Symmetries in the Hubbard model with n-fold orbital degeneracy

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The present paper studies the symmetries of the Hubbard model of electrons with generally n-fold orbital degeneracy. It’s shown $SU_d(2n)$ and $SU_c(2n)$ symmetries hold respectively for the model with completely repulsive or attractive on-site interaction and that with partly attractive interactions. An extended Lieb-Mattis transformation is given to map these two symmetries into each other. By assuming at most two electrons on each site, on the basis of which two models and for arbitrary chemical potential $\mu$ by symmetry are valid for the unified model with partly attractive interactions. But the sub-symmetry $SU_d(2n) \times SU_c(2n) \rightarrow SU_d(2n)$ another underlying symmetry $SU_c(2n) \times SU_c(2n) \rightarrow SU_c(2n)$ is also revealed for the unified model under the excluding. The symmetry is valid for the partially attractive model with chemical potential $\mu = -U$.

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I. INTRODUCTION

Recently considerable attention has been directed to the studies on correlated electrons in the present of orbital degree of freedom which is relevant to transitional-metal oxides $\beta$-Fe$_2$O$_3$, C$_{60}$ materials and artificial quantum dot arrays $\alpha$. Apart from the numerical $\gamma$ and perturbative $\delta$ works theories based on symmetries were presented for one-dimensional models of these systems. An $SU(4)$ theory describing spin systems with orbital degeneracy was proposed $\epsilon$ for a theoretical understanding of the observed unusual properties. The ground-state phase diagrams for the system with a symmetry breaking of $SU(4) \rightarrow SU(2) \times SU(2)$ were discussed in ref. $\zeta$. For the 2-fold orbital degenerate Hubbard model a recent paper $\eta$ presented the $SU(4)$ theory and showed the underlying $SU_d(4)$ symmetry of spin-orbital double and a charge $SU_c(4)$ symmetry with an extended Lieb-Mattis transformation mapping those two $SU(4)$ generators into each other. On the basis of elementary degenerate perturbative theory, it was also shown that the effective Hamiltonian is equivalent to the $SO(6)$ and $SU(4)$ Heisenberg models respectively at half-filling and quarter-filling with strong coupling. In ref. $\theta$, the one-dimensional $SU(4)$ Hubbard model is extensively studied on the basis of Bethe ansatz solution. As for the symmetry theory of one-dimensional Hubbard model without orbital degeneracy, it has been well-investigated. Yang introduced the pairing operators and so constructed the symmetry $SU(2) \times SU(2)$ $\iota$. Based on the symmetry Fabian H.L.E.\textsuperscript{f}er et. al. $\kappa$ discussed the completeness of the Bethe ansatz solutions, M.Pernici $\lambda$ showed the off-diagonal long-range order, D.B.Uglov and V.E.Korepin presented the Yangian symmetry $Y(sl(2)) \subset Y(sl(2))$ $\mu$, and Fabian H.L.E.\textsuperscript{f}er and Holger Frahm considered the density correlations $\nu$. And it has been argued that the two-dimensional single-band Hubbard model has approximate $SO(5)$ symmetry $\omega$. But the research works on the Hubbard model with orbital degeneracy are still in accumulation. In the present paper we study the symmetries of the Hubbard model of electrons with generally n-fold orbital degeneracy. We show and clarify that the $SU_d(2n)$ and $SU_c(2n)$ symmetries hold respectively, unlike in the simple Hubbard model for which both of the two symmetries are valid, for model with unified on-site interaction and that with partly attractive interactions. But the sub-symmetry $SU_d(2n) \times SU_c(2n)$ $\rho$ is found both valid for the two models and for arbitrary chemical potential $\mu$. An extended Lieb-Mattis transformation as in $\sigma$ is given to map these two symmetries into each other. By assuming at most two electrons on each site, on the basis of which the Bethe ansatz can be applied in one-dimensional model, we find the $SU_d(2n)$ and $SU_c(2n)$ symmetries both exist in each kind of the two models so we have a larger symmetry $SU_d(2n) \times SU_c(2n)$. Under the exclusion another underlying symmetry $SU_d(2n) \times SU_c(2n) \rightarrow SU_d(2n) \times SU_c(2n)$, is also revealed for the unified model with $\mu = U$ that the model totally possesses the symmetry $SU_d(2n) \times SU_c(2n) \rightarrow [SU_d(2n) \times SU_c(2n) \otimes SU_d(2n) \times SU_c(2n)]$. The underlying symmetry is also valid for the partially attractive model with $\mu = -U$.

