Distance in the Ellipticity Graph

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Abstract

I. Kapovich and M. Lustig defined the ellipticity graph, a curve complex analogue for free groups. This paper presents an algorithm that uses Stallings subgroup $X$-digraphs and Whitehead automorphisms to determine when the distance between two vertices of the ellipticity graph is two. In addition, a bound on the distance is obtained using Nielsen transformations.

1 Introduction

I. Kapovich and M. Lustig define several curve complex analogues for free groups in [1]. In this paper, we explore one of these curve complex analogues, the ellipticity graph, a bipartite graph:

Definition 1.1. Let $F$ be a free group, and let $[A * B]$ denote the equivalence class of the free splitting $A * B$ up to conjugation. Let the vertex set of the ellipticity graph $\mathcal{Z}(F)$ be

$$\{[A * B] \mid A * B \text{ is a proper free splitting of } F\} \sqcup \{w \mid w \text{ is a nontrivial cyclic word of } F\}.$$  

The vertices $[A * B]$ and $w$ are adjacent whenever $w$ has a representative in $A$ or in $B$. If this is the case, we say that $w$ is elliptic to $A * B$. (Free splittings are defined in Section 2.1.)

In Sections 3 and 4, we use subgroup $X$-digraphs and Whitehead automorphisms to prove the following:

Theorem 1.2. Let $F$ be a finitely generated free group. There is an algorithm that, given two free splittings of $F$, decides whether or not there is a nontrivial element of $F$ that is elliptic to both of them.

Theorem 1.3. Let $F$ be a finitely generated free group. There is an algorithm that, given two cyclic words of $F$, decides whether or not they are both elliptic to some free splitting of $F$.

Together, these two algorithms can decide whether or not two vertices of the ellipticity graph are a distance of two apart. In Section 5.1, we show the following:

Theorem 1.4. Let $F$ be a finitely generated free group, and let $A * B$ and $C * D$ be free splittings of $F$. If a nontrivial element $f$ in $F$ is elliptic to both $A * B$ and $C * D$, then there exists a primitive element $g$ in $F$ that is also elliptic to both $A * B$ and $C * D$.

This shows that removing the non-primitive cyclic words from the ellipticity graph does not change the distances in the graph. Finally, in Section 5.2 we prove the following bound on distances in the ellipticity graph.

Theorem 1.5. Let $F = F(X)$ be a free group with at least two generators, let $A * B$ be a free splitting of $F$ with basis $X$, let $C * D$ be an arbitrary free splitting, let $Y \neq X$ be a basis of $C * D$, and let $\phi = \nu_1 \cdots \nu_n$ be the automorphism that sends $X$ to $Y$, where the $\nu_i$ are the elementary Nielsen transformations. Then the distance between $[A * B]$ and $[C * D]$ in $\mathcal{Z}(F)$ is at most $2n$. In particular, $\mathcal{Z}(F)$ is connected.

The connectedness of the ellipticity graph for free groups with at least three generators was previously mentioned in [1].
2 Definitions and Supporting Results

2.1 The ellipticity graph

The vertices of the ellipticity graph are cyclic words and equivalence classes of free splittings. We begin by defining the notation we will use for the free group and for cyclic words.

**Notation.** For any set $X$, let $X \cup X^{-1}$ denote the set $X \times \{1, -1\}$ containing the elements of $X$ and their formal inverses.

**Definition 2.1.** If $X$ is a set, let $\Sigma = X \cup X^{-1}$. Let the **free group on $X$**, denoted $F(X)$, be the set of all freely reduced words in $\Sigma$, where the group operation is concatenation followed by free reductions. The **length** of an element $w$ of $F(X)$ is denoted $|w|_X$ or simply $|w|$.

If $w$ is any word over $\Sigma$, the corresponding element of $F(X)$, denoted $\tilde{w}$, is the word obtained by performing all possible free reductions on $w$.

**Definition 2.2.** A **cyclic word** over $\Sigma$ is a cyclically ordered set of letters of $\Sigma$ such that no two consecutive letters are inverses of each other. The **length** of a cyclic word $w$ is denoted $|w|_X$ or simply $|w|$.

**Remark.** There is a natural bijection between the cyclic words over $\Sigma$ and the conjugacy classes of $F(X)$.

**Definition 2.3.** An element $g$ of the free group $F(X)$ is **cyclically reduced** if the word’s first letter is not the inverse of the word’s last letter. Equivalently, if $w$ is the cyclic word corresponding to $g$, then $g$ is cyclically reduced if $|g| = |w|$.

Next we define free splittings of groups and the equivalence classes that are vertices of the ellipticity graph.

**Definition 2.4.** (Free splittings). Let $G$ be a group. Let $A$ and $B$ be subgroups of $G$ such that for any product of the form $a_1b_1a_2b_2\cdots a_nb_n$, where $a_i \in A$ and $b_i \in B$, if the product is equal to the identity, then there exists an $i$ such that $a_i = 1$ or $b_i = 1$. Then the **free product** of $A$ and $B$, denoted $A*B$, is the subgroup generated by $A$ and $B$. If, moreover, $A*B = G$, then the pair $(A,B)$ is a **free splitting** of $G$, and we say that $A$ and $B$ are **free factors** of $G$. If, moreover, $A$ and $B$ are both proper subgroups of $G$, then $(A,B)$ is a **proper free splitting** of $G$. We will abuse notation and use $A*B$ to refer to the free splitting $(A,B)$.

**Definition 2.5** (Equivalence classes of free splittings). If $F$ is a free group, we define an equivalence relation on the proper free splittings of $F$ by letting $A*B$ be equivalent to $C*D$ if and only if there exists an element $x \in F$ such that $A = xcx^{-1}$ and $B = xdx^{-1}$ or $A = xdx^{-1}$ and $B = xcx^{-1}$. Let $[A*B]$ denote the equivalence class containing $A*B$.

**Definition 2.6** (Ellipticity). Let $A*B$ be a free splitting. An element $g \in G$ is **elliptic** to $A*B$ if there exists $x \in G$ such that $g \in xax^{-1}$ or $g \in xbx^{-1}$.

**Remark.** If $g$ is elliptic to $A*B$, then $g$ is elliptic to every element of $[A*B]$. Therefore, we can say that $g$ is elliptic to $[A*B]$. Moreover, if $g$ is elliptic to $[A*B]$, then any element of the conjugacy class of $g$ is elliptic to $[A*B]$. If $w$ is the cyclic word corresponding to $g$, we can say that $w$ is elliptic to $[A*B]$.

