A class of minimal submanifolds in spheres

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Abstract

We introduce a class of minimal submanifolds $M^n$, $n \geq 3$, in spheres $S^{n+2}$ that are ruled by totally geodesic spheres of dimension $n-2$. If simply-connected, such a submanifold admits a one-parameter associated family of equally ruled minimal isometric deformations that are genuine. As for compact examples, there are plenty of them but only for dimensions $n = 3$ and $n = 4$. In the first case, we have that $M^3$ must be a $S^1$-bundle over a minimal torus $T^2$ in $S^5$ and in the second case $M^4$ has to be a $S^2$-bundle over a minimal sphere $S^2$ in $S^6$. In addition, we provide new examples in relation to the well-known Chern-do Carmo-Kobayashi problem since taking the torus $T^2$ to be flat yields a minimal submanifolds $M^3$ in $S^5$ with constant scalar curvature.

In several directions, this paper should be considered as a continuation of our work in [8] where a new class of minimal ruled submanifolds $M^n$ of Euclidean space $\mathbb{R}^{n+2}$, $n \geq 3$, were studied. These submanifolds lay in codimension two and may be metrically complete regardless the dimension. The rulings are of codimension two in the manifold whereas the rank, that is, the complement of the index of relative nullity, is $\rho = 4$ (unless $n = 3 = \rho$) along an open dense subset. If simply-connected, the submanifolds admit a $S^1$-parameter family of genuine isometric deformations. Hence, this class of examples should be seen as a new addition to the possible, local or global, classification of Euclidean submanifolds in codimension two that admit genuine isometric deformations; see [8] for a discussion of that open problem.

In this paper, we consider a similar construction but for the round sphere as ambient space. We obtain minimal submanifolds $M^n$ in $S^{n+2}$, $n \geq 3$, with similar properties as the ones in the Euclidean space. Notice that being ruled now means that the submanifold carries a foliation by (open subsets of) totally geodesic spheres in $S^{n+2}$ of dimension $n-2$. If the manifold is simply-connected, by taking the cones in $\mathbb{R}^{n+3}$ of the components in the associated family in $S^{n+2}$ we obtain a new class of genuinely deformable Euclidean submanifolds in codimension two but, of course, these are not complete.

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New examples of minimal submanifolds in spheres are certainly welcome since the explicitly known ones are usually quite elaborate and certainly less abundant than in the Euclidean space. Frequently, they are spheres of constant sectional curvature or products of them. On the other hand, the submanifolds here introduced can only be complete, or even compact, for dimensions $n = 3$ or $4$. If compact and according to the dimension, the submanifold must be topologically either a $\mathbb{S}^1$-bundle over $\mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbb{S}^5$ or a $\mathbb{S}^2$-bundle over $\mathbb{S}^2$ in $\mathbb{S}^6$.

The compact examples in the case of the torus are of particular interest by two quite different reasons. First, if we replace the torus by its universal cover we obtain a three-dimensional manifold that is not longer compact but has an $\mathbb{S}^1$-parameter family of isometric minimal deformations. But the compact submanifold itself only admits, at most, a finite set of isometric minimal deformations. The second reason, has to do with the well-known Chern-do Carmo-Kobayashi problem [5] that concerns compact minimal submanifolds of the sphere with constant scalar curvature. We show that if the torus considered is flat, and these were all parametrically described by Miyaoka [12], then $M^3$ has constant scalar curvature.

1. The results

This section is devoted to state the results of the paper while proofs are left for the following one. Up to the last two results, the other theorems in this paper can be seen as the “spherical version” of the results obtained in [8] for submanifolds in Euclidean space.

Let $g : L^2 \to \mathbb{S}^{n+2}$ denote a substantial oriented minimal surface. As already recalled in [8] the normal bundle $N_g L$ of $g$ splits along an open dense subset of $L^2$ as

$$N_g L = N_1^g \oplus N_2^g \oplus \cdots \oplus N_m^g, \quad m = [(n + 1)/2],$$

where each subbundle $N_s^g$, $1 \leq s \leq m$, is spanned by the corresponding $(s + 1)^{th}$-fundamental form $\alpha_g^{s+1} : TL \times \cdots \times TL \to N_g L$ and has rank two except possible the last one that has rank one if $n$ is odd.

If $L^2$ is simply-connected, there exists a one-parameter associated family of minimal isometric immersions. In fact, for each constant $\theta \in \mathbb{S}^1 = [0, \pi)$ consider the parallel orthogonal tensor field

$$J_\theta = \cos \theta I + \sin \theta J$$

where $I$ is the identity map and $J$ the complex structure determined by the metric and orientation. Then, the symmetric section $\alpha_g(J_\theta \cdot, \cdot)$ of the bundle $\text{Hom}(TL \times TL, N_g L)$ satisfies the Gauss, Codazzi and Ricci equations with respect to the same induced normal connection; see [6] for details. Therefore, there exists an isometric minimal immersion
$g_\theta: L^2 \to S^{n+2}$ whose second fundamental form is

$$\alpha_{g_\theta}(X,Y) = \phi_\theta \alpha_g(J_\theta X, Y)$$

where $\phi_\theta: N_g L \to N_{g_\theta} L$ is the parallel vector bundle isometry that identifies the normal bundles as well as each normal subbundles $N^s_g$ with $N^s_{g_\theta}$ for any $1 \leq s \leq m$.

In the sequel, let $g: L^2 \to S^{n+2}$, $n \geq 2$ be a substantial 1-isotropic surface. This means that $g$ is minimal and that the ellipse of curvature (of first order) at any point is a circle. Let $L_0$ be the open subset of $L^2$ where $\dim N^2_L(p) = 2$. It was shown in [7] that $L^2 \setminus L_0$ consists of isolated points and that the vector bundle $N^2_L|_{L_0}$ smoothly extends to a plane bundle over $L^2$, that we still denote by $N^2_g$.

Let $\pi: \Lambda_g \to L^2$ denote the vector bundle of rank $n-2$ whose fibers are the orthogonal complement in the normal bundle $N_g L$ of $g$ of its extended first normal bundle $N^2_L$. Then $F_g: M^n \to S^{n+2}$ is the submanifold of $S^{n+2}$ associated to $g$ constructed by attaching at each point of the surface $g$ the totally geodesic sphere $S^{n-2}$ whose tangent space at that point is the fiber of $\Lambda_g$, that is,

$$ (p,v) \in \Lambda_g \mapsto F_g(p,v) = \exp_g(p) v, \quad (1)$$

while dropping the singular points whenever they exist, i.e., points where the induced metric is singular. By definition $F_g$ is an $(n-2)$-ruled submanifold, that is, there is an integrable tangent distribution of dimension $n-2$ whose leaves are mapped diffeomorphically by $F_g$ onto open subsets of totally geodesic $(n-2)$-spheres of $S^{n+2}$.

