Link Invariants, Holonomy Algebras and Functional Integration

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December 18, 1992

Abstract

Given a principal $G$-bundle over a smooth manifold $M$, with $G$ a compact Lie group, and given a finite-dimensional unitary representation $\rho$ of $G$, one may define an algebra of functions on $A/G$, the “holonomy Banach algebra” $H_b$, by completing an algebra generated by regularized Wilson loops. Elements of the dual $H^*_b$ may be regarded as a substitute for measures on $A/G$. There is a natural linear map from $\text{Diff}_0(M)$-invariant elements of $H^*_b$ to the space of complex-valued ambient isotopy invariants of framed oriented links in $M$. Moreover, this map is one-to-one. Similar results hold for a C*-algebraic analog, the “holonomy C*-algebra.”

1 Introduction

Our goal in this paper is to provide a rigorous expression of some ideas from physics concerning the relation of link invariants to diffeomorphism-invariant states of gauge field theories. In quantum field theory it is common to work formally with “measures” on the space of field configurations, ignoring the difficulty of describing these measures in mathematically rigorous terms. The simplest example arises in the theory of free boson fields. Here one treats “measures” of the form

$$e^{Q(x,x)}dx$$

where $Q$ is a complex-valued symmetric bilinear form on the real vector space $V$. When $V$ is finite-dimensional and $dx$ denotes Lebesgue measure, $e^{Q(x,x)}dx$ is a Borel measure on $V$. When $V$ is infinite-dimensional, there is no Lebesgue measure on $V$, but one may still seek some reasonable way to make sense of the expression

$$\int_V f(x)e^{Q(x,x)}dx,$$

at least for some class of complex-valued functions $f$ on $V$.

One approach is as follows. Let $\mathcal{E}$, the exponential algebra, be the algebra of functions on $V$ generated by those of the form

$$f(x) = e^{Q(v,x)}$$
for $v \in V$. Note that if $V$ is finite-dimensional and $\text{Re} Q(x, x) < 0$ for all nonzero $x \in V$ we have

$$\frac{\int_V e^{Q(v, x)} e^{Q(x, x)} \, dx}{\int_V e^{Q(x, x)} \, dx} = e^{Q(v, v)/4}$$

Generalizing from this fact, for an arbitrary vector space $V$ and quadratic form $Q$ one may simply define a complex-linear functional $E: \mathcal{E} \to \mathbb{C}$, often called the “expectation value,” such that

$$E(e^{Q(v, \cdot)}) = e^{Q(v, v)/4}.$$ 

The case where $Q$ is real-valued and negative definite has been widely used in constructive quantum field theory \[5\]. In this case the relation of the expectation value to conventional integration theory is more than a merely formal one. Since

$$E(1) = 1,$$

$$f \geq 0 \implies E(f) \geq 0,$$

and

$$f \geq 0 \implies |E(fg)| \leq \|g\|_{\infty} E(f),$$

a theorem of Segal says that $\mathcal{E}$ may be embedded as a dense subspace of $L^1(X)$ for a probability measure space $X$ (unique up to a suitable equivalence relation) on which $E$ is represented as integration \[19\]. Moreover, one can complete $\mathcal{E}$ in the $L^\infty$ norm to obtain a C*-algebra $\mathcal{E}_c$, and the expectation value extends uniquely to a continuous linear functional $E: \mathcal{E}_c \to \mathbb{C}$.

On the other hand, Feynman integrals, in which $Q$ is imaginary, arise frequently in practical calculations in quantum field theory. The question of extending the expectation value $E: \mathcal{E} \to \mathbb{C}$ to a larger algebra of functions on $V$ been studied extensively in this case \[1, 9, 13\]. In this case, $E: \mathcal{E} \to \mathbb{C}$ does not extend to a continuous linear functional on the completion of $\mathcal{E}$ in the $L^\infty$ norm, as is easily checked in the case when $V = \mathbb{R}$ and $Q(v, v) = iv^2$, where the $L^\infty$ completion of $\mathcal{E}$ is the space of almost periodic functions. Since $L^\infty$ norm is the unique C*-norm on $\mathcal{E}$, it follows that $\mathcal{E}$ admits no C*-algebra completion to which $E$ extends to a continuous linear functional. One can, however, find a Banach algebra completion of $\mathcal{E}$ to which $E$ extends continuously. For example, one can complete $\mathcal{E}$ in the norm given by

$$\| \sum_i c_i e^{Q(v_i, x)} \| = \sum_i |c_i|$$

if the points $v_i$ are distinct.

In this paper we consider an extension of this idea from the linear context of the free boson field to nonabelian gauge field theories. An interesting example of a diffeomorphism-invariant gauge field theory is Chern-Simons theory. Given a compact
Lie group $G$ and a trivial principal $G$-bundle $P \to M$, the Chern-Simons functional of a connection $A$ on $P$ is given by

$$S(A) = \frac{k}{4\pi} \int_M \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

where the level $k \geq 0$ is an integer. The quantity $S(A)$ changes by an integer multiple of $2\pi$ under gauge transformations of $A$, so one may formally define a “measure”

$$e^{iS(A)} DA$$
on the space $\mathcal{A}/G$ of smooth connections modulo gauge transformations, where $DA$ is the (purely heuristic) “Lebesgue measure” on $\mathcal{A}/G$. Since $S(A)$ is invariant under the action under the group $\text{Diff}_0(M)$ of diffeomorphisms of $M$ that are connected to the identity, at a formal level we expect the Chern-Simons “measure” to be $\text{Diff}_0(M)$-invariant.

The notion of the holonomy around a loop provides an interesting relation between link invariants and such diffeomorphism-invariant “measures.” Given an oriented piecewise smooth loop $\gamma : S^1 \to M$, let $T(\gamma, A)$ denote the trace of the holonomy of the connection $A$ around $\gamma$, relative to some finite-dimensional unitary representation of $G$. The function $T(\gamma) = T(\gamma, \cdot)$ is called a Wilson loop in physics. (For an extensive review of Wilson loops see [14] and the references therein.) Since $T(\gamma, A)$ does not change when we apply a gauge transformation to $A$, we may regard $T(\gamma)$ as a function on $\mathcal{A}/G$.

Ashtekar and Isham [3] define the holonomy algebra to be the algebra of functions on $\mathcal{A}/G$ generated by piecewise smooth Wilson loops, and form the holonomy C*-algebra as the completion of the holonomy algebra in the $L^\infty$ norm. One may work with continuous linear functionals on the holonomy C*-algebra as a rigorous version of “measures” on $\mathcal{A}/G$. In particular, suppose that $L$ is an oriented link in $M$ having components $(\gamma_1, \ldots, \gamma_p)$, where the $\gamma_i$ are embedded circles. Then if $E$ is a continuous linear functional on the holonomy C*-algebra, defining

$$\mathcal{L}(L) = E(T(\gamma_1) \cdots T(\gamma_p)),$$

it is easy to see that if $E$ is invariant under the action of $\text{Diff}_0(M)$, $\mathcal{L}(L)$ is an ambient isotopy invariant of the link $L$.

