SDCA without Duality

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Abstract

Stochastic Dual Coordinate Ascent is a popular method for solving regularized loss minimization for the case of convex losses. In this paper we show how a variant of SDCA can be applied for non-convex losses. We prove linear convergence rate even if individual loss functions are non-convex as long as the expected loss is convex.

1 Introduction

The following regularized loss minimization problem is associated with many machine learning methods:

$$\min_{w \in \mathbb{R}^d} P(w) := \frac{1}{n} \sum_{i=1}^{n} \phi_i(w) + \frac{\lambda}{2} \|w\|^2.$$ 

One of the most popular methods for solving this problem is Stochastic Dual Coordinate Ascent (SDCA). [8] analyzed this method, and showed that when each $\phi_i$ is $L$-smooth and convex then the convergence rate of SDCA is $\tilde{O}\left((L/\lambda + n) \log(1/\epsilon)\right)$.

As its name indicates, SDCA is derived by considering a dual problem. In this paper, we consider the possibility of applying SDCA for problems in which individual $\phi_i$ are non-convex, e.g., deep learning optimization problems. In many such cases, the dual problem is meaningless. Instead of directly using the dual problem, we describe and analyze a variant of SDCA in which only gradients of $\phi_i$ are being used (similar to option 5 in the pseudo code of Prox-SDCA given in [6]). Following [3], we show that SDCA is a variant of the Stochastic Gradient Descent (SGD), that is, its update is based on an unbiased estimate of the gradient. But, unlike the vanilla SGD, for SDCA the variance of the estimation of the gradient tends to zero as we converge to a minimum.

For the case in which each $\phi_i$ is $L$-smooth and convex, we derive the same linear convergence rate of $\tilde{O}\left((L/\lambda + n) \log(1/\epsilon)\right)$ as in [8], but with a simpler, direct, dual-free, proof. We also provide a linear convergence rate for the case in which individual $\phi_i$ can be non-convex, as long as the average of $\phi_i$ are convex. The rate for non-convex losses has a worst dependence on $L/\lambda$ and we leave it open to see if a better rate can be obtained for the non-convex case.

Related work: In recent years, many methods for optimizing regularized loss minimization problems have been proposed. For example, SAG [5], SVRG [3], Finito [2], SAGA [1], and S2GD [4]. The best convergence rate is for accelerated SDCA [6]. A systematic study of the convergence rate of the different methods under non-convex losses is left to future work.

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We maintain pseudo-dual vectors \( \alpha_1, \ldots, \alpha_n \), where each \( \alpha_i \in \mathbb{R}^d \).

**Dual-Free SDCA**\( (P, T, \eta, \alpha^{(0)}) \)**

**Goal:** Minimize \( P(w) = \frac{1}{n} \sum_{i=1}^{n} \phi_i(w) + \frac{\lambda}{2} \|w\|^2 \)

**Input:** Objective \( P \), number of iterations \( T \), step size \( \eta \) s.t. \( \beta := \eta \lambda n < 1 \), initial dual vectors \( \alpha^{(0)} = (\alpha^{(0)}_1, \ldots, \alpha^{(0)}_n) \)

**Initialize:** \( w^{(0)} = \frac{1}{\lambda n} \sum_{i=1}^{n} \alpha_i^{(0)} \)

**For** \( t = 1, \ldots, T \)

- Pick \( i \) uniformly at random from \([n]\)
- Update: \( \alpha_i^{(t)} = \alpha_i^{(t-1)} - \eta \lambda n \left( \nabla \phi_i(w^{(t-1)}) + \alpha_i^{(t-1)} \right) \)
- Update: \( w^{(t)} = w^{(t-1)} - \eta \left( \nabla \phi_i(w^{(t-1)}) + \alpha_i^{(t-1)} \right) \)

Observe that SDCA keeps the primal-dual relation

\[
 w^{(t-1)} = \frac{1}{\lambda n} \sum_{i=1}^{n} \alpha_i^{(t-1)}
\]

