Solutions to Difference Equations Have Few Defects

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Abstract
We establish a strong form of Nevanlinna’s Second Main Theorem for solutions to difference equations

\[ f(z + 1) = R(z, f(z)), \]

with the coefficients of \( R \) growing slowly relative to \( f \), and \( \text{deg}_w(R(z, w)) \geq 2 \).

Keywords Difference equation \cdot Meromorphic function \cdot Order of growth

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Picard famously showed that a non-constant meromorphic function \( f : \mathbb{C} \to \hat{\mathbb{C}} \) omits at most two values, and Nevanlinna generalized this with his Second Main Theorem. Specifically, at each \( a \in \hat{\mathbb{C}} \) the Nevanlinna defect satisfies \( 0 \leq \delta_f(a) \leq 1 \), and has \( \delta_f(a) = 1 \) for any omitted value \( a \), and it follows from Nevanlinna’s result that the defects of \( f \) at all points of \( \hat{\mathbb{C}} \) sum to at most 2.

At least since work of Yanagihara [9, 10], there has been interest in applications of Nevanlinna Theory to difference equations of the form

\[ f(z + 1) = R(z, f(z)), \]

where \( R(z, w) \) is a rational function in \( w \), with coefficients meromorphic in \( z \). A relation of the form (1) makes it harder for \( f \) to omit values, and Yanagihara [9]...
showed that such solutions omit no values, except in some special circumstances, if
$R(z, w) = R(w) \in \mathbb{C}(w)$ is constant in $z$.

Our main result is a strong form of Nevanlinna’s Second Main Theorem for solutions to (1), and generalizes Yanagihara’s result from the case of constant coefficients, and from omitted values to defects (although see also [5], which gives a similar generalization when the coefficients are constant). We will say that $a \in \mathbb{C}$ is shift-exceptional for $R(z, w)$ if $w = a$ is a totally ramified fixed point of $w \mapsto R(z, R(z − 1, w))$, and that $a \in \mathbb{C}$ is ordinary otherwise (note that most $R$ have no exceptional points). See Sect. 1 for the definitions of the Nevanlinna characteristic function $T_f(r)$, the proximity $m_f(r; a)$, and the defect $\delta_f(a)$.

**Theorem 1** Let $f$ be a solution to (1), suppose that $\deg_w(R) \geq 2$, that the coefficients of $R$ are slow-growing relative to $f$, and that $a \in \mathbb{C}$ is ordinary for $R$. Then $\delta_f(a) = 0$ and in fact

$$m_f(r; a) = o(T_f(r))$$

as $r \to \infty$ outside of some set of finite measure.

Note that the Second Main Theorem gives

$$\sum_{i=1}^{k} m_f(r; a_i) \leq (2 + o(1))T_f(r)$$

as $r \to \infty$ outside of a set of finite length, for distinct $a_1, \ldots, a_k \in \mathbb{C}$, and so Theorem 1 can also be seen as a strengthening of this in the special case of solutions to (1). If $R$ has two exceptional points (the most possible) then Theorem 1 reduces to the usual Second Main Theorem if we include the exceptional points in our sum, but in this case $R$ is (up to change of variables over the field of coefficients) either $R(z, w) = w^d$ for $d = \pm \deg_w(R)$. The case of one exceptional point coincides with $R(z, w)$ being a polynomial in $w$ (again up to change of coordinates), and so in the general case we may apply Theorem 1 to any $a \in \mathbb{C}$. Also, just as the Second Main Theorem can be extended to the case in which the $a_i$ are functions of slow growth (relative to $f$), we note that Theorem 1 also holds in the context of such moving targets, as long we assume that $a(z)$ and $a(z + 1)$ both grow slowly.

The proof is motivated by a result of Silverman [6] on diophantine approximation in arithmetic dynamics. As noted above, this idea was already used by Ru and Yi [5] to produce similar results, under different assumptions.

