Theoretical analyses of mechanical performance of higher-order vertex-based hierarchical square cell structure subjected to compression

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Abstract. Hierarchical structure has gained great popularity in recent years, due to its significant promotion on structural strength and stiffness. In this study, the concept of high-order vertex-based hierarchy structure (VHS) was incorporated based on the new-rising multi-cell square tubes by replacing every vertex with scaled itself, layer by layer. The theoretical solution of VHS structure subjected to static compression has been conducted, based on the classical Super Folding Element. Mean crushing force (MCF, P) and energy absorption E have been calculated theoretically. The innovative finding lies that the relationship of the mean load between case lever i to lever i+1 has been determined. The relationship of the mean load between case lever i to lever i+1 mainly related to the cell number and length scale. All these achievements shield a light in aid to design novel thin-walled and light-weight energy absorption device.

1. Introduction
Lightweight materials and high specific absorption energy restrict the development of aircraft, trains, cars, ships and other transportation tools. Kinds of innovative structures were put forward in near decades. Among of them, hierarchical structure is a new hot and frontier engineering solid structure, which has been considered as an excellent potential advanced structure due to its possible wide use in kinds of industrial engineering applications. It is an excellent biological structure selected by nature for millions of years, mainly from wood, diatom, bone, teeth, and so on [1]. At present, many man-made hierarchies, as polymers, composites and sandwich cores, were developed in recently years.

Given the significance of hierarchy, crushing behaviors of these hierarchical structures are subjected to the type and order of hierarchy [2], indicating that an optimal hierarchical level could be defined. Therefore, extensive efforts were devoted to tailoring the levels of hierarchies, so as to obtain their outstanding energy absorption ability. For instance, Zhang et al. [3] explored two further strategies to improve out-of-plane crashworthiness by altering the material distribution, suggesting that further
improvement can be achieved by optimum designs of the fractal geometries. Oftadeh et al. [4] discovered that anisotropic hierarchical honeycombs with the first to fourth order can be 2.0-8.0 times stiffer than the regular honeycombs, which indicates that anisotropic hierarchical honeycombs have the larger plastic collapse strength comparing to the regular hierarchical honeycombs of the same order at certain oblique wall angles. Najafi et al. [5] found that different hierarchical orders possess diverse properties, deriving a problem that whether the increasing structural sophistication can enhance crashworthiness. Christelle and Ryan S. Elliott [6] explored the importance of nonlinear material properties in the design of hierarchical honeycomb materials, which means that the consideration of nonlinear properties and their implications in the design and optimization of hierarchical materials is significant. The above results showed that the energy absorption ability of the hierarchical honeycombs is higher than that of the regular structures and can be the ideal lightweight structure for designing crashworthy structures [7,8].

Vertex-based conception, among all the studies of hierarchical structures, is totally innovative. The concept of vertex-based hierarchy is incorporated into multi-cell square tubes by replacing every vertex with miniature cell itself [9]. The adjustment of the structure layer series can make the structural quality lighter and stronger. General hierarchical structure for crashworthiness design is mainly based on multi-cell tubes and hexagonal honeycomb structures. This notion has been largely applied in mathematics and computers. For example, Cantin et al. [10] conducted and analyzed vertex-based schemes on polyhedral meshes to approximate advection–reaction equations. Only some principle studies concentrating on the hexagonal honeycomb structures. Taking an example, Haghpanah et al. [11] investigated the in-plane collapse strength of hierarchical honeycombs under a general loading condition. These structures have offered up to almost 2.0 times strength over the regular honeycombs with the same density. Sun et al. [12] constructed the novel structure by replacing the vertex of a regular hexagonal network with a smaller hexagon and repeating this process. The comparisons of mechanical performance of different hierarchical orders have been made, suggesting that there are several mechanical merits of increasing structural sophistication [13,14].

