Some special functions identities arising from commuting operators

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Abstract. Commuting is an important property in many cases of investigation of properties of operators as well as in various applications, especially in quantum physics. Using the observation that the generalized weighted differential operator of order \( k \) and the weighted Hardy-type operator commute we derive a number of new and interesting identities involving some functions of mathematical physics.

1 Introduction

When studying two operators \( P, Q \) of quantum theory it is crucial whether the relation

\[
[P, Q] = 0, \quad \text{where} \quad [P, Q] = (-\imath \hbar)^{-1}(PQ - QP),
\]

is fulfilled or not. Such physical quantities, for which there are uniquely assigned operators \( P, Q \), are simultaneously measurable if and only if \([P, Q] = 0\). Equivalently, we say that the operators \( P, Q \) commute if the commutator \([P, Q] = 0\), i.e. \( PQ = QP \). There are many known operators for which the commutation relation is fulfilled, e.g. the class of normal operators. On the other side, non-commutative operators are source of some interesting stories in mathematics and physics, e.g. the famous Heisenberg uncertainty principle saying that it is impossible to know the momentum and position of a particle simultaneously (see [6]).

In this paper we are interested in two operators, namely the Hardy-type operator \( H_w \) defined by

\[
H_w f(x) = \frac{1}{w_1(x)} \int_x^\infty w(t) f(t) \, dt, \quad x > 0,
\]

where \(-\infty \leq \alpha \leq \infty\), \( w_1(x) = \int_x^\infty w(t) \, dt \) and \( w, f \) are real measurable, locally integrable functions, and the generalized weighted differential operator \( D^w_k \) of order \( k \) given by

\[
D^w_k f(x) = \left( \frac{w_1(x)}{w(x)} \frac{d}{dx} \right)^k f(x) = \left( \frac{w_1(x)}{w(x)} \frac{d}{dx} \right)^{k-1} \left( \frac{w_1(x)}{w(x)} f'(x) \right),
\]

\( k = 1, 2, \ldots \). Note that operators \( H_w \) are of the interest in various functional spaces mainly in connection with Hardy’s inequality, cf. [3]. The case \( \alpha = 0 \) and \( w \equiv \text{const} \) corresponds to the Hardy’s averaging operator (or, Hardy’s arithmetic mean operator), therefore we propose to say \( H_w \) the Hardy-type

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operator. On the other hand, operators $D^w_k$ appear in the theory of differential equations (see e.g. [5] and [7]). If $w_1(x)/w(x) = 1$, the resulting $k$-th order differential operator will be simply denoted by

$$D_k f(x) = \frac{d^k}{dx^k}f(x), \quad k = 1, 2, \ldots,$$

as usual. A generalization of weighted Hardy’s averaging operator is provided in [3], where the general mean-type inequality involving such operator is investigated.

The first result of this paper is the observation that the operators $D^w_k$ and $H_w$ commute. Using this fact in Section 3 we establish few interesting new identities involving some special functions of mathematical physics.

2 A note on commutation relation

Now we describe an easy observation on commutativity of operators $D^w_k$ and $H_w$. For the sake of brevity let us replace $H_w h(x)$ by $H(x)$. Then we have

$$H(x) = \frac{1}{w_1(x)} \int_\alpha^x w(t)h(t) \, dt. \quad (1)$$

Differentiating the equality (1) and then using integration by parts with $w_1(\alpha) = 0$ we obtain the identity

$$H'(x) = \frac{w(x) \int_\alpha^x w_1(t)h'(t) \, dt}{w_1(x)^2},$$

which may be written as follows

$$\frac{w_1(x)}{w(x)} H'(x) = \frac{1}{w_1(x)} \int_\alpha^x w_1(t)h'(t) \, dt$$

$$= \frac{1}{w_1(x)} \int_\alpha^x w(t) \left( \frac{w_1(t)}{w(t)} h'(t) \right) \, dt, \quad (2)$$

or in the terms of weighted differential operator as

$$D^w_1 H(x) = \frac{1}{w_1(x)} \int_\alpha^x w(t)D^w_1 h(t) \, dt.$$

Induction on $k$ gives

$$D^w_k H_w h(x) = \frac{1}{w_1(x)} \int_\alpha^x w(t)D^w_k h(t) \, dt, \quad k = 1, 2, \ldots. \quad (3)$$

