Study of Solving Electrostatics by Partial Differential Equation the Relationship between Dielectric Medium and Capacitance

Guangzhao Yang

The High School Affiliated to Renmin University of China, China

Abstract. The Maxwell equations as a fundamental equation governing the electromagnetic fields are investigated. Several ideal systems of parallel plate capacitor with various dielectric mediums are discussed. By considering these cases, the boundary conditions in the process of solving the equations and the separation of variables as a primary method to solve the equations are studied, which will be the foundation of our future work.

Keywords: partial differential equation; capacitance; dielectric medium

1. Introduction

The essay will research on the parallel capacitor. The parallel capacitor is a typical electric component, which consists of two electrodes and a dielectric medium. The properties of the capacitor are essential for electric industries; the capacitor plays an irreplaceable role in coupling circuit, which can isolate the direct current and filter circuit, which can delete specific-range of signal’s frequency.

In electrostatic, the capacitor-system shows more exciting properties that are not considered well. For example, the shape and position of the dielectric medium can affect the capacitance of the parallel capacitor dramatically. The electric field between two parallel plates will polarize the dielectric medium, which not fill the space between the capacitor, generate a sophisticatedly new field that will change the capacitance.

Usually, the dielectric medium will fill the parallel-plates capacitor to change the capacitance. However, the questions about the not-full filling (any shape of dielectric medium) are frequently neglected since the electric field produced is very complex, causing some difficulties for the estimating of capacitance. Although the application of not-full filling is minimal, the complexity of the application cannot represent that these questions are worthless. Theoretically, the discussion of these problems is inevitable in the process to investigate the electromagnetics.

The whole process will base on the Maxwell equation [1], since it is the crux of electromagnetics and certainly, solving the partial differential equation is the core methodology of this research. This essay will show the methodology of solving partial differential equations to solve the relationship between dielectric medium and capacitance in specific situations.

In the theoretical background part, the Maxwell equation in the situation of the emergence of dielectric medium will be discussed. Also, several critical concepts will be defined and clarified, which will lay a solid foundation of the following investigation. For the sake of the solution of Maxwell equation, various boundary conditions have to be utilized to determine a specific solution of an electric
field, which will be presented by the potential in order to get a Laplace equation. The useful boundary conditions are: the uniform potential on plates; the uniform field (linear potential) when there is no dielectric medium; the coherent electric displacement on the boundary of the dielectric medium; the coherent potential on the boundary of the dielectric medium; the limited potential near the center of the dielectric medium. Used the boundary condition, the specific solution of the partial differential equations can be got. In specific situation, the condition that consist infinite-uniform-parallel medium will be solved and an expression of the relationship between capacitance, the capacitor-spacing, and thickness of the dielectric medium will be gained. In the outlook part, the spherical-medium condition will be solved partially. The problem encountered will be concluded detailly to be a valuable experience for further work.

2. Theoretical background

The Maxwell equations that represent the most thorough comprehension to the electromagnetic phenomenon are the foundation of this investigation. The equations are (1):

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \frac{\mathbf{J}}{\mu_0} + \frac{\varepsilon_0}{c^2} \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

In the current paper we consider the electrostatic case, which means that the electric field between the capacitor will not change through time. Therefore \(\frac{\partial \mathbf{B}}{\partial t}\) and \(\frac{\varepsilon_0}{c^2} \frac{\partial \mathbf{E}}{\partial t}\) are equal to zero. The Maxwell equations (1) become:

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \times \mathbf{E} &= 0
\end{align*}
\]

When a dielectric medium, an insulator, is placed in the parallel-plates capacitor, the medium will be polarized. The polarization is defined as \(\mathbf{P} = Nq\mathbf{\delta}\), where \(\mathbf{\delta}\) is the displacement of charge because of polarization, and it has the following relationship between surface charge density \(\sigma_p = \mathbf{P} \cdot \mathbf{n}\) with \(\mathbf{n}\) being the unit normal vector to the surface

\[
S\sigma_p = SNq\mathbf{\delta} = q_{\text{surface}}
\]

Namely

\[
\sigma_p = Nq\mathbf{\delta} = \mathbf{P}
\]

Moreover, \(\mathbf{P}\) is assumed to be proportional to the electric field:

\[
\mathbf{P} = \chi \varepsilon_0 \mathbf{E}
\]

