Lightest Kaonic Nuclear Clusters

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We present our study of kaonic three-body $KNN$, $KK\bar{K}$ and four-body $KNNN$, $KKNN$ clusters within the framework of a potential model using the method of hyperspherical functions in momentum representation. To perform a numerical calculations for the bound state energy of the light kaonic system, we use a set of different potentials for the nucleon-nucleon and $KN$ interactions, as well as for the kaon-kaon interaction. The calculations show that a quasibound state energy is not sensitive to the $NN$ interaction, and it shows very strong dependence on the $KN$ potential. We also compare our results with those obtained using different theoretical approaches. The theoretical discrepancies in the binding energy and width for the lightest kaonic system related to the different $NN$ and $KN$ interactions are addressed.

I. INTRODUCTION

Nowadays, the study of exotic nuclear systems involving a $\bar{K}$ is an important topic in hadron physics, because the existence of kaonic nuclear states is related to kaon condensation and to physics of the core of neutron stars that by today’s understanding are built up from exotic matter: pion and kaon condensates and quark matter\textsuperscript{1,2}. Kaonic nuclei carry important information concerning the $\bar{K}−nucleon interaction in the nuclear medium. This information is very important in understanding kaon properties at finite density and in determining of the constraints on kaon condensation in high-density matter. The latter will allow one to adjust the methods developed in condensed matter physics for exciton and excitonic polariton condensates (see, for example,\textsuperscript{3,4}) to study the kaon condensation. The best way to understand the many body kaonic nuclear system is to study the simplest two-, three- and four-body clusters: $KNN$, $KK\bar{K}$, and $KNNN$, as well as double kaonic clusters, when one nucleon in the three- or four-body kaonic cluster is replaced by the $K−$meson. The light kaonic clusters $KNN$ and $KNNN$ represent three- and four-body systems and theoretically can be treated within the framework of a few-body physics approaches. In the recent past much efforts have been focused on the calculations of quasibound state energies and widths for three- and four-body kaonic clusters. A variety of methods have been used in configuration and momentum spaces, to obtain eigenvalues for energy and width of quasibound states using diverse sets of $KNN$ and $NN$ interactions. These include but are not limited by variational method approaches\textsuperscript{5–15}, the method of Faddeev equations in momentum and configuration spaces\textsuperscript{16–28}, Faddeev-Yakubovsky equations\textsuperscript{26} and the method of hyperspherical harmonics in configuration and momentum spaces\textsuperscript{28–30}. However, the predicted values for the binding energy and the width are in considerable disagreement. For example, for the $K−pp$ cluster the predicted values for the binding energy and the width are 9–95 MeV and 20–110 MeV, respectively.

On the experimental side, several experiments have been performed to search for the kaonic clusters using various nuclear reactions starting from the first measurement reported by the FINUDA collaboration for the $K−pp$ cluster\textsuperscript{31} and including the most recent report of J-PARC E15 Collaboration\textsuperscript{32}. The situation is still controversial and the existence, for example, of the $K−pp$ quasibound state has not been established yet. Thus, the theoretical and experimental study of composite systems of $K−$mesons and nucleons is still a challenging issue in nuclear physics.

Below we present a study of the lightest kaonic nuclear clusters using the method of hyperspherical functions. We focus on three- and four-body nonrelativistic calculations within the framework of a potential model for the three- and four-body kaonic clusters using the method of hyperspherical harmonics (HH) in momentum representation. Calculations for a binding energy and width of the kaonic three- and four-body system are performed using different $NN$ potentials and kaon-nucleon interactions, as well as kaon-kaon interactions. Such approach allows one to understand the key role of the kaon-nucleon interaction and the importance of nucleon-nucleon interaction in the formation of quasibound states of the kaonic three- and four-body systems.
II. THEORETICAL FORMALISM

