EXPLICIT MULTIDIMENSIONAL INGHAM–BEURLING TYPE ESTIMATES

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Abstract. Recently a new proof was given for Beurling’s Ingham type theorem on one-dimensional nonharmonic Fourier series, providing explicit constants. We improve this result by applying a short elementary method instead of the previous complex analytical approach. Our proof equally works in the multidimensional case.

1. Introduction

Let \((\omega_k)_{k \in K}\) be a family of vectors in \(\mathbb{R}^N\) satisfying the gap condition
\[
\gamma := \inf_{k \neq n} |\omega_k - \omega_n| > 0.
\]
Let \(K = K_1 \cup \cdots \cup K_m\) be a finite partition of \(K\) and set
\[
\gamma_j := \inf \{|\omega_k - \omega_n| : k, n \in K_j \text{ and } k \neq n\}, \quad j = 1, \ldots, m.
\]
We denote by \(B_R\) the open ball of radius \(R\) in \(\mathbb{R}^N\), centered at the origin, and by \(\mu\) the first eigenvalue of \(-\Delta\) in the Sobolev space \(H_0^1(B_1)\). Finally, we set
\[
R_j := \frac{2\sqrt{\mu}}{\gamma_j} \quad \text{and} \quad R_0 := R_1 + \cdots + R_m.
\]

We prove the following theorem:

Theorem 1.1. There exist two positive constants \(c_1\) and \(c_2\), depending only on \(N\), \(\gamma\), \(m\) and \(\gamma_1, \ldots, \gamma_m\), such that if \(R_0 < R \leq 2R_0\), then
\[
(1.1) \quad c_1(R - R_0)^{5m-4+2N} \sum_{k \in K} |x_k|^2 \leq \int_{B_R} \left| \sum_{k \in K} x_k e^{i\omega_k \cdot t} \right|^2 dt \leq c_2 \sum_{k \in K} |x_k|^2
\]
for every square summable family \((x_k)_{k \in K}\) of complex numbers.

Remarks.

- The estimates remain valid for all balls of radius \(R\) by translation invariance. The explicit values of \(c_1\) and \(c_2\) can be computed easily from the proof below.
- The proof remains valid if in the multidimensional case we replace \(|\omega_k - \omega_n|\) in the definition of \(\gamma\) and \(\gamma_j\) by the \(L^p\) norm \(\|\omega_k - \omega_n\|_p\) for some \(1 \leq p \leq \infty\) and we replace \(\mu\) in the statement of the theorem by the first eigenvalue of \(-\Delta\) in the Sobolev space \(H_0^1(B_1^p)\) with
  \[
  \{x \in \mathbb{R}^N : \|x\|_p < 1\}.
  \]
• In the one-dimensional case we may assume that \((\omega_k)\) is a (doubly infinite) increasing sequence satisfying
\[
\gamma := \inf (\omega_{k+1} - \omega_k) > 0.
\]
Putting
\[
\gamma'_m := \inf \frac{\omega_{k+m} - \omega_k}{m}
\]
for some \(m\), we deduce from the theorem that the estimates (1.1) hold on the interval \((-R, R)\) if \(R_0 := 2\pi/\gamma'_m\) and \(R_0 < R \leq 2R_0\). This definition of \(\gamma'_m\) was introduced in [3], [4] where it was also shown that the Ingham type condition \(2\pi R/\gamma'_m\) is equivalent to Beurling’s original condition based on the Pólya upper density. Our paper thus also provides a short elementary proof of Beurling’s theorem.

• The case \(N = 1\) of our result is stronger than a theorem recently proved in [10] by employing deeper tools of complex analysis. Contrary to a claim in [10] we show that an elementary approach used in [8] leads to explicit constants, even in higher dimensions. In order to get better constants \(c_1\) and \(c_2\) (this is not necessary in order to get the exponent \(5m - 4 + 2N\)) we use a new proof of the multidimensional Ingham type theorem given in [1], [2].

