Disentanglement Approach to Quantum Spin Ground States: Field Theory and Stochastic Simulation

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We develop an analytical and numerical framework based on the disentanglement approach to study the ground states of many-body quantum spin systems. In this approach, observables are expressed as functional integrals over scalar fields, where the relevant measure is the Wiener measure. We identify the leading contribution to these integrals, given by the saddle point field configuration. Analytically, this can be used to develop an exact field-theoretical expansion of the functional integrals, performed by means of appropriate Feynman rules. The expansion can be truncated to the desired order to obtain approximate analytical results for ground state expectation values. Numerically, the saddle point configuration can be used to compute physical observables by means of an exact importance sampling scheme. We illustrate our methods by considering the quantum Ising model in 1, 2 and 3 spatial dimensions. Our analytical and numerical results are applicable to a broad class of many-body quantum spin systems, bridging concepts from quantum lattice models, continuum field theory, and classical stochastic processes.

I. INTRODUCTION

Lattice quantum spin systems have long been of great interest to many-body physicists. Spin Hamiltonians provide phenomenological models of real-life condensed matter systems [1–3]. More recently, advancements in the fields of ultra cold atomic gases [4, 5] and trapped ions [6, 7] have made it possible to experimentally realize isolated model systems; these offer an ideal testbed to investigate fundamental concepts of quantum physics, such as quantum phase transitions and entanglement. Away from exactly solvable integrable models [8, 9], which are mostly one dimensional, analytical treatments of quantum spin systems are typically based on the spin coherent state path integral [10–15]. While path integrals frequently elude an exact evaluation, they are often useful to develop approximation schemes, including semiclassical treatments or instanton techniques [13–15]. However, the continuum limit of the coherent state path integral is mathematically subtle [16–19], and is still an area of current research [19–23]. In taking the continuum limit, the differentiability of trajectories is incorrectly assumed [10, 13, 24]; this has been reported to lead to wrong results for certain simple models [19]. The lack of generally applicable analytical techniques has also lead to the development of several numerical schemes. Monte Carlo methods [2, 25] have achieved great success for a range of systems [2, 26–34]; other applications (notably, frustrated magnets [35]) are however plagued by sign or phase problems [36–38], which have been circumvented in special cases [39–44], but whose general resolution has proved to be a hard task. More recent tensor-network approaches [45, 46] have been able to handle large or even infinite systems in one [47] and higher dimensions [48–51]; however, their applicability in the latter case is significantly restricted by the growth of entanglement and the computational cost associated with contracting higher dimensional lattices [51, 52].

In this manuscript, we consider an alternative analytical and numerical framework for quantum spin systems, based on a disentanglement approach [24, 53–55] whereby ground state expectation values are expressed as functional integrals over single-spin trajectories. Said integrals are performed with respect to the Wiener measure; they are thus straightforwardly amenable to numerical evaluation [24, 55]. Furthermore, as noted in Ref. [24], this construction does not assume the differentiability of paths, and is therefore free from the related issues that affect coherent state path integrals. In the disentanglement approach, an interacting quantum spin system is described as a statistical ensemble of non-interacting trajectories. We identify the trajectory yielding the largest contribution to a functional integral of interest; this corresponds to the saddle point field configuration, which extremizes a suitable effective action. This makes it possible to develop the disentanglement approach towards both analytical and numerical applications. As an analytical framework, the disentanglement formalism provides an exact field theoretical formulation of lattice quantum spin systems, and allows one to obtain successive approximations to observables by means of a set of Feynman rules and the corresponding diagrammatic representation. As a numerical tool, the disentanglement approach can be used to exactly formulate quantum expectation values as averages over classical stochastic processes [24, 55]. In this context, the saddle point configuration can be used to develop an exact importance sampling technique, greatly improving the performance of the method over direct sampling of the integrals.

The manuscript is organized as follows. In Section II we outline the disentanglement formalism, defining the effective action for ground state expectation values and the associated saddle point equation. We explicitly illustrate our method by obtaining the saddle point configuration for the quantum Ising model in D spatial dimension. In Section III we focus on the field-theoretical framework that arises from the disentanglement formalism, discussing how observables can be analytically computed...
by expanding the effective action around the saddle point configuration. The expansion gives rise to Feynman rules and diagrams: we explicitly exemplify this by considering the ground state energy of the quantum Ising chain. The numerical approach is presented in Section IV; we outline the importance sampling algorithm and illustrate it by applying it to the quantum Ising model in 1, 2 and 3 spatial dimensions. We also study the numerical performance of the method, showing that the importance sampling algorithm leads to a vast improvement over direct sampling. We conclude our discussion in Section V, summarizing our findings and outlining future directions.

II. DISENTANGLEMENT FORMALISM FOR QUANTUM SPIN GROUND STATES

A. Disentanglement Transformation

Consider a quantum system with Hamiltonian $\hat{H}$. The ground state $|\psi_G\rangle$ of the system can be obtained from a generic state $|\psi_0\rangle$ by performing imaginary time evolution: since at late imaginary times $\tau$ all excited states are exponentially suppressed compared to the ground state, one has

$$|\psi_G\rangle \sim \lim_{\tau \to \infty} e^{-\hat{H}\tau} |\psi_0\rangle,$$

where we set $\hbar = 1$. It is then natural to introduce the Euclidean time evolution operator $\hat{U}(\tau) = e^{-\hat{H}\tau}$. Without loss of generality, let us consider initial states $|\psi_0\rangle$ that are product states; these can be conveniently parameterized in terms of a single reference state $|\Psi\rangle = \prod_i |i\rangle_i$, where $\hat{S}_{ji}^- |i\rangle_i = 0$. The Euclidean time evolution from an arbitrary state $|\psi_0\rangle$ is then obtained by considering the modified time evolution operator

$$\hat{\mathcal{U}}(\tau) \equiv e^{-\tau\hat{H}} \hat{U}_0,$$

where the unitary operator $\hat{U}_0$ satisfies

$$\hat{U}_0 |\Psi\rangle = |\psi_0\rangle.$$

The ground state expectation value of an observable $\hat{O}$ can thus be written as

$$\mathcal{O}_G = \lim_{\tau \to \infty} \frac{\langle \Psi | \hat{\mathcal{U}}(\tau) \hat{O} \hat{\mathcal{U}}(\tau) | \Psi \rangle}{\langle \Psi | \hat{\mathcal{U}}(\tau) \hat{\mathcal{U}}(\tau) | \Psi \rangle}.$$

The denominator of Eq. (4) provides the necessary normalization, since $\hat{\mathcal{U}}(\tau)$ inherits the non-unitarity of $\hat{U}(\tau)$. All information about ground state expectation values is then encoded in the late-time behaviour of $\hat{\mathcal{U}}(\tau)$. For quantum spin systems with quadratic Hamiltonians, $\hat{\mathcal{U}}(\tau)$ can be conveniently re-expressed using a disentanglement formalism [24, 53, 54], recently applied to the real time evolution of quantum spin systems [55, 56]. In particular, let us consider a system with Hamiltonian

$$\hat{H} = -J \sum_{i,j} J_{ij}^a \hat{\xi}_i^a \hat{\xi}_j^a + \sum_i h_i^a \hat{\xi}_i^a,$$

where the spin operators $\hat{S}_j^a$ on site $j$ satisfy the SU(2) commutation relations $[\hat{S}_j^a, \hat{S}_{k}^b] = i \delta_{jk} \epsilon^{abc} \hat{S}_j^c$ with $a, b \in \{x, y, z\}$. We consider a symmetric interaction matrix $J_{ij}^a$ with interaction strength $J$ and an applied magnetic field $h_i^a$. By performing a Hubbard-Stratonovich decoupling [57, 58] followed by a Lie-algebraic disentanglement transformation [24, 53, 54, 59, 60], $\hat{\mathcal{U}}(\tau)$ can be exactly represented as a functional integral [24, 53, 54]:

$$\hat{\mathcal{U}}(\tau) = \int \mathcal{D} \varphi e^{-S_0[\varphi]} \hat{\mathcal{U}}^*(\tau),$$

where the noise action $S_0$ is given by

$$S_0[\varphi] = \frac{J}{4} \int_0^\tau d\tau' \sum_{abij} (J^{-1})^{ij}_{ab} \varphi^a_i(\tau') \varphi_b^j(\tau')$$

and the stochastic time evolution operator $\hat{\mathcal{U}}^*$ is defined as a product of on-site operators:

$$\hat{\mathcal{U}}^*(\tau) \equiv \prod_j \hat{U}_j^*(\tau) = \prod_j e^{\xi_j^+(\tau)\hat{S}_j^+ + \xi_j^-(\tau)\hat{S}_j^-}\hat{\xi}_j^-(0)\hat{\xi}_j^+(0).$$

The operators $\hat{U}_j^\pm$ have a functional dependence on the fields $\varphi = \{\varphi^a_i\}$ via the disentangling variables $\xi \equiv \{\xi_j^a\}$, which satisfy [24]

$$\xi_j^+ = \Phi_j^+ + \Phi_j^+ \xi_j^+ - \Phi_j^+ \xi_j^-;$$

$$\xi_j^- = \Phi_j^- + 2\Phi_j^- \xi_j^-;$$

where $\Phi_j^a = h_j^a + J \varphi_j^a$. The initial conditions of the disentangling variables are determined from $\hat{\mathcal{U}}(0) = \hat{U}_0$; for example, for a spin-1/2 system, the general product state $|\psi_0\rangle \equiv \prod_i (a_i, b_i)$ corresponds to the initial conditions

$$\xi_j^+(0) = a_j/b_j;$$

$$\xi_j^-(0) = -\log(b_j);$$

$$\xi_j^-(0) = -a_j^*/b_j^*.$$

For completeness, we outline the derivation of Eqs (6) and (9) in Appendix A. Eq. (6) can be seen as an exact path integral representation of the time-evolution operator. The operators inside the functional average (6) are decoupled over sites and act in a simple way on any state of interest. This allows one to formulate an exact field theoretical description of lattice spin systems, as we show in Section III. The noise action (7) can be diagonalized in terms of a new set of fields $\phi = \{\phi^a_i\}$ by performing the linear transformation $\phi_i^a = \sum_j O_{ij}^a \phi_j^a$, where $O$ is a

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1 This convention differs from that of Refs [55, 56] by a rescaling of the scalar fields $\phi_i^a$; see Appendix A.
matrix satisfying $O^T J^{-1} O / 2J = 1$. With this transformation, the noise action takes the form \cite{55}

$$S_0[\phi] \equiv \frac{1}{2} \int_0^\tau d\tau' \sum_{ai} \phi_i^a(\tau') \phi_i^a(\tau'). \tag{11}$$

Notably, due to the Gaussian nature of (11), the fields $\phi$ can also be interpreted as delta-correlated, unit-variance Gaussian white noise variables \cite{24, 53}. Thus, the functional integral in Eq. (6) can be equivalently seen as an average over the stochastic processes $\phi_i^a(\tau)$ \cite{24, 53, 55}:

$$\tilde{U}(\tau) = \langle \prod_j e^{\xi_j^+ J_s^+ \xi_j^+ J_s^+ \xi_j^+ J_s^+ \xi_j^+ J_s^+} \rangle \phi, \tag{12}$$

where the notation $\langle \ldots \rangle_\phi$ denotes averaging with respect to the noise action (11). The equations of motion Eq. (9) are then interpreted as stochastic differential equations (SDEs) for the variables $\xi_i^a$ \cite{24, 53}. By representing each of the time evolution operators in Eq. (4) using the disentangling formula (12), one can express quantum ground state expectation values as classical averages. Introducing independent sets of forwards and backwards fields, $\phi_f \equiv \{ \phi_{f,i}^a \}$ and $\phi_b \equiv \{ \phi_{b,i}^a \}$, and the associated disentangling variables $\xi_f \equiv \{ \xi_{f,i}^a \}$, $\xi_b \equiv \{ \xi_{b,i}^a \}$, one has

$$O_G = \lim_{\tau \to \infty} \frac{\langle F_0(\tau) \rangle_{\phi_f, \phi_b}}{\langle F_0(\tau) \rangle_{\phi_f, \phi_b}}, \tag{13}$$

where the classical function $F_0$ corresponding to the operator $O$ is defined by

$$F_0 \equiv \langle \psi | \hat{U}^\tau(\psi) | \hat{O} \hat{U}^\tau(\psi) | \psi \rangle, \tag{14}$$

and $F_0$ is obtained from (14) when $\hat{O}$ is replaced by the identity operator. It can be readily seen that the functions $F_0$ take the same functional form as their real time counterparts, given in Refs \cite{55, 56}. Here, in contrast to the cited references, we express all initial states in terms of a single reference state and variable initial conditions $\xi_i^a(0)$: in this way, each observable corresponds to one classical expression only, regardless of the initial state. The reference state $\langle \psi |$ was selected because it results in the simplest classical expressions \cite{55, 56}. For example, for spin-$1/2$ systems, the classical function for the normalization is given by

$$F_1(\tau) = \prod_i \left[ 1 + \xi_{f,i}^+ (\tau) \xi_{b,i}^+ (\tau) e^{-\frac{1}{2} \xi_{f,i}^+ (\tau) + \xi_{b,i}^+ (\tau)} \right], \tag{15}$$

while the longitudinal magnetization $M_z = \sum_j (\xi_j^a) / N$ corresponds to the classical function \cite{55}

$$F_{M_z}(\tau) = \frac{F_1(\tau)}{N} \sum_j \frac{1 - \xi_{f,j}^+ (\tau) \xi_{b,j}^+ (\tau)}{1 + \xi_{f,j}^+ (\tau) \xi_{b,j}^+ (\tau)}. \tag{16}$$

For any observable, the appropriate classical function can be constructed using the building blocks provided in Ref. \cite{56}. The expectation values of functions such as (15) and (16) can be evaluated numerically by averaging them over realizations of the stochastic processes $\phi_i^a$, as done in Refs \cite{55, 56} for real time evolution; the processes $\phi_i^a$ determine the time evolution of the variables $\xi_i^a$ via the SDEs (9). The key towards both analytical and numerical developments is identifying the trajectories $\phi$ which provide the largest contribution for each observable; we discuss this in the following Section.

