1. Introduction

The log canonical threshold of a complex algebraic variety $X$ is a numerical invariant measuring singularities of $X$. Suppose $X$ is a subvariety of an affine space $A$, $Y$ is the generic link of $X$ in a suitable affine extension $B$ of $A$, and $V$ is a generic complete intersection by which $Y$ is linked to $X$, see section 2 for the precise definition. The work of the third author [Niu14, Prop. 3.7] gives the following relationship on the log canonical thresholds of $X$, $Y$, and $V$,

$$lct(B,Y) \geq lct(B,V) = lct(A,X).$$

In general, equality in Equation (1) does not hold. Indeed, the log canonical threshold of the generic link of a hypersurface is always 1, whereas there are hypersurfaces whose log canonical threshold is strictly less than 1.

To the best of our knowledge, there is no known non-trivial class of varieties where equality in Equation (1) holds in general. Determinantal varieties are classical objects in algebraic geometry and commutative algebra. The goal of this article is to prove that one gets equality throughout when $X$ is a determinantal variety.

**Theorem 1.** Let $X$ be a determinantal variety and $Y$ its generic link. Then $X$ and $Y$ have the same log canonical threshold.

Our approach to prove Theorem 1 is to utilize an embedded log resolution of $X$ in an affine space [Vai84]. This resolution is explicitly described in [Joh03] and [Doc13]. In general, this embedded log resolution of $X$ does not extend to an embedded log resolution of $Y$. However, to prove our theorem it suffices to compare the order of $X$ and $Y$ along the exceptional divisors associated to the log resolution of $X$. Indeed, we can relate these numbers explicitly utilizing facts about determinantal varieties and linkage theory; cf., Lemma 2.3.

**Acknowledgments:** The authors would like to thank Roi Docampo, William Heinzer, William D. Taylor, and Bernd Ulrich for valuable discussions.

2. Notation and overview of the proof

Throughout, all varieties are assumed to be reduced and irreducible schemes of finite type over $\mathbb{C}$. We begin by reviewing the notion of generic linkage. The definition given in [HU87] is more general, but we only state our definition for determinantal varieties. Suppose $M$ is a set of variables, $I_X \subset \mathbb{C}[M]$ defines a variety $X$ of codimension $c$, and $g_1, \ldots, g_\mu$ be a generating set of $I_X$. One may
set \( G \) the column vector with entries consisting of the generators \( g_i \) of \( I_X \). Form a generic \( c \times \mu \) coefficient matrix \( U \) of new variables. In the ring \( S = \mathbb{C}[M, U] \), the ideal \( I_Y \) of entries of the matrix \( U \cdot G \) is a complete intersection of codimension \( c \) \cite{Hoc73}. Set \( I_Y = I_Y :_S I_X S \). Then \( V = \text{Spec} S/I_Y \) and \( Y = \text{Spec} S/I_Y \) are subvarieties of \( B := \text{Spec} S \). We refer to \( Y \) as the generic link of \( X \) in \( B \).

In the setting of Theorem \( \text{[I]} \) \( X \) is the variety parametrizing \( m \) by \( n \) matrices of rank at most \( r - 1 \) with entries in \( \mathbb{C} \), so its ideal \( I_X \) is the ideal of \( r \times r \)-minors of a generic \( m \times n \) matrix. We assume 
\[
m \geq n \geq r \quad \text{and} \quad c = (m-r+1)(n-r+1)\]
the codimension of \( X \) in \( \text{Spec} \mathbb{C}[M] \).

