DP-SEP: Differentially private stochastic expectation propagation

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Abstract

We are interested in privatizing an approximate posterior inference algorithm called Expectation Propagation (EP). EP approximates the posterior by iteratively refining approximations to the local likelihoods, and is known to provide better posterior uncertainties than those by variational inference. However, using EP for large-scale datasets imposes a challenge in terms of memory requirements as it needs to maintain each of the local approximates in memory. To overcome this problem, stochastic expectation propagation (SEP) was proposed, which only considers a unique local factor that captures the average effect of each likelihood term to the posterior and refines it in a way analogous to EP. Therefore in this work, we focus on developing a differentially private stochastic expectation propagation (DP-SEP) algorithm, which outputs differentially private natural parameters of the exponential-family posteriors in each step of SEP.

1 Introduction

Bayesian learning provides a level of certainty about the parameters of a model, which then provides reasoning about how certain the model is about its output through the posterior predictive distribution. Variational inference (VI) \cite{2,5} is a popular Bayesian inference method that refines a global approximation of the posterior and scales well to large applications. However, VI often underestimates the variance of the posterior and poor performance for models with non-smooth likelihoods \cite{3,12}.

In contrast, expectation Propagation (EP) is known to provide better posterior uncertainties than VI \cite{8,9}. EP constructs the posterior approximation by iterating local computations that refine approximating factors which capture each likelihood contribution to the true posterior. With large datasets, however, using EP imposes challenges as maintaining each of the local approximates in memory is costly. Stochastic Expectation Propagation (SEP) \cite{7} overcomes this challenge by iteratively refining a single approximated factor that is repeated as many times as number of datapoints that are in the dataset. This makes the algorithm suitable for large-scale datasets as it only has to maintain in memory the global approximation in contrast to EP, that needs to keep in memory each of the approximating factors.

More importantly, in terms of applying DP, the SEP algorithm is more suitable. To apply DP to EP, a difficulty arises in sensitivity analysis: at each step of the algorithm, the approximating factor that is being refined depends on the rest, and so the sensitivity of the approximated posterior depends not only the particular factor that is being refined but also the rest of the factors that contribute to the posterior. On the other hand, in every SEP step it considers a single approximating factor at a time while all the other factors are fixed to the initial value. Hence, the sensitivity analysis of the approximate posterior becomes straightforward as we demonstrate in the following sections.

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2 Background

We begin describing relevant background information.

Algorithm 1 EP
1: Choose a factor $f_n$ to refine
2: Compute the cavity distribution
   $q_{-n}(\theta) \propto q(\theta)/f_n(\theta)$
3: compute tilted distribution
   $\tilde{p}_n(\theta) \propto p(x_n|\theta)q_{-n}(\theta)$
4: moment matching
   $f_n(\theta) \leftarrow \text{proj}[\tilde{p}_n(\theta)] / q_{-n}(\theta)$
5: inclusion
   $q(\theta) \leftarrow q_{-n}(\theta)f_n(\theta)$

Algorithm 2 SEP
1: Choose a datapoint $x_n \sim D$
2: Compute the cavity distribution
   $q_{-1}(\theta) \propto q(\theta)/f(\theta)$
3: compute the tilted distribution
   $\tilde{p}_n(\theta) \propto p(x_n|\theta)q_{-1}(\theta)$
4: moment matching
   $f_n(\theta) \leftarrow \text{proj}[\tilde{p}_n(\theta)] / q_{-1}(\theta)$
5: implicit update
   $f(\theta) \leftarrow f(\theta)1 + \frac{1}{N} f_n(\theta)$
6: inclusion
   $q(\theta) \leftarrow q_{-1}(\theta)f(\theta)$

Expectation propagation (EP) and Stochastic EP (SEP)
Consider a dataset $D = \{x_n\}_{n=1}^N$ containing $N$ i.i.d samples and the parametric probabilistic model given by the prior $p_0(\theta)$ of the unknown parameters $\theta$ and the likelihood $p(x|\theta)$. The true (intractable) posterior in Bayesian inference can be computed by:

$$p(\theta|D) \propto p_0(\theta) \prod_{n=1}^N p(x_n|\theta) \approx q(\theta) \propto p_0(\theta) \prod_{n=1}^N f_n(\theta).$$

