Let $R = \mathbb{C}[x_{ij}]$ be the ring of polynomial functions in $mn$ variables where $m > n$. Set $X$ to be the $m \times n$ matrix in these variables and $I := I_n(X)$ the ideal of maximal minors of $X$. We consider the rings $R/I^{t}$; for $t \gg 0$ the depth of $R/I^{t}$ is equal to $n^2 - 1$, and we show that each local cohomology module $H_{m}^{n^2 - 1}(R/I^{t})$ is a cyclic $R$-module. We also compute the annihilator of $H_{m}^{n^2 - 1}(R/I^{t})$ thereby completely determining its $R$-module structure.

In the case that $X$ is an $n \times (n-1)$ matrix we describe a map between the Koszul complex of the $t$-powers of the maximal minors and a free resolution of $R/I^{t}$. We use this map to explicitly describe the modules $\text{Ext}_R^n(R/I^{t}, R)$ as submodules of the top local cohomology module $H_1^n(R)$. Moreover, we can realize the filtration $\bigcup \text{Ext}_R^n(R/I^{t}, R) = H_1^n(R)$ in terms of differential operators. Utilizing this description, along with an explicit isomorphism $H_1^n(R) \cong H_{m}^{n^2 - 1}(R)$, we determine the annihilator of $\text{Ext}_R^n(R/I^{t}, R)$ and hence by graded local duality give another computation of the annihilator of $H_{m}^{n^2 - 1}(R/I^{t})$.

1. Introduction

Let $I$ be a homogeneous ideal in a polynomial ring $R$. Then $I$ defines a projective variety and one may consider its thickenings, i.e., the varieties defined by the ideals $I^t$. Understanding the ideals $I^t$ is an important component of understanding the singularities of the variety defined by $I$. For example, they comprise the graded components of Rees algebras and also appear in the study of the functors $H_i^(-)$. It was shown in [BBL+19] that under certain conditions the graded components of the local cohomology modules $H_m^i(R/I^t)$ stabilize for sufficiently large $t$. This recent work has brought renewed attention to thickenings and created an interest in their homological properties and invariants.

In the case that $I = I_r(X)$ and $R = \mathbb{C}[X]$ where $X$ is matrix of indeterminates, the modules $H_m^i(R/I^t)$, $\text{Ext}_R^i(R/I^t, R)$ and $H_1^i(R)$ have been studied extensively and successfully using representation theoretic techniques. In [RWW14], [RW14] and [Rai18] Raicu–Weyman–Witt, Raicu–Weyman and Raicu described the GL-equivariant structure of $\text{Ext}_R^i(R/I^t, R)$ and $H_1^i(R)$. These results have been used by Kenkel and Li in [Ken19].
and [Li21] to study the asymptotic length of $H^i_m(R/I^t)$ and find formulas for the higher epsilon multiplicity of $I$. In a similar flavor, the regularity of $I^t$ was described in [RW14] and [Rai18] along with a classification of which GL-invariant ideals satisfy the property that $H^i_I(R) = \bigcup_t \text{Ext}^i(R/I^t, R)$.

In this paper we focus on the case that $I \subseteq \mathbb{C}[X]$ is the ideal of maximal minors of a $m \times n$ generic matrix $X$, with $m > n$ and examine $H^{n^2-1}_m(R/I^t)$. For sufficiently large $t$, $H^{n^2-1}_m(R/I^t)$ is the first non-vanishing local cohomology module of $R/I^t$ and it was shown by Li that $n^2 - 1$ is the only cohomological index to yield a nonzero finite length module [Li21]. In Proposition 2.11, we will show that $H^{n^2-1}_m(R/I^t)$ is in fact a cyclic $R$-module. This module has also been examined in the case that $X$ is 2×3 matrix in [Ken20] where Kenkel explicitly describes a generator of $[H^2_3(R/I^t)]_0$ via the Čech complex on the variables of $R$.

The aforementioned results about $H^i_m(R/I^t)$ speak about the structure of its graded components, i.e., its structure as a graded $\mathbb{C}$-vector space. Additionally, the description of $\text{Ext}^i_R(R/I^t, R)$ given in [RWW14] is as a GL-representation and a priori does not speak on its structure as an $R$-module. In this paper we study the $R$-module structure of these modules and explicitly describe this structure for certain Ext and local cohomology modules.

We proceed by investigating the modules $\text{Ext}^i_R(R/I^t, R)$ via the natural map $\text{Ext}^i_R(R/I^t, R) \to H^i_I(R)$. In the case that $I$ is the ideal of maximal minors of a generic matrix, the natural map is an injection [RWW14], hence describing $\text{Ext}^i_R(R/I^t, R)$ is equivalent to describing its image in $H^i_I(R)$. To understand the map $\text{Ext}^i_R(R/I^t, R) \to H^i_I(R)$ we first view $H^i_I(R)$ as Čech cohomology of the maximal minors and compare this to the Koszul cohomology of the powers of maximal minors in the usual way. We then examine the natural map from $\text{Ext}^i_R(R/I^t, R)$ to this Koszul cohomology. As the map from Koszul cohomology to Čech cohomology is well understood, it remains to understand the map $\text{Ext}^i_R(R/I^t, R) \to H^i([d_1 \cdots d_k]; R)$ where $d_1, \cdots, d_k$ are the maximal minors of $X$. Thus, to explicitly describe $\text{Ext}^i_R(R/I^t, R) \to H^i([d_1 \cdots d_k]; R)$ we need to describe a map of complexes $\varphi_t$ such that

$$
\begin{array}{ccc}
F_* & \longrightarrow & I^t & \longrightarrow & 0 \\
\varphi_t & \uparrow & & \uparrow & \\
K_*([d_1 \cdots d_k]; R) & \longrightarrow & (d_1, \cdots, d_k) & \longrightarrow & 0
\end{array}
$$

commutes, where $F_*$ is a free resolution of $I^t$. The utility of using this approach to study $\text{Ext}^i_R(R/I^t, R)$ is in that the module structure of $H^i_I(R)$ may be quite familiar, cf. [RW16, Main Theorem]. For example, for $i = mn - n^2 + 1$, the cohomological dimension of $I$, $H^i_I(R) \cong H^{mn}_m(R)$.

In the case that $X$ is size $n \times (n-1)$ we are able to explicitly construct a map $\varphi_t$ as above; this is the content of Section 3. Using this lift, in Section 4, we give the following description of $\text{Ext}^n_R(R/I^t, R)$ as a submodule of $H^n_I(R)$.

**Theorem (4.1).** Let $X$ be a $n \times (n-1)$ matrix of indeterminates and $R = \mathbb{C}[X]$. Set $I = (d_1, \ldots, d_n) \subseteq R$ where $d_1, \ldots, d_n$ are the maximal minors of $X$. For a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ write $d^\alpha = d_1^{\alpha_1} \cdots d_n^{\alpha_n}$. Then $\text{Ext}^n_R(R/I^t, R)$ embeds into $H^n_I(R)$ as the submodule generated by the classes

$$\left\{ \frac{1}{\prod_{i=1}^n d_i^{\alpha_i}}, \frac{1}{d^\alpha} \right\}_{|\alpha|=t-n+1}.$$
This embedding can be realized as coming from differential operators and after identifying $H^n_t(R) \otimes \mathrm{Ext}^n_n(R)$ we obtain the following key corollary.

**Corollary (4.4).** In the setting of the previous theorem, for $f \in R$, let $f^*$ denote the polynomial differential operator obtained from $f$ by replacing $x_i$ with $\partial_i$. Then for $t \geq n-1$ we have that $\mathrm{Ext}^n_n(R/I^t, R)$ embeds in $H^n_{t-1}(R)$ as the $R$-submodule generated by the classes

\[ \left\{ (d^n)^* \cdot \frac{1}{x} \right\}_{|x|=t-n+1}, \]

where $\cdot$ denotes the application of an operator.

The Weyl algebra annihilator of the class $\frac{1}{x} \in H^n_{t-1}(R)$ is well understood and in the remainder of Section 4 we use Corollary 4.4 to compute the $R$-annihilator of $\mathrm{Ext}^n_n(R/I^t, R)$. By graded duality, the annihilator of $\mathrm{Ext}^n_n(R/I^t, R)$ is the annihilator of $H^{(n-1)^2-1}_{m-1}(R/I^t)$, hence we obtain a complete description of $H^{(n-1)^2-1}_{m-1}(R/I^t)$ when $X$ is size $n \times (n-1)$ as a cyclic $R$-module generated in degree zero, see Proposition 2.11.

In the general case of maximal minors of an $m \times n$ matrix with $m > n + 1$ the map of complexes, $\varphi_t$, and with it the structure of the Ext modules, remains mysterious. However, by analyzing the GL-structure of $H^{2-1}_{m-1}(R/I^t)$ we are able to compute its annihilator as follows:

**Theorem (4.8, 5.1).** Let $X$ be a $m \times n$ matrix of indeterminates with $m > n$ and set $I = I_n(x) \subseteq R = \mathbb{C}[X]$. If $t < n$ then $H^{2-1}_{m-1}(R/I^t) = 0$. If $t \geq n$, then we have an isomorphism of graded $R$-modules:

\[ H^{2-1}_{m-1}(R/I^t) \cong R/I_\lambda, \]

where $I_\lambda$ is the GL-invariant ideal associated to the partition $\lambda = (t-n+1)$, i.e., the ideal generated by $\mathrm{GL}_m \times \mathrm{GL}_n$ orbit of $x_{1,1}^{t-n+1}$, i.e., the ideal of $t-n+1$ generalized permanents of $X$ c.f. 2.3.

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2. Background

**Notation 2.1.** Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring and $D = R[\partial_1, \ldots, \partial_n]$ be the ring of differential operators on $R$. Fix $f \in R$ and $\psi \in D$.

- For a $D$-module $M$ and an element $h \in M$ we write $\psi \cdot h$ for element obtained by acting on $h$ by $\psi$. In particular $\psi \cdot f \in R$ is the application of $\psi$ to $f$.
- We write $\psi f \in D$ for the multiplication of $\psi$ and $f$ in $D$.
- We write $f^* \in \mathcal{D}$ for $f(\partial)$, the “dual” operator to $f$ obtained by replacing $x_i$ by $\partial_i$.

Let $G$ be a group acting on a set $S$. Fix $g \in G$ and $s \in S$.

