Adwords with Unknown Budgets

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In the classic Adwords problem introduced by Mehta et al. (2005), we have a bipartite graph between advertisers and queries. Each advertiser has a maximum budget that is known a priori. Queries are unknown a priori and arrive sequentially. When a query arrives, advertisers make bids and the we (immediately and irrevocably) decide which (if any) Ad to display based on the bids and advertiser budgets. The winning advertiser for each query pays their bid up to their remaining budget. Our goal is to maximize total budget utilized without any foreknowledge of the arrival sequence (which could be adversarial). Mehta et al. (2005) gave an online algorithm for this problem with the best possible competitive ratio guarantee of \((1 - \frac{1}{e})\). The key insight in this algorithm is to decide the winner greedily based on \textit{bid prices} that account for the fraction of remaining budgets.

Motivated by applications in automated budget optimization, we consider the setting where online algorithms do not know advertiser budgets a priori. Instead, the budget of an advertiser is revealed to the algorithm only when it is exceeded. An algorithm that is oblivious to budgets gives an Ad platform the flexibility to adjust budgets in real-time which, we argue, has tangible benefits. Unfortunately, the prominent bid price approach critically relies on knowledge of budgets and it is not evident if there are budget oblivious algorithms with better performance than a naive greedy approach. We show that no deterministic algorithm can have a competitive ratio guarantee better than 0.5 and give the first budget oblivious (randomized) algorithm for Adwords with competitive ratio guarantee of at least 0.522 against an offline algorithm that knows all bids and budgets.

1. Introduction

Online advertising has emerged as the dominant marketing channel in many parts of the world. According to some estimates, in the year 2019, more than 450 billion USD were spent on online ads, which accounts for over 60% of overall expenditure on ads\footnote{Digital advertising spending worldwide from 2019 to 2024. https://www.statista.com/statistics/237974/online-advertising-spending-worldwide/}. Internet search is a prominent channel for online advertisement. In this medium, also called \textit{search ads}, advertisements are displayed alongside online search results for key words relevant to the advertiser. In 2020 alone, Google made a revenue of over 104 billion USD from “search & other”, which accounts for 70% of their total revenue from advertising and exceeds half the total revenue of parent company Alphabet\footnote{Alphabet Year in Review 2020. https://abc.xyz/investor/}. Given the significance of search advertisement, there is a wealth of work that studies the problem from different points of view. In this paper, we are interested in the following viewpoints.
From the platform’s point of view: A crucial problem is deciding which (if any) ads to show for each search query before future queries are realized. The Adwords model introduced by Mehta et al. (2007), captures the key elements of this problem. For simplicity, the model describes the problem of showing at most one ad per query.

Adwords problem: At the beginning of the planning period (typically a day), the platform has a set $I$ of advertisers along with their maximum budgets $(B_i)_{i \in I}$. Queries arrive sequentially on the platform and when a query $t$ arrives, advertisers make bids $(b_{it})_{i \in I}$. Given the bids and advertiser budgets, the platform decides immediately which ad to display along side the search results for the query. The chosen advertiser pays their bid but only up to their remaining budget i.e., advertisers do not make a total payment exceeding their budget. The objective of the platform is to maximize the total advertiser budgets utilized without any foreknowledge of the arrival sequence (which could be adversarial).

We measure the performance of an online allocation algorithm for this problem by evaluating the competitive ratio i.e., the worst case relative performance gap between the online algorithm and (optimal) offline algorithm that knows all the queries and advertiser bids.

The main algorithm design goal for this problem and its many variations is to perform better than the na"ıve greedy algorithm that, for each query, shows the ad with highest bid and (non-zero) available budget. Mehta et al. (2007) proposed the prominent bid pricing algorithm for this problem (formally discussed later on), that achieves the best possible competitive ratio of $(1 - 1/e)$. In comparison, the greedy algorithm has a competitive ratio of 0.5.

From advertiser’s point of view: The goal is often to target their customers through multiple ad campaigns and marketing channels. Starting with an overall budget, an advertiser must determine a good distribution of their budget to individual ad campaigns. For search ads, advertisers must also determine the bids for relevant key words. While these decisions play a crucial role in the success of their ad campaigns, determining a good distribution of budgets between diverse options and deciding optimal bids for specific campaigns can be an incredibly challenging task for any advertiser.

Many advertisers rely on automated bidding and budget management tools. The motivation behind these tools is to improve performance for advertisers while simplifying the usage of ad platforms (Aggarwal et al. 2019). In using a tool to manage their portfolio, the advertiser decides the overall budget, creates a portfolio of ad campaigns, and specifies a high level performance goal for the portfolio. Using these specifications, the tool aims to automatically determine (over time) a good budget distribution for the portfolio, as well as, bids for key words in each campaign.

3 See Section 6 in Mehta et al. (2007) for the generalization to multiple ads.
1.1. Adwords with Unknown Budgets: Motivation and Problem Description

While automated management of ad portfolio has become increasingly prominent, to the best of our knowledge, platforms typically optimize a daily budget distribution for the portfolio and do not perform real-time adjustments during the day to this distribution based on live performance of campaigns in the portfolio.[4] Platforms with in-house tools for automated budget optimization (such as Search Ads 360 by Google) may, in fact, have the cross-campaign data necessary to perform adjustments to budget distribution in real-time. This can have tangible benefits. For the sake of illustration, consider the following stylized example.

**Example:** Consider an advertiser with a portfolio composed of two search ad campaigns labeled \{1, 2\}. In the absence of a budget constraint, let the total expenditure (per day) in campaign 1 be a Bernoulli random variable \(X_1 \in \{0, 1\}\). Similarly, let \(X_2 \in \{0, 1\}\) represent the maximum expenditure (per day) in campaign 2. Now, suppose we have a total daily budget of 1 that needs to be distributed between the two campaigns. When \(X_1\) and \(X_2\) are identically distributed, splitting the budget evenly between the two campaigns is always optimal. When the mean of \(X_1\) (and \(X_2\)) is 0.5, this strategy uses 50% of the total budget in expectation.

Now, suppose that instead of fixing the daily budget the platform performs real-time budget adjustments using live cross-campaign data. Suppose that \(X_2 = 1 - X_1\) i.e., the expenditures in the campaigns are in perfect negative correlation. Using this knowledge in conjunction with live data, the platform can always adjust budgets on the fly to obtain a budget utilization of 100%. In general, by real-time budget adjustments the platform can achieve a utilization of \(\times E[\min\{X_1 + X_2, 1\}] \geq 50\%\).

We believe that an important rationale behind the current practice of fixing budgets at the start of each day, stems from the importance of fixed budgets in algorithms for ad allocation. For instance, consider the bid pricing algorithm of [Mehta et al. (2007)] that we mentioned earlier. This algorithm decides ad allocation greedily based on bid prices that account for the fraction of remaining budgets. Formally, given remaining budget \((B_i(t))_{i \in I}\) on arrival of query \(t\), the algorithm computes bid prices

\[
b_{it} \left(1 - e^{-B_i(t)/B_i}\right) \quad \forall i \in I,
\]

and shows the ad with the largest bid price. Thus, foreknowledge of daily budgets for all advertisers is essential for defining the algorithm. Similarly, most (if not all) algorithms for ad allocation in the literature (see [Mehta et al. (2013), Alaei et al. (2012), Devanur et al. (2019), Devanur and Hayes (2009), Mirrokni et al. (2012)]), rely critically on the knowledge of (fixed) budgets for each advertiser.

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4 Google Search Ads 360 Help. https://support.google.com/searchads/answer/7657341
The naïve greedy algorithm is the one exception to this. For each query, greedy shows the ad with highest bid and (non-zero) available budget. Therefore, it is budget oblivious i.e., does not require any advance information about budgets except the knowledge of which advertisers are still participating (have non-zero remaining budget). Recall that the central goal in designing online algorithms for ad allocation is to find algorithms that are provably better than greedy. This motivates us to consider the following problem.

Is there a budget oblivious online algorithm for Adwords that outperforms greedy?

A budget oblivious algorithm easily adapts to real-time changes in budget since it makes (randomized) ad allocation decisions for each query using only the bids and the knowledge of which advertisers have non-zero remaining budget.

Another motivation: This problem also has a strong connection to the setting of online matching with stochastic rewards [Mehta and Panigrahi 2012]. In fact, this connection previously prompted Mehta et al. (2013) to raise the same question in a different context (see Open Question 20 in Mehta et al. (2013)). We describe this relationship in more detail in Section 2.1.1.

1.2. Overview of Our Contributions

Recall that for the classic Adwords setting (with known budgets), a deterministic algorithm achieves the best possible guarantee of \(1 - \frac{1}{e}\) Mehta et al. (2007). When budgets are unknown, we establish the following upper bound.

**Theorem** (Informal). No deterministic algorithm can have competitive ratio better than 0.5 for Adwords with unknown budgets.

This implies that the greedy algorithm achieves the best possible competitive ratio guarantee in the class of deterministic algorithms. We give the first budget oblivious algorithm for ad allocation that provably outperforms greedy.

