Lattice Matter

Simon Hands\textsuperscript{a}

\textsuperscript{a}Department of Physics, University of Wales Swansea, Singleton Park, Swansea SA2 8PP, United Kingdom

I review recent developments in the study of strongly interacting field theories with non-zero chemical potential $\mu$. In particular I focus on (a) the determination of the QCD critical endpoint in the $(\mu, T)$ plane; (b) superfluid condensates in Two Color QCD; and (c) Fermi surface effects in the NJL model. Some remarks are made concerning the relation of superconductivity with the sign problem.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{qcd_phase_diagram}
\caption{Schematic view of the QCD phase diagram.}
\end{figure}

Fig. 1 summarises our current knowledge of the QCD phase diagram in the plane of temperature $T$ and quark chemical potential $\mu$. Last year Shinji Ejiri reviewed QCD simulations at $T \neq 0$, i.e. along the vertical axis. In my talk I wish to discuss what can be done in the interior of the plane. Significant progress has been made in the region to the upper left of Fig. 1 where both $\mu$ and $T$ differ from zero, which is also of direct phenomenological interest for heavy-ion collisions. I also wish to cover the lower right region describing cold dense strongly-interacting matter. In recent years there has been intense theoretical activity in this region driven by the possibility that quark matter is unstable with respect to diquark condensation $\langle qq \rangle \neq 0$, resulting in a ground state with color superconducting properties \cite{2}. Model calculations suggest that the BCS gap $\Delta$ at the Fermi surface may be as large as 100 MeV \cite{2}, comparable with the constituent quark scale, implying significant consequences for the physics of compact astrophysical objects \cite{4}. In this case simulations of QCD are to date impracticable, for reasons I will review below; however, diquark condensation and Fermi surface effects can be studied by lattice techniques in certain models.

1. Why is $\mu \neq 0$ so difficult?

For a vectorlike gauge theory in Euclidean metric the introduction of a quark chemical potential breaks the $\gamma_5$-hermiticity of the Dirac operator:

$$D(\mu) \equiv D(0) + \mu \gamma_0 = \gamma_5 D^{\dagger}(-\mu) \gamma_5.$$  \hspace{1cm} (1)

This implies that eigenvalues of $D$ are no longer pure imaginary and hence not related to each other by complex conjugation, in turn implying

$$\det M(\mu) \neq \det M^*(\mu) = \det M(-\mu).$$  \hspace{1cm} (2)

Therefore the functional measure is no longer positive definite. In principle the determinant can be factorised into a modulus $\rho$ and a phase $\phi$, and the phase included with the observable $O$ in Monte Carlo simulations via

$$\langle O \rangle \equiv \langle O e^{i\phi} \rangle \rho / \langle e^{i\phi} \rangle \rho.$$  \hspace{1cm} (3)

Unfortunately fairly general arguments suggest that $\langle e^{i\phi} \rangle \propto e^{-V}$, where $V$ is the system vol-
ume. Acquiring sufficient statistics therefore becomes exponentially difficult as the thermodynamic limit is approached. This is known as the Sign Problem, and has plagued the study of $\mu \neq 0$ since its inception. However, we should not be so surprised, since generic problems in the quantum theory of $N$ objects require $N! \sim e^N$ different complexions or wavefunctions to be examined. Perhaps a more productive way of phrasing the question would be why is vacuum QCD so easy?

Given this difficulty there are two routes forward. Firstly, one can perform a QCD simulation at $\mu = 0$ and attempt to analytically continue the results to $\mu \neq 0$. This can be done either by calculating terms in a Taylor expansion about $\mu = 0$, exemplified, for instance, by the calculation of baryon number susceptibility $\chi_{\phi} \sim \mu \sim V^{-1} \partial^2 \ln Z/\partial \mu^2 |_{\mu=0}$, or by directly reweighting configurations, as in the ‘Glasgow method’ developed by Ian Barbour and collaborators. In the former case the prospects are limited by a finite radius of convergence, dictated by the presence of a critical point in the ($\mu, T$) plane. In the latter case the effectiveness is limited by the requirement of maintaining a reasonable overlap between the trial and true ensembles, which once again becomes exponentially hard in the thermodynamic limit. Both methods have their best chance of succeeding at $T > 0$; at $T = 0$ the behaviour is horribly non-analytic, since the ground state does not change as $\mu$ increases out to to the onset of nuclear matter at $\mu_0 \approx 307$ MeV. As we shall see, the overlap problem is also considerably more tractable at $T > 0$.

