ALGEBRAIC QUANTUM FIELD THEORY AND CAUSAL
SYMmetric SPACes

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Abstract. In this article we review our recent work on the causal structure of symmetric spaces and related geometric aspects of Algebraic Quantum Field Theory. Motivated by some general results on modular groups related to nets of von Neumann algebras, we focus on Euler elements of the Lie algebra, i.e., elements whose adjoint action defines a 3-grading. We study the wedge regions they determine in corresponding causal symmetric spaces and describe some methods to construct nets of von Neumann algebras on causal symmetric spaces that satisfy abstract versions of the Reeh–Schlieder and the Bisognano-Wichmann condition.

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1. Introduction
Recent interest in causal symmetric spaces in relation with representation theory arose from their role as analogs of spacetime manifolds in the context of Algebraic Quantum Field Theory (AQFT) in the sense of Haag–Kastler, where one considers nets of von Neumann algebras \( \mathcal{M}(\mathcal{O}) \) on a fixed Hilbert space \( \mathcal{H} \), associated to open subsets \( \mathcal{O} \) in some space-time manifold \( M \) (Ha96). The hermitian elements of the algebra \( \mathcal{M}(\mathcal{O}) \) represent observables

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that can be measured in the “laboratory” $O$. In our context $M$ need not be a time-oriented Lorentzian manifold. We only assume the existence of a field $(C_m)_{m \in M}$ of pointed generating closed convex cones $C_m \subseteq T_m(M)$ that is invariant under a smooth action of a connected Lie group $G$ with Lie algebra $g$.

On de Sitter space $dS^d$ and Anti-de Sitter space $AdS^d$ the causal structure is given by a Lorentzian metric, but in general semisimple causal symmetric spaces are pseudo-Riemann but not Lorentzian (cf. Section 2). This allows us to study causality aspects of AQFT in a highly symmetric context without the need of an invariant Lorentzian form.

One typically requires the following properties:

(I) Isotony: $O_1 \subseteq O_2$ implies $\mathcal{M}(O_1) \subseteq \mathcal{M}(O_2)$.

(L) Locality: $O_1 \subseteq O_2'$ implies $\mathcal{M}(O_1) \subseteq \mathcal{M}(O_2)'$, where $O'$ is the causal complement of $O$, i.e., the maximal open subset that cannot be connected to $O$ by causal curves, and for a von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ its commutant is denoted $\mathcal{N}'$.

(RS) Reeh–Schlieder property: There exists a unit vector $\Omega \in \mathcal{H}$ that is cyclic for every $\mathcal{M}(O)$ with $O \neq \emptyset$, i.e., the subspace $\mathcal{M}(O)\Omega$ is dense in $\mathcal{H}$.

(Cov) Covariance: There exists a unitary representation $U: G \to \mathcal{U}(\mathcal{H})$ such that

$$U(g)\mathcal{M}(O)U(g)^{-1} = \mathcal{M}(gO) \quad \text{for } g \in G.$$ 

(BW) Bisognano–Wichmann property: There exists an open subset $W \subseteq M$ (called a wedge region) such that $\Omega$ is also separating for $\mathcal{M}(W)$, i.e., the map $\mathcal{M}(W) \to \mathcal{H}, A \to A\Omega$ is injective. We further assume that there exists an element $h \in g$, for which the modular operator $\Delta$ associated to the pair $(\mathcal{M}(W), \Omega)$ by the Tomita–Takesaki Theorem \cite[Thm. 2.5.14]{BR87} (cf. (1) below) satisfies $\Delta^{-it/2\pi} = U(\exp(th))$ for $t \in \mathbb{R}$.

(Vac) Invariance of vacuum: $U(g)\Omega = \Omega$ for every $g \in G$.

In this context a natural question is, to which extent such nets of von Neumann algebras exist on a causal homogeneous space $M = G/H$ of a finite-dimensional Lie group $G$, and in particular on causal symmetric spaces. To address this question, it is natural to simplify the structures by considering instead of the pair $(\mathcal{M}, \Omega)$ the corresponding real subspace $V := V_{(\mathcal{M}, \Omega)} := \overline{\mathcal{M}_h\Omega}$, where $\mathcal{M}_h = \{ M \in \mathcal{M} : M^* = M \}$. This subspace is called

- **cyclic** if $V + iV$ is dense in $\mathcal{H}$, which means that $\Omega$ is cyclic for $\mathcal{M}$.
- **separating** if $V \cap iV = \{0\}$, which means that $\Omega$ is separating for $\mathcal{M}$.
- **standard** if it is cyclic and separating, i.e., if $V \cap iV = \{0\}$ and $V + iV = \mathcal{H}$.
These three properties of real subspaces make sense without any reference to operator algebras, but they still reflect an important part of the underlying structures that can be studied in the much simpler context of real subspaces. If $V \subseteq H$ is a standard subspace, then $S_V : V + iV \to H, x + iy \mapsto x - iy$ defines a densely defined closed operator with $V = \text{Fix}(S_V)$; the Tomita operator of $V$. Its polar decomposition can be written as

(1) \[ S_V = J_V \Delta_V^{1/2} \]

where $J_V$ is a conjugation (an anti-unitary involution) and $\Delta_V$ is a positive selfadjoint operator satisfying $J_V \Delta_V J_V = \Delta_V^{-1}$. For standard subspaces of the form $V(M, \Omega)$ we thus recover the modular objects provided by the Tomita–Takesaki Theorem. In particular, the interaction between unitary group representations and nets of von Neumann algebras can already be studied in the context of nets of real subspaces. A net of real subspaces on $M$ is a family $V(O) \subseteq H$ of closed real subspaces of a complex Hilbert space $H$, assigned to open subsets $O$ of a causal manifold $M$. We consider the following properties:

(I) Isotony: $O_1 \subseteq O_2$ implies $V(O_1) \subseteq V(O_2)$.

(L) Locality: $O_1 \subseteq O'_2$ implies $V(O_1) \subseteq V(O_2)' := V(O_2) \perp \omega$, where $\omega = \text{Im} \langle \cdot, \cdot \rangle$ is the canonical symplectic form on $H$.

(RS) Reeh–Schlieder property: If $O$ is non-empty, then $V(O)$ is cyclic.

(Cov) Covariance: There exists a unitary representation $U : G \to U(H)$ such that $U(g)V(O) = V(gO)$ for $g \in G$.

(BW) Bisognano–Wichmann property: There exists an open subset $W \subseteq M$ (a “wedge region”) such that $V(W)$ is standard and an element $h \in \mathfrak{g}$ such that $\Delta_V^{-it/2\pi} = U(\exp th)$ holds for all $t \in \mathbb{R}$.

Nets of standard subspaces serve as building blocks for nets of von Neumann algebras. Applying second quantization functors (such as bosonic or fermionic second quantization; see [Si74]) to associate to each real subspace $V \subseteq H$ a pair $(\mathcal{R}(V), \Omega)$, where $\mathcal{R}(V) \subseteq B(\mathcal{F}(H))$ is a von Neumann algebra on a suitable Fock space $\mathcal{F}(H)$, and if $V$ is cyclic/separating, then the vacuum vector $\Omega$ is cyclic/separating for $\mathcal{R}(V)$. This method has been developed by Araki and Woods in the context of free bosonic quantum fields ([Ar63, Ar64, AW63]); some of the corresponding fermionic results are more recent (cf. [EO73, BJL02]). Other statistics (anyons) are developed in [Schr97] and more recent deformations are discussed in [Le15, §3]. Thus any net of real subspaces defines a free quantum field in the sense of Haag–Kastler on the corresponding Fock space. The properties listed above for the net $\mathcal{R}(V(O))$ follow from the corresponding ones for the net $V(O)$ and $\Omega$ is the canonical vacuum vector.

Conversely, any net of von Neumann algebras $\mathcal{M}(O)$ satisfying the listed properties immediately leads to the net $V(\mathcal{M}(O), \Omega)$ of standard subspaces with the corresponding properties. Having these two passages in mind, we
therefore content ourselves in the following with the discussion of nets of standard subspaces.

The current interest in standard subspaces arose in the 1990’s from the work of Borchers and Wiesbrock \[Bo92, Wi93\]. This led to the concept of modular localization in AQFT introduced by Brunetti, Guido and Longo in \[BGL02, BGL93\]; see also \[BDFS00\] and \[Le15, LL15\] for important applications of this technique.

We know from \[MN22b\] that, if \( \ker(U) \) is discrete, then the elements \( h \in \mathfrak{g} \) generating the modular group of the standard subspace \( V(W) \) in (BW) is an Euler element, i.e., \( \text{ad} \ h \) defines a 3-grading
\[
\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h), \quad \text{where} \quad \mathfrak{g}_\lambda(h) = \ker(\text{ad} \ h - \lambda \mathbb{1}).
\]
Moreover, the modular conjugation \( J = J_{\mathfrak{V}(W)} \) of \( \mathfrak{V}(W) \) satisfies
\[
JU(g)J = U(\tau_h(g)) \quad \text{for} \quad \tau_h(\exp x) = \exp(e^{\pi i \text{ad} \ h} x), \quad x \in \mathfrak{g}.
\]
It follows in particular that the unitary representation \( U : G \rightarrow U(\mathcal{H}) \) extends by \( U(\tau_h) := J \) to an anti-unitary representation of the extended group
\[
G_{\tau_h} := G \rtimes \{ \text{id}_G, \tau_h \}
\]
in the sense of \[MN21, NO17\] (cf. Subsection 4.2).

