String interactions and discrete symmetries of the pp–wave background

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Abstract

Free string theory on the plane–wave background displays a discrete $Z_2$ symmetry exchanging the two transverse $SO(4)$ rotation groups. This symmetry should be respected also at the interacting level. We show that the zero mode structure proposed in hep-th/0208148 can be completed to a full kinematical vertex, contrary to claims appeared in the previous literature. We also comment on the relation with recent works on the string–bit formalism and on the comparison with the field theory side of the correspondence.

1 Introduction

One of the main obstacles to a complete understanding of AdS/CFT duality [1] has been the lack of control on the string side of the correspondence. The presence of non–trivial R–R form and of a curved metric in the background makes the analysis of the string theory challenging already at the classical level. In fact most of the achievements made in the AdS/CFT duality are restricted to the supergravity limit ($R^4/\alpha'^2 \to \infty$), where the bulk theory becomes more tractable. However, in this regime, the field theory side is

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in a strong coupling limit ($\lambda = g_M^2 N \to \infty$). Thus, one can really check the connection between the two sides of the duality only for those quantities that are protected by some symmetry against quantum corrections. Because of this restriction one cannot really probe the dynamical and most interesting aspects of the duality.

However, in this last year significant progress has been made to lift this restriction, at least for a very particular sector of the original AdS/CFT correspondence. Berenstein, Maldacena and Nastase in [2] proposed a duality relation between a sector of large $R$-charge operators of $\mathcal{N} = 4$ SYM and type IIB string theory on the maximally supersymmetric background (plane wave) [3]

$$g_{+-} = -2, \quad g_{++} = -\mu^2 \sum_{I=1}^{8} x_I x^I, \quad g_{IJ} = \delta_{IJ}, \quad I, J = 1, \ldots, 8,$$

$$F_{+1234} = F_{+5678} = 2\mu, \quad \phi = \text{constant}. \quad (1)$$

This proposal is very interesting for two reasons. First, it has been shown [4] that IIB string theory in the plane-wave background is solvable in the light-cone gauge using the Green-Schwarz formalism. This gives a concrete tool to study a string theory on a non-flat background that captures (part of) the dynamics of a gauge theory. A second important point is that the background (1) is actually the Penrose-limit [5] of the usual $AdS_5 \times S_5$ geometry. This means that the duality proposed in [2] is just the restriction of the original Maldacena duality [1] to a particular sector. Thus what we have learned so far on the AdS/CFT duality should apply also in this new contest.

In this spirit, the proposal of [2] provides an interesting setup, where we can try to answer to some long-standing questions in the AdS/CFT duality. The two main problems are: 1) how to build an exact dictionary between states on the string side and gauge invariant operators on the Yang–Mills side, 2) how to isolate the relevant field theory dynamical quantities and to connect them to string amplitudes. From the very recent literature [6, 7, 8, 9, 10] it seems that the two problems are strictly related. If we restrict to the BMN sector of the AdS/CFT duality, we have in principle the necessary technical tools to answer to the above questions and this may lead to interesting progress also in the understanding of the full AdS/CFT duality. Here, we will focus on the study of 3-string vertex for the background of eq. (1) and, in particular, we will discuss how to implement all the kinematical symmetries of the background at the level of string interactions.

In Section 2, we begin by discussing the bosonic symmetries of the solution (1). In particular, we focus on a discrete $Z_2$ symmetry and study its realization at the level of the string spectrum and interaction. As remarked in [11], the presence of this discrete symmetry in (1) gives an important physical input in determining the string amplitudes. We then briefly recall the techniques invented in [12] and subsequently generalized to the superstring case in [13, 14] (see also Chap. 11 of [15] and references therein), since our approach is based on those results. We show how to modify the 3-vertex construction in order to accommodate the discrete parity in the interaction, thus providing an explicit counterexample to the no-go theorem presented in the Appendix C of [16].
In the discussion of Section 3, we briefly comment on the general symmetry properties of the string amplitudes and on their relevance for the comparison with the results of field theory [6, 7, 9, 10] or those derived within the string bits approach [8]. We believe that these symmetry properties are of crucial importance in fixing the exact dictionary between the string and gauge theory sides. In fact, symmetry arguments usually provide, to the duality under consideration, robust information that are less dependent on the technical details of the computations. Therefore symmetries should be the first thing to be checked among the various dual descriptions, as it was done in the original AdS/CFT proposal [1].

