Finite Automata Encoding Functions

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Abstract Finite automata are used to describe functions, geometric figures, fractals and as a tool for image compression. We introduce a new family of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ encoded as finite automata by utilizing hierarchical tensor product B–splines and the theory of FA–presentable structures. We show that some problems which appear in the framework of hierarchical tensor product B–splines can be efficiently solved using the introduced encoding.

Keywords finite automata, FA–presentable structures, encoding functions, hierarchical meshes, B–splines

1 Introduction

Finite automata are used to encode geometric figures and functions. In [7, 8] Boigelot et al. use a natural representation of points in $\mathbb{R}^d$ by $d$–tuples of infinite $\omega$–words (right–infinite words) and a Buchi automaton to encode a subset in $\mathbb{R}^d$ which consists of the points represented by the $d$–tuples accepted by this Buchi automaton (it is referred to as RVA – Real Vector Automaton). Efficient algorithms for constructing RVA representing the sets of solutions of systems of linear equations and inequalities are described in [6, 7]. Jürgensen, Staiger and Yamasaki [17, 18] use a similar approach to represent subsets of the unit cube $[0, 1]^d$ as finite automata. They obtained a characterization of convex polyhedra encodable as finite automata, see [18, Theorem 26]. Moreover, they show that for a continuously differentiable function $f : [0, 1] \rightarrow [0, 1]$ with non–constant derivative the graph $\Gamma(f)$ is not encodable as a finite automaton. For applications of finite automata for image compression and describing fractals, see, e.g., [10, 11, 22, 23].

In this manuscript we propose a family of functions encoded as finite automata by utilizing hierarchical tensor product B–splines and the theory of FA–presentable structures. Instead of representing points in $\mathbb{R}^d$ by $\omega$–words, we opt to use finite words. The main motivation for this choice is that only finite words can be stored in computer memory and used for computations. A set of finite words over a countable alphabet is at most countable. Therefore, only a countable subset of points in $\mathbb{R}^d$ can be represented by finite words. It appears then quite natural to represent by finite words points in a countable dense subset in $\mathbb{R}^d$.

As we opt to use countable subsets in $\mathbb{R}^d$, the idea to utilize the theory of FA–presentable structures comes naturally from the following fundamental fact: for a first order definition of a relation over the domain of a FA–presentable structure there is an algorithm deciding this relation. The pioneering work linking decidability of the first order theory and finite automata is due to Hodgson [14, 15]. But the systematic study of FA–presentable structures was initiated by Khoussainov and Nerode [20] and

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1FA is the abbreviation for finite automata.
Blumensath and Grädel [3, 4]. For survey articles on the theory of FA–presentable structures the reader is referred to [19, 27, 29]. The first application of FA–presentable structures for representing dense subsets in $\mathbb{R}^d$, for $d = 2$, is due to Gao et al. [13]. In [13] the authors found semiautomatic rings with automatic addition which represent dense subsets of points in $\mathbb{R}^2$ for which rotations by $30^\circ$ or $45^\circ$ are given by automatic (FA–presentable) functions. The present manuscript also considers application of FA–presentable structures, but for encoding functions.

In general, by a function we mean a mapping $f : \mathbb{R}^d \to \mathbb{R}^1$ defined by a $k$–tuple $\vec{f} = (f_1, \ldots, f_k)$, where $f_i : \mathbb{R}^d \to \mathbb{R}$ for $i = 1, \ldots, k$. If each $f_i : \mathbb{R}^d \to \mathbb{R}$ is encoded as a finite automaton, then $\vec{f}$ can be encoded as a $k$–finite automaton. So in this manuscript we discuss only the case $k = 1$. To construct a family of functions $f : \mathbb{R}^d \to \mathbb{R}$ encoded as finite automata we utilize hierarchical tensor product B–splines which are widely used in computer–aided design, engineering and computer graphics. Hierarchical tensor product B–splines are defined by a nested sequence of domains $\Omega^d = \Omega^1 \supseteq \Omega^2 \supseteq \cdots \supseteq \Omega^{N-1} \supseteq \Omega^N = \emptyset$ by the process which is usually referred to as Kraft’s selection mechanism [21]. Mokriš, Jüttler and Giannelli found a geometric condition on the domains $\Omega^1, \ldots, \Omega^{N-1}$ under which the hierarchical tensor product B–splines of degree $m$ span the whole space of $C^{m-1}$ functions which are polynomials of degree $m$, with respect to each of the $d$ variables, in every $d$–dimensional cube (cell) of the hierarchical mesh generated by these domains [24]. Such functions are referred to as splines over a hierarchical mesh.

We associate each cell of a hierarchical mesh with its barycentre. Each of these barycentres gives a coordinate $d$–tuple which we present as the $d$–tuple of strings obtained from a standard representation of elements of the group $(\mathbb{Z}/[1/b], +)$ for even base $b$, where $\mathbb{Z}/[1/b] = \{s/b^\ell \mid s, \ell \in \mathbb{Z}, \ell \geq 0\}$. For this representation of elements of $(\mathbb{Z}/[1/b], +)$ the addition is FA–recognizable (see Subsection 3). We use the same representation for the coefficients $\lambda_d$ of a spline $\sum_{s \in K} \lambda_s s$ over the hierarchical mesh (see Subsection 2), assuming that they are all in $\mathbb{Z}/[1/b]$. Finally both a hierarchical mesh and a spline over it can be described by languages of $d$–tuples and $(d+1)$–tuples of strings, respectively. Note that one can use other representations of elements of $(\mathbb{Z}/[1/b], +)$ for which the addition is FA–recognizable. Moreover, it is possible to use other countable abelian subgroups in $\mathbb{R}^d$ and their representations for which the addition is FA–recognizable. For nonstandard FA–presentations of some countable abelian subgroups of $\mathbb{Q}^d$ see [1, 25]. Note that the group $(\mathbb{Q}, +)$ cannot be used for our purposes as it is not FA–presentable [30].

Let us consider hierarchical meshes and splines over them for which the languages describing them are regular. This defines a family of hierarchical meshes and splines over them which we will call regular, see Sections 5 and 6, respectively. In particular, each of the regular splines is encoded as a (deterministic) finite automaton. We note that, by construction of splines over hierarchical meshes, a continuous function defined over them which we will call regular, see Sections 5 and 6, respectively. In particular, we provide a verification procedure for this nestedness condition.

Let us be given a deterministic finite automaton corresponding to either a hierarchical mesh $\mathcal{T}$ or a spline function $f$ over it as an input. In this manuscript we will also show that some problems arising in the framework of hierarchical tensor product B–splines can be efficiently resolved using the encodings by finite automata. Namely, we will describe the following procedures and algorithms:

a) A $N$–level hierarchical mesh $\mathcal{T}$ must be defined by a nested sequence of domains $\Omega^d = \Omega^1 \supseteq \cdots \supseteq \Omega^{N-1} \supseteq \Omega^N = \emptyset$, where $\Omega^{N-1} \neq \emptyset$. We provide a verification procedure for this nestedness condition.

b) Kraft’s selection mechanism generate a complete basis of a spline space over a hierarchical mesh $\mathcal{T}$ if a certain geometric condition (see Assumption 2) on the
shapes of the domains $\Omega^1, \ldots, \Omega^{N-1}$ is satisfied [24]. We provide a verification procedure for this geometric condition.

c) For a hierarchical mesh $T$ each basis function generated by Kraft’s selection mechanism can be naturally associated with a certain cell of this mesh (see Figure 4 for the association rule). We show that a collection of cells (encoded by strings representing their barycentres) forms a regular language and provide a procedure for constructing deterministic finite automaton accepting this language.

d) We provide an algorithm for computing $f(\mathbf{x})$, the value of a spline function $f$ at any point $\mathbf{x} \in \mathbb{Z}[1/b^d] \subset \mathbb{R}^d$ given as an input.

e) Suppose that a hierarchical mesh $T$ is refined by selecting a nonempty subdomain $\Omega^N \subseteq \Omega^{N-1}$. The domain $\Omega^N$ is composed of cells which we again encode by strings representing their barycentres. We assume that the language of such strings is regular and we are given a deterministic finite automaton recognizing this language. For the refined hierarchical mesh $T'$ the coefficients of $f$ must be updated. We provide a procedure for constructing deterministic finite automaton corresponding to $f$ with respect to $T'$.

For verification procedures in a) and b) we provide first order formulas checking the nestedness and the geometric condition in Assumption 2, respectively. Similarly, for the procedure in c) we provide the first order formula defining a regular language. A concrete implementation of an algorithm constructing a deterministic finite automaton mostly follows from Theorem 2. For an algorithm computing the value of a spline function at a given point in d) we use the fact that for a FA–recognizable function there is an algorithm computing it in linear time (see Section 3). For e) we show how to construct a FA–recognizable relation describing a spline function over the refined mesh.

The rest of the paper is organized as follows. In Section 2 we provide necessary background on splines over hierarchical meshes. In Sections 3 and 4 we recall necessary definitions and facts from automata theory and the theory of FA–presentable structures, respectively. In Section 5 we introduce regular hierarchical meshes. In Subsections 5.1, 5.2 and 5.3 we describe the above mentioned procedures a), b) and c). In Section 6 we introduce regular splines. In Subsections 6.1 and 6.2 we describe the above mentioned algorithm d) and the procedure e). In Subsection 6.3 we show simple examples of regular splines.

2 Splines with Maximum Order Smoothness

For a given integer $\ell \geq 0$, we denote by $T^\ell$ a bi–infinite knot vector:

$$T^\ell = (\ldots, t^\ell_{i-1}, t^\ell_i, t^\ell_{i+1}, \ldots),$$

where $t^\ell_i = \frac{i}{2^\ell}$ for $i \in \mathbb{Z}$. Note that $T^\ell$ is uniform with the same distance between consecutive knots equal to $\frac{1}{2^\ell}$. Let $d$ be a positive integer. We denote by $G^\ell_d$ a $d$–dimensional grid consisting of the hyperplanes $H^\ell_{j,i} = \{(x_1, \ldots, x_d) | x_j = t^\ell_i\}$ for $j = 1, \ldots, d$ and $i \in \mathbb{Z}$. For a given integer $m \geq 0$, a grid $G^\ell_d$ defines the set of tensor product B–splines $B^\ell_{d,m}$ each of which is the product:

$$P^\ell_{i_1, \ldots, i_d,m}(x_1, \ldots, x_d) = N^\ell_{i_1,m}(x_1) \ldots N^\ell_{i_d,m}(x_d),$$

where $i_1, \ldots, i_d \in \mathbb{Z}$ and, for $i \in \mathbb{Z}$, $N^\ell_{i,m}(t)$ is the $i$th B–spline basis function of degree $m$ associated to the knot vector $T^\ell$ which is recursively defined by Cox–de Boor’s formula:

$$N^\ell_{i,0}(t) = \begin{cases} 1, & t^\ell_i \leq t < t^\ell_{i+1}, \\ 0, & \text{otherwise.} \end{cases},$$

(2)
where $j = 1, \ldots, m$. Each tensor product B-spline $P_{i_1, \ldots, i_d, m}^\ell$ has local support:

$$\{(x_1, \ldots, x_d) \mid \left( P_{i_1, \ldots, i_d, m}^{t_1} \right)^{(1)}(x_1, \ldots, x_d) \neq 0 \} = (t_1^\ell, t_{i_1+m+1}^\ell) \times \cdots \times (t_d^\ell, t_{i_d+m+1}^\ell)$$

on which it takes positive values. Tensor product B-splines from $B_d^\ell,m$ are locally linear independent: for every open bounded set $U \subseteq \mathbb{R}^d$ the tensor product B-splines from $B_d^\ell,m$ having nonempty intersections of its support with $U$ are linearly independent on $U$. For introduction to B-splines we refer the reader to, e.g., [20].

For a given $\ell$ we denote by $C_d^\ell$ the collection of all closed $d$-dimensional cubes $\prod_{j=1}^d [t_j^\ell, t_{j+1}^\ell]$. Following [24] we call each of the cubes from $C_d^\ell$ a cell of the grid $G_d^\ell$ (or simply a cell). Let us consider a nested sequence of domains

$$\Omega^0 = \mathbb{R}^d \supset \Omega^1 \supset \cdots \supset \Omega^{N-1} \supset \Omega^N = \varnothing,$$

where $\Omega^{N-1} \neq \varnothing$.

**Assumption 1.** We assume that each $\Omega^\ell$, $\ell = 1, \ldots, N - 1$ is composed of cells from $C_{d-1}^{\ell-1}$. That is, for each $\ell = 1, \ldots, N - 1$ there is a subset $M \subseteq C_d^{\ell-1}$ for which $\Omega^\ell = \bigcup_{c \in M} c$.

