Reverse quantum speed limit and minimum Hilbert space norm

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Abstract

The reverse quantum speed limit (RQSL) gives an upper limit to the time for evolution between initial and final quantum states. We show that, in conjunction with the existence of a minimum time scale, the RQSL implies a lower limit to the norm of the change in a quantum state, and confirm that this limit is satisfied in two-state and ideal-measurement models. Such a lower limit is of relevance for interpretational issues in probability and for understanding the meaning of probability in Everett quantum theory.

Key words: reverse quantum speed limit, minimum Hilbert space norm, minimum time scale, discrete Hilbert space, probability, preclusion, Everett interpretation

1 Introduction

The spacetime that provides the mathematical framework within which both classical and quantum fields are defined is a continuum. Nevertheless, it has been argued for decades that the combination of quantum mechanics and general relativity implies the existence of a minimum length scale at or near the Planck length. Relativistic covariance then implies the existence of a corresponding minimum time scale comparable in magnitude to the Planck time.

Buniy, Hsu and Zee have argued that the existence of a minimum length scale implies a minimum scale to the norms of states in quantum-mechanical Hilbert space, such that components of the wavefunction with norm below this scale can be removed from the

\[^1\]A device-independent argument for such a minimum length scale is presented in [1]. For a review of this subject, see [2].
wavefunction. The existence of such a minimum Hilbert-space norm provides a solution to the problem of “maverick” states in Everett quantum theory and thus allows for an understanding of the origin and nature of probability in that theory.

Hsu has presented an estimate for the norm of the change in a quantum state in the presence of a minimum time scale. In the present note we employ the reverse quantum speed limit (RQSL) of Mohan and Pati to derive a lower limit to this norm.

2 Reverse quantum speed limit

The RQSL gives an upper limit to the time for an initial state to evolve to a final state. The expression for this limit involves the reference section , a function of the state vector and the initial time (here taken to be ) defined as

The length of the reference section for evolution from time to time is defined to be

In terms of the length, the RQSL is

where is the square root of the expectation value of the variance of the Hamiltonian in the state :

Here we will only consider the case of time-independent Hamiltonians, in which case is time-independent and the formula can be evaluated at any convenient time, in particular at .

3 Minimum Hilbert space norm

Consider the time evolution to occur over a time interval short enough so that we can approximate the derivative of the reference section during this interval as

Recently Calmet and Hsu have presented an argument for such a Hilbert-space minimum norm based on a minimum scale to angular measurement coming from quantum mechanics and general relativity.

For the generalization to time-dependent Hamiltonians, see .
Clearly the range of applicability of subsequent results will depend on the range of validity of the approximation (5); we will return to this issue below.

Using (5) in (2) with $T = \Delta t$,

$$l(\chi(t))|_0^{\Delta t} = \left[ \left( \frac{\langle \chi(\Delta t)| - \langle \chi(0)|}{\Delta t} \right) \left( \frac{|\chi(\Delta t)| - |\chi(0)|}{\Delta t} \right) \right]^\frac{1}{2} \Delta t$$

$$= \left[ (\langle \chi(\Delta t)| - \langle \chi(0)|) (|\chi(\Delta t)| - |\chi(0)|) \right]^\frac{1}{2} \Delta t$$

(6)

Using (4) in (6), along with the normalization of $|\psi(t)\rangle$, yields

$$l(\chi(t))|_0^{\Delta t} = \left[ \langle \psi(\Delta t)| \psi(\Delta t) \rangle - 2|\langle \psi(\Delta t)| \psi(0) \rangle| + 1 \right]^\frac{1}{2} \Delta t$$

(7)

Expressing $|\psi(\Delta t)\rangle$ in terms of $|\Delta \psi(\Delta t)\rangle$, the change in $|\psi(t)\rangle$, i.e.

$$|\psi(\Delta t)\rangle = |\psi(0)\rangle + |\Delta \psi(\Delta t)\rangle,$$

(8)

and defining

$$z = 1 + \langle \Delta \psi(\Delta t)| \psi(0) \rangle$$

(9)

we obtain from (7)

$$\left( l(\chi(t))|_0^{\Delta t} \right)^2 = \langle \Delta \psi(\Delta t)| \Delta \psi(\Delta t) \rangle + 2(\text{Re}(z) - |z|).$$

