Quantum Dynamics in Stochastic Mechanics

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(November 6, 2018)

We present a study of the motion that a massive particle in a non-dissipative Brownian motion in vacuum is subject. Noise source comes from vacuum fluctuations of some quantum field capable of interact with the particle. For the associated Fokker-Planck equation we do a perturbation theory which has terms presenting dynamics similar to that satisfied by the probability amplitude in Quantum Theory even though the physical interpretation of these terms be classical. In particular observables present expected values that coincide with those calculated using Quantum Mechanics.
A. Introduction

The probabilistic character of Quantum Mechanics as well as the formal similarity of the Schröedinger equation to the diffusion equation raised from long ago the question of a possible stochastic justification to Quantum Mechanics [1]. A first formal approach was developed by Nelson [2] and up to now many authors presented complementary or alternative works (a few of them listed in [3]–[8]) on the subject named usually as Stochastic Mechanics. A common goal of these theories lie on the possibility of find a stochastic process that is equivalent to Quantum Mechanics. Nevertheless no such a process was yet proposed that lent to Stochastic Mechanics, in any of its presented forms, a explicit advantage over the sophisticated epistemological structure of Quantum Theory. On another way Quantum Theory is continuously tested at deeper levels fixing its place as fundamental one although this do not invalidate the search for alternative interpretations, hopefully not subject to inconsistencies (like infinities) yet plaguing the theory. In this work we focus on a somewhat different approach where a particular kind of stochastic process is studied. We present a perturbation theory for the associated probability density function that has quantum formalism as a subjacent dynamics but still preserving the classical nature of the theory. We also discuss a possible physical origin for the noise source capable of describe the type of process we study. No full equivalence to Quantum Theory was found so this indicates that Quantum Mechanics as a fundamental theory is preserved also. Nevertheless our formulation shows explicitly that quantal aspects may appear quite naturally in a classical theory.

Once every physical process include or can generate some sort of noise the motion of a massive particle under its influence is an important subject in the study of real particle’s dynamics. For a particular class of noise - the Wiener process - a rigorous probability formalism can be obtained where a field equation - the Fokker-Planck equation (FPE) - for the probability density function of some noise-dependent variable is defined [9]. For this case noise source has an uncorrelated stationary probability distribution and when the stochastic process under study describes specifically the motion of a particle the resulting process is the Brownian motion. Due to its generality a FPE can be considered as a fundamental although its detailed predictions are dependent of the specific Wiener process considered.

The origin of noise that a particle in Brownian motion is subject usually is thermodynamic. This is the case of a particle dispersed on a fluid at some temperature. There thermal noise gives a bit of energy to the particle and dissipative effects returns it back (randomly) to the thermal bath. The net effect is a random walk equilibrated to the bath’s temperature. Here we will consider this kind of motion but with two main differences. First the particle is in vacuum and noise is generated by vacuum fluctuations of some quantum field capable of interaction with that particle. Second the motion is not dissipative. This last hypothesis seems to be at first sight unrealistic because particle may gain energy from nothing. We accept it for simplicity since this hypothesis do not invalidate the methods we present here for the general case.

One may argue against the reality of vacuum fluctuations itself. This subject has yet some controversy but its theoretical as well as experimental grounds has been sufficiently discussed in the literature [3] and we can accept it in a general basis. Moreover recent careful measurements of Casimir forces [10] demonstrated a good agreement between observation and theoretical predictions reinforcing our belief in its physical existence. Generally we even can believe that geometry changes in a physically confined systems may induce a small violation of energy conservation principle, undetectable by today’s instrument sensitivity or be possible that during molecular conformational changes some energy comes from vacuum, involving van der Walls force. We think these arguments are sufficient to justify a possible existence of a natural noise source interacting to any elementary particle coming from vacuum properties. Besides all we must also have in mind that every charged particle is subject to the cosmic background radiation field that is real and also stochastic. Therefore we think that stochastic process are inevitable to the motion of massive particles in vacuum, coming as a intrinsic part of the reality in Nature and must be considered in any precise description of this kind of motion.

