Some results on topological currents in field theory∗

Vivek M. Vyas†a and Prasanta K. Panigrahi‡b

aInstitute of Mathematical Sciences, Taramani, Chennai-600113, INDIA
bDepartment of Physical Sciences, Indian Institute of Science Education and Research (IISER) - Kolkata, Nadia-741252, INDIA

November 13, 2014

Abstract

A few exact results concerning topological currents in field theories are obtained. It is generally shown that, a topological charge can not generate any kind of symmetry transformation on fields. It is also proven that, the existence of a charge that does not generate any kind of symmetry transformation on fields, has to be of topological origin. As a consequence, it is found that in a given theory, superconductivity via Anderson-Higgs route can only occur if the gauge coupling with other fields is minimal. Several physical implications of these results are studied.

1 Introduction

Topological aspects are central to several problems in quantum field theory. In study of problems related to monopoles, instantons, anomalies and σ-models, ideas of topology are indispensable [1, 2, 3]. Insights based on topology are useful in the understanding of various phenomena in areas like superconductivity, Hall effect, QCD and cosmology [4, 5, 1]. It has been generally seen that topology often governs the global aspects in a given field theory, and provides with information which is non-perturbative in nature.

In this paper, we study aspects of currents which are of topological origin, and are conserved without resort to any symmetry principle. It is generally proven that, topological charges do not generate any symmetry transformation on the dynamical fields of the theory. Conversely, it is also proven that, if there exists a charge that does not generate any transformation on the dynamical fields of the theory, then the corresponding current must be topological in nature. One of the interesting implication of these statements is that, topological spontaneous symmetry breaking is non-existent. As a result, it is found that in a given theory, superconductivity via Anderson-Higgs route can only occur if the gauge coupling is minimal. This has several consequences in condensed matter and nuclear physics. Occurrence of topological currents in certain non-relativistic models are also studied and their physical relevance pointed out.

∗IMSc Preprint Number: IMSC/2014/11/11
†vivekmv@imsc.res.in
‡pprasanta@iiserkol.ac.in
2 Topological currents

As is well known, the dynamics of a given (classical) field theory of a field \( \phi(x) \), defined in \( n+1 \)-dimensional space-time, is governed by the equation of motion, which follows from extremisation of the action functional, \( \mathcal{S} = \int dV x dt \mathcal{L}(x, \phi(x), \partial \phi(x)) \) as per the Hamilton’s principle. Given such a theory, classical Noether theorem [7, 8, 9] dictates that, if continuous transformation of that of coordinates \( (x^\mu \to x^\mu + \delta x^\mu) \), fields \( (\phi(x) \to \phi(x) + \delta \phi(x)) \) and possibly Lagrangian (density) \( (\mathcal{L} \to \mathcal{L} + \delta \mathcal{L}) \), is such that action transforms as: \( \mathcal{S} \to \mathcal{S} + \int dV dt \partial_\mu \mathcal{L} \), then correspondingly there exists a current \( j^\mu \) via Noether theorem [11]. It must be noted that current \( j^\mu \) is a function of dynamical fields \( \phi(x) \) and \( \pi(x) \), and the statement \( d_\mu j^\mu = 0 \) holds true only for those field configurations which solve equation(s) of motion [9]. Such statements are often referred to in the literature as weak current statements [10] and we shall denote them as \( d_\mu j^\mu \equiv 0 \).

A simple but instructive example of this appears in the theory of free real scalar field \( \phi(x) \) defined in \( 1+1 \) dimensional spacetime with action \( S = \int dx dt \partial_\mu \phi \partial^\mu \phi \). Canonical conjugate of \( \phi(x) \) is \( \pi = \partial \phi / \partial t \), which obeys canonical equal time Poisson brackets: \( \{ \phi(x,t), \pi(y,t) \} = \delta(x-y) \). The equation of motion for \( \phi \) field is: \( \partial^2 \phi = 0 \). As is evident, above action is invariant under continuous field transformation: \( \phi \to \phi + \text{constant} \), and so as per Noether theorem one obtains a current \( j^\mu = \partial^\mu \phi \), which is conserved \( \partial_\mu j^\mu = \partial_\mu \partial^\mu \phi = 0 \). Note that the current conservation equation itself coincides with equation of motion and so current conservation only holds as long as \( \phi(x) \) obeys equation of motion. Hence, the statement of current conservation is a weak one: \( \partial_\mu j^\mu \equiv 0 \). Also note that conserved charge \( Q = \int dx \pi(x) \), generates field transformation \( \delta \phi(x) = \epsilon \{ \phi(x), Q \} = \epsilon \), where \( \epsilon \) is a real infinitesimal.

Unlike the above mentioned Noether currents, there exist another class of currents called topological currents, which are conserved due to topological reasons. Their conservation is not in reference to any particular action or Hamiltonian under consideration, and holds identically in general [10]. An example of such a current, in case of real scalar field in \( 1+1 \) dimensional space-time is: \( j^\mu = \epsilon^{\mu \nu} \partial_\nu \phi \). It is clear that the conservation of this current is not dependent on any particular form of action, and neither is \( \phi(x) \) required to obey the equation of motion. Also note that the conservation is insensitive to space-time geometry as well, since \( \partial_\mu j^\mu = 0 \) is independent of metric.

In what follows, we shall work with following definition of topological current:

**Definition.** In a given field theory, a topological current \( j^\mu_T \), which is in general a function of
coordinates, dynamical fields and possibly of the derivatives of dynamical fields, is a conserved current, whose conservation holds identically: \( d_\mu J^\mu_T = 0 \).

Such topological conservation, hence, holds in the strong sense, in contrast to weak sense defined above, and holds true for all dynamical field configurations, which in general do not solve equations of motion of the theory.

Since conservation of these topological currents is not in reference to any particular equation of motion, it is easy to see that, they are not connected to any continuous field transformation. This fact can be expressed as:

**Theorem 1.** Assume that there exists a relativistic field theory of a dynamical field \( \phi(x) \), which is Poincaré invariant. Let \( \pi(y) \) denote canonical momentum corresponding to \( \phi(x) \), obeying canonical Poisson bracket \( \{ \phi(\vec{x}, t), \pi(\vec{y}, t) \} = \pm \delta(\vec{x} - \vec{y}) \). If there exists a conserved topological current \( J^\mu_T \) in such a theory, then the corresponding conserved charge \( Q_T = \int dV J^\mu_T \) does not generate any transformations: \( \{ Q_T, \phi(x) \} = 0 \) and \( \{ Q_T, \pi(x) \} = 0 \).

