THE JACOBI GROUP AND THE SQUEEZED STATES - SOME COMMENTS

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ABSTRACT. The generalized coherent states attached to the Jacobi group realize the squeezed states. Imposing hermitian conjugacy to the generators of the Jacobi algebra, we find out the form of the weight function appearing in the scalar product. We show effectively the orthonormality of the base functions with respect to the scalar product. From the explicit form of the reproducing kernel, we find out the expression of the multiplier in a holomorphic representation of the Jacobi group.

1. Introduction

In this note we continue our investigation of the properties of the Jacobi group started in [1, 2] using Perelomov’s generalized coherent states [3]. The Jacobi group [4, 5, 6] – the semidirect product of the Heisenberg-Weyl group and the symplectic group – is an important object in the framework of Quantum Mechanics, Geometric Quantization, Optics [7, 8, 9, 10, 11, 12, 13, 14]. The Jacobi group was investigated by physicists under other various names, as “Schrödinger group” [15] or “Weyl-symplectic group” [14]. The squeezed states [7, 8, 9, 10] in Quantum Optics represent a physical realization of the coherent states associated to the Jacobi group.

In [1] we have constructed generalized coherent states attached to the Jacobi group, $G^J_1 = H_1 \ltimes SU(1,1)$, based on the homogeneous Kähler manifold $\mathcal{D}_1^J = H_1/\mathbb{R} \times SU(1,1)/U(1) = \mathbb{C}^1 \times \mathcal{D}_1$. Here $\mathcal{D}_1$ denotes the unit disk $\mathcal{D}_1 = \{ w \in \mathbb{C} | |w| < 1 \}$, and $H_1$ is the $(2n+1)$-dimensional real Heisenberg-Weyl (HW) group with Lie algebra $\mathfrak{h}_1$. In [1] we have also emphasized the connection of our results with those of Berndt and Schmidt [5] and Kähler [16]. In [2] we have considered coherent states attached to the Jacobi group $G^J_n = H_n \ltimes Sp(2n, \mathbb{R})$, defined on the manifold $\mathcal{D}_n^J = \mathbb{C}^n \times \mathcal{D}_n$, where $\mathcal{D}_n$ is the Siegel ball.

In the present note we follow the notation and convention of [1]. If $\pi$ is a representation of a Lie group $G$ with Lie algebra $\mathfrak{g}$, then we denote $X = d\pi(X)$, $X \in \mathfrak{g}$. In [2] we recall the the main results on holomorphic representation in differential operators of the generators of the Jacobi algebra, while [3] summarizes the information on the space of functions on which the differential operators act. [4] contains our new results summarized in the abstract. The results contained in Proposition [5] have been announced in [17].

Key words and phrases. Squeezed states, Jacobi group.
2. The Jacobi algebra

The Jacobi algebra is defined as the semi-direct sum of the Lie algebra \( \mathfrak{h}_1 \) of the Heisenberg-Weyl Lie group and the algebra of the group \( SU(1,1) \).

The Heisenberg-Weyl ideal \( \mathfrak{h}_1 = \langle is1 + xa^\dagger - \bar{x}a \rangle_{s \in \mathbb{R}, x \in \mathbb{C}} \) is determined by the commutation relations

\[
[a, K_+] = a^\dagger; \quad [K_-, a^\dagger] = a; \quad [K_+, a^\dagger] = [K_-, a] = 0; \quad [K_0, a^\dagger] = \frac{1}{2} a^\dagger; \quad [K_0, a] = -\frac{1}{2} a,
\]

where \( a^\dagger (a) \) are the boson creation (respectively, annihilation) operators, which verify the canonical commutation relations

\[
[a, a^\dagger] = I, \quad [a, I] = [a^\dagger, I] = 0,
\]

and \( K_0, K_\pm \) are the generators of \( SU(1,1) \) which satisfy the commutation relations:

\[
[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0.
\]

