The Electroweak Phase Transition, Part II: $\epsilon$-Expansion Results *

Laurence G. Yaffe†

Department of Physics, FM-15, University of Washington
Seattle, Washington 98195 USA

Detailed knowledge of the electroweak phase transition is needed to determine the viability of electroweak baryogenesis. However, as discussed in the preceding talk by Peter Arnold, standard perturbative (or mean field theory) techniques are only adequate for studying the finite-temperature electroweak phase transition when the Higgs mass is sufficiently small.

The $\epsilon$-expansion, based on dimensional continuation from 3 to $4-\epsilon$ spatial dimensions, provides an alternative systematic approach for computing the effects of (near)-critical fluctuations. After reviewing the basic strategy of the $\epsilon$-expansion and its application in simple scalar theories, I will discuss the application of the $\epsilon$-expansion to electroweak theory, describe the computation of a variety of physical quantities, and summarize the results of several tests of the validity of $\epsilon$-expansion calculations in electroweak theory.

1. The $\epsilon$-expansion in scalar theory

The $\epsilon$-expansion is based on the idea that instead of trying to solve a theory directly in three spatial dimensions, it can be useful to generalize the theory from three to $4-\epsilon$ spatial dimensions, solve the theory near four dimensions (when $\epsilon \ll 1$), and then extrapolate to the physical case of 3 spatial dimensions. Specifically, one expands physical quantities in powers of $\epsilon$ and then evaluates the resulting (truncated) series at $\epsilon = 1$. This can provide a useful approximation when the relevant long distance fluctuations are weakly coupled near 4 dimensions, but become sufficiently strongly coupled that the loop expansion parameter is no longer small in three dimensions.

Scalar $\phi^4$ theory (or the Ising model) is a classic example. In four dimensions, the long distance structure of a quartic scalar field theory is trivial; this is reflected in the fact that the renormalization group equation $\mu (d\lambda /d\mu) = \beta_0 \lambda^2 + O(\lambda^3)$, has a single fixed point at $\lambda = 0$. In $4-\epsilon$ dimensions, the canonical dimension of the field changes and the renormalization group equation acquires a linear term,$^1$

$$\mu \frac{d\lambda}{d\mu} = -\epsilon \lambda + \beta_0 \lambda^2 + O(\lambda^3).$$

---

*Based on a talk presented at the Quarks-94 conference in Vladimir, Russia, May 1994. This work was performed in collaboration with P. Arnold and is described in greater detail in reference 1.

†Research supported in part by DOE grant DE-FG06-91ER40614.

*See reference 2 for other discussions of the $\epsilon$-expansion in the context of electroweak theory.
This has a non-trivial fixed point (to which the theory flows as $\mu$ decreases) at $\lambda^* = \epsilon/\beta_0 + O(\epsilon^2)$. The fixed point coupling is $O(\epsilon)$ and thus small near four dimensions, but grows with decreasing dimension and becomes order one when $\epsilon = 1$. Near four dimensions, a perturbative calculation in powers of $\lambda$ is reliable and directly generates an expansion in powers of $\epsilon$.

The existence of an infrared-stable fixed point indicates the presence of a continuous phase transition as the bare parameters of the theory are varied. Interesting physical quantities include the critical exponents which characterize the non-analytic behavior at the transition. Performing conventional (dimensionally regularized) perturbative calculations of the appropriate anomalous dimensions, and evaluating the perturbative series at the fixed point, one finds, for example, that the susceptibility exponent (equivalent to the anomalous dimension of $\phi^2$) has the expansion

$$\gamma = 1 + 0.167 \epsilon + 0.077 \epsilon^2 - 0.049 \epsilon^3 + O(\epsilon^4),$$

while the exponent characterizing the power-law decay of the propagator at the critical point (equivalent to the anomalous dimension of $\phi$) is

$$\eta = 0.0185 \epsilon^2 + 0.0187 \epsilon^3 - 0.0083 \epsilon^4 + 0.0359 \epsilon^5 + O(\epsilon^6).$$

Adding the first three non-trivial terms in these series, and evaluating at $\epsilon = 1$, yields results which agree quite well with the best available results for these exponents. (For $\gamma$, the $\epsilon$-expansion gives 1.195, to be compared with $1.2405 \pm 0.0015$, while for $\eta$ one finds 0.029 versus 0.035.)

