Stefan Problem with Internal Heat Generation:
Comparison of Numerical Modeling and Analytical Solution

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The Stefan problem with internal heat generation can be found in various industrial applications. In particular, this problem describes melting of nuclear fuel rods during a reactor emergency. The present study reports on numerical modeling of the heat conduction equation and the Stefan equation with an application of the method of catching of the front into a space grid node. To solve numerically the heat conduction equation we realized an implicit four-node scheme using the Thomas method and iteration procedure.

1. Introduction

The first published solution to the solid-liquid phase change problem is attributed to Lamé and Clapeyron [1]. Independently, Stefan, for whom the Stefan problem is named, published an analysis of solid-liquid phase change based on data collected from ice formation on ships as they sailed the polar seas. His analysis compared favorably with the experimental data. The solid-liquid phase change is well studied [3,4] and continues to be a fruitful area of research today, especially in light of research involving phase change materials used in cooling systems.

We propose solutions to a non-homogeneous form of the Stefan problem, namely phase change under the influence of internal heat generation. Applications to this problem include geophysical systems, materials processing and cryogenic treatment of biological systems. The primary application of this work is nuclear safety and the analysis of melting of nuclear fuel rods. This paper compares the analytical solutions in a cylindrical geometry with a numerical result using the catching of the front method.

2. State of Problem and Solving Methods

2.1. Governing equation and conditions

We consider melting of a cylinder with internal heat generation \( q \) in both phases (Fig. 1). The radius of the cylinder equals \( r_0 \) and the distance from the cylinder center to the melting front equals \( s \). The outer surface of the cylinder has a temperature \( T_{\text{out}} \) and the temperature at the interphase border equals melting temperature \( T_m \). Top and bottom surfaces of the cylinder are assumed to be adiabatic, i.e. there is no heat flux along the cylinder. Heat conductivity \( k \) and heat capacity \( c_p \) are considered to be the
same in both phases. For such conditions the melting process can be described by the equation of heat conductivity and the Stefan condition for interphase border respectively:

$$\frac{1}{\eta} \frac{\partial}{\partial \eta} \left( \frac{\partial \Theta(\eta, \tau)}{\partial \eta} \right) + Q = \frac{\partial \Theta(\eta, \tau)}{\partial \tau},$$

(1)

$$\frac{\partial \Theta_m(\eta, \tau)}{\partial \eta} \bigg|_{\eta=\zeta(\tau)} + \frac{1}{St} \frac{d \zeta(\tau)}{d \tau} = \frac{\partial \Theta_m(\eta, \tau)}{\partial \eta} \bigg|_{\eta=\zeta(\tau)},$$

(2)

where $\eta = r / r_0$ – nondimensional radius of the cylinder, $\Theta(\eta, \tau) = (T(r, t) - T_0) / (T_m - T_0)$ – nondimensional temperature, $\tau = \alpha t / r_0^2$ $Q = \Phi_0 / k(T_m - T_0)$ – nondimensional internal heat generation, Stefan number $St = c_p (T_m - T_{rod}) / \Delta h_f$, $\zeta(\tau) = s(\tau) / r_0$ – nondimensional coordinate of the melting front.

![Fig. 1. Schematics of the melting problem with internal heat generation.](image)

In non-dimensional variable the boundary conditions can be written as follows:

$$\frac{\partial \Theta(0, \tau)}{\partial \eta} = 0, \quad \Theta(\zeta(\tau), \tau) = 1, \quad \Theta(1, \tau) = 0.$$  

(3)

For the initial temperature distribution, we set a constant zero value $\Theta(\eta, 0) = \Theta_0 = 0$ excluding Fig. 5, where the effect of the initial temperature is presented.

2.2. The method of catching of the front into a space grid node.

To solve numerically the heat conduction equation (1) we realized an implicit four-node scheme using the Thomas method [5] and iteration procedure. The Stefan equation (2) was modeled with an application of the method of catching of the front into a space grid node [6]. The sense of the method consists in that the distance $\Delta \zeta = d \zeta$ in equation (2), which melting front has gone for time $\Delta \tau$, is set strongly equal to the step of space grid $\Delta \eta$. Then one can express the time step $\Delta \tau = d \tau$ from equation (2) as follows:

$$\Delta \tau = \frac{\Delta \eta}{St \left( \frac{\partial \Theta_{rod}(\eta, \tau)}{\partial \eta} \bigg|_{\eta=\zeta(\tau)} - \frac{\partial \Theta_m(\eta, \tau)}{\partial \eta} \bigg|_{\eta=\zeta(\tau)} \right)}.$$

(4)

This method does not allow direct setting of time step. Nevertheless, the space step can control the time step. It is worth to note that the time step decreases with the increase of Stefan number that means more intensive melting process.

