An Optimal Measurement Strategy to Beat the Quantum Uncertainty in Correlated System

Jun-Li Li and Cong-Feng Qiao*

Uncertainty principle is an inherent nature of quantum system that undermines the precise measurement of incompatible observables and hence the applications of quantum theory. Entanglement, another unique feature of quantum physics, may help to reduce the quantum uncertainty. In this paper, a practical method is proposed to reduce the one party measurement uncertainty by determining the measurement on the other party of an entangled bipartite system. In light of this method, a family of conditional majorization uncertainty relations in the presence of quantum memory is constructed, which is applicable to arbitrary number of observables. The new family of uncertainty relations implies sophisticated structures of quantum uncertainty and non-locality, that are usually studied by using scalar measures. Applications to reduce the local uncertainty and to witness quantum non-locality are also presented.

1. Introduction

One of the distinct features of quantum mechanics is its inherent limit on the joint measurement precisions of incompatible observables, known as the uncertainty relation. The most representative uncertainty relation writes\(^{(1)}\)

\[
\Delta X \Delta Y \geq \frac{1}{2} |\langle [X, Y] \rangle|
\]

(1)

Here the product of the uncertainties of two observables \(X\) and \(Y\) is lower bounded by the expectation value of their commutator. This lower bound is state dependent and could be null which trivializes the uncertainty relation.\(^{(2)}\) The entropic form of uncertainty relations may avoid such problem, which was developed with the state independent lower bounds.\(^{(2-4)}\) A typical one of them, the Maassen and Uffink (MU) form\(^{(4)}\), goes as

\[
H(X) + H(Y) \geq \log \frac{1}{c}
\]

(2)

Here \(H(X) = - \sum_i p_i \log p_i\) denotes the Shannon entropy, \(p_i\) represents the probability of obtaining \(x_i\) while measuring \(X\), and similarly for observable \(Y\). The logarithm is assumed to have base number \(c\) if not further specified, and the symbol \(c \equiv \max_i |\langle x_i | y_j \rangle|^2\) quantifies the maximal overlap of eigenvectors \(\langle x_i |\) and \(| y_j \rangle\) of \(X\) and \(Y\), respectively. Though great efforts have been made, to find out the optimal lower bound for entropic type uncertainty relation remains a challenging task.\(^{(5)}\) It is remarkable that according to a recent study the variance and entropy types of uncertainty relations in fact are mutually equivalent.\(^{(6)}\)

Recently, people found that though uncertainty is an inherent nature of quantum physics, it is beatable in the presence of quantum memory\(^{(7)}\). In such a situation, the uncertainty relation takes the following form

\[
S(X|B) + S(Y|B) \geq \log \frac{1}{c} + S(A|B)
\]

(3)

Here \(S(X|B)\) stands for conditional von Neumann entropy, representing the uncertainty in the measurement of \(X\) on Alice(A) side, given information is stored as quantum memory on Bob(B). The term \(S(A|B)\) on the right-hand side of Equation (3) is supposed to signify the influence of entanglement on the uncertainty relation, but actually it has no business with the local measurement uncertainties, namely, \(H(X)\) or \(H(Y)\). In the literature, though a lot of effort has been made,\(^{(8-9)}\) Equation (3) is still subject to the absence of optimal lower bound in entropic uncertainty relation.\(^{(10)}\)

Contrary to the variance and entropy, a vectorized measure of uncertainty was introduced in ref.\(^{(11)}\), where the majorization employed helps to improve the entropic uncertainty relation\(^{(12)}\) and leads to a universal uncertainty relation.\(^{(13)}\) A typical majorization uncertainty relation in direct sum form goes as\(^{(14)}\)

\[
\bar{p}(x) \oplus \bar{p}(y) < \bar{s}
\]

(4)

where \(\bar{p}()\) signifies the probability distribution of the measurement outcomes. The majorization relation \(\bar{a} < \bar{b}\) is defined as \(\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i\) for two vectors with components in descending order and the equality holds for \(k = N\). Unlike variance and entropy, the upper bound \(\bar{s}\) in uncertainty relation (4) is unique and optimal which can be easily determined via the...
theory of majorization lattice. Whereas, it was proved that this limit may be violated in the situation with entangled bipartite state.\cite{13}

For a bipartite system, it is interesting to know what measurement strategy $B$ takes that can minimize the measurement uncertainty on $A$’s side. Such an optimal measurement strategy would build a maximal correlation between measurements, from which a substantial reduction on local uncertainty can be deduced. In addition to reducing the local uncertainty, one can imagine that the correlation maximality would have certain effects on other quantum information processing scenarios. For example, the Bell non-locality, quantum steering,\cite{16} and non-separability may be somehow distinguished by stronger correlations reached by optimal joint measurements in bipartite system. In statistical inference, a typical problem is to identify the extremal joint distribution that maximizes the correlation for given marginals,\cite{17} which is closely related to infer a maximal correlation for given joint measurements.

