Polynomial growth harmonic functions on groups of polynomial volume growth

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Abstract We consider harmonic functions of polynomial growth of some order $d$ on Cayley graphs of discrete groups of polynomial volume growth of order $D$ w.r.t. the word metric and prove the optimal estimate for the dimension of the space of such harmonic functions. More precisely, the dimension of this space of harmonic functions is at most of order $d^{D-1}$. As in the already known Riemannian case, this estimate is polynomial in the growth degree. More generally, our techniques also apply to graphs roughly isometric to Cayley graphs of groups of polynomial volume growth.

Keywords Groups of polynomial growth · Polynomial growth harmonic functions · Rough isometry

Mathematics Subject Classification 53C21 · 20F65 · 20F69 · 31C05 · 05C63 · 82B41

1 Introduction

In [21], Gromov proved the celebrated structure theorem for finitely generated groups of polynomial volume growth. His result says that every such group is virtually nilpotent, i.e. it has a nilpotent subgroup of finite index. Kleiner [31] proved the polynomial growth harmonic function theorem on Cayley graphs of such groups. Namely, the space of polynomial growth harmonic functions with a fixed growth rate is of finite dimension. With this result, he obtained a new proof of Gromov’s theorem. Then Shalom and Tao [46] gave a quantitative version of Kleiner’s result. Since they used Colding and Minicozzi’s original argument in [10], the
dimension estimate of polynomial growth harmonic functions they obtained is exponential in the growth degree. In this paper, we use a more delicate volume growth property proved by Pansu [43] to obtain the optimal dimension estimate analogous to the Riemannian case on Cayley graphs of groups of polynomial volume growth. This optimal estimate is polynomial in the growth degree.

The history leading to these results started in Riemannian geometry. In 1975, Yau [53] proved the Liouville theorem for harmonic functions on Riemannian manifolds with nonnegative Ricci curvature. Then Cheng-Yau [8] used Bochner’s technique to derive a gradient estimate for positive harmonic functions which implies that sublinear growth harmonic functions on these manifolds are constant. Then Yau [54, 55] conjectured that the space of polynomial growth harmonic functions with growth rate less than or equal to $d$ on Riemannian manifolds with nonnegative Ricci curvature is of finite dimension. Li and Tam [36] and Donnelly and Fefferman [17] independently solved the conjecture for manifolds of dimension two. Then Colding and Minicozzi [9–11] proved Yau’s conjecture for any dimension by using the volume doubling property and the Poincaré inequality. A simplified argument via the mean value inequality can be found in [12, 34] where the dimension estimate is asymptotically optimal. This inspired many generalizations on manifolds [7, 30, 33, 37, 38, 47, 48, 50]. Essentially, the crucial ingredients of these proofs are the volume growth property and the Poincaré inequality (or mean value inequality).

It was then found that this line of reasoning carries over to graphs. Let $(G, S)$ be a Cayley graph of a group $G$ with a finite generating set $S$. In this paper, we only consider finitely generated groups. There is a natural metric on $(G, S)$ called the word metric, denoted by $d^S$. Let $B^S_p(n) := \{x \in G | d^S(x, p) \leq n\}$ denote the closed geodesic ball of radius $n \in \mathbb{N}$ centered at $p \in G$. We denote by $|B^S_p(n)| := |\#B^S_p(n)|$ the volume (i.e. cardinality) of the set $B^S_p(n)$. When $e$ is the unit element of $G$, the volume $\beta_S(n) := |B^S_e(n)|$ of $B^S_e(n)$ is called the growth function of the group. The intensive investigation of growth functions of finitely generated groups began after Milnor’s work [40] (see [41, 52] or the survey papers [18–20]), although the notion of the growth of a group had already been introduced earlier by A. Švarc (A. Schwarz) [45]. A group $G$ is called of polynomial growth, or of polynomial volume growth, if $\beta_S(n) \leq Cn^A$, for any $n \geq 1$ and some $A > 0$, which is independent of the choice of the generating set $S$ since the metrics $d^S$ and $d^S_1$ are bi-Lipschitz equivalent for different finite generating sets $S$ and $S_1$. By Gromov’s theorem and Bass’ volume growth estimate of nilpotent groups [2], for any group $G$ of polynomial growth there are constants $C_1(S)$, $C_2(S)$ depending on $S$ and $D \in \mathbb{N}$ such that for any $n \geq 1$,

$$C_1n^D \leq \beta_S(n) \leq C_2n^D,$$

where the integer $D$ is called the homogeneous dimension or the growth degree of $G$. In this paper, since $D$ is sort of dimensional constant of $G$, we always omit the dependence of $D$ in various constants. Then it implies that for any $n \geq 1$, $p \in G$

$$|B^S_p(2n)| \leq C(S)|B^S_p(n)|,$$

which is called the volume doubling property. Moreover, by the group structure, Kleiner [31] obtained the Poincaré inequality for any Cayley graph of a finitely generated group. This inequality yields that for a Cayley group of a group of polynomial growth, by the volume doubling property (1.2), the (uniform) Poincaré inequality holds

$$\sum_{x \in B^S_p(n)} (u(x) - \bar{u})^2 \leq C(S)n^2 \sum_{x, y \in B^S_p(3n); x \sim y} (u(x) - u(y))^2,$$

where the integer $D$ is called the homogeneous dimension or the growth degree of $G$. In this paper, we only consider finitely generated groups.
where $\tilde{u} = \frac{1}{|B^S_n(1)|} \sum_{x \in B^S_n(1)} u(x)$ and by $x \sim y$ we mean they are neighbors.

The discrete Laplacian operator $L^S$ on the Cayley graph $(G, S)$ is defined as

$$L^S u(x) = \sum_{y \sim x} (u(y) - u(x)) \quad \text{for } x \in G. \quad (1.4)$$

A function $u : G \to \mathbb{R}$ is called (discrete) harmonic if $L^S u(x) = 0$ for any $x \in G$. For any $d > 0$, we denote by

$$H^d(G, S) := \{ u : G \to \mathbb{R} \mid L^S u = 0, |u(x)| \leq C(d^S, e + 1)^d \} \quad (1.5)$$

the space of polynomial growth harmonic functions of growth rate less than or equal to $d$ on $(G, S)$. Kleiner [31] (or [46]) adapted the original argument in Colding and Minicozzi [10] to show that $\dim H^d(G, S) \leq C_1(S) e^{C_2(S)d^2}$, where $\dim H^d(G, S)$ is the dimension of the linear space $H^d(G, S)$, i.e. the upper bound estimate of the dimension is exponential in the growth degree $d$. Delmotte [15] proved the polynomial growth harmonic function theorem with the estimate $\dim H^d(G, S) \leq C(S)d^{v(S)}$ for graphs satisfying the volume doubling property (1.2) and the Poincaré inequality (1.3) where $v(S)$ depends on the doubling constant in (1.2). In case of Cayley graphs, both of these results are not optimal compared to the Riemannian case. We will prove the asymptotically optimal estimate in the rest of the paper.

