The kinematic dynamo action of spiralling convective flows

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SUMMARY
We consider the kinematic production of magnetic fields in a sphere by velocity fields dominated by differential rotation and spiralling convective cells. The high magnetic Reynolds number limit of Braginsky (1964) is considered and formulae are derived allowing an $a$-effect parametrization of such flows to be easily calculated. This permits an axisymmetric system to be investigated in parallel with the direct 3-D numerical computations. Good agreement between the asymptotic and 3-D calculations is found. The 'spiralling' property typical of convective motion in rotating spheres is important in terms of dynamo action; the differential rotation coexisting with this feature is also vital. Indeed, it is the presence of both features which allows the analysis of Braginsky to be employed. With flows approximating the columnar form anticipated for rapidly rotating convection, dynamo action is relatively easily achieved for all azimuthal wavenumbers; modes of differing wavenumbers interact almost by a simple superposition. With flows of more complex latitudinal form, the mutual interactions between modes become more complicated. For columnar-type flows, dipole magnetic fields are favoured when the sense of outward spiralling is prograde and the zonal flow is eastwards, as is physically preferred.

Key words: kinematic dynamos, convection, geomagnetism.

1 INTRODUCTION
The production of the Earth's magnetic field was first linked to the homogeneous dynamo problem early in this century. Initial interest focused on the kinematic part of the problem, in which a velocity field is presupposed and its capacity to excite magnetic fields is investigated. After initial difficulties were overcome and the possibility of dynamo action by some simple, idealized flows was established, the kinematic problem has become somewhat neglected; attention has focused on the more complex magneto-hydrodynamic dynamo problem involving the non-linear interaction between field and flow. The simpler kinematic problem remains imperfectly understood, however.

Several flows of various types have been identified in the literature as being capable of self-excited dynamo action, but there is only a partial understanding of why these flows, and not others, have this capacity. Explanations often cite the flows' helicity, or behaviour observed in simpler problems—flux expulsion by closed-streamline circulation, or the concentration of flux by high $R_m$ flow, for example— but the relevance of these features is seldom truly established. The flows that have been investigated kinematically have almost always been of rather simple type, mainly for numerical reasons. It is therefore worrying that highly complicated models of the full dynamo problem are being investigated, while the kinematic dynamo properties of flows of even moderate complexity have hardly been studied in isolation.

Work on the dynamic problem has meanwhile progressed largely through the 'intermediate' approach, where the dynamo action of the velocity is conveniently parametrized in terms of a so-called $a$-effect. Whilst the $a$-effect models may constitute a valid approximation to the full dynamo process, and the intermediate studies are still necessary and useful at the present stage of knowledge, it also remains somewhat unsatisfactory that the $a$-effect is so widely, and arbitrarily, invoked.

In this paper we consider the kinematic problem for rather general flows, of a type that might be excited by convection in rapidly rotating systems. We make use of the theory of magnetic-field generation developed by Braginsky (1964) and applied by him to the Earth's core. This theory allows axisymmetric calculations appropriate to the high magnetic Reynolds number regime to be
conducted in parallel with the 3-D numerical computations. The axisymmetric calculations are carried out using rigorously derived z-effects, to some extent justifying their more arbitrary use elsewhere, and, it is hoped, leading to a better understanding of the z-effect as a parametrization of the inductive action of quite general 3-D flows.

2 MATHEMATICAL FORMALISM

The kinematic dynamo problem has been described by many authors, so we shall discuss it here only briefly. Roberts (1994) gives a thorough review; Sarson & Gubbins (1996; henceforth SG) describe the problem more briefly, with specific details of the numerical method used here. This method has been highly satisfactorily benchmarked by Holme (1997).

The kinematic problem is described by the induction equation for the time-evolution of a magnetic field in an electrically conducting fluid,

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \nabla^2 B,$$

where the magnetic field \(B\) is solenoidal, and where the velocity \(u\)—here assumed stationary and incompressible, and so also solenoidal—is prescribed. The magnetic Reynolds number arising from the non-dimensionalization, \(R_m = \mu \sigma L^2 / D\), is defined in terms (in the order given) of the magnetic permeability and electrical conductivity of the fluid, and of its velocity and length scales. We solve (1) in a unit sphere, \(r \leq 1\), with an insulating exterior; see SG for details.

Eq. (1) is linear in \(B\), and so can be solved for eigensolutions \(B \exp(\lambda t)\), with the complex growth-rate \(\lambda\) as an eigenvalue. We are interested in self-sustained magnetic fields, and so look for solutions with \(\Re \lambda = 0\). The critical magnetic Reynolds number, \(R_m^c\), at which such a solution is obtained—if it exists—is a measure of the ability of the flow to sustain dynamo action. Solutions are also characterized by their frequency at criticality, \(\omega = \Im \lambda\).

Previous kinematic calculations (e.g. Bullard & Gellman 1954; Pekeris, Accad & Shkoller 1973; Kumar & Roberts 1975; Dudley & James 1989) have employed simple, large-scale velocities. We wish to study how different components of velocity interact in the production of a magnetic field, and assume velocities of the general form

$$u = \epsilon_0 \rho_{10}^c + \epsilon_1 \rho_{10}^s + \sum_{lm} \epsilon_{lm} s_{lm}^c + \epsilon_{lm} c_{lm}^c,$$

where we have introduced toroidal and poloidal vector harmonics following the Bullard & Gellman (1954) convention,

$$\rho_{mc/s} = \nabla \times \left( s_{mc/s} (r) P_{lm}^m (\cos \phi) / \sin m \phi \right),$$

$$s_{mc/s} = \nabla \times \left( s_{mc/s} (r) P_{lm}^m (\cos \phi) / \sin m \phi \right),$$

in terms of the associated Legendre functions \(P_{lm}^m(\cos \phi)\). We drop the superscript ‘\(c\)’ for the axisymmetric (\(m = 0\)) components, the distinction between cosine and sine no longer being required.

The flow (2) contains axisymmetric components of differential rotation (\(\rho_{10}^c\)) and meridional circulation (\(\rho_{10}^s\)), as employed by Kumar & Roberts (1975; KR hereafter). These were retained following the conclusions of P. H. Roberts (1972) on their importance for dynamo action in z-effect models, and the confirmation of this property in 3-D calculations by SG. The \(s_{mc}^m\) components, of both sine and cosine \(\phi\) dependence, are investigated for various combinations of \(l\) and \(m\). The \(\epsilon\) parameters simply control the relative magnitudes of the appropriate velocity components. In what follows we fix \(\epsilon_0 = 1\), since the amplitude of the flow is already defined by \(R_m\).

Whilst it might seem limiting to restrict the flow under consideration to non-axisymmetric poloidal harmonics, these harmonics should be the principal ones excited by buoyant convection in a system with radial gravity, so are of primary interest. Although toroidal harmonics can also potentially interact with poloidal harmonics to cause dynamo action in the high-\(R_m\) regime, this possibility is not investigated in the current work.

Following KR, the radial variations of these flows were prescribed as

$$\rho_{10}^c (r) = r^l (1 - r^2),$$

$$\rho_{10}^s (r) = r^l (1 - r^2)^3,$$

$$s_{lm}^c (r) = r^l (1 - r^2)^2 \cos pr,$$

$$s_{lm}^s (r) = r^l (1 - r^2)^2 \sin pr.$$

The parameters \(p\) control the radial complexity (‘number of cells’) of the \(s_{lm}^c\) components, allowing further freedom in the flow prescribed. The cosine and sine variations in \(r\) introduce a sense of ‘spiralling’ to the convective flow, as observed in experiments (Carrigan & Busse 1983) and numerical simulations (e.g. Zhang 1992). The spiralling nature of the flow is required for dynamo generation in the asymptotic limit of Braginsky (1964); this property is discussed in some detail in Section 5. The lowest power of \(r^4\)
appearing in $\zeta^{mc}_{\ell m}(r)$ is, strictly, only appropriate for $l \leq 3$; to ensure regularity at the origin these functions should vary as $O(r^{l+1})$. The forms (7)–(8) were used for all $l$, however, for numerical convenience. No difficulties at the origin were encountered in practice; the weak singularity at an isolated point in the neighbourhood of which the amplitudes of all fields are very low obviously has little effect on the convergence of the numerical scheme.

