ECONOMIC COUPLINGS AND ACYCLIC FLOWS

DAVIDE GABRIELLI AND IDA GERMANA MINELLI

Abstract. To any coupling between two probability measures $\mu_1$ and $\mu_2$ on a finite set there is naturally associated a flow that represents the amount of mass transported to change the measure $\mu_1$ into the measure $\mu_2$. This flow is not unique since several different choices of paths are possible. We prove that also a converse statement holds when the flow is acyclic. In particular we show two different algorithms that associate to any acyclic flow having divergence coinciding with $\mu_1 - \mu_2$ a coupling between the two measures. The couplings that can be obtained in this way are called economic. In the case of a countable set the constructions are implemented with a limit procedure and the flows for which they hold need to satisfy a suitable finite decomposability condition. We show several consequences of these constructive procedures among which a third equivalent statement in Strassen Theorem and a constructive proof of the equivalence between a mass transportation problem with a geodesic cost and a minimal current problem. We illustrate the results discussing several solvable cases.

Keywords: Couplings, flows on networks, stochastic monotonicity, mass transportation.

AMS 2010 Subject Classification: 97K50, 60E15, 05C21

1. Introduction

Coupling is a commonly used and very powerful technique in probability theory [16]. Among other applications we recall its key role in estimating the rate of convergence to equilibrium for Markov processes [16], its use in problems of stochastic monotonicity through the so called Strassen Theorem [15, 21], and of course all the problems of mass transportation [18, 22].

In this paper we show and analyze a connection among coupling theory, flows on networks and decompositions of weighted graphs. We restrict to the cases of finite or countable sets. Given a coupling between two probability measures then it is easy to associate to it a flow. This is done for example choosing for any pair of vertices a path going from one to the other and associating to this path the weight corresponding to the probability of the fixed pair according to the coupling. This flow represents the amount of mass that has to be transported to transform the first marginal of the coupling into the second. This flow is not unique since several choices of the paths are possible. All the flows obtained in this way have the discrete divergence coinciding with the difference between the two marginals. Couplings for which there exists a choice of the paths such that the corresponding flow is acyclic will be called economic and the motivation of the name will be clear below.
One of the main results of this paper is that also an inverse construction holds. We will show two constructive algorithms that associate to any acyclic flow a coupling between two probability measures whose difference coincides with the divergence of the flow. In the case of an infinite countable set the flow has to satisfy an additional suitable condition of finite decomposability. This condition implies a compactness property and allows to implement the algorithms with a limit procedure.

The consequences of this correspondence are at least two. The first one is that to construct a coupling between two measures it is enough to construct a finitely decomposable flow having as divergence the difference between the two measures. The coupling will then be obtained implementing the algorithmic constructions. The second one is that for all problems on which it is important the amount of mass flown and not the exact provenance and destination it is enough to consider the flow without constructing explicitly a coupling. Stochastic monotonicity and optimal mass transportation with geodesic costs are two examples of such a situation.

One of our algorithms associates to the acyclic flow a positive measure on the space of finite self–avoiding paths and is similar (but different) to the one proposed in [17] to prove a decomposition of acyclic normal currents on metric spaces. Our construction is therefore a discrete version of the original decomposition due to S.K. Smirnov on bounded domains of $\mathbb{R}^n$ [20]. While the results in [17, 20] hold under a summability condition, our results extend to the wider class of finitely decomposable flows. We discuss two main applications.

The first one is a third equivalent statement in Strassen Theorem. Strassen Theorem states [15, 21] that one measure stochastically dominates another one if and only if there exists a coupling between the two measures that gives zero weight to pairs of elements not increasingly ordered. The partial order structure of a finite or countable partially ordered set can be described in terms of an acyclic directed graph. We show that a probability measure stochastically dominates another one if and only if there exists a finitely decomposable flow on the acyclic directed graph associated to the partial order and having as divergence the difference between the two measures. This gives a precise formulation to the intuitive statement that one measure stochastically dominates another one when it is possible to transform this second measure into the first one moving mass according to the partial order structure. This is a natural equivalent reformulation of stochastic domination and we show its power discussing some solvable examples. An important property of this statement is that stochastic domination is equivalent to the existence of a flow on a digraph encoding all the information about the partial order. We expect this reformulation could be useful on the study of problems strictly related to the geometry of a partial order [9, 10].

The second application is related to problems of mass transportation [18, 22]. Given a cost function and two fixed probability measures, the Monge-Kantorovich problem consists in optimizing over all possible couplings the expected value of the cost function. We give a constructive proof of the equivalence between the Monge-Kantorovich problem and a discrete version of a minimal current problem proposed by Beckmann [3] under the name of continuous transportation model. This equivalence holds in the case of a geodesic cost function. Also in this case we illustrate with some examples the power of this reformulation of the problem.

The structure of the paper is the following.
In section 2 we fix notation and state some of the main results of the paper. In particular, in this section we introduce basic notation concerning graphs and directed graphs, we discuss some basic properties of partially ordered sets and their description in terms of graphs and directed graphs, we introduce the notions of coupling, flow and discrete vector field and their relationship and finally we state our main results on stochastic monotonicity and mass transportation. A key role will be played by the notion of finite decomposability of a flow. In section 3 we show two algorithms that associate to any acyclic flow with divergence coinciding with the difference between two probability measures a coupling between them. We first discuss the case of a finite set, then we show how the condition of finite decomposability allows to extend the constructions, with a limiting procedure, to the infinite case. These algorithmic constructions are the remaining main results of the paper. In section 4 we prove Theorem 2.6 and discuss some examples. In section 6 we discuss some examples of the applications of Theorems 2.1 and 2.3. In section 7 we prove Theorem 2.4 and discuss some examples.

2. Notation and main results

In this section we discuss the general framework, introduce notation and state some of the main results.

2.1. Graphs digraphs and posets. We consider a finite or countable infinite set $V$. A graph with vertices $V$ is a pair $(V,E)$ where $E \subset \mathcal{P}_2(V)$ is the collection of unordered edges. With $\mathcal{P}_2(V)$ we denote the collection of all subsets of $V$ having cardinality 2. The element of $E$ connecting $x,y \in V$ is denoted by $\{x,y\}$. According to this definition our graphs do not contain loops i.e. edges of the type $\{x,x\}$. A path $\gamma$ connecting $x,y \in V$ is a sequence of vertices $\gamma := (x_0,x_1,\ldots,x_n)$ such that $x_0 = x$, $x_n = y$ and $\{x_i,x_{i+1}\} \in E$, for $i = 0,\ldots,n-1$. The integer $n$ is the length of the path and is denoted by $|\gamma|$. If there exists an $i$ such that $\{u,v\} = \{x_i,x_{i+1}\}$ we write $\{u,v\} \in \gamma$. Since the edges are unordered we identify the paths $(x_0,\ldots,x_n)$ and $(x_n,\ldots,x_0)$. The graph $(V,E)$ is connected if any pair of vertices can be connected by a path. A cycle is a path for which $x_0 = x_n$. In this case we identify all the cycles obtained starting from each $x_i$ and moving in one of the two possible orientations. A graph that contains no cycles is a forest and a tree if it is connected.

A directed graph, called shortly a digraph, with vertices $V$ is the pair $(V,E)$ where $E \subset V \times V$ is the collection of directed edges. We assume that there are not edges of the type $(x,x)$ for some $x \in V$. A directed path $\gamma$ from $x \in V$ to $y \in V$ is a sequence of vertices $\gamma := (x_0,\ldots,x_n)$ such that $x_0 = x$, $x_n = y$ and $(x_i,x_{i+1}) \in E$, for $i = 0,\ldots,n-1$. The integer $n$ is the length of the directed path and is denoted also by $|\gamma|$. If there exists an $i$ such that $(u,v) = (x_i,x_{i+1})$ we write $(u,v) \in \gamma$. Given a subset $S \subseteq V$, if $x_i \in S$ for any $i$ we write $\gamma \subseteq S$. We call $\gamma^- := x_0$ the starting point of the path and $\gamma^+ := x_n$ its final point. A directed cycle is a directed path for which $x_0 = x_n$. Given two paths $\gamma = (x_0,\ldots,x_n)$ and $\gamma' = (x'_0,\ldots,x'_k)$ such that $x_n = x'_0$ we denote by

$$\gamma \ast \gamma' := (x_0,\ldots,x_n,x'_1,\ldots,x'_k), \quad (2.1)$$

their concatenation. A path is called self-avoiding if $x_i \neq x_j$ when $i \neq j$. A digraph containing no directed cycles is called a directed acyclic graph.
Given a digraph \((V, E)\) we can construct a new digraph \((V, \overline{E})\) called its transitive closure. A pair \((x, y)\) \(\in \overline{E}\) if and only if there exists a directed path form \(x\) to \(y\). When \(|V| < +\infty\) and \((V, E)\) is an acyclic digraph we can define also a new directed acyclic graph \((V, \overline{E})\) that is called its transitive reduction. This is the minimal acyclic digraph having the same transitive closure of the original digraph \((V, E)\). This means that given any digraph \((V, F)\) such that \((V, \overline{E}) = (V, \overline{F})\) then \(\overline{E} \subseteq F\). When the original digraph \((V, E)\) is acyclic and \(|V| < +\infty\), it can be shown that \((V, \overline{E})\) is uniquely determined (see [1] section 4.3).

To any digraph \((V, E)\) we associate the graph \((V, \mathcal{E}_E)\) defined requiring that \(\{x, y\} \in \mathcal{E}_E\) if at least one between \((x, y)\) and \((y, x)\) belongs to \(E\). Conversely to any graph \((V, \mathcal{E})\) we associate the digraph \((V, \mathcal{E}_E)\) obtained splitting each edge \(\{x, y\} \in \mathcal{E}\) into the two directed edges \((x, y)\) and \((y, x)\). Note that in general \(E \subseteq E_{\mathcal{E}_E}\) and the inclusion can be strict.

A partial order relation \(\leq\) on \(V\) is determined by a subset \(S \subseteq V \times V\) satisfying some suitable properties. When \((x, y) \in S\) we write \(x \leq y\). The properties are the following

- \(x \leq x\) for any \(x \in V\) (reflexivity),
- if \(x \leq y\) and \(y \leq x\) then necessarily \(x = y\) (antisymmetry),
- if \(x \leq y\) and \(y \leq z\) then \(x \leq z\) (transitivity).

The pair \((V, \leq)\) is called a partially ordered set or simply a poset. A function \(f : V \to \mathbb{R}\) is called increasing with respect to the partial order \(\leq\) if

\[f(x) \leq f(y), \quad \forall x \leq y.\]

The easiest way to describe a partial order is using its Hasse diagram [6]. This is an unoriented graph embedded into \(\mathbb{R}^2\). The vertices of this graph are labelled by \(V\). The edges \(\{x, z\}\) are associated to pairs \((x, z)\) of distinct elements of \(V\) such that \(x \leq z\) and there are no elements \(y \neq x, z\) such that \(x \leq y \leq z\). The embedding is such that the cartesian coordinates \((X_1, X_2)\) and \((Y_1, Y_2)\) associated respectively to two distinct vertices \(x, y \in V\) such that \(x \leq y\) satisfy the inequality \(X_2 < Y_2\).

Every finite partial order can be described in terms of its Hasse diagram.

Let us now discuss how to identify a poset in terms of digraphs. An acyclic digraph \((V, E)\) induces a partial order among the vertices \(V\). Indeed we can say that \(x \leq y\) if either \(y = x\) or there exists an oriented path from \(x\) to \(y\). Clearly the reflexivity property holds. Since the graph is acyclic then also antisymmetry holds. Finally concatenating the directed paths the transitivity property can be easily checked. By definition we have that when \(x \neq y\) then \(x \leq y\) if and only if \((x, y) \in \overline{E}\) where we recall that \((V, \overline{E})\) is the transitive closure of \((V, E)\). It is clear from the above construction that any poset \((V, \leq)\) can be identified with the transitive closure \((V, \overline{E})\) of an acyclic digraph using the correspondence \(x \leq y \iff (x, y) \in \overline{E}\) when \(x \neq y\). Since the transitive closure \((V, \overline{E})\) is completely determined by the acyclic digraph \((V, E)\), a poset \((V, \leq)\) can be identified assigning an acyclic digraph. Since different acyclic digraphs can have the same transitive closure, this description is not unique. When \(|V| < +\infty\) it is uniquely identified \((V, \overline{E})\) the transitive reduction of \((V, \overline{E})\). Since \((V, \overline{E})\) is the minimal digraph having \((V, \overline{E})\) as transitive closure, it is the easiest way to describe the poset using digraphs.

It is easy to see that \((V, \overline{E})\) can be obtained from the Hasse diagram of the poset associating the edge \((x, y)\) to any edge \(\{x, y\}\) of the Hasse diagram such that \(X_2 < Y_2\) (recall that \((X_1, X_2)\) and \((Y_1, Y_2)\) are the elements of \(\mathbb{R}^2\) associated
respectively to \(x\) and \(y\). In this sense speaking of the Hasse diagram of a poset or of the associated acyclic transitive reduction \((V,E)\) is the same. See Figure 1 for an illustrative example.

When \(|V| = +\infty\), it is not always possible to describe a partial order using a Hasse diagram. Equivalently it is not always defined a transitive reduction of a countable acyclic digraph. The easiest example of this fact is given by the set of rational numbers with the usual partial order relation. Nevertheless any acyclic digraph with \(|V| = +\infty\) has a well defined transitive closure and consequently determines a partial order on \(V\). Indeed any countable partial order can be described by an acyclic digraph.

2.2. **Couplings, flows and discrete vector fields.** Given \(\mu_1\) and \(\mu_2\) two probability measures on \(V\), a coupling between them is a probability measure \(\rho\) on \(V \times V\) such that

\[
\left\{ \begin{array}{c}
\sum_{y \in V} \rho(x,y) = \mu_1(x), \quad \forall x \in V, \\
\sum_{x \in V} \rho(x,y) = \mu_2(y), \quad \forall y \in V.
\end{array} \right.
\]

We say that a coupling \(\rho\) is compatible with the partial order \(\leq\) if

\[
\rho\left( \{(x,y) : x \leq y\} \right) = 1.
\]

Given \(\mu_1\) and \(\mu_2\) two probability measures on \(V\) we say that \(\mu_2\) stochastically dominates \(\mu_1\) with respect to the partial order \(\leq\) and write \(\mu_1 \leq \mu_2\) if

\[
\mathbb{E}_{\mu_1}(f) \leq \mathbb{E}_{\mu_2}(f)
\]

for any bounded increasing function \(f\).

