LOCAL EXISTENCE OF SOLUTIONS OF SELF GRAVITATING RELATIVISTIC PERFECT FLUIDS

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Abstract. This paper deals with the evolution of the Einstein gravitational fields which are coupled to a perfect fluid. We consider the Einstein–Euler system in asymptotically flat spacetimes and therefore use the condition that the energy density might vanish or tend to zero at infinity, and that the pressure is a fractional power of the energy density. In this setting we prove a local in time existence, uniqueness and well posedness of classical solutions. The zero order term of our system contains an expression which might not be a $C^\infty$ function and therefore causes an additional technical difficulty. In order to achieve our goals we use a certain type of weighted Sobolev space of fractional order. In [4] we constructed an initial data set for these of systems in the same type of weighted Sobolev spaces.

We obtain the same lower bound for the regularity as Hughes, Kato and Marsden [13] got for the vacuum Einstein equations. However, due to the presence of an equation of state with fractional power, the regularity is bounded from above.

1. Introduction

This paper deals with the Cauchy problem for the Einstein–Euler system describing a relativistic self-gravitating perfect fluid, whose density either has compact support or falls off at infinity in an appropriate manner.

The evolution of the gravitational field is described by the Einstein equations

\begin{equation}
G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta}
\end{equation}

where $g_{\alpha\beta}$ is a semi-Riemannian metric having a signature $(-, +, +, +)$, $R_{\alpha\beta}$ is the Ricci curvature tensor, and $R$ is the scalar curvature. Both tensors are functions of the metric $g_{\alpha\beta}$ and its first and second order partial derivatives. The right-hand side of (1.1) consists of the energy–momentum tensor $T_{\alpha\beta}$, which in the case of a perfect fluid takes the form

\begin{equation}
T^{\alpha\beta} = (\epsilon + p) u^\alpha u^\beta + pg^{\alpha\beta},
\end{equation}

where $\epsilon$ is the energy density, $p$ is the pressure and $u^\alpha$ is the four-velocity vector. The vector $u^\alpha$ is a unit timelike vector, which means that it satisfies the normalization condition

\begin{equation}
g_{\alpha\beta} u^\alpha u^\beta = -1.
\end{equation}

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The Euler equations describing the evolution of the fluid take the form
\[(1.4) \quad \nabla_\alpha T^{\alpha\beta} = 0,\]
where \(\nabla\) denotes the covariant derivative associated to the metric \(g_{\alpha\beta}\). Equations (1.1) and (1.4) are not sufficient to determinate the structure uniquely, a functional relation between the pressure \(p\) and the energy density \(\epsilon\) (equation of state) is also needed. We choose an equation of state that has been used in astrophysical problems. It is the analogue of the well known polytropic equation of state in the non-relativistic theory, given by
\[(1.5) \quad p = p(\epsilon) = K\epsilon^\gamma, \quad K, \gamma \in \mathbb{R}^+, \quad 1 < \gamma.\]
The sound velocity is denoted by
\[\sigma^2 = \frac{dp}{d\epsilon},\]
and the range of the energy density \(\epsilon\) will be restricted so that the causality condition \(\sigma^2 < 1\) will hold.

The unknowns of these equations are the semi Riemannian metric \(g_{\alpha\beta}\), the velocity vector \(u^\alpha\) and the energy density \(\epsilon\). These are functions of \(t\) and \(x^a\) where \(x^a\) \((a = 1, 2, 3)\) are the Cartesian coordinates on \(\mathbb{R}^3\). The alternative notation \(x^0 = t\) will also be used and Greek indices will take the values 0, 1, 2, 3 in the following.

In the present paper we prove the well-posedness of the coupled systems (1.1), (1.2), (1.4) and (1.5) under the harmonic gauge condition in asymptotically flat spacetimes. In order to achieve this, we need to rewrite the above equations as a hyperbolic system.

In astrophysical context the density \(\epsilon\) is expected to have compact support, or tend to zero at spatial infinity in an appropriate sense. It is well known that the usual symmetrization of the Euler equations is badly behaved in cases where the density tends to zero somewhere. The coefficients of the system degenerate or become unbounded when \(\epsilon\) approaches zero. It was observed by Makino [18] that this difficulty can be to some extend circumvented in the case of a non-relativistic fluid by using a new matter variable \(w\) in place of the mass density. For this reason we introduce the quantity
\[(1.6) \quad w = M(\epsilon) = \epsilon^{\frac{\gamma-1}{2}},\]
and we call it the Makino variable.

A similar device was used by Gamblin [11] and Bezard [2] for the Euler-Poisson equations, and by Rendall [22] and Oliynyk [21] for the Einstein–Euler equations.

The common method for solving the Cauchy problem for the Einstein equations consists usually of the following steps.

1. Initial data must satisfy the constraint equations, which are intrinsic to the initial hypersurface. Therefore, the first step is to construct solutions of these constraints.
2. The second step is to use the harmonic coordinate condition and to solve the evolution equations with these initial data.
3. The last step is to prove that the harmonic coordinate condition and the solution of the constraints propagate. That means if they held on a initial hypersurface, they hold for later times.
The last step was treated in detail, for example in Fisher and Marsden [10]. The idea is to work out the condition $\nabla_\alpha G^{\alpha\beta} = 0$. Since our energy–momentum satisfy (1.4) their result can be immediately generalised for our case, but for the sake of brevity we have omitted the details.

However, the presence of the equation of state (1.5) introduces an additional step: the compatibility problem of the initial data for the fluid and the gravitational field (see (2.10)). There are three types of initial data for the Einstein–Euler system:

- The gravitational data is a triple $(\mathcal{M}, h, K_{ab})$, where $\mathcal{M}$ is a space-like manifold, $h$ is a proper Riemannian metric on $\mathcal{M}$, and $K_{ab}$ is the second fundamental form on $\mathcal{M}$ (extrinsic curvature). The pair $(h, K_{ab})$ must satisfy the constraint equations (2.9);
- The matter variables, consisting of the energy density $z$ and the momentum density $j^a$, appear on the right hand side of the constraints (2.9);
- The initial data for Makino variable $w$ and the velocity vector $u^\alpha$ of the perfect fluid.

The only type of Sobolev spaces which are known to be useful for existence theorems for the constraint equations in an asymptotically flat manifold, are the weighted Sobolev spaces $H^k,\delta$, where $k \in \mathbb{N}$ and $\delta \in \mathbb{R}$. These spaces were introduced by Nirenberg and Walker [20] and Cantor [5], and they are the completion of $C_0^\infty(\mathbb{R}^3)$-functions under the norm

$$
\|u\|_{k,\delta}^2 = \sum_{|\alpha| \leq k} \int \left( (1 + |x|)^{\delta + |\alpha|} |\partial^\alpha u| \right)^2 dx.
$$

Due to the presence of the equation of state (1.5) and the Makino variable (1.6), we have to estimate $\|w^2\|_{k,\delta}$. So it is perhaps worth discussing the estimate of the Sobolev’s norm of $u^\beta$ in more details for $\beta > 1$. For simplicity we discuss this in the ordinary Sobolev space $H^k = H^k(\mathbb{R}^3)$. The simplest case is when $\beta \in \mathbb{N}$, then $\|u^\beta\|_{H^k} \leq C(\|u\|_{L^\infty})\|u\|_{H^k}$ and there is no restriction on $k$. When $\beta \not\in \mathbb{N}$, then we obtain the same estimate, provided that $k \leq \beta$. This bound on $k$ was improved by Runst and Sickel [23] to $k < \beta + \frac{1}{2}$. Applying this to $\beta = \frac{2}{\gamma - 1}$, and taking into account the Sobolev embedding $\|u\|_{L^\infty} \leq C\|u\|_{H^k}$ for $k > \frac{3}{2}$, we get a lower and upper bound for $k$:

$$
\frac{3}{2} < k < \frac{2}{\gamma - 1} + \frac{1}{2}.
$$

The only exception is the case when $\frac{2}{\gamma - 1}$ is an integer.

Note that for certain values of $\gamma$, inequalities (1.8) possesses no integer solution. Hence, for these values of $\gamma$ it is impossible to obtain a solution to the Einstein–Euler system in Sobolev spaces of integer order. So in order to be able to solve the coupled system for the maximal range of the power $\gamma$, and, in addition, to improve the regularity of the solutions, we are considering the Cauchy problem in the weighted fractional spaces $H_{s,\delta}$, where $s$ is real number (see Definition 2.1). These spaces were introduced by Triebel [25], and they generalize $H_{k,\delta}$ to a fractional order.
In [4] the authors constructed initial data for coupled systems (1.1), (1.2) and (1.4) with the equations of state (1.5). This includes the solution to the constraint equations (2.9), as well as the solution to the compatibility problem between the matter variable \((z, j^a)\) and the fluid variables \((w, u^a)\), (2.10), in the \(H_{s,\delta}\)-spaces. Here we will establish the well-posedness of Einstein–Euler systems in the weighted fractional Sobolev spaces \(H_{s,\delta}\).

The common way to prove well-posedness is to rewrite the evolution equations as a symmetric hyperbolic system. So our first step is to use the the Makino variable (1.6) and to reduce the Euler equations (1.4) to a uniformly first order symmetric hyperbolic system. This result was announced in [3] and here we present a detailed proof of it. Our hyperbolic reduction is based on the fluid decomposition; for alternative reductions see [22].

It is well-known that the Einstein equations can be written as a system of quasi-linear waves equations under the harmonic gauge condition [6, 7, 26]. The proofs of existence and uniqueness either using second order techniques [6, 8, 12, 13, 16], or transferring the equations to a first order symmetric hyperbolic system. Fischer and Marsden used the first order techniques and obtained the well-posedness of the reduced vacuum Einstein equations in \(H^s\) and for \(s > \frac{7}{2}\) [10]. This result was improved by Hughes, Kato and Marsden [13], who obtained \((g_{\alpha\beta}, \partial_t g_{\alpha\beta}) \in H^{s+1} \times H^s\) for \(s > \frac{3}{2}\). They used second order theory, and took advantage of the specific form of the quasi-linear waves equations, namely, that the coefficients depend only on the semi-metric \(g_{\alpha\beta}\), but not on its first order derivatives.

Our aim is to prove existence and uniqueness of the reduced Einstein–Euler system (1.1), (1.2) and (1.4) with the equation of state (1.5). In addition, we would like to achieve the same regularity of the metric as in [13]. But since we have here a coupled system which one of them is a first order, the second order techniques of Hughes, Kato and Marsden in [13], is no longer available for the present problem.

In asymptotically flat spacetimes the initial metric \(g_{\alpha\beta}(0)\) differs from the Minkowski metric by a term which is \(O(1/r)\) at spatial infinity, and this term does not belong to \(H^s\). It is therefore more appropriate to consider both the constraint and evolution equations in the \(H_{s,\delta}\) spaces rather than the unweighted spaces \(H^s\). For the vacuum equations the second author obtained well-posedness of the reduced Einstein equations with \((g_{\alpha\beta}, \partial_t g_{\alpha\beta}) \in H_{s+1,\delta} \times H_{s,\delta+1}, \ s > \frac{3}{2}\) and \(\delta > -\frac{3}{2}\), see [15]. But unlike Hughes, Kato and Marsden [13], Karp treated the quasi-linear waves equations as a first order symmetric hyperbolic system.

The first order techniques have the advantage that they enable, in a convenient way, the coupling of the gravitational fields equations to other matter models, in particular, to perfect fluids. In the Appendix we explain the main idea of [15] which allows us to obtain the regularity index \(s > \frac{3}{2}\) by means of first order hyperbolic systems.

A crucial step in the proof of existence and uniqueness of any hyperbolic system is to establish energy estimates for the linearized system. In order to achieve this we define an appropriated inner–product in the \(H_{s,\delta}\) spaces, which takes into account the coefficients of the linearized system (see Section 5 ). A similar inner–product was used in [15], and here we rely on these energy estimates.

Once we have obtained the energy estimates for the linearized system, we use Majda’s iterative scheme in order to obtain existence and uniqueness of the quasi–linear symmetric
hyperbolic system [17]. This procedure uses the fact that solutions to a linear first order symmetric hyperbolic system with $C^\infty_0$ coefficients and initial data, are also $C^\infty_0$. But here we encounter a further difficulty, namely, the right hand side of (1.1) contains the fractional power $w^{\frac{2}{\gamma-1}}$, see (3.16). So even when $w \in C^\infty_0$ and $w \geq 0$, $w^{\frac{2}{\gamma-1}}$ might not be a $C^\infty$ function. We solve that problem by using the fact that that $\epsilon = w^{\frac{2}{\gamma-1}}$ satisfy a certain first order linear equation. Gamblin encountered a similar problem for the Euler–Poisson equations [11], but he solved it in a some what different way.

Our results improve the existence theory of solutions locally in time of self gravitating relativistic perfect fluids in several aspects. Rendall studied this problem in [22], but he assumed time symmetry, which means that the extrinsic curvature of the initial manifold is zero, and therefore the Einstein constraint equations are reduced to a single scalar equation. In addition, he dealt only with $C^\infty_0$-solutions. In his study of the Newtonian limit of perfect fluids, Oliynyk obtained existence locally in time in the weighted space of integer order $H_{k,\delta}$, for $k \geq 4$ [21]. Both Rendall and Oliynyk assume that the adiabatic exponent of (1.5) satisfies the condition that $\frac{2}{\gamma-1}$ is an integer.

