LINEAR PRECISION FOR PARAMETRIC PATCHES

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Abstract. We study linear precision for multi-sided parametric patches of any dimension, showing that every proper parametric patch has a unique reparametrization which has linear precision and giving a geometric criterion for when this reparametrization is rational. For toric patches, this geometric criterion is equivalent to a certain toric differential defining a birational map. While the reparametrization of a general toric patch having linear precision is not necessarily a rational function, we show that it is computed by iterative proportional fitting, a numerical algorithm from statistics.

Introduction

Bézier curves and surfaces are the fundamental units in geometric modeling. There are two basic shapes for surfaces—triangular Bézier patches and rectangular tensor product patches. Multi-sided patches are needed for some applications, and there are several control point schemes for $C^\infty$ multi-sided patches. These include the S-patches of Loop and DeRose [13], Warren’s hexagon [19], Karčiauskas’s M-patches [10], and the multi-sided toric Bézier patches of Krasauskas [12]. (Relationships between these and other patches are discussed in [11].) Parametric patches are general control point schemes for $C^\infty$ patches whose shape is a polygon or polytope. They include the patch schemes just mentioned, as well as barycentric coordinates for polygons and polytopes [18, 20].

The success and widespread adoption of Bézier and tensor-product patches is due in part to their possessing many useful mathematical properties. Some, such as affine invariance and the convex hull property, are built into their definitions and also hold for the more general parametric patches. Other properties, such as de Casteljau’s algorithm for computing Bézier patches, come from the specific form of their Bernstein polynomial blending functions. Linear precision is the ability of a parametric patch to replicate linear functions. When the blending functions of a parametric patch correspond to the vertices of a polytope, they give barycentric coordinates precisely when the patch has linear precision. In this way, blending functions for a parametric patch having linear precision are barycentric coordinates for general control point schemes.

A (parametric) patch is a collection of non-negative blending functions indexed by a finite set $\mathcal{A}$ of points in $\mathbb{R}^d$, where the common domain of the blending functions is the convex hull $\Delta$ of $\mathcal{A}$. A collection of control points in $\mathbb{R}^\ell$ indexed by $\mathcal{A}$ is used to define a

\begin{itemize}
\item 2000 Mathematics Subject Classification. 65D17, 14M25.
\item Key words and phrases. tensor product Bézier surfaces; triangular Bézier surface patches; Barycentric coordinates; Iterative proportional fitting.
\item Work of Sottile supported by NSF CAREER grant DMS-0538734, by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation, and by Peter Gritzmann of the Technische Universität München.
\end{itemize}
map from \( \Delta \) to \( \mathbb{R}^\ell \). The blending functions determine the internal structure of this map and the basic shape, \( \Delta \), of its image, while the control points determine how the image lies in \( \mathbb{R}^\ell \). Choosing the control points to be the points of \( \mathcal{A} \) gives the tautological map, and the patch has linear precision when this tautological map is the identity on \( \Delta \). We show that every patch has a unique reparametrization (composing the blending functions with a homeomorphism of \( \Delta \)) having linear precision (Theorem 1.11). This generalizes and corrects Theorem 8.5 of [16], which was for toric patches. There, the linear projection associated to the tautological map was incorrectly identified as the moment map on a toric variety. See Remark 2.7 for an elaboration.

The blending functions of a patch may be arbitrary non-negative \( C^\infty \) functions. A patch is rational if it has a reparametrization having polynomial blending functions. For a rational patch, a map given by control points corresponds to a linear projection of a projective algebraic variety associated to its blending functions. Its unique reparametrization having linear precision has rational blending functions if and only if a certain canonical map defined on this variety is a birational isomorphism (Theorem 2.8), which implies that this variety has a maximally degenerate position with respect to a canonical linear subspace given by the set \( \mathcal{A} \) (Theorem 2.9).

We apply this analysis to Krasauskas’s toric patches, which are rational. The shape of a toric patch is a pair \( (\mathcal{A}, w) \) where \( \mathcal{A} \) is a set of integer points in \( \mathbb{Z}^d \) and \( w \) is a collection of positive real numbers indexed by \( \mathcal{A} \). We show that iterative proportional fitting [4], a simple numerical algorithm from statistics, computes the blending functions which have linear precision. This algorithm was suggested to us by Bernd Sturmfels. It may form a basis for algorithms to manipulate these patches.

Krasauskas [13, Problem 3] asked whether any toric patches (besides the classical Bézier simploids [5]) admit a rational reparametrization having linear precision. A toric patch of shape \( (\mathcal{A}, w) \) corresponds to a homogeneous polynomial \( f = f_{\mathcal{A},w} \) whose dehomogenization is the sum of monomials with exponents \( \mathcal{A} \) and coefficients \( w \). The toric patch admits a rational reparametrization having linear precision if and only if the toric differential

\[
D_{\text{toric}} f := \left[ x_0 \frac{\partial}{\partial x_0} f : x_1 \frac{\partial}{\partial x_1} f : \cdots : x_d \frac{\partial}{\partial x_d} f \right] : \mathbb{CP}^d \longrightarrow \mathbb{CP}^d
\]

defines a birational isomorphism (Corollary 3.13). This analysis of linear precision is used in [9] to restrict the possibilities of which toric surface patches can have linear precision. We conjecture that the Bézier simploids are the only toric patches of any dimension with rational blending functions having linear precision.

In Krasauskas’s question about linear precision, he allowed the points \( \mathcal{A} \) indexing the blending functions of toric patches to move within their (fixed) convex hull, keeping the same blending functions (See Example 3.18). Our analysis of linear precision for general parametric patches will help to address that version of his question.

In Section 1, we define parametric patches and show that every patch has a unique reparametrization that has linear precision. In Section 2 we show that if a rational patch has linear precision, then an algebraic variety we obtain from the blending functions has an exceptional position with respect to a certain linear space. We define toric patches
in Section 3, and in Section 4 we explain how iterative proportional fitting computes the
blending functions for toric patches which have linear precision.

1. Linear precision for parametric patches

We interpret the standard definition of a mapping via control points and blending func-
tions (see for example [11, §2]) in a general form convenient for our discussion. All functions
here are smooth ($C^\infty$) where defined and real-valued unless otherwise stated. Let $\mathbb{R}_>$ be
the set of strictly positive real numbers and $\mathbb{R}_{\geq}$ the set of non-negative real numbers.

Let $\mathcal{A}$ be a finite set of points in $\mathbb{R}^d$, which we shall use as geometrically meaningful
indices. A control point scheme for parametric patches, or (parametric) patch, is a collection
$\beta = \{ \beta_a \mid a \in \mathcal{A} \}$ of non-negative functions, called blending functions. The common
domain of the blending functions is the convex hull $\Delta$ of $\mathcal{A}$, which we call the domain polytope.
We will always assume that $\Delta$ is full dimensional in that it has dimension $d$. We
assume that the blending functions do not vanish simultaneously at any point of $\Delta$. That
is, the blending functions have no base points in their domain.

A set $\{ b_a \mid a \in \mathcal{A} \} \subset \mathbb{R}^\ell$ of control points gives a map $F: \Delta \to \mathbb{R}^\ell$ defined by

$$F(x) := \frac{\sum_{a \in \mathcal{A}} \beta_a(x) b_a}{\sum_{a \in \mathcal{A}} \beta_a(x)}.$$ 

The denominator in (1.1) is positive on $\Delta$ and so the map $F$ is well-defined.

