Probabilistic Counters for Privacy Preserving Data Aggregation

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Abstract

Probabilistic counters are well-known tools often used for space-efficient set cardinality estimation. In this paper, we investigate probabilistic counters from the perspective of preserving privacy. We use the standard, rigid differential privacy notion. The intuition is that the probabilistic counters do not reveal too much information about individuals but provide only general information about the population. Therefore, they can be used safely without violating the privacy of individuals. However, it turned out, that providing a precise, formal analysis of the privacy parameters of probabilistic counters is surprisingly difficult and needs advanced techniques and a very careful approach.

We demonstrate that probabilistic counters can be used as a privacy protection mechanism without extra randomization. Namely, the inherent randomization from the protocol is sufficient for protecting privacy, even if the probabilistic counter is used multiple times. In particular, we present a specific privacy-preserving data aggregation protocol based on Morris Counter and MaxGeo Counter. Some of the presented results are devoted to counters that have not been investigated so far from the perspective of privacy protection. Another part is an improvement of previous results. We show how our results can be used to perform distributed surveys and compare the properties of counter-based solutions and a standard Laplace method.

Keywords: differential privacy, data aggregation, probabilistic counters

1. Introduction

Since Big Data related topics have been widely developed in recent years, solutions that focus on saving memory resources have become very popular. We would like to consider a standard example of such space-efficient mechanisms, namely probabilistic counters, which are used to represent the cardinality of dynamically counted events. More precisely, we would like to indicate the occurrence of \( n \) events using a very small (significantly less than \( \log n \)) number of bits. We assume that \( n \) is unknown in advance and may change. Clearly, a simple information-theoretic argument convinces
us that it is not feasible if we demand an exact representation of the number of events. Nevertheless, there are some very efficient solutions that require only $\Theta(\log \log n)$ bits and guarantee sufficient accuracy for a wide range of applications. As examples, one can point most of probabilistic counters – probabilistic structures well known in the literature since the seminal Morris’ paper [1] followed by its thorough mathematical analysis by Flajolet in [2]. They are used as building blocks in many space-efficient algorithms in the field of data mining or distributed data aggregation in networks or smart metering just to mention a few ([3] or [4]).

In this paper, we investigate probabilistic counters from the privacy-protection perspective. Our analysis is based on a differential privacy notion, which is commonly considered the only state-of-the-art approach. The differential privacy has the undeniable advantage of being mathematically rigorous and formally provable, contrary to previous anonymity-derived privacy definitions. This approach to privacy-preserving protocols can be used to give a formal guarantee for privacy that is resilient to any form of post-processing. For a survey about differential privacy properties, see [5] and references therein. Analysis of protocols based on differential privacy is usually technically complex, but by using this notion, we are immune to e.g., linkage attacks (see, for example [6, 7]).

The idea behind differential privacy is as follows: for two “neighboring” scenarios that differ only in the participation of a single user, a differentially private mechanism should provide a response chosen from very similar distributions. Roughly speaking, differential privacy is described by two parameters: $\varepsilon$ – which controls a similarity of probabilities of common events – and $\delta$ – which is related to a probability of unusual events. The smaller the parameters, the better from the privacy point of view. In effect, judging by the output of the mechanism, one cannot say if a given individual (user) was taken into account for producing a given output. Intuitively, probabilistic counters should provide a high level of differential privacy since, statistically, many various numbers of events are “squeezed” into a small space of possible output results. In the case of one counter considered in our paper (MaxGeo) counter, one can find some similar, recent results about the privacy the algorithm offers. Nevertheless, the question about the value of parameters of potential differential privacy property remains open (see the discussion in Section 2).

Moreover, an additional problem for small values of $n$ may occur whenever one can distinguish the output for $n$ events from “neighboring” cases with $n - 1$ and $n + 1$ occurrences with significant probability when two “neighboring” outputs are provided. In our paper, we provide a very precise analysis of two well known probabilistic counters from the perspective of preserving privacy. It turned out that this task is surprisingly complex from the mathematical point of view.

Our primary motivation is to find possibly accurate privacy parameters of two most fundamental probabilistic counter protocols, namely Morris Counter [1] and MaxGeo Counter [8]. Note that the second one is used for yet another popular algorithm — HyperLogLog [9]. One may realize that these two counters are relatively old; however, they, together with their modifications, are extensively used until these days. Morris Counter is often used in big data solutions, for instance, for measurement of network’s capabilities [10]. The most crucial examples of refinements of HyperLogLog algorithm are mentioned in Section 2. We claim that a high-precision analysis in the case of prob-
Probabilistic counters is particularly important. This is because even a mechanism with very good privacy parameters may cause a serious privacy breach when used multiple times (see e.g., [5]). Probabilistic counters in realistic scenarios may be used as fundamental primitives and subroutines in more complex protocols since the differential privacy property is immune to post-processing.

We also show that those two probabilistic counters can be used safely without any additional randomization, even in very demanding settings. It is commonly known that no deterministic algorithm can provide non-trivial differential privacy. However, Probabilistic counters possess inherent randomness, achieving desired privacy parameters. In other words, one can say that probabilistic counters are safe by design, and we do not need any additional privacy-oriented methods. In particular, what is most important, existing, working implementations do not need to be changed if we start demanding the provable privacy of a system.

Finally, we demonstrate how our results can be used for constructing a data aggregation protocol based on probabilistic counters that can be used in some specific scenarios until we want them to satisfy even more rigorous privacy properties.

1.1. Paper structure and results

Starting from this point, for the sake of clarity, we use an abbreviation $DP$ as a shortcut for differential privacy, while this property is described by some parameters.

The main contribution of our paper is as follows:

- We prove that the classic Morris Counter satisfies $(\varepsilon(n), \delta(n))$-DP with $\varepsilon(n) = O\left(\frac{(\log(n))^2}{n}\right)$ and $\delta(n) = O\left(\max\left\{n^{-(\ln(n))^{-1}}, n^{-1} (\ln(n))^{-c}\right\}\right)$, for any $c > 0$ (Theorem 5 in Section 4).

- We prove that the Morris Counter satisfies $(L(n), 0.00033)$-DP property (see Definition 2), where $L(n) = -\ln \left(1 - \frac{16}{n}\right) \approx \frac{16}{n}$ (Theorem 1 in Section 4). In Observation 1, we also show that the constant 16 cannot be improved.

- We prove that MaxGeo Counter satisfies $(\varepsilon, \delta)$-DP property (Definition 2 is provided in Section 3) if a number of events $n$ (in Section 4 a concept of event is clarified) is at least $\frac{\ln(\delta)}{\ln(1 - 2^{-l_\varepsilon})}$, where $l_\varepsilon = \left\lceil \log \left(\frac{\varepsilon^\varepsilon}{e-1}\right)\right\rceil$ (Theorem 6 in Section 4).

- We construct a privacy-preserving distributed survey protocol based on probabilistic counters in Section 5 and compare it with the Laplace method, which is considered as the actual state of the art of differentially private protocols and is not based on probabilistic counters.

The rest of this paper is organized as follows. First, in Section 2 we mention work related to our paper and some popular examples of other probabilistic counters, which are not considered in this paper.
In Section 3, we recall the differential privacy definition. In Section 4 we propose a general definition for probabilistic counter (Definition 3) and further, we recall the definitions of both Morris and MaxGeo Counters. Moreover, we state Fact 1, a useful reformulation of the standard definition of differential privacy for probabilistic counters. Section 4 presents our most important technical contribution, where we analyze how both counters behave under a differential privacy regime. For clarity, some proofs and lemmas are moved to the Appendix. In Section 5, we demonstrate how a probabilistic counter can be used for constructing a data aggregation protocol in a very particular, yet natural, scenario. We also compare these solutions with the standard Laplace method. Finally, in Section 6 we present conclusions and future work propositions.

2. Previous and Related Work

In our paper, we provide a detailed analysis of some probabilistic counters from the perspective of differential privacy. Differential privacy concepts have been discussed in many papers in recent years. One can also find a well developed body of literature devoted to probabilistic counters and similar structures. Therefore, we limit the related literature review to the most relevant papers.

**Differential Privacy literature.** In our paper, we focus on the inherent privacy guarantees of some probabilistic structures are defined as differential privacy. The idea of differential privacy has been introduced for the first time in [12]; however, its precise formulation in the widely used form appeared for the first time in [13]. There is a long list of papers concerning differential privacy, e.g. [14, 15, 16], to mention a few.

Most of these papers focus on a centralized (global) model, namely a database with a trusted party holding it. See that in our paper, despite the distributed setting, we have the same (non-local) trust model. In particular we assume an existence of a curator that is entitled to gather and see all participants’ data in the clear and release the computed data to a wider (possibly untrusted) audience. Comprehensive information concerning differential privacy can be found in [5].

**Probabilistic counters and their applications.** The idea of probabilistic counters, along with the well-known Morris Counter was presented in the seminal paper [1]. The aim was to construct a very small data structure for representing a huge set or events of some kind. In our paper, we concentrate on the Morris Counter analyzed in detail in [2]. The second structure discussed in our paper is MaxGeo Counter introduced and analyzed in [8]. More detailed and precise analysis can be found in [17]. The most important application of MaxGeo Counter can be found in [9], where the authors propose the well-known HyperLogLog algorithm. Its practical applications are widely described in [18]. There are several widely used improvements of HyperLogLog algorithm: HyperLogLog+ [18], Streaming HyperLogLog with sketches based either on historic inverse probability [19] or martingal estimator [20] or empirically adjusted HyperBitBit (proposed by R. Sedgewick [21]). The main goal of these adjustments is a reduction of the memory requirements (see e.g., [22] or [23]). For instance, some of the above solutions are used in database systems for queries’ optimization or for document
classification purposes. Moreover, MaxGeo counter was used in [24], for an adjustment of ANF tool, developed for data mining from extensive graphs, which enables it to answer many different questions based on some neighborhood function defined on the graph.

Unsurprisingly one of the main applications of the approximate counter is to compute the size of a database or its specific subset. A set of such applications can be found in [25]. In [26], the authors use Morris Counter for online, probabilistic, and space-efficient counting over streams of fixed, finite length. Authors of [27] proposed an application of a system of Morris Counters for flash memory devices. Another application, presented in [28], is a revisited version of Morris Counter designed for binary floating-point numbers. In [29], Morris Counter is used in a well-known problem of counting the frequency moments of long data streams. Authors of [30] focused on making probabilistic counters scalable and accurate in concurrent settings. Paper on probabilistic counters in hardware can be found in [31]. A slightly modified version of Morris Counter called Morris+ was recently introduced in [32] with the proof of its optimality in terms of accuracy–memory trade-off.

In random graphs theory, Morris Counter is usually connected to greedy structures. For instance, in an arrangement of a random labeled graph in Gilbert model $G(n, p)$, it is possible to construct a greedy stable set $S_n$, which size has the same distribution as Morris Counter $M_n$ of the base $a = (1 - p)^{-1}$ (see e.g., [33] or [34] for fundamentals of random graph theory).

There are many other birth processes that are quite similar to Morris Counter, which are applicable in a variety of disciplines like biology, physics, or the theory of random graphs. Short descriptions of such examples can be found in [35].

When talking about probabilistic counters, it is worth to mention about Bloom filters [36], which are space-efficient probabilistic data structures that are the representations of sets. There exists a probabilistic counter which approximates the number of elements represented by the given Bloom filter [37].

Other common examples of probabilistic counters are $F_p$ counters [38, 39], which approximate the $p$-th moments of frequencies of occurrences of different elements in the database.

One may note that all the mentioned probabilistic counters have equivalent versions which are consistent with Definition 3 introduced here.

Let us also mention a paper [40] where one can find numerous applications of similar constructions for creating pseudorandom sketches in Big Data algorithms.

Notice that the variety of possible applications of probabilistic counters creates an opportunity to exploit inherent differential privacy properties. However, a new challenge arises — to calculate the parameters of differential privacy for those counters, which are not connected straightforwardly with Morris or MaxGeo Counters.

**Probabilistic counters and preserving privacy.** Some probabilistic counters and similar structures were previously considered in terms of privacy preservation. We mention only the papers strictly related to the algorithms discussed in our paper (i.e., Morris Counter and MaxGeo).

The authors of [41] show that in the scenario of using different types of probabilistic counters for set cardinality estimation with the Adversary being able to extract the
intermediate values of the counter, the privacy is not preserved. Note that, in this paper, we perform data aggregation instead of cardinality estimation. Moreover, we assume the Adversary is not able to extract any intermediate values from the counter. That is, we consider a global model, while the result from [41] assumes the settings closer to the classic local model [5].

One of the main results of this submission is a careful and tight analysis of Morris Counter from the context of privacy-preserving that has not been provided so far. Our second contribution is an analogous analysis of the MaxGeo. There are a few very recent papers presenting privacy-preserving protocols that use Flajolet—Martin sketch as a building block. We concern the more general concept of MaxGeo counter, which is a core of Flajolet—Martin sketch or HyperLogLog sketch, however, it can be used in other arrangements as well. These papers in some cases provide an analysis of the privacy guaranteed by the Flajolet—Martin with the global model. In all the cases, the conclusion is positive in the sense that the protocol itself provides some level of differential privacy without adding extra randomness.