Summarizing the above we get

**Theorem 2.1** For each $k \in \mathbb{N}$ operators $D^w_k$ and $H_w$ commute, i.e.

$$D^w_k H_w = H_w D^w_k. \quad (4)$$

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Remark 2.2 The observation that operators $D^w_k$ and $H_w$ commute for each $k \in \mathbb{N}$ may be also verified using the well-known fact that the operators

$$A f(x) = f(x) + \frac{w_1(x)}{w(x)} f'(x)$$

and $H_w$ are the inverses of each other.

Remark 2.3 Note that for the weighted Hardy’s averaging operator (the case $\alpha = 0$) it is enough to consider the operators

$$C f(x) = e^{-x} \int_{-\infty}^{x} e^t f(t) \, dt,$$

and $L f(x) = f(\ln w_1(x))$ for positive function $w$. Taking into account the relations

$$LC = H_w L, \quad LD_k = D^w_k L,$$

for each $k \in \mathbb{N}$, we have the commutativity of operators $C$ and $D_k$, and hence $H_w$ and $D^w_k$.

Further, let us denote $w_k(x) = \int_0^x w_{k-1}(x) \, dx$, where $w_0(x) = w(x)$, and

$$r_k(x) = \frac{w^2_k(x)}{w_{k-1}(x)w_{k+1}(x)}, \quad k = 1, 2, \ldots .$$

Using $w_1(\alpha) = 0$ and the relation

$$w^2_k(x) \frac{d}{dx} \left( \frac{1}{w(x)} \int_{\alpha}^{x} w(t) f(t) \, dt \right) = w(x) \int_{\alpha}^{x} w_1(t) f'(t) \, dt,$$

we get

$$r_1(x) \frac{d}{dx} H_w h(x) = \frac{1}{w_2(x)} \int_{\alpha}^{x} w_1(t) D_1 h(t) \, dt.$$

When we repeat this argument $k$-times with $w_{\ell+1}(x) = \int_0^x w_\ell(t) \, dt$, for $\ell = 1, 2, \ldots , k$, we obtain

$$r_k(x) \frac{d}{dx} \left( \frac{r_{k-1}(x)}{w_{k+1}(x)} \int_{\alpha}^{x} w_k(t) D_k h(t) \, dt, \quad k = 1, 2, \ldots ,
$$

which yields the following

**Theorem 2.4** In the sense of the above notation

$$R_k H_w = H_w D_k,$$

where the operator $R_k$ is given by

$$R_k \nu(x) = r_k(x) \frac{d}{dx} \left( \frac{r_{k-1}(x)}{\nu(x)} \int_{\alpha}^{x} \left( \frac{d}{dx} \left( \frac{r_1(x)}{\nu(x)} \int_{\alpha}^{x} \left( \frac{d}{dx} \left( \frac{\nu(x)}{\nu(x)} \int_{\alpha}^{x} \right) \right) \right) \right).$$
Remark 2.5 The operator $R_k$ will be called the *quasi-differential operator of order* $k$ because it appears in connection with differential equations with quasi-derivatives, see e.g., [5]. Clearly, if $r_k(x) = 1$ for each $k \in \mathbb{N}$, then $R_k$ reduces to $D_k$ and it is easy to observe that the usual $k$-th differential operator $D_k$ commutes with $H_w$ for each $k \in \mathbb{N}$.