- We write $g \cdot s$ for the element obtained by acting on $s$ with $g$.
- We write $G \cdot s$ for the orbit of $s$. 
2.1. Dominant Weights, Partitions and Schur Functors. We begin by establishing some notation and recalling some useful facts about Schur functors, for a complete treatment see [FH91] and [Wey03]. A vector \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \) is called a dominant weight if \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). We write \( \mathbb{Z}_{\text{dom}}^n \) for the set of all dominant weights in \( \mathbb{Z}^n \) and write \( |\lambda| = \sum_{i=1}^{n} \lambda_i \) for the size of \( \lambda \). Additionally, for \( c \in \mathbb{Z} \) and \( 0 \leq d \leq n \), we write \( (c^d) \in \mathbb{Z}_{\text{dom}}^n \) for the vector with \( d \) nonzero components all equal to \( c \). A partition into \( n \) parts is a dominant weight, \( \lambda \in \mathbb{Z}_{\text{dom}}^n \), with \( \lambda_n \geq 0 \), we write \( \mathcal{P}_n \subseteq \mathbb{Z}_{\text{dom}}^n \) to be the set of all such weights. An element \( \lambda \in \mathcal{P}_n \) may be realized as a Young diagram with \( \lambda_i \) boxes in row \( i \), for example the diagram associated to \((4, 3, 1) \in \mathcal{P}_3\) is:

(1)

If \( m \geq n \), we can naturally identify an element of \( \mathcal{P}_n \) with an element of \( \mathcal{P}_m \) by adjoining zeroes, e.g., \((2, 2) \in \mathcal{P}_2\) is identified with \((2, 2, 0, 0) \in \mathcal{P}_4\). Generally we omit the trailing zeroes and would write \((2, 2) \in \mathcal{P}_4\). For \( \lambda \in \mathcal{P}_n \) we can consider its transpose partition, which is the partition associated to the transpose of the Young diagram of \( \lambda \). The transpose partition of \((4, 3, 1) \in \mathcal{P}_3\) is \((3, 2, 2, 1) \in \mathcal{P}_4\) because the transpose of (1) is:

(2)

Let \( H \) be an \( n \) dimensional \( \mathbb{C} \)-vector space. Then to each dominant weight \( \lambda \in \mathbb{Z}_{\text{dom}}^n \) we associate an irreducible representation of \( \text{GL}(H) \), denoted \( S_\lambda H \), called a Schur functor. Moreover every irreducible representation of \( \text{GL}(H) \) can be realized in this manner. For some dominant weights, Schur functors are quite familiar: there are \( \text{GL}(H) \)-equivariant isomorphisms:

\[
S_{(1^d)} H \cong \bigwedge^d H
\]

and

\[
S_{(d)} H \cong \text{Sym}^d H.
\]

For computational purposes, frequently it is sufficient to consider \( \lambda \in \mathcal{P}_n \) as we have the following \( \text{GL}(H) \)-equivariant isomorphisms:

\[
S_{\lambda+(1^n)} H \cong S_\lambda H \bigotimes^n \bigwedge H
\]

and

\[
S_{(\lambda_1, \ldots, \lambda_n)} H \cong \text{Hom}(S_{(-\lambda_n, \ldots, -\lambda_1)} H, \mathbb{C}).
\]

Lastly, the dimension of Schur functors is relatively easy to compute and is described by the following formula.

**Proposition 2.2.** [FH91] Let \( \lambda \in \mathbb{Z}_{\text{dom}}^n \), then

\[
\dim_{\mathbb{C}} S_\lambda \mathbb{C}^n = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
\]
2.2. GL-invariant ideals. Let \( F = \mathbb{C}^m \) and \( G = \mathbb{C}^n \) where \( m \geq n \). Then

\[
R := \text{Sym}(F \otimes G) = \mathbb{C}[\{x_{ij}\}] = \mathbb{C}[X]
\]

is a polynomial ring admitting a

\[
GL := \text{GL}(F) \times \text{GL}(G)
\]

action as follows: for \((g_1, g_2) \in GL\), \(g \cdot (x_{ij}) = (z_{ij})\) where \([z_{ij}] = g_1 X g_2^{-1}\). Cauchy’s formula describes the decomposition of \( R \) into irreducible representations \([\text{Wey03}]\):

\[
(3) \quad R = \bigoplus_{\lambda \in \mathcal{P}_n} S_{\lambda} F \otimes S_{\lambda} G
\]

where \( S_{\lambda} F \otimes S_{\lambda} G \) lives in degree \(|\lambda|\).

For a number \( 1 \leq l \leq n \) set \( \det_l := \det(x_{ij})_{1 \leq i,j \leq l} \), i.e., the \( l \times l \) minor in the top left corner of \( X \). Then for a partition, \( \lambda \in \mathcal{P}_n \), with \( n \) parts let \( \lambda' \) be the transpose partition and define

\[
\det_{\lambda} := \prod_{i=1}^{\lambda_1} \det_{\lambda_i}'.
\]

The \( \mathbb{C} \)-linear span of the GL orbit of \( \det_{\lambda} \) is equal to \( S_{\lambda} F \otimes S_{\lambda} G \). We set

\[
I_{\lambda} := (GL \cdot \det_{\lambda}) \subseteq R,
\]

the ideal generated by the GL orbit of \( \det_{\lambda} \). This ideal is GL-invariant. We endow \( \mathcal{P}_n \) with a partial ordering: for \( \mu, \lambda \in \mathcal{P}_n \), we say that \( \mu \geq \lambda \) if \( \mu_i \geq \lambda_i \) for all \( i \). It was shown in \([\text{dCEP80}]\) that:

\[
(4) \quad I_{\lambda} = \bigoplus_{\mu \geq \lambda} S_{\mu} F \otimes S_{\mu} G.
\]

By taking a collection of partitions \( \chi \subseteq \mathcal{P}_n \) we can form the GL-invariant ideal \( I_{\chi} = \sum_{\lambda \in \chi} I_{\lambda} \). It was proven in \([\text{dCEP80}]\) that all GL-invariant ideals may be realized in this manner for some finite subset \( \chi \subseteq \mathcal{P}_n \) and so more generally GL-invariant ideals decompose as:

\[
I_{\chi} = \bigoplus_{\mu \geq \lambda \in \chi} S_{\mu} F \otimes S_{\mu} G.
\]

**Example 2.3.** Let \( r, t \) be positive integers.

1. \( I_{(1^r)} = I_r(X) \) the ideal of \( r \times r \)-minors of \( X \).
2. \( I_{\chi_t} = I_r(X)^t \) where \( \chi_t = \{ \lambda \in \mathcal{P}_n | |\lambda| = rt, \lambda_1 \leq t \} \).
3. \( I_{(t^n)} = I_n(X)^t \) the \( t \)-th power of the ideal of maximal minors of \( X \).
4. \( I_{(t)} \), the ideal of \( t \times t \) generalized permanents of \( X \), i.e., the ideal generated by the permanent of all \( t \times t \) matrices of the form \([x_{\alpha_i, \beta_j}]\) where \( \alpha_i \leq \alpha_{i+1} \) and \( \beta_j \leq \beta_{j+1} \).

**Remark 2.4.** Notice that Cauchy’s Formula, \((3)\), says that every irreducible representation of GL contained in \( R \) appears at most once. Combining this with the classification of GL-invariant ideals of \([\text{dCEP80}]\) stated above, we have that a GL-invariant ideal is uniquely determined by its structure as a GL-representation.
2.3. A $\mathbb{C}$-Linear GL-Equivariant Pairing. Let $R$ be as above. Let $R^* = \mathbb{C}[\partial_{ij}]$ and 

$(-)^* : R \rightarrow R^*$

be the map induced by $x_{ij} \mapsto \partial_{ij}$. We can view $R$ as the coordinate ring for the space of $m \times n$ complex matrices and the GL action on $R$ described above as being induced by the GL action on this space of matrices. We now view $R^*$ as the coordinate ring for $n \times m$ matrices and hence GL acts on it as follows: for $g = (g_1, g_2) \in \text{GL}$, $g \cdot x^*_{ij} = z_{ij}$ where $[z_{ij}] = (g_1^{-1})^T X^* g_2^T$.

The action of $R^*$ on $R$ via differentiation induces a perfect pairing

$$\langle \ , \ \rangle : [R^*]_k \times [R]_k \rightarrow \mathbb{C}.$$ 

We will see below this pairing is GL-equivariant, where GL acts on $R$ and $R^*$ as above and fixes $\mathbb{C}$. A more general statement about differential operators acting on representations is known, see [LRWW17, Section 2.2], however we include the following proof for completeness.

Lemma 2.5. The pairing, $\langle \ , \ \rangle : [R^*]_k \times [R]_k \rightarrow \mathbb{C}$ is GL-equivariant.

Proof. By linearity it is enough to check this for $f^*, g$ where $f = x^\alpha$ and $g = x^\beta$ are monomials. Let $\theta = (\theta_1, \theta_2) \in \text{GL}$, it is sufficient to prove the statement for $\theta = (\theta_1, ID)$ and $\theta = (ID, \theta_2)$, where $ID$ denotes the identity. The arguments in each case are analogous so we assume that $\theta = (\theta_1, ID)$. Thus, $\theta \cdot x_{ij} = \sum_{k=1}^n x_{ik} \theta_{kj}$ and $\theta \cdot x^*_{ij} = \sum_{k=1}^n x^*_{ik} (\theta^{-1})^T_{kj} = \sum_{k=1}^n x^*_{ik} \theta_{jk}$.

