as the remaining option there is the Borovik subgroup (iii) above). In particular, we can always apply Lemmas 2.13 and 2.14.
We will look at almost simple groups in the next section. Thus in this section we determine the list for
the first option, fixed points of \((\sigma\text{-stable})\) maximal members of \(\mathcal{X}\).

By [23, Theorem 2] these are either parabolic or maximal-rank subgroups, or a short list of other (possibly
disconnected) reductive groups. The maximal-rank subgroups are tabulated in [23], and we reproduce those
subgroups here. The parabolic subgroups are well known. The result from [24] lists all subgroups that are
not almost simple, and in [27, Corollary 5] a (likely complete) list of such maximal subgroups that are almost
simple is given.

Ignoring the maximal-rank, parabolic, and exotic \(r\)-local subgroups, this leaves only the maximal sub-
groups arising from reductive maximal subgroups, which for \(E_7\) are (from [27]):

\begin{enumerate}
  \item \(A_1F_4\) (all \(p\)), \(G_2C_3\) (all \(p\)), \(A_1G_2\) (\(p \geq 3\));
  \item \(A_1A_1\) (\(p \geq 5\));
  \item \(A_2.2\) (\(p \geq 5\));
  \item \(A_1\) (\(p \geq 17\));
  \item \(A_1\) (\(p \geq 19\));
  \item \((2^2 \times D_4) \rtimes \text{Sym}(3)\) (\(p \geq 3\)).
\end{enumerate}

(The final one is disconnected, and the connected component lies in \(A_7\).) In [27, Corollary 5], it is also
stated that any maximal subgroup that is (the normalizer of) a Lie type group in characteristic \(p\) appears in
the list above, except possibly for some small-rank groups defined over small fields. All of these small-rank
possibilities except for PSL\(_2\) were eliminated for \(E_7\) in [10], and all PSL\(_2(q_0)\) except for \(q_0 = 7, 8, 25\) were
eliminated in [21]. Since there is no maximal PSL\(_2(q_0)\) for \(p \leq 7\), we see that the list above is therefore a
complete list of the maximal subgroups of \(\hat{G}\) that can arise as the fixed points of a member of \(\mathcal{X}\).

The number of classes in \(G_{\text{ad}}\) is given in [24, 27], and so our issue is whether the diagonal automorphism of
\(G\) normalizes the subgroup or fuses two classes of them. The three subgroups in (i) are easy to understand:
they are all non-split extensions over the centre, and exactly one of the two factors has a single isogeny
type. Thus they are direct products and of simply connected type. This yields subgroups PSL\(_2(q) \times F_4(q),\)
\(G_2(q) \rtimes \text{PSp}_6(q)\) and PSL\(_2(q) \times G_2(q)\) of \(G\). Since they are not groups of Lie type themselves, they must be
normalized by a diagonal automorphism, and since they are unique up to \(G\)-conjugacy, they are stabilized by
all Frobenius endomorphisms. A similar statement holds for the two classes of \(A_1\) subgroups (which cannot
be interchanged by \(\sigma\) for many reasons, for example the unipotent classes intersecting the two subgroups are
different).

(If \(q = 2, 3\) then PSL\(_2(q)\) is soluble. In this case the groups above are \(r\)-local subgroups, where \(r = 3\) for
\(q = 2\), and \(r = 2\) for \(q = 3\). We consult [5] to see when these are maximal.)

The subgroup \(A_1A_1\) is of type \(\text{SL}_2 \times \text{PGL}_2\), by considering the composition factors on \(M(E_7)\), and so we cannot have a Frobenius endomorphism that swaps the two factors. Thus the only option is PSL\(_2(q) \times \text{PGL}_2(q)\) in \(E_7(q)\), and as with the other cases, there must be a single class normalized by the diagonal
automorphism.

For \(X = (2^2 \times D_4) \rtimes \text{Sym}(3)\), this can also be written as \(D_4 \rtimes \text{Sym}(4)\). We work first in the adjoint
version of \(G\), and then see whether the subgroup lies in the derived subgroup of \(G_{\text{ad}}\). Since \(\text{Sym}(4)\) has no
outer automorphisms, if \(X\) is \(\sigma\)-stable then \(\sigma\) induces an inner automorphism on \(\text{Sym}(4)\). There are five
conjugacy classes for \(\text{Sym}(4)\), hence five \(\sigma\)-conjugacy classes. The classes given by 1 and \((1,2)(3,4)\) yield
fixed-point subgroups $D_4(q) \ltimes \text{Sym}(4)$ and $D_4(q) \ltimes D_8$ respectively; the classes given by $(1, 2)$ and $(1, 2, 3, 4)$ both give subgroups $2D_4(q)$, and $(1, 2, 3)$ yields a fixed-point subgroup $2D_4(q)\cdot 3$. Since each of $(1, 2)(3, 4)$, $(1, 2)$ and $(1, 2, 3, 4)$ yield a fixed point on the $2^2$ subgroup in the centre of the fixed points, in these three cases the $\sigma$-fixed points lie in the centralizer of an involution, so a subgroup $A_7$ or $3A_7$. In particular, they cannot be maximal. Thus the only maximal subgroups are $(2^2 \times D_4(q)) \ltimes \text{Sym}(3)$ and $3D_4(q)\cdot 3$, neither of which can lie in any other maximal subgroup.

Of course the subgroup $3D_4(q)\cdot 3$ must lie in the derived subgroup, which has index 2 in $G_{ad}$, so it remains to consider $(2^2 \times D_4(q)) \ltimes \text{Sym}(3)$. But this is the exotic 2-local subgroup from [5]. It does not appear to be explicitly stated there that there is a single class for the simple group, just for $G_{ad}$. However, in [11, Table 1], for example, it is stated that $(2^2 \times D_4(q))\cdot 3$ lies in the simple group, hence there is a single class.

Finally, for the subgroups $A_2\cdot 2$, we see one class each of $\text{PGL}_3(q)\cdot 2$ and $\text{PGU}_3(q)\cdot 2$ in the adjoint group $G_{2}$ (see, for example, [23], Corollary 7 for this statement; the proof follows the same lines as the same subgroup of $E_6$ in [12]). The question is whether these groups lie inside $G$ or intersect $G$ in $\text{PGL}_3(q)$ or $\text{PGU}_3(q)$. Notice that, by Lemma 2.8, the number of conjugacy classes of $\text{PGL}_3(q)$ and $\text{PGU}_3(q)$ put together is equal to the number of $\sigma$-conjugacy classes on the cyclic group of order 4 (which is the normalizer of $A_2$ modulo $A_2$), and this number is either 2 or 4, according as $\sigma$ inverts or centralizes the 4. Hence, either both $\text{PGL}_3(q)$ and $\text{PGU}_3(q)$ are self-normalizing in $G$ or neither is, so we may as well consider just $\text{PGL}_3(q)$.

For $p = 5$, we obtain a different answer to $p \geq 7$, because the action on $M(E_7)$ is completely different. For $p \geq 7$, the action is a sum of two 28-dimensional modules, and the centralizer in $\text{Sp}(M(E_7))$ is a torus. For $p = 5$, the action is a tilting module, of the form $L(22)/L(10), L(01), L(20), L(02)/L(22)$. This has centralizer in $\text{Sp}(M(E_7))$ a unipotent subgroup of dimension 1.

Consequently, while the centralizer for $p \geq 7$ has order $q - 1$, the centralizer for $p = 5$ has order 2$q$ (with the 2 being the centre of $\text{Sp}_{56}(k)$). This means that there is an odd number of $\varepsilon_7$-subalgebras of the Lie algebra, and thus the outer 2 on top of $\text{PSL}_3(q)$ must stabilize at least one of them. Thus $\text{PSL}_3(q)\cdot 2$ must embed in $E_7(q)$, as stated in Table 4.1. (In the supplementary materials we prove this by computer for $q = 5$ to confirm, using the argument for $p \geq 7$ below to extend to all powers of 5.)

For $p \geq 7$, we cannot determine the answer when $q$ is an odd power of $p$, but we believe the answer is that there is a single class. We have checked this for $q = 7, 11, 13, 17$ using a computer (we include the case $q = 17$ in the supplementary materials for the reader’s benefit) but can see no way to prove it in general at the moment. We can say something though. Let $H = \text{PSL}_3(q)$ and let $L = \text{PSL}_4(p)$ be a subgroup of $H$. We have $H.2 = \langle H, L.2 \rangle$, and so $H.2$ lies in $G/Z(G)$ if and only if $L.2$ does. If $q$ is an even power of $p$, then $L.2 \leq E_7(p)\cdot 2 \leq E_7(q)$, and so $H.2$ does lie inside $G/Z(G)$. If $q$ is an odd power of $p$, then the outer diagonal automorphism of $E_7(q)$ restricts to the outer diagonal automorphism of $E_7(p)$, so $L.2$ lies in $E_7(p)$ if and only if $H.2$ lies in $E_7(q)$.

Thus there are definitely two classes of $\text{PSL}_3(q)$ for $q$ an even power of $p$, and the answer is always the same for $q$ any odd power of a fixed prime $p$.

5 Determination of the almost simple groups

We now turn our attention to all almost simple groups that embed in $E_7$ for some prime $p$, but that do not come from a positive-dimensional subgroup as in the previous section. We split them up into families, and proceed through alternating groups, sporadic groups, groups of Lie type other than $\text{PSL}_2(r)$, and finally the groups $\text{PSL}_2(r)$. In each case, and for each group $H$, we consider the possible embeddings of $H$ into $G = E_7$,
| Group                                                                 | $p$ | $q$ | No. classes | Stabilizer |
|----------------------------------------------------------------------|-----|-----|-------------|------------|
| $[q^{43}] \cdot d \cdot PΩ_{12}^+(q) \cdot (q - 1)$                  | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $[q^{42}] \cdot SL_7(q) \cdot (q - 1)/d$                             | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $[q^{47}] \cdot d \cdot (PSL_2(q) \times SL_6(q)) \cdot (q - 1)/d$  | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $[q^{53}] \cdot f \cdot (SL_3(q) \times PSL_2(q) \times PSL_4(q)) \cdot (q - 1)/d$ | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $[q^{50}] \cdot (SL_4(q) \times SL_5(q)) \cdot (q - 1)/d$             | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $[q^{42}] \cdot f \cdot (Ω_{10}^+(q) \times PSL_2(q)) \cdot (q - 1)/d$ | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $[q^{27}] \cdot e \cdot (E_6(q) \times (q - 1)/de) \cdot e$          | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $d \cdot (PSL_2(q) \times PΩ_{12}^+(q)) \cdot d$                    | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $f / d \cdot PSL_6(q) \cdot g / f : 2$                               | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $f' / d \cdot PSU_8(q) \cdot g'/f' : 2$                              | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $e \cdot PSL_3(q) \times PSU_6(q) \cdot e \cdot 2$                  | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $e' \cdot PSU_3(q) \times PSU_6(q) \cdot e' \cdot 2$                | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $d^2 \cdot (PSL_2(q)^3 \times PΩ_8^+(q)) \cdot d^2 \cdot Sym(3)$    | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $(PSL_2(q)^3 \times PΩ_8^+(q)) \cdot 3$                              | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $d^3 \cdot PSL_2(q)^7 \cdot d^3 \cdot PSL_3(2)$                     | All $p$ | $q \geq 3$ | 1 | $(δ, ϕ)$ |
| $PSL_2(q^7) \cdot 7$                                                 | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $e \cdot (E_6(q) \times (q - 1)/de) \cdot e \cdot 2$                | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $e' \cdot (2E_6(q) \times (q + 1)/de') \cdot e' \cdot 2$            | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $(q - 1)^7 / d \cdot W(E_7)$                                         | All $p$ | $q \geq 5$ | 1 | $(δ, ϕ)$ |
| $(q + 1)^7 / d \cdot W(E_7)$                                         | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $PSL_2(q) \times F_4(q)$                                             | All $p$ | $q \geq 3$ | 1 | $(δ, ϕ)$ |
| $G_2(q) \times PSp_6(q)$                                             | All $p$ | All $q$ | 1 | $(δ, ϕ)$ |
| $PSL_2(q) \times G_2(q)$                                             | All $p$ | $q \geq 5$ | 1 | $(δ, ϕ)$ |
| $PSL_2(q) \times PGL_2(q)$                                           | $p \geq 5$ | All $q$ | 1 | $(δ, ϕ)$ |
| $PGL_3(q) \cdot 2$                                                   | 5    | All $q$ | 2 | $(δ, ϕ)$ |
| $PGU_3(q) \cdot 2$                                                   | 5    | All $q$ | 2 | $(δ, ϕ)$ |
| $NC(PGL_3(q))$                                                       | $p \geq 7$ | All $p^{2n+1}$ | 1? | $(δ, ϕ)$? |
| $NC(PGU_3(q))$                                                       | $p \geq 7$ | All $p^{2n+1}$ | 1? | $(δ, ϕ)$? |
| $PGL_3(q) \cdot 2$                                                   | $p \geq 7$ | All $p^{2n}$ | 2 | $(δ, ϕ)$ |
| $PGU_3(q) \cdot 2$                                                   | $p \geq 7$ | All $p^{2n}$ | 2 | $(δ, ϕ)$ |
| $PSL_2(q)$                                                           | $p \geq 17$ | All $q$ | 1 | $(δ, ϕ)$ |
| $PSL_2(q)$                                                           | $p \geq 19$ | All $q$ | 1 | $(δ, ϕ)$ |
| $(2^2 \times PΩ_8^+(q) \cdot 2^2) \cdot 3$                         | $p \geq 3$ | All $q$ | 2 | $(δ, ϕ)$ |
| $PSL_2(q)$                                                           | $p \geq 19$ | All $q$ | 1 | $(δ, ϕ)$ |
| $E_7(q_0) \cdot gcd(d, r)$                                           | All $p$ | $q = q_0$, r prime | $gcd(d, r)$ | $(δ, gcd(d, r), ϕ)$ |

Table 4.1: Maximal subgroups not in $S$ for $G = E_7(q)$. Here, $d = gcd(2, q - 1)$, $e = gcd(3, q - 1)$, $e' = gcd(3, q + 1)$, $f = gcd(4, q - 1)$, $f' = gcd(4, q + 1)$, $g = gcd(8, q - 1)$, $g' = gcd(8, q + 1)$.
whether there are any such subgroups that are not strongly imprimitive, and if so, determine the number of \(G\)-conjugacy classes, determine \(N_G(H)\), and compute the actions of outer automorphisms of \(G\) on the \(G\)-classes of subgroups \(H\), whenever \(H \leq G\).

The collection of all simple groups that embed in \(G\), other than groups of Lie type in characteristic \(p\), is given in [26]. As remarked in the last section, for Lie type groups in characteristic \(p\) we only need consider \(\text{PSL}_2(r)\) for \(r = 7, 8, 25\). For the other simple groups, we start with [26], and then remove those that were proved to always be strongly imprimitive in [29] for all simple groups, and [11] for alternating groups in particular. This yields the groups in Table 5.1.

For each of these we have to consider each set of composition factors for \(M(G)\) and \(L(G)\) that appears in the tables of [29] with ‘P’. For the three Lie type groups in characteristic \(p\), the possible module structures were described in [9].

### 5.1 Alternating groups

We rely heavily on [11] here, which deals with most cases. Let \(H \cong \text{Alt}(n)\) or \(2 \cdot \text{Alt}(n)\) with \(Z(H) = Z(G)\). If \(n = 8\) then \(H\) either stabilizes a line on \(M(E_7)\) or \(L(E_7)\), or \(p = 3\) and \(H\) contains a ‘generic unipotent element’ in the sense of [9, Definition 6.4], so is strongly imprimitive by [9]. If \(n = 9\) then for \(p \geq 5\) we always have that \(H\) is strongly imprimitive by [29]. If \(p = 3\) then either \(H\) stabilizes a line on \(L(E_7)\) or contains a generic unipotent element, and if \(p = 2\) then \(H\) stabilizes a line on \(M(E_7)\). If \(n \geq 10\) then \(H\) is strongly imprimitive by [29]. Thus \(H\) is strongly imprimitive for \(n \geq 8\).

If \(n = 5\) then for \(p = 3, 5\), \(H\) stabilizes a line on either \(M(E_7)\) or \(L(E_7)\). If \(p = 2\) then from [11, Proposition 5.4] we see that \(H\) stabilizes a line or 2-space on \(M(E_7)\), and one may then apply [9, Proposition 4.7] to obtain that \(H\) is strongly imprimitive.

If \(p = 0\) then in [11, Proposition 5.1] we see that \(H\) cannot be Lie primitive, and \(H\) is either strongly imprimitive or \(H\) has two isomorphic 2-dimensional composition factors on \(M(E_7)\) (and none of the other isomorphism type). If \(C_G(H) \neq Z(G)\) then the double centralizer \(C_G(C_G(H))\) is positive dimensional and of course \(N_{\text{Aut}^+(G)}(H)\)-stable, whence \(H\) is strongly imprimitive.

