Kinetic theory of QED plasmas in a strong electromagnetic field
II. The mean-field description

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Abstract

Starting from a general relativistic kinetic equation, a self-consistent mean-field equation for fermions is derived within a covariant density matrix approach of QED plasmas in strong external fields. A Schrödinger picture formulation on space-like hyperplanes is applied. The evolution of the distribution function is described by the one-particle gauge-invariant $4 \times 4$ Wigner matrix, which is decomposed in spinor space. A coupled system of equations for the corresponding Wigner components is obtained. The polarization current is expressed in terms of the Wigner function. Charge conservation is obeyed. In the quasi-classical limit for the Wigner components a relativistic Vlasov equation is obtained, which is presented in an invariant, i.e. hyperplane independent, form.

Key words: relativistic kinetic theory; QED plasma; hyperplane formalism; mean-field approximation

1 Introduction

In part one of this article [1] we have developed a covariant density matrix approach to kinetic theory of QED plasmas, making use of the relativistic hyperplane formalism [2–4] in the Schrödinger picture. In what follows the...
paper [1] will be referred to as I and equations from this paper will be labeled by \((I\ldots)\), where “\ldots” denotes the equation number. In the present paper we aim to derive quantum mean-field kinetic equations for the fermionic subsystem starting from a general relativistic covariant equation discussed in paper I.

Section 2 briefly recalls notations and definitions from paper I needed in further considerations. Section 3 is devoted to the derivation of a mean-field kinetic equation for the gauge-invariant fermionic Wigner function in the hyperplane formalism. We next use the spinor decomposition of the Wigner matrix and obtain a set of coupled covariant equations describing kinetic processes in different channels. The charge conservation is shown to be fulfilled. In Section 4 we discuss the quasi-classical limit in the hyperplane formalism. We derive mean-field kinetic equations for the distribution functions of particles and antiparticles, and show that these equations can be represented in a fully covariant form. Section 5 will conclude and give a short outlook. In Appendix A we show the relation between our approach and the existing mean-field theories of QED [5,6]. Finally, Appendix B gives expressions for the matrix commutation and anticommutation relations which are necessary for the spinor decomposition of the kinetic equation.

Except for the quasi-classical limit, we use the system of units with \(\hbar = c = 1\). The signature of the metric tensor is \((+,-,-,-)\).

### 2 Basic definitions

We consider a quantum plasma of charged fermions interacting through the electromagnetic (EM) field. For simplicity, we will take these fermions to be electrons and positrons, but the inclusion other fermions (say, protons) as additional Dirac fields is not a particular problem. The system is assumed to be subjected to a prescribed external EM field which is not necessarily weak.

In paper I we defined the one-particle density matrix for fermions as the average

\[
\rho_{aa'} \left( x_{\perp}, x'_{\perp}; \tau \right) = \left\langle \hat{\rho}_{aa'} \left( x_{\perp}, x'_{\perp} \right) \right\rangle_{\tau} \equiv \text{Tr} \left\{ \hat{\rho}_{aa'} \left( x_{\perp}, x'_{\perp} \right) \varrho(n, \tau) \right\}, \quad (2.1)
\]

where \(\varrho(n, \tau)\) is the nonequilibrium statistical operator of the system on a hyperplane \(\sigma_{n,\tau}\) characterized by a unit time-like normal four-vector \(n^\mu\) and a scalar parameter \(\tau = x \cdot n\) which may be interpreted as an “invariant time”. The fermionic density operator \(\hat{\rho}\) is given in the Schrödinger picture on the
Hyperplane by

\[
\hat{\rho}_{aa'}(x_\perp, x'_\perp) = -\frac{1}{2}[\hat{\psi}_a(x_\perp), \hat{\psi}_{a'}(x'_\perp)],
\]

(2.2)
a, a' being spinor indices of the Dirac field operators. The transverse components (which are space-like) of \(x = \{x^\mu\}\) and other four-vectors \(V = \{V^\mu\}\) are defined with respect to the normal \(n\) through the decomposition

\[
x^\mu = n^\mu \tau + x^\mu_\perp, \quad V^\mu = n^\mu V^\parallel + V^\mu_\perp,
\]

(2.3)

where

\[
V^\parallel = n \cdot V, \quad V^\mu_\perp = \Delta^\mu_\nu V^\nu, \quad \Delta^\mu_\nu = \delta^\mu_\nu - n^\mu n^\nu.
\]

(2.4)

Our further analysis rests heavily on the basic “equal-time” anticommutation relations for the Dirac field operators on hyperplanes [cf. Eqs.(I.3.24) and (I.3.25)]

\[
\{\hat{\psi}_a(\tau, x_\perp), \hat{\psi}_{a'}(\tau, x'_\perp)\} = \left[\gamma_\parallel(n)\right]_{aa'} \delta^3(x_\perp - x'_\perp),
\]

(2.5)

\[
\{\hat{\psi}_a(\tau, x_\perp), \hat{\psi}_{a'}(\tau, x'_\perp)\} = \{\hat{\psi}_a(\tau, x_\perp), \hat{\psi}_{a'}(\tau, x'_\perp)\} = 0,
\]

where

\[
\delta^3(x_\perp) = \int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot x} \delta(p \cdot n)
\]

(2.6)

is the three-dimensional delta function on a hyperplane \(\sigma_{\mu, \tau}\) and the matrix \(\gamma_\parallel(n)\) is defined through the following decomposition of the Dirac matrices:

\[
\gamma^\mu = n^\mu \gamma_\parallel(n) + \gamma^\mu_\perp(n),
\]

\[
\gamma_\parallel(n) = n_\mu \gamma^\mu; \quad \gamma_\perp^\mu(n) = \Delta^\mu_\nu \gamma^\nu.
\]

(2.7)

As discussed in paper I, the self-consistent mean-field approximation for the fermionic subsystem can be introduced only when the EM field variables are separated into the macroscopic condensate mode and the photon degrees of freedom. This we have shown in paper I by means of a time-dependent unitary transformation of the statistical operator and the operators of the EM field [see Eqs. (I.4.1) and (I.4.3)]. After this procedure, the effective Hamiltonian
describing the fermionic subsystem in the mean-field approximation can be

\[
\hat{H}_0^{\tau}(n) = \int_{\sigma_n} d\sigma \left( \hat{\psi} \left( -\frac{i}{2} \gamma_\perp^{\tau}(n) \nabla_\mu + m \right) \hat{\psi} + \int d\sigma \hat{j}_\mu(x_\perp) A^\mu(\tau, x_\perp) \right),
\]  

(2.8)

where the symbol \( : \hat{O} : \) shows the normal ordering in operators, and the space-like derivatives \( \nabla_\mu = \nabla_\mu - \nabla_\mu \) are defined by the relations

\[
\partial_\mu = n_\mu \frac{\partial}{\partial \tau} + \nabla_\mu, \quad \nabla_\mu = \Delta_\mu^{\nu} \partial_\nu = \Delta_\mu^{\nu} \frac{\partial}{\partial x_\perp^{\nu}}.
\]  

(2.9)

We will use the notation \( G(\tau, x_\perp) \equiv G(n\tau + x_\perp) \) for any function \( G(x) \) on the hyperplane \( \sigma_{n, \tau} \) furthermore.

