The spectrum of the 2D Black Hole
or Does the 2D black hole have tachyonic or W–hair?

Neil Marcus* and Yaron Oz*

School of Physics and Astronomy
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel-Aviv University
Ramat Aviv, Tel-Aviv 69978, ISRAEL.

Abstract

We solve the equations of motion of the tachyon and the discrete states in the background of Witten’s semiclassical black hole and in the exact 2D dilaton-graviton background of Dijkgraaf et al. We find the exact solutions for weak fields, leading to conclusions in disagreement with previous studies of tachyons in the black hole. Demanding that a state in the black hole be well behaved at the horizon implies that it must tend asymptotically to a combination of a Seiberg and an anti-Seiberg c = 1 state. For such a state to be well behaved asymptotically, it must satisfy the condition that neither its Seiberg nor its anti-Seiberg Liouville momentum is positive. Thus, although the free-field BRST cohomologies of the underlying SL(2, ℍ)/U(1) theory is the same as that of a c = 1 theory, the black hole spectrum is drastically truncated: There are no W∞ states, and only tachyons with x-momenta |p_{tach}| ≤ |m_{tach}| are allowed. In the Minkowski case only the static tachyon is allowed. The black hole is stable to the back reaction of these remaining tachyons, so they are good perturbations of the black hole, or “hair”. However, this leaves only 3 tachyonic hairs in the black hole and 7 in the exact solution! Such sparse hair is clearly irrelevant to the maintenance of coherence during black hole evaporation.

*Work supported in part by the US-Israel Binational Science Foundation, and the Israel Academy of Science. E-Mail: NEIL@HALO.TAU.AC.IL, YARONOZ@CCSG.TAU.AC.IL
1 Introduction

Black hole solutions of two-dimensional string theory [1–3] have attracted much attention in the last couple of years. Black holes are very interesting since they provide an arena for studying the interplay between gravity and quantum theory. Since string theories can be solved, at least in principle, understanding black holes in string theories could shed some light at the quantum level on deep issues such as singularities, the coherence of black hole evaporation, no hair theorems and cosmic censorship. Evidently, our real interest is in four dimensions and the hope is that the two-dimensional case is rich enough to teach us something about that.

The two-dimensional black holes are constructed using the fact that the \( SL(2, \mathbb{R})/U(1) \) gauged WZW theory at level \( k = 9/4 \) is a good background to the bosonic string [1]. The usual black hole spacetime can be found from the semi-classical approximation to this coset theory [1], or by solving the \( \beta \)-function equations to lowest order in \( \alpha' \) [2] in the sigma-model approach to strings in curved spacetime. An exact dilaton-graviton background has also been proposed by Dijkgraaf et al. by expressing the \( L_0, \bar{L}_0 \) Virasoro generators of the \( SL(2, \mathbb{R})/U(1) \) gauged WZW theory as differential operators on the \( SL(2, \mathbb{R}) \) group manifold [3], and it has been verified that this background solves the \( \beta \)-function equation perturbatively to 4 loops [4, 5]. This spacetime also has the essential properties of a black hole.

Since the black hole is a 2-dimensional solution to the string, it is not surprising that there are many relations between it and the \( c = 1 \) theory: First, the black hole tends asymptotically to a flat space with a linear dilaton, which is the target space of the \( c = 1 \) theory without a cosmological constant. In addition, the \( c = 1 \) theory can be perturbed to the black hole by the anti-Seiberg \( W_{1,0} \) discrete state. In fact, using a free-field representation of the current algebra, the spectrum of the \( SL(2, \mathbb{R})/U(1) \) gauged WZW theory has been found to be in a one-to-one correspondence with the \( c = 1 \) theory [6], with infinitely many tachyon and \( W_\infty \) states. (There is also a calculation directly in the Kac-Moody module which has extra states [7].) This correspondence has led to the expectation that the black hole hole has an infinite amount of “\( W \)-hair”, giving it infinitely many conserved quantum numbers, and that this \( W \)-hair can maintain quantum coherence during the evaporation of the black hole [8].

However, here the \( c = 1 \) theory should provide us with a warning: The BRST free-field cohomology is not necessarily the same as the spectrum of the theory. Thus, when the cosmological constant \( \mu \) is turned on, it is accepted that the spectrum of the \( c = 1 \) theory is restricted to only the Seiberg states [9], with anti-Seiberg states such as the black-hole
operator destroying the background. In the black hole, one might expect the truncation of the spectrum to be much stronger because of the classical “no-hair theorems” of black holes. By exactly solving the linearized equations of motion of the tachyon and $W_\infty$ states in the background of the black hole, we shall see that the black hole actually has no $W$–hair, and only very few tachyonic hairs, depending on exactly which version of the theory is being considered.

We should point out that perturbing the black hole solution by static tachyons in the linearized approximation sigma-model approach has been studied by various authors[10–12]. It has been claimed that the black hole structure is changed due to the perturbation— that the horizon is split into two yielding a structure similar to Reissner-Nordstrom black hole[10, 12], and that a curvature singularity develops at the horizon[12]. What is common to all these works is that the equation of motion for the tachyon in the dilaton-graviton black hole background was not completely solved, and that the study of the back reaction was based on a tachyon configuration that diverges at the horizon. Since this is an invalid solution, we disagree with their results. The stability of the black hole against tachyon perturbation was studied in a somewhat different context in [13]. We disagree with their result concerning the change of the radius of the cigar shaped black hole manifold due to the tachyon perturbation.

The paper is organized as follows: In section 2 we review the basics of the two-dimensional string black hole, making comments and establishing conventions. In section 3 we solve the equation describing a static tachyon in the approximate black hole background. We discuss the appropriate boundary conditions to impose asymptotically and at the horizon of the black hole, and see that the tachyon asymptotically becomes a mixture of the zero-momentum tachyon and the “cosmological constant” operator $\varphi e^{-\sqrt{2}\varphi}$ of the 2D string. We then study its back reaction on the dilaton-graviton system, and verify the stability of the black hole. All these calculations are performed in the linear-dilaton gauge, and are repeated in the conformal gauge in the appendix. In section 4 we study tachyons with non-zero momenta. We find that in the Minkowski black hole only static tachyons can exist, and in the Euclidean black hole only the 3 tachyons with momenta $|p_{\text{tach}}| \leq |m_{\text{tach}}|$ are well behaved. We show that the black hole is again stable to the back-reaction of these tachyons. In section 5 we study the massive $W_\infty$ states in the black hole background, after making an ansatz as to how to derive them from the asymptotic $c = 1$ states. We solve their equations of motion and see that no such states can exist in the black hole, so the black hole has no $W$–hair. In section 6 we analyse the tachyon and $W$–hair in the exact dilaton-graviton background, finding similar results to those of the black-hole, except that there are now 7 allowed tachyons with $|p_{\text{tach}}| \leq |m_{\text{tach}}|$ in the
2 Review of the black hole in 2 dimensions

2.1 The sigma-model

The chiral sector of $c = 1$ string theory describes states with ghost numbers ranging from $-1$ to $+1$ [14–16]. Those with ghost number 0, with which we shall be concerned, are the tachyons which can have an arbitrary $x$-momentum, and an infinite number of discrete states “$W_\infty$” states at fixed $x$-momentum [16,17]. Both the tachyons and the discrete states come in “Seiberg” and “anti-Seiberg” versions. Precisely which momenta (and windings) of the tachyon and which discrete states survive to the full theory depend on the radius of compactification of the $x$-coordinate of the theory [18].

There are two approaches to constructing spacetime field theories for this—or any other—string theory; string field theory (SFT) or low-energy effective actions. The SFT approach contains the infinite number of fields of the higher-dimensional strings with a large gauge-invariance, but has the disadvantage that it describes all these fields perturbatively. It is thus hard to see non-trivial geometries in this language. The effective action approach to the string is somewhat orthogonal to that of the SFT: here one considers the theory as an expansion in $\alpha'$ (set to 2 in our notation), where it should be dominated by the massless fields. The massless discrete states are then represented by a metric $G_{\mu\nu}$ and a dilaton field $\Phi$, which can be large compared to their flat-space values. (There is no perturbative axion field, since we are in two dimensions.) One can also consider the tachyonic and massive states in this framework, but again only as perturbations. This procedure has been partially carried for the tachyon, which is described by a tachyon field $T(x, \varphi)$ as in the SFT. We shall postpone a discussion of the discrete states to section 5.

To lowest order in $\alpha'$, the sigma-model action of the dilaton-graviton-tachyon system is given by [19,20]

$$S = \int d^2 x \ e^{-2\Phi} \sqrt{G} \left( R - 4(\nabla \Phi)^2 + (\nabla T)^2 + V(T) + c/3 \right) , \quad (2.1)$$

and the equations of motion for $G_{\mu\nu}$, $\Phi$ and $T$ are

$$R_{\mu\nu} - 2 \nabla_\mu \nabla_\nu \Phi + \nabla_\mu T \nabla_\nu T = 0 ,$$

$$R + 4(\nabla \Phi)^2 - 4 \nabla^2 \Phi + (\nabla T)^2 + V(T) + c/3 = 0 , \quad (2.2)$$

$$\nabla^2 T - 2 \nabla \Phi \nabla T - \frac{1}{2} V'(T) = 0 ,$$
respectively. In these formulae, $c$ is the central charge ($c = D - 26 = -24$) and $V(T)$ is the potential of the tachyon. The potential begins with the mass term $V = -2T^2$, but the existence of higher-order terms has been the subject of some controversy [21]. We shall restrict ourselves to considering only weak tachyon fields with $T^2 \ll T^3$, so the exact form of the higher-order terms shall not affect us.