**Proof.** From the definition of the current, it immediately follows that the current \( J^\mu_T \) is strongly conserved: \( d_\mu J^\mu_T = 0 \). Equation of motion for any observable \( O(x, \phi(x), \pi(x)) \) can be written as \([13]\):

\[
d_\mu O = \partial_\mu O + \{ O, P_\mu \},
\]

where \( P_\mu \) are generators of space-time translation, with \( P_0 = H \), the Hamiltonian of the theory, and \( \vec{P} = \int dV \pi \vec{\nabla} \phi \). By virtue of above equation of motion, one can write current conservation as:

\[
d_\mu J^\mu_T = \partial_\mu J^\mu_T + \{ J^\mu_T, P_\mu \} = 0.
\]

But since topological current is strongly conserved, the conservation holds even if the theory is defined with modified Hamiltonian \( \tilde{H} \):

\[
d_\mu J^\mu_T = \partial_\mu J^\mu_T + \{ J^\mu_T, \vec{P}_\mu \} = 0.
\]

This immediately implies: \( \{ H - \tilde{H}, J^0_T \} = 0 \). Considering the case when \( H - \tilde{H} = \int dV_y \phi(y) \), such that \( x_0 = y_0 = t \), one finds that: \( \int dV_y \{ \phi(\vec{y}, t), J^0_T(\vec{x}, t) \} = 0 \). Owing to translational invariance one has: \( \{ \phi(\vec{y}, t), J^0_T(\vec{x}, t) \} = F(\vec{y} - \vec{x}, t) \), where \( \int dV_y F(\vec{y} - \vec{x}, t) = 0 \). Exploiting the translational invariance of measure, this can be written as: \( \int dV_y F(\vec{y}, t) = 0 = \int dV_y F(\vec{z} + \vec{y}, t) \), where \( \vec{z} \) is an independent constant vector. This can be rewritten as:

\[
\int dV_y \{ \phi(\vec{z}, t), J^0_T(\vec{y}, t) \} = 0,
\]

or \( \{ \phi(\vec{z}, t), Q_T \} = 0 \). Working along similar lines, but now considering the case when \( H - \tilde{H} = \int dV_y \pi(y) \), it follows that \( \{ \pi(\vec{x}, t), Q_T \} = 0 \).

It can be easily seen that a similar statement will also hold true for the case of quantised theory, that is, commutator of topological charge operator \( Q_T \) with dynamical field operators is always vanishing: \( [Q_T, \phi(x)] = 0, [Q_T, \pi(x)] = 0 \). This fact was observed by Ezawa \([13]\) in context of a particular model, leading to topological superselection rules in such theories \([14]\). Later Shamir and Park offered a general proof for this statement \([16]\); however in the proof it was assumed that the topological current can always be written in a certain form. Above proof is more general, as it overcomes such an assumption. Interestingly, one notes that \( \{ H, Q_T \} = 0, \) for

---

4Here and throughout this paper, we shall work with generalised Poisson brackets, which are defined for both commuting and anticommuting c-number fields, as in Ref. \([12]\).
any Hamiltonian $H$. From Hamilton’s equation (Heisenberg equation in quantum case) one finds that $\frac{\partial Q}{\partial t} = 0$, which states that topological charges can not have any explicit time dependence.

Although above theorem was proven assuming that field $\phi(x)$ is a single component object, it is straightforward to see that it will also hold, when field $\phi(x)$ in general is a multi-component field. Apart from $1 + 1$ dimensional real scalar theory discussed above, another example of this theorem is given by the nonrelativistic theory of bosons, living on a line, governed by complex field $\psi(x, t)$. Such a theory admits three conserved topological currents: (a) $J_0 = \partial_t (\psi + \psi^\dagger)$, $J_x = -\partial_x (\psi + \psi^\dagger)$; (b) $J_0 = -i\partial_x (\psi - \psi^\dagger)$, $J_x = i\partial_t (\psi - \psi^\dagger)$ and (c) $J_0 = \partial_x (\psi^\dagger \psi)$, $J_x = -\partial_t (\psi^\dagger \psi)$. Corresponding conserved charges are given by $Q_1 = \int dx \partial_x (\psi + \psi^\dagger)$, $Q_2 = \int dx \partial_x (\psi^\dagger - \psi)$ and $Q_3 = \int dx \partial_t (\psi^\dagger \psi)$. Owing to canonical commutation relations: $[\psi(x, t), \psi^\dagger(y, t)] = \delta(x - y)$. one sees that $[Q_{(1,2,3)}, \psi(x, t)] = 0$, irrespective of the form of Hamiltonian.

Yet another example of above theorem, in case of fermions, is given by quantised field theory of spinor field $\Psi(x)$; where the conserved current, $J_\mu = \partial_\mu (\bar{\Psi} \sigma^{\mu\nu} \Psi)$, $(\sigma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu])$ is a topological current. The topological charge $Q = \int dV \partial_\mu (\bar{\Psi} \sigma^{\mu\nu} \Psi)$ commutes with both $\Psi(x)$ and $\Psi^\dagger(x)$, owing to equal-time anticommutation relation: $[\Psi(x, t), \Psi^\dagger(y, t)]_+ = \delta(x - y)$. In this light, it is very interesting to note that, the conserved charge current $J_\mu = \bar{\Psi} \gamma_\mu \Psi$, which is the Noether current corresponding to U(1) phase invariance $\Psi \rightarrow \Psi e^{i\theta}$ (where $\theta$ is a real constant), can be written using Gordon decomposition \cite{17} as:

$$J_\mu = \frac{i}{2m} \left( \bar{\Psi} \partial^\mu \Psi - \partial^\mu \bar{\Psi} \Psi \right) + \frac{1}{2m} \partial_\nu \left( \bar{\Psi} \sigma^{\mu\nu} \Psi \right).$$

The first term in above expression is the orbital current or the diamagnetic term, whereas the second term is the spin term or paramagnetic term, which is topological. Correspondingly, Noether charge $Q = \int dV J_0$ has two components, first one corresponding to orbital current: $Q_o = \int dV \frac{1}{2m} \left( \bar{\Psi} \partial^\mu \Psi - \partial^\mu \bar{\Psi} \Psi \right)$ and second one corresponding to topological current: $Q_t = \int dV \frac{1}{2m} \partial_\nu \left( \bar{\Psi} \sigma^{\mu\nu} \Psi \right)$. From this decomposition it is clear (as per Theorem \cite{1}) that, the U(1) symmetry transformation is actually generated by $Q_o$, since $Q_s$ can not give rise to such transformation. This is an important result, which tells us that, although Noether theorem provides us with an expression for the conserved charge, such a conserved charge may contain in general a contribution due to topological charges. As per Theorem \cite{1} such a contribution is harmless as far as symmetry properties are concerned.

