Crouzeix–Raviart and Raviart–Thomas
finite-element error analysis on anisotropic meshes
violating the maximum-angle condition

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded polyhedral domain. Furthermore, we assume that $\Omega$ is convex if necessary. We consider the Poisson problem as follows. Find $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

(1.1)
where \( f \in L^2(\Omega) \) is a given function. Let \( \ell \) be a nonnegative integer. \( H^\ell(\Omega) \) is a Hilbert space with scalar product \( (\varphi, \psi)_{H^\ell(\Omega)} := \sum_{|\beta| \leq \ell} (\partial^\beta \varphi, \partial^\beta \psi)_{L^2(\Omega)} \) and norm \( \| \varphi \|_{H^\ell(\Omega)} := \sqrt{(\varphi, \varphi)_{H^\ell(\Omega)}} \). We set \( L^2(\Omega) := H^0(\Omega) \) with \( (\cdot, \cdot) := (\cdot, \cdot)_{H^0(\Omega)} \) and \( \| \cdot \| := \| \cdot \|_{H^0(\Omega)} \). This paper gives error estimates for the first-order Crouzeix–Raviart (CR) finite-element approximation on anisotropic meshes in three dimensions. Anisotropic meshes have different mesh sizes in different directions. The shape regularity assumption on triangulations \( T_h \) is no longer valid on these meshes; see for example [2]. Furthermore, we do not impose the maximum-angle condition proposed in [4] during mesh partitioning. In many instances, the discussion also relates to two dimensions. We therefore discuss the problem here as uniformly valid in an arbitrary number of dimensions.

CR finite error estimates for the non-homogeneous Dirichlet Poisson problem are known. Let \( CR^{1h}_0 \) be the CR finite-element space, to be defined in Section 2.3. Let \( u \in H^1(\Omega) \) and \( u^{CR}_h \in CR^{1h}_0 \) be the exact and CR finite-element solutions, respectively. In [13, Corollary 2.2], adopting medius analysis, the estimate

\[
|u - u^{CR}_h|_{H^1(T_h)} \leq c_0 \left( \inf_{v_h \in CR^{1h}_0} |u - v_h|_{H^1(T_h)} + \text{Osc}_1(f) \right),
\]

is given, where \( |\cdot|_{H^1(T_h)} \) denotes the broken (piecewise) \( H^1 \)-semi norm defined in Section 2.2 and \( c_0 \) a positive constant independent of \( h \). Here, the oscillation \( \text{Osc}_1(f) \) is expressed as

\[
\text{Osc}_1(f) := \left( \sum_{T \in T_h} h_T^2 \left[ \inf_{f \in P^0(T)} \| f - \bar{f} \|_{L^2(T)}^2 \right] \right)^{1/2},
\]

where \( P^0(T) \) denotes the piecewise constant space in \( T \). Suppose that \( u \in H^2(\Omega) \) and oscillation \( \text{Osc}_1(f) \) vanishes. Let \( I_h u \in CR^{1h}_0 \) be the nodal interpolation of \( u \) at the midpoints of the faces. Then, from the standard interpolation error estimate (see for example [10, Corollary 1.109]), we have

\[
|u - u^{CR}_h|_{H^1(T_h)} \leq c_0 |u - I_h u|_{H^1(T_h)} \leq c_1 h |u|_{H^2(\Omega)},
\]

where \( c_1 \) represents a positive constant independent of \( h \) and \( u \) but dependent on the parameter of the simplicial mesh; see for example [10, Definition 1.107]. This parameter is bounded if the simplicial mesh sequence is shape regular. However, the situation is different without the shape-regular condition. The aim of the present paper is to deduce an analogous error estimate on anisotropic finite-element meshes. Note that very flat elements might be included in the mesh sequence. In many papers reporting on such investigations, the maximum-angle condition instead of the shape-regular condition is imposed. However, the maximum-angle condition is not necessarily needed to obtain error estimates. Recently, in the two-dimensional instance, the CR finite-element analysis of the non-homogeneous Dirichlet-Poisson problem has
Error analysis of the CR finite-element method

been investigated under a more relaxed mesh condition, [13]. The present paper extends previous research to a three-dimensional setting.

