Elementary Yet Precise Worst-case Analysis of MergeSort

A short version (SV) of a manuscript intended for future publication

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Abstract
The full version of this paper offers two elementary yet precise derivations of an exact formula

\[ W(n) = \sum_{i=1}^{n} \lceil \log i \rceil = n \lceil \log n \rceil - 2^{\lceil \log n \rceil} + 1 \]

for the maximum number \( W(n) \) of comparisons of keys performed by MergeSort on an \( n \)-element array. The first of the two, due to its structural regularity, is well worth carefully studying in its own right.

Close smooth bounds on \( W(n) \) are derived. It seems interesting that \( W(n) \) is linear between the points \( n = 2^{\lceil \log n \rceil} \) and it linearly interpolates its own lower bound \( n \log n - n + 1 \) between these points.

The manuscript (MS) of the full version of this paper, dated January 20, 2017, can be found at:

http://csc.csudh.edu/suchenek/Papers/Analysis_of_MergeSort.pdf

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Contents

1 Introduction 3
2 Some Math prerequisites 4
3 MergeSort and its worst-case behavior \( W(n) \) 5
4 An easy yet precise derivation of \( W(n) \) 7
5 Close smooth bounds on \( W(n) \) 9
6 Other properties of the recursion tree \( T_n \) 13
7 A derivation of \( W(n) \) without references to the recursion tree 16
8 Other work 16
9 Best-case analysis of MergeSort 19

Appendix A A Java code of MergeSort 20
Appendix B Generating worst-case arrays for MergeSort 20
Appendix C Notes from my Analysis of Algorithms lecture 22
1. Introduction

MergeSort is one of the fundamental sorting algorithms that is being taught in undergraduate Computer Science curricula across the U.S. and elsewhere. Its worst-case performance, measured by the number of comparisons of keys performed while sorting them, is optimal for the class of algorithms that sort inductively\(^1\) by comparisons of keys.\(^2\) Historically, it\(^3\) was the first sorting algorithm to run in \(O(n \lg n)\) time.\(^4\)

So it seems only fitting to provide an exact formula for MergeSort’s worst-case performance \(W(n)\) and derive it precisely. Unfortunately, many otherwise decent texts offer unnecessarily imprecise\(^5\) variants of it, and some with quite convoluted, incomplete, or incorrect proofs. Due to these imperfections, the fact that the worst-case performance of MergeSort is the same as that of another benchmark sorting algorithm, the binary insertion sort first described by Steinhaus in \([5]\) that is worst-case optimal in the class of inductive sorting algorithms that sort by comparisons of keys; see \([3]\) page 186.

A bottom-up version of it, invented by John Neumann.\(^3\) In the worst case.

Notable exceptions in this category are \([2]\) and \([4]\) that derive almost exact formulas, but see Section \([8]\) page 16 for a brief critique of the results and their proofs offered there.\(^6\)

Even in \([3]\).

The detailed derivations can be found in \([9]\).

Elementary derivation of an exact formula for the best-case performance \(B(n)\) of MergeSort, measured by the number of comparisons of keys performed while sorting them, has been done in \([8]\); see Section \([9]\) page 19 of this paper.

\(W(n) = \sum_{i=1}^{n} \lceil \lg i \rceil = n \lceil \lg n \rceil - 2 \lceil \lg n \rceil + 1\)

for the maximum number \(W(n)\) of comparisons of keys performed by MergeSort on an \(n\)-element array. The first of the two, due to its structural regularity, is well worth carefully studying in its own right.

\(^1\)Inductive sorting of \(n\) keys sorts a set of \(n - 1\) of those keys first, and then “sorts-in” the remaining \(n\)-th key.

\(^2\)In its standard form analyzed in this paper, MergeSort is not an inductive sorting algorithm. However, its worst-case performance, measured by the number of comparisons of keys performed while sorting them, is equal to the worst-case performance of the binary insertion sort first described by Steinhaus in \([5]\) that is worst-case optimal in the class of inductive sorting algorithms that sort by comparisons of keys; see \([3]\) page 186.

\(^3\)A bottom-up version of it, invented by John Neumann.

\(^4\)In the worst case.

\(^5\)Notable exceptions in this category are \([2]\) and \([4]\) that derive almost exact formulas, but see Section \([8]\) page 16 for a brief critique of the results and their proofs offered there.

\(^6\)Even in \([3]\).
Unlike some other basic sorting algorithms\(^9\) that run in \(O(n \lg n)\) time, MergeSort exhibits a remarkably regular\(^10\) worst-case behavior, the elegant simplicity of which has been mostly lost on its rough analyses. In particular, \(W(n)\) is linear\(^11\) between the points \(n = 2^\lceil \lg n \rceil\) and it linearly interpolates its own lower bound \(n \lg n - n + 1\)\(^12\) between these points.

