Multi-Scalar $p$-brane Solitons

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ABSTRACT

In a previous paper [10], supersymmetric $p$-brane solutions involving one dilatonic scalar field in maximal supergravity theories were classified. Although these solutions involve a number of participating field strengths, they are all equal and thus they carry equal electric or magnetic charges. In this paper, we generalise all these solutions to multi-scalar solutions in which the charges become independent free parameters. The mass per unit $p$-volume is equal to the sum of these Page charges. We find that for generic values of the Page charges, they preserve the same fraction of the supersymmetry as in their single-scalar limits. However, for special values of the Page charges, the supersymmetry can be enhanced.

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Isotropic $p$-brane solitons from supergravity theories have been extensively studied in recent years. Most of the solutions that have been found can be described in terms of a single dilatonic scalar field and a single antisymmetric tensor field strength [1-10]. Two possible kinds of solution arise, one carrying an “electric” charge for the field strength, and describing a fundamental or elementary $p$-brane, and the other carrying a “magnetic” charge, corresponding to a solitonic $p$-brane solution. (When the dimension is twice the degree of the field strength, dyonic solutions that carry both electric and magnetic charges can also arise.) In dimensions $D$ lower than 10, the scalar field might be a linear combination of the dilatonic scalar fields in the maximal supergravity theory in that dimension. At the same time, more than one of the original antisymmetric tensor fields might be non-zero in the solution, although all of them will be proportional to one another. Thus there is only one overall charge parameter characterising any given solution, with the electric or magnetic charges of the individual field strengths occurring in a fixed ratio.

Recently, some further solutions have been found in which the charges of participating field strengths are independent of one another [11-14]. This is achieved by relaxing the condition that only one linear combination of the dilatonic scalar fields is non-vanishing, and so these solutions may be characterised as multi-scalar solutions. Each of the previous single scalar solutions with more than one participating field strength therefore has such a multi-scalar generalisation. In this paper, we shall give a systematic construction of multi-scalar supersymmetric solutions in maximal supergravity theories in $4 \leq D \leq 9$. The supergravity theories that we shall consider are those that are obtained from type IIA supergravity in $D = 10$, or, equivalently, from $D = 11$ supergravity, by Kaluza-Klein dimensional reduction. For convenience, we shall consider the supergravity theories in the versions where all the field strengths have degrees $n$ that are less than or equal to 4. Some related results for 0-branes in $D = 4$ have previously been obtained in [11-14], mostly in the context of the 4-dimensional heterotic string.

Let us consider a $p$-brane solution in which a number $N$ of $n$-index ($n = 1, 2, 3, 4$) field strengths $F^\alpha$ ($\alpha = 1, 2, \ldots, N$) are involved. The relevant part of the bosonic Lagrangian for the supergravity theory is given by

$$e^{-1} \mathcal{L} = R - \frac{1}{2} (\partial \vec{\phi})^2 - \frac{1}{2n!} \sum_{\alpha} \varepsilon_{\vec{a} \cdot \vec{\phi}} (F^\alpha)^2 ,$$

(1)

where $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_{11-D})$ are the $(11 - D)$ dilatonic scalar fields. The dilatonic vectors $\vec{a}_\alpha$ follow by dimensional reduction from $D = 11$ supergravity, and can be found, for example, in [10].

In general, there are further contributions in the bosonic Lagrangian coming from the dimensional reduction of $F \wedge F \wedge A$ in $D = 11$ and from the Chern-Simons modifications that the field strengths acquire in the dimensional reduction process. We shall be concerned only with solutions where
these terms do not contribute. This imposes certain constraints on the field configurations, which were discussed in [10].

The metric ansatz is given by

$$ds^2 = e^{2A} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B} dy^m dy^m ,$$

where $x^\mu (\mu = 0, \ldots, d-1)$ are the coordinates of the $(d-1)$-brane volume, and $y^m$ are the coordinates of the $(D-d)$-dimensional transverse space. The function $A$ and $B$, as well as all the dilatonic scalars $\vec{\phi}$, depend only on $r = \sqrt{y^m y^m}$. Thus the ansatz preserves an $SO(1, d-1) \times SO(D-d)$ subgroup of the original $SO(1, D-1)$ Lorentz group.

For each $n$-index field strength $F^\alpha$, there are two different ansätze that also preserve the same subgroup, namely [1, 4]

$$F^\alpha_{m\mu_1\cdots\mu_{n-1}} = \epsilon_{\mu_1\cdots\mu_{n-1}} (e^{C^\alpha}_r) y^m_r \quad \text{or} \quad F^\alpha_{m_1\cdots m_n} = \lambda_\alpha \epsilon_{m_1\cdots m_n p} y^p_r r^{n+1} ,$$

where a prime denotes a derivative with respect to $r$. The first case gives rise to an elementary $(d-1)$-brane with $d = n-1$ for $n = 2, 3$ or 4; the second gives rise to a solitonic $(d-1)$-brane with $d = D-n-1$ for $n = 1, 2, 3$ or 4. In this paper, we consider solutions with each $F^\alpha$ having either elementary or solitonic contributions, but not both. In $D = 2n$, some $F^\alpha$ might be the duals of the original field strengths of the same degree. Thus in terms of the original field strengths, such solutions have both elementary and solitonic contributions, giving rise to dyonic solutions of the first type [1]. As we shall see later, these dyonic solutions are possible in $D = 4$, but not in $D = 8$ or $D = 6$.

