Stretching Riemannian spherical solar dynamos from differential rotation

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Abstract

Stretching solar dynamos from differential rotation in a Riemannian manifold setting is presented. The spherical model follows closely a twisted magnetic flux tube Riemannian geometrical model or flux rope in solar physics, presented previously by Ricca [Solar Physics (1997)]. The spherical model presented here present new and interesting feature concerning its connection with spherical steady solar dynamos. One of this new feature is represented by the fact that the by considering poloidal magnetic field component much weaker than its toroidal counterpart, one obtains a stretch dynamo action where the Riemannian solar spherical line element is proportional to differential rotation. This result is obtained also by using the Vainshtein-Zeldovich stretch, twist and fold (STF) method to generate dynamos. One notes that for high magnetic Reynolds of $Rm = O(10^7)$ the dynamo action is present for a corresponding small stretching factor of $K^2 = 1.6$ where $K^2 = 1$ represents the unstretched dynamo. The constant stretched dynamos considered here are shown to be Riemann-flat, where the Riemann curvature tensor vanishes. Solar cycle dynamos are therefore compatible with the Riemann-flat stretching dynamo model from solar differential rotation presented here.

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I Introduction

Stretching in convective overshooting regions in solar physics has been previously investigated by Fisher at al [1] by stretching magnetic flux tubes [2] from the differential rotation of the Sun. The differential rotation acts to strengthen magnetic fields with a non-azimuthal component by strengthening them in the azimuthal direction. It is a consensus that dynamos responsible for generating solar cycles reside in the form of flux tubes or flux tube dynamos [1]. Have been convinced that dynamos can reside in the form of such Riemannian structures as thin magnetic twisted flux tubes, Ricca [3] developed a Riemannian geometrical model of twisted magnetic flux tubes to explain inflexional and desequilibrium properties of these magnetic tubes. Other models of Riemannian curvature in material flow lines have been also displayed recently [4]. In this paper following the importance of spherical solar models [5] it is shown that it is possible to obtain a analogous Riemannian geometrical [6] model in spherical coordinates where, analytical solutions are obtained of the self-induction steady equations. The first result to be shown is that by using the Vainshtein-Zeldovich [7] stretch-twist-fold [8] generation method it is possible to obtain a ratio between the Riemannian lines elements of the stretched and unstretched spherical models and connect it to the differential rotation. Actually on a recent paper we call the attention to the fact that is possible to obtain a analytical Riemannian geometrical model of conformal stretching dynamo in highly conductively Riemannian manifold flow. It is also shown here that unstretched spherical Riemann curvature tensor vanishes, which mathematicians call a Riemann-flat manifold, which leads to a nondynamo or by the absence of dynamo action. Actually finally we show that our model is compatible with a solution given by Livermore and al [9] of the role of stretching in spherical shells, since our dynamo possesses a high order magnetic Reynolds number $Rm = O(10^7)$, whereas dynamo action considered by them has a Reynolds number of maximum $Rm = O(500)$. Other dynamo mechanism has been recently displayed in conformal stretching [10] in Riemannian manifolds. This paper is organized as follows: In section II the model is presented and the Riemannian curvature tensor is computed for the general case of non-uniform stretching. In section III the Riemann-flat uniform stretching model is addressed along with the self-induction equation for the steady dynamo solar flow also with solar physics implications and discussions are presented in section IV.
II Riemannian non-uniformly stretching

This section presents the Riemann metric of a spherical symmetric space in curvilinear coordinates \((r, \theta, \phi)\). This metric is encoded into the Riemann line element

\[
ds^2 = dr^2 + r^2\left[d\theta^2 + K^2 \sin^2 \theta d\phi^2\right]
\]

where, accordingly to our hypothesis, \(K\) is a constant uniform stretch. As one can see below, the differential rotation \([11]\) is constant along the toroidal direction and slowly decays along the azimuthal poloidal direction. Just by analogy the Riemann metric of the magnetic flux tube \([3]\) is appended

\[
dl^2 = dr^2 + r^2 d\theta_R^2 + K^2(r, s)ds^2
\]

