Abstract

In quantum theory, real degrees of freedom are usually described by operators which are self-adjoint. There are, however, exceptions to the rule. This is because, in infinite dimensional Hilbert spaces, an operator is not necessarily self-adjoint even if its expectation values are real. Instead, the operator may be merely symmetric. Such operators are not diagonalizable - and as a consequence they describe real degrees of freedom which display a form of “unsharpness” or “fuzzyness”. For example, there are indications that this type of operators could arise with the description of space-time at the string or at the Planck scale, where some form of unsharpness or fuzzyness has long been conjectured.

A priori, however, a potential problem with merely symmetric operators is the fact that, unlike self-adjoint operators, they do not generate unitaries - at least not straightforwardly. Here, we show for a large class of these operators that they do generate unitaries in a well defined way, and that these operators even generate the entire unitary group of the Hilbert space. This shows that merely symmetric operators, in addition to describing unsharp physical entities, may indeed also play a rôle in the generation of symmetries, e.g. within a fundamental theory of quantum gravity.
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1 Introduction

As a rule, real entities, or “real degrees of freedom”, are described in quantum theory through operators which are self-adjoint. There are, however, exceptions to the rule. A common feature of these exceptional degrees of freedom is that they display a form of “unsharpness” or “fuzzyness”.

To see this, let us first recall that if an observable is described, as usual, through a self-adjoint operator, then the observable is absolutely sharp in the sense that, in principle, it can be measured to arbitrarily fine resolution. This is because every self-adjoint operator possesses a spectral resolution, or “eigenbasis”. Indeed, if the system is in a $Q$-eigenstate, say $|q_n\rangle$, then the $Q$-uncertainty

$$\Delta Q(|\psi\rangle) := \langle\psi| (Q - \langle\psi|Q|\psi\rangle)^2|\psi\rangle^{1/2}$$

(1)

(for $|\psi\rangle$ normalized) vanishes:

$$\Delta Q(|q_n\rangle) = \langle q_n|(Q - \langle q_n|Q|q_n\rangle)^2|q_n\rangle^{1/2} = 0$$

(2)

Also, eigenvectors to different eigenvalues are orthogonal. Thus, if the system is localized with respect to the observable $Q$ around some value $q$ then the probability for finding it localized around any other value $q'$ vanishes. Of course, if an “eigenvalue” $q$ is in the continuous spectrum then the “eigenstate” $|q\rangle$ is nonnormalizable and must be approximated by a sequence of normalizable states. But also in this case, the uncertainty $\Delta Q$ can be made arbitrarily small, and two eigenstates to different points in the spectrum are orthogonal, then with respect to the continuum normalization.

Thus, as is well-known, any self-adjoint observable $Q$ is “sharp” in the sense that its states of maximal $Q$-localization - the eigenstates $\{|q\rangle\}$ - are orthogonal if localized around different values of $q$, and in the sense that each maximal localization state has vanishing uncertainty $\Delta Q$.

The reason why unsharp real degrees of freedom can occur is that, a priori, a real entity in a quantized theory may correspond to any operator whose expectation values are real - and, crucially, operators whose expectation values are real need not be self-adjoint:

We recall that an operator whose expectation values are real is a symmetric operator. In finite dimensional Hilbert spaces, symmetric operators are self-adjoint and self-adjoint operators are symmetric. But in infinite dimensional Hilbert spaces, symmetric operators need not be self-adjoint. This also means that they need not possess a spectral resolution and diagonalization. As a consequence, an operator $Q$ whose expectation values are real may also describe a degree of freedom which is fuzzy or unsharp, in the sense that the minimum value for $\Delta Q$ may be larger than zero, and/or those vectors which realize the minimum value for $\Delta Q$ may not be orthogonal.
A simple example in nonrelativistic quantum mechanics is the momentum operator \( p \) of the particle in a box. For simplicity, let us consider the case in one dimension, where the particle is confined, e.g., into the interval \([-L, L]\):

We recall that all physical wave functions \( \psi(x) \) vanish at the boundary of the box and that the momentum operator \( p \) acts on physical wave functions as the derivative operator \( p = -i\hbar \partial_x \). Indeed, \( p \) is an example of a merely symmetric operator: Clearly, all expectation values of \( p \) are real, and therefore \( p \) is symmetric. On the other hand, \( p \) is not self-adjoint. This is because plane waves do not vanish at the boundaries and they are therefore not physical states. This means that plane waves are not in the domain of \( p \), which implies that \( p \) has no eigenbasis and no spectral resolution.

As a consequence, \( p \) is indeed unsharp: From the uncertainty relation, since \( \Delta x \) is surely smaller than \( 2L \), we can expect that the minimum uncertainty in momentum is larger than zero. Indeed, the precise minimum value for \( \Delta p \) is \( \Delta p_{\text{min}} = \pi \hbar / 2L \). We will discuss this example in more detail below.

In general, for example in any candidate theory for a fundamental theory of quantum gravity, it appears reasonable\(^1\) to assume that entities which are described in the classical theory through real variables are described in the quantized theory through operators which are linear and whose expectation values are real. Of course, we cannot assume that all those operators are observables in the usual quantum mechanical sense, nor that they even act on the space of states. Instead, these operators may act on some Hilbert space of fields, or branes (as we will briefly discuss below), or indeed on any abstract Hilbert space.

On this level of generality we can only say that while some real degrees of freedom may be described as self-adjoint operators, others may be described by merely symmetric operators - and that correspondingly the real physical entities which they correspond to are “sharp” or “unsharp”. Interestingly, however, there exists only a limited number of types of sharpness or unsharpness, or “short-distance structures”, which can occur with operators whose expectation values are real, i.e. with symmetric operators. A classification has been outlined in \([1]\): Since the class of symmetric operators includes the self-adjoint operators, two of the possible short distance structures are lattices and continua, corresponding to the fact that self-adjoint operators occur with discrete and continuous spectra. The other extreme are the purely unsharp cases. These are described by the class of simple symmetric operators. Those are operators which are symmetric but not self-adjoint, not even on any invariant subspace. There exist subclasses of these operators which describe different types of unsharpness. In \([1]\), they have been divided into two broad classes, fuzzy-A and fuzzy-B. Technically, the two classes correspond the two possibilities of the deficiency indices being equal or unequal.

In the present paper, we are concerned with those operators which describe entities\(^1\) We will here not consider the alternative possibility of nonlinear operators.
that are unsharp of type fuzzy-$A$. Mathematically, these are the simple symmetric operators with equal deficiency indices. For example, the momentum operator of the particle in a box is of the type fuzzy-$A$.