Let us first summarise the known results for single-scalar $p$-brane solutions. In [10], the equations of motion following from (1) were solved by first truncating to a single field strength and a single scalar field, with Lagrangian:

$$e^{-1} \mathcal{L} = R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2n!} e^{a\phi} F^2 ,$$

where $\phi$ is a linear combination of the original dilatonic scalars $\vec{\phi}$, and $F$ is the canonically-normalised field strength formed from the participating field strengths $F^\alpha$. In order to be able to set the orthogonal combinations of dilatonic scalars to zero, in a manner consistent with their equations of motion, the field strengths $F^\alpha$ must be all proportional to $F$ [10]: In the generic case, where the matrix $M_{\alpha\beta} \equiv \vec{a}_\alpha \cdot \vec{a}_\beta$ is non-singular, we must have

$$a^2 = \left( \sum_{\alpha\beta} (M^{-1})_{\alpha\beta} \right)^{-1} , \quad \phi = a \sum_{\alpha\beta} (M^{-1})_{\alpha\beta} \vec{a}_\alpha \cdot \vec{\phi} ,$$

$$\quad (F^\alpha)^2 = a^2 \sum_{\beta} (M^{-1})_{\alpha\beta} F^2 .$$

(5)
When the matrix \( M_{\alpha\beta} \) is singular, one finds that the only new solution that need be considered is for the case \( \sum_{\alpha} \vec{a}_\alpha = 0 \), which gives rise to \( a = 0 \) and \( (F^{\alpha})^2 = F^2/N \) for all \( \alpha \). It is convenient to parameterise the dilaton prefactor \( a \) by

\[
a^2 = \Delta - \frac{2\bar{d}d} {D - 2},
\]

where \( \bar{d} = D - d - 2 \) and \( d\bar{d} = (n-1)(D-n-1) \). It is then straightforward to solve the equations of motion following from the Lagrangian \([8]\), using the metric ansatz \([2]\) and field strength ansätze \([3]\).

The metrics of the solutions for given degree of the field strength in dimension \( D \) are determined by the value of \( \Delta \), and are given by

\[
ds^2 = \left( 1 + \frac{k}{r^{\bar{d}}} \right) \frac{dx^\mu dx^\nu \eta_{\mu\nu}} {2^{d-2} \sum_{\alpha} S_{\alpha}^2} + \left( 1 + \frac{k}{r^{d}} \right) \frac{dy^m dy^m} {2^{D-2} \sum_{\alpha} S_{\alpha}^2},
\]

where \( k = \sqrt{\Delta} \lambda / (2\bar{d}) \). The mass per unit \( p \)-brane volume is given by \( m = \frac{1}{2}(B' - A')e^{-B} r^{d+1} \) in the limit \( r \rightarrow \infty \). The Page charge \( P \) for the canonically-normalised field strength \( F \) is given by \( \frac{1} {4\omega_{D-n}} \int_{S^{D-n}} F \) for the elementary case and by \( \frac{1} {4\omega_n} \int_{S_n} F \) for the solitonic case. Thus we have

\[
m = \frac{\lambda} {2\sqrt{\Delta}}, \quad P = \frac{1} {4\lambda}.
\]

Note that the Page charge \( P_{\alpha} \) of each individual field strength \( F^{\alpha} \) is a certain fixed multiple of \( P \) in a given solution, as determined by \([6]\).

Now we turn to the consideration of multi-scalar solutions. Substituting the ansätze \([3]\) and \([2]\) directly into the equations of motion that follow from the Lagrangian \([1]\), we find that \( \vec{\phi} \), \( A \) and \( B \) satisfy

\[
\vec{\phi}'' + \frac{\bar{d} + 1} {r} \vec{\phi} + (dA' + \bar{d}B') \vec{\phi} = -\frac{1} {2\epsilon} \sum_{\alpha} \vec{a}_\alpha S_{\alpha}^2,
\]

\[
A'' + \frac{\bar{d} + 1} {r} A' + (dA' + \bar{d}B')A' = \frac{\bar{d}} {2(D - 2)} \sum_{\alpha} S_{\alpha}^2,
\]

\[
B'' + \frac{\bar{d} + 1} {r} B' + (dA' + \bar{d}B')B' + \frac{1} {r} = -\frac{d} {2(D - 2)} \sum_{\alpha} S_{\alpha}^2,
\]

\[
d(D - 2)A'^2 + \bar{d}(dA'' + \bar{d}B'') - (dA' + \bar{d}B')^2 - \frac{\bar{d}} {r} (dA' + \bar{d}B') + \frac{1} {2} \bar{d} \vec{\phi}^2 = \frac{1} {2} \bar{d} \sum_{\alpha} S_{\alpha}^2,
\]

where \( \epsilon = 1 \) and \( -1 \) for the elementary and solitonic ansätze respectively, and the functions \( S_{\alpha} \) are given by

\[
S_{\alpha} = \lambda_{\alpha} e^{-\frac{1} {2\epsilon} \vec{a}_\alpha \cdot \vec{\phi} + dA - \bar{B} r^{\bar{d}-1}}.
\]

In the elementary case, \( \lambda_{\alpha} \) arises as the integration constant for the function \( C_{\alpha} \), given by

\[
(e^{C_{\alpha}})' = \lambda_{\alpha} e^{\vec{a}_\alpha \cdot \vec{\phi} + dA - \bar{B} r^{\bar{d}-1}}.
\]
From (10) and (11), we see that a natural solution for $B$ is to take
\[ dA + dB = 0 \]  
(15)

