Free Groups in Quaternion Algebras

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Abstract. In [6] we constructed pairs of units \( u, v \) in a \( \mathbb{Z} \)-order of a quaternion algebra over imaginary rational extensions \( \mathbb{Q}(\sqrt{-d}) \), where \( d \equiv 7 \pmod{8} \) is positive and square free integer, such that the group generated by \( u^n, v^n \) is a free group for some positive integer \( n \). Here we extend this result to any imaginary quadratic extension of \( \mathbb{Q} \), thus including matrix algebras. We show that \( \langle u^n, v^n \rangle \) is a free group for all non-zero integers \( n \), except when \( d = 2 \) yielding \( n > 1 \). The units we use in our construction belong to a new class of units, coming from Pell’s and Gauss’ equations. Also, we establish a criterion for a pair of elements to generate a free semigroup.

Introduction

Constructions of free groups is a nontrivial problem and is a main research topic. Let \( R \) be a ring with unity and \( G \) a group, denote by \( RG \) the set \( \{r_1 g_1 + \cdots + r_n g_n, r_i \in R, g_i \in G, 0 < i \leq n, n \in \mathbb{N} \} \) of formal sums in \( RG \). Clearly, \( RG \) is a unitary ring. Denote by \( U(RG) \) the group of units of the ring \( RG \) and \( U_1(RG) \) the subset of \( U(RG) \), such that, if \( u = u_1 g_1 + \cdots + u_n g_n \) is in \( U_1(RG) \), then \( u_1 + \cdots + u_n = 1 \). The set \( \{u_1, \cdots, u_n\} = : \text{supp}(u) \) is the support of \( u \in U(RG) \). In [5], Higman shows that if \( A \) is a finite abelian group, then \( U_1(\mathbb{Z}A) = A \times F \), where \( F \) is a finitely generated free abelian group. In [4], it is proved that if \( G \) is a finite non-Hamiltonian 2-group, then \( U(\mathbb{Z}G) \) contains a free group. On the other hand, a result of Tits, [11], gives necessary conditions for a pair of diagonalizable elements of \( \text{GL}(2, \mathbb{C}) \) to generate a free group. In [9], assuming the existence of nontrivial units in \( \mathbb{Z}G \) defined by bicyclic units, free groups are constructed in \( U(\mathbb{Z}G) \). In [7], pairs of units are constructed generating a free group in quaternion algebras. In [8], the same was done in finite dimensional crossed products.

In [6], finite groups \( G \) and rings \( R \) of characteristic zero, with \( U(RG) \) a hyperbolic group, were classified. Units, arising from solutions of Pell’s and Gauss’ equations, were also constructed in the algebras \( \mathbb{Q}(\sqrt{-d}) \), where \( R = \mathbb{Q}(\sqrt{-d}) \) is the ring of algebraic integers of quadratic extensions of \( \mathbb{Q} \). We coined the names Pell and Gauss units. In [6], we also use a Gauss unit of norm \(-1\) and the deep results of [3], to give a full set of generators of the unit group of \( \mathbb{Z}[\frac{1+i\sqrt{-d}}{2}]K_8 \), where \( K_8 \) is the quaternion group of order 8. Using a result of Gromov, we show that this unit group...
is a hyperbolic group with one end and whose hyperbolic boundary is the 2-dimensional euclidean sphere. From this and \([2\) Proposition III.Γ.3.20], it follows that if \(d \equiv 7 \pmod{8}\) is a positive integer, and \(u, v \in U(\{\frac{-1}{d}\})\) are Pell units with distinct supports then \((u^n, v^n)\) is a free group for a suitable positive \(n\).

The problem on determining the power \(n\), is a non-trivial task. In general, as in \([2\) Proposition III.Γ.3.20], we have results about the existence of the power \(n\). In the section 2, we construct units with suitable algebraic properties. With a pair of these units, we can use the Ping-Pong Lemma, in a non-standart way, to determine the power \(n\), hence generalizing \([6\) Theorem 5.5].

Here, we construct free groups of rank 2 in \(\mathbb{Z}\)-orders of quaternion algebras over all imaginary quadratic extensions of the rational number field, in particular those which are division rings. The pair of 2-Pell units we use is such that we can neither apply Proposition 3.20 of \([2\) nor Tits’ result because the norm of the eigenvalues of the units, when considered as matrices, is equal to one. We finish the paper with a result which gives a criterium for two elements to generate a free semigroup. We show how it is applied in our case.

