TILINGS OF AN ISOSCELES TRIANGLE

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ABSTRACT. An \( N \)-tiling of triangle \( ABC \) is a way to cut \( ABC \) into \( N \) congruent smaller triangles. The smaller triangle is the “tile.” When \( ABC \) is isosceles with base angles \( \alpha \), and not equilateral, there are only four possible tiles (aside from a tile similar to \( ABC \)): a right-angled tile with one angle \( \alpha \), a tile with angles \((\alpha, \beta, 2\alpha)\), a tile with angles \((\alpha, \beta, 2\pi/3)\), or a tile with angles satisfying \(3\alpha + 2\beta = \pi\) (and in all but the first case, with \( \alpha \) not a rational multiple of \( \pi \)). We study the first three cases in this paper.

For tilings by a right triangle, \( N \) has to be a square, or an even sum of squares, or six times a square; in particular it cannot be a prime congruent to \( 3 \mod 4 \); and all these possibilities actually occur. We prove that unless \( ABC \) is a right isosceles triangle, \( N \) has to be even.

For tilings by \((\alpha, \beta, 2\alpha)\), we show that the tile is necessarily rational (the ratios of its sides are rational), and we give a necessary condition for the existence of a tiling. This condition implies that when an isosceles and not equilateral \( ABC \) is \( N \)-tilled by such a tile, \( N \) cannot be a prime number, or even squarefree.

In the last case, when the tile has a 120 degree angle, we also prove that the tile must be rational, and find a necessary condition for the existence of a tiling. That condition rules out \( N < 36 \), but leaves open whether \( N \) can possibly be prime. The smallest known such tiling has \( N = 2736 \).

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1. INTRODUCTION

An \( N \)-tiling of triangle \( ABC \) by triangle \( T \) is a way of writing \( ABC \) as a union of \( N \) triangles congruent to \( T \), overlapping only at their boundaries. The triangle \( T \) is the “tile”. We consider here the case of an isosceles (but not equilateral) triangle \( ABC \).

Our results fit into a larger research program, begun by Laczkovich [7]. Laczkovich studied the possible shapes of tiles and triangles that can possibly be used in tilings, and obtained results that will be described below. The reader who is new to the subject may want to see examples of \( N \)-tilings for various shapes of \( ABC \); such pictures can be found in [1]. Here we give only examples relevant to the case of \( ABC \) isosceles.

First we point out that any triangle can be decomposed into \( n^2 \) congruent triangles by drawing \( n - 1 \) equally spaced lines parallel to each of the three sides of the triangle, as illustrated in Fig. 1. Moreover, the large (tiled) triangle is similar to the small triangle (the “tile”). We call such a tiling a quadratic tiling.
It follows that if we have a tiling of a triangle $ABC$ into $N$ congruent triangles, and $m$ is any integer, we can tile $ABC$ into $Nm^2$ triangles by subdividing the first tiling, replacing each of the $N$ triangles by $m^2$ smaller ones. Hence the set of $N$ for which an $N$-tiling of some triangle exists is closed under multiplication by squares.

Sometimes it is possible to combine two quadratic tilings (using the same tile) into a single tiling, as shown in Fig. 2. We will explain how these tilings are constructed. We start with a big right triangle resting on its hypotenuse, and divide it into two right triangles by an altitude. Then we quadratically tile each of those triangles. The trick is to choose the dimensions in such a way that the same tile can be used throughout. If that can be done then evidently $N$, the total number of tiles, will be the sum of two squares, $N = n^2 + m^2$, one square for each of the two quadratic tilings. On the other hand, if we start with an $N$ of that form, and we choose the tile to be an $n$ by $m$ right triangle, then we can construct such a tiling. We call these tilings “biquadratic.” More generally, a biquadratic tiling of triangle $ABC$ is one in which $ABC$ has a right angle at $C$, and can be divided by an altitude from $C$ to $AB$ into two triangles, each similar to $ABC$, which can be tiled respectively by $n^2$ and $m^2$ copies of a triangle similar to $ABC$. A larger biquadratic tiling, with $n = 5$ and $m = 7$ and hence $N = 74$, is shown at the right of Fig. 2.

If the original triangle $ABC$ is chosen to be isosceles, and is then quadratically tiled, then each of the $n^2$ triangles can be divided in half by an altitude; hence any isosceles triangle can be decomposed into $2n^2$ congruent triangles. If the original triangle is equilateral, then it can be first decomposed into $n^2$ equilateral triangles, and then these triangles can be decomposed into 3 or 6 triangles each, showing that any equilateral triangle can be decomposed into $3n^2$ or $6n^2$ congruent triangles. For example we can 12-tile an equilateral triangle in two different ways, starting with
a 3-tiling and then subdividing each triangle into 4 triangles (“subdividing by 4”),
or starting with a 4-tiling and then subdividing by 3.

There is another family of \(N\)-tilings, in which \(N\) is of the form \(3m^2\), and both the
tile and the tiled triangle are 30-60-90 triangles. We call these the “triple-square”
tilings. The case case \(m = 2\) makes \(N = 12\). There are two ways to 12-tile a
30-60-90 triangle with 30-60-90 triangle. One is to first quadratically 4-tile it, and
then subtile the four triangles with the 3-tiling of Figure 1. This produces the
first 12-tiling in Fig. 3. Somewhat surprisingly, there is another way to tile the
same triangle with the same 12 tiles, also shown in Fig. 3. The next member of
this family is \(m = 3\), which makes \(N = 27\). Two 27-tilings are shown in Fig. 4.
Similarly, there are two 48-tilings (not shown).

Figure 3. Two 12-tilings

![Two 12-tilings](image1)

Figure 4. Two 27-tilings

![Two 27-tilings](image2)

Whenever there is an \(N\)-tiling of the right triangle \(ABM\), there is a \(2N\)-tiling
of the isosceles triangle \(ABC\). Using the biquadratic tilings (see Fig. 2) and triple-
square tilings (see Fig 3 and Fig. 4), we can produce \(2N\)-tilings when \(N\) is a
sum of squares or three times a sum of squares. We call these tilings “double
biquadratic” and “hexquadratic”. For example, one has two 10-tilings and two 26-
tilings, obtained by reflecting Figs. 4 and 5 about either of the sides of the triangles
shown in those figures; and one has 24-tilings and 54-tilings obtained from Figs. 8
and 9. Note that in the latter two cases, \(ABC\) is equilateral.

In the case when the sides of the tile \(T\) form a Pythogorean triple \(n^2 + m^2 + k^2 =
N/2\), then we can tile one half of \(ABC\) with a quadratic tiling and the other half
with a biquadratic tiling. The smallest example is when the tile has sides 3, 4, and
5, and \(N = 50\). See Fig. 7. One half is 25-tiled quadratically, and the other half is
divided into two smaller right triangles which are 9-tiled and 16-tiled quadratically.
This shows that the tiling of \(ABC\) does not have to be symmetric about the altitude.

As we shall see below, the work of Laczkovich implies that there are only four
possible shapes of the tile: right-angled, \(\gamma = 2\alpha\), \(\gamma = 2\pi/3\), and \(3\alpha + 2\beta = \pi\). The
last case is taken up in another paper, since the techniques apply also to tilings of non-isosceles triangles $ABC$ with $3\alpha + 2\beta = \pi$. The first three are studied in this paper. We obtain, in the second and third case, necessary conditions on $N$, but not necessary and sufficient conditions. In the case of a right-angled tile, our conditions are necessary and sufficient.

All known examples of tilings of isosceles $ABC$ with $\alpha \neq \frac{\pi}{2}$ have $N$ even. We could prove that it must be so when the tile is right-angled, but we could not prove it in the other two cases, where indeed we know only a few tilings, all of which require $N$ with five to seven digits.

1.1. Acknowledgment. I am grateful to Miklos Laczkovich for his valuable comments on my work and especially for his simplification of the proofs of Lemmas 7.1 and 7.3 and of course for his many pioneering papers in this subject, on which this paper rests.

1.2. Definitions and notation. We first note that this paper is about triangles $ABC$ that are isosceles and not equilateral. Let that be understood; then for the rest of this paper, “isosceles” means “exactly two sides are equal.”

We give a mathematically precise definition of “tiling” and fix some terminology and notation. Given a triangle $T$ and a larger triangle $ABC$, a “tiling” of triangle $ABC$ by triangle $T$ is a set of triangles $T_1, \ldots, T_n$ congruent to $T$, whose interiors are disjoint, and the closure of whose union is triangle $ABC$.

Let $a, b,$ and $c$ be the sides of the tile $T$, and angles $\alpha, \beta,$ and $\gamma$ be the angles opposite sides $a, b,$ and $c$. The letter “$N$” will always be used for the number of triangles used in the tiling. An $N$-tiling of $ABC$ is a tiling that uses $N$ copies of some triangle $T$. The meanings of $N, \alpha, \beta, \gamma, a, b, c, A, B,$ and $C$ will be fixed throughout this paper. We do not assume $\alpha \leq \beta$ in general; although that may sometimes be justified by symmetry, we often will consider some equation such as $3\alpha + 2\beta = \pi$, in which case we do not want to assume $\alpha \leq \beta$.

2. History

Above we exhibited quadratic and biquadratic tilings in which the tile is similar to $ABC$. There are hexagonal tilings, not exhibited in this paper, but see [1] for pictures. These involve $N$ being square, a sum of two squares, or three times a square. The biquadratic tilings were known in 1964, when the paper [4] was published. This is the earliest paper on the subject of which I am aware. Snover et al. [13] took up the challenge of showing that these are the only possible values of $N$, when the tile is similar to $ABC$. The following theorem completely answers the question, “for which $N$ does there exist an $N$-tiling in which the tile is similar to the tiled triangle?”

**Theorem 2.1** (Snover et al. [13]). Suppose $ABC$ is $N$-tilied by tile $T$ similar to $ABC$. If $N$ is not a square, then $T$ and $ABC$ are right triangles. Then either

(i) $N$ is three times a square and $T$ is a 30-60-90 triangle, or

(ii) $N$ is a sum of squares $e^2 + f^2$, the right angle of $ABC$ is split by the tiling, and the acute angles of $ABC$ have rational tangents $e/f$ and $f/e$.

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1 That was Euclid’s definition of “isosceles.”

2 The simplest hexagonal tiling is attributed to Major MacMahon (1921) in the notes accompanying a plastic toy I purchased at an AMS meeting in 2012.
and these two alternatives are mutually exclusive.

Soifer’s book [14] appeared in 1990, with a second edition in 2009. He considered two “Grand Problems”: for which \( N \) can every triangle be \( N \)- tiled, and for which \( N \) can every triangle be dissected into similar, but not necessarily congruent triangles. (The latter eventually became a Mathematics Olympiad problem.) The 2009 edition has an added chapter in which the biquadratic tilings and a theorem of Laczkovich occur.

Miklos Laczkovich published six papers [6, 7, 8, 2, 9, 10] on triangle and polygon tilings. According to Soifer, the 1995 paper was submitted in 1992. Laczkovich, like Soifer, studied dissecting a triangle into smaller similar triangles, not congruent triangles as we require here. If those similar triangles are rational (i.e., the ratios of their sides are rational) then if we divide each of them into small enough quadratic subtilings, we can achieve an \( N \)-tiling into congruent triangles, but of course \( N \) may be large. Laczkovich focused primarily on the shapes of \( ABC \) (or more generally, convex polygons) and of the tile. His theorems give us an exhaustive list of the possible shapes of \( ABC \) and the tile, which we will need in our proof that there is no 7-tiling. This list can be found in [13] (of this paper). However, his theorem published in the last chapter of [14] does mention \( N \). It states that given an integer \( k \), there exists an \( N \)-tiling for some \( N \) whose square-free part is \( k \).

3. Laczkovich

A basic fact is that, apart from a small number of cases that can be explicitly enumerated, if there is an \( N \)-tiling of \( ABC \) by a tile with angles \( (\alpha, \beta, \gamma) \), then the angles \( \alpha \) and \( \beta \) are not rational multiples of \( \pi \). This theorem follows from Theorems 4.1, 5.1, and 5.3 of [7]. Laczkovich is dealing with a more general situation, tiling an arbitrary triangle by tiles that are only required to be similar, not congruent. We extract the following theorem from his results by specializing to isosceles \( ABC \) and congruent tiles.

Table 1. Possible tilings of isosceles triangles, according to Laczkovich.

| \( ABC \)          | the tile               |
|--------------------|------------------------|
| \((\beta, \beta, 2\alpha)\) | similar to \( ABC \)   |
| \((\beta, \beta, 2\alpha)\) | \(\gamma = \pi/2\)   |
| \((\alpha, \alpha, \pi - 2\alpha)\) | \(\gamma = 2\alpha\) |
| \((\alpha, \alpha, \pi - 2\alpha)\) | \(\gamma = 2\pi/3\) |
| \((\alpha, \alpha + 2\beta)\) | \(3\alpha + 2\beta = \pi\) |
| \((\beta, \beta, 3\alpha)\) | \(3\alpha + 2\beta = \pi\) |
| \((\alpha + \beta, \alpha + \beta, \alpha)\) | \(3\alpha + 2\beta = \pi\) |

**Theorem 3.1** (Laczkovich [7]). Let isosceles (and not equilateral) triangle \( ABC \) be \( N \)-tilled by a tile with angles \( (\alpha, \beta, \gamma) \). Then the possible shapes of \( ABC \) and the tile are given by Table 1. In the table, the triples giving the angles of the tile are
(α, β, γ) after a suitable permutation, i.e., they are unordered triples. In all but the first two lines, α is not a rational multiple of π.

Remark. For example, in the second line of the table, we do not list separately (α, α, 2β), as that is already covered by the entry (β, β, 2α), and the fact that we do not assume α < β.

Proof. This theorem is proved in [7], but it is not stated in quite this way; therefore we spell out in detail how this statement follows from theorems explicitly stated in [7]. Let isosceles (and not equilateral) triangle ABC be N-tiled by a tile with angles (α, β, γ). Then either all three angles are rational multiples of π, or not.

Case 1, they are not all rational multiples of π. Then by Theorem 4.1 of [7], where T in that paper is our ABC, one of six cases holds. Cases (i), (ii), and (iv) are the first three lines of our table. Case (iii) says ABC is equilateral, which we have ruled out by hypothesis. Case(v) says 3α + 2β = π and the base angles of isosceles ABC must be α or β or α + β, by Theorem 2.4 of [7]: so that is lines 5, 6, and 7 of our table. Finally, case (vi) is has the tile (α, α, α + 3β) with γ = 2π/3, which is another way of writing line 4 of the table, since if γ = 2π/3, then α + 3β = π − 2α.

Case 2, all three of (α, β, γ) are rational multiples of π. Then Theorem 5.1 of [7] applies. That theorem is about dissections into similar (rather than congruent) triangles, and according to the subsequent Theorem 5.3, the last three cases (cases (v), (vi), and (vii)) in Theorem 5.1 cannot hold for dissections into congruent triangles. Cases (i) and (ii) are the first lines of our table. Case (iii) requires ABC equilateral, which we have ruled out by hypothesis. Case (iv) has ABC a right triangle with one angle π/6, which is not isosceles and hence irrelevant here. That completes the proof.

We note in passing the following immediate consequence of Laczkovich’s theorem: If an isosceles triangle ABC is tiled by a right-angled tile (α, β, π/2), then the base angles of ABC are either equal to β or to α. That follows, because in Table 1, there is only one entry corresponding to a right-angled tile, namely the second line. Readers are invited to try to prove that directly, without appeal to Laczkovich, in order to gain a deeper appreciation for Laczkovich’s work.

4. Some number-theoretic facts

The facts in this section may not be well-known to all our readers, and their proofs are short.

Lemma 4.1. An integer N can be written as a sum of two integer squares if and only if the squarefree part of N is not divisible by any prime of the form 4n + 3.

Proof. See for example [5], Theorem 366, p. 299.

Lemma 4.2. N is a sum of two squares if and only if 2N is a sum of two squares.

Proof. The lemma follows immediately from the identities

\[(p - q)^2 + (p + q)^2 = 2(p^2 + q^2)\]

\[\left(\frac{p - q}{2}\right)^2 + \left(\frac{p + q}{2}\right)^2 = \frac{1}{2}(p^2 + q^2).\]

This lemma is also a corollary of Lemma 4.1, of course, but that is not needed.
The following lemma identifies those relatively few rational multiples of \( \pi \) that have rational tangents or whose sine and cosine satisfy a polynomial of low degree over \( \mathbb{Q} \).

**Lemma 4.3.** Let \( \theta = 2m\pi/n \), where \( m \) and \( n \) have no common factor. Suppose \( \cos \theta \) is algebraic of degree 1 or 2 over \( \mathbb{Q} \). Then \( n \) is one of 5, 6, 8, 10, 12. If both \( \cos \theta \) and \( \sin \theta \) have degree 1 or 2 over \( \mathbb{Q} \), then \( n \) is 6, 8, or 12.

**Proof.** Let \( \varphi \) be the Euler totient function. Assume \( \cos \theta \) has degree 1 or 2. By [12], Theorem 3.9, p. 37, \( \varphi(n) = 2 \) or 4. The stated conclusion follows from the well-known formula for \( \varphi(n) \). The second part of Theorem 3.9 of [12] rules out \( n = 5 \) or 10 when \( \sin \theta \) is also of degree 1 or 2.

**Lemma 4.4** (Pythagorean triangles). The integer solutions of the equation \( x^2 + y^2 = z^2 \) have the form \((x, y, z) = (m^2 - k^2, 2mk, m^2 + k^2)\) for some integers \((m, k)\).

**Proof.** See any number theory textbook. But the proof is short, so we just give it here. By the Pythagorean theorem, \((x, y, z)\) form a right triangle, with one angle \( \alpha \) such that \( x/z = \cos \alpha \) and \( y/z = \sin \alpha \). We use the Weierstrass substitution, \( t = \tan(\alpha/2) \). Then

\[
\cos \alpha = \frac{1 - t^2}{1 + t^2} \quad \sin \alpha = \frac{2t}{1 + t^2}
\]

Setting \( t = m/k \) in lowest terms, and replacing \( \sin \alpha \) and \( \cos \alpha \) by \( y/z \) and \( x/z \), we find the formulas of the lemma for \((x, y, z)\). That completes the proof.

**Lemma 4.5.** If the integer \( n \) is a sum of two rational squares then it is a sum of two integer squares.

**Proof.** Suppose \( n = (p/q)^2 + (s/t)^2 \). Then \((qt)^2n = p^2 + s^2 \). Then by Lemma 4.1 the square-free part of \( n \) is not divisible by any prime congruent to 3 mod 4. Then by a second application of Lemma 4.1 \( n \) is a sum of two integer squares. That completes the proof.

5. Tilings of an isosceles \( \triangle ABC \) by a right-angled tile: examples

**Figure 5.** A 54-tiling; \( N/2 \) is three times a square. Tile is 30-60-90.

Is it possible to have more complicated tilings without essential segments? Yes, because when two tiles share their hypotenuses, they form a rectangle, and we can just draw the diagonal of that rectangle the other way. In this way we can produce (exponentially) many different tilings, but they differ only in this trivial way. And
Figure 6. \( N \) is twice a square or twice a sum of squares.

![Figure 6](image1)

Figure 7. 50 is both twice a square and twice a sum of squares.

![Figure 7](image2)

Figure 8. \( N = 104 \), eight essential segments, base angles about 56°

![Figure 8](image3)

sometimes, as shown in Fig. 9, even those rectangles can be rotated. That figure also shows that a tiling need not necessarily include the altitude of \( ABC \).

In the tilings based on two biquadratic tilings, there are no \( c \) edges on \( AB \) and \( BC \), while in the tilings based on two quadratic tilings, there are only \( c \) edges. There are of course some hybrid tilings when a square is also a sum of squares, in which \( AB \) falls under one case and \( BC \) under the other. If \( N/2 \) is not a square (as is the case for the biquadratic tilings) then there are no \( c \) edges on \( AB \) and \( BC \), as we see in the biquadratic tilings (and prove in the next section).

All these tilings, in which \( N/2 \) is a sum of squares, involve essential segments (where tiles of different lengths occur on the two sides of an internal line). One sees such linear relations in two of the tilings illustrated in Fig. 6.
Figure 9. The altitude need not be part of the tiling.

Figure 10. $N = 2312$, $N/2 = 34^2$, $(a, b, c) = (3, 4, 5)$

6. Laczkovich’s graphs $\Gamma_a$

In trying to prove the impossibility of certain tilings directly, it is easy to become involved in complicated arguments with many cases, involving complicated diagrams. Laczkovich had the brilliant idea to abstract some of these arguments using graph theory. The definition will not be grasped immediately, but instead will require time and the study of examples to understand. But it leads to very elegant proofs of theorems that are much more complicated or impossible to prove more directly. To emphasize its importance, we devote a whole section just to the definition.

Given a tiling of a triangle $ABC$, an internal segment is a line segment connecting two vertices of the tiling that is contained in the union of the boundaries of the tiles, and lies in the interior of $ABC$ except possibly for its endpoints. A maximal
segment is an internal segment that is not part of a longer internal segment. A
segment is **terminated** at a vertex $P$ if it has tiles on both sides with vertices at $P$. (In that case there may or may not be a continuation of that segment past $P$.)

A **left-terminated segment** is an internal segment $XY$ that is terminated at $X$. A **left-maximal segment** is an internal segment $PQ$ that cannot be extended past $P$ to a longer internal segment $UPQ$. (In these two concepts, we are using directed segments; so $PQ$ is not the same as $QP$ in this context. The “left” in “left maximal” refers to the fact that $P$ is listed to the left of $Q$ in $PQ$.) A tile is supported by $XY$ if one edge of the tile lies on $XY$. The internal segment $XY$ is said to have “all $c$’s on the left” if the endpoints $X$ and $Y$ are vertices of tiles supported by $XY$ and lying on the left side of $XY$, and all tiles supported by $XY$ lying on the left of $XY$ have their $c$ edges on $XY$. Similarly for “all $c$’s on the right.” (Here again $XY$ is a directed segment, so the concept “left side” of $XY$ makes sense; but this is a different sense of the English word “left” than in “left-terminated.”)

