Submitted to “Annual Review of Condensed Matter Physics”

Dynamics of Simple Cracks

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November 2, 2009

Abstract

Cracks are the major vehicle for material failure, and often exhibit rather complex dynamics. The laws that govern their motion have remained an object of constant study for nearly a century. The simplest kind of dynamic crack is a single crack that moves along a straight line. We first briefly review current understanding of this “simple” object. We then critically examine the assumptions of the classic, scale-free, theory of dynamic fracture, and note when it works and how it may fail if certain of these assumptions are relaxed. A number of examples is provided, where the introduction of physical scales into this scale-free theory profoundly affects both a crack’s structure and the resulting dynamics.

1 Introduction

1.1 Physics of Cracks

The ability of solids to withstand mechanical forces is one of their fundamental properties. Solids divide roughly into two classes, brittle and ductile. The brittle
solids break easily, catastrophically. Ductile solids flow or bend before they break and are more resilient. There is a simple thermodynamic definition of an ideally brittle solid.

Consider any solid sample of height $b$ whose top and bottom boundaries are displaced by distances $\pm \delta/2$, as shown in Fig. 1. If the total displacement $\delta$ is sufficiently small, then the restoring force $F$ on the boundaries will be $F = CA\delta/b$, where $C$ is an elastic constant, and the change in energy of the solid must be

$$\Delta E = \frac{1}{2}CA\delta^2/b.$$  

(1)

For all solids one can also define the surface energy $\gamma$, where $2\gamma$ is the minimum energy per area needed to separate bonds and cut the solid into two pieces. For a given solid $\gamma$ may depend upon the angle of the cut, but we will assume for simplicity that the solid is oriented so that $\gamma$ has a minimum along the plane perpendicular to $\vec{F}$. Then there is a critical displacement $\delta_G$ for which the solid will have lower energy by separating into two pieces than by continuing to withstand tension. This critical displacement is given by

$$2\gamma A = \frac{1}{2}CA\delta_G^2/b \quad \Rightarrow \delta_G = \sqrt{4\gamma b/C} \Rightarrow F/A = \sqrt{4\gamma C/b},$$  

and is called the Griffith point. In the limit of large systems, where $b \to \infty$, the force per area goes to zero. This means that the approximation of Eq. 1 is as accurate as desired. Thus, there exists a universal relation between energy, force, and boundary displacement shown in Fig. 1 and given by

$$\Delta E = \begin{cases} \frac{1}{2}CA\delta^2/b & \text{for } \delta < \delta_G \\ \frac{1}{2}CA\delta_G^2/b & \text{for } \delta \geq \delta_G \end{cases} \quad F = \begin{cases} CA\delta/b & \text{for } \delta < \delta_G \\ 0 & \text{for } \delta \geq \delta_G \end{cases}$$  

(3)

For large enough samples this relation is exact.

A solid that actually obeys Eq. 3 is ideally brittle. Ceramics such as silicon, and some glasses come close. However, for most materials the whole picture is misleading or simply wrong. It is not enough for thermodynamics to say that a solid can lower its energy by dividing into two pieces. There must actually exist some physically feasible way for the separation to take place. Accomplishing this separation is the role of cracks. Cracks are nonequilibrium propagating dissipative structures. Initiating at a weak spot, they travel across solids under mechanical stress and separate them into multiple pieces. The questions of which solids are brittle, which are ductile, and how much force or energy truly is needed in order to cause a solid to fail come down to the questions of when cracks propagate and how.
1.2 Scope of this article

Two of us wrote a review 10 years ago that laid out the basic mechanical theory of cracks and then described conditions under which the tips became unstable [1]. The opportunity to write a second review raised a question of how to cover additional topics in an article that would still be fairly self-contained. We decided to focus on subjects that on the one hand represent recent developments, but on the other hand are in some ways more elementary than those we discussed before. We examine both experimental and theoretical issues that arise when one checks carefully if the theory of fracture is complete.

The fundamental theory of fracture, called Linear Elastic Fracture Mechanics (LEFM), is now a well established body of knowledge. The mathematical foundations were carefully laid by investigators such as Irwin [2], Rice [3] and Willis [4], and are documented in textbooks such as those by Broberg [5], Freund [6], and Slepyan [7]. By itself, this theory of fracture does not have a length scale within it, and it must be supplemented with some physical information about what makes cracks start to propagate that can either be obtained from experiments, or deduced from additional calculations. Here we will discuss a number of different cases where scale-free fractures interact with phenomena involving specific physical scales.

In contrast with physical theories for solid properties such as electrical conduction, physical theories of fracture are still in a primitive state. The ability to predict material properties based upon knowledge of atomic constituents or other underlying properties, is limited. We have assembled recent advances in describing detailed processes near crack tips in the hopes that, eventually, the understanding of crack tip mechanics will reach the point where crack direction and speed come under reliable experimental control. It may then be possible to design materials so that cracks begin to move at designated thresholds. At the present time these goals are not in reach, as we describe in our concluding section on conclusions and open questions.

2 Fracture Mechanics

Linear Elastic Fracture Mechanics

When a crack exists in a material, any externally applied stress undergoes a very large amplification at its tip. This was first noted by Inglis [8], who demonstrated amplification of the stresses at the tip of an elliptical hole in an otherwise uniformly stressed sheet. Irwin and Orowan [9, 2] later showed that if a crack exists in a sheet assumed to obey linear elasticity, the stress field $\sigma_{ij}$ actually becomes singular at
its (infinitely sharp) tip. The stress (see Fig. 1C) at a distance \( r \) from the tip takes the universal singular form \( \sigma_{ij} \propto K/\sqrt{r} \) where the coefficient \( K \) of the singularity is called the stress intensity factor. Depending on the symmetry of loading, there can be three independent fracture modes \([6, 5, 10]\), Mode I - fracture in pure tension, Mode II - fracture in pure in-plane shear, and Mode III - fracture by out-of-plane or “tearing”. The stress field due to each of these modes has the same universal form, and there may be, in general, three independent stress intensity factors. Here, we will only consider fracture under pure tension (Mode I).

The existence of a singular stress is, itself, insufficient to make a crack propagate, since the singularity is always cut off at least at the atomic scale, where it may or may not be large enough to snap bonds. A better way to determine if cracks can move is to note that new surface must be created and this requires energy. Defining the energy required to create a unit area as the “Fracture Energy”, \( \Gamma \), a simple generalization of Griffith’s idea \([11]\) suggests that cracks begin to move when the potential elastic energy per unit area released by a unit extension of a crack becomes equal to \( \Gamma \) (cf. Eq. 2). Even if this condition of energy balance predicts the onset of motion, once a crack starts to move the kinetic energy due to material motion around the moving crack has to be taken into account. Accounting for this energy led to the first equation of motion for a crack, which Mott initially obtained by dimensional analysis \([12, 13]\). This equation predicted that in a two-
dimensional medium of infinite extent, a crack should continuously accelerate as a function of its instantaneous length to a limiting, but finite, asymptotic velocity. This speed limit for a crack is due to physical constraints on energy transport. The crack tip requires energy to extend. If this energy has to be transported to the tip from remote parts of the medium by elastic waves, the speed of a crack must certainly be limited by the propagation speed of those waves. This asymptotic limit can be reached by either increasing the amount of energy driving the crack or by reducing the fracture energy $\Gamma$ to zero. Stroh \[14\] noted that for $\Gamma \to 0$, a crack propagating at its asymptotic velocity is equivalent to a disturbance moving along a free surface and therefore predicted the crack’s limiting velocity to be the Rayleigh wave speed, $V_R$, which is the highest speed that a wave can move along a free surface.