Convergence tests were done for space grid. At 200 nodes, the melting front distribution did not depend on the number of nodes in space grid. However, we used the space grid with 13000 nodes in order to reach the position of the interphase border in the steady-state regime.
3. Results

Fig. 2 shows profiles of non-dimensional temperature on the cylinder radius for $St = 1$ and $Q = 5$. At the non-dimensional time $\tau = 0.05$ the temperature profile belongs to the stage of initial heating. At $\tau = 0.25$ temperature in the center of the cylinder is rather close to one. This value of non-dimensional temperature corresponds to the melting temperature. Further, the melting process starts.

Temperature profile at $\tau = 0.6$ shows a characteristic temperature distribution for the active melting process of the cylinder. At $\eta = 0.22$ we can see the kink in the temperature profile that corresponds to the interphase transition.

At time $\tau = 4.16$ melting of the cylinder is finished in fact and temperature distribution corresponds to the steady-state regime. At that, the kink in the temperature profile disappears. It has two reasons which are finish of melting process that gives a zero derivative on the front position in equation (2) and adopted assumption about properties' equality of liquid and solid phases.

![Fig. 2. Temperature profiles at melting process with internal heat generation.](image)

Fig. 3a demonstrates an effect of Stefan number on movement of melting front and comparison of numerical modeling results and analytical solution of the Stefan problem in cylindrical coordinates [7]:

$$
\sum_{n=1}^{\infty} \lambda_n A_n \exp(-\lambda_n^2 \tau) J_1(\lambda_n \zeta(\tau)) + \frac{1}{St} \frac{d \zeta(\tau)}{d \tau} = 0
$$

As it is seen, the increase of Stefan number accelerates the melting process. One can allocate three-time parts which are initial heating, active melting, and steady-state regime. Analytical and numerical dependencies coincide well with each other at Stefan numbers 0.1 and 0.01. There is some discrepancy between numerical modeling and analytical solution at Stefan numbers 1.0 and 10. We explain this difference by machine error at the calculation of Bessel functions that formula (5) contains. For high Stefan numbers the second term of (5) has lower values than for small ones, consequently, the contribution of the first and third terms of (5) increases.

Fig. 3b shows dependencies of melting front position on the complex $\tau St$. Time parts of initial heating were deleted from the data. Applying this complex allows to compress the time scale and to represent all data within one order. Moreover, dependencies for $St = 0.01 - 1.0$ come down to the unity...
line practically. On the one hand, it proves that within the pointed range Stefan number and as a consequence material properties have the main influence on the melting process. On the other hand, we can conclude that at $St = 0.01 \text{--} 1.0$ steady-state regime starts from $\tau St \approx 2$. In this case, at known Stefan number one can calculate easy the total time of the active melting.

Fig. 3. Effect of Stefan number on melting front movement. a) Comparison results of numerical simulation and analytical solution [3] b) Time scaling

Fig. 4 presents the effect of internal heat generation on the interphase border position in the steady-state regime. There is no melting process at $Q < 4$. In addition, it follows from the analytical solution [7]:

$$
\xi_{\text{max}} = \sqrt{1 - \frac{4}{Q}}.
$$

(6)
The absence of melting indicates that the total heat generated within the cylinder is less than heat flux through the outer surface of the cylinder. The other limit of the presented dependence is the outer surface of the cylinder, i.e. that is the case of full melting of the cylinder. As it is seen, it can be realized at rather high values of the internal heat generation.

Fig. 4. Effect of internal heat generation on the interphase border at the steady-state regime.

Fig. 5. Influence of initial temperature on the melting front movement at different Stefan numbers.
Fig. 5 shows dependencies of melting front position on non-dimensional time for various Stefan numbers and at values of initial temperature $\Theta_0 = 0.0 0.9$. For low Stefan numbers 0.01 and 0.1 the initial temperature has a weak influence on the melting process. Only beginning from $\Theta_0 = 0.87$ it is changed. At that, on the initial time, the area with non-dimensional radius $\eta = 0.15$ melts instantly. This area of instant melting increases with the increase of the initial temperature.

At Stefan numbers 1 and 10 initial temperature of the cylinder has a strong effect on the melting process. The effect is realized through the decreasing of the time of initial heating. There is an area of instant melting at high initial temperatures as well. At that, its radius does not depend on Stefan number.

4. Conclusions
The paper presents results on the numerical modeling of the melting process of a cylinder with internal heat generation. We have demonstrated the following results:
- The method of catching of the front into a space grid node [6] allows a successful numerical model of the rod (cylinder) melting with internal heat generation. Comparison with the results of the analytical solution of Stefan Problem [7] proves it.
- Application of $t St$ leads to time scaling of melting front movement at different Stefan numbers. At that, for $St=0.01 – 1$ steady-state regime starts from $t St = 2$.
- Melting process begins at $Q > 4$, when the total internal heat generation exceeds heat flux through the outer surface.
- The initial temperature of the cylinder has a stronger influence on the melting process at higher Stefan numbers.

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