The paper is organized as follows: In the next section we define the weighted Sobolev spaces of fractional order $H_{s,\delta}$ and state the main result. Section 3 has two subsections: the first one deals with the hyperbolic reduction of the Euler equations (1.4); in the second one we spell out the matrices which describe the coupled equations (1.1), (1.2) and (1.4) as a hyperbolic system. In Section 4 we present tools and properties of the $H_{s,\delta}$-spaces which are necessary for us. We also define the corresponding product spaces. The energy estimates for the linearized system are considered in Section 5, there we also define the appropriate inner-product. In Section 6 we treat the iteration procedure. A part of the steps are standard and known, but some of them require special attention due to the specific form of the system (3.24) and the product spaces. In this Section we will use the fact that the coefficients of the first order derivatives depend only on the semi-metric $g_{\alpha\beta}$. Finally, in Section 7 we proof the main result. In the Appendix we give a heuristic idea explaining how that fact that the coefficients of the wave equations depend only on the semi–metric $g_{\alpha\beta}$ enable us to obtain the desired regularity by means of symmetric hyperbolic systems.

2. The main results

We obtain the well-posdeness in the weighted Sobolev spaces of fractional order. So we first recall their definition.

Let $\{\psi_j\}_{j=0}^\infty \subset C^\infty_0(\mathbb{R}^3)$ be a sequence of cutoff function such that, $\psi_j(x) \geq 0$ for all $j \geq 0$, supp($\psi_j$) $\subset \{x : 2^{j-2} \leq |x| \leq 2^{j+1}\}$, $\psi_j(x) = 1$ on $\{x : 2^{j-1} \leq |x| \leq 2^j\}$ for $j = 1, 2, \ldots$, supp($\psi_0$) $\subset \{x : |x| \leq 2\}$, $\psi_0(x) = 1$ on $\{x : |x| \leq 1\}$ and

$$|\partial^\alpha \psi_j(x)| \leq C_\alpha 2^{-|\alpha|j},$$

where the constant $C_\alpha$ does not depend on $j$. 

We restrict ourselves to the case $p = 2$ and denote the Bessel potential spaces by $H^s$ with the norm given by
\[
\|u\|_{H^s}^2 = \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,
\]
where $\hat{u}$ is the Fourier transform of $u$.

**Definition 2.1.** For $s, \delta \in \mathbb{R}$,
\[
(\|u\|_{H_{s,\delta}})^2 = \sum_{j=0}^{\infty} 2^{(s+\delta)2j} \|\psi_j u\|_{H^s}^2,
\]
where $f_\varepsilon(x) = f(\varepsilon x)$ denotes the scaling by a positive number $\varepsilon$. The space $H_{s,\delta}$ is the set of all temperate distributions having a finite norm given by (2.1).

The $H_{s,\delta}$-norm of a distribution $u$ in an open set $\Omega \subset \mathbb{R}^3$ is given by
\[
\|u\|_{H_{s,\delta}(\Omega)} = \inf_{f_{\Omega} = u} \|f\|_{H_{s,\delta}(\mathbb{R}^3)}.
\]

**Definition 2.2.** Let $\mathcal{M}$ be a 3 dimensional smooth connected manifold and let $h$ be a metric on $\mathcal{M}$ such that $(\mathcal{M}, h)$ is complete. We say that $(\mathcal{M}, h)$ is asymptotically flat of the class $H_{s,\delta}$ if $h \in H_{s,\delta}^0(\mathcal{M})$ and there is a compact set $S \subset \mathcal{M}$ such that:
1. There is a finite collection of charts $\{(U_i, \varphi_i)\}_{i=1}^N$ which cover $\mathcal{M} \setminus S$;
2. For each $i$, $\varphi_i^{-1}(U_i) = \{x \in \mathbb{R}^3 : |x| > r_i\}$ for some positive $r_i$;
3. The pull-back $(\varphi_i h)_{ab}$ is uniformly equivalent to the Euclidean metric $\delta_{ab}$ on $E_{r_i}$ for each $i$;
4. For each $i$, $(\varphi_i h)_{ab} - \delta_{ab} \in H_{s,\delta}(E_{r_i})$.

The $H_{s,\delta}$-norm on the manifold $\mathcal{M}$ is defined as follows. Let $U_0 \subset \mathcal{M}$ be an open set such that $S \subset U_0$ and $\overline{U_0} \in \mathcal{M}$. Let $\{\chi_0, \chi_i\}$ be a partition of unity subordinate to $\{U_0, U_i\}$, then
\[
\|u\|_{H_{s,\delta}(\mathcal{M})} := \|\chi_0 u\|_{H^s(U_0)} + \sum_{i=1}^{N} \|\varphi_i^* (\chi_i u)\|_{H_{s,\delta}(\mathbb{R}^3)}
\]
is the norm of the weighted fractional Sobolev space $H_{s,\delta}(\mathcal{M})$. For the definition of the norm $\|\chi_0 u\|_{H^s(\Omega)}$ on the manifold $\mathcal{M}$ see e.g. [1]. Note that the norm (2.2) depends on the partition of unity, but different partitions of unity result in equivalent norms. In the following we will omit the notation $\mathcal{M}$, that is, we will write $\|u\|_{H_{s,\delta}}$ instead of $\|u\|_{H_{s,\delta}(\mathcal{M})}$.

Since the principal symbol of the fields equations (1.1) is characteristic in every direction (see e.g. [9]), it is impossible to solve (1.1) in the present form. We study these equations under the **harmonic gauge condition**
\[
F^\mu = g^{\beta\gamma} \Gamma^\mu_{\beta\gamma} = 0,
\]
where $g^{\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta}$. Then the fields equations (1.1) are equivalent to the **reduced Einstein equations**
\[
g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} = H_{\alpha\beta}(g, \partial g) - 16\pi T_{\alpha\beta} + 8\pi g^{\mu\nu} T_{\mu\nu} g_{\alpha\beta},
\]
where $H_{\alpha\beta}(g, \partial g)$ is a quadratic function of the semi–metric $g_{\alpha\beta}$ and its first order derivatives.

Since $g^{\mu
u}$ has a Lorentzian signature, (2.4) is a system of quasi-linear wave equations. Taking into account the equation of state (1.5), the normalization condition (1.2), and the Makino variable (1.6), then the wave equations (2.4) become

$$ g^{\mu
u} \partial_\mu \partial_\nu g_{\alpha\beta} = H_{\alpha\beta}(g, \partial g) - 8\pi w \cdot - (1 - Kw^2) g_{\alpha\beta} + 2(1 + Kw^2)u_\alpha u_\beta \) . $$

So the unknowns of the system (2.5) coupled with the Euler equations (1.4) are the semi–metric $g_{\alpha\beta}$, the velocity vector $u^\alpha$ and the Makino variable $w$. Note that even if $w$ is a smooth function, $w \frac{\partial}{\partial t}$ might not be smooth in certain regions.

The initial data consist of the triple $(\mathcal{M}, h, K_{ab})$, where $\mathcal{M}$ is a space-like manifold, $h$ is a proper Riemannian metric on $\mathcal{M}$ and $K_{ab}$ is its second fundamental form (extrinsic curvature). The semi–metric $g_{\alpha\beta}$ takes the following data on $\mathcal{M}$:

$$
\left\{
\begin{array}{l}
g_{00}|_\mathcal{M} = -1, \quad g_{0\alpha}|_\mathcal{M} = 0, \quad g_{ab}|_\mathcal{M} = h_{ab} \quad a, b = 1, 2, 3, \\
-\frac{1}{2} \partial_0 g_{ab}|_\mathcal{M} = K_{ab}.
\end{array}
\right.
$$

The rest of the data $\partial_0 g_{ab}|_\mathcal{M}$ are determined by the harmonic gauge condition (2.3). In addition, the initial data of the velocity vector $u^\alpha$ and the Makino variable $w$ are given on $\mathcal{M}$. We denote the Minikowski metric by $\eta_{\alpha\beta}$.

**Theorem 2.3** (Main result). Let $\frac{3}{2} < s < \frac{2}{\gamma - 1} + \frac{1}{2}$ and $\delta > -\frac{3}{2}$. Assume $\mathcal{M}$ is asymptotically flat of class $H_{s+1,\delta}$, $K_{ab} \in H_{s+1,\delta+1}$, and $(u^0 - 1, u^a, w)|_\mathcal{M} \in H_{s+1,\delta+2}$. Then there exists a positive $T$, a unique semi–metric $g_{\alpha\beta}$, a unite timelike vector $u^a$ and $w$ satisfying the reduced Einstein equations (2.5) and the Euler equations (1.4) such that

$$
(g_{\alpha\beta}(t) - \eta_{\alpha\beta}) \in C([0, T], H_{s+1,\delta}) \cap C^1([0, T], H_{s,\delta+1})
$$

and

$$
(u^0 - 1, u^a, w) \in C([0, T], H_{s+1,\delta+2}) \cap C^1([0, T], H_{s,\delta+3}).
$$

**Remark 2.4** (On the differentiability). Note that we have a lower and an upper bound of the differentiability index $s$, however, in case $\frac{2}{\gamma - 1}$ is an integer, then there is no upper bound.

A necessary and sufficient condition for the equivalence between the reduced Einstein equations (2.5) and the field equations (1.1) is that the geometric data $(h, K_{ab})$ satisfy the constraint equations

$$
\left\{
\begin{array}{l}
R(h) - K_{ab}K^{ab} + (h^{ab}K_{ab})^2 = 16\pi z \\
(\nabla^b K^{ab}) - (\nabla^b (h^{bc}K_{bc})) = -8\pi j^a.
\end{array}
\right.
$$

Here $R(h) = h^{ab}R_{ab}$ is the scalar curvature with respect to the metric $h$. The right hand-side $(z, j^a)$ contains the energy density and the momentum density respectively. Thus solving the constraint equations (2.9) ensures that the solution of (2.5) satisfies the original system (1.1).
However, before we solve the constraints, we need to treat the compatibility problem between the matter variables \((z, j^a)\) and the initial data for the velocity \(u^\alpha\) and the Makino variable \(w\).

This problem can be described as follows: Let \(\bar{u}^\alpha\) denote the projection of the velocity vector \(u^\alpha\) on the initial manifold \(M\) and \(n^\alpha\) the timelike unite normal vector to \(M\). The energy density \(z\) is the double projection of \(T_{\alpha\beta}\) on \(n^\alpha\) and the momentum density \(j^a\) is once the projection of \(T_{\alpha\beta}\) on \(n^\alpha\) and once on \(M\). Applying these projections to the perfect fluid \((1.2)\) results in

\[
\begin{align*}
 z &= w^{\frac{2-\gamma}{\gamma}} (1 + (1 + Kw^2) h_{ab} \bar{u}^a \bar{u}^b) \\
 j^a &= w^{\frac{2-\gamma}{\gamma}} (1 + Kw^2) \bar{u}^a \sqrt{1 + h_{ab} \bar{u}^a \bar{u}^b}.
\end{align*}
\]

So the compatibility problem consists of solving \((2.10)\) for \(w, \bar{u}^a\), when \(z\) and \(j^a\) are given. This problem was solved in the \(H^{s,\delta}\)-spaces and for the same ranges of \(s\) and \(\delta\) in [4, Theorem 2.5].

Since, \((z, j^a) \in H^{s,\delta+2}\), it follows from Proposition 4.10 below that \((w, u^0 - 1, u^a)|_M \in H^{s,\delta+2}\). This explains the difference between the weights for the metric and for the fluid variables. We conclude using [4] that there is an initial data set \((h, K_{ab})\) and \((w, u^a)\) in the \(H^{s,\delta}\)-spaces which satisfy the constraints \((2.9)\) together with the compatibility problem \((2.10)\).

**Corollary 2.5.** Under the assumptions of Theorem 2.3 and in addition under the assumption that the initial data \((h, K_{ab})\) and \((w, u^a)\) satisfy the constraint equations \((2.9)\) and compatibility problem \((2.10)\), there exists a positive \(T\), a semi–metric \(g\), a unite timelike vector \(u^\alpha\) and \(w\) satisfying the Einstein \((1.1)\) and the Euler equations \((1.4)\) for \(t \in [0, T]\). The regularity of \(g\), \(u^\alpha\) and \(w\) are the same as in Theorem 2.3.

### 3. Symmetric Hyperbolic Systems

The main result is proved by transforming the coupled system \((2.5)\) and \((1.4)\) into a symmetric hyperbolic system. We therefore recall its definition.

**Definition 3.1** (Symmetric hyperbolic system). A first order quasi–linear \(k \times k\) system is symmetric hyperbolic system in a region \(G \subset \mathbb{R}^k\), if it is of the form

\[
L[U] = A^\alpha(U) \partial_\alpha U + B(U) = 0,
\]

where the matrices \(A^\alpha(U)\) are symmetric and for every arbitrary \(U \in G\), there exists a covector \(\xi\) such that

\[
\xi_\alpha A^\alpha(U)
\]

is positive definite. The covectors \(\xi_\alpha\) for which \((3.2)\) is positive definite, are called spacelike with respect to equation \((3.1)\).

