Infinite-Dimensional Hamiltonian Description of Dissipative Mechanical Systems

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In this paper an approach is proposed to define an infinite-dimensional Hamiltonian formalism to represent dissipative mechanical systems. This approach is based upon below viewpoints: for any non-conservative classical mechanical system and any initial condition, there exists a conservative one; the two systems share one and only one common phase curve; the value of the Hamiltonian of the conservative system is equal to the sum of the total energy of the non-conservative system on the aforementioned phase curve and a constant depending on the initial condition. We called the conservative system as the substituting conservative system. The infinite-dimensional Hamilton’s description of the ideal fluid in Lagrangian and Poisson-Vlasov equation motivate us to consider a dissipative mechanical system as a special fluid in a domain $D$ of the phase space, viz. a collection of particles in the domain. By comparing the description of the ideal fluid in Lagrangian coordinates, the Hamiltonian and the Lagrangian can be thought of as the integrals of the Hamiltonian and the Lagrangian of the substituting conservative system over the initial value space and a new Poisson bracket is defined to represent the Hamilton’s equation. The advantage of the approach is: the value of the canonical momentum density $\pi$ is identical with that of the mechanical momentum and the value of canonical coordinate $q$ is identical with that of the coordinate of the dissipative mechanical system.

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I. INTRODUCTION

Since Hamilton originated Hamilton equations of motion and Hamiltonian formalism, it has been stated in most classical textbooks that the Hamiltonian formalism focuses on solving conservative problems.

In 1960s, Hori and Brouwer (1961) utilized the classical Hamiltonian formalism and a perturbation theory to solve a non-conservative problem. They did not attempt to derive the Hamiltonian formalism for non-conservative problems. Several authors have attempted to enlarge the scope of Hamiltonian formalism to dissipative problems. Some significant works in this area were reported by Vujanovic (1970, 1978) and Djukic (1973); Djukic and Vujanovic (1975); Djukic (1975). They have proposed a technique for systems with gauge variant Lagrangian. A.Mukherjee and A.Dasgupta (2006) considered that the technique is rather algebraic in nature. To overcome the limitations, A.Mukherjee (1994) proposed a modified equation with an introduction of an additional time like variable called 'umbra time' and extending this notion to the co-kinetic kinetic, potential, complimentary energies as well as Lagrangian itself. Amalendu Mukherjee (1997) introduced a procedure for getting umbra-Lagrangian through system bond graphs and extended the basic idea of Karnopp (1977). Mukherjee (2001) consolidated this idea and presented an important idea of invariants of motion. A gauge variant Lagrangian implies a new definition of canonical momentum, which might not be identical with mechanical momentum. Some other literature of Jerrold E. Marsden (1994) and Morrison (2006, 1998), Salmon (1988) in the geometrical mechanics field focused on the conservative system or some special dissipative systems, e.g. an oscillator with gyroscopic damping. Morrison (1998) had written so: 'the ideal fluid description is one in which viscosity or other phenomenological terms are neglected. Thus, as is the case for systems governed by Newton’s second law without dissipation, such fluid descriptions posses Lagrangian and Hamiltonian descriptions.' Krecetnikov and Marsden (2007) and other researchers applied the equations as below to the problem of stability of
dissipative system,

\[ \dot{p}_i = -\frac{\partial H}{\partial q_i} + F \left( \frac{\partial r}{\partial q_i} \right) \]

\[ \dot{q}_i = \frac{\partial H}{\partial p_i}, \]

where \{q, p\} denote the coordinate and momentum, and the position vector \( r \) depends on the canonical variable \{q, p\}, i.e. \( r(q, p) \). \( H \) denotes Hamiltonian, \( F(\partial r/\partial q_i) \) denotes a generalized force in direction \( i \). Marsden considered that Eqs. (1) was composed of a conservative part and a non-conservative part. Eq. (1) apparently is not a Hamilton’s equation but only a representation of dissipative mechanical systems in the phase space.

In this paper an \( n \)-dimensional dissipative mechanical system as the following is considered:

\[ \ddot{q} + c\dot{q} + kq = 0, \]

where \( c \) denotes the damping coefficient matrix, \( k \) denotes the stiffness coefficient matrix.

In light of the proposition proposed by [Luo and Guo (2010)] an attempt is made to represent the dissipative mechanical system (2) as an infinite-dimensional Hamilton’s equation. This proposition asserts that for any non-conservative classical mechanical system and any initial condition, there exists a conservative one; the two systems share one and only one common phase curve; the Hamiltonian of the conservative system is the sum of the total energy of the non-conservative system on the aforementioned phase curve and a constant depending on the initial condition. In sec. [II] the demonstration of the proposition is first reported.

