Critical Exponents of the 3-dimensional Blume-Capel model on a cellular automaton

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Abstract

The static critical exponents of the three dimensional Blume-Capel model which has a tricritical point at $D/J = 2.82$ value are estimated for the standard and the cooling algorithms which improved from Creutz Cellular Automaton. The analysis of the data using the finite-size scaling and power law relations reproduce their well-established values in the $D/J < 3$ and $D/J < 2.8$ parameter region at standard and cooling algorithm, respectively. For the cooling algorithm at $D/J = 2.8$ value of single-ion anisotropy parameter, the static critical exponents are estimated as $\beta = 0.31$, $\gamma = \gamma' = 1.6$, $\alpha = \alpha' = 0.32$ and $\nu = 0.87$. These values are different from $\beta = 0.31$, $\gamma = \gamma' = 1.25$, $\alpha = \alpha' = 0.12$ and $\nu = 0.64$ universal values. This case indicated that the BC model exhibit an ununiversal critical behavior at the $D/J = 2.8$ parameter value near the tricritical point($D/J = 2.82$). The simulations carried out on a simple cubic lattice with periodic boundary conditions.

Keywords: Blume-Capel Model; Creutz Cellular Automaton; Finite-Size Scaling; Universality; Simple Cubic Lattice
1. Introduction

One of the interesting problems in the study of phase transitions of model systems is to determine the order of phase transition and universality of the systems. Blume-Capel model which exhibits a tricritical behavior has been of considerable interest in recent years. The Hamiltonian of the model is given by,

\[ H_I = -J \sum_{<ij>} S_i S_j + D \sum_i S_i^2 \]  

where \( s_i = -1, 0, 1 \) and the first sum is carried out over all nearest-neighboring (nn) spin pairs on a three-dimensional simple cubic lattice. The parameters of the \( J \) and \( D \) are the bilinear interaction energy and single-ion anisotropy constant, respectively. The BC model has been applied with success in many different physical situations such as the magnetic phase transitions[1,2], structural transitions[3], dilute Ising ferromagnets[4] and staged intercalation compounds[5]. It is not exactly solvable in three-dimension, but it has been studied over finite and infinite d-dimensional lattices by means of many different simulation and approximate techniques. An extensive analysis of this model for three-dimensional lattices was made using the mean-field approximation[1,2], the effective field theory[6,7], the Bethe-Peierls approximation[8,9], the series expansion methods[10,11], the self-consistent Ornstein Zernike (SCOZ) approximation [12,13], the renormalization group theory[14,15], the cluster variation method[16,17], the Monte Carlo method [18 – 22], cellular automaton[23]. Most of these analysis predict in the BC model the existence of a tricritical point at which the phase transition changes from second-order to first order for a \( D/J \) value in the interval
$2.7 < \frac{D}{J} < 2.9$ on the simple cubic lattice. The problem of identifying the tricritical point is particularly difficult in numerical simulations. Difficulties have arisen in the inaccessible of the entire region of metastable and unstable states properties of first order phase transitions.

In previous paper[23], we investigated the tricritical behavior of the 3-d Blume-Capel model using an improved algorithm from Creutz Cellular automaton (CCA) Ising model on a simple cubic lattice. The phase diagram characterizing phase transition and the tricritical point value of the model is obtained. For determining of the tricritical point, the thermodynamic quantities are computed using two different procedures which called as the standard and the cooling algorithm for the anisotropy parameter values in the interval $3 \geq \frac{D}{J} \geq -8$. The simulations confirm the existence of a tricritical point at which the phase transition changes from second-order to first order at the $\frac{D}{J} = 2.82$ value for the cooling algorithm. The simulations indicates that the cooling algorithm is a suitable procedure for the calculations near the first order phase transition region, and the cooling rate is an important parameter in the determining of the phase boundary. The CCA algorithm is first introduced for spin-1/2 Ising model by Creutz[24] which is a microcanonical algorithm interpolating between the canonical Monte Carlo and molecular dynamics techniques on a cellular automaton. The Creutz algorithm for the spin-1/2 Ising model in two and higher dimensions[25, 26] and spin-1 Ising model in two dimensions[27 – 29] has been proven to be successful in producing the value of the static critical exponents.

