METRIC THEORY OF LOWER BOUNDS ON WEYL SUMS

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Abstract. We prove that the Hausdorff dimension of the set $x \in [0,1)^d$, such that
$$\left| \sum_{n=1}^{N} \exp \left( 2\pi i \left( x_1 n + \ldots + x_d n^d \right) \right) \right| \geq c N^{1/2}$$
holds for infinitely many natural numbers $N$, is at least $d - 1/2d$ for $d \geq 3$ and at least $3/2$ for $d = 2$, where $c$ is a constant depending only on $d$. This improves the previous lower bound of the first and third authors for $d \geq 3$. We also obtain similar bounds for the Hausdorff dimension of the set of large sums with monomials $x n^d$.

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1. Introduction

1.1. Background. For an integer $d \geq 2$, let $T_d = (\mathbb{R}/\mathbb{Z})^d$ be the $d$-dimensional unit torus. For a vector $x = (x_1, \ldots, x_d) \in T_d$ and integer $N$, we consider the exponential sums

$$S_d(x; N) = \sum_{n=1}^{N} e \left( x_1 n + \ldots + x_d n^d \right),$$

which are commonly called Weyl sums, where throughout the paper we denote $e(x) = \exp(2\pi i x)$. These sums were originally introduced by Weyl to study equidistribution of fractional parts of polynomials and rose to prominence through applications to the circle method and Riemann zeta function. Despite more than a century since these sums were introduced, their behaviour is not well understood, see [4, 5].

For large values of $d$, the sharpest bounds for $S_d(x; N)$ are obtained through Vinogradov’s method of bilinear forms and produce a bound of the shape

$$S_d(x; N) \ll N^{1-c/d^2 + o(1)},$$

for a certain absolute constant $c$, provided one of the entries of $x$ has suitable rational approximation. Over the years the value of $c$ in (1.2) has been refined although improving the dependence on $d$ remains an important open problem. The $o(1)$ term in (1.2) depends on $d$ and hence the estimate (1.2) is valid for a fixed $d$. The case where $d$ grows with $N$ has attracted special attention due to connections with zero free regions of the Riemann zeta function. For this problem the sharpest estimates are due to Ford [14] and based on ideas of Arkhipov and Karatsuba [1] and Wooley [24, 25]. Progress on estimates of the type (1.2) has been through new bounds for the Vinogradov mean value theorem. A precise statement of the current sharpest estimate is given in [2, Theorem 5]. In particular, we have:

Let $x = (x_1, \ldots, x_d) \in T_d$ be such that for some $\nu$ with $2 \leq \nu \leq d$ and some positive integers $a$ and $q$ with $\gcd(a, q) = 1$ we have

$$\left| x_\nu - \frac{a}{q} \right| \leq \frac{1}{q^2}.$$

Then for any $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$|S_d(x; N)| \leq C(\varepsilon)N^{1+\varepsilon} \left( q^{-1} + N^{-1} + qN^{-\nu} \right)^{\frac{1}{d-1}}.$$
Assuming optimal parameters in (1.3), we obtain a constant $c \sim 1$ in (1.2) while heuristics predict an upper bound of the form

$$S_d(x; N) \ll N^{1-1/d+o(1)}.$$  

The average behaviour of $S_d(x; N)$ is much better understood. The recent advances of Bourgain, Demeter and Guth [3] (for $d \geq 4$) and Wooley [26] (for $d = 3$) (see also [28]), for the Vinogradov mean value theorem imply the estimate

$$\int_{T_d} |S_d(x; N)|^{2s(d)} \, dx \leq N^{s(d) + o(1)},$$

where

$$s(d) = \frac{d(d+1)}{2}$$

and is best possible up to $o(1)$. Obtaining good uniform (with respect to $d$) estimates on the $o(1)$ factor in (1.4) is still an open problem which may lead to refinements of estimates for the Riemann zeta function near the line $\Re s = 1$.

1.2. Previous results and questions. In this paper we consider the question of obtaining lower bounds for the sums (1.1). Due to their erratic behaviour for individual values of $x$ (for example, we may have $S_d(x; N) = 0$ for infinitely many values of $N$) our goal is to obtain results which hold for almost all $x$ or for a set with large Hausdorff dimension. Results of this type fall into the metric theory of Weyl sums. The first results in this direction are due to Hardy and Littlewood [16] and concern Gauss sums

$$G(x; N) = \sum_{n=1}^{N} e(xn^2).$$

To estimate the sums (1.5), Hardy and Littlewood [16] iterate a summation formula obtained through the method of contour integration which allows an asymptotic formula in terms of the continued fraction expansion of $x$. Metric results for $G(x; N)$ then follow by combining with techniques from the metric theory of numbers, such as Khinchin's work on continued fractions [17]. This idea has been expanded upon by Fiedler, Jurkat and Körner [12, Theorem 2] who give the following optimal lower and upper bounds. Suppose that $\{f(n)\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive numbers. Then for almost all $x \in T$
one has
\begin{equation}
\lim_{N \to \infty} \frac{|G(x;N)|}{\sqrt{N f(N)}} < \infty \iff \sum_{n=1}^{\infty} \frac{1}{n f(n)^{4}} < \infty.
\end{equation}

For the more general sums \( S_2(x;N) \), (which correspond to \( G(x;N) \) with a linear term in the phase) Fedotov and Klopp [11, Theorem 0.1] have obtained the following optimal lower and upper bounds. Suppose that \( \{g(n)\}_{n=1}^{\infty} \) is a non-decreasing sequence of positive numbers. Then for almost all \( x \in T_2 \) one has the following equivalence:
\begin{equation}
\lim_{N \to \infty} \frac{|S_2(x;N)|}{\sqrt{N g(\ln N)}} < \infty \iff \sum_{n=1}^{\infty} \frac{1}{g(n)^{6}} < \infty.
\end{equation}

For \( d \geq 3 \), the first and the third authors, see [6, Corollary 2.2] and [7, Appendix A], have shown that for almost all \( x \in T_d \)
\begin{equation}
|S_d(x;N)| \leq N^{1/2+\alpha(1)}, \quad N \to \infty.
\end{equation}
The first and third authors have conjectured that the exponent 1/2 is best possible, see [7, Conjecture 1.1].

From the almost all result in (1.8) one may ask how “large” is the exceptional set. For this purpose we introduce following notation.

For any \( \alpha \in (0, 1) \) and integer \( d \geq 2 \), we consider the set
\[ E_{d,\alpha} = \{ x \in T_d : |S_d(x;N)| \geq N^\alpha \text{ for infinity many } N \in \mathbb{N} \}. \]
Using this notation, the estimate (1.8) may be rephrased as: For any \( \alpha \in (1/2, 1) \) and integer \( d \geq 2 \) the set \( E_{d,\alpha} \) has zero Lebesgue measure. For \( \alpha \in (0, 1/2] \) we conjecture that the set \( E_{d,\alpha} \) has full Lebesgue measure, which is open for \( d \geq 3 \).

For sets of Lebesgue measure zero, it is common to use the Hausdorff dimension to describe their size; for the properties of the Hausdorff dimension and its applications we refer the reader to [13]. We recall that for \( U \subseteq \mathbb{R}^d \)
\[ \text{diam} U = \sup \{ \| u - v \| : u, v \in U \} \]
where \( \| w \| \) is the Euclidean norm in \( \mathbb{R}^d \).

**Definition 1.1.** The Hausdorff dimension of a set \( A \subseteq \mathbb{R}^d \) is defined as
\[ \dim A = \inf \left\{ s > 0 : \forall \varepsilon > 0, \exists \{ U_i \}_{i=1}^{\infty}, U_i \subseteq \mathbb{R}^d, \text{ such that } A \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } \sum_{i=1}^{\infty} (\text{diam} U_i)^s < \varepsilon \right\}. \]
In [7], the first and third authors have obtained a lower bound for the Hausdorff dimension of $E_{d,\alpha}$. Among other things, it is shown that for any $\alpha \in (0,1)$ and any cube $\Omega \subseteq T_d$ one has
\[
\dim (E_{d,\alpha} \cap \Omega) \geq \ell(d, \alpha)
\]
with some explicit function $\ell(d, \alpha) > 0$, which for $\alpha = 1/2$ grows like
\[
\ell \left( 2, \frac{1}{2} \right) = \frac{3}{2} \quad \text{and} \quad \ell \left( d, \frac{1}{2} \right) \sim \frac{3}{4d} \quad \text{as} \quad d \to \infty.
\]
Note the results of [7] are much sharper if one lets $\alpha \to 1$ as $d$ gets large, however in this paper we are mainly concerned with the case $\alpha = 1/2$ and take (1.9) as our comparison. It is not difficult to see that $\dim E_{d,\alpha}$ is monotonically non-decreasing, hence by (1.9)
\[
(1.10) \quad \dim E_{d,\alpha} \geq \dim E_{2,\alpha} \geq \ell \left( 2, \frac{1}{2} \right) = \frac{3}{2}.
\]
Furthermore, in [8] the first and third authors give a non-trivial upper bound for $E_{d,\alpha}$. More precisely, for any $1/2 < \alpha < 1$ we have
\[
\dim E_{d,\alpha} \leq u(d, \alpha)
\]
with some explicit function $u(d, \alpha) < d$. Moreover, if $\alpha \to 1$ then $u(d, \alpha) \to 0$. Indeed, it is expected that as $\alpha$ increases the set $E_{d,\alpha}$ becomes small. We refer the reader to [8] for more details.

