Cosmologies with a general non-canonical scalar field

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Abstract

The scalar field models with the Lagrangian $L = F(X) - V(\phi)$, which we call general non-canonical scalar field models, are investigated. We find that a special square potential (with a negative minimum) is needed to drive the linear field solution ($\phi = \phi_0 t$) in our model, while in the K-essence model ($L = -V(\phi) F(X)$) the potential should be taken as an inverse square one. Hence the cosmological evolutions for these models are totally different. The linear field solutions are found to be highly degenerate, and their cosmological evolutions are equivalent to the model where the sound speed diverges. We also study the stability of the linear field solution and find the condition for stable solutions to exist. The cosmological solution in the presence of matter and radiation is further studied by numerically solving the potential and the cosmological evolution, and the results are shown to be quite different from the case of no matter or radiation. Then we analyze the case with a constant barotropic index $\gamma$ and show that, unlike in the K-essence model, the detailed form of $F(X)$ depends on the potential $V(\phi)$, and that this constant $\gamma$ solution is stable for $\gamma_0 \leq 1$. When the potential is taken to be a constant, we find the first integral and obtain the corresponding $\gamma$, which is similar to that in the K-essence model.

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1. Introduction

The origin of dark energy is perhaps among the biggest problems faced by contemporary theoretical physicists and cosmologists (see [1] for a recent review). The construction of theoretical models and the confrontation with astrophysical observations have been continuous since 1998. Generally speaking, one can modify either the right-hand side or the left-hand side of the Einstein equation to get a reasonable explanation for the cosmic accelerating expansion,
and many candidates have actually been proposed, such as the cosmological constant [2], quintessence [3], K-essence [4], phantom [5], modifying gravity [6] and so on. Among these the scalar field models are undoubtedly a very important and popular subclass.

Over the last few years the scalar field models with non-canonical kinetic terms have been attracting much attention. These models generally have rather good motivations from purely phenomenological considerations and their theoretical structures are also common in effective field theories. For example, a non-canonical kinetic term appears in supergravity theories [7] and was also adopted in [8] to relate the present cosmic acceleration to the onset of matter domination at the scale factor \( a \simeq 10^{-4} \). In higher dimensional models the identification of \( \ln \phi \) with the volume of the internal space or some appropriate dilaton-type field also frequently leads to some non-canonical kinetic terms [9].

In a general manner the Lagrangian of the non-canonical scalar field can be parameterized as [10]

\[
L = f(\phi)F(X) - V(\phi)
\]

where \( X = \frac{1}{2} \nabla_t \phi \nabla_t \phi = \frac{1}{2} \dot{\phi}^2 \) for a spatially homogeneous scalar field. Equation (1) includes all the popular single scalar field models: it represents a conventional scalar field (phantom quintessence) for arbitrary \( f(\phi) \) (as one can always redefine the field to make \( f(\phi) \) a constant) when \( F(X) = X(\frac{F(X)}{X} = -X) \), and it reduces to the K-essence model if \( V(\phi) = 0 \). The idea of K-essence was firstly introduced in the context of inflation [11] and later was considered as an alternative to the standard quintessence model to account for the dark energy [4, 12].

Chimento found the first integral of the K-essence field equation for arbitrary functions \( F(X) \) when the potential was taken as an inverse square form or a constant [13]. Then in [14], it was found that all quintessence models can be viewed as K-essence models generated by appropriate kinetic linear \( F(X) \) functions, and some correspondences between K-essence potentials and quintessence models are also given. In this paper, we will focus on another class of models with the Lagrangian \( L = F(X) - V(\phi) \), which we believe to be as important as K-essence while its general character and roles in cosmology are not very clear. Here we want to point out that, from the original viewpoint in the literature [11, 12], our Lagrangian equation (2) also belongs to the K-essence class, which is characterized by a Lagrangian \( L = L(X, \phi) \). However, as far as we know, most works on K-essence were just based on the Lagrangian form \( L = V(\phi)F(X) \). Therefore in this paper we call our model \( L = F(X) - V(\phi) \) the general non-canonical scalar field model while referring to the model \( L = V(\phi)F(X) \) as K-essence.

