Rationality properties of manifolds containing quasi-lines

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Abstract

Let $X$ be a complex, rationally connected, projective manifold. We show that $X$ admits a modification $\tilde{X}$ that contains a quasi-line, i.e. a smooth rational curve whose normal bundle is a direct sum of copies of $O_{\mathbb{P}^1}(1)$. For manifolds containing quasi-lines, a sufficient condition of rationality is exploited: There is a unique quasi-line from a given family passing through two general points. We define a numerical birational invariant, $e(X)$, and prove that $X$ is rational if and only if $e(X) = 1$. If $X$ is rational, there is a modification $\tilde{X}$ which is strongly-rational, i.e. contains an open subset isomorphic to an open subset of the projective space whose complement is at least 2-codimensional. We prove that strongly-rational varieties are stable under smooth, small deformations. The argument is based on a convenient characterization of these varieties.

Finally, we relate the previous results and formal geometry. This relies on $\tilde{c}(X,Y)$, a numerical invariant of a given quasi-line $Y$ that depends only on the formal completion $\tilde{X}|_Y$. As applications we show various instances in which $X$ is determined by $\tilde{X}|_Y$. We also formulate a basic question about the birational invariance of $\tilde{e}(X,Y)$.

Introduction

Classical examples of rational projective manifolds are given by usually elementary, sometimes ingenious, geometric constructions of linear systems, yielding birational maps (e.g. projections from subvarieties). Related to the Lüroth problem in dimension at least three, several fairly sophisticated techniques for proving non-rationality of some Fano manifolds have been developed (see e.g. [11]). Using deformation theory of rational curves, Kollár, Miyaoka and Mori introduced in [13] the very useful class of rationally connected varieties, generalizing the classes of both rational and Fano manifolds. Rational connectedness admits several convenient characterizations and is invariant under deformations and birational isomorphism. It is therefore natural to try to understand rationality within the larger class of rationally connected manifolds.

Let $X$ be a complex projective manifold of dimension $n \geq 2$. $X$ is rationally connected if two general points of it may be joined by a rational curve. Equivalently, $X$ contains a smooth rational curve with ample normal bundle, see [13], [14]. A smooth rational curve $Y \subset X$ is called a quasi-line (see [11]) if its normal bundle is isomorphic to $\bigoplus_{1}^{n-1} O_{\mathbb{P}^1}(1)$. $X$ is called strongly-rational (see [11]) if there exists a birational map $\varphi : X \dashrightarrow \mathbb{P}^n$ which is an isomorphism from an open subset $U$ onto an open subset $V$, whose complement in $\mathbb{P}^n$ is at least 2-codimensional. Note that strongly-rational manifolds contain quasi-lines (the pull-back
of a line contained in \( V \) gives rise to a quasi-line on \( X \)). Therefore we have the diagram:

\[
\begin{align*}
\text{rational} & \quad \xrightarrow{\text{strongly-rational}} \quad \text{rationally connected} \\
& \quad \xleftarrow{\text{contains quasi-lines}} \\
\end{align*}
\]

In the first section we quote from [8] and [9] two rather general classes of examples of rational manifolds. Moreover, we prove a new rationality criterion, Theorem 1.3.

In Section 2, we show that any rationally connected manifold, after being suitably blown-up, contains quasi-lines, Theorem 2.3. Note that, in a similar vein, by [6], rational manifolds become strongly-rational after suitable blowing-ups. The proof of Theorem 2.3 applies to show the existence of almost-lines, i.e. quasi-lines \( Y \) such that \( D \cdot Y = 1 \) for some divisor \( D \) on \( X \). This completes Theorem 2.1 from [1].

In Section 3, we use quasi-lines to characterise rationality and to define, for each rationally connected manifold \( X \), a birational numerical invariant, denoted \( e(X) \). We first introduce and compute for some examples, the number \( e(X,Y) \) of quasi-lines from a given family that pass through two general points of \( X \); for instance, when \( X \) is a smooth cubic threefold in \( \mathbb{P}^4 \) and \( Y \) is a general conic, \( e(X,Y) = 6 \), see Proposition 3.2. Then, \( e(X) \) represents the minimum among \( e(X',Y') \), where \( X' \) is obtained from \( X \) by a sequence of blowing-ups with smooth centers. In Theorem 3.3, we prove that \( X \) is rational if and only if \( e(X) = 1 \). However, note that \( e(X) \) seems to be very difficult to compute. In order to get the rationality via quasi-lines, the key is to show that \( e(X,Y) = 1 \) for a certain quasi-line \( Y \), see Proposition 3.1.

Section 4 contains a convenient characterization of strongly-rational manifolds, Theorem 4.2. As a consequence we show in Theorem 4.5 that strongly-rational manifolds are stable with respect to small deformations. Note that such an invariance property is not expected to hold for rational manifolds.

Section 5 relates the preceding results to formal geometry. To each quasi-line \( Y \subset X \) we associate a “local” invariant denoted \( \tilde{e}(X,Y) \). It depends only on the formal completion \( \hat{X}|_{Y} \). Theorem 5.1 shows that \( e(X,Y) = \tilde{e}(X,Y) \cdot b(X,Y) \), where \( b(X,Y) = [K(\hat{X}|_{Y}) : K(X)] \). Here \( K(X) \) is the field of rational functions on \( X \) and \( K(\hat{X}|_{Y}) \) is the field of formal rational functions of \( X \) along \( Y \), see [7]. As applications of this formula we give various examples of instances when the formal completion of \( X \) along the quasi-line \( Y \) determines \((X,Y)\) up to isomorphism. They include the case mentioned above of a general conic on a cubic threefold, Corollary 5.9.

In the last section we address the basic question: Does the local invariant \( \tilde{e}(X,Y) \) depend only on the field \( K(\hat{X}|_{Y}) \)? A positive answer would have nice consequences, e.g. a completely new proof of the non-rationality of the smooth cubic threefold in \( \mathbb{P}^4 \).

In the Appendix, we show via a toric calculation, that a certain useful property of quasi-lines does not hold in general.

We shall work over the field of complex numbers. Unless otherwise stated, we follow the usual conventions and notation in Algebraic Geometry (see e.g. [5]).
1 Some rational varieties

Let \( X \subset \mathbb{P}^N \) be a projective manifold of dimension \( n \geq 2 \). In dimension two, the famous Castelnuovo criterion characterizes rationality by the vanishing of two numbers which are birational invariants of \( X \); in particular, rationality and rational connectedness are equivalent. For \( n \geq 3 \), deciding the rationality of \( X \) may be a quite difficult problem. Rationally connected manifolds, which are easier to understand, form a much larger class than rational ones.

Many examples of rational manifolds come from more precise biregular classification statements. We would like to exemplify this principle by two rather general results. To state them we recall some numerical invariants of \( X \).

We denote by \( g \) the sectional genus of \( X \), that is the genus of the curve got by intersecting \( X \) with \( n-1 \) general hyperplanes in \( \mathbb{P}^N \). We let \( d \) be the degree of \( X \). Finally, let \( q := h^1(X, \mathcal{O}_X) \) be the irregularity of \( X \).

**Theorem 1.1.** Assume that \( d \geq 2g - 1 \) and \( q = 0 \). Then \( X \) is rational, unless \( X \) is a cubic hypersurface, \( n \geq 3 \).

This statement is a consequence of the precise biregular classification, given in [S], Corollaries 8, 9 and 10, of all manifolds satisfying conditions \( d \geq 2g - 1 \), \( n \geq 3 \). The classification, due to Fujita, of the so called “del Pezzo manifolds”, is also used. It corresponds to the case \( g = 1 \), which includes the exception in the statement of Theorem 1.1. Note that the bound \( d \geq 2g - 1 \) is sharp. Indeed, the quartic threefold in \( \mathbb{P}^4 \) and the complete intersection of a quadric and a cubic in \( \mathbb{P}^5 \) satisfy \( d = 2g - 2 \). However, they are known, as well as the cubic threefold of \( \mathbb{P}^4 \), to be non-rational. See e.g. [11] for a discussion of these very delicate results.

The previous theorem shows that, for fixed sectional genus, regular manifolds of “high” degree are rational. On the other hand, many examples of manifolds of “small” degree are known to be rational. Moreover, note that deciding the rationality property is particularly difficult when \( X \) is a Fano manifold with \( b_2(X) = 1 \). In this direction, we quote the following recent result:

**Theorem 1.2.** (cf. [9]) Let \( X \subset \mathbb{P}^N \) be non-degenerate and assume that \( d \leq N \). Then one of the following holds:

(i) \( X \) is Fano and \( b_2(X) = 1 \), or
(ii) \( X \) is rational.

The bound \( d \leq N \) is clearly the best possible one. A hypersurface of degree \( N + 1 \) is neither rational, nor Fano. Again, the rationality comes a posteriori, using a classification result. In fact, manifolds as in (ii) may be completely described: There are 6 infinite series and 14 “sporadic” examples, see [9].

Next we prove a result that allows one to deduce the rationality directly from the existence of a suitable rational submanifold of \( X \). To the best of our knowledge, this theorem seems to have been overlooked in the classical literature. Our proof depends on Hironaka’s desingularisation theory from [6] and on basic properties of rationally connected manifolds (cf. [13]).
Theorem 1.3. Let $X$ be a projective variety and $|D|$ a complete linear system of Cartier divisors on it. Let $D_1, \ldots, D_s \in |D|$ and put $W_i := D_1 \cap \cdots \cap D_i$ for $1 \leq i \leq s$. Assume that for all $i$, $W_i$ is smooth, irreducible and has dimension $n - i$. Assume moreover that there is a divisor $E$ on $W =: W_s$ and a linear system $\Lambda \subset |E|$ such that

(i) $\varphi_\Lambda : W \to \mathbb{P}^{n-s}$ is birational, and

(ii) $|D|_W - E| \neq \emptyset$.

Then $X$ is rational.

Proof. We proceed by induction on $s$. We explain the case $s = 1$, the general induction step being completely similar. So, let $W \in |D|$ be a smooth, irreducible Cartier divisor such that $\varphi_\Lambda : W \to \mathbb{P}^{n-1}$ is birational for $\Lambda \subset |E|$, $E \in \text{Div}(W)$ and $|D|_W - E \neq \emptyset$. Note that $W$ is contained in the smooth locus of $X$. So, replacing $X$ by its desingularisation, we may assume $X$ to be smooth. As $W$ is rational, it is in particular rationally connected; so by [13], there is some smooth rational curve $Y \subset W$ with ample normal bundle. We have $Y \cdot E > 0$ since $E$ moves and $Y \cdot (D|_W - E) \geq 0$ by condition (ii). It follows that $Y \cdot D > 0$. Looking at the standard exact sequence of normal bundles, we get that $N_{Y|X}$ is ample. So, again by [13], $X$ is rationally connected and, in particular, $q(X) = h^1(X, O_X) = 0$. The standard exact sequence:

$$0 \to O_X \to O_X(D) \to O_W(D) \to 0,$$

shows that $\dim |D| = \dim |D|_W + 1 \geq \dim |E| + 1 \geq n$. Choose a pencil $(W, W') \subset |D|$, containing $W$, such that $W'|_W = E_0 + E_1$, with $E_1 \in \Lambda$ and $E_0 \geq 0$. Now, by the theory in [6], we may use blowing-ups with smooth centers contained in $W \cap W'$, such that, after taking the proper transforms of the elements of our pencil, to get:

(a) $\text{supp}(E_0)$ has normal crossing;

(b) $\Lambda$ is base-points free (so $\varphi : W \to \mathbb{P}^{n-1}$ is a birational morphism).