**Remark.** An element $g$ is elliptic to $[A*B]$ if and only if there exists $A'*B' \in [A*B]$ such that $g \in A'$.

We can now define the ellipticity graph, the main object of study in this paper.

**Definition 2.7.** Let $F$ be a free group. Let the vertex set of the **ellipticity graph** $\mathcal{Z}(F)$ be

$$\{(A*B) \mid A*B \text{ is a proper free splitting of } F\} \cup \{w \mid w \text{ is a nontrivial cyclic word of } F\}.$$ 

The vertices $[A*B]$ and $w$ are adjacent whenever $w$ is elliptic to $[A*B]$.

**Remark.** The group $\text{Aut}(F)$ acts on $\mathcal{Z}(F)$ preserving edges. Since the group of inner automorphisms $\text{Inn}(F)$ fixes $\mathcal{Z}(F)$, the group $\text{Out}(F) = \text{Aut}(F)/\text{Inn}(F)$ acts on $\mathcal{Z}(F)$.

The ellipticity graph is not locally finite, so computing distances is not easy. Nevertheless, some questions about distance in the graph can be answered.
2.2 Nielsen transformations and Whitehead automorphisms

The automorphism group of $F(X)$ is relevant to our study of distances in the ellipticity graph. We present two sets of generators for $\text{Aut}(F(X))$.

**Definition 2.8** ([3]). An elementary Nielsen transformation is an automorphism $\phi$ of $F(X)$ such that $\phi$ sends some letter $a \in X$ to either $a^{-1}$ or $ab$, where $b \in X$, and $\phi$ fixes every element of $X$ other than $a$.

**Theorem 2.9** ([3, Proposition 4.1]). If $X$ is finite, then the elementary Nielsen transformations generate $\text{Aut}(F(X))$.

The Whitehead automorphisms are a larger set of generators for $\text{Aut}(F(X))$, and can be used to determine when tuples of cyclic words are in the same orbit.

**Definition 2.10** ([3]). A Whitehead automorphism $\tau$ of $F(X)$ is an automorphism of $F(X)$ satisfying one of the following two properties.

- The automorphism $\tau$ permutes the elements of the set $X \cup X^{-1}$. In this we case will call $\tau$ a relabeling automorphism.
- The set $X \cup X^{-1}$ contains a letter $a$, called the multiplier of $\tau$, such that for all $x \in X \cup X^{-1}$, we have $\tau(x) \in \{x, xa, a^{-1}x, a^{-1}xa\}$.

The set of all Whitehead automorphisms is denoted $\Omega$.

The following theorems describe Whitehead’s algorithm.

**Theorem 2.11** ([3, Proposition 4.20]). Suppose $w_1, \ldots, w_t$ and $w'_1, \ldots, w'_t$ are cyclic words in $F$ such that for some $\alpha \in \text{Aut}(F)$ we have $\alpha(w_h) = w'_h$ for $1 \leq h \leq t$. Suppose that the sum $\sum |w'_h|$ is minimal among all sums of the form $\sum |\alpha'(w_h)|$ for $\alpha' \in \text{Aut}(F)$. Then there exist Whitehead automorphisms $\tau_1, \ldots, \tau_n$ such that $\alpha = \tau_n \cdots \tau_1$ and $\sum |(\tau_i \cdots \tau_1)(w_h)| \leq \sum |w_h|$ for $0 < i < n$ with strict inequality unless $\sum |w_h| = \sum |w'_h|$.

In other words, if a tuple of cyclic words has minimal length in its orbit, then one can arrive at this tuple from any other tuple in the orbit by a sequence of Whitehead automorphisms that strictly decrease the length of the tuple until the length is minimal and then keep the length the same.

**Corollary 2.12.** If the cyclic words $w_1, \ldots, w_t$ are such that $\sum |w_h|$ is not minimal in the orbit of $(w_1, \ldots, w_t)$ under the action of $\text{Aut}(F)$, then there exists a Whitehead automorphism $\tau$ such that $\sum |\tau(w_h)| < \sum |w_h|$.

**Theorem 2.13** ([3, Proposition 4.21]). If $w_1, \ldots, w_t$ and $w'_1, \ldots, w'_t$ are cyclic words then it is decidable whether or not there exists an automorphism $\alpha$ of $F$ such that $\alpha(w_h) = w'_h$ for $1 \leq h \leq t$.

2.3 $X$-digraphs

Stallings digraphs are an elegant way of representing a subgroup of a free group as a graph with labeled edges. One can use these graphs to produce simple algorithms for various problems, such as deciding whether a given word is in the subgroup. We present the relevant results here. For a more complete treatment of Stallings subgroup digraphs, see “Stallings Foldings and Subgroups of Free Groups” by I. Kapovich and A. Myasnikov [2].

**Definition 2.14** ([2]). If $X$ is a finite alphabet, then an $X$-labeled digraph or $X$-digraph $\Gamma$ is a 5-tuple $(V, E, o, t, \mu)$, where $V$ and $E$ are sets and $o$, $t$, and $\mu$ are functions with $o, t : E \rightarrow V$ and $\mu : E \rightarrow X$. The set $V$, also denoted $V\Gamma$, contains the vertices of $\Gamma$. The set $E$, also denoted $E\Gamma$, contains the edges of $\Gamma$. For any edge $e$, $o(e)$, $t(e)$, and $\mu(e)$ are the origin, terminus, and label of $e$, respectively.

A morphism between two $X$-digraphs is a map that sends vertices to vertices, sends edges to edges, sends the origin and terminus of an edge to the origin and terminus of the image of that edge, and preserves the labels of edges.
Definition 2.15. An $X$-digraph with base vertex is a pair $(\Gamma, v)$ where $\Gamma$ is an $X$-digraph and $v$ is a vertex of $\Gamma$. A morphism of $X$-digraphs with base vertex is a morphism of $X$-digraphs that sends the base vertex of one $X$-digraph to the base vertex of the other.

There is a bijection between certain kinds of $X$-digraphs with base vertex and subgroups of $F(X)$. In order to define this bijection, we first define paths.

Definition 2.16 (2). Let $\Sigma = X \cup X^{-1}$. To each $X$-digraph $\Gamma$ we will associate a $\Sigma$-digraph $\hat{\Gamma}$ with vertex set $\hat{V} = V$ and edge set $\hat{E} = E \cup E^{-1}$. For each edge $e$ of $\Gamma$ from $v$ to $w$ with label $x$, the edge $e^{-1}$ of $\hat{\Gamma}$ is defined to go from $w$ to $v$ with label $x^{-1}$.