There are indeed theoretical indications that short-distance structures of the type fuzzy-$A$ occur with the description of space-time at the Planck scale:

Various theoretical arguments have long indicated that space-time displays a fundamental “foaminess”, see [2], or unsharpness at very small lengths. In particular, several studies, see e.g. [3]-[11], suggest that the structure of space-time at the Planck scale, or the string scale, is characterized effectively by correction terms to the uncertainty relations and, in particular, by corrections of the type:

$$\Delta X \Delta P \geq \frac{\hbar}{2} (1 + k (\Delta P)^2 + ...)$$  \hspace{1cm} (3)

As is easily verified, for any $k > 0$, Eq.(3) implies the existence of a finite lower bound for $\Delta X$, namely:

$$\Delta X_{\min} = \hbar \sqrt{k}$$  \hspace{1cm} (4)

Here, $k$ is assumed to be a small positive constant which is related to the Planck scale, or in string theory to the string scale. We here only remark that recent studies (on large extra dimensions that are seen by gravity only) suggest that the unification and/or the Planck scale may even be as low as the TeV scale, see e.g. [12].

A positive minimum uncertainty $\Delta X_{\min} > 0$ arising from uncertainty relations of the type of Eq.(3) can be introduced as an ultraviolet cutoff in quantum field theories [13, 14]. It has also been shown that this type of cutoff may solve the transplanckian energy paradox of black hole radiation, see [15]. For general reviews of quantum gravity and string theory motivations of Eq.(3), see e.g. [16, 17]. For a recent discussion of the potential origins of Eq.(3) see e.g. [18], and for a path integral approach to modified uncertainty relations see [19].

Technically, it is clear that any operator $X$ which obeys an uncertainty relation of the type of Eq.(3) cannot possess eigenvectors, since the uncertainty $\Delta X$ would vanish for eigenvectors. Therefore, any such operator $X$ can only be symmetric but not self-adjoint. More precisely, it must be of the type fuzzy-$A$, as was first shown in [1]. We remark that operator realizations and the functional analysis of uncertainty relations of the type of Eq.(3) were first discussed in [20].

On the other hand, if we are to study those cases in which a real degree of freedom is represented not by a self-adjoint but instead by a merely symmetric operator, then we must also address the fact that self-adjoint operators often play two rôles, namely both as real degrees of freedom and also as generators of symmetries. Therefore, the question arises, whether, or how, merely symmetric operators could also be involved in the generating of symmetries. Indeed, it is known that there is an important difference in this respect between self-adjoint and merely symmetric operators. Namely, merely
symmetric operators, unlike self-adjoint operators, do not generate unitaries, at least not directly.

In the present paper, we therefore consider fuzzy-A type operators with respect to the generation of unitary transformations - and we will find that these operators possess a remarkable property:

By definition, the fuzzy-A type operators are those operators $Q$ which on the physical domain $D_Q = D_{phys}$ are simple symmetric with equal deficiency indices. For each such operator there exists a family of operators $\{Q(\alpha)\}$ which coincide with $Q$ on the physical domain $D_Q$ and which are self-adjoint. The $Q(\alpha)$ therefore generate unitaries in the usual way. We claim that these $Q(\alpha)$ - we recall that they all coincide with $Q$ on the physical domain - generate, together, all unitary operators in the Hilbert space. This shows that, in this way, operators of the type fuzzy-A can indeed relate to all aspects of symmetries in the Hilbert space on which they act.

We will also find that this result supports a conjecture made in [1, 21]. The conjecture proposes a mechanism by which those small wavelengths which are being cut off in the case of a fuzzy-A short-distance structure effectively turn into internal degrees of freedom with an isospinor structure on which unitary groups act.

2 Examples of Unsharp Degrees of Freedom

Before we discuss the theorem, let us introduce concrete examples of simple symmetric operators to which the theorem will apply.

2.1 The momentum of the particle in a box

We have already mentioned the example of the momentum operator $p = -i\partial_x$ of the particle in a box (from now on we set $\hbar = 1$). Since we will later use this example also to illustrate the new theorem on generating symmetries, let us discuss this case in more detail:

Assume the box to be the one-dimensional interval $[-L, L]$. Due to the confining box potential, all physical wave functions $\psi(x) \in D_{phys} \subset H = L^2(-L, L)$ vanish at the boundary:

$$\psi(-L) = 0 = \psi(L)$$  \hfill (5)

The expectation values of $p$ are real:

$$\langle \psi | p | \psi \rangle \in \mathbb{R}, \quad \text{for all} \quad |\psi\rangle \in D_{phys}$$  \hfill (6)

Thus, $p$ is a symmetric operator. On the other hand, since no plane wave obeys the boundary condition, Eq.5, $p$ does not possess (normalizable nor nonnormalizable) eigenvectors. Thus $p$ is not self-adjoint, instead $p$ is simple symmetric.
Even though there are no plane waves among the physical states, plane waves can of course be approximated by sequences of physical states which are approximately plane waves within most of the interval \([-L, L]\), but which also quickly decay to zero towards the boundaries, such as to always obey the boundary condition, Eq.\[3\].

One may therefore be tempted to assume that \(p\) is still “approximately” self-adjoint and should therefore describe a sharp entity. This is, however, not the case: Indeed, as we already mentioned, \(p\) is unsharp in the sense that for all physical states \(|\psi\rangle \in D_{\text{phys}}\) the momentum uncertainty is bounded from below by a fixed finite amount:

\[
\Delta p(\psi) \geq \Delta p_{\text{min}} = \frac{\pi}{2L} \quad \text{for all normalized} \; |\psi\rangle \in D_{\text{phys}}
\]

Intuitively, the reason is that the larger the part of the interval on which a physical wave function approximates a plane wave, the steeper it must decay to zero towards the boundaries. The steep decay necessarily yields a significant contribution to the action of the derivative operator \(p\). This is connected to the fact that \(p\) is a noncontinuous operator. We remark that only noncontinuous i.e. only unbounded operators can be simple symmetric and display this unsharpness. For an explanation of the unsharpness phenomena in these terms, see [21, 22].

Here, let us explicitly calculate the physical states with the lowest momentum uncertainty. To this end, we solve the variational problem of minimizing \(\Delta p(\psi)\) by minimizing \(\langle d_{\psi}^2|d_{\psi}\rangle - \langle d_{\psi}|d_{\psi}\rangle^2\) under the constraints \(\langle d_{\psi}|d_{\psi}\rangle = \rho\) and \(\langle d_{\psi}|d_{\psi}\rangle = 1\), and the boundary condition Eq.\[3\].