(10)

From the RQSL, eq. (3),

$$\left( l(\chi(t))|_0^{\Delta t} \right)^2 \geq \left( \frac{\Delta t \Delta H}{\hbar} \right)^2,$$

(11)

so, with (10), we have

$$\langle \Delta \psi(\Delta t)| \Delta \psi(\Delta t) \rangle - \left( \frac{\Delta t \Delta H}{\hbar} \right)^2 \geq 2(|z| - \text{Re}(z)).$$

(12)

But for any complex number $z$

$$|z| - \text{Re}(z) \geq 0,$$

(13)

so

$$\langle \Delta \psi(\Delta t)| \Delta \psi(\Delta t) \rangle \frac{1}{2} \geq \frac{\Delta t \Delta H}{\hbar}.$$

(14)

Now suppose that time is in fact discrete; i.e., there is a lower limit $\Delta t_{\text{min}}$ during which time evolution can take place. Then

$$\langle \Delta \psi(\Delta t_{\text{min}})| \Delta \psi(\Delta t_{\text{min}}) \rangle \frac{1}{2} \geq \text{NormLim}_{\text{RQSL}}.$$
where
\[
\text{NormLim}_{\text{RQSL}} = \frac{\Delta t_{\text{min}} \Delta H}{\hbar}.
\] (16)
That is, there is no Hamiltonian with expected variance \((\Delta H)^2\) in state \(|\psi(0)\rangle\) which will in a time \(\Delta t_{\text{min}}\) evolve \(|\psi(0)\rangle\) into a state \(|\psi(\Delta t_{\text{min}})\rangle\) closer in norm to \(|\psi(0)\rangle\) than \(\text{NormLim}_{\text{RQSL}}\).4

4 Examples

Since time appears to us to be continuous rather than discrete, we will take the approximation (5) and the conclusion (15) that follows from it be valid provided \(\Delta t_{\text{min}}\) is much smaller than the smallest time scale characterizing changes in \(|\psi(t)\rangle\). We consider two examples:

4.1 Two-state system

The Hilbert space \(\mathcal{H}_2\) is spanned by two vectors, \(|S_1\rangle\) and \(|S_2\rangle\), which we take to be normalized eigenstates of the Hamiltonian \(\hat{H}_{\mathcal{H}_2}\):
\[
\hat{H}_{\mathcal{H}_2}|S_i\rangle = E_i|S_i\rangle, \quad \langle S_i|S_j\rangle = \delta_{i,j}, \quad i, j = 1, 2.
\] (17)
The normalized initial state vector is
\[
|\psi_{\mathcal{H}_2}(0)\rangle = c_1|S_1\rangle + c_2|S_2\rangle, \quad |c_1|^2 + |c_2|^2 = 1.
\] (18)
The lower limit to the norm of the change in the state vector from \(t = 0\) to \(t = \Delta t_{\text{min}}\) is, from (4), (16), (17) and (18),
\[
\text{NormLim}_{\text{RQSL}}_{-\mathcal{H}_2} = \left[|c_1|^2 E_1^2 + |c_2|^2 E_2^2 - \left(|c_1|^2 E_1 + |c_2|^2 E_2\right)^2\right]^{\frac{1}{2}} \frac{\Delta t_{\text{min}}}{\hbar}.
\] (19)
The exact state vector for \(t \geq 0\) is, from (17), (18) and the Schrödinger equation,
\[
|\psi_{\mathcal{H}_2}(t)\rangle = c_1 \exp\left(-\frac{iE_1 t}{\hbar}\right)|S_1\rangle + c_1 \exp\left(-\frac{iE_2 t}{\hbar}\right)|S_2\rangle.
\] (20)