Observe also that the update of \( \alpha \) can be rewritten as

\[
 \alpha_i^{(t)} = (1 - \beta) \alpha_i^{(t-1)} + \beta \left( -\nabla \phi_i(w^{(t-1)}) \right),
\]

namely, the new value of \( \alpha_i \) is a convex combination of its old value and the negation of the gradient. Finally, observe that, conditioned on the value of \( w^{(t-1)} \) and \( \alpha^{(t-1)} \), we have that

\[
 \mathbb{E}[w^{(t)}] = w^{(t-1)} - \eta \left( \nabla \mathbb{E}[\phi_i(w^{(t-1)})] + \mathbb{E}[\alpha_i^{(t-1)}] \right) \\
= w^{(t-1)} - \eta \left( \nabla \frac{1}{n} \sum_{i=1}^{n} \phi_i(w^{(t-1)}) + \lambda w^{(t-1)} \right) \\
= w^{(t-1)} - \eta \nabla P(w^{(t-1)}).
\]

That is, SDCA is in fact an instance of Stochastic Gradient Descent. As we will see in the analysis section below, the advantage of SDCA over a vanilla SGD algorithm is because the *variance* of the update goes to zero as we converge to an optimum.

3 Analysis

The theorem below provides a linear convergence rate for smooth and convex functions. The rate matches the analysis given in [8], but the analysis is simpler and does not rely on duality.

**Theorem 1.** Assume that each \( \phi_i \) is \( L \)-smooth and convex, and the algorithm is run with \( \eta \leq \frac{1}{L + \lambda n} \). Let \( w^* \) be the minimizer of \( P(w) \) and let \( \alpha^*_i = -\nabla \phi_i(w^*) \). Then, for every \( t \geq 1 \),

\[
 \mathbb{E} \left[ \frac{\lambda}{2} \|w^{(t)} - w^*\|^2 + \frac{1}{2 Ln} \sum_{i=1}^{n} \| \alpha_i^{(t)} - \alpha^*_i \|^2 \right] \leq e^{-\eta \lambda t} \left[ \frac{\lambda}{2} \|w^{(0)} - w^*\|^2 + \frac{1}{2 Ln} \sum_{i=1}^{n} \| \alpha_i^{(0)} - \alpha^*_i \|^2 \right].
\]
In particular, setting \( \eta = \frac{1}{L + \lambda n} \), then after

\[
T \geq \tilde{\Omega} \left( \frac{L}{\lambda} + n \right)
\]

iterations we will have \( \mathbb{E}[P(w(T)) - P(w^*)] \leq \varepsilon \).

The theorem below provides a linear convergence rate for smooth functions, without assuming that individual \( \phi_i \) are convex. We only require that the average of \( \phi_i \) is convex. The dependence on \( L/\lambda \) is worse in this case.

**Theorem 2.** Assume that each \( \phi_i \) is \( L \)-smooth and that the average function, \( \frac{1}{n} \sum_{i=1}^{n} \phi_i \), is convex. Let \( w^* \) be the minimizer of \( P(w) \) and let \( \alpha_i^* = -\nabla \phi_i(w^*) \). Then, if we run SDCA with \( \eta = \min\{\frac{\lambda}{2L^2}, \frac{1}{2\lambda n}\} \), we have that

\[
\mathbb{E} \left[ \frac{\lambda}{2} \|w(t) - w^*\|^2 + \frac{\lambda}{2L^2 n} \sum_{i=1}^{n} \|\alpha_i^{(t)} - \alpha_i^*\|^2 \right] \leq e^{-\eta \lambda t} \left[ \frac{\lambda}{2} \|w(0) - w^*\|^2 + \frac{\lambda}{2L^2 n} \sum_{i=1}^{n} \|\alpha_i^{(0)} - \alpha_i^*\|^2 \right].
\]

It follows that whenever

\[
T \geq \tilde{\Omega} \left( \frac{L^2}{\lambda^2} + n \right)
\]

we have that \( \mathbb{E}[P(w(T)) - P(w^*)] \leq \varepsilon \).

### 3.1 SDCA as variance-reduced SGD

As we have shown before, SDCA is an instance of SGD, in the sense that the update can be written as

\( w_i(t) = w_i(t-1) - \eta u_i \), with \( u_i = \nabla \phi_i(w(t-1)) + \alpha_i^{(t-1)} \) satisfying \( \mathbb{E}[u_i] = \nabla P(w(t-1)) \).