An internal segment $XY$ is said to witness the relation $jc = \ell a + mb$ if one side has $j$ more $c$ edges than the other, and the other has $\ell$ more $a$ edges and $m$ more $b$ edges than the first. The simplest example is when $XY$ has all $c$’s on one side, and exactly $j$ of them (that is, the length of $XY$ is $jc$), and on the other side $XY$ supports $\ell$ tiles with their $a$ edges on $XY$ and $m$ tiles with their $b$ edges on $XY$ (in any order) and no other tiles, and the endpoints $X$ and $Y$ are vertices of tiles on both sides of $XY$. Similarly we use the terminology “$XY$ witnesses a relation $jb = \ell a + mc$.”

An internal segment that witnesses a relation is called an **essential segment**. The definition allows that an essential segment might have different numbers of tiles of lengths $a, b, c$ on its two sides, without necessarily having all the tiles on one side be the same length, but often it is the case that all the tiles on one side are the same length.

To be sure that you understand the concept of “essential segment”, identify the eight essential segments in Fig. 8. Also identify in that figure some internal segments $PQ$ that are not essential segments, because each side of $PQ$ supports tiles with three $b$ and two $a$ sides on $PQ$. (Those $PQ$ connect the midpoints of the sides of $ABC$ in that figure.)

The following definition is equivalent to the one given in [10, p. 346], except that there condition (iv) is automatic because of an additional assumption about the tile.

**Definition 6.1** (The directed graph $\Gamma_a$). Given a tiling of some triangle, the nodes of the graph $\Gamma_a$ are certain vertices of the tiling. A link of $\Gamma_a$ connects vertices $X$ and $Y$ if

(i) the segment $XY$ is a left-maximal internal segment having all $a$ edges on one side (say Side 1) of $XY$, and

(ii) On the other side of $XY$ (say Side 2) the first tile (the one with a vertex at $X$) does not have its $a$ edge on $XY$, and

(iii) At vertex $Y$, there is another tile supported by $XY$ on Side 1 of $XY$ with a vertex at $Y$, that does not have its $a$ edge on Side 1, and

(iv) No tile supported by $XY$ on Side 2 of $XY$ has a vertex at $Y$.

The directed graphs $\Gamma_b$ and $\Gamma_c$ are defined similarly.
Remarks for clarification. We use “link” instead of “edge” with these graphs, to avoid confusion with tile “edges.” If $XY$ is a link in $\Gamma_a$, then $XY$ does not terminate at $Y$, because there is another tile past $Y$ whose $a$ side is not on $XY$; and also because $Y$ lies on the interior of an edge of a tile on the other side of $XY$.

The reader is recommended to identify all three graphs $\Gamma_a$, $\Gamma_b$, and $\Gamma_c$ in Fig. \ref{fig:1}. Hint: $\Gamma_c$ is empty; $\Gamma_b$ has four links, starting at the midpoints of the sides of $ABC$; $\Gamma_a$ is, in this example, the same graph as $\Gamma_b$.

7. Tilings of an isosceles $ABC$ by a right-angled tile: theory

Laczkovich studied the possible shapes of tiles that can tile an isosceles triangle, but did not characterize the possible $N$. We do so in this section for right-angled tiles and $N$ even. We have two ways to tile an isosceles triangle by a right triangle: either tile each of its two halves by a quadratic tiling, in which case $N$ is twice a square, or tile each of its halves with a biquadratic tiling, in which case $N$ is twice a sum of squares. See Figs. \ref{fig:1} and \ref{fig:2}. The main theorem in this section shows that these are the only possible values of $N$. But Fig. \ref{fig:3} shows that, when $N/2$ is a square, there are also more complicated tilings.

Lemma 7.1. Suppose isosceles (or equilateral) triangle $ABC$ with base angles $\beta$ is $N$-tiled by tile $(\alpha, \beta, \pi/2)$ with sides $(a, b, c)$, and $\alpha$ is not a rational multiple of $\pi$, or $\alpha$ is an odd multiple of $\beta$. Let $PQ$ be a link in $\Gamma_c$. Then there are two adjacent tiles with vertices at $Q$ whose common boundary contains an $a$ or $b$ edge of one tile, and a $c$ edge of the other tile.

Remarks. For short, there is an $a/c$ edge at $Q$, or a $b/c$ edge at $Q$. Consider Fig. \ref{fig:1} in which $\beta = \pi/6$ and $\alpha = \pi/3$, so $\alpha$ is not an odd multiple of $\beta$. Observe that the present lemma fails in the tiling of Fig. \ref{fig:1}, showing that the hypothesis that $\alpha$ is an odd multiple of $\beta$ cannot be removed.

Proof. Let $\Delta_1, \ldots, \Delta_k$ be tiles with vertices at $Q$, numbered so that $\Delta_1$ is supported by $PQ$ (and hence has side $c$ on $PQ$), $\Delta_i$ and $\Delta_{i+1}$ are adjacent, and $\Delta_k$ has one edge extending $PQ$ past $Q$. Since $PQ$ is a link in $\Gamma_c$, there does exist such a tile $\Delta_k$, and $\Delta_k$ has its $a$ or $b$ edge extending $PQ$. (There may or may not be tiles on the other side of $PQ$ with a vertex at $Q$, but if so, we do not list them among the $\Delta_i$.)

If there are an even number of tiles with vertices at $Q$ and an $\alpha$ or $\beta$ angle at $Q$, then each has a $c$ edge ending at $Q$. Since $\Delta_1$ has its $c$ edge ending at $Q$ and $\Delta_k$ does not, the remaining odd number of $c$ edges cannot all be paired with other $c$ edges supported by the same line. Therefore, there is an $a/c$ or $a/b$ edge, as claimed. Therefore, we may assume the number of such tiles is odd. At most one tile can have its right angle at $Q$, and it cannot be $\Delta_1$.

Now, if $\alpha$ is not a rational multiple of $\pi$, then either $k = 2$ and both angles are right angles, or $k = 4$ and there are two $\alpha$ and two $\beta$ angles, or $k = 3$ and there are one each of $(\alpha, \beta, \pi/2)$. In all these cases, the above condition that there are an even number of tiles at $Q$ with an $\alpha$ or $\beta$ angle at $Q$ is fulfilled.

If $\alpha$ is an odd multiple $m$ of $\beta$, then the same argument works, as the number of tiles with vertices at $Q$ that do not have their right angle at $Q$ will still be even.

That completes the proof of the lemma.

Lemma 7.2. Suppose isosceles (or equilateral) triangle $ABC$ with base angles $\beta$ is $N$-tiled by tile $(\alpha, \beta, \pi/2)$ with sides $(a, b, c)$, and $\alpha$ is not a rational multiple of $\pi$, 

or \( \alpha \) is an odd multiple of \( \beta \). Suppose there is no relation \( jc = ua + vb \) with \( j > 0 \) and \( u, v \geq 0 \), and \( u, v, j \) integers. Then

(i) Let \( PQ \) be a link in \( \Gamma_c \). Then there is a vertex \( R \) such that \( QR \) is a link in \( \Gamma_c \).

(ii) The in-degree and out-degree of each node of \( \Gamma_c \) is exactly 1.

Remark. Note that in Fig. 3 it is not true that the in-degree of every vertex in \( \Gamma_b \) is equal to the out-degree of that vertex. That is because, in that tiling, there is a relation \( 2b = 3a \). Please take the time to verify that in Fig. 3 the conclusion of this lemma fails, but so does the hypothesis that \( \alpha \) is an odd multiple of \( \beta \); this shows the necessity of that hypothesis.

Proof. According to Lemma 7.1 there is an outgoing \( a/c \) edge from \( Q \). Let \( R \) lie on the internal segment containing that edge, as far as possible from \( Q \) such that there are only \( c \) edges on one side of \( QR \). Then \( R \) is not a vertex of a tile on the other side of \( QR \), since that would give rise to relation \( jc = ua + vb \), where \( j \) is the excess of the number of \( c \) edges on one side of \( QR \) over the other. Then, by definition of \( \Gamma_c \), \( QR \) is a link in \( \Gamma_c \). That completes the proof of (i).

Next we observe that the in-degree of each node \( Q \) of \( \Gamma_c \) is at most 1. For, if \( PQ \) is a link of \( \Gamma_c \), then \( PQ \) is part of an internal segment of the tiling that extends past \( Q \) and on one side, is not a vertex of any tile on that side. Hence no other internal segment can pass through \( Q \); hence there is no other link of \( \Gamma_c \) ending at \( Q \).

By part (i), the out-degree of each node of \( \Gamma_a \) that has positive in-degree is at least 1. Hence, the out-degree always is greater than or equal to the in-degree. But since every link has one head and one tail, the total in-degree is equal to the total out-degree. Therefore, the in-degree and out-degree are equal at every node. Therefore, if the in-degree is positive, both the in-degree and out-degree are 1. If the in-degree is zero, so must the out-degree be zero, but then that node is not in \( \Gamma_c \) at all. That completes the proof of the lemma.

**Corollary 7.3.** Suppose isosceles (or equilateral) triangle \( ABC \) with base \( AC \) and base angles \( \beta \) is \( N \)-tiled by tile \((\alpha, \beta, \pi/2)\) with sides \((a, b, c)\), and \( \alpha \) is not a rational multiple of \( \pi \), or \( \alpha \) is an odd multiple of \( \beta \). Suppose there is no relation \( jc = ua + vb \) with \( j > 0 \) and \( u, v \geq 0 \), and \( u, v, j \) integers. Then \( AC \) is composed only of \( c \) edges and there are no \( c \) edges on \( AB \) or \( BC \), or \( AB \) and \( BC \) are composed only of \( c \) edges and there is no \( c \) edge on \( AC \).

Proof. According to Lemma 7.3 each link \( PQ \) in \( \Gamma_c \) has a corresponding link \( QR \).

Suppose there is a tile with a \( c \) edge on \( AB \). Unless the entire segment \( AB \) supports only tiles with their \( c \) edges on \( AB \), there will be a segment \( PQ \) lying on \( AB \) such that \( PQ \) is composed of \( c \) edges but beyond \( Q \) there is another tile with an \( a \) or \( b \) edge on \( PQ \). But that contradicts Lemma 7.3 as there will be an outgoing link of \( \Gamma_c \) from \( Q \), but at a boundary point, there can be no incoming link. Hence if there is any \( c \) edge on \( AB \), then \( AB \) is composed entirely of \( c \) edges. Similar for \( BC \) and \( AC \).

At vertex \( A \), there is an angle \( \beta \). There must be a single tile there, with its \( \beta \) angle at \( A \), since either \( \beta < \alpha \) or \( \alpha \) is not a rational multiple of \( \pi \). The \( c \) edge of that tile must lie on \( AB \) or \( AC \). If it lies on \( AB \), then \( AB \) is composed entirely of \( c \) edges. If it lies on \( AC \), then \( AC \) is composed entirely of \( c \) edges. If \( AC \) is composed of \( c \) edges, then \( AB \) and \( BC \) do not contain any \( c \) edges, since if they contained
one, then there would also be a $c$ edge at $A$ or $C$, which is impossible since the tiles at $A$ and $C$ have only one $c$ edge, and it is on $AC$. That completes the proof of the lemma.

**Lemma 7.4.** Suppose isosceles (or equilateral) triangle $ABC$ with base angles $\beta$ at $A$ and $C$ is $N$-tilied by a tile with angles $(\alpha, \beta, \gamma)$ and sides $(a, b, 1)$. Suppose $\beta \neq \pi/6$ and $\sqrt{N/2}$ is irrational (i.e., $N$ is not twice a square). Then $a$ and $b$ belong to $\mathbb{Q}(\sqrt{N/2})$.

**Remark.** The tiling in Fig. 5 shows that the exception for $\beta = \pi/6$ is necessary.

**Proof.** Let $X = |AB|$. Twice the area of $ABC$ is the cross product of the two equal sides, which is

$$X^2 \sin 2\alpha = 2X^2 \sin \alpha \cos \alpha = 2X^2 \sin \alpha \sin \beta = 2X^2 ab$$

Twice the area of the tile is $ab$. Since $N$ tiles cover $ABC$ we have the area equation

$$2X^2 = N$$

Define

$$\lambda := \sqrt{N/2}$$

Then $X = \lambda$. Let $M$ be the midpoint of the base $AC$. Then triangle $ABM$ has a right angle at $M$, angle $\alpha$ at $B$, and angle $\beta$ at $A$, so it is similar to the tile. Therefore $AM = X \sin \alpha = \lambda \alpha$. Therefore $|AC| = 2a\lambda$. Since there is a tiling of $ABC$, there are non-negative integers $(p, q, r)$ and $(s, t, u)$ such that

$$\lambda = pa + qb + r$$
$$2a\lambda = sa + tb + u$$

We write this as a system of equations in unknowns $a, b$:

$$pa + qb = \lambda - r$$
$$(s - 2\lambda)a + tb = -u$$

The determinant $D = pt - q(s - 2\lambda)$. If $D \neq 0$, then both $a$ and $b$ belong to $\mathbb{Q}(\lambda)$, since that field contains $D$ and the right-hand sides of the system.

Now suppose, for proof by contradiction, that $D = 0$. Then since $\lambda$ is irrational, we have $q = 0$ and $pt = qs = 0$. Since $pa + qb = \lambda - r \neq 0$, we have $p \neq 0$ and hence $t = 0$. Then the two equations become

$$pa = \lambda - r$$
$$(s - 2\lambda)a = -u$$

Multiplying these two equations we have

$$(s - 2\lambda)(\lambda - r) = -pu$$
$$-2\lambda^2 + \lambda(s + 2r) - sr = -pu$$
$$-N + \lambda(s + 2r) - sr = -pu$$

Since $\lambda$ is irrational, and $s$ and $r$ are non-negative, we have $s = r = 0$. Hence $AC = u$ is made up only of $c$ edges (each of length 1) and $X = pa + qb$, so $AB$ has no $c$ edges. Since also $q = 0$ (as shown above), $X = pa$, so $AB$ also has no $b$ edges, and is therefore composed entirely of $a$ edges. A similar argument applies to show that $CA$ also is composed entirely of $a$ edges.
Therefore $a = X/p = \lambda/p$ belongs to $\mathbb{Q}(\lambda)$. We cannot, however, immediately conclude that $b$ is in $\mathbb{Q}(\lambda)$. But we can conclude that there is no relation $jc = ua + vb$ with integers $j, u, v$: Suppose there was such a relation. Since $c = 1$, that would mean $a$ is a rational multiple of $b$, and since $a$ belongs to $\mathbb{Q}(\lambda)$, so does $b$, and we are done. Therefore, as claimed, there is no relation $jc = ua + vb$.

Now consider the tiles with a vertex at $B$, where triangle $ABC$ has an angle of $2\alpha$. Since sides $AB$ and $CB$ are composed entirely of $a$ edges, the tiles at $B$ supported by $AB$ and $CB$ have their $a$ edges on $AB$ or $BC$, and hence do not have their $\alpha$ angles at $B$. In particular, the case when there is only one tile at $B$ is ruled out, since it cannot have an $a$ edge on both $AB$ and $BC$; and the case of just two tiles at $B$, both with $\alpha$ angles at $B$, is also ruled out, since neither would have an $a$ edge on $AB$ or $BC$. Therefore there must be some tiles with $\beta$ angles at $B$. Either $\alpha$, or $2\alpha$, or $2\alpha - \pi/2$ must be a multiple of $\beta$. In the latter case, $2\alpha - \pi/2 = \alpha - \beta$ is a multiple of $\beta$, so $\alpha$ is a multiple of $\beta$. In all of these cases, then, $2\alpha$ is a multiple of $\beta$, say $2\alpha = m\beta$, with $m > 1$. Then

\[
\beta = \frac{\pi}{2} - \alpha
\]
\[
2\beta = \pi - 2\alpha = \pi - m\beta
\]
\[
(m + 2)\beta = \pi
\]
\[
\beta = \frac{\pi}{m + 2} = \frac{2\pi}{2(m + 2)}
\]

Now $\cos \beta = \sin \alpha = a$ belongs to $\mathbb{Q}(\lambda)$. By Lemma 4.3, $2(m + 2)$ equal to one of 5, 6, 8, 10, 12. Therefore $m + 2$ is one of 3, 4, 5, 6. Since $m > 1$, we have $m = 3$ or $m = 4$.

Case 1, $m = 3$. Then $2\alpha = 3\beta$. Then $\alpha = 3\pi/10$ and $\beta = \pi/5$. The $2\alpha$ angle of $ABC$ at $B$ is filled with three tiles, each with their $\beta$ angle at $B$. The two tiles on $AB$ and $BC$ have their $a$ edges on $AB$ and $BC$, and their $c$ edges on the two interior segments. The $a$ edge of the middle tile must lie on one of the $c$ edges of the outer tiles. Hence, there is a $c/a$ edge emanating from the vertex. Since there are no relations $jc = ua + vb$, there is a link of $\Gamma_c$ emanating from $B$. Since $\alpha$ is an odd multiple of $\beta$, Lemma 7.2 is applicable; then there must be an incoming link at $B$, which is impossible, as links of $\Gamma_c$ cannot terminate on the boundary of $ABC$. Hence Case 1 is impossible.

Case 2, $m = 4$. Then $2\alpha = 4\beta$, $\alpha = \pi/3$ and $\beta = \pi/6$, which is ruled out by hypothesis (and has to be, because of Fig. 5). Note that Lemma 7.2 does not apply, since $2\alpha$ is an even multiple of $\beta$. That completes the proof of the lemma.

Lemma 7.5. Suppose isosceles triangle $ABC$ with base angles $\beta$ at $A$ and $C$ is $N$-tiled by a tile with angles $(\alpha, \beta, \gamma)$ and sides $(a, b, c)$. Suppose $\beta \neq \frac{\pi}{6}$ and $\sqrt{N/2}$ is irrational. Then

(i) $a$ and $b$ are rational multiples of $\lambda = \sqrt{N/2}$, and

(ii) $AC$ is composed only of $c$ edges, and there are no $c$ edges on $AB$ or $BC$.

Proof. Without loss of generality, we may assume $c = 1$. Since by hypothesis, $\sqrt{N/2}$ is irrational and $\beta \neq \frac{\pi}{6}$, we can apply Lemma 7.4 to conclude that $a$ and $b$ belong to $\mathbb{Q}(\lambda)$.

I say that

$\lambda = X = |AB|$. 

Twice the area of $ABC$ is equal on the one hand to $Nab$, and on the other to

$$X^2 \cos 2\alpha = X^2 2\sin \alpha \cos \alpha = 2X^2 ab.$$  

Hence $N = 2X^2$. But $N = 2X^2$ by definition of $\lambda$. Hence $X = \lambda$, as claimed.

Let $a = x\lambda + y$ and $b = z\lambda + w$, where $x, y, z, w$ are rational. Then for some nonnegative integers $p, q, r$, 

$$\lambda = pa + qb + r = (px + qz)\lambda + (py + qw + r).$$  

Since $\lambda$ is irrational,  

$$(1) \quad py + qw + r = 0.$$  

We also have 

$$1 = a^2 + b^2 = (x\lambda + z)^2 + (y\lambda + w)^2 = (x^2 + z^2)\frac{N}{2} + 2(xy + zw)\lambda.$$  

Since $\lambda$ is irrational we have

$$(2) \quad xy + zw = 0.$$  

Case 1: $xy \neq 0$. Then $xy$ and $zw$ have different signs. I say that $x > 0$ and $z > 0$. We will prove this by cases, according to the sign of $xy$. First suppose $xy > 0$. Then $x$ and $y$ are of the same sign. Since $a = \lambda x + y > 0$, it follows that $x, y > 0$. Since the tile at $B$ has either its $b$ or $c$ edge on $AB$, not both $q$ and $r$ are zero; hence (1) implies that $qw \leq 0$; hence $w \leq 0$. Since $zw < 0$, we have $z > 0$ and $w < 0$. Thus the claim $x > 0$ and $z > 0$ holds if $xy > 0$.

Now suppose $xy < 0$. Then $zw > 0$, and we find $z > 0$ and $w > 0$, since $b = z\lambda + w > 0$. Since not both $q$ and $r$ are zero, and $w > 0$, we have $qw + r > 0$. Then (1) implies $py < 0$; hence $y < 0$. Since $xy < 0$, we conclude $x > 0$, establishing the claim $x > 0$ and $z > 0$ also in case $xy < 0$. Thus we have proved that $xy \neq 0$ implies $x > 0$ and $z > 0$.

I say there is no relation $jc = j = ua + vb$ with non-negative integers $j, u, v$ and $j > 0$. Suppose such a relation exists; then

$$j = ua + vb$$

$$= u(x\lambda + y) + v(z\lambda + w)$$

$$= (ux + vz)\lambda + (uy + vw)$$

Since $\lambda$ is irrational, we have $ux + vz = 0$. Since $x$ and $z$ are positive and $u, v \geq 0$, this implies $u = v = 0$, which is a contradiction, since $j > 0$.

Then by Corollary 7.3, $AC$ is composed of all $c$ edges, and there are no $c$ edges on $AB$. Since the length of $AC$ is $2a\lambda$, we conclude that $a = s/(2\lambda) = (s/N)\lambda$, since $2\lambda^2 = N$. Hence $a$ is a rational multiple of $\lambda$. Since $AB$ has no $c$ edges, $r = 0$ and $X = \lambda = pa + qb$. Hence $b = (\lambda - pa)/q$ is also a rational multiple of $\lambda$. That completes the proof in Case 1.

Case 2: $xy = 0$. Then also $zw = 0$. I say this case implies $y = w = 0$. We argue by cases on whether $x = 0$ or not. If $x = 0$, then $a = y$ is rational and $y > 0$. If $z = 0$, then $b = w > 0$, contradicting (1). If $w = 0$, then $b = z\lambda$,

$$2a\lambda = sa + tb + u = -tz\lambda + (sy + u)$$

$$sy + u = 0$$

$$s = u = 0$$
and \(AB\) is composed only of \(b\) edges, which is impossible, since the tile at \(A\) cannot have its \(b\) edge on \(AB\).

On the other hand, if \(x \neq 0\) then since \(xy = 0\) we have \(y = 0\). Then \(qw + r = 0\) by (11). If \(z = 0\), then \(b = w > 0\) and therefore \(q = r = 0\), contradiction. The only remaining possibility is \(y = w = 0\), as claimed. Then \(a = x\lambda\) and \(b = z\lambda\). Then

\[
X = \lambda = pa + gb + r = (x + z)\lambda + r, \quad \text{so} \quad r = 0 \quad \text{and}
\]

\[
Y = 2a\lambda = 2x\lambda^2 = xN = sa + tb + u = (sx + tz)\lambda + u,
\]

which implies \(s = t = 0\). That completes the proof in Case 2, and also the proof of the lemma.