Analogous to Hamiltonian description of ideal fluid in Lagrangian variables and that of Poisson-Vlasov equations, we attempt to define Lagrangian and Hamiltonian as an integral over the entire initial value space. The generalized coordinates and the canonical momentum will be thought of as the function of the initial value and time. A new Poisson bracket will be defined to represent Eq. (2) as an infinite-dimensional Hamilton’s Equation. This process will be in detail presented in Sec. [III].
II. CORRESPONDING CONSERVATIVE MECHANICAL SYSTEMS

A. Common Phase Flow Curve

First we represented Eq. (2) as Eq. (1). Under general circumstances, the force $F$ is a damping force that depends on the variable set $q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n$. We denote by $F_i$ the components of the generalized force $F$.

$$F_i(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n) = F \left( \frac{\partial r}{\partial q_i} \right). \tag{3}$$

Thus we can reformulate the Eq. (2) as follows:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + F_i(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n) \tag{4}$$
$$\dot{q}_i = \frac{\partial H}{\partial p_i}.$$  

Suppose the Hamiltonian quantity of a conservative system without damping is $\hat{H}$. Thus we may write a Hamilton’s equation of the conservative system:

$$\dot{p}_i = -\frac{\partial \hat{H}}{\partial q_i} \tag{5}$$
$$\dot{q}_i = \frac{\partial \hat{H}}{\partial p_i}.$$  

We do not intend to change the definition of momentum in classical mechanics, but we do require that a special solution of Eq. (5) is the same as that of Eq. (4). We may therefore assume a phase curve $\gamma$ of Eq. (4) coincides with that of Eq. (5). The phase curve $\gamma$ corresponds to an initial condition $q_{i0}, p_{i0}$. Consequently by comparing Eq. (4) and Eq. (5), we have

$$\frac{\partial \hat{H}}{\partial q_i} \bigg|_{\gamma} = \frac{\partial H}{\partial q_i} \bigg|_{\gamma} - F_i(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n) \bigg|_{\gamma} \tag{6}$$
$$\frac{\partial \hat{H}}{\partial p_i} \bigg|_{\gamma} = \frac{\partial H}{\partial p_i} \bigg|_{\gamma},$$

where $\frac{\partial H}{\partial q_i} \bigg|_{\gamma}, \frac{\partial H}{\partial q_i} \bigg|_{\gamma}, \frac{\partial H}{\partial p_i} \bigg|_{\gamma}$ and $\frac{\partial H}{\partial p_i} \bigg|_{\gamma}$ denote the values of these partial derivatives on the phase curve $\gamma$ and $F_i(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n) \big|_{\gamma}$ denotes the value of the force $F_i$ on the phase curve $\gamma$. 
In classical mechanics the Hamiltonian $H$ of a conservative mechanical system is mechanical energy and can be written as:

$$H = \int_\gamma \left( \frac{\partial H}{\partial q_i} \right) dq_i + \int_\gamma \left( \frac{\partial H}{\partial p_i} \right) dp_i + \text{const}_1,$$

where $\text{const}_1$ is a constant that depends on the initial condition described above. The mechanical energy $H$ of the system (4) can be evaluated via Eq. (7) too. If $q_i = 0, p_i = 0$, then $\text{const}_1 = 0$. The Einstein summation convention has been used this section. Thus an attempt has been made to find $\hat{H}\Big|_\gamma$ through line integral along the phase curve $\gamma$ of the dissipative system

$$\int_\gamma \left( \frac{\partial \hat{H}}{\partial q_i} \right) dq_i = \int_\gamma \left[ \left( \frac{\partial H}{\partial q_i} \right) - F_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \right] dq_i$$

$$\int_\gamma \left( \frac{\partial \hat{H}}{\partial p_i} \right) dp_i = \int_\gamma \left( \frac{\partial H}{\partial p_i} \right) dp_i.$$  

(8)

Analogous to Eqs. (7), we have

$$\hat{H} = \int_\gamma \left( \frac{\partial \hat{H}}{\partial \hat{q}_i} \right) d\hat{q}_i + \int_\gamma \left( \frac{\partial \hat{H}}{\partial \hat{p}_i} \right) d\hat{p}_i + \text{const}_2,$$

where $\text{const}_2$ is a constant which depends on the initial condition. Substituting Eqs. (7) and Eqs. (8) into Eq. (9), we have

$$\hat{H}\Big|_\gamma = H - \int_\gamma F_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) dq_i + \text{const.}$$

(10)

where $\text{const} = \text{const}_2 - \text{const}_1$, and $H = H\Big|_\gamma$ because $H$ is mechanical energy of the non-conservative system (4). According to the physical meaning of Hamiltonian, $\text{const}_1, \text{const}_2$ and $\text{const}$ are added into Eq. (7), (9), (10) respectively such that the integral constant vanishes in the Hamiltonian quantity. Arnold (1997) had presented the Newton-Laplace principle of determinacy as, 'This principle asserts that the state of a mechanical system at any fixed moment of time uniquely determines all of its (future and past) motion.' In other words, in the phase space the position variable and the velocity variable are determined only by the time $t$. Therefore, we can assume that we have already a solution of Eq. (4)