In attempting to model the geodynamo, which generates a predominantly axisymmetric magnetic field and which is thought to be at relatively high $R_m$, we may make use of the asymptotic analysis of Braginsky (1964). Braginsky considered the dynamo action of a system ordered as

$$u = \bar{u} + R_m^{-1/2} u', \quad B = \bar{B} + R_m^{-1/2} B',$$

where $\bar{u}$, $\bar{B}$ denote the axisymmetric components, and $u'$, $B'$ the non-axisymmetric. The axisymmetric parts are dominated by the zonal ($\phi$) components, with weaker meridional (poloidal) parts:

$$\bar{u} = u\epsilon_{\phi} + R_m^{-1} u_p, \quad \bar{B} = B\epsilon_{\phi} + R_m^{-1} B_p.$$

In terms of our velocity (2), these expansions are appropriate only for $\epsilon_1 \ll 1$, $\epsilon_2 \ll 1$. To leading order in $R_m^{-1/2}$, the toroidal and poloidal parts of the azimuthally averaged induction equation can then be combined to give

$$\frac{\partial \bar{B}_e}{\partial t} = \nabla \times (u_e \times \bar{B}_e) + \nabla \times (\epsilon \bar{B}_e \epsilon_{\phi}) + \nabla^2 \bar{B}_e,$$

in terms of

$$\bar{B}_e = B\epsilon_{\phi} + B_p + \nabla \times \left( \frac{s}{2} \left( \bar{v}_p \times \epsilon_{\phi} \right) \epsilon_{\phi} \right),$$

$$u_e = u\epsilon_{\phi} + u_p + \nabla \times \left( \frac{s}{2} \left( \bar{v}_p \times \epsilon_{\phi} \right) \epsilon_{\phi} \right),$$

where

$$z = \frac{1}{s} \left\{ \left[ \bar{v}_p \times \epsilon_{\phi} \right] \epsilon_{\phi} + \left[ \bar{v}_p \times \frac{\partial \bar{v}_p}{\partial \phi} \right] \epsilon_{\phi} \right\} + 2 \left( \nabla \bar{v}_p \epsilon_{\phi} \right) \nabla \bar{v}_p \epsilon_{\phi},$$

$$\bar{v}_p = u'_{s} / u.$$

Angle brackets denote azimuthal averaging, $\partial / \partial \phi$ differentiation with respect to $\phi$ treating unit vectors as constants, and a hat indicates the inverse operation of indefinite integration with respect to $\phi$. The subscript ‘$p$’ denotes the ‘poloidal’ part of a quantity [more strictly the meridional part, which can be identified with the poloidal component defined by (4) only in the axisymmetric case]. The subscript ‘$e$’ refers to the ‘effective’ variables, introduced as defined above so that eq. (11) reduces to the simple form given, identical to that for a true axisymmetric magnetic field, but with the generation term involving $z$ now present.

For $R_m \gg 1$, expression (9) requires solutions to be dominantly axisymmetric; we refer to this state as the ‘nearly axisymmetric’, or Braginsky, regime. The purely axisymmetric nature of the asymptotic limit facilitates detailed numerical surveys.

For a velocity of the form (2), with

$$u' = \epsilon_2^{\frac{1}{2}} s^{\frac{1}{2}} + \epsilon_1 s^{\frac{3}{2}},$$

(considering only a single harmonic initially), the asymptotic prescriptions (13), (14) give

$$u_e = \ell_1 \epsilon_1^{\frac{1}{2}} s^{\frac{1}{2}} + u_{e},$$

$$u_{e} = \epsilon_2^{\frac{1}{2}} s^{\frac{1}{2}} + u_{eq},$$

$$u_{eq} = \epsilon_1 \epsilon_2^{\frac{1}{2}} \frac{l(l+1)}{m(l+m)} \left( \frac{-1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \right) f_2(r) \left( \frac{m}{r^2} \right)^{\frac{1}{2}},$$

$$z = \frac{1}{s} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} \right) \right) f_2(r) \left( \frac{m}{r^2} \right) \left( \frac{1}{\sin \theta} \right) \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} \right) \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \phi} \right) f_2(r),$$

in terms of

$$f_2(r) = \ell_1 (\sigma^{\epsilon_{\phi}}_{\ell m} - \sigma^{\epsilon_{\phi}}_{\ell m}),$$

$$f_2(r) = \left[ (\sigma^{s_{\ell m}}_{\ell m} - \sigma^{s_{\ell m}}_{\ell m}) - (\sigma^{s_{\ell m}}_{\ell m} - \sigma^{s_{\ell m}}_{\ell m}) \right] \left( \epsilon_1 \frac{f_2(r)}{f_2(r)} \right),$$

$$f_3(r) = \left( \sigma^{s_{\ell m}}_{\ell m} - \sigma^{s_{\ell m}}_{\ell m} \right) / r^2,$$

where $\sigma^m_{\ell m}$ is $s_{\ell m} / \ell_1$ and the primes denote differentiation with respect to $r$. Here $\epsilon_1 = \epsilon_1 R_m$ is a finite quantity arising in the $R_m \to \infty$, $\epsilon_1 \to 0$ limit, following from expansion (10); likewise $\epsilon_2^{\frac{1}{2}} s^{\frac{1}{2}}$ is $\epsilon_2^{\frac{1}{2}} s^{\frac{1}{2}} R_m$ is finite, from (9). Expressions (18) and (19) are given for Schmidt-normalized spherical harmonics, which we use throughout.
The above prescriptions can be used for any choice of \( l \) and \( m \) of interest. \( u_{ep} \) can be expanded as a series of terms

\[
u_{ep} = \sum_{l'=2, even}^{2l} s_{l'}^0, \quad s_{l'}^0(r) = S_l f_1(r),
\]

with the constants \( S_l \) determined from expression (18). Similarly we can rewrite (19) in the form

\[
x = \sum_{l=1, odd}^{l_{max} - 1} A_l(r)P_l^m(\cos \theta), \quad A_l(r) = A_l^{(1)} f_1(r) + A_l^{(2)} f_2(r).
\]

Values of the coefficients \( S_l \), \( A_l^{(1)} \), \( A_l^{(2)} \), for certain values of \( l \) and \( m \), are given in Table 1.

In the asymptotic limit the quantity \( \varepsilon^{m_s}_l \varepsilon^{m_c}_l \) appearing in the \( x \)-term can be taken as the critical parameter for dynamo excitation. Asymptotic calculations carried out for specified values of \( \varepsilon_1 \) and \( \varepsilon^{m_s}_l \varepsilon^{m_c}_l \) approximate the 3-D state with

\[
R_m = \varepsilon^{m_s}_l \varepsilon^{m_c}_l / (\varepsilon^{m_s}_l \varepsilon^{m_c}_l), \quad \varepsilon_1 = \varepsilon_1 / R_m,
\]

for any chosen value of \( \varepsilon^{m_s}_l \varepsilon^{m_c}_l \). The magnitude of \( \varepsilon^{m_s}_l \varepsilon^{m_c}_l \) determines the strength of the non-axisymmetric component, and hence the ‘distance from the asymptote’. The correspondence between the asymptotic and 3-D cases is described in more detail in SG and in KR, the latter work detailing a suite of calculations validating the procedure. Because of the non-linear nature of transformation (25), care must be taken in extrapolating conclusions from the Braginsky limit to the 3-D case. \( \varepsilon_1 \) does not scale directly to \( \varepsilon_1 \) but depends upon \( R_m \) also, and thus a fixed \( \varepsilon_1 \) does not correspond to a fixed \( \varepsilon_1 \), except in the case \( \varepsilon_1 = 0 \).