However, the calculations of Polyakov [16] and Witten [21] in Chern-Simons theory show that making sense of the Chern-Simons “measure”

$$e^{iS(A)} DA$$

involves further subtleties. Polyakov gave a heuristic argument to show that in the case of the defining representation of $G = U(1)$, given disjoint embedded oriented circles $\gamma_1, \gamma_2$ in $S^3$, one should be able to calculate

$$\int_{\mathcal{A}/G} T(\gamma_1, A) T(\gamma_2, A) e^{iS(A)} DA.$$
in terms of the linking number of $\gamma_1$ and $\gamma_2$. The argument breaks down when $\gamma_1$ and $\gamma_2$ intersect, since one obtains a divergent integral. If $\gamma$ is an embedded circle equipped with a framing (that is, a section of the tangent bundle of $S^3$ over $\gamma$ that is everywhere linearly independent from the tangent vector of $\gamma$), one could define

$$\int_{A/\mathcal{G}} T(\gamma, A)^2 e^{iS(A)} dA$$

to be the “self-linking number” of $\gamma$, which is an invariant of oriented framed links. However, since the self-linking number involves the framing of $\gamma$, while the integral above appears not to, this may appear rather ad hoc.

A similar problem occurs in $SU(2)$ Chern-Simons theory. Using techniques from conformal field theory Witten argued that in the case of the defining representation of $G = SU(2)$, one should be able to calculate

$$\int_{A/\mathcal{G}} T(\gamma_1, A) \cdots T(\gamma_p, A) e^{iS(A)} dA$$

in terms of the Kauffman bracket of the link in $S^3$ with components $\gamma_i$. Again, though, the Kauffman polynomial is an invariant of framed links.

In this paper we describe a modified holonomy C*-algebra, and also a holonomy Banach algebra, such that diffeomorphism-invariant continuous linear functionals on these algebras define invariants of framed links. Our holonomy algebras are completions of the algebra generated by a kind of regularized Wilson loop, or “tube.” (Similar regularizations appear in the physics literature; see [2] and the references therein.) This clarifies the general relation between framing and regularization in diffeomorphism-invariant gauge theories - a relation that is already familiar in Chern-Simons theory [4, 8, 10, 11].

Moreover, we show that diffeomorphism-invariant continuous linear functionals on our holonomy algebras are classified by the framed link invariants they define. That is, the map from such functionals to framed link invariants is one-to-one. (See Theorems 2 and 4.) One reason for working with Banach and C*-algebras is to obtain this result, which depends on approximating arbitrary products of tubes by products of nonintersecting tubes. This result is a step towards finding a useful inverse of the “loop transform” for diffeomorphism-invariant gauge field theories including general relativity [14, 18].

I would like to thank Michel Lapidus, Geoffrey Mess, Richard Palais, Vladimir Pestov, Michael Renardy, and Jim Stafney for their help.

2 The Holonomy Algebra and Tube Algebra

In all that follows, let $G$ be a compact Lie group and $\rho$ a $k$-dimensional unitary representation of $G$. Let $M$ be a $n$-manifold and $P \to M$ a principal $G$-bundle over
M. Define $\tau: G \to \mathbb{C}$ by

$$\tau(g) = \frac{1}{k} \text{tr}(\rho(g)).$$

Given a smooth connection $A$ on $P$ and a smooth loop $\gamma: S^1 \to M$, let $T(\gamma, A)$ denote the trace of the holonomy of $A$ around the loop $\gamma$, computed using the trace $\tau$. These “Wilson loops” are of special interest because they are a convenient set of gauge-invariant functions on the space of connections.

Let us write $(t, x)$ for a point in $S^1 \times D^{n-1}$. Given a point $x \in D^{n-1}$ and a map $\gamma: S^1 \times D^{n-1} \to M$, we write $\gamma^x$ for the loop in $M$ given by

$$\gamma^x(t) = \gamma(t, x).$$

We define a tube to be a pair $(\gamma, \omega)$ consisting of an embedding $\gamma: S^1 \times D^{n-1} \to M$ and a smooth complex-valued $(n-1)$-form $\omega$ compactly supported in the interior of $D^{n-1}$. We will often use $\Gamma$ to denote a tube. Given a tube $\Gamma = (\gamma, \omega)$ and a smooth connection $A$ on $P$, we define

$$T(\Gamma, A) = \int_{D^{n-1}} T(\gamma^x, A) \omega(x).$$

Let $\mathcal{A}$ denote the space of smooth connections on $P$. $\mathcal{A}$ is an affine space with a natural topology, the $C^\infty$ topology; choosing any point in $\mathcal{A}$ as an origin allows us to identify $\mathcal{A}$ with a locally convex topological vector space. This enables us to define the algebra $C^\infty(\mathcal{A})$ of smooth complex functions on $\mathcal{A}$. We define the functions $T(\gamma), T(\Gamma) \in C^\infty(\mathcal{A})$ for a loop $\gamma$ and a tube $\Gamma$ by:

$$T(\gamma)(A) = T(\gamma, A), \quad T(\Gamma)(A) = T(\Gamma, A).$$

Let $\mathbf{H}_0$ denote the unital subalgebra of $C^\infty(\mathcal{A})$ generated by the functions $T(\Gamma)$ for all tubes $\Gamma$. We wish to construct a Banach algebra completion of $\mathbf{H}_0$ in as strong a norm as possible, so that as many linear functionals on $\mathbf{H}_0$ as possible extend to continuous linear functionals on the completion. One might be tempted to give $\mathbf{H}_0$ a norm by setting

$$\|f\| = \sup_F \|F(f)\|$$

with supremum is taken over all homomorphisms $F$ of $\mathbf{H}_0$ into a Banach algebra. The problem is that the supremum might not be finite. So instead we take the supremum over a smaller class of homomorphisms, defined using a space spanned by equivalence classes of tubes, the so-called “tube space,” which we now introduce.

Let $V$ denote the complex vector space having the set of all tubes as a basis. Let $V_1 \subset V$ denote the subspace spanned of finite linear combinations

$$\sum_i c_i(\gamma, \omega_i)$$
such that
\[ \sum_i c_i \omega_i = 0. \]

Let \( V_2 \subset V \) denote the subspace spanned by elements of the form
\[(\gamma_1, \omega_1) - (\gamma_2, \omega_2)\]
where \((\gamma_1, \omega_1)\) and \((\gamma_2, \omega_2)\) are *essentially equivalent*, that is, there are embedded discs \( D_1, D_2 \subseteq D^{n-1} \) with \( \text{supp} \omega_i \subseteq D_i \), and a diffeomorphism \( \alpha: D_1 \to D_2 \) such that
\[ \omega_1 = \alpha^* \omega_2 \]
and such that for all \( x \in D_1 \), \( \gamma_1^x \) is equal to \( \gamma_2^{\alpha(x)} \) up to an orientation-preserving reparametrization of \( S^1 \).

