Large Deflection Analysis of Functionally Graded Beams Resting on a Two-Parameter Elastic Foundation

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Abstract

This paper presents a finite element procedure for the large deflection analysis of functionally graded (FG) beams resting on a two-parameter elastic foundation. The material properties of the FG beams are assumed to vary continuously in the thickness direction by a power-law distribution. Based on the strain energy expression, a shear deformable beam element, taking the effect of the material non-homogeneity and the foundation support into account, is formulated and employed in the analysis. An incremental/iterative procedure in combination with the arc-length control method is used for solving nonlinear equilibrium equations. The numerical results show that the convergence of the formulated element is fast, and the large displacements of the beams can be accurately assessed by using a few numbers of the elements. A parametric study is carried out to highlight the effect of the material non-homogeneity and the foundation support on the large deflection behavior of the beams. The influence of the aspect ratio on the large deflection response of the beam is also examined and highlighted.

Keywords: functionally graded beam; elastic foundation; large deflection analysis; finite element method

1. Introduction

Analysis of beams resting on an elastic foundation is a classical topic in the field of structural mechanics. Many investigations on the topic, using both analytical and numerical methods, have been reported in the literature. In his early and excellent monograph, Hetényi (1946) presented a number of closed-form solutions on bending and buckling of beams resting on different foundation models. Horibe and Asano (2001) studied the large deflection of beams on a two-parameter elastic foundation by employing the boundary integral equation method. Kounadis et al. (2006) investigated the post-buckling behavior of columns resting on a Winkler elastic foundation by proposing a simple mechanical model to simulate the salient features of the buckling mechanism. Based on a perturbation method, Shen (2012) studied the nonlinear vibration, bending and post-buckling of beams resting on a two-parameter elastic foundation. Patel et al. (1999) investigated the free vibration and post-buckling behavior of laminated orthotropic beams on an elastic foundation by deriving a three-node Bernoulli beam element. Nguyen (2004) presented a finite element procedure for studying the buckling and post-buckling behaviors of Timoshenko beams resting on a two-parameter foundation. Mullapudi and Ayoub (2011) proposed a finite element procedure for modeling elastic-plastic beams on a two-parameter foundation. Recently, the authors and their co-worker (Nguyen et al., 2012) investigated the post-buckling behavior of elastic-plastic beams resting on an elastic foundation by using the finite element.

Functionally graded materials (FGMs) invented by Japanese scientists in 1984 (Koizumi, 1997) have received much attention from researchers. The FGMs are formed by varying the percentage of constituents in any desired direction, and as a result the specific physical and mechanical properties of the formed material can be obtained. FGMs offer great potential for use as a structural material, and analysis of FG structures has become an important topic in structural mechanics. A comprehensive list of publications on the analysis of FG structures under different loadings is given in a review paper by Birman and Byrd (2007). Contributions that are most relevant to the present work are discussed below.

By employing solutions of the governing different equations as interpolation functions, Chakraborty et al. (2003) formulated a first-order shear deformation beam element for analyzing the thermoelastic behavior of FG beams. Singh and Li (2009) presented a mathematical model for computing the buckling loads of uniform and non-uniform axially FG columns. Ying et al. (2008) presented the exact solutions for bending and
free vibration of FG beams resting on a Pasternak foundation with the material properties varying exponentially along the thickness direction. Kang and Li (2009, 2010) proposed closed-form solutions for a nonlinear FG cantilever beam undergoing large deflection and subjected to a tip load or a tip moment. Fallah and Aghdam (2011) derived the nonlinear governing differential equation for geometrically nonlinear vibration and post-buckling analysis of FG beams resting on a nonlinear elastic foundation. Shahba et al. (2011) employed the exact shape functions of a uniformed homogeneous Timoshenko beam segment to formulate a finite element formulation for computing natural frequencies and buckling loads of tapered Timoshenko beams made of axially FGM. Piovan et al. (2012) developed a finite element model for the dynamic and buckling analyses of FG circular curved beams. Recently, the second author developed a beam element for studying the large displacement response of tapered cantilever beams made of axially FGM (Nguyen, 2013).

