Bicoloring random hypergraphs

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We study the problem of bicoloring random hypergraphs, both numerically and analytically. We apply the zero-temperature cavity method to find analytical results for the phase transitions (dynamic and static) in the 1RSB approximation. These points appear to be in agreement with the results of the numerical algorithm. In the second part, we implement and test the Survey Propagation algorithm for specific bicoloring instances in the so-called HARD-SAT phase.

I. INTRODUCTION

The hypergraph bicoloring is one of the classic combinatorial optimization problems belonging to the NP-complete class \[\text{\#P}.\] Its random version, bicoloring of random hypergraphs, is a very interesting problem for the phase transitions it shows. Indeed, varying the average connectivity of the random hypergraph, the model undergoes a transition from a phase in which all links can be properly colored to a phase in which a sizeable fraction of links are violated. Around the transition point most difficult instances accumulate.

A graph is an ensemble of sites and links between them. In a hypergraph, the links connect triplets of sites. Each site (or vertex) can be colored in two ways, say black or white, so it is natural to identify it with an Ising spin variable that can assume the values 1 or \(-1\). The link is considered to be satisfied if the three spins that share it are not all of the same color. In the following we will often refer to a link as a function node, as it is called for example in the K-SAT problem \[\text{\#P}.\] The bicoloring problem consists in finding an assignment to all spins such that all the links are satisfied. Consequently a graph will be called colorable or uncolorable.

We can write the Hamiltonian for the problem assuming that each unsatisfied link gives a positive energy and zero otherwise. The total energy is proportional to the number of unsatisfied links: a colorable hypergraph will have a zero-energy ground state, while a non colorable one will have a positive-energy ground state.

The Hamiltonian for bicoloring a hypergraph \(G\) reads

\[
\mathcal{H} = \sum_{\{i,j,k\} \in \mathcal{E}} \frac{1 + \sigma_i \sigma_j + \sigma_j \sigma_k + \sigma_i \sigma_k}{2},
\]

where \(\sigma_i = \pm 1\) are Ising variables (corresponding to the 2 available colors) and the sum runs over all the hyperedges of \(G\). Note that a factor 2 has been introduced for computational convenience.

Each term in the above sum is equal to 2 if and only if all the spins in the same interaction are parallel, that is if all the vertices connected by a hyperedge have the same color. The Hamiltonian in Eq. (1) thus counts twice the number of badly colored hyperedges. Perfect colorations correspond to zero-energy configurations.

In the present work we focus on colorability of random hypergraphs with \(N\) vertices and \(M\) hyperedges, varying the relevant parameter \(\alpha = \frac{M}{N}\). In a typical random hypergraph the connectivity of a spin (i.e. the degree of a vertex) is a random variable distributed according to a Poissonian of mean \(3\alpha\).

Analogously to random K-SAT \[\text{\#P},\] random K-XORSAT \[\text{\#P},\] and Q-coloring of random graphs \[\text{\#P},\] the random hypergraph bicoloring is expected to undergo two phase transitions increasing \(\alpha\). The first one is called “dynamical transition” and is located at \(\alpha_d\) where solutions to the problem (perfect colorations) undergo a clustering phenomenon. At this point the complexity \(\Sigma\), which counts the number of clusters of solutions, becomes non-zero. We remind that if \(\mathcal{N}(E)\) is the number of states at energy \(E\) the complexity is defined by the relation \(\mathcal{N}(E) = \exp N \mathcal{S}(\alpha, E/N)\), so it is a function of \(\alpha\) and of the energy density. In the region where the complexity becomes positive, on top of a great number of ground states there appear an even larger number of metastable states: the latter may trap and slow down linear-time coloring algorithms and local search randomized methods \[\text{\#P},\]. At present all known linear-time coloring algorithms stop converging for \(\alpha\) values well below \(\alpha_d\).

The second transition takes place at \(\alpha_c\), where the ground-state energy becomes positive: for \(\alpha < \alpha_c\) most of the hypergraphs are colorable, while for \(\alpha > \alpha_c\) most of them are not. This transition is formally equivalent to the so-called SAT/UNSAT transition of K-SAT \[\text{\#P},\] K-XORSAT \[\text{\#P},\] and Q-coloring of random graphs \[\text{\#P},\] and we will refer to it with this name, although it is also known as “COL/UNCOL” transition in the computer science literature.

Known results on the SAT/UNSAT transition are only upper and lower bounds. The best upper bound for \(\alpha_c\), found with rigorous calculation, is 2.409 \[\text{\#P},\]. The best lower bound is 3/2 \[\text{\#P},\]. In Ref. \[\text{\#P},\] it is analyzed the more general problem of bicoloring random hypergraphs with \(p\)-spin hyperlinks. However for the \(p = 3\) case the bounds...
II. NUMERICAL RESULTS

We wrote a recursive Davis-Putnam algorithm [13] to color random finite-size hypergraph in order to localize the point $\alpha_c$, that will be calculated analytically in the next sections. Here we present the numerical results, whose uncertainties are very small thanks to the average of a large number of disorder realizations. In Fig. 1(left) we show that the energy curves for different $N$ cross at $\alpha_c$. Indeed for $\alpha < \alpha_c$ $\lim_{N \to \infty} E = 0$ because all hypergraphs are colorable, while for $\alpha > \alpha_c$ $E \propto N$ and diverges for $N \to \infty$. From Fig. 1 we estimate $\alpha_c \simeq 2.1$. All the curves can be nicely collapse when plotted versus $(\alpha - \alpha_c)N^{1/2}$, see Fig. 1(right).