We take a representation of \( G^I_1 \) (cf. [1] and Proposition 1 reproduced below) such that the cyclic vector \( e_0 \) fulfills simultaneously the conditions

\[
a e_0 = 0, \quad K_- e_0 = 0, \quad K_0 e_0 = k e_0; \quad k > 0,
\]

and we take \( e_0 = \varphi_0 \otimes \phi_{k_0} \). We consider for \( Sp(2, \mathbb{R}) \approx SU(1,1) \) the unitary irreducible positive discrete series representation \( D^+_k \) with Casimir operator \( C = K_0^2 - K_1^2 - K_2^2 = k(k-1) \), where \( k \) is the Bargmann index for \( D^+_k \) [18]. The orthonormal canonical basis of the representation space of \( SU(1,1) \) consists of the vectors

\[
\phi_{km} = \left[ \frac{\Gamma(2k)}{m!\Gamma(2k+m)} \right]^{1/2} (K_+)^m \phi_{k_0}, \quad m \in \mathbb{Z}._+
\]

Also, in the Fock space \( \mathcal{F} \), we have the orthonormality \( \langle \varphi_n', \varphi_n \rangle = \delta_{nn'} \), where \( \varphi_n = (n!)^{-\frac{1}{2}}(a^\dagger)^n \varphi_0 \).

Perelomov coherent state vectors associated to the group Jacobi \( G^I_1 \), based on the manifold \( \mathcal{D}^I_1 \), are defined as

\[
e_{z,w} = e^{za^\dagger + wK_+} e_0, \quad z, w \in \mathbb{C}, \quad |w| < 1.
\]

The general scheme [19, 20, 21] associates to elements of the Lie algebra \( \mathfrak{g} \) first order differential operators, \( X \in \mathfrak{g} \rightarrow X \in \mathcal{D}_1 \), and we have [1]

**Lemma 1.** The differential action of the generators of the Jacobi algebra is given by the formulas:

\[
a = \frac{\partial}{\partial z}; \quad a^\dagger = z + w \frac{\partial}{\partial z};
\]

\[
K_- = \frac{\partial}{\partial w}; \quad K_0 = k + \frac{1}{2} z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w};
\]

\[
K_+ = \frac{1}{2} z^2 + 2kw + zw \frac{\partial}{\partial z} + w^2 \frac{\partial}{\partial w},
\]

where \( z \in \mathbb{C}, \quad |w| < 1. \)
3. A multiplier representation of the Jacobi group

We introduce the displacement operator

\[ D(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a) = \exp(-\frac{1}{2} |\alpha|^2) \exp(\alpha a^\dagger) \exp(-\bar{\alpha} a), \]

and the unitary squeezed operator of the $D^k_z$ representation of the group SU(1, 1), $S(z) = S(w) (w = \frac{1}{2} \tanh(|z|), \eta = \ln(1 - |w|^2))$:

\[ S(z) = \exp(z K_+ - \bar{z} K_-), \quad z \in \mathbb{C}; \]
\[ S(w) = \exp(w K_+ \exp(\eta K_0) \exp(-\bar{w} K_-), \quad |w| < 1. \]

We introduce also [1] the generalized squeezed coherent state vector

\[ \Psi_{\alpha, w} = D(\alpha) S(w) e_0, \quad e_0 = \varphi_0 \otimes \phi_{k0}. \]

We introduce the auxiliary operators

\[ K_+ = \frac{1}{2} (a^\dagger)^2 + K'_+, \quad K_- = \frac{1}{2} a^2 + K'_, \quad K_0 = \frac{1}{2} (a^\dagger a + \frac{1}{2}) + K'_0, \]

which have the properties

\[ K'_- e_0 = 0, \quad K'_0 e_0 = k' e_0; \quad k = k' + \frac{1}{4}, \]

\[ [K'_\sigma, a] = [K'_\sigma, a^\dagger] = 0, \quad |\sigma| = \pm 0; \quad [K'_0, K'_\pm] = \pm K'_\pm; \quad [K'_-, K'_+] = 2K'_0. \]