The second approach is to throw up our hands and use a real measure $\text{det}M M^* = \text{det}M(\mu) M(-\mu)$. Physically this has the effect of introducing conjugate quarks $q^c$ which have positive baryon charge but transform in the conjugate representation of the gauge group, leading to the possibility of gauge invariant $qq^c$ bound states. In real QCD this is a disaster since the lightest such state is degenerate with the pion, being in effect a ‘Goldstone baryon’. As $\mu$ is raised the onset of nuclear matter would thus be expected at $\mu_0 \approx m_\pi/2$ rather than the constituent quark scale $\Sigma \approx m_\pi/3$. There are models, however, such as Two Color QCD and QCD with non-zero isospin chemical potential $\mu_I \propto \mu_\pi - \mu_d$, where $qq$ Goldstone baryons are a feature, not a problem. Another class amenable to this approach are NJL-like models, where $qq^c$ states don’t couple to the Goldstone mode and hence remain at the constituent scale, so the use of a real measure doesn’t damage the physics. The common feature of all these models is that the diquark condensate which potentially forms at large $\mu$ is gauge singlet, and hence describes a superfluid, as opposed to superconducting, ground state.

2. Progress at $T > 0$

Fodor and Katz have made spectacular progress in reweighting at $T > 0$. The basic formula is

$$Z[\alpha] = \int DU \exp(-S_{\text{bos}}[U; \alpha_0]) \text{det}M[U; \alpha_0] \times \left\{ \exp(-\Delta S_{\text{bos}}[U; \alpha, \alpha_0]) \frac{\text{det}M[U; \alpha]}{\text{det}M[U; \alpha_0]} \right\}$$

where the parameter set $\alpha = \{\beta, m, \mu\}$ but importance sampling is performed using a different set $\alpha_0 = \alpha - \Delta \alpha$, with $\mu$ chosen either zero or pure imaginary to keep the determinant real. Usually reweighting is only effective if $\Delta \alpha$ is small enough to maintain a good overlap between trial and true ensembles. Fodor and Katz’ insight is that along the crossover/coexistence line between hadronic and quark-gluon plasma phases the true ensemble cannot alter very much, since it must contain contributions from both phases; hence the overlap between ensembles along the line remains high. They therefore generalise the Glasgow method by reweighting with both $\Delta \mu, \Delta \beta \neq 0$ in order to stay on this line. In practice $\beta_c(\mu)$ is identified via the real part of the lowest Lee-Yang zero $\beta_0$, which approaches the real axis as $V$ increases. Now, the nature of the transition between hadronic and QGP phases can be determined by the volume scaling of $\text{Im} \beta_0$:

$$\lim_{V \to \infty} \text{Im} \beta_0 \begin{cases} \neq 0; & \text{crossover,} \\ = 0; & \text{first order transition.} \end{cases} \quad (4)$$

The results for 2+1 flavor QCD with $m_u, m_d = 0.025$, $m_s = 0.2$ on volumes up to $8^4 \times 4$ are
shown in Fig. 2. The physical scale is set using separate simulations at $\beta_c(0)$ and $\beta_c(\mu_E)$, which yield a critical temperature $T_c(\mu = 0) = 172(3)\text{MeV}$. The main prediction is for the critical endpoint of the first order line, at

$$T_E = 160(4)\text{MeV} ; \mu_E = 242(12)\text{MeV}. \quad (5)$$

The result for $\mu_E$ is unexpectedly large, and might undermine proposals to observe it directly at RHIC [14]. However, the endpoint is expected to move to the left as the chiral limit is approached, so one should consider [6] more as a proof of principle at this stage. What should be stressed is that currently the results are limited by cpu resources rather than overlap problems as $V \to \infty$; it is certainly possible, therefore, that lattice methods can ultimately obtain an accurate determination of $(\mu_E, T_E)$.

