SELFAcceleration of electrons in one-dimensional bunches, moving in cold plasma

A.Ts. Amatuni
Yerevan Physics Institute,
Alikhanian brothers st.2, Yerevan 375036,
Republic of Armenia

Abstract
Nonlinear dynamics of the one-dimensional ultrarelativistic bunch of electrons, moving in cold plasma, is considered in multiple scales perturbative approach. A square root of the inverse Lorentz factor of the bunch electrons is taken as a small parameter. Bunch electrons momenta is changed in the first approximation. In the underdense plasma and for the model example of the combined bunch the selfacceleration of the bunch electrons can be remarkable.

1 Introduction
Nonlinear wake waves exitation in overdense plasma by relativistic electron (positron) one-dimensional bunches, when it is possible to obtain an exact analytical solution, was considered in [1]-[7]. The bunch assumed as a given one (rigid bunch approximation) in the most of these work.

The bunch influence of the exited plasma wake on electron bunch was considered numerically in [8],[9]. Some attempts of the analytical treatment of the problem have been performed in [8]-[12].

In the present work the problem of the plasma back nonlinear influence on the driving one-dimensional electron bunch is treated by the method of multiple scales [13]. The bunch is ultrarelativistic and the square root of the inverse Lorentz factor of the bunch $\epsilon \equiv \gamma_0^{-1/2} \ll 1$ is taken as a small parameter.

It is assumed that the bunch-plasma interaction takes place by two stages. First one is the formation of the stationary plasma wake field, generated by the rigid bunch and the second stage is the influence on the momenta of the bunch electrons and wake field itself by this field. This assumption is valid, when $\gamma_0 \gg 1$. Indeed, the time $\tau_s$ needed for the formation of the stationary wake in a plasma with density $n_0$, generated by the electron bunch with the density $n_b$, $n_b/n_0 < 1/2 - \Delta, 1/8 \gamma_0^{-2} \ll \Delta < 1/2$, is $\tau_s \sim \omega_p^{-1}, \omega_p^2 = 4\pi e^2 n_0/m$. The time required for the change of the bunch electrons momenta $p_0$ is $\tau_p \sim$
\( p_0/eE_0 \) where \( E_0 \) is the electric field inside the bunch. In the overdense \((n_b/n_0 < 1/2)\) plasma \( E_0 \leq \frac{mc\omega_p}{e} \); in the underdense \((n_b/n_0 \ll n_0) \) plasma \( E_0 \leq \frac{mc\omega_p}{e} \gamma_0^{-1/2} \). Hence in the overdense plasma \( \tau_s/\tau_p \lesssim \gamma_0^{-1} \), and in the underdense plasma \( \tau_s/\tau_p \lesssim \gamma_0^{-1/2} \). (For some special values of \( n_b/n_0 \approx 1/2 \) these conditions can change the form or even violate.) As in \cite{5} consider the flat electron bunch with the infinite transverse dimensions, longitudinal length \( d \) and initial homogenous charge density \( n_b \), moving in the lab system with the initial velocity \( v_0 \) through neutral cold plasma with the immobile ions.

In the work \cite{14} it was shown that when the beam is traversing the semiinfinite plasma after a few plasma wave lengths transient effects dissipate and stationary wake field regime is established in coinsidenseness with the abovementioned estimate. In what follows this moment is taken as an initial one, \( t = 0 \), and the future development for \( t > 0 \) of the bunch - plasma system is considered.

## 2 Formulation of the Problem

The considered electron bunch - cold plasma system is described by the hydrodynamic equations of the motion, continuity equations for charge densities and currents for the bunch and plasma electrons, and Maxwell equation (Coulomb law) for the electric field.

Dimensionless variables and arguments are introduced:

\[
\begin{align*}
t' &= \omega_p t, \quad z' = k_p z, \quad \omega_p^2 = \frac{4\pi n m c^2 e^2}{n}, \quad k_p = \omega_p/c, \\
E &= (4\pi n m c^2)^{1/2} E', \quad n_e' = \frac{n_e}{n}, \quad n_b' = \frac{n_b}{n} \\
\rho_e &= \frac{p_e}{p_b}, \quad \rho_b = \frac{p_b}{m c \gamma_0}, \quad \beta_e = \frac{v_e}{c}, \quad \beta_b = \frac{v_b}{c}, \\
\gamma_0 &= (1 - \beta_0^2)^{-1/2},
\end{align*}
\]

where \( p_e, v_e, n_e; p_b, v_b, n_b \) are the momenta, velocity and density of the plasma and bunch electrons subsequently, \( n \) is a normalizing constant, which is suitable to choose \( n = n_0 \) for overdense plasma and \( n = n_b \) for underdense plasma cases, \( \beta_0 = v_0/c \) is the initial velocity of the bunch electrons in the laboratory system.

The equations, which describe the considered problem are:

\[
\begin{align*}
\frac{\partial p_e}{\partial t} + \beta_e \frac{\partial p_e}{\partial z} &= -E \tag{2} \\
\frac{\partial p_b}{\partial t} + \beta_b \frac{\partial p_b}{\partial z} &= -\frac{1}{\gamma_0} E \tag{3} \\
\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z} (\beta_e n_e) &= 0 \tag{4} \\
\frac{\partial n_b}{\partial t} + \frac{\partial}{\partial z} (\beta_b n_b) &= 0 \tag{5}
\end{align*}
\]
\[
\frac{\partial E}{\partial z} = n_0 - n_e - n_b
\] (6)

