Quasi-order of clocks and their synchronism
and quantum bounds for copying timing information

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The statistical state of any (classical or quantum) system with non-trivial time evolution can be interpreted as the pointer of a clock. The quality of such a clock is given by the statistical distinguishability of its states at different times. If a clock is used as a resource for producing another one the latter can at most have the quality of the resource. We show that this principle, formalized by a quasi-order, implies constraints on many physical processes. Similarly, the degree to which two (quantum or classical) clocks are synchronized can be formalized by a quasi-order of synchronism.

Copying timing information is restricted by quantum no-cloning and no-broadcasting theorems since classical clocks can only exist in the limit of infinite energy. We show this quantitatively by comparing the Fisher timing information of two output systems to the input’s timing information. For classical signal processing in the quantum regime our results imply that a signal loses its localization in time if it is amplified and distributed to many devices.

I. QUANTUM CLOCKS

Synchronizing clocks of distant parties is an important part of modern communication. Schemes for performing synchronization with minimal physical resources have attracted attention recently (see e.g. [1] and references therein). The application of these schemes has mostly been described in the context of synchronizing clocks of rather distant parties. Here we have in mind the miniature version of the scenario and recall that time synchronization takes place on every computer chip since a common clock signal controls many circuits at once. In other words, since the clock signal triggers the execution of several gates implemented at the same time, it realizes essentially a clock synchronization.

More explicitly, one can think of the following situation: a physical signal, a light pulse or electrical pulse for instance, enters an amplifier. The amplifier produces two output signals in different directions in order to supply two distant gates with the same clock signal. To assure that both signals reach the corresponding gates at the same time is an example of time synchronization.

In the common language of communication protocols, the situation we have in mind can be described as follows. Alice (‘the source emitting the clock signal’) sends a physical system that carries information about her time to Bob. He (‘the amplifier’) copies the signal and sends each copy to another person in order to provide them with timing information. Abstractly speaking, Alice sends a quantum system with density matrix $\rho$ to Bob. The state $\rho$ evolves according to the system’s Hamiltonian $H$, i.e., it is given by

$$\rho_t := \exp(-iHt)\rho\exp(iHt)$$

at the time $t$ after Alice has prepared the state $\rho$. Here we assume that $[\rho, H] \neq 0$, such that the states $\rho_t$ are not all the same. Then the actual system state $\rho_t$ contains some information about Alice’s time. We emphasize the word ‘some’ since it is not possible to readout the parameter $t$, i.e., every measurement performed on the quantum system allows only to estimate $t$. Furthermore it is not possible for Bob to provide a third party (Carol) with exact copies of $\rho_t$. Assume the clock for instance to be a two-level system in the state $|0\rangle + \exp(-i\omega t)|1\rangle$ with unknown $t$. Then the problem of preparing two clocks of this type with the same time $t$ is the problem of cloning equatorial qubits (see [2] and references therein), which is only possible with a certain fidelity. In order to be more general, consider a finite dimensional quantum system with density matrix $\rho_t$. To produce copies of $\rho_t$ would mean to prepare states $\sigma_t$ on the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{H}$ (if $\mathcal{H}$ is the space that $\rho_t$ acts on) in such a way that the partial traces over the first and second components of $\sigma_t$ coincide both with $\rho_t$. This is usually referred to as quantum broadcasting. It is possible if and only if all the $\rho_t$ mutually commute (see [3] Subsection 4.3), a requirement that cannot be true for any non-trivial dynamics.

Can timing information be copied? Obviously it is possible to perform measurements on the unknown state $\rho_t$ in order to give optimal estimations for $t$. Then one can initialize an arbitrary large number of classical clocks according to the estimated time and distribute them to different parties. Here we are interested in the question whether the initialized classical clocks are as good as the original density matrix $\rho_t$. In some sense, they are: if one is interested in giving optimal estimations about $t$, each party is able to estimate $t$ with the

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same error probabilities as the owner of the system $\rho_t$. Then all parties have optimal information about $t$ provided that the measurement was optimal with respect to a common optimality criterion of all parties. However, there is another meaning of timing information and in this broader sense not all the information of $\rho_t$ can be copied. To illustrate this consider the equatorial qubit in the state $|0\rangle + \exp(-i\omega t)|1\rangle$ prepared by Alice. Assume Bob receives the state, performs a measurement on it and initializes his clock according to his estimated value $t'$ of $t$. Then he can prepare a new qubit in the state $|0\rangle + \exp(-i\omega t')|1\rangle$ and hope that $t' \approx t$ in order to have approximatively a copy of the original state. From Alice’s point of view, Bob is not able at all to prepare any coherent superposition of energy eigenstates with well-defined phase (relative to Alice’s time). In this sense, he has lost timing information in the measurement procedure. Adopting the latter point of view timing information is the ability to produce coherent superpositions of energy states with well-defined phase.

The paper is organized as follows. In Section II we introduce a quasi-order of clocks classifying resources with respect to their ‘worth’ for preparing coherence. As an example, we consider one mode of a light field in a Glauber state as the resource clock and investigate for which states it can provide enough timing information for producing a perfect coherent superposition in a two-level system: coherent states are only suitable in the limit of infinite phonon number but many other states in the Fock space can provide enough timing information.

In Section III we illustrate how to quantify the quality of clocks by their Fisher timing information. This concept will be useful for discussing quantum bounds on copying timing information. In Section IV we show that the Fisher timing information of two copies can only coincide with the resource’s timing information in the limit of an infinite amount of available energy. We derive bounds on the Fisher information of the copies depending on the clock’s energy. This describes quantitatively the transition from quantum timing information to classical timing information.

In Section V we show that the problem of evaluating the synchronism of two clocks can be reduced to the quality of a corresponding clock.

In this article clocks are assumed to be closed physical systems with well-known dynamical evolution, i.e.: no decoherence is considered. Hence the clock’s information about the actual time is only limited by (1) inaccurate initialization of the pointer position, (2) quantum fluctuations of the pointer position, and (3) the periodicity of the dynamics.

II. QUASI-ORDER OF CLOCKS

We characterize a quantum ‘clock’ abstractly by the pair $(\rho, H)$, where $\rho$ is the system’s density matrix and $H$ its Hamiltonian. According to the distinguishability of the time evolved states $\rho_t = \exp(-iHt)\rho \exp(iHt)$ different clocks have different worth. To classify the worth, we introduce the following quasi-order of clocks:

The clock $(\rho, H)$ is able to prepare the clock $(\hat{\rho}, \hat{H})$, formally writing

$$(\rho, H) \geq (\hat{\rho}, \hat{H})$$

if there is a completely positive trace preserving map $G$ such that $G(\rho) = \hat{\rho}$ and $G$ satisfies the covariance condition

$$G \circ \alpha_t = \hat{\alpha}_t \circ G \quad \text{for all } t \in \mathbb{R}$$

with the abbreviation

$$\alpha_t(\sigma) := \exp(-iHt)\sigma \exp(iHt)$$

for each density matrix $\sigma$ (and $\hat{\alpha}_t$ defined similarly).

We say that two clocks are equivalent if each of both majorizes the other in the sense of this quasi-ordering. The equivalence class of trivial (or ‘worthless’) clocks is given by all those with $[\rho, H] = 0$. They are majorized by all the other clocks.