2.2 The black hole

The black hole is a solution of the equations of motion (2.2) with the tachyon field set to zero. We shall consider these equations in two convenient coordinate systems: the conformal and the linear-dilaton gauges. The linear-dilaton gauge is defined by taking the dilaton to be proportional to a Liouville-like coordinate $\phi$:

$$\Phi = -\sqrt{2} \varphi ,$$

(2.3)

where we have inserted the factor of $-\sqrt{2}$ to match the background charge of the dilaton in the $c = 1$ string theory, $Q/2 = -\sqrt{-c/12} = -\sqrt{2}$ [22]. We have also chosen an $x$-coordinate so that $G_{x\varphi} = 0$, used the freedom to redefine $x \rightarrow x'(x)$ to make $\partial_x$ the Killing vector of the metric, and normalized $x$ so that the solution reduces to the flat background of the $c = 1$ theory in the weak-coupling limit $\varphi \rightarrow \infty$. Then one finds a unique one-parameter family of solutions to eqs. (2.2) with the tachyon field set to zero, given by the linear dilaton of eq. (2.3) and the metric [2] (see also section 3.2):

$$ds^2 = \frac{1}{1 - ae^{-2\sqrt{2}\varphi}} d\varphi^2 \pm \left(1 - ae^{-2\sqrt{2}\varphi}\right) dx^2 .$$

(2.4)

(Here the $+$ sign gives the Euclidean solution, and the $-$ sign the Minkowski one.) The parameter $a$ is the ADM mass of the black hole [1], and one recovers the usual linear dilaton solution of the $c = 1$ theory (without the cosmological constant) by setting $a$ to zero. The mass can usually be changed by a constant shift of the dilaton field, but here we have fixed this freedom by our definition of the Liouville coordinate in eq. (2.3).

We should stress that, as pointed out in ref. [1], the time coordinate in the black hole must be taken to be $x$, and not $\phi$ as is usually done in the Liouville form of the $c = 1$ string. This means that only the Euclidean version of eq. (2.4) can be compared to the $c = 1$ theory. Although we shall often consider the Euclidean and Minkowski metrics of eq. (2.4) together, and shall use the same notation and nomenclature for the two cases, one should bear in mind that their spacetimes are very different. The Minkowski metric is indeed that of a black hole, as can be seen by comparing the structure of its zeroes and infinities.
to those of the standard four-dimensional black hole in Schwarzschild coordinates: In terms of the coordinate

\[ z \equiv a e^{-2\sqrt{2}\phi} , \]  

(2.5)

which we shall use extensively in this paper, the asymptotic region of the black hole is given by \( z \to 0 \), the horizon is at \( z = 1 \), and the singularity is at \( z = \infty \). The “Euclidean black hole” has the same asymptotic behaviour, but the horizon at \( z = 1 \) becomes simply a coordinate singularity where the spacetime ends, and the full space is given by the “cigar” of Witten [1].

These structures can be seen more clearly in the “conformal gauge”, where the metric is taken to be proportional to the Euclidean/Minkowski metric. In the Euclidean case this coordinate system is obtained by defining

\[ u \equiv \sqrt{\frac{(1 - z)}{2z}} e^{i\sqrt{2}x} . \]  

(2.6)

Then the metric and the dilaton become [2]

\[ ds^2 = \frac{du\,d\bar{u}}{1 + 2u\bar{u}} \quad \text{and} \]

\[ \Phi = -\frac{1}{2} \log a - \frac{1}{2} \log (1 + 2u\bar{u}) , \]  

(2.7)

(2.8)

so the free parameter in this gauge is a constant shift of the dilaton. Note that \( z \) indeed ranges only from 0 to 1, since it is now defined by

\[ z = \frac{1}{1 + 2u\bar{u}} . \]  

(2.9)

The asymptotic region is again \( z \to 0 \), but now the “horizon” at \( z = 1 \) is simply the origin of the complex \( u \) plane. This leads to an important feature of the Euclidean black hole: In order to avoid a conical singularity at the origin, \( z \) must be well defined. From its definition in eq. (2.6) this means that \( \sqrt{2}x \) is an angular coordinate, so \( x \) must be compactified on a circle of radius \( R = 1/\sqrt{2} \). From the point of view of the asymptotic \( c = 1 \) theory, this means that \( x \) is compactified with one-half of the self-dual radius! The black hole thus constrains the radius of \( x \), which would be arbitrary in the Liouville theory. A similar result was found by Witten in the gauged WZW formulation of the black hole, where the radius was found to be \( \sqrt{k'/2} \) for large \( k' \) (which is unfortunately equal to 1/4). In our case, as well, the radius 1/2 is also only valid to lowest order in \( \alpha' \). This should be borne in mind when comparing the black hole states to those of the \( c = 1 \) theory. The fact that we are at 1/2 the self-dual radius means that states in the black hole
can only have $x$-momenta* which are integral multiples of $\sqrt{2}$, and so describe only the integral-spin $W_\infty$ states of Polyakov [17]. However in the exact BRST cohomologies of the gauged WZW theory [7, 6] and in the metric of ref. [3], which is supposed to describe the theory to all orders of $\alpha'$, the radius of $x$ is $R = \sqrt{2k} = 3/\sqrt{2}$, or 3 times the black-hole value. We shall return to the issue of the exact metric in section 6.

To get the Minkowski-space black hole, one Wick rotates $x \rightarrow ix$. Then $\bar{u}$ becomes independent of $u$, and should rather be called $v$, and $u$ and $v$ become null coordinates. In this form, one can see that the Minkowski metric indeed has the Penrose diagram of a black hole. $z \rightarrow 0$ is again the asymptotic region of the space. $z = 1$ now means $uv = 0$, and so gives the horizon of the black hole. There is clearly no singularity of the metric or dilaton at the horizon, so $z$ is no longer restricted to be $\leq 1$. In fact the metric can be continued up to $z = \infty$, which is the singularity of the black hole. In the Minkowski case, there is no longer any compactification of $x$, but the fact that it is compactified in the Euclidean case implies a temperature of the Minkowski black hole of $1/\sqrt{2}\pi$, or $1/3\sqrt{2}\pi$ for the exact metric.

3 The static tachyon in the black hole background

3.1 The tachyon and its boundary conditions

In order to solve for the tachyon in the black-hole background, it is useful to write its equations of motion from eqs. (2.2) in terms of the coordinates $z$ and $x$. The equation then becomes

$$z(1-z)T'' - zT' + \frac{1}{4z}T \pm \frac{1}{8z(1-z)}\ddot{T} = 0,$$

(3.1)

with primes denoting derivatives with respect to $z$, and dots derivatives with respect to $x$. Since this equation is linear and $x$ is a Killing direction of the metric, one can Fourier expand the tachyon in terms of tachyons with fixed $x$-momenta. We shall consider a general tachyon in section 4. It is instructive to first consider the static tachyon $T_{(0)}$ in detail. In the static case the Minkowski and Euclidean equations are the same, and if one lets $T_{(0)} \rightarrow \sqrt{z} F(z)$, they reduce to the hypergeometric equation:

$$z(1-z)F''(z) + (1-2z)F'(z) - \frac{1}{4}F(z) = 0.$$

(3.2)

*The winding states of the $c = 1$ theory can not be seen in the effective theory.
Thus two linearly independent solutions of the tachyon equation (3.1) are\footnote{In this case the two solutions are equal to $2/\pi \sqrt{z}$ times the complete elliptic functions $K(\sqrt{z})$ and $K'(\sqrt{z})$, respectively.} [23]:

\begin{align}
T^a_{(0)}(z) &= \sqrt{z} F\left(\frac{1}{2}, \frac{1}{2}, 1, z\right), \\
T^b_{(0)}(z) &= \sqrt{z} F\left(\frac{1}{2}, \frac{1}{2}, 1, 1-z\right), \tag{3.3}
\end{align}

with $F(\alpha, \beta, \gamma, z)$ is the standard $\, _2F_1$ hypergeometric function.

Asymptotically, the first solution $T^a_{(0)}(z)$ behaves as

\begin{equation}
T^a_{(0)} \xrightarrow{z \to 0} \sqrt{a} e^{-\sqrt{2}\varphi} + O(e^{-3\sqrt{2}\varphi}). \tag{3.4}
\end{equation}

This is the standard discrete zero-momentum tachyon which has been studied as a perturbation of the black hole solution in [10]. However, at the horizon it becomes

\begin{equation}
T^a_{(0)} \xrightarrow{z \to 1} -\frac{1}{\pi} \log\left(\frac{1-z}{16}\right) + \text{vanishing terms}, \tag{3.5}
\end{equation}

and so blows up. (It also has a cut from 1 to $\infty$ in the Minkowski black hole.) Since $T$ is a scalar quantity, one would expect that this divergence should invalidate the solution. We shall indeed see that this tachyon induces a singularity in the metric when we study the back-reaction of the tachyon on the black hole in section 3.3 and in the appendix. $T^a_{(0)}$ is therefore not a legitimate background for the tachyon in the metric of the black hole. Even if such a solution were meaningful in the full theory, with higher order terms in the potential stabilizing the tachyon, it could not be studied in our approximation of neglecting higher powers of the tachyon field.

We are thus left with the second solution $T^b_{(0)}(z)$. More precisely, we define

\begin{equation}
T_{(0)} = t_0 \sqrt{z} F\left(\frac{1}{2}, \frac{1}{2}, 1, 1-z\right), \tag{3.6}
\end{equation}

where the constant $t_0$ should be small, since we are considering a weak tachyon, and since we have solved for the tachyon in the background of the unperturbed black hole. This tachyon is a monotonically increasing function of $z$, and is well behaved at the horizon, where it tends to $t_0$. Asymptotically, it behaves as

\begin{equation}
T_{(0)} \xrightarrow{z \to 0} -\frac{\sqrt{a} t_0}{\pi} e^{-\sqrt{2}\varphi} \left(\log \frac{a}{16} - \sqrt{8} \varphi\right) + O(\varphi e^{-3\sqrt{2}\varphi}); \tag{3.7}
\end{equation}

it is thus a mixture of the usual zero-momentum tachyon and the “cosmological constant” operator of the $c = 1$ theory. We thus see that choosing a sensible boundary condition
at the horizon leads one to a combination of the two possible boundary conditions in the asymptotic region of the black hole. We shall see that this behaviour is generic, and occurs also for non-zero momenta tachyons and for the discrete states. In previous works it has been argued that the physical tachyon is not $T$, but $S \equiv e^{-\Phi}T$ [10]; for our tachyon, $S$ behaves asymptotically as $\varphi$, so it would not be considered a valid solution. However, looking at the sigma-model equations of motion of eq. (2.2), one sees that $T$ and not $S$ is the physical quantity, and we shall indeed see that the back reaction of $T(0)$ is sensible.