By simple dimension counting, the stabilizer of any 2-space on \(M(E_7)\) has dimension at least 25 (see, for example, [11, Lemma 1.4]), and so \(H\) must lie in a positive-dimensional subgroup of \(G\) of dimension at least 25. Furthermore, \(H\) has no trivial composition factors on \(L(E_7)\), so \(H\) is not contained in a parabolic subgroup of \(G\).

| Prime | Group |
|-------|-------|
| \(p \nmid |H|\) | \(\text{Alt}(6), \text{PSL}_2(r), r = 7, 11, 13, 19, 27, 29, 37, \text{PSL}_3(4), \text{PSU}_3(3), \text{PSU}_3(8), \Omega_8^+(2)\) |
| \(p = 2\) | \(J_2, \text{PSL}_2(r), r = 8, 11, 13, 19, 27, 29, 37\) |
| \(p = 3\) | \(\text{PSL}_2(r), r = 7, 11, 13, 19, 27, 37, \text{PSL}_3(4), \text{PSU}_3(8), \Omega_8^+(2)\) |
| \(p = 5\) | \(\text{Alt}(6), \text{Alt}(7), M_{22}, M_{22}, \text{Rut}, HS, \text{PSL}_2(r), r = 11, 19, 25, 29, \text{PSL}_3(4), \Omega_8^+(2), 2B_2(8)\) |
| \(p = 7\) | \(\text{PSL}_2(r), r = 7, 8, 13, 27, 29, \text{PSL}_3(4), \text{PSU}_3(3), \text{PSU}_3(8), \Omega_8^+(2)\) |
| \(p = 13\) | \(\text{PSL}_2(27)\) |
| \(p = 19\) | \(\text{PSL}_2(37), \text{PSU}_3(8)\) |

Table 5.1: Simple subgroups of \(E_7\) that could yield subgroups that are not strongly imprimitive.

\(^2\)Note that \(2B_2(8), p = 5\) is missing from [26]. It embeds in \(A_7\).
Thus if $H$ is not strongly imprimitive then $H$ cannot lie in a parabolic subgroup of $G$, and cannot lie in the centralizer of a semisimple element of $G$, and has to lie in a subgroup of dimension at least 25. Examining the maximal subgroups of $E_7$ (see, for example, [28]), this leaves only the $G_2C_3$ and $A_1F_4$ maximal subgroups that could contain $H$. If $H$ lies in $A_1F_4$ then, since $L(F_4)$ is a composition factor of the $A_1F_4$-action on $L(E_7)$, the projection of $H$ on $F_4$ has finite centralizer in $F_4$ as well. But this subgroup is unique up to $F_4$-conjugacy and contained in $A_2A_2$ (this is due to Magaard: see [13] Theorem 4.17), which centralizes an element of order 3 in $F_4$, hence in $G$.

Finally, if $H$ lies in $G_2C_3$, then we see from the list of maximal subgroups of $G_2$ [13] that the image of $H$ in $G_2$ either centralizes an element of order 2 or 3, or lies inside the maximal $A_1$ acting irreducibly as $L(6)$ on $M(G_2)$. But $A_1C_3$ has dimension 24, so cannot contain any 2-space stabilizer. Thus $H$ must be contained in some other maximal positive-dimensional subgroup of $G$, and this completes the proof.

This leaves $n = 6$ and $n = 7$. In the latter case, $H$ is always strongly imprimitive if $p \neq 5$ by [11] Propositions 7.1, 7.2, 7.4 and 7.5, so we need only look at $p = 5$. If $n = 6$ and $p = 2, 3$ then $H$ stabilizes a line on $M(E_7)$ or $L(E_7)$, or a 2-space on $M(E_7)$ in the case $H \cong 2 \cdot Alt(6)$ and $p = 3$. To obtain strong imprimitivity we can again apply [9] Proposition 4.7 for the third possibility.

Thus we are left with $Alt(7)$, $p = 5$ (which we can resolve) and $Alt(6)$, $p \geq 5$ (which we cannot), and we discuss the latter in Section 5.3.

5.1.1 Alt(7)

From [11] Section 7, we see that there are two possibilities for $H$ a cover of Alt(7), both of which require $p = 5$.

We start with the possibility of $2 \cdot Alt(7)$ in $G$. Here, the actions on $M(E_7)$ and $L(E_7)$ are

$$20 \oplus (4/14) \oplus (14/4^*), \quad 35 \oplus 10 \oplus 10^* \oplus 8.$$  

As we see from [20] Table 6.128, this is the only possible action of $2 \cdot Alt(7)$. We therefore see that the copy of $2 \cdot Alt(7)$ in $A_2A_5$ (which is $3 \cdot Alt(7)$ in $A_2$ and $6 \cdot Alt(7)$ in simply connected $A_5$, acting irreducibly on each minimal module) also has this action.

**Proposition 5.1.** If $H \cong 2 \cdot Alt(7)$ then $H$ is strongly imprimitive.

**Proof.** We utilize the subalgebra method from Section 5.2 to prove uniqueness of $H$ up to conjugacy. In order to do this, first we need to determine $H$ up to conjugacy in $Sp_{56}(k)$. Any bilinear form on $20 \oplus (4/14) \oplus (14/4^*)$ is a direct sum of a form on the 20 and a form on the other summand, thus lies inside a subgroup $Sp_{20}(k) \times Sp_{36}(k)$. The 20 supports a unique alternating form, so given a form for the 36-dimensional summand, one obtains a unique subgroup of $Sp_{56}(k)$ up to conjugacy by Proposition 2.10. Thus if $H$ is unique up to conjugacy in $Sp_{36}(k)$ with that action, then $H$ is unique up to conjugacy in $Sp_{56}(k)$, as needed.

Since this action of $H$ is the extension with quotient $4 \oplus 14$ and submodule its dual, we see that $H$ stabilizes a—necessarily totally isotropic—18-space. Thus $H$ is contained in an $A_{17}$-parabolic subgroup of $Sp_{36}(k)$. One wishes to apply Corollary 2.10 so one requires the 1-cohomology of the module $S^2(4^* \oplus 14)$, which by a computer calculation (or even a hand calculation if you so desire) is 1-dimensional. (The 1-cohomology of $S^2(4 \oplus 14)$ on the other hand is 0-dimensional, which tells you which you need to take.)

Thus $H$ is unique up to $Sp_{56}(k)$-conjugacy. Applying the subalgebra method in the supplementary materials yields a single orbit under the action of $C_{Sp_{56}(k)}(H)$. Although there are two representations of $H$ of our form, they are $Aut(H)$-conjugate, so yield a single class of subgroups of $G$.  

19
Thus $H$ is contained in $A_2A_5$ as there is also a copy of $H$ there. The subgroup $X = A_2A_5$ has summands $(10, \lambda_1)$, $(01, \lambda_5)$ and $(00, \lambda_9)$, i.e., the product of the minimal modules, its dual, and the exterior cube of $M(A_5)$. These summands have dimensions 18, 18 and 20, so that $H$ and $X$ stabilize the same 20-space. Furthermore, this is clearly stable under $N_{\text{Aut}^+(G)}(H)$, so $H$ is strongly imprimitive by Proposition 2.7.

The other possibility is for $H \cong \text{Alt}(7)$, acting on $M(E_7)$ and $L(E_7)$ as

$$(10 \oplus 10^*) \oplus 8^{\otimes 2}, \quad 10 \oplus 10^* \oplus P(8)^{\otimes 3} \oplus 8.$$ 

There is a copy of this inside $D_4$, inside the maximal-rank subgroup $A_7$ of $E_7$. It turns out that this is the only conjugacy class.

**Proposition 5.2.** If $H \cong \text{Alt}(7)$ embeds in $G$ then $H$ is strongly imprimitive.

**Proof.** As stated above, the only case to consider is $p = 5$ and $H$ acting as above. The 8 carries a symmetric form, so there is a unique symplectic form on $8^{\otimes 2}$. There is also a unique symplectic form on $W = (10 \oplus 10^*)^{\otimes 2}$, up to conjugacy. To see this, notice that the centralizer in $GL(W)$ is isomorphic to $GL_2(k) \times GL_2(k)$, and since $H$ and its centralizer in $Sp(W)$ must lie in the $A_{19}$ subgroup of $Sp(W)$, the centralizer in $Sp(W)$ is $GL_2(k)$. Finally, the set of symplectic forms on $W$ is 4-dimensional, and the non-singular ones are in natural bijection with $GL_2(k)$ (one can see this by considering the fixed points of the exterior square of $W$, or simply use a computer, and we check this in the supplementary materials). We see that the centralizer in $GL(W)$ must act transitively on the non-singular symplectic forms, and therefore $H$ is unique in $Sp(W)$ up to conjugacy. Thus $H$ is unique in $Sp_{10}(k)$ up to conjugacy as well.

We therefore may apply the subalgebra method, and prove that $H$ always stabilizes a 28-dimensional subalgebra $\mathfrak{h}$ of type $\vartheta_4$. It seems difficult to proceed further and find the subalgebra of type $\vartheta_7$ that is stabilized, or the $\vartheta_7$ above it. Thus we proceed differently, understanding subalgebras of type $\vartheta_4$ in the Lie algebra $\mathfrak{g} = \vartheta_7$.

We consider the composition factors of the action of $\mathfrak{h}$ on $\mathfrak{g}$. Using the eigenvalues of semisimple elements, one easily sees that there are exactly two sets of dimensions of composition factors, namely $35^3, 28$ (the irreducible $\vartheta_4$ inside $\vartheta_7$) and $28, 8^{\otimes 2}, 1^9$ (the $\vartheta_4$-Levi). As $H$ stabilizes the homogeneous components of the action of $\mathfrak{h}$ on $\mathfrak{g}$, these dimensions must be compatible with the composition factors of $H$ on $\mathfrak{g}$, which have dimensions $13^3, 10^2, 8^7, 6^3$. One cannot make 9 from these dimensions, so $\mathfrak{h}$ acts with factors of dimension $35^3, 28$.

Since $\mathfrak{h}$ has no trivial factor on $\mathfrak{g}$, it cannot be contained in a parabolic. Also, if $\mathfrak{h}$ is contained in a maximal subalgebra $\mathfrak{r}_1\mathfrak{r}_2$ then both $\mathfrak{r}_i$ must contain a copy of $\mathfrak{h}$. This is impossible, so $\mathfrak{h}$ can only be contained in simple subalgebras. By [33] Theorem 1.1, and the list of maximal $G$-irreducible subgroups (see, for example, [27] or [38]), we see that $\mathfrak{h}$ is contained in $\vartheta_7$. Any 8-dimensional irreducible representation of $\vartheta_4$ integrates to a representation of $D_4$, and so $\mathfrak{h}$ is $\text{Lie}(H)$ for $H$ the $G$-irreducible $D_4$.

Thus $H$ is contained in this subgroup. At this point it is relatively easy to prove strong imprimitivity, and there are multiple paths. One option is to place $H$ inside an $A_2$ subgroup $X$ of $H$ (subgroup 24 from the list in [39]). This must act semisimply on $M(E_7)$ as $(10 \oplus 10^*)^{\otimes 2} \oplus 8^{\otimes 2}$, and so $H$ is a blueprint for $M(E_7)$. Another is to note that $H$ embeds in $\Omega^+_8(5)$, and so Frobenius endomorphisms can only act as inner automorphisms. Either way, $H$ is strongly imprimitive, as needed. 

\[ \square \]
5.2 Sporadic groups

The sporadic groups that appear in [29] with ‘P’ are $J_2$ for $p = 2$, and $2 \cdot M_{12}$, $2 \cdot M_{22}$, $2 \cdot HS$ and $2 \cdot Ru$, all for $p = 5$. The groups $M_{22}$, $HS$ and $Ru$ are all irreducible on $L(E_7)$, and proved to be unique up to $G$-conjugacy in [12].

We start with $HS$ and $Ru$, and then we can do $M_{22}$ (as it is contained in $HS$). We obtain a previously (the author believes) unknown maximal subgroup $M_{12}$, and prove that $J_2$ does not yield a maximal subgroup via theoretical means.

In this section, $H$ will be a subgroup of $G$ (and $G$) and $\overline{H}$ will denote its image in $G/Z(G)$ (and $\overline{G}$).

5.2.1 $HS$

Let $H$ be a cover of $HS$, so we require $p = 5$ and $H \cong 2 \cdot HS$. Up to isomorphism there is a unique 133-dimensional simple module $M$, defined over $F_5$, and $R^H(M)$ is 1-dimensional. Thus $H$ is unique up to $G$-conjugacy.

**Proposition 5.3.** Let $H \cong 2 \cdot HS$ and $p = 5$.

(i) There is exactly one $G$-conjugacy class of subgroups $H$. If $H \leq G$ there are exactly two $G$-conjugacy classes of subgroups $H$. Furthermore, $N_G(H) = H$.

(ii) $H$ embeds in $G$ for all $q$ a power of 5.

(iii) We have that $N_{\overline{G}}(\overline{H})$ is maximal in $\overline{G}$ if and only if $\overline{G} = E_7(5)$.

**Proof.** We have already noted that $H$ is unique up to $G$-conjugacy. Since the outer automorphism of $H$ swaps the two 133-dimensional simple modules for $H$, we must have $N_G(H) = H$, and therefore there are exactly two $G$-conjugacy classes by Lemma 2.13. Since the Lie product is defined over $F_5$, $H \leq 2 \cdot E_7(5)$, and as there can be no overgroups of $H$ by Proposition 7.1, this completes the proof of maximality.

5.2.2 $Ru$

This is almost the same as $HS$. Let $H \cong 2 \cdot Ru$, so we require $p = 5$. Up to isomorphism there is a unique 133-dimensional simple module $M$, defined over $F_5$, and $R^H(M)$ is 1-dimensional. Thus $H$ is unique up to $G$-conjugacy. (The action on $M(E_7)$ has composition factors 28 and $28^*$, and is not semisimple. The easiest way to see this is that it contains $SL_2(29)$, and this is seen not to be semisimple in Section 5.4.2.)

**Proposition 5.4.** Let $H \cong 2 \cdot Ru$ and $p = 5$.

(i) There is exactly one $G$-conjugacy class of subgroups $H$. If $H \leq G$ there are exactly two $G$-conjugacy classes of subgroups $H$. Furthermore, $N_G(H) = H$.

(ii) $H$ embeds in $G$ for all $q$ a power of 5.

(iii) We have that $N_{\overline{G}}(\overline{H})$ is maximal in $\overline{G}$ if and only if $\overline{G} = E_7(5)$.

**Proof.** We have already noted that $H$ is unique up to $G$-conjugacy. Since $Out(H/Z(H)) = 1$, $N_G(H) = H$, and therefore there are exactly two $G$-conjugacy classes by Lemma 2.13. Since the Lie product is defined over $F_5$, $H \leq 2 \cdot E_7(5)$, and as there can be no overgroups of $H$ by Proposition 7.1, this completes the proof of maximality.
5.2.3 \( M_{12} \)

From [29], Tables 6.145–6.150, we see that \( M_{12} \) or \( 2 \cdot M_{12} \) always stabilizes a line on either \( M(E_7) \) or \( L(E_7) \), unless \( p = 5 \) and \( H \cong 2 \cdot M_{12} \) acts as \( 55 \oplus 78 \) on \( L(E_7) \). The space \( R^H(L(E_7)) \) is 12-dimensional, and this case can be solved using the Lie product method from Section 5.1. The corresponding factors for \( H \) on \( M(E_7) \) are \( 32, 12^2 \).

The action of \( u \in H \) of order 5 on \( L(E_7) \) has Jordan blocks \( 5^{26}, 3 \), so by [20] Table 8, we see that \( u \) belongs to the unipotent class \( A_4 + A_2 \). Since this acts on \( M(E_7) \) with blocks \( 5^{10}, 3^2 \), and on the sum \( 32 \oplus 12^{92} \) with blocks \( 5^{10}, 1^6 \), we see that \( M(E_7) \downarrow H \) has structure 12/32/12.

Note that no extension of this module, or indeed the simple modules 12 and 32, exists over \( \mathbb{F}_5 \), and one must move to \( \mathbb{F}_{25} \) in order to obtain it. Thus, while \( H \) might be contained in \( 2 \cdot E_7(5) \), the group \( H \cdot 2 \) cannot be contained in \( 2 \cdot E_7(5) \).

**Proposition 5.5.** Let \( H \cong 2 \cdot M_{12} \) and \( p = 5 \).

(i) There is exactly one \( G \)-conjugacy class of subgroups \( H \). If \( H \leq G \) there is exactly one \( G \)-conjugacy class of subgroups \( H \). Furthermore, \( N_G(H) = H \cdot 2 \).

(ii) \( H \) embeds in \( G \) for all \( q \) a power of 5.

(iii) We have that \( N_G(\bar{H}) \) is maximal in \( \bar{G} \) if and only if \( q = 5 \).

**Proof.** The action of \( H \) on \( L(E_7) \) is \( M \cong 55 \oplus 78 \), and \( R^H(M) \) is 13-dimensional. It is easy to prove, as is accomplished in the supplementary materials, that there is a unique Lie product structure on \( M \). We also show that this action extends to one of \( H \cdot 2 \).

The action of \( H \) on \( M(E_7) \) has composition factors \( 32, 12^2 \). There are two isoclinic groups \( 2 \cdot M_{12} \cdot 2 \), and for both of these the extensions of 12 and 32 to the whole group requires \( \mathbb{F}_{25} \) to be present, not just \( \mathbb{F}_5 \). (See [17] p.76) for one of these classes. Only one of these extensions results in a subgroup of \( \text{Sp}_{56}(k) \). Thus the group \( H \cdot 2 \) cannot embed in \( 2 \cdot E_7(5) \leq \text{GL}_{56}(5) \). Thus \( H \) is unique up to conjugacy, with the diagonal automorphism of \( E_7(5) \) normalizing \( \bar{H} \) (up to conjugacy).

\[ \square \]

5.2.4 \( M_{22} \)

Let \( H \cong 2 \cdot M_{22} \), so we require \( p = 5 \). Up to isomorphism there is a unique 133-dimensional simple module \( M \), defined over \( \mathbb{F}_5 \), and \( R^H(M) \) is 2-dimensional. In [12] Example 3.12 the author showed that imposing the Jacobi identity reduced this space to a line. It was also shown that this Lie product structure was invariant under \( M_{22} \cdot 2 \). Thus \( H \) is unique up to \( G \)-conjugacy, and \( |N_G(H) : H| = 2 \).