We may also consistently set to zero the $(11 - D - N)$ components of $\mathbf{\tilde{\phi}}$ that are orthogonal to the space spanned by the $N$ dilaton vectors $\mathbf{a}_\alpha$. The remaining equations are
\[
\varphi''_\alpha + \frac{\tilde{d} + 1}{r} \varphi'_\alpha = -\frac{1}{2} \epsilon \sum_\beta M_{\alpha\beta} S_{\beta}^2 ,
\]
(16)
\[
A'' + \frac{\tilde{d} + 1}{r} A' = \frac{\tilde{d}}{2(D - 2)} \sum_\alpha S_{\alpha}^2 ,
\]
(17)
\[
d(D - 2)A'^2 + \frac{1}{2} \tilde{d} \sum_\alpha (M^{-1})_{\alpha\beta} \varphi'_\alpha \varphi'_\beta = \frac{1}{2} \tilde{d} \sum_\alpha S_{\alpha}^2 ,
\]
(18)

where we have defined $\varphi_\alpha = \mathbf{a}_\alpha \cdot \mathbf{\tilde{\phi}}$. (Here we are assuming that $M_{\alpha\beta}$ is non-singular, and we shall comment on the case when it is singular later.) Note that the number of non-vanishing scalar fields $\varphi_\alpha$ is precisely the same as the number $N$ of participating field strengths. By acting on (16) with $(M^{-1})_{\alpha\beta}$, and comparing with (17), we see that it is natural to solve for $A$ by taking
\[
A = -\frac{\epsilon \tilde{d}}{D - 2} \sum_\alpha (M^{-1})_{\alpha\beta} \varphi_\alpha .
\]
(19)

The equations of motion now reduce to
\[
\sum_\beta (M^{-1})_{\alpha\beta} \left( \varphi''_\beta + \frac{\tilde{d} + 1}{r} \varphi'_\beta \right) = -\frac{1}{2} \epsilon \Lambda_{\alpha}^2 e^{-\epsilon \varphi_\alpha + 2dA} r^{-2(\tilde{d} + 1)} ,
\]
(20)
\[
d(D - 2)A'^2 + \frac{1}{2} \tilde{d} \sum_\alpha (M^{-1})_{\alpha\beta} \varphi'_\alpha \varphi'_\beta = \frac{1}{2} \tilde{d} \sum_\alpha \Lambda_{\alpha}^2 e^{-\epsilon \varphi_\alpha + 2dA} r^{-2(\tilde{d} + 1)} .
\]
(21)

As in the case of the solutions that involve only one dilatonic scalar field, the solutions here are determined completely by the structure of the dot products $M_{\alpha\beta}$ of dilaton vectors $\mathbf{a}_\alpha$ of the corresponding field strengths $F^\alpha$. Solutions exist only for $N \leq (11 - D)$. In general, the solutions of (20) and (21) are still very complicated. However, we can find simple solutions if we make the ansatz that the quantity $(-\epsilon \varphi_\alpha + 2dA)$ appearing in the exponential in $S_{\alpha}^2$ is proportional to the quantity $\sum_\beta (M^{-1})_{\alpha\beta} \varphi_\beta$ appearing on the left-hand side of (20). For this to be true, it implies that $M_{\alpha\beta}$ must take the form
\[
M_{\alpha\beta} = 4 \delta_{\alpha\beta} - \frac{2d\tilde{d}}{D - 2} .
\]
(22)

Note that the coefficient of $\delta_{\alpha\beta}$ can a priori be any constant, but it is fixed to be 4 in maximal supergravity theories, since all the dilaton vectors in such theories have magnitude $a$ given by (6) with $\Delta = 4$. We can now solve (20) and (21) completely by making the further ansatz that
$S_\alpha \propto (-\epsilon \varphi' + 2dA')$. The solution is given by

$$e^{\frac{1}{2} \epsilon \varphi - dA} = 1 + \frac{\lambda_\alpha}{d} r^{-\tilde{d}},$$

$$ds^2 = \prod_{\alpha=1}^N \left( 1 + \frac{\lambda_\alpha}{d} r^{-\tilde{d}} \right) dx^\mu dx^\nu \eta_{\mu\nu} + \prod_{\alpha=1}^N \left( 1 + \frac{\lambda_\alpha}{d} r^{-\tilde{d}} \right)^{(D-2)} dy^m dy^m. \quad (23)$$

We may now calculate the mass per unit $p$-brane volume and the Page charges for the solution, finding

$$m = \frac{1}{4} \sum_{\alpha=1}^N \lambda_\alpha, \quad P_\alpha = \frac{1}{4} \lambda_\alpha. \quad (24)$$

Note that in our derivation of the solutions, we assumed that the matrix $M_{\alpha\beta}$ is non-singular, and indeed the matrix given by (22) is non-singular in general. However, it can be singular in two relevant cases, namely $D = 5, N = 3$ and $D = 4, N = 4$ for the 2-form field strengths. In these cases, the analysis requires modification; however, it turns out that (23) continues to solve the equations of motion.