However, it may not be easy to use the estimate (1.2) on anisotropic finite-element meshes. The CR finite-element space is not in \( H^1_0(\Omega) \). Hence, an error between the exact solution and the CR finite-element approximation solution with a \( H^1 \)-broken seminorm is divided into two parts. One is an approximation error that measures how well the exact solution is approximated by the CR finite-element functions. The other is a nonconformity error term. For the former, the CR interpolation error estimates are often used; in the latter, the standard scaling argument is often used to obtain the error estimates. However, in this way, we are unable to derive the correct order on anisotropic meshes. To overcome this difficulty, we shall use the lowest-order Raviart–Thomas (RT) interpolation error estimates on anisotropic meshes. By this technique, we consequently have the error estimates in the \( H^1 \)-broken seminorm (Theorem 4) and the \( L^2 \) norm (Theorem 5) on anisotropic meshes. Furthermore, we present an error estimate for the first-order RT finite-element approximation of the Poisson problem (1.1) based on the dual mixed formulation (Theorem 7). We again emphasise that we do not impose either the shape-regular or the maximum-angle condition during mesh partitioning.

We next present the equivalence of the enriched piecewise linear CR finite-element method introduced by [14] and the first-order RT finite-element method. In two dimensions, the work [3] represents pioneering research. Marini [21] further found an expression relating RT and CR finite-element methods:

\[
\bar{\sigma}^{RT}\big|_T = \nabla \bar{u}^{CR}_h = \frac{f_0}{2} (x - x_T) \quad \text{on } T,
\]

where \( T \) denotes a mesh element, \( x_i (i = 1, 2, 3) \) the vertices of triangle \( T \), \( x_T \) the barycentre of \( T \) such that \( x_T := \frac{1}{3}(x_1 + x_2 + x_3) \), and \( \bar{\sigma}^{RT}_h \) and \( \bar{u}^{CR}_h \) respectively denote the RT and CR finite-element solutions with a given external piecewise-constant function \( f_0 \). It was recently proved [14] that the enriched piecewise-linear CR finite-element method is identical to the first-order RT finite-element method for both the Poisson and Stokes problems in any number of dimensions. In the present paper, we extend Marini’s results to three dimensions (Lemma 10).

The remainder of the present paper is organised as follows. Section 2 introduces the weak form of the continuous problem (1.1), the finite-element meshes, and finite-element spaces. Furthermore, we propose a parameter \( H \). Section 3 introduces discrete settings of the CR finite-element method for (1.1) and proposes error estimates. Section 4 proves error estimates for the first-order RT finite-element method based on the dual mixed formulation of the Poisson problem. Section 5 gives the equivalence of the RT and CR finite-element problems. Finally, Section 6 presents numerical results obtained using the Lagrange P1 element and the first-order CR element.
2 Preliminaries

2.1 Weak formulation

The variational formulation for the Poisson problem (1.1) is then as follows. Find $u \in H^1_0(\Omega)$ such that

$$a_0(u, \varphi) = (f, \varphi) \quad \forall \varphi \in H^1_0(\Omega),$$

where $a_0 : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ denotes a bilinear form defined by

$$a_0(u, \varphi) := (\nabla u, \nabla \varphi).$$

Here, we define $H^1_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ in the semi-norm $|\cdot|_{H^1(\Omega)}$. By the Lax–Milgram lemma, there exists a unique solution $u \in H^1_0(\Omega)$ for any $f \in L^2(\Omega)$ and it holds that

$$|u|_{H^1(\Omega)} \leq C_P(\Omega) \|f\|,$$

where $C_P(\Omega)$ is the Poincaré constant depending on $\Omega$. Furthermore, if $\Omega$ is convex, then $u \in H^2(\Omega)$ and

$$|u|_{H^2(\Omega)} \leq \|\Delta u\|.$$  \hfill (2.2)

The proof can be found in, for example, [12, Theorem 3.1.1.2, Theorem 3.2.1.2].

