Rethinking Randomized Smoothing for Adversarial Robustness

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Abstract
The fragility of modern machine learning models has drawn a considerable amount of attention from both academia and the public. While immense interests were in either crafting adversarial attacks as a way to measure the robustness of neural networks or devising worst-case analytical robustness verification with guarantees, few methods could enjoy both scalability and robustness guarantees at the same time. As an alternative to these attempts, randomized smoothing adopts a different prediction rule that enables statistical robustness arguments and can scale to large networks. However, in this paper, we point out for the first time the side effects of current randomized smoothing workflows. Specifically, we articulate and prove two major points: 1) the decision boundaries shrink with the adoption of randomized smoothing prediction rule; 2) noise augmentation does not necessarily resolve the shrinking issue and can even create additional issues.

1. Introduction
The vulnerability of deep neural networks to human-imperceptible adversarial perturbations has attracted great attention within the machine learning community since the seminal works (Szegedy et al., 2014; Biggio et al., 2013). This has remained an important concern for various machine learning fields, ranging for instance from computer vision (Szegedy et al., 2014) to speech recognition (Carlini & Wagner, 2018). In particular, for safety-critical applications, such as self-driving cars and surveillance, there is almost zero tolerance for erroneous decisions. Hence the machine learning model is expected to be worst-case-robust to all kinds of noises. As a result, the existence of adversarial examples in deep neural networks have motivated efforts toward neural network robustness quantification, as well as toward designing of training algorithms that can enhance such robustness (Elsayed et al., 2018; Eykholt et al., 2018; Kurakin et al., 2017).

To date, there are two popular ways to approach the problem of robustness evaluation: 1) attack evaluation and 2) formal verification. From the attack perspective, the adversary would like to develop strong adversarial attacks that are able to fool the network classifier with the smallest adversarial distortions (Carlini & Wagner, 2017; Goodfellow et al., 2015; Moosavi-Dezfooli et al., 2016; Chen et al., 2018). Whereas the purpose of formal verification methods is to guarantee that intrinsic robustness conditions will always hold. For example, one key goal, within robustness verification, is to show that no adversarial examples can ever exist within an \( \mu \)-neighborhood of the original test sample. Furthermore, ideally the formal verification algorithms should identify the largest possible \( \mu \). As a result of the shared concern of unverified models in real-life deployment, focus has shifted to seek trust-worthy and attack-agnostic robustness verification (Hein & Andriushchenko, 2017; Weng et al., 2018a; Singh et al., 2018; Zhang et al., 2018; Jordan et al., 2019). However, due to the intrinsic hardness (NP-completeness) of robustness verification problem, these certifiable verification methodologies do not scale to large networks. To cope with this, one emerging branch of studies, randomized smoothing (Cohen et al., 2019; Lecuyer et al., 2019; Li et al., 2019), proposes transforming the original classifier into a “smoothed” counterpart. This new counterpart now returns the class with the highest probability by querying isotropic Gaussian noise \( N(0, \sigma^2 I) \) corrupted data. This corresponds to applying low-pass filters (cf. GaussWeierstrass transform, Gaussian blur or Gaussian filter in signal processing.) to score functions.

Nevertheless, while converting the original classifier to a randomized smoothing classifier allows a more verifiable robustness verification for deep neural networks on large scale datasets (e.g. ImageNet), one potential problem is the trade-off of clean accuracy. We exemplify this by showing in Figure 1 that a decision-region-shrinkage issue could potentially exist. Specifically, we show that the decision regions of class 1 data shrink when the smoothing factor \( \sigma \) increases (i.e. standard deviation in the Gaussian filter), and thus the accuracy of class 1 data could eventually drop. Meanwhile, while the certified radius \( \mu \) of the smoothed

\[ \text{Figure 1: Decision Region Shrinkage Issue} \]

[Image: Figure1.png]

\[ \text{Figure 2: Randomized Smoothing Augmentation} \]

[Image: Figure2.png]

\[ \text{Figure 3: Gaussian Smoothing Example} \]

[Image: Figure3.png]
Rethinking Randomized Smoothing for Adversarial Robustness

Figure 1. An example of **bounded** decision boundary in Sec 3.2.

Figure 2. An example of **semi-bounded** decision boundary in Sec 3.3.

The decision boundary of the randomized smoothed classifier shrinks as the smoothing factor $\sigma$ increases. In the case of Fig. 1(d), 2(d) with large $\sigma$, the decision region has shrunk so much thus resulting in mis-classification (i.e. the classifier accuracy decreases). We also plot the certified radius (Equation 3) of point $A$ and $B$ and show that it may decrease as $\sigma$ increases.

**Contributions.** In this paper, we study Gaussian smoothing both theoretically and numerically. For an easier reference, two summarizing tables of our main contributions can be found as Table 1 and Table 2. To the best of our knowledge, we provide the first theoretical result showing the limitation of randomized smoothing in terms of the classification accuracy. Specifically,

1. We provide theoretical characterization and identify sufficient conditions under which Gaussian smoothing leads to a decrease in classification accuracy;

2. For the cases where Gaussian smoothing causes a decrease in accuracy, we provide theoretical lower bounds for the magnitude of the effect, and inspect numerically the behavior of the certified radius;

3. We use tools from information theory to analyze the effect of Gaussian smoothing during training (cf. data augmentation in (Cohen et al., 2019)) and conclude that it can lead to a loss in information;

4. We validate our information theoretic results in terms of classification accuracy and show that Gaussian smoothing during training may leads to low classification accuracy for large $\sigma$ on both synthetic and real datasets.

### 2. Background

The standard prediction rule of a classifier is to predict the class of an input example $x_0$ by taking the highest output of the score function (a neural network) $g(x)$:

$$\xi_A = \arg\max_j g_j(x_0)$$ (1)

Different from the standard prediction rule, randomized smoothing aggregated the results of sample points around the given input example $x_0$ to give the final prediction – i.e. adding isotropic Gaussian noises $\mathcal{N}(0, \sigma^2 I)$ to the given input example $x_0$ and take the highest probability class as the prediction:

$$\xi_A = \max_j \mathbb{P}[j = \arg\max_i g_i(x)], x \sim \mathcal{N}(x_0, \sigma^2 I).$$ (2)
Note that Equations (1) and (2) are referred as base classifier and smoothed classifier respectively. There has been many research efforts developing robustness verification techniques for the base classifier (Hein & Andriushchenko, 2017; Gehr et al., 2018; Raghunathan et al., 2018; Weng et al., 2018a; Weng et al., 2018b; Wong & Kolter, 2018)), and the main goal is to solve the following problem: given \( g, x_0, \xi_A \) and \( p \), solve

\[
\max \mu \text{ s.t. } \arg \max_j g_j(x_0 + \delta) = \xi_A, \forall ||\delta||_p \leq \mu,
\]

However, due to the intrinsic hardness of the problem (Katz et al., 2017; Weng et al., 2018a), the above approaches can hardly scale to state-of-the-art deep neural networks such as ResNet-50 and VGG-19 nets. On the other hand, it is also possible to perform robustness verification on the smoothed classifier and the problem is formulated as:

\[
\max \mu \text{ s.t. } \arg \max_j P[g_j(x_0 + \delta) = \xi_A, \forall ||\delta||_p \leq \mu,
\]

In (Lecuyer et al., 2019), the authors used the techniques in differential privacy to derive a lower bound on \( \mu > 0 \) for \( p = 1, 2 \) and the bound is further improved by (Li et al., 2019) via the tools in information theory for \( p = 2 \). Recently, (Cohen et al., 2019) prove a tighter bound of \( \mu \) for \( p = 2 \) in the following:

\[
\mu = \frac{\sigma}{2} (\Phi^{-1}(p_A) - \Phi^{-1}(\overline{p_B})),
\]

where \( \sigma \) is the smoothing factor in the Gaussian noise, \( \Phi^{-1} \) is the inverse of standard Gaussian CDF, and \( p_A \) and \( \overline{p_B} \) are the lower/upper bound on the probability with class \( \xi_A \) and \( \xi_B \) respectively, where \( \xi_A \) is the most probable class in the smoothing classifier and \( \xi_B \) is the “runner-up” class. It has been shown promising in (Cohen et al., 2019) that the verification algorithm is more scalable to large ImageNet networks with probability certificate (as the probability \( p_A \) and \( \overline{p_B} \) cannot be computed analytically and thus Monte-Carlo for estimation is applied). In practice, (Cohen et al., 2019) set \( \overline{p_B} = 1 - p_A \) and abstain when \( p_A < 0.5 \), implying that no radius can be certified in this case.

### 3. Theoretical Characterization:

#### Randomized Smoothing and Accuracy

In this section, we study the effect of lowering classification accuracy with increased randomized smoothing factor \( \sigma \). Specifically, the degradation in accuracy reflects as a rather high loss \( l(E_{z \sim N(0, \sigma^2 I)}[f(x + z), h(x)]) \), where \( l \) is the loss function, \( f(x) = 1_{c = \arg \max g(x)} \) is an indicator function of top 1 class of the base classifier, and \( h(\cdot) \) is the ground truth. We argue generally that the smoothed predictor \( f_{\sigma}(x) = E_{z \sim N(0, \sigma^2 I)}[f(x + z)] \) has an overall “drifted” output, which renders an increased difference with \( h(x) \). Before introducing our formal analysis, we will start the section by giving some necessary preliminaries in Section 3.1. Following that, we categorized network decision regions by bounded and semi-bounded and sum up discussions in Section 3.2 and Section 3.3, respectively. We refer readers to corresponding sections or the supplementary materials for their formal definitions. Importantly, the proofs of results in Section 3.1, Section 3.2, and Section 3.3 are included in Section B1, Section B2, and Section B3 in the supplementary materials, respectively.

#### 3.1. Preliminaries

In a multi-class classification problem with \( c \) classes, we set our goal based on the definitions below.

**Definition 1** (Smoothed). If we use \( f \) to denote an original neural network function with outputs in the simplex \( \Delta^c \), \( f_{\sigma} \) then its smoothed counterpart defined on \( d \)-dimensional inputs \( x \in \mathbb{R}^d \) is defined by

\[
f_{\text{smooth}}(x) = \int_{x' \in \mathbb{R}^d} f(x')p(x')dx',
\]

where \( p(x') \) is the probability density function of the filter.

**Definition 2** (Gaussian smoothing). If \( p(x') \) is the probability density function of a normally-distributed random variable with an expected value \( x \) and standard deviation \( \sigma \), then we call \( f_{\text{smooth}} \) a Gaussian-smoothed function and denote it by \( f_{\sigma} \).

\[\Delta^c = \{ z \in \mathbb{R}^c \mid \sum_{i=1}^cz_i = 1, 0 \leq z_i \leq 1, \forall i \}\]
Problem Reductions: We notice from the definition of Gaussian smoothing that the smoothed function depends on the base classifier only through the indicator function \( f(x) \). Thus, the smoothed function only depends on the partitioning of the input space created by the base classifier. Therefore we will shift our focus from the output of \( f \) to how it partitions the input space. Specially, we are interested in characterizing all possible partitions of the input space that can lead to decrease in accuracy as one applies Gaussian smoothing with a high \( \sigma \). And as the above makes the problem geometric, we subsequently recast our idea of decrease in accuracy to the mismatch partitions of input space of \( f \) and \( f_\sigma \).

However, the problem of characterizing the partitions of the space into multiple classes is intractable. So we instead focus on tracking the behaviour of the decision boundary of a single class (class 1) with respect to Gaussian smoothing. In this case, we analyze the misclassification rate for class 1 by the region size of the input space that is partitioned as \( f \). We obtain

\[
\text{Therefore we will shift our focus from the output of } f \text{ to how it partitions the input space.}
\]

3.2. Bounded Decision Region

In this section, we discuss the case when the decision region is bounded. On the whole, we aim at proving the shrinking side-effects incurred by the smoothing filter to the decision region. Formally, we say a decision region is bounded and shrinks according to the following definition:

**Definition 3 (Bounded Decision Regions).** If the decision region\(^3\) of class 1 data is a bounded set in the Euclidean space (can be bounded by a ball of finite radius), then we call these decision regions bounded decision regions.

