Relative Hitchin–Kobayashi correspondences for principal pairs

Steven B. Bradlow\textsuperscript{1,2} \\
Department of Mathematics, \\
University of Illinois, \\
Urbana, IL 61801, USA \\
\textit{E-mail:} bradlow@math.uiuc.edu

Oscar García-Prada\textsuperscript{1,3,4} \\
Departamento de Matemáticas, \\
Universidad Autónoma de Madrid, \\
28049 Madrid, Spain \\
\textit{E-mail:} oscar.garcia-prada@uam.es

Ignasi Mundet i Riera\textsuperscript{1,3,5} \\
Departamento de Matemáticas, \\
Universidad Autónoma de Madrid, \\
28049 Madrid, Spain \\
\textit{Current Address:} \\
Dep. Matemàtica Aplicada I ETSEIB, UPC \\
c/Diagonal 647 08028 Barcelona, Spain \\
\textit{E-mail:} ignasi.mundet@uam.es

\textbf{Abstract.} A principal pair consists of a holomorphic principal $G$-bundle together with a holomorphic section of an associated Kaehler fibration. Such objects support natural gauge theoretic equations coming from a moment map condition, and also admit a notion of stability based on Geometric Invariant Theory. The Hitchin–Kobayashi correspondence for principal pairs identifies stability as the condition for the existence of solutions to the equations. In this paper we generalize these features in a way which allows the full gauge group of the principal bundle to be replaced by certain proper subgroups. Such a generalization is needed in order to use principal pairs as a general framework for describing augmented holomorphic bundles. We illustrate our results with applications to well known examples.

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1. Introduction

The Hitchin–Kobayashi correspondence relates a notion of stability inspired by Geometric Invariant Theory and a set of partial differential equations coming from Gauge Theory. Originally proven for holomorphic bundles (cf. [NS, D, UY, RS]), where it relates slope stability to the Hermitian-Einstein equations, similar correspondences are known to hold in more general settings. Despite the uniformity of the pattern, each instance of the correspondence has required individualized treatment. Such ad hoc approaches inevitably obscure the underlying general principles. There is thus a clear need for a unifying framework which clarifies the origins of the common features.

The full scope of the correspondences is determined, at least in the current state of the art, by the collection of instances where they have been established. In all cases, the setting consists of holomorphic bundles with some extra prescribed holomorphic data. Called augmented holomorphic bundles, a useful illustrative example is that of the holomorphic pair \((E, \phi)\) in which \(E\) is a holomorphic bundle and \(\phi \in H^0(E)\) is a holomorphic section. Other examples which have emerged naturally in other settings or have turned out to have interesting applications are surveyed in [BDGW] and also in [S, M, AG].

A general framework intended to unify a broad class of augmented bundles was introduced by one of us in [M], building on ideas introduced in [Br]. The key elements in this framework, called principal pairs, are objects defined by (i) a principal \(K\)-bundle \(P_K \rightarrow X\), where \(X\) is a compact Kaehler manifold and \(K\) is the compact real form of a complex Lie group \(G\), and (ii) a Kaehler manifold, \(Y\), with a Hamiltonian \(K\)-action. A principal pair is then a pair \((A, \Phi)\), where \(A\) is a connection on \(P_K\), and \(\Phi\) is a section of the associated fibre bundle \(Y_K = P_K \times_K Y\). Equivalently, one can replace \(P_K\) by the principal \(G\)-bundle \(P_G = P_K \times_K G\) and consider the unique extension of the \(K\)-action to a holomorphic \(G\)-action on \(Y\) (this exists because the complex structure on \(Y\) is integrable). The principal pair is then described by a holomorphic structure on \(P_G\) together with a section of the associated bundle \(Y_G = P_G \times_G Y\).

As shown in [M], there is a natural Hitchin–Kobayashi correspondence for such pairs. Moreover, many familiar instances of Hitchin–Kobayashi correspondences on augmented bundles follow as special cases of the general result for principal pairs. There are however important examples, including coherent systems, and Higgs bundles, which do not fit this mold.

In each of these cases where the principal pairs framework turns out to be inadequate, the reason is the same. By an appropriate choice of principal bundle and Kaehler manifold, the augmented bundles can indeed be described as particular types of principal pairs. The problem is that the set of all augmented bundles of the given type corresponds to only a subset of all the corresponding principal pairs. Furthermore, the automorphism groups for the augmented bundles correspond to only a subgroup of the principal pairs automorphism groups.

Our goal in this paper is to overcome these shortcomings in the principal pairs framework. Since the automorphism group for a principal pair corresponds to the gauge group of the principal bundle, we are led to reexamine the role played by the gauge group in the theory of principal pairs. In our main result (Theorem 4.1) in Section
we show that subject to certain natural constraints, the results of [M] can indeed be generalized to allow for the full gauge group to replaced by a subgroup.

As in the proof given in [M], our proof is motivated by the analogy between the Hitchin–Kobayashi correspondence and the results of Kirwan and Kempf–Ness (see Remark 9 for an important subtlety which explains why in some cases this is only an approximate analogy). The Kirwan–Kempf–Ness results apply to finite dimensional (Geometric Invariant Theory or Kaehler) quotients, where they relate an appropriate notion of stability to the vanishing of a symplectic moment map. The key to understanding this relation comes from the re-formulation of stability as a numerical condition expressed in terms of a function called the Hilbert numerical function. This function, in turn, can be expressed in terms of an integral of the moment map.

In the setting which describes principal pairs, the required quotient construction involves the action of an infinite dimensional (gauge) group on the infinite dimensional configuration space of all holomorphic pairs. One can no longer directly apply Geometric Invariant Theory, but, as shown in [M], one can still construct an integral of the moment map. One can thus define stability in precisely the same way as dictated by the Hilbert numerical condition. The Hitchin–Kobayashi correspondence between stable orbits of the complex gauge group and orbits which contain a zero of the moment map is then revealed as a precise analog of the finite dimensional correspondence.

The generalization we require is easily accomplished in the finite dimensional setting, where it corresponds to replacing the action of a reductive group on a projective variety by the action of a reductive subgroup. In that setting it is clear how to modify the GIT notion of stability, and also how to modify the appropriate moment map. Our main task is thus to show how to make the analogous modifications in the infinite dimensional gauge theoretic setting. As described in Section 4, the resulting generalized Hitchin–Kobayashi correspondence still relates zeros of a moment map and a notion of stability, but now both are defined with respect to the action of a subgroup of a gauge group.

In order to check that the resulting equations and notions of stability correspond to those obtained by the ad hoc methods used on specific examples, it is useful to understand how they relate to their counterparts for the full gauge group. For both the equations and the stability conditions, the relationship is easily understood: the moment maps are related by a projection from the Lie algebra of the full group onto that of the subgroup, while the notions of stability — which are both formulated as a set of algebraic conditions on subobjects — are related by a restriction of the conditions to special subobjects determined by the subgroup. In sections 6 and 7 we discuss some specific types of subgroups of the gauge group and consider several special cases of principal pairs. We show how our enlarged principal pair framework encompasses them all and allows us to recover many known Hitchin–Kobayashi correspondences.

A modification of the Hitchin–Kobayashi correspondences similar to the one described in this paper, but more limited in scope, can be found in [OT]. The setting considered there corresponds only to the special case discussed in Section 6.2.
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2. Principal pairs framework

2.1. Geometric Setting. We summarize from [M] the structures we need for the definition and analysis of principal pairs.

2.1.1. Let $K$ be a compact connected Lie group, and let $G = K^C$ be its complexification. Let $k$ (resp. $g$) be the Lie algebra of $K$ (resp. $G$).

2.1.2. Fix a faithful unitary representation $\rho_a : K \to U(\mathcal{W}_a)$, where $\mathcal{W}_a$ is a finite dimensional Hermitian vector space. Denote as well by $\rho_a$ the induced holomorphic representation of $G$ on $\text{GL}(\mathcal{W}_a)$ and let also $\rho_a : g \to \text{End}(\mathcal{W}_a)$ be the induced Lie algebra representation. Let $\ast : G \to G$ be the Cartan involution. Use this to define an Hermitian form on $g$, by:

$$\langle u, v \rangle = \text{Tr}(\rho_a(u)\rho_a(v)^\ast).$$

Let $|u| = \langle u, u \rangle^{1/2}$. The restriction of $\langle , \rangle$ to $\mathfrak{k}$ will be used to identify ($K$-equivariantly) $\mathfrak{k} \simeq \mathfrak{k}^\ast$.

2.1.3. Let $X$ be a compact Kaehler manifold of complex dimension $n$. Let $\omega$ be the symplectic form of $X$, and denote as usual $\omega^{[k]} = \omega^k/k!$. Unless otherwise stated, in the integrals of functions on $X$ the volume form $\omega^{[n]}$ will be used. Let $\Lambda : \Omega^{*+2}(X) \to \Omega^*(X)$ denote the adjoint of wedging with $\omega$.

2.1.4. Let $P_K \to X$ be a $K$-principal bundle, and let $P_G = P_K \times_K G$ be the $G$-principal bundle associated to $P_K$. Let $\mathcal{G} = \Gamma(P_K \times K)$ be the gauge group of $P_K$, and similarly let $\mathcal{G}^C$ be the gauge group of $P_G$. Let $\text{ad} P_K = P_K \times_K \mathfrak{k}$ and $\text{ad} P_G = P_G \times_K g$. Then $\text{Lie} \mathcal{G} = \Omega^0(\text{ad} P_K)$ and $\text{Lie} \mathcal{G}^C = \Omega^0(\text{ad} P_G)$. Use $\langle , \rangle$ to define an $L^2$ inner product on $\Omega^0(\text{ad} P_K)$, and use this to obtain an inclusion $\text{Lie} \mathcal{G} \subset (\text{Lie} \mathcal{G})^\ast$.

2.1.5. Let $\mathcal{Y}$ be a complete Kaehler manifold with a Hamiltonian action of $K$, which we denote by $\rho : K \times \mathcal{Y} \to \mathcal{Y}$, and let

$$\mu : \mathcal{Y} \to \mathfrak{k}^\ast$$

be the corresponding moment map. Suppose that the action of $K$ on $\mathcal{Y}$ respects the complex structure, $I_\mathcal{Y} \in \text{End}(T\mathcal{Y})$, of $\mathcal{Y}$. Then, since $I_\mathcal{Y}$ is integrable, there is a unique extension of the $K$-action on $\mathcal{Y}$ to a unique holomorphic action of $G$. 
2.1.6. Let $Y_K = P_K \times_K Y$ be the bundle with fibre $Y$ associated to $P_K$. Using the extension of the $K$-action to $G$, we can think of $Y_K$ as a $Y$-bundle associated to $P_G$. In that case we write $Y_G = P_G \times_G Y$. If the distinction is not crucial, we write simply $Y$.

**Definition 2.1.** We will say that the data $(P_K, Y, \rho)$ determines a **symplectic pair type**. We will abbreviate this to $(P_K, Y_K)$ where $Y_K = P_K \times \rho Y$. There is a corresponding **complex pair type** determined by the data $(P_G, Y, \rho)$ (abbreviated to $(P_G, Y_G)$ where $Y_G = P_G \times \rho Y$).

2.2. The configuration space of pairs. Let $A$ be the space of connections on $P_K$ and let $C$ be the space of $G$-invariant almost complex structures on $P_G$ for which the projection $P_G \to X$ is a pseudoholomorphic map.

**Proposition 2.2.** (see §2.2 in [M]) There is a bijective correspondence between $A$ and $C$.

The group $G$ acts on $A$, and the group $G^C$ acts on $C$. In both cases the action is by pullback. Thus we get a natural action of $G^C$ on $A$. This action extends that of $G$. Finally, let $A^{1,1} \subset A$ be the set of connections whose curvature is of type $(1,1)$. The set $A^{1,1}$ corresponds to integrable complex structures on $P_G$, and is clearly $G^C$-invariant. The group $G$ acts on the space of sections $\Gamma(Y)$ and, since the action of $K$ on $Y$ extends to an action of $G$, we deduce that $G^C$ acts on $\Gamma(Y)$ extending the action of $G$. We now define covariant derivations of sections of $Y = Y_K = Y_G$, generalizing the usual definition in the vector bundle case (i.e., when $Y$ is a vector space and the action of $K$ on $Y$ is linear). Any connection $A \in A$ induces a projection

$$\alpha_A : TY \to TY_v = \operatorname{Ker} d\pi,$$

where $\pi : Y \to X$ is the projection ($TY_v$ is then the vertical tangent bundle). Given any $\Phi \in \Omega^0$ we define

$$d_A \Phi := \alpha_A \circ d\Phi \in \Omega^1(X; \Phi^*TY_v).$$

To generalize the notion of $\overline{\partial}$ operators on vector bundles, observe that the complex structure on $Y$ induces a structure of complex vector bundle on $TY_v \to Y$. Hence we can decompose

$$d_A \Phi = \partial_A \Phi + \overline{\partial}_A \Phi,$$

where $\partial_A \in \Omega^{1,0}(X; \Phi^*TY_v)$ and $\overline{\partial}_A \Phi \in \Omega^{0,1}(X; \Phi^*TY_v)$.

**Remark 2.3.** If $A$ is an integrable connection on $P_K$, then a section $\Phi$ satisfies the condition $\overline{\partial}_A \Phi = 0$ if and only if it is holomorphic with respect to the holomorphic structure induced by $A$ on $Y_G$.

**Definition 2.4.**

- A **symplectic pair** on $(P_K, Y_K)$ consists of the pair $(A, \Phi)$, where $A$ is an integrable connection on $P_K$ and $\Phi$ is a section of $Y_K$ such that $\overline{\partial}_A \Phi = 0$.
- A **complex pair** on $(P_G, Y_G)$ consists of the pair $(I_G, \Phi)$, where $I_G \in C$ is a $G$-invariant holomorphic structure on $P_G$ and $\Phi$ is a holomorphic section of $Y_G$. 
Definition 2.5. The configuration space of symplectic pairs on \((P_K, Y_K)\), denoted by \(\mathcal{X}(P_K, Y_K)\) or simply \(\mathcal{X}_K\), is the subspace of \(A^{1,1} \times \Gamma(Y_K)\) defined by the condition that \(\overline{\partial}_A \Phi = 0\). Similarly, the configuration space of complex pairs on \((P_G, Y_G)\), denoted by \(\mathcal{X}(P_G, Y_G)\) or simply \(\mathcal{X}_G\), is the subspace of \(\mathcal{C} \times \Gamma(Y_G)\) defined by the condition that \(\Phi\) is holomorphic with respect to \(I_G\).

Since \(Y_K = Y_G\), there is a \(G^\mathbb{C}\)-equivariant bijection between these configuration spaces.

2.3. Moment map equations. It is an observation which goes back to the work of Atiyah and Bott \([AB]\) and Donaldson \([D]\) that the set of connections \(A\) carries a natural symplectic structure, which can be defined by combining the Kaehler structure on \(X\) with the bi-invariant metric on \(\mathfrak{k}\). Taking the restriction of this symplectic structure we get a symplectic structure on the smooth locus of the set \(A^{1,1}\). On the other hand it is possible to define a symplectic structure on \(\Gamma(Y)\) using the symplectic structure on \(\mathbb{Y}\) (see §4.2 in \([M]\) for details). Combining both structures we get a symplectic structure on the smooth locus of \(A^{1,1} \times \Gamma(Y)\), whose restriction to \(\mathcal{X}(P_K, Y_K)\) is also symplectic. It turns out that the action of \(G\) on \(\mathcal{X}(P_K, Y_K)\) preserves this structure and is Hamiltonian. Its moment map

\[
\mu_G : A^{1,1} \times \Gamma(Y) \to \text{Lie } G \subset (\text{Lie } G)^*.
\]

This formula should be interpreted as follows: \(\Lambda F_A\) is a section of \(\text{ad } P_K\), and using the \(K\)-equivariant isomorphism \(\mathfrak{k} \simeq \mathfrak{k}^*\) we can also regard \(\mu(\Phi)\) as a section of \(\text{ad } P_K\). Finally, the \(L^2\) inner product in \(\Omega^0(\text{ad } P_K)\) gives the inclusion \(\Omega^0(\text{ad } P_K) \subset \Omega^0(\text{ad } P_K)^* = (\text{Lie } G)^*\). In general the moment map of a hamiltonian action is unique only up to addition of central elements in the (dual of the) Lie algebra. In particular, if \(c \in \mathfrak{k}\) is any central element, then we can take the moment map to be

\[
(2) \quad \mu_{G,c}(A, \Phi) = \Lambda F_A + \mu(\Phi) - c.
\]

2.4. Maximal weights and definition of stability.

Definition 2.6. We will say that an element \(s \in \Omega^0(\text{ad } P_G)\) is semisimple if for any \(x \in X\), taking a \(G\)-equivariant identification \((\text{ad } P_G)_x \simeq \mathfrak{g}\), \(s(x) \in \mathfrak{g}\) is semisimple. We will say that a pair \((A, \Phi) \in \mathcal{X}(P_K, E)\) is simple if there is no semisimple element in \(\Omega^0(\text{ad } P_G)\) which leaves \((A, \Phi)\) fixed by the infinitesimal action. Observe that if a pair \((A, \Phi)\) is simple, then any pair in the orbit \(G^\mathbb{C}(A, \Phi)\) is also simple.

Let \(W = P_K \times_{\rho_s} \mathbb{W}_a\). Any connection \(A \in A^{1,1}\) induces a \(\overline{\partial}\)-operator

\[
\overline{\partial}_A : \Omega^0(W) \to \Omega^{0,1}(W).
\]

Definition 2.7. Let \(\chi \in \Omega^0(\text{ad } P_K)\). We will say that \(\chi\) induces an \(A\)-holomorphic filtration if the following two conditions are satisfied: (1) all the eigenvalues of...
\[ \rho_a(\sqrt{-1} \chi) \text{ acting on } W \text{ are constant}; \ (2) \text{ if } \alpha_1 < \cdots < \alpha_r \in \mathbb{R} \text{ are the different eigenvalues of } \rho_a(\sqrt{-1} \chi), \text{ and we define} \]
\[ W^k = \bigoplus_{j \leq k} \text{Ker}(\alpha_j \text{Id} - \rho_a(\chi)) \subset W \quad ; \quad 1 \leq k \leq r \]
then the filtration \[ W^1 \subset W^2 \subset \cdots \subset W^r = W \] is holomorphic w.r.t. \( A \), that is, \[ \bar{\partial}_A(W^k) \subset \Omega^{0,1}(W^k). \]

When these conditions hold, we define the degree of \( \chi \) to be \[ \deg(\chi) = \alpha_r \deg(W) + \sum_{k=1}^{r-1} (\alpha_k - \alpha_{k+1}) \deg(W^k), \]
where for any vector bundle \( W' \) we denote \[ \deg(W') = 2\pi(c_1(W') \cup [\omega^{n-1}], [X]). \]
(Here \([\omega^{n-1}]\) denotes the cohomology class represented by the form \( \omega^{n-1} \) and \([X] \in H_{2n}(X; \mathbb{Z}) \) is the fundamental class of \( X \).)