The first term in Eq. (2.8) is the Hamiltonian of free fermions, while the second term describes their interaction with the total mean EM field in the system, \( A^\mu \). We have shown in paper I that the total field tensor \( F_\mu^{\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) satisfies Maxwell equations

\[
\partial_\mu F_\mu^{\nu}(x) = j_\nu(x) + j_\nu^{\text{ext}}(x).
\]  

(2.10)

Here \( j_\nu^{\text{ext}}(x) \) is a prescribed external current and

\[
\hat{j}_\mu(x) = \left( \hat{j}_\mu(x_\perp) \right)^\tau
\]  

(2.11)

is the mean polarization current. For the electron-positron plasma, the current operator is \( (e < 0) \)

\[
\hat{j}_\mu(x_\perp) = e \hat{\psi}(x_\perp) \gamma_\mu \hat{\psi}(x_\perp).
\]  

(2.12)

If protons are treated as a dynamical subsystem, the corresponding term must be included into the current operator.

As outlined in paper I, the total Hamiltonian of the system contains, in addition to (2.8), the Hamiltonian of free photons, \( \hat{H}_{EM}(n) \), and the term \( \hat{H}_\text{int}^{\tau}(n) \), describing the interaction between fermions and photons. In the mean-field approximation the interaction term is neglected. The derivation of the fermionic kinetic equation requires to calculate commutators of the dynamical fermion operators \( \hat{\psi} \) and \( \hat{\psi} \) with the Hamiltonian (see below). The free photon contribution \( \hat{H}_{EM}(n) \), consisting completely of the photon dynamical operators \( \partial_\nu A_\mu^{\perp} \), will not contribute to these commutators and can therefore be omitted.
This implies that the dynamics of the EM field in the mean-field approximation is completely governed by the Maxwell equations (2.10). Nevertheless, at the end of the paper we shall discuss some non-trivial relations between the mean-field description of the fermionic subsystem and the photon kinetics in QED plasmas.

To complete the list of definitions, we write down the expression for the gauge-invariant “one-time” Wigner function \[9\] on the hyperplane \(\sigma_{n,\tau}\). The Wigner function is expressed in terms of the one-particle density matrix (2.1) by

\[
W_{aa'}(x_\perp, p_\perp; \tau) = \int d^4y e^{ip\cdot y} \delta(y \cdot n) \times \exp \left\{ ie \Lambda(x_\perp + \frac{1}{2}y_\perp, x_\perp - \frac{1}{2}y_\perp; \tau) \right\} \rho_{aa'} \left( x_\perp + \frac{1}{2}y_\perp, x_\perp - \frac{1}{2}y_\perp; \tau \right),
\]

(2.13)

where

\[
\Lambda(x_\perp, x'_\perp; \tau) = \int_{x'_\perp}^{x_\perp} A_{\perp \mu}(\tau, R_\perp) dR_\perp^\mu
\]

\[
\equiv \frac{1}{i} \int_0^1 ds \left( x'^\mu_\perp - x'^{\mu}_\perp \right) A_{\perp \mu}(\tau, x'_\perp + s(x_\perp - x'_\perp))
\]

(2.14)

is the gauge function. Our immediate task will be to derive a mean-field kinetic equation for \(W\).

3 Mean-field kinetic equations

3.1 Kinetic equation for the one-particle density matrix

We start with the mean-field kinetic equation for the density matrix (2.1). Taking (2.8) as the effective Hamiltonian for the fermionic subsystem, we have

\[
\frac{\partial}{\partial \tau} \rho_{aa'}(x_\perp, x'_\perp; \tau) = -i \left[ \hat{\rho}_{aa'}(x_\perp, x'_\perp), \hat{H}_\perp(n) \right]^\tau.
\]

(3.1)

The commutator in the right-hand side is easily calculated by using the identity
\[
\hat{\rho}_{aa'}(x_\perp, x'_\perp) : \hat{\psi}_b(y_\perp) \hat{\psi}_b'(y'_\perp) : = (\gamma_\parallel)_{ab} \delta^3(x_\perp - y_\perp) \hat{\rho}_{b'b'}(y'_\perp, x'_\perp) \\
- (\gamma_\parallel)_{b'a'} \delta^3(x'_\perp - y'_\perp) \hat{\rho}_{ab}(x_\perp, y_\perp),
\] (3.2)

which follows from the anticommutation relations (2.5). After some algebra we find that Eq. (3.1) can be written in matrix notation \( \rho \equiv [\rho_{aa'}] \) as

\[
\frac{\partial}{\partial \tau} \rho(x_\perp, x'_\perp; \tau) = -im [\gamma_\parallel, \rho(x_\perp, x'_\perp; \tau)] \\
+ \left( -i\nabla_\mu + eA_\perp(\tau, x_\perp) \right) S^\mu \rho(x_\perp, x'_\perp; \tau) \\
+ \left( i\nabla'_\mu + eA_\perp(\tau, x'_\perp) \right) \rho(x_\perp, x'_\perp; \tau) S^\mu \\
- ie \left( A_\parallel(\tau, x_\perp) - A_\parallel(\tau, x'_\perp) \right) \rho(x_\perp, x'_\perp; \tau),
\]

where

\[
S^\mu = \bar{\sigma}^{\mu\nu} n_\nu, \quad \bar{\sigma}^{\mu\nu} = \frac{i}{2} \left[ \gamma^\mu, \gamma^\nu \right].
\]

Eq. (3.3) defines the mean-field dynamics of the fermionic one-particle density matrix for time-like translations with respect to the space-like plane \( \sigma_{n,\tau} \). In the special Lorentz frame where \( n^\mu = (1, 0, 0, 0) \) (which sometimes will be referred to as the “instant frame”), we have \( \tau = t \) and therefore Eq. (3.3) describes the time evolution of the one-particle density matrix. It is interesting to note that, within the mean-field description, there is no need to know the explicit form of the nonequilibrium statistical operator \( \hat{\rho}(n, \tau) \). The mean-field kinetic equation follows directly from the equation of motion for the density operator \( \hat{\rho} \) with the effective Hamiltonian (2.8).

