Strong existence and uniqueness of solutions of SDEs with time dependent Kato class coefficients

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Abstract: Consider stochastic differential equations (SDEs) in $\mathbb{R}^d$: $dX_t = dW_t + b(t, X_t)dt$, where $W$ is a Brownian motion, $b(\cdot, \cdot)$ is a measurable vector field. It is known that if $|b|^2(\cdot, \cdot) = |b|^2(\cdot)$ belongs to the Kato class $K_{d,2}$, then there is a weak solution to the SDE. In this article we show that if $|b|^2$ belongs to the Kato class $K_{d,\alpha}$ for some $\alpha \in (0, 2)$ ($\alpha$ can be arbitrarily close to 2), then there exists a unique strong solution to the stochastic differential equations, extending the results in the existing literature as demonstrated by examples. Furthermore, we allow the drift to be time-dependent. The new regularity estimates we established for the solutions of parabolic equations with Kato class coefficients play a crucial role.

Key Words: strong solution; singular drift; Kato class; maximal function; Zvonkin transformation.

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In this article, we are concerned with the strong solutions to the following stochastic differential equations (SDEs) in $\mathbb{R}^d$ ($d \geq 2$):

$$X_t = x + W_t + \int_0^t b(s, X_s) ds, \quad \forall t > 0,$$

where $W := (W_t)_{t \geq 0}$ is a $d$-dimensional standard Brownian motion on a given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual assumptions, and $b$ is a measurable $\mathbb{R}^d$-valued function on $(0, \infty) \times \mathbb{R}^d$. We stress that the assumption that the diffusion matrix is identity is just for the clarity of the exposition. Our method also works for general, non-degenerate diffusion matrices.

The strong solutions of SDEs with singular drift have been investigated by many authors. In the celebrated work [20], Zvonkin introduced a quasi-isometric transformation of the phase space that can convert a stochastic differential equation with a non-zero singular drift into a SDE without drift. This method is now called Zvonkin transformation. From then on, there are many papers devoted to extending the Zvonkin transformation in various ways to obtain the strong solutions of stochastic differential equations with singular coefficients. Krylov and Röckner in [10] showed that there exist unique strong solutions to Brownian motion with drifts $b$ that are in the class $L^p-L^q := \bigcap_{T>0} L^p((0, T); L^q(\mathbb{R}^d))$ for some $p, q \geq 2$ and $\frac{2}{p} + \frac{d}{q} < 1$. Then Xicheng Zhang in [18] extended the above work to the case that the diffusion coefficients are uniformly non-degenerate and belong to some Sobolev spaces. Recently, in the critical case when the drift $b$ belongs to $L^p-L^q$ with $\frac{2}{p} + \frac{d}{q} = 1$, Beck et al. in [2] obtained the existence and uniqueness of strong solutions for almost every starting point $x$. Xia et al. in [14] obtained the strong well-posedness of SDE (1.1) for $b \in \tilde{L}^q_p$ (see (2.2) in [14] for the precise definition of the space $\tilde{L}^q_p$) with $p, q \geq 2$. We also mention [5], [13], [14], [15] and [17] for further related works.

On the other hand, the existence and uniqueness of weak solutions to SDEs require much less conditions on the coefficients. We refer readers to [1], [6], [7] and [8] for references.

If $|b|^2$ belongs to the Kato class $K_{d,2}$, then using Karamanskii’s inequality and Girsanov transformation, it is known that there exists a weak solution to SDE (1.1). It is natural to ask the following question:

Is there a unique strong solution to SDE (1.1) when $|b|^2$ belongs to the Kato class $K_{d,\alpha}$ for some $\alpha \in (0, 2)$ arbitrarily close to 2?

The purpose of this article is to give a positive answer to the above question. We also allow the drifts to be time-dependent. It is easy to verify that the $L^p-L^q$ (or $\tilde{L}^q_p$) condition appeared in the literature mentioned above satisfy our assumption. Moreover, as the examples below show, there are plenty of functions satisfying our conditions do not fulfill the $L^p-L^q$ (or $\tilde{L}^q_p$) assumptions.

To prove the uniqueness of the strong solutions to SDEs (1.1), we apply the Zvonkin transformation. To this end we need to establish the required regularity of the solutions of the associated parabolic partial differential equations. Because of the singularity of the drift $b$, the classical Krylov’s estimates for semi-martingales (see [9, Lemma 5.1]) do not apply here. Moreover, since $b$ may not be in the class $L^p-L^q$, the Calderon-Zygmund inequality
for parabolic equations can not be used anymore. Therefore, one of the main tasks is to establish new regularity estimates for the associated parabolic equations from scratch under the Kato class conditions.

The rest of this article is arranged as follows. In Section 2, we state the main result and give some examples. In Section 3, we give several equivalent conditions for the Kato class functions. And we also obtain some useful properties for the maximal function of kato class functions. In Section 4, We consider the Kolmogorov equations associated with SDEs (1.1). We derived the existence and uniqueness of the mild solutions to the Kolmogorov equations and prove that the second derivative of the mild solutions belong to the desired Kato class. In Section 5, we prove the pathwise uniqueness of solutions of equation (1.1). To use the Zvonkin transformation, we establish the Krylov type estimates for the SDEs (1.1) and exploit the regularity estimates for the maximal function of the second derivative of the solutions of the Kolmogorov equation.

## 2 Statement of the main result

In this section, we state the main result and give some examples. We start with the definition of the Kato classes of functions:

**Definition 2.1** Given $0 < \alpha \leq 2$, a measurable function $f : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is said to be in the Kato class $K_{d,\alpha}$, if for each $\lambda > 0$,

$$\lim_{T \to 0} \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} s^{-\frac{d+2-\alpha}{2}} e^{-\frac{\lambda|x-y|^2}{2s}} |f(t+s, y)| dy ds = 0,$$

$$\lim_{T \to 0} \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} s^{-\frac{d+2-\alpha}{2}} e^{-\frac{\lambda|x-y|^2}{2s}} |f(t-s, y)| dy ds = 0,$$

where we set $f(t, x) := 0$ for $(t, x) \in (-\infty, 0) \times \mathbb{R}^d$.

The main result of the paper reads as follows.

**Theorem 2.2** Assume for any $T > 0$, $|b(t, x)|^2 I_{(0, T)}(t) \in K_{d,\alpha}$ for some $\alpha \in (0, 2)$. Then for each $x \in \mathbb{R}^d$, there exists a unique strong solution to SDE (1.1).

**Remark 2.1** By the Proposition 3.2 below, we see that for any $T > 0$, $|b(t, x)|^2 I_{(0, T)}(t) \in K_{d,\alpha}$ for some $\alpha \in (0, 2)$ is equivalent to one of the following two conditions:

(i) For any $T > 0$, there exists $\beta \in (0, 2)$ such that

$$\lim_{T \to 0} \sup_{t \in [0, T], x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} s^{-\frac{d+\beta}{2}} e^{-\frac{|x-y|^2}{2s}} |b(t+s, y)|^2 dy ds = 0.$$  

(ii) For any $T > 0$, there exist constants $p > d$ and $M > 0$ depending on $T$ such that

$$\sup_{t \in [0, T], x \in \mathbb{R}^d} \int_0^{r^2} \int_{B(x, r)} |b(t+s, y)|^2 dy ds \leq Mr^p, \ \forall 0 < r < 1,$$  

where $B(x, r)$ is the ball in $\mathbb{R}^d$ centered at $x$ with radius $r$.  

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It is easy to see that all bounded measurable functions satisfy (2.1) with \( p = d + 2 \), and that the \( L^{p_1}\)-\( L^{p_2} \) class functions with \( p_1, p_2 \geq 2 \) and \( \frac{2}{p_1} + \frac{d}{p_2} < 1 \) (Krylov-Röckner condition in \([11]\)) satisfy (2.1) with \( p = d + 2(1 - \frac{1}{p_1} - \frac{d}{p_2}) \). Now we give some examples of functions satisfying (2.1), which are not even contained in the critical case of \( L_p\)-\( L_q \) conditions.

**Example 2.3** Given \( \alpha_i > -\frac{1}{2} \) for each \( 1 \leq i \leq d \). Set \( f(x_1, x_2, \ldots, x_d) := \prod_{1 \leq i \leq d} (|x_i| \wedge 1)\alpha_i \).

If \( \sum_{1 \leq i \leq d} \alpha_i > -1 \), then \( f \) satisfies (2.1) with \( p = d + 2 + \sum_{1 \leq i \leq d} \alpha_i \). However, if \( \alpha_i \leq -\frac{1}{q} \) for some \( 1 \leq i \leq d \) and \( q > 2 \), then \( f \notin L_{loc}^q(\mathbb{R}^d) \).

**Example 2.4** Given \( \alpha \in (0, \frac{1}{2}) \). Set \( r_0 = 0 \) and \( r_i := i^{-\frac{\alpha}{2q}} \) for each \( 1 \leq i \leq d \). Let \( x_n := (2 \sum_{1 \leq i \leq n-1} r_i + r_n, 0, \ldots, 0) \in \mathbb{R}^d \). Define \( f(x) := \sum_{n \geq 1} |x - x_n|^{\alpha - 1} I_{B(x_n,r_n)}(x) \). Then one can verify that \( f \) satisfies (2.1) with \( p = d + 2\alpha \). However, \( f \notin L_{loc}^d(\mathbb{R}^d) \).

**Example 2.5** Let \( \alpha \in (-\frac{1}{2}, -\frac{1}{2d}] \) and \( g_i \) be a bounded measurable function on \( \mathbb{R} \) for each \( 2 \leq i \leq d \). Define

\[
   f(t, x_1, x_2, \ldots, x_d) := t^{-\frac{\alpha}{4}} (|x_1| \wedge 1)^\alpha \prod_{2 \leq i \leq d} g_i(x_i).
\]

Then \( f \) satisfies (2.1) with \( p = d + 2\alpha + 1 \). However, \( f \notin L^{p_1}((0,1); L^{p_2}(B(0,1))) \) for any \( p_1, p_2 \) with \( \frac{2}{p_1} + \frac{d}{p_2} \leq 1 \).

### 3 Time dependent Kato class functions

In this section, we will provide equivalent/sufficient conditions for functions to be in the Kato Class and obtain some important properties for the maximal functions. Throughout, for a measurable function \( f \) defined on \([0,T] \times \mathbb{R}^d \), we will let \( f \) vanish outside of \([0,T] \times \mathbb{R}^d \) when there is no danger of confusion. The letter \( c \) with or without subscripts stand for an unimportant positive constant, whose value may be different in different places.

First of all, using Kolomogrov-Chapman equation for the transition function of a Brownian motion it is easy to see that the following Lemma holds (see \([8]\) Proposition 1):

**Lemma 3.1** Assume \( f \in K_{d,2-\alpha} \) for some \( \alpha \in [0,2) \). Then for any \( \lambda, T > 0 \),

\[
   \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} s^{-d+\alpha} e^{-\frac{\lambda|x|}{2s}} |f(t+s,y)| dy ds < \infty.
\]

The following result is contained in \([3]\) Lemma 2.3.

