Expressive power of basic modal intuitionistic logic as a fragment of classical FOL

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Abstract. The paper treats 4 different fragments of first-order logic induced by their respective versions of Kripke style semantics for modal intuitionistic logic. In order to capture these fragments, the notion of asimulation, defined in [7], is modified and extended to yield Van Benthem type of semantic characterization of their respective expressive powers. It is shown further, that this characterization can be easily carried over to arbitrary first-order definable subclasses of classical first-order models.

Keywords. model theory, modal logic, intuitionistic logic, propositional logic, bisimulation, Van Benthem’s theorem.

It was shown in [7] and [8] that both intuitionistic first-order logic and its propositional fragment, viewed as a different fragments of classical first-order logic, admit of a full analogue of modal characterization theorem, where invariance with respect to bisimulations is replaced with invariance with respect to (first-order) asimulations respectively. The present paper extends these results onto the main versions of basic modal intuitionistic logic.

The layout of the paper is as follows. Section 1 starts with notational conventions, after which we introduce the main variants of Kripke style semantics for the basic modal intuitionistic system. All in all we consider 4 different variants of semantics, of which 2 are easily discharged by the versions of clauses employed in [7] and [8] for basic intuitionistic logic. However, for the other 2 systems their semantical characterization is less obvious, mainly due to their treatment of diamond modality.

Section 2 then starts with the main task of the present paper. It contains definitions for all the four variants of extension of basic asimulation notion to be employed in semantic characterization of their respective variants of basic modal intuitionistic logic. We also formulate here the main results of the paper, although their proofs are postponed till Sections 3 and 4. Then Section 5 is devoted to characterization of modal intuitionistic fragments of FOL modulo first-order definable classes of models. Section 6 gives conclusions and drafts directions for future work.

1See, e.g. [2 Ch.1, Th. 13].
1 Preliminaries

1.1 Notation

A formula is a formula of classical predicate logic without identity whose predicate letters are in vocabulary \( \Sigma = \{ R^2, R^2_0, R^2_1, P^1_1, \ldots, P^1_n, \ldots \} \). We assume \( \{ \bot, \to, \lor, \land, \forall, \exists \} \) as the set of basic connectives and quantifiers for this variant of classical first-order language, which we call correspondence language. A model is a classical first-order model of correspondence language. We refer to correspondence formulas with lower-case Greek letters \( \varphi, \psi, \chi \), and to sets of correspondence formulas with upper-case Greek letters \( \Gamma, \Delta \). If \( \varphi \) is a correspondence formula, then we associate with it the following finite vocabulary \( \Sigma_{\varphi} \subseteq \Sigma \) such that \( \Sigma_{\varphi} = \{ R^2, R^2_0, R^2_1 \} \cup \{ P_i \mid P_i \text{ occurs in } \varphi \} \). More generally, we refer with \( \Theta \) to an arbitrary subset of \( \Sigma \) such that \( R \in \Theta \). If \( \psi \) is a formula and every predicate letter occurring in \( \psi \) is in \( \Theta \), then we call \( \psi \) a \( \Theta \)-formula.

We refer to sequence \( x_1, \ldots, x_n \) of any objects as \( \vec{x}_n \). We identify a sequence consisting of a single element with this element. If all free variables of a formula \( \varphi \) (formulas in \( \Gamma \)) coincide with a variable \( x \), we write \( \varphi(x) \) (\( \Gamma(x) \)).

By degree of a classical first-order formula we mean the greatest number of nested quantifiers occurring in it. Degree of a formula \( \varphi \) is denoted by \( r(\varphi) \). Its formal definition by induction on the complexity of \( \varphi \) goes as follows:

\[
\begin{align*}
    r(\bot) &= r(\varphi) = 0 & \text{for atomic } \varphi \\
    r(\varphi \circ \psi) &= \max(r(\varphi), r(\psi)) & \text{for } \circ \in \{ \land, \lor, \to \} \\
    r(Qx\varphi) &= r(\varphi) + 1 & \text{for } Q \in \{ \forall, \exists \}
\end{align*}
\]

For \( k \in \mathbb{N} \), we say that \( \Theta \)-formula \( \varphi(x) \) such that \( r(\varphi) \leq k \) is a \( (\Theta, x, k) \)-formula.

For a binary relation \( S \) and any objects \( s, t \) we abbreviate the fact that \( s \stackrel{S}{\leftrightarrow} t \) by \( s \equiv t \).

We use the following notation for models of classical predicate logic:

\[
M = \langle U, \iota \rangle, \quad M_1 = \langle U_1, \iota_1 \rangle, \quad M_2 = \langle U_2, \iota_2 \rangle, \ldots, \quad M' = \langle U', \iota' \rangle, \quad M'' = \langle U'', \iota'' \rangle, \ldots,
\]

where the first element of a model is its domain and the second element is its interpretation of predicate letters. If \( k \in \mathbb{N} \) then we write \( R_k \) \( (R_{0k}, R_{1k}) \) as an abbreviation for \( \iota_k(R) \) \( (\iota_k(R_0), \iota_k(R_1)) \). If \( a \in U \) then we say that \( (M, a) \) is a pointed model. Further, we say that \( \varphi(x) \) is true at \( (M, a) \) and write \( M, a \models \varphi(x) \) if for any variable assignment \( \alpha \) in \( M \) such that \( \alpha(x) = a \) we have \( M, \alpha \models \varphi(x) \). It follows from this convention that the truth of a formula \( \varphi(x) \) at a pointed model is to some extent independent from the choice of its only free variable. Moreover, for \( k \in \mathbb{N} \) we will sometimes write \( a \models_k \varphi(x) \) instead of \( M_k, a \models \varphi(x) \).

A modal intuitionistic formula is a formula of modal intuitionistic propositional logic, where \( \{ \bot, \to, \lor, \land, \Box, \Diamond \} \) is the set of basic connectives and modal operators, and \( \{ p_n \mid n \in \mathbb{N} \} \) is the set of propositional letters. We refer to intuitionistic formulas with letters \( I, J, K \), possibly with primes or subscripts.

1.2 Definitions of basic modal intuitionistic logic

There exist different versions of basic system of modal intuitionistic logic. In these paper we only consider versions that have a Kripke-style semantics associated with
them, and we will view these versions via the lens of their respective Kripke-style semantics.

Quite naturally, Kripke-style semantics for a given version of intuitionistic modal logic is normally built as an extension of Kripke semantics for basic intuitionistic propositional logic; that is to say, the models extend Kripke models for basic propositional logic and the satisfaction clauses for $\perp, \rightarrow, \lor, \land$ are left unchanged (cf. [6, Definition 7.1 and Definition 7.2]).

The new components in the models are one or more additional binary relations between states which are needed to handle the satisfaction clauses for $\Box$ and $\Diamond$. In the most general case both modal operators are handled by separate relations $R_\Box$ and $R_\Diamond$, although not infrequently one assumes that $R_\Box$ and $R_\Diamond$ do coincide or are otherwise non-trivially related. It is also quite common to assume different conditions connecting $R_\Box$ and $R_\Diamond$ with accessibility relation $R$, like, e.g. assuming that

$$R \circ R_\Box \subseteq R_\Box \circ R.$$  

Both the condition that $R_\Box = R_\Diamond$ and the other conditions mentioned in this connection in the existing literature are easily first-order definable. Since it was shown in [7] and [8] that asimulations are easily scalable according to arbitrary first-order conditions imposed upon the models, we will first concentrate on the minimal case without any restrictions imposed and then accommodate for the possible restrictions in a trivial way, by restricting the domain and the counter-domain of asimulation relations accordingly.

As for the satisfaction clauses, employed in the existing literature on Kripke-style semantics for intuitionistic modal operators, the following variants of them seem to be the most common and general:

$$M, s \models \Box I \iff \forall t (sR_\Box t \Rightarrow M, t \models I)$$  \quad (\Box_1)

$$M, s \models \Box I \iff \forall t (sRt \Rightarrow \forall u (tR_\Box u \Rightarrow M, u \models I))$$  \quad (\Box_2)

$$M, s \models \Diamond I \iff \exists t (sR_\Diamond t \land M, t \models I)$$  \quad (\Diamond_1)

$$M, s \models \Diamond I \iff \forall t (sRt \Rightarrow \forall u (tR_\Diamond u \land M, u \models I))$$  \quad (\Diamond_2)

This gives us 4 possible choices of satisfaction clauses. In literature, these sets of satisfaction clauses are often viewed as more or less explicit manifestations of one and the same set of semantic intuitions. In this view, the reason why clauses (\Box_2) and (\Diamond_2) differ from (\Box_1) and (\Diamond_1), respectively, is that the former clauses lift up to the level of semantical definitions some desirable properties that under (\Box_1) and (\Diamond_1) are handled by restrictions on the class of models. However, in what follows, we will disregard this circumstance and will simply consider these four systems of clauses as bona fide different systems. The reason for this is that, like we said above, we find it convenient in the context of treating modal intuitionistic formulas via asimulations, to omit whatever restrictions on models that are employed to equate these systems in the existing literature on the subject.

Every of the 4 semantical choices sketched above induces a different standard translation of modal intuitionistic formulas into classical FOL thus giving a different frag-

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2This is the case, e.g., for all the systems mentioned in [9, Ch.3].
3In this form they are given, e.g. in [4, Section 4].
4E.g. the property of monotonicity. Cf. the motivation for the clause (\Box_2) given in [9, p.46].
ment of it. More precisely, for \( i, j \in \{1, 2\} \) we will denote the \((i, j)\)-standard translation, or the standard translation induced by adopting \((\Box_i)\)-clause together with \((\Diamond_j)\)-clause above, by \( ST_{ij} \).

Thus the inductive definitions of the \((i, j)\)-standard \(x\)-translations run as follows:

\[
\begin{align*}
ST_{ij}(p_n, x) &= P_n(x); \\
ST_{ij}(\bot, x) &= \bot; \\
ST_{ij}(I \land J, x) &= ST_{ij}(I, x) \land ST_{ij}(J, x); \\
ST_{ij}(I \lor J, x) &= ST_{ij}(I, x) \lor ST_{ij}(J, x); \\
ST_{ij}(I \rightarrow J, x) &= \forall y(R(x, y) \rightarrow (ST_{ij}(I, y) \rightarrow ST_{ij}(J, y))); \\
ST_{1j}(\Box I, x) &= \forall y(R_{\Box}(x, y) \rightarrow ST_{1j}(I, y)); \\
ST_{2j}(\Box I, x) &= \forall y(R(x, y) \rightarrow \forall z(R_{\Box}(y, z) \rightarrow ST_{2j}(I, z))); \\
ST_{i1}(\Diamond I, x) &= \exists y(R_{\Diamond}(x, y) \land ST_{i1}(I, y)); \\
ST_{i2}(\Diamond I, x) &= \forall y(R(x, y) \rightarrow \exists z(R_{\Diamond}(y, z) \land ST_{i2}(I, z))).
\end{align*}
\]

Standard conditions are imposed on the variables \( x, y, \) and \( z \).

2 Characterization of modal intuitionistic formulas: definitions and main results

Our aim in the present paper is to characterize the expressive power of the four fragments of correspondence language induced by the four above-mentioned versions of \((i, j)\)-standard translation of modal intuitionistic formulas via the suitable extension of the notion of asimulation for the intuitionistic propositional logic. We will begin by giving strict definitions of the four required extensions, and then formulate the two versions of our main result for all the four considered fragments of correspondence language in one full sweep.