Inevitably, perturbative expansions in powers of $\lambda$ are only asymptotic; coefficients grow like $n!$, so that succeeding terms in the series begin growing in magnitude when $n \gtrsim O(1/\lambda)$. Expansions in $\epsilon$ are therefore also asymptotic, with terms growing in magnitude beyond some order $n \gtrsim O(1/\epsilon)$. If one is lucky, as is the case in the pure scalar theory, $O(1/\epsilon)$ really means something like three or four when $\epsilon = 1$ and the first few terms of the series will be useful. If one is unlucky, no terms in the expansion will be useful. Whether or not one will be lucky cannot be determined in advance of an actual calculation.

2. Electroweak Theory

To apply the $\epsilon$-expansion to electroweak theory, one begins with the full 3+1 dimensional finite temperature Euclidean quantum field theory (in which one dimension is periodic with period $\beta = 1/T$) and integrates out all non-static Fourier components of the fields. The integration over modes with momenta of order $T$ or larger may be reliably performed using standard perturbation theory in the weakly-coupled electroweak theory. This reduces the theory to an effective 3-dimensional SU(2)-Higgs theory with a renormalization point $\mu$ which may be conveniently chosen to equal the temperature. Fermions, having no static Fourier components, are completely eliminated in the effective theory. For simplicity, the effects of a non-zero weak mixing angle and the resulting perturbations due to the U(1) gauge field are ignored. Finally, one may also integrate out the static part of the time component of the gauge field, since this field acquires an $O(gT)$ Debye-screening mass.
The renormalization group flow for an SU(2)-Higgs theory. Arrows indicate the direction of decreasing renormalization point. The dashed line is the trajectory which flows from an initial set of couplings \((g_2^2, \lambda_1)\) into the region where \(\lambda \ll g^2\).

**Fig. 1.** The renormalization group flow for an SU(2)-Higgs theory. Arrows indicate the direction of decreasing renormalization point. The dashed line is the trajectory which flows from an initial set of couplings \((g_2^2, \lambda_1)\) into the region where \(\lambda \ll g^2\).

Next, one replaces the 3-dimensional theory by the corresponding 4\(-\epsilon\) dimensional theory (and scales the couplings so that \(g_1^2/\epsilon\) and \(\lambda_1/\epsilon\) are held fixed). This is the starting point for the \(\epsilon\)-expansion. When \(\epsilon\) is small, one may reliably compute the renormalization group flow of the effective couplings. The renormalization group equations have the form:

\[
\mu \frac{d\lambda}{d\mu} = -\epsilon \lambda + (a g^4 + b g^2 \lambda + c \lambda^2) + \cdots ,
\]

\[
\mu \frac{dg^2}{d\mu} = -\epsilon g^2 + \beta_0 g^4 + \cdots .
\]

The precise values of the coefficients (and the next order terms) may be found in reference 1. These equations may be integrated analytically, and produce the flow illustrated in figure 1.

Note that a non-zero gauge coupling renders the Ising fixed point at \(\lambda = \beta_0/\epsilon\) unstable, and that no other (weakly coupled) stable renormalization group fixed point exists. Trajectories with \(g^2 > 0\) eventually cross the \(\lambda = 0\) axis and flow into the region where the theory (classically) would appear to be unstable. Such behavior is typically indicative of a first-order phase transition. To determine whether this is really the case, one must be able to perform a reliable calculation of the effective potential (or
other physical observables). As discussed in Peter Arnold’s talk, the loop expansion parameter for long distance physics is $\lambda(\mu)/g^2(\mu)$. Consequently, the best strategy is to use the renormalization group to flow from the original theory at $\mu = T$, which may have $\lambda(T)/g^2(T)$ large, to an equivalent theory with $\mu \ll T$ for which $\lambda(\mu)/g^2(\mu)$ is small. This is equivalent to the condition that one decrease the renormalization point until it is comparable to the relevant scale for long distance physics, specifically, the gauge boson mass, $M$. By doing so, one eliminates large factors of $[(M/\mu)^\epsilon - 1]/\epsilon$ which would otherwise spoil the reliability of the loop expansion. (This, of course, is nothing other than the transcription to 4−$\epsilon$ dimensions of the usual story in 4 dimensions, where appropriate use of the renormalization group allows one to sum up large logarithms which would otherwise spoil the perturbation expansion.)