**Lemma 3.2** Given a measure \( \nu \) on \( \mathbb{R}^d \) and positive constants \( \delta \) and \( r \). Then for any positive integer \( N \) with \( \frac{\delta}{N} \leq r \), we have

\[
   \sup_{x \in \mathbb{R}^d} \int_{|x-y| \geq r} e^{-\delta|x-y|^2} \nu(dy) \leq \sup_{z \in \mathbb{R}^d} \nu(B(z, \frac{1}{N})) \sum_{k \in \mathbb{Z}^d; |k| \geq 2N \delta-2} e^{-\frac{\delta^2 |k|^2}{2N}}.
\]

\[
   \sup_{x \in \mathbb{R}^d} \int \mathbb{R}^d e^{-\delta|x-y|^2} \nu(dy) \leq \sup_{z \in \mathbb{R}^d} \nu(B(z,1)) e^{\delta} \left( \sum_{m=0}^{\infty} e^{-\frac{\delta m^2}{N}} \right)^d.
\]
Given a measurable function \( f : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R} \), we introduce the following hypothesis:

**Hypothesis 3.1** (\( H_p \))  There exist constants \( p > d \) and \( M_1 > 0 \) such that

\[
\sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^r \int_{B(x,r)} |f(t + s, y)| \, dy \, ds \leq M_1 r^p, \quad \forall 0 < r < 1.
\]

**Lemma 3.3**  Assume \( f \) satisfies the Hypothesis \( H_p \). Then there exists \( M_2 > 0 \) such that for any \( 0 < r < 2 \),

\[
\sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^T \int_{B(x,r)} |f(t + s, y)| \, dy \, ds \leq M_2 T^{\frac{p-d}{d}} r^d, \quad \forall T \leq r^2, \quad (3.1)
\]

\[
\sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^T \int_{B(x,r)} |f(t + s, y)| \, dy \, ds \leq M_2 T^{r^2-p^2}, \quad \forall T > r^2. \quad (3.2)
\]

**Proof:** Let \( r \in (0, 2) \). If \( T \leq r^2 \), then \( T \in (2^{-2n}r^2, 2^{-2n+2}r^2] \) for some \( n \geq 1 \). Thus,

\[
\int_0^T \int_{B(x,r)} |f(t + s, y)| \, dy \, ds \leq \int_0^{2^{-2n}r^2} \int_{B(x,r)} |f(t + s, y)| \, dy \, ds
\]

\[
\leq 4 \sup_{t \geq 0} \int_0^{2^{-2n}r^2} \int_{B(x,r)} |f(t + s, y)| \, dy \, ds
\]

\[
\leq c 2^{nd} \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^{2^{-2n}r^2} \int_{B(x,2^{-n}r)} |f(t + s, y)| \, dy \, ds
\]

\[
\leq c 2^{nd} 2^{-np} r^p \leq c T^{\frac{p-d}{d}} r^d.
\]

If \( T > r^2 \), then \( T \in (nr^2, (n + 1)r^2] \) for some \( n \geq 1 \). Thus,

\[
\int_0^T \int_{B(x,r)} |f(t + s, y)| \, dy \, ds \leq \int_0^{(n+1)r^2} \int_{B(x,r)} |f(t + s, y)| \, dy \, ds
\]

\[
\leq c(n + 1) \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^{r^2} \int_{B(x,\frac{T}{n})} |f(t + s, y)| \, dy \, ds
\]

\[
\leq c(n + 1) r^p \leq c T r^{p-2}.
\]

Now we show that the Hypothesis \( H_p \) is an equivalent condition for the Kato class.

**Proposition 3.2**  (i) Assume there exists a constant \( p > d \) such that

\[
\sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^1 \int_{\mathbb{R}^d} s^{-\frac{d}{2}} e^{-\frac{|s|^2}{2s}} |f(t + s, y)| \, dy \, ds < \infty. \quad (3.3)
\]

Then \( f \) satisfies Hypothesis \( H_p \).

(ii) If \( f \) satisfies Hypothesis \( H_p \), then \( f \in K_{d,2-\alpha} \) for any \( \alpha \in [0, p-d) \).
Proof: First we show (i). Assume (3.3) holds, then for $0 < r < 1$, $t \geq 0$ and $x \in \mathbb{R}^d$,

$$
\int_0^r \int_{\mathbb{R}^d} s^{\frac{d}{2}} e^{-\frac{|x-y|^2}{2s}} |f(t+s,y)| dy ds \geq \int_0^r \int_{|x-y|<r} s^{\frac{d}{2}} e^{-\frac{|x-y|^2}{2s}} |f(t+s,y)| dy ds
$$

$$\geq \int_0^r \int_{|x-y|<r} s^{\frac{d}{2}} e^{-\frac{|x-y|^2}{2s}} |f(t+s,y)| dy ds
$$

$$\geq cr^{-p} \int_0^r \int_{|x-y|<r} |f(t+s,y)| dy ds.
$$

Therefore for any $0 < r < 1$,

$$\sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^r \int_{|x-y|<r} |f(t+s,y)| dy ds \leq cr^p,$$

which yields that

$$
\int_0^r \int_{|x-y|<r} |f(t+s,y)| dy ds \leq \sum_{n \geq 0} \int_0^{2^{-n-1}r} \int_{|x-y|<r} |f(t+s,y)| dy ds
$$

$$\leq c \sum_{n \geq 0} 2^{\frac{n(d-p)}{2}} \sup_{x \in \mathbb{R}^d} \int_0^{2^{-n-1}r} \int_{|x-y|<2^{-\frac{n}{2}}r} |f(t+s,y)| dy ds
$$

$$\leq c \sum_{n \geq 0} 2^{\frac{n(d-p)}{2}} r^p \leq cr^p.
$$

Hence $f$ satisfies the Hypothesis $H_p$.

Now we show (ii). Assume $H_p$ holds, and fix a positive constant $\alpha < p - d$. Then the following limits hold.

$$\lim_{r \to 0} \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^r \int_{B(x,1)} |f(t+s,y)| dy ds = 0, \quad (3.4)$$

$$\lim_{r \to 0} \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^r \int_{B(x,r) \setminus B(x,\sqrt{s})} |x - y|^{-d-\alpha} |f(t+s,y)| dy ds = 0, \quad (3.5)$$

$$\lim_{r \to 0} \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^r \int_{B(x,\sqrt{s})} s^{-\frac{d+n}{2}} |f(t+s,y)| dy ds = 0. \quad (3.6)$$

(3.4) is a direct consequence of (3.1).
Let us prove (3.5). Take $\alpha < \beta_1 < p - d$ and $d + \alpha - p + 2 < \beta_2 < 2$. Then we have

\[
\int_0^r \left( \int_{B(x, r) \setminus B(x, \sqrt{s})} |x-y|^{-d-\alpha} f(t+s, y) \right) dy ds
\]

\[
= \sum_{k \geq 0} \left( \sum_{n \geq 0} \int_{2^{-k-1} r}^{2^{-k} r} f(t+s, y) \right) dy ds
\]

\[
\leq \sum_{k \geq 0} \left( \sum_{n \geq 0} \int_{2^{-k-1} r}^{2^{-k} r} f(t+s, y) \right) dy ds
\]

Note that for any $\beta \in \mathbb{R}$, $|x-y|^\beta \leq c 2^{-n_d} r^\beta$ if $y \in B(x, 2^{-k} r) \setminus B(x, 2^{-k+1} r)$. Thus by Lemma 3.3

\[
\int_0^r \left( \int_{B(x, r) \setminus B(x, \sqrt{s})} |x-y|^{-d-\alpha} f(t+s, y) \right) dy ds
\]

\[
\leq c \sum_{k \geq 0} \left( \sum_{n \geq 0} 2^{\beta_1 (k+1)} r^{-\beta_1} r^{-n(d_1-n-\alpha)} r^{\beta_1 - d_\alpha} \int_{2^{-k-1} r}^{2^{-k} r} f(t+s, y) \right) dy ds
\]

\[
+ c \sum_{k \geq 0} \left( \sum_{n \geq 0} 2^{\beta_2 (k+1)} r^{-\beta_2} r^{-n(d_2-n-\alpha)} r^{\beta_2 - d_\alpha} \int_{2^{-k-1} r}^{2^{-k} r} f(t+s, y) \right) dy ds
\]

\[
\leq c \sum_{k \geq 0} \left( \sum_{n \geq 0} 2^{\beta_1 (k+1)} r^{-\beta_1} r^{-n(d_1-n-\alpha)} r^{\beta_1 - d_\alpha} \int_{2^{-k-1} r}^{2^{-k} r} f(t+s, y) \right) dy ds
\]

\[
+ c \sum_{k \geq 0} \left( \sum_{n \geq 0} 2^{\beta_2 (k+1)} r^{-\beta_2} r^{-n(d_2-n-\alpha)} r^{\beta_2 - d_\alpha} \int_{2^{-k-1} r}^{2^{-k} r} f(t+s, y) \right) dy ds
\]

\[
\leq c r^{\beta_1 - d_\alpha} \sum_{k \geq 0} \left( \sum_{n \geq 0} 2^{\beta_1 (k+1)} r^{-n(d_1-n-\alpha)} \right)
\]

\[
+ c r^{\beta_2 - d_\alpha} \sum_{k \geq 0} \left( \sum_{n \geq 0} 2^{\beta_2 (k+1)} r^{-n(d_2-n-\alpha)} \right)
\]

\[
\leq c r^{\beta_1 - d_\alpha} \sum_{k \geq 0} \left( \sum_{n \geq 0} 2^{\beta_1 (k+1)} r^{-n(d_1-n-\alpha)} \right)
\]

\[
+ c r^{\beta_2 - d_\alpha} \sum_{k \geq 0} \left( \sum_{n \geq 0} 2^{\beta_2 (k+1)} r^{-n(d_2-n-\alpha)} \right)
\]

\[
\leq c r^{\beta_1 - d_\alpha} \sum_{k \geq 0} \left( \sum_{n \geq 0} 2^{\beta_1 (k+1)} r^{-n(d_1-n-\alpha)} \right)
\]

\[
+ c r^{\beta_2 - d_\alpha} \sum_{k \geq 0} \left( \sum_{n \geq 0} 2^{\beta_2 (k+1)} r^{-n(d_2-n-\alpha)} \right)
\]

Hence (3.5) follows.
(3.6) follows from the following bound:

\[
\begin{align*}
\int_0^r s^{-\frac{d+n}{2}} \int_{B(x, \sqrt{s})} |f(t + s, y)| dy ds \\
&= \sum_{n \geq 0} \int_{B(x, \sqrt{s})} |f(t + s, y)| dy ds \\
&\leq \sum_{n \geq 0} 2^{(n+1)(d+n+1)} \left( \frac{1}{2} \right)^{d-\alpha} \int_{B(x, \sqrt{s})} |f(t + s, y)| dy ds \\
&\leq M_2 \sum_{n \geq 0} 2^{(n+1)(d+n+1)} \left( \frac{1}{2} \right)^{d-\alpha} 2^{-\frac{n(d+n+1)}{2}} \frac{p-d-\frac{4}{d}}{2} \\
&= cr^{p-d-\alpha} \sum_{n \geq 0} 2^{\frac{n(d+n+1)}{2}}.
\end{align*}
\]

Finally we show that \( f \in K_{d,2-\alpha} \). By Lemma 3.2 we have that for any \( t, T, r, \delta > 0 \)

\[
\begin{align*}
&\int_0^T \int_{|x-y| \geq r} e^{-\delta |x-y|^2} |f(t + s, y)| dy ds \\
&\leq \sup_{z \in \mathbb{R}^d} \int_0^T \int_{|x-y| \leq \frac{1}{2r}} |f(t + s, y)| dy ds \left( \sum_{-\infty < n < \infty} e^{-\frac{\delta n^2}{16x^2}} \right)^d,
\end{align*}
\]

where \( N \) is some positive integer larger than \( \frac{3}{7} \). Therefore together with (3.4) we get

\[
\lim_{T \to 0} \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^T \int_{|x-y| \geq r} e^{-\delta |x-y|^2} |f(t + s, y)| dy ds = 0. \tag{3.7}
\]

Taking \( 0 < T \leq r^2 < 1 \) and \( \lambda > 0 \), then we have

\[
\begin{align*}
&\sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} e^{-\frac{\lambda |x-y|^2}{2s}} |f(t + s, y)| dy ds \\
&\leq \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^r \int_{|x-y| < \sqrt{s}} e^{-\frac{\lambda |x-y|^2}{2s}} |f(t + s, y)| dy ds \\
&\quad + \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^r \int_{\sqrt{s} \leq |x-y| < r} e^{-\frac{\lambda |x-y|^2}{2s}} |f(t + s, y)| dy ds \\
&\quad + \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^r \int_{|x-y| \geq r} e^{-\frac{\lambda |x-y|^2}{2s}} |f(t + s, y)| dy ds \\
&\quad + \sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^r \int_{|x-y| \geq r} e^{-\frac{\lambda |x-y|^2}{4}} |f(t + s, y)| dy ds.
\end{align*}
\]
First letting $T \to 0$ and then $r \to 0$, by \([3.1]-(3.7)\), it follows that

\[
\lim_{T \to 0} \sup_{t \geq 0, s \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} s^{-d+\frac{3}{2}} e^{-\frac{|x-y|^2}{2s}} |f(t + s, y)| \mathrm{dy} \mathrm{ds} = 0,
\]

By a similar argument we also have

\[
\lim_{T \to 0} \sup_{t \geq 0, s \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} s^{-d+\frac{3}{2}} e^{-\frac{|x-y|^2}{2s}} |f(t - s, y)| \mathrm{dy} \mathrm{ds} = 0,
\]

We thus have proved that $f \in \mathcal{K}_{d,2-\alpha}$.

Combining with Proposition \([3.2]\) we obtain the following result.

**Corollary 3.3** A measurable function $f$ satisfies Hypothesis $H_p$ if and only if $f \in \mathcal{K}_{d,\alpha}$ for some $\alpha \in (0, 2)$.

Next result shows that if $|f|^2$ belongs to some Kato class, $|f|$ is also in some Kato class with a different index.

**Lemma 3.4** Given a measurable function $f$. If $|f|^2 \in \mathcal{K}_{d,\alpha}$ for some $\alpha \in (0, 2)$, then $f \in \mathcal{K}_{d,1-\beta}$ for any $\beta \in [0, \frac{2-\alpha}{2})$.

