A volume-weighted measure for eternal inflation

Sergei Winitzki$^{1,2}$

1Department of Physics, Ludwig-Maximilians University, Munich, Germany and 2Yukawa Institute of Theoretical Physics, Kyoto University, Kyoto, Japan

I propose a new volume-weighted probability measure for cosmological “multiverse” scenarios involving eternal inflation. The “reheating-volume (RV) cutoff” calculates the distribution of observable quantities on a portion of the reheating hypersurface that is conditioned to be finite. The RV measure is gauge-invariant, does not suffer from the “younghess paradox,” and is independent of initial conditions at the beginning of inflation. In slow-roll inflationary models with a scalar inflaton, the RV-regulated probability distributions can be obtained by solving nonlinear diffusion equations. I discuss possible applications of the new measure to “landscape” scenarios with bubble nucleation. As an illustration, I compute the predictions of the RV measure in a simple toy landscape.

I. INTRODUCTION

In cosmological scenarios such as the “recycling universe” [1] or the string-theoretic landscape [2, 3, 4], the fundamental theory does not predict with certainty the values of “constants of nature,” such as the effective cosmological constant and particle masses. The cosmological observables may vary significantly between different causally disconnected regions of the spacetime. Hence one may only hope to obtain the probability distribution of the cosmological observables. Heuristically, one would like to compute probability distributions of the cosmological parameters as measured by an observer randomly located in the spacetime. However, eternal inflation produces an infinite volume in which possible observers may find themselves. Thus one runs into an immediate difficulty of defining a “randomly chosen” location within a noncompact space.

Observers may appear only after reheating; the physics after reheating is tightly constrained by current experimental knowledge. The average number of observers produced in any freshly-reheated spatial domain is a function of cosmological parameters in that domain. Calculating that function is, in principle, a well-defined astrophysical problem that does not involve any infinities. Here I focus on obtaining the probability distribution of cosmological observables at reheating.

The set of all spacetime points where reheating takes place is a spacelike three-dimensional hypersurface [5, 6, 7] called the “reheating surface.” The hallmark feature of eternal inflation is that a finite, initially inflating spatial 3-volume typically grows to a reheating surface having an infinite 3-volume, and even (potentially) to infinitely many causally disconnected pieces of the reheating surface, each having an infinite 3-volume (see Fig. 1). This feature of eternal inflation is at the root of several technical and conceptual difficulties known collectively as the “measure problem” (see Refs. 8, 9, 10, 11, 12, 13 for reviews and discussions of this problem).

To visualize the measure problem, it is convenient to consider an initial inflating spacelike region $S$ of horizon size (an “$H$-region”) and the portion $R \equiv R(S)$ of the reheating surface that corresponds to the comoving future of $S$. If the 3-volume of $R$ were finite, the volume-weighted average of any observable quantity $Q$ at reheating would be defined simply by averaging $Q$ over $R$,

$$
\langle Q \rangle \equiv \frac{\int_R Q \sqrt{\gamma} d^3x}{\int_R \sqrt{\gamma} d^3x}, \tag{1}
$$

where $\gamma$ is the induced metric on the 3-surface $R$. However, in the presence of eternal inflation the 3-volume of $R$ is infinite with a nonzero probability $X(\phi_0)$, where $\phi = \phi_0$ is the initial value of the inflaton field at $S$. The function $X(\phi_0)$ can be computed in slow-roll inflationary models where typically $X(\phi_0) \approx 1$ [5]. The geometry and topology of the infinite reheating surface is quite complicated. For instance, the reheating surface contains infinitely many future-directed “spikes” around never-thermalizing comoving worldlines called “eternally inflating geodesics” [15, 17, 18]. In a spacetime diagram such as Fig. 1, these spikes reach out to a timelike infinity since the eternally inflating geodesics never intersect the reheating surface. It is known that the set of spikes has

---

1 Various equivalent conditions for the presence of eternal inflation were examined in more detail in Refs. 13, 14, and 5. Here I adopt the condition that $X(\phi)$ is nonzero for all $\phi$ in the inflating range.
a well-defined fractal dimension that can be computed in the Fokker-Planck approach [13].

For an infinite volume of $R$, the straightforward average $\langle R \rangle$ of a fluctuating quantity $Q(x)$ over a noncompact reheating surface $R$ is mathematically undefined. The average $\langle Q \rangle$ can be computed only after imposing a volume cutoff on the reheating surface in some way. A volume cutoff (or a “measure”) is, in effect, a physically motivated prescription that makes volume averages $\langle Q \rangle$ well-defined.