These intuitive ideas were later \[15, 16, 6, 4, 3\] shown to be rigorously correct, when a quantitative theory of dynamic fracture was developed. This complete theory is called Linear Elastic Fracture Mechanics (LEFM). Assuming that the medium always obeys linear elasticity, the theory predicts that the general form for the stress singularity at the tip of a crack as a function of the distance $r$ and angle $\theta$ from the tip of a crack moving at velocity $v$ is given by \[6\]

$$
\sigma_{ij} = \frac{K}{\sqrt{r}} f_{ij}(v, \theta),
$$

where $f_{ij}(v, \theta)$ is a known universal function and $K$ can be explicitly calculated for any externally applied loads. $K$ has the dimensions of stress $\times$ length$^{1/2}$, and thus must depend on a macroscopic geometric lengthscale. For sufficiently large samples, there is always a region around the crack tip where the singular term in Eq. 4 dominates any other contribution to the stress field. This fact can then be used to find the energy release rate, $G$, defined as the energy per unit extension per unit width that is flowing into the tip of a crack. $G$ is related to $K$ by \[6\]:

$$
G = (1 - \nu^2)K^2A(v)/E,
$$

where $\nu$ and $E$ are, respectively, the Poisson ratio and Young’s modulus of the medium. $A(v)$ is a universal function of the instantaneous crack velocity $v$. With this expression in hand, the general equation of motion for a crack can be obtained by using the energy balance criterion, equating $G$ to $\Gamma$. Energy balance, however, needs to be supplemented by an additional condition that tells us in what direction a crack will move. In general, no such first-principles condition exists. One commonly used assumption, called the “principle of local symmetry” \[17, 18\], tells us that a crack will locally align itself in a direction so as to negate any shear component at its tip. For quasistatic (“slow”) fracture this idea has been shown to be justified by symmetry considerations \[19, 20\] and has been used, for example, to quantitatively describe \[21, 22, 23, 24, 25, 26, 27, 28, 29\] experimental measurements of slow oscillating cracks in heated strips \[30\].
Other closely related path selection criteria were proposed and tested, see for example [32]. Although the “principle of local symmetry” is often assumed for rapid cracks, there is, however, neither fundamental or experimental justification for this (or any other) crack path selection criterion.

An additional tacit assumption is also used; that all of the dissipation in the system occurs within a “small” region surrounding the tip of the crack. This means that all of the complex dissipative mechanisms, inherent in the fracture process, have to occur in a region for which the surrounding stress field is entirely dominated by the singular stress field described by Eq. 4. This condition, called the condition of small-scale yielding, is, in essence, the definition of a brittle material - which extends the concept of an ideal brittle material (cf. Fig. 1A). The small zone surrounding the tip in which all nonlinear and dissipative processes occur is called the process zone. Thus, \( \Gamma \) need not necessarily be the energy cost to break a single plane of molecular bonds – it embodies all of the complex nonlinear processes that are driven by the putatively singular stress field near the crack tip. Small-scale yielding is one of the foundations of fracture mechanics, as it ensures the equivalence of the global energy balance criterion used by Mott with the locally formulated equation of motion:

\[
G = \Gamma = (1 - \nu^2)K^2(l, v)A(v)/E.
\]  

(5)

Freund [6] demonstrated that for straight semi-infinite cracks in infinite plates, under a variety of loading conditions [6], the dynamic stress intensity factor \( K(l, v) \) has a separable form \( K(l, v) = K(l)k(v) \), where \( K(l) \), which can be explicitly calculated, is solely dependent on the external loading configuration and the instantaneous value of the crack length, \( l \). Both \( k(v) \) and \( A(v) \) are universal functions of only \( v \). Freund [6] showed that, to a good approximation \( k(v)A(v) \approx 1 - v/V_R \) yielding

\[
\Gamma = G \approx \frac{(1 - \nu^2)K^2(l)}{E} \cdot (1 - v/V_R)
\]  

(6)

Inverting Eq. (6) yields the following prediction for an equation of motion:

\[
v = V_R \left[ 1 - \frac{\Gamma E}{(1 - \nu^2)K^2(l)} \right]
\]  

(7)

which is identical to the equation obtained earlier by dimensional analysis for a crack in an infinite plate.

Does this equation of motion work? Sharon et al. [33] conducted a quantitative test of Eq. 7 where the detailed dynamics of rapidly propagating cracks were measured in both PMMA (Plexiglas) and soda-lime glass, two brittle amorphous materials. The experiments measured the instantaneous values of \( l \) and \( v \) under
loading conditions whose geometry enabled a precise calculation of $K(l)$. The values of $\Gamma$ were derived using Eq. 6 for widely different experimental conditions. $\Gamma$ is a material property, but need not be velocity independent and, indeed, is generally a function of $v$. If Eq. 7 is valid, the values of $\Gamma(v)$ derived from Eq. 6 using measurements of $l$ and $v$, should all collapse onto a single curve for each material. The results of these experiments were surprising. As we see in Fig. 2, the experiments for both materials indeed collapse onto single curves for both glass and PMMA at crack velocities that are less than about $0.4V_R$ in each material. Thus, Eq. 7 is indeed in excellent quantitative agreement for $v < 0.4V_R$ for both glass and PMMA, two materials whose micro-structure is entirely different. These results agree with earlier results of Bergqvist [34] and Ravi-Chandar et al [35] that showed that cracks appear to behave in a way that is consistent with LEFM, for relatively low crack velocities.

What happens when $v > 0.4V_R$? At a critical velocity, $v_c \sim 0.4V_R$ a single crack state become unstable [38, 1, 39] to one in which the main crack is continuously sprouting frustrated microscopic branched cracks, as shown in Fig. 3C. This “micro-branching” instability causes rapid oscillations in the instantaneous crack velocity and the formation of non-trivial structure on the fracture surface. There is currently no first-principles theory for the origin of this and other instabilities, although many serious attempts have been made [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66]. It seems clear, however, that the key does not lie within the framework of LEFM. In fact, once a simple single-crack state is lost, we have little fundamental understanding of fracture dynamics. Some examples of the complexities that such non-trivial states can produce are shown in Fig. 3.