**Proof:** Let $\beta \in [0, \frac{2-\alpha}{2})$. By Hölder’s inequality we have

\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^d} s^{-\frac{d+1+\beta}{2}} e^{-\frac{|x-y|^2}{2s}} |f(t, y)| \mathrm{dy} \mathrm{ds} &\leq (\int_0^T \int_{\mathbb{R}^d} s^{-\frac{d+2-\alpha}{2}} e^{-\frac{|x-y|^2}{2s}} |f(t, y)|^2 \mathrm{dy} \mathrm{ds})^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^d} s^{-\frac{d+2+\beta}{2}} e^{-\frac{|x-y|^2}{2s}} \mathrm{dy} \mathrm{ds}\right)^{\frac{1}{2}} \\
&\leq c \left(\int_0^T \int_{\mathbb{R}^d} s^{-\frac{d+2-\alpha}{2}} e^{-\frac{|x-y|^2}{2s}} |f(t, y)|^2 \mathrm{dy} \mathrm{ds}\right)^{\frac{1}{2}}.
\end{align*}
\]

This yields that $f \in \mathcal{K}_{d,1-\beta}$ according to the condition on $|f|^2$.

Next we will prove some properties of the local Hardy-Littlewood maximal functions which will be used later. For a measurable function $f : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}^n$ and $R > 0$, let $M_R f(t, x)$ denote the local Hardy-Littlewood maximal function of $f$ given by

\[
M_R f(t, x) := \sup_{0 < \delta \leq R} \frac{1}{m(B(x, \delta))} \int_{B(x, \delta)} |f(t, y)| \mathrm{dy},
\]

where $m(B(x, \delta))$ stands for the volume of the ball $B(x, \delta)$. $M_\infty f$ is the maximal function of $f$ defined in \([12]\). Here is the result.

**Lemma 3.5** Assume $|f|^2$ satisfies the Hypothesis $H_p$. Then for any $R > 0$, $|M_R f|^2$ satisfies the Hypothesis $H_q$ for any $q \in (d, p)$.

**Proof:** Fix $(t, x_0) \in [0, \infty) \times \mathbb{R}^d$, $R > 0$ and $0 < r < 1$. Set $f_1(s, x) := f(s, x)I_{|f| \geq \frac{\alpha}{4}}I_{|x-x_0| < 2r}$ and $f_2(s, x) := f(s, x)I_{|f| \geq \frac{\alpha}{4}}I_{|x-x_0| \geq 2r}$. Then

\[
\{|M_R f > \alpha\} \subset \{M_R f_1 > \frac{\alpha}{4}\} \cup \{M_R f_2 > \frac{\alpha}{4}\}.
\]
Therefore,

\[
\int_0^r \int_{B(x_0,r)} M_R f(t + s, y)^2 dy ds = 2 \int_0^r \int_{B(x_0,r)} \int_0^{M_R f(t+s,y)} \alpha d\alpha dy ds
\]

\[
\leq 2 \int_0^r \int_{B(x_0,r)} \int_0^{M_{Rf_1} > \frac{\alpha}{4}} dy + 2 \int_0^r \int_{B(x_0,r)} \int_0^{M_{Rf_2} > \frac{\alpha}{4}} dy
\]

\[
\leq 2 \int_0^r \int_{\mathbb{R}^d} dy \int_0^{M_{Rf_1}} \alpha d\alpha + 2 \int_0^r \int_{\mathbb{R}^d} dy \int_0^{M_{Rf_2}} \alpha d\alpha
\]

\[
\leq 16 \int_0^r \int_{\mathbb{R}^d} |M_{Rf_1}|^2 dy + 2 \int_0^r \int_{\mathbb{R}^d} dy \int_0^{M_{Rf_2}} \alpha d\alpha := I_1 + I_2.
\]

For the term $I_1$, by [12, Theorem 1 in section 1], we have

\[
\int_{\mathbb{R}^d} |M_{Rf_1}|^2 dy \leq \int_{\mathbb{R}^d} |M_{\infty f_1}|^2 dy
\]

\[
\leq c \int_{\mathbb{R}^d} |f_1|^2 dy \leq c \int_{B(x_0,2r)} |f|^2 dy.
\]

Hence by Lemma 3.3,

\[
I_1 \leq c \int_0^r ds \int_{B(x_0,2r)} |f|^2 dy \leq cr^p.
\]

Now we turn to the term $I_2$. For $y \in B(x_0, r)$, if $\delta < r$, by the definition of $f_2$, we see that $\int_{B(y, \delta)} |f_2(s, z)| dz = 0$. If $r \leq \delta \leq R$, then $B(y, \delta) \subset B(x_0, r + \delta)$. Thus, for $y \in B(x_0, r)$,

\[
M_R f_2(t + s, y) = \sup_{r \leq \delta \leq R} \frac{1}{m(B(y, \delta))} \int_{B(y, \delta)} |f_2(t + s, z)| dz
\]

\[
\leq \sup_{r \leq \delta \leq R} \frac{2\alpha^{-1}}{m(B(y, \delta))} \int_{B(x_0, r + \delta) \setminus B(x_0, r)} |f(t + s, z)|^2 dz
\]

\[
\leq \sup_{r \leq \delta \leq R} \frac{2\alpha^{-1}}{m(B(y, \delta))} (r + \delta)^d \int_{B(x_0, r + \delta) \setminus B(x_0, r)} |z - x_0|^{-d} |f(t + s, z)|^2 dz
\]

\[
\leq c \alpha^{-1} \sup_{r \leq \delta \leq R} \left( \frac{r}{\delta} + 1 \right)^d \int_{B(x_0, r + R) \setminus B(x_0, r)} |z - x_0|^{-d} |f(t + s, z)|^2 dz
\]

\[
\leq c \alpha^{-1} \int_{B(x_0, r + R) \setminus B(x_0, r)} |z - x_0|^{-d} |f(t + s, z)|^2 dz := c\alpha^{-1} m_{t+s}.
\]
It follows that if \( \alpha > 2\sqrt{cm_{t+s}} \), then \( B(x_0, r) \cap \{ M_{R} f_2 > \frac{\alpha}{2} \} = \emptyset \). Hence we have

\[
I_2 = \int_0^2 ds \int_0^\infty \alpha \sigma \int_{B(x_0,r) \cap \{ M_{R} f_2 > \frac{\alpha}{2} \}} dy \\
\leq \int_0^2 ds \int_0^{\frac{2}{\alpha} \sigma m_{t+s}} \alpha \sigma \int_{B(x_0,r)} dy \\
\leq cr^d \int_0^2 m_{t+s} ds.
\]  

(3.10)

On the other hand, for any \( 0 < \beta < p - d \), by Lemma 3.3 we have

\[
\int_0^2 m_{t+s} ds = \int_0^2 \int_{B(x_0,r+R) \setminus B(x_0,r)} |x - y|^{-d} |f(t + s, y)|^2 dy ds \\
\leq r^{-\beta} \sum_{n \geq 0} \int_0^2 \int_{B(x_0,2^{-n}(r+R)) \setminus B(x_0,r)} |x - y|^{-d} |f(t + s, y)|^2 dy ds \\
\leq r^{-\beta} \sum_{n \geq 0} 2^{(n+1)(d-\beta)} (r + R)^{\beta - d} \int_0^2 \int_{B(x_0,2^{-n}(r+R)) \setminus B(x_0,r)} |f(t + s, y)|^2 dy ds \\
\leq cr^{-\beta} \sum_{n \geq 0} 2^{(n+1)(d-\beta)} (r + R)^{\beta - d} |r - d| 2^{-n(d)} (r + R)^d \leq c(r + R)^{\beta - d} r^{-d - \beta}.
\]  

(3.11)

Putting together (3.10) and (3.11) we obtain

\[
I_2 \leq cr^{p - d - \beta}.
\]  

(3.12)

Combining (3.8), (3.9) and (3.12), we find that \( |M_R f|^2 \) satisfies the Hypothesis \( H_q \) for any \( q \in (d, p) \).

Following the same argument as in the proof of [19, Lemma 3.5], we have the following Lemma:

Lemma 3.6 Let \( f \in W^{0,1}_{1,loc}((0, T) \times \mathbb{R}^d) \). Then there exists a \( dt \times dx \)-null set \( A \subset (0, T) \times \mathbb{R}^d \) such that for any \( R > 0 \) and \( (t, x), (t, y) \in (0, T) \times \mathbb{R}^d \setminus A \) with \( |x - y| \leq R \),

\[
|f(t, x) - f(t, y)| \leq 2^d |x - y| (M_R |\nabla_x f|)(t, x) + M_R |\nabla_x f|)(t, y).
\]

4 Kolmogorov equations associated with SDEs

In this section, we study the Kolmogorov equations associated with SDEs (1.1). We will provide the regularity results for the solutions which will be used in next section. Given a vector valued function \( f : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^d \), consider the following backward second order parabolic equation:

\[
\begin{dcases}
\partial_t u(t, x) + \frac{1}{2} \Delta u(t, x) + \langle b(t, x), \nabla_x u(t, x) \rangle = f(t, x), \quad \forall (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(T, x) = 0, \quad \forall x \in \mathbb{R}^d.
\end{dcases}
\]  

(4.1)

The following is the definition of the mild solution to equation (4.1):
where \( h(t, x, y) \) is the transition density function of the Brownian motion.

Fix a non-negative function \( \varphi \in C^\infty_0((0, 1) \times B(0, 1)) \) with \( \int_{R^{d+1}} \varphi(t, x)dxdt = 1 \). For any positive integer \( n \), we put \( \varphi_n(t, x) := 2^{nd+n} \varphi(2^n t, 2^n x) \).

For \( \lambda, T > 0 \) and \( h \in K_{d,1} \), define

\[
\mathcal{N}_h^\lambda(T) := \sup_{t \geq 0, x \in R^d} \int_t^{t+T} \int_{R^d} (s - t) - \frac{d+1}{2} e^{-\frac{\lambda|x-y|^2}{4(s-t)^{-d}}} |h(s, y)|dyds.
\]

We need the following Lemma.

**Lemma 4.1** Given \( h \in K_{d,1} \) and \( T > 0 \). Let \( \{f_n\}_{n \geq 1} \) be a sequence of Borel measurable functions satisfying \( \sup_{n \geq 1} \|f_n\|_{L^\infty} < \infty \) and

\[
\lim_{n \to \infty} \sup_{s, |z| \in [0,2^{-n}]} \sup_{(t,x) \in [0,T-s] \times R^d} |f_n(t + s, x + z) - f_n(t, x)| = 0. \tag{4.2}
\]

Then for any \( \lambda, R > 0 \),

\[
\sup_{n \geq 1} \mathcal{N}_{h_n}^\lambda(T) \leq \mathcal{N}_h^\lambda(T), \tag{4.3}
\]

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T, x \in B(0,R)} \int_t^T \int_{R^d} \nabla_x q(s - t, x, y) f_n(s, y) (h_n(s, y) - h(s, y))dyds = 0, \tag{4.4}
\]

where \( h_n(t, x) := \int_{R^{d+1}} \varphi_n(t - s, x - y)h(s, y)dyds \).

**Proof:** Using the Fubini’s theorem, we have

\[
\int_t^{t+T} \int_{R^d} (s - t) - \frac{d+1}{2} e^{-\frac{\lambda|x-y|^2}{4(s-t)^{-d}}} |h_n(s, y)|dyds \\
\leq \int_t^{t+T} \int_{R^d} \varphi_n(\tau, z) |h(\tau - t, y - z)|dzd\tau dyds \\
= \int_{R^{d+1}} \varphi_n(\tau, z)dzd\tau \int_t^{t+T} \int_{R^d} (s - t) - \frac{d+1}{2} e^{-\frac{\lambda|x-y|^2}{4(s-t)^{-d}}} |h(s - \tau, y - z)|dyds \\
= \int_{R^{d+1}} \varphi_n(\tau, z)dzd\tau \int_{t-\tau}^{t+T-\tau} \int_{R^d} (s + \tau - t) - \frac{d+1}{2} e^{-\frac{\lambda|x-y|^2}{4(s-t)^{-d}}} |h(s, y)|dyds \\
\leq \mathcal{N}_h^\lambda(T),
\]

which is \( \Box \).