### B. Extremal Trajectories

In the disentanglement formalism ground state expectation values are expressed in terms of classical averages, as in Eq. (13). We discussed how such averages can be computed by numerical sampling: one averages over realizations of the trajectories $\phi_i^a$ distributed according to the action (11). In this approach, which we refer to as direct sampling, it can be seen that one preferentially generates trajectories that are close to the non-interacting limit $\phi_i^a(\tau) = 0$; see Appendix B. However, the trajectories that contribute most significantly to a given functional integral may be substantially different from the non-interacting trajectories. It is therefore important to identify the dominant contributions, as also suggested in \cite{61}. For this purpose, it is convenient to work with the action (7) featuring the fields $\varphi_i^a$. An observable $O$ can be written as

$$O \equiv \langle \hat{O} \rangle = \langle f_O(\varphi) \rangle_\varphi \tag{17}$$

where $\varphi = \{ \varphi_\alpha \}$ denotes all of the HS fields: the collective index $\alpha$ runs over sites, Lie algebra generators, and sets of fields (e.g. forwards and backwards). Eq. (17) can be equivalently written as

$$O \equiv \int D\varphi e^{-S_O[\varphi]}, \tag{18}$$

which defines the effective action $S_O[\varphi] \equiv S_0[\varphi] - \log f_O[\varphi]$ for the observable $O$. The leading contribution to the integral (18) is given by the configuration $\varphi_{\text{SP}}(\tau)$ which extremizes the effective action:

$$\varphi_{\text{SP}} : \frac{\delta S_O}{\delta \varphi} |_{\varphi_{\text{SP}}} = 0. \tag{19}$$

We refer to $\varphi_{\text{SP}}$ as the saddle point (SP) field. The trajectory corresponding to the field $\varphi_{\text{SP}}$ provides the leading order (LO) approximation to a given observable, as we further discuss in Section III A.

One can view Eq. (18) as an exact continuum field theoretical formulation of the lattice quantum spin system; Eq. (18) can then be expanded about the saddle point to analytically obtain corrections beyond LO, computed by using the associated Feynman rules and diagrams. This approach is discussed and explicitly applied in Section III.
Alternatively, one can view the disentanglement approach as a numerical tool. \( \varphi_{\text{SP}} \) can then be used to perform an exact change of variables in the functional integral (18); this amounts to preferentially sampling trajectories near \( \varphi_{\text{SP}} \), which yield the largest contributions to the integral. In this approach one does not truncate to a given order in the fluctuations: the resulting equation is still exact and, due to the Gaussianity of the noise action, can also be evaluated numerically as in Refs \[55, 56\]. This is discussed in Section IV, where we show how this method can be used to numerically access much larger system sizes than it is possible when sampling according to the naive measure (11).

In the next Section, we show how the saddle point trajectory is computed; this provides the starting point for both analytical and numerical developments.

C. Ising Saddle Point Equation

We wish to identify the leading contribution to a given functional integral by solving Eq. (19). For definiteness, we illustrate this by considering the quantum Ising model in \( D \) spatial dimensions; this is given by the Hamiltonian

\[
\hat{H}_1 = -J \sum_{ij} \hat{S}^z_i \hat{S}^z_j - \Gamma \sum_{j=1}^N \hat{S}^z_j,
\]

where \((ij)\) denotes nearest-neighbor interactions. We consider a system of \( N = N_1 \times \cdots \times N_D \) spin-1/2 degrees of freedom on a \( D \)-dimensional hypercubic lattice, with periodic boundary conditions and ferromagnetic interactions \( J > 0 \). We begin by considering the one-dimensional case, and then generalize our results to general \( D \). For \( D = 1 \), the model (20) reduces to the quantum Ising chain, which is exactly solvable in terms of free fermions [62]; this allows for the exact computation of physical observables in the thermodynamic limit. In this case, the general result (9) specializes to the Euclidean Ising SDEs [24, 55, 56]

\[
\begin{align}
\dot{\xi}^+_i(\tau) &= \Gamma \frac{1}{2} \left( 1 - \xi^+_i(\tau)^2 \right) + J \xi^+_i(\tau) \varphi_i(\tau), \\
\dot{\xi}^-_i(\tau) &= -\Gamma \xi^-_i(\tau) + J \varphi_i(\tau), \\
\dot{\xi}^z_i(\tau) &= \Gamma \frac{1}{2} \exp \xi^+_i(\tau).
\end{align}
\]

A natural choice of observable is the ground state energy density \( \epsilon_G \). This can be computed using Eq. (13), according to the general formalism outlined in Section II A. Alternatively, \( \epsilon_G \) can also be obtained as

\[
\epsilon_G = - \lim_{\tau_f \to \infty} \frac{1}{N \tau_f} \log A(\tau_f),
\]

where the Euclidean Loschmidt amplitude \( A(\tau_f) \) for the initial state \( |\psi_0\rangle \) is defined as

\[
A(\tau_f) = \langle \psi_0 | \hat{U}(\tau_f) | \psi_0 \rangle.
\]

By computing \( \epsilon_G \) by means of Eq. (22), one only needs to consider a single time evolution operator, which corresponds to a single set of HS fields \( \phi^a_i \). Thus, using Eq. (22) allows us to simplify the subsequent analytical developments. As shown in Appendix C, the same results can be equivalently obtained from the more general formalism of Section II A, which involves two time evolution operators. To further simplify our calculations, we choose the all-down initial state \( |\psi_0\rangle = \otimes_j |\downarrow \rangle_j \equiv |\downarrow \rangle \); in this case, the Loschmidt amplitude is given by the functional integral [55]

\[
A(\tau_f) = \int \mathcal{D}\varphi \ e^{-S[\varphi]} e^{-\frac{\Gamma}{2} \sum_i \xi^+_i(\tau_f)},
\]

where the equation of motion of \( \xi^+_i \) is given by (21b) and the initial conditions are \( \xi^+_i(0) = 0 \). Following the discussion of Section II B, we write the Loschmidt amplitude (24) as

\[
A(\tau_f) = \int \mathcal{D}\varphi \ e^{-S[\varphi]}
\]

which defines the Euclidean Loschmidt action:

\[
S[\varphi] = \int_0^{\tau_f} d\tau \left[ \frac{1}{2} \sum_{ij} (J^{-1})_{ij} \varphi_i(\tau) \varphi_j(\tau) - \frac{\Gamma}{2} \sum_i \xi^+_i(\tau) + \sum_i \varphi_i(\tau) \right].
\]

The variables \( \xi^+_i \) featured in the action (26) are themselves functionals of \( \varphi_i \), as determined by (21). It follows that \( S[\varphi] \) cannot be written in terms of a Lagrangian involving only the fields \( \varphi \) and their time-derivatives, and
it is thus not possible to obtain Euler-Lagrange equations in the standard way. Rather, in order to obtain the saddle point field configuration, one directly extremizes the action (26) with respect to varying the field \( \phi_i \). This yields the \textit{Loschmidt saddle point equation}

\[
\varphi_i(\tau')_{SP} = \frac{1}{J} \sum_j J_{ij} \int_0^{\tau_f} d\tau \frac{\delta \xi^+_i(\tau)}{\delta \varphi_j(\tau')}_{SP} - 1, \quad (27)
\]

where we used \( \sum_j J_{ij} = 1 \forall i \) and \( \delta \xi^+_i / \delta \varphi_j \propto \delta_{ij} \). The subscript \( SP \) denotes quantities that are evaluated at the saddle point. By varying Eq. (21a) with respect to \( \varphi_i(\tau') \), one obtains

\[
\frac{\delta \xi^+_i(\tau)}{\delta \varphi_i(\tau')} = \Xi_i(\tau, \tau') = J \xi_i(\tau') \theta(\tau - \tau') e^{-J\tau \gamma(s)ds}, \quad (28)
\]

where \( \theta(\tau) \) is the Heaviside step function and we defined

\[
\gamma_i(s) \equiv \Gamma \xi^+_i(s) - J \varphi_i(s). \quad (29)
\]

Due to the translational invariance of the model (20) and of the chosen initial state, at the saddle point all \( \xi^+_i \) take the same value, \( \xi^+_i |_{SP} = \xi_{SP} \). From the translational symmetry of Eq. (27), it also follows that \( \varphi_i |_{SP} = \varphi_{SP} \) and \( \Xi_i |_{SP} = \Xi_{SP} \). Hence, in the translationally invariant case the SP equation for the field \( \varphi_{SP} \) simplifies to

\[
\varphi_{SP}(\tau') = \frac{1}{J} \int_0^{\tau_f} \Xi_{SP}(\tau, \tau') d\tau - 1. \quad (30)
\]

From Eqs. (30) and (28) one immediately obtains the boundary condition \( \varphi_{SP}(\tau_f) = -1 \); Eqs. (30), (28) and (21) further imply that \( \varphi_{SP}(\tau') \) must remain real-valued at all times. To the best of our knowledge, the functional equation (30) cannot be solved analytically. However, for the computation of quantum ground states one is only interested in the limit \( \tau_f \to \infty \). A recursive numerical solution of Eq. (30) shows that in this limit, away from a transient near \( \tau = 0 \) and a boundary region at \( \tau \lesssim \tau_f \), the SP equation is dominated by a time-independent plateau value \( \varphi_P \); see Appendix C. We may then assume that late-time plateau values, denoted by a subscript \( P \), dominate the integrals, and approximate

\[
\int_0^{\tau_f} \Xi_P(\tau, \tau') \approx \int_0^{\tau_f} \theta(\tau - \tau') J \xi^+_P e^{-\gamma_P(\tau - \tau')} = \frac{J \xi^+_P}{\gamma_P}, \quad (31)
\]

where \( \gamma_P \equiv \Gamma \xi^+_P - J \varphi_P \). Convergence of the integral in Eq. (31) requires \( \gamma_P < 0 \). Assuming that this condition is satisfied, which can be self-consistently verified a posteriori, in the \( \tau_f \to \infty \) limit Eq. (30) is reduced to an algebraic equation for \( \varphi_P \):

\[
\varphi_P = \frac{\Gamma \xi_P}{\Gamma \xi^+_P - J \varphi_P} - 1. \quad (32)
\]

This can be solved together with the condition that \( \xi^+_P \) is a fixed point of the Euclidean dynamics when \( \varphi = \varphi_P \), yielding four solutions:

\[
\xi_P^\pm = \begin{cases} J - \sqrt{J^2 - \Gamma^2}/\Gamma & \text{if } \Gamma < J \text{ and } J^2 - \Gamma^2 > 0 \text{,} \\ 1 & \text{if } \Gamma + J^2 < 0 \text{,} \\ -\sqrt{J^2 - \Gamma^2}/\Gamma & \text{if } \Gamma > J \text{ and } J^2 - \Gamma^2 > 0 \text{,} \\ 0 & \text{if } \Gamma - J^2 < 0 \text{.} \end{cases}
\]

In order for the SP field to be real valued, the first and second solutions are only acceptable when \( \Gamma < J \); they both give \( \gamma_P = J \) and the corresponding \( \xi_P \) are reciprocal to each other. The fourth solution is not acceptable as it gives \( \gamma_P < 0 \); it corresponds to a maximum of the action (26). We refer to the first and second saddle points as the \textit{small}-\( \Gamma \) SPs and to the third one as the \textit{large}-\( \Gamma \) saddle point, as they give the leading order contribution to the ground state energy in these limits; see Section III A below. Notably, the plateau values \( \varphi_P \) in Eq. (33) coincide with the effective fields acting on each spin within the mean field (MF) approximation; similarly, the disentangling variables \( \xi_P \) parameterize the mean field ground states. For comparison, we provide details of the MF solution in Appendix D. This finding has a transparent physical interpretation: the path integral (24) is a sum over configurations of non-interacting spins, i.e. product states, and the SP trajectory is the single such configuration which gives the best approximation to the ground state energy. The product state which best approximates a quantum ground state is precisely given by mean field. The first and second saddle points in (33), which have opposite \( \varphi_P \), can thus be interpreted as arising from spontaneous symmetry breaking at the mean field level. In order to compute ground state expectation values using the present method, it is therefore convenient to initialize the system in the MF ground state and subsequently perform imaginary time evolution. This is tantamount to initializing the disentangling variables at their plateau values, \( \xi^+_i(0) = \xi^+_P \), which removes the initial transient behaviour. In principle, the above discussion should be repeated for every observable, since each corresponds to a different effective action and therefore to a different SP equation. However, it can be shown that indeed the SP of the Loschmidt action is also the SP for all physical observables, as obtained from the general formalism of Section II A; see Appendix C. The findings of this Section also readily generalize to higher dimensions. As in the one-dimensional case, the SP solution corresponds to MF; a detailed derivation is provided in Appendix C. For instance, the plateau SP values for an isotropic quantum...
Ising model in $D$ spatial dimensions are given by

\[
\xi_P^+ = \begin{cases}
\frac{[DJ - \sqrt{D^2J^2 - \Gamma^2}]}{\Gamma} & D
\frac{[DJ + \sqrt{D^2J^2 - \Gamma^2}]}{\Gamma} & 1
\end{cases}
\]

\[
\varphi_P = \begin{align*}
&\frac{-\sqrt{D^2J^2 - \Gamma^2}}{DJ} & \gamma_P = \begin{cases}
DJ & \Gamma \\
D & \Gamma \\
0 & -\Gamma.
\end{cases}
\end{align*}
\]

III. FIELD THEORY

The disentanglement formalism makes it possible to represent lattice quantum spin models within an exact field theoretical framework, which does not involve a continuum limit in space or the mapping of the quantum system to a higher dimensional classical one. The disentanglement approach does also not involve coherent states, avoiding the related issues discussed in the Introduction. In the field theoretical description derived from the disentanglement formalism, the saddle point field configurations discussed in Section II C give the leading order approximation to observables, with successive corrections corresponding to higher order terms in the expansion of the path integral (18) about the saddle points. Successive corrections about the saddle points are computed using a set of Feynman rules and the associated diagrammatic representation. We illustrate this by considering the ground state energy density of the quantum Ising chain.