In this paper, a finite element procedure for large deflection analysis of FG beams resting on a two-parameter elastic foundation is presented. The material properties of the beams are assumed to vary in the thickness direction by a power-law distribution. A shear deformable beam element taking the effects of the material non-homogeneity and the foundation support into account is formulated and employed in the analysis. The element in the present work can be viewed as an extension of the element previously formulated by the second author for homogeneous beams (Nguyen, 2004). With the formulated element, the response of the beams is computed by using an incremental/iterative procedure in combination with the arc-length method. Numerical investigations are given to illustrate the accuracy of the derived formulation and to study the influence of the material and foundation parameters on the large deflection response of the beam. The effect of the aspect ratio on the large deflection response of the beam is also examined and highlighted.

2. FG Beam on Foundation

Fig. 1. shows a cantilever beam with length of $L$, width $b$, and thickness $h$, resting on an elastic foundation. The beam is assumed to be composed of two different materials, and the effective material properties of the FG beam such as Young's modulus $E$ and shear modulus $G$ are assumed to vary continuously in the thickness direction as

$$
\begin{align*}
E(z) &= (E_t - E_b) \left( \frac{z}{h} + \frac{1}{2} \right)^n + E_b \\
G(z) &= (G_t - G_b) \left( \frac{z}{h} + \frac{1}{2} \right)^n + G_b
\end{align*}
$$

(1)

where $E_b$ ($G_b$) and $E_t$ ($G_t$) are Young's (shear) moduli of the material on the beam bottom and top surfaces, respectively; $n$ is the non-negative power-law index dictating the distribution of the constituents. Fig. 2. shows the variation of the effective Young's modulus of the FG beam formed from aluminum and steel. The Young's modulus of aluminum is 70 MPa and that of steel is 210 MPa (Chakraborty et al., 2003).

The elastic foundation is assumed to be modeled by two parameters, namely $k_w$ — the stiffness of the Winkler foundation, and $k_G$ — the stiffness of a shear layer which was introduced to account for interaction between the Winkler springs (Dutta & Roy, 2002; Tanahashi, 2007). When $k_G$ is zero the foundation model returns to the traditional Winkler foundation. The load-displacement relation for the two-parameter foundation is given by (Patel et al., 1999).

$$
p(x) = k_w w - k_G \frac{\partial^2 w}{\partial x^2}
$$

(2)

where $p(x)$ is the force per unit area and $w$ is the transverse displacement. As is customary (Kounadis et al., 2006; Shen, 2012), the foundation is assumed to be always attached to the beam during the deformation process.

3. Finite Element Formulation

The finite element formulation derived in this section is an extension of the formulation previously derived by the second author (Nguyen, 2004) for a homogeneous beam. Assuming the beam is divided
into a number of elements of the length of \( \ell \). Fig. 3 shows a generic two-node beam element, where the vector of nodal displacements is given by

\[
d = \begin{bmatrix} u_i & w_i & \theta_i & u_j & w_j & \theta_j \end{bmatrix}^T
\]

(3)

where \( u_i, w_i \) and \( \theta_i \) are the axial, transverse and rotation at node \( i \); \( u_j, w_j \) and \( \theta_j \) are the corresponding quantities at node \( j \). In Eq. (3) and hereafter, a superscript \( T \) denotes the transpose of a vector or a matrix.