Next, by blowing-up the components of $\text{supp}(E_0)$, we may also suppose that $E_0 = 0$, i.e. $D|_W$ is linearly equivalent to $E$. Now, using the previous standard exact sequence and the fact that $q(X) = 0$, it follows that $|D|$ is base-points free. But we have that $D^n = (D|_W)^{n-1}_W = 1$, so $\varphi|_D$ is a birational morphism onto $\mathbb{P}^n$. \hfill $\square$

Example 1.4. Let $X \subset \mathbb{P}^{n+d-2}$ be a non-degenerate projective variety of dimension $n \geq 2$ and degree $d \geq 3$. Then $X$ is rational, unless it is a smooth cubic hypersurface, $n \geq 3$.

Indeed, we may assume $X$ to be smooth; otherwise, use a projection from a singular point. We may also suppose that $X$ is linearly normal (if not, use again a projection from one of its points). One sees easily that such a linearly normal, non-degenerate manifold $X \subset \mathbb{P}^{n+d-2}$ has anticanonical divisor linearly equivalent to $n - 1$ times the hyperplane section, i.e. they are exactly the so called “classical del Pezzo manifolds”. They were classified by Fujita in a series of papers, see e.g. [11] for a survey of his argument. As Fujita’s proof is quite long and difficult, we show how Theorem 1.3 above may be used to prove directly the rationality of $X$ if $d \geq 4$. Consider the surface $W$ got by intersecting $X$ with $n - 2$ general hyperplanes. Note that $W$ is a non-degenerate, linearly normal surface of degree $d$ in $\mathbb{P}^d$, so it is a del Pezzo surface. As such, $W$ is known to admit a representation $\varphi : W \to \mathbb{P}^2$ as the blowing-up of $9 - d$ points (in general position). Let $L \subset W$ be the pull-back via $\varphi$ of a general line in $\mathbb{P}^2$. It is easy to see that $L$ is a cubic rational curve in the embedding of $W$ into $\mathbb{P}^d$. So, for $d \geq 4$, $L$ is contained in a hyperplane of $\mathbb{P}^d$. This shows that the conditions of Theorem 1.3 are fulfilled.
for $X$, $|D|$ being the system of hyperplane sections. We also see that Theorem 1.3 is sharp, as the previous argument fails exactly for the case of cubics.

In the remaining of this section we slightly generalize the fibration Theorem 1.12 in [10]. As a consequence we get a rationality criterion, Corollary 1.9. It will be convenient to refer to a couple $(X,Y)$, where $Y$ is a smooth rational curve with ample normal bundle, as to a model, cf. [10]. Talking about a model $(X,Y)$, we shall often replace $(X,Y)$ by $(X,Y')$, where $Y'$ is a deformation of $Y$ (we write $Y' \sim Y$). A morphism of models, $(X,Y) \to (X',Y')$, is a morphism $X \to X'$ that maps $Y$ isomorphically to $Y'$.

First, we recall the above mentioned fibration theorem:

**Theorem 1.5** (cf. [10], 1.12). Let $(X,Y)$ be a model and let $D$ be a divisor such that $D \cdot Y = 1$ and $\dim |D| =: s \geq 1$. Then there exists $Y' \sim Y$ and a diagram of models

$$
(Z,Y') \longrightarrow (\tilde{X},Y') \longrightarrow (\mathbb{P}^s,l) 
$$

such that

(i) $\sigma$ is a sequence of blowing-ups,

(ii) $\varphi$ is surjective, with connected fibres,

(iii) any smooth fibre of $\varphi$ is rationally connected,

(iv) $l \subset \mathbb{P}^s$ is a line and $Z = \varphi^{-1}(l)$ is smooth, and

(v) $\tilde{Y}'$ is a section for $\varphi|_Z$.

Next, we generalize it to the case when $D \cdot Y \geq 2$. To see this, we observe the behaviour of the normal bundle of the curve when $X$ is blown-up at a point lying on the curve.

**Lemma 1.6.** Let $C \subset X$ be a smooth curve, $p \in C$ a point and $\sigma : \tilde{X} \rightarrow X$ the blowing-up of $X$ at $p$. If $\tilde{C}$ is the strict transform of $C$, then

$$
N_{\tilde{C}|\tilde{X}} \simeq \sigma^* (N_{C|X} \otimes \mathcal{O}_C(-p)).
$$

**Proof.** Let $\{U_\alpha\}$ be a covering of $C$ with open subsets of $X$, $p \in U_0$. Let $(u_1^0, \ldots, u_n^0)$ be local coordinates on $U_\alpha$ such that $u_1^0, \ldots, u_{n-1}^0$ are local equations for $C$, and $u_n$ is a local equation for $p$ along $C$. If

$$
u_i^\alpha = \sum_{j=1}^{n-1} h_{ij}^\alpha u_j^\beta
$$

for every $i = 1, \ldots, n-1$ on $U_\alpha \cap U_\beta$, then $c^\alpha_\beta = (h_{ij}^\alpha \mid_C)$ represents the transition function for $N_{C|X}$, from $U_\alpha$ to $U_\beta$. The open covering $\{U_\alpha\}$ induces an open covering $\{V_\alpha\}$ of $\tilde{C}$: for $\alpha \neq 0$, $V_\alpha = U_\alpha$. For $\alpha = 0$, the open subset $V_0$ is an open subset with local coordinates $(\xi_1, \ldots, \xi_{n-1}, u_0^n)$ such that $u_0^n = u_0^0 \xi_i$. It follows that $N_{\tilde{C}|\tilde{X}}$ is given by the transition functions $c^\alpha_\beta$, if $\alpha \neq 0$ and $\beta \neq 0$, and by $\frac{1}{u_0^n} c^\alpha_\beta$ if not. But the constant function 1 and $\frac{1}{u_0^n}$ are transition functions of $\mathcal{O}_C(-p)$ relative to the covering $\{U_\alpha\}$, hence the result. \qed
Now we can rephrase Theorem 1.5 as follows:

**Theorem 1.7.** Let \((X, Y)\) be a model such that \(N_Y|_X = \bigoplus_{i=1}^{n-1} \mathcal{O}_Y(a_j)\) with \(a_1 \leq \cdots \leq a_{n-1}\), and let \(D\) be a divisor such that \(D \cdot Y =: d > 0\) with \(a_1 \geq d\) and \(\dim |D| \geq d\). Then, there is \(\tilde{X}\) a blow-up of \(X\) and a diagram of models

\[
(Z, Y') \to (\tilde{X}, \tilde{Y}') \xrightarrow{\varphi} (\mathbb{P}^{\dim |D| - d + 1}, l)
\]

such that

(i) \(\varphi\) is surjective, with connected fibres,
(ii) any smooth fibre of \(\varphi\) is rationally connected,
(iii) \(Z = \varphi^{-1}(l)\) is smooth, and
(iv) \(\tilde{Y}'\) is a section for \(\varphi|_Z\).

**Proof.** We may suppose that \(|D|\) is free from fixed components and that \(Y\) does not meet the base locus of \(|D|\). Indeed, for the latter, a general deformation of \(Y\) avoids a closed subset of codimension \(\geq 2\) and in the decomposition of its normal bundle, \(a_1 \geq d\) (the function \(- \min a_j\) is upper-semicontinuous, see [12], Lemma II.3.9.2). We continue by blowing up \(d - 1\) points on \(Y\). We take \(X'\) to be the new variety and \(Y'\) the strict transform of \(Y\); then by the above lemma \((X', Y')\) is a model. Moreover, the divisors linearly equivalent to \(D\) through the \(d - 1\) points determine a linear system \(|D'|\) on \(X'\). We have \(D' \cdot Y' = 1\) and \(\dim |D'| \geq 1\), so Theorem 1.5 applies.

**Corollary 1.8.** Let \((X, Y)\) be a model with \(N_Y|_X = \bigoplus_{i=1}^{n-1} \mathcal{O}_Y(a_j)\), \(a_1 \leq \cdots \leq a_{n-1}\) and let \(D\) be a divisor such that \(0 < D \cdot Y =: d \leq a_1\) and \(\dim |D| \geq n + d - 1\). Then \(X\) is rational.

Finally, we state the following corollary from which Theorem 1.5 stemmed.

**Corollary 1.9.** Let \(X\) be a projective variety and \(|D|\) a complete linear system of Cartier divisors on it. Let \(D_1, \ldots, D_s \in |D|\) and put \(W_i =: D_1 \cap \cdots \cap D_i\) for \(1 \leq i \leq s\). Assume that for all \(i\), \(W_i\) is smooth, irreducible and has dimension \(n - i\). Assume moreover that there is a divisor \(E\) on \(W =: W_s\) and a smooth rational curve \(Y \subset W\) such that:

(i) \(N_Y|_W \simeq \bigoplus_{i=1}^{s-1} \mathcal{O}_{W}(a_i)\), \(0 < d \leq a_1 \leq \cdots \leq a_{s-1}\), where \(d =: E \cdot Y\),
(ii) \(\dim |E| \geq s + d - 1\), and
(iii) \(|D|_W - E| \neq \emptyset\).

Then \(X\) is rational.

**Proof.** We proceed as in the proof of Theorem 1.5. After suitable blowing-ups we get \(D \cdot Y = d\) and \(\dim |D| \geq n + d - 1\). So the above corollary applies.

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2 Existence of quasi-lines and a first application

In this section we show that a rationally connected manifold, up to blowing it up along smooth subvarieties, contains quasi-lines. The proof depends on the following considerations about elementary transforms, which may be of some independent interest.
If $M$ is a smooth variety and $V \to M$ a vector bundle, then we shall use the classical convention for the projective space that is most suitable for the present work. Accordingly, the associated projective bundle of $V$ will be $P(V) = \text{Proj}(\text{Sym} V^*)$.

