Hyperscaling violating black holes in scalar-torsion theories

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We study a gravity theory where a scalar field with potential, beyond its minimal coupling, is also coupled through a non-minimal derivative coupling with the torsion scalar which is the teleparallel equivalent of Einstein gravity. This theory provides second order equations of motion and we find large-distance non-perturbative static spherically symmetric four-dimensional solutions. Among them a general class of black hole solutions is found for some range of the parameters/integration constants with asymptotics of the form of hyperscaling violating Lifshitz spacetime with spherical horizon topology. Although the scalar field diverges at the horizon, its energy density and pressures are finite there. From the astrophysical point of view, this solution provides extra deflection of light compared to the Newtonian deflection.

I. INTRODUCTION

Teleparallelism \cite{1-8} is an equivalent approach of General Relativity with deep implications which cannot be provided by the standard metric formulation of gravity. For example, it provides a correct definition for the local energy-momentum tensor of the gravitational field which cannot be expressed in terms of the metric. This formulation is based on the use of the vierbein, along with a connection of vanishing curvature, as basic objects, not aiming to define another gravitational theory, but to describe anew Einstein’s gravity itself. Torsion turns out to be the appropriate quantity for this reformulation. The Einstein-Hilbert Lagrangian is expressed (up to boundary quantities) in terms of torsion, so that only up to first order derivatives in the dynamical quantities (vierbein/connection) appear in the new Lagrangian. This Lagrangian is a particular quadratic combination of torsion which is called torsion scalar $T$. As a result, the Einstein equations have also their torsion analogue. As the metric formulation, also the teleparallel one is both diffeomorphism and local Lorentz invariant. By employing the Weitzenböck connection as solution of the connection equation of motion (vanishing curvature, but non-vanishing antisymmetric piece of the connection when expressed in coordinates, and hence non-vanishing torsion), the vierbein remains as the single dynamical variable and only the Lagrangian lacks local Lorentz invariance. Notice that in practice the teleparallelism condition of vanishing curvature is implemented through the Weitzenböck connection, and therefore the teleparallel approach is different than other approaches (e.g. \cite{4}) where torsion introduces a new dynamical field beyond the fields (metric, vierbein) describing the Einstein sector. Not only Einstein gravity but also Gauss-Bonnet gravity has been shown to possess its teleparallel representation through another torsion scalar $T_G$. \cite{10}. Modifications of Einstein or Gauss-Bonnet gravity, different than the curvature based modifications $f(R)$ \cite{11, 12}, $f(R, G)$ \cite{13, 14} theories have been constructed based on $T$, $T_G$, i.e. $f(T)$ \cite{16, 20}, $f(T, T_G)$ \cite{21} theories have been investigated which in the covariant formulation are also diffeomorphism and local Lorentz invariant.

Beyond the purely gravitational sector, there are in nature various sorts of matter fields. Scalar fields for example are predicted by fundamental theories and may be present in the universe at large scales mixed up with baryonic matter, and so be present in ordinary stars. Spherically symmetric configurations for self-gravitating scalar fields with various types of interactions have been derived as solutions of the coupled Einstein-scalar system \cite{22, 24}. At cosmological scales the scalar field dynamics allows to investigate the features of the early universe, or a quintessence scalar field can be interpreted as the present dynamical dark energy sector \cite{25}. The appearance of a non-minimal coupling of the form $f(\phi)R$ is motivated by many reasons, such as the variability of the fundamental constants, the Kaluza-Klein compactification scheme, the low-energy limit of superstring theory, etc. A non-minimal derivative coupling between matter and gravity is the next step of generalization, particularly if the Newton’s constant depends on the gravitational field source mass \cite{26}.

In \cite{27}, a first step was made in an attempt to probe the effect of torsion to General Relativity when this is treated in its teleparallel reformulation, so the torsion scalar $T$ was coupled to a scalar field $\phi$ in four spacetime dimensions. Since for a diagonal vierbein, a simpler coupling of the form $\phi T$ does not possess spherically symmetric solutions (however for three dimensions see \cite{28}), a non-minimal derivative coupling of the form $T g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ was considered.

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This is a novel coupling which provides only second order equations of motion. The metric-based analogous coupling of the form $R g_{\mu \nu} \partial_\mu \phi \partial_\nu \phi$ gives higher order equations of motion [26], while the sector $G^{u \nu} \partial_\mu \phi \partial_\nu \phi$ of the general Horndeski Lagrangian also is a healthy theory with second order field equations [29]. Of course, the above derivative coupling between torsion and matter is not the only one that can be constructed, but other combinations of $\partial_\mu \phi \partial_\nu \phi$ with quadratic (of even higher) powers of torsion could be presented. In [27] static spherically symmetric solutions were studied for the above coupling with torsion, however after the formulation was developed, only a general class of large-distance linearized solutions around its asymptotic Anti-de Sitter (AdS) form was found. Here, we will extend the analysis and find other (general and special) large-distance solutions in the non-linearized regime however, which become of higher and higher accuracy as we restrict in particular regions of the parameter/integration constants space, which is not however fine-tuned or particularly narrow. The regime of applicability of these solutions is for any distance larger than some length scale determined by these parameters. A general such black hole solution turns out to have asymptotics which deviate from the AdS form and are of the hyperscaling violating Lifshitz form.

Our work is organized as follows. In Section II the basic elements and equations of the model appeared in [27] are briefly presented so that to make the present work autonomous. In Section III we proceed with the essential steps in order to simplify the master equation of the problem and facilitate our quest for finding large-distance non-perturbative spherically symmetric solutions of the theory. In Section IV we consider a special solution (i.e. for a particular value of the integration constant of the scalar field) and find black holes and other solutions along with the scalar field profile and the potential. In Section V a dynamical systems analysis reveals the global phase portrait of all the local solutions and serves to obtain an intuition for the approximations made in order to obtain the solutions of the next Section. In Section VI a general from the viewpoint of the number of integration constants black hole solution is found, its asymptotic behavior is analyzed and some astrophysical perspective of the solution is elaborated. Finally, Section VII is devoted to our conclusions.

II. GENERAL FORMULATION

We will consider the following non-minimal derivative coupling of a scalar field $\phi$ with torsion [27]

$$S = -\frac{1}{2\sqrt{\xi}} \int d^4 x \sqrt{g} \left( \frac{1}{2} - \xi \nabla \right) g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + V,$$

(2.1)

where $V$ is the potential of the scalar field and the parameter $\xi$ has dimensions of length square, so $\sqrt{\xi}$ introduces a new length scale. The torsion scalar $T$ is defined by

$$T = \frac{1}{4} T^{\mu \nu \lambda} T_{\mu \nu \lambda} + \frac{1}{2} T^{\mu \nu \lambda} T_{\lambda \mu \nu} - T^\nu \mu T^\lambda \lambda \mu = S^{\mu \nu \lambda} T_{\mu \nu \lambda},$$

(2.2)

where the tensor $S^{\mu \nu \lambda}$ is

$$S^{\mu \nu \lambda} = \frac{1}{2} K^{\nu \lambda \mu} + \frac{1}{2} (g^{\mu \lambda} T^\rho \nu - g^{\mu \nu} T^\rho \lambda) = -S^{\mu \nu \lambda},$$

(2.3)

and

$$K_{\mu \nu \lambda} = \frac{1}{2} (T_{\lambda \mu \nu} - T_{\nu \lambda \mu} - T_{\mu \nu \lambda}) = -K_{\nu \mu \lambda} = \omega_{\mu \nu \lambda} - \Gamma_{\mu \nu \lambda},$$

(2.4)

is the contortion tensor. The connection $\omega^a = \omega^a_{\beta \gamma} d\xi^\beta = \omega^a_{\beta c} e^c$ is assumed to satisfy the teleparallelism condition of vanishing curvature $R^a_{\beta \mu \nu} = \omega^a_{\beta \mu, \nu} - \omega^a_{\beta \mu} \omega^\mu_{\nu} - \omega^a_{\gamma \nu} \omega^\gamma_{\beta \mu} = 0$ (indices $a, b, \ldots$ refer to tangent space, while $\mu, \nu, \ldots$ are coordinate ones). In terms of an arbitrary orthonormal vierbein $e_a = e^a_{\mu} \partial_\mu$ (with dual $e^a = e^a_{\mu} dx^\mu$ and $e = \det(e^a_{\mu})$), i.e. $g_{\mu \nu} = \eta_{ab} e^a_{\mu} e^b_{\nu}$ where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric, the torsion is defined by $T^a_{\beta \gamma} = \omega^a_{\beta \gamma} - \omega^a_{\gamma \nu} e^\gamma_{\beta \mu} e^\nu_{\beta \mu}$ and is a tensor under local Lorentz transformations and under diffeomorphisms. The metric is assumed to be compatible with the connection $\omega^a_{\beta \gamma}$, i.e. $\eta_{abc} = 0$, and so $\omega_{\mu \nu \lambda} = -\omega_{\nu \mu \lambda}$, where $| denotes covariant differentiation with respect to $\omega^a_{\beta \gamma}$. The importance of the torsion scalar $T$ is that it provides after variation with respect to $e^a_{\mu}$ the Einstein equations, i.e. $cT =$ equals $\hat{R}$ up to a total divergence, where $\hat{R}$ is the Ricci scalar of the Christoffel connection $\Gamma^{\nu}_{\mu \lambda}$.

A curvature-based analogue of the above non-minimal coupling is the term $R g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi$. The basic difference between these two is that while the curvature coupling provides higher than second derivatives (and therefore ghosts), here, the torsion coupling gives only second order equations of motion. In some sense, the coupling in (2.1) can be
said that it is closer to the curvature coupling $G^{\mu\nu}\partial_\mu \phi \partial_\nu \phi$ which also provides second order equations of motion, and indeed at the level only of background cosmology these two couplings coincide.