We induct on $k$. For $k = 1$ we need to consider $f = x_{ab}$, $g = x_{cd}$. Then $\langle f^*, g \rangle$ is 1 if $(a, b) = (c, d)$ and 0 otherwise. Now consider,

$$\langle \theta \cdot f^*, \theta \cdot g \rangle = \left( \sum_{k=1}^n x^*_{ak} \theta_{bk}^{-1} \right) \cdot \left( \sum_{k=1}^n x_{ck} \theta_{kd} \right)$$

$$= \sum_{k=1}^n \sum_{l=1}^n \theta_{bk}^{-1} \theta_{ld} \left( x^*_{ak} \cdot x_{cl} \right)$$

$$= \begin{cases} 
0 & a \neq c \\
\sum_{k=1}^n \theta_{bk}^{-1} \theta_{kd} & \text{else} 
\end{cases}$$

$$= \begin{cases} 
0 & a \neq c \\
ID_{b,d} & \text{else} 
\end{cases}$$

$$= \begin{cases} 
1 & (a, b) = (c, d) \\
0 & \text{else} 
\end{cases} .$$
For $k > 1$ we may write $f = x_{ab}f'$ and $g = x_{cd}g'$, then :

\begin{align*}
(5) \quad (\theta \cdot f^*) \circ (\theta \cdot g) & = (\theta \cdot f^*) \circ (\theta \cdot x_{ab}^* \cdot (\theta \cdot g)\circ (\theta \cdot x_{cd})) \\
(6) & = (\theta \cdot f^*) \circ ((\theta \cdot x_{cd})(\theta \cdot x_{ab}^* \cdot (\theta \cdot g))) + (\theta \cdot g^*)(\theta \cdot x_{ab}^* \circ (\theta \cdot x_{cd})) \\
(7) & = (\theta \cdot f^*) \circ ((\theta \cdot x_{cd})(\theta \cdot (x_{ab}^* \cdot g^)) + (\theta \cdot g^*)(\theta \cdot (x_{ab}^* \cdot x_{cd}))) \\
(8) & = (\theta \cdot f^*) \circ (\theta \cdot (x_{cd}(x_{ab}^* \cdot g^) + g^\prime(\theta \cdot (x_{ab}^* \cdot x_{cd})))) \\
(9) & = \theta \cdot (f^* \circ (x_{cd}(x_{ab}^* \cdot g^) + g^\prime(\theta \cdot (x_{ab}^* \cdot x_{cd})))) \\
(10) & = \theta \cdot (f^* \circ (x_{ab}^* \cdot (x_{cd}g^'))) \\
(11) & = \theta \cdot (f^* \circ g),
\end{align*}

where (5) to (6) is by the product rule since $\theta \cdot x_{ab}^*$ is a linear operator. From (6) to (7) and (9) to (10) are by induction hypothesis. Finally (10) to (11) is again by the product rule.

Now given an $GL$-invariant subspace of $[R]_k$, e.g., a graded component of a $GL$-invariant ideal, we can use this pairing to analyze which polynomial operators annihilate that subspace.

**Lemma 2.6.** Let $N \subseteq [R]_k$ be a $GL$-invariant subspace and suppose $f \in [R]_j$ such that $f^* \circ N = 0$. Then the $GL$ orbit of $f$ also annihilates $N$.

**Proof.** If $j > k$ the statement is trivial so assume $j \leq k$. Since $f^* \circ N = 0$ for all $g^* \in [R^*]_{k-j}$ and $h \in N$ we have that $\langle (fg)^*, h \rangle = 0$. So for all $\varphi \in GL$ we have that $0 = \langle \varphi \cdot (fg)^*, \varphi \cdot h \rangle = \langle (\varphi \cdot f)^*(\varphi \cdot g)^*, h \rangle$.

Since the choice of $g$ was arbitrary and $N$ is $GL$-equivariant we have that for all $g^* \in [R^*]_{k-j}$ and all $h \in N$

$\langle (\varphi \cdot f)^*g^*, h \rangle = 0$.

Suppose for contradiction that $(\varphi \cdot f)^* \circ N \neq 0$, then there exists $h \in N$ such that $(\varphi \cdot f)^* \circ h \neq 0$. Thus, $(\varphi \cdot f)^* \circ h$ contains a monomial $g$ with nonzero coefficient, but then $\langle (\varphi \cdot f)^*g^*, h \rangle \neq 0$, a contradiction. 

\[ \square \]

### 2.4. GL-equivariant description of certain Ext modules.

Let $R, F, G$ and $GL$ be as defined above in Section 2.2. Let $I = I_{(1^n)}$ be the ideal generated by the maximal minors of $X$. In [RWW14] the authors gave a $GL$-equivariant description of $H_1^j(R)$ as a direct sum of irreducible $GL$-representations. They also prove a number of results about the modules $\text{Ext}_R^t(I^t, R)$, we recall two of these results below.

**Theorem 2.7.** [RWW14, Theorem 4.3] Let $m > n$ and $t \geq n$. If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ we write

$\lambda(s) = (\lambda_1, \ldots, \lambda_{m-n}, -s, \ldots, -s, \lambda_{m-n+1} + (m - n), \ldots, \lambda_n + (m - n)) \in \mathbb{Z}^m$

Writing $W(r; s)$ for the set of dominant weights $\lambda \in \mathbb{Z}^n_{dom}$ with $|\lambda| = r$ such that $\lambda(s)$ is also dominant. We have

\[ \text{Ext}_R^t(I^t, R)_r = \bigoplus_{\lambda \in W(r; s), \lambda_n \geq -t - (m-n)} S_{\lambda(s)}F \otimes S_{\lambda}G. \]
An analogous description of $\Ext^i_R(J, R)$ was computed in [Rai18] for any GL-invariant ideal $J$ not just powers of maximal minors.

We will use Theorem 2.7 in Section 5 in conjunction with graded duality to compute the GL-structure of $H^{mn-n^2+1}_m(R/I^t)$. To make use of this description it will be useful to understand how the modules $\Ext^i_R(R/I^t, R)$ sit inside $H^i_J(R)$.

**Theorem 2.8.** [RWW14, Section 4] Let $i \in \mathbb{Z}_{\geq 0}$, for all $t \geq 1$, the induced maps $\Ext^i_R(R/I^t, R) \to \Ext^i_R(R/I^{t+1}, R)$ are injective.

This immediately gives us the following:

**Corollary 2.9.**

$$H^i_J(R) = \bigcup_{t \geq 0} \Ext^i_R(R/I^t, R).$$

More generally, the pairs of GL-invariant ideals $I_{X_1}$ and $I_{X_2}$ for which $\Ext^i_R(R/I_{X_1}, R) \to \Ext^i_R(R/I_{X_2}, R)$ is injective is classified in [RW14] and [Rai18]. In particular, Theorem 2.8 and Corollary 2.9 fail for ideals of non-maximal minors.

2.5. Other facts on local cohomology of determinantal things. Let $R, I$ be as in Section 2.4 and let $\mathcal{D} = R[\{\partial_{ij}\}]$ be the Weyl algebra. The action of differentiation makes $R$ a left $\mathcal{D}$-module and formal application of the quotient rule then gives $R_a = R[\frac{\partial}{\partial a}]$ a $\mathcal{D}$-modules structure for all $a \in R$. Thus for any ideal $J = (a_1, \ldots, a_k)$, the Čech complex, $\check{C}^\bullet(a_1, \ldots, a_k; R)$, is a complex of $\mathcal{D}$-modules, hence $H^j_J(R) \cong H^j(\check{C}^\bullet(a_1, \ldots, a_k; R))$ carries the structure of a $\mathcal{D}$-modules.

**Theorem 2.10.** [Wit12, Theorem 1.2][LSW16, Theorem 1.2, Theorem 3.1] There exists a degree preserving isomorphism of $\mathcal{D}$-modules:

$$H^{mn-n^2+1}_I(X) \cong H^{mn}_m(R),$$

in particular,

$$H^{mn-n^2+1}_J(R) \cong H^{mn}_m(R).$$

As these are cyclic $\mathcal{D}$-modules, in order to describe an isomorphism as above, we just need to choose a socle generator of $H^{mn-n^2+1}_I(R)$ and $H^{mn}_m(R)$. Such a map will be constructed in section 4.1 in the case where $X$ is size $n \times (n - 1)$.

In light of Corollary 2.9, Theorem 2.10 and local duality gives us an avenue to examine $H^{n^2-1}_m(R/I^t)$. Some implications of this result to the asymptotic structure of the graded components of $H^{n^2-1}_m(R/I^t)$ has been remarked on in [BBL+19] and [Ken20]. We obtain the following result on the structure of $H^{n^2-1}_m(R/I^t)$ as an $R$-module.

**Proposition 2.11.** $H^{n^2-1}_m(R/I^t)$ is a cyclic $R$-module generated in degree 0. In other words, there exists $J \subseteq R$ such that $H^{n^2-1}_m(R/I^t) \cong R/J$.

**Proof.** First to show $H^{(n-1)^2-1}_m(R/I^t)$ is cyclic, by graded duality it is sufficient to show that $\Ext^{mn-n^2+1}_R(R/I^t, R)$ is finite length and has socle dimension at most 1. By Theorem 2.8 and Theorem 2.10 we have that $\Ext^{mn-n^2+1}_R(R/I^t, R) \hookrightarrow H^{mn}_m(R)$. Thus $\Ext^{mn-n^2+1}_R(R/I^t, R)$ is a finitely generated submodule of an Artinian module hence has finite length. Moreover, since $H^{mn}_m(R)$ has socle dimension 1 we have that $\Ext^{mn-n^2+1}_R(R/I^t, R)$ has socle dimension at most 1. That $H^{(n-1)^2-1}_m(R/I^t)$ is generated in dimension 0 follows by graded duality since $\text{Soc}(\Ext^{mn-n^2+1}_R(R/I^t, R)) = \text{Soc}(H^{mn}_m(R))$ is generated in degree $-mn$. \qed
3. A Map Between Complexes

Let $A$ be a set, we write $\#A$ for the cardinality of $A$. Let $X$ be a $n \times (n-1)$ matrix of indeterminates. For $A \subseteq \{1, \ldots, n\}$ and $H \subseteq \{1, \ldots, n-1\}$ with $\#A = \#H$, we write $X_{A,H}$ for the determinant of the submatrix of $X$ coming with rows in $A$ and columns in $H$. We will make use of the Hilbert-Burch theorem so it is convenient for this section to use signed minors: set $\Delta_i = (-1)^i X_{(\{i\} \times \nu)}$, that is to say $(-1)^i$ times the maximal minor of $X$ obtained by deleting the $i$th row.

Let $\mathbb{K}$ be a field and $R = \mathbb{K}[X]$, set $I = I_n(X) = (\Delta_1, \ldots, \Delta_n) \subseteq R$ the ideal of maximal minors of $X$. In this case the Rees algebra of $I$, 
\[ \mathcal{R}(I) := \bigoplus_{i \geq 0} \varpi^i \subseteq R[t], \]
is linear type, i.e., 
\[ \mathcal{R}(I) \cong S/(F_1, \ldots, F_{n-1}) \]
where $S = R[T_1, \ldots, T_n]$ and $F_j = \sum_{i=1}^{n} x_{ij} T_i$. Moreover $\mathcal{R}(I)$ is a complete intersection so the Koszul complex of $[F_1, \ldots, F_{n-1}] : S^{n-1} \to S$ is a resolution. In this section we will compare the linear strands of this Koszul complex to the Koszul complexes of $[\Delta_1, \ldots, \Delta_n] : R^n \to R$.