Eqs. (4-13) are written in dimensionless variables (1) and prime is omitted. Considering the ultrarelativistic bunch, introduce a small parameter \( \epsilon^2 = 1/\gamma_0 \), and all variables, entering in (2-6), let be functions of the fast \( \tilde{z} = z - \beta_0 t \) and slow \( \tau = \epsilon t, \zeta = \epsilon z \) arguments. According to the multiple scale method [13], all variables, entered in (2-6), are developed in the following series:

\[
\begin{align*}
\rho_e &= \rho_{e0}(\tilde{z}) + \epsilon \rho_{e1}(\tilde{z}, \zeta, \tau) + \epsilon^2 \rho_{e2}(\tilde{z}, \zeta, \tau) + \cdots \quad (7) \\
\beta_e &= \beta_{e0}(\tilde{z}) + \epsilon \beta_{e1}(\tilde{z}, \zeta, \tau) + \epsilon^2 \beta_{e2}(\tilde{z}, \zeta, \tau) + \cdots \quad (8) \\
\rho_b &= \rho_{b0} + \epsilon \rho_{b1}(\tilde{z}, \zeta, \tau) + \epsilon^2 \rho_{b2}(\tilde{z}, \zeta, \tau) + \cdots \quad (9) \\
\beta_b &= \beta_{b0} + \epsilon \beta_{b1}(\tilde{z}, \zeta, \tau) + \epsilon^2 \beta_{b2}(\tilde{z}, \zeta, \tau) + \cdots \quad (10) \\
n_e &= n_{e0}(\tilde{z}) + \epsilon n_{e1}(\tilde{z}, \zeta, \tau) + \epsilon^2 n_{e2}(\tilde{z}, \zeta, \tau) + \cdots \quad (11) \\
n_b &= n_{b0} + \epsilon n_{b1}(\tilde{z}, \zeta, \tau) + \epsilon^2 n_{b2}(\tilde{z}, \zeta, \tau) + \cdots \quad (12) \\
E &= E_{0}(\tilde{z}) + \epsilon E_{1}(\tilde{z}, \zeta, \tau) + \epsilon^2 E_{2}(\tilde{z}, \zeta, \tau) + \cdots \quad (13)
\end{align*}
\]

In (7-13) \( \rho_{b0}, \beta_{b0}, n_{b0} \) are the constants connected with the initially rigid electron bunch and differs from zero, when \( 0 \leq \tilde{z} = z - \beta_0 t \leq d \). The functions \( \rho_{e0}(\tilde{z}), \beta_{e0}(\tilde{z}), n_{e0}(\tilde{z}), E_{0}(\tilde{z}) \) are the solutions of the steady state (stationary) problem, and are obtained in [1], [3], [5]. Derivatives in (2-6), according to the multiple scales method are given by

\[
\begin{align*}
\frac{\partial}{\partial t} &= -v_0 \frac{\partial}{\partial \tilde{z}} + \epsilon \frac{\partial}{\partial \tau} \\
\frac{\partial}{\partial z} &= \frac{\partial}{\partial \tilde{z}} + \epsilon \frac{\partial}{\partial \zeta}
\end{align*}
\] (14)

Decompositions (7-14) correspond to the above mentioned main assumption, which, as it was seen, is valid for the ultrarelativistic bunch.

The following steps are evident—decomposition of the eqs. (2-3), using (7-14) provides the sequence of the quasilinear equations, describing the steady state regime as a zero approximation and in the next approximations—the back influence of the generated electric field on initialy rigid electron bunch and on wake wave itself.

### 3 Dynamics of the Bunch Electrons

It is necessary for the following to know the relations between \( \rho_{bi} \) and \( \beta_{bi} \) (8-14). From the definition (1) it follows that:

\[
\beta_b = \left(1 + \frac{\epsilon^4}{\rho_b} \right)^{-1/2} = \beta_0 + \epsilon^5 \beta_{b5} + \epsilon^6 \beta_{b6} + \epsilon^7 \beta_{b7} + \cdots \quad (15)
\]
It is essential to notice that in (15) the terms proportional to $\epsilon^1, \ldots, \epsilon^4$ are absent. Using (9) and (15) we have

$$\beta_{b5} = \frac{\rho_{b1}}{\beta_0^3} \quad (\rho_{b0} = \beta_0)$$

(16)

$$\beta_{b6} = -\frac{1}{2\beta_0^3} (3\rho_{b1}^2 - 2\rho_{b2}\rho_{b0})$$

(17)

$$\beta_{b7} = -\frac{1}{\beta_0^3} (3\rho_{b1}\rho_{b2} + 2\rho_{b1}^3 - \rho_{b3}\rho_{b0})$$

(18)

Zero ($\sim \epsilon^0$) and first order ($\sim \epsilon$) equations (3) for bunch electron momenta identically satisfied. The second order equations ($\sim \epsilon^2$) is

$$\frac{\partial \rho_{b1}}{\partial \tau} + \beta_0 \frac{\partial \rho_{b1}}{\partial \zeta} = -E_0(\tilde{z}),$$

(19)

where $E_0(\tilde{z})$ is the solution of the steady state problem presented in [5].

The characteristics of the eq. (19) are

$$\zeta - \beta_0 \tau = c_1; \rho_{b1} + E_0(\tilde{z})\tau = c_2;$$

Boundary condition is $\rho_{b1} = 0$, when $\tilde{z} = d$; initial condition is $\rho_{b1} = 0$, when $\tau = 0; \rho_{b1} \neq 0$, when $\epsilon d_0 < \zeta - \beta_0 \tau < \epsilon d$, where $d_0$ is the position of the end of the bunch for $t > 0; d_0 = 0$, when $t = 0$; the solution of the eq. (13) is

$$\rho_{b1} = -E_0(\tilde{z})\tau[\theta(\zeta - \beta_0 \tau - \epsilon d_0) - \theta(\zeta - \beta_0 \tau - \epsilon d)]$$

(20)

In the usual units eq. (20) reads

$$p_{b1} = -eE_0(\tilde{z})t\gamma_0^{1/2}[\theta(z - v_0 t - d_0) - \theta(z - v_0 t - d)]$$

(21)

