Feynman’s path integral seen as a Henstock integral

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Abstract. The motivation of this paper is to give mathematical formalism to the Feynman path integral using the Henstock integral. This integral saves some of Feynman’s integral difficulties and justifies Feynman’s intuition: to interpret the state function as “a sum of complex contributions, one from each path in the region”.

1. Introduction.
The interpretation of Richard Feynman, in order to solve the Schrödinger equation, (for non relativistic particle) involves taking limits of the sum over a space of paths of a functional of the action for each path. He did not provide proofs, see [1]. R. Feynman formulated quantum mechanics in terms of integrals over spaces of paths (Feynman path integrals). But the absolute value of Feynman’s integrand is not integrable. A non-absolute method of integration is required in order to allow the cancellation effects described by Feynman to come into play. So, instead of Lebesgue integration, we use the non-absolute integral of Henstock. The main objective in this work is to realize a mathematical fundament where the Feynman path integral exists as an integral over the space of all paths.

It is well known that the Schrödinger equation describes the evolution of a state function for a particle of a constant mass $m$, moving in Euclidean space $\mathbb{R}^d$ in the presence of a potential $V(x)$. The state $\psi$ at time $t = 0$ gives the initial condition for the equation and allows one to uniquely determine the state $\psi$ function at all subsequent times:

$$\frac{\partial \psi}{\partial t} = i \left[ \frac{1}{2m} \Delta - V \right] \psi$$

$$\psi_{|t=0} = \varphi,$$

where, $\hbar = 1$, $\Delta$ is the Laplacian operator in $\mathbb{R}^d$ and $V : \mathbb{R}^d \to \mathbb{R}$ is a measurable function. Suppose that we measure successive positions of a particle in one-dimensional space separated by a small time-interval $\epsilon$, denote them by $x_1, x_2, x_3...$. Letting the intervals between measurements $\epsilon$ get smaller and smaller, we would expect the sequence $x_1, x_2, x_3...$ converges to a path of the particle, represented by a function of time $x(t)$. Then the state function can be represented as

$$\psi(x, t) = \lim_{\epsilon \to 0} \int_{\mathbb{R}} e^{i \sum S(x_{i+1}, x_i)} \frac{dx_i}{A} \frac{dx_{i+1}}{A}...$$

(1)
where $\frac{1}{I}$ is the normalization factor and $S(x(t))$ is the classical action, that is the time integral of the Lagrangian along the path of the particle $S(x(t)) = \int L(\dot{x}(t), x(t))dt$. In the case when the Lagrangian has a simple form of a quadratic function of the velocity $L(\dot{x}(t), x(t)) = \frac{1}{2}m(\dot{x}(t))^2 - V(x(t))$, Feynman was able to define a state function. By Trotter’s Theorem is obtained that, under certain conditions, the state function exists as limit of product of operators, see [2], [3] and [4], among others. Thus,

$$\psi(x,t) = \lim_{n \to \infty} \left(\frac{mn}{2\pi i}\right)^{nd/2} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{i \sum_{j=1}^{m} \frac{1}{2}(x_j-x_{j-1})^2 - V(x_j)]t/n} \varphi(x_n) dx_1 \cdots dx_n$$

According to Feynman’s fundamental postulate a resulting path of a particle is a sum over all paths and it is written in the form of an integral over the space of all paths, see [5]. Thus, this integral is written in the following way

$$\text{Const} \int_{\Omega_e} e^{iS(\omega,t)} \varphi(\omega(x)) D\omega,$$

where $\Omega_e$ is the set of all paths that start at $x$. We might interpret the limit (1) as a sum over all paths. Then it would be natural to represent it as an integral with respect to some measure in the path-space. However, there exist some difficulties, for example: i) the normalization “constant” has a meaning for every finite “$n$”, but it becomes infinite as $n$ goes to infinity; ii) and $\lim_{n \to \infty} \prod_{j=1}^{n} dx_j$ corresponds to some measure on a space of all paths, but this product is infinite, thus defined this way, the measure has no firm mathematical meaning.

These difficulties are saved by interpreting the Feynman path integral as Henstock integral over the space of all paths, $\mathbb{R}^P$, and it is justified the Feynman’s intuition, to interpret the state function as “a sum of complex contributions, one from each path in the region”. These problems are discussed in detail in [6], [7], [8], [9], [10] and [11]. It is well known that other methods exist to give solution to the Schrödinger equation, however the proposal of Feynman in addition to giving solution has implications in other areas as Quantum Mechanics, Financial Mathematics, among other, see for example, [12], [13], [14].

The objective of this paper is to present the main idea of this integral and its implications in Quantum Mechanics to give mathematical formalism.