$$q_i = q_i(t)$$

$$\dot{q}_i = \dot{q}_i(t),$$

(11)
where the solution satisfies the initial condition. We can divide the whole time domain into a group of sufficiently small domains and in these domains $q_i$ is monotone, and hence we can assume an inverse function $t = t(q_i)$. If $t = t(q_i)$ is substituted into the non-conservative force $F_i|_{\gamma}$, we can assume that:

$$F_i(q_1(t(q_i)), \cdots, q_n(t(q_i)), \dot{q}_1(t(q_i)), \cdots, \dot{q}_n(t(q_i)))|_{\gamma} = \mathcal{F}_i(q_i), \quad (12)$$

where $\mathcal{F}_i$ is a function of $q_i$ alone. In Eq. (12) the function $F_i$ is restricted on the curve $\gamma$, such that a new function $\mathcal{F}_i(q_i)$ yields. Thus we have

$$\int_{\gamma} F_i dq_i = \int_{q_{i0}}^{q_i} \mathcal{F}_i(q_i) dq_i = W_i(q_i) - W_i(q_{i0}). \quad (13)$$

According to Eq. (13) the function $\mathcal{F}_i$ is path independent, and therefore $\mathcal{F}_i$ can be regarded as a conservative force. For that Eq. (12) represents an identity map from the non-conservative force $F$ on the curve $\gamma$ to the conservative force $\mathcal{F}_i$ which is distinct from $F_i$. Eq. (12) is tenable only on the phase curve $\gamma$. Consequently the function form of $\mathcal{F}_i$ depends on the aforementioned initial condition; from other initial conditions $\mathcal{F}_i$ with different function forms will yield.

According to the physical meaning of Hamiltonian, $const$ is added to Eq. (10) such that the integral constant vanishes in Hamiltonian quantity. Hence $const = -W_i(q_{i0})$. Substituting Eq. (13) and $const = -W_i(q_{i0})$ into Eq. (10), we have

$$\hat{H}|_{\gamma} = H - W_i(q_i) \quad (14)$$

where $-W_i(q_i)$ denotes the potential of the conservative force $\mathcal{F}_i$ and $W_i(q_i)$ is equal to the sum of the work done by the non-conservative force $F$ and $const$. In Eq. (14) $\hat{H}$ and $H$ are both functions of $q_i$ and $W_i(q_i)$ a function of $q_i$. Eq. (14) and Eq. (10) can be thought of as a map from the total energy of the dissipative system (4) to the Hamiltonian of the conservative system (5). Indeed, $\hat{H}|_{\gamma}$ and the total energy differ in the constant $const = -W_i(q_{i0})$. When the conservative system takes a different initial condition, if one does not change the function form of $\hat{H}|_{\gamma}$, one can consider $\hat{H}|_{\gamma}$ as a Hamiltonian quantity.
\( \dot{H} \),
\[
\dot{H} = \dot{H} \big|_{\gamma} = H - W_i(q_i)
\]  \hspace{1cm} (15)

and the conservative system \( (\dot{H}) \) can be thought of as an entirely new conservative system.

Based on the above, the following proposition is made:

**Proposition II.1.** *For any non-conservative classical mechanical system and any initial condition, there exists a conservative one; the two systems share one and only one common phase curve; the value of the Hamiltonian of the conservative system is equal to the sum of the total energy of the non-conservative system on the aforementioned phase curve and a constant depending on the initial condition.*

**Proof.** First we must prove the first part of the Proposition II.1, i.e. that a conservative system with Hamiltonian presented by Eq. (15) shares a common phase curve with the non-conservative system represented by Eq. (4). In other words the Hamiltonian quantity presented by Eq. (15) satisfies Eq. (6) under the same initial condition. Substituting Eq. (15) into the left side of Eq. (6), we have
\[
\frac{\partial \dot{H}(q_i, p_i)}{\partial q_i} = \frac{\partial H(q_i, p_i)}{\partial q_i} - \frac{\partial W_j(q_j)}{\partial q_i},
\]
\[
\frac{\partial \dot{H}(q_i, p_i)}{\partial p_i} = \frac{\partial H(q_i, p_i)}{\partial p_i} - \frac{\partial W_j(q_j)}{\partial p_i}.
\]  \hspace{1cm} (16)

It must be noted that although \( q_i \) and \( p_i \) are considered as distinct variables in Hamilton’s mechanics, we can consider \( q_i \) and \( \dot{q}_i \) as dependent variables in the process of constructing of \( \dot{H} \). At the trajectory \( \gamma \) we have
\[
\frac{\partial W_j(q_j)}{\partial q_i} = \frac{\partial (\int_{q_i_0}^{q_i} F_j(q_j) dq_j + W_i(q_{i_0}))}{\partial q_i} = F_i(q_i),
\]
\[
\frac{\partial W_j(q_j)}{\partial p_i} = 0,
\]  \hspace{1cm} (17)

where \( F_i(q_i) \) is equal to the damping force \( F_i \) on the phase curve \( \gamma \). Hence under the initial condition \( q_0, p_0 \), Eq. (6) is satisfied. As a result, we can state that the phase curve of Eq. (5) coincides with that of Eq. (4) under the initial condition; and \( \dot{H} \) represented by Eq. (15) is the Hamiltonian of the conservative system represented by Eq. (5).
Then we must prove the second part of Proposition II.1: the uniqueness of the common phase curve.