Volume cutoffs are usually implemented by restricting the consideration to a finite portion $V$ of the reheating domain $R$. After imposing a cutoff, one computes the “regularized” distribution $p(Q|V)$ of an observable $Q$ by gathering statistics over a large but finite volume $V$. The final probability distribution $p(Q)$ is then defined as

$$p(Q) \equiv \lim_{V \to \infty} p(Q|V),$$

provided that the limit exists. A cutoff prescription is a specific choice of the compact subset $V$ and of the way $V$ approaches infinity when the cutoff is removed. It has been found early on (e.g. [6, 19]) that $p(Q)$ depends sensitively on the choice of the cutoff. Without a natural mathematical definition of the measure, one judges a cutoff prescription viable if its predictions are not obviously pathological. Possible pathologies include the dependence on choice of spacetime coordinates [20, 21], the “youngness paradox” [22, 23, 24, 25], and the “Boltzmann brain” problem [26, 27, 28, 29].

The presently viable cutoff proposals fall into two rough classes that may be designated as “worldline-based” and “volume-based” measures; a more fine-grained classification of measure proposals can be found in Refs. [2, 18]. Here I propose a new volume-based measure called the “reheating-volume (RV) cutoff.”

II. REHEATING-VOLUME CUTOFF

In the RV cutoff, the reheating surface is not being restricted to an artificially chosen domain. Instead, one simply selects only those initial regions $S$ that, by rare chance, evolve into compact reheating surfaces $R$ having a finite, fixed volume $\text{Vol}(R) = V$. The ensemble $\mathcal{E}_V$ of such initial regions $S$ is a nonempty subset of the ensemble $\mathcal{E}$ of all initial regions $S$. The volume-weighted probability distribution $p(Q|\mathcal{E}_V)$ of a cosmological observable $Q$ in the ensemble $\mathcal{E}_V$ can be determined through ordinary sampling of the values of $Q$ over the finite volume $V$. The RV cutoff defines the probability distribution $p(Q)$ as the limit of $p(Q|\mathcal{E}_V)$ at $V \to \infty$, provided that the limit exists.

To develop an approach for practical computations in the RV cutoff, let us first consider the probability density $\rho(V; \phi_0) dV$ of having finite volume $\text{Vol}(R) \in [V, V + dV]$ of the reheating surface $R$ that results from a single $H$-region with initial value $\phi = \phi_0$. This distribution is normalized to the probability of the event $\text{Vol}(R) < \infty$, namely

$$\int_0^\infty \rho(V; \phi_0) dV = \text{Prob}(\text{Vol}(R) < \infty) = 1 - X(\phi_0). \quad (2)$$

The probability density $\rho(V; \phi_0)$ is nonzero since $X(\phi_0) < 1$. I call this $\rho(V; \phi_0)$ the “finitely produced reheated volume” (FPRV) distribution. This and related distributions constitute the mathematical basis of the RV cutoff.

Below I will use the Fokker-Planck (or “diffusion”) formalism to derive equations from which the FPRV distributions can be in principle computed for models of slow-roll inflation with a single scalar field. Generalizations of $\rho(V; \phi_0)$ to multiple-field or non-slow-roll models are straightforward since the Fokker-Planck formalism is already developed in those contexts (e.g. [31, 32]).

Let us now define the FPRV distribution for some cosmological observable $Q$ at reheating. Consider the probability density $\rho(V, V_0, \phi_0, Q_0)$, where $\phi_0$ and $Q_0$ are values of $\phi$ and $Q$ in the initial $H$-region, $V$ is the total reheating volume, and $V_0$ is the portion of the reheating volume where the observable $Q$ has a particular value $Q_R$. The distribution $\rho(V, V_0, \phi_0, Q_0)$ as a function of $V_0$ at fixed and large $V$ is sharply peaked around a mean value $\langle V_0 | V \rangle$ corresponding to the average volume of regions with $Q = Q_R$ within the total reheated volume $V$. Hence, although the full distribution $\rho(V, V_0, \phi_0, Q_0)$ could be in principle determined, it suffices to compute the mean value $\langle V_0 | V \rangle$. One can then expect that the limit

$$\rho(Q_R) \equiv \lim_{V \to \infty} \frac{\langle V_0 | V \rangle}{V} = \lim_{V \to \infty} \frac{\int_0^\infty \rho(V, V_0, \phi_0, Q_0) |V_0| \, dQ_0}{\text{Prob}(\text{Vol}(R) = V)} \quad (3)$$

exists and is independent of $\phi_0$ and $Q_0$. (Below I will justify this statement more formally.) The function $\rho(Q_R)$ is then interpreted as the mean fraction of the reheated volume where $Q = Q_R$. In this way, the RV cutoff yields the volume-weighted distribution for any cosmological observable $Q$ at reheating.