We remark that we do not have any plausible conjecture about the exact behaviour of $\dim E_{d,\alpha}$ for $\alpha \in (1/2,1)$.

In [8], the first and third authors also investigate the monomials
\[
\sigma_d(x; N) = \sum_{n=1}^{N} e \left( x n^d \right).
\]
For each $\alpha \in (0,1)$ let
\[
E_{d,\alpha} = \{ x \in [0,1) : |\sigma_d(x; N)| \geq N^\alpha \text{ for infinitely many } N \in \mathbb{N} \}.
\]
Similarly to $E_{d,\alpha}$, for $\alpha \in (0,1)$ and integer $d \geq 2$ the set $E_{d,\alpha}$ has positive Hausdorff dimension. Moreover for $\alpha \in (1/2,1)$ and $d \geq 2$ the set $E_{d,\alpha}$ has zero Lebesgue measure [6, Corollary 2.2]. In this paper we improve the lower bounds of $\dim E_{d,\alpha}$ and $\dim E_{d,\alpha}$ of [7] for all $\alpha \in (0,1/2)$ and $d \geq 3$. More specifically, we obtain a new lower bound for the Hausdorff dimension of a set slightly larger than $E_{d,1/2}$. In order for a direct comparison with the results of [7], we then need to consider $\dim E_{d,\alpha}$ for $\alpha$ in the open interval $(0,1/2)$. 

2. Main results

2.1. Formulations. Here we are mostly interested in the case $\alpha = 1/2$. Hence we slightly redefine the notations for $E_{d,1/2}$ and $E_{d,1/2}$. We will also require a weighted variant of $E_{d,1/2}$. In particular, for a sequence of complex weights $a = (a_n)_{n=1}^\infty$ with $|a_n| = 1$. Define

$$\sigma_{a,d}(x; N) = \sum_{n=1}^N a_n e(xn^d).$$

For integer $d \geq 2$ and a constant $c > 0$ denote

$$E_{a,c}(d) = \{x \in [0, 1) : |\sigma_{a,d}(x; N)| \geq cN^{1/2} \text{ for infinitely many } N \in \mathbb{N}\},$$

and

$$E_{a,c}(d) = \{x \in T_d : |S_{a,d}(x; N)| \geq cN^{1/2} \text{ for infinitely many } N \in \mathbb{N}\},$$

where

$$S_{a,d}(x; N) = \sum_{n=1}^N a_n e(x_1 n + \ldots x_d n^d).$$

In particular,

$$E_{e,1}(d) = E_{d,1/2},$$

where $e = (1, 1, \ldots)$. Our main results concern more general sequence $a$.

**Theorem 2.1.** For $d \geq 3$ and there exists a constant $c > 0$ that depends only on $d$, such that for any sequence of complex weights $a = (a_n)_{n=1}^\infty$ with $|a_n| = 1$ we have $\dim E_{a,c}(d) \geq 1 - 1/2d$.

Combining Theorem 2.1 with Lemma 4.1 below, we obtain a lower bound for $\dim E_{a,c}(d)$.

**Theorem 2.2.** For $d \geq 3$ there exists a constant $c > 0$ that depends only on $d$ such that $\dim E_{a,c}(d) \geq d - 1/2d$.

For the case $d \geq 3$ our arguments combine Weyl differencing with decay of integrals with polynomial phases. Since Weyl differencing with a degree 2 polynomial produces a linear phase, the case $d = 2$ requires separate treatment.

**Theorem 2.3.** There is a constant $c > 0$ such that $\dim E_{a,c}(2) \geq 1/2$.

Combining Theorem 2.3 with Lemma 4.1 below, we obtain a lower bound for $\dim E_{a,c}(2)$.

**Theorem 2.4.** There is a constant $c > 0$ such that $\dim E_{a,c}(2) \geq 3/2$. 
Theorem 2.4 provides an improvement to (1.10) by allowing weights, (however for a slightly bigger set due to the present of the constant $c$).

We remark that the results (1.6) and (1.7) give optimal bounds for the sums $G(x;N)$ and $S_2(x;N)$ respectively. However, for sums with weights, Theorem 2.3 and Theorem 2.4 give new and non-trivial lower bounds.

Theorems 2.1 and 2.2 show that these sets have nearly full Hausdorff dimension as $d \to \infty$, approaching the optimal values 1 and $d$, respectively, as $d \to \infty$.

Moreover, our results imply lower bounds for $\dim \mathcal{E}_{d,\alpha}, \mathcal{E}_{d,\alpha}$ for $\alpha \in (0,1/2)$. With notation as in (2.1), note that for each $\alpha \in (0,1/2)$ and any $c > 0$ one has

$$\mathcal{E}_{e,c}(d) \subseteq \mathcal{E}_{d,\alpha}.$$ 

Therefore we obtain that for $d = 2$,

$$\dim \mathcal{E}_{2,\alpha} \geq \dim \mathcal{E}_{e,c}(2) \geq 3/2$$

and for $d \geq 3$,

$$\dim \mathcal{E}_{d,\alpha} \geq \dim \mathcal{E}_{e,c}(d) \geq d - 1/2d.$$ 

A similar argument also works for $\mathcal{E}_{d,\alpha}$, thus for $\alpha \in (1,1/2)$ we have $\dim \mathcal{E}_{d,\alpha} \geq 1 - 1/2d$.

However, we believe that the above lower bounds are not optimal, and these sets have full Lebesgue measure.

**Conjecture 2.5.** For $d \geq 2$, $c > 0$ and $|a_n| = 1$ for all $n \in \mathbb{N}$, the sets $\mathcal{E}_{e,c}(d)$ and $\mathcal{E}_{e,c}(d)$ are of full Lebesgue measure and hence the sets $\mathcal{E}_{d,1/2}$ and $\mathcal{E}_{d,1/2}$ are also of full Lebesgue measure.

**2.2. Outline of the method.** Our approach builds on some ideas introduced by the first and third authors [8], which proceeds by finding a Cantor like subset inside $\mathcal{E}_{d,\alpha}$. One of the key new ideas is to pass to a one-dimensional problem. Consider the more general sums

$$\sigma_{a,d}(x;N) = \sum_{n=1}^{N} a_n e\left(x n^d\right),$$

where $a = (a_n)_{n=1}^{\infty}$ is an arbitrary sequence of complex weights satisfying $|a_n| = 1$. If we can show the set of $0 \leq x < 1$ such that $\sigma_{a,d}(x;N)$ is large for infinitely many $N$ has large Hausdorff dimension, then on taking

$$a_n = c(x_1 n + \cdots + x_{d-1} n^{d-1})$$

we may deduce that the set of $x \in T_d$ such that $S_d(x,N)$ is large for infinitely many $N$ has large Hausdorff dimension via a slicing argument, see Lemma 4.1 below.
To find large values of the sums (2.2) we iterate two simple results, see Lemma 3.1 and Lemma 3.5. Ignoring some technical details, Lemma 3.1 says
\[ \sigma_{a,d}(x; N) \sim C_{a,d}(y; N) \quad \text{if} \quad |x - y| \ll N^{-d} \]
and Lemma 3.5 says for most short intervals \( I \) inside any other interval \( J \) of lengths
\[ |I| \geq N^{-d+1/2} \quad \text{and} \quad |J| \geq N^{-d+2} \]
we have
\[ \int_I |\sigma_{a,d}(x; N)|^2 dx \sim |I|N. \]

To see how these two results may be iterated to construct a Cantor-like set, start with an interval \( I_0 \) of length \( N_0^{-d+1/2} \). By (2.5) with \( I = I_0 \), \( N = N_0 \), we obtain many well-separated values of \( x_j \in I_0, j = 1, \ldots, q_1 \) such that
\[ |\sigma_{a,d}(x_j; N_0)| \gg N_0^{1/2}, \quad \forall j = 1, \ldots, q_1. \]
Then, using (2.3), for each \( x_j \) we obtain subintervals \( I_j, j = 1, \ldots, q_1 \) of length \( N_0^{-d} \) such that for any \( 1 \leq j \leq q_1 \) we have
\[ |\sigma_{a,d}(x; N_0)| \gg N_0^{1/2}, \quad x \in I_j. \]

Let \( F_1 \) be the collection of the intervals \( I_j, 1 \leq j \leq q_1 \). Note that \( F_1 \) is the first step construction of the desired Cantor-like set, see Figure 2.1 with the case \( q_1 = 3 \).