2.2 Meshes, Mesh faces, Averages and Jumps

Let $\mathbb{T}_h = \{T\}$ be a simplicial mesh of $\overline{\Omega}$, made up of closed $d$-simplices, such as

$$\overline{\Omega} = \bigcup_{T \in \mathbb{T}_h} T,$$

with $h := \max_{T \in \mathbb{T}_h} h_T$, where $h_T := \text{diam}(T)$. We assume that each face of any $d$-simplex $T_1$ in $\mathbb{T}_h$ is either a subset of the boundary $\partial \Omega$ or a face of another $d$-simplex $T_2$ in $\mathbb{T}_h$. That is, $\mathbb{T}_h$ is a simplicial mesh of $\overline{\Omega}$ without hanging nodes.

**Definition 1** For any $T \in \mathbb{T}_h$, we define the parameter $H_T$ as

$$H_T := h_T^2 \min_{1 \leq i \leq 3} |L_i| \quad \text{if } d = 2,$$

where $L_i$ ($i = 1, 2, 3$) denotes edges of the triangle $T$. Further, we define the parameter $H_T$ as

$$H_T := h_T^2 \min_{1 \leq i,j \leq 6, i \neq j} |L_i||L_j| \quad \text{if } d = 3,$$

where $L_i$ ($i = 1, \ldots, 6$) denotes edges of the tetrahedra $T$. Here, $|T|$ denotes the measure of $T$. Furthermore, we set

$$H := H(h) := \max_{T \in \mathbb{T}_h} H_T.$$
We impose the following assumption.

**Assumption 1** We assume that \( \{T_h\}_{h>0} \) is a sequence of triangulations of \( \Omega \) such that

\[
\lim_{h \to 0} H(h) = 0.
\]

We adopt the concepts of mesh faces, averages and jumps in the analysis of RT and CR finite element method. Let \( \mathcal{F}_h \) be the set of interior faces and \( \mathcal{F}_h^\partial \) the set of the faces on the boundary \( \partial \Omega \). For any \( F \in \mathcal{F}_h \), we define the unit normal \( n_F \) to \( F \) as follows: (i) If \( F \in \mathcal{F}_h^\text{i} \) with \( F = T_1 \cap T_2, T_1, T_2 \in \mathbb{T}_h \), let \( n_1 \) and \( n_2 \) be the outward unit normals of \( T_1 \) and \( T_2 \), respectively. Then, \( n_F \) is either of \( \{n_1, n_2\} \); (ii) If \( F \in \mathcal{F}_h^\partial, n_F \) is the unit outward normal \( n \) to \( \partial \Omega \).

Let \( k \) be a positive integer. We then define the broken (piecewise) Sobolev space as

\[
H^k(\mathbb{T}_h) := \{ \varphi \in L^2(\Omega); \varphi|_T \in H^k(T) \forall T \in \mathbb{T}_h \}
\]

with the norm

\[
|\varphi|_{H^1(\mathbb{T}_h)} := \left( \sum_{T \in \mathbb{T}_h} \| \nabla \varphi \|^2_{L^2(T)^d} \right)^{1/2} \varphi \in H^1(\mathbb{T}_h).
\]

Let \( \varphi \in H^k(\mathbb{T}_h) \). Suppose that \( F \in \mathcal{F}_h^\text{i} \) with \( F = T_1 \cap T_2, T_1, T_2 \in \mathbb{T}_h \). Set \( \varphi_1 := \varphi|_{T_1} \) and \( \varphi_2 := \varphi|_{T_2} \). The jump and the average of \( \varphi \) across \( F \) is then defined as

\[
[[\varphi]]_F := (\varphi_1 n_1 + \varphi_2 n_2) \cdot n_F, \quad \{\varphi\}_F := \frac{1}{2}(\varphi_1 + \varphi_2).
\]