A second estimate of $\alpha_c$ can be obtained from the curves of the probability of being colorable as a function of $\alpha$ (see Fig. 2). However here the crossing point is less clear because of larger finite-size corrections.
III. THE CAVITY REPLICA SYMMETRIC SOLUTION

A. Self-consistency equations

We now study the bicoloring problem with the cavity method at zero temperature \cite{14,15}. The simplest form of the zero-temperature cavity method is the Replica Symmetric (RS) approximation, in which we suppose the system to have a single state. The basic hypothesis of the cavity method is the lack of correlation between two randomly chosen spins, because of the local tree structure of the hypergraph. Thanks to these vanishing correlations, the energy of the system for fixed $\sigma_0$ can be written as a function of the cavity fields $h_j$ and $g_j$ on the $2k$ neighbors of $\sigma_0$ \cite{16}

$E(\sigma_0) = E_0 - \sum_{j=1}^{k} \hat{w}(g_j, h_j) - \sigma_0 \sum_{j=1}^{k} \hat{u}(g_j, h_j)$. \hspace{1cm} (2)

In the case of hypergraph bicoloring the function $\hat{u}$ and $\hat{w}$ are given by

$\{
\begin{align*}
\hat{u}(h_2, h_3) &= \theta(-h_2)\theta(-h_3) - \theta(h_2)\theta(h_3) \\
\hat{w}(h_2, h_3) &= |h_2| + |h_3| - |u(h_2, h_3)|
\end{align*}
\}
\hspace{1cm} (3)

where $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ elsewhere. The $\hat{u}$ are integers and can assume the values 0, 1 or $-1$. Note that $\hat{w} = \sum |h| - |u|$ is a general relation for models with Ising type variables.

In the thermodynamic limit, we can assume the probability distributions of cavity fields $h$ and cavity biases $u$ to have well defined limits, and write for them self-consistency equations

\[
\begin{align*}
Q(u) &= \int dP(h_1) dP(h_2) \delta(u - \hat{u}(h_1, h_2)) \\
P(h) &= \sum_{k=0}^{\infty} f_{3\alpha}(k) \int dQ(u_1) \ldots dQ(u_k) \delta(h - \sum_{i=1}^{k} u_i),
\end{align*}
\]

with

$f_{3\alpha}(k) = \frac{(3\alpha)^k}{k!} e^{-3\alpha}$

As expected, these equations coincide with those obtained from a replica calculation in Ref. \cite{16}.

Exploiting system symmetries one can always write

$Q(u) = c_0 \delta(u) + \frac{1 - c_0}{2} \left[ \delta(u + 1) + \delta(u - 1) \right]$. \hspace{1cm} (5)

Analogously the distribution of cavity fields can be written as $P(h) = \sum_{i=-\infty}^{\infty} p_i \delta(h - i)$, where the coefficients $p_i$ are symmetric, i.e. $p_i = p_{-i}$. The self consistency equations can be then written in terms of $p_0$ and $c_0$ as

\[
\begin{align*}
p_0 &= e^{-3\alpha(1-c_0)} I_0(3\alpha(1-c_0)) \\
c_0 &= 1 - \frac{(1-p_0)^2}{2}
\end{align*}
\]

(6)

where $I_0(x)$ is the zero-order modified Bessel function. $c_0$ is the order parameter of the system and it satisfies the self-consistency equation

$1 - \sqrt{2(1 - c_0)} = e^{-3\alpha(1-c_0)} I_0(3\alpha(1-c_0))$. \hspace{1cm} (7)

For any $\alpha$ value a “paramagnetic” solution $c_0 = 1$ exists, for which all the cavity fields are zero. For $\alpha > \alpha_{RS} = 2.3336$, there also exists a non-trivial “glassy” solution with $c_0 < 1$. 
B. Energy density

We now compute the RS energy density, following the notation already used in [15]. We must compute $E(\alpha) = \Delta E_1 - 2\alpha \Delta E_3$ where

$$
\Delta E_3 = \int dP(h_1) dP(h_2) dP(h_3) \cdot \left[ \min_{\sigma_1,\sigma_2,\sigma_3} \left( \frac{1 + \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_1 \sigma_3}{2} - h_1 \sigma_1 - h_2 \sigma_2 - h_3 \sigma_3 \right) + |h_1| + |h_2| + |h_3| \right]
= 2 \int dP(h_1) dP(h_2) dP(h_3) \theta(h_1 h_2) \theta(h_2 h_3) = \frac{1}{2} (1 - p_0)^3 = \sqrt{2}(1 - c_0)\frac{3}{2},
$$

$$
\Delta E_1 = \sum_{k=0}^{\infty} f_{3\alpha}(k) \int dQ(u_1) \ldots dQ(u_k) \left( \sum_{i=1}^{k} |u_i| - \sum_{i=1}^{k} u_i \right)
= 3\alpha(1 - c_0) - 2e^{-3\alpha(1-c_0)} \sum_{r=1}^{\infty} r I_r \left( 3\alpha(1 - c_0) \right).
$$

If we introduce the parameter $\lambda = 3\alpha(1 - c_0)$ which satisfies the equivalent of Eq.[7] the total RS energy density can be written as follows

$$
E = \lambda - 2e^{-\lambda} \sum_r r I_r(\lambda) - \frac{2}{3} \lambda \left( 1 - e^{-\lambda} I_0(\lambda) \right).
$$

The expression [10] seems to be the same for the different models with Ising variables (like $p$-spin [17], K-SAT [18], etc.), the difference being only in the self-consistency equation for $\lambda$, where $\alpha$ is multiplied by a different constant. For example the $\alpha_{RS}$ value for the present bicoloring model is twice the value it takes in the 3-spin model [17].