In this paper, we propose a practical method to reduce the local measurement uncertainty in the presence of quantum memory. That is, given the bipartite quantum state $\rho_{AB}$ and measurement $X$ on $A$, how to perform the measurement $X'$ on $B$ to reduce the uncertainty in measuring $X$ on $A$. By virtue of the lattice theory, an optimal measurement strategy will be established. To this aim, we formulate a family of conditional majorization uncertainty relations (CMURs) in the presence of quantum memory. The CMUR is applicable to multiple, including infinitely many, observables, which enables us to study the uncertainty relation and non-locality with infinite number of measurement settings. This is a hurdle hard to surmount in other formalisms, that is, variance or entropic uncertainty relations are difficult to adapt to infinitely many observables.

2. Optimal Measurement Strategy and Applications

2.1. Optimal Measurements to Reduce the Local Uncertainty

In a bipartite system $\rho_{AB}$, the reduced density matrix $\rho_A = \text{Tr}_B(\rho_{AB})$ describes the local system $A$. The diagonal elements $\rho_{A,i} = \sum_j \rho_{ij}$ give the probability distribution of the measurements on $X$, where the unitary matrix $u_x = (|x_1\rangle, \ldots, |x_N\rangle)$ is composed of the eigenvectors of $X$. A measurement on system $B$, that is, $X'$ of dimension $N$, may also be performed, and hence results in the joint distribution

$$P(X, X') = (\tilde{p}^{(1)}(x), \tilde{p}^{(2)}(x), \ldots, \tilde{p}^{(N)}(x))$$

(5)

Here $\tilde{p}^{(i)}(x)$ represents the $i$th distribution vector of the measurements on $X$ conditioned to be $x'_{i}$. From the joint distribution $P(X, X')$, the $X$’s marginal distribution is $\tilde{p}(x) = \sum_{i=1}^{N} \tilde{p}^{(i)}(x)$. We define the vectorized measure of uncertainty for $X$ conditioned on the knowledge of $X’$ as

$$\tilde{p}(x|x’) ≡ \tilde{p}^{(1)}(x) + \tilde{p}^{(2)}(x) + \ldots + \tilde{p}^{(N)}(x)$$

(6)

where the superscript $\dagger$ indicates that the components of the summand vectors are arranged in descending order. The conditional probability distribution $\tilde{p}(x|x’)$ may be called the majorized marginal distribution. For two joint distributions $P(X, X')$ and $P(X, X')'$, $\tilde{p}(x|x’) < \tilde{p}(x|x’)$ means that $X$ has less uncertainty conditioned on the information of $X’$ than that conditioned on $X’'$. With the definition of majorized marginal distribution, one may figure out that, (1) $\tilde{p}(x|x’)$ is always less uncertain than the ordinary marginal $\tilde{p}(x)$, that is, $\tilde{p}(x) < \tilde{p}(x|x’)$; (2) The joint distribution of independent observables $X$ and $X’$ gives $\tilde{p}(x) = \tilde{p}(x|x’)$.

For all possible measurements on $X’$, the majorized marginal distributions $\tilde{p}(x|x’)$ have the following property:

Proposition 1. Given $\rho_{AB}$ and the measurement $X$ on $A$, there exists a unique least upper bound for $\tilde{p}(x|x’)$, that is,

$$\forall x’, \quad \tilde{p}(x|x’) < 3^{(i)}$$

Here $3^{(i)} = \tilde{p}(x|x’_i) \lor \tilde{p}(x|x’_j) \lor \ldots \lor \tilde{p}(x|x’_n)$ depends only on $X$ and $\rho_{AB}$. The majorized marginal $\tilde{p}(x|x’)$ has the largest sum of first $k$ components, that is, $\sum_{i=1}^{k} p_i(x|x’) = \max\{\sum_{i=1}^{k} p_i(x|x’)|\}$. Majorization relation (7) holds due to the fact that there exists a least upper bound for the join operator “∨” of a majorization lattice.\cite{14} The unique least upper bound $3^{(i)}$ is determined by the measurement set $\{X’_k = 1, \ldots, N\}$, in which each $X’_k$ gives the majorized marginal distribution of $\tilde{p}(x|x’)$ defined in Proposition 1 (see Supporting Information A). We call the measurement set $\{X’_k\}$ the optimal measurement strategy for $B$ to reduce the uncertainty of $X$. In the following, we present several typical applications based on this measurement strategy.