2 String theory in pp–wave background

2.1 Symmetries of the pp–wave string theory

The bosonic symmetries of the IIB solution (1) that are manifest in the light–cone gauge are summarized by the group $SO(4) \times SO(4) \times Z_2$. The two $SO(4)$ rotate $x^i, i = 1, 2, 3, 4$, and $x^{i'}, i' = 5, 6, 7, 8$ respectively, while the discrete $Z_2$ symmetry swaps the first and the second $SO(4)$ factors

$$Z_2 : \ (x_1, x_2, x_3, x_4) \leftrightarrow (x_5, x_6, x_7, x_8).$$

In addition to the bosonic symmetries, the background also preserves 32 supersymmetries. As usual, we require that string theory respects all the bosonic and fermionic symmetries of the background. Here we will focus in particular on the bosonic $Z_2$ symmetry, since this point has not been well appreciated in the literature so far.

The $Z_2$ transformation (2) is just a particular element of the $SO(8)$ rotation group. It is straightforward to derive its effect on the eight Majorana–Weyl fermionic coordinates that survive the light–cone constraint. In a convenient representation of the $SO(8)$ $\gamma$–matrices, the $Z_2$ action on the spinors is [11]:

$$\theta^3 \leftrightarrow \theta^4, \quad \text{and} \quad \theta^7 \leftrightarrow -\theta^8,$$

while all the other components are unchanged. It is instructive to review how the $Z_2$ symmetry of the background is realized in the string Hilbert space. Having displayed how $Z_2$ acts on the fields through (2) and (3), one needs to specify how $Z_2$ acts on the states of the Hilbert spaces. This can be achieved by specifying the action on the ground state. Two states $|0\rangle$ and $|v\rangle$ play a particular role. The state $|v\rangle$, defined by

$$a_n |v\rangle = b_n |v\rangle = 0 \quad \forall n,$$

has the minimal light–cone energy (zero) and is the true vacuum state of the theory. The state $|0\rangle$ is defined by

$$a_n |0\rangle = 0, \forall n, \quad b_n |0\rangle = 0, n \neq 0, \quad \theta_0 |0\rangle = 0,$$

3Here and in the following we use the same conventions as [16] both for the string mode expansions and for the matrices entering the kinematical constraints. Also we take $\alpha' = 2$ below.
and is not the state of minimal light–cone energy, since it has energy $4\mu$. In the limit $\mu \rightarrow 0$ both these states have vanishing energy and the same is true for all the states created by means of fermionic zero modes. In flat–space $|0\rangle$ is taken to be true vacuum, since its definition preserves the $SO(8)$ invariance of the theory. In the plane-wave background, $|0\rangle$ is related to $|v\rangle$ as follows (for example, for positive $p^+$):

$$|0\rangle = \theta_0^5 \theta_0^6 \theta_0^7 \theta_0^8 |v\rangle .$$  \hfill (6)

This, together with (3), implies that $|0\rangle$ and $|v\rangle$ have opposite $Z_2$ parity. When $\mu \neq 0$, $|v\rangle$ is the real vacuum state and thus it should be taken as $Z_2$ invariant. With this definition the world–sheet action [4] and the free string spectrum are also $Z_2$ invariant. At the interacting level, the 3–string vertex should also respect this $Z_2$ symmetry. By this we mean that two physical amplitudes related by a $Z_2$ transformation should be exactly equal, as it is for amplitudes that are connected by $SO(4) \times SO(4)$ rotations. However, the simplest generalization of the flat–space construction [13, 14] to the plane–wave background (1) considered in [17] does not satisfy this property: the physical amplitudes derived from the vertices [17, 16] are $SO(4) \times SO(4)$ invariant, but transform non–trivially under the $Z_2$ map (2). In fact, consider the vertices constructed in [17, 16]. They all have the same structure

$$|H_3\rangle = [\cdots |0\rangle_{123},$$  \hfill (7)

where $[\cdots]$ is an $SO(4) \times SO(4) \times Z_2$ invariant operator, and we have denoted $|0\rangle_{123} := |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$ for convenience. For instance, one can compute the on-shell amplitudes

$$A^{IJ} := \left( \langle v| \alpha_{n(3)}^I \alpha_{-n(3)}^J \otimes_2 \langle v| \alpha_{m(1)}^I \alpha_{-m(1)}^J \right) |H_3\rangle.$$  \hfill (8)

