DYNAMICAL LOCALIZATION FOR POLYNOMIAL LONG-RANGE HOPPING RANDOM OPERATORS ON $\mathbb{Z}^d$

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Abstract. In this paper, we prove a power-law version dynamical localization for a random operator $H_\omega$ on $\mathbb{Z}^d$ with long-range hopping. In brief, for the linear Schrödinger equation

$$i\partial_t u = H_\omega u, \quad u \in \ell^2(\mathbb{Z}^d),$$

the Sobolev norm of the solution with well localized initial state is bounded for any $t \geq 0$.

1. Introduction and the main result

From the breaking working of Anderson [4], a great deal of attention has paid to the Anderson model (a linear random Schrödinger operator) $H_\omega$ on $\mathbb{Z}^d$, where

$$H_\omega = H_0 + V_\omega.$$ The operator $H_0$ is a negative discrete Laplacian:

$$(H_0 u)(n) = -\sum_{m \in \mathbb{Z}^d: \sum_{i=1}^d |m_i - n_i| = 1} (u(m) - u(n)).$$

The potential $V_\omega$ is a multiplication operator with a function $V_\omega(n)$ on $\mathbb{Z}^d$, that $V_\omega(n)$ are independent, identically distributed random variables. We say that the operator $H_\omega$ has exponential localization, if its spectrum is pure point with exponential decay energy state. Namely, for some $\alpha > 0$, and any energy state $\psi_k$, one has

$$(1.1) \quad |\psi_k(n)| \leq C(k)e^{-\alpha |n|},$$

where $C(k) < +\infty$, and depend on energy state.

The mathematicians have development a bit knowledge about the localization of the random operator $H_\omega$. For $d = 1$, they proved that the exponential localization about the random Schrödinger operator $H_\omega$ for all energies. For $d \geq 2$, based on the multi-scale-analysis method(see [5, 13]) and fractional moment method(see [1, 3]), they can prove that the exponential localization of Anderson model at high disorder or low energies. However the physicists are more concerned with the transport properties of the model. In particular, the phenomenon is known as dynamical localization.

Considering the random operator $H_\omega$ on $\mathbb{Z}^d$ with pure point spectrum, the notion of dynamical localization can be reformulated as follows: for the Schrödinger equation on $\mathbb{Z}^d$,

$$i\partial_t u = H_\omega u, \quad u \in \ell^2(\mathbb{Z}^d)$$

with well localised initial state $u(0)$, the solution of the equation (1.2) satisfies that

$$(1.3) \quad \sup_{t} \|u(t)\|_{H^s} = \sup \left( \sum_{n \in \mathbb{Z}^d} (1 + |n|)^{2s} |u_n(t)|^2 \right)^{\frac{1}{2}} < +\infty,$$

for any $s > 0$. Hence, the dynamical localization is equivalent to the Sobolev norm of the solution is bounded for the all time.
The first rigorous proof of dynamical localization is attributed to Aizenman by employing the fractional moment method. From the point view of multi-scale analysis technique, an effective way to obtain dynamical localization is to control the constant \( C(k) \) in (1.1). In the authors introduced the SULE condition. Namely, all the energy state of random Schrödinger operator \( H_\omega \) have the form of

\[
|\psi_k(n)| \leq D(\epsilon, \omega)e^{\epsilon|nk|}e^{-\alpha|n-n_k|},
\]

where \( \epsilon > 0 \), \( D(\epsilon, \omega) \) is a finite constant that does not depend on the energy of the state, and \( n_k \) is the localization center point where \( \psi_k(n) \) has its maximum. After that, the results of SULE condition was applied in \([6, 7, 12, 15]\) to prove dynamical localization of some concrete models.

The above results focus on the models with exponential localization energy state. However, there are no results involving the models with power-law localization energy state. Namely, for some \( \alpha > 0 \), and any energy state \( \psi_k \), one has

\[
|\psi_k(n)| \leq C(k)|n|^{-\alpha},
\]

where \( C(k) \) is constant. Shi study the dimensional random operators with long-range hopping, that is

\[
H_\omega = \lambda^{-1}T + V_\omega(n)\delta_{nn'}, \quad \lambda \geq 1,
\]

where \( \lambda \) is the coupling constant describing the effect of disorder. Thorough the multi-scale analysis method, Shi proves that the energy state of random operator (1.6) exhibits power-law localization of energy state. In this paper, we try to extend the SULE condition to the power-law localization energy state and prove a new version of dynamical localization for the random operator (1.6).

Here, we make some set-up for our main results.

- The polynomial long-range hopping operator \( T \) is

\[
T(m, n) = \begin{cases} 
|n - m|^{-r}, & \text{for } m \neq n \text{ with } m, n \in \mathbb{Z}^d, \\
0, & \text{for } m = n \in \mathbb{Z}^d,
\end{cases}
\]

where \( |n| = \max_{1 \leq i \leq d} |n_i| \) and \( r > 0 \). From [9], we known that the operator \( T \) can be viewed as a negative discrete fractional Laplacian on \( \mathbb{Z}^d \).

- \( \{V_\omega(n)\}_{n \in \mathbb{Z}^d} \) is independent identically distributed \((i.i.d.)\) random variables (with common probability distribution \( \mu \)) on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) \((\mathcal{F} \sigma\text{-algebra on } \Omega \) and \( \mathbb{P} \text{ a probability measure on } (\Omega, \mathcal{F}))\).