For a boundary face \( F \in \mathcal{F}_h^\partial \) with \( F = \partial T \cap \partial \Omega \), \( [[\varphi]]_F := \varphi|_T \) and \( \{\varphi\}_F := \varphi|_T \). When \( v \) is an \( \mathbb{R}^d \)-valued function, we use the notation

\[
[[v \cdot n]]_F := (v_1 - v_2) \cdot n_F, \quad \{v \cdot n\}_F := \frac{1}{2}(v_1 \cdot n_1 + v_2 \cdot n_2)
\]

for the jump of the normal component of \( v \). For a boundary face \( F \in \mathcal{F}_h^\partial \) with \( F = \partial T \cap \partial \Omega \), \( [[v]]_F := v|_T \cdot n_F \) and \( \{v\}_F := v|_T \). Whenever no confusion can arise, we simply write \( [[v]] \) and \( \{v\} \), respectively.

We here define a broken gradient operator as follows.

**Definition 2** For \( \varphi \in H^1(\mathbb{T}_h) \), the broken gradient \( \nabla_h : H^1(\mathbb{T}_h) \to L^2(\Omega)^d \) is defined by

\[
(\nabla_h \varphi)|_T := \nabla(\varphi)|_T \forall T \in \mathbb{T}_h.
\]

Note that \( H^1(\Omega) \subset H^1(\mathbb{T}_h) \) and the broken gradient coincides with the distributional gradient in \( H^1(\Omega) \).
2.3 Finite Element Spaces and Interpolations Error Estimates

This section introduce the RT, CR and piecewise-constant finite element spaces and the interpolation error estimates proposed in [15].

2.3.1 RT finite element space

Let $T \in \mathbb{T}_h$. For any $k \in \mathbb{N}_0$, let $P_k(T)$ be the space of polynomials with degree at most $k$ in $T$.

The lowest order RT finite element space is defined by

$$\text{RT}_0(T) := \{ v; \; v(x) = p + xq, \; p \in P_0(T), \; q \in P_0(T), \; x \in \mathbb{R}^d \}.$$  

The functionals are defined by, for any $v \in \text{RT}_0(T)$,

$$\chi_i(v) := \frac{1}{|F_i|} \int_{F_i} v \cdot n_i \, ds, \quad F_i \subset \partial T, \quad 1 \leq i \leq d + 1,$$

(2.3)

where $n_i$ denotes the outer unit normal vector of $T$ along $F_i$. We set $\sum := \{ \chi_i \}_{i=1}^{d+1}$. Note that $\dim \text{RT}_0(T) = d + 1$. The triple $\{ T, \text{RT}_0(T), \Sigma \}$ is then a finite element. We define the global RT finite element space by

$$\text{RT}_0(T_h) := \{ v_h \in L^2(\Omega)^d; \; v_h|_T \in \text{RT}_0(T), \; \forall T \in \mathbb{T}_h, \; |[v_h \cdot n]|_F = 0, \; \forall F \in F_h \}.$$ 

Note that $\text{RT}_0(T_h) \subset H(\text{div}; \Omega) := \{ v \in L^2(\Omega)^d; \; \text{div} v \in L^2(\Omega) \}$.

We next define the local RT interpolation as

$$I_{RT}^T : H^1(T)^d \to \text{RT}_0(T),$$

(2.4)

using

$$\int_{F_i} (v - I_{RT}^T v) \cdot n_i \, ds = 0, \quad F_i \subset \partial T, \quad i \in \{1, \ldots, d + 1\} \quad \forall v \in H^1(T)^d.$$ 

(2.5)

Further, we define the global RT interpolation $I_h^{RT} : H(\text{div}; \Omega) \cap H^1(\mathbb{T}_h)^d \to \text{RT}_0(T_h)$ by

$$(I_h^{RT} v)|_T = I_{RT}^T(v|_T) \quad \forall T \in \mathbb{T}_h, \quad \forall v \in H^1(\Omega)^d.$$ 

(2.6)

We give the local RT interpolation error estimate.