A flow on a digraph \((V,E)\) is a map \(Q : E \to \mathbb{R}^+\). The divergence of \(Q\) at \(x \in V\) is defined by

\[
\text{div} \, Q(x) := \sum_{y : (x,y) \in E} Q(x,y) - \sum_{y : (y,x) \in E} Q(y,x).
\]

When \(|V| = +\infty\) this is not always well defined. In this case we say that the divergence of a flow \(Q\) exists and is given by (2.2) if both series appearing in the r.h.s. of (2.2) are convergent for any \(x \in V\). In words this corresponds to have that the flux exiting from any single vertex and the flux incoming towards any single vertex are both finite.
We denote by $E(Q)$ the elements $(x,y) \in E$ such that $Q(x,y) > 0$. Given a directed path $\gamma = (x_0, \ldots, x_n)$ on $(V,E)$, we associate to it the flow $Q_\gamma$ defined by

$$Q_\gamma(x,y) := \begin{cases} 1 & \text{if } (x,y) \in \gamma, \\ 0 & \text{otherwise}. \end{cases} \quad (2.3)$$

On the set of flows on a fixed digraph there is a natural partial order structure defined by $Q \leq Q'$ if $Q(x,y) \leq Q'(x,y)$ for any $(x,y) \in E$.

A discrete vector field on a graph $(V,E)$ is a function $\phi : E_E \to \mathbb{R}$ that is antisymmetric

$$\phi(x,y) = -\phi(y,x), \quad \{x,y\} \in E.$$

Given a flow $Q$ on a digraph $(V,E)$, we can naturally project it to the discrete vector field $\phi^Q$ on $(V,E_E)$ defined as follows. We first define the flow $\tilde{Q}$ on $(V,E)$ as

$$\tilde{Q}(x,y) := \begin{cases} Q(x,y) & \text{if } (x,y) \in E, \\ 0 & \text{otherwise}, \end{cases}$$

and then define

$$\phi^Q(x,y) := \tilde{Q}(x,y) - \tilde{Q}(y,x).$$

Conversely given a discrete vector field $\phi$ on a graph $(V,E)$ we search for the flows $Q$ on a digraph $(V,E)$ such that $\phi^Q = \phi$. We start defining the flow $\tilde{Q}^\phi$ on $(V,E_E)$ as

$$\tilde{Q}^\phi(x,y) := \lfloor \phi(x,y) \rfloor_+, \quad \{x,y\} \in E,$$

where $\lfloor \cdot \rfloor_+$ denotes the positive part

$$\lfloor a \rfloor_+ := \begin{cases} a & \text{if } a \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

If $E = E_E$ we can define the flow $Q^\phi$ on $(V,E)$ as $Q^\phi = \tilde{Q}^\phi$. It is easy to see that $Q^\phi$ is the minimal (with respect to the natural partial order structure) flow that is projected to the fixed discrete vector field $\phi$. Any other flow that is projected to $\phi$ can be written as

$$Q^\phi(x,y) + s(\{x,y\}),$$

where $s : E \to \mathbb{R}^+$ is an arbitrary function.

If $E \neq E_E$ the set of flows that are projected to a fixed discrete vector field $\phi$ may be empty. A necessary and sufficient condition for the non emptiness is that $\tilde{Q}^\phi(x,y) = 0$ for any $(x,y) \in E_E \setminus E$. If this condition is verified then any flow $Q$ that is projected to $\phi$ is of the form

$$Q(x,y) = \begin{cases} \tilde{Q}^\phi(x,y) & \text{if } (y,x) \in E_E \setminus E, \\ \tilde{Q}^\phi(x,y) + s(\{x,y\}) & \text{if } (y,x) \in E, \end{cases} \quad (2.4)$$

where $s : E \to \mathbb{R}^+$ is an arbitrary function. In this case we call $Q^\phi$ the flow in (2.4) corresponding to $s = 0$.

The divergence of a discrete vector field $\phi$ is defined by

$$\text{div } \phi(x) := \sum_{y : (x,y) \in E_E} \phi(x,y). \quad (2.5)$$

As before when $|V| = +\infty$ to have that (2.5) is well defined we require

$$\sum_{y : \{x,y\} \in E} |\phi(x,y)| < +\infty.$$
We use the same symbol for the divergence of a flow and a discrete vector field since, when they are well defined, we have
\[
\text{div } Q(x) = \text{div } \phi^Q(x).
\]
A discrete vector field is called of gradient type if there exists a function \( f : V \to \mathbb{R} \) such that \( \phi(x, y) = f(y) - f(x) \). In this case we use the notation \( \phi = \nabla f \).

2.3. A third equivalent statement in Strassen Theorem. We will prove a generalization of the classical Strassen Theorem [15, 21]. We distinguish the two cases when \( V \) is finite or infinite. We start with the finite case for which the statement is easier.

**Theorem 2.1.** Let \((V, \leq)\) be a finite partial order and let \((V, E)\) the acyclic digraph associated to its Hasse diagram. The following statements are equivalent.

1. \( \mu_1 \preceq \mu_2 \),
2. there exists a compatible coupling between \( \mu_1 \) and \( \mu_2 \),
3. there exists a flow \( Q \) on \((V, E)\) such that \( \text{div } Q = \mu_1 - \mu_2 \).

The classical statement of Strassen Theorem claims the equivalence between items (1) and (2). Here we show that there is a third equivalent statement that is (3). Using Strassen Theorem, a direct proof of Theorem 2.1 is simple and can be obtained by a direct application of Farkas Lemma (see for example [19] for the statement of Farkas Lemma). Since the proofs of the equivalence among the different items are interesting and constructive we will prove this Theorem discussing more details than the minimal ones. In particular in subsection 5.1.1 we will show the equivalence between (2) and (3) while in subsection 5.1.2 we will show the equivalence between (1) and (3).

It is important to observe that while statements (1) and (2) are written in terms of the partial order relation \( \leq \), the new statement (3) uses only the Hasse diagram. In this sense it is a statement that captures the essential geometric features of the partial order. In particular statement number (3) establishes that the monotonicity problem is equivalent to a flow on network problem. This is a natural formulation that allows to connect monotonicity results to the geometry of the underlying partial order (that is coded by the digraph \((V, E)\)). In statement number (3) the digraph \((V, E)\) could be substituted by any acyclic digraph having the same transitive closure.

While for statements (1) and (3) it is evident the dependence from the measures \( \mu_1 \) and \( \mu_2 \) only through their difference \( \mu_1 - \mu_2 \) this is not the case for statement number (2). This implies the not so well known fact that if there exists a compatible coupling between two probability measures \( \mu_1 \) and \( \mu_2 \) then there exists a compatible coupling also between all the pairs of measures having the same difference or even more between any pair of probability measures having as difference \( \lambda (\mu_1 - \mu_2) \) where \( \lambda \) is an arbitrary positive number. We will see for example in subsection 6.4 how these invariance properties appear naturally using item (3) of Theorem 2.1.

**Remark 2.2.** It is easy to see that statement (3) of Theorem 2.1 is equivalent to the following one

(3') there exists a time dependent flow \( Q(t) \) with \( t \in [0, 1] \) solving
\[
\begin{align*}
\partial_t \mu(t) + \text{div } Q(t) &= 0, \\
\mu(0) &= \mu_1, \mu(1) = \mu_2.
\end{align*}
\]
A result similar to Theorem 2.1 holds also when $|V| = +\infty$ but we need to introduce more notation. We say that a flow $Q$ on the digraph $(V,E)$ is finitely decomposable if there exists a countable family of finite self avoiding directed paths $\{\gamma_n\}_{n \in \mathbb{N}}$ and a summable sequence $\{q_n\}_{n \in \mathbb{N}}$ of non negative weights (this means $q_n \geq 0$ and $\sum_n q_n < +\infty$) such that

$$Q = \sum_n q_n Q_{\gamma_n}.$$  
(2.6)

In (2.6) we require just a point-wise convergence. If $|V| < +\infty$ then any flow is finitely decomposable since we have for example

$$Q = \sum_{(x,y) \in E} Q(x,y) Q(x,y).$$

A finitely decomposable flow is not necessarily summable since we have

$$\sum_{(x,y) \in E} Q(x,y) = \sum_n q_n |\gamma_n|$$  
(2.7)

and the r.h.s. of (2.7) can be diverging. Note that a finite decomposition (2.6) of a finitely decomposable flow induces naturally a finite positive measure on $\Gamma$ the countable set of all finite self-avoiding paths on $(V,E)$. This is simply

$$\sum_n q_n \delta_{\gamma_n},$$  
(2.8)

where $\delta$ is the delta measure.

Since the paths in (2.6) are self avoiding, every single path $\gamma_n$ may contribute just once to the outgoing or ingoing flux at a single site. This implies that the divergence of a finitely decomposable flow is well defined and coincides with

$$\text{div} \ Q(x) := \sum_{\{n : \gamma_n = x\}} q_n - \sum_{\{n : \gamma_n^+ = x\}} q_n.$$  
(2.9)

If we have a discrete vector field $\phi$ on a graph $(V,E)$ such that the flow $Q^\phi$ on $(V,E)$ is finitely decomposable then $\text{div} \ \phi$ is well defined and

$$\text{div} \ \phi = \text{div} \ Q^\phi.$$  

We can now state our result in the infinite case.

**Theorem 2.3.** Let $(V, \preceq)$ be a countable infinite partial order and let $(V,E)$ be a directed acyclic graph such that its transitive closure $(V, \overline{E})$ induces the partial order $\preceq$. The following statements are equivalent.

1. $\mu_1 \preceq \mu_2$,
2. there exists a compatible coupling between $\mu_1$ and $\mu_2$,
3. there exists a finitely decomposable flow $Q$ on $(V,E)$ such that $\text{div} \ Q = \mu_1 - \mu_2$.

Since to verify if a flow $Q$ is finitely decomposable is in general not easy we state a sufficient and a necessary condition. Let $(V,E)$ be an infinite digraph. An invading sequence of vertices $\{V_n\}_{n \in \mathbb{N}}$ is a sequence of subsets $V_n \subseteq V$ such that $|V_n| < +\infty$, $V_n \subseteq V_{n+1}$ and $\bigcup_n V_n = V$. Given a flow $Q$ we say that it has zero
flux towards infinity (see [4] for the original definition and related results) if there exists an invading sequence of vertices such that

\[
\lim_{n \to +\infty} \left( \sum_{x \in V_n, y \notin V_n} Q(x, y) \right) = 0.
\]

The sum in the above formula is exactly the flux exiting from \(V_n\) and as a part of the definition we require that it is finite for any \(n\). We have the following sufficient condition.

**Proposition 2.4.** Let \(Q\) be an acyclic flow on an infinite digraph \((V,E)\) such that \(\text{div} \ Q = \mu_1 - \mu_2\). If \(Q\) has zero flux towards infinity, then it is finitely decomposable.

We have also the following necessary condition.

**Proposition 2.5.** Let \(Q\) be a flow on an infinite digraph \((V,E)\). If there exists an invading sequence such that

\[
\sup \left\{ Q(x, y) : \{x,y\} \cap \{V \setminus V_n\} \neq \emptyset \right\}
\]

is not converging to 0 when \(n\) diverges, then \(Q\) is not finitely decomposable.

In section 6 we will discuss several examples and applications of the above Theorems 2.1 and 2.3.

### 2.4. Mass transportation

Consider a countable set \(V\) and a cost function \(c : V \times V \to \mathbb{R}^+\). Given two probability measure \(\mu_1\) and \(\mu_2\) on \(V\), the mass transportation problem associated to the cost function \(c\) (see for example [18, 22]) consists in the minimization problem

\[
\inf_{\rho} \left\{ \mathbb{E}_\rho(c) \right\},
\]

where the infimum is over all couplings \(\rho\) between the two probability measures \(\mu_1\) and \(\mu_2\). When the cost function \(c\) is a metric on \(V\) then the mass transportation problem (2.11) induces a metric on the space of probability measures on \(V\). This is called the Monge-Kantorovich metric associated to the cost \(c\).

We consider the cases when the set \(V\) has in addition a geometric structure determined by a digraph \((V,E)\) and a weight function \(w : E \to \mathbb{R}^+\). We then define the cost function \(c\) as

\[
c(x, y) := \inf_{\gamma} \left\{ \sum_{e \in \gamma} w(e) \right\},
\]

where the infimum is over all paths \(\gamma = (x_0, \ldots, x_n)\) such that \(x_0 = x, x_n = y\) and moreover \((x_i, x_{i+1}) \in E\) for any \(i\). We will always restrict to the case of connected digraphs i.e. digraphs for which any pair of vertices is connected by a directed path. It is easy to see that with this geodesic construction we are indeed considering all the cost functions satisfying the triangle like inequality

\[
c(x, z) \leq c(x, y) + c(y, z), \quad \forall x, y, z \in V.
\]

If the weights are also symmetric, i.e. if \(w(x, y) = w(y, x)\), then this symmetry is inherited by the cost function \(c\) that becomes a (pseudo)metric.

In the case of a bounded convex domain of \(\mathbb{R}^d\) with cost function \(c(x, y) = |x - y|\) it is known (see [18] for details) that the Monge-Kantorovich problem is equivalent to a minimal current problem proposed by Beckmann [3] under the name of continuous transportation model. The proof of this equivalence uses some duality
arguments. Here instead we give a constructive proof in the case of a countable set using the correspondence between economic couplings and finitely decomposable flows. In [17] a similar proof has been given in the framework of a general metric space but with the requirement that the measures $\mu_1$ and $\mu_2$ have bounded support. Our proof is less general but removes such a constraint.

Given a flow $Q$ and a weight $w$ on a digraph $(V,E)$ we use the notation

$$\langle Q, w \rangle_E := \sum_{(x,y) \in E} Q(x,y)w(x,y). \quad (2.13)$$

Theorem 2.6. Let $V$ be a finite or countable set and $c$ a cost function defined using a weight function $w$ on a digraph $(V,E)$ by (2.12). Then we have

$$\inf_{\rho} \left\{ \mathbb{E}_\rho(c) \right\} = \inf_{\{Q: \text{div} Q = \mu_1 - \mu_2\}} \left\{ \langle Q, w \rangle_E \right\}. \quad (2.14)$$

The infimum on the l.h.s. of (2.14) is over all couplings between the two fixed probability measures $\mu_1$ and $\mu_2$ while the infimum on the r.h.s. is over the finitely decomposable flows on $(V,E)$.

Remark 2.7. Given a discrete vector field $\phi$ on $E_E$ and $Q$ a flow such that $\phi Q = \phi$ we saw in subsection 2.2 that $Q^\phi \leq Q$ so that $\langle Q^\phi, w \rangle_E \leq \langle Q, w \rangle_E$. This means that there is then a third variational problem that is equivalent to the two in (2.14) that is

$$\inf_{\{\phi: \text{div} \phi = \mu_1 - \mu_2\}} \left\{ \langle Q^\phi, w \rangle_E \right\}. \quad (2.15)$$

The infimum is over all discrete vector fields on $E_E$ such that $Q^\phi$ is well defined and finitely decomposable. The variational problem (2.15) has the form of a minimal current problem.