It is worth mentioning that the above theorem also holds for gauge theories, which can be elegantly seen in case of Abelian gauge field $A_\mu$ defined in $2 + 1$ dimensional spacetime. Such a theory admits a topological current: $J_\mu = e^{i\phi} F_{\mu\nu}$, with topological charge $Q_T = \int d^2x e^{i\phi} F_{ij}$ (here Latin indiciess take values $1, 2$). Note that conservation of topological charge is independent of gauge choice. In covariant gauge, the equal time commutator is $[A_\mu(x), F_{ij}(y)]_{x_0 = y_0} = 0$, hence one has $[Q_T, A_\mu] = 0$. Similarly in axial gauge ($A_0 = 0$), one has $[A_i(x), F_{jk}(y)]_{x_0 = y_0} = 0$ and hence $[Q_T, A_i] = 0$.

An important implication of Theorem \cite{1} is regarding Poincaré invariance of the theory. It is a well known fact that canonical energy-momentum tensor $\Theta^{\mu\nu}$ in a relativistic quantum field theory, obtained using Noether theorem is conserved $\partial_\mu \Theta^{\mu\nu} = 0$ and is in general not symmetric $\Theta^{\mu\nu} \neq \Theta^{\nu\mu}$. It was shown by Belinfante that canonical energy-momentum tensor can always be improved by adding a specifically constructed term $\partial_\mu \chi^{\mu\nu}$ (where $\chi^{\mu\nu} = -\chi^{\nu\mu}$), such that improved energy-momentum tensor $T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\mu \chi^{\mu\nu}$, is symmetric $T^{\mu\nu} = T^{\nu\mu}$ \cite{1,18}. By construction $\partial_\mu (\partial_\nu \chi^{\mu\nu}) = 0$ identically, which ensures conservation of the improved energy-momentum tensor $\partial_\mu T^{\mu\nu} = 0$. This results in modification of canonical angular momentum tensor $M^{\mu\nu\rho}$ to improved angular momentum tensor $L^{\mu\nu\rho} = M^{\mu\nu\rho} + \partial_\rho \eta^{\mu\nu\rho}$, where $\eta$ is anti-symmetric in first two indices, and can be written in terms of $\chi$. In a given theory, if the charges constructed out of canonical energy-momentum and angular momentum tensors, obey Poincaré
algebra, then the theory is said to be Poincaré invariant. Now consider a given theory, where the charges constructed out of canonical energy-momentum and angular momentum tensors, obey Poincaré algebra. However, when one constructs charges out of improved energy-momentum and angular momentum tensors, one wonders whether these are guaranteed to obey Poincaré algebra in general or not. In other words, is it possible that for some theory, the improvements may spoil Poincaré invariance of the theory? As mentioned above, both the improvements, respectively to energy-momentum and angular momentum tensor, are topological currents. The charges constructed out of improved tensors will differ from the canonical one by a contribution coming from the topological part, which by Theorem 1 would not contribute to any commutation relations, and hence can not spoil Poincaré invariance of the theory. It is worth emphasising that, this conclusion also holds in general, that is, under any redefinition of a conserved Noether current by addition of a topological current, the commutation relations amongst Noether charges remain unaffected.

Above assertion also implies that, if in a theory the energy-momentum tensor itself is conserved topologically, then such a theory is non-dynamical. Chern-Simons theory defined in 2 + 1 dimensional space-time with action: \( \mathcal{L} = \epsilon^{\mu \nu \rho} A_\mu F_{\nu \rho} \), is an instructive example of this. For this theory, the canonical energy-momentum tensor \( \theta_{\mu \nu} \) vanishes identically and hence is conserved topologically (as per our definition). The improved energy-momentum tensor, in general, will be of the form \( T^{\mu \nu} = \epsilon^{\mu \nu \rho} \partial_\rho \phi \) (where \( \phi(x) \) is some function whose exact form is irrelevant in this discussion), which is also a topologically conserved current. As is evident, both the energy-momentum tensors lead to same conclusion that, the theory is non-dynamical.

From above discussion it is clear that, the complete set of conserved currents can be divided into exactly two equivalence classes: Noether currents and non-Noether or topological currents. Note that for any conserved charge \( Q_i \) in the theory, charge \( \lambda Q_i \) is also conserved, for each \( \lambda \in \mathbb{R} \). Further, a linear combination of any two conserved charges \( Q_{i,j} \), \( \lambda_1 Q_i + \lambda_2 Q_j \) is also a conserved charge, for all \( \lambda_{1,2} \in \mathbb{R} \). From these two properties (which are vector addition and scalar multiplication), it easy to see that, the set of all conserved charges, for a given theory, form a linear vector space \( \mathbf{V} \) under addition, and over field \( \mathbb{R} \). Interestingly, since addition of two topological charges is also a topological charge, and zero charge (which is the identity element) is topological, one sees that the set of conserved topological charges itself forms a vector space \( \mathbf{T} \), which is a subspace of \( \mathbf{V} \). Note that, all the Noether charges live in \( \mathbf{T}^\perp \), which is the complement of \( \mathbf{T} \) and a subset of \( \mathbf{V} \). Further, any two Noether charges which differ by a scalar multiple or an element in \( \mathbf{T} \) generate same transformation. So it is imperative that, one identifies all elements in \( \mathbf{V} \) which differ from each other by an element in \( \mathbf{T} \), defining a quotient space \( \mathbf{V}/\mathbf{T} \). Now one can use the fact that, quotient space \( \mathbf{V}/\mathbf{T} \) forms a linear vector space under addition with field being \( \mathbb{R} \), and construct a suitable complete basis set. Elements of such a basis set, each being a linearly independent Noether charge, will generate a distinct transformation.