Note that the integral identities (4) and (5) may be equivalently written as differential identities in the form

$$D_1 (w_1(x)D_k^w H_w h(x)) = w(x)D_k^w \left( \frac{1}{w(x)} D_1 \left( w_1(x)H_w h(x) \right) \right),$$

(7)

and

$$D_1 (w_{k+1}(x)R_k H_w h(x)) = w_k(x)D_k \left( \frac{1}{w(x)} D_1 \left( w_1(x)H_w h(x) \right) \right),$$

(8)

respectively. Indeed, the term

$$H_w h(x) = \frac{1}{w_1(x)} \int_x^x w(t)h(t) \, dt$$

is equivalent to

$$\frac{d}{dx} \left( w_1(x)H_w h(x) \right) = w(x)h(x).$$

Now the identity (4) may be rewritten as

$$\frac{d}{dx} \left( w_1(x)D_k^w H_w h(x) \right) = w(x)D_k^w h(x).$$

(9)

If we substitute

$$h(x) = w^{-1}(x) \frac{d}{dx} \left( w_1(x)|H_w h(x)\right)$$

(10)

into (9), we obtain the following differential operator identity

$$\frac{d}{dx} \left( w_1(x)D_k^w H_w h(x) \right) = w(x)D_k^w \left( \frac{1}{w(x)} \frac{d}{dx} \left( w_1(x)H_w h(x) \right) \right),$$

which corresponds to (7). Similarly, the integral identity (5) is equivalent to the following differential identity

$$\frac{d}{dx} \left( w_{k+1}(x) r_k(x) \frac{d}{dx} \left( r_{k-1}(x) \frac{d}{dx} \left( \ldots r_1(x) \frac{d}{dx} H_w h(x) \right) \right) \right)$$

$$= w_k(x)D_k h(x),$$

which may be rewritten by the use of substitution (10) and the quasi-differential operator $R_k$ to desired form (8).

3 Applications to special functions

The obtained relations (4) and (5) involve functions defined as integrals of upper limit. Such functions usually appear in the theory of special functions. Special functions, also denoted as functions of mathematical physics, have important
applications in many areas of mathematics, science and engineering. Using the observation on commuting operators we now present few applications of the above obtained relations to some special functions to derive some interesting identities and representations for them. As far as we know these identities are not known in the available literature, e.g. in [1] and [8].

In the following few examples we consider \( \alpha = 0 \) (the Hardy’s averaging operator) and polynomial weights.

**Example 3.1** Considering the functions \( h(t) = e^{-t} \) and \( w(t) = t^{a-1} \) for \( a > 0 \) we get \( w_1(x)/w(x) = x/a \). Then

\[
H_w h(x) = \frac{1}{\int_0^x t^{a-1} \, dt} \int_0^x t^{a-1} e^{-t} \, dt = \frac{a}{x^a} \gamma(a, x),
\]

where \( \gamma(a, x) \) denotes the (lower) incomplete gamma function (see [1]). For the generalized \( k \)-th derivative of \( H_w h(x) \) we get

\[
D^k w H_w h(x) = \frac{1}{a^{k-1}} \left( \frac{d}{dx} \right)^k \left( x^{-a} \gamma(a, x) \right), \quad k = 1, 2, \ldots .
\]

On the other hand, the direct calculation of \( H_w D^k w h(x) \) yields

\[
H_w D^k w h(x) = \frac{1}{x^a a^{k-1}} \int_0^x t^{a-1} \left( \frac{d}{dt} \right)^k \left( e^{-t} \right) \, dt
\]

\[
= \frac{1}{x^a a^{k-1}} \sum_{i=1}^k (-1)^{k-i+1} S_2(k, k-i+1) \int_0^x t^{a+k-i} e^{-t} \, dt
\]

\[
= \frac{1}{x^a a^{k-1}} \sum_{i=1}^k (-1)^{k-i+1} S_2(k, k-i+1) \gamma(a + k - i + 1, x),
\]

where \( S_2(n, m) \) is the Stirling number of the second kind (see [1]). By Theorem 2.1 we obtain the relation

\[
\left( \frac{d}{dx} \right)^k \left( x^{-a} \gamma(a, x) \right) = \frac{1}{x^a} \sum_{i=1}^k (-1)^{k-i+1} S_2(k, k-i+1) \gamma(a + k - i + 1, x).
\]

For \( k = 1 \) we get the well known relation \( \gamma(a + 1, x) = a \gamma(a, x) - x^a e^{-x} \). Similarly, for \( k = 2 \) we have \( \gamma(a + 2, x) = a(a + 1) \gamma(a, x) - (a + 1 + x) x^a e^{-x} \).

