Mapping class groups are linear

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Abstract

It is shown, that the mapping class group of a surface of genus \( g \geq 2 \) admits a faithful representation into the matrix group \( GL_{6g-6}(\mathbb{Z}) \). The proof is based on a categorical correspondence between the Riemann surfaces and the so-called toric AF-algebras.

Key words and phrases: Riemann surfaces, toric AF-algebras

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1 Introduction

A. The Harvey conjecture. The mapping class group has been introduced in the 1920-ies by M. Dehn [6]. Such a group, \( \text{Mod} (X) \), is defined as the group of isotopy classes of the orientation-preserving diffeomorphisms of a two-sided closed surface \( X \) of genus \( g \geq 1 \). The group is known to be prominent in algebraic geometry [9], topology [13] and dynamics [14]. When \( X \) is a torus, the \( \text{Mod} (X) \) is isomorphic to the group \( SL_2(\mathbb{Z}) \). (The \( SL_2(\mathbb{Z}) \) is called a modular group, hence our notation for the mapping class group.) A little is known about the representations of \( \text{Mod} (X) \) beyond the case \( g = 1 \). Recall, that the group is called linear, if there exists a faithful representation into the matrix group \( GL_m(R) \), where \( R \) is a commutative ring. The braid groups are known to be linear [3]. Using a modification of the argument for the braid groups, it is possible to prove, that \( \text{Mod} (X) \) is linear in the case \( g = 2 \) [4]. Whether the mapping class group is linear for \( g \geq 3 \), is an open problem, known as the Harvey conjecture [10], p.267.

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B. The Teichmüller functor. A covariant (non-injective) functor from a category of generic Riemann surfaces to a category of the so-called toric $AF$-algebras (see section 2 for a definition) was constructed in [12]. The functor, a Teichmüller functor, maps any pair of isomorphic (i.e. conformal) Riemann surfaces to a pair of the stably isomorphic (Morita equivalent) toric $AF$-algebras. Since each isomorphism of Riemann surfaces is given by an element of $Mod (X)$ [9], it is natural to ask about a representation of $Mod (X)$ by the stable isomorphisms of toric $AF$-algebras; roughly, our objective can be stated as follows.

Main problem. To study the Harvey conjecture from the standpoint of toric $AF$-algebras.

The stable isomorphisms of toric $AF$-algebras are well understood and surprisingly simple; provided the automorphism group of the algebra is trivial (this is true for a generic algebra), its group of stable isomorphism admits a faithful representation into a matrix group over the commutative ring $\mathbb{Z}$ [7]. This fact, combined with the properties of the Teichmüller functor, implies an amazingly simple positive solution to the Harvey conjecture.

Theorem 1 For every surface $X$ of genus $g \geq 2$, there exists a faithful representation $\rho : Mod (X) \to GL_{6g-6}(\mathbb{Z})$.

The structure of the note is as follows. In section 2, the preliminary facts, necessary to prove theorem 1 are brought together. Theorem 1 is proved in section 3.

2 Preliminaries

We review the toric $AF$-algebras and the Teichmüller functor on the space of generic Riemann surfaces; the reader is referred to [2], [5], [7] and [12] for details.

2.1 $AF$-algebras

A. The $C^*$-algebras. By the $C^*$-algebra one understands a noncommutative Banach algebra with an involution. Namely, a $C^*$-algebra $A$ is an algebra over the complex numbers $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$, $a \in A$, such that $A$ is complete with the respect to the norm, and
such that $||ab|| \leq ||a|| \cdot ||b||$ and $||a^*a|| = ||a||^2$ for every $a, b \in A$. If $A$ is commutative, then the Gelfand theorem says that $A$ is isometrically $*$-isomorphic to the $C^*$-algebra $C_0(X)$ of the continuous complex-valued functions on a locally compact Hausdorff space $X$. For otherwise, the algebra $A$ represents a noncommutative topological space.