We illustrate the intuition of this quasi-order in the following protocol:

Assume Alice sends a clock $(\rho, H)$ to Bob. Assume this system for the moment to be a spin 3/2 system rotating about its $z$-axis. Carol asks Bob to send her a clock showing as much about Alice’s time as possible, but she complains: ‘please do not send the spin 3/2 system, I am not able to read it out.’ Since she can only deal with spin 1/2 particles, she asks Alice to put as much information as possible about Alice’s time into a spin-1/2-particle. All that Bob can do is mathematically described by a completely positive trace preserving map $G$ from the set of density matrices of spin 3/2-systems to the set of density matrices of spin-1/2-systems. To illustrate why $G$ has to satisfy the covariance condition note that the state of the system received by Carol is given by

$$\hat{\alpha}_t \circ G \circ \alpha_t(\rho)$$

if the time $t$ has passed since the preparation of the resource state $\rho$ and Bob was running the conversion process at time $\tau$. Since Bob is not able to run the process at a well-defined time $\tau$, he applies a mixture over all $0 \leq \tau \leq t$, whether he wants or not. If we assume for simplicity that $t$ is large compared to the recurrence time of the quantum dynamics this mixture can equivalently be described by a time covariant map $\overline{G}$.

It is interesting to note that the covariance condition also appeared in a thermodynamic context in [4]. There we introduced a quasi-order classifying the thermodynamic worth of energy resources. The intuitive meaning of the covariance condition was the assumption that it is only allowed to apply energy conserving processes for converting one energy source into another, otherwise one would use additional energy sources. The reason that
the covariance condition appears in both cases is essentially the energy-time uncertainty principle: If a process is started by a negligible amount of switching energy it cannot be well-localized in time, i.e., the process can be started without using a clock.

One can easily extend the quasi-order in such a way that it is appropriate for classical and quantum systems. Then one can describe each system by a unital $C^*$-algebra $\mathcal{A}$ of observables and the states are positive functionals $\rho$ with $\rho(1) = 1$. Note that the expression $\rho(\mathcal{A})$ generalizes the expression $\text{tr}(\rho A)$ where $\rho$ is a density matrix. We shall use the latter notation if we are doing usual Hilbert space quantum mechanics and use the other notation if we discuss classical and quantum systems. The time evolution $\alpha$ on the states is given by the dual of an automorphism group on $\mathcal{A}$. A process or channel is described as follows. If the input is a state on the algebra $\mathcal{A}$ and the output a state on $\mathcal{A}$ the process $G$ is the dual of a completely positive unity-preserving map from $\mathcal{A}$ to $\mathcal{A}$.

A clock is defined as the pair $(\rho, \alpha)$ specifying the actual state and its dynamical evolution.

We set $(\rho, \alpha) \geq (\tilde{\rho}, \tilde{\alpha})$ if there is a process $G$ from the state space of $\mathcal{A}$ to the state space of $\mathcal{A}$ such that $\tilde{\rho} = G(\rho)$ and $G$ is satisfying the covariance condition

$$G \circ \tilde{\alpha}_t = \alpha_t \circ G \text{ for all } t.$$ 

A classical clock can for instance be given by a rotating pointer, i.e., a point $z$ on the unit circle $\Gamma \subset \mathbb{C}$ with the dynamical evolution

$$z \mapsto z \exp(i\omega t).$$

Then the observable algebra $\mathcal{A}$ is the set $C(\Gamma)$ of continuous functions on $\Gamma$. The dynamical evolution of the observable $f$ is given by

$$\alpha_t(f)(z) := f(z \exp(-i\omega t)).$$

A state $\rho$ on the classical algebra $C(\Gamma)$ is a probability measure on $\Gamma$. If $\rho$ is the point measure of a single point $z \in \Gamma$ the clock $(\rho, \alpha)$ has no inaccuracies at all. In the following we will use both notations $(\rho, H)$ and $(\rho, \alpha)$ for quantum systems, whereas classical clocks are always denoted by $(\rho, \alpha)$. Note that for two classical 'unit circle clocks' with the same periodicity and measures $\mu$ and $\tilde{\mu}$, respectively, the first clock majorizes the second if and only if there is a probability measure $\nu$ such that the convolution product $\mu \star \nu$ coincides with $\tilde{\mu}$.

**Time transfer from quantum to classical clocks**

Assume we have a quantum clock $(\rho, \alpha)$ rotating with the recurrence time $1$, i.e., $\rho \circ \alpha_1 = \rho$ and we want to transfer its timing information on a classical clock with the same period. Consider for instance the classical clock mentioned above with $\omega = 2\pi$. Time transfer is done by measuring the quantum system and initializing the classical pointer to the position $\exp(-i2\pi t') \in \Gamma$ according to the estimated time $t'$. The set of possible values $t'$ in this estimation might be discrete or continuous and a priori there is no restriction to the positive operator valued measure describing the possible measurements. Nevertheless one can show that each clock initialization can equivalently be performed by a time covariant measurement. Formally it is described by a family of positive operators $(M_t)_{t \in [0,1]}$ with $\int_0^1 M_t = 1$ such that $\alpha^*_t(M_t) = M_{t-1}$, where the parameter $t$ is defined modulo $1$ and $\alpha^*$ is the dual of $\alpha$.

The fact that only covariant measurements are relevant can be justified by a similar argument as above when the covariance condition for the CP-maps was motivated: covariance describes merely the fact that the time of the measurement and initialization procedure is unknown. The result that only time covariant measurements are relevant might seem astonishing at first sight, since optimal measurements of states transformed into each other by the action of a group are not in general group covariant and optimality has only be shown for irreducibility of the group action later on. This shows the difference between the following tasks: (1) Estimate the time at which the measurement has been taken place or (2) initialize a clock in such a way that one can get information about the actual time later on.

The set of covariant CP-maps $G$ from the quantum to the classical clock is in one-to-one correspondence to the set of covariant measurements. This can be seen as follows. For all set $A_t \subset \Gamma$ defined by $A_t := \{\exp(i2\pi t') \mid 0 \leq t' \leq t\}$ we postulate

$$G(\sigma)(A_t) = \int_0^t \sigma(M_{t'})dt' \quad \forall 0 \leq t \leq 1$$

for each density matrix $\sigma$. This defines a map $G$ for each covariant POVM $(M_t)$ and conversely a covariant family $(M_t)$ for each covariant $G$. This is rather intuitive since it states that the only transfer of timing information from quantum to classical clocks is given by measurements. Is the worth of such a classical clock less than the worth of the quantum clock that was measured? To see that in some sense, this is indeed the case assume one provides us with the additional information that the time $t$ is either equal to $t_1$ or to $t_1 + 1/2$. Then the qubit is always able to decide which value is true, whereas generally the classical clock can only provide probabilities for both cases.

**Time transfer from classical to quantum clocks**

The set of quantum clocks that can be prepared using a given classical clock is easy to describe. Consider the classical 'unit circle clock' $(\rho, \alpha)$ with period $1$ as above and a quantum clock $(\tilde{\rho}, \tilde{\alpha})$ with the same period. Assume
\[(\rho, \alpha) \geq (\hat{\rho}, \hat{\alpha})\]

and let \(G\) be the corresponding CP-map. Consider the point measure \(\delta_1\) on the point \(1 \in \Gamma\). Set \(\sigma := G(\delta_1)\). Since \(G\) is time covariant and preserves convex combination we have

\[G(\rho) := \int_0^1 \hat{\alpha}_t(G(\delta_1))d\rho(t).\]

Less formally speaking every procedure for obtaining a quantum clock from a classical one is completely defined by determining which quantum state \(G(\delta_1)\) one wants to prepare if the pointer of the classical clock is in position \(t = 0\). Due to the covariance condition one has to prepare the state \(\hat{\alpha}_t(G(\delta_1))\) if the clock is at position \(t\). This illustrates that a classical clock can only prepare pure superpositions of energy states if it is supported by discrete points. The quantum clock \(|0\rangle + \exp(-i2\pi t)|1\rangle\) for instance can only be prepared by the unit circle clock if \(\rho\) is a single point measure.