C. RS phase diagram

If we plot the energy [10] versus $\alpha$ we see that the energy of the non trivial solution is negative for $\alpha < 2.5906$. In the region $2.3336 < \alpha < 2.5906$ the RS solution is therefore non-physical, because the energy density of this problem must be positive by definition. In the RS approximation we have found a paramagnetic phase for $\alpha < 2.3336$ and a glassy phase for $\alpha > 2.5906$. This prediction is not correct, both quantitatively and qualitatively. The values of $\alpha$ where the transitions appear are not in agreement with numerical simulations, and there is a non-physical region.

D. Instability of evanescent field in the paramagnetic region

Before going to the 1RSB approximation, let us concentrate in this section on the RS paramagnetic region $\alpha < 2.3336$, in order to analyze the distribution of the so-called evanescent fields [12]. In the paramagnetic phase at zero temperature all the cavity fields $h_i$ are null, but considering the first order correction in temperature one can write $h_i = Th'_i$ (whence the adjective evanescent).

In terms of expectation values of spin variables, an evanescent field is the only one that can give a finite magnetization in the zero temperature limit: $m = \tanh(\beta h) \rightarrow \tanh(h')$. On the contrary, in the `strictly'-zero-temperature formalism that we use to study ground state energy, variables are either frozen, $|m| = 1$, or paramagnetic, $m = 0$, and we disregard any detailed information concerning the fluctuations of the local magnetizations of the unfrozen variables. The global probability distribution of the local magnetizations could in principle be non trivial, with some variable polarized (yet never frozen) in some preferential direction.

There are two equivalent ways of obtaining such information on the distribution of magnetizations: The first consists in writing the iterative cavity equations for such magnetizations and next taking the average over the underlying random hyper-graph. The second simply consists in computing the RS cavity equations at finite temperature assuming appropriate scaling of the cavity fields. Taking $h_i = Th'_i$ with $h'_i$ finite leads, in the $\beta \rightarrow \infty$ limit, to a distribution of evanescent fields which may describe non trivial expectations for the spins.

Following the same steps which brought us to the RS self-consistency equations [11], we can write analogously the self-consistency equations for the distributions of $h'_i = \beta h_i$ and $u'_i = \beta u_i$ in the $\beta \rightarrow \infty$ limit. These equations look
identical those in Eq.\([1]\), the only difference being the definition of the function \(\hat{u}(h_1, h_2)\), which now reads

\[
\hat{u}'(h'_1, h'_2) = \frac{\tanh(h'_1) + \tanh(h'_2)}{\tanh(h'_1) \tanh(h'_2) - 3} .
\]  

(11)

For very low \(\alpha\) the only solution to the self-consistency equations is \(P(h') = \delta(h')\). At variance with respect to other problems like for instance 3-SAT [18] in which the low \(\alpha\) phase is highly non trivial, the bicoloring problem is simple. As it happens in the \(Q\)-coloring [15] and in the 3-spin problems [4,17], the very low \(\alpha\) phase is a genuine paramagnet, with local fields concentrated around zero even at the first order in temperature.

However the solution \(P(h') = \delta(h')\) and \(Q(u') = \delta(u')\) may become unstable at a certain value of \(\alpha\), that we call \(\alpha_s\). In order to study the stability of this solution (in which local fields are uncorrelated independently of the local structure of the underlying hypergraph) it is enough to give an infinitesimal width to \(P(h')\) and check whether it increases or decreases under the iteration of Eq.\([4]\). For very small values of \(h'_1\) one can linearize the function \(\hat{u}'(h'_1, h'_2) \approx -(h'_1 + h'_2)/3\) and obtain very simple relations among the variances of \(P(h')\) and \(Q(u')\) at two consecutive iterations \((n \text{ and } n + 1)\)

\[
\langle (u')^2 \rangle_{n+1} = \frac{2}{9} \langle (h')^2 \rangle_n ,
\]

(12)

\[
\langle (h')^2 \rangle_{n+1} = 3 \alpha \langle (u')^2 \rangle_n .
\]

(13)

For \(\alpha < \alpha_s = 3/2\) the variances do not increase under iteration of the RS equations and the system is in a truly paramagnetic phase with all the magnetization identically zero.

For \(\alpha > \alpha_s\), the presence of a broad distribution of first-order corrections \(h'\) suggests the presence of a full RS\(B\) spin-glass phase at finite temperature, produced by a “replicon” instability at \(\alpha_s\). The finite-temperature phase transition at \(\alpha_s\) corresponds at \(T = 0\) to the onset of a non trivial organization of ground states, with non trivial magnetizations (unfrozen RS\(B\) scenario). We incidentally note that the value of \(\alpha_s\) coincides with the best lower bound available for \(\alpha_c\).

However, as soon as the dynamical transition is reached at \(\alpha_d \simeq 1.915\) (see next section), the system looses memory of the unfrozen RS\(B\) phase. The non-evanescent fields, \(h = \mathcal{O}(1)\), are the only ones relevant in determining the ground state energy. At the level of non vanishing fields, at \(\alpha_d\) we have a transition from RS to 1RS\(B\). At this point, the analytically disconnected solution with vanishing fields disappears. The presence of full RS\(B\) is somehow accidental and we expect for higher number of colors to disappear completely (as it happens in graph coloring [2]).