4 Note that the estimate given by Hsu [19, eq. (6)] for the norm of the change in the state, which in our notation is \(\Delta t_{\text{min}}\langle\psi(0)|\hat{H}^2|\psi(0)\rangle^{\frac{1}{2}}/\hbar\), will always be greater than or equal to \(\text{NormLim}_{\text{RQSL}}\). Hsu argues that by considering the estimates for all possible initial states in a finite-dimensional Hilbert space one may arrive at a lower bound on the norm. The minimum norm given in the present paper applies to all states with a specified value of \(\Delta H\).
So, the shortest time scale characterizing changes in $|\psi_{H_2}(t)\rangle$ is

$$\Delta t_{H_2} = \frac{2\pi \hbar}{\max_i(|E_i|)}, \quad (21)$$

in that $|\psi(t)\rangle_{H_2}$ will be approximately constant over time intervals much smaller than (21).

Using (8), (18) and (20),

$$\langle \Delta \psi_{H_2}(\Delta t) | \Delta \psi_{H_2}(\Delta t) \rangle^{1/2} = \left[ 2 - 2|c_1|^2 \cos \left( \frac{E_1 \Delta t}{\hbar} \right) - 2|c_2|^2 \cos \left( \frac{E_2 \Delta t}{\hbar} \right) \right]^{1/2}. \quad (22)$$

If

$$\Delta t \ll \Delta t_{H_2} \quad (23)$$

then

$$\langle \Delta \psi_{H_2}(\Delta t_{\text{min}}) | \Delta \psi_{H_2}(\Delta t_{\text{min}}) \rangle^{1/2} = \left[ |c_1|^2 E_1^2 + |c_2|^2 E_2^2 \right]^{1/2} \frac{\Delta t}{\hbar}, \quad (24)$$

so the norm of the change in the state vector during the minimum possible time interval $\Delta t_{\text{min}}$ (assumed, as discussed above, to satisfy (23)) is greater than or equal to the limit (19) deduced from the RQSL:

$$\langle \Delta \psi_{H_2}(\Delta t_{\text{min}}) | \Delta \psi_{H_2}(\Delta t_{\text{min}}) \rangle^{1/2} \geq \text{NormLim}_{\text{RQSL-}H_2} \quad (25)$$

### 4.2 Ideal measurement

We consider the simplest ideal measurement model, perhaps more appropriately described as a model of a “detector” rather than a “measuring device.” An observer (a human or an apparatus) starts in a ready state $|O_1\rangle$ and measures a two-state system that starts and remains in its initial state. If the observer finds that the measured system is in state $|S_1\rangle$ the observer remains in the ready state; if she finds the measured system to be in state $|S_2\rangle$ the observer transitions to a state $|O_2\rangle$.

In detail: The Hilbert space $H_D$ is spanned by the four vectors $|S_i\rangle \otimes |O_j\rangle$, $i, j = 1, 2$, where

$$\langle S_i|S_j \rangle = \langle O_i|O_j \rangle = \delta_{i,j}, \quad i, j = 1, 2. \quad (26)$$

The Hamiltonian is

$$\hat{H}_{H_D} = \hat{P}_{S_2} \otimes \hat{h}_O \quad (27)$$

where

$$\hat{P}_{S_2} = |S_2\rangle \langle S_2| \quad (28)$$

and

$$\hat{h}_O = i\kappa (|O_2\rangle \langle O_1| - |O_1\rangle \langle O_2|) \quad (29)$$
The state vector describing the observer and measured system at the initial time \( t = 0 \) has the observer in the ready state and uncorrelated with the measured system:

\[
|\psi_{HD}(0)\rangle = (c_1|S_1\rangle + c_2|S_2\rangle) \otimes |O_1\rangle,
\]

\[|c_1|^2 + |c_2|^2 = 1.\] (30)

Using (4), (16) and (26)-(30), we find the lower limit on the norm of the change in the state vector from \( t = 0 \) to \( t = \Delta t_{\text{min}} \) to be

\[
\text{NormLim}_{\text{RQSL-}H_D} = \frac{|c_2|\kappa\Delta t_{\text{min}}}{\hbar}.\] (31)