To obtain a more visual picture of the RV cutoff, consider a large number of initially identical $H$-regions having different evolution histories to the future. A small subset of these initial $H$-regions will generate finite reheating surfaces. An even smaller subset of $H$-regions will have the total reheated volume equal to a given value $V$. Conditioning on a finite value $V$ of the reheating volume, one obtains a well-defined statistical ensemble $E_V$ of initial $H$-regions. For large $V$, the ensemble $E_V$ can be pictured as a set of initial $H$-regions that happen to be located very close to some eternally inflating worldlines but do not actually contain any such worldlines (see Fig. 2). In this way, the ensemble $E_V$ samples the total reheating surface near the “spikes” where an infinite reheated 3-volume is generated from a finite initial 3-volume. It is precisely near these “spikes” that one would like to sample the distribution of observable quantities along the reheating surface. Therefore, one expects that the ensemble $E_V$
(in the limit of large \( V \)) provides a representative sample of the infinite reheating surface, despite the small probability of the event \( \text{Vol}(R) = V \). In this sense, the ensemble \( E_V \) at large \( V \) is designed to yield a controlled approximation to the infinite reheating surfaces \( R \).

The RV cutoff proposed here has several attractive features. By construction, the RV cutoff is coordinate-invariant; indeed, only the intrinsically defined 3-volume within the reheating surface \( R \) is used, rather than the 3-volume within a coordinate-dependent 3-surface. The results of the RV cutoff are also independent of initial conditions. This independence is demonstrated more formally below and can be understood heuristically as follows. The evolution of regions \( S \) conditioned on a large (but finite) value of \( \text{Vol}(R) \) is dominated by trajectories that spend a long time in the high-\( H \) inflationary regime and thereby gain a large volume. These trajectories forget about the conditions at their initial points and establish a certain equilibrium distribution of values of \( Q \) on the reheating surface. Hence, one can expect that the distribution of observables within the reheating domain \( R \) will be independent of the initial conditions in \( S \).

The “youngness paradox” arises in some volume-based prescriptions because \( H \)-regions with delayed reheating are rewarded by an exponentially large additional volume expansion. However, the RV measure groups together the \( H \)-regions that produce equal final reheated volume \( V \); a delay in reheating is not rewarded but suppressed by the small probability of a quantum fluctuation at the end of inflation. Therefore, most of these \( H \)-regions have “normal” slow-roll evolution before reheating. For this reason, the youngness paradox is absent in the RV measure. A more explicit calculation confirming this conclusion will be given in Sec. [IV].

### III. RV CUTOFF IN SLOW-ROLL INFLATION

As a first specific application, I implement the RV measure in a slow-roll inflationary model with a scalar inflaton \( \phi \) and the action

\[
S = \int \left[ -\frac{R}{16\pi G} + \frac{1}{2} (\partial_{\mu} \phi)^2 - V(\phi) \right] \sqrt{-g} d^4 x.
\]  

(4)

If the energy density is dominated by the potential energy \( V(\phi) \), the Hubble expansion rate \( H \) is approximately given by

\[
H \approx \sqrt{\frac{8\pi G}{3} V(\phi)}.
\]

(5)

In the stochastic approach to inflation,\(^2\) the semiclassical dynamics of the field \( \phi \) averaged over an \( H \)-region is a superposition of a deterministic motion with velocity

\[
\dot{\phi} = v(\phi) \equiv -\frac{1}{4\pi G} H, \phi
\]

(6)

and a random walk with root-mean-squared step size

\[
\sqrt{(\delta \phi)^2} = \frac{H(\phi)}{2\pi} \equiv \frac{2D(\phi)}{H(\phi)}, \quad D \equiv \frac{H^3}{8\pi^2},
\]

(7)

during time intervals \( \delta t = H^{-1} \). A convenient description of the evolution of the field at time scales \( \delta t \lesssim H^{-1} \) is

\[
\phi(t + \delta t) = \phi(t) + v(\phi) \delta t + \xi(t) \sqrt{2D(\phi)} \delta t,
\]

(8)

where \( \xi(t) \) is (approximately) a “white noise” variable,

\[
\langle \xi \rangle = 0, \quad \langle \xi(t) \xi(t') \rangle = \delta(t - t'),
\]

(9)

which is statistically independent for different \( H \)-regions.\(^{32}\) This stochastic process describes the evolution \( \phi(t) \) along a single comoving worldline. For convenience, we assume that inflation ends in a given \( H \)-region when \( \phi \) reaches a fixed value \( \phi = \phi_* \).