**Remark 1.2.** Positive weights, or shape parameters, $\{ w_a \in \mathbb{R}_> \mid a \in \mathcal{A} \}$ scaling the
blending functions are typically included in the definition of the map $F$. We have instead
absorbed them into our notion of blending functions. See Remarks 1.7 and 3.3.

The blending functions $\{ \beta_a \mid a \in \mathcal{A} \}$ are normalized if they form a partition of unity,

$$\sum_{a \in \mathcal{A}} \beta_a(x) = 1.$$ 

As the denominator in (1.1) is (strictly) positive on $\Delta$, we may divide each blending function
by this denominator to obtain normalized blending functions.

Redefine $\beta_a := \frac{\beta_a}{\sum_{a \in \mathcal{A}} \beta_a}$. Then $\beta_a(x) \geq 0$ and $\sum_{a \in \mathcal{A}} \beta_a(x) = 1$, for $x \in \Delta$. For normalized blending functions, the formula (1.1) becomes

$$F(x) = \sum_{a \in \mathcal{A}} \beta_a(x) b_a.$$ 

We deduce two fundamental properties of parametric patches.

- **Convex hull property.** For $x \in \Delta$, $F(x)$ is a convex combination of the control
points. Thus $F(\Delta)$ lies in the convex hull of the control points.
- **Affine invariance.** If $\Lambda$ is an affine function on $\mathbb{R}^\ell$, then $\Lambda(F(x))$ is the map with
the same blending functions, but with control points $\{ \Lambda(b_a) \mid a \in \mathcal{A} \}$. 

A patch is non-degenerate if its blending functions are linearly independent. This implies in particular that $F(\Delta)$ cannot collapse into a point unless all the control points are equal.

We make our key definition.

**Definition 1.4.** A patch $\{\beta_a \mid a \in A\}$ has linear precision if for every affine function $\Lambda(x)$ defined on $\mathbb{R}^d$,

$$\Lambda(x) = \sum_{a \in A} \beta_a(x) \Lambda(a).$$

That is, if the patch can replicate affine functions. This notion depends strongly on the positions of the points in $A$.

The tautological map $\tau$ of a given patch is the map (1.3) when the control points are taken to be the corresponding points of $A$,

$$\tau(x) := \sum_{a \in A} \beta_a(x) a.$$

By the convex hull property, $\tau(\Delta) \subset \Delta$. By affine invariance, the patch has linear precision if and only if $\tau$ is the identity function, $\tau(x) = x$. We record this fact.

**Proposition 1.6.** A patch has linear precision if and only if its tautological map is the identity map on the domain polytope $\Delta$.

**Remark 1.7.** This is a more restrictive notion of linear precision than is typically considered. It is more common to define a patch to be a collection $\{\beta_i \mid i = 1, \ldots, n\}$ of blending functions, and this has linear precision if we have

$$x = \sum_{i=1}^{n} w_i \beta_i(x) a_i / \sum_{i=1}^{n} w_i \beta_i(x),$$

for some non-negative weights $\{w_i \mid i = 1, \ldots, n\}$ and some points $A = \{a_i \mid i = 1, \ldots, n\}$ whose convex hull is the domain polytope $\Delta$ of the blending functions.

Our more restrictive definition of patches and linear precision, where we incorporate the weights and the points $A$ into our definition, allows us to derive precise criteria with which to study linear precision. Our intention is to employ these criteria to study patches in the generality in which other authors had worked.

**Example 1.8.** Fix an integer $n > 0$ and let $A := \{\frac{i}{n} \mid 0 \leq i \leq n\}$ be the set of $n+1$ equally distributed points in the unit interval. The $i$th Bernstein polynomial is $\beta_i := \binom{n}{i} x^i (1-x)^{n-i}$, which we associate to the point $\frac{i}{n} \in A$. These form a partition of unity,

$$\sum_{i=0}^{n} \beta_i(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} = (x + (1-x))^n = 1,$$

and are therefore normalized blending functions. Given control points $b_i \in \mathbb{R}^3$, formula (1.1) becomes

$$F(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} b_i.$$
which is the classical formula for a Bézier curve of degree \( n \) in \( \mathbb{R}^3 \).

This patch has linear precision. First, note that \( \frac{i}{n} \cdot \binom{n}{i} = \binom{n-1}{i-1} \). Then \( \tau(x) \) is

\[
\sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} \cdot \frac{i}{n} = \sum_{i=1}^{n} \binom{n-1}{i-1} x^i (1-x)^{n-i} = x \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j} = x.
\]

**Example 1.9.** Let \( A \) be the vertices of the hexagon, given below by its defining inequalities.

Let the blending function associated to a vertex be the product of the linear forms defining edges which do not contain that vertex.

\[
\beta_{[-1]} := (1 + x - y)(1 - y)(1 - x)(1 + y - x)
\]
\[
\beta_{[-1]} := (1 - y)(1 - x)(1 + y - x)(1 + y)
\]
\[
\beta_{[1]} := (1 - x)(1 + y - x)(1 + y)(1 + x)
\]
\[
\beta_{[1]} := (1 + y - x)(1 + y)(1 + x)(1 + x - y)
\]
\[
\beta_{[0]} := (1 + y)(1 + x)(1 + x - y)(1 - y)
\]
\[
\beta_{[1]} := (1 + x)(1 + x - y)(1 - y)(1 - x)
\]

(1.10)

The normalized blending functions have denominator \( \varphi(x, y) := 2(3 + xy - x^2 - y^2) \), which is strictly positive on \( \Delta \). These normalized blending functions have linear precision.

\[
\sum_{a \in A} \beta_a(x, y) a = \varphi(x, y) \begin{bmatrix} x \\ y \end{bmatrix}.
\]

In fact, the normalized blending functions are Wachspress's barycentric coordinates \[18\] for this hexagon.

When \( A \) is the set of vertices of \( \Delta \) (so that \( A \) is in convex position), then normalized blending functions of a patch with linear precision are barycentric coordinates for \( \Delta \).
A \textit{reparametrization} of a patch \( \{ \beta_a \mid a \in A \} \) by a homeomorphism \( \psi : \Delta \to \Delta \) is a new patch with blending functions \( \beta_a \circ \psi \) for \( a \in A \). A patch is \textit{proper} if its tautological map (1.5) is a homeomorphism. This condition is necessary for the patch to have linear precision. If a patch is proper, then reparametrizing it by the inverse \( \tau^{-1} \) of \( \tau \) gives a new patch whose tautological map is \( \tau \circ \tau^{-1} \), the identity function, and \( \tau^{-1} \) is the unique homeomorphism of \( \Delta \) with this property. We record this fundamental result about linear precision.

\textbf{Theorem 1.11.} A proper patch has a unique reparametrization which has linear precision.

Theorem 1.11 suggests that it will be fruitful to discuss patches up to reparametrization. In Section 2 we show that a natural geometric object \( X_\beta \) associated to a patch is invariant under reparametrizations and thus represents the patch up to reparametrization.

A patch is \textit{rational} if it has a reparametrization whose normalized blending functions are rational functions (quotients of polynomials). A patch has \textit{rational linear precision} if it is proper and its reparametrization having linear precision has blending functions that are rational functions. Such a patch is necessarily rational. We seek criteria which determine when a rational patch has rational linear precision.