To show \( \Box \), we first prove that for any \( \delta \in (0, T) \),

\[
\lim_{n \to \infty} \sup_{t \in [0,T-\delta], x \in B(0,R)} \int_{T-2^{-n}}^T \int_{R^d} |\nabla_x q(s - t, x, y)||h(s, y)|dyds = 0. \tag{4.5}
\]
For $r > 0$ and sufficiently large integer $n$, we have

$$
\sup_{t \in [0, T - \delta], x \in B(0, R)} \int_{T - 2n}^{T} \int_{\mathbb{R}^d} |\nabla_x q(s - t, x, y)| |h(s, y)| dy ds \\
\leq \sup_{t \in [0, T - \delta], x \in B(0, R)} \int_{T - 2n}^{T} \int_{B(0, R + r)} |\nabla_x q(s - t, x, y)| |h(s, y)| dy ds \\
+ \sup_{t \in [0, T - \delta], x \in B(0, R)} \int_{T - 2n}^{T} \int_{B(0, R + r)^c} |\nabla_x q(s - t, x, y)| |h(s, y)| dy ds \\
\leq c(\delta - 2^{-n})^{-\frac{d+1}{2}} \int_{T - 2n}^{T} \int_{B(0, R + r)} |h(s, y)| dy ds \\
+ c \sup_{t \in [0, T - \delta], x \in B(0, R)} \int_{T - 2n}^{T} \int_{B(0, R + r)^c} (s - t)^{-\frac{d+1}{2}} e^{-\frac{|x-y|^2}{8(s-t)}} |h(s, y)| dy ds \\
\leq c(\delta - 2^{-n})^{-\frac{d+1}{2}} \int_{T - 2n}^{T} \int_{B(0, R + r)} |h(s, y)| dy ds \\
+ c e^{-\frac{2r^2}{\delta}} \sup_{t \in [0, T - \delta], x \in B(0, R)} \int_{T - 2n}^{T} \int_{B(0, R + r)^c} (s - t)^{-\frac{d+1}{2}} e^{-\frac{|x-y|^2}{8(s-t)}} |h(s, y)| dy ds \\
\leq c(\delta - 2^{-n})^{-\frac{d+1}{2}} \int_{T - 2n}^{T} \int_{B(0, R + r)} |h(s, y)| dy ds \\
+ c e^{-\frac{2r^2}{\delta}} \sup_{t \in [0, T - \delta], x \in B(0, R)} t \int_{\mathbb{R}^d} (s - t)^{-\frac{d+1}{2}} e^{-\frac{|x-y|^2}{8(s-t)}} |h(s, y)| dy ds.
$$

From the definition of the Kato class and Lemma 2.1, we see that $|h(s, y)|$ is integrable on $(0, T) \times B(0, R + r)$ and

$$
\sup_{t \in [0, T - \delta], x \in B(0, R)} \int_{t}^{T} \int_{\mathbb{R}^d} (s - t)^{-\frac{d+1}{2}} e^{-\frac{|x-y|^2}{8(s-t)}} |h(s, y)| dy ds < \infty.
$$

Now first letting $n \to \infty$ and then letting $r \to \infty$ in (4.6), we obtain (4.5).

Now we show (4.4). Note that for $\alpha \in (0, 1)$, there exists constants $C_1 > 0$ and $0 < C_2 \leq 1$ such that for $0 < t_1 < t_2 \leq T$,

$$
|\nabla_x q(t_1, x_1, y) - \nabla_x q(t_2, x_2, y)| \\
\leq C_1 |x_1 - x_2|^{\alpha t_1^{-\frac{d+1+\alpha}{2}} e^{-\frac{C_2|y|^2}{2t_1}} + e^{-\frac{C_2|x-y|^2}{2t_1}}} + C_1 |t_1 - t_2|^{\alpha t_1^{-\frac{d+1+\alpha}{2}} e^{-\frac{C_2|y|^2}{2t_1}}}.
$$

Take $\delta \in (0, T)$ and a large positive integer $N$ so that $2^{-N} < \delta$. For $t \in [0, T - \delta]$, since
supp \( \varphi_n \subset (0, 2^{-n}) \times B(0, 2^{-n}) \), we have for \( n \geq N \),

\[
\left| \int_t^T \int_{\mathbb{R}^d} \nabla_x q(s-t, x, y) f_n(s, y) (h_n(s, y) - h(s, y)) dy ds \right| \\
\leq \| f_n \|_{L^\infty} \int_t^{t+\delta} \int_{\mathbb{R}^d} |\nabla_x q(s-t, x, y)|||h_n(s, y)| + |h(s, y)|| dy ds \\
+ \| f_n \|_{L^\infty} \int_{\mathbb{R}^{d+1}} \varphi_n(\tau, z) d\tau d\tau \int_{\mathbb{R}^d} |\nabla_x q(s-t + \tau, x-z, y) - \nabla_x q(s-t, x, y)|||h(s, y)|| dy ds \\
+ \int_{\mathbb{R}^{d+1}} \varphi_n(\tau, z) d\tau d\tau \int_{\mathbb{R}^d} |\nabla_x q(s-t, x, y)||f_n(s+\tau, y+z) - f_n(s, y)||h(s, y)|| dy ds \\
+ \| f_n \|_{L^\infty} \sup_{\tau \in (0, 2^{-n})} \int_{\mathbb{R}^d} |\nabla_x q(s-t, x, y)||h(s, y)|| dy ds \\
+ \| f_n \|_{L^\infty} \sup_{\tau \in (0, 2^{-n})} \int_{T-\tau}^{T} \int_{\mathbb{R}^d} |\nabla_x q(s-t, x, y)||h(s, y)|| dy ds \\
\leq \mathcal{N}_h^1(\delta) + c \int_{\mathbb{R}^{d+1}} \varphi_n(\tau, z) d\tau d\tau \int_{\mathbb{R}^d} |z|^\alpha(s-t) - \frac{d+1+\alpha}{2} (e^{-\frac{C_2|x-y|^2}{2(t-s)}} + e^{-\frac{C_2|x-y|^2}{2(t-s)}}) \\
+ \tau^\alpha(s-t) - \frac{d+1+2\alpha}{2} e^{-\frac{C_2|x-y|^2}{2(t-s)}} \||h(s, y)|| dy ds \\
+ \int_{\mathbb{R}^{d+1}} \varphi_n(\tau, z) d\tau d\tau \int_{\mathbb{R}^d} |\nabla_x q(s-t, x, y)||f_n(s+\tau, y+z) - f_n(s, y)||h(s, y)|| dy ds \\
+ c \int_{T}^{T+\delta} \int_{\mathbb{R}^d} |\nabla_x q(s-t, x, y)||h(s, y)|| dy ds \\
+ c \int_{T-2^{-n}}^{T} \int_{\mathbb{R}^d} |\nabla_x q(s-t, x, y)||h(s, y)|| dy ds \\
\leq \mathcal{N}_h^1(\delta) + c 2^{-n} [\left( \delta - 2^{-n} \right) - \frac{d+1+\alpha}{2} + (\delta - 2^{-n} - \frac{d+1+2\alpha}{2})] \sup_{x \in \mathbb{R}^d} \int_{\delta-2^{-n}}^{T} \int_{\mathbb{R}^d} e^{-\frac{C_2|x-y|^2}{2(t-s)}} |h(s, y)|| dy ds \\
+ c \sup_{\tau, |z| \in (0, 2^{-n})} \sup_{(s, y) \in (0, T-\tau] \times \mathbb{R}^d} |f_n(s+\tau, y+z) - f_n(s, y)| \\
\times \sup_{t \in [0, T-\delta]} \int_{t}^{T} \int_{\mathbb{R}^d} |\nabla_x q(s-t, x, y)||h(s, y)|| dy ds \\
+ \sup_{t \in [0, T-\delta]} \int_{T-2^{-n}}^{T} \int_{\mathbb{R}^d} |\nabla_x q(s-t, x, y)||h(s, y)|| dy ds. \tag{4.8}
\]

On the other hand, for \( t \in [T-\delta, T] \),

\[
\left| \int_t^T \int_{\mathbb{R}^d} \nabla_x q(s-t, x, y) f_n(s, y) (h_n(s, y) - h(s, y)) dy ds \right| \\
\leq \| f_n \|_{L^\infty} \int_0^{T-t} \int_{\mathbb{R}^d} |\nabla_x q(s, x, y)|||h_n(s+t, y)| + |h(s+t, y)|| dy ds \leq \mathcal{N}_h^1(\delta). \tag{4.9}
\]
Lemma 4.1 and (4.3) together implies that for all \( t \in [0, T] \) and \( n \geq N \),
\[
| \int_t^T \int_{\mathbb{R}^d} \nabla_x q(s - t, x, y) f_n(s, y)(h_n(s, y) - h(s, y)) dy \, ds |
\leq \epsilon N^3(a) + c 2^{-n a} [ (\delta - 2^{-n})^{-\frac{d+1}{2}} + (\delta - 2^{-n})^{-\frac{d+1}{2}} ] \sup_{x \in \mathbb{R}^d} \sup_{\delta - 2^{-n}} \int_t^T \int_{\mathbb{R}^d} e^{-\frac{C_3|x-y|^2}{2T}} |h(s, y)| ds dy
\]
\[+ c \sup_{\tau, |z| \in [0, 2^{-n}]} \sup_{(s, y) \in [0, T - \tau] \times \mathbb{R}^d} |f_n(s + \tau, y + z) - f_n(s, y)|
\times \sup_{t \in [0, T - \delta]} \int_t^T \int_{\mathbb{R}^d} |\nabla_x q(s - t, x, y)||h(s, y)| dy ds
\]
\[+ \sup_{t \in [0, T - \delta]} \int_{T - 2^{-n}}^T \int_{\mathbb{R}^d} |\nabla_x q(s - t, x, y)||h(s, y)| dy ds.
\]
Since \( b \in K_{d, 1} \), \( \sup_{x \in \mathbb{R}^d} \int_{\delta - 2^{-n}}^T \int_{B(x, 1)} |h(s, y)| \, dy \, ds < \infty \). Combining with Lemma 3.2 we see that
\[
\sup_{x \in \mathbb{R}^d} \int_{\delta - 2^{-n}}^T \int_{\mathbb{R}^d} e^{-\frac{C_3|x-y|^2}{2T}} |h(s, y)| \, dy \, ds < \infty.
\]
Hence, by (4.2) and (4.5), first letting \( n \to \infty \) and then letting \( \delta \to 0 \) in (4.11), we obtain (4.4).

For a matrix \( A = (a_{ij})_{1 \leq i, j \leq d} \), define \( \|A\| := \sup_{1 \leq i, j \leq d} |a_{ij}| \). Introduce
\[
b_n(t, x) := \int_{\mathbb{R}^{d+1}} \varphi_n(t - s, x - y) b(s, y) dy \, ds,
\]
\[
f_n(t, x) := \int_{\mathbb{R}^{d+1}} \varphi_n(t - s, x - y) f(s, y) dy \, ds.
\]
For \( b, f \in K_{d, 1} \), it is easy to see that \( b_n \) and \( f_n \) are bounded, smooth functions. Thus by [4] Corollary VI.4.2 and Theorem VI.4.6], for any \( T > 0 \), there exists a unique solution \( u_n \in C^{1,2}([0, T] \times \mathbb{R}^d) \) to the following parabolic equations:
\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u_n(t, x) + \frac{1}{2} \Delta u_n(t, x) + \langle b_n(t, x), \nabla_x u_n(t, x) \rangle = f_n(t, x), \\
u_n(T, x) = 0,
\end{array} \right. \quad \forall (t, x) \in (0, T) \times \mathbb{R}^d,
\end{aligned}
\]
(4.11)
Furthermore, by [4] Theorem VI.4.5] and the boundedness of \( f_n \), it is easy to see that \( u_n \in C^{0,1}_b([0, T] \times \mathbb{R}^d) \).

Now we have the following result on the existence and uniqueness of mild solution to the equation (4.1):

**Proposition 4.2** Assume that \( |b| \) and \( |f| \) belong to the Kato class \( K_{d, 1} \). Then there exists a constant \( T > 0 \), which depend only on the functions \( N^1_b(\cdot) \) and \( N^1_f(\cdot) \), such that the following statements hold:
(i) there exists a unique mild solution \( u \in C^{0,1}_b([0, T] \times \mathbb{R}^d) \) to the equation (4.1),
(ii) for any \( R > 0 \),
\[
\sup_{n \geq 1} \|u_n\|_{C^{0,1}_b([0, T] \times \mathbb{R}^d)} \vee \|u\|_{C^{0,1}_b([0, T] \times \mathbb{R}^d)} \leq M_3 N^1_f(T),
\]
(4.12)

\[
\lim_{n \to \infty} \|u_n - u\|_{C^0_b([0,T] \times B(0,R))} = 0,
\]

where \(M_3\) is some positive constant depending only on \(d\) and \(N_1(T)\).