The method we give here to solve the FPE for the Brownian motion has general validity and no fundamental difficulties are present when the more general case where not only dissipative effects as well as a larger class of Wiener process are considered. Specifically for a quantum noise coming from vacuum fluctuations we understand a stationary Wiener-type source with noise intensity proportional to a ”vacuum power” which we write as \( \mathcal{P} \equiv \frac{1}{2} \hbar \langle \omega^2 \rangle \) and with mean energy \( \mathcal{E} \equiv \frac{1}{2} \hbar \sqrt{\langle \omega^2 \rangle} \) and or generally

\[
\mathcal{P} \hbar = \mathcal{E}^2
\]  

(1)

Here \( \hbar \) is the Planck’s constant and \( \langle \omega^2 \rangle \) is the variance of the field frequencies averaged over some appropriate distribution (we assume \( \langle \omega \rangle = 0 \) since \( \omega \) and \(-\omega\) must be considered as independent fluctuations). For example in the cosmic background case where \( T \simeq 2K \) we find \( \mathcal{P} \simeq 1.15 \mu W \). Calculation of \( \langle \omega^2 \rangle \) for quantum fluctuations is not trivial because vacuum energy density diverges as \( \omega^3 \) [3] with (assumed) uniform probability distribution denying
a simple averaging process unless a physical cutoff at high frequencies exist. This must be achieved by multiplying
the instantaneous vacuum modes by the appropriate cross section (in that frequency) which falls off properly at high
energies. If doing so the instantaneous momentum transferred to the particle may be calculated resulting in a noise
source with zero mean and the desired power in as much as in the Casimir effect calculation where high frequencies
modes cancel out and the net effect comes from low frequencies modes only. This is analogous to the thermal effect
where the enormous kinetic energy of one mol of molecules at room temperature has minimal action over the center
of mass motion of a macroscopic body because the random collisions between the molecules and the body are isotropic
so the net resulting effect are small fluctuations. Nevertheless if the phenomena we describe here has to be universal
it is reasonable that $E$ should be a characteristic measurable constant in Nature even though we concentrate here only
on the effect on particle motion of a noise source with properties listed above and explicit in the perturbative series
expansion procedure we give below. Therefore besides eqn (1) detailed physical origin of $P$ (or $E$) is not relevant to
our calculations in the present stage of development of our reasoning.

B. Perturbation expansion for the Fokker-Planck equation

In sequence we give the precise mathematical formulation of the kind of motion we want to describe. Particle’s
dynamics arise from Hamilton equations plus a Langevin term acting as a stochastic force. We assume no explicit
stochastic term for position variable meaning that any random effect on it comes indirectly through coupling with
the momentum term resulting in a non-dissipative Brownian motion. More precisely we have

$$dx(t) = \frac{\partial H}{\partial p} dt$$

$$dp(t) = -\frac{\partial H}{\partial x} dt + \sqrt{2mP} dW(t)$$

Here $m$ is particle’s mass and $dW(t)$ a stationary Wiener-type stochastic variable with noise power $P$ given as above.
Rigorously speaking if the particle has a electric charge it radiates when accelerated which is a kind of an intrinsic
unremovable dissipative effect. In this work we assume these effects as small thus leaving out any radiation reaction
terms and deserving for a future work a proper inclusion of appropriate corrections.

The associated Fokker-Planck equation for the joint $\Phi(x,p,t)$ probability density distribution in phase space is
given by [3]

$$\frac{\partial \Phi(x,p,t)}{\partial t} = -\frac{\partial H}{\partial p} \frac{\partial \Phi}{\partial x} + \frac{\partial H}{\partial x} \frac{\partial \Phi}{\partial p} + mP \frac{\partial^2 \Phi}{\partial p^2}$$