One can then show that the amplitudes $A^{IJ}$ derived from the vertices of the form (7) are not invariant under $Z_2$. This can be done either by explicit computation of some examples [18] or by using the $Z_2$ action directly in (8) [11]. In particular, one finds that $A^{ij} = -A^{i'j'}$. The basic reason for this asymmetry is that either the 3–string vertex (7) or the string vacuum (4) are odd under $Z_2$ and thus the interacting theory derived from (7) does not realize the $Z_2$ symmetry explicitly.

### 2.2 Constructing the 3-string vertex

In the covariant quantization the 3–string vertex is basically determined by its transformation properties under the BRST charge. In the light–cone gauge all the world–sheet symmetries are fixed and one has to follow a different method. Following [12, 13, 14] one can use the space–time symmetries of the theory to fix the light–cone string interaction. The construction consists of two steps. First, one looks for a string vertex $|V\rangle$ realizing locally on the world–sheet all the kinematical symmetries of the light–cone algebra. Then one has to add a particular polynomial prefactor term [14] in order to respect also the dynamical part of the supersymmetry algebra. In this note, we will concentrate on the
kinematical constraints, which can be satisfied with an ansatz where the bosonic and the fermionic sector are factorized:

$$|V\rangle = \delta \left( \sum_{r=1}^{3} \alpha_r \right) |E_a\rangle |E_b\rangle . \quad (9)$$

Here $\alpha_r$ is related to the $+$ component of the string momentum ($\alpha_r = 2p^+_r$) and $|E_a\rangle$ (resp. $|E_b\rangle$) is the contribution from the bosonic (resp. fermionic) oscillators. Moreover, $|E\rangle$ and $|V\rangle$ are kets in the tensor product of the three independent Hilbert spaces describing the external strings. They have the same structure containing a bilinear exponential part acting on the vacuum$^4$. For instance $|E_a\rangle = \exp \left( \sum a^\dagger_{n(r)} X_{nm}^{rs} \hat{a}^\dagger_{m(s)} \right)|0\rangle_{123}$. In this formalism the kinematical constraints become

$$\sum_{r=1}^{3} \sum_{n \in \mathbb{Z}} \alpha_r X^{(r)}_{mn} \hat{x}_{n(r)} |E_a\rangle = 0, \quad \sum_{r=1}^{3} \sum_{n \in \mathbb{Z}} X^{(r)}_{mn} \hat{p}_{n(r)} |E_a\rangle = 0, \quad (10)$$

$$\sum_{r=1}^{3} \sum_{n \in \mathbb{Z}} \alpha_r X^{(r)}_{mn} \hat{\theta}_{n(r)} |E_b\rangle = 0, \quad \sum_{r=1}^{3} \sum_{n \in \mathbb{Z}} X^{(r)}_{mn} \hat{\lambda}_{n(r)} |E_b\rangle = 0 , \quad (11)$$

where we added the hats on the various modes to stress that they are operators acting on the string Hilbert spaces. Notice that all these constraints (anti)–commute among themselves, thanks to the identity $\sum_{r=1}^{3} \alpha_r (X^{(r)} X^{(r)T})_{mn} = 0$. Thus there is hope to find a state $|V\rangle$ satisfying all the eqs. (10)–(11). However, it is a challenging task to find the vertex by direct solution of the above constraints. The idea of [12] is to write an ansatz for $|V\rangle$ in an integrated form where one can show that the constraints hold, and then derive $|V\rangle$ in the oscillator space by performing explicitly the integrals. In the bosonic sector this procedure completely fixes the form of the matrix $N_{nm}^{rs}$ appearing in the exponential. In this case the integrated ansatz is

$$|E_a\rangle = \prod_{m} \delta \left( \sum_{r=1}^{3} \sum_{n \in \mathbb{Z}} X^{(r)}_{mn} \hat{p}_{n(r)} \right) \int [dp] \prod_{r=1}^{3} \prod_{k=-\infty}^{\infty} \psi(\hat{a}^\dagger_{k(r)}, p_{k(r)}) |V\rangle_{123}. \quad (12)$$