**Theorem** (Informal). There exists a randomized 0.522 competitive algorithm for Adwords with unknown budgets.

Our algorithm (defined in Section 4) is a natural generalization of the Perturbed-Greedy algorithm of Aggarwal et al. (2011) (further details in Section 2.1). Proving a performance guarantee for this family of algorithms faces many obstacles due to combinatorial interactions between the time varying nature of bids and randomness inherent in the algorithm, as well as, its budget oblivious nature. To address these challenges, our analysis takes an indirect path where we first show the desired guarantee for a relaxed (fractional) version of the algorithm that is not budget oblivious. To establish a performance guarantee for the relaxed algorithm, we employ a recently developed LP free approach from Goyal et al. (2021) which, in turn, builds on the classic randomized primal-dual
approach of Devanur et al. (2013). Even with this approach, proving a guarantee better than 0.5 for the relaxed algorithm requires novel structural insights into the problem (due to the time-varying bids and randomness in the algorithm). Finally, we relate the expected performance of the original algorithm with its budget aware relaxation. This step is made challenging by the fact that our original algorithm is budget oblivious and its decisions may differ substantially from the budget aware relaxation.

**Outline for rest of the paper:** Section 2 discusses the assumption of small bids and presents a resource allocation version of the Adwords setting that we use in the rest of the paper. We show our results for this generalization. In Section 2.1, we discuss related work in online matching and resource allocation. In Section 4 we first present our algorithm and main result, followed by a discussion of the key bottlenecks in proving the result. This is followed by the formal analysis in Section 4.2. Finally, Section 5 concludes our discussion and highlights interesting directions for future work.

### 2. Preliminaries

The following resource allocation setting is equivalent to the Adwords problem.

**Online Budgeted Allocation (OBA):** Consider a complete bipartite graph $G$ with vertex set $(I, T)$. Vertices $i \in I$, called *resources*, have budgets $(B_i)_{i \in I}$ and per unit rewards $(r_i)_{i \in I}$. Resource and their rewards and budgets are known to us. The remaining vertices $t \in T$, called *arrivals*, are unknown a priori and arrive sequentially. When a vertex $t \in T$ arrives, we see their bids $(b_it)_{i \in I}$. The bid vector $(b_it)_{i \in I}$ indicates that arrival $t$ is interested in up to $b_it$ units of resource $i$, for every $t \in T, i \in I$. Every arrival is interested in at most one resource. Given the bids for arrival $t$, we must immediately and irrevocably *match* the arrival to at most one resource. If arrival $t$ is matched to resource $i$, $b_it$ units of $i$ are consumed and we receive a reward $r_i$ per unit of $i$ consumed with the caveat that the total reward/revenue from $i$ can not exceed budget $r_iB_i$. In other words, the total reward from $i$ is capped at $r_iB_i$. The goal is to decide the allocation/matching for arrivals without any knowledge of future arrivals and such that the total reward is maximized.

The Adwords problem is an instance of OBA where resources correspond to ads, arrivals to queries, and arrival bids $(b_it)_{i \in I}$ represent the bid of advertisers $i \in I$ for query $t$. The per units rewards $r_i$ are set to 1 in Adwords for every $i \in I$. On the other hand, an instance of OBA with bids $b_it$, budgets $B_i$, and per unit revenues $r_i$, is equivalent to the Adwords setting with scaled bids $r_i b_it$ and budgets $r_i B_i$.

To understand a crucial but often implicit assumption in the Adwords setting, we define the bid-to-budget ratio,

$$\gamma := \max_{i \in I, t \in T} \frac{b_it}{B_i}.$$
In the Adwords setting, one typically assumes that $\gamma \to 0$. This is also called the small bid assumption. This assumption is in line with the practice of search ads, where individual bids are typically much smaller than the overall budget. Note that the $(1 - 1/e)$ guarantee for the bid pricing algorithm of Mehta et al. (2007) holds only in the small bid regime. Even in this regime, no online algorithm can have competitive ratio better than $(1 - 1/e)$. The focus of this paper is on OBA with unknown budgets in the small bid regime.

2.1. Related Work

We start with the classic setting of online bipartite matching. This problem can be viewed as a special case of OBA where every resource has unit budget and bids are binary. A bid of 1 denotes an edge in the bipartite graph and bid of 0 denotes the absence of an edge. All per unit rewards are identically 1 and the goal is to find the largest matching. Karp et al. (1990) introduced this problem and showed (among other results) that randomly ranking resources at the start and then matching every arrival to the best ranked unmatched resource is a $(1 - 1/e)$ competitive algorithm for this setting. In fact, this algorithm, called Ranking, achieves the best possible guarantee for the problem. The analysis of Ranking was clarified and considerably simplified by Birnbaum and Mathieu (2008) and Goel and Mehta (2008).

Aggarwal et al. (2011), proposed the Perturbed Greedy algorithm that is $(1 - 1/e)$ competitive for vertex weighted case where per unit rewards $r_i$ can be arbitrary. Kalyanasundaram and Pruhs (2000) considered the problem of online $b$-matching where the budget of every resource can be more than 1 and showed that as $b \to \infty$, the natural (deterministic) algorithm that balances the budget used across resources is $(1 - 1/e)$ competitive. Generalizing this setting, Mehta et al. (2007) introduced the Adwords problem and proposed the bid pricing based $(1 - 1/e)$ algorithm for Adwords under the small bid assumption. Buchbinder et al. (2007) gave a primal-dual analysis for this algorithm. Subsequently, Devanur et al. (2013) proposed the randomized primal-dual framework that can be used to show the aforementioned results in an elegant and unified way.

Very recently, Albers and Schubert (2021) and Vazirani (2021, 2022) (independently), showed that the Perturbed Greedy algorithm is $(1 - 1/e)$ competitive for online $b$-matching. Note that $b$-matching is a special case of Adwords where every bid is either 0 or 1. Interestingly, both papers use a different approach to arrive at this result. To the best of our knowledge, these approaches do not yield a performance guarantee better than 0.5 for Perturbed Greedy in the Adwords setting and this is posed as an open problem in Vazirani (2022). Vazirani (2022) also identifies a key structural property, called no surpassing, the absence of which prevents a generalization of their result to the Adwords setting. Through various numerical experiments (based on synthetic data), they demonstrate that the performance of Perturbed Greedy is at par with the Balance algorithm of Mehta et al. (2007).
It is worth noting that the Adwords/OBA setting (without the small bid assumption), generalizes each of the settings discussed above. The budget-aware greedy algorithm that matches every arrival $t \in T$ according to the following rule,

$$\text{argmax}_{i \in I} \left( \min\{b_{it}, B_i(t)\} r_i \right),$$

where $B_i(t)$ is the remaining budget of $i$ on arrival of $t$, is unconditionally 0.5 competitive for OBA. Kapralov et al. (2013) showed that no online algorithm has competitive ratio better than 0.612 for general OBA. Recently, Huang et al. (2020) gave the first algorithm with competitive ratio better than 0.5 for general OBA/Adwords i.e., without the small bids assumption.

While the body of work discussed above considers an adversarial arrival sequence, there is also a wealth of work on online matching and resource allocation in stochastic and hybrid/mixed models of arrival (for example, Goel and Mehta (2008), Feldman et al. (2009), Devanur and Hayes (2009), Karande et al. (2011), Manshadi et al. (2012), Alaei et al. (2012), Devanur et al. (2019), Mirrokni et al. (2012)). For a comprehensive review of these settings, we refer to Mehta et al. (2013).

2.1.1. Online Matching with Stochastic Rewards

Introduced by Mehta and Panigrahi (2012), this problem generalizes online bipartite matching by associating a probability of success $p_{it}$ with every edge $(i,t)$. When a match is made i.e., edge is chosen, it succeeds independently with this probability. If the match fails the arrival departs but the resource is available for future rematch. While the general case of this problem is challenging and unresolved, a well studied case (motivated by applications) is when edge probabilities are vanishingly small i.e., $p_{it} \to 0 \forall (i,t)$.

Mehta and Panigrahi (2012) showed an equivalence between the vanishing probabilities case and the following instance of Adwords with unknown budgets: (i) Bids $b_{it} = p_{it}$ i.e., bids are equal to the edge probabilities and missing edges have a bid of 0. (ii) Budgets are unknown but it is known that the budget of each resource is independently sampled and follows the exponential distribution with unit mean.

Subsequent work (Mehta et al. 2015, Goyal and Udwani 2020, Huang and Zhang 2020), further generalized and used this connection to show new results for the stochastic rewards setting with vanishing probabilities. Using this connection, our result for Adwords with unknown budgets yields a 0.522 competitive algorithm for stochastic rewards with vanishing probabilities. We note that this result does not improve the state-of-the-art for this problem in terms of the competitive ratio guarantee. However, it shows that a natural generalization of the Perturbed Greedy algorithm, which yields the optimal guarantee for the setting where all edge probabilities are 1 (more generally,

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5 This applies when the competitive ratio is evaluated against an LP upper bound on the offline problem.

6 In particular, Goyal and Udwani (2020) give an algorithm with guarantee of 0.596.
decomposable \cite{Goyal and Udwani2020}), also has a guarantee better than 0.5 for the well studied case of vanishing probabilities.

3. Upper Bound for Deterministic Algorithms

It is commonly known that for Adwords and many other online resource allocation problems, deterministic algorithms are as powerful as randomized algorithms under the small bid (or large budget) assumption. Perhaps surprisingly, we show that when budgets are unknown, randomized algorithms are provably better than their deterministic counterpart\footnote{The standard method of proving equivalence between deterministic and randomized relies on constructing a deterministic fractional algorithm that matches the expected performance of a randomized integral algorithm. This line of argument (subtly) relies on budgets being known in advance.}

In this section, we establish that no (budget oblivious) deterministic algorithm can beat the performance of greedy even for online $b$-matching, which is a special case of Adwords. In the following section, we show that the (randomized) Perturbed Greedy algorithm has strictly better competitive ratio.