The deformation at a point on the beam is defined by an angle \( \theta \), a rotation of its associated cross section, and a position vector

\[
r(x) = \{x + u(x)\}t_i + w(x)t_2
\]

(4)

where \( t_i, t_2 \) are the base unit vectors; \( 0 \leq x \leq \ell \) is measured on the straight configuration; \( u(x) \) and \( w(x) \) are the axial and transverse displacements, respectively. The strain energy for the beam element, \( U \), is given by

\[
U = U_b + U_f
\]

(5)

where \( U_b \) and \( U_f \) denote the strain energies stemming from the beam bending and foundation deformation, respectively. For the two-parameter foundation model, the strain energy \( U_f \) is given by (Shen, 2012)

\[
U_f = \frac{1}{2} \int_0^\ell k_y w^2 dx + \frac{1}{2} \int_0^\ell k_z \left( \frac{\partial w}{\partial x} \right)^2 dx
\]

(6)

Based on the first-order shear deformable beam theory, the strain energy of a beam element is given by (Shames & Dym, 1985).

\[
U_b = \frac{1}{2} \int V \left[ E(z) \varepsilon^2 + \psi G(z) \chi^2 \right] dV
\]

(7)

where \( V \) is the element volume; \( \psi \) is the shear correction factor, equal to 5/6 for the rectangular section considered in the present work; \( \varepsilon \) and \( \gamma \) respectively denote the axial and shear strains

\[
\varepsilon = \frac{\partial u}{\partial x} - z \frac{\partial \theta}{\partial x}, \quad \gamma = \frac{\partial w}{\partial x} - \theta
\]

(8)

Using Eq. (8), one can write the strain energy \( U_b \) in the form

\[
U_b = \frac{1}{2} \int \left( A_{11} \varepsilon^2 + 2A_{12} \varepsilon \gamma + A_{22} \gamma^2 + \psi A_{13} \varepsilon \chi \right) dx
\]

(9)

where \( \varepsilon = \partial u/\partial x \) and \( \gamma = -\partial \theta/\partial x \) are the membrane strain and beam curvature, respectively; \( A_{11}, A_{12}, A_{22} \) and \( A_{13} \) are respectively the axial, coupling, bending and shear rigidities, and they are defined as

\[
\begin{align*}
(A_{11}, A_{12}, A_{22}) &= \int E(z) \{1, z, z^2\} dA \\
A_{13} &= \int G(z) dA
\end{align*}
\]

(10)

where \( A \) denotes the cross-sectional area. It should be noted that in the case of homogeneous beams the coupling rigidity \( A_{12} \) will vanish, and the \( U_b \) will return to the well-known strain energy expression of the Timoshenko beam (Shames & Dym, 1985).

It should be noted that for the large displacement considered herein, the strains \( \varepsilon, \gamma \) and the curvature \( \chi \), although parameterized for convenience by the reference abscissa \( x \), take values in the current configuration, and can be defined through the position vector in accordance with Nguyen (2004) as

\[
\frac{\partial r}{\partial x} = (1 + \varepsilon_0) \mathbf{e}_1 + \gamma \mathbf{e}_2, \quad \chi = -\frac{\partial \theta}{\partial x}
\]

(11)

where

\[
\mathbf{e}_1 = \cos \theta \mathbf{t}_1 + \sin \theta \mathbf{t}_2
\]

(12)

are the unit vectors, parallel and orthogonal to the tangent of the beam axis as shown in Fig. 3.

From Eqs. (4), (11) and (12), the membrane and shear strains can be written in the forms

\[
\varepsilon_0 = \left(1 + \frac{\partial u}{\partial x}\right) \cos \theta + \frac{\partial w}{\partial x} \sin \theta - 1
\]

\[
\gamma = \left(1 + \frac{\partial u}{\partial x}\right) \sin \theta + \frac{\partial w}{\partial x} \cos \theta
\]

\[
\chi = -\frac{\partial \theta}{\partial x}
\]

(13)