2. Formalism. Finite-dimensional integration case.

We follow the notation from [9]. In order to present the Henstock integral theory over the space of all paths, we need to introduce basic definitions of the Henstock integral in the finite-dimensional case, $\mathbb{R}^n$ with $n \geq 1$. To see more about this integral consult [15], [16] and [17] among others.

**Definition 2.1.** A cell $I$ in $\mathbb{R}^n$ consists in the product $I = I(N) = I_1 \times \cdots \times I_n$, where $N = \{1, \ldots, n\}$ and each $I_j$ can have the form

$$(-\infty, a), \quad [u, v], \quad (b, \infty), \quad \text{or} \quad (-\infty, \infty).$$

The collection of all cells in $\mathbb{R}^n$ is denoted by $I(\mathbb{R}^n) = \{I(N)\}$. This definition is given by Riemman sums, thus each cell must be related or associated to a point $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$.

**Definition 2.2.** Let $I$ be a cell in $\mathbb{R}^n$. The cell is associated to $x \in \mathbb{R}^n$ if for each $j = 1, \ldots, n$ i) $x_j = -\infty$, if $I_j = (-\infty, a]$; ii) $x_j = u$ or $x_j = v$, if $I_j = [u, v]$; iii) $x_j = \infty$, if $I_j = (b, \infty)$; iv) $x_j = -\infty$ or $x_j = \infty$, if $I_j = (-\infty, \infty)$. 


The association condition means that the point \( x \) should be in the interior or on the boundary of \( I \).

**Definition 2.3.** A gauge in \( \mathbb{R}^n \) is a positive function \( \delta \) defined for \( x \in \mathbb{R}^n \). An associated point-cell pair \( (x, I) \) of \( \mathbb{R}^n \) is \( \delta \)-fine if, for each \( j \), the pair \( (x_j, I_j) \) is \( \delta \)-fine in \( \mathbb{R} \); that is,

\[
a < -\frac{1}{\delta(x)}; \quad u - v < \delta(x); \quad \text{or} \quad b > \frac{1}{\delta(x)},
\]

respectively.

A **partition** of \( \mathbb{R}^n \) is a finite collection \( \mathcal{P} \) of disjoint cells whose union is \( \mathbb{R}^n \). A **division** \( \mathcal{D} \) of \( \mathbb{R}^n \) is a finite collection of associated point-cell pairs \( (x, I) \) whose cells form a partition of \( \mathbb{R}^n \). Given a gauge \( \delta : \mathbb{R}^n \to \mathbb{R}^+ \), a **division** \( \mathcal{D} \) is \( \delta \)-fine if each \( (x, I) \in \mathcal{D} \) is \( \delta \)-fine, it is denoted by \( \mathcal{D}_\delta \). The class of all the cells in \( \mathbb{R}^n \) is denoted by \( I(\mathbb{R}^n) \).

In general, an integrand in \( \mathbb{R}^n \) is a point-cell function \( h(x, I) \) defined in the product \( \mathbb{R}^n \times I(\mathbb{R}^n) \) to \( \mathbb{C} \) (or \( \mathbb{R} \)); in particular it can be a product \( f(x)h(I) \).

**Definition 2.4.** A function \( h(x, I) \) is integrable in \( \mathbb{R}^n \) with integral \( \alpha = \int_{\mathbb{R}^n} h(x, I) \), given \( \epsilon > 0 \) there exists a gauge function \( \delta \) in \( \mathbb{R}^n \) such that, for each \( \delta \)-fine division \( \mathcal{D}_\delta \) of \( \mathbb{R}^n \), the corresponding Riemann sums satisfies

\[
\left| \alpha - \sum_{(x, I) \in \mathcal{D}_\delta} h(x, I) \right| < \epsilon,
\]

where \( \mathcal{D}_\delta \sum h(x, I) = \sum_{(x, I) \in \mathcal{D}_\delta} h(x, I) \).

**Example 2.5.** The function \( \cos(x^2) \) is not Lebesgue integrable on \((0, \infty)\). The function \( \cos(x^2)|I| \), defined on \((0, \infty) \times I([0, \infty))\), is Henstock integrable.