The condition that \( \chi \in \Omega^0(\text{ad}P_K) \) induces \( A \)-holomorphic filtration is in fact independent of the chosen representation \( \rho_a \) (provided it is faithful). Indeed, the induced holomorphic filtrations are in bijection with holomorphic reductions of the structure group of \( P_G \) to parabolic subgroups of \( G \) (see §2.7 in [M]). Finally, observe that the preceding definitions make sense when the section \( \chi \) is defined only on \( X - X_0 \) where \( X_0 \) is a closed complex subset of \( X \) and \( X - X_0 \) has codimension at least two (in this situation \( \chi \) defines a filtration of \( W|_{X_0} \), and the degree of each subbundle in the filtration is well defined, see [UY]). This will be relevant below, when we will consider elements of \( \mathfrak{h}(X_0) \).

**Definition 2.8.** Let \((\mathcal{Y}, \omega_\mathcal{Y}, I_\mathcal{Y})\) be a Kaehler manifold with a Hamiltonian action of a compact Lie group \( K \), and corresponding moment map \( \mu : \mathcal{Y} \to \mathfrak{k}^* \). Let \( x \in \mathcal{Y} \) be any point, and take an element \( s \in \mathfrak{k} \). We define the **maximal weight** \( \lambda(x; s) \) of the action of \( s \) on \( x \) to be \[ \lambda(x; s) = \lim_{t \to \infty} \langle \mu(e^{-\frac{1}{N}t}s), s \rangle \in \mathbb{R} \cup \{\infty\}. \]

Notice that the maximal weight is \( K \)-invariant in the sense that \( \lambda(x; s) = \lambda(kx; ks^{-1}) \) for any \( k \in K \) (this follows from equivariance of the moment map). For any \( \Phi \in \Gamma(Y) \) and \( \chi \in \Omega^0(\text{ad}P_K) \) we can thus define a map \( \lambda(\Phi; \chi) : X \to \mathbb{R} \cup \{\infty\} \) by using any local frame to identify \( Y_x = \mathcal{Y} \) and \( (\text{ad}P_K)_x = \mathfrak{k} \), and setting \[ \lambda(\Phi; \chi)(x) = \lambda(\Phi(x); \chi(x)). \]

**Definition 2.9.** Let \((A, \Phi)\) be any pair in \( A^{1,1} \times \Gamma(Y) \). Let \( \chi \) be a section in \( \Omega^0(\text{ad}P_K) \) which induces a \( A \)-holomorphic filtration and let \( c \) be a central element in \( \mathfrak{k} \). Define the **maximal weight**
\[ \lambda_c(A, \Phi; \chi) = \deg(\chi) + \int_X \lambda(\Phi; \chi) - \int_X \langle \chi, c \rangle. \]
Remark 2.10. We can identify $\deg(\chi)$ as a maximal weight for the action of $\chi$ on $\mathcal{A}^{1,1}$ (cf. Lemmas 4.2 and 4.3 in [3]). Thus $\lambda_c(A, \Phi; \chi)$ is in fact the maximal weight in the sense of Definition 2.8 for the action of $\chi$ on $\mathcal{A}^{1,1} \times \Gamma(Y)$, and using the moment map $\mu_G$ for product $G$-action on $\mathcal{A}^{1,1} \times \Omega^0(Y)$.

**Definition 2.11.** Let $(A, \Phi) \in \mathcal{X}(P_K, Y)$, and let $c$ be a central element in $\mathfrak{k}$. We say that $(A, \Phi)$ is $c$-stable if for

- any open subset $X_0 \subset X$ whose complementary has complex codimension $\geq 2$ and
- any $\chi \in \Omega^0(X_0; \text{ad}P_K)$ which induces a $\mathcal{A}$-holomorphic filtration, we have

\[
\deg(\chi) + \int_{X_0} \lambda(\Phi; \chi) - \int_{X_0} \langle \chi, c \rangle > 0
\]

(if the integral $\int_{X_0} \lambda(\Phi; \chi)$ is equal to $\infty$ then, since the other terms are finite numbers, we consider the left hand side as being greater than zero).

Remark 2.12. Notice that if $X_0 = X$, then [3] says $\lambda_c(A, \Phi; \chi) > 0$.

2.5. **A Hitchin–Kobayashi correspondence.** The main result in [3] describes which orbits of the $G^\mathbb{C}$ action on $\mathcal{X}(P_K, Y)$ contain solutions to the moment map equation $\mu_G(A, \Phi) = c$.

**Theorem 2.13.** (Hitchin–Kobayashi Correspondence for principal pairs) Let $c \in \mathfrak{k}$ be a central element. Let $(A, \Phi) \in \mathcal{X}(P_K, Y)$ be a simple pair. Then $(A, \Phi)$ is $c$-stable if and only if there exists a gauge transformation $g \in G^\mathbb{C}$ such that $g(A, \Phi) = (g^*A, g(\Phi))$ satisfies the generalized vortex equation

\[
\Lambda F_{g^*A} + \mu(g(\Phi)) = c
\]

Furthermore, if $(A, \Phi)$ is $c$-stable then any two such gauge transformations $g, g' \in G^\mathbb{C}$ satisfy $g'g^{-1} \in G$.

Remark 2.14. The equation [3] generalizes the vortex equations, which arise in the case $\mathbb{V} = \mathbb{C}^n$ and $K = U(n)$ acting through the fundamental representation. The vector bundle case (see Section 2.6) of this result was proved by Banfield in [34].

2.6. **The vector bundle case.** Suppose that $\mathbb{Y} = \mathbb{V}$ is a Hermitian vector space with the Kaehler structure given by the Hermitian metric, and that the action $\rho : K \times \mathbb{V} \rightarrow \mathbb{V}$ is linear, so that it comes from a morphism of groups $\rho_0 : K \rightarrow \text{U}(\mathbb{V})$. Then $\mathbb{V} = P \times_{\rho} \mathbb{V}$ is a vector bundle, and the stability condition can be greatly simplified. To explain this we need to make a definition. Let $A \in \mathcal{A}^{1,1}$, and suppose that $\chi \in \Omega^0(X_0; \text{ad}P_K)$ induces an $A$-holomorphic filtration on $W = P_K \times \rho_0 W_A$, (as in Definition 2.7) with eigenvalues $\sqrt{-1}\{\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r\}$ (here $\alpha_j \in \mathbb{R}$). Then $\sqrt{-1}\rho_0(\chi)$ induces an $A$-holomorphic filtration on $\mathbb{V}$, also consisting of subbundles spanned by eigenvectors. The eigenvalues will in general be combinations (which depend on the representation $\rho_0$) of the $\alpha_i$ (see section 3 for examples).

**Definition 2.15.** Let $V^-(\chi) \subset \mathbb{V}$ denote the subbundle spanned by the eigenvectors of the endomorphism $\sqrt{-1}\rho_0(\chi)$ with non-positive eigenvalue. By assumption $V^-(\chi)$ is a holomorphic subbundle.
Lemma 2.16. A pair \((A, \Phi) \in \mathcal{X}(P_K, Y)\) is c-stable if and only if: for any \(\chi \in \Omega^0(X_0; \text{ad}P_K)\) inducing an \(A\)-holomorphic filtration and such that \(\Phi \subset V^-(\chi)\) we have
\[
\deg(\chi) - \int_{X_0} \langle \chi, c \rangle > 0.
\]

Proof: Let \(s \in \mathfrak{t}\), and let \(V^-(s) \subset V\) be the subspace spanned by the eigenvectors of the endomorphism \(\sqrt{-1} \rho_0(s)\) with eigenvalue \(\leq 0\). Then for any \(x \in V\) we have
\[
\lambda(x; s) = \begin{cases} 0 & \text{if } x \in V^-(s) \\ \infty & \text{if } x \notin V^-(s). \end{cases}
\]
This proves the result. \(\Box\)

This result relates our general stability condition with the notion of stability given by Banfield in [Ba] in the case of vector bundle pairs.

3. Subgroups of the gauge group

3.1. The subgroup setting. Let \(\mathcal{H} \subset \mathcal{G}\) and \(\mathcal{H}^C \subset \mathcal{G}^C\) be Lie subgroups (with respect to the \(C^\infty\) topology), and consider their respective Lie algebras
\[
\mathfrak{h} = \{ s \in \Omega^0(\text{ad}P_K) | \exp(ts) \in \mathcal{H} \ \forall t \in \mathbb{R} \},
\]
\[
\mathfrak{h}^C = \{ s \in \Omega^0(\text{ad}P_G) | \exp(ts) \in \mathcal{H}^C \ \forall t \in \mathbb{R} \}.
\]

Assumptions 3.1. We make the following assumptions:
1. \(\mathfrak{h}^C = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}\),
2. the map \(\exp : \sqrt{-1} \mathfrak{h} \to \mathcal{H}^C\) induces an isomorphism \(\sqrt{-1} \mathfrak{h} \simeq \mathcal{H}^C/\mathcal{H}\).
3. \(\Omega^0(\text{ad}P_K) = \mathfrak{h}^+ \oplus \mathfrak{h}\) is a splitting of Fréchet spaces, with \(\mathfrak{h}^+\) orthogonal to \(\mathfrak{h}\) with respect to the pairing \(\int_X \langle \cdot, \cdot \rangle : \Omega^0(\text{ad}P_K) \otimes \Omega^0(\text{ad}P_K) \to \mathbb{R}\).

Remark 3.2. The first two conditions can be rephrased by saying that \(\mathcal{H}^C\) is the complexification of \(\mathcal{H}\).

Definition 3.3. Given a subgroup \(\mathcal{H}^C\), we can define \(\mathcal{H}^C\)-invariant subsets \(\mathcal{X}_\mathcal{H} \subset \mathcal{X}(P_K, Y)\). A pair \((A, \Phi) \in \mathcal{X}_\mathcal{H}\) will be called an \(\mathcal{H}\)-pair of type \((P_K, Y)\). We say that \((\mathcal{H}, \mathcal{H}^C, \mathcal{X}_\mathcal{H})\) defines a subgroup setting if
- the subgroups \(\mathcal{H} \subset \mathcal{G}\) and \(\mathcal{H}^C \subset \mathcal{G}^C\) satisfy Assumptions [3.7],
- \(\mathcal{X}_\mathcal{H} \subset \mathcal{X}(P_K, Y)\) is an \(\mathcal{H}^C\)-invariant subset, and
- \(d_A(\mathfrak{h}) \subset \mathfrak{h}^1\) where \(A\) is any connection in an \(\mathcal{H}\)-pair of type \((P_K, Y)\)
\[
\mathfrak{h}^1 = \Omega^1(X) \otimes \mathfrak{h} \subset \Omega^2(\text{ad}P_K).
\]

We will see later in Lemma [4.3] that if a connection \(A\) satisfies the condition \(d_A(\mathfrak{h}) \subset \mathfrak{h}^1\) then any other connection in the orbit \(\mathcal{G}^C A\) also satisfies it. For future reference, we make two more definitions:

Definition 3.4. Denote by \(\pi_\mathfrak{h} : \Omega^0(\text{ad}P_K) \to \mathfrak{h}\) the projection induced by the splitting \(\Omega^0(\text{ad}P_K) = \mathfrak{h}^+ \oplus \mathfrak{h}\) in (3).
Definition 3.5. If \( X_0 \subset X \) is an open subset such that \( X - X_0 \) is a complex subset of codimension at least two, define
\[
\mathfrak{h}(X_0) = \left\{ \sigma \in \Omega^0(X_0; \text{ad} P_K) \mid \forall s \in \mathfrak{h}^\perp, \int_{X_0} \langle \sigma, s \rangle = 0 \right\}.
\]

Remark 3.6. The need to formulate Definition 3.5 in this way can be understood as follows. Recall that the definition of \( c \)-stability (see Definition 2.11) uses the positivity of a certain integral defined over open subsets \( X_0 \subset X \) where \( \text{codim} \, X \setminus X_0 \geq 2 \). The integral can be interpreted as the maximal weight of the pair \((A, \Phi)\) with respect to an element in the Lie algebra for the gauge group of the restriction of \( P_K \) to \( X_0 \) (see Remark 2.12). Hence if \( \dim X \geq 2 \), it is not enough to check positivity of the maximal weights with respect only to actual elements of the Lie algebra of the gauge group.\footnote{In the Hitchin–Kobayashi correspondence for vector bundles without additional structure this translates into the need to consider reflexive subsheaves.}

We might describe the situation as follows: consider on \( X \) the topology whose open sets are of the form \( X_0 \subset X \) with \( \text{codim} \, X \setminus X_0 \geq 2 \), and let \( \mathcal{G}_{sh} \) be the sheaf such that for any open set \( X_0 \subset X \) the sections \( \Gamma(\mathcal{G}_{sh}; X_0) \) are the automorphisms of \( P_K |_{X_0} \). Then \( \mathcal{G}_{sh} \) is a sheaf of groups and, strictly speaking, the Hitchin–Kobayashi correspondence should be understood as a version of the Kempf–Ness theorem for \( \mathcal{G}_{sh} \) (acting on the sheaf defined by considering connections and sections defined over open subsets). It is only when \( \dim X = 1 \) that one can identify the sheaf \( \mathcal{G}_{sh} \) with the gauge group \( G \) and properly say that the Hitchin–Kobayashi correspondence is a theorem à la Kempf–Ness for the infinite dimensional group \( G \).

From this point of view it is clear that rather than a subgroup of the gauge group what we need is to specify a subsheaf of \( \mathcal{G}_{sh} \). Given a subgroup \( H \subset G \), Definition 3.5 is intended to provide a subsheaf \( \mathcal{H}_{sh} \subset \mathcal{G}_{sh} \) (although this is done, strictly speaking, at the level of Lie algebras), in such a way that the main result of this paper (Theorem 4.1) is a theorem à la Kempf–Ness for the sheaf \( \mathcal{H}_{sh} \).

3.2. The vortex equations for \( H \). The restriction of the Hamiltonian action of \( \mathcal{G} \) on \( \mathcal{X}(P_K, Y_K) \to H \subset \mathcal{G} \) is also Hamiltonian. Thus we can define a moment map for the \( H \)-action on the smooth locus of \( \mathcal{X}(P_K, Y_K) \). This moment map for \( H \) is the composition of \( \mu_\mathcal{G} : \mathcal{X}(P_K, Y) \to \text{Lie} \, G \subset (\text{Lie} \, \mathcal{G})^* \) with the projection \( (\text{Lie} \, \mathcal{G})^* \to \mathfrak{h}^* \) induced by the inclusion. Equivalently, it is the composition of the map \( \mu_\mathcal{G} : \mathcal{X}(P_K, Y) \to \text{Lie} \, \mathcal{G} \) with the orthogonal projection \( \pi_\mathfrak{h} : \text{Lie} \, \mathcal{G} \to \mathfrak{h} \) and the inclusion \( \mathfrak{h} \subset \mathfrak{h}^* \) induced by restricting the \( L^2 \) inner product on \( \Omega^0(\text{ad} P_K) \) to \( \mathfrak{h} \). On the other hand, the restriction of the symplectic structure on \( \mathcal{X}(P_K, Y) \) gives a symplectic structure on \( \mathcal{X}_H \). Since \( \mathcal{X}_H \) is \( H \)-invariant, we deduce that \( \mathcal{X}_H \) carries a Hamiltonian action of \( H \). Its moment map, as described above, is thus
\[
\mu_H(A, \Phi) = \pi_\mathfrak{h}(\Lambda F_A + \mu(\Phi)).
\]

Definition 3.7. Let \( c \) be a constant central element in \( \mathfrak{k} \) and let \( c_H \) be its projection onto \( \mathfrak{h} \). We say an \( H \)-pair of type \((P_K, Y)\) satisfies the \((H, c_H)\)-vortex equations if
there exists $h \in \mathcal{H}^C$ such that

$$\pi_h(\Lambda F_h(A) + \mu(h(\Phi))) = c_H$$

3.3. Stability for elements of $\mathcal{X}_\mathcal{H}$.

**Definition 3.8.** We will say that an element $s \in \mathfrak{h}^C$ is semisimple if, seen as a section in $\Omega^0(\text{ad} P_G)$, it is semisimple (see Definition 2.7). We will say that a pair $(A, \Phi) \in \mathcal{X}_\mathcal{H}$ is simple if there is no semisimple element in $\mathfrak{h}^C$ which leaves $(A, \Phi)$ fixed by the infinitesimal action. Observe that if a pair $(A, \Phi)$ is simple, then any pair in the orbit $\mathcal{H}^C(A, \Phi)$ is also simple.

**Definition 3.9.** Let $(A, \Phi) \in \mathcal{X}_\mathcal{H}$, and let $c_\mathcal{H}$ be a constant central element in $\mathfrak{h}$. We will say that $(A, \Phi)$ is $(\mathcal{H}, c_\mathcal{H})$-stable if

$$\deg(\chi) + \int_{X_0} \lambda(\Phi; \chi) - \int_{X_0} \langle \chi, c_\mathcal{H} \rangle > 0$$

whenever

- $X_0 \subset X$ is an open set such that $X - X_0$ is a complex subset of codimension at least two, and
- $\chi \in \mathfrak{h}(X_0)$ induces an $A$-holomorphic filtration.

(if the integral $\int_{X_0} \lambda(\Phi; \chi)$ is equal to $\infty$ then, since the other terms are finite numbers, we consider the left hand side as being to be greater than zero).

**Remark 3.10.** (The vector bundle case) When we are in the situation described in 2.6, the definition of $(\mathcal{H}, c_\mathcal{H})$-stability simplifies exactly in the same way as in the definition of $c$-stability.

4. **The Main Theorem**

Our Main Theorem provides the Hitchin–Kobayashi Correspondence for $\mathcal{H}$-pairs of type $(P_K, Y)$.

**Theorem 4.1.** (Main Theorem) Let $(\mathcal{H}, \mathcal{H}^C, \mathcal{X}_\mathcal{H})$ define a subgroup setting, as in Definition 3.3. Let $c_\mathcal{H}$ be a constant central element in $\mathfrak{h}$. Let $(A, \Phi) \in \mathcal{X}_\mathcal{H}$ be a simple pair. Then $(A, \Phi)$ is $(\mathcal{H}, c_\mathcal{H})$-stable if and only if there exists $h \in \mathcal{H}^C$ such that

$$\pi_h(\Lambda F_h(A) + \mu(h(\Phi))) = c_\mathcal{H}.$$  

Furthermore, if two different $h, h' \in \mathcal{H}^C$ solve equation (7), then there exists $k \in \mathcal{H}$ such that $h' = kh$.