From the expansions (23) and (24) it is evident that \( u_{ep} \) is symmetric and \( x \) anti-symmetric with respect to the equatorial plane. This property permits magnetic fields to be either equatorially symmetric (quadrupolar) or anti-symmetric (dipolar), consistent with the symmetries permitted by the full 3-D flow.

When two modes of differing \( m \) are present, their individual contributions to the asymptote may simply be added; the prescriptions (12)-(14) do not provide any intermode interaction in this case. To conduct asymptotic calculations, we need only fix the relative strengths of the two modes, \( \varepsilon^{m_s}_l \varepsilon^{m_c}_l / (\varepsilon^{m_s}_l \varepsilon^{m_c}_l) \).

| \( s_1 \) | \( S_2 \) | \( S_3 \) | \( S_5 \) | \( S_8 \) | \( S_{10} \) | \( S_{12} \) |
|---|---|---|---|---|---|---|
| 3/7 | 9/70 | 9/11 | 10/9 | 980/1287 | 35/2574 | 0 |
| 200/693 | -45/91 | 10/9 | 35/2574 | 0 |
| -50/99 | 45/143 | -10/99 |
| 75/143 | -45/286 | -180/187 | -3570/2717 | 56700/46189 |
| 75/143 | -135/286 | 120/186 | 3255/10868 | 4725/92378 |
| 105/143 | 63/442 | -210/323 | 7455/5434 | 138915/96577 |
| 21/286 | 1512/2431 | 420/3553 | 9345/10868 |
| 21/286 | 70/323 | 35/2494 | -13253/96577 |
| 21/286 | 189/442 | 35/2494 |
| 10/99 | 45/143 | 10/99 | 35/2574 | 0 |
| 10/3 | 60/7 | -60/7 |
| 10/3 | 60/7 |
| 10/3 | 60/7 |
| 10/3 | 60/7 |
| 45/22 | 315/286 | -225/26 | 36225/4862 |
| 63/13 | 3087/143 | 630/17 | 11025/5535 |
| 63/13 | 3087/143 | 630/17 | 11025/5535 |
| 21/13 | 60/7 | -2835/221 | -343035/4199 |
| 21/13 | -2835/221 | -343035/4199 |
| 21/13 | 730/221 | 105/221 | -231/4199 |
| 21/13 | 730/221 | 105/221 |
| 198/7 | 66 | 600/7 |
| 190/3 | 7010/33 | 9200/39 | 166600/429 |
| 95/3 | 1295/33 | 25/39 | 2975/429 |
| 1305/11 | 5145/11 | 8700/13 | 1656900/2431 |
| 1305/22 | 945/22 | -3375/26 | -111825/4862 |
| 2583/13 | 121257/143 | 318150/221 | 72645750/46189 |
| 2583/26 | 55125/286 | -53613/442 | -21831705/23278 |
| 2583/26 | -1715/13 | 18634/221 | -63210/4199 |

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When flow of a given azimuthal mode $m$ has a more complex latitudinal structure than the single / harmonic considered above, the interaction between the various harmonics $l_1 \neq l_2$ must be taken into account. For

$$u' = (e_m^{l_1} s_{l_1}^{m*} + e_m^{l_2} s_{l_2}^{m*}) + (e_m^{l_1} s_{l_1}^{m*} + e_m^{l_2} s_{l_2}^{m*}),$$  

(26)

we obtain the following terms, in addition to the terms arising from (18) and (19) for each $l$ individually:

$$u_{cp} = \frac{1}{m} \left[ \left( \frac{l_1 - m}{l_1 + m} \right)^{1/2} \left( F_1(\theta) \frac{\partial f_1(r)}{\partial r} + F_2(\theta) \frac{\partial f_2(r)}{\partial r} - F_2(\theta) \frac{\partial f_1(r)}{\partial r} + F_1(\theta) \frac{\partial f_2(r)}{\partial r} - \frac{2}{m} \left( \frac{l_1 - m}{l_1 + m} \right)^{1/2} \left[ F_3(\theta)f_3(r) + F_4(\theta)f_4(r) + F_5(\theta)f_5(r) + G_1(\theta)g_1(r) + G_2(\theta)g_2(r) + G_3(\theta)g_3(r) \right] \right) \right],$$

(27)

$$z = \frac{2}{m} \left[ \left( \frac{l_1 - m}{l_1 + m} \right)^{1/2} \left( F_3(\theta)f_3(r) + F_4(\theta)f_4(r) + F_5(\theta)f_5(r) + G_1(\theta)g_1(r) + G_2(\theta)g_2(r) + G_3(\theta)g_3(r) \right) \right].$$

(28)

The terms

$$F_1(\theta) = -l_2(l_2 + 1) \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial P_m^{l_2}}{\partial \theta} \right),$$

(29)

$$F_2(\theta) = l_2(l_2 + 1) \frac{\partial P_m^{l_2}}{\partial \theta},$$

(30)

$$F_3(\theta) = \frac{1}{\sin \theta} \frac{\partial P_m^{l_2}}{\partial \theta},$$

(31)

$$F_4(\theta) = l_2(l_2 + 1) \left[ (m^2 - 1) \frac{1}{\sin^2 \theta} \frac{\partial P_m^{l_2}}{\partial \theta} + \frac{\partial^2 P_m^{l_2}}{\partial \theta^2} + \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial P_m^{l_2}}{\partial \theta} \right) - l_1(l_1 + 1) \frac{\cos \theta}{\sin \theta} \frac{\partial P_m^{l_2}}{\partial \theta} P_m^{l_1} \right],$$

(32)

$$f_1(r) = \frac{\partial^2 c_m^{l_1} s_{l_1}^{m*} \sigma_{l_1}^{m*}}{\partial \theta^2} - \frac{\partial^2 c_m^{l_1} s_{l_1}^{m*} \sigma_{l_1}^{m*}}{\partial \theta^2} - l_1(l_1 + 1) \frac{\cos \theta}{\sin \theta} \frac{\partial P_m^{l_2}}{\partial \theta} P_m^{l_1},$$

(33)

$$f_2(r) = \frac{\partial^2 c_m^{l_2} s_{l_2}^{m*} \sigma_{l_2}^{m*}}{\partial \theta^2} - \frac{\partial^2 c_m^{l_2} s_{l_2}^{m*} \sigma_{l_2}^{m*}}{\partial \theta^2} - l_1(l_1 + 1) \frac{\cos \theta}{\sin \theta} \frac{\partial P_m^{l_2}}{\partial \theta} P_m^{l_1},$$

(34)

$$f_3(r) = \frac{\partial^2 c_m^{l_3} s_{l_3}^{m*} \sigma_{l_3}^{m*}}{\partial \theta^2} - \frac{\partial^2 c_m^{l_3} s_{l_3}^{m*} \sigma_{l_3}^{m*}}{\partial \theta^2} - l_1(l_1 + 1) \frac{\cos \theta}{\sin \theta} \frac{\partial P_m^{l_2}}{\partial \theta} P_m^{l_1},$$

(35)

are analogous to the earlier terms in (18)–(22); $F_m^{l_1}(\theta)$ and $f_m^{l_1}(r)$ are obtained from $F_1(\theta)$ and $f_1(r)$ by interchanging $l_1$ and $l_2$.