It is easy to see that if one chooses \((\Diamond_1)\) as a definition for the semantics of possibility, then the respective standard translation of modal propositional formulas looks almost as a notational variant of intuitionistic first-order logic. The semantics of diamond then resembles the semantics of intuitionistic existential quantifier, and the semantics of box, if interpreted according to \((\Box_2)\), resembles the semantics of intuitionistic universal quantifier. The only difference is that the binary relation of existence of object in a state is replaced by binary relations \(R_{\Box}\) and \(R_{\Diamond}\) respectively. Also, note that now these ‘quantifiers’ are based on different binary relations rather than one and the same, but the proofs given in the following section show that this little wrinkle is of no consequence:

**Definition 1.** Let \((M_1, t), (M_2, u)\) be two pointed \(\Theta\)-models. A binary relation \(A\) is called \((2, 1)\)-modal \(\langle (M_1, t), (M_2, u) \rangle_k\)-asimulation iff for any \(i, j \in \{1, 2\}\), any \(\bar{a}_m, a, c \in U_i, \bar{b}_m, b, d \in U_j\), any unary predicate letter \(P \in \Theta\), the following conditions

\(\text{Every such standard translation is an obvious extension of the well known notion of standard translation of propositional intuitionistic formulas, see e.g. }[\text{Definition 8.7}].\)
hold:

\[ A \subseteq \bigcup_{n>0} \left( (U_1^n \times U_2^n) \cup (U_2^n \times U_1^n) \right) \]  

(p-type)

\[ t \ A \ u \]  

(elem)

\[ ((\bar{a}_m, a) \ A \ (\bar{b}_m, b) \land a \models_i P(x)) \Rightarrow b \models_j P(x) \]  

(p-base)

\[ ((\bar{a}_m, a) \ A \ (\bar{b}_m, b) \land b \ R_j \ d \land m < k) \Rightarrow \]  

(p-step)

\[ (\bar{a}_m, a) \ A \ (\bar{b}_m, b) \land b \ R_j \ d \land d \ R_{\Theta} f \land m + 1 < k) \Rightarrow \]  

(p-box-2)

\[ (\bar{a}_m, a) \ A \ (\bar{b}_m, b) \land a \ \forall_{i,j} c \land m < k) \Rightarrow \]  

(p-diam-1)

**Definition 2.** Let \( (M_1, t), (M_2, u) \) be two pointed \( \Theta \)-models. A binary relation \( A \) is called \((2, 1)\)-modal \((M_1, t), (M_2, u)\)-**asimulation** iff for any \( i, j \in \{1, 2\} \), any \( a, c \in U_i, b, d \in U_j \), any unary predicate letter \( P \in \Theta \) the following conditions hold:

\[ A \subseteq (U_1 \times U_2) \cup (U_2 \times U_1) \]  

(type)

\[ t \ A \ u \]  

(elem)

\[ (a \ A \ b \land a \models_i P(x)) \Rightarrow b \models_j P(x) \]  

(base)

\[ (a \ A \ b \land b \ R_j \ d) \Rightarrow \exists c \in U_i(a \ R_i c \land c \ A \ d) \]  

(step)

\[ (a \ A \ b \land b \ R_j \ d \land d \ R_{\Theta} f) \Rightarrow \exists c, e \in U_i(a \ R_i c \land c \ R_{\Theta} e \land e \ A f) \]  

(box-2)

\[ (a \ A \ b \land a \ R_{\forall} \ c) \Rightarrow \exists d \in U_j(b \ R_{\forall} \ d \land d \ A \ c) \]  

(diam-1)

The situation changes very little when one adapts \((\square\forall)\) instead of \(\exists\) as the clause defining box. In fact, this clause is the standard one from classical modal logic, and to accommodate for this change one only needs to pick a suitable asymmetric variant of the bisimulation clause. The resulting definitions then look as follows:

**Definition 3.** Let \( (M_1, t), (M_2, u) \) be two pointed \( \Theta \)-models. A binary relation \( A \) is called \((1, 1)\)-modal \((M_1, t), (M_2, u)\)-**asimulation** iff for any \( i, j \in \{1, 2\} \), any \( \bar{a}_m, a, c \in U_i, \bar{b}_m, b, d \in U_j \) any unary predicate letter \( P \in \Theta \), the conditions [p-type], [elem], [p-base], [p-step], [p-diam-1] are satisfied together with the following condition:

\[ ((\bar{a}_m, a) \ A \ (\bar{b}_m, b) \land b \ R_j \ d \land m + 1 < k) \Rightarrow \]  

(p-box-1)

\[ \exists c \in U_i(a \ R_{\forall} \ c \land (\bar{a}_m, a, c) \ A \ (\bar{b}_m, b, d)) \]  

**Definition 4.** Let \( (M_1, t), (M_2, u) \) be two pointed \( \Theta \)-models. A binary relation \( A \) is called \((1, 1)\)-modal \((M_1, t), (M_2, u)\)-**asimulation** iff for any \( i, j \in \{1, 2\} \), any \( a, c \in U_i, b, d \in U_j \), any unary predicate letter \( P \in \Theta \) the conditions [elem], [type], [base], [step], [diam-1] are satisfied together with the following condition:

\[ (a \ A \ b \land b \ R_j \ d) \Rightarrow \exists c \in U_i(a \ R_{\forall} \ c \land c \ A \ d) \]  

(box-1)
If, instead of using clause (\(\bigcirc_2\)), one chooses clause (\(\bigcirc_3\)), things get somewhat more complicated. In order get the right extensions of the basic assimilation notion, one has to re-define asimulations as relation pairs rather than single binary relations. As our first case we consider the set of correspondence formulas induced by (2, 2)-standard translations of modal intuitionistic formulas:

**Definition 5.** Let \((M_1, t), (M_2, u)\) be two pointed \(\Theta\)-models. An ordered couple of binary relations \((A, B)\) is called (2, 2)-modal \(\langle (M_1, t), (M_2, u) \rangle_k\)-asimulation iff for any \(i, j \in \{1, 2\}\), any \(\bar{a}_m, a, c \in U_i, \bar{b}_m, b, d \in U_j\) any unary predicate letter \(P \in \Theta\), the conditions (p-type), (elem), (p-base), (p-step), (p-box-1), (p-diam-2(1)) are satisfied together with the following conditions:

\[
B \subseteq \bigcup_{n>0} ((U_1^n \times U_2^n) \cup (U_2^n \times U_1^n)) \quad \text{(p-B-type)}
\]

\[
\begin{align*}
((\bar{a}_m, a) \ A (\bar{b}_m, b) \land b R_j d \land m + 1 < k) & \Rightarrow \\
& \Rightarrow \exists e \in U_i(a R_i c \land (\bar{a}_m, a, c) B (\bar{b}_m, b, d)) \quad \text{(p-diam-2(1))}
\end{align*}
\]

\[
\begin{align*}
((\bar{a}_m, a) \ B (\bar{b}_m, b) \land a R_{i,j} c \land m < k) & \Rightarrow \\
& \Rightarrow \exists d \in U_j(b R_{o,j} d \land (\bar{a}_m, a, c) A (\bar{b}_m, b, d)) \quad \text{(p-diam-2(2))}
\end{align*}
\]

**Definition 6.** Let \((M_1, t), (M_2, u)\) be two pointed \(\Theta\)-models. An ordered couple of binary relations \((A, B)\) is called (2, 2)-modal \(\langle (M_1, t), (M_2, u) \rangle\)-asimulation iff for any \(i, j \in \{1, 2\}\), any \(a, c \in U_i, b, d \in U_j\) any unary predicate letter \(P \in \Theta\), the conditions (elem), (type), (base), (step), (box-2) are satisfied together with the following conditions:

\[
B \subseteq (U_1 \times U_2) \cup (U_2 \times U_1) \quad \text{(B-type)}
\]

\[
\begin{align*}
(a \ A b \land b R_j d) & \Rightarrow \exists c \in U_i(a R_i c \land c B d) \quad \text{(diam-2(1))}
\end{align*}
\]

\[
\begin{align*}
(a \ B b \land a R_{i,j} c) & \Rightarrow \exists d \in U_j(b R_{o,j} d \land c A d) \quad \text{(diam-2(2))}
\end{align*}
\]

The only case left is the one where one defines the satisfaction relation by using (\(\bigcirc_2\)) combined with (\(\bigcirc_1\)). The respective definitions simply re-shuffle the conditions mentioned in the previous versions:

**Definition 7.** Let \((M_1, t), (M_2, u)\) be two pointed \(\Theta\)-models. An ordered couple of binary relation \((A, B)\) is called (1, 2)-modal \(\langle (M_1, t), (M_2, u) \rangle_k\)-asimulation iff for any \(i, j \in \{1, 2\}\), any \(\bar{a}_m, a \in U_i, \bar{b}_m, b, d \in U_j\) any unary predicate letter \(P \in \Theta\), the conditions (p-type), (p-B-type), (elem), (p-base), (p-step), (p-box-1), (p-diam-2(1)), and (p-diam-2(2)) are satisfied.

**Definition 8.** Let \((M_1, t), (M_2, u)\) be two pointed \(\Theta\)-models. An ordered couple of binary relation \((A, B)\) is called (2, 2)-modal \(\langle (M_1, t), (M_2, u) \rangle\)-asimulation iff for any \(i, j \in \{1, 2\}\), any \(a \in U_i, b, d \in U_j\) any unary predicate letter \(P \in \Theta\) the conditions (elem), (B-type), (type), (base), (step), (box-1), (diam-2(1)) and (diam-2(2)) are satisfied.

Before we reach the formulation of our main results, we still need one more technical notion, that of invariance with respect to a class of relations:
Definition 9. Let \( \alpha \) be a class of relations such that for any \( A \in \alpha \) there is a \( \Theta \) and there are \( \Theta \)-models \( M_1 \) and \( M_2 \) such that \([i\text{-type}]\) holds. Then a formula \( \varphi(x) \) is said to be \emph{invariant with respect to} \( \alpha \), iff for any \( A \in \alpha \) for any corresponding \( \Theta \)-models \( M_1 \) and \( M_2 \), and for any \( a \in U_1 \) and \( b \in U_2 \) it is true that:

\[
(a \ A \ b \land a \models_1 \varphi(x)) \Rightarrow b \models_2 \varphi(x).
\]

Of course, our primary examples of \( \alpha \) will be the classes of all \((i,j)\)-modal \( k \)-asimulations for a given natural \( k \) and the classes of all \((i,j)\)-modal asimulations. In case \( j = 2 \) we identify the invariance with respect to \((A,B)\) with invariance with respect to its left projection \( A \).

It turns out that the properties of invariance with respect to these relation classes can be used to characterize modal intuitionistic fragment of FOL described by their respective \( ST_{ij} \). More precisely, one can obtain the following theorems:

Theorem 1. Let \( i,j \in \{1,2\} \). A formula \( \varphi(x) \) is equivalent to an \((i,j)\)-standard \( x \)-translation of a modal intuitionistic formula iff there exists a \( k \in \mathbb{N} \) such that \( \varphi(x) \) is invariant with respect to \((i,j)\)-modal \( k \)-asimulations.

Theorem 2. Let \( i,j \in \{1,2\} \). A formula \( \varphi(x) \) is equivalent to an \((i,j)\)-standard \( x \)-translation of an intuitionistic formula iff \( \varphi(x) \) is invariant with respect to \((i,j)\)-modal asimulations.

In what follows, we will also need to speak of \emph{basic asimulations}. Using the notation of Definitions 1 and 2 a basic \( \langle(M_1,t),(M_2,u)\rangle_k \)-asimulation (resp. a basic \( \langle(M_1,t),(M_2,u)\rangle_k \)-asimulation) is a binary relation satisfying all the conditions of Definition 1 (resp. Definition 2) but the last two.

3 Characterization of modal intuitionistic formulas: the main case

Theorems 1 and 2 admit of four different instantiations of of \( i \) and \( j \). The most difficult ones seem to be the two instantiations with \( j = 2 \) since with them the situation bears the least degree of analogy to the first-order intuitionistic logic. Therefore, in the present section consider in some detail the case \( i = j = 2 \), whereas in the next one we show how to adapt our proofs to the other cases.

In formulating our lemmas and presenting our proofs we will mimic the structure of the proofs given in [7] and [8] and will sometimes refer the reader to their respective parts in cases concerning basic asimulations and their properties.

3.1 Proof of Theorem 1

Lemma 1 (Key Lemma 1). Let \( \varphi(x) = ST_{22}(I,x) \) for some modal intuitionistic formula \( I \), and let \( r(\varphi) = k \). Let \( \Sigma_\varphi \subseteq \Theta \), let \((M_1,t),(M_2,u)\) be two pointed \( \Theta \)-models, and let \((A,B)\) be a \((2,2)\)-modal \( \langle(M_1,t),(M_2,u)\rangle_1 \)-asimulation. Then

\[
((\bar{a}_m,a) \ A \ (\bar{b}_m,b) \land m + k \leq l \land a \models_i \varphi(x)) \Rightarrow b \models_j \varphi(x),
\]

for all \( i,j \in \{1,2\} \), \((\bar{a}_m,a) \in U_{i}^{m+1} \), and \((\bar{b}_m,b) \in U_{j}^{m+1} \).
Proof. We proceed by induction on the complexity of $I$. In what follows we will abbreviate the induction hypothesis by IH.

First we note, that since $(A, B)$ is a (2, 2)-modal $\langle (M_1, t), (M_2, u) \rangle$-simulation, then $A$ is a basic $\langle (M_1, t), (M_2, u) \rangle$-simulation, which means that we can re-use Lemma 1 of [7] in order to handle the basis and the induction steps for $\bot$, $\land$, $\lor$, and $\rightarrow$. There remain the two cases which involve the modal operators:

Case 1. Let $I = \square J$. Then

$$\varphi(x) = \forall y (R(x, y) \rightarrow \forall z (R_{\square}(y, z) \rightarrow ST_{22}(J, z))).$$

Assume that:

1. $a \models_i \forall y (R(x, y) \rightarrow \forall z (R_{\square}(y, z) \rightarrow ST_{22}(J, z)))$
2. $(\bar{a}_m, a) A (\bar{b}_m, b)$
3. $m + r(\varphi(x)) \leq l$

Moreover, it follows from definition of $r$ that:

4. $r(\varphi(x)) \geq 2$
5. $r(ST(\bar{J}, \bar{z})) \leq r(\varphi(x)) - 2$

Now, consider arbitrary $d, f \in U_j$ such that $b R_j d$ and $d R_{\square_j} f$. Since (3) and (4) clearly imply that $m + 1 < l$, it follows from (2) and (p-box-2) that one can choose $c, e \in U_i$, such that:

6. $a R_i c$
7. $c R_{\square_i} e$
8. $(\bar{a}_m, a, c, e) A (\bar{b}_m, b, d, f)$

So, we reason as follows:

9. $e \models_i ST_{22}(J, z)$ (from [11], [6], and [17])
10. $m + 2 + r(ST_{22}(J, z)) \leq l$ (from [3] and [4])
11. $f \models_j ST_{22}(J, z)$ (from [8], [9], [10] by IH)

Since $d$ was chosen to be an arbitrary $R_j$-successor of $b$, and $f$ an arbitrary $R_{\square_j}$-successor of $d$, this means that

$$b \models_j \forall y (R(x, y) \rightarrow \forall z (R_{\square}(y, z) \rightarrow ST_{22}(J, z))),$$

and we are done.