**Time transfer from quantum to quantum clocks**

In order to illustrate that the quasi-order of clocks describes constraints on many physical processes, we consider a two level system initialized in its lower state \(|0\rangle\). The resource clock, i.e., the optical on the maximally achievable fidelity for preparing the desired pure state. The resource clock, i.e., the optical mode is the system \((|\psi\rangle\langle\psi|, H)\), where \(|\psi\rangle\langle\psi|\) is the coherent state (‘Glauber state’) \(|\alpha\rangle\rangle\]

\[|\psi\rangle := \sum_n f_n |n\rangle\]

with \(|f_n|^2 = 1\) and \(f_n = \exp(-|\alpha|^2/2)|\alpha^n|/\sqrt{n!}\). The Hamiltonian \(H\) acts on the photon number states \(|n\rangle\) as \(H|n\rangle = n|n\rangle\) (if the frequency is assumed to be 1). The process initializing the rotating two-level system is described by the completely positive map \(G\) from the set of density matrices on the Fock space to the density matrices on \(\mathbb{C}^2\). We write \(G\) by Krauss operators

\[G\rho = \sum_j A_j \rho A_j^\dagger.\]

with \(\sum_j A_j^\dagger A_j = 1\). Due to the covariance condition for \(G\) the operators \(A_j\) can be chosen to be time invariant up to phase factors, i.e., \(\exp(-i\tilde{H}t)A_j \exp(iHt) = \exp(-i\lambda t)A_j\) where \(\tilde{H}\) is the two-level Hamiltonian \(H(j) = j|j\rangle\rangle\]

for \(j \in \mathbb{N}\) and \(c_j, d_j \in \mathbb{C}\). The condition \(\sum_j A_j^\dagger A_j = 1\) implies

\[|c_n|^2 + |d_{n-1}|^2 = 1 \ \forall n \in \mathbb{N}.\]

Now we prove that the quasi-order gives bounds on the achievable fidelity although we will not calculate them explicitly. The proof is valid for all states satisfying \(f_1 < f_0\).

Assume that \(|\psi\rangle\rangle\]

allows arbitrarily perfect preparation of the superposition state \(|\phi\rangle\rangle\]

Then there is a sequence \(G_n\) of time covariant completely positive maps with

\[\lim_{n \to \infty} G_n(|\psi\rangle\langle\psi|) = |\phi\rangle\langle\phi| .\]

By standard compactness arguments (each \(G_n\) maps into a finite dimensional space) we can assume without loss of generality (consider appropriate subsequences of \(G_n\)) that \(G(\sigma) := \lim_n G_n(\sigma)\) exists for all density matrices \(\sigma\). The map \(G\) is completely positive and time covariant.

By assumption, we have \(G(|\psi\rangle\langle\psi|) = |\phi\rangle\langle\phi|\). Hence each \(A_j|\psi\rangle\) has to be in the span of \(|\phi\rangle\). This implies

\[c_n f_{n-1} = d_n f_n\]

for all \(n \in \mathbb{N}_0\)

where we have defined \(c_0 := 0\). We conclude \(d_0 = 0\). Due to eq. (1) this implies \(c_1 = 1\). If \(f_1 < f_0\) this would imply \(d_1 > 1\) by eq. (1) in contradiction to eq. (1).

There is a rather intuitive argument showing that coherent states with large photon numbers (i.e. large \(\alpha\)) are able to prepare the equatorial qubit almost perfectly: in the limit of infinite photon numbers the coherent state has a classical phase without any quantum uncertainty. One might draw the erroneous conclusion that the statement proven above already follows from usual phase and photon number uncertainty principle, since only infinite uncertainty of photon number allows states without phase uncertainty. To see that this argument is wrong consider the state \(|n\rangle + |n + 1\rangle\) which is a good enough clock for preparing the rotating qubit. There are also less trivial examples. The state

\[\frac{1}{\sqrt{4}}(|0\rangle + |1\rangle + |2\rangle + |3\rangle)\]

also majorizes the equatorial qubit with respect to the clock ordering. To see this define Krauss operators \(A_n\) as above with

1The fact that the state \(\frac{1}{\sqrt{4}}(|0\rangle + |1\rangle + |2\rangle)\) does not majorize the equatorial qubit can be shown by similar arguments as above.
\[ c_{2n} = d_{2n} = 1 \]
\[ c_{2n+1} = d_{n+1} = 0 \]
for \( n \in \mathbb{N}_0 \). This illustrates that the equatorial qubit can either be prepared by a quantum clock or by a perfect classical clock (here: the classical phase). Note that the rotating classical phase \( z \in \Gamma \) of the macroscopic light field is a physical example for the abstract classical clock discussed above. Note that this classical clock does only exist as a limit of quantum systems with energy increasing to infinity.

Is it a general principle of quantum mechanics that classical timing information can only exist in the limit of infinite energy? In order to discuss this question consider a density matrix \( \rho_t \) evolving according to its Hamiltonian. Broadcasting of \( \rho_t \) is only possible if the different \( \rho_t \) mutually commute. Hence one can argue that timing information can only be classical information in this example if one assumes (by prior knowledge about \( t \)) that \( t = t_1, t_2, \ldots \) such that \( [\rho_{t_i}, \rho_{t_j}] = 0 \) for all pairs \( i, j \). If the distances between the possible times \( t_i \) are decreased the required energy of the system goes to infinity: assume that \( \rho_t \) commutes with \( \rho_{t+\tau} \) and \( \rho_t \neq \rho_{t+\tau} \). Then the dynamical evolution has permuted some eigenvectors of \( \rho_t \). It has been shown in [1] that a pure state \( |\psi\rangle \) can only evolve into an orthogonal one within the time interval \( \tau \) if its mean energy is as least \( h/2\tau \).

If no prior information about the time is available one can use the quasi-order of clocks as a formal basis to derive bounds on the timing information of two clocks obtained from one resource clock. The task is to find a clock of the form

\[ (\tilde{\rho}, \tilde{H}_1 \otimes 1 + 1 \otimes \tilde{H}_2) \]

that can be produced from the original clock \( (\rho, H) \) with the following property: The subsystems \( (\tilde{\rho}_1, \tilde{H}_1) \) and \( (\tilde{\rho}_2, \tilde{H}_2) \) obtained by tracing out the right or left system, respectively, are good clocks. In contrast to usual climbing problems, the task here is not to obtain optimal copies of \( \rho \) but to put as much timing information as possible into both systems. The states \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) can differ considerably from \( \rho \). In order to investigate the maximal timing information in both components we have to quantify the information. There are many possibilities for doing so, the quantity we chose in the next section has many useful properties that justify to consider it as a resource for physical processes.