In light of above discussions, it worth enquiring whether conserve of Theorem 1 is true or not. That is, for a given charge (which can be written as surface integral of a current, with integration being done over a space-like surface) that commutes with all the dynamical fields of the theory, does it imply that the corresponding current is strongly conserved? Below it is shown that the answer to this question is affirmative.

**Theorem 2.** Assume that there exists a relativistic field theory of a dynamical field \( \phi(x) \), which is Poincaré invariant. Let \( \pi(y) \) be the canonical momentum corresponding to \( \phi(x) \), obeying canonical Poisson bracket: \( \{ \phi(\vec{x}, t), \pi(\vec{y}, t) \} = \pm \delta(\vec{x} - \vec{y}) \). If there exists a four-vector current \( j^\mu \) in the theory, such that the corresponding conserved charge \( Q = \int_\sigma ds_\mu j^\mu \) does not generate

---

Here \( \sigma \) denotes the space-like surface over which the integration is to be done \[^1\] generally taken to be a constant time surface. Here \( ds_\mu \) stands for oriented area element on the surface \( \sigma \).
any transformations: \( \{ Q_T, \phi(x) \} = 0 \) and \( \{ Q_T, \pi(x) \} = 0 \), then the corresponding current is identically conserved: \( \partial_\mu j^\mu = 0 \).

**Proof.** Since \( Q \) commutes (in Poisson bracket sense) with both \( \phi(x) \) and \( \pi(x) \), it implies that \( \{ Q_T, f(y) \} = 0 \), where \( f(y) \) is some function of \( \phi(y), \pi(y) \) and possibly their derivatives, with \( y \) being a fixed coordinate. This implies:

\[
\int ds_\mu \{ j^\mu(x), f(y) \} = 0.
\]

(3)

Defining \( B_\mu(x - y) = \{ j_\mu(x), f(y) \} \), this can be written as:

\[
\int ds_\mu B_\mu(x - y) = 0.
\]

(4)

Without loss of generality we set \( y = 0 \) in what follows. It is reasonable to assume that, the boundary conditions obeyed by the fields are such that, as \( |\vec{x}| \to \infty \) the dynamical fields go to a constant (usually zero) i.e., \( \phi(|\vec{x}| \to \infty) \to c \) and similarly \( \pi(|\vec{x}| \to \infty) \to c' \), where \( c \) and \( c' \) are constants. The same can be stated in a covariant manner: \( \phi(x_\mu x^\mu \to -\infty) \to c \) and \( \pi(x_\mu x^\mu \to -\infty) \to c' \). As a result of these conditions, functions of dynamical fields, like \( B_\mu(x) \), attain a constant value as \( x_\mu x^\mu \to -\infty \). It is beneficial to work in Euclidean space-time, by performing Wick rotation on all the vectors: \( A_0 \to iA_0, \vec{A} \to \vec{A} \). Above boundary condition, now becomes \( B_\mu \to \text{constant} \), as \( x_\mu x^\mu \to \infty \). This allows us to identify space-time infinity as a single point for such functions, which amounts to say that such functions are actually living on a space-time which is topologically a sphere.\(^6\) With this identification, above expression \(^5\) actually becomes a surface integral over a closed space-like surface\(^7\) on the space-time sphere:

\[
\oint ds_\mu B_\mu(\vec{x},t) = 0.
\]

(5)

Using the divergence theorem this can be written as a volume integral over the space-time region enclosed by the surface:

\[
\int dV \partial_\mu B_\mu = 0.
\]

(6)

Owing to the generality in the definition of \( B_\mu \), above statement can only hold, if \( \partial_\mu B_\mu(\vec{x},t) = 0 \), which in turn implies \( \partial_\mu j_\mu(\vec{x},t) = \text{constant} \). Since \( Q \) is a conserved charge, consistency requires that \( \partial_\mu j_\mu(\vec{x},t) = \text{constant} = 0 \). Wick rotating to Minkowski space-time, one finds that \( \partial_\mu j^\mu(x) = 0 \). Note that in arriving at this result, we have not assumed at any juncture that the dynamical fields which constitute \( j_\mu(x) \) solve equation of motion, and so current conservation is a strong equality and holds identically: \( \partial_\mu j^\mu \equiv 0 \).

\[\text{3 Coupling with gauge field}\]

It is well known that in a quantum field theory, in constrast to quantum mechanics, vacuum in general is not unique\(^{22}\). The celebrated Goldstone theorem\(^{23}\) for a given quantised field theory of field \( \phi(x) \) states that, if for a conserved charge \( Q \), the vacuum expectation value of Goldstone commutator \( [Q, \phi(x)] \) is nonvanishing \( \langle \text{vac} | [Q, \phi(x)] | \text{vac} \rangle \neq 0 \), then the symmetry corresponding to charge \( Q \) is said to be spontaneously broken in that vacuum, implying existence of gapless Nambu-Goldstone modes in field \( \phi(x) \). The same in general holds true for nonvanishing

\(^6\)Mathematically this is one point compactification\(^{21}\) of Euclidean space-time \( \mathbb{R}^n \) to \( n \)-sphere \( S^n \).

\(^7\)This closed space-like surface is also a sphere, albeit of one lesser dimension than the space-time sphere.
of vacuum expectation of Goldstone commutator \( \langle \text{vac}|[Q,F(x)]|\text{vac} \rangle \) for any composite field \( F(\phi(x),\Pi(x)) \), as in the case of superconductivity [22]. As shown by Nambu [24], the symmetry associated with charge \( Q \) is never lost, but is rearranged and gets manifested in such a vacuum by presence of Nambu-Goldstone modes [22].

In a given theory, if the symmetry generated by charge \( Q \) is spontaneously broken, and if a gauge field is coupled to the conserved charge \( Q \), then via Anderson-Higgs mechanism, the gauge field gets gauge invariant mass [22, 25]. In case of superconductivity, it is well known that \( Q \) corresponds to electric charge, and the electromagnetic field becomes massive [22]. Now if charge \( Q \) happens to be a topological charge, then from Theorem 1, one arrives at an interesting result:

**Theorem 3.** In a given quantised field theory of some field \( \phi(x) \), obeying appropriate canonical (anti)commutation relation: \( [\phi(\vec{x},t),\Pi(\vec{y},t)]_\pm = \mp i\delta(\vec{x}-\vec{y}) \), if there exists a conserved topological charge \( Q_T \), such that the gauge field \( A_\mu \) only couples to \( Q_T \), then gauge field \( A_\mu \) can not become massive via Anderson-Higgs mechanism in such a theory.

