Modeling the propagation of a signal through a layered nanostructure: Connections between the statistical properties of waves and random walks.

Gabriel A. Cwilich
Yeshiva University
Department of Physics
500 W 185th Street
New York, NY 10033, USA

November 13, 2018

Abstract

It is possible to discuss the propagation of an electronic current through certain layered nanostructures modeling them as a collection of random one-dimensional interfaces, through which a coherent signal can be transmitted or reflected while being scattered at each interface. We present a simple model in which a persistent random walk (the “t-r” model in 1-D) is used as a representation of the propagation of a signal in a medium with such random interfaces.

In this model all the possible paths through the system leading to transmission or reflection can be enumerated in an expansion in the number of loops described by the path. This expansion allows us to conduct a statistical analysis of the length of the paths for different geometries and boundary conditions and understand their scaling with the size of the system. By tuning the parameters of the model it is possible to interpolate smoothly between the ballistic and the diffusive regimes of propagation. An extension of this model to higher dimensions is presented. We show Monte Carlo simulations that support the theoretical results obtained.

1 Introduction

The seminal work of Anderson[1] raising the possibility that disorder can lead to non-diffusive behavior (the so called localized regime) refocused the attention of the Physics community on the problem of the propagation of waves in disordered systems. In the last two decades new theoretical ideas (like the scaling theory of localization[2], weak localization[3], universal conductance fluctuations[4] and Wigner dwelling times[5]) were advanced, and a new field (soon called Mesoscopic Physics) emerged. It reached and influenced many experimental
areas, among them electronic systems\[1\], microwaves\[7\], optics\[8\], acoustics\[9\],
geophysics\[10\], laser physics\[11\], medical physics\[12\] and atomic physics\[13\].

It has become an extremely important problem in this field to understand
what should be the signature of the propagation of a signal in the different
regimes (ballistic, diffusive, localized) since concomitant phenomena, like
absorption can complicate the interpretation of experimental results\[14\]. It is for
that reason that theoretical analyses of the characteristics of the propagation,
and in particular its statistical properties\[15\][16], are of great interests, since
those properties have become recently experimentally accessible\[6\][17].

When the inelastic scattering length in a system is large compared to its
size, the wave propagates coherently in the sense of its phase being preserved
while its direction is randomized by elastic scattering processes with the impu-
rities constituting the random medium. It is natural, then, to establish connec-
tions between the coherent propagation of a wave and the statistical theory
of random walks\[18\], where the continuum limit is known to lead to diffusive
theory\[19\]. This connection is relevant to the propagation of an electronic
current through certain layered nanostructures, since they can be modeled as
a collection of random one-dimensional interfaces, through which a coherent
signal can be transmitted or reflected while being scattered at each interface.

The connection between the effect of the disorder of the propagating
medium and the statistical randomness of the mathematical description of the diffusion
has not been satisfactorily clarified yet. One aspect that did not receive enough
attention in this approach is the fact that, since the scattering with the impu-
rities is not isotropic and the cross section is normally enhanced in the forward
direction, the statistical jumps are not independent from each other. In other
words the probability distribution of each step of these persistent random walks
(PRW) is dependent on the previous step\[20\]. The PRW are more difficult
to study than the standard random walks, where the probability distribution
of each step is independent of what happened in the previous steps. This can
be seen from the fact that, for example, while a standard random walk can be
easily mapped into a chain of non interacting spins (in the presence of a
magnetic field, if the random walk is biased), the same mapping for the PRW
leads to a chain of nearest-neighbor interacting spins (the full Ising model for
the case of a 1-D PRW)\[16\]. In particular in this work we will explore the
connection between the propagation of a wave and the 1-D version of the PRW
called the “t-r” model\[21\], in which a particle moves with probability \(t\) in the
same direction as in the previous step and with probability \(r\) reversing direction
(\(t + r = 1\)). This type of model is more suited than the standard random walk
model to explore the quasiballistic regime, which becomes more important in
transport phenomena in nanostructures.