For small $\epsilon$, the change of scale required to flow from an initial theory where $\lambda_1/g_1^2 = O(1)$ to an equivalent theory with $\lambda(\mu)/g^2(\mu) \ll 1$ is exponentially large; the ratio of scales is

$$s \equiv \frac{T}{\mu} \sim e^{\lambda_1/g_1^2} \sim e^{O(1/\epsilon)}.$$  

This is easy to see directly from the renormalization group equations (3) and (4). Since $g_1^2$ and $\lambda_1$ are (by construction) both $O(\epsilon)$, all terms on the right-hand side of the renormalization group equations are $O(\epsilon^2)$. Hence, a change in $\ln \mu$ of $O(1/\epsilon)$ is required to produce an order one change in the ratio of $\lambda/g^2$.

Given the parameters $g^2(\mu)$, $\lambda(\mu)$ and $m^2(\mu)$ of the resulting effective theory, one may use the usual loop expansion to compute interesting physical quantities. Because the change in scale is exponentially sensitive to $1/\epsilon$, the result for a typical physical quantity will have the schematic form

$$O = f[g^2(\mu), \lambda(\mu), m^2(\mu)] \left( \frac{\mu}{T} \right)^\#$$

$$\sim \epsilon^\# (1 + O(\epsilon) + \cdots) \exp \left[ \frac{\#}{\epsilon} (1 + O(\epsilon) + \cdots) \right].$$

In general, a calculation accurate to $O(\epsilon^n)$ requires an $n$-loop calculation in the final effective theory, together with $n+1$ loop renormalization group evolution.

To obtain predictions for the original theory in three spatial dimensions, one finally truncates the expansions at a given order and then extrapolates from $\epsilon \ll 1$ to $\epsilon = 1$. Just as for the simple $\phi^4$ theory, the reliability of the resulting predictions at $\epsilon = 1$ can only be tested a-posteriori.

Peter Arnold and I have carried out the above procedure for a variety of observables characterizing the electroweak phase transition at both leading and next-to-leading order in the $\epsilon$-expansion. To obtain leading order results, one must first integrate the one-loop renormalization group equations, determine the change in scale $\mu/T$, and express the renormalized parameters $g^2(\mu)$ and $\lambda(\mu)$ in terms of the initial parameters $g_1^2$ and $\lambda_1$. It is convenient to choose the final renormalization point as precisely that scale where the (minimally subtracted) value of $\lambda(\mu)$ vanishes. This
greatly simplifies the resulting formula for the effective potential. Computing the one-loop effective potential of the theory with \( \lambda(\mu) = 0 \) is easy; one finds the characteristic Coleman-Weinberg form

\[
V_{\text{eff}}(\phi) \mu^\epsilon = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} a (g\phi)^4 \left[ \ln \left( \frac{g\phi}{\mu} \right) + \text{const.} \right].
\] (7)

As \( m^2(\mu) \) (which is a function of \( T \)) varies, the minimum of the effective potential jumps discontinuously from the symmetric minimum at \( \phi = 0 \) to the asymmetric minimum where \( g\phi = O(\mu) \). In other words, the theory (for \( \epsilon \ll 1 \)) undergoes a first order phase transition, for all initial values of \( \lambda_1/g_1^2 \). Computing, for example, the scalar correlation length at the transition in the asymmetric phase yields

\[
\xi_{\text{asym}} = \frac{1}{\mu} \frac{\#}{g(\mu)} = \frac{1}{\mu} \frac{\#}{T} \frac{1}{\sqrt{\epsilon}} e^{-\#/\epsilon}
\] (8)

and

\[
g^2(\mu) = s^\epsilon g_1^2 / \left[ 1 + \beta_0(s^\epsilon - 1)g_1^2/\epsilon \right].
\] (9)