The paper is organized as follows: in section 2 we present the theoretical framework. In section 3 the linear field solution is analyzed and the potential driving this linear field solution is found. We also discuss the stability of this solution and furthermore consider the case with presence of matter and radiation. The model with divergent sound speed \( (c_s^2 \rightarrow \infty) \) is then discussed in section 4. Some explicit general non-canonical scalar field solutions are presented in section 5 and finally section 6 displays our conclusions.

2. Basic framework

Let us restrict ourselves for the time being to the cosmological settings corresponding to a flat universe described by the FRW metric, and consider a spatially homogeneous real scalar field \( \phi \) with a non-canonical kinetic energy term. The Lagrangian density is given as

\[
L = F(X) - V(\phi)
\]

where \( V(\phi) \) is the potential and \( F(X) \) an arbitrary function of \( X \). Obviously this equation is a special case of equation (1) with \( f(\phi) = \) constant. It includes the quintessence \( (F(X) = X) \)
and phantom \((F(X) = -X)\) models. In fact this Lagrangian has already appeared in other contexts. For example, Babichev considered the global topological defects in symmetry breaking models with this non-canonical kinetic term \([15]\); the production of large-scale gravitational waves in inflationary models with a nontrivial kinetic term was also discussed in \([16]\), as well as the effects of scalar fields with non-canonical kinetic terms in the neighborhood of a black hole \([17]\).

The pressure, energy density and sound speed of the scalar field could be easily obtained as follows:

\[
p = L = F(X) - V(\phi) \tag{3}
\]

\[
\rho = 2L,_{X}X - L = 2XF,_{X} - F(X) + V(\phi) = 3H^{2} \tag{4}
\]

\[
c_{s}^{2} = \frac{p,_{X}}{\rho,_{X}} = \left[1 + 2X \frac{F,_{XX}}{F,_{X}} \right]^{-1} \tag{5}
\]

where we have adopted the convention \(8\pi G = 1\) and \(F,_{X}\) is the derivative with respect to \(X\). From equations (3), (4) we can get the relation \(\rho + p = 2XF,_{X}\) and this makes the equation of state (EOS) \(w_{\phi}\) larger than \(-1\) if \(F,_{X} > 0\) and \(w_{\phi} < -1\) if \(F,_{X} < 0\). Note that Vikman has argued that, for an arbitrary scalar field Lagrangian \(p(\phi, \nabla_{\mu} \phi)\), it is physically implausible for \(w_{\phi}\) to cross the phantom line divide \((w_{\phi} = -1)\) because they are either realized by a discrete set of trajectories in the phase space or are unstable with respect to the cosmological perturbations \([18]\); however, it is pointed out in \([19]\) that this conclusion holds only for models without considering the higher derivative terms. The sound speed \(c_{s}\), as given in equation (5), is another important parameter of the scalar field model. It describes the propagation of the perturbations of the scalar field which could affect the CMB power spectrum, and thus provides crucial information complementary to the equation of state \(w\). From equation (5) it could be seen that \(c_{s}^{2} = 1\) when \(F,_{XX} = 0\). If \(c_{s}^{2}\) is constant, then equation (5) could be integrated to give

\[
F(X) = \frac{2c_{s}^{2}}{1 + c_{s}^{2}} c_{1}X \frac{1 + c_{1}}{1 + c_{2}} + c_{2} \tag{6}
\]

where \(c_{1}\) and \(c_{2}\) are integration constants. Since the potential \(V(\phi)\) does not affect the sound speed, our model has the same \(c_{s}^{2}\), equation (5), as the K-essence model, and therefore the same equation (6) when \(c_{s}^{2}\) is constant.