Throughout this paper we assume that:

- We have that \( d < r < \infty \).
- Let \( \text{supp}(\mu) = \{x : \mu(x - \epsilon, x + \epsilon) > 0 \text{ for any } \epsilon > 0\} \) be the support of the common distribution \( \mu \). We assume that \( \text{supp}(\mu) \) contains at least two points and \( \text{supp}(\mu) \) is compact:

\[
\text{supp}(\mu) \subset [-M, M], \quad 0 < M < \infty.
\]

\[1\] By Schur’s test and self-adjointness of \( T \), we get (for \( r > d \))

\[
\|T\| \leq \sup_{m \in \mathbb{Z}^d} \sum_{n \neq m} |m - n|^{-r} \leq \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |n|^{-r} < \infty,
\]

where \( \|\cdot\| \) is the standard operator norm on \( \ell^2(\mathbb{Z}^d) \).
Under above assumptions, $H_\omega$ is a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^d)$ for each $\omega \in \Omega$. Denote by $\sigma(H_\omega)$ the spectrum of $H_\omega$. A well-known result due to Pastur \cite{14} imply that there exists a set $\Sigma$ (compact and non-random) such that for $P$ almost all $\omega$, $\sigma(H_\omega) = \Sigma$.

Let us recall the Hölder continuity of a distribution defined in \cite{5}.

**Definition 1.1 (\cite{5}).** We will say a probability measure $\mu$ is Hölder continuous of order $\rho > 0$ if

$$\frac{1}{\mathcal{K}_\rho(\mu)} = \inf_{\kappa > 0} \sup_{0 <|a-b| \leq \kappa} |a-b|^{-\rho} \mu([a,b]) < \infty.$$  

In this case will call $\mathcal{K}_\rho(\mu) > 0$ the disorder of $\mu$.

The main result of this paper is the following theorem.

**Theorem 1.2.** Let $H_\omega$ be defined by (1.6) with the common distribution $\mu$ being Hölder continuous of order $\rho > 0$, i.e., $\mathcal{K}_\rho(\mu) > 0$. Assume $r \geq \max\{\frac{200d}{\rho} + 25d, 1800\}$. Fix any $0 < \kappa < \mathcal{K}_\rho(\mu)$. Then there exists $\lambda_0 = \lambda_0(\kappa, \mu, \rho, M, r, d) > 0$ such that for $\lambda \geq \lambda_0$ and for $P$ almost all $\omega \in \Omega$, there exists a positive constant $q := q(r) \leq \frac{r}{1600}$, such that for any $\phi \in \ell^2(\mathbb{Z}^d)$ satisfying $|\phi(n)| \leq C_\phi|n|^{-\theta}$, where $\theta \geq r/160$, there exists a constant $C_\phi = C_\phi(d, r, q, \theta)$ such that

$$\|X^{\theta/2}e^{-itH_\omega}\phi\|^2 \leq C_\phi, \quad \forall t \geq 0,$$

where $X$ is the usual position operator.

**Remark 1.3.**

- The core of Theorem 1.2 is to obtain a power-law version SULE condition, that is, all energy state of random operator (1.6) have the form of

$$|\psi_{k,\omega}(n)| \leq D(\epsilon, \omega)|n_k|^{r}|n-n_k|^{-\alpha},$$

where $\epsilon > 0$ and $D(\epsilon, \omega)$ is a finite constant that does not depend on energy state.

- It should emphasized that the index $q$ in Theorem 1.2 is closely related to the index $r$ of operator $T$. This is different from the case where exponential localization leads to dynamical localization, where there is no restriction on the index $q$.

## 2. Preliminary Knowledge

### 2.1. Sobolev Norm of a Matrix.

Let $X_1, X_2 \subset \mathbb{Z}^d$ be finite sets. Define

$$M_{X_1}^{X_2} = \{M = (M(k, k')) \in \mathbb{C}\}_{k \in X_1, k' \in X_2}$$

to be the set of all complex matrices with row indexes in $X_1$ and column indexes in $X_2$. If $Y_1 \subset X_1, Y_2 \subset X_2$, we write $M_{Y_1}^{Y_2} = (M(k, k'))_{k \in Y_1, k' \in Y_2}$ for any $M \in M_{X_1}^{X_2}$.

**Definition 2.1.** Let $M \in M_{X_1}^{X_2}$. Define for $s \geq s_0$ the Sobolev norms of $M$ as:

$$\|M\|_s^2 = C_0(s_0) \sum_{k \in X_1 - X_2} \left( \sup_{k_1 - k_2 = k} |M(k_1, k_2)| \right)^2 \langle k \rangle^{2s},$$

where $\langle k \rangle = \max\{1, |k|\}$ and $C_0(s_0) > 0$ is a constant depending on $s_0$. 

green's function estimate. The green's function plays a key role in spectral theory. In this subsection we present the first main result about Green's function estimate. For \( n \in \mathbb{Z}^d \) and \( L > 0 \), define the cube \( \Lambda_L(n) = \{ k \in \mathbb{Z}^d : |k - n| \leq L \} \). Moreover, write \( \Lambda_L = \Lambda_L(0) \). The volume of a finite set \( \Lambda \subset \mathbb{Z}^d \) is defined to be \( |\Lambda| = \#\Lambda \). We have \( |\Lambda_L(n)| = (2L + 1)^d \) \((L \in \mathbb{N})\) for example.

If \( \Lambda \subset \mathbb{Z}^d \), denote \( H_\Lambda = R_\Lambda H_\omega R_\Lambda \), where \( R_\Lambda \) is the restriction operator. Define the Green's function (if it exists) as

\[
G_\Lambda(E) = (H_\Lambda - E)^{-1}, \quad E \in \mathbb{R}.
\]

Let us introduce good cubes in \( \mathbb{Z}^d \).