For simplicity, it is very convenient to do computations in terms of the cone of $M^n$ in $S^{n+2} \subset \mathbb{R}^{n+3}$, and then view $M^n$ as the intersection of that cone with $S^{n+2}$. More precisely, we consider the map $G_g: \mathbb{R} \times \Lambda_g \to \mathbb{R}^{n+3}$ given by

$$G_g(s,p,v) = sg(p) + v \quad (2)$$

and set $S_G = \{0\} \times (\Lambda^*_g(p) \setminus \{0\})$ where

$$\Lambda^*_g(p) = \{(p,v) \in \Lambda_g : v \perp N^2_L(p)\}.$$ 

In next section, we show that the set of singular points of the metric induced by $G_g$ consists of the vertex $V = (0,p,0)$ and the set $S_G$. Set

$$N^{n+1} = \mathbb{R} \times \Lambda_g \setminus (V \cup S_G)$$

and denote $G = G|_{N^{n+1}}$. Thus, we have

$$M^n = \{(s,p,v) \in \mathbb{R} \times \Lambda_g \setminus S_G : s^2 + \|v\|^2 = 1\}$$

and $F_g = G|_{M^n}$ where $M^n$ is endowed with the induced metric. Observe that $M^n$ is complete (respectively, compact) if and only if $g$ is complete (respectively, compact) and $S_G$ is empty. Notice also that $S_G$ can only be empty for $n = 3, 4$. 

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In the sequel, we denote by $\mathcal{H}$ the tangent distribution orthogonal to the rulings. An embedded surface $j: L^2 \to M^n$ is called an integral surface of $\mathcal{H}$ if $j_* T_p L = \mathcal{H}(j(p))$ at every point $p \in L^2$.

**Theorem 1.** Let $g: L^2 \to S^{n+2}, n \geq 3$, be a 1-isotropic substantial surface. Then the associated immersion $F_g: M^n \to S^{n+2}$ is an $(n-2)$-ruled minimal submanifold with rank $\rho = 4$ (unless $n = 3 = \rho$) on an open dense subset of $M^n$. Moreover, the integral surface $L^2$ of $\mathcal{H}$ is totally geodesic and unique up to the one obtained by composing with the antipodal map.

Conversely, let $F: M^n \to S^{n+2}$ be an $(n-2)$-ruled minimal immersion with $n \geq 4$ and $\rho = 4$ (unless $n = 3 = \rho$) on an open dense subset of $M^n$. Assume that $\mathcal{H}$ admits a totally geodesic integral surface $j: L^2 \to M^n$ which is a global cross section to the rulings. Then the surface $g = F \circ j: L^2 \to S^{n+2}$ is 1-isotropic and $F$ can be parametrized as $F_g$.

The existence of genuine deformations is considered in the following result.

**Theorem 2.** Let $g: L^2 \to S^{n+2}, n \geq 3$, be a simply-connected 1-isotropic substantial surface. Then $F_g$ allows a smooth one-parameter family of minimal genuine isometric deformations $F_\theta: M^n \to S^{n+2}, \theta \in S^1$, such that $F_0 = F_g$ and each $F_\theta$ carries the same rulings and relative nullity leaves as $F_g$.

The relation between the second fundamental forms of members of the associated family is given next, for simplicity, in terms of their cones.

**Theorem 3.** Let $g: L^2 \to S^{n+2}, n \geq 3$, be a simply-connected 1-isotropic substantial surface. Then $G$ allows an associated smooth one-parameter family of minimal genuine isometric immersions $G_\theta: N^{n+1} \to \mathbb{R}^{n+3}, \theta \in S^1$, such that $G_0 = G$ and each $G_\theta$ carries the same rulings and relative nullity leaves as $G$.

Moreover, there is a parallel vector bundle isometry $\Psi_\theta: N G_s \to N_{G_\theta} s$ such that the relation between the second fundamental forms is given by

$$\alpha_{G_\theta}(X,Y) = \Psi_\theta(R_{-\theta} \alpha_G(X,Y) + 2\kappa \sin(\theta/2) \beta(J_{-\theta/2}X,Y))$$

where $R_\theta$ is the rotation of angle $\theta$ on $N_G N$ that preserves orientation, $\kappa$ is the radius of the ellipse of curvature of $g$ and $\beta$ is the traceless bilinear form defined by (17).

A substantial surface in even codimension $g: L^2 \to S^{n+2}$ is called pseudoholomorphic when the ellipses of curvature of any order are circles at any point. In odd codimension, the surface is called isotropic when the ellipses of curvature of any order but for the last one-dimensional normal subbundle are circles at any point.

If $g: L^2 \to S^{n+2}$ is pseudoholomorphic, then taking a rotation of angle $\theta \in S^1$ that preserves orientation in each $N^s G_s$, $s \geq 2$, induces an intrinsic isometry $S_\theta$ on $M^n$. The next result says that $F_g$ is equivariant with respect to the one-parameter family of intrinsic isometries $S_\theta$. 

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**Theorem 4.** If \( g: L^2 \rightarrow S^{n+2} \) is pseudoholomorphic, then \( F_g \circ S_{-\theta} \) is congruent to \( F_\theta \) for any \( \theta \in S^1 \).

We have that \( F_g : M^3 \rightarrow S^5 \) or \( F_g : M^4 \rightarrow S^6 \) is compact if and only if \( L^2 \) is compact and \( g \) is regular. The latter condition means that \( L_0 \) is empty and that \( N_2 \) has constant dimension. According to a result of Asperti [1] any compact regular substantial minimal surface in \( S^5 \) is a topological torus and in \( S^6 \) is a topological sphere. For both cases, there are plenty of 1-isotropic examples. In fact, the tori in \( S^5 \) include the flat ones described parametrically by Miyaoka [12] and those that are holomorphic with respect to the nearly Kaehler structure of \( S^6 \) considered in [3], [10] and [11]. Other examples of 1-isotropic surfaces in \( S^5 \) are the Legendrian surfaces given in [13].

Minimal 2-spheres in spheres have been investigated by Calabi, Barbosa and Chern among others. From their work, we know that these surfaces must be substantial in even codimension and pseudoholomorphic. It was then shown by Calabi [4] that any such surface in \( S^6 \) is regular if its area is \( 24\pi \). Then Barbosa [2] proved that the space of these surfaces is diffeomorphic to \( SO(7,\mathbb{C})/SO(7,\mathbb{R}) \), where \( SO(7,\mathbb{C}) \) denotes the set of \( 7 \times 7 \) complex matrices that satisfy \( AA^t = I \) and \( \det A = 1 \).

Concerning the set of genuine minimal isometric deformations of compact submanifolds constructed from tori we have the following result.