### 1 Notation and Background

We write $M$ for the field of meromorphic functions on $\mathbb{C}$, and $\overline{M}$ for the algebraic closure, the field of algebroid functions on $\mathbb{C}$. Given $\beta \in \mathbb{C}$, we write $\text{ord}_\beta(f)$ for the order of vanishing of $f(z)$ at $z = \beta$, and set

$$\text{ord}_\beta^+(f) = \max\{\text{ord}_\beta(f), 0\}.$$
We note the convenient property that
\[ \text{ord}_\beta(f + g) \geq \min\{\text{ord}_\beta(f), \text{ord}_\beta(g)\}, \]
with equality except perhaps when \( \text{ord}_\beta(f) = \text{ord}_\beta(g) \). We then set, as usual,
\[ n_f(r) = \sum_{|z| \leq r} \text{ord}_z^+(\frac{1}{f}), \]
and, with \( r > 0 \),
\[ N_f(r) = \int_0^r \left( n_f(t) - n_f(0) \right) \frac{dt}{t} + n_f(0) \log r \]
\[ = \sum_{0 < |z| \leq r} \text{ord}_z^+(\frac{1}{f}) \log \frac{r}{|z|} + \text{ord}_0^+(\frac{1}{f}) \log r, \]
for the Nevanlinna counting function. The proximity function is
\[ m_f(r) = \int_0^{2\pi} \log^+|f(re^{i\theta})| \frac{d\theta}{2\pi}, \]
and the characteristic function is
\[ T_f(r) = N_f(r) + m_f(r). \]
We will also set \( N_f(r; a) = N_{1/(f-a)}(r) \) for any \( a \in M \setminus \{f\} \), and similarly for \( m \) and \( T \), while \( N_f(r; \infty) = N_f(r) \).

The First Main Theorem of Nevanlinna [4] gives
\[ T_f(r; a) = T_f(r) + O_a(1) \]
for any \( a \in \mathbb{C} \), or more generally \( T_f(r; a) = T_f(r) + O(T_a(r)) \) for any \( a \in M \setminus \{f\} \).

As in Steinmetz [7], we write \( K_f \) for the field of functions of type \( S(f, r) \), that is, the field of \( g \in M \) such that \( T_g(r) = o(T_f(r)) \), except possibly on a set of finite Lebesgue measure.

We define also the Nevanlinna defect of \( f \) at \( a \in K_f \) by
\[ \delta_f(a) = \liminf_{r \to \infty} \frac{m_f(r; a)}{T_f(r)}, \]
which satisfies \( 0 \leq \delta_f(a) \leq 1 \) by definition. Note that if \( f(z) = a \) has no solutions, then \( N_f(r; a) \equiv 0 \), and so \( m_f(r; a) = T_f(r) + S(f, r) \), whence \( \delta_f(a) = 1 \).
\section*{2 Some Technical Lemmas}

We maintain the notation and conventions of the previous section. Our first lemma relates the proximity function of a rational function of \( f \) to the denominator of the rational function, and the proximity of \( f \) itself.

Our first lemma is a basic fact from commutative algebra.

\begin{lemma}
Let \( F \) be a field, and let \( P, Q \in F[T] \) be polynomials of degree \( d \) and \( e \), with no common factor. Then there exist polynomials \( A, B \in F[T] \) of degree (at most) \( e - 1 \) and \( d - 1 \) such that
\[ A(T)P(T) + B(T)Q(T) = 1. \]
Furthermore, if \( \text{Res}(P, Q) \) is the resultant of \( P \) and \( Q \), then \( \text{Res}(P, Q) \), the coefficients of \( \text{Res}(P, Q)A(T) \), and the coefficients of \( \text{Res}(P, Q)B(T) \) are all polynomials in the coefficients of \( P \) and \( Q \).
\end{lemma}

\begin{proof}
With \( a_{e-1}, \ldots, a_0, b_{d-1}, \ldots, b_0 \) as indeterminates, consider
\[ S(T) = (a_{e-1}T^{e-1} + \cdots + a_0)P(T) + (b_{d-1}T^{d-1} + \cdots + b_0)Q(T), \]
which has degree \( d + e - 1 \). Setting \( S(T) = 0 \) as polynomials, and equating coefficients, yields a system of \( d + e \) linear equations in \( d + e \) unknowns; \( \text{Res}(P, Q) \) is the determinant of the coefficient matrix.

If \( \text{Res}(P, Q) = 0 \), then there is a non-trivial solution to \( S(T) = 0 \), giving an equality
\[ \frac{P(T)}{Q(T)} = -\frac{b_{d-1}T^{d-1} + \cdots + b_0}{a_{e-1}T^{e-1} + \cdots + a_0} \]
(or the reciprocals) which contradicts the degrees of \( P \) and \( Q \) (unless they have a common factor).