**Theorem 1.** Every deterministic budget oblivious online algorithm for online $b$-matching (with variable vertex capacities) has competitive ratio at most 0.5.

**Proof.** Consider an instance of online $b$-matching with $n$ resources that have capacities $1, 2, 3, \ldots, n - 1$ and $(n - 1 + n(n - 1)/2)$. We will ensure that the offline optimal will match everything, resulting in total value $n^2 - 1$. Let ALG denote a deterministic budget oblivious online algorithm. W.l.o.g., let ALG match every arrival that can be matched. We will construct a worst case budget assignment and arrival sequence for ALG with $n$ arrival phases. In phase 1, we have $n$ arrivals with an edge to all resources. Let resource 1 be the first resource that is matched to an arrival by ALG. We let the capacity of resource 1 be 1. In phase 2 of arrivals, we have $n - 1$ arrivals that have an edge to every resource except resource 1. Let resource 2 be the first resource that is matched to two arrivals in ALG and let the capacity of resource 2 be 2. In a similar fashion, phase $k$ has $n - k + 1$ identical arrivals that have edges to resources $\{k, k + 1, \ldots, n\}$. Finally, in phase $n$, we have $n(n - 1)/2$ arrivals that have edges to resources $\{1, \ldots, n - 1\}$. The optimal solution matches the first $n - 1 + n(n - 1)/2$ arrivals to resource $n$ and matches the final batch to remaining resources. ALG cannot match any of the arrivals in the final batch. Thus, for $n \rightarrow +\infty$, we have $\text{ALG} = 0.5 \text{Opt}$. \qed
4. Randomized Budget Oblivious Algorithm and Analysis

Consider the following family of randomized algorithms with parameter $\beta > 0$.

**ALGORITHM 1:** Generalized Perturbed-Greedy (GPG)

**Inputs:** Set of advertisers $I$, parameter $\beta$;

Let $g(t) = e^{\beta(t-1)}$;

For every $i \in I$ generate i.i.d. r.v. $y_i \sim U[0, 1]$;

for every new arrival $t$ do
  Match $t$ to $i^* = \arg \max_{i \in I} b_i r_i (1 - g(y_i))$;
  if $i^*$ is out of budget update $I = I \setminus \{i^*\}$;

Algorithm 1 matches every arrival $t \in T$ greedily based on randomized bid prices $b_i r_i (1 - g(y_i)) \forall i \in I$. The uniform random variables $(y_i)_{i \in I}$, called seeds, are sampled independently for each $i \in I$. Therefore, ties between bid prices of any two (different) resources occur with a probability of 0. Algorithm 1 replaces the budget dependent factor $(1 - e^{-B_i(t)/B_i})$ with a random quantity, resulting in a budget oblivious algorithm.

Note that, Algorithm 1 is a generalization of the Perturbed Greedy (PG) algorithm in [Aggarwal et al. (2011)](https://www.journalofcomputerandsystemsciences.com/article/S0022000011000845). Recall that the PG was designed for vertex weighted online bipartite matching that, in terms of OBA, corresponds to instances where bids are binary and all budgets equal 1. In this special case, when $\beta = 1$, Algorithm 1 collapses to the PG algorithm.

Due to challenges outlined in Section 4.1, we prove a performance guarantee for Algorithm 1 by first analyzing its fractional relaxation.

**ALGORITHM 2:** Fractional GPG

**Inputs:** Set of advertisers $I$, budgets $(B_i)_{i \in I}$, parameter $\beta$;

Initialize $I(0) = I$, $g(t) = e^{\beta(t-1)}$ and $x_i(0) = 0$ for every $i \in I$;

Generate i.i.d. sample $y_i \in U[0, 1]$ for every $i \in I$;

for every new arrival $t$ do
  for $\tau \in [t, t+1)$ do
    $I(\tau) = \{i \mid x_i(\tau) < B_i\}$;
    $i^*(\tau) = \arg \max_{i \in I(\tau)} b_i r_i (1 - g(y_i))$;
    For every $i \in I(\tau)$, update $x_i(\tau)$ according to $dx_i(\tau) \frac{d\tau}{\lambda} = \mathbb{1}(i = i^*(\tau)) b_i$;
    For every $i \in I$, $x_i(t+1) - x_i(t)$ of $i$'s budget is allocated to $t$;

In the fractional setting time runs continuously from $[1, T+1)$. In other words, arrival $t \in T$ is matched fractionally during the time period $[t, t+1)$. Set $I(\tau)$ is the set of resources with available budget at time $\tau \in [1, T+1)$. Taking an infinitesimal viewpoint, at time $\tau$ we match $d\tau$ fraction of arrival $t$ to resource $i^*(\tau)$. This is the resource with the maximum bid price out of all resources in $I(\tau)$. Similar to Algorithm 1, ties between bid prices occur with a probability of 0. Now, the
allocation of \(i^*(\tau)\) to \(\tau\) uses \(b_{i^*(\tau)}* d\tau\) of resource \(i^*(\tau)\)'s budget and earns a reward of \(b_{i^*(\tau)} r_{i^*(\tau)} d\tau\). At time \(t + 1\) the process of fractionally matching arrival \(t\) ends and we start matching arrival \(t + 1\). The total amount of \(i\)'s budget that is allocated to arrival \(t\) between \([t, t + 1)\) is given by \(x_i(t + 1) - x_i(t)\). Thus, the total reward of Algorithm 2 given seed \(Y = (y_i)_{i \in I}\) is \(\sum_{i \in I} x_i(T + 1) r_i\).

Observe that Algorithm 2 is not budget oblivious. However, this is not of concern as the algorithm is only used as an intermediate step to analyze Algorithm 1. We show the following guarantee for Algorithm 1 and Algorithm 2.

**Theorem 2.** Given a competitive ratio guarantee \(\eta\) for Algorithm 2 (for some parameter value \(\beta\)), we have that Algorithm 1 (with the same value of \(\beta\)) is \(\frac{1}{1 + \gamma} \eta\) competitive.

**Theorem 3.** With \(\beta = 1.15\), Algorithm 2 is at least 0.522 competitive against optimal (integer) offline allocation. When \(\beta = 1\), Algorithm 2 is 0.508 competitive.

We start by highlighting some of the key challenges with analyzing the algorithms above. This is followed by an overview of our analysis and approach for tackling the obstacles presented below. We present the main analysis in Section 4.2.

### 4.1. Challenges in Analysis

We illustrate the main challenges at a high level via a simple example. For more details and nuanced examples, see Appendices A and B. We shall use the following labels, Algorithm 1 is \(\text{ALG}\) and Algorithm 2 is \(f\)-\(\text{ALG}\). Also, we use \(Y\) to denote the random seeds \((y_i)_{i \in I}\) in \(\text{ALG}\) and \(f\)-\(\text{ALG}\).

**Example 4.1.** Consider a snippet of an instance with resources 1 and 2 and per unit rewards \(r_1 = r_2 = 1\). We focus on consecutive arrivals \(t\) and \(t + 1\). Let bids \(b_{1t} = b_{2t} = 1\), \(b_{1t+1} = 2\) and \(b_{2t+1} = 4\). Consider an execution of \(\text{ALG}\) with \(\beta = 1\) and the random seed for resource 2 fixed at \(y_2 = 0.5\). Suppose that for every value of seed \(y_1 \in [0, 1]\), resource 1 is not matched to any arrival prior to \(t\) and only 1 unit of resource 2’s budget is available at \(t\). Observe that,

\[(i)\quad b_{it+1} r_i (1 - g(0))) < b_{jt+1} r_j (1 - g(y_j)).
\]

\[(ii)\quad b_{it} r_i (1 - g(0.5)) = b_{jt} r_j (1 - g(y_j)).\]

Consider the matching generated by \(\text{ALG}\) as we decrease \(y_1\) from 1 to 0. For \(y_1 > 0.5\), \(\text{ALG}\) matches \(t\) to 2 and this uses up all of 2’s budget. Consequently, \(t + 1\) is matched to 1 for \(y_1 > 0.5\). For \(y_1 < 0.5\), \(t\) is matched to 1 and \(t + 1\) to 2. Thus, the number of units of resource 1 matched in \(\text{ALG}\) decreases from 2 to 1 as \(y_1\) decreases. This is somewhat surprising as the bid price of resource 1 increases monotonically as we decrease \(y_1\). At a high level, there are two main reasons behind this occurrence.

The first reason is that, in general, bid prices vary with both time (due to time-varying bids) and seed values \(Y\). In the example above, even though bid price of resource 1 increases everywhere as \(y_1\) decreases, the time variation in bid ensures that at arrival \(t + 1\) the bid price of resource
2 is always higher than that of resource 1. This is the first high level challenge in analyzing the performance of ALG.