### 3.2 Kinetic equation for the Wigner function

Applying the Wigner transformation (2.13) to Eq. (3.3), we obtain in matrix notation \( W \equiv [W_{aa'}] \)

\[
D_\tau W = -im [\gamma_\parallel, W] - \frac{i}{2} D_\perp [S^\mu, W] - P_\mu \{S^\mu, W\},
\]

where we have introduced the operators

\[
D_\tau = \frac{\partial}{\partial \tau} - e \int_{-1/2}^{1/2} ds \, n^\mu F_{\mu\nu}(\tau, x_\perp - is \nabla_p) \nabla_\nu\,.
\]

\[\text{(3.6)}\]
\( \textbf{D}_{\perp \mu} = \nabla_{\mu} - e \int ds \mathcal{F}_{\perp \mu \nu} \left( \tau, x_{\perp} - is\nabla_p \right) \nabla_{p \nu} \) \hfill (3.7)

\( \text{P}_{\mu} = p_{\perp} - ie \int ds \mathcal{F}_{\perp \mu \nu} \left( \tau, x_{\perp} - is\nabla_p \right) \nabla_{p \nu} \) \hfill (3.8)

and the transverse gradient in the momentum space: \( \nabla_{p \mu}^{\perp} = \Delta^{\mu \nu} \partial / \partial p_{\perp} \nu \). The transverse part of the total field tensor is defined as

\[ \mathcal{F}_{\perp \mu \nu} = \nabla_{\mu} A_{\perp \nu} - \nabla_{\nu} A_{\perp \mu} \]  

The virtue of Eq. (3.5) is its compact and covariant form. It should be emphasized, however, that the Wigner function, the matrices \( \gamma_\parallel \), \( S_{\mu} \), and the operators (3.6) – (3.8) are defined with respect to the family of hyperplanes \( \sigma_{n, \tau} \) characterized by the normal \( n^\mu \). The fact that \( n^\mu \) is an arbitrary time-like unit vector reflects Lorentz covariance of Eq. (3.5). To see this, we note that the normal vectors of the same plane in different frames, \( n^\mu \) and \( n'^\mu \), are related by a Lorentz transformation (boost) \( n'^\mu = \Lambda_{\mu \nu} n_\nu \) which is equivalent to the transformation of space-time coordinates \( x'^\mu = \Lambda_{\mu \nu} x_\nu \). In the new Lorentz frame the invariant time parameter \( \tau \) has the same value, since \( \tau = n \cdot x = n' \cdot x' \). For any given \( n \), there exists the special “instant frame” where \( n^\mu = (1, 0, 0, 0) \) and, consequently, \( \tau = t \). In Appendix A we show that in this frame Eq. (3.5) reduces to the kinetic equation derived by Bialynicki-Birula et al. [5] on the basis of a different approach.

Despite its apparently simple structure, Eq. (3.5) is a rather complicated matrix equation. To obtain a deeper physical insight into processes described by this equation, it is convenient to expand the Wigner function in a complete basis in spinor space

\[ W = \frac{1}{4} \left( I W + \gamma_\mu W^\mu + \gamma_5 W_{(P)} + \gamma_5 \gamma_\mu W_{(A)}^\mu + \sigma_{\mu \nu} W^{\mu \nu} \right). \]  

(3.10)

Here \( I \) is the unit matrix, and \( W^\mu, W_{(P)}, W_{(A)}^\mu, W^{\mu \nu} \) are the scalar, vector, pseudo-scalar, axial-vector and tensor coefficient function of the Wigner matrix \( W \) respectively. By the trace rules in spinor space it can easily be verified that the coefficient functions can be expressed as

\[ W = \text{tr} \left( W \right), \]  

(3.11)

\[ W^\mu = \text{tr} \left( \gamma^\mu W \right), \]  

(3.12)

\[ W_{(P)} = \text{tr} \left( \gamma_5 W \right), \]  

(3.13)
\[ \mathcal{W}^\mu = \text{tr} (\gamma^\mu \gamma_5 W), \]  
\[ \mathcal{W}^{\mu\nu} = \frac{1}{2} \text{tr} (\bar{\sigma}^{\mu\nu} W), \]  
(3.14)  
(3.15)

where the symbol “tr” means the trace over spinor indices. Now a straightforward algebra allows to derive a coupled set of equations for the coefficient functions from the kinetic equation (3.5) by using Eqs. (3.11) – (3.15)

\[ D_{\tau} \mathcal{W} = 2 \left( n_\alpha P_\beta - n_\beta P_\alpha \right) \mathcal{W}^{\alpha\beta}, \]  
(3.16)

\[ D_{\tau} \mathcal{W}_{\mu} = - \left( n^\mu D_{\perp\alpha} - n_\alpha D_{\perp}^\mu \right) \mathcal{W}^{\alpha} - 2 \varepsilon_{\alpha\beta\lambda\mu} n^{\alpha} P^{\beta} \mathcal{W}^{\lambda}_{(A)} - 4 m \mathcal{W}^{\mu\alpha} n_\alpha, \]  
(3.17)

\[ D_{\tau} \mathcal{W}_{(P)} = 2 i m n_\alpha \mathcal{W}^{\alpha}_{(A)} + 2 i \varepsilon_{\alpha\beta\lambda\mu} n^{\alpha} P^{\beta} \mathcal{W}^{\lambda}_{\nu}, \]  
(3.18)

\[ D_{\tau} \mathcal{W}^{\mu}_{(A)} = - 2 \varepsilon_{\alpha\beta\lambda\mu} n^{\alpha} P^{\beta} \mathcal{W}^{\lambda}_{(A)} + 2 i m n_\alpha \mathcal{W}^{\alpha}_{(P)} - \left( n^\mu D_{\perp\alpha} - n_\alpha D_{\perp}^\mu \right) \mathcal{W}^{\alpha}_{(A)}, \]  
(3.19)

\[ D_{\tau} \mathcal{W}^{\mu\nu} = \left( n^\mu P_\nu - n^\nu P_\mu \right) \mathcal{W} - m \left( n^\mu \mathcal{W}_\nu - n^\nu \mathcal{W}_\mu \right) + i \varepsilon^{\mu\nu\alpha\beta} n_\alpha P_\beta \mathcal{W}_{(P)} 
+ \left( n^\mu D_{\perp\alpha} - n_\alpha D_{\perp}^\mu \right) \mathcal{W}^{\alpha\nu} - \left( n^\nu D_{\perp\alpha} - n_\alpha D_{\perp}^\nu \right) \mathcal{W}^{\mu\alpha}, \]  
(3.20)

where \( \varepsilon^{\mu\nu\alpha\beta} \) is the Levi-Civita tensor. In Appendix B the necessary algebra including commutator and anticommutator relations is shortly surveyed.

