Design of Oscillator Networks for Generating Signal with Prescribed Statistical Property

Tatsuo Yanagita

Department of Engineering Science, Osaka Electro-Communication University, Neyagawa 572-8530, Japan
E-mail: yanagita@osakac.ac.jp

Abstract. We design oscillator networks to generate a signal with a prescribed power spectrum. We consider networks of identical sin-wave oscillators and the Kuramoto order parameter as an output signal of the system. We use the Kullback-Leibler entropy as a measure of the distance between the power spectrum of the output signal and those of desired one. By optimizing the connection network through the Markov chain Monte Carlo method with replica exchange, we found that even oscillator network with a small number of elements can be generate a variety of time signals. The output signals include periodic and quasi-periodic signals with prescribed periods, and aperiodic signals with prescribed power spectrums.

1. Introduction
Generations of time signal with prescribed statistical properties is important for engineering and living organisms, e.g., controlling machines and robots, and waking and sleeping rhythms of animals and so on. The control signals for the motions of animals and machines is not only periodic rhythms, it sometimes need becomes aperiodic or stochastic. Complex autonomous rhythms are ubiquitous in living organisms, and signal generators are crucial roles to control organisms and machines. In many engineering applications, signals with prescribed character are usually generated by special electrical circuit, and we have to manufacture a specialized circuit for each purpose. If a simple circuit can generated a diversity of signals with prescribed properties by changing parameters, it is versatile importance for many engineering fields.

Recently, the oscillator network show varieties of dynamics, from synchronization to chaos [1, 2, 3, 4]. Even in the system with a small number of oscillator elements, it exhibits a diversity of dynamics, e.g., stationary, periodic, quasi-periodic and chaotic, depending on the coupling strength and the connection structures [5, 6]. Here, we treat oscillator network as a signal generator, and consider an inverse problem that is how to construct oscillator networks which produce output signals with a given characteristics. By designing such oscillator networks, we show that the simple dynamical system can be used as versatile signal generators with prescribed statistical properties.

Application of evolutionary learning is one possible approach to constructing networks with prescribed dynamical properties. On this issue, several methods have been explored, where network structure has been modified in response to selection pressure via learning algorithms in such a way that the system evolved towards a specified goal [7, 8, 9, 10]. In previous studies, we have applied the Markov chain Monte Carlo (MCMC) method with replica exchange...
to design synchronization-optimized networks \[9, 10\]. In these studies, we constructed large ensembles of optimal networks for synchronization, which was evaluated by the Kuramoto order parameter, and analyzed their common statistical properties. For a network of heterogeneous phase oscillators, we found the transition from the linear to bipartite-like networks by increasing the number of links \[9\]. For a network of identical phase oscillators under uncommon noise, the most optimal network is a star-like structure when the number of links is small, while an interlaced structure is preferable when the number of links is large.

The most of previous works treat the synchronization property of oscillator networks. In this study, we design oscillator networks, which generate time signal with a desired statistical property. As a statistical property, we consider the periodgram and power spectrum for an output signal of the oscillator network. The paper is organized as follows. In Sec. 2, we introduce oscillator network which is a model of oscillators occupying the nodes of an asymmetrically coupled network, and define the Kuramoto order parameter as the output signal of the system. The Kullback-Leibler entropy introduces as a measure between the periodgram (or power spectrum) of the output signal and those of desired one. The sampling method, the Markov chain Monte Carlo, is also introduced in this section. Then, construction of the learned networks and the generated time signals and their periodgram are performed in Sec. 3 while, finally, the results are discussed in Sec. 4.