IV. THE CAVEY 1RS\(B\) SOLUTION

A. Self-consistency equations: the distribution \(\rho(\eta)\)

In the previous section we have seen that the RS approximations produce a wrong solution. Here we study the system with a better approximation, the so-called “one step Replica Symmetry Breaking”.

In this approximation the scenario is a bit more complex: at \(\alpha_d < \alpha_c\) there is a clustering phenomenon so that the computation made in the RS case is only valid within each state (cluster). It must be also considered the crossing between the energy of two states, for which we use the “rewriting parameter” \(\mu\) as in [15].

The 1RS\(B\) order parameter is a distribution of distributions, whose self-consistency equations are the following

\[
\mathcal{Q}[\mathcal{Q}] = \int D\mathcal{P}[P_1] D\mathcal{P}[P_2] \delta^{(F)} \left[ Q(u) - \int dP_1(h_1) dP_2(h_2) \delta(u - \hat{u}(h_1, h_2)) \right]
\]

(14)

\[
\mathcal{P}[P] = \sum_{k=0}^{\infty} f_{\alpha}(k) \int \prod_{i=1}^{k} D\mathcal{Q}[Q_i] \delta^{(F)} \left[ \mathcal{P}(h) - \frac{1}{A_k} \int \prod_{i=1}^{k} dQ_i(u_i) e^{-\mu(|\Sigma_i u_i - \Sigma_j u_j|)} \delta(h - \sum_{i=1}^{k} u_i) \right]
\]

(15)

with \(\delta^{(F)}\) being a functional delta, and \(A_k\) normalization coefficients.

Thanks to the system symmetries the most general form for \(Q(u)\) is given by

\[
Q(u) = \eta \delta(u) + \frac{1 - \eta}{2} \left[ \delta(u + 1) + \delta(u - 1) \right],
\]

(16)

that is symmetric under \(u \leftrightarrow -u\) and with \(u \in \{-1, 0, 1\}\). The heterogeneity of the random hypergraphs is now reflected in the very different values \(\eta\) may take: e.g. isolated plaquettes certainly have \(\eta = 1\). Let us call \(\rho(\eta)\) the probability distribution function of \(\eta\). The problem will be now studied in terms of \(\rho(\eta)\), which completely determines the order parameter \(\mathcal{Q}[\mathcal{Q}]\).
Self-consistency equations (14) and (15) can be written as a single self-consistency equation for the distribution $\rho(\eta)$. In the $\mu \to \infty$ limit it reads

$$
\rho(\eta) = \sum_{k=0}^{\infty} f_{3\alpha}(k) \sum_{k'=0}^{\infty} f_{3\alpha}(k') \int \prod_{i=1}^{k} d\rho(\eta_i) \int \prod_{j=1}^{k'} d\rho(\eta'_j) \delta \left[ \eta - 1 + \frac{1}{2} \left( 1 - \prod_{i=1}^{k} \eta_i A_k \right) \left( 1 - \prod_{j=1}^{k'} \eta'_j A_{k'} \right) \right],
$$

with the normalization coefficients $A_k = 2 \prod_{i=1}^{k} \frac{1+\eta_i}{2} - \prod_{i=1}^{k} \eta_i$. Eq. (17) can be solved by a population dynamics algorithm. Starting from a population of $\eta$s randomly distributed in $[0, 1]$ we then iterate the following steps:

- take $k$ elements and compute $\eta^k$ and $A_k$, where $k$ is a Poissonian number;
- take $k'$ elements and compute $\eta^{k'}$ and $A_{k'}$, where $k'$ is a Poissonian number too;
- compute a new $\eta$ as

$$
1 - \frac{1}{2} \left( 1 - \frac{\eta^k}{A_k} \right) \left( 1 - \frac{\eta^{k'}}{A_{k'}} \right),
$$

and insert it in the population eliminating another random $\eta$.

The asymptotic distribution $\rho(\eta)$ is plotted in figure 3 (left) for different values of $\alpha$. For $\alpha > \alpha_d \simeq 1.915$ the distribution has both a trivial contribution in 1 and a non-trivial one in the $[\frac{1}{2}; 1]$ region, while for $\alpha < \alpha_d$ it collapses into a single delta function in 1.

In figure 3 (right) we plot the average value of $\eta$ versus $\alpha$, by which we immediately localize the dynamical phase transition at $\alpha_d = 1.915$. An identical curve has been calculated analytically in the more tractable case of the $p$-spin model [4].

C. Complexity

In the $\mu \to \infty$ limit the complexity is given by

$$
\Sigma = \lim_{\mu \to \infty} (-\mu \Phi) = \lim_{\mu \to \infty} \left\{ \log A_k - 2\alpha \log \left| 1 - \frac{1}{2} (1 - \eta) (1 - \frac{\eta^k}{A_k}) \right| \right\},
$$

where the averages are taken with respect to the Poissonian distribution of $k$ and with respect to $\rho(\eta)$.

The complexity curve is plotted in figure 4 we identify the critical point $\alpha_c = 2.105$ that corresponds to the SAT/UNSAT transition, as the point where the complexity vanishes.