The equations of motion (EOM) for general non-canonical scalar fields are given by

\[
\dot{\phi} + 3c_{s}^{2}H\phi + \frac{P,_{\phi}}{\rho,_{X}} = 0, \tag{7}
\]

\[
(F,_{X} + 2XF,_{XX})\phi + 3HF,_{X}\phi + V,_{\phi} = 0, \tag{8}
\]

\[
\left( \frac{\gamma}{\phi} \right) + 3H(1 - \gamma) \left( \frac{\gamma}{\phi} \right) + \frac{V,_{\phi}}{3H^{2}} = 0, \tag{9}
\]

where an overdot denotes the time derivative, and \(\gamma = \frac{\rho + p}{\rho}\) is the barotropic index. Equations (7)-(9) are different expressions of the EOM and are equivalent to each other.

### 3. The linear field model

In this section we will investigate a special case which possesses a linear field solution:

\[
\phi = \phi_{0} \cdot t. \tag{10}
\]
We shall find that a square potential $V(\phi)$ with a negative minimum is needed to drive this evolution. We will also show that this solution leads to a completely different universe for the present model in contrast to the K-essence model.

For the linear field solution, $X = \frac{1}{2} \dot{\phi}^2 \equiv \frac{1}{2} \phi_0^2$ is a constant and $\ddot{\phi} = 0$, and we could get the following equation from equations (4), (8):

$$\frac{V_{\phi}}{3F_{,X} \phi_0^2} - V(\phi) + F(X) - \phi_0^2 F_{,X} = 0, \quad (11)$$

where $F$ and $F_{,X}$ are functions of $X$ only and therefore also constant. We simply set $F$ and $F_{,X}$ to their values at $X = X_0 = \frac{1}{2} \phi_0^2$, that is, $F_0 = F(X_0)$ and $F_{,0} = F_{,X}(X_0)$. Then from equations (11) and (4) we could obtain exact solutions for the potential $V(\phi)$ and scale factor $a$:

$$V(\phi) = \frac{3}{4} F_{,0} \phi_0^2 (\phi + c)^2 + F_0 - F_{,0} \phi_0^2, \quad (12)$$

$$a = a_0 \exp \left[ -\frac{F_{,0}}{4} (\phi_0 t + c)^2 \right], \quad (13)$$

where $c$ is an integration constant. Therefore the linear field solution leads to a square potential equation (12). It is worthwhile to notice that, as found in [20], the same linear field solution in the K-essence model leads to an inverse square potential and a power law expansion of the scale factor. Thus the same linear field solution, equation (10), leads to different cosmological evolutions and may have different cosmological implications in the two models. We should emphasize here that the potential is exactly derived from equations (4), (11) and its form is unique. Moreover, in the present model different forms of $F(X)$ fall into two equivalence cases, $F_{,0} > 0$ and $F_{,0} < 0$, which respectively correspond to the state parameters $w > -1$ and $w < -1$. The phantom case describes a universe transiting from a contracting phase to an expanding one and is clearly excluded by current observations, so in the following we shall restrict ourselves to $F_{,0} > 0$. From equation (4) the energy density is given by $\rho = F_{,0} \phi_0^2 - F_0 + V(\phi)$. To ensure that $\rho$ has a positive kinetic energy term we thus require $F_{,0} \phi_0^2 - F_0 > 0$, which immediately indicates that the square potential has a negative minimum value $F_0 - F_{,0} \phi_0^2$ (see equation (12)). In fact this result is well intelligible in an expanding universe. On the one hand, if the square potential has a non-negative minimum it is well known that the scalar field $\phi$ will roll down the potential and finally cease at the minimum position $\phi = -c$, after which the linear field solution $\phi = \phi_0 t$ will no longer be valid. On the other hand it is noticed that for a potential with a negative minimum the scalar field can oddly roll up the potential from its minimum (see figure 1) and the universe then enters a contracting phase from an expanding phase [21] and the scalar field can evolve to $\infty$.