The equations of motion for the vierbein and the scalar field were found in [27] to be

$$
\left( \frac{2}{\kappa^2} - 4\xi \phi \rho \phi^\rho \right) \left[ e S_{\kappa}^{\lambda} e_b^{\kappa} \right]_{,\rho} e^b_{,\mu} + e \left( \frac{1}{2} T \phi^\lambda - S^{\nu\kappa} T_{\nu\kappa} \right) + 4\xi \left( \frac{1}{2} T \phi_{,\mu} \phi^\lambda + e S_{\nu}^{\lambda} \phi_{,\nu} \phi^\kappa \right)_{,\rho} e^b_{,\mu} + e \left( \frac{1}{2} \phi_{,\rho} \phi^\rho \delta^\lambda_{,\phi_{,\mu} \phi^\lambda} + V \delta^\lambda_{,\mu} \right) - \left( \frac{2}{\kappa^2} - 4\xi \phi_{,\rho} \phi^\rho \right) e S^{\alpha\kappa} \omega_{bdc} e_a^{\alpha} e^b_{,\mu} = 0
$$

(2.5)

$$
\left[ e(1 - 2\xi T) \phi_{,\mu} \right]_{,\rho} - e \frac{dV}{d\phi} = 0,
$$

(2.6)

where $\phi_{,\mu} = g^{\mu\nu} \phi_{,\nu}$. As for the connection $\omega^a_{\nu b c}$, we adopt the Weitzenböck solution of the equations of motion $R_{\nu b c d} = 0$, i.e. $\omega^a_{\nu b c} = 0$ in the frame $e^\mu_{,b}$ we will work with, therefore, $\omega^\lambda_{,\mu} = e_a^{\lambda} e_{,\mu}^a$ in all coordinate frames.

We are interested in extracting static spherically symmetric solutions of the four-dimensional scalar-torsion gravity described by the action (2.1). Due to the spherical symmetry, the general form of the metric is diagonal

$$
d s^2 = -N(r)^2 d t^2 + K(r)^2 - 2 dr \right)^2 + R(r)^2 d \Omega_2^2,
$$

(2.7)

where $d \Omega_2^2 = d \theta^2 + \sin^2 \theta \ d \phi^2$ is the two-dimensional sphere, and $N(r)$, $K(r)$ and $R(r)$ are three unknown functions. One could have chosen the gauge with $R(r) = r$, but it turns out to be more convenient to let the function $R(r)$ free. As a next step in order to extract a vierbein that gives rise to the above metric, we choose the simplest diagonal vierbein of the form

$$
e^a_{,\mu} = \text{diag} \left( N(r), K(r)^{-1}, R(r), R(r) \sin \theta \right).
$$

(2.8)

Substitution of (2.8) in the field equations (2.5), (2.6) provides the following set of equations of motion

$$
\frac{1}{K^2} \left( \phi'' + \frac{2V}{K^2} \right) + 2 \left( \frac{1}{\kappa^2 - 2\kappa^2} \right) \left( \frac{R''}{R} + \frac{2R'''}{R} + \frac{2R'}{R} \right) - \frac{16\xi \phi'}{R} \left( \frac{K'}{K} \phi' + \phi'' \right) = 0
$$

(2.9)

$$
\frac{1}{K^2} \left( \phi'' + \frac{2V}{K^2} \right) + 2 \left( \frac{1}{\kappa^2 - 2\kappa^2} \right) \left[ \frac{N'}{N} \left( \frac{R'}{R} + \frac{K'}{K} \right) + \frac{R'}{R} \frac{R'}{K} + \frac{N'}{N} \right] - 8\xi \phi' \left( \frac{R'}{R} + \frac{N'}{N} \right) \left( \frac{K'}{K} \phi' + \phi'' \right) = 0
$$

(2.10)

$$
\left\{ K R^2 \phi \left[ 1 + 4\xi K^2 \frac{R'}{R} \right] \right\} - \frac{N R^2 dV}{K \phi} = 0
$$

(2.11)

(2.12)

where a prime denotes differentiation with respect to $r$. Equations (2.9)-(2.12) are respectively the non-vanishing $(\lambda, \mu) = tt, rr, \theta \theta, \theta r$ components of the system (2.5). An interesting characteristic, usual in teleparallel gravity, is the appearance of the off-diagonal equation (2.12), although the metric ansatz (2.7) and the vierbein (2.8) are diagonal. This does not happen in the equations of motion derived from curvature-based actions, and in general, the extra equation imposes difficulties in finding solutions in the torsion formulation. Equations (2.9)-(2.13) have the reparametrization invariance, so for $r \to \bar{r}(r)$ and $K \to K \frac{d\bar{r}}{dr}$, $N \to \bar{N}$, $R \to R$, $\phi \to \phi$, the equations remain form invariant. This implies that one of these equations is not necessary because it arises from the others. Thus, we remain with four independent equations for four unknowns $N(r)$, $K(r)$, $\phi(r)$, $V(r)$ ($R(r)$ is not considered in the enumeration of the unknowns due to the choice of the radial gauge), so there is no arbitrary function left. As a result the previous system has basically a unique solution and this gives us a hope to extract some information on the structure and behaviour of the derived solutions not constrained by a prefixed potential.

In [27] it was shown that if

$$
x = \ln R
$$

(2.14)

$$
Y = \left( \frac{\dot{R}}{R} \right)^2,
$$

(2.15)

where a dot denotes differentiation with respect to $\phi$, the system of equations (2.9)-(2.13) reduces to the master equation for $Y(x)$

$$
2 \frac{d^2 Y}{dx^2} - \frac{1}{Y} \left( \frac{dY}{dx} \right)^2 + 2 \left( 2 - \eta^2 \right) \frac{dY}{dx} + 3Y + 2 \eta^2 - 12 \eta^2 Y \left( \frac{dY}{dx} \right)^2 + 3Y + 2 \eta^2 = 0,
$$

(2.16)
\[
\eta = \frac{\kappa^2}{2\nu^2(1 - 2\xi \kappa^2 \nu^2)}, \quad \tilde{\eta} = \frac{\kappa^2}{\nu^2(1 - 6\xi \kappa^2 \nu^2)}.
\] (2.17)

The integration constant \(\nu\) with dimensions of inverse length square is introduced through the relation \(\phi' = \frac{\kappa}{\nu}\) arising from (2.12). After equation (2.16) is solved, the potential, the scalar field profile and the metric can be found. Namely, the potential \(V(R)\) is found from

\[
V = \frac{1}{2\eta \nu^2} e^{-2x} - \frac{\nu^2}{2} - \frac{1}{2\eta} \left(\frac{dY}{dx} + 3Y\right)
\]  (2.18)

and the scalar field \(\phi(R)\) is obtained from

\[
\left(\frac{dx}{d\phi}\right)^2 = Y(x).
\] (2.19)

The lapse function \(N(R)\) is found from

\[
\left[\frac{d\ln(RN^2)}{dx}\right]^2 = \frac{Z(x)}{Y(x)}
\] (2.20)

where

\[
Z = \frac{\tilde{\eta}^2}{4\eta^2 Y} \left(\frac{dY}{dx} + 3Y + 2\nu^2\right)^2.
\] (2.21)

Finally the metric takes the form

\[
ds^2 = -N^2 dt^2 + \frac{dR^2}{\nu^2 \eta^2 R^2 Y} + R^2 d\Omega_2^2.
\] (2.22)

Note that as far as \(\xi\) and \(\nu\) remain unrelated, the coupling \(\xi\) appears only through the combination \(\xi \kappa^2 \nu^2\) and this is due to the initial interaction term \(T_{\theta \theta} = g_{\theta \theta} \partial_{\theta} \phi \partial_{\theta} \phi\).

It is also interesting to investigate the behavior of another physically meaningful quantity which is the energy-momentum tensor \(T^a_{\mu}\) of the scalar field \(\phi\). This arises by variation with respect to \(e_a^\mu\) of the scalar field action, i.e. of the second integral appearing in (2.1). One can find from (2.9)-(2.12) the on-shell components of \(T^a_{\mu}\), as \(T^t_t = T^\theta_\theta = \frac{1}{2} \frac{2V - \kappa^2 \nu^2}{2\xi \kappa^2 \nu^2 - 1}\), \(T^R_R = T^t_R = T^R_t = \frac{1}{2} \frac{2V - (1 + 8\xi \kappa^2 R^2) \kappa^2 \nu^2}{6\xi \kappa^2 \nu^2 - 1}\), which express the energy density and the pressures of the scalar field. Since \(K \phi' = \nu\), it arises \(T^t_t = T^\theta_\theta = \frac{1}{2} \frac{2V + \nu^2}{2\xi \kappa^2 \nu^2 - 1}\). We start by writing equation (2.16) as

\[
2\frac{d^2 Y}{dx^2} - \frac{1}{Y} \left(\frac{dY}{dx}\right)^2 + 2 \left(2 - \frac{\eta \nu^2}{Y}\right) \frac{dY}{dx} + 3 \left(1 - \frac{4\nu^2}{\eta^2}\right) Y + 2\nu^2 + 24\frac{\eta^2}{\eta^2 Y} \left(\frac{dY}{dx} + 3Y + 2\nu^2\right) = 0,
\] (3.1)

which also takes the form

\[
4\sqrt{Y} \frac{d^2 \sqrt{Y}}{dx^2} + 2 \left(2 - \frac{\eta \nu^2}{Y}\right) \frac{dY}{dx} + 3 \left(1 - \frac{4\nu^2}{\eta^2}\right) Y + 2\nu^2 + 24\frac{\eta^2}{\eta^2 Y} \left(\frac{dY}{dx} + 3Y + 2\nu^2\right) = 0.
\] (3.2)