2. Model and Method

We consider the following network of identical oscillators as an autonomous signal generator,

\[ \phi_i = \omega_0 + \frac{\epsilon}{N} \sum_j A_{i,j} \sin(\phi_j - \phi_i + 2\pi \alpha), \]

where \(\epsilon\) is the coupling strength and \(A_{i,j}\) specify connectivity between oscillators, and \(\alpha \in [0, 1]\) is a phase shift parameter. Because the rotation frequencies of all the oscillators are the same, we can always go into the rotational frame \(\phi_i \rightarrow \phi_i - \omega_0 t\) and, thus, eliminate the term with \(\omega_0\). Hence, without any loss of generality, we can set \(\omega_0 = 0\) in Eq. (1). It is known that when the \(\alpha > \alpha_c = 0.25\), the oscillator networks shows chaotic dynamics \[5, 6\]. The connections between oscillators are asymmetrical and weighted, and \(A_{i,j}\) takes any of \(2L + 1\) number of values, i.e.,

\[ A_{i,j} \in \{ -1, (-L + 1)/L, \ldots, -2/L, -1/L, 0, 1/L, 2/L, \ldots, (L - 1)/L, 1 \}. \]

Collective motion of oscillator networks is usually described by the Kuramoto order parameter,

\[ R(t) = \frac{1}{N} \left| \sum_{i=1}^{N} \exp(-i\phi_i) \right|. \]

We observe the real part of the Kuramoto order parameter at each time interval \(\tau\), and consider the time series,

\[ \{x_k\} = \{x_k\} |\text{Re}(R(T_0 + n\tau))\} (k = 0, \ldots, K - 1), \]

as the output signal of the system, where the initial transient time \(T_0\) is discarded. For the time sequence \(\{x_i\}\), we require a prescribed statistical property, and determine the network \(A_{i,j}\), the phase shift \(\alpha\), and the sampling interval \(\tau\). Thus the output signal is determined by the combination of system parameters, \(s = (\{A_{i,j}\}, \alpha, \tau)\), which we will call the “state” of the model. As a desired property for the signal, we consider the normalized periodgram \(p(\omega_m)\) of the time sequence \(\{x_n\}\),

\[ p(\omega_m) = \frac{|f(\omega_m)|^2}{\sum_{k=0}^{K-1} |f(\omega_k)|^2}, \]
where
\[ f(x_m) = K \sum_{k=0}^{K-1} x_k \exp(i\omega_m k), \quad (k = 0, 1, \cdots, K - 1) \] (5)
is the Fourier transformation of the time sequence, where \( \omega_m = \frac{2\pi m}{K} \). For a chaotic signal, we also consider the power spectrum as an average of the periodgram,
\[ S(\omega_m) = \langle p(\omega_m) \rangle, \] (6)
where \( \langle \cdots \rangle \) represents time average over \( J \) realizations of time sequences starting from a given initial condition. When \( J = 1 \), \( S(\omega_m) \) equals to the periodgram \( p(\omega_m) \). We represent the target of the power spectrum as \( \tilde{S}(\omega_m) \), and we assume the distance between these spectrums is defined by the Kullback-Leibler entropy,
\[ D_{KL}(\tilde{S}|S) = \sum_{k=0}^{K-1} \tilde{S}(\omega_k) \log \frac{\tilde{S}(\omega_k)}{S(\omega_k)}. \] (7)
Using the the Kullback-Leibler entropy, we define an “energy” or “cost function” of the time series generated by the oscillator network with state \( s \) as
\[ E(s) = \log \frac{D_{KL}(\tilde{S}|S)}{|S|}, \] (8)
where \( |S| = \sum_m S(\omega_m) \) is the total power of the out signal. The total power \( |S| \) in the energy function prevents that the designed signal becomes too weak. In other words, it works as a penalty for the system.

Since our goal is to construct oscillator network having smaller energy, it is convenient to borrowed from the idea of statistical mechanics in which lower energy states are studied. We sample the state \( s \) (the parameters for the oscillator network and the sampling interval) from the canonical ensemble, i.e.,
\[ q(s) = \frac{\exp[-\beta E(s)]}{\sum_s \exp[-\beta E(s)]}. \] (9)
The application of the canonical ensemble to a network is also considered in [11, 12], in which \( E \) is called the graph Hamiltonian. The sampling is carried out by the Markov chain Monte Carlo with replica exchange method, which has been previously applied to constructing the oscillator networks and several dynamical systems [13, 14, 15, 16, 17, 18, 19, 20, 9, 10].