We recall some properties of the coherent states associated to the group $G_1^*$ [1]:

**Proposition 1.** The generalized squeezed coherent state vector [9] and Perelomov coherent state vector [5] are related by the relation

\[ \Psi_{\alpha, w} = (1 - w\bar{w})^k \exp(-\frac{\bar{\alpha}}{2} \bar{z}) e_{z, w}, \text{where } z = \alpha - w\bar{w}. \]

Perelomov coherent state vector [5] was calculated in [1] as

\[ e_{z, w} = E(z, w) \varphi_0 \otimes e^{w K'_+} \phi_{k0}; \]

\[ E(z, w) \varphi_0 = e^{z a^\dagger + \frac{1}{2} (a^\dagger)^2} \varphi_0 = \sum_{n=0}^{\infty} \frac{P_n(z, w)}{(n!)^{1/2}} \varphi_n, \]

\[ P_n(z, w) = n! \sum_{p=0}^{[\frac{z}{2}]} \left( \frac{w}{2} \right)^p \frac{z^{n-2p}}{p!(n-2p)!}. \]

The base of functions $f_{nks}(z, w) = < e_{z, w}, \varphi_n \otimes \phi_{ks} >$, where $k = k' + 1/4$, $2k' = \text{integer}$, $n, s = 0, 1, \ldots, z, w \in \mathbb{C}, \quad |w| < 1$, consists of functions:

\[ f_{nks}(z, w) = \sqrt{\frac{\Gamma(s + 2k - 1/2)}{s! \Gamma(2k - 1/2)} \frac{w^n P_n(z, w)}{n!}}. \]
The composition law in the Jacobi group \( G_J^1 = HW \times SU(1, 1) \) is
\[
(g_1, \alpha_1, t_1) \circ (g_2, \alpha_2, t_2) = (g_1 \circ g_2, g_2^{-1} \cdot \alpha_1 + \alpha_2, t_1 + t_2 + \Im(g_2^{-1} \cdot \alpha_1 \bar{\alpha}_2)),
\]
where \( g^{-1} \cdot \alpha = \bar{a} \alpha - b \alpha \), and \( g \in SU(1, 1) \) is parametrized as
\[
g = \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right), \quad |a|^2 - |b|^2 = 1.
\]

Let \( h = (g, \alpha) \in G_J^1 \), \( \pi(h)_k = T(g)_k D(\alpha) \), \( g \in SU(1, 1), \alpha \in \mathbb{C} \), and let \( x = (z, w) \in D_J^1 = \mathbb{C} \times D_1 \). Then we have the formulas:
\[
\pi(h)_k \cdot e_{z,w} = (\bar{a} + \bar{b}w)^{-2k} \exp(-\lambda_1) e_{z_1,w_1},
\]
\[
2\lambda_1 = \frac{\bar{b}}{\kappa} \gamma^2 + \bar{a}(z + \gamma),
\]
\[
(z_1, w_1) = \left( \frac{\gamma}{\kappa}, \frac{aw + b}{\kappa} \right); \quad \kappa = \bar{a} + \bar{b}w; \quad \gamma = z + \alpha - \bar{\alpha}w.
\]

The space of functions \( \mathcal{H}_K \) attached to the reproducing kernel \( K(z, w; \bar{z}', \bar{w}') := (e_{z',w'}, e_{z,w'}) : D_J^1 \times \bar{D}_J^1 \to \mathbb{C} \):
\[
K(z, w; \bar{z}', \bar{w}') = (1 - w\bar{w}')^{-2k} \exp \frac{2z'z + z^2\bar{w}' + z^2w}{2(1 - w\bar{w}')},
\]
consists of square integrable functions with respect to the scalar product
\[
(f, g)_k = \Lambda \int_{z, w \in \mathbb{C}, |w| < 1} f(z, w)g(z, w)\rho(z, w)d^2z d^2w, \quad \Lambda = \frac{4k - 3}{2\pi^2},
\]
where
\[
\rho(z, w) = \rho_0(w)F(z, w),
\]
\[
\rho_0(w) = (1 - w\bar{w})^p, \quad p = 2k - 3,
\]
\[
F(z, w) = \exp \left[ -\frac{2|z|^2 + z^2\bar{w} + z^2w}{2(1 - |w|^2)} \right],
\]

