HYPER SPACE COMPLEX NUMBERS

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Abstract. A new kind of numbers called Hyper Space Complex Numbers and its algebras are defined and proved. It is with good properties as the classic Complex Numbers, such as expressed in coordinates, triangular and exponent forms and following the associative and commutative laws of addition and multiplication. So the classic Complex Number is developed from in complex plane with two dimensions to in complex space with N dimensions and the number system is enlarged also.

INTRODUCTION

It’s well known that the numbers are composed of the real and complex numbers. A pair of numbers is defined in the complex number, and forms a complex plane. There are some expressions and algebras for the complex numbers. The complex is important for science.

A new kind of number is defined in this paper on the base of the classic Complex Number and is called Hyper Space Complex Number. Different from the Hyper-complex Numbers given by Hamilton, Gelasiman or others, it is with good properties as the classic Complex Numbers, such as expressed in coordinates, triangular and exponent forms and following the associative and commutative laws of addition and multiplication. The classic Complex Number is developed from in complex plane with two dimensions to in complex space with N dimensions. When N=2, it is the classic Complex Numbers, and when N=3, it is the Space Complex Numbers. The Space Complex Number in polar coordinates system and its algebras are defined and proved in this paper also. Therefore the number system is enlarged.

1. Definition of Hyper Space Complex Numbers

1.1. Definition. A number $s$ is defined as a Hyper Space Complex Number, and expressed in anti-clockwise direction or in clockwise direction as following (See the proof afterward in this paper)

\begin{equation}
\begin{aligned}
    s &= a_0 + \sum_{k=1}^{N-1} i_k a_k \prod_{j=1}^{k-1} e^{i_j \theta_j} = a_0 + \sum_{k=1}^{N-1} i_k s_k \\
    s &= a_0 + \sum_{k=1}^{N-1} i_k a_k \prod_{j=k+1}^{N-1} e^{i_j \theta_j} = a_0 + \sum_{k=1}^{N-1} i_k s_k
\end{aligned}
\end{equation}
where, $i_k = \sqrt{-1}$ is called the unit of imaginary number and is defined in the complex plane $\mathbb{C}_{x_k}$. Because $i_k$ (k = 1,2,...,N-1) are defined in different complex planes, $i_k$ is the unit imaginary number with directions. $s_k = a_k \prod_{j=1}^{k-1} e^{i_j \theta_j}$ and $s_k = a_k \prod_{j=k+1}^{N-1} e^{i_j \theta_j}$ are Hyper Space Complex Numbers with k or N-k dimension, respectively.

The multiplication of the unit of imaginary number $i_k$ and an operator $e^{i_j \theta_j}$ of rotations for in anti-clockwise direction or in clockwise direction are (See proof afterward in this paper.)

\[
(1.2) \quad i_k \prod_{j=1}^{k-1} e^{i_j \theta_j} = i_k, k = 2, 3, ..., N - 1
\]

\[
(1.3) \quad i_k \prod_{j=k+1}^{N-1} e^{i_j \theta_j} = i_k, k = 1, 2, ..., N - 2
\]

The coefficients $a_0$ and $a_k$ are real numbers. $a_0$ is called the real part of $s$, $a_k$ is called the $k$th component of imaginary part of $s$ on axis $x_k$. $a_0$ and $a_k$ are denoted $a_0 = \text{Re } s$, $a_k = \text{Im } s(k)$.

1.2. Equal of two Hyper Space Complex numbers. Two Hyper Space Complex numbers, $s_1$ and $s_2$ are equal when and only when their real parts on axis $r$, imaginary parts on $x_k$ are equal separately. That is, for

\[
s_1 = a_{10} + \sum_{k=1}^{N-1} i_k a_{1k} \prod_{j=1}^{k-1} e^{i_j \theta_j} = a_{10} + \sum_{k=1}^{N-1} i_k s_{1k}
\]

\[
s_2 = a_{20} + \sum_{k=1}^{N-1} i_k a_{2k} \prod_{j=1}^{k-1} e^{i_j \theta_j} = a_{20} + \sum_{k=1}^{N-1} i_k s_{2k}
\]

when and only when there are $a_{10} = a_{20}$ and $s_{1k} = s_{2k}$, there is $s_1 = s_2$.

1.3. Modulus of the Hyper Space Complex Number.

\[
|s| = \sqrt{\sum_{k=0}^{N-1} a_k^2}
\]

is called the modulus of the Hyper Space Complex Number $s$.

1.4. Hyper Space conjugate complex numbers. Define two Hyper Space Complex numbers

\[
s = a_0 + \sum_{k=1}^{N-1} i_k a_k \prod_{j=1}^{k-1} e^{i_j \theta_j} = a_0 + \sum_{k=1}^{N-1} i_k s_k
\]

\[
\bar{s} = a_0 - \sum_{k=1}^{N-1} i_k a_k \prod_{j=1}^{k-1} e^{-i_j \theta_j} = a_0 - \sum_{k=1}^{N-1} i_k \bar{s}_k
\]

then $s$ and $\bar{s}$ are Hyper Space conjugate complex numbers.
1.5. **Argument of the Hyper Space Complex.** There are two groups of argument $s$ for in anti-clockwise direction or in clockwise direction.

(1.3) \[ \text{Args}(k) = \text{arctg} \frac{a_k}{\sqrt{a_0^2 + \sum_{j=1}^{k-1} a_j^2}} \quad k = 1, 2, ..., N - 1 \]

(1.4) \[ \text{Args}(k) = \text{arctg} \frac{a_k}{\sqrt{a_0^2 + \sum_{j=k+1}^{N-1} a_j^2}} \quad k = 1, 2, ..., N - 1 \]

Argument in formulas (1.3, 1.4) is called the $n$th component of argument of a Hyper Space Complex Number $s$. Therefore, there are infinite $n$th component of arguments for a Hyper Space Complex Number, but only one of them is called the principal $n$th component of argument for a Hyper Space Complex Number, and is denoted $\text{arg } s(n)$. It satisfies the condition $0 \leq \text{args}(n) < 2\pi$ and there is $\text{Arg } s(n) = \text{arg } s(n) + 2k\pi$, $k = 0, \pm 1, \pm 2, ...$.

2. **Expression of Hyper Space Complex Numbers**

As the classic complex number, there are some expressions for the Hyper Space Complex Number $s = a_0 + \sum_{k=1}^{N-1} i_k a_k \prod_{j=1}^{k-1} e^{i_j \theta_j}$

2.1. **Expression in coordinates.** According to formula (1.2), there is $s = a_0 + \sum_{k=1}^{N-1} i_k a_k$. There exists a relation between the Hyper Space Complex number and the N-dimensional right angular coordinates system.

2.2. **Expression in vector.** Let express $a_0$ and $a_k$ as projections of a vector $\overrightarrow{OP}$ on axis $r$ and axis $x_k$, then the Hyper Space Complex Number $s = a_0 + \sum_{k=1}^{N-1} i_k a_k$ is expressed by the vector $\overrightarrow{OP}$. A point is denoted $P'$, which is on the same plane formed by the axis $r$ and vector $\overrightarrow{OP}$ and is symmetry about axis $r$ with the point $P$. The Hyper Space Complex Number, $\bar{s} = a_0 - \sum_{k=1}^{N-1} i_k a_k$, which is the conjugate of $s$, is expressed by the vector $\overrightarrow{OP'}$.

2.3. **Expression in triangular.** A Hyper Space Complex number is denoted by triangular functions

(2.1) \[ s = |s| \prod_{k=1}^{N-1} (\cos \theta_k + i_k \sin \theta_k) \]

Expand expression (2.1), in anti-clockwise direction or in clockwise direction, we get

(2.2) \[ s = |s| \left( \prod_{k=1}^{N-1} \cos \theta_k + \sum_{k=1}^{N-1} i_k \sin \theta_k \prod_{j=1}^{k-1} e^{i_j \theta_j} \prod_{j=k+1}^{N-1} \cos \theta_j \right) \]

(2.3) \[ s = |s| \left( \prod_{k=1}^{N-1} \cos \theta_k + \sum_{k=1}^{N-1} i_k \sin \theta_k \prod_{j=k+1}^{N-1} e^{i_j \theta_j} \prod_{j=1}^{k-1} \cos \theta_j \right) \]

Comparing the expressions with the formula (1.1), we get two groups of formulas following and the first group of formulas will be used in discussion afterward.
\[ a_0 = |s| \prod_{k=1}^{N-1} \cos \theta_k \]

\[ a_k = |s| \sin \theta_k \prod_{j=k+1}^{N-1} \cos \theta_j, \quad k = 1, 2, ..., N - 1 \]

\[ \theta_k = \arctan \frac{a_k}{\sqrt{a_0^2 + \sum_{j=k+1}^{N-1} a_j^2}}, \quad k = 1, 2, ..., N - 1 \]

and

\[ a_0 = |s| \prod_{k=1}^{N-1} \cos \theta_k \]

\[ a_k = |s| \sin \theta_k \prod_{j=1}^{k-1} \cos \theta_j, \quad k = 1, 2, ..., N - 1 \]

\[ \theta_k = \arctan \frac{a_k}{\sqrt{a_0^2 + \sum_{j=k+1}^{N-1} a_j^2}}, \quad k = 1, 2, ..., N - 1 \]

where \( \theta_k \) is the principal argument \( \arg s(k) \) of \( s \).

