UNIQUENESS OF ANCIENT SOLUTIONS
TO GAUSS CURVATURE FLOW ASYMPTOTIC TO
A CYLINDER

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ABSTRACT. We address the classification of ancient solutions to the
Gauss curvature flow under the assumption that the solutions are con-
tained in a cylinder of bounded cross-section. For each cylinder of convex
bounded cross-section, we show that there are only two ancient solutions
which are asymptotic to this cylinder: the non-compact translating soli-
ton and the compact oval solution obtained by gluing two translating
solitons approaching each other from time $-\infty$ from two opposite ends.

1. Introduction

A one-parameter family $\Sigma_t := F(M^n, t)$ of complete convex embedded
hypersurfaces defined by $F : M^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$ is a solution of the Gauss
curvature flow (GCF in abbreviation) if $F(p, t)$ satisfies

$$\frac{\partial}{\partial t} F(p, t) = -K(p, t) \nu(p, t),$$

where $K(p, t)$ is the Gauss curvature of $\Sigma_t$ at $F(p, t)$, and $\nu(p, t)$ is the unit
normal vector of $\Sigma_t$ at $F(p, t)$ pointing outward of the convex hull of $\Sigma_t$.

The GCF was first introduced by W. Firey \cite{Firey1974} in 1974 as a model that
describes the deformation of a compact convex body $\Sigma_0$ embedded in $\mathbb{R}^{n+1}$
which is subject to wear under impact from any random angle. An example
can be a stone on a beach impacted by the sea. The probability of impact
at any point $P$ on the surface $\Sigma_t = \partial \Sigma_t$ is proportional to the Gauss curva-
ture $K$ of $\Sigma$ at $P$. W. Firey showed, assuming that a solution exists, that
the GCF shrinks smooth, compact, strictly convex and centrally symmetric
hypersurfaces embedded in $\mathbb{R}^3$ to round points. The existence of solutions in any dimension was established in 1985 by Tso [39]. Tso showed that under the assumption that the initial surface $\Sigma_0$ is smooth, compact and strictly convex the Gauss curvature flow admits a unique solution $\Sigma_t$ which shrinks to a point at the exact time $T^* := V/4\pi$, where $V$ is the volume enclosed by the initial surface $\Sigma_0$. Around the same time Chow [23] proved that, under certain restrictions on the second fundamental form of the initial surface, the Gauss curvature flow shrinks smooth compact strictly convex hypersurfaces to round points. Later, in [3] Andrews showed that the Gauss Curvature flow shrinks any compact convex hypersurface in $\mathbb{R}^3$ to a round point. For higher dimensions $n \geq 3$, P. Guan and L. Ni in [32] obtained the convergence of the flow after rescaling to a self-shrinking soliton. K. Choi and P. Daskalopoulos [19] have recently shown that the sphere is the unique self-shrinking soliton which combined with the result in [32] shows that the only finite time singularities in the $n$-dimensional GCF are the spheres.

In this work we will study the ancient solutions to the GCF, that is solutions which exist for all time $t \in (-\infty, T)$, for some $T \in (\infty, \infty]$. Shrinking and translating solitons are typical important models of ancient solutions. A shrinking soliton refers a solution which homothetically shrinks to a point. A shrinking soliton which shrinks to at spatial origin at time $t = 0$, is of the form $\Sigma_t = (-t)^{-1/(n-1)} \Sigma_0$ and $\Sigma_t$ satisfies $K = \frac{1}{(n+1)(-t)}(F, \nu)$. A translating soliton refers to a solution which moves by translation $\Sigma_t = \Sigma_0 + \omega t$, along a fixed direction $\omega \in \mathbb{R}^{n+1}$, it is defined for all $t \in (-\infty, \infty)$, and satisfies $K = (-\nu, \omega)$.

In one dimension, the GCF coincides with the Curve Shortening Flow (CSF in abbreviation) for curves embedded in $\mathbb{R}^2$. In this case, there is only one translating soliton (up to isometries and rescaling). It is called the Grim Reaper solution and is given by the graphical representation $y = t - \ln \cos x$, for $(x, t) \in (-\pi/2, \pi/2) \times (-\infty, \infty)$.

The result of P. Daskalopoulos, R. Hamilton and N. Sesum [25] reveals that shrinking and translating solitons are major building blocks of ancient solutions. It was shown in [25] that the only compact convex ancient solutions to CSF are the shrinking round sphere or the Angenent oval. The latter, looks as if it is constructed by the gluing of two Grim Reapers coming from opposite ends. It is given by the implicit equation $\cos x = e^t \cosh y$ and, as $t \to -\infty$, it is approximately the intersection of the two Grim Reapers $y = (t - \ln 2) - \ln \cos x$ and $y = -(t - \ln 2) + \ln \cos x$. Moreover recently, T. Bourni, M. Langford, and G. Tinaglia [10] completed this classification by showing that the shrinking circle, the Angenent oval, the Grim Reaper, and the stationary line are the only convex ancient solutions to the curve shortening flow.

A similar classification holds true in the two-dimensional Ricci flow. It was shown by P. Daskalopoulos, R. Hamilton, and N. Sesum in [26], that the shrinking sphere and the King solution, are the only ancient solutions
defined on $S^2$. For the *mean curvature flow* and the *Ricci flow* in higher dimensions, the classifications are done under a non-collapsing along with convexity, low entropy or certain other conditions. See [13], [8], [12], [29].

In this paper, we prove the classification of ancient solutions to the Gauss curvature flow under the assumption that the solution is contained in a cylinder of bounded cross-section. The relevance of this assumption is found in the classification of translating solitons to the GCF given by J. Urbas [40, 41] where each translator is shown to be a graph on a convex bounded domain $\Omega \subset \mathbb{R}^n$. Note also that the Grim Reaper and the Angenent oval are solutions to the curve shortening flow contained in a strip, and the rotationally symmetric steady cigar soliton and the King solution are ancient solutions of the 2-dim Ricci flow asymptotic to a fixed round cylinder. However, a significant difference between those previous results and ours in this work, is that a translator exists for each $\Omega \subset \mathbb{R}^n$ (see J. Urbas [40, 41]) and hence there are infinitely many ancient solutions. We will show the *uniqueness* of ancient solutions having *asymptotic cylinder* $\Omega \times \mathbb{R}$, according to the definition below.

**Definition 1.1** (Asymptotic cylinder of an ancient solution). Assume that $\Sigma_t, t \in (-\infty, T), T \in (-\infty, +\infty]$ is a complete ancient GCF solution such that $\bigcup_{t \in (-\infty, T)} \Sigma_t \subset \text{cl}(\Omega) \times \mathbb{R}$, for a some open bounded domain $\Omega \subset \mathbb{R}^n$, and that $\bigcup_{t \in (-\infty, T)} \Sigma_t$ is not contained in any smaller cylinder $\text{cl}(\Omega') \times \mathbb{R}$. We will then refer to the cylinder $\Omega \times \mathbb{R}$ as the *asymptotic cylinder* of the ancient solution $\Sigma_t, t \in (-\infty, T)$.

Similarly, we define the asymptotic cylinder of a time slice $\Sigma'_\tau$ by the smallest cylinder containing $\Sigma'_\tau$.

Our first result given below settles the *uniqueness* of non-compact ancient solutions.

**Theorem 1.2** (Uniqueness of non-compact ancient solutions). Given a convex bounded $\Omega \subset \mathbb{R}^n$, the translating soliton asymptotic to $\Omega \times \mathbb{R}$ is the unique non-compact ancient solution asymptotic to $\Omega \times \mathbb{R}$. This uniqueness holds up to translations along the $e_{n+1}$ direction and reflection about $\{x_{n+1} = 0\}$.

Regarding compact ancient solutions, our next result shows the existence of ancient oval solutions which are the analogue of the Angenent oval solution for curve shortening flow.

**Theorem 1.3** (Existence of ancient oval solutions). Given a convex bounded $\Omega \subset \mathbb{R}^n$ with $C^{1,1} \text{ boundary}$, there exists a compact ancient solution $\Gamma_t \subset \mathbb{R}^{n+1}$ to Gauss curvature flow which is defined for all $t \in (-\infty, T)$, it becomes extinct at $T := -\frac{2V_\Omega}{\omega_n}$, and has asymptotic cylinder $\Omega \times \mathbb{R}$. Here $V_\Omega$ is the volume under the graph of the translating soliton asymptotic to $\Omega \times \mathbb{R}$ which is finite according to Lemma 5.1. Furthermore, the solution $\Gamma_t$ satisfies the properties in Propositions 5.3 and 5.4.
Our last result shows that $\Gamma_t$ in Theorem 1.3 is unique compact ancient solution asymptotic to $\Omega \times \mathbb{R}$.

**Theorem 1.4.** Given a convex bounded $\Omega \subset \mathbb{R}^n$ with $C^{1,1}$ boundary, let $\Sigma_t$, $t \in (-\infty, T)$ be a compact ancient solution to the Gauss curvature flow in $\mathbb{R}^{n+1}$ which is asymptotic to $\Omega \times \mathbb{R}$. Assuming that the solution becomes extinct at time $T := -\frac{2V_\Omega}{\omega_n}$, there is $v \in \mathbb{R}$ such that $\Sigma_t + ve_{n+1} = \Gamma_t$, for all $t \in (-\infty, T)$, where $\Gamma_t$ is the solution constructed in Theorem 1.3.

**Remark 1.5.** In the proofs of Theorem 1.3 and Theorem 1.4, the only use of the $C^{1,1}$ assumption on $\Omega$ is to ensure that $V_\Omega$ is finite. In other words, if one can show Lemma 5.1 without such an assumption, this condition can be removed from the theorems. In two theorems, the extinction time $T = -\frac{2V_\Omega}{\omega_n}$ is chosen so that the maximum height of $\Gamma_t$, $h(t) := \max_{x \in \Gamma_t} |x_{n+1}|$, satisfies $h(t) = \lambda |t| + o(1)$ as $t \to -\infty$. Here $\lambda := \frac{\omega_n}{2|\Omega|}$ is the speed of the translating soliton asymptotic to $\Omega \times \mathbb{R}$. 

The organization of this paper is as follows: In Section 2, we give some preliminary results and define an appropriate notion of weak solution. This is needed as translating solitons defined on non-strictly convex domains are not necessarily smooth and the corresponding compact ancient solutions may not be smooth as well. Theorem 1.2-1.4 will be shown in later sections with this notion of weak solution.

In Section 3, we show Theorem 3.3, the asymptotic convergence of an ancient solution to a translating soliton, as $t \to -\infty$, if one translates the solution so that its tip is fixed. In their recent work [17, 18] the authors established the forward in time convergence of non-compact solutions asymptotic to a cylinder to the corresponding translating soliton. Theorem 3.3 is obtained as an application of this result, and hence we will refer to the results in [18] when they are needed. Theorem 1.2, the uniqueness of non-compact ancient solutions, will be shown in Section 4 as a consequence of Theorem 3.3 and [18].

In Section 5, we show Theorem 1.3, the existence of a compact ancient solution asymptotic to a given cylinder, by showing Propositions 5.3 and 5.4. These two propositions also establish additional properties of the constructed solution $\Gamma_t$ which will be used when we show the uniqueness Theorem 1.4 by comparing $\Gamma_t$ with an arbitrary ancient solution $\Sigma_t$.

In Section 6, we show Theorem 1.4, the uniqueness of a compact ancient solution asymptotic to a given cylinder. Part of our proof is inspired by the recent significant works of Bourni-Langford-Tinaglia [9, 10] where they use the rate change in the enclosed volume as a function of time to estimate the location of the tips.