By considering the formula for the volume change density \(\rho_p = -\nabla \cdot \mathbf{P}\), we can figure out easily the relationship that \(\rho_p = -\chi \varepsilon_0 \nabla \cdot \mathbf{E}\). By denoting the free change density as \(\rho_f\) and substituting the formula \(\rho_p = -\nabla \cdot \mathbf{P}\) into (2) we have:

\[
\nabla \cdot \mathbf{E} = \frac{\rho_f + \rho_p}{\varepsilon_0} \\
\nabla \cdot \mathbf{E} = \frac{\rho_f - \nabla \cdot \mathbf{P}}{\varepsilon_0} \\
\nabla \cdot \left( \mathbf{E} + \frac{\mathbf{P}}{\varepsilon_0} \right) = \frac{\rho_f}{\varepsilon_0}
\]

By defining electric displacement \(\mathbf{D}\) as:
\[ D = \varepsilon_0 E + P = (\chi + 1)\varepsilon_0 E = \kappa \varepsilon_0 E \]  

(9)

The Maxwell equation can be simplified as:

\[ \nabla \cdot D = \rho_f \]  

(10)

which gives a group of equations involving only the free change

\[ \begin{cases} 
\nabla \cdot \kappa E = \rho_f \\
\n\nabla \times E = 0 
\end{cases} \]  

(11)

3. Boundary conditions

Several boundary conditions will be utilized to solve the Maxwell equation in this problem, including the uniform electric field (liner potential) when the distance from the center of dielectric medium is infinite; the coherent electric displacement and potential at the boundary of dielectric medium; the coherent potential in the boundary of dielectric medium; the limited potential distribution in the center of dielectric medium. All of the boundary conditions will be considered in the follows.

3.1. Uniform electric field between parallel-plates capacitor

Let us consider the electric field between two parallel capacitor plates without dielectric medium. In order to obtain the results, surface integral will be used. In a cartesian coordinate system, we set two surfaces \( \Sigma_1(x, y, z): z = 0 \) with positive charge \( q \) and \( \Sigma_2(\xi, \eta, \zeta): \zeta = d \) with negative charge \(-q\), which indicates that the distance between these two plates is \( d \). The distance vector from one point \((x, y, 0) \in \Sigma_1\) to another point \((\xi, \eta, d) \in \Sigma_2\) is \( r = (\xi - x)i + (\eta - y)j + (d - 0)k \). All of the analytic geometry settings are shown in Figure 1.

Since the system is unchanged under translational transformation along x-direction and y-direction, two plates are uniformity for charge. Thus, we have \( \sigma_1 = \sigma_2 = \sigma \) with \( \sigma_1 \) and \( \sigma_2 \) the density of change in the corresponding plates. Let us imagine that there exists a test charge \( q_0 \) locating at \((x_0, y_0, z_0)\). Then the force on the charge sourcing from these two plates is:

**Figure 1.** The figure shows two parallel-plates capacitors: \( \Sigma_1(x, y, z): z = 0 \) and \( \Sigma_2(\xi, \eta, \zeta): \zeta = d \) with \( q = S\sigma_1 \) and \(-q = -S\sigma_2\) respectively. In space-coordinate system, two axis, X and Y, are shown in this figure and the third axes, Z, cannot be shown for the reason of clear presentation. The Z axes is actually out of paper from the original point of XY.
Again, since the system is unchanged under translational transformation along x-direction and y-direction, which is the symmetrical properties, $F$ must be along the z-direction. Therefore, it has

$$F = \int_{\Omega_1} \frac{q_0}{4\pi\varepsilon_0 \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \cdot [(x-x_0)i + (y-y_0)j + z_0k] \, dx \, dy - \int_{\Omega_2} \frac{q_0}{4\pi\varepsilon_0 \sqrt{(\xi-x_0)^2 + (\eta-y_0)^2 + (\zeta-z_0)^2}} \cdot [(\xi-x_0)i + (\eta-y_0)j + (d-z_0)k] \, d\xi \, d\eta$$ (12)

Calculating this integral gives us that:

$$F = -\frac{q_0\sigma}{\varepsilon_0} \mathbf{k}$$ (14)

Which gives us the electric field

$$E = -\frac{\sigma}{\varepsilon_0} \mathbf{k}$$ (15)

which gives us the potential at $(x_0, y_0, z_0)$ as

$$\phi(x_0, y_0, z_0) = -\int E \cdot d\mathbf{l} = \frac{\sigma}{\varepsilon_0} z_0$$ (16)

After a dielectric medium is placed between the capacitor, the field between the capacitor will not be influenced at the points infinitely far from the dielectric. Namely, the potential is liner at infinitely far points.

