EXISTENCE THEOREM
FOR HYPERBOLIC QUASIPERIODIC SOLUTIONS
OF LAGRANGIAN SYSTEMS ON RIEMANNIAN MANIFOLDS

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Abstract. We establish new sufficient conditions for the existence of classical hyperbolic quasiperiodic solutions for natural Lagrangian system on Riemannian manifold with time-quasiperiodic force function.

1. Introduction. On Riemannian manifold \((\mathcal{M}, \langle \cdot, \cdot \rangle)\) with Riemannian metric \(\langle \cdot, \cdot \rangle\) and Levi-Civita connection \(\nabla\), consider a natural Lagrangian system with Lagrangian density

\[
L(t\omega, x, \dot{x}) = K(\dot{x}) + W(t\omega, x).
\]

where \(K(\dot{x}) := \langle \dot{x}, \dot{x} \rangle / 2\) is the kinetic energy, \(W(\cdot, \cdot): T^k \times \mathcal{M} \mapsto \mathbb{R}\) is a smooth force function, \(T^k = \mathbb{R}^k / (2\pi \mathbb{Z}^k)\) is \(k\)-dimensional torus with natural angle coordinates \(\varphi = (\varphi_1, \ldots, \varphi_k) \mod 2\pi\), and \(\omega \in \mathbb{R}^k\) is a basis frequency vector with rationally independent components.

In [1], there was developed a variational method for detection of weak quasiperiodic solutions to system with Lagrangian density (1). Earlier this method was applied for analogous purposes in the case of Lagrangian systems in Euclidean space [2–9], as well as for Lagrangian systems on Riemannian manifold with nonpositive sectional curvature [10, 11]. In contrary to the last two papers, the authors of [1] managed to avoid the requirement concerning nonpositiveness of sectional curvature, but instead they impose additional conditions in terms of an auxiliary function. To formulate these conditions we shall use the following notations: \(\nabla_\xi, \|\xi\|, \nabla V(x)\) and \(H V(x)\) are, respectively, the covariant derivative along a tangent vector \(\xi\), the norm of this vector, the gradient of a smooth function \(V(\cdot): \mathcal{M} \mapsto \mathbb{R}\) at a point \(x\), and the Hessian at this point.

The above conditions are as follows. There exists a bounded smooth function \(V(\cdot): \mathcal{M} \mapsto \mathbb{R}\) satisfying the following conditions:

\(\textbf{C1:}\) the set \(\mathcal{D} := \{x \in \mathcal{M} : 2\lambda V(x) + \|\nabla V(x)\|^2 > 0\}\), where

\[
\lambda V(x) := \min \{\langle H V(x)\xi, \xi \rangle : \|\xi\| = 1, \xi \in T_x\mathcal{M}\}
\]

stands for minimal eigenvalue of Hessian \(H V(x)\), is nonempty and, for a noncritical value \(v \in V(\mathcal{D})\), there exists a bounded connected component \(\Omega\) of sublevel set \(V^{-1}(-\infty, v)\) such that \(\bar{\Omega} := \Omega \cup \partial \Omega \subseteq \mathcal{D}\);

\(\textbf{C2:}\) the restriction of quadratic form \(\langle H V(x)\xi, \xi \rangle\) on \(T_x\partial \Omega\) is positive definite for all \(x \in \partial \Omega\), and

\[
\min_{x \in \Omega} \{\mu V(x) - 2K^*(x)\} > 0,
\]

where

\[
\mu V(x) := \min \left\{\langle H V(x)\xi, \xi \rangle - \frac{1}{2} \langle \nabla V(x), \xi \rangle^2 : \|\xi\| = 1, \xi \in T_x\mathcal{M}\right\},
\]

and \(K^*(x)\) is maximal sectional (Riemannian) curvature over two-dimensional tangent planes of tangent space \(T_x\mathcal{M}\) (see [12 Sect. 3.6] for the definition);

\(\textbf{C3:}\) any two points \(x, y \in \Omega\) can be connected in \(\mathcal{D}\) by a geodesic segment of conformally equivalent Riemannian metric \(\langle \cdot, \cdot \rangle_V := e^V \langle \cdot, \cdot \rangle\).
Unfortunately, Condition C3 is of “noncoefficient” and nonlocal character and as a result its verification may be quite difficult. In the present paper we shall show that actually the verification of this condition is needless, since it is fulfilled automatically. This fact allows us to assert that Conditions C1, C2 together with certain additional properties of function $W(\cdot, \cdot)$ guarantee the existence of function $u(\cdot) : \mathbb{T}^k \mapsto \mathcal{M}$ which is associated with a week Besicovitch quasiperiodic solution $x(t) := u(t\omega)$ to system with Lagrangian density $L(t\omega, x, \dot{x})$ (Theorem 1). This solution has the following extremal property: it is a limit of sequence minimizing the functional

$$L[x(\cdot)] := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} L(t\omega, x(t), \dot{x}(t)) dt$$

on a metric space formed as a closure of the set of smooth uniformly quasiperiodic functions $x(\cdot) : \mathbb{R} \mapsto \Omega$ by the pseudometric

$$d_1(x_1(\cdot), x_2(\cdot)) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[ \|\dot{x}_1(t) - \dot{x}_2(t)\|^2 + \rho^2(x_1(t), x_2(t)) \right] dt.$$  

where $\rho(\cdot, \cdot) : \mathcal{M} \times \mathcal{M} \mapsto \mathbb{R}$ is the distance function on $(\mathcal{M}, \langle \cdot, \cdot \rangle)$. Basing on the approach proposed in [5] and developed in [13], we shall prove that $x(\cdot)$ is classical uniformly quasiperiodic solution of the system with Lagrangian density $L(t\omega, x, \dot{x})$ (Theorem 2). Besides, we establish that the system in variations along $x(t)$ is exponentially dichotomic on whole real axis (Theorem 3).

2. Existence theorem for weak quasiperiodic solution to Lagrangian system on Riemannian manifold.

By the Nash theorem [14], the manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ admits a smooth isometric embedding $\iota : \mathcal{M} \mapsto \mathbb{R}^n$ into Euclidean space $\mathbb{R}^n = (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ of a dimension $n > m$, so that $\langle \xi, \eta \rangle = (\iota_\ast \xi, \iota_\ast \eta)$ for any $\xi, \eta \in T_x\mathcal{M}$. Here $(\cdot, \cdot)$ is standard dot product in $\mathbb{R}^n$, $\iota_\ast$ stands for derivative of the embedding map $\iota$.

Denote by $D_\omega f(\varphi) := \frac{d}{dt}|_{t=0} f(\varphi + t\omega)$ the derivative in direction of vector $\omega$ for a function $f(\cdot) : \mathbb{T}^k \mapsto \mathbb{R}^n$ and by $\nabla W(\varphi, x)$ the gradient of the function $W(\varphi, \cdot) : \mathcal{M} \mapsto \mathbb{R}$ at point $x$ for fixed $\varphi \in \mathbb{T}^k$.

**Proposition 1.** A function $t \mapsto x(t) := u(t\omega)$ of class $C^2(\mathbb{R}; \mathcal{M})$ where $u(\cdot) \in C(\mathbb{T}^k \mapsto \mathcal{M})$ and $D_\omega^2 u(\cdot) \in C(\mathbb{T}^k \mapsto \mathcal{M})$ is a classical quasiperiodic solution of Lagrangian system on $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ with Lagrangian density (1) iff for any mapping $h(\cdot) \in C(\mathbb{T}^k; T\mathcal{M})$ such that $D_\omega \iota_\ast h(\cdot) \in C(\mathbb{T}^k; \mathbb{R}^n)$ and $h(\varphi) \in T_{u(\varphi)}\mathcal{M}$ at every point $\varphi \in \mathbb{T}^k$ the function $u(\cdot)$ satisfies the equality

$$\int_{\mathbb{T}^k} [(D_\omega \iota \circ u(\varphi), D_\omega \iota_\ast h(\varphi)) + (\iota_\ast \nabla W(\varphi, u(\varphi)), \iota_\ast h(\varphi))] d\varphi = 0$$

where $d\varphi := d\varphi_1 \wedge \ldots \wedge d\varphi_k$ is differential volume form on torus $\mathbb{T}^k$.