2. Preliminaries

In this section we collect some preliminary results. Throughout this paper, $h_{ij}$ denotes the second fundamental form. For a strictly convex solution,
one may consider the inverse $b^{ij}$ of the second fundamental form $h_{ij}$, which satisfies $b^{ik}h_{kj} = \delta^i_j$. Let us recall the unique existence of translating solitons by J. Urbas and denote them as follows.

**Definition 2.1** (Theorem of J. Urbas [40, 41]). Given a convex bounded domain $\Omega \subset \mathbb{R}^n$, we define $u_\Omega : \Omega \to \mathbb{R}$ by the graph function of the unique translating soliton which is asymptotic to $\Omega \times \mathbb{R}$, moves in the positive $e_{n+1}$ direction, and satisfies $\inf u_\Omega(\cdot) = 0$. In other words, the hypersurface given by $\partial\{ (x,x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > u(x) \}$ defines the translating soliton. The existence and the uniqueness is shown in [40, 41].

**Remark 2.2.** In the case where $\Omega$ is not a strictly convex domain, it is possible that $\limsup_{x \to x_0} u_\Omega(x) < \infty$, for some $x_0 \in \partial \Omega$, hence the hypersurface $\{ x_{n+1} = u_\Omega(x) \}$ may not be always complete. This is the reason why we denote the translating soliton by $\partial\{ (x,x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > u(x) \}$. Urbas [41] showed the existence of such solitons and their uniqueness among solutions realized in certain generalized sense. To be more specific, Urbas [41] showed if a convex function $u(x)$ defined on $\Omega$ satisfies the translating soliton equation

$$\det D^2 u = \beta (1 + |Du|^2)^{n+1}\sqrt{n+1}$$

for some $\beta > 0$ in the sense of Alexandrov, and $|\mathbb{R}^n - Du(\Omega)| = 0$, then $u = u_\Omega + C$, for some constant $C$. We will use this characterization of soliton in the proofs of Theorem 3.3.

**Definition 2.3.** For given convex bounded domain $\Omega \subset \mathbb{R}^n$, let us note the speed of associated translating soliton by

$$\lambda := \frac{1}{|\Omega|} \left[ \int_{\mathbb{R}^n} \frac{1}{(1 + |p|^2)^{n+1}} dp \right] = \frac{\omega_n}{2|\Omega|}$$

where $\omega_n = |\mathbb{S}^n|$. One can find a derivation of this $\lambda$ in [41] or the equality case of (3.4).

In [20] it was shown that a translating soliton may be weakly convex with flat sides in which case it fails to be smooth at the boundary $\partial \Omega \times \mathbb{R}$ of its asymptotic cylinder. This requires a suitable notion of weak solutions. In **Definition 2.4** we define a weak solution in such a way that it satisfies global comparisons with smooth classical solutions. The existence and the uniqueness of ancient solutions will then be shown in this class of weak solutions. Here and the remaining sections, an ancient solution is assumed to be a weak solution to the GCF in the sense of Definition 2.4 unless otherwise stated. Also, throughout the paper, we will use $\hat{\Sigma}$ to denote the closed region which is bounded by $\Sigma$.

**Definition 2.4** (Definition 2.6 in [18]). Suppose that $\hat{\Sigma}_t \subset \mathbb{R}^{n+1}$ for $t \in \bar{T_0,T_1}$ is a one-parameter family of closed convex sets with non-empty interior. $\Sigma_t = \partial \hat{\Sigma}_t \subset \mathbb{R}^{n+1}$ is a weak subsolution to the GCF if the following holds: for given a smooth strictly convex solution to the GCF $\Sigma'_t = \partial \hat{\Sigma}'_t$
defined for \( t \in [a, b] \subset [T_0, T_1] \) with initial data \( \hat{\Sigma}'_a \subset \hat{\Sigma}_a, \hat{\Sigma}'_t \subset \hat{\Sigma}_t \) holds for all \( t \in [a, b] \). We define a weak supersolution in a similar way with the opposite inclusion. \( \Sigma_t = \partial \hat{\Sigma}_t \) is a weak solution if it is both a weak sub- and super-solution. \( \Sigma_t = \hat{\Sigma}_t \) for \( t \in (-\infty, T) \) is a weak ancient solution if \( \Sigma_t, t \in [a, b] \), is a weak solution for all \(-\infty < a \leq b < T\).

The following result shows the existence and uniqueness of a weak solution starting at any convex hypersurface \( \Sigma_0 = \partial \hat{\Sigma}_0 \subset \mathbb{R}^{n+1} \) which is compact or non-compact and asymptotic to a cylinder.

**Theorem 2.5** (Theorem 2.7 in [18]). Let \( \hat{\Sigma}_0 \subset \mathbb{R}^{n+1} \) be a convex set with non-empty interior. If \( \Sigma_0 = \partial \hat{\Sigma}_0 \) is compact then there is a unique weak solution \( \Sigma_t \) to the GCF running from \( \Sigma_0 \) and defined over \( t \in [0, T) \) for some \( T < +\infty \). If \( \Sigma_0 = \partial \hat{\Sigma}_0 \) is non compact and asymptotic to a cylinder \( \Omega \times \mathbb{R} \), then there is a unique weak solution \( \Sigma_t \) to the GCF running from \( \Sigma_0 \) defined for all \( t \in [0, +\infty) \). Moreover, each time slice \( \Sigma_t \) is non-compact and asymptotic to \( \Omega \times \mathbb{R} \) for all \( t \in [0, \infty) \).

**Proof.** Let us only give an outline of the proof and refer the reader to [18] for details. Let \( \hat{\Sigma}_{i,0} \) be a sequence of convex sets with smooth strictly convex boundaries which strictly increases to \( \hat{\Sigma}_0 \). We denote by \( \Sigma_{i,t} \) the unique GCF solution with initial data \( \Sigma_{i,0} = \partial \hat{\Sigma}_{i,0} \) and we simply define \( \Sigma_t \) to be the limit of \( \Sigma_{i,t} \). Since each \( \Sigma_{i,t} \) is smooth it can be compared with any smooth solution or any weak solution. The existence and the uniqueness of a weak solution, as stated in the theorem, follow from this comparison principle.

**Lemma 2.6.** The following hold:

(i) The limit of any monotone sequence of weak solutions is a weak solution, provided that the limit is compact.

(ii) The comparison principle between weak solutions holds.

(iii) \( \partial_t \text{Vol}(\hat{\Sigma}_t) = -|\mathbb{S}^n| = -\omega_n \), for any compact weak solution \( \Sigma_t \).

**Proof.** Let us begin by showing (i). Suppose \( \partial \hat{\Sigma}_{i,t} \) is an increasing sequence of weak solutions and \( \Sigma_t = \text{cl}(\cup_i \hat{\Sigma}_{i,t}) \). Then \( \Sigma_t := \partial \Sigma_t \) is a supersolution. To show that \( \Sigma_t \) is a subsolution, suppose that \( \Sigma'_a \subset \hat{\Sigma}_a \) and assume, without loss of generality, that \( 0 \in \text{int}(\hat{\Sigma}_a) \). For \( \lambda < 1 \), \( \lambda \Sigma'_{a+\lambda^{-n+1}(a-t)} \) is a smaller solution and thus \( \lambda \Sigma'_a \subset \Sigma_{i,a} \) for large \( i > i_\lambda \) (here we use the compactness of \( \Sigma_t \)). This shows that \( \lambda \Sigma'_{a+\lambda^{-n+1}(a-t)} \subset \hat{\Sigma}_t \). By letting \( \lambda \uparrow 1 \) we obtain \( \Sigma'_t \subset \hat{\Sigma}_t \). A similar argument shows the same for decreasing sequences. (ii) and (iii) follow by approximation with smooth solutions as discussed in the proof of Theorem 2.5.

Lemma 2.6, in particular, implies that in the case that \( \Sigma_0 \) is compact, the maximal time of existence in Theorem 2.5 is given by \( T = \text{Vol}(\hat{\Sigma}_0)/\omega_n \).
The following is the Harnack inequality for the GCF and its consequence to graphical solutions.

**Theorem 2.7** (B. Chow [24], Proposition 3.2 [18]). Let $\Sigma_t$, $t \geq 0$, be a smooth compact strictly convex solution to the GCF. Then,

\[
\frac{1}{K}(\partial_t K - b^{ij} \nabla_i K \nabla_j K) \geq -\frac{n}{1+n} \frac{1}{t}.
\]

Let $x_{n+1} = u(x', t)$, be a smooth strictly convex graphical solution to the GCF which could be possibly incomplete. If the solution satisfies (2.3), then

\[
(2.4) \quad u_{tt} \geq -\frac{n}{1+n} \frac{u_t}{t}
\]

and hence, for $t_2 \geq t_1 > 0$,

\[
(2.5) \quad u_t(\cdot, t_2) \geq \left( \frac{t_1}{t_2} \right) \frac{n}{1+n} u_t(\cdot, t_1).
\]

We finish this section with the following regularity result, which roughly says that a weak GCF becomes smooth on region which are away from the initial surface. A similar property holds for other degenerate equations, such as the porous medium equation. While this property is known to hold true for GCF as well, its proof doesn’t seem to exist in the literature (see in [29] for a related result). We include it here for completeness. The proof uses some of main results shown in [18]. We suggest to the reader to skip the proof of this proposition at their first reading.

**Proposition 2.8.** Assume that $\Sigma_0 = \partial \hat{\Sigma}_0$ is a convex hypersurface which is either compact or non-compact asymptotic to a cylinder $\Omega \times \mathbb{R}$ of bounded cross-section. Let $\Sigma_t$ be a weak GCF solution starting at $\Sigma_0$. If a point $p \in \Sigma_t'$, for some $t' > 0$, is away from $\Sigma_0$ and $\hat{\Sigma}_t'$ has non empty interior, then the solution $\Sigma_t$ is strictly convex and smooth around $p$ in spacetime, i.e. there is $B_r(p) \subset \mathbb{R}^{n+1}$ such that $B_r(p) \cap \Sigma_T$ is smooth for $t \in (0, t']$. Moreover, the Harnack inequality (2.3) holds on smooth part of $\Sigma_t$.

**Proof.** Consider a sequence smooth strictly convex compact solutions $\Sigma_{i,t}$ which approximates $\Sigma_t$ from the inside in the sense that $\text{cl}(\cup_i \hat{\Sigma}_{i,t}) = \hat{\Sigma}_t$. See the proof of Theorem 2.5 which is Theorem 2.7 in [18], for the construction of such an approximation. Let us denote the graphical representations of lower part of $\Sigma_{i,t}$ by $x_{n+1} = u_i(x, t)$. Since $\Sigma_{i,t}$ are compact solutions, $\Sigma_{i,t}$ and $u_i(x, t)$ satisfy Theorem 2.7.

Let $p$ be a point in $\Sigma_t' \setminus \Sigma_0$ for some $t' > 0$. After a translation we may assume that $p = (0, 0) \in \mathbb{R}^{n+1}$. Next, since a convex hypersurface is locally a convex graph, after a rotation and renaming the index $i$, we may find positive constants $r'$, $\delta$, $L$ and $M$ with the following significance: all $\Sigma_{i,t}$ enclose the sphere of radius $r'$ centered at $(0, L) \in \mathbb{R}^{n+1}$ and the graphical representation $u_i(x, t)$ defined on $D_{r'}(0) = \{x \in \mathbb{R}^n : |x| \leq r'\}$ for $t \in [0, t']$. 