There have also been developments in Taylor expansion methods, which can deliver information on a wider range of observables. Gavai and Gupta have reexamined quark number susceptibilities in quenched QCD [15], in particular finding that $\chi_s$ increases markedly across the transition to QGP, and approaches $\chi_{u,d}$ for $T \gtrsim 2 T_c$, though all three depart significantly from ideal-gas values. These quantities are testable in ion collisions via eg. event-by-event fluctuations in charged particle yields and strangeness enhancement. QCD-TARO have looked at the response of hadron masses to a change of both baryon and isospin chemical potentials [16]. They find $\partial^2 m_\pi/\partial \mu_B^2 > 0$, and with a much larger value in the QGP phase, indicative that the pion is no longer a Goldstone mode at high $T$; by contrast $\partial^2 m_\pi/\partial \mu_T^2 < 0$ and is much larger in the hadronic phase, consistent with pion condensation in cold dense isospin-asymmetric matter [17]. Finally Ejiri has determined the curvature of the critical line $\partial^2 \beta_c(\mu)/\partial \mu^2$ by measuring the shift in the peaks for $\langle \bar{\psi}\psi \rangle$ and Polyakov loop susceptibilities [18]. For quark mass $ma = 0.2$ on $16^3 \times 4$ he obtains a value $\simeq -1.5(4)$; assuming a critical temperature $T_c(\mu = 242\text{MeV}) \approx 150\text{MeV}$, not too far from (5).

3. Two colors matter

For gauge group SU(2), the fermion determinant is trivially real since all matter representations are either real, or pseudoreal (in which case $\det M = \det \tau_2 M^* \tau_2$). Moreover, if we specialise to the case of $N$ staggered fermions in the fundamental representation, the usual $U(N) \otimes U(N)_\tau$ global symmetry at $m = \mu = 0$ is enhanced, via

$$\bar{\chi} \bar{\psi} \chi = \bar{X}_c \bar{\psi} X_o,$$

$$X_c = (\bar{\chi}_c, -\chi^{tr}_e \tau_2) \quad ; \quad X_o = \left( \begin{array}{cc} \chi_o^{tr} \tau_2 \\ -\tau_2 \chi_o \end{array} \right) \quad (6)$$

to $X \mapsto VX, \bar{X} \mapsto \bar{X}V^\dagger, V \in U(2N)$. For adjoint quarks the same relation holds with the $\tau_2$ factors replaced by 1. Now, at $\mu = 0$ we expect this global symmetry to be spontaneously broken by a chiral condensate $\langle \bar{\chi} \chi \rangle \neq 0$. For fundamental quarks the breaking pattern is $U(2N) \to O(2N)$ yielding $N(2N + 1)$ Goldstones; for adjoint the pattern is $U(2N) \to \text{Sp}(2N)$ with $N(2N - 1)$ Goldstones [19]. This differs in detail from the corresponding patterns for continuum fermions [20]. The important point is that besides the usual mesonic $q\bar{q}$ states, in general some of the Goldstones must be $qq$ baryons, as anticipated in Sec. 4. It is also possible to identify diquark con-
densates for both 2 and 3 representations:
\[
\langle \{ q q_2 \} \rangle = \frac{1}{2} \langle x^{tr} \{ \tau_2 \} \rangle \chi + \langle \{ q q_3 \} \rangle = \frac{1}{2} \langle x^{tr} \{ \tau_2 \} \rangle \chi \tag{7}
\]
where in the 3 case the antisymmetric \( \epsilon \) acts on flavor. These are related to \( \langle \chi \bar{\chi} \rangle \) by a U(2N) rotation and are gauge invariant. Since, however, \( \langle \{ q q_2 \} \rangle \)

is not invariant under the original U(1)_B, baryon charge is no longer a good quantum number.

