Phase fluctuations in superconductors: from Galilean invariant to quantum XY models

L. Benfatto, A. Toschi, S. Caprara, and C. Castellani

Dipartimento di Fisica, Università di Roma “La Sapienza” and Istituto Nazionale per la Fisica della Materia, Unità di Roma 1, Piazzale Aldo Moro, 2, 00185 Roma, Italy

We analyze the corrections to the superfluid density due to phase fluctuations within both a continuum and a lattice model for s- and d-wave superconductors. We expand the phase-only action beyond the Gaussian level and compare our results with the quantum XY model both in the quantum and in the classical regime. We find new dynamic anharmonic vertices, absent in the quantum XY model, which are responsible for the vanishing of the correction to the superfluid density at zero temperature in a continuum (Galilean invariant) model. Moreover the phase-fluctuation effects are reduced with respect to the XY model by a factor at least of order $1/(k_F\xi_0)^2$.

The issue of explaining the linear temperature dependence of the superfluid density $\rho_s$ in cuprate superconductors renewed the interest on the effects of the phase fluctuations (PF) of the order parameter in the superconducting phase at low temperature. Indeed, both quasiparticles and PF in the classical limit have been suggested as alternative explanations for the observed depletion of $\rho_s$. Recently Refs. considered the quantum effects on PF and showed that the classical limit is reached at a temperature $T_d$ which is too high to account for the low-temperature linearity of $\rho_s$ in cuprates.

A crucial point in the analyses of PF effects is the choice of the effective model for the PF. In Refs. the Gaussian action for the phase $\theta$ was derived microscopically, while the anharmonic (non-Gaussian) terms were obtained by expanding the $\cos(\theta_i - \theta_j)$ coupling term of a lattice quantum XY model, derived by coarse-graining the Gaussian action on the scale of the coherence length $\xi_0$, in powers of $(\theta_i - \theta_j) \sim \xi_0|\nabla \theta|$. In Ref. instead, also the anharmonic terms were derived microscopically, within a d-wave continuum BCS model. In this approach the interaction vertices for $\nabla \theta$ are determined by the fermion loops and are smaller than the corresponding vertices of the XY model, leading to one-loop corrections to $\rho_s$ which are much smaller than those of the quantum XY model, both at $T = 0$ and at $T > 0$. However, they found a finite correction to $\rho_s$ even at $T = 0$, which is expected within a lattice XY model, but not within a continuum (Galilean invariant) model where $\rho_s$ equals the particle density $\rho$ at $T = 0$. This indicates that something is missing in their analysis. Moreover their description of PF effects within a continuum model is too restrictive, as it neglects lattice effects, which are certainly relevant in cuprates.

In this paper we give a detailed analysis of the one-loop correction to $\rho_s$ due to PF within both a continuum and a lattice model for s- or d-wave pairing. We specifically consider the weak- to intermediate-coupling regime, even though at the end we also comment on the strong-coupling limit. We find that the microscopic derivation of the phase-only action, besides the classical (static) anharmonic terms $(\nabla \theta)^4$ considered in Ref. introduces new third- and fourth-order quantum (dynamic) interaction terms which contain the time derivative of $\theta$. These quantum terms are absent in the quantum XY model where the dynamics only appears at the Gaussian level, and induce a correction to $\rho_s$ which cancels exactly, in the continuum case, the contribution due to the classical interaction, restoring the equality $\rho_s = \rho$ at $T = 0$. On the other hand, the same cancellation does not hold in the lattice case, in which $\rho_s$ equals the average kinetic energy at $T = 0$. The inclusion of both classical and quantum interaction terms leads to a finite one-loop correction to $\rho_s$, which is however of order $1/(k_F\xi_0)^2$ with respect to the result of the quantum XY model, $k_F$ being the Fermi wave vector. In the classical regime we find that the PF correction to $\rho_s$ is smaller than within the classical XY model by the factor $\sim 1/(k_F\xi_0)^2$, for both the continuum and lattice model. The reduction of the PF effects for $k_F\xi_0 \gg 1$ is made even more pronounced in the continuum case by the inclusion of long-range Coulomb forces. The fact that the XY model generically overestimates the PF effects further supports the claim of Refs., in which, even adopting the XY model, it has been shown that the contribution of PF does not account for the linear temperature decrease of $\rho_s$ in cuprates and gives temperature-dependent small corrections when compared to the experimental data.