The log canonical threshold is a measure of singularities of pairs, for general reference see \cite{Kol97, Laz04}. It is a rational number which may be computed from data of a log resolution. Recall, an embedded log resolution of a pair \((A, X)\), where \( X \subset A \) is a subvariety and \( A \) is an affine space, is a birational projective morphism \( f: \tilde{A} \to A \) such that \( \tilde{A} \) is non-singular, \( I_X \varphi_{\tilde{A}} \) is invertible, and \( \text{Exc}(f) \cup f^{-1}(X) \) has simple normal crossing support. Here, \( \text{Exc}(f) \) denotes the set of exceptional divisors of \( f \). We may write \( I_X \varphi_{\tilde{A}} = \varphi_{\tilde{A}}(-G) \) for some divisor \( G = \sum a_i X E_i \), and we write \( K_{\tilde{A}/A} := K_{\tilde{A}} - f^* K_A = \sum k_i E_i \) for the relative canonical divisor. In this note, we only consider embedded log resolutions, so we will drop the use of the adjective ‘embedded’.

Such a log resolution of the pair \((A, X)\) gives the following characterization of the log canonical threshold
\[
\text{lct}(A, X) = \min_i \left\{ \frac{k_i + 1}{a_i X} \right\}.
\]
We use this as our definition of a log canonical threshold. When the ambient space is clear, we write simply \( \text{lct}(X) \) for \( \text{lct}(A, X) \).

**Remark 2.1.** (a) A generic link of \( X \) depends on the choice of the generating set for \( I_X \). There is a notion of equivalence of generic links \cite[Prop. 2.4]{HU85}, if \( Y \) are \( X \)'s generic links and \( Y' \) are \( X \)'s generic links.

(b) Indeed, the proof of \cite[Prop. 2.4]{HU85} shows more. In our notation, the isomorphism constructed for the equivalence of generic links fixes \( \mathbb{C}[M] \). So, if \( U \) and \( U' \) are sets of variables for generic links of \( Y \) and \( Y' \) of \( I_X \), respectively, then one has that \( \text{ord}_{\mathbb{C}[M, U]} I_Y = \text{ord}_{\mathbb{C}[M, U']} I_{Y'} \) for any \( \mathbb{C}[M] \)-ideal \( I \).

Theorem \text{[I]} holds trivially when \( \text{lct}(X) = \text{codim} X \), indeed
\[
\text{lct}(X) = \text{codim} X = \text{codim} Y \geq \text{lct}(Y),
\]
where the last inequality holds by \cite[Lem. 2.5]{Niu14} or \cite[Ex. 9.2.14 and Prop. 9.2.32(a)]{Laz04}. For the general situation, our approach is to explicitly relate the order of \( X \) calculated on a particular log resolution of \( X \) to the order of the generic link \( Y \). To do that we extend such log resolution \( f: \tilde{A} \to A \) of \( (A, X) \) to \( (B, X) \) and compare \( \text{ord}_{E_i} (I_X) \) and \( \text{ord}_{E_i} (I_Y) \). In Proposition \text{[3.2]}, we show that unless \( \text{lct}(A, X) = \text{codim} X \), the value \( a_{i, X} \) calculating \( \text{lct}(X) \) satisfies \( a_{i, X} = \text{ord}_{E_i} (I_X) = \text{ord}_{E_i} (I_Y) \).
which allows us to conclude equality of log canonical thresholds.

The log resolution of \((A,X)\) we use is described in explicit detail in [Joh03, Thm. 4.4 and Cor. 4.5].

**Theorem 2.2.** Let \((A,X)\) be the pair where \(X\) is defined by the \(r \times r\)-minors of a generic \(m \times n\)-matrix \(M\) and \(A = \text{Spec } \mathbb{C}[M]\). For \(i = 1, \ldots, r\), let

\[
\pi_i : A_i := \text{Bl}_V(I_i(M)^\sim)(A_{i-1}) \to A_{i-1},
\]

where \(A_0 = A\) and \(V(I_i(M))\) denotes the strict transform of \(V(I_i(M))\) in \(A_{i-1}\) under the map \(\pi_{i-1} \circ \cdots \circ \pi_1\). Then the composition of the blowups \(\pi := \pi_r \circ \cdots \circ \pi_1\) is a log resolution for \((A,X)\). Furthermore, denote by \(E_i\) the exceptional divisor introduced by \(\pi_i\) in \(A_r\) for \(i = 1, \ldots, r\). One has