which unless a missing dissipative term has the same structure of the Kramers’ equation [12], originally written to
describe the motion of a large mass particle in a fluid at thermal equilibrium with a bath in such a way that noise
comes from the temperature effect cited above. As already discussed this motion is unstable against infinitesimal
stochastic perturbations. In fact even if the intensity $\sqrt{2mP}$ of the Langevin term goes to zero a smeared phase space
still results. We observe that in this case the above equation reduces to the Liouville equation which admits a class
of solutions concentrated along the deterministic trajectories in phase space describing particle’s classical motion. In
fact if $H_{cl}(t) \equiv \frac{P_{cl}(t)^2}{2m} + V(x_{cl}(t))$ is the Hamiltonian of the (deterministic) Hamilton equations one class of solutions
for Liouville equation is

$$\Phi(x,p,t) = A \exp(-\beta \left| \frac{p^2 - P_{cl}(t)^2}{2m} + V(x) - V(x_{cl}(t)) \right|)$$

We see that trajectories (in phase space) no longer exist any more: position and momentum variables decouple and
information about the system is supplied by an ensemble of states in thermal equilibrium at temperature $\beta^{-1}$. This
temperature effect is the only memory of the ghost stochastic effect and is caused by the known singular small noise
expansion of the FPE [3]. Thus there is a big difference if we take Hamilton equations without noise to get a full
deterministic motion and a FPE with null noise term. In the first case the topology of phase space as represented by
Poincare maps are sets of points even when the motion is chaotic whereas a probability density function continuously
fills phase space in the another case.
The structure of the FPE allows easy calculation of some averages. For example time derivative of the expected value of the Hamiltonian is given by

\[
\frac{d}{dt} \langle H \rangle = \int_{-\infty}^{\infty} dx dp \left[ \frac{p^2}{2m} + V(x, t) \right] \Phi(x, p, t)
\]

\[
\frac{d}{dt} \langle H \rangle = P - \langle \frac{\partial V}{\partial x} \rangle + \langle \frac{\partial V}{\partial x} \rangle = P
\]

and the expected value of the momentum changes at the rate

\[
\frac{d}{dt} \langle p \rangle = -\langle \frac{\partial V}{\partial x} \rangle = \langle f \rangle
\]

following in mean the second Newton’s law. Note that the mean exchanged momentum with stochastic source is zero but a quadratic dispersion on momentum exists being this consequence of the missing dissipative effects. This dispersion increases with time meaning that, as anticipated above, the particle gains some energy from the stochastic source. Position variable is also affected although there is no noise source acting directly on it. Its mean value and the associated fluctuations change at the rate

\[
\frac{d}{dt} \langle x \rangle = \frac{\langle p \rangle}{m}
\]

\[
\frac{d}{dt} \left( \langle x^2 \rangle - \langle x \rangle^2 \right) = \frac{2}{m} (\langle px \rangle - \langle p \rangle \langle x \rangle)
\]

Thereby classical laws of motion are satisfied only in mean no matter the value of the noise intensity as shown by eqns (8) and (9). We must conclude that a field dynamics for the probability density should exist even for zero noise power and it will be shown bellow it follows exactly quantum dynamics when our perturbative schema is considered.

We develop in sequence the perturbative expansion. To do this we shall use \( \hbar \) in order to Fourier Transform \( \Phi(x, p, t) \) in momentum to the adjoint space \( \mathcal{Y} \) associated to this variable. In the present formalism this space refer to wavelengths of a mode decomposition of the probability density function. Differently from Quantum Theory here position and momentum are not related by an unitary transformation between adjacent spaces. This happens for momentum and wavelengths of waves of probability density having momentum \( p = \hbar y \). It is in this space the FPE has a form suitable to the structure of the perturbative series we look for. Nextly we write

\[
\Phi(x, p, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(x, y, t) e^{\frac{i\lambda y}{\hbar}} dy
\]

so the FPE reads

\[
\partial_t \chi(x, y, t) = \frac{\hbar}{im} \partial_y \partial_x \chi - \frac{y}{i\hbar} \frac{\partial V}{\partial x} \chi - \frac{y^2}{\tau^2} \chi
\]