The advantage of SDCA over a generic SGD is that the variance of the update goes to zero as we converge to the optimum. To see this, observe that

\[
\mathbb{E}[\|u_i\|^2] = \mathbb{E}[\|\alpha_i^{(t-1)} + \nabla \phi_i(w(t-1))\|^2] = \mathbb{E}[\|\alpha_i^{(t-1)} - \alpha_i^* + \alpha_i^* + \nabla \phi_i(w(t-1))\|^2] \leq 2\mathbb{E}[\|\alpha_i^{(t-1)} - \alpha_i^*\|^2] + 2\mathbb{E}[\|\nabla \phi_i(w(t-1)) - \alpha_i^*\|^2]
\]

Theorem 1 (or Theorem 2) tells us that the term \( \mathbb{E}[\|\alpha_i^{(t-1)} - \alpha_i^*\|^2] \) goes to zero as \( e^{-\eta \lambda t} \). For the second term, by smoothness of \( \phi_i \) we have \( \|\nabla \phi_i(w(t-1)) - \alpha_i^*\| = \|\nabla \phi_i(w(t-1)) - \nabla \phi_i(w^*)\| \leq L\|w(t-1) - w^*\| \), and therefore, using Theorem 1 (or Theorem 2) again, the second term also goes to zero as \( e^{-\eta \lambda t} \). All in all, when \( t \geq \tilde{\Omega} \left( \frac{1}{\eta \lambda} \log(1/\varepsilon) \right) \) we will have that \( \mathbb{E}[\|u_i\|^2] \leq \varepsilon \).

### 4 Proofs

Observe that \( 0 = \nabla P(w^*) = \frac{1}{n} \sum_i \nabla \phi_i(w^*) + \lambda w^* \), which implies that \( w^* = \frac{1}{\lambda n} \sum_i \alpha_i^* \).

Define \( u_i = -\nabla \phi_i(w(t-1)) \) and \( v_i = -u_i + \alpha_i^{(t-1)} \). We also denote two potentials:

\[
A_t = \frac{1}{n} \sum_{j=1}^{n} \|\alpha_j^{(t)} - \alpha_j^*\|^2, \quad B_t = \|w(t) - w^*\|^2.
\]
We will first analyze the evolution of $A_t$ and $B_t$. If on round $t$ we update using element $i$ then $\alpha_i(t) = (1 - \beta)\alpha_i(t-1) + \beta u_i$, where $\beta = \eta\lambda n$. It follows that,

\[
A_t - A_{t-1} = \frac{1}{n}||\alpha_i(t) - \alpha_i^*||^2 - \frac{1}{n}||\alpha_i(t-1) - \alpha_i^*||^2 \\
= \frac{1}{n}||\alpha_i(t-1)|| + \beta(u_i - \alpha_i^*)- (1 - \beta)||\alpha_i(t-1) - u_i|| - \beta(u_i - \alpha_i^*) - (1 - \beta)||\alpha_i(t-1) - u_i||^2 - ||\alpha_i(t-1) - \alpha_i^*||^2
\]

In addition,

\[
B_t - B_{t-1} = ||w(t) - w^*||^2 - ||w(t-1) - w^*||^2 = -2\eta(w(t-1) - w^*)^\top v_t + \eta^2||v_t||^2.
\]

The proofs of Theorem 1 and Theorem 2 will follow by studying different combinations of $A_t$ and $B_t$.

### 4.1 Proof of Theorem 2

Define

\[
C_t = \frac{\lambda}{2} \left[ \frac{1}{L^2} A_t + B_t \right].
\]

Combining (1) and (2) we obtain

\[
C_{t-1} - C_t = \frac{\eta\lambda^2}{2L^2} \left[ \lambda\alpha_i(t-1) - \alpha_i^*\right] - ||u_i - \alpha_i^*||^2 + (1 - \beta)||v_t||^2 + \frac{\lambda}{2} \left[ 2\eta(w(t-1) - w^*)^\top v_t + \eta^2||v_t||^2 \right]
\]