The results (20-21) are valid for $\tau \leq 1$ i.e. $\omega_p t \leq \gamma_0^{1/2}$, or for the distances along the plasma column $l \sim ct \leq \frac{1}{2\pi^2} \lambda_p \gamma_0^{1/2}$. From (21) follows that in underdense plasma, when $E_0(\tilde{z}) \geq 0$ inside the bunch, the bunch electrons are deaccelerated, except the front electrons, which will always have the same velocity $v_0$, due to $E_0(d) = 0$. In the overdense plasma and for bunches longer than plasma nonlinear wave length (2)

$$z_\lambda = \frac{2v_0 (1 + \alpha)^{3/2}(1 + \alpha \beta_0)^{1/2}}{\omega_p \gamma_0} E\left(\frac{\pi}{2}, k\right)$$

(22)

the electrons in the head part of the bunch, where $d - z_\lambda/2 \leq z \leq d$, and $E_0(\tilde{z}) \geq 0$, are deaccelerated and in the rear part, where $d_0 \leq \tilde{z} < d - z_\lambda/2$ and $E_0(\tilde{z}) < 0$ are accelerated. (In (22) $\alpha = \frac{n_b/n_0}{1-n_b/n_0}$, $k = (\frac{2\alpha \beta_0}{1+\alpha \beta_0})^{1/2}$, $E\left(\frac{\pi}{2}, k\right)$ is the second kind complete elliptic integral). Hence, the bunch traversing the overdense plasma contracted around the point $\tilde{z} = d - z_\lambda/2$, and traversing the underdense plasma it always expands. The quantative
estimates of these changes will be given below. They are small in the considered ultra-relativistic bunch case.

The eqs. for the electron bunch momenta in the next approximations have the following form:

\[
\frac{\partial \rho_{b2}}{\partial \tau} + \beta_0 \frac{\partial \rho_{b2}}{\partial \zeta} = -E_1(\tilde{z}, \zeta, \tau) \tag{23}
\]

\[
\frac{\partial \rho_{b3}}{\partial \tau} + \beta_0 \frac{\partial \rho_{b3}}{\partial \zeta} = -E_2(\tilde{z}, \zeta, \tau) \tag{24}
\]

\[
\frac{\partial \rho_{b4}}{\partial \tau} + \beta_0 \frac{\partial \rho_{b4}}{\partial \zeta} = -E_3(\tilde{z}, \zeta, \tau) \tag{25}
\]

\[
\frac{\partial \rho_{b5}}{\partial \tau} + \beta_0 \frac{\partial \rho_{b5}}{\partial \zeta} = -E_4(\tilde{z}, \zeta, \tau) - \beta_{b5} \frac{\partial \rho_{b1}}{\partial \tilde{z}} \tag{26}
\]

The quantities \(E_i(\tilde{z}, \zeta, \tau), i = 1, 2, 3, 4\) entered in (23), (24) must be found as solutions of the subsequent system of the equations for plasma electrons, resulting from (23), (24).

4 Continuity Equation for the Bunch Electrons

The decompositions (14), (12) of the continuity eq (4) gives

\[
\left( \frac{\partial}{\partial \tau} + \beta_0 \frac{\partial}{\partial \zeta} \right) n_{bi} = 0 \quad i = 1, 2, 3 \tag{25}
\]

Initial conditions \(n_{bi} = 0, i = 1, 2, 3\), when \(\tau = 0\), are satisfied only when \(n_{bi}(\zeta - \beta_{b0} \tau) = 0, i = 1, 2, 3\). In the fifth order approximation, eq. for \(n_{b4}\) is

\[
\left( \frac{\partial}{\partial \tau} + \beta_0 \frac{\partial}{\partial \zeta} \right) n_{b4} = -n_{b0} \frac{\partial \beta_{b5}}{\partial \tilde{z}} \tag{26}
\]

or using (17), (21),

\[
\left( \frac{\partial}{\partial \tau} + \beta_0 \frac{\partial}{\partial \zeta} \right) n_{b4} = \frac{n_{b0}}{\beta_0^2} \frac{\partial E_0}{\partial \tilde{z}} \tau \left[ \theta(\zeta - \beta_0 \tau - \epsilon d_0) - \theta(\zeta - \beta_0 \tau - \epsilon d) \right] \tag{27}
\]

The solution of eq. (27), obeing the usual boundary and initial conditions, is

\[
n_{b4} = \frac{n_{b0}}{2 \beta_0^2} \frac{\partial E_0(\tilde{z})}{\partial \tilde{z}} \tau^2 \left[ \theta(\zeta - \beta_0 \tau - \epsilon d_0) - \theta(\zeta - \beta_0 \tau - \epsilon d) \right] \tag{28}
\]

From (28) follows, that for the underdense plasma, when \(\frac{\partial E_0(\tilde{z})}{\partial \tilde{z}} < 0\) everywhere inside the bunch, \(n_{b4} < 0\) and the bunch expands. For the overdense plasma \(n_{b4}\) changes sign: on the first quarter of the plasma wave length \(z = \lambda\) (22) it is negative, then positive on the second
and third quarters of the $z_\lambda$ and again negative on the forth quarter of the $z_\lambda$. Hence in the overdense plasma the bunch of the length $d \geq z_\lambda$ contracted on the middle of the bunch. Density modulation with the period equal $z_\lambda$ for the long bunches $d > z_\lambda$ will take place. (See also [8]).