Introducing Lagrange multipliers \(k_1, k_2\), the functional to be minimized is:

\[
S = \int_{-L}^{L} dx \left\{- (\partial_x^2 d_{\psi}^*) (\partial_x d_{\psi}) + k_1 (d_{\psi}^* d_{\psi} - c_1) + k_2 (-i d_{\psi}^* \partial_x d_{\psi} - c_2)\right\},
\]

yielding the Euler-Lagrange equation:

\[
\partial_x^2 d_{\psi} + k_1 d_{\psi} - i \partial_x d_{\psi} = 0
\]

For each choice of momentum expectation value \(\langle d_{\psi}\rangle = \rho\), there is (up to a phase) one normalized and the boundary condition obeying solution:

\[
\psi_{\rho}(x) = \frac{1}{L^{1/2}} \cos \left(\frac{\pi x}{2L}\right) e^{i\rho x}
\]

These are the physical wave functions which minimize \(\Delta p\). We see that the \(\psi_{\rho}(x)\) are essentially plane waves, apart from the modulus, which approaches zero at the boundaries, as it must, being a physical state. It is clear that the modulus of the wave functions \(\psi_{\rho}(x)\) goes to zero with just the optimal steepness to minimize the
momentum uncertainty $\Delta p$.
The minimum value for the uncertainty in the momentum of a particle in the box is now readily calculated from the solutions $\psi_\rho$, as:

$$\Delta p_{\text{min}} = \sqrt{\langle \psi_\rho | p^2 | \psi_\rho \rangle - \rho^2} = \frac{\pi}{2L} \quad (12)$$

In this case here, the minimum uncertainty $\Delta p_{\text{min}}$ does not depend on the expectation value $\rho$. Note that for generic simple symmetric operators with equal deficiency indices the minimum standard deviation can depend on the expectation value:

$$\Delta Q_{\text{min}} = \Delta Q_{\text{min}}(\langle Q \rangle). \quad (13)$$

It is standard procedure to verify that the deficiency indices of $p$ are indeed equal, namely $(1, 1)$. Thus, the short-distance structure (of momentum space) is of the type fuzzy-A in the terminology of [1].

### 2.2 An “unsharp” position operator

Let us now illustrate the same phenomena with the example of a simple symmetric operator which is given explicitly in terms of an infinite dimensional matrix.

Consider the operator $Q$ which is defined as the matrix

\[
Q = \begin{pmatrix}
0 & a_1 & 0 & 0 & \ldots \\
a_1 & 0 & a_2 & 0 & \ldots \\
0 & a_2 & 0 & a_3 & \ldots \\
0 & 0 & a_3 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\quad (14)
\]

where we define the matrix elements $a_n$ through

$$a_n := \sqrt{1 + s + s^2 + \ldots s^{n-1}} \quad (15)$$

with $s$ being a constant, obeying $s \geq 1$. Of course, one may use the partial geometric series:

$$1 + s + s^2 + \ldots s^{n-1} = \frac{(s^n - 1)}{(s - 1)}. \quad (16)$$

We define the domain $D_Q$ of $Q$ to consist of all column vectors which possess an arbitrary but finite number of nonzero entries. The domain $D_Q$ is dense in the Hilbert space $H = l^2$ of all of square summable vectors, i.e. $\overline{D_Q} = H$.

Clearly,

$$Q_{ij} = Q^*_{ji}. \quad (17)$$

Thus, on its domain, $D_Q$, the expectation values of $Q$ are real, i.e. $Q$ is a symmetric operator.
Let us consider first the special case \( s = 1 \).

In the case \( s = 1 \), the matrix elements reduce to \( a_n = \sqrt{n} \). We recognize that \( Q \) is then the ordinary essentially self-adjoint quantum mechanical position operator, in its Fock space representation. Its spectrum is the real line and there exist sequences of vectors in its domain \( D_Q \) such that \( \Delta Q \) becomes arbitrarily small, i.e. \( Q \) is a sharp observable.

The situation is qualitatively different for \( s > 1 \).

It has been shown in \[20\] that if \( s > 1 \), then for all vectors in \( D_Q \) the uncertainty \( \Delta Q \) is finitely bounded from below, by:

\[
\Delta Q_{\text{min}} = \sqrt{1 - s^{-1}}
\]  

This means that for all normalized \( |\phi\rangle \in D_Q \), i.e. for all normalized vectors with an arbitrary but finite number of nonzero entries, the uncertainty in \( Q \) obeys:

\[
\Delta Q(|\phi\rangle) = \langle \phi | (Q - \langle Q | \phi \rangle)^2 |\phi\rangle^{1/2} \geq \Delta Q_{\text{min}}.
\]  

The fact that \( \Delta Q_{\text{min}} = \sqrt{1 - s^{-1}} \) is larger than zero implies that there are no (normalizable nor nonnormalizable) eigenvectors of \( Q \), i.e. that \( Q \) is not self-adjoint. \( Q \) has been shown to be simple symmetric of type fuzzy-A, see \[1, 20\].

In fact, in this example, the minimum standard deviation is a nontrivial function of the expectation value of \( Q \). For simplicity, we have only given the absolute minimum \( \Delta Q_{\text{min}} \). For the precise form of \( \Delta Q_{\text{min}}(\langle Q \rangle) \) and its derivation, see \[20\].

Finally, we remark that no finite dimensional truncation of the matrix \( Q \) could possess a nonzero minimum uncertainty \( \Delta Q_{\text{min}} > 0 \). This is because the notions of symmetry and self-adjointness only differ on infinite dimensional Hilbert spaces: Every finite dimensional symmetric matrix is also self-adjoint and therefore possesses eigenvectors \( |q\rangle \) for which, of course, \( \Delta Q(|q\rangle) = 0 \).

### 2.3 Unsharpness from noncommutativity

The statement that self-adjoint operators can always be resolved to arbitrary precision is compatible with the Heisenberg uncertainty principle. Assume, for example, that \( S \) and \( T \) are two self-adjoint observables which do not commute.

Then, there holds the uncertainty relation:

\[
\Delta S \Delta T \geq \frac{1}{2} |\langle [S, T] \rangle|\]

This implies of course that if \( \Delta T \) is smaller than some value, say \( \Delta T < t_0 \) and if, say, \( [S, T] = i1 \) then the RHS is nonvanishing, yielding \( \Delta S \geq 1/2t_0 \). Thus, in this case, \( \Delta S \) cannot be made arbitrarily small but possesses instead a finite lower bound.
\[ \Delta S_{\text{min}} = 1/2t_0. \]

This is not a contradiction to the statement that self-adjoint operators can always be diagonalized, because to require \( \Delta T \leq t_0 \) is to restrict the Hilbert space to only those states for which \( \Delta T \leq t_0 \) holds. On this restricted domain, the operator \( S \) is not self-adjoint, instead it is simple symmetric.