III. FISHER TIMING INFORMATION

In order to quantify the accuracy of a clock consider the following situation in every-day life: imagine a classical clock having such a wide pointer that one is not able to read out its time exactly. Here it is straightforward to define the time error as the pointer’s width \( w \) divided by its velocity \( v \). To translate this to the quantum clock \( (\rho, H) \) we consider an observable represented by the POVM \( M := (M_\lambda)_{\lambda \in \mathbb{R}} \) defined in such a way that \( \text{tr}(\rho M_\lambda) \) is the probability for measuring values less or equal to \( \lambda \) (Note that this definition does not coincide with the definition in Section [1] where \( \rho(M_t) \) was the probability density for measuring \( t \). Here we do not restrict our attention to covariant measurements and cannot assume that a probability density describing the measurement exists.). The expectation value of the measurement outcomes is \( \text{tr}(\rho \int \lambda \, dM_\lambda) \) and moves with the velocity

\[ v := \text{tr}(-i[H, \rho]A) \]

with \( A := \int \lambda \, dM_\lambda \). The measurement values have a distribution with standard deviation

\[ w := \sqrt{\text{tr}(\rho \int (\lambda 1 - (\text{tr}(\rho A))^2) \, dM_\lambda)}. \]

We can consider the quotient \( v/w \) of the standard deviation and the time derivative of the expectation value as an analogue of the time error above. The problem of finding a POVM minimizing this error and finding bounds on the error is a quantum estimation problem that has already been studied in the literature [1] [4]. We rephrase some of the results in our setting and relate them to our quasi-order. As already noted in [1] the optimal POVM can always be chosen to be projector-valued. We give a different proof by showing that the projector valued measurement defined by the spectral resolution of \( A \) can never be worse than the POVM \( M \). This can be seen as follows: The expectation values and their time derivatives of both measurements coincide. The variances differ by \( \text{tr}(\rho(\int \lambda^2 \, dM_\lambda) - A^2) \). Some calculations show

\[ \int \lambda^2 \, dM_\lambda - A^2 = \int (\lambda 1 - \int \mu \, dM_\mu) \, dM_\lambda (\lambda 1 - \int \mu \, dM_\mu). \]

This notation may require some explanations. Assume the POVM consists of a finite set \( Q_j \) of positive operators with \( \sum_j Q_j = 1 \) corresponding to the measurement outcomes \( \lambda_j \). Then the corresponding expression is

\[ \sum_j ((\lambda_j - \sum_i \lambda_i Q_i) Q_j (\lambda_j - \sum_i \lambda_i Q_i)). \]

In the case of an infinite number of results, \( dM_\lambda \) is the analogue of \( Q_j \) and the summation is substituted by an integral.

The right hand side of eq. (6) is a positive operator since an operator of the form \( BCB \) with self-adjoint operators \( B \) and \( C \) is always positive and an integral of positive operators is positive too.

We define the Fisher timing information of a classical or quantum clock as the inverse of the squared time error for the optimal measurement. Formally, this reads as follows.
Definition 1: Let \((\rho, \alpha)\) be a clock, i.e., \(\rho\) is a state on a C*-algebra \(\mathcal{A}\) of observables and \(\alpha := (\alpha_t)_{t \in \mathbb{R}}\) be the group defining the time evolution of states. Then its Fisher timing information \(F\) is defined as

\[
F(\rho, \alpha) := \sup_A \frac{d}{dt} \alpha_t(\rho)(A)|_{t=0}^2/\rho(A^2)
\]

where one optimizes over all self-adjoint operators \(A \in \mathcal{A}\).

To justify this terminology we show that \(F\) reduces to the usual Fisher information when applied to classical clocks. Let the state be a probability distribution on a phase space \(X\). Assume that it is given by a density function \(x \mapsto p(x)\). At time \(t\) we have the density function \(x \mapsto p_t(x)\). Let \(x \mapsto f(x)\) be an observable, i.e., a continuous real function on the phase space. The time derivative of its expectation value is given by

\[
\int f(x) \frac{d}{dt} p_t(x) dx
\]

and its variance by

\[
\int f^2(x) p_t(x) dx,
\]

if the expectation value of \(f\) is assumed to be zero without loss of generality. For technical reasons we assume \(p_t(x) \neq 0\) for all \(x\). Then we can define an inner product on the set of real continuous bounded functions on \(X\) by

\[
\langle f | g \rangle := \int f(x) g(x) p(x) dx.
\]

We have to maximize the expression

\[
\frac{\langle (\dot{p}_t / p_t) | (\dot{f}) \rangle^2}{\langle f | f \rangle}
\]

with \(\dot{p}_t = d/dt p_t\). For geometric reasons the function

\[
f := \frac{\dot{p}_t}{p_t} = \frac{d}{dt}(\ln p_t)
\]

is optimal. Then \(F\) is given by the usual Fisher information

\[
\int \frac{(\dot{p}_t(x))^2}{p_t(x)} dx,
\]

a quantity that is well-known in classical statistics. It is decisive as a bound on the error of estimators \(\mathbb{E}\), the so-called Cramér-Rao bound.

For finite dimensional quantum systems the optimization can be performed similarly: In order to maximize the expression \(tr(\dot{\rho} A)/tr(\rho A^2)\) we can introduce a bilinear form

\[
\langle A | B \rangle := tr(A \Delta_\rho B)
\]

on the set of self-adjoint operators introducing the super operator \(\Delta_\rho\) with \(\Delta_\rho B := (\rho B + B \rho)/2\). For technical reasons we assume \(\rho\) to have full rank such that \(\Delta_\rho\) has an inverse. Following \([14,13]\) we obtain the ‘symmetric logarithmic derivative’ \(\Delta_\rho^\alpha \rho\) as optimal observable with \(\dot{\rho} := i[\rho, H]\). We conclude the following.

Theorem 1: The Fisher timing information of a quantum clock \((\rho, H)\) on a Hilbert space is given by

\[
F(\rho, H) = tr(\dot{\rho} \Delta_\rho^{-1} \dot{\rho})
\]

with \(\Delta_\rho^{-1}\) and \(\dot{\rho}\) as above.

As noted in \([1]\) this expression appears also if one calculates the Bures distance between the states \(\rho\) and \(\rho_{dt}\) for infinitesimal time \(dt\):

\[
d_{\text{Bures}}(\rho, \rho_{dt}) = tr(\dot{\rho} \Delta_\rho^{-1} \dot{\rho})(dt)^2.
\]

Introducing the fidelity \(\bar{F}\)

\[
f(\rho_1, \rho_2) := \inf_M \sum_b \sqrt{\rho_1(M_b) \rho_2(M_b)},
\]

where \(M := (M_b)_b\) is an arbitrary POVM, one can write the Bure distance as

\[
d^2_{\text{Bures}}(\rho_1, \rho_2) = 2 - 2f(\rho_1, \rho_2).
\]

Since the fidelity is clearly monotone with respect to completely positive maps the Bure distance is monotone too. We conclude that Fisher timing information is monotone with respect to our quasi-order of clocks, since the state \(G(\rho)\) evolves to the state \(G(\rho_{dt})\) for every time covariant CP-map \(G\). Formally we state the following:

Lemma 1: For each two quantum or classical clocks we have:

\[
(\rho, \alpha) \geq (\tilde{\rho}, \tilde{\alpha})
\]

implies

\[
F(\rho, \alpha) \geq F(\tilde{\rho}, \tilde{\alpha}).
\]

Similar arguments show that Fisher timing information is constant according to the system’s Hamiltonian evolution, i.e., \(F(\rho, \alpha) = F(\alpha_t(\rho), \alpha)\). This justifies to consider it as a convenient measure characterizing the quality of resources for preparing time-localized signals. This is even more justified by noting that the quantity is additive with respect to the composition of independent quantum clocks:

Lemma 2: Fisher information of quantum systems is an additive quantity, i.e.,

\[
F(\rho \otimes \tilde{\rho}, H \otimes 1 + 1 \otimes \tilde{H}) = F(\rho, H) + F(\tilde{\rho}, \tilde{H})
\]
Proof: As already noted in [12] the optimal observable of the composite system is given by the sum
\[ \Delta_\rho \hat{\rho} \otimes 1 + 1 \otimes \Delta_\rho \hat{\rho} . \]
Then the statement follows immediately. \( \Box \)

Similarly, for classical clocks with a probability distributions \( p \) and \( \tilde{p} \) moving on the phase spaces \( X \) and \( X \) the optimal observable \( (x, y) \mapsto f(x, y) \) on \( X \times X \) is given by the sum
\[ f(x, y) = \frac{\tilde{p}(x)}{p(x)} + \frac{\tilde{p}(y)}{p(y)} . \]
Here the additivity of Fisher timing information follows straightforwardly. We conjecture that additivity is even true for the composition holds even for the composition of quantum with classical clocks.