3. Numerical Results

In this section, we designed oscillator networks with prescribed statistical properties. In order to estimate the energy of a network oscillator, we integrate the differential equation (1) numerically, and generate time series and calculate it’s periodgram. Using the periodgram, the estimation of energy \( E(s) \) with a state \( s \) is obtained. By using the \( E(s) \), we sample the network \( A_{i,j} \) from the Gibbs distribution \( \exp(-\beta E) \) with the inverse temperature \( \beta \) by the Markov exchange Monte Carlo with replica exchange method. 24 replicas are used, and each replica has the following inverse temperatures \( \beta_k = \kappa^{k-1} - 1 \) \( (k = 1, \cdots, 24) \).

The several types of target periodgrams or power spectrums are considered. We consider the target periodgrams with several characteristic frequencies. We also use uniform power spectrum as target spectrum in which the designed oscillator network exhibits chaotic. We have mainly carried out the numerics with the following parameters, \( N = 5, L = 8, T_0 = 500, \) and \( K = 5000 \). The details of the target spectrums and numerical results are shown as follows.
Figure 1. Typical time series and the corresponding periodigrams generated by the oscillator networks design by MCMC with prescribed periodgram are shown. The value at the top of each figure is the energy estimated through the Kullback-Leibler entropy. The time series produced by the learned oscillator network are shown in (a), and the corresponding periodgrams are shown in (b). The red broken lines are the prescribed or target periodgram. The parameters of the target periodgram in the equation (10) are $M = 1$, $\alpha = 1.0, \tilde{\omega}_1 = 20$. The number of elements of the oscillator network is $N = 3$ and the number of weighted values of the link is $L = 8$.

3.1. Design of periodic and quasi-periodic signal generators

For periodic and quasi-periodic signal generators having prescribed frequencies, we use a sum of the Lorenzian type weight function as target periodgrams as follows,

$$\tilde{S}(\omega_m) = \sum_{i=1}^{M} \frac{a_i}{|\omega_m - \tilde{\omega}_i|^\alpha + \delta},$$  \hspace{5mm} (10)$$

where $M$ is the number of characteristic frequencies and $a_i$ represents the strength of the target signal with frequency $\tilde{\omega}_i$, and $\delta$ and $\alpha$ determine the the singularity of the peak of the target periodgram at $\tilde{\omega}_i$. The typical parameters for the oscillator network are $N = 5, L = 8$, and $\epsilon = 1.5$. In Figure 1, we show typical time series and their periodograms of the designed oscillator networks for a given specified target periodgram. The value at the top of each figure is “energy” estimated through the Kullback-Leibler entropy between the obtained and prescribed periodgram. The parameters of the target periodgram in the equation (10) are $M = 1, a_1 = 1.0, \tilde{\omega}_1 = 20, \alpha = 1.0,$ and $\delta = 1.0$. In this figure, a typical time series and their corresponding periodgrams for a different energy levels are shown. We observe that the periodgrams with lower energies level are fitted well with the prescribed periodgram, which is shown in the red broken lines in Figure 1(b). The top left periodgram obtain by the “learned” oscillator network with $N = 3$ having smallest energy ($E = -3.95282$) is good agreement with the target periodgram. In particular, the value of the characteristic frequency are matched well with the target one.
The results for a target periodogram having two characteristic frequencies with $M = 2, \tilde{\omega}_1 = 10, \tilde{\omega}_2 = 20$ and the corresponding strength $a_1 = 0.5, a_2 = 0.5$, and $\alpha = 2.0, \delta = 1.0$, are shown in Figure 2. The periodogram generated by the “learned” oscillator network having the smallest energy, i.e., $E = -2.21416$ also is good agreement with the target periodogram. In this case, we use the oscillator network with 5 elements.