**Theorem 5.** Let \( g : L^2 \rightarrow S^5 \) be a regular substantial isotropic surface. Then, the set of all equally ruled minimal isometric immersions of \( M^3 \) into \( S^5 \) as \( F_g : M^3 \rightarrow S^5 \) is finite or parametrized by a circle \( S^1 \). If \( L^2 \) is compact then the set is necessarily finite.

As discussed in the introduction the last result is of independent interest.

**Theorem 6.** Let \( g \) be a flat 1-isotropic torus in \( S^5 \). Then \( F_g : M^3 \rightarrow S^5 \) is a compact minimal submanifold with constant normalized scalar curvature \( s = -1/3 \).

## 2 The proofs

In this section, we provide several proofs for \( n \geq 4 \) but similar arguments take care of the case \( n = 3 \).

First we discussed the set of singular points of \( F_g \).

**Proposition 7.** Let \( g : L^2 \rightarrow S^{n+2}, n \geq 4, \) be a substantial oriented minimal surface. Then, the set of singular points of the map \( G : \mathbb{R} \times \Lambda_g \rightarrow \mathbb{R}^{n+3} \) given by (2) consists of \( V = (0, p, 0) \) and \( S_G \).
Proof: Fix \((s_0, p_0, v_0) \in \mathbb{R} \times \Lambda_g \setminus \{V\}\). Choose a smooth orthonormal frame \(\{e_5, \ldots, e_{n+2}\}\) of \(\Lambda_g\) on a neighborhood \(U\) of \(p_0\) and set
\[
v_0 = \sum_{i \geq 1} a_i e_{i+4}(p_0).
\]
Consider the projection \(\Pi: \mathbb{R} \times \Lambda_g \to L^2\) and parametrize \(\Pi^{-1}(U)\) via the diffeomorphism \(h: U \times \mathbb{R}^{n-1} \to \Pi^{-1}(U)\) given by
\[
h(p, s, t_1, \ldots, t_{n-2}) = (s, p, \sum_{i \geq 1} t_i e_{i+4}).
\]
That \((s_0, p_0, v_0) \in S_G\) means that there exists a non-zero vector
\[
Z = X + \lambda_0 \partial/\partial s + \sum_{i \geq 1} \lambda_i \partial/\partial t_i \in \ker(G \circ h)_*(p_0, s_0, a_1, \ldots, a_{n-2})
\]
where \(X \in T_{p_0}L\). Thus,
\[
\lambda_0 g(p_0) + s_0 g_*(p_0)X + \sum_{i \geq 1} a_i \nabla^\perp_X e_{i+4}(p_0) + \sum_{i \geq 1} \lambda_i e_{i+4}(p_0) = 0.
\]
Since \(Z \neq 0\), we obtain that \(\lambda_0 = 0, s_0 = 0, X \neq 0\) and
\[
\sum_{i \geq 1} a_i \nabla^\perp_X e_{i+4}(p_0) + \sum_{i \geq 1} \lambda_i e_{i+4}(p_0) = 0.
\]
It follows that
\[
\langle v_0, \nabla^\perp_X \xi \rangle(p_0) = 0
\]
for any \(\xi \in N^g_1\). We easily conclude that \(v_0 \perp N^g_2(p_0)\). The converse is immediate. \(\blacksquare\)

In the sequel, we argue for an open set of \(L^2\) where all the normal subspaces \(N^g_r\)'s of the substantial oriented minimal surface \(g: L^2 \to \mathbb{S}^{n+2}\) have constant dimension. Choose local positively oriented orthonormal frames \(\{e_1, e_2\}\) in \(TL\) and \(\{e_3, e_4\}\) of \(N^g_1\) such that
\[
\alpha_g(e_1, e_1) = \kappa e_3 \quad \text{and} \quad \alpha_g(e_1, e_2) = \mu e_4
\]
where \(\kappa, \mu\) are the semi-axes of the ellipse of curvature. Take a local orthonormal normal frame \(\{e_5, \ldots, e_{n+2}\}\) such that \(\{e_{2r+1}, e_{2r+2}\}\) is positively oriented spanning \(N^g_r\) for every even \(r\). When \(n = 2m + 1\) is odd, then \(e_{2m+1}\) spans the last normal bundle. We refer to \(\{e_1, \ldots, e_{n+2}\}\) as an adapted frame of \(g\) and consider the one-forms
\[
\omega_{ij} = \langle \nabla e_i, e_j \rangle \quad \text{for} \quad 1 \leq i, j \leq n + 2,
\]
where \( \nabla \) denotes the Riemannian connection in the ambient space. Using that
\[
\alpha^3_g(e_1, e_1, e_1) + \alpha^3_g(e_1, e_2, e_2) = 0
\]
we easily obtain
\[
\omega_{45} = -\frac{1}{\lambda} \ast \omega_{35} \quad \text{and} \quad \omega_{46} = -\frac{1}{\lambda} \ast \omega_{36} \tag{4}
\]
where \( \lambda = \mu/\kappa \) and \( \ast \) denotes the Hodge operator, i.e., \( \ast \omega(e) = -\omega(Je) \). Here \( J \) is the complex structure of \( L^2 \) induced by the orientation. We denote by \( V = a_1 e_1 + a_2 e_2, \ W = b_1 e_1 + b_2 e_2, \ Y = c_1 e_1 + c_2 e_2 \) and \( Z = d_1 e_1 + d_2 e_2 \) the dual vector fields of \( \omega_{35}, \omega_{36}, \omega_{45} \) and \( \omega_{46} \), respectively. Then (4) is equivalent to
\[
Y = -\frac{1}{\lambda} JV \quad \text{and} \quad Z = -\frac{1}{\lambda} JW,
\]
and hence
\[
\lambda c_1 = a_2, \quad \lambda c_2 = -a_1, \quad \lambda d_1 = b_2 \quad \text{and} \quad \lambda d_2 = -b_1. \tag{5}
\]
Clearly, we have that \( G: N^{n+1} \to \mathbb{R}^{n+3} \) is an immersion and
\[
T_{(s,p,v)}N = \mathbb{R} \oplus T_{(p,v)}\Lambda_g = \mathbb{R} \oplus \mathcal{H}^G(p, v) \oplus \mathcal{V}(p, v)
\]
where \( \mathbb{R} = \text{span}\{\partial/\partial s\} \) and \( \mathcal{H}^G \) is the orthogonal complement of \( \mathcal{V} \) in \( T\Lambda_g \). Moreover, \( \mathcal{V} \) denotes the vertical bundle of \( \pi: \Lambda_g \to L^2 \) given by \( \mathcal{V} = \ker \pi_* \).