The second reason is unknown budgets, due to which we may overestimate the number of units of a resource available. In the example above, recall that only 1 unit of resource 2 is available at \( t+1 \). For \( y_1 < 0.5 \), we have \( b_{1,t+1} r_1 (1 - g(0.5)) > r_2 (1 - g(y_2)) \) i.e., at arrival \( t+1 \), resource 1 has a higher bid price than resource 2 if we account for the number of remaining units of each resource. However, ALG, being ignorant of the budget, computes a higher bid price for resource 2 at \( t+1 \).

**Overview of Our Analysis.** To address these challenges we first isolate them. We accomplish the isolation by switching over to the fractional algorithm. Notice that the second challenge disappears in the fractional version; \( f\text{-ALG} \) uses budgets and never over-estimates the bid price of a resource. However, the variation of bid prices with both time and seed \( Y \) makes it challenging to analyze even \( f\text{-ALG} \) (more details in Appendices [A] and [B]). So to prove Theorem 3 we use an LP free analysis approach developed recently in Goyal et al. (2021). It is worth noting that previous work uses the LP free approach to handle stochastic elements in online allocation problems. While our underlying problem is deterministic, the increased flexibility of the LP free framework resolves, in a very natural way, some of the difficulty we face in analyzing \( f\text{-ALG} \) (for an example, see Appendix [A]).

To prove Theorem 2 one possibility would be to establish that the behavior of ALG is very close to \( f\text{-ALG} \) on every sample path. This is where the second challenge comes to the fore. Since ALG is oblivious to budgets but \( f\text{-ALG} \) is not, there can be a substantial difference between the output of these algorithms for the same seed \( Y \) (see Example B.1 in Appendix [B]). We overcome this challenge by comparing ALG not with \( f\text{-ALG} \) on the same instance but with the performance of \( f\text{-ALG} \) on a modified instance where the budgets of resources are (slightly) higher. Next, we introduce some new notation before presenting the main analysis.

**Notation.** Let the number of resources \(|I| = n\). We extend the definition of bids to every moment \( \tau \in [1, T+1) \) as follows, \( b_{i,\tau} = b_{i,t} \) where \( \tau \in [t, t+1) \) for some arrival \( t \in T \).

We use ALG to refer to Algorithm [I] as well as its expected reward. \( \text{ALG}(Y) \) denotes the matching generated by ALG when the random seed is given by \( Y = (y_i)_{i \in I} \). Let \( \text{ALG}_i(Y) \) denote the set of arrivals matched to \( i \) in matching \( \text{ALG}(Y) \). Overloading notation, we also use \( \text{ALG}(Y) \) to denote the total reward of ALG with seed \( Y \) and \( \text{ALG}_i(Y) \) to denote the total budget of \( i \) used in \( \text{ALG}(Y) \). Let \( x_i(t, Y) \) denote the total budget of \( i \) allocated to arrivals \( t' < t \) in \( \text{ALG}(Y) \) i.e.,

\[
x_i(t, Y) = \sum_{t' < t; t' \in \text{ALG}_i(Y)} b_{i,t'} \quad \forall i \in I, t \in T, Y \in [0, 1]^n.
\]
Similarly, let $f$-Alg denote the fractional Algorithm and also its expected total reward. Let $f$-Alg$(Y)$ denote the fractional matching as well as the total reward generated by $f$-Alg given seed $Y$. Let $I(\tau,Y)$ denote the set of resources available at time $\tau$ in $f$-Alg with seed $Y$. Notice that for every $t \in T$ and $i \in I$, there is at most one interval $[t_1,t_2)$ such that every moment $\tau \in [t_1,t_2)$ is matched to $i$ and every other moment i.e., $\tau \in [t,t+1) \setminus [t_1,t_2)$, is not matched to $i$. We call the interval $[t_1,t_2)$ a segment. Since each segment corresponds to a unique resource, there are at most $n$ segments in the interval $[t,t+1)$ for every $t \in T$. Thus, for every $Y$, there are at most $nT$ segments in $f$-Alg$(Y)$. Let $f$-Alg$_i(Y)$ denote the union of all segments matched to $i$ in $f$-Alg$(Y)$ as well as the total budget of $i$ used in $f$-Alg$(Y)$. Let $x_i^f(\tau,Y)$ denote the total budget of $i$ allocated prior to time $\tau$ in $f$-Alg$(Y)$. Notice that

$$x_i^f(T+1,Y) = f$-Alg_i(Y) \quad \forall i \in I, Y \in [0,1].$$

Finally, let Opt refer to the optimal offline algorithm as well as its total reward. Since there are no unknowns in the offline problem i.e., budgets and bids are all known, Opt is deterministic. Let Opt$_i$ denote the set of arrivals matched to $i$ as well as the total fraction of $i$’s budget that is matched in Opt. More generally, we use the notation $\tau \in$ Opt$_i$, for every moment $\tau$ such that $\tau \in [t,t+1)$ for some arrival $t \in$ Opt$_i$. Note that $\text{Opt} = \sum_{i \in I} r_i \text{Opt}_i$.

### 4.2. Analysis

Consider the following linear system in variables $\lambda_i$ and $\theta_i$,

$$\sum_{t \in T} \lambda_t + \sum_{i \in I} \theta_i \leq f$-Alg \quad \text{(1)}$$

$$\sum_{t \in \text{Opt}_i} \lambda_t + \theta_i \geq \alpha r_i \text{Opt}_i \quad \forall i \in I, \quad \text{(2)}$$

$$\lambda_t \geq 0, \quad \theta_i \geq 0 \quad \forall t \in T, i \in I. \quad \text{(3)}$$

The following lemma is a special case of Lemma 3 in [Goyal et al., 2021]. For the sake of completeness, we include a proof.

**Lemma 4.** Given a solution to the system defined by (1)–(3), we have that $f$-Alg is $\alpha$ competitive against Opt.

**Proof.** Summing up inequalities (2) over all $i \in I$, we have

$$\alpha \text{Opt} = \alpha \sum_{i \in I} r_i \text{Opt}_i \leq \sum_{i \in I} \sum_{t \in \text{Opt}_i} \lambda_t + \sum_{i \in I} \theta_i \leq \sum_{t \in T} \lambda_t + \sum_{i \in I} \theta_i \leq f$-Alg.$$

\[\square\]

\[8\] It can be shown that there are at most $T + n - 1$ segments in all but the loose bound of $nT$ suffices.
In the following, we prove Theorem 3 by finding a feasible solution to the system (1)–(3) with a suitably large value α. Recall that \( I(τ, Y) \) denotes the set of resources available at time τ in \( f-{\text{ALG}}(Y) \). To define the candidate solution, we first define,

\[
λ_τ(Y) = \max_{j \in I(τ, Y)} b_{jτ}(1 - g(y_j)) \quad \forall τ \in [t, t + 1), Y \in [0, 1]^n, \tag{4}
\]

\[
θ_i(Y) = f-{\text{ALG}}_i(Y) r_i g(y_i) \quad \forall i \in I, Y \in [0, 1]^n. \tag{5}
\]

The candidate solution is,

\[
λ_i = E_Y \left[ \int_{τ=t}^{τ=t+1} λ_τ(Y) \, dτ \right] \quad \forall τ \in T \quad \text{and} \quad \theta_i = E_Y [θ_i(Y)] \quad \forall i \in I. \tag{6}
\]

For the above candidate solution, constraints (3) are obviously satisfied. Also by definition, constraint (1) holds with equality. It remains to show that inequalities (2) are satisfied. In fact, we show a stronger statement as described in the next lemma.

**Lemma 5.** Consider a resource \( i \in I \) and seed \( Y_{-i} = (y_j)_{j \in I \setminus \{i\}} \) denote the random seed in \( f-{\text{ALG}} \) for all resources except \( i \in I \). Suppose that for the candidate solution (6), we have,

\[
E_{\tilde{Y}_i} \left[ \sum_{t \in \text{OPT}_i} \int_{τ=t}^{τ=t+1} λ_τ(Y) \, dτ + θ_i(Y) \mid Y_{-i} \right] \geq α r_i \text{OPT}_i, \quad \forall Y_{-i} \in [0, 1]^{n-1}, \tag{7}
\]

for some value \( α > 0 \). Then, inequality (2) is satisfied for resource \( i \) with the same \( α \).

**Proof.** The lemma follows by taking expectation over \( Y_{-i} \) on both sides of (7). \( \square \)

For the analysis that follows, fix an arbitrary resource \( i \in I \) and seed \( Y_{-i} \). For brevity, we suppress dependence on \( Y_{-i} \) from notation and highlight only the dependence on seed \( y_i \in [0, 1] \). So \( f-{\text{ALG}}_i(y_i) \) is the matching generated by \( f-{\text{ALG}} \) when it is executed with seed \( Y = (y_i, Y_{-i}) \). Similarly, \( f-{\text{ALG}}_i(y_i) \) denotes the total amount of \( i \)'s budget allocated in \( f-{\text{ALG}}(y_i) \). Let \( λ_τ(y_i) = λ_τ(y_i, Y_{-i}) \) and \( θ_i(y_i) = θ_i(y_i, Y_{-i}) \). Let \( I(τ, y_i) \) denote the set of resources available at τ in \( f-{\text{ALG}}(y_i) \). For every τ \( \in [t, t + 1) \), we define critical threshold \( y^*_i(τ) \in [0, 1] \) such that

**Critical threshold \( y^*_i(τ) \):** \( b_{iτ} r_i (1 - g(\lambda^*_i(τ))) = \max_{j \in I(τ, 1)} b_{jτ} (1 - g(y_j)) \).