- In [11] authors consider, among others Flajolet—Martin sketch that can be seen as a particular application of MaxGeo counter. They introduce its differentially private version via trick (adding artificial utilities) and provide its accuracy when used to count the number of elements in multisets.
  Accidentally, a proof of basic theorem from [11] uses an incorrect argument (inappropriate utilization of Hoeffding’s inequality), so it is difficult to compare the results precisely. Nevertheless, the overlap of results between our paper and [11] is only partial.

- In [42], authors also consider Flajolet—Martin sketch as a subroutine. After a careful analysis, they show that it is asymptotically \((\varepsilon, \delta) = \text{negl}(\lambda)\)-DP (with respect to the numbers of different elements), when the number of elements counted by the mechanism is at least \(8K\lambda \max\left(\frac{1}{\varepsilon}, 1\right)\), where \(K\) is some accuracy parameter, \(\lambda\) is some security parameter and \(\text{negl}(x)\) is some negligible function of argument \(x\) (Theorem 4.2 in [42]). Nevertheless, the analysis does not explain how to choose parameters \(K\) and \(\lambda\) in order to obtain \((\varepsilon, \delta)\)-DP for a given \(\varepsilon\) and \(\delta\) parameters. Moreover, a consideration of asymptotic behavior (with respect to the number of unique elements \(n\)) is not relevant when the hash function restricts the possible result to the size bounded by its domain. Our analysis of MaxGeo counter provides exact (non-asymptotic) dependence between \(n\) and parameters \(\varepsilon\) and \(\delta\).

- In [43], authors also consider a privacy property of Flajolet—Martin sketch. Their deep analysis entails an algorithm which is able to determine a minimal number of unique elements needed to guarantee (almost) \((\varepsilon, \delta)\)-DP. The algorithm is quite complicated but very efficient in terms of time of execution. Nevertheless, the authors actually proved conditions that look quite similar to \((\varepsilon, \delta)\)-DP, but in fact, these conditions are slightly weaker ([43], Sec. 5.2.2). Again, this oversight makes it hard to compare the results from [43] with our contribution for MaxGeo counter.
Let us also mention that some other pseudorandom structures have been analyzed from the perspective of differential privacy. For example, in [44], the authors considered Bloom filters as a means for constructing privacy-preserving aggregation protocol.

3. Differential Privacy Preliminaries

In this section, we briefly recall differential privacy. For more details see, e.g. [5]. We denote the set of (positive) natural numbers by \( \mathbb{N} \) and the set of all integers by \( \mathbb{Z} \). Moreover, let \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For \( a, b \in \mathbb{Z} \) let us define a discrete interval \([a, b] \cap \mathbb{Z}\) by \([a : b]\). We assume that there exists a trusted curator who holds, or securely obtains, the data of individuals in a (possibly distributed) database \( x \). Every row of \( x \) consists of the data of some individual. By \( \mathcal{X} \), we denote the space of all possible rows. The goal is to protect the data of every single individual, even if all users except one collude with an adversary to breach the privacy of this single, uncorrupted user. On the other hand, the curator is responsible for producing a release – a possibly accurate response to a requested query. This response is then released to the public, which is allowed to perform statistical analysis on it. The differential privacy is, by design, resilient to post-processing attacks, so even if the adversary obtains the public release, he will not be able to infer anything about specific individuals participating in this release.

For simplicity, we interpret databases as their histograms in \( \mathbb{N}_0^{\lvert \mathcal{X} \rvert} \), so we may just focus on unique rows and the numbers of their occurrences.

**Definition 1 (Distance between databases).** The \( \ell_1 \) distance between two databases \( x, y \in \mathbb{N}_0^{\lvert \mathcal{X} \rvert} \) is defined as

\[
\|x - y\|_1 = \sum_{i \in \mathcal{X}} |x_i - y_i|,
\]

where \( x_i \) and \( y_i \) denote the numbers of occurrences of an item (an individual) \( i \) in the databases \( x \) and \( y \) accordingly.

One can easily see that \( \|x - y\|_1 \) measures how many records differ between \( x \) and \( y \). Moreover, \( \|x\|_1 \) measures the size of the database \( x \).

A privacy mechanism is a randomized algorithm used by the curator that takes a database as input and produces the output (the release) using randomization.

**Definition 2 (Differential Privacy (formulation from [5])).** A randomized algorithm \( \mathcal{M} \) with domain \( \mathbb{N}_0^{\lvert \mathcal{X} \rvert} \) is \( (\varepsilon, \delta) \)-differentially private (or \( (\varepsilon, \delta) \)-DP), if for all \( S \subseteq \text{Range}(\mathcal{M}) \) and for all \( x, y \in \mathbb{N}_0^{\lvert \mathcal{X} \rvert} \) such that \( \|x - y\|_1 \leq 1 \) the following condition is satisfied:

\[
P (\mathcal{M}(x) \in S) \leq \exp(\varepsilon) \cdot P (\mathcal{M}(y) \in S) + \delta,
\]

where the probability space is over the outcomes of the mechanism \( \mathcal{M} \).

When \( \delta = 0 \), \( \mathcal{M} \) is called \( (\varepsilon) \)-DP mechanism.

An intuition of \( (\varepsilon, \delta) \)-DP is as follows: if we choose two consecutive databases (that differ exactly on one record), the mechanism will likely return indistinguishable values. In other words, it preserves privacy with high probability, but it is admissible for a mechanism to be out of control with negligible probability \( \delta \).
Example 1. (Laplace noise) In the central model, a standard, widely used mechanism with \((\varepsilon)-DP\) property is the so-called Laplace noise. A variable \(X\) has Laplace distribution with parameter \(\lambda\) (denoted as \(X \sim L(\lambda)\)), if its probability density function is
\[
f(x) = \frac{1}{2\lambda} \exp \left(-\frac{|x|}{\lambda}\right).
\]
Then \(E(X) = 0\) and \(\text{Var}(X) = 2\lambda^2\). Let \(c(x)\) be the number of rows in \(x\), that satisfy a given property. Imagine that an aggregating mechanism is defined as follows: \(M(x) = c(x) + L(\varepsilon^{-1})\). Then \(M\) is \((\varepsilon)-DP\) (for more precise properties of Laplace noise, see \([5]\)).

4. Probabilistic Counters

This paper focuses on probabilistic counters, denoted further by \(M\). It is a stochastic process that can be interpreted as a mechanism defined on the space of all possible inputs.

Each increase of the data source being counted by the probabilistic counter is called an incrementation request. Due to the randomized nature of probabilistic counters, each may change the value of the counter, but not necessarily. We will further indicate the single incrementation request by ‘1’. For the sake of generality, we also assume that the counter can get as an input ‘0’, and in such case, it simply does nothing. This is useful for real-life scenarios, e.g., data aggregation (see Section 5). Obviously, only incrementation requests impact the counter’s final result; hence, we indicate the counter’s value after \(n\) incrementation requests by \(M_n\), and we are not considering the number of the rest of the rows.

The notion of the probabilistic counter is ambiguous in literature, so for the reader’s convenience, we provide a general definition of the probabilistic counter in the following way:

**Definition 3.** We call a stochastic process \(M\) a probabilistic counter if
\[
M_{n+1} = f(M_n, X(M_n))
\]
where \(M_n\) is the value of the counter after \(n\) incrementation requests, \(X(M_n)\) is a random variable, possibly dependent on \(M_n\), and \(f\) is an arbitrary, non-negative function.

Note that, using this definition, any probabilistic counter can be described with a tuple \(\{M_0, (f(.,.), X(.))\}\). To the best of the authors’ knowledge, Definition 3 embraces all mechanisms from literature, which are considered to be probabilistic counters. Note that probabilistic counters randomly increment their value. However, the non-negativity of \(f\) enables the probabilistic counter to reset its value when needed.

In Figure 1, one can see a graphical representation of the probabilistic counter. As mentioned, incrementation requests are indicated by ‘1’ and other rows by ‘0’ input. The dice depicts the randomness (namely \(X(M_n)\)) from Definition 3. The X-mark stands for no action.

We emphasize that the probabilistic counter depends on the number of incrementation requests. We want to show that if we reveal its final value, then it does not expose
any sensitive data about any single record. Moreover, note that if $x$ and $y$ differ only by one '0' input, then $P(M(x) \in S) = P(M_n \in S) = P(M(y) \in S)$, where $n$ is the number of incrementation requests both for $x$ and $y$. See that then the condition in Definition 2 is trivially fulfilled. Hence for our convenience, in this paper, we use to mark only the number $n$ of incrementation requests provided by individuals when talking about the probabilistic counter $M_n$. We present the following

\textbf{Fact 1.} Let $M$ be a probabilistic counter with a discrete $Range(M) = A$. Moreover assume that for all $n, m \geq 1$, such that $|n - m| \leq 1$, there exists such $S_n \subset A$ that for all $s \in S_n$

\begin{equation}
P(M_n = s) \leq \exp(\varepsilon) \cdot P(M_m = s)\end{equation}

and

\begin{equation}
P(M_n \notin S_n) \leq \delta.
\end{equation}

Then $M$ is $(\varepsilon, \delta)$-DP.

Note that, for our setting, Fact 1 is fully compatible with the intuition of regular differential privacy (Definition 2). Indeed, Fact 1 can be easily derived from the observation that any set $B \subset A$ is a disjoint union of $B \cap S_n$ and $B \cap S'_n$.

Remark that $\varepsilon$ and $\delta$ in Fact 1 can also be treated as functions of $n$ parameter. Thence, we can consider either differential privacy of $M_n$ variable, when $n$ is known or $(\varepsilon(n), \delta(n))$-DP of probabilistic counter $M$. The second variant enables us to provide precise dependence of privacy parameters of the counter as the number of incrementation requests gets large.

4.1. Morris Counter

We begin with a short description of Morris Counter (originally referred to as an approximate counter [1, 2]). Let us fix $a > 1$. Algorithm 1 is a very simple pseudocode of Morris Counter [1].

Roughly speaking, we start with $M = 1$. Each incoming incrementation request triggers a random event. This event increments the counter ($M \leftarrow M + 1$) with probability $a^{-M}$ ($r \sim \text{Uni}([0, 1])$ generates a number uniformly at random from the interval $[0, 1]$, using in practice some Pseudo Random Number Generator). Note that this approximate counting protocol can be easily distributed. Indeed, any entity who wants to increment the counter only has to send the request to increment it. These requests can be queued on the server and resolved one after another. A detailed description of
the approximate counting method can be found in [1, 2]. Throughout this article, we examine only a standard Morris Counter i.e., with the base $a = 2$.

**Fact 2.** Morris Counter can be represented in terms of the general counter from Definition 3 for $M_0 = 1$, $X(M_n) \sim B(2^{-M_n})$ and $f(x, y) = x + y$.

Morris Counter can also be defined recursively in the following way

**Definition 4.** Morris Counter is a Markov process $(M_n, n \in \mathbb{N}_0)$ which satisfies:

\[
\begin{align*}
\mathbb{P}(M_0 = 1) &= 1, \\
\mathbb{P}(M_{n+1} = l | M_n = l) &= 1 - 2^{-l}, \\
\mathbb{P}(M_{n+1} = l + 1 | M_n = l) &= 2^{-l},
\end{align*}
\]

for any $l \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Note that Definition 4 can be derived directly from a run of Algorithm 1. From now on let $\mathbb{P}(M_n = l) = p_{n,l}$. Directly from Definition 4 we get the following recursion:

\[
p_{n+1,l} = (1 - 2^{-l})p_{n,l} + 2^{-l+1}p_{n,l-1}
\]

for $l \geq 2$ and $p_{n,0} = 0$ for $n \in \mathbb{N}_0$.

**Accuracy versus Differential Privacy.** The accuracy of Morris Counter has been thoroughly analyzed in various classical papers. The first detailed analysis was proposed by Ph. Flajolet in [2]. Here we present the essence of theorems presented in this paper, which will be useful later on:

**Fact 3.** Let $M_n$ denote Morris Counter after $n$ successive incrementation requests. Then this random variable has an expected value $E(M_n) \approx \log(n) - 0.273225$ (in this paper log states for binary logarithm) and a variance $\text{Var}(M_n) \approx 0.763177$.

Realize that Fact 3 guarantees good accuracy of Morris Counter and also a high concentration of $M_n$ around its average — a characteristic desirable in order to satisfy differential privacy definition. Fact 3 justifies a definition of moving discrete intervals:

\[
I_n = \left[ \max\{1, \lceil \log(n) \rceil - 4\} : \min\{n + 1, \lceil \log(n) \rceil + 4\} \right]
\]

which will emerge as a crucial point of our further considerations of this Markov process in terms of differential privacy in this section.

```plaintext
1 M ← 1;
2 while receive request do
3     generate r ∼ Unif([0, 1]);
4     if r < a^{-M} then
5         M ← M + 1;

Algorithm 1: Morris Counter Algorithm
```
The lion’s share of applications of Morris Counter is based on counting a number of occurrences, that is, the number of incrementation requests. In order to estimate this value, we may use (3) and simply obtain
\[ E(2^{M_n}) = n + 2. \] (5)
Hence \( 2^{M_n} - 2 \) is an unbiased estimator of the number of increments \( n \). Remark that \( n \) can be saved in \( \lceil \log(n) \rceil \) bits. On the other hand, Fact 3 shows that on average, \( \log(\log(n)) + O(1) \) bits are required to store \( M_n \). As announced earlier, this is the crucial advantage of Morris Counter. Moreover, analogically to (5) we may obtain
\[ \text{Var}(2^{M_n} - 2) = \frac{n(n + 1)}{2}. \] (6)
Formulas (5) and (6) will be used in the analysis of data aggregation example in Section 5.

