ON THE FOURIER-LAPLACE TRANSFORM OF FUNCTIONALS ON A WEIGHTED SPACE OF INFINITELY DIFFERENTIABLE FUNCTIONS

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The strong dual space of linear continuous functionals on a weighted space $G$ of infinitely differentiable functions defined on the real line is described in terms of their Fourier-Laplace transforms.

1. Preliminaries and the main result.

Let $M_0 = 1, M_1, M_2, \ldots$, be an increasing sequence of positive numbers which satisfies the following conditions:

i). $M_k^2 \leq M_{k-1}M_{k+1} \forall k \in \mathbb{N};$

ii). $\lim_{k \to \infty} \left( \frac{M_{k+1}}{M_k} \right)^{\frac{1}{k}} = 1.$

iii). $\exists Q_1, Q_2 > 0$ such that $M_k \geq Q_1 Q_2^k k! \forall k \in \mathbb{Z}^+.$

Let

$$w(r) = \sup_{k \in \mathbb{Z}^+} \ln \frac{r^k}{M_k}, \quad r > 0, \quad w(0) = 0.$$  

$w$ is continuous for $r \geq 0$ [3] and $w(r) = 0$ for $r \in [0, M_1]$. From this and the condition iii) it follows that there exists $A_w > 0$ such that

$$w(r) \leq A_w r, \quad r \geq 0. \quad (1)$$

It is clear that $w(|z|)$ is a subharmonic function in the complex plane.

For $z \in \mathbb{C}$, $t > 0$ denote by $D(z, t)$ an open disk of radius $t$ about a point $z$ and by $\partial D(z, t)$ its boundary.
Let \( \alpha > 1 \) and \( \psi : \mathbb{R} \to [0, \infty) \) be a convex function satisfying the conditions:

1. \( \exists A_\psi > 0 \) such that for any \( x_1, x_2 \in \mathbb{R} \)
   \[
   |\psi(x_1) - \psi(x_2)| \leq A_\psi (1 + |x_1| + |x_2|)^{\alpha - 1}|x_1 - x_2|;
   \]  
   \( \phi \) \( \in \mathbb{R} \to [0, \infty) \) be a convex function satisfying the conditions:

2. \( \lim_{\psi(x) \to \infty} |x| = +\infty \);

3. for any \( z \in \mathbb{C}, |z| > 1 \),
   \[
   \mu_\psi(D(z, t)) \leq c_\psi |z|^{\alpha - 1}t, \quad t \in (0, |z|),
   \]
where \( \mu_\psi \) is a measure associated by Riesz with subharmonic function \( \psi(\text{Im} z) \), \( c_\psi > 0 \) is some constant.

For a function \( g : \mathbb{R} \to [0, \infty) \) such that \( \frac{g(x)}{|x|} \to +\infty \) as \( x \to \infty \) the Young transform \( g^* \) of \( g \) is defined by

\[
g^*(x) = \sup_{y \in \mathbb{R}} (xy - g(y)), \quad x \in \mathbb{R}.
\]

\( g^* \) is convex and \( \frac{g^*(x)}{|x|} \to +\infty \) as \( x \to \infty \) \([10]\). Moreover, if \( g \) is convex then the inversion formula holds \([10]\): \((g^*)^* = g\).

Let \( \varphi = \psi^* \). From (2)

\[
\varphi(x) > A_\psi |x|^{\frac{\alpha - 1}{\alpha}} - B_\psi, \quad x \in \mathbb{R},
\]
where \( A_\varphi, B_\varphi \) are some positive numbers.

As usual \( \mathcal{E}(\mathbb{R}) \) will denote the space of infinitely differentiable functions defined in \(\mathbb{R}\) with the topology of uniform convergence of functions and all their derivatives on compact subsets of \( \mathbb{R} \). \( C(\mathbb{R}) \) will denote the space of continuous functions in \( \mathbb{R} \). \( H(\mathbb{C}) \) is the space of entire functions equipped with the topology of uniform convergence on compact sets of complex plane. For a locally convex space \( X \) let \( X^* \) be the strong dual of \( X \).

Fix \( \sigma > 0 \). Let \( \{\varepsilon_m\}_{m=1}^\infty \) be an arbitrary decreasing to zero sequence of positive numbers. Set \( \theta_m(x) = \exp(\varphi(x) - m \ln(1 + |x|)), x \in \mathbb{R}, m \in \mathbb{N} \). Let

\[
G_m = \{f \in \mathcal{E}(\mathbb{R}) : \|f\|_{G,m} = \sup_{x \in \mathbb{R}, k \in \mathbb{Z}_+} \frac{|f^{(k)}(x)|}{(\sigma + \varepsilon_m)^k M_k \theta_m(x)} < \infty\}, m \in \mathbb{N}.
\]

We let \( G = \bigcap_{m=1}^\infty G_m \) and endow this vector space with its natural projective limit topology. It is clear that the definition of \( G \) doesn’t depend on the
choice of the sequence \( \{\varepsilon_m\}_{m=1}^\infty \). It is easy to show that for any \( m \in \mathbb{N} \) the canonical (inclusion) mapping \( i_{m+1,m} : G_{m+1} \to G_m \) is relatively compact in \( G_m \). So \( G \) is the space \( (M^*) \) (see for definitions [11]). From this and theorem 5 of [11] it follows that \( G^* \) is a space \( (LN^*) \).

Let \( w_m(|z|) = w((\sigma + \varepsilon_m)^{-1}|z|), z \in \mathbb{C}, m \in \mathbb{N} \). Let

\[
P_m = \left\{ f \in H(\mathbb{C}) : \|f\|_m = \sup_{z \in \mathbb{C}} \frac{|f(z)|}{\exp(\psi(Im z) + w_m(|z|))} < \infty \right\}, m \in \mathbb{N}.
\]

Let \( P \) be the union of these normed spaces with a topology of inductive limit of the spaces \( P_m \).

For \( T \in G^* \) we define the Fourier-Laplace transform \( \hat{T} \) of \( T \) by

\[
\hat{T}(z) = T(e^{-ixz}), \ z \in \mathbb{C}.
\]

The main point of our work is to characterize the dual space \( G^* \) of linear continuous functionals on \( G \) in terms of their Fourier-Laplace transforms. The similar problem was considered in [14], but the method presented there is not applicable in our case (see condition \( \Phi \ 3 \) in [14], p. 48). Our principal result is the following theorem.

**Theorem 1.** The Fourier-Laplace transform establishes topological isomorphism of the spaces \( G^* \) and \( P \).

We establish surjectivity of the Fourier-Laplace transform by using the representation of entire functions from \( P \) in the form of the Lagrange series. This idea comes from [15]. To realise this idea we construct some special entire function. For the construction of this entire function we use results of R.S. Yulmukhametov [16], [17] and M.I. Solomesch [7], [12], [13].

2. Auxiliary results.

**Lemma 1.** For any \( z_1, z_2 \in \mathbb{C} \)

\[
|w(|z_2|) - w(|z_1|)| \leq A_w e|z_2 - z_1|.
\]  

**Proof.** Let \( N(r) = \min \left\{ k \in \mathbb{Z}_+ : w(r) = \ln \frac{r^k}{M_k} \right\}, r > 0 \). Using the inequality from [15, Lemma 1.2]: \( w(r) - w(1) \geq N \left( \frac{r}{e} \right), r > 1 \), and the estimate (1), we obtain

\[
N(r) \leq A_w e, \ r > e^{-1}.
\]
From the equality (see [3], [15, Lemma 1.2])

\[ w(r) - w(1) = \int_{1}^{r} \frac{N(x)}{x} \, dx, \quad r > 1, \]

and the estimate (5) we derive that:

\[ w(r_2) - w(r_1) \leq A_w(r_2 - r_1), \quad r_2 > r_1 > 1; \]

\[ w(r_2) - w(r_1) = w(r_2) - w(1) \leq A_w(r_2 - 1) \leq A_w(r_2 - r_1), \quad r_2 > 1, r_1 \in [0, 1]. \]

Since \( w(r) = 0 \) for \( r \in [0, M_1] \) then from two last inequalities the statement of Lemma follows.

**Lemma 2.** For any \( A > 0, m \in \mathbb{N} \) \( \exists Q > 0 \) such that

\[ w_m(|z|) + A \ln(1 + |z|) \leq w_{m+1}(|z|) + Q, \quad z \in \mathbb{C}. \]

**Proof.** Let \( N(r) \) be defined as in Lemma 1. Then

\[ \frac{r^{N(r)}}{M_{N(r)}} \geq \frac{r^{N(r)+1}}{M_{N(r)+1}}. \]

Consequently, \( r \leq \frac{M_{N(r)+1}}{M_{N(r)}} \). From this, using \( ii \), for any \( \delta > 0 \) we can find \( A_\delta > 0 \) such that \( r \leq A_\delta(1 + \delta)^{N(r)} \). Thus, \( N(r) \geq \frac{\ln \frac{r}{A_\delta}}{\ln(1 + \delta)} \). Set

\[ r = \frac{|z|}{\sigma + \varepsilon_m}, \quad z \in \mathbb{C}, z \neq 0. \]

Then

\[ w_{m+1}(|z|) - w_m(|z|) \geq \ln \left( \left( \frac{|z|}{\sigma + \varepsilon_{m+1}} \right)^{N(r)} \frac{1}{M_{N(r)}} \right) - \]

\[ \ln \left( \left( \frac{|z|}{\sigma + \varepsilon_m} \right)^{N(r)} \frac{1}{M_{N(r)}} \right) = k_0(r) \ln \frac{\sigma + \varepsilon_m}{\sigma + \varepsilon_{m+1}} \geq \frac{\ln \frac{|z|}{A_\delta(\sigma + \varepsilon_m)}}{\ln(1 + \delta)} \ln \frac{\sigma + \varepsilon_m}{\sigma + \varepsilon_{m+1}}. \]

Now we can choose so small \( \delta \) and then suitable constant \( Q \geq 0 \) so that for all \( z \in \mathbb{C} \)

\[ w_{m+1}(|z|) - w_m(|z|) \geq A \ln(1 + |z|) - Q. \]

Lemma 2 is proved.