What follows is a short version (SV) of a manuscript dated January 20, 2017, of the full version version \([9]\) of this paper that has been posted at:

http://csc.csudh.edu/suchenek/Papers/Analysis_of_MergeSort.pdf

The derivation of the worst case of MergeSort presented here is roughly the same\(^13\) as the one I have been doing in my undergraduate Analysis of Algorithms class. Appendix C shows sample class notes from one of my lectures.

2. Some Math prerequisites

A manuscript of the full version \([9]\) of this paper contains a clever derivation of a well-known\(^14\) closed-form formula for \(\sum_{i=1}^{n} \lceil \lg i \rceil\). It proves insightful in my worst-case analysis of MergeSort as its right-hand side will occur on page 8 in the fundamental equality (5) and serve as an instrument to derive the respective exact formula for MergeSort’s worst-case behavior.

Lemma 2.1. For every integer \(n \geq 1\),

\[
\sum_{i=1}^{n} \lceil \lg i \rceil = \sum_{y=0}^{\lceil \lg n \rceil - 1} (n - 2^y).
\]

\(\Box\)

From this one can easily conclude that:

\(^9\)For instance, Heapsort; see \([7]\) for a complete analysis of its worst-case behavior.
\(^10\)As revealed by Theorem 5.2, page 10
\(^11\)See Figure 4, page 11
\(^12\)Given by the left-hand side of the inequality (12), page 11
\(^13\)Except for the present proof of Lemma 2.1 which I haven’t been using in my class.
\(^14\)See [3].
Corollary 2.2. For every integer $n \geq 1$,
\[ \sum_{i=1}^{n} \lceil \lg i \rceil = n \lceil \lg n \rceil - 2 \lceil \lg n \rceil + 1. \] (2)

3. MergeSort and its worst-case behavior $W(n)$

A call to MergeSort inherits an $n$-element array $A$ of integers and sorts it non-decreasingly, following the steps described below.

Algorithm MergeSort 3.1. To sort an $n$-element array $A$ do:

1. If $n \leq 1$ then return $A$ to the caller,
2. If $n \geq 2$ then
   (a) pass the first $\lfloor n/2 \rfloor$ elements of $A$ to a recursive call to MergeSort,
   (b) pass the last $\lceil n/2 \rceil$ elements of $A$ to another recursive call to MergeSort,
   (c) linearly merge, by means of a call to Merge, the non-decreasingly sorted arrays that were returned from those calls onto one non-decreasingly sorted array $A'$,
   (d) return $A'$ to the caller.

A Java code of Merge is shown on the Figure 1.15

A typical measure of the running time of MergeSort is the number of comparisons of keys, which for brevity I call $\text{comps}$, that it performs while sorting array $A$.

Definition 3.2. The worst-case running time
\[ W(n) \]

of MergeSort is defined as the maximum number of comps it performs while sorting an array of $n$ distinct16 elements.

15 A Java code of MergeSort is shown in Appendix A, Figure A.6 page 20.
16 This assumption is superfluous for the purpose of worst-case analysis as the mere presence of duplicates does not force MergeSort to perform more comps.
Figure 1: A Java code of \texttt{Merge}, based on a pseudo-code from [1]. Calls to \texttt{Boolean} method \texttt{Bcnt.incr()} count the number of comps for the purpose of experimental verification of the worst-case analysis of \texttt{MergeSort}.

Clearly, if $n = 0$ then $W(n) = 0$. From this point on, I am going to assume that $n \geq 1$\footnote{This assumption turns out handy while using expression $\log n$.}

Since no comps are performed outside \texttt{Merge}, $W(n)$ can be computed as the sum of numbers of comps performed by all calls to \texttt{Merge} during the execution of \texttt{MergeSort}. The following classic results will be useful in my analysis.

\textbf{Theorem 3.3.} \textit{The maximum number of comps performed by \texttt{Merge} on two sorted list of total number $n$ of elements is $n - 1$.}

\textit{Proof} (constructive, with Java code that generates worst cases shown in the Appendix B) in [9]. \hfill \Box

Moreover, if the difference between the lengths of merged list is not larger than 1 then no algorithm that merges sorted lists by means of comps beats \texttt{Merge} in the worst case, that is, has a lower than $n - 1$ maximum number of comps\footnote{Proof in [2], Sec. 5.3.2 page 198; the worst-case optimality of \texttt{Merge} ($n - 1$ comps) was generalized in [6] over lists of lengths $k$ and $m$, with $k \leq m$, that satisfy $3k \geq 2m - 2$.}. This fact makes \texttt{MergeSort} optimal in the intersection of the class of sorting algorithms that sort by merging two sorted lists of lengths’
difference not larger than $1^{19}$ with the class of sorting algorithms that sort by comps.

4. An easy yet precise derivation of $W(n)$

MergeSort is a recursive algorithm. If $n \geq 2$ then it spurs a cascade of two or more recursive calls to itself. A rudimentary analysis of the respective recursion tree $T_n$, shown on Figure 2, yields a neat derivation of the exact formula for the maximum number $W(n)$ of comps that MergeSort performs on an $n$-element array.