Define the vector space \( T_0 \) by
\[ T_0 = V/(V_1 + V_2). \]

We give \( T_0 \) a norm as follows:
\[ \|v\| = \inf \left\{ \sum_i \|c_i \omega_i\| : v = \sum_i c_i [(\gamma_i, \omega_i)] \right\}. \]

Here we use an \( L^1 \)-type norm on \((n-1)\)-forms on \( D^{n-1} \) given by
\[ \|\omega\| = \int_{D^{n-1}} |f(x)| \, dx \]
where \( dx \) is the standard volume form on \( D^{n-1} \) and \( F \) is the function such that \( \omega = f \, dx \). Let the *tube space* \( \mathcal{T} \) denote the Banach space completion of \( T_0 \).

There is a linear map \( T: V \to H_0 \) given by
\[ T(\Gamma)(A) = T(\Gamma, A). \]

Moreover:

**Lemma 1.** *The map* \( T: V \to H_0 \) *vanishes on* \( V_1 + V_2 \).

Proof - Note that if \( \sum c_i (\gamma_i, \omega_i) \in V_1 \), then
\[ T(\sum c_i (\gamma_i, \omega_i))(A) = \sum \int_{D^2} T(\gamma_i^x, A) c_i \omega_i(x) = 0, \]
so \( T \) vanishes on \( V_1 \). Also, if \( \Gamma_1 = (\gamma_1, f_1) \) and \( \Gamma_2 = (\gamma_2, f_2) \) are essentially equivalent tubes, with \( D_1, D_2, \alpha \) as above, then
\[ T(\Gamma_1)(A) = \int_{D_1} T(\gamma_1^x, A) \omega_1(x) \]
\[
\begin{align*}
&= \int_{D_1} T(\gamma^{\alpha(x)}_2, A)(\alpha^* \omega_2)(x) \\
&= \int_{D_2} T(\gamma^x_2, A)f_2(x) \\
&= T(\Gamma_2, A)
\end{align*}
\]

Thus \( T \) vanishes on \( V_2 \).

It follows that \( T \) factors through \( T_0 \); that is, we may define a linear map, which we also call \( T \), from \( T_0 \) to \( C^\infty(A) \) by

\[
T(\Gamma)(A) = T(\Gamma)(A).
\]

Henceforth, given a tube \( \Gamma = (\gamma, \omega) \), we will often abuse notation and write simply \( \Gamma \) or \((\gamma, \omega)\) for its equivalence class in \( T_0 \).

Now we return to the problem of putting a norm on \( H_0 \). Let \( \Lambda \) denote the class of all homomorphisms \( F: H_0 \to A \), where \( A \) is a Banach algebra, such that

\[
\|F(T(v))\| \leq \|v\|
\]

for all \( v \in T_0 \). Give \( H_0 \) a norm as follows:

\[
\|f\| = \sup_{F \in \Lambda} \|F(f)\|.
\]

Note that \( \|f\| \) is finite for all \( f \in H_0 \). Also, note that \( \iota \in \Lambda \), where \( \iota \) is the inclusion of \( H_0 \) in the C*-algebra of bounded continuous functions on \( A \), and that \( \iota(f) \neq 0 \) for all nonzero \( f \in H_0 \). Thus \( \|f\| \) is nonzero for all nonzero \( f \in H_0 \), so \( \| \cdot \| \) is indeed a norm on \( H_0 \). The completion of \( H_0 \) in this norm is a Banach algebra, the holonomy Banach algebra, \( H_b \), associated to the bundle \( P \) and the representation \( \rho \).

We will always use \( \| \cdot \| \) to denote the above norm on \( H_0 \), and use \( \| \cdot \|_\infty \) to denote the \( L^\infty \) norm on \( H_0 \) regarded as a subalgebra of \( B(A) \). By the above,

\[
\|f\|_\infty \leq \|f\|
\]

for all \( f \in H_0 \), so \( H_b \) may be regarded as a subalgebra of the bounded continuous functions on \( A \). Thus \( H_b \) is a commutative semisimple Banach algebra with unit. In Section 5, we will discuss a holonomy C*-algebra, \( H_c \). This is just the completion of \( H_0 \) in the \( L^\infty \) norm. The diffeomorphism-invariant continuous linear functionals on both \( H_b \) and \( H_c \) are classified by invariants of framed links in \( M \). Each algebra has its own technical advantages. From the viewpoint of mathematical physics, it is preferable to work with a C*-algebra as an “algebra of observables” for gauge field theories, since C*-algebras have a well-understood representation theory that is closely tied to Hilbert space theory. On the other hand, the Banach algebra approach is in a sense more conservative: \( H_b \) is a dense subalgebra of \( H_c \), so every continuous linear functional on \( H_c \) defines one on \( H_b \).
To relate the holonomy Banach algebra to link invariants it is convenient to introduce an auxiliary object, the “tube Banach algebra,” depending only on the base manifold $M$. First, note that by definition of the norm on $H_b$,

$$\|Tv\| \leq \|v\|$$

for all $v \in T_0$. It follows that $T$ extends uniquely to a continuous linear map from $T$ to $H_b$, which will also call $T$.

**Question 1.** What is the kernel of the map $T: T_0 \to H_0$?

**Question 2.** What is the kernel of the map $T: T \to H_b$?

The map $T: T \to H_b$ extends uniquely to an algebra homomorphism, which we again call $T$, from the symmetric algebra $ST$ to $H_b$. We now construct the “tube Banach algebra” and further extend $T$ to a homomorphism from this algebra to $H_b$. For this we use the following general result about completing the symmetric algebra over a Banach space. By a contraction we mean a linear map $f: X \to Y$ between Banach spaces satisfying $\|f(x)\| \leq \|x\|$. Also, in all that follows Banach algebras will be assumed to have a multiplicative unit, and homomorphisms between them will be assumed to preserve the unit.

**Lemma 2.** Let $X$ be a Banach space. Then up to isometric isomorphism there is a unique commutative Banach algebra $S_bX$ equipped with a contraction $\iota: X \to S_bX$ such that: given any contraction $F: X \to A$ where $A$ is a commutative Banach algebra $A$, there is a unique homomorphism $\tilde{F}: S_bX \to A$ that is a contraction and such that $F = \tilde{F} \iota$. Moreover, the space $\iota(X) \subseteq X$ generates a dense subalgebra of $S_bX$ naturally isomorphic to the symmetric algebra $SX$.

Proof - Uniqueness follows from the universal property of $S_bX$. For existence, we construct $S_bX$ as follows. Let $\Lambda$ be the class of homomorphisms $F: SX \to A$, $A$ a commutative Banach algebra, such that

$$\|F(x)\| \leq \|x\|$$

for all $x \in X$. We define a norm on $SX$ as follows:

$$\|f\| = \sup_{F \in \Lambda} \|F(f)\|$$

for all $f \in SX$. To show that this norm is well-defined it suffices to show that the supremum is finite, and nonzero for every nonzero $f$. Finiteness follows from the fact that $\|F(x)\| \leq \|x\|$ for every $x \in X$, $F \in \Lambda$. To prove that $\|f\| > 0$ for all nonzero $f \in SX$, it suffices to find $F \in \Lambda$ that is one-to-one. We defer this to the proof of Lemma 9 below.
Now let $S_bX$ be the completion of $SX$ in the norm we have constructed. Then $S_bX$ is a commutative Banach algebra, and the natural embedding $\iota:X \to S_bX$ has the desired universal property. It also follows from the construction that $\iota(X)$ generates a dense subalgebra of $S_bX$ isomorphic to $SX$. \qed

It follows from Lemma 2 that the homomorphism $T:S^bT \to H_b$ extends uniquely to a continuous homomorphism from $S^bT$ to $H_b$. We again call this $T$.