We will discuss in section 4 conditions to have optimality in the r.h.s. of (2.14) and discuss some applications.

3. Economic couplings and acyclic flows

In this section we illustrate two different algorithmic procedures that are the core of the constructive proofs of Theorems 2.1, 2.3 and 2.6. The result is interesting by itself and ensures that in order to construct a coupling between two probability measures $\mu_1$ and $\mu_2$ it is enough to construct a finitely decomposable acyclic flow having $\mu_1 - \mu_2$ as divergence. We start discussing the finite case and then show how to extend the construction with a limit procedure to the infinite case.

3.1. The finite case. We start considering a finite set $V$ without any additional geometric structure. Consider $\mu_1$ and $\mu_2$ two arbitrary probability measures and a coupling $\rho$ between them. To every coupling $\rho$ we can associate some flows on digraphs having as set of vertices $V$. This is done in the following way. For every pair $(x, y)$ of elements of $V$ such that $x \neq y$ we choose a path $\gamma_{(x,y)} := (x_0, \ldots, x_n)$ going from $x$ to $y$. Since on the set $V$ we did not fix any additional structure the elements $x_i$ with $i \neq 0, n$ can be arbitrary. The flow $Q$ associated to the coupling $\rho$ is then

$$Q = \sum_{\{x,y \in V: x \neq y\}} \rho(x,y)Q_{\gamma_{(x,y)}}. \quad (3.1)$$
Clearly $Q$ depends on the arbitrary choice of the paths and several flows can be obtained choosing different paths. More generally for any pair $(x, y)$ we can fix a family of paths $\gamma^i_{(x,y)}$ going from $x$ to $y$ and corresponding some weights $\rho^i(x, y)$ such that $\sum_i \rho^i(x, y) = \rho(x, y)$. We can then generalize (3.1) to

$$Q = \sum_{\{x,y\in V : x \neq y\}} \sum_i \rho^i(x, y)Q_{\gamma^i_{(x,y)}}. \tag{3.2}$$

Any flow $Q$ obtained by (3.2) is such that $\text{div} \; Q = \mu_1 - \mu_2$.

We call a coupling $\rho$ an economic coupling if there exists a choice of the paths in (3.2) such that the associated digraph $(V,E(Q))$ is acyclic. Recall that with $E(Q)$ we denote the set of edges $(x, y)$ such that $Q(x, y) > 0$. When $V$ has not any additional structure, to check whether a coupling is an economic one it is enough to consider just the elementary paths $\gamma^i_{(x,y)} = (x, y)$ for any $(x, y)$ and $i$.

Lemma 3.1. If the paths $\gamma^i_{(x,y)}$ have not constraints apart the starting and final vertices then a coupling $\rho$ is economic if and only if the flow

$$\sum_{\{x,y\in V : x \neq y\}} \rho(x, y)Q_{(x,y)} \tag{3.3}$$

is acyclic.

Proof. If the flow (3.3) is acyclic then the coupling is clearly economic and we need to prove just the converse statement. Consider an economic coupling $\rho$. This means that there exists a choice of the paths $\gamma^i_{(x,y)}$ and of the weights $\rho^i(x, y)$ such that the flow (3.2) is acyclic. We want to show that necessarily also the flow (3.3) is acyclic. Let us suppose by contradiction that (3.3) is not acyclic. This means that there exists a cycle $(x_0, \ldots, x_n)$ with $x_n = x_0$ and such that $\rho(x_i, x_{i+1}) > 0$. Then for any $i = 0, \ldots, n-1$ there exists an $m(i)$ such that $\rho^{m(i)}(x_i, x_{i+1}) > 0$. This implies that the concatenation of the paths

$$\gamma^{m(0)}_{(x_0,x_1)} \ast \gamma^{m(1)}_{(x_1,x_2)} \ast \cdots \ast \gamma^{m(n-1)}_{(x_{n-1},x_n)}$$

will constitute a cycle in the digraph $(V,E(Q))$ with $Q$ as in (3.2). This is a contradiction.

The motivation of the name economic is quite intuitive and is the following. The flows $Q$ of the form (3.2) identify the amount of mass that it is necessary to move along the different edges in the coupling $\rho$ to transform the measure $\mu_1$ into the measure $\mu_2$. If a coupling is not economic then it is possible to construct easily a new coupling having a minor amount of mass flowing. We will obtain this showing that it is possible to construct a different coupling $\tilde{\rho}$ such that $\tilde{\rho}(x, y) \leq \rho(x, y)$ for any $x \neq y$.

Lemma 3.2. Let $\rho$ be a coupling between the probability measures $\mu_1$ and $\mu_2$ such that the flow (3.3) is not acyclic. Then there exists a coupling $\tilde{\rho}$ between the same probability measures such that $\tilde{\rho}(x, y) \leq \rho(x, y)$ for any $x \neq y$ and the inequality is strict for at least one pair $(x, y)$.

Proof. Consider the flow $Q$ as in (3.3). If the digraph $(V,E(Q))$ is not acyclic this means that there exists a cycle $C = (x_0, \ldots, x_n)$ with $x_n = x_0$ such that $m := \min_i Q(x_i, x_{i+1}) > 0$. We can then consider a new flow defined by $Q - mQ_C$. The motivation of the name economic is quite intuitive and is the following. The flows $Q$ of the form (3.2) identify the amount of mass that it is necessary to move along the different edges in the coupling $\rho$ to transform the measure $\mu_1$ into the measure $\mu_2$. If a coupling is not economic then it is possible to construct easily a new coupling having a minor amount of mass flowing. We will obtain this showing that it is possible to construct a different coupling $\tilde{\rho}$ such that $\tilde{\rho}(x, y) \leq \rho(x, y)$ for any $x \neq y$.
and iterate a finite number of times this procedure up to obtain an acyclic flow \( \tilde{\rho} \) such that \( (V, E(\tilde{\rho})) \) is acyclic. Starting from \( \tilde{\rho} \) we can define

\[
\tilde{\rho}(x, y) := \begin{cases} 
\tilde{Q}(x, y) & \text{if } x \neq y, \\
\mu_1(x) - \sum_{z \neq x} \tilde{Q}(x, z) & \text{if } x = y.
\end{cases}
\]

Using \( \text{div} \tilde{Q} = \mu_1 - \mu_2 \) it is easy to see that \( \tilde{\rho} \) is a coupling between \( \mu_1 \) and \( \mu_2 \). The condition \( \tilde{\rho}(x, y) \leq \rho(x, y) \) for any \( x \neq y \) is clearly satisfied and the strict inequality holds in some cases. 

Given an economic coupling \( \rho \) between \( \mu_1 \) and \( \mu_2 \) we can associate to it \( \mathcal{F}[\rho] \) that is a subset of all the possible acyclic flows having divergence coinciding with \( \mu_1 - \mu_2 \). The subset \( \mathcal{F}[\rho] \) is obtained by (3.2) for all the possible choices of the paths and weights that give as a result an acyclic flow. What is remarkable, and is the main result of this section, is that also a converse procedure holds.

**Theorem 3.3.** Given an acyclic flow \( Q \) such that \( \text{div} Q = \mu_1 - \mu_2 \) then there exists a coupling \( \rho \) between \( \mu_1 \) and \( \mu_2 \) such that \( Q \) is obtained from \( \rho \) by (3.2) for some choice of the paths and the weights.

The proof of Theorem 3.3 is constructive and shows how to obtain explicitly such a coupling starting from the flow. We will show an algorithmic iterative procedure that identify \( \mathcal{F}^{-1}[Q] \) the subset of the probability measures on \( V \times V \) for which it is possible to get \( Q \) by (3.2) for some choices of the paths and the weights. An element of \( \mathcal{F}^{-1}[Q] \) is a coupling between two probability measures whose difference coincides with the divergence of \( Q \).

In the iterative procedure of the proof there will be several arbitrary choices in the subdivision of the mass. It could be shown that performing all the different choices it is possible to cover completely \( \mathcal{F}^{-1}[Q] \).

**Proof of Theorem 3.3.** We start with a flow \( Q \) such that \( \text{div} Q = \mu_1 - \mu_2 \) and such that \( (V, E(Q)) \) is an acyclic digraph. By Proposition 1.4.2 of [1] in any finite acyclic digraph there exists at least one vertex \( x^1 \) such that \( (y, x^1) \notin E(Q) \) for any \( y \in V \). Let us associate to every element \( x \in V \) a vector \( \eta^x = \{\eta^x_y\}_{y \in V} \in [0, 1]^V \). We will say that at site \( x \) there is \( \eta^x_y \) mass of type \( y \). Initially we fix the values of such vectors as \( \eta^x_y = \mu_1(x)\delta_x(y) \) where \( \delta_x \) is the point–mass at \( x \). We transport mass along the oriented edges according to the following iterative procedure. Consider the vertex \( x^1 \) and let flow the amount \( Q(x^1, y) \) of mass of type \( x^1 \) through any \((x^1, y) \in E(Q)\). This is possible since

\[
\sum_{y \in E} Q(x^1, y) = \text{div} Q(x^1) = \mu_1(x^1) - \mu_2(x^1) \leq \eta^{x^1}.
\]

After this operation we need to upgrade the vectors \( \eta \) into \( \eta(1) \) according to the rule

\[
\begin{align*}
\eta^{x^1}(1) &= \eta^{x^1} - \sum_y Q(x^1, y)\delta_{x^1} = \mu_2(x^1)\delta_{x^1}, \\
\eta^{y}(1) &= \eta^{y} + Q(x^1, y)\delta_{x^1}, \quad \text{if } y \neq x^1.
\end{align*}
\]

We also upgrade the original flow \( Q \) into a new flow \( Q_1 \) defined by

\[
Q_1(y, z) = \begin{cases} 
Q(y, z) & \text{if } y \neq x^1, \\
0 & \text{if } y = x^1.
\end{cases}
\]
Finally we upgrade the original acyclic digraph into the new acyclic digraph \((V \setminus \{x^1\}, E(Q_1))\).

Still applying Proposition 1.4.2 of \([1]\) we deduce that there exists a vertex \(x^2\) such that \(Q_1(y, x^2) = 0\) for any \(y\). We now transport some mass present at \(x^2\) through the edges \((x^2, y)\) such that \(Q_1(x^2, y) > 0\). By construction it holds

\[
\sum_{y \in V} \eta^x_2(1) = \mu_2(x^2) + \sum_{y} Q_1(x^2, y) \geq \sum_{y} Q_1(x^2, y).
\]

Consequently we can select nonnegative numbers \(\Delta_x(y)\) such that \(\sum_{y} \Delta_x(y) \leq \eta^x_2(1)\) for any \(x\) and \(\sum_{x} \Delta_x(y) = Q_1(x^2, y)\). For any \(x \in V\) we let flow through the edge \((x^2, y)\) an amount \(\Delta_x(y)\) of mass of type \(x\) present at \(x^2\) and consequently we transform the vectors \(\eta(1)\) into \(\eta(2)\) defined by

\[
\eta^x_2(2) = \begin{cases} 
\eta^x_2(1) - \sum_{x} \Delta_x(x), & \text{if } x = x^2, \\
\eta^x_2(1) + \Delta_x(x), & \text{if } x \neq x^2.
\end{cases}
\]

By construction we have \(\sum_{x} \eta^x_2(2) = \mu_2(x^2)\) and the vector of masses at \(x^2\) will not be modified anymore i.e. \(\eta^x(i) = \eta^x_2(2)\) for any \(i \geq 2\). As before we update also the flow and the digraph. We can now iterate this procedure up to when mass has flown across any edge in \(E(Q)\). After a finite number, \(l\), of iterations, the final configuration \(\eta(l)\) will be such that

\[
\sum_{x} \eta^x_2(l) = \mu_2(x), \quad \forall x \in V.
\]

We define now

\[
\rho(x, y) := \eta^y_2(l),
\]

and claim that it is a coupling between \(\mu_1\) and \(\mu_2\) such that it holds \([3.2]\) for some choice of the paths and weights.

First we show that \(\rho\) is a coupling between \(\mu_1\) and \(\mu_2\). By \([3.3]\) we have that the second marginal of the measure \(\rho\) coincides with \(\mu_2\). The validity of \(\sum_{x} \eta^x_2(l) = \mu_1(x)\) follows by the fact that initially there is an amount \(\mu_1(x)\) of mass of type \(x\) and all the flow construction preserves mass so that the same amount is present in the final configuration.

To show that it holds a formula like \([3.2]\) we should identify the paths and the weights from the algorithmic construction. Instead of giving a formal proof we outline a geometric construction. We represent the initial vectors \(\eta^x\) with segments of different colors having length respectively \(\mu_1(x)\) and associated to the corresponding vertices. In the first iteration we cut from the segment of color \(x^1\) some segments of length \(Q(x^1, y)\) and move them on the sites with label \(y\) respectively. In the iteration number \(i\) at site \(x^i\) there will be different segments with different colors and lengths (there can be more than one segment with the same color that followed different paths). We cut pieces of such segments and move them according to the rules outlined above. In the final configuration \(\eta(l)\) there will be several segments of several colors on each site. With \(\rho(x, y)^i\) we denote the lengths of the segments of color \(x\) present at site \(y\) in \(\eta(l)\). Every such segment followed a path from \(x\) to \(y\) that we call \(\gamma^i_{(x, y)}\). These are exactly the ingredients to prove \([3.2]\). \(\square\)

Note that the algorithmic procedure of the proof depends on the measures \(\mu_1\) and \(\mu_2\) just through their difference \(\mu_1 - \mu_2\). Choosing differently the initial vectors \(\eta\) you
can get, with the same construction, a coupling between two different probability measures having the same difference.

Remark 3.4. Note that \((V, E(Q))\) is an acyclic digraph that induces a partial order \(\leq\). Since the mass is flowing only through the edges in \(E(Q)\) it follows immediately that the coupling \(\rho\) is compatible with respect to the partial order \(\leq\). This fact holds also for the coupling constructed using the algorithm in Theorem 3.6.

To show the second algorithmic construction we start with a Lemma. Define \(V_- := \{x \in V : \mu_1(x) > \mu_2(x)\}\) and \(V_+ := \{x \in V : \mu_2(x) > \mu_1(x)\}\).