Theorem 1.11 and our preceding discussion concerns reparametrizations of a patch having linear precision. We may also alter a patch by multiplying its blending functions by positive real numbers, typically called weights, or by moving the points of \( A \) that are not vertices of the domain polytope \( \Delta \). This second change does not alter any map (1.1) given by control points, but it will change the tautological map \( \tau \) (1.5). Our analysis in the next section may be helpful in addressing whether it is possible to tune a given patch (using weights or moving the points in \( A \)) to obtain one which has rational linear precision. For example, in Example 3.18 a toric patch is tuned to achieve linear precision.

2. The geometry of linear precision

We introduce an algebraic-geometric formulation of patches to clarify the discussion in Section 1 and to provide tools with which to understand and apply Theorem 1.11. This leads us to discuss the relevance for geometric modeling of the subtle difference between rational varieties and unirational varieties. Lastly, we use this geometric formulation to give a geometric characterization of when a patch has rational linear precision. We recommend the text [2] for additional background on algebraic geometry. We first review linear projections, which are the geometric counterpart of control points.

2.1. Linear projections. We consider \( \mathbb{R}^\ell \) as a subset of the \( \ell \)-dimensional real projective space \( \mathbb{RP}^\ell \) via the embedding

\[ \mathbb{R}^\ell \ni z \mapsto [1, z] \in \mathbb{RP}^\ell. \]

A point \( [z_0, z_1, \ldots, z_\ell] \in \mathbb{RP}^\ell \) lies in this copy of \( \mathbb{R}^\ell \) if and only if \( z_0 \neq 0 \). In that case, the corresponding point of \( \mathbb{R}^\ell \) is

\[ (\frac{z_1}{z_0}, \frac{z_2}{z_0}, \ldots, \frac{z_\ell}{z_0}). \]
Let $A \subset \mathbb{R}^d$ be a finite set of points and write $\mathbb{RP}^A$ for the real projective space whose coordinates are indexed by $A$. Let $b = \{b_a \in \mathbb{R}^\ell \mid a \in A\}$ be a collection of control points, which we regard as points in $\mathbb{RP}^\ell$ so that $b = \{[1, b_a] \in \mathbb{RP}^\ell \mid a \in A\}$.

Given a point $y = [y_a \mid a \in A] \in \mathbb{RP}^A$, if the sum
\[
\sum_{a \in A} y_a \cdot (1, b_a) \in \mathbb{R}^{\ell+1}
\]
is non-zero then it represents a point $\pi_b(y)$ in $\mathbb{RP}^\ell$. This map $\pi_b$ is a linear projection
\[
\pi_b : \mathbb{RP}^A \longrightarrow \mathbb{RP}^\ell.
\]
We use a broken arrow $\longrightarrow$ to indicate that $\pi_b$ is defined only on the complement of the set $E_b \subset \mathbb{RP}^A$ where the sum \eqref{eq:sum} vanishes. This linear subspace is the center of projection.

When the control points affinely span $\mathbb{R}^\ell$, the center of projection $E_b$ has codimension $\ell + 1$ in $\mathbb{RP}^A$. The inverse image $\pi_b^{-1}(x)$ of a point $x \in \mathbb{RP}^\ell$ is the set $H \setminus E_b$, where $H \subset \mathbb{RP}^A$ is a linear subspace of codimension $\ell$ containing $E_b$. If $L$ is any $\ell$-dimensional linear subspace of $\mathbb{RP}^A$ which does not meet $E_b$, then $\pi_b$ maps $L$ isomorphically to $\mathbb{RP}^\ell$. Identifying $L$ with $\mathbb{RP}^\ell$ gives an explicit description of the map $\pi_b$. If $y \in \mathbb{RP}^A \setminus E_b$, then
\[
\pi_b(y) = L \cap \left(\frac{y}{E_b}\right),
\]
the intersection with $L$ of the linear span of $y$ and the center $E_b$. If $y \in L$, then its inverse image under $\pi_b$ is $\left(\frac{y}{E_b}\right) \setminus E_b$.

For example, Figure 1 shows the effect of a linear projection $\pi$ on a cubic space curve $C$. The center of projection is a line, $E$, which meets the curve in a point, $B$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{projection.png}
\caption{A linear projection $\pi$ with center $E$.}
\end{figure}

2.2. Geometric formulation of a patch. Consider \eqref{eq:patch} from the point of view of projective geometry. Given a patch $\beta = \{\beta_a(x) \mid a \in A\}$ with $A \subset \mathbb{R}^d$, then
\[
x \longmapsto [\beta_a(x) \mid a \in A]
\]
defines a map $\beta : \Delta \rightarrow \mathbb{RP}^A$ whose ath coordinate is the blending function $\beta_a(x)$. 
The map $\beta$ is unchanged if we multiply all blending functions by the same positive function $\varphi(x)$ defined on $\Delta$. Thus we obtain the same map if we use instead the normalized blending functions. The image $X_\beta$ of $\Delta$ under $\beta$ is a closed subset of $\mathbb{R}P^A$ (but not Zariski closed!), as $\Delta$ is compact and the map $\beta$ is continuous. It is non-degenerate (does not lie in a hyperplane) exactly when the patch is non-degenerate.

Suppose now that we have control points $b = \{b_a \in \mathbb{R}^\ell \mid a \in A\}$ and consider the composition of the map $\beta$ and the linear projection $\pi_b$

$$\Delta \xrightarrow{\beta} X_\beta \subset \mathbb{R}P^A \xrightarrow{\pi_b} \mathbb{R}P^\ell.$$ 

By our assumption on the positivity of the blending functions, formula (2.1) implies that the image lies in the standard copy of $\mathbb{R}^\ell$, and there it is given by the formula (1.1). Thus this image, $F(\Delta)$, is the image of $X_\beta$ under the projection $\pi_b$ given by the control points.

A reparametrization of the patch $\beta$ by a homeomorphism $\psi: \Delta \to \Delta$ gives a different map with the same image in $\mathbb{R}P^A$

$$\tilde{\beta} : \Delta \xrightarrow{\psi} \Delta \xrightarrow{\beta} X_\beta.$$ 

Thus $X_\beta$ is an invariant of the patch modulo reparametrization.

If we take $A$ to be control points indexed in the natural way, then the resulting linear projection $\pi_A$ is the tautological projection,

$$(2.5) \quad \pi_A : \mathbb{R}P^A \longrightarrow \mathbb{R}P^d.$$ 

The tautological map $\tau$ (1.5) is the composition

$$\Delta \xrightarrow{\beta} X_\beta \subset \mathbb{R}P^A \xrightarrow{\pi_A} \mathbb{R}P^d.$$ 

The patch is proper if this map is a homeomorphism onto its image $\Delta \subset \mathbb{R}P^d$, and the patch has linear precision if the composition is the identity map, by Proposition 1.6.

**Theorem 2.6.** Suppose that $X_\beta$ is a proper patch and $\pi_A: X_\beta \to \Delta$ is the tautological projection restricted to $X_\beta$. Then the blending functions for $X_\beta$ which have linear precision are given by the coordinates of the inverse of $\pi_A$.