Let $C$ be a smooth curve, $V \to C$ a vector bundle of rank $n$ and $\pi : P(V) \to C$ the corresponding projection. If $F$ is a fibre of $P(V)$, $\pi(F) = c$, and $L \subset F$ a hyperplane, the elementary transform of $P(V)$ with center $L$, denoted by $\text{elm}_L P(V)$, is the projective bundle $P'$ over $C$ constructed as follows:

1) Denote by $\tilde{P}$ the blow-up of $P = P(V)$ along $L$, and by $\sigma$ the projection from $\tilde{P}$ to $P$. The exceptional divisor $E$ of $\tilde{P}$ is $P(N_{L|P})$, a $\mathbb{P}^1$-bundle over $L$. The fibre $(\pi \circ \sigma)^{-1}(c)$ is the sum of two effective Cartier divisors

$$(\pi \circ \sigma)^{-1}(c) = \tilde{F} + E.$$ 

They intersect in the hyperplane of $\tilde{F}$ that corresponds to $L$. On $E$, this intersection is the exceptional divisor, when $\tilde{E}$ is seen as the blow-up of $\mathbb{P}^{n-1}$ at a point.

2) The normal bundle of $\tilde{F}$ in $\tilde{P}$ is $\mathcal{O}_{\tilde{F}}(-1)$, hence there is a contraction $\sigma' : \tilde{P} \to P'$ that sends $\tilde{F}$ to a point:

```
\[
\begin{array}{ccc}
\tilde{P} & \xrightarrow{\sigma} & P \\
\downarrow{\pi} & \quad & \downarrow{\pi'} \\
C & \xrightarrow{\sigma} & P'
\end{array}
\]
```

$P'$ maps to $C$ with all fibers isomorphic to $\mathbb{P}^{n-1}$. It follows that $P'$ is a projective bundle.

The construction of $\text{elm}_L P(V)$ is a generalization of the elementary transforms of geometrically ruled surfaces. In this case, if the base curve is the projective line, the elementary transform can be described more precisely: The Hirzebruch surface $P(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$ is transformed either in the surface $P(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d+1))$, or the surface $P(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d-1))$, depending on whether or not the point $L$ lies on the distinguished section (see [5]). The next proposition is an analogous result in arbitrary dimensions.

**Proposition 2.1.** Let $V = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$, with $a_1 \leq \cdots \leq a_n$. If $L$ is a general hyperplane in a fibre of $P(V) \to \mathbb{P}^1$, then $\text{elm}_L P(V) = P(V')$, where

$$V' = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n-1}) \oplus \mathcal{O}_{\mathbb{P}^1}(a_n - 1).$$

**Proof.** If $W$ is a sub-bundle of $V$, with quotient bundle $Q$, there is a canonical embedding $i$ of $P(W)$ in $P(V)$. For $W$ of rank $n - 1$, $P(W)$ is an effective divisor. Taking $\pi_*$ on the exact sequence

$$0 \to \mathcal{O}_V(1) \otimes \mathcal{O}_V(-P(W)) \to \mathcal{O}_V(1) \to \mathcal{O}_V(1) \otimes \mathcal{O}_W \to 0,$$

we have

$$0 \to \pi_* (\mathcal{O}_V(1) \otimes \mathcal{O}_V(-P(W))) \to V^* \to W^* \to 0.$$
It follows that \( \mathcal{O}_V(P(W)) \otimes \mathcal{O}_V(-1) \simeq \pi^* Q \), and restricting to \( P(W) \), that
\[
\mathcal{O}_W(P(W)) \otimes \mathcal{O}_W(-1) \simeq (\pi \circ i)^* Q.
\]

We shall call splitting divisors, the divisors \( \Delta_i, 1 \leq i \leq n \), corresponding to the sub-bundles
\[
W_i = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{i-1}) \oplus \mathcal{O}_{\mathbb{P}^1}(a_{i+1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n) \subset V.
\]
These \( n \) divisors have empty set-theoretic intersection and
\[
(\Delta_i)^n = c^n_i(\mathcal{O}_V(1) \otimes \mathcal{O}_V(a_i F)) = -(a_1 + \cdots + a_n) + na_i,
\]
for any \( 1 \leq i \leq n \). It is obvious that giving a projective bundle over \( \mathbb{P}^1 \) is equivalent to giving \( n \) 1-codimensional projective sub-bundles with an empty intersection. The splitting type of the bundle can be restored, up to tensoring with a line bundle, from the self intersection numbers \( \mathbb{I} \).

Claim: Any sufficiently general hyperplane \( L \subset F \) is cut out by a projective sub-bundle \( \Delta \) linearly equivalent to \( \Delta_n \).

It is sufficient to show that the restriction \( H^0(P(V), \mathcal{O}_V(\Delta_n)) \rightarrow H^0(F, \mathcal{O}_F(1)) \) is surjective. The dimension
\[
h^1(P(V), \mathcal{O}_V(\Delta_n - F)) = h^1(P(V), \mathcal{O}_V(1) \otimes \mathcal{O}_V((a_n - 1)F))
\]
\[
= h^1(\mathbb{P}^1, \pi_*(\mathcal{O}_V(1) \otimes \mathcal{O}_V((a_n - 1)F)))
\]
\[
= h^1(\mathbb{P}^1, \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(-a_i + a_n - 1)F))
\]
\[
= h^0(\mathbb{P}^1, \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i - a_n - 1)F))
\]
vanishes, since \( a_i \leq a_n \) for all \( i \), and the surjection follows. If \( p_n \in F \) is the intersection of the fibre \( F \) with the section that corresponds to \( \mathcal{O}_{\mathbb{P}^1}(a_n) \rightarrow V \), then \( \Delta_n \) does not pass through \( p_n \), contrary to the other \( \Delta_i \)'s.

Let \( L \) be a general hyperplane in \( F \), with \( p_n \notin L \), let \( \Delta \sim \Delta_n \) be the projective sub-bundle that corresponds to \( L \), and let \( P' = \text{elm}_L P \). We reconstruct the vector bundle corresponding to \( P' \) from the \( n \) 1-codimensional sub-bundles \( \Delta', \Delta'_1, \ldots, \Delta'_{n-1} \), where \( \Delta' = \sigma'(\Delta) \), with \( \Delta \) the strict transform of \( \Delta \) on \( \tilde{P} \), and the same for \( \Delta'_i, i = 1, \ldots, n - 1 \). Let \( b_1, b_2, \ldots, b_n \) be \( n \) integers such that \( (\Delta'_i)^n = -(b_1 + \cdots + b_n) + nb_n \) and \( (\Delta')^n = -(b_1 + \cdots + b_n) + nb_n \). Then \( P' = P(\mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(b_n)) \), and since for every \( 1 \leq i \leq n - 1 \), \( \sigma^* \Delta_i = \tilde{\Delta}_i \) and \( (\sigma')^* \Delta'_i = \tilde{\Delta}'_i + \tilde{F} \), we have that
\[
(\Delta_i)^n = (\tilde{\Delta}_i)^n = ((\sigma')^* \Delta'_i - \tilde{F})^n = (\Delta'_i)^n + (-1)^n \tilde{F}^n = (\Delta'_i)^n - 1.
\]
These relations together with \( \mathbb{I} \) provide a linear system of \( n - 1 \) equations, \( n \) unknowns and of rank \( n - 1 \). Now \( b_i = a_i \) for \( 1 \leq i \leq n - 1 \), and \( b_n = a_n - 1 \) is one solution, the others being obtained from this one by translations.

**Lemma 2.2.** Let \( X \) be a smooth projective \( n \)-fold and \( Y \subset X \) be a smooth rational curve with normal bundle \( N_{Y|X} = \mathcal{O}_Y(a_1) \oplus \cdots \oplus \mathcal{O}_Y(a_{n-1}) \), where \( a_1 \leq \cdots \leq a_{n-1} \). If \( Z \subset X \) is a general, smooth, 2-codimensional subvariety intersecting \( Y \) in a point, \( X' = \text{Bl}_Z X \) and \( \tilde{Y} \) the strict transform of \( Y \), then \( N_{Y'|X'} = \mathcal{O}_{\tilde{Y}}(a_1) \oplus \cdots \oplus \mathcal{O}_{\tilde{Y}}(a_{n-1} - 1) \).
Proof. We need to compare the normal bundles of $Y$ and $\tilde{Y}$. Accordingly, we first look for a comparison of the corresponding projective bundles. These are exceptional divisors of the blow-ups of $X$ and $X'$ along $Y$ and $\tilde{Y}$, respectively. Throughout the proof, the exceptional divisor of the blow-up along the subvariety $S$ will be denoted by $P_S$, and the strict transform of a subvariety $S$ on a blow-up by $\widetilde{S}$. Hence, in the next diagram, we want to compare the exceptional divisors $P_Y \subset \text{Bl}_Y X$ and $P_{\tilde{Y}} \subset \text{Bl}_{\tilde{Y}} X'$.

\[
P_{\tilde{Y}} \subset \text{Bl}_{\tilde{Y}} X' \xrightarrow{\sigma'} X' \quad \text{and} \quad P_Y \subset \text{Bl}_Y X \xrightarrow{\sigma} X
\]

Claim: $P_{\tilde{Y}} = \text{elm}_L P_Y$, where $L$ is a hyperplane in one of the fibres of $P_Y$.

To justify the claim, let $F \subset P_Y$ be the fibre over the intersection point $\{x_0\} = Z \cap Y \subset Y$, let $L$ be the hyperplane cut out by $\tilde{Z}$ on $F$ (actually on $P_Y$), and let $\epsilon : X'' \rightarrow \text{Bl}_Y X$ be the blowing-up along $\tilde{Z}$. The fibre of $\tilde{P}_Y$ above $x$ has two components: $\widetilde{F}$ and $\Xi$. Moreover, $\tilde{P}_Y$ and $P_Z$ intersect along $\Xi$. We first notice that there is a morphism $u : X'' \rightarrow X'$. Indeed, $(\sigma \circ \epsilon)^{-1}(Z)$ is a Cartier divisor, and from the universal property of the blowing-up $\rho$, we obtain $u$. Further, the universal property is used for $\sigma'$ to imply that the natural birational map from $X''$ to $\text{Bl}_{\tilde{Y}} X'$ is defined at any point of $\Xi$ not lying on $\widetilde{F}$.

\[
\begin{align*}
X'' \xrightarrow{\epsilon} & \text{Bl}_Y X \\
& \text{Bl}_{\tilde{Y}} X' \\
& \text{Bl}_Y X \\
& X
\end{align*}
\]

We restrict $v$ to $\tilde{P}_Y$. Since $\widetilde{F}$ is now a divisor, it follows that the restriction (for which we use the same symbol $v$) is defined at the generic point of $\widetilde{F}$, and establishes an isomorphism $\tilde{P}_Y - \tilde{F} \rightarrow P_Y - \{x'_0\}$. Here $x'_0$ is the point of intersection of $P_Y$ with the strict transform of the fibre of $P_Z$ over $x_0$. Using the Zariski Main Theorem, we conclude that $v$ is a morphism that contracts $\widetilde{F}$ to $x'_0$. The definition of the elementary transforms gives the claim.