Here the $\hat{p}$’s are the operators in the string mode expansion, while the $p$’s are just $c$–numbers and are integrated with the measure $[dp] = \prod_{r=1}^{3} \prod_{k=-\infty}^{\infty} dp_{k(r)}$. The operators $\psi(\hat{a}^\dagger_{k}, p_{k})$ are related to the harmonic oscillator wave-function of a state with occupation number $k$ (see, for instance, eq. (3.3) of [16]). It is easy to show that (12) satisfies both conditions (10). In order to prove the validity of the second constraint (10), one exploits the fact that the ket $\psi(\hat{a}^\dagger_{k}, p_{k})|\rangle$ is an eigenvector of the momentum operator $\hat{p}_k$. Thus one can insert the $\delta$–function in the integral and eliminate all the hats both in the integrand (12) and in the constraint. The integral becomes then of the form $\int dx \ x \delta(x)$ and is clearly vanishing. On the contrary, for the first equation in (10), it is

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$^4$In the flat space case the vertex $|V\rangle$ has a slightly different structure, since the zero–mode bosonic momenta $\hat{p}_{0(r)}$ and fermionic momenta $\hat{\lambda}_{0(r)}$ have continuous spectra. In the pp–wave case, both $\hat{p}_{0(r)}$ and $\hat{\lambda}_{0(r)}$ are rewritten in terms of oscillators and are not different from the nonzero modes.
easier to keep the $\delta$--function out of the integral and commute the constraint inside the integration. Then, one can realize the operators $\hat{x}$ as $\partial/\partial p$ and obtain an integrand that is a total derivative. The boundary terms do not contribute because of the Gaussian factor $\exp\left(-\frac{1}{2}p^2k(r)\right)$ in the $\psi(\hat{a}_k^\dagger, p_k)$.

The generalization of this procedure to the fermionic sector is straightforward, except for a subtle point in the treatment of the zero modes. The fermionic analogues, $\chi(\hat{b}_k^\dagger, \lambda_{\pm k})$, of the bosonic operators $\psi(\hat{a}_k^\dagger, p_k)$ can only be defined if the fermionic oscillators are paired. The non–zero mode creation operators can be naturally paired $(\hat{b}_k^\dagger, \hat{b}_{-k}^\dagger)$, but the oscillators $\hat{b}_0^\dagger$ have to be treated separately. So the form of $|E_b\rangle$ satisfying (11) for $m = 0$ has to be supplied by hand to the integrated ansatz. Thus we first look for a state satisfying simultaneously

$$\sum_{r=1}^{3} \lambda^a_{0(r)} |\delta\rangle = 0 , \quad \sum_{r=1}^{3} \alpha_r \theta^a_{0(r)} |\delta\rangle = 0 ,$$

In flat space, these conditions are usually solved by the following state

$$|E^0_b\rangle = \prod_{a=1}^{8} \left( \sum_{r=1}^{3} \hat{\lambda}^a_{0(r)} \right) |0\rangle_{123}.$$  \hspace{1cm} (14)

In [17] and in many subsequent papers this same zero mode structure has been adopted also in the construction of the bosonic interaction vertex in the background (1). However from the discussion of the previous section, it is clear that eq. (14) has a quite different behaviour in flat space and in the plane–wave background: in the first case one can define both the vacuum of the theory $|0\rangle$ and eq. (14) to be $SO(8)$ invariant, while in the second case either the true vacuum $|\nu\rangle$ or eq. (14) are $Z_2$–odd. However as explained in [11], there is a different solution of the constraints (13) which is $Z_2$ symmetric together with the definition $Z_2|\nu\rangle = |\nu\rangle$

$$|\delta\rangle = \prod_{a=1}^{8} \left( \sum_{r=1}^{3} \hat{\lambda}^a_{0(r)} \right) \prod_{a=1}^{8} \left( \sum_{r=1}^{3} \alpha_r \theta^a_{0(r)} \right) |\nu\rangle_{123}.$$  \hspace{1cm} (15)