where \( \lambda \equiv \mathcal{E}/\hbar c \) is the mean vacuum fluctuation wavelength and \( \tau \) the particle’s Compton length divided by the velocity of light. We stress that phase space probability density \( \Phi(x, p, t) \) is similar but has to be not confused with the Wigner function \( \Pi \) in Quantum Theory. The former satisfies exactly the Liouville equation for zero noise power. Wigner function satisfies it only approximately. Moreover in a Fourier transform on momentum variable of the Wigner function similar to eqn(11) variable \( y \) has the meaning of position. Specifically in the Wigner function the role of \( x \) and \( y \) are symmetric so they share the same physical meaning whereas in this work symbols \( x \) and \( y \) are trully independent variables that play different role. It will be shown explicitly later that in fact both functions do not coincide.

Owing to the structure of the equ(13) it is natural to think in a Taylor expansion of \( \chi(x, y, t) \) in the variable \( y \) which should be convenient because this way we may compare probability wavelengths to vacuum wavelength (that is very big) as expressed by the fraction present in the last term of this equation. This perturbative schema has the advantage that it permits a control of noise effects over field dynamics for \( \chi \). Saying differently if particle’s kinetic energy is much greater than vacuum energy \( \mathcal{E} \) its typical mode wavelength \( y \sim \hbar/p \) is very small compared to vacuum wavelength \( \lambda \) thus making noise effects negligible. But a infinite series in powers of \( y \) has no simple inverse Fourier representation generating a somewhat pathological phase space reconstruction. For vanishingly small noise we know from equ(4) that Liouville equation feels the presence of a thermal bath with temperature \( \beta^{-1} \). In the present case this effect is to be replaced by a “vacuum temperature” proportional to \( \lambda^{-1} \). Thus we insert a additional gaussian term in the perturbation series in order to regularize phase space reconstruction and to provide the appropriate thermal bath effect. More precisely we look for a general series solution of the type

\[
\chi(x, y, t) = \exp \left( -\frac{y^2}{2\lambda^2} \right) \sum_{n=0}^{\infty} a_n(x, t) y^n
\]
in such a way that eqn(12) becomes
\[
\sum_{n=0}^{\infty} \frac{\partial a_n(x, t)}{\partial t} y^n = \frac{\hbar}{i m} \sum_{n=0}^{\infty} \frac{\partial a_n(x, t)}{\partial x} \left[ n y^{n-1} - \frac{y^{n+1}}{\lambda^2} \right] - \frac{1}{\hbar} \frac{\partial V}{\partial x} \sum_{n=0}^{\infty} a_n(x, t) y^{n+1} - \frac{1}{\tau \lambda^2} \sum_{n=0}^{\infty} a_n(x, t) y^{n+2}
\]
generating the following set of differential equations for the coefficients \{a_n\}
\[
\begin{align*}
\frac{\partial a_0(x, t)}{\partial t} &= \frac{\hbar}{i m} \frac{\partial a_1(x, t)}{\partial x} \\
\frac{\partial a_1(x, t)}{\partial t} &= \frac{\hbar}{i m} \left[ \frac{\partial a_2(x, t)}{\partial x} - \frac{1}{\lambda^2} \frac{\partial a_0(x, t)}{\partial x} \right] - \frac{1}{\hbar} \frac{\partial V}{\partial x} a_0(x, t) \\
\frac{\partial a_n(x, t)}{\partial t} &= \frac{\hbar}{i m} \left[ (n+1) \frac{\partial a_{n+1}(x, t)}{\partial x} - \frac{1}{\lambda^2} \frac{\partial a_{n-1}(x, t)}{\partial x} \right] - \frac{1}{\hbar} \frac{\partial V}{\partial x} a_{n-1}(x, t) - \frac{1}{\tau \lambda^2} a_{n-2}(x, t)
\end{align*}
\]