**Proof.** The proof is straightforward. Note that for topological charge \( Q_T \), Goldstone commutator vanishes for both \( \phi \) and \( \Pi \): \( [Q_T,\phi(x)] = 0 \) and \( [Q_T,\Pi(x)] = 0 \) due to Theorem 1. As a consequence, Goldstone commutator: \( [Q_T,O(x)] \) for any composite operator \( O(\phi(x),\Pi(x)) \) also vanishes. This means that in all possible vacua, there would be no Nambu-Goldstone gapless mode associated with charge \( Q_T \), since nonvanishing of vacuum expectation value of Goldstone commutator is a necessary and sufficient condition for their existence [20, 25]. In absence of this gapless mode, gauge field \( A_\mu \) can not attain mass via Anderson-Higgs mechanism.

This conclusion was also obtained in Ref. [16]. This theorem gives rise to several interesting consequences:

**Corollary 4.** Consider a theory of massive spinors coupled to a Abelian gauge field,

\[
\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \mathcal{L}_{\text{int}}(\bar{\psi},\psi) + \mathcal{L}_g(\bar{\psi},\psi,A_\mu),
\]

where \( \mathcal{L}_{\text{int}} \) interaction terms involving only spinor fields, whereas \( \mathcal{L}_g \) stands for gauge-spinor interaction terms (It is assumed that the net action is gauge invariant under appropriate local gauge transformations). Further, assume that above Lagrangian is invariant under symmetry transformation \( \psi \rightarrow \psi e^{-i\theta} \), where \( \theta \) is a real parameter. Then in such a theory, gauge boson can gain mass via Anderson-Higgs mechanism, only if there is minimal coupling \( \bar{\psi}\gamma^\mu\psi A_\mu \) between fermions and gauge field.

**Proof.** Above theory possesses only one internal symmetry, which is generated by conserved Noether charge \( Q \), corresponding to current \( \bar{\psi}\gamma^\mu\psi \). Hence existence of any other conserved current in this theory has to be of topological origin[8]. Now if gauge field is only coupled to any of such topological currents, then as per Theorem 3, gauge field can not get mass via Anderson-Higgs mechanism.

It is straightforward to see that a similar result will also hold if one has non-Abelian gauge field coupled to fermions. As is evident this corollary puts severe constraint on the type of theory one can construct involving massive gauge bosons, where gauge boson mass comes via Anderson-Higgs route. In particular, this corollary has an interesting consequence for non-minimally coupled QED in 3 + 1 dimensional spacetime:

\[
\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + g\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu} + \mathcal{L}_{\text{int}}(\bar{\psi},\psi),
\]

\[8\] Here we are not considering currents which are conserved due to space-time symmetries.
where $\mathcal{L}_{\text{int}}$ represents other interaction terms involving only spinor fields. Here the gauge field couples to conserved spin current $J^\mu = \partial_\nu (\bar{\psi} \gamma^\mu \sigma_\nu \psi)$ which, as noted before, is topological. In light of above corollary, this model can not exhibit superconductivity via Anderson-Higgs mechanism. Such non-minimally coupled models are relevant in context of physics of neutrons, since neutrons being electrically neutral, can only interact with photons non-minimally [1]. The same also holds for systems involving Majorana fermions, which are currently studied intensely in context of condensed matter systems [5, 27]. These results may also have relevance for condensed matter systems, where the coupling of quasiparticles to electromagnetic field is only via Zeeman term, for example in models dealing with magnetism and high $T_c$ superconductivity [5].

It is not difficult to see that, similar corollary will hold for any interacting theory of scalar fields, namely gauge boson can gain mass via Anderson-Higgs mechanism only if scalar field is minimally coupled to gauge field. A similar corollary will also hold for both nonrelativistic theory of interacting bosons and fermions in general. Infact, nonrelativistic theory of interacting bosons living on a line, which was discussed earlier, has found realisation in ultracold atom experiments [28, 29]. Since the constituent atoms in these experiments are electrically neutral, the effective theory of bosons arising out of them does not couple to electromagnetic field minimally [28, 30]. In this light, above results will have relevance to these systems as well.

Apart from above theories, similar conclusions will also hold for theories involving antisymmetric tensor field $B_{\mu\nu}$ (in $3 + 1$ dimensions), wherein the coupling between gauge field $A_\mu$ and $B_{\mu\nu}$, is of $B \wedge F$ form: $\epsilon^{\mu\nu\rho\lambda} B_{\mu\nu} F_{\rho\lambda}$. As is evident this coupling is non-minimal, and the gauge field is coupled to a topological current. Hence, from Theorem 3 it follows that $A_\mu$ can not gain mass via Anderson-Higgs route. However, it is worth emphasizing that $A_\mu$ can still be massive, as seen in Ref. [31], without any spontaneous symmetry breaking whatsoever.

Above results, in certain sense, indicate the uniqueness of minimal gauge coupling in quantum field theory.

4 Extended objects and classical theory

As mentioned earlier, a quantum field theory, unlike quantum mechanics of N-point particles, can realise different vacua, which are unitarily inequivalent to each other [22, 4]. In many cases, the system realises a vacuum such that vacuum expectation value of field is non-vanishing and in general a function of spacetime $\langle \text{vac}|\Phi(x)|\text{vac} \rangle = f(x) \neq 0$. In different context, such spacetime dependent vacua are known by different names like solitons, vortices, kinks, hedgehogs, monopoles and so on [4]. Generically these are also called extended objects, since function $f(x)$ is non-vanishing in finite spatial domain, or equivalently has a finite spatial extent.

It is seen in many field theories that the extended objects carry topological charge. To be more precise, for some topological charge $Q_{\text{top}}$, in the theory, there exists a particular vacuum $|f(x)\rangle$ such that, $\Phi(x)|f(x)\rangle = f(x)|f(x)\rangle$ and $Q_{\text{top}}|f(x)\rangle \neq 0$. A well known example of such topological object is a vortex vacuum of real scalar theory in $2 + 1$ dimensions, for which the topological charge is $Q = \int d^2 x \mathbf{\nabla} \times \mathbf{\nabla} \Phi = \oint \mathbf{r} \cdot \mathbf{\nabla} \Phi$ is non-vanishing [4]. An important consequence follows from these; in event of instability of a vacuum which has non-vanishing topological charge, the theory can not realise another vacuum, which has a topological charge different than the former, since the topological charge is conserved in the theory. Hence topological charge makes such extended objects extraordinarily stable, and it gives rise to various physical phenomena [22, 4]. In many cases, such topological objects lead to fermion number and charge fractionalisation [2, 32].