Using Lemma 2 it is easy to show that for any \( m \in \mathbb{N} \) the canonical (inclusion) mapping \( \gamma_{m,m+1} : P_m \rightarrow P_{m+1} \) is relatively compact in \( P_{m+1} \). So \( P \) is a space \((LN^*)\) (see for definitions [11]).
Lemma 3. Let \( T \in G^* \), say for some \( m_0 \in \mathbb{N}, c > 0 \)

\[
|T(f)| \leq c\|f\|_{G,m_0}, \quad f \in G.
\]

Then \( T \) can be represented in the form

\[
T(f) = \sum_{k=0}^{\infty} \frac{1}{(\sigma + \varepsilon_{m_0})^k M_k} \int_{\mathbb{R}} \frac{f^{(k)}(x)}{\theta_{m_0}(x)} \, d\mu_k(x),
\]

where \( \mu_k \) are complex bounded measures in \( \mathbb{R} \) such that for all \( k \in \mathbb{Z}_+ \)

\[
\int_{\mathbb{R}} d|\mu_k|(x) \leq c.
\]

The proof of the Lemma is standard [1].

For \( z \in \mathbb{C} \) we let

\[
f_z(x) = \exp(-ix), \quad x \in \mathbb{R}.
\]

Lemma 4. For every \( z \in \mathbb{C} \) \( f_z \in G \).

Proof. For every \( z \in \mathbb{C}, m \in \mathbb{N} \) we have

\[
\|f_z\|_{G,m} = \sup_{x \in \mathbb{R}, k \in \mathbb{Z}_+} \frac{|(-iz)^k \exp(-izx)|}{(\sigma + \varepsilon_{m})^k M_k \theta_{m}(x)}
\]

\[
= \exp(w_m(|z|)) \exp(\sup_{x \in \mathbb{R}} (x \operatorname{Im} z - \ln \theta_{m}(x))), \quad z \in \mathbb{C}.
\]

Let’s obtain the upper estimate of \( \sup_{x \in \mathbb{R}} (x \operatorname{Im} z - \ln \theta_{m}(x)) \). From (3) it follows that for some \( A_m(\varphi), B_m(\varphi) > 0 \)

\[
\varphi(x) - m \ln(1 + |x|) \geq A_m(\varphi)|x|^{\frac{\alpha}{4}} - B_m(\varphi), \quad x \in \mathbb{R}.
\]

Hence there exists \( y_m > 0 \) such that for all \( z \in \mathbb{C}, |\operatorname{Im} z| > y_m, \) the supremum of \( x \operatorname{Im} z - \varphi(x) + m \ln(1 + |x|) \) over \( \mathbb{R} \) is attained at some point \( x_m \in (-2A_{m}^{-1}(\varphi)|\operatorname{Im} z|)^{\alpha-1}, (2A_{m}^{-1}(\varphi)|\operatorname{Im} z|)^{\alpha-1}) \). If \( |\operatorname{Im} z| > y_m, \) we have

\[
\sup_{x \in \mathbb{R}} (x \operatorname{Im} z - \varphi(x) + m \ln(1 + |x|)) - \sup_{x \in \mathbb{R}} (x \operatorname{Im} z - \varphi(x))
\]

\[
\leq m \ln(1 + |x_m|) \leq \ln(1 + (2A_{m}^{-1}(\varphi)|\operatorname{Im} z|)^{\alpha-1})
\]

\[
\leq m(\alpha-1) \ln(1 + |\operatorname{Im} z|) + m(\alpha-1) \ln(1 + 2A_{m}^{-1}(\varphi)) + m \ln 2.
\]

Using inversion formula for the Young transform and choosing appropriate constant \( b_m > 0 \), we have for all \( z \in \mathbb{C}, m \in \mathbb{N} \)

\[
\sup_{x \in \mathbb{R}} (x \operatorname{Im} z - \ln \theta_{m}(x)) - \psi'(\operatorname{Im} z) \leq m(\alpha-1) \ln(1 + |\operatorname{Im} z|) + b_m. \quad (6)
\]
Consequently,
\[ \|f_z\|_{G,m} \leq \exp(\psi(Im \ z) + w_{m+1}(|z|) + q_m), \ z \in \mathbb{C}. \] (7)

By Lemma 2 \( \exists q_m \geq 0 \) such that
\[ \|f_z\|_{G,m} \leq \exp(\psi(Im \ z) + w_{m+1}(|z|) + q_m), \ z \in \mathbb{C}. \] (7)

Thus, \( f_z \in G \) for all \( z \in \mathbb{C} \).

Lemma 5. Suppose that for \( T \in G^* \) there are \( c > 0, m \in \mathbb{N} \) such that
\[ |T(f)| \leq c \|f\|_{G,m}, \ f \in G. \] (8)

Then \( \hat{T} \) is entire and satisfies
\[ |
\hat{T}(z)| \leq c \exp(\psi(Im \ z) + w_{m+1}(|z|) + q_m), \ z \in \mathbb{C}, \] (9)

where \( q_m \) is the same as in (7). Moreover, \( \hat{T} \) can be represented in the form
\[ \hat{T}(z) = \sum_{k=0}^{\infty} V_k(z)z^k, \ z \in \mathbb{C}, \] (10)

where \( V_k \) are entire functions such that for some \( A > 0 \) independent of \( k \)
\[ |V_k(z)| \leq \frac{A(1 + |z|)^{m(a-1)} \exp(\psi(Im \ z))}{(\sigma + \varepsilon_m)^k M_k}, \ z \in \mathbb{C}. \] (10)

Proof. Using lemma 3 we have
\[ \hat{T}(z) = \sum_{k=0}^{\infty} \frac{(-iz)^k}{(\sigma + \varepsilon_m)^k M_k} \int_{\mathbb{R}} \frac{e^{-ixz}}{\theta_m(x)} \ d\mu_k(x), \ z \in \mathbb{C}, \] (11)

where \( \mu_k \) are complex bounded measures in \( \mathbb{R} \) such that for all \( k \in \mathbb{Z}_+ \)
\[ \int_{\mathbb{R}} d|\mu_k|(x) \leq c. \] (12)

Using the boundedness of the measures \( \mu_k \) and the estimate
\[ \left| \frac{e^{-ixz}}{\theta_n(x)} \right| \leq \exp(\psi(Im \ z) + l(\alpha - 1) \ln(1 + |Im \ z|) + b_n), \] (13)
which follows from (6) and holds for every $z \in \mathbb{C}, x \in \mathbb{R}, n \in \mathbb{N}$, it’s easy to show that

$$V_k(z) = \frac{(-i)^k}{(\sigma + \varepsilon_m)^k M_k} \int_{\mathbb{R}} e^{-ixz} \theta_m(x) \, d\mu_k(x), \quad z \in \mathbb{C}, k \in \mathbb{Z}_+,$$

is entire. The estimate (10) immediately follows from (12), (13) (set $n = m$). From this and the condition iii) it follows that the series in the right-hand side of (11) converges in the topology of $H(\mathbb{C})$. Consequently, $\hat{T}$ is entire. The estimate (9) is obtained from (7) and (8).

**Lemma 6.** Let $F \in P$ has the form $F(z) = U(z)V(z), z \in \mathbb{C}$, where entire functions $U, V$ for some $m \in \mathbb{N}, C_U, C_V > 0$ satisfy the estimates:

$$|U(z)| \leq \frac{C_U \exp(\psi(Im \ z))}{1 + |z|^2}, \quad z \in \mathbb{C};$$

$$|V(z)| \leq C_V e^{w_m(|z|)} , \quad z \in \mathbb{C}.$$

Then there exists $T \in G^*$ such that $\hat{T} = F$ and

$$|T(f)| \leq \beta_m C_U C_V \|f\|_{G,m+1}, \quad f \in G,$$

where $\beta_m > 0$ is some constant depending only on $m$.

**Proof.** From the estimate of $|U(z)|$ it follows (see, for example, [4], [5], [8]) that $\exists p \in C(\mathbb{R})$ such that:

1. $\sup_{t \in \mathbb{R}} |p(t)| \exp(\varphi(t)) < \frac{C_U}{2}$;
2. $U(z) = \int_{\mathbb{R}} p(t) e^{-izt} \, dt, \quad z \in \mathbb{C}$.