In general, noncommutativity of symmetric operators in any physical theory induces an interplay between the domains of those operators, which in turn affects whether or not they are self-adjoint or merely symmetric. Indeed, even more generally, not only kinematical but also dynamical operator equations, i.e. not only commutation relations but also operator equations of motion can affect the domains of operators, and can therefore affect whether or not these operators are symmetric or self-adjoint.

Thus, while it is a well-known and much-discussed phenomenon that the sharpness or unsharpness of real entities in quantum theory can depend on the kinematics - through uncertainty relations - it appears that there is a priori no reason to exclude the possibility that the sharpness of real entities can also change dynamically, for example in a fundamental theory of quantum gravity.

3 Unsharp Degrees of Freedom and the Generating of Unitaries

As is well-known, self-adjoint operators often act not only as real degrees of freedom, but simultaneously also as generators of symmetries. Merely symmetric operators, on the other hand, do not directly generate unitary operators.

This appears to indicate that while symmetric operators possess the interesting property of being able to describe unsharp real degrees of freedom, they should not be able to play a rôle in the generation of symmetries.

Here, we will therefore address the problem of the generation of unitary operators for the class of simple symmetric operators which describe fuzzy-A type short-distance structures, i.e. which have equal deficiency indices. This class includes our examples above, and it includes, in particular, all operators \( X \) with a finite lower bound \( \Delta X_{\text{min}} > 0 \).

We will prove the following:

For each simple symmetric operator \( X \) with equal deficiency indices, acting on a physical domain \( D_{\text{phys}} \) which is dense in a Hilbert space \( H \), there exists a family of self-adjoint operators \( X(\alpha) \) which coincide with \( X \) on the physical domain. We claim that these operators \( X(\alpha) \), together, generate the full unitary group of the Hilbert space. This result shows that, in this way, the operators of this class can relate to all aspects of symmetry in the Hilbert space on which they act.

The precise formulation of the general theorem and its proof are given in Sec.4. Before,
however, we will give a detailed illustration of the theorem in concrete examples.

### 3.1 The theorem in concrete examples

In order to demonstrate the mechanism by which simple symmetric operators are able to generate all unitaries of the Hilbert space, let us consider a concrete example in ordinary nonrelativistic quantum mechanics in one dimension.

In this case, for the particle on the real line, the operators $\mathbf{x}$ and $\mathbf{p}$ are self-adjoint and can be exponentiated to yield unitaries:

We may represent the operators $\mathbf{x}$ and $\mathbf{p}$, irreducibly, as the self-adjoint multiplication and differentiation operators $\mathbf{x}\psi(x) = x\psi(x)$ and $\mathbf{p}\psi(x) = -i\partial_x\psi(x)$ acting on a dense domain in the Hilbert space $H$ of square integrable wave functions $\psi(x)$ over the real line.

As is well-known, $\mathbf{x}$ and $\mathbf{p}$, together, generate all unitary operators $U$ on the Hilbert space $H$, via the Weyl formula

$$U = \int \int \frac{d\sigma dt}{2\pi\hbar} u(s, t) \exp[i(s\mathbf{x} + t\mathbf{p})/\hbar]$$  \hfill (21)

where the $u(s, t)$ are suitable complex-valued functions. In fact, all bounded operators $B \in B(H)$ can be generated in this way.

On the other hand, we can also represent $\mathbf{x}$ and $\mathbf{p}$ reducibly, for example, as the self-adjoint multiplication and differentiation operators $\mathbf{x}\psi_i(x) = x\psi_i(x)$ and $\mathbf{p}\psi_i(x) = -i\partial_x\psi_i(x)$ (22)

acting on a Hilbert space of wave functions $\psi_i(x)$ on the real line which possess an additional “isospinor” index, running $i = 1, ..., n$.

The scalar product of wave functions then contains an iso-sum:

$$\langle \psi | \phi \rangle = \sum_{i=1}^{n} \int_{-\infty}^{\infty} dx \psi_i^*(x)\phi_i(x)$$  \hfill (23)

Clearly, $\mathbf{x}$ and $\mathbf{p}$ are acting diagonally in the isospinor space. Therefore, $\mathbf{x}$ and $\mathbf{p}$ do not generate the $U(n)$ of the isorotations. Thus, in this case, the Weyl formula, Eq.(21), does not yield all bounded operators nor does it yield only all unitaries on the Hilbert space. Only if we supplemented the operators $\mathbf{x}$ and $\mathbf{p}$ by additional hermitean $n \times n$ matrices, $T_i$, could we generate $U(n)$ on the isospinor space and therefore all of $B(H)$.

Let us now consider again the case where the particle is confined to the interval $[-L, L]$. As we saw above, the momentum operator $\mathbf{p} = -i\partial_x$ is then no longer self-adjoint and
it is instead simple symmetric of type fuzzy-A. Therefore, $p$ then matches the conditions of our proposition.

Namely, our proposition is that for any simple symmetric operator $p$ of type fuzzy-A, e.g. the momentum of the particle in a box, there exists a one-parameter family of self-adjoint operators $p(\alpha)$, $0 \leq \alpha < 2\pi$, such that:

- each $p(\alpha)$ coincides with $p$ on the physical domain, i.e.
  $$p(\alpha)|\psi\rangle = p|\psi\rangle \quad \text{for all } |\psi\rangle \in D_{\text{phys}}$$

- the $p(\alpha)$, together, (weakly) generate the algebra $B(H)$ of bounded operators on the Hilbert space, which includes of course the full unitary group on $H$.

Indeed, we claim that, unlike in the Weyl formula, the operator $x$ is now no longer needed to generate $B(H)$, because the operators $p(\alpha)$ alone already generate $B(H)$ (even though each $p(\alpha)$ coincides with $p$ on the dense physical domain $D_p$).

Even further, we can consider the case where the wave functions of the particle in the box carry some isospinor index. Then, $p$ is again simple symmetric of type fuzzy-A and our theorem applies. We claim that there then exists a multi-parameter set of self-adjoint operators $p(u)$, which again all coincide with $p$ on physical states, and which generate all of $B(H)$! This means that there is no need to introduce isospin rotation generators $T_j$ by hand, since the $p(u)$ are able to generate all: translations, phase rotations and isorotations.

### 3.2 Generating $B(H)$ in the scalar case

To see this, we consider first the case without an isospinor index.