At this point, we are thus facing the following questions:

(Q1) What are the natural causal spaces \( M \) on which such structures exist and how does the existence of Euler elements relate to the geometry of these spaces?

(Q2) Given \( M \) and an Euler element \( h \in \mathfrak{g} \), what are the natural wedge regions \( W \subseteq M \) for which (BW) is satisfied?

(Q3) Given an anti-unitary representation \((U, \mathcal{H})\) of \( G_{\tau_h} \), how can we construct corresponding nets of standard subspaces?

(Q4) Which unitary representations \((U, \mathcal{H})\) of \( G \) occur?

For the simplicity of exposition, we restrict in the following to the special case, where \( \mathfrak{g} \) is a simple real Lie algebra. Many results are true with suitable modifications in a more general context (cf. \[NO21, Oeh22\]). The existence of an Euler element \( h \in \mathfrak{g} \) leads to a natural family of causal symmetric spaces of the form \( G/H \), where \( G \) is a connected Lie group with Lie algebra \( \mathfrak{g} \) (see Section 2 and \[MN22a\] for details). If, in addition, \( \mathfrak{g} \) is simple hermitian, i.e., isomorphic to
\[
\text{su}_{p,q}(\mathbb{C}), \quad \text{so}_{2,d}(\mathbb{R}), \quad \text{sp}_{2n}(\mathbb{R}), \quad \text{so}^*(2n), \quad \mathfrak{e}_{6(-14)} \quad \text{or} \quad \mathfrak{e}_{7(-25)}
\]
(see \[Her8\] for the notation), the Euler element specifies a causal structure on its associated minimal flag manifold \( G/P^-(h) \) (cf. Section 3) whose simply connected covering is a simple space-time manifolds in the sense of Mack–de Riese \[MdR07\]. We think of this observation as a suitable answer to (Q1).

On some of these spaces the causal curves define a global order structure with compact order intervals (they are called globally hyperbolic) and in this context one can also prove the existence of a global “time function”
with group theoretic methods (see [Ne91]). In fact, if \((G, \tau, H, C)\) is a non-compactly causal symmetric Lie group (cf. Section 2), then \(S = (\exp C)H\) is a closed subsemigroup, such that the polar map \(C \times H \to S\) is a homeomorphism and the causal ordering on \(M = G/H\) is given by \(aH \geq bH\) if \(b^{-1}a \in S\). In particular the set \(\{x \in G/H : eH \leq x\}\) is homeomorphic to the cone \(C\). On the other hand, if the center \(Z(G)\) is finite, then the compactly causal symmetric spaces \(G/H\) have closed causal curves. Hence there is no global causal ordering and in particular \(O'\) cannot be defined in terms of this order; typically \(O' = \emptyset\). We refer to the monograph [H ´O97] for more details and a complete exposition of the classification of irreducible symmetric spaces, which is also explained in [MN ´O22a] from the perspective of Euler elements.

On a superficial level, the answer to (Q2) is also rather simple. Given a causal symmetric space \(M = G/H\) with cone field \((C_m)_{m \in M}\) and the Euler element \(h \in \mathfrak{g}\), we consider on \(M\) the modular flow \(\alpha_t(gH) = \exp(th)gH\) and write \(X^h_M\) for its infinitesimal generator, a smooth vector field on \(M\).

Then its positivity domain \((5)\)
\[
W := W^+_M(h) := \{m \in M : X^h_M(m) \in C^0_m\}
\]
turns out to be the most natural candidate for a wedge region in \(M\). However, the geometry of these domains is not so easy to understand, and it took us some effort to understand these domains (see Section 3, [N ´O22a, N ´O22b, MN ´O22b]). On the wedge region \(W\), the (BW) property has the physical interpretation that the action of the modular group on the algebra \(\mathcal{M}(W)\) is considered to encode the dynamics, the flow of time, for the quantum theory on \(W\) ([Su05, §6.4]). This connects naturally with the approach of Connes and Rovelli who construct the dynamics of a quantum statistical system as a modular one-parameter group \(\Delta^t\) ([CR94]). As the flow of time should “point into the future”, the positivity domain \(W^+_M(h)\) is a natural candidate of a domain for which (BW) could be satisfied.

Covariant families of real subspaces of \(\mathcal{H}\), which are not necessarily standard are easy to construct in any anti-unitary representation \((U, \mathcal{H})\) of \(G_\tau\), using distribution vectors (cf. Section 4). To a finite-dimensional \(H\)-invariant real linear subspace \(E \subseteq \mathcal{H}\) and an open subset \(O \subseteq M = G/H\), we associate a closed real subspace of \(\mathcal{H}\) as follows. Let \(q : G \to G/H, g \mapsto gH\) be the canonical projection. Then we put
\[
(6)\quad H_E(O) := \text{span}_\mathbb{R}\{U^{-\infty}(\varphi)E : \varphi \in C^c_c(q^{-1}(O), \mathbb{R})\} \subseteq \mathcal{H}.
\]
Here \(U^{-\infty}(\varphi) : \mathcal{H}^{-\infty} \to \mathcal{H}\) is the linear smoothing operator defined by
\[
(U^{-\infty}(\varphi)\eta)(\xi) = \int_G \varphi(g)\eta(U(g^{-1})\xi) \, dg \quad \text{for} \quad \xi \in \mathcal{H}^{\infty}
\]
(cf. Section 4.1). It is easy to see that \(H_E\) defines a net of real subspaces satisfying (I) and (Cov). So the real problem is to find subspaces \(E\) such that (RS) and (BW) hold as well (cf. [N ´O21]). Typically the elements \(\eta \in E\)
satisfy a KMS-like condition of the following form: The orbit map

\[ U^\eta: \mathbb{R} \to \mathcal{H}^{-\infty}, \quad t \mapsto U^{-\infty}(\exp t \eta) \]

extends to a map on the closure of the strip

\[ S_\pi := \{ z \in \mathbb{C} : 0 < \text{Im} z < \pi \} \]

which is a holomorphic \( \mathcal{H} \)-valued map on the interior, weak-\( * \)-continuous on the closed strip and

\[ U^\eta(t + \pi i) = JU^\eta(t) \quad \text{for} \quad t \in \mathbb{R}. \]  

This implies in particular that \( v := U^\eta(\frac{\pi i}{2}) \in \mathcal{H} \) satisfies \( Jv = v \) and

\[ \eta = \lim_{t \to -\pi/2} \Delta t/2^\pi v \]

in the weak-* topology (cf. Proposition 5.1).

The locality property (L) is even more difficult to implement because for compactly causal symmetric spaces there may be closed causal curves, so that \( \mathcal{O}' \) may be empty and (L) is satisfied for trivial reasons. To understand the geometry underlying the locality condition (L) is an important problem for the future.

We write \( \partial U(x) \) for the skew-adjoint infinitesimal generator of the unitary one-parameter group \( U(\exp tx) = e^{t\partial U(x)}, \ t \in \mathbb{R} \). Presently, the best understood situation is the case where the positive cone

\[ C_U = \{ x \in \mathfrak{g} : -i\partial U(x) \geq 0 \} \]

of the representation \( U \) is non-trivial. This relates naturally to compactly causal symmetric spaces (cf. [NO22a]). If \( \mathfrak{g} \) is simple and \( U \) is irreducible, [5] specifies unitary highest/lowest weight representations. We refer to [NO21] for constructions of nets satisfying (I), (RS), (Cov) and (BW) for the left translation action of a reductive Lie group \( G \) on itself and also on the flag manifolds \( G/P^-(h) \) corresponding to simple space-time manifolds (cf. Section 6). In [Oeh22] one finds generalizations to non-reductive Lie groups. In [NO22a] one finds a construction for compactly causal symmetric spaces \( M = G/H \) (cf. Section 2), such as Anti-de Sitter space \( \text{AdS}^d \cong \text{SO}_{2,d-1}(\mathbb{R})_e/\text{SO}_{1,d-1}(\mathbb{R})_e \).

Constructions of nets on non-compactly causal spaces are developed in [FNÖ22], but we are still far from a classification of those \( H \)-invariant finite-dimensional subspaces \( \mathcal{E} \subseteq \mathcal{H}^{-\infty} \) for which the net \( \mathcal{H}_\mathcal{E} \) satisfies (I), (RS), (Cov) and (BW). However, we expect that for every irreducible unitary representation \( (U, \mathcal{H}) \) of \( G \) such subspaces exist for the corresponding non-compactly causal symmetric spaces such as de Sitter space \( dS^d \cong \text{SO}_{1,d}(\mathbb{R})_e/\text{SO}_{1,d-1}(\mathbb{R})_e \). Here is the central idea of our construction: Pick a Cartan involution \( \theta \) of \( G \) (Section 2) with \( \theta(h) = -h \) and consider the subgroup \( K := G^\theta \). Let \( \mathcal{F} \subseteq \mathcal{H} \) be a finite-dimensional \( U(K) \)-invariant subspace
consisting of $J$-fixed vectors and consider the subspace

$$E := \beta(F) \quad \text{for} \quad \beta(v) := \lim_{t \to -\pi/2} e^{it \cdot \partial U(h)} v.$$ 

Here the main difficulty is to show that $v$ is contained in the domain of the selfadjoint operators $e^{it \cdot \partial U(h)}$ for $|t| < \pi/2$ and that the limit $\beta(v)$ exists in the space of distribution vectors of $(U, \mathcal{H})$ (cf. [3] and [FNO22]).

