Primitive Vassiliev Invariants and Factorization in Chern-Simons Perturbation Theory

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ABSTRACT

The general structure of the perturbative expansion of the vacuum expectation value of a Wilson line operator in Chern-Simons gauge field theory is analyzed. The expansion is organized according to the independent group structures that appear at each order. It is shown that the analysis is greatly simplified if the group factors are chosen in a certain way that we call canonical. This enables us to show that the logarithm of a polynomial knot invariant can be written in terms of primitive Vassiliev invariants only.

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1 Introduction

Vassiliev invariants, or numerical invariants of finite type, are a set of knot invariants first proposed in [1] to classify knot types. To each knot corresponds an infinite sequence of rational numbers which have to satisfy some consistency conditions in order to be knot class invariants. This infinite sequence is divided into finite subsequences, each one characterized by its order, which form vector spaces. The number of independent elements in each finite subsequence is called the dimension of the space of Vassiliev invariants at that order.

Apart from the original definition [1] of these invariants, there are other approaches to the subject. Since their formulation in terms of inductive relations for singular knots [4, 5] and of their relation to knot invariants based on quantum groups or in Chern-Simons gauge theory [3, 4, 5, 6, 7, 8], several works have been performed to analyze Vassiliev invariants in both frameworks [3, 4, 5, 6, 7, 8, 10]. In [8, 10] it was shown that Vassiliev invariants can be understood in terms of representations of chord diagrams without isolated chords modulo the so called 4T relations (weight systems), and that using semi-simple Lie algebras weight systems can be constructed. It was also shown in [10], using Kontsevitch’s representation for Vassiliev invariants [14] that the space of weight systems is the same as the space of Vassiliev invariants. In [11] it was argued that these representations are precisely the ones underlying quantum-group or Chern-Simons invariants.

We observed in [12] that the generalization of the integral or geometrical knot invariant first proposed in [13] and further analyzed in [4], as well as the invariant itself, are Vassiliev invariants. In [12] we proposed an organization of those geometrical invariants and we described a procedure for their calculation from known polynomial knot invariants. This procedure was applied to obtaining Vassiliev knot invariants up to order six for all prime knots up to six crossings. These geometrical invariants have also been studied by Bott and Taubes [16] using a different approach. The relation of this approach to the one in [12] has been studied recently in [17].

An interesting outcome of the analysis presented in [12] is the well known fact that the Vassiliev invariants of a given knot form an algebra in the sense that the product of two invariants of orders $i$ and $j$ is an invariant of order $i+j$. Therefore the set of independent Vassiliev invariants at a given order can be divided in two subsets: those that are products of invariants of lower orders (composite invariants), an those that are not (primitive invariants). We shall call the decomposition of a Vassiliev invariant as a product of lower order Vassiliev invariants “factorization”. This phenomenon is most clearly exposed after choosing a particular kind of basis of group factors that we will call “canonical”. The detailed description of these bases and its significance to the theory of numerical knot invariants of finite type is the main goal of the present work.

More precisely, in this paper we shall show that the factorizations observed in [12] can be resummed in a single exponential, which includes only the primitive Vassiliev invariants of the knot $C$, thus disentangling the contribution of these primitive invariants to all orders in perturbation theory. This, which is the main result of this paper, can be regarded as an extension of the theorems by Birman and Lin in [4, 8, 5] where it is proven that the coefficients of the power expansion of any Chern-Simons or quantum group polynomial invariant is a Vassiliev invariant. Our main result is contained in the “Factorization Theorem” presented
in sect. 5 and it can be very simply stated as follows:

Let \( C \) be a knot and let \( \mathcal{H}_t^R(C, G) \) be a Chern-Simons or quantum group polynomial invariant associated to a compact semi-simple Lie group \( G \) and a representation \( R \) of \( G \) (normalized so that for the unknot it takes the value 1). \( \mathcal{H}_t^R(C, G) \) is a polynomial in \( t \). Let \( \mathcal{W}_x^R(C, G) \) be obtained from \( \mathcal{H}_t^R(C, G) \) by replacing the variable \( t \) by \( e^x \) and let us consider the power series expansion of \( \log \mathcal{W}_x^R(C, G) \) around \( x = 0 \):

\[
\log \mathcal{W}_x^R(C, G) = \sum_{i=0}^{\infty} w_i^c x^i. \tag{1}
\]

Then, \( w_0^c = 0 \) and each \( w_i^c, i \geq 1 \), is a primitive Vassiliev invariant relative to a canonical basis.