**Proof.** By definition of classical solution,

$$\nabla_{\dot{x}(t)} \dot{x}(t) = \nabla W(t\omega, x(t)) \quad \forall t \in \mathbb{R}.$$  

If for any point $x \in \mathcal{M}$, we define the orthogonal projection operator

$$P_x : T_{\iota(x)} \mathbb{R}^n \mapsto \iota_\ast T_x \mathcal{M},$$
the well-known property of the Levi–Civita connection \cite{12} Sect. 3.5] for any smooth field of tangent vectors $\xi(t)$ along $x(t)$ we have

$$\iota_* \nabla_{\dot{x}(t)} \xi(t) = P_{x(t)} \frac{d}{dt} \iota_* \xi(t). \quad (7)$$

Now taking into account rational independence of frequency vector components the equality (6) implies the chain of equivalent interrelations

$$P_{x(t)} \frac{d}{dt} \iota_* \dot{x}(t) = \iota_* \nabla W(t\omega, x(t)) \quad \forall t \in \mathbb{R} \quad \Leftrightarrow$$

$$P_{u(\varphi)} D_{\omega}^2 \iota_* \circ u(\varphi) = \iota_* \nabla W(\varphi, u(\varphi)) \quad \forall \varphi \in \mathbb{T}^k \quad \Leftrightarrow$$

$$\int_{\mathbb{T}^k} \left( D_{\omega}^2 \iota_* \circ u(\varphi), P_{u(\varphi)} v(\varphi) \right) d\varphi = \int_{\mathbb{T}^k} \left( \iota_* \nabla W(\varphi, u(\varphi)), P_{u(\varphi)} v(\varphi) \right) d\varphi$$

$$\forall v(\cdot) \in C\left(\mathbb{T}^k; \mathbb{E}^n\right): \quad D_{\omega} v(\cdot) \in C\left(\mathbb{T}^k; \mathbb{E}^n\right) \quad \Leftrightarrow$$

$$\int_{\mathbb{T}^k} \left( D_{\omega}^2 \iota_* \circ u(\varphi), \iota_* h(\varphi) \right) d\varphi = \int_{\mathbb{T}^k} \left( \iota_* \nabla W(\varphi, u(\varphi)), \iota_* h(\varphi) \right) d\varphi \quad (8)$$

$$\forall h(\cdot) \in C\left(\mathbb{T}^k; \mathcal{M}\right): \quad D_{\omega} \iota_* h(\cdot) \in C\left(\mathbb{T}^k; \mathbb{E}^n\right), \quad h(\varphi) \in T_{u(\varphi)} \mathcal{M} \quad \forall \varphi \in \mathbb{T}^k.$$ Integrating by parts of the left-hand side of equality (8) we get the required result. \hfill \Box

Now, basing on Proposition \ref{prop1} introduce the notion of weak quasiperiodic solution to the system with Lagrangian density \ref{lagrange} on Riemannian manifold $\mathcal{M}$. Before formulating the corresponding definition let us recall definitions of a number of functional spaces and corresponding objects.

Denote by $\mathcal{H}(\mathbb{T}^k; \mathbb{E}^n)$ the space of $\mathbb{E}^n$-valued functions on torus $\mathbb{T}^k$ Lebesgue integrable by the square of standard Euclidean norm $||\cdot|| = \sqrt{\langle \cdot, \cdot \rangle}$ in $\mathbb{E}^n$ (note that in view of the isometricity of embedding $\iota$ the norms of vectors $\xi \in T_x \mathcal{M}$ and $\iota_* \xi \in T_{i(x)} \mathbb{E}^n \sim \mathbb{E}^n$ in corresponding spaces are equal, thus we use the same symbols for these norms). For elements of $\mathcal{H}(\mathbb{T}^k; \mathbb{E}^n)$, one can define a dot product $\langle \cdot, \cdot \rangle_0 := (2\pi)^{-k} \int_{\mathbb{T}^k} \langle \cdot, \cdot \rangle d\varphi$ as well as seminorm $||\cdot||_0 = \sqrt{\langle \cdot, \cdot \rangle_0}$. By $\mathcal{H}_1(\mathbb{T}^k; \mathbb{E}^n)$ denote the space of functions $f(\cdot) \in \mathcal{H}(\mathbb{T}^k; \mathbb{E}^n)$ each of which has weak (generalized) Sobolev derivative $D_\omega f(\cdot) \in \mathcal{H}(\mathbb{T}^k; \mathbb{E}^n)$ in direction of $\omega$, i.e. $\langle f(\cdot), D_\omega g(\cdot) \rangle_0 = -\langle D_\omega f(\cdot), g(\cdot) \rangle_0$ for any continuously differentiable function $g(\cdot): \mathbb{T}^k \mapsto \mathbb{E}^n$. In the space $\mathcal{H}_1(\mathbb{T}^k; \mathbb{E}^n)$ one can naturally define a dot product $\langle \cdot, \cdot \rangle_1 := \langle \cdot, \cdot \rangle_0 + \langle D_\omega \cdot, D_\omega \cdot \rangle_0$ and a seminorm $||\cdot||_1$ generated by it. After we identify elements equal a.e., both spaces $\left(\mathcal{H}(\mathbb{T}^k; \mathbb{E}^n), \langle \cdot, \cdot \rangle_0\right)$, $\left(\mathcal{H}_1(\mathbb{T}^k; \mathbb{E}^n), \langle \cdot, \cdot \rangle_1\right)$ become Hilbertian ones (see, e.g., \cite{13,15}).

Next, for arbitrary bounded set $\mathcal{A} \subset \mathcal{M}$, denote by $\mathcal{S}_\mathcal{A}$ the space of smooth functions $u(\cdot): \mathbb{T}^k \mapsto \mathcal{A}$.

**Definition 1.** A function $u(\cdot): \mathbb{T}^k \mapsto \mathcal{M}$ is said to be of class $\mathcal{H}_1^1$, if $\iota \circ u(\cdot)$ is a strong limit in $\mathcal{H}_1^1(\mathbb{T}^k; \mathbb{E}^n)$ of a sequence $\{ \iota \circ u_j(\cdot) \}$, where $u_j(\cdot) \in \mathcal{S}_\mathcal{A}$, $j = 1, 2, \ldots$.

Note that for compact set $\bar{\mathcal{A}} \subset \mathcal{M}$ there exist positive constants $c$ and $C$ such that

$$c ||\iota(x_1) - \iota(x_2)|| \leq \rho(x_1, x_2) \leq C ||\iota(x_1) - \iota(x_2)|| \quad \forall x_1, x_2 \in \bar{\mathcal{A}}.$$ Without loss of generality one can regard functions of class $\mathcal{H}_1^1$ as taking values in $\bar{\mathcal{A}}$.

**Definition 2.** A mapping $h(\cdot): \mathbb{T}^k \mapsto T \mathcal{M}$ is called a vector field along a mapping $u(\cdot) \in \mathcal{H}_1^1$ determined by a sequence $\{ u_j(\cdot) \in \mathcal{S}_\mathcal{A} \}$ if $\iota_* h(\cdot)$ is a strong limit in $\mathcal{H}_1^1(\mathbb{T}; \mathbb{E}^n)$ of a sequence $\{ \iota_* h_j(\cdot) \}$, where $\{ h_j(\cdot): \mathbb{T}^k \mapsto T \mathcal{M} \}$ is a sequence of smooth mappings such that

$$h_j(\varphi) \in T_{u_j(\varphi)} \mathcal{M} \quad \forall j \in \mathbb{N}, \quad \sup_{j \in \mathbb{N}, \varphi \in \mathbb{T}^k} ||h_j(\varphi)|| < \infty.$$
By the well-known Riesz–Fischer theorem, a formal sum \( \sum_{n \in \mathbb{Z}^k} f_n e^{i(n, \varphi)} \) is the Fourier series of a function \( f(\cdot) \in H(T^k; \mathbb{E}^n) \) iff the series \( \sum_{n \in \mathbb{Z}^k} \| f_n \|^2 \) is convergent. On the other hand, by the Riesz-Fischer-Besicovitch theorem [16, p. 110], to the formal sum \( \sum_{n \in \mathbb{Z}^k} f_n e^{i(n, \omega)} \), one can assign (nonuniquely) a function \( t \mapsto f(t) \) of class \( L^2_{\text{loc}}(\mathbb{R}; \mathbb{E}^n) \) for which the above sum is its Fourier series and which represents an element of the space of Besicovitch quasiperiodic functions \( B^2(\mathbb{R}; \mathbb{E}^n) \). In view of the form of both Fourier series we generally use the notation \( f(t) = f(t \omega) \) for the quasiperiodic function \( f(\cdot) \), associated with a function \( f(\cdot) \) on torus. However, one should keep in mind the following factors. Firstly, when proving the Riesz-Fischer-Besicovitch theorem a Besicovitch function with prescribed Fourier series is constructed irrespectively to restriction of concrete associated function to line \( \{ \varphi = t \omega \}_{t \in \mathbb{R}} \), and, besides, two functions from \( L^2_{\text{loc}}(\mathbb{R}; \mathbb{E}^n) \) representing an element of the space \( B^2(\mathbb{R}; \mathbb{E}^n) \) (i.e. possessing common Fourier series) may differ by everywhere nonvanishing function. Secondly, if we take a function \( f(\cdot) \in H(T^k; \mathbb{E}^n) \), then we can only assert that the function \( t \mapsto f(\varphi + t \omega) \) belongs to \( B^2(\mathbb{R}; \mathbb{E}^n) \) for almost every point \( \varphi \in T^k \), but a priori it’s unknown whether the same is true for \( \varphi = 0 \). To give informal sense to the equality \( f(t) = f(t \omega) \), conversely, having a function \( f(\cdot) \in B^2(\mathbb{R}; \mathbb{E}^n) \), we think of associated function \( f(\cdot) \in H(T^k; \mathbb{E}^n) \) as being redefined on the line \( \{ \varphi = t \omega \}_{t \in \mathbb{R}} \) (which has zero Lebesgue measure) in such a way that the above equality is actually fulfilled [13, Sect. 1.5].