In [MNÖ22b, Thm. 7.1] we have seen that if $G$ is simple with trivial center (hence isomorphic to the adjoint group $\text{Ad}(G) = \text{Inn}(g)$ of inner automorphisms), then the positivity domain $W = W_M^+(h)$ in the corresponding non-compactly causal symmetric space $M = G/H$, endowed with the maximal invariant cone field, is connected and that its stabilizer subgroup $G_W = \{g \in G : gW = W\}$ is $G^h = \{g \in G : \text{Ad}(g)h = h\}$. Therefore the wedge space $W := G.W = \{g.W : g \in G\}$ (the set of wedge regions in $M$) can be identified with the adjoint orbit $O_h := \text{Ad}(G)h \subseteq \mathfrak{g}$ of the Euler element in $\mathfrak{g}$ (Theorem 3.1(3)). This connects the above construction based on distribution vectors with the abstract wedge spaces studied in [MN21]. In fact, starting with an anti-unitary representation $(U, \mathcal{H})$ of $\tau_h$ (see (4)), we can associate to the wedge region $W = W_M^+(h)$ the standard subspace

$$H_{\text{BGL}}(W) := \text{Fix}(J\Delta^{1/2}), \quad \text{where} \quad J = U(\tau_h) \quad \text{and} \quad \Delta := e^{2\pi i \cdot \partial U(h)}.$$ 

From the identity $G_W = G^h$, it then follows that we obtain a well-defined covariant extension

$$H_{\text{BGL}}(g.W) := U(g)H_{\text{BGL}}(W).$$

This is called the Brunetti–Guido–Longo (BGL) construction (cf. [BGL02], [NÖ17]). Now

$$H_{\text{BGL}}(O) := \bigcap \{U(g)H_{\text{BGL}}(W) : g \in G, O \subseteq g.W\}$$

defines a net of real subspaces satisfying (I), (Cov) and (BW). If $E \subseteq \mathcal{H}^{-\infty}$ is a real subspace for which the net $H_E$ in (6) satisfies (BW), we immediately obtain the relation

$$H_{\text{BGL}}(O) \supseteq H_E(O).$$

In particular the net $H_{\text{BGL}}(O)$ inherits (RS) from $H_E(O)$. We presently do not know for which domains $O$ and subspaces $E$ equality holds in (12) (see [MN22b] for more details). The structure of this paper is as follows: In Section 2 we discuss the connection between Euler elements and causal symmetric spaces. We also introduce the crown of the Riemannian symmetric space $G/K$. In Section 3 we then turn to the wedge regions in causal symmetric spaces. Section 4 reviews some concepts related to unitary representations that are illustrated by some examples. Standard subspaces and some of their key properties are presented in Section 5. Finally, we briefly turn to Cayley type spaces and their causal compactifications in Section 6.
We hope that this survey provides a useful presentation of the fascinating connection between causal manifold structures, representation theory and AQFT that is accessible to a wide audience.

2. Causal symmetric spaces and Euler elements

In this section we discuss causal symmetric spaces and Euler elements and in particular how Euler elements specify non-compactly causal symmetric spaces ([MNÖ22a]). First we review basic definitions and concepts related to causal symmetric spaces. On the algebraic level we deal with a symmetric Lie algebra \((\mathfrak{g}, \tau)\), i.e., \(\mathfrak{g}\) is a finite-dimensional real Lie algebra and \(\tau\) is an involutive automorphism of \(\mathfrak{g}\). We then have

\[
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{with} \quad \mathfrak{h} := \mathfrak{g}^\tau := \ker(\tau - 1) \quad \text{and} \quad \mathfrak{q} := \mathfrak{g}^{-\tau} := \ker(\tau + 1).
\]

A causal symmetric Lie algebra is a triple \((\mathfrak{g}, \tau, C)\), where \((\mathfrak{g}, \tau)\) is a symmetric Lie algebra and \(C \subseteq \mathfrak{q}\) is a pointed generating closed convex cone invariant under the group \(\text{Inn}_{\mathfrak{g}}(\mathfrak{h}) := \langle e^\text{ad}_h \rangle\). A causal symmetric Lie algebra \((\mathfrak{g}, \tau, C)\) is called compactly causal (cc for short) if the pointed generating cone \(C\) is elliptic, i.e., if its interior consists of elements \(x\) which are elliptic in the sense that \(\text{ad}_x\) is semisimple with purely imaginary spectrum. Typical examples of compactly causal Lie algebras arise from invariant pointed generating cones \(C_\mathfrak{q} \subseteq \mathfrak{g}\) satisfying \(\tau(C_\mathfrak{q}) = C_\mathfrak{q}\) by \(C := \mathfrak{q} \cap C_\mathfrak{g}\).

Duality of causal symmetric Lie algebras: Causal symmetric Lie algebras come in pairs: If \((\mathfrak{g}, \tau, C)\) is causal, then the \(c\)-dual symmetric Lie algebra is \((\mathfrak{g}^c, \tau^c, iC)\) with \(\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}\) and \(\tau^c(x + iy) := x - iy\) is also causal. This duality exchanges compactly causal and non-compactly causal spaces.

On the global level we call a quadruple \((G, \tau, H, C)\) a causal symmetric Lie group if \(G\) is a connected Lie group, \(\tau\) an involutive automorphism of \(G\), \(H \subseteq G^\tau\) an open subgroup, and \(C \subseteq \mathfrak{q}\) a pointed generating \(\text{Ad}(H)\)-invariant closed convex cone. Then \(M := G/H\) is called the associated symmetric space. We refer to [He78,Lo69] for the basic theory of symmetric spaces and to [HÖ97] for causality aspects.

The involution \(\tau\) on \(G\) induces an involution \(\tau_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}\) such that \(\exp(\tau_\mathfrak{g}(x)) = \tau(\exp(x))\) for \(x \in \mathfrak{g}\) and \((\mathfrak{g}, \tau_\mathfrak{g}, C)\) is a causal symmetric Lie algebra. For simplicity, we shall also write simply \(\tau\) instead of \(\tau_\mathfrak{g}\) on the Lie algebra. We say that \(M\) is compactly causal, resp., non-compactly causal if \((\mathfrak{g}, \tau, C)\) has this property. The space \(\mathfrak{q}\) can be identified with the tangent space of \(M\) at the origin \(x_M = eH\) by the tangent map

\[
T_0(q \circ \exp|_q) : \mathfrak{q} \to T_{x_M}(M), \quad x \mapsto \left. \frac{d}{dt} \right|_{t=0} \exp(tx).x_M.
\]

If \((G, \tau, H, C)\) is a causal symmetric Lie group, then \(G/H\) becomes a causal \(G\)-space in the sense that the diffeomorphisms \(\sigma_g(xH) := gxH\) on \(G/H\) lead to a \(G\)-invariant cone field

\[
C_{gH} := T_{x_M}(\sigma_g)C \subseteq T_{yH}(M) \quad \text{for} \quad g \in G.
\]
If $G$ is simple, as we are assuming in this article, with finite center, then, for compactly causal spaces, there are periodic causal curves, so that no global causal order exists on $M$. However, this pathology can be resolved by passing to the simply connected covering $\tilde{M}$. A typical example is Anti-de Sitter space (see Example 2.2 below). Non-compactly causal spaces always carry a global order which is *globally hyperbolic* in the sense that all order intervals are compact ([HO97, Thm. 5.3.5]).

**Cartan involutions:** An involutive automorphism $\theta$ of $g$ is called a *Cartan involution* if $\text{Inn}(g^\theta)$ is a maximal compact subgroup of the group $\text{Inn}(g)$ of inner automorphisms. We write $k = g^\theta$, $p = g^{-\theta}$, $K := G^\theta$ and note that $G/K$ is a Riemannian symmetric space (see [He78] for more on the geometry of Riemannian symmetric spaces).

If $m$ is a $\theta$-stable subspace of $g$ then we write $m_k = m \cap k$ and $m_p = m \cap p$, so that $m = m_k \oplus m_p$. If $\tau : g \to g$ is an involution, then there exists a Cartan involution $\theta$ such that $\theta \tau = \tau \theta$ ([KN96, Prop. I.5]) and we thus obtain decompositions $h = h_k \oplus h_p$ and $q = q_k \oplus q_p$.

**Classifying ncc spaces by Euler elements:** Let $h \in g$ be an Euler element. By [KN96, Cor. II.9] there exists a Cartan involution $\theta$ such that $\theta(h) = -h$. We write $E(g) \subseteq g$ for the set of Euler elements in $g$. Then $G$ acts on $E(g)$ by the adjoint action with finitely many orbits (see [MN21, Thm. 3.10] and [Kan98, Kan00] for a classification).

If $(g, \tau, C)$ is causal, then an Euler element $h \in C^0$ is said to be *causal*. For any pair $(\theta, h)$ of a Cartan involution $\theta$ and an Euler element satisfying $\theta(h) = -h$, the involution $\tau = \tau_h \theta$ with $\tau_h = e^{\pi i \text{ad} h}$ makes $h$ causal for $(g, \tau, C)$, where $C \subseteq q$ is the closed convex cone generated by $\text{Inn}_q(h)h$. In [MN022a, Thm. 4.21] this construction is used to classify irreducible non-compactly causal symmetric Lie algebras in terms of $\text{Inn}(g)$-orbits of Euler elements. In particular, for $(g, \tau, C)$ non-compactly causal, the set of causal Euler elements is non-empty and contained in a single $\text{Inn}(g)$-orbit. By duality, this also yields a classification of irreducible cc symmetric spaces. From the dual perspective, focusing on cc spaces, this classification goes back to [Ol91].