If \(\xi\) can be chosen to be the vector \((1, 0, 0, 0)\), then condition \((3.2)\) implies that the matrix \(A^0(U)\) is a positive definite matrix, and we may write system \((3.1)\) in the form

\[
A^0(U) \partial_t U = A^\alpha(U) \partial_\alpha U + B(U).
\]
3.1. The Euler equations written as a symmetric hyperbolic system. It is not obvious that the Euler equations written in the conservative form $\nabla_\alpha T^{\alpha\beta} = 0$ are symmetric hyperbolic. In fact these equations have to be transformed in order to be expressed in a symmetric hyperbolic form. Rendall presented such a transformation of these equations in [22], however, its geometrical meaning is not entirely clear and it might be difficult to generalize it to the non time symmetric case. Hence we will present a different hyperbolic reduction of the Euler equations and discuss it in some details, for we have not seen it anywhere in the literature. The basic idea is to perform the standard fluid decomposition and then to modify the equation by adding, in an appropriate manner, the normalization condition (1.3) which will be considered as a constraint equation.

The fluid decomposition method consists of the projection of equation $\nabla_\nu T^{\nu\beta} = 0$ onto $u^\alpha$ which leads to $u_\beta \nabla_\nu T^{\nu\beta} = 0$, and the projection of these equations on rest space $O$ orthogonal to $u^\alpha$ of a fluid which leads to $P_{\alpha\beta} \nabla_\nu T^{\nu\beta} = 0$, where $P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$.

Inserting this decomposition into (1.2) results in a system of the following form:

\begin{align}
(3.4a) & \quad u^\nu \nabla_\nu \epsilon + (\epsilon + p) \nabla_\nu u^\nu = 0; \\
(3.4b) & \quad (\epsilon + p) P_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^{\nu\alpha} \nabla_\nu p = 0.
\end{align}

Note that we have beside the evolution equations (3.4a) and (3.4b) the following constraint equation: $g_{\alpha\beta} u^\alpha u^\beta = -1$. We will show in Subsection 3.1.1 that this constraint equation is conserved under the evolution equation.

In order to obtain a symmetric hyperbolic system we have to modify it in the following way. The normalization condition (1.3) gives that $u_\beta \nabla_\nu u^\beta = 0$, so we add $(\epsilon + p) u_\beta u^\nu \nabla_\nu u^\beta = 0$ to equation (3.4a) and $u_\alpha u_\beta u^\nu \nabla_\nu u^\beta = 0$ to (3.4b), which results in

\begin{align}
(3.5a) & \quad u^\nu \nabla_\nu \epsilon + (\epsilon + p) P^{\nu\beta} \nabla_\nu u^\beta = 0 \\
(3.5b) & \quad \Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + \frac{\sigma^2}{(\epsilon + p)} P^{\nu\alpha} \nabla_\nu \epsilon = 0,
\end{align}

where $\sigma := \sqrt{\frac{\partial \rho}{\partial \epsilon}}$ is the speed of sound and $\Gamma_{\alpha\beta} = P_{\alpha\beta} + u_\alpha u_\beta = g_{\alpha\beta} + 2 u_\alpha u_\beta$ is a reflection with respect to the rest subspace $O$. As mentioned above, we will introduce a new matter variable which is given by (1.6). The idea which is behind this is the following: The system (3.5a) and (3.5b) is almost of symmetric hyperbolic form, it would be symmetric if we multiply the system by appropriate factors, for example, (3.5a) by $\frac{\partial \rho}{\partial \epsilon} = \sigma^2$ and (3.5b) by $(\epsilon + p)$. However, doing so we will be faced with a system in which the coefficients will either tend to zero or to infinity, as $\epsilon \to 0$. Hence, it is impossible to represent this system in a non-degenerate form using these multiplications.

The central point is now to introduce a new variable $w = M(\epsilon)$ which will regularize the equations even for $\epsilon = 0$. We do this by multiplying equation (3.5a) by $k^2 M' = k^2 \frac{\partial M}{\partial \epsilon}$. This results in the following system which we have written in a matrix form:
\[
(3.6) \begin{pmatrix}
\kappa^2 u^\nu & \kappa^2 (\epsilon + p) M' P^\nu_\beta \\
\frac{\sigma^2}{(\epsilon + p) M' P^\nu_\alpha} & \Gamma_{\alpha\beta} u^\nu
\end{pmatrix} \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

In order to obtain symmetry we have to demand that
\[
(3.7) M' = \frac{\sigma}{(\epsilon + p) \kappa},
\]
where \(\kappa \gg 0\) has been introduced in order to simplify the expression for \(w\). If we choose \(\kappa = \frac{2}{\gamma - 1 + K w^2}\), then (3.7) holds and consequently the system (3.6) is transferred to the symmetric system
\[
(3.8) \begin{pmatrix}
\kappa^2 u^\nu & \sigma \kappa P^\nu_\beta \\
\kappa \sigma P^\nu_\alpha & \Gamma_{\alpha\beta} u^\nu
\end{pmatrix} \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

The covariant derivative \(\nabla_\nu\) takes in local coordinates the form \(\nabla_\nu = \partial_\nu + \Gamma^\gamma_{\nu\delta} \partial g^\delta_{\alpha\beta}\) which expresses the fact that equations (3.8) is coupled to equations (1.1) for the gravitational field \(g_{\alpha\beta}\). In addition, from the definition of the Makino variable (1.6), we see that \(\epsilon^{-1} = w^2\), so \(\kappa = \frac{2}{\gamma - 1 + K w^2}\) and \(\sigma = \sqrt{\gamma K} w\). Thus the fractional power of the equation of state (1.5) does not appear in the coefficients of the system (3.8), and these coefficients are \(C^\infty\) functions of the scalar \(w\), the four vector \(u^\alpha\) and the gravitational field \(g_{\alpha\beta}\).

Now we want to show that \(A^0\) of our system (3.8) is indeed positive definite. In order to do it we analyze the principal symbol of this system. For each \(\xi_\alpha \in T^*_x V\) the principal symbol is a linear map from \(\mathbb{R} \times E_x\) to \(\mathbb{R} \times F_x\), where \(E_x\) is a fiber in \(T^*_x V\) and \(F_x\) is a fiber in the cotangent space \(T^*_x V\). Since in local coordinates \(\nabla_\nu = \partial_\nu + \Gamma^\gamma_{\nu\delta} \partial g^\delta_{\alpha\beta}\), the principal symbol of system (3.8) is
\[
(3.9) \xi_\nu A^\nu = \begin{pmatrix}
\frac{\kappa^2}{\kappa} (u^\nu \xi_\nu) & \sigma \kappa P^\nu_\beta \xi_\nu \\
\sigma \kappa P^\nu_\alpha \xi_\nu & (u^\nu \xi_\nu) \Gamma_{\alpha\beta}
\end{pmatrix}
\]

and the characteristics are the set of covectors \(\xi\) for which \((\xi_\nu A^\nu)\) is not an isomorphism. Hence the characteristics are the zeros of \(Q(\xi) := \det(\xi_\nu A^\nu)\).

The geometric advantages of the fluid decomposition are the following. The operators in the blocks of the matrix (3.9) are the projection on the rest hyperplane \(O\), \(P^\nu_\alpha\), and the reflection with respect to the same hyperplane, \(\Gamma_{\alpha\beta}\). Therefore, the following relations hold:
\[
\Gamma^\alpha_{\gamma\beta} = \delta^\alpha_{\beta}, \quad \Gamma^{\alpha\gamma} P^\gamma_\nu = P^{\alpha\nu} \quad \text{and} \quad P^\beta_\alpha P^\alpha_\nu = P^\nu_\beta.
\]
which yields

(3.10) \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \Gamma^\alpha_\gamma \\ 0 & 0 & 0 \end{pmatrix} (\xi_\nu A^\nu) = \begin{pmatrix} \kappa^2 (u^\nu \xi_\nu) & \sigma \kappa P^\nu_\beta \xi_\nu \\ \sigma \kappa P^\alpha_\nu \xi_\nu & (u^\nu \xi_\nu) (\delta^\alpha_\beta) \end{pmatrix}.
\]

It is now fairly easy to calculate the determinate of the right hand side of (3.10) and we have

\[
\det \begin{pmatrix} \kappa^2 (u^\nu \xi_\nu) & \sigma \kappa P^\nu_\beta \xi_\nu \\ \sigma \kappa P^\alpha_\nu \xi_\nu & (u^\nu \xi_\nu) (\delta^\alpha_\beta) \end{pmatrix} = \kappa^2 (u^\nu \xi_\nu)^3 \left( (u^\nu \xi_\nu)^2 - \sigma^2 P^\alpha_\beta \xi_\alpha \xi_\beta \right).
\]

Since \( P^\alpha_\beta \) is a projection,

\[ P^\alpha_\nu \xi_\nu P^\nu_\beta \xi_\beta = g^\nu_\beta \xi_\nu \xi_\beta P^\alpha_\nu P^\nu_\alpha = g^\nu_\beta \xi_\nu P^\nu_\beta \xi_\beta = P^\nu_\beta \xi_\nu \xi_\beta, \]

and since \( \Gamma^\gamma_\beta \) is a reflection,

\[
\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \Gamma^\alpha_\gamma \\ 0 & 0 & 0 \end{pmatrix} = \det (g^\alpha_\beta \Gamma^\gamma_\beta) = - (\det (g^\alpha_\beta))^{-1}.
\]

Consequently,

(3.11) \[
Q(\xi) := \det(\xi_\nu A^\nu) = -\kappa^2 \det(g^\alpha_\beta)(u^\nu \xi_\nu)^3 \left\{ (u^\nu \xi_\nu)^2 - \sigma^2 P^\alpha_\beta \xi_\alpha \xi_\beta \right\}
\]

and therefore the characteristic covectors are given by two simple equations:

(3.12) \[
\xi_\nu u^\nu = 0;
\]

(3.13) \[
(u^\nu \xi_\nu)^2 - \sigma^2 P^\alpha_\beta \xi_\alpha \xi_\beta = 0.
\]

Remark 3.2. The characteristics conormal cone is a union of two hypersurfaces in \( T^*_x V \).

One of these hypersurfaces is given by the condition (3.12) and it is a three dimensional hyperplane \( O \) with the normal \( u^\alpha \). The other hypersurface is given by the condition (3.13) and forms a three dimensional cone, the so called, sound cone.

Let us now consider the timelike vector \( u_\nu \) and insert the covector \( -u_\nu \) into the principal symbol (3.9), then

\[
-u_\nu A^\nu = \begin{pmatrix} \kappa^2 & 0 \\ 0 & \Gamma^\alpha_\beta \end{pmatrix}
\]

is positive definite. Indeed, \( \Gamma^\alpha_\beta \) is a reflection with respect to a hyperplane having a timelike normal. Hence, \( -u_\nu \) is a spacelike covector with respect to the hydrodynamical equations (3.8). Herewith, we have showed relatively elegant and elementary that the relativistic hydrodynamical equations are symmetric–hyperbolic.
We want now to show that the covector \( t_\alpha = (1, 0, 0, 0) \) is also spacelike with respect to the system (3.8). Since \( P^\alpha_\beta u_\alpha = 0 \), the covector \(-u_\nu\) belongs to the sound cone (3.14)
\[
(\xi_\nu u^\nu)^2 - \sigma^2 P^\alpha_\beta \xi_\alpha \xi^\beta > 0.
\]
Inserting \( t_\nu = (1, 0, 0, 0) \) the right hand side of (3.14) yields (3.15)
\[
(u^0)^2(1 - \sigma^2) - \sigma^2 g^{00}.
\]
Since the sound velocity is always less than the light speed, that is \( \sigma^2 = \frac{\partial p}{\partial \epsilon} < c^2 = 1 \), we conclude from (3.15) that \( t_\nu \) also belongs to the sound cone (3.14). Hence, the vector \(-u_\nu\) can be continuously deformed to \( t_\nu\) while condition (3.14) holds along the deformation path. Consequently, the determinant of (3.11) remains positive under this process and hence \( t_\nu A^\nu = A^0 \) is also positive definite. Thus we have proved.

**Theorem 3.3.** Let \( \epsilon \) be non-negative density function, then the Euler system (1.4) coupled with the equation of state (1.5) can be written as a symmetric hyperbolic system of the form (3.3), and where \( A^0 \) is a positive definite.

3.1.1. **Conservation of the unit length of the fluid.**

**Proposition 3.4.** The constraint condition \( g_\alpha_\beta u^\alpha u^\beta = -1 \) is conserved along stream lines \( u^\alpha \).

**Proof.** Let \( k(t) \) be a curve such that \( k'(t) = u^\alpha \) and set \( Z(t) = (u \circ k)_\beta(u \circ k)^\beta \), then we need to establish
\[
\frac{d}{dt} Z(t) = 2u_\beta \nabla_{k(t)} u^\beta = 2u^\nu u_\beta \nabla_\nu u^\beta = 0.
\]
Multiplying the four last rows of the Euler system (3.8) by \( u^\alpha \) and recalling that \( P^\nu_\alpha \) is the projection on the rest space \( O \) orthogonal to \( u^\alpha \), we have
\[
0 = u^\alpha \left( \Gamma^\nu_\alpha u^\nu \nabla_\nu u^\beta + \kappa \sigma P^\nu_\alpha \nabla_\nu w \right) = u^\alpha P^\nu_\alpha u^\nu \nabla_\nu u^\beta - u^\nu u_\beta \nabla_\nu u^\beta + \kappa \sigma u^\alpha P^\nu_\alpha \nabla_\nu w = -u^\nu u_\beta \nabla_\nu u^\beta.
\]
Therefore, if \( g_\alpha_\beta u^\alpha u^\beta = -1 \) on the initial manifold, then it holds along the stream lines \( u^\alpha \). \( \square \)

3.2. **The coupled hyperbolic system.** In this section we will transform the coupled system (2.5) and (3.8) into a symmetric hyperbolic system. We will pay attention to the fact that the system will be in a form in which we can apply the energy estimates of [15]. That allows us to obtain the same regularity for the gravitational fields as Hughes, Kato and Marsden [13] got for the Einstein vacuum equations. Note that our system is slightly different from the symmetric hyperbolic system obtained by Fisher and Marsden [10], since our system contains a constant matrix \( \xi^\alpha \) as given by (3.21).