More precisely, we will describe maps of complexes, $\varphi_r$, for each $r$ making the following diagram commute.
\[ \begin{array}{ccc}
[K_*([F_1 \ldots F_{n-1}]; S)]_t & \longrightarrow & [\mathcal{R}(I)]_t \\
\varphi_t & \uparrow & \uparrow \\
K_{r-1}([\Delta_1, \ldots, \Delta_n]; R) & \longrightarrow & (\Delta_1, \ldots, \Delta_n) \\
\end{array} \]

Where $[K_*([F_1 \ldots F_{n-1}]; S)]_t$ denotes the $t$-th linear strand of $K_*([F_1 \ldots F_{n-1}]; S)$.

First we need to establish some notation and prove a small lemma that will be helpful later.

**Notation 3.1.**

1. Let $A = \{a_1, \ldots, a_r\} \subseteq \mathbb{Z}_{\geq 1}$ with $a_1 < a_2 < \ldots < a_r$. Then set 
   \[ e_A := e_{a_1} \land e_{a_2} \land \ldots \land e_{a_r} \]
   and 
   \[ \Delta_A := \prod_{a \in A} \Delta_a. \]

2. Let $A, B$ be ordered sets of integers then 
   \[ \rho(A, B) := \begin{cases} 
0 & \text{if } A \cap B \neq \emptyset \\
(-1)^{\nu(A, B)} & \text{else}
\end{cases} \]
   where $\nu(A, B) = \#\{(a, b) \subseteq A \times B| a > b\}$.

3. Let $A \subseteq \{1, \ldots, n\}$ and $H \subseteq \{1, \ldots, n-1\}$ with $\#A = r$ and $\#H = r - 2$. Then set 
   \[ Y_{A,H,i} := \det \begin{bmatrix} r_{i,H}^c \\ Z \end{bmatrix}, \]
   where $r_{i,H}^c$ is the entries of the $i$-th row of $X$ with columns in $H^c$ and $Z$ is the submatrix of $X$ with rows in $A^c$ and columns in $H^c$.

4. For $A$ an ordered sets of integers set 
   \[ (-1)^A := (-1)^{\sum_{a \in A} a}. \]

**Lemma 3.2.**
3.3 and Corollary 3.5 will constitute the base case of this induction with \( \varphi \) addressing the first column. Then the diagram above commutes. (Note that for \( e \) every module in a square is non-zero.

Proof.

1. This is an expansion of the determinant along the first row.
2. \( \rho(\{\alpha\}, A \setminus \{\alpha\}) = (-1)^{\alpha-1} \).

Since \( A \setminus \{\alpha\} \) and \( A^c \) are disjoint we have that

\[
\nu(\{\alpha\}, A \setminus \{\alpha\}) + \nu(\{\alpha\}, A^c) = \#\{b \in A \setminus \{\alpha\} | b > \alpha\} + \#\{b \in A^c | b > \alpha\}.
\]

Our strategy will be to consider each commutative square of Diagram (13) and induct on \( t \). Theorem 3.3 and Corollary 3.5 will constitute the base case of this induction with Theorem 3.3 addressing the first \( t \) for which a every module in a square is non-zero.

**Theorem 3.3.** Consider the following diagram for \( n \geq r \geq 2 \).

\[
\begin{array}{ccc}
\bigwedge^{r-1}(R^n) & \xrightarrow{\delta} & \bigwedge^{r-2}(\bigoplus RT_i) \\
\varphi_{r-1}^{-1} & \downarrow \varphi_{r-1}^{-2} & \\
\bigwedge^r R^n & \xrightarrow{d} & \bigwedge^{r-1} R^n,
\end{array}
\]

where \( \delta \) is the map on the \( (r-1) \)-st linear strand of \( K_*([F_1 \ldots F_{n-1}];S) \) and the bottom map, \( d \), is the differential of \( K([\Delta^{r-1}_1 \ldots \Delta^{r-1}_n];R) \).

Let \((f_i)_{i=1}^{n-1}\) denote the standard \( S \)-basis of \( S^{n-1} \) and \((e_i)_{i=1}^{n}\) denote the standard \( R \)-basis for \( R^n \). Define the vertical maps as follows: let \( A, B \subseteq \{1, \ldots, n\} \) with \( \#A = r \) and \( \#B = r - 1 \), set

\[
\varphi_{r-1}^{-1}(e_A) := (-1)^{r-1} \Delta_A^{-2} \sum_{K \subseteq \{1 \ldots n-1\}, \#K = r-1} (-1)^{A+K} X_{A,K,C} \cdot f_K
\]

\[
\varphi_{r-2}^{-1}(e_B) := (-1)^{r} \Delta_B^{-3} \sum_{K \subseteq \{1 \ldots n-1\}, \#K = r-2} (-1)^{B+K} X_{B,K,C} \cdot f_K \sum_{b \in B} \Delta_B \frac{T_b}{\Delta_b}
\]

\[
= (-1)^{r} \Delta_B^{-2} \sum_{b \in B} \frac{T_b}{\Delta_b} \sum_{K \subseteq \{1 \ldots n-1\}, \#K = r-2} (-1)^{B+K} X_{B,K,C} \cdot f_K
\]

Then the diagram above commutes. (Note that for \( r = 2 \), \( \varphi_{r-1}^{-2} \) is the map that takes \( e_i \rightarrow T_i \)).

Before we begin the proof we first give an example.
Example 3.4. Suppose $n = 3$ and $t = 2$ then Diagram (13) becomes:

$$0 \longrightarrow [\Lambda^2 S^2]_0 \longrightarrow [\Lambda^1 S^2]_1 \longrightarrow [\Lambda^0 S^2]_2 \longrightarrow I^2 \longrightarrow 0$$

$$\varphi_2^r \uparrow \quad \quad \varphi_1^r \uparrow$$

$$0 \longrightarrow \Lambda^3 R^3 \longrightarrow \Lambda^3 R^2 \longrightarrow \Lambda^1 R^1 \longrightarrow (\Delta_1^3, \Delta_2^3, \Delta_3^3) \longrightarrow 0.$$  

Then, Theorem 3.3 says the left square commutes for,

$$\varphi^2_2(e_1 \wedge e_2 \wedge e_3) = (-1)^2 \Delta_1 \Delta_2 \Delta_3 ((-1)^{6+3} f_1 \wedge f_2)$$

and

$$\varphi^1_2(e_a \wedge e_b) = (-1)^{3-2} \Delta_a \Delta_b \left( \frac{T_a}{\Delta_a} + \frac{T_b}{\Delta_b} \right) \left( (-1)^{a+b+1} x_{(a,b)} x_{1} f_1 + (-1)^{a+b+2} x_{(a,b)} x_{1} f_2 \right)$$

$$= - (\Delta_b T_a + \Delta_a T_b) ((-1)^{a+b+1} x_{(a,b)} x_{1} f_1 + (-1)^{a+b+2} x_{(a,b)} x_{1} f_2)$$

$$= \begin{cases} 
- (\Delta_2 T_1 + \Delta_1 T_2) (x_{3,2} f_1 - x_{3,1} f_2) & (a,b) = (1,2), \\
- (\Delta_3 T_1 + \Delta_1 T_3) (-x_{2,2} f_1 + x_{2,1} f_2) & (a,b) = (1,3), \\
- (\Delta_3 T_2 + \Delta_2 T_3) (x_{1,2} f_1 - x_{1,1} f_2) & (a,b) = (2,3).
\end{cases}$$

One may notice that the only way to possibly complete this diagram with a map $\varphi^0_2 : \Lambda^1 R^3 \rightarrow [\Lambda^0 S^2]_2$ and have any hope that it commutes is to set $\varphi^0_2(e_i) = T_i^2$. Later, in Theorem 3.7 we will see that this is the correct choice to make the diagram commute, along with how to construct the maps for other $t$.

Proof of Theorem 3.3. To show this diagram commutes we simply compute the two compositions of maps. Fix $A \subseteq \{1 \ldots n\}$. Then,

$$\delta(\varphi_{r-1}^-(e_A)) = \delta((-1)^{r-1} \Delta_{r-2} A \sum_{K \subseteq \{1 \ldots n\} \atop \#K = r-1} (-1)^{A+K} X_{A^c K^c} f_K)$$

$$= (-1)^{r-1} \Delta_{r-2} A \sum_{K \subseteq \{1 \ldots n\} \atop \#K = r-1} (-1)^{A+K} X_{A^c K^c} \delta(f_K)$$

$$= (-1)^{r-1} \Delta_{r-2} A \sum_{K \subseteq \{1 \ldots n\} \atop \#K = r-1} (-1)^{A+K} X_{A^c K^c} (\sum_{k \in K} \rho(\{k\}, K \setminus \{k\}) F_k f_K |_{K \setminus \{k\}})$$

$$= (-1)^{r-1} \Delta_{r-2} A \sum_{K \subseteq \{1 \ldots n\} \atop \#K = r-1} (-1)^{A+K} X_{A^c K^c} (\sum_{k \in K} \rho(\{k\}, K \setminus \{k\}) (\sum_{i=1}^n x_{i k} T_i) f_K |_{K \setminus \{k\}})$$

$$= (-1)^{r-1} \Delta_{r-2} A \sum_{i=1}^n T_i \sum_{K \subseteq \{1 \ldots n\} \atop \#K = r-1} (-1)^{A+K} X_{A^c K^c} \rho(\{k\}, K \setminus \{k\}) x_{i k} f_K |_{K \setminus \{k\}}$$

$$= (-1)^{r-1} \Delta_{r-2} A \sum_{i=1}^n T_i \sum_{H \subseteq \{1 \ldots n\} \atop \#H = r-2} (-1)^{A+H} f_H \sum_{a \in H^c} (-1)^a \rho(\{\alpha\}, H) x_{i \alpha} X_{A^c (H \cup \{\alpha\})^c} f_K |_{K \setminus \{k\}}$$
Now by Lemma 3.2 (2) we know that $\rho(\{\alpha\}, H^c \setminus \alpha) \rho(\{\alpha\}, H) = (-1)^{\alpha-1}$. Hence, 
$$(-1)^{\alpha} \rho(\{\alpha\}, H) = (-1)^{\rho(\{\alpha\}, H^c \setminus \alpha)}.$$ So the above is

$$= (-1)^{r-1} \Delta_{r-2}^{A} \sum_{i=1}^{n} T_i \sum_{H \subseteq \{1, \ldots, n-1\}} \sum_{H \setminus i \in H^c} (-1)^{A+H} f_H \sum_{a \in H^c} (-1)^{\rho(\{\alpha\}, H^c \setminus \alpha)x_{ia}X_{A^c(H \cup \{a\})^c}}$$

$$= (-1)^{r} \Delta_{r-2}^{A} \sum_{i=1}^{n} T_i \sum_{H \subseteq \{1, \ldots, n-1\}} \sum_{H \setminus i \in H^c} (-1)^{A+H} f_H \sum_{a \in H^c} \rho(\{\alpha\}, H^c \setminus \alpha)x_{ia}X_{A^c(H \cup \{a\})^c}$$

Now applying Lemma 3.2 (1) we get

$$= (-1)^{r} \Delta_{r-2}^{A} \sum_{i=1}^{n} T_i \sum_{H \subseteq \{1, \ldots, n-1\}} (-1)^{A+H} f_H Y_{A,H,i}$$

$$= (-1)^{r} \Delta_{r-2}^{A} \sum_{H \subseteq \{1, \ldots, n-1\}} (-1)^{A+H} f_H \sum_{i=1}^{n} Y_{A,H,i} T_i$$

$$= (-1)^{r} \Delta_{r-2}^{A} \sum_{H \subseteq \{1, \ldots, n-1\}} (-1)^{A+H} f_H \sum_{i \in A} Y_{A,H,i} T_i.$$ 

Here the last equality follows from the fact that $Y_{A,H,i} = 0$ if $i \notin A$.