As discussed above, the momentum operator $p$, acting as $p\psi(x) = -i\partial_x \psi(x)$ on the physical wave functions $\psi \in D_{\text{phys}}$ over the interval is a simple symmetric operator. We recall that although all physical wave functions vanish at the boundary, they are a dense set in the Hilbert space of square integrables $D_{\text{phys}} = H = L^2(-L, L)$.

Let us now construct a family of operators $p(\alpha)$ which coincide with $p$ on the physical domain $D_{\text{phys}}$, but whose domain is larger and who are self-adjoint on this larger domain.

To this end, we define the operators $p(\alpha)$ by extending the domain $D_{\text{phys}}$ such as to include wave functions which are periodic up to a phase

$$\psi(-L) = e^{i\alpha} \psi(L), \quad (24)$$
where $\alpha$ is some arbitrary but fixed real number. To be precise, the domain of the self-adjoint extension $p(\alpha)$ is therefore

$$D_{p(\alpha)} := D_{phys} \cup \{\psi(x) \in D_{p^*} | \psi(-L) = e^{i\alpha}\psi(L)\},$$  \tag{25}

where $D_{p^*}$ is the domain of the adjoint operator $p^*$.

Note that $e^{i\alpha}$ must be a fixed phase in order to ensure that the boundary terms cancel in the partial integrations which are needed to show that $\langle \psi_1 | (p(\alpha) | \psi_2) \rangle = (\langle \psi_1 | p(\alpha) | \psi_2 \rangle).$

Indeed, for each fixed choice of a phase $e^{i\alpha}$ there exist eigenvectors of $p(\alpha)$, i.e. plane waves, $\psi_n^{(\alpha)}(x)$, which obey the corresponding boundary condition:

$$\psi_n^{(\alpha)}(x) = e^{i\omega_n x} \quad \text{where} \quad \omega_n = \frac{2\pi n - \alpha}{2L}, \quad n \in \mathbb{Z} \tag{26}$$

As is straightforward to check, the $\psi_n^{(\alpha)}$ form an orthonormal eigenbasis of $p(\alpha)$, and each $p(\alpha)$ is self-adjoint.

Let us now consider the implications for the generating of unitaries:

If the wave functions were not restricted to the interval, $p$ would be self-adjoint and $p$ could be exponentiated to obtain a unitary operator, say

$$U(a) := \exp(iap),$$ \tag{27}

for some $a \geq 0$. The action of this unitary is to translate wave functions by the amount $a$ to the right:

$$U(a).\psi(x) = e^{ia\partial_x} \psi(x) = \psi(x + a).$$ \tag{28}

In the case where the particle is confined to a box, however, i.e. where the Hilbert space only consists of wave functions on the interval, the operator $p$ is not self-adjoint and cannot be exponentiated: The formal expression $U(a) = \exp(ia p)$ is now not a unitary transformation, because it would translate beyond the interval boundaries, which is not defined in the Hilbert space.

Nevertheless, for the particle in a box, there exists, as we saw, a whole family of self-adjoint extensions $p(\alpha)$ of $p$. Since each $p(\alpha)$ is self-adjoint, each can be exponentiated and the resulting operator

$$U_\alpha(a) := \exp(ia p(\alpha))$$ \tag{29}

is unitary. The action of $U_\alpha(a)$ on wave functions is again to translate wave functions to the right (for $a > 0$), as in Eq.28. Now, however, due to the boundary condition, Eq.24, the part of the wave function which would be translated beyond the right interval boundary reappears into the interval from the left, with the same modulus, but phase shifted by the phase $e^{i\alpha}$.

Thus, the unitary $U_\alpha(a)$ translates the wave functions by the amount $a$ and phase
shifts the wave functions by $e^{i\alpha}$ when translating them beyond a boundary and into the interval again from the opposite boundary.

Let us consider the composition of such unitaries. Crucially, the product
\[ U_{\alpha'}(-a)U_\alpha(a) \]

is a unitary operator which does not translate wave functions. This is because the first factor translates by $a$ and the second factor translates back by the same amount. Nevertheless, since the two factors translate with different phase shifts, the product is not the identity operator. Namely, $U_{\alpha'}(-a)U_\alpha(a)$ is the unitary operator whose action is to leave the modulus of wave functions unchanged, but to phase shift the wave functions on a part of the interval.

E.g., choosing some $a \in [0, 2L]$, the action is
\[ U_{\alpha'}(-a) U_\alpha(a).\psi(x) = \begin{cases} 
\psi(x), & \text{for } x \in [-L, L-a] \\
 e^{i(\alpha-\alpha')}\psi(x), & \text{for } x \in [L-a, L] 
\end{cases} \]

By suitable composition of operators $U_\alpha(a)$ for various $a$ and $\alpha$ it is therefore possible to generate unitaries which yield arbitrary local phase rotations of wave functions.

For example, choosing some $a,b$ obeying $0 < b < a < 2L$, we form the operator:
\[ U_\alpha(-(a-b)) U_0(-b) U_\alpha(a).\psi(x) \]

\[ = \begin{cases} 
\psi(x), & \text{for } x \in [-L, L-a] \cup [L-a+b, L] \\
 e^{i\alpha}\psi(x), & \text{for } x \in [L-a, L-a+b] 
\end{cases} \]

The action of this operator is to phase rotate wave functions by $e^{i\alpha}$ in the interval $[L-a, L-a+b]$ and to leave the wave functions invariant outside that interval.

Thus, remarkably, the set of self-adjoints which coincide with $p$ on the physical domain is able to generate all translations \textit{and} also all local phase rotations, while we recall that in the case where $p$ is self-adjoint, the operator $x$ is needed order to generate phase rotations, namely through $e^{i\beta\psi(x)} = e^{i\beta\psi(x)}$.
3.3 The case with isospin

We consider again a particle constrained to the interval $[-L, L]$. The particle’s wave function $\psi_i(x)$ shall now carry an isospinor index $i = 1, \ldots, n$. The scalar product in the Hilbert space of square integrables on the interval then includes an iso-sum:

$$\langle \psi | \phi \rangle = \sum_{i=1}^{n} \int_{-L}^{L} dx \, \psi_i^*(x) \phi_i(x)$$  \hspace{1cm} (33)

Due to the box potential, the physical wave functions, $|\psi\rangle \in D_{phys}$, again obey the boundary condition

$$\psi_i(-L) = 0 = \psi_i(L), \quad (i = 1, \ldots, n)$$  \hspace{1cm} (34)

The action of $p$ is diagonal in iso-space: $p_\psi(x) = -i\partial_x \psi_i(x)$. Again, there are no plane waves in the physical domain and therefore the momentum operator on the physical domain is not self-adjoint. Instead, $p$ is simple symmetric (with deficiency indices $(n, n)$). Self-adjoint extensions $p(u)$ are now obtained by enlarging the domain of $p$ to include wave functions which obey the boundary condition

$$\psi_i(-L) = \sum_{i=1}^{n} u_{ij} \psi_j(L)$$  \hspace{1cm} (35)

where $u_{ij}$ is any unitary $n \times n$ matrix, generalizing the phase $e^{i\alpha}$ of the scalar case above. As is readily checked, the proof of self-adjointness of the $p(u)$ requires again the cancellation of the boundary terms which arise through the partial integrations needed to show that $(\langle \psi | p(u) | \phi \rangle = \langle \psi | (p(u) | \phi \rangle)$, and this cancellation is achieved exactly by the boundary conditions of the form of Eq.35.