In the present paper, we investigate the universality of the three dimen-
sional Blume-Capel model in the second order phase transition region using
the standard and cooling algorithms. For this purpose, the static critical
exponents are estimated by analyzing the data within the framework of
finite size scaling theory. At the same time, their values are calculated using
power law relations of related thermodynamic quantities. Three dimen-
sional Blume-Capel model is expected to be in the universality class of the
three dimensional Ising model for second order phase transition region with
critical exponents $\alpha = 0.12$, $\beta = 0.31$, $\gamma = 1.25$ and $\nu = 0.64$ [30]. The
simulations carried out on simple cubic lattice $L \times L \times L$ of linear dimension
$L = 8, 12, 16, 20$ and $24$ with periodic boundary conditions. The remainder
of the paper is organized as follows. The details of the model are given in
Section 2, the data are analyzed and the results are discussed in section 3
and a conclusion is given in Section 4.

2. Model

Three variables are associated with each site of the lattice. The value
of each site is determined from its value and those of its nearest- neighbors
at the previous time step. The updating rule, which defines a deterministic
cellular automaton, is as follows: Of the three variables on each site, the first
one is the Ising spin $B_i$. Its value may be 0 or 1 or 2. The Ising spin energy
for the model is given by Eq.1. In Eq.1, $S_i = B_i - 1$. The second variable
is for the momentum variable conjugate to the spin (the demon). The
kinetic energy associated with the demon, $H_K$, is an integer, which equal to
the change in the Ising spin energy for the any spin flip and its values lie in
the interval $(0, m)$. The upper limit of the interval, $m$, is equal to $24J$. The
total energy

\[ H = H_I + H_K \]  \hspace{1cm} (2)

is conserved.

The third variable provides a checkerboard style updating, and so it allows the simulation of the Ising model on a cellular automaton. The black sites of the checkerboard are updated and then their colour is changed into white; white sites are changed into black without being updated. The updating rules for the spin and the momentum variables are as follows: For a site to be updated its spin is changed one of the other two states with 1/2 probability and the change in the Ising spin energy, \(dH_I\), is calculated. If this energy change is transferable to or from the momentum variable associated with this site, such that the total energy \(H\) is conserved, then this change is done and the momentum is appropriately changed. Otherwise the spin and the momentum are not changed.

For a given total energy the system temperature is obtained from the average value of kinetic energy, which is given by:

\[
\langle E \rangle = \frac{\sum_{n=0}^{m} n e^{-nJ/kT}}{\sum_{n=0}^{m} e^{-nJ/kT}}
\]  \hspace{1cm} (3)

where \(E = H_K\). The expectation value in Eq. 3 is average over the lattice and the number of time steps. Because of the third variable, the algorithm requires two time steps to give every spin of the lattice a chance to change. Thus, in comparison to ordinary Monte Carlo simulations, two steps correspond to one full sweep over the system variables.
The initial configurations with the different total energy are obtained using standard and cooling algorithms[23]. In the standard algorithm, all the spins take the ferromagnetic ordered structure (↑↑) and the kinetic energy is, randomly, given to the lattice via the second variables in the black sites such that the kinetic energy is equal to the change in the Ising spin energy for flipped of the spin at those sites. This initialization procedure resets the starting configuration at each total energy. The cooling algorithm is divided into two basic parts, the initialization procedure and the taking of measurements. In the initialization procedure, firstly, all the spins in the lattice sites take the ferromagnetic ordered structure (↑↑) and the kinetic energy per site in the all lattice sites is equal to the maximum change in the Ising spin energy for any spin flip using the second variables. This configuration is run during the 10,000 cellular automaton time steps. In the next step, last configuration in the disordered structure has chosen as a starting configuration for the cooling run. Rather than resetting the starting configuration at each energy, it was convenient to use the final configuration at a given energy as the starting point for the next. During the cooling cycle, energy was subtracted from the spin system through the second variables ($H_k$) after the 1,000,000 cellular automaton steps.