Equations for the next approximations are

\[
\left( \frac{\partial}{\partial \tau} + \beta_0 \frac{\partial}{\partial \zeta} \right) n_{b5} = -n_{b0} \left( \frac{\partial \beta_{b6}}{\partial \tilde{z}} + \frac{\partial \beta_{b5}}{\partial \zeta} \right)
\]

(29)

\[
\left( \frac{\partial}{\partial \tau} + \beta_0 \frac{\partial}{\partial \zeta} \right) n_{b6} = -n_{b0} \left( \frac{\partial \beta_{b7}}{\partial \tilde{z}} + \frac{\partial \beta_{b6}}{\partial \zeta} \right)
\]

(30)

the solutions of the eqs. (29), using (16-18) with $\rho_{b2} = \rho_{b3} = 0$ (see below), are

\[
n_{b5} = \frac{n_{b0}}{\beta_0^3} \left[ E_0(z) \frac{\partial E_0}{\partial \tilde{z}} + 3 E_0(z) (\zeta - \epsilon d_0) \right] \times \left[ \theta(\zeta - \beta_0 \tau - \epsilon d_0) - \theta(\zeta - \beta_0 \tau - \epsilon d_0) \right]
\]

(31)

\[
n_{b6} = \frac{n_{b0}}{\beta_0^3} \left[ -\frac{3}{2} \beta_0 E_0^2(z) \frac{\partial E_0}{\partial \tilde{z}} + 3 E_0(\tilde{z}) (\zeta^2 - 2 \epsilon \zeta d_0 + \epsilon^2 d_0^2) \right] \times \left[ \theta(\zeta - \beta_0 \tau - \epsilon d_0) - \theta(\zeta - \beta_0 \tau - \epsilon d_0) \right]
\]

(32)

The second terms in the first square brackets in (30),(31) are absent for the cases, when bunch is contracted, $d_0 > 0$.

The boundary condition $n_{b5} = 0, p_{b6} = 0$ at $\tilde{z} = d$ means that the front of the bunch always moves with the constant initial velocity $v_0$ in the lab system (see also [11]). The end of the bunch, which was at $\tilde{z} = 0$, when $t = 0$, can be changed and be at $\tilde{z} = d_0 \neq 0$, when $t > 0$. From the bunch charge conservation it is possible to find $d_0(t)$ using the relation:

\[
n_{b0} d = \int_{d_0}^{d} \left( n_{b0} + \epsilon^4 n_{b4} + \epsilon^5 n_{b5} + \epsilon^6 n_{b6} + \cdots \right) d\tilde{z}
\]

(33)

An approximate expression for $d_0(t)$ follows from (32), using (28,30,31):

\[
d_0(t) = -\frac{t^2}{2 \gamma_0^2} E_0(0) - \frac{t^3}{2 \gamma_0^4} E_0^2(0) + \frac{t^4}{2 \gamma_0^6} E_0^3(0) + \frac{1}{\gamma_0^2} \int_0^d E_0(\tilde{z}) (\tilde{z} + v_0 t) d\tilde{z} + \frac{3}{\gamma_0^4} \int_0^d E_0^2(z) (z^2 + 2 \tilde{z} v_0 t + (v_0 t)^2) d\tilde{z}
\]

(33)

(in (33) all quantities are dimensionless, see [1]). The last two terms in (33) are absent, when $d_0 < 0$. From (33) it is evident that ultrarelativistic bunch practically does not change its length, passing through plasma during the time interval $t \leq \omega_p^{-1} \gamma_0^{1/2}$, up to terms of the order of $\gamma_0^{-2}$.
5 Dynamics of the Plasma Electrons

The plasma electron motion, described by eqs. (2), (4), (3) in the zero order approximation (fixed bunch, steady state regime) is considered in [3]. In the next-first, second and third-approximations, due to the solutions of eq. (23) \( n_{bi}(\zeta - \beta_0 \tau) = 0, i = 1, 2, 3 \), it is necessary to choose the solutions of eq. (2), (4), (3), which are zero inside the bunch for all values of \( \tau \):

\[ n_{ei} = 0, E_i = 0, \rho_{ei} = 0, i = 1, 2, 3. \]

The nonzero solutions, which are possible to obtain analytically for small \( |\rho_{e0}| \ll 1 \) and large \( |\rho_{e0}| \gg 1 \) values of plasma electron momenta and which satisfy the zero boundary and initial conditions, are physically meaningless.

System of the equations (2), (4), (3) in the forth order has the form

\[ \frac{\partial \rho_{e4}}{\partial z} = \frac{\rho_{e4} E_0(\tilde{z})(1 + \rho_{e0}^2)^{1/2}}{[\beta_0(1 + \rho_{e0}^2)^{1/2} - \rho_{e0}]^2} = E_4 \frac{(1 + \rho_{e0}^2)^{1/2}}{[\beta_0(1 + \rho_{e0}^2)^{1/2} - \rho_{e0}]^2}, \]

\[ \frac{\partial E_4}{\partial \tilde{z}} = -n_{e4} - n_{b4} = -n_0 \frac{\beta_0(1 + \rho_{e0}^2)^{1/2} \rho_{e4}}{[\beta_0(1 + \rho_{e0}^2)^{1/2} - \rho_{e0}]^2} - \frac{n_{b0} \partial E_0}{2 \beta_0^3 \beta_0} \beta_4 \frac{\partial E_0}{\partial \tilde{z}} \tau^2, \]

where results of zero order approximation [3], forth order continuity eq. (4), as well as relation \( \beta_{e4} = \rho_{e4}(1 + \rho_{e0}^2)^{-1/2} \) are used.

Quasilinear system of eqs. (24), (25) can be treated numerically. The analytical solution of the system (24), (25) is possible to obtain for small and large values of plasma electron momenta \( \rho_{e0}(0) \). First consider the case of the small values of \( |\rho_{e0}(\tilde{z})| \ll 1, \rho_{e0}(\tilde{z}) < 0 \), which take place behind the front of the bunch in the underdense and overdense plasma and around the \( \tilde{z} \sim z_\lambda \) in overdense plasma.