Case 2. Let $I = \Diamond J$. Then

$$\varphi(x) = \forall y (R(x, y) \rightarrow \exists z (R_{\Diamond}(y, z) \land ST_{22}(J, z))).$$

Assume that:
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\[ a \models \forall y (R(x, y) \rightarrow \exists z (R_\circ(y, z) \land ST_{22}(J, z))) \]  \hspace{1cm} (12)
\[ (\bar{a}_m, a) \ A (\bar{b}_m, b) \]  \hspace{1cm} (13)
\[ m + r(\varphi(x)) \leq l \]  \hspace{1cm} (14)

Moreover, it follows from definition of \( r \) that:
\[ r(\varphi(x)) \geq 2 \]  \hspace{1cm} (15)
\[ r(ST(J, y)) \leq r(\varphi(x)) - 2 \]  \hspace{1cm} (16)

Since (14) and (15) clearly imply that \( m + 1 < l \), it follows from (13) and (p-diam-2(1)) that one can choose a \( c \in U_i \), such that:
\[ a R_i c \]  \hspace{1cm} (17)
\[ (\bar{a}_m, a, c) \ B (\bar{b}_m, b, d) \]  \hspace{1cm} (18)

Now from (12) and (17) it follows that we can choose an \( e \in U_i \) such that
\[ e R_\circ_i e \]  \hspace{1cm} (19)
\[ e \models_i ST_{22}(J, z) \]  \hspace{1cm} (20)

Also, by (18), condition (p-diam-2(2)), and the fact that \( m + 1 < l \), we have:
\[ \exists f \in U_j (d R_\circ_j f \land (\bar{a}_m, a, c, e) \ A (\bar{b}_m, b, d, f)) \]  \hspace{1cm} (21)

We further get that:
\[ m + 2 + r(ST_{22}(J, z)) \leq l \]  \hspace{1cm} (from (14) and (16)) \hspace{1cm} (22)
\[ \exists f \in U_j (d R_\circ_i f \land (f \models_i ST_{22}(J, z))) \]  \hspace{1cm} (by IH from (20), (21), and (22)) \hspace{1cm} (23)

Since \( d \) was chosen to be an arbitrary \( R_j \)-successor of \( b \), this means that
\[ b \models_j \forall y (R(x, y) \rightarrow \exists z (R_\circ(y, z) \land ST_{22}(J, z))), \]
and we are done.

We use Lemma 1 to derive the ‘easy’ direction of (2, 2)-instantiation of Theorem 1.

**Corollary 1.** If \( \varphi(x) \) is equivalent to a (2, 2)-standard \( x \)-translation of an intuitionistic formula, then there exists \( k \in \mathbb{N} \) such that \( \varphi(x) \) is invariant with respect to (2, 2)-modal \( k \)-asimulations.

**Proof.** Let \( \varphi(x) \) be logically equivalent to \( ST_{22}(I, x) \) for some intuitionistic formula \( I \), and let \( r(ST_{22}(I, x)) = k \). Then it follows from Lemma 1 (setting \( i := 1, j := 2, m := 0, \) and \( l := k \)) that \( ST_{22}(I, x) \) is invariant with respect to (2, 2)-modal \( k \)-asimulations, and so is \( \varphi(x) \).
On our way to the inverse direction of (2, 2)-instantiation of Theorem 1 we first need a new piece of notation. For a formula \( \varphi(x) \) in the correspondence language, variable \( x \) and a natural \( l \), we denote with \( \text{int}(\varphi, x, l) \) the set of all (2, 2)-standard \( x \)-translations of intuitionistic formulas, which happen to be \((\Sigma_\varphi, x, l)\)-formulas. We use this notation to define three types of formulas which are important components in the proofs to follow. Let \( \Sigma_\varphi \subseteq \Theta \), let \( M \) be a \( \Theta \)-model and let \( a \in U \). Then:

\[
\text{tp}_l(\varphi(x), M, a) = \{ \psi(x) \in \text{int}(\varphi, x, l) \mid M, a \models \psi(x) \},
\]

\[
\overline{\text{tp}_l}(\varphi(x), M, a) = \{ \psi(x) \in \text{int}(\varphi, x, l) \mid M, a \not\models \psi(x) \},
\]

and, further:

\[
\text{imp}_l(\varphi(x), M, g) = \bigcap \{ \overline{\text{tp}_l}(\varphi(x), M, b) \mid \iota(R_\Theta)(a, b) \}
\]

We mention the following obvious link between different pieces of notation just defined. Under the above-mentioned assumptions about \( \varphi(x) \), \( l \), \( \Theta \), \( M \), \( a \), and for any \( M \) be a \( \Theta \)-model \( M' \) and \( a' \in U' \) we have:

\[
\text{tp}_l(\varphi(x), M, a) \subseteq \text{tp}_l(\varphi(x), M', a') \iff \overline{\text{tp}_l}(\varphi(x), M', a') \subseteq \overline{\text{tp}_l}(\varphi(x), M, a).
\]

We then invoke the following well known-fact about classical first-order logic:

**Lemma 2.** For any finite predicate vocabulary \( \Theta \), any variable \( x \) and any natural \( k \) there are, up to logical equivalence, only finitely many \((\Theta, x, k)\)-formulas.

This fact is proved as Lemma 3.4 in [5, pp. 189–190]. It implies that for every set of formulas, which have one of the forms \( \text{int}(\varphi, x, l) \), \( \text{tp}_l(\varphi(x), M, a) \), \( \overline{\text{tp}_l}(\varphi(x), M, a) \), \( \text{imp}_l(\varphi(x), M, a) \) there exist a finite subset collecting the logical equivalents for all the formulas in the set. Moreover, it allows us to collect logical equivalents of all (2, 2)-standard translations of intuitionistic formulas which are true together with a given formula at some pointed model in a single formula which we will call a complete conjunction:

**Definition 10.** Let \( \varphi(x) \) be a formula. A conjunction \( \Psi(x) \) of formulas from \( \text{int}(\varphi, x, k) \) is called a complete \((\varphi, x, k)\)-conjunction iff there is a pointed model \((M, a)\) such that \( M, a \models \Psi(x) \land \varphi(x) \), and for any \( \psi(x) \in \text{tp}_k(\varphi(x), M, a) \) we have \( \Psi(x) \models \psi(x) \).

The two following lemmas summarize some rather obvious properties of complete conjunctions:

**Lemma 3.** For any formula \( \varphi(x) \), any natural \( k \geq 1 \), any \( \Theta \) such that \( \Sigma_\varphi \subseteq \Theta \) and any pointed \( \Theta \)-model \((M, a)\) such that \( M, a \models \varphi(x) \) there is a complete \((\varphi, x, k)\)-conjunction \( \Psi(x) \) such that \( M, a \models \Psi(x) \land \varphi(x) \).

**Proof.** Consider \( \text{tp}_k(\varphi(x), M, a) \). This set is non-empty since \( ST_{22}(\bot \rightarrow \bot, x) \) will be true at \((M, a)\). Due to Lemma 2 we can choose in this set a non-empty finite subset \( \Gamma(x) \) such that any formula from \( \text{tp}_k(\varphi(x), M, a) \) is logically equivalent to (and hence follows from) a formula in \( \Gamma(x) \). By \( \Gamma(x) \subseteq \text{tp}_k(\varphi(x), M, a) \), we also have \( M, a \models \bigwedge \Gamma(x) \), therefore, \( \bigwedge \Gamma(x) \) is a complete \((\varphi, x, k)\)-conjunction. □
Lemma 4. For any formula \( \varphi(x) \) and any natural \( k \) there are, up to logical equivalence, only finitely many complete \((\varphi, x, k)\)-conjunctions.

Proof. It suffices to observe that for any formula \( \varphi(x) \) and any natural \( k \), a complete \((\varphi, x, k)\)-conjunction is a \((\Sigma_{\varphi}, x, k)\)-formula. Our lemma then follows from Lemma 2.

As a result, we are now able to establish the ‘hard’ right-to-left direction of Theorem 1.

Lemma 5 (Key Lemma 2). Let \( k = r(\varphi(x)) \) and let \( \varphi(x) \) be invariant with respect to \((2,2)\)-modal \( k \)-asimulations. Then \( \varphi(x) \) is equivalent to a \((2,2)\)-standard \( x \)-translation of a modal intuitionistic formula.

Proof. We may assume that both \( \varphi(x) \) and \( \neg \varphi(x) \) are satisfiable, since both \( \bot \) and \( \top \) are obviously invariant with respect to \((2,2)\)-modal \( k \)-asimulations and we have, for example, the following valid formulas:

\[
\bot \leftrightarrow ST_{22}(\bot, x), \top \leftrightarrow ST_{22}(\bot \rightarrow \bot, x).
\]

We may also assume that there are two complete \((\varphi, x, k+2)\)-conjunctions \( \Psi(x), \Psi'(x) \) such that \( \Psi'(x) \models \Psi(x) \), and both formulas \( \Psi(x) \wedge \varphi(x) \) and \( \Psi'(x) \wedge \neg \varphi(x) \) are satisfiable.

For suppose otherwise. Then take the set of all complete \((\varphi, x, k+2)\)-conjunctions \( \Psi(x) \) such that the formula \( \Psi(x) \wedge \varphi(x) \) is satisfiable. This set is non-empty, because \( \varphi(x) \) is satisfiable, and by Lemma 3 it can be satisfied only together with some complete \((\varphi, x, k+2)\)-conjunction. Now, using Lemma 3 choose in it a finite non-empty subset \( \{ \Psi_{i_1}(x), \ldots, \Psi_{i_n}(x) \} \) such that any complete \((\varphi, x, k+2)\)-conjunction is equivalent to an element of this subset. We can show that \( \varphi(x) \) is logically equivalent to \( \Psi_{i_1}(x) \lor \ldots \lor \Psi_{i_n}(x) \). In fact, if \( M, a \models \varphi(x) \) then, by Lemma 3 at least one complete \((\varphi, x, k+2)\)-conjunction is true at \((M, a)\) and therefore, its equivalent in \( \{ \Psi_{i_1}(x), \ldots, \Psi_{i_n}(x) \} \) is also true at \((M, a)\), and so, finally we have

\[
M, a \models \Psi_{i_1}(x) \lor \ldots \lor \Psi_{i_n}(x).
\]

In the other direction, if \( M, a \models \Psi_{i_1}(x) \lor \ldots \lor \Psi_{i_n}(x) \), then for some \( 1 \leq j \leq n \) we have \( M, a \models \Psi_{i_j}(x) \). Then, since \( \Psi_{i_j}(x) \models \Psi_{i_j}(x) \) and by the choice of \( \Psi_{i_j}(x) \) the formula \( \Psi_{i_j}(x) \wedge \varphi(x) \) is satisfiable, so, by our assumption, the formula \( \Psi_{i_j}(x) \wedge \neg \varphi(x) \) must be unsatisfiable, and hence \( \varphi(x) \) must follow from \( \Psi_{i_j}(x) \). But in this case we will have \( M, a \models \varphi(x) \) as well. So \( \varphi(x) \) is logically equivalent to \( \Psi_{i_1}(x) \lor \ldots \lor \Psi_{i_n}(x) \) but the latter formula, being a disjunction of conjunctions of \((2,2)\)-standard \( x \)-translations of modal intuitionistic formulas, is itself a \((2,2)\)-standard \( x \)-translation of a modal intuitionistic formula, and so we are done.