Examples

We give two examples where Fisher timing information can be calculated explicitly.

1. Let \( p_t \) be a Gauss distribution with standard deviation \( \sigma \) moving with the velocity \( v \) on the real line, i.e.,
\[ p_t(x) := \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(x - tv)^2}{2\sigma^2}\right). \]
Then \( F \) is given by
\[ F = \int \frac{d}{dt}p_t(x)^2 \frac{1}{p_t(x)} \, dx \]
\[ = \int \frac{d}{dt} \ln p_t(x)^2 p_t(x) \, dx = \frac{v^2}{\sigma^2}. \]

2. Let \( \rho := |\psi\rangle \langle \psi| \) be a pure state of a quantum system with Hamiltonian \( H \). Then the Fisher timing information is given by 4 times the variance of the energy values [12], i.e.,
\[ F = 4(\langle |\psi| H |\psi| \rangle - (\langle |\psi| H |\psi| \rangle)^2). \]

Note that the intuitive meaning of the quantity \( F \) becomes even more evident if one considers a large number \( n \) of copies of the same clock:
\[ (\rho, H)^n := (\rho \otimes \ldots \otimes \rho, H \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes H \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes H). \]
First note that the error of the large clock is decreasing in the order of \( 1/\sqrt{n} \), i.e., the error probability for distinguishing between \( \rho^{\otimes n} \) and \( \rho_1^{\otimes n} \) does neither converge to 0 nor to 1 (for details see [13][17]). Then consider observables of the type
\[ A_n := \frac{1}{\sqrt{n}} (A \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes A \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes A) , \]
where \( A \) is any self-adjoint operator acting on the single clock’s Hilbert space with \( \rho(A) = 0 \). Due to the central limit theorem, \( A_n \) is approximatively Gauss distributed for large \( n \). The standard deviation of \( A_n \) is given by \( \sqrt{n}A^2 \). Easy calculation shows that
\[ \lim_{n \to \infty} (\rho/\sqrt{n})^{\otimes n} (A_n) = t \rho(A). \]
This has a very intuitive implication: on the time scale \( 1/\sqrt{n} \) the expectation value of the Gauss distribution moves with velocity \( \rho(A) \). If one allows only measurements of the type \( A_n \) the only criterion for the distinguishability of the states \( \rho_t/\sqrt{n} \) (for different values of \( t \)) is the quotient of the velocity and \( \rho(A) \) the \( \sqrt{n}A^2 \), i.e., the square root of the Fisher timing information.

Remarkably, observables of the type \( A_n \) can be considered as approximate time operators in the following sense. Multiply the total Hamiltonian of the system by the factor \( 1/\sqrt{n} \) and note that this rescaled Hamiltonian \( H_n \) is also an observable of the type eq. (4). In a suitable sense, the commutator
\[ [A_n, H_n] = \frac{1}{n} [A, H]. \]
converges to \( \rho([A, H]) \) times the identity operator in the limit \( n \to \infty \). This is made precise in so-called non-commutative central limit theorems [16][17]. In this approach the distribution of the observables \( A_n \) is given by quasi-free states on a CCR-algebra (‘algebra of fluctuations’). In the CCR-algebra the corresponding limit observables \( A_\infty \) and \( H_\infty \) satisfy
\[ \exp(iH_\infty t)A_\infty \exp(-iH_\infty t) = A_\infty + ct1 \]
with \( c := \rho(i[H, A]) \).
This indicates the quantum estimation problem of distinguishing the states \( \rho_t^{\otimes n} \) for different \( t \) becomes more and more related to the estimation problem in a quantum Gaussian channel (for Gaussian channels see [18]).

IV. COPYING TIME-LOCALIZED SIGNALS

Since Fisher timing information seems to be a convenient quantity to measure the quality of a clock (at least one aspect of it) it is straightforward to ask whether it is possible to produce two clocks from one resource clock such that the timing information of each is the same as the resource’s timing information. Formally speaking we have a resource clock \( (\rho, H) \), and look for a clock
\[ (\sigma, H_1 \otimes 1 + 1 \otimes H_2) \]
acting on a joint Hilbert space such that the clocks $(\rho_1, H_1)$ and $(\rho_2, H_2)$ given by the partial traces over the Hilbert space components have a large amount of Fisher information and

$$(\rho, H) \geq (\sigma, H_1 \otimes 1 + 1 \otimes H_2).$$

Note that the states of the produced clocks are allowed to be highly correlated. Hence one cannot argue that the Fisher information of the resource has to be divided between both clocks, i.e., we may have

$$F(\rho_1, H_1) + F(\rho_2, H_2) > F(\rho, H).$$

Here we address the question whether it is possible that both clocks are of the same quality as the resource, i.e., whether one can have

$$F(\rho_1, H_1) = F(\rho_2, H_2) = F(\rho, H).$$

The answer is that it is never possible if the energy of the produced clocks is bounded. To see this assume that $H_1$ and $H_2$ are positive operators. Let $T_1$ and $T_2$ be the optimal time observables for $C_1 := (\rho_1, H_1)$ and $C_2 := (\rho_2, H_2)$, respectively. We assume $\rho_1(T_1) = \rho_2(T_2) = 1$ and $\rho_1(T_1) = \rho_2(T_2) = 0$ without loss of generality. Then the values of the Fisher timing information $F_1$ and $F_2$ of $C_1$ and $C_2$ are given by $1/\rho_1(T_1^2)$ and $1/\rho_2(T_2^2)$, respectively. One has

$$\rho_1(T_1^2) + \rho_2(T_2^2) \geq \frac{1}{2}(\sigma((T_1 \otimes 1 + 1 \otimes T_2)^2) + \sigma((T_1 \otimes 1 - 1 \otimes T_2)^2)).$$

Note that $\sigma(T_1 \otimes 1 + 1 \otimes T_2) = 2$. Therefore the variance of this observable can not be less than 4 divided by the Fisher information $F$ of the state $\sigma$. Due to the monotony of Fisher information with respect to the quasi-order we have $F \leq F$, where $F$ is the Fisher information of the resource clock. Hence we find

$$1/F_1 + 1/F_2 \geq 2/F + \sigma((T_1 \otimes 1 - 1 \otimes T_2)^2) \geq 2/F + V_\sigma(T_1 \otimes 1 - 1 \otimes T_2),$$

where we have introduced the notation $V_\sigma(A)$ for the variance of any observable $A$ in the state $\sigma$, i.e.,

$$V_\sigma(A) := \sigma(A^2) - (\sigma(A))^2.$$

By Heisenberg’s uncertainty relation we have

$$V_\sigma(T_1 \otimes 1 - 1 \otimes T_2) V_\sigma(H_1 \otimes 1 - 1 \otimes H_2) \geq \frac{1}{4}(\sigma((T_1 \otimes 1 - 1 \otimes T_2, H_1 \otimes 1 - 1 \otimes H_2))^2 \geq \frac{1}{4}(\sigma(T_1 \otimes 1 + 1 \otimes T_2))^2 = 1.$$

We have

$$V_\sigma(H_1 \otimes 1 - 1 \otimes H_2) \leq \sigma((H_1 \otimes 1 - 1 \otimes H_2)^2) \leq \sigma((H_1 \otimes 1 + 1 \otimes H_2)^2),$$

where the last inequality follows since $H_1$ and $H_2$ are positive operators by assumption.