Now for the other composition, 

$$\varphi_{r-1}^{n-2}(d(e_A)) = \varphi_{r-1}^{n-2}(\sum_{\beta \in A} \rho(\{\beta\}, A \setminus \{\beta\}) \Delta_{r-1}^{n-1} e_{A \setminus \{\beta\}})$$

$$= \sum_{\beta \in A} \rho(\{\beta\}, A \setminus \{\beta\}) \Delta_{r-1}^{n-1} \varphi_{r-1}^{n-2}(e_{A \setminus \{\beta\}})$$

$$= \sum_{\beta \in A} \rho(\{\beta\}, A \setminus \{\beta\}) \Delta_{r-1}^{n-1} (-1)^{r} \Delta_{A^c\setminus\{\beta\}}^{r-2} \sum_{b \in A \setminus \{\beta\}} \Delta_{b} \sum_{H \subseteq \{1, \ldots, n-1\}} (-1)^{A \setminus \{\beta\}+H} X_{(A \setminus \{\beta\})^c H^c} f_H$$

$$= (-1)^{r} \sum_{H \subseteq \{1, \ldots, n-1\}} (-1)^{A+H} f_H \sum_{\beta \in A} \sum_{b \in A \setminus \{\beta\}} (-1)^{\beta} \Delta_{b} \rho(\{\beta\}, A \setminus \{\beta\}) \Delta_{r-2}^{n-2} \Delta_{\beta} X_{(A \setminus \{\beta\})^c H^c}$$

$$= (-1)^{r} \Delta_{r-2}^{A} \sum_{H \subseteq \{1, \ldots, n-1\}} (-1)^{A+H} f_H \sum_{i \in A} \sum_{T_i} (-1)^{\gamma} \rho(\{\gamma\}, A \setminus \{\gamma\}) \Delta_{r} X_{(A \setminus \{\gamma\})^c H^c}$$

Now, set

$$\Theta := (-1)^{r} \Delta_{r-2}^{A} \sum_{H \subseteq \{1, \ldots, n-1\}} (-1)^{A+H} f_H.$$ 

So,

$$\varphi_{r-1}^{n-2}(d(e_A)) = \Theta \cdot \sum_{i \in A} \sum_{\gamma \in A \setminus \{i\}} (-1)^{\gamma} \Delta_{r} \rho(\{\gamma\}, A \setminus \{\gamma\}) X_{(A \setminus \{\gamma\})^c H^c}.$$
We now write
\[ X_{(A\setminus\{\gamma\})^c} = \rho(\{\gamma\}, A^c \setminus \{\gamma\})Y_{A,H,\gamma} = \rho(\{\gamma\}, A^c \setminus \{\gamma\}) \sum_{\alpha \in H^c} x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c} \]

Hence by Lemma 3.2 (2) we have
\[ \rho(\{\gamma\}, A \setminus \{\gamma\})Y_{(A\setminus\{\gamma\})^c} = \rho(\{\gamma\}, A \setminus \{\gamma\})\rho(\{\gamma\}, A^c \setminus \{\gamma\}) \sum_{\alpha \in H^c} x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c} \]
\[ = (-1)^{\gamma-1} \sum_{\alpha \in H^c} x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c} \]

So, returning to the original expression,
\[ \varphi_{r-1}^*\left(d(e_A)\right) = \Theta \cdot \sum_{i \in A} \Delta_i \sum_{\gamma \in A \setminus \{i\}} (-1)^{\gamma} \Delta_{\gamma} \sum_{\alpha \in H^c} x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c} \]
\[ = \Theta \cdot (-1)^{\gamma} \sum_{i \in A} \Delta_i \sum_{\gamma \in A \setminus \{i\}} x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c} \]

Using the fact that \( \sum_{i=1}^{n} \Delta_i x_{i\alpha} = 0 \) we get that \( \sum_{\gamma \in A \setminus \{i\}} \Delta_{\gamma} x_{\gamma\alpha} = -\sum_{\gamma \in A^c \cup \{i\}} \Delta_{\gamma} x_{\gamma\alpha} \). Therefore the previous line becomes
\[ = \Theta \cdot (-1)^{\gamma} \sum_{i \in A} \Delta_i \sum_{\gamma \in A^c \cup \{i\}} \Delta_{\gamma} \sum_{\alpha \in H^c} x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c} \]
\[ = \Theta \cdot \sum_{i \in A} \Delta_i \sum_{\gamma \in A^c \cup \{i\}} \Delta_{\gamma} Y_{A,H,\gamma} \]

Using Lemma 3.2 (1), we see that \( \sum_{\alpha \in H^c} x_{\gamma\alpha}X_{A^c(H\cup\alpha)^c} = Y_{A,H,\gamma} \). Thus, we have
\[ = \Theta \cdot \sum_{i \in A} \Delta_i \sum_{\gamma \in A^c \cup \{i\}} \Delta_{\gamma} Y_{A,H,\gamma} \]

Finally, using that \( Y_{A,H,\gamma} = 0 \) for \( \gamma \in A^c \), the expression simplifies to
\[ = \Theta \cdot \sum_{i \in A} \Delta_i \sum_{\gamma \in A^c \cup \{i\}} \Delta_{\gamma} Y_{A,H,\gamma} \]
\[ = (-1)^{\gamma} \Delta_A^{r-2} \sum_{i \in A} (-1)^{A^c+H} f_H \sum_{i \in A} \Delta_i \sum_{\gamma \in A^c \cup \{i\}} \Delta_{\gamma} Y_{A,H,\gamma} \]
\[ = (-1)^{\gamma} \Delta_A^{r-2} \sum_{i \in A} (-1)^{A^c+H} f_H \sum_{i \in A} Y_{A,H,i} \Delta_i \]

We have shown that
\[ \varphi_{r-1}^*(d(e_A)) = (-1)^{\gamma} \Delta_A^{r-2} \sum_{i \in A} (-1)^{A^c+H} f_H \sum_{i \in A} Y_{A,H,i} \Delta_i \]
and
\[
\delta(\varphi^{-1}_r(e_A)) = (-1)^r \Delta^{r-2}_A \sum_{H \subseteq \{1, \ldots, n\} \atop \#H = r-2} (-1)^{A+H} f_H \sum_{i \in A} Y_{A,H,i} T_i
\]
so the commutativity of the diagram is proven. \(\square\)

**Corollary 3.5.** Suppose \(\varphi^{-1}_r\) and \(\varphi^{-2}_r\) are the maps defined in Theorem 3.3. Consider the following two squares of Diagram (13) where \(t = r - 1\).

\[
\begin{array}{ccc}
0 &=& \Lambda^r(0)^{n-1} \\
\varphi^{-1}_r &
\downarrow & \\
\Lambda^{r+1} R^n &
\rightarrow & \\
\varphi^{-2}_r &
\downarrow & \\
\Lambda^r R^n &
\rightarrow & \\
\delta &
\downarrow & \\
\Lambda^{r-2}(\bigoplus RT)_1^{n-1} &
\rightarrow & \\
\delta &
\downarrow & \\
0 &=& \Lambda^r(0)^{n-1}
\end{array}
\]

This diagram commutes.

**Proof.** This follows from the injectivity of \(\delta\), since the top row is the tail of a resolution of \(I^{r-1}\), and Theorem 3.3: We have that \(\text{im}(\varphi^{-1}_r \circ d^{-1}_r) \subseteq \ker \delta = 0\). \(\square\)

**Notation 3.6.** Let \(s_e(y_1, \ldots, y_d)\) be the complete homogeneous symmetric function of degree \(e\) in \(y_1, \ldots, y_d\). For \(A = \{a_1, \ldots, a_d\} \subseteq \{1, \ldots, n\}\) define \(h_e(A) = s_e(\frac{T_{a_1}}{X_{a_1}}, \ldots, \frac{T_{a_d}}{X_{a_d}})\).

**Theorem 3.7.** For \(r > 1\) let \(\varphi^{-1}_r\) be the maps defined in Theorem 3.3 and let \(\varphi^0_0 : \Lambda^0 R^n \rightarrow [\Lambda^0 S^{n-1}]_0\) be the map \(\varphi^0_0(1) = 1\). Then for all \(t, r \geq 1\) define functions \(\varphi^t_r : \Lambda^r R^n \rightarrow [\Lambda^{r-1} S^{n-1}]_{t-r+1}\) as follows:

\[
\varphi^t_r(e_A) := \begin{cases} 
\varphi^t_r(e_A) & t < r - 1, \\
\varphi^{-1}_r(e_A) & t = r - 1, \\
\varphi^{-1}_r(e_A)(\Delta^{r-1}_A h_{t-r+1}(A)) & t > r - 1.
\end{cases}
\]

(Note that this is the same definition of \(\varphi^{-2}_r\) as in Theorem 3.3). Then

\[
\begin{array}{ccc}
\cdots &
\rightarrow & [\Lambda^{r-1} S^{n-1}]_{t-r+1} \\
\varphi^t_r &
\downarrow & \\
\cdots &
\rightarrow & \Lambda^r R^n \\
\varphi^t_r &
\downarrow & \\
\cdots &
\rightarrow & [\Lambda^{r-1} S^{n-1}]_{t-1} \\
\varphi^t_r &
\downarrow & \\
\cdots &
\rightarrow & \Lambda^r R^n \\
\varphi^t_r &
\downarrow & \\
\cdots &
\rightarrow & \Lambda^2 R^n \\
\varphi^t_r &
\downarrow & \\
\cdots &
\rightarrow & \Lambda^1 R^n \\
\varphi^t_r &
\downarrow & \\
\cdots &
\rightarrow & \Lambda^1 \cdots, \Lambda^t_n
\end{array}
\]

commutes, where the rightmost map is the natural inclusion, the top row is the \(t\)-th strand of the \(K_\bullet([F_1 \ldots F_{n-1}]; S)\) and the bottom row is \(K_\bullet([\Delta^t_1, \ldots, \Delta^t_n]; R)\).