Consider an ensemble of initial \( H \)-regions \( S_1, S_2, ..., \) where the inflaton field \( \phi \) is homogeneous and has value \( \phi = \phi_0 \) within the inflationary regime. Following the spacetime evolution of the field \( \phi \) in each of the regions \( S_j \) along comoving geodesics, we arrive at reheating surfaces \( R_j \) where \( \phi = \phi_* \). Most of the surfaces \( R_j \) will have infinite 3-volume; however, some (perhaps small) subset of \( S_j \) will have finite \( R_j \). The \( \phi_0 \)-dependent probability, denoted \( X(\phi_0) \equiv 1 - X(\phi_0) \), of having a finite volume of \( R_j \) is a solution of the gauge-invariant equation \(^{15}\)

\[
D(\phi) \dot{X}_{\phi} + v(\phi) \dot{X} + 3H(\phi) \dot{X} \ln \dot{X} = 0,
\]

(10)

\(^{32}\) See Refs. \(^{35, 36} \) for early works and Refs. \(^{10, 11} \) for pedagogical reviews.
with the boundary conditions $\tilde{X}(\phi_0) = 1$ and $\tilde{X}(\phi_{1}) = 1$ at reheating and at Planck boundaries. While $\tilde{X}(\phi) \equiv 1$ is always a solution of Eq. (10), the existence of a non-trivial solution with $0 < \tilde{X}(\phi) < 1$ indicates the possibility of eternal inflation. The gauge invariance of Eq. (10) is manifest since a change of the time variable, $\tau(t) \equiv \int^t T(\phi) dt$, results in dividing the three coefficients $D, v, H$ by $T(\phi)$ [20], which leaves Eq. (10) unchanged.

The probability distribution $p(V; \phi_0)$ can be found by considering a suitable generating function. Let us define the generating function $g(z; \phi_0)$ by

$$g(z; \phi_0) \equiv \langle e^{-zV} \rangle_{V < \infty} = \int_{0}^{\infty} e^{-zV} p(V; \phi_0) dV. \tag{11}$$

(Note that the formal parameter $z$ has the dimension of inverse volume. The parameter $z$ can be made dimensionless by a trivial rescaling which we omit.) The function $g(z; \phi_0)$ is analytic in $z$ and has no singularities for $Re z \geq 0$. Moments of the distribution $p(V; \phi_0)$ are determined as usual through derivatives of $g(z; \phi_0)$ in $z$ at $z = 0$, while $p(V; \phi_0)$ itself can be reconstructed through the inverse Laplace transform of $g(z; \phi_0)$ in $z$.

The generating function $g(z; \phi)$ has the following multiplicative property: For two statistically independent $H$-regions that have initial values $\phi = \phi_1$ and $\phi = \phi_2$ respectively, the sum of the (finitely produced) reheating volumes $V_1 + V_2$ is distributed with the generating function

$$\langle e^{-z(V_1 + V_2)} \rangle = \langle e^{-zV_1} \rangle \langle e^{-zV_2} \rangle = g(z; \phi_1) g(z; \phi_2). \tag{12}$$

This multiplicative property is the only assumption in the derivation of Eq. (10) in Ref. [15]. Hence, $g(z; \phi)$ satisfies the same equation (we drop the subscript 0 in $\phi_0$).

$$Dg_{z, \phi} + vg_{z, \phi} + 3H g \ln g = 0. \tag{13}$$

The boundary condition at $\phi_*$ is $g(z; \phi_*) = e^{-zH^{-3}(\phi_*)}$ since an $H$-region with $\phi = \phi_*$ is already reheating and has volume $H^{-3}(\phi_*)$. The boundary condition at the Planck boundary $\phi_{1}$ (or other boundary where the effective field theory breaks down) is “absorbing,” i.e. we assume that regions with $\phi = \phi_{1}$ disappear and never generate any reheating volume: $g(z; \phi_{1}) = e^{z-0} = 1$. Note that the variable $z$ enters Eq. (13) as a parameter and only through the boundary conditions. At $z = 0$ the solution of Eq. (13) is $g(0; \phi) = \tilde{X}(\phi)$. Explicit approximate solutions of Eq. (13) can be obtained using the methods developed in Ref. [15].

Let us now consider FPRV distributions of cosmological parameters $Q$. The generating function for the distribution $p(V, Q; \phi, Q)$ discussed above is

$$\tilde{g}(z, q; \phi, Q) \equiv \int \int e^{-zV - qVQ} p(V, Q; \phi, Q) dV dQ. \tag{14}$$

The equation for $g(z, q; \phi, Q)$ is derived similarly to Eq. (13) and is of the form

$$D\tilde{g}_{z, \phi} + DQ_{z} \tilde{g}_{Q, Q} + v_{\phi} \tilde{g}_{z, \phi} + v_{Q} \tilde{g}_{Q, Q} + 3H \tilde{g} \ln \tilde{g} = 0, \tag{15}$$

where $D_{\phi}, D_{Q, \phi}, v_{\phi}, v_{Q}$ are the suitable kinetic coefficients representing the “diffusion” and the mean “drift velocity” of $\phi$ and $Q$. The boundary condition at $\phi = \phi_*$ is

$$\tilde{g}(z, q; \phi_*, Q) = \exp \left[ - (z + \tilde{g}(Q_{\phi}H)^{-3}(\phi_*)) \right], \tag{16}$$

where we use the delta-symbol defined by $\delta_{Q_{\phi}H} = 1$ if $Q$ belongs to a narrow interval $[Q_{R, Q} + dQ]$ and $\delta_{Q_{\phi}H} = 0$ otherwise.