We start with a continuum BCS action at a temperature $T = \beta^{-1}$ in $d$ dimensions, $S = \int^0_\beta d\tau L$, with

$$L = \int d^d x \sum_{\sigma} c_\sigma^+ \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) c_\sigma + H_I, \quad (1)$$

$H_I = -\frac{k_F}{\hbar} \sum_{\mathbf{k},\mathbf{k}' q} \left( \gamma_{\mathbf{k}} c_{\mathbf{k}}^+ \mathbf{c}_{\mathbf{k}+\mathbf{q}}^+ \mathbf{c}_{\mathbf{k}+\mathbf{q}'} c_{\mathbf{k}'+\mathbf{q}'} - \text{h.c.} \right)$

Here $\Omega$ is the volume, $U > 0$ is the pairing interaction strength and the factor $\gamma_{\mathbf{k}}$ controls the symmetry of the order parameter. In the following, unless explicitly indicated, we set $\hbar = k_B = 1$. We perform the standard Hubbard-Stratonovich decoupling of $H_I$ and make the dependence on the phase $\theta$ of the complex order parameter $\Delta = |\Delta| e^{i\theta}$ explicit by means of the gauge transformation $c_\sigma \to c_\sigma e^{i\theta/2}$. Then, after integrating out the fermions around the supercon-
ducting saddle-point solution, and neglecting the fluctuations of $|\Delta|$, we obtain the effective action for PF, $S_{eff}[\theta] = \text{Tr} \sum_{n=1}^{\infty} \frac{1}{n!} (SG_0)^n$. $G_0$ is the mean-field Nambu Green function, and the self-energy matrix is

$$\Sigma = \left[ \frac{\theta}{2} + \frac{(\nabla\theta)^2}{8m} \right] \tau_3 + \frac{i}{2m} (\nabla\theta \cdot \nabla) \tau_0,$$  

where $\tau_i$ are the Pauli matrices, and the operator $\nabla = (\nabla - \nabla)/2$ acts on $G_0$. In the following we distinguish the “bosonic” contributions, generated by the $\tau_3$ term in the analogous to those of a boson model in the presence of condensate, the “fermionic” contributions, generated by the $\tau_0$ term, and the “mixed” ones, obtained by combinations of $\tau_0$ and $\tau_3$ terms. Both bosonic and fermionic contributions contribute to the Gaussian phase action which, in the hydrodynamic limit, reduces to the well-known form

$$S_{eff}^G[\theta] = \frac{1}{8} \sum_q (\chi_0^2 + D q^2) \theta \theta - q,$$  

where $q = (q, i\omega_n)$, $\chi$ is the compressibility and $D(T) = \hbar^2 p_s/m$ is the superfluid stiffness (in dimensional units). At Gaussian level the temperature dependence of $D$ is entirely due to the quasiparticle excitations, giving $D(T) = \frac{\theta}{2} + \frac{(\nabla\theta)^2}{8m} \tau_3 + \frac{i}{2m} (\nabla\theta \cdot \nabla) \tau_0$, and $\tau_3$ term in Eq. (3) 

![Fig. 1](image.png)

FIG. 1. (a): Bosonic diagrams in the non-Gaussian phase-only action up to fourth order. The dashed line indicates a $\theta$ insertion, the solid line represents $G_0$. The vertex with a single incoming line corresponds to the insertion of the $(\theta/2)\tau_3$ term of Eq. (3), the vertex with two incoming lines to the insertion of $(\nabla\theta)/(8m)\tau_3$. (b): Bosonic one-loop corrections to the PF Gaussian action.

In the present context, to make comparison with Ref. [8], we do not consider the effect of a dissipative term in Eq. (3) for a $d$-wave superconductor, even though this term would be essential for the low-temperature behavior and for the estimate of the temperature $T_{c_1}$ [8]. We first consider a neutral system, while the long-range Coulomb forces will be considered later.

The one-loop PF corrections to $D(T = 0)$ are induced by the anharmonic terms in $\theta$. As we discussed above, within the quantum $XY$ model, the dynamics is introduced only at mean-field level adopting the PF Gaussian propagator $P(g) = 4(\chi_0^2 + D q^2)^{-1}$ (see Eq. (3)), and the one-loop correction to $D$, coming from the purely static interaction term $\propto \xi D \sum_{n=1}^{\infty} (\partial_\theta)^4$, is

$$\delta D_{XY} = - \frac{D_0^2}{2} \frac{T}{d\Omega} \sum_q q^2 P(q) \frac{\theta}{\theta} - q \frac{g}{\sqrt{\chi_0^2 + D_q^2}} \delta D,$$  

where $g = \zeta^{d+1}/d(d + 1)$, and $\zeta \simeq 1/\xi_0$ is the PF momentum cutoff [8].