We can expand these as

$$u_{cp} = \sum_{l=1, \text{even}}^{l_1 + l_2} S_l^0, \quad l_1 + l_2 \text{ even},$$

(36)

$$= \sum_{l=1, \text{odd}}^{l_1 + l_2} S_l^0, \quad l_1 + l_2 \text{ odd},$$

$$z = \sum_{l=1, \text{even}}^{l_1 + l_2} A_l(r) P_l^0(\cos \theta), \quad l_1 + l_2 \text{ even},$$

(37)

$$= \sum_{l=1, \text{ odd}}^{l_1 + l_2 - 1} A_l(r) P_l^0(\cos \theta), \quad l_1 + l_2 \text{ odd},$$

with $S_l(r)$ and $A_l(r)$ defined as before. For $(l_1 + l_2)$ odd, $u_{cp}$ is no longer symmetric with respect to the equatorial plane and $z$ is no longer anti-symmetric, thus eliminating the possibility of purely dipolar or quadrupolar magnetic fields. The relevant constants in the expansions of $u_{cp}$ and $z$ are given in Table 2 for the interaction of an $s_2^z$ velocity with other $m = 2$ harmonics with $l \leq 6$.

The additional terms

$$G_1(\theta) = l_2(l_2 + 1) \frac{1}{\sin \theta} \frac{\partial P_m^{l_2}}{\partial \theta} - l_1(l_1 + 1) \frac{1}{\sin \theta} \frac{\partial P_m^{l_1}}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial P_m^{l_2}}{\partial \theta},$$

(38)

$$G_2(\theta) = l_2(l_2 + 1) \left[ (m^2 - 1) \frac{1}{\sin^2 \theta} \frac{\partial P_m^{l_2}}{\partial \theta} + \frac{\partial^2 P_m^{l_2}}{\partial \theta^2} \right]$$

(39)

$$- l_1(l_1 + 1) \left[ (m^2 - 1) \frac{1}{\sin^2 \theta} \frac{\partial P_m^{l_1}}{\partial \theta} + \frac{\partial^2 P_m^{l_1}}{\partial \theta^2} \right],$$

(40)

$$G_3(\theta) = l_1(l_1 + 1) \left( \frac{1}{\sin \theta} \frac{\partial P_m^{l_2}}{\partial \theta} - \frac{\partial P_m^{l_1}}{\partial \theta} \right).$$

(41)

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Table 2. Signed squares of the coefficients in the expansions of $u_p$ and $z$, for the interaction of an $s_2^2$ flow with other $s_l^m$ components with $m=2$ and $l \leq 6$.

| $s_2^2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| $x_1^4$ | 0     | -125/1323 | -5/4 | -177147/118580 | 1/5 | 1280/3267 | - | - |
| $x_2^4$ | 0     | -35/121 | -1372/845 | -1440/847 | - | 91125/143143 | - | - |
| $x_3^4$ | 0     | 0 | -55566/102245 | - | -1440/847 | - | 315/338 | - |
| $A_1^{(1)}$ | $A_1^{(1)}$ | $A_1^{(1)}$ | $A_1^{(1)}$ | $A_1^{(1)}$ | $A_1^{(1)}$ | $A_1^{(1)}$ | $A_1^{(1)}$ | $A_1^{(1)}$ |
| $A_2^{(1)}$ | $A_2^{(1)}$ | $A_2^{(1)}$ | $A_2^{(1)}$ | $A_2^{(1)}$ | $A_2^{(1)}$ | $A_2^{(1)}$ | $A_2^{(1)}$ | $A_2^{(1)}$ |
| $A_3^{(1)}$ | $A_3^{(1)}$ | $A_3^{(1)}$ | $A_3^{(1)}$ | $A_3^{(1)}$ | $A_3^{(1)}$ | $A_3^{(1)}$ | $A_3^{(1)}$ | $A_3^{(1)}$ |
| $A_4^{(1)}$ | $A_4^{(1)}$ | $A_4^{(1)}$ | $A_4^{(1)}$ | $A_4^{(1)}$ | $A_4^{(1)}$ | $A_4^{(1)}$ | $A_4^{(1)}$ | $A_4^{(1)}$ |
| $A_5^{(1)}$ | $A_5^{(1)}$ | $A_5^{(1)}$ | $A_5^{(1)}$ | $A_5^{(1)}$ | $A_5^{(1)}$ | $A_5^{(1)}$ | $A_5^{(1)}$ | $A_5^{(1)}$ |
| $A_6^{(1)}$ | $A_6^{(1)}$ | $A_6^{(1)}$ | $A_6^{(1)}$ | $A_6^{(1)}$ | $A_6^{(1)}$ | $A_6^{(1)}$ | $A_6^{(1)}$ | $A_6^{(1)}$ |
| $A_7^{(1)}$ | $A_7^{(1)}$ | $A_7^{(1)}$ | $A_7^{(1)}$ | $A_7^{(1)}$ | $A_7^{(1)}$ | $A_7^{(1)}$ | $A_7^{(1)}$ | $A_7^{(1)}$ |
| $A_8^{(1)}$ | $A_8^{(1)}$ | $A_8^{(1)}$ | $A_8^{(1)}$ | $A_8^{(1)}$ | $A_8^{(1)}$ | $A_8^{(1)}$ | $A_8^{(1)}$ | $A_8^{(1)}$ |

$g_1(r) = (e_m^m e_{mc}^e c_m^c c_{mc}^e - e_m^m e_{mc}^e c_m^c c_{mc}^e)(\ell_1^l / \ell_2^l)$,

$g_2(r) = (e_m^m e_{mc}^e c_m^c c_{mc}^e - e_m^m e_{mc}^e c_m^c c_{mc}^e)(B_1 / \rho_r)^2$,

$g_3(r) = (e_m^m e_{mc}^e c_m^c c_{mc}^e - e_m^m e_{mc}^e c_m^c c_{mc}^e)(\rho_r / \rho_r)^2$,

now also appear in the definition of $z$. These latter terms can be treated as before; they vanish, however, if $e_m^m e_{mc}^e = e_m^m e_{mc}^e$.

As this is a reasonable simplification with which to begin studies, we restrict ourselves to this case in the calculations reported here.

Tableoridal harmonics can also interact with poloidal harmonics to cause dynamo action in the high-$R_m$ regime. This possibility is not investigated in the current work.

### 3 FLOWS OF VARYING AZIMUTHAL MODE

Of considerable interest to the picture of 'columnar' convection anticipated from the hydrodynamics of rapidly rotating systems (e.g. Busse 1970) is the selection and interaction of modes of varying azimuthal wavenumber $m$. The choice of $m$ is a matter of even greater interest in magnetohydrodynamics, given the conflicting tendencies involved, with rotation favouring high $m$ but magnetic constraints preferring smaller values. Magnetocconvective studies (e.g. Busse 1976, 1983; Fearn 1979a,b; Longbottom, Jones & Hollerbach 1995; Zhang 1995) suggest that for Elsasser numbers $\Lambda = \sigma \beta^2 / (2 \Omega \rho)$—where $\beta$ is a measure of the magnetic field strength, $\Omega$ is the rotation rate, and $\rho$ the fluid density—in the range $1 \leq \Lambda \leq 10$, as estimated for the geodynamo, $m < O(10)$ is to be expected.

Within our formalism, we can approximate a columnar form for the flow of wavenumber $m$ by restricting the latitudinal structure to be sectorial (i.e. of degree $l = m$). In this section we restrict ourselves to such flows and superpositions thereof. We will consider flows of more complex form, with latitudinal structures $l > m$, in Section 4.