**Theorem 7.6.** Suppose \(ABC\) is isosceles with base angles \(\beta\), or \(ABC\) is equilateral, and \(ABC\) is tiled by triangle \(T\) similar to half of \(ABC\). If \(\alpha\) is a rational multiple of \(\pi\), then either

(i) \(N\) is even and \(N/2\) is a square, or

(ii) \(N\) is a square and \(\beta = \pi/4\) and \(ABC\) has base angles \(\pi/4\), or

(iii) \(N/2\) is three times a square and \(\beta = \pi/6\) and \(\alpha = 2\beta = \pi/3\).

**Remark.** One possible tiling under case (iii) of the theorem is illustrated in Fig. 5. \(N\) can be odd in case (ii), since half the triangle \(ABC\) is similar to \(ABC\), so quadratic tilings are allowed.

**Proof.** We begin by remarking that \(N/2\) is a rational square if and only if it is an integer square, since if it is a rational square then \(2N\) is an integer that is a rational square, hence it is an integer square. Then it is the square of an even integer \(2m\), so \(N/2 = m^2\). Hence if \(\sqrt{N/2}\) is rational, then \(N\) is even and \(N/2\) is a square. Then condition (i) holds.

Therefore we may assume that \(\sqrt{N/2}\) is irrational. We now divide into cases according as \(\beta \neq \frac{\pi}{6}\) or not.

First assume \(\beta \neq \frac{\pi}{6}\). Then by Lemma 4.4 \(a = \cos \alpha\) and \(b = \sin \alpha\) belong to \(\mathbb{Q}(\sqrt{N/2})\). By hypothesis, \(\alpha\) is a rational multiple of \(\pi\). These two facts make Lemma 4.3 applicable, so we can drastically limit the possible values of \(\alpha\). Namely, by Lemma 4.3 \(\alpha\) and \(\beta\) are odd multiples of \(2\pi/n\), where \(n\) is one of 6, 8, 12; that is, they are odd multiples of \(\frac{\pi}{3}, \frac{\pi}{4}, \text{or} \frac{\pi}{6}\). Since they are both less than \(\frac{\pi}{2}\), \(\alpha\) and \(\beta\) must be exactly \(\frac{\pi}{3}, \frac{\pi}{4}, \text{or} \frac{\pi}{6}\). Those are the values of \(\alpha\) and \(\beta\) allowed in the statement of the lemma. We arrived at that conclusion under the assumption that \(\beta \neq \frac{\pi}{6}\), but since that conclusion includes \(\beta = \frac{\pi}{6}\), it holds without that assumption. That is, we have proved outright that \(\alpha\) and \(\beta\) are equal to \(\frac{\pi}{3}, \frac{\pi}{4}, \text{or} \frac{\pi}{6}\).

It remains to show that \(N\) has one of the stated values. Let \(X = pa + rb + q\) be the length of \(AB\). Then the area equation is

\[
Nab = X^2 \cos 2\alpha = 2X^2ab
\]

so \(N = 2X^2\).
Case 1. \( \alpha = \beta = \frac{\pi}{2} \). Then with \( c = 1 \) we have \( a = b = 1 / \sqrt{2} \), so the area of each tile is \( \frac{1}{2} \). By Lemma 7.5 \( q = 0 \), so \( X = pa + rb \). Therefore

\[
\begin{align*}
X &= (p + r)(1/\sqrt{2}) \\
X^2 &= (p + r)^2/2 \\
N &= 2X^2 \quad \text{as shown above} \\
N &= (p + r)^2/2 \quad \text{by the previous two lines}
\end{align*}
\]

Hence \( 2N \) is a rational square. As remarked at the beginning of the proof, it is therefore an integer square, and hence \( N \) is even. Therefore conclusion (i) of the theorem holds. That completes Case 1.

Case 2, \( \alpha = \frac{\pi}{3} \). (This is the case when \( ABC \) is equilateral.) Then \( a = \cos \alpha = \sqrt{3}/2 \) and \( b = \sin \alpha = 1/2 \); hence \( X = |AB| = pa + qb + r \) belongs to \( \mathbb{Q}(\sqrt{3}) \). Then \( \sqrt{N/2} \) has the form \( u + v\sqrt{3} \) with \( u \) and \( v \) rational. Squaring both sides we have \( N/2 = u^2 + 3v^2 + 2uv\sqrt{3} \). Hence \( uv = 0 \). Hence either \( u = 0 \) or \( v = 0 \).

In case \( u = 0 \) then \( N/2 \) is three times a rational square (which is possible, for example see Fig. 4). Then let \( v = s/t \) with \( s \) and \( t \) relatively prime integers, so \( N/2 = 3(s/t)^2 \). Then \( 6N = (6s/t)^2 \) so \( Nt^2 = 6s^2 \). Since \( s \) and \( t \) are relatively prime, 6 divides \( N \). Hence \( N/6 = (s/t)^2 = v^2 \) is a integer that is a rational square; hence \( N/6 \) is an integer square. Hence \( N/2 \) is three times a square, so conclusion (iii) holds.

In case \( v = 0 \) then \( N/2 = u^2 \) is a rational square, so it is an integer square by the first paragraph of this proof. Hence conclusion (i) holds.

Case 3, \( \alpha = \frac{\pi}{5} \). Then \( \beta = \frac{\pi}{3} \), which is no Then \( a = \cos \alpha = \sqrt{3}/2 \) and \( b = \sin \alpha = 1/2 \); hence \( X = |AB| = pa + qb + r \) belongs to \( \mathbb{Q}(\sqrt{3}) \). Then the proof is completed verbatim as in Case 2. That completes the proof of the theorem.

Lemma 7.7. Suppose isosceles (and not equilateral) triangle \( ABC \) is \( N \)-tiled by a tile with angles \( (\alpha, \beta, \frac{\pi}{2}) \). Then the base angles of \( ABC \) are equal to \( \alpha \) or to \( \beta \).

Remark. The lemma fails for equilateral \( ABC \).

Proof. This is an immediate consequence of Laczkovich’s work: the first and second lines of Table 1 are the only ones allowing a right-angled tile, and the first line can apply to an isosceles \( ABC \) only if \( ABC \) and the tile are both right isosceles triangles.

Theorem 7.8. Suppose isosceles triangle \( ABC \) is \( N \)-tiled by a tile with angles \( (\alpha, \beta, \frac{\pi}{2}) \). Then

(i) \( N \) is a square and \( \alpha = \beta = \frac{\pi}{4} \), or

(ii) \( N \) is twice a square (possible for any such \( N \) with any right-angled tile), or

(iii) \( N = 6k^2 \) and \( \beta = \pi/6 \) or \( \alpha = \pi/6 \) (example with \( N = 54 \) in Fig. 5), or

(iv) \( N \) is an even sum of squares (so \( N/2 \) is also a sum of squares). (Possible for any such \( N \) with a suitable tile by a double biquadratic tiling as in Fig. 6).

Proof. By Lemma 7.6 the base angles of \( ABC \) are equal to \( \alpha \) or to \( \beta \). Since the conclusion of the theorem is insensitive to which angle is labeled \( \alpha \), we may assume the base angles are \( \beta \). By Theorem 7.4 the conclusion is correct when \( \alpha \) is a rational multiple of \( \pi \); indeed in that case either (i), (ii), or (iii) holds. Therefore we may assume that \( \alpha \) is not a rational multiple of \( \pi \). If \( N/2 \) is a rational square, then \( 2N \) is a rational square, and an integer, hence an integer square. Hence \( N \) is
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even, and $N/2$ is an integer. Since $N/2$ is a rational square and an integer, it is also an integer square, so $N$ is twice a square, and case (ii) of the theorem holds.

Therefore we may assume that

$$\lambda = \sqrt{N/2} \text{ is irrational.}$$

Let $X$ be the length of $AB$. I say that

$$X = \lambda$$

Twice the area of $ABC$ is

$$X^2 \cos 2\alpha = 2X^2 \cos \alpha \sin \alpha = 2X^2ab.$$ 

It is also $Nab$, since there are $N$ tiles each of area $ab/2$. Therefore $N = 2X^2$. But $N = 2\lambda^2$ by definition of $\lambda$, and both $X$ and $\lambda$ are positive. Therefore $X = \lambda$, as claimed.

Let $(a, b, c)$ be the sides of the tile; we may choose the scale so that $c = 1$. Since $\alpha$ is not a rational multiple of $\pi$, it is not equal to $\pi/6$. Since $\lambda$ is irrational, Lemma 7.5 is applicable. Therefore, side $AC$ is composed only of $c$ edges. Let $u$ be the number of those edges. Let $M$ be the midpoint of $AC$ (which may or may not be a vertex of the tiling). Then triangle $ABM$ has angle $\alpha$ at $B$ and a right angle at $M$. The length of $AM$ is $u/2$, and the length of $AB$ is $\lambda$. Therefore

$$\tan \alpha = \frac{u}{2\lambda}.$$ 

Since $\lambda$ is irrational, $\tan \alpha$ is irrational. It follows that there does not exist any linear relation $pa = qb$ with integers $p$ and $q$, for if there were, then $\tan \alpha = b/a = p/q$ would be rational. It follows that there are no relations of the form $ja = pb + qc$, $jb = pq + qc$, or $jc = pa + qb$ with $j \neq 0$. From this it follows that every internal segment in the tiling has equal numbers of $a$ edges on both sides, equal numbers of $b$ edges on both sides, and equal numbers of $c$ edges on both sides.

I say that $N$ is even. For proof by contradiction, assume $N$ is odd. Now the number of $a$ edges in the interior is even, and the number of $b$ edges in the interior is even, and there are no $a$ or $b$ edges on $AC$. Hence the number of $a$ edges on $AB$ and $BC$ together is odd, and number of $b$ edges on $AB$ and $BC$ together is odd. Suppose $AB = pa + qb$ and $BC = ra + sb$. Then $p \neq r$ and $q \neq s$, since $p + r$ is odd and $q + s$ is odd. We may suppose $p \geq r$ by relabeling $A$ and $C$ if necessary. Then

$$\frac{|AB|}{|BC|} = \frac{(pa + qb)}{(ra + sb)}$$

$$= (p - r)a = (s - q)b$$

with $p - r$ a positive integer, and hence $s - q$ also a positive integer. Since we showed above that no such relations between $a$ and $b$ exist, we have reached a contradiction. Hence $N$ is even, as claimed.

Lemma 7.6 also tells us that $a$ and $b$ are rational multiples of $\lambda$. Let $x$ and $z$ be rational numbers such that $a = x\lambda$ and $b = z\lambda$. Then the equation $1 = a^2 + b^2$ becomes

$$1 = (x^2 + z^2)\lambda^2$$

$$= (x^2 + z^2)N/2$$
Multiplying by \(2N\) we have

\[
2N = (xN)^2 + (zN)^2
\]

Thus \(2N\) is a sum of two rational squares. Then by Lemma 4.2, \(2N\) is a sum of two integer squares. Then by Lemma 4.2, \(N\) is also a sum of two squares. That completes the proof of the theorem.

**Corollary 7.9.** Suppose isosceles triangle \(ABC\) is \(N\)-tiled by a right triangle. Then \(N\) is not a prime congruent to 3 mod 4, nor is it twice such a prime, except for \(N = 6\).

**Proof.** The Corollary follows from Theorem 7.8 by Lemma 4.1.

Theorem 7.8 gives necessary and sufficient conditions on \(N\) for the existence of an \(N\)-tiling of some isosceles triangle \(ABC\) by a right-angled tile, if \(N\) is even. It remains to specify exactly which isosceles \(ABC\) can be \(N\)-tilded, when \(N\) is given. The following theorem spells it out.

**Theorem 7.10.** Given a positive integer \(N > 1\), the possible isosceles triangles \(ABC\) that can be \(N\)-tilded by a right-angled tile are as follows. Here the sides of the tile are \((a, b, c)\) and the angles are \((\alpha, \beta, \frac{\pi}{2})\).

(i) if \(N/2\) is a square, any isosceles triangle can be \(N\)-tilded (by a double quadratic tiling)

(ii) if \(N/2\) is a sum of two squares, then isosceles triangle \(ABC\) with base angles \(\beta\) can be \(N\)-tilded with tile \((\alpha, \beta, \frac{\pi}{2})\) if and only if \(\tan \beta = r/p\) where \(N/2 = r^2 + p^2\).

(iii) if \(N\) is a square, the right isosceles \(ABC\) can be \(N\)-tilded by a quadratic tiling.

(iv) if \(N\) is six times a square, then the isosceles triangle with base angles \(\frac{\pi}{6}\) can be \(N\)-tilded by the tile with \(\alpha = \frac{\pi}{6}\) and \(\beta = \frac{\pi}{6}\).

(v) If none of the above apply, then no isosceles triangle can be \(N\)-tilded by any tile.

**Remark.** Since \(N/2\) may sometimes be expressible in more than one way as a sum of two squares, there can sometimes be more than one possible \(ABC\) and tile for a given \(N\), but only finitely many. Moreover, if \(N\) is both a square and a sum of squares, there are more possibilities, as in Fig. 6. It will be very difficult to provide a full characterization of all \(N\)-tilings.

**Proof.** Ad (ii). Just divide \(ABC\) by its altitude \(BD\) and tile each half with a quadratic tiling.

Ad (iii). If \(ABC\) is \(N\)-tilded, and \(N/2\) is not a square, then by Theorem 7.8 the tile has the form mentioned. In case \(N/2 = p^2 + r^2\), there is a tiling made by combining biquadratic tilings of the two halves of \(ABC\).

Ad (iv). By Theorem 7.8 if \(N = 6k^2\) then a tiling with \(\beta = \frac{\pi}{6}\) is possible; see Fig. 6. It remains to show that no other tile is possible. Let \(N = 6k^2\). Then \(N\) is not a square, and \(N\) is not twice a square. Since \(N\) contains an odd power of 3, \(N\) is not a sum of two squares, and the same is true of \(N/2\). Hence no other case in the theorem can apply, and \(\beta = \frac{\pi}{6}\) is the only possibility.

Ad (v). By Theorem 7.8 these cases are exhaustive. That completes the proof of the corollary.
8. Possible values of $N$ in tilings with commensurable angles

We wish to add a third column to Laczkovich’s Table 1 giving the possible forms of $N$ if there is an $N$-tiling of $ABC$ by the tile in that row. For example, when $ABC$ is similar to the tile, then $N$ must be a square, so we put $n^2$ in the third column. While we are at it, we add a fourth column with a citation to the result, and delete the rows corresponding to the tilings of the equilateral triangle that we have proved impossible. The revised and extended table is Table 2. All the entries in this table except the last one give necessary and sufficient conditions on $N$ for the tilings to exist. The last one gives necessary conditions for certain tilings that probably do not actually exist, but since $ABC$ is equilateral, this question is out of scope for this paper.

| $ABC$ | the tile form of $N$ | citation |
|-------|---------------------|----------|
| $(\beta, \beta, 2\alpha)$ similar to $ABC$ | $n^2$ | [13] |
| $(\beta, \beta, 2\alpha)$ | $(\alpha, \beta, \frac{\pi}{2})$ | $e^2 + f^2$ | [13] |
| $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2})$ similar to $ABC$ | $3n^2$ | [13] |
| $(\beta, \beta, 2\alpha)$ | $(\alpha, \beta, \frac{\pi}{2})$ | $2n^2$ | Theorem 7.6 |
| $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$ | $(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$ | $6n^2$ | Theorem 7.6 |
| equilateral | $(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$ | $6n^2$ | Theorem 7.6 |
| equilateral | $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$ | $3n^2$ | Theorem 7.6 |

**Theorem 8.1.** Suppose $(\alpha, \beta, \gamma)$ are all rational multiples of $2\pi$, and triangle $ABC$ is $N$-tiled by a tile with angles $(\alpha, \beta, \gamma)$. Then $ABC$, $(\alpha, \beta, \gamma)$, and $N$ correspond to one of the lines in Table 2.

**Proof.** As discussed above, Laczkovich characterized the pairs of tiled triangle and tile, as given in Table 1. It remains to characterize the possible $N$ for each line. In several cases lines in Table 1 split into two or more lines in Table 2 which supplies the required possible forms of values of $N$. That table lists in its last column citations to the literature or theorems in this paper for each line. That completes the proof.

9. Tilings with $\gamma = 2\alpha$ and $(a, b, c)$ commensurable

In this section and the next, we take up the row of Laczkovich’s second table in which $ABC$ is isosceles with base angles $\alpha$ and is tiled by a tile with $\gamma = 2\alpha$, and $\alpha$ is not a rational multiple of $\pi$. The condition $\gamma = 2\alpha$ can also be written as $3\alpha + \beta = \pi$. Unlike the similar-looking condition $3\alpha + 2\beta = \pi$, this condition does not imply $\gamma > \pi/2$. The vertex angle of $ABC$ is then $\pi - 2\alpha = \alpha + \beta$. The tile $(4, 5, 6)$ satisfies $\gamma = 2\alpha$; this is shown below as an example of Lemma 11.2.

---

3 Again, we remind readers who may check with [7] that there are three entries in Laczkovich’s Theorem 5.1 that are shown in the subsequent Theorem 5.3 not to apply to tilings by congruent triangles, so they do not appear in our tables.
The numbers \((a, b, c)\) are called \textit{commensurable} if their ratios are rational. In that case we say the “tile is rational.” If the edges of a triangle are commensurable, then the triangle is similar to one with integer edges. The remarkable fact is that, if \(ABC\) is isosceles with base angles \(\alpha\) and vertex angle \(\alpha + \beta\), then it can be \(N\)-tiled for some \(N\) by a tile with angles \((\alpha, \beta, \gamma)\) if and only if the edges of the tile are commensurable. This fact is really two different theorems:

- If the tile is rational, there is an \(N\)-tiling for some \(N\), and
- If the tile is not rational, there is no \(N\)-tiling for any \(N\).

The first statement is due to Laczkovich \cite{Laczkovich}. We will explain his proof in this section. The second statement is proved in the next section.

Laczkovich \cite{Laczkovich} proves that an isosceles triangle with angles as described, can be dissected into triangles similar to the tile, plus one parallelogram; then using the commensurability condition, these triangles and the parallelogram can all be tiled with the same size of tile. The only problem is that the tile will have to be \textit{really tiny}.

We illustrate this with the tile \((4, 5, 6)\). The idea of Laczkovich’s construction (Fig. 3 in \cite{Laczkovich}) is shown in Fig. 11. Laczovich’s idea is to quadratically tile each triangle, and then tile the parallelogram. As Laczovich pointed out, the commensurability of the tile edges mean that with a small enough tile, this will succeed. We illustrate the idea in Fig. 12. But observe that the tiles shown in that figure will not work, because with that size of tile, we cannot continue the tiling into the next (blue) triangle, as if the tile is \((4, 5, 6)\), the boundary between light green and blue is five 6-edges, total 30, which cannot be made of 4-edges, as it would have to be to tile the blue triangle. Clearly we should have chosen a smaller tile, for example half that size. But with a tile half that size, we run into similar trouble at the next boundary. To choose the tile correctly, we introduce a variable for each triangle to count the number of tiles on each side of that triangle. Then there is a linear equation at each boundary. If we assume that the parallelogram will be tiled by \(n\) tiles on its diagonal side and \(m\) on its horizontal side, then these variables will satisfy the following equations. The equations show that everything is determined.
Figure 12. We look for a tiling starting like this

once the number of tiles on each side of the red triangle is chosen.

\[
\begin{align*}
red &= 1 \\
orange &= 4 \text{ red}/6 \\
lightgreen &= 5 \text{ orange}/4 \\
blue &= 6 \text{ lightgreen}/4 \\
green &= 5 \text{ blue}/4 \\
lightblue &= 6 \text{ blue}/4 \\
pink &= 5 \text{ lightblue}/4 \\
m &= (5 \text{ pink} - 4 \text{ orange})/6 \\
n &= (5 \text{ red})/5 \\
total &= red^2 + orange^2 + yellow^2 + blue^2 \\
&\quad + green^2 + lightblue^2 + pink^2 + 2mn
\end{align*}
\]

Solving these equations, starting with \(red = 1\), we get these answers (in the order of variables listed above):

\[
\begin{array}{cccccccccccc}
1 & 2 & 5 & 5 & 25 & 15 & 75 & 869 & 1 \\
3 & 6 & 4 & 16 & 8 & 32 & 576 & \end{array}
\]

This reveals that Fig. 12 is misleading, in that the parallelogram of Fig. 11 is not accurately tiled—the tiled parallelogram is off by less than a pixel. To get it accurately tiled requires a much smaller tile. To clear those denominators we have to start with \(red = 576\) instead of \(red = 1\). Then we get

\[
\begin{array}{cccccccccccc}
576 & 384 & 480 & 720 & 900 & 1080 & 1350 & 869 & 576 & 1 \\
\end{array}
\]

So the number of tiles required (namely the sum of the squares of the color numbers plus \(2mn\)) is 6028020.

Here matters stood for about eight years. Then, in 2024, Bryce Herdt found \(N\)-tilings for \(N = 1125\) and then 720. These tilings are exhibited in an Appendix.

For readers without colors: red is the triangle to the right of the parallelogram, and the other colors are in counterclockwise order from red.
TILINGS OF AN ISOSCELES TRIANGLE

10. No tilings with $\gamma = 2\alpha$ and $(a, b, c)$ incommensurable

In this section we rule out tilings of an isosceles triangle with base angles $\alpha$ and vertex angle $\alpha + \beta$, in case the tile edges are incommensurable. To state the point another way, if there is a tiling of such an isosceles $ABC$, then the tile must be rational.

10.1. Stars and centers. Suppose isosceles triangle $ABC$ is tiled by a tile with angles $(\alpha, \beta, \gamma)$, not a right triangle, and $\alpha$ is not a rational multiple of $\pi$. We will consider and analyze the possible configurations formed by tiles at a vertex of the tiling. We begin by ruling out certain possibilities.

Lemma 10.1. Let isosceles $ABC$ with base angles $\alpha$ (at $A$ and $C$) be $N$-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, and suppose that the tile is not a right triangle and $\alpha$ is not a rational multiple of $\pi$. Let $P$ be a vertex on the boundary of $ABC$. Then there are not two $\beta$ angles of tiles at $P$, and there are not two $\gamma$ angles of tiles at $P$.