2.4. Expression in exponent. Because of \( \cos \theta + i \sin \theta = e^{i\theta} \), according to formula (2.1), there is

\[ s = |s| \prod_{k=1}^{N-1} (\cos \theta_k + i_k \sin \theta_k) = |s| \prod_{k=1}^{N-1} e^{i_k \theta_k} \]

3. Algebra of Hyper Space Complex Numbers

As the classic complex number, there are some algebras for the Hyper Space Complex Number \( s \) expressed by formula (1.1)

3.1. Rules of operation in algebra. For Hyper Space Complex Numbers

\[ s_1 = a_{10} + \sum_{k=1}^{N-1} i_k a_{1k} \prod_{j=1}^{k-1} e^{i_j \theta_{ij}} = a_{10} + \sum_{k=1}^{N-1} i_k s_{1k} \]

\[ s_2 = a_{20} + \sum_{k=1}^{N-1} i_k a_{2k} \prod_{j=1}^{k-1} e^{i_j \theta_{2j}} = a_{20} + \sum_{k=1}^{N-1} i_k s_{2k} \]

there are

\[ s_1 \pm s_2 = (a_{10} \pm a_{20}) + \sum_{k=1}^{N-1} i_k (s_{1k} \pm s_{2k}) \]
\[s = s_1 s_2 = (a_{10} + \sum_{k=1}^{N-1} i_k s_{1k})(a_{20} + \sum_{k=1}^{N-1} i_k s_{2k})
\]
\[= a_0 + \sum_{k=1}^{N-1} i_k a_k \prod_{j=1}^{k-1} e^{i \theta_j} = a_0 + \sum_{k=1}^{N-1} i_k a_k
\]

where, according to formulas (2.4, 2.5, 2.6), for \(k=1,2,\ldots,N-1\) (See the proof afterward in this paper)

\[a_0 = a_{10} a_{20} - a_{11} a_{21} - \sum_{k=2}^{N-1} a_{1k} a_{2k} \prod_{j=1}^{k-1} \cos(\theta_{1j} + \theta_{2j})
\]

\[a_k = [a_{1k} \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2 + a_{2k}^2 \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2}}] - a_{1k+1} a_{2k+1} \sin(\theta_{1k} + \theta_{2k}) [1 + \sum_{i=1}^{N-k-2} \prod_{j=k+1}^{k+i} \cos(\theta_{1j} + \theta_{2j})]
\]

\[\theta_k = \theta_{1k} + \theta_{2k}
\]

When there are

\[s_1 = a_{10} + \sum_{k=1}^{N-1} i_k a_{1k}
\]
\[s_2 = a_{20} + \sum_{k=1}^{N-1} i_k a_{2k}
\]

we get

\[a_0 = a_{10} a_{20} - \sum_{k=1}^{N-1} a_{1k} a_{2k}
\]

\[a_k = a_{1k} \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2 + a_{2k}^2 \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2}}
\]

\[\theta_k = \theta_{1k} + \theta_{2k}
\]

or, according to formulas (2.7, 2.8, 2.9), for \(k=1,2,\ldots,N-1\)

\[a_0 = a_{10} a_{20} - a_{1N-1} a_{2N-1} - \sum_{k=1}^{N-2} a_{1k} a_{2k} \prod_{j=k+1}^{N-1} \cos(\theta_{1j} + \theta_{2j})
\]
\[ a_k = \left[ a_{1k} \sqrt{a_{20}^2 + \sum_{j=k+1}^{N-1} a_{2j}^2 + a_{2k}} \right] \]

\[-a_{1k-1} a_{2k-1} \sin(\theta_{1k} + \theta_{2k}) \left[ 1 + \sum_{i=k+2}^{N-1} \prod_{j=i-k-1}^{N-1} \cos(\theta_{1j} + \theta_{2j}) \right] \]

\[ \theta_k = \theta_{1k} + \theta_{2k} \]

When there are

\[ s_1 = a_{10} + \sum_{k=1}^{N-1} i_k a_{1k} \]
\[ s_2 = a_{20} + \sum_{k=1}^{N-1} i_k a_{2k} \]

we get

\[ a_0 = a_{10} a_{20} - \sum_{k=1}^{N-1} a_{1k} a_{2k} \]

\[ \theta_k = \theta_{1k} + \theta_{2k} \]

Therefore, there is \( s_1 s_2 = s_2 s_1 \). The associative and commutative laws of addition and multiplication exist for the Hyper Space Complex Numbers.

When \( s_1 = s \) and \( s_2 = \bar{s} \), there are

\[ s\bar{s} = (a_0 + \sum_{k=1}^{N-1} i_k a_k)(a_0 - \sum_{k=0}^{N-1} i_k a_k) = \sum_{k=0}^{N-1} a_k^2 + \sum_{k=0}^{N-1} i_k 0 = |s|^2 \]

\[ s = s_1 \div s_2 = (a_{10} + \sum_{k=1}^{N-1} i_k s_{1k}) \div (a_{20} + \sum_{k=1}^{N-1} i_k s_{2k}) \]

\[ = \frac{1}{|s_2|^2}(a_{10} + \sum_{k=1}^{N-1} i_k s_{1k})(a_{20} - \sum_{k=1}^{N-1} i_k \bar{s}_{2k}) \]

\[ = a_0 + \sum_{k=1}^{N-1} i_k a_k \prod_{j=1}^{N-1} e^{i_j \theta_j} = a_0 + \sum_{k=1}^{N-1} i_k s_k \]

where, according to formulas (2.4, 2.5, 2.6), for \( k=1,2,\ldots,N-1 \)

\[ a_0 = \frac{1}{|s_2|^2}(a_{10} a_{20} + a_{11} a_{21} + \sum_{k=2}^{N-1} a_{1k} a_{2k} \prod_{j=1}^{k-1} \cos(\theta_{1j} + \theta_{2j})) \]
\[ a_k = \frac{1}{|s_2|^2} \left\{ a_{1k} \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2} - a_{2k} \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2} \right\} \\
+ a_{1k+1} a_{2k+1} \sin(\theta_{1k} + \theta_{2k}) \left[ 1 + \sum_{i=1}^{N-k-2} \prod_{j=k+i}^{k+i} \cos(\theta_{1j} + \theta_{2j}) \right] \]

(3.18) \[ \theta_k = \theta_{1k} + \theta_{2k} \]

When there are
\[ s_1 = a_{10} + \sum_{k=1}^{N-1} i_k a_{1k} \]
\[ s_2 = a_{20} + \sum_{k=1}^{N-1} i_k a_{2k} \]

we get

(3.19) \[ a_0 = \frac{1}{|s_2|^2} (a_{10} a_{20} + \sum_{k=1}^{N-1} a_{1k} a_{2k}) \]

(3.20) \[ a_k = \frac{1}{|s_2|^2} \left\{ a_{1k} \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2} - a_{2k} \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2} \right\} \\
+ a_{1k+1} a_{2k+1} \sin(\theta_{1k} + \theta_{2k}) \left[ 1 + \sum_{i=1}^{N-1} \prod_{j=i-k-1}^{i} \cos(\theta_{1j} + \theta_{2j}) \right] \]

(3.21) \[ \theta_k = \theta_{1k} - \theta_{2k} \]

or, according to formulas (2.7, 2.8, 2.9), for \( k=1,2,...,N-1 \)

(3.22) \[ a_0 = \frac{1}{|s_2|^2} \left\{ a_{10} a_{20} + a_{1N-1} a_{2N-1} + \sum_{k=1}^{N-2} a_{1k} a_{2k} \prod_{j=k+1}^{N-1} \cos(\theta_{1j} + \theta_{2j}) \right\} \]