Figure 2: A sketch of the recursion 2-tree $T_n$ for MergeSort for a sufficiently large $n$, with level numbers shown on the left and the numbers of nodes in the respective level shown on the right. The nodes correspond to calls to MergeSort and show sizes of (sub)arrays passed to those calls. The last non-empty level is $h$. The empty levels (all those numbered $> h$) are not shown. The root corresponds to the original call to MergeSort. If a call that is represented by a node $p$ executes further recursive calls to MergeSort then these calls are represented by the children of $p$; otherwise $p$ is a leaf. The wavy line $\ldots\ldots$ represents a path in $T_n$.

$^{19}$Or, by virtue of the above-quoted result from [6], with the difference not larger than the half of the length of the shorter list plus 1.
The idea behind the derivation is strikingly simple. It is based on the observation\(^{20}\) that for every \(k \in \mathbb{N}\), the maximum number \(C_k\) of comps performed at each level \(\ell_k\) of \(T_n\) is given by this neat formula\(^{22}\)

\[
C_k = \max\{n - 2^k, 0\}.
\]

(3)

Since

\(n - 2^k > 0\) if, and only if, \(\lceil \lg n \rceil - 1 \geq k\),

(4)

the Corollary\(^{22}\) will allow me to conclude from (3) and (4) the main result of this paper:\(^{23}\)

\[
W(n) = \sum_{k \in \mathbb{N}} C_k = \sum_{k=0}^{\lceil \lg n \rceil - 1} (n - 2^k) = n\lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1 = \sum_{i=1}^{n} \lceil \lg i \rceil.
\]

(5)

The missing details\(^{24}\) in the above sketch are in [9]. Naturally, their only purpose is to prove the equality (3) for all \(k \in \mathbb{N}\), as the rest, shown in (5), easily follows from it. In particular, we get:

**The Main Theorem 4.1.** The number \(W(n)\) of comparisons of keys that MergeSort performs in the worst case while sorting an \(n\)-element array is

\[
W(n) = \sum_{i=1}^{n} \lceil \lg i \rceil = n\lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1.
\]

(6)

*Proof in [9].* 

From that we can conclude a usual rough characterization of \(W(n)\):

\[
W(n) \leq n(\lg n + 1) - 2^{\lg n} + 1 = n \lg n + n - n + 1 = n \lg n + 1
\]

and

\[
W(n) \geq n \lg n - 2^{\lg n+1} + 1 = n \lg n - 2n + 1.
\]

---

\(^{20}\)Which I prove in [9] as Theorem 4.6, page 14.

\(^{21}\)Empty or not.

\(^{22}\)It is a simplification of formulas used in derivation presented in [2] and discussed in Section 8 page 16; in particular, it does not refer to the depth \(h\) of the decision tree \(T_n\).

\(^{23}\)This is how I have been deriving it in my undergraduate Analysis of Algorithms class for some 15 years or so, now.

\(^{24}\)Which I did not show in my Analysis of Algorithms class.
Therefore,
\[ W(n) \in \Theta(n \log n). \]

The occurrence of \( \sum_{i=1}^{n} \lceil \lg i \rceil \) in (6) allows to conclude that \( W(n) \) is exactly equal\(^{25}\) to the number of comparisons of keys that the binary insertion sort, considered by H. Steinhaus in [5] and analyzed in [3], performs in the worst case. Since the binary insertion sort is known to be worst-case optimal\(^{26}\) in the class of algorithms that perform incremental sorting, MergeSort is worst-case optimal in that class\(^{27}\) too. From this and from the observation at the end of Section 3, page 6, I conclude that no algorithm that sorts by merging two sorted lists and only by means of comps is worst-case optimal in the class of algorithms that sort by means of comps as it must perform 8 comps in the worst case while sorting 5 elements\(^{28}\) while one can sort 5 elements by means of comps with no more than 7 comps.

5. Close smooth bounds on \( W(n) \)

Our formula for \( W(n) \) contains a function ceiling that is harder to analyze than arithmetic functions and their inverses. In this Section, I outline a derivation of close lower and upper bounds on \( W(n) \) that are expressible by simple arithmetic formulas. I show that these bounds are the closest to \( W(n) \) in the class of functions of the form \( n \lg n + cn + 1 \), where \( c \) is a real constant. The detailed derivation and missing proofs can be found in [9].