The tensor structure of Eqs. (3.16) – (3.20) becomes more clear if we split these equations into longitudinal and transverse components with respect to the hyperplane \( \sigma_n, \tau \). The vector and axial-vector functions are decomposed according to

\[ \mathcal{W}^\mu = n^\mu \mathcal{W}_{\parallel} + \mathcal{W}^\mu_{\perp}, \quad \mathcal{W}^\mu_{(A)} = n^\mu \mathcal{W}_{\parallel(A)} + \mathcal{W}^\mu_{\perp(A)}, \]  
(3.21)

where

\[ \mathcal{W}_{\parallel} = n_\alpha \mathcal{W}^{\alpha}, \quad \mathcal{W}_{\parallel(A)} = n_\alpha \mathcal{W}^{\alpha}_{(A)}, \]  
\[ \mathcal{W}^\mu_{\perp} = \Delta^\mu_{\alpha} \mathcal{W}^{\alpha}, \quad \mathcal{W}^\mu_{\perp(A)} = \Delta^\mu_{\alpha} \mathcal{W}^{\alpha}_{(A)}. \]  
(3.22)

The tensor function can be written as

\[ \mathcal{W}^{\mu\nu} = (\mathcal{U}^{\mu} n^\nu - \mathcal{U}^{\nu} n^\mu) + \mathcal{W}^{\mu\nu}_{\perp}, \]  
\[ \mathcal{U}^{\mu} = n_\alpha \mathcal{W}^{\mu\alpha}, \quad \mathcal{W}^{\mu\nu}_{\perp} = \Delta^\mu_{\alpha} \Delta^\nu_{\beta} \mathcal{W}^{\alpha\beta}_{\perp}. \]  
(3.23)

Then we arrive at the following set of equations:

\[ D_{\tau} \mathcal{W} = - 4 P_\alpha \mathcal{U}^{\alpha}, \]  
(3.24)
This representation for the mean-field kinetic equation will prove to be particularly convenient in the quasi-classical limit.

3.3 The charge conservation

In order to describe the picture consistently the polarization current (2.11), which enters the Maxwell equations (2.10), must be expressed in terms of the Wigner function. As shown in paper I [see (I.5.9)], the polarization current can be written in the form

\[ j_\mu(x) = e \int \frac{d^4p}{(2\pi)^3} \delta(p \cdot n) \mathcal{W}_\mu(x_\perp, p_\perp; \tau = x \cdot n), \]  

(3.32)

where \( \mathcal{W}_\mu \) is the vector component (3.12) of the Wigner function. Let us prove that the fundamental charge conservation law \( \partial_\mu j^\mu = 0 \) is satisfied in our theory.

It is convenient to define for any function \( G(x_\perp, p_\perp; \tau) \) the transformation

\[ \overline{G}(x) = \int \frac{d^4p}{(2\pi)^3} \delta(p \cdot n) G(x_\perp, p_\perp; \tau = x \cdot n). \]  

(3.33)

Then Eq. (3.32) takes a compact form

\[ j^\mu(x) = e \overline{\mathcal{W}}^\mu(x) . \]  

(3.34)

It follows easily from Eq. (3.33) that

\[ \partial_\mu \overline{G}(x) = n_\mu \left( \frac{\partial G}{\partial \tau} \right) + \nabla_\mu \overline{G}(x). \]  

(3.35)
Note also that for functions $G$ which go to zero as $|p_\perp| \to \infty$ we have

$$D_\tau G = n^\mu \partial_\mu G, \quad D_\perp^\mu G = \nabla^\mu G, \quad P_\mu G = \bar{p}_\perp^\mu G. \quad (3.36)$$

Applying the transformation (3.33) to Eq. (3.17), we obtain

$$n^\lambda \partial_\lambda j^\mu = -(n^\mu \nabla_\nu - n_\nu \nabla^\mu) j^\nu - 2e \varepsilon_\alpha^\mu \alpha_\beta^\lambda n^\alpha \left(\bar{p}_\perp^\beta W^\lambda_{(A)}\right) - 4emW^{\mu\nu} n_\nu. \quad (3.37)$$

On the other hand, we may write

$$\partial_\mu j^\mu = n_\mu n^\lambda \partial_\lambda j^\mu + \nabla_\mu j^\mu.$$ 

Combining this with Eq. (3.37), we see that $\partial_\mu j^\mu = 0$.

One can follow a similar procedure to derive other conservation laws and balance equations for local quantities like the mass current, the spin density, the magnetic moment density, and the angular momentum density. The advantage of the hyperplane formalism over the previous approaches to the mean-field QED kinetic theory [5,6] is that all the balance equations and conservation laws will have a manifestly covariant form.

4 The quasi-classical limit

4.1 The local-field approximation

In view of practical applications of the theory, it is of interest to study kinetic processes in QED plasmas depending on a slowly varying external EM field. To consider this case, we insert the constants $\hbar$ and $c$ into Eq. (3.5) and the operators (3.6) – (3.8). Then we obtain the kinetic equation

$$D_\tau W = - \frac{imc}{\hbar} \left[\gamma_1, W\right] - \frac{i}{2} D_\perp S^\mu [S^\mu, W] - \frac{1}{\hbar} P_\mu \left\{S^\mu, W\right\} \quad (4.1)$$

and the following expressions for the operators with the corresponding gradient expansions:

$$D_\tau = \frac{\partial}{\partial \tau} - \frac{e}{c} \int_{-1/2}^{1/2} ds \int n^\mu \mathcal{F}_{\mu\nu} \left(\tau, x_\perp - is\hbar \nabla_p\right) \nabla^\nu$$
\[ \frac{\partial}{\partial \tau} - \frac{e}{c} n^\mu F_{\mu \nu} \nabla^\nu + \frac{\hbar^2}{24c} \left( \nabla \cdot \nabla \right)^2 n^\mu F_{\mu \nu} \nabla^\nu + \ldots, \tag{4.2} \]

\[ D_{\perp \mu} = \nabla_{\mu} - \frac{e}{c} \int_{-1/2}^{1/2} ds \mathcal{F}_{\perp \mu \nu} \left( \tau, x_{\perp} - i \hbar \nabla_{\perp} \right) \nabla_{\nu} \]

\[ = \nabla_{\mu} - \frac{e}{c} \mathcal{F}_{\perp \mu \nu} \nabla_{\nu} + \frac{\hbar^2}{24c} \left( \nabla \cdot \nabla \right)^2 n^\mu \mathcal{F}_{\perp \mu \nu} \nabla_{\nu} + \ldots, \tag{4.3} \]

\[ P_{\mu} = p_{\perp \mu} - \frac{ie\hbar}{c} \int_{-1/2}^{1/2} s ds \mathcal{F}_{\perp \mu \nu} \left( \tau, x_{\perp} - i \hbar \nabla_{\perp} \right) \nabla_{\nu} \]

\[ = p_{\perp \mu} - \frac{e\hbar^2}{12c} \left( \nabla \cdot \nabla \right) \mathcal{F}_{\perp \mu \nu} \nabla_{\nu} + \ldots \tag{4.4} \]

The condition that the terms containing the field derivatives be small reads

\[ \bar{\lambda}_B \ll l_{EM}, \tag{4.5} \]

where \( \bar{\lambda}_B \) is the average de Broglie wave length for fermions and \( l_{EM} \) is the characteristic length for variations of the EM field in the system. For laser induced plasmas the latter quantity is roughly equal to the wave length of the external laser field. We will refer to the condition (4.5) as the local approximation, from which the operators (4.2) – (4.4) can be concluded to be

\[ D_{\tau} = \frac{\partial}{\partial \tau} - \frac{e}{c} n^\mu F_{\mu \nu} \nabla_{\nu}, \tag{4.6} \]

\[ D_{\perp \mu} = \nabla_{\mu} - \frac{e}{c} \mathcal{F}_{\perp \mu \nu} \nabla_{\nu}, \tag{4.7} \]

\[ P_{\mu} = p_{\perp \mu}. \tag{4.8} \]