It is very interesting that the universe in our model can avoid a beginning singularity. If we believe the classical cosmology to be valid when energy is below the Planck scale, then the scale factor at the beginning is very small ($a = a_0 e^{-\rho pl / 3 F_{,0} \phi_0^2}$), where $\rho pl$ is the energy density at Planck time) but does not equal zero. However, the universe cannot escape from a collapse in the future. Note that this evolutionary behavior is completely different from the same linear field solution case ($\phi = \phi_0 t$) in the K-essence model where the scale factor behaves as $a \propto t^n$ [13] and the universe was born from a singularity and will expand for ever.

We are concerned with the question whether our model can describe a suitable universe with a phase of accelerating expansion. The answer is positive because we have $\ddot{a} \propto -(\rho + 3p) = -\frac{3}{2} F_{,0} \phi_0^2 [2 - F_{,0}(\phi + c)^2] + \rho + 3p < 0$ for $\phi < \phi_1$ or $\phi > \phi_2$, where $\phi_1 = -c - \sqrt{2/F_{,0}}$, $\phi_2 = -c + \sqrt{2/F_{,0}}$. In figure 1 the potential and cosmic evolution for...
Figure 1. The potential and evolution of the universe.

the model are shown, from which we can see that the field rolls down the potential from an initial value and the universe undergoes an accelerating expansion. The universe enters a decelerating expansion phase when the field evolves to \( \phi_1 \) and then stops and turns around at \( \phi = -c \). After that the field rolls up the potential from its minimum; when \( -c < \phi < \phi_2 \) the universe undergoes an accelerating contraction and after \( \phi > \phi_2 \) the contraction becomes decelerating. Finally the universe collapses to a future singularity.

Since the potential (equation (12)) is obtained just from the linear field solution without any additional assumptions, another nontrivial question is whether an arbitrary square potential with a negative constant can always give the linear field solution. To answer this question, we consider a general square potential,

\[
V(\phi) = A(\phi + c)^2 - B,
\]

where \( A, B \) are arbitrary positive constants. In order to have a linear field solution, according to equation (12), \( A \) and \( B \) should satisfy

\[
A = \frac{3}{4} F_0^2 \phi_0^2 \quad \text{and} \quad B = F_0 - F_0 \phi_0^2,
\]

respectively, from which we can get the following constraint equation:

\[
4A = 3F_0(F_0 - B).
\]

For arbitrary \( A \) and \( B \), we can always choose an appropriate form of \( F(X) \) to make equation (15) satisfied by the values of \( F_0 \) and \( F_0 \). However, if the form of \( F(X) \) is known, equation (15) shows that \( A \) and \( B \) are no longer arbitrary, and given \( A \) the value of \( B \) is determined by equation (15) such that the potential is not arbitrary.

Another interesting feature of the linear field solution can also be seen from equation (15). Since only the first two coefficients \( (F_0, F_0) \) of the series expansion of the function \( F(X) \) around \( X = X_0 \) \([F(X) = F_0 + F_0(X - X_0) + \cdots]\) appear, the cosmic evolutions with any \( F(X) \) will be equivalent as long as their first two expansion coefficients, \( F_0, F_0 \), are the same, disregarding the higher order terms. This means that the linear field solution model is highly degenerate.

Let us now investigate the stability of the linear field solution \( \phi = \phi_0 t \). To this effect we write the field solution as

\[
\phi = \phi_0 t + \delta
\]
where $\delta$ is a first-order small value compared with $\phi_0 t$. From equations (8) and (16) we can get the following differential equation for $\delta$ up to first order,

$$\ddot{\delta} - \frac{\phi_0}{2} \left[ 3 F_0 (\phi_0 t + c) + \frac{2}{\phi_0 t + c} \right] \dot{\delta} = 0,$$

(17)

whose solution is given as

$$\delta = c_3 e^{\frac{r}{F_0} (\phi_0 t + c)^2} + c_4$$

(18)

where $c_3, c_4$ are integration constants. When $F_0 < 0$, equation (18) shows that the field solution behaves as $\phi \to \phi_0 t + c_4$ and can be stable. Note that this result is just obtained in the absence of matter and radiation. How the situation is in the presence of matter and radiation remains to be studied. We will investigate this issue in our next paper.