Having obtained the generic multi-scalar solutions for matrices $M_{\alpha\beta}$ satisfying (22), it is a simple matter to search among the dilaton vectors $\vec{a}_\alpha$ in all maximal supergravity theories for sets that have this required form of inner product. The selection of the field strengths must also satisfy the constraints imposed both by the terms coming from the dimension reduction of the $F \wedge F \wedge A$ term in $D = 11$, and by the Chern-Simons modifications to the field strengths. This problem has in fact been solved in [10], where single-scalar solutions for maximal supergravities were extensively studied. In particular, the supersymmetric solutions were classified in [10], and all these solutions have $M_{\alpha\beta}$ satisfying (22). Therefore, the multi-scalar solutions we have obtained in this paper are generalisations of the supersymmetric single-scalar solutions involving $N \geq 2$ participating field strengths, in which the Page charges of the individual field strengths are allowed to become independent free parameters. It is easy to verify that these multi-scalar solutions (23,24) reduce to the single-scalar solutions (7,8) when the Page charges are given by

$$\lambda_\alpha = \frac{\lambda}{\sqrt{N}}, \quad \text{for all } \alpha. \quad (25)$$

Having established that the multi-scalar solutions that we have obtained are generalisations of the supersymmetric single-scalar solutions, it is of interest to examine their supersymmetry properties. We shall discuss this separately for field strengths of each degree $n = 1, 2, 3, 4$. Note that in $D < 2n$ dimensions, the $n$-form field strength can be dualised to a lower degree $(D - n)$-form field strength. We shall always do this. Since we have shown that there is a one-to-one correspondence between the supersymmetric single-scalar solutions and their multi-scalar generalisations, we can classify a multi-scalar solution by the single-scalar case that it degenerates to when the individual
Page charges are set equal. Thus we may refer to [10] for a detailed classification of the supersymmetric single-scalar solutions. In this paper, we shall study the supersymmetry properties when the Page charges become independent.

There is only one 4-form field strength in any maximal supergravity theory, and thus there are no multi-scalar generalisations in this case. In fact, single-scalar solutions with only one participating field strength, of any degree, all preserve $\frac{1}{2}$ of the supersymmetry, and they admit no multi-scalar generalisations. For 3-form field strengths, as shown in [10], the corresponding inner-product matrices $M_{\alpha\beta}$ are given by

$$M_{\alpha\beta} = 2\delta_{\alpha\beta} - \frac{2(D - 6)}{D - 2}.$$  

Thus there are no multi-scalar solutions of the type we are considering for 3-form field strengths either, since this is not of the form given by (22). In fact, single-scalar solutions involving more than one 3-form field strength are all non-supersymmetric [10].

Let us now consider multi-scalar solutions with 2-form field strengths, which give rise to elementary 0-branes or solitonic $(D - 4)$-branes in $D$ dimensions. In this case, the number of participating field strengths in supersymmetric single-scalar solutions can be $N = 1, 2, 3$ or 4, and they occur in dimensions $D \leq 10, 9, 5$ and 4 respectively. In other words, for the 2-form field strengths, only these numbers $N$ of dilaton vectors $\vec{a}_\alpha$ can give rise to matrices $M_{\alpha\beta}$ of the form given in (22). Thus there are multi-scalar solutions with $N = 2, 3$ or 4 non-vanishing scalar fields and independent Page charges. The supersymmetry of these solutions can be studied using the method described in [10], namely by constructing the Bogomol’nyi matrix $\mathcal{M}$ from the Nester form for the supergravity theory. This matrix arises from the commutator of the conserved supercharges, and its zero eigenvalues correspond to unbroken components of $D = 11$ supersymmetry. From the general results in [10], the relevant terms in the Bogomol’nyi matrix for 2-form field strengths are given by

$$\mathcal{M} = m1 + \frac{1}{2}u_{ij}\Gamma_{0ij} + p_i\Gamma_{0i} + \frac{1}{2}v_{ij}\Gamma_{1\hat{2}\hat{3}ij} + q_i\Gamma_{1\hat{2}\hat{3}i},$$  

where $u_{ij}$ and $p_i$ are the electric Page charges for the 2-forms $F_{MNij}$ and $F^{(i)}_M$, coming from the dimensional reduction of the eleven-dimensional 4-form and vielbein respectively. Similarly, $v_{ij}$ and $q_i$ are the corresponding magnetic Page charges. In (27), the 0 index denotes the time coordinate of the (elementary) 0-brane, the hatted indices run over the transverse space of the $y^m$ coordinates of the (solitonic) $(D - 4)$-brane, and the $i, j, k$ indices run over the directions that are compactified in the Kaluza-Klein reduction from eleven dimensions to $D$-dimensional maximal supergravity.

There is in general more than one way to select a set of 2-form field strengths whose dilaton vectors satisfy (22); however, the supersymmetry properties for the different choices are identical,
and hence we shall present only one representative in each case. We find that the (elementary) Page charges, and the eigenvalues of the Bogomol'nyi matrix for the multi-scalar solutions, calculated from (27), are given by

\[
N = 2 : \quad \{p_1, u_{12}\} = \frac{1}{4}\{\lambda_1, \lambda_2\}, \quad \text{for } D \leq 9 \\
\mu = \frac{1}{2}\{0, \lambda_1, \lambda_2, \lambda_1 + \lambda_2\}.
\]

\[
N = 3 : \quad \{u_{12}, u_{34}, u_{56}\} = \frac{1}{4}\{\lambda_1, \lambda_2, \lambda_3\}, \quad \text{for } D \leq 5 \\
\mu = \frac{1}{2}\{0, \lambda_1, \lambda_2, \lambda_3, \lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_3\}.
\]

\[
N = 4 : \quad \{u_{12}, u_{34}, u_{56}, p^*_7\} = \frac{1}{4}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}, \quad \text{for } D = 4 \\
\mu = \frac{1}{2}\{0, \lambda_1 + \lambda_4, \lambda_2 + \lambda_4, \lambda_3 + \lambda_4, \lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\}.
\]

Here a * on a Page charge indicates that the associated field strength is dualised. Thus \(p^*_7\) is the electric charge of the dualised field strength \(F_M^{(7)}\), and so it is the magnetic charge in terms of the original undualised field strength \(F_{MN}^{(7)}\). In other words, it corresponds to a contribution \(p^*_7 \Gamma_{12347}\) in the Bogomol'nyi matrix. We shall discuss this type of dyonic solution later.