Further, by the Cauchy inequality for Taylor coefficients of entire function $V(z) = \sum_{k=0}^{\infty} v_k z^k$ we have

$$|v_k| \leq C_V \inf_{r>0} \frac{\exp(w_m(r))}{r^k} = C_V (\sigma + \varepsilon_m)^{-k} \inf_{r>0} \frac{\exp(w(r))}{r^k}, \quad k \in \mathbb{Z}_+.$$

Using the equality [3]

$$\inf_{r>0} \frac{\exp(w(r))}{r^k} = \frac{1}{M_k}, \quad k \in \mathbb{Z}_+, \quad (14)$$

we get

$$|v_k| \leq C_V (\sigma + \varepsilon_m)^{-k} M_k^{-1}, \quad k \in \mathbb{Z}_+. \quad (15)$$
Define the functional \( T \) on \( G \) by the formula

\[
T(f) = \int_\mathbb{R} p(t) \sum_{k=0}^{\infty} v_k i^k f^{(k)}(t) \, dt, \quad f \in G.
\]

It is defined correctly. Indeed, for every \( f \in G \)

\[
\sum_{k=0}^{\infty} i^k v_k f^{(k)}(t) \leq \sum_{k=0}^{\infty} v_k \| f \|_{G,m+1} (\sigma + \varepsilon_{m+1})^k M_k \theta_{m+1}(t), \quad t \in \mathbb{R}.
\]

Using (15), we have for every \( f \in G \)

\[
\sum_{k=0}^{\infty} v_k i^k f^{(k)}(t) \leq \beta_m C_U C_V \| f \|_{G,m+1}, \quad t \in \mathbb{R},
\]

where \( \beta_m = \frac{\sigma + \varepsilon_m}{\varepsilon_m - \varepsilon_{m+1}} \). Consequently,

\[
|T(f)| \leq \beta_m C_U \| f \|_{G,m+1} \int_\mathbb{R} |p(t)| \exp(\varphi(t))(1 + |t|)^{-m+1} \, dt
\]

\[
\leq \beta_m C_U C_V \| f \|_{G,m+1}, \quad f \in G.
\]

Thus, \( T \in G^* \). Obviously,

\[
\hat{T}(z) = \int_\mathbb{R} p(t) \sum_{k=0}^{\infty} v_k i^k (-iz)^k e^{-izt} \, dt = U(z)V(z) = F(z), \quad z \in \mathbb{C}.
\]

3. The space \( E(\Phi) \).

Let

\[
E(\Phi) = \{ f \in E(\mathbb{R}) : \forall n \in \mathbb{Z}_+, m \in \mathbb{N} \quad \| f \|_{n,m} = \sup_{x \in \mathbb{R}, 0 \leq k \leq n} \frac{|f^{(k)}(x)|}{\theta_m(x)} < \infty \}.
\]

It is easy to see that \( E^*(\Phi) \subseteq G^* \). In this section we study the Fourier-Laplace transform of functionals from \( E^*(\Phi) \).

If \( F \in E^*(\Phi) \) then \( \exists c > 0, n, m \in \mathbb{N} \) such that \( \forall f \in E(\Phi) \) we have \( |F(f)| \leq c \| f \|_{n,m} \). So, using (15), we have for some \( C > 0 \)

\[
|F(\exp(-ixz))| \leq C(1 + |z|)^n m^{(\alpha-1)} \exp(\psi(Imz)), \quad z \in \mathbb{C}.
\]

By Lemma 5 \( \hat{F} \) is an entire function. The description of functionals from \( E^*(\Phi) \) in terms of their Fourier-Laplace transforms will be done in theorem 2. But at first we study the density of polynomials in \( E(\Phi) \). We need the following simple lemma.
Lemma 7. Let function \( g \) defined on the real line satisfies for some constants \( A > 0, B \) the inequality
\[
g(x) > A|x|^\alpha - B, \ x \in \mathbb{R}.
\]

Then there exists constant \( C \) depending only on \( A \) and \( B \) such that for any \( m \in \mathbb{N} \)
\[
\sup_{x \in \mathbb{R}} (m \ln(1 + |x|) - g(x)) < (1 - \alpha^{-1})m \ln m + Cm.
\]

Proof. For any \( m \in \mathbb{N} \)
\[
\sup_{x \in \mathbb{R}} (m \ln(1 + |x|) - g(x)) \leq \sup_{x \in \mathbb{R}} (m \ln(1 + |x|) - A|x|^\alpha) + B
\]
\[
= B + \sup_{x \geq 0} (m \ln(1 + x) - Ax^\alpha) < B + 2m + \sup_{u > 0} (mu - Ae^{\frac{\alpha}{1-\alpha}}u)
\]
\[
= B + 2m + (1 - \alpha^{-1})m \ln m - (1 - \alpha^{-1})m \ln((1 - \alpha^{-1})Ae).
\]

Theorem 2. The polynomials are dense in \( \mathcal{E}(\Phi) \).

Proof. Let \( f \in \mathcal{E}(\Phi) \), that is \( f \in \mathcal{E}(\mathbb{R}) \) and for any \( n, m \in \mathbb{N} \) there exists \( c_{m,n} > 0 \) such that for all \( x \in \mathbb{R}, k = 0, 1, \ldots, n \)
\[
|f^{(k)}(x)| \leq c_{m,n} \theta_m(x).
\]

Let us approximate \( f \) by polynomials in \( \mathcal{E}(\Phi) \). There are three steps in the proof.

1. Let \( \gamma \in \mathcal{E}(\mathbb{R}) \) be such that \( \text{supp} \ \gamma \subseteq [-2, 2], \gamma(x) = 1 \) for \( x \in [-1, 1] \), \( 0 \leq \gamma(x) \leq 1 \ \forall x \in \mathbb{R} \). Set \( f_\nu(x) = f(x)\gamma(\frac{x}{\nu}), \ \nu \in \mathbb{N}, x \in \mathbb{R} \). Obviously, \( f_\nu \in \mathcal{E}(\Phi) \).

Let us show that \( f_\nu \rightarrow f \) in \( \mathcal{E}(\Phi) \) as \( \nu \rightarrow \infty \). Take arbitrary \( m, n \in \mathbb{N} \).

Then
\[
\sup_{x \in \mathbb{R}} \frac{|f_\nu(x) - f(x)|}{\theta_m(x)} \leq \sup_{|x| > \nu} \frac{|f(x)|}{\theta_m(x)} \leq \sup_{|x| > \nu} \frac{c_{m+1,n} \theta_{m+1}(x)}{\theta_m(x)}.
\]

Consequently, as \( \nu \rightarrow \infty \)
\[
\sup_{x \in \mathbb{R}} \frac{|f_\nu(x) - f(x)|}{\theta_m(x)} \rightarrow 0.
\]

Then,
\[
\sup_{x \in \mathbb{R}, 1 \leq k \leq n} \frac{|(f_\nu(x) - f(x))^{(k)}|}{\theta_m(x)}
\]
From this, using (16) (substituting $m$ by $m+1$), we conclude that as $\nu \to \infty$

$$\sup_{x \in \mathbb{R}, 1 \leq k \leq n} \left| \sum_{s=o}^{k-1} C_k^s f(s)(x) \nu^{s-k} \gamma^{(k-s)}(\frac{x}{\nu}) + f^{(k)}(x)(\gamma(\frac{x}{\nu}) - 1) \right|$$

$$\leq \sup_{\nu < |x| < 2\nu, 1 \leq k \leq n} \left| \sum_{s=o}^{k-1} C_k^s |f(s)(x)| \nu^{s-k} |\gamma^{(k-s)}(\frac{x}{\nu})| \right| + \sup_{|x| > \nu, 1 \leq k \leq n} \frac{|f^{(k)}(x)|}{\theta_m(x)}.$$ 

From this, using (16) (substituting $m$ by $m+1$), we conclude that as $\nu \to \infty$

$$\sup_{x \in \mathbb{R}, 1 \leq k \leq n} \frac{|(f_\nu(x) - f(x))^{(k)}|}{\theta_m(x)} \to 0.$$

From this and (17) it follows that $\|f_\nu - f\|_{n,m} \to 0$ as $\nu \to \infty$. Since $m, n \in \mathbb{N}$ are arbitrary this means that the sequence $\{f_\nu\}_{\nu=1}^\infty$ converges to $f$ in $\mathcal{E}(\Phi)$ as $\nu \to \infty$.

2. Fix $\nu \in \mathbb{N}$. Let $h(z) = \sum_{k=0}^{+\infty} a_k z^{2k}, z \in \mathbb{C}, h \not\equiv 0$, be an entire function of exponential type 1 such that $h \in L_1(\mathbb{R}), h(x) \geq 0, x \in \mathbb{R}$. For example, we may put $h(z) = \frac{\sin^2 \frac{z}{2}}{z^2}, z \in \mathbb{C}$.

By the Paley-Wiener theorem $\exists g \in C(\mathbb{R})$ with support $g \subseteq [-1, 1]$ such that

$$h(z) = \int_{-1}^{1} g(t) e^{-itz} \, dt, \quad z \in \mathbb{C}.$$ 

Since

$$h^{(k)}(z) = \int_{-1}^{1} g(t)(-it)^k e^{-itz} \, dt, \quad z \in \mathbb{C}, \; k \in \mathbb{Z}_+,$$

then

$$|h^{(k)}(x)| \leq C_g, \; x \in \mathbb{R}, \; k \in \mathbb{Z}_+,$$

where $C_g = 2 \max_{|t| \leq 1} |g(t)|$.