We assume that Eq. (5) shares two common phase curves, $\gamma_1$ and $\gamma_2$, with Eq. (4). Let a point of $\gamma_1$ at the time $t$ be $z_1$, a point of $\gamma_2$ at the time $t$ be $z_2$, and $g^t$ the Hamiltonian phase flow of Eq. (5). Suppose a domain $\Omega$ at time $t$ which contains only points $z_1$ and $z_2$, and $\Omega$ is not only a subset of the phase space of the non-conservative system (4) but also that of the phase space of the conservative system (5). Hence there exists a phase flow $\hat{g}^t$ composed of $\gamma_1$ and $\gamma_2$, and $\hat{g}^t$ is the phase flow of Eq. (4) restricted by $\Omega$. According to the following Louisville’s theorem in the book of Arnold, 1978:

**Theorem II.1.** The phase flow of Hamilton’s equations preserves volume: for any region $D$ in the phase space we have

$$
\text{volume of } g^t D = \text{volume of } D
$$

where $g^t$ is the one-parameter group of transformations of phase space

$$
g^t : (p(0), q(0)) \rightarrow (p(t), q(t))
$$

$g^t$ preserves the volume of $\Omega$. This implies that the phase flow of Eq. (4) $\hat{g}^t$ preserves the volume of $\Omega$ too. But the system (4) is not conservative, which conflicts with Louisville’s theorem; hence only a phase curve of Eq. (5) coincides with that of Eq. (4).

**B. Obtaining the Equivalent Stiffness Matrix $\tilde{K}$**

According to Proposition II.1 an attempt is made to find a new conservative mechanical system which is corresponding to the dissipative system (2) and an initial condition. Under the initial condition, the dissipative system (2) possesses a phase curve $\gamma$. As in Eq. (12) we can consider that the damping forces are equal to some conservative force on the phase curve.
\[ c_{11} \dot{q}_1 = \varrho_{11}(q_1) \ldots c_{1n} \dot{q}_n = \varrho_{1n}(q_1) \]
\[ \vdots \ldots \vdots \]
\[ c_{n1} \dot{q}_1 = \varrho_{21}(q_n) \ldots c_{nn} \dot{q}_n = \varrho_{nn}(q_n). \]

For convenience, these conservative forces can be thought of as elastic restoring forces:
\[ \varrho_{11}(q_1) = \kappa_{11}(q_1)q_1 \ldots \varrho_{1n}(q_1) = \kappa_{1n}(q_1)q_1 \]
\[ \vdots \ldots \vdots \]
\[ \varrho_{n1}(q_1) = \kappa_{n1}(q_n)q_n \ldots \varrho_{nn}(q_n) = \kappa_{nn}(q_n)q_n. \]

An equivalent stiffness matrix \( \tilde{K} \) is obtained, which is a diagonal matrix
\[ \tilde{K}_{ii} = \sum_{l=1}^{n} \kappa_{il}(q_l). \]

Consequently an \( n \)-dimensional conservative system is obtained
\[ \ddot{q} + (K + \tilde{K})q = 0 \]
which shares the common phase curve \( \gamma \) with the \( n \)-dimensional damping system (2). In this paper, the conservative system is named as substituting conservative system. The Lagrangian of Eqs. (21) is
\[ \dot{L} = \frac{1}{2} \dot{\mathbf{q}}^T \dot{\mathbf{q}} - \frac{1}{2} \mathbf{q}^T K \mathbf{q} - \int_0^\mathbf{q} (\tilde{K} \mathbf{q})^T d\mathbf{q}, \]
and the Hamiltonian of Eqs. (21) is
\[ \dot{H} = \frac{1}{2} \mathbf{p}^T \mathbf{p} + \frac{1}{2} \mathbf{q}^T K \mathbf{q} + \int_0^\mathbf{q} (\tilde{K} \mathbf{q})^T d\mathbf{q}, \]
where \( \mathbf{0} \) is a zero vector, \( \mathbf{p} = \dot{\mathbf{q}} \). \( \dot{H} \) in Eq. (23) is the mechanical energy of the conservative system (21), because \( \int_0^\mathbf{q} (\tilde{K} \mathbf{q})^T d\mathbf{q} \) is a potential function such that \( \dot{H} \) does not depend on any path of Eq. (22).

### III. Definition of a Generalized Hamilton’s Equation

In this section Proposition III.1 would be represented as a uniform infinite-dimensional Hamilton’s equation. IN infinite-dimensional Hamiltonian formalism, techniques of func-
tional derivative must be devoted. Morrison (1998) introduced the definition the functional derivative simply. We would report the introduction.