Proof. Suppose, for proof by contradiction, that two tiles have their $\beta$ angles at the same boundary vertex. Then for some nonnegative integers $u, v, w$ we have

\[
\begin{align*}
\pi &= u\alpha + (v + 2)\beta + w\gamma \\
\pi &= (u + 2w)\alpha + (v + 2)\beta & \text{since } \gamma = 2\alpha \\
0 &= (u + 2w - 3)\alpha + (v + 1)\beta & \text{since } 3\alpha + \beta = \pi \\
\beta &= \left(\frac{3 - u - 2w}{v - 1}\right)\alpha
\end{align*}
\]

Putting that value for $\beta$ into $3\alpha + \beta = \pi$ we can solve for $\alpha/\pi$:

\[
\frac{\alpha}{\pi} = 3 + \left(\frac{3 - u - 2w}{v - 1}\right) - 1
\]

But that is rational, contradicting the hypothesis that $\alpha$ is not a rational multiple of $\pi$. That completes the proof that two $\beta$ angles do not occur at the same boundary vertex.

The proof for two $\gamma$ angles is similar. First, if there are no $\beta$ angles, then

\[
\begin{align*}
\pi &= u\alpha + (w + 2)\gamma \\
\pi &= (u + 4w + 4)\alpha
\end{align*}
\]

contradiction, since $\alpha$ is not a rational multiple of $\pi$. And if there is one $\beta$ angle, then

\[
\begin{align*}
\pi &= u\alpha + (w + 2)\gamma + \beta \\
0 &= (u - 1)\alpha + (w + 1)\gamma \\
0 &= (u - 1 + 2(w + 1))\alpha \\
0 &= (u + 2w + 1)\alpha
\end{align*}
\]

contradiction, since $u + 2w + 1 > 0$. Since we already proved there cannot be more than one $\beta$, that completes the proof of the lemma.

We define the angle sum of a vertex to be the sum of the angles of the tiles sharing that vertex. Except for the vertices $A$, $B$, and $C$, that angle sum will always be either $\pi$ or $2\pi$. It will be $\pi$ if and only if the vertex lies on the interior of
the boundary of a tile or of \( ABC \). For short, we refer to a vertex with angle sum \( \pi \) as a \textit{boundary vertex}, though it need not be on the boundary of \( ABC \).

Consider a boundary vertex. A \textit{normal boundary vertex} has three tiles, with angles \( \alpha \), \( \beta \), and \( \gamma \). A \textit{star} has three \( \alpha \) angles and a \( \beta \). These are the only possibilities for a boundary vertex, since \( \alpha \) is not a rational multiple of \( \pi \), as spelled out in Lemma \textbf{[10.1]} Next, consider an interior vertex. A \textit{normal interior vertex} has two each of \( \alpha \), \( \beta \), and \( \gamma \) angles. A \textit{center} has three \( \gamma \) and two \( \beta \) angles. (For example, there is a center in Fig. \textbf{[11]} more or less in the center of the figure.) There may also be interior vertices other than centers that are not normal; these will have either \( 4\alpha + 2\beta + \gamma \) or \( 6\alpha + 2\beta \). These vertices we call \textit{interior stars}. The case of angles \( 4\alpha + 2\beta + \gamma \) we call a \textit{single interior star} and the other case is a \textit{double interior star}.

\textbf{Lemma 10.2.} Suppose isosceles triangle \( ABC \) is tiled by a tile with angles \((\alpha, \beta, \gamma)\) with \( \alpha \) not a rational multiple of \( \pi \). Let \( C \) be the number of centers and \( S \) the number of stars, counting double interior stars twice. Then \( S + 1 = C \). In particular there is at least one center.

\textit{Example.} When the tiling begun in Fig. \textbf{[11]} is completed, there will be one center and zero stars. All the vertices introduced by quadratic tilings will be normal vertices. If we then combine four copies of this tiling to create a quadratic tiling of a triangle twice the size, there will be four centers, balanced by four interior stars in the midpoints of the sides, where the three of the four copies have common vertices.

\textit{Remark.} This lemma is about the only thing we can prove about the internal structure of tilings. We use it only for the existence of at least one center.

\textit{Proof.} Each tile has one each of \( \alpha \), \( \beta \), and \( \gamma \) angles. At the vertices of \( ABC \) we have three \( \alpha \) angles and one \( \beta \) angle (just as we have at a star). Counting the vertex angles we have equal numbers of \( \alpha \), \( \beta \), and \( \gamma \) angles at each normal vertex. At each vertex we define the “excess” or “deficit” of each of \((\alpha, \beta, \gamma)\) to be the difference between the number of those angles at the vertex and the number at a normal vertex. At a star we have two excess \( \alpha \) angles and a deficit of one \( \gamma \) angles. At a single interior star the same applies; at a double interior star we have double that contribution. At a center we have an excess of one \( \gamma \) and a deficit of two \( \alpha \). At interior stars we have excesses of \( \alpha \) and \( \beta \) and deficits of \( \gamma \). Adding up the excesses and deficits from the vertices of the tiling, including \( A \), \( B \), and \( C \), we must get zero. The vertices of \( ABC \) count the same as a star. One center will “balance” one star, in the sense that the deficits and excesses add to zero. (For example, in Fig. \textbf{[11]} we have one center, and no stars; so the center balances the vertices of \( ABC \), which count as a star.)

If there are no interior stars, then the number of stars, plus one for \( ABC \), will equal the number of centers. If, however, there are interior stars, those will require additional centers to balance them, one center for each single interior star and two for each double interior star. Since we defined \( S \) by this double-counting of double interior stars, we still have \( C = S + 1 \). That completes the proof of the lemma.

\textbf{10.2. The tile is rational.} If \( Q \) is a vertex of a tiling, and \( QR \) is an internal segment of the tiling supporting a tile on one side with its \( a \) edge on \( QR \) and a vertex at \( Q \), and supporting a tile on the other side with its \( b \) edge on \( QR \) and a
vertex at $Q$, then we say $QR$ is an $a/b$ edge, or an $a/b$ edge at $Q$. Similarly for $a/c$ edge and $b/c$ edge. Note that an $a/b$ edge is also a $b/a$ edge.

**Lemma 10.3.** Let the isosceles triangle $ABC$ with base angles $\alpha$ be $N$-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, with $\alpha$ not a rational multiple of $\pi$. Let $Q$ be a center in the tiling. Then either there is an $a/b$ edge at $Q$ or there are both $a/c$ and $b/c$ edges at $Q$.

**Proof.** Assume there is no $a/b$ edge. We must prove there is an $a/c$ and a $b/c$ edge at $Q$. Each tile with a vertex at the center $Q$ has an $a$ edge ending at $Q$, since the angles at $Q$ are all $\beta$ or $\gamma$. At a center, five tiles meet, so that is a total of five $a$ edges. Since five is an odd number, these edges cannot all be paired with other $a$ edges. We have assumed there is no $a/b$ edge; hence there is an $a/c$ edge. Similarly, there are three $b$ edges ending at $Q$, belonging to the tiles with their $\gamma$ angles at $Q$. Since three is odd, these cannot each be paired with another $b$ edge. Since there are no $a/b$ edges, there is a $b/c$ edge. That completes the proof of the lemma.

**Lemma 10.4.** Let the isosceles triangle $ABC$ with base angles $\alpha$ be $N$-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$ with $\alpha$ not a rational multiple of $\pi$. Then the tiling contains essential segments with associated relations $jb = ua + vc$ and $Ja = Ub + Vc$.

**Proof.** Suppose, for proof by contradiction, that there are no such essential segments. We consider the directed graph $\Gamma_b$ defined in Definition 6.1. We wish to identify the terminal links in this graph. To that end we must consider the possible configurations that can arise when an internal segment $UQ$ of the tiling supports on the same side a series of (one or more) tiles with their $b$ edges, followed by a tile with an $a$ or $c$ edge. Let $PQV$ be the three successive vertices, with $PQ$ of length $b$ and $QV$ of length $a$ or $c$. We consider all possible configurations in which $Q$ is a vertex where three tiles on one side of a line $PQ$ each have a vertex at $Q$, contributing one angle each, so the angle sum at $Q$ is $\alpha + \beta + \gamma$ on one side of $PQ$. All these configurations are shown in Fig. 13. The figure shows that in each of those cases, there is a unique outgoing segment $QT$ that is a $b/c$ edge or a $b/a$ edge. If this segment is extended far enough, we will come to the last $b$ edge on the side that has a $b$ edge at $Q$. Since there are no essential segments, that point cannot be a vertex of a tile on the other side of $QT$, so $QT$ is an outgoing link in $\Gamma_b$.

On the other hand, if $Q$ is a star, the possible configurations are more complicated, and some of them have zero outgoing $b/c$ or $b/a$ edges, while others have two. It turns out that we do not need to make use of that fact, so we do not give a diagram of these configurations.

If $PQ$ is a link, then line $PQ$ extends past $Q$ as an interior segment of the tiling. Therefore no link terminates on the boundary, so certainly not at a boundary star. At an interior star $Q$, no interior segment of the tiling passes through $Q$, as if it did, two of the three $\gamma$ angles would lie on one side of it, leaving an angle $\beta - \alpha$ on that side, which cannot be filled by a tile. Therefore, no link can end at an interior star, since an interior star is not located on the interior of a tile edge, but the end of a link must be on the interior of a tile edge.

Therefore the out-degree of any star is $\geq$ the in-degree, since the in-degree is zero. At a center, the out-degree is at least one, and the in-degree is zero.

At a given vertex $Q$ (star or not) there can be at most one link ending at $Q$, since $Q$ lies on the interior of a tile boundary. At a normal vertex $Q$, if there is an
incoming link $PQ$, there is an outgoing link $QR$, as shown above. So out-degree $\geq$ in-degree at a normal vertex. At a star or center, there are no incoming links, and there is at least one outgoing link, so out-degree $>\text{ in-degree}$. By Lemma 10.2 there exists at least one center. Therefore the total out-degree minus in-degree is positive. But since each link has one head and one tail, the total out-degree equals the total in-degree, contradiction.

We have reached a contradiction from the assumption that there is no essential segment with associated relation of the form $jb = ua + vc$. Hence there is such a segment.
Next we prove the existence of essential segments with relations of the forms $Ja = Ub + Vc$. This is proved in the same way, using the graph $\Gamma_a$ instead of $\Gamma_b$. Again there is an outgoing link from each center, since by Lemma 10.3 there is either an $a/b$ or an $a/c$ edge at $Q$, and that edge is part of an outgoing link since there are no essential segments. Again we have to prove that at a normal boundary vertex there is a unique outgoing link. See Fig. [13] That completes the proof.

**Theorem 10.5.** Let the isosceles triangle $ABC$ with base angles $\alpha$ be $N$-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$ with $\alpha$ not a rational multiple of $\pi$. Then the tile is rational; that is, the ratios of its sides are rational.

**Remark.** If the tile is rational, then after scaling we can assume its sides are integers with no common factor.

**Proof.** By Lemma 10.4 there is an essential segment witnessing a relation $ja = ub + vc$, and another essential segment witnessing $Jb = Ua + Vc$. In matrix form we have

$$
\begin{pmatrix}
  j & -u \\
  U & -J
\end{pmatrix}
\begin{pmatrix}
  a \\
  b
\end{pmatrix} =
\begin{pmatrix}
  vc \\
  -Vc
\end{pmatrix}.
$$

This equation can be solved for $(a, b)$ provided $j/u \neq U/J$. We have

- $ja = ub + vc \leq ub$
- $j/u \leq b/a$ with equality only when $v = 0$
- $Jb = Ua + Vc \leq Ua$
- $Jb \leq Ua$ with equality only when $V = 0$
- $b/a \leq U/J$
- $j/u \leq U/J$ with equality only when $v = V = 0$

Thus: either $a/b$ is rational (when $v = 0$ or $V = 0$), or $a$ and $b$ are both rational multiples of $c$ (when the equation can be solved). In either case $a/b$ is rational. Similarly, $a/c$ is rational. That completes the proof of the theorem.
11. On the number of tiles required when $\gamma = 2\alpha$

We continue to consider tilings of isosceles $ABC$ with base angles $\alpha$ and vertex angle $\alpha + \beta$. In the previous two sections, we showed that the tile has to be rational, and that in that case, an $N$-tiling always exists, for some $N$. Next we will try to
show that some values of $N$ are impossible. We have two theorems along that line: First, $N$ cannot be a prime number. Second, $N$ has to be “at least so big”, i.e., we have a lower bound on $N$.

11.1. Characterization of the tile.

**Lemma 11.1.** Suppose $(a, b, c)$ are the integer sides of a triangle with angles $(\alpha, \beta, 2\alpha)$. Then

$$c^2 = a^2 + ab.$$ 

**Remark.** Rational triangles with $\gamma = 2\alpha$ correspond to solutions of this equation with $c < a + b$ and $b < a + c$ and $a < b + c$. For example, $(4, 5, 6)$, and $(9, 7, 12)$.

**Proof.** By the law of cosines,

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

$$= a^2 + b^2 - 2ab \cos 2\alpha \quad \text{since } \gamma = 2\alpha$$

$$= a^2 + b^2 - 2ab(2\cos^2 \alpha - 1)$$

$$= a^2 + b^2 + 2ab - 4ab \cos^2 \alpha$$

By the law of sines, $\sin \alpha/a = \sin \gamma/c = \sin 2\alpha/c = 2\sin \alpha \cos \alpha/c$, so $\cos \alpha = c/(2a)$.

Hence

$$c^2 = a^2 + b^2 + 2ab - bc^2/a$$

$$= (a + b)^2 - bc^2/a$$

$$c^2(1 + b/a) = (a + b)^2$$

$$c^2 = a(a + b)$$

$$c^2 = a^2 + ab$$

That completes the proof of the lemma.

The following lemma gives a more nuanced characterization of $(a, b, c)$. It was published in [11], but we give the short proof here.

**Lemma 11.2.** Let $(a, b, c)$ be integers with no common factor, and suppose the triangle with sides $(a, b, c)$ has angles $(\alpha, \beta, 2\alpha)$. Then $(a, b, c) = (k^2, m^2 - k^2, mk)$ for some relatively prime integers $(k, m)$, with $2k > m > k$.

Conversely, let $(a, b, c)$ be a triple of integers $(a, b, c) = (k^2, m^2 - k^2, mk)$ with $2k > m > k$ and $m$ and $k$ relatively prime. Then $(a, b, c)$ form a triangle, and it has angles $(\alpha, \beta, 2\alpha)$.

**Examples:** $(4, 5, 6)$ satisfies this equation with $k = 2$ and $m = 3$. Therefore it is an example of a tile satisfying $\gamma = 2\alpha$. Although $(1, 3, 2)$ satisfies this equation with $k = 1$ and $m = 2$, it does not correspond to a triangle.

**Remarks.** Thus $b$ and $c$ are relatively prime, but $a$ and $c$ have a common factor $k$ (if $k \neq 1$). Also, $c > a$ but not necessarily $c > b$, and $\gamma$ can be more or less than a right angle.

**Proof.** By Lemma 11.1 we have

$$c^2 = a^2 + ab$$

---

5I am indebted to Gerry Myerson for pointing out this representation of $(a, b, c)$ to me on MathOverflow.
Luthar observed that this can be written as
\[ b^2 + (2c)^2 = (2a + b)^2 \]
as is apparent upon expanding the right side. But now we can apply Lemma 4.4, according to which there are integers \((m, k)\) such that
\[
\begin{align*}
 b &= m^2 - k^2 \\
 2c &= 2mk \\
 2a + b &= m^2 + k^2 
\end{align*}
\]
These equations imply the equations to be proved. That completes the proof of the first claim of the lemma.

Conversely, suppose \((a, b, c) = (k^2, m^2 - k^2, mk)\), and \((a, b, c)\) form a triangle. Then one can check that
\[
\begin{align*}
 c^2 &= a^2 + ab \\
 &= a(a + b)
\end{align*}
\]
and we showed above that this equation characterizes \(\gamma = 2\alpha\).

Finally, if \((a, b, c) = (k^2, m^2 - k^2, mk)\), then \(b+c > a\) becomes \(m^2 - k^2 + mk > k^2\), or \(m^2 + mk > 2k^2\), which follows from \(m > k\).
\[
\begin{align*}
 a + c &> b \\
 a + c &> m^2 - k^2, \text{ or } k(k + m) > (m + k)(m - k), \text{ or } k > m - k, \\
 a + b &> c \\
 a + b &> k^2 + (m^2 - k^2) > mk, \text{ which follows from } m > k. 
\end{align*}
\]
That completes the proof of the lemma.

11.2. **Possible shapes of the tile.** In this section, we consider the possible shapes of a tile \((a, b, c)\) with \(\gamma = 2\alpha\). We begin by observing that \(a < c\) is the only obvious restriction on the ordering of the edges. As well as \(a < b\), we can have \(b < a\), as in \((9, 7, 12)\), or \(a < c < b\), as in \((9, 16, 15)\).

One way of describing the shape of a triangle is by the ratios of its sides. Here we give lower bounds on some of those ratios. Actually, we use only the bound on \(c/b\), and that only once, but it does seem necessary. We present all the bounds anyway, as they improve the reader’s mental picture of these (possible) tilings.

Empirically, these bounds are tight, though we have not proved that they are best-possible. There are apparently no positive lower bounds on \(b/a\) or \(b/c\), although again we have not proved that. When \(b/a\) is very small, an isosceles triangle \(ABC\) with \(\gamma = 2\alpha\) will be very close to equilateral, and by Laczkovich’s method (see Fig. 12), it can be tiled with trillions of tiny needle-shaped tiles. For example, \((a, b, c) = (61504, 497, 61752)\) has \(a/b > 123\), and we get \(N = 722797699088825426993\), more than a billion trillion, and the side and base of \(ABC\) are 470043721566328 and 471939059153289.

**Lemma 11.3.** Suppose \((a, b, c)\) are the integer sides of a triangle with angles \((\alpha, \beta, 2\alpha)\). Then
\[
\begin{align*}
 a/c &> 1/2 \\
 a/b &> 1/3 \\
 c/b &> 2/3
\end{align*}
\]
and if \( b < a \) we also have
\[
a/c > 1/\sqrt{2}
\]

Proof. By Lemma 11.2 there are relatively prime integers \( k, m \) with \( m/2 < k < m \), such that \( a = k^2, b = m^2 - k^2 \), and \( c = mk \). Then
\[
a/c = k^2/mk = k/m > 1/2,
\]
proving the first claim of the lemma.

We have
\[
a/b = \frac{k^2}{m^2 - k^2} > \frac{k^2}{4k^2 - k^2} \quad \text{since } 4k^2 > m^2
\]
\[
= 1/3
\]
proving the second claim.

To prove the third claim, we consider the function
\[
f(m, x) = \frac{mx}{m^2 - x^2}.
\]
Then \( f \) is monotone increasing for \( x < m \). Since \( k > m/2 \), and \( c/b = f(m, k) \), a lower bound is \( f(m, m/2) \), namely
\[
c/b > \frac{m^2/2}{m^2 - (m/2)^2}
\]
\[
= \frac{2m^2}{4m^2 - m^2}
\]
\[
= 2/3
\]
That is the third claim.

Now suppose \( b < a \). That is, \( k^2 - m^2 < k^2 \). Then \( 2k^2 < m^2 \), so \( a/c = k/m < 1/\sqrt{2} \). That completes the proof of the lemma.

11.3. The area equation.

Lemma 11.4 (Area equation). Suppose isosceles triangle \( ABC \), with base angles \( \alpha \), is \( N \)-tiled by a tile with sides \( (a, b, c) \) and angles \( (\alpha, \beta, 2\alpha) \). Let \( X \) be the length of the equal sides \( AB \) and \( BC \). Then \( X^2 = Nab \).

Proof. Let \( \gamma = 2\alpha \). The base angles of \( ABC \) are \( \alpha \), so \( \pi = 2\alpha + \angle B = \gamma + \angle B \). But also \( \pi = \alpha + \beta + \gamma \), so \( \angle B = \alpha + \beta \). Twice the area of \( ABC \) is given by the magnitude of the cross product of \( BA \) and \( BC \), namely \( X^2 \sin(\alpha + \beta) \). Twice the area of the tile is given by \( ab \sin \gamma \). Since \( \gamma = \pi - (\alpha + \beta) \), twice the area of the tile is also \( ab \sin(\alpha + \beta) \). But the area of \( ABC \) is \( N \) times the area of the tile. Hence
\[
X^2 \sin(\alpha + \beta) = Nab \sin(\alpha + \beta)
\]
Dividing both sides by \( \sin(\alpha + \beta) \), we have the area equation of the lemma. That completes the proof of the lemma.

Lemma 11.5 (Area equation, second form). Suppose isosceles triangle \( ABC \), with base angles \( \alpha \), is \( N \)-tiled by a tile with sides \( (a, b, c) \) and angles \( (\alpha, \beta, 2\alpha) \). Let \( X \) be the length of the equal sides \( AB \) and \( BC \), and let \( Y \) be the length of \( AC \). Then \( XY = Nbc \), and \( Y = (c/a)X \).
Proof. Twice the area of \( ABC \) is given by the magnitude of the cross product of \( AB \) and \( AC \), namely \( XY \sin \alpha \). Twice the area of the tile is \( bc \sin \alpha \). But the area of \( ABC \) is \( N \) times the area of the tile. Hence

\[
XY \sin \alpha = Nbc \sin \alpha \\
XY = Nbc
\]

which proves the first claim of the lemma. By Lemma 11.4 we have \( X^2 = Nab \). Dividing \( XY = Nbc \) by \( X^2 = Nab \) we have

\[
\frac{Y}{X} = \frac{Nbc}{Nab} = \frac{c}{a}.
\]

That completes the proof of the lemma.

11.4. The non-primality of \( N \). In this section, we will show that \( N \) cannot be a prime number. What is more, \( N \) cannot even be squarefree.

Lemma 11.6. Let \( ABC \) be isosceles with base angles \( \alpha \), and \( \alpha \) not a rational multiple of \( \pi \). Suppose \( ABC \) is \( N \)-tiled by a tile with \( \gamma = 2\alpha \) and integer sides \( (a, b, c) \) with no common factor. Then the squarefree part of \( N \) divides \( b \) and the squarefree part of \( b \) divides \( N \). If \( N \) is squarefree, then \( N \) divides \( b \), and \( b/N \) is a square, i.e., \( b = N\ell^2 \) for some integer \( \ell \).