Now by Proposition 2.1, the vector bundle that corresponds to $P_{\tilde{Y}}$ is determined up to tensoring with a line bundle. To finish the proof of the lemma, we apply the adjunction formula to obtain that $\deg N_{Y|X} = \deg N_{\tilde{Y}|X'} - 1$. □

Theorem 2.3. Let $X$ be a rationally connected variety and $Y \subset X$ a smooth rational curve with ample normal bundle. Then there exists a sequence of blowing-ups with smooth 2-codimensional centers $\tilde{X} \rightarrow X$ such that the strict transform $\tilde{Y}$ becomes a quasi-line.

Proof. We blow-up different well-chosen 2-codimensional smooth subvarieties such that, by Lemma 2.2, the strict transform of $Y$ becomes a quasi-line. □

We end this section by presenting an application of the above theorem. Let us recall from [1] the following definition. A quasi-line $Y \subset X$ is called an almost-line if there is a divisor...
$D \in \text{Div}(X)$ such that $D \cdot Y = 1$. The main reason for introducing this notion was the theorem below, proved in [1]:

**Theorem 2.4 (2.1 in [1]).** Let $X$ be a projective manifold of dimension at least two, $Y \subset X$ a closed, smooth, connected curve with ample normal bundle and $Y(1)$ the first infinitesimal neighbourhood of $Y$ in $X$. The following conditions are equivalent:

(i) The natural restriction map $\text{Pic}(X) \rightarrow \text{Pic}(Y(1))$ is surjective.

(ii) $Y$ is an almost-line.

For a discussion of the history and motivation of condition (i) reference [1] may be consulted. Theorem 2.4 is completed by the next proposition, a consequence of Theorem 2.3. The proposition shows that, at least birationally, the situation described in (i) occurs precisely when $X$ is rationally connected.

**Proposition 2.5.** Let $X$ be a rationally connected projective manifold. Then there is a morphism $\sigma: X' \rightarrow X$ which is a composition of blowing-ups with smooth $2$-codimensional centers, such that $X'$ contains an almost-line.

**Proof.** We proceed as in the proof of Theorem 2.3. If $Y \subset X$ is a smooth rational curve with $N_{Y|X} = \bigoplus_{1}^{n-1} O_Y(a_j)$, $a_1 \leq \cdots \leq a_{n-1}$ and $a_{n-1} > 1$, the quasi-line $Y$ constructed in loc. cit. is actually an almost-line. Indeed, if $E$ is the exceptional locus of the last blowing-up, we have $E \cdot Y = 1$. If $a_{n-1} = 1$ (i.e. $Y$ is already a quasi-line), we take $f: \mathbb{P}^1 \rightarrow Y$ to be a degree-$b$ covering, with $b \geq 2$. Applying Theorem II 3.14 in [12], we get that a general deformation of $f$ is an embedding with ample normal bundle, of numerical type $b_1 \leq \cdots \leq b_{n-1}$ and such that $b_{n-1} \geq 2$. So the previous argument applies to give the desired conclusion. \hfill $\square$

The smooth cubic threefold $X \subset \mathbb{P}^4$ is an example of a rationally connected manifold that does not contain almost-lines. To see this, let $Y \subset X$ be a quasi-line. By the adjunction formula $-K_X \cdot Y = 4$, and since $-K_X \sim 2H$, $Y$ is a conic. But the Picard group is generated by the hyperplane section $H$, hence $Y$ is not an almost-line. After blowing-up one line $l \subset X$, $\text{Bl}_l X$ becomes a conic bundle, $\pi: \text{Bl}_l X \rightarrow \mathbb{P}^2$. The pull-back of a general line in $\mathbb{P}^2$ is a surface $S$ isomorphic to $\mathbb{P}^2$ blown-up at 6 points. The surface $S$ contains a section $Y$ for $\pi|_S$ with self intersection 1. $Y$ is an almost-line on $\text{Bl}_l X$.

### 3 Rationality via quasi-lines

In this section we wish to discuss some conditions under which a rationally connected manifold is actually rational.

We start by reviewing a construction that involves the Hilbert scheme associated to a rational curve on $X$, and the universal family of this Hilbert scheme; see [12], or [1] Section 3. Let $X$ be a projective manifold and $Y \subset X$ a quasi-line. Consider the Hilbert scheme which corresponds to the Hilbert polynomial (for a certain polarisation) of $Y$ in $X$. Since $H^1(Y, N_{Y|X}) = H^1(\mathbb{P}^1, \bigoplus_{1}^{n-1} O_{\mathbb{P}^1}(1)) = 0$, the Hilbert scheme is smooth at $[Y]$, and $[Y]$ lies on a unique irreducible component, $\mathcal{H}$, of this Hilbert scheme. We denote by $\mathcal{Y}$ the universal
family over $\mathcal{H}$, and by $\pi$ and $\Phi$ the two projections.

$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\Phi} & X \\
\downarrow & \downarrow \pi & \\
\mathcal{H} & \xrightarrow{\Phi} & X
\end{array}
$$

For each $h \in \mathcal{H}$, the curve $Y_h = \Phi(\pi^{-1}(h))$ satisfies $[Y_h] = h$, and the restriction of $\Phi$ to $\pi^{-1}(h)$ is an isomorphism onto $Y_h$. In addition, there exists a neighbourhood of $[Y]$ in $\mathcal{H}$ such that for each closed point $h$ in this neighbourhood, the curve $Y_h$ is a quasi-line.

For $x \in Y$, we consider the closed subscheme $\pi(\Phi^{-1}(x))$ of $\mathcal{H}$; it contains a closed point $h$ if and only if the curve $Y_h$ passes through $x$. If $\mathcal{I}_x$ is the ideal sheaf of $x$ in $Y$, then the tangent space to $\pi(\Phi^{-1}(x))$ at $[Y]$ is identified with the space of global sections of $N_{Y/X} \otimes \mathcal{I}_x$. Since $N_{Y/X} \simeq (n-1)O_{P^1}$ and $H^1$ is trivial, the tangent space is isomorphic to $\mathbb{C}^{n-1}$ and the subscheme is smooth at $[Y]$. Let $\mathcal{H}_x$ be the unique irreducible component that contains $[Y]$. We shall use the same notation $\pi$ and $\Phi$ for the above projections restricted to the universal family $\mathcal{Y}_x \to \mathcal{H}_x$.

Notation. Let $Y$ be a quasi-line on $X$. The number of quasi-lines from the family determined by $Y$ and passing through two general points of $X$ will be denoted by $e(X,Y)$, cf. [10].

The number of quasi-lines from the family passing through one general point of $X$ and tangent to a general tangent vector at that point will be denoted by $e_0(X,Y)$.

To see that these numbers are indeed finite, we take $\xi$ a 0-dimensional subscheme of length 2 in $Y$ in such a way that $\{x\} \subset \text{supp}(\xi)$. The closed subscheme $\mathcal{H}_\xi$ of curves through $\xi$ is contained in $\mathcal{H}_x$ and, as before, its tangent space at $[Y]$ is identified with $H^0(Y, N_{Y/X} \otimes \mathcal{I}_\xi)$. This space of global sections is trivial, and this implies the finiteness of the number of quasi-lines through $\xi$. Notice that the degree of $\Phi : \mathcal{Y}_x \to X$ is equal to $e(X,Y)$.

Recall that a model is a couple $(X,Y)$ with $Y \subset X$ a smooth rational curve with ample normal bundle.

**Proposition 3.1.** Let $(X,Y)$ be a model with $Y$ a quasi-line. Then the following assertions hold:

(i) If $e_0(X,Y) = 1$, then $X$ is a unirational variety.

(ii) If $e(X,Y) = 1$, then $X$ is a rational variety.

**Proof.** Let $x \in Y$ be a fixed point. Let $\sigma : \text{Bl}_x X \to X$ be the blow-up of $X$ at $x$ and $E$ be the exceptional divisor. In the diagram

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{s} & \mathcal{Y}_x \\
\downarrow \pi & & \downarrow \Phi \\
\mathcal{H}_x & \xrightarrow{\Phi} & X
\end{array}
$$

Bl$_x$ X

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the map $\Phi$ contracts the divisor $E = \Phi^{-1}(x)$. This divisor is the image of the natural section $s : H_x \to Y_x$ that maps a point $h \in H_x$ to the point $x$ on the fibre $Y_h$ of the universal family. Since both $H_x$ and $\pi$ are generically smooth, the rational map $\sigma^{-1} \circ \Phi$ is defined at a general point of $E$. It follows that $\sigma^{-1} \circ \Phi$ maps $E$ to $E$.

The fact that the restriction of the map $\sigma^{-1} \circ \Phi$ to $E$ gives a birational isomorphism to $E$ means precisely that $e_0(x, Y) = 1$. As $Y_x$ is birationally isomorphic to $E \times \mathbb{P}^1$, (i) is proved.

To see (ii), we first show that the restriction of the rational map $\sigma$ to the image of a line in $X$ is a quasi-line isomorphic to $\mathbb{P}^n$, and some results to be established in Section 3.2. Proposition 3.2. Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold and let $Q$ be the family of all conics lying on $X$. Then:

(i) $Q$ is an irreducible family of dimension 4.
(ii) A conic $\Gamma$ corresponding to a general point of $Q$ is a quasi-line.
(iii) $e_0(X, \Gamma) = e(X, \Gamma) = 6$.

Proof. It is a classical fact that the family $D$ of lines contained in $X$ is a smooth, irreducible surface and that there are 6 lines passing through a general point of $X$, see e.g. [12], 266-270. Let $G(3, 5)$ be the Grassmannian of planes in $\mathbb{P}^4$. The incidence $\{(l, P) \mid l \subset P\} \subset D \times G(3, 5)$ is irreducible and has dimension 4. The projection of this incidence on the second factor is birational on its image which identifies to $Q$. Hence (i).

(ii) was first proved by Oxbury in [13]. See also [14], Theorem 3.2 for a more conceptual argument.

To prove (iii), we consider $l$ a general line in $\mathbb{P}^4$ and $\{x, x', y\}$ the intersection of $l$ with $X$. Clearly, there is a bijection between lines contained in $X$ and passing through $y$ and conics contained in $X$ and passing through $x$ and $x'$. Hence $e(X, \Gamma) = 6$. In a similar vein, let $x$ be a general point of $X$, $v$ a general tangent vector at $x$ and $y$ the other point of intersection of the line, $l_{x,v}$ determined by $v$, with $X$. For every line contained in $X$ and passing through $y$, the plane spanned by this line and $l_{x,v}$ cuts out a residual conic $\Gamma$ through $x$ and tangent to $v$. This shows that $e_0(X, \Gamma) = 6$, too.
The considerations made at the beginning of this section, together with Theorem 2.3, allow us to introduce the definition below.