The proof of this theorem is accomplished through the choice of a canonical basis for the independent group factors at each order in the perturbative expansion of the vacuum expectation value of the Wilson line, \( \langle \mathcal{W}_R(C, G) \rangle \), which is precisely, up to a normalization factor, \( \mathcal{H}_t^R(C, G) \), or \( \mathcal{W}_x^R(C, G) \), above. The use of these bases reveals a simplicity in the perturbative expansion that can hardly be grasped otherwise.

This paper is organized as follows. Section 1 contains an elementary exposition of Chern-Simons quantum field theory, along with the definition of the object of interest in this article: the Wilson loop operator. In section 3 we introduce the general structure of the perturbative expansion of \( \langle \mathcal{W}_R(C, G) \rangle \) and the definition of canonical bases. Section 4 presents some consequences of our having chosen a canonical basis to express the perturbative expansion, encoded in a “Master Equation”. In Section 5 these results are shown to lead to the factorization of the expansion in a single exponential. It is noteworthy that this implies that the logarithm of the polynomial invariant contains primitive Vassiliev Invariants only; the concept of primitive invariant has a simple definition in a canonical basis. Changes of basis are analyzed in section 6. Section 7 contains our conclusions.

2 Chern-Simons theory

In this section we will recall a few facts on Chern-Simons gauge theory. Let us consider a compact semi-simple Lie group \( G \) and a connection \( A \) on \( \mathbb{R}^3 \). The Chern-Simons action is defined as:

\[
S_k(A) = \frac{k}{4\pi} \int_{\mathbb{R}^3} \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{2}
\]

where \( \text{Tr} \) denotes the trace in the fundamental representation of \( G \). Given a knot \( C \), \( i.e. \), an embedding of \( S^1 \) into \( \mathbb{R}^3 \), we define the Wilson line associated to \( C \) carrying a representation \( R \) of \( G \) as:

\[
\mathcal{W}_R(C, G) = \text{Tr}_R \left[ \text{P exp} \int A \right], \tag{3}
\]
where “P” stands for path ordered and the trace is to be taken in the representation $R$. The vacuum expectation value is defined as the following ratio of functional integrals:

$$\langle W_R(C, G) \rangle = \frac{1}{Z_k} \int [DA] W_R(C, G) e^{iS_k(A)},$$

(4)

where $Z_k$ is the partition function:

$$Z_k = \int [DA] e^{iS_k(A)}.$$  

(5)

The theory based on the action (2) possesses a gauge symmetry which has to be fixed. In addition, one has to take into account that the theory must be regularized due to the presence of divergent integrals when performing the perturbative expansion of (4). Regarding these two problems we will follow the approach taken in [12]. It is known [19] that in the Landau gauge the contribution of the one-loop gauge field self-energy and one-loop gauge field vertex to the perturbative expansion of any vacuum expectation value can be traded by a shift in the parameter $k$ that multiplies the Chern-Simons action: $k \rightarrow k - C_A$, being $C_A$ the quadratic casimir in the adjoint representation of $G$. In so doing we do not need to include one loop two- or three-point gauge field subdiagrams in the perturbative expansion. Also, it has been shown [20] that higher-order corrections to the gauge field two- and three-point functions vanish.

There is one more problem emanating from perturbative quantum field theory which must be considered. Often, products of operators $A_\mu(x)A_\nu(y)$ must be considered at the same point $x = y$, where they are ambiguous. This situation can be solved [21, 15] without spoiling the topological nature of the theory. In the process one needs to introduce a framing attached to the knot which is characterized by an integer $n$. It was shown in [18] that working in the standard framing, $n = 0$, is equivalent to ignoring diagrams containing collapsible propagators in the sense explained in [18, 12].

### 3 General structure of the perturbative expansion

The facts mentioned in the previous section (exclusion of loop contributions to gauge field two- and three-point functions, and of collapsible propagators) simplify considerably the perturbative analysis of the vacuum expectation value (4). As shown in [12] the perturbative expansion of the vacuum expectation value of the Wilson line (3) has the form:

$$\langle W_R(C, G) \rangle = d(R) \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha_i^j r_{ij} x^i,$$