Considering only even $m$ and starting our study, for numerical convenience, with the lowest $m$ that allow multiple-mode interaction, we consider the special case of the velocity field (2) given by

$$u = e_0\ell_1^0 + e_1\ell_1^1 + e_2\ell_1^2 + e_3\ell_1^3 + e_4\ell_1^4 + e_5\ell_1^5,$$

where we have adopted a simplified indexing of the $e$ parameters, consistent with the studies of KR and SG. The radial functions given by (5)–(8) are employed, but with an additional parameter, $\omega$, introduced to allow a phase-shift in $\phi$ between the $m=2$ and $m=4$ modes:

$$a_2^2(r) = r^2(1 - r^2)^2 \cos \phi_1 r,$$

$$a_3^2(r) = r^4(1 - r^2)^2 \sin \phi_1 r,$$

$$a_4^2(r) = r^4(1 - r^2)^2 \cos 2\phi_1 r \cos \omega - e_3/4 \sin 2\phi_1 r \sin \omega,$$

$$a_5^2(r) = r^4(1 - r^2)^2 (e_3/2 \cos 2\phi_1 r \sin \omega + \sin 2\phi_1 r \cos \omega).$$

When $\omega = 0$, the functions given in Section 2 are regained.
The forms of $a$ and $u_{ep}$ produced by this flow in the Braginsky limit are shown in Fig. 1; as well as showing the individual contributions from the $m=2$ and $m=4$ components, this figure also shows the contributions from an $m=6$ mode, to illustrate the varying form of these quantities with increasing wavenumber.

Critical curves for the onset of dynamo action can easily be calculated in the asymptotic limit, and extrapolated to give approximations of the true 3-D behaviour, as described in Section 2. Since the 3-D calculations are more time-consuming to perform, the basic behaviour of the velocities is, wherever possible, investigated through such extrapolated results. 3-D calculations are described here only to illustrate individual solutions, or to investigate features (such as the phase-shift, $\omega$, introduced above) which cannot be investigated in the asymptote. More extensive 3-D calculations have been carried out, however, confirming the extrapolated results, and the validity of the asymptotic approximation in general. Failure to obtain a convincing 3-D solution in conjunction with the asymptotic result occurred only in very high $R_m$ cases, where the 3-D calculation was clearly breaking down due to inadequate numerical resolution.

Fig. 2 shows critical curves for the $m=4$ mode in isolation, detailing solutions of both dipole and quadrupole symmetry. There are two notable features in the behaviour exhibited. First, the preference is for dipole solutions to be excited for $R_m > 0$, quadrupole for $R_m < 0$. (Negative $R_m$ simply constitutes a reversal in the sense of the velocity.) For these simple flows this can be related to a change in sign of the product $\epsilon^m \epsilon^m_1$ (determining the sense of spiralling of the convective cells) or of $\epsilon_0$ (the sense of zonal flow), with a simultaneous reversal of $\epsilon_1$ (the meridional circulation). (See SG for a discussion of the lack of independence of the signs of the various $\epsilon_i$ parameters and of $R_m$.) Second, meridional circulation (controlled by $\epsilon_1$) strongly influences the ease of excitation and the time dependence of the preferred solution. Both of these features are consistent with previous studies of z-effect dynamos (e.g. P. H. Roberts 1972).

Fig. 2 also compares closely with results for the $m=2$ flow studied in detail by SG. This is reasonable given the similar asymptotic forms arising for sectorial modes of differing $m$ (see Fig. 1). The axisymmetric magnetic fields produced by the two modes

---

**Figure 1.** The distribution of $a$ (top row) and the streamfunction of effective poloidal flow, $u_{ep}$ (bottom row), arising in the asymptotic limit for the velocities $s_2$ (left), $s_4$ (centre) and $s_6$ (right). Contour intervals are in all cases set to one tenth of the maximum value.

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are also similar. The most notable difference between the two wavenumbers is an approximate ‘offset’ in the $\epsilon_1$ axis; $\epsilon_1 = 0$ favours an oscillatory solution for $m = 2$, a stationary solution for $m = 4$, for example. This can be explained by the relative strength of the effective meridional circulation, $u_{ep}$, in the two cases (compare the coefficients in Table 1); this quantity acts analogously to the true meridional circulation. The importance of this feature, first highlighted by SG, is therefore confirmed here.

Fig. 3 shows the critical curves with the $m = 2$ and $m = 4$ modes simultaneously present. The basic pattern is the same, with the detailed behaviour lying somewhere between the two individual cases. The lower values of $R_{cm}$ now obtained show that both modes are contributing towards dynamo action. That they should combine so effectively—almost as a simple addition of the effects—is not
surprising, however, given the similar way in which they act to generate magnetic fields, reflected in their similar asymptotic forms and their similar behaviour in isolation.

In the case considered above, the senses of spiralling of the two individual modes, controlled by the signs of \( \epsilon_2 \) and of \( \epsilon_5 \), are the same. When the two modes spiral in the opposite sense, so that the product \( \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 \) is negative, dynamo action is impeded, as the higher values of \( R_{mc} \) shown in Fig. 4 evince. This is not inconsistent with the effects of the two modes being broadly superimposed, however. In this case the two individual \( \alpha \) distributions in the asymptote are oppositely signed, and would in isolation excite magnetic fields of differing symmetry (dipole versus quadrupole). They might thus be expected to combine negatively in terms of dynamo action, and to excite fields rather different from those obtained above. This is indeed observed.

The radial dependence of the two modes has hitherto been fixed in the form \((46) - (49)\), with \( p_1 \sim p_2 \sim 3 n \). As a preliminary investigation of the importance of this factor, various alternative \( p_i \) were considered. Their effect was found to be rather slight, however, with qualitatively identical behaviour being obtained in all cases, the variation being consistent with that obtained by SG upon such radial variations in the \( m = 2 \) velocity alone. This was not unexpected, since the 'cellular' velocity structure was chosen to allow eqs (13) and (14) to assume smooth large-scale forms for all values of \( p_i \). A more thorough investigation of the importance of the radial structure should consider modes without the imposed \( \cos p_1 r \sin p_2 r \) form. Such a study has not yet been undertaken, although this point is further addressed in Section 5.

The most easily excited critical curves detailed above, obtained for the asymptotic system, have been confirmed by direct 3-D calculations. Significant discrepancies between the asymptotic and 3-D cases were found only at the higher values of \( R_{mc} \), where solutions of more complex field morphology required a finer resolution than was available for the 3-D computations. The clear manner in which the 3-D numerical resolution deteriorated as these solutions were approached (through a continuous variation in parameters from well-established solutions), and the good agreement obtained in all other cases, give us no reason to doubt the asymptotic results.

In all cases the behaviour of the system could be understood through recourse to the asymptotic system, with dynamo action depending upon the distributions of \( x \) and \( u_{ep} \) arising there in a relatively straightforward way. The presence of several modes of varying \( m \) is not in itself detrimental to dynamo action. The various modes can in fact be combined almost as a simple superposition. If the sense of spiralling of the two convective modes is the same—that is if \( \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 \) is positive, for the specific velocity (45)—then dynamo action can be obtained at lower \( R_{mc} \) than for either mode alone. If the opposite is the case, however, the two modes can interact negatively, resulting in a higher \( R_{mc} \) and a more complex magnetic field. Results consistent with all of the above have also been obtained with the addition of an \( m = 6 \) flow.

A 3-D dipole solution obtained with both \( m = 2 \) and \( m = 4 \) components of flow present is shown in Fig. 5. The axisymmetric field (Figs 5a and b) is comparable with that obtained for the KR flow (see KR or SG); the radial field (Fig. 5c) shows a somewhat more complicated non-axisymmetric structure, but the field remains dominantly large scale. The numerical convergence of this solution, and subsequent 3-D solutions shown, is addressed in Appendix A.