In addition, it is worth mentioning that for planar graphs with nonnegative combinatorial curvature, Hua et al. [27] and Hua and Jost [26] adopted a more delicate volume growth property, called the relative volume comparison, to obtain the optimal dimension estimate.

Besides Bass’ volume growth estimate (1.1), Pansu [43] proved a sharp volume growth property for groups of polynomial growth (see also Breuillard [5]). He showed that for the Cayley graph $(G, S)$ of a group $G$ of polynomial growth the following limit exists

$$\lim_{n \to \infty} \frac{\beta_S(n)}{n^D} = C(S), \quad (1.6)$$

where $D$ is the homogeneous dimension of $G$. Then it is easy to see (Lemma 2.1 below) that for any $\theta \ll 1$, there exists $R_0(\theta, S)$ such that

$$\frac{|B_p(R)|}{|B_p(r)|} \leq (1 + \theta) \left( \frac{R}{r} \right)^D, \quad (1.7)$$

for any $R \geq r \geq R_0(\theta, S)$. We call it the relative volume comparison in the large. By combining this with the mean value inequality (see Lemma 3.2), which is a result of Moser iteration (c.f. [13,16,23]), we obtain the optimal dimension estimate by Li’s argument [34,35]. This estimate is asymptotically optimal since the upper bound is achieved asymptotically in the case of finitely generated abelian groups (see [24,28,42]).

**Theorem 1.1** Let $(G, S)$ be a Cayley graph of a group of polynomial growth with the homogeneous dimension $D$. Then for $d \geq 1$

$$\dim H^d(G, S) \leq C(S)d^{D-1}. \quad \text{(1.8)}$$

In [1], Alexopoulos obtained much detailed information of polynomial growth harmonic functions on finitely generated groups of polynomial growth. By Gromov’s theorem [21], every finitely generated group $G$ of polynomial growth is virtually nilpotent (i.e. it has a nilpotent subgroup $H$ of finite index). For some torsion-free subgroup $H'$ of $H$, it can be embedded as a lattice in a simply connected nilpotent Lie group $N$. By considering the exponential coordinate of $N$, $N$ is identified with $\mathbb{R}^q$ (see [44]). Alexopoulos proved that
every polynomial growth harmonic function on $G$ when restricted to $H'$ coincides with some polynomial on $\mathbb{R}^d$. It seems not easy to calculate the precise dimension of polynomial growth harmonic functions by this method which depends on the embedding of the nilpotent subgroup into a simply connected Lie group. Instead of doing that, we give a dimension estimate by the geometric analysis methods of Colding–Minicozzi and Li described above. 

In the second half of the paper, we generalize our previous results and prove the optimal dimension estimate for polynomial growth harmonic functions on graphs (with bounded geometry) roughly isometric to Cayley graphs of groups of polynomial growth. Let $X = (V, E)$ be a graph with the natural (combinatorial) distance $d^X$. The degree of a vertex $x \in V$ is defined as $\text{deg} x := \# \{ y \in V | y \sim x \}$. A graph $X$ with bounded geometry means that $\text{deg} x \leq \Delta$ for some $\Delta > 0$ and all $x \in V$. The graph $(X, d^X)$ is called roughly isometric to a Cayley graph $(G, S, d^G)$ of a group if there is mapping $\phi : X \rightarrow G$ which, roughly speaking, is bi-Lipschitz in the large scale (see Definition 2.3, or [6,22,51]). It is well known [14,31] that the volume doubling property and the Poincaré inequality are rough-isometry invariants. Let $H^d(X) := \{ u : X \rightarrow \mathbb{R} | L^X u = 0, |u(x)| \leq C(d^X(x, p_0) + 1)^d \}$, for some $p_0 \in X$, denote the space of polynomial growth harmonic functions on $X$ with growth rate less than or equal to $d$. It turns out the choice of the reference point $p_0 \in X$ is irrelevant in the definition. If $(X, d^X)$ is roughly isometric to a Cayley graph $(G, S)$ of a group of polynomial growth, then Colding–Minicozzi and Li’s arguments yield that

$$\dim H^d(X) \leq C d^D,$$

for $d \geq 1$ where $D$ is the homogenous dimension of $G$. This is not asymptotically optimal. Since in general the relative volume comparison in the large, (1.7), on $X$ cannot be derived from the rough isometry $\phi$, it is hard to obtain the optimal dimension estimate on $X$.

A well-known result of Kanai [29] states that for manifolds or graphs the property to be parabolic (recurrent in the terminology of random walks) is a rough-isometry invariant (see also [51]). In contrast, counterexamples of Lyons [39] and Benjamini [3] showed the instability of the Liouville theorem for bounded harmonic functions under rough isometries. They constructed two roughly isometric graphs: one admits only constant bounded harmonic functions, while the other admits an infinite dimensional space of nonconstant bounded harmonic functions. That is, the dimension estimate of polynomial growth harmonic functions is not a rough-isometry invariant in general. In addition, Lee [33] obtained the optimal dimension estimate for Riemannian manifolds with bounded geometry roughly isometric to Riemannian manifolds with nonnegative Ricci curvature. His method is quite different from ours and seems not suitable for the discrete setting.

