A superpolynomial lower bound for the size of non-deterministic complement of an unambiguous automaton.

Michael Raskin
raskin@mccme.ru
LaBRI, University of Bordeaux
December 20, 2017

Abstract

Unambiguous non-deterministic finite automata have intermediate expressive power and succinctness between deterministic and non-deterministic automata.

It has been conjectured [2] that every unambiguous non-deterministic one-way finite automaton (1UFA) recognizing some language $L$ can be converted into a 1UFA recognizing the complement of the original language $L$ with polynomial increase in the number of states.

We disprove this conjecture by presenting a family of 1UFAs on a single-letter alphabet such that recognizing the complements of the corresponding languages requires superpolynomial increase in the number of states even for generic non-deterministic one-way finite automata.

We also note that both the languages and their complements can be recognized by sweeping deterministic automata with a linear increase in the number of states.

1 Introduction

In many areas of computer science, the relationship between deterministic and non-deterministic devices is a subject of significant interest. An intermediate notion between deterministic and non-deterministic computation devices is the notion of unambiguous device. Such a device can make non-deterministic choices, but it is guaranteed that for every input there is at most one accepting execution trace.

For finite automata it is known that not only non-deterministic automata can be exponentially more succinct than deterministic automata, but also that unambiguous automata can be exponentially separated [4]. The paper establishing exponential separation also defines several automata classes of limited ambiguity and provides exponential separation between some of them.

*This work was supported by the French National Research Agency (ANR project GraphEn / ANR-15-CE40-0009).
Other notions of unambiguity have been considered. Some of them (for example, structural unambiguity — for a given input word and a given target state there is at most one way from the initial state) describe a wider class of automata than unambiguity. Some are more restrictive than simple unambiguity (for example, strong unambiguity — there is a set of result states, for every input there is exactly one way to reach a result state, and the result states can be accepting or rejecting). We do not consider these notions in the present paper.

We study the problem of representing a complement of a language specified by a finite automaton. It is easy to see that replacing the set of accepting states with its complement allows to recognize the complement of a language specified by a deterministic finite automaton without increasing the number of states. Complementing a language specified by a non-deterministic finite automaton may require an exponential number of states [1].

It has been conjectured [2] that every unambiguous non-deterministic one-way finite automaton (1UFA) recognizing some language $L$ can be converted into a 1UFA recognizing the complement of the original language $L$ with polynomial increase in the number of states. The best known lower bound was quadratic [5], while the upper bounds were exponential [3]. The quadratic lower bound holds even for the single-letter alphabet.

In the present paper we show a superpolynomial lower bound for the state complexity of recognizing the complement of a language of a unambiguous finite automaton by a non-deterministic finite automaton. The construction uses only the single-letter alphabet. The family of languages used in the construction can be recognised both by succinct unambiguous finite automata and by succinct sweeping deterministic automata. Complementing the language of a sweeping deterministic finite automaton can be done without increasing space complexity, thus our construction also provides a proof of superpolynomial state complexity of converting a sweeping deterministic finite automaton to one-way non-deterministic finite automaton.

2 Basic definitions and the main result

Definition 1. A 1NFA (1-way non-deterministic finite automaton) is defined by an alphabet $\Sigma$, a set of states $Q$, an initial state $q_0 \in Q$, a subset of accepting states $Q_A \subseteq Q$ and the list of permissible transitions $T \subseteq Q \times \Sigma \times Q$.

The size of a 1NFA $A$ is the number $|A|$ of states in the definition of the automaton.

A run of a 1NFA on a word $w \in \Sigma^*$ is a list of states $q_0, q_1, \ldots, q_{|w|}$ such that all the transitions are permissible, i.e. $\forall i \in 1 \ldots |w| : (q_{i-1}, w_i, q_i) \in T$.

An accepting run of a 1NFA is a run such that the last state is accepting, i.e. $q_{|w|} \in Q_A$.