First, we will show that \( \cos(x^2) \) is not Lebesgue integrable, which means that it is not absolutely integrable on \((0, \infty)\). The function \( \cos(x^2)|I| \) has zeros at

\[
a_n := \sqrt{\frac{(2n+1)\pi}{2}},
\]

where \( n = 0, 1, 2, \ldots \), and let us define \( b_n := \int_{a_n}^{a_{n+1}} \cos(x^2)dx \). Note that \( |b_n| \to 0 \) since \( a_{n+1} - a_n \to 0 \) as \( n \to \infty \). Therefore, \( \sum_{n=0}^{\infty} b_n \) converges non-absolutely. The Lebesgue integrability of \( \cos(x^2) \) on \((0, \infty)\) is equivalent to the convergence of \( \sum |b_n| \). However,

\[
|b_n| > \frac{1}{2}(a_{n+1} - a_n) = \frac{1}{2} \sqrt{\frac{\pi}{2}} \left( \sqrt{2n+3} - \sqrt{2n+1} \right).
\]

Since \( \sum (\sqrt{2n+3} - \sqrt{2n+1}) \) diverges, it is clear that \( \cos(x^2) \) is not Lebesgue integrable on \((0, \infty)\). Now, we want to show that there exists a number \( \alpha \) such that

\[
\int_0^\infty \cos(x^2)|I| = \alpha \quad \text{where} \quad \alpha = \int_0^{a_0} \cos(x^2)dx + \sum_{n=0}^{\infty} b_n.
\]

We have that

\[
\int_0^{a_0} \cos(x^2)|I| = \int_0^{a_0} \cos(x^2)dx,
\]

first being a Henstock integral and the second a Riemann integral. We denote \( \alpha' = \sum_{n=0}^{\infty} b_n \), it is sufficient to prove that

\[
\int_0^\infty \cos(x^2)|I| = \alpha'.
\]
If $\epsilon > 0$ is given, $n'$ can be chosen sufficiently large so that $\sum_{n>n'} b_n < \epsilon/2$, and then $\delta(\infty)$ can be defined as $a_n' > 1/\delta(\infty)$. Designating by $n''$ the smallest integer $n$ for which $a_n > \delta(\infty)^{-1}$, the (infinite) group of terms $\sum_{n>n''} b_n$, though not zero, has value less than $\epsilon 2^{-n''}$.

The next step is to compare each of the remaining (finite) number of terms $b_n$ of the series $\sum b_n$, with the corresponding groups of terms of the Riemann sum, in order to use this comparison to define a positive function $\delta(x)$ when $(x \neq \infty)$. Accordingly, for $a_n \leq x \leq a_{n+1}$ and $n = 0, 1, 2, \ldots$, we choose $\delta_n(x)$ to satisfy:

$$|b_n - (D_{b_n}) \sum \cos(x^2)| |I| < \epsilon 2^{-(n+1)},$$

for all $\delta_n$ – fine division of $(a_n, a_{n+1}]$, and define $\delta(x)$ as follows:

- if $a_n < x < a_{n+1}$, then $\delta(x) < \min\{x - a_n, a_{n+1} - x\}$;
- if $x = a_n$, $\delta(x) \leq \min\{\delta_n-1(x), \delta_n(x)\}$;
- if $a_n < x < a_{n+1}$, $\delta(x) \leq \delta_n(x)$.

This ensures conformance, meaning that if $x \neq a_n$ for any $n$, and if $(x, I)$ is $\delta$ – fine, then $I$ is a subset of some $(a_n, a_{n+1}]$, and cannot intersect with two of them. Therefore, any $\delta$ – fine division $D = (x, I)$ must include division points $x = a_0, a_1, \ldots, a_m$. Any $\delta$ – fine division then gives a Riemann sum of the form

$$(D_{\delta}) \sum \cos(x^2)|I| = \sum_{n=0}^{n''} (D_{\delta}) \sum \cos(x^2)|I| = \sum_{n=0}^{n''} c_n.$$

Since

$$\left| \sum_{n=0}^{\infty} b_n - (D_{\delta}) \sum \cos(x^2)|I| \right| \leq \left| \sum_{n=0}^{n''-1} b_n - \sum_{n=0}^{n''-1} c_n \right| + \left| \sum_{n=n''}^{\infty} b_n \right|,$$

thus,

$$\left| \sum_{n=0}^{\infty} b_n - (D_{\delta}) \sum \cos(x^2)|I| \right| < \sum_{n=0}^{n''-1} \frac{\epsilon}{2^{n+1}} + \frac{\epsilon}{2} < \epsilon.$$

Therefore the function $\cos(x^2)|I|$ is Henstock integrable on $(0, \infty)$.

3. Infinite-dimensional integration case.

Let $T$ be an interval in $(0, \infty)$. $\mathbb{R}^T$ represents the set $(x_t)_{t \in T}$, the set of real-valued functions defined on $T$. We start by considering cells in $\mathbb{R}^T$. $\mathcal{N} = \mathcal{N}(T)$ denotes the class of finite subsets $N$ of $T$.