In the considered case the eqs. (24), (25) simplify due to relations, valid for \( |\rho_{e0}| \ll 1, \frac{\partial E_0(\tilde{z})}{\partial z} \approx -n_{b0}, E_0 \approx \pm \left[ \frac{2 \beta_0}{n_0} \beta_0 |\rho_{e0}| \right]^{1/2} \ll 1 \) and the resulting eq. for \( \rho_{e4} \) is:

\[ \frac{\partial^2 \rho_{e4}}{\partial \tilde{z}^2} + \frac{1}{\beta_0^2} n_{b0} + n_0) \rho_{e4} = \frac{1}{2 \beta_0^2} n_{b0}^2 \tau^2 \]

Solution of eq. (36) with the boundary conditions

\[ \rho_{e4}(\tilde{z} = d) = 0, \frac{\partial \rho_{e4}(\tilde{z} = d)}{\partial \tilde{z}} = 0 \]

which follows from \( E_4(\tilde{z} = d) = 0 \) is the following:

\[ \rho_{e4} = \frac{n_{b0}^2 \tau^2}{2 \beta_0 (n_0 + n_{b0})} \left[ 1 - \cos \left( \frac{n_{b0} + n_0}{\beta_0^2} \right)^{1/2} (d - \tilde{z}) \right] \]

(37) Then from (34), (35):

\[ E_4 \approx \beta_0 \frac{\partial \rho_{e4}}{\partial z} = -\frac{n_{b0}^2 \tau^2}{2 \beta_0 (n_0 + n_{b0})^{1/2}} \sin \left( \frac{n_0 + n_{b0}}{\beta_0} \right)^{1/2} (d - \tilde{z}) \]

(38)
for τ approximation for wake waves is valid up to terms, proportional to γ the forth order problem are: rear part of the long enough bunch for underdense plasma, the approximate solutions of at the end of the bunch (\( \tilde{z} = 0 \)), will be of the same order of magnitude, i.e. the rigid bunch approximation for wake waves is valid up to terms, proportional to γ_{0}^{-2}. Subsequent corrections to the wake wave behind the bunch, due to continuity condition at the end of the bunch (\( \tilde{z} = 0 \)), will be of the same order of magnitude, i.e. the rigid bunch approximation for wake waves is valid up to terms, proportional to γ_{0}^{-2}. For the case |\( \rho_{e0} \) | ≪ 1, which take place around \( \tilde{z} \approx \frac{\tilde{z}}{2} \) for overdense plasma and at rear part of the long enough bunch for underdense plasma, the approximate solutions of the forth order problem are:

\[
\rho_{e4} = -\frac{n_{b0} \tau^{2}}{2 \beta_{b0}^{2}} \int_{d}^{\tilde{z}} E_{0}(\tilde{z}) d\tilde{z}
\]

\[
E_{4} = -\frac{n_{b0} \tau^{2}}{2 \beta_{b0}^{2}} E_{0}(\tilde{z})
\]

\[
n_{e4} = -\frac{n_{0} n_{b0} \tau^{2}}{2(1 + \beta_{b0})^{2} \beta_{b0} |\rho_{e0}|} \int_{d}^{\tilde{z}} E_{0}(\tilde{z}) d\tilde{z}
\]

For the underdense case \( E_{0}(z) \approx |\rho_{e0}|^{1/2} \) and is large, so the quantities in (40) could be large too for \( \tau \leq 1 \); but the Lorentz factor dependence remains the same as in the case of small \( |\rho_{e0}| \ll 1 \).

Hence, only forth order corrections to plasma electron momenta, electron density and electric field inside and behind the bunch are differ from zero and are proportional to \( \gamma_{0}^{-2} \), in the considered ultrarelativistic bunch case.

6 Self-acceleration of the Bunch Electrons

The back influence of the plasma wake waves on the bunch electron momenta is calculated in the first approximation (see (24-27)) in the section 3. In the next approximations \( \rho_{bi} = 0, i = 2, 3, 4 \) due to \( E_{k} = 0, k = 1, 2, 3 \) (see section 5) and the initial condition \( \rho_{b0}(\tau = 0) = 0 \). From the sixth order equation it is possible to find from eqs. (20, 24, 39, 41) \( \rho_{b5} \) which differs from zero. For small \( |\rho_{e0}| \ll 1 \) and large \( |\rho_{e0}| \gg 1 \) it is possible to calculate \( \rho_{b5} = b(\tilde{z}) \tau^{3} \), where \( b(\tilde{z}) \) is a known function of \( \tilde{z} \). The subsequent contribution to bunch electron momenta is \( \epsilon^{5} \rho_{b5} = b(\tilde{z}) \gamma_{0}^{-4} t^{3} \), which is much smaller in considered case than first order contribution (21) \( \epsilon \rho_{b1} = -E_{0}(z) \gamma_{0}^{-1} t \), even when \( t \) approaches its limit value \( t < \omega_{p}^{-1} \gamma_{0}^{1/2} \).

Consider the first order correction (20, 21) in more detail. From (20, 21) in overdense plasma case the electrons of the rear part of the bunch with the length \( d > z_{\lambda}/2 \) accelerate and the increase of the momenta in the ordinary units is

\[
\triangle \rho_{b} = \rho_{b0} + \epsilon \rho_{b1} - \rho_{b0} = -e E_{0}(z) t
\]
The maximum of the electric field inside the bunch in overdense plasma case is \[E_{0}^{\text{max}} \approx \frac{mc}{e},\] and
\[c\Delta \rho_{b} = mc^{2}\omega_{p}t, \quad \frac{\Delta p_{b}}{\rho_{b}} = \frac{\omega_{p}t}{\gamma_{0}},\]
acceleration gradient
\[G = \frac{c\Delta p}{el} = \frac{mc^{2}}{e}\frac{2\pi}{\lambda_{p}} = \frac{\pi}{\lambda_{p}} \text{MeV/cm}, (l = ct).\]
If
\[t \leq \omega_{p}^{-1/2}\]
then
\[c\Delta p_{b} \leq 0, 5\gamma_{0}^{1/2} \text{MeV}, \quad \frac{\Delta p_{b}}{\rho_{b}} \leq \gamma_{0}^{-1/2},\]
for
\[cp_{b0} = 10\text{MeV}, \quad \gamma_{0} = 20, c\Delta p_{b} \leq 2, 23\text{MeV}, \frac{\Delta p_{b}}{\rho_{b}} \leq 22\%;\]
for
\[cp_{b0} = 100\text{MeV}, \quad \gamma_{0} = 200, c\Delta p_{b} \leq 7, 07\text{MeV}, \frac{\Delta p_{b}}{\rho_{b}} \leq 7\%.