If we define

\[
U = \sqrt{Y},
\] (3.3)

equation (3.2) takes the form

\[
\frac{d^2 U}{dx^2} + \left(2 - \frac{\eta \nu^2}{U^2}\right) \frac{dU}{dx} + 3 \left(1 - \frac{4\nu^2}{\eta^2}\right) U + \frac{\eta \nu^2}{2U} + \frac{2\nu^2}{\eta^2} \frac{dU}{dx} + \frac{2}{3} U + \frac{\nu^2}{U} = 0,
\] (3.4)
which is simpler than the initial equation (2.16). Equation (3.4) for 

\[ R \gg \frac{1}{\nu^2 \sqrt{\eta}} \]  

becomes autonomous

\[ \frac{d^2 U}{dx^2} + \left( 2 - \frac{\eta \nu^2}{U^2} \right) \frac{dU}{dx} + \frac{3}{4} \left( 1 - \frac{4 \eta^2}{\eta^2} \right) U + \frac{3 \eta^2}{\eta^2} \frac{dU}{dx} + \frac{3 \eta^2}{2U} \frac{dU}{dx} + \frac{4 \eta^2}{3U} + \frac{\eta^2}{U} = 0 . \]  

(3.6)

If

\[ \Omega = \frac{dU}{dx} + \frac{3 \eta^2}{2U} + \frac{\eta^2}{U} , \]  

(3.7)

equation (3.6) becomes

\[ \Omega \left( \Omega - \frac{3 \eta^2}{2U} - \frac{\eta^2}{U} \right) \frac{d\Omega}{dU} + \frac{1}{2} \Omega^2 - \frac{3 \eta^2}{\eta^2} U \Omega + \frac{3 \eta^2}{\eta^2} \eta^2 = 0 . \]  

(3.8)

It is convenient to define \( \alpha = \eta \nu^2, \beta = \frac{3 \eta^2}{\eta^2} > 0 \). The parameters \( \alpha, \beta \) are not independent, but they obey the relations

\[ \alpha = \frac{\kappa^2}{2(1 - 2 \Xi)}, \beta = \frac{4}{3} \left( \frac{1 - 6 \Xi}{1 - 2 \Xi} \right)^2 = \frac{4}{3} \left( \frac{1 - 6 \Xi}{1 - 2 \Xi} \right)^2 , \]  

where \( \Xi = \xi \kappa^2 \nu^2 \). It is seen that for \( \alpha > 0 \), the parameter \( \beta \) can take any positive value, while for \( \alpha < 0 \) it is \( \beta > \frac{27}{4} \). The inequality (3.5) gets the form

\[ R \gg \frac{1}{\nu \sqrt{\alpha}} = \sqrt{2} \left| \frac{1}{\sqrt{2} \xi - \frac{1}{\kappa^2 \nu^2}} \right| . \]  

(3.9)

Finally, equation (3.8) becomes

\[ \left( \Omega - \frac{3 \eta^2}{2U} - \frac{\alpha}{U} \right) \frac{d\Omega}{dU} + \frac{1}{2} \Omega^2 - \beta U \Omega + \alpha \beta \Omega = 0 . \]  

(3.10)

Equation (3.10) is also written as

\[ \left( \Omega - \frac{3 \eta^2}{2U} - \frac{\alpha}{U} \right) \frac{d\Omega}{dU} + \frac{1}{2} \Omega - \beta U + \alpha \beta \Omega = 0 , \]  

(3.11)

or also

\[ \frac{d\Omega}{dU} = \beta U - \frac{\alpha}{\Omega - \frac{3 \eta^2}{2U} - \frac{\alpha}{U}} . \]  

(3.12)

Although (3.12) is a first order differential equation, significantly and unexpectedly simpler than the initial equation (2.16), there is no known method how to solve it. However, its characteristic form with the presence of the inverse powers of \( U, \Omega \) will enable us to find “large”-distance non-perturbative solutions. As always, the word large is meant in comparison with scales and integration constants of the problem, and depending on them, the corresponding distances can be relevant for a physical situation.

Note that the system of equations (2.9)-(2.13) for vanishing scalar field \( \phi = 0 \) and a cosmological constant as the potential \( V \) has the standard Schwarzschild-(A)dS solution. A vanishing scalar field means that \( \nu = 0 \) and then it is \( \alpha = \frac{\kappa^2}{2}, \beta = \frac{4}{3}, \eta = \frac{1}{2} \), while \( \nu, \eta \) of (3.2), (3.4) become infinite. Moreover, the autonomous equation (3.6) cannot recover this limit since the inequality (3.9) is never satisfied. The reason is that the exponential factor in (3.4) becomes significant in this limit. However, although the general solutions to be derived will not be continuous deformation of Schwarzschild, this does not necessarily mean that they will not capture some interesting phenomenology, for example they could be related to galactic scales or beyond.
IV. SPECIAL SOLUTIONS WITH $\nu \to 0$

In the limit $\nu \to 0$ the theory possesses solutions, including black holes, with non trivial scalar field and potential. For $\nu \to 0$, the scalar field $\phi \to 0$ and $\dot{\phi}$ vanishes. However, the “normalized” field $\tilde{\phi} = \phi / \nu$ obeys $\tilde{\phi}' = K^{-1}$ which is a meaningful equation. Similarly, although $U \to \infty$ in (3.4), however this equation still makes sense. Indeed, if we define the “normalized” variable $\tilde{U} = \nu U$, (4.1) becomes in this limit

$$\left( \frac{d\tilde{U}}{dx} + \frac{3}{2} \tilde{U} \right) \left( \frac{d^2\tilde{U}}{dx^2} + 2 \frac{d\tilde{U}}{dx} \right) + \frac{1}{4} e^{-2x} = 0,$$

where there is no free parameter left in (4.1). Then, (2.19) becomes

$$\frac{dx}{d\tilde{\phi}} = \pm \tilde{U},$$

and equations (2.20), (2.21) are meaningful. Finally, the metric (2.22) becomes

$$ds^2 = -N^2 dt^2 + R^4 d\Omega^2.$$

The potential $V$ is found from (2.18) to be

$$V = \frac{1}{\kappa^2} \left( e^{-2x} - 3 U^2 - 2 U \frac{d\tilde{U}}{dx} \right).$$

We will find as a warm up large-distance solutions in the above limit. For $F = e^x U$ we get from (4.1)

$$\left( \frac{dF}{dx} + \frac{1}{2} F \right) \left( \frac{d^2F}{dx^2} - F \right) + \frac{1}{4} = 0,$$

If $v(F) = \frac{dF}{dx}$, then

$$\left( v + \frac{1}{2} F \right) \left( v \frac{dv}{dF} - F \right) + \frac{1}{4} = 0,$$

or equivalently

$$\frac{dv}{dF} = F - \frac{1}{2F + 4v}.$$  

The variables $F, v$ are dimensionless and we can check by numerically solving (4.7) that every solution extends to large values of $|F|, |v|$. Thus, for $|F|, |v| \gg 1$ we get from (4.7)

$$v \frac{dv}{dF} = F,$$

with general solution

$$v^2 = F^2 - \sigma,$$

where $\sigma$ is integration constant. Then,

$$F = \frac{\sigma R^2 + \tau^2}{2\tau R} \quad \text{or} \quad F = \frac{\tau^2 R^2 + \sigma}{2\tau R},$$

where $\tau$ is another integration constant. Depending on $\sigma, \tau$, this solution can indeed correspond to large values of $R$, since we see that large values of $R$ can correspond to large values of $|F|, |v|$. Integrating equation (2.21) we find $N$. Finally, rescaling the time $t$, there arise two sorts of metrics

$$ds^2 = -F^2 dt^2 + \frac{dR^2}{F^2} + R^2 d\Omega^2.$$
or

\[ ds^2 = -\frac{1}{R^2 F^2} dt^2 + \frac{dR^2}{F^2} + R^2 d\Omega_2^2. \] (4.12)

Thus, there exist four different metrics.

Depending on the constants \( \sigma, \tau \), the metric (4.11) may possess horizon. To be concrete, let us consider the first case of (4.10) with \( \tau > 0, \sigma < 0 \) and \( |\sigma| \ll 1 \). Then, it is obvious that large values of \( R \) correspond to \( F \ll 1 (or |F| \gg 1) \), which means \( R \gg \frac{\tau}{2} \). From (4.19) it is also \( |v| \gg 1 \), while the quantity \( 2F + 4v \) is large even for \( Fv < 0 \). On the other hand, the horizon of (4.11) corresponds to \( F = 0 \), which means \( R_{\text{hor}} = \frac{\sqrt{|\sigma|}}{2|\tau|} \). This value can be made arbitrarily larger than \( \frac{\tau}{2} \) by choosing \( |\sigma| \) sufficiently small, and so the existence of the horizon within our approximation has been shown. In this case, (4.11) with the first of (4.10) is the black hole

\[ ds^2 = -\left(\frac{|\sigma| R^2 - \tau^2}{4\tau^2 R^2}\right) dt^2 + \frac{4\tau^2 R^2}{(\tau^2 R^2 - |\sigma|)^2} dR^2 + R^2 d\Omega_2^2 \] (4.13)

and the solution is certainly valid outside the horizon. Asymptotically, it gets an AdS form

\[ ds^2_{\infty} = -\frac{\sigma^2}{4\tau^2} R^2 dt^2 + \frac{4\tau^2 R^2}{\sigma^2} dR^2 + R^2 d\Omega_2^2. \] (4.14)

The length scale introduced here is \( \ell_{\text{eff}} = \frac{\sigma^2}{4\tau^2} \) and the corresponding effective cosmological constant \( \Lambda_{\text{eff}} = -3\ell_{\text{eff}}^{-2} \).

