GROUP ACTIONS AND A MULTI-PARAMETER FALCONER DISTANCE PROBLEM

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Abstract. In this paper we study the following multi-parameter variant of the celebrated Falconer distance problem \([8]\). Given \(d = (d_1, d_2, \ldots, d_\ell) \in \mathbb{N}^\ell\) with \(d_1 + d_2 + \cdots + d_\ell = d\) and \(E \subseteq \mathbb{R}^d\), we define
\[
\Delta_d(E) = \left\{ \left( |x^{(1)} - y^{(1)}|, \ldots, |x^{(\ell)} - y^{(\ell)}| \right) : x, y \in E \right\} \subseteq \mathbb{R}^\ell,
\]
where for \(x \in \mathbb{R}^d\) we write \(x = (x^{(1)}, \ldots, x^{(\ell)})\) with \(x^{(i)} \in \mathbb{R}^{d_i}\).

We ask how large does the Hausdorff dimension of \(E\) need to be to ensure that the \(\ell\)-dimensional Lebesgue measure of \(\Delta_d(E)\) is positive? We prove that if \(2 \leq d_i\) for \(1 \leq i \leq \ell\), then the conclusion holds provided
\[
\dim(E) > d - \frac{\min d_i}{2} + \frac{1}{3}.
\]
We also note that, by previous constructions, the conclusion does not in general hold if
\[
\dim(E) < d - \frac{\min d_i}{2}.
\]
A group action derivation of a suitable Mattila integral plays an important role in the argument.

1. Introduction

Given a set \(E \subseteq \mathbb{R}^d\), the distance set of \(E\) is
\[
\Delta(E) = \{ |x - y| : x, y \in E \} \subseteq \mathbb{R}.
\]

Falconer \([6]\) studied how large the Hausdorff dimension of \(E\) must be to guarantee that the Lebesgue measure of \(\Delta(E)\) is positive. Falconer’s conjecture is

Conjecture 1.1. Let \(E\) be a compact subset of \(\mathbb{R}^d\), \(d \geq 2\). If \(\dim(E) > d/2\), then \(|\Delta(E)| > 0\).

Here \(| \cdot |\) is the Lebesgue measure and \(\dim(\cdot)\) is the Hausdorff dimension. In \([6]\), Falconer showed that \(d/2\) in the conjecture is best possible by constructing, for each \(0 < s < d/2\), a compact set \(E_s \subseteq \mathbb{R}^d\) such that \(\dim(E_s) = s\) and \(\dim(\Delta(E_s)) \leq 2s/d\). Falconer’s conjecture is open for all dimensions \(d \geq 2\). Partial results have been obtained by Falconer \([6]\), Mattila \([11]\), Bourgain \([2]\), and others. The best currently known result, due to Wolff \([13]\) \((d = 2)\) and Erdoğan \([5]\) \((d \geq 3)\), is

Theorem 1.2. Let \(E\) be a compact subset of \(\mathbb{R}^d\), \(d \geq 2\). If \(\dim(E) > d/2 + 1/3\), then \(|\Delta(E)| > 0\).

We will study a multi-parameter variant of Falconer’s distance problem. Given \(d = (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell\), we let \(d = d_1 + \cdots + d_\ell\). For \(x \in \mathbb{R}^d\), we write
\[
x = (x^{(1)}, \ldots, x^{(\ell)})
\]
where \(x^{(i)} \in \mathbb{R}^{d_i}\). Given a set \(E \subseteq \mathbb{R}^d\), we define the multi-parameter distance set of \(E\) to be
\[
\Delta_d(E) = \left\{ \left( |x^{(1)} - y^{(1)}|, \ldots, |x^{(\ell)} - y^{(\ell)}| \right) : x, y \in E \right\} \subseteq \mathbb{R}^\ell.
\]
Further, we let
\[
\mathcal{F}(d) = \sup \{ \dim(E) : E \subseteq \mathbb{R}^d, |\Delta_d(E)| = 0 \}.
\]
By considering (a sequence of near) maximal dimensional sets with zero-measure distance sets in one hyperplane, crossed with full boxes in the other hyperplanes, we immediately have the relation

$$\mathcal{F}(d) \geq d - d_i + \mathcal{F}(d_i)$$

for all $1 \leq i \leq \ell$. Moreover, by the construction of Falconer [6] mentioned above, we have $\mathcal{F}(d_i) \geq d_i/2$ for all $1 \leq i \leq \ell$, and so

$$\mathcal{F}(d) \geq d - \frac{\min d_i}{2}.$$ 

Our main result is

**Theorem 1.3.** Let $d = (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell$ with $2 \leq d_i$ for $1 \leq i \leq \ell$ and $d = d_1 + \cdots + d_\ell$. If $E$ is a compact subset of $\mathbb{R}^d$ with

$$\dim(E) > d - \frac{\min d_i}{2} + \frac{1}{3},$$

then $|\Delta_d(E)| > 0$.