The Hamiltonians of the three and four nonrelativistic particles for the $\bar{K}NN$ and $\bar{K}NNN$ systems, respectively, read as

$$H_3 = \hat{T}_3 + V_{N_1N_2} + V_{\bar{K}N_1} + V_{\bar{K}N_2},$$

$$H_4 = \hat{T}_4 + \sum_{1\leq i<j\leq 3} V_{N_iN_j} + \sum_{i=1}^3 V_{\bar{K}N_i},$$

where $\hat{T}_3$ and $\hat{T}_4$ are the operators of the kinetic energy for three- and four-particle system, respectively, $V_{N_iN_j}$ is the nucleon-nucleon potential and $V_{\bar{K}N_i}$ is a pairwise effective antikaon interaction with the nucleon. The effective interactions of the $\bar{K}N$, $KN$, $\bar{K}\bar{K}$ and $K\bar{K}$ two-body subsystems are discussed in detail in Refs. [5, 10, 14, 33–36]. Below, we use two effective $\bar{K}N$ interactions that were derived in different ways. The effective $\bar{K}N$ interactions can be derived phenomenologically or constructed using the chiral SU(3) effective field theory, which identifies the Tomozawa-Weinberg terms as the main contribution to the low-energy $\bar{K}N$ interaction [34]. The single-channel potential for the description of the $\bar{K}N$ interaction was derived in Refs. [3, 10] phenomenologically, using $\bar{K}N$ scattering and kaonic hydrogen data and reproducing the $\Lambda(1405)$ resonance as a $\tilde{K}^-p$ bound state at 1405 MeV, and the decay width of Lambda(1405) is also taken into account in this potential. We refer to this as the Akaishi-Yamazaki (AY) potential. The AY potential is energy independent. The other $\bar{K}N$ interaction given in Ref. [32] was derived based on the chiral unitary approach for the $s$-wave scattering amplitude with strangeness $S = -1$, and reproduces the total cross sections for the elastic and inelastic $K^-p$ scattering, threshold branching ratios, and the $\pi\Sigma$ mass spectrum associated with the $\Lambda(1405)$. Hereafter we refer to this energy-dependent potential for the parametrization [37] as the HW potential. Both potentials are constructed in the coordinate space, are local, and can be written in the one-range Gaussian form as

$$V_{\bar{K}N}(r) = \sum_{I=0,1} U^I \exp \left[ -(r/b)^2 \right] P^I_{\bar{K}N},$$

where $r$ is the distance between the kaon and the nucleon, $b$ is the range parameter and $P^I_{\bar{K}N}$ is the isospin projection operator. The values of the potential depth $U^{I=0}$ and $U^{I=1} = 1$ for each interaction are given in Refs. [5, 10] and [37], and the range parameter is chosen to be $b = 0.66$ fm for the AY potential and $b = 0.47$ fm for the HW potential.

To describe the $V_{N_iN_j}$ nucleon-nucleon interaction, we use several different $NN$ potentials: the realistic Argonne V14 (AV14) and V18 (AV18) [38, 39], the semi-realistic Malfliet and Tjon MT-I-III (MT) [40] potential, the Tamagaki G3RS potential [41], which we hereafter refer to as the T potential, and potential [42], which we refer to as the M potential. Therefore, the use of different $NN$ potentials and $\bar{K}N$ interactions allows one to perform a validity test for the lightest kaonic clusters against various $NN$ and $\bar{K}N$ interactions.

The binding energies and the wave functions of the three and four nonrelativistic particles can be obtained by solving the Schrödinger equation with the Hamiltonians (1) and (2), respectively. In our approach we use the hyperspherical harmonics method that represents a technique of solution of the Schrödinger equation to find the bound and scattering states for a few-body system. The main idea of this method is the expansion of the wave function of the corresponding nuclear system in terms of hyperspherical harmonics that are the eigenfunctions of the angular part of the Laplace operator in the six-dimensional space (three-body problem) or in the nine-dimensional space (four-body problem). The details of this method can be found in the monographs [43–45]. In our calculations we use the HH method in momentum representation [45, 46]. One starts from the Schrödinger equation for the three or four particles with the Hamiltonians (1) and (2), respectively, and rewrites this equation in the integral form in the momentum representation using the set of the Jacobi momenta $q_i$ in $3(N-1)$-dimensional momentum space. These momenta are the trees of Jacobi coordinates for three- or four-particle system $q_i = \sqrt{\frac{m_{12...i+1}}{m_{12...i}}} \left( \frac{1}{m_{12...i}} \sum_{j=1}^i m_j q_j - q_{i+1} \right)$, $i = 1, 2, ..., N-1$, where $m_j$ and $q_j$ are the particles masses and momentum vectors conjugated to the position vectors $r_j$ respectively, $m_{12...i} = \sum_{j=1}^i m_j$ and $N$ is the number of particles. After that, one introduces the set of
the hyperspherical coordinates in the momentum space given by the hyperradius \( \kappa^2 = \sum_{i=1}^{N-1} \kappa_i^2 \) and the set of angles \( \Omega_{\kappa} \), which define the direction of the vector \( \kappa \) in \( 3(N-1) \)-dimensional momentum space, as well as the symmetrized hyperspherical harmonics in momentum representation \( \Phi_{\mu}^\lambda (\Omega_{\kappa}, \sigma, \tau) \) that are written as a sum of products of spin and isospin functions and hyperspherical harmonics \([43–45]\). By expanding the wave function of \( N \) body hyperspherical harmonics depend and the integer \( \mu \) is the global momentum in the \( 3(N-1) \)-dimensional configuration space, which is the analog of angular momentum in case of \( N = 2 \). The HH are the eigenfunctions of the angular part of the \( 3(N-1) \)-dimensional Laplace operator in configuration space with eigenvalue \( L_N(L_N + 1) \), where \( L_N = \mu + 3(N-2)/2 \) and they are expressible in terms of spherical harmonics and Jacobi polynomials \([43–45]\). By expanding the wave function of \( N \) bound particles in terms of the symmetrized hyperspherical harmonics in momentum space