**Proposition 5.6.** Let \( H \cong 2 \cdot M_{22} \) and \( p = 5 \).

(i) There is exactly one \( G \)-conjugacy class of subgroups \( H \). If \( H \leq G \) there is exactly one \( G \)-conjugacy class of subgroups \( H \). Furthermore, \( N_G(H) = H \cdot 2 \).

(ii) \( H \) embeds in \( G \) for all \( q \) a power of 5.

(iii) We have that \( N_G(\bar{H}) \) is maximal in \( \bar{G} \) if and only if \( \bar{G} = E_7(5) \cdot 2 \), and \( N_G(\bar{H}) \) is a novelty maximal subgroup.

**Proof.** We have already noted that \( H \) is unique up to \( G \)-conjugacy, and that \( N_G(H) = H \cdot 2 \). Since the Lie product is defined over \( \mathbb{F}_5 \), \( H \leq 2 \cdot E_7(5) \). We need to know whether all of \( N_G(H) \) is contained in \( G = 2 \cdot E_7(5) \) or whether the diagonal automorphism of \( E_7(5) \) induces the outer automorphism of \( H \).
Note that $H$ is contained in $2 \cdot HS$ by Proposition \[\text{Proposition} \] but if one inspects \[\text{p.80}, one sees that $M_{22, 2}$ does not embed in $HS$. Both $M_{22}$ and $HS$ act irreducibly on $L(E_7)$. If $x \in E_7(5) \cdot 2$ induces an outer automorphism on $H$, and $L \cong 2 \cdot HS$ is a subgroup of $E_7(5)$ containing $H$, then $L^x$ also contains $H$.

In the supplementary materials we construct a copy of $M_{22}$, and two copies of $HS$ above it, that generate $E_7(5)$. We then add an outer automorphism of $M_{22}$ and check that the resulting group is $E_7(5) \cdot 2$, not $E_7(5)$.

This shows that $N_G(H) = H$ and there is a unique $G$-class of subgroups $H$, via Lemma \[\text{Lemma 2.13} Thus the diagonal automorphism of $E_7(5)$ must normalize $H$, hence act as the outer automorphism, as needed. \[\text{\[\Box\]}

5.2.5 $J_2$

Here, $p = 2$. From \[\text{Table 6.157}], we see that the only case of interest is $H$ acting with composition factors $36, 6^2_1, 6_2, 1^2$ on $M(E_7)$, and on $L(E_7)^5$ with factors

$$84, 14^2_1, 14_2, 6_2.$$ 

Note that $\text{Ext}^1_{L(G)}(14_1, 14_2) = \text{Ext}^1_{L(G)}(14_1, 6_1) = 0$, so $14_2$ and $6_1$ split off as summands, and there are no self-extensions for $14_1$. Thus there are only two options for $L(E_7) \downarrow H$:

$$84 \oplus 14_1^{\oplus 2} \oplus 14_2 \oplus 6_2 \quad \text{and} \quad (14_1/84/14_1) \oplus 14_2 \oplus 6_1.$$ 

The action of $v$ of order $8$ in $H$ on both of these modules is the same, with Jordan blocks $8^{13}, 6^4, 2^2$. Thus from \[\text{Lemma 6.3}], the action of $v$ on $L(E_7)$ must be $8^{13}, 6^4, 2^2, 1$, but this does not appear in \[\text{Table 8}]. Thus $H$ cannot embed in $G$ with these composition factors.

**Proposition 5.7.** If $p = 2$ and $H \cong J_2$, then $H$ stabilizes a line on either $M(E_7)$ or $L(E_7)^5$, and hence is strongly imprimitive.

5.3 Lie type not $\text{PSL}_2(r)$

Now let $H$ be a group of Lie type not of type $\text{PSL}_2(r)$, which appear in Table \[\text{5.1}]. If $H \cong \text{PSU}_3(8)$ then $H$ acts irreducibly on $L(E_7)$, and is unique up to $G$-conjugacy by \[\text{Theorem 2}]. The remaining cases are: $2 \cdot \text{PSL}_3(4)$ for $p \neq 2$; $\text{PSU}_3(3)$ for $p = 0, 7$; $\text{O}^+_{10}(2)$ for $p \neq 2$; and $^2\text{B}_2(8)$ for $p = 5$.

In this section, $H$ will be a subgroup of $G$ (and $G$) and $\tilde{H}$ will denote its image in $G/Z(G)$ (and $\tilde{G}$).

5.3.1 $\text{PSL}_3(4)$

Here the primes are $p = 3$, $p = 5$, $p = 7$ and $p \neq 2, 3, 5, 7$. From \[\text{Table 6.214–6.217}], we see that the only cover of $\text{PSL}_3(4)$ that embeds in $G$ is $H \cong 2 \cdot \text{PSL}_3(4)$. For $p \neq 3$, the action of $H$ on $L(E_7)$ is the sum of two non-isomorphic 35-dimensional modules and a 63-dimensional module. For $p = 3$ the action is slightly different: the sum of a 63-dimensional module and a module $19/1, 1, 15_1, 15_2/19$, as we shall see now.

Let $p = 3$. From \[\text{Table 6.217}] we see that composition factors of $L(E_7) \downarrow H$ are, up to automorphism of $H$,

$$63_1, 10^2, 15_1, 15_2, 1^2.$$ 

Since $\text{Ext}^1_{L(H)}(19, 1)$ has dimension 2, $H$ need not stabilize a line on $L(E_7)$. The $63_1$ has no extension with 1 or 19, so splits off as a summand. Suppose that $H$ does not stabilize a line on $L(E_7)$. Suppose first that no $15_i$ is a submodule of $L(E_7) \downarrow H$. There is a module

$$1, 1, 15_1, 15_2/19,$$ 

23
on which we may place two copies of 19. If \( u \in H \) has order 3, then \( u \) acts on the module above with blocks \( 3^{16}, 2, 1, \) and on 19 with blocks \( 3^{6}, 1. \) The action of \( u \) on the module \( 19, 19/1, 1, 15_{1}, 15_{2}/19 \) has blocks \( 3^{29}, 2, \) whence we see that there is a unique module \( 19/1, 1, 15_{1}, 15_{2}/19 \) with \( u \) acting as \( 3^{22}, 2^{2}, \) and on all others as \( 3^{23}, 1. \) As \( u \) acts projectively on 63, \( u \) acts on \( L(E_{7}) \) with blocks either \( 3^{43}, 2^{2} \) or \( 3^{44}, 1. \) Only the former appears in \([20, \text{ Table } 8]\), and \( u \) belongs to class \( 2A_{2} + A_{1}. \) Thus \( L(E_{7}) \downarrow H \) is unique up to isomorphism.

If there is at least one \( 15_{i} \) as a submodule, as all \( 15_{i} \) are \( \text{Aut}(H) \)-conjugate, we may assume that \( 15_{2} \) is a submodule. On top of \( 1, 1, 15_{1}/19 \) we may place a single 19, and the resulting module is self-dual. Thus we obtain a module

\[
(19/1, 1, 15_{1}/19) \oplus 15_{2} \oplus 63_{1}.
\]

The element \( u \) acts on this with blocks \( 3^{44}, 1, \) so this case is excluded.

If \( L(E_{7}) \downarrow H \) has both \( 15_{i} \) as submodules then \( H \) must stabilize a line on \( L(E_{7}) \), as we see from the above modules.

In all cases we have the following result.

**Proposition 5.8.** Any subgroup \( H \cong 2 \cdot \text{PSL}_{3}(4) \) in \( G \) is strongly imprimitive.

**Proof.** Suppose that we can place \( H \) inside a maximal-rank \( A_{7} \) subgroup \( X \). Note that both \( H \) and \( X \) stabilize the same unique irreducible 63-dimensional submodule of \( L(E_{7}) \). Thus \( H \) is strongly imprimitive by Proposition 2.7.

Thus we use the 53- or 54-dimensional (depending on \( p \)) space \( R^{H}(L(E_{7})) \) to prove that there is a unique \( H \)-invariant Lie product on \( L(E_{7}) \downarrow H \). In the supplementary materials we show this for each of \( p = 3, p = 5, p = 7 \) and \( p \neq 2, 3, 5, 7. \) This completes the proof. \( \square \)

### 5.3.2 \( \text{PSU}_{3}(3) \)

Let \( H \cong \text{PSU}_{3}(3) \), where the appropriate primes are \( p = 7 \) and \( p \neq 2, 3, 7 \) (as \( H \cong G_{2}(2)^{\prime} \)). From \([29, \text{ Tables } 6.219-6.220]\) we find two sets of factors labelled ‘\( P \)’ for \( p \neq 2, 3, 7 \), and three for \( p = 7 \). We can eliminate two of these five quickly. One case for \( p \neq 2, 3, 7 \) contains an irreducible summand of dimension 14, which via the inclusion of \( H \) into \( G_{2} \) admits a simple Lie algebra structure. Since \( \Lambda^{2}(14) = 14 \oplus 21_{1} \oplus 28 \oplus 28^{*} \), and only 14 from these appears in the composition factors of \( L(E_{7}) \downarrow H \), we see that the 14 forms a \( G_{2} \)-subalgebra of \( L(E_{7}) \). This is enough to prove that there is no such Lie primitive subgroup \( H \) over \( E_{7}(\mathbb{C}) \) (cf. \([6, \text{ p.146}]\)), and then we use Larsen’s \( (0,p) \)-correspondence (Theorem 2.7) to deduce the result for all \( p \neq 2, 3, 7. \) (This only shows that \( H \) is Lie imprimitive, not strongly imprimitive. However, both \( H \) and the \( G_{2} \) subgroup stabilize the unique 14-dimensional summand, so \( H \) is strongly imprimitive by Proposition 2.7.)

The other case we can immediately remove is for \( p = 7 \). Here only 26 has non-trivial 1-cohomology, and the structure of the projective is

\[
26/1/26/1, 6/26.
\]

Examining \([29, \text{ Table } 6.220]\), we see that Case 4 has four 26s and three 1s. From the structure of the projective above, it is clear that we cannot build such a module without a trivial submodule or quotient.

One of the remaining three cases is the above case with a 14-dimensional summand, but for \( p = 7 \) so we cannot use the previous method.

**Proposition 5.9.** Let \( H \cong \text{PSU}_{3}(3) \). If \( p = 7 \) and \( L(E_{7}) \downarrow H \) has composition factors

\[
26^{3}, 21_{1}, 14, 7_{1}, 6^{2}, 1
\]
then $H$ is unique up to $G$-conjugacy and lies in an $A_6$-subgroup of $G$. In particular, $H$ is a blueprint for $M(E_7)$ so is strongly imprimitive.

Proof. The composition factors of $H$ on $M(E_7)$ are $7, 7^*, 21_2, 21_2^*$. As these modules are each projective, there are no extensions between them, so $M(E_7)\downarrow_H$ is semisimple. The alternating form is unique up to $\text{Sp}_{56}(k)$-conjugacy by Proposition 2.15.

In the supplementary materials we show that $H$ is unique up to $G$-conjugacy. On the other hand, since $H$ has a 7-dimensional irreducible module it lies in the $A_6$-Levi subgroup, and must act on $M(E_7)$ as $7 \oplus 21_2 \oplus 21_2^* \oplus 7^*$. Thus this is our subgroup $H$. However, $A_6$ itself acts as the sum of four non-isomorphic modules of dimensions $7, 7, 21, 21$, so $H$ and $A_6$ stabilize the same subspaces of $M(E_7)$. Thus $H$ is a blueprint for it, as claimed.

The remaining two cases are where $H$ acts on $M(E_7)$ as $28 \oplus 28^*$ and $p \neq 2, 3, 7$ or $p = 7$. In this case there are two $G$-classes of subgroups, and the difficulty is in understanding the action of the field automorphism on these classes.

Proposition 5.10. Let $H \cong \text{PSU}_3(3)$, let $p \neq 2, 3$, and suppose that $M(E_7)\downarrow_H \cong 28 \oplus 28^*$.

(i) There is exactly one $G$-conjugacy class of subgroups $H$. If $H \leq G$ there are exactly two $G$-conjugacy classes of subgroups $H$. Furthermore, $N_G(H) = H \times Z(G)$.

(ii) $H$ embeds in $G$ if and only if $q \equiv \pm 1 \mod 8$.

(iii) If $N_G(\bar{H})$ is maximal in $\bar{G}$, then $q$ is minimal such that $H$ embeds in $G$.

Proof. The proof proceeds in stages.

Determination of $N_G(H)$ This is surprisingly difficult to do. In [2, p.15] we see that the $28 \oplus 28^*$ extends to the full automorphism group $H.2$ of $H$ as an irreducible 56-dimensional module, but this irreducible module preserves a symmetric bilinear form, not an alternating one. (The same holds for $p = 7$, see [17, p.24].) Thus either $N_G(H)$ is $H \times Z(G)$ or of the form $H \cdot 4$. The latter subgroup does embed in $\text{Sp}_{56}(p)$, as a simple computer construction proves (the subgroup is of course inside the group $(\text{PSU}_3(3) \times 2)\wr 2$). We will prove that this outer automorphism of $H$ interchanges two $e_7$-subalgebras for $p = 7$, and hence it does not lie in $G$ for all $p \geq 7$ by Proposition 2.2. We can see no ‘internal’ way to prove it does not embed in $G$.

Determination of $H$ up to conjugacy We apply the subalgebra method from Section 3.2, noting that since $M(E_7)$ is the sum of two dual modules, the alternating form is unique. In the supplementary materials we prove that for both $p = 7$ and $p \neq 2, 3, 7$ there is exactly one $G$-conjugacy class of subgroups $H$.

Action of outer automorphisms By Lemma 2.13 the diagonal automorphism must fuse two $G$-classes of subgroups $H$. It remains to decide on the action of $\sigma = F_p$. Of course, $\sigma$ normalizes the unique $G$-class, so either $\sigma$ acts as an outer automorphism on a representative $H$ (by Lemma 2.8) or as an inner automorphism, hence centralizes some conjugate of $H$ by Lemma 2.11.

First, note that the centralizer in $\text{Sp}_{56}(k)$ of $H$ acts on the $H$-invariant $e_7$-subalgebras and there are two orbits. (This must be the case as the centralizer times $H$ has index 2 in the normalizer of $H$, and there is a single $G$-class of subgroups $H$. We also prove it in characteristic 0 over the ring $\mathbb{Z}[\zeta]$, where $\zeta$ is a primitive 8th root of unity.)
We can see that \( \sigma \) acts as an outer automorphism on \( H \) if and only if it swaps these two orbits (which are also interchanged by the normalizer of \( H \)). First, by a direct calculation in the supplementary materials, we check this—and therefore that the statement holds—for \( p = 5, 7, 11, 23 \). (We must consider \( p = 7 \) separately as this divides \(|H|\), and \( p = 5, 11, 23 \) cover the congruences 3, 5 and 7 modulo 8.)

Next, we must work over an extension of \( \mathbb{Z} \). The 28-dimensional simple modules are induced from elements of order 8 in the group of 1-dimensional modules for the Borel subgroup \( 3^{1+2} \times 8 \) of \( H \). Thus there is a natural description of \( M(G) \downarrow_H \) over the ring \( \mathbb{Z}[\zeta] \), where as above \( \zeta \) is a primitive 8th root of 1. Thus in the subalgebra method, we can construct \( \mathbb{Z}[\zeta]^H \)-modules (actually \( \mathbb{Z}[\zeta]^H \)-lattices because they are free over \( \mathbb{Z}[\zeta] \)) for \( S^2(M(G) \downarrow H) \). We can also construct the modules \( 21, 21^*, 27 \) and \( 32 \oplus 32^* \) over \( \mathbb{Z}[\zeta] \), so we obtain a module for \( L(G) \downarrow_H \). Thus we can run the subalgebra method over \( \mathbb{Q}(\zeta) \) (as we need a field for Magma to construct Hom-spaces and so on).

Doing so, in the supplementary materials, yields points that generate exactly two candidate \( \zeta \)-subalgebras up to the action of the centralizer, and they are defined over \( \mathbb{Z}[\zeta] \). Since, over any algebraically closed field \( k \) of characteristic not 2, 3 or 7, there must be exactly two solutions (up to the centralizer), these must be the solutions for any such \( k \). (As a sanity check, the primes dividing the denominators of the points above when they were originally constructed, before clearing denominators, were 2 and 7.)

Thus we see that the two orbits of \( \zeta \)-invariant subalgebras are invariant under any field automorphism of \( k \) that fixes all 8th roots of unity in \( k \). Thus if \( q \equiv 1 \) mod 8 then the result is proved. It remains to understand the action of the elements of the Galois group of \( \mathbb{Q}(\zeta) \) over \( \mathbb{Q} \) on the solutions, whether they swap or stabilize the solutions.

Of course, there are four elements in this Galois group, given by \( \zeta \mapsto \zeta^i \) for \( i = 1, 3, 5, 7 \). If an element of the Galois group \( \zeta \mapsto \zeta^i \) stabilizes the two subalgebras, then any field automorphism of \( k \) that induces the same map on the roots of unity (e.g., \( F_p \) for \( p \equiv i \) mod 8) also stabilizes the two classes.

Now we use our three cases \( p = 5, 11, 23 \), which we have computed in separate files in the supplementary materials. We show that the two orbits are swapped for \( p = 5, 11 \), and so \( \zeta \mapsto \zeta^3 \) and \( \zeta \mapsto \zeta^5 \) swap the two orbits of subalgebras. On the other hand, for \( p = 23 \) the two orbits are stabilized, so \( \zeta \mapsto \zeta^{-1} \) must stabilize the two orbits. This completes the proof.

5.3.3 \( \text{PSU}_3(8) \)

Let \( H \cong \text{PSU}_3(8) \), where the appropriate primes are \( p = 3, p = 7, p = 19 \), and \( p \neq 2, 3, 7, 19 \). This subgroup acts irreducibly on both \( M(E_7) \) and \( L(E_7) \) for all \( p \neq 2 \). The character of this for \( p = 0 \) (and therefore its reduction modulo \( p \) if \( p \neq 0 \)) appears in [12] pp.64–66.