3. Results and discussion

The universality of the three dimensional Blume-Capel model is investigated in the continuous transition region using standard and cooling algorithms. At the cooling algorithm, the cooling rate is equal to 0.01$H_k$ per site for all the $D/J$ values, but the kinetic energy of the system reduced by the
different cooling amounts per site because the kinetic energy, \( H_k \), is an integer variable in the interval \((0, m)\). The computed values of the quantities are averages over the lattice and over the number of time steps (1,000,000) with discard of the first 100,000 time steps during which the cellular automaton develops.

The critical temperatures are estimated using two different ways. Firstly, the critical temperatures are estimated from the temperature variation of the Binder forth-order cumulant\,[31,32]\ for the finite lattices. The Binder fourth-order cumulant of the magnetization is given by,

\[
g_L = 1 - \langle M^4 \rangle / 3 \langle M^2 \rangle^2 .
\]  

The temperatures variations of the Binder cumulant are illustrated in Fig.1(a) for the different lattice sizes at a selected \( D/J \) value. The infinite lattice critical temperatures \( T_c(\infty, D/J) \) are obtain from the intersection of the Binder cumulants curves for the different lattice sizes in Fig1(a). Furthermore, the critical exponent \( \nu \) can be obtained using the finite size scaling relation for the Binder cumulant, which is defined by\,[32],

\[
g_L = G(\varepsilon L^{-1/\nu})
\]  

where \( \varepsilon = (T - T_c(\infty))/T_c(\infty) \). It can be seen from Fig.1(b) that the scaling data for the finite size lattices lie on a single curve near the critical temperature when the value of the correlation length critical exponent is equal to the universal value of \( \nu = 0.64 \) in the \( D/J < 3 \) and \( D/J < 2.8 \) parameter regions at the standard and cooling algorithm, respectively. The scaling
of the Binder cumulant for all second order phase regions exhibits a similar behavior except $D/J = 2.8$ value for cooling algorithm. The scaling of the Binder cumulant at $D/J = 2.8$ parameter value is shown in Fig.2 for standard and cooling algorithm. Although, the scaling data for finite size lattices lie on single curve with $\nu = 0.64$ for standard algorithm (Fig.2(a)). The data of Binder cumulant could not be scaled with $\nu = 0.64$ universal value (Fig.2(b)). On the other hand, it is scaled with $\nu = 0.87$ (Fig.2(c)). This result indicates that the value of the correlation length critical exponent at $D/J = 2.8$ is not equal the theoretical value ($\nu = 0.64$) at the cooling algorithm.

Secondly, the critical temperatures are also obtained from susceptibility maxima $T_\chi(L, D/J)$ and specific heat maxima $T_C(L, D/J)$. According to finite size scaling theory, the infinite lattice critical temperature $T_c(\infty, D/J)$ is given by,

$$T_c(\infty, D/J) = T_c(L, D/J) + aL^{-1/\nu}. \quad (6)$$

For the all $D/J$ values, infinite lattice critical temperatures are obtained from the extrapolation of susceptibility and specific heat peak temperatures to $1/L^{1/\nu} \to 0$ and from intersection of the Binder cumulant curves at finite lattices. The critical temperature values estimated from these methods are given in Table 1. The obtained values from extrapolation and Binder cumulant are in good agreement with each other. On the other hand, the obtained critical temperatures values at the cooling and the standard algorithms are not in agreement with each other above the $D/J = 2.2$ value.
as the increasing of the $D/J$ value. However, the estimated critical temperatures using the cooling algorithm are in good agreement with other estimations[6,8,10,12,16] for all $D/J$ parameter values.

The values of the static critical exponents are estimated using power-law relations and the finite-size scaling relations of thermodynamic quantities. Firstly, the critical exponents $\beta(L)$, $\gamma(L)$ and $\alpha(L)$ for each lattice are obtained from the log-log plots of the following power-law relations:

\[
M = \varepsilon^{\beta} \quad \varepsilon \to 0^- \quad (7)
\]

\[
M = (-\varepsilon)^{\beta'} \quad \varepsilon \to 0^+ \quad (8)
\]

\[
kT\chi = \varepsilon^{-\gamma} \quad \varepsilon \to 0^- \quad (9)
\]

\[
kT\chi = (-\varepsilon)^{-\gamma'} \quad \varepsilon \to 0^+ \quad (10)
\]