The definition of $\eta$ implies that $\eta \leq \lambda(1 - \beta)/L^2$, so the coefficient of $||v_t||^2$ is non-negative. By smoothness of each $\phi_i$ we have $||u_i - \alpha_i^*||^2 = ||\nabla \phi_i(w(t-1)) - \nabla \phi_i(w^*)||^2 \leq L^2||w(t-1) - w^*||^2$. Therefore,

\[
C_{t-1} - C_t \geq \eta \lambda \left[ \frac{\lambda}{2L^2}||\alpha_i(t-1) - \alpha_i^*||^2 - \frac{\lambda}{2} ||w(t-1) - w^*||^2 + (w(t-1) - w^*)^\top v_t \right].
\]

Taking expectation of both sides (w.r.t. the choice of $i$ and conditioned on $w(t-1)$ and $\alpha(t-1)$) and noting that $\mathbb{E}[v_t] = \nabla P(w(t-1))$, we obtain that

\[
\mathbb{E}[C_{t-1} - C_t] \geq \eta \lambda \left[ \frac{\lambda}{2L^2} \mathbb{E}||\alpha_i(t-1) - \alpha_i^*||^2 - \frac{\lambda}{2} ||w(t-1) - w^*||^2 + (w(t-1) - w^*)^\top \nabla P(w(t-1)) \right].
\]

Using the strong convexity of $P$ we have $(w(t-1) - w^*)^\top \nabla P(w(t-1)) \geq P(w(t-1)) - P(w^*) + \frac{1}{2} ||w(t-1) - w^*||^2$ and $P(w(t-1)) - P(w^*) \geq \frac{\lambda}{2} ||w(t-1) - w^*||^2$, which together yields $(w(t-1) - w^*)^\top \nabla P(w(t-1)) \geq \frac{\lambda}{2} ||w(t-1) - w^*||^2$. Therefore,

\[
\mathbb{E}[C_{t-1} - C_t] \geq \eta \lambda \left[ \frac{\lambda}{2L^2} \mathbb{E}||\alpha_i(t-1) - \alpha_i^*||^2 - \frac{\lambda}{2} ||w(t-1) - w^*||^2 + \frac{1}{2} ||w(t-1) - w^*||^2 \right].
\]
Taking expectation over $\lambda w$. Therefore,

\[
E[C_{t-1} - C_t] \geq \eta \lambda \left[ \frac{\lambda}{2L^2} \mathbb{E}[\|\alpha^*_i - \alpha^*_i\|^2] + \left( \frac{\lambda L^2}{2L^2} + \lambda \right) \|w^{(t-1)} - w^*\|^2 \right] = \eta \lambda C_{t-1}.
\]

It follows that

\[
\mathbb{E}[C_t] \leq (1 - \eta \lambda) C_{t-1}
\]

and repeating this recursively we end up with

\[
\mathbb{E}[C_t] \leq (1 - \eta \lambda)^t C_0 \leq e^{-\eta \lambda C_0},
\]

which concludes the proof of the first part of Theorem 2. The second part follows by observing that $P$ is $(L + \lambda)$ smooth, which gives $P(w) - P(w^*) \leq \frac{L + \lambda}{2} \|w - w^*\|^2$.

### 4.2 Proof of Theorem 1

In the proof of Theorem 1 we bounded the term $\|u_i - \alpha^*_i\|^2$ by $L^2 \|w^{(t-1)} - w^*\|^2$ based on the smoothness of $\phi_i$. We now assume that $\phi_i$ is also convex, which enables to bound $\|u_i - \alpha^*_i\|^2$ based on the current sub-optimality.

**Lemma 1.** Assume that each $\phi_i$ is $L$-smooth and convex. Then, for every $w$,

\[
\frac{1}{n} \sum_{i=1}^n \|\nabla \phi_i(w) - \nabla \phi_i(w^*)\|^2 \leq 2L \left( P(w) - P(w^*) - \frac{\lambda}{2} \|w - w^*\|^2 \right).
\]

**Proof.** For every $i$, define

\[
g_i(w) = \phi_i(w) - \phi_i(w^*) - \nabla \phi_i(w^*)^\top (w - w^*).
\]