In order to extend the optimal dimension estimate for polynomial growth harmonic functions on a graph $X$ roughly isometric to a Cayley graph $(G, S)$ of a group of polynomial growth, we borrow the idea from [26]. Since the relative volume comparison (1.7) holds on $(G, S)$ rather than on $X$, we do the dimension estimate on $G$, translate it to $X$ and obtain the dimension estimate on $X$. Precisely, for any harmonic function $u$ on $X$, we construct a function $\tilde{u}$ on $G$ (see (4.2)) which preserves the crucial property—the mean value inequality—even though $\tilde{u}$ is not necessarily harmonic anymore. We sketch the proof of the mean value inequality for $\tilde{u}$. Firstly, it is easy to obtain the mean value inequality for a harmonic function $u$ on $X$ since the volume doubling property and the Poincaré inequality are stable under rough isometries (see Lemma 4.2). Secondly, via the rough isometry $\phi : X \rightarrow G$ we can construct an injective rough isometry $\phi' : X \rightarrow G'$ where $G'$ is a modified group of $G$ with polynomial growth and the same homogenous dimension. Hence without loss of generality we assume that the rough isometry $\phi$ is injective. By the injectivity of $\phi$ and the mean value inequality of $u$, we can prove the mean value inequality in the large for $\tilde{u}$ in Theorem 4.4.
Let $P^d(G) := \{ u : G \to \mathbb{R} | |u(x)| \leq C(d^S(x, e) + 1)^d \}$ denote the space of polynomial growth functions on $G$ with growth rate less than or equal to $d$. We define the following linear operator

$$E : H^d(X) \to P^d(G),$$

$$u \mapsto Eu := \tilde{u}.$$ 

By the construction, one can see that it is injective. Hence

$$\dim H^d(X) = \dim EH^d(X).$$

It suffices to bound $\dim EH^d(X)$ on $G$. By the same argument as in the proof of Theorem 1.1, the relative volume comparison (1.7) and the mean value inequality in the large (4.5) yield the following.

**Theorem 1.2** Let $(X, d^X)$ be a graph with bounded geometry roughly isometric to a Cayley graph $(G, S, d^S)$ of a group of polynomial growth of homogenous dimension $D$. Then for $d \geq 1$

$$\dim H^d(X) \leq Cd^{D-1}.$$ 

This line of the proof is even new compared with the results in Lee [33]. As an application of Theorem 1.2, we obtain in Corollary 4.6 the optimal dimension estimate for polynomial growth harmonic functions on quasi-transitive graphs of polynomial volume growth (see [51] for the definition).

Our results provide the final answer for the control of the dimension of the space of harmonic functions of some given polynomial growth on Cayley graphs of groups of polynomial growth, and we can even achieve this control more generally in rough isometry classes of such Cayley graphs. Also, as described above, such a control has been achieved under curvature bounds [26]. We may ask, however, for which other classes of graphs such a control might conceivably be possible. In this direction, in [4], it was shown that on a percolation cluster in $\mathbb{Z}^D$, the space of harmonic functions of linear growth, i.e., $d = 1$ in our terminology, is almost surely of dimension at most $D + 1$. From the perspective of our paper, we should expect that also harmonic functions of polynomial growth for any $d$ can be controlled.

### 2 Preliminaries and notation

Let $G$ be a group. It is called finitely generated if it has a finite generating set $S$. We always assume that the generating set $S$ is symmetric, i.e. $S = S^{-1}$. The Cayley graph of $(G, S)$ is a graph structure $(V, E)$ with the set of vertices $V = G$ and the set of edges $E$ where for any $x, y \in G$, $xy \in E$ (also denoted by $x \sim y$) if $x = ys$ for some $s \in S$. The Cayley graph of $(G, S)$ is endowed with a natural metric, called the word metric (c.f. [6]): For any $x, y \in G$, the distance between them is defined as the length of the shortest path connecting $x$ and $y$, 

$$d^S(x, y) := \inf\{k \in \mathbb{N} | \exists x = x_0 \sim x_1 \sim \cdots \sim x_k = y\}.$$ 

It is easy to see that for two generating set $S$ and $S_1$ the metrics $d^S$ and $d_1^S$ are bi-Lipschitz equivalent, i.e. there exist two constants $C_1(S, S_1), C_2(S, S_1)$ such that for any $x, y \in G$

$$C_1(S, S_1)d_1^S(x, y) \leq d^S(x, y) \leq C_2(S, S_1)d_1^S(x, y).$$

Let $B^S_p(n) := \{ x \in G | d^S(p, x) \leq n \}$ denote the closed geodesic ball of radius $n$ centered at $p \in G$ on the Cayley graph $(G, S)$, and $|B^S_p(n)| := \#B^S_p(n)$ the volume of the ball $B^S_p(n)$.
By the group structure, it is obvious that $|B_p^S(n)| = |B_q^S(n)|$, for any $p, q \in G$. The growth function of $(G, S)$ is defined as $\beta_S(n) := |B_e^S(n)|$ where $e$ is the unit element of $G$. A group $G$ is called of polynomial growth if there exists a finite generating set $S$ such that $\beta_S(n) \leq Cn^A$ for some $C, A > 0$ and any $n \geq 1$. It is easy to check that this definition is independent of the choice of the generating set $S$. Thus, the polynomial growth is indeed a property of the group $G$.

Gromov [21] proved a celebrated structure theorem for groups of polynomial growth: every finitely generated group of polynomial growth is virtually nilpotent, i.e. it has a nilpotent subgroup of finite index. Moreover Van den Dries and Wilkie [49] showed that it suffices to get one-scale polynomial volume control, that is, if

$$\lim \inf_{n \to \infty} \frac{\log \beta_S(n)}{\log n} < \infty,$$

i.e. there exists a subsequence $\{n_i\}_{i=1}^\infty$ of $\{n\}$ such that for any $i \in \mathbb{N}$

$$\beta_S(n_i) \leq C_1 n_i^A,$$

then $\beta_S(n) \leq C_2 n^A$ for any $n \geq 1$. By Gromov’s theorem and Bass’ volume growth property [2] for nilpotent groups, we have for any Cayley graph $(G, S)$ of a group of polynomial growth

$$C_1(S) n^D \leq \beta_S(n) \leq C_2(S) n^D,$$

for some $D \in \mathbb{N}$ and any $n \geq 1$, where $D$ is called the homogenous dimension of $G$. Then it is easy to show the volume doubling property

$$|B_p^S(2n)| \leq C(S) |B_p^S(n)|,$$

for any $p \in G$ and $n \geq 1$.

Moreover, Pansu [43] proved a more delicate volume growth property for the Cayley graph $(G, S)$ of polynomial growth that for some $D \in \mathbb{N}$ the limit exists

$$\lim_{n \to \infty} \frac{\beta_S(n)}{n^D} = C(S) < \infty.$$

The following lemma, called the relative volume comparison in the large, is a direct consequence of Pansu’s result.