Given that \( \text{Res}(P, Q) \neq 0 \), we can then solve \( S(T) = 1 \) as a system of linear equations in the \( a_i \) and \( b_j \), using Cramer's Rule. In particular, \( \text{Res}(P, Q) \), the coefficients of \( \text{Res}(P, Q)A \), and the coefficients of \( \text{Res}(P, Q)B \) are all determinants of matrices (of side length \( d + e \)) whose entries are coefficients of \( P \) and \( Q \). It follows that these quantities are polynomials in the coefficients of \( P \) and \( Q \).
\end{proof}

The following is due to Valiron \cite{Valiron} in a special case, and Mohon'ko \cite{Mohon'ko} more generally, and is straightforward to prove from Lemma 1.

\begin{lemma} \textit{(Valiron \cite{Valiron}, Mohon'ko \cite{Mohon'ko})} \label{lemma2}
Let \( R(z, w) \in K_f(w) \), and let \( f \in M \). Then
\[ T_{R(z, f(z))}(r) = \deg_w(R)T_f(r) + S(f, r). \]
\end{lemma}

\begin{remark}
Yanagihara \cite{Yanagihara, Yanagihara2} showed that if \( R(z, w) \) is rational in both variables, and \( \deg_w(R) \geq 2 \), then any finite-order solution to \( f(z + 1) = R(z, f(z)) \) is rational.
\end{remark}

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This raises the question of describing all rational solutions, and we refer the reader here to a recent survey [1] and the author’s related note [2]. The referee points out to the author that

For convenience in the next lemma, we suppress the notational dependence on \( z \).

**Lemma 3** Let \( P(w), Q(w) \in K_f[w] \) be polynomials, with \( \deg_w(P) \geq \deg_w(Q) \). Then for \( f \) with \( Q(f) \neq 0 \),

\[
m_{P(f)/Q(f)}(r) \leq (\deg_w(P) - \deg_w(Q))m_f(r) + m_{Q(f)}(r;0) + S(f,r).
\]

**Proof** Let \( \beta \in \mathbb{C} \), let \( S(w) = c_m w^m + \cdots + c_0 \in M[w] \), and set

\[
C_\beta(S) = \max_{0 \leq i < m} \left\{ \frac{\ord_\beta \left( \frac{c_m}{c_i} \right)}{m-i}, \frac{\ord_\beta \left( c_m^{-1} \right)}{m} \right\} \geq 0. \tag{2}
\]

Then for \( f \) with \( \ord_\beta(1/f) > C_\beta(S) \), we have

\[
\ord_\beta(c_m f^m) < \ord_\beta(c_i f^i)
\]

for all \( 0 \leq i < m \), and so

\[
\ord_\beta \left( \frac{1}{S(f)} \right) = m \ord_\beta \left( \frac{1}{f} \right) - \ord_\beta(c_m) > 0.
\]

Note, on the other hand, that for any \( C \geq 0 \), the hypothesis \( \ord_\beta(1/f) \leq C \) always implies

\[
\ord_\beta \left( \frac{1}{S(f)} \right) \leq mC + \kappa_{\beta,S},
\]

with

\[
\kappa_{\beta,S} = \sum_{i=0}^m \ord_\beta \left( \frac{1}{c_i} \right) \geq 0.
\]

Suppose first that

\[
\ord_\beta \left( \frac{1}{f} \right) > C_\beta(P) + C_\beta(Q).
\]

Writing \( P(w) = a_d w^d + \cdots + a_0 \) and \( Q(w) = b_e w^e + \cdots + b_0 \), we have

\[
\ord_\beta \left( \frac{Q(f)}{P(f)} \right) = \ord_\beta(Q(f)) - \ord_\beta(P(f))
\]
\[ = - \deg(Q) \ord_\beta \left( \frac{1}{f} \right) + \ord_\beta (b_e) + \deg(P) \ord_\beta \left( \frac{1}{f} \right) - \ord_\beta (a_d) \]
\[ \geq (\deg(P) - \deg(Q)) \ord_\beta \left( \frac{1}{f} \right) - \ord_\beta (b_e^{-1}) - \ord_\beta (a_d). \] (3)

We also have \[ \ord_\beta (Q(f)) \leq \ord_\beta (b_e) \] in this case, and so
\[ \ord_\beta \left( \frac{Q(f)}{P(f)} \right) \geq (\deg(P) - \deg(Q)) \ord_\beta \left( \frac{1}{f} \right) + \ord_\beta (Q(f)) \]
\[ - (\ord_\beta (b_e) + \ord_\beta (a_d) + \ord_\beta (b_e^{-1})). \] (4)