As we discussed earlier, the Bogomol'nyi matrix is obtained from the commutator of the conserved Hermitian supercharges, and so its eigenvalues should always be non-negative at the quantum level. However, as classical solutions, where the \(\lambda_\alpha\)'s are just free parameters or integration constants, it is clear from (28) that the eigenvalues will be negative for certain choices of these parameters. In these cases, the quantum positivity argument evidently breaks down, and hence such configurations would be disallowed at the quantum level. Thus we should presumably restrict the choices of parameters so that all the eigenvalues are non-negative. Note that the last entry in the list of eigenvalues is twice the mass of the solution for all the three cases, and the restriction will rule out p-brane solutions where the charges are chosen to make the mass equal to zero. In fact, even if this mass is chosen to be positive, there still can be negative eigenvalues under certain circumstances.

Having obtained the Bogomol'nyi matrices of the multi-scalar solutions for 2-forms, it is now straightforward to analyse their supersymmetry, since the zero eigenvalues correspond to unbroken components of \(D = 11\) supersymmetry. In each of the three cases in (28), the degeneracies of each eigenvalue are equal, with the total number of eigenvalues being 32. Thus for generic values of the parameters \(\lambda_\alpha\), these 2-scalar, 3-scalar and 4-scalar solutions preserve \(\frac{1}{4}, \frac{1}{8}\) and \(\frac{1}{8}\) of the supersymmetry respectively. It is easy to see that when the \(\lambda_\alpha\) are chosen to be equal, we recover the single-scalar solutions with \(\Delta = \frac{4}{N}\), preserving the same fractions of the supersymmetry as in the generic cases.

There is a supersymmetry enhancement for certain choices of the parameters. First of all, we
note that this occurs when any of the parameters $\lambda_\alpha$ is zero, but this corresponds merely to reducing the number of participating field strengths and scalars. Thus we shall assume all the parameters $\lambda_\alpha$ are non-zero. In all the three cases in \eqref{enhancement}, there is supersymmetry enhancement when the mass $m = \frac{1}{4} \sum_\alpha \lambda_\alpha$ is zero; however, this implies that some of the eigenvalues are negative. In fact there can be no supersymmetry enhancement, while still requiring that all the eigenvalues be non-negative, for the cases $N = 2$ and $3$. The situation is different for $N = 4$, and we can have three inequivalent enhancements, given by

\[
\begin{align*}
\lambda_1 &= -\lambda, \quad \lambda_2 = \lambda, \quad \mu = \frac{1}{2} \{0, (\lambda_3 \pm \lambda)_4, (\lambda_4 \pm \lambda)_4, (\lambda_3 + \lambda_4)_8\}, \\
\lambda_1 &= -\lambda, \quad \lambda_2 = \lambda_3 = \lambda, \quad \mu = \frac{1}{2} \{0_{12}, (2\lambda)_4, (\lambda_4 - \lambda)_4, (\lambda_4 + \lambda)_12\}, \\
\lambda_1 &= -\lambda, \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda, \quad \mu = \{0_{16}, \lambda_{16}\},
\end{align*}
\]

where the subscript denotes the degeneracy of each eigenvalue. Thus these three cases preserve $\frac{1}{4}, \frac{3}{8}$ and $\frac{1}{2}$ of the supersymmetry respectively, in contrast to $\frac{1}{8}$ for generic values of the charge parameters. Note that in these cases, although the Bogomol'nyi matrices have no negative eigenvalues when the supersymmetry enhancements occur, the metrics of the solutions still seem to have naked singularities since one of the Page charges $\lambda_\alpha$ is negative. If we relax the condition that the eigenvalues of the Bogomol'nyi matrix should be non-negative, then supersymmetry enhancement can occur for $N = 2$ and $3$ as well. For $N = 2$, the solutions can also preserve a fraction $\frac{k}{8}$ of the supersymmetry, with $k = 4$ for appropriately-chosen non-vanishing $\lambda_\alpha$; for $N = 3$, we can have $k = 2$ or $3$. In the case of $N = 4$, in addition to the supersymmetry enhancements described in \eqref{enhancement}, we can have also $k = 5$ and $6$ if negative eigenvalues are allowed, in which case the solutions preserve more than $\frac{1}{2}$ of the supersymmetry.

So far we have discussed elementary solutions, where the field strengths carry electric charges. The discussion for the solitonic solutions is analogous and the conclusions are the same. In $D = 4$ dyonic solutions can occur, since the dual of a 2-form is again a 2-form. Although, in our multiscalar solutions, all the participating field strengths have the same purely elementary or purely solitonic nature, some can nevertheless be the duals of the original ones in the case $D = 4$, and therefore, in terms of the original field strengths, we can have solutions with mixed electric and magnetic charges. These were called dyonic solutions of the first type in \cite{8}. In \eqref{enhancement}, for the cases $N = 2$ and $3$, we presented solutions with purely electric charges. They also exist for purely magnetic charges, and in $D = 4$ they also exist for mixed dyonic charges of the kind we just

\footnote{This does not violate the classification of supermultiplets given in \cite{11}, since non-negativity of the commutator of supercharges was assumed there. In fact, if we require that all the eigenvalues of the Bogomol'nyi matrix be non-negative, then all the solutions preserve no more than $\frac{1}{2}$ of the supersymmetry, which is consistent with the classification in \cite{17}.}
discussed. For the case $N = 4$, which occurs only in $D = 4$, the situation is different, in that there are no purely electric or purely magnetic solutions; they are intrinsically dyonic.