(6)

where $x = \frac{2\pi i}{k-C_A}$, and $d(R)$ is the dimension of the representation $R$. The factors $\alpha_i^j$ and $r_{ij}$ in (6) incorporate all the dependence dictated from the Feynman rules apart from the dependence on $k$ which is contained in $x$. The power of $x$, $i$, represents the order in perturbation theory. Of the two factors, $r_{ij}$ and $\alpha_i^j$, the first one contains all the group-theoretical dependence, while the second all the geometrical dependence. The quantity $d_i$ denotes the number of independent group structures $r_{ij}$ which appear at order $i$. The first values of $d_i$, $\alpha_i^j$ and $r_{ij}$
are: $\alpha_j^i = r_{0,1} = 1$, $d_0 = 1$ and $d_1 = 0$. Notice that we are shifting $k$ in the definition of $x$ and therefore no diagrams with loop contributions to two and three-point functions should be considered. In addition, there is no linear term in the expansion ($d_1 = 0$) so that diagrams with collapsible propagators (isolated chords) should be ignored in the sense explained in [12]. It was proven in [12] that the quantities $\alpha_j^i$ are Vassiliev invariants of order $i$.

We introduce now some vocabulary in order to classify Feynman diagrams. We will assume that the reader is familiar with the types of trivalent Feynman diagrams appearing in Chern-Simons perturbation theory. These diagrams are trivalent graphs with a distinguished line called Wilson line which carry the representation chosen. A detailed account of Chern-Simons perturbation theory specially suited for our purposes can be found in [12].

We begin introducing the notion of “connected” loop diagram. We will say that a diagram is a connected loop diagram if it is possible to go from one propagator (or internal line) to another without ever having to go through the Wilson line. If the diagram is “disconnected” that is not possible. In this second case we say that the diagram has subdiagrams, which are the connected components of the whole loop diagram. We say that two subdiagrams are “non-overlapping” if we can move along the Wilson line meeting all the legs of one subdiagram first, and all the legs of the other second. Here, “legs” means the propagators directly attached to the Wilson line. If it is impossible to do that, the subdiagrams are “overlapping”. In fig. 1 the diagram $a$ is connected while the diagrams $b$ and $c$ are disconnected. Of these last two, diagram $b$ contains subdiagrams which are overlapping while $c$ does not.

In the general expansion (6) there are many possible choices of the independent group factors $r_{ij}$. Given all Feynman diagrams contributing to a given order in perturbation theory some of the resulting group factors might be related due to the relations among the generators $T^a$ and the structure constants $f_{ijk}$ of semi-simple groups. From a diagramatic point of view these relations are the so called STU and IHX relations [10]. Since for a semi-simple group the structure constants can be chosen antisymmetric there is no need to distinguish orientation of internal three-vertices. The group factors entering (6) are chosen to be associated to diagrams that are independent. Of course, many choices are possible. Each possible set of group factors $r_{ij}$ represents a basis. There are two simple but far-reaching facts about the bases $r_{ij}$ which we summarize in two Propositions:

**Proposition 1:** It is always possible to choose a basis such that the $r_{ij}$ come from connected diagrams, or products of connected diagrams. That is, if there are subdiagrams, they can be chosen so that they do not overlap. The value of such an $r_{ij}$ is the product of
Figure 2: Example of a canonical basis up to order 6.

the values of its subdiagrams.

**Proposition 2:** The $r_{ij}$ which are products can be chosen as products of connected $r_{ij}$'s of lower orders.

These propositions follow from two simple facts. First, using STU relations it is always possible to trade in a disconnected diagram overlapping subdiagrams by connected diagrams and disconnected diagrams containing non-overlapping subdiagrams. Second, if a loop diagram is not connected, and its subdiagrams are non-overlapping, its group factor is the product of the group factors of its subdiagrams. This last statement follows from the fact that if one cuts a Wilson line at a given point where no leg is inserted the resulting matrix is a diagonal matrix.

Propositions 1 and 2 are very important because a basis of group factors such constructed shows the following unique feature: a given connected $r_{ij}$ begets a whole family of other group factors at higher orders in which it enters as a subdiagram. A basis constructed following Propositions 1 and 2 shall be called “a canonical basis”. The basis used in [12] up to order
six is canonical. The diagrams chosen are reproduced in fig. 2.

