Three kinds of novel multi-symplectic methods for stochastic Hamiltonian partial differential equations

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Abstract

Stochastic Hamiltonian partial differential equations, which possess the multi-symplectic conservation law, are an important and fairly large class of systems. The multi-symplectic methods inheriting the geometric features of stochastic Hamiltonian partial differential equations provide numerical approximations with better numerical stability, and are of vital significance for obtaining correct numerical results. In this paper, we propose three novel multi-symplectic methods for stochastic Hamiltonian partial differential equations based on the local radial basis function collocation method, the splitting technique, and the partitioned Runge–Kutta method. Concrete numerical methods are presented for nonlinear stochastic wave equations, stochastic nonlinear Schrödinger equations, stochastic Korteweg-de Vries equations and stochastic Maxwell equations. We take stochastic wave equations as examples to perform numerical experiments, which indicate the validity of the proposed methods.

Keywords: stochastic Hamiltonian partial differential equations, multi-symplecticity, local radial basis function collocation method, splitting technique, partitioned Runge–Kutta method

1. Introduction

A common way to describe the physical and engineering phenomena in the area of fluid dynamics, nonlinear optics, and quantum field theory (see e.g., \cite{2, 11, 16, 18} and references therein) is by means of stochastic Hamiltonian partial differential equations (PDEs). Stochastic Hamiltonian PDEs, which include stochastic wave equations, stochastic Schrödinger equations, stochastic Korteweg-de Vries (KdV) equations, stochastic Maxwell equations, etc., are proposed in \cite{3, 7, 11}, and they have a prominent characteristic, that is, multi-symplectic conservation law. The multi-symplecticity is the concatenation of differential 2-forms in both space and time, decomposes neatly the different facets of the governing equation, and characterizes the geometric invariants of the solution manifold. Theoretical results concerning such multi-symplecticity reformulation can be found in \cite{3, 8, 11} and references therein.

When designing a numerical method, a basic principle is that it should inherit the intrinsic properties of the original system as much as possible. Numerical methods that are incorporated more physical and geometric properties, especially multi-symplectic methods admitting the discrete multi-symplectic conservation law, have remarkable numerical superiority to conventional numerical methods. Recently, multi-symplectic methods possessing good performance in preserving local conservation laws of the original system have been developed in the field of stochastic geometric integration of stochastic Hamiltonian PDEs (see e.g., \cite{3, 5, 7, 8, 10} and references therein). For instance, \cite{3, 10} present multi-symplectic methods for

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stochastic nonlinear Schrödinger equations by making use of the central finite difference method in spatial direction combined with the midpoint method in temporal direction.\cite{8} proposes a multi-symplectic energy-conserving method, based on the wavelet collocation method in space and the symplectic method in time, for a three-dimensional stochastic Maxwell equation with multiplicative noise.\cite{3,7} investigate multi-symplectic methods for the stochastic Maxwell equation with additive noise via the implicit midpoint method and the leapfrog method. To the best of our knowledge, there is few work on the study of constructions of multi-symplectic methods for the general stochastic Hamiltonian PDEs. The first attempt to show the multi-symplectic method for the general 1-dimensional stochastic Hamiltonian PDEs is given in \cite{21}, which takes advantage of symplectic Runge–Kutta methods with two Butcher tableaux. Our results in this paper not only present multi-symplectic partitioned Runge–Kutta methods with more Butcher tableaux, which increase diversity and flexibility of numerical methods, but also propose another two multi-symplectic methods via the local radial basis function (LRBF) collocation method and the splitting technique for the general stochastic Hamiltonian PDEs.

Inspired by the fact that the LRBF collocation method has been successfully utilized to numerically solve deterministic Hamiltonian PDEs, we apply the LRBF collocation method in space and midpoint method in time to derive the first kind of multi-symplectic method, that is, the meshless LRBF collocation midpoint method shown in Section 3. The method is made on the overlapping sub-domains, which significantly reduces the size of the collocation matrix at the cost of solving lots of small matrices, and thus leads to cost efficiency. Moreover, it performs stably, can deal with complicated irregular domains and moving boundary, and possesses a long-time tracking capability. The second strategy of constructing the multi-symplectic method is utilizing the splitting technique allowing one to deal with sequentially a deterministic Hamiltonian PDE and a stochastic system, which are simpler than the original equation. For the numerical study of deterministic Hamiltonian PDEs, a lot of reliable and efficient numerical methods preserving the multi-symplecticity have been given (see \cite{4,9,15}). In Section 4, we first adopt the multi-symplectic Runge–Kutta method applied to the stochastic system, we arrive at the second kind of multi-symplectic method, that is, the splitting multi-symplectic Runge–Kutta method. We would like to mention that the splitting method does not need to handle the interaction between the nonlinear potential and the driving stochastic process. In Section 5, we propose the third kind of multi-symplectic method for four stochastic Hamiltonian PDEs, i.e., stochastic wave equation, stochastic nonlinear Schrödinger equation, stochastic KdV equation, and stochastic Maxwell equation, by employing the partitioned Runge–Kutta method in both temporal and spatial directions. The resulting method maybe explicit for some stochastic Hamiltonian PDEs. For instance, the method based on the symplectic Euler method in both space and time is explicit for the stochastic wave equation.

The paper is organized as follows. In Section 2, we introduce the multi-symplectic conservation law and the definition of multi-symplectic method for stochastic Hamiltonian PDEs. Section 3 presents the first kind of multi-symplectic method, which is constructed by the meshless LRBF collocation method and the midpoint method. Section 4 is devoted to the second kind of multi-symplectic Runge–Kutta method based on the splitting technique and symplectic Runge–Kutta method. In Section 5, we apply the partitioned Runge–Kutta method to deriving the third kind of multi-symplectic method. We take stochastic wave equations as examples to perform numerical experiments, which indicate the validity of the proposed methods. Finally, we give a conclusion in Section 6.

2. Stochastic Hamiltonian PDEs

Stochastic Hamiltonian PDEs, as natural extensions of stochastic Hamiltonian ordinary differential equations, play important roles in the fields of fluid dynamics, nonlinear optics, plasma physics, communications and medical science and so forth. They are due to \cite{11} and given by \begin{equation}
Md\dot{z} + Kz\dot{z}dt = \nabla S_1(z)dt + \nabla S_2(z) \circ dW(t),
\end{equation}
where $x \in \mathcal{O}$, $M$ and $K$ are skew-symmetric matrices, $S_1$ and $S_2$ are smooth functions of the variable $z$, and ‘$\circ$’ stands for the Stratonovich product. Moreover, $\{W(t)\}_{t \geq 0}$ is an $\mathbb{L}_2(\mathcal{O}, \mathbb{R})$-valued $Q$-Wiener process
with respect to a normal filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) on a filtered probability space \((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P})\) and has the expansion form

\[
W(t) = \sum_{k=1}^{\infty} \sqrt{\kappa_k} \epsilon_k \beta_k(t),
\]

where \( \{(\kappa_k, \epsilon_k)\}_{k=1}^{\infty} \) is a sequence of eigenpairs of symmetric, positive definite and finite trace operator \(Q\) with orthonormal eigenvectors and \( \{\beta_k(t)\}_{k=1}^{\infty} \) is a sequence of real-valued mutually independent standard Brownian motions. Stochastic PDEs that can be rewritten as (2.1), include and are not limited to nonlinear stochastic wave equation, stochastic nonlinear Schrödinger equation, stochastic KdV equation, etc. More precisely,

1. by introducing \( v = u_t \) and \( w = u_x \), we reformulate the nonlinear stochastic wave equation with homogenous Dirichlet boundary condition

\[
du_t - u_{xx} dt + f(u) dt = g(u) \circ dW(t)
\]

into

\[
\begin{align*}
    du &= v dt, \\
    u_x &= w, \\
    dv - w_x dt &= -f(u) dt + g(u) \circ dW(t),
\end{align*}
\]

where \( f : L^2(\mathcal{O}, \mathbb{R}) \to L^2(\mathcal{O}, \mathbb{R}) \) and \( g : L^2(\mathcal{O}, \mathbb{R}) \to L^2(L^2(\mathcal{O}, \mathcal{R}), Q^\frac{1}{2}(L^2(\mathcal{O}, \mathcal{R}))) \) satisfy the global Lipschitz continuous condition with \( \mathcal{O} = [x_L, x_R], x_L, x_R \in \mathbb{R} \), and \( L^2(\mathcal{O}, \mathcal{R}), Q^\frac{1}{2}(L^2(\mathcal{O}, \mathcal{R})) \) being the separable Hilbert space of Hilbert–Schmidt operators. Denoting \( z = (u, p, v, w)^\top \), then (2.2) can be transformed into the multi-symplectic formulation

\[
Mdz + K z_x dt = \nabla S_1(z) dt + \nabla S_2(z) \circ dW(t)
\]

with

\[
M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

\[
S_1(z) = \frac{1}{2} (w^2 - v^2) - \tilde{f}(u), \quad S_2(z) = \tilde{g}(u),
\]

where \( f = \tilde{f}_u \) and \( g = \tilde{g}_u \) (see e.g., [13]).

2. consider the stochastic nonlinear Schrödinger equation under the homogenous Dirichlet boundary condition

\[
 idu + u_{xx} dt + |u|^2 u dt = u \circ dW(t)
\]

with \( \mathcal{O} = [0, 1] \) and \( i^2 = -1 \). Setting \( u = p + iq \) and letting \( v = p_x \) and \( w = q_x \), we rewrite the above equation as

\[
\begin{align*}
    dq - v_x dt &= (p^2 + q^2) p dt - p \circ dW(t), \\
    dp + w_x dt &= -(p^2 + q^2) q dt + q \circ dW(t), \\
    p_x &= v, \\
    q_x &= w.
\end{align*}
\]

Defining a state variable \( z = (p, q, v, w)^\top \), [11] presents the associated multi-symplectic form of (2.3) as follows

\[
Mdz + K z_x dt = \nabla S_1(z) dt + \nabla S_2(z) \circ dW(t)
\]
3. the stochastic Korteweg-de Vries equation with the homogenous Dirichlet boundary condition takes

\[ du + uu_x dt + \beta u_{xxx} dt = \lambda dW(t), \]

where \( \beta, \lambda > 0 \) and \( \mathcal{O} = [0, 1] \). Given new variables \( v, \rho, w \) satisfying

\[
\begin{cases}
-\frac{1}{2} d\rho - \beta w_x dt = \frac{1}{2} u^2 dt - v dt, \\
\rho_x = u, \\
\frac{1}{2} du + v_x dt = \lambda dW(t), \\
ux = w,
\end{cases}
\]

we have the compact form (see [1])

\[ M dz + K z_x dt = \nabla S_1(z) dt + \nabla S_2(z) \circ dW(t) \]

with \( z = (u, v, \rho, w)^T \),

\[
M = \begin{pmatrix}
0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
K = \begin{pmatrix}
0 & 0 & 0 & -\beta \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
\beta & 0 & 0 & 0
\end{pmatrix},
S_1(z) = \frac{1}{6} u^3 - uv + \frac{1}{2} \beta w^2, \\
S_2(z) = \lambda \rho.
\]

4. take the stochastic Maxwell equation with multiplicative noise

\[
\begin{align*}
\frac{dE(t, x, y, z)}{dt} &= \nabla \times H(t, x, y, z) - \lambda H(t, x, y, z) \circ dW(t), \\
\frac{dH(t, x, y, z)}{dt} &= -\nabla \times E(t, x, y, z) + \lambda E(t, x, y, z) \circ dW(t)
\end{align*}
\]

into account, where \( \lambda \in \mathbb{R}, \mathcal{O} \subset \mathbb{R}^3 \) is a bounded and simply connected domain with smooth boundary \( \partial \mathcal{O} \). We employ the perfectly electric conducting (PEC) boundary condition \( E \times n = 0 \) on \((0, T] \times \partial \mathcal{O} \) with \( n \) being the unit outward normal of \( \partial \mathcal{O} \) (see [3]). Denote \( u = (H^T, E^T)^T = (H_1, H_2, H_3, E_1, E_2, E_3)^T \) and \( S(u) = \frac{1}{2} \left( |E_1|^2 + |E_2|^2 + |E_3|^2 + |H_1|^2 + |H_2|^2 + |H_3|^2 \right) \). Then (2.5) can be rewritten as

\[ M du + K_1 u_{xx} dt + K_2 u_{yy} dt + K_3 u_{zz} dt = \nabla S(u) \circ dW; \]

where

\[
M = \begin{pmatrix}
0 & I_{3 \times 3} & 0 \\
I_{3 \times 3} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
K_i = \begin{pmatrix}
\mathcal{P}_i & 0 \\
0 & \mathcal{P}_i
\end{pmatrix}, \quad i = 1, 2, 3
\]

with \( I_{3 \times 3} \) being a \( 3 \times 3 \) identity matrix,

\[
\mathcal{P}_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
Analogues to the symplecticity-preserving property of stochastic Hamiltonian ordinary differential equations, [1] shows that stochastic Hamiltonian PDEs possess the multi-symplectic conservation law. In detail, denote two differential 2-forms by \( \omega = dz \wedge Mdz \) and \( \kappa = dz \wedge Kdz \), where ‘\( \wedge \)’ represents the wedge product. Then the multi-symplecticity, as a local invariant, is given by
\[
d\omega(t, x) + \partial_x \kappa(t, x) dt = 0, \quad a.s.,
\]
where (\( t_0, t_1 \)) is the local definition domain of \( z \). From the multi-symplectic conservation law (2.7) it can be found that symplecticity changes locally and synchronously both in temporal and spatial directions. We would like to remark that the word ‘local’ means that such conservative property does not depend on the specific domain or on boundary conditions of stochastic PDEs. In addition, the multi-symplectic conservation law (2.7) for stochastic Hamiltonian PDEs holds almost surely. To simplify the notation, below we shall suppress the notation ‘a.s.’ unless it is necessary to avoid confusion. The multi-symplectic conservation law