Moreover on $D_{r'}(0), -M \leq u_i(x, 0) \leq -2\delta$ and $-\delta \leq u_i(x, t')$. Our goal is to find some $\epsilon > 0$, $C_1, C_2$ such that

$$
(2.6) \quad 0 < C_2 \leq \partial_t u_i(x, t) \leq C_1, \quad \text{on } (x, t) \in D_{r'/2}(0) \times [t' - \epsilon, t'].
$$

Once we have these graphical speed bounds. Proposition 4.3, Theorem 4.4, and the argument in Corollary 4.5 in [18] can be applied to $x_{n+1} = u_i(x, t)$ on $D_{r'/4}(0) \times [t' - \frac{\epsilon}{2}, t']$ showing positive upper and lower bound on the curvature and gradient estimate (uniform in $i$). This would give a uniform smooth estimate and would show the spacetime $C^\infty$ convergence of $\Sigma_{i,t}$ to $\Sigma_t$ around $p$.

The main tool in showing (2.6) is the Harnack estimate. Let us fix a point $x' \in D_{r'/2}(0)$. First, we are going to show the upper bound for $\partial_t u_i(x', t')$. Using the spherical solution starting from the sphere of radius $r'$ centered at $(0, L) \in \mathbb{R}^{n+1}$ at time $t = t'$ as a barrier, we may find some $t'' > t'$ such that $u_i(x', t) \leq L$ on $x' \in D_{r'/2}(0)$ and $t \leq t''$. Inequality (2.5) of Theorem 2.7 yields that for any $0 < t < t''$,

$$
M + L \geq u_i(x', t'') - u_i(x', t) = \int_t^{t''} \partial_t u_i(x', s) ds \geq \partial_t u_i(x', t) \int_t^{t''} \left( \frac{s}{t} \right)^{\frac{n}{1+n}} ds,
$$

and it gives

$$
\partial_t u_i(x', t) \leq \frac{M + L}{(1 + n) t^\frac{n}{1+n} [ (t'')^\frac{1}{1+n} - t^\frac{1}{1+n} ]}.
$$

In particular, for $t \in [t'/2, t']$, we have

$$
\partial_t u_i(x', t) \leq C_1, \quad \text{for some } C_1 = C_1(M + L, t', t'', n)
$$

proving the upper bound in (2.6).

Let us now show the lower bound of $\partial_t u_i(x', t)$ in (2.6). For $0 \leq \tau \leq t'/2$, the previous upper bound implies

$$
u_i(x', t' - \tau) \geq u_i(x', t') - \tau C_1 = u_i(x', 0) + (u_i(x', t') - u_i(x', 0)) - \tau C_1 \geq u(x', 0) + \delta - \tau C_1.
$$

Hence, for $\tau \leq \min\left( \frac{t'}{2}, \frac{\delta}{2C_1} \right) =: \epsilon$, by integrating inequality (2.5) of Theorem 2.7 we obtain

$$
\frac{\delta}{2} \leq \int_0^{t' - \tau} \partial_t u_i(x', s) ds \leq (1 + n) (t' - \tau) \partial_t u_i(x', t' - \tau)
$$

which readily shows the lower bound

$$
\partial_t u_i(x', t) \geq \frac{\delta}{2(1 + n)t'} =: C_2, \quad \text{for } t \in [t' - \epsilon, t'].
$$

This proves the desired estimate which implies the smooth convergence of $\Sigma_{i,t}$ to $\Sigma_t$ around the point $p$. \hfill \Box
Figure 1. Definition 3.1

3. CONVERGENCE OF SOLUTION AROUND TIP

Throughout this section we will assume that $\Sigma_t$, $t \in (-\infty, T)$ is a weak ancient complete solution to the GCF which is asymptotic to the cylinder $\Omega \times \mathbb{R}$, as $t \to -\infty$ (see Definition 1.1). The goal is to show if we translate the solution and observe our solution around the tip region, then as $t \to -\infty$ it converges to the unique translating soliton asymptotic to $\Omega \times \mathbb{R}$.

As we mentioned earlier, an ancient solution may touch the boundary of its asymptotic cylinder $\Omega \times \mathbb{R}$ (c.f. in [20]). For this reason, some of the results in this section are written and shown in terms of $\text{int}(\hat{\Sigma}_t)$.

**Definition 3.1.** For an ancient convex GCF solution $\Sigma_t$, $t \in (-\infty, T)$, asymptotic to the cylinder $\Omega \times \mathbb{R}$, we define

$$h^+(t) := \sup_{x \in \Sigma_t} \langle x, e_{n+1} \rangle \quad \text{and} \quad h^-(t) := \inf_{x \in \Sigma_t} \langle x, e_{n+1} \rangle$$

to be the maximum and minimum heights, respectively. They are both finite if $\Sigma_t$ is compact. For the non-compact case, after reflection, we will assume that $-\infty < h^-(t) < +\infty$ and $h^+(t) = \infty$.

We also define $p^+(t)$ and $p^-(t) \in \Sigma_t$ to be the tips of $\Sigma_t$ by the condition

$$\langle p^+, e_{n+1} \rangle = h^+ \quad \text{and} \quad \langle p^-, e_{n+1} \rangle = h^-.$$  

In the non-compact case we only have one tip $p^-(t)$.

**Definition 3.2.** Let $\Sigma_t = \partial \hat{\Sigma}_t$, $t \in (-\infty, T)$ be an ancient convex GCF solution, asymptotic to the cylinder $\Omega \times \mathbb{R}$. For each $t \in (-\infty, T)$, $\text{int}(\hat{\Sigma}_t)$ can be represented as the region between two graphs $u^+(\cdot, t)$ and $u^-(\cdot, t)$ defined on some domain $\Omega_t \subset \Omega$ as follows:

$$\text{int}(\hat{\Sigma}_t) = \{(x', x_{n+1}) \in \mathbb{R}^{n+1} : u^-(x', t) < x_{n+1} < u^+(x', t) \quad \text{and} \quad x' \in \Omega_t\}.$$  

Here, $\Omega_t$ is the image of the projection of $\text{int}(\hat{\Sigma}_t)$ to the hyperplane $\{x_{n+1} = 0\}$, which is an open bounded convex set. For non-compact case, we set $u^+ = \infty$.

Note that since $\Omega \times \mathbb{R}$ is the smallest open cylinder containing $\cup_t \text{int}(\hat{\Sigma}_t)$, the domains $\Omega_t$ increase to $\Omega$, as $t \to -\infty$. For the non-compact case, the last assertion in Theorem 2.5 implies $\Omega_t = \Omega$ for all $t$. The functions $u^+(\cdot, t)$
and $u^-(\cdot,t)$ are graphical solutions to the GCF, which are defined on the domain $\Omega$. Before proceeding to the next theorem, recall the definition of translating soliton $u_\Omega$ in Definition 2.1 and the speed $\lambda$ in Definition 2.3.

**Theorem 3.3.** Let $\Sigma_t$, $t \in (-\infty, T)$, be a complete ancient convex weak solution of GCF which is asymptotic to the cylinder $\Omega \times \mathbb{R}$. Then, as $t \to -\infty$, $\Sigma_t - h^-(t)e_{n+1}$ converges locally smoothly to the unique translating soliton $\{x_{n+1} = u_\Omega(x)\}$. In the case that $\Sigma_t$ is compact, $\Sigma_t - h^+(t)e_{n+1}$ also converges locally smoothly to the translating soliton $\{x_{n+1} = -u_\Omega(x)\}$. More precisely, we have

$$u^-(x,t) - h^-(t) \to u_\Omega(x) \quad \text{and} \quad u^+(x,t) - h^+(t) \to -u_\Omega(x) \quad \text{in } C^\infty_{loc}(\Omega)$$

as $t \to -\infty$.

We need several lemmas before giving the proof of this theorem.

**Lemma 3.4.** We have $h^-(t) \to -\infty$ as $t \to -\infty$. If $\Sigma_t$ is compact, then $h^+(t) \to \infty$ as $t \to -\infty$ as well. Furthermore, in both cases we have $\cup_{t} \text{int}(\hat{\Sigma}_t) = \Omega \times \mathbb{R}$.

**Proof.** We give the proof assuming that $\Sigma_t$ is compact. The proof in the non-compact case is similar. We first show that $h^-(t) \to -\infty$ as $t \to -\infty$ by a contradiction argument. Suppose that $h_- \geq -C$, for some $C < \infty$ for all time. This implies $\cup_{t} \text{int}(\hat{\Sigma}_t) \subset \Omega \times [-C, \infty)$. Then we may find a translating soliton which is asymptotic to a slightly but strictly larger cylinder while containing $\Omega \times [-C, \infty)$. By comparing this soliton with our solution $\Sigma_t$, starting at large negative times $t_0 \ll -1$, we conclude that $\Sigma_t$ has to be empty for each $t$. This is a contradiction and hence $\lim_{t \to -\infty} h^-= -\infty$. Similarly, $\lim_{t \to -\infty} h^+ = \infty$.

Let us now see that $\cup_{t} \text{int}(\hat{\Sigma}_t) = \Omega \times \mathbb{R}$. Since $\cup_{t} \text{int}(\hat{\Sigma}_t)$ is a convex set and its boundary contains $p^+(t)$ and $p^-(t)$ which move to opposite infinities as $t \to -\infty$, it is easy to see that the sections $\cup_{t} \text{int}(\hat{\Sigma}_t) \cap \{e_{n+1} = l\}$ have to be identical and thus $\cup_{t} \text{int}(\hat{\Sigma}_t)$ is a convex open cylinder. Since, by assumption, $\cup_{t} \text{int}(\hat{\Sigma}_t)$ is contained in no smaller cylinder than $\Omega \times \mathbb{R}$, we obtain the conclusion.

From now on, we will concentrate on the convergence of $u^-(x,t) - h^-(t)$. The convergence of $u^+(x,t) - h^+(t)$ follows by a similar arguments. Lemma 3.4 and Proposition 2.8 imply the following regularity lemma.

**Lemma 3.5** (c.f. Theorem 2.7 or [24]). An ancient weak solution $\Sigma_t$, $t \in (-\infty, T)$, satisfying the assumptions of Theorem 3.3 is smooth and strictly convex away from $\partial \Omega \times \mathbb{R}$ provided that $\Sigma_t$ has non empty interior. Moreover, the Harnack inequality

$$\frac{1}{K}(\partial_i K - b^j \nabla_i K \nabla_j K) \geq 0$$

holds on all points where $\Sigma_t$ is smooth. As a consequence, we have
(i) \( \partial_t u^-(x, t) = \frac{K}{(\nu \cdot e_n+1)} \) satisfies \( \partial_t^2 u^-(x, t) \geq 0 \) on \( (x, t) \in \cup_t (\Omega_t \times \{t\}) \).

(ii) Let \( K(\nu, t) \) be the Gaussian curvature of a point on \( \Sigma_t \) parametrized by its outer unit normal \( \nu \). Then \( \partial_t K(\nu_0, t) \geq 0 \) whenever \( \Sigma_t \) is smooth around the point \( p_0 \) with \( \nu(p_0, t) = \nu_0 \).

In the following steps, we are interested in establishing lower bounds on the Gaussian curvature \( K \) for any weak ancient solution \( \Sigma_t \) satisfying the assumptions of the Theorem 3.3.

**Lemma 3.6.** Let

\[
\beta := \lim_{t \to -\infty} \partial_t h^- = \lim_{t \to -\infty} K(-e_{n+1}, t)
\]

which exists by Lemma 3.5. Then, we have \( \beta \geq \lambda \), where \( \lambda \) is the speed of the translating soliton in Definition 2.3.