It is also of our interest to discuss the cosmological solutions in the presence of matter and radiation. Below we shall use the same linear field solution equation (10) to present the effects of matter and radiation on the potential form and the cosmic evolution in the model.

When matter and radiation are added, equation (4) becomes

$$\rho = 2X F_X - F(X) + V(\phi) + \rho_m + \rho_r = 3H^2$$

(19)

where $\rho_m$ and $\rho_r$ denote respectively the energy densities of nonrelativistic (matter) and relativistic (radiation) species, which satisfy the conservation equations:

$$\dot{\rho}_m + 3H \rho_m = 0, \quad (20)$$

$$\dot{\rho}_r + 4H \rho_r = 0. \quad (21)$$

The solutions are $\rho_m = \rho_{m i} \left( \frac{a_i}{a_0} \right)^3$ and $\rho_r = \rho_{r i} \left( \frac{a_i}{a_0} \right)^4$, where $\rho_{m i}, \rho_{r i}$ are the initial energy densities at the time when the scale factor $a = a_i$. Substituting equation (10) into equation (8) we can obtain

$$H = -\frac{\dot{V}}{6F_0 X_0}$$

(22)

where $\dot{V} = dV(\phi(t))/dt$. The solution of equation (22) is given as

$$a = a_0 e^{-\frac{V(\phi(t))}{6F_0 X_0}}. \quad (23)$$

From equations (19), (22) we have

$$\frac{\dot{V}^2}{12F_0 X_0^2} - V - (2F_0 X_0 - F_0) = \rho_m + \rho_r. \quad (24)$$

Differentiating equation (24) and using equations (20)–(22) we obtain

$$\frac{\ddot{V}}{6F_0 X_0^2} - 1 = \frac{3\rho_m + 4\rho_r}{6F_0 X_0}. \quad (25)$$

Substituting equation (23) into equation (24) and equation (25) we then arrive at the differential equations for potential $V(\phi(t))$:

$$\frac{\dot{V}^2}{12F_0 X_0^2} - V - (2F_0 X_0 - F_0) = \rho_{m i} \left( \frac{a_i}{a_0} \right)^3 e^{2V/3F_0 X_0} + \rho_{r i} \left( \frac{a_i}{a_0} \right)^4 e^{2V/3F_0 X_0}$$

(26)

$$\frac{\ddot{V}}{6F_0 X_0^2} - 1 = \frac{3\rho_{m i} \left( \frac{a_i}{a_0} \right)^3 e^{2V/3F_0 X_0} + 4\rho_{r i} \left( \frac{a_i}{a_0} \right)^4 e^{2V/3F_0 X_0}}{6F_0 X_0}. \quad (27)$$
Cosmologies with a general non-canonical scalar field

Figure 2. The solid line is the square potential when there are no matter and radiation, just as in figure 1. The dashed line is the potential when matter and radiation exist, which is numerically obtained from equation (27). Note that it is not the square form.

Figure 3. The evolution of $\frac{da}{dt}$. The solid line is the case for the absence of matter and radiation. The dashed line is the evolution when matter and radiation exist. Figures 2 and 3 are both plotted with the same initial value.

Without matter or radiation (that is, with the right-hand sides of equation (26) and equation (27) vanishing), we can immediately obtain the square potential with a negative minimum (equation (12)). When matter and radiation are included, the potential is determined by a differential equation, (26) or (27). Since it is very difficult to get an exact solution for potential $V$ from these equations, we shall rely on numerical calculations to see the effects of matter and radiation on the form of the potential. In figure 2 we numerically plot the potential, as well as the evolution of $a$.