**Remark 3.2** In the literature (see [1]) there exists the following representation

\[
\gamma(a + n, x) = (-1)^n x^{a+n} \frac{d^n}{dx^n} (x^{-a} \gamma(a, x)). \quad (11)
\]

Therefore,

\[
\left( \frac{d}{dx} \right)^k \left( x^{-a} \gamma(a, x) \right) = \sum_{i=1}^k S_2(k, k-i+1) x^{k-i+1} \frac{d^{k-i+1}}{dx^{k-i+1}} (x^{-a} \gamma(a, x)).
\]
Example 3.3 Let us consider \( w(t) = t^n, \ n \in \mathbb{N}_0, \) and \( h(t) = \psi_0(t), \) where \( \psi_0(t) \) is the digamma function. Using the same method as in the previous example, we get

\[
D_k^w H_w h(x) = \frac{1}{(n+1)^{k-1}} \left( \frac{d}{dx} \right)^k \left( \frac{1}{x^{n+1}} \int_0^x t^n \psi_0(t) \, dt \right),
\]

and

\[
H_w D_k^w h(x) = \frac{1}{x^{n+1}(n+1)^{k-1}} \int_0^x t^n \left( \frac{d}{dt} \right)^k \psi_0(t) \, dt.
\]

Since \( \left( \frac{d}{dt} \right)^k \psi_0(t) = \sum_{i=1}^k S_2(k, i) t^i \psi_i(t), \) where \( \psi_n(t) = \frac{d^n}{dt^n} \psi_0(t) \) is the polygamma function, then

\[
H_w D_k^w h(x) = \frac{1}{x^{n+1}(n+1)^{k-1}} \sum_{i=1}^k S_2(k, i) \int_0^x t^{n+i} \psi_i(t) \, dt.
\]

Thus,

\[
\left( \frac{d}{dx} \right)^k \left( \frac{1}{x^{n+1}} \int_0^x t^n \psi_0(t) \, dt \right) = \frac{1}{x^{n+1}} \sum_{i=1}^k S_2(k, i) \int_0^x t^{n+i} \psi_i(t) \, dt.
\]

Finally, for \( k = 1 \) we get

\[
\int_0^x t^{n+1} \psi_1(t) \, dt = x^{n+2} \frac{d}{dx} \left( \frac{1}{x^{n+1}} \int_0^x t^n \psi_0(t) \, dt \right)
= x^{n+1} \psi_0(x) - (n+1) \int_0^x t^n \psi_0(t) \, dt,
\]

which gives a formula for the \( (n+1) \)-th moment of the trigamma function by the use of digamma function.

Example 3.4 Let \( w(t) = t^m, \ m \in \mathbb{N}_0, \) \( h(t) = B_n(t), \ n \in \mathbb{N}_0, \) where \( B_n(t) \) is the \( n \)-th Bell polynomial (see [1]). Then

\[
D_k^w H_w h(x) = \frac{1}{(m+1)^k} \left( \frac{d}{dx} \right)^k \left( \frac{1}{x^{m+1}} \int_0^x t^m B_n(t) \, dt \right),
\]

and

\[
H_w D_k^w h(x) = \frac{1}{x^{m+1}(m+1)^k} \int_0^x t^m \left( \frac{d}{dt} \right)^k \psi_0(t) \, dt.
\]