Before finishing the discussion of the 2-form solutions, we should remark that there is a further subtlety for the case of $N = 4$ scalars. As was observed in [10], the bosonic equations of motion governing the single-scalar solutions leave the signs of the Page charges for the participating field strengths undetermined. If we choose our conventions so that the mass is always given by $m = \frac{1}{4} \sum \alpha \lambda_\alpha$, then the bosonic equations allow solutions where the individual Page charges $P_\alpha$ can be either $+\frac{1}{4}\lambda_\alpha$ or $-\frac{1}{4}\lambda_\alpha$. If we calculate the eigenvalues of the Bogomol'nyi matrices, we find that in the cases $N = 1, 2, 3$, they are insensitive to the choices of the signs of the Page charges. However, for $N = 4$ the signs do matter, and there are precisely two inequivalent sets of eigenvalues that can arise. Only one of these includes zero eigenvalues, and this is the supersymmetric single-scalar solution with $N = 4$ participating field strengths, whose generalisation to 4 scalars we have discussed above. The other single-scalar solution also generalises to a 4-scalar solution, which is inequivalent to the one described above. For this case, we find that the eigenvalues are

$$\mu = \frac{1}{2}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_1 + \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_4, \lambda_1 + \lambda_3 + \lambda_4, \lambda_2 + \lambda_3 + \lambda_4\},$$

with each eigenvalue having degeneracy 4. Note that in this case none of the eigenvalues is proportional to the mass, for generic $\lambda_\alpha$. This solution reduces to supersymmetric lower-$N$ cases when any of the parameters is zero. Supersymmetry enhancements also occur for certain non-vanishing Page charges; however, if we require that all the eigenvalues be non-negative, then this 4-scalar solution is always non-supersymmetric for all allowed non-vanishing Page charges.

We now turn our attention to multi-scalar solutions for 1-form field strengths, which give rise to solitonic $(D-3)$-branes in $D$ dimensions. As has been shown in [10], the number of participating field strengths in supersymmetric single-scalar solutions can be $N = 1, 2, \ldots, 7$. The solutions with $N = 1, 2$ and 3 occur in dimensions $D \leq 9, 8$ and 6 respectively. There are two inequivalent supersymmetric solutions for $N = 4$, one of which occurs in $D \leq 6$ and the other in $D = 4$. The solutions with $N = 5, 6$ and 7 all occur in $D = 4$ only. All the solutions with $N \geq 2$ can be generalised to $N$-scalar solutions, as given in [23]. The relevant terms in the Bogomol’nyi matrix for these solutions are given by

$$\mathcal{M} = mI + \frac{1}{6}v_{ijk} \Gamma_{i\hat{2}j\hat{k}} + \frac{1}{2}q_{ij} \Gamma_{i\hat{2}j\hat{2}} + v^* \Gamma_{012} + v^*_i \Gamma_{0i},$$

where $v_{ijk}$ and $q_{ij}$ are the Page charges of the field strengths $F_{Mijk}$ and $F^{(ij)}_M$ coming from the 4-form and vielbein respectively, $v^*$ is the Page charge of the dual of the 4-form $F_{MNPQ}$ (in $D = 5$ only), and $v^*_i$ are the Page charges of the duals of the 3-forms $F_{MNPi}$ (in $D = 4$ only). All these
Page charges are magnetic, since the elementary ansatz given in (3) does not encompass 1-form field strengths. We find that the Page charges, and the eigenvalues of the Bogomol'nyi matrix, for the $N$-scalar solutions for the 1-forms are given by

$$
N = 2 : \quad \{ q_{12}, v_{123} \} = \frac{1}{4} \{ \lambda_1, \lambda_2 \} , \quad \text{for } D \leq 8 ,
$$

$$
\mu = \frac{1}{2} \{ 0, \lambda_1, \lambda_2, \lambda_1 \} ,
$$

$$
N = 3 : \quad \{ q_{12}, q_{45}, v_{123} \} = \frac{1}{4} \{ \lambda_1, \lambda_2, \lambda_3 \} , \quad \text{for } D \leq 6 ,
$$

$$
\mu = \frac{1}{2} \{ 0, \lambda_1, \lambda_2, \lambda_3, \lambda_12, \lambda_13, \lambda_23, \lambda_123 \} ,
$$

$$
N = 4' : \quad \{ q_{12}, q_{45}, v_{123}, v_{345} \} = \frac{1}{4} \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \} , \quad \text{for } D \leq 6 ,
$$

$$
\mu = \frac{1}{2} \{ 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_12, \lambda_13, \lambda_14, \lambda_23, \lambda_123, \lambda_124, \lambda_134, \lambda_1234 \} ,
$$

$$
N = 4 : \quad \{ q_{12}, q_{34}, q_{56}, v_{1237} \} = \frac{1}{4} \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \} , \quad \text{for } D = 4 ,
$$

$$
\mu = \frac{1}{2} \{ 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_12, \lambda_13, \lambda_14, \lambda_23, \lambda_123, \lambda_124, \lambda_134, \lambda_1234 \} ,
$$

$$
N = 5 : \quad \{ q_{12}, q_{34}, q_{56}, v_{1237}, v_{347} \} = \frac{1}{4} \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \} , \quad \text{for } D = 4 ,
$$

$$
\mu = \frac{1}{2} \{ 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_12, \lambda_13, \lambda_14, \lambda_23, \lambda_123, \lambda_124, \lambda_134, \lambda_1234 \} ,
$$

$$
N = 6 : \quad \{ q_{12}, q_{34}, q_{56}, v_{1237}, v_{347}, v_{567} \} = \frac{1}{4} \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \} , \quad \text{for } D = 4 ,
$$

$$
\mu = \frac{1}{2} \{ 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_12, \lambda_13, \lambda_14, \lambda_23, \lambda_123, \lambda_124, \lambda_134, \lambda_1234 \} ,
$$

$$
N = 7 : \quad \{ q_{12}, q_{34}, q_{56}, v_{1237}, v_{347}, v_{567}, v_{7}^* \} = \frac{1}{4} \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \} , \quad \text{for } D = 4 ,
$$