We also consider more complex target periodogram having three and four characteristic frequency. For the target periodogram with three characteristic frequencies, we use the following parameters, $M = 3, \tilde{\omega}_1 = 10, \tilde{\omega}_2 = 20, \tilde{\omega}_3 = 30$, and the corresponding Fourier coefficients are $a_1 = 1.0, a_2 = 0.5, a_3 = 0.33$, respectively, and other parameters are $\alpha = 2.0, \delta = 1.0$. For the target periodogram with four characteristic frequencies, we use the following parameters, $M = 4, \tilde{\omega}_1 = 5, \tilde{\omega}_2 = 10, \tilde{\omega}_3 = 20, \tilde{\omega}_4 = 40$, and the corresponding Fourier coefficients are $a_1 = 0.125, a_2 = 0.25, a_3 = 0.50, a_4 = 1.0$, respectively, and other parameters are $\alpha = 1.0, \delta = 1.0$. The times series and the corresponding periodograms are shown in Figure 3. For target periodogram with three characteristic frequency, the periodogram generated by the “optimal” oscillator network is fitted well with the target one as shown in Figure 3(a). It is, however, that in the case with four characteristic frequency, the periodogram obtain by the “optimal” oscillator network is not good agreement with the target periodogram. In this case, time series generated by the optimal oscillator network does not include the low frequency component, i.e., $\tilde{\omega}_1 = 5$, as shown in the lower panel in Figure 3(b). It is because that the number of elements of the oscillator network is too small to describe the complex time series. The time series and the corresponding periodogram generated by the optimal oscillator network with $N = 12$ is also shown in Figure 3(c). In this Figure, we notice that small modulation with long period appears in the time series and the peak at the lower characteristic frequency also emerges in the periodogram. However, the energy, $E = -0.157259$ in the case of $N = 12$, is larger.

Figure 2. The same graphs as shown in figure 1 for the target periodogram with two characteristic frequencies. The parameters for the target periodogram are $M = 2, \tilde{\omega}_1 = 10, \tilde{\omega}_2 = 20, \tilde{\omega}_3 = 30, a_1 = 1.0, a_2 = 0.5, a_3 = 0.33$. Other parameters are the same as in Fig. 1.
than the energy $E = -0.86616$ in the case of $N = 5$, and thus the learned oscillator network with smaller number of elements perform better than that with larger number of elements. This is because that numerical cost increases with the number of elements in the oscillator network since both the time integration for the oscillator network increases and the Monte Carlo sampling time increases. Thus there is a trade off between the learning capability of oscillator network which increases with $N$ and computational cost.

3.2. Designing oscillator network as white noise generator

In this subsection, we consider white noise as a typical broad-ranged target power-spectrum. The modulus of the Fourier coefficient for white noise does not depend on frequencies. Thus the power of any frequency should equivalently be included in the signal, and thus, we set the target weight of power spectrum equals one for all frequency, i.e.,

$$\tilde{S}(\omega_m) = 1/|S|. \quad (11)$$

Typical time series and the power spectrum generated by the learned oscillator networks for different energy levels are shown in Figure 4. It is clearly seen that periodic times series varies to chaotic one with decease of energy level, corresponding to how oscillator network learns. For oscillator network having smallest energy, i.e., the optimal oscillator networks exhibit chaotically and the broad-ranged power spectrum is obtained. When the number of oscillators increases, the power spectrum of the optimal oscillator network becomes broader, which means that power of frequencies involved in the time series equivalently be included. It is, however, that there is a trade off between the system size and the numerical cost for optimization as noticed previously.