As in the scalar case, while $p$ does not directly yield unitaries, each of the self-adjoint $p(u)$ which reduce to $p$ on the physical domain does generate unitaries, e.g. by exponentiation ($a$ real):

$$U_u(a) := e^{i a p(u)}$$  \hspace{1cm} (36)

The unitaries $U_u(a)$ again act on wave functions by translating them by the amount $a$, and, due to the self-adjoint extensions’ boundary conditions, any part of the wave function which hits a boundary reappears from the other side into the interval, now iso-rotated by the matrix $u$ (or by $u^{-1}$ if $a$ is negative).

It is possible to proceed as in the scalar case, composing such unitaries to translate the wave functions back and forth, using different self-adjoint extensions. It is clear that in this way arbitrary local isorotations can be generated.

Thus, the set of self-adjoint operators which reduce to $p$ on the physical domain indeed generates not only translations, which they may be expected to, but also arbitrary local phase rotations, and - if an isospinor index is present - then they even generate all local iso-rotations.

We now proceed to the proof of the theorem for the general case.
4 Theorem

4.1 Definitions

Let us recall that a symmetric operator $X$ is called simple symmetric if $X$ is not self-adjoint and if it possesses no invariant subspace such that the restriction of $X$ to this subspace yields a self-adjoint operator. Our examples above are simple symmetric.

Further, we recall that the Cayley transformed operator $S$ of a symmetric operator $X$, defined as

$$S := (X - i1)(X + i1)^{-1}$$  \hspace{1cm} (37)

is isometric. An isometric operator is called simple isometric if it cannot be reduced to an invariant subspace such that the reduced operator is unitary. It is known that a subspace reduces a symmetric operator $X$ if and only if it reduces its Cayley transform, see, for example, [28]. Note, however, that not every isometric operator is the Cayley transform of a symmetric operator.

4.2 Theorem

Let $X$ be a closed simple symmetric operator with equal deficiency indices, defined on a domain $D_X$ which is dense in a complex Hilbert space $H$. Then, the self-adjoint extensions $X(\alpha)$ of $X$ generate a $*$-algebra $\mathcal{A}$ which is weakly dense in $\mathcal{B}(H)$. Thus, in particular, the self-adjoint extensions generate the full unitary group $U(H)$ of the Hilbert space.

4.3 Outline of the proof

The first step will be to use the $X(\alpha)$ to generate a suitable set $\mathcal{M}$ of unitaries, which in turn generate an algebra $\mathcal{A}$. The proof then consists in showing that the commutant $\mathcal{A}'$ of the algebra $\mathcal{A}$ is $\mathcal{A}' = C1$. This implies that its double commutant is $\mathcal{A}'' = \mathcal{B}(H)$. The proposition then follows since, with v. Neumann, the double commutant of any $*$-algebra is its weak closure.

4.4 Proof

We begin by choosing a suitable set of unitaries which are generated by the self-adjoint extensions $X(\alpha)$ of $X$. To this end, consider the isometric Cayley transform $S$ of $X$

$$S := (X - i1)(X + i1)^{-1}$$  \hspace{1cm} (38)

with domain

$$D_S = (X + i1).D_X.$$  \hspace{1cm} (39)
We define the local group $\mathcal{T}$ as the set of all unitaries which map the deficiency space $D_S^\perp = ((X + i1)D_X)^\perp$ onto itself and which act as the identity on $D_S$, i.e.:

$$\mathcal{T} := \{ T \mid T : D_S \to D_S, \ T : D_S^\perp \to D_S^\perp, \ T|_{D_S} = 1, \ TT^\dagger = T^\dagger T = 1 \}.$$  

(40)

It is clear that the local group, $\mathcal{T}$, is isomorphic to the unitary group $U(n)$, where $n$ is the deficiency index $n := \dim(D_S^\perp)$.

Since, by assumption, both deficiency indices are equal, i.e. both spaces

$$L_\pm = ((X \pm i1)D_X)^\perp$$  

are of equal dimension, there exist unitary extensions of $S$.

Let $U$ be one of the unitary extensions of $S$:

$$U^\dagger U = UU^\dagger = 1, \ U : L_+ \to L_-, \ U|_{D_S} = S.$$  

(42)

We consider now the coset

$$\mathcal{M} := \{ M \mid M = UT, \ T \in \mathcal{T} \}$$  

(43)

of unitary extensions of $S$.

Indeed, as is well known, each unitary extension of the Cayley transform $S$ of a symmetric $X$, i.e. here each element of $\mathcal{M}$, is indeed generated, via the Cayley transform, by a self-adjoint extension $X(\alpha)$ of $X$.

We will now show that the $^*$-algebra $\mathcal{A}$ generated by $\mathcal{M}$ is weakly dense in $B(H)$. As mentioned, this follows from v. Neumann’s double commutant theorem if we can prove that only multiples of the identity operator commute with $\mathcal{M}$, i.e. with $U$ and all elements of $\mathcal{T}$.

To this end, let us consider an operator $V$ which obeys:

$$||V|| < \infty \ \text{and} \ [V,U] = 0 = [V,T], \ \forall \ T \in \mathcal{T}$$  

(44)

We need to show that $V$ is a multiple of the identity operator.