2. Proof of the theorem

We introduce some notations. We fix an eigenfunction \(H\) of \(-\Delta\) in \(H^1_0(B_1)\) corresponding to the first eigenvalue \(\mu\). We may assume that \(H > 0\) in \(B_1\). The function \(H\) is even and we extend it by zero to the whole \(\mathbb{R}^N\). We denote by \(h\) the Fourier transform of \(H\) defined by
\[
h(t) := \int_{B_1} H(x)e^{-ix\cdot t} \, dx = \int_{B_1} H(x)\cos(x\cdot t) \, dx;
\]
we observe that
\[
\min_{|t| \leq \pi/2} |h(t)|^2 > 0.
\]
Putting \(H_s(x) := H(s^{-1}x)\) where \(s\) is a given positive number, a scaling argument shows that \(H_s\) is the first eigenfunction of \(-\Delta\) in \(H^1_0(B_s)\) with the corresponding eigenvalue \(\mu_s = s^{-2}\mu\). Furthermore, we have
\[
\|H_s\|_{L^2(B_s)}^2 = s^N\|H\|_{L^2(B_1)}^2
\]
for all \(1 \leq p < \infty\), and the Fourier transform \(h_s\) of \(H_s\) is given by the formula
\[
h_s(t) = s^N h(st).
\]
We set
\[
r := \frac{R - R_0}{2m}
\]
for brevity, so that \(0 < r \leq R_0/(2m)\) by the assumptions of the theorem.

The following two lemmas are due to Ingham [6] in one dimension and to Kahane [7] in several dimensions. We recall their simple proof given in [1], [2] in order to precise the nature of the constants appearing in the estimates. Here and in the sequel the letters \(\alpha, \alpha_i\) stand for diverse positive constants depending only on \(N, \gamma, m\) and \(\gamma_m\), and the value of \(\alpha\) may be different for different occurrences.

Lemma 2.1. We have
\[
\int_{B_R} \left| \sum_{k \in K} x_k e^{i\omega_k \cdot t} \right|^2 \, dt \leq \alpha_0 \sum_{k \in K} |x_k|^2.
\]
Proof. Setting $G := H_{\gamma / 2} * H_{\gamma / 2}$ and denoting by $g$ its Fourier transform we have $g = |\hat{h}_{\gamma / 2}|^2 \geq 0$. Writing
\[
x(t) := \sum_{k \in K} x_k e^{i\omega_k t}
\]
for brevity and applying the Fourier inversion formula we obtain that
\[
\left( \min_{B_{\pi/\gamma}} g \right) \int_{B_{\pi/\gamma}} |x(t)|^2 dt \leq \int_{\mathbb{R}^N} g(t)|x(t)|^2 dt = (2\pi)^N \sum_{k,n \in K} G(\omega_k - \omega_n) x_k \overline{x_n} = (2\pi)^N G(0) \sum_{k \in K} |x_k|^2
\]
because for $k \neq n$ the vector $\omega_k - \omega_n$ lies outside the support of $G$ by our gap assumption.

This proves the lemma for $B_{\pi/\gamma}$ instead of $B_R$ with
\[
\alpha'_0 = \frac{(2\pi)^N G(0)}{\min_{B_{\pi/\gamma}} g}
\]
in place of $\alpha_0$. Since $B_R$ may be covered by at most $(1 + R\gamma / \pi)^N$ translates of $B_{\pi/\gamma}$ and $R \leq 2R_0$, the lemma follows with
\[
\alpha_0 = \left( 1 + \frac{2R_0\gamma}{\pi} \right)^N \alpha'_0.
\]

**Lemma 2.2.** We have
\[
\alpha_j r \sum_{k \in K_j} |x_k|^2 \leq \int_{B_{R_j+r}} \left| \sum_{k \in K_j} x_k e^{i\omega_k t} \right|^2 dt, \quad j = 1, \ldots, m.
\]