Using the function \( \varepsilon \) (analyzed briefly in [3] and [7]), a form of which is shown on Figure 3, given by:
\[ \varepsilon = 1 + \theta - 2^\theta \quad \text{and} \quad \theta = \lfloor \lg n \rfloor - \lg n, \quad (7) \]
one can conclude\(^{29}\) that, for every \( n > 0 \),
\[ n \lfloor \lg n \rfloor - 2^{\lfloor \lg n \rfloor} = n(\lg n + \varepsilon - 1), \quad (8) \]

\(^{25}\) [3] contains no mention of that fact.
\(^{26}\) With respect to the number of comparisons of keys performed.
\(^{27}\) Although it is not a member of that class.
\(^{28}\) They can be split in two: 1 plus 4, and follow the binary insertion sort, or 2 plus 3, and follow MergeSort.
\(^{29}\) See [7], Thm. 12.2 p. 94 for a proof.
which yields

\[ W(n) = n(\log n + \varepsilon - 1) + 1 = n \log n + (\varepsilon - 1)n + 1. \]  

(9)

\[ \delta = 1 - \log e + \log \log e \approx 0.0860713320559342 \]  

(10)

for every

\[ n = 2^{\lfloor \log n + \log \log e \rfloor} \log 2 \]  

(11)

and only such \( n \). The function \( \varepsilon \) restricted to integers never reaches the value \( \delta \). However, \( \delta \) is the supremum of \( \varepsilon \) restricted to integers.

\textbf{Proof} in \[9\]. \hfill \Box

Characterization (9) and Property 5.1 yield close smooth bounds of \( W(n) \). They are both of the form \( n \log n + cn + 1 \) and they sandwich tightly \( W(n) \) between each other. If one sees \( W(n) \) as an infinite polygon its lower bound circumscribes it and its upper bound inscribes it.

\[30\]The constant \( 1 - \log e + \log \log e \) has been known as the \textit{Erdös constant} \( \delta \). Erdös used it around 1955 in order to establish an asymptotic upper bound for the number \( M(k) \) of different numbers in a multiplication table of size \( k \times k \) by means of the following limit:

\[ \lim_{k \to \infty} \frac{\ln \frac{k \times k}{M(k)}}{\ln \ln (k \times k)} = \delta. \]

\[31\]Which it is.
Theorem 5.2. $W(n)$ is a continuous concave function, linear between the points $n = 2^{\lfloor \lg n \rfloor}$, that for every $n > 0$ satisfies this inequality:

\[ n \lg n - n + 1 \leq W(n) \leq n \lg n - (1 - \delta)n + 1 < n \lg n - 0.913n + 1, \tag{12} \]

with the left $\leq$ becoming $=$ for every $n = 2^{\lfloor \lg n \rfloor}$ and the right $\leq$ becoming $=$ for every $n = 2^{\lfloor \lg n + \lg \lg e \rfloor} \ln 2$, and only for such $n$. Moreover, the graph of $W(n)$ is tangent to the graph of $n \lg n - (1 - \delta)n + 1$ at the points $n = 2^{\lfloor \lg n + \lg \lg e \rfloor} \ln 2$, and only at such points.

Proof in [9].

The bounds given by (12) are really close\(^{32}\) to the exact value of $W(n)$, as it is shown on Figure 4 page 11. The exact value $n \lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1$ is a continuous function (if $n$ is interpreted as a real variable) despite that it incorporates discontinuous function ceiling.

\(^{32}\)The distance between them is less than $\delta n \approx 0.0860713320559342n$ for any positive integer $n$. 

\[^{32}\]
Note 5.3. It seems interesting that \( W(n) = n \lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1 \) (whether \( n \) is interpreted as a real variable or an integer variable) is linear between points \( n = 2^{\lceil \lg n \rceil} \) and linearly interpolates its own lower bound \( n \lg n - n + 1 \) between these points.

For \( n \) restricted to positive integers, the inequality (12) can be slightly enhanced by replacing the \( \leq \) symbol with \(<\), with the following result.

**Theorem 5.4.** \( 1 - \delta \) is the greatest constant \( c \) such that for every integer \( n \geq 1 \),
\[
W(n) < n \lg n - cn + 1.
\] (13)

*Proof in \([9]\).*

Theorem 5.4 can be reformulated as follows.

**Corollary 5.5.**
\[
\inf \{ c \in \mathbb{R} \mid \forall n \in \mathbb{N} \setminus \{0\}, W(n) < n \lg n - cn + 1 \} = 1 - \delta.
\] (14)

*Proof in \([9]\).*

No upper bound of \( W(n) \) that has a form \( n \lg n - cn + 1 \) can coincide with \( W(n) \) at any integer \( n \), as the following fact ascertains.

**Corollary 5.6.** There is no constant \( c \) such that for every integer \( n \geq 1 \),
\[
W(n) \leq n \lg n - cn + 1
\] (15)
and for some integer \( n \geq 1 \),
\[
W(n) = \lg n - cn + 1.
\] (16)

*Proof in \([9]\).*

In particular
\[
\inf \{ c \in \mathbb{R} \mid \forall n \in \mathbb{N} \setminus \{0\}, W(n) \leq n \lg n - cn + 1 \} = 1 - \delta.
\] (17)

Moreover, we can conclude from Theorem 5.4 the following fact.