(iii) 
\[
|u(t, y) - u(t, x)| \leq \frac{1}{2}|x - y|, \quad \forall (t, x), (t, y) \in [0, T] \times \mathbb{R}^d.
\]

**Proof:** First we show (4.12). Assume \(u\) is a mild solution to equation (4.1). Then
\[
u(t, x) = \int_t^T \int_{\mathbb{R}^d} q(s - t, x, y)[\langle b(s, y), \nabla_x u(s, y) \rangle - f(s, y)] dyds.
\]

Hence
\[
|\nabla_x u(t, x)| \leq \int_t^T \int_{\mathbb{R}^d} |\nabla_x q(s - t, x, y)||\langle b(s, y), \nabla_x u(s, y) \rangle - f(s, y)|| dyds
\leq \int_t^T \int_{\mathbb{R}^d} |\nabla_x q(s - t, x, y)||b(s, y)|| dyds \sup_{(s, y) \in [0,T] \times \mathbb{R}^d} \|\nabla_x u(s, y)\| + \int_t^T \int_{\mathbb{R}^d} |\nabla_x q(s - t, x, y)||f(s, y)|| dyds
\leq C_0 N_1(T) \sup_{(s, y) \in [0,T] \times \mathbb{R}^d} \|\nabla_x u(s, y)\| + C_0 N_1(T),
\]

for some positive constant \(C_0\) depending only on \(d\). Since \(|b|, |f| \in K_{d,1}\), take \(T > 0\) sufficiently small so that
\[
C_0 N_1(T) \leq \frac{1}{2}, \quad C_0 N_1(T) \leq \frac{1}{2d}.
\]

Combining (4.14) with (4.15) yields that
\[
\sup_{(t, x) \in [0,T] \times \mathbb{R}^d} |\nabla_x u(t, x)| \leq 2C_0 N_1(T).
\]

By (4.14) and (4.16),
\[
\sup_{(t, x) \in [0,T] \times \mathbb{R}^d} |u(t, x)| = \int_t^T \int_{\mathbb{R}^d} q(s - t, x, y)[\langle b(s, y), \nabla_x u(s, y) \rangle - f(s, y)] dyds
\leq c N_1(T) N_1(T) + c N_1(T) \leq c N_1(T),
\]

\(C_1\) is some positive constant dependent only on \(d\) and \(N_1(T)\).

By Lemma 4.1 and (4.15), we have
\[
\sup_{n \geq 1} C_0 N_{b_n}^1(T) \leq C_0 N_1^1(T) \leq \frac{1}{2}.
\]

Following the proofs for (4.16) and (4.17), we have for \(n \geq 1\),
\[
\sup_{(t, x) \in [0,T] \times \mathbb{R}^d} |\nabla_x u_n(t, x)| \vee \sup_{(t, x) \in [0,T] \times \mathbb{R}^d} |u_n(t, x)| \leq (2C_0 \vee C_1) N_1^1(T).
\]

Hence (4.12) holds.
It is easy to see that the uniqueness of the mild solution to the equation (4.1) follows from (4.12). On the other hand, since $u_n$ is the solution to the equation (4.11), (4.13) would imply that $u$ is the unique mild solution to the equation (4.1). Therefore to prove (i) we only need to show that $\{u_n\}_{n \geq 1}$ is a Cauchy sequence in $C^0_b([0, T] \times B(0, R))$ for any $R > 0$.

Recall that $u_n$ is the mild solution to the equation (4.11) satisfying

$$u_n(t, x) = \int_t^T \int_{\mathbb{R}^d} q(s - t, x, y) \left[ \langle b_n(s, y), \nabla_x u_n(s, y) \rangle - f_n(s, y) \right] dy ds.$$  

For $m, n > 0$, by Lemma 4.1 and (4.18) we see that

$$|\nabla_x u_m(t, x) - \nabla_x u_n(t, x)| \leq \int_t^T \int_{\mathbb{R}^d} |\nabla_x q(s - t, x, y)||b_n(s, y)|dy ds \sup_{(s, y) \in [0, T] \times \mathbb{R}^d} |\nabla_x u_n(s, y) - \nabla_x u_m(s, y)|$$

$$+ \left| \int_t^T \int_{\mathbb{R}^d} \nabla_x q(s - t, x, y)\langle \nabla_x u_m(s, y), b_n(s, y) - b_m(s, y) \rangle dy ds \right|$$

$$+ \left| \int_t^T \int_{\mathbb{R}^d} \nabla_x q(s - t, x, y)(f_n(s, y) - f_m(s, y)) dy ds \right|$$

$$\leq \frac{1}{2} \sup_{(s, y) \in [0, T] \times \mathbb{R}^d} |\nabla_x u_n(s, y) - \nabla_x u_m(s, y)|$$

$$+ \left| \int_t^T \int_{\mathbb{R}^d} \nabla_x q(s - t, x, y)\langle \nabla_x u_m(s, y), b_n(s, y) - b_m(s, y) \rangle dy ds \right|$$

$$+ \left| \int_t^T \int_{\mathbb{R}^d} \nabla_x q(s - t, x, y)(f_n(s, y) - f_m(s, y)) dy ds \right|.$$  

Thus

$$|\nabla_x u_m(t, x) - \nabla_x u_n(t, x)| \leq 2\left| \int_t^T \int_{\mathbb{R}^d} \nabla_x q(s - t, x, y)\langle \nabla_x u_m(s, y), b_n(s, y) - b_m(s, y) \rangle dy ds \right|$$

$$+ 2\left| \int_t^T \int_{\mathbb{R}^d} \nabla_x q(s - t, x, y)(f_n(s, y) - f_m(s, y)) dy ds \right|.  \tag{4.19}$$

On the other hand, setting $F_n(s, y) := \langle b_n(s, y), \nabla_x u_n(s, y) \rangle - f_n(s, y)$, by (4.7), for
0 ≤ t₁ < t₂ ≤ T, x₁, x₂ ∈ R^d, δ ∈ (0, T) and α ∈ (0, 1), we have

\[ |\nabla_x u(t₁, x₁) - \nabla_x u(t₂, x₂)| \]

\[ \leq \int_{t₁}^{t₂} \int_{R^d} |\nabla_x q(s - t₁, x₁, y)||F_n(s, y)|dyds \]

\[ + \int_{t₂}^{T} \int_{R^d} |\nabla_x q(s - t₁, x₁, y) - \nabla_x q(s - t₂, x₂, y)||F_n(s, y)|dyds \]

\[ \leq c N_{F_n}^1(t₂ - t₁) + \int_{t₂}^{t₂ + \delta} \int_{R^d} |\nabla_x q(s - t₁, x₁, y)||F_n(s, y)|dyds \]

\[ + \int_{t₂ + \delta}^{T} \int_{R^d} |\nabla_x q(s - t₂, x₂, y)||F_n(s, y)|dyds \]

\[ + c \int_{t₂ + \delta}^{T} \int_{R^d} |x₁ - x₂|^{\alpha}(s - t₂)^{-\frac{d+1+\alpha}{2}}(e^{-\frac{C|x₁ - y|^2}{2(s - t₂)}} + e^{-\frac{C|u(x₁, y)|^2}{2(s - t₂)}})|F_n(s, y)|dyds \]

\[ + c \int_{t₂ + \delta}^{T} \int_{R^d} |t₁ - t₂|^{\alpha}(s - t₂)^{-\frac{d+1+2\alpha}{2}}e^{-\frac{C|x₁ - y|^2}{2(s - t₂)}}|F_n(s, y)|dyds \]

\[ \leq c N_{F_n}^1(t₂ - t₁) + \int_{t₁}^{t₂ + \delta} \int_{R^d} |\nabla_x q(s - t₁, x₁, y)||F_n(s, y)|dyds + c N_{F_n}^1(\delta) \]

\[ + c(|x₁ - x₂|^{\alpha}\delta^{-\frac{d+1+\alpha}{2}} + |t₁ - t₂|^{\alpha}\delta^{-\frac{d+1+2\alpha}{2}}) \sup_{x ∈ R^d} \int_\delta^T \int_{R^d} e^{-\frac{C|\nabla_x u|^2}{2t}}|F_n(s + t₁, y)|dyds \]

\[ \leq c N_{F_n}^1(t₂ - t₁ + \delta) + c(|x₁ - x₂|^{\alpha}\delta^{-\frac{d+1+\alpha}{2}} + |t₁ - t₂|^{\alpha}\delta^{-\frac{d+1+2\alpha}{2}}), \]

where the last inequality follows from Lemma 3.2 and the fact that

\[ \sup_{t ≥ 0, x ∈ R^d, n ≥ 1} \int_\delta^T \int_{B(x, 1)}|F_n(s + t₁, y)|dyds < ∞. \]

Hence (4.2) is fulfilled by \( \nabla_x u_n \). Now using Lemma 4.1 and (4.19), we see that \( \{u_n\}_{n ≥ 1} \) is a Cauchy sequence in \( C^0_b(0, T) ∩ B(0, R) \).

(iii) follows from (4.15) and (4.19). \( \square \)

Next result gives the Hölder continuity of \( \nabla u \) with respect to the time variable.

**Proposition 4.3** Assume

\[ \sup_{t ∈ [0, T]} \int_{r^2}^{r^2} |b(t + s, y)| + |f(t + s, y)|dyds ≤ Cr^p, ∀0 < r < 1. \] (4.20)

for some constants \( p > d + 1 \) and \( C > 0 \). Then for any \( \alpha ∈ (0, \frac{p-d-1}{2} \land \frac{p-d}{d+3} \land 1) \), there exists a \( M₄ > 0 \) depending on \( d, T, \alpha, p \) and \( C \), such that

\[ |\nabla_x u(t₁, x) - \nabla_x u(t₂, x)| ≤ M₄|t₂ - t₁|^\alpha, ∀(t₁, x), (t₂, x) ∈ [0, T] × R^d. \] (4.21)

**Proof:** Assume (4.20) holds. Then by Proposition 3.2 \( |b| \) and \( |f| \) belong to the Kato class \( K_{d,1} \). Take a constant \( \alpha ∈ (0, \frac{p-d-1}{2} \land \frac{p-d}{d+3} \land 1) \). Set \( F(s, y) := \langle b(s, y), \nabla_x u(s, y) \rangle - f(s, y) \).

First we prove

\[ \sup_{0 ≤ t₁ < t₂ ≤ T, x ∈ R^d} \int_{t₁}^{t₂} \int_{R^d} (s - t₂)^{-\frac{\alpha(d+3)}{2}}(s - t₁)^{-\frac{(1-\alpha)(d+1)}{2}}e^{-\frac{(1-\alpha)|x₁ - y|^2}{4(s - t₂)}}|F(s, y)|dyds < ∞. \] (4.22)
Set \( \delta := t_2 - t_1 < 1 \). Since \( s - t_1 \in (\delta, 2\delta) \) for \( s \in (t_2, t_2 + \delta) \) and \( s - t_2 < s - t_1 < 2(s - t_2) \) for \( s \in (t_2 + \delta, T) \), we have

\[
\int_{t_2}^{t_2 + \delta} \int_{\mathbb{R}^d} (s - t_2)^{-\frac{\alpha(d + 3)}{2}} (s - t_1)^{-\frac{(1-n)(d+1)}{2}} e^{-\frac{(1-n)|x-y|^2}{4(s-t_1)}} |F(s, y)|dyds
\]

\[
\leq \int_{t_2}^{t_2 + \delta} \int_{\mathbb{R}^d} (s - t_2)^{-\frac{\alpha(d + 3)}{2}} \delta^{-\frac{(1-n)(d+1)}{2}} e^{-\frac{(1-n)|x-y|^2}{8\delta}} |F(s, y)|dyds
\]

\[+ c \int_{(t_2 + \delta) \cap \mathbb{R}^d} \int_{\mathbb{R}^d} (s - t_2)^{-\frac{\alpha(d + 3)}{2}} (s - t_2)^{-\frac{(1-n)(d+1)}{2}} e^{-\frac{(1-n)|x-y|^2}{4(s-t_2)}} |F(s, y)|dyds \quad (4.23)\]

\[
\leq \delta^{-\frac{(1-n)(d+1)}{2}} \int_0^\delta \int_{\mathbb{R}^d} s^{-\frac{\alpha(d + 3)}{2}} e^{-\frac{(1-n)|x-y|^2}{8s}} |F(t_2 + s, y)|dyds
\]

\[+ c \int_{t_2}^T \int_{\mathbb{R}^d} (s - t_2)^{-\frac{\alpha(d + 3) + 2n}{2}} e^{-\frac{(1-n)|x-y|^2}{4(s-t_2)}} |F(s, y)|dyds.
\]