**Remark 1.** Note that, since $\Omega^0 = \mathbb{R}^d$, this assumption is exactly equivalent to [24, Assumption 3.1]: each set $\Omega^0 \setminus \Omega^{\ell+1}$ for $\ell = 0, \ldots, N - 1$ must be composed of cells from $C_d^\ell$. However, in this paper we do not assume that the domains $\Omega_1, \ldots, \Omega_{N-2}$ are bounded. They can be bounded or unbounded.

A hierarchy of domains $\Omega^0 = \mathbb{R}^d \supset \Omega^1 \supset \cdots \supset \Omega^{N-1} \supset \Omega^N = \varnothing$ satisfying Assumption 1 creates a subdivision of $\mathbb{R}^d$ into the collection of cells $R^\ell \subseteq C_d^\ell$ such that $\Omega^\ell \setminus \Omega^{\ell+1} = \bigcup_{c \in M} c$ for $\ell = 0, \ldots, N - 1$. We denote the subdivision of $\mathbb{R}^d$ into the cells from $R^\ell$, $\ell = 0, \ldots, N - 1$ by $T$. $T$ is also referred to as a $d$-dimensional box-partition.

If $d = 2$, then $T$ is also called a T-mesh. We will simply call $T$ a hierarchical mesh. See Fig. 1 for an example of a 2-dimensional hierarchical mesh generated by a nested sequence of domains $\Omega^0 = \mathbb{R}^2 \supset \Omega_1 \supset \Omega_2 \supset \Omega_3 = \varnothing$. We denote by $T_d$ the collection of all $d$-dimensional cells of $T$: $T_d = \bigcup_{\ell=0}^{N-1} R^\ell$.

**Definition 1.** We denote by $S_m(T)$ the space of functions $f : \mathbb{R}^d \to \mathbb{R}$ of the class $C^{m-1}$ which are polynomials of multi-degree $(m, \ldots, m)$ in every cell from $T_d$. That is, for every $c \in T_d$, $f|_c = \sum_{i_1, \ldots, i_d=0}^{m} a_{i_1, \ldots, i_d} x_1^{i_1} \cdots x_d^{i_d}$. A function from $S_m(T)$ is called a spline with maximum order of smoothness over $T$.

In Definition 1 one can require a weaker assumption for a function $f$: the derivatives $\frac{\partial^{m-1} f}{\partial x_i^{m-1}}$ exist and continuous everywhere in $\mathbb{R}^d$ for $i = 1, \ldots, d$. In Appendix 7, we show that this weaker assumption does not affect the space of functions $S_m(T)$, see Proposition 2. Moreover, as the same proposition shows, a stronger assumption: all derivatives $\frac{\partial^{k_1+\cdots+k_d} f}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}$ exist and continuous everywhere in $\mathbb{R}^d$ for $k_i = 0, \ldots, m - 1$ and $i = 1, \ldots, d$, does not affect the space $S_m(T)$ as well.

For a given $\ell = 0, \ldots, N - 1$ let $M^\ell = \Omega^0 \setminus \Omega^{\ell+1} = \mathbb{R}^d \setminus \Omega^{\ell+1}$. Then we have a nested sequence of domains $M^{N-1} = \varnothing, M^0 = \mathbb{R}^d \setminus \Omega^1, M^1 = \mathbb{R}^d \setminus \Omega^2, \ldots, M^{N-2} = \mathbb{R}^d \setminus \Omega^{N-1}, M^{N-1} = \mathbb{R}^d$.

$$\varnothing = M^{N-1} \subseteq M^{N-2} \subseteq \cdots \subseteq M^1 \subseteq M^0 = \mathbb{R}^d.$$ (4)
By Assumption 1, each domain $M_\ell$ is composed of the cells from $C_\ell$ for $\ell = 0, \ldots, N-1$. Now we are ready to formulate Theorem 1 – the main result of [24] which states that if the domains (4) satisfy the following Assumption 2, then each $f \in S_m(T)$ can be uniquely written as $f = \sum_{\delta \in K} \lambda_\delta \delta$, where $K$, given by (5), is the collection of hierarchical tensor product B–splines obtained from Kraft’s selection mechanism. See Remark 2 for an informal description of Kraft’s selection mechanism. For complete details and proofs see Appendix 8. Theorem 1 is proved in Appendix 9.

**Assumption 2.** For a nested sequence of domains:

$$\emptyset = M^{-1} \subseteq M^{0} \subseteq M^{1} \subseteq \cdots \subseteq M^{N-2} \subseteq M^{N-1} = \mathbb{R}^d$$

we assume that for each $\ell = 0, \ldots, N-2$ the domain $M^\ell$ satisfies the condition: for each $\beta \in B_{d,m}^\ell$, $\ell = 0, \ldots, N-2$, for which $\text{supp} \beta \cap M^{\ell-1} = \emptyset \wedge \text{supp} \beta \cap M^{\ell} \neq \emptyset$, the intersection $\text{supp} \beta \cap M^{\ell}$ is connected. See also Appendix 8 where this condition is formulated in Corollary 2.

We say that a hierarchical mesh $T$ satisfies Assumption 2 if it is generated by a nested sequence of domains $\Omega^{\ell} = \mathbb{R}^d \supseteq \Omega^{1} \supseteq \cdots \supseteq \Omega^{N-1} \supseteq \Omega^{N} = \emptyset$ for which the domains $M^{\ell} = \Omega^{\ell} \setminus \Omega^{\ell+1}$, $\ell = 0, \ldots, N-1$ satisfy Assumption 2. For $\ell = 0, \ldots, N-1$, let:

$$K^{\ell} = \{ \beta \in B_{d,m}^\ell | \text{supp} \beta \cap M^{\ell-1} = \emptyset \wedge \text{supp} \beta \cap M^{\ell} \neq \emptyset \}$$

and $K = \bigcup_{\ell=0}^{N-1} K^{\ell}$. (5)

Each formal sum $\sum_{\delta \in K} \lambda_\delta \delta$ defines a function from $S_m(T)$. Moreover, each function $f \in S_m(T)$ can be uniquely represented as a formal sum $\sum_{\delta \in K} \lambda_\delta \delta$ as stated below.

**Theorem 1 ([24]).** Assume that a hierarchical mesh $T$ satisfies Assumption 2. Then for every $f \in S_m(T)$, $f = \sum_{\delta \in K} \lambda_\delta$ for some uniquely defined coefficients $\lambda_\delta$.

**Proof.** See Appendix 9 for the proof.

**Remark 2.** The equation (5) defines a procedure usually known as Kraft’s selection mechanism for generating basis functions. Informally, it can be described as follows. At the first iteration this mechanism takes all tensor product B–splines from $B_{d,m}^0$ (they are all tensor product B–splines with respect to the grid $G^d_0$ with the support overlapping with
the domain $\Omega = \mathbb{R}^d$). At the second iteration it removes all tensor product $B$–splines with the support in the domain $\Omega_1$ obtained at the previous iteration and add tensor product $B$–splines from $B^1_{1,m}$ with the support in the domain $\Omega_1$. At the third iteration it removes all tensor product $B$–splines with the support in the domain $\Omega_2$ obtained at the previous iteration and add tensor product $B$–splines from $B^2_{2,m}$ with the support in the domain $\Omega_2$ and etc. The process stops after the $N$th iteration.

3 Multitape Synchronous Finite Automata

Let us first recall the notion of finite automata and regular languages; see, e.g., [16]. Let $\Sigma$ be a finite alphabet. We say that $w$ is a string over the alphabet $\Sigma$ if $w$ is a finite sequence of symbols $\sigma_1\sigma_2\ldots\sigma_n$, where $\sigma_i \in \Sigma$ for $i = 1, \ldots, n$ and $n$ is a nonnegative integer. We denote by $|w|$ the length of the string $w$: $|w| = n$. If $n = 0$, $w$ is the empty string which we denote by $\varepsilon$. A collection of all strings over the alphabet $\Sigma$ is denoted by $\Sigma^*$. A nondeterministic finite automaton $M$ over the alphabet $\Sigma$ consists of a finite set of states $S$, a set of initial states $I \subseteq S$, a transition functions $T : S \times \Sigma \to P(S)$ and a set of accepting states $F \subseteq S$. The automaton $M$ accepts a string $w = \sigma_1\ldots\sigma_n$ if there exists a sequence of states $s_1, \ldots, s_{n+1} \in S$ for which $s_1 \in I$, $s_{i+1} \in T(s_i, \sigma_i)$ for all $i = 1, \ldots, n$ and $s_{n+1} \in F$. We say that a language $L \subseteq \Sigma^*$ is recognized by $M$ if $L$ consists of all strings accepted by $M$. A language recognized by a nondeterministic finite automaton is called regular. $M$ is called a deterministic finite automaton if $I$ has exactly one element and for each state $s \in S$ and a symbol $\sigma \in \Sigma$ the set $T(s, \sigma)$ has exactly one element. Both deterministic and nondeterministic finite automata have the same computational power - they recognize the class of regular languages.

We denote by $\Sigma_0 = \Sigma \cup \{\emptyset\}$ the alphabet $\Sigma_0 = \Sigma \cup \{\emptyset\}$; it is assumed that the padding symbol $\emptyset$ is not in the alphabet $\Sigma$. We denote by $\Sigma^k$ the Cartesian product of $k$ copies of $\Sigma_0$. Let $w_1, \ldots, w_k \in \Sigma^*$ be some strings over the alphabet $\Sigma$. The convolution $w = w_1 \otimes \cdots \otimes w_k$ of the strings $w_1, \ldots, w_k$ is the string $w$ over the alphabet $\Sigma^k \setminus \{\emptyset, \ldots, \emptyset\}$ such that for the $i$th symbol $(\sigma_1^i, \ldots, \sigma_k^i)$ of $w$ the symbol $\sigma_j^i$ is the $j$th symbol of $w_j$ if $i \leq |w_j|$ and $\emptyset$, otherwise, for $i = 1, \ldots, |w|$ and $j = 1, \ldots, k$, where $|w| = \max\{|w_j| | j = 1, \ldots, k\}$.

For example, the convolution of three strings $w_1 = 001101$, $w_2 = 10100101110$ and $w_3 = 100101$ is as follows:

$$
\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 1 & 0 & 1 & \emptyset & \emptyset & \emptyset & \emptyset \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
\end{array}
$$

For a given relation $R \subseteq \Sigma^k$, we denote by $\otimes R$ the relation:

$$
\otimes R = \{w_1 \otimes \cdots \otimes w_k | (w_1, \ldots, w_k) \in R\} \subseteq \left(\Sigma^k \setminus \{\emptyset, \ldots, \emptyset\}\right)^* .
$$

We say that a relation $R$ is FA–recognizable if $\otimes R$ is a regular language over the alphabet $\Sigma^k \setminus \{\emptyset, \ldots, \emptyset\}$. One can think of a finite automaton recognizing $\otimes R$ as a read–only $k$–tape Turing machine with the input $w_i$ written on the $i$th tape for each $i = 1, \ldots, k$ and $k$ heads moving synchronously from the left to the right until the whole input is read; after that the input is either accepted or rejected. Such an automaton is also called a $k$–tape synchronous finite automaton.

Let $f : D \to \Sigma^{n+m}$ be a function from $D \subseteq \Sigma^n$ to $\Sigma^{n+m}$ for any integers $n, m \geq 1$. We denote by $\text{Graph}(f)$ the graph of $f$, i.e., $\text{Graph}(f) = \{(\overline{n}, \overline{m}) \in \Sigma^n \times \Sigma^{n+m} | f(\overline{n}) = \overline{m}\} \subseteq \Sigma^{(n+m)}$. We say that $f$ is FA–recognizable, if $\text{Graph}(f)$ is FA–recognizable. Clearly, if $f$ is FA–recognizable, $D \subseteq \Sigma^n$ must be FA–recognizable. Assuming that $n = m = 1$, for a FA–recognizable function $f : D \to \Sigma^*$, where $D \subseteq \Sigma^*$, there is a linear–time
algorithm which for a given input \( \pi \) returns the output \( \pi = f(\pi) \), see, e.g., the proof of [12, Theorem 2.3.10]. Moreover, there is a characterization of FA–recognizable functions 
\[ f : D \to \Sigma^* \] as functions computed by a deterministic position–faithful one–tape Turing machine in linear time, see below.

A position–faithful one–tape Turing machine is a Turing machine which uses a semi–infinite tape with the left–most position containing the special symbol \( \boxempty \) which only occurs at this position and cannot be modified. The initial configuration of the tape is \( \boxempty \)}{u}{\boxempty}^{\infty} \), where \( \boxempty \) is a special blank symbol, and \( u \in \Sigma^* \) for some alphabet \( \Sigma \) with \( \Sigma \cap \{\boxempty, \boxempty\} = \emptyset \). During the computation the Turing machine operates as usual, reading and writing cells to the right of the \( \boxempty \) symbol.

