Illustrating dimensionless scaling with Hooke’s law

J J Bissell, A Ali and B J Postle

School of Physics, Engineering & Technology, University of York, York YO10 5DD, United Kingdom

E-mail: john.bissell@york.ac.uk

Abstract

Dimensionless variables and scalings are powerful tools for emphasising the universality of natural laws, but their application is typically restricted to topics in advanced physics. Here we describe how dimensionless scaling can be illustrated in an introductory physics context by considering a dimensionless form for Hooke’s law. As a simple empirical study, we verify this form for Hooke’s law using data from foundation-year degree (high-school level) experiments on springs in series and parallel.

Keywords: dimensionless scaling, nondimensionalization, dimensional analysis, Hooke’s law

Nondimensionalisation—or dimensionless scaling—is the process whereby equations governing a physical system are expressed in terms of variables intrinsic to the system [1, 2]; for example, rather than expressing the size of a system in S.I. units, one might express the size relative to some kind of characteristic system length-scale. Scaling is a powerful tool for emphasising the universality of physical laws, but its use is typically restricted to topics in advanced physics, with little or no discussion at more introductory levels [3, 4]. This is something of a missed opportunity, as the basic ideas can help clarify the concept of measurement as a kind of ratio, and enhance students’ appreciation for the universal beauty of even the most elementary phenomena [1]. Indeed, here we show how dimensionless scaling can be illustrated in an introductory physics context by investigating a dimensionless form for Hooke’s law.

Hooke’s law for the restoring force \( f \) of a spring stretched to a length \( l \) is

\[
f = k(l - l_0),
\]

(1)

where \( l_0 \) (the spring’s natural length) and \( k \) (the spring constant) are properties of the particular spring of interest [5]. Typically one uses this equation by measuring \( f \) in newtons (N), \( l \) and \( l_0 \) in meters (m), and \( k \) in units of Nm\(^{-1}\). However, after dividing equation (1) through by \( kl_0 \), we may also express Hooke’s law in the dimensionless form

\[
F = (L - 1),
\]

(2)

where

\[
F = \left( \frac{f}{kl_0} \right) \quad \text{and} \quad L = \left( \frac{l}{l_0} \right)
\]

(3)
are force-like \( F \) and extension-like \( L \) quantities without units. When Hooke’s law has been written in this way we say that it has been non-dimensionalised, or scaled.

Observe that the effect of scaling is to absorb the parameters \( k \) and \( l_0 \) into the dimensionless quantities \( F \) and \( L \). Thus, the quantity \( L \) is a ratio-measure of the extended length \( l \) of the spring compared to its intrinsic length \( l_0 \); for example, if \( L = 3 \), then we know that the spring has been stretched to a length \( l \) three-times its natural length \( l_0 \), regardless of whether the spring is long or short. Similarly, the quantity \( F \) is a measure of the restoring force \( f \) of the spring compared to \( kl_0 \); hence, if \( F = 2 \), then we know that the restoring force \( f \) is twice \( kl_0 \), regardless of whether the spring is stiff or flexible. It follows that the force-length data from any experiment on an ideal spring will—when scaled according to equation (3)—coincide with the line \( F = (L - 1) \).

A simple way to illustrate this idea is to measure the force-length characteristics of different springs, and then collate the data on dimensionless axes. Here we achieve this effect by using force-length data derived from experiments on springs configured in series and parallel. Experiments of this kind are commonplace in high-school physics classes, meaning that our procedure can be adapted as the basis for ‘further work’ exercises within the current high-school curriculum [6]. Indeed, the data reported in this article were gathered by foundation-year (high-school level) students at the University of York.

The apparatus used for such experiments is depicted in figure 1, with the system configured as: (a) a single spring; (b) two springs in parallel; and (c) two springs in series. These configurations are treated as individual ‘spring-systems’ in their own right, each with an effective spring constant \( k \) and natural length \( l_0 \). Force-length data for each type of configuration is obtained by suspending a mass \( m \), and measuring the system extension \( l \) from an arbitrary fixed point. Because the restoring force \( f \) is equal to the mass weight, i.e.

\[
    f = mg
\]

(where \( g \approx 9.81 \text{ m s}^{-2} \) is the acceleration due to gravity), a range of force-length measurements can be gathered by selecting different masses \( m \) (see table 1). As depicted in figure 2, this data may then be plotted, and the system spring constants \( k \) and natural lengths \( l_0 \) inferred by fitting linear Hooke’s law trends of the form \( f = k(l - l_0) \).

Processing our data in this way we find (see table 1): \( k = 28.1 \text{ Nm}^{-1} \) and \( l_0 = 0.102 \text{ m} \) for the single spring; \( k = 54.8 \text{ Nm}^{-1} \) and \( l_0 = 0.0966 \text{ m} \) for the springs in parallel; and \( k = 13.7 \text{ Nm}^{-1} \) and \( l_0 = 0.139 \text{ m} \) for the springs in series. With these values for \( k \) and \( l_0 \) it is then a simple step to rescale the data in terms of the dimensionless variables \( F = (f/kl_0) \) and \( L = (l/l_0) \), as listed in the final

![Figure 1](image-url)
Illustrating dimensionless scaling with Hooke’s law

**Table 1.** Force-length data for the three spring configurations, given (naively) to three significant figures. Although the error is likely to be larger than 1 part in 1000, we expect overall uncertainty to fall within the marker size used in figures 2 and 3. The values for \( k \) and \( l_0 \) are inferred from the linear Hooke’s law fits described in figure 2; these values are then used to scale the data as listed in the final two columns.