\section{The model}

We will consider a model of the propagation of a wave first introduced for nu-
merical purposes by Edrei, Kaveh and Shapiro\[22\], and later applied extensively
by Vanneste et al [23] and Sebbah [24]. Each site is represented by a scattering matrix \( S \) of dimension \( 2D \times 2D \), \( D \) being the dimensionality, connecting the \( 2D \) incoming amplitudes at time \( t \), with the \( 2D \) outgoing ones at time \( t + 1 \). These amplitudes can be regarded as residing in the bonds (2 joining each site to its nearest neighbor, one in each direction). These bonds represent free propagation between the sites, and their phase can be included in the \( S \) matrix. The wave function at time \( t \) is, thus, determined by the complex numbers associated with each of the bonds at that time. Numerically this has the advantage of letting the stationary wave being built step by step in time, from an initial input without solving a huge diagonalization problem. The matrices \( S \) representing each scatterer are unitary and symmetric reflecting, respectively, the energy conservation and time reversal symmetry inherent to the problem. If we adopt the simplifying assumption that the scatterers themselves are symmetric, these matrices simplify even further, since the following relations can be easily obtained for their elements

\[
\begin{align*}
    r^2 + t^2 + 2(D - 1)d^2 &= 1 \\
    rt \cos(\varphi_r - \varphi_t) + (D - 1)d^2 &= 0 \\
    rd \cos(\varphi_r - \varphi_d) + t d \cos(\varphi_t - \varphi_d) + (D - 2)d^2 &= 0
\end{align*}
\]

where \( \{r, \varphi_r\}, \{t, \varphi_t\} \) and \( \{d, \varphi_d\} \) are the amplitudes and phases of the elements for reflection, transmission, and turning (in 1D, \( d = 0 \)). The matrices \( S \) have, thus, only two (in 1D) or three (in \( D > 1 \)) independent parameters. For example, in 1D the scattering matrix representing one scatterer can be parameterized as:

\[
\begin{bmatrix}
    r & I & t \\
    I & t & r
\end{bmatrix}
\exp[I \varphi] \quad (r^2 + t^2) = 1
\] (2)

In this simple version of the model the distances are measured in units of the mean free path (the distance between scattering elements) and the phase velocity of the wave is one. The effect of the disorder in this model has been considered both analytically [25] and numerically [26], by assuming a distribution of values for the variable \( r \), linked to the reflectivity of the scatterers. Alternatively, one can consider the effect of disorder in the variable \( \varphi \), more closely linked to the distance between scatterers [27].

We will be interested in considering a signal incident from the left on a system composed of \( N \) such scatterers, and in monitoring what is reflected or transmitted through it at different times, i.e. the output to the left of scatterer 1 and to the right of scatterer \( N \), respectively. It is important to observe that while calculating the response of the system to an applied pulse implies adding all the paths of a certain length (that will interfere by arriving simultaneously to the boundary), the problem of determining the response to a continuous wave is simpler, since in the stationary state it is sufficient to add all the transmitting or reflecting paths as long as the phase factors associated with the propagation through each scatterer, \( \exp[I k] \), are included. \( k \) represents here the wavevector of the incident wave.
3 Loop expansion

The amplitude factor for any particular path reaching the boundaries of the system can be easily written, and we can sum all the possible paths by performing a loop expansion. For example, for a system of two scatterers the amplitude associated with the simple paths illustrated in figure 1(a) are, respectively, $(I t \exp[I k])^2$, $(I t \exp[I k])^2(r \exp[I k])^2$ and $(I t \exp[I k])^2(r \exp[I k])^4$; the total amplitude for transmission through the two scatterers, when all possible paths are considered, becomes then:

$$t_2 = \frac{(I t \exp[I k])^2}{1 - (r \exp[I k])^2} \equiv \frac{(I t \exp[I k])^2}{1 - l_1}$$ (3)

For the three scatterers in figure 1(b), the amplitude factors associated with the paths shown are $(I t \exp[I k])^3$, $(I t \exp[I k])^3(r \exp[I k])^2(I t \exp[I k])^2$ and $(I t \exp[I k])^3(r \exp[I k])^2(I t \exp[I k])^2(r \exp[I k])^4$, respectively. The “bare” loop of length 2 has an amplitude factor of $l_2 \equiv (r \exp[I k])^2(I t \exp[I k])^2$ and when fully “dressed” by all the possible loops of length 1 becomes:

$$\tilde{l}_2 = \frac{(r \exp[I k])^2(I t \exp[I k])^2}{(1 - (r \exp[I k])^2)^2} \equiv \frac{l_2}{(1 - l_1)^2}$$