![Figure 4](https://example.com/figure4.png)

**Figure 4.** Critical stability curves \( R_{mc} \) as a function of \( \varepsilon_1 \), from extrapolations of the asymptotic system with \( p_1 = p_2 = 3 \pi \), \( \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0.04 \), from calculations with 40 grid points, truncated at degree 14. Details of the plot as for Fig. 2.
The compatibility of modes of varying $m$ in exciting magnetic fields is not significantly affected by an azimuthal phase-difference $\omega$ between the modes, as can be seen from Fig. 6. The difference in $R_{mc}$ obtained for varying $\omega$ is only of the order of 5 per cent. This value achieves a maximum when the strengths of the two modes are comparable, as is the case for the figure shown; in other cases one mode acts as a relatively minor perturbation to the second, and the effect is diminished. Dynamo action is slightly favoured when the phase difference is such that equatorially outward flow is concentrated in two large cells and two small cells, rather than in four equally sized cells.

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Figure 5. Magnetic field associated with the critical solution obtained for the velocity $p_1 = p_2 = 3\pi$, $\epsilon_1 = 0.03$, $\epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 0.04$, $R_{mc} = 1815$ with 150 radial grid points and harmonics up to degree and order 20. (a) Axisymmetric toroidal field $B_\phi$ in plane of constant $\phi$. (b) Streamfunction of axisymmetric poloidal field $B_p$ in plane of constant $\phi$. (c) Radial field $B_r$ on cylindrical equidistant projection of surface $r = 1$. Solid (dashed) lines denote positive (negative) values.

Figure 6. $R_{mc}$ as a function of the azimuthal phase-difference $\omega$ between $m=2$ and $m=4$ modes of flow. The parameters $p_1 = p_2 = 3\pi$, $\epsilon_1 = 0.03$, $\epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 0.04$, have been used and the eigenvalues are Richardson-extrapolated from runs with 50, 100 and 150 grid points, for the system truncated at degree and order 14.

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4 MORE COMPLEX CELLULAR FLOWS

A convective flow may be dominated by a given azimuthal wavenumber \( m \), yet possess significant latitudinal structure within this mode. Whilst the preceding section restricted itself to non-axisymmetric flows of purely sectorial \( (l = m) \) form, this section briefly considers the effect on dynamo action of the inclusion of components of flow with \( l > m \), such modes inevitably being present in any realistic convective velocity.

A simple special case of the velocity field (2) is again considered,

\[
u = \epsilon_0 t^0 + \epsilon_1 x^0 + \epsilon_2 s_2^0 + \epsilon_3 s_2^2 + \epsilon_4 s_2^4 + \epsilon_5 f^0 + \epsilon_6 f^2 + \epsilon_7 f^4,
\]

for various choices of \( l > 2 \). The radial functions \( (5)^{(8)} \) are retained. The prescriptions (27) and (28) allow us to incorporate the interaction between components of differing degree into the axisymmetric parametrization, and the asymptotic results again agree satisfactorily with direct 3-D computations.

The addition of an \( l = 4 \) structure to the \( m = 2 \) flow is first considered. This composite flow retains the equatorial symmetry of the original, avoiding additional computational requirements. The distributions of \( \alpha \) and \( u_{ep} \) arising in the Braginsky limit are shown in Fig. 7. Critical curves obtained for the \( s_4^2 \) mode in isolation are shown in Fig. 8; results for the combined flow, with \( s_2^4 \) and \( s_2^2 \) modes both present, are given in Fig. 9.

The patterns of behaviour obtained with varying \( \epsilon_1 \) differ significantly from those obtained for the purely sectorial \( m = 2 \) flow. This was to be anticipated, however, given the more complex latitudinal structures now present, evident in the asymptotic forms as well as in the original 3-D flow. The magnetic field morphology produced is also somewhat more complex, as can be seen in Fig. 10, obtained from a 3-D calculation with both components of flow present. In comparison with Fig. 5, the toroidal flux now exhibits a

![Figure 7](image-url)
sign-change in each hemisphere. The simultaneous presence of both modes need not inhibit dynamo action, however; the global minimum of $R_{cm}$ is slightly smaller than that obtained with either mode in isolation.

Similar calculations carried out with the addition of an $l=6$ component of flow confirm this behaviour. Calculations with such higher modes in isolation show that the magnetic field structure can change significantly as the difference $l-m$ is increased, however. When several such modes are combined we might then anticipate an effect similar to that obtained in Section 3 for negative values of $\epsilon_3=\epsilon_4$; although each component can in isolation excite magnetic fields, their simultaneous presence can inhibit dynamo action.

When modes of even and odd $l$ are mixed, the symmetry of flow which permitted separate dipole and quadrupole solutions is broken, and fields asymmetric with respect to the equator are produced. The only such case we have investigated thoroughly is that given by (50) with $l=3$. The contributions to $x$ and $u_{mp}$ arising for this flow in the Braginsky limit are shown in Fig. 11. The results for the $s_2^3$ mode in isolation, and with both $s_2^3$ and $s_2^2$ modes present (only the latter case breaking the dipole/quadrupole dichotomy) are given in Figs 12 and 13 respectively.

The behaviour with varying $\epsilon_1$ again differs significantly from that obtained for the sectorial flows considered in Section 3, stationary solutions now being preferentially excited at all parameter values considered. For the $s_2^3$ mode in isolation (Fig. 12), the importance of the sign of $R_m$—and hence of the relative senses of spiralling and of zonal flow—to the preference for dipole or quadrupole modes can still be seen, however.

The $s_2^3$ velocity is no less capable of dynamo action than the sectorial flow. Nor does the interaction of the two modes inhibit dynamo action, despite their opposing equatorial symmetries. Furthermore, the magnetic fields excited by the composite flow need not differ totally from those considered previously. Fig. 14 shows the asymmetric field obtained for such a velocity; the field remains large scale and well resolved, and in some respects comparable with that of Fig. 5.

The introduction of non-sectorial components of non-axisymmetric flow need not inhibit dynamo action, therefore, at least not for the relatively small values of $l-m$ considered here. The Braginsky limit system remains a good approximation for the dynamo action of these more complex 3-D flows, and therefore remains a useful tool for exploring and comprehending their behaviour. The dynamo action of the non-sectorial flows can be rather different from that considered previously. In all cases, however, the overall topology of flow—for example the senses of spiralling and of zonal flow—remains an important determinant for the type of magnetic field excited.

Figure 8. Critical stability curves $R_{cm}$ as a function of $\epsilon_1$, from extrapolations of the asymptotic system with $l=4$, $p_2=3\pi$, $\epsilon_2=\epsilon_3=0$, $\epsilon_6=\epsilon_7=0.04$, from calculations with 40 grid points, truncated at degree 14. Details of the plot as for Fig. 2.
5 DISCUSSION

In using the forms (7) and (8) for the non-axisymmetric components of flow we have introduced a tilt to the radial motion. Although velocity fields of this form differ considerably from the actual solutions for thermal convection in a rotating sphere, they exhibit some similarities for the cases of small $l-m$ which we have emphasized in our numerical study. The tilt describes a variation of the azimuthal phase with radius, which mimics the variation of azimuthal phase with distance from the axis shown by convection columns. For $\epsilon_1^m = \epsilon_4^c = A$, for example, the non-axisymmetric flow associated with (7) and (8) can be described by

$$s^m_1(r, \phi) = Ar^4(1-r^2)^2 \sin(m\phi + pr),$$

making this phase relation explicit; the parameter $p$ can then be directly linked to the strength (and sense) of the tiltedness. Although such tiltedness was not included in the original analyses of convection in rotating spheres by Roberts (1968) and Busse (1970), it was soon recognized as an important feature of convection in connection with the generation of mean zonal flows (Gilman 1976; Busse & Hood 1982; Busse 1982). The spiralling of the columns was also demonstrated in the experiment of Carrigan & Busse (1983) and becomes dramatically enhanced with increasing Taylor numbers, as shown, for instance, in the computations of Zhang (1992).