### 4.2. MaxGeo Counter

We begin with a short description of MaxGeo Counter. Algorithm 2 shows its pseudocode. Speaking informally, for each incrementation request, the server has to generate a random variable from the geometric distribution \( \text{Geo}(1/2) \) (ranged in \( \mathbb{N} \)). The final result is the maximum taken over all these generated random variables.

```plaintext
1 C ← [1];
2 while receive request do
3     generate r ∼ Geo(1/2);
4     add r to C;
5 return max(C);
```

**Algorithm 2: MaxGeo Counter Algorithm**

**Fact 4.** *MaxGeo Counter can be represented by the general counter from Definition 3 for \( M_0 = 1 \), \( X(M_n) \sim \text{Geo}(1/2) \) and \( f(x, y) = \max(x, y) \).*

The expectation and variance of the maximum of \( n \) i.i.d. geometric variables have already been analyzed in the literature. For instance, Szpankowski and Rego [8] provided exact formulas for such variables’ expected value and variance. However, they are impractical for applications for large \( n \). Hence they also provided asymptotics (here, for a maximum of \( n \) independent \( \text{Geo}(1/2) \) distributions): \( E(M_n) = \log(n) + O(1) \) and \( \text{Var}(M_n) ≈ \frac{\pi^2}{6 \ln(2)^2} + \frac{1}{12} = 3.507048 \ldots \) and thus, similarly to Morris Counter, there is only \( \log(\log(n)) + O(1) \) bits required on average to save MaxGeo Counter after \( n \) incrementation requests.

However, for the first time, the MaxGeo Counter was used as an aggregating algorithm by Flajolet and Martin [25]. They have provided that \( E(2^{M_n}) ≈ \varphi n \), where the “magic Flajolet—Martin constant” (the name according to [45]) is defined as follows:
\[ \varphi = \frac{\exp(\gamma)}{\sqrt{2}} \cdot \frac{2}{3} \prod_{n=1}^{\infty} \left( \frac{(4n + 1)(4n + 2)}{4n(4n + 3)} \right)^{\epsilon_n} = 0.77351 \ldots , \] (7)
where $\gamma = 0.57721\ldots$ is Euler—Mascheroni constant and $\epsilon_n$ is $\{-1, 1\}$-Morse—Thue sequence ($\epsilon_n = (-1)^{\nu(n)}$, where $\nu(n)$ is the number of occurrences of digit '1' in the binary representation of number $n \in \mathbb{N}_0$).

### 4.3. Probabilistic Counting with Stochastic Averaging

Here we recall shortly more general Probabilistic Counting with Stochastic Averaging algorithm from [25]. Let $m$ be of the form $2^k$ for some $k \in \mathbb{N}$. Assume that there are $m$, initially empty lots. For each incrementation request, we connect it with one of the groups uniformly at random. Finally, we perform Algorithm 2 separately and independently for each lot, obtaining MaxGeo Counters $M[1], M[2], \ldots, M[m]$. By $\sigma_n(m)$, we denote a sum of these $m$ MaxGeo Counters after the total number of incrementation requests $n$. Let us introduce the estimator:

$$\Xi_n(m) = \left\lfloor \frac{m \cdot 2^{\sigma_n(m)}}{\varphi} \right\rfloor.$$

Then (according to [25]), for any $m = 2^k$, $k \in \mathbb{N}$,

$$E(\Xi_n(m)) = n \left( 1 + \frac{0.31}{m} + \psi_1(m, n) + o(1) \right), \text{ with } |\psi_1(m, n)| \leq 10^{-5}$$

and

$$Var(\Xi_n(m)) = n^2 \left( \frac{0.61}{m} + \psi_2(m, n) + o(1) \right), \text{ with } |\psi_2(m, n)| \leq 10^{-5}.$$

Note that averaging reduces the variance of the probabilistic counter. Remark that “Stochastic Averaging” in PCSA algorithm refers to the random choice of the number of entities in each group, and it slightly differs from the standard averaging solution via the Monte Carlo method with groups of equal size.

### 4.4. HyperLogLog

The maximum of geometric variables is also used as a primitive in well known HyperLogLog algorithm (see [9]). Therefore its privacy properties are important both from the theoretical and practical point of view. Essentially, in HyperLogLog we perform the same Stochastic Averaging as in the PCSA algorithm, but the final estimation is different:

$$\text{HyperLogLog}_n := \alpha_k m^2 \left( \sum_{j=1}^{m} 2^{-M_n[j]} \right)^{-1},$$

where $\alpha_k$ is a constant dependent only on $k$ (see [9] for more details). It is worth noting that HyperLogLog related algorithms (mentioned in Section 1) are the best-known procedures designated for cardinality estimation, and they are close to optimum [46]. According to [9], for $m = 2^k$, where $k \geq 4$,

$$E(\text{HyperLogLog}_n) = n(1 + \psi_3(n) + o(1)), \text{ with } |\psi_3(n)| < 5 \cdot 10^{-5}$$
and
\[
\text{Var}(\text{HyperLogLog}_n) = n^2 \left( \frac{\beta_m}{\sqrt{m}} + \psi_4(n) + o(1) \right)^2, \text{ with } |\psi_4(n)| < 5 \cdot 10^{-4},
\]
where \( \beta_m \xrightarrow{m \to \infty} \sqrt{2 \log(2) - 1} = 1.03896 \ldots \) and \( \beta_m \leq 1.106 \) for \( m \geq 16 \).

4.5. Morris Counter Privacy

In this subsection we investigate Morris Counter in terms of \((\varepsilon, \delta)\text{-DP}\) in order to obtain the following

**Theorem 1.** Let \( M \) denote the Morris Counter and assume \( |n - m| \leq 1 \). Then
\[
\mathbb{P}(M_n = l) \leq \left( 1 - \frac{16}{n} \right)^{-1} \cdot \mathbb{P}(M_m = l) + \delta,
\]
where \( \delta < 0.00033 \), so \( M \) is \((L(n), 0.00033)\text{-DP}\) with
\[
L(n) = -\ln \left( 1 - \frac{16}{n} \right) = \frac{16}{n} + \frac{128}{n^2} + O(n^{-3}) \leq \frac{16}{n - 8}.
\]

The proof is complicated and very technical. In order to better understand it, we are going to provide a presentation of a plan and main ideas beneath the parts of the proof.

**Roadmap of the proof.** We can divide the proof of Theorem 1 into five phases (the main results of the phases are given in brackets):

1. \( \delta \) phase (Theorem 3),
2. relations between ”special” sequences \( (P^{(c)}_k)_k \) with respect to \( c \) (Claim 1),
3. dependencies between consecutive distributions (of \( M_n \) and \( M_{n+1} \)) (Claim 2),
4. extrapolation of \( P^{(c)}_k \leq 2^{c+3} P^{(c+1)}_k \) property to \( N > n_k \) (Lemma 3),
5. \( \varepsilon \) phase (Theorem 4).

During the first phase, we consider a concentration of Morris Counter in the vicinity of its mean value. More precisely, we show that Morris Counter after \( n \) incrementation requests takes values in relatively small intervals \( I_n \) with probability \( 1 - \delta \) (note that then \( M_n \) satisfies the condition (2) for \( S_n = I_n \)), where \( I_n \) is defined as in (4) and \( \delta \) is some small constant, which arises from the proof. Note that \( I_n \) may be interpreted as confidence intervals at level \( 1 - \delta \) (see e.g. [47]). This phase is divided into lemmas 1 and 2. The first one uniformly bounds the formula for probabilities given by beneath Theorem, provided by Flajolet:
Theorem 2 (Proposition 1 from [2]). The probability $p_{n,l}$ that the Morris Counter has value $l$ after $n$ incrementation requests is

$$p_{n,l} = \sum_{j=0}^{l-1} (-1)^j 2^{-j(j-1)/2} \left(1 - 2^{-(l-j)}\right)^n \prod_{i=1}^j \left(1 - 2^{-i}\right)^{-1} \prod_{i=1}^{l-1-j} \left(1 - 2^{-i}\right)^{-1}.$$ 

Further, Lemma 1 sum up the bounds to obtain small upper bound for $\delta_1 := \mathbb{P}(M_n \leq \lceil \log(n) \rceil - 5)$. The same bounds cannot be efficiently utilized in the proof of Lemma 2. Instead, it couples $M_n$ with a process $X_n$, which increases during first $\lceil \log(n) \rceil + 1$ steps, and then follows the same rule of update, so $M_n \leq X_n$ almost surely. Therefore $\delta_2 := \mathbb{P}(M_n \geq \lceil \log(n) \rceil + 5) \geq \mathbb{P}(X_n \geq \lceil \log(n) \rceil + 5)$, which is much easier to bound from definition. Let us note that such coupling cannot be used in the proof of Lemma 1. The first phase is summarized by Theorem 3, i.e., establishes $\delta = \delta_1 + \delta_2$.

In the second phase, Lemma 6 provides that $(P_k^{(4)})$ is descending for big enough $k$ and $(P_k^{(5)})$ is ascending for big enough $k$. Let us mention that together with other results from Appendix A, one can provide these monotonicity properties for any $c \leq 4$ and any $c \geq 5$, respectively (however, it is not needed in this proof). This interesting behavior of "special" sequences is the main issue, which hindered all our attempts to prove Theorem 1. The proof of Lemma 6 utilizes purely analytical lemmas 4 and 5, which both have very technical proofs. The main idea beneath the proof of Lemma 6 is to calculate the differences between consecutive elements of the considered "special" sequences by representing them as the sums via application of Flajolet’s Theorem 2 and realizing that usually at most first ten terms of the sums are crucial (on the other hand, taking less than eight terms is rarely sufficient). This is the second issue, why the proof is so complicated. Let us notice that Theorem 2 presents an explicit formula for $\mathbb{P}(M_n = l)$, which (as we may experience in Appendix A) is not convenient to analyze. However, it is simple enough to find the values numerically (also note that recursive Definition 4 provides those probabilities easily as well. However, this approach is inefficient in terms of memory and time for a big number of requests $n$). Therefore, by precise analysis, we can finally check some sums numerically and obtain the thesis of Lemma 6 for $k \geq 15$. However, numerically, one can extrapolate it to some smaller $k$ as well (remark that this proof does not work for small values of $k$, since not only ten terms of the aforementioned sums are important). Lemma 6 can be directly used to show Claim 1 i.e. $P_k^{(4)} \leq 2^{7} P_k^{(5)}$ for $k \geq 7$ (in fact, this result is not true for $k < 7$).

The main subject of the third phase are lemmas 7 and 8. An intuition beneath them can be formulated as follows: if ratios of consecutive probabilities of distribution of $M_n$ increase almost exponentially, then the ratios of distribution of $M_{n+1}$ increase similarly (but slightly slower). The proofs utilize mainly the definition of the Morris counter. Then Claim 2 (a simple conclusion from the two mentioned lemmas) summarizes the main result of this phase.

The fourth phase gives Lemma 3, which shows that if $p_{n_k,k+c} \leq 2^{c+3} p_{n_k,k+c+1}$, for $c \in [-k : 4]$, then the same is true if we substitute $n_k$ with a bigger number (i.e. this property is increasing with respect to $n$ parameter). In order to apply this Lemma, we have to satisfy some starting conditions. We have numerically checked the appropriate
condition for \( k = 7 \) (Table 1). Therefore, the second phase of the proof of Theorem 1 justifies the assumptions of Lemma 3 with respect to \( k \) parameter, as long as \( k \geq 7 \) and the third phase let us obtain the appropriate assumptions with respect to \( c \). One can check that for \( k < 7 \), an analogous assumption is not true.

The latter phase starts with the application of Lemma 3. We attain that \( \varepsilon(n) \) (given by the formula (10)) is at most \( L(n) \) (provided in the formulation of Theorem 1), for \( k \geq 7 \) (see Theorem 4). The last piece of this puzzle is justified by a numerical evaluation for \( k < 7 \) (presented in Figure 2), what ends the \( \varepsilon \) phase, and so the whole proof.

\( \delta \) phase. Let us commence with a reminder. First of all, \( M_n \) is ranged in \( \mathbb{N}_0 \) and moreover \( I_n \subset \lceil \log(n) \rceil - 4 : \lceil \log(n) \rceil + 4 \]. We provide few facts about the concentration of the distribution of random variable \( M_n \), or more precisely about the probability that \( M_n \) will be outside the interval \( I_n \):

**Lemma 1.** Let \( M_n \) be the state of the Morris Counter after \( n \) incrementation requests. Then

\[
\delta_1 := \mathbb{P}(M_n \leq \lceil \log(n) \rceil - 5) \leq 0.000006515315 \ldots .
\]

An increasing sequence \( \prod_{i=1}^{k} (1 - 2^{-i})^{-1} \) that arose in Theorem 2 will be indicated by \( r_k \) and we denote its limit \( \prod_{i=1}^{\infty} (1 - 2^{-i})^{-1} = 3.46274 \ldots \) by \( R \).