**The crown of the Riemannian symmetric space $G/K$:** Throughout our constructions, analytic continuation of functions and orbit maps play a central role. In this respect an important tool is the *crown domain of the Riemannian symmetric space $G/K$*. To describe it, we assume that $G$ is contained in a complex Lie group $G_C$ with a Lie algebra $g_C$. Let

$$\Omega_p = \{x \in p : \sigma(\text{ad} x) \subset (-\pi/2, \pi/2)\},$$

where $\sigma(\text{ad} x)$ denotes the spectrum of $\text{ad} x$, and define the *crown of $G/K$* by

$$\Xi = G(\exp i\Omega_p)K_C \subset G_C/K_C,$$

where $K_C = K \exp(i\mathfrak{t})$.

Then $\Xi$ is open in $G_C/K_C$ by [KS01, KS05] and every eigenfunction of the algebra of $G$-invariant differential operators on $G/K$ extends to a holomorphic
function on $\Xi$ \cite{KrSc09}. Of particular interest for us is the following result of Gindikin and Krótz on the realization of ncc symmetric spaces $G/H$ in the boundary of a crown domain \cite[Lem. 3.4, Thm 3.5]{GKO02}:

**Theorem 2.1.** Let $(\mathfrak{g}, \tau, C)$ be a simple ncc symmetric Lie algebra, specified by the causal Euler element $h \in \mathfrak{q}_p$ via $\tau = \theta \tau_h$ and $G$ a connected Lie group contained in a complex group $G_C$ with Lie algebra $\mathfrak{g}_C$. Let $K_C = K \exp(\i t) \subseteq G_C$ and $s_H := \exp\left(\frac{\pi}{4} h\right)$. Then

$$H := s_H K_C s_H \cap G \subseteq G^\tau$$

is an open subgroup,

$$x_H := s_H K_C \in \partial \Xi \quad \text{and} \quad G/H \cong G.x_H.$$ 

Thus, up to covering, every ncc symmetric space can be realized in the boundary of the crown of $G/K$. So the crown can be used as a tool to translate between different symmetric spaces, in particular the Riemannian symmetric spaces $G/K$ and all ncc spaces of the form $G/H$. In particular, boundary values of holomorphic functions on $\Xi$ lead to distributions on $G/H$, and a generalization of this idea to vector bundles can be used to realize unitary representations of $G$ in spaces of distributional sections of vector bundles over $G/H$ (cf. \cite{FNÖ22}). We also refer to \cite{GKO03, GKÖ04} and \cite{NO20} for some results in this direction.

In this context, an important result by Krótz–Stanton \cite{KS04} is that, if $G$ is contained in a simply connected complex group $G_C$, for any $K$-finite vector in an irreducible unitary $G$-representation $(U, \mathcal{H})$, there exists a holomorphic extension $\hat{U}^v$ of the orbit map $U^v$

$$\hat{U}^v: G \exp(\i \Omega_{\mathfrak{p}})K_C \to \mathcal{H}.$$ 

In particular, $e^{\i t \theta U(h)} v = \hat{U}^v(\exp(\i \theta h))$ is defined for $|t| < \pi/2$ (cf. \cite{10}).

**Example 2.2.** (De Sitter and Anti-de Sitter space) An important example of a non-compactly causal irreducible symmetric space is *de Sitter space*

$$\text{(16)} \quad dS^d := \{(x_0, x_1, \ldots, x_d) \in \mathbb{R}^{1,d}: x_0^2 - x_1^2 - \cdots - x_d^2 = -1\}$$

with $G = SO_{1,d}(\mathbb{R})_e$, $H = Ge_1 \cong SO_{1,d-1}(\mathbb{R})_e$, and $C \subseteq T_{e_1}(dS^d) \cong e_1^\perp$ given by

$$C = \{(x_0, x_2, \ldots, x_{d-1}, x_d): x_0 \geq 0, x_0^2 \geq x_2^2 + \cdots + x_d^2\},$$

the closed light cone in $\mathbb{R}^{1,d-1}$.

Likewise *Anti-de Sitter space* is a compactly causal irreducible symmetric space

$$\text{(17)} \quad \text{AdS}^d := \{(x_0, x_1, \ldots, x_d) \in \mathbb{R}^{2,d-1}: x_0^2 + x_1^2 - x_2^2 - \cdots - x_d^2 = 1\}$$

with $G = SO_{2,d-1}(\mathbb{R})_e$, $H = Ge_1 \cong SO_{1,d-1}(\mathbb{R})_e$, and $C \subseteq T_{e_1}(\text{AdS}^d) \cong e_1^\perp$ given by

$$C = \{(x_0, x_2, \ldots, x_d): x_0 \geq 0, x_0^2 \geq x_2^2 + \cdots + x_d^2\}.$$
These spaces are dual to each other and can both be realized in the complex sphere
\[ dS^d_C := \{ z \in \mathbb{C}^{d+1} : z_0^2 - z_1^2 - \cdots - z_d^2 = -1 \} \cong SO_{d+1}(\mathbb{C})/SO_d(\mathbb{C}). \]
In the case of de Sitter space \( G^{dS}/H^{dS} \), the corresponding Riemannian symmetric space \( G^{dS}/K^{dS} \) is naturally realized as a \( d \)-dimensional hyperbolic space in \( dS^d_C \):
\[ \mathbb{H}^d := \{ (x_0, x) \in \mathbb{R}^{d+1} : x_0^2 - x^2 = 1, x_0 > 0 \} = SO_{1,d}(\mathbb{R})_e. ie_0 \]
isomorphic to \( SO_{1,d}(\mathbb{R})_e/SO_d(\mathbb{R}) \). The crown of \( \mathbb{H}^d \) is the domain
\[ \Xi = dS^d_C \cap (\mathbb{R}^{d+1} + iV_+) = \{ z = (z_0, z) \in \mathbb{R}^{d+1} + iV_+ : z_0^2 - z^2 = -1 \}, \]
where \( V_+ := \{ (x_0, x) \in \mathbb{R}^{d+1} : x_0^2 - x^2 > 0, x_0 > 0 \} \) is the open upper light cone (see [NO20, Prop. 3.2]). Note that the Euler element \( h \in so_{1,d}(\mathbb{R}) \) defined by \( h x = (x_1, x_0, 0, \ldots, 0) \) satisfies
\[ \exp(z h). ie_0 = i \cosh(z) e_0 + i \sinh(z) e_1 \in \Xi \quad \text{for} \quad \text{Im} z < \frac{\pi}{2}, \]
and \( \exp(\pm \frac{\pi i}{2}) h. ie_0 = \mp e_1 \in dS^d \), which exhibits \( dS^d \) as a \( SO_{1,d}(\mathbb{R})_e \)-orbit in \( \partial \Xi \) (cf. Theorem 2.1).

As described above, we expect that the embedding of de Sitter space into the boundary of \( \Xi \), combined with the Krötz–Stanton extension technique (cf. [15]), can be used to realize all irreducible unitary representations of the Lorentz group in distributional sections of vector bundles over \( dS^d \). In the particular case of irreducible representations with a \( K \)-fixed vector (spherical representations), we are simply dealing with functions. This case has been addressed with methods based on reflection positivity on the sphere \( S^d \) in [NO20]. The realization on \( \Xi \) leads a reproducing kernel space of holomorphic functions whose kernel is given by a hypergeometric function
\[ \Psi_m((z_0, z), (w_0, w)) = 2F1 \left( \lambda + \frac{d-1}{2}, -\lambda + \frac{d-1}{2}, \frac{n}{2}, 1 - \frac{z_0 w_0 + z w}{r^2} \right), \]
where \( m > 0 \) and
\[ \lambda := \sqrt{\left( \frac{d-1}{2} \right)^2 - m^2} \in i\mathbb{R} \cup \left[ 0, \frac{d-1}{2} \right). \]
For \( w = e_1 \), we obtain the \( H \)-invariant function
\[ \Psi_m((z_0, z), e_1) = 2F1 \left( \lambda + \frac{d-1}{2}, -\lambda + \frac{d-1}{2}, n, 1 + z_1 \right), \]
which has distributional boundary values on \( dS^d \) with a singularity at \( z_1 = 1 \) (cf. [GKO04, Thm. 2.2.4]). We also refer to [BM96] for closely related results by J. Bros and H. Moschella.

**Example 2.3** \( sl_2(\mathbb{R}) \cong so_{1,2}(\mathbb{R}) \cong su_{1,1}(\mathbb{C}) \). (a) The lowest dimensional example of a simple causal symmetric Lie algebra arises for the 3-dimensional
Lie algebra \( \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \) and the corresponding connected Lie group \( G = \text{SL}_2(\mathbb{R}) \) as follows. In the basis

\[
(18) \quad h = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

the element \( h \) is an Euler element with

\[ [h, e] = e, \quad [h, f] = -f \quad \text{and} \quad [e, f] = 2h. \]

Further \( \theta(x) = -x^\tau \) is a Cartan involution with \( \theta(h) = -h \), so that \( \tau := \theta h \) specifies a non-compactly causal symmetric Lie algebra with

\[
\tau: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.
\]

In particular \( \mathfrak{h} = \mathbb{R}h, \mathfrak{q} = \mathbb{R}e \oplus \mathbb{R}f \) and \( H := G^\tau = G^h \cong \mathbb{R}^\times \) is the subgroup of diagonal elements. Note that the adjoint orbit \( O_h \cong G/G^h \cong \text{SO}_{1,2}(\mathbb{R})_e/\text{SO}_{1,1}(\mathbb{R})_e \) can be identified with de Sitter space \( dS^2 \) (see [MN22a] for a detailed discussion).