We conclude

$$\sigma((T_1 \otimes 1 - 1 \otimes T_2)^2) \geq 4/\sigma((H_1 \otimes 1 + 1 \otimes H_2)^2 - (\sigma(H_1 \otimes 1 - 1 \otimes H_1))^2).$$

With eq. (5) we have

$$1/F_1 + 1/F_2 \geq 2/F + 2/\sigma((H_1 \otimes 1 + 1 \otimes H_2)^2).$$

We introduce the abbreviation

$$\langle E^2 \rangle := \rho((H_1 \otimes 1 + 1 \otimes H_2)^2)$$

for the expectation value of the square of the output’s energy. Then we can write

$$1/F_1 + 1/F_2 \geq 2/F + 2/\langle E^2 \rangle.$$

Assume one wants to have the same Fisher timing information in both outputs, i.e., $F_1 = F_2$. Then one obtains

$$1/F_1 \geq 1/F + 1/\langle E^2 \rangle.$$

This implies that the timing information of the target systems is less than the information of the resource by an amount determined by the square of the available energy. One can see that perfect copy of the timing information, i.e., $F_1 = F_2 = F$ is only possible in the limit $\langle E^2 \rangle \rightarrow \infty$. In particular if the energy of the system is bounded by a maximal energy value $E_{\text{max}}$ the loss of timing information in the copies can be expressed in terms of $E_{\text{max}}$. Consider for example the case that the output systems are coherent light pulses with frequency $\nu$. Its energy is Poisson distributed. The expectation value of its energy is $Nh\nu$ if $N$ is the average photon number. The variance is $Nh\nu$ as well. Hence we find

$$\langle E^2 \rangle = N h \nu^2 + N h \nu$$

for the expectation value of the energy squared. Then equation (6) gives a lower bound on the loss of timing information of the copies compared to the resource (depending on the mean photon number).

Assume now that the joint state $\tilde{\rho}$ of the target systems is pure. Then the covariance of the energy values coincides with the Fisher information of the joint output state. We can write

$$\sigma((H_1 \otimes 1 + 1 \otimes H_2)^2) = \langle E \rangle^2 + F(\sigma, H_1 \otimes 1 + 1 \otimes H_2),$$

with the abbreviation

$$\langle E \rangle := \tilde{\rho}(H_1 \otimes 1 + 1 \otimes H_2).$$

Since the Fisher information of the joint output state cannot be larger than $F$ we obtain
\[ 1/F_1 \geq 1/F + 1/(F + \langle E \rangle^2). \]

The copies can only obtain the full timing information of the input in the limit of infinite average energy (see Fig. 1). Since \( \langle E^2 \rangle \) can be infinite even for finite \( \langle E \rangle \) the bound for pure states is considerably tighter than for mixed states.

Note that the assumption of limited energy is indeed necessary. In the following we sketch an example of an input signal of large energy that produces two outgoing signals each of it having almost the same Fisher timing information as the input. Let \( \mathcal{H} := L^2(\mathbb{R}^+) \) be the square-integrable functions on the set of non-negative numbers. Define an operator \( A : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \equiv L^2(\mathbb{R}^+ \times \mathbb{R}^+) \) by

\[
(A \psi)(x, y) := \begin{cases} 
\psi(x + y)/\sqrt{x + y} & \text{for } x, y \geq 0 \\
0 & \text{else}
\end{cases}
\]

Since \( A \) is an isometry we have \( A^\dagger A = 1 \), hence the map

\[
G(.) := A(.)A^\dagger
\]

is completely positive and trace preserving. For both copies of \( \mathcal{H} \) assume the Hamiltonian to be given by the multiplication operator

\[
(H \psi)(x) := x \psi(x).
\]

Then \( G \) is clearly time covariant, since \( A \) maps states onto states with the same energy distribution. The operator \( A \) maps the wave function of the in-going signal on the joint wavefunction of the outgoing signals.

Assume now the in-going wave-function \( |\psi(x)\rangle^2 \) to be a Gauss distribution with expectation value \( E \) and standard deviation \( \Delta E \) such that \( E \gg \Delta E \). Its Fisher timing information is given by \( F = 4(\Delta E)^2 \) due to Example 2 in Section [I]. The wave package \( (x, y) \mapsto A \psi(x, y) \) is essentially localized on a strip of length \( \sqrt{2}E \) and width in the order of \( \Delta E \). Along the lines \( x = y + \epsilon \) with \( |\epsilon| \ll E \) the function \( (x, y) \mapsto |A \psi(x, y)|^2 \) is approximatively a Gaussian wave package with standard deviation \( \Delta E/\sqrt{2} \). Introducing the isometric Fourier transformation

\[
\mathcal{F} : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})
\]

with

\[
\mathcal{F}(f)(k) := \frac{1}{\sqrt{2\pi}} \int f(x) \exp(-ikx)dx
\]

the probability distribution according to

\[
|\mathcal{F}(A \psi(x, y))|^2
\]

is for large \( E \) essentially localized on the line \( x = y \). Along this line it is a Gauss distribution. To calculate its standard deviation note that the spread of of \( |A \psi|^2(x, y) \) along the \( x = y \)-line is \( \Delta E/\sqrt{2} \). Hence we obtain the value \( \sqrt{2}/(2\Delta E) = 1/(\sqrt{2}\Delta E) \).

To estimate the timing information of each of the signals we define an operator \( \hat{T} \) on \( L^2(\mathbb{R}) \) by

\[
\hat{T} \psi(x) := \psi(x)
\]

and an operator \( T \) on \( L^2(\mathbb{R}^+) \) by

\[
T := \mathcal{F}^\dagger \mathcal{T} \mathcal{F}.
\]

Now we will show that \( T \otimes 1 \) is an operator with the property that its variance is approximatively equal to \( 1/(4(\Delta E)^2) \) and its expectation values has time derivative 1. In order to avoid lengthy technicalities, we sketch this by hand-waving arguments.

To see that the time derivative of the expectation values of \( T \otimes 1 \) is equal to 1 note that the dynamical evolution on each Hilbert space component \( L^2(\mathbb{R}) \) is a translation by \( t \) after the time \( t \). Hence the time derivative is 1 for every state. There is, however, a difficulty in calculating the variance of the observable \( T \otimes 1 \) with respect to the state \( A \psi \). One would like to make use of the fourier transformation and calculate the variance of \( T \) by calculating the variance of \( \hat{T} \) with respect to the fourier transformed wave-package. Unfortunately this is wrong because \( \mathcal{F} \) is not unitary and we have

\[
T^2 = \mathcal{F}^\dagger \hat{T} \mathcal{F} \mathcal{F}^\dagger T \mathcal{F}^\dagger \neq \mathcal{F}^\dagger \hat{T}^2 \mathcal{F}.
\]

Note that the operator \( \mathcal{F} \mathcal{F}^\dagger \) is not the identity but the projector onto the image of \( \mathcal{F} \). However, this difference becomes irrelevant in the limit of \( E \to \infty \). The wave-function \( (\mathcal{F} \otimes \mathcal{F})A \psi \) contains essentially large positive frequencies since \( E \) is large. Hence \( (T \otimes 1)(\mathcal{F} \otimes \mathcal{F})A \psi \) contains essentially positive frequencies. We conclude that the variance of

\[
\hat{T} \otimes 1
\]

with respect to the fourier transformed wave-function \( (\mathcal{F} \otimes \mathcal{F})A \psi \) is a good approximation for the variance of
and the error goes to zero for $E \to \infty$. The variance of $T \otimes 1$ is easy to see in the limit of $E \to \infty$. Note that is is given by the variance of the $x$-values according to the probability density $|(\mathcal{F} \otimes \mathcal{F})A\psi|^2(x, y)$ which is more and more localized on the $x = y$-line for large $E$. As we have already argued, the statistical distribution along this line is Gaussian with standard deviation $1/\sqrt{2\Delta E}$. The standard deviation of the $x$-values is therefore given by $1/(2\Delta E)$ and the corresponding variance by $1/(4(\Delta E)^2)$. The Fisher timing information of each signal is therefore approximately given by $4(\Delta E)^2$, the same value as we have assumed for the in-going signal.