Proposition 5.11. Let \( H \cong \text{PSU}_3(8) \) and \( p \neq 2 \).

(i) There is a unique \( G \)-conjugacy class of subgroups isomorphic to \( H \), and \( N_G(H) = (Z(G) \times (H.3)).2 \).

(ii) If \( q \equiv \pm 1 \) mod 8 then there are exactly two \( G \)-conjugacy classes of subgroups \( H \) in \( G = E_7(q) \), and \( N_G(H) = N_O(H) \). If \( q \equiv \pm 3 \) mod 8 then there is a single \( G \)-conjugacy class of subgroups \( H \) in \( G = E_7(q) \), and \( N_G(H) = 2 \times H.3 \).

(iii) We have that \( N_{\tilde{G}}(\tilde{H}) \) is maximal in \( \tilde{G} \) if and only if \( q = p \), and furthermore \( \tilde{G} = G \) when \( p \equiv \pm 1 \) mod 8.

Proof. In [12] it is proved that \( H \) is unique up to \( G \)-conjugacy, and \( R^H(M) \) is 1-dimensional, where \( M \) is a 133-dimensional simple module for \( H \). From [7] pp.64–66 we see that the character of \( H \) on \( M(E_7) \) and
It is left to check the actions of automorphisms of \( G \) on \( H \). Since \( H \) is unique up to \( G \)-conjugacy, all automorphisms stabilize the \( G \)-class. Since the character is rational, field automorphisms must stabilize the character and thus centralize \( N_G(H) \). Thus \( H.3 \) embeds in \( E_7(p) \) for all odd \( p \), as claimed in the proposition.

The last question is whether the full group \( H.6 \) embeds in \( G = E_7(q) \), or just \( H.3 \). The unique extension \((H \times 2).2\) is easily constructible in Magma (for example, inside the wreath product), and so the extension of \( M(E_7) \) to this group can be computed. It turns out that this character is no longer rational \([7, pp.64–66]\), but has an irrationality \( \zeta_8 + \zeta_8^{-1} \), where \( \zeta_8 \) is a primitive 8th root of unity. If \( q \not\equiv \pm 1 \mod 8 \) then this does not lie in \( \mathbb{F}_q \), and so clearly \((H \times 2).2\) cannot embed in \( 2 \cdot E_7(q) \), as claimed.

If \( q \equiv \pm 1 \mod 8 \) then \((H \times 2).2\) embeds in \( \text{Sp}_{56}(q) \), at least. Since \( H \) acts irreducibly on \( M(E_7) \), there exists a unique group \( X \cong 2 \cdot E_7(q) \) in \( \text{Sp}_{56}(q) \) containing a given copy of \( H \). Thus if \( g \in \text{Sp}_{56}(k) \) normalizes \( H \) then it normalizes \( X \). Letting \( g \in \text{Sp}_{56}(q) \) normalize \( H \) and induce \( H.2 \), we see that \( g \) normalizes \( X \); but \((X,q) \) cannot be \( 2 \cdot E_7(q).2 \) because this group does not embed in \( \text{Sp}_{56}(q) \) (and only in \( \text{Sp}_{56}(q).2 \)). Thus \( g \in X \), as needed.

The groups are maximal as they are not contained in any other Lie primitive group, by Proposition 7.1 below.

\[ \quad \]
5.3.5 $2B_2(8)$

Here $p = 5$, and the composition factors of $M(E_7)\downarrow_H$ are $(14,14^*)^2$ by [29] Table 6.234. The only module action consistent with the action of a unipotent element is $(14/14^*)^2$, using [20] Table 7. (There is no extension $14^*/14$.) The composition factors on $L(E_7)$ are $14, 14^*, 35_1, 35_2, 35_3$. The 35-dimensional modules are projective, hence become summands, and the module that has the correct action of $u$ has a summand $14/14^*$. Thus the action of $H$ on $L(E_7)$ is also determined up to isomorphism.

Proposition 5.13. If $H \cong 2B_2(8)$ then any subgroup $H \leq G$ is strongly imprimitive.

Proof. Using the Lie product method on $\mathcal{R}^H(L(E_7))$, which is 37-dimensional, we obtain a unique $H$-invariant Lie product on $L(E_7)\downarrow_H$. Notice that $2.H$, which is unique up to isomorphism, possesses an 8-dimensional representation over $\mathbb{F}_5$, hence lies inside $A_7$. The exterior square of this 8-dimensional module is $14/14^*$, and so the action is correct. Notice that there are exactly $q+1$ indecomposable 28-dimensional submodules of this module over $\mathbb{F}_q$.

By a computer check, this 28-dimensional module supports a symmetric bilinear form, and hence $H$ is contained in a $D_4$ subgroup $X$ of $A_7$. This also acts as the sum of two isomorphic 28-dimensional modules, so $X$ stabilizes exactly $q+1$ different 28-dimensional subspaces of $M(E_7)$ that are definable over $\mathbb{F}_q$. Hence all indecomposable 28-dimensional $H$-invariant subspaces of $M(E_7)$ are $X$-invariant, and thus $H$ is strongly imprimitive by Proposition 2.7. \qed

5.4 Lie type $PSL_2(r)$

We come to the main issue. The groups $PSL_2(r)$ for $r \neq p$ form the largest collection of subgroups to consider, and also all of the most difficult cases, at least over the algebraic closure. (The issue for $PSU_3(3)$ was for finite fields.)

Since we have considered $Alt(5)$ and $Alt(6)$ earlier, the cases under consideration here are $PSL_2(r)$ for $r = 7, 8, 11, 13, 19, 25, 27, 29, 37$. Each of these has at least one set of composition factors labelled with ‘$P$’ in the tables in [29] (except for $r = 25$, where we need the case $p = 5$), so we will have to consider each in turn. Some of them, for example $r = 11, 25$, do not yield Lie primitive subgroups, or maximal subgroups. Most of the others do, but there are some we cannot at the moment determine either way. These are $r = 7, 8, 13$.

In this section, we let $H$ be a subgroup of $G$, and write $\bar{H}$ for the image of $H$ in $\bar{G}$ (under the natural map $G \to \bar{G}$).

5.4.1 $PSL_2(37)$

This is the first example where the version of $H$ that embeds in $G$ depends on $p$. If $p \neq 2$, write $H = SL_2(37)$, and if $p = 2$, write $H = PSL_2(37)$. Note that $H$ embeds in $G$, and $H/Z(H)$ does not for $p \neq 2, 37$. The appropriate primes here are $p = 2, p = 3, p = 19$ and $p \neq 2, 3, 19, 37$. If $p \neq 2, 3, 19, 37$, then in [29] Table 6.207 we see that the action of $H$ on $M(E_7)$ is the sum of an 18- and a 38-dimensional module (see also [19]). There are three potential 38-dimensional modules, on which an element of $H$ of order 3 acts with trace 2. One has rational character, the other two are algebraic conjugates.

Next, consider $p = 2$. In [29] Table 6.210 there are three potential Lie primitive sets of factors. As $P(18_1)$ has structure

$$18_1/1/18_2/1/18_1$$

we see that the second set of factors for $M(E_7)\downarrow_H—18_1/1/18_2$—must yield a stabilized line. The first set of factors is $18_1, 38$, so this must be semisimple. An element $u \in H$ of order 2 acts on $H$ with Jordan blocks.
$2^{27}, 1^2$, which does not appear in [29, Table 7], so this case cannot occur. The third set of factors for $M(E_7)_{g} \downharpoonright_H$ is $18^1, 18^2, 1^2$, and so $H$ must act as $P(18)$. Since $H$ embeds in the adjoint group $E_7(\mathbb{C})$, $H$ must embed in $G$, so this is the action on $M(E_7)$.

The reduction modulo 2 of the two algebraically conjugate characters for characteristic 0 is the case where $H$ stabilizes a line (these 38s remain irreducible modulo 2). Since the (connected component of the) line stabilizers on $M(E_7)$ are contained in an $E_6$-parabolic and a $D_6$-parabolic (see Proposition [21,12]), into neither of which a cover of $H$ embeds, the rational character, which reduces modulo 2 to the case not stabilizing a line, must be the case yielding an embedding of $H$ into $G$. (This agrees with the analysis of [32], which constructs the characteristic 0 case.)

For $p = 19$ there are again three possible 38-dimensional cases, but we cannot reduce modulo 2 in this case. We show in the supplementary materials using the Lie product method that there is a unique subgroup $H$ in this case, and so the action on $M(E_7)$ must be the reduction modulo 19 of that in characteristic 0. For $p = 3$, there is an ambiguity in [29, Table 6.209]; this can be resolved once one includes the action of an order 9 in $H$. This must act on the unique module $M(E_7)_{g} \downharpoonright_H = 18^1 \oplus 38$ with Jordan blocks $9^6, 1^2$. From [20], we see that $u$ lies in class $E_6(a_1)$, and acts on $L(E_7)$ with blocks $9^4, 7$. In particular, this means that $L(E_7)_{g} \downharpoonright_H$ has the structure

\[ 19_i/19_{3-i}/19_{i}/19_{3-i}/19_i/19_{3-i}/19_i, \]

and $i$ can be determined by the trace of an element of order 37 in $H$.

With this description of the actions of $H$ on $M(E_7)$ and $L(E_7)$, we can now state our result.

**Proposition 5.14.** Let $p \neq 37$. If $p = 2$ then write $H$ for the group $PSL_2(37)$, and if $p$ is odd write $H$ for the group $SL_2(37)$.

(i) There is a unique $G$-conjugacy class of subgroups $H$. When $H \leq G$, if $p \neq 2$ then there are exactly two $G$-conjugacy classes of subgroups $H$, and if $p = 2$ there is a single $G$-conjugacy class. Furthermore, $N_G(H) = H$.

(ii) $H$ embeds in $G$ if and only if $x^2 + x - 9$ splits over $\mathbb{F}_q$ (which is true if and only if $q$ is a square modulo 37).

(iii) If $N_G(\bar{H})$ is maximal in $\bar{G}$, then $q$ is minimal such that $x^2 - x + 9$ splits over $\mathbb{F}_q$, and $\bar{G} = G$ or $q = p^2$ and $\bar{G}$ induces the field automorphism on $G$.

**Proof.** The proof proceeds in stages.

**Determination of $N_G(H)$** This is easy: since the composition factors of $M(E_7)_{g} \downharpoonright_H$ are not stable under the diagonal automorphism of $H$, $N_G(H) = H$.

**Determination of $H$ up to conjugacy** For $p \neq 2, 3, 19, 37$ this is accomplished in [32], and we obtain a unique class. (We independently verify this in the supplementary materials, using the subalgebra method.) For $p = 2, 19$ one may use the space $R^H(E_7)$. The space has dimension 58 for $p = 19$ and 60 for $p = 2$. In each case one finds a unique $H$-invariant $\tau$-Lie algebra structure on $L(E_7)_{g} \downharpoonright_H$. For $p = 3$, although $R^H(L(E_7)_{g} \downharpoonright_H)$ has dimension 58, the equations are very complicated, and it seems difficult to proceed in this way. One instead uses the subalgebra method from Section [21]. Since the action of $H$ on $M(E_7)$ is $18^1 \oplus 38$ and each of these is a symplectic space, $H$ is unique up to $Sp_{56}(k)$-conjugacy by Proposition [21,15].
The subalgebra method also yields a unique $\tau_7$-subalgebra (this time up to conjugacy in the centralizer of $H$ in $\text{Sp}_{56}(k)$).

Thus there is a unique $G$-conjugacy class in all cases. The number of $G$-classes follows from Lemma 2.13.

**Action of outer automorphisms** Let $\sigma$ denote a field automorphism $F_q$. Since there is a unique $G$-class, we may choose $H$ in the class such that $\sigma$ normalizes it. Either $\sigma$ acts as an outer automorphism on $H$, so induces the diagonal automorphism, or as an inner automorphism, so up to conjugacy $H \leq G^\sigma$ by Lemma 2.11.

To decide if $\sigma$ acts as an outer or inner automorphism on the subgroup we may examine the character. Since the action of $H$ on $M(E_7)$ is the sum of the 38 and one of the two 18s that are permuted by the diagonal automorphism of $H$, $\sigma$ centralizes $H$ (up to conjugacy) if and only if $M(E_7) \downarrow_H$ is definable over $\mathbb{F}_q$, and acts as an outer automorphism on $H$ if and only if it is definable only over $\mathbb{F}_q^{\bar{a}}$. The irrationalities in the character satisfy $x^2 \pm x - 9$, as claimed in the proposition.

By Lemma 2.13 the diagonal automorphism always swaps two $G$-classes.

\[ \square \]

**5.4.2 PSL$_2(29)$**

As with the previous case, the group $H$ depends on $p$: if $p = 2$, let $H = \text{PSL}_2(29)$, and if $p \neq 2$ then let $H = \text{SL}_2(29)$. The appropriate primes here are $p = 2$, $p = 3$, $p = 5$, $p = 7$ and $p \neq 2, 3, 5, 7, 29$. If $p \neq 2, 3, 5, 7, 29$ then the action of $H$ on $M(E_7)$ is the sum of two non-isomorphic 28-dimensional modules (there are four such modules with non-rational character, on which an element of order 3 has trace 1). The same holds for $p = 2, 7$. However, for $p = 3, 5$, the two 28-dimensional composition factors are isomorphic, and there is a (unique) non-trivial self-extension. In these two cases, the action of $u$ of order $p$ on $M(E_7)$ does not appear in [20 Table 7] if $M(E_7) \downarrow_H$ is semisimple, so the module is the self-extension. In both cases, the 1-cohomology of the symmetric square of the 28 is 1-dimensional, and so there is a unique $\text{Sp}_{56}(k)$-class of subgroups $H$ with this module structure by Corollary 2.10.

The composition factors of $L(E_7) \downarrow_H$ are uniquely determined for all primes, given the factors for $M(E_7)$, by [29 Tables 6.202–6.206]. The structure of $L(E_7) \downarrow_H$ is easy to determine using these composition factors and the action of a unipotent element of order $p$ in $H$, via [20 Tables 7 and 8].

We now state our result.

**Proposition 5.15.** Let $p \neq 29$. If $p = 2$ then write $H$ for the group $\text{PSL}_2(29)$, and if $p$ is odd write $H$ for the group $\text{SL}_2(29)$.

(i) If $p \neq 5$ then there are exactly two $G$-conjugacy classes of subgroups $H$, and if $p = 5$ then there is a unique $G$-class. If $H \leq G$, and $p = 2, 5$ or $p \neq 2, 5$, then there are exactly two or four $G$-conjugacy classes of subgroups $H$ respectively. Furthermore, $H = N_G(H)$.

(ii) $H$ embeds in $G$ if and only if both $x^2 - x - 1$ and $x^2 \pm x - 7$ split over $\mathbb{F}_q$ (which is true if and only if $q \equiv 0, \pm 1 \pmod{5}$ is a square modulo 29).

(iii) If $N_G(H)$ is maximal in $\bar{G}$, then $p \neq 5$, and $q$ is minimal such that $q \equiv \pm 1 \pmod{5}$ is a square modulo 29. Either $\bar{G} = G$, or $q = p^2$ for $p \equiv \pm 1 \pmod{5}$ a non-square modulo 29 and $\bar{G}$ induces the field automorphism on $G$.

**Proof.** The proof proceeds in stages.
Determination of \( N_G(H) \)  This is easy: since the composition factors of \( L(E_7) \downarrow_H \) are not stable under the diagonal automorphism of \( H \), \( N_G(H) = H \).

Determination of \( H \) up to conjugacy  From [32], we see that for \( p \neq 2, 3, 5, 7, 29 \), each \( \text{Aut}(H) \)-class of representation yields a unique conjugacy class of subgroups \( H \) of \( G \). (We independently verify this in the supplementary materials, using the subalgebra method.) Using the subalgebra method for \( p = 2, 3, 5, 7 \), we find the same result.

If \( p = 5 \) then \( H \) is contained in a copy of \( Ru \) by Proposition [7,4] below, and this must be the same embedding as the actions on \( M(E_7) \) and \( L(E_7) \) are uniquely determined.

Action of outer automorphisms  Let \( \sigma \) denote a field automorphism \( F_q \). If \( p = 5 \) then in the supplementary materials we prove that \( H \) embeds in \( E_7(5) \), as does \( Ru \) (see Section [32.2]). Thus \( H \) is contained in a \( \sigma \)-stable copy of \( Ru \).

In all other cases, there are two \( G \)-classes, so we first decide if \( \sigma \) swaps the two classes or stabilizes them. An element of order 5 acts on the two representations with complex numbers satisfying the equation \( x^2 - x - 1 = 0 \). Thus these lie in \( F_q \) if and only if \( q \equiv \pm1 \mod 5 \). Hence \( \sigma \) interchanges the two representations if \( q \equiv \pm2 \mod 5 \) and stabilizes the two if \( q \equiv \pm1 \mod 5 \).

Thus we may assume that \( \sigma \) stabilizes the class. Since the action of the diagonal automorphism of \( H \) interchanges the two simple modules of dimension 15 (14 for \( p = 2 \)) and they appear with differing multiplicities in \( L(E_7) \), we can determine if \( \sigma \) acts as an outer automorphism on \( H \) by whether it interchanges these two modules. The irrationalities in the 15-dimensional (and 14-dimensional) modules are for elements of order 29, and they satisfy \( x^2 - x - 7 = 0 \). This equation splits over \( F_q \) if and only if \( q \) is a square modulo 29.