Similarly, for the second case of (4.10) with \( \tau < 0, \sigma < 0 \) and \( |\sigma| \ll 1 \), it is obvious that large values of \( R \) correspond to \( F \ll 1 (or |F| \gg 1) \), which means \( R \gg \frac{\sigma}{2|\tau|} \). From (4.19) it is also \( |v| \gg 1 \), while the quantity \( 2F + 4v \) is large even for \( Fv < 0 \). On the other hand, the horizon of (4.11) corresponds to \( F = 0 \), which means \( R_{\text{hor}} = \frac{\sqrt{|\sigma|}}{|\tau|} \). This value again can be made arbitrarily larger than \( \frac{\sigma}{2|\tau|} \) for \( |\sigma| \) sufficiently small, and so in this case (4.11) with the second of (4.10) forms another black hole

\[ ds^2 = -\left(\frac{\tau^2 R^2 - |\sigma|}{4\tau^2 R^2}\right) dt^2 + \frac{4\tau^2 R^2}{(\tau^2 R^2 - |\sigma|)^2} dR^2 + R^2 d\Omega_2^2 \] (4.15)

Asymptotically, it also gets another AdS form

\[ ds^2_{\infty} = -\frac{\tau^2}{4} R^2 dt^2 + \frac{4}{\tau^2} dR^2 + R^2 d\Omega_2^2 \] (4.16)

with \( \ell_{\text{eff}} = \frac{\tau}{2} \).

Concerning the scalar field \( \tilde{\phi} \), we find from (4.12) for the two black hole solutions (4.13), (4.15) respectively

\[ \tilde{\phi} - \tilde{\phi}_0 = \pm \frac{\tau}{\sigma} \ln \left(|\sigma| R^2 - \tau^2\right) \quad \text{or} \quad \tilde{\phi} - \tilde{\phi}_0 = \pm \frac{1}{\tau} \ln \left(\tau^2 R^2 - |\sigma|\right). \] (4.17)

Notice that the scalar field diverges at the position of the horizon and also asymptotically. The scalar field is a secondary hair since it does not introduce a new non-trivial integration constant (i.e. \( \phi \) is not varied independently of the black hole parameters).

Finally from (4.13), since \( \frac{d\ell}{dx} = -\frac{v - F}{R} \), we find the potential \( V \) respectively

\[ V = \frac{\sigma^2}{\kappa^2 \tau^2} F^{-2} - 1 + 2\epsilon \sqrt{1 + |\sigma| F^{-2}} \left(1 + \sqrt{1 + |\sigma| F^{-2}}\right)^2 \] or \[ V = \frac{\tau^2}{\kappa^2} F^{-2} - 1 + 2\epsilon \sqrt{1 + |\sigma| F^{-2}} \] (4.18)

where \( \epsilon \) is a \( \pm \) sign. From (4.17), converting \( R \) to \( \tilde{\phi} \), we can easily find \( F(\tilde{\phi}) \) respectively

\[ F^{-2} = \frac{4\tau^2}{|\sigma|} \left[e^\frac{\tau}{\kappa} (\tilde{\phi} - \tilde{\phi}_0) + \tau e^\frac{2\tau}{\kappa} (\tilde{\phi} - \tilde{\phi}_0)\right] \quad \text{or} \quad F^{-2} = 4\tau^2 \left[e^\frac{\tau}{\kappa} (\tilde{\phi} - \tilde{\phi}_0) - \epsilon e^\frac{2\tau}{\kappa} (\tilde{\phi} - \tilde{\phi}_0)\right]. \] (4.19)

Substituting (4.19) into (4.18) we find \( V(\tilde{\phi}) \). Note that for any solution with specific \( \sigma, \tau \) the corresponding potential \( V \) that supports the solution also depends on \( \sigma, \tau \). We could rescale \( \tilde{\phi} \) to \( \phi = \frac{\tau}{\kappa} (\tilde{\phi} - \tilde{\phi}_0) \) or \( \phi = \tau (\tilde{\phi} - \tilde{\phi}_0) \), but still
σ, τ remain in V. At spatial infinity it is $F^{-2} \to 0$, therefore the potential becomes for the two cases $V \to \frac{(2\epsilon - 1)\sigma^2}{4\kappa^2}$ or $V \to \frac{(2\epsilon - 1)\tau^2}{4\kappa^2}$, and for $\epsilon = -1$ these values coincide with the corresponding values of $\Lambda_{\text{eff}}/\kappa^2$. For $\epsilon = 1$, these values of V are positive, therefore the cosmological constant coming from the potential in the action is positive, while the effective cosmological constant $\Lambda_{\text{eff}}$ of asymptotically AdS space is negative. This discrepancy is due to the growth of a nontrivial profile of the scalar field which is coupled to the torsion and modifies the asymptotic form of the spacetime. In [27] a similar discrepancy between $\Lambda_{\text{eff}}$ and V has been found, but not for positive V which is the case here. A positive value of V has the merit that it can be related to the vacuum energy of spacetime. On the other hand, in [23] and other solutions the asymptotic value of V is the same as $\Lambda_{\text{eff}}/\kappa^2$. Note also that at the horizon the potential V is finite. Concerning the components of the energy-momentum tensor we have $T^t_t = T^R_R = T^r_r = T^\theta_\theta = -V$. Therefore, although the scalar field diverges at the horizon, its energy density and the pressures are finite there. Asymptotically, still the energy density of the scalar field and the pressures remain finite. Finally, notice that due to the higher order pole at the horizon, the temperature vanishes in analogy to the Reissner-Nordstrom solution.

V. DYNAMICAL SYSTEMS ANALYSIS

The second order autonomous differential equation (3.6) can be converted into a two-dimensional dynamical system with one function the variable $\hat{U}$ and as second function some combination containing the derivative $\frac{d\hat{U}}{dx}$, e.g. $\hat{\Omega}$. To study the system it is better to convert into dimensionless variables. We define

$$\hat{U} = \frac{U}{\kappa}, \quad \hat{\Omega} = \frac{\Omega}{\kappa}, \quad \hat{\alpha} = \frac{\alpha}{\kappa^2},$$

(5.1)

where $\hat{U} > 0$ and $\hat{\Omega}$ is real. So, instead of the equation (3.6) we have the following equivalent system

$$\frac{d\hat{U}}{dx} = \hat{\Omega} - \frac{3}{2} \hat{U} - \frac{\hat{\alpha}}{\hat{U}},$$

(5.2)

$$\frac{d\hat{\Omega}}{dx} = \beta \hat{U} - \frac{1}{2} \hat{\Omega} - \frac{\hat{\alpha} \beta}{\hat{\Omega}},$$

(5.3)

where $\beta = \frac{3}{4}(4\hat{\alpha} - 3)^2$. We can find the fixed points of this system by setting $\frac{d\hat{U}}{dx} = \frac{d\hat{\Omega}}{dx} = 0$. Then, we find these fixed points to be

$$\hat{U}_* = \sqrt{\frac{2\hat{\alpha}}{\pm 2\sqrt{3\beta} - 3}}, \quad \hat{\Omega}_* = \pm \sqrt{\frac{6\hat{\alpha} \beta}{\pm 2\sqrt{3\beta} - 3}}.$$

(5.4)

For $\hat{\alpha} < 0$, $\beta > \frac{27}{4}$ there is exactly one fixed point ($\hat{U}_* = \sqrt{\frac{2\hat{\alpha}}{2\sqrt{3\beta} + 3}}, \hat{\Omega}_* = -\sqrt{\frac{6\hat{\alpha} \beta}{2\sqrt{3\beta} + 3}}$). There are two negative eigenvalues $3\lambda_{1,2} = -2(\sqrt{3\beta} + 3) \pm \sqrt{12\beta + 6\sqrt{3\beta} + 9}$ of the linearized system, thus the fixed point is attractor
FIG. 2: Phase portrait \((\hat{U}, \hat{\Omega})\) for \(\hat{\alpha} = \frac{3}{2}, \beta = \frac{27}{4}\).

FIG. 3: Phase portrait \((\hat{U}, \hat{\Omega})\) for \(\hat{\alpha} = \frac{7}{8}, \beta = \frac{3}{16}\).

(stable node), as also seen in Fig. 1. In Fig. 1, 2, 3 the horizontal axis is \(\hat{U}\) and the vertical \(\hat{\Omega}\), and the arrows show increase of the radius \(R\). Moreover, due to the pole \(\hat{\Omega} = 0\) in (5.3), the phase portraits are separated into two quadrants, the upper one with \(\hat{\Omega} > 0\) and the lower with \(\hat{\Omega} < 0\). For \(\hat{\alpha} \in (0, \frac{1}{2}) \cup (1, +\infty), \beta > \frac{3}{4}\) there is again exactly one fixed point \((\hat{U}_*, \hat{\Omega}_*) = \sqrt{\frac{2\hat{\alpha}}{2\sqrt{3\beta}-3}}, \sqrt{\frac{6\hat{\alpha}\beta}{2\sqrt{3\beta}-3}}\). In this case, the two eigenvalues \(3\lambda_{1,2} = 2(\sqrt{3\beta-3})\pm \sqrt{12\beta-6\sqrt{3\beta}+9}\) have opposite signs, therefore the fixed point is saddle, as also seen in Fig. 2. Finally, for \(\frac{1}{2} < \alpha < 1, 0 < \beta < \frac{4}{9}\) there is no fixed point, as it is seen in Fig. 3. The values (5.4) of \(\hat{U}\) are the same found in [27] studying the asymptotic behaviour of equation (2.16). From the above analysis it becomes clear that only for \(\hat{\alpha} < 0\) (case A of [27]) the fixed point is attractor and indeed corresponds to a large-distance asymptotic solution. For \(\hat{\alpha} > 0\) (case B of [27]) the fixed point is saddle and does not correspond to an asymptotic solution. It would be interesting for \(\hat{\alpha} < 0\) to investigate beyond the linearized asymptotic behavior found in [27], also the non-perturbative regime of the solution, however, this will not be studied in the present paper. In the present paper we will find the large-distance non-perturbative solutions which correspond to large values of \(\hat{U}, \hat{\Omega}\) as shown in the upper quadrants of Fig. 1 and 2. These solutions will be now shown that are attracted by stable fixed points at infinity.