**Lemma 2.1** Let $(G, S)$ be a Cayley graph of a group of polynomial growth. Then for any $\theta \ll 1$ there exists $R_0(\theta, S)$ such that

$$\frac{|B_p^S(R)|}{|B_p^S(r)|} \leq (1 + \theta) \left(\frac{R}{r}\right)^D$$

for any $p \in G$, $R \geq r \geq R_0(\theta, S)$ where $D$ is the homogeneous dimension of $G$.

**Proof** By Pansu’s result (2.3), for any $\delta \ll 1$, there exists $R_0(\delta, S)$ such that

$$\left|\frac{B_p^S(n)}{Cn^D} - 1\right| < \delta.$$
for any \( p \in G, n \geq R_0(\delta, S) \). We denote by \([R]\) the largest integer less than or equal to \( R \in \mathbb{R} \). Hence for \( R \geq r \geq R_0(\delta, S) \)

\[
\frac{|B_p^S(R)|}{|B_p^S(r)|} \leq \frac{(1 + \delta)C[R]^D}{(1 - \delta)C[r]^D} = \frac{1 + \delta}{1 - \delta} \left( \frac{R}{r} \right)^D \left( \frac{[R]r}{r} \right)^D
\]

\[
\leq \frac{1 + \delta}{1 - \delta} \left( \frac{r}{[r]} \right)^D \left( \frac{R}{r} \right)^D
\]

\[
\leq \frac{1 + \delta}{1 - \delta} \left( \frac{R_0}{R_0 - 1} \right)^D \left( \frac{R}{r} \right)^D
\]

\[
= (1 + \theta) \left( \frac{R}{r} \right)^D,
\]

where \( \theta = \frac{1 + \delta}{1 - \delta} \left( \frac{R_0}{R_0 - 1} \right)^D - 1 \). For \( \delta \ll 1, R_0(\delta, S) \gg 1 \), we have \( \theta \ll 1 \) which proves the lemma. \( \square \)

By this lemma, it is easy to show the weak relative volume comparison that

\[
\frac{|B_p^S(R)|}{|B_p^S(r)|} \leq C(S) \left( \frac{R}{r} \right)^D
\]

(2.5)

for any \( p \in G, R \geq r \geq 1 \), where \( C(S) \) may not be close to 1.

Kleiner \cite{31} proved the Poincaré inequality for any Cayley graph \((G, S)\) that there exists a constant \( C(S) \) such that for any function \( u \) defined on \( B_p^S(3n), p \in G \) and \( n \geq 1 \),

\[
\sum_{x \in B_p^S(n)} (u(x) - \bar{u})^2 \leq C(S)n^2 \frac{|B_p^S(2n)|}{|B_p^S(n)|} \sum_{x, y \in B_p^S(3n); x \sim y} (u(x) - u(y))^2,
\]

where \( \bar{u} = \frac{1}{|B_p^S(n)|} \sum_{x \in B_p^S(n)} u(x) \). By the volume doubling property (2.2), we obtain the uniform Poincaré inequality for the Cayley graph \((G, S)\) of a group of polynomial growth.

**Lemma 2.2** Let \((G, S)\) be a Cayley graph of a group of polynomial growth. Then there exists a constant \( C_1(S) \) such that for any function \( u \) defined on \( B_p(3n), p \in G \) and \( n \geq 1 \),

\[
\sum_{x \in B_p^S(n)} (u(x) - \bar{u})^2 \leq C_1(S)n^2 \sum_{x, y \in B_p^S(3n); x \sim y} (u(x) - u(y))^2,
\]

(2.6)

where \( \bar{u} = \frac{1}{|B_p^S(n)|} \sum_{x \in B_p^S(n)} u(x) \).

For any subset \( \Omega \subset G \), we denote \( d^S(x, \Omega) := \inf\{d^S(x, y) \mid y \in \Omega\} \) for any \( x \in G \), \( \partial \Omega := \{z \in G \mid d^S(z, \Omega) = 1\} \), and \( \Omega := \Omega \cup \partial \Omega \). For any function \( u : \Omega \rightarrow \mathbb{R} \), the discrete Laplacian operator is defined on \( \Omega \) as

\[
L^S u(x) = \sum_{y \sim x} (u(y) - u(x)), \quad x \in \Omega.
\]

The function \( f \) is called harmonic (subharmonic resp.) on \( \Omega \) if \( L^S u(x) = 0 \) (\( \geq 0 \) resp.), for any \( x \in \Omega \). Let

\[
H^d(G, S) := \{ u : G \rightarrow \mathbb{R} \mid L^S u = 0, |u(x)| \leq C(d^S(x, e) + 1)^d \}
\]
denote the space of polynomial growth harmonic functions of growth rate less than or equal to $d$.

Let $X = (V, E)$ be a graph with the set of vertices, $V$, and the set of edges, $E$. For any $x, y \in V$, they are called neighbors (denoted by $x \sim y$) if $xy \in E$. The degree of a vertex $x$ is defined as $\deg x := |\{ y \in V | y \sim x \}|$. For simplicity, we only consider locally finite connected graphs (i.e. $\deg x < \infty$, for any $x \in V$) without self-loops and multiedges. We say a graph $X$ has bounded geometry if there is a constant $\Delta > 0$ such that

$$\deg x \leq \Delta, \quad \forall x \in V. \quad (2.7)$$

In other words, it has uniformly bounded degree (see Woess [51]). For any graph $X$, there is a natural metric structure $d^X(x, y) := \inf \{ k \in \mathbb{N} \mid \exists x = x_0 \sim x_1 \sim \cdots \sim x_k = y \}$, where $x, y \in V$. The closed geodesic ball is denoted by $B^X_p(n) := \{ x \in V | d^X(x, p) \leq n \}$ and the volume by $|B^X_p(n)| := \#B^X_p(n)$. Note that for graphs with bounded geometry this definition is equivalent up to a factor to the other one in graph theory, $|B^X_p(n)| := \sum_{x \in B^X_p(n)} \deg x$, see [51].

We recall the definition of rough isometries between metric spaces, also called quasi-isometries (see [6,22,51]). For a metric space $(X, d^X)$ and some subset $\Omega \subset X$, the distance function to $\Omega$ is defined as $d^X(x, \Omega) := \inf \{ d^X(x, y) | y \in \Omega \}$, for any $x \in X$.