If, on the other hand, one can take two complete \((\varphi, x, k+2)\)-conjunctions \( \Psi(x), \Psi'(x) \) such that \( \Psi'(x) \models \Psi(x) \), and formulas \( \Psi(x) \wedge \varphi(x) \) and \( \Psi'(x) \wedge \neg \varphi(x) \) are satisfiable, we reason as follows. Take a pointed \( \Sigma_{\varphi} \)-model \((M_1, t)\) such that \( t \models_1 \Psi(x) \wedge \varphi(x) \), and that any formula \( \psi(x) \in tp_{k+2}(\varphi(x), M_1, t) \) follows from \( \Psi(x) \), and take any pointed model \((M_2, u)\) such that \( u \models_2 \Psi'(x) \wedge \neg \varphi(x) \).
We can construct an \((2, 2)\)-modal \((\langle M_1, t \rangle, \langle M_2, u \rangle)_k\)-asimulation and thus obtain a contradiction in the following way. We define it as the ordered couple \((A, B)\), where for arbitrary \(i, j \in \{1, 2\}\) and \((\bar{a}_m, a) \in U^{m+1}_i, (\bar{b}_m, b) \in U^{m+1}_j\) we set:

\[
(\bar{a}_m, a) A (\bar{b}_m, b) \iff (m \leq k \land tp_{k-m+2}(\varphi(x), M_i, a) \subseteq tp_{k-m+2}(\varphi(x), M_j, b)),
\]

and, for \(B\):

\[
(\bar{a}_m, a) B (\bar{b}_m, b) \iff (m \leq k \land imp_{k-m+1}(\varphi(x), M_j, b) \subseteq imp_{k-m+1}(\varphi(x), M_i, a)).
\]

With these definitions, we can show that \(A\) is a basic \((\langle M_1, t \rangle, \langle M_2, u \rangle)_k\)-asimulation, arguing as in Theorem 1 of \([2]\). Therefore, in order to show that \((A, B)\) is a \((2, 2)\)-modal \((\langle M_1, t \rangle, \langle M_2, u \rangle)_k\)-asimulation, we only need to verify conditions \([p-B\text{-}type}\), \([p-box\text{-}2]\), \([p\text{-}diam}\text{-}2(1)]\), and \([p\text{-}diam}\text{-}2(2)]. It is clear that \(B\) satisfies \([p\text{-}B\text{-}type}\).

To verify condition \([p-box\text{-}2]\), take any \((\bar{a}_m, a)\) \((\bar{b}_m, b)\) such that \(m + 1 < k\) and any \(d, f \in U_j\) such that \(b \not\models_R^j d\) and \(d \not\models_R^j f\). In this case we will also have \(m + 2 \leq k\). Then consider \(tp_{k-m}(\varphi(x), M_j, f)\). This set is non-empty, since by our assumption we have \(k - m \geq 0\). Therefore, as we have \(r(ST_{22}(\perp, x)) = 0\), we will also have \(ST_{22}(\perp, x) \in tp_{k-m}(\varphi(x), M_j, f)\). Then, according to our Lemma \([2]\), there exists a finite non-empty set of logical equivalents for \(tp_{k-m}(\varphi(x), M_j, f)\). Choosing this finite set, we in fact choose some finite \(\{ST_{22}(J_1, x) \ldots ST_{22}(J_q, x)\} \subseteq tp_{k-m}(\varphi(x), M_j, f)\) such that

\[
\forall \psi(x) \in tp_{k-m}(\varphi(x), M_j, f)(\psi(x) \models ST_{22}(J_1, x) \lor \ldots \lor ST_{22}(J_q, x)).
\]

But then we obtain that

\[
b \not\models_{j} ST_{22}(\Box(J_1 \lor \ldots \lor J_q), x).
\]

In fact, \(d, f\) jointly falsify this boxed disjunction for \((M_j, b)\). But, given that

\[
\{ST_{22}(J_1, x) \ldots ST_{22}(J_q, x)\} \subseteq tp_{k-m}(\varphi(x), M_j, f),
\]

the standard translation of boxed disjunction under consideration must be in \(tp_{k-m+2}(\varphi, M_j, b)\). Note, further, that by \((\bar{a}_m, a)\) \((\bar{b}_m, b)\) we have

\[
 tp_{k-m+2}(\varphi(x), M_i, a) \subseteq tp_{k-m+2}(\varphi(x), M_j, b),
\]

thus:

\[
 tp_{k-m+2}(\varphi(x), M_j, b) \subseteq tp_{k-m+2}(\varphi(x), M_i, a),
\]

and therefore this boxed disjunction must be false at \((M_i, a)\) as well. But then take any \(c, e \in U_i\) such that \(a R t e, c R t e\) and \(c, e\) falsify the boxed disjunction under consideration. By choice of \(\{ST(J_1, x) \ldots ST(J_q, x)\}\) it follows that

\[
 tp_{k-m}(\varphi(x), M_j, f) \subseteq tp_{k-m}(\varphi(x), M_i, e),
\]

and thus

\[
 tp_{k-m}(\varphi(x), M_i, e) \subseteq tp_{k-m}(\varphi(x), M_j, f),
\]

But then, again by the definition of \(A\), and given the fact that \(m + 2 \leq k\), we must also have \((\bar{a}_m, a, c, e)\) \((\bar{b}_m, b, d, f)\), and so condition \([p-box\text{-}2]\) holds.
To verify condition \(\text{p-diam-2(1)}\), take any \((\bar{a}_m, a)\) \(A\) \((\bar{b}_m, b)\) such that \(m + 1 < k\) and any \(d \in U_j\) such that \(b \ R_j \ d\). In this case we will also have \(m + 2 \leq k\). Then consider \(\text{imp}_{k-m}(\varphi(x), M_j, d)\). This set is non-empty, since by our assumption we have \(k - m \geq 0\). Therefore, as we have \(r(ST_{22}(\top, x)) = 0\), we will also have \(ST_{22}(\top, x) \in \text{imp}_{k-m}(\varphi(x), M_j, d)\). Then, according to our Lemma \(\ref{lem:finite}\) there exists a finite non-empty set of logical equivalents for \(\text{imp}_{k-m}(\varphi(x), M_j, d)\). Choosing this finite set, we in fact choose some finite

\[
\{ ST_{22}(J_1, x) \ldots ST_{22}(J_q, x) \} \subseteq \text{imp}_{k-m}(\varphi(x), M_j, d)
\]

such that

\[
\forall \psi(x) \in \text{imp}_{k-m}(\varphi(x), M_j, d) (\psi(x) \models ST_{22}(J_1, x) \lor \ldots \lor ST_{22}(J_q, x)).
\]

But then we obtain that

\[
b \not\models ST_{22}(\Diamond(J_1 \lor \ldots \lor J_q), x).
\]

In fact, \(d\) falsify this disjunction for \((M_j, b)\). But, given that

\[
\{ ST_{22}(J_1, x) \ldots ST_{22}(J_q, x) \} \subseteq \text{imp}_{k-m}(\varphi(x), M_j, d),
\]

the standard translation of the modalized disjunction under consideration must be in \(\overline{tp}_{k-m+2}(\varphi(x), M_j, b)\). Note, further, that by \((\bar{a}_m, a)\) \(A\) \((\bar{b}_m, b)\) we have

\[
 tp_{k-m+2}(\varphi(x), M_i, a) \subseteq tp_{k-m+2}(\varphi(x), M_j, b),
\]

thus:

\[
 \overline{tp}_{k-m+2}(\varphi(x), M_j, b) \subseteq \overline{tp}_{k-m+2}(\varphi(x), M_i, a),
\]

and therefore the modalized disjunction must be false at \((M_i, a)\) as well. But then take any \(c \in U_i\) such that \(a \ R_i \ c\), and for every \(e\), such that \(c \ R_{0i} \ e\), we have

\[
e \not\models ST_{22}(J_1, x) \lor \ldots \lor ST_{22}(J_q, x)
\]

By choice of \(\{ ST_{22}(J_1, x) \ldots ST_{22}(J_q, x) \}\) it follows that

\[
 \text{imp}_{k-m}(\varphi(x), M_j, d) \subseteq \text{imp}_{k-m}(\varphi(x), M_i, c),
\]

But then, again by the definition of \(B\), and given the fact that \(m + 2 \leq k\), we must also have \((\bar{a}_m, a, c)\) \(B\) \((\bar{b}_m, b, d)\), and so condition \(\text{p-diam-2(1)}\) holds.

Finally, to verify condition \(\text{p-diam-2(2)}\), take any \((\bar{a}_m, a)\) \(A\) \((\bar{b}_m, b)\) such that \(m < k\) and any \(c \in U_i\) such that \(a \ R_{0i} \ c\). In this case we will also have \(m + 1 \leq k\). Then consider \(tp_{k-m+1}(\varphi(x), M_i, c)\). This set is non-empty, since by our assumption we have \(k - m \geq 1\). Therefore, as we have \(r(ST_{22}(\top \rightarrow \top, x)) = 1\), we will also have \(ST_{22}(\top \rightarrow \top, x) \in tp_{k-m+1}(\varphi(x), M_i, c)\). Then, according to our Lemma \(\ref{lem:finite}\) there exists a finite non-empty set of logical equivalents for \(tp_{k-m+1}(\varphi(x), M_i, c)\). Choosing this finite set, we in fact choose some finite \(\{ ST_{22}(I_1, x) \ldots ST_{22}(I_p, x) \} \subseteq tp_{k-m+1}(\varphi(x), M_i, c)\) such that

\[
\forall \psi(x) \in tp_{k-m+1}(\varphi(x), M_i, c) (ST_{22}(I_1, x) \land \ldots \land ST_{22}(I_p, x) \models \psi(x)).
\]
But then we obtain that
\[ ST_{22}((I_1 \land \ldots \land I_p), x) \notin \text{imp}_{k-m+1}(\varphi(x), M_i, a). \]

Note, further, that by \((\bar{a}_m, a) B (\bar{b}_m, b)\) we have
\[ \text{imp}_{k-m+1}(\varphi(x), M_j, b) \subseteq \text{imp}_{k-m+1}(\varphi(x), M_i, a), \]
thus:
\[ ST_{22}((I_1 \land \ldots \land I_p), x) \notin \text{imp}_{k-m+2}(\varphi(x), M_j, b). \]

But then take any \(d \in U_j\) such that \(b \not\in j\) and we have
\[ d \models_j ST_{22}(I_1, x) \land \ldots \land ST_{22}(I_p, x). \]

By choice of \{ \(ST_{22}(I_1, x) \ldots ST_{22}(I_p, x)\) \} it follows that
\[ tp_{k-m+1}(\varphi(x), M_i, c) \subseteq tp_{k-m+1}(\varphi(x), M_j, d), \]

But then, again by the definition of \(A\), and given the fact that \(m + 1 \leq k\), we must also have \((\bar{a}_m, a, c) A (\bar{b}_m, b, d)\), and so condition \([p\text{-diam}-2(2)]\) holds.

Therefore \((A, B)\) is a \((2, 2)\)-modal \(((M_1, t), (M_2, u))\)-asimulation, and we have got our contradiction in place. \(\square\)

We can now finish the proof of \((2, 2)\)-instantiation of Theorem 1

**Proof.** The ‘only if’ direction we have by Corollary 1. In the other direction, let \(\varphi(x)\) be invariant with respect to \(k\)-asimulations for some \(k\). If \(k \leq r(\varphi)\), then every \((2, 2)\)-modal \(r(\varphi)\)-asimulation is a \((2, 2)\)-modal \(k\)-asimulation, so \(\varphi(x)\) is invariant with respect to \((2, 2)\)-modal \(r(\varphi)\)-asimulations and hence, by Lemma 1, \(\varphi(x)\) is equivalent to a standard \(x\)-translation of an intuitionistic formula. If, on the other hand, \(r(\varphi) < k\), then set \(l := k - r(\varphi)\) and consider variables \(\bar{y}_l\) not occurring in \(\varphi(x)\). Then \(r(\forall \bar{y}_l \varphi(x)) = k\) and \(\varphi(x)\) is logically equivalent to \(\forall \bar{y}_l \varphi(x)\), so the latter formula is also invariant with respect to \(k\)-asimulations, and hence by Theorem 5 \(\forall \bar{y}_l \varphi(x)\) is logically equivalent to a standard \(x\)-translation of an intuitionistic formula. But then \(\varphi(x)\) is equivalent to this standard \(x\)-translation as well. \(\square\)

### 3.2 Proof of Theorem 2

We now turn to the proof of \((2, 2)\)-instantiation of Theorem 2. The ‘only if’ direction we again have by Corollary 1.

**Corollary 2.** If \(\varphi(x)\) is equivalent to a \((2, 2)\)-standard \(x\)-translation of an intuitionistic formula, then \(\varphi(x)\) is invariant with respect to \((2, 2)\)-modal asimulations.