**Lemma 3.5.** Let \((V, E)\) be a finite digraph. Any acyclic flow \(Q\) such that \(\text{div} Q = \mu_1 - \mu_2\) has a decomposition \((3.6)\) such that for any subset \(S \subseteq V\) it holds

\[
\sum_{\{n: \gamma_n \subseteq S\}} q_n \leq \min \left\{ \sum_{\{x \in S \cap V_+\}} (\mu_1(x) - \mu_2(x)), \sum_{\{x \in S \cap V_-\}} (\mu_2(x) - \mu_1(x)) \right\}. \tag{3.5}
\]

Moreover when \(S = V\) it holds

\[
\sum_n q_n = \frac{1}{2} \sum_x \left| \mu_1(x) - \mu_2(x) \right|. \tag{3.6}
\]

**Proof.** First of all we show that it is possible to decompose the flow in such a way that \(\gamma_n^- \in V_-\) and \(\gamma_n^+ \in V_+\) for any \(n\). Consider any finite decomposition of \(Q\) and suppose that for example there exists a site \(x \in V_-\) and a \(n^*\) such that \(\gamma_n^* = x\). Since by definition \(\mu_1(x) > \mu_2(x)\) there exist necessarily some paths \(\{\gamma_n\}_{n \in \mathcal{N}}\) of the decomposition such that \(\gamma_n^- = x\) for any \(n \in \mathcal{N}\) and moreover \(\sum_{n \in \mathcal{N}} q_n > q_{n^*}\). We can then find some weights \(\{q'_n\}_{n \in \mathcal{N}}\) such that \(\sum_{n \in \mathcal{N}} q'_n = q_{n^*}\) and \(q'_n \leq q_n\). With these weights we construct the new decomposition

\[
\sum_{n \in \mathcal{N}} \left[ q'_n Q_{\gamma_n^* \gamma_n} + (q_n - q'_n) Q_{\gamma_n} \right] + \sum_{\{n \in \mathcal{N} \cup n^*\}} q_n Q_{\gamma_n}. \tag{3.7}
\]

Since the flow is acyclic the paths obtained by concatenation are still self avoiding. Performing a finite number of times this procedure the final decomposition will have the required property.

Consider now a decomposition such that \(\gamma_n^- \in V_-\) and \(\gamma_n^+ \in V_+\) for any \(n\). This condition immediately implies that

\[
\sum_{\{n: \gamma_n^- = x\}} q_n = \mu_1(x) - \mu_2(x), \quad x \in V_-. \tag{3.8}
\]

We then have

\[
\sum_{\{n: \gamma_n^- \subseteq S\}} q_n \leq \sum_{\{x \in S \cap V_+\}} \sum_{\{n: \gamma_n^- = x\}} q_n = \sum_{\{x \in S \cap V_-\}} \left[ \mu_1(x) - \mu_2(x) \right]. \tag{3.9}
\]

A similar argument using the set \(V_+\) allows to prove \((3.6)\). When \(S = V\) the first inequality in \((3.9)\) is an equality and this concludes the proof. \(\square\)

Using the previous Lemma we can obtain a refinement of Theorem 3.3 with a different proof. The idea is to construct a finite decomposition to which it is associated by \((2.8)\) a positive measure with total mass less or equal to one on \(\Gamma\), the countable set of finite self avoiding paths.
Theorem 3.6. Let \((V, E)\) be a finite digraph. Given \(Q\) an acyclic flow such that \(\text{div} Q = \mu_1 - \mu_2\) then there exists a coupling \(\rho\) between \(\mu_1\) and \(\mu_2\) such that \(Q\) is obtained from \(\rho\) by (3.2). Moreover it holds

\[
\sum_{x \neq y} \rho(x, y) = \frac{1}{2} \sum_x |\mu_1(x) - \mu_2(x)|. \tag{3.10}
\]

Proof. The proof is constructive and uses the decomposition of Lemma 3.5. In particular we call \(q_n\) and \(\gamma_n\) respectively the weights and the corresponding paths constructed with the algorithmic procedure of Lemma 3.5. With this special decomposition we can construct the coupling immediately. We can indeed define

\[
\rho(x, y) := \begin{cases} 
\mu_1(x) = \mu_2(x) & \text{if } x = y \notin \{V_- \cup V_+\}, \\
\mu_2(x) & \text{if } y = x \in V_-, \\
\mu_1(x) & \text{if } y = x \in V_+, \\
\sum_{\gamma_n} q_n & \text{if } x \neq y.
\end{cases} \tag{3.11}
\]

Using (3.8) and the analogous formula valid for \(V_+\), it is easy to verify that \(\rho\) defined in (3.11) is a coupling between \(\mu_1\) and \(\mu_2\). Equation (3.10) follows immediately by (3.0). Finally the decomposition (3.2) coincides with the decomposition \(\sum_n q_n Q_{\gamma_n}\). □

We can add a geometric structure to the set \(V\) considering a digraph \((V, E)\). This digraph can be for example either the Hasse diagram of a partial order on \(V\) or a digraph associated to edges with finite weight on a mass transportation problem. In this case the definition of the class of the economic couplings is given as before with the only modification that the paths \(\gamma\) have to be paths on the digraph \((V, E)\).

Since the class of paths is restricted the class of the couplings that are economic will be reduced.

3.2. The infinite case. When \(|V| = +\infty\) we have results similar to the ones in the previous subsection provided that the flow is finitely decomposable. Next Theorem 3.7 is weaker that Theorem 3.6 but it can be applied to both the finite algorithmic constructions of the previous subsection. Moreover its proof contains some constructions useful in the following.

Theorem 3.7. Let \((V, E)\) be an infinite digraph and let \(Q\) be a finitely decomposable acyclic flow such that \(\text{div} Q = \mu_1 - \mu_2\). Then there exists a coupling \(\rho\) between \(\mu_1\) and \(\mu_2\) such that \(\rho(x, y) = 0\) if there exist no paths form \(x\) to \(y\) in the digraph \((V, E(Q))\).

Proof. Let \(Q\) be an acyclic finitely decomposable flow such that \(\text{div} Q = \mu_1 - \mu_2\). Let \(V_n\) be an invading sequence of vertices such that \(\cup_{k \leq n} \gamma_k \subseteq V_n\) where the paths \(\gamma_k\) are the one involved in the finite decomposition (2.0) of \(Q\).

For each \(n\) we consider the finite digraph having vertices \(V_n \cup g\) where \(g\) is a ghost site. The set of edges \(E_n\) contains all edges \((x, y) \in E\) such that \(x, y \in V_n\), moreover it contains the edges \((g, z)\) or \((z, g)\) with \(z \in V_n\) if respectively there exists an \((x, z) \in E\) such that \(x \notin V_n\) or there exists an \((z, y) \in E\) such that \(y \notin V_n\).

Starting from the flow \(Q\) on \((V, E)\) we associate to it a flow \(Q_n\) on \((V_n \cup g, E_n)\) as
follows
\[ Q_n(x, y) := \begin{cases} 
Q(x, y) & \text{if } x, y \in V_n, \\
\sum_{z \in V_n} Q(x, z) & \text{if } y = g, x \in V_n, \\
\sum_{z \in V_n} Q(z, y) & \text{if } x = g, y \in V_n. 
\end{cases} \tag{3.12} \]

The sums appearing in (3.12) are finite since \( Q \) has a well defined divergence. Clearly it holds
\[
\text{div } Q_n(x) = \begin{cases} 
\text{div } Q(x) = \mu_1(x) - \mu_2(x), & \text{if } x \in V_n, \\
\sum_{y \not\in V_n} (\mu_1(y) - \mu_2(y)), & \text{if } x = g
\end{cases}
\]

In general the flow \( Q_n \) will not be acyclic but since the digraph \( (V_n \cup g, E_n) \) is finite after a finite number of deletions of cycles we can obtain an acyclic flow \( Q^*_n \) having the same divergence of \( Q_n \). We need to analyze more carefully this deletion procedure. Consider a path \( \gamma_k \) of the finite decomposition of \( Q \) such that it exits from and enters in \( V_n \) several times. After the identification of all the sites outside \( V_n \) with the single ghost site \( g \) the path will not be anymore self-avoiding (see Figure 2). If we remove the cycles that have been created (all of which will contain the ghost site) the self-avoiding path that we obtain will exit from \( V_n \) or enter in \( V_n \) at most once. To construct \( Q^*_n \) we first consider every single path of the cyclic decomposition and remove the cycles going through the ghost site. After this deletion procedure the flow is still not necessarily acyclic. We remove the remaining cycles in an arbitrary way up to obtain the acyclic flow \( Q^*_n \). To the values of the flow \( Q^*_n \) on edges entering or exiting from the ghost site \( g \) we can contribute only the paths \( \gamma_k \) with \( k > n \). The deletion procedure outlined above guarantees that every single path \( \gamma_k \) with \( k > n \) may contribute no more than once to the total flux entering in \( g \) (that is \( \sum_{x \in V_n} Q^*_n(x, g) \)) or to the total flux exiting from \( g \) (that is \( \sum_{x \in V_n} Q^*_n(g, x) \)). This means that we have the bounds
\[
\begin{align*}
\sum_{x \in V_n} Q^*_n(x, g) & \leq \sum_{k>n} q_k, \\
\sum_{x \in V_n} Q^*_n(g, x) & \leq \sum_{k>n} q_k.
\end{align*} \tag{3.13}
\]

Let us now consider the flow \( \bar{Q}^*_n \) such that \( \bar{Q}^*_n(x, y) = Q^*_n(x, y) \) when both \( x \) and \( y \) belong to \( V_n \) and \( \bar{Q}^*_n(x, y) = 0 \) otherwise. The flow \( \bar{Q}^*_n \) can be naturally interpreted
as a flow on the original digraph \((V, E)\) and clearly it holds \(\tilde{Q}^*_n \leq Q\). We have also
\[
\text{div } \tilde{Q}^*_n(x) = \mu_1(x) - \mu_2(x) + \delta_n(x), \quad x \in V_n, 
\]
with
\[
\sum_x |\delta_n(x)| \leq 2 \sum_{k>n} q_k. \tag{3.15}
\]

We define the following sequences of positive measures on \(V\)
\[
\bar{\mu}_1^{(n)}(x) := \begin{cases} 
\mu_1(x) + \delta_n(x) & \text{if } x \in V_n, \delta_n(x) > 0, \\
\mu_1(x) & \text{if } x \in V_n, \delta_n(x) \leq 0, \\
0 & \text{if } x \not\in V_n,
\end{cases} \tag{3.16}
\]
\[
\bar{\mu}_2^{(n)}(x) := \begin{cases} 
\mu_2(x) - \delta_n(x) & \text{if } x \in V_n, \delta_n(x) < 0, \\
\mu_2(x) & \text{if } x \in V_n, \delta_n(x) \geq 0, \\
0 & \text{if } x \not\in V_n.
\end{cases} \tag{3.17}
\]

We have \(\sum_x (\bar{\mu}_1^{(n)}(x) - \bar{\mu}_2^{(n)}(x)) = 0\) and \(\text{div } \tilde{Q}^*_n = \bar{\mu}_1^{(n)} - \bar{\mu}_2^{(n)}\). We can then apply the algorithmic constructions of Theorems 3.3 or 3.6 (that work also for positive measures and not just for probability measures) obtaining from the acyclic flow \(\tilde{Q}^*_n\) a measure \(\rho^{(n)}\) on \(V \times V\) such that \(\sum_x \rho^{(n)}(x, y) = \bar{\mu}_2^{(n)}(y)\) and \(\sum_y \rho^{(n)}(x, y) = \bar{\mu}_1^{(n)}(x)\). Since the mass is transported along edges of the digraph \((V, E(Q))\) we deduce that \(\rho^{(n)}(x, y) = 0\) when \(y\) cannot be reached from \(x\) with a path on \((V, E(Q))\). We show that the sequence of positive measures \(\rho^{(n)}\) has total mass uniformly bounded and is tight. The bound on the mass follows by
\[
\sum_x \sum_y \rho^{(n)}(x, y) = \sum_x \bar{\rho}^{(n)}(x) \leq \sum_x \mu_1(x) + |\delta_n(x)| \leq 1 + 2 \sum_{k=1}^{+\infty} q_k.
\]
The tightness follows by the following argument. Fix an arbitrary \(\varepsilon > 0\) and let \(m^*\) be an integer number such that
\[
\max \{\mu_1(V^c_{m^*}), \mu_2(V^c_{m^*})\} < \varepsilon, \tag{3.18}
\]
where the upper index \(c\) denotes the complementary set. Fix also \(n^*\) such that \(2 \sum_{k=n^*}^{+\infty} q_k < \varepsilon\). Then we have for any \(n > n^*\) and \(m > m^*\)
\[
\rho^{(n)}((V_m \times V_m)^c) = \rho^{(n)}(V_m^c \times V_m) + \rho^{(n)}(V_m \times V_m^c) + \rho^{(n)}(V_m^c \times V_m^c)
\]
\[
\leq \rho^{(n)}(V \times V_m^c) + \rho^{(n)}(V_m^c \times V_m) = \bar{\mu}_2^{(n)}(V_m^c) + \bar{\mu}_1^{(n)}(V_m^c)
\]
\[
\leq \mu_2(V_m^c) + \mu_1(V_m^c) + \sum_x |\delta_n(x)| \leq 3\varepsilon.
\]
This clearly implies the tightness of the sequence.

By Prokhorov Theorem for positive measures there exists a subsequence, that we still call \(\rho^{(n)}\), that is weakly convergent. Let us call \(\rho\) its weak limit. Since by (3.15), (3.16) and (3.17) \(\bar{\mu}_i^{(n)}(x) \to \mu_i(x)\) for any \(x\) we immediately obtain
\[
\sum_y \rho(x, y) = \lim_{n \to +\infty} \sum_y \rho^{(n)}(x, y) = \lim_{n \to +\infty} \bar{\mu}_1^{(n)}(x) = \mu_1(x),
\]
so that \(\rho\) is a coupling between \(\mu_1\) and \(\mu_2\). Since \(\rho^{(n)}(x, y) = 0\) for any pairs \((x, y)\) such that there exists not a path from \(x\) to \(y\) in \((V, E(Q))\), this will be true also for the limiting measure \(\rho\). □
Remark 3.8. Theorem 3.7 is still true if the flow \( Q \) is finitely decomposable but not acyclic. This can be easily checked going through its proof. The difference with respect to the acyclic case is that one cannot hope to recover \( Q \) in the limit from the growing sequence of finite acyclic flows \( Q^*_n \).

**Theorem 3.9.** Let \((V, E)\) be an infinite digraph and let \( Q \) be a finitely decomposable acyclic flow such that \( \text{div} \, Q = \mu_1 - \mu_2 \). Then there exists a coupling \( p \) between \( \mu_1 \) and \( \mu_2 \) and such that (3.7) holds and moreover \( \sum_{x \neq y} \rho(x, y) = \frac{1}{2} \sum_x |\mu_1(x) - \mu_2(x)| \).