1. Preliminaries

We denote by \(\left(\frac{-1}{K}\right) := K[i, j : i^2 = -1, j^2 = -1, -ij = \frac{-1}{K}]\) the quaternion algebra over \(K\). Let \(K\) be an algebraic number field and \(\mathfrak{o}_K\) its ring of integers, denote by \(\left(\frac{-1}{K}\right)\), the \(\mathfrak{o}_K\)-span of \(\{1, i, j, k\}\), which is an \(\mathfrak{o}_K\)-algebra. Clearly, since \(\mathfrak{o}_K\) is a \(\mathbb{Z}\)-order of \(K\), the algebra \(\left(\frac{-1}{\mathfrak{o}_K}\right)\) is a \(\mathbb{Z}\)-order of \(\left(\frac{-1}{K}\right)\).

In \([11\) it is shown that \((z + 2, \frac{z + 2}{2} + 1)\) is a free group of the group \(M\). This is a trivial application of the Ping-Pong Lemma, \([2\) Lemma 3.19\)], which states that if \(h_1, \cdots, h_r\) are bijections of a set \(\Omega\) and there exist non-empty disjoint subsets \(A_{i,1}, A_{i,1} \cdots, A_{r,1}, A_{r,1} \subset \Omega\) such that \(h^ε_i(\Omega \setminus A_{i,ε}) \subset A_{i,-ε}\), for \(ε \in \{-1, 1\}\) and \(i = 1, \cdots, r\), then \(\langle h_1, \cdots, h_r\rangle\) generates a free subgroup of rank \(r\) in \(\text{Perm}(\Omega)\). In fact, \(h_1(z) := \frac{z}{2z + 1}\) and \(h_2(z) := z + 2\) are bijections of \(\Omega := \mathbb{R} \cup \{-\infty, \infty\}\). It is easily verified that the sets \(A_{0,1} := [-1, 0], A_{1,-1} := [0, 1], A_{2,1} := [-\infty, -1]\), and \(A_{2,-1} := [1, \infty]\) are in the conditions of the Ping-Pong Lemma.
Theorem 1.1. The group generated by the units \( u = \sqrt{-1} + (\overline{\sqrt{-1}} + i)j \) and \( w = \sqrt{-1} + (\overline{\sqrt{-1}} + i)j \) in \( \mathbb{Z}[\sqrt{-1}] \) is free.

Proof. Consider the images of \( u \) and \( w \) in \( M_2(C) \) which are \( \Psi(u) = \begin{pmatrix} \sqrt{-1} & 0 \\ 2\sqrt{-1} & \sqrt{-1} \end{pmatrix} \) and \( \Psi(w) = \begin{pmatrix} \sqrt{-1} & 2\sqrt{-1} \\ 2\sqrt{-1} & \sqrt{-1} \end{pmatrix} \). As Möbius transformations they generates the group \( \langle \frac{z}{2z+1}, z+2 \rangle \) which is free of rank two.

\[ \Box \]

2. Free Groups in Quaternion Algebras

In the sequel, \( K = \mathbb{Q}(\sqrt{-d}) \) is an imaginary quadratic extension with \( d \) positive and square-free integer. Let \( \xi \neq \psi \) be elements of \( \{1, i, j, k\} \). Suppose

\[ (2.1) \quad u := m\sqrt{-d}\xi + p\psi, \text{ where } p, m \in \mathbb{Z}, \]

is an element in \( \mathbb{Q}(\sqrt{-d}) \) having norm 1. Then

\[ (2.2) \quad p^2 - m^2d = 1, \]

i.e., the pair \((p, m)\) is a solution of Pell’s equation \( x^2 - dy^2 = 1 \) in \( \mathbb{Q}(\sqrt{d}) \). Equation \((2.2)\) implies that \( \epsilon = p + m\sqrt{d} \) is a unit in \( \mathbb{O}_Q(\sqrt{d}) \). Conversely, if \( \epsilon = x + y\sqrt{d} \) is a unit of norm 1 in \( \mathbb{O}_Q(\sqrt{d}) \), then, necessarily, \( x^2 - y^2d = 1 \), and, therefore, for any choice of \( \xi, \psi \) in \( \{1, i, j, k\} \), \( \xi \neq \psi \),

\[ (2.3) \quad y\sqrt{-d}\xi + x\psi \]

is a unit in \( \mathbb{Q}(\sqrt{-d}) \). In particular,

\[ (2.4) \quad u(\epsilon, \psi) := x + y\sqrt{-d}\psi, \quad \psi \in \{i, j, k\}. \]

Thus \( u(\epsilon, \psi) \) is a unit in \( \mathbb{Q}(\sqrt{-d}) \).