**Definition 2.2.** Fix \( \tau' > 0 \), \( \delta \in (0,1) \) and \( d/2 < s_0 < r_1 < r - d/2 \). We call \( \Lambda_L(n) \) is \((E, \delta)\)-good if \( G_{\Lambda_L(n)}(E) \) exists and satisfies

\[
\|G_{\Lambda_L(n)}(E)\|_s \leq L^{r+d}s \quad \text{for all} \quad s \in [s_0, r_1].
\]

Otherwise, we call \( \Lambda_L(n) \) is \((E, \delta)\)-bad. We call \( \Lambda_L(n) \) an \((E, \delta)\)-good (resp. \((E, \delta)\)-bad) \( L \)-cube if it is \((E, \delta)\)-good (resp. \((E, \delta)\)-bad).

**Remark 2.3.** Let \( \zeta \in (\delta, 1) \) and \( \tau' - (\zeta - \delta)r_1 < 0 \). Suppose that \( \Lambda_L(n) \) is \((E, \delta)\)-good. Then we have for \( L \geq L_0(\zeta, \tau', \delta, r_1, d) > 0 \) and \(|n' - n''| \geq L/2\),

\[
|G_{\Lambda_L(n)}(E)(n', n'')| \leq |n' - n''|^{-(1-\zeta)r_1}.
\]

Assume the following relations hold true:

\[
\begin{cases}
-(1-\delta)r_1 + \tau' + 2s_0 < 0, \\
-\xi r_1 + \tau' + \alpha \tau + (3 + \delta + 4\xi)s_0 < 0, \\
(1-2\xi - 2\alpha \tau + (5 + 4\xi + 2\delta)s_0 + s_0 < \tau',
\end{cases}
\]

where \( \alpha, \tau, \tau', r_1 > 1, \xi > 0, s_0 > d/2 \) and \( \delta \in (0,1) \).

**Definition 2.4.** We call a site \( n \in \Lambda \subset \mathbb{Z}^d \) is \((l, E, \delta)\)-good with respect to \((w.r.t) \Lambda \) if there exists some \( \Lambda_l(m) \subset \Lambda \) such that \( \Lambda_l(m) \) is \((E, \delta)\)-good and \( n \in \Lambda_l(m) \) with \( \text{dist}(n, \Lambda \setminus \Lambda_l(m)) \geq L/2 \). Otherwise, we call \( n \in \Lambda \subset \mathbb{Z}^d \) is \((l, E, \delta)\)-bad w.r.t \( \Lambda \).

Let

\[
\tau > (2p + (2 + \rho)d)/\rho.
\]

The multi-scale analysis argument on Green's function estimate is

**Theorem 2.5** (Theorem 4.4 in [16]). Let \( \mu \) be Hölder continuous of order \( \rho > 0 \) (i.e., \( K_\rho(\mu) > 0 \)). Fix \( E_0 \in \mathbb{R} \) with \( |E_0| \leq 2(\|T\| + M) \), and assume \((2.2), (2.3)\) hold true. Assume further that

\[
(1 + \xi)/\alpha \leq \delta, \quad p > \alpha d + 2\alpha p/J
\]

for \( J \in 2\mathbb{N} \). Then for \( 0 < \kappa < K_\rho(\mu) \), there exists

\[
L_0 = L_0(\kappa, \mu, p, \|T\|, r_1, M, J, \alpha, \tau, \xi, \tau', \delta, p, r_1, s_0, d) > 0
\]

such that the following holds: For \( L_0 \geq L_0 \), there is some \( \lambda_0 = \lambda_0(L_0, \kappa, \rho, p, s_0, d) > 0 \) and \( \eta = \eta(L_0, \kappa, \rho, p, d) > 0 \) so that for \( \lambda \geq \lambda_0 \) and \( k \geq 0 \), we have

\[
\mathbb{P}( \exists E \in [E_0 - \eta, E_0 + \eta], \text{ s.t. } \Lambda_{L_k}(m) \text{ and } \Lambda_{L_k}(n) \text{ are } (E, \delta)\text{-bad} ) \leq L_k^{-2p}
\]

for all \( |m - n| > 2L_k \), where \( L_{k+1} = [L_k^2] \) and \( L_0 \geq L_0 \).
2.3. Power-law localization. Recall the Poisson’s identity: Let \( \psi = \{\psi(n)\} \in \mathbb{C}^{Z^d} \) satisfy \( H_\omega \psi = E \psi \). Assume further \( G_\lambda(E) \) exists for some \( \Lambda \subset \mathbb{Z}^d \). Then for any \( n \in \Lambda \), we have
\[
(2.5) \quad \psi(n) = -\sum_{n' \in \Lambda \setminus \Lambda'} \lambda^{-1} G_\lambda(E)(n, n') T(n', n'') \psi(n'').
\]
From Shnol’s Theorem of \([11]\) in long-range operator case, to prove pure point spectrum of \( H_\omega \), it suffices to show that each \( \varepsilon \)-generalized eigenfunction belongs to \( l^2(\mathbb{Z}^d) \). In \([10]\), Shi shows that every \( \varepsilon \)-generalized (with \( 0 < \varepsilon \leq c(d) < 1 \)) eigenfunction \( \psi \) of random operator \((1.4)\) decays as \( |\psi(n)| \leq |n|^{-r/600} \) for \( |n| \gg 1 \). Specifically, Shi obtains polynomially decaying of each generalized eigenfunction of \( H_\omega \) for \( \mathbb{P} \) a.e. \( \omega \). This yields the power-law localization:

**Theorem 2.6** (Theorem 2.5 in \([10]\)). Let \( H_\omega \) be defined by \( (1.4) \) with the common distribution \( \mu \) being Hölder continuous of order \( \rho > 0 \), i.e., \( K_\rho(\mu) > 0 \). Let \( r \geq \max\{\frac{1004+23\rho}{\rho}, 331d\} \). Fix any \( 0 < \kappa < K_\rho(\mu) \). Then there exists \( \lambda_0 = \lambda_0(\kappa, \mu, \rho, M, r, d) > 0 \) such that for \( \lambda \geq \lambda_0 \), \( H_\omega \) has pure point spectrum for \( \mathbb{P} \) almost all \( \omega \in \Omega \). Moreover, for \( \mathbb{P} \) almost all \( \omega \in \Omega \), there exists a complete system of eigenfunctions \( \psi_{j,\omega} = \{\psi_{j,\omega}(n)\}_{n \in \mathbb{Z}^d}, j = 1, 2, \ldots, \), satisfying
\[
(2.6) \quad |\psi_{j,\omega}(n)| \leq C_{j,\omega}|n|^{-r/600}, \quad |n| \gg 1.
\]

**Remark 2.7.** Note that the coefficients \( C_{j,\omega} \) in \((2.6)\) depend on the selection of energy state.

**Lemma 2.8** (Lemma A.1 in \([10]\)). Let \( L \geq 2 \) with \( L \in \mathbb{N} \) and \( \Theta - d > 1 \). Then we have that
\[
(2.7) \quad \sum_{n \in \mathbb{Z}^d: |n| \geq L} |n|^{-\Theta} \leq C(\Theta, d)L^{-(\Theta-d)/2},
\]
where \( C(\Theta, d) > 0 \) depends only on \( \Theta, d \).

3. Proof of Theorem 1.2

In order to obtain dynamical localization, it’s need to control the location and the size of the boxes outside of which the eigenfunctions has “effective” decrease.

**Theorem 3.1.** For the operator \( H_\omega \) defined by \((1.4)\), we assume that \( \kappa, \lambda_0 \) and \( \psi_{j,\omega}, j \in \mathbb{N} \) satisfy Theorem 2.6. Let \( r \geq \max\{\frac{200d}{\rho} + 25d, 331d\} \). Then for \( \lambda \geq \lambda_0 \), there exists centers \( n_{j,\omega} \) associated to the eigenfunctions \( \psi_{j,\omega} \) with eigenvalues \( E_{j,\omega} \) such that for any \( \gamma \in [0, \frac{1}{100}] \) and any \( \epsilon' \in (\frac{1}{4}, \frac{1}{2}) \), there exists a constant \( C(\epsilon', \gamma) > 0 \) such that
\[
(3.1) \quad |\psi_{j,\omega}(n)| \leq C(\epsilon', \gamma)|n_{j,\omega}|^{-\gamma}|n - n_{j,\omega}|^{-\gamma}, \quad \forall n \in \mathbb{Z}^d.
\]
Here, \( C(\epsilon', \gamma) \) does not depend on \( j \) (the eigenfunction).

From Theorem 3.1, the eigenfunctions \( \psi_{j,\omega} \) are localized outside boxes of size \( |n_{j,\omega}|/2 \) around “centers” \( n_{j,\omega} \). This result is stronger than the power-law localization.

In the following, we choose appropriate parameters satisfying Remark 2.3, 2.2, 2.1. For this purpose, we can set by direct calculation that
\[
(3.2) \quad \alpha = 6, \quad \delta = \frac{1}{2}, \quad \xi = 2, \quad \zeta = \frac{19}{20}, \quad p = 13d.
\]
We define \( J = J_\varepsilon(d, \varepsilon) \) to be the smallest even integer satisfying \( p > 6d + \frac{12}{J_\varepsilon}p \). As a consequence, we can set
\[
(3.3) \quad \tau = \frac{29d}{\rho} + d, \quad s_0 = \frac{3}{4}d, \quad \tau' = \frac{87d}{\rho} + 7d,
\]
In order to prove Theorem 1.2, we need the following lemma which says that if \( \psi \) is an eigenfunction of \( H \) with eigenvalue \( E \), then \( E \) must be close to the spectrum of \( H_{\Lambda_L(n)} \) provided \( L \) is big enough and \( \Lambda_L(n) \) is centered on a maximum of \( |\psi(n)| \).

**Lemma 3.2.** There exists a constant \( L_*(d,r) \) so that if \( \psi \in \ell^2(\mathbb{Z}^d) \) is an eigenfunction of \( H \), with eigenvalue \( E \), and \( n_* \) satisfies \( |\psi(n_*)| = \sup\{ |\psi(n)|, n \in \mathbb{Z}^d \} \), then \( \Lambda_L(n_*) \) is \((E,1/2)\)-bad for all \( L \geq L_*(d,r) \).