By Lemma [2.1.3], the outer diagonal automorphism of \( G \) always swaps two \( G \)-classes. \( \square \)

5.4.3 PSL\(_2\)(27)

Again, the group \( H \) depends on \( p \) if \( p = 2 \), let \( H = \text{PSL}_2(27) \), and if \( p \neq 2 \) then let \( H = \text{SL}_2(27) \). The appropriate primes here are \( p = 2, p = 7, p = 13 \) and \( p \neq 2, 3, 7, 13 \). Examining [32] Tables 6.195–6.201, we find two options for the composition factors of \( M(E_7) \downarrow_H \) if \( p \) is odd: either two non-isomorphic 28-dimensional modules for \( p \neq 13 \) (and necessarily we have the direct sum) or \((14,14^*)^2 \) for \( p = 7, 13 \). If \( p = 2 \) then [29] Table 6.201] shows that there are three possible sets of composition factors, two of which yield strongly imprimitive subgroups. The remaining set is, again, the sum of two non-isomorphic 28-dimensional modules. For \( p \neq 13 \) there are six possible options for this sum, all algebraically conjugate. Notice that the field automorphism of \( \text{SL}_2(27) \) of order 3 acts semiregularly on the six 28-dimensional modules, and thus there are up to automorphism of \( H \) exactly two possible representations (the diagonal automorphism stabilizes all six modules).

If \( p = 7 \) and the factors on \( M(E_7) \) are \((14,14^*)^2 \) then the composition factors on \( L(E_7) \) are \( 26^5, 1^3 \), and the projective cover of \( 26 \) has composition factors \( 26^4, 1 \). Thus \( H \) always stabilizes a line in this scenario.

Thus the only case where we do not yet understand the module \( M(E_7) \) is \( p = 13 \), and the composition factors of \( M(E_7) \downarrow_H \) are \((14,14^*)^2 \). The action of \( u \in H \) of order 13 on \( 14^\pm \) has Jordan blocks 13, 1, and the action on \( 14^\pm/14^\mp \) has blocks 13\(^2\), 2. Finally, the action on \( 14^\pm/14^\mp/14^\tilde{\pm}/14^\tilde{\mp} \) has blocks 13\(^4\), 4. The actions 13\(^4\), 1\(^4\) and 13\(^4\), 4 are valid actions for \( E_7 \) (see [29] Table 7]), corresponding to 13\(^{10}\), 1\(^3\) and 13\(^{10}\), 3 on \( L(E_7) \) respectively, but 13\(^4\), 2\(^2\) does not exist.
For $L(E_7)\downarrow H$, the composition factors are $27^2, 1$ and a sum of three of the six non-isomorphic 26-dimensional modules. The $27^2, 1$ must assemble into $27/1/27$ (or else $H$ stabilizes a line on $L(E_7)$ and is strongly imprimitive by Lemma 2.6), and the 26s are projective: $u$ acts on $27/1/27$ with blocks $13^4, 3$, so $M(E_7)\downarrow H$ must be unserial.

It remains to describe the action of $H$ on $L(E_7)$ for $p \neq 13$. If $p \neq 2, 3, 7, 13$ then it is a sum of five distinct modules, of dimensions $28, 27, 26, 26, 26$. If $p = 7$ the composition factors are $28, 26^2, 1$, and from the factors of $P(26)$ given above, we see that $L(E_7)\downarrow H = 28 \oplus P(26)$. Finally, for $p = 2$ the composition factors are $28, 26_1, 26_2, 26_3, 13, 13^*, 1$. The 28- and 26-dimensional modules must split off, and by examining the action of $u \in H$ of order 4, we obtain that the $13, 13^*$ must assemble into module of the form $13/1, 13^*$ (for some choice of labels for 13 and 13*).

In all cases then, the actions of $H$ on $M(E_7)$ and $L(E_7)$ are determined uniquely up to isomorphism (given the caveat about algebraic conjugacy).

**Proposition 5.16.** Let $p \neq 3$. If $p = 2$ then write $H$ for the group $PSL_2(27)$, and if $p$ is odd write $H$ for the group $SL_2(27)$. Suppose that $H$ is not strongly imprimitive.

(i) If $p \neq 13$ then there are exactly two $G$-conjugacy classes of subgroups $H$, and if $p = 13$ then there is a unique $G$-class. If $H \leq G$, and $p = 2, 13$ or $p \neq 2, 13$, then there are exactly two or four $G$-conjugacy classes of subgroups $H$ respectively. Furthermore, $N_G(H) = H$ for $p \neq 13$ and $N_G(H) = H.3$ for $p = 13$.

(ii) $H$ embeds in $G$ if and only if $q \equiv 0, \pm 1 \text{ mod } 13$.

(iii) If $N_G(H)$ is maximal in $G$, then $q$ is minimal such that $q \equiv 0, \pm 1 \text{ mod } 13$. Either $G = G$, or $q = p^3a$ for $a = 1, 2$ and $G$ induces the field automorphism of order 3 on $G$, with $|N_G(H) : H| = 3$.

**Proof.** As usual by now, we proceed in stages.

**Determinant of $N_G(H)$** In all cases, the composition factors of $H$ on $M(E_7)$ and $L(E_7)$ are invariant under the diagonal automorphism, so in theory $N_G(H)$ could include that. However, for $p \neq 13$ the composition factors of $L(E_7)\downarrow H$ are not invariant under the field automorphism (it’s easiest to see for $PSL_2(27)$) and so $|N_G(H) / H| \leq 2$ if $p \neq 13$.

Assume first that $p \neq 13$. Note that $H$ extends to $H.2$ in $Sp_{56}(k)$. For $p \neq 2, 7, 13$ in [10], and for $p = 2, 7$ in the supplementary materials, we show that this normalizing element swaps two classes of $e_7$-subalgebras of the Lie algebra of $Sp_{56}(k)$, and hence cannot normalize it.

(For $p \neq 2, 3, 13$, we find that there is no extension of $L(E_7)\downarrow H$ to a module for $PGL_2(27)$ where the traces of the classes of involutions are those of involutions in (adjoint) $E_7$, offering another proof for these primes.)

For $p = 13$, however, this is all false. The $H$-actions on $M(E_7)$ and $L(E_7)$ are extensible to both diagonal and field automorphisms of $H$. However, the module $M(E_7)\downarrow H$, which has structure $14/14^* / 14/14^*$ is not extensible to $H.2$, since the diagonal automorphism swaps 14 and 14*, hence has 28-dimensional simple modules. However, in the supplementary materials we do find an element of order 3 in $Sp_{56}(k)$ that induces the field automorphism on $H$ and lies in $E_7(k)$.

**Determinant of $H$ up to conjugacy** From [10], we see that for $p \neq 2, 3, 7, 13$, each $Aut(H)$-class of representation yields a unique conjugacy class of subgroups of $G$. Using the subalgebra method for
In addition, since the field automorphism of $H$ the field automorphism or the product of a field and a diagonal. Thus if $L \subseteq N \subseteq H$ acts on $M$, see that $H$ inner automorphism of $A$. $q$ and $\zeta$ and contains $F$. Furthermore, by $[5]$ we have that the $3 \times 3 \times 26$ on $E_7$, since it would still lie in $F_4A_1$. If $H$ is normalized by $F_q$ and contains $L$, then $F_q$ cannot act as a diagonal automorphism, since that results in a $3 \times 26$ on $L$. In addition, since the field automorphism of $H$ does not stabilize the composition factors, $F_q$ cannot induce the field automorphism or the product of a field and a diagonal. Thus if $F_q$ normalizes $H$ then it acts as an inner automorphism of $H$. In particular, $H \leq E_7(q)$ (up to conjugacy).

Since diagonal automorphisms have order 2 and $|N_G(H) : H|$ is either 1 or 3, we see that $N_G(H) = N_G(H)$. Thus Lemma 2.13 implies that the diagonal automorphism of $G$ fuses two $G$-classes of subgroups $H$.

This completes the proof. \qed

5.4.4 PSL$_2(25)$

If $p \neq 5$ then all examples for either PSL$_2(25)$ or SL$_2(25)$ are strongly imprimitive by [29], Tables 6.191–6.194. Of course, $p = 5$ is a defining-characteristic embedding, and is one of the cases left from [9], and $H \cong SL_2(25)$. In this case, the actions of $H$ on both $M(E_7)$ and $L(E_7)$ were completely determined in [9]. We will prove that any such $H$ is strongly imprimitive, using the subalgebra method.

Proposition 5.17. If $p = 5$ and $H \cong SL_2(25)$ is a subgroup of $G$ then $H$ is strongly imprimitive.

Proof. We use the notation from [9] for the modules for SL$_2(25)$, with $2_1$ being the natural module and $3_1$, $4_1$ and $5_1$ being its symmetric powers. Write $2_2$ and so on for the image of $2_1$ under the Frobenius morphism. Write $6_{1,2} = 2_1 \otimes 3_2$, $10_{2,1} = 2_2 \otimes 5_1$, and so on.
The action of $H$ on $M(E_7)$ is

$$10_{2,1} \oplus (12_{1,2}/21, 41, 6_{1,2}, 10_{1,2}/12_{1,2}),$$

and the action on $L(E_7)$ is

$$(15_{2,1}/8_{2,1}/1, 3_2/8_{2,1}/15_{2,1}) \oplus (15_{1,2}/8_{1,2}/1, 3_1/8_{1,2}/15_{1,2}) \oplus (3_2/8_{2,1}, 16/3_2) \oplus 3_1.$$ 

Up to conjugacy in $Sp_{56}(k)$, there is a unique conjugacy class of subgroups $H$ with this action on $M(C_{28})$. To see this, first note that because the two summands have no composition factors in common, any alternating form is a sum of alternating forms for the two summands, so $H$ lies in a copy of $Sp_{10}(k) \times Sp_{46}(k)$. If the form is unique (up to conjugacy) on both factors then it is unique up to conjugacy by Proposition 2.14. Of course, this holds for the first factor because $H$ acts irreducibly on it. Let $M$ denote the 46-dimensional summand.

The 12-space in the socle is always totally isotropic; a short computer program shows that any symplectic form on the 46-space is the sum of the standard form (which vanishes on $12_{1,2}$) and a form of rank 12 (and hence vanishes on the Jacobson radical of the module). Thus the $12_{1,2}$ is always totally isotropic and $H$ lies in a $C_{11}A_{11}T_1$-parabolic subgroup of $Sp_{46}(k)$. Notice that the symplectic forms on $M$ are isomorphic to a projective line over $k$.

We clearly see a 1-dimensional unipotent subgroup that centralizes the corresponding subgroup of $GL_{46}(k)$, and since the map has rank 12 and squares to 0, any (non-trivial) element $v$ in the subgroup has Jordan blocks $2^{12}, 1^{22}$. If $v$ lies in a copy of $Sp_{46}(k)$ then its centralizer is a positive-dimensional subgroup whose reductive part has type $C_{11}D_6$ by [28] Theorem 3.1(iv). But the module $12_{1,2}$ supports a symplectic form, not an orthogonal form, and so $H$ cannot also lie in this copy of $Sp_{46}(k)$. We see therefore that $v$ must act semiregularly on the symplectic forms on $M$ that have rank 46 (and must stabilize the unique one of rank 12). This means that the unipotent subgroup regularly permutes the symplectic forms of rank 46. This means that there is a unique conjugacy class of $SL_2(25)$ subgroups acting on the minimal module $M(Sp_{46}(k))$ as $M$.

This proves that $H$ is unique up to conjugacy in $Sp_{56}(k)$, so we may choose any representative for the form. We can now apply the subalgebra method, and in the supplementary materials we see that this subgroup is unique up to $G$-conjugacy.

In [9] we stated that $H$ would be strongly imprimitive if this were the case. Let us see this. Let $X$ be the $A_1A_1$ maximal subgroup of $G$. In [27] Table 10.2 we see that for $p = 5$ the subgroup $X$ acts with summands of dimensions 10 and 46. Subgroup 19 from [38] Table 7 (with $r = 0, s = 1$) contains a copy of $SL_2(25)$ with the same actions on $M(E_7)$ and $L(E_7)$, so this is $G$-conjugate to $H$. Thus we may assume that $H \leq X$.

All 10-dimensional $H$-submodules of $M(E_7)$ are stabilized by $X$, so $H$ is strongly imprimitive by Proposition 2.17.

5.4.5 PSL$_2(19)$

This section is separated from the next one because there is a class of Lie primitive subgroups PSL$_2(19)$ and a class of Lie primitive subgroups PGL$_2(19)$. The second of these is due to Serre [35] (although uniqueness for all fields of odd characteristic is proved here) but this first class is new.

We start with considering subgroups $L$ of $G$ of the form $19 \times 9$, where $p \neq 19$. Such groups have simple $kL$-modules of dimensions 1 and 9, with the two 9-dimensional modules being dual. Suppose that $L$ acts on $M(E_7)$ with composition factors $(9, 9^*)^3$, $1_a, 1_b$, where $1_a$ and $1_b$ are two 1-dimensional modules. If $p \neq 19$
then \( L \) stabilizes a line on \( M(E_7) \): if \( p \neq 3, 19 \) then the action is semisimple, and if \( p = 3 \) then the 9-dimensional factors are projective and so break off as summands. Thus \( L \) lies in a line stabilizer, and cannot lie in a \( D_6 \)-parabolic or a subgroup of type \( F_4 \) by comparing dimensions of composition factors on \( M(E_7) \). Hence \( L \) lies in an \( E_6T_1 \)-parabolic subgroup by Proposition \[\ref{2.12}\] as \( L \) has no subgroup of index 2, so if it lies in a group \( E_6T_2 \) it lies in \( E_6 \).

If \( p \neq 3 \) then \( L \) must lie in the \( E_6T_1 \)-Levi subgroup, and if \( p = 3 \) then \( L \) acts on \( M(E_6) \) with factors of dimension 9 only, hence has no 1-cohomology on the module. Thus there is a single conjugacy class of complements in the \( E_6T_1 \)-parabolic, and again \( L \) lies in \( E_6T_1 \). There is a unique such conjugacy class of subgroups \( L \) in \( E_6 \) by \[\ref{2.12} \) Proof of Proposition \( 5.20 \), so if the modules \( 1_a \) and \( 1_b \) are trivial modules then \( L \) is uniquely determined up to conjugacy. (This is the case for \( p = 3 \).

If \( 1_a \) and \( 1_b \) have kernel of index 3 in \( L \) then \( L \) is again unique up to conjugacy: there are two such diagonal subgroups, and they must be interchanged in the subgroup \( E_6T_1 \).2 (the normalizer of the \( E_6T_1 \)-Levi subgroup in \( E_7 \)). We also see that the graph automorphism normalizes the subgroup \( L \) inside the \( E_6 \) factor, and forms a subgroup \( 19 \times 18 \) in the adjoint version of \( E_7 \). Since such a subgroup must lie in \( E_6T_1 \), the other conjugacy class of subgroups \( L \) cannot be contained in a subgroup \( 19 \times 18 \) in the adjoint version of \( E_7 \).

In particular, we see that if \( \text{PSL}_2(19) \) lies in the adjoint version of \( E_7 \) and contains one of these subgroups \( L \), then it cannot extend to \( \text{PGL}_2(19) \) unless the subgroup \( L \) is the one lying inside \( E_6 \) rather than diagonally in \( E_6T_1 \).

With these preliminaries considered, we return to groups \( \text{PSL}_2(19) \). From \[\ref{20} \) Tables 6.184–6.190] we see that for \( p \) odd any copy of \( \text{PSL}_2(19) \) is strongly imprimitive, so if \( p \) is odd, let \( H \cong \text{SL}_2(19) \). If \( p = 2 \) we let \( H \cong \text{PSL}_2(19) \), as usual.

One of the possibilities in characteristic 0 is that \( H \) acts on \( M(E_7) \) as \( 18 \oplus 18' \oplus 20 \), where \( 18 \) and \( 18' \) are two non-isomorphic (faithful) 18-dimensional modules for \( \text{SL}_2(19) \) and the 20 is the faithful module with rational character. The action on \( L(E_7) \) is

\[
18_1 \oplus 18_2^{\oplus 2} \oplus 19 \oplus 20_1 \oplus 20_2 \oplus 20_3,
\]

where \( 18_1 \) and \( 18_2 \) are two of the four non-isomorphic 18-dimensional modules and the \( 20_i \) are all three 20-dimensional modules for \( \text{PSL}_2(19) \) that have non-rational character.

For the reduction modulo 5 of this module, the two modules of dimension 18 in \( M(E_7) \) become isomorphic, and using the action of an element of order 5 in \( H \) and \[\ref{20} \) Table 7], as in previous cases, we must obtain a module of the form \( 20 \oplus (18/18) \).

For \( p = 3 \) the reduction of this and the \( \text{PGL}_2(19) \) case in the next section are the same, and we delay treatment of this until that section.

For \( p = 2 \) the two 18-dimensional modules in \( M(E_7) \) again become isomorphic, and we again are forced to have a module \( 20 \oplus (18/18) \).

We have the following proposition.

**Proposition 5.18.** Let \( p \neq 3, 19 \). If \( p = 2 \) then write \( H \) for the group \( \text{PSL}_2(19) \), and if \( p \) is odd write \( H \) for the group \( \text{SL}_2(19) \). Suppose that \( H \) is a subgroup of \( G \) and that \( M(E_7) \downarrow H \) contains a composition factor of dimension 20. If \( p = 2 \), suppose that \( H \) does not have a composition of dimension 9 on \( M(E_7) \).

(i) There are exactly two \( G \)-conjugacy classes of subgroups \( H \) unless \( p = 5 \), in which case there is exactly one. If \( H \leq G \) and \( p \neq 2, 5 \) there are exactly four \( G \)-conjugacy classes of subgroups \( H \), and if \( p = 2, 5 \) and \( H \leq G \) there are two \( G \)-conjugacy classes. Furthermore, \( H = N_G(H) \).
(ii) $H$ embeds in $G$ if and only if $q \equiv \pm 1 \pmod{5}$, and the orders of $q$ modulo 3 and 19 are both odd or both even, or $q$ is a power of 5.