The exact state vector for \( t \geq 0 \) is, from (27)-(30) and the Schrödinger equation,

\[
|\psi_{HD}(t)\rangle = c_1|S_1\rangle|O_1\rangle + c_2|S_2\rangle \left[ \cos\left(\frac{\kappa t}{\hbar}\right)|O_1\rangle + \sin\left(\frac{\kappa t}{\hbar}\right)|O_2\rangle \right].\] (32)

From (32) we see that the time scale characterizing changes in \( |\psi_{HD}(t)\rangle \) is

\[
\Delta t_{H_D} = \frac{2\pi\hbar}{|\kappa|}.\] (33)

We can arrive at the same conclusion by noting that the distinct eigenvalues of \( \hat{H}_D \) are 0, \( \kappa \) and \( -\kappa \), so

\[
\Delta t_{H_D} = \frac{2\pi\hbar}{\max(|0|, |\kappa|, | - \kappa|)}.\] (34)

From (8), (26), (30) and (32),

\[
\langle \Delta \psi_{HD}(t)|\Delta \psi_{HD}(t)\rangle^{1/2} = |c_2| \left[ 2 - 2 \cos\left(\frac{\kappa t}{\hbar}\right) \right]^{1/2}.\] (35)

For \( \Delta t \) satisfying

\[
\Delta t \ll \Delta t_{H_D}\] (36)

the norm of the change in the state vector, (35), becomes

\[
\langle \Delta \psi_{HD}(\Delta t)|\Delta \psi_{HD}(\Delta t)\rangle^{1/2} = \frac{|c_2|\kappa\Delta t}{\hbar}.\] (37)

Taking \( \Delta t \) to be the minimum possible time increment \( \Delta t_{\text{min}} \), we see from (31) and (37) that in this case the norm of the change in the state vector is equal to the limit coming from the RQSL:

\[
\langle \Delta \psi_{HD}(\Delta t_{\text{min}})|\Delta \psi_{HD}(\Delta t_{\text{min}})\rangle^{1/2} = \text{NormLim}_{\text{RQSL-}H_D}.\] (38)

Note from (32) that at time \( t_{\text{meas}} = \pi\hbar/(2\kappa) \) the ideal measurement has completed, in that the observer and observed-system states have become perfectly correlated. This confirms that the choice of Hamiltonian (27) and initial state (30) indeed models an ideal measurement. Of course \( t_{\text{meas}} \) is much longer than \( \Delta t_{\text{min}} \); see (33), (36).
5 Discussion

While the arguments and results in [3, 5, 16–20] and the present paper suggest that the usual continuous Hilbert space is not the proper arena in which to do quantum physics, they do not make this case conclusively, let alone point to a mathematical structure to serve as a replacement. Nevertheless Hilbert-space discreteness is an idea that deserves continued investigation, particularly for its potential to explain one of the most recondite conceptual problems in physics and philosophy of science, that of the meaning and nature of probability.

This issue is particularly salient for probability in the Everett interpretation of quantum mechanics. In that theory, all outcomes of quantum processes occur in parallel, and since the inception of the theory physicists have struggled to understand the sense in which probabilities can be associated with outcomes if everything that can happen does happen [7, 22–39].

For example, when the ideal measurement of Sec.4.2 is complete, at time $t = t_{meas} = \frac{\pi \hbar}{2 \kappa}$, the state vector (32) has the form

$$|\psi_{H_D}(t_{meas})\rangle = c_1|S_1\rangle|O_1\rangle + c_2|S_2\rangle|O_2\rangle.$$ (39)

In the standard (single-outcome, non-Everett) interpretation of quantum mechanics commonly presented in textbooks, a post-measurement state such as (39) is replaced stochastically by one of the states corresponding to the possible outcomes of the measurement, either $|S_1\rangle|O_1\rangle$ (“system is in state 1, observer is in the ready state 1”) or $|S_2\rangle|O_2\rangle$ (“system is in state 2, observer is in state 2”), with the respective probabilities for each of these outcomes occurring being the absolute-squared amplitudes $|c_1|^2, |c_2|^2$. This is termed “reduction of the state vector” [40, p. 236].