Now let \( f(\cdot) = \iota \circ u(\cdot) \), where \( u(\cdot) \in H^1_A \). To this function, one can assign an associated \( \mathbb{E}^n \)-valued Besicovitch quasiperiodic function. But in view of the above mentioned circumstances the following question needs to be clarified: does there exist a Besicovitch quasiperiodic function associated with \( \iota \circ u(\cdot) \) and taking values in \( \iota(A) \)? I.e., does there exist a Besicovitch quasiperiodic function associated with \( u(\cdot) \in H^1_A \) and taking values in \( A \)?

Basing on completeness of Marcinkiewicz spaces (see, [17–19]) one can show that the answer to the question is affirmative.

Now in view of Proposition 1 we can introduce the following definition.

**Definition 3.** A quasiperiodic Besicovitch function \( t \mapsto u(t \omega) \) associated with function \( u(\cdot) \in H^1_A \), where \( A \subseteq M \) is a bounded set, is called a weak solution of Lagrangian system on \( M \) with Lagrangian density (1) if

\[
(D_{\omega t} \circ u(\cdot), D_{\omega t} u(\cdot))_0 + (\iota \circ \nabla W(\cdot, u(\cdot)), \iota \circ h(\cdot))_0 = 0
\]

for any vector field \( h(\cdot) \) along \( u(\cdot) \).

The function \( u(\cdot) \) from this definition can be interpreted as an extremal of functional

\[
J[u(\cdot)] := \int_{T^k} \left[ \frac{1}{2} \| D_{\omega t} \circ u(\varphi) \|^2 + W(\varphi, u(\varphi)) \right] d\varphi
\]

on set \( H^1_A \), and the corresponding weak solution \( t \mapsto u(t \omega) \) as an extremal of functional \( L \) on set of Besicovitch quasiperiodic functions associated with functions of class \( H^1_A \).

Now let us prove an existence theorem for weak quasiperiodic solution.

**Theorem 1.** Let there exist a bounded smooth function \( V(\cdot) : M \rightarrow \mathbb{R} \) satisfying conditions C1, C2, as well as the inequalities

\[
\lambda_W(\varphi, x) + \frac{1}{2} \langle \nabla W(\varphi, x), \nabla V(x) \rangle > 0 \quad \forall (\varphi, x) \in T^k \times \Omega,
\]

\[
\langle \nabla W(\varphi, x), \nabla V(x) \rangle > 0 \quad \forall (\varphi, x) \in T^k \times \partial \Omega,
\]

where \( \lambda_W(\varphi, x) := \min \{ \langle H_W(\varphi, x) \xi, \xi \rangle : \xi \in T_xM, \| \xi \| = 1 \} \), \( H_W(\varphi, x) \) is the Hessian of function \( W(\varphi, \cdot) : M \rightarrow \mathbb{R} \) at point \( x \) for \( \varphi \in T^k \). Then the system with Lagrangian density (1) has a weak quasiperiodic solution associated with function of class \( H^1_A \).
Proof. At first, let us show that any two points of the set $\Omega := V^{-1}((−\infty, v))$ can be connected in $\Omega$ by a geodesic segment of conformally equivalent metric $\langle \cdot, \cdot \rangle_V := e^V \langle \cdot, \cdot \rangle$. Denote by $\nabla^V$ the Levi–Civita connection of Riemannian manifold $(M, \langle \cdot, \cdot \rangle_V)$, and by $\exp^V(\cdot)$ the exponential mapping at $x$ associated with $\nabla^V$. Let $x \in \Omega$ and thus $v_0 := V(x) < v$. Consider the open set

$$Z_x = \{ \xi \in T_x M : \exp^V_x(s\xi) \in \Omega \forall s \in [0, 1] \}.$$ 

In [1] p. 10, relying on well-known formula of sectional curvature for conformally equivalent metric [12, Sect. 3.6], it is observed that the inequality (2) yields nonpositiveness of sectional curvature for connection $\nabla^V$ at any point of $\Omega$ and along any two-dimensional direction. Then by the Morse–Schoenberg theorem [12, Sect. 6.2], for any $\xi \in Z_x$, the geodesic segment $\exp^V_x(s\xi)$, $s \in [0, 1]$, does not contain conjugate points. For this reason the mapping $\exp^V_x(\cdot)$ restricted to the set $Z_x$ is a local diffeomorphism and this implies that $\Xi := \exp^V_x(Z_x)$ is an open subset of the set $\Omega$.

Let us show that at the same time $\Xi$ is a closed subset of $\Omega$. Assume $\{\xi_k \in Z_x\}$ to be such a sequence that the subsequence $\{x_k = \exp^V_x(\xi_k) \in \Xi\}$ converges to a point $x_* \in \Omega$. We must show the existence of a vector $\xi_* \in Z_x$ such that $\exp^V_x(\xi_*) = x_*$. The set $Z_x$ is compact, so we may consider that $\xi_k$ converges to a point $\xi_* \in Z_x$. It remains to show that $\xi_* \notin \partial Z_x$. Suppose, conversely, that $\xi_* \in \partial Z_x$. Since for sufficiently small $\delta > 0$ and for all $t \in [0, \delta]$ we have $\exp^V_x(t\xi_*) \notin \Omega$, then $t\xi_* \in Z_x$ for all $t \in [0, \delta]$. From definition of $Z_x$ it follows that there exists $s_* \in (\delta, 1)$ such that $\exp^V_x(s_*\xi*) \in \partial\Omega$, but $\exp^V_x(s\xi_*) \in \Omega$ when $s \in [0, s_*]$. Here we take into account that the equality $s_* = 1$ is impossible since $x_* \in \Omega$ (if the equality $s_* = 1$ were valid, the vector $\xi_*$ would belong to $Z_x$ in contrary to our assumption).

Consider now the sequence $s_*\xi_k$. It converges to $s_*\xi_*$. Hence, $V \circ \exp^V_x(s_*\xi_k) \to v$ and $V \circ \exp^V_x(s_*\xi_k) < v$. Put $\hat{V}(\xi) := \exp \circ V \circ \exp^V_x(\xi)$. Then for all sufficiently large $k$ we have

$$\frac{1}{2} (e^v + e^{v_0}) < \hat{V}(s_k \xi_k) < e^v.$$ 

(10)

It is shown in [1] p. 9 that the inclusion $\hat{\Omega} \subset D$ yields the existence of $\sigma > 0$ such that

$$\frac{d^2}{ds^2} \hat{V}(s_k \xi_k) > 0 \quad \forall s \in [0, 1], \forall k \in \mathbb{N}.$$ 

Then each derivative $\frac{d}{ds} \hat{V}(s_k \xi_k)$ is monotone increasing on segment $[0, 1]$ and in addition

$$\frac{d}{ds} \hat{V}(s_k \xi_k) \geq \frac{1}{2s_*} (e^v - e^{v_0}) \quad \forall s \in [s_*, 1]$$ 

(11)

for all $k$ starting from sufficiently large number $k_*$. In fact, in opposite case, there would exists sufficiently large $k$, for which the inequalities (10) hold true together with

$$\frac{d}{ds} \hat{V}(s_k \xi_k) \leq \frac{d}{ds} \frac{1}{2s_*} \hat{V}(s_k \xi_k) < \frac{1}{2s_*} (e^v - e^{v_0}) \quad \forall s \in [0, s_*].$$

But then

$$\hat{V}(s_* \xi_k) < e^{v_0} + \frac{1}{2s_*} (e^v - e^{v_0}) s_* = \frac{1}{2} (e^v + e^{v_0})$$

in contrary to inequality (10). Now from inequality (11) we get

$$\exp \circ V(x_k) = \hat{V}(\xi_k) \geq \hat{V}(s_k \xi_k) + \frac{1}{2s_*} (e^v - e^{v_0}) (1 - s_*) \quad \forall k \geq k_*.$$ 

Letting $k$ to infinity we arrive at

$$\exp \circ V(x_*) \geq e^v + \frac{1}{2s_*} (e^v - e^{v_0}) (1 - s_*) > e^v \quad \Rightarrow \quad V(x_*) > v,$$
and this contradicts with assumption that \( x_\ast \in \Omega \), i.e., that \( V(x_\ast) < v \).