**Proof.** At first, we want to bound a lower tail of the distribution \( \delta_1 \). Here we would like to find a sufficient upper limitation for the above probability. Assume that \( l \leq \lceil \log(n) \rceil - 5 \). Consider the probability that \( M_n \) has value \( l \):

\[
p_{n,l} \leq 2^{\frac{l-1}{2}} \sum_{j=0}^{\frac{l-1}{2}} 2^{-j(-j)} (1 - 2^{-(l-j)})^n r_j r_{l-1-j} \leq R^2 \left( 1 - 2^{-l} \right)^n \sum_{j=0}^{\frac{l-1}{2}} \sqrt{2}^{-j+1} \leq R^2 \frac{2}{\sqrt{2} - 1} \exp(-n2^{-l}) = R^2(2\sqrt{2} + 2) \exp(-n2^{-l}).
\]

Remark that the above restraint is useless when \( l \geq \log(n) - 2 \), so it cannot be employed to obtain a reasonable bound for an symmetrical upper tail. However, the aforementioned formula will help us to limit the left tail of the distribution of \( M_n \):

\[
\delta_1 = \sum_{l=1}^{\lceil \log(n) \rceil - 5} \mathbb{P}(M_n = l) \leq R^2(2\sqrt{2} + 2) \sum_{l=1}^{\lceil \log(n) \rceil - 5} \exp(-n2^{-l}) \leq R^2(2\sqrt{2} + 2) \sum_{k=4}^{\infty} \exp(-2^k) \leq R^2(2\sqrt{2} + 2) \sum_{k=1}^{\infty} \exp(-16k) = R^2(2\sqrt{2} + 2) \frac{\exp(-16)}{1 - \exp(-16)} = 0.000006515315 \ldots .
\]

\( \square \)
Lemma 2. Let $M_n$ be the state of the Morris Counter after $n$ incrementation requests. Then
\[ \delta_2 := \mathbb{P}(M_n \geq \lceil \log(n) \rceil + 5) \leq 0.000325521 \ldots . \]

Proof. Actual goal is to limit the upper tail, that is $\mathbb{P}(M_n \geq \lceil \log(n) \rceil + 5)$. Consider a process $X = (X_k, k \in [0 : n])$. Let $X$ initially follow the incrementation rule $\mathbb{P}(X_k = k + 1) = 1$ for $k \in [0 : \lceil \log n \rceil + 1]$. Afterwards, let this Markov chain imitate the transition rule of Morris Counter, that is
\[ \mathbb{P}(X_{k+1} = m + 1|X_k = m) = \frac{1}{2m} = 1 - \mathbb{P}(X_{k+1} = m|X_k = m) \]
for $k \geq \lceil \log(n) \rceil + 1$. Naturally, for $k \leq \lceil \log(n) \rceil + 1$, we have $X_k \geq M_k$, so we may couple realizations of these two processes in such a way that whenever $X$ is incremented, then so is $M$, and if $M$ does not change, then $X$ does not increase as well (note that $X$ has at most the same probability of a positive incrementation as $M$ at any point of time).

To abbreviate the expressions let us denote $m = n - \lceil \log(n) \rceil - 1$ and
\[ \mu_i = \mathbb{P}(X_{k+1} = \lceil \log(n) \rceil + i + 1|X_k = \lceil \log(n) \rceil + i) = \frac{1}{2^{\lceil \log(n) \rceil + i} + 1} = 1 - \nu_i, \]
for any $i \in \mathbb{Z}$. Moreover, let us define a three-dimensional discrete simplex:
\[ S_k^{(3)} = \{ \bar{l} = (l_1, l_2, l_3) \in \mathbb{N}^3_0 : l_1 + l_2 + l_3 \leq k \} . \]

The coupling encountered above, ensures us that
\[ \delta_2 \leq \mathbb{P}(X_n \geq \lceil \log(n) \rceil + 5) = \sum_{\bar{l} \in S_k^{(3)}} \nu_1^{l_1} \nu_2^{l_2} \nu_3^{l_3} \nu_4^{l_4} \leq \sum_{\bar{l} \in S_k^{(3)}} \frac{1}{2^{3\lceil \log(n) \rceil + 9}} \]
\[ = \sum_{k=0}^{m-3} \binom{k + 3}{2} \frac{1}{2^{3\lceil \log(n) \rceil + 9}} \leq \frac{1}{210\mu^3} \sum_{k=3}^{m} k^2 - k . \]

Realize that $\sum_{k=3}^{m} \frac{m}{k} = (m - 2)(m + 3)/2$ and $\sum_{k=3}^{m} k^2 = (m - 2)(2m^2 + 7m + 15)/6$, so
\[ \delta_2 \leq \frac{1}{210\mu^3} \frac{1}{6} (m - 2)(2m^2 + 4m + 6) = \frac{1}{3 \cdot 210\mu^3} (m^3 - m - 6) \]
\[ \leq \frac{m^3}{3 \cdot 210\mu^3} \leq \frac{1}{3 \cdot 210} = 0.000325521 \ldots . \]

Note that when $m < 3$ (that is, when $n < 7$), then the above sums are empty, but on the other hand $\lceil \log(n) \rceil + 5 > n + 1$, so the inequality is trivially true. \hfill \Box

Theorem 3. The state of the Morris Counter after $n$ incrementation requests is not in the set
\[ I_n = [\max\{1, \lceil \log(n) \rceil - 4\} : \min\{n + 1, \lceil \log(n) \rceil + 4\}] \]
with probability $\delta < 0.00033$. 

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Realize that $P(M_n \in [1 : n+1]) = 1$. This observation, together with lemmas 1 and 2 bears a conclusion that

$$\delta := P(M_n \notin I_n) = \delta_1 + \delta_2 < 0.00033.$$ 

The second phase. In this part of the investigation, we try to establish $\varepsilon(n)$ parameter of DP of $M_n$. In fact it remains to examine the property (1) in the interval $I_n$, as Theorem 3 entails (2) for $S_n = I_n$. Therefore, we are interested in finding the upper bound for maximal privacy loss for any $n \in \mathbb{N}$ and $k \in I_n$, namely:

$$\varepsilon(n) = \max \left\{ \left| \ln \left( \frac{p_{n+1,k}}{p_{n,k}} \right) \right| : k \in I_n \right\}. \quad (10)$$

Actually, we may consider $'+'$ sign instead of $'\pm'$ in (10), because $|\ln(x)| = |\ln(1/x)|$. However, when $I_n \neq I_n \pm 1$, we have to behave carefully, so in particular, an additional check of privacy loss with $'-'$ sign is needed when $n$ is of a form $2^l+1$ for some $l \in \mathbb{N}$.

Claim 1. For $k \geq 7$, we have $p_{2k+1,k+4} \leq 2^7 p_{2k+1,k+5}$.

The above claim is the result of a simple application of Lemma 6 from Appendix A.

The third phase.

Claim 2. If for any given $n$, there exists an ascending and positive sequence $(\alpha_i)_{i=1}^n$ such that $(\forall i \in [1 : n]) p_{n,i} = 2^i \alpha_i p_{n,i+1}$, then there also exists an ascending and positive sequence $(\alpha_i')_{i=1}^{n+1}$ such that

$$(\forall i \in [1 : n+1]) (p_{n+1,i} = 2^i \alpha_i' p_{n+1,i+1}) \land (\forall i \in [1 : n]) (\alpha_i' < \alpha_i).$$

This claim emerges from lemmas 7 and 8 from Appendix A.

The fourth phase. We use Claim 2 to guarantee starting conditions for the next Lemma 3. However, in order to apply Lemma 3 we will use Claim 1 as well, which assumes that $n \geq 2^7+1$. Hence we would like to gather some information about the distribution of $M_{2^7+1}$. More precisely, we are interested in the behaviour of $\theta_i = p_{129,i}/p_{129,i+1}$ for $i \leq 11$, which we present in the Table 1. We briefly see a superexponential trend of proportions $\theta_i$, so a possibility to use Claim 2 for $n \geq 2^7+1$ is justified. It might seem that the choice of $n$ is arbitrary, but it occurs that the distribution of $M_{2^7+1}$ does not have satisfy the necessary assumptions for the privacy loss although $M_{2^7+1}$ still can fulfill the property of $(\varepsilon(n), \delta)$-DP with parameters given in Theorem 1.

Lemma 3. Let $k \in \mathbb{N} \setminus \{0\}$ and $n_k = 2^k + 1$. If $p_{n_k,k+c} \leq 2^{c+3} p_{n_k,k+c+1}$ for every $c$ in the interval $[-k : 4]$, then

$$(\forall N \geq n_k)(\forall c \in [-k : 4]) p_{N,k+c} < 2^{c+3} p_{N,k+c+1}.$$
Table 1: Ratios of adjacent probabilities of the distribution of $M_{2^{n+1}},$ compared with the exponential function of base $2.$

\[ \begin{array}{|c|c|c|c|} 
\hline
i & \theta_i & 2^{i-4} \theta_i & 2^{4-i} \theta_i \\
\hline
1 & 9.6205 \ldots \times 10^{-24} & 0.125 & 7.6964 \ldots \times 10^{-25} \\
2 & 1.73351 \ldots \times 10^{-29} & 0.25 & 6.93402 \ldots \times 10^{-29} \\
3 & 0.000119359 \ldots & 0.5 & 0.00023871 \ldots \\
4 & 0.0140238 \ldots & 1 & 0.0140238 \ldots \\
5 & 0.158163 \ldots & 2 & 0.0790814 \ldots \\
6 & 0.771817 \ldots & 4 & 0.192954 \ldots \\
7 & 2.07702 \ldots & 8 & 0.334628 \ldots \\
8 & 7.83367 \ldots & 16 & 0.489604 \ldots \\
9 & 20.8095 \ldots & 32 & 0.650297 \ldots \\
10 & 52.0472 \ldots & 64 & 0.813238 \ldots \\
11 & 125.065 \ldots & 128 & 0.977073 \ldots \\
\hline
\end{array} \]

PROOF. Realize that for $c = -k,$ the required inequality is trivial. Therefore we can safely consider only $c \in [-k + 1 : 4].$ We would like to prove it inductively with respect to $c$ and $N.$ Assume that for some $N \geq n_k$ and any $d \in \{c - 1, c\}$ we have $p_{N,k+d} < 2^{d+3} p_{N,k+d+1}.$ Then also

\[
p_{N+1,k+c} = p_{N,k+c} \left( 1 - 2^{-k-c} \right) + p_{N,k+c-1} 2^{-k-c+1} \\
\leq 2^{3+c} p_{N,k+c+1} \left( 1 - 2^{-k-c} \right) + 2^{3+(c-1)} p_{N,k+c} 2^{-k-c+1} \\
< 2^{3+c} \left( p_{N,k+c+1} \left( 1 - 2^{-k+c-1} \right) + 2^{-k-c} p_{N,k+c} \right) = 2^{3+c} p_{N+1,k+c+1}.
\]

If we start with $c = -k + 1,$ then the above let us prove inductively the appropriate condition for all $N \geq n_k.$ Further the thesis is followed by the induction with respect to $c.$ \[\square\]

ε phase. Claims 1 and 2, together with Table 1 enable us to apply Lemma 3 for $n = 2^k + 1$ for any $k \geq 7.$

**Theorem 4.** Let $n > 2^7 = 128$ and $k \in I_n.$ Then

\[
1 - \frac{16}{n} \leq \frac{p_{n,k+1}}{p_{n,k}} \leq 1 + \frac{16}{n}.
\]

PROOF. According to the previous discussion about the formula (10), we examine:

\[
\frac{p_{n+1,k}}{p_{n,k}} = \frac{p_{n,k} \left( 1 - 2^{-k} \right) + 2^{-k+1} p_{n,k-1}}{p_{n,k}} = 1 + 2^{-k} \left( -1 + 2^{-k+1} \frac{p_{n,k-1}}{p_{n,k}} \right).
\]

Let us denote $l = \lceil \log(n) \rceil$ and $c = k - l \in [-4 : 4].$ Then Lemma 3 bears $p_{n,k-1} \leq 2^{c+3} p_{n,k},$ so

\[
\frac{p_{n+1,k}}{p_{n,k}} \leq 1 + 2^{-l-c} (1 + 2^{c+4}) < 1 + 2^{-l+4} = 1 + \frac{16}{2^\lceil \log(n) \rceil} < 1 + \frac{16}{n}.
\]
Realize that if $n = 2^{l-1} + 1$ for some $l \in \mathbb{N}$, then a little adjustment is necessary. Indeed, let now $c - 1 = k - l \in [-4 : 4]$, and once again, Lemma 3 provides $p_{n-1,k-1} < 2^{c+2}p_{n-1,k}$. However, it still holds that:

$$\frac{p_{n,k}}{p_{n-1,k}} = 1 + 2^{-k} \left( -1 + 2^{p_{n-1,k-1}} \right) \leq 1 + 2^{-l-c+1} \left( -1 + 2^{c+3} \right) < 1 + \frac{16}{n}.$$ 

On the other hand, we obviously have inequalities $p_{n+1,k} > (1 - 2^{-l-c})p_{n,k}$ and $p_{n,k} > (1 - 2^{-l-c})p_{n-1,k}$ for any $c \in [-4 : 4]$, so both of these fractions exceed $1 - 16/n$.