Before discussing solutions of the above set of equations note that a naive interpretation can be given to some of these coefficients. In fact since
\[
\chi(x, y, t) = \int_{-\infty}^{\infty} \Phi(x, p, t) e^{-i\frac{p^2}{2\hbar}} dp
\]
we have
\[
\begin{align*}
a_0(x, t) &= \chi(x, 0, t) = \int_{-\infty}^{\infty} \Phi(x, p, t) dp \equiv |\Psi(x, t)|^2 \\
a_1(x, t) &= \left( \frac{\partial \chi}{\partial y} \right)_{y=0} = \frac{1}{\hbar} \int_{-\infty}^{\infty} p \Phi(x, p, t) dp \equiv \frac{1}{\hbar} \pi(x, t) \\
a_2(x, t) &= \frac{1}{2} \left( \frac{\partial^2 \chi}{\partial y^2} \right)_{y=0} + \frac{a_0}{2\lambda^2} = \frac{-1}{2\hbar^2} \int_{-\infty}^{\infty} p^2 \Phi(x, p, t) dp + \frac{a_0}{2\lambda^2}
\end{align*}
\]
that is \(a_0\) is the marginal probability distribution, \(a_1\) is proportional to the expected value of the momentum at point \(x\) and \(a_2\) is proportional to the kinetic energy at \(x\). We have also introduced the probability amplitude \(\Psi\) belonging to the \(L^2\) stuff of the \(L^1\)-based probability distribution \(a_0\). Integrating eqns(14)-(16) we get for \(n = 2\)
\[
\begin{align*}
\frac{d}{dt} \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx &= \frac{dN}{dt} = 0 \\
\frac{d}{dt} \int_{-\infty}^{\infty} \pi(x, t) dx &= \frac{d\langle p \rangle}{dt} = -\left( \frac{\partial V}{\partial x} a_0(x, t) \right) = \langle f(x, t) \rangle \\
-\frac{\hbar^2}{m} \frac{d}{dt} \int_{-\infty}^{\infty} a_2(x, t) dx &= \frac{1}{2m} \frac{d\langle p^2 \rangle}{dt} = \frac{\langle f(x, t) \pi(x, t) \rangle}{m} + P
\end{align*}
\]
The first of these equations indicates that the norm of \(\Psi\) is time-independent. Incidentally it also shows that the Hilbert space operator \(i\hbar \frac{\partial}{\partial t}\) satisfy \((i\hbar \partial_t \Psi, \Psi) = (\Psi, i\hbar \partial_t \Psi)\) where \((\cdot, \cdot)\) stands for Hilbert space inner product. This does not mean at the present stage that \(i\hbar \partial_t\) is hermitian since the relation was not proved to be valid in general, for any pair of Hilbert space vectors. However it is a necessary condition suggesting that this operator admits a Hermitian representation on an appropriate Hilbert space and in fact it will be constructed as the consistence of our calculations becomes closed. The second equation is similar to the Ehrenfest theorem and the third one indicates together with eqn(7) that the rise in total particle’s energy comes from its kinetic component. It’s worthwhile note that the first two equations are independent of noise intensity so concerning the non-stochastic limit their validity has the same status as the Liouville equation. In the next step we look for a field equation for \(\Psi\). Define \(L \equiv F[P] + V(x, t)\), where \(P \equiv \frac{\hbar}{\tau} \frac{\partial}{\partial x}\). We then have
\[ \Psi^* [L, P] \Psi = -\frac{\hbar}{i} \frac{\partial V}{\partial x} \Psi^* \Psi \]

and in consequence

\[ \frac{\partial}{\partial t} a_1 (x, t) = \frac{\hbar}{2m} \frac{\partial}{\partial x} \left[ 2a_2 (x, t) - \frac{1}{\lambda^2} a_0 (x, t) \right] + \frac{\Psi^* [L, P] \Psi}{\hbar^2} \]

\[ \quad \text{(19)} \]

\[ \frac{d}{dt} \int_{-\infty}^{\infty} a_1 (x, t) \, dx = \frac{1}{\hbar^2} (\Psi, [L, P] \Psi) \]

\[ \quad \text{(20)} \]