A function \( f : D \to \Sigma^* \) from a regular domain \( D \subseteq \Sigma^* \) to \( \Sigma^* \) is said to be computed by a position–faithful one–tape Turing machine, if when started with tape content being \( \boxempty \)}{u}{\boxempty}^{\infty} \), the head initially being at \( \boxempty \), the Turing machine eventually reaches an accepting state (and halts), with the tape content starting with \( \boxempty f(u) \boxempty \).

There is no restriction on the output beyond the first appearance of \( \boxempty \).

Case, Jain, Seah and Stephan showed that a function \( f : D \to \Sigma^* \), \( D \subseteq \Sigma^* \), is FA–recognizable if and only if it is computed by a deterministic position–faithful one–tape Turing machine in linear time [9]. This characterization of FA–recognizable functions
\[ f : D \to \Sigma^* \] , \( D \subseteq \Sigma^* \), clearly holds valid for FA–recognizable relations \( f : D \to \Sigma^m \)

Namely, a function \( f : D \to \Sigma^m \), for \( D \subseteq \Sigma^n \), is FA–recognizable if and only if it is computed by a deterministic position–faithful one–tape Turing machine in linear time, where the input is a string from \( \otimes D \) and the output is a string from \( \otimes \Sigma^m \).

Furthermore, their result clearly holds valid for multivalued FA–recognizable functions \( f \) if it is assumed that the number of values that \( f \) can take for each argument in \( D \) is bounded from above by some fixed constant. That is, a multivalued function \( f : D \to \Sigma^m \), for \( D \subseteq \Sigma^n \), satisfying this assumption, is FA–recognizable if and only if for a given input \( u \in \otimes D \) all values \( f(u) \) are computed by a deterministic position–faithful one–tape Turing machine in linear time.

\[ \Box \]

## 4 FA–Presented Structures

Now we are ready to discuss a key ingredient to be used in Sections 5 and 6. Let us first recall the notion of FA–presented structures as it was introduced by Khoussainov and Nerode [20]. Let
\[ A = (A; R_1^{m_1}, \ldots, R_s^{m_s}, f_1^{n_1}, \ldots, f_r^{n_r}, c_1, \ldots, c_t) \]
be a structure, where \( A \subseteq \Sigma^* \) for some alphabet \( \Sigma^* \), \( R_i^{m_i} \subseteq A^{m_i} \) are \( m_i \)–ary relations for \( i = 1, \ldots, s \), \( f_j^{n_j} : A^{n_j} \to A \) are \( n_j \)–ary functions for \( j = 1, \ldots, r \), and \( c_k \) are constants for \( k = 1, \ldots, t \). The structure \( A \) is said to be FA–presented if \( A \) is regular, the relations \( R_i^{m_i} \) and the functions \( f_j^{n_j} \) are FA–recognizable for all \( i = 1, \ldots, s \) and \( j = 1, \ldots, r \). A structure is said to be FA–presentable if it is isomorphic to a FA–presented structure.

FA–presented structures enjoy the following fundamental properties, see [20, Corollary 4.2]. There exists an effective procedure that for a given first order definition of a relation \( R \) of a FA–presented structure \( A \) yields an algorithm deciding \( R \). The first order theory of a FA–presented structure \( A \) is decidable. The proof of these two properties follows from the standard facts in automata theory which can be summarized as follows.

\[ \textbf{Theorem 2. (see [20, Theorem 4.4])} (1) Let } R_1, R_2 \text{ and } R \text{ be FA–recognizable relations. Then the relations corresponding to the expressions } (R_1 \lor R_2), (R_1 \land R_2), (R_1 \to R_2), (\neg R_1), \exists v \ R \text{ and } \forall v \ R \text{ are also FA–recognizable, where for a } k \text{–ary relation } R(v_1, \ldots, v_k), \]

\[ \text{[FA is a short for finite automata.]} \]
for $k > 1$, and a variable $v_i, i = 1, \ldots, k$:

$$\exists v R = \{(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) \mid (v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k) \in R\},$$

$$\forall v R = \{(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) \mid \forall v_i \in A ((v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k) \in R)\}.$$  

(2) The emptiness problem for finite automaton is decidable. That is, for a unary FA–recognizable relation $R(v)$ there is an algorithm which for a given deterministic finite automaton accepting $R$ decides whether $\exists v R$ is true or false. Similarly, there is an algorithm deciding whether $\forall v R$ is true or false.

(3) There exists a procedure which for deterministic multi–tape synchronous finite automata recognizing $R_1, R_2$ and a $k$-ary relation $R(v_1, \ldots, v_k)$, for $k > 1$, constructs deterministic multi–tape synchronous finite automata for recognizing the relations corresponding to the expressions ($R_1 \lor R_2$), ($R_1 \land R_2$), ($R_1 \rightarrow R_2$), ($\neg R_1$), $\exists v R$ and $\forall v R$ for $i = 1, \ldots, k$.

A brief sketch of the proof of Theorem 2 is as follows. Part (1) follows from part (3). Part (2) for $\exists v R$ is the standard fact from automata theory, see, e.g., [16, Theorem 3.7]. For $\forall v R$ it follows from the equivalency of $\forall$ and the composition $\neg \circ \exists \circ \neg$ (see the same argument used in the proof of [12, Theorem 1.4.6]). As for part (3), it is enough to show it only for the expressions ($R_1 \land R_2$), ($\neg R_1$) and $\exists v R$. For the expression ($R_1 \land R_2$) it follows from the product construction for a deterministic finite automaton accepting the intersection of two regular languages. For ($\neg R_1$) it follows from a construction of a deterministic finite automaton accepting the complement of a given regular language by swapping accepting and non–accepting states. For $\exists v R$ it follows from the standard Rabin–Scott powerset construction for converting a nondeterministic finite automaton into deterministic finite automaton; see, e.g., [16, Theorem 2.1].

5 Regular Hierarchical Meshes

Let $b$ be a positive integer divisible by 2. We denote by $\mathbb{Z}[1/b]$ the abelian group of all rational numbers of the form $\frac{s}{\ell}$ for $s, \ell \in \mathbb{Z}$ and $\ell > 0$. Each positive $z \in \mathbb{Z}[1/b]$ can be uniquely represented as the sum of its integral and fractional parts:

$$z = [z]_i + [z]_f = \sum_{i=1}^{k} \alpha_i b^{-i-1} + \sum_{i=1}^{k} \beta_i b^{-i},$$

(6)

where $\alpha_i, \beta_i \in \{0, 1, \ldots, b-1\}$ for all $i = 1, \ldots, k$ for which either $\alpha_k \neq 0$ or $\beta_k \neq 0$.

Let $\Sigma_b$ be the alphabet consisting of the symbols $\frac{\alpha}{\beta}$, where $\alpha, \beta \in \{0, 1, \ldots, b-1\}$. Now, for a given positive $z \in \mathbb{Z}[1/b]$ we represent it as a string:

$$\begin{array}{cccc}
0 & \alpha_1 & \alpha_2 & \ldots & \alpha_k \\
0 & \beta_1 & \beta_2 & \ldots & \beta_k
\end{array}$$

(7)

over the alphabet $\Sigma_b$. The first symbol $0$ indicates that $z$ is positive. Let $z \in \mathbb{Z}[1/b]$ be negative and $-z = \sum_{i=1}^{k} \alpha'_i b^{-i-1} + \sum_{i=1}^{k} \beta'_i b^{-i}$ be the decomposition of the form (6) for $-z > 0$. We represent $z$ as a string:

$$\begin{array}{cccc}
1 & \alpha'_1 & \alpha'_2 & \ldots & \alpha'_{k'} \\
1 & \beta'_1 & \beta'_2 & \ldots & \beta'_{k'}
\end{array}$$

(8)

over the alphabet $\Sigma_b$. The first symbol $1$ indicates that $z$ is negative.
For $z \in \mathbb{Z}[1/b]$ we denote by $(z)_b \in \Sigma_b^*$ the string \((7)\) if $z > 0$, the string \((8)\) if $z < 0$ and the string \(0\) if $z = 0$. For example, if $z = -\frac{27}{16}$, then $(z)_2 = \frac{1}{1} \frac{1}{1} \frac{0}{1} \frac{1}{1}$.

The language $L_b = \{(z)_b | z \in \mathbb{Z}[1/b]\}$ is regular. We denote by $\psi_b : L_b \rightarrow \mathbb{Z}[1/b]$ the bijection which maps a string $(z)_b \in L_b$ to $z \in \mathbb{Z}[1/b]$. For both cases, $L_b$ and $\psi_b$, the subscript indicates the base $b$. For $b = 2$, the representation $\psi_2$, up to minor modification, coincides with the representation of $\mathbb{Z}[1/2]$ described in [25, § 2]. Let us denote by $\text{Add}$ the graph of the addition operation in $\mathbb{Z}[1/b]$ with respect to $\psi_b$, namely, $\text{Add} = \{(u,v,w) \in \mathbb{L}_b \times \mathbb{L}_b \times \mathbb{L}_b | \psi_b(u) + \psi_b(v) = \psi_b(w)\}$. The relation $\text{Add}$ is FA–recognizable [25, § 2]. We denote by $\text{add}$ the addition operation in $\mathbb{L}_b$, that is, $\text{add} : \mathbb{L}_b \times \mathbb{L}_b \rightarrow \mathbb{L}_b$ is a two–place function for which $\text{add}(u,v) = w$ if $\psi_b(u) + \psi_b(v) = \psi_b(w)$.

For a given $d$–tuple $\tau = (z_1, \ldots, z_d) \in \mathbb{Z}[1/b]^d$ let us denote by $\tau_b$ the convolution $(z_1)_b \otimes \cdots \otimes (z_d)_b$ of strings $(z_1)_b, \ldots, (z_d)_b \in \mathbb{L}_b$. Clearly, the language $L_b^d = \{w_1 \otimes \cdots \otimes w_d | w_i \in \mathbb{L}_b, i = 1, \ldots, d\}$ is regular. We define by $\psi_b^d : L_b^d \rightarrow \mathbb{Z}[1/b]^d$ the bijection which maps a string $(\tau)_b \in L_b^d$ to $\tau \in \mathbb{Z}[1/b]^d$. For both cases, $L_b^d$ and $\psi_b^d$, the superscript indicates the dimension $d$. Let us denote by $\text{Add}_d$ the graph of the addition operation in $\mathbb{Z}[1/b]^d$ with respect to $\psi_b^d$, namely, $\text{Add}_d = \{(u,v,w) \in L_b^d \times L_b^d \times L_b^d | \psi_b^d(u) + \psi_b^d(v) = \psi_b^d(w)\}$. The relation $\text{Add}_d$ is FA–recognizable. We denote by $\text{add}_d$ the addition operation in $L_b^d$, that is, $\text{add}_d : L_b^d \times L_b^d \rightarrow L_b^d$ is a two–place function for which $\text{add}_d(u,v) = w$ if $\psi_b^d(u) + \psi_b^d(v) = \psi_b^d(w)$. Clearly, if $d = 1$, then $L_b^d = \mathbb{L}_b$, $\psi_b^d = \psi_b$, $\text{Add}_1 = \text{Add}$ and $\text{add}_1 = \text{add}$.

Let $T$ be a $d$–dimensional hierarchical mesh defined by a nested sequence of domains:

$$\Omega^0 = \mathbb{R}^d \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^{N-1} \supseteq \Omega^N = \emptyset,$$

where $\Omega^{N-1} \neq \emptyset$ and each $\Omega^\ell$, $\ell = 1, \ldots, N - 1$ is composed of cells from $\mathcal{C}_d^{\ell-1}$.

For each $d$–dimensional cube $c = \prod_{j=1}^{d} [t^j_0, t^j_1]$ we associate it with its barycentre $\tau_c = (z_1, \ldots, z_d)$, where $z_j = \frac{1}{2}(t^j_0 + t^j_1)$ for $j = 1, \ldots, d$ (see Fig. 2).

![Figure 2](image)

Figure 2: The figure shows a 2–dimensional cell and its barycentre (a black dot in the centre of the cell).

For each $\ell = 1, \ldots, N - 1$ we denote by $L_\ell \subset L_b^d$ the language:

$$L_\ell = \{(\tau)_b \mid c \in \mathcal{C}_d^{\ell-1} \land c \subseteq \Omega^\ell\}.$$

**Definition 2.** We say that a hierarchical mesh $T$ is regular if the language $L_\ell$ is regular for each $\ell = 1, \ldots, N - 1$.

The languages $L_\ell$, $\ell = 1, \ldots, N - 1$ are pairwise disjoint: $L_i \cap L_j = \emptyset$ for $i, j = 1, \ldots, N - 1$ and $i \neq j$. Let $L = L_1 \cup \cdots \cup L_{N-1}$. The following proposition shows that the hierarchical mesh $T$ is regular if and only if the language $L$ is regular.