\[
\Psi(\kappa, \Omega_{\kappa}) = \sum_{\mu, \lambda} u_{\mu}^{\lambda}(\kappa) \Phi_{\mu}^\lambda (\Omega_{\kappa}, \sigma, \tau),
\]

and substituting Eq. (1) into the corresponding integral Schrödinger equation in the momentum representation, one obtains a system of coupled integral equations for the hyperradial functions \( u_{\mu}^{\lambda}(\kappa) \) for the system of three or four particles. The detailed description of the formalism for the \( K^-pp \) cluster can be found in Ref. \([28]\). Here we expand the wave function of three bound particles in terms of the symmetrized hyperspherical harmonics \( \Phi_{\mu}^\lambda(\Omega_{\kappa}, \sigma, \tau) \) in momentum representation: \( \Psi(p, q) = \sum_{\mu, l, l_q} u_{\mu}^{l,l_q}(\kappa) \Phi_{\mu}^{l,l_q}(\Omega_{\kappa}, \sigma, \tau) \), where \( \mu \) is the grand angular momentum, \( L \) is the total orbital momentum, \( l_p \) and \( l_q \) are the angular momenta corresponding to the Jacobi momenta \( p \) and \( q \) that are conjugated to the standard Jacobi coordinates for three particles, \( \kappa \) is the hyperradius in the six dimensional momentum space, and \( \Omega_{\kappa} \) is the set of five angles which define the direction of the vector \( \kappa \). The functions \( \Phi_{\mu}^{l,l_q}(\Omega_{\kappa}, \sigma, \tau) \) are written as a sum of products of spin and isospin functions and HH, using the Raynal-Révai coefficients \([48]\). For the system \( K^-pp \) the wave function is antisymmetrized with respect to two protons, \( K^-pp \), while for the \( K^-K^-p \) system it is symmetrized with respect to two antikaons. For the hyperradial functions \( u_{\mu}^{l,l_q}(\kappa) \) we obtain the coupled integral equations. By solving the coupled integral equations one can find the hyperradial functions \( u_{\mu}^{l,l_q}(\kappa) \) for a given \( L \) and the binding energies for the \( K^-pp \) and \( K^-K^-p \) systems. For the system \( \bar{K}NN \) the wave function is antisymmetrized with respect to three nucleons, while for the \( K^-K^-p \) system it is symmetrized with respect to two antikaons and antisymmetrized with respect to two protons.