In the local-field approximation, Eqs. (3.24) – (3.31) (with inserted \( \hbar \) and \( c \)) become

\[ D_{\tau} \mathcal{W} = - \frac{4}{\hbar} p_{\perp \mu} U^\mu, \tag{4.9} \]

\[ D_{\tau} \mathcal{W}_{\parallel} = - D_{\perp \mu} \mathcal{W}^\mu, \tag{4.10} \]

\[ D_{\tau} \mathcal{W}_{\perp}^\mu = D_{\perp}^\mu \mathcal{W}_{\parallel} - \frac{2}{\hbar} \varepsilon_{\alpha \beta \lambda} n^\alpha p_{\perp}^\beta \mathcal{W}_{\parallel (A)} - \frac{4mc}{\hbar} U^\mu, \tag{4.11} \]

\[ D_{\tau} \mathcal{W}_{(P)} = i \frac{2mc}{\hbar} \mathcal{W}_{\parallel (A)} + \frac{2}{\hbar} \varepsilon_{\alpha \beta \lambda \rho} n^\alpha p_{\perp}^\beta \mathcal{W}_{\parallel (A)}, \tag{4.12} \]
\[ \text{It should be emphasized that the local approximation in the operators } D_\tau, D_\perp, \text{ and } P^\mu \text{ does not necessarily implies that all quantum effects are neglected. It can easily be seen from Eqs. (4.9) – (4.16) that some components of the Wigner function show non-analytic behavior in the limit } \hbar \to 0 \text{ describing quantum phenomena like pair production in strong fields [7,8]. This aspect of the “one-time” mean-field theory of QED was discussed by Bialynicki-Birula et al. [5] in their study of the Dirac vacuum in strong external fields. For QED plasmas the situation is somewhat similar to the problem of the Dirac vacuum, but, generally speaking, a self-consistent description of quantum effect in plasmas involves the photon kinetics. This point will be detailed in a special article.} \]

### 4.2 Quasi-classical kinetic equations for fermions

We now want to obtain the quasi-classical limit of Eqs. (4.9) – (4.16) for those components of the Wigner function which determine the polarization current (3.32).

We first notice that Eq. (4.11) allows to eliminate \( U^\mu \) in the other equations. In particular, we have

\[ p_\perp \mu U^\mu = \frac{\hbar}{4mc} p_\perp \mu \left( D^\mu \mathcal{W}_\parallel \mathcal{W}_\perp \right), \]  

which is to be inserted into Eq. (4.9). Then we can observe that, in the quasi-classical limit \( (\hbar \to 0) \), Eq. (4.15) leads to the relation

\[ \mathcal{W}_\perp^\mu = \frac{p_\perp^\mu}{mc} \mathcal{W}. \]  

Thus Eqs. (4.9) and (4.10) give a closed set of quasi-classical equations for \( \mathcal{W} \).
and $\mathcal{W}_\perp$:
\[
\begin{align*}
D_{\tau}\mathcal{W} - \frac{1}{mc^2} p_{\perp\mu} D_{\tau}(p_{\perp}^\mu \mathcal{W}) + \frac{1}{m} p_{\perp}^\mu D_{\perp\mu} \mathcal{W}_\parallel &= 0, \\
D_{\tau}\mathcal{W}_\parallel + \frac{1}{mc} D_{\perp\mu}(p_{\perp}^\mu \mathcal{W}) &= 0.
\end{align*}
\] (4.19)

With (4.6) and (4.7), it is easy to verify that in the quasi-classical limit
\[
p_{\perp\mu} D_{\tau} p_{\perp}^\mu = -\frac{e}{c} \tau_{\mu\nu} F_{\nu\rho} p_{\perp}^\rho, \quad D_{\perp\mu} p_{\perp}^\mu = 0.
\] (4.20)

Using these relations, a little algebra shows that Eqs. (4.19) can be written in a more symmetric form
\[
\begin{align*}
D_{\tau} \left( \frac{\epsilon(p_{\perp})}{mc^2} \mathcal{W} \right) + \frac{v_{\perp}^\mu}{c} D_{\perp\mu} \mathcal{W}_\parallel &= 0, \\
D_{\tau} \mathcal{W}_\parallel + \frac{v_{\perp}^\mu}{c} D_{\perp\mu} \left( \frac{\epsilon(p_{\perp})}{mc^2} \mathcal{W} \right) &= 0,
\end{align*}
\] (4.21)

where we have introduced the dispersion relation for fermions on the hyperplane
\[
\epsilon(p_{\perp}) = c\sqrt{m^2 c^2 - p_{\perp}^2}
\] (4.22)

and the transverse four-velocity
\[
v_{\perp}^\mu = \frac{\epsilon^2}{\epsilon(p_{\perp})} p_{\perp}^\mu.
\] (4.23)

We now define the distribution functions for electrons ($w$) and positrons ($\bar{w}$) on the hyperplane $\sigma_{n,\tau}$:
\[
\begin{align*}
w(x_{\perp}, p_{\perp}; \tau) &= \frac{1}{2} \left\{ \frac{\epsilon(p_{\perp})}{mc^2} \mathcal{W}(x_{\perp}, p_{\perp}; \tau) + \mathcal{W}_\parallel(x_{\perp}, p_{\perp}; \tau) \right\}, \\
\bar{w}(x_{\perp}, p_{\perp}; \tau) &= \frac{1}{2} \left\{ \frac{\epsilon(p_{\perp})}{mc^2} \mathcal{W}(x_{\perp}, -p_{\perp}; \tau) - \mathcal{W}_\parallel(x_{\perp}, -p_{\perp}; \tau) \right\}.
\end{align*}
\] (4.24)

These functions satisfy independent kinetic equations which follow from Eqs. (4.21).
In the expanded form, we have

\[
\left( \frac{\partial}{\partial \tau} + \frac{v_\perp}{c} \nabla_c \right) w - \frac{e}{c} \left( n^\mu F_{\mu\nu} + \frac{v_\perp}{c} F_{\perp\mu\nu} \right) \nabla_r^\mu w = 0, \\
\left( \frac{\partial}{\partial \tau} + \frac{v_\perp}{c} \nabla_c \right) \bar{w} + \frac{e}{c} \left( n^\mu F_{\mu\nu} + \frac{v_\perp}{c} F_{\perp\mu\nu} \right) \nabla_r^\mu \bar{w} = 0.
\]

These equations are in fact nothing more than a generalization of relativistic Vlasov equations to the case that the distribution functions for particles and antiparticles are defined on arbitrary hyperplanes \( \sigma_{n,\tau} \). In the special “instant frame”, where \( n^\mu = (1, 0, 0, 0) \) and \( x^\mu = (ct, \mathbf{r}) \), Eqs. (4.25) take the well-known form [see Appendix A]

\[
\frac{\partial w}{\partial \tau} + \mathbf{v} \cdot \nabla w + e \left[ \mathbf{E} + \frac{1}{c} (\mathbf{v} \times \mathbf{B}) \right] \cdot \frac{\partial w}{\partial \mathbf{p}} = 0,
\]

\[
\frac{\partial \bar{w}}{\partial \tau} + \mathbf{v} \cdot \nabla \bar{w} - e \left[ \mathbf{E} + \frac{1}{c} (\mathbf{v} \times \mathbf{B}) \right] \cdot \frac{\partial \bar{w}}{\partial \mathbf{p}} = 0,
\]

where \( \mathbf{v} = c^2 \mathbf{p}/\epsilon(\mathbf{p}) \) is the velocity vector, and \( \epsilon(\mathbf{p}) = c\sqrt{\mathbf{p}^2 + m^2 c^2} \) is the relativistic energy in the “instant frame”.