Choose some \( N_1 \) large enough in terms of \( N_0 \) and for each \( I_j, 1 \leq j \leq q_1 \), apply the above argument to \( I_j \) to obtain many well-separated points \( x_{j,\ell} \in I_j, 1 \leq \ell \leq q_2 \), such that \( \sigma_{a,d}(x_{j,\ell}; N_1) \) is large. Applying (2.3), we obtain intervals \( I_{j,\ell} \subseteq I_1, 1 \leq \ell \leq q_2, \) of length \( |I_{j,\ell}| = N_1^{-d} \) such that
\[ |\sigma_{a,d}(x; N_1)| \gg N_1^{1/2}, \quad x \in I_{j,\ell}. \]
Let $F_2$ be the collection of all the subintervals which arise from every interval $I_j$, see Figure 2.1 with the case $q_2 = 4$. Note that $F_2$ is the second construction of the Cantor-like set. Clearly $F_2 \subseteq F_1$.

Continuing in this way, we obtain subsets $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ and the Cantor-like set $E$ which is the intersection of these sets $F_i$, $i \in \mathbb{N}$. Observe that for $x \in E$

$$|\sigma_{a,d}(x; N_i)| \gg N_i^{1/2}, \quad i = 1, 2, \ldots.$$ 

The fact that $E$ is an intersection of intervals allows computation of the Hausdorff dimension via the mass distribution principle and we refer the reader to Section 3.5 for this part of the argument.

The Hausdorff dimension we obtain this way depends on the size of the intervals occurring in (2.3) and (2.4). To obtain further progress via this method one would need shorter intervals $I$ for which the asymptotic (2.5) holds. For example, if one could show

$$\int_I |\sigma_{a,d}(x; N)|^2 \, dx \sim |I| N \quad \text{whenever} \quad |I| = N^{-d + o(1)},$$

then it would follow that $\dim E_{a,c}(d) = 1$ and $\dim E_{a,c}(d) = d$, which as we have mentioned is what we believe to be true, see Conjecture 2.5.

3. Preliminaries

3.1. Notation and conventions. Throughout the paper, the notation $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leq cV$ for some positive constant $c$, which depend on the degree $d$ and occasionally on the small real positive parameter $\varepsilon$.

For any quantity $V > 1$ we write $U = V^{o(1)}$ (as $V \to \infty$) to indicate a function of $V$ which satisfies $|U| \leq V^\varepsilon$ for any $\varepsilon > 0$, provided $V$ is large enough. One additional advantage of using $V^{o(1)}$ is that it absorbs log $V$ and other similar quantities without changing the whole expression.

We use $\#S$ to denote the cardinality of a finite set $S$. For an interval $I$ we use $|I|$ to denote its length. For more general sets $A \subseteq \mathbb{R}^k$ we use $\lambda(A)$ to denote the Lebesgue measure of $A$.

We always identify $T_d$ with half-open unit cube $[0, 1)^d$, in particular we naturally associate the Euclidean norm $\|x\|$ with points $x \in T_d$.

We say that some property holds for almost all $x \in [0, 1)^k$ if it holds for a set $X \subseteq [0, 1)^k$ of $k$-dimensional Lebesgue measure $\lambda(X) = 1$.

We will also use $\sum_{n \leq N} a_n$ to represent the sum $\sum_{n=1}^N a_n$ when there is no confusion.
3.2. Continuity of Weyl sums. In full analogue of [6, Lemma 3.4] and [27, Lemma 2.1] we obtain:

**Lemma 3.1.** For \( d \geq 2 \) and for any real numbers \( x, y \) and sequence of complex weights \( a = (a_n)_{n=1}^\infty \) satisfying \( |a_n| = 1 \), we have

\[
\sum_{n \leq N} a_n e(xn^d) - \sum_{n \leq N} a_n e(yn^d) \ll |x - y| N^d \max_{M \leq N} \left| \sum_{n \leq M} a_n e(xn^d) \right|.
\]

**Proof.** Let 
\[
\delta = y - x.
\]
We have
\[
\sum_{n \leq N} a_n e(xn^d) - \sum_{n \leq N} a_n e(yn^d) = \sum_{n \leq N} (1 - e(\delta n^d)) a_n e(xn^d),
\]
hence by partial summation
\[
\sum_{n \leq N} a_n e(xn^d) - \sum_{n \leq N} a_n e(yn^d) = (1 - e(\delta N^d)) \sum_{n \leq N} a_n e(xn^d)
\]
\[
+ 2\pi i d \delta \int_1^N t^{d-1} e(\delta t^d) \left( \sum_{n \leq t} a_n e(xn^d) \right) dt.
\]
Observe that \( 1 - e(\delta N^d) \ll \delta N^d \), and hence
\[
\sum_{n \leq N} a_n e(xn^d) - \sum_{n \leq N} a_n e(yn^d) \ll \delta N^d \left| \sum_{n \leq N} a_n e(xn^d) \right| + \delta N^{d-1} \int_1^N \left| \sum_{n \leq t} a_n e(xn^d) \right| dt
\]
\[
\ll \delta N^d \max_{M \leq N} \left| \sum_{n \leq M} a_n e(xn^d) \right|,
\]
which concludes the proof. \( \square \)

**Corollary 3.2.** For \( d \geq 2 \) and for any real numbers \( x, y \) with \( |x - y| \ll N^{-d} \) and sequence of complex weights \( a = (a_n)_{n=1}^\infty \) satisfying \( |a_n| = 1 \),
we have
\[
\max_{M \leq N} \left| \sum_{n \leq M} a_n \, e\left( x n^d \right) \right| \ll \max_{M \leq N} \left| \sum_{n \leq M} a_n \, e\left( y n^d \right) \right| \ll \max_{M \leq N} \left| \sum_{n \leq M} a_n \, e\left( x n^d \right) \right|.
\]

**Proof.** We prove the upper bound
\[
\max_{M \leq N} \left| \sum_{n \leq M} a_n \, e\left( x n^d \right) \right| \ll \max_{M \leq N} \left| \sum_{n \leq M} a_n \, e\left( y n^d \right) \right|,
\]
the corresponding lower bound follows from symmetry. Let \( K \) be the smallest positive number such that
\[
\max_{M \leq N} \left| \sum_{n \leq M} a_n \, e\left( x n^d \right) \right| = \sum_{n \leq K} a_n \, e\left( x n^d \right).
\]
By Lemma 3.1 and \( |x - y| \ll K^{-d} \) we obtain
\[
\sum_{n \leq K} a_n \, e\left( x n^d \right) - \sum_{n \leq K} a_n \, e\left( y n^d \right) \ll |x - y| \max_{M \leq N} \left| \sum_{n \leq M} a_n \, e\left( y n^d \right) \right| \ll \max_{M \leq N} \left| \sum_{n \leq K} a_n \, e\left( y n^d \right) \right|,
\]
and hence
\[
\sum_{n \leq K} a_n \, e\left( x n^d \right) \ll \max_{M \leq N} \left| \sum_{n \leq K} a_n \, e\left( y n^d \right) \right|,
\]
which concludes the proof. \( \square \)

### 3.3. Average over small intervals

We start with a very simple identity.

**Lemma 3.3.** Let \( x \) and \( \varepsilon > 0 \) be real numbers. Let \( y_1, \ldots, y_K \) be a sequence of real numbers and let \( \beta_1, \ldots, \beta_K \) be a sequence of complex numbers. We have
\[
\int_{x}^{x+\varepsilon} \left| \sum_{k=1}^{K} \beta_k \, e\left( z y_k \right) \right|^2 \, dz = \varepsilon \sum_{k=1}^{K} |\beta_k|^2 + \sum_{1 \leq k \neq \ell \leq K} \frac{\beta_k \overline{\beta_\ell} \left( e(\varepsilon(y_k - y_\ell)) - 1 \right)}{2\pi i(y_k - y_\ell)} e(x(y_k - y_\ell)).
\]
Proof. This follows after expanding the square, interchanging summation and evaluating the integral.

Applying Lemma 3.3 to monomials of degree 2, we obtain the following \(L^2\)-type mean value estimate. It is possible to obtain a slightly sharper estimate (with error term \((\log N)^2\)) by appealing to results of Montgomery and Vaughan [21, Equation (1.9)]. Since our approach for more general monomials is an elaboration of Lemma 3.4 below we provide details (and a slightly weaker error term is inconsequential for our main results).