A 1DFA (one-way deterministic finite automaton) is a 1NFA such that for every state and every letter there is at most one permissible transition, i.e. $\forall q, q_1, q_2 \in Q, s \in \Sigma : (q, s, q_1) \in T \land (q, s, q_2) \in T \Rightarrow q_1 = q_2$.

A 1UFA (one-way unambiguous non-deterministic finite automaton) is a 1NFA such that for every word there is at most one accepting run.

A 2NFA (2-way non-deterministic finite automaton) is defined by an alphabet $\Sigma$, a set of states $Q$, an initial state $q_0 \in Q$, a subset of accepting states $Q_A \subseteq$
Q and the list of permissible transitions \( T \subseteq Q \times (\Sigma \cup \{\top, \bot\}) \times \{+1, -1, 0\} \).

We call \( \top \) and \( \bot \) endpoint markers.

A run of a 2NFA on an input word \( w \in \Sigma^* \) is a list of pairs of positions and states, \((p_0 = 1, q_0), (p_1, q_1), \ldots, (p_n, q_n)\) such that all transitions are allowed and the run ends by the only non-moving transition. The exact conditions are as follows:

1) \( p_0 = 1 \);
2) \( \forall i = 0..n : 0 \leq p_i \leq |w| + 1 \);
3) \( \forall i = 1..n : (q_{i-1}, w_{p_{i-1}}, q_i, p_i - p_{i-1}) \in T \), where we assume \( w_0 = \top \) and \( w_{|w|+1} = \bot \);
4) \( \forall i = 1..n - 1 : p_i \neq p_{i-1} \);
5) \( p_n = p_{n-1} \).

An accepting run of a 2NFA is a run such that the last state is accepting, i.e. \( q_n \in Q_A \).

A 2DFA is a 2NFA such that for every state and every letter there is at most one permissible transition, i.e. \( \forall q \in Q, s \in \Sigma : \exists q' \in Q, d \in \{-1, +1, 0\} : (q, s, q', d) \in T \).

A swNFA (sweeping 2-way deterministic finite automaton) is a 2NFA such that every state has transitions with only one direction of movement (except in case of the endpoint markers and termination transitions). Namely, 
\[ \neg \exists q_1, q_2 \in Q, s_1, s_2 \in \Sigma : (q, s_1, q_1, +1), (q, s_2, q_2, -1) \in T \]

A swDFA is a swNFA that is also a 2DFA.

Definition 2. A language \( L \) over alphabet \( \Sigma \) is an arbitrary subset \( L \subseteq \Sigma^* \).

The language recognized by a 1NFA \( A \) is the set \( L(A) \) of all words where there exists an accepting run of \( A \).

Theorem 1. There exists a sequence of 1UFA-s \( A_d \) over the single-letter alphabet such that the minimal 1NFA recognizing \( L(A_d) \) has size at least \( |A_d|^d \).

In other words, complementing a 1UFA requires more than polynomial increase in size regardless of the size of the alphabet, and the bound holds even if the complement can be representing by 1NFA.

3 Proof sketch

We will provide a brief proof outline with a randomized construction.

We consider the single-letter alphabet. We consider only 1UFA-s with the following structure: there is a set of moduli \( m_1, \ldots, m_n \) and some subset of good remainders for each modulus: \( R_i \subseteq \{0, \ldots, m_i - 1\} \). The states are \( (0) \) and \( (i \in 1..n, j \in 0..m_i - 1) \). The accepting states are \( (i, j \in R_i) \). (Actually, every 1UFA on a single-letter alphabet can be approximated this way with only a finite number of mistakes).

We will base our construction of the automaton on an orientation of the complete \( n \)-vertex graph. The oriented graph will be denoted by \( G \). We will want to ensure that for some \( k \) for every choice of \( k \) vertices in the graph there is a vertex outside the chosen subset such that all \( k \) edges between the last vertex and the chosen subset are oriented from the subset towards the last vertex. If \( n \)
is not large enough relative to \( k \), this is impossible; for large enough \( n \) a random orientation will satisfy the condition with probability close to 1. The examples large enough for \( k = 2 \) are already too large to draw in a small illustration, so the illustration will use \( n = 6 \). We will select subsets of edges with some properties, with the understanding that for larger \( n \) all possible choices have these properties.

