Smoothed dynamics in the central field problem

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Abstract

Consider the motion of a material point of unit mass in a central field determined by a homogeneous potential of the form \((-1/r^\alpha)\), \(\alpha > 0\), where \(r\) being the distance to the centre of the field. Due to the singularity at \(r = 0\), in computer-based simulations, usually, the potential is replaced by a similar potential that is smooth, or at least continuous.

In this paper, we compare the global flows given by the smoothed and non-smoothed potentials. It is shown that the two flows are topologically equivalent for \(\alpha < 2\), while for \(\alpha \geq 2\), smoothing introduces fake orbits. Further, we argue that for \(\alpha \geq 2\), smoothing should be applied to the amended potential \(c/(2r^2) - 1/r^\alpha\), where \(c\) denotes the angular momentum constant.

Keywords: central field, singular homogeneous potential, smoothing, regularized vector field, topological equivalence

1 Introduction

For large particle systems, a principal tool of investigation is computer-based simulation. In a variety of problems the interaction of the particles is determined by a potential that is undefined at collisions. A common technique in dealing with the vector field singularities is to replace the potential with a

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smooth, or at least continuous, function. This procedure is called smoothing, or, in physics terminology, softening.

Smoothing was introduced in 1963 by S.J Aarseth cf. [1], [2], in the context of numerical simulations of galaxies. Since then, smoothing has became a commonly used technique in numerical modeling of large particle systems (see for instance, [6], [10], [9] or [11]).

Understanding the modifications induced by smoothing in large particle systems still remains a challenging task. A first step is to look at systems formed by two particles, but even in this simplified context, one is faced with difficulties; see, for instance, the analysis presented in [5], where several conjectures concerning the convergence of the approximation methods are stated.

Closely related to smoothing is the concept of regularization: they both target singularities in the flow as induced by the singularities in the vector field, but the resolution is different. Smoothing modifies the vector field. Regularization relies on a qualitative analysis of the phenomena near singularity and is achieved in two distinct steps. First, new parametrizations are applied, both time-dependent and -independent, leading to a regularized vector field, that is a vector field free of singularities. The phase space in the new coordinates is extended to include the singularity set, now blown-up into a physically fictitious and invariant manifold, usually called the collision manifold. Second, analysis of the flow on the extended phase space is performed in order to decide whether solutions asymptotically reaching the collision manifold can be matched to solutions asymptotically leaving the collision manifold, while preserving good behavior with respect to initial data. If such a matching is possible, then the flow may be extended (at least continuously) to include orbits ending/starting in collision. When this extension is performed, then the problem is said to be regularized. (For more on regularization, see [8] or, from a more physical point of view, see [12].)

We also mention the paper of Bellenttini et al. [4], where regularization is seen from the different perspective of approximating collision solutions by solutions of the smoothed flow. While analyzing a system where the interaction is given by homogeneous potentials of the form $-1/r^\alpha$, $\alpha > 0$, the authors convey that their procedure leads to a larger set of regularizable problems than in the standard treatment. Moreover, the smoothing chosen is irrelevant, as long as it provides a flow free of singularities.

In this paper we question the appropriateness of smoothing when motion both near and far from collision is under scrutiny. Our analysis is performed within the class of homogeneous potentials to which a standard potential smoothing

$$U_\epsilon(r; \alpha) := -1/(r^2 + \epsilon^2)^{\alpha/2}, \quad \epsilon > 0,$$

is applied. Within negative energy levels, we focus on the topological equivalence of the non-smoothed and the smoothed flows outside the collision set. We show that for $\alpha \geq 2$ the two flows are not topologically equivalent and thus smoothing of the form (1) generates orbits that do not correspond to
orbits of the real non-smoothed motion. For this case, we introduce the idea of smoothing the amended potential and show that, with such a modification, the two flows are topologically equivalent.

Employing a technique similar to that of McGehee [8], we choose to describe the dynamics in a parametrization where the non-smoothed flow is nonsingular and where the phase space is extended to include the collision set, now blown-up into a one-dimensional manifold (this is the first step of regularization as described above). The orbits lie on compact three dimensional manifolds which are level sets of the energy integral for negative energy values. Since the regularized vector field preserves the $SO(2)$ equivariance of the original problem, dynamics can be studied in a reduced three dimensional space. Here orbits can be easily visualized as curves determined by the intersection of the two surface integrals, the energy and the angular momentum. This allows us to compare the orbital pictures of the non-smoothed versus the smoothed problem.