**Proof.** Let \(\varphi(x)\) be logically equivalent to \(ST_{22}(I, x)\) for some intuitionistic formula \(I\).
For an arbitrary \(\Theta \supseteq \Sigma_\varphi\), \(\Theta\)-models \(M_1\) and \(M_2\), and arbitrary \(t \in U_1, u \in U_2\) let \((A, B)\) be a \((2, 2)\)-modal \(((M_1, t), (M_2, u))\)-asimulation, so that we have \(t A u\). Assume that
\[ t \models_1 \varphi(x). \]
Then consider the ordered couple \((A', B')\) such that:
\[
A' = \{ \langle (\bar{a}_m, a), (\bar{b}_m, b) \rangle | \exists i, j(\{i, j\} = \{1, 2\} \land \bar{a}_m, a \in U_i \land \bar{b}_m, b \in U_j \land a \equiv A b) \};
\]
\[
B' = \{ \langle (\bar{a}_m, a), (\bar{b}_m, b) \rangle | \exists i, j(\{i, j\} = \{1, 2\} \land \bar{a}_m, a \in U_i \land \bar{b}_m, b \in U_j \land a \equiv B b) \};
\]
It is straightforward to verify that \((A', B')\) is a \((2, 2)\)-modal \(\langle (M_1, t), (M_2, u) \rangle_k\)-}\text{-}\text{asimulation for every } k \in \mathbb{N}. Moreover, we still have \(t A' u\). By Corollary \[\text{Lemma 5}\] there is a natural \(k\), such that \(\varphi(x)\) is invariant with respect to \((2, 2)\)-modal \(k\)-}\text{-}\text{asimulation}, therefore we have
\[
u \models_2 \varphi(x).
\]
Since the \((2, 2)\)-modal \(\langle (M_1, t), (M_2, u) \rangle\)-}\text{-}\text{asimulation} \((A, B)\) was chosen arbitrarily, this means that \(\varphi(x)\) is invariant with respect to \((2, 2)\)-modal asimulations.

To proceed, we need to introduce some further notions and results from classical model theory. For a model \(M\) and \(\bar{a}_n \in U\) let \([M, \bar{a}_n]\) be the extension of \(M\) with \(\bar{a}_n\) as new individual constants interpreted as themselves. It is easy to see that there is a simple relation between the truth of a formula at a sequence of elements of a \(\Theta\)-model and the truth of its substitution instance in an extension of the above-mentioned kind; namely, for any \(\Theta\)-model \(M\), any \(\Theta\)-formula \(\varphi(\bar{y}_n, \bar{w}_m)\) and any \(\bar{a}_n, \bar{b}_m \in U\) it is true that:
\[
[M, \bar{a}_n], \bar{b}_m \models \varphi(\bar{a}_n, \bar{w}_m) \iff [M, \bar{a}_n] \models \varphi(\bar{y}_n, \bar{w}_m).
\]
We will call a theory of \(M\) (and write \(Th(M)\)) the set of all first-order sentences true at \(M\). We will call an \(n\)-type of \(M\) a set of formulas \(\Gamma(\bar{w}_n)\) consistent with \(Th(M)\).

**Definition 11.** Let \(M\) be a \(\Theta\)-model. \(M\) is \(\omega\)-saturated iff for all \(k \in \mathbb{N}\) and for all \(\bar{a}_n \in U\), every \(k\)-type \(\Gamma(\bar{w}_k)\) of \([M, \bar{a}_n]\) is satisfiable in \([M, \bar{a}_n]\).

Definition of \(\omega\)-saturation normally requires satisfiability of 1-types only. However, our modification is equivalent to the more familiar version: see e.g. \[\text{Lemma 4.31, p. 73}\].

It is known that every model can be elementarily extended to an \(\omega\)-saturated model; in other words, the following lemma holds:

**Lemma 6.** Let \(M\) be a \(\Theta\)-model. Then there is an \(\omega\)-saturated extension \(M'\) of \(M\) such that for all \(\bar{a}_n \in U\) and every \(\Theta\)-formula \(\varphi(\bar{w}_n)\):
\[
M, \bar{a}_n \models \varphi(\bar{w}_n) \iff M', \bar{a}_n \models \varphi(\bar{w}_n).
\]

The latter lemma is a trivial corollary of e.g. \[\text{Lemma 5.1.14, p. 216}\].

A very useful property of \(\omega\)-saturated models is that one can define among them \((2, 2)\)-modal asimulations more or less according to the strategy assumed in the proof of \[\text{Lemma 5}\]. In order to do this, however, we need to re-define the types used in the above proof. We collect the required changes in the following definition:
**Definition 12.** Let $M$ be a $\Theta$-model, $t \in U$ and let $x$ be a variable in correspondence language. Then we define $\text{int}_x(\Theta)$ to be the set of all $\Theta$-formulas that are $(2,2)$-standard $x$-translations of modal intuitionistic formulas. We further set:

$$\text{tp}_x(M,t) = \{ \varphi(x) \in \text{int}_x(\Theta) \mid M,t \models \varphi(x) \};$$

$$\overline{\text{tp}}_x(M,t) = \{ \varphi(x) \in \text{int}_x(\Theta) \mid M,t \not\models \varphi(x) \};$$

$$\text{imp}_x(M,t) = \bigcap \{ \overline{\text{tp}}_x(M,u) \mid i(\mathcal{R}_\Box)(t,u) \}. $$

The analogue of ‘contrapositive’ scheme mentioned above holds, namely, for arbitrary models $M$, $M'$, and elements $a \in U$, and $a' \in U'$ we have:

$$\text{tp}_x(M,a) \subseteq \text{tp}_x(M',a') \iff \overline{\text{tp}}_x(M',a') \subseteq \overline{\text{tp}}_x(M,a). $$

The following lemma gives the precise version of the above statement:

**Lemma 7 (Key Lemma 3).** Let $M_1$, $M_2$ be $\omega$-saturated $\Theta$-models, let $t \in U_1$, and $t' \in U_2$ be such that $\text{tp}(M_1,t) \subseteq \text{tp}(M_2,t')$. Let $x$ be a variable in correspondence language. Then the ordered couple $(A,B)$ such that:

$$A = \{ (a,b) \mid \exists i,j \{ \{ i,j \} = \{ 1,2 \} \land \text{tp}_x(M_i,a) \subseteq \text{tp}_x(M_j,b) \} \}$$

and:

$$B = \{ (a,b) \mid \exists i,j \{ \{ i,j \} = \{ 1,2 \} \land \text{imp}_x(M_i,a) \supseteq \text{imp}_x(M_j,b) \} \}$$

is a $(2,2)$-modal $((M_1,t),(M_2,t'))$-asimulation.

**Proof.** It is evident that $B$ satisfies the $\Box$-type, and arguing as in Lemma 8 of [8], we can show that $A$ is a basic $((M_1,t),(M_2,t'))$-asimulation. So we are left to consider the last three conditions of Definition 8.

To verify condition $\Box$-2, choose any $i$, $j$ such that $\{ i,j \} = \{ 1,2 \}$, any $a \in U_i$, $b \in U_j$ such that $a \mathcal{R} b$, that is to say, $\text{tp}_x(M_i,a) \subseteq \text{tp}_x(M_j,b)$ and choose any $d$, $f \in U_j$ for which we have $b \mathcal{R}_d d$ and $d \mathcal{R}_{\Box} f$.

Consider $\overline{\text{tp}}_x(M_j,f)$. If $\{ ST_{22}(J_1,x) \ldots ST_{22}(J_q,x) \}$ is a finite subset of this type, then we have

$$b \not\models_j ST(\Box(J_1 \lor \ldots \lor J_q),x).$$

Since by contraposition of $a \mathcal{R} b$ we have that $\overline{\text{tp}}_x(M_j,b) \subseteq \overline{\text{tp}}_x(M_i,a)$, we obtain that

$$a \not\models_i ST(\Box(J_1 \lor \ldots \lor J_q),x).$$

This means that every finite subset of the type

$$\{ R(a,y), R_\Box(y,x) \} \cup \{ \neg \psi(x) \mid \psi(x) \in \overline{\text{tp}}_x(M_j,f) \}$$

is satisfiable at $[M_i,a]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([M_i,a])$ and, by $\omega$-saturation of both $M_1$ and $M_2$, it must be satisfied in $[M_i,a]$ by some $c$, $e \in U_i$. So for any such $c$ and $e$ we will have $a \mathcal{R}_i c$, $c \mathcal{R}_{\Box_i} e$ and, moreover,

$$\forall \psi \in \overline{\text{tp}}_x(M_j,f)(e \not\models_i \psi(x)).$$
Thus we have that $\overline{tp}_x(M_j, f) \subseteq \overline{tp}_x(M_i, e)$, and further, by contraposition, that $tp_x(M_i, e) \subseteq tp_x(M_j, f)$. Thus we get that $c A f$ and condition (box-2) is verified.

To verify condition (diam-2(1)), choose any $i, j$ such that $\{i, j\} = \{1, 2\}$, any $a \in U_i$, $b \in U_j$ such that $a A b$, that is to say, $tp_x(M_i, a) \subseteq tp_x(M_j, b)$ and choose any $d \in U_j$ for which we have $b R_j d$.

Consider $imp_x(M_j, d)$. If $\{ST_{22}(J_1, x) \ldots ST_{22}(J_q, x)\}$ is a finite subset of this type, then we have

$$b \not\models_j ST(\bigvee (J_1 \lor \ldots \lor J_q), x).$$

Since by contraposition of $a A b$ we have that $\overline{tp}_x(M_j, b) \subseteq \overline{tp}_x(M_i, a)$, we obtain that

$$a \not\models_i ST(\bigvee (J_1 \lor \ldots \lor J_q), x).$$

This means that every finite subset of the type

$$\{ R(a, x) \} \cup \{ \forall y(R_\bigcirc(x, y) \rightarrow \neg \psi(y)) \mid \psi(x) \in imp_x(M_j, d) \}$$

is satisfiable at $[M_i, a]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([M_i, a])$ and, by $\omega$-saturation of both $M_1$ and $M_2$, it must be satisfied in $[M_i, a]$ by some $c \in U_i$. So for any such $c$ we will have a $R_i c$ and, moreover,

$$\forall \psi(x) \in imp_x(M_j, d))(\forall e \in U_i)(R_{\bigcirc i}(c, e) \Rightarrow e \not\models_i \psi(x)).$$

Thus we have that $imp_x(M_j, d) \subseteq imp_x(M_i, c)$, and therefore, that $c B d$. Thus condition (diam-2(1)) is verified.

Finally, to verify condition (diam-2(2)), choose any $i, j$ such that $\{i, j\} = \{1, 2\}$, any $a \in U_i$, $b \in U_j$ such that $a B b$, that is to say, $imp_x(M_j, b) \subseteq imp_x(M_i, a)$ and choose any $c \in U_i$ for which we have a $R_{\bigcirc j} c$.

Consider $tp_x(M_i, c)$. If $\{ST_{22}(I_1, x) \ldots ST_{22}(I_p, x)\}$ is a finite subset of this type, then we have

$$c \models_i ST((I_1 \land \ldots \land I_p), x).$$

Therefore, the set $\{ST_{22}(I_1, x) \ldots ST_{22}(I_p, x)\}$ is disjoint from $imp_x(M_i, a)$, and thus it is also disjoint from $imp_x(M_j, b)$. Therefore, the formula $ST(\bigvee (I_1 \land \ldots \land I_p), x)$ is also verified by some $R_{\bigcirc j} b$-successor of $b$. More formally, this means that every finite subset of the type

$$\{ R_\bigcirc (b, x) \} \cup \{ \psi(x) \mid \psi(x) \in tp_x(M_i, c) \}$$

is satisfiable at $[M_j, b]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([M_j, b])$ and, by $\omega$-saturation of both $M_1$ and $M_2$, it must be satisfied in $[M_j, b]$ by some $d \in U_j$. So for any such $d$ we will have $b R_{\bigcirc j} d$ and, moreover,

$$\forall \psi(x) \in tp_x(M_i, c))(d \models_j \psi(x)).$$

Thus we have that $tp_x(M_i, c) \subseteq tp_x(M_j, c)$, and therefore, that $c A d$. The condition (diam-2(2)) is verified.

We are prepared now to prove the hard part of (2, 2)-instantiation of Theorem 2.

**Lemma 8.** Let $\varphi(x)$ be invariant with respect to (2, 2)-modal asimulations. Then $\varphi(x)$ is equivalent to (2, 2)-standard translation of a modal intuitionistic formula.
Proof. We may assume that \( \varphi(x) \) is satisfiable, for \( \bot \) is clearly invariant with respect to \((2,2)\)-modal asimations and \( \bot \leftrightarrow ST_{22}(\bot, x) \) is a valid formula. Throughout this proof, will write \( ic(\varphi(x)) \) for the following set:

\[
\{ \psi(x) \in \text{int}_x(\Sigma_c) \mid \varphi(x) \models \psi(x) \}
\]

Our strategy will be to show that \( ic(\varphi(x)) \models \varphi(x) \). Once this is done, we will apply compactness of first-order logic and conclude that \( \varphi(x) \) is equivalent to a finite conjunction of standard \((2,2)\)-modal \( x \)-translations of intuitionistic formulas and hence to a standard \( x \)-translation of the corresponding intuitionistic conjunction.

To show this, take any \( \Sigma_c \)-model \( M_1 \) and \( a \in U_1 \) such that \( a \models_1 ic(\varphi(x)) \). Then, of course, we also have \( ic(\varphi(x)) \subseteq tp_x(M_1, a) \). Such a model exists, because \( \varphi(x) \) is satisfiable and \( ic(\varphi(x)) \) will be satisfied in any model satisfying \( \varphi(x) \). Then we can also choose a \( \Sigma_c \)-model \( M_2 \) and \( b \in U_2 \) such that \( b \models_2 \varphi(x) \) and \( tp_x(M_2, b) \subseteq tp_x(M_1, a) \).