Figures 2 and 3 show that the potential as well as the cosmological evolution is quite different from the case of no matter and radiation. We should emphasize that the results are obtained on the assumption that the field still has linear solution equation (10). Figure 2 shows that the potential in the presence of matter and radiation (dashed line), numerically
obtained from equation (27), is not the square potential but steeper than the squared form (solid line). Note that the square potential (solid line) shown in figure 2 is actually the same one shown in figure 1. The reason why they seem different is that we take a different scaling of the \(x–y\) axes. The evolution of the derivative of the scale factor \(\dot{a}\) is shown in figure 3. The most different cosmological behavior is that, after a decelerating contraction, the universe will dramatically reenter an accelerating contraction phase and then collapse to a future singularity, which is due to the presence of matter and radiation.

4. The divergent model

In this section we investigate another interesting case where the sound speed is divergent \((c_s^2 \to \infty)\). From equation (5), we have

\[
1 + 2X \frac{F_{,XX}}{F_{,X}} = 0
\]

which could be integrated to give the following form of \(F(X)\):

\[
F(X) = c_5 X^{\frac{1}{2}} + c_6
\]

where \(c_5, c_6\) are integration constants. From equations (5), (29), the sound speed of this special form of \(F(X)\) diverges \((c_s^2 \to \infty)\). The same form of function \(F(X)\) is also obtained in the K-essence model with the Lagrangian \(L = -V(\phi) F(X)\) [13, 20]. This type of Lagrangian is thoroughly investigated in the model [22], which is named Cuscuton. Recently these authors further investigated the Cuscuton dark energy model and pointed out that Cuscuton could be considered as a minimal theory of evolving dark energy, or a minimal modification to a cosmological constant. Due to the lack of internal dynamics, Cuscuton only modifies (or dresses) the gravity of massive objects, and thus resembles a modified gravity theory [23].

From equations (4) and (29) the Hubble constant is given by

\[
3H^2 = V(\phi) - c_6.
\]

This equation, together with equation (8), leads to

\[
\frac{1}{2} c_5 \sqrt{3(V(\phi) - c_6)} + \frac{dV(\phi)}{d\phi} = 0
\]

from which we can immediately get the potential \(V(\phi)\):

\[
V(\phi) = \frac{1}{8} c_5^2 (\phi - c_7)^2 + c_6
\]

where \(c_7\) is an integration constant. For model building, the concrete functional forms of \(F(X)\) and the potential \(V(\phi)\) can be reconstructed respectively; however, it is interesting that the divergent model with the special Lagrangian given by equation (29) determines the potential equation (32) uniquely, which might indicate that the divergent model has some special implications. Let us recall the result obtained in section 3: if the square potential satisfies the constraint equation (12), the solution of the scalar field will be that listed in equation (10). Now from equation (29) we can easily get \(F_0 = c_5 X_0^2 + c_6, F_{,\phi} = \frac{1}{2} c_5 X_0^{-\frac{1}{2}}\), which solve \(\frac{3}{8} c_5^2 = \frac{1}{2} F_{,\phi}^2 \phi_0^2, c_6 = F_0 - F_{,\phi} \phi_0^2\) (remember that \(X_0 = \frac{1}{2} \phi_0^2\)). Then the potential equation (32) just coincides with equation (12) and means that the divergent model is degenerate with the linear field solution, i.e., these two models are kinetically isomorphic and share the same evolution of the scale factor \(a\) and the scalar field \(\phi\):

\[
\phi = \phi_0 t, \quad a = a_0 \exp \left[ -\frac{F_0}{4}(\phi_0 t + c)^2 \right].
\]
The reason for this degeneracy remains to be clarified. From the mathematical point of view the interpretation may come from the EOM equation (8) since in both models the first term of equation (8) vanishes, and thus they share the same EOM and lead to the same cosmological evolution. But what is the physical implication that the scalar field theory with an infinite sound speed is degenerate with the linear field model? This is an interesting direction for the future. Also note that a similar situation arises in the K-essence model where the linear field solution and the divergent model are isomorphic too [20].