**Definition 3.3.** Let $X$ be a rationally connected projective manifold. We denote by $e(X)$ the minimum of $e(X',Y')$ for all models $(X',Y')$, where $\sigma : X' \to X$ is a composition of blowing-ups with smooth centers, and $Y'$ is a quasi-line on $X'$.

The number $e(X)$ leads to a characterization of rational manifolds inside the class of rationally connected ones.

**Theorem 3.4.** Let $X$ be a rationally connected projective manifold.
(i) The number $e(X)$ is a birational invariant of $X$.
(ii) $X$ is rational if and only if $e(X) = 1$.

**Proof.** (i) Let $\varphi : X_1 \dashrightarrow X_2$ be a birational isomorphism between two rationally connected projective manifolds. Let $\sigma : X' \to X_2$ be a composition of blowing-ups and let $Y' \subset X'$ be a quasi-line such that $e(X_2) = e(X',Y')$. Let $\mu = \sigma^{-1} \circ \varphi : X_1 \dashrightarrow X'$. By [6], there is $\rho : X \to X_1$ which is a composition of blowing-ups such that $\mu \circ \rho : X \to X'$ is a birational morphism. Let $Y \subset X$ be the inverse image by $\mu \circ \rho$ of a general deformation of $Y'$. We have $e(X_2) = e(X',Y') = e(X,Y) \geq e(X_1)$. The opposite inequality follows by symmetry.
(ii) Clearly $e(\mathbb{P}^n) = 1$, so if $X$ is rational, then $e(X) = 1$ by (i). The converse follows from Proposition 3.1. □

The number $e(X)$ seems to be very difficult to compute. On one hand, its definition involves arbitrary blowing-ups of $X$; on the other hand, even for fixed $X$, there are in general infinitely many families of quasi-lines $Y$ on $X$ and it is not at all clear how to compute the minimum of all numbers $e(X,Y)$. To circumvent this, we shall introduce in Section 5 a “local version” of $e(X)$, denoted $\tilde{e}(X,Y)$. It is associated to a given quasi-line $Y$ and it is easier to compute. However, it is an open problem whether or not $\tilde{e}(X,Y)$ leads to a birational invariant. We refer to the discussion in the last section.

### 4 Strongly-rational manifolds

The aim of this section is to give a convenient characterization of strongly-rational manifolds, such that later on we can establish their stability with respect to small smooth deformations. We shall need the following result from [1], which is also a particular case of Theorem 1.5.

**Theorem 4.1** (4.4 in [1]). $X$ is strongly-rational if and only if $X$ contains a quasi-line $Y \subset X$ and a divisor $D$ with $D \cdot Y = 1$ and $\dim |D| \geq n$.

The argument for the converse is by induction on $n = \dim X$. It eventually says that the linear system $|D|$ has dimension $n$ and corresponds to the hyperplanes of the projective space.

**Theorem 4.2.** A manifold $X$ is strongly-rational if and only if
1) $X$ contains a quasi-line $Y$,
2) there exists a point $x \in X$ smooth on every curve $Y' \sim Y$ passing through it, and
3) $e(X,Y) = 1$.
Proof. If $X$ is strongly-rational, then $X$ contains an open subset $U$ isomorphic to an open subset $V$ of the projective space, whose complement is at least 2-codimensional. Conditions 1) and 3) follow by taking $Y$ to be the image of a line on $V$.

For 2), we consider a general point $x \in U$, a curve $Y$ as above, and $\Lambda$ the linear system of divisors that pass through $x$ and correspond to hyperplanes of $\mathbb{P}^n$. Clearly $x$ is an isolated base point of $\Lambda$. If this base locus reduces to $x$, then, for every $Y \in \mathcal{H}_x$, we can choose a divisor $D \in \Lambda$ which avoids a point in each irreducible component of $Y$. Hence $(D \cdot Y)_x = 1$.

Let $C_h(x)$ be the irreducible component that corresponds to $Y_0$, of the Chow scheme of cycles through $x$, and similarly, $C_h(x')$ to $Y_0'$. If we take $\tilde{H}_x \to \mathcal{H}_x$ to be the normalisation of $\mathcal{H}_x$, then the fundamental class of an element in $\tilde{H}_x$ provides us with a natural morphism $f : \tilde{H}_x \to C_h(x)$ that is birational and surjective. From the diagram

\[
\begin{array}{ccc}
\tilde{H}_x' & \xrightarrow{f'} & C_h(x') \\
\downarrow \sigma_* & & \downarrow \\
\tilde{H}_x & \xrightarrow{f} & C_h(x)
\end{array}
\]

it follows that for any $[Y] \in C_h(x)$, the curve that corresponds to it is smooth at $x$. This smoothness and Theorem I.6.5 in [12] imply that the morphism from the universal family to the Chow scheme, $p : \mathcal{C}_x \to C_h(x)$, is smooth at every point of the distinguished section. Then, from the commutative diagram,

\[
\begin{array}{ccc}
\tilde{Y}_x & \xrightarrow{\tilde{\pi}} & \mathcal{C}_x \\
\downarrow & & \downarrow p \\
\tilde{H}_x & \xrightarrow{f} & C_h(x)
\end{array}
\]

it follows that $\tilde{\pi}$ is smooth at every point of $\tilde{\mathcal{E}}$, and consequently the same holds for $Y_x \to \mathcal{H}_x$.

To prove the converse, we start with some remarks. The first one is that the hypotheses give rise to the diagram
with $\sigma$ being the blowing-up of $x$, $E$ the exceptional divisor, $\tilde{\Phi}$ a birational morphism and $s$ an isomorphism. To see that $\tilde{\Phi}$ is a morphism, it is sufficient to show that $\mathcal{E}$ is a Cartier divisor (see the lemma after), and to apply the universal property of a blowing-up.

The second remark is that if $F \subset \mathcal{Y}_x$ is a general fibre of $\pi$ and $y \in F - (F \cap \mathcal{E})$ is any point, then $\Phi$ is a local isomorphism at $y$. Indeed, if it is not, from the Zariski Main Theorem, $\Phi^{-1}(\Phi(y))$ is positive dimensional. Moreover $\Phi(y) = x' \neq x$, hence there are infinitely many quasi-lines through $x$ and $x'$, which is impossible.

The third remark is that there exists an effective divisor $D \subset X$ such that $D \cdot Y = 1$. Indeed, if $H \subset E \cong \mathbb{P}^{n-1}$ is a general hyperplane, we look at $D = \Phi((\tilde{\Phi}|_E \circ \pi)^*H)$. Then, the multiplicity of $x$ on $D$ is given by

$$\text{mult}_x(D) = (-1)^{n-2}(\sigma|E)_*(\tilde{D} \cdot E^{n-1}) = H^{n-1} = 1,$$

hence the point $x$ is smooth on $D$. As a consequence of the previous remark, a general deformation of $Y$ through $x$ meets $D$ only at $x$ and the intersection is transverse. We conclude that $D \cdot Y = 1$.

The fourth remark is that for a general point $p \in E$ and general hyperplanes $H_1, \ldots, H_{n-1}$ through $p$, if $D_1, \ldots, D_{n-1}$ are the divisors on $X$ corresponding to the $H_i$’s as in (2), then the $D_i$’s cut out a quasi-line transversely. It is sufficient to justify this on $\mathcal{Y}_x$, since $\Phi$ is birational. On $\mathcal{Y}_x$, this is obvious from the generic smoothness of $\pi$.

The last remark is that $\dim |D| \geq n$. Clearly $\dim |D| \geq n - 1$. If equality existed, $|D|$ would consist only of divisors that come from hyperplanes of $E$. Using the above results, the point $x$ would be an isolated base point of the linear system $|D|$ and $x$ would be smooth on a general divisor. Let $\epsilon : X' \to X$ be the blow-up of $X$ along $B^1$, the base locus of $|D|$ minus $x$, and let $|D'|$ be the strict transform of $|D|$. By Bertini, it would follow that a general divisor in $|D'|$ is smooth. Let $D_1, \ldots, D_{n-1} \in |D|$ be general divisors. Their intersection locus would contain a quasi-line $Y$ and perhaps some other closed subset disjoint from $Y$, and would be transverse along $Y$. Now $Y$ is disjoint from $B$, hence $Y' = \epsilon^{-1}(Y)$ is a quasi-line on $X'$ and in a neighbourhood of $Y'$, the strict transforms $D'_1, \ldots, D'_{n-1}$ would also intersect transversely along $Y'$. From $H^1(X', \mathcal{O}_{X'}) = 0$ and the exact sequence

$$0 \to \mathcal{O}_{X'} \to \mathcal{O}_{X'}(D') \to \mathcal{O}_{D'_1}(D') \to 0,$$

it would follow that $\dim |D'|_{D'_1} = n - 2$, and using the exact sequence of normal bundles

$$0 \to N_{Y'|D'_1} \to N_{Y'|X'} \to \mathcal{O}_{Y'}(1) \to 0,$$

that $Y'$ is a quasi-line on $D'_1$. As we have noticed, $D'_1$ is smooth, hence $H^1(D'_1, \mathcal{O}_{D'_1}) = 0$, and we could pursue the restriction procedure. After $n - 2$ steps, we would arrive at a smooth, regular surface $S$, together with a rational curve $Y' \subset S$ such that $(Y')^2 = 1$ and $\dim |Y'| \leq 1$. By Riemann-Roch, this would be impossible.

At this point, we invoke Theorem 4.1 and obtain that $X$ is strongly-rational. $\Box$

\textbf{Lemma 4.3.} \textit{Let $\mathcal{E} \hookrightarrow \mathcal{Y} \rightarrow \mathcal{H}$ be a flat morphism of schemes of relative dimension 1, with $\mathcal{H}$ integral and $\mathcal{E}$ a section. Then $\mathcal{E}$ is a Cartier divisor on $\mathcal{Y}$ if and only if $\mathcal{E}$ intersects each fibre in a smooth point of the fibre.}\n
\footnote{In fact we systematically apply Hironaka’s result to obtain a base-point free linear system through blowing-ups with smooth centers.}
Proof. The “only if” part is clear. For the converse, let us suppose that for every \( e \in \mathcal{E} \), the local ring \( \mathcal{O}_{\pi^{-1}(\pi(e)),e} \) is a discrete valuation ring and let us note \( A = \mathcal{O}_{\mathcal{H},\pi(e)} \), \( B = \mathcal{O}_{\mathcal{Y},e} \) the local rings with \( \mathfrak{m} \) and \( \mathfrak{n} \) the maximal ideals. We have the commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow \cong & & \downarrow \\
B/I & & \\
\end{array}
\]

with \( I = \mathcal{I}_{\mathcal{E},e} \). Hence, the exact sequence of \( A \)-modules

\[
0 \rightarrow I \rightarrow B \rightarrow B/I \rightarrow 0
\]
splits. Tensoring with \( A/\mathfrak{m} \) over \( A \), the sequence remains exact and gives the injection

\[
0 \rightarrow I \otimes_A A/\mathfrak{m} = I/\mathfrak{m}I \rightarrow B/\mathfrak{m}B.
\]

But \( B/\mathfrak{m}B \) is a discrete valuation ring, hence \( I/\mathfrak{m}I \) is a principal ideal. The inclusion \( \mathfrak{m}I \subseteq \mathfrak{n}I \) implies that \( I/\mathfrak{m}I \) surjects onto \( I/\mathfrak{n}I \), that \( I/\mathfrak{n}I \) is principal, and applying Nakayama’s Lemma, that \( I \) is principal. It remains to be shown that \( I \) is generated by a nonzero divisor in \( B \). Since \( B/I \cong A \) is a domain, \( I \) is a prime ideal, and the next lemma ends the proof.

