Holography and the Shannon’s First Theorem.

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Abstract

A possible link between Holography and Information Theory is presented. Using the relation between the Shannon and Boltzmann formulas Holography can be seen as the best encoding scheme.

PACS number(s): 04., 89.70.+c

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1 Introduction.

The phrase "Information is physical", enunciated by Rolf Landauer\textsuperscript{[1]} is the core of the research field known as physics of information. The author claimed that information is everywhere, and, whether it is in biological systems, in a computer, or in a black hole, inevitably has a physical form. Also information, of whatever kind, is associated with matter, radiation or fields of different type. It is impossible to get away from this sort of physical embodiment. Therefore, manipulation of information is inevitably subject to the laws of physics.

On the other hand there is a particular physical conjecture known as the Holographic Principle\textsuperscript{[2]}, which states that the number of the degrees of freedom describing the physics inside a volume (including gravity) is bounded by the area of the boundary which encloses this volume.

As information is physical we can also try the other way around, interpreting physical theories in terms of the language of the Information Theory. More precisely, in this paper I will attempt to connect the holography with one of the cornerstone of Information Theory, the Shannon noiseless coding theorem\textsuperscript{[3]}.

2 Information Theory.

The classical theory of information is due to a Bell Labs mathematician, Claude Shannon, who in two seminal works definitively laid down its principles. The works of Shannon deal with two very important tasks of information processing: (i) file compression and (ii) error correction. The Shannon noiseless coding theorem -also called source coding theorem- addresses the first issue: How much redundancy can be eliminated from a given source text without losing information? The Shannon noisy coding theorem addresses the second issue: How much redundancy must be added to a given source text in order to guarantee error free communication over a noisy channel.
In this paper we will concentrate just on the first theorem and its relation with holography.

2.1 Definitions and example

Information is discretized: it comes in irreducible packages. The elementary unit of classical information is the bit (or cbit, for classic bit), a classical system with only two states, 0 and 1. Any text can be coded into a string of bits[4], [5].

Definitions

• Let \( S := \{s_1, s_2, ..., s_n\} \) be a finite alphabet, endowed with a probability distribution \( P : s_j \rightarrow p_s(s_j) \)

• Codeword - a string of bits which represents a symbol in the alphabet.

• Code, codebook - the set of all possible codewords, e.g., the 4-bit binary code has 16 code-words

• Let \( \Gamma = (S, P) \) be a source. An encoding scheme for \( \Gamma \) is an ordered pair \( (C, f) \), where \( C \) is a code and \( f : S \rightarrow C \) is an injective function, called encoding function.

For the purposes of noiseless encoding, the measure of efficiency of an encoding scheme is its average codeword length.

Definition: The average codeword length of an encoding scheme \( (C, f) \) for a source \( \Gamma = (S, P) \) is defined by

\[
\text{AveLen}(C, f) = \sum_{i=1}^{N} p_i \text{len}(f(s_i))
\]  

(1)

where \( \text{len}(f(s_i)) \) is the length of the string that represents \( s_i \).

The Noiseless Coding Theorem, first proved by Claude Shannon in 1948, asserts that the average codeword length of any instantaneous code is bounded
from below by \( H(C) \), i.e. \( H(C) \leq \text{AveLen}(C) \), where \( H \) is the so called Shannon’s entropy

\[
H(C) = - \sum_{i=1}^{N} p_{i} \log_{2} p_{i}
\]  

(2)

The randomness of an information source can be describe by its entropy. The operational meaning of entropy is that it determines the smallest number of bits per symbol that is required to represent the total output.

**Example**

Consider an alphabet \( \{a, b, c, d\} \) with probabilities \( \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\} \). With four alternatives, it seems natural to assign two-bit binary codewords to each symbol, eg, \( \{00, 01, 10, 11\} \). If this coding scheme is used, the average codeword length is

\[
\text{AveLen}(l) = \frac{1}{2}2 + \frac{1}{4}2 + \frac{1}{8}2 + \frac{1}{8}2 = 2.
\]

(3)

However, we can do better than this. The entropy of this alphabet is

\[
H = - \left( \frac{1}{2} \log_{2} \frac{1}{2} + \frac{1}{4} \log_{2} \frac{1}{4} + \frac{1}{8} \log_{2} \frac{1}{8} + \frac{1}{8} \log_{2} \frac{1}{8} \right) = \frac{7}{4}
\]

(4)

and Shannon’s theorem says that a code exists which achieves this rate. In this case, it is easy. The code \( \{0, 10, 110, 111\} \) assigns shorter codewords to the more frequent symbols. The average codeword length is

\[
\text{AveLen}(l) = \frac{1}{2}1 + \frac{1}{4}2 + \frac{1}{8}3 + \frac{1}{8}3 = \frac{7}{4}.
\]

(5)

In the simple example above, the Shannon entropy and the average code length were both \( \frac{7}{4} \) bits/symbol. It happened this way because the probabilities were powers of \( \frac{1}{2} \). We are not so fortunate in reality and there will always be a difference between the Shannon’s entropy and the achieved average code length. This is quantified by two numbers, the *efficiency* and the *redundancy*

\[
\text{Efficiency} = \frac{H}{\text{AveLen}} \leq 1
\]