With the notations as above, we have:

Proposition 2.1. \[ 6 \]

(1) If \( 1 \notin \text{supp}(u) \), the support of \( u \), then \( u \) is a torsion unit.

(2) If \( \epsilon = x + y\sqrt{d} \) is a unit in \( \mathbb{O}_Q(\sqrt{d}) \), then

\[ u^n(\epsilon, \psi) = u(\epsilon^n, \psi) \]

for all \( \psi \in \{i, j, k\} \) and \( n \in \mathbb{Z} \).

Units of type \((2.3)\) are called 2-Pell units. Denote the norm of \( \epsilon = x + y\sqrt{d} \) by \( N(\epsilon) := \epsilon\overline{\epsilon} \), where \( \overline{\epsilon} = x - y\sqrt{d} \). If \( N(\epsilon) = -1 \), then the unit \( y\sqrt{-d}\xi + x\psi \) is also a 2-Gauss unit, see \([6]\).

Likewise, for \( N(\epsilon) = 1 \), we define the 4-Pell unit \( u := \frac{1}{2}\sqrt{-d}\xi + (\frac{1}{2}\sqrt{-d})\xi + (\frac{1}{2}\sqrt{-d})\psi + (\frac{1}{2}\sqrt{-d})\phi \),

where \( \xi, \psi, \phi \in \{1, i, j, k\} \) are different of each other.

Remark 2.2.

- We cannot define a 4-Pell unit when \( N(\epsilon) = -1 \), since the unit has norm \( \frac{2+2(x^2-y^2d)}{4} \neq 0 \).
When \( y \equiv 1 \pmod{2} \), the invertible \( \epsilon^2 \) defines a 4-Pell unit \( u := xy\sqrt{-d} + (xy\sqrt{-d})i + (x^2+j) + (y^2d)k \).

- If the invertible \( \epsilon = (2x - 1) + y\sqrt{-d} \in \mathbb{Q}(\sqrt{-d}) \) has norm \( N(\epsilon) = 1 \), we define a 3-Pell unit \( u := y\sqrt{-d} \xi + x\psi + (1 - x)\phi \in \left(\frac{-1 - 1}{y\sqrt{-d}}\right) \), where \( \xi, \psi, \phi \in \{1, i, j, k\} \) are different of each other.

Let \( \epsilon = x + y\sqrt{-d} \) be the fundamental invertible, \( u = x + (y\sqrt{-d})i \) and \( \varphi_u(z) = \frac{x - y\sqrt{-d}}{x + y\sqrt{-d}} \) its image in \( \mathcal{M} \). Since \( x, y \) are always positive, if \( N(\epsilon) = 1 \), then \( x + y\sqrt{-d} > 1 \) and \( 0 < x - y\sqrt{-d} < 1 \). Hence, we have that \( \varphi_u(z) = \rho z \) with \( \rho = \frac{x - y\sqrt{-d}}{x + y\sqrt{-d}} \in \mathbb{Q}(\sqrt{-d}) \).

Let \( \varphi(z) = \frac{a + \sqrt{-d}}{b + \sqrt{-d}} \). Define \( z_p := \frac{z}{\sqrt{-d}} \), the pole and the zero of \( \varphi \). Given a Pell unit \( w \), the points \((z_0,0),(z_p,\varphi_\infty)\), when \( \varphi \in \{\varphi_w, \varphi_w^{-1}\} \), play an important role in what follows.