It is possible to treat TCQCD analytically by ignoring all excitations except the Goldstones, and writing down an effective action in the spirit of chiral perturbation theory (\( \chi PT \)), in which the physical parameters are \( \langle \chi \bar{\chi} \rangle \) and \( m_\pi \) at \( \mu = 0 \). Remarkably, the \( \mu \)-dependent terms in the chiral Lagrangian are completely determined in terms of these two parameters by the global symmetries \( [13] \). At leading order in \( \chi PT \) a second order onset phase transition is predicted at the rescaled chemical potential \( x \equiv 2 \mu/m_\pi = 1 \):

\[
\frac{\langle \chi \bar{\chi} \rangle_0}{\langle \chi \bar{\chi} \rangle_0} = \begin{cases} 
1/x^2; & \langle qq \rangle_0 = \left\{ \begin{array}{ll}
0 & x < 1 \\
\sqrt{1 - x^2} & x > 1
\end{array} \right.
\end{cases}
\frac{N}{2m_\langle \chi \rangle_0} n_B = \begin{cases} 
0; & x < 1 \\
x(1 - \frac{1}{x^2}) & x > 1
\end{cases}
\tag{8}
\]

where the baryon charge density \( n_B = \langle \bar{\psi}\gamma_0\psi \rangle \)

and the 0 subscript denotes values at \( \mu = 0 \). The high-\( \mu \) phase is a superfluid which forms as a result of a Bose-Einstein condensation of weakly interacting diquark bosons.

Let us briefly discuss the measurement of \( \langle qq \rangle \).

In a finite volume the technicalities are identical to those involved in measuring the chiral condensate. A diquark source term \( j^{tr}\tau_2 \chi \) is introduced and the action rewritten in a Gor’kov basis \( [14] \):

\[
\mathcal{L} = (\bar{\chi}, x^{tr}) \left( \sum_{j} \left( j^{tr} \tau_2 - \frac{1}{\beta} M^{tr} \right) \right) ^{\chi x^{tr}} (\chi) \equiv \Psi^{tr} \mathcal{A} \Psi \tag{9}
\]

whence

\[
Z[j, \bar{j}] = \int \mathcal{D}[\mathcal{A}U, j, \bar{j}] e^{-S_{\text{lat}}[U]}.
\tag{10}
\]

The condensate is then defined by

\[
\langle qq (j) \rangle = \frac{1}{V} \frac{\partial \ln Z}{\partial j} = \frac{1}{2V} \langle \text{tr} \tau_2 \mathcal{A}^{-1} \rangle.
\tag{11}
\]

Of course, since \( j \) is not a physical parameter one wants the \( j \to 0 \) limit, but any favourite method for probing \( \langle \bar{\psi}\psi(m = 0) \rangle \) can be used, such as direct inversion of \( \mathcal{A}(j) \) followed by extrapolating \( j \to 0 \) \( [20] \), using a Banks-Casher relation on the eigenvalue density of \( \tau_2 \mathcal{A} \) \( [21] \), or by calculating the probability distribution function of \( \langle qq \rangle \) \( [22] \).

Since the pioneering work of Nakamura \( [23] \), there have been several studies of TCQCD with \( \mu \neq 0 \) using a variety of algorithms and actions \( [18, 22–24, 26] \). I will briefly discuss some highlights. Fig. 3 shows strong coupling results \( [22] \) for \( \langle \bar{\chi}\chi \rangle, n_B \) and \( \langle qq \rangle \) vs. \( \mu \). Together with \( \chi PT \) predictions \( [6] \), showing acceptable agreement until saturation artifacts creep in for \( \mu/m_\pi \gtrsim 0.6 \). Fig. 4 shows \( \langle \bar{\chi}\chi \rangle/\langle \chi\bar{\chi} \rangle_0 \) vs. rescaled chemical potential for three different bare quark masses in TCQCD with adjoint quarks \( [18, 22] \), together with the prediction \( [6] \). In this case \( \chi PT \) appears to work well up to \( x \approx 2 \) over a decade of quark mass. Since \( \langle qq \rangle \neq 0 \) breaks a global symmetry, we anticipate a diquark Goldstone boson for \( \mu > \mu_o \) whose mass should vanish as \( j \to 0 \) \( [14] \).