**Definition 3.1.** Let $N$ be a finite set in $\mathcal{N}$, such that $t_1 < t_2 < \ldots < t_n$. A cell in $\mathbb{R}^T$ is

$$I[N] = I(N) \times \mathbb{R}^{T \setminus N}.$$

It is helpful to emphasize the “restricted” dimensions $N$ of the cylindrical interval $I \subset \mathbb{R}^T$, written as $I[N]$, while with round brackets, is a finite-dimensional interval. The collection of all cells in $\mathbb{R}^T$ is denoted by $I(\mathbb{R}^T) = \{I[N] : N \in \mathcal{N}\}$. Note that $\mathbb{R}^{T \setminus N}$ denotes the set of real-valued functions defined on $T \setminus N$.

**Definition 3.2.** A partition of $\mathbb{R}^T$ is a finite collection $\mathcal{P}$ of disjoint cells $I[N]$ whose union is $\mathbb{R}^T$.

**Definition 3.3.** It is said that $(x, N, I[N])$ is associated in $\mathbb{R}^T$ to the corresponding finite-dimensional pair $(x(N), I(N))$, then it is associated in the finite-dimensional space $\mathbb{R}^N$. 

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Thus, an integrand in $\mathbb{R}^T$ may be expressed as $h(x, N, I[N])$, where the triple $(x, N, I[N])$ is associated if the pair $(x(N), I(N))$ is associated in $\mathbb{R}^n$. Note that $x(N) = (x(t_1), ..., x(t_n))$, $x_i = x(t_i)$ and $I_j = I_{t_j}$ for each $i = 1, ..., n$.

In addition to a condition $\delta$, on the lengths of the restricted edges $I_j$, a further condition is imposed: the dimension sets $N$ (or sets of restricted dimensions) of partitioning cells $I[N]$ should, for each associated $x$, include some minimal set of dimensions $L(x)$. That is, we require $L(x) \subseteq N$, where $L(x)$ can be made successively larger, just as $\delta(x)$ is made successively smaller in forming Riemann sums. Due to this, a gauge in $\mathbb{R}^T$ is considered as a pair of mappings $(\delta, L)$.

**Definition 3.4.** A gauge $\gamma$ in $\mathbb{R}^T$ is a pair of mappings $(\delta, L)$ such that

- $L : \mathbb{R}^T \to N$, it means, $x \to L(x) \in N$
- $\delta : \mathbb{R}^T \to (0, \infty)$, it means, $(x, N) \to \delta(x, N)$.

**Definition 3.5.** With $N = \{t_1, ..., t_n\} \in N(T)$, an associated triple $(x, N, I[N])$ is $\gamma$ – fine if, $L(x) \subset N$ and $(x_i, I_j)$ is $\delta$ – fine for $t_j \in N$, $1 \leq j \leq N$.

**Definition 3.6.** A division of $\mathbb{R}^T$ is a finite collection $\mathcal{D}$ of point-cell pairs $(x, I[N])$ such that the corresponding $(x, N, I[N])$ are associated, and the cells $I[N]$ form a partition $\mathcal{P}$ of $\mathbb{R}^T$.

**Definition 3.7.** Let be $\gamma = (L, \delta)$ a gauge function. A division $\mathcal{D}$ is a $\gamma$ – fine if each $(x, I[N]) \in \mathcal{D}$ is $\gamma$ – fine. In that case, we can denote the $\gamma$ – fine division $\mathcal{D}$ by $\mathcal{D}_\gamma$.

Suppose $h = (x, N, I)$ is a real- or complex-valued function of associated elements $(x, N, I[N])$ in the infinite-dimensional domain $\mathbb{R}^T$ and $\mathcal{D} = \{(x, I[N])\}$ is a division of $\mathbb{R}^T$, then the corresponding Riemann sum for $h$ is

$$\sum_{(x, I[N]) \in \mathcal{D}} h(x, N, I[N]) = \sum_{(x, I[N]) \in \mathcal{D}} \{h(x, N, I[N]) : (x, N, I[N]) \in \mathcal{D}\}.$$ 

**Definition 3.8.** A function $h$ of associated triples $(x, N, I[N])$ is integrable on $\mathbb{R}^T$, with integral $\alpha = \int_{\mathbb{R}T} h(x, N, I[N])$, if given $\epsilon > 0$, there exists a gauge $\gamma$ so that, for each $\gamma$ – fine division $\mathcal{D}_\gamma$ of $\mathbb{R}^T$, the corresponding Riemann sum satisfies

$$|\alpha - (\mathcal{D}_\gamma) \sum h| < \epsilon.$$ 

We will examine some of the integrands encountered in $\mathbb{R}^T$.