**Proof.** Setting $G := [(R_j + r)^2 + \Delta](H_{\gamma_j / 2} * H_{\gamma_j / 2})$ and denoting its Fourier transform by $g$ we have
\[
g(t) = [(R_j + r)^2 - |t|^2] |\hat{h}_{\gamma_j / 2}(t)|^2.
\]
We have $g \leq 0$ outside $B_{R_j+r}$ and $g \leq \alpha$ in $B_{R_j+r}$, so that writing
\[
x_j(t) := \sum_{k \in K_j} x_k e^{i\omega_k t}
\]
and applying the Fourier inversion formula we obtain that
\[
(2\pi)^N G(0) \sum_{k \in K_j} |x_k|^2 = \int_{\mathbb{R}^N} g(t)|x_j(t)|^2 dt \leq \alpha \int_{B_{R_j+r}} |x_j(t)|^2 dt.
\]
It remains to show that $G(0) \geq \alpha r$. Using the variational characterization of the eigenvalue $\mu_{\gamma_j / 2} = R_j^2$ we have
\[
G(0) = \int_{B_{\gamma_j / 2}} (R_j + r)^2 H_{\gamma_j / 2}^2 - |\nabla H_{\gamma_j / 2}|^2 \, dx
\]
\[
= ((R_j + r)^2 - \mu_{\gamma_j / 2}) \int_{B_{\gamma_j / 2}} H_{\gamma_j / 2}^2 \, dx
\]
\[
= (2R_j + r) \int_{B_{\gamma_j / 2}} H_{\gamma_j / 2}^2 \, dx
\]
\[
\geq \alpha r.
\]

\qed
In order to improve the last result we need a technical lemma, which generalizes to $N > 1$ a well-known property of the function $\frac{\sin \omega}{\omega}$. We define

$$g(\omega) := \frac{1}{V_1} \int_{B_1} e^{i\omega \cdot s} \, ds = \frac{1}{V_1} \int_{B_1} \cos(\langle \omega \cdot s \rangle) \, ds, \quad \omega \in \mathbb{R}^N$$

where $V_1$ denotes the volume of $B_1$.

**Lemma 2.3.** We have

$$\inf_{|\omega| \geq t} 1 - g(\omega) \geq \alpha_{m+1} t^2$$

for all $0 \leq t \leq R_0 \gamma/(2m)$.

**Proof.** First we observe that $g(\omega)$ depends only on $|\omega|$. Using Taylor’s formula and assuming by rotation invariance that $\omega$ is parallel to the first coordinate axis, for $\omega \to 0$ we have

$$1 - g(\omega) = \frac{1}{V_1} \int_{B_1} \left( 1 - \cos(\langle \omega \cdot s \rangle) \right) \, ds$$

$$= \frac{1}{2V_1} \int_{B_1} |\omega \cdot s|^2 \, ds + O(|\omega|^4)$$

$$= \frac{|\omega|^2}{2V_1} \int_{B_1} s_1^2 \, ds + O(|\omega|^4)$$

$$= \frac{|\omega|^2}{2NV_1} \int_{B_1} |s|^2 \, ds + O(|\omega|^4)$$

$$= \frac{|\omega|^2}{2N^2 + 4} + O(|\omega|^4)$$

because denoting the surface area of $B_1$ by $\beta$ we have

$$V_1 = \int_0^1 \beta t^{N-1} \, dt = \frac{\beta}{N} \quad \text{and} \quad \int_{B_1} |s|^2 \, ds = \int_0^1 \beta t^{N+1} \, dt = \frac{\beta}{N + 2}.$$  

There exists therefore a number $0 < t_0 < R_0 \gamma/(2m)$ such that

$$1 - g(\omega) \geq \alpha_{m+1} |\omega|^2$$

for all $\omega$ satisfying $|\omega| \leq t_0$.