4 The Master Equation

Let the gauge group be a product $G \otimes G'$, where $G$ and $G'$ are compact semi-simple groups. From a path-integral representation of the vacuum expectation values, the following identity is obvious:

$$\langle W_{R \otimes R'}(C, G \otimes G') \rangle = \langle W_R(C, G) \rangle \langle W_{R'}(C, G') \rangle .$$

When combined with the choice of the same canonical basis for all the vev’s, this equation proves to be most fruitful. In order to show this, consider an $r_{ij}$ composed of several connected subdiagrams which we denote by $r_{ij}^{(p)}$ with $p = 1, \ldots, \#(ij)$. Some of them may be identical. In a canonical basis these $r_{ij}^{(p)}$ are elements of the basis at lower orders, and therefore are associated with geometrical factors which we denote by $(\alpha^j_i)^{(p)}$. If the Lie group is simple, we know that

$$r_{ij}(G) = \prod_{p=1}^{\#(ij)} r_{ij}^{(p)}(G) ,$$

but if the Lie group is a product, we find that

$$r_{ij}(G \otimes G') = \prod_{p=1}^{\#(ij)} \left( r_{ij}^{(p)}(G) + r_{ij}^{(p)}(G') \right) .$$

Inserting eq. (6) in eq. (7) and putting things together, we arrive at the “Master Equation”:

$$\sum_{i=0}^{\infty} \sum_{j=1}^{\#(ij)} d_i \alpha^j_i(C) \prod_{p=1}^{\#(ij)} \left( r_{ij}^{(p)}(G)x^i + r_{ij}^{(p)}(G')x'^i \right) =$$

$$\left( \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \alpha^l_k(C)r_{kl}(G)x^k \right) \left( \sum_{m=0}^{\infty} \sum_{n=1}^{d_m} \alpha^m_n(C)r_{mn}(G')x'^m \right) ,$$

where $x = 2\pi i/(k - C_A)$ and $x' = 2\pi i/(k - C'_A)$. In eq. (8), (9) and (11) we have shown explicitly the fact that the group factors $r_{ij}$ depend only on the group-theoretical data while the geometrical factors $\alpha^j_i$ depend only on the knot $C$. Actually, we would ought to have indicated the representations $R$ and $R'$ in the group factors. We did not do it to avoid a cumbersome notation but it certainly should be understood. The matching of the two polynomials in $x$ and $x'$ in (11) produces an infinite string of identities relating $\alpha^j_i$’s at a given order with products of the $(\alpha^l_k)^{(p)}$ of its components. The general result is as follows. Let us consider a composite $r_{ij}$ that consists of $p_{ij}^{(k)}$ connected non-overlapping subdiagrams of some type $k$, with $k = 1, \ldots, N$. This means in particular that

$$\sum_{k=1}^{N} p_{ij}^{(k)} = \#(ij) .$$

(The only purpose of this formula is to clarify the notation). We call $r_{ij;k}$ the connected subdiagram of type $k$, which in a canonical basis is also an element of the basis at a lower
order, and therefore is associated to a geometrical factor denoted by $\alpha^j_{i,k}$. In other words, the element $r_{ij}$ contains the following connected subdiagrams:

$$
\begin{align*}
 r^{(1)}_{ij} &= r^{(2)}_{ij} = \ldots = r^{(p^{(1)}_{ij})}_{ij} \equiv r_{ij;1}, \\
r^{(p^{(1)}_{ij}+1)}_{ij} &= r^{(p^{(1)}_{ij}+2)}_{ij} = \ldots = r^{(p^{(1)}_{ij}+p^{(2)}_{ij})}_{ij} \equiv r_{ij;2},
\end{align*}
$$

etc.

The Master Equation (10) leads to a formula for the $\alpha^j_i$ associated with our composite $r_{ij}$:

**THEOREM 1**

$$
\alpha^j_i = \prod_{k=1}^{N} \frac{1}{p^{(k)}_{ij}} \left(\alpha^j_i r^{(k)}_{ij}\right).
$$

Note that this is true only for a canonical choice of basis. This general result is the key to the next sections.

5 Factorizations

When the approach described in the preceding sections was first proposed in [12], the canonical basis did not include any elements at order 1. The reason is that the only contribution to the vacuum expectation value at that order would be the framing factor, which is not intrinsic to the knot. In that paper the interest was centered in extracting numerical knot invariants from the perturbative expansion, so it was natural to exclude the framing factor and its corresponding element of the basis.

We can include the framing factor if we enlarge our basis with $r_{11} = C_2$, the quadratic casimir. The geometrical factor associated with this new group structure will be denoted by $n$ for it is the framing. We are including these new elements here in order to show the simplest application of eq. (13).