We denote the smallest ball that contains the original decision region of \( f \) by \( S_{D} (D \subseteq S_{D}) \). Similarly, we let the smallest ball that contains the smoothed decision region (the decision region of smoothed classifier) be \( S_{D_\sigma} (D_\sigma \subseteq S_{D_\sigma}) \). We remark that these smallest balls have the radii equal to the radius of the decision region\(^4\).

**Definition 4 (Shrinking of Bounded Decision Regions).** A bounded decision region is distinguished as shrunk after applying smoothing filters if the radius \( R_{\sigma} \) of \( S_{D_\sigma} \) is rigorously smaller than the radius \( R \) of \( S_{D} \), i.e. \( R_{\sigma} < R \), where \( S_{D} \) and \( S_{D_\sigma} \) are the smallest balls containing the original decision region and the smoothed decision region, respectively.

To prove the bounded decision region shrinks with Gaussian smoothing, we start by noticing the following corollary

**Corollary 1.** The smallest ball \( S_{D_\sigma} \) containing the smoothed decision region is contained within the smoothed version of \( S_{D} \), i.e. \( S_{D_\sigma} \subseteq (S_{D})_{\sigma} \).

**Theorem 1.** A bounded decision region shrinks after applying Gaussian smoothing filters with large \( \sigma \). Specifically, if \( \sigma > \frac{R\sqrt{\pi}}{\sqrt{2(d-1)}} \), then \( R_{\sigma} < R \) (cf. Definition 4).

Analysis of bounded decision regions with Gaussian smoothing. As we have proven that any bounded decision region shrinks after applying Gaussian smoothing filters, we will investigate in this part of the paper how fast the decision region (quantified by \( R_{\sigma} \)) shrinks/vanishes and how big the certified radius (\( \mu \)) can be. Specially, from Corollary 1, we have that the smallest ball \( S_{D_\sigma} \) containing the smoothed decision region is contained within the smoothed version of \( S_{D} \). Therefore we only consider the worst case when we have a ball-like decision region. Without loss of generality, we consider a case when the decision region of class 1 data characterized by the network function is exactly \( \{x \in \mathbb{R}^d \mid \|x\|_2 \leq R \} \).

Before giving our results, we acknowledge a few facts listed below:

\(^3\)Can be a disconnected or connected set.

\(^4\)The radius of a set \( S \subseteq \mathbb{R}^d \) is defined by \( \min_{x \in S} \max_{x' \in S} \|x - x'\|_2 \).

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\( \text{Rethinking Randomized Smoothing for Adversarial Robustness} \)

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\( \text{w/o loss of generality, we set the concerned class as class 1.} \)
We validate Theorem 2 for binary classification. Similar to that of $d = 20$ and the decision region in a $d = 100$ case vanishes as early as $\sigma = 0.099$. For class 1 data, the point at the origin has less than 0.5 probability to be classified as class 1. That is, $
abla f_1 = 0$ if the point at the origin has less than 0.5 probability to be classified as class 1.

For class 1 data, the point at the origin has less than 0.5 probability to be classified as class 1.

Remark. For the certifiability, the effective number of classes is 2 as (Cohen et al., 2019) treats it as a one vs all setting. Thus, the upper bound of vanishing smoothing factor for certifiability is $\sqrt{\frac{2}{d}} R \leq \frac{1}{\sqrt{2}}$ for $R = \frac{\sqrt{d}}{2}$. Therefore we see that if all the points originally classified as class 1 are inside the unit cube, then the certifiable region vanishes for $\sigma \geq \frac{1}{\sqrt{2}}$, regardless of the input-space dimension $d$.

Remark. In a multi-class case, as mentioned in the above remark, the certifiability and prediction do not follow the same setting in (Cohen et al., 2019). Therefore one would be unable to certify any radius with some smoothing factor $\sigma < \sigma_{\text{van}}$ in the multi-class case.

3.3. Semi-bounded Decision Region

In this section, we discuss the case when the decision region is semi-bounded and is not a half-space. Formally, we say a decision region is semi-bounded and shrinks according to the following definition:

Definition 5 (Unbounded Decision Regions). If for any ball there exists at least one point in the decision regions that reside outside the ball, then we call these decision regions unbounded decision regions.
**Definition 6** (Semi-bounded Decision Regions). For an unbounded decision region, if there exists any half-space \( \mathcal{H} \) (decided by a hyperplane) that contains the unbounded decision region, then we call it semi-bounded decision region. We say a semi-bounded decision region is bounded in \( v \)-direction if there exists any half-space \( \mathcal{H}_v \) denoting three different classes’ data and their decision regions, we demonstrate in the following that any “narrow” decision region is not a half-space, it will shrink.

An illustrative example of semi-bounded decision regions is shown as Figure 2, where we have 3 clusters of data points denoting three different classes’ data and their decision regions. Here we specifically concern the decision region of class 1. From Figure 2, one can see that the Gaussian-smoothed decision region of class 1 is also semi-bounded. We give the formal statement as the following corollary and its proof is given in the appendix.

**Corollary 2.** For a given semi-bounded decision region \( \mathcal{D} \) bounded in direction \( v \), we have \( \mathcal{D}_v \) is also a semi-bounded decision region in the \( v \) direction.

By Corollary 2, for a given semi-bounded decision region \( \mathcal{D} \) bounded in direction \( v \), we have \( \max_{x \in \mathcal{D}} v^T x \) and \( \max_{x \in \mathcal{D}_v} v^T x \) are finite numbers. We denote them by \( \Upsilon_{\mathcal{D}} \) and \( \Upsilon_{\mathcal{D}_v} \), respectively.

**Definition 7** (Shrinking of Semi-bounded Decision Regions). A semi-bounded decision region bounded in \( v \)-direction is distinguished as shrinked along the direction after applying smoothing filters if the upper bound of projections of the decision region onto direction \( v \) shrinks, i.e. \( \Upsilon_{\mathcal{D}_v} < \Upsilon_{\mathcal{D}} \), where \( \Upsilon_{\mathcal{D}} = \max_{x \in \mathcal{D}} v^T x \), \( \Upsilon_{\mathcal{D}_v} = \max_{x \in \mathcal{D}_v} v^T x \).

With the definition of shrinking of semi-bounded decision regions, we demonstrate in the following that any “narrow” semi-bounded decision region bounded in \( v \)-dimension will shrink along the direction (cf. Figure 2(b-d)). We quantify the size of a decision region as follows:

**Definition 8** (\( \theta, v \)-Bounding Cone for a Decision Region). A \( \theta, v \) cone is defined as a right circular cone \( C \) with axis along \( -v \) and aperture \( 20 \). Then we define the \( \theta, v \)-bounding cone \( C_{\theta,v} \) for \( \mathcal{D} \) as the \( \theta, v \) cone that has the smallest projection on \( v \) and contains \( \mathcal{D} \), i.e., \( C_{\theta,v} = \arg \min_{\mathcal{D} \subseteq C_{\theta,v}, \mathcal{D}} \Upsilon_{C_{\theta,v}} \).

**Corollary 3.** As \( \mathcal{D} \subseteq (C_{\theta,v}) \), using Lemma 1, we have that the smoothed decision region is contained within the smallest version of \( C_{\theta,v} \), i.e. \( \mathcal{D}_v \subseteq (C_{\theta,v})_\sigma \).

**Theorem 3.** A semi-bounded decision region that has a narrow bounding cone shrinks along \( v \)-direction after applying Gaussian smoothing filters with high \( \sigma \), i.e. if the region admits a bounding cone \( C_{\theta,v} \) with \( \tan(\theta) < \sqrt{\frac{2(d-1)}{c \log(c-1)}} \), then for \( \sigma > (\Upsilon_{C_{\theta,v}} - \Upsilon_{C_{\theta,v}}) \tan(\theta) \sqrt{\frac{d}{d-1}} \), \( d - 1 \), \( (d-1)^{-2} \tan^2(\theta)c \log(c-1) \), we have \( \Upsilon_{\mathcal{D}_v} < \Upsilon_{\mathcal{D}} \) (cf. Definition 7).

Concretely, the narrowness condition of the cone will be as relaxed as 0.43\( \pi = 76.7^\circ \) for MNIST dataset (LeCun, 1998), meaning that if any single class’s decision region can be bounded by a \( \theta, v \) cone with \( \theta \) being less than 76.7\(^\circ\), then shrinking effect happens. Correspondingly, this narrowness condition exists as 0.46\( \pi = 83.2^\circ \) for CIFAR10 dataset (LeCun, 1998) and 0.42\( \pi = 75.2^\circ \) for ImageNet dataset (Russakovsky et al., 2015).

**Remark.** For binary classification tasks \( (c = 2) \), according to Theorem 3, the condition for shrinking reduces to \( \tan(\theta) < \infty \) that implies \( \theta < \pi/2 \). In other words, when there are only two classes, as long as the semi-decision region is not a half-space, it will shrink.

**Analysis of the semi-bounded case with Gaussian smoothing.** As in Section 3.2, we conduct the analysis using the worst-case ball-like bounded decision region, here we correspondingly consider a solid right circular cone along the \( v \) direction. The shrinkage in this case serves as a non-trivial lower bound.

Without loss of generality, we consider a \( \theta, v \) solid right circular cone \( \{ x \in \mathbb{R}^d \mid v^T x - ||v|| \leq \cos(\theta) \leq 0 \} \) as the decision region \( \mathcal{D} \) of class 1 data, where \( -v = [0, \ldots, 0, 1]^T \in \mathbb{R}^d \).

We admit similar facts as in Section 3.2 except that 1) for class 1 data \( x \), the point on the axis (along direction \( v \)) has the highest probability to be classified as class 1; and 2) by definition, the semi-bounded decision region is unbounded and will shrink but will not vanish. Therefore we emphasize in this section only on giving the shrinking rate with respect to the smoothing factor \( \sigma \), the number of classes \( c \), the angle \( \theta \), and the data dimension \( d \) with Gaussian smoothing. A complementary study on the certified radius is also provided.

Two major theorems regarding the shrinking rate in the solid cone-like decision region are:

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6The larger the easier to fulfill.
7The decision regions in other semi-bounded cases will shrink more than our results herein.
Theorem 4. The shrinkage of class 1 decision region is proportional to the smoothing factor, i.e. $\mathcal{Y}_{\mathcal{D}_0}^c - \mathcal{Y}_{\mathcal{D}_0}^p \propto \sigma$.

With the above Theorem 4, we can fix the smoothing factor to $\sigma = 1$ and further obtain a lower bound of the shrinking rate w.r.t $c$, $\theta$, and $d$:

Theorem 5. The shrinking rate of class 1 decision region is at least
\[
\frac{\mathcal{Y}_{\mathcal{D}_0}^c - \mathcal{Y}_{\mathcal{D}_0}^p}{\sigma} > \sqrt{\frac{d-1}{c \tan^2(\theta)}}, \quad (d-1) - 2 \tan^2(\theta) c \log(c-1) \quad (d-1).\]

Since we have provided a theoretical lower bound for the shrinking rate (cf. Theorem 5), we subsequently study herein the certified radius profile of a given point on the axis as a function of narrowness $\theta$ and dimension $d$ by analyzing a binary classification ($c = 2$) case. In the specified solid cone, we consider the point on the axis and has a unit length to the origin ($x_0 = [0, \ldots, 0, 1]$). Acknowledging that the minimum distance from $x_0$ to $\theta, \psi$ cones is $\sin(\theta)$, we show in Figure 5 the scaled certified radius $\frac{\mathcal{Y}_{\mathcal{D}_0}^c}{\sin(\theta)}$ as a function of an increasing smoothing factor $\sigma$ with various levels of cone narrowness when $d = 25$. From Figure 5, one can readily verify that overall the peak scaled certified radius decreases with $\psi$, e.g. the scaled certified radius at $x_0$ can be as large as 0.84 when $\theta = 80^\circ$, while it is at most 0.49 when $\theta = 10^\circ$. Moreover, we point out that the certified radius will drop to zero when we keep increasing the smoothing factor $\sigma$ - the “narrower” (smaller $\theta$) the decision region is, the faster it will drop to zero. We discuss the effect of the input data dimension $d$ on the certified radius in the supplementary materials.