Take \( 0 < \lambda < 1 \) with \( p - (2 - \lambda)d - (2\alpha + 1) > 0 \). Then by Lemma 3.2 and Lemma 3.3 we have

\[
\int_0^\delta \int_{\mathbb{R}^d} s^{-\frac{\alpha(d + 3)}{2}} e^{-\frac{(1-n)|x-y|^2}{8s}} |F(t_2 + s, y)|dyds
\]

\[= \sum_{n \geq 0} \int_{2^{-n-1} \delta}^{2^{-n} \delta} \int_{\mathbb{R}^d} s^{-\frac{\alpha(d + 3)}{2}} e^{-\frac{(1-n)|x-y|^2}{8s}} |F(t_2 + s, y)|dyds \]

\[\leq \sum_{n \geq 0} 2^{\frac{\alpha(n+1)(d+3)}{2}} \delta^{-\frac{\alpha(d + 3)}{2}} \left( \int_{2^{-n-1} \delta}^{2^{-n} \delta} \int_{|x-y| < \frac{\delta}{2}} e^{-\frac{(1-n)|x-y|^2}{8s}} |F(t_2 + s, y)|dyds \right. \]

\[\left. + \int_{2^{-n-1} \delta}^{2^{-n} \delta} \int_{|x-y| \geq \frac{\delta}{2}} e^{-\frac{(1-n)|x-y|^2}{8s}} |F(t_2 + s, y)|dyds \right) \quad (4.24)\]

\[\leq \sum_{n \geq 0} \frac{2^{\alpha(n+1)(d+3)}}{16} \delta^{-\frac{\alpha(d + 3)}{2}} \left( \int_{2^{-n-1} \delta}^{2^{-n} \delta} \int_{|x-y| < \frac{\delta}{2}} e^{-\frac{(1-n)|x-y|^2}{16s}} |F(t_2 + s, y)|dyds \right. \]

\[\left. + e^{-\frac{(1-n)|x-y|^2}{16}} \sup_{x \in \mathbb{R}^d} \int_{2^{-n-1} \delta}^{2^{-n} \delta} \int_{B(x, \delta)} |F(t_2 + s, y)|dyds \right) \]

\[\leq c \sum_{n \geq 0} 2^{\alpha(n+1)(d+3)} \delta^{-\frac{\alpha(d + 3)}{2}} \left( 2^{\frac{(n+1)p-d}{2}} \delta^{\frac{p-d}{2}} - e^{-\frac{(1-n)|x-y|^2}{16}} \right) \]

\[\leq c \sum_{n \geq 0} 2^{\alpha(n+1)(d+3) + 3\alpha - p - d} \left( \delta^{\frac{p-d}{2}} - e^{-\frac{(1-n)|x-y|^2}{16}} \right). \]

Note that \( a \alpha + 3\alpha - p - d < 0 \). (4.24) yields that

\[
\delta^{-\frac{(1-n)(d+1)}{2}} \int_0^\delta \int_{\mathbb{R}^d} s^{-\frac{\alpha(d + 3)}{2}} e^{-\frac{(1-n)|x-y|^2}{8s}} |F(t_2 + s, y)|dyds \]

\[\leq c \sum_{n \geq 0} 2^{(n+1)(d+3) + 3\alpha - p - d} \left( \delta^{\frac{p-d}{2}} - e^{-\frac{(1-n)|x-y|^2}{16}} \right) \quad (4.25)\]
where we used the fact that $\lambda - 1 < 0$ in the last inequality. By Proposition (3.2) and the fact that $2\alpha < p - d - 1$, we see that $|F| \in K_{d,1-2\alpha}$. Therefore (4.23) and (4.23) imply
\[
\int_{t_2}^{T} \int_{\mathbb{R}^d} (s - t_2)^{-\frac{\alpha(d+3)}{2}} (s - t_1)^{-\frac{(1-\alpha)(d+1)}{2}} e^{-\frac{(1-\alpha)|x-y|^2}{4(s-t_1)}} |F(s,y)| dy ds \\
\leq c + c \int_{t_2}^{T} \int_{\mathbb{R}^d} (s - t_2)^{-\frac{d+2\alpha}{2}} e^{-\frac{(1-\alpha)|x-y|^2}{4(s-t_2)}} |F(s,y)| dy ds \leq c.
\]
(4.22) is proved.

Now we show (4.21). Noting that for $0 < t_1 < t_2$,
\[
|\nabla_x q(t_1, x, y) - \nabla_x q(t_2, x, y)| \\
\leq |t_1 - t_2| \int_{0}^{1} |\partial_t \nabla_x q(t_1 + \lambda(t_2 - t_1), x, y)| d\lambda \leq c|t_1 - t_2| t_1^{-\frac{d+3}{2}},
\]
we have
\[
|\nabla_x q(t_1, x, y) - \nabla_x q(t_2, x, y)| \\
\leq |\nabla_x q(t_1, x, y) - \nabla_x q(t_2, x, y)|^{a} |\nabla_x q(t_1, x, y) - \nabla_x q(t_2, x, y)|^{1-a} \\
\leq c|t_1 - t_2|^{a} t_1^{-\frac{\alpha(d+3)}{2}} (t_1^{\frac{(1-\alpha)(d+1)}{2}} e^{-\frac{(1-\alpha)|x-y|^2}{4t_1}} + t_2^{\frac{(1-\alpha)(d+1)}{2}} e^{-\frac{(1-\alpha)|x-y|^2}{4t_2}} ) \\
\leq c|t_1 - t_2|^{a} t_1^{-\frac{d+2\alpha}{2}} e^{-\frac{(1-\alpha)|x-y|^2}{4t_1}} + t_2^{\frac{\alpha(d+3)}{2}} t_2^{-\frac{(1-\alpha)(d+1)}{2}} e^{-\frac{(1-\alpha)|x-y|^2}{4t_2}}).
\]
(4.26)

For $0 \leq t_1 < t_2 \leq T$ with $t_2 - t_1 < 1$, using (4.22) and (4.26) we get
\[
|\nabla_x u(t_1, x) - \nabla_x u(t_2, x)| \\
\leq \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\nabla_x q(s-t_1, x, y)||F(s,y)| dy ds \\
+ \int_{t_2}^{T} \int_{\mathbb{R}^d} |\nabla_x q(s-t_1, x, y) - \nabla_x q(s-t_2, x, y)||F(s,y)| dy ds \\
\leq c|t_1 - t_2|^{a} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} (s - t_1)^{-\frac{d+2\alpha}{2}} e^{-\frac{|x-y|^2}{4(s-t_1)}} |F(s,y)| dy ds \\
+ c|t_1 - t_2|^{a} \int_{t_2}^{T} \int_{\mathbb{R}^d} (s - t_2)^{-\frac{d+2\alpha}{2}} e^{-\frac{(1-\alpha)|x-y|^2}{4(s-t_2)}} |F(s,y)| dy ds \\
+ c|t_1 - t_2|^{a} \int_{t_2}^{T} \int_{\mathbb{R}^d} (s - t_2)^{-\frac{\alpha(d+3)}{2}} (s - t_1)^{-\frac{(1-\alpha)(d+1)}{2}} e^{-\frac{(1-\alpha)|x-y|^2}{4(s-t_1)}} |F(s,y)| dy ds \\
\leq c|t_1 - t_2|^{a}.
\]

\[\blacksquare\]

Remark 4.1 Using Hölder’s inequality, it is easy to see that (4.20) holds if $|b|^2$ and $|f|^2$ satisfy the Hypothesis $H_p$.

Next, we will establish some regularities of the mild solution of equation (1.1), which plays a crucial role later. We start with the following $L^2$-estimate.
Lemma 4.2 Suppose a function \( g : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) vanishes outside \([0, T] \times B(x_0, r)\) for some \( r \in (0, 2)\), and moreover \( |g|^2 \) satisfies the Hypothesis \( H_p \). Let \( u \) be the mild solution of the following equation:

\[
\begin{cases}
\partial_t u(t, x) + \frac{1}{2} \Delta u(t, x) = g(t, x), & \forall (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(T, x) = 0, & \forall x \in \mathbb{R}^d.
\end{cases}
\]

Then \( u \in W^{1,2}_{t,x}((0, T) \times \mathbb{R}^d) \). Furthermore, there exist constants \( \alpha, \beta \in (0, p-d) \) and \( M_5 > 0 \), which are independent of \( x_0 \) and \( r \), such that for any \( 0 < a < b < T \),

\[
\int_a^b \int_{\mathbb{R}^d} \|D^2_x u\|^2 \, dx \, dt \leq M_5 \left( \int_a^b \int_{B(x_0, r)} |g(t, x)|^2 \, dx \, dt + r^d(b-a)^\alpha + r^{p-\beta} \right). \tag{4.27}
\]

**Proof:** Let \( g_n(t, x) := \int_{R^{d+1}} \varphi_n(t-s,x-y)g(s,y)dy \) and \( u_n \in C^{1,2}_b([0, T] \times \mathbb{R}^d) \) be the mild solution of the following equation:

\[
\begin{cases}
\partial_t u_n(t, x) + \frac{1}{2} \Delta u_n(t, x) = g_n(t, x), & \forall (t, x) \in (0, T) \times \mathbb{R}^d, \\
u_n(T, x) = 0, & \forall x \in \mathbb{R}^d.
\end{cases}
\]

Using the representation

\[
u_n(t, x) = \int_t^T \int_{B(x_0, r+1)} g(s-t, x, y)g_n(s, y) \, dy \, ds,
\]

we can see easily that for any \( R \geq 1 \) and \( |x-x_0| \geq R+r+1 \),

\[
\sup_{n \geq 1, t \geq 0} \left\{ \|D^2_x u_n(t, x)\| + \|\nabla_x u_n(t, x)\| + |\partial_t u_n(t, x)| \right\} \leq ce^{-\frac{a^2}{4T}}. \tag{4.28}
\]

Using the integration by parts we have

\[
\int_{\mathbb{R}^d} \|g_n(t, x)\|^2 \, dx = \int_{\mathbb{R}^d} \left( |\partial_t u_n(t, x)|^2 + \frac{1}{4} |\Delta u_n(t, x)|^2 \right) \, dx - \int_{\mathbb{R}^d} \partial_t |\nabla_x u_n(t, x)|^2 \, dx \tag{4.29}
\]

and

\[
\int_{\mathbb{R}^d} \|g_n(t, x) - g_m(t, x)\|^2 \, dx = \int_{\mathbb{R}^d} \left( |\partial_t u_n(t, x) - \partial_t u_m(t, x)|^2 + \frac{1}{4} |\Delta u_n(t, x) - \Delta u_m(t, x)|^2 \right) \, dx - \int_{\mathbb{R}^d} \partial_t |\nabla_x u_n(t, x) - \nabla_x u_m(t, x)|^2 \, dx \tag{4.30}
\]

On the other hand, \( 4.28 \) and the Green’s formula give that

\[
\sum_{i,j=1}^d \int_{\mathbb{R}^d} |\partial^2_{ij} u_n(t, x) - \partial^2_{ij} u_m(t, x)|^2 \, dx = \int_{\mathbb{R}^d} |\Delta u_n(t, x) - \Delta u_m(t, x)|^2 \, dx. \tag{4.31}
\]
Hence by (4.29)-(4.31), we have
\[
\sum_{i,j=1}^{d} \int_{0}^{T} \int_{\mathbb{R}^d} |\partial_{ij}^2 u_n(t,x) - \partial_{ij}^2 u_m(t,x)|^2 dx dt + \int_{0}^{T} \int_{\mathbb{R}^d} |\partial_t u_n(t,x) - \partial_t u_m(t,x)|^2 dx dt 
\leq 4 \int_{0}^{T} \int_{\mathbb{R}^d} |g_n(t,x) - g_m(t,x)|^2 dx dt,
\]
and
\[
\sum_{i,j=1}^{d} \int_{a}^{b} \int_{\mathbb{R}^d} |\partial_{ij}^2 u_n(t,x)|^2 dx dt + \int_{a}^{b} \int_{\mathbb{R}^d} |\partial_t u_n(t,x)|^2 dx dt 
\leq 4 \int_{a}^{b} \int_{\mathbb{R}^d} |g_n(t,x)|^2 dx dt + 4 \int_{\mathbb{R}^d} (|\nabla_x u_n(b,x)|^2 - |\nabla_x u_n(a,x)|^2) dx,
\]
for any $0 \leq a < b \leq T$.