We consider a spacetime \( (V, g_{\alpha_\beta}) \) of the type \( \mathbb{R} \times M \), where \( M \) is a Riemannian manifold, and we denote local coordinates by \( (t, x^a) \). Set
\[
h_{\alpha_\beta\gamma} = \partial_\gamma g_{\alpha_\beta},
\]
then the reduced Einstein equations (2.5) takes the form

\[
\begin{align*}
\partial_t g_{\alpha\beta} &= h_{\alpha\beta}, \\
-g^{00} \partial_t h_{\alpha\beta} &= \left\{ 2g^{0a} \partial_a h_{\alpha\beta} + g^{ab} \partial_a h_{\alpha\beta} + H_{\alpha\beta}(g, \partial g) \right. \\
&- \left. 8\pi w^{2-\tau} ((1 - Kw^2)g_{\alpha\beta} + 2(1 + Kw^2)u_\alpha u_\beta) \right\}, \\
g^{ab} \partial_t h_{_{\alpha\beta}a} &= g^{ab} \partial_a h_{\alpha\beta}. 
\end{align*}
\] (3.16)

In order to apply the energy estimates of [15], we need that the coefficients of \( \partial_t h_{\alpha\beta} \) will be independent of \( t \). This is because the specific form of the inner-product in \( H_{s,\delta} \) spaces which takes into account the matrix \( A^0 \) of the system (3.1). In Section 5 we will further clarify this issue. Therefore we divide the second row by \( -g^{00} \) and in order to preserve the symmetry of the system, we also multiply the third row by \( (-g^{00})^{-1} \). Thus the wave equations (2.5) are equivalent to the system

\[
\begin{align*}
\partial_t g_{\alpha\beta} &= h_{\alpha\beta}, \\
\partial_t h_{\alpha\beta} &= (-g^{00})^{-1} \left\{ 2g^{0a} \partial_a h_{\alpha\beta} + g^{ab} \partial_a h_{\alpha\beta} + H_{\alpha\beta}(g, \partial g) \right. \\
&- \left. 8\pi w^{2-\tau} ((1 - Kw^2)g_{\alpha\beta} + 2(1 + Kw^2)u_\alpha u_\beta) \right\}, \\
(-g^{00})^{-1} g^{ab} \partial_t h_{_{\alpha\beta}a} &= (-g^{00})^{-1} g^{ab} \partial_a h_{\alpha\beta}. 
\end{align*}
\] (3.17)

To shorten and simplify notations, we introducing the auxiliary variables

\[(v, \partial_t v, \partial_x v) = (g_{\alpha\beta} - \eta_{\alpha\beta}, \partial_t g_{\alpha\beta}, \partial_x g_{\alpha\beta}),\]

where \( \eta_{\alpha\beta} \) denotes the Minkowski metric and \( \partial_x \) denotes the set of all spatial derivatives. We also set \( (v_0)^\alpha = (1, 0, 0, 0) \) and \( W = (w, u^\alpha - v_0^\alpha) \) stands for the Makino and the fluid variables. Finally,

\[U = (v, \partial_t v, \partial_x v, W)\]

represents the unknowns of the coupled systems.

We write the matrices in a block form, \( A = (a_{ij}) \), the \( k \times k \) identity matrix is denoted by \( e_k \) and \( 0_{m \times n} \) is the zero matrix.

The coupled system (3.17) and (3.8) can be written in the form of (3.1), where \( A^\alpha \) are \( 55 \times 55 \) symmetric matrices which depends only on \( v \) and \( W \). We shall describe now the structure of these matrices:

\[
A^0(v, W) = \begin{pmatrix}
e_{10} & 0_{10 \times 10} & 0_{10 \times 30} & 0_{10 \times 5} \\
0_{10 \times 10} & e_{10} & 0_{50 \times 30} & 0_{10 \times 5} \\
0_{30 \times 10} & 0_{30 \times 10} & a_{33}^0 & 0_{30 \times 5} \\
0_{5 \times 10} & 0_{5 \times 10} & 0_{5 \times 30} & a_{44}^0
\end{pmatrix},
\] (3.18)

where

\[a_{33}^0 = \frac{1}{-g^{00}} \begin{pmatrix}g_{11}^e_{10} & g_{12}^e_{10} & g_{13}^e_{10} \\
g_{21}^e_{10} & g_{22}^e_{10} & g_{23}^e_{10} \\
g_{31}^e_{10} & g_{32}^e_{10} & g_{33}^e_{10}\end{pmatrix},\]
and \( a_{44}^0 = a_{44}^0(g_{\alpha\beta}, w, u^\alpha) \) is given by (3.8) when \( \nu = 0 \). From (3.17) we see that the coefficients of \( \partial_u U \), \( a = 1, 2, 3 \), have the form

\[
\begin{pmatrix}
0_{10 \times 10} & 0_{10 \times 40} & 0_{5 \times 5} \\
0_{40 \times 10} & a_{22}^\alpha & 0_{40 \times 5} \\
0_{5 \times 10} & 0_{5 \times 40} & a_{33}^\alpha
\end{pmatrix},
\]

where \( a_{33}^\alpha = a_{33}^\alpha(g_{\alpha\beta}, w, u^\alpha) \) is from the system (3.8) of the fluid and

\[
a_{22}^\alpha(g_{\alpha\beta}) = \frac{1}{g^{00}} \begin{pmatrix}
2g^{00}e_{10} & g^{11}e_{10} & g^{22}e_{10} & g^{33}e_{10} \\
g^{11}e_{10} & g^{22}e_{10} & g^{33}e_{10} & 0_{30 \times 30} \\
g^{33}e_{10} & 0_{30 \times 30}
\end{pmatrix}.
\]

An essential demand is that \( a_{22}^\alpha(g_{\alpha\beta}) \in H_{\alpha\beta} \), whenever \( g_{\alpha\beta} - \eta_{\alpha\beta} \in H_{\alpha\beta} \). Obviously, this does not hold for the matrix in (3.19). Therefore we need to modify these matrices by a constant matrix

\[
c_{22}^\alpha = \begin{pmatrix}
0_{10 \times 10} & \delta^{11}e_{10} & \delta^{22}e_{10} & \delta^{33}e_{10} \\
\delta^{11}e_{10} & \delta^{22}e_{10} & \delta^{33}e_{10} & 0_{30 \times 30}
\end{pmatrix},
\]

then \( (a_{22}^\alpha - c_{22}^\alpha)(v) \in H_{\alpha\beta} \) whenever \( v \in H_{\alpha\beta} \). So we set

\[
A^\alpha(v, W) = \begin{pmatrix}
0_{10 \times 10} & 0_{10 \times 40} & 0_{5 \times 5} \\
0_{40 \times 10} & a_{22}^\alpha - c_{22}^\alpha & 0_{40 \times 5} \\
0_{5 \times 10} & 0_{5 \times 40} & a_{33}^\alpha
\end{pmatrix}
\]

and a constant matrix

\[
C^\alpha = \begin{pmatrix}
0_{10 \times 10} & 0_{10 \times 40} & 0_{5 \times 5} \\
0_{40 \times 10} & c_{22}^\alpha & 0_{40 \times 5} \\
0_{5 \times 10} & 0_{5 \times 40} & 0_{5 \times 5}
\end{pmatrix}.
\]

We turn now to the lower order terms. The presence of the fractional power \( w^{2/(\gamma - 1)} \) in (3.17) causes substantial technical difficulties. We set

\[
f(v, W) := -\frac{8\pi w^{2/\gamma - 1}}{g^{00}} \left( (1 - Kw^2)g_{\alpha\beta} + 2(1 + Kw^2)u_\alpha u_\beta \right),
\]

then we can write \( B(U) \) in the form

\[
B(U) = B(U)(v, \partial_v v, \partial_x v)^T + F(v, W),
\]

where \( F(v, W) = (0, f(v, W), 0, 0)^T \) and

\[
B(U) = \begin{pmatrix}
0_{10 \times 10} & e_{10} & 0_{10 \times 10} & 0_{10 \times 10} \\
0_{10 \times 10} & b_{22} & b_{23} & b_{24} & b_{25} \\
0_{30 \times 10} & 0_{30 \times 10} & 0_{30 \times 10} & 0_{30 \times 10} & 0_{30 \times 10} \\
0_{5 \times 10} & b_{42} & b_{43} & b_{44} & b_{45}
\end{pmatrix}.
\]
The block $b_{2j}$, $j = 2, 3, 4, 5$, appear from the quadratic terms in (2.4)
\[ H_{\alpha\beta}(g, \partial g) = C_{\alpha\beta\gamma\delta\rho\sigma}^{\epsilon \zeta \eta \kappa \lambda \mu} h_{\epsilon \zeta \eta} h_{\kappa \lambda \mu} g^{\gamma \delta} g^{\rho \sigma}, \]
where $C_{\gamma\delta\beta\rho\sigma}^{\epsilon \zeta \eta \kappa \lambda \mu}$ are a combination of Kronecker deltas with integer coefficients. Thus
\[ b_{2j} = (-g^{00})^{-1} C_{\alpha\beta\gamma\delta\rho\sigma}^{\epsilon \zeta \eta \kappa \lambda \mu} h_{\epsilon \zeta \eta} g^{\gamma \delta} g^{\rho \sigma}, \mu = j - 2. \]
The block $b_{4j}$, $j = 2, 3, 4, 5$, appear from the multiplication of the reflection $\Gamma_{\alpha\beta}$ and $u^\nu$ in (3.8) with the Christoffel symbols. So its coefficients consist multiplications of $g_{\alpha\beta}$, $g^{\alpha\beta}$ and $u^\nu$.

In summary, we can write the coupled systems (3.17) and (3.8) as a symmetric hyperbolic system
\[ \mathcal{A}^0(v, W) \partial_t U = ((\mathcal{A}^a(v, W) + C^a) \partial_a U + \mathcal{B}(U) \left( \begin{array}{c} v \\ \partial_x v \end{array} \right)) + \mathcal{F}(v, W), \]
where $\mathcal{A}^0(U)$ is positive definite in the neighborhood of the initial data (2.6), $\mathcal{A}^0(0) - e_{55} = \mathcal{A}^0(0) = 0$ and $C^a$ is a constant symmetric matrix.

4. The $H_{s,\delta}$ spaces and their properties

The definition of the weighted Sobolev spaces of fractional order $H_{s,\delta}$, Definition 2.1, is due to Trieble [25]. Here we quote the propositions and properties which are needed for the proof of the main result. For their proofs see [4, 19, 25]

We start with some notations.

- Let $\{\psi_j\}$ be the sequence of functions in Definition 2.1. For any positive $\gamma$ we set
\[ \|u\|_{H^{s,\delta,\gamma}}^2 = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j^\gamma u)_{2j} \right\|_{H^s}^2 \]
and we will use the convention $\|u\|_{H^{s,\delta,1}} = \|u\|_{H^{s,\delta}}$. The subscripts $2^j$ mean a scaling by $2^j$, that is, $(\psi_j^\gamma u)_{2j}(x) = (\psi_j^\gamma u)(2^j x)$.

- For a non-negative integer $m$ and $\beta \in \mathbb{R}$, the space $C^m_{\beta}$ is the set of all functions having continuous partial derivatives up to order $m$ and such that the norm
\[ \|u\|_{C^m_{\beta}} = \sum_{|\alpha| \leq m} \sup_{\mathbb{R}^3} ((1 + |x|)^{\beta + |\alpha|} |\partial^{\alpha} u(x)|) \]
is finite.

- We will use the notation $A \lesssim B$ to denote an inequality $A \leq CB$ where the positive constant $C$ does not depend on the parameters in question.

**Proposition 4.1.** For any positive $\gamma$, there are two positive constants $c_0(\gamma)$ and $c_1(\gamma)$ such that
\[ c_0(\gamma) \|u\|_{H^{s,\delta}} \leq \|u\|_{H^{s,\delta,\gamma}} \leq c_1(\gamma) \|u\|_{H^{s,\delta}}. \]
Proposition 4.2. For any nonnegative integer $m$, positive $\gamma$ and $\delta$ there holds
\begin{equation}
\|u\|_{H_{m,\delta,\gamma}}^2 \lesssim \|u\|_{m,\delta}^2 \lesssim \|u\|_{H_{m,\delta,\gamma}}^2,
\end{equation}
where $\|u\|_{m,\delta}$ is defined by (1.7).