Suppose, on the other hand, that we have \[ \ord_\beta (1/f) \leq C_\beta (P) + C_\beta (Q). \]
Now, by Lemma 1 we can find \( A(w), B(w) \in K[w] \) of degrees \( e - 1 \) and \( d - 1 \), respectively, such that
\[ A(f)P(f) + B(f)Q(f) = 1. \]

Note that if
\[ \ord_\beta (Q(f)) > (d - 1)(C_\beta (P) + C_\beta (Q)) + \kappa_{\beta,B} \geq \ord_\beta \left( \frac{1}{B(f)} \right), \]
then \[ \ord_\beta (A(f)P(f)) = 0, \] and so
\[ \ord_\beta (P(f)) = \ord_\beta \left( \frac{1}{A(f)} \right) \leq (e - 1)(C_\beta (P) + C_\beta (Q)) + \kappa_{\beta,A}. \]

This then gives
\[ \ord_\beta \left( \frac{Q(f)}{P(f)} \right) \geq \ord_\beta (Q(f)) - (e - 1)(C_\beta (P) + C_\beta (Q)) - \kappa_{\beta,A} \]
and hence
\[ \ord_\beta \left( \frac{Q(f)}{P(f)} \right) \geq (\deg(P) - \deg(Q)) \ord_\beta \left( \frac{1}{f} \right) + \ord_\beta (Q(f)) \]
\[ - \kappa_{\beta,A} - (d - 1)(C_\beta (P) + C_\beta (Q)). \] (5)

Finally, if we have \[ \ord_\beta (Q(f)) \leq (d - 1)(C_\beta (P) + C_\beta (Q)) + \kappa_{\beta,B} \] and still \[ \ord_\beta (1/f) \leq C_\beta (P) + C_\beta (Q), \] we immediately have
\[ \ord_\beta \left( \frac{Q(f)}{P(f)} \right) \geq 0 \geq (\deg(P) - \deg(Q)) \ord_\beta \left( \frac{1}{f} \right) + \ord_\beta (Q(f)) \]
\[ - (2d - e - 1)(C_\beta (P) + C_\beta (Q)) - \kappa_{\beta,B}. \] (6)
Combining (4), (5), and (6) we have in any case

\[
\ord_\beta^+(\frac{Q(f)}{P(f)}) \geq 0 \geq (\deg(P) - \deg(Q)) \ord_\beta^+(\frac{1}{f}) + \ord_\beta^+(Q(f)) - E_\beta
\]  

for

\[
E_\beta = (2d - e - 1)(C_\beta(P) + C_\beta(Q)) + \kappa_{\beta,A} + (\ord_\beta^+(b_e) + \ord_\beta^+(a_d) + \ord_\beta^+(b_e^{-1}))
\]  

by the non-negativity of the various terms in the error.

Now, referring back to Lemma 1, not that every coefficient of \(\Res(P, Q)A\) is a polynomial in the coefficients of \(P\) and \(Q\) of at most degree \((2d - 1)!\), and so

\[
\kappa_{\beta,A} \leq (2d - 1)!(\sum_{i=0}^{d} \ord_\beta^+(\frac{1}{a_i}) + \sum_{i=0}^{e} \ord_\beta^+(\frac{1}{b_i})) + \ord_\beta^+(\Res(P, Q)),
\]

and similarly for \(\kappa_{\beta,B}\), and hence (from this and the definitions of \(C_\beta(P)\) and \(C_\beta(Q)\))

\[
\sum_{|\beta| \leq r} E_\beta \log \frac{r}{|\beta|} \leq 2N_{\Res(P, Q)}(r; 0) + O\left(\sum m_{a_i}(r) + \sum m_{b_i}(r) + m_{a_d}(r; 0) + m_{b_e}(r; 0)\right)
\]

\[
\leq 2T_{\Res(P, Q)}(r) + S(f \cdot r)
\]

\[
= S(f \cdot r),
\]

since \(\Res(P, Q)\) is itself a polynomial in the coefficients of \(P\) and \(Q\). From this and (7) we have

\[
N_{Q(f)/P(f)}(r) \geq (\deg(P) - \deg(Q))N_{f}(r) + N_{Q(r; 0)} + S(f \cdot r),
\]