**Definition 2.3** Let $(X, d^X), (Y, d^Y)$ be two metric spaces. A rough isometry is a mapping $\phi : X \rightarrow Y$ such that

$$a^{-1}d^X(x, y) - b \leq d^Y(\phi(x), \phi(y)) \leq ad^X(x, y) + b,$$

for all $x, y \in X$, and

$$d^Y(z, \phi(X)) \leq b,$$

for any $z \in Y$, where $a \geq 1, b \geq 0$. It is called an $(a, b)$-rough isometry.

From a rough isometry $\phi : X \rightarrow Y$, we can construct a rough inverse $\psi : Y \rightarrow X$. For any $y \in Y$ we choose $x \in X$ such that $d^Y(y, \phi(x)) \leq b$ and set

$$\psi(y) = x. \quad (2.8)$$

Then $\psi$ is an $(a, 3ab)$-rough isometry if $\phi$ is an $(a, b)$-rough isometry. It is obvious that the composition of two rough isometries is again a rough isometry. Hence, to be roughly isometric is an equivalence relation between metric spaces.

In this paper, we only consider rough isometries between metric spaces of graphs with bounded geometry. It is well known that the volume doubling property and the Poincaré inequality are roughly isometric invariants (see [14,31]). Hence by the volume doubling property (2.2) and the Poincaré inequality (2.6) for Cayley graphs of groups of polynomial growth, we have

**Lemma 2.4** Let $(X, d)$ be a graph with bounded geometry roughly isometric to a Cayley graph $(G, S, d^S)$ of a group of polynomial growth. Then

$$|B^X_p(2R)| \leq C|B^X_p(R)|, \quad (2.9)$$

for any $p \in X, R \geq 1$.  

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Lemma 2.5 Let \((X, d)\) be a graph with bounded geometry roughly isometric to a Cayley graph \((G, S, d^S)\) of a group of polynomial growth. Then there exist constants \(C_1(\Delta, a, b, S)\) and \(C_2(\Delta, a, b, S)\), where \(\Delta\) is defined in (2.7), such that for any \(p \in X, R \geq 1\) and any function \(u\) defined on \(B^X_p(C_1R)\) we have
\[
\sum_{x \in B^X_p(R)} (u(x) - \bar{u})^2 \leq C_2 R^2 \sum_{x, y \in B^X_p(C_1R): x \sim y} (u(x) - u(y))^2,
\]
where \(\bar{u} = \frac{1}{|B^X_p(R)|} \sum_{x \in B^X_p(R)} u(x)\).

In the sequel, for the Cayley graph \((G, S)\) we omit the dependence of the generating set \(S\), e.g. \(B_p(n) := B^G_p(n)\) if there is no danger of confusion; in other cases we denote it by \(B^X_p(n)\) if we need to distinguish it from \(B^X_p(n)\) in the graph \(X\). In addition, for the simplicity the summation over a geodesic ball is denoted by the integration with respect to the counting measure \(\int_{B^X_p(n)} u^2 := \sum_{x \in B^X_p(n)} u^2(x)\).

3 Mean value inequality and optimal dimension estimate

In this section, we obtain the mean value inequality by the volume doubling property (2.2) and the Poincaré inequality (2.6) on Cayley graphs of groups of polynomial growth. Then we use the relative volume comparison (2.4) and the mean value inequality to get the optimal dimension estimate for \(H^d(G, S)\) by Li’s argument [34,35].

Delmotte [16] and Holopainen and Soardi [23] independently carried out the Moser iteration on graphs satisfying the volume doubling property and the Poincaré inequality which implies the Harnack inequality.

Lemma 3.1 (Harnack inequality) Let \((G, S)\) be a Cayley graph of a group of polynomial growth. Then there exist constants \(C_1(S), C_2(S)\) such that for any \(p \in G, n \geq 1\), any positive harmonic function \(u\) on \(B_p(C_1n)\) we have
\[
\max_{B_p(n)} u \leq C_2 \min_{B_p(n)} u.
\]

The mean value inequality is one part of the Moser iteration (see [13,16,23]).

Lemma 3.2 (Mean value inequality) Let \((G, S)\) be a Cayley graph of a group of polynomial growth. Then there exists a constant \(C_1(S)\) such that for any \(p \in G, R \geq 1\), any harmonic function \(u\) on \(B_p(R)\) we have
\[
u^2(p) \leq \frac{C_1}{|B_p(R)|} \sum_{x \in B_p(R)} u^2(x).
\]

The dimension estimate follows from Li’s argument in [34] (see also [15,25,26,35]). We need some lemmata. For convenience, we sometimes write integrals (w.r.t. the counting measure on vertices) for the summations.

Lemma 3.3 For any finite dimensional subspace \(K \subset H^d(G, S)\), there exists a constant \(R_1(K)\) depending on \(K\) such that for any \(R \geq R_1(K)\)
\[
A_R(u, v) := \int_{B_p(R)} uv := \sum_{x \in B_p(R)} u(x)v(x)
\]
is an inner product on $K$.

Proof A contradiction argument (see [25]).

Lemma 3.4 Let $(G, S)$ be a Cayley graph of a group of polynomial growth with the homogeneous dimension $D$, $K$ be a $k$-dimensional subspace of $H^d(G, S)$. Given $\beta > 1, \delta > 0$ for any $R'_1 \geq R_1(K)$ there exists $R > R'_1$ such that if $\{u_i\}_{i=1}^k$ is an orthonormal basis of $K$ with respect to the inner product $A_{\beta R}$, then

$$\sum_{i=1}^k A_R(u_i, u_i) \geq k\beta^{-(2d+D+\delta)}.$$ 

Proof The lemma follows from the volume growth property (2.1) and the linear algebra (see [15,25,34,35]).

The next lemma follows from the mean value inequality (3.1) of Lemma 3.2 for harmonic functions.

Lemma 3.5 Let $(G, S)$ be a Cayley graph of a group of polynomial growth with the homogeneous dimension $D$, $K$ be a $k$-dimensional subspace of $H^d(G, S)$. Then for any fixed $0 < \epsilon < \frac{1}{2}$ there exist constants $C(S)$ and $R_2(\epsilon, S)$ such that for any $R \geq R_2(\epsilon, S)$ and any basis $\{u_i\}_{i=1}^k$ of $K$ we have

$$\sum_{i=1}^k A_R(u_i, u_i) \leq C\epsilon^{-(D-1)} \sup_{u \in \langle A, U \rangle} \int_{B_R(1+\epsilon)} u^2,$$

where $\langle A, U \rangle := \{w = \sum_{i=1}^k a_i u_i | \sum_{i=1}^k a_i^2 = 1\}$.

