According to the teleparallel equivalent of general relativity, curvature and torsion are two equivalent ways of describing the same gravitational field. Despite equivalent, however, they act differently: whereas curvature yields a geometric description, in which the concept of gravitational force is absent, torsion acts as a true gravitational force, quite similar to the Lorentz force of electrodynamics. As a consequence, the right-hand side of a spinless-particle equation of motion (which would represent a gravitational force) is always zero in the geometric description, but not in the teleparallel case. This means essentially that the gravitational coupling prescription can be minimal only in the geometric case. Relying on this property, a new gravitational coupling prescription in the presence of curvature and torsion is proposed. It is constructed in such a way to preserve the equivalence between curvature and torsion, and its basic property is to be equivalent with the usual coupling prescription of general relativity. According to this view, no new physics is connected with torsion, which appears as a mere alternative to curvature in the description of gravitation. An application of this formulation to the equations of motion of both a spinless and a spinning particle is made.

I. INTRODUCTION

The correct form of the gravitational coupling prescription of elementary particle fields in the presence of curvature and torsion is an old and, up to now, open problem. The basic difficulty is that, in the presence of curvature and torsion, covariance alone is not able to determine the connection entering the definition of covariant derivative. In fact, there are in principle two different possibilities. The first one is to assume that the coupling is minimal in the Cartan connection, a general connection presenting simultaneously curvature and torsion. The second one is to consider that, even in the presence of curvature and torsion, the coupling is minimal in the so called Ricci’s coefficient of rotation, the torsionless spin connection of general relativity. Notice that, in the specific case of general relativity, where torsion vanishes, the only connection present is the Ricci coefficient of rotation, and this problem does not exist. In this case, as is well known, the gravitational coupling is minimal in the Ricci coefficient of rotation.

To understand better the differences between the above two possibilities, let us leave for a while the general case characterized by the simultaneous presence of curvature and torsion, and remember that, from the point of view of teleparallel gravity, curvature and torsion are able to provide, each one, equivalent descriptions of the gravitational interaction. In fact, according to general relativity, the gravitational interaction is geometrized in the sense that all equations of motion are given by geodesics, whose solutions are trajectories that follow the curvature of spacetime. Teleparallelism, on the other hand, attributes gravitation to torsion, but in this case torsion accounts for gravitation not by geometrizing the interaction, but by acting as a force. In the teleparallel equivalent of general relativity, therefore, there are no geodesics, but force equations quite analogous to the Lorentz force equation of electrodynamics. As this teleparallel force equation is ultimately equivalent to the geodesic equation of general relativity (in the sense that both yield the same physical trajectory), the gravitational interaction can consequently be described alternatively
in terms of curvature, as is usually done in general relativity, or in terms of torsion, in which case we have the so-called teleparallel equivalent of general relativity.

The fundamental reason for gravitation to present two equivalent descriptions is related to the universality of free fall. To understand this point, let us notice first that, like any other interaction of nature, gravitation presents a description in terms of a gauge theory. Furthermore, as usual in gauge theories, the gravitational interaction in teleparallel gravity is described by a force, with contorsion playing the role of force. On the other hand, universality of free fall implies that all particles feel gravity the same. As a consequence, it becomes also possible to describe gravitation not as a force, but as a geometric deformation of the flat Minkowski spacetime. More precisely, according to this point of view, due to the universality of free fall, gravitation can be supposed to produce a curvature in spacetime, the gravitational interaction in this case being achieved by letting (spinless) particles follow the geodesics of spacetime. This is the approach used by general relativity, in which geometry replaces the concept of gravitational force, and the trajectories are determined, not by force equations, but by geodesics. Now, on account of the fact that in general relativity there is no the concept of gravitational force—which means that, except for a possible non-gravitational force, the right-hand side of the geodesic equation is always zero—the gravitational coupling prescription in this case results to be minimal in the Ricci coefficient of rotation. On the other hand, in the case of teleparallel gravity, as contorsion plays the role of a gravitational force, similarly to the Lorentz force of electrodynamics, it will always appear in the right-hand side of the corresponding teleparallel equation of motion. Therefore, the connection defining the gravitational coupling rule in teleparallel gravity, must necessarily include (minus) the contorsion tensor to account for this force. The coupling prescription in teleparallel gravity, therefore, cannot be minimal in the usual sense of the word.