In other words, Theorem 1.3 is precisely the statement that

$$\mathcal{F}(d) \leq d - \frac{\min d_i}{2} + \frac{1}{3}.$$ 

Note that Theorem 1.3 implies Theorem 1.2 by taking $\ell = 1$. Note also that a similar problem has been studied in vector spaces over finite fields by Birkbauer and Iosevich [1].

The standard approach in studying Falconer’s distance conjecture and related problems is to reduce the problem to the convergence of a so-called Mattila integral. This reduction is typically carried out via a stationary phase argument (see, for example, [2], [5], [11], [12], [13], and references therein). Our approach is notable in that we instead carry out this reduction via the group action method developed by Greenleaf, Iosevich, Liu, and Palsson [9] in the study of the distribution of simplexes in compact sets of a given Hausdorff dimension. The method has its roots in the method developed by Elekes and Sharir in [4], which was ultimately used by Guth and Katz [10] to prove the Erdős distance conjecture in the plane.

2. **Proof of Theorem 1.3**

For the entirety of the proof, we fix $d = (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell$ with $2 \leq d_i$ for $1 \leq i \leq \ell$ and $d = d_1 + \cdots + d_\ell$. We also fix a compact set $E \subseteq \mathbb{R}^d$.

The notation $A \lesssim B$ means there is a constant $C > 0$ such that $A \leq CB$; the constant may depend on $(d_1, \ldots, d_\ell)$ and $E$, but not on any other parameters. Additionally, $A \gtrsim B$ means $B \lesssim A$, and $A \approx B$ means both $A \lesssim B$ and $B \lesssim A$. For $n \in \mathbb{N}$, we let $\mathbb{O}(n)$ denote the orthogonal group on $\mathbb{R}^n$, and we note that $\mathbb{O}(n)$ is a compact group with the operator norm topology.

For each finite non-negative Borel measure $\mu$ supported on $E$, we define a measure $\nu$ on $\mathbb{R}^\ell$ by

$$\int_{\mathbb{R}^\ell} f(t) d\nu(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(|x^{(1)}(t) - y^{(1)}|, \ldots, |x^{(\ell)}(t) - y^{(\ell)}|) d\mu(x) d\mu(y),$$

and, further, for each $g = (g^{(1)}, \ldots, g^{(\ell)}) \in \prod_{i=1}^{\ell} \mathbb{O}(d_i)$, we define a measure $\nu_g$ on $\mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} f(z) d\nu_g(z) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x^{(1)} - g^{(1)} y^{(1)}, \ldots, x^{(\ell)} - g^{(\ell)} y^{(\ell)}) d\mu(x) d\mu(y).$$

We emphasize that $\nu$ and $\nu_g$ both depend on $\mu$ and that $\operatorname{supp}(\nu) \subseteq \Delta_d(E)$.

Our goal is to show that, whenever (1.1) holds, there is a choice of $\mu$ for which the Fourier transform $\hat{\nu}$ is in $L^2$. This will imply $\nu$ has an $L^2$ density with respect to Lebesgue measure on $\mathbb{R}^\ell$, and hence $|\Delta_d(E)| > 0$. 

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Our argument has two parts. In the first part, we exploit the action of the orthogonal group to show that, for any measure \( \mu \) as above,

\[
\int_{\mathbb{R}^d} |\widehat{\nu}(\eta)|^2 d\eta \lesssim \int_{\mathbb{R}^d} \prod_{i=1}^d \mathbb{O}_i(\mathbb{O}_d) \left| \widehat{\nu}(\xi) \right|^2 d\xi d\xi \approx \int_{\mathbb{R}^d} \prod_{i=1}^d \mathbb{O}_i(\mathbb{O}_d) \left| \widehat{\nu}(\xi)|\theta^{(i)}|, \ldots, |\xi^{(i)}|\theta^{(i)}\right|^2 d\theta d\xi.
\]

This is split into Lemma 2.1 and Lemma 2.2. Here \( d\mathbf{g} = dg^{(1)} \cdots dg^{(\ell)} \) is the product of the normalized Haar measures on \( \mathbb{O}_i(\mathbb{O}_d) \), \( i = 1, \ldots, \ell \), and \( d\theta = d\theta^{(1)} \cdots d\theta^{(\ell)} \) is the product of the uniform probability measures on the spheres \( S^{d_i-1}, i = 1, \ldots, \ell \).

In the second part of the argument, we use a slicing technique and a bound due to Wolff \( n = 3 \) and Erdogan \( n \geq 3 \) on the \( L^2 \) spherical average of the Fourier transform of a measure on \( \mathbb{R}^n \) to show that the multi-parameter Mattila integral

\[
\int_{\mathbb{R}^d} |\widehat{\nu}(\xi)|^2 \prod_{i=1}^d S^{d_i-1} |\widehat{\nu}(\xi)|^2 d\theta d\xi
\]

is finite for some Frostman measure \( \mu \) on \( E \) whose existence is implied by the dimension hypothesis \( \mathbf{1.1} \). This is Lemma 2.3.