For suitable positive real numbers \( m_2, m_1, n_1, n_2 \), a disjoint partition of the extended real numbers is either:

- \([\infty, m_2z_p] \cup [m_2z_p, m_1z_0] \cup [m_1z_0, m_1z_1] \cup \cdots \cup [n_1z_0, n_2z_0] \cup [n_2z_0, \infty) \) if \( z_p < z_0 < z_0' < z_p' \), or:
- \([-\infty, -m_2z_0] \cup [-m_2z_0, m_1z_p] \cup [m_1z_p, m_1z_1] \cup \cdots \cup [n_1z_0, n_2z_0] \cup [n_2z_0, \infty) \) if \( z_p < z_0 < z_0' < z_p' \), where \( z_0 \) is the zero and \( z_p \) is the pole of \( \varphi_w \), and \( z_0' \) is the zero and \( z_p' \) is the pole of \( \varphi_w^{-1} \). Our next result proves, on the conditions of the theorem, that \( \varphi_w, \varphi_w \) and the sets above satisfy the Ping-Pong Lemma. Since \( \varphi \) is a continuous mapping with positive derivative, if \( a_1 < z_0 < z_p < a_1 \) and \( A_{11} := \{a_2, a_1\} \), then \( \varphi(R \setminus A_{11}) = \{\varphi(a_2), \varphi(a_1)\} \).

In the proof of the next theorem we freely use this property.

**Theorem 2.3.** Let \( \epsilon = x + y\sqrt{-d} \) be the fundamental invertible in \( \mathbb{Q}(\sqrt{-d}) \) of norm \( N(\epsilon) = \epsilon \in \mathbb{Q}(\sqrt{-d}) \) and \( u = x + (y\sqrt{-d})i \). If \( w \) is one of the units

1. \( \frac{x \sqrt{-d}}{y \sqrt{-d}} + \frac{y \sqrt{-d}}{x \sqrt{-d}}i \),
2. \( \frac{x \sqrt{-d}}{y \sqrt{-d}} + \frac{y \sqrt{-d}}{x \sqrt{-d}} + \frac{1}{x \sqrt{-d}} + \frac{y \sqrt{-d}}{x \sqrt{-d}}j \),
3. \( x^2 - (xy \sqrt{-d})i - (y^2 \sqrt{-d})j + (xy \sqrt{-d})^2k \).

Then the group \( \langle u, w \rangle \subset \mathbb{Z}((\frac{1}{x \sqrt{-d}})) \) is a free group.

**Proof.** Consider \( \varphi_u, \varphi_w \in \mathcal{M} \) and \( \Omega := \mathbb{R} \cup \{\pm \infty\} \). We claim that there exist real numbers \( a_2 < a_1 < b_1 < b_2 \) such that

\[
A_{1,1} := [a_2, a_1], \quad A_{1,-1} := [b_1, b_2], \quad A_{2,1} := [a_1, b_2, \infty) \quad \text{and} \quad A_{2,-1} := a_1, b_1]
\]

are sets satisfying the conditions stated in the Ping-Pong Lemma.

In fact, let \( h_2(z) := \varphi_u(z) = \frac{x - y\sqrt{-d}}{x + y\sqrt{-d}} \) and \( h_1(z) := \varphi_w(z) \), where \( w \) is like in the items 1, 2, 3.

Then, since each \( A_{i,1} \), \( i = 1, 2 \) is to be an interval, or a disjoint union of two intervals, we get the following conditions.

- **first:** \( h_1(\Omega \setminus A_{1,1}) \subset A_{1,-1} \), which is equivalent to \( h_1(a_2), h_1(a_1) \in A_{1,-1} \);
- **second:** \( h_1^{-1}(\Omega \setminus A_{1,-1}) \subset A_{1,1} \), which is equivalent to \( h_1^{-1}(b_1), h_1^{-1}(b_2) \in A_{1,1} \);
- **third:** \( h_2(\Omega \setminus A_{2,1}) \subset A_{2,-1} \), which is equivalent to \( h_2(a_2), h_2(b_2) \in A_{2,-1} \);
- **fourth:** \( h_2^{-1}(\Omega \setminus A_{2,-1}) \subset A_{2,1} \), which is equivalent to \( h_2^{-1}(a_1), h_2^{-1}(b_1) \in A_{2,1} \).