4. Comments

Remark 1. If the generators of the Jacobi group \( G_J^1 \) have the differential realization given in [4], then the operator \( a (\mathbb{K}_-) \) is the hermitian conjugate of the operator \( a^1 \), (respectively, \( \mathbb{K}_+ \)), while the operator \( \mathbb{K}_0 \) is self adjoint with respect to the scalar product [24].
Proof.
Using integration by parts in the equations \((a f, g) = (f, a^\dagger g), (\mathbb{K}_- f, g) = (f, \mathbb{K}_+ g), (\mathbb{K}_0 f, g) = (f, \mathbb{K}_0 g)\) with respect to the scalar product (24), we get respectively the equations

\[(28a) \quad (w \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}) \rho = z \rho,\]
\[(28b) \quad \left( w^2 \frac{\partial}{\partial w} - \frac{\partial}{\partial \bar{w}} \right) \rho = \left( \frac{z^2}{2} + pw - zw \frac{\partial}{\partial z} \right) \rho,\]
\[(28c) \quad 2 \left( \frac{\partial}{\partial z} - \frac{\bar{w}}{w} \frac{\partial}{\partial \bar{w}} \right) \rho = \left( \frac{\bar{z}}{\partial \bar{z}} - \frac{z}{\partial z} \right) \rho.\]

But

\[(29a) \quad \frac{\partial \rho}{\partial w} = -2pw(1 - w\bar{w}) + \bar{z}^2 + 2\bar{w}|z|^2 + \bar{w}^2z^2 \frac{\rho}{2(1 - w\bar{w})^2},\]
\[(29b) \quad \frac{\partial \rho}{\partial z} = -\frac{\bar{z} + zw}{1 - |w|^2} \rho,\]

and it is check out that the function (25) - (27) verifies the conditions (28).

Remark 2. If \(w = 0\) in (28a), then we get the solution \(\rho(z) = cte^{-|z|^2}\) of the reproducing kernel for the coherent states associated o the Heisenberg-Weyl group, while if \(z = 0\) in (28b), (28c), we get the solution \(\rho(w) = c(1 - |w|^2)^p\) of the reproducing kernel for \(SU(1,1)\).

Starting from the Segal-Bargmann-Fock realization of the boson operators \(a \rightarrow \frac{\partial}{\partial z}, a^\dagger \rightarrow z\), Bargmann [22] has determined the reproducing kernel \(\rho(z, \bar{z}) = cte^{-|z|^2}\) from the relation \((zf, g) = (f, \frac{\partial g}{\partial \bar{z}})\). Now we apply his technique to the Jacobi group \(G^J_1\) and we obtain:

Proposition 2. If the generators of the Jacobi group \(G^J_1\) have the differential realization given in (6), then the conditions \((af, g) = (f, a^\dagger g), (\mathbb{K}_- f, g) = (f, \mathbb{K}_+ g), (\mathbb{K}_0 f, g) = (f, \mathbb{K}_0 g)\) with respect to the scalar product (24) impose to the function \(\rho\) to verify the equations (28), which admit the solution (25)-(27).