Does LEFM break down for $v > 0.4V_R$ if instabilities are suppressed? Recent experiments [36, 67, 37, 68] show us that, as long as its underlying assumptions are met, LEFM is entirely accurate. For example, the displacement fields surrounding a crack’s tip can be obtained by integrating Eq. 4. In particular, displacements normal to the crack’s propagation direction ($\theta = 0$), predicts a parabolic crack tip opening $u_y(r < 0, \theta = \pi) \propto K(l, v) \cdot \sqrt{r}$ behind the crack (Fig. 1C-top), whose curvature is determined by $K(l, v)$. Crack instabilities have been suppressed to up to $v \sim 0.9V_R$ in soft brittle gels [67]. In these materials, $V_R$ is 1000 times lower than in standard materials. This enabled direct measurements of the crack tip profile of rapidly propagating cracks [37]. These experiments show that, as long as we are not too close to the crack tip, LEFM provides excellent quantitative agreement with measurements. Values of $\Gamma(v)$ collapse onto a well-defined function over a very large range of velocities, as Fig. 2C shows, when calculated using values of $K(l, v)$ obtained by measuring the crack tip curvature.
Figure 2: Fracture energy, $\Gamma$, derived using Eq. 6 as a function of the mean velocity for soda-lime glass (A) and PMMA (B). Top: Velocities of cracks driven with different initial stresses and initial crack lengths as a function of their instantaneous lengths. The data were smoothed over a 1mm length, to filter out velocity fluctuations. The dashed lines indicate the highest values of $v$ used to derive the corresponding $\Gamma(v)$ (lower plots). These are 350 m/s (0.38 $V_R$) in PMMA and 1325 m/s (0.40 $V_R$) in glass. The observed data collapse to a single (material-dependent) function, $\Gamma(v)$ is a quantitative validation of the equation of motion derived for a single crack. Data were taken from [33]. Values of $V_R$ for PMMA and soda-lime glass are, respectively, 930m/s and 3,300m/s. (C) $\Gamma(v)$, derived by using the parabolic profile of the crack tip predicted by LEFM. These measurements were made possible by [36, 37] using soft brittle gels, where $V_R = 5.3$m/s.
2.1 LEFM - when does it work?

Sometimes LEFM provides an excellent quantitative description of both a crack’s motion and of the elastic fields surrounding its tip. This happens whenever the, at times subtle, fundamental assumptions underlying LEFM are valid. Let us list these assumptions:

- Only the dynamic behavior of straight single-crack states are accounted for. Once more complex dynamic states (e.g. micro-branching) occur, LEFM fails to accurately predict their behavior. In fact, there is currently no first-principles criterion for what path a crack must take.

- LEFM assumes that linear elasticity is valid away from the crack tip. In principle, for large enough samples, this assumption can always be made as accurate as desired, but in practice for samples of fixed size it must be checked.

- Small-scale yielding is observed ↔ all nonlinear and/or dissipative processes are assumed to occur in a region of negligible size at a crack’s tip. As LEFM is a scale-free theory, this tacitly means that the scales at which these complex processes occur are negligible. We will see that this tacit assumption is not necessarily justified; the existence of additional scales within the crack tip region can and does have a large effect on crack dynamics.

- A tacit assumption concerning “energy balance” is that energy has to flow into a crack’s tip, otherwise a crack cannot propagate.

In the current “state of the art” in fracture we have little fundamental understanding of the complexities of a crack’s motion once a crack decides to stray from a straight path or become unstable. These questions are hard ones, and although there is a number of possible directions to take, we have no clear answers. In the following sections we have, instead, chosen to describe what we currently know about “simple” straight cracks. We will do this by describing a number of interesting cases where theory and experiment move beyond limits constraining LEFM. By doing this we hope to provide both insights into the character of the “simple” crack state, and to, perhaps, provide a springboard for understanding more complex fracture dynamics.
3 Tales of Scales

3.1 The Importance of Small Scales near a Crack’s Tip

LEFM is a scale-free theory. Scales, however, must be introduced (either tacitly or explicitly) to obtain a closed theoretical description of a crack’s dynamics. A necessary scale, for example, is inherent in the idea of the process zone, which both cuts off the singular fields given by Eq. 4 and provides a scale below which dissipation and material nonlinearity take place. A central question in fracture is whether the details of these small scales are important. Is a proper description of these scales needed to describe how a crack behaves, or do the universal properties of fracture indicate that the details of the fracture processes that take place at these scales are irrelevant and simply serve to determine a material’s fracture energy.
There is a number of hints that the character of these small scales plays a major role in fracture dynamics. A classic way to regularize the continuum singularity at the tips of cracks is to introduce *cohesive zones*, fictitious forces pulling the crack’s faces together behind the tip in just the right way so that the singularity of the stress field disappears [72, 73]. In cohesive zone models, these fictitious forces acquire equations of motion so that they can travel along with the crack, cancelling out the singularity as they go. Results of working out such models are disquieting. It is difficult to obtain stable straight-ahead propagation of a single crack at any velocity [49, 74]. As noted by Lobkovsky and Langer [74], this entire class of models “seem[s] to be highly sensitive to details that ought to be physically unimportant”.

This point of view is reinforced by numerical simulations. For example, finite element calculations of dynamic cracks [75] with cohesive zones can provide quantitative agreement with experimental descriptions of both stable cracks and cracks undergoing the micro-branching instability described in Fig. 3C. However, it turned out that even the existence of micro-branching in numerics depends on the mesh size used in the simulations, and the experimental phenomena disappear in the limit where the mesh size goes to zero [76].

In this section, we will first describe three approaches in which a well-defined length scale at the crack’s tip occurs naturally. Two of these are continuum theories in which scales are formed either by (1) relaxing the assumption that materials are elastic up until the point where dissipation occurs or (2) the introduction of a phenomenological scalar field that represents the material damage that occurs around the tip. Both of these approaches define a new dynamic scale at the tip. A third class of models will be described in which the medium is an ideal crystal, which therefore possesses an intrinsic length-scale, the lattice size. We will see that in all of these approaches the existence of a length scale at the crack tip has important consequences on both the dynamics and form of propagating cracks.

We then show that an entire new class of single-crack solutions can arise when the Griffith condition is supplanted by the introduction of new energy scale. These solutions, which are wholly physical, surpass the speed limit set by LEFM by violating its assumption about how energy is conserved near a crack tip.