While the non-smoothed reparametrized flow includes the orbits on the collision manifold, our analysis refers only to orbits outside of it and concerns only orbit topology. We do not refer to regularization of solutions (the second step of regularization, as outlined above) and we do not focus on issues related to approximating the non-smoothed solutions by smoothed ones.

The paper is organized as follows: we begin by briefly reviewing known facts about dynamics of two particle systems. Next, we reparametrize the vector fields of the non-smoothed and smoothed problems such that the collision set of the non-smoothed problem is blown-up into the aforementioned collision manifold. Using the $SO(2)$ symmetry to reduce the phase space to three dimensions, we study relative equilibria and examine symmetries of the reduced flow. Further, we analyse and compare of the orbits of the non-smoothed and smoothed flows, drawing the conclusion that for $\alpha \geq 2$ the two flows are not topologically equivalent. In the last section we argue that for $\alpha \geq 2$, smoothing should be applied to both the potential and the rotational non-inertial term, leading to the idea of a smoothed amended potential. Moreover, we show that when such a smoothing is applied, the topological equivalence of the non-smoothed and smoothed flows outside the collision set is achieved.

2 Equations of Motion

Consider the two degree of freedom Hamiltonian system given by the system of first order ordinary differential equations:

$$\begin{align*}
\dot{q} &= \frac{\partial H}{\partial p}, \\
\dot{p} &= -\frac{\partial H}{\partial q},
\end{align*}$$

(2)
where \( q = (x, y) \in \mathbb{R}^2, \ p = (p_x, p_y) \in \mathbb{R}^2 \). The function

\[
H_\epsilon(q, p) := \frac{p^2}{2} + U_\epsilon(q; \alpha)
\]

is the Hamiltonian of the system and \( U_\epsilon \) is a “smoothed” potential

\[
U_\epsilon(q; \alpha) = -\frac{1}{(r^2 + \epsilon^2)^{\alpha/2}}
\]

where \( \epsilon \geq 0 \) is a parameter and \( r = |q| = \sqrt{x^2 + y^2} \). For \( \epsilon = 0 \) the potential \( U_0(q; \alpha) \) reduces to the classical homogeneous potential, in which case the vector field defined by (2) has a singularity at \( q = 0 \).

Since the system is Hamiltonian, it is well known that the total energy is conserved. Consequently, the level sets of \( H_\epsilon \) are invariant under the flow of (2)

\[
H_\epsilon(q(t), p(t)) = h = \text{const.}
\]

Due to radial symmetry, the angular momentum is conserved as well and we have:

\[
x(t)p_y(t) - y(t)p_x(t) = c = \text{const.}
\]

Therefore the system has two independent first integrals in involution and it is integrable by the Liouville-Arnold theorem.

Since \( U_\epsilon(q, \alpha) : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \) is real analytic, standard results of differential equation theory guarantee, for any initial data \( (q_0, p_0) = (q(0), p(0)) \in \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \), the existence and uniqueness of an analytic solution defined on a maximal interval \([0, t^*)\), where \( 0 < t^* \leq \infty \). If \( t^* < \infty \), the solution is said to experience a singularity.

For \( \epsilon = 0 \), the singularity in the vector field induces singularities in the solution. Singularities specific to particle systems are given by collisions, which occur when \( q(t) \to 0 \) as \( t \to t^* \). In [8], McGehee showed that collisions are the only possible singularities of (2). He also proved that if \( \alpha \in (0, 2) \), then \( (q_0, p_0) \) leads to a collision if and only if the angular momentum is zero (that is \( q_0 \times p_0 = 0 \)) and that if \( \alpha \geq 2 \), then the set of initial conditions leading to a collision is rather large, including for \( \alpha > 2 \) zones where the energy integral is negative.

The standard methodology in dealing with collisions in n-particle simulations is smoothing the potential by setting \( \epsilon > 0 \). Then the vector field (2) is real analytic for all initial data \( (q_0, p_0) \in \mathbb{R}^2 \times \mathbb{R}^2 \), and the associated system of differential equations admits a unique global analytic solution.