As the shear deformation is taken into account, the transverse displacement, \( w(x) \), and the rotation, \( \theta(x) \), are independent parameters, and linear functions can be employed to interpolate displacements and rotation (Crisfield, 1986) as
\[
\begin{align*}
&u = \frac{\ell - x}{\ell} u_i + \frac{x}{\ell} u_j \\
w = \frac{\ell - x}{\ell} w_i + \frac{x}{\ell} w_j \\
&\theta = \frac{\ell - x}{\ell} \theta_i + \frac{x}{\ell} \theta_j
\end{align*}
\]

(14)

The beam element based on the shape functions (14) does, however, encounter the shear-locking problem (Crisfield, 1986), and in order to overcome this problem the reduced integral, namely the one-point Gauss quadrature, is used to evaluate the strain energy of the beam element. In this context, and using Eq. (14), one can write the strain energy given by Eq. (9) in the form

\[
U_b = \frac{1}{2} \left( A_{11} E_0^2 + 2 A_{12} E_0 \bar{\varepsilon} + A_{22} \bar{\varepsilon}^2 + \psi A_{33} \bar{\varepsilon}^2 \right)
\]

(15)

where \( \bar{\varepsilon}, \bar{\varepsilon}, \bar{\varepsilon} \) are given by

\[
\bar{\varepsilon} = \left( 1 + \frac{u_j - u_i}{\ell} \right) \cos \bar{\theta} + \frac{w_j - w_i}{\ell} \sin \bar{\theta} - 1
\]

\[
\bar{\varepsilon} = \left( 1 + \frac{u_j - u_i}{\ell} \right) \sin \bar{\theta} + \frac{w_j - w_i}{\ell} \cos \bar{\theta}
\]

\[
\bar{\varepsilon} = \frac{\theta_j - \theta_i}{\ell}, \quad \text{with} \quad \bar{\theta} = \frac{\theta_i + \theta_j}{2}
\]

Using the shape functions (14), the strain energy \( U_F \) defined by Eq. (6) is as follows

\[
U_F = \frac{1}{6} k_w \left( w_i^2 + w_j^2 + w_i w_j \right) + \frac{1}{2} k_w \left( w_j - w_i \right)^2
\]

(17)

With the strain energy written in terms of the nodal displacements, one can compute the nodal internal force vector, \( f_m \), and tangent stiffness, \( k_i \), for the element by once and twice differentiating the strain energy with respect to the nodal displacements, respectively (Crisfield, 1991):

\[
f_m = \frac{\partial(U_b + U_F)}{\partial \bar{d}} = f_a + f_c + f_b + f_e + f_f
\]

\[
k_i = \frac{\partial^2(U_b + U_F)}{\partial \bar{d}^2} = k_a + k_c + k_b + k_e + k_f
\]

(18)

where the subscripts \( a, c, b, s \) and \( F \) denote the terms stemming from axial stretch, coupling, bending, shear deformation of the beam and the foundation deformation, respectively.

In the large displacement analysis, both the internal force vector \( f_m \) and tangent stiffness matrix \( k_i \) depend on the current computed nodal displacement \( \bar{d} \). The detailed expressions for the internal force vector and tangent stiffness matrix in Eq. (19) are given in the Appendix.

4. Equilibrium Equations

The derived formulation is assembled into structural nodal force vector and tangent stiffness matrix to construct the equilibrium equations, which can be written in the form (Crisfield, 1991)

\[
g(p, \lambda) = q_m(p) - \lambda f_m = 0
\]

(19)

where the residual force vector \( g \) is a function of the current structural nodal displacements \( p \), and the load-level parameter \( \lambda \); \( q_m \) is the structural nodal force vector, assembled from the formulated element force vector \( f_m \); \( f_m \) is the fixed external loading vector. Eq. (18) is imposed by the boundary conditions of the beam such as

\[
\begin{align*}
u(0) &= w(0) = \theta(0) = 0 & \text{for cantilever beams} \\
u(0) &= w(0) = w(L) = 0 & \text{for simply supported beams}
\end{align*}
\]

(20)

The system of nonlinear equations (19) can be solved by an incremental/iterative procedure based on the Newton-Raphson method (Crisfield, 1991). In order to deal with complex situations, in which the structure tangent stiffness matrix ceases to be positive definite, the spherical arc-length constraint method proposed by Crisfield (1991) is adopted.