Every edge $\hat{e}$ of $\hat{\Gamma}$ has a corresponding edge of $\Gamma$, which we will call the positive edge corresponding to $\hat{e}$.

Remark. It is clear that we can recover $\Gamma$ from $\hat{\Gamma}$ by taking the positive edges of $\hat{\Gamma}$ and restricting $o$, $t$, and $\mu$ appropriately. Therefore, we will occasionally abuse notation and refer to $\hat{\Gamma}$ as $\Gamma$.

Definition 2.17 (2). A nontrivial path $p$ in $\Gamma$ is a finite sequence of edges of $\hat{\Gamma}$ such that the terminus of each edge is the origin of the next edge. We define $o(p)$ and $t(p)$ naturally. The label $\mu(p)$ is the word over the alphabet $\Sigma$ constructed by writing the labels of the edges of $p$ in order. Note that $\mu(p)$ is not necessarily freely reduced.

A trivial path $p$ is a vertex $v$ of $\Gamma$. We have $o(p) = t(p) = v$, and the label of $p$ is the empty word.

Notation. If $p$ is a path of $\Gamma$ with $p = e_1 e_2 \cdots e_k$, then let $\bar{p}$ denote the subgraph of $\Gamma$ containing all positive edges corresponding to edges of $p$, as well as the origins and termini of all those edges. If $p$ is a trivial path of $\Gamma$ from a vertex $v$ to itself, let $\bar{p}$ denote the subgraph of $\Gamma$ containing only the vertex $v$ and no edges.

In general, we will only concern ourselves with paths that do not backtrack.

Definition 2.18 (2). A path $p$ in an $X$-digraph $\Gamma$ is reduced if it does not contain an edge $e \in E\hat{\Gamma}$ such that the next edge in the path is $e^{-1}$. A path reduction of a nonreduced path $p$ is a construction of a path $p'$ by removing of two consecutive edges of $p$ that are inverses of each other. To reduce a path $p$ means to construct a path $\bar{p}$ by performing all possible path reductions.

Remark. One can check that $o(\bar{p}) = o(p)$, $t(\bar{p}) = t(p)$, and $\mu(\bar{p}) = \mu(p)$.

We use paths to define the language of an $X$-digraph with base vertex.

Definition 2.19 (2). If $\Gamma$ is an $X$-digraph and $v$ is a vertex of $\Gamma$, then the language of $\Gamma$ with respect to $v$, denoted $L(\Gamma, v)$, is the set of all labels of reduced paths from $v$ to $v$. That is,

$$L(\Gamma, v) = \{ \mu(p) \mid p \text{ is a reduced path in } \Gamma \text{ and } o(p) = t(p) = v \}.$$

Usually, we would like our $X$-digraphs to satisfy two additional properties: We would like them to be folded and core.

Definition 2.20 (2). An $X$-digraph $\Gamma$ is folded if for every vertex $v$ and label $x$, there is at most one edge with origin $v$ and label $x$, and there is at most one edge with terminus $v$ and label $x$.

Remark. Any finite $X$-digraph can be easily transformed into a folded $X$-digraph in a way that preserves $L(\Gamma, v)$, the set of free reductions of the words in the language of the digraph. This transformation is called Stallings folding and is discussed in [2].

Definition 2.21 (2). Let $\Gamma$ be an $X$-digraph and $v$ be a vertex of $\Gamma$. The core of $\Gamma$ at $v$, denoted by $\text{Core}(\Gamma, v)$, is a subgraph of $\Gamma$ with base vertex $v$ defined by

$$\text{Core}(\Gamma, v) := \bigcup \{ \bar{p} \mid p \text{ is a reduced path in } \Gamma \text{ from } v \text{ to } v \}.$$

If $\text{Core}(\Gamma, v) = (\Gamma, v)$, we will say that $(\Gamma, v)$ is core or $\Gamma$ is a core graph with respect to $v$. 

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Corollary 2.24. If $H$ and $K$ are subgroups of $F(X)$ and finite, folded, and core $X$-digraphs.

Theorem 2.22 ([2] Lemma 3.2]). If $(\Gamma, v)$ is a folded $X$-digraph with base vertex, then $L(\Gamma, v)$ is a subgroup of $F(X)$.

Theorem 2.23 ([2] Proposition 3.8, 5.1, and 5.2]). For every subgroup $H$ of the free group $F(X)$ there is an $X$-digraph with base vertex, denoted $(\Gamma(H),1_\Gamma)$, such that $\Gamma(H)$ is folded and core with respect to $1_\Gamma$ and such that $L(\Gamma(H),1_\Gamma) = H$. Moreover, $(\Gamma(H),1_\Gamma)$ is unique up to isomorphism of $X$-digraphs with base vertex.

If $H$ is finitely generated, then $\Gamma(H)$ is a finite graph that can be constructed in finite time.

Corollary 2.24. If $(\Gamma, v)$ is folded and core, then

\[
\left(\Gamma(L(\Gamma, v)),1_{L(\Gamma,v)}\right) \cong (\Gamma, v).
\]

Proof. Both $(\Gamma(L(\Gamma, v)),1_{L(\Gamma,v)})$ and $(\Gamma, v)$ are folded, core, and have language $L(\Gamma, v)$.

We now present the applications of $X$-digraphs that are relevant to this paper. First, we describe an algorithm for recovering a basis for a subgroup from its digraph.

Theorem 2.25 ([2] Lemma 6.1]). Let $(\Gamma, v)$ be a connected folded digraph with base vertex, and let $H = L(\Gamma, v)$. Let $T$ be a spanning tree of $\Gamma$, and let $T^+$ be the set of all edges in $\Gamma$ that are not on $T$. Let $[u_1,u_2]_T$ denote the unique reduced path in $T$ from $u_1$ to $u_2$. For every edge $e \in T^+$ let $p_e = [v,o(e)]_T[e[t(e),v]]_T$. Note that $p_e$ is a reduced path in $\Gamma$. Let $Y_T := \{\mu(p_e) \mid e \in T^+\}$. Then $Y_T$ is a free basis for $H$.

Next, we describe an algorithm for determining when two subgroups are conjugate to each other.

Definition 2.26 ([2]). If $\Gamma$ is core with respect to some vertex $v$, we will define the type of $\Gamma$, denoted $\text{Type}(\Gamma)$, as follows.