Fixed \((p, v) \in \Lambda_g\), let \( \delta_v \) be the normal vector field defined in a neighborhood of \( p \) by
\[
\delta_v(q) = \sum_{j \geq 5} \langle v, e_j(p) \rangle e_j(q). \tag{6}
\]
Let \( \beta_i, 1 \leq i \leq 2, \) be the curves in \( \Lambda_g \) satisfying \( \beta_i(0) = (p, v) \) given by
\[
\beta_i(t) = (c_i(t), \delta_v(c_i(t)))
\]
where \( c_i(t) \) is a smooth curve in a neighborhood of \( p \) satisfying \( c_i'(0) = e_i(p) \). Set
\[
Y_i = \beta_i'(0) \in T_{(p,v)}\Lambda_g, \ 1 \leq i \leq 2. \tag{7}
\]
Let \( G_i, H_i \in C^\infty(\Lambda_g), \ 1 \leq i \leq 2, \) be the functions
\[
G_i = t_2 \omega_{56}^i + t_3 \omega_{57}^i + t_4 \omega_{58}^i, \quad H_i = -t_1 \omega_{56}^i + t_3 \omega_{67}^i + t_4 \omega_{68}^i
\]
where \( \omega_{ij}^k = \omega_{ij}(e_k) \) and \( t_j \in C^\infty(\Lambda_g) \) is defined by
\[
t_j(q, w) = \langle w, e_{j+4}(q) \rangle, \ 1 \leq j \leq 4.
\]
It is clear that $G_*(s, p, v)\mathcal{V} = (N^3_0(p))^{\perp} \subset N_2L(p)$ holds up to parallel identification in $\mathbb{R}^{n+3}$. The vector bundle $\mathcal{V}$ can be orthogonally decomposed as $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0$ where $\mathcal{V}^1$ denotes the plane bundle determined by

$$G_*(s, p, v)\mathcal{V}^1 = N_2^0(p).$$

Let $\{E_3, E_4\}$ and $\{E_5, \ldots, E_n\}$ be local orthonormal frames of $\mathcal{V}^1$ and $\mathcal{V}^0$, respectively, such that

$$G_*E_j = e_{j+2} \text{ for } 3 \leq j \leq n.$$

Lemma 8. The vectors $X_1, X_2 \in T_{(p,v)}\Lambda_g$ defined as

$$X_i = Y_i + G_i E_3 + H_i E_4 - \sum_{j \geq 7} \langle \nabla^{\perp}_{e_i} \delta_v, e_j \rangle E_{j-2}$$

satisfy that $X_1, X_2 \in \mathcal{H}^G(p, v)$ and that

$$G_* X_1 = s g_* e_1 - \varphi_1 e_3 - \frac{1}{\chi} \varphi_2 e_4, \quad G_* X_2 = s g_* e_2 - \varphi_2 e_3 + \frac{1}{\chi} \varphi_1 e_4$$

where $\varphi_j = t^0_1 a_j + t^0_2 b_j$ and $t^0_j = t_j(p, v)$. Moreover, the space $N_G N(s, p, v)$ is spanned by

$$\xi = g_*(t^0_1 V(p) + t^0_2 W(p)) + s e_3(p), \quad \eta = g_*(t^0_1 Y(p) + t^0_2 Z(p)) + s e_4(p).$$

In particular, if $g$ is 1-isotropic then

$$\|X_1\| = \Omega = \|X_2\| \text{ with } \langle X_1, X_2 \rangle = 0 \text{ and } \|\xi\| = \Omega = \|\eta\| \text{ with } \langle \xi, \eta \rangle = 0$$

where $\Omega^2 = s^2 + \|t^0_1 V(p) + t^0_2 W(p)\|^2$.

Proof: On one hand,

$$G_* Y_i = s g_* e_i(p) + \sum_{j \geq 3} \langle \nabla^{\perp}_{e_i} \delta_v, e_j \rangle(p)e_j(p), \quad 1 \leq i \leq 2,$$

gives

$$G_* Y_i - \sum_{j \geq 5} \langle \nabla^{\perp}_{e_i} \delta_v, e_j \rangle(p)G_* E_{j-2} = s g_* e_i(p) - \sum_{3 \leq k \leq 4} \langle \nabla^{\perp}_{e_i} e_k, \delta_v \rangle(p)e_k(p).$$

On the other hand,

$$\langle \nabla^{\perp}_{e_i} \delta_v, e_5 \rangle(p) = -t^0_2 \omega^i_{56}(p) - t^0_3 \omega^i_{57}(p) - t^0_4 \omega^i_{58}(p) = -G_i(p, v),$$

$$\langle \nabla^{\perp}_{e_i} \delta_v, e_6 \rangle(p) = t^0_1 \omega^i_{56}(p) - t^0_3 \omega^i_{57}(p) - t^0_4 \omega^i_{58}(p) = -H_i(p, v),$$

$$\langle \nabla^{\perp}_{e_i} e_3, \delta_v \rangle(p) = t^0_1 \omega^i_{35}(p) + t^0_2 \omega^i_{36}(p) = t^0_1 a_i(p) + t^0_2 b_i(p),$$

$$\langle \nabla^{\perp}_{e_i} e_4, \delta_v \rangle(p) = t^0_1 \omega^i_{45}(p) + t^0_2 \omega^i_{46}(p) = t^0_1 c_i(p) + t^0_2 d_i(p).$$

Hence,

$$G_* X_i = s g_* e_i - (t^0_0 a_i + t^0_1 b_i)e_3 - (t^0_0 c_i + t^0_2 d_i)e_4, \quad 1 \leq i \leq 2.$$

The remaining of the proof is straightforward using [4].
Lemma 9. The following equations hold:

\[ \xi_* \partial / \partial s = e_3, \quad \eta_* \partial / \partial s = e_4, \quad (9) \]

\[ \xi_* E_3 = g_* V, \quad \xi_* E_4 = g_* W \quad \text{and} \quad \xi_* = 0 \text{ on } \mathcal{V}^0, \quad (10) \]

\[ \eta_* E_3 = g_* Y, \quad \eta_* E_4 = g_* Z \quad \text{and} \quad \eta_* = 0 \text{ on } \mathcal{V}^0, \quad (11) \]

\[ \xi_* X_1 = g_*(e_1(\varphi_1) - s \kappa)e_1 + e_1(\varphi_2)e_2 + \omega_1^1 \langle t^0_1 V + t^0_2 W \rangle + G_1 V + H_1 W \]

\[ + \kappa \varphi_3 e_4 + (s \omega_4 + \lambda \kappa \varphi_1)e_4 + sa_1 e_5 + sb_1 e_6 - \varphi_1 g, \quad (12) \]

\[ \xi_* X_2 = g_*(e_2(\varphi_1) + (e_2(\varphi_2) + s \kappa)e_2 + \omega_2^1 \langle t^0_1 V + t^0_2 W \rangle + G_2 V + H_2 W \]

\[ - \kappa \varphi_2 e_3 + (s \omega_3 + \lambda \kappa \varphi_1)e_3 + sa_2 e_5 + sb_2 e_6 - \varphi_2 g, \quad (13) \]