3 The flow of the smoothed potential

3.1 Topological description of the energy surfaces

From now on, unless otherwise stated, the energy \( h \) is assumed to take negative values.
Using a technique similar to McGehee [8] we consider the following transformations \((r, p_r, \theta, p_\theta) \rightarrow (r, v, \theta, u)\) defined by
\[
\begin{aligned}
u &= p_\theta r^{\alpha/2}, \\
v &= p_r r^{\frac{\alpha-2}{2}}.
\end{aligned}
\]
This transformation is a diffeomorphism from \((0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R}\) to itself (where \(S^1\) is the unit circle). Further, we rescale the time parametrization by
\[
d\tau = r^{-\alpha + 2} dt.
\]
It is useful to keep in mind \(v\) and \(u\) are re-parametrised linear and angular momenta, respectively. The equations of motion take the form
\[
\begin{aligned}
r' &= rv \\
v' &= u^2 + \frac{\alpha}{2} v^2 - \frac{\alpha r^{\alpha+2}}{(r^2 + \epsilon^2)^{\frac{\alpha}{2} + 1}} \\
\theta' &= u \\
u' &= (\alpha - 2)uv,
\end{aligned}
\]
where prime denotes differentiation with respect to the independent variable \(\tau\). In the new coordinates the conservation of energy integral reads
\[
u^2 + v^2 - \frac{2r^\alpha}{(r^2 + \epsilon^2)^{\alpha/2}} + 2r^\alpha|\theta| = 0.
\]
From above we deduce that the phase space is foliated by the energy surfaces:
\[
\Sigma_{\epsilon, \epsilon} := \{ (r, v, \theta, u) \in [0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R} \mid F_\epsilon(r, v, \theta, u) = 0 \},
\]
where
\[
F_\epsilon(r, v, \theta, u) := u^2 + v^2 - \frac{2r^\alpha}{(r^2 + \epsilon^2)^{\alpha/2}} + 2r^\alpha|\theta|.
\]
For \(\epsilon = 0\), the transformations (3) and (4) have important consequences. First, the dynamics given by (5) has no singularity at \(r = 0\); in fact in the new coordinates the system extends analytically to all of \((r, v, \theta, u) \in [0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R}\), and hence the equations of motion are regularized. Second, the submanifold \(r = 0\) is now invariant under the flow. Third, the energy relation also extends to the set \(r = 0\) giving
\[
u^2 + v^2 - 2 = 0.
\]
Let
\[
M_0 := \{ (r, v, \theta, u) \mid r = 0, \ \theta \in S^1, \ u^2 + v^2 - 2 = 0 \},
\]
that is $M_0$ is the boundary of the extended phase space. Note that, since each energy surface has the same boundary $r = 0$, the set $M_0$ is independent of $h$. Topologically, it is a two-dimensional manifold embedded in $\mathbb{R}^3 \times S^1$ and it is diffeomorphic to a torus. We call $M_0$ the collision-ejection manifold or, simply, the collision manifold.

In the new parametrization, orbits which previously reached $r = 0$ in a finite time now tend asymptotically toward collision manifold. Further, orbits which previously passed close to collision now spend a long time near $M_0$. By continuity with respect to initial data, the flow on collision manifold, although lacking physical meaning, provides useful information about collision and near-collision solutions.

For $\epsilon > 0$, using the energy relation (6), we deduce that on each $\Sigma_{h,\epsilon}$ the radial coordinate is bounded by:

$$0 \leq r \leq R_{\epsilon}^{\text{max}} = \sqrt{\left(\frac{1}{|h|}\right)^{2/\alpha} - \epsilon^2}. \quad (8)$$

In physical space this means that the maximal distance in between points is bounded by the value $R_{\epsilon}^{\text{max}}$.

**Remark 3.1** Inequality (8) is meaningful provided $\epsilon$ is small enough, that is, if

$$\epsilon < \left(\frac{1}{|h|}\right)^{\frac{1}{\alpha}}.$$

Henceforth we will assume this condition is always satisfied.

The energy surface $\Sigma_{h,0}$ is a smooth manifold. For $\epsilon > 0$, if $\alpha \leq 1$, $\Sigma_{h,\epsilon}$ is also a smooth manifold; if $\alpha > 1$, the energy surface is not smooth at $(0,0,\theta,0)$ since $\nabla F_\epsilon$ at $(0,0,\theta,0) = (0,0,0,0)$ as it can be readily seen from

$$\frac{dF_\epsilon}{dr} = r^{\alpha-1} \left( \frac{-2\alpha}{(r^2 + \epsilon^2)^{\alpha/2}} + \frac{2r^2\alpha}{(r^2 + \epsilon^2)^{\alpha/2+1}} + 2\alpha|h| \right).$$

**Proposition 3.2** For $\epsilon > 0$ and negative energy $\Sigma_{h,\epsilon}$ is diffeomorphic to $S^2 \times S^1$ for $\alpha \leq 1$ and homeomorphic to it for $\alpha > 1$.