In the present context the tilting of the non-axisymmetric cells of motion is a necessary ingredient of dynamo action. To make this point we generalize the radial function (8) to

$$s^m_2(r) = r^4(1-r^2)^2 \sin(pr + p_0).$$

This expression reduces to the original form for $p_0 = 0$, whilst in the case $p_0 = \pi/2$ the tilt disappears and planes of symmetry of constant $\phi$ are realized.

Results for various values of $p_0$ are given in Table 3 for the $s^m_2$ velocity with $\epsilon_1 = 0, \epsilon_4 = \epsilon_5 = 0.04$. Dynamo action could not be obtained for the case $p_0 = \pi/2$. The table also shows the expected $1/\cos p_0$ variation for the critical $R_{cm}^m$ in the asymptotic limit, since only the $\sin pr$ component of (52) interacts with the $s^m_2$ term in that limit. The good agreement with the predicted asymptotic variation shows that the ‘spiralling component’ of the flow is the essential one for magnetic field generation.

The importance of the parameter $p$ for dynamo action was explained originally (Kumar & Roberts 1975) in terms of the ‘number of cells in radius’, in analogy with Cartesian periodic dynamos (e.g. G. O. Roberts 1972). Here we suggest that it is perhaps better
interpreted in terms of the tiltedness of the flow. It is worth noting though that whilst the absence of tilt prohibits dynamo action in the Braginsky limit for the simple $l=m$ case investigated above, it need not necessarily do so when modes with $l > m$ are present. The terms given by (42)–(44) then allow for non-zero $\alpha_i$ if $\mathcal{P}_l \mathcal{P}'_l \neq \mathcal{P}'_m \mathcal{P}'_m$.

Flows that resemble convection in rotating spheres should not be stationary, as we have prescribed, but rather drift with respect to the rotating reference frame. A uniformly drifting flow can be modelled simply by the addition of an overall solid body rotation; by augmenting $t_0$ by $\mathcal{J}$ in (5), we effectively solve for the field and flow in a system co-rotating at angular velocity $\mathcal{J}$. The addition of such a drift was first considered by KR, who found the effect to be slight. This was also the case for the other flows investigated here. Although this technique restricts us to a single drift rate applying uniformly to all components of flow, the result of Section 3, that an azimuthal phase difference between modes does not greatly affect dynamo action, suggests that the same conclusion should also apply for more complexly vacillating flows.

It is worth pointing out that the critical values of $R_m$ associated with these flows need not be particularly high, despite their connection with the high-$R_m$ theory of Braginsky. Solutions with $R_m \sim O(100)$ can be attained by using stronger non-axisymmetric flows than those detailed to date. Table 4 shows the variation of $R_m$ with $l = m = 4$ flow. Here we have also calculated a normalized critical magnetic Reynolds number, $R_m^* = R_m / \cos p_0$—using the root mean square velocity, $u_{rms}$, as the velocity scale $\mathcal{U}$—as a more suitable measure of the true ability of these flows to excite dynamo action. [The amplitude of our velocities was previously fixed by $\epsilon_0 = 1$ in (2).] Thus these solutions need not be considered inapplicable to the geodynamo, where an $R_m$ of $O(100)$ is to be anticipated (e.g. Gubbins & Roberts 1987).

As the strength of the non-axisymmetric flow is increased in Table 4, we move further from the Braginsky asymptote, of course. This can be seen by comparing $R_m$ with values extrapolated from the asymptotic calculation via (25), $R_m^{(ext.)}$. (The relevant critical

![Figure 10. Magnetic field associated with the critical solution obtained for the velocity with $l = 4$, $p_1 = p_2 = \pi$, $\epsilon_1 = 0.03$, $\epsilon_2 = \epsilon_3 = \epsilon_4 = 0.04$, $R_m = 1821$ with 150 radial grid points and harmonics up to degree and order 20. (a) Axisymmetric toroidal field $B_\phi$ in plane of constant $\phi$. (b) Streamfunction of axisymmetric poloidal field $B_\phi$ in plane of constant $\phi$. (c) Radial field $B_r$ on cylindrical equidistant projection of surface $r = 1$. Solid (dashed) lines denote positive (negative) values.](https://academic.oup.com/gji/article-abstract/133/1/140/590805)
value of $\varepsilon_4\varepsilon_5$ in the asymptote is 4.336.) Even at $\varepsilon_4 = \varepsilon_5 = 0.04$, the extrapolated value differs by 16 per cent from the true value; this deviation increases with the strength of the non-axisymmetric flow. For the higher values of $\varepsilon_4 = \varepsilon_5$ considered, the validity of the asymptotic approximation is clearly breaking down, with $R_m$ increasing with these parameters, at odds with the asymptotic theory. This is hardly surprising, as the strength of the non-axisymmetric flow, reflected in the variation of $u_{\text{rms}}$, is clearly now violating the

| $\varepsilon_4 = \varepsilon_5$ | 0.04 | 0.08 | 0.12 | 0.16 | 0.20 | 0.24 |
|-----------------------------|-----|-----|-----|-----|-----|-----|
| $R_m$                       | 3215.5 | 1147.6 | 705.0 | 507.4 | 406.5 | 354.7 |
| $u_{\text{rms}}$            | 0.2297 | 0.2424 | 0.2621 | 0.2874 | 0.3170 | 0.3498 |
| $R_m^{\text{(ext.)}}$       | 738.8  | 278.1  | 184.8  | 145.8  | 128.9  | 124.1  |
| $H$                         | -0.0852 | -0.2384 | -0.3762 | -0.4787 | -0.5601 | -0.6184 |
| $R_m/(H/H_0)$               | 3215.5 | 1142.9 | 728.2 | 572.3 | 489.1 | 443.0 |

Figure 11. The asymptotic forms of $u$ (top row), and of the streamfunction of effective poloidal flow, $u_{\text{ep}}$ (bottom row), associated with the interaction of the velocities $s_2$ and $s_3$. The left column shows the terms arising from $s_3$ in isolation (those for $s_2$ are shown in Fig. 1), the centre column those arising purely from the interaction; the right column shows the net form, for $\varepsilon_2 = \varepsilon_3 = \varepsilon_5$. © 1998 RAS, GJI 133, 140–158
Braginsky regime scaling (9). Despite the relatively large discrepancies in $R_{cm}$, however, the axisymmetric magnetic field morphology remains similar to its asymptotic form up to rather high values of $\epsilon_4 = \epsilon_5$; the asymptotic theory thus remains useful for these flows, even if it is no longer quantitatively accurate.

Since the asymptotic system cannot entirely explain the variation in dynamo action for these velocities with significant non-axisymmetric parts, it is of interest to consider alternative measures which might be of use. Previous workers have tried to relate the ability of flows to act as dynamos to the helicity, $h_\sim u \cdot v \cdot w$, where $\omega = \nabla \times u$ is the vorticity (e.g. Nakajima & Kono 1991; Love & Gubbins 1996); this quantity is important for magnetic field generation in the two-scale theory of Steenbeck, Krause & Rädler (1966). The helicity has most commonly been considered in a volume-integrated form. Here we consider a normalized form of this quantity,

$$H = \int u \cdot \omega \, dV / (u_{rms} = \omega_{rms}).$$

(53)
The integral is carried out over a single hemisphere (the northern) since the localized helicity is equatorially anti-symmetric. Table 4 shows this quantity with varying $\varepsilon_4 \sim 5$, and also shows the anticipated variation of $R_m$ with the latter parameters, assuming a linear relationship between $H$ and $R_m$ and taking the value at $\varepsilon_4 \sim 0.04$ as a reference. Although the relationship appears reasonable for the smallest values of $\varepsilon_4$, the agreement across the whole range is poor. Indeed, no convincing relationship between $R_m$ and $H$ could be established for all values of $\varepsilon_4 \sim 0.04$, given that the helicity remains monotonic in these parameters, whereas $R_m$ does not. This is perhaps not surprising; 3-D dynamo action is a complicated process, and no single quantity, no matter how useful in certain cases, should be expected to describe it in general.