5. Solvable general non-canonical scalar field cosmologies

In this section we will focus on some special cases where equation (9) has a first integral or can be solved exactly. Although the models we consider are quite simple, they can nevertheless lead to some important results. In addition, we will also try to find the relationships and the differences between our general non-canonical scalar field model and the K-essence model.

(A) $\gamma = \text{constant}$. In this subsection we will assume that the barotropic index $\gamma = \gamma_0 = \text{constant}$. Then the EOS $w = \gamma - 1 = \gamma_0 - 1$ is also a constant. The constant $\gamma$ kinematically leads to the cosmological solution

$$a = a_0 t^{2/3\gamma_0}, \quad \rho_\phi = \frac{4a_0^{3\gamma_0}}{3\gamma_0^2} a^{3\gamma_0}. \quad (34)$$

From the relation

$$\gamma_0 = \frac{\rho + p}{\rho} = \frac{2H}{3H^2} = \frac{2XF,X}{2XF,X - F(X) + V(\phi)} \quad (35)$$

we get the following equation:

$$\frac{2(\gamma_0 - 1)}{\gamma_0} XF,X - F(X) + V(\phi) = 0. \quad (36)$$

In [20], it is shown that $F(X) = X^{\gamma_0 - 1}$ can lead to the constant $\gamma$ with an arbitrary potential in the K-essence model. Here we show that in our model the form of $F(X)$ depends on the potential (see equation (36)), that is, to obtain the cosmological solution equation (34), the kinetic term $F(X)$ and potential term $V(\phi)$ must satisfy equation (36). For a constant potential $V_0$, we get a similar function of $F(X)$:

$$F(X) = c_8 X^{\frac{\gamma_0 - 1}{\gamma_0}} + V_0 \quad (37)$$

where $c_8$ is an integration constant.

Let us also consider whether the solution with constant $\gamma_0$ is stable. To answer this question, we let $\gamma$ vary with time. Differentiating the equation of the barotropic index $\gamma$ gives

$$\dot{\gamma} = (\gamma - 1) \left( 3H \gamma + \frac{\dot{p}}{p} \right). \quad (38)$$

We then immediately get two critical points: $\gamma_0 - 1 = 0$ or $\gamma_0$ satisfies the relation

$$3H \gamma_0 + \frac{\dot{p}}{p} = 0. \quad (39)$$

When the condition equation (39) is fulfilled, the potential $V(\phi)$ and function $F(X)$ will satisfy the relation

$$p = F(X) - V(\phi) = \frac{c_8}{a^{3\gamma_0}}. \quad (40)$$
With equations (38), (39), we can have
\[
\dot{\gamma} = 3H(\gamma - 1)(\gamma - \gamma_0) \tag{41}
\]
which could be integrated to give
\[
\gamma = \frac{\gamma_0 a^{3(1-\gamma_0)} - c_{10}}{a^{3(1-\gamma_0)} - c_{10}} \tag{42}
\]
where \(c_9, c_{10}\) are integration constants. Equation (42) indicates that if the universe is expanding and \(\gamma_0 < 1\), then the barotropic index \(\gamma\) has the asymptotic limit \(\gamma_0\). For \(\gamma_0 > 1\), on the other hand, \(\gamma\) will approach the asymptotic limit 1. The case \(\gamma_0 = 1\) should be considered separately: its solution is
\[
\gamma = 1 - \frac{c_{11}}{3 \ln a} \tag{43}
\]
where \(c_{11}\) is an integration constant. Equation (43) shows that the solution with \(\gamma_0 = 1\) is also stable in an expanding universe. Therefore, we can conclude that the solutions with constant barotropic index are attractors in the case \(\gamma_0 \leq 1\) and the solution of \(\gamma_0 = 1\) separates stable from unstable regions.