In the appendix C of [16], it is claimed that the zero mode delta-function $|\delta\rangle$ cannot be extended to include non–zero modes such that (11) are satisfied. Of course, if the zero mode structure $|\delta\rangle$ is inserted in the same integral ansatz used in flat space, the first condition in (11) is not satisfied. However, we want to stress here that the form of the integral ansatz usually employed in flat space does not have a fundamental meaning. Its main virtue is to exploit a physical requirement like momentum conservation to solve the kinematical constraints, but this ansatz does not have to be valid in all backgrounds and can be modified to satisfy possible additional symmetry requirements. In the plane–wave case, it is not difficult to deform the usual answer for flat space and to write an exponential vertex that satisfies all the constraints (10) and (11). For example,

$$|E_b\rangle = \exp\left[ \sum_{r,s=1}^{3} \sum_{m,n=1}^{\infty} \hat{b}^\dagger_{-m(r)} Q_{rs}^{mn} \hat{b}^\dagger_{n(s)} - \sqrt{2\Lambda} \sum_{r=1}^{3} \sum_{m=1}^{\infty} \tilde{Q}_m \hat{b}^\dagger_{-m(r)} \right]$$

$$\hspace{1cm} \text{as}$$

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$$\hspace{1cm} \text{as}$$

$$\sum_{r=1}^{3} \lambda^a_{0(r)} |\delta\rangle = 0 , \quad \sum_{r=1}^{3} \alpha_r \theta^a_{0(r)} |\delta\rangle = 0 .$$
\[
\times \exp \left[ \sum_{r,s=1}^{3} \sum_{m,n=1}^{\infty} \hat{b}_{m(r)}^{+} Q_{mn}^{rs} \hat{b}_{-n(s)}^{+} + \frac{\alpha}{\sqrt{2}} \Theta \sum_{r=1}^{3} \sum_{m=1}^{\infty} \hat{Q}_{m}^{r} \hat{b}_{m(r)}^{+} \right],
\]
where
\[
\Lambda := \alpha_{1} \lambda_{0(2)} - \alpha_{2} \lambda_{0(1)}, \quad \Theta := \frac{1}{\alpha_{3}} (\hat{\theta}_{0(1)} - \hat{\theta}_{0(2)}), \quad \alpha = \alpha_{1} \alpha_{2} \alpha_{3}.
\]
\[
[\cdots]' \text{ denote a summation over the positive } \Pi \text{–chirality components } (a = 1, \cdots, 4) \text{ of the spinors } b_{a}^{+}, \text{ while } [\cdots]'' \text{ denote a summation over the negative } \Pi \text{–chirality components } (a = 5, \cdots, 8). \text{ The matrices } Q \text{ are diagonal in the spinor space and we have suppressed the spinor indices. In the notation of [16] they read}
\]
\[
\hat{Q}_{mn}^{rs} := e(\alpha_{1}) \sqrt{\alpha_{s}} \left[ U^{1/2}(r) C^{1/2} N^{rs} C^{-1/2} U^{1/2}(s) \right]_{mn},
\]
\[
\hat{Q}_{m}^{r} := e(\alpha_{1}) \sqrt{\alpha_{s}} \left[ U^{1/2}(r) C^{1/2} C^{1/2} N_{m} \right].
\]

In the Appendix we explicitly show that the exponential (16) satisfies the constraints (10) and (11). In this respect the vertex presented here is on the same footing as the vertex proposed in [16]. However for the vertex (16), the symmetry under the full \( SO(4) \times SO(4) \times Z_{2} \) is compatible with the invariance of the vacuum \( |v\rangle \), while this is not true for the vertex in [16], which is built on the zero–mode structure (14). Thus, just like the other generators of the spacetime symmetry algebra, the discrete \( Z_{2} \) symmetry plays an important role in fixing the form of the string interaction. In fact, it can be used to distinguish between different forms for the kinematical part of the interaction (the one of [16] and the one in eq. (16)) which otherwise have the same properties. In our opinion, only after having correctly implemented this discrete symmetry at the level of the kinematical vertex, it is possible to consider the dynamical symmetries and to look for the supersymmetric completion of the vertex by determining the prefactor.