**Example 3.9.** Consider $f(x) = \prod_{j=1}^n 2\pi(t_j - t_{j-1})^{1/2}e^{-\frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}}$, with $x_j = x(t_j)$ for $t_j \in N \in N(T)$. In this case $f$ depends not just on the variable coordinates $x_j$ for variable $x$, but also on the variable $N = \{t_1, ..., t_n\}$, and this is something new which does not arise whenever $T$ is a finite set. Since $f$ depends explicitly on the variable elements $t_j \in T$ as well as on the variables $x_j = x(t_j)$, it is appropriate to write $f$ as $f(x, N)$. Our theory of integration will enable us to consider the integrability of $h(x, I, N) = f(x, N)\mu(I)$, with $\mu(I) = |I[N]|$ and

$$\mu(I[N]) = |I[N]| = |I[N]| = \begin{cases} \prod_{j=1}^n (v_j - u_j) & \text{if } I_j = (u_j - v_j) \\ 0 & \text{Otherwise} \end{cases}.$$ 

In the next section we will explain more about this type of integrand. The class of integrands that we are interested in consists of real- or complex-valued functions $h$ defined on associated triples $(x, N, I[N])$:

$$\mathbb{R}^T, N, I[N]) \overset{h}{\to} \mathbb{R} \text{ or } \mathbb{C}$$

$$\begin{align*}
(x, N, I[N]) & \to h(x, N, I[N]) \\
(x, N, I[N]) & \in (\mathbb{R}^T, N, I[N]) \\
x & \in I[N]
\end{align*}$$
If $T$ is a finite set, then integrands $h(x, N, I[N])$ reduce to functions of the form $h(x, I)$ where $x$ and $I$ are familiar finite-dimensional objects.

4. Fresnel Integral.

An integrand in $\mathbb{R}^T$ might then take the form $h(x, N, I[N]) = f(x)|I[N]|$ for some point function $f$, where $|I[N]|$ is given by expression (3). Because $\mathbb{R}^T$ is unbounded in each dimension it is easy to see that constant functions $f(x)$ are not integrable on $\mathbb{R}^T$ with respect to $|I[N]|$ unless, for instance, $f(x)$ is equal to zero for all $x$. That is why this approach will be based on an application of the Henstock integration technique to the Fresnel integrands. First, we will introduce one-dimensional Henstock Fresnel integral in order to present the infinite-dimensional case.

In [17], it was shown how the Feynman path integral leads to the Wiener kernel, with the purely imaginary diffusion coefficient and how this makes the corresponding measure not countably additive. The main obstacle consists in the exponentials with pure imaginary exponents. The improper Riemann integrals of such expressions are called the Fresnel integrals:

$$\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2} dx.$$  

(4)

It can be proved that the integral (4) exists as a Henstock integral, even more, is equal to $\sqrt{\frac{2\pi}{i}}$, see [17]. This result can be generalized to any complex number $c = a + ib$, where $a \leq 0$, $b \geq 0$ and $c \neq 0$, see [9].

Let us define a complex-valued function $\varphi(x)$ for $x = (x_1, 1_2, ..., x_n) \in \mathbb{R}^n$ as:

$$\varphi(x) = \varphi_{n, \frac{1}{2}}(x) = e^{\frac{i}{4}(x_1^2 + x_2^2 + ... + x_n^2)}.$$

Now we consider a “volume” function $\mu$ defined on a set $I(\mathbb{R}^n)$ of finite-dimensional cells $I(N) = I_1 \times ... \times I_n$ as the expression (3).

Consider the function $h(x, I(N)) = \varphi(x)\mu(I(N))$; we will refer to its integral as the Henstock Fresnel integral, and

$$\int_{\mathbb{R}^n} \varphi(x)\mu(I(N)) = \int_{\mathbb{R}^n} \prod_{j=1}^{n} e^{\frac{i}{2}x_j^2} |I_j| = \left(\frac{2\pi}{-i}\right)^n,$$

see [17].

We would like to redefine or normalize the Fresnel integrand so that the new integral can be considered as a probability distribution function. If a cell $I \in I(\mathbb{R}^n)$ has an associated point $x \in \mathbb{R}^n \setminus \mathbb{R}^n$, that is, $x = (x_1, ..., x_n)$ has some component $x_j = \pm \infty$, then a value zero was assigned to “volume” $\mu(I(N)) = |I|$, by convention. But since we need to construct a probability distribution function, nonzero values are assigned in all cases. Thus, we define the following function,

$$g_n(x)|I| = \begin{cases} 
\left(\frac{2\pi}{-i}\right)^n e^{\frac{i}{2}x_1^2 + ... + x_n^2} |I_1|...|I_2| & \text{if } x \in \mathbb{R}^n \\
\left(\frac{2\pi}{-i}\right)^n \prod_{j=1}^{n} \int_{I_j} e^{\frac{i}{2}x_j^2} dx_j & \text{if } x \in \mathbb{R}^n \setminus \mathbb{R}^n.
\end{cases}$$