We will select a large number \( N \gg n \) and some \( N \) different primes of roughly the same magnitude, \( P = \{ P_1, \ldots, P_N \} \). We will also pick a probability of inclusion \( p \in (0; 1) \). We will pick each modulus to independently include each prime with probability \( p \), so each modulus will include a different subset of approximately \( pN \) primes. For the single-letter alphabet the word is fully described by its length, and as the moduli are just products of some subsets of \( P \), we will only consider the list of remainders of the lengths in question modulo \( P_1, \ldots, P_N \).

Each node will correspond to one modulus. We ascribe share \( p \) of primes to each node. The primes ascribed to an edge are the primes ascribed to both ends of the edge simultaneously (a share \( p^2 \) of all primes).

We will now define the set of accepting states. The goal is to make sure that the list of remainders \((0, \ldots, 0)\) is not recognized, but there are many accepted lists with many zeros in each. The implementation idea is to treat each remainder as either «no information» or description of the only vertex that is allowed to accept. Additionally, checking all the common prime factors for two vertices and finding only zero remainders will mean that the source vertex of the edge is not allowed to accept. For the modulus \( m_i \) we will recognize the lists of remainders with the following properties: all the remainders modulo primes dividing \( m_i \) are either 0 or \( i \), there is at least one remainder equal to \( i \), and for every \( j \) such that the edge between \( i \) and \( j \) in \( G \) goes from \( i \) to \( j \) there is a prime dividing both \( m_i \) and \( m_j \) such that the remainder is \( i \).

Basically, the remainder 0 means «no information» and \( i \) means «vertex \( i \) can recognize». The vertex \( i \) recognizes the situations where it sees only 0 and \( i \) among the remainders modulo its primes, and for every outgoing edge it sees at least one \( i \) on the edge (an incoming edge can be all-zeros).

Unambiguity of the automaton is easy to verify: if two different moduli \( m_i \) and \( m_j \) can be used to recognize the same word, without loss of generality we can assume that the orientation of the edge in \( G \) is from \( i \) to \( j \). Then out of \( \approx p^2N \) primes shared by \( m_i \) and \( m_j \) there should be one giving the remainder \( i \); but this violates the condition for acceptance modulo \( m_j \).

We can say that two vertices can look at the remainders modulo their common primes, and ensure unambiguity: the vertex will not accept if it sees that some other vertex should, and in case all remainders say «no information», edge orientation is used as a tie-breaker.

The size of the 1UFA we have constructed is roughly \( n \times |P_1|^{pN} \), because we have \( n \) cycles with \( pN \) prime factors each.

Let us estimate the size of a 1NFA recognising the complement of this 1UFA. It has to recognize all the lengths divisible by the product of all primes
in $P$, and almost all of such runs have to traverse some cycle because of finiteness. Going around the cycle more times than in this selected run will still produce accepting runs with the same remainders modulo the primes dividing $C$ and arbitrary remainders modulo the other ones (by the Chinese Remainder theorem).

Pick any modulus $m_i$; if for every $m_j \neq m_i$ there is a common prime factor dividing $m_i$ and $m_j$ but not $C$, we can build an accepting run of the 1NFA such that its length has remainder zero modulo every prime dividing $C$ and remainder $i$ modulo all the other primes in $P$. This run would correspond to a word recognised by the initial 1UFA; this contradiction proves that for every modulus $m_i$ there is another modulus $m_j$ such that the edge in $G$ goes from $i$ to $j$ and every common prime factor of $m_i$ and $m_j$ divides $C$.