**B. The stable isomorphisms of $C^*$-algebras.** Let $A$ be a $C^*$-algebra deemed as a noncommutative topological space. One can ask, when two such topological spaces $A, A'$ are homeomorphic? To answer the question, let us recall the topological $K$-theory. If $X$ is a (commutative) topological space, denote by $V_C(X)$ an abelian monoid consisting of the isomorphism classes of the complex vector bundles over $X$ endowed with the Whitney sum. The abelian monoid $V_C(X)$ can be made to an abelian group, $K(X)$, using the Grothendieck completion. The covariant functor $F : X \rightarrow K(X)$ is known to map the homeomorphic topological spaces $X, X'$ to the isomorphic abelian groups $K(X)$, $K(X')$. Let $A, A'$ be the $C^*$-algebras. If one wishes to define a homeomorphism between the noncommutative topological spaces $A$ and $A'$, it will suffice to define an isomorphism between the abelian monoids $V_C(A)$ and $V_C(A')$ as suggested by the topological $K$-theory. The role of the complex vector bundle of the degree $n$ over the $C^*$-algebra $A$ is played by a $C^*$-algebra $M_n(A) = A \otimes M_n$, i.e. the matrix algebra with the entries in $A$. The abelian monoid $V_C(A) = \cup_{n=1}^{\infty} M_n(A)$ replaces the monoid $V_C(X)$ of the topological $K$-theory. Therefore, the noncommutative topological spaces $A, A'$ are homeomorphic, if the abelian monoids $V_C(A) \cong V_C(A')$ are isomorphic. The latter equivalence is called a stable isomorphism of the $C^*$-algebras $A$ and $A'$ and is formally written as $A \otimes \mathfrak{g} \cong A' \otimes \mathfrak{g}$, where $\mathfrak{g} = \cup_{n=1}^{\infty} M_n$ is the $C^*$-algebra of compact operators. Roughly speaking, the stable isomorphism between the $C^*$-algebras means that they are homeomorphic as the noncommutative topological spaces.

**C. The $AF$-algebras.** An $AF$-algebra (approximately finite $C^*$-algebra) is defined to be the norm closure of an ascending sequence of the finite dimensional $C^*$-algebras $M_n$'s, where $M_n$ is the $C^*$-algebra of the $n \times n$ matrices with the entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents a semisimple matrix algebra $M_n = M_{n_1} \oplus \ldots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots$, where $M_i$ are the finite dimensional $C^*$-algebras and $\varphi_i$ the homomorphisms between such algebras. The set-theoretic limit $A = \lim M_n$ has a natural algebraic structure given by the formula $a_m + b_k \rightarrow a + b$; here $a_m \rightarrow a, b_k \rightarrow b$ for the sequences...
\(a_m \in M_m, b_k \in M_k\). The homomorphisms \(\varphi_i\) can be arranged into a graph as follows. Let \(M_i = M_{i_1} \oplus \ldots \oplus M_{i_k}\) and \(M_{i'} = M'_{i_1} \oplus \ldots \oplus M'_{i_k}\) be the semi-simple \(C^*\)-algebras and \(\varphi_i : M_i \to M_{i'}\) the homomorphism. One has the two sets of vertices \(V_i, \ldots, V_k\) and \(V'_{i}, \ldots, V'_{k}\) joined by the \(a_{rs}\) edges, whenever the summand \(M_{i'}\) contains \(a_{rs}\) copies of the summand \(M_{i'}\) under the embedding \(\varphi_i\). As \(i\) varies, one obtains an infinite graph called a Bratteli diagram of the \(AF\)-algebra \([5]\). The Bratteli diagram defines a unique \(AF\)-algebra.

**D. The stationary \(AF\)-algebras.** If the homomorphisms \(\varphi_1 = \varphi_2 = \ldots = \text{Const}\) in the definition of the \(AF\)-algebra \(A\), the \(AF\)-algebra \(A\) is called stationary. The Bratteli diagram of a stationary \(AF\)-algebra looks like a periodic graph with the incidence matrix \(A = (a_{rs})\) repeated over and over again. Since matrix \(A\) is a non-negative integer matrix, one can take a power of \(A\) to obtain a strictly positive integer matrix – which we always assume to be the case. The stationary \(AF\)-algebra has a non-trivial group of the automorphisms \([7]\), Ch.6.