3.2 Small Scales Dominated by Nonlinear Elasticity

An essential feature of crack propagation is the high concentration of deformation in the immediate vicinity of a crack’s tip, as given by LEFM’s $\sqrt{r}$-singular fields. LEFM, however, is inherently confined to small material deformations. Therefore, any attempt to understand the physics of the process zone involves first the question of how and where LEFM breaks down near the tip of a crack. To address this
question, consider the interatomic potential by which particles interact. Adopt now the simplified assumption that the material’s response until the point of failure is completely contained within this interaction potential (i.e. neglect features such as dislocation motion, or changes in the underlying geometry of amorphous solids). Linear elasticity is described by the harmonic approximation about the equilibrium state at the bottom of the potential well. Plastic deformation, damage evolution and finally fracture correspond to interatomic separations far from the minimum of the potential, where the interaction energy decreases significantly. It is both natural and physically sound to assume that in numerous materials before one of these irreversible processes occurs, the harmonic approximation will first break down via reversible nonlinear deformation. Therefore, the first physical process that intervenes when LEFM breaks down should result from corrections that arise from nonlinear elasticity. The first generic nonlinear elastic contribution would be expected to come from quadratic corrections to linear elasticity.

To model this behavior at a macroscopic level, we expand a general stress vs. displacement-gradient relation up to second order

\[ \sigma \simeq \mu \partial u - \bar{\mu} (\partial u)^2 + \mathcal{O} \left( (\partial u)^3 \right) . \]  

(8)

For simplicity, we suppress the tensorial nature of the quantities involved and let \( \sigma \) and \( \partial u \) schematically denote stress and displacement-gradient (strain) measures, respectively (\( u \) denotes a displacement and \( \partial \) schematically denotes a spatial derivative); \( \mu \) and \( \bar{\mu} \) denote the first and second order elastic moduli, respectively. We term the theory that is based on this expansion “weakly nonlinear fracture mechanics” \[36, 77, 68\], as higher order nonlinearities are neglected in Eq. 8. To explore the implications of Eq. 8 on fracture dynamics, we schematically write the momentum balance equation as

\[ \partial \sigma = 0 , \]  

(9)

where, for simplicity, we omit the inertial term on the right hand side. We then expand the displacement \( u \) in powers of the magnitude of the (small) displacement-gradient \( \varepsilon \simeq |\partial u| \) and to second order obtain

\[ u \simeq \varepsilon \tilde{u}^{(1)} + \varepsilon^2 \tilde{u}^{(2)} \equiv u^{(1)} + u^{(2)} . \]  

(10)

Substituting Eq. 10 in Eq. 9 and expanding to first order, we obtain

\[ \mu \partial^2 u^{(1)} = 0 , \]  

(11)

which is our schematic analog of the Lamé equation \[6\]. LEFM tells us that this equation, together with the traction-free boundary conditions on the crack faces,
leads to the following asymptotic behavior near the tip of a crack \[6\]

\[\partial u^{(1)} \sim \frac{K}{\mu \sqrt{r}},\]  

(12)

where \(K\) is the stress intensity factor and \(r\) is measured from the crack tip, cf. Eq. \[4\]. In terms of the displacement-gradients Eq. \[9\] to second order, becomes

\[\mu \partial^2 u^{(2)} - \bar{\mu} \partial (\partial u^{(1)})^2 = 0.\]  

(13)

This is analogous to a Lamé equation for \(u^{(2)}\) with an effective body-force which scales as

\[\bar{\mu} \partial (\partial u^{(1)})^2 \sim \frac{\bar{\mu} K^2}{\mu^2 r^2}.\]  

(14)

By inspection, Eq. \(13\) admits a solution of the form

\[\partial u^{(2)} \sim \frac{\bar{\mu} K^2}{\mu^3 r}.\]  

(15)

We therefore see that simply accounting for quadratic contributions to material stress-strain relations already leads to the interesting conclusion that the first non-linear correction to the asymptotic LEFM fields is characterized by a \(1/r\) displacement-gradient singularity. The simple scaling-like considerations described above capture the essence of the solution, although the complete dynamic (i.e. with inertial effects) weakly nonlinear solution is, of course, much more mathematically involved \[36, 68\]. For example, the complete solution, which depends on the tip polar coordinates \(r\) and \(\theta\), shows that \(1/r\) displacement-gradients terms arise from both \(r\)-independent and \(\log (r)\) displacement contributions; the latter has an important implication as it yields significant distortion of the parabolic crack profile when the tip is approached.

Having addressed the question of how LEFM breaks down under the stated assumptions, we proceed now to consider the question of where this happens in space. According to the preceding discussion, LEFM breaks down when the second order correction \(\partial u^{(2)}\) becomes non-negligible with respect to \(\partial u^{(1)}\). A plausible estimate for this condition is obtained from the condition that the second order nonlinear contribution to the stress-strain relation becomes larger than 10% of the leading, linear, term. Thus,

\[\frac{\partial u^{(2)}}{\partial u^{(1)}} \simeq 0.1.\]  

(16)

This immediately implies that higher order corrections like \(1/r^{3/2}\) are negligibly small in this region. As the strains surrounding any crack tip are approaching a
mathematical singularity, this condition must always be satisfied at some scale. Denoting the scale at which Eq. 16 is first satisfied by \( \ell_{nl} \) and using Eqs. \( 12 \) and \( 15 \) we obtain
\[
\ell_{nl} \sim \frac{\mu^2 K^2}{\mu^4 0.1^2} .
\] (17)

This simple analysis reveals the emergence of a new length-scale associated with weak elastic nonlinearities. It is important to note that \( \ell_{nl} \) is dynamic as its value increases with the energy driving the crack \( \sim K^2/\mu \) or, equivalently, with a crack’s velocity.

Figure 4: (A) Left: An experimental picture of a crack, with a parabolic LEFM fit, cf. Fig. 2. Middle: A zoom in on the crack tip. Right: A comparison of the weakly nonlinear theory (solid line) with the measured crack tip opening profile (circles). The LEFM parabolic fit is added (dashed line). \((x',y')\) are the coordinates in the deformed (laboratory) frame. (A) A comparison of the weakly nonlinear theory (solid line) with the measured strain \( \varepsilon_{yy} \) along the symmetry line ahead of the crack tip, for three crack velocities. The LEFM predictions are added (dashed lines). See text for more details.

Is the scale \( \ell_{nl} \) in Eq. [16] physically relevant? Nonlinear elasticity in large-scale measurements is often not readily apparent. Brittle materials nearly always appear linear elastic until the onset of irreversible deformation. These measurements can be, however, quite deceiving. Failure in these tests is not due to intrinsic material behavior, but generally takes place via the activation of cracks or defects that are already in place within the material. On the other hand, cracks generally propagate into locally virgin material. The deformations near the tip of a crack are large enough to be well within the nonlinear elastic regime. As defects are ubiquitous in most macroscopic materials, these effects may be nearly undetectable in large-scale measurements. This does not mean, however, that they are either unimportant or do not commonly occur.
With this in mind, irreversible (plastic) deformation may commonly be pre-
ceded by elastic nonlinearities as the displacement-gradients increase. Thus, the
plastic deformation scale $\ell_{pl}$ is expected to satisfy

$$\ell_{nl} \gtrsim \ell_{pl}.$$  \hspace{1cm}  (18)

For strongly nonlinear elastic materials (gels, rubber etc.) we expect a scale sepa-
ration of the form $\ell_{nl} \gg \ell_{pl}$, while for other materials we cannot rule out the pos-
sibility that $\ell_{nl}$ is not very different from $\ell_{pl}$. In the former case, the small scales
near a crack’s tip are dominated by nonlinear elasticity, while in the latter case,
dissipation may play a more decisive role.