**Proof.** We argue by contradiction. By translating the solution in time if necessary, we may assume \( \Sigma_0 \) is not empty. Suppose that \( \beta < \lambda - 2\epsilon \), for some \( \epsilon > 0 \). Consider a strictly larger cylinder containing \( \Omega \times \mathbb{R} \) whose corresponding translating soliton has the speed \( \lambda - \epsilon \). Let us denote by \( \Sigma^- \) to be such a soliton moving in positive \( e_{n+1} \) direction and having \( \inf_{x \in \Sigma^-} (x, e_{n+1}) = 0 \) (namely its tip is the point \((0, 0) \in \mathbb{R}^{n+1}\)). Then there is \( C_1 > 0 \) such that \( \Sigma^- - C_1 e_{n+1} \) encloses \( \Omega \times [0, \infty) \). Therefore, \( \Sigma^- + (\lambda - \epsilon, t) e_{n+1} \) encloses \( \Sigma_t \) for \( t < 0 \). Then, the comparison principle implies that the surface \( \Sigma^- + (\lambda - \epsilon, t) e_{n+1} \) encloses \( \Sigma_0 \) for all \( t < 0 \). On the other hand, as \( t \to -\infty \), \( h^- (t) \geq \lambda - 2\epsilon t + o(t) \) by (3.3). Thus

\[-C_1 + h^- (t) - (\lambda - \epsilon, t) t \geq -C_1 - \epsilon t + o(t) \to \infty, \quad \text{as } t \to -\infty,\]

which contradicts the assumption \( \Sigma_0 \) is non-empty. \( \square \)

**Proposition 3.7.** Let \( \Sigma_t, t \in (-\infty, T) \) be an ancient solution satisfying the assumptions of the Theorem 3.3. Given any \( \Omega' \subset \subset \Omega \), there is \( t_0 < 0 \) and \( c > 0 \) such that

\[c \leq \partial_t u^- \leq c^{-1}, \quad \text{for } t \leq t_0.\]

**Proof.** Let \( \epsilon > 0 \) be such that \( \text{dist} (\Omega', \partial \Omega) = 2\epsilon > 0 \). By Lemmas 3.4 and 3.6, we may choose \( t_0 \ll -1 \) so that the following hold for \( t \leq t_0 \):

(i) \( \beta (t_0 - t) \leq h^- (t_0) - h^- (t) \leq 2\beta (t_0 - t) \).

(ii) If \( \Omega_{t,0} \) is the cross-section of \( \Sigma_t \) at \( x_{n+1} = 0 \), namely we have \( (\Sigma_t \cap \{x, e_{n+1} = 0\} =: \Omega_{t,0} \times \{0\}) \), then \( \Omega' \subset \Omega_{t,0} \) and \( \text{dist} (\Omega', \partial \Omega_{t,0}) \geq \epsilon \).

From now on, assume \( t \leq t_0 \) and let \( x' \in \Omega' \) be an arbitrary point. By the monotonicity of \( \partial_t u^- (\cdot, t) \) in \( t \) which follows from 2.4, we have \( \partial_t u^- (x', t) \leq \partial_t u^- (x', t_0) \leq \sup_{\Omega'} \partial_t u^- (x', t_0) < \infty \), which proves the upper bound.

We next show the lower bound. Since \( \Sigma_t \) is convex, it has to enclose a cone generated by the base \( \Omega_{t,0} \times \{0\} \) and the vertex \( p^- (t) \). Together with property (ii) above, this implies the bound \( u^- (x', t) \leq \frac{1}{\epsilon} h^- (t) \), where
$C = \text{diam } \Omega$ (recall that both $u^-$ and $h^-$ are negative). Using also that $h^-(t) \leq u^-(x', t)$, we conclude that for any $\tau_1 < \tau_2 \leq t_0$, we have

$$u(x', \tau_2) \geq h^-(\tau_2) \geq 2\beta (\tau_2 - t_0) + h^-(t_0)$$

and

$$u(x', \tau_1) \leq \frac{\epsilon}{C} h^-(\tau_1) \leq \frac{\epsilon}{C} (\beta (\tau_1 - t_0) + h^-(t_0)).$$

Subtracting these two inequalities and using $t_0 < 0$ and $h^-(t_0) < 0$ yields

$$u(x', \tau_2) - u(x', \tau_1) \geq \beta (2\tau_2 - \frac{\epsilon}{C} \tau_1) - \beta (2 - \frac{\epsilon}{C}) t_0 + (1 - \frac{\epsilon}{C}) h^-(t_0) \geq \beta (2\tau_2 - \frac{\epsilon}{C} \tau_1) + h^-(t_0).$$

If we choose $\tau_1 = \frac{C(2+L)}{\epsilon} \tau_2$ for $L > 0$, the monotonicity of $u^-_i$ implies

$$u^-_i (x', \tau_2) (\tau_2 - \frac{C(2+L)}{\epsilon} \tau_2) \geq u(x', \tau_2) - u(x', \frac{C(2+L)}{\epsilon} \tau_2) \geq L \beta (\tau_2) + h^-(t_0)$$

which gives

$$u^-_i (x', \tau_2) \geq \frac{L \beta}{\frac{(2+L)}{\epsilon} - 1} + \frac{h^-(t_0)}{\tau_2 - \frac{C(2+L)}{\epsilon} \tau_2}.$$

Finally, taking $L \to \infty$, we obtain the desired lower bound $u^-_i (x', \tau_2) \geq \frac{\epsilon}{C} \beta$.

$\square$

**Proposition 3.8.** Let $\Sigma_t$, $t \in (-\infty, T)$, be an ancient solution satisfying the assumptions of the Theorem 3.3. For any given sequence $\tau_i \to -\infty$, passing to a subsequence if necessary, $u^-_i (x', t) := u^- (x, t + \tau_i) - h^- (\tau_i)$ converges to $u^-_\infty (x, t) = u^- (x, t)$ in $C^\infty_{\text{loc}} (\Omega \times \mathbb{R})$. Moreover, the limiting graphical solution $x_{n+1} = u^-_\infty (x, t)$ satisfies $\partial_t u^-_\infty \equiv \beta$, where $\beta$ is as in (3.3). The solution $u^-_\infty$ represents a translating soliton which may possibly be incomplete.

**Proof.** Since we have the bounds on the graphical speed $u_t = K (\nu, e_{n+1})^{-1}$ from Proposition 3.7 we can apply Proposition 4.3 and Theorem 4.4 in [18] as they are applied in Corollary 4.5 in [18] and obtain the following: for any given $\Omega' \subset \subset \Omega$, there is $t_0 < 0$ and $C > 0$ such that

$$|Du^-|, \lambda^-_{\text{min}}, \lambda^-_{\text{max}} \leq C \text{ on } \Omega' \times (-\infty, t_0].$$

The equation of graphical GCF becomes uniformly parabolic provided we have the gradient bound, positive upper and lower curvature bounds. Thus we may pass to a limit $u^-_\infty$, $i \to \infty$, by the standard regularity theory of parabolic equations, obtaining a graphical eternal solution $u^-_\infty (x, t)$. In view of (i) in Lemma 3.5, $\partial_t u^-_\infty (x, t)$ must be independent of $t$, that is $\partial_t u^-_\infty (x, t) = \beta_\infty (x)$. Furthermore, the fact that $|Du^-_\infty (\cdot, t)|$ is bounded, globally in time, on every compact set $\Omega' \subset \subset \Omega$, implies that $\beta_\infty (\cdot)$ has to be a constant. From Lemma 3.6 we conclude that $\partial_t u^-_\infty \equiv \beta$.

$\square$

**Proof of Theorem 3.3.** By Proposition 3.8 $u^-_\infty (x, t) = u^-_{\infty, 0} (x) + \beta t$ where $u^-_{\infty, 0} (\cdot) := u (\cdot, 0)$. It remains to prove that $u^-_{\infty, 0} (\cdot) = u_\Omega (\cdot)$. By the
characterization of \( u_\Omega \) given after Definition 2.1, it suffices to show \( |\mathbb{R}^n - Du_{\infty,0}(\Omega)| \) = 0. Note that

\[
 u_t \equiv \beta = (1 + |Du_{\infty,0}|^2)^{\frac{1}{2}} \left[ \frac{\det D^2 u_{\infty,0}}{(1 + |Du_{\infty,0}|^2)^{\frac{n+2}{2}}} \right] \text{ on } \Omega.
\]

This implies

\[
\beta|\Omega| = \int_{\Omega} \frac{\det D^2 u_{\infty,0}}{(1 + |Du_{\infty,0}|^2)^{\frac{n+2}{2}}} = \int_{Du_{\infty,0}(\Omega)} \frac{1}{(\sqrt{1 + |p|^2})^{n+1}}.
\]

Let us fix an arbitrary \( x' \in \Omega \). Together with Theorem 3.3, we have

\[
\lim_{t \to \infty} u^- (x', t) = \lim_{t \to -\infty} u^- (x', t) = \lambda
\]

where \( \lambda \) is the speed of the translating soliton \( u_\Omega \). By (i) Lemma 3.5, \( u^- \equiv \lambda \) on \( (x, t) \in \Omega \times \mathbb{R} \) showing \( \Sigma_t \) is a translating soliton with the speed \( \lambda \). We may repeat the same argument in (3.4), while \( \beta \) replaced by \( \lambda \), to conclude \( |\mathbb{R}^n - Du^- (\cdot, t)(\Omega)| = 0 \) and hence \( u^- (x, t) = u_\Omega (x) + \lambda t + C \), for some constant \( C \).

\[
\square
\]

4. Uniqueness of non-compact ancient solution

We are ready to give a proof of the uniqueness of non-compact ancient solution.

**Proof of Theorem 1.2.** Let \( \Sigma_t \) be an ancient solution as in the statement of the theorem. By Theorem 2.5, the solution exist for all \( t \in (-\infty, \infty) \) and \( \Sigma_t \) is asymptotic to \( \Omega \times \mathbb{R} \) for each time slice. i.e. the domain of graphical representation of \( \Sigma_t \) does not change over time. Therefore, we may represent \( \Sigma_t \) by a graph \( \partial \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq u(x, t)\} \) for \( (x, t) \in \Omega \times (-\infty, \infty) \).

The main result, Theorem 1.1, in [18] proves the forward-in-time convergence to the soliton \( u_\Omega (x) \), namely

\[
u^- (x, t) - h^- (t) \to u_\Omega (x) \text{ in } C^\infty_{\text{loc}}(\Omega), \quad \text{as } t \to \infty.
\]

Let us fix an arbitrary \( x' \in \Omega \). Together with Theorem 3.3, we have

\[
\lim_{t \to \infty} u^- (x', t) = \lim_{t \to -\infty} u^- (x', t) = \lambda
\]

where \( \lambda \) is the speed of the translating soliton \( u_\Omega \). By (i) Lemma 3.5, \( u^- \equiv \lambda \) on \( (x, t) \in \Omega \times \mathbb{R} \) showing \( \Sigma_t \) is a translating soliton with the speed \( \lambda \). We may repeat the same argument in (3.4), while \( \beta \) replaced by \( \lambda \), to conclude \( |\mathbb{R}^n - Du^- (\cdot, t)(\Omega)| = 0 \) and hence \( u^- (x, t) = u_\Omega (x) + \lambda t + C \), for some constant \( C \).

\[
\square
\]

5. Existence of compact ancient solution

Let \( \Omega \subset \mathbb{R}^n \) is a bounded convex open domain. In this section we will construct an ancient compact solution of GCF which has asymptotic cylinder \( \Omega \times \mathbb{R} \), as \( t \to -\infty \); (see Definition 1.1).
We recall that $u_\Omega$ denotes the translator associated with the domain $\Omega$ satisfying $\inf_{x \in \Omega} u_\Omega(x) = 0$. For the construction of compact ancient solutions we need to show that the volume under the translating soliton $V_\Omega := \int_\Omega u_\Omega(x)\,dx$ is finite. Although this is expected to hold for any compact convex domain $\Omega$ with no further regularity assumptions on $\partial \Omega$, we could show this under $C^{1,1}$ boundary condition.