The cone generated by \( \text{Inn}_q(h)h \) is \( C_{ncc} := [0, \infty)e \oplus [0, \infty)f \) and the quarter plane \( C_{cc} = [0, \infty)e \oplus [0, \infty)(-f) \) is also \( \text{Ad}(H) \)-invariant. So the symmetric Lie algebra \((\mathfrak{g}, \tau, C_{ncc})\) is non-compactly causal and \((\mathfrak{g}, \tau, C_{cc})\) is compactly causal.

(b) Now replace \( \text{SL}_2(\mathbb{R}) \) by its adjoint group \( G := \text{SO}_{1,2}(\mathbb{R}) e \cong \text{Inn}(\mathfrak{g}) = \text{PSL}_2(\mathbb{R}) \). The Cartan involution \( \theta \) commutes with \( \tau \), so that \( \theta \in G^\tau \). As \( \theta(e + f) = -(e + f) \), the cone \( C_{ncc} \) is not \( G^\tau \) invariant. On the other hand, \((G, G^\tau, C_{cc})\) is still compactly causal. This shows that the causal properties of \((G, H, C)\) depend on the group \( \pi_0(H) = H/H_e \) of connected components of \( H \). Note that \( G/G^\tau_e = G/G^h \cong dS^2 \) is an example of a causal space of Cayley type (cf. Section 6).

(c) The group \( G \cong \text{PSU}_{1,1}(\mathbb{C}) \cong \text{SO}_{1,2}(\mathbb{R})_e \) acts by Möbius transformations

\[
(a \ b) \begin{pmatrix} z \\ \bar{z} \end{pmatrix} := \frac{az + b}{\bar{b}z + \bar{a}}
\]

on the unit disc \( \mathbb{D}_\tau := \{ z \in \mathbb{C}: |z| < 1 \} \) and since \( K := \{ g \in G: g.0 = 0 \} \cong \mathbb{T} \) is maximal compact, this leads to the identification \( \mathbb{D} \cong G/K \). The crown domain of \( \mathbb{D} \) can be identified with the bidisc \( \mathbb{D}^2 \) ([KS05] Thm. 7.7), into which \( \mathbb{D} \) embeds via \( z \mapsto (z, \overline{z}) \) as a totally real submanifold and \( G \) acts on \( \mathbb{D}^2 \) by \( g.(z, w) = (g.z, \overline{g.w}) \).

For the Euler element \( h := \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{su}_{1,1}(\mathbb{C}) \), the orbit map

\[
\mathbb{R} \to \mathbb{D}, \quad t \mapsto \exp(th).0 = \begin{pmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{pmatrix}.0 = \tanh(t/2)
\]

extends to a biholomorphic map

\[
(19) \quad S_{\pm \pi/2} = \{ z \in \mathbb{C}: |\text{Im } z| < \frac{\pi}{2} \} \to \mathbb{D}, \quad z \mapsto \tanh(z/2).
\]
This implies that, in the closed polydisc \( \overline{D^2} \), we have
\[
\exp \left( \frac{\pi i}{2} \hbar \right) (0, 0) = i \tan \left( \frac{\pi}{4} \right) (1, 1) = (i, i)
\]
and with \( H := \{ g \in G : g.i = i, g.i = i \} \) it follows that \( G.(i, i) \cong G/H \cong dS^2 \) is a non-compactly causal symmetric space, embedded in the Shilov boundary \( \mathbb{T}^2 \) of \( \mathbb{D}^2 \) (cf. Theorem 2.1).

**Example 2.4.** (Complex case and group case) (a) If \( g \) is a complex Lie algebra and \( \tau \) is antilinear, then \( g = \mathfrak{h}_C \) and \( \mathfrak{h} \) is a real form of \( g \), so that we have \( \tau(x + iy) = x - iy \) for \( x, y \in \mathfrak{h} \) and \( q = i\mathfrak{h} \). We assume that the involution \( \tau \) on \( g \) integrates an involution, also denoted \( \tau \), on the group \( G \). Then, with \( H = G^\tau \), the symmetric space \( G/H \) is causal if and only if there exists a pointed generating \( \text{Ad}(H) \)-invariant cone \( C \subseteq \mathfrak{h} \), which means that \( \mathfrak{h} \) is a simple hermitian Lie algebra. Then \( C \) is elliptic, so that \( iC \) is a hyperbolic cone in \( q \), and thus \( (g, \tau, iC) \) is ncc.

(b) The \( c \)-dual \( (g^c, \tau^c) \) of \( (g, \tau) \) is isomorphic to the symmetric pair \( (\mathfrak{h} \oplus \mathfrak{h}, \tau^c_g) \) with the flip involution \( \tau^c_{(x, y)} = (y, x) \). In particular
\[
(H \times H)^{\tau^c} = \text{diag}(H) = \{(a, a) : a \in H \} \cong H
\]
is the diagonal subgroup and \( q^c = \{(x, -x) : x \in \mathfrak{h} \} \). The symmetric space \( (H \times H)/\text{diag}(H) \) can be identified with the group \( H \), where the \( H \times H \)-action on \( H \) is given by \( (a, b).c = abc^{-1} \). The elliptic invariant cone \( C \subseteq \mathfrak{h} \) specifies an elliptic cone \( C^c = \{(x, -x) : x \in C \} \subseteq q^c \) and \( (H \times H, \text{diag}(H), C^c) \) is compactly causal.

**Theorem 2.5.** (Cone Extension Theorem) ([HÔ97, Thm. 4.5.8], [NÔ22b, Thm. 2.4]) If \( (g, \tau, C) \) is ncc, then there exists an invariant cone \( C_{g^c} \subseteq g^c \) in the dual Lie algebra \( g^c \) such that
\[
C = q \cap iC_{g^c}.
\]

3. Wedge regions in causal symmetric spaces

In this section we discuss several results on wedge regions associated to Euler elements in a causal symmetric space and how they can be characterized.

The first domain is the positivity domain
\[
W^+_M(h) := \{ m \in M : X^M_h(m) \in C^c_m \}
\]
of the modular flow (see [5] in the introduction).

To specify the second type of domains, we start with the complex tube domain
\[
T_M := G \exp(iC^0) \subseteq M_C := G_C/H_C
\]
(here we assume for simplicity that \( G \subseteq G_C \) and \( H = H_C \cap G \)) if \( M \) is compactly causal, and
\[
T_M := G \exp(iC^0_{\text{res}}) \subseteq T_M
\]
We first note that $C_\text{res} \subseteq C^\circ$ is specified as follows: We first note that $C^\circ = \text{Ad}(H)(C^\circ \cap q_p)$. If $h \in C^\circ \cap q_p$ is a causal Euler element, then $C^\circ_\text{res}$ is the unique $\text{Ad}(H)$-invariant domain specified by

$$C^\circ_\text{res} \cap q_p = \left\{ x \in C^\circ \cap q_p : \sigma \left( \text{ad} \left( x - \frac{\pi}{2}h \right) \right) \subseteq \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right\}$$

$$= C^\circ \cap q_p \cap \left( \frac{\pi}{2}h + \Omega_p \right).$$

For a non-compactly causal space we thus obtain for $G = \text{Inn}(g)$ a natural identification of the tube domain $T_M$ with the crown domain $\Xi$ of $G/K$ (see [14]), both realized in $G_C/K_C$ ([NO221], Thm. 5.4]).

The modular flow $\alpha_t(m) = \exp(th)m$ extends to a holomorphic flow on $M_C$, and we define the KMS wedge domain by

$$W^\text{KMS}_M(h) := \{ m \in M : (\forall z \in S_{e}) \alpha_z(m) \in T_M \}.$$ 

This terminology is inspired by the close connection with KMS conditions and standard subspaces; see [NÕ19] and Proposition 5.1 below which establishes an analogy between the pair $(\mathcal{V}, \mathcal{H})$ for a standard subspace $\mathcal{V} \subseteq \mathcal{H}$ and the pair $(W^\text{KMS}_M(h), T_M)$.

The following theorem displays some key information on wedge domains.

**Theorem 3.1.** (Wedge domains in causal symmetric spaces) Let $(G, \tau^G)$ be a connected symmetric Lie group corresponding to the causal symmetric Lie algebra $(g, \tau, C)$. Then the following assertions hold:

(a) If $(g, \tau, C)$ is compactly causal and $h \in \mathfrak{h}$ is an Euler element for which $\tau_h(C) = -C$, then $W^+_M(h) = W^\text{KMS}_M(h)$ and the connected component containing the base point $eH$ in its boundary is $G^\mathfrak{h}_e. \text{Exp}(C_+^\circ + C_+^\circ)$, where $C_{\pm} = \pm C \cap q_{\pm1}(h)$.

(b) Suppose that $(g, \tau, C)$ is non-compactly causal, $h \in \mathfrak{h}$ is an Euler element with $\tau_h(C) = -C$, $G = \text{Inn}(g)$, $G^C := \text{Inn}_{g_C}(g^C)$ and $H := G \cap G^C$.

Then $W^\text{KMS}_M(h) = G^h_e. \text{Exp}_H(C^\text{res}_C)$ is a connected component of the domain $W^+_M(h)$, where $C^\text{res}_C := \{ x_+ + x_- : x_\pm \in q_{\pm1}(h), x_+ - x_- \in C^\circ_\text{res} \}$.