which in turn gives

\[
m_{Q(f)/P(f)}(r)
\]

\[
= T_{Q(f)/P(f)}(r) - N_{Q(f)/P(f)}(r)
\]

\[
\leq \deg\left(\frac{P}{Q}\right)T_{f}(r) - (\deg(P) - \deg(Q))N_{f}(r) - N_{Q(f)(r; 0)} + S(f \cdot r)
\]

\[
= (\deg(P) - \deg(Q))T_{f}(r) - \left(\deg\left(\frac{P}{Q}\right) - \deg(Q)\right)N_{f}(r) + T_{Q}(r; 0)
\]

\[
- N_{Q(f)(r; 0)} + S(f \cdot r)
\]

\[
= (\deg(P) - \deg(Q))m_{f}(r) + m_{Q(f)}(r; 0) + S(f \cdot r).
\]
The following lemma is closely related to a lemma of Silverman [6] (see also results of Ru and Yi [5]), but it is sufficiently different that we present a self-contained proof. As it becomes slightly more convenient in the next lemma, we will set \( R_z(w) = R(z, w) \), thought of as a function of \( w \) alone, so that \( R(z, R(z - 1, w)) \) is written \( R_z \circ R_{z - 1}(w) \).

**Lemma 4** Suppose that \( R_z \circ R_{z - 1}(w) \) is not a polynomial in \( w \), and write

\[
R_z \circ \cdots \circ R_{z - k + 1}(w) = \frac{P_k(w)}{Q_k(w)}
\]

in lowest terms (where we suppress the dependence on \( z \)). Further write

\[
Q_k(w) = \prod_{i=1}^{m_k} H_{i,k}(w)^{e_{i,k}}
\]

with the \( H_{i,k} \) irreducible and distinct, and set \( e_{0,k} = \deg_w(R)^k - \deg_w(Q_k) \). Then we have

\[
e_k := \max\{e_{0,k}, e_{1,k}, \ldots, e_{m_k,k}\} = o(\deg_w(R)^k)
\]

as \( k \to \infty \).

**Proof** Let \( \varepsilon > 0 \), and set \( d = \deg_w(R) \). On the assumption that \( R_{z+1} \circ R_z(w) \) is not a polynomial, we will show that \( e_k \leq \varepsilon d^k \) once \( k \) is larger than some explicit value depending on \( \varepsilon \) and \( R \).

Over the algebraic closure, let \( g_i \) be chosen so that \( g_0 = \infty \), and \( R_{z-i}(g_{i+1}) = g_i \). Then the ramification index of \( R_z \circ \cdots \circ R_{z-k+1} \) at \( g_k \) is one of the \( e_{j,k} \), and all \( e_{j,k} \) are obtained with some such choice. Thus we are interested in bounding the ramification index

\[
e_{R_z \circ \cdots \circ R_{z-k+1}}(g_k) = e_{R_z}(g_1) \cdots e_{R_{z+k-1}}(g_k),
\]

independent of the choice of \( g_k \).

Let \( \sigma \) be the field automorphism \( f^\sigma(z) := f(z + 1) \) of \( M \), which extends to the algebraic closure \( \overline{M} \) and to the projective line over \( \overline{M} \) (by \( \infty^\sigma = \infty \)). Note that \( e_{R_z^\sigma}(h) = e_{R_z}(h^{-\sigma}) \) and so, in particular, we are interested in computing

\[
\prod_{i=1}^{k} e_{R_z^\sigma}(g_{i}^{\sigma_{i-1}}),
\]

and it will suffice to show that this is \( o(d^k) \) as \( k \to \infty \).

Suppose first that there is no \( j > 0 \) such that \( g_j = \infty \). We claim that then the values \( g_j^{\sigma_{j-1}} \) are distinct. If not, then there exist \( j \) and \( m \) with \( j \geq m > 0 \) so that
\[ g^j_{j-1} = g^j_{j-m}, \text{ or } g_{j-m} = g^m_j. \]  It then follows that
\[
 g_{j-2m} = R_z^{-(j-2m)} \circ \cdots \circ R_z^{-(j-m-1)}(g_{j-m})
 = R_z^{-(j-2m)} \circ \cdots \circ R_z^{-(j-m-1)}(g^m_j)
 = (R_z^{-(j-m)} \circ \cdots \circ R_z^{-(j-1)}(g_j))^{\sigma^m}
 = g^m_{j-m}
 = g^m_{j-2m},
\]