We can illustrate the choice of primes by drawing solid arrows when all the primes ascribed to the edge divide the length of the selected cycle. To avoid accepting all remainders being zero, we must have at least one solid outgoing arrow for every vertex.

If for every $k$ vertices in $G$ there is another one with only incoming edges from the $k$ selected ones, we can pick $\frac{k}{2}$ independent edges such that the prime factors shared by the moduli corresponding to the endpoints of each edge divide $C$. As long as we have less than $\frac{k}{2}$ edges we can consider all the endpoints, pick the vertex that doesn’t have outgoing edges to the already selected endpoints, and add the corresponding outgoing edge for this vertex.

We can pick just an independent subset of arrows. Note that our proof doesn’t work for this orientation, and the graph is too small for a proper one; we just illustrate independence of the edges.

The first edge in such a set forces $C$ to be divisible by $p^2 N$ primes from $P$; each next edge adds the share $p^2$ of the remaining primes. Therefore, the total number of primes included will be approximately $1 - (1 - p^2)^{\frac{k}{2}}$; for $k > \frac{1}{p}$ it will be more than the half of the entire amount of primes. So $C$ will be roughly $P^{\frac{p}{1-p^2}}$, only a bit less than the size of the initial 1UFA to the power $\frac{1}{2p}$.

4 Graph orientations without small inbound-covering sets

In this section we will prove existence of the orientation of the edges of a complete graph that is needed for our construction.
Definition 3. Consider a finite complete graph $G$. Consider an orientation on its edges. The orientation can also be represented as an irreflexive anticommutative relation $R \subset V(G) \times V(G)$ on its vertices. The relation $R$ holds for a pair of vertices $u$ and $v$ if the edge $(u, v)$ is oriented towards $v$. A set of vertices $S \subset V(G)$ is called an inbound-covering set if every vertex $v$ outside $S$ has at least one edge to $S$ oriented towards $S$. We can write this as $\forall v \in V(G) \setminus S \exists s \in S : R(v, s)$.

Lemma 1. For every positive integer $k$ there exist a large enough complete graph and an orientation of the graph such that the smallest inbound-covering set has size larger than $k$.

Proof. Consider a uniformly random orientation of a complete graph with $n$ vertices. Consider an arbitrary set $S \subset V(G)$. For a given vertex $v \in V(G) \setminus S$ the probability (over the choice of a random orientation) of at least one edge between $v$ and $S$ going towards $S$ is $1 - 2^{-k}$. For a given set $S$ and different vertices $v_1, v_2, \ldots \in V(G) \setminus S$ existence of an outgoing edge from $v_i$ towards $S$ is independent for different vertices, because the sets of the edges under consideration are disjoint. Therefore the probability for a given set $S$ to be inbound-covering is equal to $(1 - 2^{-k})^{n-k}$.

An upper bound can be obtained by approximating the logarithm: $\log(1 - 2^{-k}) < (-2^{-k})$ and $(1 - 2^{-k})^{(n-k)} < e^{-2^{-k} \times (n-k)}$. There are at most $n^k$ sets of vertices with size up to $k$, and the union (upper) bound for the probability of existence of an inbound-covering set of size up to $k$ is $n^k \exp(-n^{-k}) = \exp(-n^{-k} + k \log n)$.

To prove existence of an orientation without inbound-covering sets of size up to $k$ it is enough to show that this probability is less than 1.

$$\exp(-\frac{n-k}{2k} + k \log n) < 1 \iff -\frac{n-k}{2k} + k \log n < 0 \iff \frac{n-k}{\log n} > k \times 2^k \iff n > k \times 2^k \times \log n + k \quad (1)$$

Without loss of generality we can assume that $k$ is at least 8. In this case we know that $k > 2 \log k + \log 3$. Now let us pick a large enough $n$, for example $n = 3k^2 \times 2^k$.

$$k2^k \log n + k < k2^k \times (k \log 2 + 2 \log k + \log 3) + k < k2^k \times (2k) + k = 2k^2 \times 2^k + k < 3k^2 \times 2^k = n \quad (2)$$