The complete solution of the weakly nonlinear equations, which generalizes the
simple considerations above, was obtained in Refs. [36, 68]. Can these solutions
be observed in experiments? Generally, this is a near-impossible task since cracks
propagate at near sound-speed velocities with (at best) $\mu$m-scale crack tip open-
ings. In addition, the second order elastic constants are generally not well known,
since in macroscopic stress-strain tests of brittle materials failure, due to the propa-
gation of intrinsic cracks, occurs well before sufficient strains are reached. Recent
experiments, however, circumvented these problems by utilizing brittle elastomers
(gels) [37]. While the characteristic fracture phenomenology of these soft materi-
als is identical to that of other brittle amorphous materials [78], their sound speeds
are 1000 times lower. Furthermore, as these materials are very compliant, but
tough, they undergo large strains prior to fracture. This allows precise measure-
ments of their second-order elastic constants. These properties enabled the first
direct experimental measurements of the deformation near the tip of rapid cracks.

A summary of the comparison of these measurements with the explicit predictions
of the weakly nonlinear theory is presented in Fig. 4. On the left hand side of the
upper panel the LEFM parabolic crack tip profile is shown, cf. the discussion of
this point in Sect. 2 and Fig. 2. In the middle of the upper panel a zoom in on
the crack tip region is shown, where marked deviations from the LEFM parabolic
profile are observed. In [37] it was shown explicitly that this deviation is due to
dehiscence deformation and not due to an irreversible one.

On the right hand side of the upper panel we show the predictions of the com-
plete weakly nonlinear solution for the tip profile. We observe that it agrees well
with the LEFM parabolic form at large scales (i.e. small deformation), but both
qualitatively and quantitatively (with no adjustable parameters, since all elastic
properties can be measured in this material) captures the deviation from it on
smaller scales (below $\sim 200\mu$m in this example). The deviation is a direct con-
sequence of the log $(r)$ contribution to $u$, and directly demonstrates the existence
of the new terms predicted by the weakly nonlinear theory.
In the lower panel of Fig. 4 we present a comparison between the weakly non-linear theory and the measured deformation ahead of the crack tip. We focus on $\varepsilon_{yy} = \partial_y u_y$ along the symmetry line ($\theta = 0$) ahead of the crack for three propagation velocities, including one that approaches the limiting speed, $V_R$. For the two lowest velocities we observe marked deviations from the $1/\sqrt{r}$ fields of LEFM, deviations that are accurately captured by the weakly nonlinear theory (cf. [36] for more details). These results again explicitly demonstrate the validity of the weakly nonlinear theory with its $1/r$ singularity. Moreover, the scale $\ell_{nl}$ in which LEFM breaks down via elastic nonlinearities is captured by the theory. Its value, which is in the mm range, can be easily read off from the figure.

On the right hand side of the left panel we present the same comparison for a crack propagating at $v=0.78c_s$. At this very high velocity, higher than second order nonlinearities are needed to accurately describe the data. This comparison, however, highlights an important point. According to LEFM, a velocity exists ($0.73c_s$ for an incompressible material) where $\varepsilon_{yy}$ changes sign from positive to negative. In this range of velocities $\varepsilon_{yy}$ predicted by LEFM is small (due to kinematic functions [6, 37, 36] that enter as coefficients in the exact analog of Eq. 12, but do not appear in the simple derivation used above). As a result, in this range of velocities the second order term in $\varepsilon_{yy}$ becomes the dominant contribution. Due to this contribution, as seen in the figure, no change in sign occurs. Thus, the weakly nonlinear theory retains our basic physical intuition about fracture; material points straddling the symmetry line must be separated from one another ($\varepsilon_{yy} > 0$) to precipitate fracture. This contrasts with the LEFM prediction in this range of velocities where $\varepsilon_{yy} < 0$ (cf. dashed line in the figure).

In summary, the weakly nonlinear theory of dynamic fracture describes how LEFM breaks down in materials where the scale defined by elastic nonlinearity is larger than the dissipative scale. In addition to accurately describing detailed measurements of the deformation near tips of very rapid cracks [36, 68], this theory may also have further profound implications for understanding crack tip instabilities [1, 36]. This theory, by itself, does not account for either near-tip dissipation or directly offer crack path predictions. The length-scale $\ell_{nl}$ for the breakdown of LEFM may, however, play a central role in explaining symmetry breaking instabilities. There are some tantalizing hints that crack tip instabilities may occur at approximately the length-scale $\ell_{nl}$ that emerges in this theory. For example, $\ell_{nl}$ at high velocities correlates well with the geometry-independent wavelength of crack path oscillations discussed in [67, 66]. This promising line of investigation should be further explored.
3.3 Phase Field Models

LEFM accurately predicts the transport of linear elastic energy from the large scales, where external forces are applied, to the small scales near the tip of a crack (the process zone), where energy is dissipated in fracture. It is assumed that these scales are coupled solely through the $\sqrt{r}$ -- singular field. This approach, however, is inherently deficient in at least three important aspects:

- The crack tip velocity $v$ cannot be determined self-consistently within LEFM, as energy dissipation within the nonlinear near-tip region is not described in this framework.
- The path selected by a crack’s tip remains undetermined in the framework of LEFM unless supplemented by a path selection criterion (e.g. the principle of local symmetry). Hence crack tip instabilities cannot be explained by solely LEFM.
- LEFM is a scale-free theory and hence any phenomenon that involves a non-geometric length-scale is beyond its scope (e.g. some crack tip instabilities).

All of these difficulties are directly related to the need to account for the physics at the small scales near the tip of a crack, where LEFM breaks down. A comprehensive theoretical framework should explain how LEFM breaks down near the tip of a crack, predict crack tip shape and velocity selection, quantify energy dissipation near the tip of a crack and account for the path selected by cracks. Crack tip instabilities are expected to naturally emerge from such a theory.

A class of phenomenological phase field models has recently been developed to approach this problem from a continuum perspective \[79\, \text{80}\, \text{81}\, \text{82}\, \text{83}\, \text{84}\, \text{85}\, \text{86}\, \text{87}\]. In this approach, the fracture of materials is described as the gradual accumulation of damage near the tip of a crack. Damage is quantified mathematically by a scalar “order parameter” or “phase field” $\phi(x,t)$, where $\phi = 1$ corresponds to an intact material and $\phi = 0$ to a fully broken one. The physical interpretation of the phase field $\phi$ as quantifying local damage is, in fact, not necessary and most probably will not be advocated by all practitioners of this approach. The phase field can be viewed as a mathematical interpolation between LEFM and material failure, where $\phi = 1$ is favored when the linear elastic strain energy density $\varepsilon_{el}$ is smaller than some threshold $\varepsilon_c$, $\varepsilon_{el} < \varepsilon_c$, while $\phi = 0$ is favored when $\varepsilon > \varepsilon_c$. A similar approach proved to be very fruitful in the context of non-equilibrium crystal growth, when used to predict solidification micro-structures.