(5)
For example, if $n = 1$ and $J = (u, \infty)$, $u > 0$, then

$$g_1(x)|J| = \sqrt{\frac{-i}{2\pi}} \left[ \frac{1}{2} \int_{-\infty}^{\infty} e^{\frac{1}{2}iy^2} dy - \int_{0}^{u} e^{\frac{1}{2}iy^2} dy \right]$$

$$= \frac{1}{2} - \sqrt{\frac{-i}{2\pi} g_1(x)|(0, u)|}.$$

The function $g_n(x)|I|$ is integrable in $\mathbb{R}^n$ in Henstock sense and $\int_{\mathbb{R}^n} g_n(x)|I| = 1$. For more details see [9]. If we consider $g_n(x)|I|$ as an $n$-dimensional Henstock-Fresnel density function, then we define its associated $n$-dimensional probability distribution function on cells,

$$G_n(I) = \left\{ \begin{array}{ll}
\left( \sqrt{\frac{1}{2\pi}} \right)^n \int_{I} e^{\frac{1}{2}x_1^2 + \ldots + x_n^2} |I| & \text{if } x \in \mathbb{R}^n \\
\left( \sqrt{\frac{1}{2\pi}} \right)^n \prod_{j=1}^{n} \int_{I_j} e^{\frac{1}{2}x_j^2} dx_j & \text{if } x \in \mathbb{R}^n \setminus \mathbb{R}^n \end{array} \right.$$

A figure in $\mathbb{R}^n$ is the union of a finite number of cells, denoted by $E$. The collection of all figures in $\mathbb{R}^n$ is denoted as $E(\mathbb{R}^n)$.

In fact, $G_n$ is defined over the collection of figures $E(\mathbb{R}^n)$, and this is finitely additive on disjoint figures and $\int_{\mathbb{R}^n} G_n(I) = 1$; it means that is a probability distribution function, see [9, Theorem 147].

In order to show that $\int_{\mathbb{R}^n} G_n(I) = 1$ is easy to prove that if $D$ is a division of $\mathbb{R}^n$, then

$$(D) \sum G_n(I) = \left( \frac{-i}{2\pi} \right)^n \left( \frac{2\pi}{-i} \right)^n = 1.$$

Additivity follows from the definition.

4.1. Fresnel Integral in $\mathbb{R}^T$.

The second step is to define a probability distribution function over the space of all paths. In addition, it should be noted that there are increments in the integrand, see expression (1). The increments in the variables are just translations that don’t change significantly any results stated before. Thus from now on, we will use the following definitions for the density and probability distribution functions, $g^T(x(N)|I[N]|)$ and $G^T([N])$, respectively

$$g^T(x(N)|I[N]|) = \left\{ \begin{array}{ll}
\prod_{j=1}^{n} \left( \sqrt{\frac{-i}{2\pi t_j - t_{j-1}}} \right) e^{\frac{1}{2}(x_{j} - x_{j-1})^2} |I[N]| & \text{if } x \in \mathbb{R}^T \\
\prod_{j=1}^{n} \left( \sqrt{\frac{-i}{2\pi t_j - t_{j-1}}} \right) \int_{I_j} e^{\frac{1}{2}(x_{j} - x_{j-1})^2} dx_j & \text{if } x \in \mathbb{R}^T \setminus \mathbb{R}^T \end{array} \right.$$

$$G^T([N]) = \prod_{j=1}^{n} \left( \sqrt{\frac{-i}{2\pi t_j - t_{j-1}}} \right) \int_{I_j} e^{\frac{1}{2}(x_{j} - x_{j-1})^2} dx_j,$$  \hspace{1cm} (6)

where $T \subset \mathbb{R}^+$ and $N = \{t_1, t_2, \ldots, t_n\} \in \mathcal{N}(T)$ with $0 = t_0 < t_1 < t_2 < \ldots, t_n$. Here the differences $x_j - x_{j-1} = x(t_j) - x(t_{j-1})$ are the increments or transitions of the incremental or transitional Fresnel integrand.