**Lemma 5.1.** Assume that $\Omega \subset \mathbb{R}^n$ be a convex bounded open domain with $C^{1,1}$ boundary. Let $x_{n+1} = u_\Omega(x), \, x \in \Omega$, be the translating soliton associated with the domain $\Omega$ and having $\inf_{x \in \Omega} u_\Omega(x) = 0$. Then, the volume under the translating soliton is finite, i.e. we have

$$V_\Omega := \int_\Omega u_\Omega(x)\,dx < \infty.$$  

**Remark 5.2.** If this lemma is shown without $C^{1,1}$ assumption, it also proves Theorem 1.3 and Theorem 1.4 without $C^{1,1}$ assumption on $\partial \Omega$.

**Proof of Lemma 5.1.** The basic strategy is to find a supersolution of the graphical translating soliton equation

$$\frac{\det D^2 \phi}{(1 + |D\phi|^2)^{\frac{n+1}{2}}} \leq \lambda = \frac{\omega_n}{2|\Omega|},$$

which is integrable near the boundary of $\Omega$.

Assume first that $\partial \Omega$ is smooth. Before going into the details, let us recall some properties of the distance function $d(x)$ from any point $x \in \Omega$ to $\partial \Omega$. The function $d(x)$ is well defined on $\{y \in \Omega : d(y, \partial \Omega) < \lambda_0\}$, where at each $y \in \Omega$, $\lambda_{\text{max}}(y) := \max_{i=1, \ldots, n-1} \lambda_i(y)$ denotes the maximum of the principal curvatures of $\partial \Omega$ at $y$. Furthermore, $d(x)$ is a smooth function in this tubular neighborhood.

Let $x \in \Omega$ be a point in this neighborhood and $\pi(x) \in \partial \Omega$ be the point such that $|\pi(x) - x| = \text{dist}(x, \partial \Omega)$. If we denote by $\lambda_i, \, i = 1, \ldots, n-1$ the principal curvatures of the hypersurface $\partial \Omega \subset \mathbb{R}^n$ at $\pi(x)$, then with respect to the orthonormal basis $\{(e_i)_{i=1}^{n-1}, -\frac{\pi(x)-x}{|\pi(x)-x|}\}$ of $\mathbb{R}^n$, we have

$$Dd(x) = (0, \ldots, 0, 1)$$

and

$$D^2d(x) = \begin{bmatrix} \text{diag} \left( \frac{-\lambda_i}{1-\lambda_i d} \right) & 0 \\ 0 & 1 \end{bmatrix}_{n-1 \times n-1} \begin{bmatrix} 0 \\ n-1 \times 1 \end{bmatrix}.$$  

Define $\phi(x) = -L \log d(x)$ as our test function. Then in this neighborhood we have

$$\frac{\det D^2 \phi}{(1 + |D\phi|^2)^{\frac{n+1}{2}}} \leq \frac{L^n}{(1 + \frac{L^2}{2})^{\frac{n+1}{2}}} \prod_{i=1}^{n-1} \frac{\lambda_i(\pi(x))}{1 - \lambda_i(\pi(x))} d(x) \leq \frac{1}{L} \prod_{i=1}^{n-1} \frac{\lambda_i(\pi(x))}{1 - \lambda_i(\pi(x))} d(x).$$
Assume next that \( \partial \Omega \) is in \( C^{1,1} \) and take a strictly monotone increasing sequence \( \{ \Omega_m \} \) of convex domains which approximates \( \Omega \) from the inside in such a way that each \( \partial \Omega_m \) is smooth and \( \sup_{y \in \partial \Omega_m} \lambda_{\text{max}}(y) < 2\lambda_0 \). Here strictly monotone means \( \Omega_m \subset \subset \Omega_{m+1} \). We may also assume that \( \text{dist} (x, \partial \Omega) < \frac{1}{4\lambda_0} \) for all \( x \in \partial \Omega_1 \). Set \( M := \sup_{x \in \Omega_1} u(x) > 0 \) and define the functions
\[
\phi_m(x) = -\frac{(4\lambda_0)^{n-1}}{\lambda} \log \left( \frac{\text{dist} (x, \partial \Omega_m)}{\text{diam} \Omega} \right) + M, \quad \text{for } x \in \Omega_m \setminus \Omega_1.
\]
Then, by our choice of \( \Omega_m \), each \( \phi_m \) is smooth in the interior of \( \Omega_m \setminus \Omega_1 \). Furthermore, from (5.2) we have
\[
\det D^2 \phi_m \leq \frac{\lambda}{(4\lambda_0)^{n-1}} \frac{2\lambda_0}{1 - \frac{2\lambda_0}{4\lambda_0}}^{n-1} \leq \lambda.
\]

We will next compare \( \phi_n \) with \( u_\Omega \) to conclude that \( V_\Omega < \infty \). Since \( \phi_n \geq u \) on \( \partial \Omega_1 \) and it becomes infinite on \( \partial \Omega_m \), the comparison principle implies \( \phi_m \geq u \) in the interior of \( \Omega_m \setminus \Omega_1 \). Note that \( \phi_m \) converges locally uniformly to \( \phi := -\frac{(4\lambda_0)^{n-1}}{\lambda} \log(\text{dist} (x, \partial \Omega)) + M \) on \( \Omega \setminus \Omega_1 \), which implies \( \phi(x) \geq u(x) \) in this region. Since \( \phi \) is integrable on \( \Omega \setminus \Omega_1 \), this implies \( \int_{\Omega} u = V_\Omega \) is finite.

\( \square \)

**Proof of Theorem 1.3**. Recall that the speed of the translator \( u_\Omega \) defined on the domain \( \Omega \) is given by \( \lambda = \frac{\omega_n}{2 V_\Omega} \). Theorem 1.3 is implied by two propositions below.

**Proposition 5.3.** Let \( \Omega \subset \mathbb{R}^n \) be a convex bounded domain with \( C^{1,1} \) boundary. Then, there is a compact weak ancient solution \( \Gamma_t \) of the Gauss curvature flow, defined on \( t \in (-\infty, T) \) with \( T := -\frac{2\lambda_0}{\omega_n} \), such that

1. \( \text{Vol} (\dot{\Gamma}_t) = -\omega_n t - 2V_\Omega \), where \( V_\Omega \) is given by (5.1);
2. \( \Gamma_t \) has reflection symmetry with respect to \( x_{n+1} = 0 \);
3. \( \Gamma_t \) is contained in \( \Omega \times \mathbb{R} \), but not in a smaller cylinder, i.e. \( \Gamma_t \) is asymptotic to \( \Omega \times \mathbb{R} \);
4. \( \Gamma_t \) is smooth in the interior of \( \Omega \times \mathbb{R} \), for \( t < T \) and satisfies the differential Harnack inequality.

**Proof.** For our given bounded convex domain \( \Omega \), denote by \( u_\Omega \) the graph of translating soliton corresponding to the domain \( \Omega \) having speed \( \lambda := \frac{\omega_n}{2 V_\Omega} \) and satisfying \( \inf_{\Omega} u_\Omega(x) = 0 \) (see Definition 2.1). To simplify the notation, from now on we will denote \( u_\Omega(x) \) simply by \( u(x) \).

The graphs \( x_{n+1} = u(x) + \lambda t \) and \( x_{n+1} = -u(x) - \lambda t, \ t \in \mathbb{R} \), define translating solitons, moving in opposite directions and having tips at a distance \( 2\lambda |t| \) from each other. The basic idea here is to construct our solution \( \Gamma_t \) as limit of hypersurfaces which for \( t \ll -1 \) are approximated by the boundary of the region \( \{ x_{n+1} > u(x) + \lambda t \} \cap \{ x_{n+1} < -u(x) - \lambda t \} \).
To make this rigorous, for any $s < 0$ define $\hat{\Gamma}_{s,0}$ to be the convex region which is bounded between the hypersurfaces $x_{n+1} = u(x) - \lambda |s|$ and $x_{n+1} = -u(x) + \lambda |s|$. By Lemma 5.1, we can deduce that
\[ \partial_t \hat{\Gamma}_{s,0} \in \mathcal{s}, \]
that is, a weak subsolution to the GCF.

Let $\Gamma_{s,t}$, $t \in [0, \tau_s)$ with $\tau_s = \text{Vol}(\hat{\Gamma}_{s,0})/\omega_n$, be the weak solution to the GCF starting from $\hat{\Gamma}_{s,0}$. Consider the time translated solutions $\Gamma_{s,t-s}$, $t \in [s, \tau_s + s)$. For each fixed $t$, we claim that $\Gamma_{s,t-s}$, $s \leq t$, are monotone decreasing as $s \to -\infty$: for $s_1 < s_2 < 0$, by the comparison principle between $\Gamma_{s_1,t}$ and $\Gamma_{s_1,t_0}$, $\Gamma_{s_1,s_2-s_1}$ is contained in $\Gamma_{s_2,0}$. Again by the comparison principle between $\Gamma_{s_1,s_2-s_1+t}$ and $\Gamma_{s_2,t}$, we conclude $\Gamma_{s_1,t-s_1}$ is contained in $\Gamma_{s_2,t-s_2}$. In view of (1) Lemma 2.6, weak solutions $\Gamma_{s,t-s}$ monotonically converge to a weak solution $\Gamma_t$ as $s \to -\infty$.

\[ \Gamma_{s,t-s} \downarrow \Gamma_t, \quad \text{as } s \to -\infty. \]  
Since $\lim_{s \to -\infty} \text{Vol}(\hat{\Gamma}_{s,t-s}) = -t \omega_n - 2V_\Omega$, $\hat{\Gamma}_t$ has non empty interior for $t < -\frac{2V_\Omega}{\omega_n}$ and has empty interior for $t > \frac{2V_\Omega}{\omega_n}$. This defines the ancient solution $\Gamma_t$ for $t \in (-\infty, -\frac{2V_\Omega}{\omega_n})$.

We will now see that $\Gamma_t$ satisfies properties (i)-(iv) in the statement of our theorem. Properties (i) and the reflection symmetry property (ii) clearly hold by construction. Furthermore, property (iv) is just a consequence of Lemma 3.5. It remains to show property (iii). By construction, $\hat{\Gamma}_t$ is contained in $\Gamma_{t,0}$ which is contained in $\Omega \times \mathbb{R}$. Hence, $\hat{\Gamma}_t$ is contained in $\Omega \times \mathbb{R}$. Suppose there is a smaller $\Omega' \subset \Omega$ such that $\hat{\Gamma}_t$ is contained in $\Omega' \times \mathbb{R}$ for all $t$. Since $\hat{\Gamma}_t \subset \Gamma_{t,0}$,
\[ \sup_{\hat{\Gamma}_t} x_{n+1} - \inf_{\hat{\Gamma}_t} x_{n+1} \leq \sup_{\Gamma_{t,0}} x_{n+1} - \inf_{\Gamma_{t,0}} x_{n+1} = 2\lambda(-t). \]

Therefore, $\text{Vol}(\hat{\Gamma}_t) \leq 2\lambda(-t)|\Omega'| = \frac{\Omega'}{\Omega}(-t\omega_n)$. On the other hand, we know that $\text{Vol}(\hat{\Gamma}_t) = -t\omega_n - 2V_\Omega$. If $|\Omega'| < |\Omega|$, we have a contradiction by taking $t \to -\infty$ in the above inequality. This shows there is no such smaller $\Omega'$.