Here, for convenience, we have defined $\lambda_{\alpha \beta \ldots \gamma} = \lambda_\alpha + \lambda_\beta + \cdots + \lambda_\gamma$. Note that we presented only one representative set of Page charges among many possibilities for each case, since they have the identical eigenvalues. In each case, the degeneracy of each eigenvalue for a particular solution is the same, with the total number of eigenvalues being 32. The last eigenvalue in each case is twice the mass of the corresponding solution. The eigenvalues of a higher-$N$ case reduce to those of all the lower-$N$ cases when certain of the $\lambda_\alpha$’s are set to zero, and hence we shall only consider solutions with non-vanishing $\lambda_\alpha$. When $N = 5$, there are two inequivalent ways of reducing to $N = 4$, giving the two cases that we denote by $N = 4$ and $N = 4'$. Note that the eigenvalues for the first three cases are identical to those for the 2-form field strengths that we have discussed previously.

For generic values of $\lambda_\alpha$, the solutions preserve $2^{-N}$ of the supersymmetry for $N = 1, 2, 3, 4$. The $N = 4'$ solutions preserve $\frac{1}{8}$ of the supersymmetry and all the $N = 5, 6, 7$ solutions preserve $\frac{1}{16}$. It is easy to verify that when all $\lambda_\alpha$ are equal, the solutions reduce to the single-scalar solutions with $\Delta = \frac{4}{N}$, which were discussed in [10]. The single-scalar solutions preserve the same fractions
of supersymmetry as their generic multi-scalar extensions. For certain choices of $\lambda_\alpha$, the multi-scalar solutions can have supersymmetry enhancement. In particular, it occurs when the mass $m = \frac{1}{4} \sum_\alpha \lambda_\alpha$ is zero; however, as in the case of 2-form field strengths, the Bogomol’nyi matrices for the solutions then have indefinite signature. If we require that all the eigenvalues be non-negative, then for the cases $N = 2, 3$ and 4 there is no supersymmetry enhancement when the $\lambda_\alpha$’s are non-vanishing. The analysis of the supersymmetry enhancement for the case $N = 4'$ is equivalent to the $N = 4$ case for 2-form field strengths, which we have already discussed. For $N = 5, 6$ and 7, there are many ways to choose the parameters to achieve supersymmetry enhancement, and we shall not present all of them. Choosing the parameters $\lambda_\alpha$ appropriately, the $N = 5$ solutions can preserve $\frac{k}{16}$ of the supersymmetry with $k = 2, 3, 4$; similarly $k = 2, 3, 4, 5$ or 6 for the $N = 6$ case; and $k = 2, 3, 4, 5, 6$ or 7 when $N = 7$, in contrast to $k = 1$ for generic values of $\lambda_\alpha$ for each of these values of $N$. If we relax the condition that all the eigenvalues of the Bogomol’nyi matrices be non-negative, further supersymmetry enhancements can occur, which includes the massless $p$-brane solutions. From (32), it is straightforward to obtain the possible fractions of the supersymmetry that a solution can preserve, and we shall not present the details; the maximal fractions for non-vanishing Page charges turn out to be $\frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{5}{8}$ and $\frac{1}{2}$ for $N = 2, 3, 4', 4, 5, 6$ and 7 respectively.

As in the case of 2-form solutions, the eigenvalues of the Bogomol’nyi matrix are sensitive to the signs of the Page charges for certain 1-form solutions. If we choose our conventions so that the mass is always given by $m = \frac{1}{4} \sum_\alpha \lambda_\alpha$, then bosonic equations allow solutions where the individual Page charges $P_\alpha$ can either be $+\frac{1}{4} \lambda_\alpha$ or $-\frac{1}{4} \lambda_\alpha$. For the 1-form solutions with $N = 1, 2, 3$ and 4, the eigenvalues are insensitive to the choice of the signs of the Page charges. However, for the $N = 4', 5, 6$ and 7 cases, the signs do matter, and there are precisely two inequivalent sets of eigenvalues that can arise. One of these, which includes zero eigenvalues for generic $\lambda_\alpha$, is presented in (32). The other does not have zero eigenvalues for generic Page charges. For $N = 7$, we find that the eigenvalues are given by

$$
\mu = \frac{1}{2} \{ \lambda_7, \lambda_{14}, \lambda_{25}, \lambda_{36}, \lambda_{123}, \lambda_{156}, \lambda_{246}, \lambda_{345}, \lambda_{1267}, \lambda_{1357}, \lambda_{2347}, \lambda_{4567}, \lambda_{12457}, \lambda_{13467}, \lambda_{23567}, \lambda_{123456} \}.
$$