III. RESULTS OF NUMERICAL CALCULATIONS AND DISCUSSION

A. \( K^-pp \) cluster

Let’s start with the results of our calculations of the \( K^-pp \) cluster recently reported in Ref. \([28]\). Results of these calculations for the \( K^-pp \) cluster are presented in Table 1. For the calculations of the binding energy and the width with the method of HH we use as input MT, T, and AV14 potentials for the \( NN \) interaction, while for the \( \bar{K}N \) interaction we use the energy-dependent effective HW and the phenomenological AY potentials. Such an approach allowed us to examine how the \( K^-pp \) cluster’s structure depends on different choices of the \( \bar{K}N \) interactions for the same \( NN \) potential, as well as to investigate its dependence on different choices of the \( NN \) interaction for the same \( \bar{K}N \) interaction, and to understand the sensitivity of the system to the input interactions. The analysis of the calculations presented in Table 1 show that the AY potential as the \( \bar{K}N \) interaction input falls into the 46 – 47 MeV range for the binding energy of the \( K^-pp \) cluster, while the chiral HW \( \bar{K}N \) potential gives about 17 – 21.6 MeV for the binding energy. Thus, the values for the binding energy for the \( K^-pp \) system obtained for the different \( NN \) potentials are in reasonable agreement, and the ground state energy is not very sensitive to the \( NN \) interaction. However, there is a very strong dependence on the antikaon-nucleon interaction. When we employ the effective energy-dependent chiral-theory based HW potential for the \( \bar{K}N \) interaction and different \( NN \) interactions, as inputs, we predict a weakly bound \( K^-pp \) cluster. This is similar to Ref. \([14]\), where the authors employed several versions of energy-dependent effective \( \bar{K}N \) interactions derived from chiral SU(3) dynamics together with the realistic AV18 \( NN \) potential. Our calculations also confirm results reported in earlier studies employing the same type of \( \bar{K}N \) interaction \([13, 21, 29]\). The energy of the bound state, as well as the width calculated for the AY potential \([5]\) are
TABLE I: The binding energy $B$ and width $\Gamma$ for the $K^-p$ system calculated in the framework of the method of HH in the momentum representation for different interactions. $NN$ potentials: AV14 [38], MT [40] and T [41]. $KN$ interactions: AY [3] and HW [33]. $E_{K^-p}$ is two-body energy of the $K^-p$ cluster.

|                | AV14+AY | MT+AY | T+AY | AV14+HW | MT+HW | T+HW |
|----------------|---------|-------|------|---------|-------|------|
| $B$, MeV       | 46.2    | 46.5  | 46.3 | 17.2    | 20.5  | 20.6 |
| $\Gamma$, MeV  | 66.7    | 84.3  | 74.5 | 44.3    | 48.1  | 49.5 |

TABLE II: The bound state energies of $K^-p$ ($E_2$) and $K^-K^-p$ ($B$) systems, and the root-mean-square radius of the $K^-$ distribution. $\Delta E$ is the binding energy measured from the two-body threshold.

|                | $K^-K^-$ | $K^-p < r^2 >^{1/2}$, fm | $E_2$, MeV | $B$, MeV | $\Delta E$, MeV |
|----------------|---------|---------------------------|------------|-----------|-----------------|
| $V_{K^-K^-} = 0$ | AY      | 1.36                      | 30.0       | 31.7      | 1.7             |
|                | HW      | 1.96                      | 11.42      | 11.43     | 0.01            |
| $V_{K^-K^-} = 0$ | HW      | 12.21                     | 0.79       |           |                 |

more than twice as big as those obtained for the energy-dependent chiral $\tilde{K}N$ HW potential. Therefore, the highest binding energies are obtained for the phenomenological AY potential. Let’s compare our results with those obtained with different variational approaches. Our results are in good agreement with those of Ref. [10], where the binding energy and the width for the $K^-p$ cluster were calculated by employing the AY potential as the $\tilde{K}N$ interaction and T potential as the $NN$ interaction. Recently it was reported that “resonance and coupled-channel problem are key ingredients in the theoretical study of the $K^-p$” [41]. Those authors employ a coupled-channel Complex Scaling Method combined with the Feshbach method since this approach can simultaneously treat these two ingredients. Interestingly enough, their calculations [49, 50] for the binding energy and width are consistent with our results [35, 36, 51, 52]. The baryonic systems $K\bar{K}$ with two kaons were investigated in Refs. [35, 53]. We study a possible bound state of the $K^-K^-p$ cluster with $S = -2$, $I = 1/2$, $J^+ = 1/2^+$ using the effective $s$–wave AY and HW potentials assuming that this state is formed due to the strong $K^-p$ attraction. The strength of the $s$–wave $\tilde{K}\bar{K}$ interaction for the isospin $I = 0$ is zero due to Bose statistics, and we consider a weak repulsion for the isospin $I = 1$ that reproduces the scattering lengths $a_{K^+K^-} = -0.14$ fm for the range parameter $b = 0.66$ fm (AY potential ) and $b = 0.47$ fm (HW potential). The results of calculations for the binding energies for the $K^-p$ and $K^-K^-p$, the bound $K^-K^-p$ state without $K^-K^-$ interaction, and the root-mean-square radius of the $\tilde{K}$ distribution are presented in Table I. For the AY potential, the $K^-K^-p$ system is still bound even with a much stronger $K^-K^-$ repulsion, while for the HW potential there is the bound state with the energy 0.01 MeV relative to the $K^-p + K^-$ threshold. Thus, although the $\tilde{K}N$ with $I = 1$ is attractive, the attraction is not strong enough to overcome the $K^-K^-$ repulsion. For the width within the method of HH we obtain 58.6 MeV and 41.6 MeV with the AY and HW potentials, respectively. Our results for the binding energy of the $K^-K^-p$ system obtained by the method of HH are in reasonable agreement with calculations obtained using a variational method [30] and the Faddeev calculations [29].