and hence
\[
 g_{j-tm} = g^m_j
\]
for all \( 0 \leq t \leq j/m \). It also then follows that
\[
 g_{j-tm-s} = R_z^{-(j-tm-s)} \circ \cdots \circ R_z^{-(j-tm-1)}(g_{j-tm}) = g^m_{j-s},
\]
for any \( s \geq 0 \). In particular, writing \( j = tm + s \) with \( 0 \leq s < m \) and \( t \geq 0 \), we have
\[
 \infty = g_0 = g^m_{j-s},
\]
and hence \( g_{j-s} = \infty \), a contradiction because \( j - s > j - m \geq 0 \).

So in this case the \( g^j_{j-1} \) are distinct. By the Riemann–Hurwitz formula, and the fact that the arithmetic mean bounds the geometric mean, we have
\[
 e_{R_z \circ \cdots \circ R_{z-k+1}}(g_k) = \prod_{i=1}^{k} e_{R_z}(g^i_{j-1})
 \leq \left( \frac{\sum_{i=1}^{k} e_{R_z}(g^i_{j-1})}{k} \right)^k
 \leq \left( 1 + \frac{\sum_{i=1}^{k} (e_{R_z}(g^i_{j-1}) - 1)}{k} \right)^k
 \leq \left( 1 + \frac{2d - 2}{k} \right)^k
 \leq e^{2d-2}
 \leq \varepsilon d^k
\]
as soon as \( k \geq (\log \varepsilon^{-1} + 2d - 2)/\log d \).

Now suppose that there is some \( j > 0 \) such that \( g_j = \infty \), let \( m \) be the least such value of \( j \). Note that we then have \( R_z \circ \cdots \circ R_{z-(m-1)}(\infty) = \infty \), and so \( m \) depends
only on $R$. Write

$$E = e_{R_z \circ \cdots \circ R_z \circ (m-1)}(\infty)^{1/m},$$

and assume for now that $E < d$, so in fact $E \leq (d^m - 1)^{1/m}$. Let $t \leq k/m$ be the largest value with $g_{tm} = \infty$, and $s = k - tm$. If $s = 0$, then

$$e_{R_z \circ \cdots \circ R_z \circ k+1}(g_k) = \prod_{u=0}^{t-1} e_{R_z \circ \cdots \circ R_z \circ (u+1) \circ (m+1)}(\infty)$$

$$= \prod_{u=0}^{t-1} e_{R_z \circ \cdots \circ R_z \circ u+1}(\infty)^{\sigma_u m}$$

$$= E^k$$

$$\leq (d^m - 1)^{k/m}$$

$$\leq \varepsilon d^k$$

as soon as

$$k \geq m \frac{\log \varepsilon}{\log(1 - \frac{1}{d^m})}.$$  

Otherwise, by the argument above, the values $g_j^{\sigma_j - 1}$ are distinct for $tm \leq j \leq tm + s$, and so

$$e_{R_z \circ \cdots \circ R_z \circ k+1}(g_k) \leq E^{tm} \left(1 + \frac{2d - 2}{s}\right)^s \leq E^{tm} e^{2d-2}.$$  

Note that $tm \leq k$, and so

$$\log \left(E^{tm} \left(1 + \frac{2d - 2}{s}\right)^s\right) \leq \frac{k}{m} \log(d^m - 1) + (2d - 2) \leq \log(\varepsilon d^k)$$

as soon as

$$k \geq \frac{\log \varepsilon^{-1} + (2d - 2)}{\log d - \frac{1}{m} \log(d^m - 1)}.$$  

We are left with the case that $E = d$, or in other words

$$d^m = e_{R_z \circ \cdots \circ R_z \circ (m-1)}(\infty) = e_{R_z}(g_1) \cdots e_{R_z}(g_{m-1}).$$

Since $1 \leq e_{R_z}(w) \leq d$ for all $w \in \overline{M}$, and this value is attained at most twice (by the Riemann–Hurwitz formula), we have $m = 1$ or $m = 2$. If $m = 1$, then $R_z$ is a polynomial, and if $m = 2$ then $R_z \circ R_z$ is.
In any case, as long as \( R_z \circ R_{z-1}(w) \) is not a polynomial in \( w \), we have shown that 
\[
\max \{ e_0, \ldots, e_m \} \leq \varepsilon d^k, 
\]
and since \( \varepsilon > 0 \) was arbitrary, we are done. \( \square \)

Finally, we give a standard characterization of the case in which \( R_z \circ R_{z-1}(w) \) is a polynomial in \( w \).