In the Euclidean black hole, the spacetime stops at $z = 1$. In the Minkowski case, one has to continue to the singularity of the black hole, at $z \to \infty$. For large $z$, $T(0)$ behaves as

$$T(0) \xrightarrow{z \to \infty} \frac{t_0}{\pi} \log(16z),$$

so our demand that the tachyon remain weak, so that $T^2 \ll T^3$, restricts the validity of the tachyon solution to $z \ll \exp(\pi/t_0)/16$. For $t_0$ small, this means that the tachyon solution is valid to near the singularity of the black hole, although one can never actually reach the singularity.

### 3.2 The solution to the dilaton-graviton equations with tachyons

In order to find the back reaction of the tachyon on the metric we need to return to the full equations of motion of the theory, as given in eqs. (2.2). Using the 2-dimensional identity

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R,$$

the graviton and dilaton equations can be rewritten as the traceless equation

$$g_{\mu\nu} \nabla^2 \Phi - 2 \nabla_\mu \nabla_\nu \Phi = \frac{1}{2} g_{\mu\nu}(\nabla T)^2 - \nabla_\mu T \nabla_\nu T,$$

the dilaton equation

$$\nabla^2 \Phi - 2(\nabla \Phi)^2 = -T^2 - 4,$$

and the curvature scalar equation

$$R = 4(\nabla \Phi)^2 - 8 - 2T^2 - (\nabla T)^2.$$

In the linear-dilaton gauge, the dilaton is again defined by $\Phi = -\sqrt{2} \varphi$; also noting the form of the black hole metric (2.4), it is convenient to parameterize the metric by [10]

$$ds^2 = \frac{1}{f} d\varphi^2 \pm f \, dh \, dx^2.$$
Then, the two independent components of eq. (3.10) become

\[
\frac{h'}{h} = -2z T'^2 + \frac{1}{4zf^2h} \dot{T}^2 \quad \text{and} \\
\frac{f'}{f} = 2z \dot{T} T',
\]

and eq. (3.11) becomes

\[
f - zf' = 1 + \frac{T^2}{4} - z^2 f T'^2 + \frac{1}{8fh} \dot{T}^2.
\]

(3.15)

The curvature scalar equation (3.12) becomes

\[
R(f, h) = 8f - 8 - 2 T^2 - 8z^2(1 - z) T'^2 \mp \frac{\dot{T}^2}{1 - z},
\]

but one does not need its explicit form, since it follows from the Bianchi identities of the theory. Similarly, consistency between the equations for \( \dot{f} \) and \( f' \) imply the tachyon equation of motion of eq. (2.2).

If one sets the tachyon to zero, one immediately sees that \( f_0 = 1 - z \) and \( h_0 = 1 \) are solutions to the equations of motion, giving us the black-hole metric of eq. (2.4). Since we are considering weak tachyon fields (having solved for the tachyon in the background of the black hole without matter), we can substitute \( \dot{f} \to f_0 \) and \( h \to h_0 \) in the RHS of eqs. (3.14) and (3.15), and solve them perturbatively. Thus

\[
h = 1 + \int^z dz \left( -2z T'^2 + \frac{1}{4z(1 - z)^2} \dot{T}^2 \right) + A(x) \quad \text{and} \\
f = 1 - z + z \int^z dz \left( -\frac{1}{4z^2} T^2 + (1 - z) T'^2 - \frac{1}{8z^2(1 - z)} \dot{T}^2 \right) + zB(x)
\]

(3.17)

\[
= 1 - z + 2z(1 - z) \int^x dx \left( \dot{T} T' \right) + C(z).
\]

The arbitrary function \( A(x) \) in \( h \) can be fixed by demanding that \( h \to 1 \) asymptotically. Also, equating the two solutions for \( f \) fixes \( B(x) \) and \( C(z) \) up to an arbitrariness of the form \( f \to f + \alpha z \); this is a shift in the mass \( a \) of the original black hole, and has no physical meaning.

### 3.3 The back reaction of the static tachyon

The back reaction of the tachyon on the black hole is not linear, since the tachyon appears quadratically in the graviton and dilaton equations of motion. It is thus necessary
to calculate the back-reaction of a tachyon which consists of a sum of fields with different momenta. This shall be done in section 4.2. Here we shall concentrate on the back reaction of a static tachyon, in order to understand the physics of the back reaction. In this section we carry out the calculation in the linear dilaton gauge; the calculation is repeated in the conformal gauge in the appendix, where the physical picture is somewhat clearer. The reader with a particular partiality to either gauge may concentrate on the appropriate section.

Knowing the induced metric, we can now prove our statements on the boundary conditions of the tachyon that we asserted in the previous section. In the static case the metric does not depend on $x$, which is still a Killing direction, and the last equation in eq. (3.17) carries no information. The other integrations in eqs. (3.17) can be carried out explicitly, using the tachyon equation of motion of eq. (3.1). This gives

$$f = (1 - z) \left(1 + z T_{(0)} T'_{(0)}\right),$$

$$h = 1 + c - \frac{1}{2z} T^2_{(0)} + 2(1 - z) T_{(0)} T'_0 - 2z(1 - z) T^2_{(0)}.$$  \hspace{1cm} (3.18)

Now, using eq. (3.16) for the curvature scalar, one sees that the back reaction of a tachyon containing the “bad” solution $T_{(0)}^a$ of eq. (3.3) leads to a curvature singularity of the form

$$R \sim \frac{1}{1 - z}$$  \hspace{1cm} (3.19)

at the horizon. Thus, as was expected from its divergent behaviour in eq. (3.5), $T_{(0)}^a$ is not a valid solution in the black-hole background.

We thus see that one must indeed restrict oneself to our tachyon of eq. (3.6) which is well behaved at the horizon. In this case eqs. (3.18) become:

$$f = (1 - z) \left\{1 + \frac{t_0^2}{2} z F\left(\frac{1}{2}, \frac{1}{2}, 1, 1 - z\right) \times \right.$$ \hspace{1cm} (3.20)

$$\left(F\left(\frac{1}{2}, \frac{1}{2}, 1, 1 - z\right) - \frac{1}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, 1 - z\right)\right)\right\},$$

$$h = 1 + \frac{2t_0^2}{\pi^2} \frac{t_0^2}{2} z F^2\left(\frac{1}{2}, \frac{1}{2}, 1, 1 - z\right) - \frac{t_0^2}{8} (1 - z) F^2\left(\frac{1}{2}, \frac{1}{2}, 2, 1 - z\right).$$

One can now see that this resulting metric still has the structure of a black hole*: Asymptotically $f(0) = h(0) = 1$, so one has the usual flat metric. $h(z)$ is a monotonic function that slowly decreases from 1 to $h(1) = 1 - t_0^2(1/2 - 2/\pi^2) > 0$. $f(z)$ increases briefly from 1 to a maximum, from which it decreases to vanish at $z = 1$. (The arbitrariness in $f(z)$

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*See also the appendix.
was fixed so that the position of the zero of \( f(z) \) was not perturbed.) This means that \( z = 1 \) is still a horizon of the metric. In the Minkowski case, one must still go beyond the horizon: For \( z > 1 \), \( f(z) \) is negative and is never again zero, so it does not produce any further structure. This means that, contrary to the claims in refs. [11, 12], the horizon is not split and the metric does not have a Reissner-Nordstrom form. Far beyond the horizon one could worry about the new singularity of the metric where \( h(z) \) becomes zero, at \( z \sim \exp(\pi^2/t_0^2)/16 \). However, this is beyond the limit of our quadratic tachyon approximation (see the discussion after eq. (3.8)). In any case the curvature scalar is nonsingular there, and this is only a coordinate singularity. We conclude that only the original singularity \( z \to \infty \) remains.

It thus appears that the perturbed metric differs very slightly from the original black-hole metric, with the same asymptotic behaviour, horizon and singularity structure. However there is one crucial difference: In the linear dilaton gauge the ADM mass \( M \) of the black hole is defined so that \( f(z) \to 1 - M e^{-2\sqrt{2}\varphi} \) asymptotically. Recalling the definition of \( z \) in eq. (2.5), one sees that the original black hole with \( f = 1 - z \) has mass \( a \). In the perturbed black hole with

\[
f(z) \xrightarrow{z \to 0} 1 - z \left(1 - \frac{t_0^2}{4\pi^2} \left(\log^2 \frac{z}{16} + 2 \log \frac{z}{16}\right)\right),
\]

(3.21)

one sees that the tachyon has shifted the mass of the black hole to minus infinity! While this would presumably be disastrous in four dimensions, it appears to create no troubles in the dilaton-graviton theory.

We conclude that a static tachyon can live in the back hole background, thus giving hair to the black hole.