Proof For any $x \in B_R(p)$, we set $K_x = \{u \in K : u(x) = 0\}$. Obviously, $K_x$ is of codimension one in $K$. Hence there exists an orthonormal linear transformation $\varphi : K \to K$, which maps $\{u_i\}_{i=1}^k$ to $\{v_i\}_{i=1}^k$ such that $v_i \in K_x$, for $i \geq 2$. For any $x \in B_R(p)$, since $\epsilon R \geq \epsilon R_2 \geq 1$ by choosing $R_2 \geq \frac{1}{\epsilon}$, then $(1+\epsilon)R - r(x) \geq 1$ where $r(x) := d(p, x)$. Hence the mean value inequality (3.1) implies that for any $x \in B_R(p)$

$$\sum_{i=1}^k u_i^2(x) = \sum_{i=1}^k v_i^2(x) = v_1^2(x) \leq C(S) |B_{\varphi((1+\epsilon)R - r(x))}|^{-1} \int_{B_{\varphi((1+\epsilon)R - r(x))}} v_1^2 \leq C(S) |B_{\varphi((1+\epsilon)R - r(x))}|^{-1} \sup_{u \in \langle A, U \rangle} \int_{B_R(1+\epsilon)} u^2. \tag{3.2}$$

For simplicity, we denote $V_p(t) := |B_p(t)|$. By the weak relative volume comparison (2.5), we have

$$V_x((1+\epsilon)R - r(x)) \geq \frac{1}{C(S)} \left( \frac{(1+\epsilon)R - r(x)}{2R} \right)^D V_x(2R) \geq \frac{1}{C(S)} \left( \frac{(1+\epsilon)R - r(x)}{2R} \right)^D V_p(R).$$
Hence, substituting it into (3.2) and integrating over $B_p(R)$, we have

$$
\sum_{i=1}^{k} \int_{B_p(R)} u_i^2 \leq \frac{C(S)}{V_p(R)} \sup_{u \in \mathcal{A} \cup \mathcal{U}} \int_{B_p(1+\epsilon)R} u^2 \int_{B_p(R)} (1 + \epsilon - R^{-1}r(x))^{-D} dx
$$

Define $f(t) = (1 + \epsilon - R^{-1}t)^{-D}$. It suffices to bound the term

$$
\int_{B_p(R)} f(r(x)) dx = \int_{B_p(R) \setminus B_p(\frac{R}{2})} f + \int_{B_p(\frac{R}{2})} f
\leq \int_{B_p(R) \setminus B_p(\frac{R}{2})} f + CV_p \left( \frac{R}{2} \right)
\leq \int_{B_p(R) \setminus B_p(\frac{R}{2})} f + CV_p(R)
= (*) + CV_p(R)
$$

We denote by $[R]$ the largest integer less than or equal to $R$. Then for $[\frac{R}{2}] \geq R_0(\theta, S)$ (e.g. $R \geq 3R_0(\theta, S)$ is sufficient) where $R_0(\theta, S)$ is in Lemma 2.1,

$$(*) \leq \sum_{[\frac{R}{2}] \leq i \leq [R]} f(i)(V_p(i) - V_p(i - 1))
= \sum_{[\frac{R}{2}] \leq i \leq [R]-1} (f(i) - f(i + 1))V_p(i) + V_p([R])f([R]) - V_p([R/2] - 1) f([R/2])
\leq \frac{V_p(R)}{R^D}(1 + \theta)^{-1} \sum_{[\frac{R}{2}] \leq i \leq [R]-1} (f(i) - f(i + 1))i^D + V_p(R)f([R])
\leq \frac{V_p(R)(1 + \theta)^{-1}}{R^D} \left[ \sum_{[\frac{R}{2}] + 1 \leq i \leq [R]-1} f(i)(i^D - (i - 1)^D) + f \left( \left[ \frac{R}{2} \right] \right) \left[ \frac{R}{2} \right]^D \right] - f([R])([R] - 1)^D + V_p(R)f([R])
\leq f([R])V_p(R) \left[ 1 - (1 + \theta)^{-1} \left( \left[ \frac{R}{2} \right] \right)^D \right] + \frac{V_p(R)(1 + \theta)^{-1}}{R^D} \sum_{[\frac{R}{2}] + 1 \leq i \leq [R]-1} f(i)(i^D - (i - 1)^D) + CV_p(R)
= I + II + CV_p(R)
$$

For fixed $0 < \epsilon < \frac{1}{2}$, there exist $\theta = C(\epsilon) \ll 1$ and $R_1(\epsilon) \gg 1$ such that for any $R \geq R_2 \geq R_1(\epsilon)$,

$$
\left| 1 - (1 + \theta)^{-1} \left( \frac{[R] - 1}{R} \right)^D \right| \leq \epsilon,
$$

hence

$$
I \leq CV_p(R)\epsilon^{-D} \cdot \epsilon = CV_p(R)\epsilon^{-(D-1)}.
$$
For the second term,
\[ II \leq C(D)(1 + \theta)^{-1} \frac{V_p(R)}{R^D} \sum_{\lfloor \frac{R}{2} \rfloor + 1 \leq i \leq |R| - 1} f(i) i^{D-1} \]
\[ \leq C \frac{V_p(R)}{R} \sum_{\lfloor \frac{R}{2} \rfloor + 1 \leq i \leq |R| - 1} f(i) \]
\[ \leq C \frac{V_p(R)}{R} \int_{\lfloor \frac{R}{2} \rfloor}^R f(t) dt \]
\[ \leq C \epsilon^{-(D-1)} V_p(R). \quad (3.7) \]

Combining the estimates of (3.4) (3.5) (3.6) and (3.7), we obtain that for \( \theta = C(\epsilon) \), any \( R \geq R_2(\epsilon, S) = \max\{ \frac{1}{\epsilon}, 3R_0(\theta, S), R_1(\epsilon) \} \)
\[ \int_{B_p(R)} f(r(x)) dx \leq C(\epsilon^{-(D-1)} + C) V_p(R) \]
\[ \leq C \epsilon^{-(D-1)} V_p(R). \quad (3.8) \]

The lemma follows from (3.3) and (3.8). \( \square \)