4. Can Gaussian Smoothing during Training Help Improve Accuracy?

As Section 3 justifies that the degraded accuracy in Gaussian smoothing is a result of the drifting decision region, we investigate in this section from several perspectives whether the state-of-the-art workflow to deal with the degraded accuracy can effectively solve this issue. Notably, the seminal work in Gaussian smoothing (Lecuyer et al., 2019; Cohen et al., 2019) suggests to apply Gaussian smoothing during training (cf. referred as data augmentation in the literature), which essentially reduces to learning a distribution from samples corrupted with noises.

Initially, authors of (Cohen et al., 2019) justify the use of Gaussian smoothing during training from the perspective of risks. Formally, let $X$ be the input space, $Y$ be the output space, $l$ be the loss function, $f$ be a neural network, and $h$ be the ground-truth classifier, the risk in the original learning problem is defined as
\[
\mathcal{R} = \mathbb{E}_{x \in \mathcal{X}}[l(f(x), h(x))].
\]

If we let $\mathcal{D}_p$ be some probability distribution, the noise smoothing risk has the following form
\[
\mathcal{R}_{\mathcal{D}_p} = \mathbb{E}_{x \in \mathcal{X}}[l(f_{\pi}(x), h(x))]
= \mathbb{E}_{x \in \mathcal{X}}[l([\mathcal{E}_{x \sim \mathcal{D}_p}[f(x + z)], h(x))]
\]
and can be generally high with an $f$ learned from minimizing risk (4). To deal with this, current approaches (Lecuyer et al., 2019; Cohen et al., 2019) adopt noise smoothing during the training that formulates and minimizes
\[
\mathcal{R}_{\mathcal{D}_p,\text{train}} = \mathbb{E}_{x \in \mathcal{X}}[l([f(x + z), h(x))].
\]

Specially, it is argued in (Cohen et al., 2019) that when one chooses $l$ as the cross entropy and $\mathcal{D}_p = N(0, \sigma^2 I)$, risk (6) is a lower bound of risk (5) and minimizing risk (6) will approximately minimize risk (5). Nevertheless, we find justifications from this risk perspective is not sufficient as there is no guarantee on the size of the gap $\mathcal{R}_{\mathcal{D}_p,\text{train}} - \mathcal{R}_{\mathcal{D}_p}$. Therefore, we focus in this section on bringing evidences from information theory (Section 4.1) and accuracy (Section 4.2) to enable a more comprehensive understanding of what Gaussian smoothing during training does. Lastly, we discuss whether Gaussian smoothing during inference can sufficiently remedy the information loss or, in other words, retain better accuracy in Section 4.3. The proofs of results in Section 4.1, Section 4.2, and Section 4.3 are included in Section C1, Section C2, and Section C3 in the supplementary materials, respectively.

4.1. A Perspective from Information Theory

Essentially, the original learning problem $\mathcal{L}$ concerns a $X \times Y$ space with a probability density function of $p(x, y)$. The noise smoothing problem during learning $\mathcal{L}_{\mathcal{D}_p,\text{train}}$ considers a $X_{\mathcal{D}_p,\text{train}} \times Y$ space\(^3\) with a probability density function of $\psi(x, y)$, where $\psi(x, y) = \int_{x' \in \mathbb{R}^d} p(x, x') \rho(x', y) dx'$ and $p(x, x')$ is the probability density function of the smoothing distribution $\mathcal{D}_p$. Denoting the entropy of $\mathcal{Y}$ with respect to $\rho(x, y)$ as $H_{\rho}(\mathcal{Y})$ and similarly we define $H_{\psi}(\mathcal{Y})$ and conditional entropies $H_{\rho}(\mathcal{Y}|X)$, $H_{\psi}(\mathcal{Y}|X)$ and $I_{\psi}(\mathcal{X}_{\mathcal{D}_p,\text{train}}, \mathcal{Y}) = H_{\psi}(\mathcal{Y}) - H_{\psi}(\mathcal{Y}|X)$ to denote the mutual information. Now we introduce our theorem:

Theorem 6. For any non-vanishing distribution $\mathcal{D}_p$, we have that noise smoothing w.r.t $\mathcal{D}_p$ during training results in loss of information, i.e., $I_{\rho}(\mathcal{X}, \mathcal{Y}) \geq I_{\psi}(\mathcal{X}_{\mathcal{D}_p,\text{train}}, \mathcal{Y})$ with equality when $I_{\rho}(\mathcal{X}, \mathcal{Y}) = 0$.

\(^3\)We circumvent the terminology “data augmentation” in our formal discussions since data augmentation can have many different definitions in various scenarios.

\(^10\)A probability distribution $\mathcal{D}_p$ over $\mathbb{R}^d$ is non-vanishing if $\int_{x \in \mathbb{R}^d} p(x, x') dx = 1$, $p(x, x') \neq 0$, $\forall x \in \mathbb{R}^d$ (randomized smoothing with infinite support) during the training, e.g. Gaussian.
For Gaussian smoothing \((p(x,x'))\) is a Gaussian distribution centered at \(x'\) with standard deviation \(\sigma\), we use \(\psi_\sigma\) to denote the joint probability distribution over \(X_{\text{RS-train}} \times Y\) after noise smoothing.

**Corollary 4.** For Gaussian smoothing during training, when \(I_{\psi_\sigma}(X_{\text{RS-train}}, Y) \neq 0\), we have \(I_{\psi_\sigma}(X_{\text{RS-train}}, Y) < I_{\psi_{\sigma_0}}(X_{\text{RS-train}}, Y)\), if \(\sigma_1 > \sigma_0\). We also notice that for \(\sigma \to \infty, I_{\psi_\sigma}(X_{\text{RS-train}}, Y) \to 0\).

By Theorem 6 and Corollary 4, the mutual information between \(X, Y\) can be arbitrarily low for large values of \(\sigma\). As in the current “Gaussian smoothing during both training and inference” workflow, the only place learning occurs is at the first step when we learn the classifier from the smoothed distribution. So, we can learn a function that is very different from our original problem. We study the implications of these from an accuracy perspective in 4.2, 4.3.

### 4.2. A Perspective from Classifier Accuracy

Although we have proved the mutual information loss when we applied Gaussian smoothing during training, it is not intuitive what does it mean to have mutual information loss in terms of the classifiers’ behaviors. We hereby demonstrate in this part of the section by constructing a numerical example and leveraging a real-life dataset to show how different \(L_{\text{RS-train}}\) is from \(L\), i.e., how bad the accuracy of a classifier \(f_{\text{train,}\sigma}\) obtained from \(L_{\text{RS-train}}\) with Gaussian smoothing can be by the original prediction rule, i.e. \(\text{argmax}_j f_j(x)\).

**Synthetic Dataset.** \(f_{\text{train,}\sigma}\) can have arbitrarily low classification accuracy using the original prediction rule. We show this by utilizing a synthetic dataset with \(\rho(x,y)\) defined by

\[
\rho(0,1) = \frac{1}{2}; \quad \rho(-a,2) = \frac{1}{2} - \epsilon; \quad \rho(ka,2) = \epsilon, \quad (7)
\]

where \(a, k \in \mathbb{R}^+ / \{0\}\). Then the following theorem holds:

**Theorem 7.** The accuracy of \(f_{\text{train,}\sigma}\) using the original prediction rule is at most \(1/2 + \epsilon\), if \(k > \sqrt{\frac{1}{2\epsilon}} - 1\) and \(\sigma \geq a \sqrt{\frac{k(k+2)}{2mn(2(k+1))}}\).

**Real-life Dataset.** \(f_{\text{train,}\sigma}\) performs consistently worse than the original prediction rule with enlarging \(\sigma\) on CIFAR10 and ImageNet. We show the above point by evaluating models trained with Gaussian smoothing. Specially, we look into the performances of Gaussian smoothing learned classifiers\(^{11}\) with \(\sigma = \{0.12, 0.25, 0.50, 1.00\}\). To evaluate classifiers in terms of the original prediction scheme (or the original learning problem \(L\)), we simply do inference without Gaussian smoothing. As a reference, we also analyze the results when \(\sigma = 0\), i.e. no smoothing is adopted in the training. One can then readily verify that the accuracy drops

\(^{11}\)Trained models provided in (Cohen et al., 2019) are used.

| Training \(\sigma\) | 0.12 | 0.25 | 0.50 | 1.00 | 0.00 |
|---------------------|------|------|------|------|------|
| CIFAR10             | 78   | 71   | 60   | 44   | 90   |
| ImageNet            | -    | 37   | 22   | 10   | 76   |

**Table 3.** The accuracy (%) of CIFAR10 and ImageNet classifiers learned with Gaussian smoothing and inferred by the original prediction rule.

from 90% to 78% for CIFAR10 classifiers when we do Gaussian smoothing during training with \(\sigma = 0.12\) and continues to decrease when we do Gaussian smoothing during training with larger \(\sigma\) (cf. Corollary 4). Finally, the classifier can only correctly classify less than half of the images when we do Gaussian smoothing during training with \(\sigma = 1.00\). We can also infer from Table 3 that the accuracy of the reference ImageNet classifier \((\sigma = 0.00)\) is 76%, which decreases to only 10% with Gaussian smoothing during training and \(\sigma = 1.00\). By now, we have demonstrated that adopting Gaussian smoothing during the inference largely changes the learning problem and the learned classifiers exhibit noticeable degrades in the classification accuracy.

### 4.3. Gaussian smoothing (during inference) after Gaussian smoothing during training

Since according to Section 4.1, conducting Gaussian smoothing during training essentially applies a Gaussian filter on the ground-truth distribution and parts of the mutual information are stored in parameter \(\sigma\) during this process, we argue here that applying another Gaussian filter during inference also cannot completely transfer the information in \(\sigma\) back to the classifier.

**Remark.** Concretely, (Cohen et al., 2019) equivalently includes a majority vote step in between of the two filtering operators, which empirically help to reduce the mutual information loss compared with the one without this step.

Moreover, from Section 4.2 we know that incorporating Gaussian smoothing during training completely changes the learning problem and its accuracy can be arbitrarily low under the original prediction rule. We analyze here how this accuracy will be after the additional Gaussian filter during inference.

**Synthetic Dataset.** \(f_{\text{train,}\sigma}\) can have arbitrarily low classification accuracy using the Gaussian smoothing prediction rule. We show this by utilizing the same synthetic dataset (7) and prove the following theorem:

**Theorem 8.** The accuracy of \(f_{\text{train,}\sigma}\) using the Gaussian smoothing prediction rule is at most \(1/2 + \epsilon\), if \(k > \frac{\epsilon^2}{\epsilon} - 1\) and \(\sigma \geq a \sqrt{\frac{k(k+1)}{2mn(2(k+1))}}\).

**Remark.** This counterexample can be extended to several...
Table 4. The accuracy (%) of CIFAR10 and ImageNet classifiers learned with Gaussian smoothing and inferred by the Gaussian smoothing prediction rule (the same smoothing factor $\sigma$ during the training is also used during the Gaussian smoothing inference).

| training $\sigma$ | 0.12 | 0.25 | 0.50 | 1.00 | 0.00 |
|-------------------|------|------|------|------|------|
| CIFAR10           | 83   | 77   | 66   | 47   | 90   |
| ImageNet          | -    | 67   | 57   | 44   | 76   |

more general and interesting cases. Among which we can have 1) a multi-class case giving accuracy $\frac{1}{\epsilon} + \epsilon$ by having class 1 with the same distribution and the rest of the classes with distributions similar to that of class 2’s, and 2) a binary-class case where Gaussian smoothing during training does not change the optimal solution but the subsequent Gaussian smoothing during inference still gets low accuracy for high enough $\sigma$.