- for nonlinear stochastic wave equation (2.2) is
  \[
d[du \wedge dv] + \partial_x [dw \wedge du] dt = 0.
\]
- for stochastic nonlinear Schrödinger equation (2.3) is
  \[
d[dq \wedge dp] + \partial_x [dp \wedge dv + dq \wedge dw] dt = 0.
\]
- for stochastic KdV equation (2.4) is
  \[
d[d\rho \wedge du] + \partial_x [2d\rho \wedge dv + 2\beta dw \wedge du] dt = 0.
\]
- for stochastic Maxwell equation (2.5) is
  \[
d[E \wedge dH] + \partial_x [H_3 \wedge dH_2 + dE_3 \wedge dE_2] dt
  + \partial_y [dH_1 \wedge dH_3 + dE_1 \wedge dE_3] dt + \partial_z [dH_2 \wedge dH_1 + dE_2 \wedge dE_1] dt = 0.
\]

In order to keep more intrinsic properties of the original system into numerical simulations, there has been growing interest in the geometric integration of stochastic Hamiltonian PDEs, namely in the multi-symplectic method, which can more fully capture behaviors of interesting phenomena. For the purpose of numerical approximation, we let \( \Delta x, \Delta y \) and \( \Delta z \) be the mesh sizes along \( x, y \) and \( z \) directions, respectively, and \( \Delta t \) be the time step length. The temporal-spatial domain we are interested in the following sections is \([0, T] \times O := [0, T] \times [x_L, x_R] \times [y_L, y_R] \times [z_L, z_R].\) It is partitioned by parallel lines, where \( t_n = n \Delta t, \) \( x_i = x_L + i\Delta x, \) \( y_j = y_L + j\Delta y \) and \( z_k = z_L + k\Delta z \) for \( n = 0, 1, \ldots, N, i = 0, 1, \ldots, I, j = 0, 1, \ldots, J \) and \( k = 0, 1, \ldots, K.\) Now we take \( O = [x_L, x_R] \) for example and denote the approximation of the \( z(x, t) \) at the mesh point \( (x_j, t_k) \) by \( z_{j,k}, \) i.e., \( z_{j,k} \approx z(x_j, t_k).\) The numerical method for (2.2) and (2.7), can be written, respectively, as
\[
\Delta t M\delta_{i,j,k}^{z_{j,k}} + \Delta t K\delta_{i,j,k}^{z_{j,k}} = \Delta t (\nabla_z S_1(z))_{j,k} + \Delta W_j^k (\nabla_z S_2(z))_{j,k},
\]
\[
\delta_{i,j,k}^{\omega_{j,k}} + \delta_{i,j,k}^{\kappa_{j,k}} = 0,
\]
where
\[
\omega_{j,k} = dz_{j,k} \wedge M dz_{j,k}, \quad \kappa_{j,k} = dz_{j,k} \wedge K dz_{j,k}.
\]
\[ \Delta W^k = W(x_j, t_{k+1}) - W(x_j, t_k), \] and \[ \delta_1^k, \delta_2^k \] are corresponding discretizations of two partial derivatives \[ \partial_t \] and \[ \partial_x \], respectively. The numerical method \[ (2.8) \] is called a multi-symplectic method for stochastic Hamiltonian PDEs if it satisfies a discrete version of the multi-symplectic conservation law \[ (2.9) \]. In recent years, many researchers have studied various multi-symplectic methods for stochastic Maxwell equations (see e.g., [7, 8, 21]), stochastic nonlinear Schrödinger equations (see e.g., [5, 10]), etc.

In what follows, we propose three multi-symplectic methods of stochastic Hamiltonian PDEs. Soon afterwards, applications to nonlinear stochastic wave equation, stochastic nonlinear Schrödinger equation, stochastic KdV equation and stochastic Maxwell equation are given.

### 3. Meshless LRBF collocation midpoint method

In this section, we present a kind of multi-symplectic methods for stochastic Hamiltonian PDEs by exploiting the meshless LRBF collocation method in space and the midpoint method in time, respectively.

The global radial basis function collocation method, such as the Kansa’s method in [12, 13], becomes a powerful tool for numerically solving deterministic PDEs, especially deterministic Hamiltonian PDEs (see e.g., [6, 11] and references therein), since it does not need to evaluate any integral and has both high-order accuracy and geometric flexibility. A key ingredient of the global radial basis function collocation method is the radial basis function \( \varphi \), such as the Gaussian radial basis function \( \varphi(x) = e^{-c x^2} \), the multiquadric radial basis function \( \varphi(x) = \sqrt{x^2 + c^2} \), and the inverse multiquadric radial basis function \( \varphi(x) = 1/\sqrt{x^2 + c^2} \), where the shape parameter \( c \) is a constant. However, when applying the global radial basis function collocation method to solve PDEs, large scaled linear systems are needed to solve, the corresponding coefficient matrices are ill-conditioned and the results are sensitive to the shape parameter \( c \). To overcome the above problems arisen by using the global radial basis function collocation method, the LRBF collocation method was formulated by [14, 17], from which the main idea is the collocation on influence domain, and can drastically reduce the collocation matrix size at the expense of solving many small matrices with the dimension of the number of nodes included in the domain of influence for each node. Since the LRBF collocation method, as a type of meshless methods, can be employed to cope with complex geometries, complicated irregular domains including moving boundary and high-dimensional problem, it has been applied for solving many problems in engineering and applied mathematics widely (see [24]). To be specific, let \( \{x_i, f(x_i)\} \) be the scattered data with \( i = 0, 1, \ldots , L, L + 1 \), and \( L \in \mathbb{N} \). Fix \( i \in \{0, 1, \ldots , L, L + 1\} \). Given \( x_i \), there exist \( n_i \) neighboring nodes which are nearest from \( x_i \) in the influence domain \( \Omega = \{x_k\}_{k=1}^{n_i} \). For \( x \in \Omega \), the function \( f \) can be approximated by

\[ f^*(x) = \sum_{k=1}^{n_i} \alpha_k \varphi(||x - x_k||), \]

where the coefficient \( \{\alpha_k\}_{k=1}^{n_i} \) in the above equation satisfies the interpolation condition \( f^*(x) = f(x) \).

Taking \( x = x_k \) for \( k = 1, \ldots , n_i \), we obtain

\[ i f = \begin{bmatrix} \varphi(||x_1 - x_1||) & \varphi(||x_1 - x_2||) & \cdots & \varphi(||x_1 - x_{n_i}||) \\ \varphi(||x_2 - x_1||) & \varphi(||x_2 - x_2||) & \cdots & \varphi(||x_2 - x_{n_i}||) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(||x_{n_i} - x_1||) & \varphi(||x_{n_i} - x_2||) & \cdots & \varphi(||x_{n_i} - x_{n_i}||) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n_i} \end{bmatrix} \]

(3.1)

with \( i f = [f(x_1), \ldots , f(x_{n_i})]^\top \). From (3.1) it follows that \( i \alpha = (i \Phi)^{-1} i f \) and

\[ f^*(x) = [\varphi(||x - x_1||), \ldots , \varphi(||x - x_{n_i}||)] (i \Phi)^{-1} i f, \]

whose \( l \)-order differential approximation reads

\[ f^{*(l)}(x) = [\varphi^{(l)}(||x - x_1||), \ldots , \varphi^{(l)}(||x - x_{n_i}||)] (i \Phi)^{-1} i f. \]
Let $n_i = 5$ without loss of generality, that is, for each inner node $\mathbf{x}_i$, the local influence domain is

$$\Omega_i = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$$

with $\mathbf{x}_i = \mathbf{x}_i$ being the center. Based on \((3.2)\), we have the approximation of $f^{(l)}(\mathbf{x}_i)$ with $l \in \mathbb{N}_+$ as follows

$$f^{(l)}(\mathbf{x}_i) = f^{(l)}(\mathbf{x}_3)$$

$$= \left[\varphi^{(l)}(\|\mathbf{x}_3 - \mathbf{x}_1\|), \varphi^{(l)}(\|\mathbf{x}_3 - \mathbf{x}_2\|), \varphi^{(l)}(\|\mathbf{x}_3 - \mathbf{x}_4\|), \varphi^{(l)}(\|\mathbf{x}_3 - \mathbf{x}_5\|)\right](\Phi)^{-1} \mathbf{f}$$

which yields

$$f^{(l)}(\mathbf{x}_i) = \begin{bmatrix} f^{(l)}(\mathbf{x}_0) & \ldots & f^{(l)}(\mathbf{x}_{L+1}) \end{bmatrix}^\top \mathbf{D}^{(l)} \mathbf{f},$$

where $\mathbf{D}^{(l)}$, $l \in \mathbb{N}_+$, is a sparse matrix and there is not any zero between $\alpha_k$ and $\alpha_{k+1}$ for $k \in \{-2, -1, 0, 1\}$, if $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ are located next to each other in the full sequence $\{\mathbf{x}_k\}_{k=1}^{n_i}$. Especially, under the homogeneous Dirichlet boundary condition, it should be noted that if the node $\mathbf{x}_k$ is out of boundary, the corresponding coefficient $\alpha_k$ is equal to zero. Hence, in this case, the form of $l$-order differential matrix $\mathbf{D}^{(l)}$ for $l \in \mathbb{N}_+$ becomes

$$\mathbf{D}^{(l)} = \begin{bmatrix}
1d_0^{(l)} & 1d_1^{(l)} & 1d_2^{(l)} & 2d_0^{(l)} & 2d_1^{(l)} & 2d_2^{(l)} \\
2d_0^{(l)} & 2d_1^{(l)} & 2d_2^{(l)} & 3d_0^{(l)} & 3d_1^{(l)} & 3d_2^{(l)} \\
3d_{-2} & 3d_{-1} & 3d_0 & 3d_1 & 4d_2 & \cdots \\
4d_0 & 4d_1 & 4d_2 & \cdots & \cdots & \cdots \\
L-3d_{-2} & L-3d_{-1} & L-3d_0 & \cdots & \cdots & \cdots \\
L-2d_{-2} & L-2d_{-1} & L-2d_0 & \cdots & \cdots & \cdots \\
L-1d_{-2} & L-1d_{-1} & L-1d_0 & \cdots & \cdots & \cdots \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1}
\end{bmatrix}. $$

Approximating the spatial derivative in (2.1) by $\mathbf{D}^{(1)}$ of the LRBF collocation method leads to a semi-discrete method

$$MdZ_i + K \sum_{k=1}^{n_i} i d_k^{(1)} Z_k dt = \nabla S_1(Z_i) dt + \nabla S_2(Z_i) \circ dW(x_i, t),$$

where $i = 1, \ldots, I - 1$, $k = 1, \ldots, n_i$, $Z_i \approx z(x_i)$, $Z_k \approx z(x_k)$ and $i d_k^{(1)}$ is the element of $\mathbf{D}^{(1)}$. After making use of the midpoint method in time, we obtain the meshless LRBF collocation midpoint method of (2.1) as follows

$$M(Z_i^{n+1} - Z_i^*) + \Delta t K \sum_{k=1}^{n_i} i d_k^{(1)} Z_k^{n+\frac{1}{2}} = \Delta t \nabla S_1(Z_i^{n+\frac{1}{2}}) + \Delta W_i^{n} \nabla S_2(Z_i^{n+\frac{1}{2}}),$$

(3.6)
where $Z^n_t \approx z(x_i, t_n), Z_i^{n+\frac{1}{2}} \approx (z(x_i, t_n) + z(x_i, t_{n+1}))/2, Z_k^{n+\frac{1}{2}} \approx (z(x_k, t_n) + z(x_k, t_{n+1}))/2$ and $\Delta W^n_t = W(x_i, t_{n+1}) - W(x_i, t_n)$.

Applying (3.6) to the nonlinear stochastic wave equation with multiplicative noise (2.2), we derive

$$\begin{align*}
\frac{U^{n+1} - U^n}{\Delta t} &= \mathbf{V}^{n+\frac{1}{2}}, \\
\frac{D^{(1)}U^{n+\frac{1}{2}}}{\Delta t} &= \mathbf{W}^{n+\frac{1}{2}}, \\
\mathbf{V}^{n+1} - \mathbf{V}^n &= \frac{\Delta t}{2} F(U^{n+\frac{1}{2}}) + G(U^{n+\frac{1}{2}}) \Delta W^n, \\
\mathbf{W}^{n+1} &= \frac{\Delta t}{2} F(U^{n+\frac{1}{2}}) + G(U^{n+\frac{1}{2}}) \Delta W^n,
\end{align*}$$

where

$$U^{n+\frac{1}{2}} = (U^{n+1} + U^n)/2, \quad V^{n+\frac{1}{2}} = (V^{n+1} + V^n)/2, \quad W^{n+\frac{1}{2}} = (W^{n+1} + W^n)/2,$$

$$U^n = [U^n_1, \ldots, U^n_{N-1}], \quad V^n = [V^n_1, \ldots, V^n_{N-1}], \quad W^n = [W^n_1, \ldots, W^n_{N-1}],$$

$$F(U^{n+\frac{1}{2}}) = [f((U^{n+1}_i + U^n_i)/2), \ldots, f((U^{n+1}_N + U^n_N)/2)],$$

$$G(U^{n+\frac{1}{2}}) = [g((U^{n+1}_1 + U^n_1)/2), \ldots, g((U^{n+1}_N + U^n_N)/2)],$$

$$\Delta W^n = [W(x_i, t_{n+1}) - W(x_i, t_n), \ldots, W(x_{N-1}, t_{n+1}) - W(x_{N-1}, t_n)].$$

**Theorem 3.1.** The fully-discrete method (3.7) applied to the stochastic wave equation (2.1) with $S_1(z) = \frac{1}{2} (u^2 - v^2) - f(u)$ and $S_2(z) = g(u)$ admits the discrete multi-symplectic conservation law, i.e.,

$$\frac{\omega^{n+1} - \omega^n}{\Delta t} + \sum_{k=1}^{n_t} d_k^{(1)} i_{s_k}^{n+\frac{1}{2}} = 0,$$

where

$$\omega^n_i = \frac{1}{2} dZ_i^n \wedge M dZ_i^n, \quad i_{s_k}^{n+\frac{1}{2}} = dZ_i^{n+\frac{1}{2}} \wedge K d_i Z_k^{n+\frac{1}{2}}, \quad Z_i^n = (U^n_i, V^n_i, W^n_i),$$

$$i_{s_k}^{n+\frac{1}{2}} = ((U_k^n + U_{k+1}^n)/2, (V_k^n + V_{k+1}^n)/2, (W_k^n + W_{k+1}^n)/2), \quad i = 1, \ldots, I - 1, k = 1, \ldots, n_i$$

with

$$M = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad K = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}. $$