(c) Suppose that $(g, \tau, C)$ is non-compactly causal, $G = \text{Inn}(g)$, $C = C^\text{max}$ is a maximal $\text{Inn}_g(\mathfrak{h})$-invariant cone in $\mathfrak{q}$, $h \in C^\circ$ a causal Euler element and $H = K^h\exp(\mathfrak{h}_p)$. Then the following assertions hold:

(1) The positivity domain $W^+_M(h)$ is connected.
(2) If $\gamma(t) := \text{Exp}_H(th)$ is the causal geodesic defined by $h$, then $W^+_M(h)$ coincides with the order convex hull $W(\gamma)$ of $\gamma(\mathbb{R})$.
(3) The wedge space $\mathcal{W} := \{ gW : g \in G \}$ of $G$-translates in $M$ is isomorphic to the symmetric space $G/G^h \cong \mathcal{O}_h$. 
Proof. (a) This is a simplification of [NÓ22a, Thm. 6.5] which is possible because we assume that \( g \) is simple, so that the Extension Theorem 2.5 applies.

(b) [NÓ22b, Thms. 6.5, 7.1]

(c) The connectedness follows from [MNÓ22b, Thm. 7.1], the second assertion is [MNÓ22b, Cor. 7.2], and the third assertion is [MNÓ22b, Cor. 7.3].

If \( g \) is a simple hermitian Lie algebra and \( G \) is a corresponding connected Lie group, then \( g \) contains a pointed generating invariant cone \( C_g \), and \( M := G \) becomes a symmetric space on which the group \( G \times G \) acts by left and right translations. If \( h \in g \) is an Euler element, then \( \alpha_t(g) = \exp(th) g \exp(-th) \) is the corresponding modular flow, and we know from [NÓ22a, Thm. 5.2] that the corresponding wedge domain in \( G \) is

\[
W^+_G(h) = W^\text{KMS}_G(h) = C^g \exp(C_g^\circ + C_0^\circ), \quad C_\pm := \pm C_g \cap g_\pm(h).
\]

In particular, this domain is an open subsemigroup of \( G \). For a general irreducible cc space \( M = G/H \) and \( C = C_g \cap q \), the Lie algebra \( g \) is hermitian and the map

\[
Q: M \to G^{-\tau}, \quad gH \mapsto g\tau(g)^{-1},
\]

defines a covering of \( M \) onto a causal subspace of the group type space specified by \( (g, C_g) \). One thus obtains for \( C = C_g \cap q \) the corresponding results in \( M \) (cf. Theorem 3.1(a)).

For ncc spaces there are stronger results that do not require an Euler element in \( h \), which is equivalent to the modular flow on \( M \) to have a fixed point, an assumption that is crucial in Theorem 3.1(a,b). Here is another one that shows all Euler elements with non-trivial wedge spaces lie in the same orbit.

**Theorem 3.2.** ([MNÓ22b, Cor. 6.3, 6.4]) Let \((G, \tau, H, C)\) be an irreducible ncc space, \( M = G/H \) and \( h \in C_0^\circ \cap q \) be a causal Euler element with \( \tau = \theta \tau_h \). If \( h_1 \in g \) is an Euler element for which the positivity domain \( W^+_M(h_1) \) is non-empty, then \( h_1 \in \mathcal{O}_h = \text{Ad}(G)h \). In particular, \( W^+_M(-h) = \emptyset \) if \( h \) is not symmetric, i.e., \( -h \not\in \mathcal{O}_h \).

4. **Unitary and anti-unitary representations**

4.1. **Smooth vectors and distribution vectors.** Let \((U, \mathcal{H})\) be a unitary representation of the Lie group \( G \). A smooth vector is an element \( \eta \in \mathcal{H} \) for which the orbit map \( U^n : G \to \mathcal{H}, g \mapsto U(g)\eta \) is smooth. We write \( \mathcal{H}^\infty \) for the space of smooth vectors. It carries the derived representation \( dU \) of the Lie algebra \( g \) given by

\[
dU(x)\eta = \lim_{t \to 0} \frac{U(\exp tx)\eta - \eta}{t}.
\]

For write \( \partial U(x) = dU(x) \) for the infinitesimal generator of the unitary one-parameter group \((U(\exp tx))_{t \in \mathbb{R}}\), so that \( U(\exp tx) = e^{i\theta U(x)} \) for \( t \in \mathbb{R} \).
We extend the $\mathfrak{g}$-representation on $\mathcal{H}^\infty$ to a homomorphism $dU: U(\mathfrak{g}) \to \mathrm{End}(\mathcal{H}^\infty)$, where $U(\mathfrak{g})$ is the complex enveloping algebra of $\mathfrak{g}$. This algebra carries an involution $D \mapsto D^*$ determined uniquely by $x^* = -x$ for $x \in \mathfrak{g}$. For $D \in U(\mathfrak{g})$, we obtain a seminorm on $\mathcal{H}^\infty$ by

$$p_D(\eta) = \|dU(D)\eta\| \quad \text{for} \quad \eta \in \mathcal{H}^\infty.$$ 

These seminorms define a Fréchet topology on $\mathcal{H}^\infty$ for which the inclusion $\mathcal{H}^\infty \hookrightarrow \mathcal{H}$ is continuous (see [NO21, Sec. A.1] for more details).

The space $\mathcal{H}^\infty$ of smooth vectors is $G$-invariant and we denote the corresponding representation by $U^\infty$. We write $\mathcal{H}^{-\infty}$ for the space of continuous anti-linear functionals on $\mathcal{H}^\infty$. Its elements are called distribution vectors.

The group $G$ and the convolution algebra $C_c^\infty(G)$ act on $\eta \in \mathcal{H}^{-\infty}$ by

- $(U^{-\infty}(g)\eta)(\xi) := \eta(U(g^{-1})\xi)$, $g \in G$, $\xi \in \mathcal{H}^\infty$.
- $U^{-\infty}(\varphi)\eta = \int_G \varphi(g)U^{-\infty}(g)\eta \, dg$ for $\varphi \in C_c^\infty(G)$.

We have natural $G$-equivariant linear embeddings

$$\mathcal{H}^\infty \hookrightarrow \mathcal{H} \xrightarrow{\xi \mapsto \langle \cdot, \xi \rangle} \mathcal{H}^{-\infty}. \quad (22)$$

4.2. Anti-unitary representations. Let $\mathcal{H}$ be a complex Hilbert space. A surjective isometry $A: \mathcal{H} \to \mathcal{H}$ is said to be anti-unitary if $A(\lambda v) = \overline{\lambda} Av$ for $v \in \mathcal{H}$, $\lambda \in \mathbb{C}$. We write $AU(\mathcal{H})$ for the group of unitary and anti-unitary operators on $\mathcal{H}$. If $J$ is an anti-unitary involution, then $AU(\mathcal{H}) = U(\mathcal{H}) \cup JU(\mathcal{H})$ is a disjoint union.

An anti-unitary representation of a Lie group $G$ is a group homomorphism $U: G \to AU(\mathcal{H})$ for which all orbit maps $U^v(g) := U(g)v$ are continuous. We refer to [NO17] for a detailed discussion of anti-unitary extensions of unitary representations. If $\tau: G \to G$ is an involutive automorphism of the Lie group $G$, then $G_\tau = G \rtimes \{\mathrm{id}_G, \tau\}$ is a Lie group and we are interested in extensions of unitary representations of $G$ to $G_\tau$, so that $J := U(\tau)$ is anti-unitary, hence a conjugation (an anti-unitary involution). Any such $J$ commutes with $U(G^r)$ and $\mathcal{H}^J = \{v \in \mathcal{H}: Jv = v\}$ is a closed real subspace invariant under $G^r$ such that $\mathcal{H} = \mathcal{H}^J \oplus i\mathcal{H}^J$. In this sense we have $U(G^r) \subset O(\mathcal{H}^J)$ and obtain a real orthogonal representation of $G^r$ on the real Hilbert space $\mathcal{H}^J$.

Example 4.1. We recall from Example 2.3 that the group

$$G = \mathrm{SU}_{1,1}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ b & \overline{a} \end{pmatrix} : |a|^2 - |b|^2 \right\}$$

acts transitively on $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ b & \overline{a} \end{pmatrix} \cdot z = \frac{az + b}{bz + \overline{a}}$$

and the stabilizer of $0$ is $K \cong \mathbb{T}$. For $s > 0$ we consider on $\mathbb{D}$ the positive definite kernel

$$K(z, w) = \frac{1}{(1 - z\overline{w})^s}.$$
and the corresponding reproducing kernel Hilbert space $H_s \subseteq O(D)$. Note that $H_1 = H^2(D)$ is the Hardy space and $H_2 = O(D) \cap L^2(D)$ the Bergman space. Then the universal covering group $\tilde{G} \cong \tilde{SL}_2(\mathbb{R})$ of $G$ acts unitarily on $H_s$ by lifting the projective representation of $G$, given by

$$(U_s(g)f)(z) = (a + bz)^{-s}f(g^{-1}.z) \quad \text{for} \quad g = \begin{pmatrix} a & b \\
 \bar{b} & \bar{a} \end{pmatrix}$$

to $\tilde{G}$. The involution $\tau(g) = \overline{g}$ on $G$ lifts to $\tilde{G}$. It coincides with $\tau_h$ for the Euler element $h = \frac{1}{2} \begin{pmatrix} 0 & 1 \\
 1 & 0 \end{pmatrix} \in so_{1,1}(\mathbb{R}) \subseteq su_{1,1}(\mathbb{C})$. The corresponding involution on $D$ is the complex conjugation which induces an conjugation on $H_s$ by

$$J(f)(z) = \overline{f(\overline{z})}.$$ 

Then $JU_s(g)J = U_s(\tau_h(g))$, so that we obtain an anti-unitary extension of $U_s$ to $\tilde{G}_{\tau_h}$.