Due to that the dynamical system (5.3) is non-compact, there could be non-trivial fixed points at infinity, i.e. at the asymptotic region of the phase portrait \((\hat{U}, \hat{\Omega})\). These can be studied using the Poincaré projection method. We define the new variables (coordinates of phase space) by

\[
\hat{U} = \frac{\dot{\rho}}{1-\rho} \cos \hat{\theta}, \quad \hat{\Omega} = \frac{\dot{\rho}}{1-\rho} \sin \hat{\theta}
\] (5.5)
with $-\frac{3}{4} \leq \hat{\theta} \leq \frac{3}{4}$, $0 \leq \hat{r} < 1$. The upper quadrants of Fig. 1, 2, 3 correspond to $\hat{\theta} > 0$, while the lower ones to $\hat{\theta} < 0$. The limit $\hat{r} \to 1^-$ corresponds to infinite distance in phase space, $\hat{U}^2 + \hat{\Omega}^2 \to \infty$. In terms of $\hat{r}, \hat{\theta}$ the dynamical system (5.2), (5.3) becomes

$$\frac{d\hat{r}}{dx} = \frac{\hat{r} - 1}{2\hat{r}} \left\{ 2\hat{\alpha}\beta(1-\hat{r}) \right\} + \hat{r}^2 \left[ 2(1+\hat{\alpha}) + \cos 2\hat{\theta} - (\beta + 1) \right]$$

(5.6)

$$\frac{d\hat{\theta}}{dx} = \frac{1}{2\hat{r}^2} \left[ 2\hat{\alpha}(1-\hat{r}) \right] \tan(\hat{\beta} + \cot \hat{\theta}) + (\beta + 1)\hat{r}^2 \cos 2\hat{\theta} + \hat{r}^2 (\sin 2\hat{\theta} + \beta - 1)$$

(5.7)

As $\hat{r} \simeq 1^-$ the leading terms of equation (5.7) for $\frac{d\hat{\theta}}{dx}$ are

$$\frac{d\hat{\theta}}{dx} \simeq \frac{1}{2} (\beta + 1) \cos 2\hat{\theta} + \sin 2\hat{\theta} + \beta - 1$$

(5.8)

while at linear order $\frac{d\hat{r}}{dx} = 0$. The critical points $\hat{\theta}_*$ at infinity are obtained by setting $\frac{d\hat{\theta}}{dx} = 0$ in (5.8) and solving for $\hat{\theta}$, thus

$$(\beta + 1) \cos 2\hat{\theta}_* + \sin 2\hat{\theta}_* + \beta - 1 = 0$$

(5.9)

This equation has for any $\beta > 0$ two roots for $\hat{\theta}_*$, one positive and one negative. Therefore, there are always two fixed points at infinity, one in the upper quadrant of the $(\hat{r}, \hat{\theta})$ plane and the other in the lower quadrant, as are also depicted in the uncompactified plots of Fig. 1, 2, 3. Since $\frac{d\hat{\theta}}{dx}|_{\hat{\theta}_*} = 0$ we cannot rely on the linearized analysis to examine the stability of these fixed points and numerical examination is needed. Indeed, it can be seen numerically that for the parameters of Fig. 1, 2 a stable fixed point exists for $\hat{\theta}_* > 0$ (for which it is $\frac{d\hat{\theta}}{dx}(\hat{\theta}_*) > 0$) and an unstable fixed point exists for $\hat{\theta}_* < 0$ (for which it is $\frac{d\hat{\theta}}{dx}(\hat{\theta}_*) < 0$), in agreement with the phase portraits of Fig. 1, 2.

 VI. GENERAL SOLUTIONS

Equation (3.12) can be approximated in the case that $U, \Omega > 0$ are large (compared to the gravity coupling $\kappa$), or more precisely the dimensionless quantities $\hat{U}, \hat{\Omega} > 0$ are large. It is obvious from Fig. 1, 2 that large values of $U, \Omega$ correspond to large $R$, something that will be verified after the solution is found. More precisely, for

$$\left| \Omega - \frac{3}{2} U \right| \gg \frac{|\alpha|}{U} \quad \text{and} \quad \left| U - \frac{1}{2\beta} \Omega \right| \gg \frac{|\alpha|}{\Omega}$$

(6.1)

we get

$$\frac{d\Omega}{dU} \simeq \frac{2\beta U - \Omega}{2\Omega - 3U}$$

(6.2)

which is a homogeneous equation. The above two inequalities are obviously consistent for large $U, \Omega$, and moreover, these inequalities (together with (5.9)) will define the exact $R$-domain where the solution is valid. The puzzling situations with $\Omega \approx \frac{3}{2} U$ or $\Omega \approx 2\beta U$ ($U$ large) do not occur since they provide for large $R$ that $\frac{d\Omega}{dU} \to \infty$ or 0 respectively, which is seen from Fig. 1, 2 not to be the case. The above inequalities set restrictions in the allowed region of the 3-dimensional space $(x, U, \frac{dU}{dx})$ of the initial second order differential equation (3.10) where the orbits reside. Due to this, although we will find a general large-distance (non-linearized) solution with the correct (maximum) number of integration constants, the solution will fail to describe other regions in the space of initial data which also provide large-distance solutions. This is obvious from the dynamical systems analysis of the previous section.

Assuming, for example, the parameter $|\alpha|$ to be sufficiently small, which means $|\Xi| = |\kappa^2 \nu^2| \gg 1$, the inequalities can more easily be satisfied and the validity of the approximation becomes even more extended. At the same time the value of the parameter $\beta$ is very close to $27/4$. Such values $|\Xi| \gg 1$ for the parameters were argued in [27] that could in principle reduce a large value of the vacuum energy to a small effective cosmological constant. Moreover, for such $\Xi$ the inequality (3.9) becomes $R \gg 2\sqrt{|\xi|}$, so this inequality is determined by the length scale defined by the non-minimal coupling $\xi$. Additionally, since $\nu$ is related to the integration constant of the scalar field $\phi$, it is expected to take large values for macroscopic solutions, thus even for small $|\xi|$ it can be $|\Xi| \gg 1$. However, we do not restrict our analysis only in this range of parameters.
Setting
\[ \Phi = \frac{\Omega}{U} > 0, \]  
(6.3)
equation (6.2) becomes
\[ U \frac{d\Phi}{dU} + 2 \frac{\Phi^2 - \Phi - \beta}{2\Phi - 3} = 0, \]  
(6.4)
which is separable. Setting
\[ \Psi = \Psi - \frac{1}{2} \]  
(6.5)
(with \( \Psi > -\frac{1}{2} \)), equation (6.4) becomes
\[ U \frac{d\Psi}{dU} + \frac{\Psi^2 - \gamma^2}{\Psi - 1} = 0, \]  
(6.6)
where \( \gamma = \sqrt{\beta + \frac{1}{4}} > \frac{1}{2} \) (and thus \( \Psi > -\gamma \)). Equation (6.7) which makes the connection with the radial variable \( x \) (or \( R \)) takes the form
\[ \frac{d\Psi}{dx} = \frac{\Psi^2 - \gamma^2}{\Psi - 1} \left( 1 - \Psi + \frac{\alpha}{U^2} \right). \]  
(6.7)
The parameter \( \alpha \), which had temporarily disappeared in (6.2), came up here again. The quantity \( U^2 \) in (6.7) will be found as a function of \( \Psi \) by integrating (6.6). Indeed, the solution of (6.6) is
\[ U^2 = C (\Psi + \gamma)^{-\frac{1}{2}} |\Psi - \gamma|^{\frac{1}{2}}, \]  
(6.8)
where \( C > 0 \) is integration constant that distinguishes the solutions in the \((U, \Omega)\) space. In terms of \( \Omega \) the first integral (6.8) takes the form
\[ \left[ \Omega + \left( \gamma - \frac{1}{2} \right) U \right]^{1 + \frac{1}{2}} |\Omega - \left( \gamma + \frac{1}{2} \right) U|^{1 - \frac{1}{2}} = C. \]  
(6.9)
From (6.8) it is seen that only for \( \Psi \simeq \gamma > 1 \) it is \( U \to \infty \), in agreement with Fig. 1, 2. Therefore, we focus on \( \gamma > 1 \) (which means \( \beta > \frac{3}{2} \), while \( \alpha < \frac{\sqrt{3}}{3} \) or \( \alpha > \kappa^2 \)) where \( U \) can indeed extend to infinity. Furthermore, since \( \Omega = (\Psi + \frac{1}{2})U \), for such \( \gamma \) there are solutions where both \( U, \Omega \) can extend to infinity. From (6.8) it is seen that there are two classes of solutions, one with \( \Psi > \gamma \) and the other with \( \Psi < \gamma \) (these solutions are represented respectively by the left/right orbits of the upper quadrants in Fig. 1, 2). From (6.6) it is seen that for the solutions with \( \Psi > \gamma \) it is \( \Psi \) a monotonically decreasing function of \( U \). To see if the above solutions correspond to large \( R \), we set \( \Psi \simeq \gamma \) in the integral (6.8) or even the equation (6.7). Indeed, we get \( R \simeq R_0 |\Psi - \gamma|^{\frac{1}{2}} \), so the region \( R \to \infty \) in included and the metric can be extended to large distances. Obviously, \( \Psi \) is not necessarily close to the asymptotic value \( \gamma \) in the range of applicability of the solution and the constraints (3.5), (6.1) will determine the exact domain of \( R \).