**Proof.** The system (3.7) can be rewritten in the form of (3.8), and its discrete variational equation is given by

$$M \frac{dZ_i^{n+1} - dZ_i^n}{\Delta t} + \sum_{k=1}^{n_t} d_k^{(1)} K d_i Z_k^{n+\frac{1}{2}} = \nabla^2 S_1(Z_i^{n+\frac{1}{2}}) dZ_i^{n+\frac{1}{2}} + \nabla^2 S_2(Z_i^{n+\frac{1}{2}}) \Delta W_i^{n} dZ_i^{n+\frac{1}{2}}. $$

Taking wedge product of the both sides of (3.9) with $dZ_i^{n+\frac{1}{2}}$ yields

$$\frac{dZ_i^{n+1} + dZ_i^n}{2} \wedge M \frac{dZ_i^{n+1} - dZ_i^n}{\Delta t} + dZ_i^{n+\frac{1}{2}} \wedge \sum_{k=1}^{n_t} d_k^{(1)} K d_i Z_k^{n+\frac{1}{2}}$$

$$= dZ_i^{n+\frac{1}{2}} \wedge \nabla^2 S_1(Z_i^{n+\frac{1}{2}}) dZ_i^{n+\frac{1}{2}} + \frac{1}{\Delta t} dZ_i^{n+\frac{1}{2}} \wedge \nabla^2 S_2(Z_i^{n+\frac{1}{2}}) \Delta W_i^{n} dZ_i^{n+\frac{1}{2}}. $$

By the symmetry of both $\nabla^2 S_1(Z_i^{n+\frac{1}{2}})$ and $\nabla^2 S_2(Z_i^{n+\frac{1}{2}})$, we obtain

$$\frac{1}{\Delta t} \left( \frac{1}{2} dZ_i^{n+1} \wedge M dZ_i^{n+1} - \frac{1}{2} dZ_i^n \wedge M dZ_i^n \right) + \sum_{k=1}^{n_t} d_k^{(1)} dZ_k^{n+\frac{1}{2}} \wedge K d_i Z_k^{n+\frac{1}{2}} = 0,$$
which is \( \approx \) by notations \( \omega^n_k \) and \( i k^{n+\frac{1}{2}} \). This completes the proof. \( \square \)

From Theorem 3.1, it is known that the numerical method on the basis of \( \approx \) for the nonlinear stochastic wave equation possesses the discrete multi-symplectic conservation law. Now we perform numerical experiments to illustrate the validity of the proposed method \( \approx \) for the 1-dimensional stochastic wave equation in different cases: (1) \( f(u) = \sin(u), g(u) = \sin(u) \); (2) \( f(u) = \sin(u), g(u) = u \); (3) \( f(u) = u^3, g(u) = \sin(u) \). In all the numerical experiments, the expectation is approximated by taking the average over 1000 realizations. Moreover, we choose the orthonormal basis \( \{e_k\}_{k \in \mathbb{N}^+} \) and the corresponding eigenvalue \( \{q_k\}_{k \in \mathbb{N}^+} \) of \( Q \) as \( e_k = \sqrt{\frac{1}{2\pi}} \sin(k\pi x) \) and \( q_k = \frac{1}{k^2} \), respectively. And set \( x \in (-8, 8), u(0) = 0, u_t(0) = \text{sech}(x) \), and \( u_x(0) = 0 \). The radial basis function is chosen as the inverse multiquadric function \( \varphi(x) = 1 / \sqrt{1 + \|x\|^2} \), i.e., \( c = 1 \). The size of influence domain is taken as \( n = 5 \). Table 1 displays strong convergence errors against \( \Delta t = 2^{-s}, s = 1, 2, 3, 4 \) on log-log scale at time \( T = 1 \), which indicates that the meshless LRBF collocation midpoint method offers a good simulation and obtains high precision. We regard the numerical approximation obtained by a fine mesh with \( \Delta t = 2^{-10}, \Delta x = 2^{-5} \) as the exact solution. Moreover, compared with the reference line in Fig. 1 it also can be observed that the mean-square convergence order of the proposed method applied to three cases is 1 in temporal direction.

| \( \Delta t \) | \( f(u) = \sin(u), g(u) = \sin(u) \) | \( f(u) = \sin(u), g(u) = u \) | \( f(u) = u^3, g(u) = \sin(u) \) |
|------------|----------------------------------|----------------------------------|----------------------------------|
| \( 2^{-1} \) | \( 2.4213e-02 \) | \( 2.4938e-02 \) | \( 2.1630e-02 \) |
| \( 2^{-2} \) | \( 1.0905e-02 \) | \( 1.1091e-02 \) | \( 1.0581e-02 \) |
| \( 2^{-3} \) | \( 5.0449e-03 \) | \( 5.1986e-03 \) | \( 5.1289e-03 \) |
| \( 2^{-4} \) | \( 2.3927e-03 \) | \( 2.4720e-03 \) | \( 2.5391e-03 \) |

Table 1: Mean-square errors of LRBF collocation midpoint method in time.

Figure 1: Mean-square convergence order of LRBF collocation midpoint method in temporal direction in the cases of (1) \( f(u) = \sin(u), g(u) = \sin(u) \) (2) \( f(u) = \sin(u), g(u) = u \) and (3) \( f(u) = u^3, g(u) = \sin(u) \).

Remark 3.2. By using \( \approx \), we obtain a multi-symplectic method for the stochastic nonlinear Schrödinger equation with multiplicative noise \( (2.3) \) as follows

\[
\begin{align*}
\frac{P^{n+1} - P^n}{\Delta t} &= -D^{(1)}W^{n+\frac{1}{2}} - \left( (P^{n+\frac{1}{2}})^2 + (Q^{n+\frac{1}{2}})^2 \right) Q^{n+\frac{1}{2}} + Q^{n+\frac{1}{2}} \frac{\Delta W^n}{\Delta t}, \\
\frac{Q^{n+1} - Q^n}{\Delta t} &= D^{(1)}V^{n+\frac{1}{2}} + \left( (P^{n+\frac{1}{2}})^2 + (Q^{n+\frac{1}{2}})^2 \right) P^{n+\frac{1}{2}} - P^{n+\frac{1}{2}} \frac{\Delta W^n}{\Delta t}, \\
D^{(1)}P^{n+\frac{1}{2}} &= V^{n+\frac{1}{2}}, \\
D^{(1)}Q^{n+\frac{1}{2}} &= W^{n+\frac{1}{2}}.
\end{align*}
\]
where $P_i^{n+\frac{1}{2}} = \frac{P_i^n + P_i^{n+1}}{2}$, $Q_i^{n+\frac{1}{2}} = \frac{Q_i^n + Q_i^{n+1}}{2}$, $i = 1, \ldots, I - 1$, and
\[
\left( (P_i^{n+\frac{1}{2}})^2 + (Q_i^{n+\frac{1}{2}})^2 \right) Q_i^{n+\frac{1}{2}} = \left[ \left( (P_i^{n+\frac{1}{2}})^2 + (Q_i^{n+\frac{1}{2}})^2 \right) Q_i^{n+\frac{1}{2}} \right], \quad \left( (P_i^{n+\frac{1}{2}})^2 + (Q_i^{n+\frac{1}{2}})^2 \right) P_i^{n+\frac{1}{2}} = \left[ \left( (P_i^{n+\frac{1}{2}})^2 + (Q_i^{n+\frac{1}{2}})^2 \right) P_i^{n+\frac{1}{2}} \right],
\]
\[
Q_i^{n+\frac{1}{2}} \Delta W^n = \left[ Q_i^{n+\frac{1}{2}} (W(x_i, t_{n+1}) - W(x_i, t_n)) \right], \quad Q_i^{n+\frac{1}{2}} (W(x_i-1, t_{n+1}) - W(x_i-1, t_n))]
\]
\[
P_i^{n+\frac{1}{2}} \Delta W^n = \left[ P_i^{n+\frac{1}{2}} (W(x_i, t_{n+1}) - W(x_i, t_n)) \right], \quad P_i^{n+\frac{1}{2}} (W(x_i-1, t_{n+1}) - W(x_i-1, t_n))]
\]

Similar to Theorem 3.1, it can be verified that the fully-discrete method (3.10) possesses the discrete multi-symplectic conservation law
\[
\frac{\omega_i^{n+1} - \omega_i^n}{\Delta t} + \sum_{k=1}^{n_i} i d_k^{(1)} i \kappa_k^{n+\frac{1}{2}} = 0
\]
where
\[
\omega_i^n = \frac{1}{2} dZ_i^n \wedge M dZ_i^n, \quad i \kappa_k^{n+\frac{1}{2}} = dZ_k^{n+\frac{1}{2}} \wedge K dZ_k^{n+\frac{1}{2}}, \quad Z_i^n = (P_i^n, Q_i^n, V_i^n, W_i^n)^T,
\]
\[
i Z_k^{n+\frac{1}{2}} = \left( (i P_k^n + i P_k^{n+1})/2, (i Q_k^n + i Q_k^{n+1})/2, (i V_k^n + i V_k^{n+1})/2, (i W_k^n + i W_k^{n+1})/2 \right)^T
\]
with $i = 1, \ldots, I - 1, k = 1, \ldots, n_i$, and
\[
M = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad K = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Remark 3.3. For the stochastic KdV equation (2.4), making use of (3.6) yields
\[
\begin{cases}
\frac{1}{2} U_i^{n+\frac{1}{2}} - \frac{1}{2} U_i^n + D^{(1)} V_i^{n+\frac{1}{2}} = \frac{1}{2} \Delta W_i^n, \\
\frac{1}{2} P_i^{n+\frac{1}{2}} - \frac{1}{2} P_i^n + \beta D^{(1)} W_i^{n+\frac{1}{2}} = V_i^{n+\frac{1}{2}} - \frac{1}{2} (U_i^{n+\frac{1}{2}})^2, \\
- D^{(1)} P_i^{n+\frac{1}{2}} = - U_i^{n+\frac{1}{2}},
\end{cases}
\]
where $U_i^{n+\frac{1}{2}} = U_i^n + U_i^{n+1}, V_i^{n+\frac{1}{2}} = V_i^n + V_i^{n+1}, P_i^{n+\frac{1}{2}} = P_i^n + P_i^{n+1}, W_i^{n+\frac{1}{2}} = W_i^n + W_i^{n+1}, i = 1, \ldots, I - 1,$
\[
(U_i^{n+\frac{1}{2}})^2 = [(U_i^{n+\frac{1}{2}})^2, \ldots, (U_i^{n+\frac{1}{2}})^2]^T, \Delta W_i^n = [W(x_i, t_{n+1}) - W(x_i, t_n), \ldots, W(x_i-1, t_{n+1}) - W(x_i-1, t_n)]^T.
\]

In fact, the fully-discrete method (3.12) has the discrete multi-symplectic conservation law
\[
\frac{\omega_i^{n+1} - \omega_i^n}{\Delta t} + \sum_{k=1}^{n_i} i c_k^{(1)} i \kappa_k^{n+\frac{1}{2}} = 0, \quad i = 1, \ldots, I - 1
\]
with
\[
\omega_i^n = \frac{1}{2} dZ_i^n \wedge M dZ_i^n, \quad i \kappa_k^{n+\frac{1}{2}} = dZ_k^{n+\frac{1}{2}} \wedge K dZ_k^{n+\frac{1}{2}}, \quad Z_i^n = (U_i^n, V_i^n, P_i^n, W_i^n)^T,
\]
\[
i Z_k^{n+\frac{1}{2}} = \left( (i U_k^n + i U_k^{n+1})/2, (i V_k^n + i V_k^{n+1})/2, (i P_k^n + i P_k^{n+1})/2, (i W_k^n + i W_k^{n+1})/2 \right)^T,
\]
and
\[
M = \begin{pmatrix}
0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad K = \begin{pmatrix}
0 & 0 & 0 & -\beta \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
\beta & 0 & 0 & 0
\end{pmatrix}.
\]
Remark 3.4. For the stochastic Maxwell equation, by means of (3.6), we obtain

\[
\begin{align*}
\frac{(E_1)^{n+1} - (E_1)^n}{\Delta t} &= -D_z^{(1)}(H_2)^{n+\frac{1}{2}} + D_y^{(1)}(H_3)^{n+\frac{1}{2}} - \lambda(H_1)^{n+\frac{1}{2}} \frac{\Delta W^n}{\Delta t}, \\
\frac{(E_2)^{n+1} - (E_2)^n}{\Delta t} &= D_z^{(1)}(H_1)^{n+\frac{1}{2}} - D_z^{(1)}(H_3)^{n+\frac{1}{2}} - \lambda(H_2)^{n+\frac{1}{2}} \frac{\Delta W^n}{\Delta t}, \\
\frac{(E_3)^{n+1} - (E_3)^n}{\Delta t} &= -D_y^{(1)}(H_1)^{n+\frac{1}{2}} + D_x^{(1)}(H_2)^{n+\frac{1}{2}} - \lambda(H_3)^{n+\frac{1}{2}} \frac{\Delta W^n}{\Delta t}, \\
\frac{(H_1)^{n+1} - (H_1)^n}{\Delta t} &= D_l^{(1)}(E_2)^{n+\frac{1}{2}} - D_l^{(1)}(E_3)^{n+\frac{1}{2}} + \lambda(E_1)^{n+\frac{1}{2}} \frac{\Delta W^n}{\Delta t}, \\
\frac{(H_2)^{n+1} - (H_2)^n}{\Delta t} &= -D_l^{(1)}(E_1)^{n+\frac{1}{2}} + D_x^{(1)}(E_3)^{n+\frac{1}{2}} + \lambda(E_2)^{n+\frac{1}{2}} \frac{\Delta W^n}{\Delta t}, \\
\frac{(H_3)^{n+1} - (H_3)^n}{\Delta t} &= D_l^{(1)}(E_1)^{n+\frac{1}{2}} - D_x^{(1)}(E_2)^{n+\frac{1}{2}} + \lambda(E_3)^{n+\frac{1}{2}} \frac{\Delta W^n}{\Delta t},
\end{align*}
\]