By contrast, we derive here the anharmonic terms by expanding $S_{eff}$ up to fourth order in $\theta$ and evaluate the one-loop corrections to the Gaussian action. All the fermionic and mixed terms generate one-loop corrections to $D$ which vanish or cancel each other at $T = 0$, whatever is the symmetry of the gap, in agreement with the result previously discussed in Ref. [5], where however the contribution of the bosonic diagrams was incorrectly evaluated. Indeed, the bosonic $\theta$ term in Eq. (3) introduces a dynamic contribution in $S_{eff}$ also beyond the Gaussian level, as it is represented by the third- and fourth-order bosonic vertices depicted in Fig. 1a. In particular, the presence of a third-order quantum interaction vertex, neglected in Ref. [5], leads to separate cancellations between both fermionic or mixed and bosonic contributions. Let us consider the one-loop diagrams for the bosonic corrections $\delta S^B$ in Fig. 1b: the first two are the self-energy corrections to $G_0$, which shift the particle density $\rho$ in the presence of PF, at fixed $\mu$. Alternatively, at fixed density, we take these contributions into account by shifting $\mu$ with respect to its mean-field value, in order to keep $\rho$ fixed. As a consequence, the only corrections to $D$ with respect to $\rho/m$ due to PF come from the last three diagrams in Fig. 1b, i.e.

$$\delta D = - \frac{1}{8m^2 d^2 \Omega} \frac{T}{d\Omega} \sum_q q^2 P(q) \chi(q) \left[ 2 - \omega_0^2 P(q) \chi(q) \right],$$

where $\chi(q)$ is the density-density bubble, which gives the compressibility $\chi$ in the limit $\omega_n = 0, q \to 0$. Notice that in writing Eq. (5) we are relying on the fact that both the third- and fourth-order vertices needed to calculate $\delta D$ are expressed in terms of $\chi(q)$. This is the relevant consequence of the Galilean-invariant form of the bosonic $\tau_3$ term appearing in the self-energy [5,6,8]. Evaluating Eq. (5) in the hydrodynamic limit, we get

$$\delta D = \frac{1}{dm^2 \Omega} \sum_q q^2 b'(\varepsilon_q) \frac{\theta}{\theta} - q 0,$$  

where $\varepsilon_q = \sqrt{D/\chi} |q|$ is the sound mode, and $b'(x) = -\beta e^{i x}/(e^{i x} - 1)^2$. The origin of this result is made clear by considering the analytic continuation of Eq. (5).
to real frequencies, and summing the two terms. The pole at \( \omega = -\varepsilon_q < 0 \) (responsible for a finite contribution at \( T = 0 \)) is cancelled in favor of a double pole at \( \omega = \varepsilon_q > 0 \), leading to Eq. (3), where the standard Bogoliubov reduction of \( \rho_s \) in a superfluid bosonic system is recognized. Thus, by fully including the dynamic structure of the interaction for the phase, we obtain that in a Galilean-invariant system \( \rho_s = \rho \) at \( T = 0 \).

A further consequence of the microscopic derivation of the effective action is a reduction of the strength \((\sim \chi)\) of the static interaction term \((\nabla \theta)^4\) with respect to the classical \(XY\) regime for \(PF\), only the interaction term \(\delta D\) survives, leading to \(\delta D \approx -T \chi/m^2 D \xi_0^4\), qualitatively similar to the result of the classical \(XY\) model, \(\delta D_{XY} \approx -2T/d^2 \xi_0^{-2}\). Since, however, \(\chi/m^2 D \approx 1/k_F^2\), we find that

\[
\frac{\delta D}{\delta D_{XY}} \approx (k_F\xi_0)^{-2},
\]

i.e., in the classical limit \(\delta D\) is smaller than within the \(XY\) model, as far as \(k_F^{-1} < \xi_0\).