---

33 Note the \( \leq \) symbol in (17).
Corollary 5.7. $1 - \delta$ is the greatest constant $c$ such that for every integer $n \geq 1$,
\[ W(n) \leq \lceil n \log n - cn \rceil. \] \hfill (18)

Proof in [9]. \hfill \Box

Since for any integer $n \geq 1$, $W(n)$ is integer, the lower bound given by (12) yields
\[ \lceil n \log n \rceil - n + 1 \leq W(n) \leq \lceil n \log n - 0.913n \rceil. \] \hfill (19)

By virtue of Corollary 5.7 for some integers $n \geq 1$,
\[ W(n) > \lceil n \log n - 0.914n \rceil. \] \hfill (20)

Although the bounds given by (19) are tighter than those given by (12), they nevertheless involve the discontinuous ceiling function, so that they may not be as easy to visualize or analyze as some differentiable functions, thus losing their advantage over the precise formula $W(n) = n \lceil \log n \rceil - 2^\lceil \log n \rceil + 1$. Therefore, the bounds given by (12) appear to have an analytic advantage over those given by (19).

6. Other properties of the recursion tree $T_n$

This sections contains some well-known auxiliary facts that I didn’t need for the derivation of the exact formula for $W(n)$ but am going to derive from the Main Lemma 4.1 of [9] for the sake of a thoroughness of my analysis of the decision tree $T_n$.

Theorem 6.1. The depth $h$ of the recursion tree $T_n$ is
\[ h = \lceil \log n \rceil. \] \hfill (21)

Proof in [9]. \hfill \Box

Note 6.2. Theorem 6.1 allows for quick derivation of fairly close upper bound on the number of comps performed by MergeSort on an $n$-element array. Since at each level of $T_n$ less than $n$ comparisons are performed by Merge and at level $h$ no comps are performed, and there are $h = \lceil \log n \rceil$ levels below level $h$, the total number of comps is not larger than
\[ (n - 1)h = (n - 1)(\lceil \log n \rceil) < (n - 1)(\log n + 1) \in O(n \log n). \] \hfill (22)

\[^{34}\text{For instance, for } n = 11.\]
\[^{35}\text{Almost the same bounds were given in [2]; see Section 8 for more details on this.}\]
A cut of a tree $T_n$ is a set $\Gamma$ of nodes of $T$ such that every branch in $T_n$ has exactly one element in $\Gamma$.

**Theorem 6.3.** The sum of values shown at the elements of any cut of $T_n$ is $n$.

*Proof in [9].*

**Theorem 6.4.** The number of leaves in the recursion tree $T_n$ is $n$.

*Proof in [9].*

The following corollary provides some statistics about recursive calls to MergeSort.

**Corollary 6.5.** For every integer $n > 0$,

(i) $T_n$ has $2n - 1$ nodes.

(ii) The number or recursive calls spurred by MergeSort on any $n$-element array is $2(n - 1)$.

(iii) The sum $S_n$ of all values shown in the recursion tree $T_n$ on Figure 2 is equal to:

\[ S_n = n[\lg n] - 2^{[\lg n]} + 2n = n(\lg n + \varepsilon + 1). \]  \hfill (23)

(iv) The average size $A_n$ of array passed to any recursive call to MergeSort while sorting an $n$-element array is:

\[ A_n = \frac{1}{2}(1 + \frac{1}{n-1})(\lg n + \varepsilon) \approx \frac{1}{2}(\lg n + \varepsilon). \]  \hfill (24)

*Proof in [9].*

Here is a very insightful property. It states that MergeSort is splitting its input array fairly evenly so that at any level of the recursive tree, the difference between the lengths of the longest sub-array and the shortest sub-array is $\leq 1$. This fact is the root cause of good worst-case performance of MergeSort.

---

36 A maximal path.
37 The sizes of the sub-arrays passed to recursive calls at any non-empty level $k$ of the decision tree $T_n$ above the last non-empty level $h$ are the same as the sizes of the elements of the maximally even partition of an $n$-element set onto $2^k$ subsets.
Property 6.6. The difference between values shown by any two nodes in the same level of the recursion tree $T_n$ is $\leq 1$.

Proof in [9].

Property 6.6 has this important consequence that Merge is, by virtue of the observation on page 6 after the Theorem 3.3 page 6 worst-case comparison-optimal while merging any two sub-arrays of the same level of the recursion tree. Thus the worst-case of MergeSort cannot be improved just by replacing Merge with some tricky merging $X$ as long as $X$ merges by means of comparisons of keys.

Corollary 6.7. Replacing Merge with any other method that merges sorted arrays by means of comps will not improve the worst-case performance of MergeSort measured with the number of comps while sorting an array.

Proof. Proof follows from the above observation.

Since a parent must show a larger value than any of its children, the Property 6.6 has also the following consequence.

Corollary 6.8. The leaves in the recursion tree $T_n$ can only reside at the last two non-empty levels of $T_n$.

Proof. Proof follows from the Property 6.6 as the above observation indicates.

As a result, one can conclude that the recursion tree $T_n$ has the mimum internal and external path lengths among all binary trees on $2n - 1$ nodes.

Since all nodes at the level $h$ of the recursion tree $T_n$ are leaves and show value 1, no node at level $h - 1$ can show a value $> 2$. Indeed, level $h - 1$ may only contain leaves, that show value 1, and parents of nodes of level $h$ that show value $1 + 1 = 2$. This observation and the previous result allow for easy characterization of contents of the last two non-empty levels of tree $T_n$.