Hence, \( \Xi \) is open–closed (in \( \Omega \)) subset of open set \( \Omega \). Since, by assumption, the set \( \Omega \) is connected, then, as is commonly known, \( \Xi = \Omega \).

Thus for any pair of points \( x, y \in \Omega \) there exists a tangent vector \( \zeta(x, y) \in T_x M \) such that \( \exp_x^V(\zeta(x, y)) = y \). Furthermore, the geodesic segment \( \bigcup_{s \in [0, 1]} \exp_x^V(s\zeta(x, y)) \) connects points \( x \) and \( y \) and completely belongs to \( \Omega \). Moreover, it is shown in [1] Proposition 3.8 that the mapping

\[
\exp_x^V(\cdot) : \mathcal{Z}_x \mapsto \Omega
\]

is a diffeomorphism, and hence, the points \( x, y \) uniquely determine the vector \( \zeta(x, y) \), as well as the geodesic segment of metric \( \langle \cdot, \cdot \rangle_V \) connecting the points and lying in \( \Omega \).

The further proving relies on propositions 3.9–3.11 from [1] and completely duplicates reasonings, contained in proof of Theorem 4.1 and in Addendum to the paper cited. On the basis of these reasonings we can establish the following: 1) there exists a smooth mapping \( \chi(\cdot, \cdot) : [0, 1] \times \Omega \times \Omega \mapsto \Omega \) (connecting mapping) such that for fixed \( x, y \in \Omega \) the function \( \chi(\cdot, x, y) : [0, 1] \mapsto \Omega \) is a solution to equation

\[
\nabla_{x'}x' = \frac{||x'||^2}{2}\nabla V(x) \quad \left( x' := \frac{dx}{ds} \right),
\]

satisfies \( \chi(0, x, y) = x, \chi(1, x, y) = y \), and one can point out a positive number \( \varkappa > 0 \) such that for any pair of smooth functions \( x_i(\cdot) : \mathbb{R} \mapsto \Omega, i = 1, 2 \), and for any \( \varphi \in \mathbb{T}^k \) the convexity condition of Lagrangian density \( (14) \) is fulfilled

\[
\frac{d^2}{ds^2}L \left( \varphi + t\omega, \chi(s, x_1(t), x_2(t)), \frac{\partial}{\partial t} \chi(s, x_1(t), x_2(t)) \right) \geq
\]

\[
\geq \varkappa \left[ ||\nabla \xi \eta||^2 + ||\xi||^2 \left( ||\eta||^2 + 1 \right) \right],
\]

where

\[
\eta := \eta(s, t) := \frac{\partial}{\partial t} \chi(s, x_1(t), x_2(t)), \quad \xi := \xi(s, t) := \frac{\partial}{\partial s} \chi(s, x_1(t), x_2(t))
\]

are vector fields along the mapping \( (s, t) \mapsto \chi(s, x_1(t), x_2(t)) \); 2) as a consequence, for arbitrary \( u_1(\cdot), u_2(\cdot) \in S_{\Omega} \) the function \( s \mapsto J[\chi(s, u_1(\cdot), u_2(\cdot))] \) is strictly convex downward on segment \([0, 1] \); 3) a minimizing sequence \( \{ u_j(\cdot) \in S_{\Omega} \} \) for \( J|_{S_{\Omega}} \) converges to a function \( u(\cdot) \in H^1 \) in the sense that

\[
\lim_{j \to \infty} \| t \circ u_j(\cdot) - t \circ u(\cdot) \|_1 = 0,
\]

the inequality \( \inf \{ J[S_{\Omega_\delta}] = J[u(\cdot)] \) being valid for sufficiently small \( \delta > 0 \); 3) the function \( u(\cdot) \) satisfies the equality \( J[u_\ast(\cdot)h(\cdot)] = 0 \) (equivalent to \( (10) \)) for arbitrary vector field \( h(\cdot) \) along \( u(\cdot) \). This means that \( t \mapsto u(t\omega) \) is a weak quasiperiodic solution to system with Lagrangian density \( (11) \).

\[ \square \]

Remark 1. Since the inequalities satisfying by functions \( V(\cdot) \) and \( W(\cdot, \cdot) \) in accordance with conditions of Theorem \( (11) \) are strict, the result obtained is still correct for a domain \( \Omega' = V^{-1}(-\infty, v') \subset \Omega \) with arbitrary \( v' < v \) sufficiently close to \( v \). And then \( u(\cdot) \in H^1 \).

3. An existence theorem for classical quasiperiodic solution.

The main result of this paper is the following theorem.
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**Theorem 2.** Let there hold the condition of Theorem 1 and a function \( u(\cdot) \in H^1_\Omega \) defines a weak quasiperiodic solution to system with Lagrangian density \([2]\). Then \( u(\cdot) \in C(T^k; \Omega) \) and the function \( x(t) := u(t\omega) \) is a classical uniformly quasiperiodic solution to this system.

The proof of the theorem relies on propositions 2–4 given below.

At first we exploit a technique proposed in [5] to prove the following proposition.

**Proposition 2.** Let there hold the conditions of Theorem 1 and the function \( u(\cdot) \in H^1_\Omega \) defines a weak quasiperiodic solution of the system with Lagrangian density \([1]\). Then the function \( t \mapsto u(\varphi + t\omega) \) is a classical Besicovitch quasiperiodic solution to system with Lagrangian density

\[
L(\varphi + t\omega, x, \dot{x}) := K(x) + W(\varphi + t\omega, x)
\]

for almost all \( \varphi \in T^k \). This solution takes values in a compact set \( K \subset \Omega \).

**Proof.** Relying on reasonings from the proof of Theorem 1 we can conclude that the set \( \Omega \) is diffeomorphic to a domain \( \mathcal{U} \) of Euclidean space \( \mathbb{E}^m = (\mathbb{R}^m, \langle \cdot, \cdot \rangle) \). We may think of this domain as a map of the set \( \Omega \). Thus, taking into account Proposition 1, we shall consider that the function \( u(\cdot) \), existence of which is established in Theorem 1 belongs to class \( H^1_\omega (T^k; \mathbb{E}^m) \) and takes values in a compact subset \( C \subset \mathcal{U} \). We shall also consider that the embedding \( i(\cdot) \) acts from \( \mathcal{U} \) into \( \mathbb{E}^m \). The metric \( \langle \cdot, \cdot \rangle \) induce in \( \mathcal{U} \) a tensor field \( g(x) \) and the corresponding metric \( (g(x), \cdot) = \langle \cdot, \cdot \rangle \). In view of (7), for any smooth vector field of tangent vectors \( \xi(t) \) along smooth curve \( x(t) \) there hold equalities

\[
\left( \frac{d}{dt} t_x(t), \frac{d}{dt} t_\xi(t) \right) = \left( t_\varphi(t), D_x(t), t_\xi(t) \right) = \left( (t_\varphi(t), \xi(t), \xi(t)) \right),
\]

and in coordinates of domain \( \mathcal{U} \) we have

\[
\nabla \dot{\xi}(t) = \xi(t) + \Gamma_x(t) (\dot{x}(t), \xi(t)),
\]

where a bilinear mapping \( \Gamma_x : \mathbb{E}^m \times \mathbb{E}^m \mapsto \mathbb{E}^m \) smoothly depends on \( x \in \mathcal{U} \) and can be expressed via the Christoffel symbols. Therefore a sequence \( \{u_j(\cdot)\} \) which defines \( u(\cdot) \) and a sequence \( \{h_j(\cdot)\} \) according to Definition 2 defines a vector field \( h(\cdot) \) along \( u(\cdot) \) satisfy the equalities

\[
(D_{\omega}u \circ u_j(\varphi), D_{\omega}h_j(\varphi)) = (t_u D_{\omega}u_j(\varphi), t_u \nabla D_{\omega}u_j(\varphi) h_j(\varphi)) =
\]

\[
= (g(u_j(\varphi)) D_{\omega}u_j(\varphi), \nabla D_{\omega}u_j(\varphi) h_j(\varphi)) =
\]

\[
= (g(u_j(\varphi)) D_{\omega}u_j(\varphi) D_{\omega}h_j(\varphi) + \Gamma_{u_j(\varphi)} (D_{\omega}u_j(\varphi), h_j(\varphi))).
\]