In the Everett interpretation, on the other hand, there is no reduction. The two terms on the right-hand side of (39) represents two Everett worlds. In one of these worlds the outcome $|S_1\rangle|O_1\rangle$ has occurred, in the other the outcome $|S_2\rangle|O_2\rangle$ has occurred.

What can it possibly mean to say that one outcome is more or less probable than the other when in fact both have occurred?6 This question is referred to as the “incoherence problem” [14].

The incoherence problem presents itself in a particularly acute form if we consider a situation in which one or more of the outcomes is a world corresponding to a state of affairs that is assigned such small probability by the standard quantum interpretation that we would feel justified in neglecting even the possibility of its occurring. For example, such a world might be one in which large violations of the second law of thermodynamics

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6 It is precisely the existence of multiple outcomes that is critical in enabling the Everett interpretation to evade Bell’s theorem and the resulting conclusion that quantum mechanics is nonlocal; see [41, Sec. 1.1] and references therein.
occur, with heat flowing from colder to warmer bodies, gases and liquids spontaneously unmixing, etc. Such worlds are termed “maverick worlds” [6]. Of course our intuition about the nature of probability leads us to feel that we understand why any outcome with exceedingly small probability can be neglected. But, in the absence of a meaning we can assign to probability small or large, we are unable to explain why maverick outcomes are not features of our day-to-day experience.

Incorporating Hilbert-space discreteness into Everett quantum mechanics provides a resolution to the problem of maverick worlds. It implies that there is a minimum norm to state vectors, and one can then argue that maverick worlds, having norms below this minimum, are simply absent from any superposition of outcomes [16].

This might seem to still leave open the issue of how to assign meaning to probabilities larger than, say, those involving violation of the second law of thermodynamics—i.e., the probabilities encountered in the vast majority of physical situations. But in fact, preclusion of state-vector components with norms below a certain minimum holds the potential of giving meaning to these probabilities as well. For those probabilistic phenomena that are not precluded but that in fact may or may not be observed to occur, probability may be identified with subjective experience of a particular kind. In the words of de Finetti, one of the most prominent proponents of this subjective interpretation of probability: “What do we mean when we say, in ordinary language, that an event is more or less probable? We mean that we would be more or less surprised to learn that it has not happened [43, p. 174].” Furthermore, there is a substantial body of experimental evidence (see [20, Sec. 6.4] and references therein) that the subjective experience of probability is at its base a result of biological evolution [44], [20, Sec. 6.1]. And it can be argued that the biological evolution, over generation upon generation, of subjective-probabilistic expectations fit to maverick worlds will be of sufficiently small norm to be precluded, leaving only organisms with subjective expectations matching our familiar notions of probability and statistics. In a nutshell, “preclusion explains evolution which then explains subjective probability [20, p. 21].” A proof-of-concept quantum-mechanical model of this process is presented in [20, Secs. 6.2, 6.3].

In addition to thus accounting for our experiences of probabilistic phenomena within the completely-deterministic Everettian framework, Everett quantum mechanics with preclusion of states below a minimum norm addresses longstanding interpretational problems of probability per se [15]. Along with purely-subjective theories of probability, this theory has the virtue of leaving no ambiguity as to what is meant by probability; namely, the subjective reactions of organisms to probabilistic situations. At the same time, the objective feature of preclusion allows it to provide explanations for physical phenomena.

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7Kolmogorov: “We apply the theory of probability to the actual world of experiments in the following manner... If P(A) is very small, one can be practically certain that when conditions [for a repeatable experiment] are realized only once, the event A would not occur at all [42, pp. 3-4].”
in general and for biological evolution in particular, something which purely subjective theories are incapable of doing. An explanation should at the very least be “a statement of what is there in reality, and how it behaves and how that accounts for the explicanda [46].” So, the explanation of objectively-existing phenomena (including subjective probability⁸) must itself be something that exists objectively (in this theory, preclusion and the norms of quantum states).