**Remark 2.7.** This geometric perspective, where a patch is first an embedding of $\Delta$ into $\mathbb{R}P^A$ followed by a linear projection, may have been introduced into geometric modeling in [6]. It is fundamental for Krasauskas’s toric Bézier patches [12], and was reworked in the tutorial [16]. There, an analog of Theorem 1.11 was formulated. Regrettably, that proof was incomplete. This was due to the author’s confusion between the tautological projection $\pi_A$ and the moment map of a toric variety [8, Chapter 4]. This was an error of notation and not a mathematical error. This same error occurs in [12], just following Theorem 25. Theorem 1.11 corrects the proof in [16], as toric patches are proper by Theorem 25 of [12]. The point is that the tautological projection is an algebraic version of the moment map of the toric variety underlying a toric patch—not the actual moment map.
2.3. **Rational varieties.** A rational patch has a parametrization by rational functions. Clearing their denominators gives a new collection \( \{ \beta_a \mid a \in A \} \) of blending functions which are polynomials. These blending functions give a map
\[
\beta_C : \mathbb{C}^d \setminus B \longrightarrow \mathbb{C}P^A,
\]
which is defined on the complement of the set \( B \) where all the blending functions vanish. This is called the base locus of the map \( \beta_C \). This map \( \beta_C \) extends the map \( \beta : \Delta \rightarrow X_{\beta} \).

We write \( \beta \) for \( \beta_C \), \( P^A \) for \( \mathbb{C}P^A \), and in general use the same notation for maps defined on complex algebraic varieties as for their restrictions to subsets of their real points.

As with linear projections, we write \( \beta : \mathbb{C}^d \longrightarrow P^A \) to indicate that \( \beta \) is only defined on \( \mathbb{C}^d \setminus B \). Such a map between algebraic varieties which is given by polynomials and defined on the complement of an algebraic subset is called a rational map.

Let \( Y_{\beta} \) be the Zariski closure of the image of \( \beta \). One important consequence of the patch being rational is that \( X_{\beta} \) is a full-dimensional subset of the real points of the algebraic variety \( Y_{\beta} \). In particular, \( X_{\beta} \) is defined locally in \( \mathbb{R}P^A \) by the vanishing of some polynomials, and this remains true for any image \( F(\Delta) \) of a map \( F(1.1) \) given by our original blending functions and any choice of control points. This is important in modeling, for these implicit equations are used to compute intersections of patches.

Suppose that \( X_{\beta} \) has rational linear precision and that \( \beta = \{ \beta_a \mid a \in A \} \) are rational blending functions with linear precision. By Proposition 1.6, the composition
\[
\Delta \xrightarrow{\beta} X_{\beta} \xrightarrow{\pi_A} \Delta
\]
is the identity map. Since \( \Delta \) has full dimension in \( \mathbb{R}^d \), it is Zariski dense in \( \mathbb{C}^d \), and so the composition of rational functions
\[
\mathbb{C}^d \xrightarrow{\beta} Y_{\beta} \xrightarrow{\pi_A} \mathbb{C}^d
\]
is also the identity map (where it is defined).

We introduce some terminology to describe this situation. A rational map \( \varphi : Y \longrightarrow Z \) between complex algebraic varieties is a birational isomorphism if there is another map \( \psi : Z \longrightarrow Y \) such that \( \varphi \circ \psi \) and \( \psi \circ \varphi \) are the identity maps where they are defined. In particular, \( Y \) has a Zariski open subset \( U \) and \( Z \) a Zariski open subset \( V \) such that \( \varphi|_U \) is one-to-one between \( U \) and \( V \). The projection \( \pi : C \longrightarrow L \) of Figure 1 is not a birational isomorphism. Indeed, if \( x \in C \) is a point in \( \pi^{-1}(p) \), then \( p, E \overset{\pi}{=} x, E \) and this plane meets the complement \( C \setminus B \) of the base locus in \( x \) and one other point.

We deduce the algebraic-geometric version of Proposition 1.6.

**Theorem 2.8.** A patch \( \beta = \{ \beta_a \mid a \in A \} \) has rational linear precision if and only if the complexified tautological projection
\[
\pi_A : Y_{\beta} \longrightarrow \mathbb{C}P^d
\]
is a birational isomorphism.

This characterization of linear precision provides necessary conditions for a patch to have linear precision. A rational variety of dimension \( d \) is one that is birational to \( \mathbb{C}^d \).
Theorem 2.9. If a patch \( \beta = \{ \beta_a \mid a \in \mathcal{A} \} \) has rational linear precision, then

1. \( Y_\beta \) is a rational variety,
2. almost all codimension \( d \) planes \( L \) containing the center \( E_A \) of the tautological projection meet \( Y_\beta \) in at most one point outside of \( E_A \).

By condition (2), \( Y_\beta \) has an exceptionally singular position with respect to \( E_A \). Typically, \( Y_\beta \) does not meet a given codimension \( d+1 \) plane and its intersection with a given codimension \( d \) plane consists of \( \deg(Y_\beta) \) points, counted with multiplicity. By Condition (2), not only does \( Y_\beta \) meet \( E_A \), but if \( L \) has codimension \( d \) and contains \( E_A \), then most of the \( \deg(Y_\beta) \) points in \( Y_\beta \cap L \) lie in \( E_A \). We will see this in Examples 2.10 and 3.18.

Proof. Statement (1) is immediate from Theorem 2.8.

Since the birational isomorphism \( \pi_A: Y_\beta \longrightarrow \mathbb{CP}^d \) is the restriction of a linear projection with center \( E_A \), then for almost all points \( y \in Y_\beta \setminus E_A \), we have

\[
(y, E_A) \cap Y_\beta = \{y\} \cup (E_A \cap Y_\beta).
\]

Since all codimension \( d \) planes \( L \) which contain \( E_A \) have the form \((y, E_A)\) for some \( y \notin E_A \) and almost all meet \( Y_\beta \), statement (2) follows.

Consider the geometric situation of Figure 1. Suppose now that \( E \) is the line tangent to \( C \) at the point \( B \). Then \( E \cap C \) is the point \( B \), but counted with multiplicity 2. A linear projection with center \( E \) restricts to a birational isomorphism of \( C \) with \( \mathbb{P}^1 \).

2.4. Unirational varieties. The necessary condition of Theorem 2.9(1) gives an important but subtle geometric restriction on patches that have rational linear precision.

The map \( \beta \) provides a parametrization of an open subset of \( Y_\beta \) by an open subset of \( \mathbb{C}^d \). Such parametrized algebraic varieties are called unirational. Unirational curves and surfaces are also rational, but these two notions differ for varieties of dimension three and higher. Clemens and Griffiths \( \mathbb{H} \) showed that a smooth hypersurface of degree 3 in \( \mathbb{P}^4 \) is not rational (these were classically known to be unirational, e.g. by Max Noether).

Thus parametric patches of dimension at least 3 will in general be unirational and not rational. This does not occur for Bézier simploids or toric patches.

Example 2.10. Consider again the Wachspress coordinates (1.10) of Example 1.9, which have linear precision. Let \( Y_\beta \subset \mathbb{P}^5 \) be the image of the blending functions \( \Delta^0 \). We study the base locus of the tautological projection \( \pi_A \) of this patch. We first find equations for \( Y_\beta \) as an algebraic subvariety of \( \mathbb{P}^5 \).