Physically this results in long-ranged interactions
between vortex excitations in a rotating superfluid, and in propagating waves of temperature variation known as second sound. Simulations with $j = 0$ slow dramatically in this regime due to the profusion of small eigenvalues of $M$ \[18\]. Fig. 4 shows the results of simulating the Pfaffian weight \[10\] with $j \neq 0$ on a much larger system \[20\], which enable the identification of a massless mode in the zero source limit. Finally, studies of the high density regime at $T > 0$ have shown a strong first-order transition restoring the normal QGP state $\langle \bar{q}q \rangle = 0$ \[26\]. Although the TCQCD measure is obviously real, we have not discussed its positivity. In fact, it is possible to prove it positive for all cases except an odd number of adjoint staggered quarks. For $N = 1$ adjoint flavor, however, the sign problem returns \[18\]. Additionally in this case, the Goldstone count does not include any baryons, and the superfluid condensate $\langle qq \rangle$ in (11) is forbidden by the Pauli Exclusion Principle. Both facts invalidate the use of $\chi$PT. Indeed, the simplest local diquark operator that can be written is gauge-variant;

$$ q q^{sc}_i = \frac{1}{2} [\chi^{tr} t^i \chi + \bar{\chi} t^i \bar{\chi} tr] \in \mathbf{3} \text{ of SU}(2), $$

leading to the possibility of color superconductivity à la Georgi-Glashow in the high density phase. Simulations using the two-step multibosonic algorithm have been performed which reweight the ensemble taking the sign into account \[18\]. Once $\mu > \mu_o$ the sign starts to fluctuate, and by $\mu = 0.38$ at $m = 0.1$ ($\text{sgn(det}M)$) has fallen to 0.30(4) on $4^3 \times 8$ \[25\]. Remarkably, once the observables are also reweighted, all signals for the onset transition disappear, as seen in Fig. 6. This is in accord with the expectation that there is now a separation between the lightest baryon mass and the Goldstone scale, and demonstrates that $\text{sgn(det)}$ plays a decisive role in determining the ground state. Fig. 3 is probably the most expensive simulation of nothing happening to date.

4. Flatland NJL

The Nambu – Jona-Lasinio model has a long history as an effective description of the strong
interaction, and also has thermodynamic applications [27]. The Lagrangian is

\[\mathcal{L} = \bar{\psi}(\partial \psi - \mu \gamma_0 + m)\psi - \frac{g^2}{2}[(\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \tau \psi)^2]\]

(13)

At zero chemical potential and for \( m = 0 \) it has a SU(2)_L \otimes SU(2)_R \otimes U(1)_B global symmetry, which for sufficiently strong coupling \( g^2 \geq g_c^2 \) spontaneously breaks to SU(2)_I \otimes U(1)_B accompanied by the dynamical generation of a constituent mass \( \Sigma = g^2 \langle \bar{\psi} \psi \rangle \). Our current interest is the model in 2+1 dimensions, since apart from the obvious computational saving there is an interacting continuum limit at \( g^2 \to g_c^2, \Sigma \to 0 \) [28]. For \( \mu \neq 0 \), there is a strong first-order chiral symmetry-restoring transition at \( \mu_c \simeq \Sigma \) [29], which is completely separate from the Goldstone scale [12][30], but appears to coincide [31] with an onset transition separating the vacuum from a regime with \( n_B \propto \mu^2 \), suggestive of a two-dimensional Fermi surface with \( E_F \propto \mu \) in the chirally-restored phase. This is natural since the lightest baryons in the model are fermions.

An obvious question is whether U(1)_B is spontaneously broken for \( \mu > \mu_c \) by a superfluid condensate \( \langle \chi^{tr} \tau_2 \chi \rangle \neq 0 \) (note that we have surreptitiously slipped back into the notation of staggered fermions; the continuum translation is given in [32]). Superfluid condensation in this model would occur via a BCS instability at the Fermi surface, as in ^3He. To investigate this we have performed Pfaffian simulations with diquark source term \[ j_{\pm} = j_{\pm}(\chi^{tr} \tau_2 \chi \pm \bar{\chi} \tau_2 \chi^{tr}), \] measuring both condensate \( \langle qq_+ \rangle \) and associated susceptibilities \( \chi_{\pm} = \sum_x \langle qq_+(0)qq_+(x) \rangle \). Analogous to the axial Ward identity, we have

\[ \chi_{-}(j_{-} = 0) = \frac{\langle qq_+ \rangle}{j_{+}}. \]

(15)