Returning now to the general case, in addition to the above described extremal cases, given by general relativity and teleparallel gravity, there are infinitely more cases characterized by the simultaneous presence of curvature and torsion. Based on the different, but equivalent, roles played by curvature and torsion in the description of gravitation, the basic purpose of this paper will be to study the coupling rule that emerges when the equivalence alluded to above is assumed to hold also in these general cases. We will proceed as follows. In Sec. 2 we introduce the gravitational gauge potentials and the associated field strengths. Using these potentials, a new coupling prescription in the presence of curvature and torsion is proposed in Sec. 3. Its main feature is to be equivalent with the coupling prescription of general relativity. As it preserves, by construction, the equivalence between curvature and torsion of teleparallel gravity, it becomes consequently a matter of convention to describe gravitation by curvature, torsion, or by a combination of them. This means that, according to this formalism, torsion is simply an alternative way to describe gravitation, and no new physics is associated with it. This coupling prescription is then used in Sec. 4 to study the trajectory of a spinless particle in the presence of both curvature and torsion. The resulting equation of motion is a mixture of geodesic and force equation, which can be reduced either to the force equation of teleparallel gravity, or to the geodesic equation of general relativity. In Sec. 5 we apply the same coupling rule to study the motion of a spinning particle. Following Yee and Bander, we use an approach based on a Routhian formalism, according to which the equation of motion for the spin is obtained as the Hamilton equation, and the equation of motion for the trajectory is obtained as the Euler-Lagrangian equation. Again, the resulting equation of motion is a mixture of geodesic and force equation, which can be reduced either to the ordinary Papapetrou equation of general relativity or to the teleparallel equivalent of the Papapetrou equation. Finally, in Sec. 6 we present the conclusions of the paper.

II. GAUGE POTENTIALS AND FIELD STRENGTHS

According to the gauge approach to gravitation, spacetime is considered to be the base-space of the theory, whose associated indices will be denoted by the Greek alphabet \( (\mu, \nu, \rho, \cdots = 0,1,2,3) \). Its coordinates, therefore, will be denoted by \( x^\mu \), and its metric tensor by \( g_{\mu \nu} \). At each point of spacetime, there is a tangent space attached to it, given by a Minkowski space, which will be the fiber of the corresponding tangent-bundle. We will use the Latin alphabet \( (a, b, c, \cdots = 0,1,2,3) \) to denote indices related to this space. Its coordinates, therefore, will be denoted by \( x^a \), and its metric tensor by \( \eta_{ab} \), which is chosen to be \( \eta_{ab} = \text{diag}(1,-1,-1,-1) \).

Analogously to the internal gauge theories, the fundamental field representing gravitation will be assumed to be a gauge connection. In the case of gravitation, there are in principle two different connections into play. The first is the so-called spin connection \( A_\mu \), which is a connection assuming values in the Lie algebra of the Lorentz group,

\[
A_\mu = \frac{1}{2} A^{ab \mu} S_{ab},
\]

with \( S_{ab} \) a matrix generator of infinitesimal Lorentz transformations. The second is the translational connection \( B_\mu \), which is a connection assuming values in the Lie algebra of the translation group,

\[
B_\mu = B^a_\mu P_a,
\]
with $P_c = -i \partial_c$ the generator of infinitesimal translations. The components $B^a_\mu$ appear as the nontrivial part of the tetrad field $h^a_\mu$, that is,

$$h^a_\mu = \partial_\mu x^a + c^{-2} B^a_\mu,$$

(1)

where the velocity of light $c$ has been introduced for dimensional reasons. We remark that, whereas the tangent space indices are raised and lowered with the Minkowski metric $\eta_{ab}$, spacetime indices are raised and lowered with the Riemannian metric

$$g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu.$$  

(2)