To obtain the distribution $p(Q; \phi)$, we need to compute the average $\langle V_{Q_{\phi}} \rangle$ at fixed $V$. We define the auxiliary generating function

$$h(z; \phi, Q) \equiv \langle V_{Q_{\phi}} e^{-zV} \rangle_{V < \infty} = -\tilde{g}(z, q = 0; \phi, Q). \tag{17}$$

Note that $\tilde{g}(z, q = 0; \phi, Q) = g(z; \phi)$. The equation for $h(z; \phi, Q)$ then follows from Eq. (15).

$$D_{\phi} h_{z, \phi} + D_{Q} h_{z, Q} + v_{\phi} h_{z, \phi} + v_{Q} h_{z, Q} + 3H (\ln g + 1) h = 0. \tag{18}$$

This linear equation contains as a coefficient the function $g(z; \phi)$, which is the solution of Eq. (13). The boundary condition for Eq. (18) is

$$h(z; \phi_*, Q) = e^{-zH^{-3}(\phi_*)} H^{-3}(\phi_*) \delta(Q - Q_{R}). \tag{19}$$

Here we can use the ordinary $\delta$-function instead of the symbol $\delta_{Q_{\phi}H}$ because the $\delta$-function enters linearly into the boundary condition. An appropriate rescaling of the distribution $h$ by the factor $dQ$ is implied when we pass from $\delta_{Q_{\phi}H}$ to $\delta(Q - Q_{R})$.

Finally, the expectation value $\langle V_{Q_{\phi}} \rangle$ at a fixed $V$ and the limit (3) can be found using the inverse Laplace transform of $h(z; \phi, Q)$ in $z$.

The computation just outlined allows one, in principle, to obtain quantitative predictions from the RV measure. Further details and a more direct computational procedure will be given elsewhere [27]. Presently, let us analyze the limit $V \rightarrow \infty$ in qualitatively terms. The function $p(Q; V)$ is expressed as [cf. Eq. (3)]

$$p(Q; V) \equiv \frac{\langle V_{Q_{\phi}} \rangle_{V}}{V} = \frac{\int_{-\infty}^{\infty} e^{-zV} h(z; \phi, Q) dz}{\int_{-\infty}^{\infty} e^{-zV} g(z; \phi) dz}. \tag{20}$$

The asymptotic behavior of the inverse Laplace transform of $h(z; \phi, Q)$ at large $V$ is determined by the locations of the singularities of $h(z; \phi, Q)$ in the complex $z$ plane. The dominant asymptotics of the inverse Laplace transform are of the form $\propto \exp(z, V)$, where $z_*$ is the singularity with the smallest $|Re z_*|$. It can be shown that solutions of Eqs. (15) and (18) cannot diverge at finite values of $\phi$ or $Q$. Thus $g(z; \phi)$ and $h(z; \phi, Q)$ cannot have $\phi$- or $Q$-dependent singularities in $z$. Moreover, the function $g(z; \phi)$ cannot have pole-like singularities in $z$; the only possible singularities are branch points where the function $g(z; \phi)$ is finite but a derivative with respect to $z$ diverges. Furthermore, derivatives $\partial^2 h$ satisfy linear equations with coefficients depending on the derivatives $\partial^2 h^{-1}(z; \phi)$, which diverge at the singularities of $g$. 
In the terminology of Ref. [38], “terminal bubbles” are those with nonpositive value of $\Lambda$. No further transitions are possible from such bubbles because bubbles with $\Lambda < 0$ rapidly collapse while bubbles with $\Lambda = 0$ do not support tunneling instantons. The RV measure, as presently formulated, can be used directly for comparing the abundances of different bubble types.

In the terminology of Ref. [38], “terminal bubbles” are those with nonpositive value of $\Lambda$. No further transitions are possible from such bubbles because bubbles with $\Lambda < 0$ rapidly collapse while bubbles with $\Lambda = 0$ do not support tunneling instantons. The RV measure, as presently formulated, can be used directly for comparing the abundances of different bubble types.

I will merely sketch the derivation of Eq. (22), which is similar to the equations for generating functions used in the theory of branching processes (see e.g. the book [39] for a mathematically rigorous presentation). The generating function $g_j$ satisfies a multiplicative property analogous to Eq. (12). This property applies to the $n_k$ independent daughter $H$-regions created by expansion from an $H$-region of type $k$. Therefore, $g_j(z, q; k)$, which is the expectation value of $z^nq^{n'}$ in an initial $H$-region of type $k$, is equal to the product of $n_k$ expectation values of $z^nq^{n'}$ in the $n_k$ daughter $H$-regions (which may be of different types). The latter expectation value is given by the right-hand side of Eq. (22). This yields Eq. (22) after raising both sides to the power $1/n_k$.