According to Propositions 1 and 2 there is a new basis in which each $r_{kl}$ originates a family of elements of the form $r_{kl}C_2$, $r_{kl}C_2^2$ and so on. This set of elements of the new basis originated by $r_{kl}$ can be called “the $kl$-$C_2$-family”. Let us focus on a given $kl$-$C_2$-family and consider its contribution to $\langle W_R(C,G) \rangle$. Theorem 1 shows that this contribution is

$$
d(R) \left(\alpha^l_k r_{kl} x^k + \alpha^l_k r_{kl} C_2 n x^{k+1} + \alpha^l_k r_{kl} C_2^2 \frac{1}{2} n^2 x^{k+2} + \ldots \right),
$$

and the following terms are the expansion of an exponential. It follows that eq. (14) is equal to:

$$
d(R)\alpha^l_k r_{kl} x^k e^{C_2nx}.
$$

Therefore, repeating the same argument for each $kl$-$C_2$-family (which are all independent because the $r_{kl}$ are) we get:

**THEOREM 2**
\[ \langle W_R(C, G) \rangle = \langle W_R(C, G) \rangle \big|_{n=0} e^{C_2 n x} \]  

(16)

This agrees exactly with the non-perturbative result \([21]\) and can be regarded as a simplified proof of the factorization of the framing factor shown in \([18]\). Now, we can use a similar approach to factorize more structures. The idea is the same: the selection of a given connected element of our canonical basis, the “dressing” of another element of the basis with copies of our selected subdiagram (thus creating a family of diagrams in the sense explained above) and the repeated use of Theorem 1. The rest of this section dwells on this subject.

The reasoning that led to the factorization of the \(C_2\) structure can be applied without changes to any other connected \(r_{ij}\). The only peculiarity of \(C_2\) is that its addition to an element of the basis at order \(k\), say \(r_{kl}\), leads to another element at order \(k + 1\), its “first descendant” in the \(kl\)-\(C_2\)-family. If we want to factorize \(r_{21}\), the members of the \(kl\)-\(r_{21}\)-family would have orders \(k + 2, k + 4\) and so on in double steps. Proposition 2 says that our basis can be constructed so that it contains this new family.

Let \(r_{k+2q,l}\) be a member of the \(kl\)-\(r_{21}\)-family generated by \(r_{kl}\) and a \(q\)-fold insertion of \(r_{21}\), i.e.

\[ r_{k+2q,l} = r_{kl} (r_{21})^q. \]  

(17)

Theorem 1, and the fact that we have chosen a canonical basis, enables us to prove that the contribution of this family to \(\langle W_R(C, G) \rangle\) is

\[
d(R) \left( \alpha_k^l r_{kl} x^k + \alpha_k^l \alpha_2^1 r_{21} x^{k+2} + \frac{1}{2} \alpha_k^l (\alpha_2^1)^2 r_{kl} (r_{21})^2 x^{k+4} + \ldots \right) \\
= d(R) \alpha_{kl} r_{kl} x^k e^{\alpha_2^1 r_{21} x^2}. \]  

(18)

We can repeat the same steps for all \(kl\)-\(r_{21}\)-families because they are all independent; in so doing we arrive at:

**THEOREM 3**

\[ \langle W_R(C, G) \rangle = \langle W_R(C, G) \rangle \big|_{\alpha_2^1=0} e^{\alpha_2^1 r_{21} x^2}, \]  

(19)

and nothing hinders the generalization of this theorem to what can be seen as the full expression of the concept of perturbative factorization of the vacuum expectation value:

**FACTORIZATION THEOREM**

\[ \langle W_R(C, G) \rangle = d(R) \exp \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{d_i} \alpha_i^j c(C) r_{ij}^c(G) x^i \right\}, \]  

(20)

where \(r_{ij}^c\) denotes the connected elements of the basis, and \(\alpha_i^j c\) their corresponding geometrical factors. These \(\alpha_i^j c\) do not correspond uniquely to connected diagrams because the \(r_{ij}^c\) include geometric factors from both connected and disconnected Feynman diagrams. The symbol \(d_i\) stands for the number of connected elements in the canonical basis at order \(i\).
Equation (20) can be written in the form
\[
\log \left( \frac{1}{d(R)} \langle W_R(C, G) \rangle \right) = \sum_{i=1}^{\infty} \sum_{j=1}^{d_i} \alpha_j^i c(C) r_{ij}^c(G) x^i,
\]
which is the result announced in (1). Equation (21) is reminiscent of the well known fact in quantum field theory that the logarithm of the generating functional can be expanded in terms of connected diagrams only. The relevance of this formula to the theory of knot invariants comes from the identification of the vacuum expectation value of the Wilson line as a polynomial invariant. Actually we are in a much more general situation because we are considering an arbitrary semi-simple gauge group, so our vev is in some sense the most general polynomial invariant possible. Therefore, eqs. (20) or (21) prove that if a canonical basis is chosen, the logarithm of a polynomial knot invariant can be expanded in terms of the primitive Vassiliev invariants of the knot only.