Lemma 3.4. Let \(a = (a_n)_{n=1}^{\infty}\) be a sequence of complex weights such that \(|a_n| = 1\). Then for any interval \(I \subseteq T\) we have

\[
\int_I \left| \sum_{n=1}^{N} a_n e(zn^2) \right|^2 dz = N|I| + O\left((\log N)^2\right).
\]

Proof. Suppose \(I = [x, x + \varepsilon]\). Using the assumption each \(|a_n| = 1\), Lemma 3.3 implies

\[
\int_x^{x+\varepsilon} \left| \sum_{n=1}^{N} a_n e(zn^2) \right|^2 dz = N\varepsilon + \sum_{1 \leq n_1, n_2 \leq N} \frac{\alpha_{n_1} \overline{\alpha}_{n_2} \left( e(\varepsilon (n_1^2 - n_2^2)) - 1 \right)}{2\pi i (n_1^2 - n_2^2)} e(x(n_1^2 - n_2^2))
\]

\[
= N\varepsilon + O\left( \sum_{1 \leq n_2 \leq n_1 \leq N} \frac{1}{(n_1 - n_2)(n_1 + n_2)} \right).
\]

Since

\[
\sum_{1 \leq n_2 \leq n_1 \leq N} \frac{1}{(n_1 - n_2)(n_1 + n_2)} \leq \sum_{1 \leq n_1, n_2 \leq 2N} \frac{1}{n_1 n_2} \ll (\log N)^2
\]

we obtain the desired bound.

We remark that a similar argument to the proof of Lemma 3.4 yields new results for \(d \geq 3\) also. However, by this way, we do not obtain better bounds than in Theorem 2.1 and Theorem 2.2. These results require two iterations of Lemma 3.3 to obtain estimates for the variance of exponential sums from their mean.

3.4. Variance of mean values. Our main technical tool is the following.
**Lemma 3.5.** Let \( d \geq 3 \), \( N \in \mathbb{N} \) and \( M = \lfloor N/2 \rfloor \). Let \( \varepsilon_0, \varepsilon_1, x_1 \) be real numbers. For any sequence \( a = (a_n)_{n=1}^{\infty} \) of complex weights satisfying \( |a_n| = 1 \) we have

\[
\int_{x_1}^{x_1 + \varepsilon_1} \left( \int_{x_0}^{x_0 + \varepsilon_0} \sum_{M < n \leq N} a_n e(xn^d) \left| dx - \varepsilon_0(N - M) \right|^2 dx_0 \right)^2 dx_1 \leq N^{-2d+3+o(1)} \left( \varepsilon_1 + N^{-d+2} \right).
\]

**Proof.** By Lemma 3.3

\[
\int_{x_0}^{x_0 + \varepsilon_0} \left| \sum_{M < n \leq N} a_n e(xn^d) \right|^2 dx_0 \leq \sum_{M < n \leq N, m \neq n} \frac{a_m \overline{a}_n}{m^d - n^d} (e(\varepsilon_0(m^d - n^d)) - 1) e(x_0(m^d - n^d))
\]

\[
\leq \sum_{1 \leq h \leq N} \frac{1}{h} \sum_{M < n \leq N - h} \beta_{n,h} e(x_0 h P_h(n))
\]

where we have made the change of variable \( m \to n + h \) and defined

\[
P_h(n) = n^{d-1} + hn^{d-2} + \ldots + h^{d-1},
\]

\[
\beta_{n,h} = a_{n+h} \overline{a}_n (e(\varepsilon_0(h P_h(n))) - 1).
\]

Squaring, applying the Cauchy-Schwarz inequality then integrating over \((x_1, x_1 + \varepsilon_1)\) gives

(3.1)

\[
\int_{x_1}^{x_1 + \varepsilon_1} \left( \int_{x_0}^{x_0 + \varepsilon_0} \sum_{M < n \leq N - h} a_n e(xn^d) \left| dx - \varepsilon_0(N - M) \right|^2 dx_0 \right)^2 dx_1 \leq \log N \sum_{1 \leq h \leq N} \frac{1}{h} I_h,
\]

where

\[
I_h = \int_{x_1}^{x_1 + \varepsilon_1} \left| \sum_{M < n \leq N - h} \frac{\beta_{n,h}}{P_h(n)} e(x_0 h P_h(n)) \right|^2 dx_0.
\]
A second application of Lemma 3.3 and using that $|\beta_{n,h}| \ll 1$, yields

$$I_h \ll \varepsilon_1 \sum_{M < n \leq N} \frac{1}{P_h(n)^2}$$

$$+ \sum_{M < m < n \leq N} \frac{1}{P_h(m)P_h(n)} \frac{1}{|P_h(n) - P_h(m)|}.$$  \hspace{1cm} (3.2)

Since $P_h$ is a monic polynomial of degree $d-1$ with positive coefficients, if $n > m > M \gg N$ then

$$P_h(n) \gg N^{d-1} \quad \text{and} \quad |P_h(m) - P_h(n)| \gg (m - n)N^{d-2}.$$  \hspace{1cm} (3.3)

Indeed the first bound is obvious. To see that second bound holds, by the mean value theorem

$$|P_h(n) - P_h(m)| = (m - n)|P_h'(\eta)|,$$

for some $\eta \in [m, n]$. Since $\eta \geq m \gg N$ and

$$P_h'(\eta) = (d - 1)\eta^{d-2} + (d - 2)h\eta^{d-3} + \ldots + h^{d-2} \gg N^{d-2}$$

we obtain (3.3).

Now, using (3.3), we derive

$$\sum_{M < n \leq N} \frac{1}{P_h(n)^2} \ll N^{-2d+3}$$

and

$$\sum_{M < m < n \leq N} \frac{1}{P_h(m)P_h(n)} \frac{1}{|P_h(n) - P_h(m)|} \ll N^{-2(2d-1)-(d-2)} \sum_{M < m < n \leq N} \frac{1}{n-m} = N^{-3d+4} \sum_{M < m < n \leq N} \frac{1}{n-m} \ll N^{-3d+5} \sum_{1 \leq n \leq N} \frac{1}{n} \ll N^{-3d+5} \log N.$$

Substituting these inequalities in (3.2) gives

$$I_h \ll N^{-2d+3 + o(1)} \left( \varepsilon_1 + N^{-d+2} \right)$$
and combined with (3.1) yields
\[
\int_{x_1}^{x_1 + \varepsilon_1} \left| \int_{x_0}^{x_0 + \varepsilon_0} \left| \sum_{M < n \leq N} a_n e(xn^d) \right|^2 \, dx - \varepsilon_0(N - M) \right|^2 \, dx_0 \\
\leq N^{-2d+3+o(1)} \left( \varepsilon_1 + N^{-d+2} \right) \sum_{1 \leq h \leq N} \frac{1}{h} \\
\leq N^{-2d+3+o(1)} \left( \varepsilon_1 + N^{-d+2} \right),
\]
which completes the proof. \(\square\)

The main result of this subsection is the following. For two intervals \(I\) and \(J\) let \(\text{Dist}(I, J)\) denote the gap between them, that is,
\[
\text{Dist}(I, J) = \inf \{ \|x - y\| : x \in I, \ y \in J \}.
\]
We say that two intervals \(I\) and \(J\) are \(\Delta\)-separated if
\[
\text{Dist}(I, J) \geq \Delta.
\]

**Lemma 3.6.** Let \(d \geq 3\) be an integer. Let \(\tau > 0\) be a small parameter and let \(a = (a_n)_{n=1}^{\infty}\) be a sequence of complex weights satisfying \(|a_n| = 1\). For any interval \(I \subseteq [0, 1]\) and for all large enough \(N\) with
\[
|I| \geq N^{-d+2}
\]
there exists a collection of
\[
K \gg N^{d-1/2-\tau} |I|
\]
pairwise \(N^{-d+1/2+\tau}\)-separated intervals \(I_i \subseteq I, 1 \leq i \leq K\), such that
\[
|I_i| = N^{-d+1/2+\tau}
\]
and
\[
\max_{x \in I_i} \left| \sum_{\lfloor N/2 \rfloor \leq n \leq N} a_n e(xn^d) \right| \gg N^{1/2}.
\]

**Proof.** Let
\[
I = [x_1, x_1 + \varepsilon_1],
\]
for some \(x_1, \varepsilon_1\) with
\[
\varepsilon_1 = N^{-d+2+\tau}.
\]
Applying Lemma 3.5 with
\[
\varepsilon_0 = N^{-d+1/2+\tau}
\]
gives
\begin{equation}
\int_{\mathcal{I}} \left( \int_{x_0}^{x_0 + \varepsilon_0} \left| \sum_{M < n \leq N} a_n e(xn^d) \right|^2 dx - \varepsilon_0 (N - M) \right)^2 dx_0 \leq N^{-2d + 3 + o(1)}|\mathcal{I}|.
\end{equation}