Above discussion convinces one that, the study of topological charges in vacuum sector or equivalently in the classical theory is interesting in its own right, and can lead to valuable information about the theory. As an example, below we study an integrable classical field theory.
model, and show that ideas similar to quantum case, lead to important results regarding solution space of the theory.

Consider a nonrelativistic classical field theory, living on a line, involving a complex field $\psi(x,t)$ defined by the Lagrangian:

$$\mathcal{L} = \frac{i}{2} \left( \psi \partial_t \psi^* - \psi^* \partial_t \psi \right) - \frac{1}{2} \left( \partial_x \psi^* \partial_x \psi + |\psi|^4 \right).$$

The equation of motion is given by the nonlinear Schrödinger equation [33]:

$$i \partial_t \psi + \frac{1}{2} \partial_{xx} \psi - |\psi|^2 \psi = 0,$$

which appears in diverse areas like optics, plasma physics, hydrodynamics and so on [34, 35]. From the monumental work of Zakharov and Shabat [36], it became known that nonlinear Schrödinger equation is an integrable system, in the sense that, there exists infinitely many independently conserved quantities. Integrability of this system has many interesting connections with two dimensional conformal field theory and gravity [37, 38, 33]. The canonical equal-time Poisson bracket for this system is given by:

$$\{ \psi(x,t), \psi^*(y,t) \} = i \delta(x-y).$$

From the discussion in section (2) regarding nonrelativistic theory, it is clear that this theory admits, amongst others, following three topological charges:

$$Q_1 = \text{Re} \left. \psi \right|_{-\infty}^{\infty}, \quad Q_2 = \text{Im} \left. \psi \right|_{-\infty}^{\infty}, \quad \text{and} \quad Q_3 = \left. \psi^* \psi \right|_{-\infty}^{\infty}.$$ 

It is well known that above equation of motion admits, what is called kink soliton or dark soliton [39] as a solution:

$$\Psi(x,t) = \sqrt{n_0} [B \tanh(\xi) + iA] e^{-in_0 t},$$

where $\xi = \sqrt{n_0} B(x-A\sqrt{n_0}t)$ and $A^2 + B^2 = 1$ ($n_0, A, B$ are real parameters). It is straightforward to see that, this kink soliton carries $Q_1$ topological charge, infact, $Q_1[\Psi] = 2$, and $Q_{2,3}[\Psi] = 0$. Interestingly, however if one considers above solution with an additional overall phase of $-\pi$, then one sees that, $Q_2[\Psi] = 2$, and $Q_{1,3}[\Psi] = 0$. In any case, this clearly shows that *kink solitons are topological objects*, and in general carry charges $Q_{1,2}$ as defined above. It is worth mentioning that these solitons are known in the literature since 1970’s and have been extensively studied both theoretically and experimentally [35, 39, 40], however their topological nature was hitherto not understood. From above discussion, one is able to see clearly the topological nature of these solitons, and the reason behind their stability.

Having shown the topological nature of these kink solitons, a natural question arises: Whether in one spatial dimension there can be Berenzinskii-Kosterlitz-Thouless transition [41, 42], with the role of vortices being played by these kink solitons? The answer to this question turns out to be negative. As it is shown in the appendix, the energy possessed by the system in a non-topological constant background state exactly turns out the be the same as that when system has any arbitrary pairs of kink solitons. This implies that the constant background is unstable against formation of pairs of kink solitons due to any small external perturbation. Further in finite temperature case, owing to the periodicity property of multi-kink solution (see appendix), it can be seen that the free energy will not have any entropic contribution, making this assertion valid even at non-zero temperature. It must also be pointed out that, dark/kink solitons are studied in context of physics of ultra cold atoms [39, 43], and their energy has been calculated using a different method in Ref. [39, 43]. It turns out that, the energy estimated here and found in Ref. [39, 43] differ. Further it is shown in Ref. [43] that if one follows the latter estimate then the Berenzinskii-Kosterlitz-Thouless transition can be realised.
A variety of variants of nonlinear Schrödinger equation also exist in the literature, which are studied in context of various physical phenomena, which possess such extended solutions as above mentioned soliton (for example see [45, 46]). Note that since charges \( Q_{1,2,3} \) defined above are topological, their conservation holds for all variants of nonlinear Schrödinger equation (and of course for usual linear Schrödinger equation as well). In particular, we would like to point out that many solutions found in Ref. [47], while studying a certain variant of nonlinear Schrödinger equation, carry topological charge \( Q_3 \) unlike kink solitons encountered above.

5 Summary

In this paper, topological currents occurring in field theory (both classical as well as quantum) are studied. It is proven in generality that, topological charges commute with dynamical fields in any given theory, with some reasonable assumptions. It is also conversely proven that, if there exists a charge in the theory which commutes with all the dynamical fields of the theory, then the corresponding current has to be topological. Interestingly, this implies that unlike many Noether charges, topological charge can not have any explicit time dependence. As a consequence of above results, it is seen that, each conserved current can be classified exactly as either being a Noether current, or a topological current. It is clear that these two currents are of fundamentally different character, in the sense that, Noether charges generate non-trivial symmetry transformations preserving the equations of motion, unlike the ones generated by topological charges which are trivial. This gives rise to many interesting consequences. For example, any modification of Noether charges by addition of topological charges, does not alter the algebra of Noether charges, and hence symmetry properties of the theory. This implies that Noether charges, which differ by a topological charge are equivalent, and the space of all Noether charges is a quotient space.

It is also proven that, when a gauge field in a given theory, only couples to topological currents, then the gauge field can never become massive via Anderson-Higgs mechanism. This has implications to situations where the matter-gauge coupling is non-minimal, like in many models in high energy and condensed matter physics. However, it must be pointed out that, above result, in case where it is applicable, do not prohibit existence of gauge boson mass or superconductivity. It only tells that gauge boson mass can not arise via Anderson-Higgs mechanism. However, it is known that gauge boson mass can arise without Anderson-Higgs mechanism, as for example in Schwinger model, and above result does not negate such a possibility [48, 49].

A connection of extended objects with topological currents is shown, and a nonrelativistic self-interacting model is studied. It is shown that, the well known kink solitons of this model, are actually topological. Its implications to other variants of this model are also mentioned.

