On the balanced decomposition number

Tadashi Sakuma

Systems Science and Information Studies
Faculty of Education, Art and Science
Yamagata University
1-4-12 Kojirakawa, Yamagata 990-8560, Japan

Abstract

A balanced coloring of a graph $G$ means a triple $\{P_1, P_2, X\}$ of mutually disjoint subsets of the vertex-set $V(G)$ such that $V(G) = P_1 \uplus P_2 \uplus X$ and $|P_1| = |P_2|$. A balanced decomposition associated with the balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of $G$ is defined as a partition of $V(G) = V_1 \uplus \cdots \uplus V_r$ (for some $r$) such that, for every $i \in \{1, \cdots, r\}$, the subgraph $G[V_i]$ of $G$ is connected and $|V_i \cap P_1| = |V_i \cap P_2|$. Then the balanced decomposition number of a graph $G$ is defined as the minimum integer $s$ such that, for every balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of $G$, there exists a balanced decomposition $V(G) = V_1 \uplus \cdots \uplus V_r$ whose every element $V_i (i = 1, \cdots, r)$ has at most $s$ vertices. S. Fujita and H. Liu [SIAM J. Discrete Math. 24, (2010), pp. 1597–1616] proved a nice theorem which states that the balanced decomposition number of a graph $G$ is at most 3 if and only if $G$ is $\left\lfloor \frac{|V(G)|}{2}\right\rfloor$-connected. Unfortunately, their proof is lengthy (about 10 pages) and complicated. Here we give an immediate proof of the theorem. This proof makes clear a relationship between balanced decomposition number and graph matching.

keywords: graph decomposition, coloring, connectivity, bipartite matching

1Email: sakuma@e.yamagata-u.ac.jp
1 Introduction

Throughout this paper, we only consider finite undirected graphs with no multiple edges or loops. For a graph $G$, let $V(G)$ and $E(G)$ denote the vertex-set of $G$ and the edge-set of $G$, respectively. For a subset $X \subseteq V(G)$, $G[X]$ denotes the subgraph of $G$ induced by $X$, and $N_G(X)$ denotes the set $\{y \in V(G) \setminus X | \exists x \in X, \{x, y\} \in E(G)\}$. This set $N_G(X)$ is called the open neighborhood of $X$ in $G$. A subset $Y \subseteq V(G)$ is called a vertex-cut of $G$ if there is a partition $V(G) \setminus Y = X_1 \uplus X_2$ such that $|X_i| \geq 1$ and $N_{G[V(G) \setminus Y]}(X_i) = \emptyset$ ($i = 1, 2$). For other basic definitions in graph theory, please consult [2].

In 2008, S. Fujita and T. Nakamigawa [4] introduced a new graph invariant, namely the balanced decomposition number of a graph, which was motivated by the estimation of the number of steps for pebble motion on graphs. A balanced coloring of a graph $G$ means a triple $\{P_1, P_2, X\}$ of mutually disjoint subsets of $V(G)$ such that $V(G) = P_1 \uplus P_2 \uplus X$ and $|P_1| = |P_2|$. Then a balanced decomposition of $G$ associated with its balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ is defined as a partition of $V(G) = V_1 \uplus \cdots \uplus V_r$ (for some $r$) such that, for every $i \in \{1, \cdots, r\}$, $G[V_i]$ is connected and $|V_i \cap P_1| = |V_i \cap P_2|$. Note that every disconnected graph has a balanced coloring which admits no balanced decompositions. Now the balanced decomposition number of a connected graph $G$ is defined as the minimum integer $s$ such that, for every balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of $G$, there exists a balanced decomposition $V(G) = V_1 \uplus \cdots \uplus V_r$ whose every element $V_i$ ($i = 1, \cdots, r$) has at most $s$ vertices.

The set of the starting and the target arrangements of mutually indistinguishable pebbles on a graph $G$ can be modeled as a balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of $G$. Then, as is pointed out in [4], the balanced decom-
position number of $G$ gives us an upper-bound for the minimum number of necessary steps to the pebble motion problem, and, for several graph-classes, this upper bound is sharp.

In addition to the initial motivations and their applications in [4], this newcomer graph invariant turns out to have deep connections to some essential graph theoretical concepts. For example, the following conjecture in [4] indicates a relationship between this invariant and the vertex-connectivity of graphs:

**Conjecture 1. (S. Fujita and T. Nakamigawa (2008))** The balanced decomposition number of $G$ is at most $\left\lfloor \frac{|V(G)|}{2} \right\rfloor + 1$ if $G$ is 2-connected.

Recently, G. J. Chang and N. Narayanan [1] announced a solution to this conjecture.

Then especially, S. Fujita and H. Liu [3] proved the affirmation of the “high”-connectivity counterpart of the above conjecture, as follows:

**Theorem 1. (S. Fujita and H. Liu (2010))** Let $G$ be a connected graph with at least 3 vertices. Then the balanced decomposition number of $G$ is at most 3 if and only if $G$ is $\left\lfloor \frac{|V(G)|}{2} \right\rfloor$-connected.

Thus, there may be a trade-off between the vertex-connectivity and the balanced decomposition number. This interesting relationship should be investigated for its own sake.

Unfortunately, the proof of Theorem[1] in [3] is lengthy (about 10 pages) and complicated.

In this note, we give a new proof of the theorem[1]. The advantages of our proof is that it is immediate and makes clear a relationship between balanced decomposition number and graph matching.
2 A quick proof of Theorem

We show our proof of the theorem here.