Being \( a_1 \) pure imaginary we can write it as

\[ a_1 = \Psi B \Psi^* \quad \text{or} \quad \Psi^* B \Psi \quad \text{where} \quad B \text{ is necessarily a real-valued operator.} \]

Further development on defining a more specific form for this operator also generates the rules for the Hilbert space we are working on. Owing to eqn(20) where a inner product emerges naturally we choose \( B \) such that \( \bar{B} \equiv \frac{\hbar}{i} B \) is Hermitian and time-independent giving the desired structure to expected value of the momentum as shown below

\[ \int_{-\infty}^{\infty} a_1 (x, t) \, dx = \frac{1}{\hbar} \left( \Psi, \bar{B} \Psi \right) \Rightarrow \left( \Psi, \bar{B} \Psi \right) = \langle \pi \rangle \equiv \Pi (t) \]

This result is to be interpreted classically even though it presents a strong resemblance to quantum mechanical rules. The average force on the particle is then equal to

\[ \frac{d\Pi (t)}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} a_1 (x, t) \, dx = \frac{1}{\hbar^2} \left[ \left( \Psi B \partial_t \Psi \right) - \left( \Psi \partial_t \bar{B} \Psi \right) \right] \]

\[ \quad \text{(22)} \]

and when compared with eqn(20) suggest a simple inner product structure for the above equation giving to \( i\hbar \partial_t \) a possible Hermitian representation \( \mathcal{H} \) which in general is function of \( x, P, \bar{B} \) and \( t \). This option defines simultaneously a field equation for the probability amplitude since now we select those Hilbert space vectors that satisfy the equality

\[ i\hbar \partial_t \Psi = \mathcal{H} \left( x, P, \bar{B}, t \right) \Psi \]

When this equation is inserted in eqn(22) and compared to eqn(20) we get

\[ \left( \Psi, \left[ \mathcal{H}, \bar{B} \right] \Psi \right) = (\Psi, [L, P] \Psi) \]

which should be valid for all \( \Psi \) that satisfy the above field equation. So in an appropriate subspace we must have \( \left[ \mathcal{H}, \bar{B} \right] = [L, P] \) with a formal solution given by

\[ \bar{B} = P \]

\[ \mathcal{H} \Psi = L \Psi = \left[ F [P] + V (x, t) \right] \Psi \]

which must be simultaneously valid. This way we conclude that \( F [P] \) is Hermitian. Now eqn(14) should also be satisfied implying that

\[ i\hbar \partial_t |\Psi (x, t)|^2 = \Psi^* \mathcal{H} \Psi - \Psi \mathcal{H} \Psi^* = \frac{\hbar^2}{2m} \left( \Psi \partial_x^2 \Psi^* - \Psi^* \partial_x^2 \Psi \right) \]

or

\[ \Psi^{-1} \left( F [P] + \frac{\hbar^2}{2m} \partial_x^2 \right) \Psi = (\Psi^*)^{-1} \left( F [P] + \frac{\hbar^2}{2m} \partial_x^2 \right) \Psi^* \]
an equation locally valid. Since \( \Psi \) and \( \Psi^* \) are independent we have \( F[P] = -\frac{\hbar^2}{2m}\partial_x^2 \) and \( \mathcal{H} = -\frac{\hbar^2}{2m}\partial_x^2 + V \) or

\[
\hbar \partial_t \Psi(x,t) = \left[ -\frac{\hbar^2}{2m}\partial_x^2 + V(x,t) \right] \Psi(x,t) \tag{23}
\]

We have found that field equation for the probability amplitude \( \Psi(x,t) \) follows quantum dynamics although the formalism used in this work be classical. This classical signature of Stochastic Mechanics as presented here is not only conceptual but has rigorous formal support as shown below.