3 Invariant States on the Holonomy Algebra

Let Aut$(P)$ denote the group of smooth bundle automorphisms of $P$, not necessarily base-preserving. Let $\text{Aut}_0(P)$ denote the identity component of Aut$(P)$. The group Aut$(P)$ acts as diffeomorphisms of $A$, hence as isomorphisms of the algebra $H_b$. Note that we have an exact sequence

$$1 \to G \to \text{Aut}(P) \to \text{Diff}(M) \to 1$$

Since the action of $G$ on $H$ is trivial, Diff$(M)$ acts as automorphisms of $H_b$ as well. Our goal is to classify generalized “measures” on $A/G$ that are invariant under the action of Diff$_0(M)$. We will require of such “measures” $\mu$ only that the integral

$$\int_{A/G} f(A) \, d\mu(A)$$

make sense for all $f \in H_b$ (where we identify $H_b$ with an algebra of functions on $A/G$) and depend linearly and continuously on $f$. More precisely, we seek to classify $\mu \in H^*_b$ that satisfy

$$\mu(f) = \mu(gf)$$

for all $f \in H_b$, $g \in \text{Diff}_0(M)$. We will show that there is a natural one-to-one mapping from such functionals to complex-valued ambient isotopy invariants of framed oriented links in $M$. Unfortunately, the range of this mapping is unknown even in the simplest cases, so the classification program is far from complete.

The strategy is as follows. To begin with, note that the group Diff$(M)$ acts as isometries of $T$, hence as automorphisms of the tube algebra $S^bT$. Also, the homomorphism $T:S_bT \to H_b$ satisfies

$$T(ga) = gT(a)$$

for all $g \in \text{Diff}(M)$, $a \in S_bT$. It follows that every Diff$_0(M)$-invariant functional $\mu \in H^*_b$ gives rise to a Diff$_0(M)$-invariant linear functional $\mu \circ T \in S_bT$. Moreover, since $T$ has dense range, its adjoint

$$T^*: H^*_b \to (S_bT)^*$$
is one-to-one. In what follows we shall show that there is a one-to-one map from the space of $\text{Diff}_0(M)$-invariant elements of $(S_b\mathbf{T})^*$ to the space of complex-valued ambient isotopy invariants of oriented framed links in $M$. As a corollary, then, we obtain a one-to-one map from $H_b^\ast$ to this space of link invariants.

Given a set of tubes $\{\Gamma_i\}$, where $\Gamma_i = (\gamma_i, \omega_i)$, we say that the $\Gamma_i$ are nonintersecting if the embeddings $\gamma_i: S^1 \times D^{n-1} \to M$ have disjoint ranges. Let $\omega = f dx$ be a fixed but arbitrary smooth $(n-1)$-form supported in the interior of $D^{n-1}$ with $f \geq 0$ and $$\int_{D^{n-1}} \omega = 1.$$ We say that the set of tubes $\{\Gamma_i\}$ is normalized if $\omega_i = \omega$ for all $i$. The following lemma is the basis for the relation between diffeomorphism-invariant elements of $(S_b\mathbf{T})^*$ and link invariants.

**Lemma 3.** Suppose $M$ is a manifold of dimension $\geq 3$. Then elements of the form $\Gamma_1 \cdots \Gamma_p$, where the $\Gamma_i = (\gamma_i, \omega)$ are nonintersecting normalized tubes in $M$, span a subspace of the tube algebra $S_b\mathbf{T}$ that is dense in the norm topology.

Proof - By Lemma 2, products of tubes $\Gamma_1 \cdots \Gamma_p$ span a dense space of $S_b\mathbf{T}$. Since every smooth $(n-1)$-form compactly supported in the interior of $D^{n-1}$ can be approximated in the $L^1$ norm by a linear combination of $(n-1)$-forms of the form $\alpha^* \omega$, where $\alpha$ is a diffeomorphism of $D^{n-1}$, it follows that linear combinations of normalized tubes are dense in the space of tubes. Thus it suffices to show that each product of tubes $\Gamma_1 \cdots \Gamma_p$ may be approximated in the norm topology on $H_b$ by finite linear combinations

$$\sum_{\ell} \Gamma_1^\ell \cdots \Gamma_p^\ell$$

where for each $\ell$, the tubes $\Gamma_i^\ell$ are nonintersecting. We use the following fact from differential topology:

**Lemma 4.** Let $\gamma_1, \ldots, \gamma_p$ be embeddings of $S^1 \times D^{n-1}$ in $M$. Let $U \subseteq (D^{n-1})^p$ be the set of $p$-tuples $(x_1, \ldots, x_p)$ such that the ranges of the loops $(\gamma_i)^x: S^1 \to M$ for $i = 1, \ldots, p$ are disjoint. Then $U$ is an open dense set in $(D^{n-1})^p$ if $n \geq 3$.

Proof - Note that $$\gamma_1 \times \cdots \times \gamma_p: (S^1 \times D^{n-1})^p \to M^p$$ is an embedding. Let $D$ be the diagonal in $M^2$, and suppose $i \neq j$. Then $$B_{ij} = (\gamma_i \times \gamma_j)^{-1} D$$ is an $n$-dimensional submanifold of $(S^1 \times D^{n-1})^2$. Let $\pi: S^1 \times D^{n-1} \to D^{n-1}$ denote projection onto the second factor. It follows that $$(\pi \times \pi)B_{ij} \subset (D^{n-1})^2$$
has an open dense complement $C_{ij}$, since the dimension of $(D^{n-1})^2$ is greater than $n$. The set $C_{ij}$ is precisely the set of pairs $(x, y)$ such that the ranges of $\gamma_{i}^{x}$ and $\gamma_{j}^{y}$ are disjoint. Letting $p_{ij}: (D^{n-1})^p \to (D^{n-1})^2$ be given by $p_{ij}(x_1, \ldots, x_n) = (x_i, x_j)$, it follows that $p_{ij}^{-1}C_{ij}$ is open and dense in $(D^{n-1})^p$. Since $U$ is the intersection of the sets $p_{ij}^{-1}C_{ij}$ it is open and dense.