B. $K^-K^-p$ cluster

Three-body problem with two mesons and one baryon have received considerable attention in the recent literature [32, 50, 51, 52]. The baryonic systems $\tilde{K}\bar{K}N$ and $\tilde{K}KN$ with two kaons were investigated in Refs. [32, 50, 53]. We study a possible bound state of the $K^-K^-p$ cluster with $S = -2$, $I = 1/2$, $J^+ = 1/2^+$ using the effective $s$–wave AY and HW potentials assuming that this state is formed due to the strong $K^-p$ attraction. The strength of the $s$–wave $\tilde{K}\bar{K}$ interaction for the isospin $I = 0$ is zero due to Bose statistics, and we consider a weak repulsion for the isospin $I = 1$ that reproduces the scattering lengths $a_{K^+K^-} = -0.14$ fm for the range parameter $b = 0.66$ fm (AY potential ) and $b = 0.47$ fm (HW potential). The results of calculations for the binding energies for the $K^-p$ and $K^-K^-p$, the bound $K^-K^-p$ state without $K^-K^-$ interaction, and the root-mean-square radius of the $\tilde{K}$ distribution are presented in Table I. For the AY potential, the $K^-K^-p$ system is still bound even with a much stronger $K^-K^-$ repulsion, while for the HW potential there is the bound state with the energy 0.01 MeV relative to the $K^-p + K^-$ threshold. Thus, although the $\tilde{K}N$ with $I = 1$ is attractive, the attraction is not strong enough to overcome the $K^-K^-$ repulsion. For the width within the method of HH we obtain 58.6 MeV and 41.6 MeV with the AY and HW potentials, respectively. Our results for the binding energy of the $K^-K^-p$ system obtained by the method of HH are in reasonable agreement with calculations obtained using a variational method [30] and the Faddeev calculations [29].