Consider the n-fold orbital degenerate electrons with states...
relating to positive roots can be obtained by 

\[
|1\rangle = |1, \uparrow\rangle, \quad |2\rangle = |1, \downarrow\rangle, \\
...
\]

\[
|2n-1\rangle = |n, \uparrow\rangle, \quad |2n\rangle = |n, \downarrow\rangle, 
\]

where in a state \(|l, \sigma\rangle\) \(l\) denotes the \(l\)th orbital component and \(\sigma = \uparrow, \downarrow\) label the spin components. We start with a general Hamiltonian with \(n\)-fold orbital degeneracy expressed by

\[
H = -\sum_{x,x'} \sum_{\alpha} \left( t_{xx'} C^+_\alpha(x) C_\alpha(x') + t^*_{xx'} C^+_\alpha(x') C_\alpha(x) \right) + \sum_{x} \sum_{a \neq a'} U_{aa'} n_a(x) n_{a'}(x) - \mu \sum_{x,a} n_a(x)
\]

where \(C^+_\alpha(x)\) creates a fermion of state \(|a\rangle\) at site \(x\) and \(n_a(x)\) is the corresponding particle number operator. The notation of site is not restricted to one-dimensional case.

II. SU\(_D(2N)\) AND SU\(_C(2N)\) SYMMETRIES

Besides the \(U(1)\) symmetry there exist two kinds of \(SU(2n)\) symmetries for the orbital degenerate Hubbard model. We define

\[
E_{ss'} = \sum_x C^+_s(x) C_{s'}(x), \\
D_m = N_m - N_{m+1}, \quad N_m = \sum_x C^+_m(x) C_m(x),
\]

they fulfill the commutation relations

\[
[E_{ss'}, E_{tt'}] = \delta_{s',t} E_{st'} - \delta_{s,t} E_{ts'}, \\
[D_m, E_{ss'}] = (\delta_{m,s} - \delta_{m,s'} - \delta_{m+1,s} + \delta_{m+1,s'}) E_{ss'}.
\]

These operators can construct an \(SU(2n)\) Lie algebra

\[
SU_d(2n) : \{D_m, \ E_{ss'} \mid m = 1, ..., 2n-1; 1 \leq s \neq s' \leq 2n\},
\]

with \(n(2n-1)\) of \(E_{ss'}\) and there are totally \((2n)^2 - 1\) of generators. \(\{D_m\}\) forms the commuting Cardan subalgebra of rank \(2n - 1\). The \(E_{s(s+1)}\’s\) are the generators related to the simple roots. For any of \(t_{xx'}\) and \(\mu\), all generators of \(SU_d(2n)\) commute with the Hamiltonian \(H\) with unified on-site interaction \(U_{aa'} = U\), so we have \(SU_d(2n)\) symmetry of spin-orbital double in this case.

Let us define another set of operators

\[
F_{\alpha_{2k-1}} = \sum_x f(x) C^+_x C^{+\prime}_{2k-1}(x), \\
F_{\alpha_{2k}} = \sum_x f(x) C_x C^{+\prime}_{2k}(x), \\
Q_m = \frac{1}{2} \sum_x [C^+_m C_m(x) + C^{+\prime}_{m+1} C^{+\prime}_{m+1}(x) - 1],
\]

where \(k = 1, ..., n, \ m = 1, ..., 2n-1, \ f(x) = 1, \ f(x + \delta) = -f(x)\) for any site \(x\) and nearest-neighbor \(x + \delta\). The above operators can realize another \(SU(2n)\) Lie algebra which we shall denote by \(SU_c(2n)\):

\[
\{Q_m, F_{\alpha}, F_{-\alpha}\}. 
\]