(3.23) \[ a_k = \frac{1}{|s_2|^2} \left\{ a_{1k} \sqrt{a_{20}^2 + \sum_{j=k+1}^{N-1} a_{2j}^2} - a_{2k} \sqrt{a_{10}^2 + \sum_{j=k+1}^{N-1} a_{1j}^2} \right\} \\
+ a_{1k-1} a_{2k-1} \sin(\theta_{1k} + \theta_{2k}) \left[ 1 + \sum_{i=k+2}^{N-1} \prod_{j=i-k-1}^{i} \cos(\theta_{1j} + \theta_{2j}) \right] \]

(3.24) \[ \theta_k = \theta_{1k} + \theta_{2k} \]

When there are
\[ s_1 = a_{10} + \sum_{k=1}^{N-1} i_k a_{1k} \]
\[ s_2 = a_{20} + \sum_{k=1}^{N-1} i_k a_{2k} \]

we get

(3.25) \[ a_0 = \frac{1}{|s_2|^2} (a_{10} a_{20} + \sum_{k=1}^{N-1} a_{1k} a_{2k}) \]

(3.26) \[ a_k = \frac{1}{|s_2|^2} (a_{1k} \sqrt{a_{20}^2 + \sum_{j=k+1}^{N-1} a_{2j}^2} - a_{2k} \sqrt{a_{10}^2 + \sum_{j=k+1}^{N-1} a_{1j}^2}) \]

(3.27) \[ \theta_k = \theta_{1k} - \theta_{2k} \]

Let

\[ s_3 = a_{30} + \sum_{k=1}^{N-1} i_k a_{3k} \prod_{j=1}^{k-1} e^{i_j \theta_{3j}} = a_{30} + \sum_{k=1}^{N-1} i_k s_{3k} \]

there is

\[ s_1 * s_3 + s_2 * s_3 = (a_{10} + \sum_{k=1}^{N-1} i_k s_{1k}) * s_3 + (a_{20} + \sum_{k=1}^{N-1} i_k s_{2k}) * s_3 \]

\[ = [(a_{10} + \sum_{k=1}^{N-1} i_k s_{1k}) + (a_{20} + \sum_{k=1}^{N-1} i_k s_{2k})] * s_{3k} \]

\[ = [(a_{10} + a_{20}) + \sum_{k=1}^{N-1} i_k (s_{1k} + s_{2k})] * s_{3k} \]

\[ = (s_1 + s_2) * s_3 \]

Therefore, the distributive law of addition and multiplication exists for the Hyper Space Complex Numbers.

3.2. **Rules of operation in triangular.** For two Hyper Space Complex Numbers

\[ s_1 = |s_1| \prod_{k=1}^{N-1} (\cos \theta_{1k} + i_k \sin \theta_{1k}) \]

\[ s_2 = |s_2| \prod_{k=1}^{N-1} (\cos \theta_{2k} + i_k \sin \theta_{2k}) \]

(3.28) \[ s_1 s_2 = |s_1| |s_2| \prod_{k=1}^{N-1} (\cos \theta_{1k} + i_k \sin \theta_{1k}) |s_2| \prod_{k=1}^{N-1} (\cos \theta_{2k} + i_k \sin \theta_{2k}) \]

\[ = |s_1| |s_2| \prod_{k=1}^{N-1} [\cos (\theta_{1k} + \theta_{2k}) + i_k \sin (\theta_{1k} + \theta_{2k})] \]
Therefore, there is $s_1 s_2 = s_2 s_1$. The associative and commutative laws of multiplication exist in triangular form for the Hyper Space Complex Numbers.

\[(3.29)\]
\[
s_1 \div s_2 = \frac{|s_1|}{|s_2|} \prod_{k=1}^{N-1} \left( \cos \theta_{1k} + i \sin \theta_{1k} \right) \prod_{k=1}^{N-1} \left( \cos \theta_{2k} - i \sin \theta_{2k} \right)
\]
\[
= \frac{|s_1|}{|s_2|} \prod_{k=1}^{N-1} \left[ \cos(\theta_{1k} - \theta_{2k}) + i \sin(\theta_{1k} - \theta_{2k}) \right]
\]

Therefore, when $s = s_1 = s_2 = ... = s_n$, according to formula \((3.28)\), there are

\[(3.30)\]
\[
s^n = |s|^n \prod_{k=1}^{N-1} (\cos n \theta_k + i \sin n \theta_k)
\]

\[(3.31)\]
\[
s^\frac{1}{n} = |s|^\frac{1}{n} \prod_{k=1}^{N-1} \left( \frac{\cos \theta_k + 2m\pi}{n} + i \frac{\sin \theta_k + 2m\pi}{n} \right), m = 1, 2, ..., n - 1
\]

3.3. **Rules of operation in exponent.** For two Hyper Space Complex Numbers,

\[s_1 = |s_1| \prod_{k=1}^{N-1} e^{i_k \theta_1k}, s_2 = |s_2| \prod_{k=1}^{N-1} e^{i_k \theta_2k}\]

there are

\[(3.32)\]
\[
s_1 s_2 = |s_1| \prod_{k=1}^{N-1} e^{i_k \theta_1k} \times |s_2| \prod_{k=1}^{N-1} e^{i_k \theta_2k} = |s_1| \times |s_2| \prod_{k=1}^{N-1} e^{i_k (\theta_{1k} + \theta_{2k})}
\]

Therefore, there is $s_1 * s_2 = s_2 * s_1$. The associative and commutative laws of multiplication exist in exponent form for the Hyper Space Complex Numbers.

\[(3.33)\]
\[
s_1 \div s_2 = |s_1| \prod_{k=1}^{N-1} e^{i_k \theta_1k} \div \left( |s_2| \prod_{k=1}^{N-1} e^{i_k \theta_2k} \right) = \frac{|s_1| \prod_{k=1}^{N-1} e^{i_k (\theta_{1k} - \theta_{2k})}}{|s_2| \prod_{k=1}^{N-1} e^{i_k (\theta_{1k} - \theta_{2k})}}
\]

Therefore, when $s = s_1 = s_2 = ... = s_n$, there are

\[(3.34)\]
\[
s^n = |s|^n \prod_{k=1}^{N-1} e^{i_k n \theta_k}
\]

\[(3.35)\]
\[
s^\frac{1}{n} = |s|^\frac{1}{n} \prod_{k=1}^{N-1} e^{i_k \frac{\theta_k + 2m\pi}{n}}, m = 1, 2, ..., n - 1
\]
4. Proof of Hyper Space Complex Numbers

4.1. Proof of Definition of Hyper Space Complex Numbers. A Hyper Space rectangular coordinates system could be defined as a rectangular coordinates system with N dimensions and composed of N-1 complex planes or N-2 complex spaces composed of Space Complex Numbers with three dimensions. A complex plane denoted $C_{rx_k}$ is composed of axis $r$ which is defined as the real axis, and axis $x_k$ ($k=1,2,\ldots,N-1$) which is defined as the imaginary axis. The axis $r$ and $x_k$ form a Hyper Space rectangular coordinates system. On $C_{rx_k}$, any point could be denoted $P_{rx_k} = (a_0, i_k a_k)$ or expressed as $s_{rx_k} = a_0 + i_k a_k$. The ordinates of a point $P(a_0, a_1, \ldots, a_{N-1})$ on the Hyper Space rectangular coordinates system could be considered as the combination of N-1 movements of the point.

Let define $i_j = \sqrt{-1}$ as the unit of imaginary number of axis $x_j$ and define an operator of rotations and denote it $e^{i_j \theta_j}$ which means a rotation on a complex plane $C_j$.

First, when beginning, the point $P_r = a_{00}$ is on the axis $r$ with a length $r$, and is rotated $\theta_1$ angular degrees on the complex plane $C_1 = C_{rx_1}$, and a new point $P_{rx_1} = (a_{10}, i_1 a_{11})$ is formed and denoted $r e^{i_1 \theta_1}$. Then, the point $P_{rx_1} = (a_{10}, i_1 a_{11})$ is rotated $\theta_2$ angular degrees on the complex plane $C_2$ composed of point $P_{rx_1}$ and axis $x_2$, and a new point $P_{rx_1x_2} = (a_{20}, i_1 a_{21}, i_2 a_{22})$ is formed and denoted $r e^{i_1 \theta_1} e^{i_2 \theta_2}$. In this way, finally, the point $P_{rx_1x_2\ldots x_{N-2}} = (a_{N-2,0}, i_1 a_{N-2,1}, i_2 a_{N-2,2}, \ldots, i_{N-2} a_{N-2,N-2})$ is rotated $\theta_{N-1}$ angular degrees on the complex plane $C_{N-1}$ composed of point $P_{rx_1x_2\ldots x_{N-2}}$ and axis $x_{N-1}$, and a new point $P(a_0, a_1, \ldots, a_{N-1})$ is formed in the complex space $C$ and denoted $r \prod_{j=1}^{N-1} e^{i_j \theta_j}$.