C. $KK\bar{K}$ system

Recently, there has been increased interest in few-body systems constituted by two or more kaons. Particularly noteworthy is the possibility of formation of the quasibound states in a $KK\bar{K}$ system. We study the $KK\bar{K}$ system using a nonrelativistic potential model in the framework of the method of HH in momentum representation and
consider the $KK\bar{K}$ system as three interacting kaons. Once the two-body interactions for the $K\bar{K}$ and $KK$ subsystems are determined one can determine the wave function of the $KK\bar{K}$ system by solving the Schrödinger equation for the Hamiltonian $H = T_3 + V_{KK}(r_{12}) + V_{KK}(r_{13}) + V_{KK}(r_{23})$, where the potential energy is the sum of the effective $KK$ and $K\bar{K}$ interactions that are the functions of the interparticle distances $r_{ij}$. For the description of the effective kaon-kaon interactions we use the local potentials from Refs. [55] and [35] that can be written in one-range Gaussian form [56]. The set of values of the potential depth $U_A^I$ for each interaction is given in Refs. [55, 56], and the range parameter $b$ is chosen to be the same for the $KK$ and $K\bar{K}$ interactions. We choose two optimized values for the range parameter: $b = 0.47$ fm (set A) and $b = 0.66$ fm (set B). The strength of strongly attractive $s$–wave $KK$ interactions was assumed to be the same for the isospin $I = 0$ and isospin $I = 1$, while the strength of the $s$–wave $KK$ interaction for the isospin $I = 0$ is $U_{KK}^0 = 0$ due to Bose statistics and we consider a weak repulsion for the isospin $I = 1$. In Ref. [36] the $K\bar{K}$ interaction is derived under the assumption that $K\bar{K}$ forms the quasibound states $f_0(980)$ and $a_0(980)$ in $I = 0$ and $I = 1$ channel, respectively, and it reproduces the masses and widths of these resonances. The strength of the repulsive $KK$ interaction in $I = 1$ was fixed to reproduce a lattice QCD calculation [54] for the scattering length $a_{K+K+} = -0.14$ fm, and a weaker repulsion that corresponds to the scattering length $a_{K+K+} = -0.10$ fm. Results of calculations for the set of potentials A when the $KK$ interaction reproduces the scattering lengths $a_{K+K+} = -0.14$ fm and $a_{K+K+} = -0.10$ fm are denoted as A1 and A2, respectively. Correspondingly, the set of potentials B that reproduces those different scattering lengths hereafter we refer as B1 and B2. The solution of a system of coupled integral equations for the hyperradial functions allows us to construct the wave function $\Psi$ for the $KK\bar{K}$ system and to determine the binding energy $B$. A reasonable convergence for the ground state energy is reached for the grand angular momentum $\mu = 10$ and we limit our considerations to this value. The width of the bound state is evaluated from the imaginary part of the $K\bar{K}$ interactions as $\Gamma = -2 \langle \Psi | \text{Im} (V_{KK}(r_{12}) + V_{KK}(r_{23})) | \Psi \rangle$. The results of our calculations for the binding energy and the width for the $KK\bar{K}$ system along with the results obtained with a coupled-channel approach based on the solution of the Faddeev equations in momentum representation and the variational method [55] are presented in Table III. The total mass the $KK\bar{K}$ system ranges from 1463.4 to 1469.4 MeV when we consider the same $K$ meson mass $m_K = 496$ MeV as in Ref. [55]. The width falls within the 82–96 MeV range for all sets of the $K\bar{K}$ and $KK$ interactions. The quasibound state for the $KK\bar{K}$ with spin-parity $0^-$ and total isospin 1/2 is found to be below the three-kaon threshold. The comparison of our results with the results obtained with the variational method [56] shows that while the binding energy found within the HH and variational calculations are close enough, the difference for the width is more than 20%. The alternative scenario is observed for the HH and the Faddeev calculations in the momentum representation: the difference in masses is more than 40 MeV. We also perform calculations for the $KK\bar{K}$ system using $s$–wave two-body separable potentials with Yamaguchi form factors from Ref. [56] that also used in Faddeev and Faddeev-Yakubovsky calculations [26] for the $K^-pp$ and $K^-pp$ kaonic clusters. The corresponding results are presented in the last column of Table III and are very close to the results obtained using the effective local kaon-kaon interactions for the set B. Thus our calculations within three body nonrelativistic potential model predict a quasibound state for the $KK\bar{K}$ system with mass around 1460 MeV that can be associated with the $K(1460)$ resonance. Our results support the conclusion obtained through the variational calculations and a coupled-channel approach using the Faddeev equations that $K(1460)$ could be considered as a dynamically generated resonance.

### Table III: Results of calculations with the Faddeev and variational calculations from Ref. [55]

|       | Faddeev [55] | Variational with A1 [55] | A1  | A2  | B1  | B2  | Separable potential |
|-------|--------------|---------------------------|-----|-----|-----|-----|---------------------|
| Mass, MeV | 1420         | 1467                      | 1469.4 | 1468.2 | 1464.1 | 1463.8 | 1463.4             |
| Width, $\Gamma$, MeV | 50   | 110                        | 82.2 | 84.0 | 96.8 | 98.2 | –                  |
| $B$, MeV | 21            | 18.6                      | 19.8 | 23.9 | 24.2 | 24.6 | –                  |
| $\sqrt{<r^2>}$, fm | 1.6 | 1.72                      | 1.65 | 1.61 | 1.56 | 1.52 | –                  |
| $KK$ distance, fm | 2.8 | 3.2                      | 2.92 | 2.72 | 2.70 | 2.68 | –                  |
| $(KK) - \bar{K}$, fm | 1.7 | 1.78                      | 1.68 | 1.66 | 1.62 | 1.60 | –                  |
| $K\bar{K}$ distance, fm | 1.6 | 1.68                      | 1.65 | 1.64 | 1.58 | 1.55 | –                  |
| $(K\bar{K}) - K$ distance, fm | 2.6 | 2.9                      | 2.86 | 2.55 | 2.50 | 2.47 | –                  |
D. $\bar{K}NNN$ clusters

