THE GEOMETRY OF ONE-RELATOR GROUPS SATISFYING A POLYNOMIAL
ISOPERIMETRIC INEQUALITY

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ABSTRACT. For every pair of positive integers \( p > q \) we construct a one-relator group
\( R_{p,q} \) whose Dehn function is \( \simeq n^{2^\alpha} \) where \( \alpha = \log_2 (2p/q) \). The group \( R_{p,q} \) has no
subgroup isomorphic to a Baumslag–Solitar group \( BS(m,n) \) with \( m \neq \pm n \), but is not
automatic, not CAT(0), and cannot act freely on a CAT(0) cube complex. This answers a
long-standing question on the automaticity of one-relator groups and gives counterex-
amples to a conjecture of Wise.

1. Introduction

A classical topic in combinatorial and geometric group theory is one-relator groups, that
is, groups that can be defined by a presentation with only one relator. Magnus proved
that a one-relator group has solvable word problem, but the algorithmic complexity
of the word problem remains unknown. A geometric measure of this complexity is
given by the Dehn function (see [Bri02] for a survey). The Dehn function of a one-
relator group can grow very quickly: the group \( \langle a, t \mid a^{(a^t)} = a^2 \rangle \) has Dehn function
tower(\( \log_2 (n) \)), which is not bounded by any finite tower of exponents, but its word
problem is nonetheless solvable in polynomial time [MUW11]. This is conjecturally
the largest Dehn function of a one-relator group; Bernasconi proved a weaker uniform
upper bound, namely the Ackermann function [Ber94].

On the other hand, much less is known about the intricacies of the geometry of one-
relator groups satisfying a polynomial isoperimetric inequality, that is, whose Dehn
function is bounded by a polynomial. All previously known examples are hyperbolic
or more generally automatic (see [ECH+92] for background on automatic groups), and
thus have linear or quadratic Dehn function. For example, every one-relator group
with torsion is hyperbolic, and Wise has proved they are virtually special [Wis11].

A standard obstruction to a group having desirable geometry is the presence of a sub-
group isomorphic to the Baumslag–Solitar group \( BS(m,n) = \langle a, t \mid t^{-1}a^m t = a^n \rangle \) for

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some \( m \neq \pm n \): this group has a distorted cyclic subgroup, and its Dehn function is exponential. A distorted cyclic subgroup rules out being hyperbolic, acting properly cocompactly on a CAT(0) space, or acting freely on a CAT(0) cube complex (of possibly infinite dimension). A torsion-free one-relator group has geometric dimension 2 \[\text{Lyn50}\] so by a theorem of Gersten such a Baumslag–Solitar subgroup gives an exponential lower bound on Dehn function \[\text{Ger92a, Theorem C}\], further ruling out automaticity which would require a quadratic isoperimetric inequality.

It has been asked whether such Baumslag–Solitar subgroups are the only pathologies for one-relator groups:

**Question 1.** Is it true that one-relator groups with no subgroups isomorphic to \( BS(m, n) \), for \( m \neq \pm n \), are automatic?

**Conjecture 2** (Wise, \[\text{Wis14, 1.9}\]). Every [torsion-free] one-relator group with no subgroup isomorphic to \( BS(m, n) \), for \( m \neq \pm n \), acts freely on a CAT(0) cube complex.

Question 1 was articulated when the theory of automatic groups was first developing \[\text{Ger92b, Problem 11 ff.}\] and was posed more recently by Myasnikov–Ushakov–Won \[\text{MUW11, 1.5}\]. If true, it would imply that all polynomial Dehn functions of one-relator groups are linear or quadratic.

The first-named author introduced in his thesis \[\text{Gar17}\] the one-relator groups

\[
R(m, n, k, l) := \langle x, y, t \mid x^m = y^n, t^{-1}x^k t = x^l y \rangle \cong \langle x, t \mid x^m (x^{-l}t^{-1}x^k t)^{-n} \rangle
\]

for \( |m|, |n| \geq 2, k \neq 0, l \neq 0 \mod m \). He then proved that they have no distorted Baumslag–Solitar subgroups, and that they are CAT(0) precisely when \( |k| > |l + m/n| \).