Again, before proving this theorem lets return to Example 3.4.

**Example 3.8.** Suppose \(n = 3\) and \(t = 2\) then Diagram (13) becomes:

\[
\begin{array}{ccc}
0 &
\rightarrow & [\Lambda^2 S^2]_0 \\
\varphi^2_2 &
\downarrow & \\
[\Lambda^2 S^2]_2 &
\rightarrow & [\Lambda^1 S^2]_1 \\
\varphi^2_2 &
\downarrow & \\
[\Lambda^1 S^2]_1 &
\rightarrow & [\Lambda^0 S^2]_2 \\
\varphi^2_2 &
\downarrow & \\
[\Lambda^0 S^2]_2 &
\rightarrow & I^2 \\
\varphi^2_2 &
\downarrow & \\
I^2 &
\rightarrow & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 &
\rightarrow & \Lambda^3 R^3 \\
\varphi^2_2 &
\downarrow & \\
\Lambda^3 R^2 &
\rightarrow & \Lambda^1 R^3 \\
\varphi^2_2 &
\downarrow & \\
\Lambda^1 R^3 &
\rightarrow & (\Delta^2_3, \Delta^2_3, \Delta^2_3) \\
\varphi^2_2 &
\downarrow & \\
(\Delta^2_3, \Delta^2_3, \Delta^2_3) &
\rightarrow & 0.
\end{array}
\]
We saw in Example 3.4 that
\[ \varphi_2^2(e_1 \wedge e_2 \wedge e_3) = (-1)^2 \Delta_1 \Delta_2 \Delta_3((-1)^{(6+3)} f_1 \wedge f_2) \]
\[ = -\Delta_1 \Delta_2 \Delta_3(f_1 \wedge f_2). \]

Now using Theorem 3.7 we compute \( \varphi_0^2 \) and \( \varphi_2^1 \): \( \varphi_0^2(e_a) = 1 \) and \( h_2(\{a\}) = \frac{T_2^2}{\Delta_a^2} \), so
\[ \varphi_2^0(e_a) = (1)(\Delta_a^2) \left( \frac{T_2}{\Delta_a} \right) = T_2^2. \]

For \( \varphi_1^1 \) we first need to compute \( \varphi_1^1 \):
\[ \varphi_1^1(e_a \wedge e_b) = (-1)^{2-1}(\Delta_a \Delta_b) ^{2-2}((-1)^{a+b+1} X_{\{a,b\},1} f_1 + (-1)^{a+b+1} X_{\{a,b\},1} f_2) \]
\[ = \left\{\begin{array}{ll}
-(x_{3,2} f_1 - x_{3,1} f_2) & (a, b) = (1, 2) \\
-(x_{2,2} f_1 + x_{1,2} f_2) & (a, b) = (1, 3) \\
-(x_{1,2} f_1 - x_{1,1} f_2) & (a, b) = (2, 3)
\end{array}\right. \]

Now \( h_1(\{a, b\}) = \frac{T_a}{\Delta_a} + \frac{T_b}{\Delta_b} \) and we have,
\[ \varphi_2^1(e_a \wedge e_b) = \varphi_1^1(e_a \wedge e_b)(\Delta_a \Delta_b) \left( \frac{T_a}{\Delta_a} + \frac{T_b}{\Delta_b} \right) \]
\[ = \varphi_1^1(e_a \wedge e_b)(\Delta_a T_a + \Delta_b T_b) \]
\[ = \left\{\begin{array}{ll}
-(\Delta_2 T_1 + \Delta_1 T_2)(x_{3,2} f_1 - x_{3,1} f_2) & (a, b) = (1, 2) \\
-(\Delta_3 T_1 + \Delta_1 T_3)(-x_{2,2} f_1 + x_{2,1} f_2) & (a, b) = (1, 3) \\
-(\Delta_3 T_2 + \Delta_2 T_3)(x_{1,2} f_1 - x_{1,1} f_2) & (a, b) = (2, 3)
\end{array}\right. \]

which agrees with the computation in Example 3.4.

Proof of Theorem 3.7. The commutativity of the rightmost square is immediate so we are done once we show that for all \( r \geq 2 \) the following square commutes:
\[ \left[ \bigwedge^{r-1} S^{n-1} \right]_{t-r+1} \xrightarrow{\delta^{r-1}} \left[ \bigwedge^{r-2} S^{n-1} \right]_{t-r+2} \]
\[ \varphi_t^{r-1} \uparrow \quad \varphi_t^{r-2} \uparrow \]
\[ \bigwedge^r R^n \xrightarrow{d^r_t} \bigwedge^{r-1} R^n \]

If \( t < r - 2 \) the top row vanishes and both vertical maps are zero, so commutativity is clear. The case that \( t = r - 2 \) is addressed by Corollary 3.5. Finally, the case where \( t = r - 1 \) is handled by Theorem 3.3. So it is left to check the cases \( t > r - 1 \).

Before computing the two compositions we note the following key identity. Let \( A \subseteq \{1, \ldots, n\} \) with \#A = r. Using Corollary 3.5 we have that:
\[ 0 = \varphi_{r-2}^r(d_{r-2})(e_A) \]
\[ = \varphi_{r-2}^r(S_{\alpha} \rho(\{\alpha\}, A \backslash \{\alpha\}) \Delta_{\alpha}^{r-2} e_{A \backslash \alpha}) \]
\[ = \sum_{\alpha \in A} \rho(\{\alpha\}, A \backslash \{\alpha\}) \Delta_{\alpha}^{r-2} \varphi_{r-2}^r(e_{A \backslash \alpha}). \]

(14)

Now we are ready to show that the square commutes. Let \( A \subseteq \{1 \ldots n\} \) with \#A = r and set \( e = t - r + 1 \). Then,
\[ \delta^{r-1}(\varphi_t^{r-1}(e_A)) = \delta^{r-1}(\varphi_t^{r-1}(e_A)(\Delta A h_e(A))) \]
\[ = \Delta A^e h_e(A) \delta^{r-1}(\varphi_t^{r-1}(e_A)), \]
where the second equality follows from the $S$-linearity of $\delta$. Now using Theorem 3.3 we have,

\[
\begin{align*}
= & \Delta_A h_e(A) \varphi_{r-1}^{-2}(d_{r-1}^I(e_A)) \\
= & \Delta_A h_e(A) \varphi_{r-1}^{-2}(\sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_A^{-1} e_{A \setminus \{\alpha\}}) \\
= & \Delta_A h_e(A) \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_A^{-1} \varphi_{r-1}^{-2}(e_{A \setminus \{\alpha\}}) \\
= & \Delta_A h_e(A) \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_A^{-1} \varphi_{r-2}^{-2}(e_{A \setminus \{\alpha\}}) \\
= & \Delta_A h_e(A) \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_A^{-2} \varphi_{r-2}^{-2}(e_{A \setminus \{\alpha\}}) \sum_{\beta \in A \setminus \{\alpha\}} \frac{T_{\beta}}{\Delta_{\alpha}} h_e(A).
\end{align*}
\]

Now apply the fact that for $\beta \in A$, $h_{e+1}(A) = \frac{T_{\beta}}{h_e(A)} + h_{e+1}(A \setminus \{\beta\})$ to see that the above is

\[
\begin{align*}
= & \Delta_A^{e+1} \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_A^{-2} \varphi_{r-2}^{-2}(e_{A \setminus \{\alpha\}}) \sum_{\beta \in A \setminus \{\alpha\}} (h_{e+1}(A) - h_{e+1}(A \setminus \{\beta\})) \\
= & \Delta_A^{e+1} \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_A^{-2} \varphi_{r-2}^{-2}(e_{A \setminus \{\alpha\}}) \sum_{\beta \in A \setminus \{\alpha\}} h_{e+1}(A) \\
& - \Delta_A^{e+1} \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_A^{-2} \varphi_{r-2}^{-2}(e_{A \setminus \{\alpha\}}) \sum_{\beta \in A \setminus \{\alpha\}} h_{e+1}(A \setminus \{\beta\})
\end{align*}
\]

But $\sum_{\beta \in A \setminus \{\alpha\}} h_{e+1}(A) = (#A - 1)h_{e+1}(A) = (r - 1)h_{e+1}(A)$. So by identity (14),

\[
\begin{align*}
0 = & \Delta_A^{e+1} \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_A^{-2} \varphi_{r-2}^{-2}(e_{A \setminus \{\alpha\}}) \sum_{\beta \in A \setminus \{\alpha\}} h_{e+1}(A) \\
= & (r - 1)\Delta_A^{e+1} h_{e+1}(A) \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_A^{-2} \varphi_{r-2}^{-2}(e_{A \setminus \{\alpha\}}).
\end{align*}
\]

Hence we have

\[
\begin{align*}
= & -\Delta_A^{e+1} \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_A^{-2} \varphi_{r-2}^{-2}(e_{A \setminus \{\alpha\}}) \sum_{\beta \in A \setminus \{\alpha\}} h_{e+1}(A \setminus \{\beta\}) \\
= & -\Delta_A^{e+1} \sum_{\alpha \in A} h_{e+1}(A \setminus \{\alpha\}) \sum_{\beta \in A \setminus \{\alpha\}} \rho(\{\beta\}, A \setminus \{\beta\}) \Delta_A^{-2} \varphi_{r-2}^{-2}(e_{A \setminus \{\beta\}}).
\end{align*}
\]
By identity (14), we see that $0 = \sum_{\beta \in A \setminus \{\alpha\}} \rho(\{\beta\}, A \setminus \{\beta\}) \Delta_\beta^{-r} \phi^{-r-2}(e_{A \setminus \{\beta\}}) + \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_\alpha^{-r} \phi^{-r-2}(e_{A \setminus \{\alpha\}})$.