Let us now consider the effect of the Coulomb interaction between the electrons. At Gaussian level the density-density bubble \(\chi(q)\) is dressed by the random-phase series of the Coulomb potential \(V(q) = \lambda e^2/|q|d^{-1}\) (here \(\lambda\) is a constant depending on the dimension \(d\) and on the dielectric constant \(\varepsilon_\infty\)), and the sound mode \(\varepsilon_q\) of Eq. (3) is converted into the plasma mode \(\omega_q\) of the \(d\)-dimensional system. In deriving the anharmonic terms in \(S_{eff}\), we must now include the CPA density fluctuations in all the vertices. The one-loop corrections to \(D\) are formally identical to Eqs. (3)-\(4\), with \(\chi \rightarrow \chi_{LR}\). Thus at \(T = 0\) we recover again the cancellations of the bosonic diagrams, with \(\varepsilon_q \rightarrow \omega_q\). At the same time, since \(\chi_{LR}(q,0) \approx 1/V(q)\) vanishes as \(q \rightarrow 0\), the classical \(\omega_q = 0\) term in Eq. (3) gives \(\delta D_{LR} \approx -T/d(2d-1)M^2 e^2 \xi_0^{d-1}\), whereas the quantum \(XY\) model leads to the same result as the neutral case. Thus we estimate

\[
\frac{\delta D_{LR}}{\delta D_{XY}} \approx (E_F/E_C) (k_F\xi_0)^{-(d+1)},
\]

where \(E_F\) is the Fermi energy and \(E_C = \lambda k_F e^2\) is a characteristic Coulomb energy. While within the \(XY\) model the Coulomb interaction modifies only the low-temperature behavior of \(\rho_s\), within the continuum model it affects also the high-temperature classical regime.

To extend the previous results to a lattice model, we rewrite Eq. (3) introducing a hopping \(t\) between nearest-neighbor sites on a cubic lattice of spacing \(a\), so that the free-electron dispersion is \(\xi_k = -2t \sum_{q=1}^d \cos k_a a - \mu\), and we obtain the generalization of Eq. (3) to the lattice case. Since we find that the cancellation of fermionic and mixed one-loop corrections to \(D\) at \(T = 0\) still holds, we focus on bosonic corrections only. Differently from the continuum case, each insertion of a spatial derivative of \(\theta\) in the fermionic loops is associated to a same-order \(k\)-derivative of \(\xi_k\). Therefore, the \((\nabla \theta)^2\) term in Eq. (3) carries a factor \(\frac{1}{\pi} \xi_0 = \frac{1}{\pi} \frac{\partial^2 \xi_k}{\partial q^2}\). At Gaussian level, then, \(D(T = 0) = \frac{1}{\pi} \sum_{k,\alpha} \Lambda_k (1 - \xi_k/E_q)\), which, in the nearest-neighbor cubic model, equals the average kinetic energy, rather than \(\rho/m\). In the anharmonic action of Fig 1a the factor \(\Lambda_k\) appears in each vertex with two incoming \(\theta\)-lines: as a consequence, the first two diagrams of Fig. 1b, which give corrections to \(\rho\) in the continuum case, are now corrections to \(D\), which are different from (and therefore do not cancel with) those coming from the shift of \(\mu\), at fixed \(\rho\). Moreover, on the lattice, the coupling of the fermions to \(\theta\) generates higher than second-order derivatives of \(\theta\) in Eq. (3): in addition to the diagrams depicted in Fig. 1b we must include also a diagram \(\mathcal{T}\), with four derivatives of \(\theta\) incoming in the same vertex, and a factor \(\partial^2 \xi_k/\partial q^2\), proportional to the average kinetic energy. The resulting bosonic one-loop correction to \(D\) is

\[
\frac{\delta D}{\delta D} = -\frac{1}{8d} \sum_{q} P(q) \left\{ q^2 \chi^{EE}(q) + \chi^{EE} + (d-1) \chi^{EE} + \mathcal{T} - d\chi^{-1}\chi^{EE} \right. \\
- \omega_n^2 [\chi^{EE} q^2 P(q) - 2d\eta_{EE}(q) + d\chi^{-1}\chi^{EE} \eta(q)] \left. \right\}. \tag{9}
\]

The \(\chi_{ab}\) bubbles correspond to the insertion of one \((\chi^{EE})\) or two equal/different \((\chi^{EE}/\chi^{EE})\) factors \(\Lambda_\alpha; \eta_{EE} \) and \(\eta\) are the bubbles with three \(G_0\) lines and one or no factor \(\Lambda_\alpha\), respectively. \(\eta_{EE}\) corresponds to the first diagram of Fig. 1b. The first terms in the two square brackets are the lattice analogous of the two terms of the continuum case, Eq. (3), whereas the last contributions come from the shift of \(\mu\) to keep \(\rho\) fixed.