**Lemma 5** Suppose that \( R_z \circ R_{z-1}(w) \) is a polynomial in \( w \). Then either \( R_z(w) \) is a polynomial in \( w \), or

\[
R_z(w) = a_z(w - b_z)^{-d} + b_{z+1},
\]

for some \( a, b \in M \).

**Proof** Suppose that \( R_z(w) \) is not a polynomial in \( w \). In \( \overline{M} \), we may then choose some \( a_z \neq \infty \) with \( R_z(a_z) = \infty \). On the other hand, since \( R_z \circ R_{z-1}(w) \) is a polynomial, any preimage of \( \infty \) by \( R_z \) must be equal to \( R_{z-1}(\infty) \), and so \( R_{z-1}(\infty) = a_z \), and in particular \( a_z \) is the unique solution to \( R_z(w) = \infty \). It follows that \( a_z \) is in the field generated by the coefficients of \( R_z(w) \).

Set \( \mu_z(w) = w + a_{z+1} \), and \( \mu_z^{-1}(w) = w - a_{z+1} \), and write \( S_z = \mu_z^{-1} \circ R_z \circ \mu_z^{-1} \).

We have

\[
S_z(0) = R_z(0 + a_z) - a_{z+1} = \infty,
\]

while

\[
S_z(\infty) = R_z(\infty) - a_{z+1} = 0,
\]

and so the rational function \( S_z(w) \) of degree \( \deg_w(R) \) must have the form \( g_z w^{-\deg_w(R)} \) for some \( g_z \) in the field generated by the coefficients of \( R_z(w) \), and hence

\[
R_z(w) = g_z(w - a_z)^{-\deg_w(R)} + a_{z+1}.
\]

\( \square \)

**Remark 2** Let \( F \) be a field and let \( \sigma \) be an automorphism of \( F \); that is, let \( (F, \sigma) \) be a difference field. For \( R(w) \in F(w) \), one might consider the dynamical system on the projective line \( \mathbb{P}^1_F \) given by \( z \mapsto R(z)^\sigma \). As an example with \( F = M \) and \( f(z)^\sigma = f(z-1) \), fixed points of this dynamical system are precisely solutions to the difference equation (1). The proof of Lemma 5 in this context gives a classification of rational functions admitting *exceptional points* (points with finite grand orbit under the difference-dynamical system), which reduces to the well-known statement in the theory of iteration of rational functions when \( \sigma \) is trivial. Similarly, the proof of Lemma 4 generalizes to this context, too, and this generalization specializes to the lemma of Silverman [6] by taking \( \sigma \) again to be the trivial automorphism.
3 The Proof of the Main Theorem

Finally, we cite a theorem of Steinmetz [7], to set the stage for the proof of Theorem 1.

**Theorem 2** (Steinmetz [7, Satz 2]) Let $H(w) \in K_f[w]$ have distinct roots. Then for any $\delta > 0$ there exists a set $E_\delta \subseteq \mathbb{R}^+$ of finite measure such that, for $r \notin E_\delta$ we have

$$m_f(r; \infty) + m_H(f)(r; 0) = m_f(r) + m_{1/H}(f)(r) \leq (2 + \delta)T_f(r).$$

With Steinmetz’s version of the Second Main Theorem in hand, we may prove the main result.

**Proof of Theorem 1** First, we prove the theorem in the case $a = \infty$, under the hypothesis that $R_z(w) \in K_\varphi(w)$ for $\varphi(r) = o(T_f(r))$ as $r \to \infty$.

Let $\varepsilon > 0$, and let $k \geq 0$ be an integer to be chosen later. Write

$$R_z \circ \cdots \circ R_{z-k+1}(w) = \frac{P_k(w)}{Q_k(w)}$$

in lowest terms as above, and again write $Q_k(w) = \prod H_{i,k}^{e_{i,k}}(w)$, where the $H_{i,k}$ are irreducible. Note that since $\varphi$ is non-decreasing, $K_{\varphi(r-1)} \subset K_\varphi$, and in particular $P_k(w), Q_k(w), H_{i,k}(w) \in K_\varphi[w]$. Setting

$$\varepsilon_{i,j} = \begin{cases} 1 & i \leq j \\ 0 & \text{otherwise} \end{cases},$$