\[
C = \varepsilon^{-\alpha} + b^- \quad \varepsilon \to 0^- \quad (11)
\]

\[
C = (-\varepsilon)^{-\alpha'} + b^+ \quad \varepsilon \to 0^+ \quad (12)
\]

where $\varepsilon = (T - T_c(L))/T_c(L)$. The critical exponents for each lattice are computed, and these critical exponents are plotted against $L^{-1/\nu}$ for $T < Tc(\infty, D/J)$ and $T > Tc(\infty, D/J)$. The data lie on straight lines, and their extrapolations to $1/L^{1/\nu} \to 0$ give the infinite lattice critical exponents. The estimated infinite lattice critical exponents($\beta$, $\beta'$, $\gamma$, $\gamma'$, $\alpha$, $\alpha'$) are shown in Table 2, 3 and 4 for all $D/J$ values. In the $D/J < 3$ parameter region at standard algorithm and in the $D/J < 2.8$ parameter region for cooling algorithm, these values are in good agreement with theoretical ones.
However, the static critical exponents for $D/J = 2.8$ are estimated as $\beta = 0.31$, $\gamma = 1.58$, $\alpha = 0.38$ for $T < T_c(\infty)$ and $\gamma' = 1.56$, $\alpha' = 0.32$ for $T > T_c(\infty)$ at the cooling algorithm.

The finite size scaling relations of the order parameter $M$, the susceptibility $\chi$ and the specific heat $C$ are given by,

\[ M = L^{-\beta/\nu} X(\epsilon L^{-1/\nu}) \quad (13) \]

\[ kT \chi = L^{-\gamma/\nu} Y(\epsilon L^{-1/\nu}) \quad (14) \]

\[ C = L^{-\alpha/\nu} Z(\epsilon L^{-1/\nu}) \quad (15) \]

For large $x = \epsilon L^{-1/\nu}$, the infinite lattice critical behaviors must be asymptotically reproduced, that is,

\[ X(x) = B x^\beta \quad (16) \]

\[ Y(x) = C x^{-\gamma} \quad (17) \]

\[ Z(x) = A x^{-\alpha} \quad (18) \]

The finite size scaling plots of the data for the order parameter $M$ are shown in Fig.3 for selected $D/J$ values. For $\beta = 0.31$ and $\nu = 0.64$ theoretical values, the data lie on a single curve for the temperatures both below and above $T_c(\infty, D/J)$, and validate the finite size scaling theory in the $D/J < 3$
region at standard algorithm and in the $D/J < 2.8$ region at cooling algorithm. Thus, the data for M are in agreement with the universal value of $\beta = 0.31$ for $T < T_c(\infty, D/J)$. Also, the straight line passing through the data for $T > T_c(\infty, D/J)$ behaves according to Eq. 16 with $\beta' = 0.55$ (Fig3(a,b,c)). The order parameter data for $D/J = 2.8$ value are not compatible with the asymptotic form for $\beta = 0.31$ and $\nu = 0.64$, but the data lie on a single curve for $\beta = 0.31$ and $\nu = 0.87$ for the temperatures below $T_c(\infty, D/J)$ on cooling algorithm. However, the order parameter data do not scale at this parameter value for the temperatures above $T_c(\infty, D/J)$ (Fig3(d)).

The finite size scaling plots of the susceptibility are shown in Fig.4 for the selected $D/J$ values together with the straight lines describing the theoretically predicted behavior for large $x$. The scaling of the magnetic susceptibility data agrees with the asymptotic form with the critical exponents $\gamma = \gamma' = 1.25$ and $\nu = 0.64$ for both $T < T_c(\infty, D/J)$ and $T > T_c(\infty, D/J)$ in the $D/J < 3$ region at the standard algorithm and $D/J < 2.8$ region at the cooling algorithm (Fig.4(a, b, c)). The susceptibility data for $D/J = 2.8$ value are not compatible with the asymptotic form for $\gamma = 1.25$, but the data lies on a single curve for $\gamma = \gamma' = 1.6$ and $\nu = 0.87$ for $T < T_c(\infty)$ and $T > T_c(\infty)$ (Fig.4(d)).