The metric (2.22) becomes
\[ ds^2 = -N^2 dt^2 + \frac{|\Psi - \gamma|^{1-\frac{1}{2}} (\Psi + \gamma)^{1+\frac{1}{2}} dR^2}{\nu^2 C R^2} + R^2 d\Omega_2^2, \]  
(6.10)
where the relation of \( R, \Psi \) in (6.10) is found from (6.7)
\[ R(\Psi) = R_0 e^{-\frac{\Psi-\gamma}{\nu+\gamma}} \frac{d\Psi}{1-\Psi+\alpha U^2}, \]  
(6.11)
with \( R_0 > 0 \) integration constant. Integration (6.11) cannot be performed analytically for arbitrary \( \gamma \). However, we will see that this is not necessary because within our approximation, equation (6.11) becomes simplified. Metric (6.10) has an implicit form through integrating and inverting (6.11) for \( \Psi(R) \). However, we can obtain the metric (6.10) in a more explicit form without the need for inversion (although not in the standard radial gauge) as follows
\[ ds^2 = -N^2 dt^2 + \frac{(\Psi - 1)^2 d\Psi^2}{\nu^2 C |\Psi - \gamma|^{1+\frac{1}{2}} (\Psi + \gamma)^{1-\frac{1}{2}} (1-\Psi+\alpha U^2)^2} + R(\Psi)^2 d\Omega_2^2. \]  
(6.12)
Since the master equation (2.16) is of second order, there are two extra integration constants \( C, R_0 \) in the solution (6.11), beyond the integration constant \( \nu \). Therefore, after the lapse function \( N \) is found, the metric (6.10) or (6.12) will be the general approximate solution in the domain of its validity.

The lapse metric function \( N \) can be obtained from the equations (2.21), (6.20) from where we obtain

\[
\frac{d \ln (R N^2)}{dx} = \frac{c}{2 \eta} \left( \frac{1}{Y} \frac{dY}{dx} + 3Y + 2 \alpha \right),
\]  

(6.13)

with \( \epsilon = \pm 1 \) a sign symbol. Integration of (6.13) gives

\[
N^2 = \frac{\tilde{c}}{R} \left( R^3 U^2 e^{2 \alpha \int d\Phi} \right)^{\frac{2\eta}{\gamma}},
\]

(6.14)

where \( \tilde{c} > 0 \) is integration constant. Using (6.7), (6.3), (6.6) we find

\[
N^2(R) = \frac{c}{R} \left( \frac{R^3 e^{J}}{|\Psi - \gamma|^{1 - \frac{1}{\alpha}} (\Psi + \gamma)^{1 + \frac{1}{\alpha}}} \right)^{\frac{2\eta}{\gamma}}.
\]

(6.15)

where

\[
J = 2 \alpha \int \frac{\Psi - 1}{\Psi^2 - \gamma^2} \frac{d\Psi}{\alpha + (1 - \Psi) U^2} = 2 \int \frac{\Psi - 1}{\Psi^2 - \gamma^2} \frac{d\Psi}{1 + \frac{C(\Psi + \frac{1}{2})}{\alpha(1 - \Psi)|\Psi - \gamma|^{1 - \frac{1}{\alpha}} (\Psi + \gamma)^{1 + \frac{1}{\alpha}}}}
\]

(6.16)

and \( c = \tilde{c} C^{\frac{2\eta}{\gamma}} > 0 \) is a redefined integration constant. The integration constant \( c \) is not essential since it can be absorbed into a redefinition of the time coordinate \( t \).

The potential can be found from (2.18), after use of (6.7), (6.3), to be

\[
\eta V = - \left( \Psi + \frac{1}{2} \right) U^2 + \frac{\alpha}{2} + \frac{1}{2 \nu R^2}
\]

(6.17)

\[
= \frac{C(\Psi + \frac{1}{2})}{|\Psi - \gamma|^{1 - \frac{1}{\alpha}} (\Psi + \gamma)^{1 + \frac{1}{\alpha}}} + \frac{\alpha}{2} + \frac{1}{2 \nu R^2}.
\]

(6.18)

Finally, the scalar field configuration can be determined from (2.14) using again (6.7), (6.3)

\[
\phi = \phi_1 + \epsilon_1 \int \frac{\Psi - 1}{\Psi^2 - \gamma^2} \frac{1}{\Psi + \alpha U^2} \frac{d\Psi}{U}
\]

(6.19)

\[
= \phi_1 + \epsilon_1 \sqrt{C} \int \frac{\Psi - 1}{\Psi^2 - \gamma^2} \frac{1}{\Psi - \gamma|^{1 - \frac{1}{\alpha}} (\Psi + \gamma)^{1 + \frac{1}{\alpha}}} \frac{(\Psi + \gamma)^{\frac{1}{2}}}{\Psi - \gamma|^{1 - \frac{1}{\alpha}} (\Psi + \gamma)^{1 + \frac{1}{\alpha}}} \frac{d\Psi}{U}.
\]

(6.20)

where \( \epsilon_1 \) is another \( \pm \) sign and \( \phi_1 \) is an integration constant.

In terms of \( \Psi \) the inequalities (6.11) are written as

\[
\frac{(\Psi + \gamma)^{1 + \gamma^{-1}} |\Psi - \gamma|^{1 - \gamma^{-1}}}{|\Psi - 1|} \leq \frac{C}{|\alpha|}, \quad 2 \beta(\Psi + \gamma)^{1 + \gamma^{-1}} |\Psi - \gamma|^{1 - \gamma^{-1}} (\Psi + \gamma)^{-1} (\Psi - (2\gamma^2 - 1)) \leq \frac{C}{|\alpha|}.
\]

(6.21)

From the second condition of (6.21) it is seen that the branch with \( \Psi > \gamma \) has a pole at \( \Psi = 2\gamma^2 - 1 > \gamma \). Therefore choosing a value for \( \frac{C}{|\alpha|} \) both conditions (6.21) will be satisfied for a solution which is defined from \( \Psi = \gamma \) (for \( R \rightarrow \infty \)) up to some value smaller than \( 2\gamma^2 - 1 \) (that corresponds to the minimum \( R \)). If \( \frac{C}{|\alpha|} \) is sufficiently large, \( \Psi \) can approach the value \( 2\gamma^2 - 1 \). Similarly, from the first condition of (6.21) the branch with \( \Psi < \gamma \) has a pole at \( \Psi = 1 \). Therefore choosing a value for \( \frac{C}{|\alpha|} \) both conditions (6.21) will be satisfied for a solution which is defined from some value larger than \( 1 \) (that corresponds to the minimum \( R \)) up to \( \Psi = \gamma \) (for \( R \rightarrow \infty \)), since from (6.6) \( \Psi \) is now an increasing function of \( U \). If \( \frac{C}{|\alpha|} \) is sufficiently large, \( \Psi \) can approach the value \( 1 \).

Hopefully, it is obvious that the first of the conditions in (6.21) assures that the quantity \( \frac{C}{|\alpha|}(\Psi + \gamma)^{1 + \gamma^{-1}} |\Psi - \gamma|^{1 - \gamma^{-1}} \) in the integral (6.11) is negligible. Then, the integration can be done and it gives

\[
R = R_0 \left( \Psi + \gamma \right)^{\frac{1}{\gamma^{-1}}}. \quad (6.22)
\]
\[ U = \frac{\sqrt{C}}{2\gamma} \left( \frac{R}{R_0} \right)^{\gamma-1} \left[ 1-s \left( \frac{R_0}{R} \right)^{2\gamma} \right] \],
\]
(6.23)

where \( s = \text{sgn}(\Psi - \gamma) \) discerns the two branches with \( \Psi > \gamma \) or \( \Psi < \gamma \). In the branch with \( \Psi > \gamma \), it is seen from (6.22) that as \( \Psi \) increases, the radius \( R \) decreases, while in the branch with \( \Psi < \gamma \), as \( \Psi \) increases, \( R \) also increases, in accordance with the previous analysis.

Finally, the conditions (6.21) take the following form in terms of \( R \)
\[
\left( \frac{R}{R_0} \right)^{2\gamma-1} > 0
\]
(6.24)

\[
\left( \frac{R}{R_0} \right)^{2\gamma-2} > 0
\]
(6.25)

respectively. For \( s = 1 \), these inequalities say that \( R \) can become as close to \( R_0 \) as we wish, given that \( |\beta|/C \) is sufficiently small. Since \( \beta > \frac{1}{2} \), this last condition means \( |\alpha|/C < 1 \) and the inequalities (6.1) are satisfied practically for any \( R \) larger than \( R_0 \). Furthermore, if \( R_0 \gg |\nu|^{-1/2} \), the condition (6.20) is satisfied for any \( R \) outside \( R_0 \). Therefore, all the conditions have been satisfied. For example, if \( |\xi|^{1/2} \gg 1 \), the condition \( |\alpha|/C < 1 \) becomes \( 4C|\xi|^{1/2} \gg 1 \) which is satisfied if \( C \) is not particularly small; moreover, for \( R_0 \gg 2|\xi|, \) the condition (6.20) is satisfied.

To proceed with the solution, now the integral \( J \) in (6.10) is found to be
\[ J \approx \frac{1}{2} \int (\Psi^{1/2} - \Psi^{-1/2}) \Psi \] d\( \Psi \), which around the dangerous for divergence point \( \Psi = \gamma \) gives \( J \approx \frac{1}{2} \int (|\Psi - \gamma|^{-1/2} - |\Psi - \gamma|^{-2}) \Psi \) d\( \Psi \). Since \( \gamma > 1 \), it is \( |\Psi - \gamma|^{-1/2} \) finite and it will be \( J \approx 0 \) for \( |\alpha|/C \ll 1 \). Using (6.22), the metric (6.10) becomes
\[
ds^2 = -\left( \frac{R}{R_0} \right)^{\sqrt{\gamma + 1}} \left[ 1-s \left( \frac{R_0}{R} \right)^{2\gamma} \right] \sqrt{\Psi} \] d\( t^2 \) + \frac{1}{C R_0^2} \left( \frac{R}{R_0} \right)^{2\gamma} \frac{dR^2}{\left[ 1-s \left( \frac{R_0}{R} \right)^{2\gamma} \right]^2} + R^2 d\Omega_2^2,
\]
(6.26)

where \( \zeta = \epsilon \text{sgn}(1-2|\nu|(1-6\xi)) = \pm 1 \), \( \hat{C} = \frac{\epsilon c}{4\gamma} > 0 \) and \( \int = \sqrt{\gamma} \) d\( \Psi \).