A. Leading Order

We begin by considering the leading order term, given by the plateau field configurations (33) obtained in Section II C. For the remainder of this Section, it is convenient to initialize the disentangling variables at their plateau values; since $\xi_P^+(0) = \xi_P^+$ corresponds to the mean field ground state $|\text{MF}\rangle$, this is equivalent to expressing the ground state energy density in the thermodynamic limit as

\[
\epsilon_G = \lim_{\tau_f \to \infty} \lim_{N \to \infty} -\frac{1}{\tau_f N} \log \langle \psi | \hat{U}(\tau_f) | \text{MF}\rangle.
\]

Normalization of the initial state also implies $\xi_0^+(0) = \log(1 + \xi_P^+)^2).$ Eq. (35) can be written as

\[
\epsilon_G = \lim_{\tau_f \to \infty} \lim_{N \to \infty} -\frac{1}{\tau_f N} \log \mathcal{A}(\tau_f)
\]

in terms of a modified Loschmidt amplitude, given by

\[
\mathcal{A}(\tau_f) = \langle \psi | \hat{U}(\tau_f) | \psi \rangle.
\]

The modified time evolution operator $\hat{U}(\tau_f)$ in Eq. (37) is given by Eq. (2) with the condition $\hat{U}(0)|\psi\rangle = |\text{MF}\rangle$; due to this definition, the modified Loschmidt amplitude $\mathcal{A}(\tau_f)$ corresponds to the same effective action (26) as $\mathcal{A}(\tau_f)$. The analysis of the action (26) in Section II C concerns the infinite time limit and is independent of the initial conditions. In this limit, we can again assume that all integrals are dominated by the plateau values; therefore, the earlier discussion equally applies to the present case, and the two amplitudes $\mathcal{A}(\tau_f)$, $\mathcal{A}(\tau_f)$ are dominated by the same large-$\tau_f$ plateaus.

The leading order approximation to the ground state energy density in the thermodynamic limit can then be obtained as

\[
\epsilon_G \approx \lim_{\tau_f \to \infty} \lim_{N \to \infty} \frac{1}{\tau_f N} \log \sum_{\text{SP}} e^{-\mathcal{S}_P},
\]

where the sum runs over the different saddle points and the plateau action is given by

\[
\mathcal{S}_P = \frac{N J}{4} \varphi_P^2 \tau_f - \frac{\Gamma}{2} \xi_P^+ \tau_f + \frac{J}{2} \varphi_P \tau_f = \begin{cases}
-\frac{N(J^2 + \Gamma^2)}{4} & \tau_f \\
-\frac{N\Gamma^2 \tau_f}{4} & \tau_f.
\end{cases}
\]

The top solution in Eq. (39) corresponds to the two small-$\Gamma$ SPs, while the bottom solution corresponds to the large-$\Gamma$ SP. The double degeneracy of the former SP amounts to a factor of 2 multiplying one of the exponentials in (38): this does not contribute in the thermodynamic limit. Noticing that, for fixed $J$,\\n
\[
\lim_{N \to \infty} \frac{1}{N} \log \sum_{\text{SP}} e^{-N \mathcal{S}_P(\Gamma)} = -\min_{\Gamma} \bar{S}_P(\Gamma),
\]

where we defined the intensive quantities $\bar{S}_P = S_P/N$, we obtain

\[
\epsilon_G(\Gamma) \approx \min_{\Gamma} (\bar{S}_P(\tau_f) = -\max_{\Gamma} \left( \frac{J^2 + \Gamma^2}{4J} \cdot \frac{\Gamma}{2} \right),
\]

where the first solution is only valid for $\Gamma < 1$, as discussed. Consistently with the findings of Section II C, the LO result (41) is equal to the result of the mean field approximation, provided in Appendix D.

B. Higher Order Corrections and Quantum Phase Transitions

We now discuss how to obtain corrections beyond the LO saddle point result. In the presence of multiple saddle points, it is customary to integrate Gaussian fluctuations about each saddle point and add up the relative contributions [63]. Here, we assume that the expansions about different saddle points can be separately carried out and added up also for corrections beyond Gaussian. Let us discuss the conditions under which this procedure may be justified. Consider an integral whose integrand has
several saddle points. The expansion about each SP can be seen as a way of grouping contributions together: by expanding to higher and higher order, one progressively includes trajectories further and further from each SP. Adding up separate expansions around different SPs is then justified provided that there is no “overlap” the trajectories included in one expansion are not significantly contributing to any of the others. A toy example showing this is provided in Appendix C. In the present case, the requirement that there is no overlap is indeed satisfied; in the thermodynamic limit, Eq. (40) implies that one only expansion contributes for each value of $\Gamma$, and no double-counting can occur. Additionally, in order to obtain finite results, each expansion should only be considered in the region of parameter space where it is convergent. This requirement can be physically understood as accounting for the breakdown of e.g. a large-coupling expansion in the small-coupling regime. With these caveats, let us carry out the full expansions as discussed; one has

$$\epsilon_G = \lim_{\tau_f \to \infty} \lim_{N \to \infty} \frac{1}{\tau f N} \log \sum_{SP} e^{-N S'_p}$$

where, for each saddle point, the quantity $S'_p$ includes all contributions from higher order terms. Since the ground state energy density is finite and intensive, we expect $S'_p$ to be independent of the system size. By Eq. (40), this means that

$$\epsilon_G = \min_{\Gamma} (S'_p / \tau_f).$$

The above structure suggests an interpretation of quantum phase transitions in terms of the crossing of different expansions. By Eq. (43), the ground state energy is given by whichever of the series $S'_p$, obtained by expanding around the saddle points, gives the lowest value of $\epsilon_G$ for a given value of the physical parameters. For the quantum Ising model, fixing $J$, this translates to the fact that only one of the summands in (42) contributes for each choice of $\Gamma$. In the thermodynamic limit, due to the minimum function, the functional form of the ground state energy then changes abruptly at the value $\Gamma = \Gamma_c$ when two series are equal: this value can be identified as the quantum critical point. In the next Section, we will show that the expansions around the small-$\Gamma$ and large-$\Gamma$ saddle points give rise to series in $\Gamma/J$ and $J/\Gamma$ respectively; they can therefore be identified as the small-$\Gamma$ and large-$\Gamma$ perturbative expansions of the ground state energy. Within the present field-theoretical formalism, the QPT then naturally emerges as the crossing of these perturbative series. This crossing can be directly verified for the quantum Ising chain by Taylor-expanding the exact result for the ground state energy density [62]

$$\epsilon_G = -\frac{2\Gamma + J}{2\pi} E \left( \frac{8 J \Gamma (J + 2\Gamma)^2}{(J + 2\Gamma)^2} \right),$$

where $E$ is the complete elliptic integral of the second kind. Expanding Eq. (44) for small or large $\Gamma$, one finds

$$\epsilon_G = \begin{cases} \frac{J}{4} - J^2 - \frac{J^4}{16J^2} - \frac{J^6}{32J^6} - \cdots, & \Gamma < \Gamma_c, \\ \frac{J}{2} - \frac{J^2}{2} - \frac{J^4}{4} - \frac{J^6}{8} - \cdots, & \Gamma > \Gamma_c, \end{cases}$$

respectively. As shown in Fig. 1, when all terms are resummed the perturbative series (45) do indeed cross only the critical point $\Gamma_c = J/2$. The small-$\Gamma$ expansion is seen to be divergent for $\Gamma > \Gamma_c$, and therefore does not contribute to the functional integral in this regime. Similarly, the large-$\Gamma$ series does not contribute when $\Gamma < \Gamma_c$. From the present formalism, it can be readily seen that the GS energy density can only be non-analytic in the thermodynamic limit and only at the point where two series cross, due to Eq. (43).
and the effective Loschmidt action is given by
\[ S = S_0 - \int_0^{\tau_f} d\tau \left[ \frac{\Gamma}{2\gamma_P} \sum_j \xi^+_j(\tau) + \frac{J}{\gamma_P} \sum_i \varphi_i(\tau) \right]. \] (47)

It is convenient to separately consider the variations of (47) which involve functional derivatives of \( \xi^+_i \) and the term originating from the noise action \( S_0 \): as we shall see, the latter provides a simple and physically appealing propagator\(^2\). Thus, for each SP we expand the action (47) as

\[ S = S_P + S_0 - \sum_{n=2}^{\infty} T^{(n)} \] (48)

where we defined

\[ T^{(n)} = \frac{1}{n!} \sum_i \int_0^{\tau_f} d\tau_i \cdots \int_0^{\tau_f} S^{(n)}(\bar{\tau}_1, \ldots, \bar{\tau}_n) \varphi_i(\bar{\tau}_1) \cdots \varphi_i(\bar{\tau}_n) d\bar{\tau}_1 \cdots d\bar{\tau}_n, \] (49)

and we exploited translational invariance and \( \delta \xi^+_i / \delta \varphi_j \propto \delta_{ij} \). The functional integral (24) can be expanded as

\[ A(\tau_f) = \sum_{S_P} e^{-S_P} \left\langle \sum_{m=0}^{\infty} \frac{\sum_{n=2}^{\infty} T^{(n)}_m}{m!} \right\rangle_0, \] (51)

where the notation \( \langle \ldots \rangle_0 \) denotes averaging with respect to the noise action (46). Each term in Eq. (51) can be evaluated using Wick’s theorem. The propagator \( \Delta \), which accounts for interactions in the system, can be read off from the quadratic action (46) and is found to be proportional to the interaction matrix:

\[ \Delta_{ij}(\bar{\tau}, \bar{\tau}') = 2\frac{\gamma_P}{J} \mathcal{J}_{ij}(\delta(\bar{\tau} - \bar{\tau}')). \] (52)

The series obtained from (51) can then be formally re-exponentiated, giving Eq. (42). Consider the averages \( \langle \ldots \rangle_0 \) in Eq. (51). We define the order of a term \( \langle T^{(n_1)} \cdots T^{(n_m)} \rangle_0 \) to be \( l = \sum_{j=1}^{m} n_j \). Wick’s theorem implies that terms of odd order vanish identically, while

\(^2\) This is somewhat different from the standard QFT approach [64], where the propagator is obtained from the term in the action that is quadratic in the fields. In the present case, this procedure would not yield a propagator in closed form. Instead, it is convenient to obtain the propagator from \( S_0 \) and treat the remaining term of quadratic order on an equal footing to higher order terms, evaluating their contributions from Wick’s theorem.
terms of even order are obtained by summing over all the possible replacements of pairs of fields \( \phi_i(\vec{\tau}_j), \phi_j(\vec{\tau}_i) \) by propagators \( \Delta_{ji}(\vec{\tau}_i, \vec{\tau}_j) \). The evaluation of a given term in (51) is simplified by means of a diagrammatic representation and the associated Feynman rules:

- Each \( T^{(n)} \) provides a vertex and contributes a factor \( S^{(n)}/n! \). Diagrammatically, a vertex is represented as \( n \) points arranged inside a box. Each vertex is labeled by a site index, e.g. \( j \). Individual points belonging to a given vertex are additionally distinguished by a unique time label, \( \vec{\tau}_{j_1}, \ldots, \vec{\tau}_{j_n} \).

- One then sums over all possible ways of joining pairs of points by lines; a line joining the points labeled by \( (j, \vec{\tau}_j) \), and \( (k, \vec{\tau}_k) \) gives a propagator \( \Delta_{jk}(\vec{\tau}_j, \vec{\tau}_k) \).

- The resulting quantity is then integrated over all times \( \vec{\tau}_i \); the integrals run between 0 and \( \vec{\tau}_f \).

- Finally, all site indices are summed over.