(1) Set \( h_1(z) := \frac{x \sqrt{-d}}{y \sqrt{-d}} + \frac{y \sqrt{-d}}{x \sqrt{-d}} - a_2 = b_2 := \frac{3 - \frac{y \sqrt{-d}}{x \sqrt{-d}}}{y \sqrt{-d}} \) and \( -a_1 = b_1 := \frac{1 - \frac{x \sqrt{-d}}{y \sqrt{-d}}}{x \sqrt{-d}} \). The first condition:

\[
h_1(\Omega \setminus A_{1,1}) = [h_1(a_2), h_1(a_1)]\] we calculate \( h_1(a_2) = \frac{x \sqrt{-d}}{x \sqrt{-d}} + \frac{y \sqrt{-d}}{x \sqrt{-d}} \) and \( h_1(a_1) = \frac{x \sqrt{-d}}{x \sqrt{-d}} + \frac{y \sqrt{-d}}{x \sqrt{-d}} \). Since \( x^2 - y^2d = 1 \),
we have $b_1 < h_1(a_2)$. Similarly, we compute $h_1(a_1) = \frac{x^2 + 1}{2y\sqrt{d}} < \frac{3x}{2y\sqrt{d}} = b_2$, thus $h_1(\Omega \setminus A_{1,1}) \subset A_{1,-1}$.

The second condition: $h_1^{-1}(\Omega \setminus A_{1,-1}) = h_1^{-1}(b_1)$; we compute $h_1^{-1}(b_2) = -\frac{y\sqrt{d}}{2x^2 + 1} = h_1(a_2)$ and $h_1^{-1}(b_1) = -\frac{y\sqrt{d}}{2x^2 + 1}$. Clearly, $h_1^{-1}(b_2) < a_1$ and $a_2 < h_1^{-1}(b_1)$, thus $h_1^{-1}(\Omega \setminus A_{1,-1}) \subset A_{1,1}$. The third and fourth conditions: $h_2(\Omega \setminus A_{2,1}) = [h_2(a_2), h_2(b_2)]$ and clearly $h_2(b_2) = h_2(a_2)$. Denoting by $(\cdot)$ the order relation between $b_2(h_2) = \frac{3x}{2y\sqrt{d}}$ and $b_1$ we have that $\frac{3x}{2y\sqrt{d}} > \frac{y\sqrt{d}}{2x^2 + 1}$. Since the real numbers $h_2(b_2)$ and $b_1$ are positive, it follows that $\frac{3x}{y\sqrt{d}}(x + 5) < x^2 + y^2d + 2xy\sqrt{d} = \frac{x + y\sqrt{d}}{x - y\sqrt{d}}$. Thus $h_2(b_2) < b_1$ and $a_1 < h_2(a_2)$. Since $h_2^{-1}(z) = \frac{1}{p}z$ we proved that $h_2(\Omega \setminus A_{2,1}) \subset A_{2,-1}$ and $h_2^{-1}(\Omega \setminus A_{2,-1}) = [-\infty, h_2^{-1}(a_1)] \cup [h_2^{-1}(b_1), \infty] \subset A_{2,1}$.

(2) In this case the pole and root of $h_1(z) = \frac{(y\sqrt{d} + (x + 1)z - (y\sqrt{d} - (x - 1))}{(y\sqrt{d} + (x - 1)z - (y\sqrt{d} - (x + 1))}$ are, respectively $z_0 = \frac{x + y\sqrt{d} - (x + 1)}{y\sqrt{d} - (x - 1)}$ and $z_0' = \frac{x + y\sqrt{d} - (x - 1)}{y\sqrt{d} + (x + 1)}$. Since $z_0$ is positive, define the intervals by $A_{1,-1} = [a_2, a_1]$ and $A_{1,1} := [b_1, b_2]$ with $a_2 := 3z_0', a_1 := \frac{x}{z_0'}$ and $b_1 := \frac{y\sqrt{d}}{2x^2 + 1}$ and $b_2 := 3z_0$, and proceed as before proving that $h_1(\Omega \setminus A_{1,1}) = [h_1(b_2), h_1(b_1)] \subset A_{1,1}$ and $h_1^{-1}(\Omega \setminus A_{1,1}) = [h_1^{-1}(a_1), h_1^{-1}(a_2)] \subset A_{1,1}$. This is true, provide that $\frac{2x}{y\sqrt{d}} < \frac{1}{\rho}$, if $x > 2$. This holds because the congruence $y \equiv 0 \pmod{2}$ implies that $x > 2$.