A Soliton energy calculation

The Hamiltonian functional for the one dimensional system discussed in section (4), is given by:

\[
H = \int dx \left( \frac{\hbar^2}{2m} \partial_x \psi^* \partial_x \psi + \frac{g}{2}(\psi^* \psi)^2 - \mu \psi^* \psi \right).
\]  

Here, for clarity, we have retained \( \hbar \) and have also have allowed for parameters like \( m, g \) and \( \mu \); and have assumed them to be positive definite. It is easy to see that above functional leads to following equation for \( \psi \), where \( \psi \) is only space dependent:

\[
\frac{\hbar^2}{2m} \partial_x^2 \psi - g(\psi^* \psi) \psi + \mu \psi = 0.
\]
This equation admits a real constant $\psi_0$ as a solution, such that:

$$
\psi_0^* \psi_0 = \frac{\mu}{g}; \quad \psi_0 = \sqrt{\frac{\mu}{g}}.
$$

(9)

Energy stored in the system, when system is in this state, can be easily seen to be:

$$
E_0 = -\frac{\mu^2}{2g} L,
$$

(10)

where $L$ is the system size.

In order to find non-trivial solutions to equation (8), it is convenient to rewrite it using parameters $c_1 = \frac{2m\mu}{\hbar^2}$ and $c_2 = \frac{2mg}{\hbar^2}$, so that it reads:

$$
\psi_{xx} + c_1 \psi - c_2 \psi^3 = 0.
$$

(11)

This equation admits real solution, in terms of Jacobi elliptic function $sn$, of the form:

$$
\psi(x) = A \text{sn}(\frac{x - x_0}{\alpha}, k);
$$

(12)

where $x_0$ is a real constant, $k$ is the elliptic modulus ($0 \leq k \leq 1$), and real parameters $A$ and $\alpha$ are given by

$$
A^2 = \left(\frac{2k^2}{1 + k^2}\right) \frac{c_1}{c_2} \quad \text{and} \quad \alpha^2 = \frac{1 + k^2}{c_1}.
$$

(13)

As is obvious, the parameter $A$ gives the amplitude of the solution, whereas $\alpha$ defines a characteristic length scale corresponding to the solution. In general, $sn(x, k)$ function is a periodic function, and the period is given by $4K(k)$, where $K(k)$ is the complete elliptic integral of first kind. For $k = 1$, elliptic function $sn$ reduces to hyperbolic tangent function: $sn(x, 1) = \tanh(x)$, which has infinite period. As seen earlier, this is the kink soliton or dark soliton. Note that kink soliton takes value $-A$ (when $x \to -\infty$) to $A$ (when $x \to \infty$) passing through zero (at $x = 0$).

Apart from above single soliton solution, it is also possible to have a two soliton solution, wherein a kink and an anti-kink are present, so that solution attains the same value as $x \to \pm \infty$. In what follows, we shall assume that our system defined over an interval $[\frac{-L}{2}, \frac{L}{2}]$, with periodic boundary condition $|\psi(-\frac{L}{2})|^2 = |\psi(\frac{L}{2})|^2$, and shall take limit $L \to \infty$ in the end. This will not only play the role of an infrared cutoff, but will also allow us to look at global behaviour of solutions. As mentioned above, the two soliton solution (of equation (8)) consistent with this boundary condition is (12), but with period $4K(k) = L/\alpha = V$, and located at $x_0 = K(k)\alpha$. Explicitly this is given by:

$$
\psi_2(x) = A \text{sn}(\frac{x - x_0}{\alpha}, k),
$$

(14)

where $k$ is such that $4K(k) = L/\alpha$. This can be generalised to $2n$ soliton solution, that is, $n$ pairs of kink and anti-kinks, by imposing condition $4nK(k) = V$ on above solution.

Energy stored in the system, when system is in the state corresponding to such $2n$ soliton $\psi_{2n}$, is given by:

$$
E_{2n} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left( \frac{\hbar^2}{2m} \partial_x \psi_{2n}^* \partial_x \psi_{2n} + \frac{g}{2}(\psi_{2n}^* \psi_{2n})^4 - \mu \psi_{2n}^* \psi_{2n} \right).
$$

(15)

This can be evaluated to yield:

$$
E_{2n} = -\frac{\hbar^2 A^2 k^2}{2m \alpha} I,
$$

(16)
where \( I = \int_0^V dy \, sn^4(y, k) \), with \( V = 4nK(k) \). This integral can be straightforwardly evaluated using properties of elliptic functions, and one finds:

\[
I = \frac{(2 + k)}{3k^2} V - \frac{2n\pi(1 + k)}{3k^2}.
\]

Now the thermodynamic limit \( L \to \infty \) can be implemented by taking \( k \to 1 \), so that \( V \to \infty \), since \( K(k \to 1) \to \infty \). Using this result, it is straightforward to see that the energy density of \( 2n \) soliton solution is given by:

\[
\mathcal{E}_{2n} = \frac{E_{2n}}{L} = -\frac{\mu^2}{2g}.
\]

This precisely coincides with the energy corresponding to constant solution \( \psi_0 \), which shows that the constant solution \( \psi_0 \) and \( 2n \) soliton solution \( \psi_{2n} \) are energy degenerate in the thermodynamic limit.

### Acknowledgments

VMV thanks Dr. Nitin Chandra for several useful discussions.

### References

[1] V. P. Nair, *Quantum field theory: A modern perspective*. Springer, 2005.

[2] R. Jackiw and J. R. Schrieffer, *Solitons with fermion number 1/2 in condensed matter and relativistic field theories*, Nuc. Phys. B **190** (1981) 253–265.

[3] A. S. Schwarz, E. Yankowsky, and S. Levy, *Quantum field theory and topology*. Springer, 1993.

[4] H. Umezawa, *Advanced field theory: micro, macro, and thermal physics*. American Institute of Physics, 1995.

[5] A. M. Tsvelik, *Quantum field theory in condensed matter physics*. Cambridge University Press, 2007.

[6] C. Itzykson and J. B. Zuber, *Quantum field theory*. Dover Publications, 2005.

[7] Y. Kosmann-Schwarzbach, *The Noether theorems*. Springer, 2011.

[8] H. R. Brown and P. Holland, *Dynamical versus variational symmetries: understanding Noether’s first theorem*, Molecular Physics **102** (2004) 1133–1139.