Proposition 4.3. If $s_1 \leq s_2$ and $\delta_1 \leq \delta_2$, then
\begin{equation}
\|u\|_{H_{s_1,\delta_1}} \leq \|u\|_{H_{s_2,\delta_2}}.
\end{equation}

Proposition 4.4. If $u \in H_{s,\delta}$, then
\begin{equation}
\|\partial u\|_{H_{s-1,\delta+1}} \leq \|u\|_{H_{s,\delta}}.
\end{equation}

Proposition 4.5. Let $s_1, s_2 \geq s$, $s_1 + s_2 > s + \frac{3}{2}$, $s_1 + s_2 \geq 0$ and $\delta_1 + \delta_2 \geq \delta - \frac{3}{2}$. If $u \in H_{s_1,\delta_1}$ and $v \in H_{s_2,\delta_2}$, then
\begin{equation}
\|uv\|_{H_{s,\delta}} \lesssim \|u\|_{H_{s_1,\delta_1}} \|v\|_{H_{s_2,\delta_2}}.
\end{equation}

Remark 4.6. If for a fixed constant $c_0$, $u - c_0 \in H_{s_1,\delta_1}$ and $v \in H_{s_2,\delta_2}$, then we can apply the multiplication property (4.7) to $(u - c_0)v$ and obtain
\begin{equation}
\|uv\|_{H_{s,\delta}} \lesssim \left(\|u - c_0\|_{H_{s_1,\delta_1}} + |c_0|\right) \|v\|_{H_{s_2,\delta_2}}.
\end{equation}

Proposition 4.7. Let $u \in H_{s,\delta} \cap L^\infty$, $1 < \beta$, $0 < s < \beta + \frac{1}{2}$ and $\delta \in \mathbb{R}$, then
\begin{equation}
\|u^\beta\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty}) \|u\|_{H_{s,\delta}}.
\end{equation}

Proposition 4.8. Let $s' < s$ and $\delta' < \delta$, then the embedding
\begin{equation}
H_{s,\delta} \hookrightarrow H_{s',\delta'}.
\end{equation}
is compact.

Proposition 4.9. If $s > \frac{3}{2} + m$ and $\delta + \frac{3}{2} \geq \beta$, then
\begin{equation}
\|u\|_{C_{\beta}^m} \lesssim \|u\|_{H_{s,\delta}},
\end{equation}
where $\|u\|_{C_{\beta}^m}$ is given by (4.2).

Proposition 4.10. Let $F : \mathbb{R}^m \to \mathbb{R}^l$ be $C^{N+1}$-function such that $F(0) = 0$ and where $N \geq [s] + 1$. Then there is a constant $C$ such that for any $u \in H_{s,\delta}$
\begin{equation}
\|F(u)\|_{H_{s,\delta}} \leq C\|F\|_{C^{N+1}} (1 + \|u\|_{L^\infty}^N) \|u\|_{H_{s,\delta}}.
\end{equation}

Proposition 4.11.
\begin{enumerate}
\item[(a)] The class $C_0^\infty(\mathbb{R}^3)$ is dense in $H_{s,\delta}$.
\item[(b)] Given $u \in H_{s,\delta}$, $s' > s \geq 0$ and $\delta' \geq \delta$. Then for $\rho > 0$ there is $u_\rho \in C_0^\infty(\mathbb{R}^3)$ and a positive constant $C(\rho)$ such that
\begin{equation}
\|u_\rho - u\|_{H_{s,\delta}} \leq \rho \quad \text{and} \quad \|u_\rho\|_{H_{s',\delta'}} \leq C(\rho) \|u\|_{H_{s,\delta}}.
\end{equation}
\end{enumerate}
4.1. **Product spaces.** The unknown of system (3.24) is a vector valued function
\[ U = (v, \partial_t v, \partial_x v, W), \]
where \( v = g_{\alpha\beta} - \eta_{\alpha\beta} \) stands for the fields variables and \( W = (w, u^\alpha - e^{\alpha}_0) \) stands for the fluid variables. We consider it in the space
\[ (4.14) \quad X_{s,\delta} := H_{s,\delta} \times H_{s,\delta+1} \times H_{s,\delta+1} \times H_{s+1,\delta+2}, \]
with the norm (see (4.1))
\[ (4.15) \quad \|U\|_{X_{s,\delta}}^2 = \|v\|^2_{H_{s,\delta,2}} + \|\partial_t v\|^2_{H_{s,\delta+1,2}} + \|\partial_x v\|^2_{H_{s,\delta+1,2}} + \|W\|^2_{H_{s+1,\delta+2,2}}. \]

**Remark 4.12.** Note that if \( U \in X_{s,\delta} \), then \( v \in H_{s,\delta} \) and \( \partial_x v \in H_{s,\delta+1} \), so by the integral representation of the norm \( H_{s,\delta} \) (see [4, §2]), we have that
\[ (4.16) \quad \|v\|_{H_{s+1,\delta}} \lesssim \left( \|v\|_{H_{s,\delta}} + \|\partial_x v\|_{H_{s,\delta+1}} \right). \]

5. **Energy estimates**

In this section we will derive the energy estimates for a linear symmetric hyperbolic system which we have obtained by linearising (3.24). So we consider
\[ (5.1) \quad A^0 \partial_t U = (A^a + C^a) \partial_a U + B \begin{pmatrix} v \\ \partial_t v \\ \partial_x v \end{pmatrix} + F + D, \]
where \( U = (v, \partial_t v, \partial_x v, W) \), the matrices \( A^0, A^a, B \) and \( C^a \) have the same structural form as the corresponding matrices in (3.24), \( C^a \) is a constant matrix, and the vectors \( F \) and \( D \) have the form \((0, f, 0, 0)\) and \((0, d_2, d_3, d_4)\) respectively.

**Assumptions 5.1.** All the matrices have the same block structure as (3.18), (3.20) and (3.23) and satisfy:
\[ (5.2a) \quad (A^0(t, \cdot) - e_{55}), A^a(t, \cdot) \in H_{s+1,\delta}; \]
\[ (5.2b) \quad \exists c_0 \geq 1 \text{ such that } c_0^{-1}V^T V \leq V^T A^0 V \leq c_0 V^T V, \quad \forall V \in \mathbb{R}^{55}; \]
\[ (5.2c) \quad \partial_t A^0(t, \cdot) \in L^\infty; \]
\[ (5.2d) \quad b_{2j}(t, \cdot) \in H_{s,\delta+1}, \quad j = 2, 3, 4, 5; \]
\[ (5.2e) \quad b_{4j}(t, \cdot) \in H_{s,\delta+2}, \quad j = 2, 3, 4, 5; \]
\[ (5.2f) \quad F(t, \cdot), D(t, \cdot) \in H_{s,\delta+1}. \]

5.1. **\( X_{s,\delta} \)-energy estimates.** We turn now to the definition of an inner-product in the space \( X_{s,\delta} \) which takes into account the structure of the matrix \( A^0 \) of the system (5.1).

Let \( F(u) \) denote the Fourier transform of a distribution \( u \) and set
\[ \Lambda^s(u) = (1 - \Delta)^{\frac{s}{2}} = F^{-1} \left( (1 + |\xi|^2)^{\frac{s}{2}} F \right)(u). \]
The standard inner-product on the Bessel-potential spaces \( H^s \) is
\[ \langle u_1, u_2 \rangle_s = \langle \Lambda^s(u_1), \Lambda^s(u_2) \rangle_{L^2}. \]
Taking into account the term-wise definition of the norm (2.1), we define the inner-product in $H_{s,\delta}$ as follows:

\begin{equation}
(5.3)\quad \langle u_1, u_2 \rangle_{s,\delta} := \sum_{j=0}^{\infty} 2^{(\delta+\frac{3}{2})2j} \left\langle \Lambda^s \left( \psi_j^2 u_1 \right)_{2j}, \Lambda^s \left( \psi_j^2 u_2 \right)_{2j} \right\rangle_{L^2},
\end{equation}

where $(u)_{2j}$ denotes scaling by $2^j$. Hence $\langle u, u \rangle_{s,\delta} = \|u\|_{H_{s,\delta},2}^2 \approx \|u\|_{H_{s,\delta}}^2$ by Proposition 4.1.

To each component of the space

\[ X_{s,\delta} := H_{s,\delta} \times H_{s,\delta+1} \times H_{s,\delta+1} \times H_{s+1,\delta+2} \]

we assign its own inner–product. Since $A^0 = (a^0_{ij})$, where $a^0_{ij}$ is the zero matrix for $i \neq j$, $a^0_{ii}$ is the identity for $i = 1, 2$, we assign to the first two components the inner–product (5.3), while for the other terms we insert $A^0$ term–wise.

**Definition 5.2** (Inner-product in $X_{s,\delta}$). Let $U_i = (v_i, \partial_i v_i, \partial_{ij} v_i, W_i) \in X_{s,\delta}$, $i = 1, 2$ and assume that the matrix $A^0$ satisfies Assumption 5.1, then we have

- **Inner-product on $H_{s,\delta}$**: $\langle v_1, v_2 \rangle_{s,\delta}$, where is defined by (5.3);
- **Inner-product on $H_{s,\delta+1}$**: $\langle \partial_i v_1, \partial_i v_2 \rangle_{s,\delta+1}$, where is defined by (5.3);
- **Inner-product on $H_{s,\delta+1}$**:

\begin{equation}
(5.4)\quad \langle \partial_x v_1, \partial_x v_2 \rangle_{s,\delta+1, a^0_{33}} := \sum_{j=0}^{\infty} 2^{(\delta+1+\frac{3}{2})2j} \left\langle \Lambda^s \left( \psi_j^2 \partial_x v_1 \right)_{2j}, \Lambda^s \left( \psi_j^2 \partial_x v_2 \right)_{2j} \right\rangle_{L^2};
\end{equation}

- **Inner-product on $H_{s+1,\delta+2}$**:

\begin{equation}
(5.5)\quad \langle W_1, W_2 \rangle_{s+1,\delta+2, a^0_{44}} := \sum_{j=0}^{\infty} 2^{(\delta+2+\frac{3}{2})2j} \left\langle \Lambda^{s+1} \left( \psi_j^2 W_1 \right)_{2j}, \Lambda^{s+1} \left( \psi_j^2 W_2 \right)_{2j} \right\rangle_{L^2};
\end{equation}

- **Inner-product on $X_{s,\delta}$**:

\begin{equation}
(5.6)\quad \langle U_1, U_2 \rangle_{X_{s,\delta}, A^0} := \langle v_1, v_2 \rangle_{s,\delta} + \langle \partial_i v_1, \partial_i v_2 \rangle_{s,\delta+1} + \langle \partial_x v_1, \partial_x v_2 \rangle_{s,\delta+1, a^0_{33}} + \langle W_1, W_2 \rangle_{s+1,\delta+2, a^0_{44}}.
\end{equation}

We denote by $\|U\|_{X_{s,\delta}, A^0}$ the norm associated with the inner-product (5.6). By assumption (5.2b) the following equivalence holds

\begin{equation}
(5.7)\quad \|U\|_{X_{s,\delta}} \lesssim \|U\|_{X_{s,\delta}, A^0} \lesssim \|U\|_{X_{s,\delta}}.
\end{equation}

In order simplify the notation we set $U(t) = U(t, x^1, x^2, x^3)$.

**Lemma 5.3.** Let $\frac{3}{2} < s$, $\delta \geq -\frac{3}{2}$ and assume the coefficients of (5.1) satisfy Assumptions 5.1. If $U(t) \in C^\infty_c (\mathbb{R}^3)$ is a solution of (5.1), then

\begin{equation}
(5.8)\quad \frac{d}{dt} (U(t), U(t))_{X_{s,\delta}, A^0} \leq C c_0 \left( (U(t), U(t))_{X_{s,\delta}, A^0} + 1 \right),
\end{equation}

where the constant $C$ depends on the corresponding norms of the coefficients, $s$ and $\delta$. 
The corresponding energy estimates for the vacuum Einstein equations were obtained in [15]. The same techniques can be applied here with some obvious modifications. We therefore give only a short sketch of the proof.

**Sketch of the proof.** From the inner-product (5.6) we see that
\[
\frac{1}{2} \frac{d}{dt} \langle U, U \rangle_{X_{s, \delta}, \mathcal{A}^0} = \langle v, \partial_t v \rangle_{s, \delta} + \langle \partial_t v, \partial_t^2 v \rangle_{s, \delta} + \langle \partial_x v, \partial_x \partial_t v \rangle_{s, \delta+1, \mathcal{A}^0_{33}} + \langle W, \partial_t W \rangle_{s+1, \delta+2, \mathcal{A}^0_{44}}
\]
\[
+ \sum_{j=0}^{\infty} 2^{(\delta+1+\frac{3}{2})2j} \left\langle \Lambda^s \left( \psi_j^2 \partial_x v \right)_{2j}, \partial_t \left( \mathcal{A}^0_{33} \right)_{2j}, \Lambda^s \left( \psi_j^2 \partial_x v \right)_{2j} \right\rangle_{L^2}
\]
\[
+ \sum_{j=0}^{\infty} 2^{(\delta+2+\frac{3}{2})2j} \left\langle \Lambda^{s+1} \left( \psi_j^2 W \right)_{2j}, \partial_t \left( \mathcal{A}^0_{44} \right)_{2j}, \Lambda^{s+1} \left( \psi_j^2 W \right)_{2j} \right\rangle_{L^2}.
\]
By the Cauchy Schwarz inequality, we obtain
\[
\langle v, \partial_t v \rangle_{s, \delta} \leq \|v\|_{H_{s, \delta, 2}} \|\partial_t v\|_{H_{s, \delta, 2}} \leq \frac{1}{2} \left( \|v\|^2_{H_{s, \delta, 2}} + \|\partial_t v\|^2_{H_{s, \delta+1, 2}} \right),
\]
and by Assumption (5.2c) the first infinite sum is less than
\[
C \|\partial_t \mathcal{A}^0_{33}\|_{L^\infty} \sum_{j=0}^{\infty} 2^{(\delta+1+\frac{3}{2})2j} \left\langle \Lambda^s \left( \psi_j^2 \partial_x v \right)_{2j}, \Lambda^s \left( \psi_j^2 \partial_x v \right)_{2j} \right\rangle_{L^2}
\]
\[
= C \|\partial_x v\|^2_{H_{s, \delta+1, 2}}.
\]
A similar estimate holds for the second infinite sum.