Using the explicit formula

\[
B_n(t) = \sum_{i=0}^n S_2(n, i) t^i
\]

we get

\[
\frac{1}{x^{m+1}(m+1)^k} \int_0^x t^m \left( \frac{d}{dt} \right)^k B_n(t) \, dt = \frac{1}{(m+1)^k} \sum_{i=1}^n \frac{i^k}{m+i+1} S_2(n, i) x^i.
\]
Thus,
\[
\left( \frac{d}{dx} \right)^k \left( \frac{1}{x^{m+1}} \int_0^x t^m B_n(t) \, dt \right) = \sum_{i=1}^n \frac{i^k}{m+i+1} S_2(n, i) x^i.
\]
Specially, when \( m = 0 \) and \( k = 1 \) we get the relation for integral of the \( n \)-th Bell polynomial
\[
\int_0^x B_n(t) \, dt = xB_n(x) - \sum_{i=1}^n \frac{i}{i+1} S_2(n, i) x^i.
\]

Example 3.5 Consider \( w(t) = 1 \) and \( h(t) = (t^2 - 1)^k, k \in \mathbb{N} \). Then we have
\[ w_k(x) = \frac{x^k}{k!}, \quad \text{and} \quad r_k(x) = \frac{w_k^2(x)}{w_{k-1}(x)w_{k+1}(x)} = \frac{k+1}{k}, \quad k = 1, 2, \ldots. \]
A direct calculation yields
\[
R_kH_w h(x) = (k+1) \left( \frac{1}{x} \int_0^x (t^2 - 1)^k \, dt \right),
\]
and
\[
H_{w_k} D_k h(x) = \frac{k+1}{x^{k+1}} \int_0^x t^k \frac{d^k}{dt^k} (t^2 - 1)^k \, dt = \frac{2^k(k+1)!}{x^{k+1}} \int_0^x t^k P_k(t) \, dt,
\]
where the Rodrigues representation
\[
P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} (x^2 - 1)^k
\]
of Legendre polynomials \( P_k(x) \) has been used (see [1]). Then by Theorem 2.4 we get
\[
\int_0^x t^k P_k(t) \, dt = \frac{x^{k+1}}{2^k k!} \left( \frac{1}{x} \int_0^x (t^2 - 1)^k \, dt \right).
\]
However, direct calculations yield
\[
\frac{1}{x} \int_0^x (t^2 - 1)^k \, dt = \sum_{i=0}^k \frac{(-1)^i \binom{k}{i}}{2(k-i)+1} x^{2(k-i)},
\]
and therefore we have
\[
\frac{d^k}{dx^k} \left( \frac{1}{x} \int_0^x (t^2 - 1)^k \, dt \right) = \sum_{i=0}^k \frac{(-1)^i \binom{k}{i}(2(k-i))!}{(2(k-i)+1)(k-2i)!} x^{k-2i},
\]
from which follows the identity
\[
\int_0^x t^k P_k(t) \, dt = \frac{1}{2^k k!} \sum_{i=0}^k \frac{(-1)^i \binom{k}{i}(2(k-i))!}{(2(k-i)+1)(k-2i)!} x^{k-2i}.
\]
Example 3.6 As in the previous example, put \( w(t) = 1 \) and \( h(t) = \frac{2}{\sqrt{\pi}} e^{-t^2} \).

Then
\[
R_k H_w h(x) = (k + 1) \frac{d^k}{dx^k} \left( \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \right) = (k + 1) \frac{d^k}{dx^k} \left( \frac{\text{erf}(x)}{x} \right),
\]

where \( \text{erf}(x) \) is the Gauss error function. On the other hand
\[
H_{w_k} D_k h(x) = \frac{k + 1}{x^{k+1}} \int_0^x t^k \frac{d^k}{dt^k} \left( \frac{2}{\sqrt{\pi}} e^{-t^2} \right) \, dt
\]
\[
= \frac{2(-1)^k k + 1}{x^{k+1}} \int_0^x t^k H_k(t) e^{-t^2} \, dt,
\]
where \( H_k \) is a Hermite polynomial (see [1]). Therefore we get the relation
\[
\frac{2}{\sqrt{\pi}} \int_0^x t^k H_k(t) e^{-t^2} \, dt = (-1)^k x^{k+1} \frac{d^k}{dx^k} \left( \frac{\text{erf}(x)}{x} \right).
\]

