PV-Reduction of Sunset Topology with Auxiliary Vector

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ABSTRACT: Passarino-Veltman (PV) reduction method has been proved to be very useful for the computation of general one-loop integrals. However, not much progress has been made when applying to higher loops. Recently, we have improved the PV-reduction method by introducing an auxiliary vector. In this paper, we apply our new method to the simplest two-loop integrals, i.e., the sunset topology. We show how to use differential operators to establish algebraic recursion relations for reduction coefficients. Our algorithm can be easily applied to the reduction of integrals with arbitrary high-rank tensor structures. We demonstrate the efficiency of our algorithm by computing the reduction with the total tensor rank up to four.

KEYWORDS: PV-reduction, Sunset Topology, Auxiliary Vector

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1 Introduction

Calculation of Feynman integrals at the multi-loop level is of great importance for both perturbation quantum field theory and particle experiments. The general strategy is to expand arbitrary integrals to finite simpler integrals (called Master Integrals) and then do the integration of master integrals only. Thus, tensor reduction of Feynman integrals is one of the critical steps for various realistic calculations in the Standard Model. There is a large body of research for one-loop reduction [1–25]. For reducing multi-loop integrals, various ideas and methods have been developed. There are some works on two-loop tensor reduction for some special integrals [26–31], while algorithm for the reduction of general Feynman integrals was proposed in [32–37]. Among these methods, the Integration-By-Parts (IBP) method [36, 37] is widely
used and has been implemented by some powerful programs such as FIRE, LiteRed, Kira, etc [38–45]. Although its popularity, the number of IBP identities grows very fast for the increasing number of mass scalars and higher rank of tensor structure, it will become a bottleneck when dealing with complicated physical processes. Even at the one-loop level, reducing a general tensor massive pentagon with the IBP method is difficult. Thus it is always welcome to find more efficient reduction method.

PV-reduction [3] for one-loop integrals has played a significant role in general one-loop computations in history. However, when trying to generalize the method to higher loops, it becomes hard, and there are not many works in the literature. In [26, 46], the tensor two-loop integrals with only two external legs are reduced to scalar integrals with PV-reduction method. However, an extra massless propagator appears. Integrals with new topologies have been introduced as master integrals to address the problem. This results against the intuitive picture: Reduction is achieved by removing one or more propagators from the original topology. Thus the reduction and master integrals should be constrained to the original topology and its sub-topologies.

Recently, we have generalized the original PV-reduction method for one-loop integrals by introducing an auxiliary vector \( R \) in [47, 48]. Our improved PV-reduction method can be easily carried out and compute reduction coefficients analytically by employing algebraic recursion relations. With these analytical expressions, one can do many things. For example, by taking limits properly, we can solve the reduction with propagators having general higher powers [49]. When external momenta are not general, i.e., the Gram determinant becomes zero, master integrals in the basis will not be independent anymore. We can systematically study their degeneration [50].

Encouraged by the results in [47, 48], we want to see if such an improved PV-reduction method can be transplanted to higher loops. When moving to two loops, some complexities arise. The first is that we need to determine the master integrals. For one-loop, the master integrals are trivially known. For higher loops, the choice of master integrals becomes an art. With proper choice, one can reduce the computation greatly [51]. The second is that when intuitively generalizing our method to two loops, we will meet the irreducible scalar products, which can not be reduced. In fact, these two complexities are related to each other. To avoid unnecessary complexity, we will focus on the simplest nontrivial two-loop integrals, i.e., the sunset topology in this paper. With this example, we will show how to generalize our new PV-reduction method to two loops.

This paper is structured as follows. In section 2, we briefly review our previous work on one-loop tensor reduction and discuss the main idea for the reduction of sunset integrals. In section 3, we derive recursion relations for sunset integrals using differential operators and the proper choice of the master integrals in our algorithm. In section 4, we demonstrate our algorithm successfully to sunset integrals with the total rank from one to four. In section 5, we give some discussions and the plan for further studying. Technical details are collected in Appendix.
where we denote $P_j$ as propagators for one-loop integrals while $D_j$ for two-loop integrals

$$I_{n+1}^{µ_1µ_2...µ_m} \equiv \int \frac{d^D \ell}{i\pi^D/2} \frac{\ell^{µ_1}\ell^{µ_2}...\ell^{µ_m}}{P_0 \prod_{j=1}^{n-1} P_j} = \int \frac{d^D \ell}{i\pi^D/2} \frac{\ell^{µ_1}\ell^{µ_2}...\ell^{µ_m}}{(\ell^2 - M_0^2) \prod_{j=1}^{n} ((\ell - K_j)^2 - M_j^2)}, \quad (2.1)$$

one can recover its tensor structure by multiplying each index with an auxiliary vector $R_{i;µ}$. Furthermore, we can combine these $R_i$ to $R = \sum_{i=1}^{m} x_i R_i$ to simplify the expression (2.1) to

$$I_{n+1}^{(m)} = \int \frac{d^D \ell}{i\pi^D/2} \frac{(2 R \cdot \ell)^m}{P_0 \prod_{j=1}^{n} P_j}. \quad (2.2)$$

The good point using (2.2) instead of (2.1) is that we have avoided the complicated Lorentz tensor structure in the reduction process, while they can easily be retained using the expansion of $R$ and taking corresponding coefficients of $x_i$'s. Since we are using dimensional regularization, we like to keep the $D$ as a general parameter. For $D = d - 2\epsilon$, the master basis is the scalar integrals with $n \leq d + 1$ propagators, i.e., the reduction results can be written as

$$I_n^{(m)} = C_{n;n}^{(m)} I_n + C_{n\rightarrow n;i}^{(m)} I_{n;i} + C_{n\rightarrow n;ij}^{(m)} I_{n;ij} + \ldots \quad (2.3)$$

where we denote $I_{n;i;\ldots;a}$ as the integrals got by removing propagators $P_{i_1}, P_{i_2}, \ldots, P_{i_a}$ from $I_n$. For simplicity, we denote

$$s_{ij} \equiv K_i \cdot K_j, \quad s_{0i} \equiv R \cdot K_i, \quad f_i \equiv M_0^2 - M_{i}^2 + s_{ii}. \quad (2.4)$$

By the explicit permutation symmetry in (2.1), we only need to calculate $C_{n+1\rightarrow \{0,1,...,r\}}^{(m)} \equiv C_{n+1\rightarrow n+1,\tilde{r}+1,\tilde{r}+2,...,\tilde{n}}^{(m)}$, while other reduction coefficients can be got by proper replacements and momentum shifting. With the introduction of auxiliary vector $R$ in (2.2), one can expand the reduction coefficients according to their tensor structure

$$C_{n+1\rightarrow \{0,1,...,r\}}^{(m)} = \sum_{2a_0 + \sum_{k=1}^{n} a_k = m} \left\{ c_{a_1,\ldots,a_n}^{(0,1,...,r)} (m)(M_0^2)^{a_0 + r - n} \prod_{k=0}^{n} s_{0k}^{a_k} \right\}. \quad (2.5)$$

By acting two types of differential operators $D_i \equiv K_i \cdot \frac{\partial}{\partial \ell^i}$ and $T \equiv \eta^{µν} \frac{\partial}{\partial \ell^µ} \frac{\partial}{\partial \ell^ν}$ on (2.2), we get the recursion relations for the expansion coefficients $c_{a_1,\ldots,a_n}^{(0,1,...,r)}$

$$c_{(0,1,\ldots,r)}^{(0,1,\ldots,r)}(a_1, \ldots, a_n; m) = T^{-1} \tilde{G}^{-1} O^{(0,1,\ldots,r)}(a_1, \ldots, a_n; m), \quad (2.6)$$

$$c_{(0,1,\ldots,r)}^{(0,1,\ldots,r)}(2k) = \frac{2k - 1}{D + 2k - n - 2} \left[ (1 - \alpha T \tilde{G}^{-1} \alpha c_{(0,1,\ldots,r)}^{(0,1,\ldots,r)}(2k - 2) + \alpha T \tilde{G}^{-1} c_{(0,1,\ldots,r)}^{(0,1,\ldots,r)}(2k - 2) \right], \quad (2.7)$$
\[ \alpha \equiv \left( \frac{f_1}{M_0^n}, \frac{f_2}{M_0^n}, \ldots, \frac{f_n}{M_0^n} \right), \quad [c^{(0,1,\ldots,r)}(a_1, \ldots, a_n;m)]_i \equiv c_{a_1,a_2,\ldots,a_i+1,\ldots,a_n}^{(0,1,\ldots,r)}(m), \quad (2.8) \]

and

\[ [O^{(0,1,\ldots,r)}(a_1, \ldots, a_n;m)]_i \equiv m \alpha_i c_{a_1,\ldots,a_i}^{(0,1,\ldots,r)}(m-1) - m \delta_{0a_i} c_{a_1,\ldots,a_i,\ldots,a_n}^{(0,1,\ldots,r)}(m-1;\bar{i}) - (m+1 - \sum_{l=1}^{n} a_l) c_{a_1,\ldots,a_l-1,a_i,\ldots,a_n}^{(0,1,\ldots,r)}(m), \]

\[ c_{0,\ldots,0}^{(0,1,\ldots,r)}(m) \equiv \left( 0, 0, \ldots, 0, \underbrace{0}_{n-1 \text{ times}}, 0(m; \underbrace{r+1}_{n-1 \text{ times}}), \ldots, \underbrace{0}_{n-1 \text{ times}}(m; \underbrace{n}_{n-1 \text{ times}}) \right). \quad (2.9) \]

where \( \bar{i} \) indicates to drop the index \( a_i \). With the known boundary conditions, one can obtain all reduction coefficients by applying the recursions (2.6) and (2.7) iteratively.

One important point of above reduction method is that we need to use both \( D_i \) and \( T \) operators to completely fix unknown coefficients in (2.5). One simple explanation is that in (2.5) coefficients there are \((n+1)\) indices \((a_i, i = 1, \ldots, n)\) and rank \( m \), so naively \((n+1)\) relations are needed. More accurate explanation is a little tricky. Let us take the bubble, i.e., \( n = 1 \) as an example. As shown in Table 1, for rank \( m \), there are \( N_c = \lceil \frac{m+1}{2} \rceil + 1 \) unknown coefficients while the number of independent \( D \)-type equations is \( N_D = \lceil \frac{m-1}{2} \rceil \). Thus, only using the \( D \)-type operators we can fix all coefficients with odd rank \( m = (2k+1) \) by these known terms with lower rank \( m \leq 2k \). However, for \( m = 2k + 2 \), \( N_c - N_D = 1 \), thus just using \( D \)-type relations is not enough and we need to adopt the \( T \) operator to provide one extra independent relation. In fact, \( T \) provides more than just one relation, but other relations are not independent to these coming from \( D \) and can be taken as the consistent check of the reduction method.

In this paper, we want to generalize our reduction method from one-loop to two-loop integrals. Again, with simplified tensor structure, we define the general tensor integrals of

| rank \( m \) | \( N_c \) | \( N_D \) | \( N_{T∪D} \) | \( N_c - N_{T∪D} \) |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 |
| 2 | 2 | 1 | 2 | 0 |
| 3 | 2 | 2 | 2 | 0 |
| 4 | 3 | 2 | 3 | 0 |
| 5 | 3 | 3 | 3 | 0 |
| 6 | 4 | 3 | 4 | 0 |

Table 1: Number of expansion coefficients and independent equations for tensor bubble.
sunset topology as\footnote{For simplicity, we denote the scalar integral $I_{a_1,a_2,a_3}^{(r_1,r_2)} \equiv I_{a_1,a_2,a_3}^{(0,0)}$.}

\begin{equation}
I_{a_1,a_2,a_3}^{(r_1,r_2)} = \int \frac{d^D \ell_1 \; d^D \ell_2 \; (2\ell_1 \cdot R_1)^{r_1} (2\ell_2 \cdot R_2)^{r_2}}{i\pi^{D/2} i\pi^{D/2}} \frac{D_1^{(r_1,r_2)} D_2^{(r_1,r_2)} D_3^{(r_1,r_2)}}{D_1 D_2 D_3} \tag{2.10}
\end{equation}

where the propagators are

\begin{equation}
D_1 \equiv \ell_1^2 - M_1^2, \quad D_2 \equiv \ell_2^2 - M_2^2, \quad D_3 \equiv (\ell_1 + \ell_2 - K)^2 - M_3^2. \tag{2.11}
\end{equation}

In this paper we consider only the reduction with all $a_i = 1$. For $a_i \geq 2$, one can use the same strategy presented in [49]. For simplicity, we denote

\begin{equation}
\int d\ell_i(\bullet) \equiv \int \frac{d^D \ell_i}{i\pi^{D/2}}, \quad \int d\ell_{i,2}(\bullet) \equiv \int \frac{d^D \ell_1}{i\pi^{D/2}} \frac{d^D \ell_2}{i\pi^{D/2}} = \int_{1,2} (\bullet). \tag{2.12}
\end{equation}

Similar to what we do for one-loop case, we can expand reduction coefficients according to their tensor structure\footnote{We will discuss the choice of master integrals $J$ in section 3.3.}

\begin{equation}
I_{i_1,i_2,j}^{(r_1,r_2)} = C_{i_1,i_2,j}^{(r_1,r_2)} J = \sum_{i_1,i_2,j} s_{i_0} s_{i_1} s_{i_2} s_{j_0} s_{j_1} s_{j_2} \frac{r_1 - i_1 - j}{s_{00}} \frac{r_2 - i_2 - j}{s_{00'}} \bar{\alpha}_{i_1,i_2,j}^{(r_1,r_2)} J \tag{2.13}
\end{equation}

where we have written $C$, $J$ to emphasize they are vector, while other kinematic variables are

\begin{equation}
s_0 = R_1 \cdot K, \quad s_{01} = R_2 \cdot K, \quad s_{11} = K^2, \quad s_{00} = R_1^2, \quad s_{00'} = R_2^2, \quad s_{00'} = R_1 \cdot R_2. \tag{2.14}
\end{equation}

In (2.13), the expansion coefficients $\bar{\alpha}_{i_1,i_2,j}^{(r_1,r_2)}$ are actually vectors with components corresponding to different master integrals $J_i$. The summation over indices $\{i_1,i_2,j\} \geq 0$ admits to the restriction $0 \leq (r_l - i_l - j)/2 \in \mathbb{N}, l = 1, 2$. We make the indices free by setting all $\bar{\alpha}_{i_1,i_2,j}^{(r_1,r_2)} = 0$ for inappropriate indices throughout the paper, so we will drop the prime in other summations. Similar to the strategy for one-loop case, we solve expansion coefficients from lower rank levels to higher rank levels iteratively, where the \textbf{rank level} is defined as the total rank $(r_1 + r_2)$.

When we do the tensor reduction, we will reach the sub topologies where one of the propagators in (2.11) is removed. For these cases, the integrals are reduced to the product of two-one-loop tadpoles. Their tensor reduction has been solved in Appendix B. To simplify notation, we denote

\begin{equation}
I_{i_1,i_2,i}^{(r_1,r_2)} = \int \frac{d^D \ell_1 \; d^D \ell_2 \; (2\ell_1 \cdot R_1)^{r_1} (2\ell_2 \cdot R_2)^{r_2}}{i\pi^{D/2} i\pi^{D/2}} \frac{D_i}{D_1 D_2 D_3} \tag{2.15}
\end{equation}

and their reductions are written as

\begin{equation}
I_{i_1,i_2,i}^{(r_1,r_2)} = C_{i_1,i_2,i}^{(r_1,r_2)} J = \sum_{i_1,i_2,j} s_{i_0} s_{i_1} s_{i_2} s_{j_0} s_{j_1} s_{j_2} \frac{r_1 - i_1 - j}{s_{00}} \frac{r_2 - i_2 - j}{s_{00'}} \bar{\alpha}_{i_1,i_2,j}^{(r_1,r_2)} J. \tag{2.16}
\end{equation}
In (2.13), there are five indices for the expansion coefficients, i.e., \(i_1, i_2, j, r_1, r_2\). With the experience from one-loop integrals, every index needs a recursion relation, so totally five kinds of differential operators are needed to give sufficient recursion relations. From the form of (2.2) we can construct three \(T\)-type operators

\[
T_{00} \equiv \eta^{\mu\nu} \frac{\partial}{\partial R_{1\mu}} \frac{\partial}{\partial R_{1\nu}}, \quad T_{0\nu'} \equiv \eta^{\mu\nu} \frac{\partial}{\partial R_{2\mu}} \frac{\partial}{\partial R_{2\nu}}, \quad T_{0\nu'}' \equiv \eta^{\mu\nu} \frac{\partial}{\partial R_{1\mu}} \frac{\partial}{\partial R_{2\nu}}. \tag{2.17}
\]

By applying these operators on the tensor integrals (2.2), we produce combinations \(\ell_1^2, \ell_2^2, \ell_1 \cdot \ell_2\) in the numerator respectively. Using the following algebraic decomposition

\[
\ell_1^2 = D_1 + M_1^2, \quad \ell_2^2 = D_2 + M_2^2, \quad 2\ell_1 \cdot \ell_2 = D_3 + M_3^2 - D_1 - M_1^2 - D_2 - M_2^2 - K^2 + 2\ell_1 \cdot K + 2\ell_2 \cdot K \tag{2.18}
\]

we can write the result as a sum of terms with one-lower or two-lower rank levels and terms for lower topologies, thus we can establish three recursion relations for these operators with expansion (2.13).

Having discussed the operators \(T_i\), it is natural to think about following two \(D\)-type operators \(K^\mu \cdot \frac{\partial}{\partial R_{1\mu}}\) and \(K^\mu \cdot \frac{\partial}{\partial R_{2\mu}}\). However, when acting on (2.2), it gives the combinations \(\ell_1 \cdot K\) and \(\ell_2 \cdot K\). These two factors do not have simple algebraic decompositions like (2.18). Thus it is not so easy to find corresponding recursion relations. One key input in our paper is that the recursion relation of \(D\)-type operators can be established if we consider the reduction of the sub one-loop integrals first. As we will show, such a recursion relation is highly nontrivial and more discussions will be given later.

3 Recursion relations for tensor integrals of sunset topology

In this section, we first derive the recursion relations for expansion coefficients of three \(T\)-type operators. Then we establish another two recursion relations of \(D\)-type operators by reducing the left/right sub one-loop first. As we will see, in the process, one needs to use a highly nontrivial relation of the reduction coefficients of tensor bubbles. Combining the recursion relations from \(T\)-type and \(D\)-type operators, we get five relations. When using them to solve expansion coefficients, we find that not all coefficients can be fixed. It indicates that some integrals with nontrivial numerators should be recognized as master integrals.

After introducing two auxiliary vectors \(R_1, R_2\), we can define differential operators (where we use the shorthand \(\partial_A = \frac{\partial}{\partial A}\))

\[
\frac{\partial}{\partial R_{1\mu}} = 2R_{1\mu} \partial_{s_{00}} + K_{\mu} \partial_{s_{01}} + R_{2\mu} \partial_{s_{0'}} + R_{1\mu} \partial_{s_{0'}} ; \quad \frac{\partial}{\partial R_{2\mu}} = 2R_{2\mu} \partial_{s_{0'}} + K_{\mu} \partial_{s_{0'}} + R_{1\mu} \partial_{s_{0'}}. \tag{3.1}
\]

Using the two expressions and external Lorentz vectors, we can get following Lorentz invariant combinations, i.e., three \(T\)-type operators \(T_{00}, T_{0\nu'}, T_{0\nu'}'\) defined in (2.17) and six \(D\)-type
operators

\[ D_{10} \equiv K \cdot \frac{\partial}{\partial R_1^0}, \quad D_{10'} \equiv K \cdot \frac{\partial}{\partial R_1^{0'}}, \quad D_{00} \equiv R_2 \cdot \frac{\partial}{\partial R_1^0}, \quad D_{00'} \equiv R_2 \cdot \frac{\partial}{\partial R_1^{0'}.} \tag{3.2} \]

It is easy to use the definition (3.1) to find their action on reduction coefficients, for example,

\[ D_{00'} = 2 s_{00'} c_{s_{00'}}, \quad T_{00'} = 2 D_{00'} c_{s_{00}} + D_{10} c_{s_{10}} + D_{10'} c_{s_{10'}}. \tag{3.3} \]

Employing them, we can write down recursion relations for expansion coefficients.

### 3.1 Recursion relations from \( T \)-type operators

Here, we derive the recursion relations for three \( T \)-type operators. Let us start with the action of \( T_{00} \). It is easy to check

\[ T_{00} I_{i_1,i_2,i_3}^{(r_1,2,r_2)} = 4 r_1 (r_1 - 1) \left[ I_{i_1,i_2,i_3}^{(r_1-2,r_2)} + M_1 I_{i_1,i_2,i_3}^{(r_1-2,r_2)} \right] \tag{3.4} \]

by using the algebraic relation (2.18). Plugging the expansion (2.13) into the both sides, we find the LHS of the equation is

\[ \sum_{i_1,i_2,i_3,j} \left[ (r_1 - i_1 - j) (D + i_1 + J + r_1 - 2) \alpha_{i_1,i_2,i_3,j}^{(r_1,r_2)} + 2 (i_1 + 1) (j + 1) \alpha_{i_1+1,i_2+1,i_3+1}^{(r_1,r_2)} \right] \]

\[ + (i_1 + 1) (i_1 + 2) s_{i_1} \hat{\alpha}_{i_1+2,i_2,i_3}^{(r_1,r_2)} + (j + 1) (j + 2) \hat{\alpha}_{i_1,i_2,i_3,j+2}^{(r_1,r_2)} \right] s_{i_1} s_{i_2,j} s_{s_{00'},s_{00'}} \frac{r_1-2-i_1-j}{s_{00'}} \frac{r_2-i_2-j}{s_{00'}} \mathbf{J}, \tag{3.5} \]

and the RHS is

\[ 4 r_1 (r_1 - 1) \sum_{i_1,i_2,i_3,j} \left( \alpha_{i_1,j_1,j_2,j_3}^{(r_1-2,r_2)} + M_1 \alpha_{i_1,j_1,j_2,j_3}^{(r_1-2,r_2)} \right) s_{i_1} s_{i_2,j} \frac{r_1-2-i_1-j}{s_{00'}} \frac{r_2-i_2-j}{s_{00'}} \mathbf{J}. \tag{3.6} \]

where the definition of \( \alpha_{i_1,j_1,j_2,j_3}^{(r_1-2,r_2)} \) has been given in (2.16). Comparing both sides, we obtain the recursion relation for operator \( T_{00} \) with \( r_1 \geq 2 \)

\[ (i_1 + 1) (i_1 + 2) s_{i_1} \hat{\alpha}_{i_1+2,i_2,i_3}^{(r_1,r_2)} + (j + 1) (j + 2) \hat{\alpha}_{i_1,i_2,i_3,j+2}^{(r_1,r_2)} \]

\[ + (r_1 - i_1 - j) (D + i_1 + J + r_1 - 2) \alpha_{i_1,i_2,i_3,j}^{(r_1,r_2)} = 4 r_1 (r_1 - 1) \left[ \alpha_{i_1,i_2,i_3,j}^{(r_1-2,r_2)} + M_1 \alpha_{i_1,i_2,i_3,j}^{(r_1-2,r_2)} \right]. \tag{3.7} \]

Similarly, one can derive the recursion relations for operator \( T_{00'} \) with \( r_2 \geq 2 \)

\[ (i_2 + 1) (i_2 + 2) s_{i_2} \hat{\alpha}_{i_1+2,i_2+2,i_3}^{(r_1,r_2)} + (j + 1) (j + 2) \hat{\alpha}_{i_1,i_2,i_3,j+2}^{(r_1,r_2)} \]

\[ + (r_2 - i_2 - j) (D + i_2 + J + r_2 - 2) \alpha_{i_1,i_2,i_3,j}^{(r_1,r_2)} = 4 r_2 (r_2 - 1) \left[ \alpha_{i_1,i_2,i_3,j}^{(r_1-2,r_2)} + M_2 \alpha_{i_1,i_2,i_3,j}^{(r_1-2,r_2)} \right]. \tag{3.8} \]
Finally, using the algebraic relation
\[ 2\ell_1 \cdot \ell_2 = D_3 - D_1 - D_2 + 2\ell_1 \cdot K + 2\ell_2 \cdot K - f_{12} \quad (3.9) \]
with
\[ f_{12} \equiv K^2 + M^2 + M_2^2 - M_3^2 , \quad (3.10) \]
we derive recursion relations of operator \( T_{00'} \) for \( r_1 \geq 1, r_2 \geq 1 \) as
\[
T_{00'} I_{1,1,1}^{(r_1,r_2)} = 2r_1 r_2 \left[ I_{1,1,1}^{(r_1-1,r_2-1)} - f_{12} I_{1,1,1}^{(r_1-1,r_2-1)} \right] + 2r_2 D_{10} I_{1,1,1}^{(r_1,r_2-1)} + 2r_1 D_{10'} I_{1,1,1}^{(r_1-1,r_2)} ,
\]
where we have used the shorthand
\[
I_{1,1,1}^{(r_1,r_2)} = \sum_{i_1,i_2,j} s_{i_1,i_2,j}^0 \alpha_{i_1,i_2,j;3}^{(r_1,r_2)} \sum_{i_0,i_0'} s_{i_0,i_0'} s_{00} s_{00'} \alpha_{i_0,i_0'}^{(r_1,r_2)} J .
\]
Employing the expansion (2.13), we find the LHS of (3.11) is
\[
\sum_{i_1,i_2,j} \left[ (j + 1) (D + r_1 - r_2 - j - 2) \alpha_{i_1,i_2,j+1}^{(r_1,r_2)} - (i_2 + 1) (i_1 - 1 + j - r_1) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} \right) + (i_1 + j - 1 - r_1) (i_2 + 1 - r_2) \alpha_{i_1,i_2,j+1}^{(r_1,r_2)} - (i_2 + 1 + j - r_2) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} + (i_1 + 1) (i_2 + 1) s_{i_1,i_2,j}^{(r_1,r_2)} \sum_{i_0,i_0'} s_{i_0,i_0'} s_{00} s_{00'} \alpha_{i_0,i_0'}^{(r_1,r_2)} J ,
\]
and the RHS is
\[
\sum_{i_1,i_2,j} \left[ 2r_2 \left( (i_2 + 1) s_{i_1,i_2,j}^{(r_1,r_2)} + (j + 1) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} - (i_1 - 1 + j - r_1) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} \right) + 2r_1 \left( (i_2 + 1) s_{i_1,i_2,j}^{(r_1,r_2)} + (j + 1) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} - (i_1 - 1 + j - r_1) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} \right) + 2r_1 r_2 \left( \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} - f_{12} \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} \right) \right] s_{i_0,i_0'} s_{00} s_{00'} s_{00'} J .
\]
Comparing both sides of equation (3.11), we have
\[
(j + 1) (D + r_1 + r_2 - j - 2) \alpha_{i_1,i_2,j+1}^{(r_1,r_2)} - (i_2 + 1) (i_1 - 1 + j - r_1) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} + (i_1 + j - 1 - r_1) (i_2 + 1 - r_2) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} + (i_1 + 1) (i_2 + 1) s_{i_1,i_2,j}^{(r_1,r_2)} + (j + 1) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} - (i_1 - 1 + j - r_1) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} + 2r_1 \left( (i_2 + 1) s_{i_1,i_2,j}^{(r_1,r_2)} + (j + 1) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} - (i_1 + 1) (i_2 + 1 + j - r_2) \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} + 2r_1 r_2 \left( \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} - f_{12} \alpha_{i_1,i_2+1,j}^{(r_1,r_2)} \right) \right) .
\]
By now, we have obtained three $T$-type relations (3.7), (3.8) and (3.15). As discussed in the introduction, these relations are not sufficient to solve all expansion coefficients iteratively. In Table 2, we list the number of expansion coefficients (each vector $\bar{\alpha}$ is counted as one coefficient) and the independent equations given by three $T$-type recursions. Comparing two tables, we find the number of reminding unknown terms is universal: $N_\alpha - N_T = 1$. So for each rank level $r \equiv (r_1 + r_2)$ there reminds $(r + 1)$ unknown expansion coefficients. We need to find more relations to determine them.

| $N_\alpha$ | $r_1$ | 0 | 1 | 2 | 3 | 4 | 5 |
|------------|-------|---|---|---|---|---|---|
| $r_2$     |       |   |   |   |   |   |   |
| 0          | 1     | 1 | 2 | 2 | 3 | 3 |   |
| 1          | 1     | 2 | 3 | 3 | 6 | 6 |   |
| 2          | 2     | 3 | 6 | 7 | 10| 11|   |
| 3          | 2     | 4 | 7 | 10| 13| 16|   |
| 4          | 3     | 5 | 10| 13| 19| 22|   |
| 5          | 3     | 6 | 11| 16| 22| 28|   |

| $N_T$ | $r_1$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|-------|---|---|---|---|---|---|
| $r_2$ |       |   |   |   |   |   |   |
| 0     | 0     | 0 | 1 | 1 | 2 | 2 |   |
| 1     | 1     | 0 | 1 | 2 | 3 | 4 | 5 |
| 2     | 2     | 1 | 2 | 5 | 6 | 9 | 10|
| 3     | 3     | 1 | 3 | 6 | 9 | 12| 15|
| 4     | 4     | 2 | 4 | 9 | 12| 18| 21|
| 5     | 5     | 2 | 5 | 10| 15| 21| 27|

Table 2: Number of expansion coefficients (left) and independent $T$-type equations (right).

### 3.2 Recursions relation from $D$-type operators

Having used the $T$-type operators, from the experience of one-loop integrals, it is obvious that we need to consider the $D$-type operators. Naively acting $D_{10}$ and $D_{10}'$ defined in (3.2) globally to (2.2), we get $K \cdot \ell_i$ in numerator, which are irreducible scalar products and cannot be reduced further algebraically. How to get out of the deadlock? Let us look back to the two-loop integrals. Instead of doing the loop integration together, one can think it as two times integration: first, we integrate the $\ell_1$, then we carry out the integration of $\ell_2$. Using this idea, we can first do the tensor reduction of sub one-loop integrals. Let us see what we can get.

Let us start from the sub one-loop integrals. From Appendix A, the $r$-rank tensor reduction of one-loop bubbles

$$ I_2^{(r)} = \int d\ell \frac{(2\ell \cdot R)^r}{P_0 P_1} $$

(3.16)
can be written as summation of $(r-1)$-rank and $(r-2)$-rank bubbles as well as the contribution of tadpoles, where $P_0 \equiv \ell^2 - M_0^2, P_1 \equiv (\ell - K)^2 - M_1^2$. The striking point is that the Gram determinant $K^2$ only appears once in the denominator, i.e.,

$$ I_2^{(r)} = \frac{1}{(D + r - 3)s_{11}} \left[ (D + 2(r - 2))f_1 s_{01} I_2^{(r-1)}
\right.$$

$$ \left. - (r - 1)(4M_0^2 s_{01} + (f_1^2 - 4M_0^2 s_{11}) s_{00}) I_2^{(r-2)} + R_{Tad}^{(r)} \right] $$

(3.17)
where we have defined

\[ s_{11} \equiv K^2, \quad f_1 \equiv s_{11} + M_0^2 - M_1^2. \] (3.18)

For example

\[
I_2^{(2)} = \frac{D_1 s_{01}}{(D - 1)s_{11}} I_2^{(1)} - \frac{4M_0^2 s_{01}^2 + s_{00}(f_1^2 - 4M_0^2 s_{11})}{(D - 1)s_{11}} I_2 \\
+ \frac{s_{00} f_1}{(D - 1)s_{11}} I_2 \tilde{\ell} + \frac{2(D - 2)s_{01}^2 + s_{00}(s_{11} - M_0^2 + M_1^2)}{(D - 1)s_{11}} I_2 \tilde{\ell} \tilde{\ell},
\] (3.19)

where the pole of \( s_{11} \) only appears in the overall factor. Now we insert (3.17) to the two loop integration by regarding propagator \( D_1, D_3 \) as the propagators \( P_0, P_1 \) respectively, i.e., we reduce \( \ell_1 \) first

\[
\int d\ell_2 \int d\ell_1 \mathcal{I}[r_1, r_2] = \int d\ell_2 \int d\ell_1 \left[ \frac{(D + 2(r_1 - 2))\tilde{f}_1 s_{01} \mathcal{I}[r_1 - 1, r_2]}{(D + r_1 - 3)s_{11}} \\
- (r_1 - 1) \frac{(4M_0^2 s_{01}^2 + (\tilde{f}_1^2 - 4M_0^2 s_{11})s_{00}) \mathcal{I}[r_1 - 2, r_2]}{(D + r_1 - 3)s_{11}} + \frac{\mathcal{R}_{\ell_1, Tad}^{(r_1, r_2)}}{(D + r_1 - 3)s_{11}} \right]
\] (3.20)

where we have defined

\[
\mathcal{I}[r_1, r_2] \equiv \frac{(2R_1 \cdot \ell_1)^r(2R_2 \cdot \ell_1)^r}{D_1 D_2 D_3}, \quad \mathcal{I}[r_1, r_2, \tilde{\ell}] \equiv \frac{D_1 (2R_1 \cdot \ell_1)^r (2R_2 \cdot \ell_1)^r}{D_1 D_2 D_3},
\] (3.21)

and

\[
\tilde{s}_{01} = R_1 \cdot (K - \ell_2), \quad \tilde{s}_{11} = (K - \ell_2)^2, \quad \tilde{f}_1 = \tilde{s}_{11} + M_1^2 - M_2^2.
\] (3.22)

The lower-topology term \( \mathcal{R}_{\ell_1, Tad}^{(r_1, r_2)} \) in (3.20) has the form \( \frac{N_{\ell_2}}{D_2} \times \frac{(2\ell_2 \cdot R_2)^r}{D_2} \) with \( i = 1, 3 \), which is counted by \( \mathcal{I}[r_1, r_2; \tilde{\ell}] \). For example, with rank \( r = 1 \), the tadpole part is

\[
\mathcal{I}_{Tad}^{(1)} = (2 - D)R \cdot K \left[ \int \frac{d\ell}{P_0} - \int \frac{d\ell}{P_1} \right],
\] (3.23)

thus in (3.20) we have

\[
\mathcal{R}_{\ell_1, Tad}^{(1, r_2)} = (2 - D)R \cdot (K - \ell_2) \left[ \frac{(2R_2 \cdot \ell_2)^r}{D_1 D_2} - \frac{(2R_2 \cdot \ell_2)^r}{D_2 D_3} \right].
\] (3.24)

The appearance \( \tilde{s}_{11} \) in the denominator in (3.20) causes some trouble since it depends on \( \ell_2 \), while the original sunset integrals do not have such a denominator. In the paper [46], they keep this factor and regard the two-loop integrals with \( \tilde{s}_{11} \) in the denominator as a new master integral. However, according to the reduction idea, such a solution is somewhat surprising since we expect that the reduction is achieved by removing original propagators. Thus we should reach the original topology and its sub topologies at last.
Since we want to keep the original reduction picture, we will try to get rid of $\tilde{s}_{11}$ in (3.20). The idea is simple: one can multiply both sides with $(D + r_1 - 3)\tilde{s}_{11}$ before integrating $\ell_2$ and get

\[
\int_{1,2} (D + r_1 - 3)\tilde{s}_{11} \mathcal{I}[r_1, r_2] = \int_{1,2} \left( (D + 2(r_1 - 2))\tilde{f}_1 \tilde{s}_{01} \mathcal{I}[r_1 - 1, r_2] \right. \\
- (r_1 - 1)(4M_1^2\tilde{s}_{01}^2 + (\tilde{f}_1^2 - 4M_1^2\tilde{s}_{11})\tilde{s}_{00}) \mathcal{I}[r_1 - 2, r_2] + \mathcal{R}^{(r_1, r_2)}_{\ell_1, \mathcal{T}_{ad}} \right] .
\]

(3.25)

Writing

\[
\tilde{s}_{11} = (K - \ell_2)^2 = K^2 + \ell_2^2 - 2K \cdot \ell_2 = s_{11} + D_2 + M_2^2 - 2K \cdot \ell_2 ,
\]

the LHS of the equation becomes (where we have suppressed the integral sign $\int_{1,2}$ for simplicity)

\[
(D + r_1 - 3)\left( s_{11} + D_2 + M_2^2 - 2K \cdot \ell_2 \right) \mathcal{I}[r_1, r_2] \\
= (D + r_1 - 3) \left[ (s_{11} + M_2^2)\mathcal{I}[r_1, r_2] + \mathcal{I}[r_1, r_2; \hat{2}] - \frac{\mathcal{D}_{10}^{10'}}{r_2 + 1} \mathcal{I}[r_1, r_2 + 1] \right]
\]

(3.27)

where in the last term we replace the new tensor structure $2K \cdot \ell_2$ by the action of differential operator $\mathcal{D}_{10'}$ on the basic form $\mathcal{I}[r_1, r_2 + 1]$ (see (3.21)). One important point in (3.27) is that we have the $\mathcal{D}$-type action on $\mathcal{I}[r_1, r_2 + 1]$, which is what we are looking for.

Now we consider the RHS of (3.25). There are three terms: $\tilde{f}_1\tilde{s}_{01}\mathcal{I}[r_1 - 1, r_2], \tilde{s}_{01}^2\mathcal{I}[r_1 - 2, r_2], (\tilde{f}_1^2 - 4M_1^2\tilde{s}_{11})\mathcal{I}[r_1 - 2, r_2]$. For the first term, using the expression of $\tilde{f}_1$ in (3.22) and the rewriting in (3.26), we can write it as

\[
\tilde{f}_1\tilde{s}_{01}\mathcal{I}[r_1 - 1, r_2] \\
= (s_{01} - R_1 \cdot \ell_2) \mathcal{I}[r_1 - 1, r_2; \hat{2}] + f_{12}s_{01}\mathcal{I}[r_1 - 1, r_2] \\
- \left( s_{01}(2K \cdot \ell_2) + f_{12}(R_1 \cdot \ell_2) \right) \mathcal{I}[r_1 - 1, r_2] + (2K \cdot \ell_2)(R_1 \cdot \ell_2)\mathcal{I}[r_1 - 1, r_2] \\
= (s_{01} - R_1 \cdot \ell_2) \mathcal{I}[r_1 - 1, r_2; \hat{2}] + f_{12}s_{01}\mathcal{I}[r_1 - 1, r_2] \\
- \left( 2s_{01}K \cdot \ell_2 + f_{12}R_1 \cdot \ell_2 \right) \mathcal{I}[r_1 - 1, r_2] + \frac{\mathcal{D}_{10'}\mathcal{D}_{00'}}{2(r_2 + 1)(r_2 + 2)}\mathcal{I}[r_1 - 1, r_2 + 2].
\]

(3.28)

From the last equation in (3.28) one can see that the rank level of all terms will be less than $(r_1 + r_2 + 1)$, except the last term with rank level $(r_1 + r_2 + 1)$. For example, the term $f_{12}R_1 \cdot \ell_2\mathcal{I}[r_1 - 1, r_2] = \frac{f_{12}}{2(r_2 + 1)}\mathcal{D}_{00'}\mathcal{I}[r_1 - 1, r_2 + 1]$ has rank $r_1 + r_2$. Similar analysis can be done for other two terms in the RHS of (3.25), it is easy to see that they can be written as a combination of terms with rank level lower than $(r_1 + r_2 + 1)$.
Collecting all terms together, the RHS of (3.25) is
\[
-(D + 2r_1 - 4) (2s_{01}K \cdot \ell_2 + f_{12}R_1 \cdot \ell_2) \mathcal{I}[r_1 - 1, r_2] + \frac{(D + 2r_1 - 4)D_{10'}D_{00'}}{2(r_2 + 1)(r_2 + 2)} \mathcal{I}[r_1 - 1, r_2 + 2] \\
+(D + 2r_1 - 4) \left[ (s_{01} - R_1 \cdot \ell_2) \mathcal{I}[r_1 - 1, r_2; 2] + f_{12}s_{01} \mathcal{I}[r_1 - 1, r_2] \right] - (r_1 - 1) \left[ 4M_1^2(s_{01} - R_1 \cdot \ell_2)^2 + [(f_{12} + D_2 - 2K \cdot \ell_2)^2 - 4M_1^2(K - \ell_2)^2] s_{00} \mathcal{I}[r_1 - 2, r_2] \right] \\
+ R_{(r_1, r_2)}^{(r_1, r_2)}. \tag{3.29}
\]

Rearranging (3.25), we arrive
\[
\frac{(D + r_1 - 3)D_{10'}D_{00'}}{r_2 + 1} \mathcal{I}[r_1, r_2 + 2] + \frac{(D + 2r_1 - 4)D_{10'}D_{00'}}{2(r_2 + 1)(r_2 + 2)} \mathcal{I}[r_1 - 1, r_2 + 2] \\
= (D + r_1 - 3) \left[ (s_{11} + M_2^2) \mathcal{I}[r_1, r_2] + \mathcal{I}[r_1, r_2; 2] \right] + (D + 2r_1 - 4) \left[ (2s_{01}K \cdot \ell_2) \mathcal{I}[r_1 - 1, r_2] \\
-(s_{01} - R_1 \cdot \ell_2) \left( \mathcal{I}[r_1 - 1, r_2; 2] + f_{12}\mathcal{I}[r_1 - 1, r_2] \right) \right] - (r_1 - 1) \left[ 4M_1^2(s_{01} - R_1 \cdot \ell_2)^2 + [(f_{12} + D_2 - 2K \cdot \ell_2)^2 - 4M_1^2(K - \ell_2)^2] s_{00} \mathcal{I}[r_1 - 2, r_2] \right] \\
- R_{(r_1, r_2)}^{(r_1, r_2)}. \tag{3.30}
\]

The recursion relation (3.30) is the wanted $\mathcal{D}$-type recursion relation. The LHS of the equation can be written as summation of expansion coefficients with rank $(r_1, r_2 + 1)$, $(r_1 - 1, r_2 + 2)$ of rank level $(r_1 + r_2 + 1)$ while the RHS can be written as sum of expansion coefficients with rank level less than $(r_1 + r_2 + 1)$ and lower topologies. Thus, it is a recursion relation for rank level, instead of the explicit rank configuration $(r_1, r_2)$. The reason is that under the $\mathcal{D}$-type action, different rank configurations are mixed as at the LHS of (3.30).

By our recursion assumption, the RHS is considered to be known, and we can just write it as
\[
\mathcal{B}_{(r_1, r_2)}^{(r_1, r_2)} \mathcal{J} = \sum_{i_1, i_2, j} \mathcal{I}_{s_{01} s_{01} s_{00} s_{00}}^{i_1 i_2 j} \mathcal{J}_{s_{01} s_{01} s_{00} s_{00}}^{i_1 i_2 j} + \beta^{(r_1, r_2)}_{i_1 i_2 j} \mathcal{J}. \tag{3.31}
\]

The expression of (3.31) for explicit $r_1, r_2$ can be found in Appendix C. Putting the expansion (2.13) into the LHS and after some algebra, finally we arrive the algebraic recursion relation for unknown coefficients $\tilde{\alpha}$
\[
\beta^{(r_1, r_2)}_{i_1 i_2 j} = \sum_{i_1, i_2, j} \mathcal{I}_{s_{01} s_{01} s_{00} s_{00}}^{i_1 i_2 j} \mathcal{J}_{s_{01} s_{01} s_{00} s_{00}}^{i_1 i_2 j} + \beta^{(r_1, r_2)}_{i_1 i_2 j} \mathcal{J}.
\]
\[-(i_2 + 1)(i_2 + 2)(i_1 + j - r_1)s_{11}a_{i_1-1,i_2,2,j}^{r_1-1,r_2+2} \\
+ (i_2 + 1)(i_1 + j - r_1)(i_2 + j - r_2 - 2)s_{11}a_{i_1,1,i_2+1,j-1}^{r_1-1,r_2+2} \\
- (i_1 + 1)(2i_2 + 1)(i_2 + j - r_2 - 2)s_{11}a_{i_1,i_2,1,j}^{r_1-1,r_2+2} \\
+ (i_1 + 1)(i_2 + 1)(i_2 + 2)s_1^2a_{i_1-1,i_2+1,2,j}^{r_1-1,r_2+2}\]

\[= \beta_{\ell_1,i_1,i_2,j}^{r_1,r_2}. \tag{3.32} \]

Similarly, if we reduce \(\ell_2\) first, we have
\[
\begin{align*}
\frac{(D + r_2 - 3)D_{10}}{r_1 + 1}I[r_1, 1, r_2] + \frac{(D + 2r_2 - 4)D_{10}D_{00}}{2(r_1 + 1)(r_1 + 2)}I[r_1 + 2, r_2 - 1] \\
= (D + r_2 - 3)\left[\left(s_{11} + M_1^2\right)I[r_1, r_2] + I[r_1, r_2; 1]\right] + \left(D + 2r_2 - 4\right)\left[(2s_{01}K \cdot \ell_1)I[r_1, r_2 - 1] \\
- (s_{01} - R_2 \cdot \ell_1)\left(I[r_1, r_2 - 1; 1] + f_{12}I[r_1, r_2 - 1]\right)\right] - (r_2 - 1)\left[\frac{4M_2^2}{4}I[r_1, r_2 - 2] - K^{(r_1,r_2)}_{\ell_2,Tad}\right]
\end{align*}
\]

which is the dual expression of (3.30). Similarly, the lower-topology term \(R_{\ell_2,Tad}^{(r_1,r_2)}\) in the equation can be got from \(R_{Tad}^{(r_2)}\) by regarding \(P_0, P_1\) as \(D_2, D_3\) respectively and multiplying with \((2\ell_1 \cdot R_1)^{r_1}/D_1\). Again, after plugging expansion (2.13) into the LHS of (3.33) and writing the RHS as
\[B_{\ell_2}^{(r_1,r_2)} = \sum_{i_1,i_2,j} s_{01}^{i_1} s_{01}^{i_2} \tilde{s}_0^{j} s_{00}^{r_1-1} s_{00}^{r_2-1} \beta_{\ell_2;i_1,i_2,j}^{(r_1,r_2)} J, \tag{3.34} \]

we will get another algebraic recursion equation for expansion coefficients which is dual to (3.32).

By now, we have derived two \(D\)-type relations (3.30) and (3.33). Unlike the \(T\)-type relations which give a recursion relation for a particular rank configuration, \(D\)-type\(^3\) will mix them: one \(D\)-type relation (3.30) mixes ranks \((r_1, r_2 + 1)\) and \((r_1 - 1, r_2 + 2)\) while the other \(D\)-type relation (3.33) mixes ranks \((r_1 + 1, r_2)\) and \((r_1 + 2, r_2 - 1)\). When using both types, one must do reduction level by level. As shown in Table 3, one can solve all expansion coefficients for \(r_1 + r_2 > 2\) by combining \(T\)-type and \(D\)-type relations while there are still four expansion coefficients to be determined, which indicates we need more relations for \(r_1 + r_2 \leq 2\) level. As will be discussed later, this observation gives us the hint of the choice of master integrals.

### 3.3 Master integrals choice

Up to now, we have avoided giving the explicit choice of master basis in (2.13), since all recursion relations of \(T\)-type and \(D\)-type are independent of the choice. The only constraint is that they can not contain auxiliary vectors. However, as pointed out in Table 3, the reduction for some lower-rank tensor integrals is not completely clear.

\(^3\)We can understand the \(D\)-type relation from another point of view. In some sense, it likes the IBP relation coming from, for example, \(q^\mu \frac{\partial}{\partial q_1^\mu}\). However, since we have used the final one-loop reduction results, the solving of these IBP relations has been avoid. Thus the method in this paper is somehow more efficient.


| $r_1 + r_2$ | $N_{\bar{a}}$ | $N_T$ | $N_{T\cup D}$ | $N_{\bar{a}} - N_{T\cup D}$ |
|-------------|-------------|--------|----------------|-----------------------------|
| 0           | 1           | 0      | 0              | 1                           |
| 1           | 2           | 0      | 0              | 2                           |
| 2           | 6           | 3      | 5              | 1                           |
| 3           | 10          | 6      | 10             | 0                           |
| 4           | 20          | 15     | 20             | 0                           |
| 5           | 30          | 24     | 30             | 0                           |
| 6           | 50          | 43     | 50             | 0                           |

**Table 3:** Number of expansion coefficients and independent equations for several rank levels

Let us consider them one by one. For the rank level zero, it is the scalar sunset $I_{1,1,1}$ with no auxiliary vectors, so there are no differential operators that can act on it with nonzero result. Then we must regard it as a master integral, i.e., $J_1 \equiv I_{1,1,1}$. Next, we consider integrals with rank level one, for example, the integration

$$I_{1,1,1}[\ell_1] = \int \frac{d\ell_{1,2} \ell_1^\mu}{D_1 D_2 D_3}. \quad (3.35)$$

According to its tensor structure, we should have

$$I_{1,1,1}[\ell_1] = BK^\mu. \quad (3.36)$$

The logic of PV-reduction method is to solve $B$ by multiplying $K$ at both sides

$$\int \frac{d\ell_{1,2}(2\ell_1 \cdot K)}{D_1 D_2 D_3} = BK^2. \quad (3.37)$$

Now a key point appears. Unlike the one-loop case, the contraction $\ell_1 \cdot K$ can not be written as the combinations of $D_i$. Thus, we can not solve $B$ as a function of external momentum $K$, masses, and space-time dimension $D$ only. This observation is well known, and it is related to the concept of **irreducible scalar product (ISP)**. For a given $L$-loop integrals with $E + 1$ external legs, there are $\frac{L(L+1)}{2} + LE$ independent scalar products involving at least one loop momentum. If there are $N$ propagators $D_i$, $N$ scalar products can be written as the linear combination of $D_i$'s, which leads $N_{ISP} = \frac{L(L+1)}{2} + LE - N$ irreducible scalar products. For one-loop integrals, we have $L = 1$ and $N = E + 1$, so there is no ISP, and every tensor integral can be decomposed to the scalar basis. But for two-loop integrals, there are nontrivial ISPs. Thus there are multiple master integrals for a given topology. For our sunset topology, we have $L = 2, E = 1, N = 3$ and $N_{ISP} = 2$, which are given by $(\ell_1 \cdot K)$ and $(\ell_2 \cdot K)$. With the above analysis and the irreducibility of the LHS of (3.37), it is natural to take the LHS to be a master integral

$$J_2 \equiv \int \frac{d\ell_{1,2}(2\ell_1 \cdot K)}{D_1 D_2 D_3}. \quad (3.38)$$
Similarly, we get another master integral
\[ J_3 \equiv \int \frac{d\ell_1}{D_1} \frac{d\ell_2}{D_2} \frac{(2\ell_1 \cdot K)}{D_3}. \] (3.39)

In Table 3, there are two undetermined expansion coefficients for rank level \( n_e \). After taking \( J_2, J_3 \) above to be master integrals, they can be solved easily as shown in the next section.

Then we consider the rank level two integrals, for example,
\[ \int d\ell_1 d\ell_2 \frac{(2\ell_1 \cdot R_1)(2\ell_2 \cdot R_2)}{D_1 D_2 D_3}. \] (3.40)

According to its tensor structure, we have
\[ I^{(1,1)}_{1,1,1} = \tilde{\alpha}^{(1,1)}_{1,1,0} s_{01} s_{0'} + \tilde{\alpha}^{(1,1)}_{0,0,1} s_{00'} . \] (3.41)

The only differential operator that can reduce the integral is \( T_{00'} \), which gives just one equation while there are two unknown expansion coefficients. As pointed out in Table 3, there is only one undetermined expansion coefficient for rank level two. When combining the information of ISP, it is natural to take the integral obtained by acting with \( D_{10} D_{10'} \), i.e.,
\[ \int J_4 \equiv \int d\ell_1 d\ell_2 \frac{(2K \cdot \ell_1)(2K \cdot \ell_2)}{D_1 D_2 D_3}. \] (3.42)

to be another master integral. In fact, another two choices, i.e.,
\[ \int d\ell_1 d\ell_2 \frac{(2K \cdot \ell_1)^2}{D_1 D_2 D_3} \quad \text{or} \quad \int d\ell_1 d\ell_2 \frac{(2K \cdot \ell_2)^2}{D_1 D_2 D_3} \] (3.43)
should be equivalently fine.

Having chosen master integrals for sunset topology, we discuss the master integrals for lower topologies. These lower topologies are obtained by removing one of the sunset’s propagators, and the resulted topology is the product of two one-loop tadpole integrations. For one-loop integrals, we know the basis is the scalar integrals, thus it is easy to see that we have the following three master integrals
\[ J_5 \equiv \int \frac{d\ell_1}{D_2} \frac{d\ell_2}{D_3}, \quad J_6 \equiv \int \frac{d\ell_1}{D_1} \frac{d\ell_2}{D_3}, \quad J_7 \equiv \int \frac{d\ell_1}{D_1} \frac{d\ell_2}{D_2}. \] (3.44)

One technical point is that the standard form of \( J_5, J_6 \) should be written with the proper momentum shifting. For example,
\[ J_5 = \int \frac{d\ell_1}{(\ell_1^2 - M_1^2)(\ell_1 + \ell_2 - K)^2 - M_3^2)} = \int \frac{d\ell_1}{(\ell_1^2 - M_1^2)(\ell_2^2 - M_3^2)}. \] (3.45)

Our discussion on ISP and the choice of basis hints that the number \( N \) of master integrals is related to the number \( N_{isp} \) of ISP when the masses and momenta are general. Another point is that when using the FIRE to do the reduction, it will take a different master basis.
In Appendix D we will present the non-degenerate transformation matrix between these two basis, thus our choice of basis is legitimate.

Before ending this subsection, let us discuss briefly the number of master integrals when the kinematics and masses are not general, for example $K^2 = 0$ or $M_1 = M_2$. For $K^2 = 0$, but $M_1, M_2, M_3$ are different, there are only four master integrals $J_1, J_5, J_6, J_7$ and $J_2, J_3, J_4$ will be decomposed to linear combination of them. We can get the proper reduction coefficients by taking the limit carefully, as discussed in \[50\]. If $M_1 = M_2 \neq M_3$, but $K^2 \neq 0$, we will have $J_5 = J_6$ and $J_2 = J_3$, i.e., there are only five master integrals. If the reduction coefficients are not singular under the limit $M_1 \to M_2$, we just add reduction coefficients together, i.e., $c_2 J_2 + c_3 J_3 \to (c_2 + c_3) J_2$. If the reduction coefficients are singular under the limit, we need to expand $J_3$ by other master integrals with expansion coefficients as the Taylor series of $(M_1 - M_2)$ as presented in \[50\].

4 Examples

Having laid out the general reduction frame in the previous sections, we present examples to demonstrate our algorithm. In this section, we will give the exact reduction process for the rank levels from one to four. The full expansion of coefficients is collected in Appendix E and an attached Mathematica file.

4.1 Rank level one

Having chose seven master integrals $J_i$, we start the reduction for tensor sunset with rank level one. There are two integrals

$$I_{1,1,1}^{(1,0)} = \int d\ell_{1,2} \frac{(2R_1 \cdot \ell_1)}{D_1 D_2 D_3}, \quad I_{1,1,1}^{(0,1)} = \int d\ell_{1,2} \frac{(2R_2 \cdot \ell_2)}{D_1 D_2 D_3}. \quad (4.1)$$

Due to the total rank is less than 2, both $T$-type and $D$-type recursions (3.30) and (3.33) can not be used. However, we can apply operators $D_{10}$ and $D_{10'}$ on the two integrals respectively. For example, the action of $D_{10}$ on the integral $I_{1,1,1}^{(1,0)}$ gives

$$D_{10} I_{1,1,1}^{(1,0)} = \int d\ell_{1,2} \frac{(2K \cdot \ell_1)}{D_1 D_2 D_3} = J_2 = \vec{\delta}_2 J \quad (4.2)$$

where we have defined a constant vector

$$\vec{\delta}_2 = \{0, 1, 0, 0, 0, 0, 0\}. \quad (4.3)$$

The reason one can obtain the relation is that we choose $J_2$ as a master integral.

For this case, the expansion (2.13) is

$$I_{1,1,1}^{(1,0)} = \alpha_{1,0,0}^{(1,0)} s_{11} J. \quad (4.4)$$

Plugging it into the LHS of (4.2), we have the relation

$$D_{10} I_{1,1,1}^{(1,0)} = \alpha_{1,0,0}^{(1,0)} s_{11} J = \vec{\delta}_2 J. \quad (4.5)$$
Comparing both sides, we can easily solve
\[
\alpha_{1,0}^{(1,0)} = \frac{\partial_2}{s_{11}} = \frac{1}{s_{11}} \{0, 1, 0, 0, 0, 0, 0\}.
\]
Similarly we have
\[
\alpha_{0,1}^{(0,1)} = \frac{\partial_3}{s_{11}} = \frac{1}{s_{11}} \{0, 0, 1, 0, 0, 0, 0\}.
\]

### 4.2 Rank level two

There are three integrals. By the expansion (2.13), we have
\[
\begin{align*}
C^{(2,0)} &= \alpha_{0,0,0}^{(2,0)} s_{00} + \alpha_{0,0,0}^{(2,0)} s_{01}, \\
C^{(1,1)} &= \alpha_{0,1}^{(1,1)} s_{00} + \alpha_{1,1,0}^{(1,1)} s_{01}, \\
C^{(0,2)} &= \alpha_{0,0}^{(0,2)} s_{00} + \alpha_{0,2,0}^{(0,2)} s_{01}.
\end{align*}
\]

Using \((4.8)\), we have
\[
J = \hat{s}_{11}(\alpha_{0,0,0}^{(1,1)} s_{00} + \alpha_{0,1,0}^{(1,1)} s_{00} + \alpha_{1,1,0}^{(1,1)} s_{01}) J.
\]

Using (3.15), we have
\[
\frac{1}{2} \left[ D \alpha_{0,0,0}^{(1,1)} s_{11} + \alpha_{0,0,0}^{(1,1)} s_{11} \right] = \alpha_{0,0,0}^{(0,0)} s_{00} + s_{11} \alpha_{0,1,0}^{(1,0)} + s_{11} \alpha_{0,0,1}^{(0,1)}.
\]

Then we consider the action of \(D_{10} D_{10'}\)
\[
D_{10} D_{10'} I_{1,1}^{(1,1)} = \int d\ell_{1,2} \frac{(2\ell_1 \cdot K)(2\ell_2 \cdot K)}{D_1 D_2 D_3} = J_4 = \delta_4 J.
\]

Let us start from the rank \((1, 1)\). The only \(\mathcal{T}\)-type recursion relation is given by \(\mathcal{T}_{00'}\).

Using (4.10), we have
\[
\alpha_{0,0,0}^{(1,1)} + \alpha_{0,1,0}^{(1,1)} s_{11} = \delta_4 / s_{11}.
\]

Combining (4.9) and (4.12) we can solve out
\[
\begin{align*}
\alpha_{0,0,0}^{(1,1)} &= \frac{1}{D-1} \left[ 2 \left( \frac{\alpha_{0,0,0}^{(0,0)}}{\alpha_{0,0,0}^{(0,0)}}, -f_{12} \alpha_{0,0,0}^{(0,0)} + s_{11} \alpha_{0,1,0}^{(0,0)} + s_{11} \alpha_{0,0,1}^{(0,0)} \right) - \delta_4 / s_{11} \right], \\
\alpha_{0,0,1}^{(1,1)} &= \frac{1}{D-1} \left[ D \delta_4 / s_{11} - 2 \left( \frac{\alpha_{0,0,0}^{(0,0)}}{\alpha_{0,0,0}^{(0,0)}}, -f_{12} \alpha_{0,0,0}^{(0,0)} + s_{11} \alpha_{0,1,0}^{(0,0)} + s_{11} \alpha_{0,0,1}^{(0,0)} \right) \right].
\end{align*}
\]

Using the results of lower rank level and lower topologies, we get
\[
\begin{align*}
\alpha_{0,0,0}^{(1,1)} &= \frac{1}{D-1} \left\{ -2 f_{12}, 2, 2, -\frac{1}{s_{11}}, -2, -2, 2 \right\}, \\
\alpha_{0,0,1}^{(1,1)} &= \frac{1}{D-1} \left\{ 2 f_{12}, -2, -2, -\frac{D}{s_{11}}, 2, 2, -2 \right\}.
\end{align*}
\]
From above analysis, one can see the reason of taking master integral $J_4$ as discussed in previous section.

Next we consider the rank $(2, 0)$. The only $\mathcal{T}$-type relation is given by $\mathcal{T}_{00}$. Using (3.7) we have

$$2D\bar{\alpha}^{(2, 0)}_{0, 0, 0} + 2s_{11}\bar{\alpha}^{(2, 0)}_{2, 0, 0} = \bar{\alpha}^{(0, 0)}_{0, 0, 1} + M_2^2\bar{\alpha}^{(0, 0)}_{0, 0, 0}.$$  (4.15)

The $\mathcal{D}$-type relation (3.33) mix the rank $(2, 0)$ and the $(1, 1)$ and give

$$(D - 2)\bar{\alpha}^{(1, 1)}_{0, 0, 1} + \frac{1}{2}(D - 2)\bar{\alpha}^{(2, 0)}_{0, 0, 0} + (D - 2)s_{11}\bar{\alpha}^{(1, 1)}_{1, 1, 0} + \frac{1}{2}(D - 2)s_{11}\bar{\alpha}^{(2, 0)}_{2, 0, 0} = \text{Known Terms}.$$  (4.16)

Combining (4.16) and (4.15) and using (4.14), we can solve

$$\bar{\alpha}^{(2, 0)}_{0, 0, 0} = \frac{1}{D - 1} \left\{ 2 \left( f_{12} + 2M_1^2 \right), -f_{12} \frac{M_1^2 + s_{11}}{s_{11}}, -2, \frac{2}{s_{11}}, 4, 2, -2 \right\},$$

$$\bar{\alpha}^{(2, 0)}_{2, 0, 0} = \frac{-1}{(D - 1)s_{11}} \left\{ 2 \left( Df_{12} + 2M_1^2 \right), -D \frac{f_{12} + 2s_{11}}{s_{11}}, -2D \frac{M_1^2 + s_{11}}{s_{11}}, \frac{2D}{s_{11}}, 4, 2D, -2D \right\}.  (4.17)$$

By proper replacement, we have

$$\bar{\alpha}^{(0, 2)}_{0, 0, 0} = \frac{1}{D - 1} \left\{ 2 \left( f_{12} + 2M_2^2 \right), -2 \frac{M_2^2 + s_{11}}{s_{11}}, -f_{12} \frac{M_2^2 + s_{11}}{s_{11}}, -2, \frac{2}{s_{11}}, 2, 4, -2 \right\},$$

$$\bar{\alpha}^{(0, 2)}_{2, 0, 0} = \frac{-1}{(D - 1)s_{11}} \left\{ 2 \left( Df_{12} + 2M_2^2 \right), -2D \frac{M_2^2 + s_{11}}{s_{11}}, -D \frac{f_{12} + 2s_{11}}{s_{11}}, \frac{2D}{s_{11}}, 4, 2D, -2D \right\}.  (4.18)$$

### 4.3 Rank level three

There are four rank configuration and the expansion (2.13) is

$$C^{(3, 0)} = \alpha^{(3, 0)}_{1, 0, 0}s_0s_0s_{01} + \alpha^{(3, 0)}_{3, 0, 0}s_{01},$$

$$C^{(2, 1)} = \alpha^{(2, 1)}_{0, 1, 0}s_0s_1s_{01} + \alpha^{(2, 1)}_{1, 0, 1}s_0s_0s_{01} + \alpha^{(2, 1)}_{2, 1, 0}s_0s_{01}s_{01} + \alpha^{(2, 1)}_{1, 1, 0}s_0s_{01}s_{01},$$

$$C^{(1, 2)} = \alpha^{(1, 2)}_{0, 1, 1}s_0s_0s_{01} + \alpha^{(1, 2)}_{1, 0, 2}s_0s_{01}s_{01} + \alpha^{(1, 2)}_{1, 2, 0}s_0s_{01}s_{01},$$

$$C^{(0, 3)} = \alpha^{(0, 3)}_{0, 1, 0}s_0s_{01}s_{01} + \alpha^{(0, 3)}_{0, 3, 0}s_{01}s_{01}.  (4.19)$$

We list 14 linear equations of $\mathcal{T}$-type and $\mathcal{D}$-type relations for these ten expansion coefficients. There are only ten of them are linear independent and other four extra can be used as the self-consistence check. We list $\mathcal{T}$-type equations first,

- $\mathcal{T}$-type relations
These six independent equations restrict the number of unknown expansion coefficients to 4. For every rank configuration, the number of $T$ equations is always one less than the number of expansion coefficients, which agrees with our discussion.

Now we consider the $D$ relations for rank level three, which are represented by four arcs in Figure 1, where two red arcs for reducing $\ell_1$ first and other two orange arcs for reducing $\ell_2$ first. We list them below:

- **$D$-type relations by reducing $\ell_1$ first**
  - Mix rank (0,3) and (1,2)
    
    $\bar{\alpha}_{0,1,0}^{(0,3)} + 3\bar{\alpha}_{0,1,1}^{(1,2)} = \text{Known Terms}$,
    
    $2\bar{\alpha}_{0,1,0}^{(0,3)} + 3\bar{\alpha}_{0,1,1}^{(1,2)} + 6\bar{\alpha}_{1,0,0}^{(1,2)} + 3s_{111}\bar{\alpha}_{0,3,0}^{(0,3)} + 6s_{111}\bar{\alpha}_{1,2,0}^{(1,2)} = \text{Known Terms}$.

  - Mix rank (1,2) and (2,1)
    
    \[ \frac{1}{4}\bar{\epsilon}\bar{\alpha}_{0,1,1}^{(1,2)} + (D - 1)s_{111}\bar{\alpha}_{0,1,0}^{(2,1)} = \text{Known Terms}, \]
    
    \[ \frac{1}{4}D\bar{\alpha}_{0,1,1}^{(2,1)} + \frac{1}{2}D\left(\bar{\alpha}_{1,0,0}^{(1,2)} + s_{111}\bar{\alpha}_{1,2,0}^{(1,2)}\right) + (D - 1)\left(\bar{\alpha}_{1,0,0}^{(2,1)} + s_{111}\bar{\alpha}_{2,1,0}^{(2,1)}\right) = \text{Known Terms}. \]

  - Mix rank (0,3) and (2,1)

- **$D$-type relations by reducing $\ell_2$ first**
- Mix rank (3, 0) and (2, 1)
\[
3\alpha_{0,1,0}^{(2,1)} + \alpha_{1,0,0}^{(3,0)} = \text{Known Terms},
\]
\[
6\alpha_{0,1,0}^{(2,1)} + 3\alpha_{1,0,1}^{(3,0)} + 2\alpha_{1,0,0}^{(3,0)} + 6s_{11}\alpha_{2,1,0}^{(2,1)} + 3s_{11}\alpha_{3,0,0}^{(3,0)} = \text{Known Terms}. 
\] (4.26)

- Mix rank (1, 2) and (2, 1)
\[
(D - 1)s_{11}\alpha_{1,0,0}^{(1,2)} + \frac{1}{4}Ds_{11}\alpha_{1,0,1}^{(2,1)} = \text{Known Terms},
\]
\[
(D - 1)\left(\alpha_{0,1,1}^{(1,2)} + s_{11}\alpha_{1,2,0}^{(1,2)}\right) + \frac{D}{4} \left(2\alpha_{0,1,0}^{(2,1)} + \alpha_{1,0,1}^{(2,1)} + 2s_{11}\alpha_{2,1,0}^{(2,1)}\right) = \text{Known Terms}.
\] (4.27)

One can pick ten independent equations from the relations above and solve all expansion coefficients. Denoting
\[
\mathbf{x}_{3-\text{level}} = \left\{ \alpha_{1,0,0}^{(3,0)}, \alpha_{3,0,0}^{(3,0)}, \alpha_{2,1,0}^{(2,1)}, \alpha_{2,1,0}^{(2,1)}, \alpha_{2,1,0}^{(2,1)}, \alpha_{0,1,0}^{(0,3)}, \alpha_{0,1,0}^{(0,3)}, \alpha_{0,1,0}^{(0,3)}, \alpha_{0,1,0}^{(0,3)}, \alpha_{0,1,0}^{(0,3)}, \alpha_{0,1,0}^{(0,3)}, \alpha_{0,1,0}^{(0,3)}, \alpha_{0,1,0}^{(0,3)} \right\},
\] (4.28)
we have linear equations \( W \mathbf{x}_{3-\text{level}} = \mathbf{b} \) with matrix \( W \)
\[
\begin{pmatrix}
1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{s_{11}}{2} & 1 & \frac{1}{2} & s_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{D}{4} & -\frac{1}{4} & -\frac{s_{11}}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{s_{11}}{2} & 1 & \frac{1}{2} & s_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{D}{4} & -\frac{1}{4} & -\frac{s_{11}}{4} \\
0 & 0 & 0 & 1 - D & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{D}{4} & -\frac{1}{4} & -\frac{D}{4} & 0 & 0 & 1 - D & 0 & 0 \\
0 & 0 & -\frac{D}{4} & -\frac{1}{4} & -\frac{D}{4} & 0 & 0 & 0 & 1 - D & s_{11} - Ds_{11} \\
0 & 0 & -\frac{D}{4} & -\frac{1}{4} & -\frac{D}{4} & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\] (4.29)
and the known vector \( \mathbf{b} \) which corresponds to lower rank level and lower topologies. We solve these linear equations and present final results in Appendix E.

### 4.4 Rank level four

There are five rank configuration and the expansion (2.13) is
\[
C^{(4,0)} = \alpha_{0,0,0}^{(4,0)} s_{00} + \alpha_{2,0,0}^{(4,0)} s_{00} s_{01} + \alpha_{4,0,0}^{(4,0)} s_{01},
\]
\[
C^{(3,1)} = \alpha_{0,1,0}^{(3,1)} s_{00} s_{00} + \alpha_{1,1,0}^{(3,1)} s_{00} s_{01} s_{01} + \alpha_{2,0,1}^{(3,1)} s_{01} s_{00} + \alpha_{3,1,0}^{(3,1)} s_{01} s_{01},
\]
\[
C^{(2,2)} = \alpha_{0,2,0}^{(2,2)} s_{00} + \alpha_{0,2,0}^{(2,2)} s_{00} s_{01} + \alpha_{1,1,1}^{(2,2)} s_{00} s_{01} s_{01} + \alpha_{2,2,0}^{(2,2)} s_{01} s_{00} + \alpha_{2,2,0}^{(2,2)} s_{01} s_{01} + \alpha_{2,0,0}^{(2,2)} s_{00} s_{00} + \alpha_{2,0,0}^{(2,2)} s_{01} s_{01},
\]
\[
C^{(1,3)} = \alpha_{0,1,0}^{(1,3)} s_{00} s_{00} + \alpha_{0,1,0}^{(1,3)} s_{00} s_{01} + \alpha_{1,1,0}^{(1,3)} s_{00} s_{01} s_{01} + \alpha_{1,1,0}^{(1,3)} s_{00} s_{01} s_{01} + \alpha_{1,3,0}^{(1,3)} s_{00} s_{00} + \alpha_{1,3,0}^{(1,3)} s_{00} s_{01} s_{01},
\]
\[
C^{(0,4)} = \alpha_{0,0,0}^{(0,4)} s_{00} s_{00} + \alpha_{0,0,0}^{(0,4)} s_{00} s_{01} + \alpha_{0,0,0}^{(0,4)} s_{00} s_{01} + \alpha_{0,0,0}^{(0,4)} s_{00} s_{01}.
\] (4.30)
One can get 34 linear equations from $T$-type and $D$-type relations for these 20 expansion coefficients. We can pick 20 independent equations among them. There are 18 $T$-type equations

$$\begin{align*}
4\alpha_{0,0,0}^{(0,4)} + 8\alpha_{0,0,0}^{(4,0)} + 2s_{11}\alpha_{0,0,2}^{(0,4)} + 2D\alpha_{0,2,0}^{(0,4)} + 8\alpha_{0,2,0}^{(0,4)} + 12s_{11}\alpha_{0,4,0}^{(0,4)} + 2D\alpha_{0,0,1}^{(1,3)} + 4\alpha_{0,0,1}^{(1,3)} + 2s_{11}\alpha_{0,2,1}^{(1,3)}, \\
2\alpha_{1,1,0}^{(1,3)} + 4\alpha_{1,1,0}^{(1,3)} + 4\alpha_{1,1,0}^{(1,3)} + 6s_{11}\alpha_{1,3,0}^{(1,3)} + 4s_{11}\alpha_{1,3,0}^{(1,3)} + 2s_{11}\alpha_{1,1,0}^{(1,3)} + 4s_{11}\alpha_{1,1,0}^{(1,3)}, \\
2\alpha_{0,2,1}^{(1,3)} + 2\alpha_{0,2,1}^{(1,3)} + 3s_{11}\alpha_{1,3,0}^{(1,3)} + 2D\alpha_{0,0,0}^{(2,2)} + 2s_{11}\alpha_{2,0,0}^{(2,2)} + s_{11}\alpha_{2,0,0}^{(2,2)} + 2s_{11}\alpha_{2,0,0}^{(2,2)}, \\
2\alpha_{0,0,2}^{(2,2)} + 2\alpha_{0,0,2}^{(2,2)} + 2s_{11}\alpha_{2,2,0}^{(2,2)} + 2D\alpha_{0,0,0}^{(2,2)} + 2s_{11}\alpha_{2,0,0}^{(2,2)} + 2s_{11}\alpha_{2,0,0}^{(2,2)}, \\
2\alpha_{0,0,0}^{(2,2)} + 2\alpha_{0,0,0}^{(2,2)} + 2s_{11}\alpha_{2,2,0}^{(2,2)} + 2D\alpha_{0,0,0}^{(2,2)} + 2s_{11}\alpha_{2,0,0}^{(2,2)} + 2s_{11}\alpha_{2,0,0}^{(2,2)}, \\
2\alpha_{0,0,1}^{(3,1)} + 4\alpha_{0,0,1}^{(3,1)} + 2s_{11}\alpha_{1,3,0}^{(3,1)} + 2D\alpha_{1,1,0}^{(3,1)} + 4\alpha_{1,1,0}^{(3,1)} + 4s_{11}\alpha_{2,0,1}^{(3,1)}, \\
2\alpha_{0,0,1}^{(3,1)} + 4\alpha_{0,0,1}^{(3,1)} + 2s_{11}\alpha_{1,3,0}^{(3,1)} + 2D\alpha_{1,1,0}^{(3,1)} + 4\alpha_{1,1,0}^{(3,1)} + 4s_{11}\alpha_{2,0,1}^{(3,1)} + 6s_{11}\alpha_{3,1,0}^{(3,1)}, \\
2\alpha_{0,0,1}^{(4,0)} + 4\alpha_{0,0,1}^{(4,0)} + 2s_{11}\alpha_{1,3,0}^{(4,0)} + 2D\alpha_{1,1,0}^{(4,0)} + 4\alpha_{1,1,0}^{(4,0)} + 4s_{11}\alpha_{2,0,1}^{(4,0)} + 6s_{11}\alpha_{3,1,0}^{(4,0)} + D\alpha_{2,0,0}^{(4,0)} + 8\alpha_{2,0,0}^{(4,0)} + 2s_{11}\alpha_{2,0,0}^{(4,0)}.
\end{align*}$$

For 16 $D$-type equations, we have

- reducing $\ell_1$ first

$$\begin{align*}
\frac{1}{3}(D + 2)\alpha_{0,0,0}^{(0,4)} + \frac{2}{3}(D + 2)\alpha_{0,0,0}^{(1,3)} + \frac{1}{6}(D + 2)s_{11}\alpha_{0,2,0}^{(0,4)}, \\
\frac{1}{6}(D + 2)\alpha_{0,0,0}^{(0,4)} + \frac{1}{3}(D + 2)s_{11}\alpha_{0,2,0}^{(0,4)}, \\
\frac{1}{2}(D + 2)\alpha_{0,0,0}^{(0,4)} + \frac{1}{3}(D + 2)s_{11}\alpha_{0,2,0}^{(0,4)} + \frac{1}{3}(D + 2)s_{11}\alpha_{0,2,0}^{(0,4)}, \\
\frac{1}{3}D\alpha_{0,0,0}^{(1,3)} + \frac{1}{3}D\alpha_{0,0,2}^{(1,3)} + \frac{1}{6}D\alpha_{1,1,0}^{(1,3)} + (D - 1)s_{11}\alpha_{2,0,0}^{(1,3)}, \\
\frac{1}{3}D\alpha_{0,0,0}^{(1,3)} + \frac{1}{3}D\alpha_{0,0,2}^{(1,3)} + \frac{1}{6}D\alpha_{1,1,0}^{(1,3)} + \frac{1}{2}(D - 1)s_{11}\alpha_{2,0,1}^{(1,3)}, \\
\frac{1}{2}(D + 2)\alpha_{0,0,0}^{(2,2)} + \frac{1}{2}(D + 2)\alpha_{0,0,2}^{(2,2)} + D\alpha_{0,0,0}^{(3,1)} + \frac{1}{2}(D + 2)s_{11}\alpha_{0,2,0}^{(2,2)} + \frac{1}{4}(D + 2)s_{11}\alpha_{2,2,0}^{(2,2)} + Ds_{11}\alpha_{3,1,0}^{(3,1)}.
\end{align*}$$

- reducing $\ell_2$ first

$$\begin{align*}
D\alpha_{0,0,1}^{(1,3)} + \frac{1}{2}(D + 2)\alpha_{0,0,0}^{(2,2)} + \frac{1}{2}(D + 2)\alpha_{0,0,2}^{(2,2)} + Ds_{11}\alpha_{1,1,0}^{(1,3)} + \frac{1}{4}(D + 2)s_{11}\alpha_{2,2,0}^{(2,2)} + (D - 1)s_{11}\alpha_{2,0,2}^{(2,2)}, \\
Ds_{11}\alpha_{1,1,0}^{(1,3)} + \frac{1}{2}(D + 2)\alpha_{0,0,0}^{(2,2)} + \frac{1}{2}(D + 2)\alpha_{0,0,2}^{(2,2)} + Ds_{11}\alpha_{1,1,0}^{(1,3)} + \frac{1}{2}(D + 2)s_{11}\alpha_{2,2,0}^{(2,2)}, \\
(D - 1)s_{11}\alpha_{0,0,2}^{(2,2)} + s_{11}\alpha_{0,0,2}^{(2,2)} + \frac{1}{2}(D - 1)s_{11}\alpha_{1,1,1}^{(3,1)} + \frac{1}{6}Ds_{11}\alpha_{1,1,0}^{(3,1)} + \frac{1}{6}Ds_{11}\alpha_{2,0,1}^{(3,1)}, \\
(D - 1)s_{11}\alpha_{0,0,2}^{(2,2)} + s_{11}\alpha_{0,0,2}^{(2,2)} + \frac{1}{2}(D - 1)s_{11}\alpha_{1,1,1}^{(3,1)} + \frac{1}{6}Ds_{11}\alpha_{1,1,0}^{(3,1)} + \frac{1}{6}Ds_{11}\alpha_{2,0,1}^{(3,1)}.
\end{align*}$$
Figure 1: Algorithm for tensor reduction of sunset topology, where every point represents a particular tensor configuration \((r_1, r_2)\). After using the \(T\)-type relations, only one unknown coefficient is left for each point. To determine the last unknown piece, we need to use \(D\)-type relations to relate different tensor configurations with the same rank level. We have used the red lines to represent relations coming from \(D\)-type relation for \(\ell_1\), and orange lines, for \(\ell_2\) respectively (here we draw \(D\)-type relations for \(r_1 + r_2 \leq 5\)). Each line provides a nontrivial relation among unknown coefficients. One can see that for rank level \(r \geq 3\), the number of lines is equal or larger than the number of unknown coefficients, thus they will give enough equations.

\[
\begin{align*}
(D-1)\tilde{d}_{10,2,0}^{(2,2)} &+ \frac{1}{2} (D-1)\tilde{d}_{11,1,0}^{(2,2)} + \frac{1}{3} D\tilde{d}_{11,1,0}^{(3,1)} + \frac{1}{6} D\tilde{a}_{20,1}^{(3,1)} + (D-1)s_{11}\tilde{a}_{20,2,0}^{(2,2)} + \frac{1}{2} D s_{11}\tilde{a}_{31,1,0}^{(3,1)}, \\
\frac{1}{3} (D-2)\tilde{d}_{00,0,1}^{(3,1)} &+ \frac{1}{6} (D-2)\tilde{d}_{00,0,0}^{(4,0)} + \frac{1}{3} (D-2)s_{11}\tilde{d}_{11,1,0}^{(3,1)} + \frac{1}{12} (D-2)s_{11}\tilde{a}_{20,0,0}^{(4,0)}, \\
\frac{2}{3} (D-2)\tilde{d}_{00,0,1}^{(3,1)} &+ \frac{1}{3} (D-2)\tilde{d}_{00,0,0}^{(4,0)} + \frac{2}{3} (D-2)s_{11}\tilde{d}_{20,1}^{(3,1)} + \frac{1}{6} (D-2)s_{11}\tilde{a}_{20,0,0}^{(4,0)}, \\
\frac{2}{3} (D-2)\tilde{d}_{11,1,0}^{(3,1)} &+ \frac{1}{3} (D-2)\tilde{d}_{20,1}^{(3,1)} + \frac{1}{4} (D-2)\tilde{d}_{20,0,0}^{(4,0)} + (D-2)s_{11}\tilde{d}_{31,1,0}^{(3,1)} + \frac{1}{2} (D-2)s_{11}\tilde{a}_{40,0,0}^{(4,0)} \right) \\
= \text{Known Terms}.
\end{align*}
\]

Picking 20 independent equations, we can solve all expansion coefficients of rank level four. Since its expression is long, we present them in an attached Mathematica file. All coefficients have been checked with the analytic output of FIRE \[38, 39, 41\].

Before ending this section, let us summarize the algorithm to reduce tensor integrals of sunset topology to the master basis. The number of expansion coefficients for a particular
rank level $r$ is given by
\[
N_{\hat{a}}[r] = \sum_{r_1 + r_2 = r} \min\{r_1, r_2\} \sum_{j=0}^{\min\{r_1, r_2\}} N_c(r_1 - j) N_c(r_2 - j)
\]  
(4.34)
where $N_c(m) = \lfloor \frac{m^2}{2} \rfloor + 1$ is the number of expansion coefficients for a one-loop rank-$m$ tensor bubble. The recursion procedure can be nicely represented in Figure 1.

5 Conclusion and outlook

In this paper, we have taken the first step in generalizing our improved PV-reduction method with auxiliary vector $R$ to two-loop integrals, i.e., we have carried out the tensor reduction for the simplest sunset topology. For two-loop integrals, the $T$-type relations can be established straightforwardly, while the $D$-type relations are rather nontrivial. We need to consider the reduction of its sub one-loop integrals first. Combining the two types of relations, one can solve all expansion coefficients with the proper choice of master basis.

One of our motivation of the paper is to find an alternative method for higher loop tensor reduction other than IBP method. Comparing these two methods, one can find some similarities. All IBP relations are generated by following six primary relations $\int \frac{\partial}{\partial \ell_i} \cdot K_j$ with $i = 1, 2$ and $K_j = (\ell_1, \ell_2, K)$. Similarly, in our algorithm, there are also six operators $\bar{K}_j \cdot \frac{\partial}{\partial \bar{R}_i}$ with $i = 1, 2$ and $\bar{K}_j = (R_1, R_2, K)$. The one to one mapping between these two types of manipulations seems to hint a deep connection between these two methods although we are not clear at this moment. As pointed out in the paper, in our algorithm we have avoid the appearance of higher power of propagators, thus the number of equations one needs to solve is much fewer than that using standard IBP method. However, in [52] an idea has been suggested to avoid the higher power of propagators in the IBP method. It will be interesting to see if such an idea can be used in our algorithm.

There are several interesting questions coming from this work. The first one is that, in section 3, we have discussed the choice of master integrals to be evaluating the ISP. We hope that for arbitrary two-loop or higher loop integrals with general masses and momenta $K$, one can determine the master basis from this perspective.

The second question is the following. As emphasized several times, the nontrivial $D$-type relations depend crucially on the reduction relation (3.17) of one-loop integrals. We have tested the reduction of triangles and found similar reduction relation also holds for several examples. We find such a reduction relation holds for general one-loop integrals with the proof being presented in [50].

The third problem is obvious. In this paper, we focus on the sunset topology. There are other topologies for two-loop integrals. One needs to check if the algorithm present in this paper can be generalized to these cases. With results from higher topologies, we can reduce sunset with propagators having higher power using the idea given in [49]. We hope to give a positive answer to these interesting questions in our future work.
Acknowledgments

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A The reduction of one-loop tadpoles and bubbles

In this part, we will collect the reduction of one-loop tensor tadpoles and bubbles, which has been extensively used in this paper.

For the one-loop rank \( r \) tensor tadpole

\[
I_1^{(r)} \equiv \int \frac{d^D \ell}{(2\pi)^D} \frac{(2R \cdot \ell)^r}{(\ell^2 - M_0^2)},
\]

the reduction result is given by

\[
I_1^{(r)} = C_{1 \to 1}^{(r)} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - M_0^2)},
\]

where

\[
C_{1 \to 1}^{(r)} = \gamma(r, M_0)(R^2)^\frac{r}{2}, \quad \gamma(r, M_0) \equiv \frac{1 + (-)^r}{2} \frac{(r - 1)!!2^r M_0^r}{\prod_{i=1}^{D} D + 2(i - 1)}.
\]

For the one-loop rank \( r \) tensor bubble

\[
I_2^{(r)} \equiv \int \frac{d^D \ell}{(2\pi)^D} \frac{(2R \cdot \ell)^r}{((\ell - K)^2 - M_1^2)},
\]

the reduction result is given by

\[
I_2^{(r)} = C_{2 \to 2; \hat{i}}^{(r)} I_{2; \hat{i}} + \sum_{i=0,1} C_{2 \to 2; i}^{(r)} I_{2; i},
\]

where

\[
I_2 = I_2^{(0)}, \quad I_{2; \hat{i}} = \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - M_1^2)}, \quad I_{2; i} = \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - M_1^2)}.
\]

If we collect these coefficients into a list \( C_2^{(r)} \) using the notations

\[
f_1 = K^2 + M_0^2 - M_1^2, \quad s_{00} = R^2, \quad s_{01} = R \cdot K, \quad s_{11} = K^2,
\]

we will have the following results:

- For the rank \( r = 1 \):

\[
C_2^{(1)} = \left\{ \frac{f_1 s_{01}}{s_{11}}, \frac{s_{01}}{s_{11}}, \frac{s_{01}}{s_{11}} \right\}.
\]
• For the rank $r = 2$:

$$C_2^{(2)} = \frac{Df_1 s_{01}}{(D-1)s_{11}} - \frac{4M_0^2 s_{01} + s_{00}(f_1^2 - 4s_{11}M_0^2)}{(D-1)s_{11}}, \quad - \frac{Df_1 s_{01}}{(D-1)s_{11}} + \frac{s_{00} f_1}{(D-1)s_{11}},$$

$$\frac{Df_1 s_{01}}{(D-1)s_{11}} + \frac{2(D-2)s_{01}^2 + s_{00}(s_{11} - M_0^2 + M_1^2)}{(D-1)s_{11}}.$$

(A.9)

For our application, we have separated terms according to the power of pole $s_{11}$. One crucial point is that we can write $C_2^{(2)}$ as the combination of lower-rank coefficients and the contributions from lower topologies (tadpoles)

$$C_2^{(2)} = \frac{Df_1 s_{01}}{(D-1)s_{11}} C_2^{(1)} - \frac{4M_0^2 s_{01}^2 + s_{00}(f_1^2 - 4M_0^2 s_{11})}{(D-1)s_{11}} C_2^{(0)}$$

$$+ \left\{ 0, \frac{s_{00} f_1}{(D-1)s_{11}}, \frac{2(D-2)s_{01}^2 + s_{00}(s_{11} - M_0^2 + M_1^2)}{(D-1)s_{11}} \right\}.$$  (A.10)

• For the rank $r = 3$:

The three reduction coefficients are

$$C_{2\to2;1}^{(3)} = -\frac{(D+2)f_1^2 s_{01}^3}{(D-1)s_{11}^3} + \frac{3f_1^2 s_{00} s_{01}}{(D-1)s_{11}^2} + \frac{8M_0^2 s_{01}^3}{D_{s_{11}}^2} - \frac{12M_0^2 s_{01} s_{00}}{D_{s_{11}}}.$$

$$C_{2\to2;0}^{(3)} = \frac{(s_{11} + M_1^2 - M_0^2)^2(3s_{00} s_{01} s_{11} + (D + 2)s_{01}^3)}{(D-1)s_{11}^3} - \frac{4M_1^2 s_{01}(2s_{01}^2 - 3s_{00} s_{11})}{D_{s_{11}}^2},$$

$$+ 6s_{01} \frac{(s_{11} + M_1^2 - M_0^2)(s_{11} s_{00} - D_{s_{01}}^2)}{(D-1)s_{11}^2} + 12s_{01} \frac{s_{00}}{s_{11}}.$$

$$C_{2\to2}^{(3)} = \frac{f_1 (s_{01}^3 ((D+2)f_1^2 - 12M_0^2 s_{11}) + 3s_{00} s_{11} s_{01} (4M_0^2 s_{11} - f_1^2))}{(D-1)s_{11}^3}.$$  (A.11)

By matching the terms with different powers of pole $s_{11}$, we can expand $C_2^{(3)}$ as

$$C_2^{(3)} = \frac{(D+2)f_1 s_{01}}{D_{s_{11}}} C_2^{(2)} - 2 \left( \frac{f_1^2 - 4M_0^2 s_{11}}{D_{s_{11}}} \right) s_{00} + \frac{8M_0^2 s_{01}^3}{D_{s_{11}}},$$

$$\left\{ 0, -\frac{4M_0^2 s_{00} s_{01}}{D_{s_{11}}}, \frac{4(-M_0^2 + 2M_1^2 + s_{11}) s_{00} s_{01}}{D_{s_{11}}} + \frac{4(D-2)s_{01}^3}{D_{s_{11}}} \right\}.$$  (A.12)

One technical point of expansion (A.12) is that there are other different expansions, for example

$$C_2^{(3)} = \frac{(D+2)f_1 s_{01}}{D_{s_{11}}} C_2^{(2)} - 2s_{00}(M_0^2 - M_1^2)^2 - 8M_0^2 s_{01}^2 C_2^{(1)}$$

$$+ \frac{2s_{01}}{D_{s_{11}}} \left\{ s_{00}(-f_1^2 + f_1(3M_0^2 + M_1^2)), s_{00}(s_{11} - 4M_0^2 - 2M_1^2),$$

$$s_{00}(s_{11} + 6M_1^2) + 2(D - 2)s_{01}^2 \right\}.$$  (A.13)
The difference between (A.13) and (A.12) is whether the coefficient of the bubble in the last term is zero or not. For simplicity of application, we prefer the form of (A.12). The choice of bubble coefficient to be zero for the remaining term has fixed the expansion uniquely.

- **For the rank** $r = 4$:

After some algebra, we have

$$C_2^{(4)} = \frac{(D + 4)f_1 s_{01}}{(D + 1)s_{11}} C_2^{(3)} - \left[ \frac{3(f_1^2 - 4M_0^2 s_{11})s_{00}}{(D + 1)s_{11}} + \frac{12M_0^2 s_{01}^2}{(D + 1)s_{11}} \right] C_2^{(2)}$$

$$+ \left\{ 0, \frac{12M_0^2 (M_0^2 - M_1^2 + s_{11}) s_{00}^2}{D(D + 1)s_{11}}, \frac{12 (-M_0^2 + 3M_1^2 + s_{11}) s_{00}s_{01}^2}{(D + 1)s_{11}} + \frac{8(D - 2)s_{01}^4}{(D + 1)s_{11}} \right\}.
$$

- **For the rank** $r = 5$:

$$C_2^{(5)} = \frac{(D + 6)f_1 s_{01}}{(D + 2)s_{11}} C_2^{(4)} - \left[ \frac{4(f_1^2 - 4M_0^2 s_{11})s_{00}}{(D + 2)s_{11}} + \frac{16M_0^2 s_{01}^2}{(D + 2)s_{11}} \right] C_2^{(3)}$$

$$+ \left\{ 0, \frac{-48M_0^4 s_{00}^2 s_{01}}{D(D + 2)s_{11}}, \frac{32 (-M_0^2 + 4M_1^2 + s_{11}) s_{00}s_{01}^3}{(D + 2)s_{11}} + \frac{16(D - 2)s_{01}^5}{(D + 2)s_{11}} \right\}.
$$

From these examples, one can observe the general expansion for any rank $r > 0$ to be

$$I_2^{(r)} = \frac{(D + 2(r - 2))f_1 s_{01}I_2^{(r-1)} - (r - 1)(4M_0^2 s_{01}^2 + (f_1^2 - 4M_0^2 s_{11})s_{00})I_2^{(r-2)} + I_{Tad}^{(r)}}{(D + r - 3)s_{11}}
$$

(A.16)

where $I_{Tad}^{(r)}$ is the contribution of tadpoles containing no poles of $s_{11}$. Relation (A.16) is one key relation used in the paper, so it is desirable to give proof. Note that reduction coefficients are completely determined by recursion relations produced by $D$-type operators and a $T$-type operator. We can prove (A.16) by arguing that it is a solution to all $D$-type recursions and $T$-type recursion relation in (2.7). We will do the proof recursively: assuming (A.16) is valid for all $r < r_0$, then it is also true for $r_0$.

Let us start from $D$-type recursion relations, i.e., the differential equation generated by $D_1$ for the bubble. Acting with $K \partial_R$ on both sides of (A.12) and using $2K \cdot \ell = D_0 - D_1 + f_1$. The LHS is

$$D_1 I_2^{(r_0)} = r_0 f_1 I_2^{(r_0-1)} + r_0 I_2^{(r_0-1)} = r_0 f_1 I_2^{(r_0-1)} + O(tad).
$$

(A.17)

The term $r_0 I_2^{(r_0-1)}$ can be reduced to scalar tadpole without any poles from Gram determinant $s_{11}$, and for simplicity, we just denote the contribution of tadpole topology as $O(tad)$.
One can prove (while acting on the RHS we get
\[ T \text{ad} \] becomes
\[
\frac{1}{(D + r_0 - 3)s_{11}} \left[ (D + 2(r_0 - 2)) f_1(s_{11} I_2^{(r_0 - 1)} + (r_0 - 1)s_{01} f_1 I_2^{(r_0 - 2)}) 
- (r_0 - 1)(2f_1^2 s_{01}) I_2^{(r_0 - 2)} - (r_0 - 1)(4M_0^2 s_{01}^2 + (f_1^2 - 4M_0^2 s_{11}) s_{00})(r_0 - 2)f_1 I_2^{(r_0 - 3)} + O(\text{tad}) \right],
\]
(A.18)

To require (A.18) be equal to (A.17), one need following equation
\[
I_2^{(r_0 - 1)} = \frac{(D + 2(r_0 - 3)) f_1 s_{01} I_2^{(r_0 - 2)} - (r_0 - 2)(4M_0^2 s_{01}^2 + (f_1^2 - 4M_0^2 s_{11}) s_{00}) I_2^{(r_0 - 3)} + O(\text{tad})}{(D + r_0 - 4)s_{11}},
\]
(A.19)

which is just (A.16) with \( r = r_0 - 1. \)

Then we consider the \( T \)-type recursion. Acting with \( T \) on the LHS of (A.16), we get
\[
TI_2^{(r_0)} = 4r_0(r_0 - 1)M_0^2 I_2^{(r_0 - 2)} + O(\text{tad}),
\]
(A.20)

while acting on the RHS we get
\[
\frac{1}{(D + r_0 - 3)s_{11}} \left[ (D + 2(r_0 - 2)) f_1 \left[ 2(r_0 - 1)f_1 I_2^{(r_0 - 2)} + 4s_{01}(r_0 - 1)(r_0 - 2)M_0^2 I_2^{(r_0 - 3)} \right] 
- (r_0 - 1)(4M_0^2 \left[ 2s_{11} I_2^{(r_0 - 2)} + 4(r_0 - 2)s_{01} f_1 I_2^{(r_0 - 3)} + 4(r_0 - 2)(r_0 - 3)s_{01}^2 M_0^2 I_2^{(r_0 - 4)} \right] 
+ (f_1^2 - 4M_0^2 s_{11}) \left[ 2(D + 2r_0 - 4) I_2^{(r_0 - 2)} + 4(r_0 - 2)(r_0 - 3)M_0^2 s_{00} I^{(r_0 - 4)} \right] \right) + O(\text{tad}) \right].
\]
(A.21)

One can prove (A.20) is equal to (A.21) by employing the recursion relation (A.16) for rank \( r_0 - 2. \)

One crucial point of the above proof is that, in principle, one can establish recursion relations of \( T_{\text{rad}}^{(r)} \) for general \( r. \) Thus we can solve it and get the complete relation (A.16) with explicit expression for \( T_{\text{ad}}^{(r)} \). This result will give another recursive way to analytically compute reduction coefficients, which may be more efficient than the one given in [48]. Furthermore, knowing \( T_{\text{rad}}^{(r)} \), one may write down the analytic expression of \( \tilde{\beta}^{(r_1, r_2)}_{i_1;i_2} \) in (3.32) and its dual form. Then we can get the algebraic recursion relation for all coefficients for any rank level, just like what has been done in [47, 48]. It is obvious that with higher and higher rank levels, all available programs, like FIRE, Kira, Reduce [40–43] will become harder and harder to handle analytically. However, the method proposed in this paper can still work with fewer efforts.
B Reduction of lower topologies \( I_{1,1,1}^{(r_1, r_2)} \)

Here we give the reduction results for integrals got by removing the \( i \)-th propagator in the
tensor sunsets, i.e., \( I_{1,1,1}^{(r_1, r_2)} \). We denote

\[
I_{1,1,1}^{(r_1, r_2)} = C_i^{(r_1, r_2)} J.
\]

(B.1)

The reduction coefficients \( C_i^{(r_1, r_2)} \) can be obtained using the results (A.3). Among the three
lower topologies, \( I_{1,1,1}^{(r_1, r_2)} \) is trivial, which is given by

\[
I_{1,1,1}^{(r_1, r_2)} = \sum_{i=0}^{\ell_1 - \ell_2 - \ell_3 + K} \frac{d\ell_1 d\ell_2 (2\ell_1 \cdot R_1)^{r_1} (2\ell_2 \cdot R_2)^{r_2}}{(\ell_1^2 - M_1^2)(\ell_2^2 - M_2^2)} \gamma(r_1, M_1) \gamma(r_2, M_2) (R_1^2)^{r_1} (R_2^2)^{r_2} \int \frac{d\ell_1 d\ell_2}{(\ell_1^2 - M_1^2)(\ell_2^2 - M_2^2)} \gamma(r_1, M_1) \gamma(r_2, M_2) (R_1^2)^{r_1} (R_2^2)^{r_2} J_7,
\]

(B.2)

from which we can easily read out \( C_i^{(r_1, r_2)} \) in (B.1). For \( I_{1,1,1}^{(r_1, r_2)} \), because the subtlety discussed
in (3.45), we need to shift \( \ell_1 \) in the numerator. The computation details are:

\[
I_{1,1,1}^{(r_1, r_2)} = \int \frac{d\ell_1 d\ell_2 (2\ell_1 \cdot R_1)^{r_1} (2\ell_2 \cdot R_2)^{r_2}}{(\ell_1^2 - M_1^2)(\ell_1^2 - M_1^2)} \int \frac{d\ell_1 d\ell_2 (2\ell_1 \cdot R_1 - 2\ell_2 \cdot R_1 + 2K \cdot R_1)^{r_1} (2\ell_2 \cdot R_2)^{r_2}}{(\ell_1^2 - M_1^2)(\ell_1^2 - M_1^2)} \gamma(r_1, M_1) \gamma(r_2, M_2) (R_1^2)^{r_1} (R_2^2)^{r_2} \int \frac{d\ell_1 d\ell_2}{(\ell_1^2 - M_1^2)(\ell_1^2 - M_1^2)} \gamma(r_1, M_1) \gamma(r_2, M_2) (R_1^2)^{r_1} (R_2^2)^{r_2} J_7.
\]

(B.3)

Similarly, we have

\[
I_{1,1,1}^{(r_1, r_2)} = \sum_{i=0}^{r_1} \sum_{j=0}^{r_2} \left( \frac{r_2}{i} \right) \gamma(i + j, M_3) \gamma(r_1 + j, M_1) \frac{(-1)^j r_2^{r_1}}{(r_2 + j)!} (2R_1 \cdot K)^{r_1 - i} (R_1^2)^{r_1 - i} \int \frac{d\ell_1 d\ell_2}{(\ell_1^2 - M_1^2)(\ell_1^2 - M_1^2)}
\]

(B.3)
\( \times (R_2 \cdot \partial_{R_2})^j (R_1^2)^{\frac{r_1+1}{2}} \int \frac{d\ell_1 d\ell_2}{(\ell_1^2 - M_1^2)(\ell_2^2 - M_2^2)}. \) (B.4)

C  Lower rank/topology terms in D-type relation

To write down D-type recursion relation for given \( r_1, r_2 \), we need calculate \( B_{\ell_1}^{(r_1,r_2)}, B_{\ell_2}^{(r_1,r_2)} \) defined in (3.31). Here we list some results. For the rank \((1, r_2)\) and \((r_1, 1)\), the general forms are

\[
B_{\ell_1}^{(1,r_2)} = (D-2) \left[ (s_{11} + M_1^2)C^{(1,r_2)}_2 + C_2^{(1,r_2)} + \frac{s_{01}D_{10}}{r_2+1}C^{(0,r_2+1)}_2 \right. \\
- s_{01} \left( C^{(r_2)}_{1,2-3} + f_{12}C^{(0,r_2)}_2 \right) + \left. \frac{T_{00}}{2(r_2+1)} \left( C^{(r_2+1)}_{1,2-3} + f_{12}C^{(0,r_2+1)}_2 \right) \right],
\]

\[
B_{\ell_2}^{(r_1,1)} = (D-2) \left[ (s_{11} + M_1^2)C^{(r_1,1)}_1 + C_1^{(r_1,1)} + \frac{s_{01}D_{10}}{r_1+1}C^{(r_1+1,0)}_1 \right. \\
- s_{01} \left( C^{(r_1)}_{1,2-3} + f_{12}C^{(0,r_1)}_2 \right) + \left. \frac{T_{00}}{2(r_1+1)} \left( C^{(r_1+1,0)}_{1,2-3} + f_{12}C^{(r_1+1,0)}_2 \right) \right].
\]

One can also write down expression for \((2, r_2)\), \((3, r_2)\) etc. The expression will become longer and longer. Since in examples the rank level is up to four, we just list the terms we used in the examples.

\[
B_{\ell_1}^{(1,0)} = -\frac{1}{2} (D-2)s_{01} \left[ 2f_{12}\alpha^{(0,0)}_{0,0,0} - f_{12}\alpha^{(0,1)}_{0,0,1} + 2\alpha^{(0)}_{0,0,0,2} - \alpha^{(0,1)}_{0,0,0,2} - 2\tilde{\alpha}^{(1,0)}_{1,0,0,2} - 2M_2^2\tilde{\alpha}^{(1,0)}_{1,0,0,0} - 2s_{11}\tilde{\alpha}^{(1,0)}_{1,0,0,0} + 2\tilde{\delta}_5 - 2\tilde{\delta}_7 \right],
\]

\[
B_{\ell_1}^{(1,1)} = \frac{1}{2} (D-2)s_{00} \left[ f_{12}\alpha^{(0,2)}_{0,0,0} + \alpha^{(0,2)}_{0,0,2} + 2\alpha^{(1,1)}_{0,0,1,2} + 2(M_2^2 + s_{11})\alpha^{(1,1)}_{0,0,1} + \frac{4}{D} (\tilde{\delta}_5 - \tilde{\delta}_7) M_2^2 \right. \\
+ \frac{1}{2} (D-2) s_{00} s_{01} \left[ -2f_{12}\alpha^{(0,1)}_{0,1,0} + f_{12}\alpha^{(0,2)}_{0,1,2} - 2\alpha^{(0,1)}_{0,1,0,2} + 2\alpha^{(0)}_{0,1,0,2} + 2\tilde{\alpha}^{(1,1)}_{1,1,0,2} \\
+ 2M_2^2\tilde{\alpha}^{(1,1)}_{1,1,0,0} + 2s_{11}\tilde{\alpha}^{(1,1)}_{1,1,0,0} + 2\tilde{\alpha}^{(1,0)}_{0,0,0} \right],
\]

\[
B_{\ell_1}^{(1,2)} = \frac{1}{6D} (D-2) s_{00} s_{01} \left[ -6Df_{12}\alpha^{(2,0)}_{0,0,0} + DF_{12}\alpha^{(3,0)}_{0,0,1,0} - 6D\alpha^{(0,2)}_{0,0,0,2} + D\tilde{\alpha}^{(1,0)}_{0,0,1,0} \\
+ 6D\alpha^{(1,2)}_{1,0,0,2} + 6DM_2^2\alpha^{(1,2)}_{1,0,0,0} + 2Ds_{11}\alpha^{(1,2)}_{1,0,0,0} + 6Ds_{11}\alpha^{(1,2)}_{1,1,0,0} - 24\tilde{\delta}_5 M_2^2 + 24\tilde{\delta}_7 M_2^2 \right. \\
+ \frac{1}{6} (D-2) s_{00} s_{01} \left[ -6f_{12}\alpha^{(0,2)}_{2,0,0} + 3f_{12}\alpha^{(0,3)}_{0,3,0} - 6\alpha^{(0,2)}_{2,0,0,2} + 3\tilde{\alpha}^{(0,3)}_{0,3,0,2} \\
+ 6\tilde{\alpha}^{(2,0)}_{1,2,0,2} + 6M_2^2\tilde{\alpha}^{(1,2)}_{1,2,0,0} + 6s_{11}\tilde{\alpha}^{(1,2)}_{1,2,0,0} + 6s_{11}\tilde{\alpha}^{(1,2)}_{1,2,0,0} + 4\alpha^{(0,3)}_{0,0,1,0} \right],
\]

\[
B_{\ell_1}^{(2,0)} = \frac{1}{2} s_{01} \left[ -Df_{12}\alpha^{(1,0)}_{1,0,0} + \frac{1}{2} Df_{12}\alpha^{(1,1)}_{1,1,0} - D\alpha^{(1,0)}_{1,0,0,2} + \frac{1}{2} D\tilde{\alpha}^{(1,1)}_{1,1,0,2} + (D - 1)\tilde{\alpha}^{(2,0)}_{2,0,0,2} + Ds_{11}\tilde{\alpha}^{(1,1)}_{1,1,0,0} \\
+ (D - 1)(M_2^2 + s_{11})\tilde{\alpha}^{(2,0)}_{0,0,1} + D\tilde{\alpha}^{(1,1)}_{0,0,0,0} - 4M_2^2\alpha^{(0,1)}_{0,0,0,0} + M_2^4\alpha^{(0,2)}_{0,0,2,0} + 4\tilde{\delta}_5 - 2\tilde{\delta}_5 D \right].
\]
\[B_{\ell_1}^{(2,1)} = s_{00}^2 \left[ 4M_1^2 \alpha_{0,0,1}^{(0,0)} - 4M_1^2 \alpha_{0,2,0}^{(0,2)} + M_1^2 \alpha_{0,3,0}^{(0,3)} - Df_{12} \alpha_{1,1,0}^{(1,1)} - \frac{1}{4} \alpha_0^{(1,0)} + M_1^2 \alpha_{0,1,0}^{(1,2)} + \frac{1}{2} D \alpha_{0,0,0}^{(2,0)} + (D - 1) \alpha_0^{(2,0)} + (D - 1)(M_2^2 + s_{11}) \alpha_0^{(2,0)} \right] \]

\[+ \frac{s_0^2}{2} \left[ \frac{1}{2} D \left( f_{12} \alpha_{0,1,0}^{(1,1)} + \alpha_{0,0,1}^{(1,2)} \right) + \frac{1}{4} D \alpha_{0,0,0}^{(2,0)} + (D - 1) \alpha_0^{(2,0)} + (D - 1)(M_2^2 + s_{11}) \alpha_0^{(2,0)} \right] \]

\[+ 2f_{12} \alpha_{0,0,2}^{(0,0)} - 2s_{11} \left( f_{12} - 2M_2^2 \right) \alpha_0^{(1,0)} + f_{12} \alpha_{0,0,0}^{(0,0)} - 4M_1^2 \alpha_{0,0,0}^{(0,0)} - M_1^2 \alpha_{0,0,0}^{(0,0)} - 2s_{11} \alpha_{0,1,0}^{(0,1)} \]

\[+ \frac{1}{4} s_{11} \alpha_{0,2,0}^{(0,2)} - 4M_1^2 \alpha_{1,1,0}^{(0,0)} - 4M_1^2 \alpha_{0,1,0}^{(0,0)} + M_1^2 \alpha_{0,0,0}^{(0,0)} + s_{11} \left( s_{11} \alpha_{0,2,0}^{(0,2)} + \alpha_0^{(0,2)} \right) \]

\[+ \frac{4 \delta_5^2 M_2^2}{D} + \delta_5 M_1^2 - 3 \delta_5 M_2^2 - \delta_5 M_2^2 - \delta_7 M_1^2 + \delta_7 M_3^2 - \delta_5 s_{11} - \delta_7 s_{11} \right], \]

\[B_{\ell_1}^{(3,0)} = s_{00}^3 \left[ - \frac{1}{2} \left( f_{12} \alpha_{0,1,0}^{(1,1)} - 4M_1^2 \alpha_{0,2,0}^{(0,2)} - 4M_2^2 \alpha_{0,0,0}^{(0,1)} - 4M_2^2 s_{11} \alpha_{0,0,1}^{(0,1)} + \frac{1}{3} M_2^2 \alpha_{0,3,0}^{(0,3)} - s_{11} \alpha_{0,0,1}^{(0,1)} - 4M_2^2 \alpha_{0,0,0}^{(0,1)} \right) \]

\[+ (D - 1) \left( M_2^2 + s_{11} \right) \alpha_{0,0,1}^{(2,1)} - M_2^2 \alpha_{0,0,1}^{(2,1)} - 2f_{12} \alpha_{0,1,0}^{(1,1)} - 2f_{12} \alpha_{0,3,0}^{(1,3)} - \frac{1}{2} Df_{12} \alpha_{0,0,1}^{(1,1)} + \frac{1}{4} Df_{12} \alpha_{0,0,1}^{(1,1)} + \frac{1}{2} Ds_{11} \alpha_{0,0,1}^{(1,1)} \right) \]

\[+ \frac{1}{2} \left( f_{12} \alpha_{0,2,0}^{(0,2)} - 2s_{11} \alpha_{0,2,0}^{(0,2)} - 2s_{11} \alpha_{0,2,0}^{(0,2)} + s_{11} \alpha_{0,2,0}^{(0,2)} + \frac{1}{4} Df_{12} \alpha_{0,0,1}^{(1,1)} + \frac{1}{4} Ds_{11} \alpha_{0,0,1}^{(1,1)} \right) \]

\[+ \frac{1}{4} s_{11} \alpha_{0,3,0}^{(0,3)} + \frac{1}{4} s_{11} \alpha_{0,3,0}^{(0,3)} + \frac{1}{4} Df_{12} \alpha_{0,1,1}^{(1,2)} + \frac{1}{4} Df_{12} \alpha_{0,1,1}^{(1,2)} + \left( D - 1 \right) \alpha_{0,1,0}^{(2,1)} + \frac{4 \delta_5^2}{D} \alpha_{0,1,0}^{(2,1)} + \frac{4 \delta_5^2}{D} \alpha_{0,1,0}^{(2,1)} \right], \]

\[B_{\ell_1}^{(3,0)} = s_{00}^3 \left[ - \frac{1}{2} \left( f_{12} \alpha_{2,0,0}^{(2,0)} + \frac{1}{2} D + 2 \right) f_{12} \alpha_{2,1,0}^{(2,1)} - \frac{1}{2} \left( D + 2 \right) \alpha_{2,0,0}^{(2,0)} + \frac{1}{2} \left( D + 2 \right) \alpha_{2,1,0}^{(2,1)} \right] \]

\[+ \frac{1}{4} \left[ - \frac{1}{2} \left( f_{12} \alpha_{2,0,0}^{(2,0)} - 4M_1^2 \alpha_{2,2,0}^{(0,2)} + 4M_1^2 \alpha_{2,1,0}^{(0,2)} + 4 \delta_5^2 \alpha_1^{(1,2)} + 8 \delta_5 M_1^2 - 8 \delta_5 M_1^2 \right] \right], \]

\[\text{C}(2)\]
For the dual cases $B^{(a,b)}_{\ell_2}$, one can obtain them from above results with proper replacements.

D Transfer matrix between tensor MIs and scalar MIs

In section 3.3, we have discussed the choice of master basis. The basis chose by FIRE6 is

$$F = \{ I_{1,2,1}, I_{1,2,2}, I_{1,1,1}, I_{1,1,0}, I_{0,1,0}, I_{0,1,1} \} ,$$

and the basis used in this paper is

$$J_1 = \int d\ell_{1,2} \frac{1}{D_1 D_2 D_3}, J_2 = \int d\ell_{1,2} \frac{2\ell_1 \cdot K}{D_1 D_2 D_3}, J_3 = \int d\ell_{1,2} \frac{2\ell_2 \cdot K}{D_1 D_2 D_3},$$

$$J_4 = \int d\ell_{1,2} \frac{(2\ell_1 \cdot K)(2\ell_2 \cdot K)}{D_1 D_2 D_3}, J_5 = \int d\ell_{1,2} \frac{2\ell_1 \cdot K}{D_2 D_3}, J_6 = \int d\ell_{1,2} \frac{2\ell_2 \cdot K}{D_1 D_3}, J_7 = \int d\ell_{1,2} \frac{2\ell_1 \cdot K}{D_1 D_2} .$$

The transform matrix $J = TF$ is

$$T_{1,1-7} = \{ 0, 0, 0, 1, 0, 0, 0 \} ,$$

$$T_{2,1-7} = \left\{ \frac{4M_1 (M_1 - s_{11})}{3(D - 2)}, \frac{2M_2 (3M_1 - M_2 - 3M_3 + s_{11})}{3(D - 2)}, \frac{2M_3 (3M_1 - 3M_2 - M_3 + s_{11})}{3(D - 2)}, \right. $$

$$\left. \frac{2(-2(D-3)M_1 + (D-3)M_2 + DM_3 + Ds_{11} - 3M_3 - 2s_{11})}{3(D - 2)}, -\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3} \right\} ,$$

$$T_{3,1-7} = \left\{ \frac{2M_1 (-M_1 + 3M_2 - 3M_3 + s_{11})}{3(D - 2)}, \frac{4M_2 (M_2 - s_{11})}{3(D - 2)}, \frac{2M_3 (-3M_1 + 3M_2 - M_3 + s_{11})}{3(D - 2)}, \right. $$

$$\left. \frac{2((D-3)M_1 - 2(D-3)M_2 + DM_3 + Ds_{11} - 3M_3 - 2s_{11})}{3(D - 2)}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \right\} ,$$

$$T_{4,1} = \frac{4(2D-3)M^2_{1s_{11}}}{3(D-2)(3D-4)} + \frac{4M^2_{1s_{11}}((3D-4)M^2_{1} + (4D-7)M^2_{2} + (1-2D)M^2_{3})}{3(D-2)(3D-4)},$$

$$- \frac{4M^4_{1}((D-1)M^2_{1} - 5(D-1)M^2_{2} + (7D-11)M^2_{3})}{3(D-2)(3D-4)},$$

$$T_{4,2} = \frac{4(3-2D)M^2_{2s_{11}}}{3(D-2)(3D-4)} + \frac{4M^2_{2s_{11}}((7-4D)M^2_{1} + (3D-4)M^2_{2} + (1-2D)M^2_{3})}{3(D-2)(3D-4)},$$

$$- \frac{4M^4_{2}(-5(D-1)M^2_{1} + (D-1)M^2_{2} + (7D-11)M^2_{3})}{3(D-2)(3D-4)},$$

$$T_{4,3} = \frac{2(5D-6)M^2_{3s_{11}}}{3(D-2)(3D-4)} - \frac{4M^2_{3s_{11}}((7-4D)(M^2_{1} + M^2_{2}) + (3D-4)M^3_{3})}{3(D-2)(3D-4)},$$

$$+ \frac{2M^4_{3}(12 - 9D)(M^4_{1} + M^4_{2})}{3(D-2)(3D-4)},$$

$$+ \frac{2M^2_{3}(2M^2_{1} (3D-4)M^2_{2} + (7-4D)M^2_{3}) + (D-2)M^2_{3} + 2(7-4D)M^2_{2}M^2_{3})}{3(D-2)(3D-4)},$$

$$T_{4,4} = \frac{(6-4D)s_{11}}{12 - 9D} - \frac{2s_{11}((-7D^2 + 26D - 20)M^2_{1} + (8D^2 - 33D + 32)(M^2_{1} + M^2_{2}))}{3(D-2)(3D-4)},$$

$$+ \frac{2(D-3)((-D-2)M^2_{1} + (7D-12)(M^2_{1} + M^2_{2})M^2_{2} + 2(D-1)(M^2_{1} + M^2_{2}) - 8(D-1)M^2_{1}M^2_{3})}{3(D-2)(3D-4)},$$

$$T_{4,5} = \frac{(6-4D)s_{11}}{12 - 9D} - \frac{2((D-1)M^2_{1} + (D-1)M^2_{2} + (D-2)M^2_{3})}{9D-12} ,$$
\[ T_{4,6} = \frac{(7D - 10)M_1^2 + (8 - 5D)M_2^2 + (D - 2)M_3^2}{9D - 12} + \frac{(6 - 5D)s_{11}}{9D - 12}, \]
\[ T_{4,7} = \frac{(8 - 5D)M_1^2 + (7D - 10)M_2^2 + (D - 2)M_3^2}{9D - 12} + \frac{(6 - 5D)s_{11}}{9D - 12}, \]
\[ T_{5,1-7} = \{0, 0, 0, 0, 0, 0, 1\}, \]
\[ T_{6,1-7} = \{0, 0, 0, 0, 0, 1, 0\}, \]
\[ T_{7,1-7} = \{0, 0, 0, 0, 1, 0, 0\}. \]

One can check that with general masses and \( K \), it is non-degenerate.

### E Results

Here we collect the reduction results from rank level one to three for reference. It is too long to write in text form for rank level four, so we collect them in an attached Mathematica file.