Explicit values for the unspecified constants above, plus the explicit (but rather involved) expression for \( s^\epsilon \) as a function of \( g_1^2/\epsilon \) and \( \lambda_1/\epsilon \), may be found in ref. 1. Similar lowest-order results were also found for the scalar correlation in the symmetric phase, the free energy difference between the symmetric and asymmetric phases \( \Delta F(T) \), the latent heat \( \Delta Q = -T d\Delta F/dT|_{T_c} \), the surface tension \( \sigma \) between symmetric and asymmetric phases at \( T_c \), the bubble nucleation rate \( \Gamma_N(T) \) below \( T_c \), and the baryon violation (or sphaleron) rate \( \Gamma_B(T_c) \).

The lowest order \( \epsilon \)-expansion predictions differ from the results of standard one-loop perturbation theory (performed directly in three space dimensions) in several interesting ways. First, the \( \epsilon \)-expansion predicts a stronger first order transition than does one-loop perturbation theory (as long as \( M_H < 130 \text{ GeV} \)). The correlation length at the transition is smaller, and the latent heat larger, than the perturbation theory results. The size of the difference depends on the value of the Higgs mass; see reference 1 for quantitative results. Naively, one would expect that a stronger first order transition would imply a smaller baryon violation rate (since a larger effective potential barrier between the co-existing phases should decrease the likelihood of thermally-activated transitions across the barrier). This expectation is wrong (in essence, because it unjustifiably assumes that the shape of the barrier remains unchanged). Along with predicting a strengthening of the transition, the \( \epsilon \)-expansion predicts a larger baryon violation rate. This occurs because the baryon violation rate is exponentially sensitive to the sphaleron mass (= the electroweak barrier height),

\[
\Gamma_B \propto \exp -S_{\text{sphaleron}},
\]

and the sphaleron mass depends inversely on \( \epsilon \),

\[
S_{\text{sphaleron}} = \frac{\#}{g^2(\mu)} = O(1/\epsilon).
\]
Hence, unlike other observables, the exponential sensitivity to $1/\epsilon$ in the baryon violation rate does not arise solely from an overall power of the scale factor $\mu/T$.

Note that an increase in the baryon violation rate (compared to standard perturbation theory) will make the constraints for viable electroweak baryogenesis more stringent; specifically, the (lowest order) $\epsilon$-expansion suggests that the minimal standard model bound $M_H \lesssim 35-40$ GeV derived using one-loop perturbation theory in ref. 9 should be even lower, further ruling out electroweak baryogenesis in the minimal model.

3. Testing the $\epsilon$-expansion

As mentioned earlier, in general there is no way to know, in advance of an actual calculation, how many terms (if any) in an $\epsilon$-expansion will be useful when results are extrapolated to $\epsilon = 1$. Therefore, in order to assess the reliability of $\epsilon$-expansion, one must try to test predictions for actual physical quantities. For the electroweak theory, three types of tests are possible.

A. $\lambda \ll g_1^2$. In the limit of a light (zero temperature) Higgs mass, or equivalently small $\lambda_1/g_1^2$, the loop expansion in three dimensions is reliable. Hence, although this is not a realistic domain, one may easily test the reliability of the $\epsilon$-expansion in this regime by comparing with direct three-dimensional perturbative calculations. Table 1 summarizes the fractional error for various physical quantities produced by truncating the $\epsilon$-expansion at leading, or next-to-leading, order before evaluating at $\epsilon = 1$, in the light Higgs limit. Although the lowest-order results often error by a factor of two or more, all but one of the next-to-leading order results are correct to better than 10%. (The free energy difference at the limit of metastability, $\Delta F(T_0)$, has the most poorly behaved $\epsilon$-expansion. However, if one instead computes the logarithm of this quantity, then the next-to-leading order result is correct to within 17%. The baryon violation rate is not shown because, due to the way its $\epsilon$-expansion was constructed, the result is trivially the same as the three-dimensional answer when $\lambda_1 \ll g_1^2$. See ref. 9 for details.)