Clearly, since $\phi_i$ is $L$-smooth so is $g_i$. In addition, by convexity of $\phi_i$ we have $g_i(w) \geq 0$ for all $w$. It follows that $g_i$ is non-negative and smooth, and therefore, it is self-bounded (see Section 12.1.3 in [7]):

\[
\|\nabla g_i(w)\|^2 \leq 2Lg_i(w).
\]

Using the definition of $g_i$, we obtain

\[
\|\nabla \phi_i(w) - \nabla \phi_i(w^*)\|^2 = \|\nabla g_i(w)\|^2 \leq 2Lg_i(w) = 2L \left( \phi_i(w) - \phi_i(w^*) - \nabla \phi_i(w^*)^\top (w - w^*) \right).
\]

Taking expectation over $i$ and observing that $P(w) = \mathbb{E}\phi_i(w) + \frac{1}{2} \|w\|^2$ and $0 = \nabla P(w^*) = \mathbb{E}\nabla \phi_i(w^*) + \lambda w^*$ we obtain

\[
\mathbb{E}\|\nabla \phi_i(w) - \nabla \phi_i(w^*)\|^2 \leq 2L \left[ P(w) - \frac{\lambda}{2} \|w\|^2 - P(w^*) + \frac{\lambda}{2} \|w^*\|^2 + \lambda w^*^\top (w - w^*) \right] = 2L \left[ P(w) - P(w^*) - \frac{\lambda}{2} \|w - w^*\|^2 \right].
\]

\[\square\]
We now consider the potential
\[ D_t = \frac{1}{2L} A_t + \frac{\lambda}{2} B_t. \]
Combining (1) and (2) we obtain
\[
D_{t-1} - D_t \geq \eta \lambda \left[ \frac{1}{2L} \left( \|\alpha_i^{(t-1)} - \alpha_i^*\|^2 - \|u_i - \alpha_i^*\|^2 \right) + \left( \frac{1 - \beta}{2L} - \frac{\eta}{2} \right) \|v_t\|^2 + \left( \lambda - \frac{\eta}{2} \|v_t\|^2 \right) \right],
\]
where in the last inequality we used the assumption
\[ \eta \leq \frac{1}{L + \lambda n} \Rightarrow \eta \leq \frac{1 - \beta}{L}. \]
Take expectation of the above w.r.t. the choice of \( i \), using Lemma [1] using \( \mathbb{E}[v_t] = \nabla P(w^{(t-1)}) \), and using convexity of \( P \) that yields \( P(w^*) - P(w^{(t-1)}) \geq (w^* - w^{(t-1)})^\top \nabla P(w^{(t-1)}) \), we obtain
\[
\mathbb{E}[D_{t-1} - D_t] \geq \eta \lambda \left[ \frac{1}{2L} \left( \mathbb{E}[\|\alpha_i^{(t-1)} - \alpha_i^*\|^2 - \mathbb{E}[u_i - \alpha_i^*\|^2] \right) + \left( \frac{1 - \beta}{2L} - \frac{\eta}{2} \mathbb{E}[v_t]\right) \mathbb{E}[v_t] \right] \
\geq \eta \lambda \left[ \frac{1}{2L} \mathbb{E}[\|\alpha_i^{(t-1)} - \alpha_i^*\|^2 - \left( P(w^{(t-1)}) - P(w^*) - \frac{\lambda}{2} \|w^{(t-1)} - w^*\|^2 \right) + (w^{(t-1)} - w^*)^\top \nabla P(w^{(t-1)}) \right] \
\geq \eta \lambda \left[ \frac{1}{2L} \mathbb{E}[\|\alpha_i^{(t-1)} - \alpha_i^*\|^2 + \lambda \frac{1}{2} \|w^{(t-1)} - w^*\|^2 \right] = -\eta \lambda D_{t-1}
\]
This gives \( \mathbb{E}[D_t] \leq (1 - \eta \lambda) D_{t-1} \leq e^{-\eta \lambda} D_{t-1} \), which concludes the proof of the first part of the theorem. The second part follows by observing that \( P \) is \( (L + \lambda) \) smooth, which gives \( P(w) - P(w^*) \leq \frac{L + \lambda}{2} \|w - w^*\|^2 \).

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