The primitive Vassiliev invariants \( \alpha_j^i C \) have been computed up to order six for all prime knots up to six crossings [12] and for arbitrary torus knots [22]. It was conjectured in [12] that there exist a normalization for the \( \alpha_j^i C \) in which they are integer-valued. The integral expressions for \( \alpha_j^1 \) and \( \alpha_j^3 \) were first presented in [15] and in [12] respectively. Properties of these two primitive Vassiliev invariants have been studied in [7] and in [23].

6 Change of basis

In this section we want to investigate to what extent the previous results are independent of the basis chosen. We are thus led to consider changes of basis. First we treat changes of canonical basis. Let \( B \) and \( B' \) be two different canonical bases, being \( r_{ij} \) and \( r'_{ij} \) its elements. The most general change of canonical basis is
\[
r'_{ij} = N_j^k r_{ik},
\]
where \( N \) is a \( d_i \times d_i \) matrix yet to be determined. We are assuming that the vectors in this space are written as columns. To see that (22) is the most general change between \( B \) and \( B' \), consider a possible extra term in the right hand side. It has to be a product of several \( r_{kl} \in B \) such that the sum of the orders of its factors is \( i \). But by definition of canonical basis such a product is also an element of \( B \) at order \( i \) and therefore the extra term can be absorbed into the first term. This shows that eq. (22) is indeed the most general change of canonical basis.

It is elementary to prove that, in order to preserve the independence of the elements of \( B \) at each order, the matrix \( N \) has to be non-singular:
\[
\det N \neq 0.
\]

We now analyze the effect of the change of basis (22) on the geometric factors \( \alpha_j^i \). The vectors in this space of numerical factors are written as rows. The starting point is the

\[\text{1There is an } N \text{ at each order } i, \text{ but we are not indicating this fact.}\]
invariance of the vacuum expectation value under changes of basis. Therefore, at each order \( i \) it is true that,

\[
\sum_{j=1}^{d_i} \alpha_j^i r_{ij} = \sum_{j=1}^{d_i} \alpha'_j r'_{ij},
\]

and it follows that the \( \alpha_j^i \) transform "contravariantly":

\[
\alpha'_j = \alpha_k^i \left( N^{-1}\right)_k^j.
\]  

In general, at order \( i \) the change of canonical basis involves a matrix \( N \in GL(d_i, Q) \). We do not know how to characterize these matrices completely. Only a small subset of the whole linear group is relevant. For example we can discard \( N \)'s which are mere permutations, or diagonal, since they do not lead to essentially new bases. More important, the elements of the basis \( B' \) will be linear combinations of the elements of \( B \), but these linear combinations must be interpretable as new diagrams because we want \( B' \) to be canonical. In other words, we would have to investigate which linear combinations of independent diagrams can be written as a single diagram. The resulting subgroup of \( GL(d_i, Q) \) would be equivalent to the space of independent canonical bases at order \( i \).

The \( N \)'s have some properties that do not depend on these details. Let us order the elements of a canonical basis \( B \) at order \( i \) as follows:

\[
B_i = \{ r_{i,1}^c, \ldots, r_{i,d_i}^c, r_{i,d_i+1}, \ldots, r_{i,d_i} \},
\]

i.e. the connected elements before, and the disconnected elements after. A given disconnected element of \( B \) can be written as a product of elements of \( B \) of lower orders,

\[
r_{ij} = r_{kl} r_{i-k,s},
\]

where \( l \) and \( s \) depend implicitly on \( j \), but this will not be relevant in what follows. If we write the identity (27) in a new canonical basis \( B' \), it reads

\[
N^q_p r'_{ip} = N^q_l r'_{kq} N^t_s r'_{i-k,t}.
\]

Note that the \( N \)'s operate on different spaces. What we have on the right hand side is a linear combination of elements of \( B' \) at order \( i \), because all these products of \( r_{kq}' \)'s times \( r_{i-k,t}' \)'s must be elements of \( B' \) (it is canonical). On the left hand side we have other elements of the same basis at the same order. Therefore eq. (28) is a contradiction unless the \( r'_{ip} \) are themselves products and, therefore, correspond to disconnected diagrams. The conclusion is that in a change of canonical basis, the disconnected \( r' \)'s in \( B \) come from the disconnected \( r \)'s in \( B' \).