Let $\int_{-\infty}^{+\infty} h(x) \, dx = A$. For $\lambda > 0$ we set

$$f_{\nu,\lambda}(x) = \frac{\lambda}{A} \int_{\mathbb{R}} f_\nu(y) h(\lambda(x - y)) \, dy, \; x \in \mathbb{R}.$$ 

It is easy to see that $f_{\nu,\lambda} \in \mathcal{E}(\Phi).$
Let us show that $f_{\nu,\lambda} \to f_\nu$ in $\mathcal{E}(\Phi)$ as $\lambda \to +\infty$. Take arbitrary $m, n \in \mathbb{N}$. For any $k \in \mathbb{Z}^+, x \in \mathbb{R}$

$$f^{(k)}_{\nu,\lambda}(x) - f^{(k)}_\nu(x) = \frac{\lambda}{A} \int_{\mathbb{R}} (f^{(k)}_\nu(y) - f^{(k)}_\nu(x)) h(\lambda(x - y)) \, dy$$

$$= \frac{\lambda}{A} \int_{|y-x| \leq \lambda^{-\frac{2}{3}}} (f^{(k)}_\nu(y) - f^{(k)}_\nu(x)) h(\lambda(x - y)) \, dy$$

$$+ \frac{\lambda}{A} \int_{|y-x| > \lambda^{-\frac{2}{3}}} (f^{(k)}_\nu(y) - f^{(k)}_\nu(x)) h(\lambda(x - y)) \, dy = I_{1,k}(x) + I_{2,k}(x).$$

Let $K_n = \max_{x \in \mathbb{R}, 0 \leq k \leq n+1} |f^{(k)}_\nu(x)|$. Obviously,

$$|I_{1,k}(x)| \leq \frac{2C_gK_n}{A} \lambda^{-\frac{1}{3}}, \quad x \in \mathbb{R}, \quad k = 0, 1, \ldots, n; \quad (19)$$

$$|I_{2,k}(x)| \leq \frac{2K_n}{A} \int_{|t| > \lambda^\frac{1}{3}} h(t) \, dt, \quad x \in \mathbb{R}, \quad k = 0, 1, \ldots, n. \quad (20)$$

Let $\varepsilon > 0$ be arbitrary and $\Theta_m = \inf_{x \in \mathbb{R}} \theta_m(x)$, $m \in \mathbb{N}$. Choose $\lambda(\varepsilon) > 0$ so that $\frac{1}{A} \int_{|t| > \lambda^\frac{1}{3}} h(t) \, dt < \frac{\varepsilon \Theta}{4K_n}$ and $2C_gK_n\lambda^{-\frac{1}{3}} < \frac{A \varepsilon \Theta}{2}$ for $\lambda > \lambda(\varepsilon)$. Then from (19) and (20) it follows that $\|f_{\nu,\lambda} - f_\nu\|_{n,m} < \varepsilon$ for all $\lambda > \lambda(\varepsilon)$. Consequently, by the semi-norm $\| \cdot \|_{n,m}$ $f_{\nu,\lambda} \to f_\nu$ as $\lambda \to +\infty$. Since $n, m \in \mathbb{N}$ are arbitrary, it means that $f_{\nu,\lambda} \to f_\nu$ in $\mathcal{E}(\Phi)$ as $\lambda \to +\infty$.

3. Fix $\lambda > 0, \nu \in \mathbb{N}$. Let us approximate $f_{\nu,\lambda}$ by the polynomials in $\mathcal{E}(\Phi)$.

Let $h(x) = \sum_{k=0}^{+\infty} a_k x^{2k}, x \in \mathbb{R}$. Let $P_{2N}(x) = \sum_{k=0}^{N} a_k x^{2k}, x \in \mathbb{R}, N \in \mathbb{N}$. Using the Taylor formula and the estimate (18), we have

$$|h(x) - P_{2N}(x)| \leq \frac{C_g|x|^{2N+1}}{(2N+1)!}, \quad x \in \mathbb{R}. \quad (21)$$

Set

$$Q_{2N}(x) = \frac{\lambda}{A} \int_{-L}^{L} f_\nu(y)P_{2N}(\lambda(x - y)) \, dy, \quad x \in \mathbb{R}, N \in \mathbb{N},$$

where $L > 0$ is chosen so that $\text{supp} \ f_\nu \subseteq [-L, L]$. $Q_{2N}$ is a polynomial of the degree not more than $2N$.  

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Let us show that the sequence \( \{Q_{2N}\}_{N=1}^{\infty} \) converges in \( E(\Phi) \) to \( f_{\nu,\lambda} \) as \( N \to \infty \). We take arbitrary \( m, n \in \mathbb{N} \). For any \( k \in \mathbb{Z}_+ \)

\[
f_{\nu,\lambda}^{(k)}(x) - Q_{2N}^{(k)}(x) = \frac{\lambda}{A} \int_{-L}^{L} f_{\nu}^{(k)}(y)(h(\lambda(x - y)) - P_{2N}(\lambda(x - y))) \, dy, \quad x \in \mathbb{R}.
\]

Using (18), (21) we have for all \( x \in \mathbb{R}, N \in \mathbb{N}, k = 0, 1, \ldots, n \)

\[
|f_{\nu,\lambda}^{(k)}(x) - Q_{2N}^{(k)}(x)| \leq K_n C_2 \frac{2L\lambda^{2N+2}(|x| + L)^{2N+1}}{A (2N+1)!}.
\]

Thus, there exist \( C_1 > 0 \) depending on \( n \) and \( C_2 \) such that for any \( N \in \mathbb{N} \)

\[
\|f_{\nu,\lambda} - Q_{2N}\|_{n,m} \leq C_1 C_2^{2N+2} \frac{2L\lambda^{2N+2}}{(2N+1)!} \sup_{x \in \mathbb{R}} \frac{(1 + |x|)^{2N+1}}{\theta_m(x)}.
\]

Since \( \varphi \) satisfies the conditions of Lemma 7 then there exists positive number \( C_3 \) such that for all \( m, N \in \mathbb{N} \)

\[
\sup_{x \in \mathbb{R}}((2N + m + 1) \ln(1 + |x|) - \varphi(x))
\]

\[
\leq (1 - \alpha^{-1})(2N + m + 1) \ln(2N + m + 1) + C_3(2N + m + 1).
\]

From this estimate and the inequality (22) it follows that there exist positive numbers \( C_4, C_5 \) depending on \( m, n \) such that for all \( N \in \mathbb{N} \)

\[
\|f_{\nu,\lambda} - Q_{2N}\|_{n,m} \leq C_4 C_5^N \frac{(2N+1)^{(2N+1)(1-\alpha^{-1})}}{(2N+1)!}.
\]

The right-hand side of the last inequality tends to 0 as \( N \to \infty \). This means that \( f_{\nu,\lambda} \) is approximated by polynomials in \( E(\Phi) \) since \( m, n \in \mathbb{N} \) are arbitrary.

The density of polynomials in \( E(\Phi) \) follows from steps 1) – 3).

**Theorem 3.** Let \( U \in H(\mathbb{C}) \) for some \( C > 0, N \in \mathbb{N} \) satisfies the inequality

\[
|U(z)| \leq C(1 + |z|)^N \exp(\psi(Imz)), \quad z \in \mathbb{C}.
\]

Then there exists the unique functional \( T \in E^*(\Phi) \) such that \( T(\exp(-izx)) = U(z), z \in \mathbb{C} \) and

\[
|T(f)| \leq AC\|f\|_{N+2,2}, \quad f \in E(\Phi),
\]

where \( A > 0 \) doesn’t depend on \( U \).
**Proof.** Let us take real numbers $\lambda_0, \lambda_1, \ldots, \lambda_{N+1}$ not equal to each other and zero. Choose numbers $a_0, a_1, \ldots, a_{N+1}$ so that an entire function

$$g(z) = U(z) - \sum_{k=0}^{N+1} a_k \exp(-i\lambda_k z), \ z \in \mathbb{C},$$

satisfies the following condition: $g^{(n)}(0) = 0, n = 0, 1, \ldots, N+1$. It is clear that these numbers are uniquely defined. From the estimate on $U$ we have

$$|U_k(0)| \leq c_1 C k = 0, 1, \ldots, N+1,$$

where $c_1 > 0$ depends only on $N$. Then

$$|a_k| \leq c_2 C, \ k = 0, 1, \ldots, N+1,$$

where $c_2 > 0$ depends only on $N$. Using the second condition on $\psi$ we have for some $c_3 > 0$ depending only on $N$

$$|g(z)| \leq c_3 C(1 + |z|)^N \exp(\psi(Im z)), \ z \in \mathbb{C}.$$

Let $h(z) = \frac{g(z)}{(-iz)^{N+2}}$. Obviously,

$$|h(z)| \leq \frac{c_4 C \exp(\psi(Im z))}{1 + |z|^2}, \ z \in \mathbb{C},$$

where $c_4 > 0$ depends only on $N$.

As in Lemma 6 $\exists p \in C(\mathbb{R})$ such that:

1. $\sup_{t \in \mathbb{R}} |p(t)| \exp(\varphi(t)) < \frac{c_4 C}{2}$;
2. $h(z) = \int_{\mathbb{R}} p(t) e^{-izt} dt, \ z \in \mathbb{C}$.

Define functional $T$ on $\mathcal{E}(\Phi)$ by the formula

$$T(f) = \sum_{k=0}^{N+1} a_k f(\lambda_k) + \int_{\mathbb{R}} p(t) f^{(N+2)}(t) dt, \ f \in \mathcal{E}(\Phi).$$

It is easy to show that for some $A > 0$ depending only on $N$ $|T(f)| \leq AC\|f\|_{N+2,2}, \ f \in \mathcal{E}(\Phi)$. Consequently, $T \in \mathcal{E}^*(\Phi)$. Obviously, $T(\exp(-izx)) = U(z), z \in \mathbb{C}$.