| observable ratio                  | LO    | NLO   |
|-----------------------------------|-------|-------|
| asymmetric correlation length     | $\xi_{\text{asym}}$ | 0.14  | -0.06 |
| symmetric correlation length      | $\xi_{\text{sym}}$  | 0.62  | -0.08 |
| latent heat                       | $\Delta Q$        | -0.23 | 0.04  |
| surface tension                   | $\sigma$          | -0.40 | -0.02 |
| free energy difference            | $\Delta F(T_0)$   | -0.76 | -0.44 |

Table 1. The fractional error in the $\epsilon$-expansion results, when computing prefactors through leading order (LO) and next-to-leading order (NLO) in $\epsilon$, when $\lambda_1 \ll g_1^2$. 6
B. $\lambda \gtrsim g^2$. When $\lambda/g^2$ is $O(1)$, the three-dimensional loop expansion is no longer trustworthy. However, one may still test the stability of $\epsilon$-expansion predictions by comparing $O(\epsilon^n)$ and $O(\epsilon^{n+1})$ predictions — provided, of course, one can evaluate at least two non-trivial orders in the $\epsilon$-expansion. For most physical quantities this is not (yet) possible; determining the lowest-order behavior of the prefactor in expansion (3) requires a one-loop calculation in the final effective theory together with a two-loop evaluation of the (solution to the) renormalization group equations. A consistent next-to-leading order calculation requires a two-loop calculation in the final theory together with three-loop renormalization group evolution. Although two-loop results for the effective potential and beta functions are known, three loop renormalization group coefficients in the scalar sector are not currently available. Nevertheless, by taking suitable combinations of physical quantities one can cancel the leading dependence on the scale ratio $\mu/T$ and thereby eliminate the dependence (at next-to-leading order) on the three loop beta functions. For example, the latent heat depends on the scale as $\Delta Q \sim (\mu/T)^{2+\epsilon}$ while the scalar correlation length $\xi \sim (\mu/T)^{-1}$. Therefore, the combination $\xi^2 \Delta Q$ cancels the leading $\mu/T \sim e^{O(1/\epsilon)}$ dependence and thus requires only two-loop information for its next-to-leading order evaluation. The result of the (rather tedious) calculation may be put in the form

$$\xi_{\text{asym}}^2 \Delta Q = T^{1-\epsilon} f(f_1^2, \lambda_1) \left[1 + \delta + O(\epsilon^2)\right],$$

where $\delta$, the relative size of the next-to-leading order correction, is plotted in figure 2. The correction varies between roughly $\pm 30\%$ for (zero temperature) Higgs masses up to 150 GeV. This suggests that the $\epsilon$ expansion is tolerably well behaved for these masses. For larger masses the correction does not grow indefinitely, but is bounded by $80\%$, suggesting that the $\epsilon$ expansion may remain qualitatively useful even when it does not work as well quantitatively.

![Fig. 2. The relative size of the next-to-leading order correction to $\xi_{\text{asym}}^2 \Delta Q$ in the $\epsilon$-expansion. The values are given as a function of the (tree-level) zero-temperature Higgs mass in minimal SU(2) theory ($N = 2$) with $g = 0.63$.](image)
The comparatively small size of $\delta$ becomes more impressive when one examines the size of the different pieces which contribute. One may separate the effects produced by the second order shift in the mass parameter of the effective theory $\delta m^2(\mu)$, the shift in the effective gauge coupling $\delta g^2(\mu)$, the change in scale $\delta(\mu/T)$, and the change in the final evaluation of the latent heat. At $m_h(0) = 80$ and 250 GeV, the four different contributions are

\begin{align*}
\delta (80 \text{ GeV}) &= -0.45 + 4.88 - 0.94 - 3.79 = -0.30, \\
\delta (250 \text{ GeV}) &= -4.42 + 34.83 - 20.00 - 9.66 = 0.75,
\end{align*}

respectively. These large cancellations clearly underscore the importance of examining physical quantities, rather than unphysical ones such as $q^2(s)$ or $s^\varepsilon$, when testing the $\epsilon$-expansion.