The specific heat of an infinite lattice for the Ising model is well described by[33],

\[ C/k = A e^{-a} + b^\pm \]  

(19)
where $b^\pm$ express the nonsingular part of the specific heat. The finite size scaling plots of the singular portion of the specific heat $(C/k - b^\pm)$ are shown in Fig.5 and Fig.6 for the selected $D/J$ values. The scaling data of specific heat lies on a single curve with the universal value of $\alpha = \alpha' = 0.12$ for the temperatures both below and above $T_c(\infty, D/J)$ in the $D/J < 3$ region at the standard algorithm and in the $D/J < 2.8$ region at the cooling algorithm. The data of the specific heat obtained using cooling algorithm for $D/J = 2.8$ scales with $\alpha = \alpha' = 0.32$ and $\nu = 0.87$ at $T < T_c(\infty, D/J)$ and $T > T_c(\infty, D/J)$. The values of $b^\pm$ for $T < T_c(\infty, D/J)$ and $T > T_c(\infty, D/J)$ are given in Table 4. For the order parameter, the susceptibility and the specific heat, estimated critical exponents using finite size scaling theory are in good agreement with estimated values using power law relations.

To get another estimation for these critical exponents, the finite size scaling relations at $T = T_c(\infty)$ are used. The finite size scaling relations of the order parameter and the susceptibility at $T_c$ are given by,

$$M = L^{-\beta/\nu} \quad (20)$$

$$kT\chi = L^{\gamma/\nu} \quad (21)$$

The value of the order parameter and the magnetic susceptibility at $T_c(\infty, D/J)$ are determined and the slope obtained from the log-log plot of the scaling relation corresponding to these quantities gives $\beta/\nu$ and $\gamma/\nu$. The estimated values are given in Table 5. The values of $\beta$ and $\gamma$ obtained from $\beta/\nu$ and $\gamma/\nu$ using $\nu = 0.64$ are in agreement with theoretical values.
in the $D/J < 3$ region at standard algorithm and in the $D/J < 2.8$ region at the cooling algorithm. Furthermore, for $D/J = 2.8$ parameter value, the value of $\beta$ obtained from $\beta/\nu$ is 0.31 using $\nu = 0.87$, and $\gamma$ obtained from $\gamma/\nu$ using $\nu = 0.87$ is equal to 1.36 at the cooling algorithm.

4. Conclusion

The three dimensional Blume-Capel model is simulated using two different procedure which called the standard and cooling algorithms on a cellular automaton. To determine the universality of the Blume-Capel model which has a tricritical point at $D/J = 2.82$ at the cooling algorithm, the static critical exponents ($\alpha, \beta, \gamma$) are estimated using power-law relations and finite-size scaling relations of the related thermodynamic quantities. Furthermore, the value of correlation length critical exponent $\nu$ is obtained using the finite-size scaling relations of the Binder cumulant. At the standard algorithm, the model exhibits a continuous phase transition which compatible with the universal Ising critical behavior in the $D/J < 3$ parameter region. The estimated values of the static critical exponents ($\beta = 0.31$, $\beta' = 0.55$, $\gamma = \gamma' = 1.25$, $\alpha = \alpha' = 0.12$ and $\nu = 0.64$) are independent on single-ion anisotropy parameter. On the other hand, The cooling algorithm calculations show that the BC model is compatible with universal Ising critical behavior in only the $D/J < 2.8$ parameter region. Although, the phase transition is continuous for $D/J = 2.8$ value near the tricritical point($D/J = 2.82$).The static critical exponents are not equal to universal values for second order phase transition. Also, the estimated values ($\beta = 0.31$, $\gamma = \gamma' = 1.6$, $\alpha = \alpha' = 0.32$ and $\nu = 0.87$) of static critical exponents for $D/J = 2.8$ are different from tricrit-
ical point critical exponents[10] \((\beta = 0.25, \gamma = 1, \alpha = 0.5 \text{ and } \nu = 0.5)\). This case indicates that the BC model exhibit an ununiversal critical behavior at the \(D/J = 2.8\) parameter value.