**Real-life Dataset.** $f_{\text{train},\sigma}$ performs consistently worse by the Gaussian smoothing prediction rule with enlarging $\sigma$ on CIFAR10 and ImageNet. Using the same trained models as in Section 4.2, we highlight in this part the degraded performance in classification accuracy even if we infer the label with Gaussian smoothing. By referring to Table 4, we can see that the classification accuracy under Gaussian smoothing prediction rule, although decreases slower than that in Section 4.2 (Table 3) and exhibits more resilience to the smoothing operator during training, still significantly decreases with the enlarging smoothing factor $\sigma$ used during training. Specially, the CIFAR10 network accuracy degrades from 90% to only 47% (compared with 44% in Table 3) when we do Gaussian smoothing during both training and prediction with $\sigma = 1.00$. Similar conclusion can be drawn by referring to ImageNet results, from which we see that inference with smoothing retains better accuracy while is still visibly worse than the reference accuracy.

**5. Conclusions**

In this paper, we provide a theoretical characterization showing that Gaussian smoothing (Cohen et al., 2019) during inference can potentially lead to a significant decrease in the classification accuracy, even when it is included in the training phase. In addition, we observe that the smoothing during inference is very sensitive to the distribution of the data and can have wildly different effects on different classes depending on the data geometry. A similar analysis could be extended to other smoothing functions in addition to Gaussian smoothing. Our analysis on Gaussian smoothing can potentially help choose smoothing distributions that are more robust to different data geometries.

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A. Recap: Definitions and Contribution Grids

A.1. Definitions

Definition 1 (Smoothed). If we use $f$ to denote an original neural network function with outputs in the simplex $\Delta_c = \{ z \in \mathbb{R}^c | \sum_{i=1}^c z_i = 1, 0 \leq z_i \leq 1, \forall i \}$, then its smoothed counterpart defined on $d$-dimensional inputs $x \in \mathbb{R}^d$ is defined by

$$f_{\text{smooth}}(x) = \int_{x' \in \mathbb{R}^d} f(x') p(x') dx',$$

where $p(x')$ is the probability density function of the filter.

Definition 2 (Gaussian smoothing). If $p(x')$ is the probability density function of a normally-distributed random variable with an expected value $x$ and standard deviation $\sigma$, then we call $f_{\text{smooth}}$ a Gaussian-smoothed function and denote it by $f_{\sigma}$.

Definition 3 (Bounded Decision Regions). If the decision region (disconnected or connected) of class 1 data is a bounded set in the Euclidean space (can be bounded by a ball of finite radius), then we call these decision regions bounded decision regions.

Definition 4 (Shrinking of Bounded Decision Regions). A bounded decision region is distinguished as shrunk after applying smoothing filters if the radius $R_{\sigma}$ of $S_{D_{\sigma}}$ is rigorously smaller than the radius $R$ of $S_D$, i.e. $R_{\sigma} < R$, where $S_D$ and $S_{D_{\sigma}}$ are the smallest balls containing the original decision region and the smoothed decision region, respectively.

Definition 5 (Unbounded Decision Regions). If for any ball there exists at least one point in the decision regions that reside outside the ball, then we call these decision regions unbounded decision regions.

Definition 6 (Semi-bounded Decision Regions). For an unbounded decision region, if there exists any half-space $\mathcal{H}$ (decided by a hyperplane) that contains the unbounded decision region, then we call it semi-bounded decision region. We say a semi-bounded decision region is bounded in $v$-direction if there exists any $k \in \mathbb{R}/\infty$ such that for $\forall x \in \mathcal{D}$, $v^T x < k$.

Definition 7 (Shrinking of Semi-bounded Decision Regions). A semi-bounded decision region bounded in $v$-direction is distinguished as shrunk along the direction after applying smoothing filters if the upper bound of projections of the decision region onto direction $v$ shrinks, i.e. $\Upsilon^v_{D_{\sigma}} < \Upsilon^v_{D}$, where $\Upsilon^v_{D} = \max_{x \in \mathcal{D}} v^T x$, $\Upsilon^v_{D_{\sigma}} = \max_{x \in \mathcal{D}_{\sigma}} v^T x$.

Definition 8 ($\theta,v$-Bounding Cone for a Decision Region). A $\theta,v$ cone is defined as a right circular cone $C$ with axis along $-v$ and aperture $2\theta$. Then we define the $\theta,v$-bounding cone $C_{\theta,v}^\mathcal{D}$ for $\mathcal{D}$ as the $\theta,v$ cone that has the smallest projection on $v$ and contains $\mathcal{D}$, i.e., $C_{\theta,v}^\mathcal{D} = \arg \min_{\mathcal{C} \subseteq C_{\theta,v}} \Upsilon^v_{\mathcal{C}}$.

A.2. Contribution grids

| Table S5. A look-up table of theoretical (T) and numerical (N) contributions in Section 3. |
|---------------------------------------------------------------|
| region geometry | shrinking | vanishing rate $\sigma_{\text{van}}$ | shrinking rate | certified radius |
| bounded          | T (Thm. 1) | T - lower bnd. (Thm. 2) | N - lower bnd. (Fig. 3) | N - case study |
| semi-bounded     | T (Thm. 3) | not applicable | T - lower bnd. (Thm.5) | N - case study |

| Table S6. A look-up table of theoretical (T) and numerical (N) contributions in Section 4. |
|---------------------------------------------------------------|
| information loss accuracy: $f_{\text{train},\sigma}$ w/o smoothing inference | accuracy: $f_{\text{train},\sigma}$ w/ smoothing inference |
| T (Thm. 6) | T (Thm. 7) & N (Tab. 3, 4) | T (Thm. 8) & N (Tab. 3, 4) |
B. Proofs for Theoretical Characterization: Randomized Smoothing and Accuracy

B.1. Proofs for Sec 3.1

Lemma 1. For any two original decision regions \( A, B \), if we have that \( A \subseteq B \), then we also have that \( A_\sigma \subseteq B_\sigma \), where \( A_\sigma \) and \( B_\sigma \) are the decision regions of the Gaussian-smoothed functions.

Proof. Recalling that decision regions \( A_\sigma \) and \( B_\sigma \) satisfy \( D_\sigma = \{ x \in \mathbb{R}^d \mid f_D^\sigma(x)_1 \geq \frac{1}{c} \} \) for \( D = A, B \). Therefore for \( \forall x \in A_\sigma \), we have \( f_A^\sigma(x) \geq \frac{1}{c} \). And

\[
f_B^\sigma(x)_1 = \int_{x' \in \mathbb{R}^d} f_B^\sigma(x')p(x')dx' = \int_{x' \in B} \mathbb{1}_{x' \in B} p(x')dx' > \int_{x' \in A} p(x')dx'
\]

\[
= \int_{x' \in \mathbb{R}^d} \mathbb{1}_{x' \in A} p(x')dx' = \int_{x' \in \mathbb{R}^d} f_A^\sigma(x')p(x')dx'
\]

\[
= f_A^\sigma(x)_1 \geq \frac{1}{c},
\]

implying \( x \in B_\sigma \). That said, we have that if \( x \in A_\sigma \), then \( x \in B_\sigma \), making \( A_\sigma \subseteq B_\sigma \). \( \square \)

B.2. Proofs for Sec 3.2

Corollary 1. The smallest ball \( S_{D_\sigma} \), containing the smoothed decision region is contained within the smoothed version of \( S_D \), i.e. \( S_{D_\sigma} \subseteq (S_D)_\sigma \).

Proof. As we have \( D \subseteq S_D \), from Lemma 1 we get \( D_\sigma \subseteq (S_D)_\sigma \). Then by isotropy we have that \( (S_D)_\sigma \) is also a ball centered at the same point as \( S_D \). As \( S_{D_\sigma} \) is the smallest ball containing \( D_\sigma \), we have that \( S_{D_\sigma} \subseteq (S_D)_\sigma \). \( \square \)

We also need another important definition for the coming theorem, the regularized Gamma function:

Definition 9 (Regularized Gamma Function). The lower regularized gamma functions \( Q(s, x) \) is defined by

\[
Q(s, x) = \frac{\int_0^x t^{s-1}e^{-t}dt}{\int_0^\infty t^{s-1}e^{-t}dt}
\]

Moreover, it is well-known that

\[
Q\left(\frac{d}{2}, \frac{R^2}{2\sigma^2}\right) = \int_{x' \in \mathbb{R}^d, \|x'\|_2 \leq R} (2\pi\sigma^2)^{-\frac{d}{2}} e^{-\frac{x'^\top x'}{2\sigma^2}} dx'.
\]

We also give a short proof of this in the proof of Theorem 2. For the number of dimensions \( d \), we summarize the lemma based on regularized Gamma functions below.

Lemma 2. For \( \forall d, c \in \mathbb{N}^+ \), \( Q\left(\frac{d}{2}, \frac{d}{2}\right) < \frac{1}{c} \) holds.

Proof. To prove \( Q\left(\frac{d}{2}, \frac{d}{2}\right) < \frac{1}{c} \), by definition 9, we aim at proving \( \int_0^\infty t^{\frac{d}{2}-1}e^{-t}dt > c \cdot \int_0^\infty t^{\frac{d}{2}-1}e^{-t}dt \) (\( \forall d \in \mathbb{N}^+ \)). For \( c = 1 \), this is clearly true as \( t^{\frac{d}{2}-1}e^{-t} \geq 0 \) is true for \( t \geq 0 \). Then we show it also holds for \( c \geq 2 \).

Let \( g(t) = t^{\frac{d}{2}-1}e^{-t} \), we have \( g'(t) = t^{\frac{d}{2}-2}e^{-t}(x - 1 - t) \). Therefore \( g(t) \) is increasing when \( t \leq x - 1 \) and decreasing when \( t > x - 1 \). Thus, giving us two equations

\[
\int_x^\infty t^{\frac{d}{2}-1}e^{-t}dt > \min\{x^{\frac{d}{2}-1}e^{-x}, \left(\frac{x}{c}\right)^{\frac{d}{2}-1}e^{-\frac{x}{c}}\}(c - 1)x
\]

\[
\frac{x}{c}^{\frac{d}{2}-1}e^{-\frac{x}{c}} > \int_0^\infty t^{\frac{d}{2}-1}e^{-t}dt
\]
So, we see that for any $x, c$ if we have $x^{x-1}e^{-x} \geq \left(\frac{1}{c}\right)^{x-1}e^{-\frac{x}{c}}$ then $\int_{x}^{c} x^{x-1}e^{-t}dt > (c - 1) \cdot \int_{0}^{\frac{x}{c}} t^{x-1}e^{-t}dt \Leftrightarrow \int_{0}^{x} t^{x-1}e^{-t}dt > c \cdot \int_{0}^{\frac{x}{c}} t^{x-1}e^{-t}dt$. Using $t^{x-1}e^{-t} \geq 0, \forall x \int_{0}^{\infty} t^{x-1}e^{-t}dt \geq \int_{0}^{x} t^{x-1}e^{-t}dt$. So, we have $\int_{0}^{x} t^{x-1}e^{-t}dt > c \cdot \int_{0}^{\frac{x}{c}} t^{x-1}e^{-t}dt$ as needed. So, for any $x, c$ it is sufficient to show

$$x^{x-1}e^{-x} \geq \left(\frac{x}{c}\right)^{x-1}e^{-\frac{x}{c}}$$

in order to prove $\int_{0}^{x} t^{x-1}e^{-t}dt > c \cdot \int_{0}^{\frac{x}{c}} t^{x-1}e^{-t}dt$. The inequality can be re-written as $(x - 1) \log(c) > \frac{e - 1}{c} x$ or $(1 - \frac{1}{c}) > (1 - \frac{1}{c}) \frac{1}{\log(c)}$. We observe that $(1 - \frac{1}{c}) \frac{1}{\log(c)}$ is a decreasing function of $c$ for $c \geq 1$ and $(1 - \frac{1}{c})$ is an increasing function of $x$.

For $x \geq 4, c \geq 2$, we see $(1 - \frac{1}{c}) > 1 - \frac{1}{4} = 0.75 > (1 - \frac{1}{2}) \frac{1}{\log(2)} \geq (1 - \frac{1}{2}) \frac{1}{\log(2)}$.

For $x \geq \frac{3}{2}, c \geq 20$, we have $(1 - \frac{1}{c}) \geq 1 - \frac{1}{2} > (1 - \frac{1}{20}) \frac{1}{\log(20)} \geq (1 - \frac{1}{2}) \frac{1}{\log(2)}$.