The results are unexpected: fig. 7 shows a log-log plot of \( \langle qq_+ \rangle \) vs. \( j_+ \) from data from volumes up to \( 32^3 \) extrapolated to the zero temperature limit \( L_t \to \infty \). The high density data (\( \mu_c \simeq 0.65 \) for our choice of coupling) suggest a power-law equation of state

\[ \langle qq_+ \rangle \propto j^\alpha, \]

(16)

with \( \alpha = \alpha(\mu) \) falling in the range 0.2 - 0.3 for the values of \( \mu \) examined. This is reinforced by the
susceptibility ratio \( R = |\chi^+ / \chi^-| \) plotted in fig. 8; using (13) and (14) it is easy to show that \( R(j) \) should take the constant value \( \alpha \). The plateaux in fig. 8 display this behaviour, with approximately the same \( \alpha(\mu) \) as those from direct fits to (16).

The strange behaviour of figs. 7, 8 is strongly reminiscent of the low temperature 2d XY model, which exists in a critical state for a range of \( T \) with continuously varying exponents \( \delta(T) \), \( \eta(T) \), which we can define analogously (the arrow denotes a 2d vector):

\[
\langle qq \rangle \propto j^\delta ; \quad \langle qq(0)qq(\vec{x}) \rangle \propto \frac{1}{|\vec{x}|^\eta}.
\]

We therefore conjecture that NJL_{2+1} in its large-\( \mu \) phase describes a 2d critical system. The condensate is washed out by long-wavelength fluctuations as \( j \to 0 \), but long range phase coherence is maintained via (17). Intriguingly, the most precise tests of the 2d XY universality class are from experiments performed on thin films of superfluid \(^4\)He [34]. The superfluid current is related to the phase \( \theta(x) \) of \( qq(x) \) via

\[
\vec{J}_s = K_s \vec{\nabla} \theta.
\]

There is a beautiful topological argument [35] that the only way to change the circulation \( \kappa = \oint \vec{J}_s \cdot d\vec{l} \) around a periodic volume is to create a vortex – anti-vortex pair and translate one of them around the universe in the perpendicular direction until they reannihilate, thereby increasing \( \kappa \) by a quantum \( 2\pi K_s / L \). Since, however, the energy required increases logarithmically with pair separation, the circulation is metastable, thereby demonstrating superfluidity.

Why, then, does the NJL exponent \( \delta(\mu) \approx 3 - 5 \) differ from the XY value \( \delta(T) \geq 15 \)? Standard dimensional reduction does not apply since the limit \( L_t \to \infty \) is needed, implying that the fermion modes do not decouple. Further insight is gained from studying the fermion spectrum via the Gor’kov propagator \( \mathcal{G} = \mathcal{A}^{-1} \) [33]. To probe the Fermi surface, we need to analyse non-zero momenta \( \vec{k} \); indeed the mass gap extracted from the decay of \( \mathcal{G} (\vec{k}) \) in Euclidean time initially decreases as \( k \gtrsim k_F \), corresponding to vacating holes in the Fermi Sea, before rising for \( k \geq k_F \) indicating particle excitations. The resulting quasiparticle dispersion relation \( E(k) \) is shown in Fig. 9. The detailed form of \( E(k) \) in-

\[
E(k) \quad \mu=0.8 \\
\beta_s=0.67 \quad K_s=0.72 \\
\beta_s=1.0 \quad K_s=0.8 \\
\mu=0.8 \text{ free field}
\]

Figure 9. \( E(k) \) for both free and interacting quasiparticles at \( \mu = 0.8 \).
\( v_F = \partial E/\partial k |_{k=k_F} \approx 0.7c \). This is characteristic of a relativistic Fermi liquid with a repulsive interaction between quasiparticles with parallel momenta \[83\]. Most importantly, there is no evidence for a BCS gap \( \Delta \neq 0 \); certainly \( \Delta \sim \Sigma \) can be excluded. The origin of the new universality class may therefore be attributed to the presence of massless fermions at the Fermi surface. The overall conclusion is that NJL\(_{2+1}\) describes a relativistic thin film gapless superfluid.