This inequality proves that a random orientation doesn’t create any inbound-covering sets of sizes up to $k$, and therefore there exists an orientation where the smallest inbound-covering set has more than $k$ vertices.

\[\Box\]

5 Building a set of remainders from a graph orientation

In this section we will describe how an orientation of the edges of a complete graph allows to define a set of words.
As we work over the single-letter alphabet, defining a set of words is the same as defining a set of lengths. We will define a set of lengths by selecting a large set of distinct primes, selecting some moduli defined as products of subsets of the selected primes, and finally selecting some accepted remainders for each selected modulus. Each modulus will correspond to one vertex of the graph. We will use the Chinese remainder theorem to represent the remainders modulo each selected modulus as a sequence of remainders over the primes in the factorisation of the modulus.

Let us fix an orientation of the edges of a complete graph with \(n\) vertices. Let the relation \(R\) represent this orientation. Let us fix the parameter \(b \in \mathbb{N}\). This parameter is the inverse of \(p\) in the proof overview; the total number of primes is \(b\) times larger than the number of primes used in any single selected modulus. Let \(N\) be \(b^n\), and let \(P = \{P_0, \ldots, P_{N-1}\}\) be a set of distinct primes within a factor of \(1 + \frac{1}{b}\) of each other. (By the Prime Number Theorem for large enough \(N\) all these primes can be chosen to be less than \(2N^2 \log N\)).

The modulus \(m_i, i \in \{1, \ldots, n\}\) is the product of the primes \(P_j\) such that \(j\) has 0 as the \(i\)-th base-\(b\) digit. Each \(m_i\) is the product of \(bN\) primes; the greatest common divisor of \(m_i\) and \(m_j\) is the product of \(b^2N\) primes, etc. Acceptable remainders \(r \mod m_i\) are given by the following conditions:
1) \(r\) can only have remainders 0 and \(i\) modulo the primes in \(m_i\) (\(\forall q \in P: q|m_i \Rightarrow r \mod q \in \{0, i\}\));
2) \(r\) has at least one non-zero remainder modulo some prime in \(m_i\), i.e. \(m_i \nmid r\);
3) for every outgoing edge from the vertex \(i\) to, say, vertex \(j\) there is a prime in \(\gcd(m_i, m_j)\) such that the remainder is \(i\) (\(\forall j: R(i, j) \Rightarrow \exists q \in P: q|m_i, q|m_j, r \mod q = i\)).

**Lemma 2.** For every integer \(t\) there is at most one selected modulus \(m_i\) such that \(t \mod m_i\) is an acceptable remainder modulo \(m_i\).

The remainder \(t = 0\) is not acceptable modulo any selected modulus.

**Proof.** Note that 0 is not acceptable because it is divisible by every modulus.

Let \(m_i\) and \(m_j\) be two distinct selected moduli such that \(t \mod m_i\) is acceptable modulo \(m_i\) and \(t \mod m_j\) is acceptable modulo \(m_j\). Without loss of generality we can assume that the edge between \(i\) and \(j\) goes from \(i\) to \(j\), i.e. \(R(i, j)\). In this case there is a prime \(q\) dividing both \(m_i\) and \(m_j\) such that \(t \mod q = i\). But \(q|m_i\) and \((t \mod m_j) \mod q \notin 0, j\).

**Lemma 3.** Assume that the graph orientation has no inbound-covering sets of size up to \(k\). If for some integer \(m\) there are no such \(l\) and \(i\) such that \(ml \mod m_i\) is an acceptable remainder modulo \(m_i\), then \(m\) has to be divisible by at least \(N(1 - (1 - \frac{12}{b})^k)\) of the primes in \(P\).

**Proof.** By the Chinese Remainder Theorem, for every \(i\) we can pick \(l\) such that for every prime \(q \in P\) no dividing \(m\), the remainder \(ml \mod q\) is equal to \(i\). Let us denote such \(l\) by \(L(m, i)\).

