Abstract. A symbolic method is discussed which can be used to obtain the asymptotic bias and variance to order $O(1/n)$ for estimators in stationary time series. Using this method the bias to $O(1/n)$ of the Burg estimator in AR(1) and AR(2) models is shown to be equal to that of the least squares estimators in both the known and unknown mean cases. Previous researchers have only been able to obtain simulation results for this bias because this problem is too intractable without using computer algebra.

Keywords. Asymptotic bias and variance; autoregression; autoregressive spectral analysis; symbolic computation.

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1. INTRODUCTION AND SUMMARY

Tjøstheim and Paulsen (1983, Correction 1984) showed that the Yule-Walker estimates had very large mean-square errors in the AR(2) case when the parameters were near the admissible boundary and that this inflated mean square error was due to bias. This result was demonstrated by Tjøstheim and Paulsen (1983) in simulation experiments as well as by deriving the theoretical bias to order $O(1/n)$. It was also mentioned by Tjøstheim and Paulsen (1983, p.397, §5) that the bias results from simulation experiments for the Burg estimates were similar to those obtained for least squares estimates but that they had not been able to obtain the theoretical bias term. Using computer algebra we are now able to compute the bias for the Burg estimator and to show that to order $O(1/n)$ it is equal to the bias for the least squares estimator in the case of AR(1) and AR(2) models in both the known and unknown mean cases.

As pointed out by Lysne and Tjøstheim (1987), the Burg estimators have an important advantage over the least squares estimates for autoregressive spectral estimation since Burg estimates always lie in the admissible parameter space whereas the least squares estimates do not. Burg estimators are now frequently used in autoregressive spectral estimation (Percival and Walden, 1993, §9.5) since they provide better resolution of sharp spectral peaks. As the Yule-Walker estimators, the Burg estimators may be efficiently computed using the Durbin-Levinson recursion. Our result provides further justification for the recommendation to use the Burg estimator for autoregressive spectral density estimation as
well as for other autoregressive estimation applications.

It has been shown that symbolic algebra could greatly simplify derivations of asymptotic expansions in the IID case (Andrews and Stafford, 1993). Symbolic computation is a powerful tool for handling complicated algebraic problems that arise with expansions of various types of statistics and estimators (Andrews and Stafford, 2000) as well as for exact maximum likelihood computation (Currie, 1995; Rose and Smith, 2000). Cook and Broemeling (1995) show how symbolic computation can be used in Bayesian time series analysis. Smith and Field (2001) described a symbolic operator which calculates the joint cumulants of the linear combinations of products of discrete Fourier transforms. A symbolic computation approach to mathematical statistics is discussed by Rose and Smith (2002). In the following sections, we develop a symbolic computation method that can be used to solve a wide variety of time problems involving linear time series estimators for stationary time series. Our symbolic approach is used to derive the theoretical bias to $O(1/n)$ of the Burg estimator for the AR(2) model.

2. ASYMPTOTIC EXPECTATIONS AND COVARIANCES

Consider $n$ consecutive observations from a stationary time series, $z_t, t = 1, ..., n$, with mean $\mu = E(z_t)$ and autocovariance function $\gamma_k = \text{Cov}(z_t, z_{t-k})$. If the mean is known, it may, without loss of generality be taken to be zero. Then one of the unbiased estimators of autocovariance
\( \gamma(m - k) \) may be written as

\[
S_{m,k,i} = \frac{1}{n + 1 - i} \sum_{t=1}^{n} z_{t-m}z_{t-k},
\]

(1)

where \( m, k \) and \( i \) are non-negative integers with \( \max(m, k) < i \leq n \). If the mean is unknown, a biased estimator of \( \gamma(m - k) \) may be written as

\[
\bar{S}_{m,k,i} = \frac{1}{n} \sum_{t=1}^{n} (z_{t-m} - \bar{z}_n)(z_{t-k} - \bar{z}_n),
\]

(2)

where \( \bar{z}_n \) is the sample mean.

**Theorem 1.** Let the time series \( z_t \) be the two-sided moving average,

\[
z_t = \sum_{j=-\infty}^{\infty} \alpha_j e_{t-j},
\]

(3)

where the sequence \( \{\alpha_j\} \) is absolutely summable and the \( e_t \) are independent \( N(0, \sigma^2) \) random variables. Then for \( i \leq j \),

\[
\lim_{n \to \infty} n \text{Cov}(S_{m,k,i}, S_{f,g,j}) = \sum_{h=-\infty}^{\infty} T_h,
\]

(4)

where

\[
T_h = \gamma(g-k+i-j+h)\gamma(f-m+i-j+h)+\gamma(f-k+i-j+h)\gamma(g-m+i-j+h).
\]

**Theorem 2.** Let a time series \( \{z_t\} \) satisfy the assumptions of Theorem 1. Then

\[
\lim_{n \to \infty} n \mathbb{E} (\bar{S}_{m,k,i} - \gamma(m - k)) = -|i - 1| \gamma(m - k) - \sum_{h=-\infty}^{\infty} \gamma(h)
\]