We next provide some extra properties of the solution $\Gamma_t$ constructed above. Those properties will be used in the proof of our uniqueness Theorem 1.4 in the next section.

**Proposition 5.4.** The constructed ancient solution $\Gamma_t$, $t \in (-\infty, T)$, with $T := -\frac{2V_\Omega}{\omega_n}$, satisfies $(\sup_{\hat{\Gamma}_t} |x_{n+1}| - \lambda|t|) \uparrow 0$ as $t \to -\infty$.

**Proof.** The Harnack inequality implies that the speed of each tip of $\Gamma_t$ is greater than $\lambda$, hence the quantity $\sup_{\Gamma_t} |x_{n+1}| - \lambda|t|$ decreases, as $t$ increases. In addition, the solution $\hat{\Gamma}_t$ is contained in $\hat{\Gamma}_{t,0}$, by construction, where $\hat{\Gamma}_{t,0}$
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is given in the proof of Proposition 5.3. Hence \( \sup_{t \to -\infty} |x_{n+1}| \leq \lambda |t| \), i.e. \( \lim_{t \to -\infty} (\sup_{t \to -\infty} |x_{n+1} - \lambda|) := L \leq 0 \). If \( L < 0 \), this implies that the solution \( \hat{\Gamma}_t \) is contained in \( \Omega \times [-L + \lambda|t|, L + \lambda|t|] \). By the convergence of solution to the translating soliton around the tips, we have that

\[
\lim_{t \to -\infty} (\sup_{t \to -\infty} |x_{n+1}| - \lambda |t|) := L \leq 0. \tag{5.5}
\]

Since \( 2[(L + \lambda|t|)\Omega - V_\Omega] = \omega_n |t| + 2L|\Omega| - 2V_\Omega \), this contradicts Proposition 5.3 (i). This proves the assertion.

6. Uniqueness of compact ancient solution

Given a bounded domain \( \Omega \subset \mathbb{R}^n \) we recall that \( u_\Omega \) denotes the translator associated with the domain \( \Omega \) satisfying \( \inf_{x \in \Omega} u_\Omega(x) = 0 \). Let us recall that if \( \Omega \) is \( C^{1,1} \),

\[
V_\Omega := \int_\Omega u_\Omega(x) \, dx < \infty
\]

by Lemma 5.1. We have shown in the previous section that there exists a compact ancient solution \( \Gamma_t, t \in (-\infty, -2V_\Omega) \) which is asymptotic to \( \Omega \times \mathbb{R} \) and which becomes extinct at time \( T := -2V_\Omega \). We will next show that \( \Gamma_t \) is unique up to translations in space along the axis \( e^{n+1} \) and translations in time.

Let us briefly outline the proof of the Theorem which will be given below. As we stated in Theorem 1.4, our goal is to show that any given compact ancient solution \( \Sigma_t \) asymptotic to \( \Omega \times \mathbb{R} \) which becomes extinct at time \( t = -2V_\Omega \) is same as \( \Gamma_t \), the solution constructed in the previous section, up to a translation in \( e^{n+1} \) direction. For now, let us set aside to deal with this translation. The main step in our proof is to show the inclusion

\[
\hat{\Gamma}_t \subset \hat{\Sigma}_t, \quad \text{for all } s < -1.
\]

Recall that \( \Gamma_t \) was obtained as the limit of \( \Gamma_{s,t-s} \) as \( s \to -\infty \), where \( \Gamma_{s,t} \) is the GCF running from \( \Gamma_{s,0} \) and \( \Gamma_{s,0} \) is the compact surface obtained from the gluing of two translators so that the distances from each tip to the origin is equal to \( |s| \lambda \). Thus, it would have been sufficient to show that \( \hat{\Gamma}_{s,0} \subset \hat{\Sigma}_s \), for all \( s \ll -1 \). However, this is unlikely to hold in general. Instead, it suffices to find a family of convex sets \( \hat{K}_s \subset \hat{\Sigma}_s \) satisfying \( \hat{K}_s \subset \hat{\Gamma}_{s,0} \) and \( \text{Vol}(\hat{\Gamma}_{s,0} \setminus \hat{K}_s) \to 0 \), as \( s \to -\infty \). If \( K_{s,0} \) is the GCF from \( K_s \), then \( \hat{K}_{s,t-s} \subset \hat{\Sigma}_t \) for all \( s \ll -1 \). Meanwhile, \( \hat{K}_{s,t-s} \subset \hat{\Sigma}_{t-s} \) and \( \text{Vol}(\hat{K}_{s,t-s} - \text{Vol}(\hat{\Gamma}_{s,t-s}) = \text{Vol}(K_{s,0} - \text{Vol}(\hat{\Gamma}_{s,0}) \to 0 \) as \( s \to -\infty \), showing that \( K_{s,t-s} \to \hat{\Gamma}_t \) as \( s \to -\infty \). This proves \( \hat{\Gamma}_t \subset \hat{\Sigma}_t \). In this argument, we used the following two properties in a strong way:

(i) \( \partial_t (\text{Vol}(\hat{\Sigma}_t)) = -\omega_n \), for any GCF solution \( \Sigma_t = \partial_t \hat{\Sigma}_t \), and
(ii) if two convex sets $M_1, M_2$ satisfy $M_1 \subset M_2$ and $\text{Vol}(\hat{M}_1) = \text{Vol}(\hat{M}_2)$, then $\hat{M}_1 = \hat{M}_2$.

Let us next describe how we find such a family $\hat{K}_s$. Instead of the translator $u_0$ in the domain $\Omega$, we will consider a hypersurface $x_{n+1} = u_{\epsilon}(x)$ on $(1 + \epsilon)^{-\frac{1}{n}} \Omega$ which is the translator of the same speed $\lambda$ on the domain $(1 + \epsilon)^{-\frac{1}{n}} \Omega \setminus B_\epsilon(0)$ (see in Lemmas 6.1 and 6.2 below). When the domain shrinks from $\Omega$ to $(1 + \epsilon)^{-\frac{1}{n}} \Omega$, the associated translator speed larger that $\Lambda$, but we can adjust the speed to be equal to $\lambda$ by subtracting a small ball $B_\epsilon(0)$ from $(1 + \epsilon)^{-\frac{1}{n}} \Omega$. If we glue two such hypersurfaces at distance $s$, then the convergence of tip regions to the translator and the comparison principle from $-\infty$ time imply that $\Sigma_s$ contains such a hypersurface as $s \ll -1$ (see in Lemma 6.3). Let $\hat{K}_s$ be the best possible (meaning the smallest $\epsilon_s$) convex set which can be inserted in $\Sigma_s$ by the argument above. We want $\text{Vol}(\hat{K}_{s,0} \setminus \hat{K}_s) \to 0$. Roughly,

$$\text{Vol}(\hat{K}_{s,0}) \approx \text{Vol}(\Omega \times [-|s|\lambda, |s|\lambda]) - 2V_\Omega = 2\lambda |s| \text{Vol}(\Omega) - 2V_\Omega$$

and

$$\text{Vol}(\hat{K}_s) \approx \text{Vol}((1 + \epsilon_s)^{-\frac{1}{n}} \Omega \times [-|s|\lambda, |s|\lambda]) - 2V_s = 2\lambda |s| \text{Vol}((1 + \epsilon_s)^{-\frac{1}{n}} \Omega) - 2V_s.$$ 

Here, $V_\epsilon$ denotes the volume under the surface $x_{n+1} = u_\epsilon(x)$ and it converges to $V_\Omega$, as $\epsilon \to 0$ (see in Lemma 6.1). Since $\text{Vol}((1 + \epsilon_s)^{-\frac{1}{n}} \Omega) \approx (1 - \epsilon_s) \text{Vol}(\Omega)$ for small $\epsilon_s$, we need $\epsilon_s = o(|s|^{-1})$ to approximate the volume of $\hat{K}_{s,0}$ by $\hat{K}_s$ as $s \to -\infty$. A stronger statement of this assertion will be shown in Proposition 6.6.

We will now give the detailed proof of Theorem 1.4. Without loss of generality we may assume that $0 \in \Omega$ and that $u_\Omega(\cdot) = u_0(0) = 0$. Let us fix $r_0 > 0, R_0 > 0$ such that $B_{2r_0}(0) \subset \Omega \subset B_{R_0}(0)$. We begin with a few preliminary results, where $\eta$ denotes a standard cut off function supported in $B_1 \subset \mathbb{R}^n$ such that $\int \eta \, dx = 1$.

**Lemma 6.1.** Let $\epsilon_0 := \min\left(\frac{r_0}{2}, 1\right)$ and $\lambda = \frac{\omega_n}{2|\eta'|}$. Given $\epsilon \in (0, \epsilon_0)$, there is a unique convex solution $u_\epsilon : \Omega_\epsilon := (1 + \epsilon)^{-\frac{1}{n}} \Omega \to \mathbb{R}$ to the equation

$$\sqrt{1 + |Du_\epsilon(x)|^2} \ K(u_\epsilon, x) = \lambda \left(1 + \epsilon^{1+\eta}(\epsilon^{-1} x)\right),$$

satisfying the conditions

$$\inf_{\Omega_\epsilon} u_\epsilon = 0 \quad \text{and} \quad Du_\epsilon((1 + \epsilon)^{-\frac{1}{n}} \Omega) = \mathbb{R}^n.$$ 

Moreover, $V_\epsilon := \int_{(1 + \epsilon)^{-\frac{1}{n}} \Omega} u_\epsilon(x) \, dx \to V_\Omega$, as $\epsilon \to 0$.

**Proof.** The result of Urbas in [40, 41] guarantee the existence of a unique solution of equation (6.1) satisfying the required conditions. In addition, standard regularity estimates for equations of Monge-Ampère type imply
that as $\epsilon \to 0$, $u_{\epsilon}(x)$ converges to the translator $u_{\Omega}(x)$ having $\inf_{\Omega} u(x) = 0$ and the convergence is in the $C_{lo}^{\infty}$ sense. The convergence of $V_\epsilon \to V_{\Omega}$ easily follows, since the proof of $V_\epsilon \to V_{\Omega}$ can be applied uniformly to the solutions $u_\epsilon$ and gives

$$
\sup_{\epsilon < \min\left(\frac{1}{2}, 1\right)} \int_{\{x \in (1+\epsilon)^{-\frac{1}{2}} \Omega : \text{dist}(x, \partial(1+\epsilon)^{-\frac{1}{2}} \Omega) \leq \delta\}} u_\epsilon(x) \, dx = o(1) \quad \text{as} \quad \delta \to 0.
$$

Since $\Sigma_t$ converges to the translating soliton near tip regions, there is $\tau_0 < -1$ and $M > 0$ such that

$$
|Du^+(x, t)|, |Du^-(x, t)| \leq M, \quad \text{on} \quad B_{\tau_0}(0) \times (-\infty, \tau_0].
$$

In particular, this implies $|Du_{\Omega}(x)| \leq M$ on $x \in B_{\tau_0}(0)$. We will use $\tau_0$ and $M$ in the remaining of this section. Also recall that we have assumed $u_{\Omega}(0) = \inf u_{\Omega}(\cdot) = 0$ in this section.