2.1. Exploiting the Action of the Orthogonal Group.

**Lemma 2.1.** For any finite non-negative Borel measure \( \mu \) supported on \( E \),

\[
\int_{\mathbb{R}^d} |\widehat{\nu}(\eta)|^2 d\eta \lesssim \int_{\mathbb{R}^d} \prod_{i=1}^d \mathbb{O}_i(\mathbb{O}_d) |\widehat{\nu}(\xi)|^2 d\xi d\xi.
\]

**Proof.** We begin by fixing approximate identities on \( \mathbb{R}^\ell \) and \( \mathbb{R}^d \) as follows. We choose \( \phi \in C_0^\infty(\mathbb{R}^\ell) \) with \( \phi \geq 0 \), \( \text{supp}(\phi) \subseteq [-1,1]^{\ell} \), and \( \int \phi(x) dx = 1 \), and the associated approximate identity is \( \phi_\epsilon(x) = \epsilon^{-\ell} \phi(\epsilon^{-1} x) \) for \( \epsilon > 0 \). Similarly, we choose \( \psi \in C_0^\infty(\mathbb{R}^d) \) with \( \psi \geq 0 \), \( \text{supp}(\psi) \subseteq [-1,1]^d \), \( \int \psi(x) dx = 1 \), and \( \psi \geq \frac{1}{2} \) on \( [-\frac{1}{2}, \frac{1}{2}]^d \), and the associated approximate identity is \( \psi_\epsilon(x) = \epsilon^{-d} \psi(\epsilon^{-1} x) \) for \( \epsilon > 0 \).

Since \( \phi_\epsilon * \nu \to \widehat{\nu} \) and \( \psi_\epsilon * \nu_\mathbf{g} \to \widehat{\nu}_\mathbf{g} \) uniformly as \( \epsilon \to 0 \), Plancherel’s theorem tells us that Lemma 2.1 will be proved upon establishing that, for all \( \epsilon > 0 \),

\[
(2.1) \quad \int_{\mathbb{R}^\ell} (\phi_\epsilon * \nu)^2(t) dt \lesssim \int_{\mathbb{R}^d} \prod_{i=1}^d \mathbb{O}_i(\mathbb{O}_d) (\psi_\epsilon * \nu_\mathbf{g})^2(z) d\mathbf{g} dz,
\]

where \( c > 0 \) is a constant depending only on the diameter of \( E \).

For \( \epsilon > 0 \) and \( \mathbf{g} \in \prod_{i=1}^\ell \mathbb{O}_i(\mathbb{O}_d) \), we define the sets

\[
D(\epsilon) = \left\{ (u,v,x,y) \in E^4 : \left| x^{(i)} - y^{(i)} - u^{(i)} - v^{(i)} \right| \leq \epsilon \quad \forall 1 \leq i \leq \ell \right\},
\]

\[
G(\epsilon, \mathbf{g}) = \left\{ (u,v,x,y) \in E^4 : \left| x^{(i)} - y^{(i)} - g^{(i)}(u^{(i)} - v^{(i)}) \right| \leq \epsilon \quad \forall 1 \leq i \leq \ell \right\}.
\]

We will establish (2.1) by proving the following three inequalities:

\[
(2.2) \quad \int_{\mathbb{R}^\ell} (\phi_\epsilon * \nu)^2(t) dt \lesssim \epsilon^{-\ell} \mu^4(D(2\epsilon)),
\]

\[
(2.3) \quad \epsilon^{-\ell} \mu^4(D(\epsilon)) \lesssim \epsilon^{-d} \int \prod_{i=1}^d \mathbb{O}_i(\mathbb{O}_d) \mu^4(G(\epsilon, \mathbf{g})) d\mathbf{g},
\]

\[
(2.4) \quad \epsilon^{-d} \mu^4(G(\epsilon/4, \mathbf{g})) \lesssim \int_{\mathbb{R}^d} (\psi_\epsilon * \nu_\mathbf{g})^2(z) dz,
\]

where \( \mu^4 \) denotes the product measure \( \mu \times \mu \times \mu \times \mu \), and \( c = 2 \max \{2 \text{diam}(E), 1\} \) in (2.3).