The kinetic equations (4.25) describe the evolution of the distribution functions with respect to the time-like variable \( \tau \). Note, however, that \( w \) and \( \bar{w} \) can also be regarded as functions of the space-time point \( x \) and the four-vector \( p_\perp \), according to

\[
w(x, p_\perp) \equiv w(x_\perp, p_\perp; \tau = x \cdot n), \quad \bar{w}(x, p_\perp) \equiv \bar{w}(x_\perp, p_\perp; \tau = x \cdot n).
\]

This interpretation of the distribution functions allows to put Eqs. (4.25) into a more compact form. First we note that

\[
v_\perp^\mu F_{\perp\mu\nu} \nabla_r^\nu = v_\perp^\mu F_{\mu\nu} \nabla_r^\nu,
\]

which follows directly from the definition of \( F_{\perp\mu\nu} \), Eq. (3.9). We next consider the relations

\[
u^\mu \partial_\mu w(x, p_\perp) = u^\mu \nabla_\mu w(x, p_\perp) + (u \cdot n) \left( \frac{\partial}{\partial \tau} w(x_\perp, p_\perp; \tau = x \cdot n) \right)_{\tau = x \cdot n},
\]

\[
u^\mu \partial_\mu \bar{w}(x, p_\perp) = u^\mu \nabla_\mu \bar{w}(x, p_\perp) + (u \cdot n) \left( \frac{\partial}{\partial \tau} \bar{w}(x_\perp, p_\perp; \tau = x \cdot n) \right)_{\tau = x \cdot n},
\]

(4.28)
where the time-like unit vector $u^\mu$ is defined as

$$u^\mu = \frac{\epsilon(p)}{mc^2} \left( n^\mu + \frac{v^\mu}{c} \right).$$

(4.29)

Elimination of the $\tau$-derivatives between Eqs. (4.25) and (4.28) and some rearrangement leads to the equations

$$u^\mu \left( \partial_\mu - \frac{e}{c} F_{\mu\nu}(x) \nabla_\nu \right) w(x,p_\perp) = 0,$$

$$u^\mu \left( \partial_\mu + \frac{e}{c} F_{\mu\nu}(x) \nabla_\nu \right) \bar{w}(x,p_\perp) = 0.$$  

(4.30)

One can verify, e.g., by going to the “instant frame”, that (4.29) is the four-velocity of a particle with the four-momentum $p^\mu$ satisfying the mass-shell condition $p^2 = m^2 c^2$. Equations (4.30) may thus be interpreted as the evolution equations for the distribution functions with respect to the invariant proper time.

In order to guarantee a self-consistent description of the plasma in the quasiclassical approximation, the polarization current (3.32) must be expressed in terms of the distribution functions $w$ and $\bar{w}$. To do this, we recall the quasiclassical result (4.18) and write

$$W^\mu = n^\mu W_\parallel + \frac{p^\mu}{mc} W.$$  

(4.31)

Elimination of $W_\parallel$ and $W$ with the aid of Eqs. (4.24) gives

$$W^\mu(x,p_\perp) = \frac{mc^2}{\epsilon(p_\perp)} \left\{ u^\mu(p_\perp) w(x,p_\perp) - u^\mu(-p_\perp) \bar{w}(x,-p_\perp) \right\},$$

(4.32)

so that the polarization current (3.32) takes the form (with the inserted Planck's constant)

$$j^\mu(x) = e \int \frac{d^3p}{(2\pi\hbar)^3} \delta(p \cdot n) \frac{mc^2}{\epsilon(p_\perp)} u^\mu(p_\perp) \left[ w(x,p_\perp) - \bar{w}(x,p_\perp) \right].$$

(4.33)

By using Eqs. (4.30), it can easily be verified that the above expression for the current is consistent with the conservation law $\partial_\mu j^\mu = 0$. 

15
4.3 The invariant quasi-classical distribution function for fermions

It is interesting that the polarization current (4.33) can be rewritten in a form where the four-vector $n$ does not appear. First we notice that the delta function $\delta(p \cdot n)$ in Eq. (4.33) may be replaced by $\delta (p \cdot n - \epsilon(p_\perp)/c)$ because other functions in the integrand do not depend on $p_\parallel = p \cdot n$. Then, according to the identity

$$\int d^4p \frac{\delta(p \cdot n - \epsilon(p_\perp)/c)}{\epsilon(p_\perp)} (\cdots) = \frac{2}{c} \int_{p^\mu > 0} d^4p \delta(p^2 - m^2c^2) (\cdots) , \quad (4.34)$$

we can rewrite Eq. (4.33) as

$$j^\mu(x) = 2e mc \int_{p^\mu > 0} \frac{d^4p}{(2\pi\hbar)^3} \delta(p^2 - m^2c^2) w^\mu(p_\perp) [w(x, p_\perp) - \bar{w}(x, p_\perp)]. \quad (4.35)$$

Finally, from Eqs. (4.23) and (4.29) follows

$$u^\mu(p_\perp) = \frac{p^\mu}{mc} - \frac{n^\mu}{mc} [p \cdot n - \epsilon(p_\perp)/c]. \quad (4.36)$$

With the mass-shell constraint $p^2 = m^2c^2$ we have $w^\mu = p^\mu/mc$, so that Eq. (4.35) becomes

$$j^\mu(x) = 2e \int \frac{d^4p}{(2\pi\hbar)^3} p^\mu [f(x, p) - \bar{f}(x, p)] \quad (4.37)$$

after introducing the mass-shell distribution functions for particles and antiparticles:

$$f(x, p) = \Theta(p^0) \delta(p^2 - m^2c^2) w(x, p_\perp),$$

$$\bar{f}(x, p) = \Theta(p^0) \delta(p^2 - m^2c^2) \bar{w}(x, p_\perp), \quad (4.38)$$

where $\Theta(p^0)$ is the unit step function. The mean-field kinetic equations for these functions can be derived from Eqs. (4.30). We will give only the derivation of the equation for $f(x, p)$ since the equation for $\bar{f}(x, p)$ is obtained analogously.
Multiplying the first of Eqs. (4.30) by \( \Theta(p^0) \delta(p^2 - m^2c^2) \) and again using the fact that on the mass-shell \( u^\mu = p^\mu/mc \), we have

\[
\Theta(p^0) \delta(p^2 - m^2c^2) p^\mu \left( \partial_\mu - \frac{e}{c} F_{\mu\nu} \nabla_\nu \right) w = 0.
\]  
(4.39)