In the general case, the spin connection $A^a_\mu$ is a Cartan connection, that is, a connection presenting simultaneously curvature and torsion. The curvature and torsion tensors are given respectively by

$$R^c_{\mu\nu} = \partial_\mu A^c_{\nu\sigma} - \partial_\nu A^c_{\mu\sigma} + A^c_{\alpha\mu} A^\alpha_{\nu\sigma} - A^c_{\alpha\nu} A^\alpha_{\mu\sigma},$$

(3)

and

$$T^c_{\mu\nu} = \partial_\mu h^c_\nu - \partial_\nu h^c_\mu + A^c_{b\mu} h^b_\nu - A^c_{b\nu} h^b_\mu.$$  

(4)

Seen from a holonomous spacetime basis, the Cartan connection is $\Gamma^\rho_{\mu\nu} = h^c_\rho (\partial_\nu h^c_\mu + A^c_{b\nu} h^b_\mu) \equiv h^c_\rho c D_\nu h^c_\mu,$

(5)

whose inverse relation reads

$$A^a_{b\mu} = h^a_\rho (\partial_\mu h^\rho_b + \Gamma^\rho_{\nu\mu} h^\nu_b) \equiv h^a_\rho c \nabla_\mu h^\rho_b.$$  

(6)

In these expressions, $c D_\nu$ and $c \nabla_\mu$ are the corresponding Cartan covariant derivatives. In terms of $\Gamma^\rho_{\mu\nu}$, the curvature and torsion tensors acquire respectively the forms

$$R^\rho_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\lambda\nu} - \partial_\nu \Gamma^\rho_{\lambda\mu} + \Gamma^\rho_{\theta\mu} \Gamma^\theta_{\lambda\nu} - \Gamma^\rho_{\theta\nu} \Gamma^\theta_{\lambda\mu},$$

(7)

and

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu}.$$  

(8)

In the specific case of general relativity, the spin connection is the Ricci coefficient of rotation $\overset{\circ}{A}^a_{b\nu}$, which in terms of the Levi-Civita connection $\overset{\circ}{\Gamma}^\rho_{\mu\nu}$ reads

$$\overset{\circ}{A}^a_{b\nu} = h^a_\rho (\partial_\nu h^\rho_b + \overset{\circ}{\Gamma}^\rho_{\nu\mu} h^\nu_b) \equiv h^a_\rho \overset{\circ}{\nabla}_\mu h^\rho_b,$$

(9)

with $\overset{\circ}{\nabla}_\nu$ the usual Levi-Civita covariant derivative. Now, as is well known, the Cartan connection and the Ricci coefficient of rotation are related by

$$\overset{\circ}{A}^a_{b\nu} = A^a_{b\nu} - K^a_{b\nu},$$

(10)

where

$$K^a_{b\nu} = \frac{h^c_{\nu}}{2} (T^a_{c\nu} + T^a_{b\nu} - T^a_{b\nu})$$

(11)

is the contorsion tensor, with $T^a_{b\nu}$ the torsion of the Cartan connection. Finally, we notice that, as a consequence of the relation (10), the curvature $\overset{\circ}{R}^c_{d\mu\nu}$ of the connection $\overset{\circ}{A}^c_{\mu\nu}$ can be decomposed as

$$\overset{\circ}{R}^c_{d\mu\nu} = R^c_{d\mu\nu} - Q^c_{d\mu\nu},$$

(12)

where

$$Q^c_{d\mu\nu} = \overset{\circ}{D}_\mu K^c_{d\nu} - \overset{\circ}{D}_\nu K^c_{d\mu} + K^c_{a\mu} K^a_{d\nu} - K^c_{a\nu} K^a_{d\mu}$$