The whole perturbative series can now be recursively calculated. Expected values of any (classical) observable \( f(x,p,t) \) is given by

\[
\langle f \rangle = \sum_n \int a_n(x,t) \tilde{f}(x,y,t)^* y^n \exp \left( \frac{-\hbar^2}{2\lambda^2} \right) dx dy \tag{24}
\]

where \( \tilde{f}(x,y,t) = \int f(x,p) e^{-i\mathcal{H}p} \). This way all predictions supported on the classical realism can be obtained although satisfying quantum dynamics exactly as present on the first two coefficients of the perturbative series. Higher order terms are also influenced by quantum dynamics but it is not clear that in the full phase space reconstruction interferences cancel out in order to recover the classical character of the theory. If this was the case averages calculated by eqn(24) will result on the values expected by classical theories. However it is noticeable that this doesn’t happens and concerning eqn(24) some results from Quantum Theory are equivalent to the present formulation of the Stochastic Mechanics. For example the expected value of the (classical) Hamiltonian is, using eqns(15) and(18)

\[
\langle H \rangle = \left\langle \frac{p^2}{2m} + V(x,t) \right\rangle = \frac{-\hbar^2}{m} \int a_2(x,t) dx + \frac{\hbar^2}{2m\lambda^2} \int a_0(x,t) dx + \int V(x,t) a_0(x,t) dx \tag{25}
\]

The value of \( a_2 \) is obtained by direct integration of eqn(15) and use of the field equation for \( \Psi \), the Shröedinger equation. The result is

\[
a_2(x,t) = \frac{1}{8} \left[ \Psi \frac{\partial^2\Psi^*}{\partial x^2} + \Psi^* \frac{\partial^2\Psi}{\partial x^2} - 2 \left| \frac{\partial\Psi}{\partial x} \right|^2 \right] + a_0(x,t) \tag{25}
\]

Using the condition that norm of the probability amplitude is equal to one the integral of eqn (23) is

\[
\frac{-\hbar^2}{m} \int a_2(x,t) dx = \frac{-\hbar^2}{2m} \int \Psi^* \frac{\partial^2\Psi}{\partial x^2} dx - \frac{\hbar^2}{2m\lambda^2}
\]

resulting that \( \langle H \rangle = \langle H \rangle_{MQ} \) exactly, where \( \langle H \rangle_{MQ} \) is the expected value of the Hamiltonian calculated by Quantum Mechanics for the same problem. In particular for stationary states of bound systems the classical energy becomes quantized. Consequently in the context presented here it appears that Stochastic Mechanics is a larger class of dynamical problem than Quantum Mechanics which seems to be a subset of the former. In fact Stochastic Mechanics gives a maximum possible information in phase space through a true independence between position and momentum variables that is not present in Quantum Theory.

But the above results show also that we have a disagreement between both theories. Remember that Wigner function in phase space satisfies Liouville equation only approximately \cite{13} whereas our probability density satisfy it exactly for zero noise power. More generally Wigner function for pure states in the adjoint space is given by

\[
\tilde{W}(x,y,t) = \sum_{n=0}^{\infty} a_{wn}(x,t) y^n \tag{24}
\]

The first two coefficients of this series coincide with ours given by eqns (17) and (21) but the value of the second order coefficient given by

\[
a_{w2}(x,t) = \frac{1}{4} \left[ \Psi(x,t) \frac{\partial^2\Psi(x,t)^*}{\partial x^2} + \Psi^*(x,t) \frac{\partial^2\Psi(x,t)}{\partial x^2} - \left| \frac{\partial\Psi(x,t)}{\partial x} \right|^2 \right]
\]
differ from the corresponding coefficient $a_2 (x, t)$ calculated in this work, even for infinite vacuum wavelength, and shown in eqn (23). This conclusively shows that Wigner function $W (x, p, t)$ and $\Phi (x, p, t)$ are not equal.

