DYNAMICAL EFFECTS OF CHERN-SIMONS TERM

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Abstract

We study how the Chern-Simons term effects the dynamically generated fermion mass in (2+1)D Quantum Electrodynamics in the framework of large $N$ expansion. We find that when the Chern-Simons term is present half of the fermions get mass $M + m$ and half get $M - m$. The parity-preserving mass $m$ is generated only when $N < \tilde{N}_c$. Both the critical number, $\tilde{N}_c$, of fermion flavor and the magnitude, $m$, reduce when the effect of the Chern-Simons term dominates.

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1 Motivation

One of the most drastic effects of the Chern-Simons term, $\mathcal{L}_{CS} = \kappa \epsilon_{\mu \nu \lambda} A^\mu \partial^\nu A^\lambda$, in field theories, is the generation of fractional spin. The Chern-Simons term induces fractional spin to a particle coupled to the Chern-Simons gauge field, $A_\mu$ \cite{1}: the spin is given as

$$s = \frac{e^2}{4\pi \kappa}.$$  \hspace{1cm} (1)

Due to this effect, the Chern-Simons theory is not only interesting, field theoretically, but also has some applications in the condensed matter systems like the fractional or integer quantum Hall system \cite{2}. The Chern-Simons term is topological in a sense that it does not involve a metric and thus does not contribute to energy-momentum tensor of the system. But, it modifies the equations of motion and breaks parity and Time-reversal symmetry, which must have dynamical significances. One nice example for this dynamical effect is the existence of a stable vortex solution found recently \cite{3} in a system described by a Lagrangian density,

$$\mathcal{L} = \frac{1}{2} |D_\mu \phi|^2 - V(|\phi|) + \mathcal{L}_{CS},$$  \hspace{1cm} (2)

where $D_\mu = \partial_\mu - ieA_\mu$ and $V(|\phi|)$ is a $\phi^6$ potential. In (2+1)-dimensions, the kinetic energy of the static soliton is scale-invariant, while the potential energy is not. Therefore, without a gauge field or other balancing force, the static soliton is unstable against collapsing to the center of the soliton \cite{4}. Namely, it is energetically favorable for the soliton to collapse to $\phi = v$, where $v$ is the minimum of the potential. One would think that adding the Chern-Simons term does not do any good in stabilizing the soliton, since it does not contribute to the energy of the soliton. But, this is not true, since not only the energy of the soliton has now a term depending on the gauge field, $\frac{1}{2} A^2_\mu |\phi|^2$, but also the Chern-Simons term modifies the equation of motion, and thus a stable vortex solution is possible.
In this talk, I would like to present another effect of the Chern-Simons term \[5\], namely the dynamical effect of the parity-noninvariance of the Chern-Simons term.

2 Parity and Mass in \((2 + 1)\)-dimensions

Consider a \((2 + 1)D\) QED, described by a Lagrangian density

\[
\mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F_{\mu \nu}^2 + \mathcal{L}_{\text{mass}}
\]

where \(\psi\) is a two-component spinor and the gamma matrices are chosen as \(\gamma^0 = \sigma^3, \gamma^1 = i\sigma^1, \gamma^2 = i\sigma^2\). The mass terms for the fermion and the photon are

\[
\mathcal{L}_{\text{mass}} = -m \bar{\psi} \psi + \kappa \epsilon_{\mu \nu \lambda} A^\mu \partial^\nu A^\lambda
\]

In \((2+1)\)-dimensions, parity is defined to be a coordinate transformation, \(P : x = (x, y, t) \mapsto x' = (-x, y, t)\), under which the fields transform as following:

\[
\begin{align*}
A^{0,2}(x) & \mapsto A^{0,2}(x') = A^{0,2}(x) \\
A^1(x) & \mapsto A^1(x') = -A^1(x) \\
\psi(x) & \mapsto \psi'(x') = e^{i\delta} \sigma^1 \psi(x)
\end{align*}
\]

One can therefore easily see that the both mass terms are odd under parity (and also under time-reversal). If either of the mass terms is absent at tree level, it will be generated radiatively, since the parity, which forbids the mass term, is broken by the other mass term explicitly. For example, when the (topological) mass term for the gauge field is absent, the fermion mass term will generate it radiatively with a coefficient \(\kappa = \frac{e^2}{8\pi} \frac{m}{|m|} \) \[6\]. Similarly, when the fermion mass term is absent, the Chern-Simons term will generate it at one-loop; one needs a counter-term to remove the divergence in the fermion mass, \(\delta m = -\frac{6}{\pi} \frac{e^2}{\kappa} |M|\), where \(M\) is the Pauli-Villas regulator \[7\].
When the number of the fermion flavors is even, the system has another obvious
discrete symmetry, $Z_2$, which interchanges half of the fermions with another half;
for $i = 1, \cdots, \frac{N}{2}$, $Z_2$ mixes the fermions fields as

$$
\psi_i(x) \rightarrow \psi_{\frac{N}{2}+i}(x)
$$

$$
\psi_{\frac{N}{2}+i} \rightarrow \psi_i(x)
$$

(7)