5. Numerical Examples

Based on the formulated element and the described numerical procedure, a computer code was developed and employed in analyzing various FG beams subjected to different loadings in this Section. The beam considered herewith is assumed to be composed of aluminum and steel with Young's moduli as stated in Section 2. A Poisson's ratio \( \nu = 0.3 \) is used for both the constituent materials. In other words, the geometric data of the beam used in the analysis are as follows: \( L = 5 \) m, \( b = 0.12 \) m, \( h = 0.1 \) m. In order to facilitate discussion of the numerical results, the following dimensionless parameters are introduced (Nguyen, 2004) as

\[
k_1 = k_w \frac{L^3}{E_s I} \quad , \quad k_2 = k_w \frac{L^3}{\pi^2 E_o I}
\]

(21)

5.1 Formulation Verification

A cantilever beam without foundation support subjected to a tip moment \( M \) with analytical solutions derived by Kang and Li (2010) is first analyzed. The expressions for the axial displacement and the transverse displacement at the free end derived by Kang and Li are as follows

\[
u_L = \frac{A_{22}}{L} \sin \left( \frac{M}{A_{22}} \right) - L
\]

\[
w_L = \frac{A_{22}}{L} \left[ 1 - \cos \left( \frac{M}{A_{22}} \right) \right]
\]

(22)

where \( A_{22} \) is the effective bending rigidity of the FG beam defined in Eq. (10).
which is the buckling load of a pure aluminum cantilever beam on a two-parameter foundation (Shen, 2012). The effect of the index on the large deflection response of the beam is clearly seen in the figure, where a beam associated with a higher index endures larger tip displacements, regardless of the applied load. This is because, as shown in Eq. (1), the beam with smaller index contains more steel and thus it is stiffer. The effect of the first foundation parameter and the second foundation parameter on the behavior of the cantilever FG beam are respectively shown in Figs. 6 and 7. for . The figures show a reduction in the tip displacements of the beam with an increase in the foundation parameters, regardless of the applied load level.

Fig.4. Tip Displacements Versus Tip Moment of Cantilever Beam without Foundation Support

| nELE | w/L   | w/L   |
|------|-------|-------|
| 4    | 0.8664| 0.7115|
| 8    | 0.8684| 0.7008|
| 12   | 0.8687| 0.6989|
| 16   | 0.6989| 0.6982|
| 20   | 0.8689| 0.6079|
| Kang & Li (2010) | 0.8690| 0.6973|

Fig.4. shows the tip displacements versus the tip moment of the cantilever beam computed by two and ten elements for various values of the index . In the figure, the moment is normalized by . For comparison, the analytical solutions according to Eq. (22) are illustrated by small red circles. As seen in the figure, when the displacements are small, a mesh of two elements can give satisfactory results, but the numerical error steadily increases as the displacements increase. The numerical results computed by ten elements are in good agreement with the analytical solutions, regardless of the index . The tip displacements listed in Table 1. obtained by different numbers of elements, , for an index show good convergence of the proposed element, where both the axial and transverse displacements steadily converge towards the analytical solution by increasing the number of elements. A slight difference in the computed displacements of the present work with that of the analytical solution (Kang & Li, 2010) might be the result of the different beam theory used herein.