Since the closure of $X$ implies the closure of the deficiency space $D_S^\perp$ and of $D_S$, we can use $H = D_S \oplus D_S^\perp$ to write $V$ and the elements $T \in \mathcal{T}$ in block form:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \quad V = \begin{pmatrix} V_{D_S D_S} & V_{D_S D_S^\perp} \\ V_{D_S^\perp D_S} & V_{D_S^\perp D_S^\perp} \end{pmatrix}$$  

(45)

Here, $t = T|_{D_S^\perp}$, i.e. $t : D_S^\perp \to D_S^\perp$ and, e.g., $V_{D_S D_S^\perp} : D_S^\perp \to D_S$. In this notation, $[T,V] = 0$ reads:

$$\begin{pmatrix} 0 & V_{D_S D_S^\perp}(1-t) \\ (t-1)V_{D_S^\perp D_S} & [t,V_{D_S^\perp D_S}] \end{pmatrix} = 0$$  

(46)
Eq. 46 holds for all \( T \in \mathcal{T} \), and in particular it holds for unitaries \( t : D_{\hat{S}}^\perp \to D_{\hat{S}}^\perp \) for which the value 1 is a regular point, e.g. \( t = -1 \). Thus, \( V_{D_{\hat{S}}D_{\hat{S}}^\perp} = 0 \) and \( V_{D_{\hat{S}}^\perp D_{\hat{S}}} = 0 \).

Further, \( \mathcal{T} \) is the full unitary group on \( D_{\hat{S}}^\perp \). It is therefore irreducibly represented on \( D_{\hat{S}}^\perp \). Thus, \( t, V_{D_{\hat{S}}^\perp D_{\hat{S}}} = 0 \), \( \forall t \) implies with Schur that \( V \) acts on \( D_{\hat{S}}^\perp \) as a multiple of the identity, i.e. \( V_{D_{\hat{S}}^\perp D_{\hat{S}}} = \lambda I \) where \( \lambda \in \mathbb{C} \). In block matrix form, \( V \) therefore reads:

\[
V = \begin{pmatrix}
V_{D_{\hat{S}}D_{\hat{S}}^\perp} & 0 \\
0 & \lambda I
\end{pmatrix}
\]  

(47)

Consider now the kernel

\[ K := \ker(V - \lambda I). \]  

(48)

By construction, \( D_{\hat{S}}^\perp \subset K \) and \( K^\perp \subset D_{\hat{S}} \). As the kernel of a closed operator, \( K \) is closed. We wish to show that in fact \( K = H \) and \( K^\perp = \emptyset \), which is to say that \( V = \lambda I \).

To this end, let us assume the opposite, namely that \( K^\perp \neq \emptyset \).

We can then use \( H = K^\perp \oplus K \) to write both \( V \) and \( U \) in a new block form:

\[
V = \begin{pmatrix}
V_{K^\perp K^\perp} & 0 \\
V_{KK^\perp} & \lambda I
\end{pmatrix}, \quad U = \begin{pmatrix}
U_{K^\perp K^\perp} & U_{K^\perp K} \\
U_{KK^\perp} & U_{KK}
\end{pmatrix}
\]  

(49)

The relation \([V, U] = 0\) now reads:

\[
\begin{pmatrix}
\ldots, & (V_{K^\perp K^\perp} - \lambda I)U_{K^\perp K} \\
\ldots, & V_{KK^\perp}U_{K^\perp K}
\end{pmatrix} = \begin{pmatrix}
0, & 0 \\
0, & 0
\end{pmatrix}
\]  

(50)

On the other hand, \( U_{K^\perp K^\perp}K \subset K^\perp \), i.e. the range of \( U_{K^\perp K^\perp} \) is not in the kernel of the operator \((V - \lambda I)\):

\[
(V - \lambda I)|w\rangle = \begin{pmatrix}
(V_{K^\perp K^\perp} - \lambda I)|w\rangle \\
V_{KK^\perp}|w\rangle
\end{pmatrix} \neq 0, \quad \forall |\omega\rangle \neq 0, |\omega\rangle \in U_{K^\perp K^\perp}K
\]  

(51)

Thus, the existence of any nonzero vector \(|w\rangle \in K^\perp \) in the range \( U_{K^\perp K^\perp} \) would contradict Eq. 51. Consequently, the range of \( U_{K^\perp K^\perp} \) is empty, i.e. \( U_{K^\perp K^\perp} = 0 \).

Therefore, \( K \) is an invariant subspace for \( U \). Since also \([U^{-1}, V] = 0\), it follows analogously that \( K \) is an invariant subspace for \( U^{-1} \). Thus, \( K \) and \( K^\perp \) both reduce \( U \):

\[
U = \begin{pmatrix}
U_{K^\perp K^\perp} & 0 \\
0 & U_{KK}
\end{pmatrix}
\]  

(52)
Since \( U|_{D_S} = S \) and \( K^\perp \subseteq D_S \) we have \( U_{K^\perp K^\perp} = S_{K^\perp K^\perp} \). This implies that \( K^\perp \) is an invariant subspace for \( S \), on which \( S \) is unitary. However, the simplicity of \( X \) implies that also \( S \) is simple, i.e. \( S \) does not have any invariant subspace on which it would be unitary.

Thus, in fact, \( K^\perp = \emptyset \) and \( K = H \). Consequently, \( V = \lambda 1 \), which had to be shown.

With von Neumann this implies that the weak closure of the \( * \)-algebra \( A \) generated by \( 1, U \) and the elements of \( T \) is the algebra \( B(H) \) of all bounded operators on the Hilbert space, and \( B(H) \) includes of course all unitaries. We recall that this means that for each bounded operator \( B \in B(H) \) there exist sequences of operators \( B_n \in A \) such that

\[
\lim_{n \to \infty} \langle \psi | B - B_n | \phi \rangle = 0 \quad \forall \ \langle \psi \rangle, \langle \phi \rangle \in H.
\]

Thus, for any simple symmetric \( X \) with equal deficiency indices the set of self-adjoint operators which coincide with \( X \) on its domain generate indeed (e.g. via generating the coset \( M \)) the full unitary group of the Hilbert space.

### 4.5 A corollary

As we mentioned before, in finite dimensional Hilbert spaces every symmetric operator, i.e. every operator whose expectation values are real, i.e. every matrix obeying \( X_{ij} = X_{ji}^* \), is also self-adjoint. Therefore, in finite dimensional Hilbert spaces, there are no simple symmetric operators, i.e. our theorem cannot be applied.

Let us add, however, that the above proof yields as a corollary that any simple isometric operator with equal deficiency indices has the property that its unitary extensions, together, generate all unitaries and \( B(H) \). And indeed, there exist simple isometric operators also in finite dimensional Hilbert spaces.