If we define a new parity, $P_4 \equiv PZ_2$, combining the old one with $Z_2$, then the
fermions can have “parity($P_4$)-invariant” mass, $m_i \bar{\psi}_i \psi_i$, with

$$
m_i = \begin{cases} 
m, & \text{if } 1 \leq i \leq \frac{N}{2} \\
-m, & \text{if } \frac{N}{2} + 1 \leq i \leq N
\end{cases}
$$

(8)

With this form of fermion mass, the Chern-Simons term will not be generated ra-
diatively. We call this “parity($P_4$)-even mass”. On the other hand, this $P_4$-invariant
fermion mass can be generated dynamically due to a non-perturbative effect, though
$L_{\text{mass}}$ is not present in the Lagrangian; namely $P$ is spontaneously broken, while $P_4$
is not. Appelquist et. al. [8] showed, using $1/N$-expansion, that, if $1/N > 1/N_c$
with $N_c = 32/\pi^2$, the fermions condensate and thus the parity-even fermion mass is
generated dynamically and the mass is given as

$$
m_{\text{even}} = \alpha e^{-\frac{2\pi^2}{16N}}
$$

(9)

When the Chern-Simons term is present, this parity-even mass will be affected.
As described below, due to the Chern-Simons term, $1/N_c$ increases (one needs a
stronger interaction to form a fermion condensate) and the magnitude of the parity-
even fermion mass decreases.

3 Gap Equation

First, we will examine the pattern of the spontaneous breaking of parity in the
$(2 + 1)D$ QED with $N$ complex two-component spinors, and then elaborate on the
dynamical mass generation for fermions. An order parameter for the spontaneous breaking of parity is the vacuum condensate of the fermion bilinear, \( \langle \bar{\psi} \psi(x) \rangle \), which can be determined once one finds the asymptotic behavior of the fermion propagator \[9\]. We use the \( 1/N \)-expansion technique, since it is a non-perturbative technique and also the IR-divergence of \((2 + 1)D\) QED softens in the large \( N \) limit \[10\].

At the leading order in the \( 1/N \) expansion, the Dyson-Schwinger gap equation is, in Euclidean notation,

\[
- (Z(p) - 1) \not{p} + \Sigma(p) = \frac{\alpha}{N} \int \frac{d^3 k}{(2\pi)^3} D_{\mu\nu}(p - k) \gamma_{\nu} \frac{Z(k) \not{k} - \Sigma(k)}{Z^2(k) k^2 + \Sigma^2(k) \gamma_{\mu}},
\]  

(10)

where \( D_{\mu\nu}^{-1} = \Delta_{\mu\nu}^{-1} - \Pi_{\mu\nu} \) and \( \Delta_{\mu\nu} \) is the free Landau gauge propagator and \( \alpha \equiv e^2 N \) is kept fixed while \( N \to \infty \). \( \Sigma \) is the fermion self energy. The vacuum polarization tensor is given as

\[
\Pi_{\mu\nu} = \Pi_{\text{even}}(p) \left( \delta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \right) + \Pi_{\text{odd}}(p) \epsilon_{\mu\nu\lambda} p^\lambda.
\]

(11)

From the equation (10), taking trace over the gamma matrix, we get

\[
\Sigma(p) = \frac{\alpha}{N} \int \frac{d^3 k}{(2\pi)^3} \frac{2\Pi_1(p - k)}{(p - k)^2} \frac{\Sigma(k)}{k^2 + \Sigma^2(k)} + \frac{\alpha}{N} \int \frac{d^3 k}{(2\pi)^3} \frac{(p - k) \cdot k}{|p - k|^3} \frac{\Pi_2(p - k)}{k^2 + \Sigma^2(k)}
\]

(12)

where \( \Pi_1 \) and \( \Pi_2 \) contain the quantum corrections to the parity-even part and the parity-odd part of the photon propagator;

\[
D_{\mu\nu}(p) = \frac{\delta_{\mu\nu} - p_{\mu} p_{\nu}/p^2}{p^2} \Pi_1(p) + \frac{\epsilon_{\mu\nu\lambda} p^\lambda}{|p|^2} \Pi_2(p),
\]

(13)

with

\[
\Pi_1(p) = \frac{1 - \Pi_{\text{even}}(p)/p^2}{(\Pi_{\text{even}}(p)/p^2)^2 + (\kappa - \Pi_{\text{odd}}(p))/p^2},
\]

(14)

\[
\Pi_2(p) = -\frac{(\kappa - \Pi_{\text{odd}}(p))/|p|}{(\Pi_{\text{even}}(p)/p^2)^2 + (\kappa - \Pi_{\text{odd}}(p))/p^2}.
\]

(15)

In the large-\( N \) approximation the dynamically generated mass will be at most of order of \( 1/N \), compared to the scale of the theory, \( \Lambda \). (\( \Lambda \) is of same order as \( \alpha \) or
κ.) In the region, $\Sigma(p) \ll p$, the vacuum polarization tensor takes a simple form;

\begin{align}
\Pi_{\text{even}}(p) &= -\frac{\alpha}{16} |p| \\
\Pi_{\text{odd}}(p) &= \frac{1}{N} \sum_{i=1}^{N} M_i \frac{\alpha}{4 |p|}
\end{align}

where $M_i \simeq \Sigma_i(0)$, the mass of the $i$-th fermion. (In general $\Sigma(p)$ will depend on the flavor but we will suppress the index $i$ for simplicity.)