Theorem 1 We have the following estimate such that

$$\| I_{RT}^T v - v \|_{L^2(T)^d} \leq C_{RT}^I H_T |v|_{H^1(T)^d} \quad \forall T \in \mathbb{T}_h, \quad \forall v \in H^1(T)^d,$$

(2.7)

where $C_{RT}^I$ is a positive constant independent of $H_T$.

Proof The proof can be found in [15] Theorem 3. □

The global RT interpolation error estimate is obtained as follows.

Corollary 1 Let $\{ \mathbb{T}_h \}$ be a family of conformal meshes satisfying Assumption 4. Then, there exists $C_g^{RT} > 0$, independent of $H$, such that

$$\| I_h^{RT} v - v \|_{L^2(\Omega)^d} \leq C_g^{RT} H |v|_{H^1(\Omega)^d} \quad \forall v \in H^1(\Omega)^d.$$ 

(2.8)
2.3.2 CR finite element space

In introducing a nonconforming method, we define the following CR finite element space as

$$CR_{10}^h := \left\{ \varphi_h \in L^2(\Omega); \varphi_h|_T \in P^1(T) \ \forall T \in \mathcal{T}_h, \ \int_F [\varphi_h]|_F ds = 0 \ \forall F \in \mathcal{F}_h \right\}.$$  

Using the barycentric coordinates $\lambda_i : \mathbb{R}^d \to \mathbb{R}, i = 1, \ldots, d+1$, we define the local basis functions as

$$\theta_i(x) := d \left( \frac{1}{d} - \lambda_i(x) \right), \quad 1 \leq i \leq d+1. \quad (2.9)$$

For $i = 1, \ldots, d+1$, let $F_i$ be the face of $T$ and $x_{F_i}$ the barycentre of face $F_i$. We then define the local CR interpolation operator as

$$I^CR_T : W^{1,1}(T) \ni \varphi \mapsto I^CR_T \varphi := \sum_{i=1}^{d+1} \left( \frac{1}{|F_i|} \int_{F_i} \varphi ds \right) \theta_i \in P^1. \quad (2.10)$$

Because the trace of a function in $W^{1,1}(T)$ is in $L^1(\partial T)$, $\frac{1}{|F_i|} \int_{F_i} \varphi ds$ is meaningful. Further, it holds that

$$\frac{1}{|F_i|} \int_{F_i} (I^CR_T \varphi - \varphi) ds = 0, \quad i = 1, \ldots, d+1. \quad (2.11)$$

We define the global CR interpolation $I^CR_h : W^{1,1}(\Omega) \to CR_{10}^h$ by

$$(I^CR_h \varphi)|_T = I^CR_T (\varphi|_T) \quad \forall T \in \mathcal{T}_h, \quad \forall \varphi \in W^{1,1}(\Omega). \quad (2.12)$$

We give the local CR interpolation error estimate.

**Theorem 2** We have the following estimates such that

$$||I^CR_T \varphi - \varphi||_{L^2(T)} \leq C^CR_{T,L^2} g_T^2 |\varphi|_{H^2(T)} \quad \forall T \in \mathcal{T}_h, \quad \forall \varphi \in H^2(T), \quad (2.13)$$

$$||I^CR_T \varphi - \varphi||_{H^1(T)} \leq C^CR_{T,H^1} H_T |\varphi|_{H^2(T)} \quad \forall T \in \mathcal{T}_h, \quad \forall \varphi \in H^2(T). \quad (2.14)$$

Here, $C^CR_{T,L^2}$ and $C^CR_{T,H^1}$ are positive constants independent of $h_T$ and $H_T$.