The proof will follow very closely that of Theorem 2.19 in [M] (which in turn relies on [Br]). Consequently we will sketch the main steps and will only explain in detail the new features of the proof. Note that there are many papers proving similar results, and we refer to [M] for a partial list of them.
4.1. Preliminaries. To begin with, it is convenient to complete our spaces by means of suitable Sobolev norms. Take \( p > 2n \). We use the \( L^p_2 \) norm on \( \mathcal{G}^C \) and the \( L^p_1 \) norm on \( \Gamma(Y) \). Let \( A_0 \in \mathcal{A} \) be any smooth connection and write

\[
A_{L^p_1} = A_0 + L^p_1(T^*X \otimes \text{ad}P_K) .
\]

With this choices, \( (\mathcal{G}^C)_{L^p_2} \) is a Hilbert Lie group which acts smoothly on \( A_{L^p_1} \) and on \( \Gamma(Y)_{L^p_1} \). We have \( \text{Lie}(\mathcal{G})_{L^p_2} = L^p_2(\text{ad}P_K) \), and \( (\mathcal{G}^C)_{L^p_2} \) is the complexification of \( (\mathcal{G})_{L^p_2} \). We also take the completions \( \mathcal{H}_{L^p_2} \) and \( \mathcal{H}_{L^p_2}^C \) of \( \mathcal{H} \) and \( \mathcal{H}^C \) with respect to the \( L^p_2 \) norm, which are Lie subgroups of \( (\mathcal{G})_{L^p_2} \) and \( (\mathcal{G}^C)_{L^p_2} \) respectively. The Lie algebra of \( \mathcal{H}_{L^p_2} \) (resp. \( \mathcal{H}_{L^p_2}^C \)) is the completion \( \mathfrak{h}_{L^p_2} \) (resp. \( \mathfrak{h}_{L^p_2}^C \)) of \( \mathfrak{h} \) (resp. \( \mathfrak{h}^C \)) with respect to the \( L^p_2 \) norm. Observe that \( \mathcal{H}_{L^p_2} \) (resp. \( \mathcal{H}_{L^p_2}^C, \mathfrak{h}_{L^p_2}, \mathfrak{h}_{L^p_2}^C \)) is the closure of \( \mathcal{H} \) (resp. \( \mathcal{H}^C, \mathfrak{h}, \mathfrak{h}^C \)) in \( (\mathcal{G})_{L^p_2} \) (resp. \( (\mathcal{G}^C)_{L^p_2}, L^p_2(\text{ad}P_K), L^p_2(\text{ad}P_G) \)). Since the splitting \( \Omega(\text{ad}P_K) = \mathfrak{h}^+ \oplus \mathfrak{h} \) is assumed to be of Fréchet sets, it follows that \( \pi_\mathfrak{h} \) extends to a continuous operator

\[
(\pi_\mathfrak{h})_{L^p_2} : L^p_2(\text{ad}P_K) \to \mathfrak{h}_{L^p_2}.
\]

Most of the time we will avoid writing the Sobolev subscripts, and the Sobolev norms will be implicitly assumed. Recall (see [2]) that \( \mathcal{X}_\mathcal{H} \) admits on its smooth locus a Kaehler structure which is preserved by the action of \( \mathcal{H} \). Furthermore, the action of \( \mathcal{H} \) admits a moment map \( \mu_\mathcal{H} : \mathcal{X}_\mathcal{H}^0 \to \mathfrak{h}^* \cong \mathfrak{h} \), where \( \mathcal{X}_\mathcal{H}^0 \) is the smooth locus in \( \mathcal{X}_\mathcal{H} \). It follows from the general results in [1] that for any central \( c_\mathcal{H} \in \mathfrak{h} \) there exists a function \( \Psi : \mathcal{X}_\mathcal{H} \times \mathcal{H}^C \to \mathbb{R} \), called the integral of the moment map, which satisfies the following key properties:

1. If \( s \in \mathfrak{h} \), \( \Psi((A, \Phi), e^{\sqrt{-1}ts}) = \int_0^1 \langle \mu_\mathcal{H}(e^{\sqrt{-1}ts}(A, \Phi)) - c_\mathcal{H}, s \rangle dt \), and if \( h \in \mathcal{H} \), then \( \Psi((A, \Phi), ke^{is}) = \Psi((A, \Phi), e^{is}) \).
2. If \( g, h \in \mathcal{H}^C \), then \( \Psi((A, \Phi), g) + \Psi(g(A, \Phi), h) = \Psi((A, \Phi), hg) \).
3. Let \( \Psi_{(A, \Phi)} : \mathcal{H}^C \to \mathbb{R} \) be the restriction of \( \Psi \) to \( (A, \Phi) \times \mathcal{H}^C \). The element \( g \in \mathcal{H}^C \) is a critical point of \( \Psi_{(A, \Phi)} \) if and only if \( (A', \Phi') = g(A, \Phi) \) satisfies \( \Lambda F_{A'} + \mu(\Phi') = c_\mathcal{H} \).

Note that (1) defines \( \Psi \) and that (3) follows from (1) and (2). In view of the Kaehler structure on \( \mathcal{X}_\mathcal{H} \) and the action of \( \mathcal{H} \), it makes sense to define maximal weights in the same way as we did for the action of \( \mathcal{G} \) on \( \mathcal{X} \) in 2.4. Thus, as in Definition 2.9, we define

\[
\lambda_{c_\mathcal{H}}(A, \Phi; \chi) = \text{deg}(\chi) + \int_X \lambda(\Phi; \chi) - \int_X \langle \chi, c_\mathcal{H} \rangle .
\]

Furthermore, this also makes sense in the singular locus of \( \mathcal{X}_\mathcal{H} \) because \( \mathcal{X}_\mathcal{H} \subset A \times \Gamma(Y) \) and the latter space is smooth and Kaehler. By Lemma 4.3 in [1], we have the following.

**Lemma 4.2.** Let \( (A, \Phi) \) be a pair in \( \mathcal{X}_\mathcal{H} \). Take \( \chi \in \mathfrak{h} \) and \( c_\mathcal{H} \) a constant central element in \( \text{Lie}(\mathcal{H}) \). If the maximal weight \( \lambda_{c_\mathcal{H}}((A, \Phi); \chi) < \infty \), then \( \chi \) induces an \( A \)-holomorphic filtration and

\[
\lambda_{c_\mathcal{H}}((A, \Phi); \chi) = \text{deg}(\chi) + \int_X \lambda(\Phi; \chi) - \int_X \langle \chi, c_\mathcal{H} \rangle .
\]
In view of the Kaehler interpretation of the equation \( \pi_b(\Lambda F_A + \mu(\Phi)) = c_H \), the uniqueness claim of the theorem now follows from general results on convexity of the integral of the moment map (see (3) of Proposition 3.3 and Theorem 5.4 in [M]). The next two lemmas show that the property \( d_A^h \subset \mathfrak{h}^1 \) depends only on the \( \mathcal{H}^C \) orbit of \( A \).

**Lemma 4.3.** Let \( A \in \mathcal{A} \) be such that \( d_A^h \subset \mathfrak{h}^1 \). Then for any \( g \in \mathcal{H}^C \) we have \( d_{g(A)}^h \subset \mathfrak{h}^1 \).

**Proof:** Use the decomposition \( \Omega^1(\text{ad} P_G) = \Omega^{1,0}(\text{ad} P_G) \oplus \Omega^{0,1}(\text{ad} P_G) \) to split the covariant derivative as \( d_A = \partial_A + \tilde{\tau}_A \). Then \( d_{g(A)} = (g^*)^{-1} \partial_A g^* + g \tilde{\tau}_A g^{-1} \). From this it follows easily that \( d_{g(A)}(\mathfrak{h}^C) \subset \Omega^1(X) \otimes \mathfrak{h}^C \). But since \( d_{g(A)} \) is a \( K \)-connection (i.e., it is compatible with Cartan involution) we deduce from the above inclusion that \( d_{g(A)}^h \subset \mathfrak{h}^1 \).

**Lemma 4.4.** Suppose that \( A \in \mathcal{A} \) satisfies \( d_A^h \subset \mathfrak{h}^1 \). Then \( \Delta_A^{\mathfrak{h}} = d_A^h \subset \mathfrak{h} \).

**Proof:** It suffices to check that \( d_A^h \subset \mathfrak{h} \). Let \( (\mathfrak{h}^\perp)^1 = \Omega^1(X) \otimes \mathfrak{h}^\perp \). Given \( a, b \in \Omega^0(\text{ad} P_K) \), integration by parts gives

\[
\int_X \langle d_A a, b \rangle = - \int_X \langle a, d_A b \rangle,
\]

from which we deduce that \( d_A^h \subset (\mathfrak{h}^\perp)^1 \). Let now \( h \in \mathfrak{h}^1 \) and \( l \in \mathfrak{h}^\perp \). Then

\[
\int_X \langle d_A h, l \rangle = \int_X \langle h, d_A l \rangle = 0,
\]

so that \( d_A^h \subset (\mathfrak{h}^\perp)^1 = \mathfrak{h} \).

**4.2. Stability implies existence of solutions.** This section and the next one provide only a sketch of the proof, with the emphasis on the modifications required in the proof for the case \( \mathcal{H} = \mathcal{G} \). See [B], [M] for more details. Let \( (A, \Phi) \in \mathcal{X}_h \) be a simple and \( (\mathcal{H}, c_H) \)-stable pair. Let us prove that there is a solution of (7) in the \( \mathcal{H}^C \) orbit of \( (A, \Phi) \). Let \( \text{Met}_2^p = \sqrt{-1} \mathfrak{h}_{L_B^p} \), and consider the functional

\[
\Psi_h : \text{Met}_2^p \to \mathbb{R}
\]

\[
s \mapsto \Psi_h((A, \Phi), e^s).
\]

The elements \( s \in \text{Met}_2^p \) moving the pair \( (A, \Phi) \) to a solution of equation (7) are precisely the critical points of \( \Psi_h \). For technical reasons it is convenient to restrict ourselves to the set

\[
\text{Met}_{2,B}^p = \{ s \in \text{Met}_2^p \mid \| \pi_b(\Lambda F_{e^s A} + \mu(e^s \Phi) - c) \|_{L^p} \leq B \},
\]

for some \( B \geq 0 \). To do this safely it is necessary to check that the critical points of the restriction of \( \Psi_h \) to \( \text{Met}_{2,B}^p \) are also critical points of \( \text{Met}_2^p \). It is here that one needs the pair \( (A, \Phi) \) to be simple and the inclusion \( \Delta_h(A) \subset \mathfrak{h} \) for any \( h \) and \( A \) given by Lemmas 1.3 and 4.4 (see §6.2.1 in [M]). The key step of the proof is to deduce from \( (\mathcal{H}, c_H) \)-stability the existence of positive constants \( C_1, C_2 \) such that for any \( s \in \text{Met}_{2,B}^p \)

\[
(8) \quad |s|_{C^0} \leq C_1 \Psi_h(s) + C_2.
\]

This can be done following word by word §6.2.1 in [M] (and note that it is here that one needs to use \( A \in \mathcal{A}^{1,1} \), to invoke a theorem of Uhlenbeck and Yau on weak \( L_1^p \).
bundles; see \([\mathcal{B}, \mathcal{M}, \mathcal{U}, \mathcal{Y}]\). Finally, it follows from the inequality that there exists some \(s \in \mathcal{M} \) which is a critical point of \(\Psi_h\), and hence gives a solution to equation (9) (see \([\mathcal{B}, \mathcal{M}]\)) (to prove this one takes a minimizing sequence \(\{s_j\} \subset \mathcal{M}\) and uses the inequality to deduce convergence—in a suitable sense—of a subsequence).

Furthermore, one can prove that \(s\) is smooth using elliptic regularity (see \([\mathcal{B}]\)).

4.3. Existence of solution implies stability. Let \((A, \Phi) \in \mathcal{X}_H\) be a simple pair, and assume that there exists \(h \in \mathcal{H}^c\) solving equation (7). Our aim is to prove that \((A, \Phi)\) is \((\mathcal{H}, c_H)\)-stable. In view of Lemma 4.2, this is equivalent to showing that for any open \(X_0 \subset X\) whose complement in \(X\) has complex codimension \(\geq 2\) and any \(\chi \in \mathfrak{h}(X_0)\) we have

\[
\lambda_{c_H}((A, \Phi); \chi) > 0.
\]

This is done in two steps. First one checks that \(\lambda_{c_H}(h(A, \Phi); \chi) > 0\) for any \(\chi \in \mathfrak{h}(X_0)\), i.e., that \(h(A, \Phi)\) is indeed \((\mathcal{H}, c_H)\)-stable. In particular, this implies, using the same methods as in 4.2, that we have an inequality of the type (8). In the second step one uses this inequality to relate the maximal weights at \(h(A, \Phi)\) to those at \((A, \Phi)\), thus deducing from the \((\mathcal{H}, c_H)\)-stability of \(h(A, \Phi)\) that \((A, \Phi)\) has also to be \((\mathcal{H}, c_H)\)-stable.

See §6.3 in \([\mathcal{M}]\) for more details.

5. Stability Simplification Conditions (SSC)

For some choices of the data \((K, P_K, \mathcal{Y}, \rho)\) the general stability condition 2.11 can be simplified. Classical examples are the stability condition for holomorphic vector bundles and for holomorphic pairs. In both cases the general stability condition, which refers to any filtration of the vector bundle by holomorphic subbundles (or in general reflexive subsheaves), can in fact be reduced to the same condition considered only for subbundles. Our aim in this section is to find a general condition (which we call Stability Simplification Condition) which implies the possibility of simplifying the stability condition. Our results are similar in spirit to those by Schmitt in \([\mathcal{S}]\).

5.1. Statement of the conditions.

**Definition 5.1.** Fix data \((K, P_K, \mathcal{Y}, \rho)\). For any open subset \(U \subset X\) and subset \(S \subset \Omega^0(\text{ad} P_K)\) we denote by \(S^\perp|_U\) the image of \(S^\perp \subset \Omega^0(X; \text{ad} P_K)\) under the restriction map \(\Omega^0(X; \text{ad} P_K) \to \Omega^0(U; \text{ad} P_K)\).

We say that the Stability Simplification Condition (or SSC for short) applies if there is a subset \(S \subset \Omega^0(\text{ad} P_K)\) such that for any \(A \in \mathcal{A}^{1,1}\):

- (SSC1) for any \(X_0 \subset X\) whose complement has complex codimension at least two, any \(\sigma \in \Omega^0(X_0; \text{ad} P_K)\) inducing a \(\overline{\partial}_A\) holomorphic filtration can be written

  \[
  \sigma = s_1 + \cdots + s_r,
  \]

  where each \(s_i\) belongs to the orthogonal complement of \(S^\perp|_{X_0}\) and induces a \(\overline{\partial}_A\) holomorphic filtration.

- (SSC2) for any pair \((A, \Phi) \in \mathcal{X}(P_K, \mathcal{Y})\) the condition \(\lambda(\Phi; \sigma) > 0\) is equivalent to the condition \(\lambda(\Phi; s_i) > 0\) for any \(i\).
If we are in the situation described in 2.6, i.e. if $\mathbb{Y} = V$ is a vector space, then we can replace condition (SSC2) by the following one.

- (SSC2') for any pair $(A, \Phi) \in \mathcal{X}(P_K, V)$ the condition $\Phi \subset V^-(\sigma)$ is equivalent to the condition $\Phi \subset V^-(s_i)$ for any $i$.

**Remark 5.2.** The need to formulate (SSC1) in this way can be traced back to the same considerations as those behind Definition 3.5 (see Remark 3).

The following is almost a tautology.

**Lemma 5.3.** Let $(A, \Phi) \in \mathcal{X}(P_K, \mathbb{Y})$ be any pair, and let $c \in \mathfrak{k}$ be a central element. Then $(A, \Phi)$ is $c$-stable if and only if for any $X_0 \subset X$ whose complementary has complex codimension $\geq 2$ and any $s \in S|X_0$ inducing an $A$-holomorphic filtration we have

$$\deg(s) + \int_{X_0} \lambda(\Phi; s) - \int_{X_0} \langle s, c \rangle > 0.$$ 

**Remark 5.4.** If $\mathbb{Y} = V$ is a vector space, then there is an obvious simplification of Lemma 5.3 using the results in 2.6. The details are left to the reader.

The following lemma will be useful later to prove that SSC holds in some situations.

**Lemma 5.5.** If SSC holds for $(K_i, P_K, \rho_i, \mathbb{Y})$ for $i = 1, 2$ and the actions $\rho_1, \rho_2$ of $K_1, K_2$ on $\mathbb{Y}$ commute, then $\rho_1 \times \rho_2$ defines an action of $K_1 \times K_2$ on $\mathbb{Y}$ and SSC holds for $(K_1 \times K_2, P_1 \times P_2, \rho_1 \times \rho_2, \mathbb{Y})$.

**Proof:** It follows from the fact that if $x \in \mathbb{Y}$ and $s_i \in K_i$ for $i = 1, 2$, then we have this equality between maximal weights:

$$\lambda(x; s_1 + s_2) = \lambda(x; s_1) + \lambda(x; s_2).$$

□

**Example 5.6.** Take $\mathbb{Y} = \mathbb{C}^n$, $K = U(n)$ and $\rho$ the fundamental representation; the resulting principal pairs correspond to rank $n$ vortices. We then can take the set $S$ to be the set of endomorphisms $\chi$ of the vector bundle $V = P_K \times_K \mathbb{C}^n$ such that:

1. the characteristic polynomial $P(\chi_x)$ of $\chi$ acting on the fibre over $x \in X$ does not depend on $x$,
2. $P(\chi_x)$ has at most two different roots, and both come from the set $\{0, \pm \sqrt{-1}\}$,
3. for any root $\alpha$ of $P(\chi_x)$ the set $\text{Ker}(\chi - \alpha \text{Id})$ is in fact a subbundle of $V$.

**Example 5.7.** If we take $\mathbb{Y} = S^2 \mathbb{C}^n$ with the obvious linear action of $K = U(n)$ then the resulting principal pairs are the so called quadric bundles. Then we can take $S$ to be the set of endomorphisms which induce holomorphic filtrations of length at most three (i.e., the sections $\chi \in \Omega^0(\text{ad}P_K)$ belonging to $S$ have at most three different eigenvalues). This is proved in [GGM].
In both examples, the reason why one can restrict to filtrations of lengths two and three is the following. Let us fix a pair \((A, \Phi)\) and consider a \(\overline{\partial}_A\)-holomorphic filtration

\[
\mathcal{V} = (0 \subset V_1 \subset \cdots \subset V_k = V)
\]

induced by any section of \(\text{ad} P_K\). Any choice of real numbers \(\lambda = (\lambda_1 < \cdots < \lambda_k)\) determines a negative subbundle \(V_\lambda^- \subset V\). Moreover, there is a unique section \(\chi_{\mathcal{V}, \lambda}\) in \(\Omega^0(\text{ad} P_K)\) inducing the filtration \(\mathcal{V}\) and having \(\sqrt{-1}\lambda\) as set of eigenvalues. With

\[
f(\lambda) = \deg(\chi_{\mathcal{V}, \lambda}) - \int \langle \chi_{\mathcal{V}, \lambda}, c \rangle,
\]

the stability condition becomes: \(f(\lambda) > 0\) whenever \(\Phi \subset V_\lambda^-\). Now let

\[
\Lambda = \Lambda(\mathcal{V}, \Phi) = \{\lambda \mid \Phi \subset V_\lambda^-\}.
\]

This is a convex polytope. Since \(f(\lambda)\) is a linear function, it is thus enough to check the stability condition for the choices of \(\lambda\) belonging to any subset \(\Lambda' \subset \Lambda\) whose convex hull is \(\Lambda\). To prove that SSC applies in the above two examples one has to study what are the possible sets \(\Lambda\) which can appear and find in each case suitably simple subsets \(\Lambda'\). See the proof of Proposition 5.3 for more details.