\[ \eta_* X_1 = g_*(e_1(\psi_1) + (s \lambda \kappa)e_1 + \omega_1^1 \langle t^0_1 V + t^0_2 W \rangle - \sigma G_1 JV - \sigma H_1 JW \]

\[ -(s \omega_4 - \kappa \psi_1)e_3 + \lambda \kappa \psi_2 e_4 + s \sigma a_1 e_5 + s \sigma b_1 e_6 - \psi_1 g, \quad (14) \]

\[ \eta_* X_2 = g_*(e_2(\psi_1) + (s \lambda \kappa)e_1 + e_2(\psi_2)e_2 + \omega_2^1 \langle t^0_1 V + t^0_2 W \rangle - \sigma G_2 JV - \sigma H_2 JW \]

\[ -(s \omega_3 + \kappa \psi_2)e_3 + \lambda \kappa \psi_1 e_4 - s \sigma a_1 e_5 - s \sigma b_1 e_6 - \psi_2 g \quad (15) \]

where \( \sigma = 1/\lambda \) and \( \psi_j = t^0_j c_j + t^0_2 d_j, \quad j = 1, 2. \)

Proof: We compute at \( (s, p, v) \in N^{n+1} \). Let \( \gamma(t) = (s, p, v(t)) \) be a curve in \( N^{n+1} \) such that \( v(0) = v \), and thus \( \gamma'(0) \in \mathcal{V}(p, v) \). We have that

\[ \xi_* \gamma'(0) = \langle Dv/\partial t(0), e_5(p) \rangle g_* V(p) + \langle Dv/\partial t(0), e_6(p) \rangle g_* W(p), \]

or equivalently, that

\[ \xi_* \gamma'(0) = \langle G_* \gamma'(0), e_5(p) \rangle g_* V(p) + \langle G_* \gamma'(0), e_6(p) \rangle g_* W(p). \]

From this we obtain (10). Similarly, we have (11).

To obtain (12) to (15) one has to use Lemma 8 and the Gauss and Weingarten formulas for \( g \). We only argue for (12) since the proof of the other equations is similar. We have from (8) and (10) that

\[ \xi_* X_i = \xi_* Y_i + G_i g_* V + H_i g_* W, \quad 1 \leq i \leq 2. \]

In view of (7) and since

\[ \xi \circ \beta_i(t) = t^0_1 g_* V(c_i(t)) + t^0_2 g_* W(c_i(t)) + s\epsilon_3(c_i(t)), \]

we obtain in terms of the connection in \( L^2 \) that

\[ \xi_* Y_i = t^0_1 \langle g_* \nabla_{e_i} V + \alpha_g(e_i, V) \rangle (p) + t^0_2 \langle g_* \nabla_{e_i} W + \alpha_g(e_i, W) \rangle (p) \]

\[ + (-1)^i s \kappa(p) g_* e_i(p) + s \nabla_{e_i}^\perp e_3(p), \]

and (12) follows by a direct computation. \( \blacksquare \)
Lemma 10. The second fundamental form of $G$ in terms of the orthonormal frame

$$E_0 = \partial/\partial s, \quad E_i = X_i/\Omega, \quad i = 1, 2, \quad \text{and} \quad G_sE_j = e_{j+2}, \quad 3 \leq j \leq n,$$

vanishes along $\mathcal{V}^0$ and restricted to $\text{span}\{E_0\} \oplus \mathcal{H}^G \oplus \mathcal{V}^1$ is given by

$$A_\xi = \begin{bmatrix} 0 & \varphi_1 & \varphi_2 & 0 & 0 \\ \varphi_1 & h_1 + \kappa & h_2 & r_1 & s_1 \\ \varphi_2 & h_2 & -h_1 - \kappa & r_2 & s_2 \\ 0 & r_1 & r_2 & 0 & 0 \\ 0 & s_1 & r_2 & 0 & 0 \end{bmatrix}, \quad A_\eta = \begin{bmatrix} 0 & \varphi_2 & -\varphi_1 & 0 & 0 \\ \varphi_2 & h_2 & \kappa - h_1 & r_2 & s_2 \\ -\varphi_1 & \kappa - h_1 & -h_2 & -r_1 & -s_1 \\ 0 & r_2 & -r_1 & 0 & 0 \\ 0 & s_2 & -s_1 & 0 & 0 \end{bmatrix}$$

where $\varphi_i\Omega = \varphi_i$, $r_i\Omega = -s_{ai}$, $s_i\Omega = -sb_i$ and

$$h_i = -\frac{s}{\Omega^2} \left( t_1(e_1(a_1) - a_2B_i - b_1\omega_{56}^i) + t_2(e_1(b_1) - b_2B_i + a_1\omega_{56}^i) + t_3(a_1\omega_{57}^i + b_1\omega_{67}^i) + t_4(a_1\omega_{58}^i + b_1\omega_{68}^i) \right)$$

with $B_i = \omega_{12}^i + \omega_{34}^i$, $i = 1, 2$.

Proof: Since $g$ is 1-isotropic, then (12) to (15) hold for $\psi_1 = \varphi_2$ and $\psi_2 = -\varphi_1$. On the other hand, a straightforward computation shows that the Ricci equations

$$\langle R^\perp(e_1, e_2)e_\alpha, e_\beta \rangle = 0$$

for $\alpha = 3, 4$ and $\beta = 5, 6$ are equivalent to

$$e_1(a_2) - e_2(a_1) + a_1B_1 + a_2B_2 - b_1\omega_{56}^1 + b_1\omega_{56}^2 = 0,$$
$$e_1(b_2) - e_2(b_1) + b_1B_1 + b_2B_2 + a_2\omega_{56}^1 - a_1\omega_{56}^2 = 0,$$
$$e_1(a_1) + e_2(a_2) - a_2B_1 + a_1B_2 - b_1\omega_{56}^1 - b_2\omega_{56}^2 = 0,$$
$$e_1(b_1) + e_2(b_2) - b_2B_1 + b_1B_2 + a_1\omega_{56}^1 + a_2\omega_{56}^2 = 0,$$

and for $\alpha = 3, 4$ and $\beta = 7, 8$ are equivalent to

$$a_2\omega_{57}^1 - a_1\omega_{57}^2 + b_2\omega_{67}^1 - b_1\omega_{67}^2 = 0,$$
$$a_2\omega_{58}^1 - a_1\omega_{58}^2 + b_2\omega_{68}^1 - b_1\omega_{68}^2 = 0,$$
$$a_1\omega_{57}^1 + a_2\omega_{57}^2 + b_1\omega_{67}^1 + b_2\omega_{67}^2 = 0,$$
$$a_1\omega_{58}^1 + a_2\omega_{58}^2 + b_1\omega_{68}^1 + b_2\omega_{68}^2 = 0.$$

We thus have that

$$\langle A_\xi E_i, E_j \rangle = -\langle G_sE_i, \xi E_j \rangle \quad \text{and} \quad \langle A_\eta E_i, E_j \rangle = -\langle G_sE_i, \eta E_j \rangle, \quad 0 \leq i, j \leq n,$$

and the result follows by a straightforward computation. \[\square\]
Proof of Theorem 1: We first prove the converse. If \( F: M^n \rightarrow S^{n+2}, n \geq 4, \) is an \((n-2)\)-ruled minimal immersion with rank \( \rho = 4 \) everywhere, then the tangent bundle splits as \( TM = \mathcal{H} \oplus \mathcal{V} \), where \( \mathcal{H} \) is orthogonal to the rulings. Moreover, we have that \( \mathcal{V} \) splits as \( \mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_0 \) with the fibers of \( \mathcal{V}_0 \) being the relative nullity leaves.