\(\{Q_m\}\) is the Cardan subalgebra. The \(F_{\alpha_{2k-1}}\) and \(F_{\alpha_{2k}}\) are the generators related to the simple roots, other generators relating to positive roots can be obtained by \(F_{\alpha_{n+\alpha_j}} = [F_{\alpha_j}, F_{\alpha_j}]\), the generators with negative roots will be \(F_{-\alpha} = (F_{\alpha})^\dagger\). If we assume that the on-site coupling \(U_{aa'} = U\) for the states labeled by \(a, a'\) with different spin components while \(U_{aa'} = -U\) for states with the same spin components

\[
U_{aa'} = U \text{ for odd-even pair } a, a', \\
U_{aa'} = -U \text{ for odd-odd or even-even pair } a, a',
\]
and the amplitudes \( t_{xx'} \) of odd-neighbor hopping are real and those of even-neighbor hopping are imaginary

\[
\begin{align*}
  t_{xx'}^+ &= t_{xx'}, \text{ when } x-x' \text{ is odd neighbor}, \\
  t_{xx'}^- &= -t_{xx'}, \text{ when } x-x' \text{ is even neighbor},
\end{align*}
\]

we will have the following relations

\[
[H, F_{\alpha_{2k}}] = 2(\mu - U)F_{\alpha_{2k}}, \quad [H, F_{\alpha_{2k-1}}] = -2(\mu - U)F_{\alpha_{2k-1}}.
\]

(9)

If the chemical potential \( \mu = U \), the Hamiltonian will commute with all the generators of \( SU_c(2n) \) so the model has the charge \( SU_c(2n) \) symmetry. In terms of the partially attractive (8), it can be easily proved that such a Hamiltonian with \( \mu = U \) has the half-filled form

\[
H = -\sum_{x,x'} \sum_a \left( t_{xx'} C_a^+(x) C_a(x') + t_{xx'}^- C_a^+(x') C_a(x) \right) + \sum_x \sum_{a \neq a'} U_{aa'} (n_a(x) - \frac{1}{2}) (n_{a'}(x) - \frac{1}{2}).
\]

(10)

The usual Hubbard model with the nearest-neighbor hopping is included in the class of (8). Whether \( f(x) \) can be well defined depends on the lattice structures, if in a lattice any sum of the nearest-neighbor pair \( \delta + \delta' \) is not a nearest-neighbor and there are even sites on every perpendicular directions, \( f(x) \) can be well defined as

\[
f(x) = \exp(i \pi \cdot x), \quad \pi = (\pi, \pi, ...), \quad x = (x_1, x_2, ...),
\]

(12)

\( x_i \) is the \( i \)th components of the site coordinate in the lattice basis. Such lattices (bipartite lattice) can be simple squared, centered squared in 2-dimension; simple cubic and body-centered cubic in 3-dimension. For 1-d case and even total sites, \( f(x) = \exp(\pi x) \).

It should be noted that the \( SU_d(2n) \) does not commute with the partially attractive Hamiltonian which has only the \( SU_c(2n) \) symmetry, neither does \( SU_d(2n) \) with the unified \( U \) model which possesses the \( SU_d(2n) \) symmetry. The two kinds of symmetries can be mapped into each other by an extended Lieb-Mattis transformation

\[
C_a(x) \mapsto \exp(i \pi \cdot x) C_a^+(x) \text{ for even } a, \\
C_a(x) \mapsto C_a(x) \text{ for odd } a.
\]

(13)

The transformation leaves the hopping term (11) invariant and changes the sign of \( U_{aa'} \) with odd-even pair \( a,a' \) in (11) so maps the model (11) into the unified \( U_{aa'} = -U \) model.

The particle number of each states and the total spin can be expressed by

\[
N_{2n} = \left( N_e - \sum_{m=1}^{2n-1} mD_m \right) / 2n,
\]

\[
N_j = \sum_{m=j}^{2n-1} D_m + \left( N_e - \sum_{m=1}^{2n-1} mD_m \right) / 2n, \quad j < 2n,
\]

(14)

\[
S_{\text{total}} = \sum_{m=1}^{n} D_{2m-1} / 2,
\]

(15)

where \( N_e \) is the number of total electrons.