By [9, Theorem 168] the function $G^T$ in the expression (6) is integrable on $\mathbb{R}^T$ and defines a distribution function. Note that in the classical sense, this function would not be considered as a probability distribution function, because it can take negative or complex values. However, in
the formulation of Quantum Mechanics, complex-valued functions play the role of probability distribution functions. Thus, in this context, we can refer \( G^T(I) \) as a probability distribution function; it means that,

\[
\int_{\mathbb{R}^T} G^T(I[N]) = 1,
\]

and \( G^T \) is finitely additive on disjoint sets \( E \in E(\mathbb{R}^T) \). In order to show that the expression (7) holds, it is enough to prove that:

\[
(\mathcal{D}) \sum G^T = \prod_{j=1}^p \left( \int_{\mathbb{R}} e^{\frac{i}{2}x_j^2} dx \right) = 1.
\]

Let \( \mathcal{D} \) be a division of \( \mathbb{R}^T \), choose any \( (x, I[N]) \in \mathcal{D} \), \( N = \{t_1, \ldots, t_n\} \), so \( I = I[N] \) is restricted in \( n \) dimensions. \( \mathcal{D} \) can be composed of cells \( I[N] \) with variable \( N \), the partition \( \mathcal{P} \) consists of cells \( J[M] \), each having the same \( M \). In other words, \( \mathcal{D} \) gives a partition \( \mathcal{P} \) of \( \mathbb{R}^T \) such that each \( I[N] \) of \( \mathcal{D} \) is the union of a finite number of cells \( J[M] \in \mathcal{P} \), and each cell \( J \) of \( \mathcal{P} \) is restricted in the same dimensions \( M = \{\tau_1, \ldots, \tau_p\} \), see [9, Partially Regular Partitions]. By additivity of the finite-dimensional Riemann and extended Riemann integrals involved,

\[
(\mathcal{D}) \sum G^T(I[N]) = (\mathcal{P}) \sum G^T(J[M]).
\]

The latter Riemann sum is easy to compute, because \( M \) is the same for each \( J \in \mathcal{P} \) and we get the expression (8).

On the other hand, let us note that

\[
\int_{\mathbb{R}^T} G^T(I[N]) = G^T(\mathbb{R}^T) = 1.
\]

As the indefinite integral of an integrable function is an additive function on figures, see [9, Theorem 16], we get the result.

**Example 4.1.** Let us consider the function \( g_0(x, N) \) as,

\[
g_0(x, N) = e \left( \frac{i}{2} \sum_{j=1}^n \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} \right) \prod_{j=1}^n (2\pi i (t_j - t_{j-1}))^{-1/2},
\]

when \( x \in \mathbb{R}^T \).

Note that \( g_0(x, N) = g^T(x(N)) \) if \( x \in \mathbb{R}^T \). We redefine \( g^T \) as \( g_0 \) because we emphasize the free particle case, it means \( V = 0 \). In [8, Proposition 68] it is shown that \( g_0 \) is Henstock Fresnel integrable. The sketch of the proof is as follows. Let \( \mathcal{D} \) a division of \( \mathbb{R}^T \) and \( M \) the dimension set of \( I, (x, I) \in \mathcal{D} \). Let \( M = \{s_1, \ldots, s_{m-1}\}, s_0 = \tau' \) and \( s_m = \tau \). For \( y \in \mathbb{R}^m \), let

\[
g(y) = e^{\frac{i}{2} \sum_{j=1}^m \frac{(y_j - y_{j-1})^2}{s_j - s_{j-1}}} \prod_{j=1}^m \left( 2\pi i (s_j - s_{j-1}) \right)^{-1/2}, \quad y_0 = \xi', \ y_m = \xi.
\]

Note that

\[
\frac{(y_{j+1} - y_j)^2}{s_{j+1} - s_j} + \frac{(y_j - y_{j-1})^2}{s_j - s_{j-1}} = \frac{y_{j+1}^2}{s_{j+1} - s_j} + \frac{y_j^2}{s_j - s_{j-1}} - 2y_j \frac{(y_{j+1} - s_{j+1})(y_j - s_j - y_{j-1})}{s_{j+1} - s_j - s_{j-1}} + \frac{y_{j+1}(s_j - s_{j-1})}{s_{j+1} - s_j - s_{j-1}} \quad (9)
\]