The normal space of the surface \( g = F \circ j \) at \( x \in L^2 \) is given by

\[
N_gL(x) = F_*(j(x))V \oplus N_F M(j(x)).
\]

Being \( j \) is totally geodesic, we have

\[
\alpha_g(X,Y) = \alpha_F(j_* X, j_* Y)
\]

for all \( X, Y \in TL \). This and our assumptions imply that \( g \) is minimal.

Let \( \pi: \Lambda_g \rightarrow L^2 \) denote the subbundle of the normal bundle of \( g \) whose fiber at \( x \in L^2 \) is \( F_*(j(x))V \). We consider the cone \( CF: \mathbb{R} \times M^n \rightarrow \mathbb{R}^{n+3} \) given by

\[
CF(t,p) = tF(p).
\]

Observe that

\[
CF(t,p) - CF(u(t,p), j(x)) = CF(t,p) - u(t,p)g \circ \pi(p) \in F_*(j(x))V
\]

for any \( p \in M^n \), where \( x = \pi(p) \), since \( p \) and \( j(x) \) belong to the same leaf of \( \mathcal{V} \) and

\[
u(t,p) = t/(F(p), g \circ \pi(p)).
\]

Since \( CF \) maps locally diffeomorphically the leaves of \( \mathcal{V} \) onto affine subspaces, it follows that the map \( T: \mathbb{R} \times M^n \rightarrow \mathbb{R} \times \Lambda_g \) given by

\[
T(t,p) = (u(t,p), \pi(p), CF(t,p) - u(t,p)g \circ \pi(p))
\]

is a local diffeomorphism. Clearly the immersion \( \tilde{G} = CF \circ T^{-1} \) satisfies

\[
\tilde{G}(s, x, v) = sg(x) + v,
\]

i.e., \( \tilde{G} = G_g \) is of the form \( \mathbf{[2]} \). Identifying locally \( \mathbb{R} \times M^n \) with \( \mathbb{R} \times \Lambda_g \) via \( T \), we have that \( CF = G_g = G \) and \( j \) is the zero section of \( \Lambda_g \), i.e., we have the parametrization given by \( \mathbf{[2]} \). The horizontal and the vertical bundles satisfy

\[
G_*(s, p, v)\mathcal{V} = (N_1^g(p))^\perp \subset N_gL(p), \quad G_*(s, p, v)\mathcal{H}^G \subset g_* T_p L \oplus (\Lambda_g(p))^\perp,
\]

\[
N_GN(s, p, v) \subset g_* T_p L \oplus (\Lambda_g(p))^\perp
\]

and now \( \mathbf{[16]} \) yields \( N_1^g = \Lambda_g^\perp \).
It remains to see that $g$ is 1-isotropic. For an adapted frame $\{e_1, \ldots, e_{n+2}\}$ of $g$ set
\[ g_{ij} = \langle G_* X_i, G_* X_j \rangle \]
and
\[ b^i_{ij} = \langle \xi_* X_i, G_* X_j \rangle, \quad b^h_{ij} = \langle \eta_* X_i, G_* X_j \rangle, \quad i, j = 1, 2. \]

Using Lemma 8 and Lemma 9 we find that
\[ g_{11} = s^2 + \varphi_1^2 + \sigma^2 \varphi_2^2, \quad g_{12} = (1 - \sigma^2) \varphi_1 \varphi_2, \quad g_{22} = s^2 + \varphi_2^2 + \sigma^2 \varphi_1^2, \]
and
\[ b^\xi_{11} = s(e_1(\varphi_1) - s \kappa - \omega_1^2 \varphi_2 + G_1 a_1 + H_1 b_1) - \kappa \varphi_1^2 - \sigma \varphi_1 (s \omega_{34}^1 + \mu \varphi_2), \]
\[ b^\xi_{12} = s(e_1(\varphi_2) + \omega_1^2 \varphi_1 + G_2 a_2 + H_1 b_2) - \kappa \varphi_1 \varphi_2 + \sigma \varphi_1 (s \omega_{34}^1 + \mu \varphi_2), \]
\[ b^\xi_{21} = s(e_2(\varphi_1) - \omega_2^2 \varphi_2 + G_2 a_1 + H_2 b_1) + \kappa \varphi_1 \varphi_2 - \sigma \varphi_2 (s \omega_{34}^2 + \mu \varphi_1), \]
\[ b^\xi_{22} = s(e_2(\varphi_2) + s \kappa + \omega_2^2 \varphi_1 + G_2 a_2 + H_2 b_2) + \kappa \varphi_2^2 + \sigma \varphi_1 (s \omega_{34}^2 + \mu \varphi_1) \]
and
\[ b^\eta_{11} = s(e_1(\psi_1) - \omega_1^2 \psi_2 + \sigma G_1 a_2 + \sigma H_1 b_2) + s \omega_{34}^1 \varphi_1 - \kappa (\varphi_1 \psi_1 + \varphi_2 \psi_2), \]
\[ b^\eta_{12} = s(e_1(\psi_2) - \mu - \omega_1^2 \psi_1 - \sigma G_2 a_1 - \sigma H_1 b_1) + s \omega_{34}^1 \varphi_2 + \kappa (\varphi_1 \psi_2 - \varphi_2 \psi_1), \]
\[ b^\eta_{21} = s(e_2(\psi_1) - \mu - \omega_2^2 \psi_2 + \sigma G_2 a_2 + \sigma H_2 b_2) + s \omega_{34}^2 \varphi_1 + \kappa (\varphi_1 \psi_2 - \varphi_2 \psi_1), \]
\[ b^\eta_{22} = s(e_2(\psi_2) + \omega_2^2 \psi_1 - \sigma G_2 a_1 - \sigma H_2 b_1) + s \omega_{34}^2 \varphi_2 + \kappa (\varphi_1 \psi_1 + \varphi_2 \psi_2). \]

From our assumptions, we have
\[ g_{11} b^\xi_{22} - g_{12} (b^\xi_{12} + b^\xi_{21}) + g_{22} b^\xi_{11} = 0 \quad \text{and} \quad g_{11} b^\eta_{22} - g_{12} (b^\eta_{12} + b^\eta_{21}) + g_{22} b^\eta_{11} = 0. \]

Viewing these as polynomials were the coefficients of $t_1^4, t_2^4$ and $t_1^2 t_2^2$ must vanish gives
\[ (\lambda^2 - 1)(a_1^2 + a_2^2)(a_1^2 - a_2^2) = 0 = (\lambda^2 - 1)(b_1^2 + b_2^2)(b_1^2 - b_2^2) \]
and
\[ (\lambda^2 - 1)a_1 a_2(a_1^2 + a_2^2) = 0 = (\lambda^2 - 1)b_1 b_2(b_1^2 + b_2^2). \]

Hence $\lambda = 1$ since, otherwise, we would have that $\omega_{35} = \omega_{36} = \omega_{45} = \omega_{46} = 0$, and that is a contradiction.