The mathematical formulation of these ideas starts with the following expression for the potential energy density as a function of $\phi$ and the displacement field
\[ u = \frac{\kappa}{2} |\nabla \phi|^2 + g(\phi) (\mathcal{E}_{el} - \mathcal{E}_c) \] 

(19)

\( \mathcal{E}_{el} \) is the linear elastic energy density, which is given in terms of linear elastic strain tensor \( \varepsilon \), as

\[ \mathcal{E}_{el} = \frac{\lambda}{2} \varepsilon_{ii}^2 + \mu \varepsilon_{ij}, \quad \varepsilon_{ij} = \frac{1}{2} (\partial_j u_i + \partial_i u_j) \] 

(20)

\( \lambda, \mu \) are the Lamé constants. Two of the phenomenological quantities introduced by phase field models appear here: \( \kappa \), a coefficient with dimensions of energy per unit length and \( g(\phi) \), a monotonic function that quantifies the degree of strain softening due to material degradation. \( g(\phi) \) satisfies \( g(0) = 0, g(1) = 1, g'(0) = g'(1) = 0, \) and \( \lim_{\phi \to 0} g(\phi) \sim \phi^\alpha \), with \( \alpha > 2 \). The latter condition was shown to ensure the existence of crack-like solutions \cite{81}. We note that \( \phi \) in Eq. (19) does not break the rotational invariance near the crack tip, unlike lattice or cohesive zone approaches that inevitably introduce some near-tip anisotropy. This anisotropy is questionable when amorphous materials, which account for a large part of the available experimental data on rapid fracture \cite{1}, are considered. Note also that Eq. (19) is symmetric with respect to tension and compression, though it is obvious that fracture occurs locally under tension. This issue was addressed in \cite{83} by introducing an ad-hoc asymmetry between tension and compression.

The equations of motion for a phase field model are variationally derived from the total potential energy \( \mathcal{E} = \int \mathcal{E} \, dV \) and read \cite{82, 83}:

\[ \rho \partial_{tt} u_i = \partial_j \left( \frac{\partial \mathcal{E}}{\partial H_{ij}} \right) - \frac{\partial \mathcal{E}}{\partial u_i}, \quad H_{ij} \equiv \partial_j u_i \] 

(21)

\[ \chi^{-1} \partial_t \phi = \partial_j \left( \frac{\partial \mathcal{E}}{\partial \zeta_j} \right) - \frac{\partial \mathcal{E}}{\partial \phi}, \quad \zeta_j \equiv \partial_j \phi \] 

(22)

where \( \rho \) is the mass density. Translational invariance implies that \( \mathcal{E} \) is independent of \( u \), i.e. that \( \frac{\partial \mathcal{E}}{\partial u_i} = 0 \). Noting that \( \frac{\partial \mathcal{E}}{\partial u_i} = \sigma_{ij} \), Eq. (21) is, in fact, the usual linear momentum balance equation, \( \rho \partial_{tt} u_i = \partial_j \sigma_{ij} \).

The dynamics of \( \phi \) in Eq. (22) are assumed to be dissipative and hence first order in time. The last phenomenological parameter \( \chi \), needed to close the theory, has the dimensions of volume per unit energy per unit time. Equations (21)-(22) are four coupled nonlinear equations that can be solved numerically when the external boundary conditions are specified. The simple choice of \( g(\phi) = 4\phi^3 - 3\phi^4 \), which satisfies all the required conditions mentioned above, is commonly used in the literature \cite{81, 82, 83}. It is important to note that the usual traction-free boundary conditions on the crack faces \cite{6} are not needed here since these are automatically
satisfied due to strain softening when $\phi \to 0$. Moreover, fracture in the phase field models is described by a diffuse interface, which is a significant technical advantage, since one does not have to track a moving boundary, which is computationally expensive. The crack faces in the phase field models are rather arbitrarily defined as a contour of fixed $\phi$, usually $\phi = 1/2$.

A simple dimensional analysis, based on the phenomenologically introduced parameters $\kappa$, $\varepsilon_c$ and $\chi$, shows that the size of the region where LEFM breaks down, the process zone, scales as

$$\xi \sim \sqrt{\frac{\kappa}{\varepsilon_c}},$$  

(23)

the fracture energy $\Gamma$ scales as

$$\Gamma \sim \sqrt{\kappa \varepsilon_c},$$  

(24)

and the timescale for energy dissipation near the tip of the crack scales as

$$\tau \sim \frac{1}{\mu \chi}.$$  

(25)
Therefore, in light of the discussion at the beginning of this Section, the phase field models include a regularizing cut-off length-scale $\xi$ and near-tip dissipation, resulting in a well-defined and self-consistent mathematical description of a fracture problem. Thus, crack tip shape, path and velocity selection are expected to emerge naturally from the numerical solution of Eqs. 21-22 in this class of models. We note that the variational structure of the phase field models allows a natural generalization of common energy-momentum tensors and configurational forces in fracture mechanics [87].

Several groups have explored numerical solutions of the phase field equations in various configurations. The results exhibit a rather rich phenomenology that is at least in qualitative agreement with experimentally observed phenomena; these include two-dimensional steady-state crack motion in a strip geometry above the Griffith threshold [81, 85], branching instabilities [80, 82, 86], rapid [83] and quasi-static [29] oscillations and complex three-dimensional crack front dynamics. In Fig. 5 we summarize some of these results.

The phase field approach seems a very good mathematical method for solving quasi-static fracture problems that are completely controlled by LEFM and by geometric length-scales, cf. [30]. In these cases it provides a proper crack tip regularization and dissipation, and offers a computationally efficient way to track complex crack configurations. For example, the morphology of a quasi-static crack propagating under thermal stresses was satisfactorily described in the framework of a phase field model [29]. In this example, all the pertinent length-scales were geometric in nature and the crack path was accurately described by the principle of local symmetry. On the other hand, in situations where the crack tip physics has explicit macroscopic manifestations [88, 67, 66], a number of important open questions remains. The phase field methodology does not offer a physically realistic description of the deformation and dissipative processes near crack tips. It is therefore difficult to directly relate the phase field $\phi(x,t)$ itself or the phenomenological parameters $\kappa$ and $\chi$ to experimental data. Therefore, this approach cannot, at present, offer testable predictions in these situations. By coupling directly to linear elasticity, it also does not incorporate elastic nonlinearities (hence does not conform with Eq. [18]). Future work is needed to further elucidate the relation between the phase field models and the physics of fracture. For example, incorporating nonlinear elastic effects as discussed in Sect. 3.2 may be an interesting future line of investigation.