5.2. **SSC in the \(H\) case.** If SSC holds for \((K, P_K, \rho, \mathcal{Y})\) and the following refinement of (1) is true: \((1')\) any \(\sigma \in \mathfrak{h}\) can be written as \(\sigma = s_1 + \cdots + s_r\), where each \(s_i\) belongs to \(S \cap \mathfrak{h}\) inducing an \(A\)-holomorphic filtration, then we say that SSC holds for the action of \(H\). The following generalization of Lemma 5.3 holds then true.

**Lemma 5.8.** Let \((A, \Phi) \in \mathcal{X}(P_K, \mathcal{Y})\) be any pair, and let \(c_H \in \mathfrak{h}\) be a central element. Then \((A, \Phi)\) is \((H, c_H)\)-stable if and only if for any \(X_0 \subset X\) whose complement has complex codimension at least two and any \(s \in S|_{X_0} \cap \mathfrak{h}\) inducing a \(A\)-holomorphic filtration we have

\[
\deg(s) + \int_{X_0} \lambda(\Phi; s) - \int_{X_0} \langle s, c_H \rangle > 0.
\]

6. **Subgroups determined by the structure group**

In many examples, the subgroups of the gauge groups come from subgroups of the structure groups of the principal bundles. In this section we describe three types of such subgroups. While there are undoubtedly other mechanisms which can produce subgroups of this kind, the three we describe account for a surprisingly broad range of examples.

6.1. **Reduction of the structure group.** Suppose that the principal \(K\)-bundle \(P_K\) admits a reduction of the structure group to \(H\). Let \(P_H\) be the corresponding principal bundle. Thus \(P_K = P_H \times_H K\) and the Adjoint bundle of \(P_H\), i.e.

\[
\text{Ad} P_H = P_H \times_{\text{Ad}} H,
\]

of vector subbundles if \(\text{dim} X = 1\), or subsheaves in general if \(\text{dim} X > 1\)
is a subbundle of $\text{Ad}P_K = P_K \times _H K$. The gauge group of $P_H$, i.e. $\mathcal{G}(P_H) = \Omega^0(\text{Ad}P_H)$, can thus be viewed as a subgroup of $\mathcal{G} = \Omega^0(\text{Ad}P_K)$. The subgroup of $\mathcal{G}$ corresponding to $H$ is given, in this case, by

$$\mathcal{H} = \mathcal{G}(P_H)$$

Similarly, the complexification $\mathcal{H}^C$ is the (complex) gauge group of the $H^C$-principal bundle $P_H \times _H H^C$.

**Lemma 6.1.** The subgroup $\mathcal{H}$ and its complexification satisfy the conditions in section (3.1). If $A$ is a connection on $P_K$ which comes from a connection on $P_H$, i.e. if $A$ is an $H$-connection, then $d_A(h) \subset h^1$.

**Proof:** The first statement is obvious. With respect to a local frame, any connection on $P_K$ can be written as $dA = d + a$ where $a$ is a 1-form with values in the Lie algebra of $K$. If $A$ is an $H$-connection, then we can assume that $a$ takes its values in the Lie algebra of $H$. The result follows from this. \qed 

Set

$$\mathcal{X}_H = \{(A, \Phi) \in \mathcal{X}(P_K, Y) | A \text{ is an } H-\text{connection}\},$$

(11)

**Lemma 6.2.** The subspace $\mathcal{X}_H \subset \mathcal{X}(P_K, Y)$ is invariant under the action of $\mathcal{H}$ and also under the extension of this action to $\mathcal{H}^C$.

**Proof:** Since $A$ is an $H$-connection, it can be viewed as a connection on $P_H$. Moreover, $Y_K$ can be described as a an associated bundle to $P_H$, i.e. as $Y = P_H \times _H Y$. Thus the pair $(A, \Psi)$ can equally well be treated as principal pairs of type $(P_H, Y)$. The results follows from this. \qed 

Given a principal pair type $(P_K, Y_K)$ in which the structure group of $P_K$ reduces to $H$, we can thus apply the Main Theorem to any simple pair $(A, \Psi)$ in $\mathcal{X}_H$. Such pairs can, however, be viewed as a pair of type $(P_H, Y_H)$. The two points of view are equivalent:

**Proposition 6.3.** Treating the pair $(A, \Phi)$ as a pair of type $(P_H, Y_H)$ is equivalent to treating it as an $\mathcal{H}$-pair of type $(P_K, Y_K)$. More precisely: the stability notions coincide, as do the generalized vortex equations, and the Hitchin–Kobayashi correspondence for pairs of type $(P_H, Y_H)$ is equivalent to the Main Theorem applied to $H$-pairs of type $(P_K, Y_K)$.

**Remark 6.4.** Consider, for example, the case in which $H$ is a subgroup of $U(n)$ and $Y = \mathbb{C}^n$. The bundle $E = P_H \times _H Y$ can then be viewed as a rank $n$ vector bundle with a reduction of its structure group from $U(n)$ to $H$. In this case Proposition 6.3 says that principal pairs of type $(P_H, Y_H)$ can equally well be viewed as special holomorphic pair on the vector bundle $E = P_H \times \mathbb{C}^n$; namely as holomorphic pairs which are compatible with the reduction of structure group of $E$ from $U(n)$ to $H$.

6.2. **Normal subgroups.** Let $H$ be a normal subgroup of $K$ and suppose that its complexification $H^C$ is a normal subgroup of $G$. Even if the structure group of $P_K$ does not reduce to $H$, we can define the subbundle

$$P_K \times _{\text{Ad}} H \subset \text{Ad}P_K,$$

(12)
We thus get a subgroup
\[
\mathcal{H} = \Omega^0(P_K \times \text{Ad} H) \subset \mathcal{G}
\]
(13)
with complexification
\[
\mathcal{H}^C = \Omega^0(P_G \times \text{Ad} H^C) \subset \mathcal{G}^C.
\]
(14)

**Lemma 6.5.** Subgroups of this sort satisfy the conditions (1)-(3) in section (3.1). Furthermore, given any connection, \(A\), on \(P_K\), we get \(d_A(\mathfrak{h}) \subset \mathfrak{h}^1\).

**Proof:** The fact that \(d_A(\mathfrak{h}) \subset \mathfrak{h}^1\) follows directly from the fact that if \(H \subset K\) is a normal subgroup, then the Lie algebra of \(H\) is an ideal in the Lie algebra of \(K\). \(\square\)

**Remark 6.6.** The main examples we have in mind (see Section 7) all have real structure groups of the form
\[
K = K_1 \times K_2 \times \cdots \times K_n.
\]
(15)
The normal subgroups of \(K\) are obtained by restricting to the identity element in some of the factors.

### 6.3. Constant gauge transformations

Suppose that \(P_K = X \times K\), i.e. suppose that \(P_K\) is the trivial principal \(K\)-bundle. Then \(K\) (or indeed any subgroup \(H \subset K\)) embeds in \(\mathcal{G}\) as the group of constant gauge transformations. In this case, given \(H \subset K\), we may take
\[
\mathcal{H} = H, \quad \mathcal{H}^C = H^C.
\]
(16)
The requirements of section (3.1) are certainly satisfied by subgroups of this kind. We can define an \(\mathcal{H}^C\)-invariant subspace in \(\mathcal{X}(P_K, Y)\) by fixing the connection to be the trivial connection on the trivial bundle, i.e. we can define
\[
\mathcal{X}_\mathcal{H} = \{(0, \Phi) \in \mathcal{X}(P_K, Y) \mid 0 \text{ denotes the trivial connection}\}.
\]
(17)
Since the connection in \(\mathcal{X}_\mathcal{H}\) is trivial, the requirements of Definition 3.3 are satisfied by \((\mathcal{H}, \mathcal{H}^C, \mathcal{X}_\mathcal{H})\), i.e. \(d_A(\mathfrak{h}) \subset \mathfrak{h}^1\) for all connections which occur in \(\mathcal{X}_\mathcal{H}\).

**Remark 6.7.** We can generalize this situation to the case in which \(K = K_1 \times K_2\) (or indeed a product of more than two factors), and \(P_K\) is a fibre product (say \(P_{K_1} \times P_{K_2}\)) with one of the factors trivial. An example of this sort arises in the description of Coherent Systems (see Section 7.3).

### 7. Examples

Subgroups of the gauge group occur naturally when the gauge group is a product of two or more groups. In this section we describe some examples of this sort. Thus we consider principal pairs for which the complex structure group is a product, say
\[
G = G_1 \times G_2 \times \cdots \times G_p,
\]
(18)
and the principal \(G\) bundle is a fibre product, say
\[
P_G = P_{G_1} \times P_{G_2} \times \cdots \times P_{G_p},
\]
(19)
where $P_{G_i}$ is a principal $G_i$-bundle. If the compact real form of $G_i$ is $K_i$, then the compact real form of $G$ is

$$K = K_1 \times K_2 \times \cdots \times K_p.$$ 

The real principal bundle obtained by a reduction of structure group from $G$ to $K$ is thus

$$P_K = P_{K_1} \times P_{K_2} \times \cdots \times P_{K_p}.$$ 

The natural subgroups of the gauge groups in such examples are determined by restricting to the identity element in one or more of the factors in $G_1 \times G_2 \times \cdots \times G_p$. To complete the specification of a principal pair type we need a Kaehler manifold together with a Hamiltonian $K$-action. In this paper we restrict ourselves to examples in which the Kaehler manifold is a vector space $V$. In most of our examples $V$ has the form

$$V = V_1 \otimes V_2 \otimes \cdots \otimes V_p$$

where for $1 \leq i \leq p$ each $V_i$ is a complex vector space of dimension $n_i$. Let $\langle \ , \rangle_i$ be a hermitian inner product on $V_i$ and let $\omega_i$ be the corresponding Kaehler form. Thus

$$\omega_i(x, y) = \frac{1}{2\sqrt{-1}}(\langle x, y \rangle_i - \langle y, x \rangle_i).$$

We let $\langle \ , \rangle$ be the hermitian inner product on $V$ determined by the inner products on the $V_i$. Thus if, for $1 \leq i \leq p$, the collections $\{e^1_i, e^2_i, \ldots, e^{n_i}_i\}$ is a unitary frame for $V_i$, then the tensor products $\{e^1_i \otimes e^2_i \otimes \cdots \otimes e^{n_i}_i\}$ form a unitary frame for $V$. We let $\Omega$ be the corresponding Kaehler form. The principal pairs we consider are then of the form $(P_K, V_K)$, with $V_K = P_K \times U_1 V_1$. Furthermore in this section we consider only examples in which the $K$ action on $V$ arises from a representations $\rho : K \to U$, where $U$ denotes the group of unitary transformations on $(V, \langle \ , \rangle)$. If we let $E_i = P_{K_i} \times U_i V_i$ denote the vector bundle associated to $P_{K_i}$ by the standard representation of $U_i$ on $V_i$, then from the holomorphic point of view, the pairs $(P_K, V_K)$ are equivalent to

- a collection of holomorphic bundles $\mathcal{E}_1, \ldots, \mathcal{E}_p$ together with
- a section of an associated holomorphic bundle $V_K$.

Here $\mathcal{E}_i$ denotes the holomorphic bundle obtained by putting a holomorphic structure on the smooth bundle $E_i$, and similarly $V_K$ denotes the holomorphic bundle obtained by putting a holomorphic structure on the smooth bundle $V_K$. The form of the generalized vortex equations on pairs of this type, and in particular the equations which result from passing to a subgroup of the gauge group (as in our Main Theorem), is the result of the following observations. Notice that a connection on $P_K$ is equivalent to a $p$-tuple of connections on $P_{K_1}, \ldots, P_{K_p}$. Writing $A = (A_1, \ldots, A_p)$, we see that the corresponding curvature term in the generalized vortex equations (cf. (1)) is of the form

$$\Lambda F_A = (\Lambda F_{A_1}, \ldots, \Lambda F_{A_p}),$$

where $\Lambda F_A$ takes its values in $\text{Lie } G = \bigoplus \text{Lie } G_i$ (and each $\Lambda F_{A_i}$ has its values in $\text{Lie } G_i$).

The other term in the vortex equation is described in the following lemma.

**Lemma 7.1.** Let $G_i$ be the gauge group for $P_{K_i}$, and write $G = G_1 \times \cdots \times G_p$. Fix faithful unitary representations $\rho_{a,i} : K \to U(W_{a,i})$, where $W_{a,i}$ are finite dimensional
Hermitian vector spaces. As in Section 2.4, use these to define inner products on Lie $G_i$, and hence to get inclusions $\text{Lie } G_i \subset (\text{Lie } G_i)^*$. Let

$$\mu_i : \Omega^0(V_K) \hookrightarrow \text{Lie } G_i \subset (\text{Lie } G_i)^*$$

be the moment map for the action of $G_i$. Then the moment map for $G$ is the map

$$\mu : \Omega^0(V_K) \hookrightarrow \bigoplus_{i=1}^P \text{Lie } G_i \subset \bigoplus_{i=1}^P (\text{Lie } G_i)^* = (\text{Lie } G)^*$$

given by

$$\mu(\Phi) = (\mu_1(\Phi), \ldots, \mu_P(\Phi)).$$

7.1. Example 1 (Tensor product bundles). Consider the case$^3$ where

- $G_i = \text{GL}(n_i)$ and $K_i = \text{U}(n_i)$ for $i = 1, 2$, so
- $G = \text{GL}(n_1) \times \text{GL}(n_2)$, and correspondingly $K = \text{U}(n_1) \times \text{U}(n_2)$,
- $V_i = \mathbb{C}^{n_i}$ for $i = 1$ and $2$ and $\rho_i : G_i \rightarrow \text{GL}(n_i)$ is the standard representation on $V_i$.

We take $V = V_1 \otimes V_2$ and let $\rho = \rho_1 \otimes \rho_2$ be the tensor product representation of $G = \text{GL}(n_1) \times \text{GL}(n_2)$ on $V$. Thus $V_G = P_G \times_\rho V$ is a tensor product of vector bundles, i.e.

$$V_G = V_1 \otimes V_2,$$

where $V_i = P_{\text{GL}(n_i)} \times_\rho V_i$ is the rank $n_i$ vector bundle associated to $P_{\text{GL}(n_i)}$. A holomorphic structure on $P_G$ is thus equivalent to a holomorphic structure on $V_G$ such that the resulting holomorphic vector bundle, $\mathcal{V}$, is a tensor product of holomorphic bundles, i.e. $\mathcal{V} = V_1 \otimes V_2$ (where $V_i$ denotes holomorphic bundle obtained by putting a holomorphic structure on $V_i$). Hence With $P_G$ and $V_G$ as above, a principal pair of type $(P_G, V_G)$ is equivalent to a pair $(\mathcal{V}, \Phi)$, where $\mathcal{V}$ is holomorphic bundle of the form $V_1 \otimes V_2$ and $\Phi$ is a holomorphic section of $\mathcal{V}$. We now show how our Main Theorem leads to a Hitchin–Kobayashi correspondence for pairs on $V_1 \otimes V_2$ but for which one of the tensor factors is regarded as fixed.$^4$ The complex gauge group $G = \text{GL}(n_1) \times \text{GL}(n_2)$ has a normal subgroup defined by

$$H^C = \text{GL}(n_1) \times \{1\}.$$

Let $H^C$ be the corresponding subgroup of $G^C$, as in Section 6.2. This is the complexification of the subgroup $H \subset G$ defined by $H = \text{U}(n_1) \times \{1\}$. The corresponding Lie algebras are the following:

$$\text{Lie } G = \{(s_1, s_2) \in \Omega^0(\text{End } V_1) \oplus \Omega^0(\text{End } V_2) \mid s_1 + s_1^* = 0, \ s_2 + s_2^* = 0 \},$$

and

$$\mathfrak{h} = \{(s_1, s_2) \in \text{Lie } G \mid s_2 = 0\}.$$

$^3$There is an obvious generalization of this example to tensor products with more than two factors. Since no new ideas are involved, we discuss only the simplest case.

$^4$Of course we can identify $V_1 \otimes V_2$ with $\text{Hom}(V_2^*, V_1)$ (or, for that matter, with $\text{Hom}(V_1^*, V_2)$). This may tempt one to interpret a pair $(\mathcal{V}_1 \otimes \mathcal{V}_2, \Psi)$ as a holomorphic triple (as in $[\text{BG}]$) on $(\mathcal{V}_1, \mathcal{V}_2^*)$, but one should resist the temptation. The two types of augmented bundle differ precisely in the fact that the bundles underlying the pair $(\mathcal{V}_1 \otimes \mathcal{V}_2, \Psi)$ are $\mathcal{V}_1$ and $\mathcal{V}_2$, while those underlying the triple are $\mathcal{V}_1$ and $\mathcal{V}_2^*$. We will return to triples in the next section.
Proposition 7.3. Let $\mathcal{H}$ and $\mathcal{X}_H \subset \mathcal{X}(P_K, E_K)$ be as above. Define the auxiliary representation $\rho_a : K \rightarrow U(\mathbb{C}^n) \oplus \mathbb{C}^{n^2}$ using the standard representations, and thereby fix an inclusion $\text{Lie} \mathcal{G} \subset (\text{Lie} \mathcal{G})^*$. Let $c_\mathcal{H} := -\sqrt{-1}(cI_1, 0) \in \mathfrak{h}$, where $c \in \mathbb{R}$ is any number and $I_1 \in \text{End} \mathbb{V}_1$ is the identity. Then

1. Fix a system of local unitary frames for $\mathbb{V}_2$, say $\{e_i^a\}_{i=1}^{n_2}$ over $U_\alpha \subset X$, where $\{U_\alpha\}$ is a suitable open cover of $X$. For any $x \in U_\alpha$ write

$$
\Phi(x) = \sum_{i=1}^{n_2} \varphi_i^\alpha(x) \otimes e_i^\alpha(x)
$$

where the $\varphi_i^\alpha(x)$ are locally defined sections of $\mathbb{V}_1$. When applied to any $((A_1, A_2^0), \Phi) \in \mathcal{X}_H$, and for any $x \in U_\alpha$, the $(\mathcal{H}, c_\mathcal{H})$-vortex equation takes the form

$$
-\sqrt{-1} \Lambda F_{A_1}(x) + \sum_{i=1}^{n_2} \varphi_i^\alpha(x) \otimes (\varphi_i^\alpha(x))^* = cI_1.
$$

2. Let $((A_1, A_2^0), \Phi) \in \mathcal{X}_H$ and denote by $\mathbb{V}_1$ the holomorphic bundle obtained by considering on $\mathbb{V}_1$ the $\overline{\partial}$-operator given by $A_1$. Assume that $\text{Vol} X = 1$ (the general case can be reduced to it by rescaling appropriately).