Proof of Theorem. In order to prove the if part, let us define the following new bipartite graph $H$ from a given balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of a graph $G$:

1. The partite sets of $H$ are $V_1(H) := P_1 \uplus X_1$ and $V_2(H) := P_2 \uplus X_2$, where each $X_i := \{(x, i) \mid x \in X\}$ ($i = 1, 2$) is a copy of the set $X(\subseteq V(G))$.

2. The edge set $E(H)$ of $H$ is defined as follows:

$$E(H) := \{\{p_1, p_2\} \mid p_1 \in P_1, p_2 \in P_2, \{p_1, p_2\} \in E(G)\} \cup \{\{p_1, (x, 2)\} \mid p_1 \in P_1, x \in X, \{p_1, x\} \in E(G)\}$$

$$\cup \{\{(x, 1), p_2\} \mid x \in X, p_2 \in P_2, \{x, p_2\} \in E(G)\}$$

$$\cup \{\{(x, 1), (x, 2)\} \mid x \in X\}.$$

Then clearly, the balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of $G$ has a balanced decomposition $V(G) = V_1 \uplus \cdots \uplus V_r$ whose every element $V_i (i = 1, \ldots, r)$ consists of at most 3 vertices, if and only if the graph $H$ has a perfect matching. Then we use here the famous “Hall’s Marriage Theorem” [5], as follows.

Lemma 2. (P. Hall (1935)) Let $G$ be a bipartite graph whose partite sets are $V_1(G)$ and $V_2(G)$. Suppose that $|V_1(G)| = |V_2(G)|$. Then $G$ has a perfect matching if and only if every subset $U$ of $V_1(G)$ satisfies $|U| \leq |N_G(U)|$.

Now, suppose that $H$ does not have any perfect matching. Then, from lemma 2, $\exists A \subseteq P_1, \exists B \subseteq X_1, |N_H(A \cup B)| \leq |A| + |B| - 1$. Let $C := P_2 \setminus N_H(A \cup B)$ and $D := X_2 \setminus N_H(A \cup B)$. Then, by symmetry, $|N_H(C \cup D)| \leq |C| + |D| - 1$ also holds. Furthermore, by the definition of $H$, $|B| \leq |X_2 \setminus D|$
and \(|D| \leq |X_1 \setminus B|\) hold, and hence \(0 \leq |X| - |B| - |D| \leq |A| + |C| - |P_1| - 1 = |A| + |C| - |P_2| - 1\) satisfies. Please see Figure 1 which shows this situation. The vertex-cut of \(V(G)\) corresponding to the set \((P_1 \setminus A) \cup (P_2 \setminus C) \cup (X_1 \setminus B)\) separates \(G[C]\) from its remainder. By symmetry, the vertex-cut of \(V(G)\) corresponding to the set \((P_1 \setminus A) \cup (P_2 \setminus C) \cup (X_2 \setminus D)\) separates \(G[A]\) from its remainder. Hence if \(G\) is \(\left\lfloor \frac{|V(G)|}{2} \right\rfloor\)-connected, \(|V(G)| - 1 \leq 2(|P_1| - |A| + |P_2| - |C'|) + (|X| - |B|) + (|X| - |D|) = (|P_1| + |P_2| + |X|) - 2((|A| + |C| - |P_1|) - (|X| - |B| - |D|)) - (|X| - |B| - |D|) \leq |V(G)| - 2\), a contradiction.

The proof of the only if part is given by a construction of special balanced colorings, which is the same as the original one in [3]. We will transcribe the construction only for the convenience of readers.

Suppose that \(G\) is not \(\left\lfloor \frac{|V(G)|}{2} \right\rfloor\)-connected. And let \(Y\) denote a minimum vertex-cut of \(G\). Note that \(2|Y| \leq |V(G)| - 2\). Then \(G[V(G) \setminus Y]\) is divided

![Figure 1: The bipartite graph \(H\) which has no perfect matching.](image-url)
into two graphs \( G_1 \) and \( G_2 \) such that \(|V(G_i)| \geq 1\) and \( N_{G[V(G) \setminus Y]}(V(G_i)) = \emptyset \) \((i = 1, 2)\). Without loss of generality, we assume that \(|V(G_1)| \leq |V(G_2)|\).

Let \( l \) denote the number \( \min\{|Y|, |V(G_1)| - 1\} \). Suppose an arbitrary balanced coloring \( V(G) = P_1 \uplus P_2 \uplus X \) of \( G \) such that \(|Y \cap P_1| = l\) and \(|Y \cap P_2| = |Y| - l\) and \(|V(G_1) \cap P_2| = l + 1\) and \(V(G_1) \cap P_1 = \emptyset\). Then, it is easy to see that every balanced decomposition associated with such a balanced coloring has at least one component whose vertex-size is at least 4, that is, the balanced decomposition number of \( G \) is at least 4. ■

References

[1] G. J. Chang and N. Narayanan, On a conjecture on the balanced decomposition number, \textit{preprint}, 2011.

[2] R. Diestel, \textbf{Graph Theory}, Third Edition, Springer-Verlag, 2005.

[3] S. Fujita and H. Liu, The balanced decomposition number and vertex connectivity, \textit{SIAM Journal on Discrete Mathematics}, \textbf{24} (2010), 1597–1616.

[4] S. Fujita and T. Nakamigawa, Balanced decomposition of vertex-colored graph, \textit{Discrete Applied Mathematics}, \textbf{156} (2008), 3339–3344.

[5] P. Hall, On Representatives of Subsets, \textit{Journal of the London Mathematical Society}, \textbf{10} (1935), 26–30.