Corollary 6.9. For every $n \geq 2$:

(i) there are $2^h - n$ leaves, all showing value 1, at the level $h - 1$, 

38Cf. [3], Sec. 5.3.1 Ex. 20 page 195.
Suchenek: Elementary Yet Precise Worst-case Analysis of MergeSort (SV) 16

(ii) there are \( n - 2^{h-1} \) non-leaves, all showing value 2, at the level \( h - 1 \), and

(iii) there are \( 2n - 2^h \) nodes, all leaves showing value 1, at the level \( h \) of the recursion tree \( T_n \), where \( h \) is the depth\(^40\) of \( T_n \).

Proof in [9]. \qed

7. A derivation of \( W(n) \) without references to the recursion tree

In order to formally prove Theorem 4.1 without any reference to the recursion tree, I use here the well-known\(^41\) recurrence relation

\[
W(n) = W(\lfloor n/2 \rfloor) + W(\lceil n/2 \rceil) + n - 1 \text{ if } n \geq 2
\]

\[
W(1) = 0
\]

that easily follows from the description (Algorithm 3.1 page 5) of MergeSort, steps 2a, 2c and Theorem 3.3. I am going to prove, by direct inspection, that the function \( W(n) \) defined by (6) satisfies equations (25) and (26).

The details of the proof are in [9].

8. Other work

Although some variants of parts of the formula (6) appear to have been known for quite some time now, even otherwise precise texts offer derivations that leave room for improvement. For instance, the recurrence relation for MergeSort analyzed in [4] asserts that the least number of comparisons of keys performed outside the recursive calls, if any, that suffice to sort an array of size \( n \) is \( n \) rather than \( n - 1 \). This seemingly inconsequential variation results in a solution \( W(n) = \sum_{i=1}^{n-1} \lceil \lg i \rceil + 2 \) on page 2, Exercise 1.4, rather than the correct formula \( W(n) = \sum_{i=1}^{n} \lceil \lg i \rceil \) derived in this paper. (Also,

\(^{39}\)This value shows in the lower right corner of Figure 2 page 7 of a sketch of the recursion tree \( T_n \); it was not need needed for the derivation of the main result (6) page 8, included for the sake of completeness only.

\(^{40}\)The level number of the last non-empty level of \( T_n \).

\(^{41}\)For instance, derived in [1] and [2].

\(^{42}\)I saw \( W(n) = \sum_{i=1}^{n} \lfloor \lg i \rfloor \) on slides that accompany [4].
the relevant derivations presented in [4], although quite clever, are not nearly as precise and elementary as those presented in this paper.) As a result, the fact that MergeSort performs exactly the same number of comparisons of keys as does another classic, binary insertion sort, considered by H. Steinhaus and analyzed in [3], remains unnoticed.

Pages 176 – 177 of [2] contain an early sketch of proof of

$$W(n) = nh - 2^h + 1,$$  \hspace{1cm} (27)

where $h$ is the depth of the recursion tree $T_n$, with remarkably close\(^{43}\) bounds given by (28) page 18. It is similar\(^{44}\) to a simpler derivation based on the equality [3], presented in this paper in Section 4 and outlined in [5] page 8 (except for the $\sum_{i=1}^{n} \lceil \lg i \rceil$ part), which it predates by several years.

The [2]'s version of the decision tree $T_n$ (Figure 4.14 page 177 of [2], shown here on Figure 5) was a re-use of a decision tree for the special case of $n = 2^\lceil \lg n \rceil$, with an ambiguous, if at all correct\(^{45}\), comment in the caption that “[w]henever a node size parameter is odd, the left child size parameter is rounded up\(^{46}\) and the right child size is rounded down\(^{47}\)” The proof of the fact, needed for the derivation in [2], that $T_n$ had no leaves outside its last two levels (Corollary 6.8 page 15, not needed for the derivation presented in Section 4) was waved with a claim “[w]e can\(^{48}\) determine that [...]”

---

\(^{43}\)Although not 100 percent correct.

\(^{44}\)The idea behind the sketch of the derivation in [2] was based on an observation that

$$W(n) = \sum_{i=0}^{h-2} (n - 2^i) + \frac{n - B}{2},$$

where $B$ was the number of leaves at the level $h - 1$ of the decision tree $T_n$; it was sketchily derived from the recursion tree shown on Figure 5 and properties stated in the Corollary 6.9 page 15 (with only a sketch of proof in [2]) not needed for the derivation presented in Section 4.

\(^{45}\)It may be interpreted as to imply that for any level $k$, all the left-child sizes at level $k$ are the same and all the right-child sizes at level $k$ are the same, neither of which is a valid statement.

\(^{46}\)Should be: down, according to (25) page 16.

\(^{47}\)Should be: up, according to (25) page 16.