If $v$ does not have degree one in $\Gamma$, then $\text{Type}(\Gamma)$ is defined to be $\Gamma$.

Otherwise, let $p$ be the unique non-trivial reduced path originating at $v$ such that vertices of $\bar{p}$ other than $o(p)$ and $t(p)$ have degree two in $\Gamma$, and $t(p)$ has degree greater than two. Then $\text{Type}(\Gamma)$ is defined to be the subgraph of $\Gamma$ constructed by removing from $\Gamma$ all edges and all vertices of $\bar{p}$, except for $t(p)$.

Equivalently, if $\Gamma$ is core with respect to some vertex $v$, then

\[
\text{Type}(\Gamma) := \bigcap_{u \in V_T} \text{Core}(\Gamma, u).
\]

Theorem 2.27 ([2] Proposition 7.7]). If $H$ and $K$ are subgroups of $F(X)$, then $H$ is conjugate to $K$ if and only if $\text{Type}(\Gamma(H))$ and $\text{Type}(\Gamma(K))$ are isomorphic as $X$-digraphs.

Finally, we present the construction of the digraph corresponding to the intersection of two subgroups.

Definition 2.28. If $\Gamma$ and $\Delta$ are $X$-digraphs, then the product graph, denoted $\Gamma \times \Delta$, is defined as follows.

\[
V(\Gamma \times \Delta) := V\Gamma \times V\Delta,
E(\Gamma \times \Delta) := \{(e,f) \in E\Gamma \times E\Delta \mid \mu(e) = \mu(f)\},
\]

\[
o(e,f) := (o(e),o(f)),
t(e,f) := (t(e),t(f)),
\mu(e,f) := \mu(e) = \mu(f).
\]

Remark. The product graph $\Gamma \times \Delta$ is not necessarily connected, even if both $\Gamma$ and $\Delta$ are connected.
Theorem 2.29 ([2], Lemma 9.3). If $(\Gamma, u)$ and $(\Delta, v)$ are two folded $X$-digraphs with base vertex, then $\Gamma \times \Delta$ is folded and

$$L(\Gamma \times \Delta, (u, v)) = L(\Gamma, u) \cap L(\Delta, v).$$

Theorem 2.30 ([2], Proposition 9.4). If $H$ and $K$ are two subgroups of $F(X)$, then

$$(\Gamma(H \cap K), 1_{H \cap K}) \cong \text{Core}(\Gamma(H) \times \Gamma(K), (1_H, 1_K)).$$

3 Free Splittings with Common Elliptic Element

We aim to determine whether or not the distance in the ellipticity graph between two given classes of free splittings is two. In order to do this, we first answer the following question.

Question. Given a finitely generated free group $F$ and two finitely generated subgroups $H$ and $K$ of $F$, how can we determine whether or not there exists a nontrivial element $g$ of $F$ such that $g$ is conjugate to an element of $H$ and $g$ is conjugate to an element of $K$?

Notation. Given two subgroups $H$ and $K$, let $C_{H,K}$, or simply $C$, denote the set

$$C := \bigcup_{x, y \in F} xHx^{-1} \cap yKy^{-1}.$$ 

The set $C$ is the set of all elements of $F$ that are conjugate to an element of $H$ and conjugate to an element of $K$. In particular, if $g \in C$, then every element in the conjugacy class of $g$ is also in $C$.

We aim, given $H$ and $K$, to decide whether or not $C = \{1\}$. We first establish the following lemma.

Lemma 3.1. If $\Gamma$ is a folded $X$-digraph with base vertex $v$ and there is a nontrivial cyclically reduced element $g$ of $L(\Gamma, v)$, then $v$ has degree at least two.

Proof. Let $x$ be the first letter of $g$, and let $y$ be the last letter of $g$. There is a reduced path in $\Gamma$ from $v$ to $v$ such that the label of the first edge $e$ of the path is $x$ and the label of the last edge $f$ of the path is $y$. Note that $e$ and $f^{-1}$ have origin $v$ and labels $x$ and $y^{-1}$, respectively. Since $g$ is cyclically reduced, $x \neq y^{-1}$. Thus $e$ and $f$ cannot be equal, so there are at least two edges originating from $v$.

Corollary 3.2. If $\Gamma$ is folded and a core graph with respect to $v$, and $L(\Gamma, v)$ contains a nontrivial cyclically reduced element, then Type$(\Gamma) = \Gamma$.

Proof. This is a straightforward application of Definition 2.26.

Lemma 3.3. There exists a nontrivial element of $C$ if and only if there exists a vertex $u$ of Type$(\Gamma(H))$ and a vertex $v$ of Type$(\Gamma(K))$ such that

$$L(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K)), (u,v)) \neq 1.$$ 

Proof. If $g$ is a nontrivial element of $C$, then let $g'$ be a cyclically reduced element of the conjugacy class of $g$. We know that $g' \in C$. Therefore there exist $x$ and $y$ such that $g' \in xHx^{-1}$ and $g' \in yKy^{-1}$. By Theorem 2.27, Type$(\Gamma(H))$ is isomorphic to Type$(\Gamma(xHx^{-1})) = \Gamma(xHx^{-1})$. Similarly, Type$(\Gamma(K))$ is isomorphic to $\Gamma(yKy^{-1})$. Therefore, there exists a vertex $u$ of Type$(\Gamma(H))$ and a vertex $v$ of Type$(\Gamma(K))$ such that

$$(\text{Type}(\Gamma(H)), u) \cong (\Gamma(xHx^{-1}), 1_{xHx^{-1}})$$ 

and

$$(\text{Type}(\Gamma(K)), v) \cong (\Gamma(yKy^{-1}), 1_{yKy^{-1}}).$$ 

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Using Theorem \[2.30\] we have that
\[
g' \in xHx^{-1} \cap yKy^{-1}
\]
\[
= L(\Gamma(xHx^{-1} \cap yKy^{-1}), 1_{xHx^{-1} \cap yKy^{-1}})
\]
\[
= L(\text{Core}(\Gamma(xHx^{-1}) \times \Gamma(yKy^{-1})), 1_{xHx^{-1} \times 1_{yKy^{-1}}}))
\]
\[
= L(\Gamma(xHx^{-1}) \times \Gamma(yKy^{-1}), 1_{xHx^{-1} \times 1_{yKy^{-1}}}))
\]
\[
= L(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K)), (u, v))
\]
Since \(g'\) is nontrivial, \(L(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K)), (u, v)) \neq 1\).
Conversely, assume there does exist a choice of \(u\) and \(v\) such that \(L(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K)), (u, v))\) contains a nontrivial element \(g\). Then by Theorem \[2.29\] we have that
\[
g \in L(\text{Type}(\Gamma(H)), u) \cap L(\text{Type}(\Gamma(K)), v).
\]
By Corollary \[2.24\]
\[
\text{Type} \left( \Gamma \left( L(\text{Type}(\Gamma(H)), u) \right) \right) \cong \text{Type}(\text{Type}(\Gamma(H))) = \text{Type}(\Gamma(H)).
\]
Therefore by Theorem \[2.27\] the subgroup \(L(\text{Type}(\Gamma(H)), u)\) is conjugate to the subgroup \(H\). Similarly, \(L(\text{Type}(\Gamma(K)), v)\) is conjugate to \(K\). Thus, \(g \in xHx^{-1} \cap yKy^{-1}\) for some \(x, y \in F\), so \(g\) is a nontrivial element of \(C\).