Moreover, using the MacLaurin series representation
\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (2n+1)} x^{2n+1},
\]
we get
\[
\int_0^x t^k H_k(t) e^{-t^2} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^{n+k} (2n)!}{(2n+1)n!(2n-k)!} x^{2n-k}, \quad x > 0.
\]

Example 3.7 Let \( h(t) = t^ke^{-t}, \ k \in \mathbb{N}_0 \) and \( w(t) = 1 \). Then
\[
R_k H_w h(x) = (k + 1) \frac{d^k}{dx^k} \left( \frac{1}{x} \int_0^x t^k e^{-t} \, dt \right) = (k + 1) \frac{d^k}{dx^k} \left( \frac{\gamma(k+1,x)}{x} \right),
\]
and
\[
H_{w_k} D_k h(x) = \frac{k + 1}{x^{k+1}} \int_0^x t^k \frac{d^k}{dt^k} \left( t^k e^{-t} \right) \, dt
\]
\[
= \frac{(k + 1)!}{x^{k+1}} \int_0^x t^k e^{-t} L_k(t) \, dt,
\]
where the Rodrigues representation
\[
L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x})
\]
for the Laguerre polynomials \( L_k \) is used. Then
\[
\int_0^x t^k e^{-t} L_k(t) \, dt = \frac{x^{k+1}}{k!} \frac{d^k}{dx^k} \left( \frac{\gamma(k+1,x)}{x} \right).
\]

According to [11] we have
\[
\frac{\gamma(k+1,x)}{x} = (-1)^k x^k \frac{d^k}{dx^k} \left( \frac{\gamma(1,x)}{x} \right) = (-1)^k x^k \frac{d^k}{dx^k} \left( \frac{1-e^{-x}}{x} \right),
\]
and therefore
\[
\frac{x^{k+1}}{k!} \frac{d^k}{dx^k} \left( \frac{\gamma(k+1,x)}{x} \right) = (-1)^k x^{k+1} \sum_{i=0}^{k} \binom{k}{i} \frac{x^i}{i!} \frac{d^{k+i}}{dx^{k+i}} \left( \frac{1 - e^{-x}}{x} \right).
\]

Since
\[
\frac{d^{k+i}}{dx^{k+i}} \left( \frac{1 - e^{-x}}{x} \right) = (-1)^{k+i}(k+i)! \frac{1}{x^{k+i+1}} + (-1)^{k+i+1} \sum_{j=0}^{k+i} T(k+i,j) e^{-x},
\]
where \(T(n,k)\) are the permutation coefficients giving number of permutations of \(n\) things \(k\) at a time, then
\[
\int_0^x t^k e^{-t} L_k(t) \, dt = \sum_{i=0}^{k} \frac{(-1)^{i+1}}{i!} \binom{k}{i} \left[ e^{-x} \sum_{j=0}^{k+i} T(k+i,j) x^{k+i-j} - (k+i)! \right].
\]

In the next three examples we derive special function relations involving \(\alpha = +\infty\), resp. \(\alpha = -\infty\), and non-polynomial weights.