(13)

is a tensor written in terms of torsion only.
III. GRAVITATIONAL COUPLING PRESCRIPTION

The gravitational coupling prescription states that all ordinary derivatives be replaced by Lorentz covariant derivatives:

$$\partial_a \rightarrow D_a = h^\nu_a D_\nu.$$  \hfill (14)

In the presence of curvature and torsion, however, Lorentz covariance alone is not able to determine the connection entering the definition of covariant derivative, and consequently neither the form of the gravitational coupling rule. In fact, due to the affine character of the connection space, one can always add an arbitrary tensor to a given connection without destroying the covariance of the corresponding derivative. In order to determine the gravitational coupling rule, therefore, besides the covariance requirement, some further ingredients turn out to be necessary.

In the presence of curvature and torsion, it is usual to assume that gravitation is minimally coupled to the Cartan connection. This means to take the Fock-Ivanenko covariant derivative

$$\overset{\circ}{D}_\nu = \partial_\nu - \frac{i}{2} A^{ab}_\nu S_{ab},$$  \hfill (15)

as defining the coupling of any field to gravitation. On account of the identity 10, it can alternatively be written as

$$D_\nu = \partial_\nu - \frac{i}{2} (A^{ab}_\nu - K^{ab}_\nu) S_{ab},$$  \hfill (18)

where, to indicate that now we are in a general case, we have dropped the “ball” notation from the covariant derivative symbol. In this form, it represents the gravitational coupling prescription in the presence of curvature and torsion, which is of course equivalent to the coupling prescription 17 of general relativity. Its spacetime version, denoted by $\nabla_\nu$, when applied to a spacetime vector $V^\rho$ reads

$$\nabla_\nu V^\rho = \partial_\nu V^\rho + (\Gamma^\lambda_{\mu\nu} - K^\lambda_{\mu\nu}) V^\mu.$$  \hfill (19)

These covariant derivatives are easily seen to satisfy the relation $\nabla_\nu V^\rho = h^\rho_a D_\nu V^a$, with $V^a = h^a_\rho V^\rho$.

As already discussed, the covariant derivative 13 can be interpreted in the following way. Since contorsion plays the role of a gravitational force, like the electromagnetic Lorentz force it will always appear in the right-hand side of the equations of motion. Accordingly, when a connection presents torsion, the corresponding coupling prescription must necessarily include (minus) the contorsion tensor to account for the gravitational force it represents. This is the reason why gravitation turns out to be minimally coupled to the torsionless Ricci coefficient of rotation $\overset{\circ}{A}^a_{b\nu}$, but not to the Cartan connection $A^a_{b\nu}$.

As a consistency requirement, let us remark that the spin connection 10 presents two extremal limits. The first corresponds to a vanishing torsion, in which case it reduces strictly to the torsionless Ricci coefficient of rotation $\overset{\circ}{A}^a_{b\nu}$ of general relativity. The second corresponds to a vanishing curvature, that is, to the teleparallel equivalent of general relativity, in which $A^a_{b\nu} = 0$, and consequently the spin connection reduces to the teleparallel spin connection, given by minus the contorsion tensor 17:

$$\overset{\circ}{A}^a_{b\nu} = 0 - K^a_{b\nu}. \hfill (20)$$
IV. SPINLESS PARTICLES

As a first application of the above gravitational coupling prescription, let us consider the specific case of a spinless particle of mass $m$. The idea is to obtain the equation of motion from first principles, and then to compare with that obtained by using the gravitational coupling prescription.

In analogy with the electromagnetic case, the action integral describing a spinless particle interacting with a gravitational field, in the context of a gauge theory, is given by

$$ S = \int_a^b \left[ -m c \, d\sigma - \frac{1}{c^2} B^a_{\mu} \mathcal{P}_a \, dx^\mu \right], $$

(21)

where $d\sigma = (\eta_{ab} dx^a dx^b)^{1/2}$ is the flat space invariant interval, and $\mathcal{P}_a$ is the Noether charge associated with the invariance of the action under translations. In other words, $\mathcal{P}_a = m c u_a$ is the particle (tangent space) four-momentum, with $u^a$ the corresponding anholonomous four-velocity.