For suppose otherwise. Then for any \( \Sigma_c \)-model \( M \) such that \( U \subseteq N \) and any \( c \in U \) such that \( M, c \models \varphi(x) \) we can choose a modal intuitionistic formula \( I(M, c) \) such that \( ST_{22}(I(M, c), x) \) is in \( tp_x(M, c) \) but not in \( tp_x(M, a) \). Then consider the set

\[
S = \{ \varphi(x) \} \cup \{ \neg ST_{22}(I(M, c), x) \mid M, c \models \varphi(x) \}
\]

Let \( \{ \varphi(x), \neg ST_{22}(I_1, x), \ldots, \neg ST_{22}(I_q, x) \} \) be a finite subset of this set. If this set is unsatisfiable, then we must have \( \varphi(x) \models ST_{22}(I_1, x) \lor \ldots \lor \neg ST_{22}(I_q, x) \), but then we will also have \( (ST_{22}(I_1, x) \lor \ldots \lor ST_{22}(I_q, x)) \subseteq ic(\varphi(x)) \subseteq tp_x(M_1, a) \), and hence \( ST_{22}(I_1, x) \lor \ldots \lor ST_{22}(I_q, x) \) will be true at \((M_1, a)\). But then at least one of \( ST_{22}(I_1, x) \ldots ST_{22}(I_q, x) \) must also be true at \((M_1, a)\), which contradicts the choice of these formulas. Therefore, every finite subset of \( S \) is satisfiable, and, by compactness, \( S \) itself is satisfiable as well. But then, by the Löwenheim-Skolem property, we can take a \( \Sigma_c \)-model \( M' \) such that \( U' \subseteq N \) and \( g \in U' \) such that \( S \) is true at \((M', g)\), and this will be a model for which we will have both \( M', g \models ST_{22}(I(M', g), x) \) by choice of \( I(M', g) \) and \( M', g \not\models ST_{22}(I(M', g), x) \) by satisfaction of \( S \), a contradiction.

Therefore, we will assume in the following that some \( \Sigma_c \)-model \( M_2 \) and some \( b \in U_2 \) are such that \( a \models_1 ic(\varphi(x)) \), \( b \models_2 \varphi(x) \), and \( tp_x(M_2, b) \subseteq tp_x(M_1, a) \). According to Lemma 6 there exist \( \omega \)-saturated elementary extensions \( M', M'' \) of \( M_1 \) and \( M_2 \), respectively. We have:

\[
M_1, a \models \varphi(x) \iff M', a \models \varphi(x) \tag{24}
\]

\[
M'', b \models \varphi(x) \tag{25}
\]

Also, since \( M_1, M_2 \) are elementarily equivalent to \( M', M'' \), respectively, we have

\[
\text{tp}_x(M'', b) = \text{tp}_x(M_2, b) \subseteq \text{tp}_x(M_1, a) = \text{tp}_x(M', a).
\]

By \( \omega \)-saturation of \( M', M'' \) and Lemma 7 the ordered couple \((A, B)\) such that:

\[
A = \{ \langle c, d \rangle \mid \exists \mu, \mu' (\{ \mu, \mu' \} = \{ M', M'' \} \land \text{tp}_x(\mu, c) \subseteq \text{tp}_x(\mu', d) \})
\]

\[
B = \{ \langle c, d \rangle \mid \exists \mu, \mu' (\{ \mu, \mu' \} = \{ M', M'' \} \land \text{imp}_x(\mu, c) \supseteq \text{imp}_x(\mu', d) \})
\]

is a \((2,2)\)-modal \( \langle M'', b \rangle, (M', a) \)-asimulation. But then by (24) and invariance of \( \varphi(x) \) we get \( M', a \models \varphi(x) \), and further, by (25) we conclude that \( M_1, a \models \varphi(x) \). Therefore, \( \varphi(x) \) in fact follows from \( ic(\varphi(x)) \).

\[\square\]

Theorem 2 now follows from Corollary 2 and Lemma 8.
4 Other cases

We now briefly show how to obtain the proofs for the other three instantiations of Theorems 1 and 2. The general scheme of the proofs in these cases is very similar to the proofs given in the previous section. The main difference is that in the other cases we need to assume different sets of conditions in the definitions of modal \( k \)-asimulations and modal asimulations \textit{simpliciter}. This affects the three key lemmas of the previous section, namely, Lemmas 1, 5, and 7, in that some parts of their proofs become irrelevant and some new parts need to be supplied instead. Accordingly, when treating the other three instantiations of our main results below, we mainly concentrate on how to revise the proofs of these three lemmas.

4.1 Case \( i = 1, j = 2 \)

In order to obtain the proofs of Theorems 1 and 2 one needs to revise the proofs given in Section in the following way:

\textit{Ad Key Lemma 1:}

Revise the inductive step in case where \( I = \Box J \) as follows:

In this case we have

\[
\varphi(x) = \forall y(R_\Box(x, y) \rightarrow ST_{12}(J, y)).
\]

Assume that:

\[
a \models_i \forall y(R_\Box(x, y) \rightarrow ST_{12}(J, y))
\]
\[
(\bar{a}_m, a) A (\bar{b}_m, b)
\]
\[
m + r(\varphi(x)) \leq l
\]

Moreover, it follows from definition of \( r \) that:

\[
r(\varphi(x)) \geq 1
\]
\[
r(ST(J, y)) \leq r(\varphi(x)) - 1
\]

Now, consider arbitrary \( d \in U_j \) such that \( b R_{\Box_j} d \). Since \( 28 \) and \( 29 \) clearly imply that \( m < l \), it follows from \( 27 \) and \( \text{p-box-1} \) that one can choose a \( c \in U_i \), such that:

\[
a R_{\Box_i} c
\]
\[
(\bar{a}_m, a, c) A (\bar{b}_m, b, d)
\]

So, we reason as follows:

\[
c \models_i ST_{12}(J, y)
\]
\[
m + 1 + r(ST_{12}(J, y)) \leq l
\]
\[
d \models_j ST_{12}(J, y)
\]

Since \( d \) was chosen to be an arbitrary \( R_{\Box_j} \)-successor of \( b \), this means that

\[
b \models_j \forall y(R_\Box(x, y) \rightarrow ST_{12}(J, y))),
\]
and we are done.

Ad Key Lemma 2:
Replace the verification of condition \( \text{[p-box-2]} \) with the following verification of condition \( \text{[p-box-1]} \):

Take any \((\bar{a}_m, a)\) \(A(\bar{b}_m, b)\) such that \(m < k\) and any \(d \in U_j\) such that \(b \not\in J_{\bar{a}_m, a}\). In this case we will also have \(m + 1 \leq k\). Then consider \(\bar{t}_k(x)\). This set is non-empty, since by our assumption we have \(k - m \geq 0\). Therefore, as we have \(r(ST_{12}(\bot, x)) = 0\), we will also have \(ST_{12}(\bot, x) \in \bar{t}_k(x)\). Then, according to Lemma 2 there exists a finite non-empty set of logical equivalents for \(\bar{t}_k(x)\). Choosing this finite set, we in fact choose some finite set \(\{ST_{12}(J_1, x) \ldots ST_{12}(J_q, x)\} \subseteq \bar{t}_k(x)\) such that

\[
\forall \psi(x) \in \bar{t}_k(x), M_j, d(\psi(x) \models ST_{12}(J_1, x) \lor \ldots \lor ST_{12}(J_q, x)).
\]

But then we obtain that

\(b \not\models_j ST_{12}(\Box(J_1 \lor \ldots \lor J_q), x)\).

In fact, \(d\) falsifies this boxed disjunction for \((M_j, b)\). But, given that

\[
\{ST_{12}(J_1, x) \ldots ST_{12}(J_q, x)\} \subseteq \bar{t}_k(x), M_j, d,
\]

the standard translation of boxed disjunction under consideration must be in \(\bar{t}_k(x)\). Note, further, that by \((\bar{a}_m, a)\) \(A(\bar{b}_m, b)\) we have

\[
\bar{t}_k(x), M_j, a \subseteq \bar{t}_k(x), M_j, b,
\]

thus:

\[
\bar{t}_k(x), M_j, b \subseteq \bar{t}_k(x), M_i, a,
\]

and therefore this boxed disjunction must be false at \((M_i, a)\) as well. But then take any \(c \in U_i\) such that \(c \not\models_i c\) falsifies the disjunction under consideration. By choice of \(\{ST_{12}(J_1, x) \ldots ST_{12}(J_q, x)\}\) it follows that

\[
\bar{t}_k(x), M_j, d \subseteq \bar{t}_k(x), M_i, c,
\]

and thus

\[
\bar{t}_k(x), M_i, c \subseteq \bar{t}_k(x), M_j, d.
\]

But then, again by the definition of \(A\), and given the fact that \(m + 1 \leq k\), we must also have \((\bar{a}_m, a, c)\) \(A(\bar{b}_m, b, d)\), and so condition \(\text{[p-box-1]}\) holds.

Ad Key Lemma 3:
Replace the verification of condition \(\text{[box-2]}\) with the following verification of condition \(\text{[box-1]}\):

Choose any \(i, j\) such that \(\{i, j\} = \{1, 2\}\), any \(a \in U_i, b \in U_j\) such that \(a A b\), that is to say, \(\bar{t}_x(M_i, a) \subseteq \bar{t}_x(M_j, b)\) and choose any \(d \in U_j\) for which we have \(b \not\models J_{\bar{a}_m, a}\). Consider \(\bar{t}_x(M_j, d)\). If \(\{ST_{12}(J_1, x) \ldots ST_{12}(J_q, x)\}\) is a finite subset of this type, then we have

\(b \not\models_j ST_{12}(\Box(J_1 \lor \ldots \lor J_q), x)\).
Since by contraposition of \( a A b \) we have that \( \overline{tp}_x(M_j, b) \subseteq \overline{tp}_x(M_i, a) \), we obtain that
\[
a \nmid_i ST_{12}(\square(J_1 \lor \ldots \lor J_q), x).
\]
This means that every finite subset of the type
\[
\{ R_{\square}(a, x) \} \cup \{ \neg \psi(x) \mid \psi(x) \in \overline{tp}_x(M_j, d) \}
\]
is satisfiable at \([M_i, a]\). Therefore, by compactness of first-order logic, this set is consistent with \( Th([M_i, a]) \) and, by \( \omega \)-saturation of both \( M_1 \) and \( M_2 \), it must be satisfied in \([M_i, a]\) by some \( c \in U_i \). So for any such \( c \) we will have \( a R_{\square} c \) and, moreover,
\[
\forall \psi \in \overline{tp}_x(M_j, d)(c \nmid_i \psi(x)).
\]
Thus we have that \( \overline{tp}_x(M_j, d) \subseteq \overline{tp}_x(M_i, c) \), and further, by contraposition, that \( tp_x(M_i, c) \subseteq tp_x(M_j, d) \). Thus we get that \( c A d \) and condition \( \text{box-2} \) is verified.

### 4.2 Case \( i = 2, j = 1 \)

The changes in three key lemmas in this case will look as follows:

**Ad Key Lemma 1:**

Revise the inductive step for the case \( I = \Box J \) as follows:

In this case we have
\[
\varphi(x) = \exists y(R_\Box(x, y) \land ST_{21}(J, y)).
\]
Assume that:
\[
a \models_i \exists y(R_\Box(x, y) \land ST_{21}(J, y)) \tag{36}
\]
\[
(\bar{a}_m, a) A (\bar{b}_m, b) \tag{37}
\]
\[
m + r(\varphi(x)) \leq l \tag{38}
\]
Moreover, it follows from definition of \( r \) that:
\[
r(\varphi(x)) \geq 1 \tag{39}
\]
\[
r(ST(J, y)) \leq r(\varphi(x)) - 1 \tag{40}
\]
Now, by \( \text{38} \) choose a \( c \in U_i \) such that
\[
a R_{\Box} c \tag{41}
\]
\[
c \models_i ST_{21}(J, y) \tag{42}
\]
Since \( \text{38} \) and \( \text{39} \) clearly imply that \( m < l \), it follows from \( \text{37} \) and \( \text{p-diam-1} \) that one can choose a \( d \in U_j \), such that:
\[
b R_{\Box} d \tag{43}
\]
\[
(\bar{a}_m, a, c) A (\bar{b}_m, b, d) \tag{44}
\]
So, we get that:

\[ m + 1 + r(ST_21(J, y)) \leq l \]  
(from (38) and (40))  
(45)

\[ d \models_j ST_21(J, y) \]  
(from (42), (44), (45) by IH)  
(46)

Finally, from (43) and (46) infer that:

\[ b \models_j \exists y(R_\bigcirc(x, y) \land ST_21(J, y)), \]

and we are done.