We start by proving (2.2).
For $t \in \mathbb{R}^\ell$, we have

$$\phi_x * \nu(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_x \left( t_i - |x^{(i)} - y^{(i)}|, \ldots, t_\ell - |x^{(\ell)} - y^{(\ell)}| \right) d\mu(x)d\mu(y)$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \epsilon^{-2} \prod_{i=1}^\ell \chi \left\{ |t_i - |x^{(i)} - y^{(i)}|| \leq \epsilon \right\} d\mu(x)d\mu(y),$$

where $\chi_A$ denotes the indicator function of a set $A$. Therefore, by the triangle inequality,

$$\int_{\mathbb{R}^\ell} (\phi_x * \nu)^2(t) dt$$

$$\leq \epsilon^{-2\ell} \prod_{i=1}^\ell \chi \left\{ |t_i - |x^{(i)} - y^{(i)}|| \leq \epsilon \right\} \chi \left\{ |t_i - |u^{(i)} - v^{(i)}|| \leq \epsilon \right\} d\mu^4(u, v, x, y) dt$$

$$\leq \epsilon^{-2\ell} \prod_{i=1}^\ell \chi \left\{ |t_i - |x^{(i)} - y^{(i)}|| \leq \epsilon \right\} \chi \left\{ |x^{(i)} - y^{(i)}| - |u^{(i)} - v^{(i)}|| \leq 2\epsilon \right\} d\mu^4(u, v, x, y) dt$$

For fixed $x^{(i)}, y^{(i)} \in \mathbb{R}^d$, the set of $t_i \in \mathbb{R}$ with $|t_i - |x^{(i)} - y^{(i)}|| \leq \epsilon$ has Lebesgue measure $\approx \epsilon$. Thus integrating out $dt$ in the last integral yields (2.2).

Now we prove (2.4).

Our choice of $\psi$ guarantees that $\psi_x \geq \frac{1}{4} \epsilon^{-d}$ on $[-\frac{1}{2} \epsilon, \frac{1}{2} \epsilon]^d$. Thus, for all $z \in \mathbb{R}^d$,

$$\psi_x * \nu_G(z) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi_x(z^{(1)} - (x^{(1)} - g^{(1)} y^{(1)}), \ldots, z^{(\ell)} - (x^{(\ell)} - g^{(\ell)} y^{(\ell)})) d\mu(x)d\mu(y)$$

$$\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \epsilon^{-d} \prod_{i=1}^\ell \chi \left\{ |z^{(i)} - (x^{(i)} - g^{(i)} y^{(i)})| \leq \frac{\epsilon}{2} \right\} d\mu(x)d\mu(y).$$

Therefore, by the triangle inequality,

$$\int_{\mathbb{R}^d} (\psi_x * \nu_G)^2(z) dz$$

$$\geq \epsilon^{-2d} \prod_{i=1}^\ell \chi \left\{ |z^{(i)} - (x^{(i)} - g^{(i)} y^{(i)})| \leq \frac{\epsilon}{2} \right\} \chi \left\{ |z^{(i)} - (y^{(i)} - g^{(i)} y^{(i)})| \leq \frac{\epsilon}{2} \right\} d\mu^4(u, v, x, y) dz$$

$$\geq \epsilon^{-2d} \prod_{i=1}^\ell \chi \left\{ |z^{(i)} - (x^{(i)} - g^{(i)} y^{(i)})| \leq \frac{\epsilon}{4} \right\} \chi \left\{ |(x^{(i)} - g^{(i)} y^{(i)}) - (y^{(i)} - g^{(i)} y^{(i)})| \leq \frac{\epsilon}{4} \right\} d\mu^4(u, v, x, y) dz$$

$$= \epsilon^{-2d} \prod_{i=1}^\ell \chi \left\{ |z^{(i)} - (x^{(i)} - g^{(i)} y^{(i)})| \leq \frac{\epsilon}{4} \right\} \chi \left\{ |x^{(i)} - y^{(i)} - g^{(i)} y^{(i)} - v^{(i)}| \leq \frac{\epsilon}{4} \right\} d\mu^4(u, v, x, y) dz.$$

For fixed $x^{(i)}, u^{(i)} \in \mathbb{R}^d$ and $y^{(i)} \in \mathbb{O}(d_i)$, the set of $z^{(i)} \in \mathbb{R}^d$ with $|z^{(i)} - (x^{(i)} - g^{(i)} u^{(i)})| \leq \epsilon/4$ has Lebesgue measure $\approx \epsilon^d$. Thus integrating out $dz$ in the last integral yields (2.4).

Finally we prove (2.5).

Consider a fixed $1 \leq i \leq \ell$. For the action of $\mathbb{O}(d)$ on $\mathbb{R}^d$, the orbit of $e_{d_i}$ is $\text{Orb}(e_{d_i}) = \{ ge_{d_i} : g \in \mathbb{O}(d_i) \} = S^{d_i-1}$. We view the sphere $S^{d_i-1}$ as a metric space with the Euclidean metric from $\mathbb{R}^d$. We fix a cover of $S^{d_i-1}$ by balls of radius $\epsilon$ such that the number of balls in the cover is $N(\epsilon, i) \approx \epsilon^{-(d_i-1)}$ and such that the cover has bounded overlap (that is, each set in the cover intersects no more than $C$ other sets in the cover, where $C$ is a constant independent of $\epsilon$). We let $T^{(i)}_m$ for $m = 1, \ldots, N(\epsilon, i)$ denote the preimages of the balls with respect to the orbit map $g \mapsto ge_{d_i}$ from $\mathbb{O}(d)$ to $S^{d_i-1}$. Of course, the cover $\{ T^{(i)}_m : 1 \leq m \leq N(\epsilon, i) \}$ of $\mathbb{O}(d)$ also has bounded overlap. Moreover, since the image of the Haar measure on $\mathbb{O}(d_i)$ with respect to the orbit map is exactly the uniform probability measure on $S^{d_i-1}$, each $T^{(i)}_m$ has measure $\approx \epsilon^{d_i-1}$. 
For each non-zero \(w \in \mathbb{R}^d\), we define the conjugation (change of basis) map \(\zeta_w : \mathbb{O}(d_i) \rightarrow \mathbb{O}(d_i)\) by \(\zeta_w(g) = pgp^{-1}\), where \(p\) is a fixed but arbitrary transformation in \(\mathbb{O}(d_i)\) such that \(pe_{d_i} = w/|w|\). For each \(\epsilon > 0\) and \(g \in \prod_{i=1}^{\ell} \mathbb{O}(d_i)\), we define
\[
M(\epsilon) = \{(m_1, \ldots, m_\ell) \in \mathbb{N}^\ell : 1 \leq m_i \leq N(\epsilon, i) \quad \forall 1 \leq i \leq \ell\},
\]
\[
G'(\epsilon, g) = \left\{(u, v, x, y) \in E^4 : ||(x(i) - y(i)) - \zeta_u(i)\cdot v(i))(u(i) - v(i))|| \leq \epsilon \quad \forall 1 \leq i \leq \ell\right\}.
\]

**Claim.** For any collection of transformations \(g_m(i) \in T_m^{(i)}, 1 \leq i \leq \ell, 1 \leq m_i \leq N(\epsilon, i),\) we have
\[
D(\epsilon) \subseteq \bigcup_{m \in M(\epsilon)} G'(\epsilon, g_m),
\]
where \(c = 2\max\{2\text{diam}(E), 1\}\) and \(g_m(i) = (g_m^{(1)}, \ldots, g_m^{(\ell)}).
\]

**Proof of Claim.** Let \(u, v, x, y \in E\). It suffices to consider a fixed \(1 \leq i \leq \ell\). Let \(w = u(i) - v(i)\) and \(z = x(i) - y(i)\). Assume \(||z| - |w|| < \epsilon\). If \(w = 0\) or \(z = 0\), then \(||z| - |w|| < \epsilon\) for all \(g \in \mathbb{O}(d_i)\), and we are done. Assume \(w\) and \(z\) are non-zero. Choose \(g \in \mathbb{O}(d_i)\) such that \(g(w/|w|) = z/|z|\), and hence \(||z - gw|| = ||z| - |w|| < \epsilon\). Define \(g_0 = \zeta_w^{-1}(g)\). We know \(g_0 \in T_m^{(i)}\) for some \(1 \leq m_i \leq N(\epsilon, i)\). Since \(g_m(i) \in T_m^{(i)}\) also, we have \(|g_0e_{d_i} - g_m(i)e_{d_i}| < 2c\). By the definition of \(\zeta_w\), the previous inequality is equivalent to \(|gw - \zeta_w(i)w| < 2\epsilon|e_{d_i}|\epsilon\). Therefore, by the triangle inequality, \(||z - \zeta_w(i)w|| \leq \epsilon + 2\epsilon|w|\epsilon \leq 2\max\{2\epsilon|w|, 1\}\) \epsilon. To conclude, we note that \(||w|| = |u(i) - v(i)| \leq |u - v| = \text{diam}(E)\). \(\square\)

For each \(m \in M(\epsilon)\), we choose \(g_m(i) = (g_m^{(1)}, \ldots, g_m^{(\ell)}) \in \prod_{i=1}^{\ell} T_m^{(i)}\) such that
\[
\mu^i(G'(\epsilon, g_m)) \lesssim \epsilon^{-d_1-1} \cdots \epsilon^{-d_{\ell-1}} \int_{T_m^{(i)}} \mu^i(G'(\epsilon, g)) \, dg.
\]

Such a choice is possible because the average of a set must be larger than at least one element of the set. Then, using that \(\epsilon^{-\ell} \epsilon^{-d_1-1} \cdots \epsilon^{-d_{\ell-1}} = \epsilon^{-d}\), the claim implies
\[
\epsilon^{-\ell} \mu^i(D(\epsilon)) \leq \epsilon^{-\ell} \sum_{m \in M(\epsilon)} \mu^i(G'(\epsilon, g_m)) \lesssim \epsilon^{-d} \sum_{m \in M(\epsilon)} \int_{T_m^{(i)}} \mu^i(G'(\epsilon, g)) \, dg.
\]

Expanding things out, the integral on the right equals
\[
\int_{T_m^{(i)}} \int_{E_4} \chi \left\{ \left| (x(i) - y(i)) - \zeta_u(i)\cdot v(i)\right| \leq \epsilon \quad \forall 1 \leq i \leq \ell \right\} \, d\mu^4(u, v, x, y) \, dg
\]
\[
= \int_{E_4} \int_{T_m^{(i)}} \chi \left\{ \left| (x(i) - y(i)) - g(i)(u(i) - v(i))\right| \leq \epsilon \quad \forall 1 \leq i \leq \ell \right\} \, d\mu^4(u, v, x, y) \, dg.
\]
Thus, noting that \(\zeta_w(i)T_m^{(i)} : 1 \leq m_i \leq N(\epsilon, i)\) is a bounded overlap cover of \(\mathbb{O}(d_i)\) for each non-zero \(w \in \mathbb{R}^d\), we obtain
\[
\epsilon^{-\ell} \mu^i(D(\epsilon)) \lesssim \epsilon^{-d} \int_{E_4} \int_{\mathbb{O}(d_i)} \chi \left\{ \left| (x(i) - y(i)) - g(i)(u(i) - v(i))\right| \leq \epsilon \quad \forall 1 \leq i \leq \ell \right\} \, dg \, d\mu^4(u, v, x, y)
\]
\[
= \epsilon^{-d} \int_{\mathbb{O}(d_i)} \mu^i(G'(\epsilon, g)) \, dg.
\] \(\square\)

**Lemma 2.2.** For any finite non-negative Borel measure \(\mu\) supported on \(E\),
\[
\int_{\mathbb{O}(d_1)} \int_{\mathbb{R}^d} |\tilde{\mu}(\xi)|^2 \, d\xi \, dg \approx \int_{\mathbb{R}^d} |\tilde{\mu}(\xi)|^2 \int_{\mathbb{O}(d_1)} \int_{\mathbb{R}^d} |\tilde{\mu}(\xi)|^2 \, d\theta \, d\xi.
\]
Proof. By the definition of \( \nu_\mathbf{g} \), we have
\[
\hat{\nu}_\mathbf{g}(\xi) = \hat{\mu}(\xi)\hat{\mu}(-g^{(1)}T\xi^{(1)}, \ldots, -g^{(\ell)}T\xi^{(\ell)}),
\]
where \( T \) indicates transpose. Therefore
\[
\int_{\mathbb{R}^d} \int_{\prod_{i=1}^\ell O(d_i)} |\hat{\nu}_\mathbf{g}(\xi)|^2 \, dg \, d\xi = \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \int_{\prod_{i=1}^\ell O(d_i)} |\hat{\mu}(g^{(1)}\xi^{(1)}, \ldots, g^{(\ell)}\xi^{(\ell)})|^2 \, dg \, d\xi.
\]
We now consider the inner integral on the right for fixed non-zero \( \xi \in \mathbb{R}^d \). By a change of variable and the translation invariance of the Haar measures,
\[
\int_{\prod_{i=1}^\ell O(d_i)} |\hat{\mu}(g^{(1)}\xi^{(1)}, \ldots, g^{(\ell)}\xi^{(\ell)})|^2 \, dg = \int_{\prod_{i=1}^\ell O(d_i)} |\hat{\mu}(g^{(1)}e_{d_i}\xi^{(1)}, \ldots, g^{(\ell)}e_{d_i}\xi^{(\ell)})|^2 \, dg,
\]
where \( e_{d_i} = (0, \ldots, 0, 1) \in \mathbb{R}^{d_i} \). The stabilizer subgroup of \( O(d_i) \) for \( e_{d_i} \) is \( \text{Stab}(e_{d_i}) = \{ g \in O(d_i) : ge_{d_i} = e_{d_i} \} \). As \( O(d_i) \) is compact and \( \text{Stab}(e_{d_i}) \) is closed, \( \text{Stab}(e_{d_i}) \) is compact. We equip \( \text{Stab}(e_{d_i}) \) with its normalized Haar measure. The quotient space \( O(d_i)/\text{Stab}(e_{d_i}) \) is homeomorphic to the sphere \( S^{d_i-1} \). The measure on \( O(d_i)/\text{Stab}(e_{d_i}) \) is the image of the uniform probability measure on \( S^{d_i-1} \); it is a left-invariant Radon measure. Putting all this together, by the quotient integral formula (see, for example, [3], [5]), the last integral above equals a constant multiple of
\[
\int_{\prod_{i=1}^\ell O(d_i)/\text{Stab}(e_{d_i})} \int_{\prod_{i=1}^\ell \text{Stab}(e_{d_i})} |\hat{\mu}(g^{(1)}h^{(1)}e_{d_i}\xi^{(1)}, \ldots, g^{(\ell)}h^{(\ell)}e_{d_i}\xi^{(\ell)})|^2 \, dh \, dg
\]
\[
= \int_{\prod_{i=1}^\ell O(d_i)/\text{Stab}(e_{d_i})} \int_{\prod_{i=1}^\ell \text{Stab}(e_{d_i})} |\hat{\mu}(g^{(1)}e_{d_i}\xi^{(1)}, \ldots, g^{(\ell)}e_{d_i}\xi^{(\ell)})|^2 \, dh \, dg
\]
\[
= \int_{\prod_{i=1}^\ell O(d_i)/\text{Stab}(e_{d_i})} |\hat{\mu}(g^{(1)}e_{d_i}\xi^{(1)}, \ldots, g^{(\ell)}e_{d_i}\xi^{(\ell)})|^2 \, dg
\]
\[
= \int_{\prod_{i=1}^\ell S^{d_i-1}} |\hat{\mu}(\xi^{(1)}\theta^{(1)}, \ldots, \xi^{(\ell)}\theta^{(\ell)})|^2 \, d\theta.
\]
\[\square\]