Theorem 4 only provides $\varepsilon(n) \leq -\ln \left( 1 - \frac{16}{n} \right)$ for $n > 128$ (compare with (10)). However, in the Figure 2 we may briefly see that the above inequality is true for smaller number of requests $n$ as well.

Having all the technical lemmas, we are now ready to prove Theorem 1.

**Proof.** Suppose that $S_n = I_n$ in Fact 1. Then from theorems 3 and 4 we can easily see that $\mathbb{P}(M_n \notin S_n) < 0.00033$ and

$$(\forall m \in \{n - 1, n + 1\}) \ (\forall l \in S_n) \ \mathbb{P}(M_n = l) \leq \left( 1 - \frac{16}{n} \right)^{-1} \cdot \mathbb{P}(M_m = l),$$

hence from Fact 1 we obtain the main result. \qed

![Figure 2](image.png)

**Figure 2:** Exact values of $\varepsilon(n)$ parameter for $n \leq 160$ compared with plots of sequences $-\ln(1 - 16/n)$ and $-\ln(1 - 8/n)$.

In the Figure 2 one can see that values of $\varepsilon(n)$ are strictly between sequences $-\ln(1 - 8/n)$ and $-\ln(1 - 16/n)$ for $n \in [17 : 160]$. We can also observe
that $\varepsilon(n) \approx 2^{4-[\log(n)]}$ in this interval. Note that $[\log(n)] \leq 4$ for $n \leq 16$, so $[\log(n)] - 4 < 1$, but $M$ is always positive. This can justify the chaotic behavior of the process for $n \leq 16$. Nevertheless, Figure 2 affirms the quality of $\varepsilon(n)$ parameter established in Theorem 1. Moreover, we present the following

Observation 1. The constant 16 in Theorem 1 cannot be improved. See that

$$\frac{p_{34,1}}{p_{32,1}} = \frac{1}{2} = 1 - \frac{16}{32}.$$  

4.6. General result on Morris’ Counter privacy

In this part we show that the Morris’ Counter guarantees privacy with both parameters tending fast to zero. The analysis is based on the observations from the previous case. However, instead of $I_n$, let us now consider intervals

$$ J_n(c) = \left[ \max\{1, [\log(n)] - [c \log(\log(n))]\} : \min\{n+1, [\log(n)] + [c \log(\log(n))]\} \right], $$

where $c$ is some positive constant such that $[c \log(\log(n))] \geq 1$, for big enough $n$.

We claim the foregoing:

Theorem 5. Let $M$ denote the Morris Counter. If $c > 0$ satisfies $[c \log(\log(n))] \geq 1$, then $M$ is $(\varepsilon(n), \delta(n))$-DP with parameters $\varepsilon(n) = O\left(\frac{[\log(n)]^2}{n}\right)$ and $\delta(n) = O\left(\max\{n^{-\varepsilon^{-1}}, \frac{1}{n^2}\}\right)$.

Remark that $[c \log(\log(n))] \geq 1$ can be always guaranteed, when the $n$ is big enough.

Proof: For our convenience, let us denote $\rho := [c \log(\log(n))]$. We assume that $\rho \geq 1$. First we show that $\delta_1^* := \mathbb{P}(M_n \leq [\log(n)] - \rho - 1) = O\left(n^{-\varepsilon^{-1}}\right)$. The proof is analogous to the one of Lemma 1 (we omit the similar parts). Indeed,

$$ \delta_1^* = \sum_{l_1=1}^{[\log(n)] - \rho - 1} \mathbb{P}(M_n = l_1) \leq R^2(2\sqrt{2} + 2) \sum_{k=\rho}^{\infty} \exp(-2^k) \leq R^2(2\sqrt{2} + 2) \sum_{k=1}^{\infty} \exp(-2^k) = R^2(2\sqrt{2} + 2) \frac{\exp(-(\ln(n))^{\varepsilon^{-1}})}{1 - \exp(-(\ln(n))^{\varepsilon^{-1}})} = O\left(n^{-\varepsilon^{-1}}\right). $$

Now, we are going to prove $\delta_2^* := \mathbb{P}(M_n \geq [\log(n)] + \rho + 1) = O\left(n^{-1} (\ln(n))^{-c}\right)$. We use similar notation and technique as in proof of Lemma 2, but this time we utilize $(\rho - 1)$-dimensional discrete simplex:

$$ S^{(\rho-1)}_k = \left\{ \bar{l} = (l_1, l_2, \ldots, l_{\rho-1}) \in \mathbb{N}_0^{\rho-1} : \sum_{i=1}^{\rho-1} l_i = k \right\}. $$
We couple \((M_n)_n\) with the same process \((X_n)_n\) as in Lemma 2. Roughly speaking, \(X_0 = 1\) and \(X_n\) almost always increments by 1 for until \(n = \lfloor \log(n) \rfloor + 1\) and further it follows the same rule of incrementation as Morris counter. Then

\[
\phi_2^* \leq \Pr(X_n \geq \lfloor \log(n) \rfloor + \rho + 1) = \sum_{i \in S^{(\rho-1)}_{m-n+\rho+1}} \prod_{i=1}^{\rho-1} \nu_i^{l_i} \mu_i \leq \sum_{i \in S^{(\rho-1)}_{m-n+\rho+1}} \prod_{i=1}^{\rho-1} \mu_i
\]

\[
= \sum_{i \in S^{(\rho-1)}_{m-n+\rho+1}} 2^{-[(\rho-1)\lfloor \log(n) \rfloor + \sum_{i=2}^\rho i]}
\]

\[
= \sum_{k=0}^{m-\rho+1} \binom{k + \rho - 1}{\rho} 2^{-[(\rho-1)\lfloor \log(n) \rfloor + \frac{(\rho+2)(\rho-1)}{2}]}
\]

\[
\leq m^{\rho-2} n^{1-\rho} 2^{-\frac{\rho^2+\rho-2}{2}} \leq n^{-1} 2^{-\rho+1} = O(n^{-1} (\log(n))^{-c})
\]

Therefore \(\Pr(M_n \notin J_n(c)) = \phi_1^* + \phi_2^* = O\left(\max\left\{n^{-(\log(n))^{c-1}}, n^{-1} (\log(n))^{-c}\right\}\right)

Further we would like to consider fractions \(\frac{p_{n+1,k}}{p_{n,k}}\) for \(k \in J_n(c)\) as in the proof of Theorem 4. Indeed

\[
1 - 2^{-k} \leq \frac{p_{n+1,k}}{p_{n,k}} = 1 - 2^{-k} + 2^{-k+1} \frac{p_{n,k-1}}{p_{n,k}}
\]

We are going to use another formula from [2]. For any \(n \in \mathbb{N}\) and \(k \in [1 : n + 1]\),

\[
p_{n,k} = 2^{-\frac{k(k-1)}{2}} \sum_{l \in S^{(k)}_{n-k+1}} \prod_{i=1}^{k} (1 - 2^{-i})^{l_i}
\]

Let us denote the above sum by \(\varsigma_k(n - k + 1)\). Let us note that

\[
2^{-k+1} \frac{p_{n,k-1}}{p_{n,k}} = 2^{-k+1} \frac{2^{-\frac{k(2-k)}{2}} \varsigma_{k-1}(n - k + 2)}{2^{-\frac{k(k-1)}{2}} \varsigma_k(n - k + 1)} = \frac{\varsigma_{k-1}(n - k + 2)}{\varsigma_k(n - k + 1)}
\]

Realize that \(\varsigma_{k-1}(n - k + 2) \leq \varsigma_{k-1}(n - k + 1) \sum_{i=1}^{k-1} (1 - 2^{-i}) = \varsigma_{k-1}(n - k + 1) \frac{(k - 2 + 2^{k-1})}{(k-1)}\). This follows from the fact that each summand of \(\varsigma_{k-1}(n - k + 2)\) can be obtained from some summands of \(\varsigma_{k-1}(n - k + 1)\) by multiplication by one of the terms \((1 - 2^{-i})\). Moreover, note that \(\varsigma_k(n - k + 1)\) has \((n-k+1+k-1)\) summands and similarly, \(\varsigma_{k-1}(n - k + 1)\) has \((n-1)\) summands. One can briefly see that a function \(f(i) = (1 - 2^{-i})\) is increasing, hence

\[
\varsigma_k(n - k + 1) \geq \sum_{l \in S^{(k)}_{n-k+1}} (1 - 2^{-k+1})^{l_{k-1}+l_k} \prod_{i=1}^{k-2} (1 - 2^{-i})^{l_i}
\]

\[
= \sum_{l \in S^{(k-1)}_{n-k+1}} (l_{k-1} + 1) \prod_{i=1}^{k-1} (1 - 2^{-i})^{l_i}
\]
Due to the monotonicity of $f$, one can use cascading substitutions: some of $f(k)$ by $f(k - 1)$, then some of $f(k - 1)$ by $f(k - 2)$ etc., in order to balance the numbers of all the occurring summands, what provides:

$$\varsigma_k(n - k + 1) \geq \sum_{i \in S^{(k-1)}_{n-k+1}} \frac{(n)}{(k-1)} \prod_{i=1}^{k-1} (1 - 2^{-i})^{l_i} = \frac{n}{k-1} \varsigma_{k-1}(n - k + 1).$$

Therefore $\varsigma_{k-1}(n - k + 2) \leq \frac{(k-2+2^{k-1})(k-1)}{n} \varsigma_k(n - k + 1)$ and finally we obtain

$$\frac{p_{n+1,k}}{p_{n,k}} \leq 1 - 2^{-k} + \frac{(k-2+2^{k-1})(k-1)}{n} < 1 + \frac{(k-1)^2}{n}.$$

When $k \in J_n(c)$, then

$$\exp \left(-O \left(\frac{\log(n)}{n}\right)\right) = 1 - 2^{-\left[\log(n)\right] + \left[\log \ln(n)\right]}$$

$$\leq \frac{p_{n+1,k}}{p_{n,k}} < 1 + \frac{(\left[\log(n)\right] + \log \ln(n))^2}{n} = \exp \left(O \left(\frac{\left(\log(n)\right)^2}{n}\right)\right).$$

This shows that $\varepsilon(n) = O \left(\frac{\left(\log(n)\right)^2}{n}\right)$. \(\square\)

### 4.7. MaxGeo Counter Privacy

In this subsection, we present a theorem that shows the privacy guarantees of MaxGeo Counter. Assume that we have $n$ incrementation requests. In the case of MaxGeo Counter it means that we generate random variables $X_1, \ldots, X_n$, where $X_i \sim \text{Geo}(1/2)$ are pairwise independent. Ultimately the result of the counter is maximum over all $X_i$'s, namely $X = \max(X_1, \ldots, X_n)$. We present the following

**Theorem 6.** Let $M$ denote the MaxGeo Counter, and $n$ denote the number of incrementation requests. Consider $m$ such that $|n - m| \leq 1$. Fix $\varepsilon > 0$ and $\delta \in (0, 1)$ and let

$$l_\varepsilon = \left[\log \left(\frac{e^\varepsilon}{n-\varepsilon - 1}\right)\right].$$

If

$$n \geq \frac{\ln(\delta)}{\ln (1 - 2^{-l_\varepsilon})} \left(\approx - \frac{\ln(\delta)}{\varepsilon}\right),$$

then

$$\mathbb{P} \left(M_n \in S\right) \leq \exp(\varepsilon) \cdot \mathbb{P} \left(M_m \in S\right) + \delta,$$

so $M$ is $(\varepsilon, \delta)$-DP.
Proof. We have $n$ incrementation requests, which influence the value of MaxGeo Counter $M$. Thence the result of the mechanism is $X = \max(X_1, \ldots, X_n)$, where $X_i \sim \text{Geo}(1/2)$ are pairwise independent. First, we observe that if $n = m$, the counter trivially satisfies differential privacy, as the probability distribution of $X$ does not change. From now on, we assume that $|n - m| = 1$. See that

$$P(X \leq l) = \prod_{i=1}^{n} P(X_i \leq l) = (P(X_1 \leq l))^n = \left(1 - \frac{1}{2^l}\right)^n = \left(\frac{2^l - 1}{2^l}\right)^n.$$ 

Furthermore

$$P(\max(X_1, \ldots, X_n) = l) = P(X = l) = P(X \leq l) - P(X \leq (l - 1)) =$$

$$= \left(\frac{2^l - 1}{2^l}\right)^n - \left(\frac{2^{l-1} - 1}{2^{l-1}}\right)^n = \frac{(2^l - 1)^n - (2^l - 2)^n}{2^n}.$$ 

Now we need to calculate the following expression

$$\frac{P(\max(X_1, \ldots, X_n) = l)}{P(\max(X_1, \ldots, X_n, X_{n+1}) = l)} = \frac{(2^l - 1)^n - (2^l - 2)^n}{2^n} = \frac{(2^l - 1)^n - (2^l - 2)^n + 1}{2^n}.$$ 

For fixed $\varepsilon$ we need to satisfy the following inequality

$$\ln \left(\frac{P(\max(X_1, \ldots, X_n) = l)}{P(\max(X_1, \ldots, X_n, X_{n+1}) = l)}\right) \leq \varepsilon,$$

which gives

$$\ln \left(1 + \frac{1}{2^l - 1}\right) \leq \varepsilon.$$ 

We can see from (12) that the greater $l$ is, the smaller $\varepsilon$ can be. Moreover, inequality (12) is true for $l \geq l_\varepsilon$. Therefore, we must ensure $P(X \leq l_\varepsilon) \leq \delta$. See that

$$P(X \leq l_\varepsilon) = (1 - 2^{-l_\varepsilon})^n.$$
It is easy to see that the above decreases when $n$ increases. Then

$$(1 - 2^{-l_e})^n \leq \delta \iff n \geq \frac{\ln(\delta)}{\ln(1 - 2^{-l_e})} \approx \frac{\ln(\delta)}{\varepsilon},$$

where the approximation is a result of substitution of $l_e$ without ceiling.