and

$$J_{i,k}(w) = \prod_{i=1}^{m_k} H_{i,k}^{e_{i,k}}(w),$$

we see that $J_{i,k}$ has no repeated factors, and $Q_k(w) = \prod_{i=1}^{\max\{e_{1,k}, \ldots, e_{m_k,k}\}} J_{i,k}(w)$. For simplicity, we set $J_{i,k}(w) \equiv 1$ if $i > \max\{e_{1,k}, \ldots, e_{m_k,k}\}$, and we will also set $e_{0,k} = \deg w(R_k^k - \deg(Q_k)$, and write again $e_k = \max\{e_{0,k}, \ldots, e_{m_k,k}\}$. By Lemma 4, we may choose $k$ sufficiently large so that

$$e_k < \frac{\varepsilon \deg w(R)^k}{6}. \quad (9)$$

Note that $m_{st}(r) \leq m_s(r) + m_t(r)$ for any $s$ and $t$, and so we can bound the proximity function of $Q_k(f)$ in terms of those of $J_{i,k}(f)$, specifically as

$$m_{Q_k(f)} \leq \sum_{i=1}^{e_k} m_{J_{i,k}(f)}(r).$$
It follows from Lemma 3, Steinmetz’s Theorem 2 with \( \delta = 1 \), and the non-negativity of the proximity function that

\[
m_{f}(r) = m_{R_{z-1} \circ \cdots \circ R_{z-k}(f(z-k))}(r)
\leq (\deg(R_{z-1} \circ \cdots \circ R_{z-k}) - \deg(Q_k))m_{f(z-k)}(r) + m_{Q_k, f(z-k)}(r; 0)
+ S(f, r)
\leq \sum_{i=1}^{\epsilon_k} (m_{f(z-k)}(r) + m_{J_{i,k} f(z-k)}(r)) + S(f, r)
\leq 3\epsilon_k T_{f(z-k)}(r) + S(f, r),
\]

(10)

for \( r \) outside of some set \( E_1 \) of finite measure, depending on \( R \) and \( k \). On the other hand, \( f(z) = R_z \circ \cdots \circ R_{z-k+1}(f(z-k)) \), and so Lemma 2 gives

\[
T_{f}(r) = \deg(R_z \circ \cdots \circ R_{z-k+1})T_{f(z-k)}(r) + S(f, r)
= \deg_{w}(R)^k T_{f(z-k)}(r) + S(f, r),
\]

(11)

since the \( R_{z-j}(w) \) all have the same degree in \( w \) and (for \( j \geq 0 \)) all have coefficients in \( K_{\varphi} \). Thus we have from (9), (10), and (11) that

\[
m_{f}(r) \leq \frac{\epsilon \deg_{w}(R)^k}{2} T_{f(z-k)}(r) + S(f, r)
\leq \frac{\epsilon}{2} T_{f}(r) + S(f, r)
\leq \epsilon T_{f}(r)
\]

for \( r \) sufficiently large and not in some set of finite length. This completes the proof in the case \( a = \infty \).

Now let \( a \in K_f \) be arbitrary, but ordinary for \( R_z(w) \), suppose that \( a(z+1) \in K_f \), and set

\[
\mu_z(w) = \frac{1}{w} + a_z, \text{ so } \mu_z^{-1}(w) = \frac{1}{w - a_z}.
\]

By definition, \( m_{f}(r; a) = m_{\mu_z^{-1}(f)}(r) \). Let \( S_z(w) = \mu_z^{-1+1} \circ R_z \circ \mu_z(w) \), and \( g = \mu_z^{-1}(f) \). Note that we then have \( g(z+1) = S_z(g(z)) \) from (1). We also have, from Lemma 2 (or an appropriate moving-targets version of the first main theorem) that

\[
T_{g}(r) = T_{f}(r) + O(T_a(r)).
\]

We note also that if \( R_z(w) \) has coefficients in \( K_f \), then \( S_z \) has coefficients in \( K_f \). We may apply the previous case of the result to obtain

\[
m_{f}(r; a) = m_{g}(r)
\]
\[ T_g(r) + S(f, r) \leq \frac{\varepsilon}{2} T_f(r) + S(f, r) \]
\[ \leq \varepsilon T_f(r) \]

for \( r \) sufficiently large and outside of an exceptional set of finite length. \( \square \)

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