(B) \(V(\phi) = V_0\). When the potential \(V(\phi)\) is a constant (\(=V_0\), equation (9) has a first integral:
\[
\frac{\dot{\phi}}{\phi} = \frac{c_{12}}{a^6 H^2}. \tag{44}
\]
The corresponding barotropic index can be written in a more convenient form:
\[
\gamma = \left[1 + a^6 F_X(V_0 - F) / 9c_{12}^2\right]^{-1} \tag{45}
\]
where \(c_{12}\) is an arbitrary integration constant. Equation (44) is formally the same as the first integral for the K-essence model with a constant potential [13, 24, 25], where the barotropic index, \(\gamma = \left[1 + V_0 a^6 F_F X / 72c_{12}^2\right]^{-1}\), is very similar to our result equation (45). By constructing the appropriate form of \(F(X)\), they show that this model can interpolate between dark matter (\(\gamma \approx 1\)) at early times and dark energy (\(\gamma \approx 0\)) at late times. The K-essence models with a constant potential were thoroughly studied for their exquisite role in unifying the dark matter and dark energy [13, 24–26]. For a constant potential the K-essence Lagrangian can be written as \(L_k = -V_0 F_k(X)\) while our Lagrangian is \(L_g = F_g(X) - V_0\). They are actually equivalent if we define \(F_g(X) = V_0(1 - F_k(X))\) so that our model can easily reproduce the K-essence models with constant potential. We can also construct a large set of models which unify the dark matter and dark energy. From equation (45) these models are precisely generated by the set of functions \(F(X)\) which at early times satisfy the condition \(a^6 F_F X(V_0 - F) \ll 1\). Let us for simplicity investigate an exactly solvable model with \(F(X) = 1 - \sqrt{1 - 2X}\), which is considered as a nonlinear Born–Infeld (NLBI) scalar field theory in [27]. The energy density of the NLBI scalar field is \(\rho = 1/\sqrt{1 - 2X} - 1 + V(\phi)\). When the potential is a constant \(V_0\), we can obtain the exact solution from equation (44):
\[
\phi^2 = \frac{9c_{12}^2}{a^6 + 9c_{12}^2} \tag{46}
\]
Then we have
\[
\rho = \sqrt{1 + \frac{9c_{12}^2}{a^6} + (V_0 - 1)}, \tag{47}
\]
\[
c_i^2 = 1 - \phi^2 = \frac{a^6}{a^6 + 9c_{12}^2}. \tag{48}
\]
Equations (47), (48) show that the energy density behaves as dark matter at early time (for small $a, \rho \propto a^{-3}$ and $c_s^2 \simeq 0$) and as dark energy at late time (for large $a, \rho \simeq V_0 = \text{const}$ and $c_s^2 \simeq 1$). Note that there are also other early papers where the scalar field models unifying the dark matter and dark energy are studied [28].

6. Conclusion

In this paper we have investigated the general non-canonical scalar field model as a candidate of dark energy. We find that a special square potential (with a negative minimum) is needed to drive the linear field solution in our model, while it should be taken as an inverse square form in the K-essence model to obtain the same solution. Our results show that the linear field solution in the present model is highly degenerate, and has the same cosmological evolution as the divergent sound speed model. The stability of this linear field solution is also studied and the condition for stability found. Furthermore, we discuss the cosmological solutions in the presence of matter and radiation by numerically solving the potential and the cosmological evolution, and find the result to be quite different from the case without matter or radiation. The model with a constant barotropic index $\gamma$ is then investigated and our result shows that, unlike in the K-essence model, the detailed form of $F(X)$ depends on the potential $V(\phi)$. We analyze the stability of this constant $\gamma$ solution and find it to be stable for $\gamma_0 \leq 1$. When the potential is taken as a constant, we find the first integral and obtain the corresponding $\gamma$, which is similar to that in the K-essence model. With a simple form of $F(X)$ we show that the model can also be considered as a candidate for unifying dark matter and dark energy. This work may shed some light on the study of the scalar field theory and the exploration of the dark energy problem.

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Cosmologies with a general non-canonical scalar field