\[
I_X \mathcal{O}_{A_r} = \mathcal{O}_{A_r}(\sum a_{i,X} E_i),
\]

where \(a_{i,X} = r - i + 1\) for \(i = 1, \ldots, r\). From this resolution it follows that

\[
lct(X) = \min_{0 \leq t \leq r-1} \frac{(m-t)(n-t)}{r-t}. \tag{2}
\]

For \(i = 1, \ldots, r\), denote by \(X_i\) and \(V_i\) the strict transforms of \(X\) and \(V\) in \(A_i\) respectively. When the ambient space is clear, we denote by \(\mathcal{O}_W\) the ideal defining a subscheme \(W\) in the ambient space. Recall that \(a_{i,X} = \text{ord}_{E_i}(I_X)\). We will use the following results in linkage theory to prove Proposition 3.2

**Lemma 2.3.** Let \(S = R[M]\), where \(R\) is a commutative noetherian ring containing an infinite field, \(M\) is an \(m \times n\) matrix of intermediates with \(m \geq n\), and \(I_X\) is the ideal of \(S\) generated by \(r \times r\) minors of \(M\). The ring \(S\) is a standard graded ring generated by the entries of \(M\) over \(R\). Write \(c = (m-r+1)(n-r+1)\) the codimension of \(I_X\). Let \(f_1, \ldots, f_c\) be general linear combinations of the generators \(I_X\), for which we may assume that \(\text{codim}(f_1, \ldots, f_c) = c\) and \(f_i\) are forms of degree \(r\). Set \(I_Y = (f_1, \ldots, f_c)\) and \(I_Y = I_Y \cap S I_X\). We have the following statements.

(a) The graded canonical module of \(S/I_X\), denoted by \(\omega_{S/I_X}\), is generated in degree \((r-1)m\).

(b) \(\omega_{S/I_X}(mn - rc) \cong I_Y/I_Y\).

(c) The ideal \(I_Y/I_Y\) is generated in degree \(rc - m(n-r+1)\). In particular, the ideal \(I_Y\) can be generated by the elements of \(I_Y\) of degree \(r\) and elements of degree

\[
rc - m(n-r+1) = (n-r+1)(m-r)(r-1).
\]

**Proof.** For item (a), see [BH92, Bottom of p.5], and item (b) follows from [Mig98, Prop. 5.2.6] with \(rc\) and \(mn - 1\) for \(r\) and \(n\) in the proposition, respectively.

Combining (a) and (b), we conclude that the ideal \(I_Y/I_Y\) is generated in degree

\[
(r-1)m - (mn - rc) = rc - m(n-r+1).
\]

Notice that with \(c = (m-r+1)(n-r+1)\), we obtain

\[
rc - m(n-r+1) = r(m-r+1)(n-r+1) - m(n-r+1)
= (n-r+1)(rm - r^2 + r - m)
= (n-r+1)(m-r)(r-1). \quad \square
\]
3. Extension of the log resolution of $X$ and comparison of orders

We assume the notation and setup of Theorem 2.2. In this section, we provide an explicit description of the numbers $\operatorname{ord}_{E_i}(I_Y)$ in terms of $m, n, r$ and $i$. To do this first we extend the log resolution of the pair $(A, X)$ to $B := A \times \operatorname{Spec} \mathbb{C}[U]$. Since $B \to A$ is flat, the log resolution of $(A, X)$ extends naturally to $B$. By abuse of notation, we call by $X$ the subvariety of $B$ defined by the equations defining $X$ in $A$ and similarly for $E_i, \pi$ and $\pi_i$.