If we divide \( \beta_{[-1]} \beta_{[0]} + 2 \beta_{[0]} \beta_{[-1]} + \beta_{[1]} \beta_{[-1]} \) by the product of the linear forms defining the hexagon we get

\[
(1 - x)(1 + y - x) + 2(1 + x)(1 - x) + (1 + x)(1 + x - y) = 4 - 2xy,
\]

which is symmetric in \( x \) and \( y \), and so

\[
(2.11) \quad \beta_{[-1]} \beta_{[0]} + 2 \beta_{[0]} \beta_{[-1]} + \beta_{[1]} \beta_{[-1]} = \beta_{[-1]} \beta_{[1]} + 2 \beta_{[0]} \beta_{[-1]} + \beta_{[1]} \beta_{[-1]}.
\]

If we let \((y_a \mid a \in \mathcal{A})\) be natural coordinates for \( \mathbb{P}^A \), then (2.11) gives the quadratic polynomial which vanishes on \( Y_\beta \),
which is the product of a linear and a quadratic form.

Cyclically permuting the vertices of the hexagon gives two other quadratics that vanish on $Y$, but these three sum to 0. There is another quadratic polynomial vanishing on $Y$, defining $E$

There is an additional cubic relation among the blending functions, which gives a cubic polynomial that vanishes on $Y$

These relations, three independent quadratics and one cubic, define $Y$ as an algebraic subset of $\mathbb{P}^3$. The quadratic equations define $Y$, together with the 2-dimensional linear space cut out by the three linear equations.

Thus $Y$ has degree $7 = 2^3 - 1$. A general codimension 2 plane $L$ containing the center $E_A$ of the tautological projection $\pi_A$ will meet $Y$ in at most 1 point outside of $E_A$, by Theorem 2.9(2). Since $L \cap Y$ has degree at least 7, the other 6 points must lie in the base locus $E_A \cap Y$, which is in fact a reducible cubic plane curve.

The three linear forms (2.13) also define the tautological projection, $\pi_A$, so this linear subspace is the center of projection $E_A$, and the base locus $B$ is defined in $E_A$ by the cubic (2.12) defining $Y$. Let us parametrize the center $E_A$ as follows. For $[a, b, c] \in \mathbb{P}^2$, set

Then we have

which is the product of a linear and a quadratic form.

3. Toric Patches

Toric patches, which were introduced by Krassauskas [12], are a class of patches based on certain special algebraic varieties, called toric varieties. For a basic reference on toric varieties as parametrized varieties, see the book [17], particularly Chapters 4 and 13, and the articles [3] [16], which are introductions aimed at people in geometric modeling. We first define toric patches, give some examples, reinterpret Theorem 2.8 for toric patches, and then state the main open problem about linear precision for toric patches.
3.1. Toric Patches. We regard elements \( a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \) as exponent vectors for monomials in the variables \( x = (x_1, \ldots, x_d) \)

\[
x^a := x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}.
\]

Let \( A \subset \mathbb{Z}^d \) be a finite set of integer vectors whose convex hull \( \Delta \) is a polytope of dimension \( d \) in \( \mathbb{R}^d \). This implies that \( A \) affinely spans \( \mathbb{R}^d \). Suppose that \( w = \{w_a \in \mathbb{R}_+ \mid a \in A\} \) is a set of positive weights indexed by \( A \). These data \( (A, w) \) define a map \( \varphi_{A, w} : (\mathbb{C}^*)^d \to \mathbb{P}^A \),

\[
(3.1) 
\varphi_{A, w} : x \mapsto [w_a x^a \mid a \in A].
\]

Let \( Y_{A, w} \) be the Zariski closure of the image of \((\mathbb{C}^*)^d\) under the map \( \varphi_{A, w} \). When the weights \( w_a \) are equal, this is the toric variety parametrized by the monomials in \( A \). In general, it is the translate of that toric variety by the element \( w \) of the positive real torus \( \mathbb{R}^A_+ \), which acts on \( \mathbb{P}^A \) by scaling the coordinates.

Let \( X_{A, w} \subset \mathbb{R}P^A \) be the closure \( \varphi_{A, w}(\mathbb{R}^d_+) \). This is the non-negative part of the translated toric variety \( Y_{A, w} \). That is, if \( \mathbb{R}P^A_+ \subset \mathbb{P}^A \) is the set of points having non-negative (real) homogeneous coordinates, then \( X_{A, w} = Y_{A, w} \cap \mathbb{R}P^A_+ \). It is homeomorphic to the convex hull \( \Delta \) of the vectors in \( A \), as a manifold with corners [8 § 4]

The set \( A \) is primitive if it affinely spans \( \mathbb{Z}^d \). We may assume that \( A \) is primitive, for there is always some primitive set \( A' \subset \mathbb{Z}^d \) with \( Y_{A, w} = Y_{A', w} \). First note that translating every vector in \( A \) by a fixed vector \( a' \) multiplies each coordinate of the map \( \varphi_{A, w} \) by the monomial \( x^{a'} \), which does not change \( \varphi_{A, w} \) as a map to projective space. Thus we may assume that \( A \) contains the zero vector. Since \( A \) affinely spans \( \mathbb{R}^d \), it generates a subgroup \( \mathbb{Z}A \) of \( \mathbb{Z}^d \) which is isomorphic to \( \mathbb{Z}^d \). Let \( b_1, \ldots, b_d \in \mathbb{Z}A \) be the basis corresponding to the basis of \( \mathbb{Z}^d \) under an isomorphism \( \mathbb{Z}A \sim \mathbb{Z}^d \). If we let \( A' \subset \mathbb{Z}^d \) be the image of \( A \) under this isomorphism and \( B := \{b_1, \ldots, b_d\} \), then the map \( \varphi_{A, w} \) factors

\[
(\mathbb{C}^*)^d \xrightarrow{\chi_B} (\mathbb{C}^*)^d \xrightarrow{\varphi_{A', w}} \mathbb{P}^A,
\]

where \( \chi_B \) is the surjective map which sends \( x \) to \((x^{b_1}, \ldots, x^{b_d})\). Note that \( \varphi_{A, w} \) is injective but that \( \chi_B \) has fibres of size \(|\mathbb{Z}^d/\mathbb{Z}A|\). Thus \( \varphi_{A, w} \) is not proper unless \( A \) is primitive.

**Definition 3.2.** A toric patch of shape \((A, w)\) is any patch \( \{\beta_a \mid a \in A\} \) such that the image, \( X_{\beta} \), of \( \Delta \) under the map \((2.4)\) given by the blending functions is equal to the non-negative part \( X_{A, w} \) of the translated toric variety \( Y_{A, w} \).

**Remark 3.3.** Our geometric perspective that the fundamental object is the image \( X_{\beta} \) requires us to absorb the weights \( w \) into our definition and distinguishes patches with different choices of weights.

**Example 3.4.** Let \( A := \{0, 1, \ldots, n\} \subset \mathbb{Z} \) and let \( w \in \mathbb{R}^{n+1}_+ \) be any set of weights. Then the map \( \varphi_{A, w} \) is

\[
x \mapsto [w_0, w_1x, w_2x^2, \ldots, w_nx^n].
\]
The map \( x \mapsto \frac{x}{n-x} \) sends \( \Delta = [0, n) \) to \([0, \infty)\). Precomposing \( \varphi_{A,w} \) with this map gives a toric patch of shape \((A, w)\).

\[
[0, n] \ni x \mapsto \left[ w_0, w_1 \left( \frac{x}{n-x} \right), w_2 \left( \frac{x}{n-x} \right)^2, \ldots, w_n \left( \frac{x}{n-x} \right)^n \right] = \left[ w_0(n-x)^n, w_1x(n-x)^{n-1}, w_2x^2(n-x)^{n-2}, \ldots, w_nx^n \right].
\]

Observe that if we choose weights \( w_i = \binom{n}{i} \), substitute \( ny \) for \( x \), and remove the common factors of \( n^n \), then we obtain the blending functions for the Bézier curve of Example 1.8.