Recall that $u_n$ converges to $u$ uniformly on the compact set of $[0, T] \times \mathbb{R}^d$ by Proposition 4.2. Hence (4.32) implies that $u \in W^{2,2}_a((0, T) \times \mathbb{R}^d)$ and
\[
\lim_{n \to \infty} \int_{a}^{b} \int_{\mathbb{R}^d} \|D_x^2 u_n(t,x) - D_x^2 u(t,x)\|^2 dx dt = 0.
\]
By Proposition 4.2 and (4.28) we also have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} |\nabla_x u_n(t,x) - \nabla_x u(t,x)|^2 dx = 0, \quad \forall t \in [0, T].
\]
Thus, by (4.31) and (4.35), letting $n \to \infty$ in (4.33) we get
\[
\sum_{i,j=1}^{d} \int_{a}^{b} \int_{\mathbb{R}^d} |\partial_{ij}^2 u(t,x)|^2 dx dt
\leq 4 \int_{a}^{b} \int_{\mathbb{R}^d} |g(t,x)|^2 dx dt + \int_{\mathbb{R}^d} (|\nabla_x u(b,x)|^2 - |\nabla_x u(a,x)|^2) dx
\leq 4 \int_{a}^{b} \int_{B(x_0,r)} |g(t,x)|^2 dx dt + \int_{|x-x_0| \geq 2r} |\nabla_x u(b,x)|^2 dx 
\] 
\[+ \int_{B(x_0,2r)} |\nabla_x u(b,x) - \nabla_x u(a,x)|(|\nabla_x u(b,x)| + |\nabla_x u(a,x)|) dx 
\leq c \int_{a}^{b} \int_{B(x_0,r)} |g(t,x)|^2 dx dt + \int_{|x-x_0| \geq 2r} |\nabla_x u(b,x)|^2 dx + (b - a)\alpha r^d),
\]
where the last inequality follows from Proposition 4.2 and Proposition 4.3 and $\alpha \in (0, p - d)$ is the constant appeared in Proposition 4.3.

On the other hand, using Hölder’s inequality and the fact that $|g|^2$ satisfies the Hypothesis $H_p$, it is easy to see that $g$ satisfies the Hypothesis $(H'_q)$ with $q := \frac{p + d + 2}{2} > d + 1$. Therefore
we have for \( t \in [0, T] \),

\[
\int_{|x-x_0| \geq 2r} |\nabla_x u(t, x)|^2 \, dx \\
\leq \int_0^T \int_{|y-x_0| < r} \int_{|z-x_0| < r} |\nabla_x q(s - t, x, y)||g(s, y)||\nabla_x g(t, z)| \, dz \, dy \, ds \\
\times t \int_{|z-x_0| \geq 2r} |\nabla_x q(t - t, x, z)||g(t, z)| \, dz \, dy \, dx \\
\leq c \int_0^{T-t} \int_{|y-x_0| < r} \int_{|z-x_0| < r} (s-t) \frac{1}{2} e^{-\frac{s^2}{2r}} (\tau - t)^{-\frac{1}{2}} e^{-\frac{\tau^2}{2r}} |g(s, y)||g(\tau, z)| \, dz \, dy \, ds \\
\times \int_{|z-x_0| \geq 2r} (s-t) \frac{4}{7} (\tau - t)^{-\frac{4}{7}} e^{-\frac{(s-x)^2}{2r}} e^{-\frac{\tau^2}{2r}} \, dx
\]

(4.37)

For \( s \in (0, r^2) \), by (3.1) we have

\[
\int_{s}^{(T-t)/\sqrt{2}} \int_{|z-x_0| < r} \tau^{-\frac{1}{2}} e^{-\frac{s^2}{2r}} (s + \tau)^{-\frac{1}{2}} |g(\tau, t, z)| \, dz \, d\tau \\
\leq \sum_{n \geq 1} \int_{ns}^{(n+1)s} \int_{|z-x_0| < r} \tau^{-\frac{1}{2}} (s + \tau)^{-\frac{1}{2}} |g(\tau, t, z)| \, dz \, d\tau \\
\leq \sum_{n \geq 1} \int_{ns}^{(n+1)s} \int_{|z-x_0| < r} n^{-\frac{1}{2}} (n+1)^{-\frac{1}{2}} s^{-\frac{d+2q}{2}} |g(\tau, t, z)| \, dz \, d\tau \\
\leq cs^{-\frac{d+2q}{2}} s^{-\frac{d+2q}{2}} \sum_{n \geq 1} n^{-\frac{1}{2}} (n+1)^{-\frac{1}{d}} \leq cs^{-\frac{2d+1-q}{d}} r^d.
\]

Therefore

\[
I \leq cr^d \int_0^{r^2} \int_{|y-x_0| < r} s^{-\frac{1}{2}} e^{-\frac{s^2}{2r}} s^{-\frac{d+1}{2}} |g(s, t, y)| \, dy \, ds \\
\leq cr^d r^{-2d-2+q} \int_0^{r^2} \int_{|y-x_0| < r} |g(s, t, y)| \, dy \, ds
\]

(4.38)

\[
\leq cr^{-d+2+q} = cr^p.
\]

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For the term II, we only need to consider the situation: \( r^2 < T - t \). Take some constant \( \beta \in (0, p - d) \). For \( s \in (r^2, T - t) \), by (3.2) we have

\[
\int_s^{T-t} \int_{|z-x_0|<r} \tau^{-\frac{1}{2}} e^{-\frac{2}{s\tau}} (s + \tau)^{-\frac{d+\beta}{2}} |g(\tau + t, z)|dzd\tau \\
\leq (2T)^\frac{3}{2} \sum_{n \geq 1} \int_{(n+1)s}^{(n+1)s} \int_{|z-x_0|<r} \tau^{-\frac{1}{2}} (s + \tau)^{-\frac{d+\beta}{2}} |g(\tau + t, z)|dzd\tau \\
\leq c \sum_{n \geq 1} n^{-\frac{1}{2}} (n + 1)^{-\frac{d+\beta}{2}} \int_{(n+1)s}^{(n+1)s} \int_{|z-x_0|<r} |g(\tau + t, z)|dzd\tau \\
\leq cs^{-\frac{d+\beta}{2}} \frac{1}{s^{q-2}} \sum_{n \geq 1} n^{-\frac{1}{2}} (n + 1)^{-\frac{d+\beta}{2}} \leq cs^{-\frac{d+\beta}{2}} r^{q-2}.
\]

Therefore

\[
II \leq cr^{q-2} \int_{r^2}^{T-t} \int_{|y-x_0|<r} s^{-\frac{1}{2}} e^{-\frac{2}{s\tau}} s^{\frac{d+\beta}{2}} |g(s + t, y)|dyds \\
\leq cr^{q-2-d-\beta} \sum_{n \geq 1} \int_{nr^2}^{(n+1)r^2} n^{-\frac{d+\beta}{2}} \int_{|y-x_0|<r} |g(s + t, y)|dyds \\
\leq cr^{q-2-d-\beta} \sum_{n \geq 1} n^{-\frac{d+\beta}{2}} = cr^{p-\beta}. \tag{4.39}
\]

Putting (4.36)-(4.39) together, we obtain (4.27). \( \square \)

Now we can state the desired regularity estimate for the solution of the parabolic equation (4.1).

**Proposition 4.4** Assume \(|b|^2\) and \(|f|^2\) satisfy Hypothesis \( H_p \). Let \( u \in C_b^{0,1}([0, T] \times \mathbb{R}^d) \) be the mild solution to the equation (4.1). Then \( \|D_2^2u\|^2 \) satisfies Hypothesis \( H_q \) for some \( q \in (d, p) \).

**Proof:** For any \( x_0 \in \mathbb{R}^d \) and \( r \in (0, 1) \), set \( F(t, x) := f(t, x) - \langle b(t, x), \nabla_x u(t, x) \rangle \) and \( F_r(t, x) := F(t, x)I_{B(x_0, 2r)}(x) \). Let \( u_r \) be the mild solution to the following equation:

\[
\begin{align*}
\partial_t u_r(t, x) + \frac{1}{2} \Delta u_r(t, x) &= F_r(t, x), \quad \forall (t, x) \in (0, T) \times \mathbb{R}^d, \\
u_r(T, x) &= 0, \quad \forall x \in \mathbb{R}^d.
\end{align*}
\]

Then by Lemma 4.2 there exist constants \( C > 0 \) and \( q \in (d, p \wedge (d+2)) \), which are independent of \( x_0, r \), such that any \( t \in [0, T] \),

\[
\int_t^{t+r^2} \int_{B(x_0, r)} \|D_2^2u_r(s, x)\|^2dxds \leq C \int_t^{t+r^2} \int_{B(x_0, 2r)} |F_r(s, x)|^2dxds + r^q \leq Cr^q. \tag{4.40}
\]

Set \( \bar{F}_r(t, x) := F(t, x) - F_r(t, x) \). Then \( \bar{u}_r := u - u_r \) is the solution of the following equation

\[
\begin{align*}
\partial_t \bar{u}_r(t, x) + \frac{1}{2} \Delta \bar{u}_r(t, x) &= \bar{F}_r(t, x), \quad \forall (t, x) \in (0, T) \times \mathbb{R}^d, \\
\bar{u}_r(T, x) &= 0, \quad \forall x \in \mathbb{R}^d.
\end{align*}
\]

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Therefore,
\[ \bar{u}_r(t, x) = \int_t^T \int_{\mathbb{R}^d} q(s - t, x, y) \tilde{F}_r(s, y) dy ds. \]

By Proposition 3.2 and Lemma 3.4 it is easy to see that \( \tilde{F}_r \in \mathbf{K}_{d,1-\alpha} \) for \( \alpha = \frac{2 - d}{2} \). Noting
\[ \| D_x^2 q(t, x, y) \| \leq c |x - y|^{\alpha - 1} \left( 1 - \frac{d+1+\alpha}{2} e^{-\frac{|x-y|^2}{4(\alpha-\beta)}} \right), \]
it follows that \( \bar{u}_r(t, x) \) is twice differentiable with respect to \( x \) for \( x \in B(x_0, r) \) and
\[ \| D_x^2 \bar{u}_r(t, x) \| \leq c \int_t^T \int_{B(x_0, 2r)^c} |x - y|^{\alpha - 1} (s - t)^{-\frac{d+1+\alpha}{2}} e^{-\frac{|x-y|^2}{4(\alpha-\beta)}} \tilde{F}_r(s, y) dy ds \]
\[ \leq c r^{\alpha - 1} \int_t^T \int_{\mathbb{R}^d} (s - t)^{-\frac{d+1+\alpha}{2}} e^{-\frac{|x-y|^2}{4(\alpha-\beta)}} \tilde{F}_r(s, y) dy ds \leq cr^{\alpha - 1}, \]
where the last inequality follows from Lemma 3.1. Thus
\[ \int_t^{t+2r^2} \int_{B(x_0, r)} \| D_x^2 \bar{u}_r(s, x) \|^2 |d x| ds \leq cr^{d+2\alpha} = cr^q. \] (4.41)
Hence, by (4.40) and (4.41), we see that \( \| D_x^2 u \|^2 \) satisfies Hypothesis H_q.

5 Existence and uniqueness of strong solutions

After the preparations, in this section, we are ready to show that there exists a unique strong solution to the SDE (1.1). First of all, by [6, Theorem 3] and [16, Theorem A] we have the following result on the weak solution of SDE (1.1):

**Theorem 5.1** Assume \( |b| \in \mathbf{K}_{d,1} \). Then for each \( x \in \mathbb{R}^d \), there exists a unique weak solution to the following SDE:
\[ X_t = x + W_t + \int_0^t b(s, X_s) ds, \forall t > 0. \]

Moreover, the solution admits a continuous transition density \( p(t, x, s, y) \) satisfying that for any \( T > 0 \), there exist positive constants \( M_6 \) and \( M_7 \) such that for any \( 0 < s - t \leq T \) and \( x, y \in \mathbb{R}^d \),
\[ p(t, x, s, y) \leq M_6(s - t)^{-\frac{d}{2}} e^{-\frac{M_7|x-y|^2}{2(s-t)}}. \] (5.1)

To carry out Zvonkin transformation for SDEs (1.1), we need the following Krylov’s type convergence.