(3.14)

where

\[
(E_j)^{n+\frac{1}{2}} = \frac{(E_j)^{n+1} + (E_j)^n}{2}, \quad (H_j)^{n+\frac{1}{2}} = \frac{(H_j)^{n+1} + (H_j)^n}{2},
\]

\[
(E_j)^n = [(E_1)^n, \ldots, (E_j)^n, \ldots, (E_d)^n]^T, \quad (H_j)^n = [(H_1)^n, \ldots, (H_j)^n, \ldots, (H_d)^n]^T, \quad j = 1, 2, 3,
\]

\[
\Delta W^n = [W(x_1, t_{n+1}) - W(x_1, t_n), \ldots, W(x_{d-1}, t_{n+1}) - W(x_{d-1}, t_n)]^T.
\]

In the three-dimensional case, \(D_z^{(1)}, D_y^{(1)} \) and \(D_x^{(1)} \) are 1-order differential approximations of partial derivatives \(\partial x, \partial y \) and \(\partial z \) of LRBF collocation method in (3.3), and the corresponding elements in above three matrices are denoted by \(d_{x,j}^{(1)}, d_{y,j}^{(1)}, d_{z,j}^{(1)} \) for \(i \in \{1, \ldots, I - 1\} \) and \(k \in \{1, \ldots, n_i\} \). The fully-discrete method (3.14) satisfies

\[
\frac{\omega_i^{n+1} - \omega_i^n}{\Delta t} + \sum_{k=1}^{n_i} d_{x,j}^{(1)}z_{i,k}^{n+\frac{1}{2}} + \sum_{k=1}^{n_i} d_{y,j}^{(1)}z_{i,k}^{n+\frac{1}{2}} + \sum_{k=1}^{n_i} d_{z,j}^{(1)}z_{i,k}^{n+\frac{1}{2}} = 0, \quad i = 1, \ldots, I - 1, \quad (3.15)
\]

where

\[
\omega_i^n = \frac{1}{2} M Z_i^n \wedge M dZ_i^n, \quad d_{x,j}^{n+\frac{1}{2}} = dZ_i^{n+\frac{1}{2}} \wedge K_j d_i Z_k^{n+\frac{1}{2}}, \quad Z_i^n = ((H_1)_i^n, (H_2)_i^n, (H_3)_i^n, (E_1)_i^n, (E_2)_i^n, (E_3)_i^n)^T,
\]

\[
i Z_k^{n+\frac{1}{2}} = ((H_1)_k^n + i (H_1)_k^{n+1})/2, ((H_2)_k^n + i (H_2)_k^{n+1})/2, ((H_3)_k^n + i (H_3)_k^{n+1})/2, ((E_1)_k^n + i (E_1)_k^{n+1})/2,
\]

\[
((E_2)_k^n + i (E_2)_k^{n+1})/2, ((E_3)_k^n + i (E_3)_k^{n+1})/2)^T,
\]

and

\[
M = \begin{pmatrix}
0 & -I_{3 \times 3} \\
I_{3 \times 3} & 0
\end{pmatrix}, \quad K_j = \begin{pmatrix}
\mathcal{D}_j & 0 \\
0 & \mathcal{D}_j
\end{pmatrix}, \quad j = 1, 2, 3
\]

with \(I_{3 \times 3} \) being a \(3 \times 3 \) identity matrix,

\[
\mathcal{D}_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad \mathcal{D}_2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad \mathcal{D}_3 = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

4. Splitting multi-symplectic Runge–Kutta method

In this section, we propose the second kind of multi-symplectic methods for (2.1) via the splitting technique, which avoids the interaction between the nonlinear drift coefficient and the driving process. This splitting technique allows us to handle a deterministic Hamiltonian PDE directly, and thus some existing deterministic multi-symplectic method can be exploited. Motivated by the fact that multi-symplectic Runge–Kutta methods are a class of efficient derivative-free numerical methods, we concentrate on the splitting multi-symplectic Runge–Kutta method for stochastic Hamiltonian PDEs.
Now we begin our study with the multi-symplectic Runge–Kutta method for deterministic Hamiltonian PDEs

\[ Mdz + K_zdt = \nabla S_1(z)dt. \]

Applying \( s\)-stage and \( r\)-stage symplectic Runge–Kutta methods, i.e., \((c, A, b)\) and \((\tilde{c}, \tilde{A}, \tilde{b})\) as follows

\[
\begin{array}{c|ccc|c|ccc|c}
  c_1 & a_{11} & \ldots & a_{1s} & \tilde{c}_1 & \tilde{a}_{11} & \ldots & \tilde{a}_{1r} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_s & a_{s1} & \ldots & a_{ss} & \tilde{c}_s & \tilde{a}_{s1} & \ldots & \tilde{a}_{sr} \\
  b_1 & \ldots & b_s & b_1 & \ldots & b_r
\end{array}
\]

(4.2)

where \( s, r \geq 1 \), to (4.1) in space and time, respectively, the resulting fully-discrete method is as follows:

\[
\begin{align*}
Z^k_m &= z_i^k + \Delta x \sum_{n=1}^s a_{mn} \delta_x^{m,k} Z^k_n, \quad \forall \ i = 0, 1, \ldots, s, \\
z_{i+1}^k &= z_i^k + \Delta x \sum_{m=1}^s b_m \delta_t^{m,k} Z^k_m, \quad \forall \ i = 0, 1, \ldots, s, \\
Z^k_m &= z^p_m + \Delta t \sum_{j=1}^r \tilde{a}_{kj} \tilde{d}_t^{m,j} Z^j_m, \quad \forall \ p = 0, 1, \ldots, r, \\
z_{p+1}^m &= z_p^m + \Delta t \sum_{k=1}^r \tilde{b}_k \tilde{d}_x^{m,k} Z^k_m, \quad \forall \ p = 0, 1, \ldots, r, \\
M \delta_x^{m,k} Z^k_m + K \delta_t^{m,k} Z^k_m &= \nabla z S_1(z^k_m),
\end{align*}
\]

where \( \delta_t^{m,k} \) and \( \delta_x^{m,k} \) are discretizations of two partial derivatives \( \partial_t \) and \( \partial_x \), respectively, and

\[
b_m b_n - b_m a_{mn} - b_n a_{nm} = 0 \quad \text{and} \quad \tilde{b}_k \tilde{b}_j - \tilde{b}_k \tilde{a}_{kj} - \tilde{b}_j \tilde{a}_{jk} = 0 \quad (4.4)
\]

for all \( m, n = 1, \ldots, s \) and \( k, j = 1, \ldots, r \). It can be verified that the above stochastic numerical method admits the discrete multi-symplectic conservation law

\[
\frac{\omega^{p+1} - \omega^p}{\Delta t} + \kappa_{i+1} - \kappa_i = 0,
\]

where \( \omega^p = \frac{1}{2} \sum_{m=1}^s b_m dz_m^p \wedge M dz_m^p \) and \( \kappa_i = \frac{1}{2} \sum_{k=1}^r \tilde{b}_k dz_x^k \wedge K dz_x^k \) for \( p = 0, 1, \ldots, r \) and \( i = 0, 1, \ldots, s \). Applying the splitting technique to (2.1) in temporal direction, and then we obtain a deterministic Hamiltonian PDE with random input and a stochastic system on \( t \in [t_m, t_{m+1}] \) as follows

\[
\begin{align*}
M \dot{z} + K \dot{z}_x dt &= \nabla S_1(z) dt, \\
\dot{z}(t_m) &= z(t_m), \quad \text{and} \quad K \dot{z}_x = 0, \\
M dz &= \nabla S_2(z) \circ dW(t), \\
z(t_m) &= \underline{z}(t_m).
\end{align*}
\]

(4.5)

By choosing symplectic methods for the stochastic system and combining (4.3), we obtain the splitting multi-symplectic Runge–Kutta method satisfying the discrete multi-symplectic conservation law. Now we construct the splitting multi-symplectic Runge–Kutta method for the nonlinear stochastic wave equation, stochastic nonlinear Schrödinger equation, stochastic KdV equation and stochastic Maxwell equation, one after the other.

We first focus on the nonlinear stochastic wave equation (2.2) and propose the associated splitting multi-symplectic Runge–Kutta method. In detail, we decompose (2.2) on \([t_0, t_1]\) into a deterministic Hamiltonian
PDE with random input

\[
\begin{align*}
\mathbf{u}_t &= \mathbf{v}, \\
\mathbf{u}_x &= \mathbf{v}, \\
\mathbf{v}_i - \mathbf{v}_x &= -f(\mathbf{v}), \\
\mathbf{v}(t_0) &= u(t_0), \quad \mathbf{v}(t_0) = v(t_0),
\end{align*}
\]

(4.6)

and a stochastic system

\[
\begin{align*}
u_x &= 0, \quad w_x = 0, \\
u_t &= 0, \\
dv &= g(u) \circ dW(t), \\
u(t_0) &= \mathbf{v}(t_0), \quad v(t_0) = \mathbf{v}(t_1).
\end{align*}
\]

(4.7)

By making use of \(s\)-stage and \(r\)-stage symplectic Runge–Kutta methods \(12\) with \(s, r \geq 1\) to approximate \(4.0\), together with the application of the symplectic Euler method to the stochastic system \(4.7\), we obtain the following fully-discrete method

\[
\begin{align*}
U_i^m &= u_0^m + \Delta x \sum_{j=1}^{s} a_{ij} \mathbf{W}_j^m, \quad \mathbf{W}_i^m = u_0^m + \Delta x \sum_{j=1}^{s} a_{ij} \delta_x \mathbf{W}_j^m, \\
\mathbf{v}_i^m &= u_0^m + \Delta x \sum_{i=1}^{r} b_i \mathbf{W}_i^m, \quad \mathbf{v}_i^m = w_0^m + \Delta x \sum_{i=1}^{r} b_i \delta_x \mathbf{W}_i^m, \\
U_i^m &= u_0^m + \Delta t \sum_{n=1}^{m} \bar{a}_{nm} \mathbf{V}_n^m, \quad \mathbf{V}_n^m = v_0^m + \Delta t \sum_{n=1}^{m} \bar{a}_{nm} (\delta_x \mathbf{W}_n^m - f(U_i^m)), \\
\mathbf{v}_i^m &= u_0^m + \Delta t \sum_{m=1}^{r} \bar{b}_m \mathbf{V}_m^m, \quad \mathbf{v}_i^m = v_0^m + \Delta t \sum_{m=1}^{r} \bar{b}_m (\delta_x \mathbf{W}_m^m - f(U_i^m)), \\
u_i^m &= \mathbf{v}_i^m, \quad w_i^m = \mathbf{v}_i^m,
\end{align*}
\]

(4.8a-4.8f)

where \(i = 1, \ldots, s, \ m = 1, \ldots, r, \delta_x\) is the discretization of the partial derivative \(\partial_x\), and \(U_i^m \approx u(c_i \Delta x, \bar{c}_m \Delta t)\), \(u_0^m \approx u(c_0 \Delta x, d_0 \Delta t)\), \(u_0^m \approx u(0, \bar{c}_0 \Delta t)\), \(u_0^m \approx u(0, d_0 \Delta t)\), \(\mathbf{v}_i^m \approx \mathbf{v}(\Delta x, \bar{c}_m \Delta t)\), etc., with \(c_i = \sum_{j=1}^{s} a_{ij}, \ bar{c}_m = \sum_{n=1}^{m} \bar{a}_{nm}\).

**Theorem 4.1.** Assume that the symplectic condition \(4.4\) or equivalently,

\[ BA + A^T B - bb^T = 0, \]

where \(B = \text{diag}(b)\) and \(\bar{B} = \text{diag}(\bar{b})\), holds. Then the fully-discrete method \(4.8a-4.8f\) admits the discrete multi-symplectic conservation law

\[ \sum_{i=1}^{s} \frac{b_i}{\Delta t} (du_i^1 \wedge du_i^1 - du_i^0 \wedge du_i^0) - \sum_{m=1}^{r} \frac{\bar{b}_m}{\Delta x} (du_i^m \wedge dw_i^m - du_i^0 \wedge dw_i^0) = 0, \]

where \(s, r \in \mathbb{N}_+\).

**Proof.** By utilizing \(4.8a-4.8f\), we obtain

\[
\begin{align*}
\sum_{i=1}^{s} \frac{b_i}{\Delta t} (du_i^1 \wedge du_i^1 - du_i^0 \wedge du_i^0) - \sum_{m=1}^{r} \frac{\bar{b}_m}{\Delta x} (du_i^m \wedge dw_i^m - du_i^0 \wedge dw_i^0) \\
= \frac{1}{\Delta t} \sum_{i=1}^{s} b_i (d\mathbf{v}_i^1 \wedge d\mathbf{v}_i^1 - du_i^0 \wedge du_i^0) = \frac{1}{\Delta x} \sum_{m=1}^{r} \bar{b}_m (d\mathbf{v}_i^m \wedge d\mathbf{v}_i^m - du_i^0 \wedge dw_i^0).
\end{align*}
\]
For fixed $i \in \{1, \ldots, s\}$ and $m \in \{1, \ldots, r\}$, taking advantage of (4.8a) leads to
\[
\begin{align*}
d\iota_1^m & \land d\iota_2^m = d\iota_1^m \land d\iota_2^m + \Delta t \sum_{m=1}^r b_m \, d\iota_1^m \land d (\delta_x W_i^m - f(U_i^m)) \\
& \quad + \Delta t \sum_{m=1}^r b_m b_i d\iota_v^m \land d (\delta_x W_i^m - f(U_i^m)).
\end{align*}
\] (4.9)

Applying $dU_i^m = d\iota_1^m + \Delta t \sum_{n=1}^s \tilde{a}_{nm} d\iota_v^m$ and $dV_i^m = d\iota_1^m + \Delta t \sum_{n=1}^s \tilde{a}_{nm} d (\delta_x W_i^m - f(U_i^m))$ to (4.10), we get
\[
\begin{align*}
d\iota_1^m & \land d\iota_1^m \\
& = d\iota_1^m \land d\iota_1^m + \Delta t \sum_{l=1}^r \tilde{b}_l d\iota_1^l \land d (\delta_x W_i^l - f(U_i^l)) - \Delta t \sum_{m,l=1}^r \tilde{b}_m \tilde{a}_{ml} d\iota_v^m \land d (\delta_x W_i^m - f(U_i^m)) \\
& \quad - \Delta t \sum_{m,l=1}^r \tilde{b}_m \tilde{a}_{ml} d\iota_v^m \land d (\delta_x W_i^m - f(U_i^m)) + \Delta t \sum_{m,l=1}^r \tilde{b}_m \tilde{b}_l d\iota_v^m \land d (\delta_x W_i^m - f(U_i^m)).
\end{align*}
\] (4.10)