**Proposition 1.** The language $L$ is regular if and only if each language $L_\ell$, $\ell = 1, \ldots, N - 1$ is regular.
The inclusion is true for the structure $\Omega$ composed of cells from $\mathbb{E}$ which contain the evaluation of $\Phi$ of the form $\mathbb{C}$ for which $c \subseteq \Omega \ell$ $\ell \subseteq \Omega$ $\subseteq \Omega$. Let us be given regular languages $L_\ell$, $\ell = 1, \ldots, N - 1$ representing domains $\Omega_1, \ldots, \Omega_N$ composed of cells from $C_0^d, \ldots, C_{N-1}^d$, respectively. How can one verify that the domains $\Omega_1, \ldots, \Omega_N$ are nested: $\Omega_1 \supseteq \cdots \supseteq \Omega_N$?

In order to verify nestedness one has to verify that for each $\ell = 2, \ldots, N - 1$: $\Omega_\ell \subseteq \Omega_{\ell - 1}$. Let $c \in C_{\ell - 1}^d$ be a cell for which $c \subseteq \Omega_{\ell - 1}$, and only if there exists a cell $c' \in C_{\ell - 2}^d$ for which $c \subseteq c'$ and $c' \subseteq \Omega_{\ell - 1}$. The inclusion $c \subseteq c'$ holds if and only if there exists a vector $\pi = (\pm \frac{1}{2}, \ldots, \pm \frac{1}{2}) \in \mathbb{Z}[1/\beta]^d$ for which $\pi_c + \pi = \pi_{c'}$, that is, $((\pi_c)_b, (\pi)_b, (\pi_{c'})_b) \in Add_d$. There are exactly $2^d$ vectors of the form $(\pm \frac{1}{2}, \ldots, \pm \frac{1}{2})$. We denote these vectors by $\pi_1, \ldots, \pi_k$, where $k = 2^d$. Let $s^i_\ell = (\pi_i)_b \in L^i_\ell$ for $i = 1, \ldots, k$.

Therefore, $c \subseteq c'$ if and only if for a first order formula:

$$\Phi_\ell = add_d(u, s^i_\ell) \in L_{\ell - 1} \lor \cdots \lor add_d(u, s^k_\ell) \in L_{\ell - 1},$$

the evaluation of $\Phi_\ell$ is true for $u = (\pi_c)_b$ and the constants $s^i_\ell, \ldots, s^k_\ell$. Therefore, $\Omega_\ell \subseteq \Omega_{\ell - 1}$ if and only if the following first order sentence:

$$\Upsilon_\ell = \forall u (u \in L_\ell \rightarrow \Phi_\ell)$$

is true for the structure $(L^i_\ell; add_d, L_\ell, L_{\ell - 1}, s^i_\ell, \ldots, s^k_\ell)$. Thus, the domains $\Omega_1, \ldots, \Omega_N$ are nested if the first order sentence:

$$\Upsilon_2 \land \cdots \land \Upsilon_{N - 1}$$
is true for the structure \((C^d_b, \text{add}_d, L_1, \ldots, L_{N-1}, s^1, \ldots, s^{N-1})\).

Let \(M_1, \ldots, M_{N-1}\) be deterministic finite automata recognizing the languages \(L_1, \ldots, L_{N-1}\), respectively. We denote by \(m_1, \ldots, m_{N-1}\) the number of states of the automata \(M_1, \ldots, M_{N-1}\), respectively. For given \(1 \leq \ell \leq N - 1\) and \(1 \leq i \leq k\), using carrying which is a part of the standard addition algorithm in \(\mathbb{Z}[1/b]\), from the automaton \(M_{\ell-1}\) one can construct a deterministic finite automaton \(M_{\ell-1, i}\) recognizing the unary relation \(\text{add}_d(u, s^i_1) \in L_{\ell-1}\). It can be seen that the number of states of \(M_{\ell-1, i}\) is \(O(m_{\ell-1})\). Therefore, using the product construction, one can construct a deterministic finite automaton recognizing the unary relation \(\Phi_\ell\) for which the number of states is \(O(m^{\ell}_{\ell-1})\). Therefore, one can construct a deterministic finite automaton recognizing the unary relation \(u \in L_\ell \rightarrow \Phi_\ell\) for which the number of states is \(O(m^{\ell}_{\ell-1} \cdot m^{\ell}_{\ell-1-1})\). Since \(\forall \rightarrow \exists \rightarrow \) and the emptiness problem for a deterministic finite automaton with \(n\) states can be solved in \(O(n^3)\) time, there is an algorithm deciding whether or not \(T_\ell\) is true in \(O(m^{\ell}_{\ell-1} \cdot m^{\ell}_{\ell-1-1})\) time. Thus, there is a polynomial-time algorithm that for given deterministic finite automata \(M_1, \ldots, M_{N-1}\) decides whether or not \(T_2 \land \cdots \land T_{N-1}\) is true.

### 5.2 Verification of Assumption 2

Now for given regular languages \(L_\ell, \ell = 1, \ldots, N - 1\) how one can verify the condition of Theorem 1, that is, Assumption 2, which ensures that for the collection of tensor product B–splines \(K\), see (5), and every \(f \in S_m(T)\), \(f = \sum \lambda_\delta \delta\) for some uniquely defined coefficients \(\lambda_\delta\)?

In order to verify Assumption 2 one has to verify that for each \(\ell = 0, \ldots, N - 2\) the domain \(M^\ell = \mathbb{R}^d \setminus \Omega^{\ell+1}\) satisfies the following: for each \(\beta \in B^{d}_{d-1, m}\), for which \(\text{supp} \beta \cap M^\ell \neq \emptyset\), the intersection \(\text{supp} \beta \cap M^\ell\) is connected.

Each \(\beta \in B^d_{d-1, m}\) we associate with one of the \((m + 1)^d\) cells from \(C^d_b\) composing \(\text{supp} \beta\), depending on the parity of \(m + 1\): if \(m + 1\) is odd then we associate \(\beta\) with the central cell of \(\text{supp} \beta\), if \(m + 1\) is even then we associate \(\beta\) with the cell which has the central vertex of \(\text{supp} \beta\) as its lower left corner\(^1\); for explanation see Fig. 4. For a given \(\beta \in B^d_{d-1, m}\) we denote by \(c_\beta \in C^d_b\) the associated cell.

![Figure 4](image-url)

Figure 4: The figure on the left shows the support of \(\beta \in B^d_{d-1, 4}\) with the associated cell \(c_\beta\) shaded in gray. The figure on the right shows the support of \(\beta \in B^d_{d-1, 3}\) with the associated cell \(c_\beta\) shaded in gray; this cell has the central vertex of \(\text{supp} \beta\) (shown as a black dot) as its lower left corner.

For a given \((i_1, \ldots, i_d) \in \mathbb{Z}^d\) let \(\overline{t}_{i_1 \ldots i_d}\) be the vector \(\overline{t}_{i_1 \ldots i_d} = (\frac{i_1}{b}, \ldots, \frac{i_d}{b}) \in \mathbb{Z}[1/b]^d\). Let \(t_{i_1 \ldots i_d} = (\overline{t}_{i_1 \ldots i_d})_b \in L^d_b\). For a given \(m \geq 0\) we denote \(I_m\) the set

---

1 We use the term lower left corner in the context of the case \(d = 2\). If \(d \neq 2\), we use the term lower left corner of a \(d\)-dimensional cell \([0, 1]^d\) for the vertex \((0, \ldots, 0) \in \mathbb{R}^d\).
$I_m = \{(i_1, \ldots, i_d) \in \mathbb{Z}^d \mid \frac{m}{2} \leq i_k \leq \frac{m+1}{2}, k = 1, \ldots, d \}$ if $m+1$ is odd and $I_m = \{(i_1, \ldots, i_d) \in \mathbb{Z}^d \mid \frac{m}{2} \leq i_k \leq \frac{m+1}{2}, k = 1, \ldots, d \}$ if $m+1$ is even.

Let $\bar{t} = (i_1, \ldots, i_d) \in I_m$. We denote by $\Phi_{t,m,\bar{t}}$ the following first order formula:

$$\Phi_{t,m,\bar{t}} = add_d(u, t_1, \ldots, t_d) \in L_{\ell+1}.$$ 

The condition that for a given $\beta \in B^d_{0,m}$ the intersection $\text{supp} \beta \cap M^d \neq \emptyset$ holds if and only if the evaluation of the following formula:

$$\Psi_\ell = \bigvee_{\bar{t} \in I_m} \neg \Phi_{t,m,\bar{t}}$$

is true for $u = (\pi_{c,j})_b$ and the constants $t_1, \ldots, t_d$. Moreover, the condition that the intersection $\text{supp} \beta \cap M^d$ is connected can be encoded by a first order formula as follows. Every possible nonempty intersection $\text{supp} \beta \cap M^d$ corresponds to a nonempty subset $J \subseteq I_m$ (see Fig. 5 for illustration) for which the evaluation of the following first order formula:

$$\Psi_{\ell,J} = \left( \bigwedge_{\bar{t} \in J} \neg \Phi_{t,m,\bar{t}} \right) \land \left( \bigwedge_{\bar{t} \in I_m \setminus J} \Phi_{t,m,\bar{t}} \right)$$

is true for $u = (\pi_{c,j})_b$, the constants $t_1, \ldots, t_d$, and the domain $L^d_b$.

We denote by $\mathcal{J}_m$ the collection of all nonempty subsets $J \subseteq I_m$ that correspond to connected intersections. For example, in Fig. 5 the intersection on the left corresponding to the set $J = \{(-2,-1), (-1,-1), (0,-1), (1,-1), (2,-1), (-1,0), (1,0), (0,1)\}$ is connected, so $J \in \mathcal{J}_m$; the intersection on the right corresponding to the set $J' = \{(-2,-1), (-1,-1), (0,-1), (1,-1), (2,-1), (-1,0), (0,0), (1,0), (0,1), (0,-2), (2,2)\}$ is not connected, so $J' \notin \mathcal{J}_m$.

![Figure 5](image.png)

Figure 5: The figure on the left shows the support of some tensor product B–spline from $B^d_{2,4}$ and its intersection with $M^d$ shaded in gray which is connected. The figure on the right shows the support of some tensor product B–spline from $B^d_{2,4}$ and its intersection with $M^d$ shaded in gray which is not connected.

For given $d > 0$ and $\ell \geq 0$, we denote by $\tilde{L}^d_{\ell} \subset L^d_{c,\ell}$ the language $\tilde{L}^d_{\ell} = \{(\pi_c)_b \mid c \in C^d_{\ell}\}$.

For example, if $b = 2$, the language $\tilde{L}^d_{\ell}$ consists of all convolutions of $d$ strings of the form $u_i \otimes v_i \in L^d_{\ell}, i = 1, \ldots, d$ for which $v_i = r_i s_i$, where $r_i \in \{0,1\}^*$, $|r_i| = \ell$ and $s_i \in \{0\}^*$. A language $\tilde{L}^d_{\ell}$ is regular, see also Proposition 1. Finally, the condition that for every $\beta \in B^{d,\ell}_{0,m}$ such that $\text{supp} \beta \cap M^d \neq \emptyset$ the intersection $\text{supp} \beta \cap M^d$ is connected holds if and only if the following first order formula:

$$\chi_\ell = \forall u((u \in \tilde{L}^d_{\ell} \land \Psi_\ell) \rightarrow \bigvee_{J \in \mathcal{J}_m} \Psi_{\ell,J})$$
is true for the structure \((L^d_0; add_d, \bar{L}^d_0, L_{+1}, \{t^\ell_\ell | \ell \in I_m\})\). Therefore, Assumption 2 holds for the domains \(\mathcal{M}^0, \ldots, \mathcal{M}^{N-2}\) if the first order sentence:

\[
\chi_0 \land \cdots \land \chi_{N-2}
\]

is true for the structure:

\[
\left( L^d_0; add_d, \bar{L}^d_0, \ldots, \bar{L}^d_{N-2}, L_1, \ldots, L_{N-1}, \{t^\ell_\ell | \ell \in I_m, \ell = 0, \ldots, N - 2\} \right).
\]

Similarly to Subsection 5.1, there is a polynomial-time algorithm that for given deterministic finite automata \(M_1, \ldots, M_{N-1}\) decides whether or not \(\chi_0 \land \cdots \land \chi_{N-2}\) is true.

### 5.3 Regularity for \(K\)

At each level \(\ell = 0, \ldots, N - 1\) the collection of tensor product B–splines functions \(K^\ell\) generated by Kraft’s selection mechanism, see (5), corresponds to the collection of cells \(K^\ell = \{c_\beta | \beta \in K^\ell \subseteq C_0\}\) according to the rule for associating a tensor product B–spline \(\beta\) with the corresponding cell \(c_\beta\), described in Subsection 5.2. We denote by \(\hat{L}_\ell\) the language \(L_\ell = \{(\pi_\ell)_b | \pi_\ell \in K^\ell \subseteq \bar{L}^d_\ell\}\). Below we will show that the languages \(\hat{L}_0, \ldots, \hat{L}_{N-1}\) are regular.