(1)

EP is an iterative algorithm that produces a simpler and tractable approximating posterior distribution, $q(\theta)$, by refining the approximating factors $f_n(\theta)$. The process that EP follows to refine iteratively these factors can be depicted in four steps. As shown in Algorithm 1, we initialize the approximating factors and form the cavity distribution $q_{-n}(\theta)$ by taking the n-th approximating factor out from the approximated posterior (i.e. $q_{-n}(\theta) \propto q(\theta)/f_n(\theta)$). On second step, the tilted distribution, $\tilde{p}_n(\theta)$, is computed by including the corresponding likelihood term to the cavity distribution: $\tilde{p}_n(\theta) \propto q_{-n}(\theta)p(x_n|\theta)$. The third step updates the approximating factor by minimizing the Kullback-Leibler (KL) divergence between the tilted distribution and $q_{n}(\theta)f_n(\theta)$ in order to capture the likelihood term contribution to the posterior. When the approximating distribution belongs to the exponential family, the KL minimization is reduced to moment matching step, denoted by: $f_n(\theta) \leftarrow \text{proj}[\tilde{p}_n(\theta)] / q_{-1}(\theta)$.

A major difference between EP and SEP is that SEP constructs an approximate posterior, $q(\theta)$, by iteratively refining $N$ copies of an unique factor, $f(\theta)$, such that $\prod_{n=1}^N p(x_n|\theta) \approx f(\theta)^N$. The intuition behind SEP is that the approximating factor captures the average effect of a likelihood term on the posterior distribution since updates are performed analogously to EP. Similar to EP, as shown in Algorithm 2, SEP algorithm starts by initializing the approximating factor and computing the cavity distribution by removing one copy of the approximating factor from the approximate posterior: $q_{-1}(\theta) \propto q_{-1}(\theta)/f(\theta)$. Then, it calculates the tilted distribution in the same way as EP by $\tilde{p}(\theta) \propto q_{-1}(\theta)p(x_n|\theta)$. In the third step, SEP minimizes the KL-divergence between the tilted distribution and $q_{-1}(\theta)f_n(\theta)$ to find an intermediate factor approximate, $f_n(\theta)$.  

Differential privacy
Given privacy parameters $\epsilon \geq 0, \delta \geq 0$ randomized algorithm, $\mathcal{M}$, is said to be $(\epsilon, \delta)$-DP [14] if for all possible sets of mechanism’s outputs $S$ and for all neighboring datasets $D, D'$ differing in an only single entry (d($D, D') \leq 1$), the following inequality holds: $\Pr[\mathcal{M}(D) \in S] \leq e^\epsilon \cdot \Pr[\mathcal{M}(D') \in S] + \delta$. The definition states that the amount of information revealed by a randomized algorithm about any individual’s participation is limited. In this work we will use the Gaussian mechanism to privatize the natural parameters of the posterior distribution and use [13] for computing the cumulative privacy loss.
We reproduced Mixture of Gaussian for clustering problem presented in the SEP paper and tested it.

The following propositions state that (1) the sensitivity of the natural parameters is \( \epsilon \)
\[ \frac{\gamma C}{\sqrt{N}} \]
and (2) the cluster identity variables are sampled from a categorial uniform distribution \( p_J = \frac{1}{4} \) on DP-SEP. The problem considers a synthetic dataset containing \( N = 1000 \) datapoints drawn from \( J = 4 \) Gaussians with the following assumptions: each mean is sampled from a Gaussian distribution \( p(\mu_j) = N(\mu; \mathbf{m}, I) \), each mixture component is isotropic \( p(x|\mathbf{h}_0) = N(x; \mu_{h_0}, 0.5^2 I) \) and the cluster identity variables are sampled from a categorical uniform distribution \( p(h_0 = j) = \frac{1}{4} \). Figure[1] visualizes the posterior means after 100 iterations for the true labels. EP, SEP and DP-SEP at different values of \( \epsilon \) with clipping norm set to \( C = 1 \). For SEP and DP-SEP we fixed the damping value, \( \gamma = 1 \), i.e., \( \gamma/N = 1/1000 \). The figure shows that for a restrictive privacy regime \( \epsilon = 1 \), the clusters
Figure 1: Mean posterior approximation for the Gaussian components (black rings indicate 98% confidence level). The top row shows the true labels (left), EP (middle) and SEP (right). The bottom row shows the labels for DP-SEP with $\delta = 10^{-5}$ and $\epsilon = 1, 5.5, 50$.