3.2. The coherent electric displacement and potential in the boundary of dielectric medium

This boundary condition is available by the utilizing of Gauss law in space. $\Omega_1$ and $\Omega_2$ as shown in the figure.

![Figure 2](image)

**Figure 2.** The gauss law will be utilized on the boundary of two area, $\Omega_2$ and $\Omega_1$. The $\Omega_2$ is the dielectric medium with $\kappa\varepsilon_0$. On the boundary with area $S$, the surface-polarized charge is $q$ and the surface charge density is $\sigma$. 

4
In order to find the relationship between dielectric displacements, the Gauss surface \( \Sigma \), which consists of \( \Sigma_1 \), \( \Sigma_2 \) (parallel to the boundary \( \partial \Omega \) and have same area) and \( \delta \) (perpendicular with the boundary \( \partial \Omega \) and has infinitesimal area and distance from \( \partial \Omega \)), is constructed. According to the Gauss law, the following equations can be figured out that

\[
\oint_{\Sigma} \mathbf{D} \cdot d\mathbf{S} = S \sigma_f
\]

namely

\[
\int_{
\Sigma_1} \mathbf{D} \cdot d\mathbf{S} + \int_{\Sigma_2} \mathbf{D} \cdot d\mathbf{S} + \int_{\delta} \mathbf{D} \cdot d\mathbf{S} = S \sigma_f
\]

Considering the limit that \( \delta \to 0 \), we have finally

\[
\int_{
\Sigma_1} \mathbf{D} \cdot d\mathbf{S} + \int_{\Sigma_2} \mathbf{D} \cdot d\mathbf{S} = S \sigma_f
\]

That is:

\[
S(D_1 - D_2) \cdot \mathbf{n} = S \sigma_f
\]

Which leads to:

\[
D_1^+ - D_2^+ = \sigma_f
\]

For the case when there do not exist any free changes on the surface of the medium, we obtain that

\[
D_1^+ = D_2^+
\]

By using the relation between \( \mathbf{E} \) and \( \mathbf{D} \), we have

\[
\varepsilon_0 E_1^+ = \kappa \varepsilon_0 E_2^+
\]

that is

\[
\varepsilon_0 \frac{\partial \Phi_1(r)}{\partial r} \bigg|_{r=\partial \Omega(r)} = \kappa \varepsilon_0 \frac{\partial \Phi_2(r)}{\partial r} \bigg|_{r=\partial \Omega(r)}
\]

Now let us consider the boundary condition for the parallel components of the electric field, which can be obtained by using the equation \( \nabla \times \mathbf{E} = 0 \). \( \Omega_1 \) and \( \Omega_2 \) are shown in the figure.

![Figure 3](image-url)

**Figure 3.** The loop theorem is used to prove this boundary condition. \( \Omega_1 \) is the vacuum with \( \varepsilon_0 \) and \( \Omega_2 \) is the region with dielectric medium with \( \kappa \varepsilon_0 \). The loop curve is constructed around the boundary \( \partial \Omega \) and goes through both \( \Omega_1 \) and \( \Omega_2 \).
For the sake of finding the relationship between the potential in both sides, the loop curve $l$ is composed by: $l_1$ and $l_2$, following the left-hand rule in integration and paralleling to the surface; $\delta_1$ and $\delta_2$, with infinitesimal length, perpendicular to the surface connecting $l_1$ and $l_2$ to form a loop. The loop theorem can be expressed as:

$$\oint E \cdot dl = 0$$

(25)

Since $\delta_1$ and $\delta_2$ is infinitesimal, the electric field on both of them can be neglected. Therefore

$$\int_{l_1} E \cdot dl + \int_{l_2} E \cdot dl = 0$$

(26)

which gives us

$$l \cdot (E_1 \parallel - E_2 \parallel) = 0$$

(27)

namely

$$E_1 \parallel = E_2 \parallel$$

(28)

This relation gives us the relation between the potential on both side of the $\partial \Omega$

$$\mathbf{v} \cdot \nabla \phi|_{\partial \Omega^-} = \mathbf{v} \cdot \nabla \phi|_{\partial \Omega^+}, \forall \mathbf{v} \parallel \partial \Omega$$

(29)

There is no point charge in the boundary. Thus, the potential is continuous, namely

$$\phi|_{\partial \Omega^-} = \phi|_{\partial \Omega^+}$$

(30)