First we note that the language \(\hat{L}_0\) is defined by the formula:

\[
\Theta_0 = u \in \bar{L}^d_0 \land \Psi_0,
\]

where \(\Psi_0\) is given by (9). That is, \(\hat{L}_0\) is the language of strings \(u\) from \(L^d_0\) for which the evaluation of the formula \(\Theta_0\) is true. The formula \(\Psi_0\) verifies whether the intersection of \(\text{supp} \beta\) for \(\beta \in B^0_d\) with \(\mathcal{M}^0\) is nonempty. If it is nonempty, then \(\beta \in K^0\).

For given \(\ell > 0\), \((i_1, \ldots, i_d) \in I_m\), and \(j = 1, \ldots, k\), where \(k = 2^d\), we denote by \(r^\ell_{i_1 \ldots i_d}\) the constant vectors \(r^\ell_{i_1 \ldots i_d} = r^\ell_{i_1 \ldots i_d} + \pi_{i+1}^\ell\). Let \(r^\ell_{i_1 \ldots i_d} = (r^\ell_{i_1 \ldots i_d})_b \in L^d_\ell\).

For a given \(\ell > 0\), let:

\[
\Theta_\ell = u \in \bar{L}^d_\ell \land \Psi_\ell \land \bigwedge_{\ell \in I_m} \bigvee_{j=1}^k \text{add}_d(u, r^\ell_{i_1 \ldots i_d}) \in L_\ell.
\]

The formula \(\Theta_\ell\) defines the language \(L_\ell\) for \(\ell = 1, \ldots, N - 2\). For \(\beta \in B^\ell_d\), the formula \(\Psi_\ell\) verifies whether the intersection of \(\text{supp} \beta\) with \(\mathcal{M}^\ell\) is nonempty. The formula \(\bigwedge_{\ell \in I_m} \bigvee_{j=1}^k \text{add}_d(u, r^\ell_{i_1 \ldots i_d}) \in L_\ell\) verifies whether \(\text{supp} \beta \subseteq \Omega^\ell\). If \(\beta \in B^\ell_d\), \(\text{supp} \beta \cap \mathcal{M}^\ell \neq \emptyset\) and \(\text{supp} \beta \subseteq \Omega^\ell\), then \(\beta \in K^\ell\). For a given \(\ell > 0\), let:

\[
\Gamma_\ell = u \in \bar{L}^d_\ell \land \bigwedge_{\ell \in I_m} \bigvee_{j=1}^k \text{add}_d(u, r^\ell_{i_1 \ldots i_d}) \in L_\ell.
\]

Clearly, the formula \(\Gamma_{N-1}\) defines the language \(\hat{L}_{N-1}\). For \(\beta \in B^{N-1}_d\) the formula \(\bigwedge_{\ell \in I_m} \bigvee_{j=1}^k \text{add}_d(u, r^{N-1}_{i_1 \ldots i_d}) \in L_{N-1}\) verifies whether \(\text{supp} \beta \subseteq \Omega^{N-1}\). If \(\beta \in B^{N-1}_d\), then \(\beta \in K^{N-1}\).

Since the languages \(L_0, \ldots, \hat{L}_{N-1}\) are defined by the first order formulae \(\Theta_0, \Theta_1, \ldots, \Theta_{N-2}, \Gamma_{N-1}\), they must be regular for regular hierarchical meshes. Moreover, similarly to the argument in Subsection 5.1, for given deterministic finite automata \(M_1, \ldots, M_{N-1}\) one can construct deterministic finite automata \(\tilde{M}_0, \ldots, \tilde{M}_{N-1}\) recognizing the languages \(\hat{L}_0, \ldots, \hat{L}_{N-1}\), respectively. Furthermore, these automata \(\tilde{M}_0, \ldots, \tilde{M}_{N-1}\) are constructed from the automata \(M_1, \ldots, M_{N-1}\) in polynomial time.
6 Regular Splines

Let $T$ be a regular hierarchical $d$-dimensional mesh defined by a nested sequence of domains $\Omega^0 = \mathbb{R}^d \supset \Omega^1 \supset \cdots \supset \Omega^{N-1} \supset \Omega^N = \emptyset$, where $\Omega^{N-1} \neq \emptyset$. Let $f = \sum_{(\ell, \beta) \in K} \lambda_\beta \beta$ be a spline in $S_m(T)$ defined by some coefficients $\lambda_\beta$, $\beta \in K$, where $K = \bigcup_{\ell=0}^{N-1} K^\ell$ is obtained by Kraft’s selection mechanism, see the equation (5) and Remark 2. Recall that by Theorem 1, if $T$ satisfies Assumption 2, then each spline in $S_m(T)$ can be written as the infinite sum $\sum_{\beta \in K} \lambda_\beta \beta$. Each $\beta \in K^\ell$ is associated with the cell $c_\beta \in K^\ell \subseteq L^\ell$ which is then associated with $(\tau_\beta)_b \in \tilde{L}_\ell$, see the notation in Subsection 5.3.

**Definition 3.** We say that a spline $f \in S_m(T)$ is regular if the coefficients $\lambda_\beta \in \mathbb{Z}[1/b]$ for all $\beta \in K$ and the relation $S_f = \{(\tau_\beta)_b \mid \beta \in K\} \subset L^\ell_b \times L_b$ is FA-recognizable.

For a given $\ell = 0, \ldots, N-1$, we denote by $S_f^\ell$ the relation:

$$S_f^\ell = \{(\tau_\beta)_b \mid \beta \in K^\ell\}.$$  

Similarly to Proposition 1, a spline $f \in S_m(T)$ is regular if and only if each of the relation $S_f^\ell$ is FA-recognizable for $\ell = 0, \ldots, N-1$.

By Theorem 2, from the automata $M_1, \ell$ and $M_2, \ell$ recognizing the relations $S_{f_1}^\ell$ and $S_{f_2}^\ell$, respectively. We denote by $m_{1, \ell}$ and $m_{2, \ell}$ the number of states of $M_1, \ell$ and $M_2, \ell$, respectively. Let $f = f_1 + f_2$. The relation $S_f^\ell \subset L^\ell_b \times L_b$ is defined by the formula:

$$S_f^\ell = \exists v_1 \exists v_2 (S_{f_1}^\ell (u, v_1) \land S_{f_2}^\ell (u, v_2) \land \text{Add}(v_1, v_2, v)).$$

That is, the evaluation of $S_f^\ell$ is true for $u \in L^\ell_b$ and $v \in L_b$ if and only if $(u, v) \in S_f^\ell$. By Theorem 2, from the automata $M_1, \ell$ and $M_2, \ell$ recognizing the relations $S_{f_1}^\ell$ and $S_{f_2}^\ell$, respectively. We can construct a deterministic finite automaton recognizing the relation $S_f^\ell$. However, the existential quantifiers in $S_f^\ell$ require us to use the Rabin–Scott powerset construction which may lead to exponential growth in the number of states. In order to avoid this we propose to present $f = f_1 + f_2$ by the relation:

$$S_f^\ell = \{(u, v_1, v_2, v) \in L^\ell_b \times L_b \times L_b \mid S_{f_1}^\ell (u, v_1) \land S_{f_2}^\ell (u, v_2) \land \text{Add}(v_1, v_2, v)\}.$$

One then can construct a deterministic finite automaton recognizing $S_f^\ell$ for which the number of states is $O(m_{1, \ell} \cdot m_{2, \ell})$. The same approach works for multiplication by a constant $\mu$. For example, if $f$ is then multiplied by $\mu$, we can present $\mu f$ by the relation:

$$S_{\mu f}^\ell = \{(u, v_1, v_2, v, w) \in L^\ell_b \times L_b \times L_b \times L_b \mid S_f^\ell (u, v_1, v_2, v) \land R_\mu (v, w)\}.$$

6.1 Computing Values of a Regular Spline

Let $f \in S_m(T)$ be a regular spline given by a FA-recognizable relation $S_f$. For each $\ell = 0, \ldots, N-1$, let us be given be a deterministic finite automaton $M_\ell$ recognizing
the relation $S_f$. For a given point $\pi = (x_1, \ldots, x_d) \in \mathbb{Z}[1/b]^d$, how one can compute the value $f(\pi)$?

Let $R_f \subseteq L_b^d \times L_b^d \times L_b$ be the relation that contains all triples $((\pi)_b, (\pi_{\beta(b)})_b, (\lambda_{\beta(b)})_b)$ of strings $(\pi)_b \in L_b^d$, $(\pi_{\beta(b)})_b \in L_b^d$ and $(\lambda_{\beta(b)})_b \in L_b$ for $\beta \in K^f$ such that $\pi \in \supp \beta$:

$$R_f = \{((\pi)_b, (\pi_{\beta(b)})_b, (\lambda_{\beta(b)})_b) \mid \pi \in \mathbb{Z}[1/b]^d, \beta \in K^f \land \pi \in \supp \beta \}.$$

Let $\overline{\pi} = (y_1, \ldots, y_d) = \pi_{\beta(b)}$. The condition $\pi \in \supp \beta$ for $\beta \in K^f$ is true if and only if the inequalities:

$$-m + \frac{2}{2^r+1} < x_i - y_i < m - \frac{m+1}{2^r+1} < x_i - y_i < \frac{m+1}{2^r+1}$$

hold for all $i = 1, \ldots, d$, if $m$ is odd and even, respectively, see Fig. 6.

![Figure 6](image)

**Figure 6:** The figure on the left shows the support of a spline $\beta \in B_{2,3}^f$, the points $\pi \in \supp \beta$, $\overline{\pi} = \pi_{\beta(b)}$ and the lower left corner of $\supp \beta$ – the point $\overline{\pi}$. The figure on the right shows the support of a spline $\beta \in B_{2,4}^f$, the points $\pi \in \supp \beta$, $\overline{\pi} = \pi_{\beta(b)}$ and the lower left corner $\overline{\pi}$.

For $\pi = (r_1, \ldots, r_d) \in \mathbb{Z}[1/b]^d$ and $\pi = (s_1, \ldots, s_d) \in \mathbb{Z}[1/b]^d$ we say that $\pi < \overline{\pi}$ if $r_i < s_i$ for all $i = 1, \ldots, d$. Let $R_{\overline{\pi}}^\ell$ be the relation $R_{\overline{\pi}}^\ell = \{((\pi)_b, (\pi_{\beta(b)})_b) \mid \pi, \overline{\pi} \in \mathbb{Z}[1/b]^d, \pi < \overline{\pi}\}$. The relation $R_{\overline{\pi}}^\ell$ is FA-recognizable. Since $Add_d$ and $R_{\overline{\pi}}^\ell$ are FA-recognizable, the relation given by the inequalities (12) is FA-recognizable. Therefore, since $S_f$ is FA-recognizable, $R_f$ is FA-recognizable.

We denote by $\overline{\pi}_{\beta(b)}$ the lower left corner $\overline{\pi} = (q_1, \ldots, q_d)$ of $\supp \beta$, see Fig. 6. We have that $y_i - q_i = \frac{m+1}{2^r+1}$ and $y_i - q_i = \frac{m+1}{2^r+1}$ for all $i = 1, \ldots, d$, if $m$ is odd and even, respectively. Since $Add_d$ is FA-recognizable, the relation $Q_d^\ell = \{((\pi_{\beta(b)})_b \mid \beta \in B_{d,m}^f\}$ is FA-recognizable.

Now let $R_f^\ell \subseteq L_b^d \times L_b^d \times L_b \times L_b$ be the following relation:

$$R_f^\ell = \{((\pi)_b, (\pi_{\beta(b)})_b, (\lambda_{\beta(b)})_b, (\pi - \overline{\pi}_{\beta(b)})_b) \mid \pi \in \mathbb{Z}[1/b]^d, \beta \in K^f \land \pi \in \supp \beta\}.$$

Since the relations $R_f^\ell$, $Q_d^\ell$ and $Add_d$ are FA-recognizable, the relation $R_f^\ell$ is FA-recognizable. From automata recognizing the relations $R_f^\ell$, $Q_d^\ell$, $Add_d$ and the automaton $M_d$ one can construct a deterministic finite automaton recognizing the relation $R_f^\ell$ for which the number of states is $O(m_d)$, where $m_d$ is the number of states of $M_d$.