Now, variation of the action (21) yields the equation of motion

$$ \frac{du^a}{ds} + A_{a\beta\nu} u^\beta u^\nu = K_{a\beta\nu} u^\beta u^\nu. $$

(22)

As $A_{a\beta\nu}$ is a connection presenting simultaneously curvature and torsion, this equation is a mixture of geodesic and force equation, with contorsion appearing as a force on its right-hand side. On the other hand, as is well known, the equation of motion for a free particle is

$$ \frac{du^a}{d\sigma} \equiv u^c \partial_c u^a = 0. $$

(23)

A comparison between the above equations shows that the equation of motion (22) can be obtained from the free equation of motion (23) by replacing $\partial_c \rightarrow \mathcal{D}_c = h^\nu_c \mathcal{D}_\nu$, with $\mathcal{D}_\nu$ the covariant derivative written in the vector representation.

The third term of the action (25) represents the coupling of the particle’s spin with the gravitational field. Notice that, according to this prescription, the spin of the particle couples minimally to the Ricci coefficient of rotation.

V. SPINNING PARTICLES

We consider now the motion of a classical particle of mass $m$ and spin $s$ in a gravitational field presenting curvature and torsion. The action integral describing such particle, in the context of a gauge theory, is

$$ S = \int_a^b \left[ -m c \, d\sigma - \frac{1}{c^2} B^a_{\mu} \mathcal{P}_a \, dx^\mu + \frac{1}{2} (A^a_{\mu} - K^a_{\mu}) S_{ab} \, dx^\mu \right], $$

(25)

where $S_{ab}$ is the Noether charge associated with the invariance of the action under Lorentz transformations. In other words, $S_{ab}$ is the spin angular momentum density, which satisfies the Poisson relation

$$ \{ S_{ab}, S_{cd} \} = \eta_{ac} S_{bd} + \eta_{bd} S_{ac} - \eta_{ad} S_{bc} - \eta_{bc} S_{ad}. $$

(26)

The third term of the action (25) represents the coupling of the particle’s spin with the gravitational field. Notice that, according to this prescription, the spin of the particle couples minimally to the Ricci coefficient of rotation.
since, on account of the relation \( A^a b - K^a b = \tilde{A}^a b \), In the presence of curvature and torsion, therefore, the Routhian arising from the action \( 24 \) is

\[
\mathcal{R}_0 = -mc \sqrt{u^2} \frac{d\sigma}{ds} - \frac{1}{c^2} B^a_\mu \mathcal{P}_a u^\mu - \frac{1}{2} (A^a b - K^a b) S_{ab} u^\mu, 
\]

where the weak constraint \( \sqrt{u^2} \equiv \sqrt{u_a u^a} = \sqrt{u^a u^a} = 1 \) has been introduced in the first term. The equation of motion for the particle trajectory is obtained from

\[
\frac{\delta}{\delta x^\mu} \int \mathcal{R}_0 ds = 0, 
\]

whereas the equation of motion for the spin tensor follows from

\[
\frac{dS_{ab}}{ds} = \{\mathcal{R}_0, S_{ab}\}. 
\]

Now, the four-velocity and the spin angular momentum density must satisfy the constraints

\[
S_{ab} S^{ab} = 2s^2 \quad \text{(30)}
\]

\[
S_{ab} u^a = 0 \quad \text{(31)}
\]

Unfortunately, the equations of motions obtained from the Routhian \( \mathcal{R}_0 \) do not satisfy the above constraints. There are basically two different ways of including these constraints in the Routhian. Here, we are going to follow the method used by Yee and Bander \[7\] to take them into account, which amounts to the following. First, a new expression for the spin is introduced,

\[
\tilde{S}_{ab} = S_{ab} - S_{ac} u^c u_b - S_{cb} u^c u_a, \quad \text{(32)}
\]