Now we prove the uniqueness. Suppose that $S \in \mathcal{E}^*(\Phi)$, $S(\exp(-izx)) = 0 \ \forall z \in \mathbb{C}$. Let us show that $S = 0$. Let $P_{N,\xi}(x) = \sum_{\nu=0}^{N} \frac{\xi^\nu}{\nu!} x^\nu, \xi, x \in \mathbb{R}, N \in \mathbb{Z}_+$. We show at first that for any $R > 0$ the sequence of polynomials $\{P_{N,\xi}\}_{N=0}^{\infty}$ converges in $\mathcal{E}(\Phi)$ to $e^{\xi x}$ uniformly by $\xi \in [-R, R]$ as $N \to \infty$. Let $n, m \in \mathbb{N}$ be arbitrary. From the Taylor formula

$$|e^{\xi x} - P_{N-k,\xi}(x)| \leq \frac{|\xi|^{N-k+1} |x|^{N-k+1}}{(N-k+1)!} \max(1, e^{\xi x}), \ k = 0, 1, \ldots, N,$$
then as \( N \geq n \)
\[
\sup_{x \in \mathbb{R}, 0 \leq k \leq n} \frac{|(e^{\xi x})^{(k)} - P_{N, k}^{(k)}(x)|}{\theta_m(x)} = \sup_{x \in \mathbb{R}, 0 \leq k \leq n} \frac{\xi^k |e^{\xi x} - P_{N - k, k}(x)|}{\theta_m(x)}
\]
\[
\leq \frac{|\xi|^{N+1}}{(N - k + 1)!} \sup_{x \in \mathbb{R}} \frac{|x|^{N-k+1} \max(1, e^{\xi x})}{\theta_m(x)}
\]
\[
\leq \frac{R^{N+1}}{(N - n + 1)!} \sup_{x \in \mathbb{R}} \frac{(1 + |x|)^{N+1} \max(1, e^{\xi x})}{\theta_m(x)}.
\]
Since \( \exists A, B > 0 \) depending on \( R \) such that for all \( \xi \in [-R, R], x \in \mathbb{R}, \phi(x) - \xi x \geq A|x|^\alpha - B \), then, using Lemma 7, we can find constant \( Q > 1 \) independent of \( \xi \in [-R, R] \) such that
\[
\sup_{x \in \mathbb{R}} \frac{(1 + |x|)^{N+1} \max(1, e^{\xi x})}{\theta_m(x)} \leq Q^{N+m+1}(N + m + 1)^{(1-\alpha^{-1})(N+m+1)}.
\]
From these two last estimates it follows that
\[
\sup_{x \in \mathbb{R}, 0 \leq k \leq n} \frac{|(e^{\xi x})^{(e_k)} - P_{N, k}^{(k)}(x)|}{\theta_m(x)} \to 0
\]
uniformly by \( \xi \in [-R, R] \) as \( N \to \infty \). Since \( n, m \) are arbitrary, then it means that for any \( R > 0 \) \( \{P_{N, k}\}_{N=0}^\infty \to e^{\xi x} \) in \( \mathcal{E}(\Phi) \) uniformly by \( \xi \in [-R, R] \) as \( N \to \infty \). From this we conclude that \( \sum_{\nu=0}^\infty \frac{S(x^{\nu})}{\nu!} \xi^\nu \) converges uniformly on compacts from \( \mathbb{R} \) to \( S(\exp(\xi x)) \). By assumption for any \( \xi \in \mathbb{R} \) \( S(\exp(\xi x)) = 0 \). Consequently, \( S(x^{\nu}) = 0 \) for all \( \nu \in \mathbb{Z}_+ \). Since the polynomials are dense in \( \mathcal{E}(\Phi) \) then \( S = 0 \).

The theorem is proved.

4. The proof of theorem 1.

In this section at first we briefly describe a special case of M. I. Solomesch’s general result [7], [12].

Let \( f \) be an arbitrary entire function with zeros \( \{\lambda_j\}_{j=1}^\infty \subset \mathbb{C} \) of corresponding multiplicities \( \{m_j\}_{j=1}^\infty \). The Weierstrass representation of \( f \) has the form [2, p. 17]
\[
f(\lambda) = g(\lambda) \prod_{j=1}^\infty \left( 1 - \frac{\lambda}{\lambda_j} \right)^{m_j} \exp \left( P_j \left( \frac{\lambda}{\lambda_j} \right) \right), \lambda \in \mathbb{C},
\]
where \( g \) is an entire function without zeros, \( P_j \) are special polynomials, \( j \in \mathbb{N} \).
Let \( t = \{t_j\}_{j=1}^{\infty} \) be a sequence of complex numbers such that for any \( j \in \mathbb{N} \), \( \lambda_j + t_j \neq 0 \). Put

\[
f_t(\lambda) = g(\lambda) \prod_{j=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_j + t_j}\right)^{m_j} \exp \left( P_j \left( \frac{\lambda}{\lambda_j} \right) \right), \lambda \in \mathbb{C}.
\]

Now we can formulate the result of M.I. Solomesch [7],[12, chapter 2, §3].

**Theorem S.** Let the disks \( D(\lambda_j, r_j) \) be pairwise disjoint, \( \lambda_j + t_j \in D(\lambda_j, r_j) \), \( r_j > 0, j \in \mathbb{N} \), and \( \sum_{j=1}^{\infty} \frac{m_j|t_j|}{r_j} < \infty \).

Then \( f_t \) is an entire function and for \( \lambda \) outside \( \bigcup_{j=1}^{\infty} D(\lambda_j, r_j) \)

\[
|\ln f_t(\lambda)| - |\ln f(\lambda)| \leq C,
\]

where \( C > 0 \) is some constant.

**Proof of theorem 1.**

Obviously, the map \( J : T \to \hat{T} \), \( T \in G^* \), is linear and by Lemma 5 acts from \( G^* \) to \( P \).

\( J \) is continuous. Indeed, if \( T \in G^* \), then for some \( m \in \mathbb{N} \) \( T \in G^*_{m} \).

Hence,

\[
|T(f)| \leq \|T\|_{G^*_{m}} \|f\|_{G_{m}}, \quad f \in G,
\]

where \( \|T\|_{G^*_{m}} \) is a norm of \( T \) in \( G^*_{m} \). By Lemma 5

\[
|\hat{T}(z)| \leq \|T\|_{G^*_{m}} \exp(\psi(\text{Im } z) + w_{m+1}(|z|) + q_m), \quad z \in \mathbb{C},
\]

where \( q_m > 0 \) is some constant. Consequently,

\[
\|\hat{T}\|_{m+1} \leq e^{q_m} \|T\|_{G^*_{m}}, \quad T \in G^*_{m}.
\]

This means that \( J \) is continuous.

At first we prove that \( J \) is injective. That is, if for \( T \in G^* \), \( \hat{T} \equiv 0 \), then \( T(f) = 0 \ \forall f \in G \). We will follow the scheme of the paper [14, see the proof of theorem 2.8]. By Lemma 3 \( T \) is represented in the form

\[
T(f) = \sum_{k=0}^{\infty} \frac{1}{(\sigma + \varepsilon_m)^k M_k} \int_{\mathbb{R}} \frac{f^{(k)}(x)}{\theta_m(x)} \, d\mu_k(x), \quad f \in G,
\]

where \( \mu_k \) are complex bounded measures in \( \mathbb{R} \) such that \( \forall k \in \mathbb{Z}_+ \)

\[
\int_{\mathbb{R}} d|\mu_k|(x) \leq \|T\|_{G^*_{m}}. \quad \text{Note that functional}
\]

\[
T_k(f) = \frac{1}{(\sigma + \varepsilon_m)^k M_k} \int_{\mathbb{R}} \frac{f(x)}{\theta_m(x)} \, d\mu_k(x), \quad f \in \mathcal{E}(\Phi),
\]

\[
\int_{\mathbb{R}} d|\mu_k|(x) \leq \|T_k\|_{G^*_{m}}.
\]

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is in $\mathcal{E}^*(\Phi)$. By Lemma 5 $\hat{T}(z) = \sum_{k=0}^{\infty} V_k(z)z^k$, where entire functions $V_k(z) = (-i)^k T_k(\exp(-iz))$ for some $B_m > 0$ independent of $k$ satisfy the estimate

$$|V_k(z)| \leq \frac{B_m(1 + |z|)^{m(\alpha - 1)} \exp(\psi(\text{Im } z))}{(\sigma + \varepsilon_m)kM_k}, \quad z \in \mathbb{C}. \quad (23)$$

Consider $H(z, u) = \sum_{k=0}^{\infty} V_k(z)u^k$, $z, u \in \mathbb{C}$. It is easy to see that $H(z, u)$ is entire. From (23) we have an estimate for $H(z, u)$

$$|H(z, u)| \leq \frac{B_m(\sigma + \varepsilon_m)}{\varepsilon_m - \varepsilon_{m+1}}(1 + |z|)^{m(\alpha - 1)} \exp(\psi(\text{Im } z) + w_{m+1}(|u|)), \quad z, u \in \mathbb{C}.\tag{24}$$

By our hypothesis $H(z, z) = 0$, $z \in \mathbb{C}$. Hence, $H(z, u) = (z - u)S(z, u)$, where $S$ is entire. Note that $S$ satisfies the estimate

$$|S(z, u)| \leq A_S(1 + |z|)^{m(\alpha - 1)} \exp(\psi(\text{Im } z) + w_{m+1}(|u|)), \quad z, u \in \mathbb{C},$$

where $A_S > 0$ is some constant. It is obvious when $|z - u| \geq 1$ and for $|z - u| < 1$ it can be obtained by applying the maximum principle and by using the inequality (4). Then we expand $S$ in a power series in $u$:

$$S(z, u) = \sum_{k=0}^{\infty} S_k(z)u^k.$$

By Cauchy’s inequality for the coefficients in a power series expansion, equality (14), inequality (24), we have

$$|S_k(z)| \leq \frac{A_S}{(\sigma + \varepsilon_{m+1})kM_k}(1 + |z|)^{m(\alpha - 1)} \exp(\psi(\text{Im } z)), \quad k \in \mathbb{Z}_+, z \in \mathbb{C}.\tag{25}$$