Examples of the above diagrammatic representation are given in Figs 2, 3 and 4, discussed below. In more usual field theories, such as \( \phi^4 \), vertices are typically represented by single points [64]; in the above rules, this would correspond to setting all \( \vec{\tau}_j \) to the same value. The fact that here vertices consist of separate points is due to the non-locality in time of the action (48). To simplify the evaluation of higher order terms, we identify two classes of diagrams which do not contribute. The first class includes diagrams where lines join points within the same vertex: these are self-interaction diagrams. An example, originating from \( \langle T^{(3)}T^{(3)}T^{(2)} \rangle_0 \), is shown in Fig. 2(a). Such diagrams feature at least one term of the form \( \Delta_{ji} \propto J_{ii} \), which is identically zero for the Hamiltonian (20) at hand. The second class of non-contributing terms includes disconnected diagrams, in which the vertices joined by internal lines form disjointed clusters. This class includes the diagram in Fig. 2(b), which is produced by \( \langle T^{(4)}T^{(4)} \rangle_0 \). It is easy to see that, due to the Feynman rules and the form of the propagator (52), a connected cluster of vertices gives a contribution proportional to \( N \). A term with \( m \) disconnected clusters of vertices is then proportional to \( N^m \). The origin of these terms can be understood by considering the expansion

\[
e^{-NS'} = 1 - NS' + \frac{1}{2!} N^2 S'' + \ldots .
\]

The terms \( \ldots_0 \) in (51), obtained from Wick’s theorem, correspond to the right-hand side of (53). The ground state energy must be intensive and finite in the thermodynamic limit; this implies that terms proportional to higher powers of \( N \) must cancel out when exponentiating the series in (51) to obtain Eq. (42). This is precisely what happens in Eq. (53). Since the desideratum here is \( S' \), we only need to consider the terms proportional to \( N \); these are given by connected diagrams. Finally, we note that a diagram with an odd number \( m \) of vertices \( T^{(2)} \) always vanishes: any such diagram is either self-interacting (Fig. 2(c)), or it gives rise to a term \( \propto \text{Tr} J'' \), which vanishes for the quantum Ising model (Fig. 2(d)).

In Section III E, we apply the Feynman rules derived in this Section to compute higher order corrections to the ground state energy of the quantum Ising chain. Before turning to this explicit example, in the next Section we complete our theoretical overview by considering how the terms in Eq. (51) depend on the physical parameters of the model, elucidating the relation between the expansion about the saddle points and perturbation theory.

### D. Relation to Perturbation Theory

In order to understand the nature of the terms produced by the expansion (51) we need to consider the higher variations of the action, \( S^{(n)} \) with \( n \geq 2 \). It is convenient to compute these variations by initially imposing a time ordering \( \vec{\tau}_n > \cdots > \vec{\tau}_1 \), and then symmetrizing the result with respect to the times \( \vec{\tau}_i \). With said ordering, one obtains for the second variation

\[
S^{(2)}(\vec{\tau}_1, \vec{\tau}_2) = -\left( \frac{J}{\gamma P} \right)^2 \int_0^{\vec{\tau}_f} \frac{\Gamma}{\gamma P} \Xi(s_1, \vec{\tau}_1) \left[ \frac{\Gamma}{\gamma P} \int_{\vec{\tau}_2}^{\vec{s}_1} \Xi(s, \vec{\tau}_2) ds + \theta(\vec{s}_1 - \vec{\tau}_2) \right] ds_1|_{\vec{\tau}_f},
\]

where we defined \( \Xi(s, \vec{\tau}_i) = \frac{\Gamma}{\gamma P} \Xi(s, \vec{\tau}_i) \) to make the dependence on physical parameters manifest. Eq. (54) shows that all variations \( S^{(n)} \) with \( n \geq 2 \) can be expressed in terms of integrals of the first variation \( \Xi \). Schematically, one obtains \( S^{(n+1)} \) from \( S^{(n)} \) by summing over all possible ways of replacing

\[
\Xi \rightarrow -\frac{\Gamma}{\gamma P} \int \Xi + \theta
\]

and multiplying by \( J/\gamma P \). When evaluated at the SP, each \( \Xi \) gives a factor of \( \xi_P \) and an exponential depending on the dimensionless times \( \vec{\tau}_i \) only. Therefore, the \( n \)-th variation (with \( n \geq 2 \)) evaluated at the plateau must be
of the form

$$S^{(n)}(\tilde{\tau}_1, \ldots, \tilde{\tau}_n) = \left( \frac{J}{\gamma_p} \right)^n \sum_{m=1}^{\infty} C_{n,m}(\tilde{\tau}) \left( \frac{\Gamma}{\gamma_p} \right)^m,$$

(55)

where $C_{n,m}(\tilde{\tau})$ are dimensionless functions depending only on the $\tilde{\tau}_i$, which do not involve any factor of $\Gamma$, $J$ or $\gamma_p$. Using the form (55) of higher variations, it is possible to determine the structure of the terms in Eq. (51). Terms of odd order $l = 2m + 1$ do not contribute due to Wick’s theorem; see the discussion in Section III C. Any given term of even order $l = 2m$ features a product of variations, whose orders add up to $2m$, and $m$ propagators, each of which carries a factor $\gamma_p/J$. Bringing everything together and substituting the SP values (33), a general $2m$-th order term $T^{2m}$ can be written as

$$T^{2m} = \frac{\sum_{n=1}^{\infty} \tilde{C}_{n,2m} \left( \frac{\Gamma}{J} \right)^{2n} }{D_{2m} \left( \frac{\Gamma}{J} \right)^m},$$

(56)

where the top and bottom solutions refer to the small-$\Gamma$ and large-$\Gamma$ expansion respectively, and $\tilde{C}_{n,2m}$ and $D_{2m}$ are dimensionless constants which do not depend on $J$ or $\Gamma$. The series of even powers of $(\Gamma/J)$ in the former case arises from Taylor expanding $\Gamma \tilde{x}/\gamma_p$. Eq. (56) thus shows that the expansions around the saddle points give rise to series in $(\Gamma/J)^2$ and $J/\Gamma$. These can be identified with the perturbative series by the following argument. Eq. (42) is valid for any value of $\Gamma$ and, due to the thermodynamic relation (43), only one expansion at a time contributes. Consider the $\Gamma \to 0$ limit; in this case, the LO term (39) of the small-$\Gamma$ expansion gives the exact value of the ground state energy. For finite but sufficiently small $\Gamma/J \ll 1$, the small-$\Gamma$ expansion will still be the dominant one and give the ground state energy $\epsilon_G$. The small-$\Gamma$ expansion must therefore be equal to the $\Gamma/J$ perturbative series for $\epsilon_G$, as they are both series in $\Gamma/J$ and they both add up to $\epsilon_G$. A symmetric argument holds for the large-$\Gamma$ series in the corresponding limit. Eq. (56) shows that the large-$\Gamma$ expansion (bottom case) is in order-by-order correspondence to the perturbative series: terms of order $2m$ are proportional to $(\Gamma/J)^m$. On the other hand, the small-$\Gamma$ expansion (top case of Eq. (56)) is not in one-to-one correspondence to perturbation theory: one needs to sum over $m$ in order to retrieve the perturbative series in $\Gamma/J$, since each of the terms in Eq. (56) may in principle contain various powers of $\Gamma/J$. The different behavior of the two expansions is due to the nature of the plateau configuration or, equivalently, the MF ground state. For large $\Gamma$, this is just the $\Gamma = \infty$ ground state $|\bar{\psi}\rangle$; the large-$\Gamma$ expansion is thus equivalent order-by-order to the perturbative series around $\Gamma = \infty$. On the other hand, for small $\Gamma$ the MF ground state is not simply given by the $\Gamma = 0$ ground state $|\psi_0\rangle$. Consider for instance the MF magnetization, given in Appendix D; this can be expanded as a Taylor series featuring all even powers of $(\Gamma/J)$. An expansion around the MF ground state is therefore not expected to be in order-to-order correspondence with a perturbative expansion around $\Gamma = 0$.

One more comment is due concerning even and odd powers in the two expansions. From expanding the exact ground state energy of the quantum Ising chain as in (45), we see that the perturbative expression for $\epsilon_G/\Gamma$ around $\Gamma = 0$ features only even powers of $\Gamma/\Gamma$ and, similarly, the perturbative expansion of $\epsilon_G/\Gamma$ around $\Gamma = \infty$ contains only even powers of $1/\Gamma$. This result is immediately retrieved from Eq. (56) for the small-$\Gamma$ expansion, and is due to spontaneous symmetry breaking at the MF level. However, odd powers of $J/\Gamma$ are not excluded a priori in the large-$\Gamma$ expansion. The necessary cancellation must therefore originate from the vanishing of the $D_{2m}$ coefficient in (56) when $m$ is odd. We explicitly show an example of such cancellation when computing higher-order corrections to $\epsilon_G$ in Section III F.

We have thus determined the structure of the terms produced by expanding about the saddle points, and clarified the relation of such expansions to perturbation theory. In summary, the full small-$\Gamma$ and large-$\Gamma$ expansions are respectively equal to the full perturbative series around $\Gamma = 0$ and $\Gamma = \infty$. This correspondence is satisfied order-by-order for the large-$\Gamma$ expansion, and only when resumming the whole series for the small-$\Gamma$ expansion. This analysis completes our overview of the present field theoretical approach; in the next Section, we apply the concepts discussed so far to compute corrections to the ground state energy of the quantum Ising chain.

### E. Example: NLO and NNLO Corrections to the Ground State Energy

In order to illustrate the machinery introduced in the previous Sections, we compute the next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) corrections to the ground state energy density of the quantum Ising chain. The lowest order correction is naively given by $\langle T^{(3)} \rangle_0$; this term however vanishes, since the corresponding diagram is self-interacting: see the discussion in Section III C. The NLO correction is then given by the next-higher term, which is of order four:

$$T^{(4)} = \frac{1}{2} \langle T^{(2)} T^{(2)} \rangle_0 + \langle T^{(4)} \rangle_0.$$

(57)

The second term on the right hand side of Eq. (57) vanishes similarly to $\langle T^{(2)} \rangle_0$; the remaining term corresponds to the diagrams in Fig. 3, and gives

$$\frac{1}{2} \langle T^{(2)} T^{(2)} \rangle_0 = T_f N \left\{ \frac{\Gamma^4}{J^2} \right\}.$$

(58)

where the top and bottom solutions are obtained from the small-$\Gamma$ and large-$\Gamma$ SPs respectively. Thus, the NLO
approximation to the ground state energy to is given by

$$\epsilon_G \approx -\max_{\Gamma} \left( \frac{J}{4} + \frac{\Gamma^2}{4J} + \frac{\Gamma^4}{32J^3} \frac{\Gamma}{2} + \frac{J^2}{32\Gamma} \right).$$  \hspace{1cm} (59)$$

As discussed in Section III B, each of the two series in Eq. (59) can only be considered within its radius of convergence. In practice, when one does not have access to the full series, the radius of convergence can be estimated by imposing that each term be smaller than the lower-order one. In the present case, this criterion indicates that the small-\( \Gamma \) series is valid for \( \Gamma < 1 \), while the large-\( \Gamma \) series is valid for \( \Gamma > 1/4 \). Including the NLO correction as discussed provides an improvement over the LO approximation for all values of \( \Gamma \); see Fig. 5. Consistently with the discussion in Section III D, Eq. (59) matches the result of second-order perturbation theory about \( \Gamma = \infty \), which is in one-to-one correspondence with the expansion around the large-\( \Gamma \) SP. On the other hand, in order to match the term of order \( \Gamma^3/J^3 \) of the perturbative series in (45), one needs to include higher order contributions from the small-\( \Gamma \) SP expansion: this is again consistent with the earlier discussion. The next-higher correction to the ground state energy, NNLO, is of order 6, and is given by

$$T^{(6)} = \frac{1}{3!} (T^{(2)} T^{(2)})_0 + \frac{1}{2!} (T^{(3)} T^{(3)})_0.$$  \hspace{1cm} (60)$$

The first term in Eq. (60) vanishes because it features an odd number of \( T^{(2)} \) vertices; see the discussion in Section III C and in particular Figs 2(c-d). The non-vanishing diagrams are shown in Fig. 4. They can be evaluated to give

$$T^{(6)} = \tau_j N \left\{ \Gamma^4 \frac{1}{64J^3} - \Gamma^6 \frac{1}{64J^4} \right\}.$$  \hspace{1cm} (61)$$

where again the top result corresponds to the small-\( \Gamma \) expansion and the bottom result to the large-\( \Gamma \) expansion. Eq. (61) shows that the NNLO correction from the large-\( \Gamma \) expansion vanishes. This was anticipated in Section III D, and is due to the fact that the SP expansions and the perturbative series must coincide; by Eq. (56), the large-\( \Gamma \) NNLO correction would be proportional to \( (J/\Gamma)^3 \), but no such term appears in the perturbative series (45): the coefficient multiplying \( (J/\Gamma)^3 \) must therefore vanish. Including also the NNLO corrections (61) leads to a further improvement in the approximation to the GS energy, as shown in the inset of Fig. 5.

We have thus illustrated how higher order corrections beyond the saddle point result can be analytically obtained, providing explicit examples. For simplicity, in this Section we focused on the ground state energy, but an analogous procedure can be carried out for other observables by expanding the appropriate effective action \( S_G \), defined as in Eq. (18).