Now let $\Gamma_i = (\gamma_i, \omega_i), 1 \leq i \leq p$, be a collection of tubes in $M$. Let $K \subset (D^{n-1})^p$ be the intersection of $\supp \omega_1 \times \cdots \times \supp \omega_p$ with the complement of the set $U$ given in Lemma 4. $K$ is compact and nowhere dense. Choose $\epsilon > 0$. Then we may cover $\supp \omega_i \subset D^{n-1}$ with embedded discs $D_{ij}, 1 \leq j \leq k$, and write

\[ \omega_i = \sum_{j=1}^{k} \omega_{ij} \]

where $\omega_{ij}$ is a smooth form supported in the interior of $D_{ij}$, in such a way that the sum of

\[ \|\omega_{1j_1}\| \cdots \|\omega_{pj_p}\| \]

over all $(j_1, \ldots, j_p)$ with

\[ D_{1j_1} \times \cdots \times D_{pj_p} \cap K \neq \emptyset \]

is $\leq \epsilon$. Let $J$ denote the set of such $p$-tuples $(j_1, \ldots, j_p)$, and let $\Gamma_{ij} = (\Gamma_i, f_{ij})$. Then

\[ \|\Gamma_{1j_1} \cdots \Gamma_{pj_p}\| \leq \|f_{1j_1}\| \cdots \|f_{pj_p}\|, \]

so writing

\[ \Gamma_1 \cdots \Gamma_p = \sum_{(j_1, \ldots, j_p) \in J} \Gamma_{1j_1} \cdots \Gamma_{pj_p} + \sum_{(j_1, \ldots, j_p) \notin J} \Gamma_{1j_1} \cdots \Gamma_{pj_p}, \]

it follows that

\[ \|\Gamma_1 \cdots \Gamma_p - \sum_{(j_1, \ldots, j_p) \notin J} \Gamma_{1j_1} \cdots \Gamma_{pj_p}\| \leq \epsilon. \]

Given $(j_1, \ldots, j_p) \notin J$, there are nonintersecting tubes $\Gamma'_{1j_1}, \ldots, \Gamma'_{pj_p}$ such that $\Gamma'_{ij}$ is essentially equivalent to $\Gamma_{ij}$. Moreover,

\[ \|\Gamma_1 \cdots \Gamma_p - \sum_{(j_1, \ldots, j_p) \notin J} \Gamma'_{1j_1} \cdots \Gamma'_{pj_p}\| \leq \epsilon. \]

as desired.

It follows that in dimension $\geq 3$ any continuous linear functional $\mu: S_b T \to \mathbf{C}$ is determined by its values on products of nonintersecting normalized tubes. Next, note that nonintersecting tubes determine framed oriented links in $M$ as follows. Given a tube $(\gamma, \omega)$ in $M$, the loop $\gamma^0: S^1 \to M$ is an embedded circle, where $0 \in D^{n-1}$ is the
origin. This embedded circle acquires an orientation from the standard orientation on $S^1$, and its conormal bundle has a frame given by

$$\{(d\gamma)(t,0)(0,e_j)\}_{j=1}^{n-1}$$

where $e_j$ are the standard basis of the tangent space at the origin of $D^{n-1}$. If $\Gamma_1, \ldots, \Gamma_p$ are nonintersecting tubes, the embedded circles $\gamma_j^\theta$ are disjoint, so we obtain a framed oriented link with $p$ components. Let us write this link as $L(\Gamma_1, \ldots, \Gamma_p)$.

Recall that two oriented framed links $L, L'$ in $M$ are said to be ambient isotopic if there exists $f \in \text{Diff}_0(M)$ taking $L$ to $L'$, including orientation and framing. We will write this simply as $fL = L'$.

**Lemma 5.** Let $\Gamma_1, \ldots, \Gamma_p$ be nonintersecting normalized tubes in the manifold $M$, and let $\Gamma'_1, \ldots, \Gamma'_p$ also be nonintersecting normalized tubes in $M$. Then the framed oriented links $L = L(\Gamma_1, \ldots, \Gamma_p)$ and $L' = L(\Gamma'_1, \ldots, \Gamma'_p)$ are ambient isotopic if and only if, possibly after some reordering of the indices of the $\gamma_i$, there exists $f \in \text{Diff}_0(M)$ such that

$$f\Gamma_i = \Gamma'_i$$

for all $i$.

Proof - It is clear that if there exists $f \in \text{Diff}_0(M)$ with $f\Gamma_i = \Gamma'_i$ for all $i$ (up to a reordering of indices), then $L = L(\Gamma_1, \ldots, \Gamma_p)$ and $L' = L(\Gamma'_1, \ldots, \Gamma'_p)$ are ambient isotopic as framed oriented links.

For the converse, suppose the framed oriented links $L$ and $L'$ are ambient isotopic. Let $B$ denote the disjoint union of $p$ copies of $S^1$, $D$ denote the disjoint union of $p$ copies of $S^1 \times D^{n-1}$, and $E$ denote the disjoint union of $p$ copies of $S^1 \times \mathbb{R}^{n-1}$. We regard $E$ as a vector bundle over $B$ and $D$ as the unit disc bundle. Let $g, g': D \to M$ denote the embeddings defined by the nonintersecting tubes $\{\Gamma_i\}$ and $\{\Gamma'_i\}$, respectively. Then, possibly after a reordering of indices, there exists $f \in \text{Diff}_0(M)$ with

$$f \circ g|_B = g'|_B$$

and

$$d(f \circ g)|_B = dg'|_B.$$ 

Let us write simply $g$ for $f \circ g$. Then we need to show that there is a diffeomorphism $F \in \text{Diff}_0(M)$ such that $F \circ g = g'$. By the isotopy extension theorem it suffices to find an isotopy $g_i: D \to M$ such that $g_0 = g$ and $g_1 = g'$. Using tubular neighborhoods of $gD, g'D$ we can extend $g, g'$ to embeddings $G, G': E \to M$, and it clearly suffices to find an isotopy $G_i: E \to M$ with $G_0 = G$ and $G_1 = G'$. Since $G$ and $G'$ agree on $B$, as do $dG$ and $dG'$, the construction used in the proof of the uniqueness of tubular neighborhoods (in [12], for example) gives the desired isotopy $G_i$. 

We thus obtain:
Theorem 1. Let \( S_bT \) be the tube Banach algebra of \( M \), and suppose that \( \dim M \geq 3 \). Suppose that \( \mu \in (S_bT)^* \) is invariant under the action of \( \text{Diff}_0(M) \). Then there is a complex-valued ambient isotopy invariant \( \mathcal{L}_\mu \) of framed oriented links in \( M \) given by

\[
\mathcal{L}_\mu(L) = \mu(\Gamma_1 \cdots \Gamma_p)
\]

where \( \Gamma_1, \cdots, \Gamma_p \) are any choice of normalized nonintersecting tubes for which \( L = L(\Gamma_1, \ldots, \Gamma_p) \). Moreover, the map \( \mu \mapsto \mathcal{L}_\mu \) is one-to-one.

Proof - By Lemma 5, \( \mathcal{L}_\mu \) is well-defined and an ambient isotopy invariant of framed oriented links in \( M \). If \( \mathcal{L}_\mu = 0 \), then \( \mu \) vanishes on all products of normalized nonintersecting tubes, so by Lemma 4, \( \mu = 0 \). \( \square \)

We remark that it is easy to show that \( L_\mu \) is independent of the choice of \((n - 1)\)-form \( \omega \) used in the definition of normalized tube.