On the other hand, the N-1 movements of the point $P_r = a_{00}$ could be in other ways. In another way, first, point $P_r = a_{00}$ is rotated $\theta_{N-1}$ angular degrees on the complex plane $C_{rx_{N-1}}$, and a new point $P_{rx_{N-1}} = (a_{N+1,0}, i_{N-1} a_{N+1,N-1})$ is formed and denoted $r e^{i_{N-1} \theta_{N-1}}$. Then, the point $P_{rx_{N-1}}$ is rotated $\theta_{N-2}$ angular degrees on the complex plane composed of point $P_{rx_{N-1}}$ and axis $x_{N-2}$, and a new point $P_{rx_1x_2\ldots x_{N-2}} = (a_{N+2,0}, i_{N-2} a_{N+2,N-2}, i_{N-1} a_{N+2,N-1})$ is formed and denoted $r e^{i_{N-1} \theta_{N-2}} e^{i_{N-2} \theta_{N-1}}$. In this way, finally, the point $P_{rx_1x_2\ldots x_{N-2}}$ is rotated $\theta_1$ angular degrees on the complex plane composed of point $P_{rx_1x_2\ldots x_{N-2}}$ and axis $x_1$, and a new point $P(a_0, a_1, \ldots, a_{N-1})$ is formed in the complex space $C$ and denoted $r \prod_{j=1}^{N-1} e^{i_j \theta_j}$.

As defined above, $i_j = \sqrt{-1}$ is the unit of imaginary number of axis $x_j$ and an operator of rotations denoted $e^{i_j \theta_j}$ means the rotation on $C_j$. Because $C_j$ is perpendicular or orthogonal to axis $x_k$ ($k=j+1,j+2,\ldots,N-1$), axis $x_k$ could be considered as a axis of the rotation on $C_j$. Because there is no change for any point on axis $x_k$, when it rotates any angular degrees around axis $x_k$, therefore, in anti-clockwise direction, there is

\begin{equation}
(4.1) \quad i_k e^{i_j \theta_j} = i_k, j = 1, 2, \ldots, N - 2, k = j + 1, j + 2, \ldots, N - 1
\end{equation}

or in clockwise direction, there is

\begin{equation}
(4.2) \quad i_k e^{i_j \theta_j} = i_k, j = 2, \ldots, N - 1, k = 1, 2, \ldots, j - 1
\end{equation}

and for a real number $r$, there is
(4.3) \[ re^{i\theta_j} = r(\cos\theta_j + i_j \sin\theta_j) \]

In these movements discussed above, \( \prod_{j=1}^{k-1} e^{i_j \theta_j} \) or \( \prod_{j=k+1}^{N-1} e^{i_j \theta_j} \) means a point \( P \) in a complex space. The point \( P \) and axis \( r \) compose a complex plane \( C_{k-1} \). Because axis \( r \) and axis \( x_k \) are perpendicular or orthogonal to axis \( x_k \), and the point \( P \) is on the complex plane \( C_{r x_k-1} \) composed by axis \( r \) and axis \( x_k \), therefore, \( C_{k-1} \) is perpendicular or orthogonal to axis \( x_k \). Hence axis \( x_k \) could be considered as a axis of the rotation on \( C_{k-1} \). Because there is no change for any point on axis \( x_k \), when it rotates any angular degrees around axis \( x_k \), therefore, in anti-clockwise direction, there is

(4.4) \[ i_k \prod_{j=1}^{k-1} e^{i_j \theta_j} = i_k, \quad k = 2, 3, ..., N - 1 \]

or in clockwise direction, there is

(4.5) \[ i_k \prod_{j=k+1}^{N-1} e^{i_j \theta_j} = i_k, \quad k = 1, 2, ..., N - 2 \]

So a Hyper Space Complex Number \( s \) could be denoted

(4.6) \[ s = |s| \prod_{k=1}^{N-1} e^{i_k \theta_k} \]

or

(4.7) \[ s = |s| \prod_{k=1}^{N-1} (\cos\theta_k + i_k \sin\theta_k) \]

or, in anti-clockwise direction

(4.8) \[ s = a_0 + \sum_{k=1}^{N-1} i_k a_k \prod_{j=1}^{k-1} e^{i_j \theta_j} = a_0 + \sum_{k=1}^{N-1} i_k s_k \]

or, in clockwise direction

(4.9) \[ s = a_0 + \sum_{k=1}^{N-1} i_k a_k \prod_{j=k+1}^{N-1} e^{i_j \theta_j} = a_0 + \sum_{k=1}^{N-1} i_k s_k \]

where \( a_k \) means the length and direction on axis \( x_k \), \( \prod_{j=1}^{k-1} e^{i_j \theta_j} \) or \( \prod_{j=k+1}^{N-1} e^{i_j \theta_j} \) means the contributions of the perpendicular or orthogonal vectors on axis \( x_j \) to \( x_k \) and form components of \( a_k \). Because these components of \( a_k \) are from different axis \( x_j \), the addition of these components of \( a_k \) should follow the law of addition of vectors and not be simple additions of these components of \( a_k \).

A complex space \( C_{r x_1 x_2 ... x_{N-1}} \) is formed by Hyper Space Complex Numbers.
4.2. Proof of Coefficients of Multination of Hyper Space Complex Numbers. According to formula (1.2) and formulas (2.4, 2.5, 2.6), let us expand Equation (3.2) and there is,

\[(4.10) \quad s = s_1 s_2 = (a_{10} + \sum_{k=1}^{N-1} i_k s_{1k})(a_{20} + \sum_{k=1}^{N-1} i_k s_{2k})\]

\[= (a_{10} + \sum_{k=1}^{N-2} i_k s_{1k})(a_{20} + \sum_{k=1}^{N-2} i_k s_{2k})\]

\[+ i^{N-1}[s_{1N-1}(a_{20} + \sum_{k=1}^{N-2} i_k s_{2k}) + s_{2N-1}(a_{10} + \sum_{k=1}^{N-2} i_k s_{1k})] - s_{1N-1}s_{2N-1}\]

\[= (a_{10} + \sum_{k=1}^{N-3} i_k s_{1k})(a_{20} + \sum_{k=1}^{N-3} i_k s_{2k})\]

\[+ \sum_{k=N-2}^{N-1} i_k[s_{1k}(a_{20} + \sum_{j=1}^{k-1} i_j s_{2j}) + s_{2k}(a_{10} + \sum_{j=1}^{k-1} i_j s_{1j})]\]

\[= a_{10}a_{20} - \sum_{k=1}^{N-1} a_{1k}s_{2k}\]

\[+ \sum_{k=1}^{N-1} i_k[s_{1k}(a_{20} + \sum_{j=1}^{k-1} i_j s_{2j}) + s_{2k}(a_{10} + \sum_{j=1}^{k-1} i_j s_{1j})]\]

\[= a_{10}a_{20} - \sum_{k=1}^{N-2} a_{1k+1} a_{2k+1} \cos(\theta_{1k} + \theta_{2k}) \prod_{j=1}^{k-1} e^{i j (\theta_{1j} + \theta_{2j})}\]

\[+ \sum_{k=1}^{N-1} i_k[a_{1k}] \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2} + a_{2k} \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2} \prod_{j=1}^{k-1} e^{i j (\theta_{1j} + \theta_{2j})}\]

\[+ \sum_{k=1}^{N-2} i_k a_{1k+1} a_{2k+1} \sin(\theta_{1k} + \theta_{2k}) \prod_{j=1}^{k-1} e^{i j (\theta_{1j} + \theta_{2j})}\]

\[+ \sum_{k=1}^{N-1} i_k[a_{1k}] \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2} + a_{2k} \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2} \prod_{j=1}^{k-1} e^{i j (\theta_{1j} + \theta_{2j})}\]

\[+ \sum_{k=1}^{N-2} i_k a_{1k+1} a_{2k+1} \sin(\theta_{1k} + \theta_{2k}) \prod_{j=1}^{k-1} e^{i j (\theta_{1j} + \theta_{2j})}\]

In this way, we get
\[ a_{10}a_{20} - a_{11}a_{21} - \sum_{k=2}^{N-1} a_{1k}a_{2k} \prod_{j=1}^{k-1} \cos(\theta_{1j} + \theta_{2j}) \]

\[ + \sum_{k=1}^{N-1} i_k [a_{1k} \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2 + a_{2k}^2} \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2}] \prod_{j=1}^{k-1} e^{i_j(\theta_{1j} + \theta_{2j})} \]

\[ - \sum_{k=1}^{N-2} i_k a_{1k+1}a_{2k+1} \sin(\theta_{1k} + \theta_{2k}) \prod_{j=1}^{k-1} e^{i_j(\theta_{1j} + \theta_{2j})} \]

\[ [1 + \sum_{i=1}^{N-k-2} \prod_{j=k+1}^{k+i} \cos(\theta_{1j} + \theta_{2j})] \]