| Configuration | \( m/\text{kg} \) | \( f/\text{N} \) | \( l/\text{m} \) | \( l_0/\text{m} \) | \( k/\text{Nm}^{-1} \) | \( L = l/l_0 \) | \( F = f/(k l_0) \) |
|---------------|-----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Single        | 0.100           | 0.98           | 0.134          | 0.102          | 28.1           | 1.32           | 0.34           |
|               | 0.200           | 1.96           | 0.173          | 0.102          | 28.1           | 1.70           | 0.69           |
|               | 0.300           | 2.94           | 0.209          | 0.102          | 28.1           | 2.06           | 1.03           |
|               | 0.400           | 3.92           | 0.241          | 0.102          | 28.1           | 2.37           | 1.38           |
|               | 0.500           | 4.90           | 0.278          | 0.102          | 28.1           | 2.73           | 1.72           |
|               | 0.600           | 5.88           | 0.309          | 0.102          | 28.1           | 3.04           | 2.06           |
| Parallel      | 0.100           | 0.98           | 0.114          | 0.097          | 54.8           | 1.18           | 0.19           |
|               | 0.200           | 1.96           | 0.132          | 0.097          | 54.8           | 1.37           | 0.37           |
|               | 0.300           | 2.94           | 0.150          | 0.097          | 54.8           | 1.55           | 0.56           |
|               | 0.400           | 3.92           | 0.170          | 0.097          | 54.8           | 1.76           | 0.74           |
|               | 0.500           | 4.90           | 0.187          | 0.097          | 54.8           | 1.94           | 0.93           |
|               | 0.600           | 5.88           | 0.202          | 0.097          | 54.8           | 2.09           | 1.11           |
| Series        | 0.100           | 0.98           | 0.211          | 0.139          | 13.7           | 1.51           | 0.51           |
|               | 0.200           | 1.96           | 0.282          | 0.139          | 13.7           | 2.02           | 1.02           |
|               | 0.300           | 2.94           | 0.354          | 0.139          | 13.7           | 2.54           | 1.54           |
|               | 0.400           | 3.92           | 0.424          | 0.139          | 13.7           | 3.04           | 2.05           |
|               | 0.500           | 4.90           | 0.497          | 0.139          | 13.7           | 3.57           | 2.56           |

**Figure 2.** Restoring force \( f \) plotted against displacement \( l \) for three spring-system configurations (see table 1): a single spring (closed circles); two springs in parallel (squares); and two springs in series (triangles). Linear Hooke’s law fits of the form \( f = k(l - l_0) \) are used to infer the corresponding system values for \( k \) and \( l_0 \) (solid lines), with \( k \) given by the gradient, and \( l_0 \) given by the abscissa (i.e. \( x \)-axis) intercept.

**Figure 3.** Dimensionless force \( F = (f/(k l_0)) \) plotted against dimensionless displacement \( L = (l/l_0) \) according to the normalised data listed in table 1 (cf figure 2). When the data are scaled in this way, all values lie on the straight line \( F = (L - 1) \) (solid line).

In this way the universality of Hooke’s law is demonstrated elegantly, and concisely, with data presented in a fashion that permits direct comparison between systems operating at different scales of interest. Notice that no information is lost in this process; the corresponding S.I. measurements can be recovered from each data point in figure 3 by computing \( l = (l_0 L) \), and \( f = (k l_0 F) \).
To summarise, then, we have explored a dimensionless form for Hooke’s law as a way of illustrating nondimensionalisation—or scaling—in an introductory physics context. Although the use of nondimensionalisation is typically reserved for topics in advanced physics, we have shown that it can be applied quite readily to high-school level laboratories. An obvious educational application of our study, therefore, is as a model ‘further-work’ exercise for students conducting Hooke’s law experiments on springs in series and parallel. Indeed, our work with undergraduate foundation-year (high-school level) students suggests that introducing dimensionless scaling at an early stage in the physics curriculum can help to clarify the concept of measurement as a form of ratio, and encourage deeper perspectives on the universality of natural laws.

Data availability statement
All data that support the findings of this study are included within the article (and any supplementary files).

Received 12 November 2021, in final form 14 December 2021
Accepted for publication 22 December 2021
https://doi.org/10.1088/1361-6552/ac45c9

References
[1] Richards M J 1971 An ABC of dimensional analysis Phys. Educ. 6 244–9
[2] Bissell J J 2012 Dimensional analysis and dimensional reasoning Ways of Thinking, Ways of Seeing ed C C Bissell and C Dillon (Berlin: Springer)
[3] For example, laboratory astrophysics is based on the principle that astrophysical phenomena can be studied in terrestrial laboratories by experimenting on appropriately scaled plasmas
[4] Bissell J J and Bhamidimarri S K 2020 Introducing the trapezoidal pendulum: dynamics of coupled pendula suitable for distance teaching Phys. Educ. 55 065008
[5] Lowe T L and Rounce J F 2002 Calculations for A-Level Physics (Oxford: Oxford University Press)
[6] Serna J D and Joshi A 2011 Studying springs in series using a single spring Phys. Educ. 46 33–40

J J Bissell is a lecturer and independent scholar based at the University of York. His primary research interests are in physics and mathematics. He holds heretical views about the purpose of higher education, and the voice of natural philosophy in the conversation of mankind.

Ahmed Ali is a foundation year physics student at the University of York hoping to progress onto the undergraduate degree programme in Mathematics and Physics. He is returning to full time education as a mature student after working in hospitality and social care.

Bradley J Postle is a student at the University of York taking foundation year physics. His current objective is to progress onto the undergraduate degree programme in Theoretical Physics. Ultimately, he hopes to make the transition to mathematics and physics research.

March 2022

Phys. Educ. 57 (2022) 023008