Suppose $\varepsilon > 0$ is small and let $\mathcal{S} \subseteq \mathcal{I}$ denote the set of $x_0$ satisfying
\begin{equation*}
\int_{x_0}^{x_0 + \varepsilon_0} \left| \sum_{M < n \leq N} a_n e(xn^d) \right|^2 dx - \varepsilon_0 (N - M) \geq N^{-d + 3/2 + \varepsilon}.
\end{equation*}
The Cauchy-Schwarz inequality and (3.6) imply
\begin{equation*}
(\lambda(\mathcal{S})N^{-d + 3/2 + \varepsilon})^2 
\leq \lambda(\mathcal{S}) \int_{\mathcal{I}} \left( \int_{x_0}^{x_0 + \varepsilon_0} \left| \sum_{M < n \leq N} a_n e(xn^d) \right|^2 dx - \varepsilon_0 (N - M) \right)^2 dx_0
\leq N^{-2d + 3 + o(1)}|\mathcal{I}|\lambda(\mathcal{S}).
\end{equation*}
For sufficiently large $N$ this gives
\[ \lambda(\mathcal{S}) \leq \frac{N^{o(1)}|\mathcal{I}|}{N^{2\varepsilon}} \leq \frac{|\mathcal{I}|}{2}. \]

Hence for the set $\mathcal{A} = \{x \in \mathcal{I} : x \notin \mathcal{S}\}$ we have
\begin{equation}
\lambda(\mathcal{A}) \geq \frac{|\mathcal{I}|}{2}.
\end{equation}
With $\varepsilon_0$ as in (3.5), for each $\alpha \in \mathcal{A}$ let $\mathcal{B}_\alpha$ denote the interval
\[ \mathcal{B}_\alpha = [\alpha, \alpha + \varepsilon_0] \]
so that
\[ \mathcal{A} \subseteq \bigcup_{\alpha \in \mathcal{A}} \mathcal{B}_\alpha. \]
For an interval $\mathcal{J} = [x - r, x + r]$ denote $\mathcal{J}^{x5} = [x - 5r, x + 5r]$. Applying the Vitali Covering Theorem [10, Theorem 1.24] to the collection $\mathcal{B}_\alpha$, $\alpha \in \mathcal{A}$, there exists a subset $\mathcal{A}_1 \subseteq \mathcal{A}$ such that
\begin{equation}
\mathcal{A} \subseteq \bigcup_{\alpha \in \mathcal{A}} \mathcal{B}_\alpha \subseteq \bigcup_{\alpha \in \mathcal{A}_1} \mathcal{B}_\alpha^{x5}
\end{equation}
and for all $\alpha, \beta \in \mathcal{A}_1$ with $\alpha \neq \beta$ we have $\mathcal{B}_\alpha \cap \mathcal{B}_\beta \neq \emptyset$. Combining (3.7) with (3.8) we conclude

\begin{equation}
|\mathcal{I}| \ll \left| \bigcup_{\alpha \in \mathcal{A}_1} \mathcal{B}_\alpha \right| \ll \sum_{\alpha \in \mathcal{A}_1} |\mathcal{B}_\alpha|.
\end{equation}

It follows that $\mathcal{A}_1$ is a finite set. Note that there exists a subset $\mathcal{A}_2 \subseteq \mathcal{A}_1$ such that $\# \mathcal{A}_2 \gg \# \mathcal{A}_1$ and for all $\alpha, \beta \in \mathcal{A}_2$ with $\alpha \neq \beta$ we have

$$\text{Dist}(\mathcal{B}_\alpha, \mathcal{B}_\beta) \geq N^{-d+1/2+\tau},$$

which establishes the desired $N^{-d+1/2+\tau}$-separation. Combining this with (3.9) we derive

$$|\mathcal{I}| \ll \sum_{\alpha \in \mathcal{A}_2} |\mathcal{B}_\alpha| \ll N^{-d+1/2+\tau} \# \mathcal{A}_2,$$

which establishes the desired bound on

$$K = \# \mathcal{A}_2 \gg N^{d-1/2-\tau}|\mathcal{I}|.$$

It remains to show (3.4). Let $\alpha \in \mathcal{A}_2$ then

$$\left| \int_{\alpha}^{\alpha+\varepsilon_0} \left| \sum_{M<n\leq N} a_n e(xn^d) \right|^2 \, dx - \varepsilon_0(N-M) \right| \leq N^{-d+3/2+\varepsilon}.$$

Recalling the choice of $\varepsilon_0$ in (3.5) and that $M = \lfloor N/2 \rfloor$, after choosing $\varepsilon < \tau$, for large enough $N$ we obtain

$$\varepsilon_0(N-M) \geq 2N^{-d+3/2+\varepsilon}$$

and hence we conclude

$$\varepsilon_0 \max_{x \in I_0} \left| \sum_{M<n\leq N} a_n e(xn^d) \right|^2 \geq \int_{\alpha}^{\alpha+\varepsilon_0} \left| \sum_{M<n\leq N} a_n e(xn^d) \right|^2 \, dx \gg \varepsilon_0 N.$$

Changing the numbering of intervals $\mathcal{B}_\alpha$ from elements of $\mathcal{A}_2$ to $\mathcal{B}_i$, $i = 1, \ldots, K$, $K = \# \mathcal{A}_2$ we complete the proof. \hfill \Box

### 3.5. Hausdorff dimension of a class of Cantor sets

A typical way to obtain a lower bound for the Hausdorff dimension of some given set is to determine the Hausdorff dimension of a Cantor-like subset via the mass distribution principle, see [13, Chapter 4].

In this subsection we formulate a class of Cantor sets which is motivated by iterating the construction of Corollary 3.2. For convenience we introduce the following definition.
Definition 3.7 ($\mathcal{I}(N,M,\delta)$-patterns). Let $\mathcal{I}$ be an interval and $1 \leq M \leq N$ with $N \geq 2$ be natural numbers and let $|\mathcal{I}|/N \geq \delta > 0$. We divide the interval $\mathcal{I}$ into $N$ smaller subintervals of equal length. Among these $N$ subintervals, we choose $M$ distinct subintervals and denote them as $\mathcal{J}_1, \ldots, \mathcal{J}_M$. For each $\mathcal{J}_k$ with $1 \leq k \leq M$ we pick some subinterval $\mathcal{I}_k \subseteq \mathcal{J}_k$ with length $|\mathcal{I}_k| = \delta$. The resulting configuration of these $M$ intervals $\mathcal{I}_k$, $1 \leq k \leq M$ is called an $\mathcal{I}(N,M,\delta)$-pattern. See Figure 3.1 for an example.

Remark 3.8. We also use the notation $\mathcal{J}(N,M,\delta)$ when the above process is applied to the interval $\mathcal{J}$.

![Figure 3.1](image.png)

Figure 3.1. A sample of the $\mathcal{I}(N,M,\delta)$-pattern with $N = 6, M = 4$ and some positive $\delta$. The union of the intervals $\mathcal{I}_i$, $1 \leq i \leq 4$, forms the $\mathcal{I}(6,4,\delta)$-pattern.

We construct Cantor sets by iterating the above $\mathcal{I}(N,M,\delta)$-patterns.

Let $(M_k), (N_k)$ be two sequence natural numbers with $1 \leq M_k \leq N_k$ and $N_k \geq 2$ for all $k \in \mathbb{N}$. Let $(\delta_k)$ be a sequence of positive numbers with $\delta_0 = 1$ and $\delta_k \leq \delta_{k-1}/N_k$ for all $k \in \mathbb{N}$.

We start from the unit interval $\mathcal{I}_0 = [0,1]$. We take a $\mathcal{I}_0(N_1,M_1,\delta_1)$-pattern inside of the interval $\mathcal{I}_0$. Let $\mathcal{C}_1$ be the collection of these $M_1$-subintervals. More precisely, let

$$\mathcal{C}_1 = \{\mathcal{I}_i : 1 \leq i \leq M_1\}.$$ 

Note that each subinterval $\mathcal{I}_i$, $1 \leq i \leq M_1$, has length $\delta_1$. For each $\mathcal{I}_i$ we take a $\mathcal{I}_i(N_2,M_2,\delta_2)$-pattern inside of $\mathcal{I}_i$, and we denote these subintervals of $\mathcal{I}_i$ by $\mathcal{I}_{i,j}$ with $1 \leq j \leq M_2$. Let

$$\mathcal{C}_2 = \{\mathcal{I}_{i,j} : 1 \leq i \leq M_1, 1 \leq j \leq M_2\}.$$ 

Note that the choices of $\mathcal{I}_i(N_2,M_2,\delta_2)$-pattern and $\mathcal{I}_j(N_2,M_2,\delta_2)$-pattern are independent for $i \neq j$. 
Suppose that we have $C_k$ which is a collection of
\[ \#C_k = \prod_{i=1}^{k} M_i \]
intervals of length $\delta_k$. For each of these intervals $I \in C_k$ we select a $\mathcal{I}_i(N_{k+1}, M_{k+1}, \delta_{k+1})$-pattern inside of $I$. Let $C_{k+1}$ be the collection of these intervals, that is
\[ C_{k+1} = \{ I_{i_1, \ldots, i_{k+1}} : 1 \leq i_1 \leq M_1, \ldots, 1 \leq i_{k+1} \leq M_{k+1} \} . \]
Our Cantor-like set is defined by
\[ \mathcal{F} = \bigcap_{k=1}^{\infty} \mathcal{F}_k , \]
where
\[ \mathcal{F}_k = \bigcup_{I \in \mathcal{C}_k} I . \]

There are uncountably many possible configurations for the above construction, we let $\Omega((N_k), (M_k), (\delta_k))$ denote the set of all possible configurations.