Theorem 2. Let \( P \) be a principal \( G \)-bundle over a manifold \( M \) of dimension \( \geq 3 \), and \( \rho \) a finite-dimensional unitary representation of \( G \). Let \( H_b \) be the holonomy Banach algebra associated to this data. The map from \( \text{Diff}_0(M) \)-invariant elements of \( H_b^* \) to complex invariants of framed oriented links in \( M \) given by

\[
\mu \mapsto \mathcal{L}_{T^*\mu}
\]

is one-to-one.

Proof - This follows immediately from the theorem and the fact that \( T^*: H_b^* \rightarrow (S_bT)^* \) is one-to-one. \( \square \)

We are left with two questions:

Question 3. What is the range of the map \( \mu \mapsto \mathcal{L}_\mu \) given in Theorem 1?

Question 4. What is the range of the map \( \mu \mapsto \mathcal{L}_{T^*\mu} \) given in Theorem 2?

In particular, it would be interesting to know whether the link invariants given by Theorem 2 include those given by Chern-Simons theory when \( M \) is compact. It would also be interesting to know whether the link invariants given by this theorem form a complete set of invariants as \( P \) ranges over all bundles with compact gauge group and \( \rho \) ranges over all finite-dimensional unitary representations. These questions are related, via recent conjectures concerning Vassiliev invariants [6, 7, 20].

We defer a serious study of the above questions to future work, contenting ourselves here with two remarks. First, recall that the theory of links is rather trivial in dimension \( > 3 \). More precisely, the ambient isotopy classes of framed oriented knots in an oriented manifold of dimension \( > 3 \) are in one-to-one correspondence (although not
naturally) with elements of $\pi_1(M) \times \mathbb{Z}_2$. Here $\pi_1(M)$ records the homotopy class of the knot, while $\mathbb{Z}_2$ records the framing, essentially because $\pi_1(SO(n-1)) \simeq \mathbb{Z}_2$. Similarly, ambient isotopy classes of framed oriented links correspond to unordered $n$-tuples of elements in $\pi_1(M) \times \mathbb{Z}_2$. It thus appears that diffeomorphism-invariant “measures” on the space of connections modulo gauge transformations are more easily classified in dimension $> 3$ than in dimension $3$, where one hopes that Chern-Simons theory provides interesting examples of such measures.

Second, there is a simple example of Theorem 2 involving the moduli space of flat connections. This has been treated already in the $SU(2)$ case by Ashtekar and Isham [3], working with their holonomy $C^*$-algebra. Let $M$ be any manifold, $G$ a compact Lie group, and $P \rightarrow M$ a $G$-bundle over $M$. Let $A_0$ denote the space of smooth flat connections on $P$. Then $A_0/G$ is a subspace of $A/G$ that may be identified with $\text{Hom}(\pi_1(M), G)/G$, where $G$ acts by conjugation. A finite Borel measure $d\mu$ on $A_0/G$ defines a linear functional $\mu_0: H_0 \rightarrow \mathbb{C}$ as follows:

$$\mu_0(f) = \int_{A_0/G} f(A) \, d\mu(A).$$

Since

$$|\mu_0(f)| \leq c \|f\|_{\infty}$$

for some $c > 0$, where

$$\|f\|_{\infty} = \sup_{A \in A} |f(A)|,$$

and since this $L^\infty$ norm is weaker than the norm in which we complete $H_0$ to obtain $H_b$, it follows that $\mu_0$ extends uniquely to a continuous linear functional $\mu: H_b \rightarrow \mathbb{C}$. Since the group $\text{Diff}_0(M)$ acting on $A/G$ fixes $A_0/G$, the functional $\mu$ is $\text{Diff}_0(M)$-invariant. Thus by Theorem 2 the functional $\mu$ determines a link invariant $L_{T^*\mu}$. This link invariant only depends on the homotopy classes of the components of the link.

4 C*-algebraic considerations

In this section we describe $C^*$-algebraic analogs of the results on holonomy Banach algebras. Recall that $H_0$ denotes the unital subalgebra of $C^\infty(A)$ generated by the functions $T(\Gamma)$ for all tubes $\Gamma$. We claim that $H_0$ is a sub-$*$-algebra of $C^\infty(A)$, the latter being a $*$-algebra under pointwise complex conjugation. Given an embedded solid torus $\gamma: S^1 \times D^{n-1} \rightarrow M$, define the embedded solid torus $\overline{\gamma}: S^1 \times D^{n-1} \rightarrow M$ by

$$\overline{\gamma}(t, x) = \gamma(-t, x),$$

where we identify $S^1$ with $\mathbb{R}/\mathbb{Z}$. Given a tube $\Gamma = (\gamma, f)$, let

$$\overline{\Gamma} = (\overline{\gamma}, \overline{f})$$
where $\overline{\omega}$ denotes the pointwise complex conjugate of the differential form $\omega$. Then we have

$$
T(\Gamma, A) = \int_{D^{n-1}} T(\gamma^x, A) \overline{\omega}(x)
$$

where we use the fact that the complex conjugate of $\text{tr}(g)$ is $\text{tr}(g^{-1})$ for $g$ unitary. It follows that $H_0$ is a sub-$*$-algebra of $C^\infty(A)$.

Note that the functions $T(\Gamma, \cdot)$ are bounded, so the $L^\infty$ norm on $H_0$, given by

$$
\|f\|_{\infty} = \sup_{A \in A} |f(A)|,
$$

is well-defined, and the completion of $H_0$ in this norm is a $C^*$-algebra. We call this $C^*$-algebra the holonomy $C^*$-algebra associated to the bundle $P$ and the representation $\rho$, and denote it by $H_c$. The holonomy algebra may be regarded as the closure of $H_0$ in the $C^*$-algebra of bounded continuous functions on $A$.

It is worth noting the differences between the above holonomy $C^*$-algebra and that defined by Ashtekar and Isham \[3\] in the special case where $G = SU(2)$ and $\rho$ is the defining (2-dimensional) representation. In this case,

$$
T(\overline{\gamma}, A) = T(\gamma, A)
$$

for all piecewise smooth loops $\gamma$ and connections $A$, where $\overline{\gamma}$ is $\gamma$ with its orientation reversed, so one does not need to reverse orientations to define the $*$-algebra structure of the holonomy algebra. However, using orientation-reversed loops one could define holonomy $C^*$-algebras of the Ashtekar-Isham type quite generally. Namely, the algebra generated by functions of the form $T(\gamma)$, where $\gamma$ is a piecewise smooth loop, is a sub-$*$-algebra of the bounded continuous functions on $A$, and its completion in the $L^\infty$ norm is a $C^*$-algebra. It is easy to see that any $\text{Diff}_0(M)$-invariant continuous linear functional on this $C^*$-algebra determines a link invariant. However, such link invariants will not depend on a framing. Thus we do not expect Chern-Simons theory to define continuous linear functionals on this type of holonomy $C^*$-algebra. Moreover, one does not expect the map from $\text{Diff}_0(M)$-invariant continuous linear functionals to link invariants to be one-to-one in this case, because there is no analog of Lemma 3.