Comparing with

\[ s = a_0 + \sum_{k=1}^{N-1} i_k a_k \prod_{j=1}^{k-1} e^{i_j \theta_j} = a_0 + \sum_{k=1}^{N-1} i_k s_k \]

So, we get

\[ a_0 = a_{10}a_{20} - a_{11}a_{21} - \sum_{k=2}^{N-1} a_{1k}a_{2k} \prod_{j=1}^{k-1} \cos(\theta_{1j} + \theta_{2j}) \]

\[ s_k = \{ [a_{1k} \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2 + a_{2k}^2} \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2}] \prod_{j=1}^{k-1} e^{i_j(\theta_{1j} + \theta_{2j})} \}

\[ -a_{1k+1}a_{2k+1} \sin(\theta_{1k} + \theta_{2k}) \prod_{j=1}^{k-1} e^{i_j(\theta_{1j} + \theta_{2j})} \]

\[ [1 + \sum_{i=1}^{N-k-2} \prod_{j=k+1}^{k+i} \cos(\theta_{1j} + \theta_{2j})] \]

\[ a_k = [a_{1k} \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2 + a_{2k}^2} \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2}] \prod_{j=1}^{k-1} e^{i_j(\theta_{1j} + \theta_{2j})} \]

\[ -a_{1k+1}a_{2k+1} \sin(\theta_{1k} + \theta_{2k}) \prod_{j=1}^{N-k-2} \prod_{j=k+1}^{k+i} \cos(\theta_{1j} + \theta_{2j}) \]

\[ \theta_k = \theta_{1k} + \theta_{2k} \]

When there are

\[ s_1 = a_{10} + \sum_{k=1}^{N-1} i_k a_{1k} \]

\[ s_2 = a_{20} + \sum_{k=1}^{N-1} i_k a_{2k} \]

we get
\( a_0 = a_{10}a_{20} - \sum_{k=1}^{N-1} a_{1k}a_{2k} \) \hspace{1cm} (4.15)

\( s_k = a_{1k}(a_{20} + \sum_{j=1}^{k-1} i_ja_{2j}) + a_{2k}(a_{10} + \sum_{j=1}^{k-1} i_ja_{1j}) \) \hspace{1cm} (4.16)

\[
\begin{align*}
&= (a_{1k} \prod_{j=1}^{k-1} e^{i_j \theta_{1j}}) \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2 \prod_{j=1}^{k-1} e^{i_j \theta_{2j}}} \\
&\quad + (a_{2k} \prod_{j=1}^{k-1} e^{i_j \theta_{2j}}) \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2 \prod_{j=1}^{k-1} e^{i_j \theta_{1j}}} \\
&= [a_{1k} \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2} + a_{2k} \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2}] \prod_{j=1}^{k-1} e^{i_j (\theta_{1j} + \theta_{2j})} \\
&\quad \prod_{j=1}^{k-1} e^{i_j \theta_{1j}}
\end{align*}
\]

\( a_k = a_{1k} \sqrt{a_{20}^2 + \sum_{j=1}^{k-1} a_{2j}^2} + a_{2k} \sqrt{a_{10}^2 + \sum_{j=1}^{k-1} a_{1j}^2} \) \hspace{1cm} (4.17)

\( \theta_k = \theta_{1k} + \theta_{2k} \) \hspace{1cm} (4.18)

According to formula (1.2) and formulas (2.7, 2.8, 2.9), Let expand Equation (3.2) and there is,

\( s = s_1s_2 = (a_{10} + \sum_{k=1}^{N-1} i_k s_{1k})(a_{20} + \sum_{k=1}^{N-1} i_k s_{2k}) \)

\[
\begin{align*}
&= (a_{10} + \sum_{k=2}^{N-1} i_k s_{1k})(a_{20} + \sum_{k=2}^{N-1} i_k s_{2k}) \\
&\quad + \sum_{k=2}^{N-1} i_k [s_{1k}(a_{20} + \sum_{j=2}^{N-1} i_j s_{2j}) + s_{2k}(a_{10} + \sum_{j=2}^{N-1} i_j s_{1j})] \\
&\quad - s_{11}s_{21} \\
&= (a_{10} + \sum_{k=3}^{N-1} i_k s_{1k})(a_{20} + \sum_{k=3}^{N-1} i_k s_{2k}) \\
&\quad + \sum_{k=3}^{N-1} i_k [s_{1k}(a_{20} + \sum_{j=k+1}^{N-1} i_j s_{2j}) + s_{2k}(a_{10} + \sum_{j=k+1}^{N-1} i_j s_{1j})] \\
&\quad - \sum_{k=3}^{N-1} s_{1k}s_{2k}
\end{align*}
\]

In this way, we get
Comparing with

\[ s = s_1 s_2 = a_{10} a_{20} - \sum_{k=1}^{N-1} s_{1k} s_{2k} \]

\[ + \sum_{k=1}^{N-1} i_k [s_{1k} (a_{20} + \sum_{j=k+1}^{N-1} i_j s_{2j}) + s_{2k} (a_{10} + \sum_{j=k+1}^{N-1} i_j s_{1j})] \]

\[ = a_{10} a_{20} - a_{1} a_{2} N - 1 \sum_{k=1}^{N-2} a_{1k} a_{2k} \prod_{j=k+1}^{N-1} \cos(\theta_{1j} + \theta_{2j}) \]

\[ + \sum_{k=1}^{N-1} i_k [a_{1k} (a_{20} + \sum_{j=k+1}^{N-1} a_{2j}) + \sum_{j=k+1}^{N-1} a_{1j} a_{2j} \prod_{j=k+1}^{N-1} e^{i j (\theta_{1j} + \theta_{2j})} \]

\[ - \sum_{k=2}^{N-1} i_k a_{1k-1} a_{2k-1} \sin(\theta_{1k} + \theta_{2k}) \prod_{j=k+1}^{N-1} e^{i j (\theta_{1j} + \theta_{2j})} \]

\[ [1 + \sum_{i=k+2}^{N-2} \prod_{j=i-k-1}^{i-1} \cos(\theta_{1j} + \theta_{2j})] \]

\[ \text{So, we get} \]

\[ (4.20) \quad a_0 = a_{10} a_{20} - a_{1} a_{2} N - 1 \sum_{k=1}^{N-2} a_{1k} a_{2k} \prod_{j=k+1}^{N-1} \cos(\theta_{1j} + \theta_{2j}) \]

\[ (4.21) \quad s_k = [(a_{1k} \sqrt{a_{20}^2 + \sum_{j=k+1}^{N-1} a_{2j}^2} + a_{2k} \sqrt{a_{10}^2 + \sum_{j=k+1}^{N-1} a_{1j}^2} \]

\[ - a_{1k-1} a_{2k-1} \sin(\theta_{1k} + \theta_{2k}) [1 + \sum_{i=k+2}^{N-1} \prod_{j=i-k-1}^{i-1} \cos(\theta_{1j} + \theta_{2j})] \prod_{j=k+1}^{N-1} e^{i j (\theta_{1j} + \theta_{2j})} \]

\[ (4.22) \quad a_k = [a_{1k} \sqrt{a_{20}^2 + \sum_{j=k+1}^{N-1} a_{2j}^2} + a_{2k} \sqrt{a_{10}^2 + \sum_{j=k+1}^{N-1} a_{1j}^2} \]

\[ - a_{1k-1} a_{2k-1} \sin(\theta_{1k} + \theta_{2k}) [1 + \sum_{i=k+2}^{N-1} \prod_{j=i-k-1}^{i-1} \cos(\theta_{1j} + \theta_{2j})] \]

\[ (4.23) \quad \theta_k = \theta_{1k} + \theta_{2k} \]