We have the following general extension theorem for groups with Lie algebra $sl_2(\mathbb{R})$ ([MN21, Thm. 4.24]):

**Theorem 4.2.** Let $h \in sl_2(\mathbb{R})$ be an Euler element and $G$ a connected Lie group with Lie algebra $sl_2(\mathbb{R})$. Then every unitary representation $(U, H)$ of $G$ extends to an anti-unitary representation of $G_{\tau_h}$ on the same Hilbert space. This extension is unique up to isomorphism.

5. Standard subspaces

There are two natural ways to construct standard subspaces. One is by using a cyclic separating vector $\Omega$ of a von Neumann algebra $M$, so that $V(M, \Omega) = \overline{M_\Omega}$ is standard. The other is to use anti-unitary representations and the Brunetti–Guido–Longo construction mentioned in the introduction (cf. [BGL02], [NO17]). It is based on the following observation.

We have already seen that every standard subspace $V$ determines a pair $(\Delta_V, J_V)$ of modular objects and that $V$ can be recovered from this pair by $V = \text{Fix}(J_V\Delta_V^1)$. We thus obtain a representation theoretic parametrization of the set of standard subspaces of $H$ (cf. [BGL02], [NO17]): Each standard subspace $V$ specifies an anti-unitary representation $U^V$ of $\mathbb{R}$ by

$$(23) \quad U^V(e^t) := \Delta_V^{-it/2\pi} \quad \text{and} \quad U^V(-1) := J_V.$$ 

This defines a bijection between the set of standard subspaces and the set of anti-unitary representations of $\mathbb{R}$ on $H$.

The following proposition ([NOØ21, Prop. 2.1]) characterizes the elements of a standard subspace $V$ specified by the pair $(\Delta, J) = (\Delta_V, J_V)$ in terms of analytic continuation of orbit maps of the unitary one-parameter group $(\Delta^t)_{t \in \mathbb{R}}$ and the real space $H^J$. 
Proposition 5.1. Let $V \subseteq \mathcal{H}$ be a standard subspace with modular objects $(\Delta, J)$,
\[
S_\pi := \left\{ z \in \mathbb{C} : 0 < \text{Im} \, z < \pi \right\} \quad \text{and} \quad S_{\pm \pi/2} := \left\{ z \in \mathbb{C} : |\text{Im} \, z| < \frac{\pi}{2} \right\}.
\]
For $\xi \in \mathcal{H}$, we consider the orbit map $\alpha^\xi : \mathbb{R} \to \mathcal{H}, t \mapsto \Delta^{-it/2\pi} \xi$. Then the following are equivalent:

(i) $\xi \in V$.

(ii) $\xi \in D(\Delta^{1/2})$ with $\Delta^{1/2} \xi = J \xi$.

(iii) (KMS-like condition) The orbit map $\alpha^\xi : \mathbb{R} \to \mathcal{H}$ extends to a continuous map $S_\pi \to \mathcal{H}$ which is holomorphic on $S_\pi$ and satisfies $\alpha^\xi(\pi i) = J \xi$.

(iv) There exists an element $\eta \in \mathcal{H}^J$ whose orbit map $\alpha^\eta$ extends to a continuous map $S_{\pm \pi/2} \to \mathcal{H}$ which is holomorphic on the interior and satisfies $\alpha^\eta(-\pi i/2) = \xi$, i.e., $\eta \in \mathcal{H}^J \cap D(\Delta^{1/4})$.

Let $(U, \mathcal{H})$ be an anti-unitary representation of $G_{\tau_h} = G \rtimes \{\text{id}, \tau_h\}$. Then
\[
J = U(\tau_h) \quad \text{and} \quad \Delta := e^{2\pi i \partial U(h)}
\]
specify a standard subspace $V := \text{Fix}(J \Delta^{1/2})$. To construct elements in $V$, we want to apply (iv) above. So we start with a $J$-fixed $K$-finite vector $v$. As $(\frac{\pi}{2}, \frac{\pi}{2}) h \subseteq \Omega_p$, the Krötz–Stanton Theorem (cf. [KS04] and [L5]) shows that the orbit map $\alpha^v(t) = U(\exp \theta h)v$ extends holomorphically to the open strip $S_{\pm \pi/2}$, but unfortunately the limit
\[
\beta(v) = \lim_{t \to \frac{\pi}{2}} \alpha^v(it) = \lim_{t \to -\frac{\pi}{2}} e^{it \partial U(h)}v
\]
does not exist in $\mathcal{H}$. However, we expect that it always exists in the space $\mathcal{H}^{-\infty}$ of distribution vectors; see [FNÖ22] for first steps in this direction.

Example 5.2. We consider the strip $\mathcal{S} := S_\pi := \{ z \in \mathbb{C} : 0 < \text{Im} \, z < \pi \}$ and its Hardy space
\[
H^2(\mathcal{S}) := \left\{ f \in \mathcal{O}(\mathcal{S}) : \|f\|^2 := \sup_{z \in \mathcal{S}} \int_{\mathbb{R}} |f(z + t)|^2 \, dt < \infty \right\},
\]
where $\mathcal{O}(\mathcal{S})$ is the space of holomorphic functions on $\mathcal{S}$. This is a reproducing kernel Hilbert space with kernel
\[
K(z, w) = \frac{i}{4\pi \sinh \left( \frac{z - w}{2} \right)} \quad \text{satisfying} \quad K(z, z) = \frac{1}{4\pi \sin(\text{Im} \, z)}.
\]
The group $\mathbb{R}$ acts unitarily on $H^2(\mathcal{S})$ by $(U_t F)(z) = F(z + t)$ (cf. [ANS22]). Writing $U_t = e^{itP}$ with the selfadjoint operator $P = -i \frac{d}{dz}$ on $L^2(\mathbb{R})$ and $\Delta := e^{-2\pi P}$. We are interested in the standard subspace corresponding to $\Delta$ and the conjugation defined by $(JF)(z) := F(i\pi + z)$ via $\mathcal{V} := \text{Fix}(J \Delta^{1/2})$. The evaluation functions $K_w(z) := K(z, w)$ satisfy $J K_w = K_{\pi i + w}$, so that, for $y \in \mathbb{R}$ and $w = \frac{\pi}{2} + y$ we have $K_w \in H^2(\mathcal{S})^J$. Consider $\eta := K_{\frac{\pi}{2}}$. For the unitary one-parameter group $(U_t)_{t \in \mathbb{R}}$, we have $U_t K_w = K_{w-t}$, so that
the orbit map \( \alpha^\eta(t) := K_{\frac{\eta}{2} - t} \) of \( \eta \) has an analytic extension \( \mathcal{S}_{\pm \pi/2} \to H^2(\mathcal{S}) \) given by \( \alpha^\eta(z) = K_{\frac{\eta}{2} - z} \). It follows in particular that

\[
\|\alpha^\eta(it)\|^2 = K\left(\frac{\pi i}{2} + y + it, \frac{\pi i}{2} + y + it \right) = \frac{1}{4\pi \sin \left(\frac{\pi}{2} + it\right)} \to \infty
\]

for \( |t| \to \frac{\pi}{2} \). So the condition in Proposition 5.1(iv) is not satisfied. However, one can show that \( \alpha^\eta(-\frac{\pi i}{2}) \in H^2(\mathcal{S})^{-\infty} \) defines a distribution vector of the unitary one-parameter group \( (U_t)_{t \in \mathbb{R}} \). It corresponds to the tempered distribution on \( \partial \mathcal{S} \), defined on \( \mathbb{R} \) by

\[
D(x) = \lim_{\varepsilon \to 0^+} \frac{i}{4\pi \sinh \frac{x+i\varepsilon}{2}} = \frac{x}{4\pi \cosh \left(\frac{x}{4}\right)} \cdot \lim_{\varepsilon \to 0^+} \frac{i}{x+i\varepsilon}
\]

and on \( \pi i + \mathbb{R} \) by

\[
D(\pi i + x) = \frac{i}{4\pi \sinh \left(\frac{x+i\pi}{2}\right)} = \frac{x}{4\pi \cosh \left(\frac{x}{4}\right)}.
\]

Then the distribution kernel \( D(x - y) \) on \( \partial \mathcal{S} \) is positive definite, taking boundary values yields an isomorphism of \( H^2(\mathcal{S}) \) with the corresponding Hilbert subspace \( \mathcal{H}_D \subseteq \mathcal{D}(\partial \mathcal{S}) \) and the standard subspace \( \mathcal{V} \) is generated by the space \( C^\infty_c(\mathbb{R}, \mathbb{R}) \) of real valued test functions on the lower boundary \( \mathbb{R} \subseteq \partial \mathcal{S} \).