FIG. 3: Left: Probability distribution $\rho(\eta)$ for $\alpha = 2.0$ and $\alpha = 2.1$. Note the trivial contribution in 1. Right: Average value of $\eta$ versus $\alpha$. This value is exactly 1 for $\alpha < \alpha_d = 1.915$. 

B. $\mu \to \infty$ limit
D. Energy density and 1RSB phase diagram

In order to evaluate free energy $\Phi$ we must generalize the computation to for finite values of $\mu$.

The self-consistency equation for general $\mu$ is

$$
\rho(\eta) = \sum_{k=0}^{\infty} f_{3a}(k) \sum_{k'=0}^{\infty} f_{3a}(k') \int \prod_{i=1}^{k} d\rho(\eta_i) \prod_{j=1}^{k'} d\rho(\eta'_j) \delta \left[ \eta - 1 + \frac{1}{2} \left( 1 - \frac{a_k}{A_k} \right) \left( 1 - \frac{a_{k'}}{A_{k'}} \right) \right]
$$

(19)

where $a_k$ is the coefficient of the delta function in 0 of the distribution $P(k)(h)$ computed by the convolution of $k$ biases $u$, and $A_k$ is its normalization factor. To compute quickly the $P(k)(h)$ we can use a recursive relation:

$$
P^{(k)}(h) = \int dQ_k(u_k) dP^{(k-1)}(g) \delta(h - g - u_k) e^{-\mu(|u_k| + |g| - |g+u_k|)}.
$$

(20)

The free energy is given by $\Phi = \Phi_1 - 2\alpha \Phi_2$ with

$$
\Phi_1 = -\frac{1}{\mu} \log(A_k),
$$

$$
\Phi_2 = -\frac{1}{\mu} \log \left( 1 - \frac{1}{2}(1 - \eta) \left( 1 - \frac{a_k}{A_k} \right) \left( 1 - e^{-2\mu} \right) \right).
$$

(21)

For $\alpha > \alpha_c$, $\Phi$ has a maximum at a finite value of $\mu$: it means that the ground state has positive energy. Otherwise for $\alpha < \alpha_c$ $\Phi$ is always negative, converging toward zero for $\mu \to \infty$: it corresponds to a zero-energy ground state.

The energy density is calculated as

$$
E = \frac{\partial}{\partial \mu}(\mu \Phi) = -\frac{1}{\mu} \frac{\partial A_k}{\partial \mu} + 2\alpha \frac{(1 - \eta)(1 - \frac{a_k}{A_k})e^{-2\mu}}{1 - \frac{1}{2}(1 - \eta)(1 - \frac{a_k}{A_k})(1 - e^{-2\mu})}.
$$

(22)

As we did before, rather than computing the derivative of the $A_k$, we can write a recursive equation for the probability distribution $R^{(k)}(h) \equiv \frac{\partial}{\partial \mu} P^{(k)}(h)$:

$$
R^{(k)}(h) = \int dQ_k(u_k) dg[R^{(k-1)}(g) + (|h| - |g| - |u_k|)P^{(k-1)}(g)] \delta(h - g - u_k) e^{-\mu(|g| + |u_k| - |h|)}.
$$

(23)

Injecting this calculation in the population dynamics algorithm provides directly the curve $E(\mu) = \frac{\partial}{\partial \mu}(\mu \Phi)$. The ground state energy is obtained as the point where $E(\mu)$ and $\Phi(\mu)$ coincide.
FIG. 5: Energy density of random hypergraph bicoloring: comparison among finite-size numerical results and analytical 1RSB solution.

FIG. 6: Complexity versus energy at $\alpha = 2.5$: note the non-physical upper branch. Along the physical lower branch at the threshold energy $E_{th} = 0.062$ the complexity is maximal, while it becomes zero at the ground state energy.

The ground state energy density is compared to the numerical results in figure 5. This curve must be considered a $N \to \infty$ limit of the finite $N$ curves that we obtained numerically.

Another interesting curve that we can compute is the complexity versus the energy, that we plot parametrically in $\mu$ using $E(\mu)$ and $\Phi(\mu)$ (see Fig. 6). The curve $\Sigma = \mu(E - \Phi)$ has two branches: the lower one is physical one and represents the true complexity.

The last quantity we display in Fig. 4 is $E_{th}$ versus $\alpha$, that is simply the maximum of $E(\mu)$.

Summarizing the 1RSB results we get the following scenario.

There is a “paramagnetic” phase for $\alpha < \alpha_d = 1.915$, where there are no metastable states and we conjecture the existence of linear algorithms for coloring the generic hypergraph. The cavity fields are zero, so the spins are not forced to be black or white. In the so-called HARD-SAT region $\alpha_d < \alpha < \alpha_c = 2.105$ the generic hypergraph is still colorable, but the presence of many states makes the coloring procedure very difficult. In each ground state there is a core of spins for which there is a particular pattern of coloring: because of the existence of an exponentially larger number of metastable states, it is very difficult for local search algorithm to color the core in the right way. For $\alpha > \alpha_d$ the 1RSB approximation becomes less valid when high energy states are considered. Most likely, the curve $E_{th}$ would slightly change if a better approximation would be used.

These 1RSB results are expected to be a very good approximation of the exact analytical solution, as it happens in the majority of similar combinatorial optimization problems. For the $p$-spin model an exact solution has been
found that is identical to the 1RSB one \[4, 21\].