2.2. The Conditional Majorization Uncertainty Relation

The uncertainty relations behave as the constraints on the probability distributions of two or more incompatible measurements. Variance and entropy are scalar measures of the distribution uncertainty (disorder or randomness in the language of statistics), while the majorization relation provides a lattice structured uncertainty measure.\cite{14} For bipartite system, the measurement uncertainty of one party may be reduced based on its entangled partner’s side information. Considering Equation (7) we have the following Corollary.

Corollary 1. For two measurements $X$ and $Y$ on $A$, we have a family of CMURs

$$\tilde{p}(x|x’) \ast \tilde{p}(y|y’) < 3^{(r)}$$

(8)

Here $X’$ and $Y’$ are measurements on $B$, and $3^{(r)} = 3^{(i)} \ast 3^{(i)}$ with $\{\ast\}$ being a set of binary operations that preserve the majorization which include direct sum, direct product, and vector sum, that is, $\{\oplus, \otimes, \ast\} \subset \{\ast\}$. The proof of Corollary 1 is deferred to the Supporting Information B for simplicity. In fact, relation (8) is not restricted to observable pairs. Arbitrary number of observables apply here by the multiple application of the operations in $\{\ast\}$. An alternative definition of conditional majorization can be found in ref. [18], where convex functions are used explicitly for the characterization. Next,
we show how the CMUR presented here affects quantum measurements, namely, on the uncertainty relation with entangled quanta and the witness of quantum steering.

2.2.1. Break the Quantum Constraint

Considering the operation of direct sum in Corollary 1, we have

$$\bar{p}(x|x') \oplus \bar{p}(y|y') < 3^{(0)} = \bar{3}^{(v)} \oplus \bar{3}^{(i)}$$

(9)

This provides an upper bound for incompatible measurements $X$ and $Y$ on $A$ in the presence of quantum memory. We know for single particle the optimal upper bound for the direct sum majorization uncertainty relation of $X$ and $Y$ is \[14\]

$$\bar{p}(x) \oplus \bar{p}(y) < \bar{3}$$

(10)

where \(\bar{3}\) depends only on the local observables. To compare these two bounds, we employ the bipartite qubit state

$$|\psi_\xi\rangle = \cos \xi |00\rangle + \sin \xi |11\rangle, \ \xi \in [0, \pi/4]$$

(11)

and local observable

$$\sigma_n = \sigma (\theta, \phi) = \sigma_x \cos \theta + \sigma_y \sin \theta \cos \phi + \sigma_z \sin \theta \sin \phi$$

(12)

According to Proposition 1, for measurement \(\sigma_n\) applying on $A$, the optimal measurement \(\sigma_{n'} = \sigma (\theta', \phi')\) performed by $B$ is achieved by performance (see Supporting Information A for details)

$$\tan \theta' = \tan \theta \sin (2\xi), \phi' = \phi,$$

(13)

and the corresponding optimal upper limit for the majorized marginal distribution is

$$\bar{p}(\sigma_n | \sigma_{n'}) < \bar{3}^{(s)} = \left( \frac{1}{2} + \frac{1}{2} \sqrt{\cos^2 \theta + \sin^2 \theta \sin (2\xi)} \right)$$

(14)

Taking $X = \sigma (\theta, 0)$ and $Y = \sigma (\theta, \pi)$ with commutator $[X, Y] = i \sigma_x \sin (2\theta)$, the upper bound \(\bar{3}^{(s)}\) in relation (9) can be constructed via (14). The relationship between \(\bar{3}^{(s)}\) (with the presence of quantum memory) and \(\bar{3}\) (single particle state) for different degrees of entanglement is illustrated in Figure 1 in Lorenz curves. Two distributions satisfy \(\bar{3}^{(s)} < \bar{3}\), if and only if the Lorenz curve of \(\bar{3}^{(s)}\) is everywhere below that of \(\bar{3}\). The state \(|\psi_\xi\rangle\) is entangled when \(\xi > 0\) and reaches the maximal entanglement at \(\xi = \pi/4\). In the whole range of \(\xi \in (0, \pi/4]\), the conditional uncertainty relation (9) has the upper bound \(\bar{3}^{(s)} < \bar{3}\), see Figure 1. That is, the quantum limit is broken in the presence of entanglement.