Proof. We consider for the functions \(\frac{\partial \rho}{\partial z}, \frac{\partial \rho}{\partial \bar{z}}\) the linear system of equations consisting of equation (28a) and his complex conjugate. It has as solution the equation (29b). Now we consider the linear system of equations (28b), (28c) in \(\frac{\partial \rho}{\partial w}\) and \(\frac{\partial \rho}{\partial \bar{w}}\). We introduce the solution (29b) for \(\frac{\partial \rho}{\partial z}\), and we obtain the equation (29a). But equation (29b) admits the solution

\[(30) \quad \rho(z, w, \bar{z}, \bar{w}) = \rho_0(w, \bar{z}, \bar{w})F(z, w),\]

with \(F(z, w)\) given by (27). Introducing (30) in (29a), we get the differential equation

\[
\frac{\partial \rho_0}{\partial w} = -\frac{p\bar{w}}{1 - w\bar{w}}\rho_0,
\]

which has the solution (26).

We shall verify explicitly that
Proposition 3. The system of vectors (17) is orthonormal with respect to the scalar product (24)-(27), i.e.

\[(f_{nks}, f_{mkr}) = \delta_{nm} \delta_{sr}, \quad k > \frac{1}{2}\]

Let now \(k \in \mathbb{R}, \quad |k| < \frac{1}{2}\). Let \(f = \sum_{nr} a_{nr} P_n w^r, \quad g = \sum_{ms} b_{ms} P_m w^s\). Then the scalar product (24)-(27) is replaced by

\[(f, g) = \sum_{n,r=0} \bar{a}_{nr} b_{nr} \frac{r! \Gamma(2k-1/2)}{\Gamma(r+2k-1/2)}.\]

The reproducing kernel (23) admits the series expansion

\[K(z, w; \bar{z}, \bar{w}') = \sum_{n,m} f_{nkm}(z, w) \bar{f}_{nkm}(z', w').\]

Proof. Firstly we proof the second assertion. We can write down [1]

\[P_n(z, w) = \left( \frac{i}{\sqrt{2}} \right)^n w^n H_n\left( \frac{-iz}{\sqrt{2}w} \right).\]

We use the summation formula (see, e.g. eq. (1.110) in [23])

\[(1-x)^{-q} = \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{\Gamma(q+m)}{\Gamma(q)}.\]

We use the summation relation of the Hermite polynomials (Mehler formula, cf. equation 10.13.22 in [24])

\[\sum_{n=0}^{\infty} (\frac{s}{2})^n H_n(x) H_n(y) = \frac{1}{\sqrt{1-s^2}} \exp \frac{2xys - (x^2 + y^2)s^2}{1-s^2}, \quad |s| < 1,\]

and we get the second part of the assertion of the Proposition.

Let us introduce the generating function

\[G_t(z, w) := \exp(zt + \frac{1}{2} wt^2) = \sum_{p,q \geq 0} \frac{z^p w^q}{p!q!} t^{p+2q} = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{q = 0}^{[n/2]} \frac{z^p w^q}{2^q q!(n-2q)!},\]

\[G_t(z, w) = \sum_{n \geq 0} \frac{t^n}{n!} P_n(z, w).\]

Now we calculate the scalar product (24)-(27) of two generating functions \(G_t\)

\[(G_t, G_t) = \Lambda \int_{z, w \in \mathbb{C}, |w| < 1} \rho_0(w) \exp[A(z, w)] d^2 z d^2 w,\]

where

\[A(z, w) = -\frac{2|z|^2 + z^2 \bar{w} + \bar{z}^2 w}{2(1-|w|^2)} + zt + z\bar{t} + \frac{1}{2} wt^2 + \frac{1}{2} \bar{w} t^2.\]