Next we show that $g(\omega) \to 0$ if $|\omega| \to \infty$. Since the characteristic function of $B_1$ may be approximated in $L^1(\mathbb{R}^N)$ by step functions, it suffices to show that

$$\int_I e^{i\omega \cdot s} \, ds \to 0 \quad \text{as} \quad |\omega| \to \infty$$

for each fixed $N$-dimensional interval $I = [-a, a]^N$. (A translation of $I$ does not change the absolute value of the integral.) Using the inequality

$$\max_j |\omega_j| \geq \frac{|\omega|}{\sqrt{N}}$$

we have

$$\left| \int_I e^{i\omega \cdot s} \, ds \right| = a^N \prod_{j=1}^N \sin \omega_j a \geq a^N \frac{\sqrt{N}}{|\omega| a} \to 0.$$

Since $g$ is continuous and $1 - g(\omega) > 0$ for all $\omega \neq 0$, it follows that

$$\inf_{|\omega| \geq t_0} 1 - g(\omega) > 0.$$

Therefore, by diminishing the constant $\alpha_{m+1}$ in (2.1) we may also assume that

$$1 - g(\omega) \geq \alpha_{m+1} \left( \frac{R_0 \gamma}{2m} \right)^2$$

whenever $|\omega| \geq t_0$. 


It follows from (2.1) and (2.2) that
\[ 1 - g(\omega) \geq \alpha_{m+1} t^2 \quad \text{whenever} \quad |t| \leq \min \left\{ |\omega|, \frac{R_0 \gamma}{2m} \right\}, \]
and this is equivalent to the statement of the lemma. \( \square \)

The following result is an adaptation of a method due to Haraux [5].

**Lemma 2.4.** Add an arbitrary element \( k' \in K \) to some \( K_j \) and denote the enlarged set by \( K_j' \). Then we have

\[
\alpha_j r \sum_{k \in K_j'} |x_k|^2 \leq \int_{B_{R_j+r}} \left| \sum_{k \in K_j'} x_k e^{i \omega_k t} \right|^2 dt.
\]