5.2 Cantilever Beam Under a Transverse Tip Load

The large deflection response of an FG cantilever beam on the elastic foundation under a transverse tip load is computed. In Fig.5., the effect of the power-law index on the large deflection of the beam is depicted for . In the figure, the applied load is normalized by

\[
P_0 = \pi^2 E_h I / 4L^2 \left(1 + k_\Gamma \frac{L^4}{\pi^4 E_h I} + k_\zeta \frac{L^2}{\pi^2 E_h I} \right)
\]

5.3 Cantilever Beam Under Eccentric Axial Load

A cantilever FG beam subjected to an eccentric axial load as shown in the lower-left corner of Fig.8. is considered. The load is assumed to act at the highest point of the free end section, as previously considered by Nguyen (2013).
5. Beam with Different Aspect Ratios

In order to study the effect of the aspect ratio, \( L/h \), the deflection of a cantilever FG beam subjected to a transverse tip load is computed with different values of the index \( n \) and the foundation parameters. Keeping all the data as above, and for \( L/h = 50, 25, 20, 15 \) and \( 10 \), the computation is performed with different values of the beam height, \( h = 0.1, 0.2, 0.25, 0.33 \) and \( 0.5 \) m, respectively.

Table 2 lists the deflection of the beam corresponding to a normalized load \( P/P_0 = 10 \) with various values of the aspect ratio. As seen in the Table, the deflection of the beam steadily increases when the aspect ratio is reduced, regardless of the index \( n \) and the foundation parameters. Examining the Table in more detail one can see that the effect of the aspect ratio is lessened by the presence of the elastic foundation, regardless of the index \( n \). For example, for \( n = 5 \) the difference between the deflections of the beam with an aspect ratio of 50 and 10 is 2.31% for the beam without the foundation support, but this value reduces to 1.13% for the beam resting on the Winkler foundation with \( k_1 = 50 \), and to 0.84% for the beam resting on the two-parameter elastic foundation with \((k_1, k_2) = (50, 0.5)\). The numerical result obtained in this Sub-section shows the ability of the formulated element to model the shear deformation effect on the large deflection behavior of the FG beams.

6. Conclusions

A finite element procedure for large displacement of an FG beam resting on a two-parameter foundation has been presented. The material properties of the beam are assumed to vary in the thickness direction by a power-law distribution. Based on the strain energy expressions, the shear deformable beam element was formulated and employed in the analysis.

The numerical results have shown that the derived formulation shows fast convergence, and the large displacements of the beams can be accurately assessed by using a few numbers of the elements. A parametric study has been carried out on the influence of the material non-homogeneity, the foundation support and
the aspect ratio on the large displacement behavior of FG beams. It can be deduced that although the examples were shown for the cantilever beam only, the finite element formulation of the present work was derived independently with the boundary conditions, and thus it is possible to analyze the FG beam with other boundary conditions also.

Acknowledgement
The financial support from Vietnam NAFOSTED to the second author is gratefully acknowledged.

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Table 2. Deflection at Free End of Cantilever FG Beam on Elastic Foundation Under a Transverse Tip Load with Different Aspect Ratios ($P/P_o = 10$)

| $(k_x, k_y)$ | $n$ | 50 | 25 | 20 | 15 | 10 |
|-------------|-----|----|----|----|----|----|
| (0,0)       | 0.2 | 0.8068 | 0.8081 | 0.8091 | 0.8113 | 0.8174 |
|             | 1.0 | 0.8387 | 0.8400 | 0.8410 | 0.8434 | 0.8505 |
|             | 5.0 | 0.8583 | 0.8608 | 0.8626 | 0.8666 | 0.8781 |
| (50,0)      | 0.2 | 0.7217 | 0.7223 | 0.7228 | 0.7238 | 0.7268 |
|             | 1.0 | 0.7579 | 0.7584 | 0.7588 | 0.7598 | 0.7629 |
|             | 5.0 | 0.7798 | 0.7809 | 0.7818 | 0.7836 | 0.7886 |
| (50,0.5)    | 0.2 | 0.6848 | 0.6853 | 0.6857 | 0.6865 | 0.6887 |
|             | 1.0 | 0.7240 | 0.7243 | 0.7246 | 0.7253 | 0.7275 |
|             | 5.0 | 0.7478 | 0.7486 | 0.7492 | 0.7505 | 0.7541 |
Appendix