**Proof** In the space $(r,u,v) \in [0,\infty) \times \mathbb{R}^2$, $H_\epsilon(r,v,\theta,u) = 0$ describes a surface of revolution. This surface is orientable, compact, connected and of genus 0. It is smooth for $\alpha \leq 1$, and thus it is diffeomorphic to a two-sphere $S^2$. It is not smooth for $\alpha > 1$ and therefore it is homeomorphic to a sphere. Consequently, for $\alpha \leq 1$ ($\alpha > 1$), $\Sigma_{h,\epsilon}$ is diffeomorphic (homeomorphic) to $S^2 \times S^1$. □

In the new coordinates, the conservation of the angular momentum trans-
lates into the presence of invariant surfaces of the form

\[ \Gamma_c = \begin{cases} 
(r, v, \theta, u) \in [0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R} & \text{if } \alpha \leq 2 \\
(r, v, \theta, u) \in [0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R} & \text{if } \alpha > 2 
\end{cases} \]

if \( \alpha \leq 2 \)

\[ \{ (r, v, \theta) \in [0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R} | u = c r^{\frac{\alpha-2}{2}} \} \]

if \( \alpha > 2 \).

Taking into account energy conservations, we deduce that the phase space is foliated by invariants sets obtained as intersections \( \Sigma_{h,\epsilon} \cap \Gamma_c \).

### 3.2 Reduced dynamics

We now return to the system of equations (5). In order to study the dynamics it is convenient to exploit the fact that \( \theta \) does not appear explicitly in the equations. This allows us to reduce the four-dimensional phase space by factoring out the flow by \( S^1 \). The three dimensional reduced phase space is described by \( (r, v, u) \in [0, \infty) \times \mathbb{R} \times \mathbb{R} \) and a vector field given by:

\[
\begin{align*}
    r' &= rv \\
    v' &= u^2 + \frac{\alpha}{2} v^2 - \frac{\alpha r^{\alpha+2}}{(r^2 + \epsilon^2)^{\frac{\alpha}{2} + 1}} \\
    u' &= (\frac{\alpha-2}{2}) uv.
\end{align*}
\]

Defining

\[ f_\epsilon(r) := \frac{2r^\alpha}{(r^2 + \epsilon^2)^{\alpha/2}} - 2r^\alpha |h|, \]

the energy relation (6) reads:

\[ u^2 + v^2 = f_\epsilon(r). \]

With a slight abuse in notation, we choose to call the energy surfaces in the reduced space

\[ \Sigma_{h,\epsilon} := \{ (r, v, u) \in [0, \infty) \times \mathbb{R} \times \mathbb{R} | u^2 + v^2 - f_\epsilon(r) = 0 \}. \]

Likewise, the angular momentum invariant sets given by (9) are:

\[ \Gamma_c = \begin{cases} 
(r, v, u) \in [0, \infty) \times \mathbb{R} \times \mathbb{R} & \text{if } \alpha \leq 2 \\
(r, v, u) \in [0, \infty) \times \mathbb{R} \times \mathbb{R} & \text{if } \alpha > 2 
\end{cases} \]

if \( \alpha \leq 2 \)

\[ \{ (r, v, \theta) \in [0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R} | u = c r^{\frac{\alpha-2}{2}} \} \]

if \( \alpha > 2 \).

### 3.3 Relative equilibria

We now study the existence and nature of the equilibria of the reduced dynamics. Such fixed points which, in addition, are located outside the collision manifold (that is, those with \( r > 0 \)) correspond in the unreduced space to relative equilibria. In physical space, they represent uniform circular motions around the center of mass.
Case $\epsilon = 0$. Fixed points with $r = 0$ are found on the collision manifold $M_0$ and are situated at $C_{\pm} = (0, \pm \sqrt{2}, 0)$. Direct calculations show that they are saddles. In physical space they correspond to radial fall to or ejection from $M_0$. The radial coordinate of fixed points with $r > 0$ is given by:

$$2|h| r^{\alpha} = (2 - \alpha).$$

This equation has a unique solution

$$r_e = \left(\frac{2 - \alpha}{2|h|}\right)^{\frac{1}{\alpha}}$$

for $\alpha < 2$ and no positive solutions if $\alpha \geq 2$. Thus relative equilibria are present only for $\alpha < 2$ and they are of the form $R_{\epsilon}^{\pm} = (r_{\epsilon}, 0, \pm u_{\epsilon})$, where $u_{\epsilon}$ may be determined by substituting $r = r_{\epsilon}$ and $v = 0$ into the energy relation (11). A direct verification shows that they are centres.

This is not surprising, since it is well-known that motion in an attractive potential of the form $1/r^{\alpha}$, $\alpha < 2$ possesses such equilibria in the reduced space (corresponding to circular orbits in the physical space) as long as the angular momentum is non-zero (see [7]).