(a) The pair $((A_1, A_2^0), \Phi)$ is $(\mathcal{H}, c_\mathcal{H})$-stable if and only if: all coherent subsheaves of $\mathbb{V}_1 \otimes \mathbb{V}_2$ of the form $\mathbb{V}' \otimes \mathbb{V}_2$ satisfy:

$$
\mu(\mathbb{V}') < c
$$

(where as usual $[\mathbb{V}], \mu(\mathbb{V}) = \text{deg}(\mathbb{V})/\text{rk}(\mathbb{V})$), and if $\Phi \in H^0(\mathbb{V}' \otimes \mathbb{V}_2)$ then

$$
\mu(\mathbb{V}_1/\mathbb{V}') > c.
$$

(b) The pair $((A_1, A_2^0), \Phi)$ is simple if and only if there is no holomorphic non-trivial splitting $\mathbb{V}_1 = \mathbb{V}_1' \oplus \mathbb{V}_2''$ such that $\Phi \in H^0(\mathbb{V}_1' \otimes \mathbb{V}_2)$.

Remark 7.4. Though the local sections $\varphi_i^\alpha$ depend on the choice of local unitary frames $\{e_i^a\}_{i=1}^{n_2}$, the expressions $\sum_{i=1}^{n_2} \varphi_i^\alpha(x) \otimes (\varphi_i^\alpha(x))^*$ do not and are thus globally defined. This can be checked directly as follows. Let $\{\tilde{e}_i^\alpha\}_{i=1}^{n_2}$ be another unitary frame for $\mathbb{V}_2$, related to $\{e_i^a\}_{i=1}^{n_2}$ by $\tilde{e}_i^\alpha = T_{ji} e_j^\alpha$. Since both frames are unitary, the elements $T_{ji}$

---

5We apologize for the excessive use of the letter $\mu$
define a unitary matrix. Writing \( \Phi(x) = \sum_{i=1}^{n_2} \tilde{\varphi}_i^\alpha(x) \otimes \tilde{e}_i^\alpha(x) = \sum_{i=1}^{n_2} T_{ji} \tilde{\varphi}_i^\alpha(x) \otimes e_i^\alpha(x) \), we see that \( \tilde{\varphi}_i^\alpha = T_{ji} \varphi_j^\alpha \). Thus

\[
\sum_{i=1}^{n_2} \varphi_i^\alpha(x) \otimes (\varphi_i^\alpha(x))^* = \sum_{j,k=1}^{n_2} T_{ji} T_{kj} \tilde{\varphi}_j^\alpha(x) \otimes (\tilde{\varphi}_k^\alpha(x))^* = \sum_{j=1}^{n_2} \tilde{\varphi}_j^\alpha(x) \otimes (\tilde{\varphi}_j^\alpha(x))^*
\]

where we have used the unitarity of \( T \) in the last equality.

**Proof: (of Proposition 7.3)** We first analyze the equations. Recall from (6) that the \((\mathcal{H}, c_\mathcal{H})\) vortex equations for a point \((A, \Phi)\) in \(\mathcal{X}_\mathcal{H}\) are given by

\[
\pi_h(\Lambda F_A + \mu(\Phi)) = c_\mathcal{H}.
\]

Given a connection \(A = (A_1, A_2)\) on \(P_G = P_{GL(n_1)} \times P_{GL(n_2)}\), we get

\[
\Lambda F_A = (\Lambda F_{A_1}, \Lambda F_{A_2}),
\]

hence \(\pi_h(\Lambda F_A) = \Lambda F_{A_1}\). The term \(\mu(\Phi)\) in (23) is determined by the \(U(n_1) \times U(n_2)\) moment map on \(\mathbb{C}^{n_2} \otimes \mathbb{C}^{n_1}\), with respect to the usual Kaehler structure. Denoting this too by \(\mu\), we have

\[
\mu(p) = (\mu_1(p), \mu_2(p))
\]

where for \(i = 1, 2\), the maps \(\mu_i : \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_1} \longrightarrow u(n_i) \cong u(n_i)\) are the moment maps for the action of \(U(n_1) = U(n_1) \times \{1\}\) and \(U(n_2) = \{1\} \times U(n_1)\) respectively. Thus, as in Lemma 7.1,

\[
\mu(\Phi) = (\mu_1(\Phi), \mu_2(\Phi)).
\]

Consequently, \(\pi_h(\mu(\Phi)) = \mu_1(\Phi)\). To obtain (23) it thus remains to evaluate the \(U(n_1)\) moment map \(\mu_1\). By Propositions 8.1 and 8.3 in the Appendix, this can be described as follows. Fixing a unitary basis, say \(\{e_i\}_{i=1}^{n_2}\) for \(\mathbb{C}^{n_2}\), we can write any \(x \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}\) as \(x = \sum_{i=1}^{n_2} x_i \otimes e_i\), where the \(x_i\) are vectors in \(\mathbb{C}^{n_1}\). The moment map \(\mu_1(x)\) is given by

\[
\mu_1(\sum_{i=1}^{n_2} x_i \otimes e_i) = -\sqrt{-1} \sum_{i=1}^{n_2} x_i \otimes x_i^*,
\]

where \(x_i^*\) denotes the dual element in \((\mathbb{C}^{n_1})^*\). Equivalently, representing vectors in \(\mathbb{C}^{n_1}\) by column vectors and using row vectors to represent their duals, the terms \(x_i \otimes x_i^*\) become matrices \(x_i x_i^*\) in \(\text{Hom}(\mathbb{C}^{n_1}, \mathbb{C}^{n_2})\). Notice that the formula in (28) is independent of the choice of unitary basis for \(\mathbb{C}^{n_2}\), and is also \(U(n_1)\) invariant. In order to describe the corresponding map

\[
\mu_1 : \Omega^0(\mathcal{V}_1 \otimes \mathcal{V}_2) \longrightarrow \text{Lie} \mathcal{G}_1
\]

we may thus pick local frames for the bundles and apply (29) directly to these. This leads immediately to the term \(\sum_{i=1}^{n_2} \varphi_i^\alpha(x) \otimes (\varphi_i^\alpha(x))^*\) in (23).

We now show what stability with respect to the subgroup \(\mathcal{H}\) means. For convenience we will assume that \(\dim X = 1\) (in the general case we should take into account that the filtrations appearing in the definition of stability could be in principle defined only in the complementary of analytic subsets of \(X\) of codimension \(\geq 2\), hence we
should consider reflexive subsheaves and not only about subbundles). By Definition 3.3 (together with §2.6) the pair \((A_1, A_0), \Phi\) is \((\mathcal{H}, c_H)\)-stable if and only if, for any \(\chi \in \mathfrak{h}\) inducing a holomorphic filtration such that \(\Phi \subset (\mathcal{V}_1 \otimes \mathcal{V}_2)^-(\chi)\), the following inequality holds

\[
\deg \chi - \int_X \langle \chi, c_H \rangle > 0 ,
\]

where \(\deg \chi\) is given in Definition 2.7. This can be re-formulated more concretely as follows. We observe that any \(\chi \in \mathfrak{h}\) is of the form \((\chi_1, 0)\). If \(\chi\) induces a \(A\)-holomorphic filtration of \(W = P_K \times_{\rho_a} U(\mathbb{C}^n_1 \oplus \mathbb{C}^n_2)\) then it has fibrewise constant eigenvalues, among which zero is included. Suppose the eigenvalues of \(\sqrt{-1}\chi\) are the real numbers \(\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r\), with \(\alpha_j = 0\). If the filtration of \(W\) is

\[
0 \subset W_0 \subset W_1 \subset \cdots \subset W_r
\]

then

\[
W_i = \begin{cases} 
V_{1,i} & \text{if } 1 \leq i \leq j - 1 \\
V_{1,i} \oplus V_2 & \text{if } i \geq j 
\end{cases}
\]

where the \(V_{1,i}\) are the terms in the holomorphic filtration

\[
0 \subset V_{1,1} \subset \cdots \subset V_{1,r} = \mathcal{V}_1
\]
determined by \(\chi_1 \in \text{Lie} \mathcal{G}_1\). We deduce that

\[
\deg \chi = \alpha_r \deg W + \sum_{k=1}^{r-1} (\alpha_k - \alpha_{k+1}) \deg W_k
\]

\[
= \alpha_r (\deg \mathcal{V}_1 + \deg \mathcal{V}_2) + \sum_{k=1}^{j-1} (\alpha_k - \alpha_{k+1}) \deg \mathcal{V}_{1,k} + \sum_{k=j}^{r-1} (\alpha_k - \alpha_{k+1}) (\deg \mathcal{V}_{1,k} + \deg \mathcal{V}_2)
\]

\[
= \alpha_r \deg \mathcal{V}_1 + \sum_{k=1}^{r-1} (\alpha_k - \alpha_{k+1}) \deg \mathcal{V}_{1,k} + \alpha_j \deg \mathcal{V}_2
\]

\[
= \alpha_r \deg \mathcal{V}_1 + \sum_{k=1}^{r-1} (\alpha_k - \alpha_{k+1}) \deg \mathcal{V}_{1,k}
\]

where in the last line we have used the fact that \(\alpha_j = 0\). Also, since \(\text{Vol} X = 1\), we get

\[
\int_X \langle \chi, c_H \rangle = \int_X \langle \chi_1, c \rangle = c(\alpha_r \text{rk} \mathcal{V}_1 + \sum_{k=1}^{r-1} (\alpha_k - \alpha_{k+1}) \text{rk} \mathcal{V}_{1,k}) .
\]

Thus the condition (31) is equivalent to

\[
\deg(\alpha) > 0
\]

where

\[
\deg(\alpha) := \alpha_r (\deg \mathcal{V}_1 - c \text{rk} \mathcal{V}_1) + \sum_{k=1}^{r-1} (\alpha_k - \alpha_{k+1}) (\deg \mathcal{V}_{1,k} - c \text{rk} \mathcal{V}_{1,k}) .\]

\(^6\)The fact that the inequalities are not strict means that one can obtain some \(\chi\) with different choices of filtrations and weights
To determine which filtrations must satisfy (35) we need to identify when the condition \( \Phi \in H^0((\mathcal{V}_1 \otimes \mathcal{V}_2)^- (\chi)) \) is satisfied. Since the action of \( \rho(\chi) \) on \( \mathcal{V}_1 \otimes \mathcal{V}_2 \) is given by \( \chi_1 \otimes \text{Id}_{\mathcal{V}_2} \), its eigenvalues are \( \{-\sqrt{1} \alpha_1, \ldots, -\sqrt{1} \alpha_r\} \), occurring with multiplicities determined by the ranks of \( \{\mathcal{V}_{1,1} \otimes \mathcal{V}_2, \mathcal{V}_{1,2} \otimes \mathcal{V}_2, \ldots, \mathcal{V}_{1,r} \otimes \mathcal{V}_2\} \).

In this case we thus associate to \( \chi \) the integers

\[
(37) \quad p(\alpha) = \max\{i \mid \alpha_i \leq 0\} \\
(38) \quad p(\chi) = \min\{i \mid \Phi \in H^0(\mathcal{V}_{1,i} \otimes \mathcal{V}_2)\}
\]

Then \( (\mathcal{V}_1 \otimes \mathcal{V}_2)^- (\chi) = \mathcal{V}_{1,p(\chi)} \otimes \mathcal{V}_2 \) and furthermore, \( \Phi \in H^0((\mathcal{V}_1 \otimes \mathcal{V}_2)^- (\chi)) \) if and only if \( p(\chi) \leq p(\alpha) \). The stability condition can thus be re-formulated as follows:

A pair corresponding to a holomorphic bundle \( \mathcal{V}_1 \otimes \mathcal{V}_2 \) and a section \( \Phi \in H^0(\mathcal{V}_1 \otimes \mathcal{V}_2) \) is \((\mathcal{H}, c_H)\)-stable if and only if the following holds: Take any holomorphic filtration \( 0 \subset \mathcal{V}_{1,1} \subset \cdots \subset \mathcal{V}_{1,r} = \mathcal{V}_1 \). Set

\[
\Lambda = \{\alpha \in \mathbb{R}^r \mid \alpha_i \leq \alpha_{i+1} \text{ for } 1 \leq i \leq r-1 \text{ and } p(\alpha) \geq p(\chi)\},
\]

where \( p(\chi) \) and \( p(\alpha) \) are as above. Then, for any \( \alpha \in \Lambda \) we have

\[
\deg(\alpha) > 0
\]

where \( \deg(\alpha) \) is as in (36).

We now show that the stability simplification condition applies. Given a holomorphic filtration \( \chi = (\chi_1,0) \), let \( p = p(\chi) \). Let \( e_1, \ldots, e_r \) be the canonical basis of \( \mathbb{R}^r \). Define, for any \( 1 \leq i \leq p, f_i = -\sum_{k \leq i} e_k \) and, for any \( p < j \leq r, g_j = \sum_{k \geq j} e_k \). It is straightforward to check that

\[
\Lambda = \mathbb{R}_{\geq 0} f_1 + \cdots + \mathbb{R}_{\geq 0} f_r + \mathbb{R}_{\geq 0} g_{p+1} + \cdots + \mathbb{R}_{\geq 0} g_r.
\]

Also, for any \( \alpha \in \Lambda \), we get

\[
(39) \quad \chi_\alpha = \sum_{i=1}^p x_i \chi(f_i) + \sum_{i=p+1}^r y_i \chi(g_i)
\]

\[
(40) \quad \deg(\alpha) = \sum_{i=1}^p x_i \deg(f_i) + \sum_{i=p+1}^r y_i \deg(g_i).
\]

Here \( \chi(f_i) = (\chi_1(f_i),0) \) where \( \chi_1(f_i) \) denotes the element in \( \text{Lie} \mathcal{G}_1 \) whose eigenvalues are \( -\sqrt{-1}\{-1,0\} \), with the multiplicity of the first being \( \sum_{k=1}^r \text{rank}(\mathcal{V}_{1,k}) \). Similarly \( \chi(g_i) = (\chi_1(g_i),0) \) where \( \chi_1(g_i) \) denotes the element in \( \text{Lie} \mathcal{G}_1 \) whose eigenvalues are \( -\sqrt{-1}\{0,1\} \), with the multiplicity of the non-zero eigenvalue being \( \sum_{k=i}^r \text{rank}(\mathcal{V}_{1,k}) \).

As in Example \ref{ex:subbundle}, we define a subset \( S \subset \mathfrak{h} \) by the conditions

1. the characteristic polynomial \( P((\chi_1)_x) \) of \( \chi_1 \) acting on the fibre over \( x \in X \) does not depend on \( x \),
2. \( P((\chi_1)_x) \) has at most two different roots, both of which come from the set \( \sqrt{-1}\{-1,0,1\} \),
3. for any root \( \alpha \) of \( P((\chi_1)_x) \) the set \( \text{Ker}(\chi_1 - \alpha \text{Id}) \) is in fact a subbundle of \( \mathcal{V}_1 \).

\footnote{In later examples the corresponding integers will be defined slightly differently}
Then $\chi(f_i)$ and $\chi(g_i)$ are in $S$ and the above computations show that condition (SSC1) in Definition 3.1 applies. To verify the condition (SSC2) we use the fact that the eigenvalues for $\chi(f_i)$ are both non-positive while those for $\chi(g_i)$ are 0 and 1. Thus
\[(V_1 \otimes V_2)^{-}(\chi(f_i)) = V_1 \otimes V_2, \text{ while } (V_1 \otimes V_2)^{-}(\chi(g_i)) = V_{1,-1} \otimes V_2.\]
Notice in particular that $(V_1 \otimes V_2)^{-}(\chi(g_{p+1})) = (V_1 \otimes V_2)^{-}(\chi)$. It remains to interpret the stability condition as applied to elements in $S$. Consider an element $\sigma \in S$ which defines a filtration $0 \subset \mathcal{V} \subset V_1$. Suppose first that the eigenvalues are $\alpha_1 = 0$ and $\alpha_2 = 1$ (i.e. an element of the form $\chi(f_i)$). Then $p(\sigma) = 1$ or 2 (depending on whether $\Phi \in H^0(\mathcal{V} \otimes V_2)$ or not) but $p(\alpha) = 2$. Thus $p(\sigma)$ is always less than or equal to $p(\alpha)$. Moreover,
\[(41) \quad \deg(\sigma) > 0 \iff \mu(\mathcal{V}) < c.\]

If $\alpha_1 = -1$ and $\alpha_2 = 0$, then $p(\sigma) \leq p(\alpha)$ if and only if $\Phi \in H^0(\mathcal{V} \otimes V_2)$, and
\[(42) \quad \deg(\sigma) > 0 \iff \mu(V_1/\mathcal{V}) > c.\]

The description of stability in (a) follows directly from these observations. The character-
ization of simple pairs in (b) is straightforward and is left as an exercise to the
reader.