**Ad Key Lemma 2:**

Replace the verification of conditions [p-diam-2(1)] and [p-diam-2(2)] with the following verification of condition [p-diam-1]:

Take any \((\bar{a}_m, a)\) \(A (\bar{b}_m, b)\) such that \(m < k\) and any \(c \in U_i\) such that \(a R_\bigcirc i\ c\). In this case we will also have \(m + 1 \leq k\). Then consider \(tp_{k-m+1}(\varphi(x), M_i, c)\). This set is non-empty, since by our assumption we have \(k - m + 1 \geq 1\). Therefore, as we have \(r(ST_21(\bot \rightarrow \bot, x)) = 1\), we will also have \(ST_21(\bot \rightarrow \bot, x) \in tp_{k-m+1}(\varphi(x), M_i, c)\). Then, according to Lemma 2 there exists a finite non-empty set of logical equivalents for \(tp_{k-m+1}(\varphi(x), M_i, c)\). Choosing this finite set, we in fact choose some finite \{ST_21(I_1, x) \ldots ST_21(I_p, x)\} \subseteq tp_{k-m+1}(\varphi(x), M_i, c)\) such that

\[ \forall \psi(x) \in tp_{k-m+1}(\varphi(x), M_i, c)(ST_21(I_1, x) \land \ldots \land ST_21(I_p, x) \models \psi(x)). \]

But then we obtain that

\[ a \models_i ST_21(\bigtriangleup(I_1 \land \ldots \land I_p), x). \]

In fact, \(c\) verifies this modalized conjunction for \((M_i, a)\). But, given that

\[ \{ST_21(I_1, x) \ldots ST_21(I_p, x)\} \subseteq tp_{k-m+1}(\varphi(x), M_i, c) , \]

the standard translation of modalized conjunction under consideration must be in \(tp_{k-m+2}(\varphi(x), M_i, a)\). Note, further, that by \((\bar{a}_m, a)\) \(A (\bar{b}_m, b)\) we have

\[ tp_{k-m+2}(\varphi(x), M_i, a) \subseteq tp_{k-m+2}(\varphi(x), M_j, b), \]

and therefore this modalized conjunction must be true at \((M_j, b)\) as well. But then take any \(d \in U_j\) such that \(b R_\bigcirc j d\) and \(d\) verifies the conjunction under consideration. By choice of \{ST_21(I_1, x) \ldots ST_21(I_p, x)\} it follows that

\[ tp_{k-m+1}(\varphi(x), M_i, c) \subseteq tp_{k-m+1}(\varphi(x), M_j, d), \]

But then, again by the definition of \(A\), and given the fact that \(m + 1 \leq k\), we must also have \((\bar{a}_m, a, c)\) \(A (\bar{b}_m, b, d)\), and so condition [p-diam-1] holds.

**Ad Key Lemma 3:**

Replace the verification of conditions [diam-2(1)] and [diam-2(2)] with the following verification of condition [diam-1]:

Choose any \(i, j\) such that \(\{i, j\} = \{1, 2\}\), any \(a \in U_i\), \(b \in U_j\) such that \(a A b\), that is to say, \(tp_x(M_i, a) \subseteq tp_x(M_j, b)\) and choose any \(c \in U_i\) for which we have \(a R_\bigcirc i c\).
Consider $tp_x(M_i,c)$. If $\{ ST_{21}(I_1,x) \ldots ST_{21}(I_p,x) \}$ is a finite subset of this type, then we have
$$a \models ST_{21}(\Diamond(I_1 \land \ldots \land I_p),x).$$
By $tp_x(M_i,a) \subseteq tp_x(M_j,b)$, we obtain that
$$b \models ST_{12}(\Diamond(I_1 \land \ldots \land I_p),x).$$
This means that every finite subset of the type
$$\{ R(\Diamond) \circ (b,x) \} \cup \{ \psi(x) \mid \psi(x) \in tp_x(M_i,c) \}$$
is satisfiable at $[M_j,b]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([M_j,b])$ and, by $\omega$-saturation of both $M_1$ and $M_2$, it must be satisfied in $[M_j,b]$ by some $d \in U_j$. So for any such $d$ we will have $b R_{\Diamond j} d$ and, moreover,
$$\forall \psi \in tp_x(M_i,c)(d \models_j \psi(x)).$$
Thus we have that $tp_x(M_i,c) \subseteq tp_x(M_j,d)$. Thus we get that $c A d$ and condition (diam 1) is verified.

Another important revision of the proofs given in Section for the case $i = 2, j = 1$ is omission of every reference to relation $B$, since asimulations are now being defined as single relations rather than ordered couples of relations.

Finally, in order to accommodate the proofs in Section to the case $i = j = 1$ one just needs to combine the revisions given in the present sections in a straightforward way.

### 5 Characterization modulo first-order definable classes of models

Theorem establishes a criterion for the equivalence of first-order formula to a standard translation of intuitionistic formula on arbitrary first-order models. But one may have a special interest in a proper subclass $\kappa$ of the class of first-order models viewing the models which are not in this subclass as irrelevant, non-intended etc. In this case one may be interested in the criterion for equivalence of a given first-order formula to a standard translation of an intuitionistic predicate formula over this particular subclass. It turns out that if some parts of this subclass are first-order axiomatizable then only a slight modification of our general criterion is necessary to solve this problem.

To tighten up on terminology, we introduce the following definitions:

**Definition 13.** Let $\kappa$ be a class of models. Then:

1. $\kappa(\Theta) = \{ M \in \kappa \mid M \text{ is a } \Theta\text{-model} \}$;
2. $\kappa(\Theta)$ is first-order axiomatizable iff there is a set $Ax$ of $\Theta$-sentences, such that a $\Theta$-model $M$ is in $\kappa$ iff $M \models Ax$;
3. A set $\Gamma$ of $\Theta$-formulas is $\kappa$-satisfiable iff $\Gamma$ is satisfied in some model of $\kappa$;
4. A $\Theta$-formula $\varphi$ $\kappa$-follows from $\Gamma$ ($\Gamma \models^\kappa \varphi$) iff $\Gamma \cup \{ \neg \varphi \}$ is $\kappa$-unsatisfiable;
5. Θ-formulas \( \varphi \) and \( \psi \) are \( \kappa \)-equivalent iff \( \varphi \models^\kappa \psi \) and \( \psi \models^\kappa \varphi \).

It is clear that for any class \( \kappa \), such that \( Ax \) first-order axiomatizes \( \kappa(\Theta) \), any set \( \Gamma \) of \( \Theta \)-formulas and any \( \Theta \)-formula \( \varphi \), \( \Gamma \) is \( \kappa \)-satisfiable iff \( \Gamma \cup Ax \) is satisfiable, and \( \Gamma \models^\kappa \varphi \iff \Gamma \cup Ax \models \varphi \).

We say, further, that a formula \( \varphi(x) \) is \( \kappa \)-invariant with respect to \((i,j)\)-modal asimulations (where \( i,j \in \{1,2\} \)) iff it is invariant with respect to the class of \((i,j)\)-modal asimulations connecting models in \( \kappa \).

Now for the criterion of \( \kappa \)-equivalence to an \((i,j)\)-standard translation of modal intuitionistic formula:

**Theorem 3.** Let \( \kappa \) be a class of first-order models such that \( \kappa(\Theta) \) is first-order axiomatizable for all finite \( \Theta \), and let \( \varphi(x) \) be \( \kappa \)-invariant with respect to \((i,j)\)-modal asimulations. Then \( \varphi(x) \) is \( \kappa \)-equivalent to an \((i,j)\)-standard translation of a modal intuitionistic formula.

*Proof.* Let \( Ax \) be the set of first-order sentences that axiomatizes \( \kappa(\Sigma_0) \). We may assume that \( \varphi(x) \) is \( \kappa(\Sigma_0) \)-satisfiable, otherwise \( \varphi(x) \) is \( \kappa \)-equivalent to \( ST_{ij}(\bot, x) \) and we are done. In what follows we will write \( ic_{\kappa}(\varphi(x)) \) for the set

\[
\{ \psi(x) \in int_x(\Sigma_\xi) \mid \varphi(x) \models^\kappa \psi(x) \}
\]

Our strategy will be to show that \( ic_{\kappa}(\varphi(x)) \models^\kappa \varphi(x) \). Once this is done we will conclude that

\[
Ax \cup ic_{\kappa}(\varphi(x)) \models \varphi(x).
\]

Then we apply compactness of first-order logic and obtain that \( \varphi(x) \) is equivalent to a finite conjunction \( \bigwedge \Psi(x) \) of formulas from this set. But it follows then that \( \varphi(x) \) is \( \kappa \)-equivalent to the conjunction of the set \( ic_{\kappa}(\varphi(x)) \cap \Psi(x) \). In fact, by our choice of \( ic_{\kappa}(\varphi(x)) \) we have

\[
\varphi(x) \models^\kappa \bigwedge \{ ic_{\kappa}(\varphi(x)) \cap \Psi(x) \},
\]

And by \( \Psi(x) \subseteq Ax \cup ic_{\kappa}(\varphi(x)) \) we have

\[
Ax \cup (ic_{\kappa}(\varphi(x)) \cap \Psi(x)) \models \varphi(x)
\]

and hence

\[
ic_{\kappa}(\varphi(x)) \cap \Psi(x) \models^\kappa \varphi(x).
\]

To show that \( ic_{\kappa}(\varphi(x)) \models^\kappa \varphi(x) \), take any \( \Sigma_\xi \)-model \( M_1 \) and \( a \in U_1 \) such that \( M_1 \in \kappa \) and \( a \models_in(\varphi(x)) \). Then, of course, we will also have \( ic_{\kappa}(\varphi(x)) \subseteq tp_{\kappa}(M_1, a) \). Such a model exists, because \( \varphi(x) \) is \( \kappa(\Sigma_0) \)-satisfiable and \( ic_{\kappa}(\varphi(x)) \) will be \( \kappa \)-satisfied in any \( \Sigma_\xi \)-model satisfying \( \varphi(x) \). Then we can also choose a \( \Sigma_\xi \)-model \( M_2 \) and \( b \in U_2 \) such that \( M_2 \in \kappa \), \( b \models \varphi(x) \), and \( tp_{\kappa}(M_2, b) \subseteq tp_{\kappa}(M_1, a) \).

For suppose otherwise. Then for any \( \Sigma_\xi \)-model \( M \in \kappa \) such that \( U \subseteq \mathbb{N} \) and any \( c \in U \) such that \( M,c \models \varphi(x) \) we can choose a modal intuitionistic formula \( I_{(M,c)} \) such that \( ST_{ij}(I_{(M,c)}, x) \) is in \( tp_{\kappa}(M,c) \) but not at \( tp_{\kappa}(M_1, a) \). Then consider the set

\[
S = \{ \varphi(x) \} \cup \{ \neg ST_{ij}(I_{(M,c)}, x) \mid M \in \kappa, M,c \models \varphi(x) \}
\]
Let \( \{ \varphi(x), \lnot ST_{ij}(I_1, x), \ldots, \lnot ST_{ij}(I_q, x) \} \) be a finite subset of this set. If this set is \( \kappa \)-unsatisfiable, then we must have

\[
\varphi(x) \models_{\kappa} ST_{ij}(I_1, x) \lor \ldots \lor ST_{ij}(I_q, x),
\]

but then we will also have

\[
(ST_{ij}(I_1, x) \lor \ldots \lor ST_{ij}(I_q, x)) \in ic_{\kappa}(\varphi(x)) \subseteq tp_x(M_1, a),
\]

and hence \((ST_{ij}(I_1, x) \lor \ldots \lor ST_{ij}(I_q, x))\) will be true at \((M_1, a)\). But then at least one of \(ST_{ij}(I_1, x)\) \ldots \(ST_{ij}(I_q, x)\) must also be true at \((M_1, a)\), which contradicts the choice of these formulas. Therefore, every finite subset of \( S \) is \( \kappa \)-satisfiable. But then every finite subset of the set \( S \cup \text{Ax} \) is satisfiable as well. By compactness of first-order logic \( S \cup \text{Ax} \) is satisfiable, and, by L"owenheim-Skolem property of first-order logic, there is a \( \Sigma_{\varphi} \)-model \( M' \) and \( g \in U' \) such that \( U' \subseteq \mathbb{N} \) and \((M', g)\) satisfies \( S \cup \text{Ax} \). But then we must have \( M' \in \kappa \), since \( M' \) is a \( \Sigma_{\varphi} \)-model satisfying the set of axioms for \( \kappa(\Sigma_{\varphi}) \).

For this model and for this element in it we will have both \( M', g \models ST_{ij}(I_{(M',g)}, x) \) by choice of \( I_{(M',g)} \) and \( M', g \not\models ST_{ij}(I_{(M',g)}, x) \) by the satisfaction of \( S \), a contradiction.