C. $N_{\text{scalar}} \gg 1$. If one generalizes the scalar sector of the standard model to include a large number $N$ of complex scalar fields, with a global $U(N)$ symmetry, then one may expand physical results in powers of $1/N$ and compare the resulting large-$N$ predictions with those of the $\epsilon$-expansion. In brief, the result is that the $\epsilon$-expansion does not work well when $N \gg 1$. However, the $\epsilon$-expansion alerts one to its own failure by producing next-to-leading order corrections that are significantly larger than the leading-order result when $\epsilon = 1$. For example, the “tricritical slope”\footnote{When $N$ is sufficiently large, an infrared stable fixed point and a tricritical fixed point appear. If $\lambda/g^2$ is sufficiently large then the theory flows to the stable fixed point (and thus has a second order transition). If $\lambda/g^2$ is sufficiently small then the theory undergoes a first order phase transition. The tricritical slope is the slope of the line separating the two domains.} has the asymptotic forms

\begin{align*}
\frac{\lambda}{q^2} &= \frac{3}{N} \left[ 54 - 126 \epsilon + O(\epsilon^2) \right] + O\left( \frac{1}{N^2} \right), \\
&= \frac{3}{N} \left( \frac{96}{\pi^2} \right) + O\left( \frac{1}{N^2} \right).
\end{align*}

The lowest order $\epsilon$-expansion result, $162/N$, differs from the correct large-$N$ result \cite{14} by more than a factor of 5, but the expansion \cite{13} is obviously poorly behaved and unreliable at $\epsilon = 1$.

Altogether, the available information suggests that the $\epsilon$-expansion can be a useful approximation for the standard model (or other gauge theories containing a small number of scalar fields). Most importantly, the $\epsilon$-expansion predicts that the bounds for viable electroweak baryogenesis are even more stringent than suggested by a one-loop analysis in three dimensions. Clearly, calculations of additional physical quantities at next-to-leading order in the $\epsilon$-expansion should be performed to further confirm the reliability of the method.
References

1. P. Arnold and L. Yaffe, *The $\epsilon$-expansion and the electroweak phase transition*, Phys. Rev. D49, 3003–3032 (1994).

2. M. Alford and J. March-Russell, *Nucl. Phys.* B417, 527 (1993); M. Gleisser and E. Kolb, *Phys. Rev.* D48, 1560 (1993).

3. K. Wilson and M. Fischer, *Phys. Rev. Lett.* 28, 40 (1972); K. Wilson and J. Kogut, *Phys. Reports* 12, 75–200 (1974), and references therein.

4. S. Gorishny, S. Larin, F. Tkachov, *Phys. Lett.* 101A, 120 (1984).

5. J. Le Guillou, J. Zinn-Justin, *Phys. Rev. Lett.* 39, 95 (1977); *ibid., J. Physique Lett.* 46, L137 (1985); *ibid., J. Physique* 48, 19 (1987); *ibid., J. Phys. France* 50, 1365 (1989); B. Nickel, *Physica A* 177, 189 (1991).

6. C. Baillie, R. Gupta, K. Hawick and G. Pawley, *Phys. Rev.* B45, 10438 (1992); and references therein.

7. K. Chetyrkin, A. Kataev, F. Tkachov, *Phys. Lett.* 99B, 147 (1981); 101B, 457(E) (1981).

8. B. Halperin, T. Lubensky, and S. Ma, *Phys. Rev. Lett.* 32, 292 (1974); J. Rudnick, *Phys. Rev.* B11, 3397 (1975); J. Chen, T. Lubensky, and D. Nelson, *Phys. Rev.* B17, 4274 (1978); P. Ginsparg, *Nucl. Phys.* B170 [FS1], 388 (1980).

9. M. Dine, R. Leigh, P. Huet, A. Linde and D. Linde, *Phys. Lett.* B238, 319 (1992); *Phys. Rev.* D46, 550 (1992); M. Shaposhnikov, JETP Lett. 44, 465 (1986); *ibid., Nucl. Phys.* B287, 757 (1987); *ibid., Nucl. Phys.* B299, 707 (1988).
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9410295v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9410295v1