Based on (4.11), we obtain
\[
\begin{align*}
d\iota_1^m & \land d\iota_1^m = d\iota_1^m \land d\iota_1^m + \Delta t \sum_{l=1}^r \tilde{b}_l d\iota_1^l \land d (\delta_x W_i^l).
\end{align*}
\] (4.11)

Similarly, from (4.8b) it follows that
\[
\begin{align*}
d\iota_1^m & \land d\iota_1^m = d\iota_1^m \land d\iota_1^m + \Delta t \sum_{i=1}^s b_i d\iota_0^m \land d (\delta_x W_i^m) \\
& \quad + \Delta t \sum_{i=1}^s b_i dW_i^m \land d\iota_0^m + \Delta t \sum_{i,k=1}^s b_i b_k dW_i^m \land d (\delta_x W_k^m).
\end{align*}
\]

By means of (4.8a), we derive
\[
\begin{align*}
dU_i^m & = d\iota_1^m + \Delta t \sum_{j=1}^s a_{ij} dW_j^m, \quad dV_i^m = d\iota_1^m + \Delta t \sum_{j=1}^s a_{ij} d (\delta_x W_j^m),
\end{align*}
\]
which yields
\[
\begin{align*}
d\iota_1^m & \land d\iota_1^m = d\iota_1^m \land d\iota_1^m + \Delta t \sum_{i=1}^s b_i dU_i^m \land d (\delta_x W_i^m) - \Delta t \sum_{i,k=1}^s b_i b_k dW_i^m \land d (\delta_x W_k^m) \\
& \quad - \Delta t \sum_{i,k=1}^s b_i a_{ik} dW_i^m \land d (\delta_x W_k^m) + \Delta t \sum_{i,k=1}^s b_i b_k dW_i^m \land d (\delta_x W_k^m) \\
& = d\iota_0^m \land d\iota_0^m + \Delta t \sum_{i=1}^s b_i dU_i^m \land d (\delta_x W_i^m).
\end{align*}
\] (4.12)

Combining (4.11) and (4.12), we deduce
\[
\begin{align*}
\frac{1}{\Delta t} \sum_{i=1}^s b_i \left( d\iota_1^m \land d\iota_1^m - d\iota_1^m \land d\iota_1^m \right) & = \frac{1}{\Delta t} \sum_{m=1}^r \tilde{b}_m \left( d\iota_1^m \land d\iota_1^m - d\iota_0^m \land d\iota_0^m \right) \\
& = \sum_{i=1}^s \sum_{l=1}^r \tilde{b}_l b_i dU_i^l \land d (\delta_x W_i^l) - \sum_{m=1}^r \tilde{b}_m b_i dU_i^m \land d (\delta_x W_i^m) = 0,
\end{align*}
\]
which completes the proof. \qed
Example 4.2. If $s = r = 1$, based on symplectic Runge–Kutta methods

$$\frac{1}{2} \quad \frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{1}$$

we get a numerical method for the nonlinear stochastic wave equation as follows

$$U_1^1 = u_0^1 + \Delta x \frac{1}{2} W_1^1, \quad W_1^1 = u_0^1 + \Delta x \frac{1}{2} \delta_x W_1^1,$$

$$\bar{u}_1^1 = u_0^1 + \Delta x \bar{W}_1^1, \quad \bar{w}_1^1 = u_0^1 + \Delta x \bar{\delta}_x W_1^1,$$

$$U_1^1 = u_0^1 + \Delta t \frac{1}{2} V_1^1, \quad V_1^1 = v_0^1 + \Delta t \frac{1}{2} (\delta_x W_1^1 - f(U_1^1)),$$

$$\bar{u}_1^1 = v_0^1 + \Delta t \bar{V}_1^1, \quad \bar{v}_1^1 = v_0^1 + \Delta t (\delta_x W_1^1 - f(U_1^1)),$$

$$u_1^1 = \bar{u}_1^1, \quad w_1^1 = \bar{w}_1^1, \quad v_1^1 = \bar{v}_1^1 + g(\bar{u}_1^1) \Delta W_1^1.$$ (4.13)

Similar to the numerical experiments in Section 3, we apply the above multi-symplectic method to approximating the 1-dimensional stochastic wave equation in three cases: (1) $f(u) = \sin(u), g(u) = \sin(u)$; (2) $f(u) = \sin(u), g(u) = u$; (3) $f(u) = u^3, g(u) = \sin(u)$. Here, we take $x \in (-\pi, \pi)$, set $u(0) = 0$, $u_t(0, x) = x \sin(x), u_x(0) = 0$, and let the orthonormal basis $\{\phi_k\}_{k \in \mathbb{N}^+}$ and the corresponding eigenvalue $\{q_k\}_{k \in \mathbb{N}^+}$ of $Q$ be $\phi_k = \frac{1}{\sqrt{2\pi}} \sin(kx)$ and $q_k = \frac{1}{k^2}$, respectively. Table 2 shows the mean-square error against $\Delta t = 2^{-s}, s = 2, 3, 4, 5$ on log-log scale at time $T = 1$. We regard the numerical approximation obtained by a fine mesh with $\Delta t = 2^{-5}, \Delta x = 2^{-7.2}$ as the exact solution. It can be found from Fig. 2 that the proposed numerical method has accuracy of mean-square order 1 in temporal direction.

| $\Delta t$ | $L^2$ error | $L^2$ error | $L^2$ error |
|------------|--------------|--------------|--------------|
| $2^{-2}$   | 5.4462e-02  | 5.7036e-02  | 5.8488e-02  |
| $2^{-3}$   | 2.8150e-02  | 2.8546e-02  | 2.9977e-02  |
| $2^{-4}$   | 1.3499e-02  | 1.4167e-02  | 1.4498e-02  |
| $2^{-5}$   | 6.4268e-03  | 6.7747e-03  | 6.8146e-03  |

Figure 2: Mean-square convergence order of (4.13) in temporal direction in the cases of (1) $f(u) = \sin(u), g(u) = \sin(u)$ (2) $f(u) = \sin(u), g(u) = u$ and (3) $f(u) = u^3, g(u) = \sin(u)$.

If $\bar{f}(u)$ is at most quadratic, then the fully-discrete method (4.8a)–(4.8f) under the symplectic condition (4.4) preserves the discrete averaged energy evolution law. This property is illustrated by Fig. 3 from which plots the quantity $\frac{1}{N} \sum_{i=1}^{N} (v_i^n)^2 + (w_i^n)^2 + 2f(u_i^n), n = 1, \ldots, N$, for (4.13) in two cases: (1) $f(u) = 0$ and (2) $f(u) = u$, respectively. The reference line (black line) in Fig. 3 stands for the averaged energy evolution law of the exact solution. It can be observed that (4.13) preserves perfectly the averaged energy evolution.
law. In detail, when $g(u) = 1,$ (4.13) reproduces the linear growth of the averaged energy, which coincides with the theoretical results.

Now we turn to the stochastic nonlinear Schrödinger equation (2.3). Repeating the similar procedures as in the case of the stochastic wave equation, we first split the stochastic nonlinear Schrödinger equation into on $[t_0, t_1]$ a deterministic system with random input

$$\begin{align*}
q_t - v_x &= (p^2 + q^2) p, \\
\overline{p}_t + \overline{q}_x &= -(p^2 + q^2) \overline{q}, \\
\overline{p}(t_0) &= p(t_0), \quad \overline{q}(t_0) = q(t_0),
\end{align*}$$

and a stochastic system

$$\begin{align*}
p_x &= 0, \quad q_x = 0, \\
v_x &= 0, \quad w_x = 0, \\
dq &= -p \circ dW(t), \\
dp &= q \circ dW(t), \\
\overline{p}(t_0) &= \overline{q}(t_0), \quad q(t_0) = q(t_0),
\end{align*}$$

Using $s$-stage and $r$-stage Runge–Kutta methods (4.2) with $s, r \geq 1$ to discretize (4.14), together with the symplectic Euler method applied to (4.15), yields the fully-discrete method

$$\begin{align*}
P^m_i &= p^0_m + \Delta x \sum_{j=1}^s a_{ij} V^m_j, \\
V^m_i &= v^0_m + \Delta x \sum_{j=1}^s a_{ij} \delta_x V^m_j, \\
Q^m_i &= q^0_m + \Delta x \sum_{j=1}^s a_{ij} W^m_j, \\
W^m_i &= w^0_m + \Delta x \sum_{j=1}^s b_{ij} \delta_x W^m_j, \\
\overline{P}^m_i &= \overline{p}^0_m + \Delta x \sum_{n=1}^r b_{in} \delta_x \overline{V}^n_i, \\
\overline{V}^m_i &= \overline{v}^0_m + \Delta x \sum_{n=1}^s a_{in} \delta_x \overline{V}^n_i, \\
\overline{Q}^m_i &= \overline{q}^0_m + \Delta x \sum_{n=1}^r \tilde{a}_{in} \delta_x (\overline{P}^n_i + (P^m_i)^2 + (Q^m_i)^2) \overline{P}^m_i, \\
\overline{Q}^m_i &= \overline{q}^0_m + \Delta x \sum_{n=1}^r \tilde{a}_{in} \delta_x (\overline{Q}^n_i + (P^m_i)^2 + (Q^m_i)^2) \overline{Q}^m_i,
\end{align*}$$

(4.16)

Figure 3: Averaged energy evolution of (4.13) (left: $f(u) = 0,$ right: $f(u) = u$) with $\Delta t = 1/20,$ $h = \pi/20.$
\[ \mathbf{\tilde{p}}_i^0 = \mathbf{q}_i^0 + \Delta t \sum_{m=1}^r \hat{b}_m \left( \delta_{ij} V_i^m + ((P_i^m)^2 + (Q_i^m)^2) P_i^m \right), \]
\[ \mathbf{\tilde{p}}_i^1 = \mathbf{p}_i^0 + \Delta t \sum_{m=1}^r \hat{b}_m \left( -\delta_{ij} W_i^m - ((P_i^m)^2 + (Q_i^m)^2) Q_i^m \right), \]

\[ p_i^m = \mathbf{p}_i^m, \quad q_i^m = \mathbf{q}_i^m, \quad u_i^m = \mathbf{u}_i^m, \quad w_i^m = \mathbf{w}_i^m, \]

where \( P_i^m \approx p(c_i \Delta x, \tilde{c}_m \Delta t), \quad \mathbf{p}_i^0 \approx p(c_i \Delta x, 0), \quad \mathbf{p}_i^1 \approx p(c_i \Delta x, \Delta t), \quad \mathbf{p}_i^m \approx p(0, \tilde{c}_m \Delta t), \quad \mathbf{p}_i^m \approx p(\Delta x, \tilde{c}_m \Delta t), \quad \mathbf{p}_i^m \approx p(\Delta x, \tilde{c}_m \Delta t), \) etc., with \( c_i = \sum_{j=1}^s a_{ij}, \quad \tilde{c}_m = \sum_{j=1}^s \tilde{a}_{mn}, \quad i = 1, \ldots, s, \quad m = 1, \ldots, r. \) Similar to Theorem 4.1, we obtain that the fully-discrete method (4.16) preserves the discrete multi-symplectic conservation law.

**Theorem 4.3.** Under the symplectic condition (4.4), the fully-discrete method (4.16) preserves the discrete multi-symplectic conservation law

\[ \sum_{i=1}^s \frac{b_i}{\Delta x} (dp_i^1 \wedge dq_i^1 - dq_i^0 \wedge dp_i^0) + \sum_{i=1}^s \frac{b_i}{\Delta x} (dp_i^m \wedge dv_i^m - dp_i^0 \wedge dv_i^0 + dq_i^m \wedge dw_i^m - dq_i^0 \wedge dw_i^0) = 0. \]

Analogously, in the case of the stochastic KdV equation with additive noise (2.4), we first decompose it on \( t \in [t_0, t_1] \) into a deterministic system with random input

\[ \begin{cases} \mathbf{u}_t + 2\mathbf{u}_x = 0, \quad \mathbf{p}_t + 2\mathbf{p}_x = 2\mathbf{v} - \mathbf{\tilde{u}}, \\ \mathbf{u}_x = \mathbf{w}, \\ \mathbf{p}_x = \mathbf{n}, \\ \mathbf{u}(t_0) = u(t_0), \quad \mathbf{p}(t_0) = \rho(t_0), \end{cases} \]

and a stochastic system

\[ \begin{cases} v_x = 0, \quad w_x = 0, \\ u_x = 0, \quad \rho_x = 0, \\ du = 2\lambda \circ dW(t), \\ \rho_t = 0, \\ u(t_0) = \mathbf{n}(t_1), \quad \rho(t_0) = \mathbf{p}(t_1). \end{cases} \]

Next, we take advantage of \( s, r \)-stage symplectic Runge–Kutta methods, where \( s, r \geq 1 \), to numerically solve the deterministic Hamiltonian PDE (4.17) and use symplectic Euler method to approximate (4.18), respectively. The resulting numerical method on \( t \in [t_0, t_1] \) is as follows

\[ V_i^m = v_i^0 + \Delta x \sum_{j=1}^s a_{ij} \delta_{x} V_j^m, \quad W_i^m = w_i^0 + \Delta x \sum_{j=1}^s a_{ij} \delta_{x} W_j^m, \]
\[ U_i^m = u_i^0 + \Delta x \sum_{j=1}^s a_{ij} W_j^m, \quad \mathbf{p}_i^0 = p_i^0 + \Delta x \sum_{j=1}^s a_{ij} U_j^m, \]
\[ \mathbf{\tilde{p}}_i^m = v_i^0 + \Delta x \sum_{i=1}^s b_i \delta_{x} V_i^m, \quad \mathbf{\tilde{w}}_i^m = w_i^0 + \Delta x \sum_{i=1}^s b_i \delta_{x} W_i^m, \]
\[ \mathbf{\tilde{u}}_i^m = u_i^0 + \Delta x \sum_{i=1}^s b_i W_i^m, \quad \mathbf{\tilde{\rho}}_i^m = \rho_i^0 + \Delta x \sum_{i=1}^s b_i U_i^m, \]

where \( s, r \geq 1 \), to numerically solve the deterministic Hamiltonian PDE (4.17) and use symplectic Euler method to approximate (4.18), respectively. The resulting numerical method on \( t \in [t_0, t_1] \) is as follows

\[ \sum_{i=1}^s \frac{b_i}{\Delta x} (dp_i^1 \wedge dq_i^1 - dq_i^0 \wedge dp_i^0) + \sum_{i=1}^s \frac{b_i}{\Delta x} (dp_i^m \wedge dv_i^m - dp_i^0 \wedge dv_i^0 + dq_i^m \wedge dw_i^m - dq_i^0 \wedge dw_i^0) = 0. \]
Theorem 4.4. Assume that the symplectic condition \( \text{(4.4)} \) holds. Then the fully-discrete method \( \text{(4.5)} \) preserves the discrete multi-symplectic conservation law

\[
\sum_{i=1}^{s} \frac{b_i}{\Delta t} (\partial \rho_i^m \wedge \partial u_i^m - \partial \rho_i^0 \wedge \partial u_i^0) + \sum_{i=1}^{r} \frac{b_m}{\Delta x} (\partial \rho_i^m \wedge \partial u_i^m - \partial \rho_i^0 \wedge \partial u_i^0 + \beta \partial w_i^m \wedge \partial u_i^0 - \partial w_i^0 \wedge \partial u_i^0) = 0.
\]