**Proof of Theorem 1.1** For any \( k \)-dimensional subspace \( K \subset H^d(G, S) \), we set \( \beta = 1 + \epsilon \), for fixed small \( 0 < \epsilon < \frac{1}{2} \). By Lemma 3.4, for any \( \delta > 0 \), there exists infinitely many \( R > R_1(K) \) such that for any orthonormal basis \( \{ u_i \}_{i=1}^k \) of \( K \) with respect to \( A(1+\epsilon) R \), we have
\[ \sum_{i=1}^k A_R(u_i, u_i) \geq k(1 + \epsilon)^{-(2d+D+\delta)}. \]

Lemma 3.5 implies that for sufficiently large \( R \)
\[ \sum_{i=1}^k A_R(u_i, u_i) \leq C(S) \epsilon^{-(D-1)}. \]

Setting \( \epsilon = \frac{1}{2d} \), and letting \( \delta \to 0 \), we obtain
\[ k \leq C(S) \left( \frac{1}{2d} \right)^{-(D-1)} \left( 1 + \frac{1}{2d} \right)^{2d+D+\delta} \leq C(S) d^{D-1}. \quad (3.9) \]
which proves the theorem. \( \square \)

### 4 Rough isometries and optimal dimension estimate

In this section we obtain the optimal dimension estimate for polynomial growth harmonic functions on graphs with bounded geometry roughly isometric to Cayley graphs of groups of polynomial growth. The strategy is similar to that in [26]. Let \((X, d^X)\) be a graph with bounded geometry roughly isometric to a Cayley graph \((G, S, d^G)\) of a group of polynomial growth. Since the rough isometry does not preserve the optimal volume growth condition (2.4), it is hard to get the optimal dimension estimate by the argument based on the graph \( X \). Instead of doing that, we construct functions on \((G, S)\) from harmonic functions on \( X \).
which are not harmonic on \((G, S)\) but satisfy the mean value inequality in the large \((4.5)\). By the same arguments as in Sect. 2, we obtain the optimal dimension estimate for functions on \((G, S)\) which implies the dimension estimate on \(X\).

In order to preserve the mean value inequality for the later construction, we need the following lemma which says that we can always construct a new injective rough isometry \(\phi' : X \to G'\) from a rough isometry to a Cayley graph \((G, S)\) of a group of polynomial growth \(\phi : X \to G\), where \(G'\) is also of polynomial growth with same homogenous dimension as \(G\).

**Lemma 4.1** Let \((X, d_X)\) be a graph with bounded geometry roughly isometric to a Cayley graph \((G, S, d^S)\) of a group of polynomial growth, \(\phi : X \to G\) be the rough isometry. Then there exist a group \(G'\) of polynomial growth with same homogenous dimension as \(G\) and an injective rough isometry \(\phi' : X \to G'\) which is constructed from \(\phi\).

**Proof** Since \(X\) is a graph with bounded geometry, \(\deg x \leq \Delta\) for all \(x \in X\). Assume that \(\phi : X \to G\) is an \((a, b)\)-rough isometry which is generally not injective. For any \(y \in \phi(X)\), by rough isometry \(\text{diam}(\phi^{-1}(y)) \leq ab\). Hence by bounded geometry of \(X\),

\[
\#(\phi^{-1}(y)) \leq \Delta^{ab+1} =: q(\Delta, a, b). \quad (4.1)
\]

Define a group \(G' := G \times \mathbb{Z}_q\) where \(\mathbb{Z}_q = \{0, 1, \ldots, q - 1\}\) is a finite cyclic group and \(\times\) is the direct product of groups (see [32]). Let \(S' := S \times \{0\} \cup \{e, \pm 1\}\), where \(e\) is the unit element of \(G\). It is easy to see that \(S'\) is a generating set of \(G'\) and \((G', S')\) are of polynomial growth with same homogenous dimension. By \((4.1)\), it is straightforward to define an injective rough isometry \(\phi' : X \to G' = G \times \mathbb{Z}_q\) such that

\[\pi_G \circ \phi' = \phi,\]

where \(\pi_G\) is a standard projection map forgetting the second factor of \(G'\). For the construction of \(\phi'\), it suffices to map the set \(\phi^{-1}(y)\) injectively into \(\{y\} \times \mathbb{Z}_q\) for any \(y \in \phi(X)\). \(\square\)

By this lemma, we always assume that (to keep the notation) there is an injective \((a, b)\)-rough isometry \(\phi : X \to G\) to a Cayley graph of a group of polynomial growth. The Laplacian operator \(L^X\) is defined for any function \(u\) on \(X\) as

\[L^X u(x) := \sum_{y \sim x} (u(y) - u(x)).\]

We denote by \(H^d(X) := \{u : X \to \mathbb{R} | L^X u = 0, |u(x)| \leq C(d^X(x, p_0) + 1)^d\}\) for some \(p_0 \in X\), the space of polynomial growth harmonic functions on \(X\) with growth rate less than or equal to \(d\), by \(P^d(G) := \{u : G \to \mathbb{R} | |u(x)| \leq C(d^S(x, e) + 1)^d\}\) the space of polynomial growth functions on \(G\) with growth rate less than or equal to \(d\). For a fixed injective rough isometry \(\phi : X \to G\), we construct a function \(\tilde{u}\) on \(G\) from a function \(u : X \to \mathbb{R}\) as follows. For any \(y \in \phi(X)\), let \(\tilde{u}(y) := u(\phi^{-1}(y))\). For \(y \in G \setminus \phi(X)\), let \(W_y := \phi^{-1}(\phi(X) \cap B^G_y(y)) \subset X\) and

\[\tilde{u}(y) := \frac{1}{|W_y|} \sum_{x \in W_y} u(x), \quad (4.2)\]

where \(|W_y| := \#W_y\). We define

\[E : H^d(X) \to P^d(G)\]

\[u \mapsto Eu := \tilde{u}.\]
It is easy to see that $E$ is an injective linear operator. Hence
\[ \dim H^d(X) = \dim EH^d(X). \]

To estimate $\dim H^d(X)$, it suffices to bound the dimension of $EH^d(X)$. The advantage of estimating $\dim EH^d(X)$ on $G$ is the good volume growth property (2.4). Next we shall prove the crucial property (mean value inequality) of $Eu = \tilde{u}$ for any harmonic function $u$ on $X$.

As in Lemma 3.2, the mean value inequality for harmonic functions on $X$ follows from the volume doubling property (2.9) and the Poincaré inequality (2.10) (see [13, 16, 23]).