If for any \(m\) there is no \(m_i\) such that \(\gcd(m_i, m_j)\) \(|\) \(m \times L(m, i)\) would be acceptable modulo \(m_i\), in contradiction with the assumption of the theorem. Therefore for every \(m_i\) there is at least one such \(m_j\) that \(\gcd(m_i, m_j)\) \(|\) \(m\). Let us call edges \(i, j\) such that \(\gcd(m_i, m_j)\) \(|\) \(m\) controlled edges.
If some vertex $i$ has only incoming controlled edges, $m \times L(m, i)$ would still be acceptable modulo $m_i$, because the vertex $i$ has to have outgoing edges to avoid being an inbound-covering set.

Let us picked $\lceil \frac{k}{2} \rceil$ controlled edges that do not share vertices. As long as we haven’t picked the desired number of edges, there are fewer than $k$ endpoints of the selected edges. The set of these endpoints cannot be an inbound-covering set, so there is a vertex other than these endpoints that has only incoming edges from the selected endpoints. This vertex has to have an outgoing controlled edge, which we can add to the set of the picked controlled edges.

Now that we have $\lceil \frac{k}{2} \rceil$ independent controlled edges, we can count the number of prime factors of $m$ needed for controlling these edges. The prime $P_j$ not being included in any of the $\lceil \frac{k}{2} \rceil$ edges means that there are $\lceil \frac{k}{2} \rceil$ pairs of positions in the base-$b$ representation of $j$, and in every pair of positions there should be at least one non-zero digit. There are $N(1 - (1 - \frac{1}{b^2})^{\frac{k}{2}})$ such primes, and $m$ is divisible by all of them.

\section{The main result and its proof}

\begin{theorem}
There exists a sequence of 1UFA-s $A_d$ over the single-letter alphabet such that the minimal 1NFA recognising $L(A_d)$ has size at least $|A_d|^6$. In other words, complementing a 1UFA requires more than polynomial increase in size regardless of the size of the alphabet, and the bound holds even if the complement can be representing by 1NFA.

The languages $L(A_d)$ and $\overline{L(A_d)}$ can also be recognised by a swDFA of size $|A_d| + o(|A_d|)$.
\end{theorem}

\textbf{Proof.} Let us fix $d$. Let $b = 2d, k = 2b^2, n = 3k^2 \times 2b, N = b^n$.

By lemma 1, there exists an orientation of complete graph with $n$ vertices without inbound-covering sets of size up to $k$.

Consider the set of moduli and corresponding acceptable remainders as built in section 5.

A 1UFA can recognize all the lengths having acceptable remainder modulo some modulus by guessing the modulus then tracking the remainder. This construction requires no more than $O(n) \times P_N^{N-1}$ states. This automaton will be unambiguous by lemma 2. This is the automaton $A_d$.

A swDFA can go through the word $n$ times calculating the remainder modulo the next modulus each time. This construction also requires $O(n) \times P_N^{N-1}$, and can be used to recognize the complement of the language.

A 1NFA recognizing the complement of the language has to have a cycle, because the complement is infinite. There is a cycle that can be traversed in a run recognizing some length divisible by the product of all the primes in $P$. The product of all the primes in $P$ has remainder zero modulo every modulus $m_i$ in the construction. As we can consider the runs of the 1NFA that traverse the cycle multiple times without changing anything else, no multiple of the cycle length can be acceptable modulo any of the moduli $m_i$.