Similarly, for the stochastic Maxwell equation with multiplicative noise \( \text{(4.20)} \), we decompose it on \( t \in [t_0, t_1] \) into a deterministic PDE with random initial value

\[
\begin{array}{l}
(E_1)_t + (\overline{H}_2)_x - (\overline{H}_3)_y = 0, (\overline{E}_2)_t + (\overline{H}_3)_x = 0, (\overline{E}_3)_t + (\overline{H}_2)_y = 0, (\overline{H}_2)_t + (\overline{E}_1)_x = 0, (\overline{H}_3)_t + (\overline{E}_2)_x = 0, \\
(E_1)(t_0) = E_1(t_0), (\overline{H}_1)(t_0) = H_1(t_0), i = 1, 2, 3.
\end{array}
\]

and a stochastic system

\[
\begin{array}{l}
\partial_t H_x = 0, \partial_t H_y = 0, \partial_t H_z = 0, \\
\partial_t E_x = 0, \partial_t E_y = 0, \partial_t E_z = 0, \\
H_1 = \lambda E \circ dW(t), E_t = -\lambda H \circ dW(t), \\
(\overline{H})(t_0) = (\overline{H})(t_1), E(t_0) = (\overline{E})(t_1).
\end{array}
\]

By exploiting \( s \)-stage and \( r \)-stage symplectic Runge–Kutta methods to discretize \( \text{(4.20)} \) and symplectic Euler method to discretize \( \text{(4.21)} \), we obtain the numerical method on \( t \in [t_0, t_1] \) as follows

\[
\begin{array}{l}
\partial_t (H)_k^{m, n} = \partial_t (H)_k^{0, n} + \Delta x \sum_{j=1}^{s} a_{kj}^1 \partial_t (\delta_x H)_j^{m, n}, \quad \partial_t (E)_k^{m, n} = \partial_t (E)_k^{0, n} + \Delta x \sum_{j=1}^{s} a_{kj}^1 \partial_t (\delta_x E)_j^{m, n}, \\
\partial_t (\overline{H})_k^{m, n} = \partial_t (\overline{H})_k^{0, n} + \Delta y \sum_{j=1}^{s} a_{kj}^2 \partial_t (\delta_y H)_j^{m, n}, \quad \partial_t (\overline{E})_k^{m, n} = \partial_t (\overline{E})_k^{0, n} + \Delta y \sum_{j=1}^{s} a_{kj}^2 \partial_t (\delta_y E)_j^{m, n}, \\
\partial_t (\overline{H})_k^{m, n} = \partial_t (\overline{H})_k^{0, n} + \Delta z \sum_{j=1}^{s} a_{kj}^3 \partial_t (\delta_z H)_j^{m, n}, \quad \partial_t (\overline{E})_k^{m, n} = \partial_t (\overline{E})_k^{0, n} + \Delta z \sum_{j=1}^{s} a_{kj}^3 \partial_t (\delta_z E)_j^{m, n}, \\
(\overline{H})_k^{m, n} = (\overline{H})_k^{0, n} + \Delta t \sum_{i=1}^{r} \tilde{a}_{m} \left(-\partial_t (\delta_x E)_k^{i, n} - \partial_t (\delta_y E)_k^{i, n} - \partial_t (\delta_z E)_k^{i, n}\right), \\
(\overline{E})_k^{m, n} = (\overline{E})_k^{0, n} + \Delta t \sum_{i=1}^{r} \tilde{a}_{m} \left(\partial_t (\delta_x H)_k^{i, n} + \partial_t (\delta_y H)_k^{i, n} + \partial_t (\delta_z H)_k^{i, n}\right), \\
\partial_t (\overline{H})_k^{m, n} = \partial_t (\overline{H})_k^{0, n} + \Delta x \sum_{k=1}^{s} b_k^1 \partial_t (\delta_x H)_k^{m, n}, \quad \partial_t (\overline{E})_k^{m, n} = \partial_t (\overline{E})_k^{0, n} + \Delta x \sum_{k=1}^{s} b_k^1 \partial_t (\delta_x E)_k^{m, n}.
\end{array}
\]
Runge–Kutta methods, are powerful tools for the construction of symplectic methods for solving stochastic differential equations. Theorem 4.5.

Remark 4.6.

The discrete multi-symplectic conservation law

\[ \Delta x, \Delta y, \Delta z, \Delta \cdot \cdot \cdot \text{conservation} \]

where \( \delta_x, \delta_y, \delta_z \) are discretizations of partial derivatives \( \partial_x, \partial_y, \partial_z \), \( E_{0,0}^0 \approx E(c_1^{(1)} \Delta x, c_1^{(2)} \Delta y, c_1^{(3)} \Delta z, 0) \), \( E_{0,0}^1 \approx E(c_1^{(1)} \Delta x, c_1^{(2)} \Delta y, c_1^{(3)} \Delta z, \Delta t) \), \( \mathbf{E}_{1,0}^0 \approx E[0, c_1^{(2)} \Delta y, c_1^{(3)} \Delta z, \Delta t] \), \( \mathbf{E}_{1,0}^1 \approx E[c_1^{(1)} \Delta x, 0, c_1^{(3)} \Delta z, \Delta t] \), \( \mathbf{E}_{1,0}^2 \approx E[c_1^{(1)} \Delta x, c_1^{(2)} \Delta y, 0, \Delta t] \), \( \mathbf{E}_{1,0}^3 \approx E[c_1^{(1)} \Delta x, c_1^{(2)} \Delta y, c_1^{(3)} \Delta z, \Delta t] \), etc., with \( c_1 = \sum_{j=1}^n a_{1j} \), \( c_2 = \sum_{j=1}^n a_{2j} \), \( c_3 = \sum_{j=1}^n a_{3j} \), \( c_m = \sum_{j=1}^n a_{mj} \), \( 1 \leq j, k, l, n \leq s, 1 \leq t, m \leq r \). Moreover, the noise increment \( \Delta W_{k,n}^1 \) is defined as \( W(t, x, y, z) - W(t, x, y, z) \).

Theorem 4.5. Suppose that the symplectic condition \( \mathbb{I} \) holds. Then the fully-discrete method \( \mathbb{I} \) preserves the discrete multi-symplectic conservation law

\[
\frac{1}{\Delta t} \sum_{k=1}^s \sum_{l=1}^s \sum_{n=1}^s b_l^{(2)} b_l^{(3)} (d\mathbf{E}^m_{k,l,n} \wedge d\mathbf{H}^m_{k,l,n} - d\mathbf{E}^m_{k,l,n} \wedge d\mathbf{H}^m_{k,l,n}) \\
+ \frac{1}{2\Delta x} \sum_{m=1}^r \sum_{k=1}^s \sum_{l=1}^s \tilde{b}_m b_l^{(1)} b_l^{(3)} (d\mathbf{U}^m_{k,l,n} \wedge K_1 d\mathbf{U}^m_{k,l,n} - d\mathbf{U}^m_{k,l,n} \wedge K_1 d\mathbf{U}^m_{k,l,n}) \\
+ \frac{1}{2\Delta y} \sum_{m=1}^r \sum_{k=1}^s \sum_{l=1}^s \tilde{b}_m b_l^{(1)} b_l^{(3)} (d\mathbf{U}^m_{k,l,n} \wedge K_2 d\mathbf{U}^m_{k,l,n} - d\mathbf{U}^m_{k,l,n} \wedge K_2 d\mathbf{U}^m_{k,l,n}) \\
+ \frac{1}{2\Delta z} \sum_{m=1}^r \sum_{k=1}^s \sum_{l=1}^s \tilde{b}_m b_l^{(1)} b_l^{(3)} (d\mathbf{U}^m_{k,l,n} \wedge K_3 d\mathbf{U}^m_{k,l,n} - d\mathbf{U}^m_{k,l,n} \wedge K_3 d\mathbf{U}^m_{k,l,n}) = 0
\]

with \( (\mathbf{U}^m_{k,l,n})^\top, (\mathbf{E}^m_{k,l,n})^\top \) and \( (\mathbf{H}^m_{k,l,n})^\top \). We would like to mention that, in the framework of splitting multi-symplectic Runge–Kutta method, other methods of multi-symplectic methods can be used to discretize the deterministic Hamiltonian PDE. By combining with the symplectic Euler method applied to the stochastic subsystem, one can obtain a class of multi-symplectic methods.

5. Multi-symplectic partitioned Runge–Kutta method

As we know, symplectic partitioned Runge–Kutta methods, which are the generalizations of symplectic Runge–Kutta methods, are powerful tools for the construction of symplectic methods for solving stochastic
Hamiltonian ordinary differential equations numerically. For separate stochastic Hamiltonian ordinary differential equations, some symplectic partitioned Runge–Kutta method is explicit, which reduces the computational cost. In this section, we construct the third kind of multi-symplectic methods, i.e., multi-symplectic partitioned Runge–Kutta methods for nonlinear stochastic wave equation, stochastic nonlinear Schrödinger equation, stochastic KdV equation and stochastic Maxwell equation by means of the symplectic partitioned Runge–Kutta method in both spatial and temporal directions. Further, we present the multi-symplectic conditions.

For the nonlinear stochastic wave equation (2.2), we proceed to take advantage of s-stage partitioned Runge–Kutta method \((c^{(1)}, A^{(1)}, b^{(1)})\) and \((c^{(2)}, A^{(2)}, b^{(2)})\), i.e.,

\[
\begin{align*}
&c^{(1)}_1 \quad a^{(1)}_{11} \quad \ldots \quad a^{(1)}_{1s} \quad c^{(1)}_2 \quad a^{(2)}_{11} \quad \ldots \quad a^{(2)}_{1s} \\
&\vdots \quad \vdots \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \vdots \quad \vdots \\
&c^{(1)}_s \quad a^{(1)}_{s1} \quad \ldots \quad a^{(1)}_{s1} \quad c^{(2)}_s \quad a^{(2)}_{s1} \quad \ldots \quad a^{(2)}_{s1} \\
&\bar{b}^{(1)}_1 \quad \ldots \quad \bar{b}^{(1)}_s \\
&\bar{b}^{(2)}_1 \quad \ldots \quad \bar{b}^{(2)}_s
\end{align*}
\]  

(5.1)

in the spatial direction, and \(r\)-stage partitioned Runge–Kutta method \((\bar{c}^{(1)}, \bar{A}^{(1)}, \bar{b}^{(1)})\), \((\bar{c}^{(2)}, \bar{A}^{(2)}, \bar{b}^{(2)})\), i.e.,

\[
\begin{align*}
&\bar{c}^{(1)}_1 \quad \bar{a}^{(1)}_{11} \quad \ldots \quad \bar{a}^{(1)}_{1r} \quad \bar{c}^{(2)}_1 \quad \bar{a}^{(2)}_{11} \quad \ldots \quad \bar{a}^{(2)}_{1r} \\
&\vdots \quad \vdots \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \vdots \quad \vdots \\
&\bar{c}^{(1)}_r \quad \bar{a}^{(1)}_{r1} \quad \ldots \quad \bar{a}^{(1)}_{r1} \quad \bar{c}^{(2)}_r \quad \bar{a}^{(2)}_{r1} \quad \ldots \quad \bar{a}^{(2)}_{r1} \\
&\bar{b}^{(1)}_1 \quad \ldots \quad \bar{b}^{(1)}_r \\
&\bar{b}^{(2)}_1 \quad \ldots \quad \bar{b}^{(2)}_r
\end{align*}
\]  

(5.2)

together with an \(r\)-stage Runge–Kutta method \((\bar{c}, \bar{A}, \bar{b})\) in the temporal direction, respectively, where \(s, r \in \mathbb{N}_+\). The resulting fully-discrete method is as follows

\[
\begin{align*}
U_i^m &= u_i^0 + \Delta t \sum_{n = 1}^r \bar{a}^{(1)}_{nm} V_i^n, \quad V_i^m = v_i^0 + \Delta t \sum_{n = 1}^r \bar{a}^{(2)}_{nm} (\delta_x W_i^m - f(U_i^m)) + \Delta W_i^1 \sum_{n = 1}^r \bar{a}_{nm} g(U_i^m), \\
u_i^1 &= u_i^0 + \Delta t \sum_{m = 1}^r \bar{b}^{(1)}_{nm} V_i^m, \quad v_i^1 = v_i^0 + \Delta t \sum_{m = 1}^r \bar{b}^{(2)}_{nm} (\delta_x W_i^m - f(U_i^m)) + \Delta W_i^1 \sum_{m = 1}^r \bar{b}_{nm} g(U_i^m), \\
U_i^m &= u_i^0 + \Delta x \sum_{j = 1}^s \bar{a}^{(1)}_{ij} W_j^m, \quad W_i^m = w_i^0 + \Delta x \sum_{j = 1}^s \bar{a}^{(2)}_{ij} \delta_x W_j^m, \\
u_i^1 &= u_i^0 + \Delta x \sum_{j = 1}^s \bar{b}^{(1)}_{ij} W_j^m, \quad w_i^1 = w_i^0 + \Delta x \sum_{j = 1}^s \bar{b}^{(2)}_{ij} \delta_x W_j^m,
\end{align*}
\]