**Proof.** Let \( \psi \in \ell^2(\mathbb{Z}^d) \) be as in the lemma, so \( n_* \) exists. Suppose that \( \Lambda_L(n_*) \) is \((E,1/2)\)-good for some \( k \geq k_1 = k_1(d,r) \) sufficiently large. Applying the identity (2.5) at the point \( n_* \), one has that

\[
|\psi(n_*)| \leq \sum_{n' \in \Lambda_L(n_*)} C(d)|G_{\Lambda_L(n_*)}(E)(n_*, n')| \cdot |n' - n''|^{-r}|\psi(n'')|
\]

where

\[
(V) = \sum_{|n' - n_*| \leq L_k/2, \ |n'' - n_*| > L_k} C(d, s_0)L_k^{r'} + \frac{1}{2} |n' - n''|^{-r}|\psi(n_*)|,
\]

\[
(VI) = \sum_{L_k/2 < |n' - n_*| \leq L_k, \ |n'' - n_*| > L_k} C(d)|n' - n_*|^{-\frac{d}{20}|n' - n''|^{-r}|\psi(n_*)|}.
\]

When \( |n' - n_*| \leq L_k/2 \) and \( |n'' - n_*| > L_k \), one has that \( |n' - n''| \geq |n'' - n_*| - |n' - n_*| > |n'' - n_*|/2 \). By (2.5), we have that

\[
(V) \leq \sum_{|n'' - n_*| > L_k} C(r, d, s_0)L_k^{d + r' + \frac{1}{2} s_0 |n'' - n_*|^{-r}}|\psi(n_*)| \leq C(d, r, s_0)L_k^{-\frac{d}{20} + r' + \frac{1}{2} s_0 + \frac{1}{2} d}|\psi(n_*)|,
\]

For the term \((VI)\), one has that

\[
(VI) = ( + \sum_{L_k/2 < |n'' - n_*| \leq L_k \atop L_k < |n' - n_*| < 2L_k} C(d, r)\sum_{L_k/2 < |n'' - n_*| \leq L_k \atop |n' - n_*| \geq 2L_k} C(d, r)|n' - n_*|^{-\frac{d}{20} |n'' - n_*|^{-r}}|\psi(n_*)|)
\]

\[
\leq \sum_{L_k/2 < |n' - n_*| \leq L_k} C(d, r)L_k^{\frac{d}{20}} |n' - n_*|^{-\frac{d}{20} |\psi(n_*)|} + C(d, r_1, r)L_k^{-\frac{d}{20} + \frac{1}{2} s_0 + \frac{1}{2} d} |\psi(n_*)| \leq C(d, r_1, r)L_k^{-\frac{d}{20} + \frac{1}{2} s_0 + \frac{1}{2} d} |\psi(n_*)|.
\]

Recalling (3.2) and (3.3), one has that

\[
-r + \frac{r'}{2} + \frac{1}{2} s_0 + \frac{3}{2} d < \frac{-r}{25}, \quad -\frac{r_1}{20} + 2d < \frac{-r}{25}.
\]
Hence, for large enough $k$ (depending on $d$ and $r$),

$$|\psi(n_*)| \leq L_k^{-\beta} |\psi(n_*)| < |\psi(n_*)|,$$

which is impossible. Therefore $\Lambda_{L_k}(n_*)$ is $(E, 1/2)$-bad for all $k \geq k_1(d, r)$. The lemma is proved. \hfill \Box

Now we can give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Under the hypotheses of Theorem 3.1 and Theorem 2.6, $H_\omega$ has power-law localization for $\omega \in \Omega$. This means that there exists $\Omega_1 \subset \Omega$, $\mu(\Omega_1) = 1$ so that for all $\omega \in \Omega_1$, $\sigma_\omega(H_\omega) = 0$ and for all eigenvalues $E_\omega$, the corresponding eigenfunction $\psi_\omega$ is $\ell^2$ and satisfies (2.6) with $\|\psi_\omega\| = 1$. Our ultimate goal is to control the constant $C_{\omega, \gamma}$ in (2.6). More precisely, for each $0 < \gamma \leq r/400$, we try to show that $n_{j, \omega}$ can be chosen so that $C_{\omega, \gamma}$ grows slower than $|n_{j, \omega}|^{\epsilon \gamma}$ for $1/3 \leq \epsilon < 1/2$.

The outline of the proof is a little complicated, for the convenience of reader, we will divide the main proof into three parts. Firstly, we fix $L_0 = [L_0]$, $\lambda_0$, $\eta$ and $I = [E_0 - \eta, E_0 + \eta]$ in Theorem 2.5. Recalling Theorem 2.5, we have for $\lambda \geq \lambda_0$ and $k \geq 0$,

$$\mathbb{P}(\exists \omega \in I, \text{ s.t. both } \Lambda_{L_k}(m) \text{ and } \Lambda_{L_k}(n) \text{ are } (E, \delta)\text{-bad}) \leq L_k^{-2p}$$

for all $|m - n| > 2L_k$, where $L_{k+1} = [L_k^2]$ and $L_0 \gg 1$.

**Step one.** For any $k \geq 0$, we define the set

$$(3.5) \quad A_{k+1}(n_0) = \Lambda_{2L_{k+1}}(n_0) \setminus \Lambda_{L_k}(n_0)$$

and the event:

$$(3.5) \quad \mathbf{E}_k(n_0) = \{\exists E, \exists n \in A_{k+1}(n_0), \text{ s.t. both } \Lambda_{L_k}(n_0) \text{ and } \Lambda_{L_k}(n) \text{ are } (E, 1/2)\text{-bad}\}.$$ 

From Theorem 2.5,

$$\mathbb{P}(\mathbf{E}_k(n_0)) \leq \sum_{n \in A_{k+1}(n_0)} L_k^{-2p} \leq C(d)(4L_k^n + 1)^d L_k^{-2p} \leq C(d)L_k^{-2p+\alpha d}.$$ 