Set \( y^*_i(τ) = 0 \) if no such value exists and \( y^*_i(τ) = 1 \) if the set \( I(τ, 1) \) is empty. Due to the monotonicity of \( g(t) = e^β(t-1) \), we have a unique value of \( y^*_i(τ) \).

**Lemma 6.** Given \( i \in I \) and seed \( Y_{-i} \), for every \( y_i \in [0, 1], τ \in [1, T + 1) \), we have

\[
λ_τ(y_i) \geq λ_τ(1) \geq b_{iτ} r_i (1 - g(y^*_i(τ))).
\]
Proof. Given \( i \) and \( Y_{-i} \), consider an arbitrary seed \( v_i \in [0,1] \) and a moment \( \tau \in [1,T+1) \). By definition of \( y_c^i(\tau) \), we have \( \lambda_c(1) \geq b_\tau r_i (1 - g(y_c^i(\tau))) \). It remains to show that \( \lambda_c(y_i) \geq \lambda_c(1) \). We claim that this follows from,

\[
\lambda_c(y_i) = \max_{j \in I(\tau,y_i)} b_j(1 - g(y_j)) \geq \max_{j \in I(\tau,1) \setminus \{i\}} b_j(1 - g(y_j)) = \max_{j \in I(\tau,1)} b_j(1 - g(y_j)) = \lambda_c(1).
\]

It remains to show that \( I(\tau,1) \setminus \{i\} \subseteq I(\tau,y_i) \).

To see this, observe that given the claim above we have,

\[
\tau \in I(\tau,1) \setminus \{i\} \subseteq I(\tau,y_i).
\]

For the sake of contradiction, let \( y_i \) be such that \( I(\tau,1) \setminus \{i\} \nsubseteq I(\tau,y_i) \). Let \( \tau_0 \leq \tau \) be the earliest time such that there exists a resource \( i_0 \in I(\tau_0,1) \setminus (I(\tau_0,y_i) \cup \{i\}) \) i.e., \( i_0 \neq i \) is available at \( \tau_0 \) in \( f{-}\text{Alg}(1) \) but unavailable at \( \tau_0 \) in \( f{-}\text{Alg}(y_i) \). This occurs only if \( i_0 \) is matched at some time \( \tau'_0 < \tau_0 \) in \( f{-}\text{Alg}(y_i) \), but not matched at \( \tau'_0 \) in \( f{-}\text{Alg}(1) \). Now, the following statements are true.

(i) \( I(\tau'_0,1) \setminus \{i\} \subseteq I(\tau'_0,y_i) \). This follows from the definition of \( \tau_0 \) and the fact that \( \tau'_0 < \tau_0 \).

(ii) \( i_0 \in I(\tau'_0,1) \setminus \{i\} \). Follows from \( i_0 \in I(\tau_0,1) \setminus \{i\} \) and the fact that \( I(\tau_0,1) \subseteq I(\tau'_0,1) \).

Thus, \( i_0 \in I(\tau'_0,1) \setminus \{i\} \subseteq I(\tau'_0,y_i) \). Now, at every moment, \( f{-}\text{Alg} \) picks an available resource with highest bid price. So if \( \tau'_0 \) is matched to \( i_0 \) in \( f{-}\text{Alg}(y_i) \), then it must also be matched to \( i_0 \) in \( f{-}\text{Alg}(1) \) (the bid prices of \( i_o \) do not change). This contradicts the definition of \( \tau'_0 \). \( \square \)

Recall that a segment in \( f{-}\text{Alg} \) is a contiguous time interval such that every moment in the interval is matched to the same resource. As we discussed, there are a finite number of segments in \( f{-}\text{Alg}(1) \). Observe that every moment in a given segment has the same critical threshold value.

We define the set

\[
V = \{ v \mid \exists t \in \text{OPT}_i \text{ and } \tau \in [t, t+1), \text{ s.t. } y_c^i(\tau) = v \}.
\]

Notice that \( V \) has finitely many distinct values (at most one value per segment). Let

\[
b(v) := \int_{\tau \in \text{OPT}_i} b_\tau \mathbb{1}(y_c^i(\tau) = v) \, d\tau = \sum_{t \in \text{OPT}_i} b_t \int_{\tau = t}^{t+1} \mathbb{1}(y_c^i(\tau) = v) \, d\tau \quad \forall v \in V,
\]

i.e., \( b(v) \) is the cumulative bid on \( i \) from segments in \( f{-}\text{Alg}(1) \) that have critical threshold value \( v \) and correspond to arrivals in \( \text{OPT}_i \). Recall that we use \( \text{OPT}_i \) to also denote the total fraction of \( i \)'s budget that is matched in \( \text{OPT} \). Observe that,

\[
\sum_{v \in V} b(v) = \text{OPT}_i.
\]

Let

\[
B(y_i) = \sum_{v \geq y_i; v \in V} b(v) \quad \forall y_i \in [0,1].
\]
First, we establish a refined lower bound on \( \sum f \). Case II: Combining this with the lower bound from Lemma 6 we have, 

\[
\text{Case I desired.}
\]

Proof. Consider \( f \) in \( \text{set } \mathcal{I} \). Given Lemma 8. Proof. Follows from Lemma 6 and the definition of set \( \mathcal{I} \) and values \( b(v) \). 

Lemma 8. Given \( i \in I \) and seed \( Y_{-i} \), let \( \mathbb{1}(y_i \leq v) \) indicate the event \( y_i \leq v \). Then, for every \( y_i \in [0,1] \), we have 

\[
\sum_{t \in \text{Opt}_i} \int_{t}^{t+1} \lambda_{t}(y_i) \, d\tau + \theta_{i}(y_i) \geq r_i \sum_{v \in \mathcal{V}} b(v) \left( 1 - g(v) + \mathbb{1}(y_i \leq v) \left( \min \left\{ g(y_i), \frac{g(v) - g(y_i)}{1 - g(y_i)} \right\} \right) \right).
\]

Proof. Consider \( i \in I \) and \( Y_{-i} \in [0,1]^{n-1} \) and fix an arbitrary seed \( y_i \in [0,1] \). Recall that, \( f_{\text{Alg}}(y_i) \) denotes both, the set of moments matched to \( i \) in \( f_{\text{Alg}}(y_i) \), as well as, the total budget of \( i \) used in \( f_{\text{Alg}}(y_i) \). When bids \( b_{jt} \in \{0, b_j \times b_t \} \) for every \( j \in I, t \in T \), it can be shown that \( f_{\text{Alg}}(y_i) \geq B(y_i) \), and this implies a competitive ratio of \( (1 - 1/e) \) for \( f_{\text{Alg}} \). See Appendix C for a formal proof. In general, \( f_{\text{Alg}}(y_i) \) may be strictly smaller than \( B(y_i) \) with constant probability (see Example B.2). In fact, there exist examples where \( \frac{A_{\text{Alg}}(y_i)}{B(y_i)} \rightarrow 0 \) for some \( y_i \in (0,1) \). Keeping this in mind, we now consider two cases. Combining inequalities derived in individual cases gives us the desired.

Case I: \( f_{\text{Alg}}(y_i) \geq B(y_i) \). Thus, 

\[
\theta_{i}(y_i) \geq r_i B(y_i) g(y_i) = r_i \sum_{v \in \mathcal{V}} b(v) g(y_i).
\]

Combining this with the lower bound from Lemma 6 we have, 

\[
\sum_{t \in \text{Opt}_i} \int_{t}^{t+1} \lambda_{t}(y_i) \, d\tau + \theta_{i}(y_i) \geq r_i \sum_{v \in \mathcal{V}} b(v) (1 - g(v) + g(y_i)).
\]

Case II: \( f_{\text{Alg}}(y_i) < B(y_i) \) \( (\leq B_i) \) i.e., resource \( i \) is available at every moment in \( f_{\text{Alg}}(y_i) \). First, we establish a refined lower bound on \( \sum_{t \in \text{Opt}_i} \int_{t}^{t+1} \lambda_{t}(y_i) \, d\tau \), followed by a lower bound on \( \theta_{i}(y_i) \).