3.3. The limited potential distribution

In this paper, we restrict the shape of dielectric medium not to be convex graphics. Thus, the medium $\Omega$ in this investigation cannot be simplified into point charge. According to the Gauss law

$$\nabla \cdot E|_{\Sigma \Omega} = \frac{\rho}{\kappa \varepsilon_0}$$

(31)

Which gives us that

$$\nabla^2 \phi|_{\Sigma \Omega} = \frac{\rho}{\kappa \varepsilon_0}$$

(32)

with $\nabla \phi = E$. The charge density is limited by $\rho < \infty$. Therefore:

$$\phi(r, \theta, z)|_{r \to 0} \neq \pm \infty$$

(33)

From (33), it can also be concluded that in any space of $r \to 0$ between capacitance, the potential cannot be infinite. This condition is very helpful for the solving of partial differential equation since it can simplify the equation dramatically.

Finally, the potential on conductor plates must be equal to a constant. Otherwise, the system cannot keep electrostatic balance. In the current paper, the considered conductor plates locate on the surface given by $z = z_0$. Thus, we have the boundary condition for the potential

$$\phi(r, \theta, z_0) = C$$

(34)

4. The process of solving partial differential equations

In electrostatic capacitor, the electric field inside and outside the dielectric medium, denoted by $E_i$ and $E_o$ respectively, can be expressed as:

$$\begin{cases}
\nabla \cdot E_i = \frac{\rho}{\kappa \varepsilon_0} \\
\nabla \cdot E_o = \frac{\rho}{\varepsilon_0}
\end{cases}$$

(35)
where $\rho$ is the volume density of the free charges. However, for the cases considered in the current paper, there are not free charges in the dielectric medium and the space between medium and the plates. Therefore, the right-hand sides of the above equations are zero. In addition, since $\nabla \times \mathbf{E} = 0$, it results in that $\mathbf{E} = \nabla \phi$ for some potential $\phi$. Therefore, the Maxwell equations becomes:

$$\begin{aligned}
\nabla^2 \phi_i(r, \theta, z) &= 0 \\
\nabla^2 \phi_o(r, \theta, z) &= 0
\end{aligned} \tag{36}$$

By using the separation of variable [2], we separate the potential $\phi$ as

$$\begin{aligned}
\phi_i &= P(r)\Theta(\theta)Z(z) \\
\phi_o &= P(r)\Theta(\theta)Z(z)
\end{aligned} \tag{37}$$

The Maxwell equations read

$$\begin{aligned}
\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \theta}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} &= 0 \\
\frac{P''(r)\Theta(\theta)Z(z)}{P(r)} + \frac{P'(r)\Theta(\theta)Z(z)}{rP(r)} + \frac{\Theta''(\theta) + \nu^2\Theta(\theta)}{\Theta(\theta)} + \frac{Z''(z)}{Z(z)} &= 0
\end{aligned} \tag{38}$$

In this way the partial differential equation is transferred into three ordinary equations:

$$\begin{aligned}
\frac{Z''(z)}{Z(z)} &= k^2 \\
\Theta''(\theta) &= -\nu^2 \\
\frac{P''(r)}{P(r)} + \frac{P'(r)}{rP(r)} &= -k^2 - \frac{\nu^2}{r^2}
\end{aligned} \tag{39}$$

The $k$ and $\nu$ is the proper values of the equation. $k$ is positive and real. The solutions are

$$\begin{aligned}
Z(z) &= e^{\pm k z} \\
\Theta(\theta) &= e^{\pm \nu \theta} \\
P(r) &= \begin{cases} 
J_\nu(kr) = \frac{kr}{2^\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+\nu+1)} \left( \frac{kr}{2} \right)^{2j} \\
J_{-\nu}(kr) = \frac{kr}{2^\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j-\nu+1)} \left( \frac{kr}{2} \right)^{2j} \\
N_\nu(kr) = \lim_{\nu \to \nu} \frac{J_\nu(kr) \cos \nu \pi - J_{-\nu}(kr)}{\sin \nu \pi}
\end{cases} \tag{40}
\end{aligned}$$

$J_\nu(kr)$ and $J_{-\nu}(kr)$ are called the Bessel functions and $N_\nu(kr)$ is called the Neumann function. The general solution is the linear combination of two different solutions. Thus

$$\begin{aligned}
Z(z) &= A e^{\nu z} + B e^{-\nu z} \\
\Theta(\theta) &= C \cos \nu \theta + D \sin \nu \theta \\
P(r) &= E J_\nu(kr) + F N_\nu(kr)
\end{aligned} \tag{41}$$