We are grateful to Lucian Bădescu for pointing out the next lemma to the first named author.

**Lemma 4.4.** Let \( B \) be a local Noetherian ring. If \( I \subseteq B \) is a principal prime ideal of height 1, then \( I \) is generated by a nonzero divisor.

**Proof.** Let \( I = (b) \) and let \( \beta \in B \) such that \( \beta b = 0 \). Then either \( \beta \notin I \), or \( \beta \in I \). In the former case, considering the localisation \( B_I \), we get \( b = 0 \), a contradiction. In the second case, \( \beta = \beta_1b \). The same argument applied to \( \beta_1 \) yields \( \beta_1 = \beta_2b \), i.e. \( \beta \in I^2 \). We may continue the process as long as \( \beta_n \notin I \) and, if this happens, infer that \( \beta \in I^n \), for every positive integer \( n \). By the Krull Intersection Theorem, \( \beta = 0 \).

In the sequel, we shall refer to the set-up

\[
\begin{array}{ccc}
X_0 & \rightarrow & X' \\
\downarrow & & \downarrow p \\
\{0\} & \rightarrow & B
\end{array}
\]

where \( X \) is a smooth variety, \( B \) a smooth affine curve and \( p \) is a proper and smooth morphism, as to a small deformation of the smooth projective variety \( X_0 \).

**Theorem 4.5.** Strongly-rational varieties are stable with respect to small deformations.

Note that such a property is not expected to hold for rational manifolds. Indeed, several examples of smooth rational cubic four-folds in \( \mathbb{P}^5 \) are known to exist, but the general one is expected to be non-rational.

**Proof.** We shall show that the three conditions of Theorem 4.2 are stable with respect to small deformations. For the first one the result is well known, see \textit{e.g.} [1] Proposition 3.10, and the following lemmas deal with the remaining two conditions.
Lemma 4.6. Condition 2) is stable with respect to small deformations.

Proof. Let \( p : \mathcal{X} \to B \) be a small deformation of \( X_0 \). \( X_0 \) contains a point \( x \) and a quasi-line \( Y \) through \( x \), such that every other \( Y' \in \mathcal{H}_x \) is smooth at \( x \).

After a base change, if necessary, we can assume that \( \mathcal{X} \to B \) has a section \( s \) through \( x \). If \( \mathcal{H}_s \) is the relative Hilbert scheme over \( B \), of quasi-lines through some point of \( s(B) \), we have the diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\Phi} & \mathcal{X} \\
\downarrow{\pi} & & \downarrow{p} \\
\mathcal{H}_s & \xrightarrow{q} & \mathcal{X} \\
\downarrow{\pi_0} & & \downarrow{\pi_0} \\
B & \xrightarrow{\Phi_0} & X_0 \\
\end{array}
\]

which over \( 0 \in B \), becomes

\[
\begin{array}{ccc}
\mathcal{E}_0 & \xrightarrow{\Phi_0} & X_0 \\
\downarrow{\pi_0} & & \downarrow{\pi_0} \\
B & \xrightarrow{\Phi_0} & X_0 \\
\end{array}
\]

Every fibre of \( \pi_0 \) is smooth at the point of intersection with \( \mathcal{E}_0 \).

Let \( \mathcal{U} \subset \mathcal{Y}_s \) be the subset of those \( y \in \mathcal{Y}_s \) at which \( \pi \) is smooth. Since \( \pi \) is flat, \( \mathcal{U} \) is an open subset. The hypothesis yields \( \mathcal{E}_0 \subset \mathcal{U} \), and since \( q \circ \pi \) is proper, \( \mathcal{Y}_s - \mathcal{U} \) is sent onto a closed subset of \( B \). We conclude that \( \mathcal{E}_b \) is contained in \( \mathcal{U} \) for every \( b \) in a neighbourhood of \( 0 \). \( \square \)

Lemma 4.7. Condition 3) is stable with respect to small deformations.

Proof. We consider a small deformation and the diagram above. As before, when restricted to \( 0 \), it becomes

\[
\begin{array}{ccc}
\mathcal{Y}_0 & \xrightarrow{\Phi_0} & X_0 \\
\downarrow & & \downarrow \\
\mathcal{H}_0 & \xrightarrow{\Phi_0} & X_0 \\
\end{array}
\]

with \( \Phi_0 \) a birational morphism. To see that \( \Phi_b \) is birational in a neighbourhood of \( 0 \), one may consider a smooth curve \( C \subset \mathcal{X} \) through \( x \), the intersection of general very ample divisors that contain \( x \), and its pre-image \( C' \subset \mathcal{Y}_s \). The restrictions \( p|_C : C \to B \) and \( (q \circ \pi)|_{C'} : C' \to B \) are proper and quasi-finite, and hence finite. Consequently \( \Phi|_{C'} : C' \to C \) is finite. But the fibre above \( x \) has length 1 and the result is established. \( \square \)

5 Quasi-lines and formal geometry

If \( Y \) is a closed subscheme of \( X \), the theory of formal functions of \( X \) along \( Y \) was developed by Zariski and Grothendieck as an algebraic substitute for a complex tubular neighbourhood of \( Y \) in \( X \). \( \hat{X}|_Y \) denotes the formal completion of \( X \) along \( Y \), which is the ringed
space with topological space $Y$ and sheaf of rings $O_{\tilde{X}|Y} = \varinjlim O_X/I^n$, $I$ is the sheaf of ideals defining $Y$ in $X$.

In [7], Hironaka and Matsumura have introduced and studied $K(\tilde{X}|Y)$, the ring of formal rational functions of $X$ along $Y$. In good cases it is a field that contains the field $K(X)$ of rational functions of $X$. We recall the following definitions from [7]: $Y$ is G2 in $X$ if $K(\tilde{X}|Y)$ is a field and the field extension $K(X) \subset K(\tilde{X}|Y)$ is finite. $Y$ is G3 in $X$ if the inclusion $K(X) \subset K(\tilde{X}|Y)$ is an isomorphism.

**Notation.** If $Y$ is G2 in $X$, we denote by $b(X,Y)$ the degree of the field extension $K(X) \subset K(\tilde{X}|Y)$.

We also recall the following two results that will be repeatedly used in the sequel. They are stated in the particular case of quasi-lines, which is enough for our purposes. Since the normal bundle of a quasi-line $Y \subset X$ is ample, $Y$ is G2 in $X$, see [4].

**The Hartshorne-Gieseker Construction** (see [3], Theorem 4.3). If $Y \subset X$ is a quasi-line, then there is a morphism of models $(X',Y') \to (X,Y)$ of degree $b(X,Y)$, étale along $Y'$, and such that $Y'$ is G3 in $X'$.

**Gieseker’s Theorem** (see [3]). Let $(X_i,Y_i)$ be two models with $Y_i \subset X_i$ a quasi-line, $i = 1,2$. Assume that $Y_i$ is G3 in $X_i$ and $\tilde{X}_1|Y_1 \simeq \tilde{X}_2|Y_2$ as formal schemes. Then there are Zariski open subsets $U_i \subset X_i$ containing $Y_i$ and an isomorphism $U_1 \simeq U_2$ that sends $Y_1$ to $Y_2$.

The following definition is similar to Definition 3.3. Here, for a given quasi-line, we consider étale neighbourhoods instead of its Zariski neighbourhoods.

**Definition.** Let $Y \subset X$ be a quasi-line. The number $\tilde{e}(X,Y)$ is the minimum of $e(X',Y')$, where $X'$ is a projective manifold, $Y' \subset X'$ is a quasi-line, $f : X' \to X$ is a generically finite morphism, étale along $Y'$, and $f(Y') = Y$.

The following result shows a useful relationship between the geometry of quasi-lines and formal geometry. It will play a key role in the sequel.

**Theorem 5.1.** If $Y \subset X$ is a quasi-line, then

$$e(X,Y) = \tilde{e}(X,Y) \cdot b(X,Y).$$

**Proof.** We first show that if $f : (X',Y') \to (X,Y)$ is a generically finite morphism of models, with $Y$ and $Y'$ quasi-lines, étale along $Y'$, then $e(X,Y) = \deg f \cdot e(X',Y')$. Let $y' \in Y'$ be a fixed point and let $y = f(y')$. For $x \in X$ a general point, we denote by $x'_1, \ldots, x'_d$ the points of the fibre over $x$, where $d = \deg f$. We consider the quasi-lines on $X'$ equivalent to $Y'$, that pass through $y'$ and $x'_i$, for some $1 \leq i \leq d$. Their images on $X$ are quasi-lines through $y$ and $x$, and are equivalent to $Y$. The induced map on Chow schemes $f_* : \text{Ch}_{Y'}(X') \to \text{Ch}_y(X)$ is injective when restricted to the open sets parametrising quasi-lines. This comes from the fact that the considered quasi-lines on $X'$ do not intersect the ramification divisor of $f$. It follows that this restriction of $f_*$ is also surjective, giving the equality.
Next we choose an $f$ as above, such that $e(X', Y') = \tilde{e}(X, Y)$. We claim that $Y'$ is G3 in $X'$. We can apply the Hartshorne-Gieseker construction to get $g : (X'', Y'') \to (X', Y')$ as above, with $\deg g = b(X', Y')$. Then, by the previous step,

$$\tilde{e}(X, Y) = e(X', Y') = b(X', Y') \cdot e(X'', Y''),$$

and from the definition of $\tilde{e}(X, Y)$ it follows that $b(X', Y') = 1$.

To finish the proof we consider the following diagram associated to $f$,

$$
\begin{array}{ccc}
K(X) & \longrightarrow & K(\hat{X}|_{Y'}) \\
\downarrow & & \downarrow \cong \\
K(X') & \longrightarrow & K(\hat{X}'|_{Y'})
\end{array}
$$

and conclude that $\deg f = b(X, Y)$. Note that the right vertical isomorphism comes from the fact that $f$ being étale along $Y'$, induces an isomorphism between $\hat{X}|_Y$ and $\hat{X}'|_{Y'}$, see [3], Lemma 4.5.