For $\frac{3}{2} \leq x < 4 \to 3 \leq d < 8$ and $2 \leq c < 20$, we numerically verify the values of $Q\left(\frac{d}{2}, \frac{c}{2}\right)$ to see the inequality is satisfied. Thus, for $d \geq 3, c \geq 2$ we have the inequality.

For $d = 2$, we have $Q\left(\frac{d}{2}, \frac{c}{2}\right) = Q\left(1, \frac{1}{c}\right)$. This has a closed form solution $Q(1, x) = 1 - e^{-x}$. So, we need to show that for $c \geq 2, 1 - e^{-\frac{x}{c}} < \frac{1}{c}$ or $e^{\frac{x}{c}} < \frac{c}{e}$ or $\frac{1}{c} < \log\left(1 + \frac{1}{c}\right)$. But we know that for $x > -1, x \neq 0$, $\log(1 + x) > \frac{x}{x + 1}$, so $\log\left(1 + \frac{1}{c}\right) > \frac{1}{1 + \frac{1}{c}} = \frac{1}{c}$. Which concludes the proof for $d = 2, c \geq 2$.

Theorem 1. A bounded decision region shrinks after applying Gaussian smoothing filters with large $\sigma$, i.e. if $\sigma > \frac{R_{S}}{\sqrt{2(d-1)}}$, then $R_{S} < R$, where $R$ and $R_{S}$ are the radii of $S_{D}$ and $S_{D_{\sigma}}$, the smallest balls bounding the original decision region and the smoothed decision region, respectively.

Proof. Considering the ball $S_{D}$, we see that from Corollary 1, $D_{\sigma} \subseteq (S_{D})_{\sigma}$. Thus, we see that by the definition of radius $R_{S_{D_{\sigma}}} \leq R_{S_{D}}$. It is sufficient to show that for large $\sigma, R_{S_{D_{\sigma}}} < R_{S_{D}}$. Then we observe that due to the isotropic nature of Gaussian smoothing, $(S_{D})_{\sigma}$ is also a sphere concentric to $S_{D}$. So, it is sufficient to show that for a point $x$ at distance $R_{S_{D}}$ from the center $x_{0}$ of the sphere, $f_{\sigma}(x) < \frac{1}{c}$. Without loss of generality consider $D$ to be the origin-centered sphere of radius $R$ and $x = [0, \ldots, 0, R]^{T}$. It is sufficient to show for large $\sigma$ $f_{\sigma}(x) < \frac{1}{c}$. By definition 2, we have

$$f_{\sigma}(x) = \int_{\|x'\|_{2} \leq R} f(x') p(x') dx'$$

$$= \int_{\|x'\|_{2} \leq R} (2\pi)^{-\frac{d}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(x'-x)^{T} \Sigma^{-1}(x'-x)} dx'$$

$$= \int_{\|x'\|_{2} \leq R} (2\pi \sigma^{2})^{-\frac{d}{2}} e^{-\frac{1}{2\sigma^{2}}(x'-x)^{2}} dx'. \quad (8)$$

Then substituting the value of $x$, we get the equation.

$$f_{\sigma}(x) = \int_{\|x'\|_{2} \leq R} (2\pi \sigma^{2})^{-\frac{d}{2}} e^{-\frac{1}{2\sigma^{2}}(x'-x)^{2}} dx'$$

$$= \int_{-R}^{R} \int_{\sum_{k=1}^{d-1} x_k'^{2} \leq R^{2} - x_d'^{2}} (2\pi \sigma^{2})^{-\frac{d}{2}} e^{-\frac{1}{2\sigma^{2}}(x'-x)^{2}} dx'_{d-1} e^{-\frac{(x_d'-x_d)^{2}}{2\sigma^{2}}} dx_{d}'$$

$$< \int_{-R}^{R} \int_{\sum_{k=1}^{d-1} x_k'^{2} \leq R^{2}} (2\pi \sigma^{2})^{-\frac{d}{2}} e^{-\frac{1}{2\sigma^{2}}(x'-x)^{2}} dx'_{d-1} e^{-\frac{(x_d'-x_d)^{2}}{2\sigma^{2}}} dx_{d}'$$

$$= \left(\int_{-R}^{R} (2\pi \sigma^{2})^{-\frac{d}{2}} e^{-\frac{(x_d'-x_d)^{2}}{2\sigma^{2}}} dx_{d}'\right) \cdot \int_{\sum_{k=1}^{d-1} x_k'^{2} \leq R^{2}} (2\pi \sigma^{2})^{-\frac{d}{2}} e^{-\frac{1}{2\sigma^{2}}(x'-x)^{2}} dx'_{d-1}$$

$$= (\Phi\left(\frac{2R}{\sigma}\right) - \Phi(0)) \cdot Q\left(\frac{d-1}{2}, \frac{R^{2}}{2\sigma^{2}}\right)$$

$$= (\Phi\left(\frac{2R}{\sigma}\right) - \Phi(0)) \cdot Q\left(\frac{d-1}{2}, \frac{R^{2}}{2\sigma^{2}}\right)$$
\[ < \frac{1}{2} \cdot Q \left( \frac{d-1}{2}, \frac{R^2}{2\sigma^2} \right). \]

Using Lemma 2 we get that for \( d \geq 3 \), if \( \frac{R^2}{2\sigma^2} \leq \frac{d-1}{c} \), then we have \( \frac{1}{2} \cdot Q \left( \frac{d-1}{2}, \frac{R^2}{2\sigma^2} \right) < \frac{1}{c} \). Now, \( \frac{R^2}{2\sigma^2} < \frac{d-1}{c} \) gives

\[ \sigma > \frac{R\sqrt{c}}{\sqrt{2(d-1)}}. \]

For class 1 data \( x \), the point at the origin has the highest probability to be classified as class 1, i.e. \( f_\sigma(x)_1 = \int_{x' \in \mathbb{R}^d} f(x') p(x') dx' = \int_{\|x'\|_2 \leq R} (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2} \|x'\|^2} dx' \leq f_\sigma(0)_1 \).

**Lemma 3.** Assume the decision region of class 1 data is \( \{ x \in \mathbb{R}^d \mid \|x\|_2 \leq R \} \), the point at the origin has the highest probability to be classified as class 1 by the gaussian-smoothed classifier \( f_\sigma \), i.e. \( f_\sigma(x)_1 \leq f_\sigma(0)_1 \).

**Proof.** We do the proof by mathematical induction and begin by giving \( d = 1 \) case. For \( \forall R > 0 \) and \( d = 1 \), Equation (8) reduces to

\[ f_\sigma(x)_1 = \int_{-R}^{R} (2\pi \sigma^2)^{-\frac{1}{2}} e^{-\frac{(x'-x)^2}{2\sigma^2}} dx' \]

\[ = e^{-\frac{(x-x')^2}{2\sigma^2}} \int_{-R}^{R} \sum_{k=1}^{d} x_k^2 \leq R^2 dx' \]

\[ = e^{-\frac{(x-x')^2}{2\sigma^2}} \int_{-R}^{R} \sum_{k=1}^{d-1} x_k^2 \leq R^2 - x_d^2 dx' \]

\[ \leq \int_{-R}^{R} \sum_{k=1}^{d-1} x_k^2 \leq R^2 - x_d^2 dx' \]

\[ = \int_{\sum_{k=1}^{d-1} x_k^2 \leq R^2} e^{-\frac{(x-x')^2}{2\sigma^2}} dx' \]

\[ = \int_{\sum_{k=1}^{d-1} x_k^2 \leq R^2} e^{-\frac{(x-x')^2}{2\sigma^2}} dx' dx_d \]

\[ \leq \int_{\sum_{k=1}^{d-1} x_k^2 \leq R^2} e^{-\frac{(x-x')^2}{2\sigma^2}} dx' \]

\[ = \int_{\sum_{k=1}^{d-1} x_k^2 \leq R^2} e^{-\frac{(x-x')^2}{2\sigma^2}} dx' \]

\[ \leq \frac{1}{2} \cdot Q \left( \frac{d-1}{2}, \frac{R^2}{2\sigma^2} \right). \]

where the first inequality comes from the assumption that the conclusion holds for \( d - 1 \) dimensional case with equality if and only if \( x_1 = \ldots x_{d-1} = 0 \), and the second inequality comes from an one dimensional observation with equality precisely when \( x_d = 0 \). This concludes our proof.

Since the value of \( f_\sigma(0)_1 \) depends on the radius \( R \) of the decision region, the dimension \( d \), and the smoothing factor \( \sigma \), we denote \( f_\sigma(0)_1 \) by \( q(R, d, \sigma) \), i.e. \( q(R, d, \sigma) := f_\sigma(0)_1 \).
Theorem 2 (Vanishing Rate in the Ball-like Decision Region Case). The decision region of class 1 data vanishes at smoothing factor $\sigma_{\text{van}} > \frac{R_{\text{C}}}{\sqrt{d}}$.

Proof. Noticing that the surface area of a $d$-dimensional ball of radius $r$ is proportional to $r^{d-1}$, we can therefore write out the probability of the point at the origin be classified as class 1 as

$$q(R,d,\sigma) = \frac{\int_0^R \int_0^{r-1} \frac{1}{2\pi^{d/2}/d} e^{-\frac{r^2}{2\sigma^2}} dr}{\int_0^{\infty} \int_0^{r-1} \frac{1}{2\pi^{d/2}/d} e^{-\frac{r^2}{2\sigma^2}} dr}$$

$$= \frac{\int_0^R \int_0^{r-1} e^{-\frac{r^2}{2\sigma^2}} dr}{\int_0^{\infty} \int_0^{r-1} e^{-\frac{r^2}{2\sigma^2}} dr}$$

$$= \frac{\int_0^R e^{\frac{r^2}{2\sigma^2}} \int_0^{\infty} \frac{1/2}{2\pi^{d/2}/d} e^{-\frac{t^2}{2\sigma^2}} dt \int_0^{\infty} \frac{1/2}{2\pi^{d/2}/d} e^{-\frac{t^2}{2\sigma^2}} dt}{\int_0^R \int_0^{r-1} e^{-\frac{r^2}{2\sigma^2}} dr}$$

$$= \frac{\int_0^R e^{\frac{t^2}{2\sigma^2}} \frac{1/2}{2\pi^{d/2}/d} e^{-\frac{t^2}{2\sigma^2}} dt \int_0^{\infty} \frac{1/2}{2\pi^{d/2}/d} e^{-\frac{t^2}{2\sigma^2}} dt}{\int_0^R \int_0^{r-1} e^{-\frac{r^2}{2\sigma^2}} dr}$$

$$= Q(\frac{d}{2\sigma^2})$$

Now let $\sigma = \sqrt{\frac{d}{2}} R$ yields $q(R, d, \sqrt{\frac{d}{2}} R) = Q(\frac{d}{2\sigma^2})$. By Lemma 2, we then have $Q(\frac{d}{2\sigma^2}) < \frac{1}{2}$, implying the decision region of class 1 data has already vanished and making $\sigma = \sqrt{\frac{d}{2}} R$ an upper bound of the vanishing smoothing factor. □

B.3. Proofs for Sec 3.3

Corollary 2. For a given semi-bounded decision region $D$ bounded in direction $v$, we have $D_v$ is also a semi-bounded decision region in the $v$ direction.

Proof. For a given semi-bounded decision region $D$ bounded in direction $v$, we get a hyperplane and a half-space $H$ such that $D \subseteq H$, cf. Definition 6. According to Lemma 1, we have $D_v \subseteq H_v$ and we have $H_v$ is a half-space parallel to $H$ by the isotropy of Gaussian distribution. Therefore we have $D_v$ is also a semi-bounded decision region in the $v$ direction. □

Corollary 3. As $D \subseteq C^p_{\theta,v}$ using Lemma 1, we have that the smoothed decision region is contained within the smoothed version of $C^p_{\theta,v}$, i.e. $D_v \subseteq (C^p_{\theta,v})_v$. □

Lemma 4. If the decision region of class 1 data is $D = \{x \in \mathbb{R}^d \mid v^T x + \|v\|\|x\| \cos(\theta) \leq 0\}$, where $v = [0, \ldots, 0, 1]^T \in \mathbb{R}^d$ and $\theta \in (-\pi, \pi]$, then after smoothing among the set of points $S_a$ with the same projection on $v$ the point on the axis has the highest probability of being in class 1. For $S_a = \{x \mid v^T x = a\}$, we have argsup$_x \epsilon S_a f_\sigma(x) = a \cdot v$. Moreover if $a_1 > a_2$, then $f_\sigma(a_1 \cdot v) < f_\sigma(a_2 \cdot v)$.