[9] P. J. Olver, *Applications of Lie groups to differential equations*, vol. 107. Springer, 2000.

[10] J. Rosen, *Noether’s theorem in classical field theory*, Ann. Phys. **69** (1972) 349–363.

[11] J. Schwinger, *The theory of quantized fields. I*, Phys. Rev. **82** (1951) 914.

[12] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* Princeton University Press, 1992.

[13] Y. Takahashi and H. Umezawa, *Relativistic quantization of fields*, Nuc. Phys. **51** (1964) 193–211.
[14] Z. Ezawa, Quantum soliton operators for vortices and superselection rules, Phys. Rev. D 18 (1978) 2091.
[15] A. Wightman, Superselection rules; old and new, Il Nuovo Cimento B 110 (1995) 751–769.
[16] Y. Shamir and S. H. Park, On topological symmetries and the Goldstone theorem, Phys. Lett. B 258 (1991) 179–182.
[17] J. J. Sakurai, Advanced quantum mechanics. Addison-Wesley, 1967.
[18] W. Greiner and J. Reinhardt, Field quantization. Springer, 1996.
[19] S. Deser, R. Jackiw, and S. Templeton, Topologically massive gauge theories, Ann. Phys. 140 (1982) 372–411.
[20] P. R. Halmos, Finite dimensional vector spaces. No. 7. Princeton University Press, 1947.
[21] J. R. Munkres, Topology: a first course, vol. 23. Prentice-Hall Englewood Cliffs, NJ, 1975.
[22] H. Umezawa, H. Matsumoto, and M. Tachiki, Thermo field dynamics and condensed states. North-Holland, 1982.
[23] J. Goldstone, A. Salam, and S. Weinberg, Broken symmetries, Phys. Rev. 127 (1962) 965–970.
[24] Y. Nambu, Quasi-particles and gauge invariance in the theory of superconductivity, Phys. Rev. 117 (1960) 648.
[25] N. Nakanishi and I. Ojima, Covariant operator formalism of gauge theories and quantum gravity. World Scientific, 1990.
[26] F. Strocchi, Spontaneous symmetry breaking in local gauge quantum field theory; the Higgs mechanism, Comm. Math. Phys. 56 (1977) 57–78.
[27] X.-L. Qi, T. L. Hughes, S. Raghu, and S.-C. Zhang, Time-reversal-invariant topological superconductors and superfluids in two and three dimensions, Phys. Rev. Lett. 102 (2009) 187001.
[28] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of Bose-Einstein condensation in trapped gases, Rev. Mod. Phys. 71 (1999) 463.
[29] T. Kinoshita, T. Wenger, and D. S. Weiss, Observation of a one-dimensional Tonks-Girardeau gas, Science 305 (2004) 1125–1128.
[30] H. D. Politzer, Light incident on a Bose-condensed gas, Phys. Rev. A 43 (1991) 6444–6446.
[31] M. Leblanc, R. MacKenzie, P. Panigrahi, and R. Ray, Induced b ∧ f term and photon mass generation in 3 + 1 dimensions, Int. J. Mod. Phys. A 9 (1994) 4717–4726.
[32] K. Rao, N. Sahu, and P. K. Panigrahi, Fermion number fractionization, Resonance 13 (2008) 738–751.
[33] A. Das, Integrable models, vol. 30. World Scientific, 1989.
[34] E. Kuznetsov, A. Rubenchik, and V. Zakharov, Soliton stability in plasmas and hydrodynamics, Phys. Rep. 142 (1986) 103–165.
[35] Y. S. Kivshar and B. Luther-Davies, *Dark optical solitons: physics and applications*, Phys. Rep. 298 (1998) 81–197.

[36] V. Zakharov and A. Shabat, *Interaction between solitons in a stable medium*, Sov. Phys. JETP 37 (1973) 823–828.

[37] A. M. Polyakov, *Quantum gravity in two dimensions*, Mod. Phys. Lett. A 2 (1987) 893–898.

[38] J.-M. Lina and P. K. Panigrahi, *Lax pair and cosmological constant for induced 2d-gravity in the light-cone gauge*, Mod. Phys. Lett. A 6 (1991) 3517–3524.

[39] D. Frantzeskakis, *Dark solitons in atomic Bose–Einstein condensates: from theory to experiments*, J. Phys. A 43 (2010) 213001.

[40] L. D. Carr, C. W. Clark, and W. P. Reinhardt, *Stationary solutions of the one-dimensional nonlinear Schrödinger equation. I. Case of repulsive nonlinearity*, Phys. Rev. A 62 (2000) 063610.

[41] V. L. Berezinskii, *Destruction of long-range order in one-dimensional and two-dimensional systems having a continuous symmetry group I. Classical systems*, Soviet JETP 32 (1971) 493.

[42] J. M. Kosterlitz and D. J. Thouless, *Ordering, metastability and phase transitions in two-dimensional systems*, J. Phys. C 6 (1973) 1181.

[43] A. D. Jackson and G. M. Kavoulakis, *Lieb mode in a quasi-one-dimensional Bose-Einstein condensate of atoms*, Phys. Rev. Lett. 89 (2002) 070403.

[44] V. M. Vyas, S. Gautam, and P. K. Panigrahi, *On Berezinskii-Kosterlitz-Thouless phase transition in quasi-one dimensional Bose-Einstein condensate*, arXiv:1106.0402.

[45] T. Raju, C. Kumar, and P. Panigrahi, *On exact solitary wave solutions of the nonlinear Schrödinger equation with a source*, J. Phys. A 38 (2005).

[46] V. M. Vyas, P. Patel, P. K. Panigrahi, C. N. Kumar, and W. Greiner, *Chirped chiral solitons in the nonlinear Schrödinger equation with self-steepening and self-frequency shift*, Phys. Rev. A 78 (2008) 021803.

[47] Alka, A. Goyal, R. Gupta, C. N. Kumar, and T. S. Raju, *Chirped femtosecond solitons and double-kink solitons in the cubic-quintic nonlinear Schrödinger equation with self-steepening and self-frequency shift*, Phys. Rev. A 84 (2011) 063830.

[48] J. Schwinger, *Gauge invariance and mass. II*, Phys. Rev. 128 (1962) 2425–2429.

[49] A. Das, *Field theory: a path integral approach*, vol. 52. World Scientific, 1993.

[50] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. No. 55. Courier Dover Publications, 1972.