Recently Faddeev-Yakubovsky calculations\cite{Ref26} were made for the four-particle $K^{-}ppn$ and $K^{-}K^{-}pp$ kaonic clusters, where the quasibound states were treated as bound states by employing real $s$–wave two-body separable potential models for the $\bar{K}K$ and the $\bar{K}$–nucleon interactions as well as for the $NN$ interaction. Fully four-body nonrelativistic realistic calculations of $\bar{K}NNN$, and $\bar{K}KNN$ quasibound states within the method of HH in configuration space, using realistic $NN$ potentials and subthreshold energy dependent chiral $\bar{K}N$ interactions, were presented in Ref.\cite{Ref29}. Giving that below we present the results of our calculations for the $\bar{K}NNN$, and $\bar{K}KNN$ quasibound states in the framework of method of HH in momentum representation using AV18\cite{Ref39} and M\cite{Ref42} $NN$ potentials and AY and HW $\bar{K}N$ interactions as inputs. To find the binding energies with above mentioned set of potentials, we solve a system of coupled integral equations for the hyperradial functions $w_{R}^{\lambda}(x)$. In the calculations we limit our consideration with the value $\mu_{\text{max}}=10$ getting a reasonable convergence for the binding energy. Using the wave function obtained for $\mu_{\text{max}}=10$ the width is evaluated through the expression $\Gamma = -2\langle \Psi \mid \text{Im} v_{R_{N}} \mid \Psi \rangle$, where $v_{R_{N}}$ sums over all pairwise $\bar{K}N$ interactions. In Table IV we present our results for $\bar{K}NNN$ cluster that we compare with those obtained via different methods. The results of our calculations for the energy and the width show dependence on the $NN$ potentials and on the $\bar{K}N$ interactions. However, this dependence is dramatically different: for the same $\bar{K}N$ interaction and different $NN$ potentials the ground state energy and the width change by about $3–15\%$, while for the same $NN$ potential and different $\bar{K}N$ interaction the energy changes by a factor of more than 3 and the width changes by more than twice.

For the comparison let’s mention that the authors of Ref.\cite{Ref29} obtained 18.5 MeV and 31.0 MeV for the binding energy and the width of the $K^{-}ppn$ cluster, while calculation within the Faddeev-Yakubovsky equations\cite{Ref26} with separable potential models for the $\bar{K}$–nucleon and the nucleon-nucleon interactions leads to very deep ground state energy $–69$ MeV. The comparison of our results for the $K^{-}ppn$ and $K^{-}ppp$ clusters obtained for AV18 $NN$ interaction and HW $\bar{K}N$ interaction with calculations\cite{Ref29} within the variational HH method for the AV14 $NN$ interaction and shallow chiral $\bar{K}N$ interaction shows that they are very close. The predictions\cite{Ref9} for the binding energy and the width for the kaonic clusters studied based on a framework of antisymmetrized molecular dynamics and employing the $T$ potential as a $NN$ interaction and AY potential as a $\bar{K}N$ interaction are presented in the last column of Table IV.

Based on the results of our calculations, we can conclude that the pairwise $\bar{K}N$ interaction plays a major role in the formation of the kaonic bound state and we found that $\bar{K}N$ effective interactions\cite{Ref13,Ref14} based on chiral SU(3) dynamics\cite{Ref33} leads to a relatively modest binding for the $K^{-}ppn$ and $K^{-}ppp$ clusters that confirms the calculations in\cite{Ref29}.