By theorem 2 there exist $\Phi_k \in \mathcal{E}^*(\Phi)$ such that $\Phi_k(\exp(-izx)) = S_k(z), z \in \mathbb{C}$ and

$$|\Phi_k(f)| \leq \frac{K}{(\sigma + \varepsilon_{m+1})kM_k}\|f\|_{N+2,2}, \quad f \in \mathcal{E}(\Phi), \tag{25}$$

where $N = [m(\alpha - 1)] + 1$, constant $K > 0$ doesn’t depend on $k \in \mathbb{Z}_+$. Since $H$ is represented in the form

$$H(z, u) = zS_0(z) + \sum_{k=1}^{\infty} (zS_k(z) - S_{k-1}(z))u^k,$$

then $V_0(z) = zS_0(z), V_k(z) = zS_k(z) - S_{k-1}(z), k \in \mathbb{N}$. By theorem 2 $T_0(f) = i\Phi_0(f), (-i)^k T_k(f) = i\Phi_k(f') - \Phi_{k-1}(f), f \in \mathcal{E}(\Phi)$. According to $ii) \forall \delta >
$0 \exists A_\delta > 0 : \forall n \in \mathbb{N} \quad M_{n+1} \leq A_\delta (1 + \delta)^n M_n$. Then from (25) we have for $f \in G$

$$|\Phi_n(f^{(n+1)})| \leq \frac{K}{(\sigma + \varepsilon_{m+1})^n M_n} \sup_{x \in \mathbb{R}, 0 \leq k \leq N+2} \frac{|f^{(n+1)}(x)|}{\theta_2(x)}$$

$$\leq \frac{K}{(\sigma + \varepsilon_{m+1})^n M_n} \sup_{x \in \mathbb{R}, 0 \leq k \leq N+2} \frac{\|f\|_{G,m+2}(\sigma + \varepsilon_{m+2})^{n+k+1} M_{n+k+1} \theta_{m+2}(x)}{\theta_2(x)}$$

$$\leq K_\delta \|f\|_{G,m+2} \left( \frac{(\sigma + \varepsilon_{m+2})(1 + \delta)^{(k+1)}}{\sigma + \varepsilon_{m+1}} \right)^n,$$

where $K_\delta = K \sup_{0 \leq k \leq N+2} \left( \frac{(\sigma + \varepsilon_{m+2})^{k+1} A_\delta^{k+1} (1 + \delta)^{(k+1)k}}{\sigma + \varepsilon_{m+1}} \right)$. Choose $\delta$ so small that $\frac{(\sigma + \varepsilon_{m+2})(1 + \delta)^{(k+1)}}{\sigma + \varepsilon_{m+1}} < 1$. Then $\Phi_n(f^{(n+1)}) \to 0$ as $n \to \infty$. Therefore, for any $f \in G$

$$T(f) = \sum_{k=0}^{\infty} T_k(f^{(k)}) = i \Phi_0(f) + \sum_{k=1}^{\infty} (i^{k+1} \Phi_k(f^{(k+1)}) - i^k \Phi_{k-1}(f^{(k)}))$$

$$= \lim_{n \to \infty} i^{n+1} \Phi_n(f^{(n+1)}) = 0.$$

Now let us prove that $J$ is surjective. Let an entire function $F \in P$ for some $m \in \mathbb{N}, c > 0$ satisfies

$$|F(z)| \leq c \exp(\psi(Im z) + w_m(|z|)), \quad z \in \mathbb{C}. \quad (26)$$

We wish to show that there is $F \in G^*$ with $\hat{F} = F$.

Since functions $\psi(Im z), w_{m+3}(|z|)$ satisfy the conditions of the theorem 6 of [16] (see also [17, theorem 4]), then there are entire functions $L$ and $Y$ such that:

(L1). All the zeros $\{\lambda_j\}_{j=1}^{\infty}$ of $L$ are simple and the disks $D(\lambda_j, d_1|\lambda_j|^{1-\alpha})$ are disjoint for some $d_1 > 0$.

(L2). Outside the set $\bigcup_{j=1}^{\infty} D(\lambda_j, d_1|\lambda_j|^{1-\alpha})$

$$|\psi(Im z) - \ln |L(z)|| \leq A \ln(1 + |z|) + A_0, \quad (27)$$

where $A, A_0$ are some positive numbers.

(Y1). All the zeros $\{\mu_k\}_{k=1}^{\infty}$ of $Y$ are simple and the disks $D(\mu_k, d_2)$ are disjoint for some $d_2 > 0$. 

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(Y2). Outside the set $\bigcup_{k=1}^{\infty} D(\mu_k, d_2)$

$$|w_{m+3}(|z|) - \ln|Y(z)|| \leq B\ln(1 + |z|) + B_0,$$  \hspace{1cm} (28)

where $B, B_0$ are some positive real numbers.

It can be assumed without loss of generality that $|\mu_k| > \max(1, 4d_2)$ for all $k \in \mathbb{N}$ and $|\lambda_j| > \max(2(d_1 + d_2), 1)$ for all $j \in \mathbb{N}$.

At first, having functions $L$ and $Y$, we construct an entire function $N \in P$, which outside some exceptional disks satisfies the estimate

$$|N(z)| \geq C_N \exp(\psi(Im\, z) + w_{m+1}(|z|)),$$  \hspace{1cm} \text{where } C_N \text{ is some positive number.}

By the maximum principle applied to $Y$ in disks $D(\mu_k, d_2), k \in \mathbb{N}$, and using inequality (4), we get from (28)

$$|Y(z)| \leq B_1 \exp(w_{m+3}(|z|) + B\ln(1 + |z|)), \ z \in \mathbb{C},$$

where $B_1$ is some positive number. Whence, using Lemma 2, we have for some $B_2 > 0$

$$|Y(z)| \leq B_2 \exp(w_{m+4}(|z|)), \ z \in \mathbb{C}. \hspace{1cm} (29)$$

For any $\nu > 0, b \in (0, d_2)$ let $\Omega(b, \nu) = \bigcup_{k=1}^{\infty} D(\mu_k, b|\mu_k|^{-\nu})$. Let us estimate $|Y(z)|$ from below outside the set $\Omega(b, \nu)$. For any $k \in \mathbb{N}$

$$Y_k(z) = \frac{z - \mu_k}{Y(z)}, \ z \in \overline{D(\mu_k, d_2)}.$$  \hspace{1cm} \text{where } Y_k(z) \text{ is holomorphic in } \overline{D(\mu_k, d_2) \text{ and has no zeros there.}}

By the maximum principle

$$|Y_k(z)| \leq \frac{d_2}{|Y(z_0)|}, \ z \in \overline{D(\mu_k, d_2)},$$

where $z_0$ is a point of $\in \partial D(\mu_k, d_2)$ where the minimum of $|Y(z)|$ over $\overline{D(\mu_k, d_2)}$ is attained. From this, using (28) and (4), we obtain

$$|Y_k(z)| \leq C_1(1 + |z|)^B \exp(-w_{m+3}(|z|)), \ z \in D(\mu_k, d_2),$$

where $C_1 > 0$ doesn’t depend on $k$. Hence

$$|Y(z)| \geq \frac{|z - \mu_k| \exp(\nu w_{m+3}(|z|))}{C_1(1 + |z|)^B}, \ z \in D(\mu_k, d_2).$$
Thus, for \( z \in D(\mu_k, d_2) \setminus D(\mu_k, b|\mu_k|^{-\nu}) \)

\[
|Y(z)| \geq \frac{b \exp(w_{m+3}(|z|))}{C_1|\mu_k|^\nu(1 + |z|)^B}.
\]

Since for \( z \in D(\mu_k, d_2) \) \( |\mu_k| \leq \frac{4|z|}{3} \), then

\[
|Y(z)| \geq \frac{C_2 \exp(w_{m+3}(|z|))}{(1 + |z|)^{B+\nu}}, \quad z \in D(\mu_k, d_2) \setminus D(\mu_k, b|\mu_k|^{-\nu}), \tag{30}
\]

where \( C_2 > 0 \) doesn't depend on \( k \).

From (28) and (30) we get

\[
|Y(z)| \geq \frac{C_3 \exp(w_{m+3}(|z|))}{(1 + |z|)^{B+\nu}}, \quad z \notin \Omega(b, \nu),
\]

where \( C_3 > 0 \) is some constant. Using Lemma 2, we get

\[
|Y(z)| \geq C_4 \exp(w_{m+2}(|z|)), \quad z \notin \Omega(b, \nu), \tag{31}
\]

where \( C_4 > 0 \) is some constant depending on \( b, \nu \).

Now we need to choose positive numbers \( b, \nu \) and sequence \( t = \{t_j\}_{j=1}^\infty \subset \mathbb{C} \) such that:

1). \( \text{for every } j \in \mathbb{N} \quad D(\lambda_j + t_j, b|\lambda_j + t_j|^{-\nu}) \subset D(\lambda_j, d_1|\lambda_j|^{1-\alpha}) \);
2). \( \Omega(b, \nu) \cap \left( \bigcup_{j=1}^{\infty} D(\lambda_j + t_j, b|\lambda_j + t_j|^{-\nu}) \right) = \emptyset \);
3). \( \sum_{j=1}^{\infty} |t_j||\lambda_j|^{\alpha-1} < \infty \).

We need only slightly to develop the scheme which was introduced in \([13]\) (see also \([12, \text{Chapter 2, } \S 4]\)).

Let \( n_\mu(r) \) is the number of zeros of \( Y \) of modulos at most \( r, r > 0 \). From the estimates (29), (1) it follows that \( Y \) is an entire function of order 1 and finite type. Consequently, for some \( A_\mu > 1 \) \( n_\mu(r) \leq A_\mu r, \quad r > 0 \) \([2, \$ 5]\).

Choose \( \nu \) and \( b \) so that \( \nu > 2\alpha + 1 \), \( 0 < b < \min(d_2, d_1 2^{-\nu+6}A_\mu^{-2}) \).