Case $\epsilon > 0$ It is immediate that the origin $O := (0, 0, 0)$ is a fixed point. The remaining solutions, if they exist, are of the form $R_{\epsilon}^{\pm} = (r_{\epsilon}, 0, \pm u_{\epsilon})$. Setting $v = 0$ in the second equation of (10), we obtain

$$u^2 - \alpha \frac{r^{\alpha+2}}{(r^2 + \epsilon^2)^{\frac{\alpha}{2}+1}} = 0$$

and, since by the energy relation

$$u^2 - f_{\epsilon}(r) = u^2 - \frac{2r^{\alpha}}{(r^2 + \epsilon^2)^{\frac{\alpha}{2}}} + 2r^{\alpha}|h| = 0,$$

we have to solve

$$-\alpha \frac{r^{\alpha+2}}{(r^2 + \epsilon^2)^{\frac{\alpha}{2}+1}} = -\frac{2r^{\alpha}}{(r^2 + \epsilon^2)^{\frac{\alpha}{2}}} + 2r^{\alpha}|h|.$$ 

This amounts to studying the zeroes of the function $r^{\alpha}/(r^2 + \epsilon^2)^{(\alpha/2+1)}h(r)$ where

$$h(r) := (2 - \alpha)r^2 + 2\epsilon^2 - 2|h|(r^2 + \epsilon^2)^{\frac{\alpha}{2}+1}.$$ 

If $\epsilon$ is sufficiently small, $h(0) = 2\epsilon^2(1 - |h|\epsilon^\alpha) > 0$ and $h(r) \to -\infty$ as $r \to \infty$. This implies that $h(r)$ has at least one zero for $r > 0$. The derivative is $h'(r) = 2rg(r)$ with

$$g(r) := (2 - \alpha) - 2|h|(\frac{\alpha}{2} + 1)(r^2 + \epsilon^2)^{\frac{\alpha}{2}}.$$ 

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Fig. 1. The reduced orbit space for \( \epsilon = 0 \) and \( \alpha = 1 \) outside the collision manifold, i.e. on the domain \( r > 0 \). The energy surface \( \Sigma_{h,0} \) is of a shape of a hemisphere. The orbits are given by the intersections of \( \Sigma_{h,0} \) with the angular momentum surfaces (not represented here). Easily noticed, in this case all orbits are periodic.

If \( 0 < \alpha < 2 \), \( g(r) \) is positive for \( r < r^* \) and negative for \( r > r^* \) where

\[
 r^* = \sqrt{\left( \frac{2 - \alpha}{2|h|\left(\frac{\alpha}{2} + 1\right)} \right)^{\frac{2}{\alpha}} - \epsilon^2}.
\]

Consequently, \( h(r) \) increases for \( r < r^* \) and decreases for \( r > r^* \), and since \( h(0) > 0 \), the equation \( h(r) = 0 \) has an unique solution. If \( \alpha \geq 2 \), \( g(r) < 0 \) thus \( h'(r) < 0 \), \( h(r) \) is a monotone decreasing function and \( h(r) = 0 \) has a unique positive solution.

Hence, for any \( \alpha > 0 \) the reduced smoothed flow admits two relative equilibria located at \( R^\pm_\epsilon = (r_\epsilon, 0, \pm u_\epsilon) \).

3.4 Symmetries

Observe that the plane \( u = 0 \) is an invariant manifold. The dynamics is symmetric with respect to transformations of the form:

\[
S : (r, v, u, \tau) \rightarrow (r, -v, u, -\tau).
\]

The invariance under this symmetry implies that if \( \gamma(\tau) \) is a solution of (5) then also \( S(\gamma(\tau)) \) is a solution. We have the following:

**Proposition 3.3** If an orbit crosses the plane \( v = 0 \) in one point it is \( S \)-symmetric. If it crosses the plane \( v = 0 \) in two (distinct) points (with \( r > 0 \)) it is \( S \)-symmetric and periodic.

**Proof** The first part of the theorem follows from the uniqueness of solutions. The second part follows since the solution is \( S \)-symmetric and closed (and does not intersect a critical point). □
Recall that for each fixed level $h < 0$ of the energy, the reduced flow is constrained to the two-dimensional surface $\Sigma_{h,\epsilon}$. Since the angular momentum is conserved, the orbits are determined by the intersection of $\Sigma_{h,\epsilon}$ with the two-dimensional angular momentum surfaces $\Gamma_c$ (see Figure 1). In other words, having $\epsilon \geq 0$ fixed, to each pair energy-angular momentum pair $(h, c)$ it corresponds a unique orbit

$$\gamma_{h,c} : = \Sigma_{h,\epsilon} \cap \Gamma_c.$$

Using Proposition 3.3, it is sufficient to study the intersection of each orbit with the plane $v = 0$. Note that, on a given orbit, points of the form $(r, 0, u)$ with $r > 0$ are turning points. Thus, for a periodic orbit, the two cuts with $v = 0$ are the points where the relative distance $r$ in between particle attains its maximum and its minimum. For a non-periodic orbit, the cut $v = 0$ corresponds to the maximal relative distance.