This new tensor satisfies the Poisson relation \( 26 \) with the metric \( \eta_{ab} - u_a u_b / u^2 \). A new Routhian, which incorporates the above constraints, is obtained by replacing all the \( S_{ab} \) in Eq. \( 27 \) by \( \tilde{S}_{ab} \), and by adding to it the term

\[
\frac{d u^a}{ds} S_{ab} u^b. 
\]

The new Routhian is then found to be

\[
\mathcal{R} = -mc \sqrt{u^2} \frac{d\sigma}{ds} - \frac{1}{c^2} B^a_\mu \mathcal{P}_a u^\mu + \frac{1}{2} (A^a b - K^a b) S_{ab} u^\mu - \frac{d u^a}{ds} S_{ab} u^b u^2, 
\]

where

\[
\frac{d u^a}{ds} = u^\mu D_\mu u^a, 
\]

with \( D_\mu \) given by Eq. \( 18 \).

Using the Routhian \( 33 \), the equation of motion for the spin is found to be

\[
\frac{D S_{ab}}{D s} = (u_a S_{bc} - u_b S_{ac}) \frac{D u^c}{D s}, 
\]

which coincides with the corresponding result of general relativity. Making use of the Lagrangian formalism, the next step is to obtain the equation of motion defining the trajectory of the particle. Through a tedious but straightforward calculation, it is found to be

\[
\frac{D}{D s} \left( mc u^c \right) + \frac{D}{D s} \left( \frac{D u^a}{D s} S_{ac} \right) \frac{D u^c}{D s} = \frac{1}{2} (R_{ab}^{cd} - Q_{ab}^{cd}) S_{ac} S_{bd}. 
\]

Using the constraints \( 30, 31 \), it is easy to verify that

\[
\frac{D u^a}{D s} S_{ac} = -u_a \frac{D S_{ac}}{D s}. 
\]
As a consequence, Eq. (35) acquires the form

$$\frac{D}{Ds} \left( mcu_c + u^a \frac{DS_{ca}}{Ds} \right) = \frac{1}{2} (R^{ab}_{\quad cd} - Q^{ab}_{\quad cd}) S_{ab} u^d. \tag{36}$$

Defining the generalized four-momentum

$$P_c \equiv mcu_c + u^a \frac{DS_{ca}}{Ds},$$

we get

$$\frac{DP_c}{Ds} = \frac{1}{2} (R^{ab}_{\quad cd} - Q^{ab}_{\quad cd}) S_{ab} u^d. \tag{37}$$

This is the equation governing the motion of the particle in the presence of both curvature and torsion. It is written in terms of a general Cartan connection, as well as in terms of its curvature and torsion. It can be rewritten in terms of the torsionless Ricci coefficient of rotation only, in which case it reduces to the ordinary Papapetrou equation. It can also be rewritten in terms of the teleparallel spin connection \(^2\) in which case it reduces to the teleparallel equivalent of the Papapetrou equation,

$$\frac{DP_c}{Ds} = -\frac{1}{2} Q^{ab}_{\quad cd} S_{ab} u^d, \tag{38}$$

with \(Q^{ab}_{\quad cd}\) given by Eq. (13). Notice that the particle’s spin, in this case, couples to a curvature-like tensor, which is a tensor written in terms of torsion only.