Note that for a given $\pi \in \mathbb{Z}[1/b]^d$, there exist at most $(m+1)^d$ tensor product B-splines $\beta \in K^f$ for which $\pi \in \supp \beta$. So $R_f^\ell$ can be seen as a multivalued function that for a given input $\pi$ returns at most $(m+1)^d$ pairs $((\lambda_{\beta(b)}), (\pi, \overline{\pi}_{\beta(b)})_b)$ as an output. Since $R_f^\ell$ is FA-recognizable, this multivalued function is computed in linear time by a deterministic one-tape Turing machine, see Section 3.
We denote by $K^\ell_\pi$ the set $\{\beta \in K^\ell | \pi \in \text{supp } \beta\}$ and by $K_\pi$ the set $\bigcup_{\ell=0}^{N-1} K^\ell_\pi$. After all pairs $((\lambda_\beta)_\pi, (\pi - \lambda_\beta)_\beta)$ for which $\pi \in \text{supp } \beta$, where $\beta \in K_\pi$, are computed, the value of the spline $f(\pi) = \sum_{\beta \in K_\pi} \lambda_\beta \delta(\pi)$ at the point is obtained from the formula for $N_{0,m}^\ell (t)$ by evaluating $N_{0,m}^\ell (\pi - \lambda_\beta)$ for every $\beta \in K^\ell_\pi$. Note that $K_\pi$ is a finite set containing at most $N(m+1)^d$ elements, so there are at most $N(m+1)^d$ terms in the sum $\sum_{\beta \in K_\pi} \lambda_\beta \delta(\pi)$.

For illustration below we provide concrete formulae for evaluating $N_{0,m}^\ell (t)$ for $m = 1, 2, 3$. If $\ell > 0$, $N_{0,m}^\ell (t) = N_{0,m}^\ell (2^\ell t)$ for $t \in (0, \frac{m+1}{m'})$. By (2)–(3) one can obtain that (see, e.g., [28]):

$$N_{0,1}^0 (t) = \begin{cases} t, & 0 \leq t < 1, \\ 2 - t, & 1 \leq t < 2, \\ 0, & \text{otherwise}. \end{cases}$$

$$N_{0,2}^0 (t) = \begin{cases} \frac{1}{2} t^2, & 0 \leq t < 1, \\ -(t-1)^2 + 2(t-1) + \frac{1}{2}, & 1 \leq t < 2, \\ \frac{1}{2} (3-t)^2, & 2 \leq t < 3, \\ 0, & \text{otherwise}, \end{cases}$$

$$N_{0,3}^0 (t) = \begin{cases} \frac{1}{6} t^3, & 0 \leq t < 1, \\ \frac{1}{6} \big(-3(t-1)^3 + 3(t-1)^2 + 3(t-1) + 1\big), & 1 \leq t < 2, \\ \frac{1}{6} \big(3(t-2)^3 - 6(t-2)^2 + 4\big), & 2 \leq t < 3, \\ \frac{1}{6} \big(-3(t-3)^3 + 3(t-3)^2 - 3(t-3) + 1\big), & 3 \leq t < 4, \\ 0, & \text{otherwise}. \end{cases}$$

It follows from the formulae (13) and (14) that for $m = 1, 2$ and $b$ divisible by 2, if $\pi \in \mathbb{Z}[1/b]$, then $f(\pi) \in \mathbb{Z}[1/b]$. However, in order to guarantee the same for $m = 3$, it is required that $b$ is divisible by 6. That is, it follows from (15) that for $m = 3$ and $b$ divisible by 6, if $\pi \in \mathbb{Z}[1/b]$, then $f(\pi) \in \mathbb{Z}[1/b]$.

If one applies the standard long multiplication algorithm to evaluate $N_{0,m}^\ell (\pi - \pi_\beta)$ and then multiply it by $\lambda_\beta$ for each $\beta \in K_\pi$, the total computational complexity for evaluating $f(\pi)$ for a given input $\pi$ is quadratic; though it can be reduced if one applies a faster multiplication algorithm.

### 6.2 Refining Regular Hierarchical Meshes and Splines

Now let us refine a regular hierarchical mesh $\mathcal{T}$ by selecting a nonempty subdomain $\Omega^N \subseteq \Omega^{N-1}$ composed of cells from $C^N_d$. We assume that the language $L_N$ corresponding to the subdomain $\Omega^N$, according to the rule described in Section 5, is regular. So a hierarchical $d$–dimensional mesh $\mathcal{T}'$ defined by a nested sequence of domains $\Omega^0 = \mathbb{R}^d \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^{N-1} \supseteq \Omega^N \supseteq \Omega^{N+1} = \emptyset$ is regular.

Let $\mathcal{K}' = \bigcup_{\ell=0}^N \mathcal{K}^\ell$ be the collection of tensor product B–splines generated by Kraft’s selection mechanism for the hierarchical mesh $\mathcal{T}'$. If the languages $\hat{L}_0, \ldots, \hat{L}_{N-1}$, corresponding to the collections $\mathcal{K}^0, \ldots, \mathcal{K}^{N-1}$, respectively, are given as an input, how one does obtain the languages $\hat{L}_0, \ldots, \hat{L}_N$ corresponding to the collections $\mathcal{K}^0, \ldots, \mathcal{K}^N$, respectively?

Since $\mathcal{K}^0 = \mathcal{K}^0, \ldots, \mathcal{K}^{N-2} = \mathcal{K}^{N-2}$, we have that $\hat{L}_0' = \hat{L}_0, \ldots, \hat{L}_{N-2}' = \hat{L}_{N-2}$. A tensor product B–spline $\beta \in \mathcal{K}^{N-1}$ if and only if $\beta \in \mathcal{K}^{N-1}$ and $\text{supp } \beta \cap \mathcal{M}^{N-1} \neq \emptyset$, where $\mathcal{M}^{N-1} = \mathbb{R}^d \setminus \Omega^N$. The condition $\text{supp } \beta \cap \mathcal{M}^{N-1} \neq \emptyset$ is verified by the formula $\Psi_{N-1}$, see (9). So the language $\hat{L}_{N-1}$ is defined by the formula $u \in \hat{L}_{N-1} \land \Psi_{N-1}$. The formula $\Gamma_N$, see (10), verifies the condition that a tensor product B–spline $\beta \in \mathcal{K}^N$. 

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So the language $\hat{L}_N$ is defined by the formula $\Gamma_N$. Thus, all languages $\hat{L}_0, \ldots, \hat{L}_N$ are regular. Let $\hat{M}_{N-1}$ and $M_N$ be deterministic finite automata recognizing the languages $\hat{L}_{N-1}$ and $L_N$, respectively. One can construct deterministic finite automata recognizing $\hat{L}_{N-1}$ and $\hat{L}_N$ for which the number of states is $O(\hat{m}_{N-1} \cdot m_N)$ and $O(m_N)$, respectively, where $\hat{m}_{N-1}$ and $m_N$ are the number of states of $\hat{M}_{N-1}$ and $M_N$, respectively.

Let $f \in S_m(T)$ be regular spline given by a FA-recognizable relation $S_f$. How one does obtain a relation $S'_f$ for the spline function $f$ over the hierarchical mesh $T'$?

We have that $f = \sum_{\ell=0}^{N-1} \sum_{\beta \in K^d} \lambda_{\beta} \beta = \sum_{\ell=0}^{N-2} \sum_{\beta \in K^{d-1}} \lambda_{\beta} \beta + \sum_{\beta \in K^{d-1}} \lambda_{\beta} \beta + \sum_{\beta \in K^m} \lambda_{\beta} \beta$. Therefore, $S'_f = S^0_f, \ldots, S^{m-2}_f = S^{m-2}_N$ and $S^{m-1}_f = \{(\pi, v) \in S^{m-1}_{\hat{N}-1} | \pi \in \hat{L}'_{N-1}\}$. Clearly, $S^0_f, \ldots, S^{m-1}_f$ are FA-recognizable. Let $M_{f,N-1}$ be a deterministic finite automaton recognizing $S^{m-1}_f$. One can construct a deterministic finite automaton recognizing $S^{m-1}_f$ for which the number of states is $O(m_{f,N-1} \cdot m_N)$, where $m_{f,N-1}$ is the number of states of $M_{f,N-1}$. Below we will show that the coefficients $\lambda_{\beta} \in \mathbb{Z}[1/b]$ for all $\beta \in K^m$ and the relation $S^{m-1}_f = \{(\pi, v) | (\beta, \beta) \in \mathbb{K}^m \}$ is FA-recognizable.

For any given $\delta \in B^{m-1}_{f,m}$ each $\beta \in B_d^m$ for which supp $\beta \subset supp \delta$ corresponds to a multi-index $J_{\beta} = (j_1, \ldots, j_d)$, where $0 \leq j_k \leq m+1$ for all $k = 1, \ldots, d$, that determines the position of supp $\beta$ inside supp $\delta$, see Fig. 7 for explanation. For given $\delta \in B^{m-1}_{f,m}$ and a multi-index $\bar{J} = (j_1, \ldots, j_d)$ we denote by $\beta_{J,\bar{J}}$ the tensor product B-spline $\beta \in B^m_d$ for which $J_{\beta_{J,\bar{J}}} = \bar{J}$. Note that for the barycentres $\tau_{c_{\beta}}$ and $\tau_{c_{\beta}}$ the following holds:

$$\tau_{c_{\beta}} - \tau_{c_{\beta}} = \left( \frac{m+1}{2} \cdot \frac{1}{2}, \ldots, \frac{m+1}{2} \cdot \frac{1}{2} \right), \text{ if } m \text{ is even},$$

$$\left( \frac{m+2}{2} \cdot \frac{1}{2}, \ldots, \frac{m+2}{2} \cdot \frac{1}{2} \right), \text{ if } m \text{ is odd.} \quad (16)$$

Figure 7: The left figure shows the support of a spline $\delta_1 \in B^{m-1}_{2,4}$, the supports of $\beta_1, \gamma_1 \in B^m_{2,4}$ (two hatched rectangles) and the points $p_1 = \tau_{c_{\beta_1}}, q_1 = \tau_{c_{\gamma_1}}$, and $\tau_1 = \tau_{c_{\beta_1}}$. The indices $J_{\beta_1}, J_{\beta_1}, \gamma_1$ are $J_{\beta_1} = (0,1)$ and $J_{\beta_1, \gamma_1} = (4,3)$. The right figure shows the support of a spline $\delta_2 \in B^{m-1}_{2,3}$, the supports of $\beta_2, \gamma_2 \in B^m_{2,3}$ (two hatched rectangles) and the points $p_2 = \tau_{c_{\beta_2}}, q_2 = \tau_{c_{\gamma_2}}$, and $\tau_2 = \tau_{c_{\beta_2}}$. The indices $J_{\beta_2}, J_{\beta_2}, J_{\gamma_2, \gamma_2}$ are $J_{\beta_2} = (0,1)$ and $J_{\beta_2, \gamma_2} = (4,3)$.

We denote by $I_{d,m}$ the set of multi–indices $I_{d,m} = \{(j_1, \ldots, j_d) | 0 \leq j_k \leq m+1, k = 1, \ldots, d\}$. For each $\delta \in B^{m-1}_{d,m}$ we have that $\delta = \sum_{\bar{J} \in I_{d,m}} \lambda_{\bar{J}} \beta_{\bar{J}}$. It can be verified directly from Boehm’s knot insertion formula for B–splines [5] that for $d = 1$: $\lambda_j = \frac{1}{m+1} \binom{m+1}{j}$, $j = 0, \ldots, m+1$, where $\binom{m+1}{j} = \binom{m+1}{j} \binom{m+1}{j-1}$ are the binomial coefficients. So all coefficients $\lambda_j, j = 0, \ldots, m+1$ belong to $\mathbb{Z}[1/2]$. By the definition of multivariate
One can construct a deterministic finite automaton recognizing $S$ may lead to exponential growth in the number of states. In order to avoid it, instead of the value $\lambda$ being regular. This follows from the partition of unity property for B–splines: $S$ is FA–recognizable. Since multiplication by a constant in $R$ is FA-recognizable, the relation $d,m$ is FA-recognizable.

\[ Q_{j,\mathbf{\gamma}} = \{(\mathbf{\tau}_{\mathbf{c}})_b, (\lambda_\beta)_b) \mid \beta \in K^{N}, \delta \in K^{N-1}, ((\mathbf{\tau}_{\mathbf{c}})_b, (\lambda_\beta)_b) \in R_{N,\mathbf{\gamma}}, ((\mathbf{\tau}_{\mathbf{c}})_b, (\lambda_\beta)_b) \in S_j^{N-1}\} \]

is FA-recognizable. Since multiplication by a constant in $Z[1/b]$ is FA-recognizable, see (11), we finally obtain that:

$$S_j^{N} = \{(\mathbf{\tau}_{\mathbf{c}})_b, (\lambda_\beta)_b) \mid \beta \in K^{N}, (\mathbf{\tau}_{\mathbf{c}})_b, (\lambda_\beta)_b) \in Q_{j,\mathbf{\gamma}} \in I_{d,m} \land \lambda_\beta = \sum_{(\delta,\gamma) \in \Delta_\beta} \lambda_\delta \lambda_\gamma \} \quad (17)$$

is FA-recognizable.