**Proof.** Recalling the proof of Theorem 3.6 we need just to show that there exists a finite decomposition of \( Q \) such that \( \gamma_n^- \in V_- \) and \( \gamma_n^+ \in V_+ \) for any \( n \).

We construct inductively a sequence \( M^{(n)} \) of positive measures with finite support on \( \Gamma \) as follows. We call a collection of self-avoiding paths *stable* if there are no sites \( x \in V \) such that at \( x \) there are both some paths of the collection starting and some paths ending. This means that if a collection of paths is stable and there is a path starting at \( x \) then every path of the collection cannot end at \( x \). Likewise if there is a path ending at \( x \) then every path of the collection cannot start at \( x \). We define \( M^{(1)} := q_1 \delta_{\gamma_1} \). Given \( M^{(n)} \) we define inductively \( M^{(n+1)} \) as \( M^{(n+1)} := M^{(n)} + q_{n+1} \delta_{\gamma_{n+1}} \). If the support of \( M^{(n+1)} \) is not stable then after a finite number of transformations like (3.7) we obtain a positive measure \( M^{(n+1)} \) whose support is stable. It is enough to apply the transformations (3.7) twice and in correspondence of \( \gamma_{n+1}^+ \), the starting and ending points of the new path added.

This is because by definition the stability constraints are satisfied by the paths in the support of \( M^{(n)} \). Analyzing all the possible configurations it is possible to show that

\[
\sum_{\gamma \in \Gamma} \left| M^{(n)}(\gamma) - M^{(n+1)}(\gamma) \right| \leq 6q_{n+1}.
\]

(3.18)

This means that for \( n > m \) we have

\[
\sum_{\gamma \in \Gamma} \left| M^{(n)}(\gamma) - M^{(m)}(\gamma) \right| \leq 6 \sum_{k=n+1}^m q_k,
\]

(3.19)

and since \( q_n \) is a summable sequence we deduce that the sequence \( \{M^{(n)}\} \) is a Cauchy sequence on \( L^1(\Gamma) \). Since \( L^1(\Gamma) \) is complete we obtain that there exists \( M \in L^1(\Gamma) \) such that

\[
\lim_{n \to +\infty} \sum_{\gamma \in \Gamma} \left| M^{(n)}(\gamma) - M(\gamma) \right| = 0.
\]

(3.20)

We end the proof showing that

\[
\sum_{\gamma} M(\gamma)Q_\gamma
\]

(3.21)

is the required finite decomposition of \( Q \). We have that (3.21) is a finite decomposition of \( Q \) since

\[
\sum_{\gamma : \gamma \ni (x, y)} M(\gamma) = \lim_{n \to +\infty} \sum_{\gamma : \gamma \ni (x, y)} M^{(n)}(\gamma) = \sum_{k : \gamma_k \ni (x, y)} q_k = Q(x, y).
\]

Consider now for example \( x \in V_- \) and let \( \gamma \) be a path such that \( \gamma^+ = x \). We have that \( \sum_{\gamma} M^{(n)}(\gamma)Q_\gamma \) is a finite decomposition of the flow \( Q^{(n)} := \sum_{k=1}^n q_kQ_{\gamma_k} \) so that there exists a \( n^* \) for which for any \( n > n^* \) we have \( \text{div} \, Q^{(n)}(x) > 0 \). Since the
support of $M^{(n)}$ is a stable set of paths we have $M^{(n)}(\tilde{\gamma}) = 0$ for any $n > n^*$ and consequently also $M(\tilde{\gamma}) = \lim_{n \to +\infty} M^{(n)}(\tilde{\gamma}) = 0$. The other cases can be proved similarly.

\[ \square \]

4. Proof of Propositions 2.4 and 2.5

In this section we give the proofs of the auxiliary Propositions 2.4 and 2.5. These are useful to individuate finitely decomposable flows.

Proof of Proposition 2.4. Consider the invading sequence $V_n$ such that the outgoing flux towards infinity $\sum_{x \in V_n, y \notin V_n} Q(x, y)$ is converging to zero when $n$ diverges. Since

\[ \sum_{x \in V_n, y \notin V_n} Q(x, y) - \sum_{x \in V_n, y \in V_n} Q(x, y) = \mu_1(V_n) - \mu_2(V_n), \quad (4.1) \]

then also the incoming flux from infinity $\sum_{x \notin V_n, y \in V_n} Q(x, y)$ is converging to zero when $n$ diverges. All the series in (4.1) are convergent since $|V_n| < +\infty$ and the series appearing in the definition of $\text{div} \, Q$ (2.2) are supposed to be summable.

For each $n$ we consider the finite digraph having vertices $V_n \cup g_- \cup g_+$ where $g_\pm$ are ghost sites and having edges $E_n$. The set of edges $E_n$ contains all edges $(x, y) \in E$ such that $x \in V_n$, moreover it contains edges of type $(g_-, z)$ or $(z, g_+)$ with $z \in V_n$ if respectively there exists an $(x, z) \in E$ such that $x \notin V_n$ or there exists an $(z, y) \in E$ such that $y \notin V_n$. Starting from the flow $Q$ on $(V, E)$ we associate to it a flow $Q_n$ on $(V_n \cup g_- \cup g_+, E_n)$ as follows

\[ Q_n(x, y) := \begin{cases} Q(x, y) & \text{if } x, y \in V_n, \\ \sum_{z \in V_n} Q(x, z) & \text{if } y = g_+, x \in V_n, \\ \sum_{z \in V_n} Q(z, y) & \text{if } x = g_-, y \in V_n, \end{cases} \]

and $Q_n(x, y) = 0$ in the remaining cases. Clearly it holds

\[ \text{div} \, Q_n(x) = \text{div} \, Q(x) = \mu_1(x) - \mu_2(x), \quad x \in V_n, \]

and moreover since $Q$ is acyclic then also $Q_n$ is acyclic. Since $Q_n$ is an acyclic finite flow it has a finite decomposition like in Lemma 3.3 and we have

\[ Q_n = \sum_m q^{(n)}_m Q_{\gamma^{(n)}_m}, \quad (4.2) \]

for suitable weights $q^{(n)}_m$ and paths $\gamma^{(n)}_m$. The paths $\gamma^{(n)}_m$ are self–avoiding paths on the digraph $(V_n \cup g_- \cup g_+, E_n)$ but to every path $\gamma$ on this digraph it can be easily associated a self–avoiding path $\tilde{\gamma}$ on the original digraph $(V, E)$. This is done simply transforming any edge $(g_-, x) \in \gamma$ into an arbitrary edge $(y, x) \in E(Q)$ and any edge $(x, g_+) \in \gamma$ into an arbitrary edge $(x, y) \in E(Q)$. After this identification we obtain an acyclic finitely decomposable flow on $(V, E)$

\[ \tilde{Q}_n := \sum_m q^{(n)}_m Q_{\tilde{\gamma}^{(n)}_m}. \quad (4.3) \]

By construction it holds

\[ \tilde{Q}_n(x, y) = Q(x, y), \quad (4.4) \]

for any $n$ big enough so that $x, y \in V_n$. 
Recall that $\Gamma$ is the countable set of all finite self-avoiding paths on the digraph $(V,E)$. Let also $\Gamma_n \subseteq \Gamma$ be the subset of all the paths $\gamma \subseteq V_n$. To the decomposition (4.3) we associate by (2.8) a positive and finite measure on $\Gamma$ given by

$$M^{(n)} := \sum_m q_m^{(n)} \delta_{\gamma_m}.$$  

(4.5)

The sequence of measures $\{M^{(n)}\}_{n \in \mathbb{N}}$ is a sequence of positive and finite measures on $\Gamma$. We now show that this sequence of measures is tight and has total mass uniformly bounded. Recall that the coefficients $q_m^{(n)}$ in (4.5) are the same of (4.2) so that they satisfy the properties stated in Lemma 3.5. By (3.6) we have

$$\sum_{\gamma \in \Gamma} M^{(n)}(\gamma) = \sum_m q_m^{(n)} \leq 1,$$

and this guarantees that the total mass of $\{M^{(n)}\}_{n \in \mathbb{N}}$ is uniformly bounded. Moreover we have

$$M^{(n)}(\Gamma \setminus \Gamma_k) = \sum_{\gamma : \hat{\gamma}^{(n)}_m \subseteq V \setminus V_k} q_m^{(n)} + \sum_{\gamma : \hat{\gamma}^{(n)}_m \cap V_k \neq \emptyset, \hat{\gamma}^{(n)}_m \cap (V \setminus V_k) \neq \emptyset} q_m^{(n)}$$

$$\leq \sum_{x \notin V_k} |\mu_1(x) - \mu_2(x)| + \sum_{\{x \in V_k, y \notin V_k\}} Q(x,y) + \sum_{\{x \in V_k, y \in V_k\}} Q(x,y).$$

(4.6)

Using the condition of zero flux towards infinity and (4.1) we have that all the terms in the r.h.s. of (4.6) converge to zero when $k \to +\infty$. Since the estimate in (4.6) is uniform in $n$ we deduce the tightness of the sequence of measures $\{M^{(n)}\}_{n \in \mathbb{N}}$. By Prokhorov Theorem for positive finite measures, we can then extract a weak converging subsequence that we still call $M^{(n)}$ and call $M := \sum_m q_m \delta_{\gamma_m}$ its limit. This is a finite and positive measure on $\Gamma$. The function that associate to any path $\gamma$ the value 1 if $(x, y) \in \gamma$ and zero otherwise is continuous and bounded on $\Gamma$ endowed of the discrete topology. By (4.1) we deduce that if we construct the flow $Q := \sum_m q_m Q_{\gamma_m}$ then we have that $Q(x,y) = \lim_{n \to +\infty} Q_n(x,y) = Q(x,y)$. This is the required finite decomposition of the flow $Q$.

Proof of Proposition 2.5. Let us suppose by contradiction that $\sum_n q_n Q_{\gamma_n}$ is a finite decomposition of $Q$ and that (2.10) does not converge to zero for an invading sequence. This means that there exists an $\varepsilon$ and an infinite sequence of edges $\{e_i\}_{i \in \mathbb{N}}$ such that $Q(e_i) > \varepsilon$ for any $i$. Let $n^*$ be such that $\sum_{n > n^*} q_n < \varepsilon$. Let $e_i^*$ such that $e_i^* \notin \cup_{n \leq n^*} \gamma_n$. Then we have

$$\varepsilon < Q(e_i^*) = \sum_{n > n^*} q_n Q_{\gamma_n}(e_i^*) < \varepsilon,$$

a contradiction. \hfill \Box

5. A third equivalent statement in Strassen Theorem

In this section we prove Theorems 2.1 and 2.3

5.1. The finite case. In this subsection we prove Theorem 2.1. We prove more implications than the minimal ones since we think that the different equivalences are interesting and reflect some geometric interpretations and structures.
5.1.1. **Couplings and flows.** In this subsection we prove the equivalence between the statements (2) and (3) in Theorem 2.1.

(2) ⇒ (3) Suppose that there exists a compatible coupling $\rho$ between $\mu_1$ and $\mu_2$. If $x \leq y$ and $x \neq y$ there exists at least one directed path in $(V,E)$ going from $x$ to $y$. Fix one of them arbitrarily and call it $\gamma_{(x,y)}$. Recalling definition (2.3), we claim that the flow

$$Q := \sum_{\{x,y \in V : x \neq y\}} \rho(x,y)Q_{\gamma_{(x,y)}},$$

(5.1)

satisfies $\text{div} \, Q = \mu_1 - \mu_2$. Indeed

$$\text{div} \, Q(x) = \sum_{y : (x,y) \in E} Q(x,y) - \sum_{y : (y,x) \in E} Q(y,x)$$

(5.2)

$$= \sum_{y : x \leq y} \rho(x,y) - \sum_{y : y \leq x} \rho(y,x) = \mu_1(x) - \mu_2(x).$$

(5.3)

In the above chain of equalities the first one follows by definition. The second one follows by the fact that we can consider only paths $\gamma$ starting or ending at $x$ since all the others appear with a plus and a minus sign and do not contribute. The last equality follows since $\rho$ is a compatible coupling between $\mu_1$ and $\mu_2$.

(3) ⇒ (2): This follows directly by Theorem 3.3 or Theorem 3.6 and Remark 3.4 applied to the flow $Q$ on the acyclic digraph $(V,E)$.

5.1.2. **Stochastic monotonicity and flows.** In this section we prove the equivalence between statements (1) and (3) of Theorem 2.1 completing the proof of the Theorem. Indeed this equivalence follows by the classic statement of Farkas Lemma.

Farkas Lemma (see for example [19] volume A section 5.4) states that given $A$ an $n \times m$ matrix and $b \in \mathbb{R}^n$ then there exists a $x \in (\mathbb{R}^+)^m$ solution of $Ax = b$ if and only if for any $y \in \mathbb{R}^n$ such that $A^T y \in (\mathbb{R}^+)^m$ it holds $y \cdot b \geq 0$ (with $\cdot$ we denotes the Euclidean scalar product).

We prove one direction of the implication with an elementary argument and invoke Farkas Lemma for the other implication.

(3) ⇒ (1) The proof of this implication follows by a discrete integration by parts. Let $Q$ be a flow such that $\text{div} \, Q = \mu_1 - \mu_2$ and let $f$ be an increasing function. Then we have

$$\mu_2(f) - \mu_1(f) = \sum_{x \in V} (\mu_2(x) - \mu_1(x)) f(x)$$

$$=- \sum_{x \in V} \text{div} \, Q(x) f(x) = \sum_{(x,y) \in E} Q(x,y)(f(y) - f(x)) \geq 0.$$  

(5.4)

The inequality holds since any term in the last sum is non negative. Indeed $Q$ is a flow and is positive by definition and $f$ is increasing so that $f(y) \geq f(x)$ for any $(x,y) \in E$.

(1) ⇒ (3) We prove this implication using Farkas Lemma. Let us introduce a $|V| \times |E|$ matrix $A$ whose rows are labeled with vertices $V$ and whose columns are labeled by the edges $E$. This matrix is the adjacency matrix of $(V,E)$. It is defined requiring that the element corresponding to the row $x$ and the column $(x,y)$ is equal to $+1$, the element corresponding to the row $y$ and the column $(x,y)$ is equal to $-1$ and all the remaining elements in the column $(x,y)$ are set equal to 0. With
this definition, given a flow $Q$ we have
$$\text{div } Q(x) = AQ(x).$$
Moreover given a function $f : V \rightarrow \mathbb{R}$ we have that
$$-A^T f(x, y) = f(y) - f(x).$$
The function $f$ is increasing if and only if $-A^T f(x, y) \geq 0$ for any $(x, y) \in E$. The implication now follows applying Farkas Lemma with the matrix $A$ coinciding with the adjacency matrix and taking the vector $b = \mu_1 - \mu_2$.