We see that the factor associated with a “bare” loop of length $n$ is simply $l_n \equiv (r \exp[I k])^2(I t \exp[I k])^{2(n-1)}$. In terms of it we can obtain a recursive
expression for the general “dressed” loop of order \( n \)

\[
\tilde{l}_1 = l_1 \quad ; \quad \tilde{l}_2 = \frac{l_2}{(1 - \tilde{l}_1)^2} \quad ; \quad \tilde{l}_3 = \frac{l_3}{(1 - \tilde{l}_1)^4(1 - \tilde{l}_2)^2}
\]

\[
\tilde{l}_n = \frac{l_n}{(1 - \tilde{l}_1)^2(n-1)(1 - \tilde{l}_2)^2(n-2)\ldots(1 - \tilde{l}_{n-1})^2}
\]  

(4)

It is possible now to add all the paths through a chain of \( N \) scatterers, the amplitude of the transmission through the system, and express it in terms of the loops:

\[
t_1 = I t \exp[It] \quad ; \quad t_2 = (I t \exp[It])^2 \quad ; \quad t_3 = (I t \exp[It])^3 \quad ; \quad t_n = \frac{(I t \exp[It])^n}{(1 - \tilde{l}_1)(n-1)(1 - \tilde{l}_2)(n-2)\ldots(1 - \tilde{l}_{n-1})}
\]  

(5)

As expected, \( \tilde{l}_n \) is proportional to \( t_n^2 \). A very similar expansion yields, for the reflection amplitudes:

\[
r_1 = r \exp[It] \quad ; \quad r_2 = r_1 + (I t \exp[It])^2 (r \exp[It]) \quad ; \quad r_3 = r_2 + (I t \exp[It])^4 (r \exp[It]) \quad ; \quad r_n = r_{n-1} + \frac{(I t \exp[It])^2(n-1)(r \exp[It])}{(1 - \tilde{l}_1)^2(n-3)(1 - \tilde{l}_2)^2(n-5)\ldots(1 - \tilde{l}_{n-2})^2(1 - \tilde{l}_{n-1})}
\]  

(6)

The expression for the amplitude of reflection through the system admits a clear interpretation in terms of a sum of all the paths that penetrate up to a certain depth in the system: \( r_1 \) corresponds to the paths that do not enter into the system at all, \( r_2 - r_1 \) represents the paths that penetrate up to a depth 1, \( r_3 - r_2 \) are the paths that penetrate up to a depth 2, etc.

The connection of this expansion for the transmission and reflection amplitudes of the propagating wave with the PRW is provided by the fact that we can write exactly the same expansion in loops in the “t-r” model, for all the possible paths of a random walker leading to transmission or reflection. Here transmission means the random walker arriving to a point further to the right of position \( N \), and reflection the walker reaching a point to the left of position 1. In particular, if we ignore the phase factors \( \exp[It] \) and the imaginary unit in
front of $t$, the amplitude factors discussed above become the probabilities of the corresponding PRW paths, and Equations (3), (4) and (5) describe the “dressed” loops, probability of transmission and probability of reflection in the case of a PRW as well, since for a random walk the total probability of transmission (reflection) is simply the sum of the probabilities of each of the paths leading to transmission (reflection). Of course in each case, for a PRW the connection between $t$ and $r$ is not provided by (2), but by the simple expression $t + r = 1$.

It is easy to show now that the probability of transmission for a PRW becomes, then:

$$t_n = \frac{t}{1 + (N - 1)r}$$

and the probability of reflection becomes simply $r_n = 1 - t_n$. Identifying $N$ with the length of the system $L$, and taking into account that in this model $\langle \cos(\vartheta) \rangle = t - r$, we can see that in the limit of large $N$ the transmission tends to the diffusive limit $t(L) = (t^* / L)$.

In the case of a wave, the transmission or reflection probabilities are given by the modulus square of the expressions calculated in (5) and (6). This implies sums of the type:

$$\left| \sum A_i \right|^2 = \sum |A_i|^2 + \sum_{i \neq j} A_i A_j^*$$

It is easy to see that, when adding the transmitting paths’ amplitudes, if we consider the contribution of only the first term in the expression above (neglecting the correlations between the paths) we will reobtain (7), where the roles of $t$ and $r$ are played by the transmission coefficient $t^2 \equiv T$, and the reflection coefficient $r^2 \equiv R$. It is precisely the overlap between paths (the second term) that leads to corrections to the diffusion theory results.