For $1/3 \leq \epsilon < 1/2$, we define

$$F_k = \bigcup_{|n_0| \leq L_k^{1/\epsilon'}} \mathbf{E}_k(n_0).$$

Then

$$\mathbb{P}(F_k) \leq \sum_{|n_0| \leq L_k^{1/\epsilon'}} \mathbb{P}(\mathbf{E}_k(n_0)) \leq C(d, \epsilon') L_k^{-2p+\alpha d(1+\epsilon')},$$

where $p$ and $\alpha$ are defined in (3.2). It is easy to verified that $\sum_{k=0}^\infty \mathbb{P}(F_k) < \infty$. Then, the Borel-Cantelli lemma implies that

$$\mathbb{P} \left( \lim_{m \to \infty} \bigcup_{k \geq m} F_k \right) = 0,$$

so that the set

$$\Omega_2 = \{\omega \in \Omega_1 : \exists \tilde{k}_1 = \tilde{k}_1(p, d, \epsilon'), \text{ s.t. } \forall k \geq \tilde{k}_1, \omega \notin F_k\}$$

has full measure.
Now pick $\omega \in \Omega_1 \cap \Omega_2$, which will be kept fixed throughout the rest of the proof. Let $\psi_{j,\omega}$ be the eigenfunction of energy $E_{j,\omega}$, and $n_{j,\omega}$ be a point where $|\psi_{j,\omega}(n_{j,\omega})|$ is maximal. Note that such a point exists, since $\omega \in \Omega_1$ and $\psi_{j,\omega} \in L^2(\mathbb{Z}^d)$. Furthermore, we define the integers

\begin{align}
(3.6) \quad \bar{k}_2(\epsilon', m) &= \min\{k \geq 0 \text{ such that } |m|^{\epsilon'} < L_{k+1}\}, \quad m \in \mathbb{Z}^d, \\
(3.7) \quad \bar{k}_2 &= \bar{k}_2(p, d, \epsilon', n_{j,\omega}) = \max\{\bar{k}_1, \bar{k}_2(\epsilon', n_{j,\omega})\}.
\end{align}

For any $k \geq \bar{k}_2$, we see that $\omega \notin E_k(n_{j,\omega})$ from the definition of $F_k$. By \((3.5)\), $\forall k \geq \bar{k}_2$ and $\forall n \in A_{k+1}(n_{j,\omega})$, either $\Lambda_{L_8}(n_{j,\omega})$ or $\Lambda_{L_5}(n)$ is $(E_{j,\omega}, 1/2)$-good. Applying Lemma 3.2 there is an integer

$$k_2 = \max\{k_1, \tilde{k}_2\} = \max(k_1, \tilde{k}_2, \bar{k}_2),$$

where $k_1 = k_1(d, r)$ and $\tilde{k}_2 := \bar{k}_2(p, d, r, \epsilon')$ does not depend on $j$, such that for any energy $E_{j,\omega}$,

$$\Lambda_{L_5}(n) \text{ is } (E_{j,\omega}, 1/2)\text{-good}, \quad \forall n \in A_{k+1}(n_{j,\omega}), \quad \forall k \geq \bar{k}_2.$$

**Step two.** Let us apply the Possion’s identity \((2.5)\) at the point $n \in A_{k+1}(n_{j,\omega})$. Similarly to the proof of Lemma 3.2 and recalling \((3.2)-(3.4)\), for any $k \geq \bar{k}_2$, one has that

$$|\psi_{j,\omega}(n)| \leq \sum_{n' \in \Lambda_{L_8}(n)} |G_{\Lambda_{L_5}(n)}(E_{j,\omega})(n, n')| |n' - n|^\epsilon |\psi_{j,\omega}(n')|$$

\begin{align*}
&\leq C(r, d, s_0)L_k^{\frac{3}{2} + |r'| + \frac{1}{2} s_0 + \frac{d}{2}} + C(d, r, r_1)L_k^{\frac{3}{2} + 2d} \\
&\leq C(d, r)L_k^{-\frac{2d}{2}}.
\end{align*}

Set

\begin{equation}
(3.8) \quad \tilde{A}_{k+1}(n_{j,\omega}) = \Lambda_{\frac{3}{2}L_k}(n) \setminus \Lambda_{\frac{3}{2}L_k}(n) \subset A_{k+1}(n_{j,\omega}).
\end{equation}

If $n \in \tilde{A}_{k+1}(n_{j,\omega})$, one has that $L_k \geq (\frac{2}{3}|n - n_{j,\omega}|)^{\frac{d}{2}}$, and

$$|\psi_{j,\omega}(n)| \leq C(d, r)|n - n_{j,\omega}|^{-\frac{2d}{2}}.$$ 

Hence, one can find $k_3 = \max\{\tilde{k}_3, k_2\}$, where $\tilde{k}_3 = \tilde{k}_3(d, r)$ is independent of $j$, such that

$$|\psi_{j,\omega}(n)| \leq |n - n_{j,\omega}|^{-\frac{2d}{2}}, \quad \forall k \geq k_3.$$

Then for any $0 < \gamma \leq r/160$,

$$|\psi_{j,\omega}(n)| \leq |n - n_{j,\omega}|^{-\gamma}, \quad \forall n \in \tilde{A}_{k+1}(n_{j,\omega}), \quad \forall k \geq k_3.$$

Since $\bigcup_{k \geq k_3} \tilde{A}_{k+1}(n_{j,\omega}) = \{n \in \mathbb{Z}^d : |n - n_{j,\omega}| > \frac{4}{3}L_{k_4}\}$, there exists $k_4 = \max\{\tilde{k}_4, k_3\}$, where $\tilde{k}_4$ does not depend on $j$, such that

$$|\psi_{j,\omega}(n)| \leq |n - n_{j,\omega}|^{-\gamma}, \quad \forall n : |n - n_{j,\omega}| \geq L_{k_4}.$$