7.2. Example 2 (Fixed-$\mathcal{E}_2$ Triples). In this example we take

- $G = \text{GL}(n_1) \times \text{GL}(n_2)$,
- $K = \text{U}(n_1) \times \text{U}(n_2)$,
- $V_1 = \mathbb{C}^{n_1}$ and $\rho_1 : \text{GL}(n_1) \rightarrow \text{GL}(V_1)$ is the standard representation
- $V_2 = \mathbb{C}^{n_2}$, but $\rho_2 : \text{GL}(n_2) \rightarrow \text{GL}(V_2)$ is the dual representation, i.e.
  \[\rho_2(C) \cdot v = (C^{-1})^t v.\]

We take $\mathcal{V} = V_1 \otimes V_2$ and let $\rho = \rho_1 \otimes \rho_2$ be the tensor product representation of
\[G = \text{GL}(n_1) \times \text{GL}(n_2)\]
on $\mathcal{V}$. Equivalently, we can take $\mathcal{V} = V_1 \otimes V_2^* = \text{Hom}(V_2, V_1)$, and regard $\rho$ as the representation $\rho : G \rightarrow \text{GL}(\mathcal{V})$ given by
\[(43) \quad \rho(C_1, C_2)(T) = C_1 \circ T \circ C_2^{-1}.\]

Remark 7.5. If we set $E_i = P_{\text{GL}(n_i)} \times_{\rho_i} \mathcal{V}_i$ for $i = 1, 2$, i.e. if we let $E_i$ be the rank $n_i$ vector bundle associated to $P_{\text{GL}(n_i)}$ then with $P_G$ and $\mathcal{V}$ as above, a principal pair of type $(P_G, V_G)$ is equivalent to the triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$, where $\mathcal{E}_i$ is holomorphic bundle with the
topological type of $E_i$, and $\Phi$ is a holomorphic section of $\mathcal{E}_1 \otimes \mathcal{E}_2^* = \text{Hom}(\mathcal{E}_2, \mathcal{E}_1)$. That is, principal pairs of this sort correspond to holomorphic triples in the sense of $\text{[BC]}.\]

Exactly as in $\text{[5,2]}$ we have two natural subgroups of $K$, namely those corresponding
to the normal subgroups
\[(44) \quad H_1 = \text{U}(n_1) \times \{1\}, \]
\[(45) \quad H_2 = \{1\} \times \text{U}(n_2)\]
in $K$. We denote the resulting subgroups of $G$ by $\mathcal{H}_1$ and $\mathcal{H}_2$, and denote their Lie algebras by $\mathfrak{h}_1$ and $\mathfrak{h}_2$. The complexifications, i.e. the subgroups of $\mathcal{G}^\mathbb{C}$ corresponding to the subgroups $\text{GL}(n_1) \times \{1\}$ and $\{1\} \times \text{GL}(n_2)$, are denoted by $\mathcal{H}_1^\mathbb{C}$ and $\mathcal{H}_2^\mathbb{C}$, with
the obvious Lie algebras. As in the previous example, if we fix an integrable connection, say $A_2^0$, on $P_{U(n_2)}$ then we can define a subspace

$$(46) \quad \mathcal{X}_1 = \{(A, \Phi) \in \mathcal{X}(P_K, V_K) \mid A = (A_1, A_2^0)\}$$

in $\mathcal{X}(P_K, V_K)$. Equivalently, using the correspondence between integrable connections on $P_K$ and holomorphic structures on $P_G$, $\mathcal{X}_1$ can be regarded as a subspace of $\mathcal{X}(P_G, V_G)$.

**Proposition 7.6.** The subspace $\mathcal{X}_1$ is an $H_1$-invariant subset in $\mathcal{X}(P_K, V_K)$ and the corresponding subspace of $\mathcal{X}(P_G, V_G)$ is an $H_1^C$-invariant subspace. The data $(\mathcal{H}_1, H_1^C, \mathcal{X}_1)$ determines a subgroup setting.

**Remark 7.7.** If we let $P_{GL(n_2)}$ have the holomorphic structure determined by $A_2^0$ on $P_{U(n_2)}$, and let $E_2$ have the holomorphic structure determined by that on $P_{GL(n_2)}$, then the $H_1^C$-orbits in $\mathcal{X}_1$ correspond to isomorphism classes of triples $(E_1, E_2, \Phi)$ in which the bundle $E_2$ is fixed.

**Remark 7.8.** There is, of course, an analog to Proposition 7.6 for fixed $E_1$-triples, i.e. in which one of the holomorphic bundles is regarded as fixed, are called fixed triple. More specifically, the objects described in that proposition may be called fixed $E_2$-triples. If the fixed bundle is $E_1 = \mathcal{F}$, then the triple $(\mathcal{F}, \mathcal{E}_2, \Phi)$ can be described as a holomorphic bundle (i.e. $\mathcal{E}_2$) together with a morphism to the fixed bundle (i.e. $\Phi : \mathcal{E}_2 \to \mathcal{F}$). Since these constitute a special case of the framed modules studied by Huybrechts and Lehn [HL], we refer to them as framed bundles. If we fix the bundle $\mathcal{E}_2$, we obtain an object which can be described as a bundle together with a morphism from a fixed holomorphic bundle. Objects of this type, in which the fixed bundle is a trivial rank $k$ bundle, provide a description of coherent systems (see Section 7.3).

**Proposition 7.10.** Let $\mathcal{X}_1 \subset \mathcal{X}(P_K, V_K)$ be as above, and let $\mathcal{H} := \mathcal{H}_1$ be the subgroup corresponding to $U(n_1) \times \{1\} \subset K$. Let $\mathfrak{h} = \text{Lie} \mathcal{H}$ and define $\pi_\mathfrak{h} : \text{Lie} \mathcal{G} \to \mathfrak{h}$ to be the projection to the first factor. Use the standard representations to define an auxiliary representation $\rho_\mathfrak{h} : K \to U(\mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2})$, and thereby get an inclusion $\text{Lie} \mathcal{G} \subset (\text{Lie} \mathcal{G})^*$ (as in Section 2.4). Let $c_\mathcal{H} := -\sqrt{-1}(cI_1, 0)$, where $c \in \mathbb{R}$ is any real number (then $c_\mathcal{H}$ is a central element in $\mathfrak{h}$).

1. When applied to any $((A_1, A_2^0), \Phi) \in \mathcal{X}_1$, the $(\mathcal{H}, c_\mathcal{H})$-vortex equation takes the form

$$ (47) \quad \sqrt{-1} \Lambda F_{A_1} + \Phi \Phi^* = cI_1. $$
2. Suppose that the volume $\text{Vol} X = 1$. A point $((A_1, A_0), \Phi) \in \mathcal{X}_1$ is $(\mathcal{H}, c_\mathcal{H})$-stable if and only if the corresponding triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ satisfies the following condition: for all coherent subsheaves $\mathcal{E}_1' \subset \mathcal{E}_1$

$$\mu(\mathcal{E}_1') < c,$$

and if $\Phi(\mathcal{E}_2) \subset \mathcal{E}_1'$ then

$$\mu(\mathcal{E}_1/\mathcal{E}_1') > c.$$

The pair $((A_1, A_0), \Phi) \in \mathcal{X}_1$ is simple if there is no holomorphic nontrivial splitting $\mathcal{E}_1 = \mathcal{E}_1' \oplus \mathcal{E}_1''$ such that $\Phi(\mathcal{E}_2) \subset \mathcal{E}_1'$.

Proof: We first analyze the equations. The proof of (1) is essentially the same as the proof of (1) in Proposition 7.3, except that in this case the term $\mu(\Phi)$ in equation (25) is determined by the moment map for $\text{U}(n_1) \times \text{U}(n_2)$ on $\text{Hom}(\mathbb{C}^{n_2}, \mathbb{C}^{n_1})$ (with respect to the usual Kaehler structure). But, denoting this too by $\mu$, we have

$$\mu(T) = -\sqrt{-1}(T^*T, -TT^*)$$

where $T$ is in $\text{Hom}(\mathbb{C}^{n_2}, \mathbb{C}^{n_1})$. The result now follows as in Proposition 7.3. To prove the statement on stability we follow exactly the same scheme as in the proof of Proposition 7.3. We consider elements $\chi_\alpha = (\chi_\alpha, 0) \in \mathfrak{h}$ with eigenvalues $-\sqrt{-1}\{\alpha_1, \ldots, \alpha_r\}$, and corresponding filtrations $0 \subset \mathcal{E}_{1,1} \subset \cdots \subset \mathcal{E}_{1,r}$. We identify filtrations $\chi(f_i)$ and $\chi(g_i)$, define the set $S \subset \mathfrak{h}$ in the same way as before, and verify that SSC applies. Notice that the eigenvalues for $\rho(\chi_\alpha)$ on $\text{Hom}(\mathcal{E}_2, \mathcal{E}_1')$ are still $-\sqrt{-1}\{\alpha_1, \ldots, \alpha_r\}$. The filtration determined by the corresponding eigen-subbundles is

$$0 \subset \text{Hom}(\mathcal{E}_2, \mathcal{E}_{1,1}) \subset \text{Hom}(\mathcal{E}_2, \mathcal{E}_{1,2}) \subset \cdots \text{Hom}(\mathcal{E}_2, \mathcal{E}_{1,r}) = \text{Hom}(\mathcal{E}_2, \mathcal{E}_1).$$

We thus define $p(\alpha)$ exactly as in (37), and set

$$p(\chi) = \min\{i \mid \Phi \in H^0(\text{Hom}(\mathcal{E}_2, \mathcal{E}_{1,i})) \}$$

In particular,

- $p(\alpha(f_i)) = 2$ while $p(\chi(f_i))$ is either one or two. Hence $p(\chi(f_i)) \le p(\alpha(f_i))$ is always satisfied.
- $p(\alpha(g_i)) = 1$ while $p(\chi(g_i))$ is either one or two. Hence $p(\chi(g_i)) \le p(\alpha(g_i))$ is satisfied if $\Phi \in H^0(\text{Hom}(\mathcal{E}_2, \mathcal{E}_{1,i}))$, i.e. $\Phi(\mathcal{E}_2) \subset \mathcal{E}_{1,i}$.

The statements in (2) of the Proposition now follow, as in the proof of Proposition 7.3, by evaluating the condition $\text{deg} (\sigma) > 0$ for those elements $\sigma \in S$ which are of the type defined by $\chi(f_i)$ and $\chi(g_i)$.

Remark 7.11. We can reformulate the stability condition in Proposition 7.11 (2) as follows. For any $\alpha \in \mathbb{R}$ define the $\alpha$-degree and $\alpha$-slope of a triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ to be defined to be

$$\text{deg}_\alpha(\mathcal{E}_1, \mathcal{E}_2, \Phi) = \text{deg}(\mathcal{E}_1) + \text{deg}(\mathcal{E}_2) + \alpha \text{rk}(\mathcal{E}_2),$$

$$\mu_\alpha(\mathcal{E}_1, \mathcal{E}_2, \Phi) = \frac{\text{deg}_\alpha(\mathcal{E}_1, \mathcal{E}_2, \Phi)}{\text{rk}(\mathcal{E}_1) + \text{rk}(\mathcal{E}_2)} = \frac{\text{deg}(\mathcal{E}_1 \oplus \mathcal{E}_2)}{\text{rk}(\mathcal{E}_1) + \text{rk}(\mathcal{E}_2)} + \alpha \frac{\text{rk}(\mathcal{E}_2)}{\text{rk}(\mathcal{E}_1) + \text{rk}(\mathcal{E}_2)}.$$
then a direct calculation shows

\[ \mu(\mathcal{E}_1) < c \iff \mu_a(\mathcal{E}_1, 0, 0) < \mu_a(\mathcal{E}_1, \mathcal{E}_2, \Phi) \]

\[ \mu(\mathcal{E}_1/\mathcal{E}_1') > c \iff \mu_a(\mathcal{E}_1', \mathcal{E}_2, \Phi) < \mu_a(\mathcal{E}_1, \mathcal{E}_2, \Phi) \]

The condition in 7.10(2) is thus equivalent to the condition that all subtriples of the form \((\mathcal{E}_1', \mathcal{E}_2, \Phi)\) or \((\mathcal{E}_1', 0, \Phi)\) satisfy the \(\alpha\)-stability condition for triples with \(\alpha\) defined by (50), i.e. it is equivalent to the usual triples stability property, but applied only to subtriples in which \(\mathcal{E}_1'\) is either the whole \(\mathcal{E}_1\) or zero.

7.3. Example 3 (Coherent Systems). We consider a special case of Example 2, i.e. of the situation in Section 7.2. Switching notation slightly (to highlight the different roles played by the two factors) we take

- \(G = \text{GL}(n) \times \text{GL}(k)\),
- \(K = \text{U}(n) \times \text{U}(k)\),
- \(\mathbb{V} = \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)\), with \(\rho : G \to \text{GL}(\mathbb{V})\) given by \(\rho(B, C)(T) = B \circ T \circ C^{-1}\).

We again take a principal \(G\)-bundle of the form \(P_G = P_{\text{GL}(n)} \times P_{\text{GL}(k)}\), but we now impose the restriction that \(P_{\text{GL}(k)}\) should be the trivial bundle, i.e.

\[ P_{\text{GL}(k)} = X \times \text{GL}(k) \]

If, moreover, we fix the trivial holomorphic structure on \(P_{\text{GL}(k)}\), then the remaining data in a pair on \((P_G, V_G = P_G \times_\rho \mathbb{V})\) determines (a) a holomorphic structure on \(P_{\text{GL}(n)}\) and (b) a holomorphic section of \(V_G\). Equivalently, replacing the principal bundles with their associated vector bundles, we see that such pairs correspond to choices for \((\mathcal{E}, \Phi)\), where \(\mathcal{E}\) is a holomorphic bundle with \(P_{\text{GL}(n)}\) as its frame bundle and \(\Phi\) is a holomorphic bundle map \(\Phi : \mathcal{O}^k \to \mathcal{E}\). If we fix an identification

\[ H^0(\mathcal{O}^k) = \mathbb{C}^k \]

then \(\Phi\) defines a map from \(\mathbb{C}^k\) into \(H^0(\mathcal{E})\). We thus arrive at an interpretation of such principal pairs as coherent systems, where a coherent system consists of a holomorphic vector bundle \(\mathcal{E}\) together with a linear subspace in \(H^0(\mathcal{E})\) (cf. \(\Xi\)N, \([\text{LeP}]\)). More precisely, it is given by a vector bundle, \(\mathcal{E}\), and a homomorphism \(u : \mathbb{V} \to H^0(\mathcal{E})\), where \(\mathbb{V}\) is a fixed \(k\)-dimensional vector space. If we fix an identification \(\mathbb{V} \simeq H^0(\mathcal{O}^k)\), then \(u\) defines a map \(\Phi : \mathcal{O}^k \to \mathcal{E}\) and the coherent system \((\mathcal{E}, u)\) determines the triple \((\mathcal{E}, \mathcal{O}^k, \Phi)\). Conversely, any such triple determines a coherent system in which \(\mathbb{V} = H^0(\mathcal{O}^k)\) and \(u\) is the map induced map by \(\Phi\). Changing the identification of \(\mathbb{V}\) with \(H^0(\mathcal{O}^k)\) is equivalent to changing \(u\) by the action of an element in \(\text{GL}(\mathbb{V}) \simeq \text{GL}(k, \mathbb{C})\).

We thus get a bijective correspondence between coherent systems modulo the action of \(\text{GL}(k)\) on \(\mathbb{V}\), and triples \((\mathcal{E}, \mathcal{O}^k, \Phi)\) modulo the action of \(\text{GL}(k)\) on \(\mathcal{O}^k\). Triples of this sort are thus precisely the principal pairs described above. Notice that since \(P_{\text{U}(k)}\) is the trivial \(\text{U}(k)\)-bundle, we can identify larger subgroups than those which appear in section 7.2. In particular, as in (56) we can define subgroups of \(\mathcal{G}^\mathbb{C}(P_G)\) and \(\mathcal{G}(P_K)\) respectively by

\[ \mathcal{H}^\mathbb{C} = \mathcal{G}(P_{\text{GL}(n)}) \times \text{GL}(k) \quad \text{and} \quad \mathcal{H} = \mathcal{G}(P_{\text{U}(n)}) \times \text{U}(k). \]
Moreover, if we take the trivial holomorphic structure on $P_{GL(k)}$, then the resulting subspace of $\mathcal{X}_H \subset \chi(P_G, V_G)$ is $\mathcal{H}^C$-invariant. Let $\mathfrak{h} = \text{Lie } \mathcal{H}$. We define the projection

$$\pi_\mathfrak{h} : \text{Lie } \mathcal{G} = \text{Lie}(\mathcal{G}(P_{U(n)})) \oplus \text{Lie}(\mathcal{G}(P_{U(k)})) \to \text{Lie}(\mathcal{G}(P_{U(n)})) \oplus \mathfrak{u}(k)$$

by integrating the second component over the base manifold $X$, i.e.

$$\pi(g_1, g_2) = (g_1, \int_X g_2).$$

(53)

As in the previous examples:

**Proposition 7.12.** The subgroups $\mathcal{H}$ and $\mathcal{H}^C$, together with the subspace $\chi_H$, define a subgroup setting in which our Main Theorem applies.

In view of the above discussion on the relation between the points in $\mathcal{X}_H$ and coherent systems, we see that the $\mathcal{H}^C$ orbits in $\mathcal{X}_H$ correspond to isomorphism classes of coherent systems. The result of our Main Theorem, which relates the moment map for the $\mathcal{H}$-action on $\mathcal{X}_H \subset \mathcal{X}(P_K, V_K)$ to stability with respect to $\mathcal{H}^C$, can thus be interpreted as a Hitchin–Kobayashi correspondence for coherent systems.

**Proposition 7.13.** Let $\mathcal{H}$, $\mathcal{H}^C$ and $\mathcal{X}_H \subset \mathcal{X}(P_G, V_G)$ be as above. Use the standard representations to define an auxiliary representation $\rho_a : K \to U(\mathbb{C}^n \oplus \mathbb{C}^k)$, and thereby an inclusion $\text{Lie } \mathcal{G} \subset (\text{Lie } \mathcal{G})^*$ (as in Section 2.1). Let $c_H := -\sqrt{-1}(c_1 I_n, c_2 I_k)$, where $c_1, c_2 \in \mathbb{R}$ are arbitrary real numbers. Let $((A, 0), \Phi)$ be a point in $\mathcal{X}_H$, where $A$ is a connection on $P_{GL(n)}$ and $0$ denotes the trivial connection on $P_{GL(k)}$. Fix a global frame $\{e_i\}_{i=1}^k$ for $\mathcal{O}^k$ and define $\phi_i = \Phi(e_i)$. Let $S = \text{Span}\{\phi_1, \ldots, \phi_k\}$ be the subspace of $H^0(\mathcal{E})$ spanned by the image of the induced map $\Phi : H^0(\mathcal{O}^k) \to H^0(\mathcal{E})$.