A. Introduction of Functional Derivative and Canonical Hamiltonian

Description of the Ideal Fluid in Lagrangian variables

Consider a functional $K[u]$. The first change in $K$ induced by $\delta u$ is called the first variation, $\delta K$, and is given by

$$\delta K[u; \delta u] := \lim_{\varepsilon \to 0} \frac{K[u + \varepsilon u] - K[u]}{\varepsilon} = \frac{d}{d\varepsilon} K[u + \varepsilon u]|_{\varepsilon=0} =: \int_{x_0}^{x_1} \delta u \frac{\delta K}{\delta u(x)} =: \langle \delta K, \delta u \rangle$$

The quantity $\delta K/\delta u(x)$ of Eq. (24) is the functional derivative of the functional $K$. Consider a now a more general functional, one of the form

$$\hat{F}[u] = \int_{x_0}^{x_1} \hat{F}(x, u, u_x, u_{xx}, \ldots) dx$$

where $\hat{F}$ is an ordinary, sufficiently differentiable, function of its arguments. Note $u_x = du/dx$, etc. This first variation of Eq. (25) yields

$$\delta \hat{F}[u, \delta u] = \int_{x_0}^{x_1} \left( \frac{\partial \hat{F}}{\partial u} \delta u + \frac{\partial \hat{F}}{\partial u_x} \delta u_x + \frac{\partial \hat{F}}{\partial u_{xx}} \delta u_{xx} + \cdots \right) dx,$$

which upon integration by parts becomes

$$\delta \hat{F}[u, \delta u] = \int_{x_0}^{x_1} \delta u \left( \frac{d}{dx} \frac{\partial \hat{F}}{\partial u} - \frac{d}{dx} \frac{\partial \hat{F}}{\partial u_x} + \frac{d^2}{dx^2} \frac{\partial \hat{F}}{\partial u_{xx}} \delta u_{xx} - \cdots \right) dx$$

$$+ \left( \frac{\partial \hat{F}}{\partial u_x} \delta u + \cdots \right) |_{x_0}^{x_1}.$$
conditions are called natural. Assuming, for one reason or the other, that the boundary term vanishes, Eq. (27) becomes

$$\delta \hat{F}[u; \delta u] = \langle \frac{\partial \hat{F}}{\partial u}, \delta u \rangle,$$  \hspace{1cm} (28)

where the functional derivative

$$\frac{\delta F}{\delta u} = \frac{\partial \hat{F}}{\partial u} - \frac{d}{dx} \frac{\partial \hat{F}}{\partial u_x} + \frac{d^2}{dx^2} \frac{\partial \hat{F}}{\partial u_{xx}} - \ldots.$$  \hspace{1cm} (29)

The main objective of the calculus of variations is the extremization of functionals. A common terminology is to call a function $\hat{u}$, which is a point in the domain, an extremal point if $\delta \hat{F}[u]/\delta u|_{u=\hat{u}} = 0$. It could be a maximum, a minimum, or an inflection point. If the extremal point $\hat{u}$ is a minimum or maximum, then such a point is called an extremum.

An example is the functional defined by evaluating the function $u$ at the point $x$. This can be written as

$$u(x') = \int_{x_0}^{x_1} \delta(x - x')u(x)dx,$$  \hspace{1cm} (30)

where $\delta(x - x')$ is the Dirac delta function and where we have departed from the [] notion. Applying the definition of Eq. (24) yields

$$\frac{\delta u(x')}{\delta u(x)} = \delta(x - x').$$  \hspace{1cm} (31)

This is the infinite-dimensional or continuum analog of $\partial x_i/\partial x_j = \delta_{ij}$, where $\delta_{ij}$ is is the Kronecker delta function. Eq. (30) shows why it is sometimes useful to display the argument of the function in the functional derivative.

The generalizations of the above ideas to functionals of more than one function and to more than a single spatial variable are straightforward. An example is given by the kinetic energy of a three-dimensional compress-ideal fluid,

$$T(\rho, v) = \int_D \frac{1}{2} \rho v^2 d^3x$$  \hspace{1cm} (32)

where the velocity has three rectangular components $\mathbf{v} = \{v_1, v_2, v_3\}$ that depend upon $\mathbf{x} = \{x_1, x_2, x_3\} \in D$ and $v^2 = \mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + x_3^2$. The functional derivatives are

$$\frac{\delta T}{\delta v_i} = \rho v_i, \quad \frac{\delta T}{\delta \rho} = \frac{v^2}{2}.$$  \hspace{1cm} (33)
For a more general functional $\hat{F}[\psi]$, where $\psi(x) = (\psi_1, \psi_2, \cdots, \psi_n)$ and $x = (x_1, x_2, \cdots, x_n)$, the analog of Eq. (24) is

$$\delta \hat{F}[\psi; \delta \psi] = \int_D \delta \psi_i \frac{\delta \hat{F}}{\delta \psi_i(x)} d^n x = \langle \frac{\delta \hat{F}}{\delta \psi}, \delta \psi \rangle.$$  \hfill (34)

Salmon (1988) and Morrison (1998) described the Hamiltonian formalism of ideal fluid in Lagrangian variables in detail. In order to state the infinite-dimensional formalism for dissipative mechanical system, we repeat the representation of Salmon (1988) and Morrison (1998).