5.2. The infinite case. We prove in this section Theorem 2.3. As before we discuss more implications than the minimal ones. We discuss only the differences with respect to the finite case.

5.2.1. Couplings and flows. In this subsection we prove the equivalence between the statements (2) and (3) of Theorem 2.3.

$(2) \Rightarrow (3)$ Consider a compatible coupling $\rho$. We can construct as in the finite case the flow
$$Q = \sum_{\{x, y \in V : x \neq y\}} \rho(x, y)Q_{\gamma(x, y)}. \tag{5.5}$$
The decomposition (5.5) guarantees that $Q$ is finitely decomposable. By absolute summability the same computation of the finite case allows to conclude the proof.

$(3) \Rightarrow (2)$ This implication coincides with the statement of Theorem 3.7.

5.2.2. Stochastic monotonicity and flows. In this subsection we prove the equivalence between items (1) and (3) in Theorem 2.3. This corresponds to prove an infinite dimensional version of Farkas Lemma.

$(3) \Rightarrow (1)$ If $Q$ is just finitely decomposable then computation (5.4) does not apply. Indeed the last term could be not absolutely summable. We need to use (2.9) and using the summability of the weights $q_n$, we have for any increasing function $f \in L^\infty(V)$
$$\mu_2(f) - \mu_1(f) = -\sum_{x \in V} f(x) \text{div } Q(x) = \sum_n q_n (f(\gamma_n^+) - f(\gamma_n^-)) \geq 0.$$ The last identity follows easily from the fact that $f$ is increasing so that $f(\gamma_n^+) \geq f(\gamma_n^-)$.

$(1) \Rightarrow (3)$ We start with a preliminary result. Let $D \subseteq L^1(V)$ be the subset of functions that can be obtained as divergence of a finitely decomposable flow. The subset $D$ is clearly convex and moreover we have the following result.

**Lemma 5.1.** The subset $D \subseteq L^1(V)$ is closed in the strong topology of $L^1(V)$.

**Proof.** We need to show that if $f^{(n)} \in D$ is converging to $f \in L^1(V)$ in the strong topology then necessarily $f \in D$. Since $f^{(n)} \in D$ then there exists a sequence of finitely decomposable flows $Q^{(n)}$ such that $\text{div } Q^{(n)} = f^{(n)}$. We need to show that there exists a finitely decomposable flow $Q$ such that $\text{div } Q = f$. We have to obtain $Q$ with a limit procedure but we can not use directly the flows $Q^{(n)}$ since they could have any converging subsequence. First of all since the digraph $(V, E)$ is acyclic then also the flows $Q^{(n)}$ are acyclic. We can then apply the construction in Theorem 3.9 obtaining a finite decomposition $Q^{(n)} = \sum_k g^{(n)}_k Q_{\gamma^{(n)}_k}$ whose support
is a stable collection of paths (recall the definition in the proof of \(3.3\)) and such that
\[
\sum_k q_k^{(n)} = \frac{1}{2} \sum_x |f^{(n)}(x)| .
\] (5.6)

For any pair of different vertices \(x \leq y\) we fix once for all a path going from \(x\) to \(y\) and call it \(\gamma_{(x,y)}\). We construct a new sequence of flows \(\tilde{Q}^{(n)} = \sum_k q_k^{(n)} Q_k\) where we defined
\[
\tilde{\gamma}_k^{(n)} := \gamma_{(\gamma_k^{(n)} - , \gamma_k^{(n)})} .
\]
This means that we substitute the path \(\gamma_k^{(n)}\) with the canonical one going from its starting point to its final one. We have \(\text{div} \tilde{Q}^{(n)} = \text{div} Q^{(n)} = f^{(n)}\). By \(2.8\), the sequence of flows \(\{\tilde{Q}^{(n)}\}_{n \in \mathbb{N}}\) induces a sequence \(\{M^{(n)}\}_{n \in \mathbb{N}}\) of positive and finite measures on \(\Gamma\). Using Prokhorov Theorem we show that this sequence is relatively compact. First we show that the total mass is uniformly bounded. Using \(5.6\) we have
\[
\sum_{\gamma} M^{(n)}(\gamma) = \sum_k q_k^{(n)} = \frac{1}{2} \sum_x |f^{(n)}(x)| .
\] (5.7)

Since the sequence \(f^{(n)}\) is converging to \(f \in L^1(V)\) in the strong topology of \(L^1\) we have that the r.h.s. of \(5.7\) is converging to \(\frac{1}{2} \sum_x |f(x)|\) and this clearly implies that the l.h.s. of \(5.7\) is uniformly bounded.

Let \(V_n\) be an invading sequence of vertices and define \(\tilde{V}_n := \cup_{x,y \in V_n} \gamma_{(x,y)}\). We define also \(\Gamma_n := \{\gamma : \gamma \subseteq \tilde{V}_n\}\). By definition and using the fact that the support of \(M^{(n)}\) is a stable configuration of paths we have
\[
M^{(n)}(\Gamma \setminus \Gamma_k) = \sum_{\{\gamma : \{\gamma - , \gamma+\} \cap V_k^c \neq \emptyset\}} M^{(n)}(\gamma) \leq \sum_{x \not\in V_k} f^{(n)}(x) .
\] (5.8)

It is now an easy task to deduce tightness from \(5.8\) the convergence of \(f^{(n)}\) to \(f\) and the summability of \(f\).

Let \(M = \sum_k q_k \gamma_k\) be the positive measure on \(\Gamma\) obtained as a weak limit of a subsequence that we still call \(M^{(n)}\). If we define the flow \(Q = \sum_k q_k Q_k\), using the weak convergence of \(M^{(n)}\) to \(M\) and the strong convergence of \(f^{(n)}\) to \(f\), that implies the point-wise one, we have
\[
\text{div} Q(x) = \sum_{k : \gamma_k^x = x} q_k - \sum_{k : \gamma_k^x = \emptyset} q_k
\]
\[
= \lim_{n \to +\infty} \left( \sum_{k : \gamma_k^{(n)} = x} q_k^{(n)} - \sum_{k : \gamma_k^{(n)} = \emptyset} q_k^{(n)} \right)
\]
\[
= \lim_{n \to +\infty} f^{(n)}(x) = f(x) .
\]

This means that \(Q\) is a finite decomposable flow such that \(\text{div} Q = f\) and consequently \(f \in D\). \(\square\)

Let us suppose that \(\mu_2(f) - \mu_1(f) \geq 0\) for any increasing function \(f \in L^\infty(V)\). This can be written as \(\langle f, \mu_2 - \mu_1 \rangle_V \geq 0\) where \(\mu_2 - \mu_1 \in L^1(V)\) and \(\langle , \rangle_V\) is the
$L^\infty(V)$, $L^1(V)$ dual paring defined by
\[
\langle f, g \rangle_v := \sum_{x \in V} f(x)g(x), \quad f \in L^\infty(V), g \in L^1(V).
\] (5.9)

Let $Q_e$ for $e \in E$ be the flow defined by (2.3) for the elementary path $\gamma$ given by the single edge $e$. A function $f \in L^\infty(V)$ is increasing if and only if $\nabla f \in L^\infty(E)$ is such that
\[
\langle \nabla f, Q_e \rangle_E \geq 0, \quad \forall e \in E,
\] (5.10)
where $\langle \cdot, \cdot \rangle_E$ is the dual pairing for functions on edges.

We need to show that $\mu_1 - \mu_2 \in D$. Let us suppose by contradiction that this is not the case. Since $D$ is convex and closed in the strong topology, by Hahn-Banach Theorem we deduce that there exists an $f^* \in L^\infty(V)$ such that
\[
\begin{cases}
\langle f^*, \text{div} Q \rangle_v < 0, & \forall Q \text{ finitely decomposable}, \\
\langle f^*, \mu_1 - \mu_2 \rangle_v > 0.
\end{cases}
\] (5.11)

In particular we have that
\[-\langle f^*, \text{div} Q_e \rangle_v = \langle \nabla f^*, Q_e \rangle_E > 0, \quad \forall e \in E,
\]
that means $f^*$ is increasing. This fact together with the second inequality in (5.11) gives a contradiction.

6. Applications

In this section we discuss some applications of Theorems 2.1, 2.3.

6.1. The one dimensional case. We discuss the simplest countable partial order that is $\mathbb{Z}$ with the usual partial order relation. We want to get the well known necessary and sufficient conditions to have $\mu_1 \preceq \mu_2$ using item (3) of Theorem 2.3. In this case the partial order can be described by its Hasse diagram corresponding to the acyclic digraph $(\mathbb{Z}, E)$ where $E = \{(x, x+1) \mid x \in \mathbb{Z}\}$. The condition $\text{div} Q = \mu_1 - \mu_2$ reads
\[Q(x, x+1) - Q(x-1, x) = \mu_1(x) - \mu_2(x),
\]
and with a finite telescopic sum for any $y < x$ we get
\[Q(x, x+1) - Q(y, y+1) = \sum_{z=y+1}^{x} (\mu_1(z) - \mu_2(z)).
\] (6.1)

By Proposition 2.3 a necessary condition to have that $Q$ is finitely decomposable is that $\lim_{y \to -\infty} Q(y, y+1) = 0$. Taking the limit $y \to -\infty$ in (6.1) we then get
\[Q(x, x+1) = \sum_{z=-\infty}^{x} (\mu_1(z) - \mu_2(z)).
\] (6.2)

This means that there is at most one finitely decomposable flow having divergence equal to $\mu_1 - \mu_2$ that is (6.2). Consider the invading sequence $V_n := \{-n, \ldots, n\}$. The flux exiting from $V_n$ coincides with $Q(n, n+1)$ that by (6.2) is converging to zero when $n \to +\infty$. By Proposition 2.3 $Q$ is finitely decomposable. The last requirement that $Q$ has to satisfy to be a flow is $Q(x, x+1) \geq 0$ for any $x \in \mathbb{Z}$. This condition reads
\[\sum_{z=-\infty}^{x} (\mu_1(z) - \mu_2(z)) = F_1(x) - F_2(x) \geq 0, \quad \forall x \in \mathbb{Z},
\] (6.3)
where $F_i(x) := \sum_{z=-\infty}^{x} \mu_i(z)$ is the distribution function of the measure $\mu_i$. Equation (6.3) is the well known classic condition.

We discussed Theorem 2.3 in the case of a countable set, but the validity of the result should be more general. Let us discuss an example. Consider $X_1$ and $X_2$ two real valued random variables with distribution respectively $\mu_1$ and $\mu_2$. The necessary and sufficient condition to have that $X_2$ stochastically dominates $X_1$ is

$$F_1(x) \geq F_2(x), \quad \forall x \in \mathbb{R},$$

where $F_i := \int_{(-\infty,x]} d\mu_i(y)$ are the distribution functions.

Let us show that condition (6.4) is equivalent to the existence of a positive current going to zero at $\pm \infty$ and such that $\text{div} j = \mu_1 - \mu_2$. Since $\mu_1$ and $\mu_2$ are measures this last identity has to be interpreted in a weak sense. We search for a $j \in L^\infty(\mathbb{R})$ such that for any test function $f \in C^1(\mathbb{R})$ with compact support it holds

$$\int_{\mathbb{R}} \nabla f \cdot j \, dx = \int_{\mathbb{R}} f \, (\mu_2 - \mu_1).$$

(6.5)

The currents that satisfy (6.5) are of the form

$$j(x) = F_1(x) - F_2(x) + c,$$

where $c$ is an arbitrary constant. Since we require that $j$ converges to zero at infinity this fixes the value of the constant as $c = 0$. This means that condition (6.4) is equivalent to the condition $j(x) \geq 0$.

6.2. Finite and infinite tree. We consider the case of partial orders described by digraphs $(V,E)$ such that the associated graph $(V,E_E)$ is a tree. We discuss both the finite and the infinite case.

Let us start with the finite case. Removing one edge of $E_E$ the graph is divided in two connected components. If the edge that has been removed is $\{x,y\}$ and $x \leq y$ we call $T_-(x,y)$ the connected component containing $x$ and $T_+(x,y)$ the connected component containing $y$. Using a discrete Gauss Green identity we get that there is a unique solution to the equation $\text{div} Q = \mu_1 - \mu_2$ that is

$$Q(x,y) = \sum_{z \in T_-(x,y)} (\mu_1(z) - \mu_2(z)).$$

(6.6)

The left hand side of (6.6) is the flux from $T_-(x,y)$ to $T_+(x,y)$ while the right hand side is the sum of the divergences in $T_-(x,y)$. Since $Q$ has to be a flow on $(V,E)$ it must be positive and this gives

$$\sum_{z \in T_-} (\mu_1(z) - \mu_2(z)) \geq 0, \quad \forall e \in E_E$$

(6.7)

that is the necessary and sufficient condition to have $\mu_1 \preceq \mu_2$.

If $(V,E_E)$ is an infinite tree then the equation $\text{div} Q = \mu_1 - \mu_2$ has not an unique solution. If $Q$ is a finitely decomposable flow then a discrete Gauss-Green formula holds in the following form. Let $V' \subseteq V$, where $|V'|$ can be also infinite, then we have

$$\sum_{x \in V'} \text{div} Q(x) = \sum_{n : \gamma_n \in V'} q_n - \sum_{n : \gamma_n^+ \in V'} q_n.$$
The r.h.s. of the above formula coincides with
\[
\sum_{\{n: \gamma_n \in V', \gamma_n \notin V'\}} q_n - \sum_{\{n: \gamma_n \notin V', \gamma_n \in V'\}} q_n. \tag{6.9}
\]
If moreover both series \(\sum_{x \in V', y \notin V'} Q(x, y)\) and \(\sum_{x \notin V', y \in V'} Q(x, y)\) are convergent then \(6.9\) coincides with
\[
\sum_{x \in V', y \notin V'} Q(x, y) - \sum_{x \notin V', y \in V'} Q(x, y). \tag{6.10}
\]
For an infinite tree if we remove one edge \(\{x, y\}\) from the graph \((V, E)\) we have as before the two components \(T_1^{\{x, y\}}\). We apply the discrete Gauss-Green identities \(6.8\), \(6.9\) with \(V' = T_1^{\{x, y\}}\). Since in this case the summability requirements to get \(6.10\) are clearly satisfied we can use it obtaining that the identity \(6.6\) still holds. This means that there is at most one finitely decomposable solution to the equation \(\text{div} Q = \mu_1 - \mu_2\) that is given by \(6.6\). Indeed, as in subsection 6.1 using Proposition 2.4 it can be easily shown that this solution is indeed finitely decomposable. It remain to impose the positivity of \(Q\) that gives the same condition \(6.7\) of the finite case.