2.2. Estimating the Multi-Parameter Mattila Integral.

Lemma 2.3. If
\[
\dim(E) > d - \frac{\min d_i}{2} + \frac{1}{3},
\]
then there exists a finite non-negative Borel measure \( \mu \) supported on \( E \) satisfying
\[
\int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \prod_{i=1}^\ell S^{d_i-1} |\hat{\mu}(\xi^{(1)}\theta^{(1)}, \ldots, \xi^{(\ell)}\theta^{(\ell)})|^2 \, d\theta \, d\xi < \infty.
\]

For the proof of Lemma 2.3, we need two lemmas. The first is an estimate for the \( L^2 \) spherical average of the Fourier transform due to Wolff [13] \((n = 2)\) and Erdoğan [5] \((n \geq 3)\).

Lemma 2.4. Let \( \lambda \) be a finite compactly supported Borel measure on \( \mathbb{R}^n \). If \( t, \epsilon > 0 \) and
\[
\frac{n}{2} \leq \alpha \leq \frac{n+2}{2},
\]
then
\[
\int_{S^n} |\hat{\lambda}(t\theta)|^2 \, d\theta \leq C \epsilon^{-\frac{n+2-\alpha}{2}} + t \, I_\alpha(\lambda),
\]
where
\[
I_\alpha(\lambda) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{-\alpha} d\lambda(x) d\lambda(y) = C_{n,\alpha} \int_{\mathbb{R}^n} |\hat{\lambda}(\xi)|^2 |\xi|^{-n+\alpha} \, d\xi.
\]
The second lemma that we need examines the behavior of Frostman-type measures on Cartesian products of differently-sized balls from coordinate hyperplanes. Here we let $B^d_\delta(x)$ denote the ball in $\mathbb{R}^d$ of radius $\delta$ centered at $x$, while we let $R^d_\delta(x)$ denote the box $x + [-\delta, \delta]^d \subseteq \mathbb{R}^d$.

**Lemma 2.5.** Suppose $0 < s \leq d$ and $\mu$ is a finite Borel measure on $\mathbb{R}^d$ satisfying
\[ \mu\left(B^d_\delta(x)\right) \lesssim \delta^s \]
for all $x \in \mathbb{R}^d$ and $\delta > 0$. If $x = (x^{(1)}, \ldots, x^{(\ell)}) \in \mathbb{R}^d$ and $\delta_1, \ldots, \delta_\ell > 0$ with $\delta_i \leq \delta_1$ for all $1 \leq i \leq \ell$, then
\[ \mu\left(B^d_{\delta_i}(x^{(1)}) \times \cdots \times B^d_{\delta_i}(x^{(\ell)})\right) \lesssim \delta_i^{s(d-\delta_i)}. \]
In particular, if $x = (x^{(1)}, \ldots, x^{(\ell)}) \in \mathbb{R}^d$ and $0 < \delta_1, \ldots, \delta_\ell \leq 1$, then
\[ \mu\left(B^d_{\delta_i}(x^{(1)}) \times \cdots \times B^d_{\delta_i}(x^{(\ell)})\right) \lesssim \prod_{i=1}^{\ell} \delta_i^{s(d-\delta_i)}. \]

**Proof.** For technical ease, we proceed using boxes instead of balls, noting that the results are equivalent. Fixing $x \in \mathbb{R}^d$ and $\delta_1, \ldots, \delta_\ell > 0$ with $\delta_i \leq \delta_1$ for all $1 \leq i \leq \ell$, we see that $R = R^d_{\delta_i}(x^{(1)}) \times \cdots \times R^d_{\delta_i}(x^{(\ell)})$ is precisely obtained by stretching $R^d_{\delta_j}(x)$ by a factor of $\delta_i/\delta_j$ in each hyperplane, so in particular $R$ is contained in $\prod_{j \neq i} [\delta_i/\delta_j]^d$ translated copies of $R^d_{\delta_j}(x)$.