(3) We have that $h_1(z) = \frac{x + y\sqrt{d}}{z - y\sqrt{d}} = \frac{x - y\sqrt{d}}{z + y\sqrt{d}}$. Set $b_1 := \frac{y\sqrt{d}}{2x^2 + 1}$, $b_2 := 3z_0 = \frac{x}{z_0}' = \frac{b_1}{\rho}$, $a_1 := \frac{x}{z_0}' = \frac{b_1}{\rho}$, and $a_2 := 3z_0' = \frac{b_1}{\rho}$, where $\rho = \frac{x + y\sqrt{d}}{x + y\sqrt{d}}$ and the intervals $A_{1,-1} := [a_2, a_1]$ and $A_{1,1} := [b_1, b_2]$. As above, we prove that $h_1(\Omega \setminus A_{1,1}) = [h_1(b_2), h_1(b_1)] \subset A_{1,1}$ and $h_1^{-1}(\Omega \setminus A_{1,1}) = [h_1^{-1}(a_1), h_1^{-1}(a_2)] \subset A_{1,1}$. As before, since $\frac{6x^2}{y\sqrt{d}} < \frac{1}{\rho}$ when $x > 1$, we prove the inclusions.

Since all conditions are satisfied we have, by the Ping-Pong Lemma, that $\langle u, w \rangle$ is a free group. This concludes the proof of the theorem. \hfill \Box

A natural question that can be raised is whether the previous theorem still holds if the norm of the fundamental invertible is $-1$. The answer is positive for a 2-Gauss unit with $x \neq 1$. When $\epsilon = 1 + \sqrt{2}$, the same calculations as before can be used to show that $\langle u^2, w \rangle$ is a free group. To see this, we apply the the Ping-Pong Lemma using the following data: $-a_2 = b_2 := 2z_0 = 2\sqrt{2}$, $-a_1 = b_1 := \frac{y\sqrt{d}}{2x^2 + 1}$, $A_{1,1} := [a_2, a_1]$, $A_{1,-1} := [b_1, b_2]$, $A_{2,1} := [-\infty, a_2] \cup [b_2, \infty]$ and $A_{2,-1} := [b_1, b_2]$. Thus it is easily checked, by contradiction, that the only possible choices are $-a_1 = b_1 := \sqrt{2} - 1$ and $-a_2 = b_2 = \sqrt{2} + 1$. However for these values the condition $h_2(a_2) = \varphi_w(a_2) < b_1$ fails to hold.

Corollary 2.4. If $\mathcal{N}(x + y\sqrt{d}) = -1$ and $x \neq 1$ then $\langle u, w \rangle$ is free, where $w$ is the 2-Gauss unit $y\sqrt{d} + xk$.

PROOF. From $x^2 - y^2d = -1$ we have that $y\sqrt{d} > x$. We apply the proof of the previous theorem using the following data: $-a_2 = b_2 := \frac{3x + \sqrt{2}}{2\sqrt{2}}$ and $-a_1 = b_1 := \frac{1}{\sqrt{d} - 2\sqrt{2}}$. The condition $x \neq 1$ is equivalent to $\epsilon \neq 1 + \sqrt{2}$.

\hfill \Box

We make use of the algebraic properties of the Pell units. This allow us a precise control of their images in $\mathcal{M}$ and to apply the Ping-Pong Lemma in a non-standard way.
Next, we give a criterion for two elements \( \varphi_1, \varphi_2 \) to generate a free semigroup. Note that we do not use the Ping-Pong Lemma. For an application \( \varphi : V \rightarrow V \), we say that \( U \subset V \) is invariant under \( \varphi \) if, and only if, the restriction of \( \varphi \) on \( U \) is a subset of \( U \).

**Lemma 2.5.** Let \( V \) be a set of infinite cardinality and let \( \varphi_1, \varphi_2 : V \rightarrow V \) be injective maps of infinite order. If \( U \not\subseteq V \) is invariant under \( \varphi_1 \) and \( \varphi_2 \), and \( x_0 \in V \setminus U \) is a fixed point of \( \varphi_1 \) such that \( \varphi_2(x_0) \in U \), then \( \langle \varphi_1, \varphi_2 \rangle \) is a free semigroup.