At the same time, the calculated critical temperature values at the cooling and the standard algorithms are not in agreement with each other above the \(D/J = 2.2\) value as the increasing of the \(D/J\) value. However, the estimated critical temperatures using the cooling algorithm are in good agreement with other estimations[10,33] for all \(D/J\) parameter values. The thermodynamic quantities at the standard and cooling algorithms behave according to power-law relations with the different critical exponents for \(D/J = 2.8\). In addition, the phase transition occurs at different critical temperature\(T_c^{\text{gl}} = 2.382 \pm 0.001\) at standard algorithm and \(T_c^{\text{gl}} = 1.61 \pm 0.02\) at cooling algorithm). The expected behavior for the BC model has not arisen at the standard algorithm since it does not produced metastable states in the first order phase transition region. The calculations indicate that the cooling algorithm is a suitable procedure for the calculations near the tricritical point, and the BC model exhibits an ununiversal critical behavior at the \(D/J = 2.8\) parameter value.

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Figure Captions

Fig.1. For the cooling algorithm, (a) The Binder cumulant as a function of $kT/J$, (b) finite size scaling plots of the Binder cumulant for $\nu = 0.64$.

Fig.2. Finite size scaling plots of the Binder cumulant at $D/J = 2.8$ (a) $\nu = 0.64$ for standard algorithm, (b) $\nu = 0.64$ for cooling algorithm, (c) $\nu = 0.87$ for cooling algorithm.

Fig.3. Finite size scaling plots of the order parameter, (a) at $D/J = 2.9$ for standard algorithm, (b) $D/J = 1$ for cooling algorithm, (c) $D/J = 2.8$ for standard algorithm and (d) $D/J = 1$ for cooling algorithm.

Fig.4. Finite size scaling plots of the susceptibility $(\varepsilon = (T-T_c(\infty))/T_c(\infty))$
for $T < T_c$, $\varepsilon' = (T - T_c(\infty))/T$ for $T > T_c$), (a) at $D/J = 2.8$ for standard algorithm, (b) $D/J = 2.85$ for standard algorithm, (c) $D/J = 1$ for cooling algorithm and (d) $D/J = 2.8$ for cooling algorithm, $\varepsilon'$ for $T < T_c$ and $T > T_c$.

Fig.5. Finite size scaling plots of the specific heat at $T < Tc(\infty, D/J)$, (a) at $D/J = 2.8$ for standard algorithm, (b) $D/J = 2.8$ for cooling algorithm, (c) $D/J = 2.9$ for standard algorithm and (d) $D/J = 1$ for cooling algorithm.

Fig.6. Finite size scaling plots of the specific heat at $T > Tc(\infty, D/J)$, (a) at $D/J = 2.8$ for standard algorithm, (b) $D/J = 2.8$ for cooling algorithm, (c) $D/J = 2.9$ for standard algorithm and (d) $D/J = 1$ for cooling algorithm.

Table Captions

Table 1. The estimated infinite lattice critical temperatures $T_c(\infty, D/J)$ for single-ion anisotropy parameter values $(D/J)$.

Table 2. The estimated infinite lattice critical exponents of the order parameter. $\beta(\infty)$ values are obtained from extrapolation of $\beta(L)$ to $1/L^{1/\nu} \rightarrow 0$.

Table 3. The estimated infinite lattice critical exponents of the susceptibility. $\gamma(\infty)$ values are obtained from extrapolation of $\gamma(L)$ to $1/L^{1/\nu} \rightarrow 0$.

Table 4. The estimated infinite lattice critical exponents of the specific heat. $\alpha(\infty)$ values are obtained from extrapolation of $\alpha(L)$ to $1/L^{1/\nu} \rightarrow 0$. $b^\pm$ values are obtained from finite size scaling theory.