\(^{48}\)This I do not doubt.
Although $h$ was claimed in [2] to be equal to $[\lg(n+1)]$ \[49\](and not to the correct $[\lg n]$ given by the equality \[21\] page 13, a fact not needed for the derivation presented in Section [4], somehow the mostly correct conclusion\[50\] was inferred from it, however, with no details offered - except for a mention that a function $\alpha$ that satisfies $h = \lg n + \lg \alpha$, similar to function $\varepsilon$ shown on Figure 3 page 10, was used. It stated that (Theorem 4.6, page 177, in [2]):

$$[n \lg n - n + 1] \leq W(n) \leq [n \lg n - 0.914 n]. \quad (28)$$

It follows from \[19\] page 13 that the constant 0.914 that appears in (28) is incorrect. It was a rounding error\[51\] I suppose, that produced a false upper bound\[52\].

---

\[49\] Which claim must have produced an incorrect formula $n[\lg(n+1)] - 2^{[\lg(n+1)]} + 1$ for $W(n)$ and precluded concluding the neat characterization $W(n) = \sum_{i=1}^{n} [\lg i]$.  

\[50\] Almost identical with \[19\] page 13 except for the constant 0.914.  

\[51\] Of $1 - \delta$, where $\delta$ is given by \[10\] page 10.  

\[52\] For instance, if $n = 11$ then \texttt{MergeSort} performs 29 comparisons of keys while the
9. Best-case analysis of MergeSort

It turns out that derivation of minimum number $B(n)$ of comps performed by MergeSort on an $n$-element array is a bit more tricky. A formula

$$\frac{n}{2} \left( \lceil \lg n \rceil + 1 \right) - \sum_{k=0}^{\lceil \lg n \rceil} 2^k \text{Zigzag} \left( \frac{n}{2^{k+1}} \right),$$

where

$$\text{Zigzag}(x) = \min(x - \lfloor x \rfloor, \lceil x \rceil - x),$$

has been derived and thoroughly analyzed in [8]. It has been also demonstrated in [8] that there is no closed-form formula for $B(n)$.

Incidentally, as it was pointed out in [8], $B(n)$ is equal to the sum $A(n, 2)$ of bits in binary representations of all integers $< n$.
Appendix A. A Java code of MergeSort

Figure A.6 shows a Java code of MergeSort.

```java
public static int[] Sort(int[] A, int lo, int hi)
    // input constrain: lo <= hi
    {
        if ((hi - lo) == 0)
            { //
                return (int[] A[lo]);
                int[] C = {A[lo]};
                return C;
            }
            int splitPoint = (lo + hi)/2; // floor of ...
            return MergeSort(A, lo, splitPoint),
            Sort(A, splitPoint + 1, hi));
    }
}
```

Figure A.6: A Java code of MergeSort. A code of Merge is shown on Figure 1.

Appendix B. Generating worst-case arrays for MergeSort

Figure B.7 shows a self-explanatory Java code of recursive method unSort that given a sorted array A reshuffles it, in a way resembling InsertionSort\(^{53}\) onto a worst-case array for MergeSort.

For instance, it produced this array of integers between 1 and 500:

1, 500, 2, 3, 4, 7, 5, 6, 8, 15, 9, 10, 11, 14, 12, 13, 16, 31, 17, 18, 19, 22, 20, 21, 30, 24, 25, 26, 29, 27, 28, 32, 62, 33, 34, 35, 38, 36, 37, 39, 46, 40, 41, 42, 45, 43, 44, 47, 61, 48, 49, 50, 53, 51, 52, 54, 60, 55, 56, 57, 59, 58, 63, 124, 64, 65, 66, 69, 67, 68, 70, 77, 71, 72, 73, 76, 74, 75, 78, 92, 79, 80, 81, 84, 82, 83, 85, 91, 86, 87, 88, 90, 89, 93, 123, 94, 95, 96, 99, 97, 98, 100, 107, 101, 102, 103, 106, 104, 105, 108, 122, 109, 110, 111, 114, 112, 113, 115, 121, 116, 117, 118, 120, 119, 125, 249, 126, 127, 128, 131, 129, 130, 132, 139, 133, 134, 135, 138, 136, 137, 140, 155, 141, 142, 143, 146, 144, 145, 147, 154, 148, 149, 150, 153, 151, 152, 156, 186, 157, 158, 159, 162, 160, 161, 163, 170, 164,

\(^{53}\)Although not with InsertionSort’s sluggishness; the number of moves of keys it performs is only slightly more than the minimum number\(^{29}\) of comps performed by MergeSort on any \(n\)-element array.
public static void unSort(int[] A, int lo, int hi)
// turns A onto a worst-case array
// input constrain: lo <= hi and A is sorted
{
    if ((hi - lo) <= 1) return; // already worst
    int splitPoint = (lo + hi)/2; // floor of ...
    int max = A[hi]; // will be sent to left array
    for (int i = hi; i > splitPoint; i--)
        A[i] = A[i-1]; //A[hi] is now 2nd largest
    A[splitPoint] = max;
    unSort(A, lo, splitPoint); // still sorted
    unSort(A, splitPoint + 1, hi); // still sorted
}

Figure B.7: A Java code of unSort that, given a sorted array A, reshuffles it onto a worst-case array for MergeSort. Its structure mimics the Java code of MergeSort shown on Figure A.6.