Ultimately it means that for fixed privacy parameters $(\varepsilon, \delta)$ we can calculate the minimum number of incrementation requests necessary to satisfy given privacy parameters. This can be done by artificially adding them before actually collecting data. Of course, it has to be taken into account that the initially added value should be subtracted from the final estimation of the appropriate cardinality before publication, and this change can impact the precision of the estimation. If we can perform such a preprocessing, then for every $(\varepsilon, \delta)$, we can easily know how many artificial counts have to be added. Notice that a similar approach may also be utilized for Morris Counter.

Note that from a differential privacy perspective, both the PCSA algorithm and HyperLogLog are arbitrary postprocessing performed on $m$ MaxGeo counters. Moreover, as each response goes to one counter only, they are independent of each other, so we can use the parallel composition theorem (see [5]). This gives us the following

**Observation 2.** Assume we have $k$ MaxGeo Counters $M[1], \ldots, M[m]$, which are used either in HyperLogLog or PCSA algorithm. If $j$th MaxGeo Counter is $(\varepsilon_j, \delta_j)$-DP then the chosen algorithm is $(\max_i \varepsilon_i, \max_j \delta_j)$-DP.

### 4.8. Comparison of Morris and MaxGeo Counters

In this subsection, we compare data aggregation algorithms’ privacy and storage properties based on one of the investigated counters or the standard Laplace method.

We start with auxiliary remarks for the privacy of MaxGeo Counter. For instance, see that if $\delta$ and $n$ are fixed, then from Theorem 6 and $l_e \leq \lceil \ln \left(1 + \varepsilon^{-1}\right)\rceil$ we obtain that

$$\varepsilon(n) \geq 2^{\left\lfloor \frac{-\log \left(1 - \frac{\delta}{n}\right)}{1 - \frac{\delta}{n}} \right\rfloor} - 1 =: \varepsilon_0(n).$$

(13)

We want to optimize $\varepsilon(n)$, so we are going to consider $\varepsilon_0(n)$ defined as the right-hand side of (13). In order to limit it let us consider the following function of $x \in \mathbb{R}_+$:

$$\psi(x, \delta) := \left(\left(1 - \delta^x\right)^{-1} - 1\right)^{-1} = -\frac{\ln(\delta)}{x} + \frac{\ln(\delta)^2}{2x^2} - \frac{\ln(\delta)^3}{6x^3} + O(x^{-4}).$$

(14)

Naturally, then $\varepsilon_0(n) \geq \psi(n, \delta) = -\ln(\delta)/n + O(n^{-2})$. Since $\psi$ is decreasing with respect to $x$, we will consider when $\varepsilon_0(n)$ changes. More precisely, consider a minimal $k$ such that $\varepsilon_0(n) < \psi(n - k, \delta) \leq \varepsilon_0(n - k)$, which occurs to be the neat upper bound for $\varepsilon(n)$. However, since $\varepsilon_0(n)$ is the non-ascending step function, we realize that

$$\varepsilon_0(n - k) \geq \left(2^{\left\lfloor \frac{-\log \left(1 - \frac{\delta}{n - k}\right)}{1 - \frac{\delta}{n - k}} \right\rfloor} - 1\right)^{-1} = -\frac{2\ln(\delta)}{n} + \frac{3\ln(\delta)^2}{n^2} - \frac{13\ln(\delta)^3}{3n^3} + O(n^{-4}).$$

(13)
If we denote $\phi(n, \delta) := \left( 2^{\left\lfloor -\log\left(1-\delta^{\frac{1}{2}}\right)\right\rfloor - 1} \right)^{-1}$, then we can sum up our recent considerations shortly by: $\psi(n, \delta) \leq \varepsilon(n) < \phi(n, \delta)$. Thence, in the case, when we fix parameter $\delta = 0.00033$, we obtain

$$
\frac{8.0164 \ldots}{n} + \frac{32.13147 \ldots}{n^2} + O(n^{-3}) \leq \varepsilon(n) \leq \frac{16.0328 \ldots}{n} + \frac{192.789 \ldots}{n^2} + O(n^{-3}).
$$

On the other hand, from Theorem 1, we know that when $\delta = 0.00033$, then for Morris Counter (with $\varepsilon(n)$ defined by (10)) the quite similar relation holds:

$$
\varepsilon(n) \leq -\ln \left( 1 - \frac{16}{n} \right) = \frac{16}{n} + \frac{128}{n^2} + O(n^{-3}).
$$

Therefore, Morris and MaxGeo Counters behave quite similarly under comparable conditions, and Figure 3 confirms this observation. Indeed, in Figure 3 we may see that the difference between the values of $\varepsilon(n)$ parameters for both counters shrinks as $n$ gets bigger.

![Figure 3: Values of $\varepsilon(n)$ parameters for Morris and MaxGeo Counters compared with boundaries for $\varepsilon(n)$ for MaxGeo Counter: the lower one — $\psi(n, \delta)$ and the upper one — $\phi(n, \delta)$ ($n \leq 160$ and $\delta = 0.00033$).](image)

Realize that previous conclusions remain true if $\delta(n)$ is not constant. This short observation enables us to obtain a more general result:

**Fact 5.** Let $\delta(n) = n^{-c}$ for some constant $c > 0$. Then

$$
\varepsilon(n) \leq \phi(n, \delta(n)) = \frac{2c \ln(n)}{n} + \frac{3c^2 \ln(n)^2}{n^2} + O\left( \frac{c^3 \ln(n)^3}{n^3} \right)
$$

and MaxGeo Counter is $(\phi(n, \delta(n)), \delta(n))$-DP for any $n \in \mathbb{N}$.  

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Notice that in this case, both sequences of parameters tend to 0, which may be used as an advantage in applications, especially when we expect that the total number of incrementation requests will be very large. However, we emphasize that this requires $\delta(n)$ to be negligible.

5. Privacy-Preserving Survey via Probabilistic Counters

In this section, we present an example scenario for data aggregation using probabilistic counters. We assume there is a server (alternatively, we call it aggregator) and a collection of nodes (e.g., mobile phone users), and we want to perform a boolean survey with a sensitive question. Namely, each user sends '0' if his/her answer is no and '1' if the answer is yes. We assume that the connections between users and the server are perfectly secure and the data can safely get to the trusted server. This can be performed using standard cryptographic solutions. The goal of the server is to publish the sum of all '1' responses in a privacy-preserving way. Such a goal could obviously be achieved by simply collecting all the data and adding an appropriately calibrated, Laplace noise (see [5]). However, we aim to show that probabilistic counters have inherently sufficient randomness to be differentially private without any auxiliary randomizing mechanism.

We can present the general scenario in the following way:

1. each user sends his/her bit of data to the server using secure channels,
2. server plugs the data points sequentially into the counter,
3. if the data point is '1', the counter receives incrementation request, otherwise, the data is ignored,
4. each incrementation request is being processed by the counter and may lead (depending on randomness) to an increase of the value of the counter,
5. when all data is processed, the value of the counter is released to the public.

Note that we assume that the Adversary has access only to the released value. We also released just the counter value itself, which does not estimate '1' responses. Such estimation is a function of released value, which is different for Morris or MaxGeo Counter. There can also be various ways to estimate the actual number using counter value. However, this does not matter for our case, as differential privacy is, conveniently, fully resilient to post-processing (see [5]). The graphical depiction of our considered scenario is presented in Figure 4.

Adversary. Our assumptions about the Adversary are the same as in most differential privacy papers. Namely, he may collude with any subset of the participants (e.g., all except the single user whose privacy he wants to breach). On the other hand, the aggregator is trusted. See that even though we have a distributed system in mind, this is, in fact, a central differential privacy scenario. We do not assume pan-privacy. It means that the algorithm’s internal state is not subject to the constraints of differential privacy. Obviously, if the Adversary would know the internal state of the counter at any
time or could observe whether, after receiving data from a specific user, the server has
to perform computations to potentially increment the counter (implying a ‘1’ response)
or not, he would easily violate the privacy. We also do not assume privacy under
continual observation. The survey is not published iteratively but one time only after it
is finished. To sum it up, the Adversary cannot

• extract or tamper with the internal state of the counter,
• extracts any information from the server or channels between users and the
  server.

The Adversary can

• collude with any subset $C$ of the participants (e.g. know their data or make them
  all send ‘0’ to the server) in order to breach the privacy of user not belonging to
  $C$,
• obtain the final result of the aggregation and perform any desired post-processing
  on it.

Note that, in light of our theorems 1 and 6, both Morris Counter and MaxGeo Counter
preserve differential privacy in such a scenario. Assume at least $n$ users holding ‘1’,
therefore at least $n$ incrementation requests. See that we can either know it based
on domain knowledge (e.g., we expect that at least some fraction of users will send
‘1’ based on similar surveys) or add $x$ counts to the counter artificially initially. The
number $x$ should be chosen according to the maximal amenable value of $\varepsilon$ parameter
for a given application, but we recommend to chose rather small values of $x$. Obviously,
in the case of artificial counts, it has to be taken into account when estimating the final
sum. Using Morris Counter we obtain $(L(n), 0.00033)$-DP with

$$ L(n) = - \ln \left( 1 - \frac{16}{n} \right) \leq \frac{16}{n - 8}. $$

For instance,
Example 2. Consider a result of Morris counter with a small number $n$ of incrementation requests (for example, a number of respondents suffering from a rare sickness). Therefore, we will likely demand the $\varepsilon$ parameter to be at most some threshold, e.g. 1. Therefore, from Theorem 1, we should add $x$ counts where $L(x) \leq 16/(x - 8) \leq 1$, so $x \geq 24$. Note that we do not include $n$ in the above formula since it is not known in advance. Therefore, using Morris Counter, the above survey is at least $(L(n + 24), 0.00033)$-DP. However, the estimator should be modified as well, i.e., $M'_n = M_{n+24}$, so $n' = \max\{2^{M'_n} - 26, 0\}$ (since $2^{M'_n}$ may be smaller than 26).

On the other hand, using MaxGeo Counter for a given $\varepsilon$ and $\delta$ we get $(\varepsilon, \delta)$-DP as long as $n \geq \frac{\ln(\delta)}{\ln(1 - 2^{-l_\varepsilon})}$, where $l_\varepsilon = \lceil \log (1 + 1/\varepsilon) \rceil$. Here we present an example. Let us consider the following example. Assume that we have at least $n = 200$ incrementation requests. From Theorem 1, we have $L(n) \leq 16/(n - 8) \leq 0.08334$. Therefore, using Morris Counter, the above survey is $(0.08334, 0.00033)$-DP.

On the other hand, using MaxGeo Counter for a given $\varepsilon$ and $\delta$ we get $(\varepsilon, \delta)$-DP as long as $n \geq \frac{\ln(\delta)}{\ln(1 - 2^{-l_\varepsilon})}$, where $l_\varepsilon = \lceil \log (1 + 1/\varepsilon) \rceil$. Here we present an example. Let us consider the following example. Assume that we have at least $n = 200$ incrementation requests. From Theorem 1, we have $L(n) \leq 16/(n - 8) \leq 0.08334$. Therefore, using Morris Counter, the above survey is $(0.08334, 0.00033)$-DP.

On the other hand, using MaxGeo Counter for a given $\varepsilon$ and $\delta$ we get $(\varepsilon, \delta)$-DP as long as $n \geq \frac{\ln(\delta)}{\ln(1 - 2^{-l_\varepsilon})}$, where $l_\varepsilon = \lceil \log (1 + 1/\varepsilon) \rceil$. Here we present an example. Let us consider the following example. Assume that we have at least $n = 200$ incrementation requests. From Theorem 1, we have $L(n) \leq 16/(n - 8) \leq 0.08334$. Therefore, using Morris Counter, the above survey is $(0.08334, 0.00033)$-DP.