Since the graph of $\Sigma_{h,\epsilon}$ is symmetric with respect to the horizontal axis $u = 0$, we can restrict our domain to $u \geq 0$. Note that from (13) we have that $u \geq 0$ if and only if $c \geq 0$. So, fixing $h < 0$, the orbits are given by the intersection of

$$u_{h,\epsilon}(r) := +\sqrt{f_c(r)}, \quad \epsilon \geq 0,$$

with the curves $u_c(r)$ as defined by

$$\begin{cases} 
 u_c(r) r^{\frac{\alpha}{2}} = c & \text{if } \alpha \leq 2, \quad c \geq 0, \\
 u_c(r) = cr^{\frac{\alpha-2}{2}} & \text{if } \alpha > 2, \quad c \geq 0.
\end{cases}$$

If this intersection is void, then there is no orbit, whereas if the $u_{h,\epsilon}(r)$ and $u_c(r)$ intersect in one (non-tangential) point, then the corresponding orbit is a fall to/escape from collision. Further, if the $u_{h,\epsilon}(r)$ and $u_c(r)$ have two distinct points in common, then the corresponding orbit is periodic. Finally, if $u_{h,\epsilon}(r)$ and $u_c(r)$ are tangent to each other, then the point of tangency corresponds to a relative equilibrium.

### 3.5 Topological equivalence of the reduced flows

Let $\epsilon > 0$ and consider the intersections of $u_{h,0}(r)$ and $u_{h,\epsilon}(r)$ with $u_c(r)$, where $c \geq 0$.

**Case** $\alpha < 2$. For $c > 0$, all orbits are periodic, exception being the fixed points $R^+_0$ and $R^+_\epsilon$. For $c = 0$, all the orbits leading to/ejecting from collision (see Figure 2). Note that outside the collision set $c = 0$, the real and smoothed flows are topologically equivalent.

**Case** $\alpha = 2$. The angular momentum curves become horizontal lines $u_c(r) = c$ (see Figure 3). All orbits of the real flow are either leading to/escaping from collision, whereas all orbits of the smoothed flow are periodic. The real and smoothed flows are not topologically equivalent.
Fig. 2. Intersections of $u_{h,0}(r)$ (top solid curve) and $u_{h,\epsilon}(r)$ (bottom solid curve) with $u_c(r)$ (dashed curves). Case $\alpha < 2$.

Fig. 3. Intersections of $u_{h,0}(r)$ (top solid curve) and $u_{h,\epsilon}(r)$ (bottom solid curve) with $u_c(r)$ (dashed curves). Case $\alpha = 2$.

**Case $\alpha > 2$.** First, note that all orbits of the real flow are all leading to/escaping from collision (see Figure 4).

To determine the nature of the smoothed flow, we have to solve $u_{h,\epsilon}(r) = u_c(r)$. That is $\sqrt{f_c(r)} = cr^{\alpha/2}$, and further

$$r^{\alpha} \left[ \frac{c^2}{r^2} - \frac{2}{(r^2 + \epsilon^2)^{\alpha/2}} - 2h \right] = 0.$$  \hspace{1cm} (14)

If $c = 0$, the previous relation becomes

$$\frac{1}{(r^2 + \epsilon^2)^{\alpha/2}} = |h|.$$

The positive solution is unique and it is given by $r_m := \sqrt{(1/|h|)^{2/\alpha} - \epsilon^2}$. This solution corresponds to an ejection-collision orbit with no spin.
If \( c \neq 0 \), let us consider \( \chi(r) := \frac{r^2}{\epsilon^2} - \frac{2}{(r^2 + \epsilon^2)\alpha} \). The positive solutions of \( \chi(r) = 0 \) are given by the positive solutions of \( \psi(r) := -\left(\frac{2}{\alpha}\right)^{\frac{2}{\alpha}} r^{\frac{4}{\alpha}} + r^2 + \epsilon^2 = 0 \). Note that \( \psi(0) = \epsilon^2 \), and \( \psi(r) \to \infty \) as \( r \to \infty \). Also, \( \psi(r) \) has a unique minimum attained for \( r = r_m := \left(\frac{2}{\alpha}\left(\frac{2}{\alpha}\right)^{2/\alpha}\right)^{\frac{\alpha}{2\alpha-4}} \). Moreover \( \psi(r_m) = \left(\frac{2}{\alpha}\left(\frac{2}{\alpha}\right)^{2/\alpha}\right)^{\frac{1}{2\alpha-4}} \left(\frac{2}{\alpha}\right)^{2/\alpha} [2/\alpha - 1] + \epsilon^2 < 0 \) provided \( \epsilon \) is sufficiently small. It follows that \( \psi(r) \) has two zeroes and thus \( \chi(r) \) has two zeroes.