Besides the analytical field theoretical formalism outlined in this Section, the disentanglement method can alternatively be used as a numerical tool; we discuss this approach in the following Section.
IV. IMPORTANCE SAMPLING

A. Optimal Measure Transformation

As anticipated in Section II B, the saddle point trajectory can also be used to implement an importance sampling numerical algorithm. For numerical applications, it is convenient to use the diagonal form (11) of the noise action, involving the fields \( \phi_i^a \). The corresponding SP values can be readily determined from \( \phi_i^a \) using \( \phi_i^a |_{\text{SP}} = \sum_k O_{ij}^{ab} |\phi_j^b |_{\text{SP}} \). The key step of the proposed approach consists in using \( \phi_{\text{SP}} = \{ \phi_i^a |_{\text{SP}} \} \) to perform a change of variables

\[
\phi_i^a = (\phi_{\text{SP}})_i^a + \phi_i^a
\]

in the functional integral for a given observable:

\[
\langle \hat{O} \rangle = e^{-S_0[\phi_{\text{SP}}]} \int D\phi' e^{-S_0[\phi']} e^{-\int d\tau \phi[\phi_{\text{SP}} + \phi']} f_O[\phi_{\text{SP}} + \phi'],
\]

where \( \phi' = \{ \phi_i^a \} \) and \( \phi_{\text{SP}} \cdot \phi' = \sum_i (\phi_{\text{SP}})_i^a \phi_i^a \). Compared to more usual path integral approaches, here we do not truncate Eq. (63) to a given order in the fluctuations. Instead, due to the Gaussianity of the noise action \( S_0 \), the full Eq. (63) can be evaluated numerically in the spirit of the stochastic approach of Refs [55, 56]. This amounts to averaging a biased function over realizations of Gaussian-distributed stochastic processes \( \phi' \):

\[
\langle \hat{O} \rangle = e^{-S_0[\phi_{\text{SP}}]} \int D\phi' e^{-S_0[\phi']} e^{-\int d\tau \phi[\phi_{\text{SP}} + \phi']} f_O[\phi_{\text{SP}} + \phi'] \phi'.
\]

In contrast, when Eq. (18) is sampled directly according to (11), trajectories close to \( \phi(\tau) = 0 \) are sampled preferentially, even though they may give a small contribution to the integral. Sampling Eq. (63) does not constitute a semiclassical approximation: the saddle point trajectory \( \phi_{\text{SP}} \) is used to perform an exact change of variables, which biases the sampling towards important trajectories; the exactness of Eq. (18) is then fully preserved in Eq. (63). The change of variables leading to Eq. (63) can be seen as a particular choice of a measure transformation, also known as Girsanov transformations [63, 66]. In the context of stochastic processes, such transformations can be regarded as the continuum version of importance sampling.

B. Numerical Results

To illustrate our method, we apply the measure transformation approach to the \( D \)-dimensional quantum Ising model (20) for \( D \in \{1, 2, 3\} \) by numerically computing different observables from stochastic simulations. In our numerical simulations and in the remainder of this Section we set \( J = 1 \). In the stochastic approach, ground state expectation values are computed according to Eq. (4). By appropriately choosing \( U_0 \) in Eq. (3), any initial state \( |\psi_0 \rangle \) can be considered. Following the discussion of Section II C, it is convenient to choose the initial state to be the mean-field ground state for the desired value of \( \Gamma, |\text{MF} \rangle \). As anticipated, this is equivalent to initializing the system at the plateau SP configuration, \( \xi^i_1(0) = \xi^i_3(0) = \log(1 + |\xi^i_2|^2) \). For observables computed from (4), the plateau values are fixed points of the saddle point equation, such that one may perform the change of variables (62) with \( \phi_{\text{SP}}(\tau) = \phi_P \); see Appendix C. The resulting SDEs are solved using the Euler scheme [66]. We begin by considering the \( D = 1 \) case, corresponding to the quantum Ising chain. We consider the imaginary time evolution of the ground state longitudinal magnetization \( M_z \), transverse magnetization \( M_x \equiv \sum_{i=1}^N \langle S^z_i \rangle / N \) and nearest-neighbor longitudinal correlations \( C_{zz} \equiv \sum_{ij} \langle S^z_i S^z_j \rangle / N \). As per our general discussion, these quantities are given by Eq. (13), where the numerator includes the appropriate stochastic function for each observable. The stochastic functions are given by Eq. (16) for \( M_z \) and by

\[
F_{M_z} = \frac{F_1}{N} \sum_i \frac{\xi^i_1(\tau) + \xi^i_3(\tau)}{1 + \xi^i_2(\tau)^2},
\]

\[
F_{C_{zz}} = \frac{F_1}{N} \sum_{ij} \left( \frac{1}{1 + \xi^i_2(\tau)^2} \right) \left( \frac{1}{1 + \xi^i_2(\tau)^2} \right),
\]

for \( M_z \) and \( C_{zz} \) respectively [56].

The ground state energy density is obtained from Eqs (65) and (66) as

\[
\epsilon_G = -\Gamma M_z - JC_{zz}.
\]

In Fig. 6 we compare our numerical results for a system of size \( N = 101 \) to imaginary time evolution performed directly in the thermodynamic limit using iTEBD [47]. We find excellent agreement across the imaginary time range we consider. The value of energy \( \epsilon(\tau_f) \) obtained at the stopping time \( \tau_f = 4 \) of our simulations is within 0.0015\% of the exact ground state result obtained from free fermions. The vertical bars in Fig. 6 show the statistical uncertainty associated with averaging over stochastic trajectories. For each quantity, the bars are obtained by partitioning the data set into \( n_B \) batches of independent simulations and computing the associated standard deviation \( \sigma \); the fluctuations are then estimated as the standard error \( \sigma/\sqrt{n_B} \). To show the improvement of importance sampling according to Eq. (63) over direct sampling using the naive measure (11), in Fig. 7 we directly compare the performance of the two approaches. We fix the physical parameters to \( N = 15, \Gamma = 0.4 \) and compute the Euclidean time evolution of the energy using the same time step and number of simulations; the results obtained from direct and importance sampling are shown in panels (a) and (b) respectively. It is clear that the importance sampling algorithm produces far better results for the same computational cost. This is further discussed in Section IV C, where we study the behaviour
We consider a system with $N = 101$ spins, initialized in the mean field ground state for $\Gamma = 0$ and evolved with the same $\Gamma$ using (a) direct sampling and (b) importance sampling. We compute the Euclidean time evolution of the ground state energy from $2 \times 10^4$ simulations, performed using the same time step $\Delta \tau = 0.001$ for both methods; the corresponding results are compared to ED (solid line). It can be seen that the importance sampling method produces significantly better results for the same computational cost. The bars show the much faster growth of fluctuations for direct compared to importance sampling. Both simulations took approximately one minute on a desktop computer. Fluctuations in the two methods are further discussed in Fig. 10.

The GS energy within 0.1% of the exact value obtained from ED. We observe that the stopping time $\tau_f$ that can be accessed for a given number of simulations decreases with the dimensionality of the system, due to the faster growth of fluctuations; this behavior is associated with the increase of connectivity in higher dimensions, and is further investigated in the next Section.

### Fluctuations

Having demonstrated the applicability of the method to higher dimensional systems, we now turn to investigating its numerical performance, quantitatively comparing the direct and importance sampling schemes and assessing the practical applicability of the numerical stochastic approach. The performance of the method is closely related to the behavior of fluctuations in the stochastic quantities. For a given number of simulations, the growth of fluctuations ultimately determines the time scale beyond which physical results are not correctly reproduced; therefore, an increasing number of simulations is needed for averages to converge as the stopping time is increased. As discussed in Ref. [56], the central limit theorem implies that the fluctuations in the observable $O$ computed...
The nearest neighbor correlation function $C$ comes deterministic: the variance result is exact, and the importance sampling scheme be-
tem. On the other hand, in the classical limit the SP
where $\sigma$, the variance of the corresponding stochastic quantity $f_\mathcal{O}$:
\begin{equation}
\sigma^2(f_\mathcal{O}) \equiv \langle |f_\mathcal{O}|^2 \rangle_\phi - |\langle f_\mathcal{O} \rangle_\phi|^2.
\end{equation}

The variance $\sigma^2$ is therefore directly related to the num-
ber of simulations required to obtain the desired degree of accuracy. We illustrate this by considering the nor-
malization function (15). The behaviour of this quantity is found to also be representative of other observables,
due to the similar functional form of the correspond-
ing stochastic functions; see for example Eq. (16). In
the classical case of the $D$-dimensional Ising model with
$\Gamma = 0$, the SDEs (21) are exactly solvable. For direct sampling, one obtains
\begin{equation}
\sigma^2(\tau) = e^{2N\alpha D J \tau} - e^{N\alpha D J \tau}
\end{equation}
where $N D$ is the total number of interactions in the sys-

tem. On the other hand, in the classical limit the SP result is exact, and the importance sampling scheme be-
comes deterministic: the variance $\sigma^2$ vanishes altogether and a single trajectory is sufficient to give the exact re-
sult. For finite $\Gamma$, the behaviour of fluctuations can be
investigated numerically. As shown in Fig. 10, we find
that the functional form
\begin{equation}
\sigma^2 = \alpha e^{\beta \tau},
\end{equation}
with $\beta \approx 2DN$, captures the behaviour of fluctuations. Thus, the exponential growth of fluctuations with $N, D$
and $\tau$, which we found for direct sampling in the clas-
cical case, survives for finite $\Gamma$, and also applies to im-
portance sampling. This is consistent with the numerical analysis carried out in Ref. 56 for real time evolution,
and with the argument of Refs 24, 61 suggesting that a large deviation principle may be at play with respect to
the system size $N$. However, direct and importance sam-
ping differ substantially in the prefactor $\alpha$ multiplying
the exponential. We find that $\alpha$ depends heavily on $\Gamma$, as
shown in the tables of Fig. 10. For direct sampling, one
has $\alpha = O(1)$ for all $\Gamma$. On the other hand, for im-
portance sampling, $\alpha$ is gradually increased from $\alpha = 0$ as $\Gamma$ is increased. For small to intermediate field strengths
$\Gamma \approx O(1)$, $\alpha$ can be orders of magnitude smaller for im-
portance sampling than for direct sampling. This allows
the importance sampling algorithm to access times and
systems sizes well beyond the reach of direct simulations.

Thus, although the importance sampling scheme does
not fully eliminate the exponential dependence of fluctu-
ations on time and the system size, it can significantly ex-
tend the regime of applicability of the stochastic method
before fluctuations become sizeable. This is promising in
view of real time applications: a generalization of the
importance sampling approach might allow the stochas-

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig8}
\caption{Imaginary time evolution for the 2D quantum Ising model. We show the ground state energy (main panel), the longitudinal and transverse magnetization, $M_x$ and $M_z$, and the nearest neighbor correlation function $C_{zz}$ (insets, top to bottom) for a $5 \times 5$ system. The system is initialized in the mean field ground state for $\Gamma = 1$ and evolved with the same value of $\Gamma$. We compare our results, obtained by solving the SDEs and applying the importance sampling scheme (dots), to exact diagonalization (lines), finding good agreement. At the stopping time $\tau_f = 1.8$, the relative error between our estimate of the ground state energy and the true value obtained from ED (horizontal dashed line) is of order $10^{-5}$. Our results were obtained from $5 \times 10^7$ realizations with $\Delta\tau = 0.01$. The bars showing the statistical uncertainty were estimated by dividing the data set into 100 batches and are not visible in the insets.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig9}
\caption{Imaginary time evolution for the 3D quantum Ising model. We consider the same observables of Figs 6 and 8 for a $3 \times 3 \times 3$ system. The system is initialized in the mean field ground state for $\Gamma = 2$ and evolved using the same value of $\Gamma$. Again, the results obtained by solving the SDEs and using importance sampling (dots) are in good agreement with ED (lines). The SDE estimate for the ground state energy at the stopping time $\tau_f = 0.8$ is within $0.1\%$ of the true ground state energy, obtained from ED (dashed horizontal line). Our results were obtained from $7 \times 10^6$ realizations of the stochastic process, with $\Delta\tau = 0.005$. Fluctuations were estimated by dividing the data into 100 batches of independent simulations.}
\end{figure}
importance sampling algorithms. Fluctuations are measured by
FIG. 10. Growth of fluctuations in the direct and the im-
portance sampling approach, panels (b-d), were obtained from $10^8$ simulations; these were sufficient, due to the smaller extent of fluctuations.

ground state is a product state, as in the classical limit. Thus, both the presence of fluctuations and the growth of entanglement signal the departure from a product state; whether a direct connection between these exists will be investigated in future work.

Finally, we comment on the computational cost of the stochastic method. In general, the runtime of a given stochastic simulation scales linearly with $N$, $\tau_f$ and inversely with the time step $\Delta \tau_f$. For instance, the results of Fig. 6 were computed from $10^3$ batches of $10^5$ simulations each; each batch takes approximately 0.8 hours on 16 cores. Due to the growth of fluctuations, the main computational cost of the method is not associated with the runtime of individual simulations, but with the growing number of simulations that are required to attain a given accuracy. This leads to a trade-off between accessible time scales and system sizes [56].

In summary, in this Section we have shown that ground-state expectation values can be numerically computed within the disentanglement formalism as expectation values over classical stochastic trajectories. This stochastic approach can be made more efficient by employing an importance sampling scheme, based on identifying the relevant saddle point configuration and preferentially sampling trajectories that are close to it. While the numerical performance of the importance sampling scheme for computing ground state expectation values is currently inferior to better established numerical techniques, such as quantum Monte Carlo [26, 29] or tensor network algorithms [49], this method substantially extends the regime of applicability of the stochastic approach, making it a viable numerical technique. In particular, the applicability of the approach in higher dimensions, demonstrated in this Section, motivates further developments and the generalization of the present work to real time evolution, in view of non-equilibrium applications.