Example 4. Let $\varepsilon = 0.5$ and $\delta = 1/D^2$, where $D$ is the the number of all survey participants. After using our theorem and straightforward calculations, we have $n \geq 7 \ln(D)$. Say we will have $\lceil \exp(20) \rceil$ participants. Then if we have at least 140 incrementation requests, we satisfy $(0.5, 1/D^2)$-DP.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{Method} & \textbf{Laplace noise} & \textbf{Morris counter} & \textbf{MaxGeo counter} \\
\hline
\textbf{Estimator} & $L(n/16)$ & $M_n$ & $M_n$ \\
\hline
\textbf{Var}(\hat{n}) & $\frac{n^2}{128}$ & $n^2 + 2^{M_n} - 2$ & $\left\lfloor \frac{1}{\varphi} 2^{M_n} \right\rfloor$ \\
\hline
\textbf{Avg. memory} & $\log(n) + O(1)$ & $\log(\log(n)) + O(1)$ & $\log(\log(n)) + O(1)$ \\
\hline
\end{tabular}
\caption{A summary of data aggregation techniques. The standard one is based on the Laplace method, and the rest are based on probabilistic counters. Recall that $\delta = 0.00033$ and $\varphi = 0.77351 \ldots$}
\end{table}

In Figure 5, we may briefly see that probabilistic counters may be used for data aggregation in order to decrease the memory usage in exchange for a slight increase of $\delta$ parameter of differential privacy and wider confidence intervals (lower accuracy). In recent years Big Data related problems become very popular. Note that this kind of application makes major use of memory. When the server aggregates many different data, standard solutions may cause a serious problem with data storage, which can be encountered by using the idea based on a probabilistic counter instead.
Example 5. Imagine that 100 million people take part in a general health survey with 100 yes/no sensitive questions.

For every question, we would like to provide an estimation of the number of people who answered yes, but we want to guarantee differential privacy property at a reasonable level. Realize that if the number of yes answers is very small for some questions (e.g., when the question is about a very rare disease), then the number of no answers may be counted instead.

According to Figure 5, if we use the Laplace method, then we may need approximately $100 \log(10^8) = 2657.54\ldots$ bits to store the counters. However, if we use Morris Counter instead, about $100 \log(\log(10^8)) = 473.20\ldots$ bits are needed. Let us note that all terms $O(1)$ in the ”Average memory” row of Figure 5 are bounded by 1. Hence their impact is negligible from a practical point of view.

One may also complain about the heavy use of Pseudo Random Number Generator that probabilistic counters make. However, this problem may be resolved by generating the number of incrementation requests, which have to be forgotten until the next update of the counter by utilizing appropriate geometric distributions (see, for instance, [48] for a similar approach applied to reservoir sampling algorithm). This way, the use of PRNG can be substantially reduced.

6. Conclusions and Future Work

In this paper, we have investigated probabilistic counters from the privacy-protection perspective. We have shown that Morris Counter and MaxGeo Counter inherently guarantee differential privacy from the mechanism itself, provided that there is at least a small, fixed number of incrementation requests. Otherwise, the counter has too low value, and intuitively, the result is not randomized enough. We have also shown that the constant in our Morris Counter result cannot be improved further.

We have shown how to perform data aggregation, namely a distributed survey, in a privacy-preserving manner using probabilistic counters. We clarified that this type of solution is especially efficient when one cares about memory resources, like in many Big Data related problems. Note that the security model in this paper was somewhat optimistic. Unfortunately, in such a setting, there is little incentive to use them other than when we already have them deployed and working as aggregators due to e.g., memory-efficiency requirements. This would, however, change tremendously if we would weakened these assumptions. This seems a very promising way to continue our research from this paper. Namely, we focused on privacy and can still not weaken the security assumptions and allow the Adversary to extract information from channels between users and the aggregator. That would put us in the so-called, Local Model, where each user is responsible for the data randomization. However, such an approach requires us to be able to perform probabilistic counter in an oblivious manner which, to the best of our knowledge, was not explored before.

In Subsection 4.1, we have mentioned the generalization of Morris Counter (for bases $a > 1$). Analysis of privacy properties of such variants of Morris Counters and various probabilistic counters presented, for example, in [28], [49] may also be promising direction of further research.
In this paper, we focused on the standard definition of differential privacy. However, there is also an issue of preservation of differential privacy for requests given by a group of $k$ users, that can be described in a language of so-called $(\varepsilon, \delta) - k$-DP (see [5]). A group of people may tend to behave in the same manner, so they may send $k$ requests in a row. Especially this “group” may be represented by a single person colluding with the Adversary. It is worth mentioning that this type of generalization creates an opportunity to modify probabilistic counters so that each incrementation request executes the update request multiple times to reduce the variance of the rescaled estimator. Intuitively, this extension should be especially efficient in preserving standard differential privacy property when $\varepsilon(n) = \frac{c}{n} + o(n^{-1})$ (as a parameter of standard differential privacy), because both $c$ and $n$ should scale with $k$ linearly. Hence the next challenging problem is showing that Morris and MaxGeo Counters satisfy $k$-DP property with similar privacy parameters.

Morris Counter and MaxGeo Counter are considered the most popular probabilistic counters. However, the results of this paper shed new light on the properties of the probabilistic counter in general. There is a possibility to provide analogous differential privacy properties for other probabilistic counters. Moreover, this paper enables the provision of differentially private algorithms for other applications, especially those based on Morris or MaxGeo Counter. For instance, in Section 4 we mentioned PCSA and HyperLogLog Counters together with their variances, which can be manually adjusted to applications. A proper choice of $m$ parameter implies an exchange of the memory usage for the accuracy of the estimation. We have mentioned that these counters’ differential privacy parameters can be obtained via Observation 2. However, such a direct result may not be satisfying. Hence a more precise calculation is needed. For instance, Observation 2 may be utilized again with some concentration inequalities.

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**Appendix A. Technical Lemmas and Proofs Related to Differential Privacy of Morris Counter**

For the sake of completeness, we present here proofs of all technical lemmas that are not directly connected to Theorem 1. Some of computations are supported by Wolfram Mathematica ver.11.3 ([50]). Whenever we obtain a result in this manner, we
indicate it by \( W \) sign. Usually results are precise, however in some cases, final forms are attained numerically.

We often struggle with expressions of a pattern \( 1 - 1/y \), so let us denote this function as \( a(y) \) to abbreviate formulas.

Next two lemmas are useful in a proof of Lemma 6:

**Lemma 4.** Let \( c > 1/x \). Then

\[
a(2cx)^{2y} \geq a(cx)^{y-1} \left( a(cx) + \frac{y}{4c^2x^2} \right)
\]

and

\[
a(cx)^y \geq a(2cx)^{2y-2} \left( a(2cx)^2 - \frac{y}{4c^2x^2} \right).
\]

**Proof.**

\[
a(2cx)^{2y} - a(cx)^y = \left( 1 - \frac{1}{cx} + \frac{1}{4c^2x^2} - 1 + \frac{1}{cx} \right)^{y-1} \sum_{i=0}^{y-1} a(2cx)^i a(cx)^{y-i-1}.
\]

Hence we obtained two inequalities: \( a(2cx)^{2y} - a(cx)^y \geq \frac{y}{4c^2x^2} a(cx)^{y-1} \)

and \( a(2cx)^{2y} - a(cx)^y \leq \frac{y}{4c^2x^2} a(2cx)^{2(y-1)} \), which imply the thesis of this Lemma. \( \square \)

**Lemma 5.** Let \( s \leq \log(x/4) \). Then \( a(2^{-s}x)^{2x+1} < \exp(-2^{s+1}) \) and

\[
a(2^{-s}x)^{x-1} > \exp(-2^{s}) \left( 1 - \frac{2^{2s-1} - 2^s}{x} - \frac{2^{2s-7} + 2^{4s-3}}{x^2} \right).
\]

**Proof.** Let \( f_1(x; s) := a(2^{-s}x)^{2x+1} = \exp(-2^{s+1}) \left( 1 - O \left( x^{-1} \right) \right) \). Realize a simple fact, that \( z \ln(z) \geq z - 1 \) for \( 0 < z \leq 1 \). Hence

\[
\left( 1 - \frac{2^s}{x} \right)^{-2s} \frac{\partial f_1(x; s)}{\partial x} = \frac{2^s(2x+1)}{x^2} + 2 \left( 1 - \frac{2^s}{x} \right) \ln \left( 1 - \frac{2^s}{x} \right) > \frac{2^s}{x^2} > 0
\]

and in a consequence \( a(2^{-s}x)^{2x+1} < \exp(-2^{s+1}) \) for any reasonable \( s \). Moreover, let

\[
D(x; s) := 1 - \frac{2^{2s-1} - 2^s}{x} - \frac{2^{2s-7} + 2^{4s-3}}{x^2},
\]

so we can attain:

\[
f_2(x; s) := \frac{a(2^{-s}x)^{x-1}}{D(x; s)} = \exp(-2^{s}) \left( 1 + O \left( x^{-2} \right) \right).
\]
Then, in a similar way

\[
D(x; s)^2 \left(1 - \frac{2^s}{x}\right)^{-x+1} \frac{\partial f_2(x; s)}{\partial x} =
\]

\[
D(x; s) \left(\frac{2^s (x - 1)}{x^2 (1 - \frac{2^s}{x})} + \ln \left(1 - \frac{2^s}{x}\right)\right) - \left(\frac{2^{2s-6} + 2^{4s-2}}{x^3} + \frac{2^{2s-1} - 2^s}{x^2}\right)
\]

\[
< \left(\frac{2^{2s-1} - 2^s}{x^2}\right) + \frac{2^{3s} - 2^{2s}}{x^3 (1 - \frac{2^s}{x})} - \frac{2^{3s}}{3x^3}\right) - \left(\frac{2^{2s-6} + 2^{4s-2}}{x^3} + \frac{2^{2s-1} - 2^s}{x^2}\right)
\]

\[
= \frac{2^{3s} - 2^{2s}}{x^3 (1 - \frac{2^s}{x})} - \frac{2^{3s}}{3x^3} - \frac{2^{2s-6} + 24s-2}{x^3}.
\]

Let \(d := 1 - 2^s / x\) and realize that \(d \in [3/4, 1)\) and \(2^{s-1} + d (-2^s / 3 - 2^{s-6} - 2^{2s-2}) > 0\). Indeed, if we put \(z = 2^s\), then we attain a quadratic inequality in \(z\) variable, with determinant \(\Delta = 1 - \frac{5d}{3} + \frac{5d^2}{6}\), that is negative for \(d \in [3/4, 1)\).

Hence \(\frac{\partial f_2(x; s)}{\partial x} < 0\) and consequently

\[
a(2^{-s} x)^{x-1} > \exp(-2^s) \left(1 - \frac{2^{2s-1} - 2^s}{x} - \frac{2^{2s-7} + 2^{4s-3}}{x^2}\right)
\]

for any reasonable \(s\).

\[\square\]

**Lemma 6.**

a) Sequence \((p_{2^k+1,k+4})_{k=2}^\infty\) is descending.

b) Sequence \((p_{2^k+1,k+5})_{k=3}^\infty\) is ascending.

**PROOF.** Let \(x = 2^k\) and \(t \in \{0, 1\}\). In advance we define

\[
\kappa(k, t) := (-1)^{k+4+t} 2^{-\left(k+4+t\right)\left(k+3+1\right)} 2^{k+4+t 2^{-2x-1}}
\]

and

\[
\tau(k, t) := 1[2 \mid (k + t)] (-1)^{k+t+3} 2^{-\left(k+1+3\right)\left(k+1+2\right)} 2^{k+t+3} \left(\frac{3}{4} 2^{x+1} - \left(\frac{1}{2}\right)^{x+2}\right),
\]

where \(1[\text{cond}]\) denote the indicator of the condition \(\text{cond}\). Realize that for \(t \in \{0, 1\}\) and \(k \geq 5\), \(|\tau(k, t) + \kappa(k, t)| < 2^{-50} < 10^{-15}\). Now, consider the differences between
the consecutive elements of sequences:

\[ p_{2k+1+k+5+t} - p_{2k+1,k+4+t} < \kappa(k,t) + \sum_{i=0}^{k+3+t} (-1)^i 2^{-i(k-1)} t_i r_i r_{k+t+4-i} \left[ \left( 1 - \frac{2^{k+i+1}}{x} \right)^{2x+1} - \left( 1 - \frac{2^{k-i+1}}{x} \right)^{2x+1} \right] \]

\[ = \sum_{i=0}^{k+2+i} \left[ 2^{-i(i-1)} t_i r_i r_{k+t+4-2i} \left[ a(2^{k+i-2i})^{2x+1} - a(2^{k-i-2i})^{2x+1} \right] + (\tau + \kappa)(k,t) \right] \]

Let us define \( u_t = 2^{k+5+t-2i} \) and

\[ W_t(i) = a(2^{i+1}) t_i r_i r_{k+t+4-2i} \left[ a(2^{k+i-2i})^{2x+1} - a(2^{k-i-2i})^{2x+1} \right] + (\tau + \kappa)(k,t) \]

and consider an upper bound of the last term:

\[ W_t(i) \leq a(2^{i+1}) t_i r_i r_{k+t+4-2i} \left[ a(2^{k+i-2i})^{2x+1} - a(2^{k-i-2i})^{2x+1} \right] + (\tau + \kappa)(k,t) \]

\[ = a(2^{i+1}) t_i r_i r_{k+t+4-2i} \left[ a(2^{k+i-2i})^{2x+1} - a(2^{k-i-2i})^{2x+1} \right] + (\tau + \kappa)(k,t) \]