E. $\bar{K}KNN$ cluster

A decade ago in Ref.\cite{Ref7} a deeply bound double kaonic $K^{-}K^{-}pp$ cluster was predicted to be deeply bound with binding energy of 117 MeV and width 35 MeV. Barnea, Gal and Liverts\cite{Ref29} perform a HH calculation in configuration space for the $K^{-}K^{-}pp$ systems based on the shallow chiral $\bar{K}N$ interaction model with the self-consistent energy dependence taken into account and obtained very shallow bound states with a binding energy that is substantially smaller than earlier prediction\cite{Ref7}. The same tendency can be observed from Table IV which presents the results of

| TABLE IV: The binding energy $B$ and width $\Gamma$ for the $\bar{K}NNN$ system calculated in the framework of the method of HH in the momentum representation for different interactions with results from Refs.\cite{Ref0,Ref26} and \cite{Ref29}. The parity $\pi$ includes the eigen parity of antikaon. |
|---|---|---|---|---|---|---|---|
| $J^\pi$ | $T$ | $B$, MeV | $\Gamma$, MeV |
| $K^{-}ppn$ | $\frac{1}{2}^-$ | 0 | 92.1 | 28.6 |
| | | $\Gamma$, MeV | 84.4 | 30.8 |
| $K^{-}ppn$ | $\frac{1}{2}^-$ | 1 | 64.6 | 17.2 |
| | | $\Gamma$, MeV | 74.2 | 27.1 |
| $K^{-}ppp$ | $\frac{3}{2}^+$ | 1 | 101.9 | 25.8 |
| | | $\Gamma$, MeV | 87.9 | 31.2 |
IV. CONCLUSIONS

Within the framework of a potential model for the kaonic clusters $\bar{K}NN$, $\bar{K}KN$, $KK\bar{K}$, $\bar{K}N\bar{N}$, and $\bar{K}\bar{K}NN$ we perform nonrelativistic three- and four-body calculations using the method of hyperspherical harmonics in the momentum representation. We examine how the binding energy and width of the $K^{-}pp$ cluster depends on different choices of the $KN$ and $NN$ interactions. Our consideration includes the realistic Argonne V14 [38], the semi-realistic MT [40] and T [41] potentials as inputs for the $NN$ interaction and we employ the phenomenological AY potential and HW potential constructed based on chiral SU(3) dynamics, as inputs for the $KN$ interaction. For all types of considered $NN$ interactions, our calculations predict deeply bound states for the AY $\bar{K}N$ interaction and a relatively shallowly bound $K^{-}pp$ cluster for the effective chiral $KN$ interaction. Moreover, the $K^{-}pp$ cluster is the most strongly quasibound three-body system. The results of our calculations show that the binding energy of the $K^{-}pp$ system depends entirely on the ansatz for the $KN$ interaction and substantially changes when we use the AY and HW $KN$ interaction. In regard to the sensitivity of the binding energy to the details of the $NN$ potentials in Ref. [33] found that as long as the $K^{-}pp$ system is only weakly bound, the dependence on different types of $NN$ interactions is weak. In fact, our study confirms this conclusion and, moreover, shows that the dependence on different types of $NN$ interactions is also weak if the $K^{-}pp$ system is strongly bound.

The strong AY $\bar{K}N$ interaction is responsible for the formation of the $K^{-}K^{-}p$ system and this cluster is still bound even with a much stronger $\bar{K}\bar{K}$ repulsion, while the HW potential leads to the bound state with energy of only 0.01 MeV relative to the $K^{-}p + K^{-}$ threshold. The mass (binding energy) of the $KK\bar{K}$ system slightly depends on the sets of parameters that determine $\bar{K}\bar{K}$ and $KK$ interactions and the width falls into the 82- 96 MeV range for all sets of these parameters. There is reasonable agreement between these results, the mass obtained using separable AMY [50] interactions and the variational calculation [52]. Our results for the $KK\bar{K}$ system support the conclusion that $K(1460)$ could be considered as a dynamically generated resonance.

Based on the results of our calculations for four-particle kaonic systems we also can conclude that the pairwise $\bar{K}N$ interaction plays a major role in the formation of the kaonic bound state and the effective chiral $\bar{K}N$ interaction gives relatively modest binding for the $K^{-}ppm$, $K^{-}pp$ and $K^{-}K^{-}pp$ clusters.

All our calculations with the effective chiral $\bar{K}N$ interaction show that the width is always larger than the binding energy. In some cases the width is more than twice as large as the binding energy. Only for some four-particle kaonic clusters when the input for the $\bar{K}N$ interaction is the AY potential, the binding energy is larger than the width. As a consequence, perhaps, we are facing a situation where it is hard to identify the resonances which would make the experimental observation challenging.

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