In the Hamiltonian description, a fluid is described as a collection of fluid particles or elements. Suppose the coordinate of a fluid particle at time $t$

$$q = q(a, t),$$  \hfill (35)

where $q = \{q_1, q_2, q_3\}, a = \{a_1, a_2, a_3\}$ is the coordinate of the particle at the initial time $t = t_0$. We assume that $a$ varies over a fixed domain $D$, which is completely filled with fluid, and that the functions $q$ map $D$ onto itself.

In Lagrangian variables $a$ the Lagrangian quantity of the fluid particle is considered as Lagrangian density

$$L_f(q, \dot{q}, \partial q/\partial a, t) = \frac{1}{2} \rho_0 \dot{q}^2 - \rho_0 E(s_0, \rho_0/J) - \phi,$$  \hfill (36)

where $\rho_0 = \rho_0(a)$ is a given initial density distribution, $\dot{q}$ is the velocity of the fluid particle, a shorthand $\dot{q}^2 = \delta_{ij} q_i q_j$ is used, $E$ is the energy per unit mass, $s_0$ is the entropy per unit mass at the time $t_0$, $J = \det(\partial q^i/\partial a^j)$, $\phi$ is a potential function for external conservative forces. The intensive quantities, pressure and temperature, are obtained as follows:

$$T = \frac{\partial U}{\partial s}(s, \rho), \quad p = \rho^2 \frac{\partial U}{\partial \rho}(s, \rho)$$  \hfill (37)

Therefore, we have the Lagrangian functional of the fluid particles of the domain $D$:

$$L_f[q, \dot{q}] = \int_D L_f d^3 a = \int_D \left[ \frac{1}{2} \rho_0 \dot{q}^2 - \rho_0 E(s_0, \rho_0/J) - \phi \right] d^3 a,$$  \hfill (38)
where $d^3a = da_1da_2da_3$. Thus the action functional is given by

$$S_f[\mathbf{q}] = \int_{t_0}^{t_1} dt \int_D L_f[\mathbf{q}, \dot{\mathbf{q}}] d^3a = \int_{t_0}^{t_1} dt \int_D \left[ \frac{1}{2} \rho_0 \mathbf{q}^2 - \rho_0 \mathbf{E} - \phi \right] d^3a \quad (39)$$

Observe that this action functional is like that for finite-degree-of-freedom systems, as treated above, except that the sum over particles is replaced by integration over $D$, i.e.,

$$\int_D d^3a \leftrightarrow \sum_i \quad (40)$$

By a Legendre transform, we have a canonical momentum density

$$\varpi_i(\mathbf{a}, t) = \frac{\delta L_f}{\delta \dot{q}_i(\mathbf{a}, t)} = \rho_0 \dot{q}_i, \quad (41)$$

and a generalized Hamiltonian quantity

$$H_f[\mathbf{q}, \varpi] = \int_D [\varpi \dot{\mathbf{q}} - L_f] d^3a = \int_D \left[ \frac{\varpi^2}{2\rho_0} + \mathbf{E} + \phi \right] d^3a, \quad (42)$$

where $\rho_0 \dot{q}_i^2/2 + E + \phi = \mathcal{H}_f$ can be consider as a Hamiltonian density. A generalized Hamilton’s equation is

$$\dot{\varpi}_i = -\frac{\delta H_f}{\delta q_i}, \quad \dot{q}_i = \frac{\delta H_f}{\delta \varpi_i}. \quad (43)$$

These equations can also be written in terms of the Poisson bracket (see Morrison (1998)),

$$\{F, G\} = \int_D \left[ \frac{\delta F}{\delta q} \cdot \frac{\delta G}{\delta \varpi} - \frac{\delta G}{\delta q} \cdot \frac{\delta F}{\delta \varpi} \right] d^3a \quad (44)$$

eviz.,

$$\varpi_i = \{ \varpi_i, H_f \}, \quad \dot{q}_i = \{ q_i, H_f \} \quad (45)$$

Here $\delta q_i(\mathbf{a})/\delta q_j(\mathbf{a}') = \delta_{ij}\delta(\mathbf{a} - \mathbf{a}')$ has been used, where $\delta(\mathbf{a} - \mathbf{a}')$ is a three-dimensional Dirac delta function(recall Eq. (31)).