Example. In Fig. 12, we indicated a tiling with tile \((4, 5, 6)\) and \( N = 669780 \). Factoring that number, we find \( N = 2^2 \cdot 3^2 \cdot 5 \cdot 61^2 \). So the squarefree part of \( N \) is \( b = 5 \), in accordance with this lemma. This provides a check on the computation of the value of \( N \), since it is not at all apparent from the construction of the tiling that \( 5 \) has to be the squarefree part of \( N \).

Proof. By Lemma 11.2 there exist relatively prime integers \( m, k \) with \( 0 < m/2 < k < m \) such that

\[
a = k^2, b = m^2 - k^2, c = km.
\]

Let \( X \) be the length of the equal sides \( AB \) and \( BC \), and \( Y \) the length of \( AC \). Then

\[
(9) \quad X^2 = Nab \quad \text{by Lemma 11.4} \\
(10) \quad XY = Nab \quad \text{by Lemma 11.5}
\]

Squaring both sides of (10), we have

\[
X^2Y^2 = N^2b^2c^2
\]

Dividing by (9),

\[
Y^2 = \frac{X^2Y^2}{X^2} = \frac{Nbc^2}{a}
\]

We know \( a \) divides \( c^2 \), since \( c = km \) and \( a = k^2 \), so \( c^2/a = m^2 \). Then

\[
Y^2 = Nbm^2.
\]

Then \( N \) divides \( Y^2 \). Then the squarefree part of \( N \) divides \( bm^2 \). But I say that actually the squarefree part of \( N \) divides \( b \), not just \( bm^2 \).

Let \( p \) be a prime dividing \( N \) to an odd power \( p^{2k+1} \), and let \( p^j \) be the highest power dividing \( m \). Then \( p^{2j+2k+1} \) divides \( Y^2 \), so \( p^{j+k+1} \) divides \( Y \), so \( p^{2j+2k+2} \) divides \( Y^2 \), so \( p^{2j+1} \) divides \( bm^2 \). If \( p \) does not divide \( b \), then \( p^{j+1} \) divides \( m \).
contradiction. Therefore \( p \) divides \( b \). Since \( p \) was any prime dividing \( N \) to an odd power, it follows that the squarefree part of \( N \) divides \( b \), as claimed.

Now let \( p \) be a prime dividing \( b \) to an odd power \( p^{2j+1} \). Then \( p^{2j+1} \) divides \( Y^2 \), so \( p^{2j} \) divides \( Y \), so \( p \) divides \( Nc^2/a \). But \( b \) is relatively prime to \( c^2/a = m^2 \), so \( p \) divides \( N \). Therefore the squarefree part of \( p \) divides \( N \).

If \( N \) is squarefree, then \( b/N \) is an integer. Since \( Y^2 = Nbm^2 \), we have

\[
\frac{b}{N} = \left( \frac{Y}{Nm} \right)^2.
\]

Therefore \( b/N \) is a rational square. Since \( b/N \) is an integer, it is also an integer square.

That completes the proof of the lemma.

**Theorem 11.7.** Let \( ABC \) be isosceles with base angles \( \alpha \), and \( \alpha \) not a rational multiple of \( \pi \). Suppose \( ABC \) is \( N \)-tiled by a tile with angles \( (\alpha, \beta, 2\alpha) \) and sides \( (a, b, c) \). Then \( N \) is not squarefree. In particular, \( N \) is not prime.

**Proof.** By Theorem 10.5 the tile is rational, so we can assume without loss of generality that \( (a, b, c) \) are integers with no common factor. By Lemma 11.2 there exist relatively prime integers \( m, k \) with \( 0 < m/2 < k < m \) such that

\[
a = k^2, \quad b = m^2 - k^2, \quad c = km.
\]

Let \( X \) be the length of the equal sides \( AB \) and \( BC \), and \( Y \) the length of \( AC \). Then by Lemma 11.4

\[
X^2 = Nab
\]

\[
= N^2a \quad \text{since } b = N\ell^2, \text{ by Lemma 11.6}
\]

\[
= N^2k^2\ell^2 \quad \text{since } a = k^2
\]

Taking the square root of both sides, we have

\[
X = Nk\ell \quad \tag{11}
\]

The tiling gives rise to a relation

\[
X = pa + qb + rc \quad \tag{12}
\]

\[
Nk\ell = pk^2 + q(m-k)(m+k) + rkm \quad \text{since } X = Nk\ell \text{ by (11)}
\]

By (11), \( X \equiv 0 \mod k \). Taking the last equation \mod \( k \) we find

\[
0 \equiv qm^2 \mod k
\]

Since \( k \) and \( m \) are relatively prime, we can divide by \( m^2 \):

\[
q \equiv 0 \mod k \quad \tag{14}
\]

Putting \( N\ell^2 = b = (m-k)(m+k) \) into (13), we have

\[
k(m-k)(m+k)/\ell = pk^2 + q(m-k)(m+k) + rkm
\]

\[
0 = pk^2 + (q-k/\ell)(m-k)(m+k) + rkm
\]

Then \( k/\ell \) is necessarily an integer. Since \( m-k > 0 \), we have \( q-k/\ell \leq 0 \), and either \( q-k/\ell < 0 \) or \( p = r = 0 \). We argue by cases:

Case 1, \( q-k/\ell < 0 \). Then \( q < k/\ell \leq k \). Then by (14), we have \( q = 0 \).

Therefore, no tile supported by \( AB \) or \( BC \) has its \( b \) edge on \( AB \) or \( BC \), since a relation \( X = pa + qb + rc \) would arise from each of \( AB \) or \( BC \) (although perhaps
with different coefficients \((p, q, r)\)). However, at \(B\) there are two tiles, one with an \(\alpha\) angle and one with a \(\beta\) angle at \(B\). Renaming \(A\) and \(C\) if necessary, we may assume the tile with \(\alpha\) at \(B\) is supported by \(AB\). Since each tile supported by \(AB\) has its \(a\) or \(c\) edge on \(AB\), each of those tiles has a \(\beta\) angle at one of its vertices on \(AB\). But there are \(\alpha\) angles at \(A\) and at \(B\). Then by the pigeonhole principle, one of the vertices on \(AB\) is a vertex of two tiles with \(\beta\) angles there. But that contradicts Lemma 10.1. That disposes of Case 1.

Case 2, \(p = r = 0\). Then every tile supported by the side \(Z = AB\) or \(BC\) that gave rise to \((12)\) has its \(b\) edge on \(Z\). Hence every tile supported by \(Z\) has a \(\gamma\) angle on \(Z\). There are no \(\gamma\) angles at \(A\), \(B\), or \(C\), so by the pigeonhole principle, there must be a boundary vertex with two \(\gamma\) angles. But there cannot be two \(\gamma\) angles at the same boundary vertex, since the only integer relations between the angles are \(\alpha + \beta + \gamma = \pi\) and \(3\alpha + \beta = \pi\). This contradiction shows that Case 2 is impossible. That completes the proof of the theorem.

11.5. The number of tiles on a side of \(ABC\). We wish to show that, given \(N\), we can calculate a finite set of triangles and a finite set of possible tiles \((a, b, c)\), such that if there is an \(N\)-tiling of some isosceles \(ABC\) with base angles \(\alpha\) by some tile with \(\gamma = 2\alpha\) and \(\alpha\) not a rational multiple of \(\pi\), then \(ABC\) and the tile are in those finite sets.

It will be important for that proof to have an upper bound on the number of tiles on the sides \(AB\) and \(BC\) of isosceles triangle \(ABC\), in terms of \(N\). This section is devoted to that problem.

We can count either the tiles supported by \(AB\), or the tiles with an edge or a vertex on \(AB\). At a given boundary vertex, there can be three tiles or there can be four tiles, as \(\pi = \alpha + \beta + \gamma = 3\alpha + \beta\). So the two ways of counting tiles “on a side” differ, but by a bounded factor.

One might initially think that such a bound should be on the order of \(\sqrt{N}\), but that idea is based on the picture in which \(ABC\) is not long and narrow. If we consider the case when \(\alpha\) is tiny, so \(AB\) and \(BC\) are almost half as long as \(BC\) and the triangle has comparatively little interior, maybe most of the tiles touch the boundary! In that case, neglecting for the moment the fact that some tile edges may be a lot larger than others, we would expect almost a quarter of the tiles to have an edge or vertex on \(AB\), and a quarter on \(BC\), and half on \(AC\). The bound we actually prove is that one of \(AB\) or \(BC\) must support fewer than \((N - 1)/4\) tiles. The number of tiles supported by \(AB\) should be about half the number of tiles with edges or vertices on \(AB\), neglecting the vertices with four instead of three tiles.

One illustration of the difficulty is the case when \(b\) is tiny, and \(a\) and \(c\) are almost equal. We already mentioned the example \((a, b, c) = (61504, 497, 61752)\). Then angle \(\beta\) is tiny, and \(\alpha\) and \(\beta\) are both close to \(\pi/3\), so \(ABC\) is nearly equilateral, and the tile is needle-shaped, long and narrow. Note that this is not at all the situation considered above when \(ABC\) itself is long and narrow. But in this situation, \(AB\) might be tiled by millions of tiles with their tiny \(b\) edges on \(AB\), while \(BC\) might be tiled with relatively fewer tiles with their long \(c\) or \(a\) edges on \(BC\). So there is no obvious relation between the number of tiles supported by one side and the number supported by another.

The difference between \(N/4\) and \(N/2\) and \((N - 1)/4\) may not seem important at first, but \((N - 1)/4\) enables us to prove that \(N\) cannot be twice a prime, while the
others mentioned do not, though they might suffice for \( N \) not being a prime. Furthermore, the better the bound, the more candidate values of \( N \) can be ruled out because they violate certain simple conditions. We conjecture that all \( N \) that correspond to tilings are not squarefree; but there are certainly not-squarefree numbers \( N \) that we cannot yet rule out.

We need to “get off the ground” by a close analysis of the case when \( N \) is very small. In particular, what is the smallest number of tiles that can be supported by the base \( AC \)? We show that at least four tiles are required. (That already shows \( N > 7 \).) A few of our lemmas will be proved also for tilings with \( \gamma = 2\pi/3 \); for example at least three tiles must be supported by \( AC \) in that case.

After these preliminary remarks, we plunge into the technical lemmas.

**Lemma 11.8.** Let isosceles \( ABC \) with base angles \( \alpha \) (at \( A \) and \( C \)) be \( N \)-tilled by a tile with angles \( (\alpha, \beta, 2\alpha) \), and suppose that the tile is not a right triangle and \( \alpha \) is not a rational multiple of \( \pi \). Then no tile has one vertex on \( AB \) and another on \( BC \).

**Proof.** By Theorem 3.1, \( \alpha \) is not a rational multiple of \( \pi \), and \( \pi \) cannot be expressed as a linear combination of \( \alpha \), \( \beta \), and \( \gamma \), except in the way determined by the vertex angles of \( ABC \). (That is, \( \pi = 3\alpha + \beta \) if \( \gamma = 2\alpha \), or \( \pi = 3\alpha + 3\beta \) if \( \gamma = 2\pi/3 \).) By Theorem 10.5 the tile is rational, so we may assume its sides are integers \((a, b, c)\) with no common factor, with \( a \) opposite angle \( \alpha \) and \( b \) opposite \( \beta \).

Suppose some tile has an edge \( EF \) with \( E \) on \( AC \) and \( F \) on \( BC \). Consider the triangle \( BEF \). Since it has the same angle at \( B \) as triangle \( ABC \), namely \( \alpha + \beta \), its angles at \( E \) and \( F \) must each be \( \alpha \). Then the north side of \( EF \) cannot be covered by a single tile, since if it were, that tile would have two \( \alpha \) angles, one at \( E \) and another at \( F \). Therefore the north side of \( EF \) supports at least two tiles. By hypothesis, \( EF \) is an edge of a single; that tile, say Tile 1, must lie on the south side of \( EF \).

Since the tile is rational by Theorem 10.5 we may assume without loss of generality that \((a, b, c)\) are integers with no common factor. In particular, none of \((a, b, c)\) is an integer multiple of another. Since the south side of \( EF \) is equal to one tile edge, the north side cannot be composed of all \( a \) edges, or all \( b \) edges, or all \( c \) edges, since then the edge on the south would be an integer multiple of the edge on the north.

Suppose Tile 1 has its \( a \) edge on \( EF \). Since \( a < c \), there are no \( c \) edges on the north side of \( EF \). Hence north of \( EF \) are only \( b \) edges, so \( a \) is an integer multiple of \( b \), contradicting the fact that \( a \) and \( b \) are relatively prime (by Lemma 11.2).

Similarly, if Tile 1 has its \( b \) edge on \( EF \), then \( b \) is an integer multiple of \( a \), contradiction. Finally, if Tile 1 has its \( c \) edge on \( EF \), then since \( a + b > c \), either all the tiles on the north of \( EF \) supported by \( EF \) have their \( a \) edges on \( EF \), or they all have their \( b \) edges on \( EF \). Then \( c \) is an integer multiple of \( a \) or \( c \) is an integer multiple of \( b \). We argue by cases:
Case 1, $c$ is a multiple of $a$, say $c = pa$. Then

$$c^2 = p^2a^2 = a^2 + ab$$

by Lemma 11.1

$$p^2a = a + b$$

by the previous two lines

$$a = 1$$

since $a$ and $b$ are relatively prime

$$k = 1$$

where $a = k^2$ and $b = m^2 - k^2$, by Lemma 11.2

Then $a = 1$ and $c = mk = m$ and $b = m^2 - 1 = c^2 - 1$, so $a + b = m^2 = c^2 = c$, so $(a, b, c)$ do not form a triangle, contradiction. That completes Case 1.

Case 2, $c$ is a multiple of $b$, say $c = pb$. Then

$$pb^2 = c^2$$

$$c^2 = a^2 + ab$$

by Lemma 11.1

$$p^2b^2 = a^2 + ab$$

by the previous two lines

$$a^2 \equiv 0 \mod b$$

$$a \equiv 0 \mod b$$

since $a$ and $b$ are relatively prime

But then $b$ divides $a$. Since $a$ and $b$ are relatively prime, that implies $b = 1$. By Lemma 11.2 there are relatively prime $(k, m)$ such that $b = m^2 - k^2$. Then $m^2 = k^2 + 1$, which is impossible, since $k > 0$. That completes Case 2.

These contradictions complete the proof of the lemma.

**Lemma 11.9.** Let isosceles $ABC$ with base angles $\alpha$ (at $A$ and $C$) be $N$-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, with $\alpha$ not a rational multiple of $\pi$. Let $T$ be a tile supported by $AC$ but not having a vertex at $A$ or $C$. Then $T$ does not have a vertex on $AB$ or $BC$.

**Proof.** Let $PQ$ be the edge of $T$ that lies on $AC$. Let $R$ be the third vertex of $T$. We must show $R$ does not lie on $AB$ or $AC$. Suppose, for proof by contradiction that it does. By renaming $A$ and $C$ if necessary, we can assume that $R$ lies on $AB$. Since $\alpha$ is not a rational multiple of $\pi$, there is only one tile, say $T_1$, with a vertex at $A$. The interior edge of $T_1$ connects $AB$ with $AC$. That is not a shared edge with $T$, since at least three tiles meet at $P$. See Fig. 15

**Figure 15.** $RPQ$ cannot be a single tile as $RP$ is too long.

Fig. 15, although already counterfactual, is not counterfactual enough, as it shows a gap between $T_1$ and $T$, which we must prove must be there, before we can prove that “$RP$ would be too long.” That is, $T$ might share a vertex with $T_1$. We will begin by showing that cannot happen.

I say that $P$ does not share a vertex on $AC$ with $T_1$. Suppose, for proof by contradiction, that it does. Then $P$, the western vertex of $T$ on $AC$, is also the
eastern vertex of $T_1$ on $AC$. Let $S$ be the third vertex of $T_1$, so $S$ lies on $AB$. See Fig. 16.

**Figure 16.** What if $T_1$ and $T$ share a vertex on $AC$?

Consider triangle $SPR$. The side $SP$ has length $a$, because that is the edge of $T_1$ opposite its $\alpha$ angle. Since $a = mk$ and $b = m^2 - k^2$, $a$ and $b$ are relatively prime, so $a$ cannot be expressed as a sum of $b$ edges; since $a < c$ that means that $SP$ supports only one tile on the east, sharing an $a$ edge with $T_1$. Call that tile $T_2$. Then $T_2$ and $T_1$ each have a $\beta$ or $\gamma$ angle at $S$. At $P$, $T_1$ has a $\beta$ or $\gamma$ angle, since its $\alpha$ angle is at $A$. By Lemma 10.1, $T_2$ and $T_1$ do not both have either a $\beta$ or $\gamma$ angle at $P$. $T_2$ does not have its $\alpha$ angle at $P$, since $PS$ is its $a$ edge. So one of $T_2$ and $T_1$ has a $\beta$ angle at $P$, and the other has a $\gamma$. Then $T$ has its $\alpha$ angle at $P$, and exactly those three tiles have a vertex at $P$. Since $T$ has its $\alpha$ angle at $P$, its side $PS$ is equal to $b$ or $c$. Now triangle $PSR$ has one side equal to $a$ (namely $PS$), side $PR$ equal to $b$ or $c$ (since $T$ has its $\alpha$ angle at $P$), and its angle at $P$ is either $\beta$ or $\gamma$, since $T$ has $\alpha$ at $P$ and $T_1$ has $\beta$ or $\gamma$. Therefore $PSR$ is congruent to the tile, by the ASA congruence theorem. But $T_1$ has angle $\beta$ or $\gamma$ at $S$, so angle $PSR$ is $\alpha + \gamma$ or $\alpha + \beta$, contradicting the fact that $PSR$ is congruent to the tile. That proves that $T$ and $T_1$ do not share a vertex on $AC$.

Now I say that $T$ does not share a vertex with $T_1$ on $AB$ either. Suppose to the contrary that it does. Then $R$ is the eastern vertex on $AB$ of $T_1$ as well as the northern vertex of $T$. See Fig. 17.

**Figure 17.** What if $T_1$ and $T$ share a vertex on $AB$?

Then triangles $T_1$ and $T$ have the same height (measured from $AC$) and since they are each one tile, they are congruent; so they have the same area. Hence they have the same base. The base of $T_1$ is $b$ or $c$, since it has its $\alpha$ angle at $A$. Then the base $PQ$ of $T$ is also $b$ or $c$.

Since the edge $RP$ is not shared with $T_1$, as we have already proved, there is another tile between $T_1$ and $T$ with a vertex at $R$. Since the third vertex of $T$, namely $Q$, does not lie on $AB$, there are at least four tiles meeting at $R$. Therefore there are three $\alpha$ angles and one $\beta$ angle at $R$. The $\beta$ angle must belong to $T_1$, since it has its $\alpha$ angle at $A$. Then $T$ has its $\alpha$ angle at $R$. Hence its base $PQ$ is opposite its $\alpha$ angle. Hence $PQ = a$. But we showed above that $PQ$ is $b$ or $c$. That is a contradiction; that proves (as claimed) that $T$ does not share a vertex with $T_1$ on $AB$. 


Let $X$ be the length of $AR$ and $Y$ the length of $RP$. Since $R$ is not a vertex of $T_1$, $AR$ is composed of at least two tile edges. One of those edges is not $a$, since $T_1$ has its $\alpha$ angle at $A$. I say that also one of them is not $b$. For suppose $AR$ supports only $b$ edges of tiles. Then for some integer $\ell$, $X = \ell b$. Each of those tiles has its $\alpha$ angle to the west and its $\gamma$ angle to the east. Then at $R$, the tile to the west of $R$ has its $\gamma$ angle at $R$. Then there are exactly three tiles with vertices at $R$. Since $RQ$ is an interior segment, $T$ is the middle one of those three. Then triangle $ARP$ is similar to the tile, since it has one $\alpha$ and one $\gamma$ angle. Triangle $ARP$ has $\alpha$ at $A$, and $\gamma$ at $P$, and therefore $\beta$ at $P$. But angle $ARP$ is the supplement of angle of $T$, which is impossible, as neither $\alpha$, $\beta$, nor $\gamma$ is the supplement of $\beta$. Hence, as claimed, one of the edges supported by $AR$ is not $b$.

We now intend to reach a contradiction by showing that the length of $RP$ is more than the length of any tile edge $a$, $b$, or $c$. That will be a contradiction, because $RPQ$ is a tile, so $RP$ has to be equal to $a$, $b$, or $c$.

Let $x$ and $y$ be two of $(a, b, c)$, not necessarily distinct, but one of $x$ and $y$ is not $a$, and one is not $b$. Then I say that $x + y > a$, $x + y > b$, and $x + y > c$. If we prove that, we can take $x$ and $y$ to be two of the edges supported by $RP$, and since $Y$ is a tile edge, and thus equal to $a$, $b$, or $c$, we will have $RP > Y$ as claimed.

Since $x + y > x$, and $x + x > x$, we are done unless the third edge is distinct from the two to be added. If the two to be added are distinct, we are done, because two sides of a triangle are together greater than the third. That leaves the cases $a + a > b$, $b + b > c$, $a + a > c$, $b + b > a$, $c + c > b$, $c + c > a$. The cases $c + c > a$ and $b + b > a$ follow from $c > a$ and $b > a$. We can drop $a + a > b$ and $a + a > c$ because we know one of two edges to be added is not $a$. We can drop $b + b > a$ and $b + b > c$ because we know that one of two edges to be added is not $b$. That leaves only $c + c > b$ still to prove. By Lemma 11.3, we have $c/b > 2/3$. Hence $c + c > (4/3)b > b$. That completes the proof of the lemma.

Remark. $b + b > c$ is not generally true. We have seen that $b$ can be much smaller than $c$. Hence we had to show that $RP$ could not support only $b$ edges.

**Lemma 11.10.** Let isosceles $ABC$ with base angles $\alpha$ (at $A$ and $C$) be $N$-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, with $\alpha$ not a rational multiple of $\pi$. Then there are at least four tiles supported by $AC$. If the tile instead has angles $(\alpha, \beta, 2\pi/3)$ instead of $(\alpha, \beta, 2\alpha)$, then there are at least three tiles supported by $AC$.