Let us introduce the notations: \( \Delta(z) = |z|, \quad z \in \mathbb{C}, \quad D_{k,Y} = D(\mu_k, b|\mu_k|^{-\nu}), \quad D_{j,L} = D(\lambda_j, d_1|\lambda_j|^{1-\alpha}), \quad k, j \in \mathbb{N} \).

Let \( j \in \mathbb{N} \) be arbitrary but fixed. For \( k \in \mathbb{N} \) such that \( \Delta(D_{k,Y}) \cap \Delta(D_{j,L}) \neq \emptyset \) we have \( |\mu_k| - |\lambda_j| < l \), where \( l = d_1 + d_2 \). Sum \( s \) of lengths of all intervals \( \Delta(D_{k,Y}) \) intersecting \( \Delta(D_{j,L}) \) is estimated as follows:

\[
s = \sum_{|\lambda_j| - 1 < |\mu_k| < |\lambda_j| + 1} 2b|\mu_k|^{-\nu} \leq 2bA_\mu(|\lambda_j| + l) \leq 2bA_\mu(|\lambda_j| + l) \leq \frac{2bA_\mu(|\lambda_j| + l)}{(|\lambda_j| - l)^\nu}.
\]
Since $|\lambda_j| > 2l$ for all $j \in \mathbb{N}$, then $s \leq 2^{\nu+2}bA_\mu|\lambda_j|^{1-\nu}$. Put $\sigma_j = 2^{\nu+2}bA_\mu|\lambda_j|^{1-\nu}$. There exists an interval $[|\lambda_j| + k_0\sigma_j, |\lambda_j| + (k_0 + 1)\sigma_j)$, where $k_0 + 1 \leq 4n(|\lambda_j| + l)$, not intersecting $\bigcup_{k=1}^\infty \Delta(D_k,Y)$. Put $t_j = (k_0 + \frac{1}{2})\sigma_j \exp(i \arg \lambda_j)$. Thus, the choice of the points $t_j$ is the same as in [12], [13]. Now we estimate $|t_j|:

\sum_{j=1}^\infty |t_j||\lambda_j|^\alpha - 1 \leq 2^{\nu+5}bA_\mu^2 \sum_{j=1}^\infty |\lambda_j|^{\alpha - 1} < \infty,$

(32)

Consequently,

$$\sum_{j=1}^\infty |t_j||\lambda_j|^\alpha - 1 \leq 2^{\nu+5}bA_\mu^2 \sum_{j=1}^\infty |\lambda_j|^{\alpha - 1} < \infty,$$

since $\nu - \alpha - 1 > \alpha$ and the convergence exponent of the zeros of $L$ doesn’t exceed $\alpha$. This means that the condition 3) holds.

Due to (32), choice of $\nu$, $b$ and since $|\lambda_j| > 1$ for all $j \in \mathbb{N}$, the condition 1) holds too. By the construction disk $D(\lambda_j + t_j, \sigma_j)$ doesn’t intersect $\Omega(b, \nu)$. Since $\sigma_j = 2^{\nu+1}bA_\mu|\lambda_j|^{1-\nu} > b(|\lambda_j| + |t_j|)^{-\nu} = b|\lambda_j + t_j|^{-\nu}$, then disk $D(\lambda_j + t_j, b|\lambda_j + t_j|^{-\nu})$ doesn’t intersect $\Omega(b, \nu)$ too. Thus, condition 2) is satisfied.

Denote $D(\lambda_j + t_j, b|\lambda_j + t_j|^{-\nu})$ by $D'_{j,L}$, $j \in \mathbb{N}$.

According to theorem S $L_t$ satisfies

$$|\ln |L_t(z)| - |L(z)|| \leq C, \ z \notin \bigcup_{j=1}^\infty D_{j,L},$$

where $C > 0$ is some constant. From this and (27) it follows that for $z \notin \bigcup_{j=1}^\infty D_{j,L}$

$$|\ln |L_t(z)| - \psi(Im z)| \leq A \ln(1 + |z|) + A_0 + C. \quad (33)$$

Note that from (2) for $z_1, z_2 \in D_{j,L}$

$$|\psi(Im z_1) - \psi(Im z_2)| \leq A_1, \quad (34)$$

where $A_1 > 0$ is some constant independent of $j \in \mathbb{N}$.

Applying the maximum principle to $L_t$ in every disk $D_{j,L}$ and using (33), (34), we obtain

$$|L_t(z)| \leq A_2(1 + |z|)^A \exp(\psi(Im z)), \ z \in \bigcup_{j=1}^\infty D_{j,L}, \quad (35)$$
where \( A_2 > \exp(A_0 + C) \) is some constant.

From (33) and (35) it follows that (35) holds for all \( z \in \mathbb{C} \). Using the inequalities (27), (33), (34), it is not hard to show (see, for example, [8, Chapter 2, §4]) that outside \( \bigcup_{j=1}^{\infty} D'_{j,L} \)

\[
|L_t(z)| \geq \exp(\psi(Im \ z) - A_3 \ln(1 + |z|) - A_4),
\]

where \( A_3, A_4 \) are some positive numbers. Thus, the necessary estimates of \(|L_t|\) from above in \( \mathbb{C} \) and from below outside \( \bigcup_{j=1}^{\infty} D'_{j,L} \) are established.

By the choice of \( b, \nu \) the sums of the radii of the disjoint exceptional disks \( D'_{j,L} \) and \( D_{k,Y} \) are finite. Hence there exists a sequence \( \{l_n\}_{n=1}^{\infty} \) of circles \( \{l_n : |z| = R_n\} \), \( R_n \to \infty \) as \( n \to \infty \), which doesn’t intersect the disks \( D'_{j,L} \) and \( D_{k,Y} \).

Put

\[
\mathcal{L}(z) = \frac{L_t(z)}{\prod_{j=1}^{N}(z - (\lambda_j + t_j))}, \quad z \in \mathbb{C},
\]

where \( N \in \mathbb{N} \) is such that \( N > A + 2 \). Then from (35), (36) for some positive constants \( A_5, A_6 \)

\[
|\mathcal{L}(z)| \leq \frac{A_5 \exp(\psi(Im \ z))}{1 + |z|^2}, \quad z \in \mathbb{C};
\]

\[
|\mathcal{L}(z)| \geq \frac{A_6 \exp(\psi(Im \ z))}{(1 + |z|)^{A_3 + N}}, \quad z \notin \bigcup_{j=N+1}^{\infty} D'_{j,L}.
\]

Let \( \mathcal{N}(z) = \mathcal{L}(z)Y(z), \quad z \in \mathbb{C} \). Let \( \{a_n\}_{n=1}^{\infty} \) be the zeros of \( \mathcal{N} \) ordered by non-decreasing modulus. By (31), (38) and Lemma 2 for \( z \) outside the set \( \bigcup_{n=1}^{\infty} D(a_n, b |a_n|^{-\nu}) \)

\[
|\mathcal{N}(z)| \geq C_{\mathcal{N}} \exp(\psi(Im \ z) + w_{m+1}(|z|)),
\]

where \( C_{\mathcal{N}} \) is some positive number. In particular, this estimate holds on the circles \( l_n, n \in \mathbb{N} \).

Let \( K \) be an arbitrary compact in \( \mathbb{C} \) and \( k_0 \in \mathbb{N} \) be such that \( \overline{K} \subset D(0, R_{k_0}) \). For \( k \geq k_0 \) let

\[
I(z) = \int_k^{\infty} \frac{F(\xi)}{\mathcal{N}(\xi)(\xi - z)} \ d\xi, \quad z \in K.
\]
Then
\[ I(z) = 2\pi i \sum_{|a_n|<R_k} \frac{F(a_n)}{N'(a_n)(a_n - z)} + 2\pi i \frac{F(z)}{N(z)}, \quad z \in K. \]  
(40)

Using (26) and (39) for \( z \in K \) we have
\[ |I(z)| \leq \int_{l_k} C_N \frac{c \exp(\psi(Im \, \xi) + w_m(|\xi|))}{dist(K, l_k) \exp(\psi(Im \, \xi) + w_{m+1}(|\xi|))} \, |d\xi|. \]

Letting \( k \to \infty \) in the right-hand side of this inequality and taking Lemma 2 into account, we get
\[ I(z) = 0, \quad z \in K. \]

Letting \( k \to \infty \) in (40), we obtain
\[ F(z) = \lim_{k \to \infty} \sum_{|a_n|<R_k} \frac{F(a_n) \, N'(z)}{N'(a_n) \, z - a_n}, \quad z \in K. \]  
(41)

Since compact \( K \) was arbitrary, then (41) holds for all \( z \in \mathbb{C} \).

Consider the series
\[ \sum_{n=1}^{\infty} \frac{F(a_n) \, N'(z)}{N'(a_n) \, z - a_n}, \quad z \in \mathbb{C}. \]  
(42)

We wish to show that it converges uniformly on every compact set of complex plane. At first for any \( n \in \mathbb{N} \) we estimate \( \frac{N'(z)}{z - a_n}, \quad z \in \mathbb{C} \). We consider two cases.