In recent years, more and more scholars have begun to apply the design of the hierarchical structure to the thin wall structure, which makes the mechanical properties of the newly designed thin-walled structure greatly improved. In Ref. [12], it is pointed out that adding the hierarchical structure to the honeycomb will further improve its good-density specific elastic and the SEA properties. A new self-similar hierarchical hexagonal column (HHC) is designed to improve the crashworthiness of structures [15-18]. By replacing each vertex of a regular hexagonal network with a smaller hexagonal topology and constructing a hierarchical honeycomb, and then repeating this process, Sun constructs a fractal honeycomb with a high order structure level and studies its breaking mechanism [12]. Recent studies have shown that the MCF and SEA of thin-walled structures with hierarchical structure are significantly improved.

The research model of this paper selects a novel hierarchical structure, vertex-based hierarchical structure (VHS), which combines the VHS with the multi-cell thin-walled square tube (the thin-walled square tube with the VHS, VHST, Fig.1), and based on the Super Folding Element theory, the excellent mechanical properties of the VHST are revealed from the theoretical point of view. With the aim to enhance crashing behavior and energy absorption capacity without sacrificing the weight, this novel structure should not only maintain the perfect mechanical performance as square-like structure, but also obtain a longer plateau stage. The former constructive works around the multi-cell vertex confirms that the vertex-based multi-cell square tubes remarkably updates the mechanical performance but it is still unclear for different cell patterns.

This study, differing from the former literatures, focuses on the investigations of the general formulas for hierarchical structure, especially emphasizing on high order structures, in terms of the folding half-wave length, mean force and energy absorption. Majority of these works addressed the explorations of mechanical merits of this novel structure performing in the axial compression.
2. Geometric characterization

For a given HSV structure who is degenerated into the multi-cell tubes with $k \times k$ cells, Fig. 2 shows the hierarchical relationship in detail, in which $h_0$ and $h_1$ are the smallest cell edge length for $Z_i$ and $Z_{i+1}$, respectively. Likewise, $t_i$ and $t_{i+1}$ are the thickness. $\lambda$ is defined as the ratio of $h_{i+1}$ to $h_i$, which is used to characterize the geometric scale ($0 < \lambda < 1$), accordingly.

By using the principle of cut and fill method, the total quality of the $n$ level can be calculated from the relationship

$$M_n = \sum_{i=0}^{n} M_i - \sum_{i=0}^{n-1} \Delta M_i = (k+1)^2 t_i h_0 \left[ \sum_{i=2}^{n} \lambda_i \left( 4^i - 4^{i-1} \right) + 4 \lambda_1 \right] + 2k(k+1)t_0 h_0 (1-\lambda_1)$$

If $\lambda$ obeys a functional relationship:

$$\lambda_n = \frac{\lambda_1}{2^{n-1}} (0 < \lambda < 1)$$

Considering the weight of the structure keeps the same in the scaling process, then the equation can be constructed as

$$2k(k+1)t_0 h_0 = (k+1)^2 t_i h_0 \lambda (3 \cdot 2^n - 2) + 2k(k+1)t_0 h_0 (1-\lambda)$$

Hence, $t_i$ can be calculated as

$$t_i = \frac{1}{1.5 \cdot 2^n \left( 1 + \frac{1}{K} \right) \lambda + 1 - (2 + \frac{1}{K}) \lambda} t_0 \ (i \geq 0)$$

3. Analytical solution

Directly predicting the crush response of multi-cell column is a huge challenge. The essential idea to overcome this is trying to divide the section into a number of representative constituent elements, and then successfully determine the energy absorption of each individual element to finally sum up the
contribution. In general, for multi-cell square tubes, their sub-sections can be categorized as the corner part (I) and T-shape part (II), as shown in Fig.3(a) ~ (b) [9,19,20]. Assuming a given multi-cell structure, it is consisting of x corner part, y T-shape part. The relationships between x and y about k are as follows:

![Diagram](image)

Fig.3 High-order hierarchical structure of (a) cell and (b) energy dissipation patterns.