(5.3a)–(5.3d)

where \(U_i^m \approx u(c_i^{(1)} \Delta x, c_i^{(1)} \Delta t), u_i^0 \approx u(c_i^{(1)} \Delta x, 0), u_i^1 \approx u(c_i^{(1)} \Delta x, \Delta t), u_i^m \approx u(0, c_i^{(1)} \Delta t), u_i^m \approx u(\Delta x, c_i^{(1)} \Delta t),\) etc., with \(c_i^{(1)} = \sum_{j = 1}^r \bar{a}^{(1)}_{ij}, c_i^{(2)} = \sum_{j = 1}^r \bar{a}^{(2)}_{ij}\) for \(i = 1, \ldots, s, m = 1, \ldots, r.\)

**Theorem 5.1.** Suppose that

\[
\begin{align*}
&\bar{a}_{nm} \bar{b}^{(1)}_m + \bar{a}^{(1)}_{nm} \bar{b}^{(1)}_n - \bar{b}^{(1)}_m \bar{b}^{(1)}_n = 0, \\
&\bar{a}_{nm} \bar{b}^{(2)}_m + \bar{a}^{(1)}_{nm} \bar{b}^{(2)}_n - \bar{b}^{(1)}_m \bar{b}^{(2)}_n = 0, \\
&\bar{a}_{ij} \bar{b}^{(2)}_j + \bar{a}^{(1)}_{ij} \bar{b}^{(2)}_j - \bar{b}^{(1)}_i \bar{b}^{(2)}_j = 0,
\end{align*}
\]

(5.4a)–(5.4c)

for \(1 \leq i, j \leq s, 1 \leq m, n \leq r.\) Then the fully-discrete method (5.3a)–(5.3d) admits the discrete multi-symplectic conservation law

\[
\sum_{i = 1}^r \bar{b}_{ij}^{(2)} \frac{1}{\Delta x} (du_i^1 \wedge dv_i^1 - du_i^0 \wedge dv_i^0) - \sum_{m = 1}^r \bar{b}_{ij}^{(2)} \frac{1}{\Delta x} (du_i^m \wedge dv_i^m - du_i^0 \wedge dv_i^0) = 0.
\]

(5.5)
Proof. From (5.3b) it follows that
\[(du^i_0 \land dv^i_0 - du^m_0 \land dv^m_0)\]
\[= (du^i_0 + \Delta t \sum_{m=1}^{r} \tilde{b}^{(1)}_m dV^m_i) \land (dv^i_0 + \Delta t \sum_{m=1}^{r} \tilde{b}^{(2)}_m d(\delta_x W^m_i - f(U^m_i)) + \Delta W^1_i \sum_{m=1}^{r} \tilde{b}^m d g(U^m_i)) - du^i_0 \land dv^i_0.\]
Based on (5.3a) we derive
\[\begin{align*}
du^i_0 &= dU^m_i - \Delta t \sum_{n=1}^{s} \tilde{a}^{(1)}_{nm} dV^n_i, \quad dv^i_0 = dV^m_i - \Delta t \sum_{n=1}^{s} \tilde{a}^{(2)}_{nm} d(\delta_x W^m_i - f(U^m_i)) - \Delta W^1_i \sum_{nm}^{r} \tilde{a}^m d g(U^m_i),
\end{align*}\]
which implies
\[\frac{1}{\Delta t} (du^i_0 \land dv^i_0 - du^m_0 \land dv^m_0) = \sum_{m=1}^{r} \tilde{b}^{(2)}_m dU^m_i \land d(\delta_x W^m_i).\]

Similarly, by means of (5.3c), we derive
\[\frac{1}{\Delta x} (du^m_i \land dv^m_i - du^m_0 \land dv^m_0) = \sum_{i=1}^{s} \tilde{b}^{(1)}_i dU^m_i \land d(\delta_x W^m_i).\]

By utilizing (5.4c), we obtain
\[\frac{1}{\Delta x} (du^m_i \land dv^m_i - du^m_0 \land dv^m_0) = \sum_{i=1}^{s} \tilde{b}^{(2)}_i dU^m_i \land d(\delta_x W^m_i).\]

Combining (5.6) and (5.7), we have
\[\sum_{i=1}^{s} \tilde{b}^{(2)}_i \left(\frac{1}{\Delta t} (du^i_0 \land dv^i_0 - du^m_0 \land dv^m_0) - \frac{1}{\Delta x} (du^m_i \land dv^m_i - du^m_0 \land dv^m_0)\right) = 0,
\]
which completes the proof.

Example 5.2. Let \(s = r = 1\) and the Butcher tableaux in both (5.1) and (5.2) be \(\frac{1}{2}\)\(\frac{1}{2}\), we obtain an explicit numerical method for the nonlinear stochastic wave equation (5.8) as follows:

\[\begin{align*}
U^1_i &= u^0_i + \frac{\Delta t}{2} V^1_i, \quad V^1_i = v^0_i + \frac{\Delta t}{2} (\delta_x W^1_i - f(U^1_i)) + \frac{\Delta W^1_i}{2} g(U^1_i),

w^1_i &= u^1_i + \Delta x W^1_i, \quad W^1_i = w^0_i + \frac{\Delta x}{2} \delta_x W^1_i,

u^1_i &= u^0_i + \Delta x W^1_i, \quad w^1_i = w^0_i + \Delta x \delta_x W^1_i.
\end{align*}\]
Now we perform experiments by applying (5.8) to the 1-dimensional nonlinear stochastic wave equation, and consider the same problem as in Example 4.2. Table 3 shows the mean-square convergence error against $\Delta t = 2^{-s}, s = 2, 3, 4, 5$ on log-log scale at time $T = 1$. The exact solution is regarded as the numerical approximation obtained by a fine mesh with $\Delta t = 2^{-8}, \Delta x = 2^{-7}\pi$. Fig. 4 shows that the mean-square convergence order of the proposed numerical method is 1 in time.

| $\Delta t$ | $L^2$ error $f(u) = \sin(u), g(u) = \sin(u)$ | $L^2$ error $f(u) = \sin(u), g(u) = u$ | $L^2$ error $f(u) = u^3, g(u) = \sin(u)$ |
|------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| $2^{-2}$   | 2.7559e-02                                   | 2.9672e-02                                   | 2.7987e-02                                   |
| $2^{-3}$   | 1.4317e-02                                   | 1.4617e-02                                   | 1.4566e-02                                   |
| $2^{-4}$   | 7.1487e-03                                   | 7.4123e-03                                   | 7.3600e-03                                   |
| $2^{-5}$   | 3.6446e-03                                   | 3.5495e-03                                   | 3.7130e-03                                   |

Figure 4: Mean-square convergence order of (5.8) in temporal direction in the cases of (1) $f(u) = \sin(u), g(u) = \sin(u)$ (2) $f(u) = \sin(u), g(u) = u$ and (3) $f(u) = u^3, g(u) = \sin(u)$.

Analogous to the nonlinear stochastic wave equation, for the stochastic nonlinear Schrödinger equation (2.3), based on $s$-stage partitioned Runge–Kutta methods

$$
\begin{align*}
&\begin{bmatrix} c_1^{(1)} & a_{11}^{(1)} & \ldots & a_{1s}^{(1)} \\
& \vdots & \ddots & \vdots \\
&\end{bmatrix} \\
&\begin{bmatrix} b_1^{(1)} & \ldots & b_s^{(1)} \\
& \vdots & \ddots & \vdots \\
&\end{bmatrix}
\end{align*}
$$

and $r$-stage partitioned Runge–Kutta methods

$$
\begin{align*}
&\begin{bmatrix} c_1^{(1)} & \tilde{a}_{11}^{(1)} & \ldots & \tilde{a}_{1r}^{(1)} \\
& \vdots & \ddots & \vdots \\
&\end{bmatrix} \\
&\begin{bmatrix} b_1^{(1)} & \ldots & b_r^{(1)} \\
& \vdots & \ddots & \vdots \\
&\end{bmatrix}
\end{align*}
$$

where $s, r \in \mathbb{N}_+$, we deduce the following fully-discrete method

$$
Q_i^m = q_i^0 + \Delta t \sum_{n=1}^r \tilde{a}_{nm}^{(1)} \left( \delta_{i} V_i^n + (Q_i^n)^2 + P_i^n \right) - \Delta W_i^1 \sum_{n=1}^r \tilde{a}_{nm}^{(1)} P_i^n, \\
P_i^m = p_i^0 + \Delta t \sum_{n=1}^r \tilde{a}_{nm}^{(2)} \left( -\delta_{i} W_i^n - (Q_i^n)^2 + P_i^n \right) + \Delta W_i^1 \sum_{n=1}^r \tilde{a}_{nm}^{(2)} Q_i^n,
$$

with $Q_i^m$ and $P_i^m$ representing the numerical approximations of $Q_i$ and $P_i$, respectively.
\[ q^1_t = q^0_t + \Delta t \sum_{m=1}^{r} \tilde{b}^{(1)}_m \delta_x V^m_t + ((P^m_t)^2 + (Q^m_t)^2)P^m_t - \Delta W^1_i \sum_{m=1}^{r} b^{(1)}_m P^m_t, \]
\[ p^1_t = p^0_t + \Delta t \sum_{m=1}^{r} \tilde{b}^{(2)}_m (-\delta_x W^m_t - ((P^m_t)^2 + (Q^m_t)^2)Q^m_t) + \Delta W^1_i \sum_{m=1}^{r} b^{(2)}_m Q^m_t, \]
\[ V^m_t = v^m_0 + \Delta x \sum_{j=1}^{s} \tilde{a}^{(1)}_{ij} \delta_x V^m_j, \quad W^m_t = w^m_0 + \Delta x \sum_{j=1}^{s} \tilde{a}^{(2)}_{ij} \delta_x W^m_j, \]
\[ P^m_t = p^m_0 + \Delta x \sum_{j=1}^{s} b^{(3)}_{ij} V^m_j, \quad Q^m_t = q^m_0 + \Delta x \sum_{j=1}^{s} \tilde{a}^{(4)}_{ij} W^m_j, \]
\[ v^m_1 = v^m_0 + \Delta x \sum_{i=1}^{s} \tilde{b}^{(1)}_i \delta_x V^m_i, \quad w^m_1 = w^m_0 + \Delta x \sum_{i=1}^{s} \tilde{b}^{(2)}_i \delta_x W^m_i, \]
\[ p^m_1 = p^m_0 + \Delta x \sum_{i=1}^{s} b^{(3)}_i V^m_i, \quad q^m_1 = q^m_0 + \Delta x \sum_{i=1}^{s} \tilde{b}^{(4)}_i W^m_i, \]

where \( Q^m_0 \equiv q(c^4(Dx, \tilde{c}^1(\Delta t), q^1_0 \equiv q(c^4(Dx, \Delta t), q^m_0 \equiv q(0, \tilde{c}^m(\Delta t), q^m_1 \equiv q(Dx, \tilde{c}^m(\Delta t), \)

etc., with \( c^4 = \sum_{a=1}^{s} a^{(4)}_a, \tilde{c}^1 = \sum_{m=1}^{r} a^{(1)}_m, \) for \( i = 1, \ldots, s, m = 1, \ldots, r. \) Similar to Theorem 5.1, we obtain the following result.

**Theorem 5.3.** If the following conditions

\[ \tilde{a}^{(1)}_{mn,ij} \tilde{b}^{(2)}_n - \tilde{b}^{(1)}_m \tilde{b}^{(2)}_n = 0, \]
\[ \tilde{a}^{(1)}_{mn,ij} \tilde{b}^{(2)}_n - \tilde{b}^{(1)}_m \tilde{b}^{(2)}_n = 0, \]
\[ \tilde{a}^{(1)}_{mn,ij} \tilde{b}^{(2)}_n - \tilde{b}^{(1)}_m \tilde{b}^{(2)}_n = 0, \]
\[ \tilde{a}^{(1)}_{mn,ij} \tilde{b}^{(2)}_n - \tilde{b}^{(1)}_m \tilde{b}^{(2)}_n = 0, \]
\[ \tilde{a}^{(3)}_{ij} b^{(3)}_j - \tilde{a}^{(1)}_{ij} b^{(3)}_j = 0, \]
\[ \tilde{a}^{(2)}_{ij} b^{(4)}_j - \tilde{a}^{(1)}_{ij} b^{(4)}_j = 0, \]
\[ \tilde{a}^{(2)}_{ij} b^{(4)}_j - \tilde{a}^{(1)}_{ij} b^{(4)}_j = 0, \]
\[ b^{(1)}_i = \tilde{b}^{(1)}_i, \quad \tilde{b}^{(1)}_m = \tilde{b}^{(1)}_m, \]

for \( 1 \leq i, j \leq s, 1 \leq m, n \leq r, \) hold, the fully-discrete method (5.11) possesses the discrete multi-symplectic conservation law

\[ \sum_{i=1}^{s} \tilde{b}^{(1)}_i \frac{1}{\Delta t} (dq^i_t \land dp^1_t - dq^0_t \land dp^0_t) + \sum_{m=1}^{r} \tilde{b}^{(1)}_m \frac{1}{\Delta x} (dp^m_t \land dw^0_t - dq^0_t \land dw^0_t) = 0. \]

In the case of the stochastic KdV equation (2.4), exploiting similar procedures that applying s-stage partitioned Runge–Kutta methods (5.9), and r-stage Runge–Kutta methods (5.10) with \( s, r \in \mathbb{N}_+ \) yields the following multi-symplectic method

\[ U^m_t = u^0_t + \Delta t \sum_{n=1}^{r} (-2\delta_x V^m_n) \tilde{a}^{(1)}_{nm} + \sum_{n=1}^{r} 2\lambda \Delta W^1_i \tilde{a}_{nm}, \]
\[ P^m_t = p^0_t + \Delta t \sum_{n=1}^{r} (-2\delta_x W^m_n + 2V^m_n - (U^m_n)^2) \tilde{a}^{(2)}_{nm}, \]