5. A conjecture about superconductivity

All the systems we have been able to study at high density (ie. those with a real measure) exhibit superfluidity. What are the prospects for studying the breaking of a local symmetry? We have already seen that the only variant of TC-QCD with a potentially superconducting solution, namely \( N = 1 \) adjoint staggered flavor, is afflicted with a sign problem. The most promising microscopic model of high-\( T_c \) superconductors, namely the Hubbard model with on-site repulsion away from half-filling, also has an intractable sign problem in precisely the regime of interest \[83\]. A more familiar example in a particle physics context is technicolor, in which condensation of quark pairs from different representations of a gauge group force its dynamical breakdown; since this requires chiral fermions, in complex representations, a sign problem of some sort seems inevitable. An exotic \( 2+1d \) example is "\( \tau_3 \)-QED" \[83\]. This exhibits planar superconductivity by generating an electromagnetic photon mass by mixing with a second "statistical" photon \( a_\mu \) via a mixed Chern-Simons interaction generated by fermion loops; for a non-zero CS coupling dynamical mass generation through a gauge-invariant condensate \( \langle \bar{\psi} \psi \rangle \neq 0 \) is required. In Euclidean space, however, the CS term is pure imaginary and the resulting effective action complex; this also follows from the bare action since in this case the coupling \( i\bar{\psi} \gamma_3 \gamma_5 \psi \) implies \( \{ \gamma_5, D \} \neq 0 \), in turn implying \( \det M \neq \det M^* \). Finally, of course, there is QCD itself.

The following conjecture suggests itself: any system which exhibits the spontaneous breaking of a local symmetry by a pairing mechanism has a sign problem when formulated in terms of local gauge covariant degrees of freedom.

6. Summary

Significant progress – both technical and psychological – has been made in QCD(\( \mu \)), resulting in the first non-trivial LGT prediction in the \( (\mu, T) \) plane. For once we have been lucky; the high-\( T \) low-\( \mu \) region where simulations can probe is precisely the regime of direct relevance to RHIC phenomenology. I anticipate much activity in this area in the coming year. At \( T = 0 \) a quantitative description of nuclear or quark matter regrettably seems as elusive as ever – however there are at least two model systems where LGT simulations can now be said to be doing condensed matter physics \textit{ab initio}, ie. with matter formed from the fundamental quanta of the theory. Two Color QCD, a confining theory with no Fermi surface, describes the condensation of tightly-bound bosons, and thus resembles superfluid \( ^4\text{He} \). The NJL model, a theory without confinement but possessing a Fermi surface, displays unexpectedly interesting behaviour in \( 2+1d \) and certainly has the potential to exhibit a fully-fledged BCS mechanism in \( 3+1d \), providing a relativistic analogue of superfluid \( ^3\text{He} \). Simulations of NJL\(_{3+1}\) will furnish non-trivial tests of model calculations of color superconductivity \[2,3\]. Ultimately, though, a microscopic description of the superconducting state may actually require a sign problem.

Acknowledgements

This work is partially supported by EU contract ERBFMRX-CT97-0122. It is my pleasure to thank all my collaborators, but especially Susan Morrison, for their hard work and inspiration in equal measure.