For determining the Hausdorff dimension of such a set, we use the following mass distribution principle, see [13, Theorem 4.2].

**Lemma 3.9.** Let $\mathcal{X} \subseteq \mathbb{R}$ and $\mu$ a measure on $\mathbb{R}$ such that
\[ \mu(\mathcal{X}) > 0 . \]
If there exists $c, \delta > 0$ such for any interval $B(r)$ of length $r$ with $0 < r < \delta$ we have
\[ \mu(B(r)) \leq cr^s , \]
then $\dim \mathcal{X} \geq s . \]

We believe the following general result is of independent interest and may find some other applications.

**Lemma 3.10.** Using above notation, moreover suppose that
\[ M_k \geq c N_k , \quad k \in \mathbb{N} , \]
for some constant $c > 0$, then for any $\mathcal{F} \in \Omega((N_k), (M_k), (\delta_k))$ we have
\[ \dim \mathcal{F} = \lim inf_{k \to \infty} \frac{\log \prod_{i=1}^{k} M_i}{\log(1/\delta_k)} . \]
Proof. It is convenient to define
\[ P_k = \prod_{i=1}^{k} M_i. \]

Let
\[ s = \liminf_{k \to \infty} \frac{\log P_k}{\log(1/\delta_k)}. \]

For any \( \varepsilon > 0 \) there exists a subsequence \( k_n, \ n \in \mathbb{N} \) such that
(3.10) \[ P_{k_n} \leq \delta_{k_n}^{-s-\varepsilon} \]
for all large enough \( n \).

Observe that for each \( k_n \) the set \( \mathcal{F} \) is covered by \( P_{k_n} \) intervals and each of them has length \( \delta_{k_n} \). Combining with (3.10) we have
\[ \delta_{k_n}^{s+2\varepsilon} P_{k_n} \leq \delta_{k_n}^{\varepsilon}. \]

Thus the definition of Hausdorff dimension implies that \( \dim \mathcal{F} \leq s+2\varepsilon \).

By the arbitrary choice of \( \varepsilon > 0 \) we obtain that \( \dim \mathcal{F} \leq s \).

Now we use the mass distribution principle to obtain a lower bound for \( \dim E \). Thus we first construct a measure on \( \mathcal{F} \). For each \( k \) let \( \nu_k \) be a probability measure on \([0,1] \) such that
\[ \nu_k(I) = \frac{1}{\# \mathcal{C}_k} = \frac{1}{P_k^{-1}}, \quad \forall I \in \mathcal{C}_k, \]
where \( \mathcal{C}_k \) is the corresponding collection of \( \# \mathcal{C}_k = P_k \) intervals as in the above. The measure \( \nu_k \) weakly converges to a measure \( \mu \), see [19, Chapter 1].

Let \( 0 < t < s \) then for all large enough \( k \) we have
(3.11) \[ P_k \geq \delta_k^{-t}. \]

For any interval \( B(r) \) with \( 0 < r < 1 \) there exists \( k \in \mathbb{N} \) such that
\[ \delta_{k+1} < r \leq \delta_k. \]

Since the value \( \delta_{k+1} \) maybe quite smaller than the value \( \delta_k \), we do a case by case argument according to the value of \( r \).

**Case 1:** Suppose that \( \delta_k/N_{k+1} \leq r < \delta_k \). Since the interval \( B(r) \) intersects at most \( 3rN_{k+1}/\delta_k \) disjoint intervals of equal length \( \delta_k/N_{k+1} \), and inside each of these intervals there exists at most one interval of \( \mathcal{C}_{k+1} \), we obtain that
\[ \nu_{k+1}(B(r)) \ll \frac{rN_{k+1}}{\delta_k P_{k+1}}. \]
Applying the condition $M_k \geq cN_k$, the estimate \((\text{3.11})\) and the assumption $r < \delta_k$, we obtain

$$\nu_{k+1}(B(r)) \ll \frac{r}{\delta_k P_k} \ll \frac{r}{\delta_k} \delta_k^t = r\delta_k^{t-1} \ll r^t.$$ 

**Case 2:** Suppose that $\delta_{k+1} \leq r \leq \delta_k/N_{k+1}$. Note that the interval $B(r)$ intersects at most two intervals with equal length $\delta_k/N_{k+1}$ and thus meets at most two intervals of $\mathfrak{C}_{k+1}$. Combining with \((\text{3.11})\) and the assumption $\delta_{k+1} \leq r$, we have

$$\nu_{k+1}(B(r)) \leq \frac{2}{P_{k+1}} \ll \delta_{k+1} \leq r^t.$$ 

Putting **Case 1** and **Case 2** together, we conclude that

\((\text{3.12})\)  

$$\nu_{k+1}(B(r)) \ll r^t.$$ 

Note that for $\delta_{k+1} \leq r < \delta_k$ we have

$$\mu(B(r)) \leq \nu_{k+1}(B(3r)).$$ 

By \((\text{3.12})\) we obtain $\mu(B(r)) \ll r^t$. Applying Lemma 3.9, we arrive at $\dim \mathcal{F} \geq t$. By the arbitrary choice of $t < s$ we obtain that $\dim \mathcal{F} \geq s$, which finishes the proof. \qed

We formulate the following result which fits into our application immediately.

**Corollary 3.11.** Using above notation, suppose that

$$M_k \geq cN_k, \quad k \in \mathbb{N}$$

for some constant $c > 0$, and $M_k$ tends to infinity rapidly such that

$$\lim_{k \to \infty} \log \frac{\prod_{i=1}^{k-1} M_i}{\log M_k} = 0.$$ 

Then for any $\mathcal{F} \in \Omega((N_k), (M_k), (\delta_k))$ we have

$$\dim \mathcal{F} = \lim \inf_{k \to \infty} \frac{\log M_k}{\log(1/\delta_k)}.$$ 

4. **Proofs of Main Results**

4.1. **Proof of Theorem 2.1.** We intend to find a Cantor set inside of $\mathcal{E}_{a,c}(d)$ then apply results of Section 3.5 to obtain the desired lower bound of $\dim \mathcal{E}_{a,c}(d)$.

For the construction of the Cantor set, we start from the unit interval $I = [0, 1]$ and some large number $N$. Applying Lemma 3.6 to
the interval \( \mathcal{I} \) and the number \( N \), we obtain a collection (taking \( M_1 \) instead of \( K \)) of
\[
M_1 \gg N^{d-1/2-\tau} |\mathcal{I}|
\]
pairwise \( N^{-d+1/2+\tau} \)-separated intervals \( \mathcal{I}_i, 1 \leq i \leq M_1 \), satisfying
\[
|\mathcal{I}_i| = N^{-d+1/2+\tau}
\]
such that there exists some \( x_i \in \mathcal{I}_i \) satisfying
\[
\sum_{\lfloor N/2 \leq n \leq N} a_n e \left( x_i n^d \right) \gg N^{1/2}.
\]
Note that (4.2) implies
\[
\max_{Q \leq N} \left| \sum_{n=1}^{Q} a_n e \left( x_i n^d \right) \right| \gg N^{1/2}.
\]
Indeed, suppose that (4.3) is false. Then both
\[
\left| \sum_{n=1}^{N} a_n e \left( x_i n^d \right) \right| \ll N^{1/2} \quad \text{and} \quad \left| \sum_{n=1}^{\lfloor N/2 \rfloor} a_n e \left( x_i n^d \right) \right| \ll N^{1/2}
\]
and hence by the triangle inequality
\[
\left| \sum_{\lfloor N/2 \leq n \leq N} a_n e \left( x_i n^d \right) \right| \ll \left| \sum_{n=1}^{N} a_n e \left( x_i n^d \right) \right| + \left| \sum_{n=1}^{\lfloor N/2 \rfloor} a_n e \left( x_i n^d \right) \right| \ll N^{1/2}
\]
contradicting (4.2) for a suitable choice of implied constants.