(5)

and

\[
\lim_{n \to \infty} n \text{Cov}(\bar{S}_{m,k,i}, \bar{S}_{f,g,j}) = \sum_{h=-\infty}^{\infty} T_h,
\]

(6)

where

\[
T_h = \gamma(g-k+i-j+h)\gamma(f-m+i-j+h)+\gamma(f-k+i-j+h)\gamma(g-m+i-j+h).
\]
These two theorems may be considered as the extensions of Theorem 6.2.1 and Theorem 6.2.2 of Fuller (1996). Letting \( p = m - k \) and \( q = f - g \), the left side of (4) or (6) can be simplified,

\[
\sum_{h=-\infty}^{\infty} T_h = \sum_{h=-\infty}^{\infty} \gamma(h)\gamma(h - p + q) + \gamma(h + q)\gamma(h - p).
\]

There is a wide variety of estimators which can be written as a function of the autocovariance estimators, \( S_{m,k,i} \) or \( \bar{S}_{m,k,i} \), such as, autocorrelation estimator, least squares estimator, Yule-Walker estimator, Burg estimator, etc. The asymptotic bias and variance may be obtained by the Taylor expansion. Unfortunately, in the most cases, those expansions include a large number of expectations and covariances of the autocovariance estimators. It is too intractable manually. Theorems 1 and 2 provide the basis for a general approach to the symbolic computation of the asymptotic bias and variance to order \( O(1/n) \) for those estimators. The definition of (1) or (2) allows an index set \( \{m, k, i\} \) to represent an estimator so that Theorem 1 or 2 can be easily implemented symbolically.

3. BIAS OF BURG ESTIMATORS IN AR(2)

The stationary second-order autoregressive model may be written as

\[ z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t, \]

where \( a_t \) are normal and independently distributed with mean zero and variance \( \sigma^2 \) and parameters \( \phi_1 \) and \( \phi_2 \) are in the admissible region, \( |\phi_2| < 1, \phi_1 + \phi_2 < 1 \) and \( \phi_2 - \phi_1 < 1 \). The Burg estimate for \( \phi_2 \) may be obtained directly from Percival and Walden (1993, eqn. 416d) and then the estimate for \( \phi_1 \) may be obtained using the
Durbin-Levinson algorithm. After simplification, these estimates may be written as

\[ \hat{\phi}_2 = 1 - \frac{CD^2 - 2ED^2}{CD^2 + 8F^2G - 4FHD}, \hat{\phi}_1 = \frac{2F}{D} (1 - \hat{\phi}_2) \tag{8} \]

where

\[ C = \frac{1}{n-2} \sum_{t=3}^{n} (z_t^2 + z_{t-2}^2), \quad D = \frac{1}{n-1} \sum_{t=2}^{n} (z_t^2 + z_{t-1}^2), \quad E = \frac{1}{n-2} \sum_{t=3}^{n} (z_t^2 z_{t-2}^2), \]

\[ F = \frac{1}{n-1} \sum_{t=2}^{n} (z_t z_{t-1}), \quad G = \frac{1}{n-2} \sum_{t=3}^{n} z_{t-1}^2, \quad H = \frac{1}{n-2} \sum_{t=3}^{n} (z_t z_{t-1} + z_{t-2} z_{t-1}). \]

Using a Taylor series expansion of \( \hat{\phi}_1 \) and \( \hat{\phi}_2 \) about \( \mu_A = E(A) \), where \( A = C, D, E, F, G \) and \( H \), the bias to order \( O(1/n) \) may be expressed in terms of asymptotic expectations of products and cross products involving \( C, D, E, F, G \) and \( H \). There are six squared terms and fifteen cross product terms involved in each expansion, that is, it is required to compute and simplify for each of these twenty one expansions involving \( C, D, E, F, G \) and \( H \). These terms may all be written in terms of the unbiased estimate of the autocovariance, \( S_{m,k,i} \). The required asymptotic expectations of each term in the expansions are obtained by Theorem 1, that is,

\[ \lim_{n \to \infty} \frac{1}{n} \text{Cov} (S_{m,k,i}, S_{f,g,j}) = \sum_{h=\infty}^{\infty} T_h, \tag{9} \]

where \( T_h = \gamma(h) \gamma(h-p+q) + \gamma(h+q) \gamma(h-p), p = m - k, q = f - g \) and

\[ \gamma(h) = \frac{\zeta_2^{1+h} - \zeta_1 \zeta_2^{1+h} + \zeta_1 \zeta_2^{1+h} \left( \zeta_2^2 - 1 \right)}{\left( \zeta_1^2 - 1 \right) (\zeta_1 - \zeta_2) (\zeta_1 \zeta_2 - 1) (\zeta_2^2 - 1)}, \tag{10} \]