4 Moving Tachyons

4.1 The tachyon and its boundary conditions

We now turn to a general tachyon with momentum \( p \), so that \( T \sim T_{(p)}(z) e^{\sqrt{2}i\varphi x} \). (The \( \sqrt{2} \) is for convenience.) The tachyon equation of motion (3.1) now becomes

\[
z(1 - z) T''_{(p)} - z T'_{(p)} + \frac{1}{4z} T_{(p)} + \frac{p^2}{4z(1 - z)} T_{(p)} = 0.
\]

(4.1)

We have seen that the solutions are constrained by their behaviour at the horizon \(( z \to 1)\). In the Minkowski case (the positive sign in eq. (4.1)), one can see that the two possible behaviours of the tachyon there are \( T_{(p)} \sim (1 - z)^{\pm p/2} \). Such tachyons correspond to
the “principal continuous series” representation of the $SL(2, \mathbb{R})$ algebra in refs. [3, 7]. However these solutions oscillate wildly around $z = 1$ and are not continuous there, so they must be discarded. We immediately conclude that only static tachyons can exist in the Minkowski black hole.

In the Euclidean case, the solutions of eq. (4.1) behave as a linear combination of $(1 - z)^{+p/2}$. This means that the generic solution blows up there, and one obtains a unique solution by demanding that the tachyon be well defined at the horizon. Its form is found by letting $T_{(p)}(z) \rightarrow z^{(1+|p|)/2} (1 - z)^{|p|/2} F(z)$, transforming eq. (4.1) into the hypergeometric equation

$$z(1-z) F''(z) + (1 + |p|) (1 - 2z) F'(z) - \frac{1}{4} (1 + 2|p|)^2 F(z) = 0 .$$

The tachyon that is well behaved at the horizon is then given by [23]

$$T_{(p)}(z) = t_p \frac{z^{1+|p|}}{2} (1 - z)^{\frac{|p|}{2}} F\left(\frac{1}{2} + |p|, \frac{1}{2} + |p|, 1 + |p|, 1 - z\right).$$

Its asymptotic behaviour can be seen by using the transformation property of the hypergeometric function:

$$F(\alpha, \beta, \gamma, 1 - z) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} F(\alpha, \beta, \alpha + \beta - \gamma + 1, z) +$$

$$z^{\gamma - \alpha - \beta} (1 - z)^{1 - \gamma} \frac{\Gamma(\gamma) \Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha) \Gamma(\beta)} F(1 - \alpha, 1 - \beta, \gamma - \alpha - \beta + 1, z),$$

turning $T_{(p)}$ into

$$T_{(p)}(z) = - \frac{t_p}{\sin \pi |p|} z^{\frac{1+|p|}{2}} (1 - z)^{\frac{|p|}{2}} F\left(\frac{1}{2} + |p|, \frac{1}{2} + |p|, 1 + |p|, 1 - z\right) +$$

$$t_p \frac{\Gamma(1 + |p|) \Gamma(|p|)}{\Gamma^2 \left(\frac{1}{2} + |p|\right)} z^{\frac{1-|p|}{2}} (1 - z)^{-\frac{|p|}{2}} F\left(\frac{1}{2} - |p|, \frac{1}{2} - |p|, 1 - |p|, z\right).$$

One can then see that asymptotically the tachyon has the form:

$$T \overset{z \to 0}{\longrightarrow} A e^{-\sqrt{2}(1+|p|) \varphi} e^{\sqrt{2}ipx} + B e^{-\sqrt{2}(1-|p|) \varphi} e^{\sqrt{2}ipx},$$

for some non-zero constants $A$ and $B$, and so becomes a linear combination of the anti-Seiberg and Seiberg tachyons of the $c = 1$ theory. This immediately leads us to our most important result for tachyons with momentum: If $|p| > 1$ the Seiberg component of the tachyon blows up asymptotically, so there is no tachyon solution that is well-behaved both asymptotically and at the horizon.
The discrete tachyons at $p = \pm 1$ need to be studied separately; they are interesting both because they are on the boundary of the allowed range of momenta, and because if one indeed takes the radius of $x$ to be $1/\sqrt{2}$, as suggested by the metric, only tachyons with $p = 0$ and $p = \pm 1$ remain as viable solutions on the black hole. The $p = \pm 1$ tachyons are given by eq. (4.3):

$$T_{(\pm 1)} = z\sqrt{1-z} F\left(\frac{3}{2}, \frac{3}{2}, 2, 1-z\right).$$

Their asymptotic behaviour can not be found from eq. (4.5), which breaks down, but by examining its limit one can see that they tend to $4/\pi$ asymptotically, and so are well defined.

It is thus reasonable to conclude that only tachyons with $|p| \leq 1$ can exist in the Euclidean black hole; we shall immediately confirm this by studying the back reaction of a general tachyon.

### 4.2 Back reaction

In general, a moving tachyon breaks the black hole killing symmetry and introduces an explicit dependence of the metric (3.13) on $x$. The metric is still given by eqs. (3.17), where $T$ is now a linear combination of the tachyons of eq. (4.3) with different momenta:

$$T = \sum_p T_{(p)}(z) e^{\sqrt{2}i p x}.$$  

(4.8)

The integrations for $f$ can again be carried out explicitly, giving

$$f = (1-z) \left( 1 + z \sum_p T_{(p)} T'_{(-p)} + z \sum_{p+q\neq 0} \frac{p T_{(p)} T'_{(q)} + q T_{(q)} T'_{(p)}}{p+q} e^{\sqrt{2}i(p+q)x} \right).$$  

(4.9)

$f$ clearly vanishes at the horizon, and asymptotically its $(p, q)$ term is proportionate to $T_{(p)} T_{(q)}$. The $h$ integration in eq. (3.17) can not be done explicitly, but one can see that $h$ is finite at the horizon and has the same asymptotic behaviour as $f$. Using eq. (3.16), one can also see that the curvature scalar $R$ is finite at the horizon, and has the same asymptotic behaviour as $f$. Thus if the tachyon has components with momenta $|p| > 1$, so that $T_{(p)}$ blows up asymptotically, the resulting metric is sick. For tachyons with momenta $|p| \leq 1$, one sees that the modified metric is still that of a black hole—it is asymptotically flat as $z \to 0$, and has a horizon at $z = 1$.

We conclude that tachyons with $|p| > 1$ do not exist on the black hole. In the Euclidean black hole (the 3) tachyons with momenta $|p| \leq 1$ are good solutions, perturbing the black hole and giving it “hair”. The Minkowski black hole supports only the static tachyon.
5 Discrete states

5.1 The BRST cohomology of the $SL(2, \mathbb{R})/U(1)$ theory

Having examined tachyons in the black-hole metric in some detail, we would now like to consider what happens to the other states in the black hole. Since we are thinking of the black hole as the solution of a string theory, one would expect to have an infinite number of massive states arising from the mode-expansion of the string, and since the theory is two-dimensional and is intimately related to the $c = 1$ string, one would expect these states to be discrete states at fixed momenta. Studying these states is not a trivial extension of the tachyonic case, since the approach of coupling fields to the sigma-model effective action of the string that we used there has not been developed for massive states. We shall thus have to use a more indirect approach.

One might expect that the correct calculation to do would be to actually find the states of the $SL(2, \mathbb{R})/U(1)$ CFT underlying the black hole. In fact, the complete analysis of the BRST cohomology of the $SL(2, \mathbb{R})/U(1)$ CFT has already been carried out. This has been done in two different approaches: First, the current algebra can be bosonized, using the Wakimoto free-field representation of the $SL(2, \mathbb{R})$ algebra [24]. The bosonization of the $U(1)$ leads to an $x$ coordinate of radius $R = \sqrt{2k} = 3/\sqrt{2}$, and the bosonization of the $SL(2, \mathbb{R})$ currents lead to a Liouville-like field $\varphi$ with the appropriate background charge. The operators in the theory are then very similar to those of the $c = 1$ theory with radius $R = 3/\sqrt{2}$, and one can show that the spectra of the two theories are identical [6]. In the figure we show the states in one chiral sector of the theory with the same ghost number as that of the tachyon. The other approach is to carry out the analysis directly in the current algebra [7]. In that case $p_x$ and $p_\varphi$ are defined in terms of the $J$ and $M$ of the $SL(2, \mathbb{R})$ algebra. Here one finds that there are extra states in addition to those of the bosonized theory; these are also shown in the figure.

It would thus appear that we know which massive states appear in the black hole. However, there are many issues to settle. The first is which (if either) of the two approaches discussed above is relevant to the black hole. Here it may be appropriate to recall another case where this issue has arisen—in Polyakov’s light-cone gauge approach to the $c = 1$ string [25]. In this gauge, the $c = 1$ theory is written as a matter field $x$ coupled to a (somewhat strange) $SL(2, \mathbb{R})/U(1)$ theory. It was found there that the current-algebra analysis again leads to the presence of extra states in the theory [26]; since this theory should be equivalent to the Liouville theory in the conformal gauge, this strongly suggests that the bosonized approach is preferred. It may also be worth noting that at the first
Figure 1: The free-field and Kac-Moody chiral cohomology states of the $SL(2, \mathbb{R})/U(1)$ current algebra at $k = 9/4$, split into Seiberg and anti-Seiberg states by the line through $p_\varphi = -1$. The discrete states common to both cohomologies are denoted by large circles $\odot$ for the integral momenta states, and “blobs” $\bullet$ for states with half-integral momenta. The extra discrete states of the Kac-Moody cohomology are depicted by small circles $\circ$. Tachyons with no windings are depicted by large circles for integral momenta, and blobs for momenta in $\mathbb{Z}/3$, and the pure-winding tachyons are depicted by plus signs. The box encloses those states for which neither their Seiberg and their anti-Seiberg Liouville momenta are positive.
level with $p = 0$, where the bosonized approach leads to one state and the Kac-Moody approach two, the string contains only the single “discrete graviton state” $\alpha_{\mu}^0 \tilde{\alpha}_{-1}$.