(iii) If $N_G(H)$ is maximal in $G$, then $q$ is minimal such that $H$ embeds in $G$, and either $\tilde{G} = G$ or $q = p^2$ for $p \equiv \pm 1 \pmod{5}$, and $\tilde{G}$ induces the field automorphism on $G$.

Proof. The proof proceeds in stages.

**Determination of $N_G(H)$** Since the odd part $L \cong 19 \times 9$ of the Borel subgroup $19 \times 18$ of $H$ acts on $M(E_7)$ with two non-trivial 1-dimensional composition factors (and six 9-dimensional ones) we see from the discussion at the start of this section that $L$ is not contained in a group $19 \times 36$ (as $G$ is simply connected). Thus $N_G(H) = H$ for all primes.

**Determination of $H$ up to conjugacy** In the supplementary materials, for $p = 2, p = 5$ and $p \neq 2, 3, 5, 19$ we provide a complete proof of the uniqueness of this case.

For $p \neq 2, 3, 5, 19$ and $p = 5$ we use the subalgebra method from Section 5.2. While it is clear that the form is unique for $p \neq 2, 3, 5, 19$, for $p = 5$ it is not clear.

As in previous cases, we must prove that the alternating form on $M(E_7)_{19}$ is unique up to conjugacy in $\text{Sp}_{56}(k)$. Since the two summands have no composition factors in common, the form is the sum of a form for each factor, and can be scaled independently by Proposition 2.14. The first summand has a unique form; the second yields a subgroup of $\text{Sp}_{56}(k)$, lying in the $A_1T_1$-parabolic subgroup. The 1-cohomology of the module $S^2(18)$ is 1-dimensional, so $H$ is unique up to conjugacy by Corollary 2.10.

For $p = 2$ things are worse because the form is not unique up to $\text{Sp}_{56}(k)$-conjugacy, and in fact there are infinitely many such classes in $G$. We thus must use the modified subalgebra method where we search in $\text{SL}_{56}(k)$ instead of $\text{Sp}_{56}(k)$. We still obtain a unique class of subgroups, however. (We also provide in the supplementary materials a proof using $\text{Sp}_{56}(k)$, but with a 1-parameter family of symplectic forms instead.)

**Action of outer automorphisms** Since $N_G(H) = H$, diagonal automorphisms of $G$ fuse two $G$-classes of subgroups by Lemma 2.13. Thus we must decide on the action of the field automorphism $\sigma = F_p$.

For $p = 2$ we require $F_4$, and for $p = 5$ we require $F_5$ for $M(E_7)$ and $L(E_7)$ to be defined. In general, irrationalities in the Brauer characters of $M(E_7)$ and $L(E_7)$ occur only for elements of order 5, 10, 20, and all satisfy $x^2 \pm 2x - 4$ (for $M(E_7)$) and $x^2 - x - 1$ (for $L(E_7)$). These equations split over $\mathbb{F}_q$ if and only if $q \equiv \pm 1 \pmod{5}$.

Thus we may replace $\sigma = F_p$ by $\sigma = F_{p^a}$ if $p \equiv \pm 1 \pmod{5}$, and assume that $\sigma$ stabilizes the $G$-class of subgroups. Choose $H$ in the class such that $\sigma$ normalizes $H$. We now must determine if $\sigma$ centralizes $H$ (up to conjugacy) or acts as the outer diagonal automorphism on $H$. This is equivalent to asking if $\sigma$ centralizes (up to conjugacy) the subgroup $L \cong 19 \times 9$ of $H$, or equivalently if $L \leq E_7(p^a)$ for a certain $a$.

We have already seen in the discussion at the start of this section that $L$ lies diagonally in an $E_6T_1$-Levi subgroup; let $L_0$ denote the projection of $L$ onto the $E_6$ factor. The subgroup $L_0$ is unique in $E_6$, so the question is whether this lies in $E_6(q)$, $E_6(q)$, or both or neither. Since the graph automorphism of $E_6$ cannot centralize $L_0$ (as it does not lie in $F_4$ or $C_4$) we have that it must act non-trivially on it. Since there is a unique $E_6$-class of subgroups $L_0$, any field automorphism of also normalizes it (up to conjugacy). Thus exactly one of the field and the graph-field must centralize $L_0$, so $L_0$ lies in exactly one of $E_6(q)$ and $E_6(q)$ for each $q$. If $d$ is the order of $q$ modulo 19 (thus $d$ is one of 1, 2, 3, 6, 9, 18), we claim that $L_0$ lies in $E_6(q)$
if and only if $d$ is odd. We first construct $L_0$ inside $E_0(q)$ for $d$ odd, then by duality $L_0$ lies in $^2E_0(q)$ for $d$ even.

Thus assume that $d$ is odd. The result clearly holds for $d = 1$, as $L_0$ lies inside the normalizer of a split torus. For the other cases, note that the orders of $q$ and $q^2$ modulo 19 are the same, and $L_0$ definitely lies in $E_0(q^2)$. On the other hand, the Sylow 19-subgroups of $E_0(q)$ and $E_0(q^2)$ are the same, and the automizers of the Sylow 19-subgroups are the same (as they only depend on $d$). Thus $L_0$ lies in $E_0(q)$ if and only if it lies in $E_0(q^2)$, completing the proof.

We now use this to decide if $L$ lies in $E_7(q)$. If $q \equiv 1 \mod 3$ then $L$ must lie in $E_6(q).q - 1$, and this occurs if and only if $L_0$ lies in $E_6(q)$, so $d = 1, 3, 9$. If $q \equiv -1 \mod 3$ then $L$ must lie in $^2E_6(q).(q + 1)$, and this occurs if and only if $L_0$ lies in $^2E_6(q)$, so $d = 2, 6, 18$. Thus $L$ embeds in $E_7(q)$ if and only if the orders of $q$ modulo 3 and 19 are both odd or both even. In the other case it does not, and so we must move to $E_7(q^2)$, as claimed in the proposition.

\[ \square \]

5.4.6 PSL\(_2(19)\) (Serre embedding)

This time, as we will see that $p = 2$ does not occur, let $H \cong \text{SL}_2(19)$. This case is similar to the previous one, but there are some important distinctions. Here, the action of $H$ on $M(E_7)$ has composition factors of dimensions 10, 10, 18, 18. The two 18-dimensional modules are the same as with the previous embedding, and the two 10-dimensional modules are 10 and $10^*$. The appropriate primes are $p = 3, p = 5$ and $p \neq 2, 3, 5, 19$.

(‘The case $p = 2$ is missing as $10^\pm$ reduces modulo 2 with factors 1 and $9^\pm$, so $H$ stabilizes a line on $M(E_7)$.)

If $p = 5$ then as with the previous case the two 18-dimensional modules become isomorphic, and as before the embedding must have structure $(18/18) \oplus 10 \oplus 10^*$. If $p = 3$ then $M(E_7)^{\downarrow H}$ is semisimple.

We have the following proposition.

\textbf{Proposition 5.19.} Let $H \cong \text{SL}_2(19)$ and $p \neq 2, 19$. Suppose that $H$ is a subgroup of $G$ and that $M(E_7)^{\downarrow H}$ contains a composition factor of dimension 10.

\begin{itemize}
  \item[(i)] There are exactly two $G$-conjugacy classes of subgroups $H$ unless $p = 5$, in which case there is exactly one. In all cases $N_G(H) = H.2$. If $H \leq G$ and $p \neq 5$ then there are exactly four $G$-conjugacy classes of subgroups $H$ if $q \equiv \pm 1, \pm 9 \mod 40$, and if $q \not\equiv \mp 1, \pm 9 \mod 40$ there are two $G$-conjugacy classes. If $q = 5^{2n+1}$ then there is exactly one $G$-class of subgroups $H$, and if $q = 5^{2n}$ then there are exactly two.
  \item[(ii)] $H$ embeds in $G$ if and only if $q \equiv 0, \pm 1 \mod 5$. $H.2$ embeds in $G$ if and only if $q \equiv 0, \pm 1 \mod 5$ and $q \equiv \pm 1 \mod 8$.
  \item[(iii)] If $N_G(H)$ is maximal in $G$, then $q$ is minimal such that $H$ embeds in $G$, and either $\bar{G} = G$, or $q \not\equiv \pm 1 \mod 8$ and $G$ is $G$ extended by the diagonal automorphism.
\end{itemize}

\textit{Proof.} The proof proceeds in stages.

\textbf{Determination of $N_G(H)$} Since this case appears in \cite{35} for $p \neq 2, 3, 5, 19$, and there it is shown that PGL\(_2(19)\) embeds in the adjoint version of $G$ with these composition factors for $H$, we see that $N_G(H) = H.2$ for all primes $p \neq 2, 19$ by Proposition 2.2.

\textbf{Determination of $H$ up to conjugacy} In the supplementary materials, for $p = 3, p = 5$ and $p \neq 2, 3, 5, 19$ we provide a complete proof of the uniqueness of this case. In all cases we use the subalgebra method from Section 3.2. The form is unique in all cases, following the same method as in the previous case for $H$. 

37
Action of outer automorphisms We first decide on the action of the field automorphism $\sigma = F_p$.

For $p = 5$ we require $F_5$ for $M(E_7)$ and $L(E_7)$ to be defined. For $p \neq 5$, irrationalities in the Brauer characters of $M(E_7)$ and $L(E_7)$ occur only for elements of order 5, 10, 20, and all satisfy $x^2 \pm 2x - 4$ (for $M(E_7)$) and $x^2 - x - 1$ (for $L(E_7)$). These equations split over $\mathbb{F}_q$ if and only if $q \equiv \pm 1 \pmod{5}$.

Thus we may replace $\sigma = F_p$ by $\sigma = F_{p^2}$ if $p \equiv \pm 2 \pmod{5}$, and assume that $\sigma$ stabilizes the $G$-class of subgroups. Choose $H$ in the class such that $\sigma$ normalizes $H$. Thus $\sigma$ acts as an inner automorphism on $H_1 = N_G(H)$ and thus by Lemma 2.11 we may assume that $\sigma$ centralizes $H_1$ in the adjoint group (as PGL$_2(19)$ is its own automorphism group). Thus $H \leq G = G^\sigma$ in this case.

It remains to understand the action of the diagonal automorphism, whether it induces the diagonal automorphism on $H$ or fuses two classes of subgroups $H$ in $G$. Equivalently, it remains to understand if $H:2$ lies in $G$, or whether $H = N_G(H)$.

The character of the extension of $M(E_7)\downarrow_H$ can be seen in [21 p.11], or can be found from the matrices in the supplementary materials. Elements of order 8 in $H:2$ act on $M(E_7)$ with Brauer character $\pm \sqrt{8}$. Thus if $\sqrt{8}$ does not lie in $\mathbb{F}_q$ then $H:2$ cannot embed in $G$. We may assume therefore that $q \equiv \pm 1 \pmod{8}$ (which is equivalent to $\sqrt{8}$ lying in $\mathbb{F}_q$ by quadratic reciprocity).

Since $H$ is centralized by $\sigma$ (by assumption), $N_G(H)$ is normalized by $\sigma$. Notice that, since $\mathbb{F}_q$ contains the character values of the two classes of elements of order 8, $\sigma$ cannot induce an automorphism that swaps these two classes. It is an easy exercise, or a quick check with Magma, that Out($H:2$) has order 2, and any outer automorphism indeed swaps these classes. Thus $\sigma$ induces an inner automorphism on $H:2$, so $\sigma$ centralizes some conjugate of $H:2$ by Lemma 2.11 This completes the proof.

5.4.7 PSL$_2(11)$

Let $p \neq 11$ be a prime. The sets of composition factors for PSL$_2(11)$ and SL$_2(11)$ on $M(E_7)$ and $L(E_7)$ are in [29 Tables 6.167–6.173]. From these, we see that if $p$ is odd then PSL$_2(11)$ is strongly imprimitive, and so we may assume that $H \cong$ SL$_2(11)$ for $p$ odd, and $H \cong$ PSL$_2(11)$ for $p = 2$.

We start by proving that $H$ is always strongly imprimitive for $p = 5$, despite being a case labelled ‘P’ in [28 6.170]. Using this we show the same thing for $p \neq 2, 3, 5$. In addition, for $p = 2$ we can also show that $H$ is strongly imprimitive, leaving only $p = 3$. The proof for this case is much more complicated, and requires both a computer calculation and theoretical arguments.

Proposition 5.20. If $p \neq 3$ then $H$ is strongly imprimitive.

Proof. We start with $p = 5$. If $H$ is not strongly imprimitive then $H \cong$ SL$_2(11)$ and the composition factors of $L(E_7)\downarrow_H$ are (up to automorphism) 11$^3$, 10$^3_1$, 5$^5$, 1$^5$. (See [29 Tables 6.169–6.170].) Although this has pressure 3, any such module must have a trivial submodule. To see this, note that the projective cover of 11 is 11/11/11/11, so for every two trivials we need three 11s in order to not to have a trivial submodule. Since this is not the case, $H$ stabilizes a line on $L(E_7)$. Thus $H$ lies in either a parabolic or maximal-rank subgroup, by Lemma 2.11.

We now show that $H$ cannot lie in such a subgroup, and therefore no such $H$ embeds in $G$ with these factors on $L(E_7)$. The corresponding factors on $M(E_7)$ are 10$_4$, 10$_5$, (6, 6$^*$)$^3$. (It is not important which two modules 10$_4$ and 10$_5$ are.) Notice that the minimal faithful degree for $H$ is 6 and no faithful simple module for $H$ carries a symmetric form. Thus $H$ does not embed in $X$ with $Z(H) \leq Z(X)$ for $X$ of type $A_n$ for $n \leq 5$ and $n = 7$, or of type $D_n$ for $n \leq 5$.

If $H$ embeds in a parabolic subgroup then there is some copy of $H$ with the same composition factors in the Levi subgroup $X$. From above we see that $X$ is of type $D_6$ or $A_5$, and if $X = D_6$ then $H$ embeds in $G$.
into $A_5$. There are two classes of $A_5$-Levi subgroup of $G$, and they have composition factors as given in, for example, [38] Table 21. One of these lies in $A_7$ and clearly cannot work. The other does work for $M(E_7)$, with $H$ acting irreducibly on $M(A_5)$. However, this subgroup has eight trivial factors on $L(E_7)$, so this is not correct. (It is the other option in [29] Table 6.170.)

If $H$ embeds in a maximal-rank subgroup of $G$ then the same arguments apply, and we see that $H$ lies in $D_6.A_1$ (and then inside the $D_6$-Levi subgroup) or in $A_2.A_5$ (and then inside the $A_5$-Levi subgroup). Since these have already been considered, we prove the result.

We now complete the case $p \neq 2, 3, 5$, so $p = 0$. From [29] Tables 6.167–6.168 there is a unique possible set of composition factors that is labelled $\cdot P \cdot$, and the reduction modulo 5 of these factors for $M(E_7)$ and $L(E_7)$ would be the case above that we have excluded. Thus it does not exist either by Proposition 2.2.

Finally, for $p = 2$, we just show that the two potential cases in [29] Table 6.173 are in fact strongly imprimitive. The projective cover of 5 is

\[ \frac{5}{1, 5^*/5}, \]

and so in any module with trivial factors we require three 5-dimensional factors for every trivial factor (except remember to remove one trivial from $L(E_7) \downarrow_H$ as $L(E_7)^g \neq L(E_7)$). There are ten 5-dimensional factors and five trivial factors in the case from that table, and so $H$ stabilizes a line on $L(E_7)^g$ and is strongly imprimitive as well by Lemma 4.40.

For the rest of this section, let $p = 3$. From [29] Tables 6.171 and 6.172 we find exactly two cases labelled with $\cdot P \cdot$, both for $\text{SL}_2(11)$. Letting $10_1$ denote the 10-dimensional module for $\text{PSL}_2(11)$, the projective cover of $10_1$ is $10_1/1/10_1$. Thus in order for $L(E_7) \downarrow_H$ not to have a trivial submodule we need two copies of $10_1$ for every 1. This is not the case in Row 2 of [29] Table 6.172, so $H$ always stabilizes a line on $L(E_7)$ in that case. The remaining one has factors

\[ 12^1_1, 12^2_2, 10^0_1, 5^*, 1^3. \]

The 12, are projective so split off, we must have $P(10_1)^{\oplus 3}$ in order to avoid stabilizing a line (and hence being strongly imprimitive by Lemma 2.40), and so it remains to deal with 5 and 5*.

We have a projective module of dimension 123 and two 5-dimensional factors, hence at least 43 Jordan blocks of size 3 in the action of $u$ in $H$ of order 3 on $L(E_7)$. We see using [29] Table 8] that this is only consistent with $u$ coming from class $2A_1 + A_1$, acting as $3^{43}, 2^2$ on $L(E_7)$ and $3^{18}, 1^2$ on $M(E_7)$.

The actions of $u$ on $5 \oplus 5^*$ and $5/5^*$ are $3^2, 2^2$ and $3^3, 1$ respectively, so we see that we have the semisimple case. The composition factors of $M(E_7) \downarrow_H$ are $12^3_1, 10^2_2, 6, 6^*$. The actions of $u$ on $10^2_2$ and $10_2/10_2$ are $3^1, 2^4$ and $3^0, 1^2$ respectively, so the second case must be the correct module. Thus $H$ acts on $M(E_7)$ and $L(E_7)$ as the modules

\[12^0_2 \oplus 6 \oplus 6^* \oplus (10_2/10_2), \quad P(10_1)^{\oplus 3} \oplus 12^0_2 \oplus 12^0_2 \oplus 12^0_2 \oplus 5 \oplus 5^*.\]

(This cannot lie in $D_6$ as the 12, stabilize a symplectic form.) This is the reduction modulo 3 of the nonexistent case for $p = 0$ in the proof above, but of course this does not prove that $H$ cannot exist for $p = 3$. It does however show that any such $H$ is Lie primitive.