### III. \( SU^{(E)}_d(N) \otimes SU^{(O)}_d(N) \) SYMMETRY

Unlike in the transitional Hubbard model for which both of the two symmetries are valid for on-site attractive and repulsive interactions, as we can see in previous section, for the orbital-degenerate Hubbard model the \( SU_d(2n) \) symmetry of spin-orbital double only holds for the unified \( U \) Hubbard model with arbitrary chemical potential \( \mu \) whereas the charge \( SU_c(2n) \) symmetry merely exists in the other partly-attractive half-filled model of which the chemical potential is \( \mu = U \). But considering that \( SU_d(2n) \) and \( SU_c(2n) \) share some common generators \( E_{2\nu,2\nu'} \) and \( E_{2\nu-1,2\nu'-1} \), we will find the shared sub-symmetry \( SU^{(E)}_d(n) \otimes SU^{(O)}_d(n) \) yields for the two models with
\[ SU_d^{(e)}(n) : \{ D_{2\nu, 2\nu + 2}, E_{2\nu, 2\nu} | \nu \leq n - 1, 1 \leq \nu' \neq \nu'' \leq n \}, \]
\[ SU_d^{(o)}(n) : \{ D_{2\nu-1, 2\nu+1}, E_{2\nu-1, 2\nu+1} | \nu \leq n - 1, 1 \leq \nu' \neq \nu'' \leq n \}, \]

where \( D_{m,m+2} = N_m - N_{m+2} \). Especially, we shall illustrate that the symmetry is valid for the partially attractive model with arbitrary chemical potential \( \mu \).

As the hopping term in the Hamiltonian and the chemical potential term \( \mu \sum_n n_a(x) \) are \( SU_d(2n) \) invariant, whether the Hubbard model possesses \( SU_d(2n) \) symmetry depends on the commutation relation of the on-site interacting term and the \( SU_d(2n) \) generators

\[
[E_{m}^{k}, \sum_{a \neq a'} U_{aa'} n_a(x) n_{a'}(x)] = C_{m}^{+}(x) C_{m+k}(x) \left( \sum_{a' \neq m+k} n_{a'}(x) U_{m+k,a'} - \sum_{a' \neq m} n_{a'}(x) U_{m,a'} \right) + \left( \sum_{a \neq m+k} n_{a}(x) U_{a,m+k} - \sum_{a \neq m} n_{a}(x) U_{a,m} \right) C_{m}^{+}(x) C_{m+k}(x)
\]

where

\[ E_{m}^{k} = \sum_{x} C_{m}^{+}(x) C_{m+k}(x). \]

Surely for the case of unified \( U_{aa'} = U \) we easily find that the above commutation vanishes so that we have the \( SU_d(2n) \) symmetry in this case, as is obtained in the second section. Although for the partially attractive model the above commutator does not go null for all \( k \)'s and consequently we do not have \( SU_d(2n) \) symmetry, the case with even \( k \) will be an exception. From the partially attractive [8] we find for even \( k \)’s

\[ U_{a,m+k} = U_{a,m} = -U, \]

\[
\left( \sum_{a \neq m+k} n_{a}(x) U_{a,m+k} - \sum_{a \neq m} n_{a}(x) U_{a,m} \right) = n_{m}(x) U_{m,m+k} - n_{m+k}(x) U_{m+k,m}
\]

\[ = (n_{m}(x) - n_{m+k}(x))(-U). \]

Then eq. (17) becomes

\[
[E_{m}^{k}, \sum_{a \neq a'} U_{aa'} n_a(x) n_{a'}(x)] = C_{m}^{+}(x) C_{m+k}(x) \left( n_{m}(x) - n_{m+k}(x) \right) (-U) + (n_{m}(x) - n_{m+k}(x)) C_{m}^{+}(x) C_{m+k}(x) (-U)
\]

\[ = (-C_{m}^{+}(x) C_{m+k}(x) + C_{m+k}(x) C_{m+k}(x)) (-U) = 0 \]