Having the terms in the perturbative series next step is phase space reconstruction. This task is not trivial because involves the whole series above displayed but if momentum of the mechanical system under study is much larger than $p_{\text{vac}} \equiv \hbar/\lambda$ the gaussian weight may induce a strong convergence in the perturbative series and truncation has some sense. Thus we expect rapid convergence for sufficiently high energy of the mechanical system compared to the mean vacuum fluctuation energy $\mathcal{E}$. In this case the result up to second order is

$$\Phi (x, p, t) \simeq G (x, p, t) \exp \left( -\frac{1}{2} \left( \frac{\lambda p}{\hbar} \right)^2 \right)$$

$$G (x, p, t) \equiv a_0 (x, t) \sqrt{2\pi} \lambda - \frac{i\lambda^3 p}{2\hbar} a_1 (x, t) \frac{\sqrt{\pi}}{\sqrt{3}}$$

$$\frac{\lambda^3}{3\hbar} a_2 (x, t) \left[ 1 - \frac{1}{3} \left( \frac{\lambda p}{\hbar} \right)^2 \right]$$

The above result shows that as an approximation, negative phase space probability also exist in our theory as it happens with Wigner function in Quantum Mechanics. Of course exact probability distribution should be non-negative and depends on calculation of all terms in the series for $\chi$. It may happens that coverage be not uniform a fact that complicate experimental verification since an oscillating series presents intrinsic measurements difficulties. Interestingly more, a new effect rises out from eqn(26) as shown in its exponential prefactor where a low energy phenomena control a high energy cutoff in phase space, being this a solution somewhat different from those usually found to eliminate ultraviolet divergences in field theory. Even though of fundamental importance this effect will not considered here for further discussions.

C. Concluding Remarks

In this work a non-dissipative Brownian motion was considered and a perturbative series for the associated Fokker-Planck equation presented. Noise source was assumed to be vacuum fluctuations of some quantum field that couples to particle’s classical dynamics. Surprisingly the first two terms of this series accommodates quantum dynamics quite naturally with properties and structure of its Hilbert space stuff. Nevertheless our approach is purely classical, explicit in the physical meaning of all mathematical objects treated here. Moreover our probability density function and Wigner function differ suggesting that both theories can be compared at a experimental level specially if phase space tomography measurements with massive particles becomes effectiveness.

In one line of reasoning this may indicate that our approach do not describe a truly quantal phenomena since a disagreement with Wigner function occur. But an alternative way may be given. Wigner function was originally constructed as a mathematical object without any explicit classical equivalence in what concern its (classically-defined) measurement. Its simultaneous dependence on position and momentum variables indicates it lives naturally in phase space. In addition it also satisfies approximately Liouville equation but since it can present negative values no support exists for an interpretation it is the quantum version of phase space probability density function.

Quite differently, in the formalism presented here phase space dynamics is real by construction and the interpretation of all mathematical objects are philosophically self-consistent since they satisfy all paradigms of Classical Mechanics reasoning. Even the probability waves and the associated wavelengths $\chi$ have clear physical meaning. In addition phase space reconstruction shown in eqn(26) presents in its lower order expansion negative probability values thus disclaiming the interpretation where this phenomenon is a mark of the Quantum Theory. In fact it is known from long that small noise perturbation series of the FPE may present negative terms. These considerations demands that experiment should definitively test whether dynamics in phase space as presented in this work is correct or not.

The most serious objection against the formulation of Stochastic Mechanics we present is probably the possibility of a material body gain some energy from vacuum. This reasoning breaks the naive classical paradigms we claim to give to the rest of the theory. It is the price paid for get so smoothly quantum dynamics from a purely classical theory. We have argued that although vacuum power seems unphysical at first sight may in fact be a real phenomenon with true detection chance. If real is reasonable think that dynamics of elementary particles are then correctly described by an appropriate FPE as discussed here. Vacuum temperature must be so small that usual quantum phenomena behave like high temperature ones and this in our interpretation means that typical wavelengths of the probability density are much smaller than vacuum wavelength so the whole perturbation series can be approximated by its stationary Liouville solution. Consequently important quantum results like atomic spectra can be understood inside our classical
reasoning. Moreover our formalism open tips to think about experimental detection of alternative theories that not only embraces Quantum Mechanics as well test its foundations at a more precise level. These considerations are, in our assessment, sufficient to take account in serious way the possibility that non-dissipative Brownian motion driven by vacuum fluctuations be a real phenomena turning out Stochastic Mechanics a faithful description of Nature.

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