The first case. Let \( a_n \) be the zero of \( \mathcal{L} \). Then for some \( j \in \mathbb{N} \; a_n = \lambda_j + t_j \), hence
\[ \frac{N'(z)}{z - a_n} = \frac{\mathcal{L}(z)}{z - (\lambda_j + t_j)} \, Y(z), \quad z \in \mathbb{C}. \]

If \( z \notin D'_{j,L} \), then by (37)
\[ \left| \frac{\mathcal{L}(z)}{z - (\lambda_j + t_j)} \right| \leq A_5 \exp(\psi(Im \, z)) \frac{|\lambda_j + t_j|^\nu}{1 + |z|^2} \frac{\nu}{b}. \]  
(43)

If \( z \in D'_{j,L} \), then by the maximum principle
\[ \left| \frac{\mathcal{L}(z)}{z - (\lambda_j + t_j)} \right| \leq \max_{\xi \in \partial D'_{j,L}} \left| \frac{\mathcal{L}(\xi)}{\xi - (\lambda_j + t_j)} \right|. \]

Let maximum in the right-hand side of the last inequality is attained at a point \( \xi_0 \in \partial D'_{j,L} \). Then, again using the inequality (37),
\[ \left| \frac{\mathcal{L}(z)}{z - (\lambda_j + t_j)} \right| \leq \frac{|\mathcal{L}(\xi_0)||\lambda_j + t_j|^\nu}{b} \leq A_5 \exp(\psi(Im \, \xi_0)) \frac{|\lambda_j + t_j|^\nu}{1 + |\xi_0|^2} \frac{\nu}{b}. \]
Taking into account that for $z \in D'_{j,L} |\xi_0| \geq |z| - 1$, and using (34), we obtain

$$\left| \frac{L(z)}{z - (\lambda_j + t_j)} \right| \leq \frac{A_7 \exp(\psi(Im \ z)) |\lambda_j + t_j|^{\nu}}{1 + |z|^2} b,$$  \hspace{1cm} (44)

where $A_7 > 0$ is some constant independent of $j$. Thus, from (43), (44) we have

$$\left| \frac{L(z)}{z - a_n} \right| \leq \frac{A_8 \exp(\psi(Im \ z)) |a_n|^{\nu}}{1 + |z|^2},$$  \hspace{1cm} (45)

where $A_8 = b^{-1} \max(A_5, A_7)$. From (29), (45) we get in the first case

$$\left| \frac{N(z)}{z - a_n} \right| \leq A_9 |a_n|^{\nu} \exp(\psi(Im \ z) + w_{m+4}(|z|)), \ z \in \mathbb{C},$$  \hspace{1cm} (46)

where $A_9 = A_8 B_2$.

The second case. Let $a_n$ be the zero of $Y$. Then for some $k \in \mathbb{N} a_n = \mu_k$. Hence

$$\frac{N(z)}{z - a_n} = \frac{Y(z)}{z - \mu_k} L(z), \ z \in \mathbb{C}.$$  

If $z \notin D_{k,Y}$, then from (29)

$$\left| \frac{Y(z)}{z - \mu_k} \right| \leq b^{-1} B_2 \exp(w_{m+4}(|z|)) |\mu_k|^{\nu}, \ z \in \mathbb{C}.$$  \hspace{1cm} (47)

If $z \in D_{k,Y}$, then by the the maximum principle

$$\left| \frac{Y(z)}{z - \mu_k} \right| \leq \max_{\xi \in \partial D_{k,Y}} \left| \frac{Y(\xi)}{\xi - \mu_k} \right|.$$  

Let maximum in the right-hand of the last inequality is attained at a point $\xi_0 \in \partial D_{k,Y}$. Then, again using (29), we have

$$\left| \frac{Y(z)}{z - \mu_k} \right| \leq b^{-1} |Y(\xi_0)||\mu_k|^{\nu} \leq b^{-1} B_2 \exp(w_{m+4}(|\xi_0|)) |\mu_k|^{\nu} .$$

For $z \in D_{k,Y} \ |\xi_0| \leq |z| + 1$, so from the last inequality, using (4), we get

$$\left| \frac{Y(z)}{z - \mu_k} \right| \leq B_3 \exp(w_{m+4}(|z|)) |\mu_k|^{\nu}.$$  \hspace{1cm} (48)
where $B_3 > 0$ doesn’t depend on $k$. From (47), (48) it follows that

$$\left| \frac{Y(z)}{z-a_n} \right| \leq B_4 |a_n|^\nu \exp(w_{m+4}(|z|)), \quad (49)$$

where $B_4 = \max(b^{-1}B_2, B_3)$ doesn’t depend on $n$. From (37) and (49) in the second case we obtain

$$\left| \frac{N(z)}{z-a_n} \right| \leq A_{10} |a_n|^\nu \exp(\psi(Im \, z) + w_{m+4}(|z|)), \quad z \in \mathbb{C}, \quad (50)$$

where $A_{10} = A_5 B_4$. From (46) and (50) we conclude that in both cases

$$\left| \frac{N(z)}{z-a_n} \right| \leq A_{11} |a_n|^\nu \exp(\psi(Im \, z) + w_{m+4}(|z|)), \quad z \in \mathbb{C}, \quad (51)$$

where $A_{11} = \max(A_9, A_{10})$.

From the representation

$$\frac{1}{N'(a_n)} = \frac{1}{2\pi i} \int_{|\xi-a_n|=b|a_n|^{-\nu}} \frac{d \xi}{N(\xi)},$$

the inequality (39) and by using (4) and (34), we get

$$\left| \frac{1}{N'(a_n)} \right| \leq A_{12} |a_n|^{-\nu} \exp(-\psi(Im \, a_n) + w_{m+1}(|a_n|)),$$

where constant $A_{12} > 0$ doesn’t depend on $n \in \mathbb{N}$. From this and (26) we obtain

$$\left| \frac{F(a_n)}{N'(a_n)} \right| \leq cA_{12} |a_n|^{-\nu} \exp(w_m(|a_n|) - w_{m+1}(|a_n|)), \quad n \in \mathbb{N}.$$

By Lemma 2 there exists constant $Q \geq 0$ such that $w_m(|a_n|) - w_{m+1}(|a_n|) < -(\alpha + 1) \ln(1 + |a_n|) + Q \quad \forall n \in \mathbb{N}$. Consequently,

$$\left| \frac{F(a_n)}{N'(a_n)} \right| \leq A_{13} (1 + |a_n|)^{-(\alpha+\nu+1)}, \quad n \in \mathbb{N}, \quad (52)$$

where constant $A_{13} > 0$ doesn’t depend on $n$.

From (51) and (52) for every $z \in \mathbb{C}$ we have

$$\sum_{n=1}^\infty \left| \frac{F(a_n)}{N'(a_n)} \right| \frac{N(z)}{z-a_n} \leq A_{11} A_{13} \sum_{n=1}^\infty (1 + |a_n|)^{-(\alpha+1)} \exp(\psi(Im \, z)+w_{m+4}(|z|)).$$
Since the convergence exponent of the zeros of \( \mathcal{N} \) doesn’t exceed \( \alpha \), then
\[
\sum_{n=1}^{\infty} (1 + |a_n|)^{-(\alpha+1)} < \infty
\]  
(53)

Consequently, the series (42) converges uniformly on every compact subset of \( \mathbb{C} \). Hence by (41) ((41) holds for all \( z \in \mathbb{C} \)) we have
\[
F(z) = \sum_{n=1}^{\infty} \frac{F(a_n) \mathcal{N}(z)}{N'(a_n) z - a_n}, z \in \mathbb{C}.
\]  
(54)

Now we wish to define \( \mathcal{F}_n \in G^* \) such that \( \hat{\mathcal{F}}_n(z) = \frac{\mathcal{N}(z)}{z - a_n} \), \( z \in \mathbb{C}, n \in \mathbb{N} \). There are two cases.

The first case. \( a_n \) is the zero of \( \mathcal{L} \). Hence, \( \frac{\mathcal{N}(z)}{z - a_n} = \frac{\mathcal{L}(z)}{z - a_n} Y(z), z \in \mathbb{C} \). By (29), (45), Lemma 6 there exists \( \mathcal{F}_n \in G^* \) such that \( \hat{\mathcal{F}}_n(z) = \frac{\mathcal{N}(z)}{z - a_n}, z \in \mathbb{C} \) and
\[
|\mathcal{F}_n(f)| \leq \beta_{m+4} A_8 B_2 |a_n|^{\nu} \|f\|_{G,m+5}, f \in G.
\]

The second case. \( a_n \) is the zero of \( \mathcal{Y} \). Hence, \( \frac{\mathcal{N}(z)}{z - a_n} = \frac{\mathcal{Y}(z)}{z - a_n} \mathcal{L}(z), z \in \mathbb{C} \).

By (37), (49), Lemma 6 there is \( \mathcal{F}_n \in G^* \) such that \( \hat{\mathcal{F}}_n(z) = \frac{\mathcal{N}(z)}{z - a_n}, z \in \mathbb{C} \) and
\[
|\mathcal{F}_n(f)| \leq \beta_{m+4} A_5 B_1 |a_n|^{\nu} \|f\|_{G,m+5}, f \in G.
\]

Thus, in both cases for every \( n \in \mathbb{N} \) there is \( \mathcal{F}_n \in G^* \) such that
\[
\hat{\mathcal{F}}_n(z) = \frac{\mathcal{N}(z)}{z - a_n}, z \in \mathbb{C},
\]  
(55)

and
\[
|\mathcal{F}_n(f)| \leq H |a_n|^{\nu} \|f\|_{G,m+5}, f \in G,
\]  
(56)

where constant \( H > 0 \) doesn’t depend on \( n \). Put
\[
\mathcal{F}(f) = \sum_{n=1}^{\infty} \frac{F(a_n)}{N'(a_n)} \mathcal{F}_n(f), f \in G.
\]

Because of (52), (53), (56) \( \mathcal{F} \) is defined correctly and it belongs to \( G^* \). \( \hat{\mathcal{F}} = F \). Thus, \( J \) is surjective.

By the open mapping theorem for the spaces \( (LN^*) \) ([9], [6, p. 12]) \( J \) establishes topological isomorphism of the spaces \( G^* \) and \( P \). Theorem 1 is proved.

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