### 2.2 Teichmüller functor

**A. The Jacobi-Perron continued fractions.** Let \(\lambda = (\lambda_1, \ldots, \lambda_n)\) be a vector with the non-negative real entries, such that \(\lambda_1 \neq 0\); consider a projective class \((1, \theta_1, \ldots, \theta_{n-1})\) of \(\lambda\), where \(\theta_{i-1} = \frac{\lambda_i}{\lambda_1}\) for \(1 \leq i \leq n\). The continued fraction

\[
\begin{pmatrix}
1 \\
\theta_1 \\
\vdots \\
\theta_{n-1}
\end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_{1}^{(1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(1)}
\end{pmatrix} \cdots \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_{1}^{(k)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(k)}
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
\vdots \\
1
\end{pmatrix},
\]

where \(b_{j}^{(k)} \in \mathbb{N} \cup \{0\}\), is called the Jacobi-Perron fraction. To recover the integers \(b_{1}^{(k)}\) from the vector \((\theta_1, \ldots, \theta_{n-1})\), one has to repeatedly solve the following system of equations: \(\theta_1 = b_{1}^{(1)} + \frac{1}{\theta_{n-1}'}, \theta_2 = b_{2}^{(1)} + \frac{b_{1}^{(1)}}{\theta_{n-1}'}, \theta_{n-1} = b_{n-1}^{(1)} + \frac{\theta_{n-2}'}{\theta_{n-1}'}\), where \((\theta_{1}', \ldots, \theta_{n-1}')\) is the next input vector. Thus, each vector \((\theta_1, \ldots, \theta_{n-1})\) gives rise to a formal Jacobi-Perron continued fraction, which can be convergent or divergent. We let \(A^{(0)} = \delta_{ij}\) (the Kronecker delta) and \(A^{(k+n)} = \sum_{j=0}^{n-1} b_{i}^{(k)} A^{(\nu+j)}\), \(b_0^{(k)} = 1\), where \(i = 0, \ldots, n-1\) and \(k = 0, 1, \ldots, \infty\). The Jacobi-Perron continued fraction of vector \((\theta_1, \ldots, \theta_{n-1})\) is
said to be convergent, if \( \theta_i = \lim_{k \to \infty} \frac{A_i^{(k)}}{A_0^{(k)}} \) for all \( i = 1, \ldots, n - 1 \). Unless \( n = 2 \), the convergence of the individual Jacobi-Perron fraction is a difficult problem; however, it is known that the Jacobi-Perron fractions converge for a generic subset of the vectors in the space \( \mathbb{R}^{n-1} \).

**B. The toric AF-algebras.** Denote by \( T_S(g) \) the Teichmueller space of genus \( g \geq 1 \) with a distinguished point \( S \). Let \( q \in H^0(S, \Omega^{g,2}) \) be a holomorphic quadratic differential on the Riemann surface \( S \) such that all zeroes of \( q \) (if any) are simple. By \( \tilde{S} \) we mean a double cover of \( S \) ramified over the zeroes of \( q \) and by \( H^{odd}_i(\tilde{S}) \) the odd part of the integral homology of \( \tilde{S} \) relative to the zeroes. Note that \( H^{odd}_1(\tilde{S}) \cong \mathbb{Z}^n \), where \( n = 6g - 6 \) if \( g \geq 2 \) and \( n = 2 \) if \( g = 1 \). It follows from the Main Theorem of \([11]\), that \( T_S(g) - \{pt\} \cong Hom(H^{odd}_1(\tilde{S}); \mathbb{R}) - \{0\} \), where \( 0 \) is the zero homomorphism.