**Example 3.8** Let \(\alpha = +\infty\) and \(w(t) = e^{-t}\). Then \(w_k(t) = e^{-t}\) for each \(k \in \mathbb{N}\), and therefore \(r_k(t) = 1\) for each \(k \in \mathbb{N}\). Choosing \(h(t) = t^{a-1}, a > 0\), according to Remark 2.5 we have
\[
e^x \int_x^\infty e^{-t} \frac{d^k}{dt^k} (t^{a-1}) \, dt = \frac{d^k}{dx^k} \left( e^x \int_x^\infty e^{-t} t^{a-1} \, dt \right) = \frac{d^k}{dx^k} (e^x \Gamma(a,x)),
\]
where \(\Gamma(a,x) = \Gamma(a) - \gamma(a,x) = \int_x^\infty e^{-t} t^{a-1} \, dt\) is the (upper) incomplete gamma function (see [II]). Since
\[
\frac{d^k}{dx^k} (e^x \Gamma(a,x)) = e^x \sum_{i=0}^{k} \frac{d^i}{dx^i} \Gamma(a,x),
\]
then we finally get the relation
\[
\sum_{i=0}^{k} \frac{d^i}{dx^i} \Gamma(a,x) = (a-1)_k \Gamma(a-k,x), \quad a > k,
\]
where \((x)_n = x(x-1) \ldots (x-(n-1))\) is the falling factorial. Observe that the case \(k = 1\) in the last relation corresponds to the well-known relation
\[
\Gamma(a+1,x) = a \Gamma(a,x) + x^a e^{-x}.
\]

**Example 3.9** Let \(\alpha = -\infty\). Consider functions \(w(t) = e^t\) and \(f(t) = \exp \left( -\frac{(t-m)^2}{2\sigma^2} \right)\), where \(-\infty < m < \infty\), and \(0 < \sigma < \infty\). Then by Remark 2.5 we have
\[
\frac{d^k}{dx^k} \left( -e^x \int_{-\infty}^x e^{-\frac{(t-m)^2}{2\sigma^2}} \, dt \right) = -e^x \int_{-\infty}^x e^t \frac{d^k}{dt^k} e^{-\frac{(t-m)^2}{2\sigma^2}} \, dt.
\]
Note that the integral \( \int_{-\infty}^{x} e^{-\frac{(t-m)^2}{2\sigma^2}} \, dt \) is the normal distribution function with mean \( m \) and standard deviation \( \sigma \). Since

\[
\int_{-\infty}^{x} e^{-\frac{(t-m)^2}{2\sigma^2}} \, dt = \sigma \sqrt{\frac{\pi}{2}} \left( 1 + \text{erf} \left( \frac{x-m}{\sqrt{2\sigma}} \right) \right),
\]

see [1], then by Leibnitz rule we have

\[
d^k \frac{d}{dx} \left( e^{-x} \int_{-\infty}^{x} e^{-\frac{(t-m)^2}{2\sigma^2}} \, dt \right) = \sigma \sqrt{\frac{\pi}{2}} e^{-x} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{d^{k-i}}{dx^{k-i}} \text{erf} \left( \frac{x-m}{\sqrt{2\sigma}} \right).
\]

Using the relation

\[
\frac{d^{n+1}}{dx^{n+1}} \text{erf}(x) = (-1)^n \frac{2}{\sqrt{\pi}} H_n(x) e^{-x^2}, \quad n \in \mathbb{N},
\]

(see [1]) where \( H_n \) is the Hermite polynomial, yields

\[
d^k \frac{d}{dx} \left( e^{-x} \int_{-\infty}^{x} e^{-\frac{(t-m)^2}{2\sigma^2}} \, dt \right) = (-1)^k \sigma \sqrt{\frac{\pi}{2}} e^{-x} \left[ \text{erf} \left( \frac{x-m}{\sqrt{2\sigma}} \right) - \frac{2}{\sqrt{\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \sum_{i=0}^{k-1} \binom{k-1}{i} (\sqrt{2\sigma})^{1-k+i} H_{k-i-1} \left( \frac{x-m}{\sqrt{2\sigma}} \right) \right].
\]