**Proof.** Writing
\[ x(t) := \sum_{k \in K_j'} x_k e^{i \omega_k t} \]
and introducing the function
\[ y(t) := x(t) - \frac{1}{V_1} \int_{B_1} e^{-ir \omega_k \cdot s} x(t + rs) \, ds \]
we have
\[ y(t) = \sum_{k \in K_j} (1 - g(r \omega_k - r \omega_k')) x_k e^{i \omega_k t} =: \sum_{k \in K_j} y_k e^{i \omega_k t}. \]
Since \( |r \omega_k - r \omega_k'| \geq r \gamma \) for all \( k \in K_j \), using Lemmas 2.2 and 2.3 we have
\[
\alpha_j \alpha_{m+1} \gamma^4 r^5 \sum_{k \in K_j} |x_k|^2 \leq \alpha_j r \sum_{k \in K_j} |y_k|^2 \leq \int_{B_{R_j+r}} |y(t)|^2 dt.
\]
Furthermore,
\[
|y(t)|^2 \leq 2|x(t)|^2 + 2 \frac{1}{V_1} \left| \int_{B_1} e^{-ir \omega_k \cdot s} x(t + rs) \, ds \right|^2
\]
\[
\leq 2|x(t)|^2 + 2 \frac{2}{V_1} \int_{B_1} |x(t + rs)|^2 \, ds
\]
\[
= 2|x(t)|^2 + 2 \frac{2}{V_r} \int_{t+B_r} |x(s)|^2 \, ds
\]
where \( V_r \) denotes the volume of \( B_r \), so that
\[
\int_{B_{R_j+r}} |y(t)|^2 dt \leq 2 \int_{B_{R_j+r}} |x(t)|^2 dt + \frac{2}{V_r} \int_{B_{R_j+r}} \int_{t+B_r} |x(s)|^2 \, ds \, dt
\]
\[
= 2 \int_{B_{R_j+r}} |x(t)|^2 dt + \frac{2}{V_r} \int_{B_{R_j+r}} \int_{(s+B_r) \cap B_{R_j+r}} |x(s)|^2 \, dt \, ds
\]
\[
\leq 2 \int_{B_{R_j+r}} |x(t)|^2 dt + 2 \int_{B_{R_j+2r}} |x(s)|^2 \, ds
\]
\[
\leq 4 \int_{B_{R_j+2r}} |x(s)|^2 \, ds.
\]
Combining this with (2.3) we conclude that
\[
\alpha_j \alpha_{m+1} \gamma^4 r^5 \sum_{k \in K_j} |x_k|^2 \leq 4 \int_{B_{R_j+2r}} |x(t)|^2 dt.
\]
It remains to estimate $x_k$. We have
\[
V_{R_j+2r}|x_k|^2 = \int_{B_{R_j+2r}} \left| x_k e^{i\omega_k t} \right|^2 dt
\]
\[
\leq 2 \int_{B_{R_j+2r}} |x(t)|^2 + \left| \sum_{k \in K_j} x_k e^{i\omega_k t} \right|^2 dt
\]
\[
\leq 2 \int_{B_{R_j+2r}} |x(t)|^2 dt + 2\alpha_0 \sum_{k \in K_j} |x_k|^2.
\]
Combining with (2.4) we get finally that
\[
\alpha_j^2\gamma_j^4 r^5 \sum_{k \in K_j'} |x_k|^2 \leq \left( 4 + 2\frac{\alpha_j^2\gamma_j^4 r^5}{V_{R_j+2r}} + \frac{8\alpha_0}{V_{R_j+2r}} \right) \int_{B_{R_j+2r}} |x(t)|^2 dt
\]
\[
\leq \alpha \int_{B_{R_j+2r}} |x(t)|^2 dt. \quad \Box
\]

We recall the following classical result on biorthogonal sequences.

**Lemma 2.5.** Let $(f_k)_{k \in K}$ be a family of vectors in a Hilbert space $H$. Assume that
\[
c'_1 \sum_{k \in K} |x_k|^2 \leq \left\| \sum_{k \in K} x_k f_k \right\|^2 \leq c'_2 \sum_{k \in K} |x_k|^2
\]
for all finite linear combinations of these vectors, with two positive constants $c'_1$ and $c'_2$. Then there exists another family $(y_k)_{k \in K}$ of vectors in $H$ such that
\[
(y_k, f_n) = \delta_{kn}
\]
for all $k, n \in K$. Moreover, both families are bounded in $H$:
\[
\|f_k\| \leq \sqrt{c'_2} \quad \text{and} \quad \|y_k\| \leq 1/\sqrt{c'_1}
\]
for all $k$.

**Proof.** The formula
\[
y_n := \|f_n - w_n\|^{-2} (f_n - w_n)
\]
where $w_n$ denotes the orthogonal projection of $f_n$ onto the closed linear subspace of $H$ spanned by the remaining vectors $f_k$, defines a family with the required properties. \[\Box\]

The proof of the theorem can now be completed by following Kahane’s method [7]. Applying Lemmas 2.2 and 2.5 we define for each $k \in K_j$ a function $\psi_{k,j} \in L^2(B_{R_j+r})$ satisfying $\psi_{k,j}(\omega_k) = 1$, $\psi_{k,j}(\omega_n) = 0$ for all $n \in K_j \setminus \{k\}$, and (using Hölder’s inequality)
\[
\|\psi_{k,j}\|_{L^2(B_{R_j+r})}^2 \leq \frac{V_{R_j+r}}{\alpha_j r} \leq \frac{\alpha}{r}.
\]
Analogously, using Lemmas 2.4 and 2.5 we define for each $k \in K \setminus K_j$ a function $\psi_{k,j} \in L^2(B_{R_j+2r})$ such that $\psi_{k,j}(\omega_k) = 1$, $\psi_{k,j}(\omega_n) = 0$ for all $n \in K_j$, and
\[
\|\psi_{k,j}\|_{L^1(B_{R_j+2r})}^2 \leq \frac{V_{R_j+2r}}{\alpha_j r^5} \leq \frac{\alpha}{r^5}.
\]
Setting
\[
\rho_k := \psi_{k,1} \cdots \psi_{k,m}
\]
we get $\rho_k \in L^2(B_{R-r})$ satisfying $\rho_k(\omega_n) = \delta_{kn}$ for all $n, k \in K$, and
\[
\|\rho_k\|_{L^2(\mathbb{R}^N)}^2 \leq \alpha \leq \frac{\alpha}{r^{m-4}}, \quad (2.5)
\]
Now we introduce the functions
\[ G := H_{r/2} * H_{r/2}, \quad g := |h_{r/2}|^2, \quad G_k(t) := G(t)e^{i\omega_k t}, \]
and for every
\[ x(t) = \sum_{k \in K} x_k e^{i\omega_k t} \]
we define
\[ y(t) = (2\pi)^{-N} \sum_{k \in K} x_k (\rho_k * G_k)(t). \]
A straightforward computation and Plancherel’s formula show that
\[
(y, x)_{L^2(B_r)} = g(0) \sum_{k \in K} |x_k|^2 = h(0)^2 \left( \frac{r}{2} \right)^{2N} \sum_{k \in K} |x_k|^2
\]
and
\[ \|y\|_{L^2(B_r)}^2 = (2\pi)^{-N} \int_{\mathbb{R}^N} \left| \sum_{k \in K} x_k \hat{\rho}_k(\omega)g_k(\omega) \right|^2 d\omega \]
where \( g_k \) denotes the Fourier transform of \( G_k \). Since \( g_k \geq 0 \), using (2.5) it follows that
\[
r^{5m-4}\|y\|_{L^2(B_r)}^2 \leq \alpha \int_{\mathbb{R}^N} \left| \sum_{k \in K} x_k g_k(\omega) \right|^2 d\omega.
\]
Using Plancherel’s equality again, this is equivalent to the inequality
\[
r^{5m-4}\|y\|_{L^2(B_r)}^2 \leq \alpha \int_{\mathbb{R}^N} |G(t)\sum_{k \in K} |x_k|e^{i\omega_k t}|^2 dt.
\]
Since \( G(t) \) vanishes outside \( B_r \) and
\[ \|G\|_{L^\infty(\mathbb{R}^N)} \leq \|H_{r/2}\|_{L^2(B_{r/2})}^2 = \alpha r^N, \]
it follows that
\[
r^{5m-4}\|y\|_{L^2(B_r)}^2 \leq \alpha r^{2N} \int_{B_r} \left| \sum_{k \in K} x_k e^{i\omega_k t} \right|^2 dt.
\]
Applying Lemma 2.1 we conclude that
\[
r^{5m-4}\|y\|_{L^2(B_r)}^2 \leq \alpha r^{2N} \sum_{k \in K} |x_k|^2.
\]
Combining (2.6) and (2.7) with the Cauchy–Schwarz inequality
\[
\left| (y, x)_{L^2(B_r)} \right|^2 \leq \|y\|_{L^2(B_r)}^2 \|x\|_{L^2(B_r)}^2
\]
we obtain that
\[
r^{4N} \left( \sum_{k \in K} |x_k|^2 \right)^2 \leq \alpha r^{2N+4-5m} \left( \sum_{k \in K} |x_k|^2 \right) \|x\|_{L^2(B_r)}^2
\]
and hence
\[
r^{5m-4+2N} \sum_{k \in K} |x_k|^2 \leq \alpha \|x\|_{L^2(B_r)}^2
\]
as stated.
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