V. CONCLUSIONS

In this manuscript, we have shown that the disentanglement formalism [24, 53–56] provides a broadly applicable analytical and numerical framework for studying many-body quantum spin ground states. In the disentanglement approach, ground state expectation values of quantum spin systems are exactly expressed as functional integrals over unconstrained scalar fields. This field-theoretical description is amenable to both analytical treatment and numerical evaluation in terms of classical stochastic processes. We have shown that the dominant contribution to observables can be analytically obtained as the saddle point of a suitable effective action. Higher order corrections to a given observable can then be analytically computed by expanding to the desired order about the saddle point, as we have explicitly shown for the quantum Ising model. This approach also reveals a description of quantum phase transitions in terms of the crossing of expansions about different saddle points. Alternatively, the saddle point trajectory can be used to perform an exact measure transformation, which results in an importance sampling numerical technique. The main drawback of the numerical application of the disentanglement method is associated with the exponential growth of fluctuations with time and the system size; this growth is however significantly mitigated by the importance sampling approach, which involves preferentially sampling in the vicinity of the saddle point configuration. Our findings equally apply to higher dimensional systems, as we explicitly showed by considering the 2D and 3D quantum Ising model. The disentanglement approach provides a rather general tool, which connects concepts from lattice spin systems, field theory and clas-
sical stochastic processes, and can be used for both analytical and numerical applications. The present approach can also be readily generalized to finite temperatures by considering evolution to finite Euclidean time. While the efficiency of the disentanglement method as a numerical technique is at present inferior to the state of the art for quantum ground states, several directions for further developments can be envisaged, including cluster approaches [68–71] or developing connections to tensor network-based methods [45, 46]. The direct relation between exact analytical expressions and numerical sampling afforded by the disentanglement approach may also prove useful in further developing the method, taking advantage of the physical understanding of the system at hand to devise problem-specific approximations or sampling methods; possible applications include the study of frustrated magnets [35, 53], which pose severe challenges to existing techniques. Finally, the applicability of the disentanglement formalism to higher dimensional systems suggests that a real time generalization of the present work may provide a useful analytical and numerical tool to study non-equilibrium quantum dynamics beyond the limitations of current approaches.

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Appendix A: Disentanglement Transformation

In order to make the manuscript self-contained, in this Appendix we recapitulate the key steps of the disentanglement formalism and provide additional details on the higher-dimensional case.

1. General Case

In the disentanglement approach, the time evolution operator

$$
\hat{U}(\tau) \equiv e^{-\tau \hat{H}} = e^{\tau \left( J \sum_{i,j} \hat{\varphi}_a^{ij} \hat{S}^a_i \hat{S}^a_j + \sum_i \hat{b}_i^a \hat{b}_i^a \right)}
$$

(A1)

is expressed as

$$
\hat{U}(\tau) = \int D\varphi e^{-S_0[\varphi] \prod_i e^{\xi^+_i(\tau) \hat{S}^+_i \xi^-_i(\tau) \hat{S}^-_i}}
$$

(A2)

where $S_0$ is given by Eq. (7) and the disentangling variables $\xi^\pm_i$ satisfy Eqs (9). Eq. (A2) is obtained in a two-step process [24, 53–55]. First, interactions are decoupled thanks to the Hubbard-Stratonovich transformation [57, 58]. Following Trotter decomposition of the exponential in Eq. (A1), at each time slice one has

$$
e^{\Delta \tau J \sum_{i,j} \hat{\varphi}_a^{ij} \hat{S}^a_i \hat{S}^a_j} \approx C \prod_{\alpha} \int d\varphi_\alpha e^{-\frac{i}{\hbar} \Delta \tau \sum_{i,j} (J^{-1})^{\alpha \beta}_{ij} \varphi_\alpha^{ij} + \Delta \tau \sum_{ij} \varphi_\alpha^{ij} \hat{S}^\alpha_i \hat{S}^\beta_j},
$$

(A3)

where $C$ is a normalization constant and we neglected terms $O(\Delta \tau^2)$. Eq. (A3) is an operatorial identity; the fields $\varphi_\alpha^{ij}$ are in general complex, and chosen in such a way as to make the integral in Eq. (A3) convergent [56]. It is convenient to rescale the fields as $\varphi_\alpha^{ij} \to J \varphi_\alpha^{ij}$, in order to make them dimensionless. Applying this rescaling and taking the continuum limit, Eq. (A3) yields

$$
\hat{U}(\tau) = \int D\varphi e^{-S_0[\varphi] \prod_i e^{\xi^+_i(\tau) \hat{S}^+_i \xi^-_i(\tau) \hat{S}^-_i}}
$$

(A4)

where the symbol $\mathbb{T}$ denotes time ordering. Eq. (A4) describes a system of non-interacting spins under the effect of complex valued stochastic fields $\varphi_\alpha^{ij}$ [24, 53]. Since interactions are decoupled inside the integral, the evolution of each spin occurs over its (complexified) Bloch sphere. Time-ordered exponentials can then be expressed in terms of ordinary exponentials by means of a Lie-algebraic disentanglement transformation, also known as Wei-Norman-Kolokolov transformation [24, 53, 54, 59, 60]. Namely, at each lattice site one has

$$
\mathbb{T} e^{\sum_{a} \int_0^\tau \left[ \sum_i [h_i^a(\tau') + J \varphi_i^a(\tau')] \hat{S}^a_i \right] d\tau'} = e^{\xi^+_i(\tau) \hat{S}^+_i \xi^-_i(\tau) \hat{S}^-_i}
$$

(A5)

This amounts to parameterizing the trajectory of each spin on its Bloch sphere in terms of a set of coordinates.
\( \xi^a \), termed the disentangling variables. Eq. (A5) can be seen as the defining equation of \( \xi^a \); differentiating both sides of (A5) and equating the coefficients that multiply the spin operators yields the SDEs (9). Alternatively, these can be obtained from differential geometry [24]. The initial conditions \( \xi^a(0) = 0 \) are fixed by the requirement \( \hat{U}(0) = 1 \). The discussion of this Section can be readily generalized to the modified time evolution operator \( \hat{U}(\tau) \), given by Eq. (2); the initial conditions of the disentangling variables are then given by (10).

2. Details on the Disentanglement Transformation in Higher Dimensions

While the formalism outlined in the previous Section is fully general, in this Section we show in greater detail how the disentanglement transformation works in higher dimensional settings of particular physical interest, providing useful formulae for analytical and numerical applications. Let us consider a Hamiltonian describing a system on a \( D \)-dimensional hypercubic lattice:

\[
\hat{H} = - \sum_{ij} \sum_{ab} J^{ab}_{ij} \hat{S}_i^a \hat{S}_j^b - \sum_i n_i^a \hat{S}_i^a. \tag{A6}
\]

We focus on the case where \( J \) only couples spins along the lattice axes, i.e. the sites \( i, j \) coupled by \( J_{ij} \) differ by a single index \( i_d \). With this choice of \( J \), we can write

\[
J^{ab}_{ij} = \sum_d J_d (J^d_{ij})^{ab} \prod_{d' \neq d} \delta_{ij_d, i_{d'}}, \tag{A7}
\]

where \( J_d \) are interaction strengths and the matrices \( J^d \) couple spins along the dimension \( d \). \( J^d \) can be seen as \( 3N_d \times 3N_d \) matrices \( J^d_{\alpha\beta} \) by introducing multi-component indices \( \alpha = \{i_d, a\}, \beta = \{j_d, b\} \). One has

\[
\sum_{ij} \sum_{ab} J^{ab}_{ij} \hat{S}_i^a \hat{S}_j^b = \sum_{d=1}^{D} \sum_{i} J_d \sum_{j_d} (J^d_{i_{d}j_d})^{ab} \hat{S}^b_{j_d} \hat{S}^a_{i_d}. \tag{A8}
\]

Exponentiating the interaction term in (A6) and considering an infinitesimal time slice, one obtains

\[
e^{\Delta \tau \sum_{i,j=ab} J^{ab}_{ij} \hat{S}_i^a \hat{S}_j^b} = \prod_{d=1}^{D} \prod_{i \neq j_d} e^{\Delta \tau \sum_{i,j_d=ab} J_d (J^d_{ij})^{ab} \hat{S}^b_{j_d} \hat{S}^a_{i_{d}}} \cdot \tag{A9}
\]

We can apply the HS transformation to each term separately:

\[
e^{\Delta \tau \sum_{i,j=ab} (J^d_{ij})^{ab} \hat{S}^a_{i} \hat{S}^b_{j}} \approx C \int \frac{d\varphi^d}{\varphi^d} e^{\frac{-\Delta \tau}{2} \sum_{i,j=ab} (J^d_{ij})^{ab}(\varphi^d)^a_{i} (\varphi^d)^b_{j} + \Delta \tau \sum_{i,j=ab} (\varphi^d)^a_{i} (\varphi^d)^b_{j} \hat{S}^a_{i} \hat{S}^b_{j}} \cdot \tag{A10}
\]

Taking the continuum limit and rescaling \( \varphi^d \rightarrow J_d \varphi^d \), this yields

\[
e^{-\Delta \tau \hat{H}} = \int \mathcal{D}\varphi e^{-S_0[\varphi]} \prod_{\tau_0} d\tau [\sum_d J_d \sum_i \varphi^d_i \hat{S}^a_i + \Delta \tau \sum_{i,j=ab} (\varphi^d)^a_{i} (\varphi^d)^b_{j} \hat{S}^a_i \hat{S}^b_j] \tag{A11}
\]

where the noise action in \( D \) dimensions is given by

\[
S_0[\varphi] = \frac{1}{4} \int_0^\tau d\tau \sum_d J_d \sum_{i,j=ab} [(J^d)^{-1}]_{i,j_d}^{ab} (\varphi^d)^a_{i} (\varphi^d)^b_{j} \delta_{ij_{d-1}, i_{d+1}} \cdot \tag{A12}
\]

It can be seen that for a \( D \) dimensional system with \( N = N_1 \times \cdots \times N_D \) spins, one needs in general to introduce \( 3DN \) Hubbard-Stratonovich fields. The individual time ordered exponentials in Eq. (A11) can then be expressed in terms of ordinary exponentials, as done in Appendix A.1. As in the 1D case, a change of variables can be performed to make the noise action \( S_0 \) diagonal. Let us introduce the notation \( i_d = i_1, i_{d-1}, i_{d+1}, \ldots, i_D \) and diagonalized by the transformation

\[
(\varphi^d)^a_i = (\varphi^d)^{\alpha}_{i_d \alpha} = \sum_{\beta=1}^{3N_d} (O^d)^{\alpha \beta} (\varphi^d)^{\beta}_{i_d \beta}, \tag{A13}
\]

where \( O^d \) is a \( 3N_d \times 3N_d \) matrix defined as for the 1D case, but in terms of \( J^d \). Using Eq. (A13), we obtain

\[
e^{-\Delta \tau \sum_{i,j=ab} (J^d_{ij})^{ab} \hat{S}^a_{i} \hat{S}^b_{j}} \approx \int \mathcal{D}\varphi e^{-S_0[\varphi]} \prod_{\tau_0} d\tau \sum_i \sum_d \sum_{\alpha,\beta} (O^d)^{\alpha \beta} (\varphi^d)^{\beta}_{i_d \beta} \cdot \tag{A14}
\]

with

\[
S_0[\varphi] = \frac{1}{2} \int_0^\tau d\tau \sum_i \sum_{\alpha} [\varphi^d]^a_i [\varphi^d]^a_i. \tag{A15}
\]
Appendix B: Euclidean Time Dynamics

In this Appendix we study the Euclidean time dynamics of the disentangling variables (9), which encode the quantum system in the disentangling formalism.

1. Ising SDEs

The stochastic representation (12) of the Euclidean time evolution operator is formally exact; this implies that the statistics of the classical disentangling variables $\xi = \{\xi_i^\alpha\}$ contain all the information about the corresponding quantum problem. In the case of real time evolution, this observation was drawn upon in Refs [55, 56] to numerically investigate the relation between fluctuations in the disentangling variables and dynamical quantum phase transitions [72, 73]. Here we consider the imaginary time behavior of the disentangling variables, which encodes information about the ground state of the corresponding quantum problem. For definiteness, we consider the quantum Ising model, given by the Hamiltonian (20). For the one-dimensional quantum Ising chain, the general result (9) specializes to the Euclidean Ising SDEs [24, 55, 56]

$$\dot{\xi}_i^+(\tau) = \frac{\Gamma}{2}(1 - \xi_i^+)^2 + J\xi_i^+ \sum_j O_{ij} \phi_j, \quad (B1a)$$

$$\dot{\xi}_i^-\phi_j, \quad (B1b)$$

$$\dot{\xi}_i^- (\tau) = \frac{\Gamma}{2} \exp \xi_i^-, \quad (B1c)$$

which are here expressed in terms of the fields $\phi_j$ that diagonalize the noise action (7). To the best of our current knowledge, Eqs (B1) are only exactly solvable in the classical ($\Gamma = 0$) and non-interacting ($J = 0$) cases. This was discussed in Ref. [56] for real time evolution and in the special case $\xi_i^+(0) = 0$; here we consider Euclidean time and general initial conditions, as it is relevant for our current purposes. For the present discussion, we focus on the one-dimensional case, which is sufficient to illustrate the relevant properties of the disentangling variables; the higher-dimensional version of Eq. (B1) is given by Eq. (C8) in Appendix C.

$$\frac{d}{d\tau} g_i(\lambda, \tau) = \lambda^2 \left(1 - \xi_i^+\right)^2 g_i(\lambda, \tau) + \lambda \xi_i^+ \sum_j O_{ij} \phi_j g_i(\lambda, \tau) + \frac{1}{2} \lambda^2 \xi_i^+ \sum_j O_{ij} O_{ij} g_i(\lambda, \tau). \quad (B5)$$

It can be easily shown that the matrix $OO^T$ is proportional to $J$ and hence has no diagonal term [56]; this implies that the Ito drift term proportional to $\sum_j O_{ij} O_{ij}$

2. Exactly Solvable Limits

In the classical case with $\Gamma = 0$, the non-linear term in Eq. (B1a) vanishes and $\xi_i^+$ performs driftless geometric Brownian motion. This is exactly solvable, giving

$$\xi_i^+(\tau) = \xi_i^+(0) \exp \left[ \sum_j O_{ij} \int_0^\tau \phi_j(s) ds \right], \quad (B2)$$

where we used $(OO^T)_{ij} \phi_j = 0$. In the classical limit, $\xi_i^-$ is decoupled from $\xi_i^+$ and satisfies Brownian motion:

$$\xi_i^- (\tau) = \xi_i^- (0) + \int_0^\tau \sum_j O_{ij} \phi_j(s) ds, \quad (B3)$$

while $\xi_i^- (\tau) = \xi_i^- (0)$.