Consider a special case where the sequence \( \{h_j(\cdot)\} \) is defined by a unique smooth mapping \( h(\cdot) : T^k \mapsto \mathbb{E}^m \), i.e., the vector \( h_j(\cdot) \) has an origin at point \( u_j(\varphi) \) and an end at point \( u_j(\varphi) + h(\varphi) \). Taking into account that \( \{u_j(\cdot)\} \) is uniformly bounded and converges in \( H^1_\omega (T^k; \mathbb{E}^m) \) and also that \( \nabla W(\varphi, x) = g^{-1}(x) W'_x(\varphi, x) \), one can present the property (5) of function \( u(\cdot) \) in the form

\[
\int_{T^k} [(g(u(\varphi)) D_{\omega}u(\varphi) D_{\omega}h(\varphi) + \Gamma_u(\varphi) (D_{\omega}u(\varphi), h(\varphi))) + (W'_x(\varphi, u(\varphi)), h(\varphi))] \, d\varphi = 0. \tag{15}
\]

Just as in [4], in the integral from the left-hand side, we perform a change of variables \( \varphi \to (\tau, y) \) by formula

\[
\varphi = Q(\tau, y) := \sum_{i=1}^{k-1} y_i e_i + \tau e_k,
\]
Since the equality (18) is fulfilled for any function \(v(\tau, y) := u(Q(\tau, y)), \quad w(\tau, y) := h(Q(\tau, y)), \quad I(y) := \{\tau \in \mathbb{R} : Q(\tau, y) \in K\},\)
and applying the Fubini theorem to the left-hand side of the equality
\[
\int_K \left[ (g(u(\varphi))D_\omega u(\varphi), D_\omega h(\varphi) + \Gamma_{u(\varphi)}(D_\omega u(\varphi), h(\varphi))) + (W_x'(\varphi, u(\varphi), h(\varphi))) \right] d\varphi = 0, \tag{16}
\]
we get
\[
\int_Y \int_{I(y)} \left[ (\|w\|^2 g(v(\tau, y)), v(\tau, y), w(\tau, y)) \right] d\tau dy + \int_Y \int_{I(y)} (W_x'(Q(\tau, y), v(\tau, y)), w(\tau, y)) d\tau dy = 0, \tag{17}
\]
where \(v(\tau, u)\) stands for Sobolev generalized partial derivative by variable \(\tau\), and \(\dot{w}(\tau, y) = \partial w(\tau, y)/\partial \tau\) is classical partial derivative by this variable. Since \(u(\varphi) \in H^1_\omega([T; E^m])\), then
\[
\int_Y \int_{I(y)} \|v(\tau, y)\|^2 + \|v'(\tau, y)\|^2 d\tau dy < \infty.
\]
If we now denote by \(Y\) the orthogonal projection of \(k\)-dimensional cube \(K := [0, 2\pi]^k\) onto the hyperplane with basis \(\{e_i\}_{i=1}^{k-1}\), then the Fubini theorem together with Theorem 6 [20] p. 399 implies the following assertions: there exists a set \(Y' \subset Y\) such that \(\text{mes} Y' = \text{mes} Y\) and for any \(y \in Y'\) the function \(v(\cdot, y) : I(y) \mapsto C\) is absolutely continuous on \(I(y)\), has classical derivative \(\dot{v}(\tau, y) = v'(\tau, y)\) a.e. on this segment, satisfies the condition
\[
\int_{I(y)} \|v(\tau, y)\|^2 + \|v'(\tau, y)\|^2 d\tau < \infty \quad \forall y \in Y',
\]
and, as a consequence, belongs to Sobolev space \(H^1(I(y); E^m)\).

By introducing a “momentum”
\[
p(\tau, y) = g(v(\tau, y))\dot{v}(\tau, u),
\]
we represent the equality (17) in the form
\[
\int_Y \int_{I(y)} (p(\tau, y), \dot{w}(\tau, y)) d\tau dy = -\int_Y \int_{I(y)} (G_v(\tau, y)(\dot{v}(\tau, y), \dot{v}(\tau, y)) + \|\omega\|^{-2} W_x'(Q(\tau, y), v(\tau, y)), w(\tau, y)) d\tau dy, \tag{18}
\]
where a family of bilinear mappings \(G_x(\cdot, \cdot) : E^m \times E^m \mapsto E^m\) smoothly dependent on \(x \in U\) is defined by
\[
(g(x) a, \Gamma_x(b, c)) = (G_x(a, b, c)) \quad \forall a, b, c \in E^m.
\]
Since the equality (18) is fulfilled for any function \(w(\tau, y)\) from space \(C_0^\infty(\mathbb{R}^{-1}(K) ; E^m)\) formed by smooth functions having their supports in interior of the set \(Q^{-1}(K) = \bigcup_{y \in Y} I(y)\), then the above equality implies that the function \(p(\tau, y)\) has integrable in \(Q^{-1}(K)\) Sobolev generalized derivative
\[
p'(\tau, y) = G_v(\tau, y)(\dot{v}(\tau, y), \dot{v}(\tau, y)) + \|\omega\|^{-2} W_x'(Q(\tau, y), v(\tau, y)). \tag{19}
\]
By using Theorem 6 [20, p. 399] again, we arrive at conclusion that there exists a set $Y'' \subset Y'$ such that $\operatorname{mes} Y'' = \operatorname{mes} Y'$ and for any $y \in Y''$ there holds the equality

$$p(\tau, y) = p(\tau_0, y) + \int_{\tau_0}^{\tau} \left[ G_{v(s,y)}(\dot{v}(s, y), v(s, y)) + \|\omega\|^{-2} W'_2(Q(s, y), v(s, y)) \right] ds,$$

(20)
on segment $I(y)$ where $\tau_0 \in I(y)$ is a fixed point. Hence, for any $y \in Y''$, the functions $\tau \mapsto p(\tau, y)$ and $\tau \mapsto \dot{v}(\tau, y) = p(\tau, y)/g(v(\tau, y))$ are absolutely continuous on the segment $I(y)$, and each of these functions has an ordinary partial derivative by $\tau$ a.e. on $I(y)$. Then the function $\tau \mapsto v(\tau, y)$ is continuously differentiable, and from (20) it follows that the function $\tau \mapsto p(\tau, y)$ is continuously differentiable. Then it becomes clear that $\tau \mapsto \dot{v}(\tau, y)$ is continuously differentiable as well. Thus, for a.e. $y \in Y$ the function $\tau \mapsto v(\tau, y)$ is twice continuously differentiable and satisfies the equality (19) everywhere on $I(y)$. Having determined from this equality the function $\dot{v}(\cdot, \cdot)$ of variables $\tau, y$, we see that it is integrable on $Q^{-1}(K)$.

Finally, let us show that for almost all $y \in Y$ the functions $\tau \mapsto v(\tau, y)$ and $\tau \mapsto p(\tau, y)$ have the properties described above not only on $I(y)$, but also on any segment of real axis $\mathbb{R}$. Represent the space $\mathbb{E}^k$ in the form of union of cubes: $\mathbb{E}^k = \bigcup_{m \in \mathbb{Z}^k} (K + 2\pi m)$. Then $\mathbb{R} = \bigcup_{m \in \mathbb{Z}} I_m(y)$ where

$$I_0(y) := I(y), \quad I_m(y) := \{ \tau \in \mathbb{R} : Q(\tau, y) \in K + 2\pi m \}.$$

Among all $I_m(y)$, there are sets of zero measure and, in particular, empty sets. Put

$$Y_m := \{ y \in Y : \operatorname{mes} I_m(y) > 0 \}, \quad M := \{ m \in \mathbb{Z}^k : Y_m \neq \emptyset \}.$$

It is clear that for a fixed $m$ the set $Y_m$ consists only of those $y \in Y$ for which the line $L(y)$ defined in $\mathbb{E}^k$ by equation $\varphi = Q(\tau, y)$, $\tau \in \mathbb{R}$, intersects the cube $K + 2\pi m$ by segment which is not reduced to single-point set, and $M$ contains only those integer vectors $m$, for which the corresponding cube $K + 2\pi m$ intersects at least with one line $L(y)$, where $y \in Y$, and the intersection is a segment or a point. It is easily seen that rational independence of components of frequency vector $\omega$ implies that any nonempty set $Y_m$ is open. Besides, $\operatorname{mes}(Y \setminus Y_0) = 0$.

Having introduced the orthoprojectors

$$\operatorname{pr}_Y(a) := \sum_{i=1}^{k-1} (a, \varepsilon_i)\varepsilon_i, \quad \operatorname{pr}_T(a) := (a, \varepsilon_k)\varepsilon_k \quad \forall a \in \mathbb{E}^k,$$

we get the following equalities [1]

$$f(\tau, y) := F(Q(\tau, y)) = F(Q(\tau, y) - 2\pi m) = F(Q(\tau - \operatorname{pr}_T(2\pi m), y - \operatorname{pr}_Y(2\pi m))) =$$

$$= f(\tau - \operatorname{pr}_T(2\pi m), y - \operatorname{pr}_Y(2\pi m))$$

for arbitrary function $F : \mathbb{T}^k \mapsto \mathbb{E}^m$, and then for any $m \in M$ and any $y \in Y_m$ we have

$$y - \operatorname{pr}_Y(2\pi m) \in Y_0, \quad I_m(y) - \operatorname{pr}_T(2\pi m) = I(y - \operatorname{pr}_Y(2\pi m)),$$

(21)

Thus, for any $y \in Y_m$, properties of function $\tau \mapsto f(\tau, y)$ on segment $I_m(y)$ are completely defined by properties of function $\tau \mapsto f(\tau, y - \operatorname{pr}_Y(2\pi m))$ on segment $I(y - \operatorname{pr}_Y(2\pi m))$ of line $L(y - \operatorname{pr}_Y(2\pi m))$ contained in cube $K$.

Denote by $Y_z$ the set of such $y \in Y$ for which the function

$$\tau \mapsto \nu(\tau, y) := \|v(\tau, y)\|^2 + \|v'_f(\tau, y)\|^2$$

We identify a function on torus $\mathbb{T}^k = \mathbb{R}^k/2\pi\mathbb{Z}^k$ with its natural lift into the space $\mathbb{E}^k$ covering this torus.
is not locally integrable on \( \mathbb{R} \), or, what is the same thing, for any \( y \in Y_* \) there exists at least one \( m \in \mathbb{M} \) such that \( y \in Y_m \) and

\[
\int_{I_m(y)} \nu(\tau, y) d\tau = \int_{I(y - \text{pr}_Y(2\pi m))} \nu(\tau, y - \text{pr}_Y(2\pi m)) d\tau = \infty. \tag{22}
\]

Then \( Y_* \subseteq \bigcup_{m \in \mathbb{M}} Y_m \) and therefore \( Y_* = \bigcup_{m \in \mathbb{M}} (Y_* \cap Y_m) \). Obviously,

\[ \text{mes}(Y_* \cap Y_m) = \text{mes}([Y_* \cap Y_m] - \text{pr}_Y(2\pi m)) \]

and in view of (21), (22) we have \([Y_* \cap Y_m] - \text{pr}_Y(2\pi m) \subseteq Y_* \cap Y_0 \). Since for almost all \( y \in Y \) the function \( \tau \mapsto \nu(\tau, y) \) is integrable on \( I_0(y) = I(y) \), then \( \text{mes}(Y_* \cap Y_0) = 0 \), and therefore

\[ \text{mes}(Y_*) \leq \sum_{m \in \mathbb{M}} \text{mes}(Y_* \cap Y_m) = \sum_{m \in \mathbb{M}} \text{mes}([Y_* \cap Y_m] - \text{pr}_Y(2\pi m)) \leq \sum_{m \in \mathbb{M}} \text{mes}(Y_* \cap Y_0) = 0. \]

Thus, for almost all \( y \in Y \) the function \( \tau \mapsto \nu(\tau, y) \) is locally integrable on \( \mathbb{R} \). From this it follows that for almost all \( y \in Y \) the function \( \tau \mapsto v(\tau, y) \) is absolutely continuous and has integrable derivative on any segment \( I_m(y), m \in \mathbb{M} \).

Arguing in the same way as above, we can easily prove that for almost all \( y \in Y \) the function \( \tau \mapsto v(\tau, y) \) is twice continuously differentiable on each \( I_m(y), m \in \mathbb{M} \). Now let us take into account that in view of (15) integration over cube \( K \) in the right-hand side of equality (16) can be replaced by integration over cube \( K + s\varepsilon_k \) with arbitrary \( s \in \mathbb{R} \). According to equality (17) the segment \( I(y) \) can be replaced by \( I(y)+s \). Then it becomes clear that the function \( \tau \mapsto v(\tau, y) \) is twice continuously differentiable on each segment \( I_m(y)+s, m \in \mathbb{M} \), and hence, on all real axis. In addition the equality (19) holds true for all real \( \tau \). By substituting \( \tau = y_k + \|\omega\| t \) in (19), we arrive at conclusion that for any \( y_k \in \mathbb{R} \) and almost all \( y \in Y \) the function \( t \mapsto u \left( \sum_{i=1}^{k} y_k \varepsilon_i + t \omega, t \right), t \in \mathbb{R}, \) is a classical solution of Lagrangian system

\[
\frac{d}{dt} (g(x) \dot{x}) = \frac{\partial}{\partial x} \left[ (g(x) \dot{x}, \dot{x}) + W(\varphi + t \omega, x) \right]
\]

with \( \varphi = \sum_{i=1}^{k} y_i \varepsilon_i \in K, \) i.e., of system generated on the map \( U \) by Lagrangian density (14). It remains only to observe that the orthogonal transformation which assigns to a point \( (y_1, \ldots, y_k) \in Q^{-1}(K) \) the point \( \varphi = \sum_{i=1}^{k} y_i \varepsilon_i \in K \) is measure-preserving. \( \square \)

Let us fix a point \( \varphi_0 \in \mathbb{T}^k \) in such a way that the function \( t \mapsto x(t) := u(\varphi_0 + t \omega) \) be a classical solution from Proposition 2. If the metric \( \langle \cdot, \cdot \rangle \) were Euclidean, i.e., \( g(x) \equiv \text{const} \), then the immediate consequence of Landau inequality (21) would be the boundedness of solution derivative on whole real axis. However, in general case the substantiation of boundedness for the derivative needs another approach.

**Proposition 3.** The function \( \|\dot{\varphi}(\cdot)\| \) is bounded on real axis.

**Proof.** The vector field \( \xi(t) := \dot{x}(t) \) along the curve \( \gamma \) defined by equation \( x = x(t), t \in \mathbb{R}, \) satisfies the identity

\[ \nabla_{\dot{x}(t)} \xi(t) \equiv \nabla W(\varphi_0 + t \omega, x(t)). \]

By applying the derivative \( \nabla_{\dot{x}(t)} \) to both sides, one ascertains that \( \xi(\cdot) \) is a solution of linear inhomogeneous system

\[ \nabla^2_{\dot{x}} \xi = H_W(\varphi_0 + t \omega, x(t)) \xi + \left[ \nabla \frac{\partial}{\partial t} W(\varphi_0 + t \omega, x) \right]_{x=x(t)}, \]
and at the same time satisfies the system

\[
\nabla_\gamma^2 \xi = H_W(\varphi_0 + t\omega, x(t))\xi - r(\|\dot{x}(t)\|)R(\dot{x}(t), \xi)\dot{x}(t) + \left[ \nabla \frac{\partial}{\partial t} W(\varphi_0 + t\omega, x) \right]_{x=x(t)},
\]

where \( R \) is the curvature tensor for Levi-Civita connection, and \( r(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \) is an arbitrary continuous function (it is sufficiently to observe that \( R(\xi, \xi) = 0 \)). If we require that \( r(s) = O(s^{-2}) \) for \( |s| \to \infty \), then the right-hand side of the system (23) will be bounded with respect to \( t \in \mathbb{R} \) for any field \( \xi \) bounded by norm.

Let \( \Xi_s^t \) be an evolution operator for the linear system \( \nabla_\gamma^s(\xi) = 0 \). By means of this operator, the parallel translation of vectors along \( \gamma \) is carried out. Namely, for any vector \( \xi_s \in T_{x(s)}\mathcal{M} \) the result of its parallel translation from a point \( x(s) \) to point \( x(t) \) along the curve \( \gamma \) is the vector \( \Xi_s^t \xi_s \). Under the parallel translation the dot product for pair of vectors stays the same. Therefore the operator \( \Xi_s^t \) is orthogonal with respect to metric \( \langle \cdot , \cdot \rangle \).