**Proof** The proof can be found in [15, Theorem 2]. \boxed{}

**Remark:** The inequality (2.14) can be improved by replacing $H_T$ with $h_T$.

The global CR interpolation error estimates are obtained as follows.

**Corollary 2** Let $\{\mathcal{T}_h\}$ be a family of conformal meshes satisfying Assumption 7. Then, there exist $C^CR_{g,L^2}, C^CR_{g,H^1} > 0$, independent of $H$ and $h$, such that

$$||I^CR_h \varphi - \varphi|| \leq C^CR_{g,L^2} g^2 |\varphi|_{H^2(\Omega)} \quad \forall \varphi \in H^2(\Omega), \quad (2.15)$$

$$||I^CR_h \varphi - \varphi||_{H^1(\Omega)} \leq C^CR_{g,H^1} H |\varphi|_{H^2(\Omega)} \quad \forall \varphi \in H^2(\Omega). \quad (2.16)$$
2.3.3 Piecewise-constant finite element space

We define the standard piecewise constant space as
\[ M_0^h := \{ q_h \in L^2(\Omega); q_h|_T \in P^0(T) \forall T \in \mathcal{T}_h \}. \]

The local \( L^2 \)-projection \( \Pi_0^T q \) from \( L^2(T) \) into the space \( P^0(T) \) is defined by
\[ \int_T (\Pi_0^T q - q) dx = 0 \forall q \in L^2(T). \tag{2.17} \]

Note that \( \Pi_0^T q \) is the constant function equal to \( \frac{1}{|T|} \int_T q dx \). We also define the global \( L^2 \)-projection \( \Pi_0^h \) to the space \( M_0^h \) by
\[ (\Pi_0^h q)|_T = \Pi_0^T (q|_T) \forall T \in \mathcal{T}_h, \forall q \in L^2(\Omega). \tag{2.18} \]

The error estimate of the \( L^2 \)-projection is as follows.

\textbf{Theorem 3} We have the error estimate of the \( L^2 \)-projection such that
\[ \| \Pi_0^T q - q \|_{L^2(T)} \leq C_L^2 P h_T |q|_{H^1(T)} \forall T \in \mathcal{T}_h, \forall q \in H^1(T). \tag{2.19} \]

Here, \( C_L^2 P \) is a positive constant independent of \( h_T \).

\textbf{Proof} The proof can be found in [15, Theorem 2]. \hfill \Box

The global error estimate of the \( L^2 \)-projection is obtained as follows.

\textbf{Corollary 3} Let \( \{ \mathcal{T}_h \} \) be a family of conformal meshes satisfying Assumption 1. Then, there exists \( C_L^2 P > 0 \), independent of \( h \), such that
\[ \| \Pi_0^h q - q \| \leq C_L^2 P h |q|_{H^1(\Omega)} \forall q \in H^1(\Omega). \tag{2.20} \]

Between the RT interpolation \( I_{RT}^h q \) and the \( L^2 \)-projection \( \Pi_0^h \), the following relation holds:

\textbf{Lemma 1} For any \( v \in H^1(\Omega)^d \), it holds that
\[ \text{div} (I_{RT}^h v) = \Pi_0^h (\text{div} v). \]

That is to say, the diagram
\[
\begin{array}{ccc}
H^1(\Omega)^d & \xrightarrow{\text{div}} & L^2(\Omega) \\
I_{RT}^h & \downarrow \text{div} \quad \Pi_0^h & \downarrow \Pi_0^h \\
RT_h & \xrightarrow{\text{div}} & M_0^h
\end{array}
\]
commutes.

\textbf{Proof} The proof of this lemma is found in [7]. \hfill \Box
The following relation plays an important role in the CR finite element analysis on anisotropic meshes.