### 3.4 Atomic Resolution of Small Scales

In some materials, details of material structure down to the atomic scale play a decisive role in determining how details of the fracture process work out. This
statement is most evidently true in crystals, which have natural cleavage planes. No matter how big a sample of mica may be, it cracks easily along one easy plane, and resists fracture along other directions [89]. One way to characterize the effects of the atomic scale is to say that $\Gamma$ depends upon orientation relative to crystal planes, with cusp-like minima in special directions. The surface energy of crystals depends upon orientation in precisely this way [90], and fracture energy must have the same general behavior. The fracture of crystals can be strongly anisotropic, even in a material that at large scales has completely isotropic elastic behavior [91].

While fracture energy has the same qualitative behavior as surface energy in a crystal, the two are not equal. Fracture energy cannot be less than surface energy because by definition surface energy describes the minimum free energy needed to induce a solid to separate along some plane. However it can be greater and in general it is. When cracks travel through a crystal, the motion through the periodic unit cells excites phonons. In general, any object moving at velocity $v$ through a crystal with phonon dispersion relation $\omega(k)$ excites phonons that obey the Cherenkov condition [92] $\omega(k) = vk$, and cracks are no exception. While this fact has been confirmed repeatedly in theoretical and numerical investigations, we are unaware of any direct experimental evidence.

The obvious way to evade exciting phonons so that fracture and surface energy coincide is for the crack to move very slowly. Considering this possibility raises a number of interesting problems. Thomson, Hsieh, and Rana [93] showed that if one takes a crystal at zero temperature and very slowly pulls its edges apart, when the stored energy reaches the Griffith point, the crystal does not break. The strained crystal is meta-stable. To see why, imagine that a crack moves along a plane in a strip loaded as in Fig. 1B. Let $E(l)$ give the minimum-energy atomic configuration subject to the constraint that the crack tip be at position $l$. The crack location is easy to define as a continuously varying function of atomic positions. Finding the function $E(l)$ in practice is demanding [94, 95], but in principle one can see immediately what shape it has to have, as shown in Fig. 6. First $E(l)$ must vary linearly with $l$, with positive slope when $\delta$ is below the Griffith point and negative slope when it is stretched above the Griffith point. In addition, $E(l)$ must have a periodic component corresponding to different locations in the unit cell. Right at the Griffith point the crack tip sees a corrugated potential, is necessarily trapped in a local minimum, and does not propagate. The crystal must be stretched by an extra amount so that the slope of $E(l)$ is negative everywhere. Thomson, Hsieh, and Rana [93] called this phenomenon lattice trapping.

This discussion of linear stability does not immediately rule out the possibility that a crack could move very slowly just above the Griffith threshold if it could only get started. Exact solutions for cracks running in lattices found by Slepyan [96, 7]
Figure 6: (A) Schematic view of how the energy of a crystal varies as a function of the location $l$ of a crack tip within a fracture plane. The potential energy surface is corrugated, and therefore the crack cannot travel spontaneously at the Griffith point when the elastic energy recovered per length equals the surface energy cost. (B) Exact solution for crack velocity measured relative to wave speed $V_R$ versus loading $\delta$ over Griffith loading $\delta_G$ in a two-dimensional crystal at zero temperature, showing lattice trapping, velocity gap, and the point where steady states become dynamically unstable. Past the point of instability steady states are technically impossible, but the deviations are on an atomic scale that would be hard to detect experimentally.

and Marder [44] show that this cannot happen either. Every time a crack moves to a new unit cell it slides down the corrugated potential and causes the atoms there to vibrate at frequencies on the order of $c/a$ where $c$ is a sound speed and $a$ is a lattice spacing. If the atom vibrates many times it radiates energy which is lost to crack motion. Therefore the most energy-efficient crack motion requires the crack to get to a new cell during a time on the order of a vibrational period. This means that its velocity must be a fraction of the sound speed, and if the velocity drops below a threshold it cannot move at all, as shown in Fig. 6B. Thus there is a finite velocity gap for cracks moving in crystals at zero temperature. Either cracks move at a finite fraction of the sound speed and spend energy on phonons or they do not move at all.

All of these conclusions must be modified at nonzero temperature. If the energy of thermal fluctuations is comparable to the size of the corrugations in Fig. 6A, then the corrugations should be washed out and the crack should behave as one would expect if the material is continuous down to the finest scales.

Attempts to follow through on all these ideas and compare with experiment has proceeded the furthest in single-crystal silicon, where the situation is not completely settled. The minimum energy plane in silicon is (111), and theoretical
Figure 7: Comparison of experiment and theory for the fracture of silicon along the (111) plane, showing the energy $\Gamma(v)$ needed for propagation at various speeds. The experiments were carried out at room temperature in silicon single crystals [97]. Deegan established an energy gradient in a strip so that the curve could be mapped out in a single run. The three upper theoretical curves are molecular dynamics simulations using the Modified Embedded Atom Method (MEAM) at temperatures of 0 and 77K, and the Inadvertently Modified Stillinger Weber (IMSW) potential [98]. Lattice trapping is so large for these potentials that they disagree substantially with experiment. However a tight-binding computation of Bernstein and Hess is in agreement [99].
estimates for the surface energy $2\gamma$ range from 1 to 2.5 J/m$^2$ \cite{100}. Experimental measurements of the surface energy are in the range of 2-5 J/m$^2$ \cite{100}, but as dynamically propagating cracks have been observed with energies as low as 2.25 ±0.25 J/m$^2$ \cite{97}, the lower end of the range is most likely. Both experimental and numerical estimates of $\Gamma(v)$ are available, as shown in Fig. 7. One might expect based upon Fig. 6A that the detailed relationship between energy available for fracture $\Gamma$ and crack speed $v$ might depend rather sensitively upon details of the potential energy surface, particularly upon the height of the corrugations, and this is the case. The mechanical behavior of silicon at the atomic scale is often described by classical empirical potentials \cite{98}, such as the one due to Stillinger and Weber \cite{101}, or the Modified Embedded Atom Method \cite{102}. All of these empirical potentials appear to have corrugations that are too large, and therefore produce substantially more lattice trapping and larger velocity gaps than are seen in experiment. The potentials are constrained by considerations of crystalline symmetry, fracture energy, and low-energy properties like sound speed. No one has yet managed to guess potentials for silicon that gets all these things right and the fracture properties as well, although the Modified Embedded Atom Method is so far the best. However, an elaborate calculation of Bernstein and Hess \cite{99} that couples tight-binding quantum mechanics near the crack tip to empirical potentials further away does seem to capture all of the observed experimental features well. It is a bit surprising that such effort must be expended to get the energy at which fracture begins right within a factor of 2.