**Lemma 3.4.** We have that \(L(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K)), (u, v)) = 1\) for all vertices \((u, v)\) if and only if \(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K))\) is acyclic.

**Proof.** If \(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K))\) is acyclic, then for any vertex \((u, v)\) of \(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K))\), the only reduced path from \((u, v)\) to \((u, v)\) is the trivial path, so \(L(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K)), (u, v)) = 1\). Conversely, assume that \(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K))\) has a cycle. Let \((u, v)\) be a vertex on this cycle, and let \(p\) be a reduced path from \((u, v)\) to \((u, v)\) going once around the cycle. Then \(\mu(p) \neq 1\) and \(\mu(p) \in L(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K)), (u, v))\), so \(L(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K)), (u, v)) \neq 1\).

The above lemmas establish the following two facts.

**Proposition 3.5.** Given a finitely generated free group \(F\) and two finitely generated subgroups \(H\) and \(K\) of \(F\), there exists a nontrivial element of \(F\) conjugate to both an element of \(H\) and an element of \(K\) if and only if the graph \(\text{Type}(\Gamma(H)) \times \text{Type}(\Gamma(K))\) has a cycle.

Since these graphs are finite, this problem is decidable. We can now prove Theorem \[1.2\]

**Theorem 3.6.** If \(F\) is a finitely generated free group, then it is decidable whether or not two classes of proper free splittings are distance two in the ellipticity graph.

**Proof.** Let \([A \ast B]\) and \([C \ast D]\) be the two classes of proper free splittings. The two classes are distance two if and only if there exists a nontrivial cyclic word elliptic to both splittings. Such a cyclic word exists if and only if there exists a nontrivial element \(g\) of \(F\) such that \(g\) is conjugate to both an element of either \(A\) or \(B\) and an element of either \(C\) or \(D\). Since free factors of finitely generated free groups are finitely generated, we can decide whether or not such a \(g\) exists in each of these four cases. Therefore, we can decide whether or not \([A \ast B]\) and \([C \ast D]\) are distance two in \(\mathbb{Z}(F)\).

**Remark.** If \(A \ast B\) and \(C \ast D\) are two free splittings, we can choose a free basis \(X_A\) for \(A\) and a free basis \(X_B\) for \(B\). We can then define a free basis for \(F\) by \(X := X_A \cup X_B\). Then \(\Gamma(A)\) and \(\Gamma(B)\) each contain exactly one vertex, so the product graphs are easy to construct. For example, \(\Gamma(A) \times \Gamma(C)\) is isomorphic to the subgraph of \(\Gamma(C)\) with all edges with labels in \(X_B\) removed.
4 Elements Elliptic to Common Free Splitting

We now turn to the distance two problem for cyclic words.

**Question.** If \( F \) is a free group with free basis \( X \) and \( v \) and \( w \) are two nontrivial cyclic words of \( F \), how can we determine whether or not there exists a proper free splitting \( A \ast B \) of \( F \) such that both \( v \) and \( w \) are elliptic to \( A \ast B \)?

We begin by defining some shorthand for talking about pairs of cyclic words.

**Notation.** If \( w \) is a cyclic word, let \( \mathcal{L}(w) \) denote the subset of \( X \) containing all letters \( x \) such that either \( x \) or \( x^{-1} \) appears in \( w \). Essentially, \( \mathcal{L}(w) \) denotes the letters in \( X \) used by \( w \). Let \( \Lambda(w) \) denote \( \mathcal{L}(w) \setminus (\mathcal{L}(w))^{-1} \).

**Definition 4.1.** We call a pair \((v, w)\) frugal if \( v \) and \( w \) do not use all letters of \( X \), that is, \( \mathcal{L}(v) \cup \mathcal{L}(w) \neq X \).

We call \((v, w)\) disjoint if \( v \) and \( w \) do not share any letters, that is, \( \mathcal{L}(v) \cap \mathcal{L}(w) = \emptyset \). We call \((v, w)\) good if it is frugal or disjoint. Let the length of a pair \((v, w)\) be defined to be \(|v| + |w|\). We say that a pair \((v, w)\) has minimal length if its length is minimal in its orbit under \( \text{Aut}(F) \). In other words, \((v, w)\) has minimal length if \(|v| + |w| \leq |\phi(v)| + |\phi(w)| \) for all \( \phi \in \text{Aut}(F) \).

We can reduce the problem of finding a free splitting adjacent to two cyclic words to the problem of finding an automorphism of \( F \) satisfying certain properties.

**Lemma 4.2.** Let \( v \) and \( w \) be nontrivial cyclic words. Then there exists a proper free splitting \( A \ast B \) of \( F \) such that both \( v \) and \( w \) are elliptic to \( A \ast B \) if and only if there exists an automorphism \( \phi \) of \( F \) such that \((\phi(v), \phi(w))\) is good.

**Proof.** If there exists a proper free splitting \( A \ast B \) of \( F \) such that both \( v \) and \( w \) are elliptic to \( A \ast B \), let \( \phi \) be an automorphism that sends \( A \) to \( \langle X_1 \rangle \) and \( B \) to \( \langle X_2 \rangle \) for some subsets \( X_1 \) and \( X_2 \) of \( X \). Since \( A \ast B \) is a free splitting, \( X_1 \sqcup X_2 = X \). Since the free splitting is proper, we know that \( X_1 \) and \( X_2 \) are both proper subsets of \( X \). It is clear that \( \phi(v) \) and \( \phi(w) \) are elliptic to \( \phi(A) \ast \phi(B) = \langle X_1 \rangle \ast \langle X_2 \rangle \).