2.2.2. Compare to the Conditional Entropic Uncertainty Relation

To compare with the existing result on the entropic uncertainty relation in the presence of quantum memory, relation (3), we transform the CMUR (8) into the entropic form. In practice this is quite straightforward by applying an arbitrary Schur’s concave function to \(\bar{p}(x|x') < \bar{3}^{(s)}\) and \(\bar{p}(y|y') < \bar{3}^{(s)}\). For the Shannon entropy $H(\cdot)$ on the direct product form of Equation (8), we have

$$H(\bar{p}(x|x')) + H(\bar{p}(y|y')) \geq H(\bar{3}^{(s)})$$

(15)

Here $\bar{3}^{(s)} = \bar{3}^{(v)} \otimes \bar{3}^{(i)}$. Figure 2 exhibits the behaviors of the lower bounds in Equations (3) and (15) with the quantum state $|\psi_\xi\rangle$ being the form of (11) and observables $X = \sigma (\theta, 0)$ and $Y = \sigma (\theta, \pi)$. The uncertainties of $X$ and $Y$ on $A$’s side are evaluated for the reduced density matrix $\rho_A = \operatorname{Tr}_B [\rho_A |\psi_\xi\rangle \langle \psi_\xi|]$. Note, $H(X) + H(Y)$ for $\rho_A$ is always greater than zero. However, when beating the uncertainties with quantum memory of $|\psi_\xi\rangle$ based on Equation (3), the right-hand side of Equation (3) is log \(\frac{1}{2} + \operatorname{Tr}[\rho_A \log \rho_A]\) with $\rho_B = \operatorname{Tr}_A [\rho_A |\psi_\xi\rangle \langle \psi_\xi|]$ being the reduced density matrix for $B$. The lower bounds are mostly negative (dot-dashed lines in Figure 2).
Figure 2. The decrease of the lower bound for the entropic uncertainty relation with quantum memory. The uppermost dashed lines give the values of $H(X) + H(Y)$ for $\rho_A = \text{Tr}_{\mathcal{B}}[\rho_{\mathcal{AB}}]$. The solid lines represent the reduced lower bounds of $H(\vec{s}^{(0)})$ under the optimal measurement strategy, and the dot-dashed lines represent the lower bounds of $\log \frac{\xi}{\pi} + S(A|B)$. The lower bounds are plotted for $\xi = \frac{\pi}{4}, \frac{\pi}{8}, 0$ in different colors which correspond to different degrees of entanglement of $|\psi\rangle$. The lower bounds $H(\vec{s})$ are greater (thus tighter) than $\log \frac{\xi}{\pi} + S(A|B)$ for each value of $\xi$, respectively. Note, a large portion of the dot-dashed lines are negative and hence meaningless.

for the parameters $\xi \in [0, \pi/4]$ in this case. In comparison, the reduced lower bounds for local uncertainties in Equation (15) are realistic and physically reachable (the solid lines in Figure 2).

2.3. Uncertainty Relation with Infinite Number of Observables

Corollary 1 can be easily generalized to arbitrary number of observables. Taking the vector sum operation as an example, for $M$ observables $X_i$ on $A$ we have

$$\sum_{i=1}^{M} \bar{p}(x_i|x'_i) < \bar{s}^{(+)\text{}}$$

(16)

where $\bar{s}^{(+)\text{}} = \sum_{i=1}^{M} \bar{s}^{(+)}(i)$. Relation (16) represents a conditional uncertainty relation in the presence of quantum memory for $M$ incompatible observables. For single particle states, there is

$$\bigoplus_{i=1}^{M} \bar{p}(x_i) < \bar{\bar{s}}$$

(17)

Here $\bar{\bar{s}}$ is a real vector of dimension $NM$ with components arranged in descending order. By partitioning the vector $\bar{s}$ into $N$ disjoint sections as $\bar{s} = (s_1, \ldots, s_M; s_{M+1}, \ldots, s_{2M}; \ldots)$, we may get an $N$-dimensional vector $\bar{\bar{s}}$ with the components of $\bar{\bar{s}} = \sum_{i=1}^{M} s_{(i-1)M+1}$. Then relation (17) leads to

$$\sum_{i=1}^{M} \bar{p}(x_i) < \bar{\bar{s}}$$

(18)

where the vector $\bar{\bar{s}}$ is the aggregation of $\bar{s}$.

Relation (18) represents the quantum prediction of relation (16) but without quantum memory.