Here the Neumann function is introduced since $J_\nu$ and $J_{-\nu}$ are not linear independent for integers $\nu$. When $\nu$ is not an integer, $N_\nu$ can be replaced by $J_{-\nu}$. By using the L’Hospital’s rule, we can obtain the limit for integers $\nu$, which gives us the Neumann function.
\[ N_v(kr) = \frac{2}{\pi} J_v(kr) \ln \frac{kr}{\varepsilon} - \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left( \frac{kr}{\varepsilon} \right)^{2j-v} - \frac{1}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(v+j)(v+j+1)} \left( \frac{kr}{\varepsilon} \right)^{2j+v} \] (45)

The general solution of the Laplace equation now is:
\[
\begin{cases}
\phi_l = \sum_{v=0}^\infty \int_0^\infty d\mu_v(A_{kl}e^{kix} + B_{kl}e^{-kix})(C_{vl} \cos v\theta + D_{vl} \sin v\theta)(E_{klv}J_v(kr) + F_{klv}N_v(kr)) \\
\phi_o = \sum_{v=0}^\infty \int_0^\infty d\mu_v(A_{ko}e^{kox} + B_{ko}e^{-kox})(C_{vo} \cos v\theta + D_{vo} \sin v\theta)(E_{ko}J_v(v\theta) + F_{ko}N_v(v\theta)(kr))
\end{cases}
\] (46)

with some suitable measure \( \mu_k \) determined by the boundary condition.

After gaining the general solution, we need to apply the boundary conditions to figured out the specific solutions in a specific problem.

Boundary conditions:
\[
\begin{align*}
\phi(r, \theta, z) \bigg|_{r=0} &\neq \pm \infty \\
\phi(r, \theta, z) \bigg|_{r=\pm \infty} &= az + b \\
\phi_o(r, \theta, z_0) &= C \\
\frac{\partial \phi_o(r)}{\partial r} \bigg|_{r=\partial \Omega(r)} &= \kappa \epsilon \frac{\partial \phi_l(r)}{\partial r} \bigg|_{r=\partial \Omega(r)} \\
\phi \bigg|_{\partial \Omega^-} &= \phi \bigg|_{\partial \Omega^+} \\
\mathbf{v} \cdot \nabla \phi \bigg|_{\partial \Omega^-} &= \mathbf{v} \cdot \nabla \phi \bigg|_{\partial \Omega^+}, \mathbf{v} \parallel \partial \Omega
\end{align*}
\] (47)

For the problems considered in the current paper, a boundary condition is that \( \phi < \pm \infty \) as \( r \to 0 \).

However, the existence of Neumann function does not fit this boundary condition because of the fact that \( \lim_{r \to 0} \frac{2}{\pi} J_v(kr) \ln \frac{kr}{\varepsilon} = -\infty \). Therefore, \( F_{kv} \) must be equal to zero. This way, the solution is simplified as:
\[
\phi = \sum_{v=0}^\infty \int_0^\infty d\mu_v(A_k'e^{kz} + B_k'e^{-kz})(C_v \cos v\theta + D_v \sin v\theta)J_v(kr)
\] (48)

By using the linear-potential (uniform-electric-field) condition:
\[
\lim_{r \to \pm \infty} \sum_{v=0}^\infty \int_0^\infty d\mu_v(A_k'e^{kz} + B_k'e^{-kz})(C_v \cos v\theta + D_v \sin v\theta)J_v(kr) = az + b
\] (49)

This boundary condition can determine the existence of certain coefficient, after the measure \( \mu_k \) is determined by specific boundary condition.

Another second boundary condition considered in the current paper is the constant potential on the plates. Assuming that the parallel plate locates on some surface given by \( z = z_0 \), we have
\[
\sum_{v=0}^\infty \int_0^\infty d\mu_v(A_k'e^{kz_0} + B_k'e^{-kz_0})(C_v \cos v\theta + D_v \sin v\theta)J_v(kr) = C
\] (50)

By using the orthogonality of the Bessel functions:
\[
\int_0^\infty xJ_v(kx)J_v(k'x)dx = \frac{1}{k} \delta(k' - k)
\] (51)

we have that
\[
\sum_{v=0}^\infty \int_0^\infty d\mu_k \delta(k - k')(A_k'e^{kz_0} + B_k'e^{-kz_0})(C_v \cos v\theta + D_v \sin v\theta) = C \int_0^\infty r J_v(kr')dr
\] (52)

In specific condition, the integral can be calculated and the relationship between \( A_k' \) and \( B_k' \) can be figured out.