**Remark 4.** Similarly to Remark 3, the use of the identity $\lambda_\beta = \sum_{(\delta,\gamma) \in \Delta_\beta} \lambda_\delta \lambda_\gamma$ in (17) may lead to exponential growth in the number of states. In order to avoid it, instead of $S_j^{N}$ one can use $\mathcal{S}_j^{N}$ defined below. For a given $\beta \in K^{N}$ and $\mathbf{\gamma} \in I_{d,m}$, let $\lambda_\beta,\mathbf{\gamma} = \lambda_\beta$ if $(\delta,\gamma) \in \Delta_\beta$ for some $\delta$ and $0$, otherwise. Clearly, we have $\lambda_\beta = \sum_{\mathbf{\gamma}} \lambda_\beta,\mathbf{\gamma} \lambda_\gamma$. Now let us define $\mathcal{S}_j^{N} \subset \mathcal{L}_b \times (\prod_{\mathbf{\gamma} \in I_{d,m}} \mathcal{L}_b) \times \mathcal{L}_b$ as:

$$\mathcal{S}_j^{N} = \{(\mathbf{\tau}_{\mathbf{c}})_b, (\lambda_\beta,\mathbf{\gamma}(0,\ldots,0))_b, \ldots, (\lambda_\beta,\mathbf{\gamma}(m+1,\ldots,m+1))_b, (\lambda_\beta)_b) \mid \beta \in K^{N}, \lambda_\beta = \sum_{\mathbf{\gamma}} \lambda_\beta,\mathbf{\gamma} \lambda_\gamma \}.$$ 

One can construct a deterministic finite automaton recognizing $\mathcal{S}_j^{N}$ for which the number of states is polynomial in $m_{f,N-1}$ and $m_N$.

### 6.3 Examples

First we note that a spline $f \in \mathcal{S}_m(T)$ over a regular hierarchical mesh $T$ with bounded support $\text{supp} f$ is regular.

Let $T^{0}_{d}$ be a mesh defined by the grid $G^{0}_{d}$. A constant function over $T^{0}_{d}$ which takes the value $\lambda \in Z[1/b]$ for every point $x \in \mathbb{R}^d$ is a regular spline in $\mathcal{S}_m(T^{0}_{d})$ for $m \geq 0$.

This follows from the partition of unity property for B–splines: $\sum_{i=1}^{\infty} N_{i,m}^{0} = 1$ for $m \geq 0$. Moreover, a linear function $\sum_{i=1}^{d} \lambda_i x_i$, for $\lambda_i \in Z[1/b]$, $i = 1, \ldots, d$ is a regular spline in $\mathcal{S}_m(T)$ for $m \geq 1$. Since the collection of regular splines is closed under...
taking the sum and multiplication by a constant $\lambda \in \mathbb{Z}[1/b]$, it is enough to prove it for the functions $f_i(x) = x_i$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $i = 1, \ldots, d$. In order to prove the latter, it is enough only to show that the linear function $f(t) = t, \ t \in \mathbb{R}$ is a regular spline in $S_m(T_1^0)$. This follows from the identity $\sum_{i=-\infty}^{\infty} c_{i,m} N_{i,m}(t) = t$ for $m \geq 1$, where $c_{i,m} = i + \frac{m+1}{2}$. This identity is proved by induction. For $m = 1$, we recall that (see (13)):

$$N_{i,1}(t) = \begin{cases} t - i, & i \leq t < i + 1, \\ i + 2 - t, & i + 1 \leq t < i + 2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, for $t \in [i, i + 1]$, we have $\sum_{i=-\infty}^{\infty} c_{i,1} N_{i,1}(t) = (i + 1)N_{i,1}(t) + ((i - 1) + 1)N_{i-1,1}(t) = (i+1)(t-i) + i(i+1-t) = t$. The inductive step follows from the Cox–de Boor’s formula (3) as follows. Assume that $\sum_{i=-\infty}^{\infty} c_{i,m} N_{i,m}(t) = t$ holds for some $m \geq 1$.

By (3), $\sum_{i=-\infty}^{\infty} c_{i,m+1} N_{i,m+1}(t) = \sum_{i=-\infty}^{\infty} c_{i,m+1} \left( \frac{t + m+1}{m+1} N_{i,m}(t) + \frac{i + m+1 - t}{m+1} N_{i+1,m}(t) \right) = \sum_{i=-\infty}^{\infty} c_{i,m+1} \left( \frac{1}{m+1} + c_{i-1,m+1} \right) N_{i,m}(t) = \sum_{i=-\infty}^{\infty} \left( \frac{1}{m+1} + \frac{m+1-i}{m+1} \right) N_{i,m}(t) = t$. From the formula $c_{i,m} = i + \frac{m+1}{2}$, it is enough to prove that $f(t) = t$ is a regular spline $S_m(T_1^0)$.

It follows from Section 6.2 that constant and linear functions are regular splines in $S_m(T_1^0)$, for $m \geq 0$ and $m \geq 1$, respectively, for every regular hierarchical mesh $T$.

![Figure 8](image-url)

Figure 8: The left figure shows the spline $g(t) = \sum_{j=-\infty}^{\infty} c_j N_{j,3}(t)$, where $c_j = 1$ if $j$ is even and $c_j = -1$ if $j$ is odd. The right figure shows the spline $h(t) = \sum_{j=0}^{\infty} c_j' N_{j,3}(t) + \sum_{j=-\infty}^{-1} c_j' N_{j+4,3}(t)$, where $c_j' = j + 1$ for $j \geq 0$ and $c_j' = -j$ for $j \leq -1$.

We show other two simple examples below. Let us consider a spline $g(t) = \sum_{j=-\infty}^{\infty} c_j N_{j,3}(t)$, where $c_j = 1$ if $j$ is even and $c_j = -1$ if $j$ is odd, see Fig.8 (left).

Clearly, $g$ is a regular spline in $S_3(T_1^0)$ as well as in $S_3(T)$ for every one–dimensional regular hierarchical mesh $T$. Now let $T'$ be a one–dimensional hierarchical mesh generated by the domains $\Omega_1 = \Omega_2 = \bigcup_{i=0}^{\infty} (2i, 2i+1] \cup [-2i-1, -2i]$. A spline function $h(t) = \sum_{j=0}^{\infty} c_j' N_{j,3}(t) + \sum_{j=-\infty}^{-1} c_j' N_{j+4,3}(t)$, where $c_j' = j + 1$ for $j \geq 0$ and $c_j' = -j$ for $j \leq -1$, is a regular spline in $S_3(T')$ as well as in
for every one-dimensional regular hierarchical mesh \( T' \) generated by domains 
\[
\Omega^0 = \mathbb{R}^1 \supseteq \Omega^1 \supseteq \Omega^2 \supseteq \cdots \supseteq \Omega^{N-1} \supseteq \Omega^N = \emptyset
\]
for \( N > 4 \), where \( \Omega_1 \) and \( \Omega_2 \) are as above.

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Appendix A

Let us consider two $d$-dimensional cubes $c_1 = \prod_{j=1}^{d} [y_j', y_j'']$ and $c_2 = \prod_{j=1}^{d} [z_j', z_j'']$ in $\mathbb{R}^d$, i.e., $y_j' < y_j''$ and $z_j' < z_j''$ for $j = 1, \ldots, d$. Suppose that the cubes $c_1$ and $c_2$ are adjacent such that the intersection $c_1 \cap c_2$ is a $(d-1)$-dimensional cube, see Fig. 9 for the case of two $2$-dimensional cubes in $\mathbb{R}^2$. This intersection $c_1 \cap c_2$ is contained in some $(d-1)$-dimensional hyperplane $x_i = x_0$ for some integer $i \in [1, d]$ and a constant $x_0 \in \mathbb{R}$. Let $f : c_1 \cup c_2 \to \mathbb{R}$ be a function such that $f|_{c_1} = p_1$ and $f|_{c_2} = p_2$ for some polynomials $p_1(x_1, \ldots, x_d)$ and $p_2(x_1, \ldots, x_d)$ of multi-degree $(m, \ldots, m)$.

Lemma 1 (2)]. The derivative $\frac{\partial^k f}{\partial x_i^k}$ exists and continuous everywhere in $\text{dom } f = c_1 \cup c_2$ for some integer $0 \leq k \leq m - 1$ if and only if $p_1 - p_2 = \lambda (x_i - x_0)^{k+1}$, where $\lambda$ is a polynomial.
Figure 9: Adjacent 2-dimensional cells $c_1$ and $c_2$ for which the intersection $c_1 \cap c_2$ is the closed 1-dimensional line segment with the endpoints $(y''_1, z''_2)$ and $(z'_1, y'_2)$.

**Proposition 2.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function which is a polynomial of multi-degree $(m, \ldots, m)$ in every cell from $\mathcal{T}_d$. Suppose that the derivatives $\frac{\partial^{m-1} f}{\partial x_1^{j_1} \cdots \partial x_d^{j_d}}$ exist and continuous everywhere in $\mathbb{R}^d$ for $i = 1, \ldots, d$. Then the derivatives $\frac{\partial^{i+1} f}{\partial x_1^{j_1} \cdots \partial x_d^{j_d}}$ exist and continuous for $k_i = 0, \ldots, m - 1$ and $i = 1, \ldots, d$.

**Proof.** Let $c_1$ and $c_2$ be two adjacent cells from $\mathcal{T}_d$ for which the intersection is a $(d-1)$-dimensional cube contained in some hyperplane $x_i = x_0$. Let $p_1$ and $p_2$ be the polynomials of multi-degree $(m, \ldots, m)$ which are the restrictions of $f$ to the cells $c_1$ and $c_2$, respectively. Since the derivative $\frac{\partial^{m-1} f}{\partial x_i}$ exists and continuous, by Lemma 1 we obtain that $p_1 - p_2 = \lambda(x_i - x_0)^m$, where $\lambda$ is a polynomial which depends on the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d$. Therefore, for every $d$-tuple of nonnegative integers $(k_1, \ldots, k_d)$ for which $k_i \leq m - 1$ we have that $\frac{\partial^{i+1} f}{\partial x_1^{j_1} \cdots \partial x_d^{j_d}} = \mu(x_i - x_0)^{m-k_i}$ for some polynomial $\mu$. So, by Lemma 1, for each of such $d$-tuples $(k_1, \ldots, k_d)$ the derivative $\frac{\partial^{i+1} f}{\partial x_1^{j_1} \cdots \partial x_d^{j_d}}$ exists and continuous on $c_1 \cup c_2$. Considering all pairs of adjacent cells from $\mathcal{T}_d$ which intersect in $(d-1)$-dimensional cubes we see that the derivative $\frac{\partial^{i+1} f}{\partial x_1^{j_1} \cdots \partial x_d^{j_d}}$ exists and continuous everywhere in $\mathbb{R}^d$ for every $d$-tuple of nonnegative integers $(k_1, \ldots, k_d)$ for which $k_j \leq m - 1$ for all $j = 1, \ldots, d$. \(\blacksquare\)

**Corollary 1.** For every function $f \in S_m(\mathcal{T})$ the derivatives $\frac{\partial^{i+1} f}{\partial x_1^{j_1} \cdots \partial x_d^{j_d}}$ exist and continuous for $k_i = 0, \ldots, m - 1$ and $i = 1, \ldots, d$.

**Proof.** The statement of the corollary directly follows from Proposition 2. \(\blacksquare\)

**Remark 5.** We note that Proposition 2 and Corollary 1 hold valid for every subdivision of $\mathbb{R}^d$ into $d$-dimensional axis-aligned cubes, not necessarily the ones obtained from the grids $G_{\ell, d}$, $\ell = 0, \ldots, N - 1$ and a nested sequence of domains $\Omega^0 = \mathbb{R}^d \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^{N-1} \supseteq \Omega^N = \emptyset$ satisfying Assumption 1.

### 8 Appendix B

**Definition 4 ([24]).** Let $c_1$ and $c_2$ be two different cells from $C_{\ell, d}$ for some $\ell \geq 0$ such that the intersection $c_1 \cap c_2$ is non-empty. Let $p_1(x_1, \ldots, x_d)$ and $p_2(x_1, \ldots, x_d)$ be two polynomials of multi-degree $(m, \ldots, m)$. We say that $p_1|_{c_1} \sim p_2|_{c_2}$ if $\frac{\partial^{i+1} f}{\partial x_1^{j_1} \cdots \partial x_d^{j_d}}(x_1, \ldots, x_d)$
Let $p(x_1, \ldots, x_m)$ be a polynomial of multi-degree $(m, \ldots, m)$ and $c \in C_d^\ell$ be a cell. The restriction $p|_c$ always can be expressed as a linear combination $p|_c = \sum_{\beta \in B_{d,m}^\ell} \lambda^\beta(c(p|_c)\beta|_c$, where $\lambda^\beta(c(p|_c)$ denote the coefficients of this linear combination.

Clearly, $\lambda^\beta(c(p|_c) = 0$ if $c \not\subseteq \text{supp } \beta$. That is, a coefficient $\lambda^\beta(c(p|_c)$ can be nontrivial only if $c \subseteq \text{supp } \beta$, so in the formal sum $\sum_{\beta \in B_{d,m}^\ell} \lambda^\beta(c(p|_c)\beta|_c$ only at most $(m+1) \times (m+1)$ terms can be nontrivial. The following lemma characterizes the relation $p|_{c_1} \sim p|_{c_2}$ in terms of the coefficients $\lambda^\beta(c_p|_{c_1})$ and $\lambda^\beta(c_p|_{c_2})$.