## 4 Statistics of the paths

In the case of the PRW, knowing the probabilities $p_i$ and the lengths $l_i$ of all the paths both in transmission and reflection, we are now in a position to study some of their statistical properties. Defining a generating function $F_{tr}(s) = \sum p_i s^{l_i}$, where we sum over all the transmitted paths, and a similar definition for $F_{ref}(s)$ where the sum is over all reflected paths, we can calculate the average length of the paths as

$$\langle l \rangle = \lim_{s \to 1} \frac{F'(s)}{F(s)},$$

obtaining for the average length:

$$\langle l \rangle_{tr} = N \frac{1 + (N - 2)r + \frac{1}{2}(N - 2)(N - 1)r^2}{[1 + (N - 1)r][1 - r]}$$

(9)
In the diffusive limit discussed above, these expressions reproduce the well-known results of diffusive theory \[28\]: 
\[
\langle l \rangle_{tr} \rightarrow \frac{L^2}{l^*}.
\]
Analogously, \( \langle l \rangle_{ref} \rightarrow (2/3)L \), independent of the mean free path. The expressions (9) and (10) are exact, within the model, and remain valid in the diffusive as well as the ballistic limit \( r \rightarrow 0 \), and can be used to interpolate between them. This can be of interest in any situation in which the ballistic paths play an important role as in medical imaging.

In a similar way one can express in terms of \( F(s) \) the variance of the path length distribution, in transmission or reflection

\[
\langle (l - \langle l \rangle)^2 \rangle = \lim_{s \rightarrow 1} \frac{F''(s) + F'(s)}{F(s)} - \left[ \frac{F'(s)}{F(s)} \right]^2
\]

obtaining the expressions:

\[
\text{var}_{tr} = \frac{N(N^2 - 1)r^2}{3} \frac{[N + \frac{2}{3}(N - 2)(2N - 1)r + \frac{2}{15}(N - 1)(N - 2)(N - 3)r^2]}{(1 - r)^2[1 + (N - 1)r]^2}
\]

\[
\text{var}_{ref} = \frac{(N^2 - 1)P(r)}{3(1 - r)^2[1 + (N - 1)r]^2}
\]

\( P(r) \) is \( 1 + (2N - 3)r + \frac{2}{15}(14N^2 - 30N + 19)r^2 + \frac{4}{105}(2N - 1)(N - 1)(N - 2)r^3 \) here. These exact expressions, which again allow us to extrapolate between the diffusive and ballistic regimes, in the diffusive limit tend to \( (L^2/l^*)^2 \) and \( (L^3/l^*) \) respectively. In the case of transmission the variance scales with the size of the system as the square of the average length (non-self averaging property) while in the case of reflection it scales even faster than it, exhibiting the well-known noisy character of reflection \[28\].

Figures 2, 3 and 4 exhibit the results of numerical simulations of a persistent random walker on a system of size \( N = 100 \). The transmission and reflection probabilities, the average length of the transmitted and reflected paths and the variance for the path lengths in transmission and reflection are represented as a function of the transmission parameter \( t \). In all cases the agreement with the expression 7 and (9) to (12) is excellent.

An interesting problem is how to define the probabilistic weights to perform a similar statistical analysis of the length of the paths in the case of a wave, since the amplitudes are complex numbers with a phase, and the different paths interfere with each other. It makes sense to use the “probabilities” (in the sense of the moduli of the complex amplitudes squared) as those probabilistic
Figure 2: Fraction of transmitted paths (•) and of reflected paths (o) for different values of the transmission parameter $t$ in a numerical simulation of up to 210,000 random walkers on a system of size $N = 100$. The full lines correspond to equation (7) for $t_{100}$, the probability of transmission, and $r_{100}$ the probability of reflection.

Figure 3: Average length of (a) the transmitted paths and (b) the reflected paths for the same simulation. The full lines correspond to the equations (9) and (10) for $N = 100$. 
Figure 4: Standard deviation of the distribution of the length of (a) the transmitted paths and (b) the reflected paths for the same simulation. The full lines correspond to the square root of equation (11) and equation (12) for \( N = 100 \).

weights. As the discussion at the end of section 3 shows, those weights describe the diffusive behavior of the system. Adopting this ansatz we can define again the generating function as in the case of the PRW, and we can reobtain all the results in expressions (9) to (12), again replacing \( t \) and \( r \) by the transmission coefficient \( T \) and the reflection coefficient \( R \), respectively.