By lemma 3 this implies that the cycle length is divisible by at least $N(1 - (1 - \frac{1}{b^2})^{\frac{k}{2}})$ primes from $P$. As $(1 - \frac{1}{b^2})^{\frac{k}{2}} = (1 - \frac{1}{b})^b < \exp(-\frac{1}{b})^b = \frac{1}{b}, N(1 -
\[(1 - \frac{1}{b^2}) > N \times (1 - \frac{1}{b^2}) > 0.6N.\] The total size of the 1NFA cannot be less than the size of this cycle, which has to be at least \(P_0^{0.6N}N.\)

We now only need to verify that \((O(n) \times P_{\frac{N}{N-1}}^N)^d < P_0^{0.6N} \). But indeed, for large enough \(d\) we have \(P_0 > N \gg n \gg d\) and

\[(O(n) \times P_{\frac{N}{N-1}}^N)^d < ((O(n) \times (1 + \frac{1}{N})P_0)\frac{N}{2})^d < O(n)^{\frac{N}{2}} \exp(d) < P_0^{0.6N} \]

In case of \(d\) not large enough, we can replace the automaton with the automaton for the smallest large enough \(d\).

Note 1. The size of \(A_d\) is \(2^{2^{2^{\Theta(1)}}}\).

It is sometimes impossible to recognize the complement of the language of a 1UFA of size \(z\) by a 1NFA of size less than \(z^{(\log \log \log z)^{\Theta(1)}}\).

Proof. Let us write down the dependencies between parameters. We know that \(b\) is linear in \(d\), \(k\) is quadratic in \(d\), \(n = 2^{\Theta(d^2)}\), \(N = b^{n} = 2^{\Theta(d^2)}\), \(N = b^{n} = 2^{\Theta(d^2)}\). The primes in \(P\) are all \(\Theta(N^2 \log N)\). Then the size of the automaton \(A_d\) is \(\Theta(n \times P_0^{\frac{N}{2}}) = P_0^{\Theta(\frac{N}{2})} = (N \log N)^{\Theta(\frac{N}{2})} = 2^{\Theta(d^2)} \times 2^{\Theta(d^2)} = 2^{2^{\Theta(d^2)}} = 2^{2^{2^{\Theta(1)}}}\)

The second claim is just a restatement of the same fact.

7 Conclusion and further directions

We have constructed a counterexample to the conjecture that the complement of a language recognized by a 1UFA can be recognized by a 1UFA with polynomial increase in the number of states. Moreover, in our example the language and its complement are easy to recognize by a \(swDFA\) with approximately the same number of states, but the complement requires superpolynomial number of states in the recognizing 1NFA even without the requirement of unambiguity. The example only uses the single-letter alphabet.

The construction provides a relatively weak kind of superpolynomial growth. It would be interesting to improve the lower bound. It seems likely that the number of primes used in the construction could be reduced, making the growth faster.

The question about exponential separation in the case of a general alphabet remains open. We hope that disproving the conjecture will inspire new results in this area.

8 Acknowledgements

The author is grateful to Gabriele Puppis for numerous useful discussions.

References

[1] Jean-Camille Birget. Partial orders on words, minimal elements of regular languages and state complexity. Theor. Comput. Sci., 119(2):267–291, 1993.
[2] Thomas Colcombet. Unambiguity in automata theory. In Jeffrey Shallit and
Alexander Okhotin, editors, *Descriptional Complexity of Formal Systems -
17th International Workshop, DCFS 2015, Waterloo, ON, Canada, June 25-
27, 2015. Proceedings*, volume 9118 of *Lecture Notes in Computer Science*,
pages 3–18. Springer, 2015.

[3] Jozef Jirásek Jr., Galina Jirásková, and Juraj Sebej. Operations on un-
ambiguous finite automata. In Srecko Brlek and Christophe Reutenauer,
editors, *Developments in Language Theory - 20th International Conference,
DLT 2016, Montréal, Canada, July 25-28, 2016, Proceedings*, volume 9840
of *Lecture Notes in Computer Science*, pages 243–255. Springer, 2016.

[4] Hing Leung. Descriptive complexity of nfa of different ambiguity. *Int. J.
Found. Comput. Sci.*, 16(5):975–984, 2005.

[5] Alexander Okhotin. Unambiguous finite automata over a unary alphabet.
*Inf. Comput.*, 212:15–36, 2012.