**Step three.** Using the fact that $|\psi_{j,\omega}(n)| \leq 1$ for all $n \in \mathbb{Z}^d$, one has

\begin{equation}
(3.9) \quad |\psi_{j,\omega}(n)| \leq C(\epsilon', \gamma)L_{k_4}^\gamma |n - n_{j,\omega}|^{-\gamma}, \quad \forall n \in \mathbb{Z}^d.
\end{equation}

Now, we try to control the $j$-dependence of the constant $L_{k_4}^\gamma$. Note that the only $j$-dependence of $k_4$ comes from $\tilde{k}_2(\epsilon', n_{j,\omega})$. Suppose $\sup\{|n_{j,\omega}|\} < \infty$, then $k_4$ can be chosen $j$-independently, so that we actually obtain a uniform localization for the all energy state

$$|\psi_{j,\omega}(n)| \leq C(\epsilon', \gamma)|n - n_{j,\omega}|^{-\gamma}, \quad \forall n \in \mathbb{Z}^d.$$
But the following Lemma 3.3 contradicts this first possibility. So, in fact, \( \sup \{ |n_{j,\omega}| \} = \infty \), and for \( j \) sufficiently large, one has

\[
k_4 = \hat{k}_2(\epsilon', n_{j,\omega}),
\]

and recalling the definition of \( \hat{k}_2(\epsilon', n_{j,\omega}) \) in (3.6),

\[
L_{k_4} \leq |n_{j,\omega}|^{\epsilon'}.
\]

Inserting this in (3.9) yields the announced result. Theorem 3.1 is proved.

We also need to control the growth of \( |n_{j,\omega}| \) with \( j \), which is given by the following preliminary lemma:

**Lemma 3.3.** Assume that \( n_{j,\omega} \) are defined in Theorem 3.1. Then one can order \( |n_{j,\omega}| \) in increasing order such that for \( j \) larger enough,

\[
|n_{j,\omega}| \geq C|j|^\frac{3}{d}.
\]

**Proof.** From Theorem 3.1. \( \{ \psi_{j,\omega} : j = 1, 2, \ldots \} \) is a complete normalized orthogonal system of \( \ell^2(\mathbb{Z}^d) \) and each \( n_{j,\omega} \) is chosen so that \( |\psi_{j,\omega}(n_{j,\omega})| = \sup \{ |\psi_{j,\omega}(n)|, n \in \mathbb{Z}^d \} \). Therefore we have

\[
\sum_{n \in \mathbb{Z}^d} |\psi_{j,\omega}(n)|^2 = 1, \quad \forall j = 1, 2, \ldots,
\]

(3.10)

\[
\sum_{j=1}^{\infty} |\psi_{j,\omega}(n)|^2 = 1, \quad \forall n \in \mathbb{Z}^d.
\]

(3.11)

Fix \( 0 < \varepsilon < 1 \). For \( j \in \mathbb{N} \setminus \{0\} \), by using Theorem 3.1. and (2.7), one has that

\[
\sum_{|n-n_{j,\omega}| \geq \varepsilon L} |\psi_{j,\omega}(n)|^2 \leq C(\epsilon', \gamma) \sum_{|n-n_{j,\omega}| \geq \varepsilon L} |n_{j,\omega}|^{2\epsilon' \gamma} |n-n_{j,\omega}|^{-2\gamma} \leq C(d, \epsilon', \gamma) \varepsilon^{-2\gamma - \frac{d}{2}} |n_{j,\omega}|^{2\epsilon' \gamma} L^{-\gamma + \frac{d}{2}}.
\]

Assume \( |n_{j,\omega}| \leq L \). Then

\[
\sum_{|n| \geq (1+\varepsilon)L} |\psi_{j,\omega}(n)|^2 \leq \sum_{|n-n_{j,\omega}| \geq \varepsilon L} |\psi_{j,\omega}(n)|^2 \leq C(d, \epsilon', \gamma) \varepsilon^{-2\gamma - \frac{d}{2}} L^{-\gamma + 2\epsilon' \gamma + \frac{d}{2}}.
\]

From (3.11),

\[
(2(1+\varepsilon)L + 1)^d = \sum_{|n| \leq (1+\varepsilon)L} |\psi_{j,\omega}(n)|^2 \geq \sum_{|n-n_{j,\omega}| \leq L} \min_{|n| \leq (1+\varepsilon)L} \sum_{|n| \leq (1+\varepsilon)L} |\psi_{j,\omega}(n)|^2 \geq \# \{ j : |n_{j,\omega}| \leq L \} \left( 1 - \max_{|n| \leq (1+\varepsilon)L} \sum_{|n| \geq (1+\varepsilon)L} |\psi_{j,\omega}(n)|^2 \right) \geq \# \{ j : |n_{j,\omega}| \leq L \} \left( 1 - C(d, \epsilon', \gamma) \varepsilon^{-2\gamma - \frac{d}{2}} L^{-\gamma + 2\epsilon' \gamma + \frac{d}{2}} \right).
\]

Choose \( \varepsilon = 1/2 \), \( \epsilon' = 1/3 \) and \( \gamma = r/160 \). Since \( r \geq 331d \), we have \( -\gamma + 2\epsilon' \gamma + \frac{d}{2} \leq -\frac{3}{160}d \). Then there exists \( L_0 = L_0(d, r) \) large enough, such that

\[
\# \{ j : |n_{j,\omega}| \leq L \} \leq C(d) L^d, \quad \forall L \geq L_0,
\]

(3.12)
where $C(d)$ is independent of $L$ and $j$. This tells us that $L \geq L_0$, $N(L) = \#\{j : |n_j,\omega| \leq L\}$ is finite and we can reorder the eigenfunctions so $|n_j,\omega|$ is increasing. Therefore, one has

$$|n_j,\omega| \geq c(d)j^{\frac{1}{6}},$$

for $j$ large enough (depending on $d$ and $r$).