Note that \( 2i \leq k + 2 + t \), so \( 8/x \leq u_t \) and in consequence \( 6u_t - 28/x > 20/x > 0 \). Moreover

\[ u_t(3u_t + 1) - \frac{u_t + 1}{x} \geq u_t \left( \frac{24}{x} + 1 \right) - \frac{u_t + 1}{x} \geq \frac{7}{x} + \frac{183}{x^2} > 0. \]
Hence

\[ W_t(i) < a(2^{2i+1}) \exp \left( -\frac{2}{u_t} \right) \frac{1}{x} \left( 3u_t + 1 - \frac{u_t + 1}{x u_t^2} - \frac{1}{x^2 u_t^2} \right) \]

\[ - 2^{-2i} a \left( \frac{u}{4} x \right) \exp \left( -\frac{4}{u_t} \right) D(x; 2i - 3 - t) \frac{1}{x} \left( \frac{6u_t + 4}{u_t^2} - \frac{32}{x u_t^2} + \frac{32}{x^2 u_t^3} \right) \]

\[ = a(2^{2i+1}) \exp \left( -\frac{2}{u_t} \right) \frac{1}{x} \left( 3u_t + 1 - \frac{u_t + 1}{x u_t^2} - \frac{1}{x^2 u_t^2} \right) \]

\[ - 2^{-2i} \left( \frac{4}{u_t} \right) \left( \frac{6u_t + 4}{u_t^2} - \frac{32 + 48u_t + 28u_t^2}{u_t^3 x} - \frac{128 + 64u_t - \frac{703}{4} u_t^2 + \frac{259}{4} u_t^3}{u_t^4 x^2} \right) \]

\[ + \frac{512 + 128u_t - 1150u_t^2 + \frac{900}{4} u_t^3}{u_t^7 x^3} \geq \frac{4608 - 1024u_t + 530u_t^2}{u_t^7 x^4} + \frac{4096 + 16u_t^4}{u_t^8 x^5} \]

Denote the above latter upper bound of \( W_t(i) \) by \( U_t(x; u_t(i)) \).

Analogically we would like to establish a lower bound of \( W_t(i) \):

\[ W_t(i) \geq a(2^{2i+1}) \left( a(u_t x) \left( \frac{u_t}{2} x \right)^{x-1} \left( a \left( \frac{u_t}{2} x \right) + \frac{1}{u_t x} \right) - a \left( \frac{u_t}{2} x \right)^{x+2} \right) \]

\[ - 2^{-2i} a \left( \frac{u}{2} x \right) \left( \frac{u_t}{2} x \right)^{2x+1} - a \left( \frac{u}{2} x \right)^{2x+2} \frac{a \left( \frac{u}{2} x \right)^2 - x + 2}{\frac{u_t^2}{x^2} x^2} \]

\[ = a(2^{2i+1}) a \left( \frac{u_t}{2} x \right)^{x-1} \frac{1}{x} \left( 3u_t + 1 - \frac{10u_t + 1}{u_t x} + \frac{8}{u_t x^2} \right) \]  \[ (A.1) \]

\[ - 2^{-2i} a \left( \frac{ut}{2} x \right)^{2x+2} \frac{1}{x} \left( \frac{6u_t + 4}{u_t^2} - \frac{4u_t + 8}{u_t^3 x} - \frac{8}{u_t^4 x^2} \right) \]

Now from 8/x ≤ u_t we attain

\[ u_t(3u_t + 1) - \frac{10u_t + 1}{x} = u_t \left( \frac{7u_t}{4} + \frac{7}{8} \right) + (10u_t + 1) \left( \frac{u_t}{8} - \frac{1}{x} \right) \]

\[ \geq u_t \left( \frac{7u_t}{4} + \frac{7}{8} \right) > 0 \]  \[ (A.2) \]

and

\[ u_t(6u_t + 4) - \frac{4u_t + 8}{x} - \frac{8}{x^2} \geq \frac{48u_t}{x} + \frac{32}{x} - \frac{4u_t + 8}{x} - \frac{8}{x^2} \geq \frac{24}{x} + \frac{344}{x^2} > 0 . \]
Hence
\[
W_t(i) > a(2^{2i+1}) \exp \left( -\frac{2}{u_t} \right) D(x; 2i - 4 - t) \frac{1}{x} \left( \frac{3u_t + 1}{u_t^2} - \frac{10u_t + 1}{u_t^3 x} + \frac{8}{u_t^3 x^2} \right)
- 2^{-2i} \exp \left( -\frac{4}{u_t} \right) a \left( \frac{u_t}{2} \right) \frac{1}{x} \left( \frac{6u_t + 4}{u_t^2} - \frac{4u_t + 8}{u_t^3 x} - \frac{8}{u_t^3 x^2} \right)
\]
\[
= a(2^{2i+1}) \exp \left( -\frac{2}{u_t} \right) \left( \frac{3u_t + 1}{u_t^2} - 2 + 5u_t + 4u_t^2 \right) \frac{2 + 4u_t - \frac{575}{32} u_t^2 + \frac{387}{32} u_t^3}{u_t^4 x^2}
+ \frac{2 + 20u_t - \frac{511}{32} u_t^2 + \frac{201}{16} u_t^3}{u_t^4 x^3} - \frac{16 + \frac{1}{2} u_t^2}{u_t^4 x^4}
- 2^{-2i} \exp \left( -\frac{4}{u_t} \right) \left( \frac{6u_t + 4}{u_t^2} - \frac{16u_t + 16}{u_t^3 x} + \frac{16}{u_t^3 x^2} + \frac{16}{u_t^3 x^3} \right).
\]

Denote the latter lower bound for \(W_t(i)\) by \(L_t(x; u_t(i))\). Now we show that \(W_t(i) > 0\) for \(i \geq 1\). Indeed, from Inequalities A.1 and A.2 we obtain
\[
W_t(i) > \frac{a(2^{2i+1}) x^2}{u_t^4} \left( a(2^{2i+1}) \frac{14u_t + 7}{8} - 2^{-2i} (6u_t + 4) \right) \tag{A.3}
\]
If \(i \geq 2\), then \((A.3) \geq \frac{1}{256} (31 (14u_t + 7) - 96u_t - 64) > 0\).
In the last case, when \(i = 1\), then \(u_t \geq 8\), so \((A.3) \geq \frac{1}{64} (7 (14u_t + 7) - 96u_t - 64) = \frac{2u_t - 15}{64} \geq \frac{1}{64}\).

Thanks to the property \(W_t(i) > 0\) for \(i \geq 1\), we may subtly neutralize the influence of \(r_{k+5-i}\) in the considered sum:
\[
\sum_{i=0}^{\lfloor \frac{k+5}{2} \rfloor} 2^{-i(2i-1)} r_{2i+1} r_{k+5-i} W_0(i) < \sum_{i=0}^{\lfloor \frac{k+5}{2} \rfloor} 2^{-i(2i-1)} r_{2i+1} W_0(i).
\]

Naturally we may consider \(U_0(x; u_0(i))\) instead of \(W_0(i)\) numerically for \(i \leq 4\):
\[
\sum_{i=0}^{4} 2^{-i(2i-1)} r_{2i+1} U_0(x; u_0(i)) = \frac{W}{x^2} - \frac{0.00407163}{x^3} - \frac{0.0298032}{x^4} + \frac{0.0198815}{x^5} - \frac{0.00785419}{x^6},
\]
so for \(x \geq 2^{15} \) \((k \geq 15)\), \(\sum_{i=0}^{4} 2^{-i(2i-1)} r_{2i+1} W_0(i) \leq -3.53741 \cdot 10^{-6}\). Moreover we may bound \(W_0(i)\) by \(a(2^{5-2i}) x^{2+1}\) for the rest of the sum:
\[
\sum_{i=5}^{\lfloor \frac{k+5}{2} \rfloor} 2^{-i(2i-1)} r_{2i+1} a(2^{5-2i}) x^{2+1} \leq \frac{R}{1 - 2^{-21}} \exp \left( -\frac{64}{R} \right) = 1.5784 \ldots \cdot 10^{-41},
\]
so \(p_{2^{k+1}+1,k+5} - p_{2^{k+1},k+4} < 0\) for \(k \geq 15\).
However, according to Theorem 2, we also present the numerical values of the sequence \(\{p_{2^{k+1}+1,k+4}\}_{k=2}^{14}\) in the Table A.2. We can now easily see that for any \(k \geq 2\) we attained \(p_{2^{k+1}+1,k+5} - p_{2^{k+1},k+4} < 0\).
Moreover, realize that $r_{k+5}/r_{k+3} < 1.1$ for any $k \geq 3$, so

$$\sum_{i=0}^{1} 2^{-i(2i-1)} r_{2i+1,1.1-i} L_1(x; u_1(i)) \frac{W}{x} - \frac{0.00326251\ldots}{x^2} + \frac{0.000219133\ldots}{x^3} - \frac{3.50924875\ldots \cdot 10^{-7}}{x^4}$$

For any possible $x \geq 8$ ($k \geq 3$), $\sum_{i=0}^{1} 2^{-i(2i-1)} r_{2i+1,1.1-i} L_1(x; u_1(i)) > 0.0015$.

We already know that $W_1(i)$ are positive for $i > 1$, so $p_{2^{k+1},1,k+6} - p_{2^{k+1},1,k+5} > 0$ for all $k \geq 3$.

We can use Theorem 2 once again to see that $p_{2^{k+1},1,10}/p_{2^{k+1},1,11} = 129.454\ldots > 2^7$ and $p_{2^{k+1},1,11}/p_{2^{k+1},1,12} = 125.065\ldots < 2^7$. Together with Lemma 6 we may easily attain Claim 1 and we instantly see that this Claim cannot be extended continuously for $k < 7$.

**Lemma 7.** Let $2 \leq l \leq n$ and assume that $p_{n,l-i} = 2^{2^{-i}} \alpha_i p_{n,l-i+1}$ for $i \in [0:2]$ and $p_{n+1,l-j} = 2^{2^{-j}} \alpha_j p_{n+1,l-j+1}$ for $j \in [0:1]$.

If $0 \leq \alpha_2 < \alpha_1 < \alpha_0$, then $0 < \alpha_1' < \alpha_0$.

**Proof.** Realize that $p_{n+1,l-i+1} = p_{n,l-i+1}(1 - 2^{-l+i-1} + 2^{-l+i-2} \alpha_i)$ for $i \in [0:2]$, so for $j \in [0:1]$,

$$\alpha_j' = \frac{p_{n+1,l-j}}{2^{2^{-j}} p_{n+1,l-j+1}} = \frac{p_{n,l-j}(1 - 2^{-l+j} + 2^{-l+j-2} \alpha_{j+1})}{2^{2^{-j}} p_{n,l-j+1}(1 - 2^{-l+j-1} + 2^{-l+j-2} \alpha_j)}$$

$$= \frac{\alpha_j(1 - 2^{-l+j} + 2^{-l+j-2} \alpha_{j+1})}{1 - 2^{-l+j-1} + 2^{-l+j+2} \alpha_j}.$$

Assume that $\alpha_1' \geq \alpha_0$. Then

$$A := \alpha_1(1 - 2^{-l+1} + 2^{-l+2} \alpha_2)(1 - 2^{-l+1} + 2^{-l+2} \alpha_0) \geq \alpha_0(1 - 2^{-l+1} + 2^{-l+2} \alpha_1)^2 =: B.$$

However, contrary to the assumption,

$$A = \alpha_1(1 - 2^{-l+1} + 2^{-l} - 2^{-l-1} + 2^{-l+2}(\alpha_0 + \alpha_2) - 2^{-2l+3} \alpha_0$$

$$- 2^{-2l+1} \alpha_2 + 2^{-2l+4} \alpha_0 \alpha_2)$$

$$< \alpha_0(1 - 2^{-l+1} + 2^{-2l}) + \alpha_1(2^{-l+2}(2\alpha_0) + 2^{-2l+4} \alpha_0 \alpha_1)$$

$$< \alpha_0(1 - 2^{-l+1} + 2^{-2l} + \alpha_1(2^{-l+3} + 2^{-2l+3} + 2^{-2l+4} \alpha_1)) = B.$$
Lemma 8. If for some $n \in \mathbb{N}$, $p_{n,n} = 2^n \eta_n p_{n,n+1}$ and $p_{n+1,n+1} = 2^{n+1} \eta_{n+1} p_{n+1,n+2}$, then $\eta_n < \eta_{n+1}$.

Proof.

\[ 0 = p_{n+1,n+1} - 2^{n+1} \eta_{n+1} p_{n+1,n+2} = p_{n,n+1} (1 - 2^{-n-1}) + p_{n,n} 2^{-n} \]
\[ - \eta_{n+1} p_{n,n+1} = p_{n,n+1} (1 - 2^{-n-1} + \eta_n - \eta_{n+1}) , \]

but $1 - 2^{-n-1} > 0$, so $\eta_n < \eta_{n+1}$. \qed