**Lemma 2** It holds that

\[(v_h, \nabla_h \psi_h) + (\text{div} v_h, \psi_h) = 0 \quad \forall v_h \in RT_0^1, \quad \forall \psi_h \in H_0^1(\Omega) + CR_{h0}^1. \quad (2.21)\]

**Proof** For any \(v_h \in RT_0^1\) and \(\psi_h \in H_0^1(\Omega) + CR_{h0}^1\), using Green formula and the fact \(v_h \cdot n_F \in P_0(F)\) for any \(F \in F_h\), we can derive

\[
(v_h, \nabla_h \psi_h) + (\text{div} v_h, \psi_h) = \sum_{T \in T_h} \int_{\partial T} (v_h \cdot n_T) \psi_h ds = \sum_{F \in F_h} \int_F ([[v_h \cdot n_F]) \psi_h] ds = \sum_{F \in F_h} \int_F ([v_h \cdot n_F] \psi_h) ds = 0.
\]

\[\Box\]

### 2.4 Discrete Poincaré Inequality on Anisotropic Meshes

We propose the discrete Poincaré inequality on anisotropic meshes.

**Lemma 3** (Discrete Poincaré inequality on anisotropic meshes) Assume that \(\Omega\) is convex. If \(H \leq 1\), there exists \(C(\Omega)\), independent of \(h\), \(H\), and the geometry of meshes, such that

\[\|\varphi_h\| \leq C(\Omega)|\varphi_h|_{H^1(\Omega_h)} \quad \forall \varphi_h \in CR_{h0}^1. \quad (2.22)\]

**Proof** Let \(\varphi_h \in CR_{h0}^1\). We consider the dual problem. Find \(z \in H^2(\Omega) \cap H_0^1(\Omega)\) such that

\[-\Delta z = \frac{\varphi_h}{\|\varphi_h\|} \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \partial \Omega.
\]

We then have a priori estimates:

\[|z|_{H^1(\Omega)} \leq C_P, \quad |z|_{H^2(\Omega)} \leq 1,
\]

where \(C_P\) is the Poincaré constant. We use the duality argument to show the target inequality. That is to say, we have

\[\|\varphi_h\| = \frac{1}{\|\varphi_h\|} (\varphi_h, \varphi_h) = (-\Delta z, \varphi_h) = (-\text{div} \nabla z, \varphi_h)
\]

\[= (-\text{div} \nabla z, \varphi_h - I_{h}^{RT}(\varphi_h)) - (\nabla z - I_{h}^{RT}(\nabla z), \nabla h \varphi_h) + (\nabla z, \nabla h \varphi_h)
\]

\[\leq \|\Delta z\|\|\varphi_h - I_{h}^{RT}(\varphi_h)\| + \|\nabla z - I_{h}^{RT}(\nabla z)\|\|\varphi_h\|_{H^1(\Omega_h)} + |z|_{H^1(\Omega)} \|\varphi_h\|_{H^1(\Omega_h)}
\]

\[\leq c (h + H|\nabla z|_{H^1(\Omega)} + C_P) |\varphi_h|_{H^1(\Omega_h)},
\]
which leads to
\[ \| \varphi_h \| \leq c(2 + C_p) |\varphi_h|_{H^1(\Omega_h)} \quad \text{if} \ H \leq 1. \]

We here used
\[
- \int_\Omega \text{div}(\nabla z) \varphi_h \, dx = \int_\Omega (\Pi_h^0 \text{div}(\nabla z) - \text{div}(\nabla z)) \varphi_h \, dx - \int_\Omega (\Pi_h^0 \text{div}(\nabla z)) \varphi_h \, dx
\]
\[
- \int_\Omega (\text{div} I_h^{RT}(\nabla z)) \varphi_h \, dx = \int_\Omega \text{div}(\nabla z) (\varphi_h - \Pi_h^0 \varphi_h) \, dx
\]
\[
- \int_\Omega (\nabla z - I_h^{RT}(\nabla z)) \cdot \nabla h \varphi_h \, dx + \int_\Omega \nabla z \cdot \nabla h \varphi_h \, dx,
\]
where
\[
\int_\Omega (\text{div} I_h^{RT}(\nabla z)) \varphi_h \, d = \sum_{T \in T_h} \int_{\partial T} n_T \cdot I_h^{RT}(\nabla z) \varphi_h ds - \int_\Omega I_h^{RT}(\nabla z) \cdot \nabla h \varphi_h \, dx
\]
\[
= \int_\Omega (\nabla z - I_h^{RT}(\nabla z)) \cdot \nabla h \varphi_h \, dx - \int_\Omega \nabla z \cdot \nabla h \varphi_h \, dx.
\]