Note that these two Riemann metrics are similar, however while Ricca’s metric helps us to investigate the topology of solar loops, the present metric helps us to investigate differential rotation of sun and other higher density stars. Let us now consider the magnetohydrodynamic (MHD) equations

\[
\nabla \cdot \mathbf{B} = 0
\]

which also has its solenoidal counterpart in the incompressibility of dynamo flow case, as

\[
\nabla \cdot \mathbf{v} = 0
\]

and self induction equation

\[
\partial_t \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}) = \eta \nabla^2 \mathbf{B}
\]

Here we consider the non-radial fields as \(\mathbf{B} = e_\phi B_\phi(r, \theta) + e_\theta B_\theta(r, \theta)\) which is the sum of the toroidal and poloidal magnetic field components respectively. Note that since the solar flow considered is steady, and highly conductive, or null magnetic diffusivity, \((\eta = 0)\), this equation reduces to

\[
\partial_t \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}) = 0
\]

Since the dynamo is also considered to be steady, the dynamo equation reduces to

\[
\nabla \times (\mathbf{v} \times \mathbf{B}) = 0
\]
A so-called "poor" solutions of this simple equation can be obtained by considering solutions such as
\[ \mathbf{v} \times \mathbf{B} = \nabla \phi \]  
where \( \phi \) is a scalar component. A particular case of this is given by the case where \( \mathbf{v} \) and \( \mathbf{B} \) are parallel. These cases are of no interest for us here and instead we shall address the case of expanding the LHS of equation (7) as
\[ (\nabla \cdot \mathbf{v}) \mathbf{B} = (\nabla \cdot \mathbf{B}) \mathbf{v} \]  
and further develop this equation in the background of stretched curved Riemannian three-dimensional spherical space given by Riemann metric (1).

Before we dig into these equations and solve them let us consider here the Riemannian curvature of two general cases, namely when \( K^2 \) stretching factor depends upon either on coordinate \(-\theta\) or on coordinate \(-\phi\). In the second case as is easily computed with the tensor package of general relativistic package simply adapted for the three-dimensional Riemannian space, in any computer software, one obtains that the Riemann tensor vanishes, while in the first case the Riemann curvature reads
\[ R_{\theta\phi\theta\phi} = -\frac{r^2}{2} [\sin^2 \theta \frac{dK^2}{d\theta} + \sin^2 \theta \frac{d^2K^2}{d\theta^2} - \frac{1}{2K^2} \left( \frac{dK^2}{d\theta} \right)^2 \sin^2 \theta] \]  
note that in the uniform stretching case, where \( K^2 = \text{constant} \) this curvature vanishes what justifies to call such a solar dynamo manifold in spherical coordinates, Riemann-flat. Actually from expression (2), and since Riccas stretching factor \( K^2 = K^2(r,s) \) is non-uniform it is easy to show that the Riemann curvature tensor components in the case of twisted magnetic flux tube, with strong folding (curvature), as in dynamo theory is
\[ R_{1313} = R_{rsrs} = -\frac{1}{4K^2} [2K^2 \partial_r A(r,s) - A^2] = -\frac{1}{2} \frac{K^4}{r^2} = -\frac{1}{2} r^2 \kappa^4 \cos^2 \theta \]  
and
\[ R_{2323} = R_{\theta s\theta s} = -\frac{r}{2} A(r,s) = -K^2 \]  
which confirms that the stretch of twisted flux tube dynamos are really non-uniform as in Fisher at al cases.
III The spherical steady solar dynamo

Let us first write the expression for solenoidal vector $B$ in this Riemannian spherical metric stretched background as

$$
\nabla \cdot B = \frac{1}{Kr^2 \sin \theta} \left[ \partial_\theta (B_\theta Kr \sin \theta) + \partial_\phi (B_\phi r) \right] = 0 \quad (13)
$$

which reduces to

$$
\partial_\theta (B_\theta \sin \theta) + \partial_\phi \left( \frac{1}{K} B_\phi \right) = 0 \quad (14)
$$

by considering that $B_\phi$ toroidal component does not depend on coordinate $\phi$ $\partial_\phi B_\phi$ vanishes and the absence of last term reduces equation for the poloidal component to

$$
\partial_\theta (B_\theta \sin \theta) = 0 \quad (15)
$$

which yields the immediate solution

$$
B_\theta = B_\theta(r) \csc \theta \quad (16)
$$

which is a periodic solution. The equations above and below are computed by using the Riemannian line element in the form

$$
ds^2 = (h_i dx^i)^2 \quad (17)
$$

where the metric coefficients are

$$
h_1 = 1 \quad (18)
$$
$$
h_2 = r \quad (19)
$$
$$
h_3 = r K \sin \theta \quad (20)
$$