With the change of variables

\[y = \frac{z + tw - \bar{t}}{\sqrt{1-|w|^2}}; \quad (z = y \sqrt{1-|w|^2} - tw + \bar{t}),\]
we get successively
\[ A(z, w) = -|y|^2 - \frac{1}{2}(y^2 \bar{w} + \bar{y}^2 w) + |t|^2, \]
\[ (G_t, G_t) = \Lambda \exp(|t|^2) \int_{|w| < 1} \rho_0(w) I(w) d^2 w, \]
\[ I(w) = \int_C \exp[-|y|^2 - \frac{1}{2}(y^2 \bar{w} + \bar{y}^2 w)] d^2 y = \pi (1 - |w|^2)^{-1/2}, \]
\[ (G_t, G_t) = \pi \Lambda \exp(|t|^2) \int_{w \in \mathbb{C}, |w| < 1} (1 - |w|^2) \rho_0(w) d^2 w. \]
For any \( k' > 1/2 \) we denote \( G_{ts}(z, w) = G_t(z, w)^{w^s}, \) and we have
\[ (G_{ts}, G_{tr}) = \pi \Lambda \exp(|t|^2) \int_{w \in \mathbb{C}, |w| < 1} \bar{w}^s w^r (1 - |w|^2)^{2k - 3 + 1/2} d^2 w. \]
Now we change the variable: \( \Re(w) = \rho \cos \theta, \, \Im(w) = \rho \sin \theta \) and then we put \( \rho^2 = x. \)
We apply formula
\[ \int_0^1 t^{x-1}(1-t)^{y-1} dt = B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \, \Re x, \Im x > 0, \]
where \( x = r + 1, \, y = 2k - 3/2. \) Finally, we have
\[ (P^w_{m, w^p}, P^{m, w^p}) = \frac{r!\Gamma(2k - 1/2)}{\Gamma(r + 2k - 1/2)} \delta_{nm} \delta_{ps}, \]
i.e. equation (31).

**Remark 3.** If the reproducing kernel \( K \) is given by (23), then the multiplier in the representation (20) of the Jacobi group \( G_1^J \) has the expression (20)-(22).

**Proof.**
We recall the relations (cf. Prop. IV.1.9 p. 104 in [6])
\[ K(h \cdot x, h \cdot x') = J(h, x)K(x, x')J(h, x'), \]
where
\[ \pi(g)k f(x) = J(g^{-1}, x)^{-1} f(g^{-1} \cdot x), \]
and
\[ \pi(h)k \cdot e_x = \lambda(h, x)e_{x_1}, \, x = (z, w); \, x_1 = (z_1, w_1). \]
In the case of the Jacobi group, we get
\[ (e_{z_1, w_1}, e_{z'_1, w'_1}) = (1 - \bar{w}_1 w'_1)^{-2k} \exp E', \]
where
\[ E' = \frac{2\gamma' + \tilde{\gamma}}{\kappa' \kappa} + \frac{\bar{w} + \bar{b} + \tilde{w} + \tilde{b}}{\kappa' \kappa} + \frac{\bar{w} + \bar{b} \cdot \kappa' \kappa}{\kappa' \kappa}; \, 1 - \bar{w}_1 w'_1 = \frac{1 - \bar{w} w'}{\kappa \kappa'}. \]
We do the splitting
\[ E' = E + P + Q, \]
and, if we take
\[ P = \frac{1}{2} \left[ \frac{\bar{b}}{\kappa} \gamma'^2 + \frac{b}{\kappa} \gamma^2 + \bar{\alpha}(z' + \gamma') + \alpha(\bar{z} + \bar{\gamma}) \right], \]
then, we get \( Q = 0 \). We have
\[ K(h \cdot x, h \cdot x') = \kappa^{2k} \exp \left( \frac{1}{2} \left[ \frac{\bar{b}}{\kappa} \gamma'^2 + \bar{\alpha}(z' + \gamma') \right] \right) K(x, x') \exp \left( \frac{1}{2} \left[ \frac{b}{\kappa} \gamma^2 + \alpha(\bar{z} + \bar{\gamma}) \right] \right) \]

So for \( h = (g, \alpha), g \in SU(1, 1), x = (w, z) \), from (39), we get
\[ J(h, x) = (\bar{b}w + \bar{\alpha})^{2k} \exp \left( \frac{1}{2} \left[ \frac{\bar{b}}{\kappa} \gamma'^2 + \bar{\alpha}(z + \gamma) \right] \right). \]

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