The most difficult part is the estimate
\[
\left| \langle \partial_t v, \partial_t^2 v \rangle_{s, \delta} + \langle \partial_x v, \partial_x \partial_t v \rangle_{s, \delta+1, \mathcal{A}^0_{33}} \right| \lesssim \|U\|^2_{X_{s, \delta}},
\]
and here it is essential to use the assumption that \( \mathcal{A}^0 \in H_{s+1, \delta} \) and \( s > \frac{3}{2} \). We refer to [15, §4] for the details of this estimate. The estimate of \( \langle W, \partial_t W \rangle_{s+1, \delta+2, \mathcal{A}^0_{44}} \) relies on similar ideas to those of (5.9), but it is simpler, since the regularity of the corresponding coefficients of (5.1) and \( W \) is \( (s+1) \).

Having collected the estimates of all the terms, we have
\[
\frac{1}{2} \frac{d}{dt} \langle U, U \rangle_{X_{s, \delta}, \mathcal{A}^0} \lesssim \|U\|^2_{X_{s, \delta}}
\]
and by the equivalence (5.7) we obtain (5.8).

**Remark 5.4.** In order to use the fact that \( U(t) \) satisfies system (5.1) in the proof of Lemma 5.3, we have to replace the expression \( \mathcal{A}^0_{33} \Lambda^s \left( \psi_j^2 \partial_x v \right) \) which appears in the inner–product (5.4), by \( \Lambda^s \left( \psi_j^2 \mathcal{A}^0_{33} \partial_x v \right) \). This is achieved by using the Kato-Ponce commutator for [24, §3.6] in several steps. In one of them we apply it to the commutator
\[
\left( \Psi_k \mathcal{A}^0_{33} \right)_{2j} \left( \Lambda^s \partial_x \right) \left( \psi_j^2 \partial_t v \right)_{2j} - \left( \Lambda^s \partial_x \right) \left( \Psi_k \mathcal{A}^0_{33} \psi_j^2 \partial_t v \right)_{2j},
\]
where \( \Psi_k = \psi_k \left( \sum_j \psi_j \right)^{-1} \). Since \( \Lambda^s \partial_x \) is a pseudodifferential operator of order \( s + 1 \), it follows from [24, §3.6] that the \( L^2 \) norm of (5.10) is bounded by \( \| (\psi_j^s \partial_t v)_{2j} \|_{H^s} \) and \( \| (\Psi_k (a_{33}^0 - e_{30}))_{2j} \|_{H^s} \), see [15, §4].

However, had the coefficients of \( \partial_t (\partial_t v) \) been depended on \( t \), the inner-product \( \langle \partial_t v_1, \partial_t v_2 \rangle_{s, \delta + 1} \) in (5.6) would have been replaced by

\[
\sum_{j=0}^\infty 2^{(4+1)^j} \left( \Lambda^s \left( \psi_j^2 \partial_t v_1 \right)_{2j}, \left( a_{33}^{0} \right)_{2j}, \Lambda^s \left( \psi_j^2 \partial_t v_2 \right)_{2j} \right)_{L^2}
\]

for a certain \( 10 \times 10 \) matrix \( a_{33}^{0} \). And now we would have needed to make the same commutation as above, but with the operator \( \Lambda^s \partial_t \) instead of \( \Lambda^s \partial_x \). Since \( \Lambda^s \partial_t \) is a pseudodifferential operator of order \( s \), this required that \( s - 1 > \frac{3}{2} \) and we would not have gotten the desired regularity. We therefore transfered the system (3.16) to (3.17).

Note that for the inner–product (5.4) we use the operator \( \Lambda^s \) for \( \partial_t v, \partial_x v \), while for the fluid variables \( W \), we have the operator \( \Lambda^s+1 \) in the inner–product (5.5). Therefore, for the Euler equations we can perform the operation mentioned above under the condition \( s > \frac{3}{2} \).

### 5.2. \( L^2_\delta \)-energy estimates.

The \( L^2_\delta (\mathbb{R}^3) \)-space is the closure of all continuous functions under the norm

\[
\| u \|_{L^2_\delta}^2 = \int_{\mathbb{R}^3} (1 + |x|)^2 |u(x)|^2 dx.
\]

This norm is equivalent to the norm \( \| u \|_{H^0, \delta} \) (see [25]). Similar to (4.14), we set

\[
Y_\delta = L^2_\delta \times L^2_{\delta+1} \times L^2_{\delta+1} \times L^2_{\delta+2}
\]

and \( \| U \|_{Y_\delta}^2 = \| v \|_{L^2_\delta}^2 + \| \partial_t v \|_{L^2_{\delta+1}}^2 + \| \partial_x v \|_{L^2_{\delta+1}}^2 + \| W \|_{L^2_{\delta+2}}^2 \).

We also define the inner-product on \( Y_\delta \) in accordance with to the system (5.1):

\[
\langle U_1, U_2 \rangle_{Y_\delta, \mathcal{A}^0} = \int \left( 1 + |x| \right)^{2\delta} v_1^T v_2 dx + \int \left( 1 + |x| \right)^{2(\delta+1)} (\partial_t v_1)^T (\partial_t v_2) dx
\]

\[
+ \int \left( 1 + |x| \right)^{2(\delta+1)} (\partial_x v_1)^T a_{33}^0 (\partial_x v_2) dx + \int \left( 1 + |x| \right)^{2(\delta+2)} W_1^T a_{44}^0 W_2 dx
\]

and the associated norm \( \| U \|_{Y_\delta, \mathcal{A}^0}^2 = \langle U, U \rangle_{Y_\delta, \mathcal{A}^0} \).

**Lemma 5.5.** Assume the coefficients of (5.1) satisfy Assumptions 5.1. If \( U(t) \in X_1, \delta \) is a solution of (5.1), then

\[
\frac{d}{dt} \langle U(t), U(t) \rangle_{Y_\delta, \mathcal{A}^0} \leq C_0 \left( \| U(t) \|_{L^2_{\delta+1}}^2 + \| \mathcal{F} \|_{L^2_{\delta+1}}^2 + \| \mathcal{D} \|_{L^2_{\delta+1}}^2 \right),
\]

where the constant \( C \) depends upon the \( L^\infty \)-norms of \( \mathcal{A}^0, \partial_\alpha \mathcal{A}^0 \) and \( \mathcal{B} \).
6. The iteration process

In this section we adopt Majda’s iterative scheme [17] in order to prove the well-posedness of the coupled hyperbolic system (3.24) in the $H_{s,\delta}$-spaces. A similar approach was carried out in [15] for the vacuum Einstein equations, but in the presence of a perfect fluid there are additional difficulties. Since the density is not strictly positive, the zero order term $B$ which contains the fractional power of the Makino variable $w^{2/(\gamma - 1)}$ is not necessarily $C^\infty$. Hence we could not apply the standard existence theory for symmetric hyperbolic systems.

We denote the initial data by $U_0 = (\phi, \varphi, \partial_x \phi, W_0)$, where $W_0 = (w_0, (u_0)^\alpha - e_0^\alpha)$ represents the initial data for the Makino variable $w$ and the four velocity vector $u^\alpha - e_\alpha^0$.

**Theorem 6.1.** Let $\frac{3}{2} < s < \frac{2}{\gamma - 1} + \frac{1}{2}$, $\delta > -\frac{3}{2}$. Assume $U_0 \in X_{s,\delta}$, $w_0 \geq 0$ and that there exists a positive constant $\mu$ such that

\[
\frac{1}{\mu} V^T V \leq V^T A^0(\phi, W_0) V \leq \mu V^T V \quad \text{for all } V \in \mathbb{R}^{55}.
\]

Then there exists a positive constant $T$ and a unique solution $U(t) = (v(t), \partial_t v(t), \partial_x v(t), W(t))$ to the system (3.24) such that $U(0, x) = U_0(x)$,

\[
U \in C([0, T], X_{s,\delta}) \quad \text{and} \quad W \in C^1([0, T], H_{s,\delta+3}).
\]

The proof of Theorem 6.1 will be carried out in several steps:

1. Setting up the iterative scheme;
2. Proving that the fractional power $(w^k)^{2/(\gamma - 1)}$ is a $C_0^\infty$-function;
3. Boundedness of the iteration sequence in the $X_{s,\delta}$-norm;
4. Weak converges of the iteration sequence in the $X_{s,\delta}$ norm;
5. Uniqueness of the solution obtained;
6. Continuity in the corresponding norm.

A part of the above proofs are standard, but some of them require a special attention due to the specific form of the system (3.24) and use of the space $X_{s,\delta}$. Moreover, the fact that the matrices $A^\alpha = A^\alpha(v, W)$ are not dependent on the derivative of the semi-metric plays an essential role here.

**Step 1.** From condition (6.1) and the embedding into the continuous, Proposition 4.9, we see that there is a bounded domain $G \subset \mathbb{R}^{55}$ containing the initial value $U_0$ and a constant $c_0 \geq 1$ such that

\[
\frac{1}{c_0} V^T V \leq V^T A^0(v, W_0) V \leq c_0 V^T V
\]

for all $U = (v, \partial_t v, \partial_x v, W) \in G$ and $V \in \mathbb{R}^{55}$. By means of the density properties of $H_{s,\delta}$, Proposition 4.11, there is a sequence

\[
\{U_0^k\}_{k=0}^{\infty} = \{(\phi^k, \varphi^k, \partial_x \phi^k, w_0^k, (u_0^\alpha)^k - e_0^\alpha)\}_{k=0}^{\infty} \subset C^\infty_0(\mathbb{R}^3),
\]

and a positive constant $R$ such that

\[
\|U_0^0\|_{X_{s+1,\delta}} \leq C \|U_0\|_{X_{s,\delta}}.
\]
(6.6) \[ \|U - U_0^0\|_{X_{s,\delta}} \leq R \Rightarrow U \in G \]

and

(6.7) \[ \|U_k - U_0\|_{X_{s,\delta}} \leq \frac{R2^{-k}}{4c_0}. \]

The iterative scheme is defined as follows: let \( U^0(t, x) = U_0^0(x) \) and \( U^{k+1}(t, x) = (u^{k+1}(t, x), \partial_t v^{k+1}(t, x), \partial_x v^{k+1}(t, x), W^{k+1}(t, x)) \) be a solution of the linear Cauchy problem

\[
\begin{align*}
\mathcal{A}(v^k, W^k) \partial_t U^{k+1} & = (\mathcal{A}(v^k, W^k) + C^a) \partial_0 U^{k+1} \\
& + \mathcal{B}(U^k) \begin{pmatrix} \partial_t v^{k+1} \\ \partial_x v^{k+1} \end{pmatrix} + \mathcal{F}(v^k, W^k),
\end{align*}
\]

(6.8) \[ U^{k+1}(0, x) = U_0^{k+1}(x) \]

where \( \mathcal{F}(v^k, W^k) = (0, f(v^k, W^k), 0, 0) \), \( f(v^k, W^k) \) is given by (3.22) and \( W^k = (w^k, (u^\alpha)^k - \epsilon_0^\alpha) \).

**Step 2.** The iterative method relies on the fact that solutions of linear symmetric hyperbolic systems with \( C_0^\infty \) coefficients and initial data, are also \( C_0^\infty \). However, even if \( w^k \geq 0 \) and \( w^k \in C_0^\infty \), it does not guarantee that \( (w^k)^{2/(\gamma-1)} \) is a \( C_0^\infty \)-function. Since the function \( f(v^k, W^k) \) contains the term \( (w^k)^{2/(\gamma-1)} \), we must assure that it is a \( C_0^\infty \)-function.

**Proposition 6.2.** Let \( u \in H_{s,\delta} \) be non-negative and \( \beta > 0 \). Then there is a sequence \( \{u^k\} \subset C_0^\infty \) such that \( u^k \rightarrow u \) in the \( H_{s,\delta} \)-norm and \( (u^k)^\beta \in C_0^\infty \).

**Proof.** Let \( \varepsilon > 0 \), then by Proposition 4.11 there is \( u^\varepsilon \in C_0^\infty \) with \( \|u - u^\varepsilon\|_{H_{s,\delta}} < \varepsilon \). Take now a positive number \( M \) so that supp\( (u^\varepsilon) \subset \{|x| \leq M\} \), and let \( \chi_M \) be the cut-off function satisfying \( \chi_M(x) = 1 \) for \( |x| \leq M \) and \( \chi_M(x) = 0 \) for \( |x| \geq M+1 \). For any positive number \( \varrho \) we set

\[ u^{\varepsilon, \varrho}(x) = \chi_R(x) (u^\varepsilon(x) + \varrho). \]

Then \( (u^{\varepsilon, \varrho})^\beta \in C_0^\infty \), since \( (u^\varepsilon + \varrho) > 0 \) and \( \chi_M \) is a cut-off function. Moreover, \( u^{\varepsilon, \varrho} - u^\varepsilon = \chi_M \varrho \), hence \( u^{\varepsilon, \varrho} \rightarrow u^\varepsilon \) in the \( H_{s,\delta} \)-norm as \( \varrho \rightarrow 0 \).