In what follows we define a “tube $C^*$-algebra” and extend the results of Sections 2 and 3 to the $C^*$-algebraic setting. The algebra $ST$ has a natural $*$-algebra structure making $T: ST \to H_0$ a $*$-homomorphism. To see this, note the following:

**Lemma 6.** Let $K: V \to V$ be the conjugate-linear map given by $K(\Gamma) = \overline{\Gamma}$ for all tubes $\Gamma$. Then $KV_1 \subseteq V_1$ and $KV_2 \subseteq V_2$. 
Proof - For \( V_1 \), given a linear combination of tubes \( v = \sum c_i(\gamma, \omega_i) \) such that \( \sum c_i f_i = 0 \), note that \( \sum c_i \omega_i = 0 \) implies \( Kv \in V_1 \). For \( V_2 \), note that if \( \Gamma \) and \( \Gamma' \) are essentially equivalent tubes, so are \( \Gamma \) and \( \Gamma' \).

We thus obtain a conjugate-linear map \( K: T_0 \to T_0 \).

**Lemma 7.** The map \( K: T_0 \to T_0 \) is norm-preserving.

**Lemma 8.** The map \( K: T_0 \to T_0 \) is norm-preserving.

Proof - Given \( v \in T_0 \), for any \( \epsilon > 0 \) we may write
\[
v = \sum c_i(\gamma, \omega_i)
\]
with
\[
\sum_i \|c_i \omega_i\| \leq \|v\| + \epsilon.
\]
We then have
\[
Kv = \sum \bar{c}_i(\bar{\gamma}, \bar{\omega}_i)
\]
thus
\[
\|Kv\| \leq \sum_i \|\bar{c}_i \omega_i\| \leq \|v\| + \epsilon.
\]
Since \( \epsilon \) was arbitrary it follows that \( \|Kv\| \leq \|v\| \). Since \( K^2 = 1 \), \( \|Kv\| = \|v\| \).

It follows that \( K \) extends uniquely to a continuous (in fact norm-preserving) conjugate-linear map \( K: T \to T \). Thus the symmetric algebra \( ST \) becomes a \( * \)-algebra in a unique manner such that
\[
v^* = Kv
\]
for \( v \in T \). This implies that \( T: ST \to H_c \) is a \( * \)-homomorphism.

Next we use a general construction. Given a Banach space \( X \), we define a conjugation \( \kappa: X \to X \) to be a continuous conjugate-linear map with \( \kappa^2 = 1 \). We define a \( * \)-contraction \( f: X \to A \) from a Banach space \( X \) with conjugation \( \kappa \) to a Banach \( * \)-algebra \( A \) to be a linear map with \( f(\kappa x) = f(x)^* \) and \( \|f(x)\| \leq \|x\| \) for all \( x \in X \). Then we have:

**Lemma 9.** Let \( X \) be a Banach space with conjugation \( \kappa \). Then up to isomorphism there is a unique commutative \( C^* \)-algebra \( S_cX \) equipped with a \( * \)-contraction \( \nu: X \to S_cX \) such that given any commutative \( C^* \)-algebra \( A \) and \( * \)-contraction \( F: X \to A \), there exists a unique homomorphism \( \tilde{F}: S_cX \to A \) for which the \( F = \tilde{F} \). Moreover the subspace \( \iota(X) \subseteq X \) generates a dense subalgebra of \( S_cX \) naturally isomorphic to the symmetric algebra \( SX \).
Proof - Uniqueness follows from the universal property. For existence, we construct $S_cX$ as follows. First, make $SX$ into a $*$-algebra in the unique manner such that $x^* = \kappa(x)$ for all $x \in X$. Let $\Lambda$ be the class of $*$-homomorphisms $F: SX \to A$, $A$ a commutative $C^*$-algebra, such that

$$\|F(x)\| \leq \|x\|$$

for all $x \in X$. We define a norm on $SX$ as follows:

$$\|F\| = \sup_{F \in \Lambda} \|F(f)\|$$

for all $f \in SX$. To show that this norm is well-defined it suffices to show that the supremum is finite and nonzero for every nonzero $f$. Finiteness follows from the fact that $\|F(x)\| \leq \|x\|$ for every $x \in X$, $F \in \Lambda$. To prove that $\|f\| > 0$ for all nonzero $f \in SX$, we construct a faithful representation $F \in \Lambda$ as follows.

Let $X_r \subseteq X$, the "real part" of $X$, be the closed real subspace $X_r = \{ x \in X : \kappa(x) = x \}$. Given $x \in X$, define $\text{Re}(x) = (x + \kappa(x))/2 \in X_r$ and $\text{Im}(x) = (x - \kappa(x))/2i \in X_r$. Elements of the real dual $X_r^*$ may be identified with elements of $X^*$ as follows:

$$\ell(x) = \ell(\text{Re}(x)) + i\ell(\text{Im}(x))$$

for $\ell \in X_r^*$ and $x \in X$. This allows us to identify $X_r^*$ with the real subspace of $X^*$ consisting of functionals $\ell$ such that $\ell(x)$ is real for all $x \in X_r$.

Let $B$ denote the closed unit ball in $X_r^*$ with respect to the $X^*$ norm, and give $B$ the topology in which $\ell_\alpha \to \ell$ if $\ell_\alpha(x) \to \ell(x)$ for all $x \in X$. Then there is a unique homomorphism $F: SX \to C(B)$ with

$$F(x)(\ell) = \ell(x)$$

for all $x \in X, \ell \in B$. Note that $F$ is a $*$-homomorphism because

$$F(x^*)(\ell) = \ell(\kappa(x)) = \ell(\text{Re}(\kappa(x))) + i\ell(\text{Im}(\kappa(x))) = \ell(\text{Re}(x)) - i\ell(\text{Im}(x)) = \overline{F(x)(\ell)}$$

Note also that $F \in \Lambda$ since

$$\|F(x)\| = \sup_{\ell \in B} |\ell(x)| \leq \|x\|.$$

We claim that $F$ is faithful. Suppose that $f \in SX$ is nonzero. Then $f$ is a polynomial in elements of some finite-dimensional subspace $V \subseteq X$. Let $V_r = V \cap X_r$,
and identify $V'_r$ as a real subspace of $V^*$ just as we identified $X'_r$ with a real subspace of $X^*$. Then since we are in a finite-dimensional situation we can find $\ell \in V'_r$ with $\|\ell\| \leq 1$ and $f(\ell) \neq 0$. By the Hahn-Banach theorem we can extend $\ell$ to an element in $X'_r \cap B$, which we again call $\ell$. It follows that $F(f)(\ell) \neq 0$, so $F(f) \neq 0$.

Now let $S_cX$ be the completion of $SX$ in the norm we have constructed. Then $S_cX$ is a C*-algebra, and the universal property follows immediately. It also follows from the construction that $\iota(X)$ generates a subalgebra of $S_cX$ isomorphic to $SX$.

We call $S_cT$ the tube C*-algebra. Note that it depends only on the base manifold $M$. Also, since any isomorphism of commutative C*-algebras is an isometric $*$-isomorphism, it is unique up to isometric $*$-isomorphism.