The transverse gradient in the momentum space, \( \nabla_\nu \), can be represented as

\[
\nabla_\nu = \partial_\nu - n_\nu \frac{\partial}{\partial p_\parallel},
\]  
(4.40)

where \( \partial_\nu = g^\nu_\lambda \partial/\partial p^\lambda \) and \( p_\parallel = p \cdot n \). Since \( w \) does not depend on \( p_\parallel \), the operator \( \nabla_\nu \) in Eq. (4.39) may be replaced by \( \partial_\nu \). Finally, using the relations

\[
\partial_\nu \left[ \Theta(p^0) \delta(p^2 - m^2c^2) \right] = 2\Theta(p^0) p^\nu \frac{\partial\delta(p^2 - m^2c^2)}{\partial p^2}
\]

and \( p^\mu F_{\mu\nu} p^\nu = 0 \), Eq. (4.39) takes the form

\[
p^\mu \left( \partial_\mu - \frac{e}{c} F_{\mu\nu}(x) \partial_\nu \right) f(x,p) = 0.
\]  
(4.41)

The analogous covariant kinetic equation for antiparticles reads

\[
p^\mu \left( \partial_\mu + \frac{e}{c} F_{\mu\nu}(x) \partial_\nu \right) \bar{f}(x,p) = 0.
\]  
(4.42)

Formally, Eq. (4.41) coincides with the well-known relativistic kinetic equation for charged particles in a prescribed electromagnetic field (see, e.g., [10]).

Here one comment is in order. We see that the invariant distribution functions \( f(x,p) \) and \( \bar{f}(x,p) \) satisfy kinetic equations (4.41) and (4.42) which do not give any indication of the family of hyperplanes \( \sigma_{n,\tau} \) used in the derivation of these equations. Note, however, that a unique solution of these equations exists only if \( f(x,p) \) and \( \bar{f}(x,p) \) are specified on some space-like surface \( \sigma \) in Minkowski space. To formulate this “initial condition”, we have to recall Eqs. (4.38) which relate the invariant distribution functions to the functions \( w \) and \( \bar{w} \) defined on the family of hyperplanes \( \sigma_{n,\tau} \). Lorentz invariance of the theory manifests itself by associating \( f(x,p) \) and \( \bar{f}(x,p) \) with an arbitrary family of hyperplanes in order to fix the “initial condition”.

17
Based on the general density matrix approach to QED plasmas [1], we have derived kinetic equations for the fermionic subsystem in the mean-field approximation. The general mean-field expression given by Eq. (3.5) is a covariant generalization of the kinetic equation derived previously by Bialynicki-Birula et al. [5]. Their result is reproduced in the “instant frame”, where the normal vector is given by \( n^\mu = (1, 0, 0, 0) \). The covariant structure of Eq. (3.5) is particularly convenient for the spinor decomposition which allows to separate kinetic processes in different channels. Another advantage of this equation is that it can be used to perform further approximations in a covariant form. For instance, the quasi-classical limit of the mean-field kinetic equation is presented.

Applications of relativistic mean-field theories have been discussed in different contexts. For instance, in heavy-ion collisions a relativistic kinetic equation in an “instant frame” is solved using the relativistic Landau-Vlasov method [12]. Present ultra-relativistic heavy-ion collisions demand a consistent relativistic approach to nonequilibrium evolution [7]. Laser-plasma interactions are most often treated within particle in cell (PIC) simulations [13], which follow from classical mean-field approximations to the relativistic kinetic equation.

Spectral information is not contained in the description presented here. In [14,15], for instance, the relation between the one-time and two-time Wigner function in the instant frame is discussed. Applying an energy moment expansion of the two-time Wigner function, the one-time Wigner function is given by the lowest moment, whereas the spectral information is contained in higher moments.

There are different ways to go beyond the approximations presented in this paper. Quantum corrections to the quasi-classical kinetic equation can be taken into account by expanding Eqs. (3.24) – (3.31) in terms of Planck’s constant. This implies to consider non-local fluctuations in the plasma at a length scale less than the de Broglie wavelength. Quantum effects associated with particle-antiparticle coherence, like for instance pair production caused by strong fields [7,8], can give significant corrections to the Wigner function in the treatment of QED plasmas under extreme conditions. Pair creation in current laser-plasma experiments is realized through a bremsstrahlung conversion of MeV electrons into MeV photons [11]. For the description of such effects the photon kinetics has to be included self-consistently into the picture presented here.

Furthermore, improving the mean-field approximation, one can consider collisions in the plasma. This can be done systematically by expanding the collision terms for the photons and electrons (compare Eq. (I:5.34) and (I:5.35))
in terms of the fine structure constant $\alpha$. First order effects related to emission and absorption of photons is subject of forthcoming studies.

Appendix A

*Kinetic equation for the matrix Wigner function in the “instant frame”*

We consider Eq. (3.5) in the “instant frame”, where $n^\mu = (1, 0, 0, 0)$ and $\tau = x^0 = t$ ($c = \hbar = 1$). Introducing the usual space-time notation $x^\mu = (t, \mathbf{r})$, we find

$$x^\mu_\perp = (0, \mathbf{r}), \quad x_\perp^\mu = (0, -\mathbf{r}), \quad p^\mu_\perp = (0, \mathbf{p}), \quad p_\perp^\mu = (0, -\mathbf{p}).$$

The transverse four-gradients with respect to space-time and the momentum variables, $\nabla_\mu$ and $\nabla_\mu^\mu$, are written in the “instant frame” as

$$\nabla_\mu = (0, \nabla), \quad \nabla^\mu = (0, -\nabla), \quad \nabla_\mu^\mu = (0, \partial_\mu), \quad \nabla^\mu = (0, -\partial_\mu),$$

where $\nabla = \partial/\partial \mathbf{r}$ and $\partial_\mu = \partial/\partial \mathbf{p}$. For definiteness, Cartesian components of all three-dimensional vectors and gradients will be written with upper Latin indices running from 1 to 3. For instance, $\nabla^i = \partial/\partial r^i$ and $\partial^i_\mu = \partial/\partial p^i$. Summation over repeated Latin indices is implied.

The total electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, are defined in the “instant frame” as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (A.1)$$

or,

$$\mathbf{E}^i = \mathcal{F}_{0i}, \quad \mathbf{B}^i = \varepsilon^{ijk} \nabla^j A^k, \quad (A.2)$$

where $\varepsilon^{ijk}$ is the three-dimensional antisymmetric symbol with $\varepsilon^{123} = 1$. Note also that, in our notation, the non-zero components of the tensor (3.9) are now given by $\mathcal{F}_{ij}^\perp = \mathcal{F}_{\perp ij} = - (\nabla^i A^j - \nabla^j A^i)$.