6.3. The exact geometry of the inequality. We consider the case \(|V| < +\infty\). Let us call \(\Lambda^1(\mathcal{E})\) the vector space of discrete vector fields on a graph \((V, E)\). This is a \(|\mathcal{E}|\) dimensional vector space. Some elementary and classical results are the following (see [9] or [12] for a short introduction). It holds the orthogonal decomposition
\[
\Lambda^1(\mathcal{E}) = \Lambda^1_y(\mathcal{E}) \oplus \Lambda^1_d(\mathcal{E}),
\]
where \(\Lambda^1_y(\mathcal{E})\) is the \(|V| - 1\) dimensional subspace of gradient discrete vector fields and \(\Lambda^1_d(\mathcal{E})\) is the \(|\mathcal{E}| - |V| + 1\) dimensional subspace of divergence free discrete vector fields. The orthogonality is with respect to the scalar product
\[
\langle \phi, \psi \rangle_{E_\mathcal{E}} := \sum_{(x, y) \in E_\mathcal{E}} \phi(x, y) \psi(x, y).
\]
If the original graph \((V, E)\) is a tree then \(\Lambda^1_d(\mathcal{E}) = \emptyset\) and all the discrete vector fields are of gradient type. A basis for \(\Lambda^1_d(\mathcal{E})\) is obtained choosing a suitable collection of cycles. Fix \((V, T)\) a spanning tree of \((V, E)\), in particular \(|T| = |V| - 1\). For any \(e \in \mathcal{E} \setminus T\) there exists a unique cycle in \((V, T \cup e)\). Choosing arbitrarily the orientation for each of them we obtain \(|\mathcal{E}| - |V| + 1\) directed cycles \(C_i, i = 1, \ldots, |\mathcal{E}| - |V| + 1\) on \((V, E_\mathcal{E})\). We can then construct the flows \(Q_{C_i}\) on \((V, E_\mathcal{E})\) and then the discrete vector fields \(\phi^{Q_{C_i}}\). Note that \(\text{div} Q_{C_i} = \text{div} \phi^{Q_{C_i}} = 0\). The vector fields \(\phi^{Q_{C_i}}\) are a basis for \(\Lambda^1_y(\mathcal{E})\). This means that given \(g: V \to \mathbb{R}\) such that \(\sum_x g(x) = 0\) and given any \(\phi^*\) such that
\[
\text{div} \phi^* = g \tag{6.11}
\]
then all the discrete vector fields having divergence coinciding with \(g\) are given by
\[
\phi^* + \sum_i \alpha_i \phi^{Q_{C_i}}, \tag{6.12}
\]
where the \(\alpha_i\) are arbitrary real numbers. The vector field \(\phi^*\) can be fixed for example of gradient type \(\phi^* = \nabla h^*\) where \(h^*: V \to \mathbb{R}\) solves a discrete Poisson equation
\[
\text{div} \nabla h^* = g. \tag{6.13}
\]
The solution of (6.13) is defined up to an additive constant that does not affect the value of $\phi^*$. 

**Remark 6.1.** In the case $|V| = +\infty$ the existence of a decomposition with finite cycles of a divergence free discrete vector field is a delicate issue. Indeed it is not always possible a decomposition like (6.12) where the $C_i$ are finite cycles. It is possible to show that such a decomposition exists for the divergence free discrete vector fields $\phi$ such that $Q^\phi$ is finitely decomposable. In this case $\sum |\alpha_i| < +\infty$ and the $C_i$ are finite cycles. This could be proven using the cyclic decomposition in [11] for flows having zero flux towards infinity.

Let $(V, E)$ be the transitive reduction of a finite acyclic digraph and let $\mu_1$ and $\mu_2$ be two probability measures on $V$. The set of discrete vector fields on $(V, E)$ having divergence equal to $\mu_1 - \mu_2$ is given by (6.12) where $\phi^* = \nabla h^*$ and $h^*$ solves (6.13) with $g = \mu_1 - \mu_2$. Let $\mathcal{A} \subseteq \Lambda^1 (E_\mathcal{E})$ be the cone defined by

$$\mathcal{A} := \{ \phi \in \Lambda^1 (E_\mathcal{E}) : \phi(x, y) \geq 0, \forall (x, y) \in E \}. \quad (6.14)$$

Item (3) of Theorem 2.1 implies that $\mu_1 \preceq \mu_2$ if and only if

$$\left\{ \phi^* + \text{Span} \left( \{ \phi^Qc_i \}_{i=1}^{\left| E \right|-|V|+1} \right) \right\} \cap \mathcal{A} \neq \emptyset. \quad (6.15)$$

This condition can be read also as

$$\nabla h^* \cap \mathcal{A}^\bot \neq \emptyset,$$

where $\mathcal{A}^\bot$ is the orthogonal projection of $\mathcal{A}$ on the subspace $\Lambda^1_\bot (E_\mathcal{E})$. In terms of the measures this can be written as

$$\mu_1 - \mu_2 \in \text{div} (\mathcal{A}^\bot).$$

The r.h.s. denotes the set of functions on vertices that can be obtained as divergence of a vector field in $\mathcal{A}^\bot$.

For edges $(x, y) \in E_\mathcal{E}$ that do not belong to any cycle of the basis, condition (6.15) requires $\phi^*(x, y) \geq 0$ where $(x, y) \in E$. For the remaining edges, condition (6.16) requires that there exists an $(\alpha_1, \ldots, \alpha_{|E|-|V|+1}) \in \mathbb{R}^{\left| E \right|-|V|+1}$ such that

$$\phi^*(x, y) + \sum_{i=1}^{\left| E \right|-|V|+1} \alpha_i \phi^Qc_i(x, y) \geq 0, \quad \forall (x, y) \in E',$$

where $E'$ is the set of edges $(x, y) \in E$ such that $(x, y)$ belongs to at least one cycle of the basis. Condition (6.16) states that a polyhedron on $\mathbb{R}^{\left| E \right|-|V|+1}$ obtained as the intersection of $E'$ half-spaces (one for each $(x, y) \in E'$) is not empty. The interesting feature is that it is a geometric problem on a space of dimension equal to the number of independent cycles of the Hasse diagram.

Consider for example the Hasse diagram of Figure 3 (left) having one single cycle and such that $E' = E$. Since the Hasse diagram has only one independent cycle the stochastic monotonicity condition will reduce to a one dimensional problem. Choosing arbitrarily one orientation we can label vertices as $V := \{1, 2, \ldots, n\}$ and the edges as $E_\mathcal{E} := \{ \{x, x+1\} \}_{x=1}^{n}$ where the sum is modulo $n$. Given $\mu_1$ and $\mu_2$ two probability measures on $V$ the discrete vector fields having divergence $\mu_1 - \mu_2$ are of the form

$$\phi^* := \phi^* + \alpha \phi^Qc,$$  

(6.17)
where \( C := (1, 2, \ldots n, 1) \), \( \alpha \) is an arbitrary real number and \( \phi^* \) is any given discrete vector field such that \( \text{div} \phi^* = \mu_1 - \mu_2 \). We can fix for example

\[
\phi^*(x, x + 1) = \sum_{y=1}^{x} (\mu_1(y) - \mu_2(y)), \quad x = 1, \ldots, n. \tag{6.18}
\]

Let \( E^+ := \{(x, y) \in E : (x, y) \in C\} \) and \( E^- \) the complementary set. Conditions (6.10) become

\[
\begin{align*}
\phi^*(x, y) + \alpha & \geq 0, \quad (x, y) \in E^+, \\
\phi^*(x, y) - \alpha & \geq 0, \quad (x, y) \in E^-,
\end{align*}
\tag{6.19}
\]

that are equivalent to the single inequality

\[
\max_{(x, y) \in E^+} \left\{ \sum_{z=1}^{x} (\mu_2(z) - \mu_1(z)) \right\} \leq \min_{(x, y) \in E^-} \left\{ \sum_{z=1}^{y} (\mu_2(z) - \mu_1(z)) \right\}. \tag{6.20}
\]

Condition (6.20) is a necessary and sufficient condition to have \( \mu_1 \leq \mu_2 \) on a poset like the one on the left of Figure 3. In the special case of the Hasse diagram of an elementary lattice (the poset on the right of Figure 3) condition (6.20) becomes

\[
|\mu_1(C) - \mu_2(C)| + |\mu_1(B) - \mu_2(B)| \leq (\mu_1(A) - \mu_2(A)) - (\mu_1(D) - \mu_2(D)). \tag{6.21}
\]

We stress that it is straightforward to get conditions (6.20) and (6.21) using item (3) of Theorem 2.1, while it is not so simple to get this compact necessary and sufficient conditions using instead items (1) or (2).

### 6.4. A generalization of Holley inequality

Holley inequality [14] gives a sufficient condition to have \( \mu_1 \leq \mu_2 \) when the partial order \((V, \leq)\) is a lattice \( [0] \). For simplicity we consider the special lattice \( V := \{0, 1\}^N \) with the usual partial order. An element \( \eta \in V \) has the form \( \eta = (\eta(1), \ldots, \eta(N)) \) with \( \eta(i) \in \{0, 1\} \). The Hasse diagram associated is \((V, E)\) where \((\eta, \eta') \in E\) if and only if \( \eta' \) is obtained by \( \eta \) transforming one single coordinate of \( \eta \) from 0 to 1. The digraph on the right of Figure 3 is the Hasse diagram for this partial order when \( N = 2 \).

Given \( \eta, \xi \in V \) we call \( \eta \vee \xi \) and \( \eta \wedge \xi \) the elements of \( V \) defined by

\[
(\eta \vee \xi)(i) := \max \{\eta(i), \xi(i)\}, \quad (\eta \wedge \xi)(i) := \min \{\eta(i), \xi(i)\}.
\]

The Holley condition is

\[
\mu_2(\eta \vee \xi)\mu_1(\eta \wedge \xi) \geq \mu_2(\eta)\mu_1(\xi) \quad \forall \eta, \xi \in V, \tag{6.22}
\]
with in addition the condition that the measures $\mu_i$ are strictly positive. In [14] the fact that condition (6.22) is a sufficient condition to have $\mu_1 \preceq \mu_2$ is obtained using item (2) of Theorem 2.1 and a dynamic argument. See also [7], [8] for a detailed analysis of the relation among Holley inequality, FKG inequality and mass transportation.

Applying item (3) of Theorem 2.1 we can easily generalize Holley’s result. We denote by $\mathcal{M}^+(V)$, $\mathcal{M}(V)$ respectively the positive and the signed measures on $V$. We call $H \subset \mathcal{M}^+(V) \times \mathcal{M}^+(V)$ the subset of all pairs $(m_1, m_2)$ of strictly positive measures satisfying the Holley condition

$$m_2(\eta \lor \xi)m_1(\eta \land \xi) \geq m_2(\eta)m_1(\xi), \quad \forall \eta, \xi \in V. \quad (6.23)$$

Given $A \subseteq \mathcal{M}^+(V)$ we denote by $\overline{A} \subseteq \mathcal{M}(V)$ the subset of signed measures defined by

$$\overline{A} := \{ \Delta \in \mathcal{M}(V) : \exists (m_1, m_2) \in A, \text{ s.t. } m_1 - m_2 = \Delta \}. \quad (6.24)$$

Note that given $\Delta \in \mathcal{M}(V)$ then all the $(m_1, m_2) \in \mathcal{M}^+(V) \times \mathcal{M}^+(V)$ such that $m_1 - m_2 = \Delta$ are of the form

$$\begin{cases} m_1 = [\Delta]_+ + m, \\ m_2 = [-\Delta]_+ + m, \end{cases}$$

(recall that $[.]_+$ denotes the positive part) where $m$ is an arbitrary element of $\mathcal{M}^+(V)$.

Consider $(m_1, m_2) \in H$ such that $m_1 - m_2 = \mu_1 - \mu_2$. In particular $m_1$ and $m_2$ have the same total mass $\sum_x m_1(x) = \sum_x m_2(x) = M$. The probability measures $(\frac{m_1}{M}, \frac{m_2}{M})$ satisfy the Holley condition (6.22) and using the Holley construction we get a compatible coupling $\rho^*$ between them. If we associate to any pair $x \leq y$ a path $\gamma_{(x,y)}$ from $x$ to $y$ on the Hasse diagram $(V, E)$ the flow $\sum_{x \neq y} M\rho^*(x, y)Q_{\gamma_{(x,y)}}$ has divergence $\mu_1 - \mu_2$. Applying item (3) of 2.1 we deduce that $\mu_1 \preceq \mu_2$. This means that a sufficient condition to have $\mu_1 \preceq \mu_2$ is that

$$(\mu_1 - \mu_2) \in \overline{H}.$$ 

Clearly if $(\mu_1, \mu_2) \in H$ then $(\mu_1 - \mu_2) \in \overline{H}$ so that this latter condition is stronger than the original Holley condition.

It could be interesting to develop a dynamic argument like the original one of Holley working directly with the flows.

7. Mass transportation

In this section we prove Theorems 2.6 and discuss some solvable examples.

Proof of Theorem 2.6. For any pair $x \neq y$ of elements of $V$ let $\gamma_{(x,y)}$ be a path in $(V, E)$ such that

$$\sum_{e \in \gamma_{(x,y)}} w(e) \leq c(x, y) + \epsilon.$$ 

Given any coupling $\rho$ we can then construct the flow $Q$ as in (3.1) using these special paths. We have

$$\langle Q, w \rangle E = \sum_{x \neq y} \rho(x, y) \sum_{e \in \gamma_{(x,y)}} w(e) \leq \sum_{x \neq y} \rho(x, y) (c(x, y) + \epsilon) \leq \mathbb{E}_\rho (\epsilon) + \epsilon.$$
Since ε is arbitrary we obtain that the r.h.s. of (2.14) is less or equal to the l.h.s.. It remains to prove the converse inequality.