We will be interested in the solution with \( s = 1 \) (\( \Psi > \gamma \)) which presents a pole. As explained above this solution is valid for distances immediately outside \( R_0 \) and for this reason, \( R_0 \) for \( \zeta = 1 \) will be called horizon. The metric (6.20) in this case takes the form
\[
ds^2 = -\left( \frac{R}{R_0} \right)^{\sqrt{\gamma + 1}} \left[ 1-s \left( \frac{R_0}{R} \right)^{2\gamma} \right] \sqrt{\nu} \] d\( t^2 \) + \frac{1}{C R_0^2} \left( \frac{R}{R_0} \right)^{2\gamma} \frac{dR^2}{\left[ 1-s \left( \frac{R_0}{R} \right)^{2\gamma} \right]^2} + R^2 d\Omega_2^2,
\]
(6.27)

where for convenience we remind that \( \hat{C} > 0, \gamma = \sqrt{\beta+1} > 1 \) and \( \beta = \frac{3}{2}(\frac{4\beta}{3} - 3)^2 > \frac{3}{2} \) with \( \alpha < \frac{\sqrt{2}}{2} \) or \( \alpha > \kappa^2 \) (the parameter \( \alpha \) is related to the coupling \( \xi \) and the scalar field integration constant \( \nu \) through the combination \( \xi \alpha^2 \nu^2 \)). The analysis of the curvature invariants reveals that these are all finite at the horizon \( R_0 \) and diverge at infinity. The integration constant \( R_0 \) is expected to be related with the mass of the black hole. Asymptotically the metric (6.27) takes the form
\[
ds^2 = -\left( \frac{R}{R_0} \right)^{\sqrt{\gamma + 1}} \left[ 1-s \left( \frac{R_0}{R} \right)^{2\gamma} \right] \] d\( t^2 \) + \frac{1}{C R_0^2} \left( \frac{R}{R_0} \right)^{2\gamma} \frac{dR^2}{\left[ 1-s \left( \frac{R_0}{R} \right)^{2\gamma} \right]^2} + R^2 d\Omega_2^2,
\]
(6.28)

This metric defines an asymptotic behaviour different than that of AdS space. Similarly to AdS, the lapse function also goes to infinity for large distances, but here the scaling behavior is different. More precisely, note that the exponent \( 2\gamma \) is larger than 2 and approaches the value 2 as \( \gamma \) approaches 1. On the contrary, the exponent \( \sqrt{\gamma+1} - 1 \) is smaller than 2 and positive, and approaches also the value 2 as \( \gamma \) approaches 1. Therefore, for \( \gamma \) very close to 1 the asymptotic metric (6.28) gets close to AdS, while as \( \gamma \) departs from the value 1, the metric gets a different structure. For example, for \( \gamma = 7/2 \), the lapse function in (6.28) becomes proportional to \( R \) (to give an astrophysical perspective, the metric (6.28) seems to provide extra attraction at large distances, while fittings of linear potentials for exponential galactic disks have been shown to explain the almost flat galactic rotation curves and such potentials yield galactic stability without the need of dark matter). For a circular orbit at constant radius \( R \) the conditions \( \epsilon N^2 = 1 - \frac{R}{2N^2} \frac{dN^2}{dR} \), \( j^2 = \frac{R^3}{2N^2} \frac{dN^2}{dR} \) have to be satisfied, where the constant \( E \) is the energy of the moving particle.
\(E > 0\) for material particles, \(E = 0\) for photons) and the constant \(j\) is related to the angular momentum. For the metric (6.28) both conditions are satisfied since \(\gamma > 1\), therefore at large distances circular orbits are supported.

Making the transformation

\[
\rho = R^\gamma,
\]

the metric (6.27) takes the form

\[
ds^2 = \rho^{-\vartheta}\left[-\rho^{2\vartheta}\left(1-\frac{R_0^{2\vartheta}}{\rho^2}\right)\sqrt{\rho^2 + \frac{R_0^{2(\vartheta-1)}}{C\gamma^2}} \frac{d\rho^2}{\rho^2 (1-\frac{R_0^{2\vartheta}}{\rho^2})^2} + \rho^2 d\Omega_2^2\right],
\]

where

\[
0 < \vartheta = 2\left(1 - \frac{1}{\gamma}\right) < 2, \quad 0 < z = 1 - \frac{1}{2\gamma} \left[3 - \sqrt{\frac{3(2\gamma+1)}{2\gamma-1}}\right] < 1
\]

and \(\hat{t} = R_0^{\frac{1}{2}}\left[1 - \sqrt{\frac{2\gamma+1}{2\gamma-1}}\right]^{-1}t\). The metric (6.30) is a hyperscaling violating black hole \([32]\) with spherical horizon topology. A hyperscaling violating black hole is a generalization of the Lifshitz black hole where \(\vartheta = 0\) (in our solution the metric can never asymptote to a Lifshitz spacetime). The Lifshitz metric arises as solution of gravity theories with negative cosmological constant coupled to appropriate matter with the simplest such theory also including an abelian gauge field \([34]\) (a pure Einstein gravity with cosmological constant cannot produce an anisotropy in spacetime). Such metrics have also been found as solutions in string theory and supergravities which arise from string constructions \([35]\). Effective gravity theories with a Maxwell as well as a dilaton field (in general a scalar field with a non-trivial potential) are quite rich and have been shown to contain hyperscaling violating solutions \([36]\). Note that the majority of the Lifshitz or hyperscaling violating Lifshitz solutions and the corresponding black holes in the literature have planar (horizon) topology and are assumed to have direct correspondence with condensed matter physics through the AdS/CFT conjecture (for a spherical horizon topology to be obtained an extra gauge field should be added). On the contrary, the spherical symmetry found here may offer to the solution some significance at local astrophysical objects at large distance scales. In general it is not easy for a given theory to possess Lifshitz or hyperscaling violating solutions, and as referred, for example the introduction of some extra matter source or higher order gravity theories are required. Here our scalar-torsion theory is an additional case which provides such solutions and notably these solutions, and as referred, for example the introduction of some extra matter source or higher order gravity theories are required.

According to the standard notation, \(\hat{\vartheta}\) is the hyperscaling violation exponent, while \(z\) is the dynamical critical exponent which indicates the anisotropy between time and space. For our solution the values that these parameters can take are seen from the conditions (6.31). The asymptotic form of (6.30) is

\[
ds_\infty^2 = \rho^{-\vartheta}\left[-\rho^{2\vartheta}d\hat{t}^2 + \frac{R_0^{2(\vartheta-1)}}{C\gamma^2} \frac{d\rho^2}{\rho^2 (1-\frac{R_0^{2\vartheta}}{\rho^2})^2} + \rho^2 d\Omega_2^2\right].
\]

The scaling transformation \(\hat{t} \rightarrow \lambda^2 \hat{t}, \rho \rightarrow \lambda^{-1}\rho, x_i \rightarrow \lambda x_i\) does not act as an isometry for the metric (6.32), so (6.32) is not scale invariant, but it transforms conformally as \(ds_\infty^2 \rightarrow \lambda^2 ds_\infty^2\).

In the context of AdS/CFT, a non-vanishing \(\vartheta\) indicates a hyperscaling violation in the dual field theory. In the four-dimensional framework we are working, theories with hyperscaling at finite temperature have an entropy density which scales with temperature as \(S \sim T^{\frac{\vartheta}{\gamma}}\). For hyperscaling violation there is a modified relationship \(S \sim T^{\frac{\vartheta}{\gamma}}\), indicating that the system lives in an effective dimension \(d_{\text{eff}} = 2 - \vartheta\). For the present solution it is \(0 < d_{\text{eff}} = \frac{2}{\gamma} < 2\).

The potential accompanying the black hole (6.27) is found from (6.18) to be

\[
V = -\frac{\hat{C}}{2\alpha} \left(\frac{R}{R_0}\right)^{2(\gamma-1)} \left[1 - \left(\frac{R_0}{R}\right)^{2\gamma}\right] \left[2\gamma + 1 + (2\gamma - 1) \left(\frac{R_0}{R}\right)^{2\gamma}\right] \frac{\nu^2}{2} + \frac{1}{2\alpha R^2}.
\]

It is seen that at the horizon the potential is finite, while at infinity it diverges as \(V \sim (R/R_0)^{2(\gamma-1)}\). Finally, the
scalar field associated with the metric \( (6.27) \) is found \(^1\) to be

\[
\phi = \phi_1 - \frac{\epsilon_1 \nu}{(\gamma - 1) \sqrt{C}} \left( \frac{R_0}{R} \right)^{\gamma - 1} 2F_1 \left( \frac{\gamma - 1}{2\gamma}, 1; \frac{3\gamma - 1}{2\gamma}; \left( \frac{R_0}{R} \right)^{2\gamma} \right).
\]  

(6.34)

Since the parameter \( \gamma \) depends on the integration constant \( \nu \) of the scalar field, the scalar field \( (6.34) \) is a primary hair.

The scalar field diverges at the horizon and is finite at infinity. This behaviour at infinity is different than the behaviour found in \(^2\) for AdS asymptotics, where the scalar field evolves logarithmically with distance. At infinite distance the scalar field behaves to dominant order as \( \phi - \phi_1 \approx - \frac{\epsilon_1 \nu}{(\gamma - 1) \sqrt{C}} \left( \frac{R_0}{R} \right)^{\gamma - 1} \). Then, we can find the corresponding behaviour of the potential \( V(\phi) \) for \( \phi - \phi_1 \approx 0 \) to be

\[
V(\phi) \approx - \frac{2\gamma + 1}{2(\gamma - 1)^2} \frac{\nu^2}{\alpha(\phi - \phi_1)^2}.
\]  

(6.35)

Thus, the potential \( V(\phi) \) close to the origin \( \phi - \phi_1 = 0 \) is very steep and can be either positive or negative depending on the value of \( \alpha \). At the opposite limit of distances close to the horizon, the potential gets an almost constant value \( V \approx \frac{\nu^2}{2} \).