Even once one has decided which BRST cohomology is preferred, there are still clearly several differences between the states of the black hole and the spectrum of the $SL(2, \mathbb{R})/U(1)$ CFT. First, as we have already mentioned, the black hole solution has been derived only at the $O(1/k^2)$ approximation to the CFT, while the BRST analysis is exact. One implication of this is that the radius of $x$ in the black hole is $R = 1/\sqrt{2}$, instead of $R = 3/\sqrt{2}$, so states in the black hole have only “integral momenta” $p = n$, instead of $p = n/3$. One could, of course, also expect other changes to the spectrum. This problem will be rectified in our discussion of the exact $SL(2, \mathbb{R})/U(1)$ metric in section 6. Another problem that is basic to all effective space-time approaches to the string is that one can see only modes with momenta, and not the intrinsically “stringy” winding states. This is unavoidable, but may be more of a technical problem than a deep issue in the theory.

A more interesting problem, and one which we shall hope to examine in our space-time approach, is that the BRST cohomology analysis may simply not be directly relevant to the physical spectrum of the theory. In either approach the BRST analysis is carried out using a free field representation of the $SL(2, \mathbb{R})$ current algebra which, although useful for simplifying the analysis, obscures the underlying geometry of the theory. We have already seen that the BRST tachyons do not appear directly in the spectrum of the black hole, and that its tachyonic spectrum is far smaller than that indicated by the BRST analysis. Another example of such a situation is again the $c = 1$ string; here the BRST analysis leads to a doubling of the spectrum into Seiberg and anti-Seiberg states. It is now generally accepted that, while this may be true for the case of zero cosmological constant $\mu$, if $\mu \neq 0$ the anti-Seiberg states are not in the spectrum of the $c = 1$ theory, since they induce a catastrophic change to the background of the theory. For example, the $W^-_{1,0} W^-_{1,0}$ operator changes the $c = 1$ theory into the black hole!

5.2 Discrete states in a nontrivial metric

In the rest of this section, we shall thus attempt to analyse the discrete states in a way that is related to our analysis of the tachyon. The most direct way of doing this would be to find the physical states in the black hole from the BRST cohomology in the black hole background: $Q |\Psi\rangle = 0$, $|\Psi\rangle \equiv |\Psi\rangle + Q |\chi\rangle$. This is well defined since any background of the string leads to a nilpotent BRST operator. Equivalently, one could look at the equations of motion and gauge invariance of the linearized string field theory.
in this background. However, neither of these methods have been developed for string backgrounds that are more complicated than small perturbations of flat space. We shall thus use a poor-mans approach, using the simplifying fact that, asymptotically, the black hole reduces to the $c = 1$ background of a flat metric with a linear dilaton.

Let us thus briefly recall some facts about the discrete states of the $c = 1$ theory. We shall be concerned only with the relative cohomology states with the same ghost number as that of the tachyon, since only these can correspond to physical states in the spacetime. Since, as we have stated above, the space-time approach can not deal with winding states, this leaves us with only the Seiberg and anti-Seiberg states $W_{\infty}$ states:

$$\overline{W}_{J,M}^{+}W_{J,M}^{+}$$ and $$\overline{W}_{J,M}^{-}W_{J,M}^{-},$$

(5.1)

where, \textit{a priori}, $J$ is a positive integer or half integer, and $|M| < J$. However, if we define states with behaviour $e^{\sqrt{2}p_x\varphi}e^{\sqrt{2}ipx}$ to have $x$-momentum $p$ and “Liouville momentum” $p_\varphi$, one sees that the above states have $p = M$ and $p_\varphi = \pm J - 1$. Since we are at one half of the self-dual radius in the black hole, or at three halves the self-dual radius in the exact metric, $M$ and $J$ are restricted to being integral in our case. The states of eq. (5.1) can be built up using only $x$ oscillators. Since these states are in the BRST cohomology, and since they do not contain any ghost oscillators, they satisfy the Virasoro constraints

$$\overline{L}_n = L_n = 0$$ for $n \geq 1$ and $\overline{L}_0 = L_0 = 1$. On states without winding, this last equation becomes

$$p^2 + N_L - p_\varphi^2 - 2p_\varphi = 1,$$

(5.2)

where $N_L$ is the left-handed (say) level of the state measured from the tachyon, and its two solutions give the Seiberg and anti-Seiberg Liouville momenta

$$p_{S,AS} = -1 \pm \sqrt{p^2 + N_L}$$

(5.3)

d of the discrete state.

From a string field theory point of view, each of the discrete states comes from a linear combination of many fields. To give a concrete example, the discrete states at level 2 (which do not exist in our case since $J = 1/2$) are

$$\overline{W}_{2,\frac{1}{2}}^{+}W_{2,\frac{1}{2}}^{+} = \frac{1}{4} \left( \frac{i}{\sqrt{2}} \partial^2 x + (\partial x)^2 \right) \left( \frac{i}{\sqrt{2}} \partial^2 x + (\partial x)^2 \right) e^{\sqrt{2}(\frac{1}{2} - 1)\varphi} e^{\sqrt{2}(\frac{1}{2})x},$$

(5.4)

and the analogous $\overline{W}_{2,\frac{1}{2}}^{-}W_{2,\frac{1}{2}}^{-}$ states with $p = -1/2$. These states come from expanding the string field to level $2^*,$

$$\Psi \sim A_{\mu,\lambda} \alpha_{\mu} \alpha_{\lambda} + A_{\mu,\lambda} \alpha_{\mu} \alpha_{\lambda} + A_{\lambda,\mu} \alpha_{\lambda} \alpha_{\mu} + A_{\mu,\mu} \alpha_{\mu} \alpha_{\mu} + A_{\mu,\nu} \alpha_{\mu} \alpha_{\nu} + A_{\nu,\nu} \alpha_{\nu} \alpha_{\nu},$$

(5.5)

*Here, for simplicity, we ignore terms containing $b_{-1}c_{-1}$ or $\bar{b}_{-1}\bar{c}_{-1}$, since they do not appear in the discrete states.
and using the (linearized) BRST condition $Q\Psi = 0$ and the equivalence $\Psi \equiv \Psi + Q\chi$. This leave one with only the non-vanishing components

$$A_{xx,xx} \sim A_{xx,x} \sim A_{x,xx} \sim e^{\sqrt{2}(\pm \frac{1}{2} - 1)} e^{\sqrt{2i}(\pm \frac{1}{2})x}.$$ (5.6)

In principle, one would now like to carry out the same calculation in the black-hole background. Note that the $L_0 = 1$ equation of (5.2) can be written covariantly on each field $A_{\mu_1...\mu_j}$ of the string field as

$$\nabla^2 A_{\mu_1...\mu_j} - 2 \nabla \Phi \cdot \nabla A_{\mu_1...\mu_j} - m^2 A_{\mu_1...\mu_j} = 0,$$ (5.7)

where $m$ is the mass of the field, given by

$$N_L = 1 + \frac{m^2}{2}.$$ (5.8)

It is thus reasonable to take eq. (5.7) to be the correct equation of the theory in a nontrivial background, although this is by no means proven, since terms involving the Riemann tensor could also appear. It is encouraging, however, that no such terms have been found in the effective sigma model couplings of the tachyon [4]. Note that the covariant derivatives in eq. (5.7) mix the different components of the $A_{\mu_1...\mu_j}$’s if one has a non-trivial metric. This means that the discrete states in the black hole can not have only $x$-components nonvanishing, and must also have nonvanishing $\phi$- and mixed components, which should be suppressed asymptotically.

One could also covariantize the $L_n = 0$ equations, which should all be compatible, since the conformal invariance of the background ensures the existence of a Virasoro algebra. However, since the resulting BRST analysis would be very difficult, we shall rather think of eq. (5.7) as being the equation of motion of the theory in Siegel gauge, where all the fields of the string are decoupled, and the kinetic operator is simply $L_0 - 1$. This equation of motion can be derived from the effective sigma model action in this gauge:

$$S = \int d^2x e^{-2\Phi \sqrt{G}} \left( R - 4(\nabla \Phi)^2 + c/3 + \sum_{A'_{\phi}} \left( (\nabla A_{\mu_1...\mu_j})^2 + m^2 A_{\mu_1...\mu_j}^2 \right) \right),$$ (5.9)

where the sum is over all of the massive states of the string, as well as the tachyon. (The massless states are, of course, treated exactly in the original sigma-model terms, and are not included in the sum.)

Being in Siegel gauge means that one has lost all information about the gauge-invariance of the theory. We shall restore this information by making the ansatz that the physical states of the theory are found by solving the equation of motion of eq. (5.7),
and demanding that the solutions tend asymptotically to the discrete states of the $c = 1$ theory. Equivalently, one can take the discrete states in the cohomology of the asymptotic $c = 1$ theory and continue them into the black hole using eq. (5.7). It is reasonable to expect that this procedure does indeed reproduce the spectrum of the black hole. This is supported by the fact that the free-field $SL(2, \mathbb{R})/U(1)$ cohomology for the black hole is, at least formally, in a one-to-one correspondence with that of the $c = 1$ theory, so one would not expect this procedure to break down. However, as in the tachyon case, we shall see that the constraint that the resulting states be well-behaved over the entire black-hole spacetime will drastically truncate the spectrum.