In the non-interacting limit $J = 0$, Eqs (B1) become deterministic and solvable, yielding

$$\xi_i^+(\tau) = \xi_i^+(0) + \frac{1 - \xi_i^+(0)}{\xi_i^+(0) + \coth(\Gamma \tau/2)}, \quad (B4a)$$

$$\xi_i^- (\tau) = \xi_i^- (0) - 2 \log \left[ \cosh(\Gamma \tau/2) + \xi_i^+(0) \sinh(\Gamma \tau/2) \right], \quad (B4b)$$

$$\xi_i^- (\tau) = \xi_i^- (0) + \frac{\exp[\xi_i^+(0)]}{\xi_i^+(0) + \coth(\Gamma \tau/2)}. \quad (B4c)$$

3. Moments of the Disentangling Variables

In the general case with finite $\Gamma$ and $J$, the SDEs (B1) cannot be solved exactly to the best of our knowledge. However, analytical insights about (B1) can be still obtained, as we discuss presently. Of particular interest is the behavior of the variables $\xi_i^+$: as observed in Refs [24, 55, 56], these play a key role, being the primary source of non-linearity in (B1) (the variable $\xi_i^-$ is seldom needed to compute observables) and the sole disentangling variable whose equation of motion is autonomous, not involving any other $\xi_i^\alpha$. The stationary probability distribution attained at late times by $\xi_i^+(\tau)$ was obtained in Refs [24, 61]. Additional information is encoded in the moment-generating function $G_i(\lambda, \tau)$ of $\xi_i^+(\tau)$, which satisfies $\partial^n G_i(\lambda, \tau)|_{\lambda=0} = \langle \xi_i^{+n}(\tau) \rangle$ and gives access to the Euclidean time-dependent moments of $\xi_i^+$. To compute this, we define the stochastic function $g_i(\lambda, \tau) \equiv e^{\lambda \xi_i^+(\tau)}$, such that $G_i(\lambda, \tau) \equiv \langle g_i(\lambda, \tau) \rangle$. The equation of motion of $g_i(\tau)$ is obtained by applying the Ito chain rule [66, 74]:

$$\frac{d}{d\tau} g_i(\lambda, \tau) = \lambda^2 \left(1 - \xi_i^+\right)^2 g_i(\lambda, \tau) + \lambda \xi_i^+ \sum_j O_{ij} \phi_j g_i(\lambda, \tau) + \frac{1}{2} \lambda^2 \xi_i^+ \sum_j O_{ij} O_{ij} g_i(\lambda, \tau). \quad (B5)$$
gives no contribution. It is also convenient to write \( \xi^+_i g_i = \frac{d}{d\tau} g_i \). With these simplifications, we obtain
\[
\frac{d}{d\tau} g_i(\lambda, \tau) = \left[ \lambda \frac{\Gamma}{2} \left( 1 - \frac{\partial^2}{\partial \lambda^2} \right) + \sum_j O_{ij} \phi_j \frac{\partial}{\partial \lambda} \right] g_i(\lambda, \tau).
\]  
(\text{B6})

Considering the expectation value of Eq. (B6) and using the property of Ito calculus \( \langle g_i(\tau) \phi_j(\tau) \rangle_\phi = 0 \) \( \forall i, j \) we obtain the partial differential equation satisfied by the moments of the properties of the Ito calculus.

\[
\frac{\partial}{\partial \tau} G_i(\lambda, \tau) = \left[ \lambda \frac{\Gamma}{2} \left( 1 - \frac{\partial^2}{\partial \lambda^2} \right) \right] G_i(\lambda, \tau),
\]  
(\text{B7})

with initial conditions \( G_i(0, \tau) = G_i(\lambda, 0) = 1 \). Eq. (B7) can be solved exactly, yielding

\[
G_i(\lambda, \tau) = \exp \left[ \lambda \left( \xi^+_i(0) + \frac{1 - \xi^{+_2}(0)}{\xi^{+_1}(0) + \coth(\Gamma \tau/2)} \right) \right].
\]  
(\text{B8})

This result predicts that all moments of \( \xi^+_i(\tau) \) are given by powers of the deterministic trajectory obtained in the non-interacting case with \( J = 0 \): the moments of each individual \( \xi^+_i \) contain no information about the interacting quantum system. All information is therefore encoded in the correlations between variables at different sites. We note that the findings of the present Section do not apply to real time evolution: in that case, the moments of \( \xi^+_i(t) \) for non-zero \( J \) differ from the non-interacting result. This discrepancy can be traced back to the failure of analytic continuation of Eq. (B8) to real time.

4. Joint Probability Distribution

Since the information about interactions is contained in the joint statistics of the \( \xi^+_i \) variables, we investigate the joint probability distribution \( P[\xi^+] \equiv P[\{\xi^+_i\}] \). The stochastic process \( \xi^+_i \) has drift and diffusion

\[
\begin{align*}
a_i(\xi^+_i) &= \frac{\Gamma}{2} (1 - \xi^{+_2}), \quad \text{(B9a)} \\
B_{ij}(\xi^+_i) &= \xi^+_i O_{ij} \quad \text{(B9b)}
\end{align*}
\]

respectively. The probability distribution of its realizations is given by \([24, 75-77]

\[
P[\xi^+] = C_\xi e^{-I[\xi^+]},
\]  
(\text{B10})

where \( C_\xi \) is a normalization constant and

\[
I[\xi^+] = \int_0^\tau d\tau' L(\xi^+, \dot{\xi}^+),
\]  
(\text{B11})

\[
L(\xi^+, \dot{\xi}^+) = \frac{1}{2} \sum_{ij} \left[ (\dot{\xi}^+_i - a_i(\xi^+_i) B_{ij}^{-1}(\xi^+) \dot{\xi}^+_j - a_j(\xi^+_j) B_{ji}^{-1}(\xi^+) \dot{\xi}^+_i) \right],
\]  
(\text{B12})

\[
B_{ij}(\xi^+) = \sum_k B_{ik}(\xi^+_i) B_{jk}(\xi^+_j).
\]  
(\text{B13})

Eqs (B9) give \( B_{ij}(\xi^+) = 2 J \eta_{ij} \eta^+ \xi^+_j \) and

\[
L(\xi^+, \dot{\xi}^+) = \frac{\Gamma}{2} \left( 1 - \xi^{+_2} \right) J^{-1}_\eta \left[ \xi^+_j - \frac{\Gamma}{2} (1 - \xi^{+_2}) \right].
\]  
(\text{B14})

Eq. (B10) provides the measure when the stochastic expression for an observable is expressed as a path integral over the variables \( \xi^+_i \) rather than the fields \( \phi_i \) [24]. When sampling according to the distribution (B10), the likeliest trajectory is obtained by extremizing the weight \( I[\xi^+] \) with respect to \( \xi^+_i(\tau) \). By solving the corresponding Euler-Lagrange equations, we readily see that the dominant trajectory is the non-interacting solution (B4a). Therefore, when applying the stochastic approach using direct sampling [55, 56], one typically samples trajectories which are nearly non-interacting.

We note that in the large \( \Gamma \) limit Eq. (B10) takes a large deviation form [78]. Since \( \Gamma \) multiplies time in \( \xi^+_i \), we rescale time as \( \tilde{\tau} = \tau \Gamma \). The corresponding rescaled stochastic equation for \( \xi^+_i \) is

\[
\dot{\xi}^+_i(\tilde{\tau}) = \frac{1}{2} (1 - \xi^{+_2}) + \epsilon \xi^+_i \sum_j O_{ij} \phi_j,
\]  
(\text{B15})

where we have defined the noise strength \( \epsilon \equiv 1/\Gamma \). The limit \( \Gamma \to \infty \) is therefore equivalent to the small-noise limit of (B15). Stochastic differential equations in the limit of small noise are described by the Freidlin-Wentzell (FW) large deviation theory [79, 80]: \( \xi^+_i(\tau) \) obeys a large deviation principle (LDP), with rate \( e^{-\psi} \) and rate function \( I[\xi^+] \equiv \epsilon^{2} I[\xi^+] \). In this small-\( \epsilon \) limit, the trajectories \( \xi^+_i(\tilde{\tau}) \) are approximately Gaussian distributed around the likeliest trajectory \( \xi^+_{N1} \) [78]:

\[
P[\xi^+_{N1}] \sim e^{-\frac{1}{2} \sum_{ij} \eta_{ij} \eta^+ \xi^+_j (\tilde{\tau})^2} \prod_{\tilde{\tau}} (\xi^+_i(\tilde{\tau}) - \xi^+_{N1}(\tilde{\tau})) d\tilde{\tau},
\]  
(\text{B16})

where the second variation \( \mathcal{I}^{(2)}_{ij} \) is given by

\[
\mathcal{I}^{(2)}_{ij} \equiv \frac{\delta^2 I}{\delta \xi^+_{ij}(\tilde{\tau}) \delta \xi^+_{ij}(\tilde{\tau})} |_{\xi^+_{N1}}.
\]  
(\text{B17})

Thus, trajectories that deviate significantly from the non-interacting limit are exponentially suppressed. The large deviation formalism also applies to real time evolution, where again the dominant trajectory is given by the deterministic result \( \xi^+_i(\tau) \). However, in contrast to \( \xi^+_{N1}(\tau) \), \( \xi^+_i(\tau) \) has an infinite number of singularities as a function of time [56]. This leads to a breakdown of the expansion about the saddle point, which can be expected to have consequences for the sampling. Even for large \( \Gamma \), regions in time that are close to the singularities in the saddle point trajectory are expected to be associated with enhanced fluctuations, leading to difficulties in sampling. This observation may lie at the root of the enhanced fluctuations of the disentangling variables found in the vicinity of dynamical quantum phase transitions [72, 73], reported in [55, 56].
Appendix C: Saddle Point Equation

In this Appendix, we provide details on the saddle point equation discussed in Section II C, including its numerical solution, its generalization to other observables and the higher dimensional case. We also provide a toy example of an integral for which several saddle points exist.