Note that replacing \( x \) by \( ny \) also replaces \( i \in A \) by \( \frac{i}{n} \).

**Example 3.5.** Krasauskas [12] generalized the classical Bézier parametrization to any polytope with integer vertices. A polytope \( \Delta \) is defined by its facet inequalities

\[ \Delta = \{ x \in \mathbb{R}^d \mid h_i(x) \geq 0, i = 1, \ldots, N \}. \]

Here, \( \Delta \) has \( N \) facets and for each \( i = 1, \ldots, N \), \( h_i(x) = \langle v_i, x \rangle + c_i \) is the linear function defining the \( i \)th facet, where \( v_i \in \mathbb{Z}^d \) is the (inward oriented) primitive vector normal to the facet and \( c_i \in \mathbb{Z} \). (Compare these to the functions in Example 1.9)

Let \( A \subset \Delta \cap \mathbb{Z}^d \) be any subset of the integer points of \( \Delta \) which includes its vertices. Let \( w = \{ w_a \mid a \in A \} \subset \mathbb{R}_+ \) be a collection of positive weights. For every \( a \in A \), Krasauskas defined toric Bézier functions

\[
\beta_a(x) := w_a h_1(x)^{h_1(a)} h_2(x)^{h_2(a)} \cdots h_N(x)^{h_N(a)}.
\]

For \( x \in \Delta \), these parametrize a patch \( X_\beta \).

Krasauskas defined a toric patch to be any reparametrization of such a patch. Our definition of a toric patch (Definition 3.2) agrees with Krasauskas’s, when \( A = \Delta \cap \mathbb{Z}^d \) consists of all the integer points in a polytope: Observe that the map \( \beta \) in (3.6) is the composition of a map \( \hat{H} : \Delta \to \mathbb{R}^N \) given by \( x \mapsto (h_1(x), \ldots, h_N(x)) \) with a rational map \( \varphi : \mathbb{C}^N \to \mathbb{R}^P \) given by

\[
\varphi : (u_1, \ldots, u_N) \mapsto [w_a u_1^{h_1(a)} \cdots u_N^{h_N(a)} \mid a \in A].
\]

This map \( \varphi \) factors through the map \( \varphi_{A,w} \). Indeed, define a map

\[
f_\Delta : (\mathbb{C}^\times)^N \to (\mathbb{C}^\times)^n
\]

\[
u = (u_1, \ldots, u_N) \mapsto t = (t_1, \ldots, t_n) \; \text{ where } \; t_j := \prod_{i=1}^{N} u_i^{<v_i,e_j>}
\]

Then

\[
\prod_{i=1}^{N} u_i^{h_i(a)} = \prod_{i=1}^{N} u_i^{c_i} \cdot \prod_{i=1}^{N} u_i^{<v_i,a>} = u^c \cdot t^a.
\]

And so \( \varphi(u) = u^c \cdot \varphi_{A,w}(f_\Delta(u)) \). This shows that \( X_\beta \subset X_{A,w} \), and Krasauskas [13 §4] shows that this inclusion is an equality.

**Example 3.7.** A standard \( d \)-simplex of degree \( n \) is the simplex in \( \mathbb{R}^d \),

\[ n\Delta_d := \{ x \in \mathbb{R}^d \mid x_1 + \cdots + x_d \leq n \text{ and } x_i \geq 0 \}. \]
For a point \( a = (a_1, \ldots, a_d) \) in \( n\Delta_d \cap \mathbb{Z}^d \) set \( |a| := a_1 + \cdots + a_d \) and let \( w_a := \binom{n}{a} = \frac{n!}{a_1! \cdots a_d!(n-|a|)!} \) be the multinomial coefficient. This gives a system of weights for \( A = n\Delta \cap \mathbb{Z}^d \).

Then Krasauskas’s toric Bézier patch for \( n\Delta_d \) has blending functions

\[
\beta_a := \binom{n}{a}(n - \sum_i x_i)^{n-|a|} \prod_{i=1}^d x_i^{a_i}.
\]

If we substitute \( ny_i \) for \( x_i \) and then remove the common factors of \( n^a \), we recover the classical Bernstein polynomials [7, §4.1].

**Example 3.8.** The Bézier simploids [5] are toric patches based on products of Bézier simplices. Suppose that \( d_1, \ldots, d_m \) and \( n_1, \ldots, n_m \) are positive integers. Set \( d := d_1 + \cdots + d_m \). Write \( \mathbb{R}^d \) as a direct sum \( \mathbb{R}^{d_1} \oplus \cdots \oplus \mathbb{R}^{d_m} \) and suppose that for each \( i = 1, \ldots, m \), the scaled simplex \( n_i\Delta_{d_i} \) lies in the \( i \)th summand \( \mathbb{R}^{d_i} \). Consider the Minkowski sum

\[
\Delta := n_1\Delta_{d_1} + n_2\Delta_{d_2} + \cdots + n_m\Delta_{d_m} \subset \mathbb{R}^d,
\]

which is just the product of the simplices \( n_i\Delta_{d_i} \).

If \( x_i := (x_{i,1}, \ldots, x_{i,d_i}) \) are coordinates on \( \mathbb{R}^{d_i} \), then

\[
\Delta = \{ x = (x_1, \ldots, x_m) \mid x_{i,j} \geq 0 \text{ and } x_{i,1} + \cdots + x_{i,d_i} \leq n_i \quad i = 1, \ldots, m \}.
\]

Let \( A := \Delta \cap \mathbb{Z}^d \) be the set of integer points of \( \Delta \). Given an integer point \( a = (a_1, \ldots, a_m) \in A \), set \( w_a := \prod_{i=1}^m \binom{n_i}{a_i} \), the product of multinomial coefficients. This gives a system of weights, and let \( Y_{A,w} \) be the resulting toric patch. Krasauskas’s toric patch of shape \( \Delta \) has blending functions

\[
\beta_a(x) := \prod_{i=1}^m \beta_{a_i}(x),
\]

where \( \beta_{a_i}(x_i) \) for \( x_i \in \mathbb{R}^{d_i} \) are the Bernstein polynomials. The resulting patch is called a Bézier simploid [5].

We state our main result about linear precision for toric patches.

**Theorem 3.9.** Let \( A \subset \mathbb{Z}^d \) be a collection of primitive exponent vectors, \( w \in \mathbb{R}^A \) be a system of weights, and define the Laurent polynomial

\[
f = f_{A,w} := \sum_{a \in A} w_a x^a.
\]

A toric patch of shape \( (A, w) \) has rational linear precision if and only if the rational function

\[
\psi_{A,w} : \mathbb{C}^d \rightarrow \mathbb{C}^d \text{ defined by}
\]

\[
\frac{1}{f} \left( \frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \ldots, \frac{\partial}{\partial x_d} f \right)
\]

is a birational isomorphism.