Since \( \phi(v) \) is elliptic to \( \langle X_1 \rangle \ast \langle X_2 \rangle \), it has a representative in either \( \langle X_1 \rangle \) or \( \langle X_2 \rangle \). Assume without loss of generality that \( \phi(v) \) has a representative \( g \) in \( \langle X_1 \rangle \). Since all the letters of \( \phi(v) \) are in the word \( g \), we know \( \mathcal{L}(\phi(v)) \subseteq X_1 \). Similarly, either \( \mathcal{L}(\phi(w)) \subseteq X_1 \) or \( \mathcal{L}(\phi(w)) \subseteq X_2 \). If \( \mathcal{L}(\phi(w)) \subseteq X_1 \), then \((\phi(v), \phi(w))\) is frugal, and if \( \mathcal{L}(\phi(w)) \subseteq X_2 \), then \((\phi(v), \phi(w))\) is disjoint.

Conversely, if there exists an automorphism \( \phi \) of \( F \) such that \( \mathcal{L}(\phi(v)) \cup \mathcal{L}(\phi(w)) \neq X \), then let \( X_1 = \mathcal{L}(\phi(v)) \cup \mathcal{L}(\phi(w)) \) and let \( X_2 = X \setminus X_1 \). If, on the other hand, there exists an automorphism \( \phi \) of \( F \) such that \( \mathcal{L}(\phi(v)) \cap \mathcal{L}(\phi(w)) = \emptyset \), then let \( X_1 = \mathcal{L}(\phi(v)) \) and let \( X_2 = X \setminus X_1 \).

In both cases \( \langle X_1 \rangle \ast \langle X_2 \rangle \) is a proper free splitting of \( F \), so let \( A = \phi^{-1}(\langle X_1 \rangle) \) and \( B = \phi^{-1}(\langle X_2 \rangle) \). Both \( \phi(v) \) and \( \phi(w) \) are elliptic to \( \langle X_1 \rangle \ast \langle X_2 \rangle \), so both \( v \) and \( w \) are elliptic to \( A \ast B \).

In order to determine whether or not there is a good pair in the orbit \((v, w)\), we prove some results about Whitehead automorphisms.

**Lemma 4.3.** If \( w \) is a cyclic word and \( \tau \) is a Whitehead automorphism with multiplier \( a \notin \Lambda(w) \), then either \( \tau(w) = w \) or \( |\tau(w)| > |w| \).

**Proof.** Let \( w = w_1w_2 \cdots w_l \) where \( w_i \) are letters. Then \( \tau(w) \) is the cyclic word \( w_1a^{\epsilon_1}w_2a^{\epsilon_2} \cdots w_la^{\epsilon_l} \) for some exponents \( \epsilon_i \). Since \( a \neq w_i \neq a^{-1} \) for all \( i \), this cyclic word is reduced. If all of the exponents \( \epsilon_i \) are zero, then \( \tau(w) = w \). If one of the exponents \( \epsilon_i \) is nonzero, then the cyclic word \( w_1a^{\epsilon_1}w_2a^{\epsilon_2} \cdots w_la^{\epsilon_l} \) has at least one more letter than the cyclic word \( w_1w_2 \cdots w_l \), so \( |\tau(w)| > |w| \).

**Corollary 4.4.** If \( \tau \) is a Whitehead automorphism with multiplier \( a \) and \( |\tau(w)| \leq |w| \), then \( \mathcal{L}(\tau(w)) \subseteq \mathcal{L}(w) \).

**Proof.** If \( a \in \Lambda(w) \), then \( \mathcal{L}(\tau(w)) \subseteq \mathcal{L}(w) \). If \( a \notin \Lambda(w) \), then by Lemma 4.3 since it cannot be that \( |\tau(w)| > |w| \), we have that \( \tau(w) = w \), so \( \mathcal{L}(\tau(w)) = \mathcal{L}(w) \).
Lemma 4.5. If there is a good pair \((v, w)\) of cyclic words that does not have minimal length, then there exists a Whitehead automorphism \(\tau\) such that \((\tau(v), \tau(w))\) is also a good pair and such that the length of \((\tau(v), \tau(w))\) is smaller than the length of \((v, w)\).

Proof. Since \((v, w)\) does not have minimal length, by Corollary 4.12 there exists a Whitehead automorphism \(\tau_1\) such that \(|\tau_1(v)| + |\tau_1(w)| < |v| + |w|\). Clearly \(\tau_1\) is not a relabeling automorphism.

If \((v, w)\) is frugal, then let \(\tau = \tau_1\). We will show that \((\tau(v), \tau(w))\) is frugal. Let the letter \(a\) be the multiplier of the Whitehead automorphism \(\tau\). If \(a\) is not in \(\Lambda(v) \cup \Lambda(w)\), then by Lemma 4.3 \(|\tau(v)| \leq |v|\) and \(|\tau(w)| \leq |w|\), which is a contradiction. Thus \(a \in \Lambda(v) \cup \Lambda(w)\), so \(\Lambda(\tau(v)) \subseteq \mathcal{L}(v) \cup \mathcal{L}(w)\), and \(\mathcal{L}(\tau(w)) \subseteq \mathcal{L}(v) \cup \mathcal{L}(w)\).

If, on the other hand, \((v, w)\) is not frugal, then it must be disjoint. Therefore, \(X = \mathcal{L}(v) \cup \mathcal{L}(w)\).

Assume without loss of generality that the multiplier of the Whitehead automorphism \(\tau_1\) is in \(\Lambda(v)\). Let the Whitehead automorphism \(\tau\) be defined on the generator set \(X\) as follows.

\[
\tau(x) := \begin{cases} 
\tau_1(x) & \text{if } x \in \mathcal{L}(v) \\
 x & \text{if } x \in \mathcal{L}(w)
\end{cases}
\]

By Lemma 4.3 since \(a \notin \Lambda(w)\), we know that \(|\tau_1(w)| \geq |w|\). Since \(|\tau_1(v)| + |\tau_1(w)| < |v| + |w|\), we have that \(|\tau_1(v)| < |v|\). It is clear that \(\tau(v) = \tau_1(v)\) and \(\tau(w) = w\). Therefore, \(|\tau(v)| + |\tau(w)| < |v| + |w|\). Moreover, \(\mathcal{L}(\tau(v)) = \mathcal{L}(\tau_1(v)) \subseteq \mathcal{L}(v)\) and \(\mathcal{L}(\tau(w)) = \mathcal{L}(w)\). Therefore, \((\tau(v), \tau(w))\) is disjoint.