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\[ u_i^1 = u_i^0 + \Delta t \sum_{m=1}^{r} (-2\delta_x V_i^m) \tilde{b}_m^{(1)} + \sum_{m=1}^{r} 2\lambda \Delta W_i^1 \tilde{b}_m, \]
\[ \rho_i^1 = \rho_i^0 + \Delta t \sum_{m=1}^{r} (-2\beta \delta_x \mathcal{W}_i^m + 2V_i^m - (U_i^m)^2) \tilde{b}_m^{(2)}, \]
\[ U_i^m = u_i^m + \Delta x \sum_{j=1}^{s} a_{ij}^{(1)} \mathcal{W}_j^m, \quad \mathcal{P}_i^m = \rho_i^m + \Delta x \sum_{j=1}^{s} a_{ij}^{(2)} U_j^m, \]
\[ V_i^m = v_i^m + \Delta x \sum_{j=1}^{s} a_{ij}^{(3)} \delta_x V_j^m, \quad \mathcal{W}_i^m = u_i^m + \Delta x \sum_{j=1}^{s} a_{ij}^{(4)} \delta_x \mathcal{W}_j^m, \]
\[ w_i^1 = w_i^0 + \Delta x \sum_{j=1}^{s} b_{ij}^{(1)} \mathcal{W}_j^m, \quad \rho_i^1 = \rho_i^0 + \Delta x \sum_{j=1}^{s} b_{ij}^{(2)} U_j^m, \]
\[ v_i^m = v_i^0 + \Delta x \sum_{j=1}^{s} b_{ij}^{(3)} \delta_x V_j^m, \quad w_i^m = w_i^0 + \Delta x \sum_{j=1}^{s} b_{ij}^{(4)} \delta_x \mathcal{W}_j^m, \]
where
\[ \tilde{a}_{m,n}^{(1)} \tilde{b}_n^{(2)} + \tilde{b}_m^{(1)} \tilde{b}_n^{(1)} - \tilde{b}_m^{(1)} \tilde{b}_n^{(2)} = 0, \]
\[ \tilde{a}_{m,n} \tilde{b}_n^{(2)} + \tilde{b}_m \tilde{a}_m^{(2)} - \tilde{b}_m \tilde{b}_n^{(2)} = 0, \]
\[ a_{ij}^{(3)} b_{j}^{(2)} + b_{j}^{(3)} a_{ij}^{(2)} - b_{j}^{(2)} b_{j}^{(3)} = 0, \]
\[ a_{ij}^{(4)} b_{j}^{(4)} + b_{j}^{(4)} a_{ij}^{(4)} - b_{j}^{(4)} b_{j}^{(4)} = 0, \]
\[ b_{i}^{(2)} = b_{i}^{(3)}, \quad b_{i}^{(4)} = b_{i}^{(4)}, \quad \tilde{b}_{i}^{(1)} = \tilde{b}_{i}^{(2)}. \]

and \( U_i^m \approx u(c_i^{(1)} \Delta x, \tilde{c}_m^{(1)} \Delta t), u_i^0 \approx u(c_i^{(1)} \Delta x, 0), u_i^1 \approx u(c_i^{(1)} \Delta x, \Delta t), u_i^0 \approx u(0, \tilde{c}_m^{(1)} \Delta t), u_i^m \approx u(\Delta x, \tilde{c}_m^{(1)} \Delta t), \)
etc., with \( c_i^{(1)} = \sum_{j=1}^{s} a_{ij}^{(1)}, \tilde{c}_m^{(1)} = \sum_{m=1}^{r} a_{i}^{(1)} \) for \( i = 1, \ldots, s, \ m = 1, \ldots, r. \) Making use of the same arguments as in the proof of Theorem 5.1, the associated discrete multi-symplectic conservation law reads
\[ \sum_{i=1}^{s} \tilde{b}_i^{(2)} \wedge du_i^1 - \Delta t \rho_i^0 \wedge du_i^0 \]
\[ + \sum_{m=1}^{r} \frac{\tilde{b}_m^{(2)}}{\Delta x} (2d\rho_i^1 \wedge du_i^1 - 2d\rho_i^0 \wedge dw_i^0 + 2\beta dw_i^1 \wedge du_i^m - 2\beta dw_i^0 \wedge du_i^m) = 0. \]

For the stochastic Maxwell equation (2.24), adopting \( s \)-stage partitioned Runge–Kutta methods with Butcher tableaux \( (c^{(1)}, A^{(1)}, b^{(1)}), \) \( (c^{(2)}, A^{(2)}, b^{(2)}) \) in \( x \) direction, \( (c^{(3)}, A^{(3)}, b^{(3)}) \) and \( (c^{(4)}, A^{(4)}, b^{(4)}) \) in \( y \) direction, which are presented in (5.5), \( (c^{(5)}, A^{(5)}, b^{(5)}) \) and \( (c^{(6)}, A^{(6)}, b^{(6)}) \) in \( z \) direction as follows
\[
\begin{array}{cccccccc}
  c_1^{(5)} & a_{11}^{(5)} & \ldots & a_{1s}^{(5)} & c_1^{(6)} & a_{11}^{(6)} & \ldots & a_{1s}^{(6)} \\
  \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
  c_s^{(5)} & a_{s1}^{(5)} & \ldots & a_{ss}^{(5)} & c_s^{(6)} & a_{s1}^{(6)} & \ldots & a_{ss}^{(6)} \\
  b_{1}^{(5)} & b_{2}^{(5)} & \ldots & b_{s}^{(5)} & b_{1}^{(6)} & b_{2}^{(6)} & \ldots & b_{s}^{(6)} \\
\end{array}
\]
and \( r \)-stage partitioned Runge–Kutta methods (5.10) in the temporal direction, respectively, where \( s, r \in \mathbb{N}_+. \)

The resulting numerical method is as follows
\[
\mathcal{D}_1(\mathbf{H})_{kn}^{m} = \mathcal{D}_1(\mathbf{H})_{0n}^{m} + \Delta x \sum_{j=1}^{s} a_{kj}^{(1)} \mathcal{D}_1(\delta_x \mathbf{H})_{jln}^{m}, \quad \mathcal{D}_2(\mathbf{H})_{kn}^{m} = \mathcal{D}_2(\mathbf{H})_{0n}^{m} + \Delta y \sum_{j=1}^{r} a_{kj}^{(2)} \mathcal{D}_2(\delta_y \mathbf{H})_{kjn}^{m},
\]
\[D_3(H)_k^n = D_3(H)_k^n + \Delta z \sum_{j=1}^{s} a_j^{(1)} D_3(\delta_{j}\bar{H})_{k,j}, \quad D_3(E)_k^n = D_3(E)_k^n + \Delta y \sum_{j=1}^{s} a_j^{(5)} D_3(\delta_{j}\bar{E})_{k,j}, \quad D_3(E)_k^n = D_3(E)_k^n + \Delta z \sum_{j=1}^{s} a_j^{(6)} D_3(\delta_{j}\bar{E})_{k,j},\]

\[(H)_k^n = (H)_k^n + \Delta t \sum_{i=1}^{r} \tilde{a}_{m(i)} - (D_2(\delta_x E)_k^n) - (D_2(\delta_y E)_k^n) + \lambda \Delta W_{1,k}^{(1)} \sum_{i=1}^{r} \tilde{a}_{m(i)}(E)_{k,n}, \quad (5.16)\]

\[D_1(H)_k^n + \Delta x \sum_{k=1}^{s} b_k^{(1)} D_1(\delta_x H)_k^n, \quad D_2(H)_k^n = D_2(H)_k^n + \Delta y \sum_{k=1}^{s} b_k^{(2)} D_2(\delta_y H)_k^n, \quad D_3(H)_k^n = D_3(H)_k^n + \Delta z \sum_{n=1}^{s} b_n^{(3)} D_3(\delta_z H)_k^n, \]

\[D_3(E)_k^n = D_3(E)_k^n + \Delta y \sum_{k=1}^{s} b_k^{(5)} D_3(\delta_x E)_k^n, \quad D_3(E)_k^n = D_3(E)_k^n + \Delta z \sum_{n=1}^{s} b_n^{(6)} D_3(\delta_y E)_k^n, \]

\[(H)_k^n = (H)_k^n + \Delta t \sum_{m=1}^{r} \tilde{b}_{m(1)} - (D_2(\delta_x E)_k^n) - (D_3(\delta_z E)_k^n) + \lambda \Delta W_{1,k}^{(1)} \sum_{m=1}^{r} \tilde{b}_{m(1)}(E)_{k,n}, \quad (5.16)\]

where

\[a_{m(i)}^{(1)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(1)} b_{m(i)}^{(1)} - \tilde{b}_{m(i)}^{(1)} = 0, \quad a_{m(i)}^{(2)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(1)} - \tilde{b}_{m(i)}^{(1)} = 0, \]

\[a_{m(i)}^{(1)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(1)} - \tilde{b}_{m(i)}^{(2)} = 0, \quad a_{m(i)}^{(2)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(1)} - \tilde{b}_{m(i)}^{(1)} = 0, \]

\[a_{m(i)}^{(1)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(2)} - \tilde{b}_{m(i)}^{(2)} = 0, \quad a_{m(i)}^{(2)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(2)} - \tilde{b}_{m(i)}^{(2)} = 0, \]

\[a_{m(i)}^{(1)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(2)} - \tilde{b}_{m(i)}^{(2)} = 0, \quad a_{m(i)}^{(2)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(2)} - \tilde{b}_{m(i)}^{(2)} = 0, \]

\[a_{m(i)}^{(1)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(2)} - \tilde{b}_{m(i)}^{(2)} = 0, \quad a_{m(i)}^{(2)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(2)} - \tilde{b}_{m(i)}^{(2)} = 0, \]

\[\tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(1)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(1)} - \tilde{b}_{m(i)}^{(1)} = 0, \quad \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(2)} + \tilde{a}_{m(i)}^{(2)} b_{m(i)}^{(2)} - \tilde{b}_{m(i)}^{(2)} = 0, \]

\[b_k^{(1)} = b_k^{(4)}, \quad b_k^{(2)} = b_k^{(5)}, \quad b_n^{(3)} = b_n^{(6)}, \quad \tilde{b}_{m(i)}^{(2)} = \tilde{b}_{m(i)}^{(2)}, \]

\[H_{kln} = H(c_k^{(1)} \Delta x, c_l^{(2)} \Delta y, c_m^{(3)} \Delta z, 0), \quad H_{kln} = H(c_k^{(1)} \Delta x, c_l^{(2)} \Delta y, c_m^{(3)} \Delta z, c_m^{(1)} \Delta t), \quad H_{kln} = H(c_k^{(1)} \Delta x, c_l^{(2)} \Delta y, c_m^{(3)} \Delta z, c_m^{(1)} \Delta t), \quad H_{kln} = H(c_k^{(1)} \Delta x, c_l^{(2)} \Delta y, c_m^{(3)} \Delta z, c_m^{(1)} \Delta t), \quad H_{kln} = H(c_k^{(1)} \Delta x, c_l^{(2)} \Delta y, c_m^{(3)} \Delta z, 0), \quad H_{kln} = H(c_k^{(1)} \Delta x, c_l^{(2)} \Delta y, c_m^{(3)} \Delta z, c_m^{(1)} \Delta t), \]

e tc., with \(c_k^{(1)} = \sum_{j=1}^{s} a_{k,j}^{(1)}, c_l^{(2)} = \sum_{j=1}^{s} a_{k,j}^{(2)}, c_m^{(3)} = \sum_{j=1}^{s} a_{k,j}^{(3)}, c_m^{(1)} = \sum_{m=1}^{r} a_{m,m}^{(1)}, \) and \(1 \leq j, k, l, n \leq s, \) \(1 \leq i, m \leq r.\) Similar to the proof of Theorem 5.1, this fully-discrete method (5.16) satisfies the following discrete multi-symplectic conservation law

\[
\frac{1}{\Delta t} \sum_{k=1}^{r} \sum_{l=1}^{s} \sum_{n=1}^{s} \left( b_{k}^{(1)} b_{l}^{(5)} b_{n}^{(3)} (d(H)^{1}_{k,n} \wedge d(H)^{0}_{k,n}) + b_{k}^{(2)} b_{l}^{(3)} b_{n}^{(5)} (d(U)^{1}_{k,n} \wedge K_1 d(U)^{0}_{k,n} - d(U)^{0}_{k,n} \wedge K_1 d(U)^{0}_{k,n}) + b_{k}^{(3)} b_{l}^{(1)} b_{n}^{(3)} (d(U)^{0}_{k,n} \wedge K_2 d(U)^{0}_{k,n} - d(U)^{0}_{k,n} \wedge K_2 d(U)^{0}_{k,n}) \right)
\]
\[ + \frac{1}{2\Delta z} \sum_{m=1}^{r} \sum_{k=1}^{s} \sum_{l=1}^{s} f_{m}^{(1)} b_{k}^{(1)} b_{l}^{(2)} (d(U)^{m}_{ktl} \wedge K_{3} d(U)^{m}_{kh0} - d(U)^{m}_{kh0} \wedge K_{3} d(U)^{m}_{kh0}) = 0. \]

6. Conclusions

In this paper, three novel multi-symplectic methods are proposed to numerically solve stochastic Hamiltonian PDEs. We prove that the meshless LRBF collocation midpoint method, the splitting multi-symplectic Runge–Kutta method and the multi-symplectic partitioned Runge–Kutta method preserve the discrete multi-symplectic conservation law almost surely. In general, these proposed multi-symplectic methods are always implicit, and have better numerical stability in the numerical implementation. Unlike the splitting multi-symplectic Runge–Kutta method and the multi-symplectic partitioned Runge–Kutta method, the meshless LRBF collocation midpoint method has high-order accuracy and does not require connection between nodes of the simulation domain, which leads to the liberty in selecting space nodes. Due to the geometric structure preserved property of the numerical method for subsystems, the splitting multi-symplectic Runge–Kutta method also has the superiority in preserving the averaged energy evolution law of some stochastic wave equations, as shown in Section 4. The multi-symplectic partitioned Runge–Kutta method based on symplectic Euler method for some separate stochastic Hamiltonian PDEs, such as stochastic wave equations, is always explicit, which reduces computational cost. We take the stochastic wave equation as an example to perform numerical experiments, which indicates the validity of the proposed methods. In fact, there are still many problems of interest which remain to be solved, such as 1) to systematically construct explicit multi-symplectic methods for nonlinear stochastic Hamiltonian PDEs; 2) to propose numerical methods preserving both the multi-symplecticity and physical properties of stochastic Hamiltonian PDEs; 3) to prove theoretically the strong convergence order of accuracy for the proposed three numerical methods applied to stochastic Hamiltonian PDEs. We attempt to study these problems in our future work.

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