165, 166, 169, 167, 168, 171, 185, 172, 173, 174, 177, 175, 176, 178, 184, 179, 180, 181, 183, 182, 187, 248, 188, 189, 190, 193, 191, 192, 194, 201, 195, 196, 197, 200, 198, 199, 202, 216, 203, 204, 205, 208, 206, 207, 209, 215, 210, 211, 212, 214, 213, 217, 247, 218, 219, 220, 223, 221, 222, 224, 231, 225, 226, 227, 230, 228, 229, 232, 246, 233, 234, 235, 238, 236, 237, 239, 245, 240, 241, 242, 244, 243, 250, 499, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 263, 261, 262, 265, 280, 266, 267, 268, 271, 269, 270, 272, 279, 273, 274, 275, 278, 276, 277, 281, 311, 282, 283, 284, 287, 285, 286, 288, 295, 289, 290, 291, 294, 292, 293, 296, 310, 297, 298, 299, 302, 300, 301, 303, 309, 304, 305, 306, 308, 307, 312, 373, 313, 314, 315, 318, 316, 317, 319, 326, 320, 321, 322, 325, 323, 324, 327, 341, 328, 329, 330, 333, 331, 332, 334, 340, 335, 336, 337, 339, 338, 342, 372, 343, 344, 345, 348, 346, 347, 349, 356, 350, 351, 352, 355, 353, 354, 357, 371, 358, 359, 360, 363, 361, 362, 364, 370, 365, 366, 367, 369, 368, 374, 498, 375, 376, 377, 380, 378, 379, 381, 382, 383, 384, 387, 385, 386, 389, 404, 390, 391, 392, 395, 393, 394, 396, 403, 397, 398, 399, 402, 400, 401, 405, 435, 406, 407, 408, 411, 409, 410, 412, 419, 413, 414, 415, 418, 416, 417, 420, 434, 421, 422, 423, 426, 424, 425, 427, 433, 428, 429, 430, 432, 431, 436, 497, 437, 438, 439, 442, 440, 441, 443, 450, 444, 445, 446, 449, 447, 448, 451, 465, 452, 453, 454, 457, 455, 456, 458, 464, 459, 460, 461, 463, 462, 466, 496, 467, 468, 469, 472, 470, 471, 473, 480, 474, 475, 476, 479, 477, 478, 481, 495, 482, 483, 484, 487, 485, 486, 488, 494, 489, 490, 491, 493, 492.
It took my MergeSort 3,989 comps to sort it. Of course,
\[ 500\lceil \lg 500 \rceil - 2^{\lceil \lg 500 \rceil} + 1 = 4,500 - 512 + 1 = 3,989. \]

Appendix C. Notes from my Analysis of Algorithms lecture

Below are some of the class digital notes I wrote while lecturing Analysis of Algorithms in Spring 2012, with some comments added after class. Figure 4.14 (decision tree) is from the course textbook [2], page 177, showing a decision tree for MergeSort. Note: This figure is copyrighted by Addison Wesley Longman (2000). I used it transformatively in my class for nonprofit education, criticism, and comment purposes only, and not for any other purpose, as prescribed by U.S. Code Title 17 Chapter 1 para 107 that established the “fair use” exception of copyrighted material.
Suchenek: Elementary Yet Precise Worst-case Analysis of MergeSort

**Recall:** \( \max \{ n \leq 2^i \} = \lfloor x \rfloor \)

So, the total number of comps in all merges is (exactly!):

\[
W(n) = \sum_{i=0}^{\lfloor \log_2 n \rfloor} \left( n - 2^i \right) = \sum_{i=0}^{\lfloor \log_2 n \rfloor} \left( 2 \cdot 2^i - i \right) = \left( \lfloor \log_2 n \rfloor + 1 \right) n - \left( 2 \cdot 2^{\lfloor \log_2 n \rfloor + 1} - 1 \right)
\]

\[
= n \left( 2^{\lfloor \log_2 n \rfloor + 1} - 2^i + 1 \right)
\]

The number of comps by MergeSort in the worst case when testing \( n \) elements:

\[
w(n) = m \left( 2^{\lfloor \log_2 n \rfloor + 1} - 2^i + 1 \right).
\]

The above formula is the same as in the textbook (p 117), the last line of equation (4.6), except that ours has been derived exactly without approximations or simplifying assumptions.
Below is the improved recursion tree (of the Figure 4.14 page page 177 of [2]) that I used in class in Spring 2012.

In Spring 2010 and before, I was deriving the equality (3) on page 8 during my lectures directly from the recurrence relation (25), (26) on page 16.

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Suchenek: Elementary Yet Precise Worst-case Analysis of MergeSort

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