So the above expression is equal to

$$-\Delta_A^{e+1} \sum_{\alpha \in A} h_{e+1}(A \setminus \alpha)(-\rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_\alpha^{-r} \phi^{-r-2}(e_{A \setminus \{\alpha\}}))$$

$$= \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_\alpha^{-r} \phi^{-r-2}(e_{A \setminus \{\alpha\}}) \Delta_{e+1} h_{e+1}(A \setminus \alpha)$$

$$= \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_\alpha^{-2+r+1} \phi^{-r-2}(e_{A \setminus \{\alpha\}}) \Delta_{e+1} h_{e+1}(A \setminus \alpha)$$

$$= \sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_\alpha^{-1} \phi^{-r-2}(e_{A \setminus \{\alpha\}})$$

$$= \phi^{-r-2}(\sum_{\alpha \in A} \rho(\{\alpha\}, A \setminus \{\alpha\}) \Delta_\alpha^{-1} e_{A \setminus \{\alpha\}})$$

$$= \phi^{-r-2}(d_t(e_A)).$$

The results of this section are highly specialized to the case that $X$ is size $n \times (n - 1)$, in all other cases the Rees algebra of the ideal of maximal minors is substantially less nice and it is much more difficult to access a resolution of $I^t$, cf. [ABW81]. However, this is the only specialized aspect of this argument. Due to the elementary computational nature of the proof, Theorem 3.7 holds for any grade 2 perfect ideal of linear type with mild assumptions assumptions on the ambient ring.

4. The $n \times (n - 1)$ Case

For this section let $X$ be a $n \times (n - 1)$ matrix of indeterminates, $R = \mathbb{C}[X]$ and $I = I_{n-1}(X)$. We write $d_i$ for the determinant of the matrix obtained by deleting the $i$-th row of $X$. As noted in Section 2.2 $R \cong \text{Sym}(F \otimes G)$ where $F = \mathbb{C}^n$, $G = \mathbb{C}^{n-1}$ and $\text{GL} = \text{GL}_n \times \text{GL}_{n-1}$ acts on $R$.

4.1. The Cyclic Local Cohomology Module. By Proposition 2.11, we have that $H_m^{n-2-1}(R/I^t) = H_m^{n-2-n}(R/I^t)$ is a cyclic $R$-module. Define $J_t$ to be the ideal such that $H_m^{n-2-n}(R/I^t) \cong R/J_t$.

We will utilize the lift constructed in Section 3 to describe the modules $\text{Ext}_R^n(R/I^t, R)$ as submodules of $H_t^n(R)$. After constructing an isomorphism of $D$-modules $H_t^n(R) \rightarrow H_m^{n(n-1)}(R)$ we obtain a description of $\text{Ext}_R^n(R/I^t, R)$ as a submodule of $H_m^{n(n-1)}(R)$ which we can use to directly compute $\text{ann}_R \text{Ext}_R^n(R/I^t, R) = J_t$.

4.2. Description of $\text{Ext}_R^n(R/I^t, R)$. Let $\Delta_i = (-1)^i d_i$ and $S = \text{Sym}(R^{n-1})$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
\Delta_{\alpha} & \rightarrow & 0 \\
\text{d}_t(e_{\alpha}) & \rightarrow & 0 \\
\phi_{\alpha} & \rightarrow & 0 \\
\end{array}
$$
\[ R \left[ \frac{1}{\Delta_1}, \ldots, \frac{1}{\Delta_n} \right] \longrightarrow H^n(\hat{C}^*(\Delta_1, \ldots, \Delta_n; R)) \cong H^n_t(R) \longrightarrow 0 \]

(15)

\[ R \longrightarrow H^n(K^*(\Delta_1, \ldots, \Delta_n; R)) \longrightarrow 0 \]

By Corollary 2.9, the composition of vertical maps on the right is injective. Moreover \( \psi_t \) is induced by the map \( \varphi_t^{n-1} : R \cong \wedge^n R^n \rightarrow [\wedge^{n-1}(S)^{n-1}]_{n-1} \cong [S]_{t-n+1} \) described in Theorem 3.3 and Theorem 3.7. This map is zero for \( t < n - 1 \). For \( t = n - 1 \), we have \( \varphi_{n-1}^{n-1} \), and hence \( \psi_{n-1} \) is multiplication by the constant:

\[ (-1)^{n-1}(\prod_{i=1}^{n} \Delta_i)^{-2}. \]

Thus, the image of \( \Ext^n_R(R/I^{n-1}, R) \) is generated by \( \prod_{i=1}^{n} \Delta_i)^{-2} = \frac{1}{\prod_{i=1}^{n} \Delta_i} \) in \( H^n_t(R) \). For \( t \geq n \) we see that for \( |\alpha| = t - n + 1 \),

\[ \psi_t(T^\alpha) = (-1)^{n-1}(\prod_{i=1}^{n} \Delta_i)^{-2}(\prod_{i=1}^{n} \Delta_i)^{t-n+1} \frac{1}{\Delta^\alpha} = (-1)^{n-1}(\prod_{i=1}^{n} \Delta_i)^{t-1} \frac{1}{\Delta^\alpha}. \]

Since \( d_i \) and \( \Delta_i \) agree up to sign the above discussion proves the following:

**Theorem 4.1.** Under the embedding \( \Ext^n_R(R/I^t, R) \hookrightarrow H^n_t(R) \) of Diagram (15), \( \Ext^n_R(R/I^t, R) \) is the submodule generated by

\[ \left\{ \frac{1}{\prod_{i=1}^{n} d_i} \cdot \frac{1}{d^\alpha} \right\}_{|\alpha|=t-n+1}. \]

Recall that \( H^n_t(R) \) is a cyclic \( \mathcal{D} \)-module. The following result allows us to describe the images of the modules \( \Ext^n_R(R/I^t, R) \) in \( H^n_t(R) \) in a manner related to the \( \mathcal{D} \)-module structure of \( H^n_t(R) \).

**Proposition 4.2.** [LRWW17, Remark 3.8] [L17] Let \( \underline{s} = (s_1, \ldots, s_n) \) and \( s = \sum s_i \). For each \( i \), we have

\[ d_i^s \cdot (d_i \cdot d^\underline{s}) = (s_i + 1)(s + 2)(s + 3) \cdots (s + n)d^\underline{s}. \]

This proposition immediately gives us the following.

**Proposition 4.3.** Under the embedding induced by Diagram (15), for \( t \geq n - 1 \), we have

\[ \Ext^n_R(R/I^t, R) = \sum_{|\alpha|=t-n+1} R \cdot (d^\alpha)^\star \cdot \frac{1}{\prod_{i=1}^{n} d_i}. \]

By Theorem 2.10 the \( \mathcal{D} \)-modules \( H^n_t(R) \) and \( H_m^{n(n-1)}(R) \) are isomorphic cyclic \( \mathcal{D} \)-modules. To describe a \( \mathcal{D} \)-isomorphism between them it is sufficient to choose a socle generator of \( H^n_t(R) \) and of \( H_m^{n(n-1)}(R) \). Choose

\[ \frac{1}{\prod_{i=1}^{n} d_i} \in \text{Soc}(H^n_t(R)) \]
and \[ \frac{1}{\overline{z}} := \frac{1}{\prod_{ij} x_{ij}} \in \text{Soc}(H_m^{n(n-1)}(R)). \]

We observe the image of \( \text{Ext}_R^n(R/I^n, R) \) in \( H_m^{n(n-1)}(R) \) under the map induced by \( \prod_{i=1}^t d_i \mapsto \frac{1}{\overline{z}} \).

**Proposition 4.4.** For \( t \geq n - 1 \), we have \[ \text{Ext}_R^n(R/I^n, R) \cong \sum_{|\alpha| = t-n+1} R \cdot (d^\alpha)^* \cdot \frac{1}{\overline{z}}, \] where we write \( \frac{1}{\overline{z}} \) for the class in \( H_m^{n(n-1)}(R) \).

**Example 4.5.** Let \( n = t = 3 \). Then, \[ \text{Ext}_R^3(R/I^3, R) \cong \sum_{i=1}^3 R \cdot (d_i)^* \cdot \frac{1}{x_{1,1} x_{1,2} x_{2,1} x_{2,2} x_{3,1} x_{3,2}}. \]

Thus, \( \text{Ext}_R^3(R/I^3, R) \subseteq H_m^{(n-1)}(R) \) is generated as an \( R \)-module by

\[ \frac{1}{\overline{z}} \left( \frac{1}{x_{2,1} x_{3,2}} - \frac{1}{x_{2,2} x_{3,1}} \right), \]

\[ \frac{1}{\overline{z}} \left( \frac{1}{x_{1,1} x_{3,2}} - \frac{1}{x_{1,2} x_{3,1}} \right), \]

and

\[ \frac{1}{\overline{z}} \left( \frac{1}{x_{1,1} x_{2,2}} - \frac{1}{x_{1,2} x_{2,1}} \right). \]

Using this description \( \text{Ext}_R^n(R/I^n, R) \), we can utilize the \( D \)-module structure of \( H_m^{n(n-1)}(R) \) to describe the annihilator of \( \text{Ext}_R^n(R/I^n, R) \). Recall from Section 2.3 that \( R^* = \mathbb{C}[[\partial_{ij}]] \) and for a polynomial \( f \in R \) we write \( f^* = f(\{\partial_{ij}\}) \in R^* \). For an element \( f \in R \) we can form the \( R^* \) module generated by \( f \), where \( R^* \) acts by differentiation.

**Proposition 4.6.** Let \( t \geq n - 1 \). Then \[ (\text{ann}_R \text{Ext}_R^n(R/I^n, R))^* = \text{ann}_R \sum_{|\alpha| = t-n+1} R^* \cdot d^\alpha. \]

**Proof.** Let \( \zeta = \frac{1}{\overline{z}} \in H_m^{n(n-1)}(R) \) then \( H_m^{n(n-1)}(R) = D \cdot \zeta \) and \( \text{ann}_D \zeta = D \cdot \text{m} \). Now \( f \in \text{ann}_R \text{Ext}_R^n(R/I^n, R) \) if and only if for all \( |\alpha| = t-n+1 \) we have \( f d^\alpha \cdot \zeta = 0 \). Now \( f d^\alpha \cdot \zeta = 0 \) if and only if \( f d^\alpha \in D \cdot \text{m} \). So, applying the Fourier transform which sends \( x_{ij} \mapsto \partial_{ij} \), \( \partial_{ij} \mapsto -x_{ij} \), we have that \( f d^\alpha \in D \cdot \text{m} \) if and only if \( f^* d^\alpha \in D \cdot (\text{m}^*) \) if and only if \( f^* \cdot d^\alpha = 0 \). Hence \( f \in \text{ann}_R \text{Ext}_R^n(R/I^n, R) \) if and only if \( f^* \in \text{ann}_R \sum_{|\alpha| = t-n+1} R^* \cdot d^\alpha. \)

4.3. **The annihilator of** \( \text{Ext}_R^n(R/I^n, R) \). Recall from Section 2.3 that for all \( k \geq 0 \) there exists a GL-equivariant pairing \( \langle , \rangle : [R^*]_k \times [R]_k \rightarrow \mathbb{C} \) induced by differentiation.