\[
\frac{y_{j+1}^2 + y_j^2}{s_{j+1} - s_j} + \frac{y_j^2 - 2y_j}{s_j - s_{j-1}} - \frac{y_{j+1}(s_j - s_{j-1})}{s_{j+1} - s_j - s_{j-1}} \quad (10)
\]
Since \( \int_{0}^{\infty} e^{iy^2} dy = \frac{1}{2} \sqrt{\frac{-2\pi}{i}} \), completing squares and by a change of variable we have that
\[
\int_{-\infty}^{\infty} (9) dy_j = \int_{-\infty}^{\infty} (10) dy_j = \left( \frac{-2\pi}{i} \frac{(s_{j+1} - s_j)(s_j - s_{j-1})}{s_{j+1} - s_{j-1}} \right)^{1/2} e^{i \frac{(y_{j+1} - y_{j-1})^2}{2 s_{j+1} - s_{j-1}}}
\]
Thus,
\[
\int_{\mathbb{R}^{m-1}} g(y) dy = \left( \frac{-2\pi}{i} (\tau - \tau') \right)^{-1/2} e^{i \frac{(\xi - \xi')^2}{2(\tau - \tau')}}.
\]
It is possible to prove that for any figure \( E \) in \( \mathbb{R}^T \), \( \int_E g(y) dy \) exists and \( \int_{\mathbb{R}^{m-1} \setminus E(M)} g(y) dy \) < \( \varepsilon \), where \( E(M) \) corresponds to those \((x, I) \in D\) for which \( y(M) \) has not infinite component. Therefore, we have that
\[
\int_{\mathbb{R}^T} g(x, N, I[N]) = \left( \frac{-2\pi}{i} (\tau - \tau') \right)^{-1/2} e^{i \frac{(\xi - \xi')^2}{2(\tau - \tau')}}.
\]
Observe that \( T = (\tau', \tau) \). To prove that \( g_0 \) is not absolutely integrable it is enough to observe that the function \( e^{iu^2} \) is not Lebesgue integrable, according to the proof of Proposition 68 from [8]. Let us write
\[
\quad e^{iu^2} = \cos(u^2) + i \sin(u^2).
\]
As
\[
\int_{-\infty}^{\infty} e^{iy^2} dy = \sqrt{i\pi},
\]
we have
\[
\int_{(0, \infty)} \cos(u^2) du = \int_{(0, \infty)} \sin(u^2) du = \frac{1}{2} \sqrt{\pi}.
\]
These integrals exist as extended Riemann integrals, but not as Lebesgue integrals. The graphs of \( \cos(u^2) \), \( \sin(u^2) \) oscillate periodically with constant amplitude 2 but with period decreasing to zero as \( u \to \infty \).

As the complex number \( c = \frac{i}{2} \) in the function \( g_0 \), we get the one-dimensional free particle propagator function of Quantum Mechanics. If \( c = -\frac{i}{2} \), then
\[
\int_{\mathbb{R}^T} g_0(x, N, I[N]) = \left( \frac{2\pi}{i} (\tau - \tau') \right)^{-1/2} e^{-i \frac{(\xi - \xi')^2}{2(\tau - \tau')}}
\]
and gives us the probability density function or diffusion function of Brownian motion.

Now we turn to the general case when \( V \) is not the function zero. Let \( V \) be a real-valued function of a single real variable.

\[
\phi_V(\xi, \tau) = \int_{\mathbb{R}^T} e^{i \frac{n}{2} \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} V(x_{j-1})(t_j - t_{j-1})} \prod_{j=1}^{n} \left( \frac{2\pi}{i} \frac{(t_j - t_{j-1})}{(t_j - t_{j-1})} \right)^{-1/2} |I[N]|
\]
\[
= \int_{\mathbb{R}^T} g_V(x, N)|I[N]|.
\]

Then \( \phi_V \) exists if \( V \) is a continuous functions. Now let
\[
G_V(J[N]) := \int_{J[N]} g_V(x, N)|I[N]|.
\]
Note that $G_V(J)$ corresponds to Feynman’s probability amplitude. 

Thus, a solution to Schrödinger equation is given by Henstock integral. Let $T = (τ’, τ) \subset R^+$ and $V$ is continuous at $y = ξ$, $τ = t$. If 

$$\psi_1(ξ, τ) = \int_{R^T} e^{(-\frac{i}{2}\sum_{j=1}^{n} V(x(t_{j-1}), t_{j-1})(t_{j}-t_{j-1}))} \prod_{j=1}^{n} \frac{1}{(2\pi i(t_{j} - t_{j-1}))^{1/2}} e^{\frac{(x_{j} - x_{j-1})^2}{t_{j} - t_{j-1}}} |I[N]|$$

exists for each $(ξ'', τ'')$ in a neighborhood of $(ξ, τ)$, then $ψ_1(ξ, τ)$ satisfies the Schrödinger equation, see [9, Theorem 221], where this integral is in Henstock sense.

5. Conclusions

In this work it is observed that there exists an equivalence between the Schrödinger equation solutions according to the perturbation theory (Trotters Theorem) and Henstock Theory. This follows from the uniqueness of the solution to the Schrödinger equation. On the other hand, the Henstock integral solves the problem that the absolute value of the Feynman integrand is not integrable because this approximation does not require the measure concept, for which the absolute value of the integrand does not need to be integrable.

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