**Lemma 6.2.** For $\epsilon \in (0, \epsilon_0)$, $u_\epsilon$ defined in Lemma 6.1 satisfies

$$
u_\epsilon(x) + M\epsilon \geq u_{\Omega}(x) \quad \text{for all} \quad x \in (1+\epsilon)^{-\frac{1}{2}} \Omega.
$$

**Proof.** Note $u_\epsilon(x) + M\epsilon \geq M\epsilon \geq u(x)$ on $B_\epsilon$ and it becomes infinity at $\partial(1+\epsilon)^{-\frac{1}{2}} \Omega$. They are both solutions to the translating soliton equation of speed $\lambda$ in the domain $(1+\epsilon)^{-\frac{1}{2}} \Omega \setminus B_\epsilon$. Hence, the comparison principle implies the lemma.

For the next lemma let us define $d(t) = \min(|u^+(0, t)|, |u^-(0, t)|)$. $u^+(0, t)$ and $u^-(0, t)$ are very similar to $h^+(t)$ and $h^-(t)$, respectively (recall Definition 3.1) in the following sense: since $u^-(x, t) - h^-(t)$ and $u^+(x, t) - h^+(t)$ converges to $u_{\Omega}(x)$ and $-u_{\Omega}(x)$ as $t \to -\infty$, respectively, and $\inf u_{\Omega} = u(0) = 0$, we have $|u^-(0, t) - h^-(t)| = o(1)$ and $|u^+(0, t) - h^+(t)| = o(1)$ as $t \to -\infty$. Moreover, $\partial_t u^-(0, t) \geq \lambda$ and $\partial_t u^+(0, t) \leq -\lambda$.

**Lemma 6.3.** Let $\epsilon \in (0, \epsilon_0)$ be a fixed given number. For a solution $\Sigma_t$ satisfying the assumptions of Theorem 1.3 set $d(t) = \min(|u^+(0, t)|, |u^-(0, t)|)$. Then, the solution $\Sigma_t$ encloses the convex body

$$
\hat{K}_{t, \epsilon} := \{(x', x_{n+1}) \in \mathbb{R}^{n+1} : |x_{n+1}'| \leq -u_\epsilon(x') - M\epsilon + d(t)\}
$$

for all $t \leq \min(\epsilon_0, \tau_0)$, where $\tau_0$ is a time satisfying (6.2) and

$$
t_\epsilon := \sup\{ t \cap (1+\epsilon)^{-\frac{1}{2}} \Omega \times \{0\} \subset \hat{\Sigma}_t \cap \{x_{n+1} = 0\} \}.
$$

**Proof.** We apply again the comparison principle. Since $\partial_t u^- (0, t) \geq \lambda$ and $\partial_t u^+ (0, t) \leq -\lambda$ (this is due to the convergence to the soliton shown in Theorem 3.3 and the Harnack inequality in Lemma 3.5 (i)), we have that $K_{t, \epsilon} := \partial_t \hat{K}_{t, \epsilon}$ is a supersolution of the Gauss curvature flow except the cross-section $\Sigma_t \cap \{x_{n+1} = 0\}$ and the two tip regions which are components of $(B_\epsilon(0) \times \mathbb{R}) \cap K_{t, \epsilon}$. From the choice of $\tau_0$ and $d(t)$, we have $u_\epsilon(x) + M\epsilon - d(t) \geq
and let \( \Sigma \) asymptotic to \( \epsilon \) obtain (6.4) by observing GCF. Moreover, on its boundary \( \partial(\Omega) \)
where \( r \) parameter family of (incomplete) hypersurfaces \( \{ \text{(Ancient entasis)} \} \).

Lemma 6.4
Hence, \( \Sigma \) plays the role of an inner barrier if

\[
(6.3) \quad r_t \leq r_{xx}(1 + r_x^2)^{-\frac{n+1}{2}} r^{-(n-1)}.
\]

Lemma 6.4 (Ancient entasis). For given \( \epsilon > 0 \) and \( L > 1 \), consider 1-parameter family of (incomplete) hypersurfaces \( \{ \Sigma_t^\epsilon \}_{t \leq 0} \) defined by

\[
\Sigma_t^\epsilon = \{(x_1,\ldots,x_{n+1}) : |(x_1,\ldots,x_n)| = r(x_{n+1},t), |x_{n+1}| \leq -t\},
\]

where \( r(x,t) \) on \( |x| \leq -t \) is defined by

\[
r(x,t) = 2\epsilon \left( 2 - \exp \frac{t}{L} \cosh \frac{x}{L} \right),
\]

where \( L = (2\epsilon)^{-(n-1)} \). Then \( \{ \Sigma_t^\epsilon \}_{t \leq 0} \) is an (incomplete) inner barrier to the GCF. Moreover, on its boundary \( |x_{n+1}| = |t| \), there holds

\[
(6.4) \quad 2\epsilon \leq r(|t|,t) = r(-|t|,t) \leq 3\epsilon.
\]

Remark 6.5. Note that \( \Sigma_t^\epsilon \) has rotationally symmetry about \( x_{n+1} \)-axis and has reflection symmetry about \( \{x_{n+1} = 0\} \).

Proof. Since \( r_{xx} \leq 0 \), \( (1 + r_x^2) \geq 1 \), and \( r \geq 2\epsilon \), we have

\[-r_{xx}(1 + r_x^2)^{-\frac{n+1}{2}} r^{-(n-1)} \leq -(2\epsilon)^{-(n-1)} r_{xx}.\]

Therefore, combining with \( L = (2\epsilon)^{-(n-1)} \) yields (6.3). In addition, we can obtain (6.4) by observing \( \frac{1}{2} \leq e^\tau \cosh(\frac{\tau}{2}) \leq 1 \).

Proposition 6.6. Let \( \Omega \) be a bounded convex domain with \( C^{1,1} \) boundary and let \( \Sigma_t = \partial \Omega_t \), for \( t \in (-\infty,T) \), be a compact ancient solution which is asymptotic to \( \Omega \times \mathbb{R} \). Then there exist large positive \( C \), \( L < \infty \) such that

\[
\sup_{x \in \partial \Omega \times \{0\}} d(x, \Sigma_t \cap \{x_{n+1} = 0\}) \leq Ce^{t/L}.
\]

Here \( L \) depends only on \( \Omega \), and \( C \) depends only on \( \Omega \), \( \Sigma_t \).
Proof. After a rescaling, we may assume $|\Omega| = \frac{\lambda}{4}$ so that the translator asymptotic to $\Omega \times \mathbb{R}$ has the speed $\lambda = 2$. Then, by the Harnack Lemma 3.5 (i), we have

\begin{equation}
(6.5) \quad |\partial_t u^+(x, t)| \geq 2, \quad |\partial_t u^-(x, t)| \geq 2,
\end{equation}

where $u^+(x, t)$ and $u^-(x, t)$ are the maximum and minimum height functions of $\Sigma_t$ from Definition 3.2.

Next, due to the $C^{1,1}$ boundary assumption, there is some constant $\epsilon > 0$ such that at each point $p \in \partial \Omega$, we have $B_{6\epsilon}(p+6\epsilon n_p) \subset \Omega$, where $n_p$ denotes the unit inward pointing normal vector to $\partial \Omega$ at $p$. Namely, given $p \in \partial \Omega$, $\Omega$ has an inscribed ball of radius $6\epsilon$ tangent at $p$.

We denote the 0-level set of $\hat{\Sigma}$ by $\hat{\Omega}_t := \{x \in \mathbb{R}^n : (x, 0) \in \hat{\Sigma}_t\}$. Since $\hat{\Sigma}_t$ increases to $\Omega$ as $t \to -\infty$, given $\delta \in (0, \epsilon]$ there is some $t_\delta \ll T$ such that

\begin{equation}
(6.6) \quad \sup_{x \in \partial \hat{\Omega}_t} d(x, \hat{\Omega}_t) \leq \delta.
\end{equation}

Now, we claim that given $p \in \partial \Omega$ and $\delta \in (0, \epsilon]$, $\Sigma_t$ encloses $\Sigma^e_{t-t_\delta} + (p_\delta, 0)$ for $t \leq t_\epsilon$, where $p_\delta := p + (4\epsilon + \delta)n_p$ and $\Sigma^e_{t}$ is the ancient entasis defined in Lemma 6.4. To prove the claim, we will apply the comparison principle by showing that the Dirichlet and initial boundaries of $\Sigma^e_{t-t_\delta} + (p_\delta, 0)$ are enclosed by $\Sigma_t$.

We first consider the Dirichlet condition. By (4.4), we have

\[ \partial \Sigma^e_{t-t_\delta} + (p_\delta, 0) \subset B_{3\epsilon}(p_\delta) \times \{|\pm|t-t_\epsilon|\}. \]

Therefore, it is enough to show

\begin{equation}
(6.7) \quad B_{3\epsilon}(p_\delta) \times [-|t-t_\epsilon|, |t-t_\epsilon|] \subset \hat{\Sigma}_t,
\end{equation}

for $t \leq t_\epsilon$. Indeed, (6.6) implies $B_{3\epsilon}(p_\delta) \subset \hat{\Omega}_t$, and thus we have $u^+(x, t_\epsilon) \geq 0$ and $u^-(x, t_\epsilon) \leq 0$ for $|x-p_\delta| \leq 3\epsilon$. Therefore, (6.5) yields $u^+(x, t) \geq 2|t-t_\epsilon|$ and $u^-(x, t_\epsilon) \leq -2|t-t_\epsilon|$ in $B_{3\epsilon}(p_\delta)$ for $t \leq t_\epsilon$. This gives us (6.7).

To deal with the initial condition, we notice that

\[ \Sigma^e_{t-t_\delta} + (p_\delta, 0) \subset B_{4\epsilon}(p_\delta) \times [-|t-t_\epsilon|, |t-t_\epsilon|]. \]

On the other hand, (6.6) yields $B_{4\epsilon}(p_\delta) \subset \hat{\Omega}_t$ so that we obtain $u^+(x, t_\delta) \geq 0$ and $u^-(x, t_\delta) \leq 0$ in $B_{4\epsilon}(p_\delta)$. Hence, (6.5) again leads to $u^+(x, t) \geq 2|t-t_\delta|$ and $u^-(x, t_\delta) \leq -2|t-t_\delta|$ in $B_{4\epsilon}(p_\delta) \times (-\infty, t_\delta]$. Therefore, there is some $\bar{t} \ll t_\epsilon$ depending only on $t_\epsilon, t_\delta$ such that

\[ B_{4\epsilon}(p_\delta) \times [-|t-t_\epsilon|, |t-t_\epsilon|] \subset \hat{\Sigma}_{\bar{t}} \]

holds for $t \leq \bar{t}$.

Since $\Sigma^e_{t-t_\delta} + (p_\delta, 0)$ satisfies the boundary conditions as an inner barrier of $\Sigma_t$, the comparison principle guarantees $\Sigma^e_{t-t_\delta} + (p_\delta, 0) \subset \hat{\Sigma}_{\bar{t}}$ for $t \leq t_\epsilon$. 
Passing $\delta$ to 0 and interpreting the containment among zero level sets, we conclude
\[
d(p, \tilde{\Omega}) \leq d(p, \pi(\Sigma^c_{t-t_{\epsilon}}) + p + 4en_p) = 4\epsilon - r(0, t - t_{\epsilon}) = 2\epsilon e^{\frac{t_{\epsilon}}{L}}
\]
where $\pi(\Sigma^c_{t-t_{\epsilon}}) = \{ x : (x, 0) \in \Sigma^c_{t-t_{\epsilon}} \}$. This completes the proof, since $\epsilon$, $L = (2\epsilon)^{(n-1)}$, and $t_{\epsilon}$ are independent on $p$. \hfill $\square$

**Remark 6.7.** The exponential decay in Proposition 6.6 is sharp in the sense that the non-compact translators satisfies similar bounds (as in the proof of Lemma 5.1) and the Grim Reaper in $\mathbb{R}^2$

\[
sin y = e^t \cosh x
\]

has no better decay.