**Proof.** Suppose that the reduced word \( \varphi = \varphi_{r_1}^s \cdots \varphi_{r_k}^{s_k} \) is the identity map. If \( r_k = 1 \), then, since \( \varphi \) is a reduced word and \( U \) is invariant under both maps, \( x_0 = \varphi(x_0) = \varphi_{r_1}^s \cdots \varphi_{r_k}^{s_k}(x_0) = \varphi_{r_1}^s \cdots \varphi_{r_{k-1}}^{s_{k-1}}(x_0) \in U \). If we have that \( r_k = 2 \), then \( x_0 = \varphi(x_0) \in U \), since \( \varphi_2(x_0) \in U \). In any case we have a contradiction because \( x_0 \notin U \). \( \square \)

We now give an application of the above result in the context of the Pell units. Here we do not need to put any restriction on the norm of the fundamental invertible \( \epsilon \), unless, like the 4-Pell unit, when the restriction \( \mathcal{N}(\epsilon) = 1 \) is necessary.

**Theorem 2.6.** Let \( \epsilon = x + y\sqrt{d} \) be the fundamental invertible of \( \mathbb{Q}(\sqrt{d}) \) and \( u = x + (y\sqrt{-d})i \). If \( w \) is one of the units

\[(1): y\sqrt{-d} + xk; \]
\[(2): \frac{x-1}{2} + \frac{y\sqrt{-d}}{2}x + \frac{y\sqrt{-d}}{2}k; \]
\[(3): \frac{x^2 - (x+y\sqrt{-d})}{2} - y^2d + (x\sqrt{-d})k. \]

Then \( \langle u, w \rangle \subset U((\frac{1}{s_k})) \) is a free semigroup.

**Proof.** Consider \( \varphi_1, \varphi_2 \in \mathcal{M}, V \) the set of extended real numbers, \( U \) the set of positive real numbers and \( x_0 = 0 \). We claim that the conditions of the previous lemma are satisfied by these data.

In fact, item (1): \( \varphi_1(z) = \varphi_w(z) = \frac{z-y\sqrt{d}}{x+y\sqrt{d}}z \) clearly keeps \( U \) invariant and fixes \( x_0 = 0 \notin U \).

\[\varphi_2(z) = \frac{z-y\sqrt{d}}{x+y\sqrt{d}}z \] has a pole at \( \frac{y\sqrt{-d}}{x} \), a zero at \( \frac{-x}{y\sqrt{d}} \) and \( \varphi_\infty = \frac{y\sqrt{-d}}{x} \). If \( \mathcal{N}(\epsilon) = 1 \), then \( \varphi_2(0) = \frac{x}{y\sqrt{d}} > y\sqrt{-d} > 0 \). Hence \( \varphi_2(0) \in U \) and \( \varphi_2(U \cup \{0\}) = \{ \frac{x}{y\sqrt{d}} \} \subset U \). If \( \mathcal{N}(\epsilon) = -1 \), then \( \frac{x}{y\sqrt{d}} > y\sqrt{-d} = \varphi_2(0) \). Hence \( \varphi_2(0) \in U \) and \( \varphi_2(U \cup \{0\}) = \{ \frac{x}{y\sqrt{d}} \} \subset U \).

Item (2), we set \( \varphi_2(z) := \varphi_{w_1}^{-1}(z) = \frac{(y\sqrt{-d}(x+1))z + (y\sqrt{-d}(x-1))}{(y\sqrt{d}+x)(y\sqrt{d}+x+1)} \). Let \( z_p \) and \( z_0 \) be the pole and the zero of \( \varphi_2 \), clearly \( z_0 < z_p \). Also, \( \varphi_\infty = \frac{(y\sqrt{-d}(x-1))}{(y\sqrt{d}+x)(y\sqrt{d}+x+1)} < \frac{x}{y\sqrt{d}} = \varphi_2(0) \). Hence \( \varphi_2(U) = (\varphi_\infty, \varphi_2(0)) \subset U \). For the item (3), define \( \varphi_2(z) := \varphi_w^{-1}(z) = (z^2-x\sqrt{-d})z + (y^2d+x\sqrt{-d}) \). The proof follows as in the item before.

The conditions of the previous lemma being met, it follows that \( \langle \varphi_1, \varphi_2 \rangle \) is a free semigroup. \( \square \)

Let \( u \) and \( v \) be units in \( (\frac{1}{s_i}) \). It follows from this theorem that if \( u = 1 + (\sqrt{-2})i \) and \( w = \sqrt{-2} + k \) then the semigroup \( < u, w > \) is free while we cannot say that these units generate a free group.

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