Further it is easy to verify that the operators (3.6) – (3.8) can be written as

$$D_t = D_t, \quad D_{\perp \mu} = (0, \mathbf{D}), \quad P_{\perp \mu} = (0, -\mathbf{P}), \quad (A.3)$$

where
\[
D_t = \frac{\partial}{\partial t} + e \int_{-1/2}^{1/2} ds \mathbf{E} (t, r + is\partial_p) \cdot \partial_p, \quad (A.4)
\]
\[
D = \nabla + e \int_{-1/2}^{1/2} ds \mathbf{B} (t, r + is\partial_p) \times \partial_p, \quad (A.5)
\]
\[
P = p - ie \int_{-1/2}^{1/2} s ds \mathbf{B} (r + is\partial_p) \times \partial_p. \quad (A.6)
\]

Finally, in the “instant frame” we have \( \gamma_\parallel = \gamma^0 \equiv \beta \), so that the matrices \( S^\mu \) [see Eq. (3.4)] can be written in terms of the Dirac \( \alpha \)-matrices as

\[
S^\mu = (0, -i\alpha). \quad (A.7)
\]

Putting expressions (A.3) and (A.7) into Eq. (3.5), it is convenient to rewrite this equation for the modified Wigner function [5]

\[
\tilde{W} = W\gamma^0, \quad (A.8)
\]

which implies that the fermionic density operator is defined as [cf. Eq. (2.2)]

\[
\hat{\rho}_{aa'}(r, r') = -\frac{1}{2} [\hat{\psi}_a(r), \hat{\psi}^\dagger_{a'}(r')].
\]

Then, in terms of \( \tilde{W} \), Eq. (3.5) becomes

\[
D_t \tilde{W} = -im [\beta, \tilde{W}] - \frac{1}{2} D \cdot \{\alpha, \tilde{W}\} - iP \cdot [\alpha, \tilde{W}]. \quad (A.9)
\]

This is the mean-field kinetic equation derived by Bialynicki-Birula et al. [5].

**Quasi-classical kinetic equations in the “instant frame”**

We now aim to show that, in the “instant frame”, the kinetic equations (4.25) take the form (4.26). First we note that in this frame the transverse four-velocity (4.23) is written as \( v^\mu_{\perp} = (0, \mathbf{v}) \), where \( \mathbf{v} = c^2 p / \epsilon(p) \) is the three-dimensional velocity vector of a particle with the energy \( \epsilon(p) = c \sqrt{p^2 + m^2 c^2} \).

Then, using the above expressions for the transverse four-gradients in the
“instant frame”, we find that
\[ n^\mu F_{\mu \nu} \nabla^\nu p = -F_{\mu 0} \partial^\mu = -E \cdot \frac{\partial}{\partial p}, \]
\[ v^\mu F_{\perp \mu \nu} \nabla^\nu p = -v^i F_{ij} \partial^j = - (v \times B) \cdot \frac{\partial}{\partial p}. \]  

(A.10)

Finally, in the “instant frame” we have the obvious relation
\[ \frac{\partial}{\partial \tau} + \frac{v^\mu}{c} \nabla_\mu = \frac{1}{c} \left( \frac{\partial}{\partial t} + v \cdot \nabla \right). \]  

(A.11)

Insertion of expressions (A.10) and (A.11) into (4.25) leads to the kinetic equations (4.26).

**Appendix B**

Here we give some basic relations, which are used to transform the matrix kinetic equation (3.5) into Eqs. (3.16) – (3.20). We follow the notation of [16].

The totally antisymmetric Levi-Civita tensor \( \varepsilon^{\mu \nu \alpha \beta} \) is defined through even and odd permutations of \( \mu \nu \alpha \beta \) with
\[ \varepsilon^{0123} = -\varepsilon_{0123} = 1. \]  

(A.1)

The relation between co- and contravariant components follows from the metric \( g_{\mu \nu} = \text{diag}(1, -1, -1, -1) \)

From the Dirac algebra (see e.g. [16]) we can calculate the commutator and anticommutator relations appearing in the different spinor channels of Eq. (3.5)

\[ [\gamma_\mu, I] = 0, \quad [\gamma_\mu, \gamma_\nu] = -2i \tilde{\sigma}_{\mu \nu}, \quad [\gamma_\mu, \gamma_5] = -2 \gamma_5 \gamma_\mu, \]
\[ [\gamma_\mu, \gamma_5 \gamma_\nu] = -2 g_{\mu \nu} \gamma_5, \quad [\gamma_\mu, \tilde{\sigma}_{\mu' \nu'}] = 2i \left( g_{\mu \mu'} \gamma_{\nu'} - g_{\mu' \nu} \gamma_{\mu'} \right), \]
\[ [\tilde{\sigma}_{\mu \nu}, I] = 0, \quad [\tilde{\sigma}_{\mu \nu}, \gamma_5] = 0, \quad [\tilde{\sigma}_{\mu \nu}, \gamma_5 \gamma_\mu] = 2i \left( g_{\nu \mu'} \gamma_5 \gamma_\mu - g_{\mu \mu'} \gamma_5 \gamma_\nu \right), \]
\[ [\tilde{\sigma}_{\mu \nu}, \tilde{\sigma}_{\mu' \nu'}] = -2i \left( g_{\mu \mu'} \tilde{\sigma}_{\nu \nu'} + g_{\nu \nu'} \tilde{\sigma}_{\mu \mu'} - g_{\mu \nu'} \tilde{\sigma}_{\nu \mu'} - g_{\nu \mu'} \tilde{\sigma}_{\mu \nu'} \right), \]
\[ \{ \tilde{\sigma}_{\mu \nu}, I \} = 2 \tilde{\sigma}_{\mu \nu}, \quad \{ \tilde{\sigma}_{\mu \nu}, \gamma_\mu \} = 2 \varepsilon_{\mu \nu \alpha \beta} \gamma_5 \gamma^\alpha, \]
\[ \{ \tilde{\sigma}_{\mu \nu}, \gamma_5 \} = 2 \varepsilon_{\mu \nu \alpha \beta} \gamma_5 \gamma^\alpha, \]
\[
\{\bar{\sigma}_{\mu\nu}; \gamma_5\} = i\varepsilon_{\mu\nu\mu'\nu'} \bar{\sigma}^{\mu'\nu'}, \quad \{\bar{\sigma}_{\mu\nu}; \gamma_5\gamma_\mu'\nu'\} = 2\varepsilon_{\mu\nu\mu'\alpha}\gamma^\alpha,
\]

\[
\{\bar{\sigma}_{\mu\nu}, \bar{\sigma}_{\mu'\nu'}\} = 2 \left( g_{\mu\mu'} g_{\nu\nu'} - g_{\mu\nu'} g_{\mu\nu} \right) I + 2i\varepsilon_{\mu\nu\mu'\nu'} \gamma_5.
\]

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