Corollary 4.6. If an orbit under the action of \(\text{Aut}(F)\) contains a good pair, then it contains a good pair that has minimal length.

Proof. If \((v, w)\) is a good pair then we can repeatedly apply Whitehead automorphisms to it such that the image is a good pair of smaller length, until we obtain a good pair that has minimal length.

We can now prove Theorem 1.3.

Theorem 4.7. It is decidable whether or not two cyclic words \(v\) and \(w\) or \(F\) are both elliptic to some proper free splitting of \(F\).

Proof. By Theorem 2.13 for any pair of cyclic words, we can determine whether or not they are in the same orbit as \((v, w)\) under the action of \(\text{Aut}(F)\). We can determine the minimal length of a pair in the orbit of \((v, w)\). There are finitely many pairs of that length, so for each pair of that length we can check whether or not it is in the orbit of \((v, w)\) and whether or not it is good. Therefore, we can determine whether or not there exists a good pair in the orbit of \((v, w)\) that has minimal length.

If each such pair exists, then by Lemma 4.2 there exists a proper free splitting \(A \ast B\) of \(F\) such that both \(v\) and \(w\) are elliptic to \(A \ast B\). Conversely, if no such pair exists, then by the contrapositive of Corollary 4.6 the orbit of \((v, w)\) does not contain a good pair, so there does not exist a proper free splitting such that both \(v\) and \(w\) are elliptic to it.

This algorithm requires finding all pairs of minimal length in the orbit of \((v, w)\) and determining whether or not any of them are good. We will show that it suffices to check just one pair of minimal length.

Lemma 4.8. If \((v, w)\) is a good pair of minimal length and \(\tau\) is a Whitehead automorphism such that \((\tau(v), \tau(w))\) also has minimal length, then \((\tau(v), \tau(w))\) is also good.

Proof. If \(\tau\) is a relabeling automorphism then we have that \(\Lambda(\tau(v)) = \tau(\Lambda(v))\) and \(\Lambda(\tau(w)) = \tau(\Lambda(w))\), so \((\tau(v), \tau(w))\) is good. We will henceforth consider the case where \(\tau\) is not a relabeling automorphism. Let the multiplier of \(\tau\) be \(a\).

Let \((v, w)\) be frugal. If \(a \in \Lambda(v) \cup \Lambda(w)\), then we have \(\mathcal{L}(\tau(v)) \subseteq \mathcal{L}(v) \cup \mathcal{L}(w)\) and \(\mathcal{L}(\tau(w)) \subseteq \mathcal{L}(v) \cup \mathcal{L}(w)\), so \((\tau(v), \tau(w))\) is frugal. If, on the other hand, \(a \notin \Lambda(v) \cup \Lambda(w)\), then by Lemma 4.3 \(|\tau(v)| \geq |v|\) and \(|\tau(w)| \geq |w|\). Since \(|\tau(v)| + |\tau(w)| = |v| + |w|\), we know that \(|\tau(v)| = |v|\) and \(|\tau(w)| = |w|\). Again using Lemma 4.3 we conclude that \(\tau(v) = v\) and \(\tau(w) = w\), so \((\tau(v), \tau(w))\) is good.
If \((v, w)\) is not frugal, then it is disjoint, so \(X = \mathcal{L}(v) \cup \mathcal{L}(w)\). Without loss of generality, assume that \(a \in \Lambda(v)\). Then \(\mathcal{L}(\tau(v)) \subseteq \mathcal{L}(v)\). Assume for contradiction that \(|\tau(v)| < |v|\). Then define the Whitehead automorphism \(\tau'\) on the generating set \(X\) as follows.

\[
\tau'(x) := \begin{cases} 
\tau(x) & \text{if } x \in \mathcal{L}(v), \\
 x & \text{if } x \in \mathcal{L}(w).
\end{cases}
\]

We have that \(|\tau'(v)| < |v|\) and \(|\tau'(w)| = |w|\), so \(|\tau'(v)| + |\tau'(w)| < |v| + |w|\), which contradicts the assumption that \((v, w)\) is of minimal length. Therefore, \(|\tau(v)| \geq |v|\). By Lemma 4.3, \(|\tau(w)| \geq |w|\). Since \(|\tau(v)| + |\tau(w)| = |v| + |w|\), we conclude that \(|\tau(w)| = |w|\). Again using Lemma 4.3, \(\tau(w) = w\), so \(\mathcal{L}(\tau(w)) = \mathcal{L}(w)\), and so \((\tau(v), \tau(w))\) is disjoint.

**Proposition 4.9.** If an orbit under the action of \(\text{Aut}(F)\) contains a good pair, then every pair in the orbit of minimal length is good.

**Proof.** By Corollary 4.6 there exists a good pair \((v, w)\) of minimal length in the orbit. By Theorem 2.11 if \((v', w')\) is a pair in the orbit of \((v, w)\) of minimal length then we can get from \((v, w)\) to \((v', w')\) by composing a sequence of Whitehead automorphisms such that, after each automorphism, the image of \((v, w)\) is of minimal length. Therefore, by Lemma 4.8, the image of \((v, w)\) is good after each automorphism, so \((v', w')\) is good.

Thus, we can check whether or not \(v\) and \(w\) are elliptic to a common proper free splitting by applying length decreasing Whitehead automorphisms until we arrive at a pair \((v', w')\) of minimal length. The words \(v\) and \(w\) are elliptic to a common proper free splitting if and only if the pair \((v', w')\) is good.

## 5 General Distances

### 5.1 Primitive elements in intersections of free factors

We begin with a definition.

**Definition 5.1.** An element \(g\) of the free group \(F(X)\) is *primitive* if it is the image of a letter under an automorphism of \(F\). That is, \(g\) is primitive if there exists \(\phi \in \text{Aut}(F(X))\) and a letter \(x \in X\) such that \(g = \phi(x)\). A cyclic word is *primitive* if a representative of the corresponding conjugacy class is primitive.

We aim to prove that if the intersection of two free factors is nontrivial, then it contains a primitive element. More precisely, we aim to show that if \(A \ast B\) and \(C \ast D\) are free splittings of a free group \(F\) such that \(A \cap C \neq 1\), then we can find a primitive element in \(A \cap C\).

**Notation.** Let \(X_A\) be a free basis for \(A\), and let \(X_B\) be a free basis for \(B\), and let \(X = X_A \cup X_B\).