**Lemma 3.1.** Let $i \in \{1, \ldots, r - 1\}$. If $A_i \to A$ is the composition of blow-ups in Theorem 2.2 then the strict transform $Y_i$ of $Y$ in $A_i$ is the link of $X_i$ by $V_i$.

**Proof.** Write $W$ for the link of $X_i$ by $V_i$ in $A_i$. The proof is essentially [Niu14, Claim 3.1.2 part (3)].

As $V$ has two components $X$ and $Y$, the center of $\pi_i$ lies in $X_i$ and $V_i$, and the strict transform of $V$ has two components $X_i$ and $Y_i$. So $W$ contains one of the components of $V_i$, but it does not contain $X_i$. Thus, it must contain $Y_i$. As both $W$ and $Y_i$ are irreducible and have same the codimension, they must be equal. $\Box$

By [Joh03, Lemma 4.1], to understand the log resolution of $(B, X)$ one essentially needs to understand one affine chart in each blowup $\pi_i$. On such an affine cover of

$$\pi_i \times \operatorname{Spec} \mathbb{C}[U] : A_i \times \operatorname{Spec} \mathbb{C}[U] \to A_{i-1} \times \operatorname{Spec} \mathbb{C}[U],$$

locally $X_i$ is the generic determinantal variety defined by the $r_i \times r_i$ minors of $m_i \times n_i$ matrix, where $m_i = m - i + 1, n_i = n - i + 1, r_i = r - i + 1$. On such an affine cover, the strict transform $X_i$ may require a smaller number of variables than $X$.

**Proposition 3.2.** For $i = 1, \ldots, r$, we have

$$\operatorname{ord}_{E_i}(I_Y) = \min\{r_i, (n_i - r_i + 1) (m_i - r_i) (r_i - 1)\}.$$

**Proof.** Consider the following diagram.

$$\begin{array}{ccc}
\text{Bl}_Z(A_{i-1}) & \to & E_i = \pi_i^{-1}(Z) \\
\pi_i \downarrow & & \downarrow \pi_i^{-1} \\
A_{i-1} & \to &
\end{array}$$

where $Z$ denotes the strict transform of $V(I_i(M))$ in $A_{i-1}$. Thus $\pi_i^{-1}(Z) = E_i$, $Z$ is non-singular in $A_{i-1}$ and locally defined by the entries of a generic $m_i \times n_i$ matrix. Furthermore, the $r_i$ by $r_i$ minors of this generic matrix locally define $X_i$ in $A_{i-1}$. These properties are preserved when we extend the log resolution of $X$ in Theorem 2.2 to $B$.

Note that $\operatorname{ord}_{E_i}(I_Y) = \operatorname{ord}_{E_i}(\mathcal{I}_{Y_{i-1}}) = \sup\{q \in \mathbb{N} \cup \{0\} \mid \mathcal{I}_{Y_{i-1}} \subset (\mathcal{I}_Z)^q\}$, where $(\mathcal{I}_Z)^0 := \mathcal{O}_{A_{i-1}}$. Such an inclusion of ideal sheaves can be checked locally. We choose an affine cover of $A_{i-1}$ as in [Joh03, Sec. 4.2]. In each affine chart, the ideal $\mathcal{I}_{Y_{i-1}}$ is defined by the $r_i$ by $r_i$ minors of a generic matrix $M'$ of size $m_i$ by $n_i$ and $\mathcal{I}_Z$ is defined by variables $I_i(M')$. We note that the equations defining $\mathcal{I}_{V_{i-1}}$ is the strict transform of the equations defining $\mathcal{I}_V$ which we used to construct the generic link $Y$ of $X$. Thus, $\mathcal{I}_{V_{i-1}}$ is not a generic linear combination of the minors defining $\mathcal{I}_{X_{i-1}}$. However, thanks to Remark 2.1b, to compute the order, we may assume that $\mathcal{I}_{V_{i-1}}$ are the generic linear combinations of the $r_i$ by $r_i$ minors of $M'$. Let $I_{X_{i-1}}, I_{V_{i-1}}$, and $I_Z$ denote these ideals, respectively, and let $S_{i-1}$ denote the coordinate ring of this affine chart. It then follows that
Localizing $S_{i-1}$ homogeneously at the ideal $I_Z$, i.e., $S_{i-1}(I_Z)$, does not change the order of an ideal with respect to $I_Z$. Thus, after applying this homogeneous localization, we may assume that $S_{i-1} = R[M']$ for some ring $R$. By Lemma 4.2, the ideal $I_{Y_i}$ is generated by elements of degrees $r_i$ and $(n_i - r_i + 1)(m_i - r_i)(r_i - 1)$, and such generating degree is uniform in each affine chart. Thus, we have $\text{ord}_{E_i} = \min\{r_i, (n_i - r_i + 1)(m_i - r_i)(r_i - 1)\}$. □