For plasma densities \(n_{p} = 10^{13} \div 10^{15} \text{cm}^{-3}\), accelerating gradient is \(G \approx (1 \div 10) \frac{Mv}{cm}\).

It is possible to increase the value of selfaccelerated bunch electrons momenta using the combined bunch in the general form suggested by M.L. Petrosian (private communication).

First part of the bunch has a density \(n_{b}^{(1)} \gg \frac{1}{2}n_{0}\) (underdense case), and the density of the second part is \(n_{b}^{(2)} < \frac{1}{2}n_{0}\) (overdense case). The first part of the bunch has a length \(d^{(1)}, 0 \leq \tilde{z} \leq d^{(1)}\) and the second part \(d^{(2)}, -d^{(2)} \leq \tilde{z} \leq 0\). Then on the end of the first part of the bunch electric field in dimensionless variables is \([5]\):

\[\frac{1}{2}E_{0}^{2}(0) = (n_{b}^{(1)} - n_{0}) \left[ (1 + \beta_{0}^{2} \rho_{e0}(0))^{1/2} - 1 \right] - n_{b}^{(1)} \beta_{0} \rho_{e0}(0)\]

When \(\rho_{e0}(\tilde{z} = d) = 0, E_{0}(\tilde{z} = d) = 0\) and increases, when \(|\rho_{e0}|\) increases \((\rho_{e0} < 0)\). The largest value of \(|\rho_{e0}|\) can be estimated from total momenta conservation, which has the form:

\[n_{b}^{(1)} \rho_{e0} d^{(1)} \gamma_{0} \geq \int_{0}^{d_{1}} \rho_{e0}(\tilde{z}) n_{e0}(\tilde{z}) d\tilde{z} \approx \frac{n_{0} \beta_{0}}{1 + \beta_{0}} \frac{|\rho_{e0}^{\text{max}}| d^{(1)}}{2} \approx \frac{n_{0} d^{(1)} |\rho_{e0}^{\text{max}}|}{4}\]

It was taken into account, that \([5]\)

\[n_{e0}(\tilde{z}) = \frac{n_{0} \beta_{0} (1 + \beta_{0}^{2})^{1/2}}{\beta_{0} (1 + \beta_{0}^{2})^{1/2} - \rho_{e0}} \to \frac{n_{0} \beta_{0}}{1 + \beta_{0}}\]

\[d^{(1)} \approx \frac{n_{0} \beta_{0}^{2}}{1 + \beta_{0}^{2}} = \frac{n_{0} \beta_{0}}{2}\]
when $|\rho_{e0}| \gg 1;(n_{e0}\rho_{e0})$ dependence on $\tilde{z}$ is approximated as linear one. From (42) follows that

$$|\rho_{e0}^{\text{max}}| \leq \frac{4n_b^{(1)}\rho_{e0}\gamma_0}{n_0}$$

The electric field in the second part of the combined bunch, where $n_b^{(2)} < \frac{1}{2}n_0$ is given by the expression:

$$E_0(\tilde{z}) = \pm \left\{ E_0^2(0) + 2 \left( n_0 - n_b^{(2)} \right) \times \left[ \left( 1 + \rho_{e0}(0) \right)^{1/2} - \left( 1 + \rho_{e0}^{(2)} \right)^{1/2} \right] - 2n_b^{(2)}n_0(\rho_{e0} - \rho_{e0}(0)) \right\}^{1/2} = \pm \left[ 2 \left( n_0 - n_b^{(2)} \right) \left( a - \left( 1 + \rho_{e0}^{(2)} \right)^{1/2} - \alpha^{(2)}\beta_{0}\rho_{e0}(0) \right) \right]^{1/2},$$

where

$$\alpha^{(2)} = \frac{n_b^{(2)}}{n_0 - n_b^{(2)}},$$

$$a \equiv 1 + \frac{n_b^{(1)} - n_b^{(2)}}{n_0 - n_b^{(2)}} \left[ \left( 1 + \rho_{e0}(0) \right)^{1/2} - 1 \right] - \frac{n_b^{(1)} - n_b^{(2)}}{n_0 - n_b^{(2)}} \beta_{0}\rho_{e0}(0);$$

$a \to 1$, when $\rho_{e}(0) \to 0$, $a > 1$, when $\rho_{e}(0) < 0$, and

$$a \approx \frac{n_b^{(1)} - n_b^{(2)}}{n_0 - n_b^{(2)}} (1 + \beta_0)|\rho_{e0}| \gg 1,$$

when $|\rho_{e0}(0)| \gg 1$.

Electric field $E_0(\tilde{z})$ (43) is equal zero at

$$\rho_{\tilde{z}} = -\frac{\alpha^{(2)}\beta_0}{1 - (\alpha^{(2)}\beta_0)^2} \pm \left[ \left( \frac{\alpha^{(2)}\beta_0}{1 - (\alpha^{(2)}\beta_0)^2} \right)^2 + \frac{a^2 - 1}{1 - (\alpha^{(2)}\beta_0)^2} \right]^{1/2}$$

and when $|\rho_{e0}(\tilde{z})| > |\rho_{-}|$ ($\rho_{-}$ is the root of (44) with minus sign in (47)) the bunch electron selfacceleration can take place in the region of the bunch, where $E_0(\tilde{z}) < 0.(E_0(\tilde{z}) < 0$, when $\rho_{e0}(\tilde{z})$ decreases with increasing $\tilde{z}$ [4]. Field $|E_0(\tilde{z})|$ (44) has a maximum value, when

$$\rho_{e} = \rho_{e}^{\text{max}} = -\frac{\alpha^{(2)}\beta_0}{1 - (\alpha^{(2)}\beta_0)^2} \frac{1}{2}$$