**Proposition 5.2.** Let \(A \ast B\) and \(C \ast D\) be two free splittings of \(F\) such that \(A \cap C \neq 1\). Then \(A \cap C\) contains a primitive element of \(F\).

**Proof.** I would like to thank I. Kapovich for this short proof of the above proposition.

With respect to the basis \(X\), construct \(\Gamma(A)\) and \(\Gamma(C)\). It is clear that \(\Gamma(A)\) contains a single vertex and loops from that vertex to itself for each element of \(X_A\). Therefore the projection map

\[
\pi: (\Gamma(A) \times \Gamma(C), (1_A, 1_C)) \to (\Gamma(C), 1_C)
\]

is bijective on vertices and injective on edges.

By Theorem 2.30, the digraph with base vertex \((\Gamma(A \cap C), 1_{A \cap C})\) is isomorphic to \(\text{Core}(\Gamma(A) \times \Gamma(C), (1_A, 1_C))\), which is a subgraph of \((\Gamma(A) \times \Gamma(C), (1_A, 1_C))\). Therefore we have an injective morphism of digraphs

\[
\iota: (\Gamma(A \cap C), 1_{A \cap C}) \to (\Gamma(C), 1_C).
\]
Let $T$ be a spanning tree of $\Gamma(A \cap C)$. Since $A \cap C \neq 1$, there must be a positive edge $e$ not on the tree $T$. Consider the path $p = qer$ where $q$ is the unique reduced path in $T$ from $1_{A \cap C}$ to $o(e)$ and $r$ is the unique reduced path in $T$ from $t(e)$ to $1_{A \cap C}$. Since $\iota$ is injective, $\iota(T)$ is a tree in $\Gamma(C)$. Extend $\iota(T)$ to a spanning tree $S$ of $\Gamma(C)$. Since $S$ is a tree, $\iota(e)$ is not on $S$. We also know that $\iota(q)$ is the unique reduced path in $S$ from $1_C$ to $o(\iota(e))$ and that $\iota(r)$ is the unique reduced path in $S$ from $t(\iota(e))$ to $1_C$. Then by Theorem \ref{thm:distance2} $\mu(\iota(p))$ is in a basis for $C$. Since $C$ is a free factor, we know that an element of any basis of $C$ is primitive, so $\mu(\iota(p))$ is primitive. Since $\mu(p) = \mu(\iota(p))$ and $\mu(p) \in A \cap C$, we know that $A \cap C$ contains a primitive element. 

**Corollary 5.3.** Let $[A * B]$ and $[C * D]$ be classes of proper free splittings. If a nontrivial element $f$ in $F$ is elliptic to both of these classes, then there exists a primitive element $g$ in $F$ also elliptic to both classes.

Corollary \ref{cor:distance} motivates the following definition.

**Definition 5.4.** Let $Z'(F)$ be the subgraph of a subgraph of the ellipticity graph $Z(F)$ containing all classes of proper free splittings but only the primitive cyclic words.

**Corollary 5.5.** The inclusion $Z'(F) \rightarrow Z(F)$ is an isometric embedding.

### 5.2 A bound on distances in $Z(F)$ and connectedness

In \cite{IL}, I. Kapovich and M. Lustig mention that if $F$ has at least three generators, then $Z(F)$ is connected. We present a proof of this result by proving a bound on the distance between two splittings in terms of elementary Nielsen transformations.

**Definition 5.6.** A basis of a free splitting $A * B$ is the union of a basis of $A$ and a basis of $B$.

**Remark.** A basis of a free splitting is also a basis of the free group $F$.

**Lemma 5.7.** Let $F = F(X)$ be a free group with at least two generators, let $A * B$ be a free splitting with basis $X$, and let $\nu$ be an elementary Nielsen transformation. If $C * D$ is a free splitting with basis $\nu(X)$, then $[A * B]$ and $[C * D]$ are distance at most two in $Z(F)$.

**Proof.** Since $|X| \geq 2$, an elementary Nielsen transformation must fix some $x \in X$. Since $A * B$ has basis $X$, either $x \in A$ or $x \in B$, so $x$ is elliptic to $A * B$. Moreover, $x$ is an element of $\nu(X)$, which is a basis for $C * D$. Thus either $x \in C$ or $x \in D$, so $x$ is elliptic to $C * D$. Therefore, $[A * B]$ and $[C * D]$ are distance at most two in $Z(F)$. 

**Theorem 5.8.** Let $F = F(X)$ be a free group with at least two generators, let $A * B$ be a free splitting of $F$ with basis $X$, let $C * D$ be an arbitrary free splitting, let $Y \neq X$ be a basis of $C * D$, and let $\phi = \nu_1 \cdots \nu_n$ be the automorphism that sends $X$ to $Y$, where the $\nu_i$ are elementary Nielsen transformations. Then the distance between $[A * B]$ and $[C * D]$ in $Z(F)$ is at most $2n$. In particular, $Z(F)$ is connected.

**Proof.** We proceed by induction on $n$. The base case $n = 1$ is shown in Lemma \ref{lem:distance}. Now assume that $n > 1$, and assume that the theorem is true when $\phi$ is the product of $n - 1$ elementary Nielsen transformations. Let $\phi' = \nu_1 \cdots \nu_{n-1}$, and let $C' * D' = \phi'(A) * \phi'(B)$, so $C' * D'$ is a splitting with basis $\phi'(X)$. By our inductive hypothesis, the distance between $[A * B]$ and $[C' * D']$ is at most $2(n-1)$. Since $C * D$ has basis $\phi'\nu_n(X)$, we know that $\phi'^{-1}(C) * \phi'^{-1}(D)$ has basis $\nu_n(X)$. By Lemma \ref{lem:distance} the distance between $[A * B]$ and $[\phi^{-1}(C) * \phi^{-1}(D)]$ is at most two. Since automorphisms of $F$ induce isometries of $Z(F)$, we know that the distance between $[\phi'(A) * \phi'(B)] = [C' * D']$ and $[C * D]$ is at most two. Therefore, the distance between $[A * B]$ and $[C * D]$ is at most $2n$. 

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References

[1] Ilya Kapovich and Martin Lustig, *Geometric intersection number and analogues of the curve complex for free groups*, Geom. Topol. 13 (2009), no. 3, 1805–1833. MR MR2496058

[2] Ilya Kapovich and Alexei Myasnikov, *Stallings foldings and subgroups of free groups*, Journal of Algebra 248 (2002), no. 2, 608–668.

[3] Roger Lyndon and Paul Schupp, *Combinatorial group theory*, Springer, 1977.