4. PROOF OF THEOREM 1

The following lemma, whose proof is elementary and left to the reader, will be useful for the proof of Theorem 1.

**Lemma 4.1.** For fixed integers $1 \leq r \leq n \leq m$, $(n - r + 1)(m - r)(r - 1) \leq r$ if and only if

(a) $r = 1$,

(b) $m = r$, or

(c) $n = r$ and $m = r + 1$.

In particular, for $i = 1, \ldots, r - 1$, set $m_i = m - i + 1, n_i = n - i + 1, r_i = r - i + 1$, then

$$(n - r + 1)(m - r)(r - 1) \leq r \iff (n_i - r_i + 1)(m_i - r_i)(r_i - 1) \leq r_i.$$ 

Note that case (b) is the determinant of a square matrix, and case (c) is the maximal minors of $m$ by $m - 1$ matrices. In case (c), the generic link is also determinantal of size $m + 1$ by $m$.

**Proposition 4.2** (cf. [Joh03, Theorem 6.4]). Let $X = \text{Spec} \ C[M]/I_r$ be the variety defined by the $r \times r$-minors of a generic $m \times n$-matrix $M$, and let $Y$ be the generic link of $X$. If (a) $r = 1$, (b) $m = r$, or (c) $n = r$ and $m = r + 1$, i.e., the three cases in Lemma 4.1, then $\text{lct}(X) = \text{codim} X$. In particular, if $(n - r + 1)(m - r)(r - 1) \leq r$, then $\text{lct}(X) = \text{lct}(Y) = \text{codim} X$.

**Proof.** Apply the formula, in Equation (2) in Theorem 2.2 with Lemma 4.1 □

Now, we are ready to proof our main theorem.

**Proof of Theorem 1.** If $(n - r + 1)(m - r)(r - 1) \leq r$, then we are done by Proposition 4.2. Without loss of generality, we may assume that $(n - r + 1)(m - r)(r - 1) \geq r$.

Suppose $a_iX$ with $1 \leq i \leq r$ is the order computing the log canonical threshold of $X$, specifically $\text{lct}(X) = \frac{k_i+1}{a_iX}$. Once we have shown that

$$\text{lct}(X) = \text{codim} X \quad \text{if } i = r \text{ and}$$

$$\text{ord}_{E_i} (I_X) \leq \text{ord}_{E_i} (I_Y) \quad \text{otherwise},$$

the equality $\text{lct}(X) = \text{lct}(Y)$ follows by Equation (1). If $i = r$, then $\text{lct}(X) = \text{codim} X$. Hence $\text{lct}(Y) = \text{lct}(X)$ by Equation (1). Without loss of generality assume $1 \leq i < r$. In this range of $i$, by Lemma 4.1, we have

$$(n - r + 1)(m - r)(r - 1) \geq r \iff (n_i - r_i + 1)(m_i - r_i)(r_i - 1) \geq r_i,$$

where $m_i = m - i + 1, n_i = n - i + 1$, and $r_i = r - i + 1$. Therefore, by Proposition 3.2, we conclude that

$$\text{ord}_{E_i} (I_Y) = r_i.$$ 

Since $\text{ord}_{E_i} (I_X) = r_i$, this completes the proof. □
Corollary 4.3. In the setting of Theorem if $(n - r + 1)(m - r)(r - 1) \geq r$, then $\text{ord}_{E_i}(I_Y) = 0$ and $\text{ord}_{E_i}(I_X) = \text{ord}_{E_i}(I_Y)$ for $i = 1, \ldots, r - 1$.