Proof. For the first part of the proof consider the set of points $S_a = \{x \mid v^T x = a\}$. For any point $x$ is $S_a$, we see that

$$f_\sigma(x)_1 = \int_{x' \in \mathbb{R}^d} f(x')_1 p(x') dx'$$

$$= \int_{x' \mid ||x'|| \cos(\theta) \leq 0} \left(2\pi \sigma^2\right)^{-\frac{d}{2}} e^{-\frac{(x'-a)^T (x'-a)}{2\sigma^2}} dx'$$

$$= (2\pi \sigma^2)^{-\frac{d}{2}} \int_{-\infty}^{0} \left(2\pi \sigma^2\right)^{-\frac{d-1}{2}} e^{-\frac{(x'-a_{d-1})^2}{2\sigma^2}} dx'_1 \cdots \int_{-\infty}^{0} \left(2\pi \sigma^2\right)^{-\frac{1}{2}} e^{-\frac{(x'_d-a)^2}{2\sigma^2}} dx'_d$$

$$\leq (2\pi \sigma^2)^{-\frac{d}{2}} \int_{-\infty}^{0} \left(2\pi \sigma^2\right)^{-\frac{d-1}{2}} e^{-\frac{(x'_d-a_{d-1})^2}{2\sigma^2}} dx'_1 \cdots \int_{-\infty}^{0} \left(2\pi \sigma^2\right)^{-\frac{1}{2}} e^{-\frac{(x'_d-a)^2}{2\sigma^2}} dx'_d$$

$$= f_\sigma(av)_1.$$

Rethinking Randomized Smoothing for Adversarial Robustness
We observe that we only need to check the point $G$. Gaussian smoothing filters with high
$a > 0$. We have that for any $x_j$ such that $a_1 > a_2$. Then
\[
\begin{align*}
 f_\sigma(x_1) &= (2\pi\sigma^2)^{-\frac{d}{2}} \int_{-\infty}^{-a_1} \int_{x_{d-1}^2 + (a_1)^2} \int_{(a_2)^2} \int_{x_d} e^{-\frac{x_{d-1}^2 + (a_1)^2}{2\sigma^2} - \frac{x_d}{2\sigma^2}} \, dx_1 \, dx_{d-1} - e^{-\frac{x_d^2}{2\sigma^2}} \, dx_d \\
&= f_\sigma(x_2).
\end{align*}
\]

Lemma 5. \( \forall a > 0, k \geq 1, \frac{\Phi(-a)}{\Phi(-ka)} \geq e^{-\frac{(a-1)^2}{a}}. \)

Proof. Consider the function $h(x) = \frac{\sqrt{2\pi} \Phi(-x)}{e^{-x^2/2}}$ and we will show in the following that it is strictly decreasing for $x > 0$. Alternatively, we take the derivative with respect to $x$,
\[
\frac{d}{dx} h(x) = \frac{\sqrt{2\pi} x \Phi(-x)}{e^{-x^2/2}} - 1,
\]
and show that it is negative for $x > 0$. Since $e^{-x^2/2} > 0$, it is sufficient to show that $\sqrt{2\pi} x \Phi(-x) - e^{-x^2/2} < 0$. Combining that 1) $\sqrt{2\pi} x \Phi(-x) - e^{-x^2/2}$ is increasing as
\[
\frac{d}{dx} \left( x \Phi(-x) - e^{-x^2/2} \right) = \Phi(-x) - \frac{x e^{-x^2/2}}{\sqrt{2\pi}} - \frac{-xe^{-x^2/2}}{\sqrt{2\pi}}
\]
and 2) $\sqrt{2\pi} x \Phi(-x) - e^{-x^2/2} \rightarrow 0$ when $x \rightarrow \infty$, we have that $\sqrt{2\pi} x \Phi(-x) - e^{-x^2/2} < 0$. As $h(x)$ is strictly decreasing we have that for any $a > 0$ and $k > 1$, $ka > a$. Thus,
\[
\frac{\sqrt{2\pi} \Phi(-a)}{e^{-a^2/2}} \geq \frac{\sqrt{2\pi} \Phi(-ka)}{e^{-(ka)^2/2}}.
\]

Rearranging the terms gives the inequality.

Theorem 3. A semi-bounded decision region that has a narrow bounding cone shrinks along $v$-direction after applying Gaussian smoothing filters with high $\sigma$, i.e. if the region admits a bounding cone $C_{D,v}$ with $\tan(\theta) < \sqrt{\frac{(d-1)}{e \log(e-1)}}$. then for
\[
\sigma > (\gamma_{C_{D,v}} - \gamma_{D}) \tan(\theta) \sqrt{\frac{e}{d-1} \cdot \frac{2(d-1)}{(d-1) - 2 \tan^2(\theta) - \log(e-1)}}, \gamma_{D,v} < \gamma_{D}.
\]

Proof. In this derivation we assume without loss of generality, $v = [0, \ldots, 0, 1]^T \in \mathbb{R}^d$ (It is always possible to orient the axis to make this happen). From Corollary 3, we can see that $D_\sigma \subseteq (C_{D,v})_\sigma$ which gives us $\gamma_{D,v} = \max_{x \in D_\sigma} v^T x \leq \max_{x \in (C_{D,v})_\sigma} v^T x = \gamma_{D,v}$. Then to show that $\gamma_{D,v} < \gamma_{D}$ it is sufficient to show that $\gamma_{C_{D,v}} < \gamma_{D,v}$. We observe that we only need to check the point $x$ on the axis of the cone at distance $\gamma_{C_{D,v}} - \gamma_{D,v}$ from the tip $x_0$ of the cone, i.e., $x = x_0 - (\gamma_{C_{D,v}} - \gamma_{D,v})v$. If $x$ is not classified as Class 1 then by Lemma 4, we have that
\[
\gamma_{C_{D,v}} < v^T x = v^T (x_0 - (\gamma_{C_{D,v}} - \gamma_{D,v})v)
\]
and
\[
\gamma_{D,v} = v^T x_0 - (\gamma_{C_{D,v}} - \gamma_{D,v})v^T v.
\]
Then we see that giving the cone is narrow enough, we have the required shrinking if we have $x_0$ be the origin. By definition 2, we have

$$f_\sigma(x_1) = \int_{x' \in \mathbb{R}^d} f(x')_1 p(x') dx'$$

$$= \int_{x'_1 + \|x'\| \cos(\theta) \leq 0} (2\pi \sigma^2)^{-\frac{d}{2}} e^{-\frac{(x'_1-x_1)^2}{2\sigma^2}} dx'$$

$$= (2\pi \sigma^2)^{-\frac{d}{2}} \int_{-\infty}^{0} e^{-\frac{(x'_1)^2}{2\sigma^2}} dx'_1 \ldots dx'_d e^{-\frac{(x'_d-x_d)^2}{2\sigma^2}} dx'_d$$

$$= (2\pi \sigma^2)^{-\frac{d}{2}} \int_{-\infty}^{0} Q\left(\frac{d-1}{2}, \frac{\tan^2(\theta) x'_d^2}{2\sigma^2}\right) e^{-\frac{(x'_d-x_d)^2}{2\sigma^2}} dx'_d$$

Substitute $X_d = \frac{x_d}{\sigma}, X'_d = \frac{x'_d}{\sigma}$

$$= (2\pi)^{-\frac{d}{2}} \int_{-\infty}^{0} Q\left(\frac{d-1}{2}, \frac{\tan^2(\theta) X_d^2}{2}\right) e^{-\frac{(X_d-x_d)^2}{2\sigma^2}} dX'_d$$

Let $M \leq \sqrt{\frac{d-1}{c\tan^2(\theta)}}, k = \frac{M}{X_d}$

$$= (2\pi)^{-\frac{d}{2}} \int_{M}^{0} Q\left(\frac{d-1}{2}, \frac{\tan^2(\theta) X_d^2}{2}\right) e^{-\frac{(X_d-x_d)^2}{2\sigma^2}} dX'_d$$

$$+ (2\pi)^{-\frac{d}{2}} \int_{-\infty}^{M} Q\left(\frac{d-1}{2}, \frac{\tan^2(\theta) X_d^2}{2}\right) e^{-\frac{(X_d-x_d)^2}{2\sigma^2}} dX'_d$$

$$< (\Phi(-X_d) - \Phi(M-X_d)) Q\left(\frac{d-1}{2}, \frac{\tan^2(\theta) M^2}{2}\right) + \Phi(M-X_d)$$

$$< \frac{\Phi(-X_d) - \Phi(M-X_d)}{c} + \Phi(M-X_d)$$

$$= \frac{1}{c} + \frac{(c-1)\Phi((k-1)X_d) - \Phi(X_d)}{c}$$

Then we see that using Lemma 5, we see that we see that if $e^{\frac{X_d^2((k-1)^2-1)}{2}} \geq c - 1$ then $(c-1)\Phi((k-1)X_d) \leq \Phi(X_d)$. So, we need to satisfy the inequalities for some $k$:

$$\sqrt{\frac{2\log(c-1)}{(k-1)^2-1}} \leq -X_d \leq \sqrt{\frac{d-1}{k^2c\tan^2(\theta)}}.$$
So, we need that $\frac{-x_d}{\sigma} = \sqrt{\frac{d-1}{k_c \tan^2(\theta)}}$ for some suitable $k$. Thus we need $\sigma = -x_d \tan(\theta) \sqrt{\frac{c}{d-1} k}$ for some suitable $k$.

Including the constraint on $k$ and substituting the value for $x_d$, we get that shrinking always happens for

$$\sigma \geq (\Upsilon_{c_{D^5}} - \Upsilon_D) \tan(\theta) \sqrt{\frac{c}{d-1} \cdot \frac{2(d-1)}{(d-1) - 2 \tan^2(\theta) c \log(c-1)}}.$$ 

\[ \square \]

**Theorem 4.** The shrinkage of class 1 decision region is proportional to the smoothing factor, i.e. $\Upsilon_D - \Upsilon_D^\alpha \propto \sigma$.

**Proof.** In this case we assume a cone-like decision region which can be represented as $\mathcal{D} = \{ x \in \mathbb{R}^d \mid v^T x + \|v\|_2 \cos(\theta) \leq 0 \}$ with $v = [0, \ldots, 0, 1]^T$ without loss of generality. By Lemma 4, we see that in order to get bounds on $\Upsilon_D$, we only need to analyze the value of $f_\sigma(x)$ for points $x$ along the axis of the cone. Then we see that for a general point $x = av$ along the axis of the cone, using the same ideas as in proof of Theorem 3, we have

$$f_\sigma(x)_1 = \int_{x' \in \mathbb{R}^d} f(x') \, p(x') \, dx'$$

$$= (2\pi \sigma^2)^{-\frac{d}{2}} \int_{-\infty}^{0} \int_{x_{d-1} = 0}^{x_d - x^2 \leq \tan^2(\theta) x^2_d} e^{-\frac{x_{d-1}^2 + x^2}{2\sigma^2}} \, dx_1 \cdots \, dx_{d-1} = e^{-\frac{(x_d - a)^2}{2\sigma^2}} \, dx_d.$$

Substitute $A = \frac{a}{\sigma}, x'_d = \frac{x_d}{\sigma}$

$$= (2\pi)^{-\frac{d}{2}} \int_{-\infty}^{0} Q\left(\frac{d-1}{2}, \tan^2(\theta) x^2_d, x^2_d \right) e^{-\frac{(x'_d - A)^2}{2}} \, dx'_d$$

$$= f_1(Av)_1 = f_1\left(\frac{1}{\sigma} x\right)_1.$$ 

Using the equation above we see that for smoothing by a general $\sigma$,

$$\Upsilon_{\mathcal{D}_x} = \sup_{x \in \mathcal{D}} \frac{v^T x}{f_\sigma(x)} = \sup_{x : f_\sigma(x) \geq \frac{1}{2}} \frac{v^T x}{f_\sigma(x)} = \sup_{x' : f_\sigma(x') \geq \frac{1}{2}} \frac{v^T x'}{f_\sigma(x')} = \sigma \Upsilon_{\mathcal{D}_1}.$$ 