Set \( y_*(t) := \Xi_s^0 \xi(t) \equiv \Xi_s^0 \xi(t) \). Since

\[
\nabla_\gamma\xi(t) = \lim_{s \to 0} \frac{1}{s} \left[ \Xi_s^0 \xi(t + s) - \xi(t) \right] = \lim_{s \to 0} \frac{1}{s} \left[ \Xi_s^0 \xi(t + s) \Xi_s^0 y_*(t + s) - \Xi_s^0 y_*(t) \right] = \Xi_s^0 \dot{y}_*(t),
\]

then, taking into account (24), we have

\[
\dot{y}_*(t) = \Xi_s^0 \nabla W(\varphi_0 + t\omega, x(t)),
\]

and from this it follows, in particular, that

\[
\|y_*(t)\| = O(|t|), \quad |t| \to \infty.
\]

Further, since the vector field \( \xi(t) \) along the curve \( \gamma \) satisfies the system (23) and

\[
\nabla_\gamma^2 \Xi_s^t y_*(t) = \nabla_\gamma \Xi_s^0 \dot{y}_*(t) = \Xi_s^0 \ddot{y}_*(t),
\]

then \( y_*(t) \) is a solution of the inhomogeneous system

\[
\ddot{y} = A(t) y + h(t)
\]

where

\[
A(t) := \Xi_s^0 \left[ H_W(\varphi_0 + t\omega, x(t))\Xi_s^0 y - r(\|\xi(t)\|)R(\xi(t), \Xi_s^0 y)\xi(t) \right],
\]

\[
h(t) := \Xi_s^0 \left[ \nabla \frac{\partial}{\partial t} W(\varphi_0 + t\omega, x) \right]_{x=x(t)}.
\]

Consider now the corresponding homogeneous systems

\[
\nabla_\gamma^2 \eta = H_W(\varphi_0 + t\omega, x(t))\eta - r(\|\xi(t)\|)R(\xi(t), \eta)\xi(t),
\]

\[
\ddot{y} = A(t) y
\]

and show that under appropriate choice of function \( r(\cdot) \) the last ones are exponentially dichotomic. Define the function

\[
\mathcal{F}(t, \eta, \nabla_\xi \eta) := \langle \nabla_\xi \eta, \eta \rangle + \frac{r(\|\xi(t)\|) \|\eta\|}{2} \|\nabla V(x(t)), \xi(t)\rangle.
\]

By calculating its derivative along solutions of the system (26), one can establish the following inequality:

\[
\frac{d}{dt} \mathcal{F}(t, \eta, \nabla_\xi \eta) \geq \langle H_W \eta, \eta \rangle + \frac{r(\|\xi\|) \|\eta\|}{2} \langle \nabla W, \nabla V \rangle + \langle \nabla_\xi \eta \rangle - r(\|\xi\|) \|\xi\| \|\nabla_\xi \eta\| \|\eta\| \langle \nabla V, \varepsilon \rangle + \frac{r(\|\xi\|) \|\xi\| \|\eta\|}{2} \|\eta\|^2 \left[ \langle H_V \varepsilon, \varepsilon \rangle - 2K^* \right] - \frac{1}{2} \|r'(\|\xi\|)\|\xi\| \langle \nabla W, \varepsilon \rangle \langle \nabla V, \varepsilon \rangle \|\eta\|^2,
\]
where $\varepsilon := \xi / \|\xi\|$. Now we set
\[
    r(s) := \begin{cases} 
        1, & s \in [0, B], \\
        B^2/s^2, & s > B,
    \end{cases}
\]
where $B > 1$, and assign $\|\nabla \xi \| = z_1$, $\|\eta\| = z_2$. Then on the set of those $t$ for which $\|\xi(t)\| \leq B$ we have
\[
    \frac{d}{dt} F(t, \eta, \nabla \xi \eta) \geq \left[ \lambda_W + \frac{1}{2} \langle \nabla W, \nabla V \rangle \right] z_2^2 + z_1^2 - \langle \nabla V, \varepsilon \rangle \|\xi\| z_2 + \frac{\|\xi\|^2 z_2^2}{2} \left[ \langle H_V \varepsilon, \varepsilon \rangle - 2K^* \right].
\]
It is easily seen that the fulfillment of conditions of Theorem 1 assures the existence of positive $\alpha_1$ and $\alpha_2$ such that
\[
    \lambda_W + \frac{1}{2} \langle \nabla W, \nabla V \rangle \geq \alpha_1 \quad \text{and} \quad z_1^2 - \langle \nabla V, \varepsilon \rangle \|\xi\| z_2 + \left[ \frac{1}{2} \langle H_V \varepsilon, \varepsilon \rangle - K^* \right] \|\xi\|^2 z_2^2 \geq \alpha_2 \left( z_1^2 + \|\xi\|^2 z_2^2 \right),
\]
from what it follows that
\[
    \frac{d}{dt} F(t, \eta, \nabla \xi \eta) \geq \alpha_1 \|\eta\|^2 + \alpha_2 \|\nabla \xi \eta\|^2
\]
if $\|\xi(t)\| \leq B$.

Set
\[
    C := \max \left\{ \max_{x \in \Omega} \|\nabla V(x)\|, \max_{(\varphi, x) \in T^k \times \Omega} \|\nabla W(\varphi, x)\|, \max_{(\varphi, x) \in T^k \times \Omega} \|H_W(\varphi, x)\| \right\}
\]
and choose $B$ so large that $\alpha_2 B^2 \geq 1 + C(1 + 3C/2)$. Then on the set of those $t$ for which $\|\xi(t)\| > B$ we get
\[
    \frac{d}{dt} F(t, \eta, \nabla \xi \eta) \geq \|\nabla \xi \eta\|^2 - B \|\nabla V, \varepsilon\| \|\nabla \xi \eta\| \|\eta\| + \frac{B^2 \|\eta\|^2}{2} \left[ \langle H_V \varepsilon, \varepsilon \rangle - 2K^* \right] - \\
    \left[ C + 3C^2/2 \right] \|\eta\|^2 \geq \alpha_2 (z_1^2 + B^2 z_2^2) - \left[ C + 3C^2/2 \right] z_2^2 \geq \alpha_2 z_1^2 + z_2^2.
\]
From this it follows that there exists $\alpha > 0$ such that the derivative of quadratic form of variables $y$, $\dot{y}$
\[
    F(t, \Xi_0^t y, \Xi_0^t \dot{y}) = \langle \dot{y}, y \rangle + \frac{r(\|\xi(t)\|)}{2} \langle \nabla V(x(t)), \xi(t) \rangle \|y\|^2
\]
along solutions of the system (27) satisfies the inequalities
\[
    \frac{d}{dt} F(t, \Xi_0^t y, \Xi_0^t \dot{y}) \geq \alpha \left[ \|\Xi_0^t \dot{y}\|^2 + \|\Xi_0^t y\|^2 \right] = \alpha \left[ \|\dot{y}\|^2 + \|y\|^2 \right]
\]
and hence is positive definite. At the same time the form $F(t, \Xi_0^t y, \Xi_0^t \dot{y})$ is, obviously, nondegenerate and has bounded coefficients. As it follows from [22], the existence of quadratic form with such properties guarantees the exponential dichotomy of liner system (27) on the whole real axis, and the inhomogeneous system (25) in view of boundedness of $\|h(t)\|$ possesses a unique bounded solution. Furthermore, any other solution of this system exponentially increases either for $t \to \infty$, or for $t \to -\infty$. Since above we have established the estimate (24), then the solution $y_*(t)$ is bounded on $\mathbb{R}$. Thus the function $\|\dot{x}(t)\| = \|\Xi_0^t \dot{y}_*(t)\| = \|y_*(t)\|$ is bounded on $\mathbb{R}$. \qed
Proposition 4. If the conditions of Theorem 1 holds true, then for any \( \varphi \in \mathbb{T}^k \) the system with Lagrangian density (14) cannot have more than one solution \( x(\cdot) : \mathbb{R} \rightarrow \mathcal{K} \) such that \( \sup_{t \in \mathbb{R}} \| \dot{x}(t) \| < \infty \), where \( \mathcal{K} \subset \Omega \) is the compact set from Proposition 2.