V. SINGLE SAMPLE ANALYSIS AND THE SP ALGORITHM

An innovative and useful reformulation of the cavity equations has been proposed in ref. \[7\]. The self-consistency equations are used to study single random problem instances and allow to get a microscopic information about the behavior of the single spins in the stable and metastable states of given energy density. The method, called Survey Propagation (SP), is general and provides the core ingredient of a new efficient algorithm \[3, 7, 22\] for finding ground states within the glassy phase. Here we will apply and check SP for the bicoloring problem. This problem is half-way between the random K-SAT problem and the random K-XORSAT (or p-spin) problem. Since the SP algorithm does work for random K-SAT \[7\], but it does not seem to work for random K-XORSAT, we believe of primary importance to check its performances on the random hypergraph bicoloring problem.

The iterative equations for the probability distributions of cavity fields that we have used in the previous sections to find the phase diagram were implementing at the same time a population dynamics process and an averaging over the random realizations. However, the equations can be easily iterated over specific realizations, that is avoiding the averaging step. In such a formulation the order parameter becomes the full list of the cavity fields over the entire graph. From the cavity fields one may determine the bias of each spin in all metastable states of given energy density and this information can be used for algorithmic purposes. The underlying hypothesis for the exactness of the single-sample formalism is the validity of the so called clustering condition within states: cavity fields should be uncorrelated within states and we expect this to be approximatively true thanks to the fact that the most numerous loops in the graph have a length that diverges as \(\log N\).

In order to set up an appropriate formalism for the single sample analysis, we resort to the factor graph representation \[23\] of the bicoloring problem: variables are represented by \(N\) circular “variable nodes” labeled with letters \(i, j, k, \ldots\) whereas links (which carry the interaction energy) are represented by \(M\) square “function nodes” labeled by \(a, b, c, \ldots\) (see Fig. 7). Function nodes have connectivity 3, variable nodes have a Poisson connectivity of average \(3\alpha\) and the overall graph is bipartite. The energy function can be trivially written as the sum over function nodes of their energies.

Following ref. \[1\], we call “messages” the \(\hat{u}\) terms which represent the contribution to the cavity fields coming from the different connected branches of the graph. In the message-passing language (typical of error correcting codes algorithms \[24\]) one may describe the SP equations as follows. In the replica symmetric approximation, the messages arriving at a node are added up and then sent to a function node. Next, the function node transforms all input signals into a new message which is sent to the descendant variable node. At the 1RSB level, the messages along the links

FIG. 7: Factor Graph representation of an energy minimization problem.
FIG. 8: Iterative equations as message-passing procedure.

of the factor graph are u-surveys of usual messages over the various possible states of the system at a given value of the energy (which is fixed by the reweighting parameter $\mu$). While the method is not restricted to zero temperature, at $T = 0$ it assumes a particularly simple form because messages can take only few values, 3 in our case, and the u-survey are given by the probabilities of these values. The u-surveys are parametrized by 2 real numbers and the SP can be implemented easily. Each edge $a \rightarrow j$ from a function node to a variable node $j$ carries a u-survey $Q_{a \rightarrow j}(u)$.

The algorithm finds these u-surveys and all the cavity fields $P_{i \rightarrow a}(h)$. Very schematically, the procedure works as follows. All the u-surveys $Q_{a \rightarrow i}(u)$ are initialized randomly. Next, function nodes are selected sequentially at random and the u-surveys are updated according to the equations:

$$P_{i \rightarrow a}(h) = C_{i \rightarrow a} \int du_1...du_k Q_{b_1 \rightarrow i}(u_1)...Q_{b_k \rightarrow i}(u_k) \delta \left( h - \sum_{a=1}^{k} u_a \right) \exp \left( \mu \left( \sum_{a=1}^{k} u_a - \sum_{a=1}^{k} |u_a| \right) \right)$$  \hspace{1cm} (24)

$$Q_{a \rightarrow i}(u) = C_{a \rightarrow i} \int dg dh P_{j \rightarrow a}(g) P_{\ell \rightarrow a}(h) \delta \left( u - \hat{u}(g, h) \right)$$  \hspace{1cm} (25)

where the function $\hat{u}(g, h)$ is the one defined in Eq. (3). In the above expressions, $C_{i \rightarrow a}, C_{a \rightarrow i}$ are normalization constants and the labels $b_i$ identify the $k$ neighboring function nodes different from $a$ connected to site the variable node $i$ (see Fig. 8).

Parameterizing the u-surveys as

$$Q_{a \rightarrow i}(u) = (1 - \eta_{a \rightarrow i}^+ - \eta_{a \rightarrow i}^-) \delta(u) + \eta_{a \rightarrow i}^+ \delta(u - 1) + \eta_{a \rightarrow i}^- \delta(u + 1)$$  \hspace{1cm} (26)

the above set of equations define a non-linear map over the $\eta$s.