**Remark 3.11.** The map (3.10) has an interesting reformulation in terms of toric derivatives. The \( i \)th toric derivative of a Laurent polynomial \( f \) is \( x_i \frac{\partial}{\partial x_i} f \). The toric differential
$D_{\text{torus}} f$ is the vector whose components are the toric derivatives of $f$. Thus the map $\psi_{A,w}$ is the logarithmic toric differential $D_{\text{torus}} \log f$.

In the proof below, we show that the map $\psi_{A,w}$ comes from a map $\mathbb{C}^d \to \mathbb{CP}^d$,

\begin{equation}
(3.12) \quad x \mapsto \left[ f : x_1 \frac{\partial}{\partial x_1} f : x_2 \frac{\partial}{\partial x_2} f : \cdots : x_d \frac{\partial}{\partial x_d} f \right]
\end{equation}

Let $F_{A,w}$ be the homogenization of $f$ with respect to a new variable $x_0$. Then

$$\deg(F_{A,w}) = \sum_{i=0}^d x_i \frac{\partial}{\partial x_i} F_{A,w}.$$ 

Thus, after homogenizing and a linear change of coordinates, the map (3.12) is defined by the formula

$$D_{\text{torus}} F_{A,w} = \left[ x_0 \frac{\partial}{\partial x_0} F_{A,w} : x_1 \frac{\partial}{\partial x_1} F_{A,w} : \cdots : x_d \frac{\partial}{\partial x_d} F_{A,w} \right].$$ 

This last form of the map (3.12) gives an appealing reformulation of Theorem 3.9 which we feel is the most useful for further analysis of linear precision for toric patches.

**Corollary 3.13.** The toric patch of shape $(A, w)$ has linear precision if and only if the map $D_{\text{torus}} F_{A,w} : \mathbb{CP}^d \to \mathbb{CP}^d$ is a birational isomorphism.

**Proof of Theorem 3.9.** By Theorem 2.8, a toric patch of shape $(A, w)$ has linear precision if and only if the complexified tautological projection

$$\pi_A : Y_{A,w} \to \mathbb{CP}^d$$ 

is a birational isomorphism. Precomposing this with the defining parametrization of $Y_{A,w}$,

$$\varphi_{A,w} : (\mathbb{C}^*)^d \to Y_{A,w} (\text{which is a rational map } \mathbb{C}^d \to Y_{A,w})$$,

gives a map

\begin{equation}
(3.14) \quad x \mapsto \left[ w_a x_a \mid a \in A \right] \mapsto \sum_{a \in A} w_a x_a (1, a).
\end{equation}

The initial (0th) coordinate of this composition is

$$\sum_{a \in A} w_a x_a$$

which is the polynomial $f_{A,w}$. For $i > 0$ the $i$th coordinate of this composition is

$$\sum_{a \in A} a_i w_a x_a = x_i \frac{\partial}{\partial x_i} f_{A,w}.$$ 

If we divide the composite map (3.14) by its 0th coordinate, $f_{A,w}$, we obtain the map $\psi_{A,w}$. This map is birational if and only if $\pi_A$ is birational, as $\varphi_{A,w}$ is birational.

**Example 3.15.** Let $A := \{0, 1, \ldots, n\} \subset \mathbb{Z}$ as in Example 3.4. Let $w \in \mathbb{R}_{>0}^{n+1}$ be any set of weights. Then the polynomial $f = f_{A,w}$ is

$$f(x) = w_0 + w_1 x + \cdots + w_n x^n.$$
Suppose that this toric patch has rational linear precision. By Theorem 3.9, the logarithmic toric differential \( xf'(x)/f(x) \) is a birational isomorphism. Thus the fraction \( xf'(x)/f(x) \) reduces to a quotient of linear polynomials and so \( f(x) \) is necessarily a pure power, say \( a(x + \alpha)^n \) with \( \alpha \neq 0 \).

We conclude that the weights are \( w_i = \alpha^{n-i} \binom{n}{i} \), which are not quite the weights in the Bézier curve [1.8]. If we rescale \( x \) (which does not change \( Y_{\mathcal{A},w} \), setting \( y = x/\alpha \), then the weights become \( |\alpha|^n \binom{n}{i} \)). Removing the common factor of \( |\alpha|^n \), reveals that this toric patch is the classical Bézier curve of Example [1.8].

We conjecture the analog of this example holds in any dimension.

**Conjecture 3.16.** Suppose that \( \mathcal{A} \) consists of the integer points in a product \( \Delta \) of scaled standard simplices. The only toric patch of shape \( (\mathcal{A},w) \) having rational linear precision is the Bézier simploid on \( \Delta \).

We also make the stronger conjecture.

**Conjecture 3.17.** The only toric patches of any shape having rational linear precision are the Bézier simploids.

The first steps in this direction are taken in [9].

**Example 3.18.** We discuss Krasauskas’s version [13] of Karčiauskas’s pentagonal \( M \)-patch [10], which has linear precision. Let \( \mathcal{A} \) be the integer points in the pentagon shown below, and let \( \{w_a \mid a \in \mathcal{A}\} \) be the indicated weights.

These define a toric patch \( X_{\mathcal{A},w} \) and translated toric variety \( Y_{\mathcal{A},w} \). This subvariety of \( \mathbb{P}^7 \) has degree 7 and is defined by 14 quadratic equations, all of the form

\[
y_a y_b - y_c y_d \quad \text{where} \quad a, b, c, d \in \mathcal{A} \quad \text{with} \quad a + b = c + d.
\]

This has a monomial parametrization

\[
\varphi_{\mathcal{A},w} : (s, t) \mapsto [s^{a_1} t^{a_2} \mid a = (a_1, a_2) \in \mathcal{A}].
\]

Its composition with the tautological projection \( \pi_{\mathcal{A}} \) gives a map \( g : \mathbb{P}^2 \to \mathbb{P}^2 \) of degree 4, with 3 base points at \((-1, -1), (-1, -\frac{1}{2}), \text{ and } (-\frac{1}{2}, -1)\).

We tune this patch (or rather the projection \( \pi_{\mathcal{A}} \)) by moving the non-extreme points of \( \mathcal{A} \) within the pentagon. Let \( \mathcal{B} \) consist of the vertices of the pentagon, together with the
three non-vertices

\[
\begin{array}{cccc}
0, 2 & (1, 2) & (0, \frac{6}{7}) & \left(\frac{5}{7}, \frac{6}{7}\right) & (2, 1) \\
0, 0 & (0, \frac{6}{7}) & \left(\frac{5}{7}, 0\right) & (2, 0) \\
\end{array}
\]

Lattice points \( \mathcal{B} \)

The composition of \( \varphi_{A,w} \) with this new tautological projection \( \pi_{B} \) gives the rational map

\[
(s, t) \mapsto \left(\frac{2s(2s + t + 3)}{(s + 1)(2s + 2t + 3)}, \frac{2t(2t + 3)}{(t + 1)(2s + 2t + 3)}\right).
\]

This is given by quadrics and has 3 base points \((-1, -1), (0, -\frac{3}{2}), \) and \((-\frac{3}{2}, 0), \) and so it has degree \(1 = 2 \cdot 2 - 3, \) and is therefore a birational isomorphism. By Theorem 2.8 the tuned toric patch (with the lattice points \( A \) replaced by \( \mathcal{B} \)) has linear precision.