Furthermore, since the intervals \( \mathcal{I}_i, 1 \leq i \leq M_1 \), are \( N^{-d+1/2+\tau} \)-separated, that is
\[
\text{Dist}(\mathcal{I}_i, \mathcal{I}_j) \geq N^{-d+1/2+\tau}, \quad 1 \leq i < j \leq M_1,
\]
we obtain that
\[
|x_i - x_j| \geq N^{-d+1/2+\tau}, \quad 1 \leq i < j \leq M_1.
\]

We now set
\[
N_1 = \left\lceil N^{d-1/2-\tau} \right\rceil + 1
\]
and divide the interval \([0, 1]\) into \( N_1 \) subintervals of equal length \( N_1^{-1} \). Note that the choice of \( N_1 \) makes sure that the length of the subinterval is slightly smaller than \( N^{-d+1/2+\tau} \).

For each \( 1 \leq i \leq M_1 \), among the above \( N_1 \) subintervals there is an interval \( \mathcal{J}_i \) containing \( x_i \). Indeed if \( x_i \) meets two of them then we choose one only. By (4.4) we conclude that \( \mathcal{J}_k \) and \( \mathcal{J}_\ell \) are separated for
all $1 \leq k < \ell \leq M \ell`. In fact what we need in the following construction is that $J_k \neq J_\ell$ for $1 \leq k < \ell \leq M \ell$. For each $J_k$, the estimate (4.3) and Corollary 3.2 imply that there exists a subinterval $\tilde{J}_i \subseteq J_k$ with length $\delta_1 = N^{-d-\tau}$ such that 
\[
\max_{Q \leq N} \left| \sum_{n=1}^{Q} a_n e (x n^d) \right| \gg N^{1/2}, \quad \forall x \in \tilde{J}_i.
\]

We now note that the collection of intervals $\tilde{J}_i$, $1 \leq i \leq M \ell$, forms a $I(N_1, M_1, \delta_1)$-pattern as in Definition 3.7.

Let 
\[
C_1 = \{ \tilde{J}_i : i = 1, \ldots, M \ell \}.
\]

Moreover, by (4.1) and (4.5) we have $M \ell \gg N_1$ where the implied constant is absolute.

Suppose we have constructed a sequence $C_1, \ldots, C_k$ where $C_k$ is a union of disjoint intervals $I_i$, $1 \leq i \leq \#C_k$, of equal length $\delta_k$. We next construct a set $C_{k+1}$ which is a union of disjoint intervals of equal length $\delta_{k+1}$ for suitable $\delta_{k+1}$.

Let $L_k$ satisfy 
\[
(4.6) \quad \delta_k \geq L_k^{-d+2},
\]
which is chosen so our parameters in the construction of $C_{k+1}$ satisfy the conditions of Lemma 3.6. For each interval $I \in C_k$, we use a similar argument to the above construction of $C_1$. To be precise, let 
\[
N_{k+1} = \lceil \delta_k L_k^{d-1/2-\tau} \rceil + 1.
\]

We divide the interval $I$ into $N_{k+1}$ subintervals of equal length $\delta_k N_{k+1}^{-1}$. Note that the choice of $N_{k+1}$ make sure that the length of the subinterval is slightly smaller than $L_k^{-d+1/2+\tau}$.

For the interval $I$ and $L_k$, applying Lemma 3.6, we conclude that among these $N_{k+1}$ intervals, there are $M_{k+1}$ intervals $J_{I,1}, \ldots, J_{I,M_{k+1}}$ of length $L_k^{-d+1/2+\tau}$ such that for each $1 \leq \ell \leq M_{k+1}$ there is a $x_\ell \in J_{I,\ell}$ satisfying 
\[
\max_{Q \leq L_k} \left| \sum_{n=1}^{Q} a_n e (x_\ell n^d) \right| \gg L_k^{1/2}.
\]

Furthermore, 
\[
N_{k+1} \geq M_{k+1} \gg L_k^{d-1/2-\tau} \delta_k \gg N_{k+1}.
\]

For each $x_\ell$, $1 \leq \ell \leq M_{k+1}$, by Corollary 3.2 there exists a subinterval $\tilde{J}_{I,\ell} \subseteq J_{I,\ell}$ such that 
\[
|\tilde{J}_{I,\ell}| = \delta_{k+1} = L_k^{-d-\tau}
\]
and

\[
(4.7) \quad \max_{Q \leq L_k} \left| \sum_{n=1}^{Q} a_n e(xn^d) \right| \gg L_k^{1/2}, \quad \forall x \in \mathcal{F}_{I, \ell}.
\]

Thus the collection of $\mathcal{F}_{I, \ell}$ forms a $\mathcal{I}(N_{k+1}, M_{k+1}, \delta_{k+1})$ pattern.

Let $\mathcal{C}_{k+1}$ be the collection of these $\mathcal{I}(N_{k+1}, M_{k+1}, \delta_{k+1})$ patterns with $\mathcal{I} \in \mathcal{C}_k$. Our desired Cantor set is defined as

\[
\mathcal{F} = \bigcap_{k=1}^{\infty} \mathcal{F}_k,
\]

where

\[
\mathcal{F}_k = \bigcup_{\mathcal{I} \in \mathcal{C}_k} \mathcal{I}.
\]

Note that the set $\mathcal{F}$ is an element of $\Omega((N_k), (M_k), (\delta_k))$ as defined in Subsection 3.5.

Now we are going to show that

\[
(4.8) \quad \mathcal{F} \subseteq \mathcal{E}_{a,c}(d).
\]

Let $x \in \mathcal{F}$ then $x \in \mathcal{F}_{k+1}$ for all $k \in \mathbb{N}$. The estimate (4.7) implies that there exists $Q_k$ such that

\[
L_k^{1/2} \ll Q_k \leq L_k,
\]

and

\[
\left| \sum_{n=1}^{Q_k} a_n e(xn^d) \right| \gg |Q_k|^{1/2}.
\]

For each $k$ we choose $L_k$ large enough such that

\[
(4.9) \quad Q_1 < Q_2 < \ldots,
\]

which implies

\[
\sum_{n=1}^{Q} a_n e(xn^d) \gg Q^{1/2}
\]

for infinitely many $Q \in \mathbb{N}$ and hence we have (4.8). Therefore we obtain

\[
(4.10) \quad \dim \mathcal{E}_{a,c}(d) \geq \dim \mathcal{F}.
\]

Note that for each $k$ we can choose $L_k$ even larger such that the conditions (4.6), (4.9) hold, and moreover

\[
\lim_{n \to \infty} \frac{\log \prod_{i=1}^{n} N_i}{\log N_{n+1}} = 0.
\]
Applying Corollary 3.11 we obtain that
\[
\dim \mathcal{F} = \liminf_{k \to \infty} \frac{\log N_k}{\log(1/\delta_k)} = \frac{d - 1/2 - \tau}{d + \tau}.
\]
By (4.10) we derive
\[
\dim \mathcal{E}_{a,c}(d) \geq \frac{d - 1/2 - \tau}{d + \tau}.
\]
Since this holds for any \( \tau > 0 \), we obtain \( \dim \mathcal{E}_{a,c}(d) \geq 1 - 1/2d \).

4.2. Proof of Theorem 2.2. We first formulate an equivalent version of [20, Proposition 6.6] in the following.

**Lemma 4.1.** Let \( \mathcal{A} \subseteq \mathbb{R}^d \), \( d \geq 2 \), \( t > 0 \) and \( \mathcal{V} \) be a \((d - 1)\)-dimensional subspace. Suppose that
\[
\lambda \left( \{ a \in \mathcal{V} : \dim(\mathcal{A} \cap (a + \mathcal{V}^\perp)) \geq t \} \right) > 0,
\]
where \( \mathcal{V}^\perp \) is the orthogonal complement space. Then we have
\[
\dim \mathcal{A} \geq d - 1 + t.
\]

We now turn to the proof of Theorem 2.2. Let
\[
\mathbf{x} = (\mathbf{x}, x_d) = (x_1, \ldots, x_d) \in \mathbb{T}_d.
\]
Denote \( a_n e\left(x_1 n + \ldots x_d n^d\right) = b_n(\mathbf{x}) e\left(x_d n^d\right) \), and hence
\[
S_{a,d}(\mathbf{x}; N) = b_n(\mathbf{x}) e\left(x_d n^d\right).
\]
Clearly \( |b_n(\mathbf{x})| = 1 \) for all \( n \in \mathbb{N} \). Theorem 2.1 implies
\[
\dim \left( \left\{ x_d \in \mathbb{T} : \left| \sum_{n=1}^{N} b_n(\mathbf{x}) e\left(x_d n^d\right) \right| \geq cN^{1/2} \right\} \right) \geq 1 - 1/2d.
\]

For \( \mathbf{x} \in \mathbb{T}_{d-1} \) denote
\[
\ell_{\mathbf{x}} = \{ \mathbf{x} + (0, \ldots, 0, t) : t \in \mathbb{R} \}.
\]
Applying (4.11) and (4.12) we conclude that
\[
\dim (\mathcal{E}_{a,c}(d) \cap \ell_{\mathbf{x}}) \geq 1 - 1/2d.
\]
Moreover, this holds for all \( \mathbf{x} \in \mathbb{T}_{d-1} \). Combining with Lemma 4.1 we obtain \( \dim \mathcal{E}_{a,c}(d) \geq d - 1/2d \) which finishes the proof.
4.3. **Proof of Theorem 2.3.** The proof is similar to the proof of Theorem 2.1, so we only give a sketch. In particular, we first find a Cantor-like subset of \( E_{a,c}(2) \) and then apply Corollary 3.11 to obtain the dimension of the Cantor set which gives a lower bound on the dimension of \( E_{a,c}(2) \).