On the other hand,

\[
\frac{d^k}{dt^k} e^{-\frac{(t-m)^2}{2\sigma^2}} = (-1)^k (t-m+\sigma^2)^k \sigma^{-2k} e^{-\frac{(t-m)^2}{2\sigma^2}} + t, \quad t > 0,
\]

and therefore

\[
e^{-x} \int_{-\infty}^{x} e^t \frac{d^k}{dt^k} e^{-\frac{(t-m)^2}{2\sigma^2}} \, dt = (-1)^k \sigma^{-2k} e^{-x} \int_{-\infty}^{x} (t-m+\sigma^2)^k e^{-\frac{(t-m)^2}{2\sigma^2}} \, dt.
\]

Summarizing the above computations we obtain

\[
\int_{-\infty}^{x} (t-m+\sigma^2)^k e^{-\frac{(t-m)^2}{2\sigma^2}} \, dt
\]

\[
= \sigma \sqrt{\frac{\pi}{2}} \left[ \text{erf} \left( \frac{x-m}{\sqrt{2\sigma}} \right) - \frac{2}{\sqrt{\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \sum_{i=0}^{k-1} \binom{k-1}{i} (\sqrt{2\sigma})^{1-k+i} H_{k-i-1} \left( \frac{x-m}{\sqrt{2\sigma}} \right) \right].
\]

**Example 3.10** Let \( \alpha = -\infty \). If \( w(t) = t e^{-t^2/2} \), then \( w_1(t)/w(t) = -1/t \) for \( t > 0 \). Putting \( h(t) = \sin t/t \) we get

\[
\left( \frac{1}{x} \frac{d}{dx} \right)^k \left( e^{x^2/2} \int_{-\infty}^{x} e^{-x^2/2} \sin t \, dt \right) = e^{x^2/2} \int_{0}^{x} t^{1-k} e^{-t^2/2} j_k(t) \, dt,
\]

where \( j_k \) is the spherical Bessel function of the first kind and the relation

\[
\left( \frac{1}{t} \frac{d}{dt} \right)^k \frac{\sin t}{t} = \frac{j_k(t)}{t^k}
\]
was used (see [1]). Since

\[ \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \sin t \, dt = \frac{1}{2} \sqrt{\frac{\pi}{2e}} \left( \text{erf} \left( \frac{x+i}{\sqrt{2}} \right) - \text{erf} \left( \frac{x-i}{\sqrt{2}} \right) \right), \]

where \( i \) is the imaginary unit, then we have

\[ \int_{0}^{x} t^{1-k} e^{-\frac{t^2}{2}} j_k(t) \, dt = \frac{1}{2} \sqrt{\frac{\pi}{2e}} \left( -\frac{1}{x} \frac{d}{dx} \right)^k \left[ e^{\frac{x^2}{2}} \left( \text{erf} \left( \frac{x+i}{\sqrt{2}} \right) - \text{erf} \left( \frac{x-i}{\sqrt{2}} \right) \right) \right]. \]

Similarly, if we choose \( h(t) = t^{m+1} j_m(t) \), \( m \) is integer, and use the Rayleigh’s relation (see [1])

\[ \left( -\frac{1}{t} \frac{d}{dt} \right)^k (t^{m+1} j_m(t)) = t^{m-k+1} j_{m-k}(t), \]

the we get

\[ \int_{0}^{x} t^{m-k+2} e^{-\frac{t^2}{2}} j_{m-k}(t) \, dt = e^{-\frac{x^2}{2}} \left( -\frac{1}{x} \frac{d}{dx} \right)^k \left[ e^{\frac{x^2}{2}} \int_{0}^{x} t^{m+2} e^{-\frac{t^2}{2}} j_m(t) \, dt \right]. \]

### 4 Concluding remarks

As far as authors know many of the above obtained identities does not appear in the available literature and therefore we suppose they are new. By all means the method of their acquirement seems to be new and interesting from the viewpoint of applicability of some commutation relations for special operators. This method also provides a way how to obtain other possible identities involving non-elementary functions (not necessarily special ones) useful in pure mathematics, theoretical physics as well as applied sciences.

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