Bring the coherent electric displacement into the equation, the relationship between the coefficient of both outside and inside equation can be got:
And, the further relationship will be gained after the boundary condition of parallel component of $E$:

$$v \cdot \nabla \left[ \sum_{v=0}^{\infty} \int_0^{\infty} d \mu_k (A_{ki} e^{kiz} + B_{ki} e^{-kiz}) (C_{vi} \cos v \theta + D_{vi} \sin v \theta) f_v(kr) \right]_{r=\partial \Omega(r)} =$$

$$\sum_{v=0}^{\infty} \int_0^{\infty} d \mu_k (A_{ki} e^{kiz} + B_{ki} e^{-kiz}) (C_{vi} \cos v \theta + D_{vi} \sin v \theta) f_v(kr) \bigg|_{r=\partial \Omega(r)}$$

Moreover, continuity of the potential because there does not exist any point charge on the interface leads to

$$\sum_{v=0}^{\infty} \int_0^{\infty} d \mu_k (A_{ki} e^{kiz} + B_{ki} e^{-kiz}) (C_{vi} \cos v \theta + D_{vi} \sin v \theta) f_v(kr) \bigg|_{r=\partial \Omega(r)} =$$

$$\sum_{v=0}^{\infty} \int_0^{\infty} d \mu_k (A_{ki} e^{kiz} + B_{ki} e^{-kiz}) (C_{vi} \cos v \theta + D_{vi} \sin v \theta) f_v(kr) \bigg|_{r=\partial \Omega(r)}$$

Then, a specific solution can be got.

The potential difference between the two plates can be calculated by:

$$\Delta \varphi = \phi_o(r, \theta, z_0) - \phi_o(r, \theta, z_0)$$

The capacitance can be figured out by

$$\left\{ \begin{array}{l}
C = \frac{q \sigma}{\Delta \varphi} \\
C_0 = \frac{\varepsilon_0 S}{D}
\end{array} \right. \quad (57)$$

Then, the relationship between $C$ and $C_0$ can be got.

5. An example of Solving the Maxwell equations

To solve the equation, several steps will be processed. This section will consider the problem of an infinite, parallel and uniform dielectric medium between the parallel capacitance and solve the Maxwell equations of this problem.

The situation is shown on figure 4.
There are no free charges in both dielectric medium and the vacuum space. Therefore
\[
\begin{align*}
\nabla^2 \phi_i &= 0, \quad b \leq z \leq b + d \\
\nabla^2 \phi_{o1} &= 0, \quad 0 < z < b \\
\nabla^2 \phi_{o2} &= 0, \quad b + d < z < D
\end{align*}
\]

(58)

Since the whole system is invariant under the translation transform along x-direction and y-direction, the potential depends only z coordinate, namely,
\[
\begin{align*}
\phi_i &= Z(z) \\
\phi_{o1} &= Z(z) \\
\phi_{o2} &= Z(z)
\end{align*}
\]

(59)

Then the Laplace equation becomes a second order ODE \( \phi''(x) = 0 \), solving which gives us
\[
\begin{align*}
\phi_i &= A_i z + B_i \\
\phi_{o1} &= A_{o1} z + B_{o1} \\
\phi_{o2} &= A_{o2} z + B_{o2}
\end{align*}
\]

(60)

By using the boundary conditions proved previously and the fact that \( \phi \) must be continuous, we have
\[
\begin{align*}
\epsilon_0 \frac{\partial \phi_i}{\partial z} |_{z=b} &= k \epsilon_0 \frac{\partial \phi_i}{\partial z} |_{z=b} \\
\epsilon_0 \frac{\partial \phi_o}{\partial z} |_{z=b+d} &= k \epsilon_0 \frac{\partial \phi_o}{\partial z} |_{z=b+d} \\
\phi_o |_{z=b} &= \phi_i |_{z=b} \\
\phi_o |_{z=b+d} &= \phi_i |_{z=b+d}
\end{align*}
\]

(61)

Without loss of generality, define the potential at \( \Sigma_2 (r, \theta, z): z = 0 \) is set to be
\[
\phi_{o1}(0) = 0
\]

(62)

Substitute the first two coherent-electric-displacement equations in (61) into the general solution. It gives us
\[
\begin{align*}
\kappa \epsilon_0 A_i &= \epsilon_0 A_{o1} \\
\kappa \epsilon_0 A_i &= \epsilon_0 A_{o2} \\
\kappa A_i &= A_{o1} = A_{o2}
\end{align*}
\]