**Lemma 10.** The map $T: T_0 \to H_0$ satisfies $\|Tv\|_\infty \leq \|v\|$ for all $x \in T_0$.

**Proof** - Given $v \in T_0$, for any $\epsilon > 0$ we may write 

$$v = \sum_i c_i(\gamma_i, \omega_i)$$

in such a way that 

$$\sum_i \|c_i\omega_i\| \leq \|v\| + \epsilon.$$ 

Then 

$$\|Tv\|_\infty = \|\sum_i c_i T(\gamma_i, \omega_i)\|_\infty$$

$$\leq \sup_{A \in A} \left| \sum_i c_i \int_{D^{n-1}} T(\gamma_i^x, A) \omega_i(x) \right|$$

$$\leq \sum_i \|c_i\omega_i\|$$

$$\leq \|v\| + \epsilon$$

Since $\epsilon$ is arbitrary we have $\|Tv\| \leq \|v\|$. Thus $T$ extends uniquely to a map from $T$ to $H$. \hfill \square

It follows from Lemmas 9 and 10 that there is a unique homomorphism from the tube algebra to the holonomy algebra extending the map $T: T \to H_c$. We write this homomorphism as 

$$T: S_cT \to H_c.$$ 

This is our final form of the map $T$. Since $H_0$ is dense in $H_c$, the range of $T$ is dense. It follows from C*-algebra theory that $T$ is onto.

Since $S_cT$ is the completion of $ST$ in a weaker norm than $S_bT$, we have a one-to-one homomorphism with dense range 

$$S_bT \hookrightarrow S_cT.$$ 

We have analogs of Theorems 1 and 2 for the tube C*-algebra and holonomy C*-algebra:
Theorem 3. Let $S_c T$ be the tube $C^*$-algebra of $M$, and suppose that $\dim M \geq 3$. Suppose that $\mu \in (S_c T)^*$ is invariant under the action of $\text{Diff}_0(M)$. Then there is a complex-valued ambient isotopy invariant $L_\mu$ of framed oriented links in $M$ given by

$$L_\mu(L) = \mu(\Gamma_1 \cdots \Gamma_p)$$

where $\Gamma_1, \cdots, \Gamma_p$ are any choice of normalized nonintersecting tubes for which $L = L(\Gamma_1, \cdots, \Gamma_p)$. Moreover, the map $\mu \mapsto L_\mu$ is one-to-one.

Proof - This follows from Theorem 1 and the fact that the homomorphism $S_b T \to S_c T$ has dense range. $\square$

Theorem 4. Let $P$ be a principal $G$-bundle over a manifold $M$ of dimension $\geq 3$, and $\rho$ a finite-dimensional unitary representation of $G$. Let $H_c$ be the holonomy $C^*$-algebra associated to this data. The map from $\text{Diff}_0(M)$-invariant elements of $H^*$ to complex invariants of framed oriented links in $M$ given by

$$\mu \mapsto L_{T^*\mu}$$

is one-to-one.

Proof - This follows from Theorem 2 and the fact that the homomorphism $H_b \to H_c$ has dense range. $\square$

And again, we have open questions:

Question 5. What is the range of the map $\mu \mapsto L_\mu$ given in Theorem 3?

Question 6. What is the range of the map $f \mapsto L_{T^*f}$ given in Theorem 4?

We conclude with some remarks on the loop transform. By the Gelfand-Naimark theorem, the holonomy $C^*$-algebra $H_c$ is isomorphic to the algebra of continuous functions on some compact Hausdorff space $X$. Since any point in $A/G$ determines a pure state on $H_c$, hence a point of $X$, and since any element of $H_c$ that is annihilated by all such pure states must vanish, we may regard $X$ as a compactification of $A/G$, and write

$$X = \overline{A/G}.$$ Elements of $H_c^*$ are in one-to-one correspondence with finite regular Borel measures on $\overline{A/G}$. The map

$$T^*: H_c^* \to (S_c T)^*$$

may be regarded as assigning to each such measure on $\overline{A/G}$ its “loop transform,” or, since we are working with tubes, its “tube transform.” We have shown that this version of the loop transform is one-to-one. To further develop the loop representation of gauge theories it would be useful to have an inverse loop transform. For this, one would like an answer to the following:

Question 7. What is the range of the map $T^*: H_c^* \to (S_c T)^*$?
References

[1] S. Albeverio and R. Høegh-Krohn, Oscillatory integrals and the method of stationary phase in infinitely many dimensions, with applications to the classical limit of quantum mechanics, I, Invent. Math. 40 (1977) 59-106.

[2] A. Ashtekar, Recent developments in classical and quantum theories of connections including general relativity, Syracuse University preprint (1992).

[3] A. Ashtekar and C. J. Isham, Representations of the holonomy algebra of gravity and non-abelian gauge theories, Syracuse University and Imperial College preprint (1991).

[4] S. Axelrod and I. Singer, perturbation theory for Chern-Simons theory, Massachusetts Institute of Technology preprint (1992).

[5] J. Baez, I. Segal and Z. Zhou, An Introduction to Algebraic and Constructive Quantum Field Theory, Princeton U. Press, Princeton, 1992.

[6] D. Bar-Natan, On the Vassiliev knot invariants, Harvard University preprint (1992).

[7] J. Birman, New points of view in knot theory, Columbia University preprint (1992).

[8] B. Brügmann, R. Gambini, J. Pullin, Jones polynomials for intersecting knots as physical states for quantum gravity, University of Utah preprint (1992).

[9] R. H. Cameron and D. A. Storvik, Some Banach algebras of analytic Feynman integrable functions, in Analytic Functions, Kozubnik, 1979, Springer Lecture Notes in Math., vol. 798, New York, Springer, 1980.

[10] P. Cotta-Ramusino, E. Guadagnini, M. Martellini, and M. Mintchev, Quantum field theory and link invariants, Nuc. Phys. B330 (1990) 557-574.

[11] E. Guadagnini, M. Martellini, and M. Mintchev, Wilson lines in Chern-Simons theory and link invariants, Nuclear Phys. B330 (1990) 575-607.

[12] M. Hirsch, Differential Topology, New York, Springer-Verlag, 1976.

[13] G. W. Johnson, The equivalence of two approaches to the Feynman integral, J. Math. Phys. 23 (1982) 2090-2096.

[14] R. Loll, Loop approaches to gauge field theory, Syracuse University preprint (1992).
[15] R. Palais, On the local triviality of the restriction map for embeddings, Comm. Math. Helv., 34 (1960), 306-312.

[16] A. M. Polyakov, Fermi-Bose transmutations induced by gauge fields, Phys. Rev. Lett. A3 (1988) 325-328.

[17] C. P. Rourke and B. J. Sanderson, Introduction to Piecewise-Linear Topology, New York, Springer-Verlag, 1972.

[18] C. Rovelli and L. Smolin, Loop representation for quantum general relativity, Nucl. Phys. B331 (1990) 80-152.

[19] I. E. Segal and R. Kunze, Integrals and operators, New York, Springer-Verlag, 1978.

[20] V. A. Vassiliev, Cohomology of knot spaces, in Theory of Singularities and its Applications, ed V. I. Arnold, Cambridge, University of Cambridge Press, 1990.

[21] E. Witten, Quantum field theory and the Jones polynomial, Comm. Math. Phys. 121 (1989) 351-399.