Thus \( \chi(r) - 2h \) has two solutions when \( h < 0 \) and \( h \) is larger than the minimum of \( \chi(r) \), one solution when \( h \) is equal to the minimum of \( \chi(r) \) and zero solutions otherwise. We deduce that for \( h \) in the domain of interest (that is for non-void intersections of \( u_{h,\epsilon}(r) \) with \( u_c(r) \)) and for \( c \neq 0 \), the orbits for the smoothed flow are periodic, provided that \( \epsilon \) is small enough. Since the real flow consists of only collision/ejection orbits, the real and the smoothed flows are not topologically equivalent.

4 The flow of the smoothed amended potential

In the previous section we have shown that for \( \alpha \geq 2 \) the flow associated to the smoothed potential is not topologically equivalent to the real flow. To understand why this is the case we return to the initial set-up of the problem.

Recall the Hamiltonian of the real flow

\[
H_0(x, y, p_x, p_y) = \frac{1}{2} \left( p_x^2 + p_y^2 \right) - \frac{1}{\sqrt{(x^2 + y^2)^\alpha}}
\]  

(15)
or in polar coordinates:

\[ H_0(r, \theta, p_r, p_\theta) = \frac{1}{2} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) - \frac{1}{r^\alpha} \]  

(16)

It is clear from the expression above that the behaviour near collision is dominated by the term \(1/r^{\max(2, \alpha)}\). It follows that for \(\alpha < 2\), modifying the potential only does not change the main features of the flow near collision. This is not the case for \(\alpha \geq 2\). Here, smoothing of the dominant term allows the centrifugal term \(p_\theta^2/r^2\) to dominate near collision and to change the character of the motion.

Recall that for the motion of a point in a central field on the plane, the distance from the centre of the field varies as in the one dimensional problem with a potential:

\[ V(r) := \frac{c^2}{2r^2} + U(r), \]

where \(c := p_\theta(t) = \text{const.}\) is the conserved angular momentum (see, for instance [3]). The function \(V(r)\) is usually called the amended or effective potential. The regions of motion together with the orbits’ type (i.e. bounded or unbounded) are determined by the inequality \(V(r) \leq h\).

When smoothing is applied, the preservation, as much as possible, of the character of the orbits is necessary. For the homogeneous interaction with \(\alpha \geq 2\), this may be achieved by smoothing the amended potential:

\[ V_\epsilon(r) := \frac{c^2}{2(r^2 + \epsilon^2)} - \frac{1}{(r^2 + \epsilon^2)^{\alpha/2}}. \]

As readily seen from Figure 5, the orbits are of similar type in both non-smoothed and smoothed problems, as long as \(c\) is chosen so that \(V_\epsilon(0) < 0\). The regions of motion are determined by \(V_\epsilon(0) < h < 0\) and so, to have a non-void orbit set, \(c\) must be chosen such that:

\[ |c| \leq \sqrt{\frac{2(1 - |h|\epsilon^\alpha)}{\epsilon^{\alpha - 2}}}. \]  

(17)

To decide the topological character or the orbits, we return to initial set-up and consider the smoothed Hamiltonian

\[ H_\epsilon(r, \theta, p_r, p_\theta) := \frac{1}{2} \left( p_r^2 + \frac{p_\theta^2}{(r^2 + \epsilon^2)} \right) - \frac{1}{(r^2 + \epsilon^2)^{\alpha/2}}, \quad \alpha \geq 2. \]  

(18)

Now we perform a study similar to one in the previous section. Omitting the details, the main steps are:
Fig. 5. The non-smoothed amended potential $V_0$ and the smoothed amended $V_\epsilon$ potential for $\alpha \geq 2$.