Finally, denote by \( \lambda = (\lambda_1, \ldots, \lambda_n) \) the image of a basis of \( H^{odd}_1(\tilde{S}) \) in the real line \( \mathbb{R} \), such that \( \lambda_1 \neq 0 \). (Note that such an option always exists, since the zero homomorphism is excluded.) We let \( \theta = (\theta_1, \ldots, \theta_{n-1}) \), where \( \theta_i = \lambda_{i-1}/\lambda_1 \). Recall that, up to a scalar multiple, the vector \((1, \theta) \in \mathbb{R}^n\) is the limit of a generically convergent Jacobi-Perron continued fraction:

\[
\begin{pmatrix}
1 \\
\theta
\end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix}
0 & 1 \\
I & b_1
\end{pmatrix} \cdots \begin{pmatrix}
0 & 1 \\
I & b_k
\end{pmatrix} \begin{pmatrix}
0 \\
1
\end{pmatrix},
\]

where \( b_i = (b_1^{(i)}, \ldots, b_{n-1}^{(i)})^T \) is a vector of the non-negative integers, \( I \) the unit matrix and \( \mathbb{I} = (0, \ldots, 0, 1)^T \). We introduce an AF-algebra, \( \mathcal{A}_\theta \), via the Bratteli diagram, shown in Fig.1. (The numbers \( b_j^{(i)} \) of the diagram indicate the multiplicity of edges of the graph.) Let us call \( \mathcal{A}_\theta \) a toric AF-algebra.

Note that in the \( g = 1 \) case, the Jacobi-Perron fraction coincides with the

\footnote{To be precise, the theorem mentions a local homeomorphism \( h : Hom(H^{odd}_1(\tilde{S}); \mathbb{R}) - \{0\} \to T_S(g) - \{pt\} \) \([11]\), p.222. Since \( T_S(g) \) is simply connected, \( h \) extends to a global homeomorphism between the two spaces. Indeed, let \( \lambda \in Hom(H^{odd}_1(\tilde{S}); \mathbb{R}) \). It is easy to see, that \( \lambda = 0 \) corresponds to the distinguished point \( S \in T_S(g) \), while \( \lambda = \infty \) represent the boundary of the space \( T_S(g) \); thus, every ball \( |\lambda| < C \) is homotopy equivalent to the ball \( |\lambda| < \infty \). Note also, that we are interested in \( q \)'s with the generic (simple) zeroes; the higher order zeroes – which can be an obstacle in the construction of global coordinates – are excluded. The interested reader can consult \([14]\), p.425 (the last paragraph) for the details; the mentioned there piecewise linear integral structure breaks \( T_S(g) \) into a finite number of cones issued from \( S \) and in these terms our construction means that we take a cone and extend it (by the linearity) to the entire \( T_S(g) \). The cones differ from each other by a permutation on the set \( \lambda = (\lambda_1, \ldots, \lambda_n) \); distinct permutations correspond to the different coordinates in the space \( T_S(g) \).}
regular continued fraction and \( \mathbb{A}_\theta \) becomes the Effros-Shen \( AF \)-algebra of a noncommutative torus \( [8] \).

Figure 1: The Bratteli diagram of a toric \( AF \)-algebra of genus 2.

C. \textbf{The Teichmueller functor.} Denote by \( V \) the maximal subset of \( T_S(g) \), such that for each Riemann surface \( R \in V \), there exists a convergent Jacobi-Perron continued fraction. Let \( F \) be the map which sends the Riemann surfaces into the toric \( AF \)-algebras according to the formula \( R \mapsto \mathbb{A}_\theta \). Finally, let \( W \) be the image of \( V \) under the mapping \( F \).

\textbf{Lemma 1 (12)} \textit{The set} \( V \) \textit{is a generic subset of} \( T_S(g) \) \textit{and the map} \( F \) \textit{has the following properties:}

(i) \( V \cong W \times (0, \infty) \) \textit{is a trivial fiber bundle, whose projection map} \( p : V \to W \) \textit{coincides with} \( F \);

(ii) \( F : V \to W \) \textit{is a covariant functor, which maps isomorphic Riemann surfaces} \( R, R' \in V \) \textit{to stably isomorphic toric} \( AF \)-\textit{algebras} \( \mathbb{A}_\theta, \mathbb{A}_{\theta'} \in W \).

\textbf{3 Proof of theorem 1}

As before, let \( W \) denote the set of toric \( AF \)-algebras of genus \( g \geq 2 \). Let \( G \) be a finitely presented group and \( G \times W \to W \) its action on \( W \) by the stable
isomorphisms of toric $AF$-algebras; in other words, \( \gamma(\mathbb{A}_\theta) \otimes \mathbb{R} \cong \mathbb{A}_\theta \otimes \mathbb{R} \) for all \( \gamma \in G \) and all \( \mathbb{A}_\theta \in W \). The following preparatory lemma will be important.