Finally, we can give the complete proof of Theorem 1.2.

**Proof of Theorem 1.2** Let $\phi \in L^2(\mathbb{Z}^d)$ be such that, for some constant $C_\phi > 0$ and $\theta \geq r/160$, $|\phi(n)| \leq C_\phi |n|^{-\theta}$. We have to bound $\|X^{q/2}e^{-iH_Lt}\phi\|$, for some $q > 0$ and all $t > 0$. Since $e^{-iH_Lt}\phi = \sum_j e^{-iE_{j,\omega}t}(\phi, \psi_{j,\omega})\psi_{j,\omega}$ and $\|e^{-iH_Lt}\phi\| = \|\phi\|$, one has that

$$\|X^{q/2}e^{-iH_Lt}\phi\|^2 = \langle X^{q/2}e^{-iH_Lt}\phi, X^{q/2}e^{-iH_Lt}\phi \rangle$$

$$\leq \sum_j |\langle \phi, \psi_{j,\omega} \rangle| |\langle X^q \psi_{j,\omega}, e^{-iH_Lt}\phi \rangle|$$

$$\leq \|\phi\| \sum_j |\langle \phi, \psi_{j,\omega} \rangle| \|X^q \psi_{j,\omega}\|.

From Theorem 3.1 we can choose $\epsilon' = 1/3$. Then there exists a constant $C(\gamma) > 0$ such that

$$|\psi_{j,\omega}(n)| \leq C(\gamma)|n_j,\omega|^{\frac{1}{2}}|n - n_j,\omega|^{-\gamma}, \quad \forall n \in \mathbb{Z}^d.

One has that

$$\|X^q \psi_{j,\omega}\|^2 = \sum_{n \in \mathbb{Z}^d} |n^q \psi_{j,\omega}(n)|^2 \leq C(\gamma)|n_j,\omega|^{\frac{2}{q}} \sum_{n \in \mathbb{Z}^d} |n|^{2q}|n - n_j,\omega|^{-2\gamma},$$

where

$$\sum_{n \in \mathbb{Z}^d} |n|^{2q}|n - n_j,\omega|^{-2\gamma} \leq \sum_{|n - n_j,\omega| \geq 2|n_j,\omega|} |n|^{2q}|n - n_j,\omega|^{-2\gamma} + \sum_{|n - n_j,\omega| < 2|n_j,\omega|} |n|^{2q}$$

$$\leq C(d, q, \gamma)|n_j,\omega|^{-\gamma + q + \frac{d}{2}} + C(d, q)|n_j,\omega|^{2q+d}$$

$$\leq C(d, q, \gamma)|n_j,\omega|^{2q+d}.

Therefore

$$\|X^q \psi_{j,\omega}\| \leq C(d, r, q, \gamma)|n_j,\omega|^{q + \frac{d}{2} + \frac{d}{4}}.$$

Moreover, from the assumption of $\phi$, we have

$$|\langle \phi, \psi_{j,\omega} \rangle| \leq \sum_{n \in \mathbb{Z}^d} |\phi(n)||n_j,\omega(n)| \leq C_\phi(\gamma)|n_j,\omega|^{\frac{1}{2}} \sum_n |n|^{-\theta}|n - n_j,\omega|^{-\gamma}$$

where

$$\sum_n |n|^{-\theta}|n - n_j,\omega|^{-\gamma} \leq \sum_{|n - n_j,\omega| < |n_j,\omega|/2} |n|^{-\theta} + \sum_{|n_j,\omega| - |n - n_j,\omega| \leq 2|n_j,\omega|} |n - n_j,\omega|^{-\gamma} + \sum_{|n - n_j,\omega| \geq 2|n_j,\omega|} |n|^{-\theta}|n - n_j,\omega|^{-\gamma}$$

$$\leq C(d, \theta)|n_j,\omega|^{-\theta + d} + C(d, \gamma)|n_j,\omega|^{-\gamma + d} + C(d, \theta, \gamma)|n_j,\omega|^{-\frac{d}{2} - \frac{d}{2} + \frac{d}{4}}$$

$$\leq C(d, \theta, \gamma)|n_j,\omega|^{-\gamma + d}.

Therefore

$$|\langle \phi, \psi_{j,\omega} \rangle| \leq C_\phi(d, \theta, \gamma)|n_j,\omega|^{-\frac{2\gamma}{2} + d}.$$
Choose $0 < q \leq \gamma/10$ and $\gamma = r/160$. Since $r \geq 1800d$, we have $-\frac{2}{3} + q + \frac{3d}{2} < -\frac{11d}{10}$. Recalling Lemma 3.3, one has that
\[
\sum_j |\langle \phi, \psi_j, \omega \rangle| \|X^q \psi_j, \omega\| \leq C_\phi(d, r, q, \theta) \sum_j |n_j, \omega|^{-\frac{\gamma}{3} + q + \frac{3d}{2}}
\]
\[
\leq C_\phi(d, r, q, \theta) \sum_j |n_j, \omega|^{-\frac{11d}{10}}
\]
\[
\leq C_\phi(d, r, q, \theta) \sum_j |j|^{-\frac{11d}{10}} \leq C_\phi(d, r, q, \theta).
\]
Therefore
\[
\|X^{q/2}e^{-iHt}\phi\|^2 \leq C_\phi(d, r, q, \theta), \quad \forall t \geq 0.
\]
The proof of Theorem 1.2 is finished. \hfill \square

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