**Lemma 5.1** Assume \( |b| \in \mathbf{K}_{d,1} \). Let \( X \) be a weak solution to the SDE (1.1) with \( X_0 = x \). Then for any \( h \in \mathbf{K}_{d,1} \) we have
\[ \lim_{n \to \infty} E[ \sup_{0 \leq t \leq T} | \int_0^t h(s, X_s) ds - \int_0^t h_n(s, X_s) ds |^2 ] = 0, \]
where \( h_n(t, x) := \int_{\mathbb{R}^{d+1}} \varphi_n(t - s, x - y) h(s, y) dy ds \).
Proof: We first show that for $t > 0$,

$$\lim_{n \to \infty} E\left[\int_0^t h_n(s, X_s)ds\right] = E[\int_0^t h(s, X_s)ds]. \quad (5.2)$$

For any $\varepsilon > 0$, by Lemma 1.1 and (5.1), there exists $0 < \delta < t$ such that

$$\sup_{n \geq 1} \left| \int_0^\delta \int_{R^d} p(0, x, s, y)(|h_n(s, y)| + |h(s, y)|)dyds \right| \leq \varepsilon.$$  \quad (5.3)

Also by Lemma 3.2 and (5.1) one can choose $R > 0$ large enough so that

$$\leq c\delta^{-\frac{d}{2}} \sup_{x \in R^d} \int_{\frac{\delta}{2}}^{\delta+\frac{\delta}{2}} \int_{|x-y| \geq R-\frac{\delta}{2}} e^{-\frac{M_\tau |x-y|^2}{4t}} |h(s, y)|dyds \leq \varepsilon.$$  \quad (5.4)

Take a positive integer $N$ such that $2^{-N} < \frac{\delta}{2}$. It follows from (5.3), (5.4) and (5.1) that for $n \geq N$ and $R > 0$,

$$\left|E\left[\int_0^t (h_n(s, X_s) - h(s, X_s))ds\right]\right|$$

$$\leq \sup_{n \geq 1} \int_0^\delta \int_{R^d} p(0, x, s, y)(|h_n(s, y)| + |h(s, y)|)dyds$$

$$+ \left| \int_{R^{d+1}} \varphi_n(\tau, z)drdz \int_{\frac{\delta}{2}}^{\delta+\frac{\delta}{2}} \int_{|x-y| \geq R} (p(0, x, s + \tau, y + z) + p(0, x, s, y))h(s, y)dyds \right|$$

$$+ \left| \int_{R^{d+1}} \varphi_n(\tau, z)drdz \int_{\frac{\delta}{2}}^{\delta+\frac{\delta}{2}} \int_{R^d} p(0, x, s, y)|h(s, y)|dyds \right|$$

$$+ \left| \int_{R^{d+1}} \varphi_n(\tau, z)drdz \int_{\frac{\delta}{2}}^{\delta+\frac{\delta}{2}} \int_{R^d} p(0, x, s, y)|h(s, y)|dyds \right|$$

$$\leq \sup_{n \geq 1} \int_0^\delta \int_{R^d} p(0, x, s, y)(|h_n(s, y)| + |h(s, y)|)dyds$$

$$+ \sup_{|\tau| \leq \frac{\delta}{2}} \left| \int_{|x-y| \geq R} (p(0, x, s + \tau, y + z) + p(0, x, s, y))h(s, y)dyds \right|$$

$$+ \left| \int_{R^{d+1}} \varphi_n(\tau, z)drdz \int_{\frac{\delta}{2}}^{\delta+\frac{\delta}{2}} \int_{B(x, R)} (p(0, x, s + \tau, y + z) - p(0, x, s, y))h(s, y)dyds \right|$$

$$+ \left| \int_{R^{d+1}} \varphi_n(\tau, z)drdz \int_{\frac{\delta}{2}}^{\delta+\frac{\delta}{2}} \int_{R^d} p(0, x, s, y)|h(s, y)|dyds \right|$$

$$+ \int_0^\delta \int_{R^d} p(0, x, s, y)|h(s, y)|dyds + \int_{t-2^n}^t \int_{R^d} p(0, x, s, y)|h(s, y)|dyds$$

$$\leq 3\varepsilon + \sup_{|\tau| \leq 2^{-n}, (s, y) \in \left(\frac{\delta}{2}, t]\times B(x, R)} |p(0, x, s, y) - p(0, x, s, y)| \int_0^\delta \int_{B(x, R)} |h(s, y)|dyds$$

$$+ \int_{t-2^n}^t \int_{R^d} p(0, x, s, y)|h(s, y)|dyds.$$
Letting $n \rightarrow \infty$, by the continuity of $p(0, x, \cdot)$ on $(0, \infty) \times \mathbb{R}^d$, we get

$$\lim_{n \rightarrow \infty} |E[\int_0^t (h_n(s, X_s) - h(s, X_s))ds]| \leq 3\varepsilon.$$  

Since $\varepsilon$ is arbitrary, we obtain (5.2).

Now let $v_n(t, x) := \int_0^T \int_{\mathbb{R}^d} q(s-t, x, y) h_n(s, y)dyds$ and $v(t, x) := \int_0^T \int_{\mathbb{R}^d} q(s-t, x, y) h(s, y)dyds$. Then $v_n$ and $v$ are the solutions of the following equations:

$$
\begin{cases}
\partial_t v_n(t, x) + \frac{1}{2} \Delta v_n(t, x) = -h_n(t, x), & \forall (t, x) \in (0, T) \times \mathbb{R}^d, \\
v_n(T, x) = 0, & \forall x \in \mathbb{R}^d.
\end{cases}
$$

Applying Ito’s formula we obtain

$$v_n(t, X_t) = v_n(0, x) + \int_0^t \langle \nabla v_n(s, X_s), dW_s \rangle + \int_0^t \langle \nabla v_n(s, X_s), b(s, X_s) \rangle ds - \int_0^t h_n(s, X_s)ds.$$  

(5.5)

Combining Proposition 4.2 and (5.2), letting $n \rightarrow \infty$ in (5.5) we get

$$v(t, X_t) = v(0, x) + \int_0^t \langle \nabla v(s, X_s), dW_s \rangle + \int_0^t \langle \nabla v(s, X_s), b(s, X_s) \rangle ds - \int_0^t h(s, X_s)ds.$$  

(5.6)

Using the Gaussian upper bound estimate of the density function of $X$ and $|b| \in \mathbb{K}_{d,1}$, one can easily see that

$$E[\int_0^T |b(s, X_s)|ds] < \infty.$$  

(5.7)

Set $\tau_R := \inf\{ t \geq 0 : |X_t| \geq R \}$ for $R > 0$. Then by Proposition 1.2, (5.5)-(5.7) and Burkholder’s inequality we have

$$\lim_{n \rightarrow \infty} E[\sup_{0 \leq t \leq T} |\int_0^t h(s, X_s)ds - \int_0^t h_n(s, X_s)ds|^2]$$

$$\leq 2 \lim_{n \rightarrow \infty} E[\sup_{0 \leq t \leq T} |v(t, X_t) - v_n(t, X_t)|^2] + 4 \lim_{n \rightarrow \infty} E[\int_0^T |\nabla v(s, X_s) - \nabla v_n(s, X_s)|^2ds]$$

$$+ \lim_{n \rightarrow \infty} E[\int_0^t |\nabla v(s, X_s) - \nabla v_n(s, X_s)| |b(s, X_s)|ds|^2]$$

$$\leq c(P[T > \tau_R] + \lim_{n \rightarrow \infty} \|v - v_n\|_{C_{0,1}^0(0, T \times B(0, R))}) = cP[T > \tau_R].$$

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Let $R \to \infty$ to obtain

$$
\lim_{n \to \infty} E[\sup_{0 \leq t \leq T} |\int_0^t h(s, X_s)ds - \int_0^t h_n(s, X_s)ds|^2] = 0.
$$

\[\square\]

Next result is the Zvonkin transform of SDEs (1.1).

**Proposition 5.2** Assume $|b|, |f| \in K_{d,1}$. Let $X$ be a solution to the SDE (1.1) and $u$ the mild solution to the equation (4.1). Then we have for any $0 < t \leq T$,

$$u(t, X_t) = u(0, x) + \int_0^t \langle \nabla_x u(s, X_s), dW_s \rangle + \int_0^t f(s, X_s)ds. \quad (5.8)$$

**Proof:** Let $u_n$ be the classical solution to equation (1.1). Then by the Ito’s formula,

$$u_n(t, X_t) = u_n(0, x) + \int_0^t \langle \nabla_x u_n(s, X_s), dW_s \rangle$$

$$\quad + \int_0^t \langle \nabla_x u_n(s, X_s), b(s, X_s) - b_n(s, X_s) \rangle ds + \int_0^t f_n(s, X_s)ds. \quad (5.9)$$

By Lemma 5.1 there exists a subsequence $\{n_k\}_{k \geq 1}$ such that for $P$-a.e. $\omega \in \Omega$ and $t \in [0, T]$,

$$\lim_{k \to \infty} \int_0^t b_{n_k}(s, X_s(\omega))ds = \int_0^t b(s, X_s(\omega))ds.$$

Since $\nabla_x u(s, X_s)$ is continuous with respect to $s$, it follows that

$$\lim_{k \to \infty} \int_0^t \langle \nabla_x u(s, X_s), b_{n_k}(s, X_s) - b(s, X_s) \rangle ds = 0.$$

Combining this with Proposition 4.2 and Lemma 5.1 letting $n \to \infty$ in (5.9) we get (5.8).

\[\square\]

Now we are ready to state the main result:

**Theorem 5.3** Assume for any $T > 0$, $|b(t, x)|^2 I_{(0,T)}(t) \in K_{d,\alpha}$ for some $\alpha \in (0, 2)$. Then for each $x \in \mathbb{R}^d$, there exists a unique strong solution to SDE (1.1).

**Proof:** By the Yamada-Watanabe theorem and Theorem 5.1 it suffices to prove the pathwise uniqueness of the solution to SDE (1.1). Assume $X$ and $Y$ are solutions to the SDE (1.1) with $X_0 = Y_0 = x$. For any given $T_0 > 0$, set $b_{T_0}(t, x) := b(t, x) I_{t<T_0}$. Then by Corollary 3.3 and Lemma 3.4, $|b_{T_0}|^2$ satisfies Hypothesis $H_p$ and $|b_{T_0}| \in K_{d,1}$. Let $u$ be the unique mild solution to the following parabolic equation:

$$
\begin{cases}
\partial_t u(t, x) + \frac{1}{2} \Delta u(t, x) + \langle b_{T_0}(t, x), \nabla_x u(t, x) \rangle = b_{T_0}, & \forall (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(T, x) = 0, & \forall x \in \mathbb{R}^d.
\end{cases}
$$

Letting $v(t, x) := x - u(t, x)$, by Proposition 4.2 there exist $0 < c_1 < c_2$ and constant $T > 0$ such that for $t \in [0, T]$ and $x, y \in \mathbb{R}^d$,

$$c_1 |x - y| \leq |v(t, x) - v(t, y)| \leq c_2 |x - y|. \quad (5.10)$$
Applying Proposition 5.2, we obtain
\[ v(t, X_t) - v(t, Y_t) = \int_0^t \langle \nabla_x u(s, X_s) - \nabla_x u(s, Y_s), dW_s \rangle. \]

Let \( \tau_R := \inf\{ t \leq T \wedge T_0 : |X_t - x| \vee |Y_t - x| \geq \frac{R}{2} \} \). Then by Lemma 3.6 and (5.10), for any \( t \in [0, \tau_R] \),
\[
|X_t - Y_t|^2 \leq c|v(t, X_t) - v(t, Y_t)|^2 \\
\leq c \int_0^t |v(s, X_s) - v(s, Y_s)|^2 \langle \nabla_x u(s, X_s) - \nabla_x u(s, Y_s), dW_s \rangle \\
+ c \int_0^t \| \nabla_x u(s, X_s) - \nabla_x u(s, Y_s) \|^2 \, ds \\
\leq c \int_0^t |v(s, X_s) - v(s, Y_s)|^2 \langle \nabla_x u(s, X_s) - \nabla_x u(s, Y_s), dW_s \rangle \\
+ c \int_0^t |X_s - Y_s|^2 (|M_R| \| D_x^2 u \| (s, X_s))^2 + |M_R| \| D_x^2 u \| (s, Y_s))^2) \, ds. \tag{5.11}
\]

By Lemma 3.5 and Proposition 4.4, we know that \( |M_R| \| D_x^2 u \| \) satisfies the Hypothesis \( H_q \) for some \( q < p \). In particular, \( |M_R| \| D_x^2 u \| ) \in K_{d,2} \) by Corollary 3.3. Using the upper bound of the density function \( p(t, x, s, y) \) in (5.1), we see that for any \( t > 0 \),
\[
E[ \int_0^t (|M_R| \| D_x^2 u \| (s, X_s))^2 + |M_R| \| D_x^2 u \| (s, Y_s))^2) \, ds ] < \infty.
\]

Applying the stochastic Gronwall inequality (see e.g. [11, Theorem 4]), we deduce from (5.11) that for any \( R > 0 \),
\[
E[ \sup_{t \leq \tau_R} |X_t - Y_t|^2 ] = 0.
\]

Letting \( R \to \infty \), we see that
\[
E[ \sup_{t \leq T \wedge T_0} |X_t - Y_t|^2 ] = 0.
\]

Since the constant \( T \) is independent of the initial value \( x \), using standard arguments, we conclude that \( X_t = Y_t \) for all \( t \in [0, T_0] \) \( P \)-a.e. Since \( T_0 \) is arbitrary, we have proven the pathwise uniqueness. \( \blacksquare \)

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