3.5 Supersonic States: a New Energy Scale

The preceding sections showed some effects of integrating length scales with LEFM. An additional assumption of LEFM (cf. Sect. 2) was that energy must flow into a crack’s tip to enable it to propagate. Here we will examine the consequences of discarding this assumption.

The Rayleigh wave speed as the upper limit for a crack was a consequence of this assumption. One can ask what limits apply to crack speeds if both the tendency to instability is suppressed and the above assumption discarded. Suppressing instabilities can be arranged by having the fracture move along a weak interface, or by finding a material which for some reason does not permit crack branching to occur. Cracks travelling along weak interfaces have been of greatest interest in the study of earthquakes. Andrews \cite{103} showed numerically that shear cracks on a weak interface could travel faster than the shear wave speed. This problem was studied further by Burridge, Conn, and Freund \cite{104}, who showed that LEFM permits shear cracks to travel precisely at a speed given by $\sqrt{2}$ times the shear wave speed, but that the range of possible speeds is broadened by cohesive forces near
the crack tip.

Since we have maintained that LEFM must be supplemented with information from smaller scales in order to make predictions about crack velocities, we should explain how this theory has been employed to make predictions about the limiting speeds of cracks. One can use LEFM to compute the energy travelling from far elastic fields in to the crack tip assuming that the crack is travelling at velocity \( v \). For velocities \( v \) lower than the Rayleigh wave speed, this energy flux is always positive. For velocities above the Rayleigh wave speed, however, it is either imaginary or negative. There is one special case, of shear cracks travelling at \( \sqrt{2} \) times the shear wave speed, where the energy flux just vanishes, and this is the physically allowed value. Cracks in tension, by this logic, are forbidden to travel faster than the shear wave speed.

Nevertheless, Buehler, Abraham, and Gao [59] observed such cracks in molecular dynamics simulations, and Petersan et al. [105] observed them in experiments in rubber. They evade the apparent limitations of linear elastic theory because they correspond to a completely different scaling theory.

In LEFM, the scaling parameter is \( \delta_G \), the total displacement of the boundaries at which the energy reaches the Griffith threshold and there is enough elastic potential energy available per unit length to create new surface. There is another critical extension, however, \( \delta_c \), which is the extension at which the solid would be stretched so much that the bonds between adjacent material points would give way completely. For most brittle materials, reaching \( \delta_c \) requires stretching the solid to around 20\% more than its original length. In rubber, it requires stretching the solid to 8 or 10 times its original length.

If a crack tip avoids going unstable, then eventually when the boundary extension \( \delta \) reaches a fraction of \( \delta_c \), the crack speed becomes supersonic. The signature of the supersonic states lies in their behavior as the size of the system increases. For conventional subsonic cracks, cracks in two systems behave in the same way when \( \delta_G \) is the same. Thus if one strip is twice the height of another, cracks will have the same velocity in the two strips if the second is stretched by a factor of \( \sqrt{2} \) times as much as the first. For supersonic cracks, strain itself is the controlling variable, and energies rise to a new scale. Thus in the supersonic case cracks in the second strip will look like those in the first when it is stretched to a height \( \delta \) twice as much. Fig. 8 shows both both analytical calculations and experiments in rubber where these scaling behaviors are observed. The analytical calculations are carried out in exactly solvable lattice models [106] and the experiments are carried out in rubber. The calculations in the figure are for a simple case of anti-plane shear (Mode III), but they have been extended to fracture in tension as well [107]. Rubber may seem an odd experimental setting for fracture mechanics, but the relation between force and displacement is nearly linear over wide range of displacements,
Figure 8: (A) Four different views of Neo–Hookean crack velocities, showing that, depending upon how they are scaled and displayed, one focuses either upon conventional subsonic fractures or supersonic ruptures. In the limit of infinite system height $N$, the two different types of solutions are separated by an infinitely long plateau at the wave speed. The final panel simply re-plots the data of the third panel with a different range of velocities visible, emphasizing that if the macroscopic limit is taken holding strain constant, it appears that all cracks are supersonic. (B) Experiments in rubber at 85 C. The temperature needs to be elevated because at room temperature rubber undergoes a transition above extension ratios of 3.5 where its fracture energy increases enormously. The experiments are carried out in strip geometries with four different system heights. Velocities are scaled by the wave speed $c = 21.9$ m/s. The dimensionless extension $\delta/\delta_c$ is estimated by the square root of the energy scaled by the Griffith energy of $5580 \text{ J/m}^2$. The vertical extension ratio $\lambda_y$, the ratio of the stretched height to the original height, is scaled by 10, which is a rough estimate of the critical extension $\delta/\delta_c$ obtained by comparing slopes in right and left panels. Experimental results courtesy of Hepeng Zhang, Johnathan Niemczura, and K. Ravi-Chandar.
and the strongest deviations from linear behavior are confined to a small region near the tip as required by small-scale yielding.

The fundamental reason that cracks normally travel slower than all wave speeds is that they are only able to propagate by absorbing all the potential energy stored in the sample, all the way out to the boundaries. In the case of supersonic cracks, the solid has been stretched so much that energy sufficient to snap bonds is located with a fixed distance of the tip. Thus the crack speed is no longer limited by the time needed to transport energy to the tip from far away. Continuum elastic solutions for supersonic cracks show that their tip must now be a source of energy, a fact that has been used to argue that they cannot exist. In fact, enough energy is stored near the tip of a supersonic crack that it can break bonds with enough left over to travel outwards. A final observation is that in samples strained enough to support supersonic cracks, most of the energy in the sample is released through contraction of material after the crack has passed. This energy must be absorbed by some mechanism; in rubber, it is absorbed by bulk dissipation, which is particularly strong along a shock line that develops in the wake of the crack.

4 Conclusions and Unsolved Problems

We took a careful look at the underlying assumptions that describe our current view (LEFM) of how the simplest cracks evolve, and noted a few possible “cracks” in its underlying structure.

- LEFM is a scale-free theory, but a fundamental understanding of the physics of fracture requires description at the scales where rupture processes occur.

- Fracture processes are usually not explained by thermodynamic ground state arguments, and they are not even necessarily controlled by flow of energy to the crack tip.

- The major unsolved problems of fracture concern the way the crack line decides how and where to move. Problems include the reasons cracks change direction while in motion, and conditions for crack fronts to remain essentially planar or to create complex ramified surfaces.

The resolution of these issues may be intimately related to both our understanding of the proper “ground state” of this system and its stability. We have seen that there may not be a single solution to this problem, and different classes of physically viable solutions may exist. We surmise that a fundamental description of the stability of such solutions for “simple” cracks could be closely linked to how to properly
take into account the scales that define them. These scales may be dynamic entities, as we have seen, in sections 3.2 and 3.3 or static ones, as in sections 3.4 and 3.5. They may simply determine the character of the ground state, or possess, themselves, intrinsic complex dynamics that may undermine ground-state stability. These issues define current active research directions. Perhaps their resolution (in the coming decade) will provide the basis of our next review.

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