In Fig.3(a), there are four kinds of angle elements involved. They are 1 represents corner element, 2, 3, and 4 represent T-shape element. According to the evolution law, the corner element only exists in the complete square structure, and the T-shape element is equal to the number of levels. The T-shape element in the smallest square cell is defined as $T_n$, the second being $T_{n-1}$, and so on. The law of evolution shows that the number of square structures with square tubes at the $N$ level is $4^N(k+1)^2$, and the number of each element in the n levels’ VHST can be summed up (see Table 1), which is to explain that the law in the table is only satisfied when n is more than 2.

Table 1 Corner and T-shape number for a given high-order hierarchical structure.

| Corner | $T_n$ | $T_{n-1}$ | $T_2$ | $T_1$ |
|--------|-------|-----------|-------|-------|
| $4^n(k+1)^2$ | $0.5 \cdot 4^n(k+1)^2$ | $0.5 \cdot 4^{n-1}(k+1)^2$ | $8(k+1)^2$ | $4k(k+1)$ |

These four types of energy would be dissipated by three members as corner, T-shape and crisscross sections. Based on the former studies and the reference [9,19], the energy of each part can be added, corresponding results can be expressed as below:

$$E_c = 9.28M_0 \frac{Hb}{t} + 2\pi M_0 c_1 + 4.44M_0 \frac{H^2}{b}$$  \hspace{1cm} (6)

$$E_T = 9.28M_0 \frac{Hb}{t} + q\pi M_0 c_2 + 7.21M_0 \frac{H^2}{b}$$  \hspace{1cm} (7)

$$E_{int} = mE_c + \sum_{i=1}^{n} l_i E_i$$  \hspace{1cm} (8)

The m in the formula represents the number of corner element; $l_i$ is the number of T-shape element, where $i$ represents the hierarchy. The external force $W_{ext}$ is a folding unit of the tube structure.

$$W_{ext} = 2P''H = E_{int}$$  \hspace{1cm} (9)

$$L_n = \frac{1}{4^n} \left[ 3\lambda \cdot 2^n - 2\lambda + \frac{2k(1-\lambda)}{1+k} \right]$$  \hspace{1cm} (10)

$$H = 0.284t_0 2^{\frac{n}{3}} \cdot \sqrt{\frac{L_n}{\lambda}} \cdot \frac{h_0}{t_0}$$  \hspace{1cm} (11)

$$P'' = 14.513(k+1)^2 \sigma_0 \frac{a}{2^{\frac{n}{3}}} \cdot \frac{h_0}{t_0}$$  \hspace{1cm} (12)
4. Conclusions

Based on the theoretical deduction of the mechanical properties of thin-walled square tubes with a vertex structure of N at the front, some obvious conclusions can be drawn.

The Folded half wave length and the MCF of the n levels’ VHST are: \[ P_n = 14.513(k+1)^2 \sigma_0 t_0^2 \frac{L}{h_0} \frac{L}{h_0} \frac{L}{h_0} \]

When the mass of the same section increases with the increase of the number of layers \( n \), the MCF \( P \) of the VHST increases exponentially, and the increase of the number of levels of \( N \) can rapidly enhance the impact resistance of the structure.

To sum up the above work, it can be seen that when compared with the original the multi-cell thin-walled square tube, the mechanical properties of the VHST are greatly improved and the impact resistance of the structure has been improved effectively. But there is still a lot of work to be studied at the moment: The \( P \) in the format 1.1.23 increases exponentially with the level of \( n \). In fact, the impact resistance of the structure is upper limit. The growth of \( P \) inevitably converges to a certain value with the growth of \( N \), which needs to be studied urgently; it is still necessary to continue to study how to select the height ratio \( \lambda \) (not the specific rule selected in the paper) to make the VHST achieve the best impact resistance; what the relationship between dimensionless \( \lambda \) and height ratio \( \lambda \) is, and how to make the value of \( \lambda \) more conducive to enhance the SEA need further research.

It is believed that with the development of 3D printing technology, the conclusions drawn in this paper will be applied in the research of VHST’s impact resistance and energy absorption characteristics.

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