In this case we have $\Upsilon_\mathcal{D} = 0$ by construction, so $\Upsilon_D - \Upsilon_{\mathcal{D}_x} = 0 - \sigma \Upsilon_{\mathcal{D}_1} = \sigma \cdot (-\Upsilon_{\mathcal{D}_1}) \propto \sigma$. 

\[ \square \]

With the above Theorem 4, we can fix the smoothing factor to $\sigma = 1$ and further obtain a lower bound of the shrinking rate w.r.t. $c, \theta$, and $d$:

**Theorem 5.** The shrinking rate of class 1 decision region is at least $\sqrt{\frac{(d-1)}{c \tan^2(\theta)}} \cdot (d-1) - 2 \tan^2(\theta) c \log(c-1)$, i.e.

$$\sqrt{\frac{(d-1)}{c \tan^2(\theta)}} \cdot (d-1) - 2 \tan^2(\theta) c \log(c-1) \leq \Upsilon_{\mathcal{D}_x}^{\prod_{i=1}^d} - \Upsilon_{\mathcal{D}_x}^{\prod_{i=1}^{d+1}}.$$

**Proof.** As in Theorem 4, we assume a cone at origin along $v = [0, \ldots, 0, 1]^T$ given by $\mathcal{D} = \{ x \in \mathbb{R}^d \mid v^T x + \|v\|_2 \cos(\theta) \leq 0 \}$. Following the same proof idea as Theorem 4, we see that the rate is given by the value $-\Upsilon_{\mathcal{D}_1}$. So, we try to get a bound on the value of $-\Upsilon_{\mathcal{D}_1}$. To establish a lower bound we show that for the point $x = av, f_1(x)_1 < \frac{1}{2}$. Then by Lemma 4 we have $\Upsilon_{\mathcal{D}_1} < a$ or $-\Upsilon_{\mathcal{D}_1} > -a$.

Using the same procedure as in the proof of Theorem 3, we get that if $x$ satisfies the two inequalities

$$\sqrt{\frac{2 \log(c-1)}{(k-1)^2 - 1}} \leq -v^T x \leq \sqrt{\frac{d-1}{k^2 c \tan^2(\theta)}}$$
for suitable real $k$, then we have $f_1(x)_1 < \frac{1}{c}$. So, we need $v^Tx = -\sqrt{\frac{d-1}{k^2c\tan^2(\theta)}}$ for some $k$ such that $\frac{2\log(c-1)}{d-1} \leq x \leq \sqrt{\frac{d-1}{k^2c\tan^2(\theta)}}$. The constraint on $k$ can be re-written as $k \geq \frac{2(d-1)}{(d-1)-2\tan^2(\theta)c\log(c-1)}$. Taking $k$ to be lower bound, we get that for

$$-a = -v^Tx = \sqrt{\frac{d-1}{c\tan^2(\theta)}} \cdot \frac{(d-1)-2\tan^2(\theta)c\log(c-1)}{2(d-1)}$$

$f_1(x)_1 \leq \frac{1}{c}$. So, we get that the rate is $-\gamma_{D_1} \geq -a \geq \sqrt{\frac{d-1}{c\tan^2(\theta)}} \cdot \frac{(d-1)-2\tan^2(\theta)c\log(c-1)}{2(d-1)}$. \qed
C. Proofs for “Can Gaussian Smoothing during Training Help Improve Accuracy?”

C.1. Proofs for Sec 4.1

**Theorem 6.** For any non-vanishing distribution $D_\rho$ \footnote{A probability distribution $D_\rho$ over $\mathbb{R}^d$ is non-vanishing if $\int_{x \in \mathbb{R}^d} p(x, x')dx = 1$, $p(x, x') \neq 0$, $\forall x \in \mathbb{R}^d$ (randomized smoothing with infinite support) during the training, e.g. Gaussian.}, we have that noise smoothing w.r.t $D_\rho$ during training results in loss of information, i.e., $I_\rho(\mathcal{X}, Y) \geq I_\psi(\mathcal{X}_{\text{RS-train}}, Y)$ with equality when $I_\rho(\mathcal{X}, Y) = 0$.

**Proof.** To prove $I_\rho(\mathcal{X}, Y) = H_\rho(Y) - H_\rho(Y|\mathcal{X}) \geq H_\psi(Y) - H_\psi(Y|\mathcal{X}) = I_\psi(\mathcal{X}_{\text{RS-train}}, Y)$, we start by giving $H_\psi(Y) = H_\rho(Y)$ since $\forall y \in Y$,

$$
\psi(y) = \int_{x \in \mathbb{R}^d} \psi(x, y)dx
= \int_{x \in \mathbb{R}^d} \int_{x' \in \mathbb{R}^d} d(x, x')\rho(x', y)dx'dx
= \int_{x' \in \mathbb{R}^d} \rho(x', y) \int_{x \in \mathbb{R}^d} d(x, x')dx'dx
= \int_{x' \in \mathbb{R}^d} \rho(x', y)dx' = \rho(y).
$$

Since we consider non-vanishing distributions, we have $d(x, x') \neq 0$, $\forall x, x'$, then using the log sum inequality we see that

$$
\int_{x' \in \mathbb{R}^d} \psi(x', y)log \frac{\psi(x', y)}{\psi(x')} dx'
= \int_{x' \in \mathbb{R}^d} \int_{x \in \mathbb{R}^d} d(x', x)\rho(x, y)dxlog \frac{\int_{x \in \mathbb{R}^d} d(x', x)\rho(x, y)dx}{\int_{x \in \mathbb{R}^d} d(x', x)\rho(x)dx} dx
\leq \int_{x \in \mathbb{R}^d} \int_{x' \in \mathbb{R}^d} d(x', x)\rho(x, y)log \frac{\int_{x \in \mathbb{R}^d} d(x', x)\rho(x, y)dx}{\int_{x \in \mathbb{R}^d} d(x', x)\rho(x)dx} dx'dx
= \int_{x \in \mathbb{R}^d} \rho(x, y)log \frac{\rho(x, y)}{\rho(x)} dx,
$$

$$
H_\rho(Y|X) = - \int_{y \in Y} \int_{x \in \mathbb{R}^d} \rho(x, y)log \frac{\rho(x, y)}{\rho(x)} dx dy
\geq \int_{y \in Y} \int_{x' \in \mathbb{R}^d} \psi(x', y)log \frac{\psi(x', y)}{\psi(x')} dx'dy = H_\psi(Y|\mathcal{X}_{\text{RS-train}})
$$

with equality when $\rho(x, y) = k(y)\rho(x)$, $\forall x \in \mathbb{R}^d$, $k \in \mathbb{R}$. Then $\rho(y) = \int_{x \in \mathbb{R}^d} \rho(x, y)dx = \int_{x \in \mathbb{R}^d} k(y)\rho(x)dx = k(y)$ that renders

$$
H_\rho(Y|X) = - \int_{y \in Y} \int_{x \in \mathbb{R}^d} \rho(x, y)log \frac{\rho(x, y)}{\rho(x)} dx dy
\geq \int_{y \in Y} \int_{x \in \mathbb{R}^d} \rho(x)log(\rho(y)) dx dy
= \int_{y \in Y} \rho(y)log(\rho(y)) dy
= H_\rho(Y)
$$

and $I_\rho(\mathcal{X}, Y) = H_\rho(Y) - H_\rho(Y|\mathcal{X}) = 0$.

**Corollary 4.** For Gaussian smoothing during the training, when $I_{\psi_1}(\mathcal{X}_{\text{RS-train}}, Y) \neq 0$, we have $I_{\psi_0}(\mathcal{X}_{\text{RS-train}}, Y) < I_{\psi_1}(\mathcal{X}_{\text{RS-train}}, Y)$, if $\sigma_0 > \sigma_1$. 

Proof. We observe that \((\mathcal{X}^\sigma_{\text{RS-train}}, \mathcal{Y}) = \{(x + z, y) \mid (x, y) \sim (\mathcal{X}, \mathcal{Y}) \wedge z \sim \mathcal{N}(0, \sigma^2 I)\}\). Thus with
\[
(\mathcal{X}^{\sigma_1}_{\text{RS-train}}, \mathcal{Y}) = \{(x + z_1, y) \mid (x, y) \sim (\mathcal{X}^{\sigma_1}_{\text{RS-train}}, \mathcal{Y}) \wedge z_1 \sim \mathcal{N}(0, \sigma_1^2 I)\}
\]
we see that if \(\sigma_0 > \sigma_1\), then \(\mathcal{X}^{\sigma_0}_{\text{RS-train}} = (\mathcal{X}^{\sigma_1}_{\text{RS-train}})^{\sigma_2}_{\text{RS-train}}\), where \(\sigma_2 = \sqrt{\sigma_0^2 - \sigma_1^2}\). Therefore from Theorem 6, we have that
\[
I_{\psi_0}(\mathcal{X}^{\sigma_0}_{\text{RS-train}}, \mathcal{Y}) < I_{\psi_1}(\mathcal{X}^{\sigma_1}_{\text{RS-train}}, \mathcal{Y}).
\]

\[\square\]

C.2. Proofs for Sec 4.2

**Theorem 7.** The accuracy of \(f_{\text{train}, \sigma}\) using the original prediction rule is at most \(1/2 + \epsilon\), if \(k > \sqrt{\frac{1}{2\epsilon} - 1}\) and \(\sigma \geq a\sqrt{\frac{k(k+2)}{2ln(2\epsilon(k+1)^2)}}\).

Proof. At \(x = -a\), the probability is
\[
\psi(-a, 1) = \int_{x' \in \mathbb{R}^d} d(-a, x') \rho(x', 1) dx' = d(-a, 0) \rho(0, 1) = \frac{1}{\sqrt{2\pi \sigma^2}} \left[ e^{-\frac{a^2}{2\sigma^2}} \right],
\]
\[
\psi(-a, 2) = \int_{x' \in \mathbb{R}^d} d(-a, x') \rho(x', 2) dx' = d(-a, -a) \rho(-a, 2) + d(-a, ka) \rho(ka, 2) = \frac{1}{\sqrt{2\pi \sigma^2}} \left[ 1 - \epsilon + e^{-\frac{(ka+1)a^2}{2\sigma^2}} \right].
\]

Therefore, to examine when will we have \(\psi(-a, 1) > \psi(-a, 2)\), we see that the following should be satisfied
\[
\frac{1}{2} e^{-\frac{a^2}{2\sigma^2}} > \frac{1}{2} - \epsilon + e^{-\frac{(k+2)ka^2}{2\sigma^2}} e^{-\frac{a^2}{2\sigma^2} \epsilon}
\]
\[
\iff 2\epsilon + 1 > 2\epsilon + e^{\frac{a^2}{2\sigma^2}} - 2e^{\frac{a^2}{2\sigma^2}} + 2e^{-\frac{(k+2)ka^2}{2\sigma^2} \epsilon}
\]
\[
\iff 2\epsilon(1 - e^{-\frac{a^2}{2\sigma^2} \epsilon}) > (1 - 2\epsilon)(e^{\frac{a^2}{2\sigma^2}} - 1)
\]
\[
\iff e^{\frac{a^2}{2\sigma^2}} - e^{-\frac{(k+2)ka^2}{2\sigma^2} \epsilon} > \frac{1}{2\epsilon}
\]
\[
\iff \frac{1 - (e^{-\frac{a^2}{2\sigma^2}})^{(k+2)k+1}}{1 - e^{-\frac{a^2}{2\sigma^2}}} = \sum_{i=0}^{(k+2)k} (e^{-\frac{a^2}{2\sigma^2}})^i > \frac{1}{2\epsilon}
\]
\[
\iff (k+1)^2 e^{-\frac{a^2}{2\sigma^2} (k+2)k} \geq \frac{1}{2\epsilon}
\]
\[
\iff \sigma \geq a\sqrt{\frac{k(k+2)}{2ln(2\epsilon(k+1)^2)}}, k > \sqrt{\frac{1}{2\epsilon} - 1}.
\]

That’s said, with large enough \(\sigma\), the probability density function \(\psi(x, y)\) on \(\mathcal{X}^\sigma_{\text{RS-train}} \times \mathcal{Y}\) is severely drifted. In other words, the optimal classifier \(L_{\text{RS-train}}\) learned is \(\psi(x, y)\), which misclassified \(x = -a\) from the perspective of \(L\) and yields an accuracy of at most \(1/2 + \epsilon\).
C.3. Proofs for Sec 4.3

Lemma 6. \( \Phi[x] + \Phi[\frac{1}{x}] \geq 1.5 \) with equality holds iff \( x \in \{0, \infty\} \).