6 CONCLUSIONS

For flows of the type considered, dominated by strong zonal flows and with spiralling convective cells, the high $R_m$ asymptotic limit of Braginsky (1964) provides a good approximation for the production of a magnetic field by a large variety of otherwise diverse 3-D velocities. This allows axisymmetric calculations to be made in parallel with the 3-D calculations, greatly reducing the computational workload involved in surveying such flows for dynamo action. Only for flows deviating strongly from the Braginsky regime scalings does the asymptotic system fail to model the 3-D behaviour qualitatively.

For flows of relatively simple latitudinal structure, resembling the columnar form preferred by convection in rapidly rotating systems, dynamo action is relatively easily attained for all azimuthal wavenumbers $m$. The form of magnetic field excited does not vary greatly with wavenumber. The interaction between modes of different wavenumber is correspondingly straightforward, and can almost be viewed as a simple superposition. Whilst our calculations show that modes spiralling in opposite senses can interact to the detriment of dynamo action, it is reasonable to expect that the sense of spiralling of a real convective flow be determined by physical factors, such as the Coriolis force and the boundary curvature, which act similarly for all wavenumbers. For flows of more complicated latitudinal form, dynamo action is still relatively easily achieved; the resultant magnetic fields can be quite different in structure, however, and the interaction between differing modes can therefore be quite complex.

Both the strong zonal flows and the spiralling structure of the convective cells are important for the dynamo action of all these velocities. These features correspond closely to those demonstrated by Zhang (1992) to dominate in a numerical study of convection.
in the rapidly rotating, moderately low Prandtl number regime appropriate to the Earth. The type of magnetic field preferentially excited—dipolar or quadrupolar—can be linked to the senses of the spiralling and of the zonal flow. In this respect it is also encouraging that the prograde spiralling and prograde (eastwards) zonal flows found in most numerical simulations of spherical convection are of the sense required for the preferential excitation of dipolar magnetic fields.

The importance of meridional circulation for the dynamo action of these flows—anticipated from the 2D models of P. H. Roberts (1972), and previously noted in 3-D by SG—is once more apparent. Mechanisms which can induce such circulation are therefore of some interest for the geodynamo; Ekman suction is one obvious candidate (e.g. Fearn 1994). The important role played by the effective meridional circulation arising from the non-axisymmetric convection, considered in detail by SG, makes this an issue of secondary importance, however.

In cases where dynamo action is obtained only at very high values of $R_m$—where the resultant magnetic field is typically of rather complex morphology—numerical difficulties are often encountered in the 3-D calculations. Such cases are particularly common for composite flows, when the various components would in isolation produce magnetic fields of rather different form. That such numerical problems are encountered even for the simple idealized flows employed here highlights the need for a more complete understanding of these kinematic processes, before more complicated 3-D dynamical calculations can be confidently undertaken. The simpler asymptotic system remains well-behaved in such cases, however, and might usefully be employed as a diagnostic tool, allowing difficulties of this nature to be identified. Velocity fields which cause numerical difficulties are often characterized by spatially complex distributions of $x$ and $u_{mp}$ in the asymptotic limit, for example.

Further work in elucidating the inductive action of a wider range of velocities is required, however. General non-axisymmetric toroidal flows should be incorporated into the analysis. More complex forms of differential rotation should also be considered, introducing the further possibility of dynamo action in layers of concentrated regeneration, as elucidated by Braginsky (1964). The simple velocity fields arbitrarily employed might then be replaced by specific flows arising from convective models. Although this has not yet proven practical, the kinematic conclusions obtained with our idealized velocities remain useful in understanding the general mechanisms occurring in magnetic field excitation, and in diagnosing some of the difficulties arising in this process.

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APPENDIX A: NUMERICAL CONVERGENCE

Given the history of numerical difficulties associated with the dynamo problem, it is wise to give some details of the convergence of our solutions with increasing numerical resolution.

The critical curves shown are all derived from calculations in the Braginsky limit. For this axisymmetric system, the degree of truncation of our spherical harmonic expansion, \( L \), and the number of radial grid points, \( N \), could be made high enough to ensure convergence at relatively low computational cost. In practice, truncation level \((L, N) = (14, 40)\) sufficed for reasonable resolution for the velocities investigated here, and all the critical curves shown were calculated at this truncation. For the most easily excited solutions, this gives an accuracy of several significant figures, so that calculations at higher resolution would not perceptibly change the critical lines plotted. For the less easily excited solutions of the more complex velocity fields, convergence is somewhat poorer, although it was always possible to make \( L \) as high as might be desired, the convergence attained is quite reasonable, with the solutions in Table A1 all having converged to at least three significant figures. For this reason we feel satisfied with the results obtained for \( L = 14 \) in the case of Fig. 14, where the loss of equatorial symmetry did not allow us to proceed to higher \( L \).

Table A1. \( R_c \) with varying numerical resolution for the 3-D solutions illustrated in Figs 5, 10 and 14. Parameters \( p_1 = p_3 = 3\pi \) in all cases. The \( O(\delta^2) \) columns give higher-order values obtained by Richardson extrapolation.

| \( L, N \) | 50 | 100 | 150 | \( O(\delta^2) \) |
|---|---|---|---|---|
| \( 8 \) | 1838.99 | 2012.79 | | |
| \( 10 \) | 1752.56 | 1809.49 | 1821.13 | 1830.69 |
| \( 12 \) | 1747.75 | 1803.69 | 1815.13 | 1824.52 |
| \( 14 \) | 1747.43 | 1803.29 | 1814.72 | 1824.10 |
| \( 16 \) | 1747.38 | 1803.23 | 1814.65 | 1824.04 |
| \( 18 \) | 1747.38 | 1803.23 | 1814.65 | 1824.03 |
| \( 20 \) | 1838.95 | 2012.79 | | |

Velocity (45), \( \epsilon_1 = 0.03, \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 0.04 \)

| \( L, N \) | 50 | 100 | 150 | \( O(\delta^2) \) |
|---|---|---|---|---|
| \( 8 \) | 1579.57 | 1593.46 | 1596.31 | 1556.99 |
| \( 10 \) | 1594.04 | 1620.81 | 1626.09 | 1630.39 |
| \( 12 \) | 1703.34 | 1745.32 | 1753.95 | 1761.06 |
| \( 14 \) | 1726.36 | 1769.92 | 1778.86 | 1786.22 |
| \( 16 \) | 1759.18 | 1808.78 | 1819.07 | 1827.55 |
| \( 18 \) | 1761.17 | 1810.87 | 1821.18 | 1829.67 |
| \( 20 \) | 1760.92 | 1810.76 | 1821.11 | 1829.64 |

Velocity (50), \( \epsilon_1 = 0.03, \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 0.04 \)

| \( L, N \) | 50 | 100 | 150 | \( O(\delta^2) \) |
|---|---|---|---|---|
| \( 8 \) | 1413.08 | 1444.42 | 1450.99 | 1456.41 |
| \( 10 \) | 1559.20 | 1584.10 | 1589.16 | 1593.32 |
| \( 12 \) | 1535.38 | 1556.45 | 1560.70 | 1564.17 |
| \( 14 \) | 1532.22 | 1552.58 | 1556.67 | 1560.02 |

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