The violation of the relation $\bar{s}^{(+\text{})} < \bar{\bar{s}}$ implies a violation of relation (17), which then implies the quantum steering in a bipartite state. The correlation matrix of a bipartite state $\rho_{\mathcal{AB}}$ is defined as $T_{ij} = \text{Tr}[\rho_{\mathcal{AB}} \sigma_i \sigma_j]$, where $\sigma_i$ are three Pauli operators. According to the CMUR (16), we have

**Corollary 2.** If a bipartite qubit state $\rho_{\mathcal{AB}}$ is non-steerable, then

$$R_C(\tau_1^2, \tau_2^2, \tau_3^2) \leq \frac{1}{2}$$

(19)

Here $R_C$ signifies certain elliptic integral and $\tau_1 \geq \tau_2 \geq \tau_3$ are singular values of the correlation matrix $T$ of $\rho_{\mathcal{AB}}$.

The definition of $R_C$ can be found in ref. [20] (No.19.16.3 DLMF of NIST and is also presented in Supporting Information C).
Figure 3. The criterion of steerability for two, three, and infinite number of projective measurements. For state $\rho_\xi$, if $A$ cannot steer $B$, then we have: $p \leq (1 + \sin^2(2\xi))^{-1/2}$ for two-measurement settings; $p \leq (1 + 2\sin^2(2\xi))^{-1/2}$ for three-measurement settings; $R_G(\tau_1^2, \tau_2^2, \tau_1^2) \leq 1/2$ for infinite-measurement settings.

We consider a typical mixed and entangled state $\rho_\xi$:

$$\rho_\xi = \frac{1 - p}{2} \rho_A^\perp \otimes 1 + p |\psi_\xi\rangle\langle \psi_\xi|, \quad p \in [0, 1]. \quad (20)$$

where $\rho_A^\perp = \text{Tr}_B[|\psi_\xi\rangle\langle \psi_\xi|]$ and $B$ can steer $A$ whenever $p > 1/2$. For measurement $X = \sigma(\theta, \phi)$ on $A$, Proposition 1 tells that the optimal measurement $X' = \sigma(\theta', \phi')$ for $B$ to reduce the local uncertainty of $X$ is

$$\begin{cases} 
\text{if } \cos \theta \in [0, \frac{p \tan(2\theta)}{(1-p)^2}] & \text{then } \tan \theta' = \tan \theta \sin(2\xi) \text{ and } \phi' = \phi \\
\text{if } \cos \theta \in (\frac{p \tan(2\theta)}{(1-p)^2}, 1] & \text{then } \theta', \phi' \text{ can be arbitrary}
\end{cases} \quad (21)$$

On the other hand, for measurement $X' = \sigma(\theta', \phi')$ on $B$, the optimal measurement $X = \sigma(\theta, \phi)$ of $A$ to reduce the uncertainty of $X'$ is (see Supporting Information D)

$$\begin{cases} 
\text{if } \cos \theta' \in [0, 1] & \text{then } \tan \theta = \tan \theta' \sin(2\xi) \text{ and } \phi = \phi'
\end{cases} \quad (22)$$

Evidently, the bipartite state $\rho_\xi$ is asymmetric from the optimal measurement point of view, though Corollary 2 is insensitive to this asymmetry.

It is remarkable that the steerability is greatly improved in the case of infinite number of measurements, which is illustrated in Figure 3. For state $\rho_\xi$, with infinite number of measurements Corollary 2 tells that if $A$ cannot steer $B$, the parameters $\xi - p$ will fall into the region of $R_G(\tau_1^2, \tau_2^2, \tau_1^2) \leq 1/2$ in Figure 3, where $\tau_i$ are the singular values of the correlation matrix of $\rho_\xi$. And in two- and three-measurement settings,$^{[21]}$ $A$ cannot steer $B$ predicts a much larger region than that of infinite-measurement setting. The mismatch of those areas indicates that although the state does not show steerability in less-measurement settings, it turns out to be steerable when more measurements are taken.

3. Conclusion

In conclusion, by virtue of the majorization lattice, we developed a systematic procedure to reduce the local uncertainties via entanglement. On account of this scheme, a practical measurement strategy was proposed and also a new class of CMURs was constructed, which is applicable to any number of observables. In the presence quantum memory, it is found that the CMUR may break the quantum constraints on the measurement uncertainties of individual quanta. That is, a substantial reduction of local uncertainty on one partite can be fulfilled by performing some designated measurements on its entangled partner. Comparing to the conditional entropic uncertainty relation, the CMUR in its entropic forms may give out reduced and physically nontrivial lower bounds. Moreover, the CMUR can be employed to witness the steerability of bipartite states with infinite-measurement settings, to which the usual uncertainty relations, like variance or entropy type of uncertainty relation, are inapplicable. Last, the measure of uncertainty we constructed is a novel formalism in
materializing the uncertainty principle of quantum theory, we believe its application to quantum processing deserves further investigation.

Supporting Information
Supporting Information is available from the Wiley Online Library or from the author.

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Conflict of Interest
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