**Lemma 2** For each \( \mathbb{A}_\theta \in W \), there exists a representation \( \rho_{\mathbb{A}_\theta} : G \to GL_{6g-6}(\mathbb{Z}) \).

*Proof.* The proof of the lemma is based on the following well known criterion of the stable isomorphism for the (toric) $AF$-algebras: a pair of such algebras \( \mathbb{A}_\theta, \mathbb{A}_\theta' \) are stably isomorphic if and only if their Bratteli diagrams coincide, except (possibly) a finite part of the diagram, see [7], Theorem 2.3. (Note, that the order isomorphism between the dimension groups, mentioned in the original text, translates to the language of the Bratteli diagrams as stated.)

Let \( G \) be a finitely presented group on the generators \( \{\gamma_1, \ldots, \gamma_m\} \) subject to relations \( r_1, \ldots, r_n \). Let \( \mathbb{A}_\theta \in W \). Since \( G \) acts on the toric $AF$-algebra \( \mathbb{A}_\theta \) by stable isomorphisms, the toric $AF$-algebras \( \mathbb{A}_{\theta_1} := \gamma_1(\mathbb{A}_\theta), \ldots, \mathbb{A}_{\theta_m} := \gamma_m(\mathbb{A}_\theta) \) are stably isomorphic to \( \mathbb{A}_\theta \); moreover, by transitivity, they are also pairwise stably isomorphic. Therefore, the Bratteli diagrams of \( \mathbb{A}_{\theta_1}, \ldots, \mathbb{A}_{\theta_m} \) coincide everywhere except, possibly, some finite parts. We shall denote by \( \mathbb{A}_{\theta_{\max}} \in W \) a toric $AF$-algebra, whose Bratteli diagram is the maximal common part of the Bratteli diagrams of \( \mathbb{A}_\theta \) for \( 1 \leq i \leq m \); such a choice is unique and defined correctly because the set \( \{\mathbb{A}_\theta\} \) is a finite set. By the definition of a toric $AF$-algebra, the vectors \( \theta_i = (\theta_1^{(\max)}, \ldots, \theta_{6g-7}^{(\max)}) \) are related to the vector \( \theta_{\max} = (1, \theta_1^{(\max)}, \ldots, \theta_{6g-7}^{(\max)}) \) by the formula:

\[
\begin{pmatrix}
1 \\
\theta_1^{(i)} \\
\vdots \\
\theta_{6g-7}^{(i)}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & \ldots & 0 & 1 \\
0 & 1 & \ldots & 0 & b_1^{(i)(i)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 1 & b_{6g-7}^{(i)(i)}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \ldots & 0 & 1 \\
0 & 1 & \ldots & 0 & b_1^{(k)(i)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 1 & b_{6g-7}^{(k)(i)}
\end{pmatrix}
\begin{pmatrix}
1 \\
\theta_1^{(\max)} \\
\vdots \\
\theta_{6g-7}^{(\max)}
\end{pmatrix}
\]

The above expression can be written in the matrix form \( \theta_i = A_i \theta_{\max} \), where \( A_i \in GL_{6g-6}(\mathbb{Z}) \). Thus, one gets a matrix representation of the generator \( \gamma_i \), given by the formula \( \rho_{\mathbb{A}_\theta}(\gamma_i) := A_i \).