First of all we observe that if ρ and Q are related by (3.2) and the paths are in (V, E) then by (2.12) we have
\[ \langle Q, w \rangle_E \geq \mathbb{E}_\rho (c) . \] (7.1)

Consider a finitely decomposable flow Q having divergence \( \mu_1 - \mu_2 \). Recalling Remark 3.3 we can apply the construction of Theorem 3.7 obtaining the finite sequence of acyclic flows \( \hat{Q}_n \leq Q \). This sequence of acyclic flows is related by a formula like (3.2) to a sequence of measures \( \rho^{(n)} \) on \( V \times V \) weakly convergent to a coupling \( \rho \) between \( \mu_1 \) and \( \mu_2 \). We then have, using (7.1), for any \( n \)
\[ \langle Q, w \rangle_E \geq \langle \hat{Q}_n, w \rangle_E \geq \mathbb{E}_{\rho_n} (c) . \] (7.2)

Since \( \rho_n \) converges to \( \rho \) by Fatou Lemma we have also
\[ \lim \inf_{n \to +\infty} \mathbb{E}_{\rho_n} (c) \geq \mathbb{E}_\rho (c) . \] (7.3)

Equations (7.2) and (7.3) imply that the l.h.s. of (2.14) is less or equal to the r.h.s.. This concludes the proof. \( \square \)

We illustrate the above Theorem with some examples. Let us consider two probability measures \( \mu_1 \) and \( \mu_2 \) on \( \mathbb{Z} \) and compute their Monge-Kantorovich distance \( d \) with respect to the cost function \( c(x, y) = |x - y| \). This corresponds to consider the digraph \( (\mathbb{Z}, E) \) where \( E \) is the set of all pairs \( (x, y) \) such that \( |x - y| = 1 \) and the weight function \( w \) is identically equal to 1. Proceeding as in subsection 6.1 we can show that there exists an unique discrete vector field \( \phi \) on \( (V, E) \) having divergence equal to \( \mu_1 - \mu_2 \) and such that \( Q^\phi \) is finitely decomposable, this is \( \phi(x, x+1) := F_1(x) - F_2(x) \) (recall that \( F_i \) is the distribution function of the measure \( \mu_i \)). Since \( Q^\phi(x, y) = |\phi(x, y)|_+ \) and \( |a|_+ + [-a]_+ = |a| \), applying Remark 2.7 we obtain immediately the classic formula
\[
\begin{align*}
d(\mu_1, \mu_2) &= \sum_{x \in \mathbb{Z}} \left( [F_1(x) - F_2(x)]_+ + [F_2(x) - F_1(x)]_+ \right) \\
&= \sum_{x \in \mathbb{Z}} |F_1(x) - F_2(x)| .
\end{align*}
\]

We can generalize the above computation to the non homogeneous but symmetric case in which \( w(x, x+1) = w(x+1, x) = w(\{x, x+1\}) \). Also in this case we apply Remark 2.7 and use the fact that there is just one possible vector field to be considered on \( (\mathbb{Z}, E) \) and we get
\[
\begin{align*}
d(\mu_1, \mu_2) &= \sum_{x \in \mathbb{Z}} w(\{x, x+1\}) |F_1(x) - F_2(x)| .
\end{align*}
\]

Finally we can also deal with the general case relaxing the symmetry condition \( w(x, x+1) = w(x+1, x) \). Still applying Remark 2.7 we get the general formula
\[
\begin{align*}
\inf_{\rho} \mathbb{E}_\rho (c) &= \sum_{x \in \mathbb{Z}} \left( w(x, x+1) |F_1(x) - F_2(x)|_+ + w(x+1, x) |F_2(x) - F_1(x)|_+ \right) .
\end{align*}
\]

The infimum is over all couplings between \( \mu_1 \) and \( \mu_2 \). In this case the result is not a metric.

We discuss some finite cases (i.e. \( |V| < +\infty \)) where the optimal flow in Theorem 2.6 can be explicitly computed. In the infinite case even the existence of an optimal
flow is guaranteed only adding some extra conditions. Recall that we are assuming that the digraph \((V, E)\), where \(E\) is the set of edges with finite weight, is a connected digraph. In the finite case the minimization problem on the r.h.s. of (2.14) is a much studied variational problem under the name of \textit{Transportation problem}. Here we give a brief summary referring to [2] for more informations and details. It is a minimization problem of an affine function over a convex set. The convex set is a non empty polyhedron and is compact in the case of an acyclic digraph. If \(Q\) is an extremal flow of this polyhedron then it can be proved (see [2]) that \((V, E(\phi^Q))\), where \(E(\phi^Q)\) is the set of unoriented edges \(\{x, y\}\) of \(E_E\) such that \(\phi(x, y) \neq 0\), is a spanning forest of \((V, E_E)\). The minimum is obtained on a face.

Using Remark [2] a necessary and sufficient condition to have that \(Q\) is a minimizer is that \(Q = Q^\phi\) for a discrete vector field \(\phi\) such that that for any divergence free discrete vector field \(\phi^d\) the real function \(\langle Q^{\phi+\alpha\phi^d}, w \rangle_E\) of the real parameter \(\alpha\) satisfies

\[
0 \in \partial^- \left( \langle Q^{\phi+\alpha\phi^d}, w \rangle_E \right) (0),
\]

where \(\partial^-\) denotes the sub-differential.

Our first example concerns the computation of the Monge-Kantorovich distance between two probability measures on a ring. Let \((V_n, E_n)\) be the graph constituted by one single cycle with \(n\) sites. More precisely \(V_n := \{1, 2, \ldots, n\}\) and \(E_n := \{\{x, x + 1\}\}\) where the sum is modulo \(n\). Given \(\mu_1\) and \(\mu_2\) two probability measures on \(V_n\) the discrete vector fields having divergence \(\mu_1 - \mu_2\) are of the form

\[
\phi^\alpha := \phi^* + \alpha \phi^{C},
\]

where \(C := (1, 2, \ldots, n, 1)\), \(\alpha\) is an arbitrary real number and \(\phi^*\) is any given discrete vector field such that \(\text{div}\ \phi^* = \mu_1 - \mu_2\). We can fix for example

\[
\phi^*(x, x + 1) = \sum_{y=1}^n (\mu_1(y) - \mu_2(y)), \quad x = 1, \ldots, n.
\]

Consider \((V_n, E_n)\) with a weight function \(w\). Using (7.4) we get that the minimizers of the r.h.s. of (2.14) are the flows of the form \(Q^{\phi^\alpha}\) with \(\alpha\) satisfying the conditions

\[
\begin{align*}
\sum_{y: \phi^\alpha(y, y+1) \geq 0} w(y, y+1) &\geq \sum_{y: \phi^\alpha(y, y+1) < 0} w(y+1, y), \\
\sum_{y: \phi^\alpha(y, y+1) > 0} w(y, y+1) &\leq \sum_{y: \phi^\alpha(y, y+1) \leq 0} w(y+1, y).
\end{align*}
\]

Indeed given a discrete vector field \(\phi^\alpha\) like in (7.5) we have that

\[
\langle Q^{\phi^\alpha}, w \rangle_E = \sum_{x=1}^n |\phi^*(x, x + 1) + \alpha|_+ w(x, x + 1) + [-\phi^*(x, x + 1) - \alpha]_+ w(x + 1, x).
\]

For each value of \(\alpha\) we can compute the left \(\frac{\partial}{\partial \alpha}\) and right \(\frac{\partial}{\partial \alpha}\) derivatives of

\[
\left(\frac{\partial}{\partial \alpha} \langle Q^{\phi^\alpha}, w \rangle_E \right)
\]

getting

\[
\left(\frac{\partial}{\partial \alpha} \langle Q^{\phi^\alpha}, w \rangle_E \right) = \sum_{y: \phi^\alpha(y, y+1) \geq 0} w(y, y+1) - \sum_{y: \phi^\alpha(y, y+1) < 0} w(y+1, y),
\]

\[
\left(\frac{\partial}{\partial \alpha} \langle Q^{\phi^\alpha}, w \rangle_E \right) = \sum_{y: \phi^\alpha(y, y+1) > 0} w(y, y+1) - \sum_{y: \phi^\alpha(y, y+1) \leq 0} w(y+1, y).
\]

Since the sub-differential is the following interval

\[
\partial^- \left( \langle Q^{\phi^\alpha}, w \rangle_E \right) = \left[ \frac{\partial}{\partial \alpha^-} \langle Q^{\phi^\alpha}, w \rangle_E, \frac{\partial}{\partial \alpha^+} \langle Q^{\phi^\alpha}, w \rangle_E \right],
\]
condition (7.10) follows.

Remark 7.1. If we have a digraph \((V_n, E_n)\) such that \(E_n \neq E_{\mathcal{E}_n}\) then conditions (6.17) are still true but you need to add the extra condition that there exists the flow \(Q^{\phi_0}\) on \((V_n, E_n)\). In this case in formulas (6.17) can appear a \(w(x, y)\) with \((x, y) \notin E_n\). You have to interpret it as \(+\infty\).

As a second example we discuss a result on Monge–Kantorovich distance between pairs of probability measures on a lattice one of which stochastically dominates the other. The original result is in [11], [13]. We show how to get it immediately using Theorem 2.6. Consider, as in subsection 6.4, the lattice \((V, E)\) diagram like in subsection 6.4. Consider two probability measures \(\mu_1 \preceq \mu_2\) on \(V\). The counterpart of this result for flows says that any flow in \((V, E)\) having divergence equal to \(\mu_2 - \mu_1\) is a minimizer for the r.h.s. of (2.14). The Monge–Kantorovich distance will coincide with the total mass \(\sum_{(\eta, \eta') \in \mathcal{E}} Q(\eta, \eta')\) of any such a flow.

The proof of this fact follows immediately from the following characteristic of the graph. Any cycle \(C = (x_0, x_1, \ldots, x_n, x_0)\) in \((V, E_{\mathcal{E}_n})\) has a length \(|C|\) that is an even number. Moreover the number of indices \(i = 0, \ldots, n\) such that \((x_i, x_{i+1}) \in E\) coincides with the number of indices such that \((x_i, x_{i+1}) \notin E\) (the sum in the indices is modulo \(n\)) and is \(\frac{|C|}{2}\). From this observation we deduce immediately the following fact. Let \(\phi\) be any divergence free discrete vector field on \((V, E_{\mathcal{E}_n})\). Using the cyclic decomposition (6.12) with \(\phi^i = 0\) we have

\[
\sum_{(x,y) \in E} \phi^d(x,y) = \sum_i \alpha_i \sum_{(x,y) \in \mathcal{E}} \phi^{Qc_i}(x,y) = 0. \tag{7.9}
\]

Consider \(Q\) a flow on \((V, E)\). Then it is of the form \(Q^{\phi}\) with \(\phi\) a discrete vector field on \((V, E_{\mathcal{E}_n})\) satisfying

\[
\phi(x,y) \geq 0, \quad \forall (x,y) \in E. \tag{7.10}
\]

An elementary computation gives

\[
\begin{align*}
\left. \frac{\partial}{\partial \alpha^+} (Q^{\phi + \alpha^d w}, w) \right|_{(0)} &= \sum_{\phi(x,y)>0} \phi^d(x,y) + \sum_{\phi(x,y)=0} \phi^d(x,y) \geq 0, \\
\left. \frac{\partial}{\partial \alpha^-} (Q^{\phi + \alpha^d w}, w) \right|_{(0)} &= \sum_{\phi(x,y)>0} \phi^d(x,y) - \sum_{\phi(x,y)=0} \phi^d(x,y) \leq 0. \tag{7.11}
\end{align*}
\]

The inequalities in (7.11) follows by (7.9) and (7.10) and they imply (7.4).

Acknowledgements. We thank L. Ambrosio for pointing out reference [17].
REFERENCES

[1] J. Bang-Jensen, G. Gutin digraphs. Theory, algorithms and applications Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2001
[2] A. Barvinok A course in convexity. Graduate Studies in Mathematics 54 American Mathematical Society 2002
[3] M. Beckmann A continuous model of transportation, Econometrica 20, 643-660, (1952)
[4] L. Bertini, A. Faggionato, D. Gabrielli Large deviations of the empirical flow for continuous time Markov chains Preprint [arXiv:1210.2004]
[5] N. Biggs Algebraic graph theory. Cambridge Tracts in Mathematics, 67 Cambridge University Press, London, 1974
[6] G. Birkhoff Lattice theory. American Mathematical Society Colloquium Publications, 25 American Mathematical Society, 1979.
[7] L.A. Caffarelli Monotonicity properties of optimal transportation and the FKG and related inequalities Comm. Math. Phys. 214 (2000), no. 3, 547–563
[8] L.A. Caffarelli Erratum: "Monotonicity properties of optimal transportation and the FKG and related inequalities" [Comm. Math. Phys. 214 (2000), no. 3, 547–563; Comm. Math. Phys. 225 (2002), no. 2, 449-450
[9] P. Dai Pra, P.Y. Louis, I.G. Minelli Realizable monotonicity for continuous-time Markov processes Stochastic Process. Appl. 120 (2010), no. 6, 959-982
[10] J.A. Fill, M. Machida Stochastic monotonicity and realizable monotonicity Ann. Probab. 29 (2001), no. 2, 938-978
[11] A.B. Kirillov, D.C. Radulescu, D.F. Styer Vasserstein distances in two-state systems J. Statist. Phys. 56 (1989), no. 5-6, 931–937
[12] D. Gabrielli, C. Valente Which random walks are cyclic? ALEA, Lat. Am. J. Probab. Math. Stat. 9, 231-267 (2012)
[13] A. Galves, N.L. Garcia, C. Prieur Perfect simulation of a coupling achieving the $\bar{d}$-distance between ordered pairs of binary chains of infinite order J. Stat. Phys. 141 (2010), no. 4, 669–682
[14] R. Holley, Remarks on the FKG inequalities, Commun.Math. Phys., 36 (1974), pp. 227–231
[15] T. Lindvall On Strassen’s theorem on stochastic domination Electron. Comm. Probab. 4 (1999), 51–59
[16] T. Lindvall Lectures on the coupling method Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, New York, (1992)
[17] E. Paolini, E. Stepanov Decomposition of acyclic normal currents in a metric space. J. Funct. Anal. 263 (2012), no. 11, 3353-3390
[18] F. Santambrogio Introduction to Optimal Transport Theory lecture notes for the Summer School Optimal transportation : Theory and applications , Grenoble, 2009, to appear in a special volume by Cambridge University Press
[19] A. Schrijver Combinatorial Optimization Polyhedra and Efficiency, Algorithms and Combinatorics 24, Springer-Verlag (2003)
[20] S.K. Smirnov Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows. (Russian) Algebra i Analiz 5 (1993), no. 4, 206–238; translation in St. Petersburg Math. J. 5 (1994), no. 4, 841-867
[21] V. Strassen The existence of probability measures with given marginals Ann. Math. Statist. 36 (1965) 423–439
[22] C. Villani Optimal transport. Old and new Grundlehren der Mathematischen Wissenschaften 338 Springer-Verlag, Berlin, (2009)

DAVIDE GABRIELLI
DISIM, UNIVERSITY OF L’AQUILA
VIA VETOIO, 67100 COPPITO, L’AQUILA, ITALY
E-mail address: gabrielli@univaq.it

IDA GERMANA MINELLI
DISIM, UNIVERSITY OF L’AQUILA
VIA VETOIO, 67100 COPPITO, L’AQUILA, ITALY
E-mail address: ida.minelli@univaq.it