- first, we apply transformations similar to (3) and (4):

$$
\begin{align*}
  u &= p_\theta(r^2 + \epsilon^2)^{\frac{\alpha-2}{4}}, \\
  v &= p_r(r^2 + \epsilon^2)^{\frac{\alpha}{4}};
\end{align*}
$$

and introduce a new time parametrization via:

$$
  d\tau = r^{-\frac{\alpha+2}{4}} dt;
$$

- the new vector field is regularized and is given by:

$$
\begin{align*}
  r' &= vr^{\frac{\alpha+2}{4}}(r^2 + \epsilon^2)^{-\frac{\alpha}{4}}, \\
  v' &= r^{\frac{\alpha+4}{4}}(r^2 + \epsilon^2)^{-\frac{\alpha+4}{4}}(u^2 + \frac{\alpha}{2}v^2 - \alpha), \\
  \theta' &= ur^{\frac{\alpha+4}{4}}(r^2 + \epsilon^2)^{-\frac{\alpha+4}{4}}, \\
  u' &= \left(\frac{\alpha-2}{2}\right)r^{\frac{\alpha+2}{4}}(r^2 + \epsilon^2)^{-\frac{\alpha+4}{4}}uv;
\end{align*}
$$

- since $\theta$ does not appear explicitly in the $(r, v, u)$ equations above, we reduce the phase space to three dimensions by factoring out the flow by $S^1$;

- in the reduced space $(r, v, u)$ the energy surfaces take the form

$$
  \Sigma_{h,\epsilon} = \{(r, v, u) \mid u^2 + v^2 = 2 \left(1 - |h|(r^2 + \epsilon^2)^{\alpha/2}\right)\};
$$

they are surfaces of revolution and, as it may be easily noticed, their topology is identical;

- the angular momentum surfaces are given by
Fig. 6. (a) Intersections of $u_{h,0}(r)$ (top solid curve) and $u_{h,\epsilon}(r)$ (bottom solid curve) with $u_c(r)$ (dashed curves). Case $\alpha = 2$. (b) Intersections of $u_{h,0}(r)$ (top solid curve) and $u_{h,\epsilon}(r)$ (bottom solid curve) with $u_c(r)$ (dashed curves). Case $\alpha > 2$.

\[ \Gamma_c = \{(r, v, u) \mid u = c(r^2 + \epsilon^2)^{\frac{\alpha-2}{4}}\}; \]

- by symmetry, each orbits may be represented by its intersection with the plane $v = 0$. Therefore, it is sufficient to study the intersections of the curves

\[ u_{h,\epsilon}(r) = \sqrt{f_{\epsilon}(r)} := \sqrt{2(1 - |h|(r^2 + \epsilon^2)^{\alpha/2})}, \]

with

\[ u_{c,\epsilon}(r) := c(r^2 + \epsilon^2)^{\frac{\alpha-2}{4}}, \quad c \geq 0. \]

For $\alpha \geq 2$ all orbits are non-periodic, leading to or ejecting from the collision manifold $M_\epsilon := \{(r, v, u) \mid r = 0, u^2 + v^2 = 2\left(1 - |h|\epsilon^{\alpha/2}\right)\}$. For $\alpha = 2$, the angular momentum curves $u_{c,\epsilon}(r)$ are horizontal lines (see Figure 5), whereas for $\alpha > 2$, they are increasing and have initial value $u_{c,\epsilon}(0) = c\epsilon^{\frac{\alpha-2}{4}}$ (see Figure 6). The admissible values for $c$ are obtained by requiring $u_{c,\epsilon}(0) \leq u_{h,\epsilon}(0)$. Thus we obtain $c\epsilon^{\frac{\alpha-2}{4}} \leq \sqrt{2(1 - |h|\epsilon^{\alpha})}$, which, after some algebra, becomes condition (17). More importantly, the real and smoothed flows are topologically equivalent.

5 Conclusions

In this paper we discussed the topological equivalence of the non-smoothed and the smoothed flows given by motion with a homogeneous potential of the form $-1/r^\alpha$, $\alpha > 0$. The analysis was performed outside the collision set. We deduced that for $\alpha < 2$ the two flows are topologically equivalent, and showed that this is not the case for $\alpha \geq 2$. For the latter situation, we introduced the idea of smoothing the amended potential and showed that, with such a modification, the two flows are topologically equivalent.
Smoothing of the amended potential in a two degree of freedom system might be considered a first approach to a more general problem: given a mechanical system with symmetry and with a singular potential, what is the best way to apply smoothing? Our investigation shows that the presence of non-inertial terms has to be treated carefully. Significant modifications of the global orbital picture might appear due to interplay of the potential and centrifugal forces, especially when the system passes close to a degenerate (e.g. for $N$-body problems, a collinear) configuration. These issues will be discussed elsewhere.

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