The process is iterated until convergence is reached and finally the stable set of u-surveys are used to compute the $N$ local field $\{P_i(H_i)\}$ distributions and the free energy $\Phi(\mu)$. We have:

$$P_i(H) = C_i \int \prod_{a \in V(i)} du_a Q_{a \rightarrow i}(u_a) \delta \left( H - \sum_{a \in V(i)} u_a \right) \exp \left( \mu \left( \sum_{a \in V(i)} u_a - \sum_{a \in V(i)} |u_a| \right) \right)$$  \hspace{1cm} (27)

with $C_i$ being the normalization constant and $V(i)$ the set of function nodes connected to variable $i$. The free-energy reads

$$\Phi(\mu) = \frac{1}{N} \left( \sum_{a=1}^{M} \Phi_a^f(\mu) - \sum_{i=1}^{N} \Phi_i^v(\mu)(n_i - 1) \right),$$  \hspace{1cm} (28)
FIG. 9: Free energy $\phi(\mu)$ for different samples of size $N = 10000$ and $\alpha = 2.05, 2.1, 2.2$. 

where

$$
\Phi_{f}(\mu) = -\frac{1}{\mu} \log \left\{ \int \prod_{i \in V(a)} \left[ \prod_{b \in V(i) - a} Q_{b \rightarrow i}(u_{b \rightarrow i}) du_{b \rightarrow i} \right] \exp \left[ -\mu \min_{\{\sigma_i, i \in V(a)\}} \left( E_a - \sum_{i \in V(a)} \left[ \sum_{b \in V(i) - a} u_{b \rightarrow i} \sigma_i + \sum_{b \in V(i) - a} |u_{b \rightarrow i}| \right] \right) \right\}
$$

$$
\Phi_{v}(\mu) = -\frac{1}{\mu} \log \left\{ \int \prod_{a \in V(i)} du_{a} Q_{a \rightarrow i}(u_{a}) \exp \left[ \mu(\sum_{a \in V(i)} u_{a} - \sum_{a \in V(i)} |u_{a}|) \right] \right\} = -\frac{1}{\mu} \log(C_i) \quad (29)
$$

In the above expressions, $V(a)$ identifies the set of variable nodes connected to the function node $a$ and $E_a$ is its energy (i.e. the link energy).

The complexity $\Sigma(\mu) = \partial \Phi(\mu)/\partial (1/\mu)$ and the energy density $\epsilon(\mu) = \partial(\mu \Phi(\mu))/\partial \mu$ of states can also be estimated over single instances. Fig. 9 shows the free energy $\phi(\mu)$ of single graphs with $N = 10000$ vertexes as a function of $\mu$ for different values of the average connectivity $\alpha$. Fig. 10 shows the ground state energies and threshold energies for single instances at different $\alpha$. Similar data can be produced for the complexity. The agreement with the averaged calculations of the previous sections is indeed remarkable already for relatively small values of $N$ (as it should be expected from the self-averaging property of the free-energy).

Once the information concerning the effective local fields acting on the single spin variables becomes available a decimation procedure for finding ground states can be easily implemented. We have done one such implementation for the $\mu \to \infty$ case, with the scope of finding perfect colorings in the dynamical region just below $\alpha_c$. In this regime, the expression of the nonlinear map simplifies considerably. From eqs. \[ \text{[eqs.]} \] we find

$$
\eta_{a \rightarrow i}^+ = \prod_{j \in V(a) \setminus i} \left[ \frac{\Pi_{j \rightarrow a}^-}{\Pi_{j \rightarrow a}^+ + \Pi_{j \rightarrow a}^- + \Pi_{j \rightarrow a}^0} \right],
$$

$$
\eta_{a \rightarrow i}^- = \prod_{j \in V(a) \setminus i} \left[ \frac{\Pi_{j \rightarrow a}^+}{\Pi_{j \rightarrow a}^- + \Pi_{j \rightarrow a}^+ + \Pi_{j \rightarrow a}^0} \right], \quad (30)
$$
FIG. 10: Ground state energy and threshold energy for a single sample of size \( N = 10000 \) at different connectivities.

where

\[
\Pi^+_{j \rightarrow a} = \prod_{b \in V(j) \setminus a} \left(1 - \eta^-_{b \rightarrow i}\right) - \prod_{b \in V(j) \setminus a} \eta^0_{b \rightarrow i} \\
\Pi^-_{j \rightarrow a} = \prod_{b \in V(j) \setminus a} \left(1 - \eta^+_{b \rightarrow i}\right) - \prod_{b \in V(j) \setminus a} \eta^0_{b \rightarrow i} \\
\Pi^0_{j \rightarrow a} = \prod_{b \in V(j) \setminus a} \eta^0_{b \rightarrow i}
\]

(31)

The value of \( \eta^0_{a \rightarrow j} \) can be calculated by normalization. Other relevant quantities such as the biases of variables and the complexity also acquire a simple form. Upon defining the bias \( W_{i}^{\pm,0} \) of a variable as the probability of picking up a cluster of ground states at random and find that variable frozen in some preferential direction, that is \( W_i^+ \equiv \text{Prob}(H_i > 0), \quad W_i^0 \equiv \text{Prob}(H_i = 0), \quad W_i^- \equiv \text{Prob}(H_i < 0) \), we have:

\[
W_i^+ = \frac{\hat{\Pi}^+}{\Pi^+_i + \Pi^-_i + \Pi^0_i} \\
W_i^- = \frac{\hat{\Pi}^-}{\Pi^+_i + \Pi^-_i + \Pi^0_i} \\
W_i^0 = 1 - W_i^{(+)} - W_i^{(-)}.
\]

(32)

with

\[
\hat{\Pi}^+_i = \prod_{a \in V(i)} \left(1 - \eta^-_{a \rightarrow i}\right) - \prod_{a} \eta^0_{a \rightarrow i} \\
\hat{\Pi}^-_i = \prod_{a \in V(i)} \left(1 - \eta^+_{a \rightarrow i}\right) - \prod_{a} \eta^0_{a \rightarrow i} \\
\hat{\Pi}^0_i = \prod_{a \in V(i)} \eta^0_{a \rightarrow i}
\]