(63)

(64)

Substituting the defined potential into the general solutions (61) leads to
\[
\phi_{o1}(0) = B_{o1} = 0
\]

(65)

Moreover, by considering the relation between the potential and the electric field, we have
\[
\begin{align*}
A_i &= -E_i = -\frac{\sigma}{\epsilon_0} \\
A_o &= -E_o = -\frac{\sigma}{\epsilon_0}
\end{align*}
\]

(66)

(67)

which means:
\[
\kappa A_i = A_{o1} = A_{o2} = -E_o
\]

(68)

\[
\frac{\sigma}{\epsilon_0} = \frac{E_o}{k} = \frac{\sigma}{k \epsilon_0}
\]

(69)

Finally, it is concluded that
\[ \begin{align*}
\phi_i &= -\frac{\sigma}{\kappa\varepsilon_0}z + B_i \\
\phi_{o1} &= -\frac{\sigma}{\varepsilon_0}z \\
\phi_{o2} &= -\frac{\sigma}{\varepsilon_0}z + B_{o2}
\end{align*} \]

(70)

The boundary condition of coherent potential (50) gives us

\[ \begin{align*}
-\frac{ab}{\kappa\varepsilon_0} + B_i &= -\frac{ab}{\varepsilon_0} \\
-\frac{\sigma(b+d)}{\kappa\varepsilon_0} + B_i &= -\frac{\sigma(b+d)}{\varepsilon_0} + B_{o2}
\end{align*} \]

(71)

Which results in \( B_i = \frac{ab(1-k)}{\kappa\varepsilon_0} \). Substitute this result back to the equation to obtain the last undetermined value:

\[ -\frac{\sigma(b+d)}{\kappa\varepsilon_0} + \frac{\sigma(b-kb)}{\kappa\varepsilon_0} = -\frac{\sigma(b+d)}{\varepsilon_0} + B_{o2} \]

(72)

that is

\[ \frac{\sigma(b+d)}{\varepsilon_0} - \frac{\sigma(b+d)}{\kappa\varepsilon_0} + \frac{\sigma(b-kb)}{\kappa\varepsilon_0} = B_{o2} \]

(73)

\[ B_{o2} = \frac{\sigma(kd-d)}{\kappa\varepsilon_0} \]

(74)

With the expression of potential, the voltage \( \Delta \varphi \) can be expressed:

\[ \Delta \varphi = -(\phi_{o1}(0) - \phi_{o2}(D)) = \frac{\sigma(kD-kd+d)}{\kappa\varepsilon_0} \]

(75)

Then, the capacitance in this situation can be expressed:

\[ C = \frac{q}{\Delta \varphi} = \frac{s\sigma \kappa \varepsilon_0}{\sigma(kD-kd+d)} = \frac{s\kappa \varepsilon_0}{kD-kd+d} \]

(76)

Since \( C_0 = \frac{\varepsilon_0 S}{D} \), the expression about the change of capacitance is:

\[ C = C_0 \frac{kD}{kD-kd+d} \]

(77)

6. Conclusion

In the current paper, the fundamental equations of electromagnetic, namely the Maxwell equations, are clarified and the equations when the emergence of the dielectric medium is discussed, which is very useful for the solving of the partial differential equations. The needed boundary conditions in the process of solving equation are all proved by basic electric principle, which means these boundary conditions can be used directly. Then, the detailed process to solve the famous mathematical equation, the Laplace equation, combined with the physical boundary condition, is discussed. The primary method is the separation of variables and solving the eigenequations. The general cylinder solution is gained and simplified by several boundary conditions and the orthogonality of Bessel function, forming a general framework of solving the Laplace equation in a dielectric medium. A specific problem about the infinite, parallel, and uniform dielectric medium between the parallel capacitance is considered as an example by the framework above. The relationship between the characteristic properties, thickness, of the dielectric medium, the spacing between plates, and the capacitance is:

\[ C = C_0 \frac{kD}{kD-kd+d} \]

Moreover, various conditions given by other shapes of dielectric mediums will be investigated as our further works.
7. Outlook: Spherical medium

The spherical medium is shown in the figure 5.