Note that this does not mean that the original state $|\psi\rangle$ would have been cloned approximatively. The joint state $A\psi$ is highly entangled as the measurement values of the time observables $T \otimes 1$ and $1 \otimes T$ are almost perfectly correlated. Hence the reduced states for both systems are highly mixed. This shows that Fisher timing information can approximatively be copied although the coherence has mostly been destroyed.

V. CLOCK SYNCHRONISM AND RELATIVE TIMING INFORMATION

So far we have only discussed timing information with respect to an ‘absolute’ clock that is not considered explicitly. In most of the applications the absolute time is less relevant than the fact that two parties have the same time, i.e., their clocks are synchronized. It is obvious that the problems of clock synchronization and measuring the absolute time are strongly related. Assume for instance that Alice has two clocks, one showing the absolute time without any error and the other showing the time approximatively. She sends the inaccurate clock to Bob. Then the clocks of Alice and Bob are more or less synchronized and the quality of the synchronism is directly given by the quality of the transferred clock. Conversely, assume that Alice and Bob have a clock each showing approximatively the same time but the relation between the time of these clocks to the absolute time is totally unclear. Now we tell Alice the absolute time. She will use this information to calculate the difference between the true time and her clock. In our formal setting it is given by

$$p(z_A | z_B) = q(z_A z_B^{-1})$$

with a probability density $q : \Gamma \to \mathbb{R}^+$. Due to the time invariance of the joint state we have $p(z_B | z_A) = p(z_A | z_B)$. If the clocks are well synchronized the probability density $q$ is essentially supported around a specific value $z \in \Gamma$ expressing the fact that the difference between Alice’s and Bob’s clock is approximately known to both. Now we tell Alice the absolute time $t_0$. She will use this information to calculate the difference between the true time and her clock. In our formal setting it is given by $z_A \exp(-i\omega t_0)$. Alice sends this value to Bob. He can turn his pointer around according to this difference value. The position of his pointer will now be distributed according to the probability density $q$.

For quantum systems the situation seems to be more difficult at first sight: Alice’s and Bob’s clocks may be classically correlated or entangled. If the synchronization is given by quantum correlations, it is not clear to what extent synchronism is destroyed by measurements on Alice’s or Bob’s clock. Up to now, it is not even clear how to define synchronism at all. Here we suggest a formal definition of synchronism that allows to compare the quality of synchronization of different bipartite systems in the same way as we have compared clocks.

**Definition 2** A synchronism is a joint state $\rho$ on a bipartite system $A \times B$ with separate time evolutions $\alpha$ and $\beta$ on $A$ and $B$ such that $\rho$ is invariant with respect to $\alpha \otimes \beta$. Formally, we will denote a synchronism by $(\rho, \alpha, \beta)$.

Synchronism can be compared as follows.

**Definition 3** The bipartite system $A \times B$ is said to be better than or equally synchronized as the system $A \times \tilde{B}$, formally denoted by

$$(\rho, \alpha, \tilde{\beta}) \geq (\hat{\rho}, \hat{\alpha}, \tilde{\beta})$$

if there is a process $G$ converting $\rho$ to $\hat{\rho}$ satisfying the covariance condition.
The intuitive meaning of the covariance condition is that the communication protocol implementing the joint transformation $G$ on the bipartite system is invariant with respect to a relative time displacement between Alice’s actions and Bob’s actions, i.e., Alice and Bob do not require external synchronized clocks in order to implement $G$. The reason for restricting the attention to stationary states $\rho$ is that a non-trivial time evolution of the joint state would mean that the bipartite system carries some information about the absolute time. This would be confusing in the following considerations. It is natural to define an equivalence relation on synchronizations by

\[
(r, \alpha, \beta) \equiv (\tilde{r}, \tilde{\alpha}, \tilde{\beta})
\]

if

\[
(r, \alpha, \beta) \geq (\tilde{r}, \tilde{\alpha}, \tilde{\beta})
\]

and

\[
(\tilde{r}, \tilde{\alpha}, \tilde{\beta}) \geq (r, \alpha, \beta).
\]

We observe:

**Observation 1** For two synchronizations we have

\[
(r, \alpha, \beta) \geq (\tilde{r}, \tilde{\alpha}, \tilde{\beta})
\]

if and only if

\[
(r, \alpha^{-1} \otimes \beta) \geq (\tilde{r}, \tilde{\alpha}^{-1} \otimes \tilde{\beta})
\]

with respect to the quasi-order of clocks.

This is trivial from the mathematical point of view. Note that

\[
(\rho, \alpha^{-1} \otimes \beta) \geq (\tilde{\rho}, \tilde{\alpha}^{-1} \otimes \tilde{\beta})
\]

if and only if

\[
(\rho, \alpha^{-1/2} \otimes \beta^{1/2}) \geq (\tilde{\rho}, \tilde{\alpha}^{-1/2} \otimes \tilde{\beta}^{1/2})
\]

since these clocks are moving with half of the velocity than those above. Observe furthermore that we have

\[
(\rho, \alpha^{-1/2} \otimes \beta^{1/2}) \equiv (\rho, 1 \otimes \beta)
\]

since we can define a process $G$ by

\[
G := \int_0^1 \alpha_t \otimes \beta_t dt,
\]

satisfying $G(\rho) = \rho$ and $G \circ (\alpha^{-1/2} \otimes \beta^{1/2}) = (1 \otimes \beta) \circ G$ and $(\alpha^{-1/2} \otimes \beta^{1/2}) \circ G = G \circ (1 \otimes \beta)$.

We would like to give an intuitive meaning to some of those clocks. For doing so, we show in the following that the clock $(\rho, 1 \otimes \beta)$ is the best clock that Bob can obtain if we tell Alice the absolute time and allow Alice and Bob to communicate by a classical and quantum channel without transferring clocks. This point has to be clarified. If Alice would send a rotating two-level system to Bob this would be a clock transfer. But we do allow Alice to send a degenerated quantum system, i.e., to send quantum information without sending a clock. This description may be a good approximation for the more realistic physical situation that the systems that Alice can send to Bob evolve on a time scale that is considerably slower than the time scale of the considered synchronization.

The following protocol allows Bob to obtain the clock $(\rho, 1 \otimes \beta)$ if one tells Alice the absolute time.

1. Tell Alice that the true time is $t_0 > 0$.

2. Alice implements the transformation $\alpha_{t_0}$ on her system and puts her part of the joint state into a physical system that is equal to her original one with the only difference that its time evolution is trivial. In the classical situation, she uses a system with the same phase space as the original, in the quantum case an isomorphic Hilbert space with the Hamiltonian $H = 0$ can be used. She sends this ‘frozen clock’ to Bob. This has not necessarily to be performed by transferring a real physical system. In the quantum case Alice can teleport this clock using prior entanglement on degenerated systems. In the classical case she can phone Bob and tell him the state of her frozen clock.