Since \( i \) is available at every moment in \( f_{\text{Alg}}(y_i) \), we have \( \forall \tau \in S(y_i) \), 

\[
\lambda_{\tau}(y_i) \geq b_{i\tau} \, r_i (1 - g(y_i)) = b_{i\tau} \, r_i (1 - g(y_i^\tau)) + g(y_i^\tau(\tau)) - g(y_i).
\]
Combining this with the lower bound from Lemma 3 we get
\[
\frac{1}{r_i} \sum_{t \in \text{Opt}_i} \int_{\tau=t}^{t+1} \lambda_r(y_i) \, d\tau \geq \sum_{v \in V} b(v) \left[ 1 - g(v) + \mathbb{1}(y_i \leq v) \left( g(v) - g(y_i) \right) \right]. \tag{8}
\]

Next, from the definition of $\theta_i(y_i)$ and Lemma 9 (shown subsequently), we have
\[
\theta_i(y_i) = r_i \, f\text{-ALG}_i(y_i) \, g(y_i) \geq r_i \sum_{v \in V; \ v \geq y_i} b(v) \left( g(v) - g(y_i) \right) / \left( 1 - g(y_i) \right).
\tag{9}
\]

Combining (8) and (9) we get,
\[
\sum_{t \in \text{Opt}_i} \int_{\tau=t}^{t+1} \lambda_r(y_i) \, d\tau + \theta_i(y_i)
\geq r_i \sum_{v \in V} b(v) \left[ 1 - g(v) + \mathbb{1}(y_i \leq v) \left( g(v) - g(y_i) \right) + \frac{g(v) - g(y_i)}{1 - g(y_i)} \right]
\geq r_i \sum_{v \in V} b(v) \left[ 1 - g(v) + \mathbb{1}(y_i \leq v) \left( g(v) - g(y_i) \right) / \left( 1 - g(y_i) \right) \right]
\]

For a given $y_i$, we could be in the worse of the two cases above. Thus, we have the following combined lower bound,
\[
\sum_{t \in \text{Opt}_i} \int_{\tau=t}^{t+1} \lambda_r(y_i) \, d\tau + \theta_i(y_i)
\geq r_i \sum_{v \in V} b(v) \left[ 1 - g(v) + \min \left\{ g(y_i), \mathbb{1}(y_i \leq v) \left( g(v) - g(y_i) \right) / \left( 1 - g(y_i) \right) \right\} \right] \geq r_i \sum_{v \in V} b(v) \left[ 1 - g(v) + \mathbb{1}(y_i \leq v) \left( \frac{g(v) - g(y_i)}{1 - g(y_i)} \right) \right].
\]

\[\square\]

**Lemma 9.** Consider a resource $i \in I$ and seed vector $(y_i, Y_{-i}) \in [0,1]^n$ such that $f\text{-ALG}_i(y_i) < B(y_i)$. Then, we have $f\text{-ALG}_i(y_i) \geq \sum_{v \in V, v \geq y_i} b(v) \left( g(v) - g(y_i) \right) / \left( 1 - g(y_i) \right)$.

**Proof.** Given resource $i$ and seed $Y_{-i}$, for every $y_i \in [0,1]$, define
\[
\lambda_{\text{net}}(y_i) = \int_{1}^{T+1} \lambda_r(y_i) \, d\tau.
\]

Using Lemma 6 we have, $\lambda_{\text{net}}(y_i) \geq \lambda_{\text{net}}(1) \ \forall y_i \in [0,1]$. Now, fix a seed $y_i$ such that $f\text{-ALG}_i(y_i) < B(y_i)$. Observe that the main claim follows from the following upper and lower bounds on $\lambda_{\text{net}}(y_i) - \lambda_{\text{net}}(1)$.
\[
r_i \, f\text{-ALG}_i(y_i) \left( 1 - g(y_i) \right) \geq \lambda_{\text{net}}(y_i) - \lambda_{\text{net}}(1) \geq r_i \sum_{v \in V, v \geq y_i} b(v) \left( g(v) - g(y_i) \right).
\]
Proof of lower bound: Since $f\text{-Alg}_i(y_i) < B(y_i) \leq B_i$, we have that $i$ is available at every moment in $f\text{-Alg}(y_i)$. Thus, in $f\text{-Alg}(y_i)$, every $\tau \in S(y_i)$ is matched to a resource $j$ such that,
\[
b_{j\tau} r_j (1 - g(y_j)) \geq b_{i\tau} r_i (1 - g(y_i)).\]
From this, we have for every $\tau \in S(y_i)$,
\[
\lambda_{\tau}(y_i) - \lambda_{\tau}(1) \geq b_{i\tau} r_i (g(y_i^c(\tau)) - g(y_i)),
\]
where we used the following facts (i) For $y_i^c(\tau) > 0$, we have $\lambda_{\tau}(1) = b_{i\tau} r_i (1 - g(y_i^c(\tau)))$ and (ii) For $y_i^c(\tau) = 0$ and $\tau \in S(y_i)$, we have $y_i = 0$.
Integrating over all segments in $S(y_i)$, we get
\[
\int_{\tau \in S(y_i)} (\lambda_{\tau}(y_i) - \lambda_{\tau}(1)) \, d\tau \geq r_i \sum_{v \in V, v \geq y_i} b(v) (g(v) - g(y_i)).
\]
Finally, from Lemma 6 we have that $\lambda_{\text{net}}(y_i) - \lambda_{\text{net}}(1) \geq \int_{\tau \in S(y_i)} (\lambda_{\tau}(y_i) - \lambda_{\tau}(1)) \, d\tau$, completing the proof of the lower bound.

Proof of upper bound: We start by observing that for every seed $y \in [0, 1]$ of resource $i$,
\[
\lambda_{\text{net}}(y) = f\text{-Alg}_i(y) r_i (1 - g(y)) + \sum_{j \in I \setminus \{i\}} f\text{-Alg}_j(y) r_j (1 - g(y_j)).
\]
Therefore,
\[
\lambda_{\text{net}}(y_i) - \lambda_{\text{net}}(1) = f\text{-Alg}_i(y_i) r_i (1 - g(y_i)) + \sum_{j \in I \setminus \{i\}} [f\text{-Alg}_j(y_i) - f\text{-Alg}_j(1)] r_j (1 - g(y_j)),
\]
where we used the fact that $g(1) = 1$ for every value of $\beta > 0$. Now, the desired upper bound on $\lambda_{\text{net}}(y_i) - \lambda_{\text{net}}(1)$ follows from the claim that,
\[
f\text{-Alg}_j(y_i) \leq f\text{-Alg}_j(1) \quad \forall j \in I \setminus \{i\}.
\]
The claim is obviously true when $f\text{-Alg}_j(1) = B_j$, so let $f\text{-Alg}_j(1) < B_j$ i.e., resource $j$ is available at every moment in $f\text{-Alg}(1)$. Therefore, in $f\text{-Alg}(1)$, every moment $\tau \in [1, T+1)$ that is not matched to $j$ must be matched to a resource with (strictly) higher bid price (since ties between bid prices do not occur except on a probability 0 set of seed values) i.e.,
\[
\lambda_{\tau}(1) > b_{j\tau} r_j (1 - g(y_j)).
\]
Since $\lambda_{\tau}(y_i) \geq \lambda_{\tau}(1)$ (from Lemma 6), we have that every moment $\tau$ that is not matched to $j$ in $f\text{-Alg}(1)$, is not matched to $j$ in $f\text{-Alg}(y_i)$ either. Therefore, $f\text{-Alg}_j(y_i) \leq f\text{-Alg}_j(1)$. \qed
Proof of Theorem 3. Let $\alpha = \min_{v \in [0,1]} \left[ 1 - g(v) + \int_0^v \left( \min \left\{ g(y), \frac{g(v) - g(y)}{1 - g(y)} \right\} \right) dy \right]$. Now, taking expectation over the randomness in seed $y_i$ on both sides of the inequality in Lemma 8, we have

$$E_{y_i} \left[ \sum_{t \in \text{OPT}_i} \int_{t-\tau}^{t+1} \lambda_r(y_i) d\tau + \theta_i(y_i) \right] \geq \sum_{v \in V} b(v) \left[ 1 - g(v) + \int_0^v \left( \min \left\{ g(y), \frac{g(v) - g(y)}{1 - g(y)} \right\} \right) dy \right],$$

$$\geq \alpha r_i \sum_{v \in V} b(v) = \alpha r_i \text{OPT}_i.$$ 

It remains to lower bound $\alpha$. We do this step numerically. For $g(x) = e^{x-1}$, we obtain $\alpha > 0.508$ (minimum at $x = 0.586$) and for $g(x) = e^{1.15(x-1)}$ we obtain $\alpha > 0.522$ (minimum at $x = 0.789$, see Figure 1).

This completes the proof of Theorem 3 i.e., establishes the desired guarantees for $f$-ALG. To establish the same guarantees for ALG and prove Theorem 2, we now link the performance of the two algorithms via a modified instance of the problem.

From Fractional to Integral Algorithm

Since ALG is oblivious to budgets but $f$-ALG is not, there can be a substantial difference between the output of these algorithm for the same seed $Y$ (see Example B.1 in Appendix B). Therefore, we compare ALG not with $f$-ALG on the same instance but with the performance of $f$-ALG on a modified instance where the budgets of resources are increased as described next.

Budget augmentation: Consider an instance where the budget of every item is augmented as follows,

$$B^a_i = B_i + \max_{t \in T} b_{it} \quad \forall i \in I.$$
Let $\text{Opt}^a$ denote the expected total reward of optimal offline on an instance with augmented budgets $B_i^a$. Since an allocation that is feasible in the original instance is also feasible after augmenting the budgets, we have

$$\text{Opt}^a \geq \text{Opt}.$$ 

Let $f$-$\text{Alg}^a$ denote the expected total reward of $f$-$\text{Alg}$ on the augmented instance. From the competitive ratio guarantee for $f$-$\text{Alg}$, we have

$$f$-$\text{Alg}^a \geq \alpha \text{Opt}^a \geq \alpha \text{Opt}.$$ 

**Proof of Theorem 2.** Fix a random seed $Y$. Let $\text{Alg}(Y)$ denote the total reward of $\text{Alg}$ on the original instance with seed $Y$. Let $f$-$\text{Alg}^a(Y)$ denote the total reward of $f$-$\text{Alg}$ on the augmented instance. We will show that,

$$\text{Alg}(Y) \geq \frac{1}{1 + \gamma} f$-$\text{Alg}^a(Y) \quad \forall Y \in [0, 1]^n. \quad (10)$$

To see that this proves the theorem, take expectation over $Y$ on both sides to obtain,

$$\text{Alg} \geq \frac{1}{1 + \gamma} f$-$\text{Alg}^a.$$ 

Then, using $f$-$\text{Alg}^a \geq \alpha \text{Opt}$ completes the proof.