We perform the \(\omega_n\) sum in Eq. (9) with the PF Gaussian propagator \(P(q)\), and retain the leading order in \(\zeta \sim 1/\xi_0\), calculating the \(\chi_{ab}\) bubbles at zero incoming momentum, and carefully evaluating the small \(q\) limit for \(\eta_{EE}\) and \(\eta\). We thus obtain an overall correction to \(D\) which is \(finite\) at \(T = 0\) in the lattice case,

\[
\frac{\delta D}{\delta D} = -\frac{q}{4\sqrt{\chi D}} \left[ 3\chi^{EE} + (d-1)\chi^{EE} + \mathcal{T} - 2\chi^{-1}\chi^{EE} \right. \\
- \omega_n^2 \left[ \eta_{EE} - \chi^{-1}\chi^{EE} \eta \right]. \tag{10}
\]

At \(T = 0\), \(\chi = \frac{1}{\sqrt{\Lambda}} \sum_{k} N_k\), with \(N_k = \Delta k^2/E_k^2\); \(\chi_{ab} = \frac{1}{\sqrt{\Lambda}} \sum_{k} \Gamma_{ab} N_k\), with \(\Gamma_{ab} = \Lambda_\alpha, \Gamma^{EE} = \Lambda_\alpha^2, \Gamma^{EE} = \Lambda_\alpha^2 \Lambda_\beta (\alpha \neq \beta)\); \(\eta_{EE} = \frac{1}{\sqrt{\Lambda}} \sum_{k} M_k\), with \(M_k = \Delta k^2/E_k^2\); \(\eta = \frac{1}{\sqrt{\Lambda}} \sum_{k} M_k\); \(\mathcal{T} = a^2 \mathcal{D}(T = 0)\). In the limit \(a \rightarrow \infty\), keeping \(2a^2t = 1/m\) finite, \(\Lambda_\alpha \rightarrow 1/m, \chi^{EE}; \chi^{EE} \rightarrow \chi/m^2, \eta_{EE} \rightarrow \eta/m\), while \(\mathcal{T} = a^2 \mathcal{D} \rightarrow 0\). Thus, Eq. (10) recovers the continuum (Galilean-invariant) result \(\delta D = 0\).
We next turn again to the issue of the comparison between $\delta D$ and $\delta D_{XY}$ in the lattice case. At $T = 0$ Eq. (1) leads to an estimate of $\delta D/\delta D_{XY}$ of the same order of Eq. (3), within a numerical factor. This is also true in the classical limit, both for the neutral and the charged system. As we discussed above, at high temperature only the static correction to $D$ contributes, with a coefficient controlled in the neutral case by $\chi_{EE}, \chi_{EE}$ and $\mathcal{T}$, which plays the same role (and has the same estimate) of $\chi/m^2$ in the continuum case. The presence of Coulomb forces does not change qualitatively this conclusion, and introduces only minor quantitative corrections, since the RPA expressions for $\chi_{EE}, \chi_{EE}$ have a finite limit for $q \to 0$, contrary to the continuum case, e.g., $\chi_{EE}^{LR} = \chi_{EE} - \chi_{EE}^2 V(q)/[1 + V(q)] \approx (\chi_{EE} - \chi_{EE}^2)/\chi + \chi_{EE}^2 q^{d-1}/\chi e^2$. As a consequence, Eq. (3) gives a proper estimate in the lattice case, even in the presence of long-range interactions.

According to Eq. (7) the PF-induced depletion of $\rho_a$ can be quite small at weak and intermediate coupling (particularly in the BCS limit) both in the quantum and the classical regime, while it is of the order predicted by the $XY$ model in the strong-coupling limit, $k_F \xi \Theta \approx 1$. In this case the corrections to $\rho_a$ due to PF are sizeable (even though less important than the quasiparticle contribution in determining the low-temperature dependence (3)). The evaluation of $\chi_{ab, \eta_{EE}}$ for $t/U \ll 1$ would lead to the conclusion that all contributions coming from the dynamic vertices are subleading with respect to those which arise from the static interaction $\sum_{a=1}^{d} (\partial_a \theta)^4$. As a consequence, at strong coupling only the static interaction survives, analogously to what is assumed in the quantum $XY$ model. Moreover, in the case of $s$-wave pairing, the value of the coefficient of $\sum_{a=1}^{d} (\partial_a \theta)^4$ is exactly the same of the $XY$ model (4), with $\xi_0$ substituted by the lattice spacing $a$, in agreement with the strong-coupling expectation. For $d$-wave pairing the coefficient of the static interaction changes only by a numerical factor, in units of $\alpha$, with respect to the $XY$ model (4).