Thus there is a rational parametrization \( \beta: \mathcal{D} \rightarrow X_{A,w} \) whose composition with the tautological projection \( \pi_{B} \) is the identity map. Indeed, Krassaukas [13] gives the following modification of the toric Bézier functions and shows that the map \( \pi_{B} \circ \beta \) is the identity.

\[
\begin{align*}
\beta_{[0]} & := 3(2 - s)^2(3 - s - t)^3(2 - t)^2 \\
\beta_{[1]} & := 5(2 - s)(3 - s - t)^2(2 - t)^2s(3 - s - \frac{t}{2}) \\
\beta_{[2]} & := 5(3 - \frac{s}{2} - t)t(2 - s)^2(3 - s - t)^2(2 - t) \\
\beta_{[1]} & := 7(3 - \frac{s}{2} - t)t(2 - t)(3 - s - t)(2 - s)s(3 - s - \frac{t}{2}) \\
\beta_{[2]} & := 2(3 - s - t)(2 - t)^2s^2(3 - s - \frac{t}{2})^2 \\
\beta_{[0]} & := 2(3 - \frac{s}{2} - t)^2t^2(2 - s)^2s(3 - s - t) \\
\beta_{[1]} & := 2(3 - \frac{s}{2} - t)t(2 - t)s^2(3 - s - \frac{t}{2})^2 \\
\beta_{[2]} & := 2(3 - \frac{s}{2} - t)^2t^2(2 - s)s(3 - s - \frac{t}{2})
\end{align*}
\]

(3.19)

This is a modification of the toric Bézier functions for the pentagon. If, in the definition (3.6), we replace the form \( s \) by \( s(3 - s - \frac{t}{2}) \) and the form \( t \) by \( (3 - \frac{s}{2} - t)t \), then we get these modified blending functions (3.19).

Let us consider the geometry of linear precision for this tuned patch in the spirit of Section 2. The projection \( \pi_{B} \) is defined by the three linear forms.

\[
\begin{align*}
X & := \frac{6}{5}y_{[1]} + 2y_{[2]} + \frac{5}{7}y_{[1]} + 2y_{[2]} + y_{[2]} \\
Y & := \frac{6}{5}y_{[1]} + 2y_{[2]} + \frac{5}{7}y_{[1]} + 2y_{[2]} + y_{[2]} \\
Z & := y_{[0]} + y_{[1]} + y_{[1]} + y_{[1]} + y_{[2]} + y_{[2]} + y_{[2]} + y_{[2]}
\end{align*}
\]
Its center $E_B$ is defined by the vanishing of these three forms. A general linear subspace $L$ of codimension 2 containing $E_B$ is defined by equations of the form

$$X = xZ \quad \text{and} \quad Y = yZ.$$

This subspace meets $Y_{A,w}$ in four points. One point lies outside of $E_B$ while three points lie in the center $E_B$. We give the three points.

![Diagram](image)

We write the coordinate $y_a$ of a point $y \in \mathbb{RP}^A$ at the position of the point $a$.

### 4. Iterative proportional fitting for toric patches

In algebraic statistics, the image of $\mathbb{R}^d_>$ under the map $\varphi_{A,w}$ (3.1) is known as a toric model [15, §1.2]. They are more commonly called log-linear models, as the logarithms of the coordinates of $\varphi_{A,w}$ are linear functions in the logarithms of the coordinates of $\mathbb{R}^d_>$, or discrete exponential families as the coordinates of $\varphi_{A,w}$ are exponentials in the logarithms of the coordinates of $\mathbb{R}^d_>$.

We identify the non-negative orthant $\mathbb{RP}^A_\geq$ with the probability simplex

$$\Delta^A := \{ y \in \mathbb{R}^d_+ \mid |y| := \sum_{a \in A} y_a = 1 \},$$

and so we may also regard $X_{A,w}$ as a subvariety of $\Delta^A$.

The tautological map $\pi_A$ (2.5) appears in algebraic statistics. Given data $q \in \Delta^A$, Lemma 4 in [4] states that the maximum likelihood estimate (for the model $X_{A,w}$) is the unique point $p \in X_{A,w}$ such that

$$\pi_A(p) = \pi_A(q).$$

Thus inverting the tautological projection is necessary for computing this maximum likelihood estimate.

Darroch and Ratcliff [4] introduced the numerical algorithm of iterative proportional fitting, also known as generalized iterative scaling, for computing this inverse. Their algorithm requires that the data $A$ be in a normal, homogeneous form.

Observe first that the toric patch $X_{A,w}$ does not change if we translate all elements of $A$ by a fixed vector $b$, $(a \mapsto a + b)$, so we may assume that $A$ lies in the positive orthant $\mathbb{R}^d_>$. Scaling the exponent vectors in $A$ by a fixed positive scalar $t \in \mathbb{R}_>$ also does not change...
X_{A,w} as x \mapsto x^t is a homeomorphism of \( \mathbb{R}_> \) which extends to a homeomorphism of \( \mathbb{R}^d \).

Thus we may assume that \( A \) lies in the standard simplex \( \Delta_d \) in \( \mathbb{R}^d \):

\[
\Delta_d = \{ x \in \mathbb{R}^d \mid x \geq 0 \text{ and } |x| \leq 1 \}.
\]

Lastly, we lift this to the probability simplex \( \Delta_d^+ \in \mathbb{R}_{\geq 1}^d \),

\[
\Delta_d^+ := \{ y \in \mathbb{R}^{1+d} \mid y_i \geq 0 \text{ and } |y| = 1 \},
\]

by

\[
A \ni a \mapsto a^+ := (1 - |a|, a) \in \Delta_d^+.
\]

Since for \( t \in \mathbb{R}_> \) and \( x \in \mathbb{R}^d \),

\[
(t, tx)^{a^+} = t^{1-|a|}(tx)^a = tx^a,
\]

we see that replacing \( A \) by its lifted version \( A^+ \) also does not change \( X_{A,w} \). We remark that this is just a way to homogenize the data for our problem.

Since these are affine transformations, the property of affine invariance for patches shows that the affine map which is the composition of these transformations intertwines the original tautological projection with the new one.

We describe the algorithm of \textit{iterative proportional fitting}, which is Theorem 1 in [4].

**Proposition 4.1.** Suppose that \( A \subset \Delta_d \) and \( y \in \text{conv}(A) \). Then the sequence

\[
\{ p^{(n)} | n = 0, 1, 2 \ldots \}
\]

whose \( a \)-coordinates are defined by \( p_a^{(0)} := w_a \) and, for \( n \geq 0 \),

\[
p_a^{(n+1)} := p_a^{(n)} \cdot \frac{y^a}{(\pi_A(p^{(n)}))^a},
\]

converges to the unique point \( p \in X_{A,w} \) such that \( \pi_A(p) = \pi_A(y) \).

We remark that if \( A \) is not homogenized then to compute \( \pi_A^{-1}(y) \) for \( y \in \text{conv}(A) \), we first put \( A \) into homogeneous form \( A^+ \) using an affine map \( \psi \), and then use iterative proportional fitting to compute \( \pi_A^{-1}(\psi(y)) = \pi_A^{-1}(y) \). We also call this modification of the algorithm of Proposition 4.1 iterative proportional fitting. Thus iterative proportional fitting computes the inverse image of the tautological projection, which, by Theorem 2.6 gives the unique parametrization of \( X_{A,w} \) having linear precision.

**Corollary 4.2.** Iterative proportional fitting computes the unique parametrization of a toric patch having linear precision.

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