Concerning the components of the energy-momentum tensor \( T^\mu_\nu \), although the scalar field diverges at the horizon, it is obvious that its energy density \( T^i_i \) and the pressures \( T^R_R, T^a_a \) are finite there. Asymptotically these components of \( T^\mu_\nu \) diverge.

To summarize, the most significant solution found in this section is described by the metric forms \( (6.27), (6.30) \) with the asymptotic structures \( (6.28), (6.32) \). The solution develops an horizon \( R_0 \) and \( R \) is practically defined outside \( R_0 \) when \( R_0 \gg |\nu|^{-1}|\alpha|^{-1/2} \) and \( |\alpha|^2 |\nu|^2 < 1 \). The solution is supported by the potential \( (6.38) \) with the scalar field profile \( (6.34) \).

The Hawking temperature \( T \) of the black hole is determined by the periodicity of the Euclidean metric \( ds_E^2 = g_{tt} d\tau^2 + g_{RR} dR^2 + R^2 d\Omega_2^2 = N^2 dt^2 + K^{-2} dr^2 + R^2 d\Omega_2^2 \) obtained by the analytic continuation \( t = -\tau \). Thus, \( T \) is given by the standard formulae \( 4\pi T = (dg_{tt}/dR)/\sqrt{g_{tt}} \big|_{R_0} = (\sqrt{K^2/N^2} dN^2/dR) \big|_{R_0} = \sqrt{K^2/N^2} dK^2/dR \big|_{R_0} \), given that \( K^2 \) vanishes at the horizon \( R_0 \). Due to the higher order pole at the horizon, the temperature vanishes in analogy to the extremal Reissner-Nordström solution.

From the astrophysical point of view, the motion of a freely falling photon in the static isotropic gravitational field \( (2.7) \) is described by the equation \( (d\hat{t}/d\hat{\phi})^2 = R^4 K^2 (\frac{1}{N^2} - \frac{1}{R^2}) \). Since the field is isotropic, the orbit of the particle can be considered to be confined to the equatorial plane \( \theta = \frac{\pi}{2} \). At the distance \( R_* \), closest approach to the center it is \( \frac{d\hat{t}}{d\hat{\phi}} = 0 \), thus \( J^2 N(R_*)^2 = R_*^2 \). The deflection of the orbit from the direction of initial incidence at infinite distance is \( \Delta \varphi = 2(\varphi(R_*) - \varphi[\infty] - \pi) \), where \( \varphi[\infty] \) indicates the incident direction. The larger the quantity \( (d\hat{t}/d\hat{\phi})^2 \) in the previous differential equation, the larger the deflection angle is. It can be easily seen that at large distances \( (d\hat{t}/d\hat{\phi})^2 \) for the metric \( (6.27) \) is bigger than that of the Schwarzschild metric. Thus, there is for our solution an extra deflection of light compared to the Newtonian deflection. The situation of increased deflection compared to that caused by the luminous matter has been well observed in galaxies or clusters of galaxies. In general, the metric \( ds^2 = -R^2 \rho^2 d\Omega_2^2 + \frac{dR^2}{R^2 \rho^2} + R^2 d\Omega_2^2 = \rho^{-\vartheta} (-\rho^{2\vartheta} d\Omega_2^2 + \frac{dR^2}{R^2 \rho^2} + \rho^2 d\Omega_2^2) \), \( \vartheta = 2 - \frac{\pi}{2} \), \( z = 1 + \frac{R^2}{R_0^2} \) \( (\rho = R^6) \) has an extra deflection of light at large distances if \( a < 1, b - a > 0 \) which mean \( (2 - \vartheta)(1 - z) > 0 \), \( (2 - \vartheta)(2 - 2z + \vartheta) > 0 \). These conditions are obviously satisfied for the solution found here.

VII. CONCLUSIONS

In this work we have extended our previous analysis \(^2\) on the quest of finding spherically symmetric solutions of a scalar-torsion theory. More precisely, we treat the torsion not as an independent field but the teleparallel condition is imposed as a constraint, therefore the corresponding connection is assumed to have vanishing curvature. This last

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\(^1\) From the Gauss' recursion formula \( c(c + 1) F_2 F_1 (a, b; c; z) - c(c + 1) F_2 F_1 (a, b; c + 1; z) - abz F_2 F_1 (a + 1, b + 1; c + 2; z) = 0 \) (p. 1010 of \(^3\)) and \( F_2 F_1 (a, b; a; z) = (1 - z)^{-b} \) \(^4\), we get the relation \( (a + 1) F_2 F_1 (a, b; a + 1; z) + bz F_2 F_1 (a + 1, b + 1; a + 2; z) = (a + 1)(1 - z)^{-b} \). Since \( 2F_2 F_1 (a, b; c; z) = \frac{ab}{\Gamma(a + 1) \Gamma(b + 1)} F_2 F_1 (a + 1, b + 1; c + 2; z) \) \(^5\), we get \( \frac{1}{\Gamma(1 - \alpha)} 2F_2 F_1 (a, b; a + 1; \mu z) = az^{-(1 - \alpha)} (1 - \mu z)^{-b} \). For \( (6.34) \) it is also used the transformation formula \( 2F_2 F_1 (a, a; a + 1; \frac{1}{a - 1}) = (1 - z)^a 2F_2 F_1 (a, 1; a + 1; z) \) \(^6\).

\(^2\) From the Gauss’ recursion formula \( c(c + 1) F_2 F_1 (a, b; c; z) - c(c + 1) F_2 F_1 (a, b; c + 1; z) - abz F_2 F_1 (a + 1, b + 1; c + 2; z) = 0 \) (p. 1010 of \(^3\)) and \( F_2 F_1 (a, b; a; z) = (1 - z)^{-b} \) \(^4\), we get the relation \( (a + 1) F_2 F_1 (a, b; a + 1; z) + bz F_2 F_1 (a + 1, b + 1; a + 2; z) = (a + 1)(1 - z)^{-b} \). Since \( 2F_2 F_1 (a, b; c; z) = \frac{ab}{\Gamma(a + 1) \Gamma(b + 1)} F_2 F_1 (a + 1, b + 1; c + 2; z) \) \(^5\), we get \( \frac{1}{\Gamma(1 - \alpha)} 2F_2 F_1 (a, b; a + 1; \mu z) = az^{-(1 - \alpha)} (1 - \mu z)^{-b} \). For \( (6.34) \) it is also used the transformation formula \( 2F_2 F_1 (a, a; a + 1; \frac{1}{a - 1}) = (1 - z)^a 2F_2 F_1 (a, 1; a + 1; z) \) \(^6\).
condition has been implemented by adopting the Weitzenböck connection whose coordinate components are a function of the vierbein, so our torsion is the torsion of the Weitzenböck connection and the dynamical object of the theory is solely the vierbein. The theory consists of the Einstein gravity in its teleparallel representation supplemented by a minimally coupled scalar field with potential. The novel additional term of the theory is a non-minimal derivative coupling of the scalar field with the torsion scalar (which is a particular quadratic combination of the torsion that provides under variation with respect to the vierbein the Einstein tensor). The equations of motion are second order differential equations and the theory (action and field equations) is both diffeomorphism and local Lorentz invariant, while after adopting the Weitzenböck connection in order to proceed, the local Lorentz invariance is abolished.

The master equation found in [27], determining the four-dimensional spherically symmetric solutions, has here been elaborated and appropriately approximated in order to find large-distance solutions. The solutions found here are non-linearized, so they are valid at any distance larger than some length scale defined by the parameters/integration constants of the problem (the region of the parameters/integration constants is not fine-tuned or particularly narrow). A dynamical systems analysis has been performed and elucidates in the space of dynamical variables describing all the local solutions the regions where the large-distance solutions inhabit. This offers an intuition for the generality of the approximation method applied in order to obtain the general solutions.

Special solutions have been found, among which black holes also, when the integration constant of the scalar field takes a particular value. Asymptotically, these solutions get an AdS form. The corresponding scalar field is a secondary hair. There are branches where the asymptotic value of the potential coincides with the effective cosmological constant of AdS; also there are other branches where the cosmological constant coming from the potential is positive (therefore it is different than the effective cosmological constant of AdS) and this is due to the growth of a non trivial profile of the scalar field at infinity after interacting with torsion.

Probably the most interesting solutions found are branches of general (from the point of view of the number of integration constants) spherically symmetric solutions. These solutions are attracted by stable fixed points at the asymptotic region of the phase portrait of the dynamical system. Among them, there is the general black hole solution described by the metrics (6.27), (6.30) with their asymptotic forms (6.28), (6.32). The form (6.30) clearly shows that the solution is a hyperscaling violating black hole with positive hyperscaling violation exponent and dynamical critical exponent. Note that the solution obtained here is a general solution of our theory, while the majority of the Lifshitz or hyperscaling violating solutions of differing theories in the literature are special. Note also that the topology of the horizon here is spherical, while most of the existing Lifshitz or hyperscaling violating black holes have planar horizon topology. Due to the spherical symmetry the solution found may have astrophysical significance, e.g. may possess extra attraction at large distances. Actually, we have found for a freely falling photon in the static isotropic gravitational field an increase of its deflection compared to the Newtonian deflection. The scalar field \( \phi \) accompanying the solution is a primary hair which diverges at the horizon and is finite at infinity. Notice, however, that in all the black hole solutions found in this work, although the scalar field diverges at the location of the horizon, its energy density and the pressures are finite there.

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