where \( h \geq 0 \), \( \zeta_1 \) and \( \zeta_2 \) are the roots, assumed distinct, of the polynomial \( \zeta^2 - \phi_1 \zeta - \phi_2 = 0 \). The order \( n^{-1} \) coefficient of the covariance expansion of
\( S_{m,k,i} \) and \( S_{f,g,j} \) given in eqn. (9) may be evaluated symbolically by defining an operator of \( S_{m,k,i} \) and \( S_{f,g,j} \), \( \text{LCOV} \{m, k, i\} \{f, g, j\} \). To illustrate this symbolic method consider the evaluation of \( \lim_{n \to \infty} n \text{Cov}(2C, H) \) which is one of the twenty one order \( n^{-1} \) expansion coefficients involving \( C, D, E, F, G \) and \( H \) mentioned above. It may be obtained by

\[
\lim_{n \to \infty} n \text{Cov}(2C, H) = 2 \{ \text{LCOV} \{(0, 0, 3) + \{2, 2, 3\}\{(0, 1, 3) + \{2, 1, 3\}\}) \}
\]

\[
= 2 \{ \text{LCOV} \{(0, 0, 3)\{0, 1, 3\}\} + \text{LCOV} \{(0, 0, 3)\{2, 1, 3\}\} + \text{LCOV} \{(2, 2, 3)\{0, 1, 3\}\} + \text{LCOV} \{(2, 2, 3)\{2, 1, 3\}\} \},
\]

since \( C = S_{0,0,3} + S_{2,2,3} \), \( H = S_{0,1,3} + S_{2,1,3} \), and \( \text{LCOV}[:] \) follows the linearity and the distributive law.

After algebraic simplification the biases to order \( O(1/n) \) are found to be

\[
\mathbb{E}(\hat{\phi}_1 - \phi_1) = -(\zeta_1 + \zeta_2)/n \quad \text{and} \quad \mathbb{E}(\hat{\phi}_2 - \phi_2) = (3\zeta_1\zeta_2 - 1)/n.
\]

More simply, in terms of the original parameters we have

\[
\mathbb{E}(\hat{\phi}_1 - \phi_1) = -\phi_1/n \quad (11)
\]

and

\[
\mathbb{E}(\hat{\phi}_2 - \phi_2) = -(1 + 3\phi_2)/n. \quad (12)
\]

We verified, using the same approach, that eqns. (11) and (12) also hold for the case of equal roots of the polynomial \( \zeta^2 - \phi_1\zeta - \phi_2 = 0 \).

For the stationary second-order autoregressive model with an unknown mean, the Burg estimators can be written as the same ratio function of the biased estimators of the autocovariances, \( \overline{S}_{m,k,i} \), as was given in eqn. (8). The symbolic approach is similar to the known mean case, but includes one more inner product associated with the biases of those autocovariance
estimators, $\mathfrak{S}_{m,k,i}$. The required asymptotic biases and covariances of $\mathfrak{S}_{m,k,i}$ are obtained by Theorem 2. The biases to order $O(1/n)$ are found to be

$E(\hat{\phi}_1 - \phi_1) \doteq ((\zeta_1\zeta_2 - \zeta_1 - \zeta_2) - 1)/n$ and $E(\hat{\phi}_2 - \phi_2) \doteq (4\zeta_1\zeta_2 - 2)/n$.

That is

$$E(\hat{\phi}_1 - \phi_1) \doteq -(\phi_2 + \phi_1 + 1)/n$$

and

$$E(\hat{\phi}_2 - \phi_2) \doteq -(2 + 4\phi_2)/n.$$  (13)

Once an estimator of a stationary time series is written as a well defined function composed of $\mathfrak{S}_{m,k,i}$ or $\mathfrak{S}_{m,k,i}$, by expanding it by a Taylor series, the estimate bias and variance to order $n^{-1}$ may be obtained by Theorem 1 or 2 with symbolic computation. This approach can be applied in the bias derivation of the Burg estimator, $\hat{\rho}$, in the first order autoregressive model, AR(1). We have obtained that its bias to order $n^{-1}$ is $-2\rho/n$ in a zero mean case and $-(1 + 3\rho)/n$ in an unknown mean case. Therefore, for both of AR(1) and AR(2) cases, the biases to order $n^{-1}$ of the Burg estimators are the same as the least squares estimation for a known mean case as well as for an unknown mean case. These results are consistent with the simulation study reported by Tjøstheim and Paulsen (1983).

4. CONCLUDING REMARKS

In addition to deriving the bias for the Burg estimator, we used our computer algebra method to verify the bias results reported by Tjøstheim & Paulsen (Correction, 1984) and we also carried out simulation experiments which provided an additional check on our results (Zhang,
Mathematica (Wolfram, 1999) notebooks with the complete details of our derivations are available on request from the authors.

Since many quadratic statistics in a stationary time series can be expressed in terms of $S_{m,k,i}$ or $S_{m,k,i}$, our computer algebra approach can be applied to derive their laborious moment expansions to order $O(1/n)$. As examples, using our method, we can easily obtain the results by Bartlett (1946), Kendall (1954), Marriott and Pope (1954), White (1961) and Tjøstheim and Paulsen (1983).

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