**B. Derivation of Hamiltonian Description of Dissipative Mechanical Systems**

The Hamiltonian description of the ideal fluid is infinite-dimensional, and the Hamiltonian quantity and Lagranian is the integrals over the domain $D$ in the initial configuration.
In addition, Morrison (1980) proposed the Hamiltonian description of Poisson-Vlasov equations with Hamiltonian quantity, which is an integral over the phase space. These ideas of Salmon (1988), Morrison (1998) and Morrison (1980) motivate us to consider the mechanical system \( (2) \) as a special fluid which is a collection of fluid particles in the phase space. In general case Hamilton’s quantity is an energy function. Although the total energy of the oscillator with damping is conservative, the total energy depends on the initial condition. Consequently there is a path-dependency problem. It is well known that the energy per unit mass \( E \) is the origin of the pressure in the fluid. The mechanical system \( (2) \) describes that a particle moves in the configuration space. One can also consider that individual particles of the special fluid moves without interaction. Therefore, one can assume that no internal energy function \( E \) exists in the Lagrangian density of the system \( (2) \); the Lagrangian variable of the special fluid in a fixed domain \( D \) is

\[
\mathbf{a} = (\mathbf{q}_0, \dot{\mathbf{q}}_0) = (q_0^1, \ldots, q_0^n, \dot{q}_0^1, \ldots, \dot{q}_0^n)
\]

(46)

; the coordinate of a particle in the configuration space is

\[
\mathbf{q} = \mathbf{q}(\mathbf{a}, t) = (q_1(\mathbf{a}, t), \ldots, q_n(\mathbf{a}, t));
\]

(47)

\( \rho_0 = 1 \). By comparing the generalized Hamilton’s equation \( (13) \) and Hamilton’s equation in odd dimensional phase space, one can find that the Hamiltonian density does not need to satisfy the path indepency requirement fully, according to Eq. \( (29) \) we have

\[
\dot{\mathbf{q}}_i(\mathbf{a}) = -\frac{\delta H_f}{\delta q_i(\mathbf{a})} = -\frac{\partial H_f}{\partial q_i(\mathbf{a})},
\]

where \( q_i(\mathbf{a}) \) is the value of \( q_i \) on the path of the particle \( \mathbf{a} \) in the configuration space. Therefore, analogous to Eq. \( (36) \), one can consider \( \hat{L} \) in Eq. \( (22) \) as a Lagrangian density of the system \( (2) \)

\[
\mathcal{L} = \hat{L} = \frac{1}{2} \dot{\mathbf{q}}^T \dot{\mathbf{q}} - \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q} - \int_0^q (\mathbf{K} \mathbf{q})^T d\mathbf{q},
\]

(48)

and consider \( \hat{H} \) in Eq. \( (23) \) as a Hamiltonian density of the system \( (2) \)

\[
\mathcal{H} = \hat{H} = \frac{1}{2} \mathbf{p}^T \mathbf{p} + \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q} + \int_0^q (\mathbf{K} \mathbf{q})^T d\mathbf{q},
\]

(49)
where \( q_i(a) \) is the value of \( q_i \) on the path of the particle \( a \) in the phase space, such that one can avoid the afore-mentioned path-dependency problem. Thus the Lagrangian functional of Eq. (2) can be presented as following:

\[
L[q, \dot{q}] = \int_D \left[ \frac{1}{2} \dot{q}^T \dot{q} - \frac{1}{2} q^T K q - \int_0^q (\tilde{K} q)^T d q \right] d^{2n} a,
\]

(50)

where \( d^{2n} a = d_n q_0 d_n \dot{q}_0 = d_n q_1 d_n \dot{q}_0 \ldots d_n q_n d_n \dot{q}_0 \). The Lagrangian functional Thus the action functional can be presented as following:

\[
S[q] = \int_{t_0}^{t_1} L[q, \dot{q}] dt = \int_{t_0}^{t_1} dt \int_D \left[ \frac{1}{2} \dot{q}^T \dot{q} - \frac{1}{2} q^T K q - \int_0^q (\tilde{K} q)^T d q \right] d^{2n} a
\]

(51)

According to Hamiltonian theorem, we have the functional derivatives \( \delta S/\delta q(a, t) = 0 \), according to the generalization Eq. (29):

\[
\frac{\delta S}{\delta q(a, t)} = \frac{\partial L}{\partial q(a, t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}(a, t)} = -\ddot{q}(a, t) - kq - \tilde{K} q = 0
\]

(52)

The equation above implies that under the initial condition \( a \) a conservative system exists, the control equation of which is Eq. (21), the phase curve of which coincides with that of the oscillator with damping. Define a canonical momentum density for the dissipative system (2) is

\[
\pi_i(a, t) = \frac{\delta L}{\delta \dot{q}_i(a)} = \dot{q}_i
\]

(53)

which is a functional derivative, while classical canonical momentum is defined as a partial derivative. By a Legendre transform, we have the generalized Hamiltonian \( \hat{K} \) is

\[
\hat{K}[\pi, q] = \int_D d^{2n} a \left[ \pi \cdot \dot{q} - \mathcal{L} \right] = \int_D d^{2n} a \left[ \frac{1}{2} q^T p + \frac{1}{2} q^T \tilde{K} q + \int_0^q (\tilde{K} q)^T d q \right],
\]