Therefore,
\[ \mu(R) \lesssim \delta_i^{s} \prod_{j \neq i} \left[\delta_i/\delta_j\right]^d, \]
and the lemma follows. □

**Proof of Lemma 2.6.** Given the hypotheses of the lemma, we let $s = \dim(E)$, we define $\epsilon > 0$ by
\[ 4\epsilon = s - \left( d - \frac{\min d_i}{2} + \frac{1}{3} \right), \]
and we let $\mu$ be any finite non-negative Borel measure supported on $E$ satisfying
\[ \mu\left(B^d_\delta(x)\right) \lesssim \delta^{s-\epsilon} \]
for all $x \in \mathbb{R}^d$ and $\delta > 0$. The existence of $\mu$ is guaranteed by Frostman’s Lemma (see, for example, [7], [12]).

We will estimate the integral in (2.5) by iteratively applying Lemma 2.4 to “Fourier slice” measures $\lambda_i$ on $\mathbb{R}^d$, defined for fixed $\xi^{(1)}, \ldots, \xi^{(i-1)}, \xi^{(i+1)}, \ldots, \xi^{(\ell)}$ by
\[ \widehat{\lambda}_i(\xi^{(i)}) = \widehat{\mu}\left(\xi^{(1)}, \ldots, \xi^{(i)}, \xi^{(i+1)}, \ldots, \xi^{(\ell)}\right). \]
Indeed, the integral in (2.5) is
\[ \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \prod_{i=2}^{\ell} \int_{S^{d_i-1}} |\widehat{\lambda}_i(\xi^{(1)})| |\theta^{(1)}(\xi^{(i)})| \cdots |\theta^{(\ell)}(\xi^{(\ell)})| |\theta^{(1)}| \cdots |\theta^{(\ell)}| d\theta^{(1)} \cdots d\theta^{(\ell)} d\xi \]
\[ \lesssim \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \prod_{i=2}^{\ell} \int_{S^{d_i-1}} |\xi^{(1)}|^{-\frac{d_i+2\alpha_i-2}{4} + \epsilon} \left( \int_{\mathbb{R}^{d_i}} |\widehat{\lambda}_i(\eta^{(1)})| \cdots |\theta^{(1)}(\xi^{(i)})| \cdots |\theta^{(\ell)}(\xi^{(\ell)})| |\theta^{(1)}| \cdots |\theta^{(\ell)}| d\eta^{(1)} \cdots d\eta^{(\ell)} \right) d\xi \]
\[ \vdots \]
\[ \lesssim \left( \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \prod_{i=1}^{\ell} |\xi^{(i)}|^{-\frac{d_i+2\alpha_i-2}{4} + \epsilon} d\xi \right) \left( \int_{\mathbb{R}^d} |\widehat{\mu}(\eta)|^2 \prod_{i=1}^{\ell} |\eta^{(i)}|^{-d_i+\alpha_i} d\eta \right), \]
provided that $d_i^2 \leq \alpha_i \leq \frac{d_i^2 + 2}{2}$ for all $1 \leq i \leq \ell$. Expressing the integrals from the last line on the space side (using the “Fourier slice” measures and (2.6)), we obtain a constant multiple of
\[ \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \prod_{i=1}^{\ell} |x^{(i)} - y^{(i)}|^{-\frac{d_i+2\alpha_i-2}{4} + \epsilon - d} d\mu(x) d\mu(y) \right) \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \prod_{i=1}^{\ell} |x^{(i)} - y^{(i)}|^{-\alpha_i} d\mu(x) d\mu(y) \right). \]
By decomposing dyadically into regions where $2^{-j_i} \leq |x^{(i)} - y^{(i)}| \leq 2^{-j_i} + 1$ and then applying Lemma 2.5 and (2.7), we see that convergence of the integrals is implied by convergence of the sums

$$\prod_{i=1}^{\ell} \sum_{j_i=0}^{\infty} 2^{-j_i} \left( d_i + 2\alpha_i - s - (d - d_i) - 2\epsilon \right) \quad \text{and} \quad \prod_{i=1}^{\ell} \sum_{j_i=0}^{\infty} 2^{-j_i} (-\alpha_i + s - (d - d_i) - \epsilon).$$

Convergence of the former sum is equivalent to $d_i + 2\alpha_i - s - (d - d_i) - 2\epsilon > d_i + 2\epsilon$ for all $1 \leq i \leq \ell$. Convergence of the latter sum is equivalent to $\alpha_i < s - (d - d_i) - \epsilon$ for all $1 \leq i \leq \ell$. Recalling the definition of $\epsilon$ and the requirement that $\frac{1}{4} \leq \alpha_i \leq \frac{d_i + 2\epsilon}{2}$ for all $1 \leq i \leq \ell$, we see that all inequalities can be satisfied by setting

$$\alpha_i = \min \left\{ s - (d - d_i) - 2\epsilon, \frac{d_i + 2\epsilon}{2} \right\}.$$

□

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