If the generating functions $g_j$ are known, the distribution $p_j(n, n'; k)$ can be recovered by computing derivatives of $g_j(z, q; k)$ at $z = 0$ and $q = 0$. Further, the mean fraction of $H$-regions of type $j$ at fixed total number $n$ of terminal $H$-regions is found as

$$p(j|n) = \langle n' | n \rangle_n = \frac{\partial^n \delta q g_j(z = 0, q = 1; k)}{n \partial^n \delta q g_j(z = 0, q = 1; k)}.$$  (23)

Then the RV cutoff defines the probability of terminal type $j$, among all the possible terminal types, through the limit

$$p(j) = \lim_{n \to \infty} p(j|n) = \lim_{n \to \infty} \frac{\partial^n \delta q g_j(z = 0, q = 1; k)}{n \partial^n \delta q g_j(z = 0, q = 1; k)},$$  (24)

similarly to Eq. (3). Again one expects that the limit exists and is independent of the initial bubble type, as long as the initial bubble is not of terminal type.

As a specific example requiring fewer calculations, let us consider a toy model with only three bubble types. There is one de Sitter ($\Lambda > 0$) vacuum labeled $j = 3$ that can decay into two possible anti-de Sitter terminal bubbles labeled $j = 1$ and $j = 2$. The growth rate $n_k$ and the nucleation probabilities $\Gamma_{31}$ and $\Gamma_{32}$ are assumed known. To mimic interesting features of the landscape, let us also assume that there is a period of slow-roll inflation inside the bubbles 1 and 2, generating respectively $N_1$ and $N_2$ additional $e$-folds of inflationary expansion after nucleation. Hence, the model is determined by the parameters $n_k, \Gamma_{31, 32}, N_1, N_2$. For convenience we define $\Gamma_{33} = 1 - \Gamma_{31} - \Gamma_{32}$.

We now perform the calculations for the RV cutoff in this simple model. There are only two generating functions, $g_1(z, q; k)$ and $g_2(z, q; k)$, that need to be considered. The only meaningful initial value is $k = 3$ (i.e., the initial bubble is of de Sitter type) since the two other bubble types do not lead to eternal inflation. Hence, we will suppress the argument $k$ in $g_j(z, q; k)$. We also need the following system of nonlinear algebraic equations,

$$g_j^{1/n_k}(z, q; k) = \sum_{i=1}^{N_T} \Gamma_{k,i} z^i q^{i/j} + \sum_{i=N_T+1}^{N} \Gamma_{k,i} g_j(z, q; i),$$  (22)
to modify Eq. (22) to take into account the additional expansion inside the terminal bubbles. Let us denote the volume expansion factors by

$$Z_1 \equiv e^{3 N_1}, \quad Z_2 \equiv e^{3 N_2}. \quad (25)$$

Then the functions $g_1$ and $g_2$ are solutions of

$$g_1^{1/n_3} = \Gamma_3 z_1 q z_1 + \Gamma_{32} z_2 q z_2 + \Gamma_{33} g_1, \quad (26)$$

$$g_2^{1/n_3} = \Gamma_3 z_1 z_1 + \Gamma_{32} z_2 z_2 + \Gamma_{33} g_2. \quad (27)$$

An explicit solution of these equations is impossible for a general $n_3$ (barring the special cases $n_3 = 2, 3, 4$). Nevertheless, sufficient information about the limit (24) can be obtained by the following method. Introduce the auxiliary function $F(x)$ as the solution $F > 0$ of the algebraic equation

$$F^{1/n_3} = x + \Gamma_3 F, \quad (28)$$

choosing the branch connected to the value $F(0) = 0$. (It is straightforward to see that Eq. (28), has at most two positive solutions, and that $F(x)$ is always the smaller solution of the two.) The generating functions $g_1$ and $g_2$ are then expressed through $F(x)$ as

$$g_1(z, q) = F(\Gamma_3 z_1 z_1 + \Gamma_{32} z_2 z_2),$$

$$g_2(z, q) = F(\Gamma_3 z_1 z_1 + \Gamma_{32} z_2 z_2). \quad (30)$$

The limit (24) involves derivatives of these functions of very high order with respect to $z$. We note that the function $F(x)$ is analytic; thus the functions $g_1$ and $g_2$ are also analytic in $z$.

To evaluate the high-order derivatives, we need an elementary result from complex analysis. The asymptotic growth of high-order derivatives of an analytic function $f(z)$ is determined by the location of the singularities of $f(z)$ in the complex $z$ plane. For instance, we may expect an expansion around the singularity $z_s$ nearest to $z = 0$, such as

$$f(z) = c_0 + c_1 (z - z_s)^s + ..., \quad (31)$$

where $s \neq 0, 1, 2, ...$ is the power of the leading-order singularity, and the omitted terms are either higher powers of $z - z_s$ or singularities at points $z'_s$ located further away from $z = 0$. The singularity structure (31) yields the large-$n$ asymptotics with the leading term

$$\frac{d^n f}{dz^n}_{z=0} \approx c_1 (z - z_s)^s \frac{\Gamma(n - s)}{\Gamma(-s)} z_s^n. \quad (32)$$

This formula enables one to evaluate large-$n$ limits such as Eq. (24).

To proceed, we need to determine the location of the singularities of $F(x)$. Since $F(x)$ is obtained as an intersection of a curve $F^{1/n_3}$ and a straight line $x + \Gamma_3 F$, there will be a value $x = x_s$ where the straight line is tangent to the curve. At this value of $x$ the function $F(x)$ has a singularity of the type

$$F(x) = F(x_s) + F_1 \sqrt{x - x_s} + O(x - x_s), \quad (33)$$

where $F_1$ is a constant that can be easily determined; we omit further details that will not be required below. The value of $x_s$ is found from the condition that $dF/dx$ diverge at $x = x_s$. The value of $dF/dx$ at $x \neq x_s$ is found as the derivative of the inverse function, or by taking the derivative of Eq. (28),

$$\frac{dF}{dx} = \frac{1}{\frac{1}{n_3} F^{n_3 - 1} - \Gamma_3}. \quad (34)$$

This expression diverges at the values

$$F(x_s) = (n_3 \Gamma_3)^{-\frac{n_3 - 1}{n_3}}, \quad (35)$$

$$x_s = (n_3 - 1) \frac{n_3}{\Gamma_3} F(x_s) = \Gamma_3 \frac{n_3 - 1}{n_3}. \quad (36)$$

Note that $\Gamma_3 \approx 1$ and $n_3 \gg 1$, hence $x_s$ is a constant of order $1$.

Rather than compute the limit (23) directly, we will perform an easier computation of the ratio of the mean number of bubbles of types 1 and 2 at fixed total number $n$ of terminal bubbles,

$$\frac{n'(1)}{n'(2)} \equiv \frac{\partial q g_1}{\partial g_1} \bigg|_{q=1} = (\Gamma_3 z_1 z_1 + \Gamma_{32} z_2 z_2) \frac{n_3 - 1}{n_3}. \quad (37)$$

The derivatives $\partial q g_1$ and $\partial q g_2$ can be evaluated directly through Eqs. (29)–(30). For instance, we compute $\partial q g_1$ as

$$\frac{\partial q g_1(z, q)}{\partial q} \bigg|_{q=1} = F'(\Gamma_3 z_1 z_1 + \Gamma_{32} z_2 z_2) \frac{n_3 - 1}{n_3}. \quad (38)$$

It is clear that the functions $\partial q g_1$ and $\partial q g_2$ have a singularity at $z = z_s$, corresponding to the singularity $x = x_s$ of the function $F(x)$, where $z_s$ is found from the condition

$$\Gamma_3 z_1 z_1 + \Gamma_{32} z_2 z_2 = x_s. \quad (39)$$

Let us analyze this equation in order to estimate $z_s$. If the nucleation rates $\Gamma_{31}$ and $\Gamma_{32}$ differ by many orders of magnitude, we may expect that one of the terms in Eq. (39), say $\Gamma_{31} z_1 z_1$, dominates. Then the value $z_s$ is well approximated by

$$z_s \approx \left(\frac{x_s}{\Gamma_{31}}\right)^{1/Z_1}. \quad (40)$$

This approximation is justified if the first term in Eq. (39) indeed dominates, which is the case if

$$\left(\frac{\Gamma_{32}}{x_s}\right)^{1/Z_2} < \left(\frac{\Gamma_{31}}{x_s}\right)^{1/Z_1}. \quad (41)$$

If the reversed inequality holds, we can relabel the bubble types 1 and 2 and still use Eq. (40). If neither Eq. (41) nor the reversed inequality hold, the approximation (40) for $z_s$ can be used only as an order-of-magnitude estimate.
To be specific, let us assume that the condition (41) holds.