5.3 The solution of the discrete states in the black hole

To find discrete states in the target space of the black hole, we now have to solve the equation of motion eq. (5.7) for the various fields $A_{\mu_1...\mu_j}$ of the string field theory, and demand that they asymptotically approach the $A_x$’s of the $c = 1$ discrete states. Since eq. (5.7) becomes a set of coupled second order differential equations in the $(x, \varphi)$ or $(x, z)$ coordinate system, we shall rather work in the conformal gauge defined by eqs. (2.7) and (2.8). Here, as is well known from using complex coordinates on Riemann surfaces, the covariant derivatives do not mix the $u$ and $\bar{u}$ components of fields, and indeed the only nonvanishing Christoffel symbols are

$$\Gamma_{uu}^u = -\frac{2\bar{u}}{1 + 2u\bar{u}}, \quad (5.10)$$

and its complex conjugate. Eq. (5.7) then splits into separate equations for each component of $A_{\mu_1...\mu_j}$, which we denote by $A_{u^n\bar{u}^m}$, somewhat symbolically since the order of the indices is important. The complication of using these coordinates is that, in order to match the asymptotic form of the discrete state to those of the $c = 1$ theory, one has to return to the $(x, \varphi)$ coordinates. Using eq. (2.6), this transformation is given, again somewhat symbolically, by

$$A_{x^s\varphi^t} \xrightarrow{\varphi \to \infty} \sum_{s-t} i^{s(n-m)} z^{-(n+m)/2} (1 - z)^{(s-t)/2} e^{i\sqrt{2}(n-m)x} A_{u^n\bar{u}^m}.$$  

Eq. (5.7) is still a second order partial differential equation for the $A_{u^n\bar{u}^m}(u, \bar{u})$’s, but one can reduce it to an ordinary differential equation using the fact that $x$ is a Killing direction of the metric. We thus choose the fields $A_{u^n\bar{u}^m}$ to have a well-defined $x$-momenta:
In the Euclidean case

\[ A_{u^\alpha \vec{a}^m} \equiv A_{u^\alpha \vec{a}^m}^{(p)}(z) e^{\sqrt{2}i(p-n+m)x} \]
\[ = A_{u^\alpha \vec{a}^m}^{(p)} \left( \frac{1}{1+2u\bar{u}} \right) \left( \frac{u}{\bar{u}} \right)^{(p-n+m)/2}, \]

(5.12)

where the peculiar factor of \( p-n+m \) in the momentum dependence is chosen because, as is seen from their definition in eqs. (2.6), the coordinate \( u \) "has momentum 1". With this convention, the linear-dilaton-gauge fields \( A_{x^\varphi} \) of eq. (5.11) will indeed have momenta \( p \). In the last line of eq. (5.12), we have used the conformal-gauge definitions of \( z \) and \( x \) from eqs. (2.6) and (2.9). Now substituting the definition of eqs. (5.12) into eq. (5.7), and using the Christoffel symbols of eq. (5.10), one finally obtains the hypergeometric-type equation:

\[
z(1-z) \frac{\partial^2}{\partial z^2} A_{u^\alpha \vec{a}^m}^{(p)} - (z + (n+m)(1-z)) \frac{\partial}{\partial z} A_{u^\alpha \vec{a}^m}^{(p)} - \frac{(p+m-n)^2}{4(1-z)} A_{u^\alpha \vec{a}^m}^{(p)} \]
\[+ \frac{1}{4z} ((n+m+1)^2 - p^2 - N_L) A_{u^\alpha \vec{a}^m}^{(p)} - mn A_{u^\alpha \vec{a}^m}^{(p)} = 0. \]

(5.13)

The Minkowski case is obtained by letting \( p \to -ip \); then at the horizon the solutions of eq. (5.13) tend to a linear combination of \( A_{u^\alpha \vec{a}^m}^{(p)} \sim (1-z)^{\pm(ip-n+m)/2} \), and one can again conclude that only static states can exist.

In the Euclidean case the solutions to eq. (5.13) have the form \((1-z)^{\pm(p-n+m)/2}\) around the horizon. One can not demand that \( A_{u^\alpha \vec{a}^m}^{(p)} \) be finite since, being a tensor, it is not a physical quantity. However its "norm" \( |A_{u^\alpha \vec{a}^m}^{(p)}|^2 = (g^{\alpha \bar{\alpha}})^{n+m} A_{u^\alpha \vec{a}^m}^{(p)} A_{\bar{\alpha} \bar{\alpha} \vec{a}^m}^{(p)*} \) is physical, and must be finite. Since \( g^{\alpha \bar{\alpha}} = 2/z \) is simply a constant at the horizon, we conclude that \( A_{u^\alpha \vec{a}^m}^{(p)} \sim |A_{u^\alpha \vec{a}^m}^{(p)}| \) must behave as \((1-z)^{p-n+m}/2\) at the horizon. Eq. (5.13) therefore has a unique solution, which is given by:

\[ A_{u^\alpha \vec{a}^m}^{(p)} = z^{\alpha_+} (1-z)^{(p-n+m)/2} F(\beta_+, \beta_-, 1 + |p-n+m|, 1-z) \],

(5.14)

with

\[
2\alpha_\pm = 1 + n + m \pm \sqrt{p^2 + N_L} \quad \text{and} \quad 2\beta_\pm = 1 + |p-n+m| \pm (n-m) + \sqrt{p^2 + N_L}.
\]

(5.15)

Using the transformation of eq. (4.4), one finds that \( A_{u^\alpha \vec{a}^m}^{(p)} \) has the asymptotic behaviour

\[ A_{u^\alpha \vec{a}^m}^{(p)} \xrightarrow{z \to 0} A \ z^{\alpha_+} + B \ z^{\alpha_-}. \]

(5.16)

The interpretation of this behaviour is not obvious, since the metric \( g^{u\bar{u}} = 2/z \) diverges asymptotically. However, since the physical quantity \( |A_{u^\alpha \vec{a}^m}| \) behaves as:

\[ |A_{u^\alpha \vec{a}^m}| \xrightarrow{z \to 0} C \ z^{(1+\sqrt{p^2+N_L})/2} + D \ z^{(1-\sqrt{p^2+N_L})/2}, \]

(5.17)
one sees that one has a well-behaved discrete state in the black hole only if $p^2 + N_L \leq 1$.

Since the level $N_L$ is a non-negative integer, this means that either $N_L = 0$, $|p| \leq 1$ or $N_L = 1$, $p = 0$.

The physical meaning of this constraint can be seen by returning to the asymptotic discrete state $A_{x^{n+m}}$ defined via eq. (5.11). This has the form

$$
A_{x^{n+m}} \xrightarrow{z \to 0} E z^{(1+\sqrt{p^2+N_L})/2} e^{i\sqrt{2} p x} + F z^{(1-\sqrt{p^2+N_L})/2} e^{i\sqrt{2} p x} = G e^{\sqrt{2} p_S^A \varphi} e^{i\sqrt{2} p x} + H e^{\sqrt{2} p_S^A \varphi} e^{i\sqrt{2} p x},
$$

(5.18)

where in the last equation we have used the Liouville momenta $p_S^{A,S}$ of eq. (5.3). This means that the discrete state $A_{x^{n+m}}$ is again a combination of the anti-Seiberg and Seiberg discrete states of the $c = 1$ theory, and the condition $p^2 + N_L \leq 1$ is simply the condition that neither the Seiberg nor the anti-Seiberg Liouville momenta is positive. Examining the figure, one can see that the only states satisfying this condition are the discrete tachyons (with $N_L = 0$ and $|p| \leq 1$) that we have already found, and the massless “discrete graviton” state(s), with $N_L = 1$ and $p = 0$. However, in our approach to the string-field states we do not add extra linearized gravitons and dilatons to the action, since all the massless fields are already included in the fully nonlinear sigma-model action around which we perturb. “Dilaton-graviton hair” with $N_L = 1$ simply corresponds to the number of free parameters in the black-hole solution, and we have seen that the only freedom in the 2D black hole solution is a constant shift of the dilaton, corresponding to the only parameter, the ADM mass $a$.

We concluded that the black hole has no $W$-hair!

6 Tachyon and W–hair in the exact $SL(2, \mathbb{R})/U(1)$ background

6.1 The exact $SL(2, \mathbb{R})/U(1)$ metric and dilaton

The effective space-time black hole background of eqs. (2.3) and (2.4) corresponds to the semiclassical $O(1/k')$ approximation of the $SL(2, \mathbb{R})/U(1)$ gauged WZW conformal field theory*. (Here $k' = k - 2$, where $k$ is the central charge of the $SL(2, \mathbb{R})$ Kac-Moody algebra.) An exact effective space time background for states without windings was proposed by Dijkgraaf et al. [3] for arbitrary $k$. It is derived by expressing the $L_0, \bar{L}_0$ Virasoro generators of the $SL(2, \mathbb{R})/U(1)$ gauged WZW theory as differential operators on the $SL(2, \mathbb{R})$ group manifold, and identifying the operator $L_0 + \bar{L}_0$ with the target space Laplacian of the sigma model.

*This can be easily seen, for instance, by comparing central charges of the theories[1].
After rescaling the $r$ and $x$ coordinates to asymptotically approach the the $c = 1$ theory, the thus derived metric and dilaton take the Euclidean and Minkowski forms:

$$ds^2 = dr^2 \pm \beta^2(r) \, dx^2 \quad \text{and} \quad \Phi = \Phi_0 - \frac{1}{2} \log \left( \frac{\sinh \left( \sqrt{2/k'} r \right)}{\beta(r)} \right),$$

(6.1)

with $\beta(r)$ given by

$$\beta(r) = \left( \frac{k}{k'} \coth^2 \left( \frac{r}{\sqrt{2k'}} \right) - \frac{2}{k'} \right)^{-\frac{1}{2}}.$$ (6.2)

It has been verified up to three loops in [4] and up to four loops in [5] that the dilaton-graviton background of eq. (6.1) is indeed conformally invariant. For large $k$, this background reduces to the dilaton-graviton black hole. For the background to give a 2-dimensional solution to the string, one needs to take $k = 9/4$, which we shall do from now on.