When there are
\[ s_1 = a_{10} + \sum_{k=1}^{N-1} i_k a_{1k} \]
\[ s_2 = a_{20} + \sum_{k=1}^{N-1} i_k a_{2k} \]

we get

\[ (4.24) \]
\[ a_0 = a_{10} a_{20} - \sum_{k=1}^{N-1} a_{1k} a_{2k} \]

\[ (4.25) \]
\[ s_k = a_{1k} (a_{20} + \sum_{j=k+1}^{N-1} i_j a_{2j}) + a_{2k} (a_{10} + \sum_{j=k+1}^{N-1} i_j a_{1j}) \]
\[ = (a_{1k} \prod_{j=k+1}^{N-1} e^{i j \theta_{1j}}) \left[ \sqrt{a_{20}^2 + \sum_{j=k+1}^{N-1} a_{2j}^2} e^{i j \theta_{2j}} \right] + (a_{2k} \prod_{j=k+1}^{N-1} e^{i j \theta_{2j}}) \left[ \sqrt{a_{10}^2 + \sum_{j=k+1}^{N-1} a_{1j}^2} e^{i j \theta_{1j}} \right] \]
\[ = [a_{1k} \sqrt{a_{20}^2 + \sum_{j=k+1}^{N-1} a_{2j}^2} + a_{2k} \sqrt{a_{10}^2 + \sum_{j=k+1}^{N-1} a_{1j}^2}] \prod_{j=k+1}^{N-1} e^{i j (\theta_{1j} + \theta_{2j})} \]

\[ (4.26) \]
\[ a_k = a_{1k} \sqrt{a_{20}^2 + \sum_{j=k+1}^{N-1} a_{2j}^2} + a_{2k} \sqrt{a_{10}^2 + \sum_{j=k+1}^{N-1} a_{1j}^2} \]

\[ (4.27) \]
\[ \theta_k = \theta_{1k} + \theta_{2k} \]

5. **Definition of Space Complex Numbers In Polar Coordinates System**

5.1. **Definition.** A number \( s \) is defined as a space complex number, and expressed as following (See the proof afterward in this paper)

\[ (5.1) \]
\[ s = a + ib + jce^{i\theta} = a + ib + j(c_r + ic_i) \]

where, \( i = \sqrt{-1} \) is called the unit of imaginary number and is defined in the complex plane \( C_{xy} \), \( j = i^*i \) is called the unit of slave imaginary number, \( i_j = \sqrt{-1} \) is called the unit of imaginary number and is defined in the complex plane \( C_{yz} \). Because \( i \) and \( i_j \) are defined in different complex planes, \( i \) and \( i_j \) are the units imaginary number with directions. And \( j \) is a unit of slave imaginary number with direction also. Therefore, \( j = i * i_j \neq -1 * -1 = -1 \).

The coefficients \( a, b \) and \( c \) are real numbers, and \( a \) is called the real part of \( s \), \( b \) is called the master imaginary part of \( s \) and \( c \) is called the slave imaginary part of \( s \). The numbers \( a, b \) and \( c \) are denoted \( a = Re s, b = Imm s \) and \( c = Ims s \).

The coefficient \( c_r \) is a component of \( c \) and means a point on axis \( x \) with length \( c_r \) rotates \( \frac{\pi}{2} \) angular degrees to axis \( z \), and \( c_i \) is a component of \( c \) and means a
point on axis y with length $c_1$ rotates $\frac{\pi}{2}$ angular degrees to axis z. Because these components of $c$ are from different axis, the addition of these components of $c$ should follow the law of addition of two vectors.

5.2. Power of the unit of slave imaginary number.

\begin{equation}
\tag{5.2}
j = i \ast i_j, j^2 = -1, j^3 = -j, j^4 = 1
\end{equation}

\begin{equation}
\tag{5.3}
j^{4n+1} = j, j^{4n+2} = -1, j^{4n+3} = -j, j^{4n} = 1, n = \pm 1, \pm 2, ...
\end{equation}

(See the proof afterward in this paper.)

5.3. Multination of real part and complex exponent $e^{ij\phi}$. By the proof afterward in this paper, when a real part multiplied by a complex exponent $e^{ij\phi}$, the complex exponent $e^{ij\phi}$ is an operator of rotations, and there is

\begin{equation}
\tag{5.4}
a e^{ij\phi} = a
\end{equation}

The multiplication of the unit of imaginary number $j$ and an operator $e^{i\theta}$ of rotations is (See proof afterward in this paper.)

\begin{equation}
\tag{5.5}
ej^{i\theta} = j
\end{equation}

5.4. Equal of two Space Complex numbers. Two Space Complex numbers, $s_1$ and $s_2$ are equal when and only when their real parts on axis x, imaginary parts on y and z are equal separately. That is, for

\begin{align*}
s_1 &= a_1 + ib_1 + j c_1 e^{i\theta_1} = a_1 + ib_1 + j (c_{1r} + ic_{1i}) \\
s_2 &= a_2 + ib_2 + j c_2 e^{i\theta_2} = a_2 + ib_2 + j (c_{2r} + ic_{2i})
\end{align*}

when and only when there are $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, and $\theta_1 = \theta_2$ or $c_{1r} = c_{2r}$ and $c_{1i} = c_{2i}$, there is $s_1 = s_2$.

5.5. Modulus of the Hyper Space Complex Number.

\[ |s| = \sqrt{a^2 + b^2 + c^2} \]

is called the modulus of the Space Complex Number $s$.

5.6. Space conjugate complex numbers. Define two Space Complex numbers

\begin{align*}
s &= a + ib + j ce^{i\theta} \\
\bar{s} &= a - ib - j ce^{-i\theta}
\end{align*}

then $s$ and $\bar{s}$ are Space conjugate complex numbers.

5.7. Argument of the Space Complex Number.

\begin{equation}
\tag{5.6}
\text{Args} = \arctan \frac{r}{a}, r = \sqrt{b^2 + c^2}
\end{equation}

is called the argument or master argument of a space complex $s$. Therefore, there are infinite arguments for a space complex, but only one of them is called the principal argument for a space complex, and is denoted Arg $s$. It satisfies the condition $0 \leq \text{args} < 2\pi$ and there is Arg $s = \text{arg } s + 2k\pi, k = 0, \pm 1, \pm 2, ...$.
5.8. **Slave argument of the Space Complex Number.**

\[(5.7) \quad \text{ArgIm} s = \arctg \frac{c}{b}\]

is called the slave argument of a space complex \(s\). Therefore, there are infinite arguments for a space complex, but only one of them is called the slave principal argument for a space complex, and is denoted \(\text{arg Im} s\). It satisfies the condition \(0 \leq \text{argIm} s < 2\pi\) and there is \(\text{Arg Im} s = \text{arg Im} s + 2k\pi, k = 0, \pm 1, \pm 2, \ldots\).

6. **Expression of Space Complex Numbers In Polar Coordinates System**

As the classic complex number, there are some expressions for the Space Complex Number \(s\) expressed by formula (5.1).

6.1. **Expression in coordinates.** According to formula (5.5), there is \(s = a + ib + jc\). There exists a relation between the Space Complex number and the 3-dimensional right angular coordinates system.

6.2. **Expression in vector.** Let express \(a, b\) and \(c\) as projections of a vector \(\vec{OP}\) on axis \(x, y\) and \(z\), then the Space Complex Number \(s = a + ib + jc\) is expressed by the vector \(\vec{OP}\). A point is denoted \(\vec{P}'\), which is on the same plane formed by the axis \(x\) and vector \(\vec{OP}\) and is symmetry about axis \(x\) with the point \(P\). The Space Complex Number, \(\bar{s} = s = a - ib - jc\), which is the conjugate of \(s\), is expressed by the vector \(\vec{OP}'\).

6.3. **Expression in triangular.** A Space Complex number is denoted by triangular functions

\[(6.1) \quad s = |s|(\cos \theta + isin \theta \cos \varphi + jsin \theta \sin \varphi)\]

Compares with the expression \(s = a + ib + jc\), there are.

\[(6.2) \quad a = |s| \cos \theta\]

\[(6.3) \quad b = |s| \sin \theta \cos \varphi\]

\[(6.4) \quad c = |s| \sin \theta \sin \varphi\]

These are the coordinates of a polar coordinates system. Therefore, there are

\[(6.5) \quad \theta = \arctg \frac{\bar{r}}{a}, r = \sqrt{b^2 + c^2}\]

\[(6.6) \quad \varphi = \arctg \frac{c}{b}\]

where \(\theta\) is the master principal argument \(\text{arg} s\) of \(s\) and \(\varphi\) is the slave principal argument \(\text{arg Im} s\) of \(s\).
6.4. **Expression in exponent.** According to formula (5.4.6.1), there is

(6.7) \[ s = |s|(\cos \theta + is \sin \theta \cos \varphi + j \sin \theta \sin \varphi) \]

\[ = |s|(e^{i\varphi}) \cos \theta + is \sin \theta \cos \varphi + j \sin \theta \sin \varphi \]

\[ = |s|(\cos \theta + is \sin \theta)e^{i\varphi} = |s|e^{i\theta}e^{i\varphi} \]