Fix a small parameter \( \tau > 0 \). Let \( I \) be an interval and \( N \in \mathbb{N} \). Divide \( I \) into \( N \) subintervals in a natural way and denote them as \( I_1, \ldots, I_N \). Applying Lemma 3.4 to each interval \( I_k \), we obtain that there exists \( x_k \in I_k \) such that

\[
\left| \sum_{n=1}^{Q_N} a_n e\left(x_k n^2\right) \right| \gg Q_N^{1/2},
\]

where \( Q_N \) is the smallest natural number such that

\[
N^{-1} \geq Q_N^{-1+\tau}.
\]

Note that \( Q_N \) is nearly the same size as \( N \).

For each \( x_k \), applying Corollary 3.2 with \( d = 2 \) we obtain that there exists an interval \( J_k \subseteq I_k \) with length \( |J_k| = Q_N^{-2+\tau} \) such that

\[
\left| \sum_{n=1}^{Q_N} a_n e\left(x n^2\right) \right| \gg Q_N^{1/2}, \quad \forall x \in J_k.
\]

Note that the collection of intervals \( J_k, 1 \leq k \leq 1, \) forms an \( \mathcal{I}(N, N, Q_N^{2-\tau}) \)-pattern as in Definition 3.7.

By iterating the above construction inside the initial interval \([0, 1]\), together with a rapidly increasing sequence of numbers

\[
N_1 < N_2 < \ldots,
\]

we obtain the desired Cantor-like set. Indeed suppose that we have constructed \( \mathcal{C}_k \) which is a collection of disjoint intervals with equal length \( \delta_k \). Then let \( N_{k+1} \) be large enough in terms of \( N_1, \ldots, N_k \). For instance, the following condition is sufficient for our application

\[
\log N_{k+1} \geq N_1 N_2 \ldots N_k.
\]

We divide each interval \( J \in \mathcal{C}_k \) into \( N_{k+1} \) subintervals in a natural way. Applying the same argument as above to the interval \( J \) and \( N_{k+1} \) we conclude that there exists a \( \mathcal{J}(N_{k+1}, N_{k+1}, \delta_{k+1}) \)-pattern \( \mathcal{A} \subseteq \mathcal{J} \) such that

\[
N_{k+1}^{-2} \leq \delta_{k+1}^{1+o(1)},
\]

and

\[
\left| \sum_{n=1}^{Q_{N+1}} a_n e\left(x n^2\right) \right| \gg Q_{N+1}^{1/2}, \quad \forall x \in \mathcal{A}.
\]
Let $C_{k+1}$ be a collection of the $\mathcal{J}(N_{k+1}, N_{k+1}, \delta_{k+1})$-pattern inside each interval $\mathcal{J} \in C_k$, see Remark 3.8. The Cantor-like subset is defined as

$$C = \bigcap_{k=1}^{\infty} C_k.$$ 

By (4.13) and (4.14) we conclude that for each $k \in \mathbb{N}$ the set $C_{k+1}$ contains nearly $N_{k+1}^{1+o(1)}$ intervals with equal length nearly $N_{k+1}^{-2}$. Combining with Corollary 3.11 we conclude that

$$\dim C \geq 1/2,$$

which finishes the proof.

4.4. **Proof of Theorem 2.4.** Applying Theorem 2.3, Lemma 4.1 and a similar argument to the proof of Theorem 2.2, we obtain the desired result.

5. **Comments**

Throughout the paper we restrict $|a_n| = 1$ for all $n \in \mathbb{N}$. However, our methods work for general complex sequence as well. Since we looking for lower bounds of exponential sums, a necessary condition for the sequence is that they are not so small. For instance for sequences $a$ with

$$\sum_{n=1}^{\infty} |a_n| < \infty$$

we are not able to derive any “interesting” lower bound. On the other hand, it seems that for any sequence $a_n$ such that

$$a_n = n^{o(1)}, \quad \forall n \in \mathbb{N}, \quad \text{and} \quad \sum_{N/2 \leq n \leq N} |a_n| \geq N^{1+o(1)}$$

for some absolute constant $c$, our methods yield the same bounds as in our main results.

Furthermore, it is easy to see that all our bounds, without any changes in the argument, extend to the intersections of the sets $\mathcal{E}_{a,c}(d)$ and $\mathcal{E}_{a,c}(d)$ with arbitrary intervals $\mathcal{I} \subseteq \mathcal{T}$ and cubes $\Omega \subseteq \mathcal{T}_d$, respectively. The only change is that in the construction of $\mathcal{I}(N, M, \delta)$-patterns we now have to start with $\mathcal{I}_0 = \mathcal{J}$ rather than $\mathcal{I}_0 = [0,1]$ as in Section 3.5. That is, we have

$$\dim (\mathcal{E}_{a,c}(d) \cap \mathcal{J}) \geq 1 - 1/2d,$$

$$\dim (\mathcal{E}_{a,c}(d) \cap \Omega) \geq d - 1/2d,$$

(5.1)
for \( d \geq 3 \) and also
\[
\dim (\mathcal{E}_{a,c}(2) \cap \mathcal{I}) \geq 1/2, \\
\dim (\mathcal{E}_{a,c}(2) \cap \mathcal{Q}) \geq 3/2, 
\]
thus showing that the sets \( \mathcal{E}_{a,c}(d) \) and \( \mathcal{E}_{a,c}(d) \) are “everywhere rich”.

The bounds (5.1) and (5.2) also have an alternative interpretation in terms of the local Hausdorff dimension, introduced by Jürgensen and Staiger [15], see also [9,18,22,23]. Namely, given a set \( \mathcal{F} \subseteq \mathbb{R}^d \), we define its local Hausdorff dimension at \( \mathbf{x} \in \mathbb{R}^d \) as
\[
\dim_{\text{loc}} (\mathbf{x}, \mathcal{F}) = \lim_{r \downarrow 0} \dim \left( \mathcal{F} \cap B(\mathbf{x}, r) \right),
\]
where \( B(\mathbf{x}, r) \) is a ball of radius \( r \) centred at \( \mathbf{x} \) and \( r > 0 \) is monotonically decreases to 0. Then (5.1) and (5.2) mean the existence of uniform lower bounds on the local Hausdorff dimension of the corresponding sets at any point \( \mathbf{x} \in \mathbb{T}_d \).

We also observe that our method works without any substantial changes for a much large class of exponential sums. Namely, given a function \( f : \mathbb{N} \to \mathbb{Z} \) on the set of positive integers, we consider the sums
\[
T_{a,f}(x; N) = \sum_{n=1}^{N} a_n \mathbf{e}(xf(n)).
\]
For a constant \( c > 0 \) we now define the set
\[
\mathcal{E}_{a,c,f} = \{ x \in [0, 1) : |T_{a,f}(x; N)| \geq cN^{1/2} \text{ for infinitely many } N \in \mathbb{N} \},
\]
Let \( \Delta_h \) denote the difference operator
\[
\Delta_h f(n) = f(n + h) - f(n),
\]
and then as usual we write
\[
\Delta^2_h f(n) = \Delta_h (\Delta_h f(n)) = f(n + 2h) - 2f(n + h) + f(n).
\]
Then our method gives a lower bound on the Hausdorff dimension of the set \( \mathcal{E}_{a,c,f} \) for functions \( f : \mathbb{N} \to \mathbb{Z} \) such that for some fixed real positive \( \vartheta \) and \( \rho \), we have
\[
|f(n)| \leq n^{\vartheta+o(1)} \quad \text{and} \quad |\Delta^2_h f(n)| \geq n^{\rho+o(1)}.
\]

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