Proof. By the proof of Theorem it suffices to show the assertion $\text{ord}_{E_i}(I_Y) = 0$. Notice that $X_r$ is non-singular, and it is the center of the blow-up $\pi$. Since $Y_r$ and $X_r$ are geometrically linked, $X_r \not\subset Y_r$. This shows that $I_Y \mathcal{O}_{X_r} = \mathcal{O}_{X_r}$, hence $\text{ord}_{E_i}(I_Y) = 0$. \hfill \Box

Remark 4.4. The reader should be warned not to jump to the incorrect conclusion that one has $\text{ord}_{E_i}(I_X) = \text{ord}_{E_i}(I_Y)$ in general. Let

$$M = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{pmatrix}$$

and $\Delta_i$ denote the signed minor of $M$ after deleting the $i$th row for $i = 1, 2, 3$. Then $I_Y$ is generated by two elements

$$v_1 = u_{11}\Delta_1 + u_{12}\Delta_2 + u_{13}\Delta_3, \quad v_2 = u_{21}\Delta_1 + u_{22}\Delta_2 + u_{23}\Delta_3,$$

and $I_Y = I_Y : I_X$ is generated by the $3 \times 3$ minors of the matrix

$$\begin{pmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{pmatrix}.$$ 

Since the ideal generated by the variables in $x_{i,j}$, say $m$, is the center of the blow up

$$\pi : A := \text{Bl}_{V(I(M))}(A) \to A := \text{Spec } \mathbb{C}[M, U],$$

and $I_Y \not\subset m^2$. Thus $\text{ord}_{E_i}(I_X) = 2$, but $\text{ord}_{E_i}(I_Y) = 1$.

References

[BH92] Winfried Bruns and Jürgen Herzog. On the computation of $a$-invariants. *Manuscripta Math.*, 77(2-3):201–213, 1992.

[Doc13] Roi Docampo. Arcs on determinantal varieties. *Trans. Amer. Math. Soc.*, 365(5):2241–2269, 2013.

[Hoc73] Melvin Hochster. Properties of Noetherian rings stable under general grade reduction. *Arch. Math. (Basel)*, 24:393–396, 1973.

[HU85] Craig Huneke and Bernd Ulrich. Divisor class groups and deformations. *Amer. J. Math.*, 107(6):1265–1303, 1985.

[HU87] Craig Huneke and Bernd Ulrich. The structure of linkage. *Ann. of Math. (2)*, 126(2):277–334, 1987.

[Joh03] Amanda Johnson. *Multiplier ideals of determinantal ideals*. ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–University of Michigan.

[Kol97] János Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[Mig98] Juan C. Migliore. *Introduction to liaison theory and deficiency modules*, volume 165 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1998.

[Niu14] Wenbo Niu. Singularities of generic linkage of algebraic varieties. *Amer. J. Math.*, 136(6):1665–1691, 2014.

[Vai84] Israel Vainsencher. Complete collineations and blowing up determinantal ideals. *Math. Ann.*, 267(3):417–432, 1984.
MATHEMATICAL SCIENCES, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, AR 72701
E-mail address: yk009@uark.edu

MATHEMATICAL SCIENCES, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, AR 72701
E-mail address: lem016@uark.edu

MATHEMATICAL SCIENCES, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, AR 72701
E-mail address: wenboniu@uark.edu