To find the physical mass, we have to solve

\[ p^2 + \Sigma^2(p) = 0 \text{ at } p^2 = -m^2_{\text{phy}} \]

But, since $\Sigma(p)$ is quite small, compared to the scale of the theory ($\alpha$ or $\kappa$), we may take

\[ m_{\text{phy}} \simeq \Sigma(0). \]

Then, the gap equation (10) becomes

\[ \Sigma_i(0) = \frac{\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \left( \frac{2\Pi_1(k)M_i}{k^2(k^2 + M_i^2)} - \frac{\Pi_2(k)}{|k| (k^2 + M_i^2)} \right) \]

Note that the first term depends on the flavor while the second term does not. The parity ($P_4$) is maximally broken when the second term is dominant, which happens precisely when the Chern-Simons term is dominant. On the other hand, if Chern-Simons term is not present, the mass will be generated in a parity-invariant way. Namely, half the fermions get positive mass $m$ and the other half negative mass $-m$.

Therefore, when both of the Chern-Simons term and the Maxwell term are present, it is reasonable to assume the pattern of the fermion mass as

\[ M_i = M + m_i, \]

with

\[ m_i = \begin{cases} 
m_i, & i = 1, \ldots, N - L, \\
-m, & i = N - L + 1, \ldots, N
\end{cases} \]
By plugging this Ansatz into Eq. (20), we obtain

\[ M = \frac{\alpha}{2\pi^2 N} \int_m^\Lambda dk \frac{k^2}{(k^2 + M^2)} \left( \frac{\alpha}{16} \right)^2 + \kappa^2 \approx \frac{\Lambda}{2\pi^2 N} \left( \frac{\alpha}{16} \right)^2 + \kappa^2 \]  

(23)

and

\[ m_i = \frac{1}{\pi^2 N} \left( \frac{\alpha}{16} \right)^2 + \kappa^2 \int_m^\Lambda dk \left( \frac{16m_i}{k} - \frac{64}{k} \theta m \right), \]  

(24)

where \( \theta = 1 - 2L/N \). For Eq. (24) to have a consistent solution, \( \theta = 0 + O(1/N) \), which yields, upon integration,

\[ m = \Lambda \exp(-4N/\tilde{N}_c) \]  

(25)

where \( \tilde{N}_c = N_c/(1+(16\kappa/\alpha)^2) \). The value for the parity-violating mass \( M \) is a perturbative one in the \( 1/N \) expansion, while the parity-preserving mass is nonperturbative.

The effect of the Chern-Simons term is now clear. It induces a parity-violating mass perturbatively and it decreases in a nonperturbative way the magnitude of the parity-preserving mass \( m \). Since \( \theta = 0 \), half the fermions get (positive) mass \( M + m \) and the other half \( M - m \). The pattern of the flavor-symmetry breaking is same whether the Chern-Simons term is present or not.

### 4 Conclusion

In conclusion, we find that, when the Chern-Simons term is present, the parity tends to be maximally broken; the magnitude of the parity-even (four-component) mass for the fermions gets smaller, and the critical number for the generation of the parity-even mass decreases. But, in the large-\( N \) limit, the pattern of the flavor-symmetry breaking does not depend on the Chern-Simons term.
References

[1] See, for example, “Fractional Statistics and Anyon Superconductivity”, edited by F. Wilczek, World Scientific Publishing Co. Pte. Ltd.

[2] B.I. Halperin, Phys. Rev. Lett 52 (1984) 2390.

[3] J. Hong, Y. Kim, and P.Y. Pac, Phys. Rev. Lett. 64 (1990) 2230; R. Jackiw and E. Weinberg, ibid. 2234.

[4] D.K. Hong, Journ. of Kor. Phys. Soc. Vol. 25, No. 3 (1992) 187.

[5] D.K. Hong and S.H. Park, Phys. Rev. D 47 (1993) 3651.

[6] A.N. Redlich, Phys. Rev. D 29 (1984) 2366.

[7] D.K. Hong, T. Lee, and S.H. Park, in preparation.

[8] T. Appelquist, M. Bowick, D. Karabali, and L. Wijewardhana, Phys. Rev. D 33 (1986) 3704; R. Pisarski, Phys. Rev. D 29 (1984) 2423; T. Appelquist, M. Bowick, D. Karabali, and L. Wijewardham, Phys. Rev. D 33 (1986) 3774; R. Pisarski, Phys. Rev. D 29 (1984) 2423.

[9] A. Cohen and H. Georgi, Nucl. Phys. B 314 (1989) 7.

[10] T. Appelquist and R. Pisarski, Phys. Rev. D 23 (1981) 2305; T. Appelquist and U. Heinz, ibid. 25 (1981) 2169; 25 (1982) 2620.