The next corollary shows that $\tilde{e}(X, Y)$ depends only on the formal completion of $X$ along $Y$.

**Corollary 5.2.** Let $(X, Y)$ and $(X', Y')$ be two models with $Y$ and $Y'$ quasi-lines. If $\hat{X}|_Y \cong \hat{X}'|_{Y'}$ as formal schemes, then $\tilde{e}(X, Y) = \tilde{e}(X', Y')$.

**Proof.** We use Hartshorne-Gieseker’s construction and suppose that $Y$ and $Y'$ are G3. As the formal completions are isomorphic, it follows that $Y$ and $Y'$ have isomorphic Zariski neighbourhoods, by Gieseker’s Theorem. Hence $e(X, Y) = e(X', Y')$. \hfill $\square$

**Corollary 5.3.** Let $Y \subset X$ be a quasi-line. Then

$$e_0(X, Y) \leq \tilde{e}(X, Y) \leq e(X, Y).$$

**Proof.** The second inequality comes directly from the theorem. As for the first one, it is enough to notice, as in the proof of the above theorem, that $e_0(X, Y)$ is preserved by étale covers over $Y$. \hfill $\square$

The connection between $e(X, Y)$ and $\tilde{e}(X, Y)$ gives the following characterisation of the G3 property.

**Corollary 5.4.** Let $Y \subset X$ be a quasi-line. $Y$ is G3 in $X$ if and only if $\tilde{e}(X, Y) = e(X, Y)$. In particular, if $e_0(X, Y) = e(X, Y)$, then $Y$ is G3.

As a very special case, a quasi-line $Y \subset X$ with $e(X, Y) = 1$ is G3. This generalizes the fact, first noticed by Hironaka, that a line in the projective space is G3.

It is easy to see that when $X$ has dimension 2 and $Y \subset X$ is a quasi-line, the formal completion $\hat{X}|_Y$ is isomorphic to $\mathbb{P}^2|_{\text{line}}$. However, in higher dimensions, the situation is completely different, as shown by the following example. See also [10].
Example 5.5. Let $E$ be a vector bundle over $\mathbb{P}^2$ associated to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \to E \to J_{p,q}(2) \to 0,$$

where $p$ and $q$ are two distinct points in $\mathbb{P}^2$. Note that, $p$ and $q$ fixed, $E$ lives in a 2-dimensional family. Let $Y \subset P(E)$ be the quasi-line, lying over a line in $\mathbb{P}^2$, given by the construction in [10], Proposition 4.2. We claim that for two models of this type, $(P(E), Y)$ and $(P(E'), Y')$, the isomorphism $\hat{P}(E)|_Y \simeq \hat{P}(E')|_{Y'}$ holds only if the bundles $E$ and $E'$ are isomorphic.

To justify the claim, we note that by the following lemma, $e(P(E), Y) = e(P(E'), Y') = 1$. Hence, by Corollary 5.3 $Y$ and $Y'$ are G3 in $P(E)$ and $P(E')$, respectively. By Gieseker’s result, there exist open subsets $U \subset P(E)$ and $U' \subset P(E')$ and an isomorphism $\varphi : U \to U'$, such that $\varphi(Y) = Y'$. Moreover, the complements of $U$ and $U'$ are at least 2-codimensional by Lemma 4.4 in [10], and $\varphi$ induces an isomorphism on the Picard groups. The formula for the canonical class shows that $\varphi^*\mathcal{O}_{P(E')}(1) = \mathcal{O}_{P(E)}(1)$. Using the exact sequence that defines the vector bundles, we infer that $\varphi$ is an isomorphism outside the fibers over the fixed points $p$ and $q$. Now, if $D$ and $D'$ are the pull-backs of a line in $\mathbb{P}^2$ on $P(E)$ and $P(E')$ respectively, standard computations with intersection numbers show that $\varphi^*\mathcal{O}_{P(E')}(D') = \mathcal{O}_{P(E)}(D)$. Here we use that a general $D$ avoids the indeterminacy locus of $\varphi$. It follows that $\varphi$ is an isomorphism between the two projective bundles over $\mathbb{P}^2$. As the Chern classes of the two vector bundles are the same, the claim is proved.

Lemma 5.6. Let $(X, Y)$ be a model with $Y$ a quasi-line, $E$ be a rank $r \geq 2$ vector bundle over $X$ and $\pi : P(E) \to X$ the canonical projection. If there exists $Y' \subset P(E)$ a quasi-line that projects isomorphically onto $Y$, then $e(P(E), Y') = e(X, Y)$.

Proof. Let $x' \in Y' \subset P(E)$ and $x = \pi(x')$. Consider the induced map $\pi_* : \text{Ch}_x(P(E)) \to \text{Ch}_{x'}(X)$. The open subset of $\pi_*^{-1}(Y)$ that corresponds to quasi-lines through $x'$ is irreducible of dimension $r - 1$. This follows from Proposition 4.1. in [10]: these quasi-lines correspond to lines in a certain $\mathbb{P}^r$, passing through a given point. In particular, there is only one such curve passing through a second point.

Now, let $x''$ be a general point of $P(E)$. The choice of $x''$ implies that any quasi-line from the family determined by $Y'$ that passes through $x'$ and $x''$ is mapped by $\pi$ to a quasi-line equivalent to $Y$ and passing through $x$ and $\pi(x'')$. □

The above lemma allows us to give a similar example with a rank $r$ vector bundle over $\mathbb{P}^r$.

Example 5.7. Let $r \geq 2$, $X = P(T_{\mathbb{P}^r})$ and $Y \subset X$ be an almost-line as in [10], Proposition 4.2. Let also $X'$ be the projective space of dimension $2r - 1$, and $Y' \subset X'$ be a line. $Y$ is G3 in $X$, but the formal completions $\hat{X}|_Y$ and $\hat{X}'|_{Y'}$ are not isomorphic$^2$.

Indeed, by the above lemma, $e(X, Y) = 1$. Hence $Y$ is G3 in $X$ by Corollary 5.3. If the formal completions were isomorphic, by Gieseker’s result, $Y$ and $Y'$ would have isomorphic Zariski neighbourhoods. The complements of these neighbourhoods would be at least 2-codimensional by Lemma 4.4 in [10], hence $X$ and $X'$ would have isomorphic Picard groups. This is absurd.

This example is relevant in connection with the following proposition:

$^2$For $r = 2$, this was proved in [10] by an ad-hoc argument. Herbert Kurke informed the first named author that he independently proved the assertion about the formal completions by a completely different method.
Proposition 5.8. Let $X$ be a projective manifold. The following conditions are equivalent:

(i) $X$ is strongly-rational;
(ii) $X$ contains a curve $Y$ in such a way that
   (a) $\hat{\mathbb{X}}|_Y$ is isomorphic (as formal schemes) to $\hat{\mathbb{P}}^n|_{\text{line}}$, and
   (b) $Y$ is G3 in $X$.

The equivalence follows from Gieseker’s result, using the fact that a line in $\mathbb{P}^n$ is G3. The Example 5.7 shows that at least for $n$ odd, $n \geq 3$, there are models $(X,Y)$ with $\dim X = n$ that satisfy condition (b), but not condition (a). On the other hand, the examples constructed in [1], 2.7 satisfy condition (a), but not (b). So the two conditions, both very hard to verify in practice, are independent.

Our last corollary is a kind of “formal Torelli theorem” for cubic threefolds.

Corollary 5.9. Let $X$ and $X'$ be smooth cubic threefolds in $\mathbb{P}^4$, and let $\Gamma \subset X$ and $\Gamma' \subset X'$ be general conics. If $\hat{\mathbb{X}}|_{\Gamma} \simeq \hat{\mathbb{X}}'|_{\Gamma'}$, then there exists an isomorphism $\varphi : X \to X'$ such that $\varphi(\Gamma) = \Gamma'$.

Proof. We have noticed in Proposition 3.2 that such a conic is a quasi-line with $e_0(X, \Gamma) = e(X, \Gamma) = 6$. We apply Corollary 5.4 to deduce that $\Gamma$ and $\Gamma'$ are G3. By Gieseker’s result, $\Gamma$ and $\Gamma'$ have isomorphic Zariski neighbourhoods. As $X$ and $X'$ are Fano manifolds with $b_2 = 1$, the birational isomorphism extends to an isomorphism, see e.g. [10], Proposition 1.17.

□

6 Some questions

A first question arises from the fact that we do not know of any example of a quasi-line $Y \subset X$ for which $e_0(X, \Gamma) = e(X, \Gamma) < \bar{e}(X,Y)$, cf. Corollary 5.3. Note that if equality holds, then $b(X,Y)$ is constant in the family of quasi-lines determined by $Y$. In particular, if $Y$ is G3, then every $Y' \sim Y$ is G3.

More importantly, we would like to ask the following:

Question. If $Y \subset X$ is a quasi-line, is $\bar{e}(X,Y)$ determined by the field of formal rational functions $K(\hat{\mathbb{X}}|_Y)$?

More precisely, if two models have $K(\hat{\mathbb{X}}|_Y) \simeq K(\hat{\mathbb{X}}'|_Y)$ as $\mathbb{C}$-extensions, does it follow that $\bar{e}(X,Y) = \bar{e}(X',Y')$? The question may be reformulated as follows: If $(X,Y)$ and $(X',Y')$ are two models with $Y$ and $Y'$ quasi-lines which are G3 in $X$ and $X'$ respectively, and $X$ is birational to $X'$, is it true that $e(X,Y) = e(X',Y')$? The equivalence of the two formulations comes from the Hartshorne-Gieseker construction. We note that a positive answer to this question would be analogous to the birational invariance of $e(X)$ established in Theorem 3.4. However, $\bar{e}(X,Y)$ is much easier to compute than $e(X)$.

A relevant particular case of the above problem concerns rational manifolds.

Question. Let $X$ be a rational projective manifold and $Y \subset X$ be a quasi-line, G3 in $X$. Is it true that $e(X,Y) = 1$?
This question appears as a natural converse of the facts proved in Proposition 5.1 and in Corollary 5.4. If \( e(X,Y) = 1 \), then \( X \) is rational and \( Y \) is \( G3 \) in \( X \).

Recall from the proof of Corollary 5.9 that a general conic \( \Gamma \) lying on a smooth cubic threefold \( X \subset \mathbb{P}^4 \) is a quasi-line, is \( G3 \) in \( X \) and has \( e(X,\Gamma) = 6 \). It follows that a positive answer to the above question would yield a completely new proof of the non-rationality of \( X \).