1. When applied to $((A, 0), \Phi)$, the $(\mathcal{H}, c_H)$-vortex equation takes the form

$$\sqrt{-1}\Lambda F_A + \Phi \Phi^* = c_1 I_n$$

$$\langle \phi_i, \phi_j \rangle_{L^2} = -c_2 I_k$$

where the inner product $\langle , \rangle_{L^2}$ is on $H^0(\mathcal{E})$, and $I_k$ denotes the $k \times k$ identity matrix. There are no solutions unless $c_1$ and $c_2$ satisfy the constraint

$$\deg(\mathcal{E}) = c_1 \text{rk}(\mathcal{E}) + c_2 k.$$  

(55)

2. Assume that $c_1$ and $c_2$ satisfy the constraint (53). Then a point $((A, 0), \Phi) \in \mathcal{X}_H$ is $(\mathcal{H}_H, c_H)$-stable if and only if for any coherent subsheaf $\mathcal{E}' \subset \mathcal{E}$:

$$\frac{\deg(\mathcal{E}')}{\text{rk}(\mathcal{E}')} + \alpha \frac{k'}{\text{rk}(\mathcal{E}')} < c_1,$$

where $k' = \dim(H^0(\mathcal{E}') \cap S)$ and $\alpha$ is defined by

$$\frac{\deg(\mathcal{E})}{\text{rk}(\mathcal{E})} + \alpha \frac{\dim(S)}{\text{rk}(\mathcal{E})} = c_1.$$  

(56)

Proof: We have to compute $\pi_\mathfrak{h}(\Lambda F_{(A,0)} + \mu(\Phi))$. If we fix a frame $\{e_i\}$ for $\mathcal{O}^k$, and define sections of $\mathcal{E}$ by

$$\phi_i = \Phi(e_i)$$

(58)
then with respect to this frame, the endomorphism \( \Phi^* \Phi \) is given by the matrix whose \((ij)\) element is \( \phi_i \phi_j \). We thus find that

\[
\int_X \sqrt{-1} \Phi^* \Phi = \sqrt{-1} \langle \phi_i, \phi_j \rangle_{L^2}.
\]

At a point \((A, 0, \Phi) \in \chi\), where \( A \) is a connection on \( P_{GL(n)} \) and 0 denotes the trivial connection on \( P_{GL(k)} \), we thus get

\[
\begin{align*}
\pi_h(\Lambda F_{(A, 0)} + \mu(\Phi)) &= \pi_h(\Lambda F_{A}, 0) + \pi_h(-\sqrt{-1} \Phi^* \Phi, +\sqrt{-1} \Phi^* \Phi) \\
&= (\Lambda F_{A} - \sqrt{-1} \Phi^* \Phi), \sqrt{-1} \langle \phi_i, \phi_j \rangle_{L^2}).
\end{align*}
\]

The equations (54) follow from this. The constraint on \( c_1 \) and \( c_2 \) is obtained by integrating the trace of the first equation in (54) and adding the result to the trace of the second equation. To prove that the stability conditions one has to follow the same scheme as in Proposition 7.3. By our choice of auxiliary representation, the vector bundle \( W = P_K \times_{\rho_a} (\mathbb{C}^n \oplus \mathbb{C}^k) \) can be identified as \( W = \mathcal{E} \oplus \mathcal{O}^k \). Any holomorphic filtration induced by \( \chi \in \Omega^0(\text{ad } P_K) \) is of the form

\[
0 = W^0 \subseteq W^1 \subseteq \cdots \subseteq W^r = W
\]

with \( W^j = \mathcal{E}^j \oplus \mathcal{F}^k \), where \( \mathcal{E}^j \subset \mathcal{E} \) are subbundles defining a filtration

\[
0 = \mathcal{E}^0 \subseteq \mathcal{E}^1 \subseteq \cdots \subseteq \mathcal{E}^r = \mathcal{E}
\]

and \( \mathcal{F}^j \subset \mathcal{O}^k \) are subbundles defining a filtration

\[
0 = \mathcal{F}^0 \subseteq \mathcal{F}^1 \subseteq \cdots \subseteq \mathcal{F}^r = \mathcal{O}^k.
\]

Suppose now that \( \chi \) is in \( \mathfrak{h} \). Then \( \mathcal{F} = \mathcal{O}^{k} \) for some \( k_j \leq k \), so that the filtration of \( \mathcal{O}^{k} \) is of the form

\[
0 = \mathcal{O}^{k_0} \subseteq \mathcal{O}^{k_1} \subseteq \cdots \subseteq \mathcal{O}^{k}.
\]

Hence, if the eigenvalues of \(-\sqrt{-1} \chi\) are \( \alpha_1 \leq \alpha_2 \leq \cdots \alpha_r \), then

\[
\deg \chi = \alpha_r \deg \mathcal{E}^1 + \sum_{j=1}^{r-1} (\alpha_j - \alpha_{j+1}) \deg \mathcal{E}^j.
\]

Notice however that with \( c_H := -\sqrt{-1}(c_1 I_n, c_2 I_k) \), (and Vol \( X = 1 \)), we get

\[
\int_X \langle \chi, c_H \rangle = c_1(\alpha_r \text{rk } \mathcal{E} + \sum_{j=1}^{r-1} (\alpha_j - \alpha_{j+1}) \text{rk } \mathcal{E}_{j}) + c_2(\alpha_r k + \sum_{j=1}^{r-1} (\alpha_j - \alpha_{j+1}) k_j).
\]

The eigenvalues for \(-\sqrt{-1} \rho(\chi)\) on \( V = \text{Hom}(\mathcal{O}^{k}, \mathcal{E}) \) are \( \alpha_i - \alpha_j \). As in the proof of Proposition 7.3, we can apply (SSC) and show that it is enough to consider \( \chi \in \mathfrak{h} \) with at most two eigenvalues, all of which come from the set \( \sqrt{-1} \{ -1, 0, 1 \} \). We may thus assume that the corresponding filtrations are of the form

\[
0 \subseteq \mathcal{E}' \oplus \mathcal{O}^k \subseteq \mathcal{E} \oplus \mathcal{O}^k,
\]

and the eigenvalues are either \((-\sqrt{-1}, 0)\) or \((0, \sqrt{-1})\). In both cases the eigenvalues for \(-\sqrt{-1} \rho(\chi)\) on \( V = \text{Hom}(\mathcal{O}^{k}, \mathcal{E}) \) are \((-1, 0, 1)\) and the condition \( \Phi \in H^0(V^-) \) is
equivalent to $\Phi(\mathcal{O}^k) \subset \mathcal{E}'$. If the eigenvalues are $(-1,0)$ then the stability condition $\text{deg} \chi - \int_X \langle \chi_\alpha, c_H \rangle > 0$ is equivalent to

\begin{equation}
(63) \quad \text{deg}(\mathcal{E}') - c_1 \text{rk}(\mathcal{E}') - c_2 k' < 0.
\end{equation}

If the eigenvalues are $(0, \sqrt{-1})$ then the condition $\text{deg} \chi - \int_X \langle \chi_\alpha, c_H \rangle > 0$ is equivalent to

\begin{equation}
(64) \quad \text{deg}(\mathcal{E}') - c_1 \text{rk}(\mathcal{E}') - c_2 k' < \text{deg}(\mathcal{E}) - c_1 \text{rk}(\mathcal{E}) - c_2 k.
\end{equation}

However by (53), $\text{deg}(\mathcal{E}) - c_1 \text{rk}(\mathcal{E}) - c_2 k = 0$. Thus (64) is the same as (63). Defining $\alpha$ as in (57), and using (55), it follows immediately that this condition is the same as (53).

\section*{7.4. Example 4 (Twisted Triples)}

Examples 1 and 2 can be combined to describe objects which might be called twisted triples; by this we mean a fixed triple, but in which one of the bundles is only partially fixed. Such objects arise naturally as a result of dimensional reduction when the reduction is on a ‘twisted’ orbit space, as in [BGK]. They are also the simplest example of the twisted quiver bundles considered in [AG]. Higgs bundles may be viewed as a special case.

\textbf{Definition 7.14.} \textit{A twisted triple consists of two holomorphic bundles }$\mathcal{E}_1$\textit{ and }$\mathcal{E}_2$\textit{, plus a bundle map}

\begin{equation}
(65) \quad \Phi : \mathcal{E}_2 \otimes \mathcal{F} \longrightarrow \mathcal{E}_1
\end{equation}

\textit{where }$\mathcal{F}$\textit{ is a fixed (‘twisting’) holomorphic bundle.}

Writing $\tilde{\mathcal{E}}_2 = \mathcal{E}_2 \otimes \mathcal{F}$, we can think of the twisted triple as a triple $(\mathcal{E}_1, \tilde{\mathcal{E}}_2, \Phi)$ in which the second bundle is partially fixed. Alternatively, we can replace $\Phi$ by either of the equivalent maps (which, by abuse of notation we also denote by $\Phi$)

\begin{equation}
(66) \quad \Phi : \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \otimes \mathcal{F}^*
\end{equation}

or

\begin{equation}
(67) \quad \Phi : \mathcal{F} \longrightarrow \mathcal{E}_1 \otimes \mathcal{E}_2^*.
\end{equation}

The twisted triple is then be viewed as a triple $(\mathcal{E}_1 \otimes \mathcal{F}^*, \mathcal{E}_2^*, \Phi)$ in which the first bundle is partially fixed, or as a triple $(\mathcal{E}_1 \otimes \mathcal{E}_2^*, \mathcal{F}, \Phi)$ in which the second bundle is fixed and the first bundle is a tensor product. Adopting the first description (i.e. the one given in Definition 7.14), we get a description in terms of principal pairs if we take $p = 3$ in (18) and (19) and consider the case where

- $G = \text{GL}(n_1) \times \text{GL}(n_2) \times \text{GL}(n_3)$,
- $K = \text{U}(n_1) \times \text{U}(n_2) \times \text{U}(n_3)$,
- $\mathbb{V}_1 = \mathbb{C}^{n_1}$ and $\rho_1 : \text{GL}(n_1) \longrightarrow \text{GL}(\mathbb{V}_1)$ is the standard representation, and
- for $i = 2, 3, \mathbb{V}_i = \mathbb{C}^{n_i}$ and $\rho_i : \text{GL}(n_i) \longrightarrow \text{GL}(\mathbb{V}_i)$ is the dual of the standard representation.
We take $V = V_1 \otimes V_2 \otimes V_3$ and let $\rho = \rho_1 \otimes \rho_2 \otimes \rho_3$ be the tensor product representation of $G$ on $V$. As in Example 1 (in section 7.4) $V_G = P_G \times_\rho V$ is then a tensor product of vector bundles; in this case $V_G = V_1 \otimes V_2^* \otimes V_3^* = \text{Hom}(V_2 \otimes V_3, V_1)$, where $V_i$ denotes the standard vector bundle associated to the principal $\text{GL}(n_i)$ bundle $P_{\text{GL}(n_i)}$, and $V_i^*$ is the dual of $V_i$. A holomorphic structure on $P_G$ is thus equivalent to a holomorphic structure on $V_G$ such that the resulting holomorphic vector bundle $\mathcal{V} = V_1 \otimes V_2^* \otimes V_3^*$ (where $V_i$ denotes holomorphic bundle obtained by putting a holomorphic structure on $V_i$). The data in a principal pairs of this type is clearly equivalent to the defining data in a twisted triple, but without any extra conditions imposed on the 'twisting bundle'. Indeed we can make the correspondence explicit if we let $E$ and $V$ be the standard (holomorphic) vector bundles associated to $P_{\text{GL}(n_1)}$ and $P_{\text{GL}(n_2)}$ respectively, and let the 'twisting bundle' $\mathcal{F}$ be the standard vector bundle associated to $P_{\text{GL}(n_3)}$. Depending on whether $\rho_i$ is the standard representation or its dual, $\mathcal{V}_i$ is identified with either $\mathcal{E}_i$ or its dual $\mathcal{E}_i^*$, and the section $\Phi$ becomes the map $\Phi$ in (65), (66) or (67). To impose the constraint on $\mathcal{F}$, we consider the normal subgroup of $K$ defined by

\[(68) \\ H = U(n_1) \times U(n_2) \times 1.\]

Let $\mathcal{H}$ be the corresponding subgroup of $G$, as in Section 3.2 and let $\mathcal{H}^C$ be its complexification. If we fix a connection, say $A_1^0$, on $P_f = P_{U(n_1)}$, then we get a subspace $\mathcal{X}_H \subset \mathcal{X}(P_K, V_K)$ defined by the condition that the connections on $P_K$ is of the form

\[(69) \\ A = (A_1, A_2, A_1^0).\]

**Lemma 7.15.** The subspace $\mathcal{X}_H$ is $\mathcal{H}$-invariant. The corresponding complex subspace of $\mathcal{X}(P_G, V_G)$ is $\mathcal{H}^C$-invariant and defines the configuration space of twisted triples. The $\mathcal{H}^C$ orbits correspond to the isomorphism classes of twisted triples. The set $(\mathcal{H}, \mathcal{H}^C, \mathcal{X}_H)$ determines a subgroup setting.

**Proposition 7.16.** Let $\mathcal{H} \subset G$ and $\mathcal{X}_H \subset \mathcal{X}(P_K, V_K)$ be as above. Define an auxiliary representation $\rho_\alpha : K \rightarrow U(C^n) \oplus C^n \oplus C^n)$ using the standard representations, and thereby fix an inclusion $\text{Lie} \mathcal{G} \subset (\text{Lie} \mathcal{G})^*$ (as in Section 2.7). Fix a central element $c_H := -\sqrt{-1}(c_1 I_1, c_2 I_2, c_3 I_3)$ in the Lie algebra of $\mathcal{H}$ and assume that

\[n_1 c_1 + n_2 c_2 = \deg(\mathcal{E}_1) + \deg(\mathcal{E}_2),\]

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are the vector bundles associated to the principal $\text{GL}(n_1)$ and $\text{GL}(n_2)$ bundles. (1) Fix a system of local unitary frames for $V_3$, say $\{e_i^\alpha\}_{i=1}^{n_3}$ over $U_\alpha \subset X$, where $\{U_\alpha\}$ is a suitable open cover of $X$. Let $\{f_1^\alpha\}_{i=1}^{n_3}$ be the dual frame for $V_3^*$. For any $x \in U_\alpha$ write

\[\Phi(x) = \sum_{i=1}^{n_3} \Phi_i^\alpha(x) \otimes f_i^\alpha(x)\]

---

This is equivalent to setting $V = \text{Hom}(C^n, C^n, C^n)$ and taking the representation $\rho(C_1, C_2, C_3)(T) = C_1 \cdot T \cdot (C_2 \otimes C_3)^{-1}$
where the $\Phi^\alpha_i(x)$ are locally defined sections of $V_1 \otimes V_2^* = \text{Hom}(V_2, V_1)$. Then when applied to any $((A_1, A_2, A_f), \Phi) \in X_H$, the $(\mathcal{H}, c_H)$-vortex equation takes the form

\begin{align}
\sqrt{-1} \Lambda F_{A_1} + \sum_{j=1}^{n_f} \Phi_j \Phi_j^* &= c_1 I_1, \\
\sqrt{-1} \Lambda F_{A_2} - \sum_{j=1}^{n_f} \Phi_j^* \Phi_j &= c_2 I_2.
\end{align}

The $\Phi_j$ (here viewed as elements in $\text{Hom}(V_2, V_1)$) depend on the choice of local frame for $\mathcal{F} = \mathcal{V}_3$, but the expressions $\sum_{j=1}^{n_f} \Phi_j \Phi_j^*$ and $\sum_{j=1}^{n_f} \Phi_j^* \Phi_j$ do not. Let $(\mathcal{E}_1, \mathcal{E}_2 \otimes \mathcal{F}, \Phi)$ be the twisted triple corresponding to the point $((A_1, A_2, A_f), \Phi)$ in $X_H$. Then $((A_1, A_2, A_f), \Phi)$ is $(\mathcal{H}, c_H)$-stable if and only if for every choice of coherent subsheaves $\mathcal{E}_1' \subset \mathcal{E}_1$ and $\mathcal{E}_2' \subset \mathcal{E}_2$ such that $\Phi(\mathcal{E}_2' \otimes \mathcal{F}) \subset \mathcal{E}_1'$, the inequality

$$\frac{\text{deg}(\mathcal{E}_1' \oplus \mathcal{E}_2')}{\text{rk}(\mathcal{E}_1' \oplus \mathcal{E}_2')} + \alpha \frac{\text{rk}(\mathcal{E}_2')}{\text{rk}(\mathcal{E}_1' \oplus \mathcal{E}_2')} < \frac{\text{deg}(\mathcal{E}_1 \oplus \mathcal{E}_2)}{\text{rk}(\mathcal{E}_1 \oplus \mathcal{E}_2)} + \alpha \frac{\text{rk}(\mathcal{E}_2)}{\text{rk}(\mathcal{E}_1 \oplus \mathcal{E}_2)}$$

is satisfied. Here $\alpha$ is determined by the identity

$$\frac{\text{deg}(\mathcal{E}_1 \oplus \mathcal{E}_2)}{\text{rk}(\mathcal{E}_1 \oplus \mathcal{E}_2)} + \alpha \frac{\text{rk}(\mathcal{E}_2)}{\text{rk}(\mathcal{E}_1 \oplus \mathcal{E}_2)} = c_1.$$ 

Proof: The form of the $(\mathcal{H}, c_H)$ vortex equations follows in the same way as in Examples 1 and 2. For more details on the precise form of the terms involving $\Phi$ see the Appendix. The characterization of the stability condition follows by exactly the same methods as in the previous examples; it is a very minor modification of the calculations in the proofs of Proposition 7.3 and Proposition 7.13. We thus omit the details.

Our Main Theorem thus reduces to the Hitchin–Kobayashi correspondence for twisted triples, as in [BGK].

Remark 7.17. 1. If $\mathcal{F}$ is the trivial rank $k$ bundle $\mathcal{O}^k$, then we can fix a global frame for it. The maps $\Phi_j : \mathcal{E}_2 \to \mathcal{E}_1$ are then globally defined and the twisted triple $(\mathcal{E}_1, \mathcal{E}_2 \otimes \mathcal{F}, \Phi)$ can equivalently be described as a pair of bundles together with $k$ prescribed maps between them.

2. The example of the twisted triples can be generalized in various ways if we allow $p > 3$, i.e. if we admit more than three bundles. Some such examples are useful for the description of quiver representations. The computation of the appropriate moment map is essentially the same as that in Proposition 7.10 but involves more complicated notation. The key is the Moment Map Lemma (with subsequent generalizations) given in the Appendix. The description of the stability condition is likewise more complicated but involves no new ideas.

7.5. Example 5 (GL($m$)-Higgs bundles). In this example we consider the case where

- $G = \text{GL}(m) \times \text{GL}(n)$ and $K = \text{U}(m) \times \text{U}(n)$, with $n = \text{dim}_\mathbb{C}(X)$,
\[ V_1 = \text{Hom}(\mathbb{C}^m, \mathbb{C}^m), \] and \( \rho_1 : \text{GL}(m) \rightarrow \text{GL}(V_1) \) is the representation \( \rho_1(A)B = ABA^{-1}, \)

\[ V_2 = \mathbb{C}^n, \] and \( \rho_2 : \text{GL}(n) \rightarrow \text{GL}(V_2) \) is the standard representation on \( \mathbb{C}^n. \)

We also stipulate that

\( P_{\text{GL}(n)} \) is the frame bundle for the holomorphic cotangent bundle \((T_X^*)^{1,0}\) (thus, as a smooth bundle, we can identify \( P_{\text{GL}(n)} \times_{\rho_2} \mathbb{C}^m = (T_X^*)^{1,0}. \))

As in the previous examples, we take \( V = V_1 \otimes V_2 \) and let \( \rho = \rho_1 \otimes \rho_2 \) be the tensor product representation of \( G \) on \( V. \) The data set which determines a principal pair on \((P_G, V_G = P_G \times_\rho V)\) is thus equivalent to the data in \((\mathcal{E}, \mathcal{T}, \Theta), \) where \( \mathcal{E} \) is a holomorphic bundle with the topological type of the standard vector bundle associated to \( P_{\text{GL}(m)} \), \( \mathcal{T} \) is a holomorphic bundle with the topological type of \((T_X^*)^{1,0}, \) and \( \Theta \) is a holomorphic section of \( \text{End}(\mathcal{E}) \otimes \mathcal{T}. \) Suppose now that the holomorphic structure on \( P_{\text{GL}(n)} \) is chosen such that the identification \( P_{\text{GL}(n)} \times_{\rho_2} \mathbb{C}^n = (T_X^*)^{1,0} \) is an identification of holomorphic bundles. With \( \mathcal{T} \) fixed to be \((T_X^*)^{1,0}, \) the remaining data in principal pairs on \((P_G, V_G)\) determine pairs \((\mathcal{E}, \Theta), \) where \( \Theta \) is a holomorphic section of \( \text{End}(\mathcal{E}) \otimes (T_X^*)^{1,0}, \) i.e. the principal pairs correspond to Higgs bundles on \( P_{\text{GL}(m)}. \) Equivalently, if \( E \) is the standard vector bundle associated to \( P_{\text{GL}(m)}, \) we say that \((\mathcal{E}, \Theta)\) defines a Higgs bundle on \( E. \) To impose the constraint on \( \mathcal{T}, \) we consider the normal subgroup of \( K \) defined by

\[ H = U(m) \times 1. \]

Let \( \mathcal{H} \) be the corresponding subgroup of \( \mathcal{G}, \) as in Section 5.2 and let \( \mathcal{H}^C \) be its complexification. If we fix a connection, say \( A^0, \) on \( P_{U(n)}, \) then we get a subspace \( \mathcal{X}_H \subset \mathcal{X}(P_K, V_K) \) defined by the condition that the connections on \( P_K \) is of the form

\[ A = (A, A^0). \]

Lemma 7.18. The subspace \( \mathcal{X}_H \) is \( \mathcal{H} \)-invariant. The corresponding complex subspace of \( \mathcal{X}(P_G, V_G) \) is \( \mathcal{H}^C \)-invariant and defines the configuration space of Higgs bundles on \( E. \) The \( \mathcal{H}^C \) orbits correspond to the isomorphism classes of Higgs bundles. The set \((\mathcal{H}, \mathcal{H}^C, \mathcal{X}_H)\) determines a subgroup setting.