**Lemma 4.2** Let $(X, d^X)$ be a graph with bounded geometry roughly isometric to a Cayley graph $(G, S, d^S)$ of a group of polynomial growth. Then there exists a constant $C_1$ such that for any $p \in X$, $R \geq 1$, any harmonic function $u$ on $B_p^X(R)$ we have
\[ u^2(p) \leq \frac{C_1}{|B_p^X(R)|} \sum_{x \in B_p^X(R)} u^2(x). \]  
\[ (4.3) \]

The proof of the following lemma is essentially as same as that of Lemma 2.4.

**Lemma 4.3** Let $\phi : X \to G$ be an injective $(a, b)$-rough isometry to a Cayley graph $(G, S)$ of a group of polynomial growth. Then there exist constants $C_1 < 1$, $C_2$ and $R_1$ such that for any $y \in G$, $p \in X$ satisfying $d^S(\phi(p), y) \leq b$ and $R \geq R_1$ we have
\[ |B_p^X(R)| \geq C_2|B_y^G(C_1 R)|. \]
\[ (4.4) \]

**Proof** For fixed $y \in G$, $p \in X$ satisfying $d^S(\phi(p), y) \leq b$, by (2.8) we can construct an inverse $(a, 3ab)$-rough isometry $\psi = \psi_{y, p} : G \to X$ from $\phi$ such that $\psi(y) = p$. We claim that $\psi(B_y^G(C_1 R)) \subset B_p^X(R)$ for some $C_1 < 1$ and any $R \geq R_1$. For any $q \in \psi(B_y^G(C_1 R))$ ($C_1$ to be chosen), $q = \psi(z)$ for some $z \in B_y^G(C_1 R)$. Then
\[ d^X(p, q) = d^X(\psi(y), \psi(z)) \leq ad^S(y, z) + 3ab \]
\[ \leq aC_1 R + 3ab \leq R, \]

if we choose $C_1 = \frac{1}{2a}$, $R_1 = 6ab$ which yields the claim.

By the bounded geometry of $G$, for any $s \in X$, $\#\{\psi^{-1}(s)\} \leq C$. Hence
\[ |B_p^X(R)| \geq |B_y^G(C_1 R)| \geq \frac{1}{C}|B_y^G(C_1 R)|. \]

The lemma follows.

The mean value inequality for $Eu = \tilde{u}$ follows from the previous two lemmata and the injectivity of the rough isometry $\phi$.

**Theorem 4.4** (Mean value inequality in the large) Let $\phi : X \to G$ be an injective $(a, b)$-rough isometry to a Cayley graph of a group of polynomial growth. Then there exist constants $C, R_0$ such that for any harmonic function $u$ on $X$, any $y \in G$ and $R \geq R_0$ we have
\[ \tilde{u}^2(y) \leq \frac{C}{|B_y^G(R)|} \sum_{x \in B_y^G(R)} \tilde{u}^2(x). \]
\[ (4.5) \]
Proof By the group structure of $G$, for any $y \in G$, $|B_y^G(b)| = \beta(b) \leq C(b)$. The bounded geometry of $X$ implies that $\#\phi^{-1}(B_y^G(b)) \leq C(\Delta, a, b)$. Then by the definition of $\tilde{u}$ (4.2), there exists $p \in W_y$ (e.g. the maximum point for $u^2$ in $W_y$) such that for any $R \geq R_1$

$$\tilde{u}^2(y) \leq C(\Delta, a, b)u^2(p)$$

$$\leq \frac{C}{|B_p^X(R)|} \sum_{x \in B_p^X(R)} u^2(x) \text{ by (4.3)}$$

$$\leq \frac{C}{|B_p^G(C_1 R)|} \sum_{x \in B_p^G(C_1 R)} u^2(x) \text{ by (4.4)}.$$

For any $z \in \phi(B_p^X(R))$, $z = \phi(q)$ where $q \in B_p^X(R)$. Then

$$d^G(z, y) \leq d^G(z, \phi(p)) + d^G(\phi(p), y) \leq a d^X(p, q) + b + b$$

if we choose $C = 2a$, $R \geq \frac{2b}{a}$ which implies that $\phi(B_p^X(R)) \subset B_p^G(CR)$. Hence, for $R \geq R_2 = \max\{1, R_1, \frac{2b}{a}\}$

$$\tilde{u}^2(y) \leq \frac{C}{|B_p^G(C_1 R)|} \sum_{x \in B_p^G(C_1 R)} u^2(x)$$

$$\leq \frac{C}{|B_p^G(C_1 R)|} \sum_{w \in B_p^G(C_1 R) \cap \phi(X)} \tilde{u}^2(w)$$

$$\leq \frac{C}{|B_y^G(CR)|} \sum_{w \in B_y^G(CR)} \tilde{u}^2(w),$$

where we use the injectivity of $\phi$ and the volume doubling property (2.2) on $G$. The theorem follows by choosing $R_0 = CR_2$. \hfill \Box

Proof of Theorem 1.2 Since we have obtained the mean value inequality in the large (4.5), combining with the relative volume comparison in the large (2.4), we prove the theorem by the same argument as in Sect. 2 (see [26]). \hfill \Box

As an application, we obtain the dimension estimate for the quasi-transitive graphs (see Woess [51]). Let $(X, d^X)$ be a locally finite, connected graph. An automorphism of $X$ is a self-isometry of $(X, d^X)$. We denote by $Aut(X)$ the set of automorphisms of $X$. The graph $X$ is called vertex transitive if $Aut(X)$ acts transitively on $X$, i.e. the factor (quotient) graph $X/Aut(X)$ has only one orbit. It is called quasi-transitive if $Aut(X)$ acts with finitely many orbits. It is easy to see that transitive and quasi-transitive graphs possess bounded geometry. We recall a result in [51, Theorem 5.11].

Lemma 4.5 Let $(X, d^X)$ be a quasi-transitive graph whose growth function satisfies $|B_p^X(n)| \leq C n^A$ for infinitely many $n$ and some $A > 0$. Then $X$ is roughly isometric to a Cayley graph of some finitely generated nilpotent group with homogenous dimension $D$. 
By Theorem 1.2 and Lemma 4.5, we obtain the following corollary.

**Corollary 4.6** Let \((X, d_X)\) be a quasi-transitive graph whose growth function satisfies 
\[ |B_p^X(n)| \leq Cn^A \]
for infinitely many \(n\) and some \(A > 0\). Then
\[ \dim H^d(X) \leq Cd^{D-1}, \]
for \(d \geq 1\) where \(D\) is the homogenous dimension of the nilpotent group in Lemma 4.5.

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