Our first task when analysing this case is to determine the number of such classes in $\text{Sp}_{56}(k)$, where $k$ is an algebraically closed field of characteristic 3.

**Lemma 5.21.** Let $M(E_7) \downarrow_H$ be as above. There is exactly one conjugacy class of subgroups of $X = \text{Sp}_{56}(k)$ isomorphic to $H$, with action on $M(X)$ as above.
Proof. Let $H$ be as above. We first examine the possible symplectic forms on $6 \oplus 6^*$, $10_2/10_2$ and $12_4^{5,2}$, and then put them together.

Of course, there is a unique symplectic form on $6 \oplus 6^*$. For $10_2/10_2$, a quick calculation (e.g., on a computer) shows that, although $10_2$ supports a symplectic form, there is a 2-space of symplectic forms on $10_2/10_2$, and the socle lies in the radical of all of them. In particular, this means that the socle is degenerate and so the subgroup lies in an $A_6$-parabolic subgroup. (Another way to see this is that if the socle were non-degenerate, $H$ would lie in a subgroup $Sp_{10} \times Sp_{10}$, which acts semisimply on $M(Sp_{20})$.) We may now apply Corollary 2.10 since the statement about the symmetric square is true in this case, to see that $H$ uniquely embeds (up to conjugacy) in $Sp_{20}(k)$.

The last summand is $12_4^{5,2}$. Here an individual submodule $12_4$ is either totally isotropic or non-degenerate. In the first case, this places $H$ into an $A_{11}$-Levi subgroup, and then into a $C_6$-subgroup of $A_{11}$. The alternative places $H$ inside a diagonal $C_6$ inside the $C_6C_6$ maximal-rank subgroup. This yields two potential embeddings of $H$ into $Sp_{24}(k)$. One may check over $F_8$ that there are two distinct classes of subgroups $H$ with this action in $Sp_{24}(k)$. (It is easier to check that there are two non-conjugate symplectic forms, because they yield different orders of centralizers in $Sp_{24}(9)$.) However, over $F_{81}$ these two subgroups are conjugate. (An example of this computation is given in the supplementary materials.)

We can also show this theoretically: if $V = W_1 \oplus W_2$ with $W_1, W_2$ non-degenerate and isomorphic as modules, with $u_1, v_1 \in W_1$ such that $(u_1, v_1) \neq 0$, then let $u_2, v_2 \in W_2$ be images of $u_1, v_1$ under an isomorphism. Then $(u_1 + \lambda u_2, v_1 + \lambda v_2) = (1 + \lambda^2)(u_1, v_1)$, which is zero if $\lambda$ is a square root of $-1$. Thus the $C_6$ inside $C_6C_6$ lies in $A_{11}$.

Since each of these summands shares no composition factors with any others, $H \leq Sp_{56}(k)$ lies inside a product of symplectic groups, $Sp_{12} \times Sp_{20} \times Sp_{24}$. As there are one, one and two forms respectively for these subgroups, one obtains exactly one conjugacy class of subgroups using Proposition 2.14.

For this class, one must show that there is no subgroup $H$ of $E_7(k)$ arising from that symplectic form. The proof of this is computer-based, and in the supplementary materials, but because it is significantly harder than most proofs here, we give an outline of the proof. We use the subalgebra method from Section 3.2.

(i) Let $W_1$ denote the subspace 5 of $L(E_7)_4 H$, $W_2$ be $5^*$, $W_3$ be the 24-dimensional subspace $12_1^{3,2}$ and $W_4$ be the 36-dimensional subspace $12_2^{3,3}$. Let $a$ and $b$ lie in $W_1$, and $c, d$ lie in $W_2$. There are ten copies each of $W_1$ and $W_2$ in $S^2(M(E_7)_4 H)$.

(ii) The commutator $[a, d]$ projects onto $W_3$ and $W_4$, so let $[a, d]_3$ and $[a, d]_4$ denote these projections. We show that $[[a, d]_3, b]$ and $[[a, d]_4, c]$ have zero image on $W_1$ and $W_2$, yielding some equations in the 20 variables for the location of $W_1 \oplus W_2$. This eliminates entirely four of those 20 variables.

(iii) Using the centralizer of $H$ in $Sp_{56}(k)$, we may assume various combinations of variables are either 0 or 1. Together with a few linear relations, this allows us to construct some candidates for $W_1$ or $W_2$ (or both). This includes linear pencils of subspaces as candidates.

(iv) By checking specific elements in these candidates, we find nilpotent elements with incorrect Jordan normal form on $S^2(M(E_7))$. Thus these candidates cannot be correct. This imposes more restrictions on the variables.

(v) Products of elements in $W_1 \oplus W_2 \oplus W_3 \oplus W_4$ always seem to lie in a specific subspace of $S^2(M(E_7))$, whose homogeneous $12_3$- and $12_4$-components consist of a particular subspace of dimension 72, with
three factors from each $12_i$. This 72-space is abelian and commutes with $W_1$. Thus in order not to have a 29-dimensional abelian subalgebra (contradicting Proposition 2.3), the intersection of this 72-space with $W_3 \oplus W_4$ must be a single copy of either $12_3$ or $12_4$.

(vi) Using this extra information, deduce a final contradiction using much more solving of equations, and knowing that we cannot lie in the candidate subspaces.

Thus we obtain the following result.

**Proposition 5.22.** Let $H$ be a central extension of $\text{PSL}_2(11)$ and $p \neq 11$. Then $H$ is strongly imprimitive.

In the supplementary materials we show this for where the $12_2^{\oplus 2}$ comes directly from both $A_{11}$ and $C_6C_6$. Of course, these are conjugate, so they give two independent proofs of the same statement.

6 Cases not determined

In this section we discuss the progress made for groups $\text{PSL}_2(r)$ for $r = 7, 8, 9, 13$. Although we are not able to completely eliminate them, significant progress can be made in all cases, and some primes are eliminated for each.

6.1 $\text{PSL}_2(7)$

The appropriate primes here are $p = 2, p = 3, p = 7$ and $p \neq 2, 3, 7$. For $p = 2$ this the group $\text{PSL}_3(2)$, a defining-characteristic case of rank 2, and was proved to be strongly imprimitive in [10] Proposition 9.1. The groups $\text{PSL}_2(7)$ and $\text{SL}_2(7)$ were proved to always be strongly imprimitive for $p = 3$ in [12] Proposition 4.7.

For $p = 7$ there were two cases that could not be proved to be strongly imprimitive in [9] Propositions 13.4 and 14.4: for $\text{PSL}_2(7)$ in $G$, the actions on $M(E_7)$ and $L(E_7)$ can be

$$7^{\otimes 4} \oplus P(3)^{\otimes 2} \quad \text{and} \quad 7^{\otimes 5} \oplus P(5)^{\otimes 6} \oplus P(3)$$

respectively. For $\text{SL}_2(7)$ in $G$ (with centres coinciding), the actions on $M(E_7)$ and $L(E_7)$ can be

$$P(6)^{\otimes 2} \oplus P(4) \oplus 6^{\otimes 2} \quad \text{and} \quad 7^{\otimes 5} \oplus P(5)^{\otimes 3} \oplus 5^{\otimes 3^{\otimes 3}}$$

respectively.

For $p \neq 2, 3, 7$, from [29] Tables 6.160 and 6.161] there is again one option for $\text{PSL}_2(7)$ and one for $\text{SL}_2(7)$, if the group is not strongly imprimitive. These are

$$8^{\otimes 2} \oplus 7^{\otimes 4} \oplus (3^{\otimes 3^{*}})^{\otimes 2} \quad \text{and} \quad 8^{\otimes 7} \oplus 7^{\otimes 5} \oplus 6^{\otimes 6} \oplus (3^{\otimes 3^{*}})$$

for $\text{PSL}_2(7)$, and

$$8^{\otimes 2} \oplus 6_2^{\otimes 2} \oplus 6_3 \oplus (4^{\otimes 4^{*}})^{\otimes 2} \quad \text{and} \quad 8^{\otimes 7} \oplus 7^{\otimes 5} \oplus 6^{\otimes 3^{\otimes 3}} \oplus (3^{\otimes 3^{*}})^{\otimes 4}$$

for $\text{SL}_2(7)$. (The modules $6_2$ and $6_3$ are swapped by the outer automorphism.)
6.2 $\text{PSL}_2(8)$

Let $H \cong \text{PSL}_2(8)$. The appropriate primes are $p = 2$, $p = 3$, $p = 7$ and $p \neq 2, 3, 7$. For all but $p = 2$, we can prove that $H$ is strongly imprimitive.

Proposition 6.1. If $H \cong \text{PSL}_2(8)$ and $p \neq 2$ then $H$ is strongly imprimitive.

Proof. Since $\text{PSL}_2(8) \cong \mathcal{G}_2(3)'$, if $p = 3$ then $H$ is strongly imprimitive by [10] Proposition 9.4. If $p = 0$ then $H$ stabilizes a line on $L(E_7)$, as we see from [29] Table 6.164, and thus $p = 7$ remains. Here the simple modules are 1, 8 and 7-dimensional projective modules.

The projective cover of 8 is $8/1/8/1/8/1/8$, so in any module we need five 8s for every four trivials, otherwise $H$ stabilizes a line or hyperplane. We see that this does not occur for $H$ from [29] Table 6.165, so $H$ stabilizes a line on $L(E_7)$. In particular, $H$ is strongly imprimitive by Lemma 2.6.

For $p = 2$ there is a case, left open in [3] Proposition 12.5 where $M(E_7)$ is the sum

$$P(4_{1,2}) \oplus P(4_{1,3}) \oplus P(4_{2,3}) \oplus 8,$$

where the $4_{i,j}$ are the 4-dimensional simple modules. Each $P(4_{i,j})$ has a 2-space of alternating forms, so $M(E_7) \downarrow H$ has a 3-parameter space of alternating forms. The action on $L(E_7)^\circ$ is currently undetermined. Its composition factors are

$$8^2, (4_{1,2}, 4_{1,3}, 4_{2,3})^4, (2_1, 2_2, 2_3)^9, 1^{14}.$$

Since $u \in H$ of order 2 acts projectively on $M(E_7)$, $u$ acts on $L(E_7)^\circ$ with blocks either $2^{53}, 1^{27}$ (class $(3A_1)^\sigma$, which is not possible) or $2^{63}, 1^6$ (class $4A_1$).

Such a subgroup embeds in the normalizer of a maximal torus (and certainly exists in $E_7(q)$, as it exists in $3^7 \rtimes W(E_7)$, which is a subgroup of $E_7(2)$), and that copy at least stabilizes a line on $L(E_7)^\circ$.

We cannot prove that any such subgroup $H$ must stabilize a line on $L(E_7)^2$, although that appears to be the case. We can, however, prove it at least for $q = 4$, and hence also for $q = 2$. (This was already proved in [2], but their proof appears not to generalize in any reasonable way.) In theory the proof could extend to all $q$, but would need more sophisticated ideas, possibly involving intersections of multiple parabolics.

In general, if $L$ denotes a Borel subgroup of $H$, then the permutation module of $H$ on the cosets of $L$ is the sum $1 \oplus 8$. Thus

$$\dim(L(E_7)^\circ)_L = \dim(L(E_7)^\circ)_H + 2,$$

since there are two copies of 8 in $L(E_7)^\circ \downarrow H$. The next result completes the proof for $q = 2, 4$ by showing that the $L$-fixed space has dimension at least 3, thus proving that $H$ stabilizes a line on $L(E_7)^\circ$.

Proposition 6.2. All copies of $L$ in $E_7(4)$ that act as $kL$ on $M(E_7)$ lie in an $A_5$-parabolic subgroup of $E_7(4)$, and centralize at least a 3-space on $L(E_7)^\circ$.

Proof. We will first show that $L$ must be contained in an $A_5$-parabolic subgroup of $G = E_7(4)$. At that point we use a computer to enumerate all copies of $L$ with the correct traces, first in the $A_5$-Levi subgroup and then in an $A_5$-parabolic subgroup of $E_6$. It turns out that, with some obvious constraints, there is a single option for the actions of $L$ on $M(E_6)$ and $L(E_6)$. Furthermore, there is no extension of these modules with top $L(E_6)$ and socle $M(E_6)^\circ$ that does not centralize a 3-space, proving the result.

Since $L$ acts on $M(E_7)$ as the module $kL$, it centralizes a line on it, and this line is not complemented. This means that this line must be centralized by either an $E_6$-parabolic subgroup or a group $U \cdot C_5(q)$. (It
cannot be in a group $U \cdot F_4$, since that group centralizes a 2-space on $M(E_7)$. We will eliminate the latter by showing that the trace of an element $x \in L$ of order 7, which is 0 on $M(E_7)$, cannot work.

First, we consider the collection of elements of order 7 in $D_6$ whose action on $M(E_7)$ has trace 0, so they can be conjugated into $L$. By a computer calculation, the trace on $M(D_6)$ is either $-2$ or, for choice of 7th root of unity, $3\zeta_7 + 3\zeta_7^{-1} + \zeta_7^2 + \zeta_7^{-2} + 4$. Since $C_5$ acts on $M(D_6)$ with two trivial composition factors, the former is impossible as it has no eigenvalue 1. The latter is missing two eigenvalues, and no submodule of the free module $kL$ can be constructed with four trivial composition factors and missing one non-trivial module and its dual. Thus $L$ cannot act on $M(D_6)$ with this trace for an element of order 7, and $\tilde{L}$ lies inside an $E_6$-parabolic subgroup of the algebraic group $G$. (Notice not an $E_6T_1$-parabolic, since $L$ centralizes the line.)

The subgroup $\tilde{L}$ cannot lie in a $D_5$-parabolic subgroup. If it did, then $L$ would lie inside a $D_5$-parabolic subgroup. One of these centralizes a 2-space on $M(E_7)$, the other has a submodule of dimension 10 modulo the centralized line. Thus we look for submodules of $kL$ of dimension 11. The trace of $x$ on $M(D_5)$ must be $3\zeta_5^2 + 3\zeta_5^{-2} + \zeta_5^3 + \zeta_5^{-3} + 2$, so again two submodules cannot occur. Using the labelling $i$ for the module on which $x$ acts with eigenvalues $\zeta^i$ (so that the trivial module becomes 0 in this notation), the largest submodule of $kL$ with these factors is

$$(35/0) \oplus (3/5/2) \oplus (2/3) \oplus (5/023/4) \oplus (2/34/5).$$

We must have all three copies of 0, so need the 4 underneath one of them. This means we cannot have the 4 in the last term, so we cannot have the top 2 on that either. This leaves three 2s, all of which must occur in our submodule. But now both 3 and 4 are submodules but neither is a summand, so this module cannot be self-dual.

The consequence of this is that $\tilde{L}$ cannot centralize a line on $M(E_6)$ (for then it would lie in a $D_5$-parabolic, rather than a $D_5T_1$-parabolic).

If $q$ is not a power of 8 then $L$ has composition factors of dimension 1 and 3 on $M(E_6)$. Note also that, under these conditions, $L$ cannot embed in $SL_2(q)$. Letting $\tilde{L}$ denote the image of $L$ in $E_6(q)$, if $\tilde{L}$ embeds in a parabolic subgroup of $E_6(q)$, then it cannot be $D_5T_1$ (as this has a 1-dimensional submodule or quotient on $M(E_6)$), or $A_4A_1T_1$ (as $\tilde{L}$ then lies in the $D_5T_1$-parabolic). If $\tilde{L}$ lies in the $A_2A_2A_1T_1$-parabolic then it lies in the $A_5T_1$-parabolic, and so we may assume that. Furthermore, since $7 \nmid (q - 1)$, $L$ lies inside the $A_5$-parabolic subgroup.

For reductive subgroups, if $L$ lies in $A_2A_2A_2$ then $q$ is a power of 8, since otherwise $L$ cannot embed in $A_2$. If $L$ embeds in $A_1A_5$ then it embeds in $A_5$, so we are in the $A_5$-parabolic again. The only other subgroup is $A_2G_2$, and again $L$ does not lie in either $A_2^2(q)$ or $G_2(q)$ when $q$ is not a power of 8. Thus $L$ lies in the $A_5$-parabolic. This completes the proof of the first statement.

The composition factors of $A_5$ on $M(E_6)$ are $L(\lambda_1)$ twice and $L(\lambda_1)$, the exterior square of the dual of $L(\lambda_1)$. The composition factors of $\tilde{L}$ on $M(E_6)$ are four copies of all non-trivial modules and three trivial modules, and it is an easy calculation that $\tilde{L}$ must act on $M(E_6)$ with factors one of each of the two 3-dimensional modules.

Now we simply enumerate all $F_4L$-modules with these composition factors (there are seven of them, including the semisimple case), and then on a computer construct the group $q^{1+20} \cdot \tilde{L}$, the preimage of $\tilde{L}$ in the $A_5$-parabolic subgroup, for each option for $\tilde{L}$. This group is small enough for $q = 4$ to enumerate all subgroups $L$, so one now has a list of all possible $L$. 

43
We can exclude all those whose action \( M \) on \( M(E_6) \) has a trivial submodule or quotient. We can also exclude those for which \( \text{Ext}^1(M,k) \) is zero. This is because the structure of \( M(E_7) \downarrow_{E_6} \) is \( 1/M^*/M/1 \), and one sees from the action of \( L \) that the non-split extension in the first two layers must remain non-split on restriction to \( L \). We can also exclude with \( \text{soc}(M) \) of dimension greater than 9, because that is the dimension of the socle of \( M(E_7) \downarrow_L \) modulo its fixed point.