In this paper we consider a subfamily of these groups: define

\[
R_{p,q} := R(2, 2, 2q, 2p - 1) \cong \langle x, y, t \mid x^2 = y^2, t^{-1}x^{2q} t = x^{2p - 1} \rangle.
\]

**Theorem A.** Let \( p > q \) be positive integers. The one-relator group \( R_{p,q} \) has Dehn function \( \simeq n^{2\alpha} \) where \( \alpha = \log_2 (2p/q) \). In particular, it has no subgroup isomorphic to a Baumslag–Solitar group \( BS(m, n) \) with \( m \neq \pm n \), but is not automatic and not CAT(0).

This answers Question 1 negatively. The key observation is that \( R_{p,q} \) is virtually a **tubular group**. A group is tubular if it splits as a finite graph of groups with \( \mathbb{Z}^2 \) vertex groups and \( \mathbb{Z} \) edge groups.

**Proof of Theorem A.** It is demonstrated in Theorem 3 below that \( R_{p,q} \) has an index two subgroup that is isomorphic to the Brady–Bridson snowflake (tubular) group \( G_{p,q} \). These groups are discussed in full in Section 2 but the salient fact is that \( G_{p,q} \) has Dehn function \( \simeq n^{2\alpha} \) where \( \alpha = \log_2 (2p/q) > 1 \); as the Dehn function is invariant up to finite index subgroups the first part of the statement holds. In contrast, automatic
and CAT(0) groups have at most quadratic Dehn function. Since $R_{p,q}$ is of geometric dimension 2 we conclude from Gersten’s theorem that there are no such Baumslag–Solitar subgroups as their presence would force at least exponential Dehn function.

\[ \square \]

**Remark 3.** Jack Button has shown that for odd $q \geq 3$, the group $R_{1,q}$ is not residually finite, giving the first examples of one-relator groups that are CAT(0) (by [Gar17, Theorem G]) but not residually finite [But17]. The proof shows that $G_{1,q}$ is non-Hopfian, which implies that $G_{1,q}$ and thus $R_{1,q}$ are not equationally Noetherian, resolving [Bau99, Problem 1]; in fact one can extend Button’s surjective endomorphism of $G_{1,q}$ with non-trivial kernel to show that $R_{1,q}$ itself is non-Hopfian, resolving [Bau99, Problem 7]. Button has informed us that he also has completely determined for which $p$ and $q$ the group $R_{p,q}$ is residually finite.

In [Wis14], Wise classified the tubular groups that act freely on CAT(0) cube complexes. In Section 3 we apply this classification to disprove Conjecture 2. The Dehn function is a quasi-isometry invariant, so there are infinitely many quasi-isometry types of counterexamples to Question 1 and Conjecture 2.

### 2. Virtually snowflake groups

In [BB00] Brady and Bridson studied the groups

$$G_{p,q} := \langle a, b, s, t \mid [a, b], s^{-1}a^q s = a^p b, t^{-1}a^q t = a^p b^{-1} \rangle$$

defined for positive integers $p$ and $q$. Due to the suggestive nature of their van Kampen diagrams, these are called snowflake groups. The main theorem of their paper states that for $p \geq q$, the Dehn function of $G_{p,q}$ is $\simeq n^{2\alpha}$ where $\alpha = \log_2(2p/q)$. This gives the Dehn function of $R_{p,q}$, via the following:

**Theorem B.** The snowflake group $G_{p,q}$ is an index 2 subgroup of the one-relator group $R_{p,q}$.

**Proof.** First, we re-write the presentation of $R$ to exploit the fact that $\langle x, y \mid x^2 = y^2 \rangle$ is the fundamental group of the Klein bottle: we map $x \mapsto a$ and $y \mapsto ab$ to get

$$R_{p,q} \cong \langle a, b, t \mid a^{-1}bab, t^{-1}a^2 bt = a^pb \rangle.$$

Let $X$ be the graph of spaces for $R_{p,q}$ with a vertex space a Klein bottle and edge space a cylinder. We can assume that the attaching maps are geodesics in the Klein bottle as in Figure 1. Let $X' \to X$ be the index two regular cover corresponding to the map to $\mathbb{Z}/2$ defined by $a \mapsto 1$, $b \mapsto 0$ and $t \mapsto 0$, indicated in Figure 1 on the Klein bottle subspace this is just the oriented double cover.
The fundamental group of $X'$ has the presentation
\[ \langle x, y, s, t \mid [x, y], s^{-1}x^q s = x^p y, t^{-1}x^q t = x^p y^{-1} \rangle \]
where $x$ and $y$ are the generators of the fundamental group of the torus (corresponding to $a^2$ and $b$ respectively). This group is none other than $G_{p,q}$. \[\square\]

**Corollary C.** The set of exponents $\rho$ such that $n^\rho$ is the Dehn function of a one-relator group is dense in $[2, \infty)$.

**Remark 4.** [MUW11, Problem 1.4] asks whether quadratic Dehn function implies that a one-relator group is automatic. The snowflake group $G_{1,1}$ is Gersten’s non-CAT(0) free-by-cyclic group introduced in [Ger94]. It has been announced that this group is not automatic [BR06], which would settle this remaining problem as well.

### 3. Non-cubulated examples

In [Wis14], Wise gave a necessary and sufficient condition for a tubular group to act freely on a CAT(0) cube complex. The condition is the existence of an equitable set which permits the construction of immersed walls in the graph of spaces associated to the group. A dual cube complex is then obtained from the corresponding wallspace.

**Proposition 5.** Let $p$ and $q$ be positive integers. The snowflake group $G_{p,q}$ acts freely on a CAT(0) cube complex if and only if $p \leq q$. 

![Figure 1. $G_{3,1}$ as an index 2 subgroup of $R_{3,1}$. The two blue arrows in each cylinder are attached along $a$ and the orange arrows along geodesics representing (lifts of) $a^6b$.](image-url)
Proof. The existence of a free action of $G_{p,q}$ on a CAT(0) cube complex is equivalent to the existence of an equitable set: $S = \{(u_1, v_1), \ldots, (u_k, v_k)\} \subseteq \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $[\mathbb{Z}^2 : (S)] < \infty$ and

$$\sum_i \#[(q,0), (u_i, v_i)] = \sum_i \#[(p, 1), (u_i, v_i)], \quad \sum_i \#[(q,0), (u_i, v_i)] = \sum_i \#[(p, -1), (u_i, v_i)]$$

where $\#[(a,b), (c,d)]$ denote the “intersection number” $|ad - bc|$. Thus the problem reduces to solving

$$(\star) \quad \sum_i |qv_i| = \sum_i |pv_i - u_i| = \sum_i |pv_i + u_i|.$$ 

If $p \leq q$, then a solution is $\{(q, 1), (q, -1)\}$. If $p > q$, there is no solution:

$$\sum_i |pv_i - u_i| + |pv_i + u_i| \geq \sum_i |(pv_i - u_i) + (pv_i + u_i)| = 2 \sum_i |pv_i| \geq 2 \sum_i |qv_i|$$

and equality can only hold in this last inequality if all $v_i = 0$, in which case some $u_i \neq 0$ and $(\star)$ clearly cannot hold. □

**Corollary D.** Let $p > q$ be positive integers. Then the one-relator group $R_{p,q}$ has no subgroup isomorphic to $BS(m,n)$ for $m \neq \pm n$ but does not act freely on a CAT(0) cube complex.

**Remark 6.** One can also deduce that for $p > q$ the group $G_{p,q}$ does not act freely on a CAT(0) cube complex from the fact that $G_{p,q}$ has a cyclic subgroup with distortion $n^\alpha$ [BB00] Corollary 2.3 whereas cyclic subgroups are undistorted in groups admitting such actions by [Hag07, Theorem 1.5] (which was generalized to finitely generated virtually abelian subgroups in [Woo17]).

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