For large value of $|\rho_{e0}(0)| \gg 1$, $a \gg 1$ from (46) $\rho_{-} \approx -\frac{a}{1 - \alpha^{(2)}\beta_0}$ and conditions $|\rho_{-}| > |\rho_{e0}(0)|$, $|\rho_{-}| > |\rho_{e}^{\text{max}}|$ are always fulfilled. As a consequence, $E_0(\tilde{z})$ in the second part of the combined bunch decreases from $E(0) > 0$ to zero, then becomes negative, reaches its minimum value (it modulus is maximum) and then tends to zero at $\tilde{z} = -\tilde{z}_{\lambda}^{(2)}$. Maximum value of the modulus of the electric field is

$$E_0^{\text{max}} \approx \left[ 2(n_0 - n_b^{(2)})a \right]^{1/2} \approx \left[ 4(n_b^{(1)} - n_b^{(2)})|\rho_{e0}(0)| \right]^{1/2}$$

$$\text{(49)}$$
Unfortunately, the plasma electron momenta largest value \([14]\) is overestimated. More restrict estimate follows from the condition \(n_e(\tilde{z}) \geq 0\) \([13]\). It is so for any \(\rho_{e0} < 0\), which is the case everywhere inside the bunch, but in wake field \(\rho_{e0}\) can be positiv too, and from \(n_{e0}(\tilde{z}) > 0\) follows that \(0 \leq \rho_{e0}(\tilde{z}) \leq \beta \gamma_0\) instead of condition \([14]\). If \(\rho_{e0}(\tilde{z}) > \beta \gamma_0\), \(n_e\) became negative, which means that basic assuption on the existence of the steady state regime is violated and this case needs the development of a new approach, which is nonstationary from the beginning.

It is possible to see that in wake field

\[
- \left[ \left( \frac{1}{n_0} A \right)^2 - 1 \right]^{-1/2} \leq \rho_{e0}(\tilde{z}) \leq \left[ \left( \frac{1}{n_0} A \right)^2 - 1 \right]^{1/2} \tag{50}
\]

and subsequently

\[
1 \leq \left( \frac{1}{n_0} A_0 \right) \leq \gamma_0, A = (n - n_b^{(2)}) a + n_b^{(2)} [1 + \rho_{e0}(-d_2)]^{1/2} - n_b^{(2)} \beta_0 \rho_{e0}(-d_2), \tag{51}
\]

where "\( a \)" is given by \([15]\) and \([46]\) for large value of \(|\rho_{e0}(0)| \gg 1\). If

\[
|\rho_{e0}(-d_2)| < 1, n_b^{(1)} \gg n_b^{(2)}
\]

\[
\frac{1}{n_0} A \approx 2 \frac{n_b^{(1)}}{n_0} |\rho_{e0}(0)| \leq \gamma_0 \tag{52}
\]

and

\[
E_{0}^{\text{max}} = [4(n_b^{(1)} - n_b^{(2)})|\rho_{e0}|]^{1/2} \leq (2n_0\gamma_0)^{1/2} \tag{53}
\]

The increase of the bunch electron momenta and acceleration gradient in the ordinary units then are

\[
\frac{\Delta p_b}{p_b} \leq \left( \frac{2 \gamma_0}{\rho_0} \right)^{1/2} \omega_p t < (2)^{1/2} \tag{54}
\]

\[
G = \frac{c \Delta p_b}{e l} \leq \frac{2\pi (2pcp_0mc^2)^{1/2}}{\lambda_p} e^{-1}, l = ct. \tag{55}
\]

In \([54], [55]\), it is taken into account that these relations are valid for \(\omega_p t \leq \gamma_0^{1/2}\) and subsequent length of the plasma column must be \(l \sim ct \leq \lambda_p \gamma_0^{1/2}\). So the total increase of the momenta is not exited the initial value of that, but acceleration gradient increases proportionally to \(\gamma_0^{1/2}\) and for example for \(n_0 = 3 \cdot 10^{13} \div 3 \cdot 10^{15} \text{cm}^{-3}\) and for \(\gamma_0 = 20, 0(\mathcal{E}_0 = 10, 0 \text{Mev}); \gamma_0 = 200, 0(\mathcal{E}_0 = 100, 0 \text{Mev}); \gamma_0 = 2000, 0(\mathcal{E}_0 = 2 \text{Gev}), G \text{ is } G = (31, 5 \div 315)\frac{M}{cm}; (100 \div 1000)\frac{M}{cm}; (223 \div 2230)\frac{M}{cm}; \text{ subsequently.}

Selfacceleration of the electrons from the second part of the combined bunch is accompanied by deacceleration at the same extend the electrons from the rear of the first
part of the combined bunch. In the both cases the change of the velocities according to \((15,16,20,53)\) is negligible for \(\gamma \gg 1\)

\[
\beta_b = \beta_0 \pm \gamma_0^{-2}(2n_0)^{1/2}
\]

Hence, the combined bunch will not change essentially its shape and density distribution during the time of selfacceleration. The estimate \((53)\) of the maximum electric field on the end of the first part of combined bunch is true for long enough length \(d_1\). Using results obtained in \([15]\) for \(|\rho_0| \gg 1\) and \(1 \ll n^{(1)}_b \ll 2\gamma_0^2\), it is possible to estimate the length \(d_1\), which in ordinary units is

\[
d_1 \leq \frac{\lambda_p n_0}{2\pi n^{(1)}_b \gamma_0^{1/2}}
\]

For

\[
\gamma_0 = 20, 200, 2000; n_0 = 3 \cdot 10^{11}; n_b = 3 \cdot 10^{12}\text{cm}^{-3};
\]

\[
d_1 \leq 0, 07\lambda_p = 0, 44\text{cm}; 0, 22\lambda_p = 1, 38\text{cm}; 0, 7\lambda_p = 4, 4\text{cm}
\]

subsequently.