obtained by DP-SEP are overlapping. As we increase the privacy loss, the performance of DP-SEP gets closer to the non-private ones (SEP and EP) and the ground truth. The posterior from DP-SEP exhibits a higher uncertainty than the other non-private methods due to the added noise to the mean and covariance during training.

In Table 4 we also provide a quantitative analysis of the results above in terms of F-norm of the difference between the ground truth parameters and the estimated parameters by each method. In addition, we use KL divergence between the ground truth posterior and the posterior obtained by each method. Under the mixture of Gaussians model, there is no closed form KL divergence. We instead use a proxy to the KL divergence in the following way: We first pair two Gaussians in terms of their mean locations (i.e., from a given Gaussian in ground truth, which estimated Gaussian is closest in terms of the mean estimate), and the compute the KL divergence between the paired Gaussians and averaged over those KL divergences across four paired Gaussians.

| Method          | F-norm | KL-divergence (proxy) |
|-----------------|--------|-----------------------|
| SEP ($\epsilon = \infty$) | 0.0007 | 4.3524                |
| DP-SEP ($\epsilon = 50$)    | 0.5650 | 516.8689              |
| DP-SEP ($\epsilon = 5.5$)   | 1.6237 | 955.0533              |
| DP-SEP ($\epsilon = 1$)     | 4.2722 | 4162.9041             |

5 Conclusions

In this work we have introduced DP-SEP, the first method for performing differentially private stochastic expectation propagation. The current experimental results follow the sensitivity we derived, i.e., the noise scales with the number of datapoints in the dataset. We will continue testing this algorithm to other models such as linear and logistic regression. A meaningful step to pursue is a theoretical understanding of the effect of noise added for privacy to the accuracy of the posterior estimates under these models.

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A Appendix

A.1 Proof of Prop. 1

Proof. Consider two neighboring databases, \( \mathcal{D}, \mathcal{D}' \) differing only in the \( x_n, n \in \{1, \cdots, N\} \) datapoint with respective updated approximated posterior parameters \( \theta_{\text{new}} \) and \( \theta'_{\text{new}} \).

\[
\Delta_2 \theta_{\text{new}} = \max_{D, D'} \| \theta_{\text{new}} - \theta'_{\text{new}} \|_2
\]

\[
= \max_{D, D'} \left\| \left( \frac{\gamma}{N} \theta_f + \left( N - \frac{\gamma}{N} \right) \theta_f + \theta_0 \right) - \left( \frac{\gamma}{N} \theta' f_n + \left( N - \frac{\gamma}{N} \right) \theta_f + \theta_0 \right) \right\|_2
\]

\[
= \frac{\gamma}{N} \max_{D, D'} \| \theta_{fN} - \theta'_{fN} \|_2 \text{ by the triangle inequality:}
\]

\[
\leq \frac{\gamma}{N} \max_{D, D'} \left( \| \theta_{fN} \|_2 + \| - \theta'_{fN} \|_2 \right) = \frac{\gamma}{N} \max_{D, D'} \left( \| \theta_{fN} \|_2 + \| \theta'_{fN} \|_2 \right) \leq \frac{\gamma}{N} (C + C') = \frac{2C\gamma}{N}
\]

A.2 Proof of Prop. 2

Proof. Due to the Gaussian mechanism, the natural parameters after each perturbation are DP. By composing these with the subsampled RDP composition \[13\], the final natural parameters are \((\epsilon, \delta)\)-DP, where the exact relationship between \((\epsilon, \delta), T\) (how many repetitions SEP runs), \(N\) (how many datapoints a dataset has), and \(\sigma\) (the privacy parameter) follows the analysis of \[13\].