Proposition 7.19. Use the standard representations to define an auxiliary representation \( \rho_a = \rho_{a,m} \oplus \rho_{a,n} : K \rightarrow U(\mathbb{C}^m \oplus \mathbb{C}^n), \) and thereby an inclusion \( \text{Lie} \mathcal{G} \subset (\text{Lie} \mathcal{G})^* \) (as in Section [2.4]). Identifying \( \text{Lie} \mathcal{H} \) with \( \text{Lie} \mathcal{G}(P_{U(n)}), \) the \( \mathcal{H} \)-moment map, restricted to points in \( \mathcal{X}_H, \) is given by

\[ \mu_H(A, A^0, \Theta) = \Lambda F_A - \sqrt{-1} \Lambda [\Theta, \overline{\Theta}]. \]

Here \( \overline{\Theta} \) is defined as follows. Given any \( x \in X, \) we can fix a local unitary frame, say \( \{e_i\}_{i=1}^n, \) for \((T_X^*)^{1,0}. \) We can then write \( \Theta = \sum_{i=1}^n \Theta_i \otimes e_i, \) where the \( \Theta_i \) are locally defined sections of \( \text{End}(E). \) Then

\[ \overline{\Theta} = \sum_{i=1}^n \overline{\Theta_i} \otimes e_i \]
where the adjoints $\Theta_i^*$ are with respect to the metric on $E$.

**Proof:** (of Proposition 7.19) The first term, i.e. $\Lambda F_A$, is obtained in the usual way as a result of the projection from the Lie algebra of the full gauge group onto the Lie algebra of $H$. The second term can be understood as follows. The moment map for the $H$-action on $\Omega(\text{End}(E) \otimes (T_X^*)^{1,0})$ is, by Proposition 8.3 in the Appendix,

$$\mu(\Theta) = \sum_{i=1}^{n} \mu_1(\Theta_i)$$

where $\mu_1$ is the moment map for the action of $U(m)$ on $\Omega(\text{End}(E) \otimes (T_X^*)^{1,0})$. But, since the moment map for the conjugation action of $U(m)$ on $\text{Hom}(\mathbb{C}^m, \mathbb{C}^m)$ is

$$\mu_1(A) = -\sqrt{-1}[A, A^*],$$

it follows that

$$\mu(\Theta) = -\sqrt{-1}\sum_{i=1}^{n}[\Theta_i, \Theta_i^*].$$

On the other hand, remembering that $\Theta$ behaves like a 1-form, we get

$$\Lambda[\Theta, \Theta^*] = \sum_{j=1}^{n} \sum_{i=1}^{n} \Theta_i \Theta_j^* \Lambda e_i \wedge e_j + \Theta_j^* \Theta_i \Lambda e_j \wedge e_i$$

$$= \sum_{j=1}^{n} \Theta_i \Theta_j^* - \Theta_j^* \Theta_i$$

$$= \sum_{j=1}^{n}[\Theta_i, \Theta_j^*],$$

where we have used the fact that $\Lambda e_i \wedge e_j = -\Lambda e_j \wedge e_i = 0$ if $i \neq j$, and also that $\Lambda e_i \wedge e_i = -\Lambda e_i \wedge e_i = 1$. $\Box$

We thus get, as an immediate corollary:

**Corollary 7.20.** Given a central element $c_H = -\sqrt{-1}(0, c_m I_m)$ in Lie $H$,

1. a point in $X_H$ thus satisfies the $(H, c_H)$-vortex equations if and only if it satisfies the conditions

$$\sqrt{-1} \Lambda F_A + \Lambda[\Theta, \Theta^*] = c_m I_m,$$

2. there are no solutions unless $c_m = \mu(E)$.

**Proof.** (1) follows immediately from proposition 7.19. (2) follows from (1) by integrating the trace of the $(\Theta, \Theta^*)$ and observing that the trace of $[\Theta, \Theta^*]$ is zero. $\Box$

Furthermore,

**Proposition 7.21.** A point in $X_H$ is $(H, c_H)$-stable if and only if the corresponding Higgs bundle $(E, \Theta)$ satisfies the condition

$$\mu(E') < \mu(E)$$

for all $\Theta$-invariant coherent subsheaves $E' \subset E$. 

Proof: We proceed as in the proof of Proposition \[7.23\]. With auxiliary representation \( \rho \) as above in Proposition \[7.19\], we consider elements \( \chi_\alpha = (\chi_{1,\alpha}, 0) \in \mathfrak{g} \) where \( \chi_{1,\alpha} \) has eigenvalues \(-\sqrt{-1}\{\alpha_1, \ldots, \alpha_r\}\) and corresponding \( A \)-holomorphic filtrations

\[
0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r
\]

of \( \mathcal{E} = P_{U(m)} \times_{\rho_{a,m}} \mathbb{C}^m \). The eigenvalues for \( \rho(\chi) \) on \( V = \text{End}(\mathcal{E}) \otimes (T^*_X)^{1,0} \) are then the differences \(-\sqrt{-1}(\alpha_i - \alpha_j)\). The negative subbundle \( V^- \) is thus determined by the condition \( \alpha_i \leq \alpha_j \). It follows that the condition \( \Theta \in H^0(V^-) \) is equivalent to the condition \( \Theta(\mathcal{E}_j) \subset \mathcal{E}_j \otimes (T^*_X)^{1,0} \) for \( 1 \leq j \leq r \). We now identify filtrations \( \chi (f_i) \) and \( \chi (g_i) \) and define the set \( S \subset \mathfrak{g} \) in exactly the same way as in proof of Proposition \[7.23\]. The verification that (SSC1) applies follows precisely as before. The verification of (SSC2') follows from the above characterization of the condition \( \Theta \in H^0(V^-) \). Suppose now that \( \sigma \in S \) defines a filtration \( 0 \subset \mathcal{E}' \subset \mathcal{E} \). Whether the eigenvalues of \( \sqrt{-1}\sigma \) are \( \alpha_1 = -1 \) and \( \alpha_2 = 0 \), or \( \alpha_1 = 0 \) and \( \alpha_2 = -1 \), the eigenvalues of \( \sqrt{-1}\rho(\sigma) \) on \( V = \text{End}(\mathcal{E}) \otimes (T^*_X)^{1,0} \) are \((-1, 0, 1)\). The condition \( \Theta \in H^0(V^-) \) is equivalent to the condition \( \Theta(\mathcal{E}') \subset \mathcal{E}' \otimes (T^*_X)^{1,0} \). If the eigenvalues of \( \sqrt{-1}\sigma \) are \( \alpha_1 = -1 \) and \( \alpha_2 = 0 \), then \( \deg(\sigma) > 0 \) is equivalent to the condition

\[
\mu(\mathcal{E}') < c_m,
\]

while if the eigenvalues are \( \alpha_1 = 0 \) and \( \alpha_2 = 1 \) then the condition is equivalent to

\[
\mu(\mathcal{E}/\mathcal{E}') > c_m.
\]

However by Corollary \[7.21\] (2) we may assume \( c_m = \mu(\mathcal{E}) \), and hence both conditions are equivalent to

\[
\mu(\mathcal{E}') < \mu(\mathcal{E}).
\]

This completes the proof. \( \square \)

Combining Corollary \[7.21\] and Proposition \[7.24\] our Main Theorem thus becomes the usual Hitchin–Kobayashi correspondence for Higgs bundles. We remark that Higgs bundles with more general structure groups (cf. \[H\]) can be treated in a similar manner, but we leave the details to the reader.

8. Appendix: Moment Map Lemma

For \( 1 \leq i \leq p \) let \( \mathbb{V}_i \) be a complex vector space of dimension \( n_i \). Let \( \langle \ , \ \rangle_i \) be a hermitian inner product on \( \mathbb{V}_i \) and let \( \omega_i \) be the corresponding Kaehler form. Thus

\[
\omega_i(x, y) = \frac{1}{2\sqrt{-1}}(\langle x, y \rangle_i - \langle y, x \rangle_i).
\]

Let \( \mathbb{V} \) be the tensor product \( \mathbb{V} = \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \cdots \otimes \mathbb{V}_p \), and let \( \langle \ , \ \rangle \) be the hermitian inner product determined by the inner products on the \( \mathbb{V}_i \). Thus

\[
\langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, y_1 \otimes y_2 \otimes \cdots \otimes y_p \rangle = \prod_{i=1}^p \langle x_i, y_i \rangle_i.
\]

Let \( \Omega \) be the corresponding Kaehler form. In this Appendix we compute moment maps for some Hamiltonian actions on \( (\mathbb{V}, \Omega) \).
Proposition 8.1. Let $U_i$ be the group of unitary transformations on $(\mathbb{V}_i, \langle \cdot, \cdot \rangle_i)$, and use the inner product $\langle \cdot, \cdot \rangle$ to identify the Lie algebra of $U_i$ with its dual. Then $U_i$ acts symplectically on $\mathbb{V}_i$, with moment map
\[ \mu_i : \mathbb{V}_i \rightarrow \text{Lie}(U_i) = u_i \]
given by
\[ \mu_i^V(x) = -\sqrt{-1}x x^T \]
or equivalently
\[ \mu_i^V(x) = -\sqrt{-1}x \otimes x^*. \]

(In (84) we regard a vector $x \in \mathbb{V}_i$ as a $n_i \times 1$ column whose entries are the components of $x$ with respect to a unitary frame for $\mathbb{V}_i$. Then the transpose $x^T$ is a $1 \times n_i$ row and $x^T x$ is a $n_i \times n_i$ matrix, i.e. an endomorphism of $\mathbb{V}_i$. In (86) the vector $x^*$ is the element corresponding to $x$ under the duality $\mathbb{V}_i \cong \mathbb{V}_i^*$ determined by the metric. Thus $x \otimes x^*$ is in $\mathbb{V}_i \otimes \mathbb{V}_i^*$, which we can identify with the endomorphisms of $\mathbb{V}_i$.) If any other group, $K_i$ acts symplectically on $\mathbb{V}_i$ via a faithful representation
\[ \rho_i : K_i \hookrightarrow U_i \]
then we get a moment map for the $K_i$ action, denoted by
\[ \mu_{K_i}^V : \mathbb{V}_i \rightarrow \text{Lie}(K_i)^* \]

Proposition 8.2. Fix an inner product on $\text{Lie}(K_i)$ such that $\rho_{i*} : \text{Lie}(K_i) \rightarrow u_i$ is an isometry, and use this to identify $\text{Lie}(K_i) \cong (\text{Lie}(K_i))^*$. Then we get
\[ \mu_{K_i}^V = \pi_{K_i} \circ \mu_i^V, \]
where $\pi_{K_i} : u_i \rightarrow \rho_{i*}(\text{Lie}(K_i))$ is orthogonal projection onto the linear subspace.

Each group $U_i$ acts symplectically on $(\mathbb{V}, \Omega)$ via the action
\[ A_i(x_1 \otimes x_2 \otimes \cdots \otimes x_p) = x_1 \otimes x_2 \otimes \cdots A_i x_i \otimes \cdots \otimes x_p. \]

More generally, each group $K_i$ acts symplectically on $(\mathbb{V}, \Omega)$ via
\[ k(x_1 \otimes x_2 \otimes \cdots \otimes x_p) = x_1 \otimes x_2 \otimes \cdots \rho_i(k)x_i \otimes \cdots \otimes x_p. \]

To describe the moment maps for these actions, it is convenient to fix unitary bases \[ \{e^{(i)}_j\}_{j=1}^{n_i} \] for each $\mathbb{V}_i$ and write $X \in \mathbb{V}$ as
\[ X = \sum_{i_1, i_2, \ldots, i_p} X_{i_1i_2\ldots i_p} e^{(1)}_{i_1} \otimes e^{(2)}_{i_2} \otimes \cdots \otimes e^{(p)}_{i_p}. \]

For any $1 \leq i \leq p$ we can think of $\mathbb{V}$ as a tensor product $\hat{\mathbb{V}}_i \otimes \mathbb{V}_i$, where $\hat{\mathbb{V}}_i$ is the tensor product of all the $\mathbb{V}_1, \ldots, \mathbb{V}_p$ except for $\mathbb{V}_i$, and write $X$ as
\[ X = \sum_{j=1}^{n_i} X_j \otimes e^{(i)}_j, \]
where $X_j$ are vectors in $\hat{\mathbb{V}}_i$. 

Proposition 8.3. The moment map $\mu_{K_i}^V : V \rightarrow \text{Lie}(K_i)$ is given by

\begin{equation}
\mu_{K_i}^V(X) = \sum_{j=1}^{n_i} \mu_{K_i}^V(X_j).
\end{equation}

In particular, the moment map for the action of $U_i$ on $V$ is

\begin{equation}
\mu_i^V(X) = -\sqrt{-1} \sum_{j=1}^{n_i} X_j \overline{X}_j
\end{equation}

\begin{equation}
= -\sqrt{-1} \sum_{j} \sum_{j'} x_{i_1i_2...i_p}^{j} \overline{x}_{i_1i_2...i_p}^{j'} e_j^{(i)} \otimes (e_j^{(i)})^*,
\end{equation}

where the sum in $\sum_j$ is over all the indices except the $j$'th one.

Remark 8.4. Notice that, while the definition of the $X_j$ depends on the choices of unitary bases, the combination $\sum_{j=1}^{n_i} \mu_{K_i}^V(X_j)$ does not.

We consider a few special cases, which come up in interesting examples.

8.1. $p = 1$. In this case we recover the basic moment map given in (83).

8.2. $p = 2$ (triples). Suppose that $V = V_1 \otimes V_2^*$. We can identify

\begin{equation}
V_1 \otimes V_2^* = \text{Hom}(V_2, V_1)
\end{equation}

and interpret any element $\Phi = \sum_{i,j} \Phi_{ij} e_i^{(1)} \otimes (e_j^{(2)})^*$ as a map

\begin{equation}
\Phi : V_2 \rightarrow V_1.
\end{equation}

Indeed the map is given by

\begin{equation}
(e_j^{(2)}) \mapsto \sum_i \Phi_{ij} e_i^{(1)}.
\end{equation}

Lemma 8.5. Under this identification the $U_1$ moment map given by (24) becomes

\begin{equation}
\mu_1(\Phi) = -\sqrt{-1} \Phi \Phi^*.
\end{equation}

We can also compute the moment map for $U_2$. Switching the roles of $V_1$ and $V_2^*$, and using the dual action on $V_2^*$, we get

\begin{equation}
\mu_2(\Phi) = \sqrt{-1} \Phi^T \Phi = \sqrt{-1} \Phi^* \Phi.
\end{equation}

The transpose comes from writing

\begin{equation}
\Phi = \sum_{i,j} \Phi_{ij} e_i^{(1)} \otimes (e_j^{(2)})^* = \sum_{i,j} \Phi_{ji} (e_i^{(2)})^* \otimes e_j^{(1)}
\end{equation}

while the change of sign and conjugation come from the dual group action. Notice that the moment maps (23) and (100) are none other than the projections onto $u_1$ and $u_2$ respectively of the moment map for the $U_1 \times U_2$ action on Hom($V_2, V_1$).
8.3. $p = 3$ (twisted triples). Suppose that $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{W} \otimes \mathcal{V}_2^*$. Under the identification
\begin{equation}
\mathcal{V}_1 \otimes \mathcal{W} \otimes \mathcal{V}_2^* = \text{Hom}(\mathcal{V}_2, \mathcal{V}_1 \otimes \mathcal{W})
\end{equation}
we can interpret vectors in $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{W} \otimes \mathcal{V}_2^*$ as maps
\begin{equation}
\Phi : \mathcal{V}_2 \longrightarrow \mathcal{V}_1 \otimes \mathcal{W}.
\end{equation}
Indeed, writing
\begin{equation}
\Phi = \sum_{i,j,k} \Phi_{ijk} e_i^{(1)} \otimes f_j \otimes (e_k^{(2)})^*,
\end{equation}
where $\{f_j\}$ is a frame for $\mathcal{W}$, the corresponding map is given by
\begin{equation}
(e_k^{(2)}) \mapsto \sum_{i,j} \Phi_{ijk} e_i^{(1)} \otimes f_j.
\end{equation}
Defining $\phi_j \in \mathcal{V}_1 \otimes \mathcal{V}_2^*$ by
\begin{equation}
\phi_j = \sum_{i,k} \Phi_{ijk} e_i^{(1)} \otimes (e_k^{(2)})^*,
\end{equation}
we can write
\begin{equation}
\Phi = \sum_j \phi_j \otimes f_j.
\end{equation}

**Lemma 8.6.** Under these identifications the $U_1$ and $U_2$ moment map given by (94) and (93) become
\begin{align}
\mu_1(\Phi) &= -\sqrt{-1} \sum_j \phi_j \phi_j^* \\
\mu_2(\Phi) &= \sqrt{-1} \sum_j \phi_j^* \phi_j.
\end{align}
Notice that, while the definition of the $\phi_j$ depend on the choices of unitary bases, the endomorphisms $\sum_j \phi_j \phi_j^*$ and $\sum_j \phi_j^* \phi_j$ are invariantly defined. These quantities correspond exactly to the terms which arise in the coupled twisted vortex equations for twisted triples (cf. [BGR]).

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