Fix an arbitrary seed $Y$ and a resource $i \in I$. Let $f$-$\text{Alg}^a_i(Y)$ denote the total revenue from resource $i$ in $f$-$\text{Alg}(Y)$ with augmented budgets, Similarly, $\text{Alg}_i(Y)$ is the total revenue from $i$ in $\text{Alg}(Y)$. To prove (10), it suffices to show that

$$f$-$\text{Alg}^a_i(Y) \leq B_i^a \frac{\text{Alg}_i(Y)}{B_i}. \quad (11)$$

We establish this in cases.

**Case I:** All $B_i$ of $i$’s budget is allocated in $\text{Alg}(Y)$ i.e., $\text{Alg}_i(Y) = B_i$. Then, (11) follows by observing that $f$-$\text{Alg}^a_i(Y) \leq B_i^a = B_i \frac{\text{Alg}_i(Y)}{B_i}$.

**Case II:** Some of $i$’s budget is available after $T$ in $\text{Alg}(Y)$. In this case, we show more strongly that $f$-$\text{Alg}^a_i(Y) \leq \text{Alg}_i(Y)$. Let $x_j^{a,f}(t,Y)$ denote the remaining budget of resource $j \in I$ at time $t$ in $f$-$\text{Alg}^a(Y)$. Define

$$z_j^{a,f}(t,Y) = \max\{0, B_j - x_j^{a,f}(t,Y)\} = (B_j - x_j^{a,f}(t,Y))^+ \quad \forall j \in I.$$ 

Recall that $x_j(t,Y)$ is the budget of $j$ remaining after arrival $t - 1$ in $\text{Alg}(Y)$. We define

$$z_j(t,Y) = (B_j - x_j(t,Y))^+ \quad \forall j \in I.$$ 

As some of $i$’s budget is unused in $\text{Alg}(Y)$, we have $z_i(T + 1, Y) > 0$. Now, observe that to prove $f$-$\text{Alg}^a_i(Y) \leq \text{Alg}_i(Y)$, it suffices to show $z_i^{a,f}(T + 1, Y) \geq z_i(T + 1, Y)$. In fact, we show more strongly that

$$z_j^{a,f}(t,Y) \geq z_j(t,Y) \quad \forall j \in I, t \in T.$$
This is true at \( t = 1 \). Suppose this is true just prior to arrival of \( t \). Using this we show the inequality also holds just prior to arrival of \( t + 1 \). Then by induction we have the desired.

Given \( z_{j}^{a,f}(t,Y) \geq z_{j}(t,Y) \), we have

\[
\{ j \mid z_{j}(t,Y) > 0 \} \subseteq \{ j \mid z_{j}^{a,f}(t,Y) > 0 \},
\]

where the LHS is the set of resources available in \( \text{Alg}(Y) \) at \( t \). Let \( e \) be the resource matched to \( t \) in \( \text{Alg}(Y) \) and \( w \) denote the resource matched at moment \( t \) in \( f-\text{Alg}^{a}(Y) \). Since both \( \text{Alg} \) and \( f-\text{Alg} \) match greedily according to bid prices, from (12) we have

\[
b_{w}t_{w}(1 - g(y_{w})) \geq b_{e}t_{e}(1 - g(y_{e})).
\]

Since there are no ties between bid prices of different resources for every seed \( Y \) (except a probability 0 set), if \( w \neq e \) then \( z_{w}(t,Y) = 0 \) i.e., \( \text{Alg}(Y) \) has used up the budget of resource \( w \) prior to arrival of \( t \). Thus,

\[
z_{w}^{a,f}(t + 1,Y) \geq z_{w}(t + 1,Y) = 0.
\]

Let \( \tau \) be the first moment in \( [t,t+1) \) (if any), where \( f-\text{Alg}^{a}(Y) \) matches to a resource in \( \{ j \mid z_{j}(t,Y) > 0 \} \). Observe that this resource must be \( e \). Now, as \( \tau \in [t,t+1) \) and \( z_{e}^{a,f}(t,Y) \geq z_{e}(t,Y) \), we have that

\[
z_{e}^{a,f}(t + 1,Y) \geq z_{e}(t + 1,Y).
\]

Further, since \( z_{e}^{a,f}(\tau,Y) > 0 \), at least \( \max_{t \in T} b_{e,t} \) of \( e \)'s augmented budget is available in \( f-\text{Alg}^{a}(Y) \) at \( \tau \). Hence, in \( f-\text{Alg}^{a}(Y) \), no resource from the set \( \{ j \mid j \neq e, z_{j}(t,Y) > 0 \} \) is matched (fractionally) during \( [t,t+1) \) i.e.,

\[
z_{q}^{a,f}(t + 1,Y) = z_{q}^{a,f}(t,Y) \geq z_{q}(t,Y) = z_{q}(t + 1,Y) \quad \forall q \in \{ j \mid j \neq e, z_{j}(t,Y) > 0 \}.
\]

This completes the proof.

\[\square\]

**Remark:** When every bid is either 0 or 1 (\( b \)-matching case), it is easy to see that \( f-\text{Alg} \) and \( \text{Alg} \) are identical on every sample path. Since \( b \)-matching is a special case of Adwords with decomposable bids \( (b_{it} \in \{0, b_{i} \times b_{i}\} \forall i \in I, t \in T) \), and \( f-\text{Alg} \) is \( (1 - 1/e) \) competitive for Adwords with decomposable bids (Appendix [C]), we have that, \( \text{Alg} \) is \( (1 - 1/e) \) competitive for \( b \)-matching (with arbitrary budgets).
5. Conclusion
Motivated by the possibility of using cross-campaign information to make real-time (automated) budget distribution, we considered the classic Adwords setting with unknown budgets. We showed that a natural generalization of Perturbed Greedy algorithm, that computes random bid prices for resources without using budget information, is 0.522 competitive. This is the first result that improves on the guarantee of 0.5 obtained by the greedy algorithm. To show the result, first, we analyze the fractional version of the algorithm using recent innovations in analysis of online matching algorithms, alongside various novel structural insights. Then, we relate the performance of the fractional algorithm on a budget augmented instance with the integral algorithm on the original instance.

New Directions and Open Problems
• The (real-time) budget adjustment/distribution problem: To the best of our knowledge, no previous work has studied this problem. So the first challenge here is finding a good model for the problem.
• Budget oblivious algorithms with strong guarantees in other arrival models: Practical instances of the Adwords problem are not adversarial (nor entirely predictable). Designing budget oblivious algorithms in other models, where the arriving sequence is stochastic or mixed, would be a crucial step in making online budget adjustments viable.
• Improving the guarantee for Adwords with unknown budgets: Is there a \((1 – 1/e)\) competitive algorithm for the problem? In particular, what is the true guarantee for \(\text{Alg}\)? We did not seek to try trade off functions \(g(\cdot)\) that are not exponential. So it is possible that choosing a different function family leads to improved guarantees. Note that for the special case where bids \(b_{it}\) are decomposable i.e., \(b_{it} \in \{0, bi \times bt\} \forall i \in I, t \in T\), \(\text{Alg}\) (with \(\beta = 1\)) is \(\frac{1}{1+\gamma}(1 – 1/e)\) competitive. If bids are binary, \(\text{Alg}\) is \((1 – 1/e)\) competitive for arbitrary budgets.

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Appendix A: Obstacles with Using Classic Primal-Dual Analysis

The primal-dual framework of [Buchbinder et al., 2007] and [Devanur et al., 2013] is one of the most versatile and general technique for proving guarantees for online matching and related problems. To describe the framework, consider the following primal and dual problems that upper bound the optimal offline solution for OBA.

**Primal:** \[
\min \sum_{i \in I, t \in T} r_i b_{it} x_{it}
\]
\[
s.t. \sum_{t \in T} b_{it} x_{it} \leq B_i \quad \forall i \in I,
\]
\[
\sum_{i \in I} x_{it} \leq 1 \quad \forall t \in T,
\]
\[
x_{it} \geq 0 \quad \forall i \in I, t \in T.
\]

**Dual:** \[
\min \sum_{t \in T} \lambda_t + \sum_{i \in I} B_i \theta_i
\]
\[
s.t. \lambda_t + b_{it} \theta_i \geq b_{it} r_i \quad \forall i \in I, t \in T
\]
\[
\lambda_t, \theta_i \geq 0 \quad \forall t \in T, i \in I.
\]

**Primal-dual certificate [Devanur et al., 2013]:** To prove \(\alpha\) competitiveness for Alg, it suffices to find a set of non-negative values \(\lambda_t, \theta_i\) such that,

(i) \(\lambda_t + b_{it} \theta_i \geq \alpha b_{it} r_i, \forall i \in I, t \in T\),

(ii) \(\delta \text{Alg} \geq \sum_{t \in T} \lambda_t + \sum_{i \in I} B_i \theta_i\).