It is again useful to define a coordinate $z$, in this case by

$$z \equiv \text{sech}^2 \sqrt{2r}.$$ (6.3)

This transforms the metric and dilaton of eqs. (6.1) to

$$\Phi = \frac{1}{2} \log \left( \frac{9z}{a} \right) - \frac{1}{4} \log (1 + 8z) \quad \text{and} \quad ds^2 = \frac{1}{8z^2(1-z)} \, dz^2 \pm \frac{1-z}{1+8z} \, dx^2,$$

(6.4)

where we have written the constant shift of the dilaton in terms of the ADM mass $a$. These are of the same form as the black hole solution of eqs. (2.3) and (2.4), except for the factors of $1 + 8z$ (and of 9). It is clear that the metric has a horizon at $z = 1$, corresponding to $r = 0$, and that the asymptotic region of the metric at $r \to \infty$ maps to $z = 0$. In the Euclidean case, one is again restricted to $0 \leq z \leq 1$. In the Minkowski case the metric has a different behaviour from that of the black hole metric as $z \to \infty$: Since $g_{xx}$ no longer diverges, the metric has only a coordinate singularity there. It was argued in ref. [27] that the spacetime could therefore be maximally extended by linking together infinitely many copies of the metric through $z = \infty$. However it is important to note that the dilaton blows up as $z \to \infty$. Since the dilaton is an integral part of the 2-dimensional gravity theory, we shall regard $z \to \infty$ as a singularity of the spacetime, and shall not allow anything to pass through this point. We shall also see that matter fields such as the tachyon blow up as $z \to \infty$. 22
Since we are provided with a conjectured exact background of the string, and since it has a structure very similar to that of the black hole, it is natural to ask whether the conclusions of the previous sections concerning the tachyon and W–hair remain valid also in this exact background. In the rest of this section we shall carry out the required analysis in order to answer this question. In the exact theory one can not study the back-reaction of the fields explicitly, since the dilaton-graviton action is an infinite series in \( \alpha' \). However, it is reasonable to argue that any field with a finite well-behaved stress tensor has a valid back-reaction on the metric and dilaton, and thus leads to a sensible perturbation of the theory.

We should, perhaps, remind the reader that “stringy states” with nontrivial windings can not be described in our target space approach. In principle, one should be able to describe states with winding, but no momenta, using the dual metric of ref. [3]. This is simply given by the region of eqs. (6.4) between \( z = -1/8 \) and \( z = 0 \), and describes a space with a naked singularity (at \( z = -1/8 \)). However it is known that there are difficulties with describing the string by this dual metric. For example, as was noted in ref. [3], one can not see the periodicity of \( x \) from the target space metric. From our point of view we would not find any restrictions on the fields in this dual space, since there is no horizon to fix their boundary conditions.

### 6.2 Static and non-static \( SL(2, \mathbb{R})/U(1) \) tachyons

If one assumes that the effective action of the tachyon still has the form

\[
S = \int d^2x \, e^{-2\Phi} \sqrt{G} \left( (\nabla T)^2 - 2T^2 \right),
\]

(6.5)

the tachyon equation of motion is now given by

\[
z(1-z) T'' - z T' \pm 1 + \frac{8z}{8z(1-z)} \ddot{T} \pm \frac{1}{4z} T = 0,
\]

(6.6)

in agreement with the equation derived from the gauged WZW theory [3]. Note, however, that the issue of higher order terms in the tachyon potential still exists for the exact metric. If one now considers a tachyon with momentum \( p \), \( T \sim T_{(p)}(z) e^{\sqrt{T}ipx} \), the equation of motion becomes

\[
z(1-z) T''_{(p)} - z T'_{(p)} - \left( \frac{\pm p^2 - 1}{4z} \right) \ddot{T}_{(p)} \pm \frac{9p^2}{4(1-z)} T_{(p)} = 0.
\]

(6.7)

In the Minkowski metric, the solutions of this equation behave as \( T_{(p)} \sim (1-z)^{\pm 3ip/2} \) around the horizon, so one is again restricted to having only static tachyons. In the
Euclidean case, the solutions behave as \( T(p) \sim (1 - z)^{\pm 3|p|/2} \), so demanding good behaviour at the horizon leads one to the unique well-behaved solution:

\[
T(p)(z) = t_p z^{1 + |p|/2} (1 - z)^{3|p|/2} F\left(\frac{1}{2} + 2|p|, \frac{1}{2} + 2|p|, 1 + 3|p|, 1 - z\right).
\]

This has a form very similar to that of the black hole tachyon of eq. (4.3), and one is led to similar conclusions for the exact tachyon. Thus, using eq. (4.4), one again sees that \( T(p) \) has both anti-Seiberg and Seiberg components asymptotically,

\[
T \xrightarrow{z \to 0} A e^{-\sqrt{2}(1+|p|)\varphi} e^{\sqrt{2}ipx} + B e^{-\sqrt{2}(1-|p|)\varphi} e^{\sqrt{2}ipx},
\]

so demanding finite asymptotic behaviour again restricts one to tachyons with \(|p| \leq 1\).

In the exact solution \( x \) has radius \( 3/\sqrt{2} \), as is also seen directly from the gauged WZW theory, so momenta are restricted to be multiples of \( 1/3 \). We thus conclude that in the background of the exact solution to the Euclidean \( SL(2, \mathbb{R})/U(1) \) theory, the only tachyons that are well behaved are \( T(0), T(\pm 1/3), T(\pm 2/3) \) and \( T(\pm 1) \), and that only the static tachyon can exist in the Minkowski solution.

### 6.3 \( SL(2, \mathbb{R})/U(1) \) W–hair

In section 5 we showed that W–hair does not exist for the black hole. We argued there that the problem could be reduced to solving the Siegel-gauge equation of motion (neglecting possible higher terms in the potential)

\[
\nabla^2 A_{\mu_1...\mu_j} - 2\nabla \Phi \cdot \nabla A_{\mu_1...\mu_j} - m^2 A_{\mu_1...\mu_j} = 0
\]

for a general massive field \( A_{\mu_1...\mu_j} \) with mass \( m \), and matching the solution to the discrete states of the \( c = 1 \) theory. Here we would like to repeat this procedure in the exact dilaton-graviton background of eqs. (6.1). As in the black-hole case, a difficulty with this procedure is that in the \((x, z)\) coordinates the various components of \( A_{\mu_1...\mu_j} \) mix. This could be solved, as before, by solving eq. (6.10) in conformal coordinates but, while this could certainly be carried out, it is somewhat messy. Since our final conclusions are based only on the fact that the asymptotic fields have both anti-Seiberg and Seiberg components, we shall simplify our lives by restricting ourselves to examining only scalar pieces of the \( A_{\mu_1...\mu_j} \)'s, made by contracting them with \( G^{\mu\nu} \)'s and \( \varepsilon^{\mu_1...\mu_j} \). This will be sufficient to show that the discrete states can not exist in the exact \( SL(2, \mathbb{R})/U(1) \) background.

We thus define, generically,

\[
A \equiv G^{\mu_1\mu_2} ... \varepsilon^{\mu_1...\mu_n|1} ... A_{\mu_1...\mu_j}.
\]
Since $A$ is a scalar, eq. (6.10) reduces to

$$z(1 - z) A'' - z A' - \left( \frac{1}{4z} \left( \pm p^2 + \frac{m^2}{2} \right) \pm \frac{9p^2}{4(1 - z)} \right) A = 0,$$

(6.12)

for an $A(p)$ with the usual momentum dependence $A \sim A(z) e^{\sqrt{2} ipx}$. Around the horizon, the Minkowski-space solutions to this equation again behave as $A(z) \sim (1 - z)^{\pm 3p/2}$; so only static discrete states can exist. In the Euclidean case the solutions behave as $A(p) \sim (1 - z)^{\pm 3|p|/2}$ around the horizon. Since $A(p)$ is a scalar quantity it is physical, and must be regular everywhere. Therefore, there is a unique well-behaved solution for $A(p)$, which is given by:

$$A(p)(z) = a_p z^{\frac{1}{2} \sqrt{p^2 + NL}} (1 - z)^{\frac{3|p|}{2}} \sqrt{2} F(\beta, \beta, 1 + 3|p|, 1 - z),$$

(6.13)

where $\beta$ is now given by

$$\beta = \frac{1}{2} \left( 1 + 3|p| + \sqrt{p^2 + NL} \right).$$

(6.14)

As in section 5, we have defined the level $N_L$ by $N_L = 1 + \frac{m^2}{2}$. As expected, this solution exhibits a combination of anti-Seiberg and Seiberg behaviour asymptotically:

$$A \xrightarrow{z \to 0} C z^{\frac{1}{2} \sqrt{p^2 + NL}} e^{\sqrt{2} ipx} + D z^{\frac{1}{2} \sqrt{p^2 + NL}} e^{\sqrt{2} ipx}$$

$$\quad = E e^{\sqrt{2} p^A \phi} e^{\sqrt{2} ipx} + F e^{\sqrt{2} p^B \phi} e^{\sqrt{2} ipx},$$

(6.15)

and so blows up except when $N_L = 0$ and $|p| \leq 1$, or when $N_L = 1$ and $p = 0$.

We can now reach our conclusion that one can not have discrete states in the exact solution. Since the potential discrete states are sums of terms of the form $A_{\phi}$ asymptotically, the "$A$'s" obtained by contracting the $A_{\mu_1...\mu_j}$'s with $G^{\mu\nu}$'s are necessarily nonvanishing. We have shown that such $A$'s can only exist for $N_L = 0$ and $|p| \leq 1$, giving the discrete tachyons of the previous section, and for $N_L = 1$ and $p = 0$. This last case exists only for the "discrete graviton" which, again is not taken as a linearized perturbation, since the full nonlinear dilaton-graviton system has been solved.

We conclude that W–hair does not exist on the exact $SL(2, R)/U(1)$ target space, and that the black hole is completely described by its ADM mass, and the 7 discrete tachyons with $|p| \leq 1$.

## 7 Conclusions

In this paper we have studied the tachyon and $W_\infty$ states in the 2D dilaton-graviton black hole of Witten, and in the exact metric proposed by Dijkgraaf et al. to describe the
$SL(2, \mathbb{R})/U(1)$ conformal field theory. We have used the effective sigma-model approach for the tachyon, and have developed a mixed Siegel-gauge string field theory/sigma model approach for the massive states. We again warn the reader that one can not deal with winding states from these space-time points of view.