However the situation is more involved, specifically for the $s$-wave pairing symmetry. Let us consider the negative-$U$ Hubbard model. In order to analyze the strong-coupling limit we need to include (i) the RPA fluctuations, also induced by $U$, in the particle-hole channel, and (ii) the fluctuations of $|\Delta| \sim (U/2) \sqrt{\rho(2 - \rho)}$, which fluctuates because $\rho$ fluctuates (8). When this analysis is carried out the contributions from the dynamical vertices are not subleading, and those from the static vertices do not reproduce the $XY$ result by themselves. Nevertheless, at $T = 0$ the inclusion of both dynamic and static corrections to $D$ leads again to $\delta D = \delta D_{XY}$, with $\xi_0$ substituted by $a$. This result holds also in the classical regime, even though only approximately and provided the particle density is not too small (15).

Acknowledgments: We thank S. De Palo, C. Di Castro, M. Grilli, and A. Paramekanti, for many useful discussions and suggestions.

[1] W. N. Hardy, D. A. Bonn, D. C. Morgan, R. Liang, and K. Zhang, Phys. Rev. Lett. 70, 3999 (1993); C. Panagopoulos and T. Xiang, Phys. Rev. Lett. 81, 2336 (1998).
[2] P. A. Lee and X. G. Wen, Phys. Rev. Lett. 78, 4111 (1997); A. J. Millis, S. M. Girvin, L. B. Ioffe, and A. I. Larkin, J. Phys. Chem. Solids 59, 1742 (1998); J. Mesot, M. R. Norman, H. Ding, M. Randeria, J. C. Campuzano, A. Paramekanti, H. T. Fretwell, A. Kaminski, T. Takeuchi, T. Yokoya, T. Sato, T. Takahashi, T. Mochiku, and K. Kadowaki, Phys. Rev. Lett. 83, 840 (1999).
[3] E. Roddick and D. Stroud, Phys. Rev. Lett. 74, 1430 (1995); V.J. Emery and S.A. Kivelson, Phys. Rev. Lett. 74, 3253 (1995); E.W. Carlson, S. A. Kivelson, V. J. Emery, and E. Manousakis, Phys. Rev. Lett. 83, 612 (1999).
[4] A. Paramekanti, M. Randeria, T. V. Ramakrishnan, and S. S. Mandal, Phys. Rev. B 62, 6786 (2000).
[5] L. Benfatto, S. Caprara, C. Castellani, A. Paramekanti, and M. Randeria, Phys. Rev. B 63, 174513 (2001).
[6] H. J. Kwon, A. T. Dorsey, and P. J. Hirschfeld, Phys. Rev. Lett. 86, 3875 (2001).
[7] For the analogy with the functional treatment of superfluid systems, see V. N. Popov, Functional Integrals in Quantum Field Theory and Statistical Physics, Reidel, Dordrecht (1983).
[8] See S. De Palo, C. Castellani, C. Di Castro, and B. K. Chakravarty, Phys. Rev. B 60, 564 (1999), and references therein.
[9] S. G. Sharapov, H. Beck, and V. M. Loktev, cond-mat/0012511 (unpublished).
[10] At low $T$, $|\Delta|$ couples to $\theta$ via its coupling to $\rho$. This coupling, which is irrelevant in the BCS limit, can be considered by including the fluctuations of $|\Delta|$ in the density bubbles.
[11] Spatial variations of $\theta(x)$ at distances smaller than $\xi_0$ must not be included. $\xi_0$ is well defined both in $s$- and $d$-wave superconductors through the spatial decay of the correlation function for $|\Delta|$.
[12] Ian J. R. Aitchison, Ping Ao, D. J. Thouless, and X. M. Zhu, Phys. Rev. B 51, 6531 (1995).
[13] The details of calculations will appear in L. Benfatto, et al., in preparation.
[14] At strong coupling in the $s$-wave case $\chi_{EE}$ and $D$ give the same contribution, while $\chi_{EE}$ is subleading. In the $d$-wave case $\chi_{EE}$ is not subleading, even if numerically less important than $\chi_{EE}$.
[15] The discussion of this issue (and of the relation between $\xi_0$ and $a$ at small density) is beyond the scope of the present paper, and will be reported elsewhere.