To understand the obstacles in using this framework to analyze Algorithm 1 (and Algorithm 2) for OBA, we start by defining a natural candidate solution for the system based on the decisions of Algorithm 1.
Let \( \text{ALG} \) denote Algorithm \( \Pi \) as well as its expected reward. In \( \text{ALG} \), let \( g(t) = e^{t-1} \) i.e., \( \beta = 1 \). Given that we have external randomness in \( \text{ALG} \) through \( Y = (y_i)_{i \in T} \), we shall define variables \( \lambda^Y_i, \theta^Y_i \) and subsequently set \( \lambda_i = E_Y[\lambda^Y_i] \) and \( \theta_i = E_Y[\theta^Y_i] \). Inspired by \cite{Devanur et al. (2013)}, we set \( \lambda^Y_i \) and \( \theta^Y_i \) as follows. Initialize all dual variables to 0. Conditioned on \( Y \), for any match \( (i,t) \) in \( \text{ALG} \) set,

\[
\lambda^Y_i = b_{ir}r_i(1 - g(y_i)) \text{ and increment } \theta^Y_i \text{ by } \frac{b_{ir}}{B_i}g(y_i).
\]

Clearly, \( \lambda^Y_i \) is set uniquely since \( \text{ALG} \) offers at most one resource \( i \) to arrival \( t \), and \( \theta^Y_i \) takes a non-zero value only if it is also accepted by some \( t \), and if this occurs \( \theta^Y_i \) is never re-set. The following lemma (stated without proof) declares that this candidate solution satisfies constraint (ii) with \( \delta \) close to 1 in the small bid regime.

\textbf{Lemma 10.} For the candidate solution given by \( \Pi \), constraint (ii) in the primal-dual certificate is satisfied with \( \delta = 1 + \gamma \).

Unfortunately, there exist instances such that constraints (i) do not hold for any value of \( \alpha > 0.5 \). For example, suppose that the first arrival has a non-zero bid exclusively from resource \( i \). In fact, let \( b_{i1} = 1 \). Consequently, the first arrival is always matched to \( i \) and we have, \( \lambda^Y_i = r_i(1 - g(y_i)) \forall Y \in [0,1]^n \). Subsequent arrivals have higher bids from many resources other than \( i \) such that they are matched to \( i \) with very small probability. Thus, \( \theta_i \approx \frac{1}{r_i} \int_0^1 g(x) dx \), and for \( B_i \rightarrow +\infty \), we have

\[
E_Y[\lambda^Y_i] + \theta_i \approx r_i \left( \int_0^1 (1 - g(x)) dx + \frac{1}{B_i} \int_0^1 g(x) dx \right) \rightarrow r_i \int_0^1 (1 - g(x)) dx.
\]

Notice that when \( g(x) = e^{x-1} \), we have \( \int_0^1 (1 - g(x)) dx = g(0) = 1/e < 0.5 \).

Now, let \( \text{OPT} \) denote the offline solution and let \( \text{OPT} \), denote the set of arrivals matched to \( i \) in \( \text{OPT} \). In contrast to the primal-dual scheme, the LP free scheme from Section 4.2 only imposes the following linear combination of the LP constraints,

\[
\sum_{t \in \text{OPT}_i} \lambda_t + B_i \times \theta_t \geq \alpha r_i \text{OPT}_i,
\]

where we summed constraints (i) over all arrivals in \( \text{OPT}_i \) and scaled \( \theta_t \) in accordance with the LP free system (the counterpart of constraint (ii) in the LP free system replaces \( B_i \theta_t \) with \( \theta_t \)). Since \( i \) is never matched to any arrival after the first one, we have that, \( \lambda_i \geq b_{i1} r_i (1 - g(0)) \forall t \geq 2 \). Thus,

\[
\sum_{t \in \text{OPT}_i} \lambda_t + \theta_t \geq (1 - 1/e) r_i \left( \sum_{t \in \text{OPT}_i, t \geq 2} b_{it} \right) + r_i \int_0^1 g(x) dx \geq (1 - 1/e) r_i \text{OPT}_i.
\]

\textbf{Appendix B: Other Examples to Demonstrate Obstacles}

The first example demonstrates an obstacle that arises primarily out of the fact that Algorithm \( \Pi \) is both randomized and budget oblivious.

\textbf{Example B.1.} Consider an instance with \( n \) resources \( \{1, \ldots, n\} \), with budget \( n-1 \) for resources \( j \in [n-1] \) and budget \( (n-1)^{1.98} \) for resource \( n \). Let per unit rewards \( r_i = 1 \forall i \in [n] \). We focus on a snippet of this instance by considering arrivals \( \{t + 1, \ldots, t + 2n - 2\} \subset T \). We execute Algorithm \( \Pi \) (with \( \beta = 1 \)) on this instance with seed \( Y_n \) for resources \( j \in [n-1] \) fixed, and observe the change in output as seed \( y_n \) varies. Suppose that exactly 1 unit of budget is available at arrival \( t+1 \) for every resource \( j \in [n-1] \) and every value of \( y_n \in [0,1] \). Further, \( \forall y_n \in [0,1] \), all of resource \( n \)'s budget is available at \( t+1 \). The bids are as follows.
Lemma 11. Given $i \in I$, seed $Y_{-i}$, and decomposable bids $b_{it} \in \{0, b_i b_t\}$ for $i \in I$, $t \in T$, we have,

$$E_{y_i}[\theta_i(y_i) \mid Y_{-i}] + \sum_{t \in \mathbb{R}^T} \int_{\tau = t}^{t+1} E_{y_i}[\lambda_i(y_i) \mid Y_{-i}] d\tau \geq r_i \sum_{v \in V} b(v) \left[ 1 - g(v) + \int_0^v g(x)dx \right].$$
Proof. It suffices to show that,
\[ \theta_i(y_i) \geq r_i B(y_i) g(y_i) \quad \forall y_i \in [0,1). \]  
(14)
Taking expectation over \( y_i \sim U[0,1] \) on both sides, we have,
\[ E_{y_i}[\theta_i(y_i) | Y_{-i}] \geq r_i \sum_{v \in V} \left( b(v) \int_0^v g(x) dx \right). \]
Combining the inequality above with the lower bound in Corollary 7 gives us the desired.

Notice that, if \( b_i = 0 \), then (14) is trivially true. Further, all arrivals \( t \in T \) where \( b_i = 0 \) can be ignored.
W.l.o.g., let \( b_i > 0 \) and \( b_t > 0 \ \forall t \in T \). Then, to show (14), it suffices to show that \( f\text{-ALG}_i(y_i) \geq B(y_i) \ \forall y_i \in [0,1) \) (see the definition of \( \theta_i(y_i) \) in (5)). For the sake of contradiction, consider a value \( y_i = y_0 \in [0,1) \) such that \( f\text{-ALG}_i(y_0) < B(y_0) \leq B_t \). There exists a moment \( \tau \in \text{OPT}_i \) such that, \( y_i^\tau(\tau) \geq y_0 \), but \( \tau \) is not matched to \( i \) in \( f\text{-ALG}(y_0) \). Since, \( f\text{-ALG}(y_0) < B_t \), we have that, \( \tau \) is matched to a resource \( i_1 \) such that,
\[ b_{i_1} r_{i_1} (1 - g(y_i)) > b_i r_i (1 - g(y_i)) \geq b_{i_1} r_{i_1} (1 - g(y_i^\tau(\tau))), \]
here the first inequality is strict w.h.p. (ties occur w.p. 0). Using the decomposability of bids, we have,
\[ b_{i_1} r_{i_1} (1 - g(y_i)) > b_i r_i (1 - g(y_i)). \]
Therefore, \( i_1 \) is preferred over \( i \) at all arrivals. We say that \( i_1 \) is better than \( i \), or \( i_1 \succ i \) (in \( f\text{-ALG}(y_0) \)). Now, notice that \( i_1 \) must be unavailable at \( \tau \) in \( f\text{-ALG}(1) \). Therefore, there exists a moment \( \tau_{i_1} \) prior to \( \tau \) such that, \( \tau_{i_1} \) is matched to \( i_1 \) in \( f\text{-ALG}(1) \) but in \( f\text{-ALG}(y_0) \), \( \tau_{i_1} \) it is matched to a resource \( i_2 \) that is better than \( i_1 \). Repeating this argument a number of times, we get a sequence of moments \( \tau > \tau_{i_1} > \cdots > \tau_{i_k} \) and resources \( i \prec i_1 \prec \cdots \prec i_k \). The number of resources is finite, so w.l.o.g., \( i_k = i_\ell \) for some \( \ell < k \) and \( k \leq n \), contradiction. \[ \square \]

When \( g(x) = e^{x-1} \), we have \( \forall i \in I, Y_{-i} \in [0,1]^{n-1}, \)
\[ \sum_{t \in \text{OPT}_i} \int_{\tau=t}^{t+1} E_{y_i}[\lambda_r(y_i) | Y_{-i}] d\tau + E_{y_i}[\theta_i(y_i) | Y_{-i}] \geq r_i \sum_{v \in V} b(v) \left( 1 - e^{-v} + e^{-v} - e^{-1} \right) = (1 - e^{-1}) r_i \sum_{v \in V} b(v), \]
\[ = (1 - e^{-1}) r_i \text{OPT}_i. \]
Therefore, \( f\text{-ALG} \) with \( g(x) = e^{x-1} \) is \((1 - 1/e)\) competitive for decomposable probabilites.