**Lemma 2** ([24]). Let $c_1, c_2$ and $p_1, p_2$ be two cells and two polynomials from Definition 4. Then $p|_{c_1} \sim p|_{c_2}$ if and only if for every $\beta \in B_{d,m}^\ell$, for which $\beta|_{c_1 \cap c_2} \neq 0$, $\lambda^\beta(c_p|_{c_1}) = \lambda^\beta(c_p|_{c_2})$.

**Definition 5** ([24]). Let $M$ be a collection of cells $M \subseteq C_d^\ell$ for some $\ell \geq 0$ and $M$ be the domain covered by the cells from $M$: $M = \bigcup_{c \in M} c$. We denote by $S_m(M)$ the space of functions $f : M \rightarrow \mathbb{R}$ which are polynomials of multi-degree $(m, \ldots, m)$ in every cell from $M$ and for every pair of cells $c_1, c_2 \in M$ having nonempty intersection $c_1 \cap c_2 \neq \varnothing$: $f|_{c_1} \sim f|_{c_2}$.

**Definition 6** ([24]). Let $\beta \in B_{d,m}^\ell$ and $M \subseteq C_d^\ell$ for some $\ell \geq 0$. The coefficient graph $\Gamma_\beta$ is defined as follows. The vertices of $\Gamma_\beta$ are the cells $c \in M$ for which $c \subseteq \text{supp } \beta$. Two vertices $c_1$ and $c_2$ in $\Gamma_\beta$ are connected by an edge if $\beta|_{c_1 \cap c_2} \neq 0$.

**Figure 10:** The figure on the left shows the cells of $M$ (shaded in gray), the closure of the support $\text{supp } \beta$ (bounded by red line segments) and the cells of $M$ which are subsets of $\text{supp } \beta$ (these cells are labeled by $c_1, \ldots, c_8$). The figure on the right shows the graph $\Gamma_\beta$ which consists of three connected components with the sets of vertices $\{c_2\}$, $\{c_1, c_3\}$ and $\{c_4, c_5, c_6, c_7, c_8\}$.

See Fig. 10 for an example of a graph $\Gamma_\beta$ for $d = 2$ and $m = 3$ (in this case the support of a tensor product B–spline $\beta$ is composed of $4 \times 4$ cells).

Let $M \subseteq C_d^\ell$ and $M = \bigcup_{c \in M} c$ be the domain covered by the cells from $M$.

**Proposition 3** ([24]). Let $f : M \rightarrow \mathbb{R}$ be a function which is a polynomial of multi-degree $(m, \ldots, m)$ in every cell of $M$. Then $f \in S_m(M)$ if and only if $\lambda^{\beta_1}(f|_{c_1}) = \lambda^{\beta_2}(f|_{c_2})$ for all $\beta \in B_{d,m}^\ell$ and $c_1, c_2$ belonging to the same connected component of $\Gamma_\beta$.

**Proof.** Assume that there exist $\beta \in B_{d,m}^\ell$ and two cells $c_1, c_2$ in the same connected component of $\Gamma_\beta$ for which $\lambda^{\beta_1}(f|_{c_1}) \neq \lambda^{\beta_2}(f|_{c_2})$. Then there exist two vertices (cells)
Theorem 3 (\cite{24})

Assuming that \( \lambda_{\ell}^B(f|c_1) = \lambda_{\ell}^B(f|c_2) \) for all \( \beta \in B_{d,m} \) and \( c_1, c_2 \) belonging to the same connected component of \( \Gamma_B \). Therefore, by Lemma 2, for every \( c, c' \in M \) having nonempty intersection \( c \cap c' \neq \emptyset \), \( f|c \sim f|c' \) which implies that \( f \in S_m(M) \).

Let \( M \subseteq C_d \), \( \mathcal{A} = \bigcup_{c \in M} c \) and \( \beta \in B_{d,m} \). Following the notation in \cite{24} we denote by \( K(\Gamma_B) \) the set of connected components of a graph \( \Gamma_B \). For a given connected component \( \Phi \in K(\Gamma_B) \) we denote by \( \beta_\Phi \) the function \( \beta_\Phi : \mathbb{R}^d \to \mathbb{R} \) defined as follows:

\[
\beta_\Phi(\pi) = \sum_{\ell \in V(\Phi)} \beta(\pi) \chi_\ell^*(\pi),
\]

(18)

where \( V(\Phi) \) is the set of vertices (cells) of the graph \( \Phi \), \( \pi = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and \( \chi_\ell^*(\pi) \) is a normalized characteristic function: \( \chi_\ell^*(\pi) = \frac{1}{\sqrt{\lambda_{\ell}^B(\pi)}} \) if \( \pi \in c \) and \( \chi_\ell^*(\pi) = 0 \), otherwise. Clearly, \( \beta_\Phi(\pi) = \beta(\pi) \) if \( \pi \in \mathcal{A} \) so \( \beta_\Phi|_\mathcal{A} = \beta|_\mathcal{A} \); \( \beta_\Phi(\pi) = 0 \) if \( \pi \notin \mathcal{A} \).

For given \( \ell \geq 0, d > 0, m > 0 \) and \( M \subseteq C_d' \) we denote by \( \Delta \) the collection of \( \beta_\Phi \) for which \( \Phi \) is finite. Following the notation in \cite{24} we extend \( \beta_\Phi \) to \( \Phi \in K(\Gamma_B) \):

\[
\Delta = \{ \beta_\Phi |_\mathcal{A} \mid \beta \in B_{d,m}, \Phi \in K(\Gamma_B) \}.
\]

It follows from the local linear independence of \( B_{d,m} \) that \( \Delta \) is locally linear independent. Assuming that \( M \) is finite, \cite[Theorem 2.12]{24} shows that \( \Delta \) is a basis of a vector space \( S_m(M) \). When \( M \) is finite, \( \Delta \) is finite so \( S_m(M) \) is \( \lambda \)-dimensional vector space. If \( M \) is infinite, then \( \Delta \) is infinite. Nevertheless, for any collection of coefficients \( \lambda_\delta \) is correctly defined as for every \( \pi \in M \) there are only finitely many \( \delta \in \Delta \) for which \( \delta(\pi) \neq 0 \). So \( \sum_{\delta \in \Delta} \lambda_\delta \) is a function from \( S_m(M) \). Theorem 3 below shows that every function \( f \in S_m(M) \) can be uniquely represented as a formal sum \( \sum_{\delta \in \Delta} \lambda_\delta \). The proof of this theorem repeats the argument of \cite[Theorem 2.12]{24}.

Theorem 3 (\cite{24}). For every \( f \in S_m(M) \), \( f = \sum_{\delta \in \Delta} \lambda_\delta \) for some uniquely defined coefficients \( \lambda_\delta \), \( \delta \in \Delta \).

Proof. Let us consider \( f \in S_m(M) \). For every cell \( c \in M \) we have that \( f|c = \sum_{\beta \in B_{d,m}} \lambda_\beta(f|c) \beta|c \). Therefore, \( f(\pi) = \sum_{\beta \in B_{d,m}} \lambda_\beta(f|c) \beta(\pi) \) for \( \pi = (x_1, \ldots, x_d) \in c \).

Also, we have that \( f(\pi) = \sum_{\ell \in V(\Phi)} f(\pi) \chi_\ell^*(\pi) \) for \( \pi \in \mathcal{A} \). Therefore,

\[
f(\pi) = \sum_{c \in M} \sum_{\beta \in B_{d,m}} \lambda_\beta(f|c) \beta(\pi) \chi_\ell^*(\pi) = \sum_{\beta \in B_{d,m}} \sum_{(\pi, \ell) \in \Gamma(\Phi)} \lambda_\beta(f|c) \beta(\pi) \chi_\ell^*(\pi) = \sum_{\beta \in B_{d,m}} \sum_{\Phi \in K(\Gamma_B)} \sum_{\ell \in V(\Phi)} \lambda_\beta(f|c) \beta(\pi) \chi_\ell^*(\pi)
\]

for \( \pi \in \mathcal{A} \). By Proposition 3, for fixed \( \beta \) and \( \Phi \in K(\Gamma_B) \), the coefficients \( \lambda_\beta(f|c) \) are the same for all \( c \in V(\Phi) \). Let us denote it by \( \lambda_\beta(f) \). Therefore, by (18), we obtain \( f(\pi) = \sum_{\beta \in B_{d,m}} \sum_{\Phi \in K(\Gamma_B)} \lambda_\beta(f) \beta(\pi) \) for \( \pi \in \mathcal{A} \). Changing \( \beta \) to \( \delta \) and \( \lambda_\beta \) to \( \lambda_\delta \), in the latter identity we conclude that \( f = \sum_{\delta \in \Delta} \lambda_\delta \). The uniqueness of coefficients \( \lambda_\delta \) immediately follows from the local linear independence of \( \Delta \). \( \Box \)
Let us denote by $\Delta^*$ the collection of $\beta|_{\mathcal{M}}$ for all $\beta \in B_{d,m}^\ell$, for which $\text{supp } \beta \cap \mathcal{M} \neq \emptyset$:

$$\Delta^* = \{ \beta|_{\mathcal{M}} | \beta \in B_{d,m}^\ell, \text{supp } \beta \cap \mathcal{M} \neq \emptyset \}.$$ 

The analogue of [24, Corollary 2.13] which includes the case of infinite domains is as follows.

**Corollary 2 ([24]).** If for each $\beta \in B_{d,m}^\ell$, for which $\text{supp } \beta \cap \mathcal{M} \neq \emptyset$, the intersection $\text{supp } \beta \cap \mathcal{M}$ is connected, then every $f \in S_m(\mathcal{M})$ is equal to $f = \sum_{\delta \in \Delta^*} \lambda_\delta \delta$ for some uniquely defined coefficients $\lambda_\delta, \delta \in \Delta^*$.

**Proof.** If the assumption of the corollary is satisfied, then $\Delta = \Delta^*$. So the corollary follows directly from Theorem 3.

### 9 Appendix C

For $\ell = 0, \ldots, N - 1$ we denote by $\mathcal{M}^\ell$ the collection of cells from $\mathcal{C}^\ell$ covering a domain $\mathcal{M}^\ell$: $\mathcal{M}^\ell = \bigcup_{c \in \mathcal{M}^\ell} c$. Let us consider $f \in S_m(T)$. By Corollary 1, $f|_{\mathcal{M}^0} \in S_m(\mathcal{M}^0)$. Therefore, by Corollary 2, $f|_{\mathcal{M}^0} = \sum_{\delta \in \mathcal{K}^0} \lambda_\delta \delta|_{\mathcal{M}^0}$ for some coefficients $\lambda_\delta, \delta \in \mathcal{K}^0$. We denote by $f_0$ the function $f_0 = \sum_{\delta \in \mathcal{K}^0} \lambda_\delta \delta$. Let consider now the function $(f - f_0)|_{\mathcal{M}^1}$. By Corollary 1, $(f - f_0)|_{\mathcal{M}^1} \in S_m(\mathcal{M}^1)$. Therefore, by Corollary 2 and the identity $(f - f_0)|_{\mathcal{M}^0} = 0$, we obtain that $(f - f_0)|_{\mathcal{M}^1} = \sum_{\delta \in \mathcal{K}^1} \lambda_\delta \delta|_{\mathcal{M}^1}$ for some coefficients $\lambda_\delta, \delta \in \mathcal{K}^1$. We denote by $f_1$ the function $f_1 = \sum_{\delta \in \mathcal{K}^1} \lambda_\delta \delta$. Clearly, the function $(f - f_0 - f_1)|_{\mathcal{M}^2} \in S_m(\mathcal{M}^2)$ and $(f - f_0 - f_1)|_{\mathcal{M}^1} = 0$, so $(f - f_0 - f_1)|_{\mathcal{M}^2} = \sum_{\delta \in \mathcal{K}^2} \lambda_\delta \delta|_{\mathcal{M}^2}$ for some coefficients $\lambda_\delta, \delta \in \mathcal{K}^2$. We define $f_2 = \sum_{\delta \in \mathcal{K}^2} \lambda_\delta \delta$.

We repeat the process until all functions $f_i = \sum_{\delta \in \mathcal{K}^i} \lambda_\delta \delta, i = 0, \ldots, N - 1$ are constructed. Finally, we have the identity $(f - \sum_{i=0}^{N-1} f_i)|_{\mathcal{M}^{N-1}} = 0$ which implies that $f = \sum_{i=0}^{N-1} f_i$, so $f = \sum_{\delta \in \mathcal{K}} \lambda_\delta \delta$. The uniqueness of coefficients $\lambda_\delta$ follows from local linear independence of tensor product B–splines.