5 Higher dimensions

All this analysis and the expansion in terms of loops can be easily generalized to a system in a higher number of dimensions. To fix ideas let us consider, instead of a row of \( N \) scatterers, a strip of \( M \) rows of \( N \) scatterers each, where the wave (or the random walker) is incident from left. For this system, the amplitudes of transmission or reflection through one column become (instead of \( t \) and \( r \)) the \( M \times M \) matrices \( \hat{T}_1 \) and \( \hat{R}_1 \), whose element \( (\hat{T}_1)_{ij} \) is the amplitude for a walker incident from the left on row \( i \) to exit to the right on row \( j \), and an analogous definition for \( (\hat{R}_1)_{ij} \). These amplitudes are obtained by adding all the possible paths that lead from the input \( i \) to the output \( j \) and are confined to one single column of the medium. It is, then, not difficult to prove that the expansion in loops remains valid as long as we replace in it \( t \) and \( r \) by these matrices, and all the products are understood as products of matrices.

Similarly, the transmission \( \hat{T}_N \) and the reflection \( \hat{R}_N \) through the strip of length \( N \) are obtained through the same replacement from the expressions (11) and (12) respectively.

If the signal impinging on the system from the left is represented by an \( M \)-component vector \( \mathbf{I} \), the output vector \( \mathbf{O} \) simply becomes, for the case of
Figure 5: Transmission through a system of length $N=8$ as a function of $t$, for the cases of a 1-D chain, a strip of width $M = 2$ (reflecting and absorbing boundary conditions) and for a 2-D periodic system. The two-dimensional systems correspond to the case $d = r$. The inset illustrates the “snaking” effect for $t \to 0$.

Transmission

$$O_j = \sum_i I_i (\hat{T}_N)_{ij}, \quad (13)$$

Since in this model the matrices $\hat{T}_1$ and $\hat{R}_1$ have exactly the same symmetries, they will have the same eigenvectors; those will also be the eigenvectors of the complicated matrices $\hat{T}_N$ and $\hat{R}_N$, since they are expressed (through the loop expansion) as products of $\hat{T}_1$ and $\hat{R}_1$. The elements of the matrix $\hat{T}_N$ involved in (13) can be evaluated in terms of those eigenvectors and the eigenvalues of $\hat{T}_1$ and $\hat{R}_1$ through standard algebraic techniques. The problem is, then, essentially reduced to solving the eigenvalue problem of an $M \times M$ matrix.

To conclude we will illustrate the treatment of the two dimensional problem by considering the particularly simple case of periodic boundary conditions in the lateral sides of the system (rows 1 and $M$). We will calculate the total transmission through a strip of length $N$. Then,

$$T_N \equiv \sum_j O_j = \sum_{i,\alpha,j} I_i \lambda_N^\alpha E_i^\alpha E_j^\alpha \quad (14)$$

where the $\lambda_N^\alpha$ stand for the eigenvalues of the matrix $\hat{T}_N$, and the $E^\alpha$ for the corresponding eigenvectors of the matrices $\hat{T}_1$ and $\hat{R}_1$. From the symmetry of
the problem readily follows that one of the eigenvectors is $E_o = \frac{1}{\sqrt{M}}(1, 1, 1, \ldots, 1)$ and all the others cancel when summed over $j$ in (14) by orthogonality, leading to $T_N = \lambda^o_N \left( \sum_i I_i \right)$. The eigenvalue $\lambda^o_N$ can be simply obtained by replacing in the expansion (15) or its equivalent for a PRW the coefficients $t$ and $r$ by the eigenvalues $\lambda^o_{1,T}$ and $\lambda^o_{1,R}$ (associated with $E_o$) of the matrices $\widehat{T}_1$ and $\widehat{R}_1$. Those eigenvalues are simply $\sum_j (\widehat{T}_1)_{ij}$ for any row $i$, and a similar expression for $\widehat{R}_1$. For the case of a PRW \[29\], (where the walker has a probability $t$ of moving forward, a probability $r$ of moving back, and a probability $d$ of turning to either side, with $t + r + 2d = 1$), these eigenvalues can be obtained writing all the possible paths starting from position $i$ that lead to transmission (reflection) through any row of a 1-column system:

$$\sum_j (T_1)_{ij} = t + 2d^2 \left( \frac{1}{1-t} \right) (1 + \frac{r}{1-t} + \left( \frac{r}{1-t} \right)^2 + \ldots ) \equiv t + \frac{2d^2}{1-r-t}$$

The expansion in this expression corresponds to all paths with no reflection in the vertical direction, one reflection in the vertical direction, etc.). This is simply $\lambda^o_{1,T} = t + d$. Analogously we obtain $\lambda^o_{1,R} = r + d$. The total transmission for a system of length $N$ becomes, then:

$$T_N = \left( \sum_i I_i \right) \frac{(t + d)}{1 + (N - 1)(r + d)}$$

Figure 5 illustrates the transmission through a strip of length 8, for periodic, absorbing and reflecting boundary conditions, and for a 1-D chain of the same length, in the specific case $r = d$. The inset illustrates the effect of “snaking”, the existence of a non-zero total transmission even in the case of a zero forward transmission $t$ at the level of an individual scattering.