4. Conclusions

We have shown that the oscillator network with a small number of elements can generate variety of time signal, i.e., periodic, quasi-periodic, and aperiodic when the connection network

![Figure 3. Typical time series and the corresponding periodgrams generated by the optimal oscillator networks for the quasi-periodic target periodgrams are shown. (a) The parameters for the target periodgrams with three characteristic frequencies are $M = 3$, $\omega_1 = 10$, $\omega_2 = 20$, $\omega_3 = 30$, $a_1 = 1.0$, $a_2 = 0.5$, $a_3 = 0.33$, $\alpha = 2.0$, $\delta = 1.0$. (b) For the target periodgram with four characteristic frequencies, we use $M = 4$, $\omega_1 = 5$, $\omega_2 = 10$, $\omega_3 = 20$, $\omega_4 = 40$, $a_1 = 0.125$, $a_2 = 0.25$, $a_3 = 0.5$, $a_4 = 1.0$, $\alpha = 1.0$, $\delta = 1.0$. (c) In the case of the oscillator network with $N = 12$. Other parameters are the same as in (b).](image-url)
is designed so as to fit with a given target periodogram or a power spectrum through the Markov chain Monte Carlo method with replica exchange. Although the elements considered here are identical and sinusoidal-oscillator, the designed oscillator networks with only five elements can generate quasi-periodic time series with three characteristic frequencies. Furthermore, the designed oscillator-network also can produce aperiodic time-series whose power of any frequency is almost equivalently.

It should be noted that the simple and small oscillator networks can generate given characteristic frequencies only by changing the connection network with preserving the dynamics of oscillator elements. In other words, when the realtime leaning applied, we can obtain the desired time signals with same architecture by changing the system parameters. Such a “learning capability” of oscillator networks can be capable of signals generator for many engineering applications.

Acknowledgments
This study has been partially supported by JSPS KAKENHI Grant No. 15K05221 and the Volkswagen Foundation (Germany).

References
[1] Nakagawa N and Kuramoto Y 1993 Progress of Theoretical Physics 89 313–323
[2] Topaj D and Pikovsky A 2002 Physica D: Nonlinear Phenomena 170 118 – 130 ISSN 0167-2789
[3] Maistrenko Y, Popovych O and Tass P 2005 Desynchronization and chaos in the kuramoto model Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems (Lecture Notes in Physics vol 671) (Springer Berlin Heidelberg) pp 285–306 ISBN 978-3-540-24289-5
[4] Popovych O V, Maistrenko Y L and Tass P A 2005 Phys. Rev. E 71(6) 065201
[5] Maistrenko Y, Popovych O, Burylko O and Tass P A 2004 Phys. Rev. Lett. 93(8) 084102
[6] Ashwin P, Burylko O and Maistrenko Y 2008 Physica D: Nonlinear Phenomena 237 454 – 466 ISSN 0167-2789
[7] Ipsen M and Mikhailov A S 2002 Phys. Rev. E 66 046109
[8] Moyano L G, Abramson G and Zanette D H 2001 Eur. Phys. J. B 22 223–228
[9] Yanagita T and Mikhailov A S 2010 Phys. Rev. E 81 056204
[10] Yanagita T and Mikhailov A S 2012 Phys. Rev. E 85 056206
[11] Park J and Newman M E J 2004 Phys. Rev. E 70(6) 066117
[12] Garrido A 2011 Symmetry 3 1–15 ISSN 2073-8994
[13] Cho A E, Doll J D and Freeman D L 1994 Chem. Phys. Lett. 229 218–224
[14] Bolhuis P G, Dellago C and Chandler D 1998 Faraday Discuss. 110 421–436
[15] J H Vlugt T and Smit B 2001 PhysChemComm 4 11–17
[16] Kawasaki M and Sasa S I 2005 Phys. Rev. E 72 037202
[17] Sasa S I and Hayashi K 2006 Europhys. Lett. 74 156–162
[18] Giardiná C, Kurchan J and Peliti L 2006 Phys. Rev. Lett. 96 120603
[19] Tailleur J and Kurchan J 2007 Nature Physics 3 203–207
[20] Yanagita T and Iba Y 2009 J. Stat. Mech. P02043