If the initial joint state is $\rho$ (at the time 0) then Bob will obtain the state

\[
(1 \otimes \beta_{t_0}^{-1}) \circ (\alpha_{t_0^{-1}} \otimes 1) \circ (\alpha_{t_0 \otimes 1} \otimes \beta_{t_0})(\rho) = (1 \otimes \beta_t)(\rho)
\]

at the time $t$. Formally he has the clock $(\rho, 1 \otimes \beta)$. To see that this is the best clock that Bob can obtain we show first that each clock defines a synchronism as follows. Assume that Bob has an arbitrary clock $(\sigma, \gamma)$ with period 1, i.e., at the time $t$ the state is $\gamma_t(\sigma)$. Alice can read the true time $t \in [0, 1)$ on her clock without any error. Now we consider an observer who does not know anything about the true time. From his point of view the state of the joint system is stationary and given by an equal distribution over all states of the form $\exp(i2\pi t) \otimes \gamma(\rho)$. This is a non-trivial synchronism of the bipartite system. If Bob starts with the clock $(\sigma, \gamma) = (\rho, 1 \otimes \beta)$ we can obtain the synchronism $(\rho, \alpha, \beta)$ by the following protocol (obtained by reversing the scheme above). (1) Bob sends the left part of the state $\rho$ on a stationary system to Bob. (2) Alice puts the left part of the state $\rho$ on a system with dynamical evolution $\alpha$ and applies an additional transformation $\alpha_{t_0}$ to the system according to the present value $t_0$ of her absolute clock. Although we omit the formal proof this protocol shows that the following two situations are equivalent with respect to their synchronism:
1. Bob has the clock \((\rho, 1 \otimes \beta)\) and Alice has a perfect classical clock.

2. The state \(\rho\) is the joint state of Alice and Bob with the time evolution \(\alpha\) and \(\beta\) on Alice’s and Bob’s system, respectively.

These considerations define a one-to-one correspondence between equivalence classes of synchronism and equivalence classes of clocks. The fact that the clock obtained from a synchronism can be used to recover the original synchronism shows that \((\rho, 1 \otimes \beta)\) is indeed the best achievable clock from the synchronism \((\rho, \alpha, \beta)\). Otherwise the better clock \((\sigma, \gamma)\) would allow to obtain a better synchronism than the resource \((\rho, \alpha, \beta)\).

The difference between the clock that Bob can obtain if we tell Alice the true time and the clock Alice can obtain if we tell Bob the time can easily be understood by comparing \((\rho, \alpha^{-1/2} \otimes \beta^{1/2})\) to \((\rho, \alpha^{1/2} \otimes \beta^{-1/2})\). Formally, Alice’s clock is the time-reversed of Bob’s clock. Consider the classical unit circle clocks above. Then Bob obtains the probability density \(q\) whereas Alice would obtain the probability density \(z \mapsto q(z^{-1})\). The clock defined by such a distribution is equivalent to a clock with probability density \(q\) with time-reversed dynamical evolution.

The arguments above show that the problem of evaluating synchronizations can completely be reduced to the problem of evaluating clocks. One could for instance use Fisher timing information to describe the accuracy of synchronism.

**VI. CONCLUSIONS**

The main idea of the quasi-order of clocks is to consider coherent superpositions of energy eigenstates as a resource carrying information about the time at which the state has been prepared. The source of timing information can also be a sender transmitting a signal, for instance a time-localized pulse. The quasi-order of clocks describes the fact that nobody can change the signal in such a way that the information about the sender’s time is increased except he receives additional hints about the sender’s time. This implies lower bounds on the achievable fidelity if a two-level system should be driven in a superposition state by a coherent laser field.

Furthermore the quasi-order gives restrictions to the possibility of copying timing information: If a device produces two outgoing signals from an input signal, the timing information of the outgoing signals cannot be larger than the original one. This does not mean necessarily that the timing information of each of both outgoing signals is less than the original one. The outgoing pulses might be perfectly correlated in time such that the timing information of each component coincides with the information inherent in both. However, perfect correlations are only possible in the limit of infinite signal energy. Otherwise at least one of the signals will be less accurately localized in time. We quantified the time accuracy of signals by the statistical distance of the actual state and the state after an infinitesimal time and proved a lower bound on the loss of accuracy in the signal cloning process. Since this lower bound depends on the total energy of the outgoing signals we have found a constraint for low-power signal processing.

Taking a thermodynamic point of view, one might state that coherent superpositions of energy states are a resource which have to be supplied by a source of timing information.

In the same way as timing information is a resource for a single party system the relative timing information or synchronism of a bipartite system is a resource if no exchange of clocks is allowed. The problem of classifying bipartite systems with respect to their synchronism is shown to be equivalent to the problem of classifying clocks provided that it is allowed to exchange stationary quantum or classical systems.

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[1] V. Giovannetti, S. Lloyd, L. Maccone, and M. Shahriar. Physical limits to clock synchronization. *LANL preprint*, quant-ph/0110154.

[2] H. Fang, K. Matsumoto, X. Wang, and M. Wadati. Quantum cloning machines for equatorial qubits. *LANL preprint*, quant-ph/0101010.

[3] C. Fuchs. Distinguishability and accessible information in quantum theory. *LANL preprint*, quant-ph/9601020, 1995.

[4] D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth. Thermodynamic cost of reliability and low temperatures: Tightening landauer’s principle and the second law. *Int. Jour. Theor. Phys.*, 39(12):2217–2753, 2000. *LANL preprint*, quant-ph/0002048.
[5] O. Bratteli and D. Robinson. *Operator algebras and quantum statistical mechanics*. Springer, New York, 1987.

[6] A. Peres and P. Scudo. Covariant quantum measures may not be optimal. LANL preprint quant-ph/0107114.

[7] E. Davies. Information and quantum measurement. *IEEE Trans. Inform. Theory*, IT-24(5):596–599, 1978.

[8] M. Sasaki, S. Barnett, R. Josza, M. Osaki, and O. Hirota. Accessible information and optimal strategies for real symmetrical quantum sources. LANL preprint, quant-ph/9812062v3.

[9] D. Walls and G. Milburn. *Quantum optics*. Springer, Heidelberg, 1994.

[10] N. Margolus and L. Levitin. The maximum speed of dynamical evolution. LANL preprint quant-ph/9710043 v2.

[11] S. Braunstein and C. Caves. Statistical distance and the geometry of quantum states. *Phys. Rev. Lett.*, 72(22):3439, 1994.

[12] B. Braunstein, C. Caves, and G. Milburn. Generalized uncertainty relations: Theory, examples and lorentz invariance. *Ann. Phys.*, 247:135–173, 1996.

[13] A. Holevo. *Probabilistic and Statistical Aspects of Quantum Theory*. North-Holland, Amsterdam, 1982.

[14] C. Helstrom. *Quantum Detection and Estimation Theory*. Academic Press, New York, 1976.

[15] H. Cramér. *Mathematical methods of statistics*. Princeton University, Princeton, 1946.

[16] D. Goderis, A. Verbeure, and P. Vets. Non-commutative central limits. *Prob. Th. Rel. Fields*, 1989.

[17] D. Goderis and P. Vets. Central limit theorem for mixing quantum systems. *Comm. Math. Phys.*, 1989.

[18] A. Holevo. Statistical decision theory for quantum systems. *J. Multivar. Analys.*, 1973.