This result can be summarized in the following symbolic representation of a matrix \( N \) valid for a change of canonical basis:

\[
N = \begin{pmatrix}
\begin{array}{c|c}
C \rightarrow C & NC \rightarrow C \\
\hline
0 & NC \rightarrow NC
\end{array}
\end{pmatrix}
\]  

(29)
where $C$ and $NC$ mean “connected” and “non-connected” respectively. In this notation a change of basis would be written as

$$\begin{pmatrix} r^c \\ r^{nc} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r'^c \\ r'^{nc} \end{pmatrix}$$  \hfill (30)

where $C = 0$. The blocks in the diagonal are square matrices because all canonical bases must have the same number $d_i$ of connected elements at a given order $i$. These matrices have an interesting property: they form a subgroup of $GL(d_i, \mathbb{Q})$. The inverse of a given element is

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}$$  \hfill (31)

and the determinant is

$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A \det D.$$  \hfill (32)

Therefore, the matrix is non-singular if and only if its diagonal blocks are non-singular. We are assuming that the matrix represents a valid change of basis, thus nothing is singular and the inverses in eq. (31) do exist.

### 6.1 Diagrammatic interpretation

We can sharpen the previous result by analyzing the constraints that the diagrammatic origin of the $r$’s imposes on their algebra. For example, the product of two $r$’s is interpretable as a diagram in an obvious way (actually, as an equivalence class of diagrams). The big constraint is on the sum of two $r$’s. We are now investigating the algebra of diagrams which are either connected or disconnected and non-overlapping, independently of their being independent or not, i.e. of being elements of a canonical basis or not. We shall call these diagrams generally $r$, and will not consider diagrams with overlapping subdiagrams at all.

The question is: when is the sum of two $r$’s interpretable as a diagram? The answer comes in two parts:

**Lemma**

1. Let $r_1$ and $r_2$ be two different connected diagrams of the same order $i$. Their sum or difference $r_1 \pm r_2$ exists as a diagram of order $i$ if and only if $r_1$ and $r_2$ are two of the terms in an STU or IHX relation. The sign in $\pm$ depends on which two terms of those relations are $r_1$ and $r_2$ related to.

2. If $r_1$ and $r_2$ are not connected, they can differ only in a single subdiagram. Again, the subdiagrams that are different must be two of the terms in an STU or IHX relation.

More complicated linear combinations of $r$’s that are interpretable as diagrams can always be decomposed in elementary steps, each of them satisfying the Lemma. We introduce now a new concept. A linear combination of $r$’s that can be interpreted as a diagram shall be called a “valid” linear combination. To be completely rigorous, a valid linear combination
should be written in an unambiguous way by using parentheses to indicate which \( r \)'s are added to which other \( r \)'s and in what order. For example,

\[
    r_1 + r_2 - r_3 = (r_1 + r_2) - r_3 \quad \text{or} \quad r_1 + (r_2 - r_3) \quad \text{or} \quad (r_1 - r_3) + r_2.
\]

(33)

Which one of the three possibilities is the good one depends on the diagrams. In this sense, the addition of \( r \)'s is commutative but not associative. This may be a minor point, because the \textit{numerical values} of the \( r \)'s can be added freely, but we are now focusing on the formal properties of the \( r \)'s as elements of an “algebra”\(^\text{2}\). In general, when we write a valid linear combination of \( r \)'s we shall assume that there is an ordering of the additions such that each step complies with the Lemma. This ordering depends on the particular diagrams in the linear combination and thus will not be indicated in general.

### 6.2 Arithmetic of diagrams

The previous subsection establishes the rules for an arithmetic of diagrams. The \( r \)'s are elements of an algebraic structure in which we can always multiply, but not always add or subtract (see the Lemma above). As for the division, an expression like \( r_1/r_2 \) is an \( r \) only if \( r_2 \) is a subdiagram of \( r_1 \), in which case we say that \( r_2 \) divides \( r_1 \). There is a neutral element for the product: the empty diagram. Therefore no diagrammatic interpretation exists for \( 1/r_2 \).

The set of \( r \)'s with the addition is not even a group, for not all diagrams can be added. This lack of structure precludes an abstract formulation of the algebra of group factors \( r \).