We can now compute the ratio (47) asymptotically for large \( n \) using Eqs. (33) and (32). The singularities of the functions \( g_1 \) and \( g_2 \) are directly due to the singularity of the function \( \sqrt{\Gamma} \). Then the dominant singularity structure of the function \( \Gamma_1 \) is found as

\[
\frac{\partial g_1(z, q)}{\partial q} \bigg|_{q=1} \approx \frac{\Gamma_{31} Z_1 z^{\frac{Z_1}{2}}}{2\sqrt{\frac{\partial}{\partial z}(\Gamma_{31} z^{Z_1} + \Gamma_{32} z^{Z_2})}} \bigg|_{z=z^*}. \tag{42}
\]

This fits Eq. (31), where \( f(z) = \partial_q g_1(z, q = 1) \) and \( s = -1/2 \). After canceling the common \( n \)-dependent factors, we obtain

\[
\frac{p(1)}{p(2)} = \lim_{n \to \infty} \left\langle \frac{n'(1)}{n'(2)} \right\rangle = \frac{\Gamma_{31} Z_1 z^{\frac{Z_1}{2}}}{\Gamma_{32} Z_2 z^{\frac{Z_2}{2}}} \bigg|_{z=z^*}. \tag{43}
\]

This is the final probability ratio found by applying the RV cutoff to a toy model landscape containing two terminal vacua. Substituting the approximation (40), which assumes the condition (41), we can simplify Eq. (43) to a more suggestive form

\[
\frac{p(1)}{p(2)} \approx \frac{\Gamma_{31} Z_1}{\Gamma_{32} Z_2} \left( \frac{\Gamma_{31}}{x_*} \right)^{1+Z_1/Z_2}. \tag{44}
\]

We can interpret this as the ratio of nucleation probabilities \( \Gamma_{31}/\Gamma_{32} \) times the ratio of volume expansion factors, \( Z_1/Z_2 \), times a certain “correction” factor. As we have seen, the correction factor is actually a complicated function of all the parameters of the landscape. The correction factor takes the simple form

\[
\left( \frac{\Gamma_{31}}{x_*} \right)^{1+Z_1/Z_2}, \tag{45}
\]

only if the condition (41) holds.

We note that the result (44) is similar to but does not exactly coincide with the results obtained in previously studied volume-based measures. For comparison, the volume-based measures proposed in Refs. [38] and [13] both yield

\[
\frac{p(1)}{p(2)} = \frac{\Gamma_{31} Z_1}{\Gamma_{32} Z_2}, \tag{46}
\]

which is readily interpreted as the ratio of nucleation probabilities enhanced by the ratio of volume factors. The “holographic” measure (40), which is not a volume-based measure, gives the ratio

\[
\frac{p(1)}{p(2)} = \frac{\Gamma_{31}}{\Gamma_{32}} \tag{47}
\]

do not depend on the number of e-folds after nucleation. While the discrepancy between volume-based and worldline-based measures is to be expected, the correction factor that distinguishes Eq. (44) from Eq. (46) is model-dependent and may be either negligible or significant depending on the particular model.

As a specific example, consider bubbles that nucleate with equal probability, \( \Gamma_{31} = \Gamma_{32} \ll 1 \), but have very different expansion factors, \( Z_1 \gg Z_2 \). Then the condition (41) holds and the “correction” factor is given by Eq. (45), which is an exponentially large quantity of order \( \Gamma_{31} \). Qualitatively this means that the RV measure rewards bubbles with a larger slow-roll expansion factor even more than previous volume-based measures.

On the other hand, if \( Z_1 = Z_2 \) but \( \Gamma_{31} \gg \Gamma_{32} \), the “correction” factor disappears and we recover the result found in the other measures.

To conclude, we note that the result (43) does not depend on the durations of time spent during slow-roll inflation inside the terminal bubbles, but only on the number of e-folds gained. This confirms that the RV measure does not suffer from the youngness paradox.

So far we were able to apply of the RV measure to the comparison of the abundances of terminal vacua. One expects that, with some more effort, the RV measure can be extended to arbitrary observables in landscape models. Further work will clarify the advantages and limitations of the new measure.

Acknowledgments

The author is grateful to Cedric Deffayet, Jaume Garriga, Takahiro Tanaka, and Alex Vilenkin for valuable discussions. Part of this work was completed on a visit to the Yukawa Institute of Theoretical Physics (University of Kyoto). The stay of the author at the YITP was supported by the Yukawa International Program for Quark-Hadron Sciences.

[1] J. Garriga and A. Vilenkin, Recycling universe, Phys. Rev. D57, 2230 (1998), astro-ph/9707292.
[2] R. Bousso and J. Polchinski, Quantization of four-form fluxes and dynamical neutralization of the cosmological constant, JHEP 06, 006 (2000), hep-th/0004134.
[3] L. Susskind, The anthropic landscape of string theory (2003), hep-th/0302219.
[4] M. R. Douglas, The statistics of string / M theory vacua, JHEP 05, 046 (2003), hep-th/0303194.
[5] A. Borde and A. Vilenkin, Eternal inflation and the initial singularity, Phys. Rev. Lett. 72, 3305 (1994), gr-qc/9312022.
[6] A. Vilenkin, Making predictions in eternally inflating universe, Phys. Rev. D52, 3365 (1995), gr-qc/9505031.