Therefore, for any given \( \Sigma_{\varphi} \)-model \( M_1 \) such that \( M_1 \in \kappa \) and for any \( a \in U_1 \) satisfying \( ic_{\kappa}(\varphi(x)) \) we can choose a \( \Sigma_{\varphi} \)-model \( M_2 \) and \( b \in U_2 \) such that \( M_2 \in \kappa \), \( b \models_2 \varphi(x) \), and \( tp_x(M_2, b) \subseteq tp_x(M_1, a) \). Then, reasoning exactly as in the proof of Theorem 3, we conclude that \( a \models_1 \varphi(x) \). Therefore, \( \varphi(x) \) in fact \( \kappa \)-follows from \( ic_{\kappa}(\varphi(x)) \).

**Theorem 4.** Let \( \kappa \) be a class of first-order models such that for all finite \( \Theta \) class \( \kappa(\Theta) \) is first-order axiomatizable. Then a formula \( \varphi(x) \) is \( \kappa \)-invariant with respect to \((i, j)\)-modal asimulations iff it is \( \kappa \)-equivalent to an \((i, j)\)-standard translation of a modal intuitionistic formula.

**Proof.** From left to right our theorem follows from Theorem 3. In the other direction, assume that \( \varphi(x) \) is \( \kappa \)-equivalent to \( ST_{ij}(I, x) \) and assume that for some \( \Theta \) such that \( \Sigma_{\varphi} \subseteq \Theta \), some \( \Theta \)-models \( M_1, M_2 \), and some \( a \in U_1 \) and \( b \in U_2 \) such that \( M_1, M_2 \in \kappa \), \( A \) is a \((i, j)\)-modal \(((M_1, a), (M_2, b))\)-asimulation and \( a \models_1 \varphi(x) \). Then, since \( ST_{ij}(I, x) \) is \( \kappa \)-equivalent to \( \varphi(x) \) and \( M_1 \) is in \( \kappa \), we also have \( a \models_1 ST_{ij}(I, x) \). By Theorem 2 it follows that \( b \models_2 ST_{ij}(I, x) \), but since \( ST(I, x) \) is \( \kappa \)-equivalent to \( \varphi(x) \) and \( M_2 \) is in \( \kappa \), we also have \( b \models_2 \varphi(x) \). Therefore, \( \varphi(x) \) is \( \kappa \)-invariant with respect to asimulations.

Theorems 3 and 4 imply that \((i, j)\)-modal asimulations as criteria for equivalence to a standard translation of a modal intuitionistic formula are easily scalable down to any first-order definable class of models. As all the conditions on intended models for intuitionistic modal logic, that were presented in the existing literature, seem to be easily first-order definable, this means that the criterion for equivalence of a formula in correspondence language to an \((i, j)\)-standard translation of modal intuitionistic formula over the class of intended models will be just invariance with respect to \((i, j)\)-modal asimulations between the intended models.
6 Conclusion

In this paper we have defined and vindicated 4 different versions of modal asimulations, capturing the 4 different fragments of classical first-order logic induced by the corresponding systems of satisfaction clauses for modal intuitionistic logic. In doing so, we were concentrating on different variants of Kripke semantics for intuitionistic modalities, present in the existing literature.

However, it is easy to see, that our approach is but an instance of a quite general algorithm that can be easily generalized to deal with a much wider class of extensions of intuitionistic propositional logic. It is not clear at the moment, how far such a generalization is able to reach; we only mention here one conjecture to give reader a flavor of what can be expected in the way of future work along the research lines presented above.

It is easy to see that all the intuitionistic modalities considered above are instances of a general scheme: the effect of the modality on its only argument is always a superposition of ‘guarded’ quantifiers, that is to say, quantifiers, restricted by a binary relation which is a part of Kripke model for this modality. To be more precise, let us call a generalized intuitionistic modality any modality $\mu I$ whose induced standard translation looks as follows:

$$ST(\mu I, x) = Q_n y_n (R_n(y_n, y_{n-1} \circ_n Q_{n-1}(\ldots Q_1 y_1(R_1(y_1, x) \circ_1 ST(I, x))\ldots))),$$

where $n \geq 1$, and for all $i$ such that $1 \leq i \leq n$ we assume $Q_i \in \{\forall, \exists\}$. Further, $\circ_i = \Rightarrow$ if $Q_i = \forall$ and $\circ_i = \land$ if $Q_i = \exists$.

Then our conjecture is that to capture the extension of intuitionistic propositional logic by modality $\mu$, one generally needs to define asimulation as a tuple $(A_1, \ldots, A_{k_\mu})$ of binary relations, where $k_\mu \leq n$ is the number of quantifier alternations in $ST(\mu I, x)$. The set of conditions on basic asimulation will then have to be enlarged by $k_\mu$ new conditions $r_1, \ldots, r_{k_\mu}$, each having one of the two following forms: either

$$a_1 A_p b_1 \land \bigwedge_{s=1}^m (i_s(S_s)(b_s, b_{s+1})) \Rightarrow \exists a_2 \ldots a_m \in U_i \bigwedge_{s=1}^m (i_s(S_s)(a_s, a_{s+1})) \land a_m A_q b_m,$$

or

$$a_1 A_p b_1 \land \bigwedge_{s=1}^m (i_s(S_s)(a_s, a_{s+1})) \Rightarrow \exists b_2 \ldots b_m \in U_j \bigwedge_{s=1}^m (i_j(S_s)(b_s, b_{s+1})) \land a_m A_q b_m,$$

for all $a_1, \ldots, a_m \in U_i$ and all $b_1, \ldots, b_m \in U_j$. Here, $i, j$ have the same meaning as in Definition 2 and $\{S_1, \ldots, S_m\} \subseteq \{R_1, \ldots, R_n\}$. Moreover, we will ensure that in the last condition $r_{k_\mu}$ we will have $A_p = A_1$ and that the form of $r_{k_\mu}$ will be the former of the above forms if $Q_n$ is $\forall$ and the latter if $Q_n$ is $\exists$.

More precisely, we define $r_1, \ldots, r_{k_\mu}$ by induction on $n$ as follows.

If $n = 1$ and $Q_1 = \forall$, we set:

$$r_1 := (a_1 A_1 b_1 \land \iota_j(R_1)(b_1, b_2)) \Rightarrow \exists a_2 \in U_i(\iota_i(R_1)(a_1, a_2) \land a_2 A_j b_2),$$

\[\tag{6}\]The relations $R_1, \ldots, R_n$ need not be all different.
and if $Q_1 = \exists$, we set

$$r_1 := (a_1 A_1 b_1 \land \epsilon_i(R_1)(a_1, a_2)) \Rightarrow \exists b_2 \in U_j(\epsilon_j(R_1)(b_1, b_2) \land a_2 A_j b_2)).$$

Incidentally, the fact that these singleton sets of conditions are adequate already follows from the above proofs concerning $[\star]$ and $[\square]$.

Assume, further, that $n = s + 1$ for $s \geq 1$.

As induction hypothesis, we suppose that the set of conditions $r_1, \ldots, r_{k_{\mu^-}}$ for $\mu^-$, where

$$ST(\mu^- I, x) = Q_{n-1}(\ldots Q_1 y_1(R_1(y_1, x) \circ_{1} ST(I, x)) \ldots)$$

is already defined.

To define $r_1, \ldots, r_{k_{\mu^-}}$, we need to distinguish the following cases:

**Case 1.** $Q_n = Q_{n-1}$. Then $k_{\mu} = k_{\mu^-}$ and we do not need to introduce new conditions. Instead, we transform condition $r_{k_{\mu^-}}$ in the following way:

**Case 1.1** If $Q_n = Q_{n-1} = \forall$, then by induction hypothesis condition $r_{k_{\mu^-}}$ has the form:

$$a_1 A_1 b_1 \land \bigwedge_{s=1}^{m} (\epsilon_j(S_s)(b_s, b_{s+1})) \Rightarrow \exists a_{m+1} \ldots a_m \in U_i(\bigwedge_{s=1}^{m} (\epsilon_i(S_s)(a_s, a_{s+1})) \land a_m A_q b_m).$$

We then replace $r_{k_{\mu^-}}$ by the following rule $r_{k_{\mu}}$, where we assume $a' \in U_i, b' \in U_j$:

$$a' A_1 b' \land (\epsilon_j(R_n)(b', b_1) \land \bigwedge_{s=1}^{m} (\epsilon_j(S_s)(b_s, b_{s+1})) \Rightarrow$$

$$\Rightarrow \exists a_1 \ldots a_m \in U_i(\epsilon_i(R_n)(a', a_1) \land \bigwedge_{s=1}^{m} (\epsilon_i(S_s)(a_s, a_{s+1})) \land a_m A_q b_m).$$

**Case 1.2** If $Q_n = Q_{n-1} = \exists$, then by induction hypothesis condition $r_{k_{\mu^-}}$ has the form:

$$a_1 A_1 b_1 \land \bigwedge_{s=1}^{m} (\epsilon_i(S_s)(a_s, a_{s+1})) \Rightarrow \exists b_2 \ldots b_m \in U_j(\bigwedge_{s=1}^{m} (\epsilon_j(S_s)(b_s, b_{s+1})) \land a_m A_q b_m).$$

We then replace $r_{k_{\mu^-}}$ by the following rule $r_{k_{\mu}}$, where we assume $a' \in U_i, b' \in U_j$:

$$a' A_1 b' \land (\epsilon_j(R_n)(a', a_1) \land \bigwedge_{s=1}^{m} (\epsilon_j(S_s)(a_s, a_{s+1})) \Rightarrow$$

$$\Rightarrow \exists b_1 \ldots b_m \in U_j(\epsilon_j(R_n)(b', b_1) \land \bigwedge_{s=1}^{m} (\epsilon_j(S_s)(a_s, a_{s+1})) \land a_m A_q b_m).$$

**Case 2.** $Q_n \neq Q_{n-1}$. Then $k_{\mu} = k_{\mu^-} + 1$ and we need to increase the number of conditions by one. We do this as follows:

**Case 2.1** If $Q_n = \forall, Q_{n-1} = \exists$, then by induction hypothesis condition $r_{k_{\mu^-}}$ has the form:

$$a_1 A_1 b_1 \land \bigwedge_{s=1}^{m} (\epsilon_i(S_s)(a_s, a_{s+1})) \Rightarrow \exists b_2 \ldots b_m \in U_j(\bigwedge_{s=1}^{m} (\epsilon_j(S_s)(b_s, b_{s+1})) \land a_m A_q b_m).$$
we then replace $r_{k\mu}$ by the following condition $r'_{k\mu}$:

$$a_1 A_{k\mu} b_1 \land \bigwedge_{s=1}^{m} (t_i(S_s)(a_s, a_{s+1})) \Rightarrow \exists b_2 \ldots b_m \in U_j \big( \bigwedge_{s=1}^{m} (t_j(S_s)(b_s, b_{s+1})) \land a_m A_q b_m \big),$$

and we add the following new condition $r_{k\mu}$:

$$a_1 A_1 b_1 \land t_j(R_n)(b_1, b_2) \Rightarrow \exists a_2 \in U_i(t_i(R_n)(a_1, a_2) \land a_2 A_{k\mu} b_2).$$

Case 2.2 If $Q_n = \exists$, $Q_{n-1} = \forall$, then by induction hypothesis condition $r_{k\mu}$ has the form:

$$a_1 A_1 b_1 \land \bigwedge_{s=1}^{m} (t_i(S_s)(a_s, a_{s+1})) \Rightarrow \exists b_2 \ldots b_m \in U_j \big( \bigwedge_{s=1}^{m} (t_j(S_s)(b_s, b_{s+1})) \land a_m A_q b_m \big),$$

we then replace $r_{k\mu}$ by the following condition $r'_{k\mu}$:

$$a_1 A_{k\mu} b_1 \land \bigwedge_{s=1}^{m} (t_i(S_s)(a_s, a_{s+1})) \Rightarrow \exists b_2 \ldots b_m \in U_j \big( \bigwedge_{s=1}^{m} (t_j(S_s)(b_s, b_{s+1})) \land a_m A_q b_m \big),$$

and we add the following new condition $r_{k\mu}$:

$$a_1 A_1 b_1 \land t_j(R_n)(b_1, b_2) \Rightarrow \exists a_2 \in U_i(t_i(R_n)(a_1, a_2) \land a_2 A_{k\mu} b_2).$$

It is straightforward to verify that our systems of rules for both (2) and (3) were generated according to the Cases 1.1 and 2.1 of this inductive definition, respectively. Note, however, that this general scheme is not always the most effective one. For example, consider an extension of intuitionistic propositional logic, where both (3) and (2) are available. This would allow to considerably simplify the corresponding notion of asimulation, indeed, one could get rid in this case of the second binary relation $B$ and define asimulations as appropriate type of single binary relation $A$.

The other thing worth noting is that the above mentioned scheme for defining asimulations which capture expressive powers of generalized intuitionistic modalities is neither the only nor the most abstract among the generalizations that naturally come to mind in this respect.

Therefore, in our future work, we hope both to provide substantiation for the above-defined scheme concerning the generalized intuitionistic modalities and further pursue the manifold research opportunities opening along this research line.

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To be inserted.

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