As an illustration, let us consider the simple case of the two dimensional Hilbert space spanned by normalized vectors \( e_1, e_2 \). We define a linear operator, \( S \), as the map which maps \( S : e_1 \to e_2 \). Clearly, \( S \) is not unitary, because of its limited domain \( D_S := \mathbb{C}e_1 \) and range \( \mathbb{C}e_2 \). Also, \( S \) does not have any invariant proper subspace. \( S \) is norm preserving where it is defined. Thus, \( S \) is a simple isometric operator. The dimensions of its deficiency spaces, i.e. of the orthogonal complements of its domain and range are both 1, i.e. they are equal. Thus, \( S \) is an operator to which the corollary of our theorem applies. The claim is that the unitary extensions of \( S \) generate all \( 2 \times 2 \) matrices, including of course the unitaries.

To see this, we begin by choosing one unitary extension \( U \) of \( S \), e.g.:

\[
U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] (53)
The elements $T(\alpha)$ of the local group $\mathcal{T}$ of all unitaries which act as the identity on $D_S$ and which act as a unitary on $D_S^\perp$ are of the form

$$T(\alpha) := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

(54)

where $e^{i\alpha}$ is any arbitrary phase. Thus, each unitary extension of $S$ is of the form $U T(\alpha)$ for some $\alpha$. Indeed, the algebra generated by $1, U$ and the unitary extensions $T(\alpha)$ is all of $M_2(\mathbb{C})$, as is clear because it contains for example the Pauli matrices:

$$\sigma_1 = U, \quad \sigma_2 = i U T(\pi), \quad \sigma_3 = T(\pi).$$

(55)

On the other hand, we recall that simple symmetric operators only exist in infinite dimensional Hilbert spaces. Indeed, in our 2-dimensional example here, the inverse Cayley transform $X$ of $S$ does exist,

$$X = i (S + 1)(S - 1)^{-1} = -i \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

(56)

but $X$ is clearly not symmetric. In general, the inverse Cayley transform of a simple isometric operator is not necessarily a simple symmetric operator. In particular, it cannot be simple symmetric in finite dimensional Hilbert spaces. On the other hand, the fact that, vice versa, the Cayley transform of a simple symmetric operator is always a simple isometric operator is what we used in the theorem.

## 5 Conclusions and Outlook

Our subject of investigation is any physical entity, within a quantized theory, which is real in the sense that it is described by an operator $Q$ whose expectation values are real.

Our first conclusion has been that any such real entity or “real degree of freedom” can only be “sharp” or “unsharp” in a few well-defined ways. Namely, the physical entity is sharp if the operator $Q$ is self-adjoint. In this case, its possible short-distance structures are lattices and continua. On the other hand, the physical entity is unsharp if the operator $Q$ is merely symmetric. Then, its possible short-distance structures are what we call fuzzy-A and fuzzy-B. All other possibilities are mixtures of these. The sharpness or unsharpness of a real entity can depend on the kinematics of the theory, e.g.through commutation and uncertainty relations. A priori, the unsharpness of a real entity can also be a function of the dynamics, e.g. through operator equations of motion. Several properties of the fuzzy-A and fuzzy-B short-distance structures are discussed in [1]. A more detailed classification is in preparation.

Secondly, and this has been the main subject of the present work, we considered that
self-adjoint operators often not only represent real degrees of freedom but that they can also act as generators of symmetries. We therefore investigated in which way also operators that describe fuzzy degrees of freedom could generate symmetries.

To this end, we focussed on the class of operators of the type fuzzy-A. We found that these possess a remarkable property: If, on the physical domain, \( D_{\text{phys}} \), an operator \( Q \) is of the type fuzzy-A (i.e. simple symmetric with equal deficiency indices), then there exists a set of self-adjoint operators \( \{Q(\alpha)\} \) (the self-adjoint extensions) which all agree with \( Q \) on the physical domain. We showed that the operators \( Q(\alpha) \), together, generate all unitaries and all bounded operators in the Hilbert space. Thus, in this way, at least the fuzzy operators of type fuzzy-A can indeed play a rôles in all aspects of symmetries in the Hilbert space in which they act.

In our investigation of the properties of physical entities which are described by operators whose expectation values are real, we did not make any assumptions about the interpretation of these operators, nor about the underlying physical theory. Therefore, our conclusions, firstly about the possible types of sharp- and unsharpness and secondly about these operator’s ability to generate symmetries, apply to all linear operators which describe real degrees of freedom - for example in candidate theories for a fundamental theory of quantum gravity.

Indeed, for example the matrix model for M-theory, see e.g. [24], does employ symmetric operators, \( X_i \), to encode space-time information. In this case, the matrix elements of the \( X_i \) are interpreted in terms of coordinates of \( D0 \)-branes. Initially, the \( X_i \) are finite dimensional, say \( N \times N \) matrices. The quantization and the necessary limit \( N \to \infty \) are highly nontrivial, but it is clear that the resulting operators will still be at least symmetric. The short-distance structure which they describe will therefore fall into the classification outlined in [1]. The \( X_i \) are in general noncommutative, which is of course a kinematical source for fuzzyness, but there may also exist dynamical causes for fuzzyness of the \( X_i \). It is clear that if, or when, these operators are of type fuzzy-A, then our present results show how they relate to the unitary group of the Hilbert space on which they act.

Studies in the context of quantum groups, see e.g. [25, 26], and in the wider field of noncommutative geometry, have yielded new approaches to building models for space-time at the Planck scale, see e.g. [27]-[29]. Some of this work has been shown to be related to string theory, see e.g. [30]. As far as these models of space-time apply linear operators to describe real entities we are covering these operators. It should be very interesting to investigate the rôles of the present results in this context.

On the other hand, even on the level of generality on which we have been working here, more conclusions can likely be drawn: Indeed, one of the examples which we gave for our theorem indicates a particular direction for further investigation:

We discussed the case of the simple symmetric differential operator \( p = -i\delta_{ij}\partial_x \) which acts on a domain of wave functions \( \psi_i(x) \) with an isospinor index \( i = 1, \ldots, n \), defined
over the interval \([-L, L]\). There exists a whole \(U(n)\)-family of self-adjoint operators \(p(u)\) which coincide with \(p\) on its domain. We showed that, even though \(p\) itself acts diagonally on the isospinor space, the operators \(p(u)\) are able to generate all unitaries in the Hilbert space - which includes, in particular, also all isospinor rotations.

We can interpret this result as providing an example for a conjecture made in [1, 21]. The conjecture is that simple symmetric operators with equal deficiency indices \((n, n)\) always induce isospinor structures of dimension equal to the deficiency index.

The conjecture also yields an intuitive physical interpretation of the effect of an ultraviolet cutoff of the type fuzzy-A: Namely, those short wavelengths which are being cut off at short distances are being turned effectively into internal degrees of freedom associated with a unitary isospinor structure. This would mean that local gauge group structures could have their origin in what we here called the local unitary group on the deficiency spaces. Work in this direction is in progress.

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