**Proof.** First assume $\gamma = 2\alpha$. We start by proving that at least three tiles are supported by the base $AC$. Suppose, for proof by contradiction, that only two tiles are supported. Those two tiles have their $\alpha$ angles at $A$ and $C$, and their $a$ edges both end at the shared vertex $P$ on $AC$. Without loss of generality we may assume that Tile 1, with edge $AP$, has its $\beta$ angle at $P$, and Tile 2, with edge $PC$, has its $\gamma$ angle at $P$, since two $\beta$ angles or two $\gamma$ angles at $P$ is impossible. Then the remaining angle to be filled at vertex $P$ is $\alpha$. By Theorem 11.4, $\alpha$ is not a rational multiple of $\pi$, and hence not a multiple of $\beta$. Since $\alpha$ is not a multiple of $\beta$, the gap must be filled by a single tile, Tile 3, with its $\alpha$ angle at $P$. Then Tile 3 has its $c$ side either against Tile 1 or Tile 2, but that is impossible since those edges are of length $a$ and terminate at the boundary, and $c > a$. Hence there are indeed at least three tiles supported by $AC$. Note that this argument works for both cases, $\gamma = 2\alpha$ and $\gamma = 2\pi/3$.

Now suppose there are exactly three tiles supported by $AC$. Not all three tiles on $AC$ can have their $c$ edges on $AC$, since then each would have a $\beta$ angle on $AC$, etc.
and by the pigeonhole principle, there would be a vertex on $AC$ with two $\beta$ angles. If the tiles at $A$ and $C$ both have their $b$ edges on $AC$, then they have their $\gamma$ angles on $AC$, so the middle tile on $AC$ cannot have a $\gamma$ angle on $AC$, so it has its $c$ edge on $AC$. Therefore the possible values of $Y$ are exactly $2c + b$, $2c + a$, $2b + c$, and $b + c + a$. So far, $\gamma$ could be $2\alpha$ or $2\pi/3$.

Now we assume $\gamma = 2\alpha$. Then Lemma 11.5 applies, yielding $X = (a/c)Y$. Therefore, the possible values of $X$ are exactly

$$2a + (ba/c), 2a + a^2/c, 2ab/c + a, (a + b + c)(a/c)$$

By Lemma 11.2, $a = k^2$, $b = m^2 - k^2$, $c = mk$, where $k$ and $m$ are relatively prime. Note that

$$(a + b + c)(a/c) = (k^2 + (m^2 - k^2) + mk)(k/m) = (m + k)k = a + c.$$ 

Then the possible values of $X$ are

$$2k^2 + k(m^2 - k^2)/m, 2k^2 + k^3/m, 2k(m^2 - k^2)/m, (m + k)k$$

Since $m > k \geq 1$, none of the first three can be an integer. Therefore

$$X = (m + k)k = a + c.$$ 

The question now is, what are the tiles supported by $AB$, the sum of whose edges on $AB$ is $a + c$? One possibility is that there are exactly two tiles supported by $AB$, one with its $a$ edge on $AB$ and the other with its $c$ edge. However, that is impossible, since the tiles at $A$ and $C$ both have their $\alpha$ angles at $A$ or $C$, and hence their $a$ edges do not lie on $AC$. So $X$ has some other configuration of tiles supported. Then $X = ua + vb + wc = a + c$ for some nonnegative integers $u, v, w$. Then $u$ or $w$ must be zero.

If $u = w = 0$ then $vb = a + c$, i.e., $v(m^2 - k^2) = k^2 + km$. Dividing by $m + k$ we have $v(m - k) = k$. Since $k$ is relatively prime to $m - k$, $k$ divides $v$. But also $v$ divides $k$. Hence $v = k$. Then $AC$ is composed of $k$ tile edges of length $b$. Then each edge has a $\gamma$ angle on $AC$. Since there are $\alpha$ angles at $A$ and $C$, then by the pigeonhole principle there is a vertex with two $\gamma$ angles. But that is impossible, since the only relations between the angles are $\alpha + 3\beta = \pi$ and $\alpha + \beta + \gamma = \pi$. Thus we have ruled out the case $u = w = 0$.

Now suppose $v = w = 0$. Then $ua = a + c$, i.e., $(u - 1)a = c$, or $(u - 1)k^2 = km$, so $u - 1 = m$. Then $ma = c = km$, so $m = k$, contradiction. Hence not both $v$ and $w$ are zero.

I say that $w \neq 0$. To prove that, suppose $w = 0$. Then $u \neq 0$ and $v \neq 0$ and $ua + vb = a + c$. Then

$$uk^2 + v(m^2 - k^2) = k^2 + km$$

$$(u - 1)k^2 + vm^2 = km$$

Then $k$ divides $v$, since $k$ is relatively prime to $m$. Since $v \neq 0$, we have $v \geq k$.

$$km = (u - 1)k^2 + vm^2$$

$$\geq vm^2 \quad \text{since } u - 1 \geq 0$$

$$\geq km^2 \quad \text{since } v \geq k$$

$$> km \quad \text{since } m > k \geq 1$$

$$km > km$$
But that is a contradiction, reached on the assumption $w = 0$. Therefore $w \neq 0$, as claimed.

Then

\[ua + vb + wc = a + c\]
\[uk^2 + v(m^2 - k^2) + wkm = k^2 + km\]
\[uk^2 + vm^2 + (w - 1)km = (v + 1)k^2\]
\[(u - 1)k^2 + vm^2 + (w - 1)km = vk^2\]

I say that $u = 0$. If $u \neq 0$, then all the terms on the left are non-negative, and $vm^2 > vk^2$ since $m > k$, so the equation is impossible. Hence $u = 0$, as claimed. Then

\[v(m^2 - k^2) + wkm = k^2 + km\]
\[vm^2 + (w - 1)km = (v + 1)k^2\]

Since $m > k$, we have $vm^2 > vk^2$, and since $w \neq 0$ we have $(w - 1)km > (w - 1)k^2$. If $w \neq 1$ then the left side of (15) is greater than the right, contradiction. Hence $w = 1$. Then

\[a + c = ua + vb + wc = vb + c\]

since $w = 1$ and $u = 0$. Then $a = vb$, which is impossible since $a$ and $b$ are relatively prime and not zero. This contradiction depends only on the assumption that there are exactly three tiles supported by $AC$. Since we already proved there are at least three tiles supported by $AC$, we have now proved there are at least four tiles supported by $AC$. That completes the proof of the lemma.

**Lemma 11.11.** Let isosceles $ABC$ with base angles $\alpha$ (at $A$ and $C$) be $N$-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, with $\alpha$ not a rational multiple of $\pi$. Then one of the sides $AB$ or $BC$ supports strictly less than $(N - 1)/4$ tiles. If the tile instead has angles $(\alpha, \beta, 2\pi/3)$, then one of the sides supports strictly less than $N/4$ tiles.

**Remarks.** This bound is not used in our tiling non-existence theorem that $N$ cannot be squarefree. But it is crucial to the theorem that, given $N$, there is an explicitly computable set of possible $ABC$ and tiles.

**Proof.** Let $P$ be a vertex of the tiling lying on the interior of a side of $ABC$. If only two tiles meet at $P$ then they cannot have different angles, since any two of $(\alpha, \beta, \gamma)$ make together less than $\pi$. But if they have the same angle at $P$, that angle would be a right angle, contrary to hypothesis. Therefore at least three tiles meet at each such vertex $P$.

Let $n$ and $m$ be, respectively, the total number of tiles with an edge or vertex on $AB$, and the total number of tiles with an edge or vertex on $BC$. First suppose the tile has $\gamma = 2\alpha$. By Lemma 11.10 $AC$ supports at least four tiles. The middle two do not touch $AB$ or $BC$ even in a vertex, by Lemma 11.9, and Lemma 11.8, and between the tiles that have no vertex at $A$ or $C$ there is (at least) a fifth tile, having only a vertex on $AC$, which also does not touch $AB$ or $BC$, by Lemma 11.9. Then $n + m \leq N - 3$, since $N$ is the total number of tiles, but at least three have a side or vertex on $AC$ and do not touch $AB$ or $BC$, and by Lemma 11.8 no tile contributes to both $n$ and $m$. Therefore either $n \leq (N - 3)/2$ or $m \leq (N - 3)/2$. Relabeling $A$ and $C$ if necessary, we can assume without loss of generality that $n \leq (N - 3)/2$. Now let $p$ and $q$, respectively, be the number of tiles supported by $AB$, and the number of tiles with one and only one vertex on $AB$. Then $q \geq p - 1$ (it might
be strictly greater if some vertices have more than three tiles sharing that vertex). Therefore

\[ p - 1 \leq q \]
\[ p + q = n \leq (N - 3)/2 \]
\[ p \leq (N - 3)/2 - q \]
\[ \leq (N - 3)/2 - (p - 1) \]
\[ 2p \leq (N - 3)/2 + 1 = (N - 1)/2 \]
\[ 2p \leq (N - 1)/2 \]
\[ p \leq (N - 1)/4 \]

We now will establish strict inequality in place of \( \leq \). Suppose that \( p = (N - 1)/4 \). Then there are exactly four tiles supported by \( AC \), and at all vertices on \( AC \) except \( A \) and \( C \), there are just three tiles meeting, since any more would introduce a strict inequality \( p - 1 < q \), instead of \( p - 1 \leq q \). Therefore each boundary vertex has one \( \alpha \), one \( \beta \), and one \( \gamma \) angle. The corner tiles have their \( \alpha \) angles at \( A \) and \( C \).

The two tiles adjacent to the corner tiles, with only a vertex on \( AC \), do not have their \( \alpha \) angles on \( AC \), since their \( \alpha \) sides must match the \( \alpha \) sides of the corner tiles. (It is impossible that \( a \) is an integer multiple of \( b \), since \( a \) is relatively prime to \( b \), by Lemma 11.2.) Therefore the middle vertex on \( AC \) must have two \( \alpha \) angles, contradiction. (In other words, if \( AC \) supports only four tiles, there must be a “boundary star” on \( AC \).) Hence, as claimed, \( p < N/4 \). That completes the proof in case the tile is \((\alpha, \beta, 2\alpha)\).

Now suppose the tile is \((\alpha, \beta, 2\pi/3)\) and \( p = N/4 \). Then we start with \( n + m \leq N - 1 \) instead of \( N - 3 \), since Lemma 11.10 gives us only three tiles supported by \( AC \) instead of four. The result is \( p \leq N/4 \) instead of \( p \leq (N - 1)/4 \). Then there must be exactly three tiles supported by \( AC \) and each boundary vertex has one \( \alpha \), one \( \beta \), and one \( \gamma \) angle, or else there would be strict inequality in the computation.

That completes the proof of the lemma.

11.6. What is the least \( N \) permitting a tiling? The smallest explicitly-known such tiling has \( N = 1125 \), with tile \((4, 5, 6)\), as shown in Fig. 23. We will show below that \( N \geq 45 \). There is a swath of ignorance between 45 and 1125.

Until now, we have no a priori estimate on the size of \((a, b, c)\). For example, there is no a priori reason why we could not have \( N < 100 \) and \((a, b, c)\) each greater than a million. We now provide such a bound.

**Lemma 11.12.** Suppose isosceles triangle \( ABC \) is \( N \)-tiled by tile \((a, b, c)\) with \( \gamma = 2\alpha \). Then \( a, b, \) and \( c \) are less than \( m^2 \), where \( m \) is as in Lemma 11.2 and satisfies

\[ m < N + \frac{(N + 1)^2}{16}. \]

**Remark.** The form of the bound is not very beautiful, but we want to use it for fairly small \( N \), and its asymptotic value is not of interest. We only care that there is some explicit and reasonably-sized bound.

**Proof.** By Lemma 11.2 there exist relatively prime integers \( m \) and \( k \leq m \) such that \( a = k^2 \), \( b = m^2 - k^2 \), and \( c = mk \). Let \( X = |AB| \) be the length of the two
equal sides of $ABC$. The tiling provides integers $p, q, r$ such that $X = pa + qb + rc$. Then

\[
\begin{align*}
X^2 & = Nab \\
X^2 & = N(k^2)(m^2 - k^2) \\
X & = pk^2 + q(m^2 - k^2) + rmk \\
X & \equiv (p - q)k^2 \mod m \\
X^2 & \equiv (p - q)^2k^4 \mod m \\
N(k^2)(m^2 - k^2) & \equiv (p - q)^2k^4 \mod m \\
-Nk^4 & \equiv (p - q)^2k^4 \mod m \\
-N & \equiv (p - q)^2 \mod m, \text{ since } \gcd(k, m) = 1 \\
N + (p - q)^2 & \equiv 0 \mod m
\end{align*}
\]

Therefore $m$ divides $N + (p - q)^2$. Since $N + (p - q)^2$ is positive, that implies $m \leq N + (p - q)^2$.

By Lemma 11.11, we may assume $p + q + r < (N + 1)/4$, since that is true on one of the two sides of $ABC$ of length $X$. In particular, each of $p, q, r$ is $< (N + 1)/4$.

Hence

\[
|p - q| < \frac{(N + 1)/4}{(N + 1)^2/16}
\]

By (16), we have

\[
m < N + \frac{(N + 1)^2}{16}.
\]

Since $k \leq m$, both $a$ and $b$ are $\leq m^2$. Hence both $a$ and $b$ are bounded by

\[
\left( N + \frac{(N + 1)^2}{16} \right)^2.
\]

That completes the proof of the lemma.

By a boundary tiling of $ABC$ by the tile $(a, b, c)$, we mean a placement of tiles supported by the boundary of $ABC$ touching every point of the boundary of $ABC$. Let $X$ be the side $AB$, and $Y$ the base $AC$, of isosceles $ABC$. A boundary tiling of $ABC$ provides integers $(p, q, r)$ and $(u, v, w)$ with

\[
\begin{align*}
X & = pa + qb + rc \\
X^2 & = Nab \quad \text{the area equation} \\
Y & = ua + vb + wc
\end{align*}
\]

We use the phrase possible boundary tiling to mean a way of writing $X$ and $Y$ in this form, with integers $(p, q, r)$ and $(u, v, w)$ satisfying Lemma 11.11 and $(a, b, c)$ satisfying Lemma 11.12. Of course, a tiling gives rise to a boundary tiling, but not every boundary tiling can be completed to a tiling, let alone every “possible” boundary tiling. Each “possible boundary tiling” might correspond to many different ways of arranging the tiles (in different orders) on the boundary, but there will be only finitely many ways. We note that a boundary tiling might use different $(p, q, r)$ on the two sides $AB$ and $BC$; we shall return to that point below.
Lemma 11.13. Let isosceles triangle $ABC$ with base $AC$ and vertex $B$ have base angles $\alpha$, not a rational multiple of $\pi$. Let $(a, b, c)$ be integers with no common factor forming a triangle with angles $(\alpha, \beta, 2\alpha)$. Let $X$ be the length of side $AB$ Suppose there is an $N$-tiling of $ABC$ by $(a, b, c)$ whose boundary tiling on $AB$ or $BC$ corresponds to $X = pa + qb + c$. Then $q \neq 0$ and $r \neq 0$.

Proof. First suppose $q = 0$. That means there are no $b$ edges on $AC$, so every tile supported by $AC$ has a $\beta$ angle on $AC$. But since the top and bottom tiles have their $\alpha$ angles at $C$ and $A$, and no vertex has two $\beta$ angles, this violates the pigeonhole principle. Now suppose $r = 0$. That means there are no $c$ edges on $AC$, so every tile supported by $AC$ has a $\gamma$ angle on $AC$. But since the top and bottom tiles have their $\alpha$ angles at $C$ and $A$, and no vertex has two $\gamma$ angles, this violates the pigeonhole principle. That completes the proof of the lemma.

Lemma 11.14. Given $N$, there is a finite set $\Delta$ of tiles $(a, b, c)$ having integer sides with no common factor and angles $(\alpha, \beta, \gamma)$, and for each tile in $\Delta$, a finite number of representations

\[ X = pa + qb + rc \]
\[ Y = ua + vb + wc \]

such that if isosceles triangle $ABC$ with base angle $\alpha$ can be $N$-tiled with some tile, then the tile belongs to $\Delta$ and the boundary representations determined by the tiling are among those allowed for that tile.

Example. With $N = 36$, there are just two possible tiles: $(9, 16, 15)$ and $(16, 9, 20)$, as we will show below, and the possible boundary tilings are given in Table 3.

Proof. Let $N$ be given. Then the number of possible boundary tilings is finite (and one can easily loop through them), by definition of “boundary tiling.” By Lemma 11.2 and Lemma 11.12 every $N$-tiling gives rise to a possible boundary tiling.

We spell out the algorithm implicit in the preceding lemmas: Given $N$, we loop through all $(k, m)$ satisfying Lemma 11.12 and with $k$ and $m$ relatively prime. There are only finitely many $(k, m)$ to loop through, because according to Lemma 11.12 we have an explicit bound on $m$ in terms of $N$. Since $k < m$, both $m$ and $k$ are bounded in terms of $N$.

For each such $(k, m)$, we compute the tile

\[(a, b, c) = (k^2, m^2 - k^2, mk).\]

We reject triples $(a, b, c)$ that either

- cannot form triangles because one side is greater than or equal to the sum of the other two, or
- the squarefree part of $N$ is not equal to the squarefree part of $b$

Then $X$ is defined by the area equation $X^2 = Nab$, and $Y$ is defined by the area equation $XY = Nbc$. We reject triples $(a, b, c)$ if either

- $X$ is not an integer, or
- $Y$ is not an integer

\[\text{The condition } k \text{ and } m \text{ relatively prime is important, because it results in } a \text{ and } b \text{ being relatively prime, which is assumed in Lemma 11.11. Without it, we got some spurious possible boundary tilings with tiles like } (46, 45, 54), \text{ which cannot correspond to real tilings because Lemma 11.11 would be violated.}\]
Then we loop through all triples \((p, q, r)\) satisfying the bound of Lemma 11.11 such that

\[ X = pa + qb + rc \]

We can immediately reject \((p, q, r)\) if \(q = 0\) or \(r = 0\), by Lemma 11.13. Now \(Y = (c/a)X\). We need to check whether \(Y\) can be expressed in the form \(ua + vb + wc\). There are only finitely many possibilities for \((u, v, w)\), so we can check that. If \(Y\) cannot be so expressed, then we can reject this \((a, b, c)\).

Otherwise, we output the possible boundary tiling given by

\[ X = pa + qb + rc \]
\[ Y = ua + vb + wc. \]

That completes the proof of the lemma.

The algorithm as described above eliminates \(N < 20\), but finds possible boundary tilings for \(N = 20, 28, 36, 44, 45\). Below we will discuss improvements to the algorithm and eliminate some of these values.

**Theorem 11.15.** Given \(N\), it is decidable by a computation whether there exists an \(N\)-tiling of some (any) triangle \(ABC\) by a tile with \(\gamma = 2\alpha\), (where \(ABC\) has base angles \(\alpha\)).

**Proof.** For a fixed \(ABC\) and tile, it is (in principle) computationally decidable whether there is a tiling: By Lemma 11.14, there are finitely many possible boundary tilings so in principle you can all the possible ways of arranging tiles on the boundary, and check by backtracking search whether the boundary tiling can be completed to an \(N\)-tiling, just like solving a jigsaw puzzle. That completes the proof.

11.7. **Ruling out more values of \(N\).** The problem of constructing an \(N\)-tiling divides into two parts: first construct an \(ABC\) and a tile \((a, b, c)\), and a possible boundary tiling

\[ X = pa + qb + rc \]
\[ Y = ua + vb + wc. \]

Then, use backtracking search to either find an \(N\)-tiling, or show that there is none.

The second part of this (the part involving backtracking) is not a trivial program, and even if coded, it would probably take too long when \(N\) is large. But the first algorithm (searching for a possible boundary tiling) is very easy to implement, as no geometry is involved, just some simple linear equations. We have already described that algorithm in Lemma 11.14.

**Lemma 11.16.** Let isosceles triangle \(ABC\) with base angles \(\alpha\) be \(N\)-tiled by an integer-sided triangle with angles \((\alpha, \beta, 2\alpha)\). Then \(N \neq 20\).

**Remark.** It seems that \(N = 20\) is the only value of \(N\) (among those passed by Lemma 11.14) that can be rejected this way and is not rejected by simpler arguments; at least, it’s the only one less than 1000.

---

\(^7\)We coded the algorithm twice, once in SageMath, which offers unlimited precision integers, and once in C, taking care to use 64-bit integers in C. We got the same results from both implementations.
Proof. Let $n$ be the minimum possible number of tiles supported on the boundary; then there must be at least $n-1$ tiles with a side or vertex on the boundary, thanks to there being that many gaps between the tiles, and no double-counting of tiles filling those gaps, because of Lemma 11.8 and Lemma 11.9. Then if $n + (n - 2)$ exceeds $N$, we can reject that possible boundary tiling.

When $N = 20$, the tile $(4, 5, 6)$ leads to $X = 20 = a + 2b + c$, so $X$ supports 4 tiles, and 20 cannot be written as a sum of fewer tiles. Then $Y = 30 = 5c$, $n = 11$, $n + (n - 1) = 21 > 20$. So this possible tile is rejected. Since that is the only possible tile for $N = 20$, there is no 20-tiling. That completes the proof.

Lemma 11.17. Let isosceles triangle $ABC$ with base $AC$ and vertex $B$ have base angles $\alpha$, not a rational multiple of $\pi$. Let $(a, b, c)$ be integers with no common factor forming a triangle with angles $(\alpha, \beta, 2\alpha)$. Let $X$ and $Y$ be the lengths of side $AB$ and base $AC$ respectively. Suppose

$$X = pa + qb + c$$

and

$$Y = ua + vb + c \text{ or } ua + vb + 2c$$

Then there is no $N$-tiling of $ABC$ by $(a, b, c)$ whose boundary tilings correspond to those representations.

Remark. So, if the only possible boundary tilings are as in the lemma, then there is no $N$-tiling of $ABC$ by $(a, b, c)$ at all.

Example. Consider $N = 28$. Consider $(a, b, c) = (9, 7, 12)$. By the area equations (10) and (11), if there were a 28-tiling of an isosceles $ABC$ with base angles $\alpha$, we would have $X = 42$ and $Y = 56$. Then $X = a + 3b + c$ and $Y = 2a + 2b + 2c$. The lemma shows we cannot have a tiling corresponding to these representations of $X$ and $Y$. Below we will show that these are the only possible decompositions of $X$ and $Y$ realizable in a tiling, thus ruling out a 28-tiling.