**VI. CONCLUSIONS**

Differently from all other interactions of nature, where covariance does determine the gauge connection, and consequently also the corresponding coupling prescription, Lorentz covariance alone is not able to determine the form of the gravitational coupling prescription. The basic reason for this indefiniteness, which occurs only in the simultaneous presence of curvature and torsion, is the affine character of the connection space. To understand this point, it is important to recall the difference between absence of torsion, which happens in “internal” gauge theories, and presence of a vanishing torsion, which happens in general relativity. In both cases, as torsion in one way or another does not appear, the covariance requirement is able to determine the corresponding coupling prescription. However, when torsion does not vanish, which is a possibility for gravitation, the connection defining the covariant derivative becomes indefinite. By using then arguments of consistency with teleparallel gravity, according to which contorsion plays the role of gravitational force, we proposed here that gravitation be minimally coupled to matter only through the Ricci coefficient of rotation \(\overset{o}{A}_{\nu}^a\), the spin connection of general relativity. As it relates to a general Cartan connection \(A_{\nu}^a\) according to

\[\overset{o}{A}_{\nu}^a = A_{\nu}^a - K_{\nu}^a, \tag{39}\]

it becomes a matter of convention to describe the gravitational coupling prescription in terms of the connection \(A_{\nu}^a\), or in terms of the connection \(A_{\nu}^a - K_{\nu}^a\). As a consequence, it becomes also a matter of convention to describe gravitation by curvature, torsion, or by a combination of them. Each one of the infinitely possible cases—characterized by different proportions of curvature and torsion—corresponds to different choices of the spin connection, and all of them are equivalent to the spin connection of general relativity.

An important point of the gravitational coupling prescription is that, in contrast with the Cartan coupling prescription, it results equivalent to apply it in the Lagrangian or in the field equations. Furthermore, in the specific case of the electromagnetic field, the coupling prescription does not violate the \(U(1)\) gauge invariance of Maxwell theory. In fact, analogously to what happens in general relativity, the coupled electromagnetic field strength \(F_{\mu\nu}\) has the form

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{40}$$

which is clearly a gauge invariant tensor.
As a first example, we have applied this minimal coupling prescription to determine the trajectory of a spinless particle in the presence of both curvature and torsion. The resulting equation of motion, which coincides with that obtained from first principles, is a mixture of geodesic and force equations, with the contorsion of the Cartan connection playing the role of gravitational force. How much of the interaction is described by curvature (geometry), and how much is described by contorsion (force), is a matter of convention. In fact, through an appropriate choice of the spin connection, the particle’s equation of motion can be reduced either to the pure force equation (22) of teleparallel gravity, or to the pure geodesic equation (24) of general relativity. Of course, any intermediate case with the connection presenting both curvature and torsion is also possible, being the choice of the connection a kind of “gauge freedom” in the sense that the resulting physical trajectory does not depend on this choice. It is important to remark also that the equivalence between the equations of motion (22) and (24) means essentially that the gravitational force represented by contorsion can always be geometrized in the sense of general relativity. As a consequence, the equivalence principle holds equally for anyone of the above mentioned intermediate cases. On the other hand, by considering that the particle’s spin couples minimally to the spin connection (10), we have obtained the equation of motion (37). In terms of the Ricci coefficient of rotation, it reduces to the ordinary Papapetrou equation

\[
\frac{\mathcal{D} \mathcal{P}_\mu}{\mathcal{D} S} = -\frac{1}{2} R^{ab \mu \nu} S_{ab} u^\nu
\]

(41)

In terms of the teleparallel spin connection, it reduces to Eq. (38), which is the teleparallel version of the Papapetrou equation.

Finally, it is important to notice that, due to the equivalence between curvature and torsion implied by the coupling prescription (18), in contrast with Einstein-Cartan, as well as with other gauge theories for gravitation, no new physics is associated with torsion in this approach. In fact, torsion appears simply as an alternative way of representing the gravitational field. This result means that, at least in principle, general relativity can be considered a complete theory in the sense that it does not need to be complemented with torsion. However, it should be remarked that the description of gravitation in terms of torsion presents some formal advantages in relation to the description in terms of curvature. For example, like in any gauge theory, the equations of motion are not given by geodesics, but by force equations which, similarly to the Lorentz force of electrodynamics, do not rest on the universality of free fall [19]. In the absence of universality, therefore, which is a possibility at the quantum level, the teleparallel gauge approach may become fundamental.

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