Proof. Use reductio ad absurdum: suppose that there exists a pair of different solutions \( x_i(\cdot) : \mathbb{R} \rightarrow \mathcal{K} \) such that \( \sup_{t \in \mathbb{R}} \| \dot{x}_i(t) \| < \infty \), \( i = 1, 2 \). Making use of connecting mapping \( \chi \) (see the end of Sect.2) and introducing the notations

\[
\eta(s, t) := \frac{\partial}{\partial t} \chi (s, x_1(t), x_2(t)), \quad \xi(s, t) := \frac{\partial}{\partial s} \chi (s, x_1(t), x_2(t)),
\]

define the function

\[
l(t) := \langle \eta(s, t), \xi(s, t) \rangle \bigg|_{s=0}^{s=1} = \langle \dot{x}_2(t), \xi(1, t) \rangle - \langle \dot{x}_1(t), \xi(0, t) \rangle
\]

This function is bounded on \( \mathbb{R} \), and it turns out that its derivative can be represented in the form

\[
\dot{l}(t) = \frac{\partial}{\partial s} L (\varphi + t \omega, \chi (s, x_1(t), x_2(t)), \eta(s, t)) \bigg|_{s=0}^{s=1} =
\]

\[
= \int_0^1 \frac{\partial^2}{\partial s^2} L (\varphi + t \omega, \chi (s, x_1(t), x_2(t)), \eta(s, t)) \, ds.
\]

Taking into account (13) we have

\[
\dot{l}(t) \geq \kappa \int_0^1 \left[ \| \nabla \xi \|^2 + (\| \eta \|^2 + 1) \| \xi \|^2 \right] \, ds.
\]

Thus, the function \( l(\cdot) \) is nondecreasing and its boundedness assures convergence of the integrals

\[
\int_{-\infty}^0 \int_0^1 \| \xi(s, t) \|^2 \, ds \, dt < \infty, \quad \int_0^1 \int_0^1 \| \xi(s, t) \|^2 \, ds \, dt < \infty.
\]

(28)

Note that \( \chi'_s(0, x, y) \neq 0 \) once \( x \neq y \). In fact, otherwise the mapping \( s \mapsto \chi(s, x, y) \) would be a solution of equation (12) which satisfies the initial conditions \( \chi|_{s=0} = x, \chi'_s|_{s=0} = 0 \), but only constant solution \( \chi(s, x, y) \equiv x \) has such a property. This contradicts the equality \( \chi(1, s, y) = y \). Taking into account that the equality \( x_1(t) = x_2(t) \) can be valid only on a discrete set, we have \( \xi(0, t) \neq 0 \). But then there hold the strict inequalities

\[
\limsup_{t \to -\infty} l(t) < l(0) < \liminf_{t \to -\infty} l(t).
\]

(29)

It is not hard to show that the second inequality (28) assures the existence of sequence \( t^+_k \to \infty \) such that max \{ \( \|\xi(0, t^+_k)\|, \|\xi(1, t^+_k)\| \} \to 0 \). In the same way, one can prove that the first inequality in (28) guarantees the existence of sequence \( t^-_k \to -\infty \) along which \( \|\xi(0, t)\| \) and \( \|\xi(1, t)\| \) simultaneously tend to zero. Then taking into account boundedness of \( \|\dot{x}_i(t)\| \) on \( \mathbb{R} \), we get

\[
\lim_{k \to \infty} l(t^+_k) = 0.
\]

This equality contradicts (29). Namely, the case \( l(0) \geq 0 \) (the case \( l(0) < 0 \) contradicts the existence of sequence \( \{t^+_k\} \) (the existence of sequence \( \{t^-_k\} \)). \( \square \)
Proof of Theorem 2. Apply the Amerio theorem (see, e.g., [23, p. 437]). On the map \( \mathcal{U} \subset \mathbb{R}^m \) of domain \( \Omega \subset \mathcal{M} \), the system with Lagrangian density \( L(\varphi_0 + t\omega, x, \dot{x}) \) takes the form of a second order system, which is equivalent to an \( 2m \)-dimensional normal first order system on phase space \( \mathcal{U} \times \mathbb{R}^m \). Since components of frequency vector \( \omega \) are rationally independent, then the so-called \( H \)-class of system with Lagrangian density \( L(\varphi_0 + t\omega, x, \dot{x}) \) is formed by a family of systems with Lagrangian densities \( L(\varphi + t\omega, x, \dot{x}) \) parametrized by points of torus \( \varphi \in \mathbb{T}^k \). From Propositions 2–4 it follows that in \( C \times \mathbb{R}^m \) (\( C \) is the image of compact set \( \mathcal{K} \) on the map) the system with Lagrangian density \( L(\varphi_0 + t\omega, x, \dot{x}) \) has a unique classical bounded Besicovitch quasiperiodic solution \( t \mapsto u(\varphi_0 + t\omega) \) and each system from its \( H \)-class has a unique bounded solution in \( \mathcal{C} \times \mathbb{R}^m \). Thus, all requirements of the Amerio theorem are fulfilled and \( u(\varphi_0 + t\omega) \) is a uniformly almost periodic function. In view of its Fourier series, this function is uniformly quasiperiodic with frequency basis \( \omega \). Then \( u(\cdot) \in C(\mathbb{T}^k \mapsto \mathcal{C}) \) and for any \( \varphi \in \mathbb{T}^k \), in particular for \( \varphi = 0 \), the function \( t \mapsto u(\varphi + t\omega) \) is a classical quasiperiodic solution of the system with Lagrangian density \( L(\varphi + t\omega, x, \dot{x}) \).

Theorem 3. Suppose that the conditions of Theorem 2 holds true and let \( x(\cdot) \in C^2(\mathbb{R} \mapsto \mathcal{K}) \) be a quasiperiodic solution of system with Lagrangian density \( L(t\omega, x, \dot{x}) \). Then the system in variations along this solution is exponentially dichotomic on \( \mathbb{R} \).

Proof. Let \( t \mapsto x(t, s), s \in (-\delta, \delta) \), be a family of solutions to system with Lagrangian density \( L(t\omega, x, \dot{x}) \). Let this family smoothly depends on parameter \( s \) and \( x(t, 0) = x(t) \). Then \( \xi(t, s) := x'_1(t, s), \eta(t, s) := x'_2(t, s) \) define vector fields along the mapping \( (t, s) \mapsto x(t, s) \) and there holds the Lagrange equation

\[
\nabla_{\xi(t,s)}(t, s) = \nabla W(t\omega + \varphi, x(t, s)).
\]

Calculate here the covariant derivative \( \nabla_{\eta(t,s)} \) from both sides, take into account the equalities \([12, \text{Sect. 3.6}]\)

\[
\nabla_{\eta}\xi = \nabla_{\xi}\eta, \quad \nabla_{\eta}\nabla_{\xi}\eta - \nabla_{\xi}\nabla_{\eta}\xi = R(\xi, \eta)\xi,
\]

where \( R \) is curvature tensor f Levi–Civita connection, and set \( s = 0, \xi(t) := \xi(t, 0), \eta(t) := \eta(t, 0) \). Then we arrive at conclusion that the field \( \eta(t) \) satisfies the system in variations along \( x(t) \):

\[
\nabla^2_{\xi(t)} \eta = \nabla_{\eta} \nabla W(t\omega + \varphi, x(t)) - R(\xi(t), \eta)\xi(t).
\]

(30)

Since in this case the function \( t \mapsto \|\xi(t)\| \) is bounded, then we can make use of reasonings from the proof of Proposition 3 with \( r(s) \equiv 1 \) to show the exponential dichotomy of system (30). 

4. Concluding remarks. The authors of papers [5, 13] restrict themselves to proof of the fact that classical solutions generated by weak solutions of certain classes of systems on Euclidean space are Besicovitch quasiperiodic functions, while by our main result we shows that under conditions of Theorem 2 such weak solutions are actually classical uniformly quasiperiodic ones. It is also worth to note that the paper [13] deals with quasiperiodic systems in \( \mathbb{R}^m \) of rather general form

\[
\frac{d^p q}{dt^p} = F(t\omega, q, \dot{q}, \ldots, \frac{d^{p-1} q}{dt^{p-1}}),
\]

but under boundedness condition of right-hand side with respect to derivatives \( q^{(j)}, j = 1, \ldots, p - 1 \). Unfortunately, this condition makes impossible application of results from [13], concerning classical Besicovitch quasiperiodic solutions, to Lagrangian systems on Riemannian manifold with nonconstant metric tensor \( g(x) \).
Application of our results to concrete mechanical systems requires constructing the auxiliary function $V(\cdot)$. One of the ways to find such a function by means of averaged force function

$$\bar{W}(x) := \frac{1}{(2\pi)^k} \int_{\mathbb{T}^k} W(\varphi, x) d\varphi$$

has been offered in [1].

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