We now prove the direct statement. Since $F_g = G|_M$, we obtain that $g = F_g \circ j$ where $j : L^2 \to M^n$ is given by $j(x) = (\pm 1, x, 0)$. Clearly, we have that $j$ is an integral surface of the distribution orthogonal to the rulings that is totally geodesic and a global cross section to the rulings. Up to uniqueness of the integral surface the proof follows from Lemma 10.
Assume that there exists a second integral surface \( \tilde{j} : L^2 \to M^n \). Set \( \tilde{g} = F_g \circ \tilde{j} \) and let \( \tilde{T} : \mathbb{R} \times M^n \to \mathbb{R} \times \Lambda_{\tilde{g}} \) be the local diffeomorphism given by

\[
\tilde{T}(t, p) = (\tilde{u}(t, p), \pi(p), CF(p) - \tilde{u}(t, p)\tilde{g} \circ \pi(p))
\]

where

\[
\tilde{u}(t, p) = t/(F(p), \tilde{g} \circ \pi(p)).
\]

Then \( \tilde{T} \circ T^{-1} : \mathbb{R} \times \Lambda_g \to \mathbb{R} \times \Lambda_{\tilde{g}} \) is given by

\[
\tilde{T} \circ T^{-1}(s, x, v) = (\tilde{s}, x, v + sg(x) - \tilde{g}(x)),
\]

where \( T^{-1}(s, x, v) = (t, p) \) and \( \tilde{s} = \tilde{u}(t, p) \). Hence \( \Lambda_g \) and \( \Lambda_{\tilde{g}} \) can be identified by parallel translation. Using that \( sg(x) - \tilde{g}(x) \in \Lambda_g(x) \), we obtain that \( \tilde{g} = \pm g \). ~\( \blacksquare \)

The vertical bundle \( \mathcal{V} \) of the submersion \( \pi \) given by \( \mathcal{V} = \ker \pi_* \) can be orthogonally decomposed as \( \mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0 \) on an open dense subset of \( L^2 \), where \( \mathcal{V}^1 \) denotes the plane bundle determined by \( N_{g}^1 \). In fact, this holds if \( N_{g}^1 \) and \( N_{g}^2 \) are subbundles, which we can assume without loss of generality. In the sequel, we consider the orthogonal decomposition of the tangent bundle of \( N^{n+1} \) given by

\[
TN = \text{span}\{\partial/\partial s\} \oplus \mathcal{H}^G \oplus \mathcal{V}
\]

where we identify isometrically (and use the same notation) the subbundle \( \mathcal{V} \) tangent to the rulings with the corresponding normal subbundle to \( g \). Then, it follows from the proof that the relative nullity leaves of \( G \) are identified with the fibers of \( \text{span}\{\partial/\partial s\} \oplus \mathcal{V}^0 \).

Let \( \mathcal{J} \) denote the endomorphism such that \( \mathcal{J}|_{\mathcal{H}^G} : \mathcal{H}^G \to \mathcal{H}^G \) is the almost complex structure in \( \mathcal{H}^G \) determined by the orientation and restricted to \( \text{span}\{\partial/\partial s\} \oplus \mathcal{V} \) is the identity and set

\[
\mathcal{J}_\theta = \cos \theta I + \sin \theta \mathcal{J}.
\]

**Proof of Theorem** For each \( \theta \in \mathbb{S}^1 \) consider the submanifold \( G_\theta : N^{n+1} \to \mathbb{R}^{n+3} \) defined by

\[
G_\theta(s, p, v) = sg_\theta(p) + \phi_\theta v
\]

where \( \phi_\theta : N_g L \to N_{g_\theta} L \) is the parallel vector bundle isometry that identifies the normal subbundles of \( g \) and of \( g_\theta \).

In the sequel, corresponding quantities of \( G_\theta \) are denoted by the same symbol used for \( G \) marked with \( \theta \). That \( G_\theta \) is isometric to \( G \) is immediate. Since the tangent frame \( \{e_1, e_2\} \) has been fixed, we have for the adapted frames of \( g_\theta \) that

\[
e_3^\theta = \phi_\theta \circ R^1_\theta e_3 \quad \text{and} \quad e_4^\theta = \phi_\theta \circ R^1_\theta e_4
\]

where \( R^1_\theta \) is the rotation of angle \( \theta \) on \( N_1^2 \). We complete the adapted frame choosing

\[
e_j^\theta = \phi_\theta e_j, \quad 5 \leq j \leq n + 2.
\]
Clearly, it holds that $\omega^\theta_{34} = \omega_{34}$ and $\omega^\theta_{ij} = \omega_{ij}$ for $i, j \geq 5$. Moreover,

$$
\omega^\theta_{35} = \cos \theta \omega_{35} - \sin \theta \ast \omega_{35} \quad \text{and} \quad \omega^\theta_{36} = \cos \theta \omega_{36} - \sin \theta \ast \omega_{36}.
$$

Hence, the dual vector fields of $\omega^\theta_{36}$ and $\omega^\theta_{36}$ are given, respectively, by

$$V^\theta = J_{\ast \theta}V \quad \text{and} \quad W^\theta = J_{\ast \theta}W.$$ 

Thus,

$$
a^\theta_i = a_i \cos \theta + a_2 \sin \theta, \quad a^\theta_2 = a_2 \cos \theta - a_1 \sin \theta,
$$

$$
b^\theta_i = b_i \cos \theta + b_2 \sin \theta, \quad b^\theta_2 = b_2 \cos \theta - b_1 \sin \theta.
$$

It follows from (6), (7) and (8) that

$$X^\theta_i = X_i, \quad i = 1, 2.$$ 

By Lemma 8, the normal bundle of $G_\theta$ is spanned by

$$\xi_\theta = g_\theta J_{\ast \theta}(t_1V + t_2W) + s\phi_\theta \circ R^1_\theta e_3, \quad \eta_\theta = -g_\theta J_{\pi/2-\theta}(t_1V + t_2W) + s\phi_\theta \circ R^1_\theta e_4.$$ 