(54)

where \( q = q(a, t) \). Thus the generalized Hamilton’s equations of the dissipative system (2) are

\[
\dot{\pi}_i = -\delta \hat{K}/\delta q_i, \quad \dot{q}_i = \delta \hat{K}/\delta \pi_i
\]

(55)

**Definition III.1.** For two functionals \( F[\pi(a), q(a)] \) and \( G[\pi(a), q(a)] \) in a domain \( D \) of the phase space exists a functional

\[
\{F, G\}[\pi(a), q(a)] = \int_D \left[ \frac{\delta F}{\delta q(a')} \cdot \frac{\delta G}{\delta \pi(a')} - \frac{\delta G}{\delta q(a')} \cdot \frac{\delta F}{\delta \pi(a')} \right] d^{2n} a,
\]

(56)
where the functional derivative $\delta F / \delta q^{(a')}$ is defined analogues to Eq. (24) and Eq. (34) as:

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} [q^{(a')} + \varepsilon \delta q^{(a')}, \pi^{(a')}] = \int_D \frac{\delta F}{\delta q^{(a')}} d^2 n a
\]

The Hamilton’s equations can also be represented in terms of the Poisson bracket (56) viz.,

\[
\dot{\pi}_i = \{\pi_i, \hat{K}\}, \dot{q}_i = \{q_i, \hat{K}\}. \tag{57}
\]

Expand $\{\pi_i, \hat{K}\}$, we have

\[
\{\pi_i(a), \hat{K}\} = \frac{\delta \pi_i(a)}{\delta q_j^{(a')}} \frac{\delta \hat{K}}{\delta \pi_j^{(a')}} - \frac{\delta \pi_i(a)}{\delta \pi_j^{(a')}} \frac{\delta \hat{K}}{\delta q_j^{(a')}} = -\delta_{ij} \delta(a - a') \frac{\delta \hat{K}}{\delta q_j^{(a')}} = -\frac{\delta \hat{K}}{\delta q_i(a)}, \tag{58}
\]

Here $\delta q_i(a) / \delta q_j^{(a')} = \delta_{ij} \delta(a - a')$ has been used, where $\delta(a - a')$ is a three-dimensional Dirac delta function (recall Eq. (31)). Analogous to Eq. (40), we have

\[
\int_D d^2 n a \leftrightarrow \sum_i, \hat{K} = \sum_i \mathcal{H}(a) \tag{59}
\]

According to Eq. (59), from Eq. (58) we can derive

\[
\dot{\pi}_i(a) = \{\pi_i(a), \hat{K}\} = -\frac{\delta \hat{K}}{\delta q_i(a)} = -\frac{\partial \mathcal{H}(a)}{\partial q_i(a)} \tag{60}
\]

In the similar way

\[
\dot{q}_i(a) = \{q_i(a), \hat{K}\} = \frac{\delta \hat{K}}{\delta \pi_i(a)} = \frac{\partial \mathcal{H}(a)}{\partial \pi_i(a)} \tag{61}
\]

Therefore, we can assert that Eq. (60) and Eq. (61) describes a phase curve which is a common phase curve of the dissipative system and a conservative system under the initial condition $a$. 
From the Hamilton’s equation (55), we can derive the total energy conservative principle

\[ \delta K = \int_D \left[ \frac{\delta \dot{K}}{\delta q_i(a)} \delta q_i(a) + \frac{\delta \dot{K}}{\delta \pi_i(a)} \delta \pi_i(a) \right] d^2n a \]

\[ = \int_D \left[ \frac{\delta \dot{K}}{\delta q_i(a)} \frac{dq_i(a)}{dt} + \frac{\delta \dot{K}}{\delta \pi_i(a)} \frac{d\pi_i(a)}{dt} \right] d^2n a \]

\[ = \int_D \left[ \frac{\delta \dot{K}}{\delta q_i(a)} \frac{\delta \dot{K}}{\delta \pi_i(a)} dt - \frac{\delta \dot{K}}{\delta \pi_i(a)} \frac{\delta \dot{K}}{\delta q_i(a)} dt \right] d^2n a \]

\[ = 0 \]

**IV. CONCLUSION**

The following conclusions can be drawn. The infinite-dimensional description (53), (54), (55), (56), (57) can describe a dissipative mechanical system based on the proposition II.1: For any non-conservative classical mechanical system and any initial condition, there exists a conservative one; the two systems share one and only one common phase curve; the value of the Hamiltonian of the conservative system is equal to the sum of the total energy of the non-conservative system on the aforementioned phase curve and a constant depending on the initial condition. In fact, if the generalized Hamilton’s equation (55) and (57) is constrained at a initial condition \( a \), the generalized Hamilton’s equation is a phase curve of the afore-mentioned conservative system (21). As the classical Hamilton’s equation represents the conservation of mechanical energy principle, the generalized Hamilton’s equation (55, 57) describes the conservation of total energy principle. One can assert that the generalized Hamilton’s equation (55, 57) are the generalization of the classic Hamilton’s equations.

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