**Example 5.3.** (The group case, \[N\ell\,O21\]) Let \( G \) be a hermitian Lie group with Cartan involution \( \theta \) and \((U, \mathcal{H})\) be a non-trivial unitary representation of \( G \) whose positive cone

\[
C_U = \{ x \in \mathfrak{g} : -i\partial U(x) \geq 0 \}
\]

is generating. Then \( U \) extends to a representation of the semigroup \( S_U = G \exp(iC_U) \) that is holomorphic on the interior of \( S_U \). We assume that \( h \in \mathfrak{g} \) is an Euler element and that \( U \) extends to \( G_{\tau_h} \) by \( U(\tau_h) = J \) (cf. \[N\ell\,O17\]). Then the standard subspace \( \mathcal{V} := \mathcal{V}(h, \tau_h) \) associated to \( J \) and \( \Delta = e^{2\pi i \cdot \partial U(h)} \) as in \([\text{\color{red}{III}}]\) permits a non-trivial endomorphism semigroup

\[
S_{\mathcal{V}} := \{ g \in G : U(g)\mathcal{V} \subseteq \mathcal{V} \} = G_{h, \tau_h} \exp(C),
\]

where \( C = C_+ + C_- \) for \( C_\pm := \pm C_U \cap \mathfrak{g}_{\pm 1}(h) \) (\[N\ell\,O19\],[Ne21]). On the other hand, the interior of this semigroup \( S_{\mathcal{V}}^0 \) is a union of connected components of the wedge domain of the modular flow \( \alpha_t(g) = \exp(th)g\exp(-th) \). In \([N\ell\,O21]\) we use this fact to describe real subspaces \( \mathcal{E} \subseteq \mathcal{H}^{-\infty} \) for which \( \mathcal{H}_E(S_U^0) \), defined by \([\text{\color{red}{III}}]\) equals \( \mathcal{V} \).

To construct elements of \( \mathcal{V} \) in the spirit of Proposition 5.1, one can also start with a \( J \)-fixed vector \( v \in \mathcal{H}^J \) and the subsemigroup

\[
S_{U, h}^{\tau_h} = G_{\tau_h} \exp(i(C_+ - C_-))
\]

of fixed points of the antiholomorphic extension \( \tau_h \) of \( \tau_h \) to \( S_U \). Then

\[
U((S_{U, h}^{\tau_h})^0)\mathcal{H}^J \subseteq \mathcal{H}^J \cap \mathcal{D}(\Delta^{1/4})
\]
follows easily from \( \alpha_{\frac{i}{\pi}} ((S_U^{\pm h})^\circ) = S_0^\circ \), so that \( \Delta^{-1/4} U ((S_U^{\pm h})^\circ) \mathcal{H}^J \subseteq \mathcal{V} \). We refer to [NO22a] for a detailed discussion of the group case (see Example 2.4) and for the construction of nets of standard subspaces satisfying (I), (Cov), (RS) and (BW).

Example 5.4. A low dimensional example of a solvable group, where many of these phenomena can be observed is the affine group

\[ G = \text{Aff}(\mathbb{R}) e = \mathbb{R} \times \mathbb{R}_+^\times \]

with Euler element \( h = (0,1) \in \mathfrak{g} = \mathbb{R} \times \mathbb{R} \) (generator of dilations), \( \tau_h(b,a) = (-b,a) \) (corresponding to point reflection in 0), and we write \( x = (1,0) \) for the generator of translations that satisfies \([h,x] = x\). For a positive energy representation of \( G_{\tau_h} \) we then have \( C_U = [0,\infty) x = C_+ \) and for \( \mathcal{V} = \mathcal{V}(h,\tau_h) \) we have \( S_\mathcal{V} = \exp(\mathbb{R}h) \exp(C_+) \). If \( H := \{0\} \times \mathbb{R}_+^\times \) is the dilation group, then \( M := \mathbb{R} \cong G/H \) is a flat symmetric space, the modular flow is given by \( \alpha_\tau(p) = e^i p \) and \( W^+_M(h) = (0,\infty) \) is the corresponding wedge domain.

A typical example arises for the Hardy space \( H^2(C_+) \) of the upper half plane \( C_+ = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \) with the reproducing kernel

\[ K(z,w) = \frac{1}{2\pi} \frac{i}{z-w} \]

on which \( G \) acts by

\[ (U(b,a)f)(z) := a^{-1/2} f(a^{-1}(z+b) \quad \text{and} \quad (Jf)(z) := f(-\bar{z}). \]

We may identify \( H^2(C_+) \) by its boundary value map in \( L^2(\mathbb{R}) \), considered as a Hilbert space \( \mathcal{H}_D \) of (tempered) distributions with distribution kernel of the form \( D(x-y) \), where

\[ D(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \frac{i}{z+i\varepsilon}. \]

From [NOO21, (3.42)] it follows that the standard subspace \( \mathcal{V} \subseteq \mathcal{H}_D \) is generated by the real subspace \( e^{-\frac{i}{\pi}} C_c^\infty((0,\infty),\mathbb{R}) \subseteq C_c^\infty(\mathbb{R}, \mathbb{C}) \). Note that the support of these test functions is restricted to the wedge region \( W^+_M(h) = (0,\infty) \).

6. Cayley type spaces and causal compactifications

An irreducible causal symmetric space \((G,H,C)\) is said to be of Cayley type if there exists an Euler element \( h \in \mathfrak{g} \) such that \( \tau = \tau_h \) and \( H = G^h \). As \( H \) always contains the center, we assume in the following that \( Z(G) = \{e\} \), i.e., \( G \cong \text{Inn}(\mathfrak{g}) \). Then \( h \in \mathfrak{h} = \mathfrak{g}^\tau \), so that \( h \) is not a causal Euler element.

Suppose first that \((G,\tau,H,C)\) is non-compactly causal and that \( h \in \mathfrak{h} \) is an Euler element with \( \tau_h = \tau \). Then \( \mathfrak{h} = \mathfrak{g}_0(h) \) and \( \mathfrak{q} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_{-1}(h) \). Let \( C_{\pm} = (\pm C) \cap \mathfrak{g}_{\pm 1}(h) \). These are \( H \)-invariant pointed generating cones in \( \mathfrak{g}_{\pm 1}(h) \). The cone \( C = C_h := C_+ - C_- \) is hyperbolic and \( C_\pm := C_+ + C_- \) is elliptic. In fact, one can characterize (irreducible) Cayley type spaces by
the existence of $\text{Inn}_q(h)$-invariant pointed generating elliptic and hyperbolic cones in $q$ [HO09].

The Lie algebras $p^\pm(h) := g_0(h) \oplus g_{\pm 1}(h)$ are maximal parabolic subalgebras. The corresponding maximal parabolic subgroups are

$$P^\pm(h) = G^h \exp(\mathfrak{g}_\pm(h)).$$

As $G \subseteq \text{Aut}(g)$, we have $(G^h)_{\text{e}} \subset G^h \subset G^h$. Hence $G/G^h$ is a symmetric space which has an open dense orbit in the compact manifold

$$X := G/P^+(h) \times G/P^-(h).$$

The cones $C_\pm \subseteq g_{\pm 1}(h) \cong g/p^\pm(h)$ are $\text{Ad}(P^\pm(h))$-invariant, hence define $G$-invariant causal structures on the flag manifolds $G/P^\pm(h)$. In particular, the product $X$ is a causal $G$-manifold, and the embedding

$$G/H \hookrightarrow G/P^+(h) \times G/P^-(h), \quad gH \mapsto (gP^+(h), gP^-(h))$$

is causal for $C_\pm$, resp., $C_\epsilon$ if $G/P^-(h)$ carries the causal structure defined by $C_\pm$ and $G/P^+(h)$ carries the causal structure defined by $-C_\pm$, resp., $C_\pm$ (see Neu09, O09, NO22b for details).

The simply connected coverings of the causal spaces $G/P^\pm(h)$ coincide with the simple space-time manifolds in the sense of Mack–de Riese [MdR07]. As $g_{\pm 1}(h)$ carries the structure of a euclidean Jordan algebra, the spaces $G/P^\pm(h)$ can also be considered as conformal compactifications of simple euclidean Jordan algebras. Nets of standard subspaces on the causal manifolds $G/P^\pm(h)$ satisfying (I), (Cov), (RS) and (BW) have been constructed from unitary highest weight representations in [NO21]. The wedge domain in $G/P^-(h)$ is given by $W := W^+_G(h) = \exp(C^+_\pm)P^-(h)$ which is diffeomorphic to the open cone $C^+_\pm \subseteq g_1(h)$, considered as an open subset of $G/P^-(h)$.

From [MN22b, §7] we infer that

$$G_W = \{ g \in G : g.W = W \} = G^h,$$

so that the wedge space $W := \{ g.W : g \in G \}$ of $G/P^-(h)$ can also be identified with the adjoint orbit $O_h$ of $h$ (cf. MN21).

In physics the example $g = so_{2, d}(\mathbb{R})$ is of particular importance. In this case any Euler element $h$ leads to $\mathfrak{h} := \mathfrak{g}^h \cong \mathbb{R}h \oplus so_{1,d-1}(\mathbb{R})$, $g_1(h) \cong \mathbb{R}^d$ is $d$-dimensional Minkowski space and $G/P^-(h) \cong (S^1 \times S^{d-1})/\{\pm 1\}$ is its conformal compactification. Here $W$ is the space of conformal wedge domains in Minkowski space. For $d = 4$, we have in particular

$$G/P^-(h) \cong (S^1 \times S^3)/\{\pm 1\} \cong (T \times SU_2(\mathbb{C}))/\{\pm 1\} \cong U_2(\mathbb{C})$$

on which the group $G = SU_{2,2}(\mathbb{C})$ with Lie algebra $su_{2,2}(\mathbb{C}) \cong so_{2,4}(\mathbb{R})$ acts by conformal maps.
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