A straightforward computation yields that the map $\Psi_\theta : N_{G_\theta} \to N_{G_\theta}$ given by

$$\Psi_\theta \xi = \xi_\theta \quad \text{and} \quad \Psi_\theta \eta = \eta_\theta$$

is a parallel vector bundle isometry. The shape operators of $G_\theta$ vanish on $V^\theta$ and restricted to $\text{span}\{\partial/\partial s\} \oplus H^G \oplus V^1$ and with respect to $\{E_1, \ldots, E_n\}$ they are given by

$$A^\theta_{\xi_\theta} = \begin{bmatrix}
0 & \tilde{\varphi}^\theta_1 r^\theta_1 \\
\tilde{\varphi}^\theta_1 r^\theta_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\varphi^\theta_2 s^\theta_2 & 0 & 0 & 0
\end{bmatrix}, \quad A^\theta_{\eta_\theta} = \begin{bmatrix}
0 & \tilde{\varphi}^\theta_2 r^\theta_1 \\
\tilde{\varphi}^\theta_2 r^\theta_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\varphi^\theta_1 s^\theta_1 & 0 & 0 & 0
\end{bmatrix},$$

where $\tilde{\varphi}^\theta_1 = \varphi^\theta_1$, $r^\theta_i \Omega = -s a^\theta_i$, $s^\theta_1 \Omega = -s b^\theta_i$, $i = 1, 2,$ and

$$
\varphi^\theta = \varphi_1 \cos \theta + \varphi_2 \sin \theta, \quad \varphi^\theta = -\varphi_1 \sin \theta + \varphi_2 \cos \theta,
$$

$$
h^\theta_1 = h_1 \cos \theta + h_2 \sin \theta, \quad h^\theta_2 = -h_1 \sin \theta + h_2 \cos \theta.
$$

Let $L_\theta : T^1 N \to T^1 N$ be such that $L_\theta|_{\text{span}\{\partial/\partial s\} \oplus V} = 0$ and $L_\theta|_{H^G} : H^G \to H^G$ is the reflection given by

$$L_\theta|_{H^G} = \begin{bmatrix}
-\sin(\theta/2) & \cos(\theta/2) \\
\cos(\theta/2) & \sin(\theta/2)
\end{bmatrix}$$

with respect to the tangent frame $\{E_1, E_2\}$. It follows easily that

$$A^\theta_{\Psi_\theta \xi} = A_{R_\theta \xi} - 2\kappa \sin(\theta/2) L_\theta \quad \text{and} \quad A^\theta_{\Psi_\theta \eta} = A_{R_\theta \eta} - 2\kappa \sin(\theta/2) J \circ L_\theta.$$
By direct computation, we obtain
\[ \alpha_{G_{\theta}}(X,Y) = \Psi_{\theta} \left( R_{-\theta} \alpha_{G}(X,Y) - \frac{2K}{\Omega^2} \sin(\theta/2) \left( \langle L_{\theta} X, Y \rangle \xi + \langle L_{\theta} J X, Y \rangle \eta \right) \right). \]

Now let \( \beta \) be the symmetric section of \( Hom(TN \times TN, N_G N) \) with nullity \( V \) given by
\[ \beta(E_1, E_1) = \frac{1}{\Omega^2} \xi = -\beta(E_2, E_2), \quad \beta(E_1, E_2) = -\frac{1}{\Omega^2} \eta, \quad (17) \]
and the proof of (3) follows easily.

Finally, that the isometric deformations \( G_{\theta} \) of \( G \) are genuine is immediate from Lemma \( \textbf{10} \) since the shape operators of \( G \) have rank four for any normal direction along an open dense subset of \( N^{n+1} \).

**Proof of Theorem 2:** Given \( \theta \in S^1 \), denote \( F_\theta = G_{\theta}|_M \) where \( G_{\theta}(s, p, v) = sg_{\theta}(p) + \phi_{\theta} v. \)

That \( F_\theta \) allows a one-parameter family of minimal isometric immersions \( F_\theta : M^n \to S^{n+2} \), \( \theta \in S^1 \), such that \( F_0 = F_g \) and each \( F_\theta \) carries the same ruling and relative nullity leaves as \( F_g \) is a consequence of Proposition \( \textbf{3} \).

**Proof of Theorem 4:** It is completely analogous to the proof of Theorem 6 in [8].

**Proof of Theorem 5:** Let \( \bar{F} : M^3 \to S^5 \) be a ruled isometric minimal immersion with the same rulings as \( F_g \) and set \( \bar{g} = \bar{F} \circ j \). From the proof of Theorem \( \textbf{1} \) we have that the surface \( \bar{g} \) is isometric to \( g \) and isotropic. Hence, the set of all minimal isometric immersions of \( M^3 \) into \( S^5 \) with the same rulings as \( F_g \) can be identified with the set of all isotropic immersions of \( L^2 \) into \( S^5 \). The proof now follows from the results in [9].

**Proof of Theorem 6:** Using Lemma \( \textbf{8} \) and Lemma \( \textbf{11} \), we have that the squared length of the second fundamental form of \( G \) is given by
\[ \|\alpha_G\|^2 = \frac{4}{\Omega^4} \left( (1 - K)\Omega^2 + \varphi_1^2 + \varphi_2^2 + \Omega^2 \sum_{i=1,2} (h_i^2 + r_i^2 + s_i^2) \right). \]
It follows that
\[ \|\alpha_G\|^2(s, p, v) = \frac{4}{\Omega^2} \left( 2 - K + h_1^2 + h_2^2 + \frac{s^2}{\Omega^2} (\|V\|^2 + \|W\|^2 - 1) \right). \quad (18) \]

By Corollary 4 in [15] any 1-isotropic torus in \( S^5 \) is regular, hence \( M^3 \) is compact. On the other hand, we have that \( g \) is \( O(6) \)-congruent to a holomorphic curve in the nearly Kaehler sphere \( S^6 \); see [14] or [15]. Choose local orthonormal frame \( \{e_1, e_2, e_3, e_4, e_5\} \) such that
\[ \alpha_{g}(e_1, e_1) = \sqrt{1/2} e_3, \quad \alpha_{g}(e_1, e_2) = \sqrt{1/2} e_4, \]
\[ \alpha_3^g(e_1, e_1, e_1) = \kappa_1 e_5, \quad \alpha_3^g(e_1, e_1, e_2) = 0, \]
where \( \kappa_1 = \sqrt{1/2} \) by Theorem 5 in [15]. Hence, we have that \( V = e_1 \). From Lemma 6 in [13] we obtain \( h_1 = h_2 = 0 \). Now (18) gives

\[ \|\alpha_G\|^2(s, p, v) = \frac{8}{\Omega^2} = \frac{8}{s^2 + r_1^2}, \]

and hence \( \|\alpha_F\|^2 = 8 \). 

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