Proof. Suppose that such a tiling exists. By hypothesis, the decomposition $Y = ua + vb + wc$, with $w = 1$ or 2, corresponds to the tiles supported by $AC$, and the decomposition $X = pa + qb + c$ corresponds to the tiles supported by $AB$ and the tiles supported by $BC$. That is, there is only one $c$ edge on $AB$, only one $c$ edge on $BC$, and at most two $c$ edges on the base $AC$.

The tile $T_1$ at $A$ has its $\alpha$ angle at $A$. Suppose, for proof by contradiction, that $T_1$ has its $c$ edge on $AB$. Since there is only one $c$ edge on $AB$, all the rest of the tiles supported by $AB$ have a $\gamma$ angle on $AB$. No tile has its $\gamma$ angle at $B$, since the total angle at $B$ is $\alpha + \beta$, which cannot be written in any other way as a rational linear combination of $\alpha$ and $\beta$.

Therefore the top tile on $AB$, with a vertex at $B$, has its $\gamma$ angle to the south on $AB$. Since there cannot be two $\gamma$ angles at any boundary vertex, all the tiles on $AB$ above $T_1$ have their $\gamma$ angles are to the south. Let the next tile (that is, next to $T_1$) supported by $AB$ be $T_3$, and let the one between $T_1$ and $T_3$ be $T_2$; and let $P$ be the shared vertex on $AB$ of these three tiles. Then $T_1$ has its $\beta$ angle at $P$, and $T_3$ has its $\gamma$ angle at $P$. Hence $T_2$ has its $\alpha$ angle there. Then the $a$ edge of $T_2$ is not shared with $T_1$. Since $a < c$, and the $a$ edge of $T_1$ has both endpoints on the boundary of $ABC$, the $c$ edge of $T_2$ is not shared with $T_1$. Hence it must be the $b$ edge of $T_2$ that is shared with $T_1$. Then $b < a$. The remaining part of the $a$ edge of $T_1$, namely $a - b$, must be composed of $b$ edges, since its length is less than $a$ and
less than \( b \). Then \( a \) is a multiple of \( b \), say \( a = kb \). By Lemma 11.11, \( c^2 = a^2 + ab \). Then \( c^2 = k^2b^2 + kb^2 \), so \( b \) divides \( c \) as well as \( a \), contradiction, since without loss of generality we may assume \( a, b, \) and \( c \) have no common divisor. This contradiction shows that \( T_1 \) does not have its \( c \) edge on \( AB \) (as was assumed at the beginning of this paragraph). Therefore, \( T_1 \) has its \( c \) edge on the base \( AC \), rather than on \( AB \).

Now let \( R \) be the eastern vertex of \( T_1 \), lying on \( AC \), and as before let \( T_2 \) be the tile sharing the northeast edge of \( T_1 \), which is the \( a \) edge. As before \( T_2 \) cannot share its \( b \) or \( c \) edge with \( T_1 \), and therefore has its \( a \) edge there, so \( R \) is a vertex of both \( T_1 \) and \( T_2 \). Then the angle of \( T_2 \) at \( R \) is not \( \alpha \), and the angle of \( T_1 \) at \( R \) is \( \beta \). At \( R \) the tile angles are either \((\beta, \gamma, \alpha)\) or \((\beta, \alpha, \alpha, \alpha)\). In either case the next tile on \( AC \), with western vertex at \( R \), has angle \( \alpha \) at \( R \).

Now this whole argument can be repeated, using vertex \( C \) instead of \( A \). Therefore the tile on \( AC \) at \( C \) has its \( c \) edge on \( AC \), and the next tile west of that has its \( \alpha \) angle to the east; call that vertex \( S \).

Then the two tiles at \( A \) and \( C \) account for both \( c \) edges that can occur on \( AC \). Then every tile supported by \( RS \) does not have its \( c \) edge on \( AC \), and therefore has a \( \gamma \) angle on \( AC \). Since these \( \gamma \) angles do not occur at the endpoints of \( RS \), the number of \( \gamma \) angles exceeds the number of vertices where they can occur. By the pigeonhole principle, there must be a vertex with two \( \gamma \) angles. But that is a contradiction. That completes the proof of the lemma.

**Theorem 11.18.** Let isosceles triangle \( ABC \) with base angles \( \alpha \) be \( N \)-tiled by a triangle with angles \((\alpha, \beta, 2\alpha)\), and \( \alpha \) not a rational multiple of \( \pi \). Then \( N \geq 45 \).

**Proof.** \( N = 20 \) is eliminated by Lemma 11.16. By Lemma 11.14 the only possible tiles are the ones shown in Table 3.

| \( N \) | \((a,b,c)\) | \( X \) | \( Y \) |
|---|---|---|---|
| 28 | (9,7,12) | \( 42 = a + 3b + c \) | \( 56 = 2a + 2b + 2c \) |
| 36 | (9,16,15) | \( 72 = a + 3b + c \) | \( 120 = a + 6b + c \) |
| 36 | (16,9,20) | \( 72 = a + 4b + c \) | \( 90 = 2a + 2b + 2c \) |
| 44 | (25,11,30) | \( 110 = a + 5b + c \) | \( 132 = 2a + 2b + 2c \) |
| 45 | (4,5,6) | \( 30 = a + 4b + c \) | \( 45 = a + b + 6c \) |
| 45 | (4,5,6) | \( 30 = 2a + 2b + 2c \) | \( 45 = a + b + 6c \) |

It remains to eliminate 28, 36, and 44. According to Lemma 11.17 given \( N \) and \((a,b,c)\), if the representations of \( X \) and \( Y \) as combinations of \((a,b,c)\) involve only one \( c \) edge in \( X \) and one or two in \( Y \), then there is no corresponding \( N \)-tiling. Adding that easily-checkable condition to the algorithm in Lemma 11.14 we find that \( N = 28, 36, \) and 44 are eliminated, leaving \( N = 45 \) as the least value for which possible boundary tilings are found.

For the benefit of the reader who prefers pencil and paper to computer code, we give a direct argument for each of those three values of \( N \), with a few steps left to the reader’s pencil:

We first take up the case \( N = 28 \), which was discussed in the example following the statement of Lemma 11.17. That lemma shows that no tiling is possible corresponding to the representations of \( X \) and \( Y \) given in Table 3. There cannot be other representations with more \( c \) edges, since no multiple of 12 cannot be made.
from (some of) one 9 and three 7s, and no multiple of 12 can be made from (some of) two 9s and two 7s, and \(a\) and \(b\) are relatively prime.

It remains to consider representations with fewer \(c\) edges. In this case that would mean zero \(c\) edges; but that would contradict Lemma 11.13. Hence no 28-tiling of this isosceles triangle \(ABC\) by this \((a, b, c)\) exists.

We turn to the case \(N = 36\). Suppose, for proof by contradiction, there is a 36-tiling by \((9, 16, 15)\). By Table 3 \((X, Y) = (72, 120)\). I say that 72 = \(a + 3b + c\) is the only possible representation of 72 that could occur in the tiling. Suppose \(72 = pa + qb + rc\). By Lemma 11.13, \(q\) and \(r\) are not zero. Suppose \(r \geq 3\). Then \(11 = pa + (q - 1)b + (r - 3)c\), which is impossible. If \(r = 2\), then \(26 = pa + (q - 1)b\); but that is impossible. By Lemma 11.13, \(r \neq 0\). Then \(r\) must be 1, so \(57 = pa + qb\).

We have \(p \leq 6\) since \(6 \cdot 9 > 57\); so \(57 - pa\) is a multiple of 16. One can check that the only possibility is \(p = 1\), \(q = 3\), which gives the known decomposition \(X = a + 3b + c\). (Another way of looking at this is that, if \(r > 1\), we must be able to “trade in” some of one \(a\) and three \(b\)s for a number of \(c\)s, i.e., make a multiple of 15 out of (some of) one 9 and three 16s. But that is impossible.)

We also must show that \(Y = a + 6b + c\) is the only decomposition of \(Y = 120\) that could occur in the tiling.

Then by Lemma 11.17, those representations for \(X\) and \(Y\) in Table 3 do correspond to the tiling. Therefore there is no 36-tiling by the tile \((9, 16, 15)\).

Table 3 also has a second entry for \(N = 36\), namely \((a, b, c) = (16, 9, 20)\), with \(X = 72 = a + 4b + c\) and \(Y = 90 = 2a + 2b + 2c\). One can check with pencil and paper that it is not possible to make a multiple of 20 with up to two \(a\) edges and up to four \(b\) edges. Then by Lemma 11.17, no such tiling exists. Since Table 3 represents the results of Lemma 11.14, there is no 36-tiling of any isosceles \(ABC\).

Turning to \(N = 44\), the tile would have to be \((25, 11, 30)\). To apply Lemma 11.17, given the representations of \(X\) and \(Y\) in the table, it suffices to check (with pencil and paper) that no multiple of 30 can be made of up to one \(a\) edges and up to five \(b\) edges, or up to two \(a\) edges and up to two \(b\) edges. That completes the proof of the theorem.

We note that the technique does not extend to \(N = 45\). After that the next possibilities left open are 63, 64, 72.

12. Tilings of an isosceles triangle by a tile \((\alpha, \beta, 2\pi/3)\)

In this section we take up the tilings of an isosceles triangle (and not equilateral) \(ABC\) with base angles \(\alpha\) or \(\beta\), by a tiling with angles \((\alpha, \beta, 2\pi/3)\), where \(\alpha\) is not a rational multiple of \(\pi\). Let \(\gamma = 2\pi/3\). We could insist that the base angles are called \(\alpha\), but then we may have to speak of tilings by \((5, 3, 7)\) instead of \((3, 5, 7)\), so it is convenient to allow the base angles to be \(\beta\) sometimes. If the base angles are \(\alpha\), then the vertex angle is \(\pi - 2\alpha = \alpha + 3\beta\).

Laczkovich proved [7, Theorem 2.5] that there exist tiles that can be used to tile some such \(ABC\), but \(N\) constructed by his method can be large. He proceeds by first constructing a dissection of \(ABC\) into similar rational triangles and parallelograms. Fig. 18 shows such a preliminary dissection. Then to get a tiling by congruent triangles, we have to choose a very small tile such that if each of the visible triangles is tiled quadratically, then every shared edge is an integer multiple of the tile edges. For example if the red triangle will get \(p^2\) tiles and the light
blue triangle will get \( q^2 \) tiles, then we must satisfy \( pb = qa \), in this case \( 5p = 3q \). There will be another such equation on every shared boundary. To clear all the denominators we will have to use a large number of tiles.

\[ \text{Figure 18. With tile (3,5,7), } N \text{ would be 1878500, too large to draw.} \]

\[ \text{Figure 19. With this configuration, } N \text{ is 3681860–even larger.} \]

Fig. 19 shows that a slightly different arrangement of the similar triangles and parallelograms can make a difference in the resulting number of tiles. In one figure, the parallelogram is tiled using \( a \) and \( c \) edges on the boundary; in the other figure, the parallelogram is tiled using \( a \) and \( b \) edges on the boundary. That makes the equations at the boundary different, even though the other boundary conditions are the same and the areas of the two parallelograms are equal.

Nevertheless, these tilings are too large to draw. In 2024, Bryce Herdt discovered a 2673-tiling, which is exhibited in the Appendix. This is presently the smallest known tiling of an isosceles triangle by a tile with \( \gamma = 2\pi/3 \).

12.1. The tile is rational. Suppose given an \( N \)-tiling of some triangle \( ABC \) by a tile with angles \((\alpha, \beta, \gamma)\) and sides \((a, b, c)\). In this section we will prove that the tile has to be rational, i.e., the ratios of the sides are all rational, so after a suitable scaling, they will be integers. The proof uses the graphs \( \Gamma_c \) introduced by Laczkovich and described above; several preparatory lemmas will be developed first.

Definition 12.1. A \( c/a \) segment is a left-terminated interior segment \( PQ \) of the tiling supporting two tiles on opposite sides of \( PQ \), each with a vertex at \( P \), one with its \( c \) edge on \( PQ \) and one with its \( a \) or \( b \) edge on \( PQ \). The segment is said to “emanate from \( P \).” Similarly for \( c/b \) segment and \( a/b \) segment.

Remarks. The point \( Q \) serves only to indicate the direction of the segment; it can be any point on that ray.
Lemma 12.2. Let $ABC$ be an isosceles triangle with base angles $\alpha$, tiled by tile $(\alpha, \beta, \gamma)$ with $\gamma = 2\pi/3$. Let $PQR$ be an internal segment of the tiling with only $c$ edges on one side of $PQ$, such that the tile on that side supported by $QR$ with a vertex at $Q$ has an $a$ or $b$ edge on $PQ$. Then there is a $c/a$ or $c/b$ segment emanating from $Q$.

Proof. We only consider tiles on the one side of $PQ$ mentioned in the lemma. There are two cases: Either there are three tiles with a vertex at $Q$, or there are six.

Case 1, three tiles at $Q$. Then one of them has its $\gamma$ angle at $Q$, and hence no $c$ edge at $Q$. The other two have a $c$ edge ending at $Q$. One of those lies on $PQ$. The other does not lie on $QR$, by hypothesis. Since there is no other $c$ edge ending at $Q$, that third $c$ edge forms either a $c/b$ segment or a $c/a$ segment.

Case 2, six tiles at $Q$. Then all six angles are $\alpha$ or $\beta$. Each of the six tiles has a $c$ edge ending at $Q$. One lies on $PQ$ and five lie on interior segments. Since five is odd, one of those $c$ edges is not paired with another $c$ edge, and hence constitutes a $c/a$ segment or a $c/b$ segment. That completes the proof of the lemma.

Lemma 12.3. Suppose isosceles triangle $ABC$ with base angles $\alpha$ is $N$-tiled by $(\alpha, \beta, 2\pi/3)$, with $\alpha$ not a rational multiple of $\pi$. Then there is a relation

$$jc = pa + qb$$

with nonnegative integers $p, q, j$ and $j > 0$.

Proof. Suppose, for proof by contradiction, that there is no such relation. Then, by Lemma 12.2, if $PQ$ is a link in the graph $\Gamma_c$, then there is a $c/b$ or $c/a$ segment emanating from $Q$. Extend that segment to the maximal segment $QR$ supporting only tiles with $c$ edges on $QR$. Since there is no relation $jc = pa + qb$, $R$ cannot be the vertex of a tile on the other side of $QR$. Therefore $QR$ is a link in $\Gamma_c$.

Therefore the out-degree of every node $Q$ in $\Gamma_c$ is at least one. But the in-degree of $\Gamma_c$ is always at most one. Since the total out-degree is equal to the total in-degree, it follows that every node of $\Gamma_c$ has both in-degree and out-degree equal to 1. Since no link of $\Gamma_c$ can terminate on the boundary of $ABC$, there can be no links of $\Gamma_c$ emanating from a vertex on the boundary of $ABC$.

I say there is at least one $c$ edge on $AC$. For if not, every tile supported by $AC$ has its $\gamma$ angle at a vertex on $AC$. Since $\gamma > \pi/2$, there cannot be two $\gamma$ angles at any one vertex. But $\gamma$ angles do not occur at $A$ or $C$, where there are only $\alpha$ angles. Then there is one more $\gamma$ angle on $AC$ than possible vertices to receive them, so by the pigeonhole principle, some vertex on $AC$ has two $\gamma$ angles, contradiction. Therefore, as claimed, there is at least one $c$ edge on $AC$. Similarly, there is at least one $c$ edge on $AB$ and at least one $c$ edge on $BC$. Since there is a single tile at $A$ with its $\alpha$ angle at $A$, the $b$ edge of that tile lies on $AC$ or on $AB$. Then there exists a segment $PQ$ lying on $AB$ or on $AC$ supporting only tiles with $c$ edges on $PQ$, and with a $b$ edge beyond $Q$. Then by Lemma 12.2 there is a $c/a$ segment or a $c/b$ segment emanating from the boundary point $Q$, say $QR$. Choose $R$ as far as possible from $Q$ such that $QR$ bounds only $c$ tiles on one side. Then $R$ is not a vertex of a tile on the other side of $QR$, since that would give rise to a relation $jc = pa + qb$ with $j > 0$. Hence $QR$ is a link in $\Gamma_c$. But that is a contradiction, since $Q$ is on the boundary of $ABC$. That completes the proof of the lemma.
Lemma 12.4. Suppose isosceles triangle $ABC$ with base angles $\alpha$ is $N$-tiled by $(\alpha, \beta, 2\pi/3)$, with $\alpha$ not a rational multiple of $\pi$. Then there is a relation

$$ja = pb + qc$$

with nonnegative integers $p, q, j$ and $j > 0$ and $p > 0$.

Proof. Suppose, for proof by contradiction, that there is no such relation. A center is a vertex of the tiling where three tiles meet, each having its $\gamma$ angle at that vertex. A star is a vertex $P$ where six tiles lying on one side of a line through $P$ have three $\alpha$ and three $\beta$ angles at $P$. A star can occur on the boundary of $ABC$ or in the interior. A double star is a vertex where twelve tiles meet, six with $\alpha$ angles and six with $\beta$ angles. The three vertices of $ABC$ together have six angles, three $\beta$ and three $\alpha$, the same count as a star. Let $S$ be the number of stars (counting a double star as two), and $C$ the number of centers. Now let us calculate the number of $\alpha$ angles, plus the number of $\beta$ angles, minus twice the number of $\gamma$ angles. At each vertex other than stars, centers, and $A$, $B$, and $C$, we get zero. At each center we get $-6$. At each star we get 6 (and 12 at double stars). Adding them up we get $6S - 6C + 6$, where the final 6 is for $A$, $B$, and $C$ together. Since the total number of $\alpha$ is $N$, the total number of $\beta$ is $N$, and the total number of $\gamma$ is $N$, we get zero for the grand total. That is, $6S - 6C + 6 = 0$. Then $S = C - 1$. (For example, in Fig. 19 we see one center and no stars.)

Now we consider the graph $\Gamma_a$. Every center has an out-link, since at a center $P$ there are three tiles, each with an $a$ edge and a $b$ edge at $P$. Since 3 is odd, one of the $a$ edges shares a segment with one of the $b$ edges, i.e., an $a/b$ edge emanates from $P$. (For example, note the center in Fig. 19.) Let $Q$ be the farthest point from $P$ along that segment such that $PQ$ supports only $a$ tiles on one side, say the “left” side. If $Q$ were vertex of a tile on the other side, we would have a relation $ja = pb + qc$, and $p$ would be positive since there is a $b$ edge on the “right” side of $PQ$. Since by hypothesis, there is no such relation, $Q$ is not a vertex of a tile on the other side. Then $PQ$ is a link in $\Gamma_a$. On the other hand, the in-degree of a center is zero, since no segment of the tiling passes through $P$.

At a star $Q$ on an internal segment $PQ$, six tiles meet, providing six $c$ edges, three $a$ edges, and three $b$ edges. There could be an incoming link at $Q$, if the tile on $PQ$ at $Q$ has its $a$ edge there, the tile past $Q$ does not have its $a$ edge on $PQ$ extended, and the other two $a$ edges are not on the same segment. The in-degree of $\Gamma_a$ can never exceed 1, since it is impossible for two lines of the tiling to cross at $Q$ when a link ends at $Q$. At the vertices $A$, $B$, and $C$ the in-degree is zero, since a link cannot terminate on the boundary. At $A$ and $C$ the out-degree is zero since there are no interior edges. At $B$ there might be outgoing links, or not.

Now we calculate the out-degree minus the in-degree vertex by vertex. At centers it is 1. At stars it is 0 or -1 (or -2 possibly at double stars). Let $t$ be the total out-degree minus in-degree at stars; then $0 \geq t \geq -S$. At $A$ and $C$ it is zero. At $B$ it is non-negative, say $n_B$. At all other vertices it is zero. The total of out-degree minus in-degree is then $C + t + n_B \geq C - S$. Since $S = C - 1$, the total out-degree minus in-degree is $\geq 1$. On the other hand, it is zero since every link has a head and a tail. This contradiction completes the proof of the lemma.

Theorem 12.5. Let $ABC$ be an isosceles (and not equilateral) triangle with base angles $\alpha$. Suppose $ABC$ is tiled by a tile $(\alpha, \beta, 2\pi/3)$ with $\alpha$ not a rational multiple of $\pi$. Then the tile is rational.
Proof. Suppose \(ABC\) is tiled as in the lemma. By Lemma 12.3 and Lemma 12.4, there are relations
\[
j c = p a + q b \quad \text{with nonnegative integers } j, p, q \text{ and } j > 0
\]
\[
J a = P b + Q c \quad \text{with nonnegative integers } J, P, Q \text{ and } J > 0 \text{ and } P > 0
\]
Dividing by \(c\) we have
\[
j = p (a/c) + q (b/c)
\]
\[
-Q = -J (a/c) + P (b/c)
\]
Case 1, \(q \neq 0\). The equations can be solved for \((a/c)\) and \((b/c)\), provided the determinant \(pP + Jq \neq 0\). Since \(q \neq 0\) and \(J \neq 0\), and \(p \geq 0\) and \(q \geq 0\), the determinant is not zero.

Case 2, \(q = 0\). Then \(a/c = j/p\) is rational by the first equation, and \(b/c\) is rational by the second equation, since \(P \neq 0\).

That completes the proof of the theorem.

12.2. The Diophantine equation \(c^2 = a^2 + b^2 + ab\). Let \((a, b, c)\) be the sides of a triangle with angles \((\alpha, \beta, 2\pi/3)\). According to the law of cosines, we have
\[
c^2 = a^2 + b^2 - 2ab \cos(2\pi/3)
\]
\[
c^2 = a^2 + b^2 + ab \quad \text{since } \cos(2\pi/3) = -1/2
\]
Therefore this Diophantine equation determines the possible rational triangles with a \(2\pi/3\) angle.

Lemma 12.6. Suppose \(c^2 = a^2 + b^2 + ab\), and \((a, b, c)\) are integers with no common factor. Then \((a, b, c)\) are pairwise relatively prime.

Proof. If prime \(p\) divides any two of \((a, b, c)\) then it also divides the third one.