**Acknowledgement** 1 Charles Wizenfeld assisted with the numerical simulations. I benefited from helpful discussions with Juanjo Saenz. The hospitality of the Departamento de Fisica de la Materia Condensada, where the last part of this work was written is gratefully acknowledged. Partially supported by the Summer Faculty Fellowship of the Office of the Vice President of Academic Affairs of Yeshiva University.

**References**

[1] Anderson P W, *Phys. Rev.* **109**, 1492 (1958)

[2] Abrahams E, Anderson P W, Licciardello D C and Ramakrishnan T V, *Phys. Rev. Lett.* **16**, 984 (1979)

[3] Van Albada M P and Lagendijk A, (1985), *Phys. Rev. Lett.* **55**, 2692.
[4] Lee P A, in STATPHYS 16, (H. E. Stanley, ed.), North Holland (1986)
[5] Cwilich G and Fu Y, Phys. Rev. B46, 12015 (1992), Van Albada M P ,Van Tiggelen B et al, Phys. Rev. Lett. 66, 3132 (1991)
[6] Stone D, Phys. Rev. Lett 54, 2692 (1985), Webb R A ,Washburn S et al, Phys. Rev. Lett. 54, 2696 (1985)
[7] Stoytchev M and Genack A Z, Phys. Rev. Lett 79, 309 (1997), Garcia N and Genack A Z , Phys. Rev. Lett 63, 1678 (1989)
[8] Soukulis C in Diffuse Waves in Complex Media (J. P. Fouque ed.), 93-107, Kluwer (1998), Yablonovitch E and Leung K M, Nature 351, 278 (1991)
[9] Fink M, Time Reversed Acoustics, Phys. Today 50, 34, March 1997.
[10] Campillo M et al, in Diffuse Waves in Complex Media (J. P. Fouque ed.), 383-404, Kluwer (1998)
[11] Wiersma D, Van Albada M P and Lagendijk A , Nature 373, 203 (1995), Beenakker C, Phys. Rev. Lett. 81, 1829 (1998)
[12] Wang L et al., Science 253, 769 (1991), and also Appl. Optics 32, 5043 (1993).
[13] Kaiser R, in Diffuse Waves in Complex Media (J. P. Fouque ed.), 249-288, Kluwer (1998)
[14] Wiersma D, Bartolini P and Lagendijk A, Nature 390, 691 (1997)
[15] Kogan E and Kaveh M, Phys. Rev. B48, 9404 (1993) and B52, 3813 (1995), Kogan E et al. Physica A200, 469 (1993).
[16] Cwilich G in OSA Proceedings on Advances in Optical Imaging and Photon Migration (R. Alfano, ed.),7-11, Opt. Soc. Of America (1994)
[17] Sebbah P, Legrand O and Genack A Z, Phys. Rev. E59, 2406 (1999)
[18] Feller W, An Introd. to Probability Theory and its Applications, John Wiley (1968)
[19] Weiss G H and Rubin R J, Adv. in Chemical Physics 52, 363 (1983).
[20] Weiss G H, J. Stat. Phys. 15, 157 (1976)
[21] Martines A S, Doctoral Thesis, Université Joseph Fourier (1995)
[22] Edrei I, Kaveh M and Shapiro B, Phys. Rev. Lett. 62, 2120 (1989)
[23] Vanneste C P, Sebbah P and Sornette D, Europhysics Letters 17, 715 (1992)
[24] Sebbah P, Doctoral Thesis, Université de Nice-Sophia Antipolis (1993)
[25] Abrahams E and Stephen M J, *J. Phys.* C13, L377 (1980)

[26] Andereck B S, Abrahams E, *J. Phys.* C13, L383, 1980

[27] Unpublished

[28] Pine D J, Weitz D A, Maret G, Wolf P E, Herbolzheimer E and Chaikin P M in *Scattering and Localization of classical waves in random media* (Ping Sheng ed.),312-372, World Scientific, (1990).

[29] The analogous results for the amplitudes of a wave will be published elsewhere.