![Figure 5. The Maxwell equation will be solved in the condition of spherical medium. The plates are \( \Sigma_1(\mathbf{r}, \theta, z): z = \frac{D}{2} \) and \( \Sigma_2(\mathbf{r}, \theta, z): z = -\frac{D}{2} \) with \( q = S\sigma \) and \( -q = -S\sigma \) respectively. The dielectric medium is placed on original point with radius \( a \): \( \Sigma_3(\mathbf{r}, \theta, z): z^2 = a^2 - r^2 \) and \( \kappa\varepsilon_0 \).](image.png)

According to the Maxwell equation (35), we have

\[
\begin{align*}
\nabla^2 \phi_i(\mathbf{r}, \theta, z) &= 0, 0 < r < a \\
\nabla^2 \phi_o(\mathbf{r}, \theta, z) &= 0, a < r, a \leq \frac{D}{2} \\
\phi_i &= P(r)\Theta(\theta)Z(z) \\
\phi_o &= P(r)\Theta(\theta)Z(z)
\end{align*}
\] (78)

The symmetries of the system lead to the fact that the potential is independent \( \theta \). Therefore, \( \nu = 0 \), which leads to the following general solutions:

\[
\begin{align*}
\phi_i(\mathbf{r}, z) &= \int_0^\infty d\mu_k (A_{k'i} e^{ik'z} + B_{k'i} e^{-ik'z}) J_0(kr) \\
\phi_o(\mathbf{r}, z) &= \int_0^\infty d\mu_k (A_{ko} e^{ikz} + B_{ko} e^{-ikz}) J_0(kr)
\end{align*}
\] (80)

Now we can consider the boundary conditions, which will give us

\[
\begin{align*}
\phi_o(\mathbf{r}, z)|_{z=\pm\frac{D}{2}} &= \pm C \\
\phi_o(\mathbf{r}, z)|_{r=\infty} &= az + b \\
\phi_i(\mathbf{r}, z)|_{r=a} &= \phi_o(\mathbf{r}, z)|_{r=a} \\
\varepsilon_0 \frac{\partial \phi_o}{\partial r}|_{r=a} &= \kappa\varepsilon_0 \frac{\partial \phi_i}{\partial r}|_{r=a}
\end{align*}
\] (81)

Using the first boundary condition, according to (49) and (50):

\[
\int_0^\infty d\mu_k \left( A_{ko} e^{\frac{D}{2}k} + B_{ko} e^{\frac{-D}{2}k} \right) \frac{1}{k} \delta(k - k') = \pm C \int_0^\infty r J_0(k'r) dr
\] (82)
Assume that the measure of $k$ is all positive real number:

$$\int_{0}^{\infty} dk \left( A_{k_0} e^{\frac{D}{2}k} + B_{k_0} e^{-\frac{D}{2}k} \right) \frac{1}{k} \delta(k - k') = \pm C \int_{0}^{\infty} r J_0(k'r')dr \quad (83)$$

According to the recurrence of Bessel function, the integration in the right side can be calculated:

$$\int x^n J_{v-1}(x) dx = x^n J_v(x) + C \quad (84)$$

$$\left( A_{k} e^{\frac{D}{2}k} + B_{k} e^{-\frac{D}{2}k} \right) \frac{1}{k} = \pm \frac{C}{k^2} \int_{0}^{\infty} kr J_0(kr) dkr \quad (85)$$

Combine the (84) and (85):

$$k \left( A_{k} e^{\frac{D}{2}k} + B_{k} e^{-\frac{D}{2}k} \right) = \pm C [kr J_1(kr)]_{0}^{\infty} \quad (86)$$

The right side of the equation is equal to (the Bessel function is calculated by L’Hospital’s rule):

$$\pm C [kr J_1(kr)]_{0}^{\infty} = \pm C \left( \lim_{kr \to \infty} kr J_1(kr) - \lim_{kr \to 0} kr J_1(kr) \right) \quad (87)$$

$$\begin{cases} 
\lim_{kr \to \infty} kr J_1(kr) = \frac{(kr)^2}{2 \Gamma(1+1)} \\
\lim_{kr \to 0} kr J_1(kr) = kr \frac{2}{\sqrt{\pi kr}} \cos(kr - \frac{3\pi}{4}) 
\end{cases} \quad (88)$$

When $kr \to \infty$, $\cos(kr - \frac{3\pi}{4})$ is not defined. Therefore, the assumption about “$k$ is all positive real number” is wrong. The determination of measure $\mu_k$ and getting the specific solution are the further work.

References

[1] Jackson, J. (2004). Classical Electrodynamics (3rd ed.). China, Beijing: Higher Education Press.

[2] Zhu, X., Zhang, W., Zou, M., Fan, Y. and Di, Z. (n.d.). Bessel Function and the Application. Shaanxi University of Science and Technology, faculty of science.