Detail expressions for the nodal forces and the tangent stiffness matrices in Eq. (18) are as follows

\[ f_0 = A_{13} \bar{e}_0 \left\{ -c - s \frac{\ell}{2} \bar{\gamma} c s \frac{\ell}{2} \bar{\gamma} \right\}^T \]

\[ f_i = A_{23} \bar{e}_0 \left\{ 0 0 1 0 0 -1 \right\}^T \]

\[ f_i = A_{33} \bar{e}_0 \left\{ 0 0 1 0 0 -1 \right\}^T + A_{43} \bar{e}_0 \left\{ -c - s \frac{\ell}{2} \bar{\gamma} c s \frac{\ell}{2} \bar{\gamma} \right\}^T \]

\[ f_s = \psi A_{33} \bar{e}_0 \left\{ s - c - \frac{\ell}{2} (1 + \bar{\varepsilon}) - s c - \frac{\ell}{2} (1 + \bar{\varepsilon}) \right\}^T \]

\[ f_F = \frac{\ell}{4} k_w \left\{ w_i + w_j \right\} \left\{ 0 1 0 0 1 0 \right\}^T \]

\[ + \frac{1}{\ell} k_G \left\{ w_i - w_j \right\} \left\{ 0 1 0 0 -1 0 \right\}^T \]

\[ k_s = \begin{bmatrix} c^2 & s^2 & \text{sym.} \\ \ell a_1 & \ell a_2 & \frac{\ell^2}{4} a_3 \\ -c^2 & -sc & \frac{\ell}{2} a_1 c^2 \\ -sc & -s^2 & \frac{\ell}{2} a_2 sc & s^2 \\ \ell a_3 & -\ell a_2 & -\ell a_1 & \frac{\ell^2}{4} a_3 \end{bmatrix} \]

\[ k_F = \frac{\ell}{4} k_w \left\{ 0 1 \text{ sym.} \right\} \]

\[ k_F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \psi A_{43} \bar{e}_0 \left\{ 0 0 1 0 0 -1 \right\}^T + \frac{\ell}{2} A_{43} \bar{e}_0 \left\{ 0 0 c 0 0 s \right\} \]

\[ + \frac{\ell}{2} \bar{e}_0 \left\{ 0 0 \frac{\ell}{2} \bar{\gamma} 0 0 -\frac{\ell}{2} \bar{\gamma} \right\} \]

\[ + \frac{\ell}{2} A_{43} \bar{e}_0 \left\{ 0 0 c 0 0 -c \right\} \]

\[ + \frac{\ell}{2} A_{43} \bar{e}_0 \left\{ 0 0 \frac{\ell}{2} \bar{\gamma} 0 0 -\frac{\ell}{2} \bar{\gamma} \right\} \]

\[ + \frac{\ell}{2} A_{43} \bar{e}_0 \left\{ 0 0 c 0 0 -c \right\} \]

\[ + \frac{\ell}{2} A_{43} \bar{e}_0 \left\{ 0 0 \frac{\ell}{2} \bar{\gamma} 0 0 -\frac{\ell}{2} \bar{\gamma} \right\} \]

where

\[ s = \sin \bar{\theta} \quad c = \cos \bar{\theta} \]

\[ a_1 = (s\bar{\gamma} - c\bar{\varepsilon}) \quad a_2 = (c\bar{\varepsilon} + s\bar{\gamma}) \]

\[ a_3 = \bar{\gamma}^2 - \bar{\varepsilon}(1 + \bar{\varepsilon}) \quad a_4 = c\bar{\varepsilon} - s(1 + \bar{\varepsilon}) \]

\[ a_5 = s\bar{\gamma} + c(1 + \bar{\varepsilon}) \quad a_6 = (1 + \bar{\varepsilon})^2 - \bar{\gamma}^2 \]