where we have used relations: \( n_{m} C_{m} = C_{m}^{+} n_{m} = 0, n_{m} C_{m}^{+} = C_{m}^{+}, C_{m} n_{m} = C_{m} \). \( E_{m}^{k} \) with even \( k \)'s and odd (or even) \( m \)'s correspond to the \( SU_d^{(e)}(n) \) (or \( SU_d^{(o)}(n) \)) generators. As a result, the \( SU_d^{(e)}(n) \otimes SU_d^{(o)}(n) \) generators in [6] commute with the Hamiltonian of the partially attractive model for any chemical potential \( \mu \). Therefore the \( SU_d^{(e)}(n) \otimes SU_d^{(o)}(n) \) symmetry is shared by the two models both for arbitrary \( \mu \).

**IV. AT MOST TWO ELECTRONS ON EACH SITE**

The application of Bethe ansatz method to the 1-dimensional degenerate Hubbard model is based on such an assumption that prevents scattering process involving three or more electrons on one site [41213]. For the traditional Hubbard model the configurations of more than two electrons on one site are excluded automatically by the Pauli principle. In the continuum limit and for small densities or \( U \gg t \) in the lattice model, the unwanted configurations in degenerate Hubbard model become negligible, so the Hamiltonian with three-electron configurations excluded will describe the system well. If we exclude more than two electrons on each site, we will find that the Hamiltonian has both \( SU_d(2n) \) and \( SU_c(2n) \) symmetries and furthermore a larger symmetry \( SU_d(2n) \times SU_c(2n) \). In addition, we will find an underlying symmetry \( (SU_c^{(e)}(2)_{p} \times \ldots \times SU_c^{(e)}(2)_{p}) \otimes (SU_c^{(o)}(2)_{p} \times \ldots \times SU_c^{(o)}(2)_{p}) \).

Consider \( U_{aa'} = U \) case, the Hamiltonian reads
where \( < x, x' > \) represents the nearest neighbor sites and let us define
\[
\mathcal{F}_{\alpha zk^{-1}} = \sum_x \exp(i\pi \cdot x) \mathcal{P} C_{2k-1}^{\pm}(x) C_{2k}^{\pm}(x) \mathcal{P},
\]
(19)

\[
\mathcal{F}_{\alpha zk} = \sum_x \exp(i\pi \cdot x) \mathcal{P} C_{2k}(x) C_{2k+1}(x) \mathcal{P},
\]
(20)

where the operator \( \mathcal{P} \) projects onto the subspace of states having at most two electrons on each site \[^{[22]}\). Other generators can be given from the \( \mathcal{F}_{\alpha zk^{-1}} \) and \( \mathcal{F}_{\alpha zk} \) as in the \( SU_c(2n) \) in section II. The \( \mathcal{P} \) operator excludes such terms as \( n_s C_s^+ C_s^\dagger, C_s C_s n_s n_s' \) with three different \( s, s' \) and \( s'' \) so that
\[
[\mathcal{P} C_s^+ C_{s'}^+(x) \mathcal{P}, \sum_{x' \neq a'} \sum_{a} n_a(x') n_{a'}(x')] = -2\mathcal{P} C_s^+ C_{s'}^+(x) \mathcal{P},
\]
(21)

thus we have the relations similar to \([^{[19]}\]

\[
[\mathcal{H}, \mathcal{F}_{\alpha zk}] = 2(\mu - U) \mathcal{F}_{\alpha zk}, \quad [\mathcal{H}, \mathcal{F}_{\alpha z k^{-1}}] = -2(\mu - U) \mathcal{F}_{\alpha z k^{-1}}.
\]
(22)

Set \( \mu = U \) and the \( \mathcal{F}_{\alpha z k^{-1}} \) and \( \mathcal{F}_{\alpha zk} \) will commute with the \( \mathcal{H} \). \( \mathcal{F}_{\alpha z k^{-1}}, \mathcal{F}_{\alpha zk} \) and \( N_s - N_s' \) can generate the \( SU_c(2n) \) symmetry so that \( \mathcal{H} \) has both \( SU_d(2n) \) and \( SU_c(2n) \) symmetries,

\[
SU_c(2n) : \{ Q_m, \quad \mathcal{F}_a \},
\]
(23)

\[
SU_d(2n) : \{ D_m, \quad \mathcal{E}_{ss'} \}
\]
(24)

where \( Q_m = \mathcal{P} Q_m \mathcal{P} \) and the \( SU_d(2n) \) generators are correspondingly revised to be

\[
\mathcal{E}_{ss'} = \mathcal{P} E_{ss'} \mathcal{P}, \quad D_m = \mathcal{P} D_m \mathcal{P}.
\]
(25)