For a detailed exposition of probability in the version of Everett quantum theory with preclusion outlined here, see [20, Secs. 6.1, 7, 8]. Other approaches to solving the Everett probability problem via discrete Hilbert space are presented in [16–19]. Much more work needs to be done to place minimum Hilbert space norms on a concrete foundation, but the completion of such a program should prove highly consequential.

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References

[1] Calmet, X., Graesser, M. and Hsu, S. D. H., “Minimum length from quantum mechanics and classical general relativity,” Phys. Rev. Lett. 93, 211101 (2004); arXiv:hep-th/0405033.

[2] S. Hossenfelder, “Minimal length scale scenarios for quantum gravity,” Living Rev. Relativity, 16, 2 (2013).

[3] R. V. Buniy, S. D. H. Hsu and A. Zee, “Is Hilbert space discrete?” Phys. Lett. B 630, 68-72 (2005); arXiv:hep-th/0508039.

[4] M. Faizal, M. M. Khalil and S. Das, “Time crystals from minimum uncertainty,” Eur. Phys. J. C 76, 30 (2016); arXiv:1501.03111.

[5] X. Calmet and S. D. H. Hsu, “Fundamental limit on angular measurements and rotations from quantum mechanics and general relativity,” arXiv:2108.11990 (2021).

[6] B. S. DeWitt, “Quantum mechanics and reality,” Physics Today 23, 30-35 (1970). Reprinted in [8].

⁸Subjective reactions are themselves objectively-existing phenomena. “The subjective opinion, as something known by the individual under consideration is, at least in this sense, something objective and can be a reasonable subject of a rigorous study” [47, p. 5].

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[7] H. Everett III, “‘Relative state’ formulation of quantum mechanics,” *Rev. Mod. Phys.* 29 454-462 (1957). Reprinted in [8].

[8] B. S. DeWitt and N. Graham, eds., *The Many Worlds Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1973).

[9] M. C. Price, “The Everett FAQ,” https://www.hedweb.com/everett/ (1995).

[10] J. A. Barrett, *The quantum mechanics of minds and worlds* (Oxford University Press, Oxford, 1999).

[11] L. Vaidman, “Many-worlds interpretation of quantum mechanics,” *The Stanford Encyclopedia of Philosophy* (Fall 2018 Edition), E. N. Zalta, ed., https://plato.stanford.edu/archives/fall2018/entries/qm-manyworlds/.

[12] C. Hewitt-Horsman, “An introduction to many worlds in quantum computation,” *Found. Phys.* 39, 826-902 (2009); arXiv:0802.2504.

[13] S. Saunders, J. Barrett, A. Kent and D. Wallace, eds., *Many Worlds? Everett, Quantum Theory & Reality*, (Oxford University Press, Oxford, 2010).

[14] D. Wallace, *The Emergent Multiverse: Quantum Theory According to the Everett Interpretation*, (Oxford University Press, Oxford, 2012).

[15] D. Wallace, “The Everett interpretation,” in R. Batterman, ed., *The Oxford Handbook of Philosophy of Physics* (Oxford University Press, Oxford, 2013).

[16] R. V. Buniy, S. D. H. Hsu and A. Zee, “Discreteness and the origin of probability in quantum mechanics,” *Phys.Lett. B* 640, 219-223 (2006); arXiv:hep-th/0606062.

[17] S. D. H. Hsu, “On the origin of probability in quantum mechanics,” *Mod. Phys. Lett. A* 27, 1230014 (2012); arXiv:1110.0549.

[18] S. D. H. Hsu, “The measure problem in no-collapse (many worlds) quantum mechanics,” *Int. J. Mod. Phys. D* 26, 1730008 (2017); arXiv:1511.08881.

[19] S. D. H. Hsu, “Discrete Hilbert space, the Born rule, and quantum gravity,” *Mod. Phys. Lett. A* 36, 2150013 (2021); arXiv:2007.12938.

[20] M. A. Rubin, “Probability, preclusion and biological evolution in Heisenberg-picture Everett quantum mechanics,” *Int. J. Mod. Phys. A* 36, 2150117 (2021); arXiv:2011.10029.