Note also that the vorticities of differential rotation obeys the following equation

$$
\nabla \cdot \vec{\omega} = 0 \quad (21)
$$

which yields the following constraints on differential rotation

$$
K^{-1} \partial_\theta \omega_\phi + \sin \theta \partial_\theta \omega_\theta = 0 \quad (22)
$$

Assuming that component $\omega_\theta$ does not depend on coordinate $\phi$ one obtains

$$
\omega_\phi = -K \phi \sin \theta \partial_\theta \ln \omega_\theta \quad (23)
$$
From the self-induction equation a long but straightforward computation only tells us that the relation between toroidal and poloidal components is given by

$$\frac{B_\phi}{B_\theta} = \frac{v_\phi}{v_\theta} = \frac{\omega_\phi}{\omega_\theta}r$$ \hspace{1cm} (24)

which implies

$$\frac{B_\phi}{B_\theta} = \frac{\omega_\phi}{\omega_\theta}$$ \hspace{1cm} (25)

From the Vainshtein et al [13] fractal Riemannian geometry and stretching

$$\frac{B_\phi^2}{B_\theta^2} = \frac{ds^2}{ds_0^2}$$ \hspace{1cm} (26)

a similar expression to this one was obtained by Fisher et al in the case of flux tubes which is given by

$$\frac{B_\phi^2}{B_\theta^2} = \frac{l^2}{l_0^2}$$ \hspace{1cm} (27)

where $l$ is the length of magnetic flux tube. From the Riemann metrics of stretched and unstretched solar models one yields

$$\frac{ds^2}{ds_0^2} = 1 + (K^2 - 1)r^2\sin^2\theta \frac{\omega_\phi^2}{v_0^2}$$ \hspace{1cm} (28)

Taking into account the expression for the differential rotation as $\omega = \frac{d\phi}{dt}$ and $v_0 = \frac{ds_0}{dt}$ which yields

$$\frac{ds^2}{ds_0^2} = 1 + (K^2 - 1)r^2\sin^2\theta \frac{\omega_\phi^2}{v_0^2}$$ \hspace{1cm} (29)

Vainshtein et al [13] also considered the relation between the ratio of squared magnetic fields and the magnetic Reynolds number as

$$\frac{B_\phi^2}{B_\theta^2} = (Rm)^n$$ \hspace{1cm} (30)

Comparing this equation with expression (27) yields

$$\frac{B_\phi^2}{B_\theta^2} = 1 + (K^2 - 1)r^2\sin^2\theta \frac{\omega_\phi^2}{v_0^2}$$ \hspace{1cm} (31)
From expressions (29) and (30) one obtains $n = 0$ corresponds to $K^2 = 1$ or the unstretched Riemannian metric. This corresponds to the nondynamo action since the toroidal does not amplify w.r.t poloidal component. For $n = 1$ the relation (30) can be computed at the surface of the Sun, where $R_{\text{sun}} \approx 10^{10} \text{cm}$ and the Reynolds number can be considered as high as $10^{10}$. In this case the relation between the two components of the magnetic field is given by

$$\frac{B_\phi^2}{B_\theta^2} = 1 + (K^2 - 1) R_{\text{sun}} \frac{\omega_\phi^2}{\omega_0^2}$$

(32)

where one has compute it at the $\theta = \frac{\pi}{2}$, and thus

$$\frac{B_\phi^2}{B_\theta^2} = (K^2 - 1) 10^{10} \omega_\phi^2 \omega_0^2$$

(33)

in the strong stretch limit where $K^2 \gg 1$. Taking into account that in the Sun the differential rotation is about 26 per cent [13] of solar rotation one obtains

$$\frac{\omega_\phi^2}{\omega_0^2} \approx 10^{-2}$$

(34)

$$\frac{B_\phi}{B_\theta} = K \times 10^4$$

(35)