Table 5. The $\beta/\nu$ and $\gamma/\nu$ exponents are obtained from finite size scaling relations at $T_c(\infty, D/J)$.
### Table 1

| $D/J$ | $T_\beta^c(\infty)$ | $T_\chi^c(\infty)$ | $T_\lambda^c(\infty)$ | $T_\beta^\mu(\infty)$ | $T_\chi^\mu(\infty)$ | $T_\lambda^\mu(\infty)$ |
|-------|----------------------|----------------------|------------------------|------------------------|------------------------|------------------------|
| -2    | 3.63±0.01            | 3.62±0.01            | 3.60±0.02              | 3.63±0.01              | 3.63±0.02              | 3.64±0.02              |
| 0     | 3.20±0.01            | 3.20±0.02            | 3.19±0.02              | 3.20±0.01              | 3.20±0.02              | 3.20±0.01              |
| 1     | 2.88±0.01            | 2.88±0.01            | 2.88±0.02              | 2.88±0.01              | 2.88±0.03              | 2.87±0.01              |
| 2     | 2.41±0.01            | 2.40±0.01            | 2.39±0.02              | 2.42±0.01              | 2.41±0.02              | 2.43±0.01              |
| 2.2   | 2.13±0.01            | 2.13±0.03            | 2.13±0.03              | 2.27±0.02              | 2.28±0.02              | 2.27±0.01              |
| 2.4   | 2.17±0.01            | 2.17±0.03            | 2.17±0.03              | 2.11±0.02              | 2.10±0.02              | 2.12±0.02              |
| 2.6   | 2.40±0.01            | 2.42±0.02            | 2.40±0.02              | 1.93±0.03              | 1.95±0.03              | 1.93±0.05              |
| 2.8   | 2.38±0.01            | 2.39±0.01            | 2.38±0.02              | 1.61±0.05              | 1.60±0.02              | 1.61±0.02              |
| 2.82  | 2.31±0.01            | 2.30±0.02            | 2.31±0.02              | 1.59±0.03              | 1.61±0.03              | 1.60±0.04              |
| 2.85  | 2.36±0.01            | 2.36±0.02            | 2.36±0.02              | 1.39±0.01              | 1.41±0.02              | 1.41±0.02              |
| 2.9   | 2.38±0.01            | 2.37±0.03            | 2.39±0.03              | 1.30±0.01              | 1.29±0.02              | 1.28±0.02              |

### Table 2

| $D/J$ | $\beta^c$ | $\beta^\mu$ | $\beta^c$ | $\beta^\mu$ |
|-------|-----------|-------------|-----------|-------------|
| -2    | 0.29±0.02 | 0.58±0.01  | 0.30±0.03 | 0.54±0.02  |
| 0     | 0.29±0.06 | 0.57±0.02  | 0.31±0.02 | 0.57±0.01  |
| 1     | 0.30±0.06 | 0.55±0.01  | 0.31±0.02 | 0.54±0.01  |
| 2     | 0.30±0.01 | 0.57±0.01  | 0.31±0.01 | 0.52±0.01  |
| 2.2   | 0.31±0.02 | 0.57±0.06  | 0.31±0.03 | 0.51±0.03  |
| 2.4   | 0.32±0.01 | 0.57±0.01  | 0.31±0.02 | 0.51±0.04  |
| 2.6   | 0.32±0.06 | 0.55±0.06  | 0.31±0.03 | 0.52±0.01  |
| 2.8   | 0.33±0.01 | 0.56±0.02  | 0.31±0.02 | -           |
| 2.82  | 0.33±0.06 | 0.57±0.06  | -         | -           |
| 2.85  | 0.33±0.01 | 0.56±0.01  | -         | -           |
| 2.9   | 0.34±0.01 | 0.58±0.01  | -         | -           |

18
Table 3.

| D/J | standard algorithm | cooling algorithm |
|-----|-------------------|-------------------|
|     | \( \gamma \)      | \( \gamma' \)    | \( \gamma \)    | \( \gamma' \)    |
| -2  | 1.27±0.06         | 1.21±0.06        | 1.24±0.03       | 1.21±0.03        |
| 0   | 1.26±0.01         | 1.18±0.02        | 1.24±0.04       | 1.22±0.02        |
| 1   | 1.29±0.01         | 1.15±0.02        | 1.26±0.06       | 1.21±0.04        |
| 2   | 1.26±0.01         | 1.18±0.02        | 1.22±0.02       | 1.21±0.03        |
| 2.2 | 1.26±0.01         | 1.13±0.01        | 1.26±0.04       | 1.26±0.03        |
| 2.4 | 1.26±0.01         | 1.13±0.01        | 1.24±0.03       | 1.24±0.02        |
| 2.6 | 1.26±0.01         | 1.20±0.02        | 1.23±0.05       | 1.21±0.06        |
| 2.8 | 1.25±0.02         | 1.25±0.01        | 1.58±0.04       | 1.56±0.05        |
| 2.82| 1.19±0.06         | 1.25±0.01        | -               | -                |
| 2.85| 1.25±0.03         | 1.12±0.02        | -               | -                |
| 2.9 | 1.27±0.06         | 1.16±0.02        | -               | -                |