It is useful to mention that the considered mechanism of selfacceleration has much in common with the selfacceleration of electrons in passive resonant systems (resonant cavities, wave guides) which was widely investigated both theoretically and experimentally in the seventies (see e.g. \([16]\), \([17]\) and review in \([18]\); review of more recent results is presented in \([19]\)).

7 Conclusion

The multiple scale perturbative approach \([13]\) is applied to the one-dimensional problem of the buck influence of the plasma wake field on bunch electrons and wakes itself for the ultrarelativistic bunch. Inverse square root of the bunch electrons Lorentz factor is chosen as a small parameter \(\epsilon = \gamma_0^{-1/2}\). It is shown that plasma wake fields characteristics experiend a perturbations only in the forth order terms, proportional to \(\epsilon^4 = \gamma_0^{-2}\), so are small for considered ultrarelativistic case. It means that rigid bunch approximation is valid for ultrarelativistic bunch up to the terms, proportional to \(\gamma_0^{-2}\).

The momenta of the bunch electrons changes in the first order approximation, eqs. \((20,21)\), density and bunch length in the forth order, eqs. \((28,32,33)\). The bunch traversing the overdense plasma contracted around the point \(\bar{z} = d - z_\lambda/2\), and expands when traverses underdense plasma.

The change in the bunch electrons momenta is more essential. In the cases of underdense plasma \((n_b < \frac{1}{2}n_0)\) or in the model example considered a combined bunch (first part of the bunch with the uniform density \(n^{(1)}_b \gg \frac{1}{2}n_0\), the second part with the density \(n^{(2)}_b \gg \frac{1}{2}n_0\)) the remarkable selfacceleration of the bunch electrons can take place.
Predicted selfacceleration can be tested experimentally and it may be of the practical importance. It is worthwhile to mention, that Langmuir already noticed, that the beam, which has passed the plasma column, contains a significant portion of electrons with energies higher than the initial energy.

In the recent times various groups (see e.g. [20],[21],[22]) observed the effect of self-acceleration experimentally.

It is possible to hope, that the present work will stimulate more systematic study of the selfacceleration of the electrons of the bunches, moving in plasma.

As it follows from the presented consideration the different approaches to the problem out of the presented frame of steady state regime taken as a first order approximation, may offer a new possibility to selfaccelerate the electrons of the bunches, moving in plasma.

8 Acknowledgement

Author is indebted to M.L. Petrosian, S.G. Arutunian, S.S. Elbakian, A.G. Khachatrian, E.V. Sekhpossian for helpful discussions, suggestions and comments.

References

[1] Amatuni A.Ts., Magomedov M.R., Sekhpossian E.V., Elbakian S.S. Physika Plasmi 5,85(1979) (Sov. J. Plasma Physics 5,49(1979)).

[2] Ruth R.D., Chao A.W., Morton P.L., Wilson P.B. Part. Acc. 17,171,(1958).

[3] Amatuni A. Ts., Sekhpossian E.V., Elbakian S.S. Physica Plasmi 12,1145,(1986).

[4] Rosenzweig J.B. Phys. Rev. 58,555,(1987).

[5] Amatuni A. Ts., Sekhpossian E.V., Elbakian S.S., Abramian R.D. Part. Acc. 41,153,(1993)

[6] Bilikmen S., Nasih R.M. Physica Scripta 47,204,(1993).

[7] Bazylev V.A., Golovin V.V., Tulupov A.V., Schep T.J., van Amersfoot Proc. EPAC-94, London, 1994.

[8] Balakirev V.A., Sotnikov G.V., Fainberg Ja.B., Physica Plasmi 22,165,(1996).

[9] Rukhadze A.A., Bogdankevich L.S., Rosinsky S.E., Rukhlin V.G. ”Physics of High Current Relativistic Beams”, Atomizdat, M. 1980 (in Russian)

[10] Kovtun R.I., Rukhadze A.A. JTEP 58,1709,(1970)

[11] Kovalenko V.P., Pergamentshik V.M., Starkov V.N., Physika Plasmi, 11,417,(1985)
[12] Amatuni A. Ts., Sekhpossian E.V., Elbakian S.S. 
Proc. XIII Int. Conf. on High Energy Particles Acc., Novosibirsk, v.1, p. 175, (1987)

[13] Nayfeh A.H. ”Perturbation Methods” ch. 6, J. Willey and Sons Inc., 1973

[14] Mtingwa S.K. FNAL report, FN-452,0104.000 April, 1987

[15] Amatuni A. Ts., Sekhpossian E.V., Elbakian S.S. YerPhi-1366(60)-91, Yerevan, 1991

[16] Kazansky L.N., Kisletsov A.V., Lebedev A.N., Atomnaya Energia 30, 27, 1971

[17] Serov V.L., Barishev A.I. Izv. AN Armenian SSR, Ser. Phys. 7, 406, (1972)

[18] Didenko A.N., Grigoriev V.P., Usov Yu. P. 
”Powerful electron beams and their application”, 
ch. 5, Atomizdat M. (1977) (in Russian)

[19] Amatuni A. Ts., Elbakian S.S., Laziev E.M. et al. 
Fizika elementarnikh chastits i atomnogo jadra, 
Dubna 20, 1246, (1989). Part. Acc. 32, 221, (1990).

[20] Nezlin M.V. ”Beam Dynamics in Plasma”, Energoizdat M., 1982 (in Russian)

[21] Fainberg Yu. B. Physika Plasmi 11, 1398, (1985)

[22] Berezin A.K., Kiselev V.A., Fainberg Yu. B. Ukrainen Physical Journal 24, 94, (1979)