It also can be similarly shown that both of the two symmetries hold for the \( \mathcal{P} \)-modified model with partially attractive \( U_{a a'} = U - U \). As the two symmetries both hold for each of the models we can construct the larger symmetry \( SU_d(2n) \times SU_c(2n) \) for each of them. If we do not exclude more than two electrons on each site, these two symmetries respectively belong to different models.

Besides \( SU_d(2n) \times SU_c(2n) \) there exist some less obvious symmetries. Define

\[
\mathcal{F}_{2k,2k'}^{(+)} = \sum_x \exp(i\pi \cdot x) \mathcal{P} C_{2k}(x) C_{2k'}^{\dagger}(x) \mathcal{P}, \quad \mathcal{F}_{2k,2k'}^{(-)} = (\mathcal{F}_{2k,2k'}^{(+)})^{\dagger},
\]
(26)

\[
\mathcal{F}_{2k-1,2k'-1}^{(+)} = \sum_x \exp(i\pi \cdot x) \mathcal{P} C_{2k-1}(x) C_{2k'-1}^{\dagger}(x) \mathcal{P}, \quad \mathcal{F}_{2k-1,2k'-1}^{(-)} = (\mathcal{F}_{2k-1,2k'-1}^{(+)})^{\dagger},
\]
(27)

such generators are not included in \(^{[23]}\) which contains pairing operators \( \mathcal{F}_{2k,2k'}^{(\pm)} \) only with odd-even pair \( ss' \). It can be easily verified that these operators also commute with the unified \( U \) model \(^{[3]}\) with \( \mu = U \) so we find an underlying symmetry \( SU_c^{(e)}(2p) \times ... \times SU_c^{(e)}(2p) \otimes (SU_c^{(o)}(2p) \times ... \times SU_c^{(o)}(2p) \) with

\[
SU_c^{(e)}(2p) : \{ Q_{2k,2k'}, \quad \mathcal{F}_{2k,2k'}^{(+)} = \mathcal{F}_{2k,2k'}^{(-)} \}
\]
(28)

\[
SU_c^{(o)}(2p) : \{ Q_{2k-1,2k'-1}, \quad \mathcal{F}_{2k-1,2k'-1}^{(+)}, \quad \mathcal{F}_{2k-1,2k'-1}^{(-)} \}
\]
(29)

There are respectively \( C_n^2 = n(n-1)/2 \) of the \( SU_c^{(e)}(2p) \) and \( SU_c^{(o)}(2p) \) symmetries. The extended Lieb-Mattis transformation \(^{[3]}\) maps the above symmetry into itself. An revised Lieb-Mattis transformation mapping into
the corresponding $SU^{(e)}(2)_P$ and $SU^{(o)}(2)_P$ will involve a third kind of Hamiltonians with partially attractive $U_{aa'}$, which differs from what we discussed before. So finally we have the symmetry $SU_d(2n) \times SU_c(2n) \times \left[(SU^{(e)}_c(2)_P \times \ldots \times SU^{(e)}_c(2)_P) \otimes (SU^{(o)}_c(2)_P \times \ldots \times SU^{(o)}_c(2)_P)\right]$ for $[18]$ with $\mu = U$. But for $U_{aa'} = U, -U$ case the underlying symmetry is valid for another chemical potential. The Hamiltonian under that exclusion is

$$\mathcal{H}' = - \sum_{<x,x'>} \sum_a t_{PC_a}^+(x)C_a(x')\mathcal{P} + \sum_x \sum_{a \neq a'} U_{aa'}n_a(x)n_{a'}(x) - \mu \sum_x n_a(x)$$