Table 4

| D/J | standard algorithm | cooling algorithm |
|-----|-------------------|-------------------|
|     | \( \alpha \)      | \( \alpha' \)    | \( b^- \)       | \( b^+ \)       | \( \alpha \)    | \( \alpha' \)    | \( b^- \)       | \( b^+ \)       |
| -2  | 0.13±0.03         | 0.12±0.01        | -0.0005         | -0.025          | 0.12±0.02       | 0.12±0.02        | -0.005          | -0.028          |
| 0   | 0.12±0.02         | 0.11±0.01        | -0.005          | -0.08           | 0.12±0.02       | 0.12±0.03        | -0.005          | -0.08           |
| 1   | 0.13±0.03         | 0.12±0.03        | 0               | -0.54           | 0.13±0.03       | 0.11±0.02        | -0.01           | -0.15           |
| 2   | 0.12±0.02         | 0.12±0.03        | -0.007          | -0.4            | 0.13±0.05       | 0.12±0.01        | -0.09           | -0.4            |
| 2.2 | 0.11±0.06         | 0.12±0.03        | -0.006          | -0.81           | 0.13±0.05       | 0.12±0.04        | -0.3            | -0.6            |
| 2.4 | 0.11±0.02         | 0.12±0.02        | 0               | -1.35           | 0.12±0.03       | 0.12±0.03        | 0              | -0.65           |
| 2.6 | 0.11±0.03         | 0.11±0.01        | 0               | -18             | 0.13±0.04       | 0.13±0.05        | -0.15           | -1.6            |
| 2.8 | 0.12±0.02         | 0.10±0.01        | 0               | -5.5            | 0.38±0.07       | 0.32±0.07        | -5             | 0               |
| 2.82| 0.12±0.02         | 0.12±0.01        | 0               | -7.5            | -              | -              | -              | -               |
| 2.85| 0.11±0.02         | 0.11±0.01        | -0.02           | -12             | -              | -              | -              | -               |
| 2.9 | 0.13±0.02         | 0.11±0.01        | -0.06           | -15             | -              | -              | -              | -               |
| $D/J$ | $\beta/\nu$ | $\gamma/\nu$ | $\beta/\nu$ | $\gamma/\nu$ |
|------|-------------|-------------|-------------|-------------|
| $-2$ | 0.49±0.04   | 1.95±0.03   | 0.48±0.06   | 1.94±0.04   |
| $0$  | 0.49±0.05   | 1.95±0.03   | 0.48±0.02   | 1.93±0.04   |
| $1$  | 0.52±0.02   | 1.92±0.02   | 0.48±0.03   | 1.93±0.03   |
| $2$  | 0.44±0.07   | 1.86±0.03   | 0.47±0.03   | 1.66±0.03   |
| $2.2$| 0.50±0.03   | 1.91±0.03   | 0.49±0.02   | 1.92±0.01   |
| $2.4$| 0.49±0.03   | 1.95±0.02   | 0.50±0.06   | 1.71±0.02   |
| $2.6$| 0.48±0.03   | 1.82±0.03   | 0.48±0.04   | 1.95±0.09   |
| $2.8$| 0.50±0.05   | 1.82±0.03   | 0.36±0.05   | 1.56±0.07   |
| $2.82$| 0.49±0.04  | 1.89±0.01   | -           | -           |
| $2.85$| 0.49±0.02  | 1.87±0.02   | -           | -           |
| $2.9$| 0.49±0.03   | 1.84±0.03   | -           | -           |
Figure 2:
Figure 3:
Figure 5:
Figure 6: