Determining the State of a Nonlinear Flexible Multibody System using an Unscented Kalman Filter

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Abstract This paper describes an estimator incorporating the Unscented Kalman Filter (UKF) technique and multibody system dynamics, to determine the state of the flexible multibody applications. The dynamic equation of the flexible mechanism is formed using a set of non-linear equations as functions of reference and modal coordinates. Since both reference and modal coordinates have no physical meaning, their information is not able to be obtained directly from sensors. Thus, a novel technique is proposed in this work that can successfully translate physical measurements collected by sensors into non-physical modal coordinates. To validate the performance of the proposed modeling technique to apply a UKF to determine the state of a nonlinear flexible multibody system, simulation were carried out for a four-bar mechanism case study to compare the simulation data and UKF data.

Index Terms Deflection, Flexible simulation models, Nonlinear Kalman Filter, State Observer, Unscented Kalman Filter

Nomenclature

(α, κ, β) Scaling parameters of the UKF
(ûma) Master nodal coordinates
(λ) Vector of Lagrange multipliers
(F) Discontinuous function form of the measurement sensitivity matrix
(Σ) Sigma points
(Q) Process noise covariance
(R) Measurement noise covariance
(I) Relation between the scaling parameters and number of states
(ûP) Vector of position with of point P respect to the origin of the floating frame of reference
(û0P) Vector of undeformed position of an particle P with respect to the floating frame in floating frame
(ûP) Displacement of the particle P within the floating frame of reference due to the deformation
(ûQ) Deformation vectors for the particle Q on a flexible beam
(û0Q) Initial values of the estimated state variables
(û̂) Estimated sate variables
(ŷ) Measurement mean
(Φma) Master modal matrix
(Φsl) Slave modal matrix
(Φt) Translational modal transformation matrix
(Φ̂l) Skew symmetric matrix
(Cq) Jacobian matrix
(f) Function of the first-order ODE equations
(h) Measurement sensitivity matrix
(K) Kalman gain
(P) Error covariance matrix
(Pxy) Cross covariance
(Pyy) Measurement covariance
(P0) Initial values of the error covariance matrix
(q) Global coordinates
(q̇) Dependent coordinates
(q̇̇) Independent coordinates
(Qc) Vector of external force
(Qv) Vector of quadratic velocity
(r) Vector of position with respect to global frame
(u) Gaussian white measurement noise
(uc) Gaussian white process noise
(u(r)) Covariance weight
(w̃) Mean weight
(x) State variables
(x0) Initial values of state variables
(y) A function which includes measurements
(e) Number of elements
(h1, h2, h3) Sensors’ measurements
(k) Time step
(L) Number of states
(N) Noise distribution
(n) Number of degrees of freedom before applying constraints
(nμ) Number of constraint equations
(nxμ) Number of modal coordinates
(P) Particle name
(Q) Arbitrary point on the flexible link in four-bar mechanism
(t) Time
(tμ) Initial value for the covariance weight
(tμ0) Initial value for the mean weight
(γ) Propagated sigma points
(Ω) Covariance matrix
(A) Rotation matrix of the floating frame with respect to the global frame
(C) Vector of the constraint equations
(M) Mass matrix
(p) Modal coordinates
(R) Position vector of floating frame with respect to global frame

I. INTRODUCTION

Multibody system dynamic models describe the real-world behaviors of physical systems using equations of motion. In general, a working physical multibody system, parameters such as position and velocity can be measured with sensors. However, there are some parameters, such as valve leakages in hydraulic systems that cannot be easily measured. To determine the status of these parameters, state estimation theories can be coupled with simulation models. Available measurement data can be fed into state estimation procedure to determine, with reasonable accuracy, the status of a number of unmeasurable parameters. Therefore, the uncertain actual dynamic states of a simulation model can be estimated using a limited number of measurements from the real system. To effectively control the number of degrees of freedom (DOF) in any system including flexible multibody systems, precise knowledge of at least a significant number of system states is required. However, state variables may not be directly measurable in practical application [1]. Therefore, state observers can and should be used to estimate unmeasured state variables of control interest. The Kalman filter is one type of state estimator that is commonly applied for state/parameter estimation.

Many types of nonlinear Kalman filters have been applied in multibody models to estimate its dynamic states, determining parameters such as velocity and accelerations [2], [3]. The extended Kalman filter (EKF) has been used to estimate position and velocity, in multibody systems with a variety of formulations [3], [4]. It has been used to estimate position for a rigid multibody four-bar mechanism including a spring-damper system [5], [6] and [7] proposed a modified EKF-based state estimation algorithm to predict the joint position and velocity of a robotic manipulator. The Unscented Kalman filter (UKF) has also been used for state estimation in many studies for both rigid and flexible multibody systems [8]- [9]. The UKF was used to improve the estimates of kinematic state and unknown external forces in a rigid slider-crank mechanism simulation [10]. They employed both the EKF and the Spherical Simple Unscented Kalman filter (SS-UKF). A sliding mode estimator and a sliding mode controller for the vibration control of a one-link flexible arm system is proposed in [11], which proved that the non-linear observer can predict state variables from motor angular position.

Control of a rotary flexible link, using a feedback linearization technique has also been investigated [12]. In that study, a Linear Extended State Observer (LESO) was used for state estimation [13], [14]. A flexible-link was studied to estimate large deformations [1] where an Equivalent Rigid Link System (ERLS) was used to describe a full-order dynamic rigid-body model based on the flexible linkage. Other researchers have also used an ERLS to construct the governed equations of motion [15], [16]. They have worked on stiff mechanical systems using a discrete-time version of an Augmented Extended Kalman Filter (AEKF) [17]. Other studies can also be found in the literature regarding this topic [18], [19].

In the UKF method, sigma points [20] are employed in nonlinear equations to calculate covariance and mean values. And the UKF approach has been shown to be more efficient than the EKF for simultaneous state and parameter estimation [21], [22]. However, because of the large number of degrees of freedom in a flexible multibody system, modal reduction methods are commonly used to decrease the number of DOFs, which results a reduction in the number of states in the Kalman filter procedure [1]. In this study, modal reduction and modal expansion techniques have been used to construct computationally-efficient flexible multibody models [16], [23]. In previous researches, numerous sensors have been used to determine system states, such as position and damping factor [17], [24]. In addition, most studies found in the literature have focused on state estimation for rigid multibody systems [3], [25]. There are relatively few studies of state estimation in flexible multibody systems. In some previous researches [26], [27] to study a flexible multibody...
The absolute nodal coordinate formulation (ANCF) and the floating frame reference formulation (FFRF) are the most applicable approaches that can be applied to a wide range of problems. The ANCF is often used to model the flexible bodies which undergo large deformation. In this method, the nodal displacement and slope vectors are employed as the nodal coordinates to result a constant mass matrix but a nonlinear expression of elastic forces in the dynamic equations of motion. Recently, ANCF has been applied for piecewise-cable element modelling to describe the deformation of wire applications [28], [29]. Another method is FFRF, which is employed to model multibody systems subjected to large reference motion with small deformation [30]. Due to the separation of the reference and elastic coordinates, a simple expression for the elastic forces but a highly nonlinear expression for the inertia forces [31] is used in the dynamic equations of motion. In this work, the flexible bodies were described using the FFRF with the modal reduction technique, to reduce the state variables in the dynamics equations.

The objective of this study is to estimate the states of the flexible multibody systems using a UKF observer. The FFRF is coupled with the UKF to estimate the state variables.

A novel transformation method is proposed to translate the physical measurements into non-physical modal coordinates. Simulations are carried out for a four-bar mechanism case study [32], [33]. The rest of the paper is organized as follows, Section 2 describes the extraction method for the equations of motion for the flexible multibody system and the for state space representation. The UKF procedure is explained in Section 3. Section 4 introduces the procedure for coupling the Kalman filter with the flexible multibody system, and Section 5 introduces the case example and reveals the numerical results of the simulations. Conclusions are presented in the final section.

II. FLOATING FRAME OF REFERENCE FORMULATION

In multibody applications, the FFRF is used to describe structural flexibility. In this approach, deformation is defined with respect to the floating frame of reference as shown in Fig. 1. The floating frame of the reference itself can experience large magnitude translations and rotations with respect to the global frame.

The global position of an arbitrary particle \( P \) within a deformable body \( i \) in the FFRF, as shown in Fig. 1 can be written as:

\[
\bar{r}^P = R + A\bar{u}^P = R + A(\bar{u}^P_o + \Phi_t p) \tag{1}
\]

where \( R \) is the position vector of the floating frame of reference with respect to the global frame, \( A \) is the rotation matrix of the floating frame of reference, \( \bar{u}^P \) is the position vector of particle \( P \) with respect to the floating frame of reference origin, \( \bar{u}^P_o \) is the undeformed position vector of a particle \( P \), and \( \bar{u}^P_f \) is the displacement of particle \( P \) within the floating frame of reference due to deformation. In Eq. (1), the displacement vector \( \bar{u}^P_f \) is given by:

\[
\bar{u}^P_f = \Phi_t p \tag{2}
\]

where \( \Phi_t \) is the translational modal transformation matrix with columns composed of deformation modes that are computed using, for example, eigenvalue analysis. Vector \( p \) describe the modal coordinates. The number of modal coordinates is \( n_{m} \), which is less than the number of degrees of freedom.

According to Eq. (1), the deformable body can be described by a set of reference coordinates as follows,

\[
q = \begin{bmatrix} R^T & \phi & p^T \end{bmatrix}^T, \tag{3}
\]

where \( \phi \) is the orientation of the floating frame of reference.

Equations of motion can be written using independent coordinate systems, which makes it possible to convert the system to state space forms [34]. According to the structure of the dynamic system and the number of degrees of freedom, the coordinate of a mechanical system, \( q \), can be partitioned into dependent and independent coordinates, \( q_d \) and \( q_i \), respectively, as:

\[
q = \begin{bmatrix} q_d^T & q_i^T \end{bmatrix} \tag{4}
\]

where the number of dependent coordinates is equal to the number of constraint equations \( n_c \). Later, the dependent coordinates can be defined based on the independent coordinates. Moreover, the number of independent coordinate is \( n - n_c \), where \( n \) is the number of coordinates. In multibody dynamics, kinematic joints are defined by constraint equations, which are expressed as follows,

\[
C(q^d, q^i, t) = 0 \tag{5}
\]
The constraint equation from Eq. (5) can be rewritten based on the virtual changes in the vector of system coordinates as:

$$ C^d_q \delta q^d + C^i_q \delta q^i = 0 $$  \hspace{1cm} (6)

where $C^d_q$ and $C^i_q$ are the partitioned part of the constraint Jacobian matrix related to the dependent and independent coordinates, respectively.

Subsequently, the virtual change of the mechanical system coordinates can be written as:

$$ \delta q = \begin{bmatrix} \delta q^d \\ \delta q^i \end{bmatrix} = B^i \delta q^i $$  \hspace{1cm} (7)

where matrix $B^i$ is given by:

$$ B^i = \left[ - (C^d_q)^{-1} C^i_q \right] $$  \hspace{1cm} (8)

where $I$ is the identity matrix with the size of $n_c \times n_c$. Thereafter, the second derivative of $\delta q$ can be written as:

$$ \ddot{q} = B^i \ddot{q}^i + D $$  \hspace{1cm} (9)

where matrix $D$ is expressed as follows,

$$ D = \left[ - (C^d_q)^{-1} (C^i_q \dot{q}^i + 2C_q \dot{q} + C_H) \right] $$  \hspace{1cm} (10)

where 0 is a null matrix with the size of $n_c \times 1$.

The equations of motion can be formulated according to the principle of virtual work, in which virtual work done by both externally applied forces and inertial forces must equal zero. In addition, the kinematic constraints are accounted for with the help of Lagrange multipliers. The equation of motion can be written with respect to the global coordinates.

$$ \delta q^T (M \ddot{q} + C^T_q \lambda - Q_v - Q_e) = 0 $$  \hspace{1cm} (11)

where $M$ is the mass matrix, $C_q$ is the Jacobian matrix of constraint equations, $\lambda$ is the vector of Lagrange multipliers, $Q_v$ is the quadratic velocity vector and $Q_e$ is the external force vector, which comprises of external and elastic forces.

Substituting Eqs. (7) and (9) into Eq. (11) results in:

$$ \delta q^T B^i (M B^i \ddot{q}^i + M D + C^T_q \lambda - Q_v - Q_e) = 0 $$  \hspace{1cm} (12)

where the virtual velocities $\delta q^i$ includes the independent coordinates and matrix $B^i$ is the null space of the Jacobian matrix. Because $C_q B = 0$, the vector of Lagrange multipliers is eliminated from Eq. (12). Considering that the virtual velocities $\delta q^i$ are independent the following can be written:

$$ B^i (M B^i \ddot{q}^i + M D + C^T_q \lambda - Q_v - Q_e) = 0 $$ (13)

Given the independent coordinates $q^i$, the dependent coordinates $\ddot{q}^i$ can be found by solving a set of nonlinear constraint equations according to Eq. (5) using the Newton-Raphson method. This results in the full set of global coordinates $q$. In addition, the velocity coordinates $\dot{q}$ can be determined from Eq. (7). Next, the generalized external force vector $Q_v$ and the generalized quadratic velocity vector $Q_e$ are computed. Finally, according to Eq. (13), the independent accelerations $\ddot{q}^i$ can be determined.

### III. DESIGNING STATE ESTIMATION FOR FLEXIBLE MULTIBODY SYSTEM

#### A. MODEL FORMULATION

The following paragraphs explain how the flexible multibody equations from Section 2 are adapted to fit the Kalman filter structure. The acceleration of the independent coordinates from Eq. (13) can be rewritten as follows,

$$ \ddot{q}^i = (B^i M B^i)^{-1} (B^i (Q_v + Q_e) - B^i M D) $$  \hspace{1cm} (14)

In the Kalman filter, only independent positions and velocities are integrated during the simulation, therefore, defining the states as $x = \begin{bmatrix} \dot{q}^i \\ \ddot{q}^i \end{bmatrix}$, based on Eq. (14), it can be concluded that [35]:

$$ \begin{bmatrix} \dot{\ddot{q}}^i \\ \ddot{q}^i \end{bmatrix} = \begin{bmatrix} (B^i M B^i)^{-1} (B^i (Q_v + Q_e) - B^i M D) \\ 0 \end{bmatrix} $$  \hspace{1cm} (15)

in which $\dot{x}$ is the time derivative of $x$ and $f(x,t)$ represents the dynamic system function.

The procedure represented by the simplified flow diagram in Fig. 2 clarifies the connection between the multibody system and the Kalman filter. Measurements data provides the Outputs from the Real System. To be noted, the multibody system dynamics is connected to the Kalman filter via the function $f(x,t)$ from Eq. (15). In other words, the independent coordinates $\ddot{q}^i$ and the time derivatives of $\dot{q}^i$ are the inputs to the Kalman filtering. Expressly, these inputs are the defined states for the UKF, which are the sensor measurement data outputs.

![FIGURE 2: Simplified flow diagram for a time step of the UKF and its connection to the multibody model, $P$ is the error covariance matrix, and $x$ represents the state variables.](image-url)
employed in the Kalman filter procedure for the next iterations estimations. Box $H$ represents the discontinuous virtual measurement model, which is a function of the simulation model coordinates. In each iteration, $k$, the Kalman filter will estimate the data ($\hat{x}_k$, and $P_k$). Furthermore, the estimated data ($\hat{x}_k$, and the error covariance matrix $P_k$) feeds back into the Kalman filter for the next step.

Note that the measurements are taken from real system. In case of simulation, they are taken from the simulation model and Gaussian noise is added to replicate the actual measurement process. Kalman filter is an estimation tool and requires estimation model (multibody) and measurements (from real system) to process. Thereafter in this paper, the procedure of the states estimation of flexible multibody systems with UKF will be discussed.

### B. UNSCENTED KALMAN FILTER

The UKF is a type of linear regression Kalman filter proposed for nonlinear systems [36]. Because the discrete-time form of the filter is most suitable for hardware implementation, the UKF procedure is formulated as discrete time. To compensate for process and measurement noises, the state space representation shown in Eq. (15) can be applied as a non-linear form as follows,

$$\begin{bmatrix} \dot{\hat{q}}^i \\ \dot{\hat{q}}^j \end{bmatrix} = f(\hat{q}^i, \hat{q}^j, t) + w(t)$$

where $w$ is the process noise that occurs during the simulation of the system. Moreover, measurement data must be defined through an equation to represent the linear relationship between the sensors and system states [34]. Therefore, the output function of the system $y$ can be expressed as:

$$y(t) = h(\hat{q}^i, \hat{q}^j) + v(t)$$

where $h$ is the measurement sensitivity matrix, a linear difference equation with independent states in this study, and $v(t)$ is the measurement noise. Noises $w(t)$ (see Eq. (16)) and $v(t)$ are assumed to be additive white zero-mean Gaussian noises with distribution of $w(t) \sim N(0, Q(t))$ and $v(t) \sim N(0, R(t))$. $N$ is the noise distribution, $Q$ is the process noise covariance matrix, and $R$ is the measurement noise covariance matrix.

The next step is to discretize the continuous-time system given in Eq. (16). This is done using the Runge kutta fourth-order method. Therefore, for the Eq. (16) and Eq. (17), the discontinuous system can be expressed as:

$$\begin{align*}
x_k &= \mathcal{F}(x_{k-1}) + w_{k-1} \\
y_k &= H(x_k) + v_k
\end{align*}$$

where $k$ is the time step. $\mathcal{F}$ and $H$ symbolize the discontinuous form of the functions $f(x, t)$ and $h(x)$, respectively.

Considering the dynamic system given in Eq. (18), the algorithm is initialized with the values $(k = 0)$ of the error covariance matrix $P_{0|0}$ and the state estimation $\hat{x}_{0|0}$. The algorithm starts with $k = 1$, and a function of the estimations is defined through the nonlinear system function to determine a certain number of sample points which are called sigma points. Based on the assumptions of $\hat{x}_{0|0}$ and $P_{0|0}$ the sigma points are calculated for $k = 1$ as follows:

$$\mathcal{X}_{k-1|k-1}^{(j)} = \begin{bmatrix} \hat{x}_{k-1|k-1} \\ \hat{x}_{k-1|k-1} + \sqrt{L + T} \sqrt{P_{k-1|k-1}} \\ \hat{x}_{k-1|k-1} - \sqrt{L + T} \sqrt{P_{k-1|k-1}} \end{bmatrix}$$

where $L$ is the number of states and $j = 0, ..., 2L$, $P_{k-1|k-1}$ is the matrix square root of the Cholesky decomposition using its lower triangular matrix, $T = \alpha^2(L + k)$. Scaling factors $\kappa$ control the weighting of the sigma points. Scaling factor $\alpha$ explains the extension of sigma points’ propagation around the mean value. The values $\kappa$ and $\alpha$ are usually defined by the user. Normally $\kappa=0$ is an appropriate choice and $0 < \alpha \leq 1$ controls the distribution size of generated sigma points [34], [37].

Afterwards, the sigma points are propagated through the integration step of the multibody simulation, Eq. (18),

$$\gamma_{k|k-1}^{(j)} = \mathcal{F} \left( \mathcal{X}_{k-1|k-1}^{(j)} \right), \quad j = 0, 1, \ldots, 2L$$

Thereafter, the mean and covariance should be calculated. The predicted covariance and mean are calculated based on the results of the sigma points as follows,

$$\begin{align*}
\hat{x}_{k|k-1} &= \sum_{j=0}^{2L} w_j^{(m)} \gamma_{k|k-1}^{(j)} \\
P_{k|k-1} &= \sum_{j=0}^{2L} \left( \gamma_{k|k-1}^{(j)} - \hat{x}_{k|k-1} \right) \left( \gamma_{k|k-1}^{(j)} - \hat{x}_{k|k-1} \right)^T + Q_{k-1}
\end{align*}$$

where $w_0^{(m)} = \Gamma/(L + \Gamma)$, $w_0^{(c)} = w_0^{(m)} + (1 - \alpha^2 + \beta)$, $w_j^{(c)} = w_j^{(m)} = \frac{1}{2(L + \Gamma)}$, $\beta$ is a secondary scaling factor. The value of $\beta$ is considered to be 2 for Gaussian noise distributions.

Next, predicted measurement points are computed with the predicted state points $\gamma_{k|k-1}^{(j)}$. At this stage, determining the prediction state variables is accomplished by propagating sigma points in the linear measurement sensitivity matrix,

$$\mathcal{Y}_{k|k-1}^{(j)} = H \left( \gamma_{k|k-1}^{(j)} \right), \quad i = 0, 1, \ldots, 2L$$

The predicted measurement $\hat{y}_{k|k-1}$ is the mean of the predicted measurement points, $(\hat{y}_j)$, is calculated next. Specifically, these are the predicted state variables with the scaling factor,

$$\hat{y}_{k|k-1} = \sum_{j=0}^{2L} w_j^{(m)} \mathcal{Y}_{k|k-1}^{(j)}$$
The innovation covariance and the cross covariance for the predicted measurement can then be computed as:

\[
P_{xy}^k = \sum_{j=0}^{2L} w_j(\mathbf{y}_{k|k-1} - \hat{x}_{k|k-1})(\mathbf{y}_{k|k-1} - \hat{\mathbf{y}}_{k|k-1})^T
\]

\[
P_{yy}^k = \sum_{j=0}^{2L} \left\{ w_j^c (\mathbf{y}_{k|k-1} - \hat{\mathbf{y}}_{k|k-1}) \right\} + \mathbf{R}_k
\]

where \( \mathbf{R}_k \) is the covariance for the measurement noise. To rectify the states and covariance matrix, the Kalman gain matrix should be calculated [2],

\[
K_k = P_{xy}^k(P_{yy}^k)^{-1}
\]

This is then used to calculate the a posteriori estimated state and covariance matrix:

\[
\hat{x}_{k} = \hat{x}_{k|k-1} + K_k (y_{k} - \hat{y}_{k|k-1})
\]

\[
P_k = P_{k|k-1} - K_k P_{yy}^k K_k^T
\]

Calculation of the a posteriori estimated state and covariance matrix begins with \( k = 1 \), which then increases incrementally for each step. The equations from Eq. (19) to Eq. (26) are calculated for each new time step. This loop is repeated until the stop criteria is reached.

IV. SENSOR MODELS FOR FLEXIBLE MULTIBODY DYNAMICS

A. TRANSFORM TRANSVERSE BEAM DEFLECTION MEASUREMENT INTO MODAL COORDINATES

As discussed in Section 2, the input to the Kalman filter are the independent states \( x = [q^T \dot{q}^T]^T \). These independent states \( x \) can come from sensor measurement data or from simulation results.

When using the floating frame of reference formulation, the modal coordinates \( p \) must be chosen as independent coordinates. However, as discussed in Section 2, modal coordinates have non-physical meaning and it is the representation of deformation modes. Therefore, use sensor data as input for the Kalman filter, each physical measurement must be converted into modal coordinates definition.

Linear encoders are used to measure the positions of the end nodes \( A, B \), as shown in Fig. 3. This helps to define an artificial body deformation frame \( (\bar{\mathbf{i}}, \bar{\mathbf{j}}) \). In addition, several linear encoders are used to measure the positions of the arbitrary points \( Q^1, Q^2, ..., Q^m \). The number of linear encoders use to measure \( Q \) is equal to the number of the modal coordinates for the beam in the floating frame of reference formulation.

According to the geometric representation from Fig. 3, the transverse deformation for point \( Q^m \) can be obtained with the help of an artificial body deformation frame \( (\bar{\mathbf{i}}, \bar{\mathbf{j}}) \), as:

\[
\bar{\mathbf{u}}_f^{Q^m} = \bar{\mathbf{j}}^T r^A_{f,y} Q^m
\]

where \( r^A_{f,y} \) is the distance vector from the end node \( A \) to the point \( Q^m \), and unit vector \( \bar{\mathbf{j}} \) gives the direction of the transverse deformation. The unit vectors \( \bar{\mathbf{i}} \) and \( \bar{\mathbf{j}} \) in the right of Fig. 3 are related as follows,

\[
\bar{\mathbf{i}} = \frac{r^A - r^B}{\|r^A - r^B\|}, \quad \bar{\mathbf{j}} = \bar{\mathbf{i}}_2\bar{\mathbf{i}}
\]

where \( \bar{\mathbf{i}}_2 \) is a skew symmetric matrix as:

\[
\bar{\mathbf{i}}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

A set of deformation vectors for arbitrary points on the flexible beam can be collected as master nodal coordinate as follows,

\[
\bar{\mathbf{u}}_f^{ma} = \begin{bmatrix} \bar{\mathbf{u}}_{f,y}^{Q^1} \\ \bar{\mathbf{u}}_{f,y}^{Q^2} \\ \vdots \\ \bar{\mathbf{u}}_{f,y}^{Q^m} \end{bmatrix} \in \mathbb{R}^{m \times 1},
\]

where \( \bar{\mathbf{u}}_{f,y}^{Q^1} \) to \( \bar{\mathbf{u}}_{f,y}^{Q^m} \) are the deformation in the \( Y \) direction computed according to Eq. (27).

The modal expansion technique has been used to convert the sensor data. There are some modal expansion methods [38], [39] that have been used in previous researches, such as expanding the size of the test mode to the size of the analysis mode, or applying the test-analysis mode technique. In this study, the first approach was taken. Master and slave modal matrices were introduced to convert the sensor measurement data to modal (non-physical) data, see Fig. 4.
As the figure shows, the modal matrix is partitioned into the master and slave matrices, $\Phi_t^{m \alpha}$, and $\Phi_t^{s \alpha}$, respectively. Therefore, the nodal coordinates can be written in two parts. Subsequently, following equations can be written:

$$ \ddot{\bar{u}}_f^{m \alpha} = \Phi_t^{m \alpha} p $$  

(31)

Thereafter the modal coordinates can be written as follows,

$$ p = (\Phi_t^{m \alpha})^{-1} \ddot{\bar{u}}_f^{m \alpha} $$  

(32)

The modal matrix, $\Phi_t^{m \alpha}$ can be partitioned according to the nodal locations of points $Q^1$ to $Q^m$. The matrix $\Phi_t^{m \alpha}$ should be non-singular and invertible.

The UKF technique offers the opportunity to determine the precise value of parameters in each time step in the presence of system noises [40], [41].

**B. IMPLEMENTING THE UNSCENTED KALMAN FILTER IN FLEXIBLE MBD**

The flowchart in Fig. 5 illustrates how the FFRF and the Kalman filter method are coupled. It depicts the state estimation procedure using a flexible simulation model and the Kalman filter.

Referring to the flowchart in the figure:

- The equations of motion for the flexible simulation model receive the initial conditions.
- The sensors produce the measured data, $(h_1, h_2, h_3)$ as three sensors measure the position of three particles on the flexible beam, and the measured data is used to calculate the deflection of the nodes.
- The sensor data is defined based on the nodal coordinates, and state estimation in the Kalman filter is defined based on the independent (nodal) coordinates. Therefore, before using the data in the Kalman filter procedure, it is converted into the data determined in the modal coordinates. An initial guess is made, and the error covariance matrix is determined in the Kalman filter procedure. Employing the initial conditions, sigma points will be generated based on Eq. (19).
- The sigma points are propagated through the integration steps of the simulation model, see Eq. (20). As it pointed out earlier in this section, the calculated weights $\omega_i^{(m)}$ and $\omega_i^{(c)}$ are used to calculate the a priori mean and covariance matrix, see Eq. (21).
- The sigma points are then propagated through the integration steps of the simulation model with the linear $H$ function, see Eq. (22).
- Afterwards, the predicted measurements, innovation covariance and cross covariance are calculated using, Eq. (23), and Eq. (24).
- Next, the sigma points are rectified based on the measured data coming from the sensors, and propagated via the $H$ function. To rectify the states and the covariance matrix, Kalman gain is calculated using Eq. (25).
- At the final stage of the Kalman filter procedure, the estimated states and the covariance matrix are updated using Eq. (26).
- Eventually the estimated states are converted from modal to nodal coordinates.

**V. NUMERICAL RESULTS**

A rigid-flexible four-bar mechanism was used as the case study to demonstrate using Unscented Kalman filter for state estimation with a flexible multibody system.

**A. FOUR-BAR MECHANISM**

Fig. 6 illustrates the four-bar mechanism model. It comprises a flexible beam and two rigid bodies 1 and 2. In this case example, the equations of motion are derived using the (FFRF) [42]. The flexible beam of four-bar mechanism consists of four elements. Table 1 gives the specifications for each beam of the four-bar mechanism.

The boundary condition between each body is the revolute joint. When running the simulation, the deflection of the flexible beam is the focus of consideration. Three rotary
encoders have been employed at points $J_2$, $J_3$, and $f$ to measure the angles at the each time step. The simulation was carried out for two scenarios. In the first scenario, the initial conditions for the simulation model, $x_0$, were considered to be the same as the initial conditions for the UKF, $\hat{x}_0$, See Fig. 5.

To have a performance comparison of the proposed UKF algorithm, a conventional EKF algorithm is employed. To this end, since the EKF algorithm is based on the first-order Taylor expansion of a nonlinear system, hence the Jacobian matrix of the system is calculated.

Figs. 7 and 8 illustrate the estimation results of the deflection of the middle node at the flexible beam, for two scenarios. In both, the noise statistics are considered to be

\[
Q = \begin{pmatrix} 10^{-8} \end{pmatrix} \cdot I_{42 \times 42} \quad \text{and} \quad R = \begin{pmatrix} 10^{-6} \end{pmatrix} \cdot I_{3 \times 3},
\]

and the initial error covariance is set to be $P_0 = 10I_{42 \times 42}$. As it mentioned earlier, in the first scenario the initial conditions of the state variables were assumed to be the same for the simulation and the Kalman filter, but for the second scenarios, different initial conditions were assumed.

Fig. 7a shows the simulated and measured deflection of the midpoint of beam 3, when the initial condition for the UKF and EKF procedure is considered to be equal to the initial condition of the simulation model. It can be seen that due to the high nonlinearity nature of the system, the conventional EKF algorithm is not able to track the simulation model. However, the proposed UKF is able to track the simulation model with high accuracy. Fig. 7b is a zoomed in view showing more detail. It is obvious that although both filters have the same initial conditions as the simulation model, EKF is not able to converge to the true simulation values whilst the UKF converges to the simulation model after $t = 0.05$ s.

As it was seen from Fig. 7 the EKF algorithm is not able to provide accurate estimations and failed to track the simulation model even with identical initial conditions. As mentioned in the literature, since EKF is based on first-order Taylor expansion, for severe nonlinear systems it results in larger errors. Regarding the flexible four-bar mechanism of this paper, the EKF algorithm failed completely for the next simulation scenarios when the initial conditions of the real system and the estimation algorithms differed. For that reason, in scenarios with different initial conditions, only the estimation of the UKF is depicted.

Fig. 8 illustrates the middle-node deflection, but the defined initial conditions for the Kalman filter are different than the initial conditions for the simulation model. The effect of the different initial conditions is clearly observable. The zoomed-in values for this simulation are shown in Fig. 8b. It took nearly 0.35 seconds for the Kalman filter to converge to the true values of the simulation model. Beyond 0.35 seconds, the estimation results are accurate.

The node next to and after the middle node was also studied with respect to different initial conditions. Fig. 9 shows its simulated and measured deflections. The Kalman filter in both scenarios accurately estimated the state variable even after increasing the scale factor. When the initial conditions for the Kalman filter is different than initial conditions for the simulation, the Kalman filter also estimated the state variables accurately.
VI. CONCLUSION

The application of an Unscented Kalman filter in a flexible multibody system to estimate state variables was evaluated by analyzing a rigid-flexible four-bar mechanism case example. The flexible equations of motion were constructed using the floating frame reference formulation. This study was designed to demonstrate a procedure for coupling the flexible multibody systems and the Kalman filter using a minimum sensor set. The approach was evaluated by applying it to two different scenarios. Simulation results verified its ability to precisely estimate system state variables. In addition, the effect of changes to the initial conditions for the Kalman filter was investigated. Results showed that increasing the difference between initial conditions for the simulation model and the Kalman filter procedure increases root mean square error. Ultimately, the results for the four-bar mechanism simulation demonstrated that increasing initial condition scale increases convergence time.

As for the future work of the paper, the main intention would be to decrease the number of sigma-points which are generated at each time sample. Generating $2L + 1$ with $L = 42$ as the number of states, results in high computational time. Hence, methods should be studied to reduce the number of sigma-points while preserving the performance accuracy of the UKF. On the other hand, utilizing adaptation mechanisms will also be studied to increase the robustness of the UKF with respect to unknown and time-varying noise statistics with application to the systems with flexible links.

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FIGURE 9: The deformation displacement in the y-direction for the node after the middle node at beam 3 with different initial conditions

(a) Deformation displacement with $\hat{x}_0 = x_0$

(b) Zoomed-in values with $\hat{x}_0 = x_0$

(c) Deformation displacement with $\hat{x}_0 = 3.2 \times x_0$

(d) Zoomed-in values with $\hat{x}_0 = 3.2 \times x_0$

FIGURE 10: The root mean square for different scenarios.

FIGURE 11: The Convergence time for different scenarios.
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