Thus we may assume that \( \{w^k\}_{k=0}^\infty \), the \( C_0^\infty \) approximation of the initial data of the Makino variable \( w_0 \) in (6.4), satisfies \( (w_0^k)^{2/(\gamma-1)} \in C_0^\infty \). We turn now showing that for \( t \geq 0 \)

\[ e^k(t, x) = (w^k)^{\frac{2}{\gamma - 1}}(t, x) \]

is also a \( C_0^\infty \)-function.

**Proposition 6.3.** For each integer \( k \geq 0 \), \( e^k(t, \cdot) \in C_0^\infty(\mathbb{R}^3) \).
Proposition 6.4. There are positive constants \( T^* \) and \( L \) such that
\[
\sup\left\{ T : \sup_{0 \leq t \leq T} \| U^k(t) - U_0^k \|_{X_{s,\delta}} \leq R \right\} \geq T^* \quad \text{for all } k
\]
and
\[
\sup\limits_{0 \leq t \leq T^*} \| \partial_t W^k \|_{H_{s,\delta+3}} \leq L \quad \text{for all } k.
\]
Proof. We prove it by induction. Set \( V^{k+1} = U^{k+1} - U_0 \), then
\begin{equation}
\mathcal{A}^0(v^k, W^k) \partial V^{k+1} = (\mathcal{A}^0(v^k, W^k) + \mathcal{C}^0) \partial V^{k+1} + \mathcal{B}(U^k) \begin{pmatrix}
\frac{v^{k+1} - \phi_0^0}{\partial_x v^{k+1} - \partial_x \phi_0^0} \\
\partial_x v^{k+1} - \partial_x \phi_0^0
\end{pmatrix},
\end{equation}
where
\begin{equation}
\mathcal{D}^k = (\mathcal{A}^0(v^k, W^k) + \mathcal{C}^0) \partial U_0^0 + \mathcal{B}(U^k) \begin{pmatrix}
\phi_0^0 \\
\partial_x \phi_0^0
\end{pmatrix}
\end{equation}
and \( V^{k+1}(0, x) = U_0^{k+1}(x) - U_0^0(x) \).

In order to apply the energy estimate, Lemma 5.3, to (6.13) we have to check that Assumptions 5.1 are satisfied and the corresponding norms are independent of \( k \). Clearly all the matrices have the same structure. From (6.3) we see that condition 5.2b holds. By the induction hypothesis,
\begin{equation}
\|v^k(t) - \phi_0^0\|_{H_{s, \delta+2}^2} + \|\partial_x v^k(t) - \partial_x \phi_0^0\|_{H_{s, \delta+1}} \leq \|U^k(t) - U_0^0\|_{X_{s, \delta}} \leq R^2,
\end{equation}
therefore by Remark 4.12, \( \|v^k(t) - \phi_0^0\|_{H_{s+1, \delta}} \lesssim R \). Applying the Moser type estimate, Proposition 4.10, we have
\begin{equation}
\|\mathcal{A}^0(v^k, W^k) - e_5\|_{H_{s+1, \delta}} \lesssim \left( \|v^k\|_{H_{s+1, \delta}}^2 + \|W^k\|_{H_{s+1, \delta}}^2 \right)
\end{equation}
\begin{equation}
\lesssim \|U^k - U_0^0\|_{X_{s, \delta}}^2 + \|U_0^0\|_{X_{s, \delta}}^2 \lesssim \left( R^2 + \|U_0^0\|_{X_{s, \delta}}^2 \right).
\end{equation}

In a similar way we get that \( \|\mathcal{A}^0(v^k, W^k)\|_{H_{s+1, \delta}} \) is bounded by a constant depending on \( R \). By Propositions 4.3, 4.5, 4.10, Remark 4.6 and the structure of the matrix \( \mathcal{B} \) in (3.23), we obtain that \( \|b_{2j}(U^k)\|_{H_{s+1}^2} \lesssim \|U^k\|_{X_{s, \delta}} \lesssim (R + \|U_0^0\|_{X_{s, \delta}}) \) and a similar estimate holds for \( \|b_{2j}(U^k)\|_{H_{s+1}^2} \) for \( j = 2, 3, 4, 5 \).

We recall the \( \mathcal{F}(v^k, W^k) = (0, f(v^k, W^k), 0, 0) \), where \( f(v^k, W^k) \) is given by (3.22). Applying again Propositions 4.5, 4.10 and Remark 4.6, we obtain that
\begin{equation}
\|f(v^k, W^k)\|_{H_{s, \delta+1}} \lesssim \left( \|v^k\|_{H_{s, \delta+1}}^{\frac{2}{\gamma-1}} + \|U^k\|_{H_{s, \delta+1}} \right).
\end{equation}
Now, for \( \frac{3}{2} < s < \frac{2}{\gamma-1} + \frac{1}{2} \), we apply the estimate for the fractional power, Proposition 4.7, and together with Propositions 4.3 and 4.9, we have
\begin{equation}
\left( \|v^k\|_{H_{s, \delta+1}}^{\frac{2}{\gamma-1}} \right)_{H_{s, \delta+1}} \lesssim \|w^k\|_{H_{s, \delta+1}} \lesssim \|w^k\|_{H_{s, \delta+2}}.
\end{equation}
Using similar arguments as in the previous estimates, we conclude that \( \|f(v^k, W^k)\|_{H_{s, \delta+1}} \) is bounded by a constant depending on \( R \) but not on \( k \).

The required estimate for \( \mathcal{D}^k \) follows from the multiplicity property (4.7) and the estimates which we already obtained for \( \|\mathcal{A}^0(v^k, W^k)\|_{H_{s+1, \delta}} \) and \( \|\mathcal{B}(U^k)\|_{H_{s, \delta+1}} \).
It remains to verify (5.2c); note that by the induction hypothesis (6.11), condition (6.6) and the embedding (4.11), we have
\[
\| \partial_t \mathcal{A}^0(v^k, W^k) \|_{L^\infty} \leq \sup_G \left| \frac{\partial \mathcal{A}^0}{\partial v}(v, W) \right| \| \partial_t v^k \|_{L^\infty} \\
+ \sup_G \left| \frac{\partial \mathcal{A}^0}{\partial W}(v, W) \right| \| \partial_t W^k \|_{L^\infty} \lesssim \left( \| \partial_t v^k \|_{H_{s,\delta+1}} + \| \partial_t W^k \|_{H_{s,\delta+3}} \right).
\]
Since \( \| \partial_t W^k \|_{H_{s,\delta+3}} \) is bounded by hypothesis (6.12), we see that \( \| \partial_t \mathcal{A}^0(v^k, W^k) \|_{L^\infty} \) is also bounded by a constant depending on \( R \) but not on \( k \).

We can now apply Lemma 5.3, and with the combination of Gronwall’s inequality, condition (6.7) and the equivalence (5.7), we conclude that there is a constant \( C \) depending on \( R \) such that
\[
\sup_{0 \leq t \leq T} \| V^k(t) \|_{X_{s,\delta}}^2 \leq e^{C \sigma T} \left( \frac{R^2}{8} + C \sigma T \right).
\]
Hence \( T^* = \sup \{ T : e^{C \sigma T} \left( \frac{R^2}{8} + C \sigma T \right) < R^2 \} > 0 \).

We show now (6.12). It follows from the structure of the matrices \( \mathcal{A}^0, \mathcal{A}^a \) and \( \mathcal{B} \) (see (3.18), (3.20) and (3.23)) that
\[
\partial_t W^{k+1} = (a_{44}^a(v^k, W^k))^{-1} \left[ a_{33}^a(v^k, W^k) \partial_a W^{k+1} + b_{4(a+1)}(U^k) \partial_a v^{k+1} \right].
\]
Since the \( \| v^k(t) \|_{H_{s+1,\delta}}, \| W^k(t) \|_{H_{s+1,\delta+2}} \) and \( \| U^k(t) \|_{X_{s,\delta}} \) are bounded by a constant independent of \( k \), it follows from (6.15) and by means of calculus in the \( H_{s,\delta} \)-spaces that \( \| \partial_t W^{k+1} \|_{H_{s,\delta+3}} \) is also bounded by a constant independent of \( k \).

**Step 4.** Having shown the boundedness of \( U^k \) in \( X_{s,\delta} \)-norm, we use the compact embedding, Proposition 4.8, and conclude that there is \( U \in X_{s',\delta'} \) such that \( \| U^k(t) - U(t) \|_{X_{s',\delta'}} \to 0 \) for any \( s' < s \) and \( \delta' < \delta \). Furthermore, by Remark 4.6, \( v^k(t) \to v(t) \) in \( H_{s'+1,\delta'} \)-norm. Thus if we choose \( \frac{3}{2} < s' < s \) and \( -\frac{3}{2} < \delta' < \delta \), then by the embedding (4.11),
\[
v^k(t) \to v(t), \quad W^k(t) \to W(t) \quad \text{in} \quad C^1
\]
and
\[
\partial_t v^k(t) \to \partial_t v(t), \quad \partial_t W^k(t) \to \partial_t W(t) \quad \text{in} \quad C^0.
\]
Thus, \( U(t) = (v(t), \partial_t v(t), \partial_x v(t), W(t)) \) is a solution of the system (3.24).

**Proposition 6.5.** For any \( \Phi \in X_{s,\delta} \),
\[
\lim_k \langle U^k(t), \Phi \rangle_{X_{s,\delta}} = \langle U(t), \Phi \rangle_{X_{s,\delta}}
\]
uniformly for \( 0 \leq t \leq T^* \), and where \( \langle \cdot, \cdot \rangle_{X_{s,\delta}} \) denote the inner-product (5.6) with \( \mathcal{A}^0 \) being the identity matrix.
As a consequence of the weak convergence (6.16), we have
\[ \|U(t)\|_{X_{s,\delta}} \leq \liminf_k \|U_k(t)\|_{X_{s,\delta}}. \]

For the proof of Proposition 6.5 see [15, §5].

**Step 5.** Here we shall prove uniqueness. Our method relies on the $L^2_\delta$-energy estimates, Lemma 5.5. A similar approach was used in [15] for the vacuum Einstein equation, we shall therefore emphasize the estimates of the terms which do not appear in the vacuum equations.

**Proposition 6.6.** Suppose $U_1(t), U_2(t) \in X_{s,\delta}$ are two solutions of (3.24) with the same initial data, then $U_1(t) \equiv U_2(t)$.

**Proof.** Let $V(t) = U_1(t) - U_2(t)$, then it satisfies the linear system
\[
A^0(v_1, W_1)\partial_t V = A^0(v_1, W_1)\partial_a V + B(U_1)V + \mathcal{F}(v_1, W_1) - \mathcal{F}(v_2, W_2) + D,
\]
where
\[
D = - (A^0(v_1, W_1) - A^0(v_2, W_2)) \partial_t U_2
+ (A^0(v_1, W_1) - A^2(v_2, W_2)) \partial_a U_2 + (B(U_1) - B(U_2)) U_2.
\]

Applying Lemma 5.5 to system (6.17), we have
\[
\frac{d}{dt} \|V(t)\|_{Y_{s,\delta,0}}^2 \lesssim \left( \|V(t)\|_{Y_{s,\delta,0}}^2 + \|\mathcal{F}(v_1, W_1) - \mathcal{F}(v_2, W_2)\|_{L^2_{s+1}}^2 + \|D\|_{L^2_{s+1}}^2 \right).
\]

Thus, our main task is to bound the two last terms on the right hand side of (6.19) by $\|U_1(t) - U_2(t)\|_{Y_{s,\delta}}^2$.

Staring with the first one, we have that
\[
\|\mathcal{F}(v_1, W_1) - \mathcal{F}(v_2, W_2)\|_{L^2_{s+1}} = \|f(v_1, W_1) - f(v_2, W_2)\|_{L^2_{s+1}}
\]
and
\[
f(v_1, W_1) - f(v_2, W_2)
= \left( \frac{\partial f}{\partial v} (\tau v_1 + (1 - \tau) v_2, \tau W_1 + (1 - \tau) W_2) \right) (v_1 - v_2)
+ \left( \frac{\partial f}{\partial W} (\tau v_1 + (1 - \tau) v_2, \tau W_1 + (1 - \tau) W_2) \right) (W_1 - W_2)
\]
for some $\tau \in [0, 1]$. Note that $v_1, v_2$ belong to $L^2_{s}$, while we have to take the $L^2_{s+1}$-norm of the above expressions. However, since the Makino $w \in H_{s+1,\delta+2} \subset H_{s,\delta+2}$, we get from (3.22), and Propositions 4.7 and 4.5 that $\frac{\partial f}{\partial v} \in H_{s,\delta+2}$. Therefore, by the equivalence
Thus time with τ

The other term is somewhat easier to treat, since \( |\partial f| \|v_1 - v_2\|_{H_{0,\delta}} \lesssim \|v_1 - v_2\|_{L^2_\delta} \).