It reduces to the $N = 6, 5$ and $4'$ cases when $\lambda_4$, $\lambda_5$ and $\lambda_6$ are successively set to zero. In each of these four cases, none of the eigenvalues is proportional to the mass, for generic $\lambda_\alpha$. When all $\lambda_\alpha$’s are set equal, the solutions reduce to single-scalar solutions that break all the supersymmetry. If we require that all the eigenvalues be non-negative, the $N = 4'$ solutions will always be non-supersymmetric for all non-vanishing $\lambda_\alpha$. However, for the $N = 5, 6$ and 7 cases, there can
be supersymmetry enhancement for certain non-vanishing choices of $\lambda_\alpha$. Choosing the parameters $\lambda_\alpha$ appropriately, the $N = 5$ solutions can preserve $\frac{k}{16}$ of the supersymmetry with $k = 1, 2, 3, 4$; similarly $k = 1, 2, 3, 4, 5, 6$ for $N = 6$ and $k = 1, 2, 3, 4, 5$ for $N = 7$. Thus although the corresponding single-scalar solutions are non-supersymmetric, their multi-scalar generalisations can be supersymmetric. Further supersymmetry enhancements can occur if we relax the condition that all the eigenvalues of the Bogomol’nyi matrix be non-negative. It is straightforward to enumerate the possibilities from the eigenvalues given above.

To conclude, in this paper we have shown that the supersymmetric single-scalar $p$-brane solutions in the maximal supergravity theories with $N$ participating field strengths for $N = 1, 2, \ldots, 7$ can be extended to $N$-scalar solutions. The $N$ Page charges, which were equal in the single-scalar solutions, become independent free parameters. For the 4-form and 3-form field strengths, $N = 1$; for the 2-form field strengths $N = 1, 2, 3, 4$; and for the 1-form field strengths $N = 1, 2, \ldots, 7$. We summarise the 2-form and 1-form solutions in the following table:

| Dim. | 2-Forms | 1-Forms |
|------|---------|---------|
| $D = 10$ | $N = 1$ | $p = 0, 6$ |
| $D = 9$ | $N = 2$ | $p = 0, 5$ | $N = 1$ | $p = 6$ |
| $D = 8$ | $p = 0, 4$ | $N = 2$ | $p = 5$ |
| $D = 7$ | $p = 0, 3$ | | $p = 4$ |
| $D = 6$ | $p = 0, 2$ | $N = 3, 4'$ | $p = 3$ |
| $D = 5$ | $N = 3$ | $p = 0, 1$ | | $p = 2$ |
| $D = 4$ | $N = 4$ | $p = 0$ | $N = 4, 5, 6, 7$ | $p = 1$ |

Table 1: Multi-scalar $p$-brane solutions

Here we list the highest dimensions where $p$-brane solutions with the indicated numbers $N$ of field strengths first occur. They then occur also at all lower dimensions. All these solutions preserve certain fractions of the $D = 11$ supersymmetry for generic values of the Page charges. Further supersymmetry enhancement can occur for certain choices of non-vanishing Page charges, which sometimes gives rise to solutions whose Bogomol’nyi matrix has indefinite signature. In some cases, however, supersymmetry enhancement can occur while still avoiding negative eigenvalues in the Bogomol’nyi matrix. For all the multi-scalar $p$-brane solutions, the mass is equal to the sum of the Page charges. Thus in principle it is possible to have massless $p$-brane solutions. In fact, when the mass is set to zero the supersymmetry of the solutions is enhanced. However, the Bogomol’nyi matrix will have an indefinite signature. Since supersymmetry enhancement occurs only when some of the Page charges are negative, it follows from (23) that the metric seems to have naked
singularity regardless of whether the Bogomol’nyi matrix has non-negative eigenvalues or not. It
seems that all the Page charges have to be positive in order to ensure that the metric does not
have any naked singularities. The status of the naked singularities in these solutions is unclear. On
the other hand, we would certainly expect that, owing to the quantum positivity argument, only
solutions with non-negative eigenvalues of the Bogomol’nyi matrix are acceptable. In particular,
this seems to cast doubt on the validity of the massless super $p$-brane solutions in the spectrum of
the theory.

In this paper, we have been primarily concerned with purely elementary or purely solitonic multi-
scalar solutions. In $D = 6$ and $D = 4$, dyonic solutions can also occur. There are two different
types of dyonic solutions. In dyonic solutions of the first type, each individual field strength carries
either electric or magnetic charge but not both. On the other hand, in dyonic solutions of the
second type, each individual field strength carries both electric and magnetic charges. In $D = 6$,
one can only construct dyonic solutions \[18\] of the second type, with just one 3-form field strength
involved \[10\], and hence there is no multi-scalar extension in this case. In $D = 4$, one can construct
dyonic solutions of both the first and second types. We presented the multi-scalar generalisations
of single-scalar dyonic solutions of the first type for all $N = 2, 3$ and 4 field strength cases. There
are two single-scalar dyonic solutions of the second type in $D = 4$, with $N = 2$ and 4 \[10\]. It would
be of interest to generalise these to multi-scalar solutions.

For the non-supersymmetric single-scalar solutions discussed in \[10\], the dot products of the
dilatonic vectors $\vec{a}_\alpha$ do not satisfy \[22\], and thus multi-scalar solutions of the kind we have discussed
in this paper cannot occur. In fact, the multi-scalar solutions for these cases have a much more com-
plicated form. Although the metrics of all these solutions can still be asymptotically Minkowskian
as $r \to \infty$, the metric structure will become very complicated near the origin. We showed that
for the supersymmetric multi-scalar solutions, owing to the fact that the Page charges become
independent, supersymmetry enhancement can occur for appropriately-chosen Page charges. It
would be of interest to know whether the multi-scalar generalisations of the non-supersymmetric
single-scalar solutions can also become supersymmetric for certain choices of Page charges.

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