On the Gravitational Energy Shift for matter waves

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Abstract

The gravitational energy shift for photons is extended to all mass-equivalent energies $E = mc^2$, obeying the quantum condition $E = h\nu$. On an example of a relativistic binary system, it was shown that the gravitational energy shift would imply, in contrast to Newtonian gravity, the gravitational attraction between full mass-equivalent energies. The corresponding space-time metric becomes exponential. A good agreement was found with all results of weak field tests of General relativity. The strong field effects in a binary system can be easily studied. A long standing problems of Pioneer and other flyby anomalies were also discussed in connection with the violation of total energy conservation. It was shown that relatively small energy non-conservation during the change of the orbit type could explain these persistent anomalies.

1 Introduction

The gravitational energy shift for photons is now a well established and experimentally tested phenomena[1], predicted by Einstein’s Equivalence principle [2]. However, Einstein also established the well known mass-energy relation , $E = mc^2$, according to which all types of energies should be subjected to a gravitational energy shift. The corresponding gravitational frequency shift for all matter waves is then obtained by making use of the quantum condition $E = h\nu$. This extension has some interesting consequences worth exploring here.

As a particular example ,let us consider an isolated, compact self-gravitating binary system with point-like proper masses $M_0$ and $m_0 << M_0$. In the rest frame of $M_0$, the gravitational field around $M_0$ is approximately static, spherically symmetric, and isotropic. The motion of the mass $m_0$ is then described
by the following Lagrangian:

\[ L = -m_0c^2 \frac{d\tau}{dt}, \]

where

\[ d\tau = dt \sqrt{g_{tt}(r) - g_{rr}(r)\tilde{\beta}^2} = dt_g \sqrt{1 - \tilde{\beta}_g^2} \]

links together three different clocks showing: the proper time, \( \tau \), the observer’s time, \( t \), and the gravitational time, \( t_g \), respectively. The components of space-time metric different from zero are \( g_{tt} \) and \( g_{rr} = g_{tt}^2[3] \). Different \( \tilde{\beta} \)-symbols denote dimensionless velocities, \( \tilde{\beta} = \vec{v}/c \) and \( \tilde{\beta}_g = g_{rr}\tilde{\beta} \), respectively. Note that \( c\tilde{\beta}_g = d\vec{r}_g/dt \) where \( d\vec{r}_g = \sqrt{g_{rr}}d\vec{r} \) and \( dt_g = \sqrt{g_{tt}}dt \).

In the 4-vector notation, \( p^\mu = (E/c, p^i) \), the relativistic energy \( E \), and the 3-momenta \( p^i \) of the mass \( m_0 \) are defined as: \( E = mc^2 = m_0c^2 dt/d\tau \) and \( p^i = mv^i = m_0 v^i dt/d\tau \). The corresponding canonical 4-momenta, \( p_\mu = (H/c, p_i) \) are obtained from the Lagrangian using:

\[ p_i = \frac{\partial L}{\partial \dot{v}^i} = g_{rr}p^i, \]

and \( H = p_i v^i - L \) to find:

\[ H = g_{tt}E = \sqrt{g_{tt}} \sqrt{p_i p^i c^2 + m_0^2 c^4}. \]

The orbits of \( m_0 \) are characterized by two constants (integrals) of motion: the total energy, \( H \) and the orbital angular momentum, which are both derived from the Lagrangian equation of motion, or more directly from the equivalent mass-shell condition: :

\[ HE = p_i p^i c^2 + m_0^2 c^4. \]

In the polar coordinate system, \((r, \theta, \varphi)\), the equatorial motion \((\theta = \pi/2)\), of the mass \( m_0 \) is described by:

\[ \left( \frac{dr}{d\tau} \right)^2 = c^2 (h^2 - g_{tt} - g_{tt}^2 \frac{l^2}{r^2}) \]

where \( h = H/m_0 c^2 \) and \( c g_{tt} l_\varphi = r^2 d\varphi / d\tau \) are two integrals of motion expressing the conservation of total energy and angular momentum. Stable bound
and unbound orbits are determined by solving the algebraic equation:

\[ h^2 = g_{tt} + \frac{g_{tt} l^2_{\varphi}}{r^2}. \]  

(7)

For a given \( g_{tt} \), the strong field orbital characteristics are best studied in the \((u, \varphi)\) coordinates, where \( u = R_g/r \) and \( R_g = GM_0/c^2 \) denotes the gravitational radius of \( M_0 \). In these coordinates (6) becomes

\[ \left( \frac{du}{d\varphi} \right)^2 + u^2 = \left( h^2 g_{rr} - 1 \right) \frac{g_{rr}}{l^2} \]  

(8)

where \( R_g l = l_{\varphi} \).

In the next Section, we are going to show how the gravitational energy shift can be used to obtain an explicit form for the metric \( g_{tt}(r) \).

## 2 Gravitational energy shift for matter waves

The total gravitational energy shift consists of two different contributions: one due to the position of the mass \( m_0 \) in the gravitational field, and other due to its relative motion. This becomes obvious if we express the relativistic energy of the mass \( m_0 \) in the form:

\[ E = \frac{E_g}{\sqrt{1 - \beta_g^2}}, \]  

(9)

where \( E_g = m_0 c^2 / \sqrt{g_{tt}} \) and \( \beta_g = g_{rr} \beta \).

Then, the total gravitational energy shift, defined as \( \Delta E/E \) is

\[ \frac{\Delta E}{E} = \frac{\Delta E_g}{E_g} + \frac{\beta_g \Delta \beta_g}{1 - \beta_g^2}, \]  

(10)

where \( \Delta E_g/E_g \) denotes a pure gravitational energy shift defined as a ratio between \( E_g \)-energies at two different radial distances of the mass \( m_0 \) from \( M_0 \):

\[ \frac{E_g(r_2)}{E_g(r_1)} = \frac{\Delta E_g}{E_g} + 1 = \sqrt[4]{g_{tt}(r_1)/g_{tt}(r_2)}. \]  

(11)
In the weak field approximation, \( g_{tt} = 1 - 2u + \ldots \), a well known expression

\[
\frac{\Delta E_g}{E_g} = \frac{R_g}{r_1r_2}(r_1 - r_2) + \ldots
\]  

is obtained. The full gravitational frequency shift is obtained by replacing \( E = E_g/\sqrt{1 - \beta_g^2} \) with \( h\nu \) and \( E_g \) with \( h\nu_g \). The higher order terms in (12) become important at relativistic velocities, \( \beta_g \sim 1 \), and in strong gravitational fields, when an explicit form of \( g_{tt} \) is needed. The Einstein’s general relativity offers one solution, the Schwarzschild metric [4], but there are many alternative solutions that all give the same weak-field results [5].

However, we have found one more or less obvious solution for \( g_{tt} \), based on the infinitesimal form, of the gravitational energy shift, \( dE_g/E_g = d(R_g/r) = du \). The formal integration of this equation yields to \( E_g = m_0c^2\exp(u) \) and to

\[
g_{tt} = e^{-2u}.
\]  

(13)

We recognize here the exponential metric, first introduced by Yilmaz [11].

The most interesting feature of this solution is the observation that the gravitational energy shift also implies the form of the radial gravitational force which is according to

\[
dE_g = -dU_g = drF_g = E_g du,
\]  

where \( U_g = m_0c^2(1 - e^u) \) is the gravitational potential energy of the form

\[
F_g = -G e^u \frac{M_0 m_0}{r^2}.
\]  

(15)

It modifies Newtonian gravity at short distances by allowing for a variation of the gravitational constant \( G \) with distance, \( G(r) = Ge^u \) [10]. This radial force can also be interpreted as a force acting between two full mass-equivalent energies, in our case \( M_0c^2 \) and \( E_g = m_0e^u c^2 \). In the following, we shall apply this new force to the problem of explaining the Pioneer and flyby anomalies.

### 3 On the Pioneer and Flyby Anomalies

The Pioneer anomaly refers to a systematic observation [7] that NASA’s two Pioneer 10 and 11 spacecrafts, launched in the early 1970s, are slowing down more than expected on their ways out of the solar system. This anomalous
deceleration towards the Sun is almost constant and uniform of a magnitude $a_P = (8.741.33) \times 10^{-10}$ ms$^{-2}$, for heliocentric distances greater than 20 AU [7]. This anomaly remained largely unexplained within the currently accepted Newton-Einstein laws of gravitation [9]. Any pure gravitational explanation of the Pioneer anomaly should also face a difficult problem: of agreeing with the very precise cartographic data of the solar system. Similar temporal anomalies have been observed during several planetary flyby missions [8]. It is interesting that all anomalous accelerations occurred after the spacecrafts have changed their orbital parameters by means of planetary flybys, during which the energy can not be conserved. This temporal energy transfer can be studied within our relativistic binary model applied to the Sun (Earth)-spacecraft system. During the flyby energy transfer the spacecraft radial acceleration consists of two contributions: one coming from the energy conserving part and the other coming from the energy non-conserving part, in the form $a = a_C + a_A$ where

$$a_C = \frac{1}{2} \frac{\partial}{\partial r} \left( \frac{dr}{dt} \right)^2$$
$$a_A = \frac{1}{2} \frac{dh}{dr} \frac{\partial}{\partial h} \left( \frac{dr}{dt} \right)^2$$

where connection with $\vec{v}^2$ is given by

$$\vec{v}^2 = c^2(1 - \frac{2u}{\eta^2})g_{tt}^2 = \left( \frac{dt}{\eta} \right)^2 + c^2g_{rr}^2\left( \frac{dr}{\eta} \right)^2.$$ 

The energy transfer during the change of the orbit type is accompanied with a temporal non-conservation of total energy, it is not difficult $h$, resulting in an anomalous acceleration,

$$2a_A = -c^2[1 - 6u + (6\beta + 12 + \beta^2)u^2 + \ldots] \frac{d}{dr}(h^{-2}),$$

where we have used $g_{tt} = 1 - 2u + 2\beta u^2 + O(u^3)$. The exponential metric is characterized by $\beta = 1$. In the case of a Pioneer anomaly the required relative violation of total energy is of the order of $10^{-16}$, or more precisely $\Delta h/h \sim 1.36h^2 \times 10^{-16}$. There exist, however, many other, more or less artificial, attempts to explain these flyby anomalies [7].

4 Discussion and Conclusion

In this letter we studied the implications of extending the gravitational energy shift for photons to all matter waves. The infinitesimal form of the gravitational energy shift implies that the gravitational attractive force should in fact act between all mass-equivalent energies. The corresponding space-time metric
is found to be exponential. A long time ago this metric was first introduced by Yilmaz [11] in an attempt to modify the Einstein’s field equations of general relativity. However, the Yilmaz theory was sharply criticized[12] on various grounds [13] as being ill defined and without an event horizon occurrence [14].

In our Lagrangian approach, the exponential metric appears as one of the solutions leading to the observed gravitational energy shift. Neither use of general relativity nor the Yilmaz theory was made. In fact we used only the theory of special relativity and the law of conservation of total energy in the binary system. Our particular solution was based on the observational fact that a proper mass \( m_0 \) is always observed as a gravitational mass \( m_g = m_0e^u \) in a gravitational field. As a consequence the gravitational attraction appears as acting between full mass-equivalent energies.

It is also well known that exponential metric belongs to a large class of alternative theories of gravity, which all agree with the solar observational tests. In contrast to Einstein’s general relativity, the exponential metric does not predict formation of an event horizon[14].

We also tried to see whether a new force of gravity, \( F_g = F_Ne^u \) was able to explain the Pioneer and flyby anomalies. Unfortunately, the anomalies are too large to be explained by modifying the Newtonian gravity without violating the well established cartographic data of the solar system. Both, exponential and Schwarzschild metrics give unsatisfactory answers, \( a = a_N(1 + 2u + ...) \). However, both Pioneer spacecrafts and other flybys have changed their orbital types from bound to unbound states and vice versa. During this maneuver the energy transfer took place violating the law of energy conservation. Taking this into account, we produced an explicit formula by which the Pioneer anomaly can be explained. The same mechanism works for flyby anomalies.

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