Thus
\[
\|f(v_1, W_1) - f(v_2, W_2)\|^2_{L^2_{\delta+1}} \lesssim \|U_1 - U_2\|^2_{Y_3}.
\]

We shall now estimate the first term of \( D \) in (6.18). From the structure of \( A^0(v, W) \) we see that
\[
(A^0(v_1, W_1) - A^0(v_2, W_2)) \partial_t U_2 = (a^0_{33}(v_1) - a^0_{33}(v_2)) \partial_t \partial_x v_2 + (a^0_{44}(v_1, W_1) - a^0_{44}(v_2, W_2)) \partial_t W_2,
\]
and furthermore,
\[
a^0_{33}(v_1) - a^0_{33}(v_2) = \frac{\partial a^0_{33}}{\partial v} (\tau v_1 + (1 - \tau)v_2)(v_1 - v_2)
\]
for some \( \tau \in [0, 1] \). At this stage we apply the calculus in the \( H_{s,\delta} \)-spaces. The multiplication property (4.7) is used twice, once with \( s_1 = s, s_2 = s - 1 \) and \( s = 0 \), and the second time with \( s_1 = s, s_2 = 1 \) and \( s = 1 \). All these result in
\[
\| (a^0_{33}(v_1) - a^0_{33}(v_2)) \partial_t \partial_x v_2 \|_{L^2_{\delta+1}} \lesssim \| (a^0_{33}(v_1) - a^0_{33}(v_2)) \partial_t \partial_x v_2 \|_{H_{0,\delta+1}}
\]
\[
\lesssim \| (a^0_{33}(v_1) - a^0_{33}(v_2)) \|_{L^2_{\delta+1}} \| \partial_t \partial_x v_2 \|_{H_{s,\delta+1}}
\]
\[
\lesssim \| (a^0_{44}(v_1, W_1) - a^0_{44}(v_2, W_2)) \partial_t W_2 \|_{L^2_{\delta+1}} \lesssim \left( \| \partial a^0_{33} \|_{H_{s,\delta+1}} \| (v_1 - v_2) \|_{H_{1,\delta}} + \| \partial a^0_{44} \|_{L^\infty} \| (W_1 - W_2) \|_{L^2_{\delta+1}} \right) \times \| \partial_t W_2 \|_{H_{s,\delta+3}}.
\]
We recall that \( \| \partial_t W_2 \|_{H^{s, \delta + 3}} \) is bounded by (6.12) and \( \| (v_1 - v_2) \|_{H^{1, \delta}} \leq \| (v_1 - v_2) \|_{L^2_3}^2 + \| (\partial_x v_1 - \partial_x v_2) \|_{L^2_{2,1}}^2 \), hence,

\[
\| (A^0(v_1, W_1) - A^0(v_2, W_2)) \|_{L^2_{3+1}}^2 \leq \| U_1 - U_2 \|_{Y_3}^2.
\]

The remaining terms can be estimated by similar methods. Thus we conclude from (6.19) and the equivalence \( \| V \|_{Y_3, A_0} \simeq \| V \|_{Y_3} \) that

\[
\frac{d}{dt} \| V(t) \|_{Y_3, A_0}^2 \leq \| V(t) \|_{Y_3, A_0}^2,
\]

and since \( V(0) \equiv 0 \), Gronwall’s inequality implies that \( V(t) \equiv 0 \).

**Step 6.** Since \( X_{s, \delta} \) is a Hilbert space it suffices to show that

\[
\limsup_{t \to 0^+} \| U(t) \|_{X_{s, \delta, A_0}} \leq \| U(0) \|_{X_{s, \delta, A_0}}
\]

in order to establish the continuity in the norm. Here \( A_0^{\alpha} \) depends on the initial data \( \phi \) and \( W_0 \), that is, \( A_0^{\alpha} = A_0^{\alpha}(\phi, W_0) \). The proof of (6.20) relies on the same arguments as in [17] and we therefore leave it out. This completes the proof of Theorem 6.1.

**7. Proof of the main result**

The proof of the main result, Theorem 2.3, actually follows from Theorem 6.1, we just have to check whether the initial data of the gravitational fields and of the fluid satisfy the assumptions of Theorem 6.1.

We recall that \( v(t) = g_{\alpha \beta}(t) - \eta_{\alpha \beta} \), so setting \( \phi = v(0) \), we have by the assumptions of Theorem 2.3 that \( \phi \in H_{s+1, \delta} \), and by (2.6), \( a_{\alpha \beta}^0(\phi) \) is a positive definite matrix. The initial data for \( \partial_t v \) is given by \( \phi = \partial_t g_{\alpha \beta}(0), \partial_t g_{ab}(0) = -2K_{ab} (a, b = 1, 2, 3) \), where \( K_{ab} \) is the second fundamental form and by the assumptions of Theorem 2.3 it belongs to \( H_{s, \delta + 1} \). The remaining data, \( \partial_t g_{\alpha 0}(0) \), are determined by the harmonic gauge (2.3). Since we assume that \( (g_{\alpha \beta}(0) - \eta_{\alpha \beta}, K_{ab}) \in H_{s+1, \delta} \times H_{s, \delta + 1} \), it follows from Propositions 4.5 and 4.10 that also \( \partial_t g_{\alpha 0}(0) \in H_{s, \delta + 1} \).

The construction of initial data for the fluid variable was studied in details in [4, Theorem 2.6], from which it follows that \( W_0(\cdot) = W(0, \cdot) \in H_{s+1, \delta + 2} \). Since \( u^\alpha \) is a unite timelike vector, it follows from Theorem 3.3 that \( a_{\alpha 4}^0(W_0) \) is a positive definite matrix. Therefore \( A_0^0(\phi, W_0) \) satisfies condition (6.1), and hence all the assumptions of Theorem 6.1 are fulfilled.

So as a consequence of (6.2) and Remark 4.12, \( v(t) \in C([0, T], H_{s+1, \delta}) \). Hence (2.7) holds. Also, by (6.2), (2.8) holds too. That completes the proof.

**8. Appendix**

The classical paper of Hughes, Kato and Marsden [13] established the short time existence of the vacuum Einstein equations by solving a second order quasi-linear hyperbolic system whose solutions \( (g_{\alpha \beta}, \partial_t g_{\alpha \beta}) \) belong to \( H^{s+1} \times H^s \) for \( s > \frac{3}{2} \). On the other hand,
Fisher and Marsden treated the Einstein vacuum equation by means of the theory of symmetric hyperbolic system. However, they only obtained the regularity of $s > \frac{5}{2}$.

In [15] we generalized the result of [13] to the $H^{s,\delta}$ spaces, treating however, the Einstein equations as a symmetric hyperbolic system. Since the techniques of [15], and in particular the energy estimates, play an essential role in the present paper, we outline its main idea that enables us to obtain the same regularity as in [13].

We present a heuristic argument explaining the essential idea. First, if a function $v$ satisfies a wave equation, then the vector $V = (v, \partial_t v, \partial_x v)$ satisfies a symmetric hyperbolic system. The general condition for existence and uniqueness in the $H^s(\mathbb{R}^3)$ spaces is $s > \frac{5}{2}$. Hence, we have by this method that $\partial_t v, \partial_x v \in H^s$ for $s > \frac{5}{2}$.

However, in our case we improve this regularity to $(v, \partial_t v, \partial_x v) \in H^{s+1}$ for $s > \frac{3}{2}$. This is because we do not consider a general quasi-linear symmetric hyperbolic system where the matrices $A^a(V)$ depend on $V$, but a system in which the matrices $A^a(v)$ only depend on $v$ but not on its derivatives.

To see how this fact allows us to improve the regularity of the solution we will derive energy estimates for the linearized symmetric hyperbolic system. For the sake of clarity we consider a simple hyperbolic system

\begin{equation}
\partial_t V = A^a(v) \partial_a V,
\end{equation}

then its linearized form is

\begin{equation}
\partial_t \tilde{V} = \tilde{A}^a \partial_a \tilde{V}.
\end{equation}

Note that in each iteration we solve the linear system (8.2) with $\tilde{A}^a = \tilde{A}^a(v^k)$, and since $V^k = (v^k, \partial_t v^k, \partial_x v^k) \in H^s$, $v^k \in H^{s+1}$ and hence $\tilde{A}^a = \tilde{A}^a(v^k) \in H^{s+1}$ by Moser type estimates.

The crucial step is to derive the energy estimate

\begin{equation}
\frac{d}{dt} \left( \frac{1}{2} \| V \|^2_{H^s} \right) \leq C \| V \|^2_{H^s}
\end{equation}

for $s > \frac{3}{2}$ and whenever $V$ satisfies the linear system (8.2). We recall that $\| V \|_{H^s} = \| \Lambda^s V \|_{L^2}$, where $\Lambda^s$ is the pseudodifferential operator $(1 - \Delta)^{\frac{s}{2}}$.

One of the basic tools for obtaining (8.3) are commutator’s estimates. Here we shall use the following Pseudodifferential operators version of Kato–Ponce estimate [24, §3.6]: Let $P$ be a differential operator in the class $OPS^1_{1,0}$, then

\begin{equation}
\| P(fg) - fP(g) \|_{L^2} \leq C \left\{ \| \nabla f \|_{L^\infty} \| g \|_{H^{s-1}} + \| f \|_{H^s} \| g \|_{L^\infty} \right\},
\end{equation}

for any $f \in H^s \cap C^1$ and $g \in H^{s-1} \cap L^\infty$. 

The standard way to obtain (8.3) is to differentiate $\|V\|_{H^s}^2$ with respect to time, to insert the differential equation (8.2) and then apply a suitable commutator which leads to

$$\frac{1}{2} \frac{d}{dt} \|V\|_{H^s}^2 = \langle \Lambda^s(V), \Lambda^s (\partial_t V) \rangle_{L^2} = \left\langle \Lambda^s(V), \Lambda^s \left( \tilde{A}^a \partial_a V \right) \right\rangle_{L^2}$$

(8.5)

and then the first term is taking care by integration by parts and the second one is by applying the above Kato–Ponce estimate to the operator $\Lambda^s$.

But this procedure results in a term of the form $\|\partial_a V\|_{L^\infty}$ which contains $\|\partial_a \partial_x v\|_{L^\infty}$. In order to estimate it by $\|\partial_a \partial_x v\|_{H^{s-1}} \lesssim \|\partial_x v\|_{H^s}$ we need to require that $s - 1 > \frac{3}{2}$, and hence we do not get the desired result.

We circumvent this difficulty by writing

$$\tilde{A}^a \partial_a V = \partial_a \left( \tilde{A}^a V \right) - \partial_a \tilde{A}^a V,$$

and making the commutation

$$\Lambda^s \left( \partial_a \left( \tilde{A}^a V \right) \right) = \left( \Lambda^s \partial_a \right) \left( \tilde{A}^a V \right) - \tilde{A}^a \left( \Lambda^s \partial_a \right) \left( V \right) + \Lambda^s \left( \tilde{A}^a \partial_a \right) \left( V \right),$$

which we insert into the first row of equation (8.5). Then we have to estimate three terms:

$$I = \left\langle \Lambda^s(V), \left( \Lambda^s \partial_a \right) \left( \tilde{A}^a V \right) - \tilde{A}^a \left( \Lambda^s \partial_a \right) \left( V \right) \right\rangle_{L^2},$$

(8.9)

$$II = \left\langle \Lambda^s(V), \tilde{A}^a \left( \Lambda^s \partial_a \right) \left( V \right) \right\rangle_{L^2}$$

and

$$III = \left\langle \Lambda^s(V), \Lambda^s \left( \left( \partial_a \tilde{A}^a \right) V \right) \right\rangle_{L^2}.$$

For the first term we apply the Kato-Ponce commutator (8.4). However, this time we do it for the operator $(\Lambda^s \partial_a)$ which has order $s + 1$, and hence

$$|I| \leq \|V\|_{H^s} \left\| \Lambda^s \partial_a \left( \tilde{A}^a V \right) - \tilde{A}^a \left( \Lambda^s \partial_a \right) \left( V \right) \right\|_{L^2}$$

(8.11)

$$\lesssim \|V\|_{H^s} \left\{ \|\nabla \tilde{A}^a\|_{L^\infty} \|\partial_a V\|_{H^{s+1}} + \|\tilde{A}^a\|_{H^{s+1}} \right\}.$$

So by Sobolev embedding theorem, we see that $|I| \lesssim \|\tilde{A}^a\|_{H^{s+1}} \|V\|_{H^s}^2$. Likewise, since $H^s$ is an algebra for $s > \frac{3}{2}$,

$$|III| \lesssim \|V\|_{H^s} \|\partial_a \tilde{A}^a\| \|V\|_{H^s} \lesssim \|V\|_{H^s}^2 \|\partial_a \tilde{A}^a\|_{H^{s+1}} \lesssim \|\tilde{A}^a\|_{H^{s+1}} \|V\|_{H^s}^2.$$

Since $\Lambda^s \partial_a = \partial_a \Lambda^s$ and $\tilde{A}^a$ is symmetric, we obtain a similar estimate for $II$ by using integration by parts. Hence we conclude that the energy estimate (8.3) holds. Note that in the estimate of all three terms above we have used the fact that $\tilde{A}^a \in H^{s+1}$. 


For the general case where $A^0 \neq I$, one has to define an appropriated inner-product which takes into account the matrix $A^0$. Details for the vacuum equations in the weighted spaces $H_{s,\delta}$ and a positive definite $A^0$ can be found in [15, §4] and only slight modifications are needed in order to extend the energy estimates of [15] to the coupled system (3.24).

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