(6)

\[
\text{Redundancy} = 1 - \text{Efficiency}.
\]

(7)
3 Holography

Physicists have the standard vision that the degrees of freedom of the world consist of fields filling space. Improving this scenario some theorists proposed that a small distance cutoff will be required in order to make sense of quantum gravity. Therefore according to that hypothesis the world could be seen as three dimensional discrete lattice with spacing order of the Plank length, $l_p$. But recently, ’t Hooft and Susskind [2] suggested an idea even more audacious. According to them the combination of Quantum Mechanics and Gravity requires the three dimensional world to be an image of data that can be stored on a two dimensional projection much like a holographic image. This description only requires one discrete degree of freedom per Planck area and yet it is rich enough to describe all three dimensional phenomena. Therefore it is obvious that the assumption of these ideas imply a radical decrease in the number of degrees of freedom for describing the Universe.

The inspiration for the holographic principle comes from the black hole physics where the entropy of a black hole is given by the Bekenstein-Hawking formula [6,7]

$$S = \frac{A}{4G} = \frac{A}{l_p^{2}} \log 2, \quad (8)$$

where $A$ is the horizon area. This result was obtained when they realized the striking resemblance between the laws of black hole mechanics and the laws of thermodynamics. Later, a great amount of work has been done in order to find a precise statistical mechanical interpretation of black hole entropy. One could derive the Bekenstein-Hawking formula by counting black hole microstates and using the Boltzmann entropy formula

$$S = ln W, \quad (9)$$

where $W$ is the number of microstates of the system.

This counting is not a simple task because is quite difficult to identified the black hole microstates. However great progress in this direction has been reported [8].
Therefore, the basic idea of holography is that the entropy scales like area, or in other words, the total number of states is of order

\[ W_A \sim e^A. \]  

(10)

This is a very dramatic change of view because, from the quantum field theory experience, generally one expects that if the energy density is bounded then the maximum entropy is proportional to the volume of space and this means

\[ W_V \sim e^V. \]  

(11)

Therefore, for our purposes, the most important conclusion that we can extract from the holographic principle is that the radical reduction of the degrees of freedom necessary for the description of the universe can be expressed in the following simple way,

\[ W_V \rightarrow W_A. \]  

(12)

4 Holography as the best Code

The tools and concepts developed in the above sections will help to show the main idea of this work. Thus, the language of Information Theory will be used to deal with Holography.

Firstly we should point out the relation between the Boltzmann and Shannon formulas. Despite the fact that they had different origins they are conceptually equivalent\[^9\] and the formal relation is very simple,\[^2\]

\[ S = \frac{1}{\log_2 e} H. \]  

(13)

Here we ignore the units of the thermodynamical entropy and we consider just the bit units.

\[^2\]The relation can be obtained when in (2) the outcomes are equally likely, that is, when \( p_i = \frac{1}{W} \).
This relation allows to establish the proper bridge between the two main topics of this work.

Now we arrive to our point. First we consider a gravitational system with a corresponding ‘physical alphabet’ \( F \). This ‘alphabet’ can be thought as the source of the dynamic of the system. The next step is to consider the codes.

Let us suppose that there exits an ‘holographic code’ \( A \) for this gravitational system and the corresponding average codeword length is

\[
AveLen(A) = \frac{A}{4} \log_2 e. \tag{14}
\]

Similarly, if instead of that we take a ‘volumetric code’ \( V \), we assume that

\[
AveLen(V) = V \log_2 e. \tag{15}
\]

On the other hand, if the Holographic Principle is satisfied, the Boltzmann entropy is \( S = A/4 \). Therefore, using (13) we get

\[
H = \frac{A}{4} \log_2 e. \tag{16}
\]

Obtaining,

\[
H = AveLen(A). \tag{17}
\]

Consequently, with the support of the Noiseless Coding Theorem we can conclude that the ‘holographic code’ is the best encoding scheme that can be constructed.

In this context a volumetric quantum field theory can be seen as a highly inefficient encoding scheme where

\[
\text{Efficiency} = \frac{A}{V} \sim 1/R \ll 1. \tag{18}
\]

5 Conclusions

Fermat pointed out that nature is economical, and that light will therefore travel along the path that takes the least time. This is quite a good rule but
there are some instances where it is erroneous and the modern statement is extended: the ray path along which light travels is such that the time taken holds a stationary value (i.e. minimum, maximum or constant). Extending this argument to physics of information we could speculate that Holography is a manifestation of the economical character of nature. The minimum amount of information is used to describe the Universe.

However, that is not the end of the story. We can go on further in the same direction studying also the possible connections of Holography with the Shannon’s Second Theorem. Also the arguments presented here could be refined using the Quantum Source Coding Theorem, introduced by Schumacher[10]

Taking into account that the main idea of this work is encoded on the title I will finish with the hope that I introduced the less possible redundancy.

ACKNOWLEDGMENT: The author is grateful to Elcio Abdalla for reading the manuscript and useful suggestions. This work was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP).

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