1. Numerical Solution

The saddle point equation (30) for the Loschmidt amplitude can be solved recursively, exploiting the intuition that the saddle point field configuration \( \varphi_{SP}(\tau') \equiv \varphi_{SP}(\tau'|\tau_f) \) should change little if \( \tau_f \) is increased by a small amount \( \Delta \tau \). In practice, one assumes

\[
\varphi_{SP}(\tau'|\tau_f + \Delta \tau) \approx \varphi_{SP}(\tau'|\tau_f) \tag{C1}
\]

for \( \tau' < \tau_f + \Delta \tau \). The field \( \varphi_{SP}(\tau'|\tau_f) \) is then used to compute \( \xi_i^{\tau}|_{SP} \) and \( \Xi_j|_{SP} \). Using these quantities, one can in turn produce a better approximation of \( \varphi_{SP}(\tau'|\tau_f + \Delta \tau) \) according to the saddle point equation (30). This procedure can be iterated until the field configuration has converged to a desired level of accuracy. The convergence of the recursion is determined by defining a quantity \( \varepsilon \) which measures how much the approximate saddle point field varies after an iteration of the algorithm. For example, a suitable definition is

\[
\varepsilon \equiv \frac{1}{k} \sum_{m=1}^{k} \left| \varphi_{SP}(\tau_m|\tau_f + \Delta \tau) - \varphi_{SP}(\tau_m|\tau_f + \Delta \tau) \right| \tag{C2}
\]

where \( \varphi_{SP} \) and \( \bar{\varphi}_{SP} \) are the old and updated estimates of the SP field respectively, evaluated at the discrete times \( \tau_m \). Convergence is then defined as \( \varepsilon < \varepsilon^* \), where \( \varepsilon^* \) is a threshold of choice. The runtime of this recursive algorithm scales quadratically with the number of time steps \( n \); this is because for each \( 1 < k < n \) one needs to perform \( k \) calculations in order to compute \( \xi_i^{\tau}|_{SP} \), so that summing over all \( k \) the total number of calculations to perform is of order \( n(n+1)/2 \). In principle, the computational cost is further increased by having to repeat each step multiple times to attain convergence. However, for reasonable values of the threshold \( \varepsilon^* \), numerical evaluation shows that the recursive algorithm has rapid convergence, typically requiring only 1–2 iterations. From recursively solving the SP equation, we find that for sufficiently large \( \tau_f \) the value \( \varphi_{SP}(\tau'|\tau_f) \) with \( \tau' \ll \tau_f \) no longer changes with \( \tau_f \), settling to a value \( \varphi_{SP}(\tau'|\infty) \equiv \varphi_{SP}(\tau') \); this is illustrated in Fig. 11(a). Because of this, when recursively solving the SP equation one only needs to update the SP configuration at the times \( \tau' \) such that \( \varphi_{SP}(\tau'|\tau_f) \neq \varphi_{SP}(\tau') \) to a desired level of precision; this speeds up the recursive solution significantly. The SP equation (30) prescribes that the value of the saddle point field at the end time is always \( \varphi_{SP}(\tau_f|\tau_f) = -1 \). Thus, the saddle point field \( \varphi_{SP}(\tau'|\tau_f) \) cannot attain a steady state, i.e. for finite \( \tau_f \) there exists no time scale \( \tau_{SS} \) such that \( \partial_{\tau'} \varphi_{SP}(\tau'|\tau_f) \approx 0 \ \forall \ \tau' > \tau_{SS} \). However, the numerical results show that for sufficiently large \( \tau_f \) the SP field \( \varphi_{SP}(\tau'|\tau_f) \) attains a plateau value at times \( 0 \ll \tau' \ll \tau_f \); this is illustrated in Fig. 11(b). The extent of the plateau grows as \( \tau_f \) is increased; since the action is extensive in time, the plateau value provides the dominant contribution to observables in the large \( \tau_f \) limit. The plateau value of \( \varphi_{SP} \) can be found analytically, as discussed in Section II C; the analytical results are in perfect agreement with the numerical solution.

2. Saddle Point for General Observables

In the disentanglement formalism, the Euclidean time evolution of a given observable is given by

\[
\langle \psi_0 | \hat{U}(\tau) \hat{O} \hat{U}^\dagger(\tau) | \psi_0 \rangle = \int \mathcal{D} \phi e^{-S_0[\phi] + F_{\phi}[\phi]}, \tag{C3}
\]

where \( \phi = \{ \phi_{f,i}, \phi_{b,i} \} \) collectively denotes the two sets of HS fields introduced to decouple the two time-evolution operators, and the classical function \( F_{\phi} \) is given by Eq. (14). As discussed in Section II B, the trajectory yielding the largest contribution to the functional integral in Eq. (C3) can be found by extremizing the effective
action

\[ S_O \equiv S_0[\phi] - \log f_O[\phi]. \]  

(C4)

\[ S_{i} = \frac{1}{2} \int_{0}^{\tau_f} dt \sum_{i} \left[ \int_{j} J_{ij}^{-1} [\varphi_{f,i} \varphi_{f,j} + \varphi_{b,i} \varphi_{b,j}^*] - \Gamma_{\psi_{f,i}}^+ - \Gamma_{\psi_{b,i}^*} + J_{\varphi_{f,i}} + J_{\varphi_{b,i}}^* \right] - \sum_{i} \log \left[ 1 + \xi_{f,i}^+(\tau_f) \xi_{b,i}^* (\tau_f) \right]. \]  

(C5)

Consider the normalization function, corresponding to setting \( \hat{O} = 1 \) in (C3) and given by Eq. (15). For the 1D quantum Ising model, the effective action for this quantity is given by

By varying Eq. (C5), we obtain the SP equations for the normalization:

\[ J \sum_{j} J_{ij}^{-1} \varphi_{f,j}(\tau')|_{SP} = \Gamma \int_{0}^{\tau_f} \Xi_{f,i}(\tau, \tau') d\tau|_{SP} - J + \frac{2 \xi_{b,i}^+ \Xi_{i}(\tau_f, \tau')}{1 + \xi_{f,i}^+(\tau_f) \xi_{b,i}^* (\tau_f)}|_{SP}. \]  

(C6)

The same equation is satisfied by \( \varphi_{b,i}|_{SP} \), with the replacement \( f \leftrightarrow b \). By direct substitution, one readily verifies that the plateau of the Loschmidt amplitude SP is a fixed point of Eq. (C6) at all times. Thus, choosing the mean field ground state as the initial state eliminates both the transient and the late-time behaviour of Eq. (C6) (in contrast, for any initial state, the solution of Eq. (30) deviates from the plateau at late times due to the boundary condition \( \phi(\tau_f) = -1 \). More generally, local observables expressed in a translationally invariant way correspond to the stochastic functions

\[ f_O(\tau_f) = f_1(\tau_f) \sum_{i} \bar{f}_{O,i}(\tau_f) \]  

(C7)

where \( \bar{f}_{O,i}(\tau_f) \) is a function of \( \xi_{f,i}^+(\tau_f) \), only. For instance, for the magnetization one has \( f_{M,i} = (1 - \xi_{f,i}^+ \xi_{b,i}^*)/(1 + \xi_{f,i}^+ \xi_{b,i}^*) \); see Eq. (16). It can be readily seen that the SP equation obtained by extremizing the effective action for (C7) differs by Eq. (C6) by a term proportional to \( 1/N \). Furthermore, the extra term is also proportional to \( \Xi_{SP}(\tau_f, \tau) \); at the plateau, one has \( \Xi_{f}(\tau_f, \tau) \propto e^{-(\Gamma \xi_{f}^+ - J_{\varphi} \phi(\tau_f - \tau))} \), so that the extra term is inconsequential as \( \tau_f \to \infty \). Thus, the SP equation for any local observable differs from (C6) by a term which is suppressed both as \( \tau_f \to \infty \) and as \( N \to \infty \): for analytical and numerical applications, the plateau SP trajectory obtained for the Loschmidt amplitude can be equally used for all other ground state expectation values.

3. Higher Dimensions

For the \( D \)-dimensional quantum Ising model, the Euclidean SDEs are given by

\[ \dot{\xi}_{d}^{+} = \frac{\Gamma}{2} (1 - \xi_{d}^{+ 2}) + \xi_{d}^{+} \sum_{d=1}^{D} J_{d}\varphi_{d}, \]  

(C8a)

\[ \dot{\xi}_{d}^{-} = -\Gamma \xi_{d}^{+} + \sum_{d=1}^{D} J_{d}\varphi_{d}, \]  

(C8b)

\[ \dot{\varphi}_{d}^{d}(\tau')|_{SP} = \frac{\Gamma}{J_{d}} \sum_{i_{d}} J_{i_{d}d} \int_{0}^{\tau_f} \Xi_{i_{1},...i_{d}}^{d}(\tau'|_{SP}) d\tau - 1. \]  

(C10)

The functional derivative \( \Xi_{f}^{d}(\tau, \tau') \) can be obtained by varying the effective action with respect to \( \varphi_{d}^{d}(\tau') \) is then given by

\[ \Xi_{f}^{d}(\tau, \tau') = J_{d}\xi_{f}^{d}(\tau')\theta(\tau - \tau')e^{\int_{\tau}^{\tau_f} \sum_{s=1}^{D} J_{d}\varphi_{d}^{d}(s) \sum_{s=1}^{D} J_{d}\varphi_{d}^{d}(s)}. \]  

(C11)

For a translationally invariant system one has \( \xi_{f}^{d}|_{SP} = \xi_{SP}^{d}, \Xi_{i_{d}}^{d}|_{SP} = \Xi_{SP}^{d}, \varphi_{d}|_{SP} = \varphi_{d}^{d} \), such that the SP equation simplifies to

\[ \varphi_{d|SP}^{d}(\tau') = \frac{\Gamma}{J_{d}} \int_{0}^{\tau_f} \Xi_{SP}^{d}(\tau, \tau')|_{SP} d\tau - 1. \]  

(C12)

For a fully isotropic system with \( J_{1} = \cdots = J_{D} = J \), one additionally has \( \varphi_{SP}^{d} = \varphi_{SP} \) and the SP equations further
FIG. 12. Expansions of integrals in the presence of more than one saddle point. We consider the integrand \( f(x) \) defined in Eq. (C18), comparing the exact value (full line), the approximation obtained by truncating Eq. (C21) to Gaussian order (dash-dotted line), and the approximation obtained from Eq. (C21) with \( n = 10 \) (dashed line). (a) For \( a = 1.5 \), the saddle points at \( x_{\text{SP}} = \pm a \) are close to each other and the expansion (C21) produces worse results for \( n = 10 \) (37% error) than for the Gaussian approximation (3% error). (b) For \( a = 5 \), the saddle points are well separated and the higher order expansion closely approximates the integrand, as shown in the inset. This leads to a better performance for the \( n = 10 \) approximation, which gives the correct result within 1.4%, compared to an error of 3.7% for the Gaussian approximation. For the present example, both expansions eventually break down as \( n \) is increased due to their asymptotic nature.

We consider the integral

\[
I(a) = \int_{-\infty}^{\infty} f(x)dx,
\]

(C17)

\[
f(x) = C_a e^{-S(x)},
\]

(C18)

\[
S(x) = \frac{x^4}{4a^2} - x^2/2,
\]

(C19)

where \( C_a \) is a normalization constant defined by \( I(a) = 1 \). Extremization of \( S(x) \) with respect to \( x \) yields two minima, \( x_{\text{SP}} = \pm a \). One can then expand the action around each SP as

\[
S = S_{\text{SP}} + \frac{1}{2^n} S_n^{(2)} (x - x_{\text{SP}})^2 + S^h,
\]

(C20)

where \( S_n^{(2)} \) is the second variation evaluated at the SP and \( S^h \) includes all contributions of higher order. We then approximate Eq. (C17) as

\[
I(a) \approx C_a \sum_{\text{SP}} e^{-S_{\text{SP}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} S_n^{(2)} (x - x_{\text{SP}})^2} \left[ 1 + \sum_{m=3}^{n} \alpha_m (x - x_{\text{SP}})^m \right] dx,
\]

(C21)

where the coefficients \( \alpha_m \) are obtained by Taylor expanding \( e^{S^h} \). For \( n \leq 2 \), none of the \( \alpha_m \) is included and thus Eq. (C21) reduces to the evaluation of Gaussian fluctuations around the SP. To show how well the approximation (C21) captures the true value of \( I(a) \), in Fig. 12 we compare the exact and approximate integrands for different values of \( a, n \). We find that the approximation (C21) gets increasingly accurate as the SPs are better spaced out. This is an example of the “small overlap” condition discussed in the main text: one can separately expand about the two saddle points and add up the individual contributions of the expansions, provided that regions (in this case, along the \( x \) axis) which contribute significantly to one integral give negligible contribution to the other.

### Appendix D: Mean Field Approximation

To aid comparison with the results of the main text, here we outline the derivation of the mean field (MF) ground state for the \( D \)-dimensional quantum Ising model (20). The MF approach consists in approximating the ground state by the product state which minimizes the energy of the system. The ground state is thus parameterized via the variational ansatz

\[
|\text{MF}\rangle = \otimes_i (\cos \theta |\uparrow\rangle_i + \sin \theta |\downarrow\rangle_i).
\]

(D1)

This ansatz gives a ground state energy density

\[
\epsilon_{\text{MF}}(\theta) = -\frac{\Gamma}{2} \sqrt{1 - \cos(2\theta)^2} - \frac{1}{4} JD \cos(2\theta)^2.
\]

(D2)
Minimizing this with respect to \( x \equiv \cos(2\theta) \), one gets three solutions:

\[
x = \pm \frac{\sqrt{D^2 J^2 - \Gamma^2}}{DJ}, \quad \text{(D3a)}
\]

\[
x = 0, \quad \text{(D3b)}
\]

where the first solution is only valid for \( \Gamma < DJ \). For each value of \( \Gamma \), one then chooses the solution in (D3) which minimizes \( \epsilon \). This yields the mean-field approximation to the ground state energy density:

\[
\epsilon_{MF} = \begin{cases} 
-\frac{D^2 J^2 + \Gamma^2}{4DJ} & \text{for } \Gamma < DJ, \\
-\frac{\Gamma^2}{2} & \text{for } \Gamma \geq DJ.
\end{cases} \quad \text{(D4)}
\]

Within the MF approximation, the ground state magnetization is then given by

\[
m_{MF} = \begin{cases} 
\pm \frac{\sqrt{(DJ-\Gamma)(DJ+\Gamma)}}{2DJ} & \text{for } \Gamma < DJ, \\
0 & \text{for } \Gamma \geq DJ.
\end{cases} \quad \text{(D5)}
\]

The MF approximation predicts a quantum phase transition at \( \Gamma_{MF}^C = DJ \). The same results may be obtained by writing \( \hat{S}_i^z = m^z + \delta \hat{S}_i^z \) and neglecting quadratic fluctuations, \( \delta \hat{S}_i^z \delta \hat{S}_j^z \approx 0 \). The definition \( m^z \equiv \langle \hat{S}_i^z \rangle \) then gives a self-consistency condition. The MF results provided in this Section correspond to the SP result given in the main text; in particular, \( \varphi_P = 2Dm_{MF} \) is precisely the effective field felt by each spin (i.e. the mean field).