- **rank** \( r_1 + r_2 = 1 \)

\[ \alpha_{1,0,0}^{(1,0)} = \frac{1}{s_{11}} \{0, 1, 0, 0, 0, 0\}, \quad \alpha_{0,1,0}^{(0,0)} = \frac{1}{s_{11}} \{0, 0, 1, 0, 0, 0\}. \]  

- **rank** \( r_1 + r_2 = 2 \)

\[ \alpha_{0,0,1}^{(1,1)} = \frac{1}{D - 1} \left\{ -2f_{12}, 2, 2, -\frac{1}{s_{11}}, -2, -2, 2 \right\}, \]
\[ \alpha_{1,1,0}^{(1,1)} = \frac{1}{(D - 1)s_{11}} \left\{ 2f_{12}, -2, -2, \frac{D}{s_{11}}, 2, 2, -2 \right\}, \]
\[ \alpha_{0,0,0}^{(2,0)} = \frac{1}{D - 1} \left\{ 2\left( f_{12} + 2M_1^2 \right), \frac{f_{12}}{s_{11}} - 2, -\frac{2}{\left( M_1^2 + s_{11} \right)}, \frac{2}{s_{11}}, 4, 2, -2 \right\}, \]
\[ \alpha_{2,0,0}^{(2,0)} = \frac{-1}{(D - 1)s_{11}} \left\{ 2\left( Df_{12} + 2M_1^2 \right), \frac{-D\left( f_{12} + 2s_{11} \right)}{s_{11}}, \frac{-2D\left( M_1^2 + s_{11} \right)}{s_{11}}, \frac{2D}{s_{11}}, 4, 2D, -2D \right\}, \]
\[ \alpha_{0,0,0}^{(0,2)} = \frac{1}{D - 1} \left\{ 2\left( f_{12} + 2M_2^2 \right), -\frac{2}{\left( M_2^2 + s_{11} \right)}, \frac{-f_{12}}{s_{11}} - 2, \frac{2}{s_{11}}, 2, 4, -2 \right\}, \]
\[ \alpha_{0,2,0}^{(0,2)} = \frac{-1}{(D - 1)s_{11}} \left\{ 2\left( Df_{12} + 2M_2^2 \right), -\frac{-2D\left( M_2^2 + s_{11} \right)}{s_{11}}, \frac{-D\left( f_{12} + 2s_{11} \right)}{s_{11}}, \frac{2D}{s_{11}}, 2D, 4, -2D \right\}. \]  

- **rank** \( r_1 + r_2 = 3 \)

There are ten expansion coefficients for rank-3 level.

\[ \alpha_{1,0,0}^{(3,0)} = \frac{6\left( 4f_{12}\left( (D - 1)M_1^2 + (1 - 2D)M_2^2 \right) + 3Df_{12}^2 - 8M_1^2M_2^2 \right)}{(D - 1)(3D - 2)s_{11}} + \frac{12(D - 2)f_{12}}{(D - 1)(3D - 2)}. \]
\[
\frac{6D (-3f_{12} + 2M_f^2 + 4M_D^2)}{(D - 1)(3D - 2)s_{11}} - \frac{3 \left( (4 - 8D)f_{12}M_f^2 + (3D - 2)f_{12}^2 + 8(D - 1)M_f^2 M_D^2 \right)}{(D - 1)(3D - 2)s_{11}^2} - \frac{12(D - 2) \left( 6f_{12}^2 + (5D - 4)f_{12} + 4(1 - 2D)M_f^2 \right)}{3D^2 - 5D + 2} - \frac{6 \left( 3Df_{12} + 2(D - 2)M_f^2 + 4(1 - 2D)M_D^2 \right)}{(D - 1)(3D - 2)s_{11}} - \frac{12(D - 2) \left( 6 (4D - 2)f_{12} + (D - 2)M_f^2 + 4(1 - 2D)M_D^2 \right)}{(D - 1)(3D - 2)s_{11}^2} + \frac{6(D - 2)}{(D - 1)(3D - 2)s_{11}},
\]
\[
12 (D (2D^2 - 5D + 4) M_f^2 - 4 \left( (D^2 - 3D + 1) M_f^2 + (D - 1)^2 M_D^2 \right) - 2D f_{12}) + \frac{12 (4D^2 - 11D + 8)}{(D - 1)(3D^2 - 8D + 4)},
\]
\[
6 (D (3D^2 - 2D - 4) f_{12} - 2 \left( D^2 + 2D - 4 \right) M_f^2 - 4D (2D^2 - 4D + 1) M_f^2 + 4(D - 1) D M_D^2) + \frac{12(D - 2)(6D - 6)}{(D - 1)(3D^2 - 8D + 4)s_{11}} - \frac{6 \left( 3D^2 f_{12} + 2(2D - 1) \left( (D - 2)M_f^2 - 2D M_D^2 \right) \right)}{(D - 1)D(3D - 2)s_{11}} - \frac{12(D - 2)}{3D^2 - 5D + 2}
\]

\[(E.3a)\]

\[
\frac{\alpha_{3,0,0}}{\alpha_{3,0}} = \left\{ - \frac{2(D + 2) \left( 4f_{12} \left( (D - 1) M_f^2 + (1 - 2D) M_D^2 \right) + 3Df_{12}^2 - 8M_f^2 M_D^2 \right)}{(D - 1)(3D - 2)s_{11}} - \frac{4(D - 2)(2D + 4)f_{12}}{(D - 1)(3D - 2)s_{11}},
\]
\[
\frac{4 \left( (2D^2 - 7D + 2) M_f^2 + 6D(D + 2)f_{12} - 8D(D + 2)M_f^2 \right)}{(D - 1)(3D - 2)s_{11}^2} + \frac{4(D - 2)(2D + 2)}{(D - 1)(3D - 2)s_{11}},
\]
\[
\frac{(D + 2) \left( (4(1 - 2D)f_{12} M_f^2 + (3D - 2)f_{12}^2 + 8(D - 1) M_f^2 M_D^2 \right)}{(D - 1)(3D - 2)s_{11}^2} - \frac{4(D - 2)(2D + 2)}{(D - 1)(3D - 2)s_{11}},
\]
\[
\frac{2(D + 2) \left( (5D - 4)f_{12} + 4(1 - 2D) M_D^2 \right)}{(D - 1)(3D - 2)s_{11}^2} + \frac{2(D + 2) \left( 3Df_{12} + 2(D - 2) M_f^2 + 4(1 - 2D) M_D^2 \right)}{(D - 1)(3D - 2)s_{11}^2},
\]
\[
\frac{8 - 2D^2}{(3D^2 - 5D + 2)s_{11}^2} - \frac{2(D + 2) \left( (4D - 2)f_{12} + (D - 2) M_f^2 + 4(1 - 2D) M_D^2 \right)}{(D - 1)(3D - 2)s_{11}^2},
\]
\[
\frac{4 \left( (2D^2 - 7D + 2) M_f^2 + 6D(D + 2)f_{12} - 8D(D + 2)M_f^2 \right)}{(D - 1)(3D - 2)s_{11}^2} + \frac{4(D - 2)(2D + 2)}{(D - 1)(3D - 2)s_{11}},
\]
\[
\frac{(D + 2) \left( (4(1 - 2D)f_{12} M_f^2 + (3D - 2)f_{12}^2 + 8(D - 1) M_f^2 M_D^2 \right)}{(D - 1)(3D - 2)s_{11}^2} - \frac{4(D - 2)(2D + 2)}{(D - 1)(3D - 2)s_{11}},
\]
\[
\frac{2(D + 2) \left( (5D - 4)f_{12} + 4(1 - 2D) M_D^2 \right)}{(D - 1)(3D - 2)s_{11}^2} + \frac{2(D + 2) \left( 3Df_{12} + 2(D - 2) M_f^2 + 4(1 - 2D) M_D^2 \right)}{(D - 1)(3D - 2)s_{11}^2},
\]
\[
\frac{8 - 2D^2}{(3D^2 - 5D + 2)s_{11}^2} - \frac{2(D + 2) \left( (4D - 2)f_{12} + (D - 2) M_f^2 + 4(1 - 2D) M_D^2 \right)}{(D - 1)(3D - 2)s_{11}^2},
\]

\[(E.3b)\]

\[
\frac{c_{0,1,0}^{(2,1)}}{c_{0,1}^{(2,1)}} = \left\{ \frac{4D f_{12}}{3D^2 - 5D + 2} + \frac{-4 f_{12} \left( (D - 1) M_f^2 + (1 - 2D) M_D^2 \right) - 2 f_{12}^2 + 8 M_f^2 M_D^2}{(D - 1)(3D - 2)s_{11}} + \frac{4D}{3D^2 - 5D + 2},
\]
\[
\frac{2 M_f^2 \left( (1 - 2D) f_{12} + 2(D - 1) M_f^2 \right)}{(D - 1)(3D - 2)s_{11}^2} + \frac{2 \left( 2(D - 1) M_f^2 - 2 D M_D^2 + f_{12} \right)}{(D - 1)(3D - 2)s_{11}} + \frac{4D}{3D^2 - 5D + 2},
\]
\[
\frac{2 M_f^2 \left( (D - 1) f_{12} + 2(1 - 2D) M_f^2 \right)}{(D - 1)(3D - 2)s_{11}^2} + \frac{2 \left( 4(D - 1) M_f^2 + 2(1 - 2D) M_D^2 \right)}{(D - 1)(3D - 2)s_{11}} + \frac{2D(2D - 5)}{(D - 1)(3D^2 - 8D + 4)},
\]
\[
\frac{(3D^2 - 5D + 2)s_{11}}{(D - 1)(3D - 2)s_{11}^2} + \frac{4(D - 2)(2D + 2)}{(D - 1)(3D - 2)s_{11}} + \frac{4D}{(D - 1)(3D^2 - 8D + 4)}
\]

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\[
\alpha^{(2,1)}_{1,0,1} = \left\{ \begin{array}{ll}
\alpha_{1,0,1}^{(2,1)} &= \frac{16 f_{12}}{2 - 3 D} - \frac{4 (2 (D - 1) M_1^2 + 2 (1 - 2 D) M_2^2 + f_{12})}{(D - 1) (3 D - 2) s_{11}} + \frac{16}{3 D - 2} \\
-4 \left( \frac{2 (D - 1) M_1^2 + 2 (1 - 2 D) M_2^2 + f_{12}}{(D - 1)(3 D - 2) s_{11}} \right) + \frac{4 (2 (D - 1) M_1^2 + 2 (1 - 2 D) M_2^2 + f_{12})}{(D - 1)(3 D - 2) s_{11}}
\end{array} \right.
\]

\[
\alpha^{(2,1)}_{2,1,0} = \left\{ \begin{array}{ll}
\alpha_{2,1,0}^{(2,1)} &= \frac{2 (D + 2) (2 (D - 1) M_1^2 - 2 D M_2^2 + f_{12})}{(D - 1)(3 D - 2) s_{11}} - \frac{4 (D - 2)^2 f_{12}}{(D - 1)(3 D - 2) s_{11}} \\
+ \frac{2 (D + 2) (2 (D - 1) M_1^2 - 2 D M_2^2 + f_{12})}{(D - 1)(3 D - 2) s_{11}} + \frac{4 (D - 2)^2 f_{12}}{(D - 1)(3 D - 2) s_{11}} \\
- \frac{2 (D + 2) (2 (D - 1) M_1^2 - 2 D M_2^2 + f_{12})}{(D - 1)(3 D - 2) s_{11}} - \frac{4 (D - 2)^2 f_{12}}{(D - 1)(3 D - 2) s_{11}} \\
+ \frac{2 (D + 2) (2 (D - 1) M_1^2 - 2 D M_2^2 + f_{12})}{(D - 1)(3 D - 2) s_{11}} - \frac{4 (D - 2)^2 f_{12}}{(D - 1)(3 D - 2) s_{11}} \\
- \frac{2 (D + 2) (2 (D - 1) M_1^2 - 2 D M_2^2 + f_{12})}{(D - 1)(3 D - 2) s_{11}} + \frac{4 (D - 2)^2 f_{12}}{(D - 1)(3 D - 2) s_{11}} \\
+ \frac{2 (D + 2) (2 (D - 1) M_1^2 - 2 D M_2^2 + f_{12})}{(D - 1)(3 D - 2) s_{11}} - \frac{4 (D - 2)^2 f_{12}}{(D - 1)(3 D - 2) s_{11}} \\
+ \frac{2 (D + 2) (2 (D - 1) M_1^2 - 2 D M_2^2 + f_{12})}{(D - 1)(3 D - 2) s_{11}} - \frac{4 (D - 2)^2 f_{12}}{(D - 1)(3 D - 2) s_{11}}
\end{array} \right.
\]

\[
\alpha^{(0,3)}_{0,1,0} = \left\{ \begin{array}{ll}
\alpha_{0,1,0}^{(0,3)} &= \frac{6 ((4 - 8 D) f_{12} M_1^2 + 4 (D - 1) f_{12} M_2^2 + 3 D f_{12}^2 - 8 M_1^2 M_2^2)}{(D - 1)(3 D - 2) s_{11}} + \frac{12 (D - 2) f_{12}}{(D - 1)(3 D - 2)} - \frac{12 (D - 2)}{3 D^2 - 5 D + 2}
\end{array} \right.
\]
\[ 
\frac{2(4 - 3D)Df_{12} - 4(D - 1)\left((D - 4)M_1^2 + DM_2^2\right)}{(D - 1)D(3D^2 - 8D + 4)s_{11}} + \frac{2D(2D - 5)}{(D - 1)(3D^2 - 8D + 4)}
+ \frac{2\left(7D^2 - 20D + 8\right)M_1^2 + D\left((-2D^2 + 5D - 4)M_2^2 + DM_2^2\right) + 2Df_{12}}{(D - 1)D(3D^2 - 8D + 4)s_{11}}
+ \frac{4D}{3D^2 - 5D + 2} 
\]
\[
+ \left(\frac{2(1 - 2D)M_1^2 + 2(D - 1)M_2^2 + f_{12}}{(D - 1)(3D^2 - 2)s_{11}}\right)
\]
\[= (E.3h)\]

\[ 
\alpha_{0,1,1}^{(1,2)} = \left\{ \begin{array}{l}
\frac{8f_{12} \left( (2D - 1)M_1^2 - (D - 1)M_2^2 \right) - 4f_{12}^2 + 16M_1^2 M_2^2}{(D - 1)(3D - 2)s_{11}} + \frac{16f_{12} + 4M_2^2 ((D - 1)f_{12} + 2(1 - 2D)M_1^2)}{(D - 1)(3D - 2)s_{11}}
+ \frac{4\left(2 - 4D\right)M_1^2 + 4(D - 1)M_2^2 + f_{12}}{(D - 1)(3D - 2)s_{11}} + \frac{\left(1 - 2D\right)f_{12} + 2(D - 1)M_1^2}{(D - 1)(3D - 2)s_{11}}
+ \frac{8}{3D - 2} - \frac{2\left(Df_{12} + 4 - 8D\right)M_1^2 + 4(D - 1)M_2^2}{(D - 1)(3D - 2)s_{11}}
+ \frac{8D \left(2D^2 - 4D + 1\right)M_1^2 + 4\left(4 - 3D\right)Df_{12} - 8(D - 1)\left((D - 4)M_2^2 + DM_2^2\right)}{(D - 1)D(3D^2 - 8D + 4)s_{11}} + \frac{40 - 16D}{3D^2 - 8D + 4} 
\end{array} \right. 
\]
\[
+ \frac{48 - 28D}{3D^2 - 8D + 4} \left(\frac{2(1 - 2D)M_1^2 + 2(D - 1)M_2^2 + f_{12}}{(D - 1)(3D - 2)s_{11}} + \frac{16}{3D - 2}\right)
\]
\[= (E.3i)\]

\[ 
\alpha_{1,2,0}^{(1,2)} = \left\{ \begin{array}{l}
\frac{2(D + 2)\left(-2f_{12} \left( (2D - 1)M_1^2 - (D - 1)M_2^2 \right) + f_{12}^2 - 4M_1^2 M_2^2\right)}{(D - 1)(3D - 2)s_{11}^3} - \frac{4(D - 2)f_{12}^2}{(D - 1)(3D - 2)s_{11}^3}
+ \frac{4(D - 2)^2}{(D - 1)(3D - 2)s_{11}^3}
+ \frac{2(D - 2)\left(2(D + 2)\left(-2DM_1^2 + 2(D - 1)M_2^2 + f_{12}\right) + (D - 1)\left(2Df_{12} + 4(1 - 2D)M_1^2 + 4(D - 1)M_2^2\right)\right)}{(D - 1)(3D - 2)s_{11}^3}
+ \frac{2(D + 2)\left(2(D - 1)f_{12} - 2(D - 1)M_2^2\right) \left(2(D + 2)\left(Df_{12} + 4(1 - 2D)M_1^2 + 4(D - 1)M_2^2\right)\right)}{(D - 1)(3D - 2)s_{11}^3}
+ \frac{2(D + 2)^2(D + 2)\left(Df_{12} + 4(1 - 2D)M_1^2 + 4(D - 1)M_2^2\right)}{(D - 1)(3D - 2)s_{11}^3}
\end{array} \right. 
\]
\[
\frac{2(D - 2)^2}{(D - 1)(3D - 2)s_{11}^3} + \frac{4(D^3 - 5D^2 + 13D - 10)}{(D - 1)(3D^2 - 8D + 4)s_{11}^3}
+ \frac{2(D + 2)\left(2(D^2 - 4D + 1)M_1^2 + (4 - 3D)Df_{12} - 2(D - 1)\left((D - 4)M_2^2 + DM_2^2\right)\right)}{(D - 1)(3D^2 - 8D + 4)s_{11}^3}
\]
\[
- \frac{4D^3 + 38D^2 - 76D + 48}{(D - 1)(3D^2 - 8D + 4)s_{11}^3} + \frac{2(D^2 \left(7D^2 - 14D + 8\right)M_2^2)}{(D - 1)(3D^2 - 8D + 4)s_{11}^3}
\]
\[
+ \frac{2(D + 2)\left(2(D^2 - 7D + 6) + (D^3 + 10D^2 - 40D + 16)M_2^2 - D \left(2D^3 + 5D^2 - 22D + 16\right) M_2^2\right)}{(D - 1)(3D^2 - 8D + 4)s_{11}^3}
\]
\[
- \frac{4(D - 2)^2}{(D - 1)(3D - 2)s_{11}^3}
\]
\[
= (E.3j) 
\]
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