**Proposition 4.7.** Let \( k \geq 1 \), \( \lambda = (k+1) \) and \( N = [I]_{(a-1)k} = \sum_{|\alpha| = k} \mathbb{C} \cdot d^\alpha \). Then for all \( f \) in the GL-orbit of \( \det_\lambda \), \( f^* \cdot N = 0 \).

**Proof.** \( \det_\lambda = x_{1,1}^{t+1} \) so for all \( |\alpha| = t \) we have that \( (\det_\lambda)^* \cdot d^\alpha = 0 \). The claim then follows from Lemma 2.6.

We are now ready to prove Theorem 5.1 in the \( n \times (n-1) \) case.
**Theorem 4.8.** If \( t \leq n - 2 \) then \( J_t = R \), for \( t \geq n - 1 \),
\[
\text{ann}_R \text{Ext}^n_R(R/I^t, R) = J_t = I_{(t-n+2)}.
\]

**Proof.** In the case that \( t \leq n - 2 \) we have that \( \text{projdim}_R(R/I^t) < n \) so clearly
\[
\text{ann}_R \text{Ext}^n_R(R/I^t, R) = R.
\]

For \( t \geq n - 1 \), first, we claim that
\[
I_{(t-n+2)} \subseteq \text{ann}_R \text{Ext}^n_R(R/I^t, R).
\]

Let \( f \in I_{(t-n+2)} \), then by Proposition 4.7, \( f^* \bullet d^\alpha = 0 \) for all \( |\alpha| = t - n + 1 \). Thus \( f^* \in \text{ann}_R \sum_{|\alpha| = t - n + 1} R^* \cdot d^\alpha \), so by Proposition 4.6, \( f \in \text{ann}_R \text{Ext}^n_R(R/I^t, R) \).

Now for the other inclusions we note that \( \text{Ext}^n_R(R/I^t, R) \) is GL-equivariant hence \( \text{ann}_R \text{Ext}^n_R(R/I^t, R) \) is a GL-invariant ideal. As was noted in Subsection 2.2, [dCEP80] proved that every GL-invariant ideal is of the form \( I_\chi = \sum_{\lambda \in \chi} I_\lambda \) for some finite collection of incomparable partitions \( \chi \).

Suppose for the sake of contradiction that \( I_{(t-n+2)} \subseteq \text{ann}_R \text{Ext}^n_R(R/I^t, R) \) and set \( I_\chi = \text{ann}_R \text{Ext}^n_R(R/I^t, R) \) where \( \chi \) is a collection of incomparable partitions. Thus there exists a partition \( \mu \in \chi \) such that either \( (t-n+2) > \mu \) or \( (t-n+2) \) is incomparable to \( \mu \).

In either case we have that \( ((t-n+1)^{n-1}) \geq \mu \), hence \( I_{((t-n+1)^{n-1})} \subseteq \text{ann}_R \text{Ext}^n_R(R/I^t, R) \).

In particular this implies that
\[
\text{det}((t-n+1)^{n-1}) = d_{n}^{t-n+1} \in \text{ann}_R \text{Ext}^n_R(R/I^t, R).
\]

However, this is a contradiction because by Theorem 4.1 we have that
\[
\prod_{i=1}^n d_i d_{n-i+1} \in \text{Ext}^n_R(R/I^t, R)
\]

but
\[
d_{n}^{t-n+1} \cdot \left( \prod_{i=1}^n \frac{1}{d_i} \right) = \prod_{i=1}^n \frac{1}{d_i} \neq 0.
\]

\( \square \)

In the next section we will generalize Theorem 4.8 to maximal minors of arbitrary size matrices using graded duality and results from [RWW14]. However, this approach does not recover a more general version of Theorem 4.1, the description of the Ext modules embedded in local cohomology, which would seem to be much more subtle. For \( X \) an arbitrary size \( m \times n \) matrix, we have that \( H^{mn-n^2+1}_{I_m(X)}(R) \) is not just a cokernel of a map of the Čech complex on the maximal minors of \( X \). Hence, writing down a description of a socle generator or even a non-zero element of \( H^{mn-n^2+1}_{I_m(X)}(R) \) is non-trivial. This makes even conjecturing an explicit description of the submodules \( \text{Ext}^{mn-n^2+1}_R(R/I_n(X)^t, R) \subset H^{mn-n^2+1}_{I_m(X)}(R) \equiv H^{mn}_m(R) \) challenging.

### 5. The General Case

We return to the setting of Section 2.2: Let \( F = \mathbb{C}^m \) and \( G = \mathbb{C}^n \) where \( m \geq n \). Then
\[
R := \text{Sym}(F \otimes G) = \mathbb{C}\{x_i\} = \mathbb{C}[X] \text{ and } \text{GL} := \text{GL}(F) \times \text{GL}(G).
\]

Fix \( I \) to be the ideal of \( n \times n \) minors of \( X \).

**Theorem 5.1.** Let \( R, I \) be as above and set \( m \) to be the homogeneous maximal ideal. Then
\[
H^{n^2-1}_{m^2-1}(R/I^t) \cong R/J_t,
\]
where \( J_t = R \) for \( t < n \), and for \( t \geq n \), \( J_t = I_{(t-n+1)} \), i.e., the ideal generated by the GL orbit of \( x_{11}^{n} \).
Proof. By graded duality we have the following isomorphism:

\[ H^{n^2-1}_m(R/I^t) \cong \text{Hom}_R(\text{Ext}^{mn-n^2+1}_R(R/I^t, R), H^{mn}_m(R)). \]

The GL structure of \( H^{mn}_m(R) \) is given by,

\[ H^{mn}_m(R) = \bigoplus_{\lambda \in \mathbb{Z}^n_{\lambda_1 \leq \cdots \leq \lambda_n}} S_{\lambda(n)} F \otimes S_{\lambda} G, \]

where \( S_{\lambda(n)} F \otimes S_{\lambda} G \) lives in degree \(|\lambda|\). We begin describing the GL structure of \( H^{n^2-1}_m(R/I^t) \) by first analyzing a single graded component.

\[
[H^{n^2-1}_m(R/I^t)]_r = [\text{Hom}_R(\text{Ext}^{mn-n^2+1}_R(R/I^t, R), H^{mn}_m(R))]_r \\
= \text{Hom}_C([\text{Ext}^{mn-n^2+1}_R(R/I^t, R)]_{-rn-r}, [H^{mn}_m(R)]_{-rn}) \\
= \text{Hom}_C([\text{Ext}^{mn-n^2+1}_R(R/I^t, R)]_{-rn-r}, (\bigwedge^m F)^{-n} \otimes (\bigwedge^n G)^{-m}) \\
= \text{Hom}_C([\text{Ext}^{mn-n^2+1}_R(R/I^t, R)]_{-rn-r}, \mathbb{C}) \otimes (\bigwedge^m F)^{-n} \otimes (\bigwedge^n G)^{-m}.
\]

Now by Theorem 2.7 we have that

\[ \text{Ext}^{mn-n^2+1}_R(R/I^t, R)]_{-rn-r} = \bigoplus_{\lambda \in A(r)} S_{\lambda(n)} F \otimes S_{\lambda} G, \]

where

\[ A(r) = \{ \lambda \in \mathbb{Z}^n | \sum_{i=1}^n \lambda_i = -mn - r \text{ and } -m \geq \lambda_1 \geq \cdots \geq \lambda_n \geq -t - (m - n) \}. \]

Dualizing into \( \mathbb{C} \) we get that

\[ \text{Hom}_C([\text{Ext}^{mn-n^2+1}_R(R/I^t, R)]_{-rn-r}, \mathbb{C}) = \bigoplus_{\lambda \in A(r)} S_{\lambda + (-m^a) + (m^b)} F \otimes S_{\lambda} G, \]

where

\[ B(r) = \{ \lambda \in \mathbb{Z}^n | \sum_{i=1}^n \lambda_i = mn + r \text{ and } t + (m - n) \geq \lambda_1 \geq \cdots \geq \lambda_n \geq m \}. \]

With this we can now describe the decomposition of \( H^{n^2-1}_m(R/I^t) \) into irreducible GL-representations:

\[
[H^{n^2-1}_m(R/I^t)]_r = \bigoplus_{\lambda \in B(r)} S_{\lambda + (-m^a) + (m^b)} F \otimes S_{\lambda} G \otimes (\bigwedge^m F)^{-n} \otimes (\bigwedge^n G)^{-m} \\
= \bigoplus_{\lambda \in B(r)} S_{\lambda + (-m^a) + (m^b)} F \otimes S_{\lambda + (-m^a)} G \\
= \bigoplus_{\lambda \in B(r)} S_{\lambda + (-m^a)} F \otimes S_{\lambda + (-m^a)} G \\
= \bigoplus_{\lambda \in B(r)} S_{\lambda} F \otimes S_{\lambda} G.
\]
Thus by Cauchy’s formula (3) we see that $J_t$ as a GL-representation is a direct sum of terms $S_{\lambda} F \otimes S_{\lambda}$ not present in the above direct sum. Hence by Remark 2.4 and Formula (4) we have that
\[
J_t = \bigoplus_{\lambda \in \mathbb{Z}^n_{\text{dom}}, \lambda_1 \geq t-n+1, \lambda_n \geq 0} S_{\lambda} F \otimes S_{\lambda} G = I_{(t-n+1)}.
\]

Comments on Characteristic $p > 0$. The description of these local cohomology modules in characteristic $p > 0$ is almost completely unknown. While the results of Section 3 are not dependent on characteristic, the approach used for the $n \times (n-1)$ case fails completely. Since $I$ is Cohen-Macaulay of height $(m-n+1)$, we have that $H_{I}^{m-n-2+1}(R) = 0$ so extracting information from the maps $\text{Ext}_{I}^{m-n-2+1}(R/I^t, R) \to H_{I}^{m-n-2+1}(R)$ is challenging.

Computer computations in Macaulay2 [GS] show that in prime characteristic the modules $H_{I}^{n-2}(R/I^t)$ are not always cyclic and may have generators in multiple degrees. In [Ken20] it was shown that the degree 0 component of $H_{I}^{n-2}(R/I^t)$ can have arbitrarily large vector space dimension, suggesting these modules may have arbitrarily many generators.

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