3 Discussion

The main result of this note is to adapt the usual construction of the light–cone string interaction to the plane–wave case. In particular, we showed that it is actually possible to construct a 3–string vertex that satisfies all the kinematical constraints and gives, at the same time, \( Z_{2} \)–invariant on–shell amplitudes. Since we are considering a \textit{discrete} symmetry, the transformation properties of the various physical quantities can not depend on any continuous parameter. Thus the \( Z_{2} \)–symmetry should appear also in the perturbative Yang–Mills computations, which are valid in the large \( \mu \) limit. However, while this symmetry is manifest in the string setup, its realization on the Yang–Mills side is much less understood. At the present the only explicit computation with operators mixing the two \( SO(4) \)'s is the one of [19]. In that analysis, it turns out that the results involving some two impurity operators are surprisingly (from the field theory point of
view) symmetric. In [19] this behaviour was explained by arguing that the two impurity operators considered were connected by a supersymmetry transformation. This idea has been expanded in various places in the recent literature and it has been proposed that all the 256 two impurity operators are connected by the supersymmetry transformations generated by the 16 supercharges commuting with the light–cone Hamiltonian $^5$. Here we just notice that this observation is nicely consistent with the 3–string vertex presented in this note, since it yields more symmetric result among some two impurity operators. On the contrary, all the vertices of the form (7) imply $A^{ij} = -A^{i'j'}$ and $A_{ij}^{i'} = 0$, which is a puzzling result if all the two impurity states are in the same long supermultiplet.

Of course, the $Z_2$ invariance of the Yang–Mills results needs to be more thoroughly tested. At the moment a systematic approach is quite difficult since we do not yet have a clear and general recipe to compare string and field theory computations. Moreover, recently there has been a radical change of perspective in the plane-wave/CFT correspondence. In the early dynamical comparisons between string and field theory, the idea was to keep valid, also at the interacting level, the dictionary between string states and YM operators proposed in [2]. In this framework each computation represents an independent test of the duality. However more recently it has been proposed that the dictionary has to be adjusted in order to have on the Yang–Mills side orthogonal operators [6]. Of course there are many different field theory basis satisfying this requirement. A particular “string theory” basis on the YM side has been singled out using results of the 3–string vertex. Clearly, if this approach is correct, the results of 3–string amplitudes do not always represent an independent check of the duality.

Another dual description of string theory in the plane–wave background is the string bit model proposed in [20]. In this framework, already now it is possible to deal with a larger class of two impurity operators. In fact the string bit results of [8] are valid for all the states with two bosonic oscillators. Of course, also in this context one has the problem of fixing the dictionary between the spectra of string theory and of string bits. However, the symmetry properties of the 3–point interaction do not seem to be sensitive to this problem; and quite interestingly it seems that the string bit interactions are $Z_2$ symmetric. If this is correct, the agreement between the interaction vertex of [17] and the string bit computations (see the second paper in [8]) cannot be extended to the bosonic operators with both vector indices in the second $SO(4)$. This suggests that the $Z_2$ transformation properties of the amplitudes can represent a check of the relation among the various dual descriptions that can be done without the need of fixing the precise dictionary between operators and states. Of course it would be very interesting to see whether this pattern also appears in the computation on Yang–Mills side of the correspondence.

$^5$See, for example, the version 2 of the second paper in [6].
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Appendix A: The fermionic constraints

In this appendix we will show that the matrices $\bar{Q}_{mn}^{rs}$ and $\bar{Q}_m^r$ do indeed satisfy the fermionic kinematical constraints of eq. (11). For $m = 0$ the constraints reduce to eq. (13) and are automatically solved by the way the zero mode part of the vertex is defined. Notice that for $m \neq 0$ the zero modes appear in (11) only through the combinations $\Theta$ and $\Lambda$, which anticommute with the zero mode structure (15). Thus the way to proceed is to get rid of the annihilation operators in $\hat{\lambda}_n$ and $\hat{\theta}_n$, by commuting them through the exponential part of the vertex and reducing to a set of equations involving only the operators $\hat{b}_n^{(r)}$, $\Theta$ and $\Lambda$. The coefficients of such operators then provide the final constraint equations. Since the matrices $\bar{Q}'$’s appearing in the vertex are diagonal in the spinor space, we can write a set of equations holding for both positive and negative $\Pi$–chirality spinor (namely for each fixed index $a = 1, \ldots, 8$). In the notation of [16], we have:

$m > 0$

\[
B + \sum_{r=1}^{3} e(\alpha_r) \sqrt{|\alpha_r|} A^{(r)} C^{-1/2}_{(r)} U^{1/2}_{(r)} \bar{Q}^r = 0 ,
\]

(20)

\[
\sqrt{|\alpha_s|} A^{(s)} C^{-1/2}_{(s)} U^{-1/2}_{(s)} + \sum_{r=1}^{3} e(\alpha_r) \sqrt{|\alpha_r|} A^{(r)} C^{-1/2}_{(r)} U^{1/2}_{(r)} \bar{Q}^{rs} = 0 ,
\]

(21)

\[
\sqrt{|\alpha_s|} A^{(s)} C^{-1/2}_{(s)} U^{1/2}_{(s)} - \sum_{r=1}^{3} e(\alpha_r) \sqrt{|\alpha_r|} A^{(r)} C^{-1/2}_{(r)} U^{-1/2}_{(r)} \bar{Q}^{sr T} + \alpha B \bar{Q}^{s T} = 0.\]

(22)

For the positive (resp. negative) $\Pi$–chirality spinors the first two equations come from the constraint on $\hat{\lambda}_n$ (resp. $\hat{\theta}_n$) in (11), while the third one comes from the equation involving $\hat{\theta}_n$ (resp. $\hat{\lambda}_n$).

$m < 0$

\[
\sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|}} A^{(r)} C C^{-1/2}_{(r)} U^{-1/2}_{(r)} \bar{Q}^r = 0 ,
\]

(23)

9
\[
A^s C_{(s)} C^{-1/2} - e(\alpha_s) \sqrt{|\alpha_s|} \sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|}} A^{(r)} C_{(r)} C^{-1/2} U^{1/2} U^{-1/2} Q_{rs} U^{-1/2} = 0 , \tag{24}
\]

\[
A^s C_{(s)} C^{-1/2} + e(\alpha_s) \sqrt{|\alpha_s|} \sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|}} A^{(r)} C_{(r)} C^{-1/2} U^{1/2} Q^{rs} U^{1/2} T U_{(s)} = 0 . \tag{25}
\]

In this case the situation is reversed. Eqs. (23) and (24) descend from the $\hat{\theta}_n$ constraint for the positive $\Pi$–chirality modes and from the $\hat{\lambda}_n$ one for the negative modes. Vice-versa, the third equation comes from $\hat{\lambda}_n$ for the positive $\Pi$–chirality modes and from $\hat{\theta}_n$ for the negative ones.

The proof for the non-zero mode constraints is rather tedious but straightforward. To give an idea of how it works we will explicitly solve the constraint in eq.(22). Inserting in the second term of eq.(22) the expression for $\bar{Q}_{rs}$, we obtain

\[
\sum_{r=1}^{3} e(\alpha_s) \frac{\alpha_r}{\sqrt{|\alpha_s|}} A^{(r)} C_{(r)} C^{-1/2} C^{-1/2} N_{rs} C^{1/2} U_{(s)}^{1/2} . \tag{26}
\]

Then using the identity (this and similar identities follow from imposing the bosonic constraints, see for instance [16])

\[
\sum_{r=1}^{3} \alpha_r A^{(r)} C_{(r)} C^{-1/2} C^{-1/2} N_{rs} C^{1/2} = \alpha B \left[ C_{(s)}^{1/2} N^s \right]^T + \alpha_s A^s C_{(s)}^{-1/2} , \tag{27}
\]

and the definition of $Q_{rm}$, eq.(19), we can rewrite eq.(26) as

\[
\sqrt{|\alpha_s|} A^s C_{(s)} C^{-1/2} U_{(s)}^{1/2} + \alpha B Q^s T , \tag{28}
\]

which cancel the other two terms in eq.(22).

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