Remembering $0 \in \Omega$, Proposition 6.6 implies that there exists large positive $L$ such that
\[
(6.8) \quad \epsilon_t := \inf\{ \epsilon > 0 : (1 + \epsilon)^{-\frac{1}{2}} \Omega \subset \Omega_t \} = O(e^{t/L}), \quad \text{as } t \to -\infty.
\]
Instead of this exponential decay, $\epsilon_t = o(|t|^{-1})$ will be sufficient to conclude the proof of Theorem 1.4.

**Proof of Theorem 1.4.** We begin by choosing a number $t_1 \ll -1$ such that the cross-section $\tilde{\Omega}_{t_1} := \{ x \in \mathbb{R}^n : (x, 0) \in \tilde{\Omega}_{t_1} \}$ contains the origin, and hence $\epsilon_t$ defined in (6.8) is a finite number for each $t \leq t_1$. Recall our notation $\Gamma_t = \partial \tilde{\Gamma}_t$ and $\Sigma_t = \partial \tilde{\Sigma}_t$, where $\tilde{\Gamma}_t$ is given in Proposition 5.3. As the key step, we claim
\[
(6.9) \quad \tilde{\Gamma}_{t-t_1} \subset \tilde{\Sigma}_t \quad \text{for all } t < t_1 - 2\omega^{-1}_n V_\Omega.
\]

Note first that, by Lemma 6.3 and the definition of $\epsilon_t$, we have that
\[
(6.10) \quad \tilde{K}_{t, \epsilon_t} \subset \tilde{\Sigma}_t
\]
for $t \leq t_1$ where $t_1 := \min(t_1, t_0, \sup\{ t : \epsilon_t < \epsilon_0 \})$. Here, $t_0$ and $\epsilon_0$ are defined in (6.2) and Lemma 6.1 respectively. Furthermore since $\partial_t u^-(0, t) \geq \lambda$ and $\partial_t u^+(0, t) \leq -\lambda$ by the Harnack, we obtain
\[
(6.11) \quad \tilde{K}'_t := \{ (x', x_{n+1}) : |x_{n+1}| \leq -u_{t_{\epsilon}}(x') - M\epsilon_t + \lambda|t - t_1| \} \subset \tilde{K}_{t, \epsilon_t},
\]
for all $t \leq t_1$. Next, recalling $\Gamma_{s,0}$ from the proof of Proposition 5.3, we have
\[
(6.12) \quad \tilde{K}'_t \subset \tilde{\Gamma}_{t-t_1,0}
\]
for $t \leq t_1$ by Lemma 6.2. Moreover, we have
\[
(6.13) \quad \text{Vol}(\tilde{\Gamma}_{t-t_1,0}) - \text{Vol}(\tilde{K}'_t) \to 0, \quad \text{as } t \to -\infty.
\]
Indeed, points in $\tilde{\Gamma}_{t-t_1,0} \setminus \tilde{K}'_t$ belong to one of two cylinders either $\Omega \setminus \Omega_{t_1} \times [-\lambda|t - t_1|, \lambda|t - t_1|]$ or $\Omega_{t_1} \times [-\lambda|t - t_1|, \lambda|t - t_1|]$. As a consequence of (6.8) (which is a consequence of Proposition 6.6), the volume of the former cylinder converges to zero. Moreover, the volume of $\tilde{\Gamma}_{t-t_1,0} \setminus \tilde{K}'_t$ inside of
the later cylinder converges to zero since \( \lim_{t \to -\infty} V_{t_1} = V_{\Omega} \) by Lemma 6.1. Now (6.13) follows by combining these two.

On the other hand, we denote the GCFs running from \( K'_{t_0} = \partial \hat{K}'_t \) and \( \Gamma_{s,0} = \partial \hat{\Gamma}_{s,0} \) by \( K'_{t,s} = \partial \hat{K}'_{t,s} \) and \( \Gamma_{s,t} = \partial \hat{\Gamma}_{s,t} \), respectively. Since (6.10) and (6.11) gave us \( \hat{K}'_t \subset K'_{t_1} \subset \hat{\Sigma}_t \), the comparison principle implies

\[
\hat{K}'_{t_2-t} \subset \hat{\Sigma}_{t_2}, \quad \text{for all } t_2 \geq t.
\]

We remember (6.12) and apply (6.13) to obtain

\[
\text{Vol}(\hat{\Gamma}_{t-t_1,t_2-t}) - \text{Vol}(\hat{K}'_{t,t_2-t}) = \text{Vol}(\hat{\Gamma}_{t-t_1,0}) - \text{Vol}(\hat{K}'_t) \to 0
\]
as \( t \to -\infty \). In addition, we have \( \Gamma_{t-t_1,t_2-t} \cap \Gamma_{t_2-t_1} \) as \( t \to -\infty \) by (5.4). Therefore, we conclude that

\[
\hat{\Gamma}_{t_2-t_1} \subset \hat{\Sigma}_{t_2}
\]
for all \( t_2 < t_1 - \frac{2V_{\Omega}}{\omega_n} \), which proves our claim (6.9).

We will now use the inclusion (6.9) to conclude the proof of our uniqueness theorem. First, since \( \hat{\Gamma}_{t-t_1} \subset \hat{\Sigma}_t \) and \( \partial_t \text{Vol}(\hat{\Sigma}_t) = \partial_t \text{Vol}(\hat{\Gamma}_{t-t_1}) = -\omega_n \), we obtain

\[
\text{Vol}(\hat{\Sigma}_t \setminus \hat{\Gamma}_{t-t_1}) \text{ is constant in time.}
\]

We recall the definitions of heights \( h^\pm \) of \( \Sigma_t \) given in Definition 3.1. Since \( \hat{\Gamma}_{t-t_1} \subset \hat{\Sigma}_t \), \( \partial_t h^+(t) \leq -\lambda \) and \( \partial_t h^-(t) \geq \lambda \), we obtain that, as \( t \to -\infty \), \( h^+(t) - \lambda |t| \) increases and \( h^-(t) + \lambda |t| \) decreases. In addition, by Theorem 3.3, Proposition 5.4, and (6.15) we have

\[
\lim_{t \to -\infty} h^+(t) - \lambda |t| = C^+, \quad \lim_{t \to -\infty} h^-(t) + \lambda |t| = C^-,
\]

for some constants \( C^+ \) and \( C^- \). Next, recall \( \hat{\Gamma}_{t-t_1} \subset \hat{\Gamma}_{t-t_1,0} \) by (5.4) and \( \text{Vol}(\hat{\Gamma}_{t-t_1,0} \setminus \hat{\Gamma}_{t-t_1}) \) converges to 0 as \( t \to -\infty \) by (5.3) and Proposition 5.3 (i). Thus \( \text{Vol}(\hat{\Sigma}_t \setminus \hat{\Gamma}_{t-t_1}) = \text{Vol}(\hat{\Sigma}_t \setminus \hat{\Gamma}_{t-t_1,0}) + o(1) \) as \( t \to -\infty \). Observe

\[
\text{Vol}(\hat{\Sigma}_t \setminus \hat{\Gamma}_{t-t_1,0}) = \int \left[(u_{\Omega}(x) - \lambda |t-t_1| - u^-(x,t))_+ dx
\right.
\]
\[
+ \int [u^+(x,t) + (u_{\Omega}(x) - \lambda |t-t_1|)_+ dx
\]

where we interpret \( u^- = \infty \) and \( u^+ = -\infty \) outside of their domain. By the dominated convergence theorem, the two integrals above converge to \((-C^- - \lambda t_1)|\Omega| \) and \((C^+ - \lambda t_1)|\Omega| \), respectively, as \( t \to -\infty \). Indeed, by (6.16), the first integrand is bounded by \( u_{\Omega}(x) - C^- - \lambda t_1 \in L^1(\Omega) \) by Lemma 5.1 and the integrand converges locally uniformly to the constant \(-C^- - \lambda t_1 \) by Theorem 3.3. The same argument works for the second integral. This proves

\[
\text{Vol}(\hat{\Sigma}_t \setminus \hat{\Gamma}_{t-t_1,0}) = (C^+ - C^- - 2\lambda t_1)|\Omega| + o(1),
\]
and we conclude $\text{Vol}(\hat{\Sigma}_t \setminus \hat{\Gamma}_{t-t_1}) = (C^+ - C^- - 2\lambda t_1)|\Omega|$ as this is constant. Since both of $\Sigma_t$ and $\Gamma_t$ become extinct at $T = -\frac{2V_\Omega}{\omega_n}$, we have
\[
(C^+ - C^- - 2\lambda t_1)|\Omega| = \text{Vol}(\hat{\Sigma}_{T+t_1}) = -\omega_n t_1,
\]
namely $C^+ = C^- =: C$.

It remains to show that $\Sigma_t = \hat{\Gamma}_t + C e_{n+1}$ and the proof follows from what we have already done. We may summarize the first part of the proof as follows: if there exist constants $t_0, \tau_1$, and a decreasing function $\epsilon_t = (|t|^{-1})$ as $t \to -\infty$, such that $\hat{\Sigma}_t$ contains $\hat{K}_{t,\epsilon_t, t_0}$ (defined in (6.8)),
\[
\hat{K}_{t,\epsilon_t, t_0} := \{(x, x_{n+1}) : |x_{n+1}| \leq -u_{\epsilon_t}(x) - M\epsilon_t + \lambda|t-t_0|\}
\]
for all $t < \tau_1$, then $\hat{\Gamma}_{t-t_0} \subset \hat{\Sigma}_t$. By looking at the intersection between $\Sigma_t$ and $\{x_{n+1} = C\}$, we may define $\epsilon'_t$ in the same way as $\epsilon_t$ is defined in (6.8), that is
\[
\epsilon'_t := \inf\{\epsilon > 0 : (1 + \epsilon)^{-\frac{1}{n}}\Omega \subset \Omega'_t\}
\]
where $\Omega'_t$ is the cross-section of $\hat{\Sigma}_t$ by $\{x_{n+1} = C\}$. Similarly to Proposition 6.6, we have $\epsilon'_t = o(|t|^{-1})$. Next, for each small $\delta > 0$, Lemma 6.3 and (6.16) imply that there is $\tau_\delta \ll -1$ such that $\hat{K}_{t,\epsilon'_t, -\delta} + C e_{n+1} \subset \hat{\Sigma}_t$ for all $t < \tau_\delta$. Then, as we obtained (6.14), we can derive $\hat{\Gamma}_{t+\delta} + C e_{n+1} \subset \hat{\Sigma}_t$ by repeating the argument from (6.11) to (6.14) after replacing $t_1$ by $-\delta$. Taking $\delta \to 0$, we get $\hat{\Gamma}_t + C e_{n+1} \subset \Sigma_t$. Since $\text{Vol}(\hat{\Gamma}_t) = \text{Vol}(\hat{\Sigma}_t) = -\omega_n t - 2V_\Omega$, we finally conclude that $\hat{\Gamma}_t + C e_{n+1} = \Sigma_t$.

\[\square\]

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