REFERENCES

1. S. Ejiri, Nucl. Phys. Proc. Suppl. 94 (2001) 19.
2. D. Bailin and A. Love, Phys. Rep. 107 (1984) 325;
M.G. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. B422 (1998) 247; Nucl. Phys.
B537 (1999) 443; R. Rapp, T. Schäfer, E.V. Shuryak and M. Velkovsky, Phys. Rev. Lett. 81 (1998) 443; M.G. Alford, Nucl. Phys. Proc. Suppl. 73 (1999) 161.
3. J. Berges and K. Rajagopal, Nucl. Phys. B538 (1998) 443; M.G. Alford, Nucl. Phys. Proc. Suppl. 73 (1999) 161.
4. M.G. Alford, J. Berges and K. Rajagopal, Nucl. Phys. B571 (2000) 269.
5. S. Gottlieb, W. Liu, D. Toussaint, R.L. Renken and R.L. Sugar, Phys. Rev. Lett. 59 (1987) 2247.
6. I.M. Barbour and A.J. Bell, Nucl. Phys. B372 (1992) 385; I.M. Barbour, J.B. Kogut and S.E. Morrison, Nucl. Phys. Proc. Suppl. 53 (1997) 456.
7. M.A. Halász, Nucl. Phys. A642 (1998) 324c.
8. A. Gocksch, Phys. Rev. D37 (1988) 1014; M.A. Stephanov, Phys. Rev. Lett. 76 (1996) 4472.
9. E. Dagotto, F. Karsch and A. Moreo, Phys. Lett. B169 (1986) 421; E. Dagotto, A. Moreo and U. Wolff, Phys. Lett. B186 (1987) 395.
10. D.T. Son and M.A. Stephanov, Phys. Rev. Lett. 86 (2000) 592.
11. J.B. Kogut, M.A. Stephanov, D. Toublan, J.J.M. Verbaarschot and A. Zhitnitsky, Nucl. Phys. B582 (2000) 477.
12. I.M. Barbour, S.J. Hands, J.B. Kogut, M.-P. Lombardo and S.E. Morrison, Nucl. Phys. B557 (1999) 327.
13. Z. Fodor and S.D. Katz, hep-lat/0104001, hep-lat/0106002.
14. M.A. Stephanov, K. Rajagopal and E.V. Shuryak, Phys. Rev. D60:114028 (1999).
15. R.V. Gavai and S. Gupta, Phys. Rev. D64:074506 (2001).
16. S. Choe et al (QCD-TARO collaboration), hep-lat/0107003.
17. S. Ejiri, these proceedings.
18. S.J. Hands, I. Montvay, S.E. Morrison, M. Oevers, L. Scorzato and J. Skullerud, Eur. Phys. J. C17 (2000) 285.
19. S.E. Morrison and S.J. Hands, in Strong and Electroweak Matter ’98, p. 364 (1999), hep-lat/9902012.
20. J.B. Kogut, D.K. Sinclair, S.J. Hands and S.E. Morrison, hep-lat/0105026.
21. E. Bittner, M.-P. Lombardo, H. Markum, and R. Pullirsch, Nucl. Phys. Proc. Suppl. 94 (2001) 445.
22. R. Aloisio, V. Azcoiti, G. Di Carlo, A. Galante and A.F. Grillo, Phys. Lett. B493 (2000) 189; Nucl. Phys. B606 (2001) 322.
23. A. Nakamura, Phys. Lett. B149 (1984) 391.
24. J. Klatke and K. Mütter, Nucl. Phys. B342 (1990) 764; S.J. Hands, J.B. Kogut, M.-P. Lombardo and S.E. Morrison, Nucl. Phys. B558 (1999) 327; Y. Liu, O. Miyamura, A. Nakamura and T. Takaishi, hep-lat/0009009.
25. S.J. Hands, I. Montvay, L. Scorzato and J. Skullerud, hep-lat/0109029.
26. J.B. Kogut, D. Toublan and D.K. Sinclair, Phys. Lett. B514 (2001) 77.
27. S.P. Klevansky, Rev. Mod. Phys. 64 (1992) 649.
28. B. Rosenstein, B.J. Warr and S.H. Park, Phys. Rep. 205 (1991) 59.
29. B. Rosenstein, B.J. Warr and S.H. Park, Phys. Rev. D39 (1989) 3088; S.J. Hands, A. Kocić and J.B. Kogut, Nucl. Phys. B390 (1993) 355.
30. S.J. Hands, S. Kim and J.B. Kogut, Nucl. Phys. B442 (1995) 364.
31. J.B. Kogut and C.G. Strouthos, Phys. Rev. D63:054502 (2001).
32. S.J. Hands and S.E. Morrison, Phys. Rev. D59:116002 (1999).
33. S.J. Hands, B. Lucini and S.E. Morrison, Phys. Rev. Lett. 86 (2001) 753; hep-lat/0109001.
34. D.R. Nelson, in Phase Transitions and Critical Phenomena, Vol 7 (1983) p.1, eds. C. Domb and J.L. Lebowitz.
35. J.M. Kosterlitz and D.J. Thouless, J. Phys. C6 (1973) 1181.
36. G. Baym and I.E. Chin, Nucl. Phys. A262 (1976) 527.
37. R.L. Sugar, Nucl. Phys. Proc. Suppl. 17 (1990) 39.
38. N. Dorey and N.E. Mavromatos, Nucl. Phys. B386 (1992) 614.