This shows that, since $K^2 > 1$ to obtain a stretch that induces an expressive amplification in the toroidal component with the dynamo action one must have a bound value of the for this expression of at least

$$\frac{B_\phi}{B_\theta} \geq 10^4$$

(36)

This expression is well within observational data since for example a poloidal field of $0.5G$ could be amplified till $3 \times 10^3 G$ in solar corona, this data allows to obtain a magnetic Reynolds from expression (26) as high as $Rm \approx 6 \times 10^7$ which is much higher than the usual Reynolds number used in numerical simulations [14]. Note that from this value $\frac{B_\phi}{B_\theta} \approx 6 \times 10^3$ and formula (28) one obtains a very small value for stretching factor of $K^2 \approx 1.6$ as a very small deviation of the stretching which is however able to yield a strong steady dynamo action. This seems to be distinct from the result obtained by Livermore, Hughes and Tobias where a best stretching seems not produce a
efficient dynamo action. But as shall be discussed next, this is only due to a high Reynolds number considered here. Comparison between equation (31) and Ricca’s equation

$$\frac{B^2_s}{B^2_\theta} \geq \frac{K^2r}{T_w}$$

shows that they are very similar knowing that Tw represents the twist of the magnetic flux tube. Of course a nonuniform stretch is also present here. As was mention by Livermore et al, one of the reasons that it is useful to use spherical coordinates is that, most of the astrophysical and planetary objects are of spherical morphology. Note that also in the case of flux tube non-uniform stretching the Riemannian stretch of the line elements of twisted flux tube and the thin one $K^2 := 1$

$$ds^2_0 = dr^2 + r^2d\theta^2 + ds^2$$

is given by

$$\frac{dl^2}{ds^2_0} = 1 + (K^2 - 1)\frac{ds^2}{ds^2_0}$$

This expression shows that the stretch of the twisted flux tube depends on the stretch of magnetic flux tube axis length ds. Besides by writting this expression as

$$\frac{dl^2}{ds^2_0} = 1 + (K^2)\frac{ds^2}{ds^2_0} \approx 1 + (r^2\kappa^2\cos\theta)\frac{ds^2}{ds^2_0}$$

which shows that the stretch also depends upon the Frenet curvature and in the helical solar tube, where the Frenet curvature and torsion coincides, this stretch depends also on the torsion of the flux tube. Actually the non-uniform carachter of the flux tube was actually discussed by Fisher et al who argued that: "Most likely stretching mechanism is radial differential rotation, in the overshooting layer, implying that flux tube does not lie at a single depth and its properties are therefore uniform". This is exactly what happens in the Riemannian model of solar dynamo presented here.
IV Conclusions

A particular solution of self-induction and MHD equations, is found representing a steady dynamo on flat uniformly stretched Riemannian manifold following earlier Riemannian models of dynamos and magnetic flux tubes in other systems of coordinates. It is shown that the Riemann space is flat when the solar dynamo is unstretched, in close analogy with the elastic stars in Einstein’s general relativity. These ideas can help us to build pseudo-Riemannian stretched relativistic models in analogy with the 3D counterpart investigated here. The absence of the radial component $B_r$ stems from the fact that in Ricca’s work [3] it is also absent and since the present paper assumes a analogous model for the solar dynamo in spherical coordinates this assumption seems to be justifiable. Besides the assumption of vanishing radial magnetic field component simplifies a great deal of computation avoiding that one has to use numerical computation and that a analytical solution can be obtained here. One important feature to test the model discussed here from the solar physics point of view is to include ohmic losses and diffusion as done by Livermore et al [9]. Though they found that stretching in dynamo flows is not always a factor of enhancement and may even turn the dynamo inefficient, they basically obtained their result for low Reynolds numbers between $20 < Rm < O(10^2)$ while here one deals with magnetic Reynolds numbers of the order of $Rm = O(10^7)$. High Rm numbers are actually related to the fast dynamos widely investigated in the literature. In a future publication we shall address this and other issues connected with spherical dynamos Riemannian manifolds in detail. The solar cycle dynamos [15] which stretches the poloidal magnetic fields to toroidal magnetic ones by differential rotation are therefore compatible with the spherical Riemann-flat model discussed here.

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