The simple example below points out the difficulty in constructing a counterexample to the above question. Consider \( \Gamma \) a conic in \( \mathbb{P}^3 \), a fixed point \( p \in \Gamma \), a general line \( l \) and a general smooth curve \( C \) both meeting \( \Gamma \). Take \( X \) to be the blow-up of \( \mathbb{P}^3 \) with center \( p, l \) and \( C \). The proper transform of the conic becomes a quasi-line \( Y \). Remark that \( \bar{\varepsilon}(X,Y) \) is independent of the choice of the curve \( C \), the formal completion \( \bar{X}|_Y \) being the same. When \( C \) is a line, we easily find out that \( e(X,Y) = 1 \). However, when \( C \) is irrational, \( e(X,Y) \) must be greater than one. This comes from the proof of Proposition 5.1. If \( e(X,Y) = 1 \), the exceptional divisor lying over \( C \) should be rational. Hence, in this case, \( Y \) is not \( G3 \) in \( X \).

**APPENDIX: A TORIC EXAMPLE**

If \( (X,Y) \) is a model, with \( Y \) an almost-line, one may ask the following question, related to the hypothesis of Theorem 4.5 (see [10], Remark 1.14): Can we find a linear system \( |D| \) on \( X \) such that \( X \) is not \( G3 \) on \( X \)?

Recall from the proof of Corollary 5.1 that a general conic \( \Gamma \) lying on a smooth cubic threefold \( X \subset \mathbb{P}^4 \) is a quasi-line, is \( G3 \) in \( X \) and has \( e(X,\Gamma) = 6 \). It follows that a positive answer to the above question would yield a completely new proof of the non-rationality of \( X \).

The simple example below points out the difficulty in constructing a counterexample to the above question. Consider \( \Gamma \) a conic in \( \mathbb{P}^3 \), a fixed point \( p \in \Gamma \), a general line \( l \) and a general smooth curve \( C \) both meeting \( \Gamma \). Take \( X \) to be the blow-up of \( \mathbb{P}^3 \) with center \( p, l \) and \( C \). The proper transform of the conic becomes a quasi-line \( Y \). Remark that \( \bar{\varepsilon}(X,Y) \) is independent of the choice of the curve \( C \), the formal completion \( \bar{X}|_Y \) being the same. When \( C \) is a line, we easily find out that \( e(X,Y) = 1 \). However, when \( C \) is irrational, \( e(X,Y) \) must be greater than one. This comes from the proof of Proposition 5.1. If \( e(X,Y) = 1 \), the exceptional divisor lying over \( C \) should be rational. Hence, in this case, \( Y \) is not \( G3 \) in \( X \).

**APPENDIX: A TORIC EXAMPLE**

If \( (X,Y) \) is a model, with \( Y \) an almost-line, one may ask the following question, related to the hypothesis of Theorem 4.5 (see [10], Remark 1.14): Can we find a linear system \( |D| \) on \( X \) such that \( D \cdot Y = 1 \) and \( \dim |D| \geq 1 \)?

We refer to [2] for basic notions on toric varieties and recall here the following facts we shall need about Cartier divisors on the toric variety \( X(\Sigma) \), where \( \Sigma \) is a fan in the lattice \( N \subset \mathbb{R}^n \):

1. The closure of the orbit corresponding to a ray in the fan is an irreducible, effective, toric Weil divisor on the variety.

2. Let \( D = \sum_i a_i D_i \) be a toric Weil divisor, where \( i \) runs over the rays in the fan, \( D_i \) is the closure of an orbit corresponding to the \( i \)th ray, \( \rho_i \cap N = v_i \mathbb{Z} \), and \( a_i \in \mathbb{Z} \). \( D \) is a Cartier divisor if and only if the map \( \psi_D(v_i) = -a_i \) can be extended to a piecewise \( \mathbb{Z} \)-linear map on \( \Sigma \).

3. The Picard group for a toric variety is spanned by the classes of the toric divisors.

4. Let \( D \) be a Cartier toric divisor and \( \psi_D \) be its associated piecewise \( \mathbb{Z} \)-linear map. The dimension of the space of global sections of \( \mathcal{O}_{X(\Sigma)}(D) \) equals the number of integer points in the polyhedron \( P_D = \{ u \in N^* \otimes \mathbb{R} \mid u \geq \psi_D \} \).

5. Let \( f : X(\Sigma_1) \to X(\Sigma_2) \) be a toric map, where \( \Sigma_i \subset N_i, i = 1,2 \). Giving a toric map is equivalent to giving a lattice homomorphism \( f : N_1 \to N_2 \) such that \( f(\Sigma_1) \subset \Sigma_2 \). The pull-back of a toric Cartier divisor \( D_2 \) on \( X(\Sigma_2) \) is the toric Cartier divisor characterised by the piecewise \( \mathbb{Z} \)-linear map \( \psi_{D_2} \circ f \).

The example we are considering is the following: Let \( U_{n+1} \) be the cyclic group of order \( n+1 \) acting on \( \mathbb{P}^n \) by \( \zeta \circ [x_0,x_1,\ldots,x_n] = [x_0,\zeta x_1,\ldots,\zeta^n x_n] \), and let \( \pi \) be the quotient map \( \mathbb{P}^n \to X = \mathbb{P}^n / U_{n+1} \). The projective space and \( X \) are toric varieties, and \( \pi \) is a toric map. To see this, let \( N \) be the canonical lattice in \( \mathbb{R}^n \) spanned by \( e_1, e_2,\ldots, e_n \) over \( \mathbb{Z} \), and let \( N' \subset N \) be the sub-lattice spanned by \((n+1)e_1, e_2,\ldots, e_n \). The vectors \( v_1 = (n+1)e_1 - 2e_2 - 3e_3 - \cdots - ne_n \), \( v_i = e_i \) for \( i = 2,\ldots,n \) and \( v_{n+1} = -(n+1)e_1 + e_2 + 2e_3 + \cdots + (n-1)e_n \) form the fans \( \Sigma_{N'} \subset N' \).
and $\Sigma_N \subset N$, with maximal cones the $n + 1$ simplicial cones. Then the projective space is the toric variety $X(\Sigma_N')$ and $X = X(\Sigma_N)$. The quotient $N/N' \simeq U_{n+1}$ acts naturally on $X(\Sigma_N')$, the action being the one considered above. The toric map induced by the inclusion $\Sigma_N' \subset \Sigma_N$ is the quotient map.

Let $H \subset \mathbb{P}^n$ be the hyperplane corresponding to the piecewise $\mathbb{Z}$-linear map $\psi_H(v_j) = -\delta_{2j}$ defined on $\Sigma_N$. This map is not the restriction of a piecewise $\mathbb{Z}$-linear map on $\Sigma_N$. But, if we consider the open subset $U \subset X$ corresponding to the sub-fan $\Delta \subset \Sigma_N$, where $\Delta$ is the union of all the rays in $\Sigma_N$, then $\psi_H$ defines a divisor $D$ on $U$. We note by $\psi_D = \psi_H|_\Delta$ the map associated to $D$.

**Lemma A.1.** $h^0(U, \mathcal{O}_U(D)) = 1$.

**Proof.** We have to count the number of integer points in the polyhedron $P_D$. Since

$$P_D \cap N^* = \{ u = \sum_{j=1}^n \alpha_j e^*_j \mid u \geq \psi_D \} \cap N^*$$

and $u \geq \psi_D$ is equivalent to

$$\begin{cases} (n+1)\alpha_1 - \sum_{j=1}^n j \alpha_j \geq 0 \\ \alpha_2 \geq -1 \\ \alpha_j \geq 0 \text{ for every } j \geq 3 \end{cases}$$

we conclude that $P_D \cap N^* = \{0\}$. \hfill \Box

**Remark.** The open subset $U \subset X$ equals $X$ minus its $n + 1$ singular points and $\pi^{-1}(U)$ is the projective space minus the fixed points for the $U_{n+1}$-action. Moreover $\pi^* \mathcal{O}_U(D) = \mathcal{O}_{\mathbb{P}^n}(H)|_{\pi^{-1}(U)}$, and since

$$H^0(\pi^{-1}(U), \mathcal{O}_{\mathbb{P}^n}(H)|_{\pi^{-1}(U)}) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H))$$

as $U_{n+1}$-representations, the space of global sections for $\mathcal{O}_U(D)$ is isomorphic to the $U_{n+1}$ trivial sub-representation of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H))$ which is 1-dimensional.

Let $\tilde{X} \to X$ be a toric desingularization of $X$ obtained by taking a smooth sub-division $\tilde{\Sigma}$ of $\Sigma_N$.

**Lemma A.2.** If $\psi_{\tilde{D}}$ is a piecewise $\mathbb{Z}$-linear extension of $\psi_D$ to $\tilde{\Sigma} \subset N$, then the space of global sections of $\mathcal{O}_{\tilde{X}}(\tilde{D})$ is of dimension $\leq 1$.

**Proof.** This is clear, since from $\Sigma_N \subset \tilde{\Sigma}$ and $\psi_{\tilde{D}}|_{\Sigma_N} = \psi_D$, we infer that $P_{\tilde{D}} \subset P_D$. \hfill \Box

Going back to the problem of finding effective divisors $\tilde{D}$ on $\tilde{X}$ such that $\tilde{D} \cdot g^* Y = 1$, where $Y$ is the almost line $\pi(L) \subset U$, $L = \{ x_0 = x_1, x_2 = x_3, x_j = 0 \text{ otherwise} \}$, the answer is the following:

**Corollary A.3.** If $\tilde{D} \subset \tilde{X}$ is a divisor such that $\tilde{D} \cdot g^* Y = 1$, then $h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{D})) \leq 1$. 

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Proof. We have the following commutative diagram,

\[
\begin{array}{ccc}
\widetilde{\mathbb{P}}^n & \xrightarrow{f} & \mathbb{P}^n \\
\widetilde{\pi} & \downarrow & \downarrow \pi \\
\widetilde{X} & \xrightarrow{g} & X
\end{array}
\]

where \(\widetilde{\mathbb{P}}^n\) is the toric variety \(X(\widetilde{\Sigma}_N')\), with \(\widetilde{\Sigma}_N'\) the fan spanned by \(\widetilde{\Sigma}\) in \(N'\). Then

\[
n = \widetilde{\pi}^*(\widetilde{D} \cdot g^*Y) = \widetilde{\pi}^* \cdot n f^*L,
\]

hence \(\widetilde{\pi}^* \cdot f^*L = 1\). Since \(f_* f^*L = L\), it follows that \(f_* \widetilde{\pi}^* \widetilde{D} = H\) and that the function \(\psi_{\widetilde{\pi}, \widetilde{D}}\) defined on \(\widetilde{\Sigma}_{N'}\) should send to 1 exactly one of the \(n+1\) vectors \(v_j, j = 1, \ldots, n+1\), and to 0 the remaining \(n\). Hence, the piecewise \(\mathbb{Z}\)-linear map \(\psi_{\widetilde{D}}\), modulo an \(SL(n, \mathbb{Z})\) transformation, is the map in lemma A.2, and the result follows from that lemma. \(\square\)

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