After solving their linearized equations of motion exactly, we have seen that choosing states to be well behaved at the horizon of the black hole forces them to be in a combination of Seiberg and anti-Seiberg states asymptotically. For these states to be well behaved asymptotically, they then need to satisfy the condition that neither the Seiberg nor the anti-Seiberg Liouville momenta can be positive. We thus disagree with previous works on the subject, which all worked only with anti-Seiberg states that blow up on the horizon. Our condition means that there are no $W_\infty$ states in the black hole, and that only the tachyons with $|p| \leq 1$ are good states. Because the radius of $x$ in the black hole is $1/\sqrt{2}$, one has only the three tachyons $T_{(0)}$ and $T_{(\pm1)}$. In the exact metric the radius is $3/\sqrt{2}$, so one has the four additional states $T_{(\pm1/3)}$ and $T_{(\pm2/3)}$. In the Minkowski black holes, only the static tachyon survives.

The black hole is stable to the back-reaction of these tachyons, so they can be regarded as being hair of the black hole. It is very interesting that this spectrum is so much sparser than the free-field BRST cohomology of the underlying $SL(2, \mathbb{R})/U(1)$ conformal field theory, which contains an infinite number of tachyon and $W_\infty$ states. We regard this as being similar to the truncation of the spectrum of the $c = 1$ theory to the Seiberg sector, once the cosmological constant is turned on and the Liouville geometry of the theory becomes relevant. As far as the black hole is concerned, this has the implication that the black hole is uniquely described by its mass and the various tachyonic perturbations. This small amount of hair is clearly useless for maintaining quantum coherence during any black-hole evaporation. From the point of view of the underlying $SL(2, \mathbb{R})/U(1)$ conformal field theory, it is interesting to speculate that the specific tachyon perturbations that we have picked out may be related to integrable perturbations of the theory.

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Appendices

A The back reaction of the static tachyon in the conformal gauge

In this section we derive the back reaction of the tachyon on the black hole in the Minkowski version of the conformal gauge, in which the global structure of the metric is most easily seen. The metric in these coordinates is defined to be

\[ ds^2 = e^\sigma du dv . \]  

(A.1)

In the following we shall need the Christoffel symbols and the Ricci tensor, the only non-vanishing components of which are

\[ \Gamma_{uu} = \partial_u \sigma , \]

\[ \Gamma_{vv} = \partial_v \sigma \quad \text{and} \]

\[ R_{uv} = \partial_u \partial_v \sigma . \]  

(A.2)

As we saw in eqs. (2.7) and (2.8), the dilaton-graviton black hole solution in these coordinates is given by

\[ \sigma = - \log(1 + 2uv) , \]

\[ \Phi = - \frac{1}{2} \log a - \frac{1}{2} \log(1 + 2uv) , \]  

(A.3)

with the horizon and singularity located at \( uv = 0 \) and \( uv = -1/2 \), respectively, and the asymptotic region at \( uv \to \infty \). Since the dilaton and the components of black hole metric depend only on the product \( uv \), the system has a Killing symmetry. In these coordinates \( z \) is defined by

\[ z = \frac{1}{1 + 2uv} , \]  

(A.4)

and, as before, the asymptotic region is at \( z = 0 \), the horizon at \( z = 1 \) and the singularity at \( z \to \infty \).

We now turn to finding the backreaction of a tachyon on the metric in this gauge. For simplicity, we shall consider only a static tachyon (although the general case could be considered as in section 5.3); then since \( T_{(0)} \) depends only on \( z, \sigma \) and \( \Phi \) also continue to do so after the backreaction. The graviton \( \beta \)-function equation in (2.2) gives rise to the two equations:

\[ \partial_z \left( z(1-z) \partial_z (\sigma - 2\Phi) \right) + z(1-z) T_{(0)}^2 = 0 \quad \text{and} \]

(A.5)

\[ \Phi'' + \frac{2}{z} \Phi' - \Phi' \sigma' - \frac{1}{2} T_{(0)}^2 = 0 , \]

(A.6)
and the dilaton $\beta$–function equation in (2.2) becomes

$$\partial_z (z(1 - z)\partial_z (\sigma - 4\Phi)) + z(1 - z)(\frac{4\Phi'^2}{T^2_{(0)}}) - \frac{1}{z^2} \left(1 + \frac{1}{4z} T^2_{(0)}\right) e^\sigma = 0 .$$ 

(A.7)

The prime denotes differentiation with respect to $z$. Before turning on the tachyon field, one can see that the general solution of these equations* is equivalent to the black hole of (A.3): First, eq. (A.5) can be integrated, giving $\sigma - 2\Phi = \gamma + \alpha \log(z/(1 - z))$. This "$\alpha$–term", which implies a coordinate divergence at the horizon, can be eliminated by the conformal coordinate transformation $u' = u^{1-\alpha}$, $v' = v^{1-\alpha}$. Now eqs. (A.5), (A.6) and (A.7) can be combined to give a first-order differential equation for $\sigma$, which can be integrated. This gives the black hole of (A.3).

Now, to find the back reaction of the tachyon, first note that the $z$ defined here is the same as that of the linear-dilaton gauge (see the transformations of eq. (2.6)), so the tachyon is again given by eq. (3.6):

$$T_{(0)} = t_0 \sqrt{z} F\left(\frac{1}{2}, \frac{1}{2}, 1, 1 - z\right) .$$

(A.8)

The back reaction of the tachyon on the black hole is found by expanding eqs. (A.5), (A.6) and (A.7) around the black hole, i.e. $\sigma \rightarrow \log(z) + \tilde{\sigma}$ and $\Phi \rightarrow \frac{1}{2} \log(z/a) + \tilde{\Phi}$, and substituting for the tachyon. This gives the equations:

$$\partial_z \left(z(1 - z)\partial_z (\tilde{\sigma} - 2\tilde{\Phi})\right) + z(1 - z) T^2_{(0)} = 0 ,$$

$$\tilde{\Phi}' - \frac{1}{2z}(\tilde{\sigma}' - 2\tilde{\Phi}') - \frac{1}{2} T^2_{(0)} = 0 ,$$

$$z(1 - z)(\tilde{\sigma}'' - 4\tilde{\Phi}'') + (1 - 2z) \tilde{\sigma}' + 4z \tilde{\Phi}' - \frac{1}{z}\tilde{\sigma} + z(1 - z) T^2_{(0)} - \frac{1}{4} T^2_{(0)} = 0 .$$

(A.9)

The first equation of eq. (A.9) can again be solved by integration. Using the tachyon equation of motion eq. (3.1), one finds

$$\sigma - 2\Phi = \log(a) + \frac{t_0^2}{2} - \frac{1}{2z} T^2_{(0)} + (1 - z) T_{(0)} T'_{(0)} ,$$

(A.10)

up to an "$\alpha$–term" which can be set to zero, as before, and a constant which we fix by demanding that $\sigma - 2\Phi = \log a$ at $z = 1$. Note that, unlike that unperturbed black hole, one can no longer have $\sigma = 2(\Phi - \Phi_0)$ everywhere when tachyons are present. This is in contrast to studies of "2D gravity", where the black hole is coupled to conformal matter that does not see the dilaton field, and one can choose the $\sigma \equiv 2\Phi$ as a gauge [28]. Here

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*We show this in the static case, but it should be true in general, as seen in the linear-dilaton gauge in [10]. (See also eqs. (3.14) and (3.15).)
the tachyon couples to the dilaton, as expected from the effective action of the string, making this impossible.

The conformal factor \( \sigma \) is once again found by combining the remaining equations to get a first-order differential equation, which can be integrated. Remarkably, this integration can be done explicitly, giving

\[
\sigma = \log (z) + 2 \left( \frac{t_0}{\pi} \right)^2 z - \frac{1}{2z} T^2_{(0)} + 2 (1 - z) T_{(0)} T'_{(0)} - 2z (1 - z)^2 T'^2_{(0)} . \tag{A.11}
\]

(The constant of integration, corresponding to a shift of the black-hole mass, is fixed by demanding that \( \sigma \) have no term linear in \( z \) for large \( z \).) The resulting dilaton field is given by

\[
\Phi =\frac{1}{2} \log \left( \frac{z}{a} \right) + \left( \frac{t_0}{\pi} \right)^2 z - \frac{t_0^2}{4} + \frac{1}{2}(1 - z) T_{(0)} T'_{(0)} - z(1 - z)^2 T'^2_{(0)} . \tag{A.12}
\]

One can now easily see the geometric structure of this spacetime. First, examining the form of \( \sigma \) and \( \Phi \), one sees that there are no special points other than \( z = 0, 1 \) and \( \infty \). As \( z \to 0 \), the scalar curvature\(^{\dagger} \) from eq. (A.2)

\[
R = -8z - 2 \left( \frac{t_0}{\pi} \right)^2 (1 - z) + \frac{1}{4z} T^2_{(0)} - z(1 - z)(1 - 2z) T'^2_{(0)} \tag{A.13}
\]

becomes

\[
R \xrightarrow{z \to 0} -8z \left( 1 - \left( \frac{t_0}{\pi} \right)^2 \left( 3 + \log \frac{z}{16} \right) \right) , \tag{A.14}
\]

so the spacetime is asymptotically flat, with a linear dilaton field. \( z = 1 \) is the horizon of the black hole, since it corresponds to \( uv = 0 \). Unlike the linear-dilaton gauge, where this is a coordinate singularity, both \( \sigma \) and \( \Phi \) are finite there, so one sees that there is no singularity at the horizon. There are then no singularities until \( z \to \infty \), the original singularity of the black hole.

We conclude that while the (static) tachyon perturbation changes the conformal factor and the form of the dilation, the global structure of the black hole is unchanged. The static tachyon is thus sensible hair for the black hole.

\(^{\dagger}\)The reader who wishes to compare this result with the linear-dilaton gauge result of eqs. (3.16) and (3.18) should be warned that after the tachyon perturbation the “\( z \)'s” of the two gauges are no longer identical!
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