Proof. Let \( f(x) = \Phi[x] + \Phi[\frac{1}{x}] \). We observe that \( f(x) = f(1/x) \) by definition. So, it is sufficient to show that for \( x \) in the interval \((1, \infty)\), \( f(x) \geq 1.5 \) with equality at \( x \to \infty \). We prove this by showing that in the interval \((1, \infty)\), \( f(x) \) is strictly decreasing and \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \Phi(x) + \Phi(1/x) = \Phi(\infty) + \Phi(0) = 1 + 0.5 = 1.5 \). To show \( f(x) \) is strictly decreasing we proceed by taking the derivative wrt \( x \),

\[
\frac{d}{dx} f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-\frac{1}{x^2}}}{x^2\sqrt{2\pi}}
\]

we show that for the interval \((1, \infty)\) this derivative is less than 0. So, we need to show that

\[
\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-\frac{1}{x^2}}}{x^2\sqrt{2\pi}} < 0
\]

\[
\Leftrightarrow x^2 e^{-\frac{x^2}{2}} < e^{-\frac{1}{x^2}}
\]

\[
\Leftrightarrow \log(x^2) + \frac{1}{2x^2} < x^2
\]

Let \( t = \log(x^2), x \geq 1 \to t > 0 \)

\[
\Leftrightarrow 2t < e^t - e^{-t}
\]

This holds for \( t > 0 \) as we have that at \( t = 0 \). \( 2 \cdot 0 = 0 = e^0 - e^0 \) and \( 2t \) increases at a rate of 2 while \( e^t - e^{-t} \) increases at a rate of \( e^t + e^{-t} > 2 \cdot \sqrt{e^t} \cdot e^{-t} = 2 \) as \( t < 1 \to e^t \neq e^{-t} \). Finally for \( x = 1 \), we calculate \( f(x) \approx 1.6829 > 1.5 \).

Theorem 8. The accuracy of \( f_{\text{train}, \sigma} \) using the Gaussian smoothing prediction rule is at most \( 1/2 + \epsilon \), if \( k > \frac{e^2}{\epsilon} - 1 \) and \( \sigma \geq a \sqrt{\frac{k(k+1)}{2ln(2\epsilon(k+1))} - \frac{2\epsilon}{\epsilon^2}} \).

Proof. At \( x = -a \), we see that if the decision region for class 1 is \([-(a+c), ka + d]\), then the probability after smoothing is

\[
g(-a, 1) = \int_{x' \in \mathbb{R}^d} d(-a, x') \psi(x', 1) dx'
\]

\[
= \int_{-(a+c)}^{ka+d} d(-a, x') dx'
\]

\[
= \int_{-\infty}^{ka+d} d(-a, x') dx' - \int_{-\infty}^{-(a+c)} d(-a, x') dx'
\]

\[
= \Phi(\frac{ka}{\sigma} + d + a) - \Phi(\frac{-c}{\sigma})
\]

\[
\geq \Phi(\frac{k+2a}{\sigma}) - \Phi(\frac{-c}{\sigma}) \quad \text{(if } d \geq 0 \text{)}
\]

\[
\geq \Phi(\frac{k+2a}{\sigma}) - \Phi(-\frac{\sigma}{k+2a}) \quad \text{(if } c \geq \frac{2\sigma^2}{(k+2)a} \text{)}
\]

\[
> 0.5. \quad \text{(by Lemma 6)}
\]

That’s said, the prediction accuracy will be at most \( 1/2 + \epsilon \) if \( d \geq 0 \) and \( c \geq \frac{2\sigma^2}{(k+2)a} \) are true. We now check for \( d \geq 0 \): for \( x \in [0, ka/\sigma] \), we have

\[
\psi(x, 1) = \int_{x' \in \mathbb{R}^d} d(x, x') \rho(x', 1) dx'
\]

\[
= d(x, 0) \rho(0, 1)
\]
Therefore we see that $\psi$ implies $2ka$.

$$\psi(x, a, 2) = \int_{x' \in \mathbb{R}^d} d(x, x') \rho(x', 2) dx' = \psi(x, 2),$$

implying $x \in [0, k\sigma^2]$ belongs to class 1 for the naive bayes classifier. Therefore the decision region for class 1 extends at least to $\frac{ka}{2} + d$ with $d \geq 0$. Next, we check for $c \geq \frac{2\sigma^2}{(k+2)a}$: at $x = -a - \frac{2\sigma^2}{(k+2)a}$, the probability is

$$\psi(-a - \frac{2\sigma^2}{(k+2)a}, 1) = \int_{x' \in \mathbb{R}^d} d(-\frac{2\sigma + a}{(k+2)a}, x') \rho(x', 1) dx'$$

$$= \int_{x' \in \mathbb{R}^d} \left[ \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x')^2}{2\sigma^2}} \right] dx' = \psi(-a - \frac{2\sigma^2}{(k+2)a}, 2)$$

Therefore we see that $\psi(-a - \frac{2\sigma^2}{(k+2)a}, 1) > \psi(-a - \frac{2\sigma^2}{(k+2)a}, 2)$ if

$$(1 - 2\epsilon)e(\frac{\epsilon}{2})^2 \frac{2k}{2k} + 2\epsilon e^{-\frac{k(k+2)}{2}(\frac{\epsilon}{2})^2 - \frac{2k}{2k^2}} < 1$$

$$\Leftrightarrow (1 - 2\epsilon)[e(\frac{\epsilon}{2})^2 \frac{2k}{2k} - 1] < 2\epsilon[1 - e^{-\frac{k(k+2)}{2}(\frac{\epsilon}{2})^2 - \frac{2k}{2k^2}}]$$

$$\Rightarrow \frac{1}{2\epsilon} < \frac{1 - e^{-\frac{k(k+2)}{2}(\frac{\epsilon}{2})^2 - \frac{2k}{2k^2}}}{e(\frac{\epsilon}{2})^2 \frac{2k}{2k} - 1}$$

$$\Rightarrow \frac{1}{2\epsilon} < \frac{\tau l - \tau^{-k(k+2)} l^{-k}}{\tau l - 1}$$

(let $\tau = e(\frac{\epsilon}{2})^2 \frac{2k}{2k}$, $l = e(\frac{\epsilon}{2})^2$)

$$\Rightarrow \frac{1}{2\epsilon} < \frac{\tau^{-k(k+2)} l^{-k} - 1}{\tau l - 1}$$

$$\Rightarrow \frac{1}{2\epsilon} < \frac{\tau^{-k(k+2)} l^{-k} - 1}{\tau l - 1}$$

$$\Rightarrow \frac{1}{2\epsilon} < \frac{(\Sigma_{i=0}^{k}(\tau^{k+1})^{-1}) \frac{\tau^{k+1} l^{-1}}{\tau l - 1} \tau^{-k}}{\tau l - 1}$$

$$\Rightarrow \frac{1}{2\epsilon} \leq \frac{(\Sigma_{i=0}^{k}(\tau^{k+1})^{-1}) \tau l - \tau^{-k}}{\tau l - 1}$$

$$\Rightarrow 0 < \ln(\tau) \leq \ln(2\epsilon(k+1)) - k\ln(l) = \ln(2\epsilon(k+1)) - \frac{2k}{k+1}$$

$$\Rightarrow \frac{2\epsilon}{\sigma^2} \frac{1}{2} \leq \ln(2\epsilon(k+1)) - \frac{2k}{k+1}, k > \frac{2}{\epsilon} - 1$$
\[ \Leftrightarrow \sigma \geq a \sqrt{\frac{k(k+1)}{2\ln(2\epsilon(k+1)) - \frac{2k}{k+2}}}, \quad k > \frac{\epsilon^2}{\epsilon} - 1. \]

These conclude our proof. \qed
D. Additional Analysis

D.1. Shrinking effect for unidimensional data

**Bounded decision region.** Without loss of generality, let the decision region be interval $D = [-R, R]$. By the symmetric nature of Gaussian smoothing, we see that $D_\sigma$ is also an interval of the form $[-a, a]$. We claim that for large $\sigma$, $a < R$ and for even larger $\sigma$, $D_\sigma$ disappears. Formally, we do the analysis as follows.

For the shrinking, we check the value of $f_\sigma(R)_1$. By definition 2, we see that $f_\sigma(R)_1 = \Phi(\frac{2R}{\sigma}) - \Phi(0)$ and if

$$\sigma > \frac{2R}{\Phi^{-1}(\frac{1}{2} + \frac{1}{\epsilon})},$$

$f_\sigma(R) < \frac{1}{\epsilon}$ is true. Thus, the bounded decision region of unidimensional data shrinks with smoothing factor $\sigma > \frac{2R}{\Phi^{-1}(\frac{1}{2} + \frac{1}{\epsilon})}$.

For the vanishing rate, we check the value of $f_\sigma(x)_1$ at $x = 0$. Now since $f_\sigma(0)_1 = \Phi(\frac{R}{\sigma}) - \Phi(-\frac{R}{\sigma})$, we have that if

$$\sigma > \frac{R}{\Phi^{-1}(\frac{1}{2} + \frac{1}{2\epsilon})},$$

$f_\sigma(0)_1 < \frac{1}{\epsilon}$ is true, i.e., $D_\sigma$ vanishes.

**Semi-bounded decision region.** In a unidimensional case, our definition of semi-bounded regions degenerates into an interval $I$ of the form $[a, \infty)$. In this case, Theorem 5 gives a trivial bound of 0 for the shrinkage of the decision region, suggesting that no shrinking happens. However, we emphasize that in practice, shrinking might still happen and more detailed analysis is left for future work.

D.2. Bounded decision region behaviors

![Graph](image-url)  
*Figure S6.* The vanishing smoothing factor $\sigma_{\text{van}}$ with an increasing input-space dimension in the exemplary adversarial ball.

The vanishing smoothing factors $\sigma_{\text{van}}$ with different data dimensions implied by Figure 3 of the main text together with the theoretical lower bound found in Theorem 2 is given as Figure S6.

Figure S7 shows the certified radius behavior as a function of the distance of points from the origin (y-axis) and the smoothing factor $\sigma$ (x-axis) for dimension $d = 30$. The contour lines in Figure S7 mark the certified radius of points under Gaussian smoothing. It is notable that points closer to the origin generally have larger certified radii and the certified radius of the point at the origin (y-axis $y = 0$) drops to zero at vanishing smoothing factor $\sigma_{\text{van}} = 0.184$ as specified in Figure S6. Specifically, one can readily verify that the certified radii of points closer to the origin increase with the growing smoothing factor $\sigma$ but begin to decrease at certain point, which is coherent with our observations through Figure 4 of the main text. Conducting similar experiments for different dimensions completes the maximum certified radius vs. data dimension relationship as shown in Figure S8.
Figure S7. The certified radius of smoothed classifiers with an increasing input-space dimension when $d = 30$.

Figure S8. The maximum certified radius with an increasing input-space dimension in the exemplary case.

D.3. Semi-bounded decision region certified radius behaviors w.r.t data dimensions

In Figure S9, we show the unscaled certified radius $\mu$ as a function of an increasing smoothing factor $\sigma$ for different input data dimension $d$ with fixed narrowness $\theta = 45^\circ$. One can then see similar trend as told in Figure 4 of the main text in the bounded decision region case, the maximum certified radius (the peak) also decreases with the increasing dimension.
Figure S9. The unscaled certified radius $\mu$ of a point on the axis $v$ for different input data dimension $d$. 