We need to know which is the diagram that a given \( r \) represents in order to ascertain if it can be added to another \( r \) or not. Nevertheless we can prove some theorems which have a bearing in our investigation.

**THEOREM 4**

Let \( r_1 \) and \( r_2 \) be two diagrams of the same order \( i \) that can be added or substracted, and \( r_3, r_4 \) be two diagrams whose orders add to \( i \). Then,

\[
    r_1 \pm r_2 = r_3 r_4 \iff (r_3 | r_1 \text{ and } r_3 | r_2) \text{ or } (r_4 | r_1 \text{ and } r_4 | r_2)
\]

(34)

where \( a|b \) means that \( a \) divides \( b \). The proof follows from the Lemma. An immediate generalization of this theorem is that a valid linear combination of \( r \)'s is disconnected non-overlapping if and only if so is each \( r \) in the linear combination. A similar conclusion holds for valid linear combinations of connected diagrams. We want to gather these important conclusions in three remarks:

1. No valid linear combination of \( r \)'s includes connected and disconnected non-overlapping diagrams at the same time.

2. A valid linear combination of connected diagrams is a connected diagram.

\(^2\)Strictly speaking they do not form an algebra, whence the quotation marks.
3. A valid linear combination of disconnected non-overlapping diagrams is a disconnected non-overlapping diagram. All of them have the same number of subdiagrams.

We can say that “a valid linear combination of diagrams conserves the number of components”. A closely related result is that a valid change of canonical basis must be represented by a block-diagonal matrix:

**THEOREM 5**

Let \( N \) be a matrix corresponding to a change of canonical basis at order \( i \), written in the form

\[
N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]  

where \( A \) is a non-singular \( \hat{d}_i \times \hat{d}_i \) matrix, and \( D \) a non-singular \( (d_i - \hat{d}_i) \times (d_i - \hat{d}_i) \) matrix. Then,

\[
B = C = 0.
\]

To prove this theorem notice that under a change of basis the connected \( r \)'s transform as (see eq. (30) for notation)

\[
r^c = Ar^c + Br^{mc}.
\]

Given that the l.h.s. is a diagram we observe that the r.h.s. is a valid linear combination of diagrams. But the Lemma implies that all diagrams in the r.h.s. must be connected, or all of them disconnected non-overlapping, i.e. either \( A=0 \) or \( B=0 \). It is clear from the Lemma that the only option is \( B=0 \). A similar argument establishes that \( C=0 \), which we already knew from previous considerations, see eq. (29).

The final picture for a valid \( N \) is

\[
N = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}
\]

with \( A \) and \( D \) non-singular square matrices. In words, the connected \( r \)'s transform independently from the disconnected \( r \)'s; they never mix in a change of canonical basis.

In particular this shows that eq. (21) is consistent under a change of canonical basis. No matter what canonical basis we choose, the only relevant diagrams are those that contribute to the \( r^c_{ij} \). As for the change from a canonical \( B \) to a non-canonical \( B' \), we have little to say. The concept of factorization as described here only makes sense for canonical bases, and only in this case we have eq. (20).

We regard canonical bases as privileged systems of reference in which the perturbative expansion is at its simplest.

### 7 Conclusions

We have shown that the perturbative expansion of the vev of a Wilson line in Chern-Simons quantum field theory shows a striking simplicity if presented in terms of a canonical basis for the group factors. These bases provide a simple characterization of primitive Vassiliev
invariants: they are the geometric factors associated to the connected elements of a canonical basis.

Within this framework it is possible to factorize the whole perturbative expansion in separate contributions from each primitive invariant. Each of these factors can be resummed in an exponential. The relevance of this result for the theory of numerical knot invariants is that the logarithm of a polynomial invariant contains only the primitive invariants.

This work opens a variety of investigations. One should study further the algebraic structure of the set of canonical bases. As follows from the discussion in sect. 6, at each order $i$ this set is characterized by a subgroup of $GL(\hat{d}_i, \mathbb{Q})$. Methods to find the groups corresponding to each order should be investigated.

Another important extension of our work consists of the study of factorization in the context of $n$-component links. Vassiliev invariants for $n$-component links have not been much studied and it is very likely that some of the ideas behind factorization can be used in the organization of their structure. The extension of the concept of canonical basis and primitiveness should be explored in that case following a similar analysis to the one presented in this paper. Work on this and other aspects of Vassiliev invariants associated to $n$-component links will be reported elsewhere.

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