(30)

where the partially attractive interaction $U_{aa'}$ is also defined by [8]. Compared with [10] the commutation relations are different

$$[\mathcal{H}', \mathcal{F}_{2k,2k'}^{(+)}] = -2(\mu + U)\mathcal{F}_{2k,2k'}^{(+)}$$

(31)

$$[\mathcal{H}', \mathcal{F}_{2k,2k'}^{(-)}] = 2(\mu + U)\mathcal{F}_{2k,2k'}^{(-)}$$

(32)

The different sign of $U$ comes from $U_{2k,2k'} = -U$ while in eq. [10] it’s $U_{2k,2k+1} = U$. Therefore the symmetry $(SU^{(e)}_c(2)_P \times \ldots \times SU^{(e)}_c(2)_P) \otimes (SU^{(o)}_c(2)_P \times \ldots \times SU^{(o)}_c(2)_P)$ holds for $\mu = -U$. It should be noted the symmetry is valid under the exclusion, without the $P$-exclusion its generators will commute with neither of the two kinds of models. And unlike $SU^{(o)}_d(n) \otimes SU^{(e)}_d(n) \subset SU_d(2n)$ in section III, none of the $SU^{(e)}_c(2)_P$ or $SU^{(o)}_c(2)$ is any sub-symmetry of $SU_c(2n)$.

Considering the commutations $\mathcal{F}_{2k}$ and $\mathcal{E}_{ss'}$, we have

$$[SU^{(ss')}_d(2), SU^{(ss')}_c(2)] = 0$$

(33)

$$[SU^{(ss')}_d(2), SU^{(ss')}_{c'}(2)] = 0,$$

for separate pairs $(ss')$ and $(s''s'''$),

$$[SU^{(ss')}_d(2), SU^{(ss')}_{c'}(2)] \neq 0,$$

(34)

where $SU^{(ss')}_d(2)$ and $SU^{(ss')}_c(2)$ are sub-symmetries involving only the states $s$ and $s'$. The whole symmetry cannot be written in a direct product $SU_d(2n) \otimes SU_c(2n)$ but $SU_d(2n) \times SU_c(2n)$. This is also a difference from the single band Hubbard model of which the symmetry in our notation is a direct product of the two $SU(2)$s: i.e., $SO(4) \simeq SU_d(2) \otimes SU_c(2)$. Therefore for the one-dimensional model [18] which can be solved by the Bethe ansatz [22], the discussion on the completeness of the Bethe ansatz solution and the off-diagonal long-range order will be expected quite different since in the single band case it’s based on the vanishing commutation of the $SU_d(2)$ and $SU_c(2)$.

V. BRIEF SUMMARY

In summary, we studied the symmetries of the Hubbard model of $n$-fold orbital degenerate electrons. We show and clarify that the $SU_d(2n)$ and $SU_c(2n)$ symmetries hold respectively for the model with unified on-site interaction and that with partly attractive interactions. An extended Lieb-Mattis transformation is given to map these two symmetries into each other. But the sub-symmetry $SU^{(e)}_d(n) \otimes SU^{(o)}_d(n)$ is found to be possessed by the two models and both for arbitrary chemical potential $\mu$. By excluding more than two electrons on the same sites we find the $SU_d(2n)$ and $SU_c(2n)$ symmetries both exist in each kind of the two models, so we have an enlarged symmetry $SU_d(2n) \times SU_c(2n)$. Under this exclusion, another underlying symmetry $(SU^{(e)}_c(2)_P \times \ldots \times SU^{(e)}_c(2)_P) \otimes (SU^{(o)}_c(2)_P \times \ldots \times SU^{(o)}_c(2)_P)$ is also found for the unified $U$ model with chemical potential $\mu = U$, and consequently this model has the symmetry $SU_d(2n) \times SU_c(2n) \times \left[(SU^{(e)}_c(2)_P \times \ldots \times SU^{(e)}_c(2)_P) \otimes (SU^{(o)}_c(2)_P \times \ldots \times SU^{(o)}_c(2)_P)\right]$. The underlying symmetry is valid for the partially attractive model with chemical potential $\mu = -U$. 

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