[21] B. Mohan and A. K. Pati, “Reverse quantum speed limit: How slow quantum battery can discharge?” arXiv:2006:14523 (2020).
[22] B. S. DeWitt, “The many-universes interpretation of quantum mechanics,” in Proceedings of the International School of Physics “Enrico Fermi” Course IL: Foundations of Quantum Mechanics, (Academic Press, Inc., New York, 1972). Reprinted in [8].

[23] N. Graham, “The measurement of relative frequency,” in [8].

[24] Y. Okhuwa, “Decoherence functional and probability interpretation,” Phys. Rev. D48, 1781-1784 (1993).

[25] A. Kent, “Against many-worlds interpretations,” Int. J. Mod. Phys. A5 1745 (1990); arXiv:gr-qc/9703089.

[26] D. Deutsch, “Quantum theory of probability and decisions,” Proc. Roy. Soc. Lond. A 455, 3129-3137 (1999); quant-ph/9906015.

[27] H. Barnum, C. M. Caves, J. Finkelstein, C. A. Fuchs and R. Schack, “Quantum probability from decision theory?” Proc. Roy. Soc. Lond. A 456, 1175-1182 (2000); arXiv:quant-ph/9907024.

[28] A. I. M. Rae, “Everett and the Born rule,” Studies in History and Philosophy of Modern Physics 40 243-250 (2009); arXiv:0810.2657.

[29] S. Saunders, “Chance in the Everett interpretation,” in [13], 181-205.

[30] D. Papineau, “A fair deal for Everettians,” in [13], 206-226.

[31] D. Wallace, “How to prove the Born rule,” in [13], 227-263.

[32] H. Greaves and W. Myrvold, “Everett and evidence,” in [13], 264-304.

[33] A. Kent, “One world versus many: The inadequacy of Everettian accounts of evolution, probability, and scientific confirmation,” in [13], 307-334.

[34] D. Albert, “Probability in the Everett interpretation,” in [13], 355-368.

[35] H. Price, “Decisions, decisions, decisions: Can Savage salvage Everettian probability?,” in [13], 369-390.

[36] W. H. Zurek, “Quantum jumps, Born’s rule, and objective reality,” in [13], 409-432.

[37] R. Schack, “The principal principle and probability in the many-worlds interpretation,” in [13], 467-475.
[38] L. Vaidman “Probability and the Many-Worlds interpretation of quantum theory,” in Proceedings of the Conference ”Quantum Theory, Reconsideration of Foundations” : Växjö (Smaland), Sweden, 17-21 June, 2001 (Växjö University Press, Växjö, Sweden, 2001); arXiv:quant-ph/0111072

[39] C. T. Sebens and S. M. Carroll, “Self-locating uncertainty and the origin of probability in Everettian quantum mechanics.” The British Journal for the Philosophy of Science, 69, 25-74 (2018); arXiv:1405.7577.

[40] L. E. Ballentine, Quantum Mechanics: A Modern Development (World Scientific Publishing Co. Pte. Ltd., Singapore, 1978).

[41] M. A. Rubin, “Observers and locality in Everett quantum field theory,” Found.Phys. 41, 1236-1262 (2011) arXiv:0909.2673

[42] A. N. Kolmogorov, Foundations of the Theory of Probability, 2nd English ed. (Chelsea Publishing Company, New York, 1956).

[43] B. de Finetti, “Probabilism: A critical essay on the theory of probability and on the value of science,” Erkenntnis 31, 169-223 (1989).

[44] B. O. Koopmans, “The axioms and algebra of intuitive probability,” Annals of Mathematics, 2nd series, 41, 269-292 (1940).

[45] A. Hájek, “Interpretations of Probability,” The Stanford Encyclopedia of Philosophy (Fall 2019 Edition), E. N. Zalta ed., https://plato.stanford.edu/archives/fall2019/entries/probability-interpret/.

[46] D. Deutsch, “The logic of experimental tests, particularly of Everettian quantum theory,” Studies in History and Philosophy of Modern Physics 55 24-33 (2016).

[47] B. de Finetti, Theory of Probability: A Critical Introductory Treatment (John Wiley & Sons Ltd., West Sussex, 2017).