Using a computer, we impose these restrictions on the set of such \( L \). We cannot compute conjugacy in \( E_6(4) \), but we can show that all of the \( L \)-actions on \( M(E_6) \) and \( L(E_6) \) are now isomorphic. Thus we can choose one and work with that. Notice that \( L(E_7)^\circ \), viewed as a \( kE_6 \)-module, as socle \( M(E_6)^* \) and second layer \( L(E_6) \). Let \( M \) denote the action of \( L \) on \( M(E_6) \). A direct-sum decomposition of the action of \( L \) on \( L(E_6) \) always has two copies of the projective \( P(1) \), two summands with no fixed point, and a single 19-dimensional summand \( M' \) with a 1-dimensional fixed space.

We find that the group \( \text{Ext}^1(M',M^*) \) is 3-dimensional, and we simply construct all 64 points of this over \( \mathbb{F}_4 \). Each has a fixed point. Together with the two copies of \( P(1) \) that must become summand in any extension, this shows that \( L \) always centralizes a 3-space on \( L(E_7)^\circ \), as needed.

\[ \square \]

### 6.3 PSL\(_2\)(9)

In Section 5.1.1 we noted that \( \text{Alt}(6) = \text{PSL}_2(9) \) and \( 2 \cdot \text{Alt}(6) = \text{SL}_2(9) \) are strongly imprimitive for \( p = 2, 3 \), and this is not known for \( p = 5 \) and \( p \neq 2, 3, 5 \). The open cases are enumerated in [11], and we just list them now.

For \( p = 0 \) and \( H \cong \text{Alt}(6) \), there is a unique possible set of composition factors, namely

\[
10^{\oplus 4} \oplus 8^{\oplus 2} \quad \text{and} \quad 10^{\oplus 2} \oplus 9^{\oplus 3} \oplus 8^{\oplus 4} \oplus 5^{\oplus 3} \oplus 5_2^{\oplus 3}.
\]

For \( p = 0 \) and \( H \cong 2 \cdot \text{Alt}(6) \), there are two sets of composition factors, both acting on \( L(E_7) \) as

\[
10^{\oplus 5} \oplus 9^{\oplus 3} \oplus 8^{\oplus 4} \oplus 8_2^{\oplus 3}.
\]

The two actions on \( M(E_7) \) are

\[
8_4^{\oplus 2} \oplus 10_2^{\oplus 3} \oplus 10_3 \quad \text{and} \quad 8_4^{\oplus 2} \oplus 10_2 \oplus 10_3^{\oplus 3}.
\]

(Here \( 8_4 \) is a specific choice of one of the two 8-dimensional faithful irreducible modules, and this depends on the choice of \( 8_1 \). The two cases correspond to the fact that an outer automorphism of \( H \) acts as \((8_1,8_2)(8_3,8_4)(10_2,10_3)\).) These appear to be missing from Frey’s table [11 Table 14].

For \( p = 5 \), we have the restrictions of the two possibilities for \( \text{Alt}(7) \) and \( 2 \cdot \text{Alt}(7) \) in Section 5.1.1, which appear in [11 Proposition 6.1]. Thus \( H \cong \text{Alt}(6) \) can act on \( M(E_7) \) and \( L(E_7) \) as

\[
10^{\oplus 4} \oplus 8^{\oplus 2} \quad \text{and} \quad 10^{\oplus 2} \oplus 5_1^{\oplus 3} \oplus 5_2^{\oplus 3} \oplus P(8)^{\oplus 3} \oplus 8.
\]

Alternatively, \( H \cong 2 \cdot \text{Alt}(6) \) can act as

\[
10_2^{\oplus 3} \oplus 10_3 \oplus (4_1/4_2) \oplus (4_2/4_1) \quad \text{and} \quad 10_1^{\oplus 5} \oplus P(8)^{\oplus 3} \oplus 8.
\]

We have not attempted it here, but the presence of no 5i modules in the actions on \( L(E_7) \) in three of the five cases might mean that the subalgebra method can be successful. The \( \text{Alt}(6) \) cases though look out of reach.
6.4 \(\text{PSL}_2(13)\)

The appropriate primes here are \(p = 2, p = 3, p = 7\) and \(p \neq 2, 3, 7, 13\). From [29, Tables 6.174–6.180], we find a number of possible sets of composition factors for both \(\text{PSL}_2(13)\) for \(p = 2, 3, 7\) and for \(\text{SL}_2(13)\) for \(p \neq 2\).

Suppose first that \(p\) is odd and \(H \cong \text{PSL}_2(13)\). If \(p = 3\), then the simple \(kH\)-modules in the principal block are 1 and 13. Furthermore, \(P(13) = 13/1/13\). Thus if there are not at least twice as many 13s as 1s in \(L(E_7)\downarrow H\) then \(H\) stabilizes a line on \(L(E_7)\). In fact, [29, Table 6.178] shows that there are at least six 13s and at least four 1s, so \(H\) stabilizes a line on \(L(E_7)\) in all cases.

If \(p = 7\) then the simple \(kH\)-modules in the principal block are 1 and 12. Furthermore, \(P(12) = 12/12/1, 12/12\). Thus if there are not at least four times as many 12s as 1s in \(L(E_7)\downarrow H\) then \(H\) stabilizes a line on \(L(E_7)\). In fact, [29, Table 6.176] shows that there are at most six 12s and at least three 1s, so \(H\) stabilizes a line on \(L(E_7)\) in all cases.

Thus we may assume that \(H \cong \text{PSL}_2(13)\) for \(p = 2\), and \(H \cong \text{SL}_2(13)\) for \(p \neq 2\). For \(p = 2\), and for one case for \(p \neq 2, 3, 7, 13\), we are able to prove that \(H\) is strongly imprimitive. But we make no progress for \(p = 3, 7\), and cannot deal with the second possible set of composition factors for \(p \neq 2, 3, 7, 13\).

**Proposition 6.3.** If \(p = 2\), or \(p \neq 2, 3, 7, 13\) and \(H\) does not act on \(M(E_7)\) and \(L(E_7)\) as

\[
14_2^{\oplus 2} \oplus 14_5 \oplus 14_6 \quad \text{and} \quad 14_1 \oplus 14_2^{\oplus 4} \oplus 13 \oplus 12_1 \oplus 12_2 \oplus 12_4 \oplus 7_1 \oplus 7_2
\]

respectively (this is Row 2 from [29, Table 6.175]), then \(H\) is strongly imprimitive.

**Proof.** We consider \(p \neq 2, 3, 7, 13\) first. From [29, Table 6.175], either \(H\) is certainly strongly imprimitive or the composition factors of \(H\) on \(M(E_7)\) and \(L(E_7)\) are one of two possibilities. One is as in the proposition, and the other is

\[
12_1 \oplus 12_5 \oplus 12_6 \oplus 14_4 \oplus 6_1 \quad \text{and} \quad 14_1 \oplus 14_2^{\oplus 4} \oplus 13 \oplus 12_1 \oplus 12_2 \oplus 12_3 \oplus 7_2^{\oplus 2}.
\]

Because \(M(E_7)\downarrow H\) is multiplicity-free we may use Proposition [2.13] to see that the embedding of \(H\) into \(\text{Sp}_{56}(k)\) is unique. Then it is a standard application of the subalgebra method to prove uniqueness of \(H\) up to \(\text{G}\)-conjugacy. The last part is to notice that \(H\) lies in the \(G_2C_3\) positive-dimensional subgroup, acting irreducibly on both \(M(G_2)\) and \(M(C_3)\). (If one fixes a 7-dimensional simple module for the action on \(M(G_2)\), then the two options for \(M(C_3)\) yield the two sets of composition factors we are considering, so both exist in \(G_2C_3\).) Since \(G_2C_3\) acts on \(M(E_7)\) with composition factors of dimensions 42 and 14, we see that it stabilizes the unique 14-space stabilized by \(H\), and \(H\) is strongly imprimitive by Proposition [2.7].

For \(p = 2\), the simple modules in the principal block are 1, 6\(_1\) and 6\(_2\), and the projective cover of 6\(_i\) is

\[
6_1/1/6_3/1/6_i.
\]

Thus we need at least three modules of dimension 6 for every two trivial composition factors in \(L(E_7)^\circ \downarrow H\) in order for \(H\) not to stabilize a line on the module. Of the two rows of [29, Table 6.180] labelled ‘P’, one does not satisfy this, so \(H\) stabilizes a line on \(L(E_7)^\circ\) in this case.

The other option is that \(M(E_7)\downarrow H\) has composition factors one each of all three (projective) 12-dimensional modules, and then \(6_2^2, 6_2, 1^2\), which must assemble into \(P(6_1)\) if \(H\) does not stabilize a line on \(M(E_7)\). The corresponding factors on \(L(E_7)^\circ\) are

\[
14^4, 12_1, 12_2, 12_3, 6_1, 6_3^3, 1^2,
\]

45
and again to avoid stabilizing a line we must have \( P(6_2) \oplus 6_2 \). In order for the action of an element of order 4 in \( H \) to match up between \( M(E_7) \) and \( L(E_7) \) (using [20] Tables 7 and 8), the 14s must assemble into \( P(14) \oplus 14^{\oplus 3} \). This determines the action on \( L(E_7)^3 \) uniquely. In order to apply the subalgebra method, we need a symplectic form on the 56-dimensional module. Given a form on the \( P(6_1) \), we can apply Proposition [2.14] to obtain a unique alternating form (up to conjugacy in \( \text{Sp}_{56}(k) \)) for the whole module. The space of alternating forms on \( P(6_1) \) is 2-dimensional (so up to scalar multiples it is a projective line), and so in the supplementary materials we introduce a parameter in the form, allowing us to deal with all forms simultaneously.

We find a unique conjugacy class of subgroups up to \( G \)-conjugacy. But this set of factors is the reduction modulo 2 of the set of factors mentioned earlier in this proof in characteristic 0. Thus we find \( H \) inside \( G_2C_3 \). For \( p = 2 \), the composition factors of \( G_2C_3 \) on \( M(E_7) \) have dimensions 36, 8, 6, and 6, with one of dimension 6 being a submodule. Since \( H \) stabilizes a unique 6-space, this proves that \( H \) is again strongly imprimitive via Proposition [2.4].

It remains to discuss \( \text{SL}_2(13) \) for \( p = 3, 7 \). Using [29] Table 6.177 and the structure of \( P(12) \) given earlier in this section, when \( p = 7 \) either \( H \) stabilizes a line on \( L(E_7) \) or the action on \( L(E_7) \) must be

\[
14^{\oplus 2}_1 \oplus 14_4 \oplus 7_2 \oplus 7_\epsilon \oplus P(12),
\]

where \( \epsilon = 1, 2 \). If \( \epsilon = 1, 2 \) then the action of \( H \) on \( M(E_7) \) is

\[
14^{\oplus 2}_2 \oplus 14_4 \oplus 14_5 \text{ and } 14_3 \oplus P(6_1)
\]

respectively. The space of alternating forms on \( P(6_1) \) is 4-dimensional, which makes the computations much harder in this case.

For \( p = 3 \), there are a number of different configurations for the action on the Lie algebra. Using [29] Table 6.179, there are two sets of composition factors for \( H \cong \text{SL}_2(13) \) that need not stabilize a line on \( L(E_7) \). The first has composition factors \( 14^4 \) on \( M(E_7) \). If \( u \in H \) has order 3, then \( u \) acts on 14 with blocks \( 3^4, 1^2 \), so \( u \) acts with at least sixteen blocks of size 3 on \( M(E_7) \). The only allowed action, according to [29] Table 7, is \( 3^{18}, 1^2 \), and therefore \( H \) acts on \( M(E_7) \) as \( P(14) \oplus 14 \). The other set of composition factors is \( 14, 12_4, 12_5, 12_6, 6_1 \), which as it is multiplicity-free must be semisimple. Again, \( u \) acts on this with blocks \( 3^{18}, 1^2 \).

There are two such unipotent classes: \( 2A_2 + A_1 \). They act on \( L(E_7) \) with blocks \( 3^{42}, 1^7 \) and \( 3^{43}, 2^2 \) respectively. In each case the module is a sum of \( P(13) \oplus 12_4 \oplus 12_2 \oplus 12_5 \), and a module with factors either \( \tau_1^4, \tau_2^4 \) or \( \tau_1^7, \tau_2^7 \). There are six ways of distributing these factors so that the unipotent action is as claimed.

7 Proof of maximality

As with the proof given in the previous paper in this series [12], the first sections of this article just found some conjugacy classes of subgroups, and asserted that they are maximal. Let \( \mathcal{S} \) denote the set of subgroups enumerated in Table [1.1] (which of course depends on the prime \( p \)). If the assertion that \( H \in \mathcal{S} \) is maximal is false, then an obstruction to this (i.e., a subgroup \( X \) such that \( H < X < G \)) must come either from \( \mathcal{S} \) as well, or from one of the other maximal subgroups, which are enumerated in Table [1.1] below.

Our first proposition starts to examine the case where \( X \) lies in \( \mathcal{S} \).
Proposition 7.1. Let $H$ be one of the groups $M_{12}$, $M_{22}$, $HS$, $Ru$, $PSL_2(r)$ for $r = 19, 27, 29, 37$, $PSU_3(3)$ and $PSU_3(8)$. Let $X$ be one of these groups or in Table \ref{tab:exotic}. If $H$ is isomorphic to a subgroup of $X$ then the pair $(H, X)$ appears in the table below:

\[
\begin{array}{|c|c|}
\hline
H & \text{Possibilities for } X \\
\hline
M_{12}, HS, Ru, PSL_2(r), r = 19, 27, 37, PSU_3(8) & \text{None} \\
M_{22} & HS \\
PSL_2(29), PSU_3(3) & Ru \\
\hline
\end{array}
\]

Proof. The easiest way to prove this is to look through the Atlas \cite{Atlas} for each $X$ checking if there exists $H$ (or a subgroup with $H$ as a subgroup) as a maximal subgroup, and then use transitivity. One finds the following:

- If $X \cong \text{Alt}(6), M_{12}, M_{22}, PSL_2(r), PSU_3(3), PSU_3(8)$ then there is no possible $H$;

- If $X \cong HS$ then $H \cong M_{22}$;

- If $X \cong Ru$ then $H \cong PSL_2(29), PSU_3(3)$.

To obtain the proposition one simply ‘inverts’ this list. 

Thus we must exclude the options for $H$ and $X$ above. If $H \cong M_{22}$ then we have already dealt with the option $X \cong HS$ in Proposition 5.6. If $H \cong PSL_2(29)$, so $X \cong Ru$, then $p = 5$ and this was already in Proposition 5.16. If $H \cong PSU_3(3)$ and $X \cong 2.Ru$, then the restriction of 28 to $H$ is $7 \oplus 21$, not irreducible. Thus it is not an obstruction to the copy of $H$ we have claimed is maximal. This completes the proof that no element of $S$ is an obstruction to $H \in S$ being maximal.

It remains to consider as possible obstructions the exotic $r$-local subgroups, and the members of $\mathcal{X}$. The only exotic $r$-local subgroup has connected component $D_4$, and this is contained in $A_7$, so we may ignore it, and so we need only consider members of $\mathcal{X}$. Thus assume that $X \in \mathcal{X}$ contains $H$. By Lemma 2.1 unless $H \cong PSU_3(3)$, we may assume that $X$ is connected.

If $X$ is an $E_6$-parabolic subgroup then $X$ stabilizes a line on $M(E_7)$. Thus $H$ cannot lie in $X$ as we do not have such actions. If $X$ is $F_4.A_1$ then either some cover of $H$ has a representation of dimension 2 (which it does not) or $H \leq F_4 \leq E_6$. Thus we may assume that the reductive part of $X \in \mathcal{X}$ is a classical group. In particular, some cover of $H$ must have a faithful representation of dimension at most 12 (there is no $D_7$ subgroup), and at most 8 if it is not self-dual.

If $H$ is one of $M_{22}, HS$ and $Ru$, so $p = 5$, then the minimal faithful dimension of any cover of $H$ is greater than 12, so $H$ cannot lie in a member of $\mathcal{X}$. If $H \cong PSU_3(8)$, so $p$ is odd, then the minimal dimension is 56, so this certainly does not lie in a member of $\mathcal{X}$. Similarly, if $H$ is $PSL_2(r)$ for $r = 19, 27, 29, 37$, then the minimal faithful dimension is $(r - 1)/2$, but these simple modules, and those of dimension $(r + 1)/2$, are not self-dual. The smallest self-dual module has dimension $r - 1$, and so there is no such $X$ for these groups either.

If $H \cong M_{12}$ then $H$ acts on $L(E_7)$ with factors of dimension 55 and 78. The only maximal positive-dimensional subgroups of $G$ acting with two composition factors are $A_7$ and $A_2$, and neither of these acts with the correct dimensions, so $H$ is Lie primitive.

This leaves $H \cong PSU_3(3)$. For $H = PSU_3(3)$, so $p \neq 2, 3$, $H$ acts on $M(E_7)$ with two factors of dimension 28, so if $H \leq X^o$ then $X^o$ either acts irreducibly or with two dual factors. Thus $H$ lies in $A_7$ or $A_2$, with
the latter clearly impossible. If $H$ lies in $A_7$ though then $H$ stabilizes a line on $M(A_7)$ (as $H$ has only a 7-dimensional simple module), so acts on $\Lambda^2(M(A_7))$ with factors of dimension 7 and 21, not 28.

Thus $H \not\leq X^e$, in which case $X$ must be the normalizer of a maximal torus. However, in this case, $X$ has a composition factor of dimension at most 7 on $L(E_7)$ (a subspace of the 0-weight space) but $H$ has no such composition factor. Thus $H$ cannot lie in $X$, and so $H$ is Lie primitive.

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