(33)

For the complexity we have:

\[
\Sigma = \frac{1}{N} \left( \sum_{a=1}^M \Sigma_a - \sum_{i=1}^N (n_i - 1) \Sigma_i \right)
\]

(34)
where
\[
\Sigma_a = \log \left[ \prod_{j \in V(a)} (\Pi_{i \rightarrow j}^+ + \Pi_{j \rightarrow a}^- + \Pi_{j \rightarrow a}^0) - \prod_{j \in V(a)} \Pi_{j \rightarrow a}^+ - \prod_{j \in V(a)} \Pi_{j \rightarrow a}^- \right] \tag{35}
\]

\[
\Sigma_i = \log \left[ \tilde{\Pi}_{i}^+ + \tilde{\Pi}_{i}^- + \tilde{\Pi}_{i}^0 \right] \tag{36}
\]

With the list of the biases on hand, the following simple decimation procedure to find ground state configurations can been implemented:

1. \(\{\eta\} \leftarrow \text{random}\)
2. SP
   - (a) Iterate eqs. \[(24,25)\] until a fixed \(\{\eta^*\}\) point is reached
3. Compute the biases \(W_i^+ = \text{Prob}(H_i > 0), W_i^0 = \text{Prob}(H_i = 0), W_i^- = \text{Prob}(H_i < 0)\), following eq. \[(27)\].
4. For \(B_i = W_i^+ - W_i^-\), Choose \(i\) such that \(|B_i|\) is maximum.
5. IF \(|B_i| < \epsilon\) for all \(i\) then STOP (paramagnetic state) and output the reduced sub-problem.
6. FIX \(\sigma_i = 1\) if \(B_i > 0\), \(\sigma_i = -1\) otherwise.
7. GOTO 2

One should notice that along the decimation procedure some of the variable are fixed and therefore new types of links appear. The corresponding new function nodes will have an energy which is inherited by the 3-body interaction by fixing one of the variables. Once decimation has started, the bicoloring problems becomes a mixture of graph and hypergraph bicoloring.

The behavior of the algorithm on sufficiently large \((n > 10^3)\) random bicoloring instances is the following:

- for low \(\alpha\) \((\alpha < \alpha_d)\), the variables turn out to be all paramagnetic (zero bias).
- in the dynamical region the biases are non-trivial and the decimation procedure fixes many variables leading to sub-problems which are paramagnetic and easily solved by a greedy heuristic. Very close to \(\alpha_c\), the decimation procedure may fail in finding solutions in the first run. In this region the algorithm can be improved in many ways, e.g. by a random restart or a backtrack or a different decimation strategy. In any case we can not exclude the existence of a threshold close to \(\alpha_c\) where the decimation procedure stops converging.

For small \(N\) the structural “rare events” of the random hyper-graph, like links sharing more than one variable or other types of short loops, require an appropriate (in principle simple) modification of the SP iterations \[(24)\]. More in general, the presence of loops of different length scales may introduce correlations which may require further non-trivial generalization of the whole SP procedure.

VI. CONCLUSIONS

In this work we have given a very detailed description of the random hypergraph bicoloring problem, both on the average-case and on single samples.

After having defined the statistical model corresponding to this problem, we have applied the cavity method to solve it: results in the RS and the 1RSB approximations have been presented.

Increasing the connectivity \(\alpha\) the model undergoes several phase transitions, which can be summarized as follows:

- for \(\alpha < \alpha_s\) the model is in a genuine paramagnetic phase, all the magnetizations are identically null;
- at \(\alpha = \alpha_s\) a “replicon” instability takes place, which manifests at finite temperature with the onset of spin-glass order (full RSB);
- for \(\alpha_s < \alpha < \alpha_d\) the presence of a full RSB phase at finite temperatures is reflected in the ground states by finite values for the spin magnetizations;
• at $\alpha = \alpha_d$ a clustering transition takes place among the ground states. They split in an exponentially large number of clusters. Within each cluster a finite fraction of variables is completely frozen (backbone);

• for $\alpha_d < \alpha < \alpha_c$ the model has a non-zero complexity and an exponentially large number of metastable states, which may block local-search algorithms. Although the very strong correlations among variables the ground state energy is still zero and the problem is colorable on average;

• at $\alpha = \alpha_c$ the COL/UNCOL phase transition takes place;

• for $\alpha > \alpha_c$ the ground state energy is positive and the problem can not be colored on average.

In the second part of this work we have applied the Survey Propagation algorithm to problem instances taken from the HARD-COL region ($\alpha_d < \alpha < \alpha_c$), finding in polynomial time solutions to the problem. So we have verified that the SP algorithm works properly also for this model, which is harder than the 3-sat problem [7]. Indeed this model, at variance with K-SAT, has no local biases which could in principle be exploited by a smart algorithm.

Next steps in this line of research will be to consider random hard combinatorial problems endowed with some non-trivial local structure of the underlying graph. This constitutes a conceptual challenge that will bring the algorithmic and analytical tools developed for sparse graphs closer to what is found in the real-world version of the same class of models [25].

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[26] Local fields will turn out to be integer valued rather than fractional.
[27] For the unphysical one there is not still a precise interpretation [12], however it seems not to have any physical meaning.
[28] In the algorithmic formalism we need a more general parametrization of surveys with respect to the one used in the first sections. As we shall see, along the decimation process the symmetries of surveys are lost.