7. **Algebra of Space Complex Numbers In Polar Coordinates System.**

As the classic complex number, there are some algebras for the Space Complex Number \( s \) expressed by formula (5.1)

7.1. **Rules of operation in algebra.** For Space Complex Numbers

\[ s_1 = a_1 + ib_1 + j c_1 e^{i\theta_1} = a_1 + ib_1 + j(c_{1r} + ic_{1i}) \]

\[ s_2 = a_2 + ib_2 + j c_2 e^{i\theta_2} = a_2 + ib_2 + j(c_{2r} + ic_{2i}) \]

there are

(7.1) \[ s_1 \pm s_2 = (a_1 \pm a_2) + i(b_1 \pm b_2) + j[(c_{1r} \pm c_{2r}) + i(c_{1i} \pm c_{2i})] \]

(7.2) \[ s = s_1 * s_2 = (a_1 + ib_1 + j c_1 e^{i\theta_1}) * (a_2 + ib_2 + j c_2 e^{i\theta_2}) \]

\[ = a_1 a_2 - b_1 b_2 + (j c_1 e^{i\theta_1})(j c_2 e^{i\theta_2}) \]

\[ + i(b_1 a_2 + b_2 a_1) \]

\[ + j[c_1 e^{i\theta_1}(a_2 + ib_2) + c_2 e^{i\theta_2}(a_1 + ib_1)] \]

\[ = a_1 a_2 - b_1 b_2 - c_{1r} c_{2r} + c_{1i} c_{2i} \]

\[ + i(b_1 a_2 + b_2 a_1 - c_{1r} c_{2i} - c_{1i} c_{2r}) \]

\[ + j[c_1 e^{i\theta_1}(a_2 + ib_2) + c_2 e^{i\theta_2}(a_1 + ib_1)] \]

\[ = a + ib + j c e^{i\theta} \]

where, according to formulas (6.2, 6.3, 6.4) and let \( r_1 = \sqrt{a_1^2 + b_1^2}, r_2 = \sqrt{a_2^2 + b_2^2} \)

\[ c e^{i\theta} = c_1 e^{i\theta_1} e^{i\theta_2} \]

\[ = (c_{1r} r_2 + c_{2r} r_1) e^{i(\theta_1 + \theta_2)} \]

\[ c_r + ic_i = (c_{1r} + ic_{1i})(a_2 + ib_2) + (c_{2r} + ic_{2i})(a_1 + ib_1) \]

\[ = c_{1r} a_2 - c_{1i} b_2 + i(c_{1i} a_2 + c_{1r} b_2) + c_{2r} a_1 - c_{2i} b_1 + i(c_{2i} a_1 + c_{2r} b_1) \]

\[ = (c_{1r} a_2 + c_{2r} a_1) - (c_{1i} b_2 + c_{2i} b_1) + i(c_{1i} a_2 + c_{2i} a_1 + c_{1r} b_2 + c_{2r} b_1) \]

So, we get

(7.3) \[ a = a_1 a_2 - b_1 b_2 - c_{1r} c_{2r} + c_{1i} c_{2i} \]

(7.4) \[ b = b_1 a_2 + b_2 a_1 - c_{1r} c_{2i} - c_{1i} c_{2r} \]

(7.5) \[ c = c_{1r} r_2 + c_{2r} r_1 \]
\[ \theta = \theta_1 + \theta_2 \]

\[ c_r = c_1r_2 + c_{2r}a_1 - c_1b_2 - c_{2r}b_1, \]
\[ c_i = c_1r_2 + c_{2i}a_1 + c_1b_2 + c_{2i}b_1 \]

Therefore, there is \( s_1s_2 = s_2s_1 \). The associative and commutative laws of addition and multiplication exist for the Space Complex Numbers.

When \( s_1 = s \) and \( s_2 = \bar{s} \), there are
\[ s \ast \bar{s} = (a + ib + jce^{i\theta}) \ast (a - ib - jce^{-i\theta}) = a^2 + b^2 + c^2 + i0 + j0 = |s|^2 \]

\[ s = s_1 \div s_2 = (a_1 + ib_1 + jc_1e^{i\theta_1}) \div (a_2 + ib_2 + jc_2e^{i\theta_2}) \]
\[ = \frac{1}{|s_2|^2} (a_1 + ib_1 + jc_1e^{i\theta_1}) \ast (a_2 - ib_2 - jc_2e^{-i\theta_2}) \]
\[ = \frac{1}{|s_2|^2} (a_1a_2 + b_1b_2 + c_1r_2 + c_{1i}c_{2i}) \]
\[ + i \frac{1}{|s_2|^2} (b_1a_2 - b_2a_1 + c_1r_2 - c_{1i}c_{2i}) \]
\[ + j \frac{1}{|s_2|^2} [c_1e^{i\theta_1}(a_2 - ib_2) - c_2e^{-i\theta_2}(a_1 + ib_1)] \]
\[ = a + ib + jce^{i\theta} \]

where, according to formulas (6.2, 6.3, 6.4),
\[ ce^{i\theta} = \frac{1}{|s_2|^2} [c_1e^{i\theta_1}(a_2 - ib_2) - c_2e^{-i\theta_2}(a_1 + ib_1)] \]
\[ = \frac{1}{|s_2|^2} [c_1e^{i\theta_1}r_2e^{-i\theta_2} - c_2e^{-i\theta_2}r_1e^{i\theta_1}] \]
\[ = \frac{1}{|s_2|^2} (c_1r_2 - c_2r_1)e^{i(\theta_1 - \theta_2)} \]

\[ c_r + ic_i = (c_1r_2 + ic_{1i})(a_2 - ib_2) - (c_2r_1 + ic_{2i})(a_1 + ib_1) \]
\[ = c_1r_2a_2 + c_{1i}b_2 + ic_1r_2a_1 - c_2r_1b_2 - c_{2i}b_1 + ic_2r_1a_1 - c_{2i}b_1 \]
\[ = (c_1r_2a_2 - c_2r_1a_1) + (c_{1i}b_2 + c_{2i}b_1) + i[(c_1r_2a_1 - c_2r_1a_1) - (c_1b_2 + c_2b_1)] \]

So, we get

\[ a = \frac{1}{|s_2|^2} (a_1a_2 + b_1b_2 + c_1r_2 + c_{1i}c_{2i}) \]
\[ b = \frac{1}{|s_2|^2} (b_1a_2 + b_2a_1 + c_1r_2 - c_{1i}c_{2i}) \]
\[ c = \frac{1}{|s_2|^2} (c_1r_2 - c_2r_1) \]
\[ \theta = \theta_1 - \theta_2 \]
Let
\[
s_3 = a_3 + i b_3 + j c_3 e^{i \theta_3} = a_3 + i b_3 + j(c_3 r + i c_3 i)
\]
there is
\[
s_1 * s_3 + s_2 * s_3 = [a_1 + i b_1 + j(c_1 r + i c_1 i)] * s_3
\]
\[
+ [a_2 + i b_2 + j(c_2 r + i c_2 i)] * s_3
\]
\[
= [(a_1 + a_2) + i(b_1 + b_2) + j(c_1 r + c_2 r + i c_1 i + i c_2 i)] * s_3
\]
\[
= (s_1 + s_2) * s_3
\]
Therefore, the distributive law of addition and multiplication exists for the Space Complex Numbers.

7.2. Rules of operation in triangular. For two Space Complex Numbers
\[
s_1 = |s_1|(\cos \theta_1 + i \sin \theta_1 \cos \varphi_1 + j \sin \theta_1 \sin \varphi_1)
\]
\[
s_2 = |s_2|(\cos \theta_2 + i \sin \theta_2 \cos \varphi_2 + j \sin \theta_2 \sin \varphi_2)
\]
According to \(i \sin \theta \cos \varphi + j \sin \theta \sin \varphi = i \sin \theta e^{i \varphi}\) and the formular(5.4), there is

\[
s_1 * s_2 = |s_1||s_2| [\cos(\theta_1 + \theta_2)
\]
\[
+ i \sin(\theta_1 + \theta_2)(\cos(\varphi_1 + \varphi_2))
\]
\[
+ j \sin(\theta_1 + \theta_2)(\sin(\varphi_1 + \varphi_2))]
\]
Therefore, there is \(s_1 * s_2 = s_2 * s_1\). The associative and commutative laws of multiplication exist in triangular form for the Space Complex Number in polar coordinates system.