The map \( \rho_{\mathbb{A}_\theta} : G \to GL_{6g-6}(\mathbb{Z}) \) extends to the rest of the group \( G \) via its values on the generators; namely, for every \( g \in G \) one sets \( \rho_{\mathbb{A}_\theta}(g) = A_1^{k_1} \cdots A_m^{k_m} \), whenever \( g = \gamma_1^{k_1} \cdots \gamma_m^{k_m} \). Let us verify, that the map \( \rho_{\mathbb{A}_\theta} \) is a well defined homomorphism of groups \( G \) and \( GL_{6g-6}(\mathbb{Z}) \). Indeed, let us
write \( g_1 = \gamma_1^{k_1} \ldots \gamma_m^{k_m} \) and \( g_2 = \gamma_1^{s_1} \ldots \gamma_m^{s_m} \) for a pair of elements \( g_1, g_2 \in G \); then their product \( g_1g_2 = \gamma_1^{k_1} \ldots \gamma_m^{k_m} \gamma_1^{s_1} \ldots \gamma_m^{s_m} = \gamma_1^{l_1} \ldots \gamma_m^{l_m} \), where the last equality is obtained by a reduction of words using the relations \( r_1, \ldots, r_n \).

One can write relations \( r_i \) in their matrix form \( \rho_A \theta(r_i) \); thus, one gets the matrix equality \( A_1^{l_1} \ldots A_m^{l_m} = A_1^{k_1} \ldots A_m^{k_m} A_1^{s_1} \ldots A_m^{s_m} = \rho_{\mathcal{A}_\theta}(g_1)\rho_{\mathcal{A}_\theta}(g_2) \) for \( \forall g_1, g_2 \in G \), i.e. \( \rho_{\mathcal{A}_\theta} \) is a homomorphism. Lemma 2 follows.

Lemma 3 Let \( \mathcal{A}_\theta \in W_{\text{aper}} \) and \( G \) be free on the \( \mathcal{A}_\theta \). Then \( \rho_{\mathcal{A}_\theta} \) is a faithful representation.

\begin{proof}
Since the action of \( G \) is free, to prove that \( \rho_{\mathcal{A}_\theta} \) is faithful, it remains to show, that in the formula \( \theta_i = A_i \theta_{\max} \), it holds \( A_i = I \), if and only if, \( \theta_i = \theta_{\max} \), where \( I \) is the unit matrix. Indeed, it is immediate that \( A_i = I \) implies \( \theta_i = \theta_{\max} \). Suppose now that \( \theta_i = \theta_{\max} \) and, let to the contrary, \( A_i \neq I \). One gets \( \theta_i = A_i \theta_{\max} = \theta_{\max} \). Such an equation has a non-trivial solution, if and only if, the vector \( \theta_{\max} \) has a periodic Jacobi-Perron fraction; the period of such a fraction is given by the matrix \( A_i \). This is impossible, since it has been assumed, that \( \mathcal{A}_{\theta_{\max}} \in W_{\text{aper}} \). The contradiction finishes the proof of lemma 3. \( \square \)

Let \( W_{\text{aper}} \subset W \) be a set consisting of the toric \( AF \)-algebras, whose Bratteli diagrams do not contain periodic (infinitely repeated) blocks; these are known as non-stationary toric \( AF \)-algebras and they are generic in the set \( W \) endowed with the natural topology. We call the action of \( G \) on the toric \( AF \)-algebra \( \mathcal{A}_\theta \in W_{\text{aper}} \) free, if \( \gamma(\mathcal{A}_\theta) = \mathcal{A}_\theta \) implies \( \gamma = \text{Id} \).

Lemma 4 The pre-image \( F^{-1}(W_{\text{aper}}) \) is a generic set in the space \( T(g) \).

\begin{proof}
Note, that the set of stationary toric \( AF \)-algebras is a countable set. The functor \( F \) is a surjective map, which is continuous with respect to the natural topology on the sets \( V \) and \( W \). Therefore, the pre-image of the complement of a countable set is a generic set. \( \square \)
To finish the proof, consider the set $U \cap F^{-1}(W_{aper})$; this is a non-empty set, since it is the intersection of two generic subsets of $T(g)$. Let $R$ be a point (a Riemann surface) in the above set. In view of lemma 1, group $G$ acts on the toric $AF$-algebra $\mathbb{A}_R = F(R)$ by the stable isomorphisms. By the construction, the action is free and $\mathbb{A}_R \in W_{aper}$. In view of lemma 3 one gets a faithful representation $\rho = \rho_{\mathbb{A}_R}$ of the group $G = \text{Mod}(X)$ into the matrix group $GL_{6g-6}(\mathbb{Z})$. Theorem 1 follows. □

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