Lemma 12.7. Suppose \((a, b, c)\) are integers with no common factor that are the sides of a triangle with angles \((\alpha, \beta, 2\pi/3)\). Then \(2b + a\) is relatively prime to each of \(a, b,\) and \(c\), except that if \(a\) is even, \(2\) divides both \(a\) and \(2b + a\).

Proof. By Lemma 12.6, \(2b + a\) is relatively prime to \(b\) and \(a\), with the exception mentioned in the statement. It remains to prove \(2b + a\) is relatively prime to \(c\). Suppose, for proof by contradiction, that \(p\) is a prime that divides both \(c\) and \(2b + a\). Then \(p\) is not 2, since then \(c\) and \(a\) would both be even, contradicting Lemma 12.6.

Suppose, for proof by contradiction, that \(p = 3\). Then mod 3 we have \(2b + a \equiv 0\).

Adding \(b\) to both sides we have \(3b + a \equiv b\). But \(3b \equiv 0\), so \(a \equiv b\). Now \(c = a^2 + b^2 + ab = (a + b)^2 - ab \equiv a^2 \mod 3\), since \(a \equiv b\). Since \(p\) divides \(a^2\), we have \(p|a^2\) and hence \(p|a\). Hence \(a\) and \(b\) are both divisible by 3, contradiction, since \((a, b)\) are relatively prime. Hence \(p \neq 3\).

Then we have, mod p,
\[
c \equiv 0
\]
\[
c^2 \equiv a^2 + b^2 + ab
\]
\[
2b + a \equiv 0
\]
Substituting \(c = 0\) in the last two equations we have
\[
0 \equiv a^2 + b^2 + ab \quad \text{(17)}
\]
\[
2b + a \equiv 0 \quad \text{(18)}
\]
From the second equation we have \(a \equiv -2b\). Since \(a\) and \(b\) are relatively prime, and \(p \neq 2\), this implies that neither \(a\) nor \(b\) is divisible by \(p\). Substituting \(a = -2b\) in (17), we have

\[
0 \equiv 4b^2 + b^2 - 2b^2 = 3b^2 \\
0 \equiv b^2 \quad \text{since } p \neq 3 \\
0 \equiv b
\]

Then \(a \equiv -2b \equiv 0\). Hence \(p\) divides both \(a\) and \(b\), contradiction, since \(a\) and \(b\) are relatively prime. That completes the proof of the lemma.

Remark. Using the techniques of [3], Corollary 6.3.15, p. 353, we are able to parametrize the solutions of \(c^2 = a^2 + b^2 + ab\) by two integer parameters \((s, t)\) or one rational parameter \(s/t\). Having worked this out, and used it in preliminary versions, in the end I found simpler proofs without it. Nevertheless I mention the reference in case it may be useful to somebody.

12.3. The area equation for an isosceles tiling with \(\gamma = 2\pi/3\).

Lemma 12.8. Let isosceles triangle \(ABC\) with base angles \(\alpha\) be \(N\)-tiled by a tile with angles \((\alpha, \beta, 2\pi/3)\). Suppose \(\alpha\) is not a rational multiple of \(\pi\). Let \(X\) be the length of the equal sides \(AB\) and \(BC\), and \(Y\) the length of the base \(AC\). Then the area equation is

\[
X^2(2b + a) = Nbc^2
\]

and another form of the area equation is

\[
XY = Nbc
\]

Proof. By the law of cosines,

\[
a^2 = b^2 + c^2 - 2bc \cos \alpha \\
\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc} \\
= \frac{b^2 + (a^2 + b^2 + ab) - a^2}{2bc} \\
= \frac{2b^2 + ab}{2bc} \\
= \frac{2b + a}{2c}
\]

(19)

\[
\cos \alpha = \frac{2b + a}{2c}
\]

Twice the area of \(ABC\) is \(X^2 \sin(\pi - 2\alpha) = X^2 \sin 2\alpha\). Twice the area of the tile is \(bc \sin \alpha\). Equating the area of \(ABC\) to \(N\) times the area of the tile, we have

\[
X^2 \sin 2\alpha = Nbc \sin \alpha \\
2X^2 \sin \alpha \cos \alpha = Nbc \sin \alpha \\
2X^2 \cos \alpha = Nbc
\]

Substituting for \(\cos \alpha\) the value from (19),

\[
2X^2 \left(\frac{2b + a}{2c}\right) = Nbc \\
X^2(2b + a) = Nbc^2
\]
That completes the proof of the first formula of the lemma.

To prove the second form: twice the area of $ABC$ is $XY \sin \alpha$. Twice the area of the tile is $bc \sin \alpha$. Therefore $XY = Nbc$. That completes the proof of the lemma.

12.4. A necessary condition.

Lemma 12.9. Let isosceles triangle $ABC$ with base angles $\alpha$ be $N$-tiled by a tile with angles $(\alpha, \beta, 2\pi/3)$ and sides $(a, b, c)$. Suppose $\alpha$ is not a rational multiple of $\pi$. Then

(i) $2b + a$ divides $N$, and $Nb/(2b + a)$ is a square, say $m^2$.

(ii) The side and base of $ABC$ are given by

\[
X = mc \\
Y = m(2b + a)
\]

Remarks. This lemma gives us an a priori bound on $(a, b, c)$, namely $2N$, since $c^2 = a^2 + b^2 + ab \leq (a + b)^2 \leq (2b + a)^2 \leq (2N)^2$. Also, if $N$ is prime, $N = 2b + a$, and $b = m^2$. It is unknown if this actually can happen.

Proof. Let $X$ be the length of the equal sides $AB$ and $BC$. According to Lemma 12.8,

\[
X^2(2b + a) = Nb^2.
\]

By Lemma 12.6, $a, b,$ and $c$ are pairwise relatively prime. By Lemma 12.7, if $a$ is odd, then $2b + a$ is relatively prime to each of $a, b,$ and $c$. On the other hand, if $a$ is even, then $b$ and $c$ are odd, so $2b + a$ is relatively prime to $c$ and $b$. Thus, whatever the parity of $a, 2b + a$ is relatively prime to $b$ and $c$. Then by the area equation, $2b + a$ divides $N$.

According to the area equation,

\[
\left(\frac{X}{c}\right)^2 = \left(\frac{Nb}{2b + a}\right)
\]

Therefore $Nb/(2b + a)$ is a rational square, and since it is an integer, it is an integer square, say $m^2$. That completes the proof of part (i).

Ad (ii). Since $X/c$ and $m$ are positive and have equal squares, they are equal, so $X = cm$ as claimed. We compute $Y$:

\[
cos \alpha = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{by the law of cosines}
\]

\[
= \frac{2b^2 + ab}{2bc} \quad \text{since } c^2 = a^2 + b^2 + ab
\]

\[
= \frac{2b + a}{2c}
\]

\[
Y = 2X \cos \alpha \quad \text{where } X = |AB|
\]

\[
= \frac{X}{c}(2b + a)
\]

By part (i), $c$ divides $X$, so the right-hand side is an integer. That completes the proof of the lemma.

Example 1. In the tiling whose construction begins with Fig. 19, we have $N = 75140, (a, b, c) = (3, 5, 7)$, so $2b + a = 13$, and $Nb/(2b + a) = 170^2$, so $m = 170, X = mc = 1190,$ and $Y = m(2b + a) = 2210$. 
Example 2. With $N = 33$, $(a, b, c) = (5, 3, 7)$, $2b + a = 11$, $Nb/(2b + a) = 15$, $m = 3$, $X = mc = 21$, and $Y = m(2b + a) = 33$. We think that no such tiling exists, although the present lemma does not rule it out. In principle one can “just check all the possibilities”, but that is easier said than done.

Example 3. With $N = 37$, $(a, b, c) = (5, 16, 19)$, $2b + a = 37$, $Nb/(2b + a) = 16$, $m = 4$, $X = mc = 76$, and $Y = m(2b + a) = 148$. See Fig. 20. We prove in Theorem 12.11 that no such tiling exists. The method of proof does not depend on 37 being prime and does not extend to $N = 71$.

12.5. Ruling out small values of $N$.

**Theorem 12.10.** If there is an $N$-tiling of some isosceles triangle $ABC$ with base angles $\alpha$ by a tile with angles $(\alpha, \beta, 2\pi/3)$, then $N$ is at least 33. If $N \leq 200$ then $N$ is one of the values shown in Table 4, and the side and base of $ABC$ must be as given in the table.

| $N$ | $(a, b, c)$ | $(X, Y)$ |
|-----|-------------|---------|
| 33  | $(5, 3, 7)$  | $(21, 33)$ |
| 37  | $(5, 16, 19)$ | $(76, 148)$ |
| 46  | $(7, 8, 13)$  | $(52, 92)$ |
| 65  | $(3, 5, 7)$   | $(35, 65)$ |
| 71  | $(39, 16, 49)$| $(196, 284)$ |
| 74  | $(56, 9, 61)$ | $(183, 222)$ |
| 130 | $(16, 5, 19)$| $(95, 130)$ |
| 132 | $(5, 3, 7)$   | $(42, 66)$ |
| 148 | $(5, 16, 19)$| $(152, 296)$ |
| 154 | $(8, 7, 13)$  | $(91, 154)$ |
| 184 | $(7, 8, 13)$  | $(104, 184)$ |
| 193 | $(143, 25, 157)$| $(785, 965)$ |

**Remarks.** We do not suggest that tilings for $N$ in the table do, or do not, exist, only that they are not ruled out by the simple considerations of area and boundary tiling.

The prime numbers 37, 71, and 193 are not ruled out immediately, and two of those are congruent to 3 mod 4. Hence the possibility of $N$ prime for this kind of tiling is not ruled out by the area equation and boundary-tiling conditions; but at least the cases 7, 11, 19 are eliminated, which is required for a proof that there are no $N$-tilings of any triangle for those values of $N$.

Actually, we are able to rule out $N = 37$; see Theorem 12.11 below. But the argument is special to $N = 37$, and does not appear to have anything to do with the primality of 37.

**Proof.** Let the positive integer $N$ be given, and suppose there is an $N$-tiling of some isosceles $ABC$ by a tile $(a, \beta, 2\pi/3)$. By Theorem 12.5 the tile is rational, so we may suppose its sides are integers $(a, b, c)$ with no common divisor. According to Lemma 12.9 $2b + a$ divides $N$ (so $a$ and $b$ are at most $N$), and $Nb/(2b + a)$ is a square, say $m^2$. Then, since the tile has a $2\pi/3$ angle, $c$ is determined by the
equation $c^2 = a^2 + b^2 + ab$. If $c$ is not an integer, then we do not consider $(a, b, c)$ further. Also if $(a, b, c)$ is not a triangle, because the sum of two of its sides is less than the third, we do not consider it further. Table 4 was computed by running this algorithm for $N \leq 200$. There are no entries for $N < 33$. That completes the proof of the theorem.

We note that it would be a waste of time to compute the length of the base $Y$ and reject $(a, b, c)$ in case $Y$ is not an integer, because $Y$ always has to be an integer $m(a + 2b)$, by Lemma 12.9. Similarly, it would be a waste of time to look for possible boundary tilings in the hope of rejecting some tiles, since by Lemma 12.9 with $m = X/3$ we always have $X = mc$ and $Y = ma + 2mb$.

**Theorem 12.11.** There is no 37-tiling of an isosceles triangle with base angles $\alpha$, using a tile with $\gamma = 2\pi/3$.

**Proof.** By Theorem 12.10, the tile would have to be $(a, b, c) = (5, 16, 19)$. Then the area equation can be used to show that $(X, Y) = (76, 148)$. That makes the altitude of $ABC$ equal to $10\sqrt{3} = 17.32$. If there is a tiling, there must be a four tiles at $B$, three of which have their $\beta$ angles at $B$, and the other its $\alpha$ angle there. Number those tiles 1 to 4 starting from $AB$ and ending at $BC$. Renaming $A$ and $C$ if necessary, we may assume that the $\alpha$ angle at $B$ belongs to Tile 1 or Tile 2, so Tile 3 and Tile 4 have their $\beta$ angle at $B$. Those two tiles each have a $c$ edge. The case when Tile 4 has its $c$ edge in the interior is shown in Fig. 20. In that position,

![Figure 20. The tile is inside $ABC$, but just barely.](image)

a tile barely fits into triangle $ABC$, and its eastern $b$ edge cannot be matched by another tile’s $b$ edge, for that tile would not be inside $ABC$. Nor can tiles be laid there with $a$ edges; so this case is impossible. Therefore Tile 4 has its $c$ edge on $BC$, and shares its $a$ edge with Tile 3. See Fig. 21.

![Figure 21. No 37-tiling: Tile 4 with its $c$ edge on the boundary.](image)

Tile 3 cannot have its $c$ edge on the west, as a $c$ edge emanating from $B$ at that angle would extend past $AC$. (Its $y$-coordinate, with $AC$ on the $x$-axis, would be $-0.866$.) Therefore it has its $c$ edge on the east, next to the $a$ edge of Tile 4, leaving an impossible situation, as a $b$ or $c$ edge will not fit inside $ABC$ on the east of Tile 3, nor will any number of $a$ edges. That contradiction completes the proof.
Remark. The next case of prime $N$ to consider would be $N = 71$. It does not seem fruitful to continue this game by hand; and in this paper, we abstain from the attempt to establish non-existence results by computer search, because of the difficulty of establishing the correctness of such results beyond a shadow of doubt. It is true that we used a computer in Theorem 12.10 but only in the most trivial way: a doubtful reader could easily replicate Table 4, perhaps even by hand.

12.6. Given $N$, find the possible tiles and $ABC$.

**Theorem 12.12.** Given $N$, we can efficiently compute a finite set $\Delta$ of $(a, b, c, X)$, such that if there is an $N$-tiling of some isosceles triangle $ABC$ with base angles $\alpha$ by a tile with $\gamma = 2\pi/3$, then the tile is $(a, b, c)$ and the side of $ABC$ is $X$, for some $(a, b, c)$ in $\Delta$.

Remark. Then by backtrackng search, applied to each tile $(a, b, c)$ and isosceles triangle $ABC$ with base angles $\alpha$ and side $X$ with $(a, b, c, X)$ in $\Delta$, we can determine (in principle) if any $N$-tiling of any isosceles $ABC$ exists. But we do not undertake that in this paper; see the previous remark.

**Proof.** Let $N$ be given. The algorithm given in the proof of Theorem 12.10 determines the possible tiles $(a, b, c)$, in such a way that $N/(2b+1)$ is a square, say $m^2$. Then $X = mc$ must be the side of triangle $ABC$, if there is any $N$-tiling of isosceles $ABC$ by $(a, b, c)$, and $Y = m(2b+a)$ is the base, by Lemma 12.9. That completes the proof of the theorem.

13. Open problems

The methods and results of this paper leave us still unable to answer some interesting questions. Here we list several. In the following, as elsewhere in this paper, “isosceles” means “isosceles and not equilateral.”

(i) What is the smallest $N$ such that some isosceles triangle with base angles $\alpha$ can be $N$-tiled by a tile of the form $\gamma = 2\alpha$? The smallest such tiling so far explicitly constructed has 1125 tiles, but for all we know there is a 45-tiling. In fact, we do not even know the smallest $N$ such that some isosceles triangle can be tiled by the tile with sides $(4, 5, 6)$ (which is the tile used in the 1125-tiling).

(ii) Is it possible to $N$-tile some isosceles triangle with $N$ a prime number, when the tile has $\gamma = 2\pi/3$? If it is possible, $N$ has to be at least 71. (For right-angled tiles, it is possible when $N$ is congruent to 1 mod 4, but not when $N$ is congruent to 3 mod 4; when $\gamma = 2\alpha$ it is never possible.)

(iii) Find easily checkable necessary and sufficient conditions on $N$ for the existence of $N$-tilings of some isosceles $ABC$ with $\gamma = 2\alpha$ or $\gamma = 2\pi/3$. Or, determine the existence or non-existence of such tilings, one $N$ at a time, using exhaustive computer search. You can use Table 3 for your initial test data.

14. Conclusions

We have studied the possible tilings of an isosceles (and not equilateral) triangle $ABC$ by a tile that is a right triangle, or by a tile of the form $(\alpha, \beta, 2\alpha)$ where the base angles of $ABC$ are equal to $\alpha$. In the case of a tile $(\alpha, \beta, 2\alpha)$, we derived a necessary condition from the area equation, and we made use of directed graphs inspired by Laczkovich to prove that the tile is necessarily rational.
We analyzed the case of a right-angled tile thoroughly enough to give a complete characterization of the possible values of \( N \) for which some isosceles \( ABC \) can be \( N \)-tiled. Namely, Theorem 7.8 says \( N \) is twice a square or twice an even sum of squares, except of course for the right isosceles triangle, which can be quadratically \( N \)-tiled for any square \( N \), including odd squares.

In the case of a tile with \( \gamma = 2\alpha \), we gave a necessary condition, using the area equation and the law of cosines for the tile. That this necessary condition is not trivial is shown by our proof that \( N \) cannot be prime. \( N = 45 \) is the least number for which we do not know whether a tiling exists, and 1125 is the smallest \( N \) for which we are certain that there does exist a tiling.

Finally, in the case of a tile with \( \gamma = 2\pi/3 \), we gave a necessary condition and an algorithm to check it. There are no such tilings for \( N < 33 \). There is one for \( N = 75140 \). Between those two values of \( N \), there are many values of \( N \) satisfying our necessary conditions, for which we do not know whether tilings exist.

In all the possible cases of Laczkovich’s tables, we have been able to show (either in this paper or in unpublished work) that given \( N \), there is a finite set \( \Delta \) of tiles \((a, b, c)\) and triangles \( ABC \) such that either there is no \( N \)-tiling falling under that line of the table, or one of the finite set permits an \( N \)-tiling. Hence, there is (in principle) an algorithm, albeit inefficient, for determining if there is an \( N \)-tiling. The inefficiency arises from the exponentially large number of ways of trying to place \( N \) tiles of a specific shape into a specific \( ABC \).

See the previous section for a list of open problems.

15. Appendix: Tilings found by Bryce Herdt

In 2024, Bryce Herdt found several new tilings with tiles \((4,5,6)\) (for which \( \gamma = 2\alpha \)) and \((3,5,7)\) (for which \( \gamma = 2\pi/3 \)). These tilings dramatically lowered the \( N \) for the “smallest known tiling”, both of isosceles triangles and of equilateral triangles. Here we exhibit Herdt’s tilings.

Consider Fig. 11. The calculations after that figure give the number of tiles required on each edge in that figure. All those numbers are divisible by 6, except \( m = 869 \). Therefore, if we shrink the diagram by a (linear) factor of 6, we can still tile all the triangles in the figure with \((4,5,6)\), and the yellow parallelogram will be 869 by 480 (since it was 576 \cdot 5, after shrinking it is 576 \cdot 5/6 = 480). That 869 by 480 parallelogram cannot be tiled with tiles all in the same orientation. But, as Herdt pointed out to me (in 2024), often a parallelogram can be divided into two smaller parallelograms, which can be tiled with tiles in different orientations. Fig. 22 illustrates the technique. In the case at hand, 869 = 425+444 = 85b+74c, while 480 is divisible by \( b \) and \( c \) (5 and 6). The final value of \( N \) will be 6028020/6^2 = 167445. That is, unfortunately, still too large to draw in a space smaller than several meters.

Figure 22. Decomposing a parallelogram with top \( pc + qb \) and side \( bc \).
Starting from a different dissection of the triangle into triangles and parallelograms, Herdt was able to construct the 1125-tiling shown in Fig. 23. Observe how the parallelogram-dissection technique has been applied twice, in both the lower left and the lower right corners of the tiling.

**Figure 23.** Herdt’s 1125-tiling by (4, 5, 6)

Dissecting $ABC$ into similar triangles and parallelograms in a different way, Herdt was able to reduce $N$ still further, to 720, as shown in Fig. 24. 

**Figure 24.** Herdt’s 720-tiling by (4, 5, 6)

Herdt also found new tilings with the tile (3, 5, 7), which has $\gamma = 2\pi/3$. He began by finding a 1215-tiling of an equilateral triangle of side 135; this was also done by his technique of decomposing parallelograms, starting with a known 10935-tiling. Then he flanked that equilateral triangle by two triangles similar to (3, 5, 7), producing the tiling shown in Fig. 25. Here $N = 1215 + 2 \cdot 27^2 = 2673$. This is presently the smallest known tiling of an isosceles triangle by a tile with $\gamma = 2\pi/3$. 

**Figure 25.** Herdt’s 2673-tiling by (3, 5, 7)
Figure 25. Herdt’s 2673-tiling by $(3, 5, 7)$. Here $\gamma = 2\pi/3$.

References

[1] Michael Beeson. No triangle can be cut into seven congruent triangles. 2018. Available on ArXiv and the author’s website.

[2] Steve Butler, Fan R. K. Chung, Ronald L. Graham, and Miklós Laczkovich. Tiling Polygons with Lattice Triangles. Discrete and Computational Geometry, 44:896–903, 2010.

[3] Henri Cohen. Number Theory, Volume I: Tools and Diophantine Equations. Springer, 2007.

[4] Solomon W. Golomb. Replicating figures in the plane. The Mathematical Gazette, 48:403–412, 1964.

[5] G. H. Hardy and E. M. Wright. The Theory of Numbers. Clarendon Press, fourth edition, 1960.

[6] M. Laczkovich. Tilings of polygons with similar triangles. Combinatorica, 10:281–306, 1990.

[7] M. Laczkovich. Tilings of triangles. Discrete Mathematics, 140:79–94, 1995.

[8] M. Laczkovich. Tilings of Polygons with Similar Triangles, II. Discrete and Computational Geometry, 19:411–425, 1998.

[9] M. Laczkovich and G. Szekeres. Tilings of the Square with Similar Rectangles. Discrete and Computational Geometry, 13:569–572, 1995.

[10] Miklós Laczkovich. Tilings of convex polygons with congruent triangles. Discrete and Computational Geometry, 38:330–372, 2012.

[11] R. S. Luthar. Integer-sided triangles with one angle twice another. College Mathematics Journal, 15(1):55–56, January 1984.

[12] Ivan Niven. Irrational Numbers. Number 11 in Carus Mathematical Monographs. Mathematical Association of America, 1967.

[13] Stephen L. Snover, Charles Waiveris, and John K. Williams. Rep-tiling for triangles. Discrete Mathematics, 91:193–200, 1991.

[14] Alexander Soifer. How Does One Cut a Triangle? Springer, 2009.

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