Same as above, there is

\[
s_1 ÷ s_2 = \frac{|s_1|}{|s_2|} [\cos(\theta_1 - \theta_2)
\]
\[
+ i \sin(\theta_1 - \theta_2)(\cos(\varphi_1 - \varphi_2))
\]
\[
+ j \sin(\theta_1 - \theta_2)(\sin(\varphi_1 - \varphi_2))]
\]
Therefore, when \(s = s_1 = s_2 = ... = s_n\), according to formula (7.14), there are

\[
s^n = |s|^n (\cos n \theta + i \sin n \theta \cos n \varphi + j \sin n \theta \sin n \varphi)
\]

\[
s^\frac{1}{n} = |s|^\frac{1}{n} (\cos \frac{\theta + 2m\pi}{n}
\]
\[
+ i \sin \frac{\theta + 2m\pi}{n} \cos \frac{\varphi + 2m\pi}{n}
\]
\[
+ j \sin \frac{\theta + 2m\pi}{n} \sin \frac{\varphi + 2m\pi}{n})
\]
7.3. Rules of operation in exponent. For two Space Complex Numbers,

\[ s_1 = |s_1| e^{i\theta_1} e^{ij\varphi_1} \]
\[ s_2 = |s_2| e^{i\theta_2} e^{ij\varphi_2} \]

there is

\[
(7.18) \quad s_1 \ast s_2 = |s_1| |s_2| e^{i(\theta_1 + \theta_2)} e^{ij(\varphi_1 + \varphi_2)}
\]

Therefore, there is \( s_1 \ast s_2 = s_2 \ast s_1 \). The associative and commutative laws of multiplication exist in exponent form for the Space Complex Number in polar coordinates system.

\[
(7.19) \quad s_1 \div s_2 = |s_1| |s_2| e^{i(\theta_1 - \theta_2)} e^{ij(\varphi_1 - \varphi_2)}
\]

Therefore, when \( s = s_1 = s_2 = \ldots = s_n \), there are

\[
(7.20) \quad s^n = |s|^n e^{i\theta} e^{ijn\varphi}
\]

\[
(7.21) \quad s^{\frac{1}{n}} = |s|^{\frac{1}{n}} e^{i\frac{\theta + 2m\pi}{n}} e^{ij\frac{\varphi + 2m\pi}{n}}
\]

\[ m = 1, 2, \ldots, n - 1 \]

8. Proof of Space Complex Numbers In Polar Coordinates System

8.1. Proof of Definition of Space Complex Numbers In Polar Coordinates System. A polar coordination could be composed of two complex planes, one is the complex plane \( C_{xy} = \mathbb{C}(x, iy) \) composed of axis x and axis y and another is the complex plane \( C_{yz} = \mathbb{C}(y, jz) \) composed of axis y and axis z. On \( C_{xy} \), any point could be denoted \( P_{xy} = (a, ib) \) or expressed as \( s_{xy} = a + ib \). On \( C_{yz} \), any point could be denoted \( P_{yz} = (ib, jc) \) or expressed as \( s_{yz} = ib + jc \). The ordinates of a point \( P_{xyz} = P(a, ib, jc) \) on a polar coordinates system, could be considered as the combination of two movements of the point.

First, when beginning, the point \( P_z = a_0 \), is on the axis x, and is rotated \( \theta \) angular degrees on the complex plane \( C_{xy} \), and a new point \( P_{xy} = (a, ib_1) \) is formed. Then, the point \( P_{xy} \) is rotate \( \varphi \) angular degrees on the complex plane \( C_{yz} \), and a new point \( P_{yz} \) is formed, and at the same time, a new point \( P_{xyz} = \mathbb{P}(a, ib, jc) \) is formed in the complex space. At the second movement, the rotation of the point \( P_{xy} \) could be considered as the rotation of the complex plane \( C_{xy} \) around axis x for \( \varphi \) angular degrees, or the self-rotation of its component a on axis x around axis x and the rotation of its component b on axis y around axis x on complex plane \( C_{yz} \) for \( \varphi \) angular degrees. Therefore, a point, \( P_{xyz} = \mathbb{P}(a, ib, jc) \), is formed by the two movements, and a space complex number s is denoted \( s = a + i(b + ic) = a + ib + ijc \). Let denote \( j = ijc \), there is \( s = a + ib + ijc = a + ib + jc \). and s is called space complex number. A complex space \( C_{xyz} \) is formed by Space...
Complex Numbers. So a Space Complex Number in polar coordinates system could be denoted \( s = a + ib + jce^{i\theta} \).

8.2. **Proof of power of slave imaginary unit.** For the polar coordinates system, as discussed above, the slave imaginary unit \( j = i * i_j \), is an expression of the axis \( z \), which is in the complex plane \( C_{yz} \) or in the complex space \( C_{xyz} \). At the same time, the slave imaginary unit \( j = i * i_j \) is also in the complex plane \( C_{zz} \), and could be considered as that a point \( P_x \) on axis \( x \) rotes 90 angular degrees from axis \( x \) to axis \( z \) around axis \( y \). Therefore the square power of the slave imaginary unit \( j \) means that a point \( P_x \) on axis \( x \) rotes 180 angular degrees from plus axis \( x \) to minus axis \( x \) around axis \( y \) and there is \( j^2 = -1 \). So, there are

\[
\begin{align*}
8.1. \quad j &= i * i_j, j^2 = -1, j^3 = -j, j^4 = 1 \\
8.2. \quad j^{4n+1} &= j, j^{4n+2} = -1, j^{4n+3} = -j, j^{4n} = 1, n = \pm 1, \pm 2, ...
\end{align*}
\]

8.3. **Proof of multiplication of real part on axis \( x \) and complex exponent \( e^{i\varphi} \).** In a polar coordinates system discussed above, on the complex plan composed of axis \( y \) and \( z \), let define an operator of rotations and denote it \( e^{i\varphi} \) which means a rotation \( \varphi \) around axis \( x \). Because there is no change for any point on axis \( x \) when it rotes any angular degrees around axis \( x \), there is \( ae^{i\varphi} = a \) for a point \( P_x = (a, 0, 0) \) on axis \( x \).

8.4. **Proof of multiplication of imaginary number \( j \) on axis \( z \) and an operator \( e^{i\theta} \).** As defined above, an operator of rotations denoted \( e^{i\theta} \) means the rotation on \( C_{xy} \). Because \( C_{xy} \) is perpendicular to axis \( z \), axis \( z \) could be considered as a axis of the rotation on \( C_{xy} \). Because there is no change for any point on axis \( z \), when it rotates any angular degrees around axis \( z \), therefore, there is

\[
8.3. \quad je^{i\theta} = j
\]

and for a real number \( r \), there is

\[
8.4. \quad re^{i\theta} = r(\cos\theta + i\sin\theta)
\]

9. **Duality Property of Hyper Space Complex Numbers**

In this section, we will discuss the duality property of hyper space complex numbers.

Let us rewrite the expression of hyper space complex numbers in formula(1.1).

\[
c_2 = a_0 + i_1 a_1 \\
c_3 = a_0 + i_1 a_1 + i_2 a_2 e^{i_1 \theta_1} = c_1 + i_2 \frac{a_2}{|c_1|} c_1 \\
c_4 = a_0 + i_1 a_1 + i_2 a_2 e^{i_1 \theta_1} + i_3 a_3 e^{i_1 \theta_1} e^{i_2 \theta_2} = c_2 + i_3 \frac{a_3}{|c_2|} c_2 \\
\ldots
\]
In the expressions above, we can see that for a real number \( a_0 \), a dual imaginary number \( i_1 a_1 \) can be taken to form a complex number \( c_2 \). The complex number \( c_2 \) can be considered as a complex plane, or a real plane in two dimensions.

For the real plane \( c_2 \), a dual imaginary plane \( i_2 \frac{a_2}{|c_2|} \) can be taken to form a 3-dimensional space complex number \( c_3 \). The space complex number \( c_3 \) can be considered as a 3-dimensional complex space, or a real 3-dimensional space.

For the real 3-dimensional space \( c_3 \), a dual imaginary 3-dimensional space \( i_3 \frac{a_3}{|c_3|} \) can be taken to form a 4-dimensional hyper space complex number \( c_4 \). The hyper space complex number \( c_4 \) can be considered as a 4-dimensional hyper complex space, or a real 4-dimensional hyper space.

So, for real \( n \)-dimensional hyper space \( c_n \), a dual imaginary \( n \)-dimensional hyper space \( i_n \frac{a_n}{|c_n|} \) can be taken to form a \( n+1 \) dimensional higher order hyper space complex number \( c_{n+1} \). The higher order hyper space complex number \( c_{n+1} \) can be considered as a \( n+1 \) dimensional hyper complex space, or a real \( n+1 \) dimensional hyper space. The duality property of hyper space complex numbers described the duality property of numbers system.