\[\square\]

3 CR Finite Element Approximation

3.1 Finite Element Approximation

The CR finite element problem is to find \( u_h^{CR} \in CR_{h0}^1 \) such that
\[
a_{0h}(u_h^{CR}, \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in CR_{h0}^1, \quad (3.1)
\]
where \( a_{0h} : (CR_{h0}^1 + H_0^1(\Omega)) \times (CR_{h0}^1 + H_0^1(\Omega)) \rightarrow \mathbb{R} \) is defined by
\[
a_{0h}(\psi_h, \varphi_h) := \sum_{T \in T_h} \int_T \nabla \psi_h \cdot \nabla \varphi_h \, dx = (\nabla \psi_h, \nabla \varphi_h).
\]

This problem is nonconforming because \( CR_{h0}^1 \not\subset H_0^1(\Omega) \).

For the CR approximate solution \( u_h^{CR} \in CR_{h0}^1 \) of (3.1), we have the a priori estimate, using (2.22),
\[
|u_h^{CR}|^2_{H^1(\Omega_h)} \leq \|f\| \|u_h^{CR}\| \leq C(\Omega) \|f\| |u_h^{CR}|_{H^1(\Omega_h)}.
\]

By the Lax–Milgram lemma, there exists a unique solution \( u_h^{CR} \in CR_{h0}^1 \) for any \( f \in L^2(\Omega) \).
3.2 Classical Error Analysis

The starting point for error analysis is the Second Strang Lemma, e.g. see [10, Lemma 2.25],

\[ |u - u_{h}^{CR}|_{H^{1}(\mathcal{T}_{h})} \leq 2 \inf_{v_{h} \in CR_{h}^{1}} |u - v_{h}|_{H^{1}(\mathcal{T}_{h})} + \sup_{\varphi_{h} \in CR_{h}^{1}} \frac{a_{0h}(u, \varphi_{h}) - (f, \varphi_{h})}{|\varphi_{h}|_{H^{1}(\mathcal{T}_{h})}}. \] (3.2)

The first term of the inequality (3.2) is estimated as follows. Using the CR interpolation error estimate (2.16), we have, for any \( u \in H^{2}(\Omega) \),

\[ \inf_{v_{h} \in CR_{h}^{1}} |u - v_{h}|_{H^{1}(\mathcal{T}_{h})} \leq |u - I_{h}^{CR}u|_{H^{1}(\mathcal{T}_{h})} \leq cH|u|_{H^{2}(\Omega)}. \] (3.3)

From the standard scaling argument, we have a consistency error inequality, e.g., see [10, Lemma 3.36].

**Lemma 4 (Asymptotic Consistency)** Let \( u \in H^{1}_{0}(\Omega) \cap H^{2}(\Omega) \) be the solution of the homogeneous Dirichlet Poisson problem [1.1]. It then holds that

\[ \frac{a_{0h}(u, \varphi_{h}) - (f, \varphi_{h})}{|\varphi_{h}|_{H^{1}(\mathcal{T}_{h})}} \leq c \left( \sum_{T \in \mathcal{T}_{h}} \frac{h_{T}^{4}}{(\min_{F \in \partial T_{h}} \ell_{F})^{2}} |u|_{H^{2}(T)}^{2} \right)^{1/2} \forall h, \forall \varphi_{h} \in CR_{h}^{1}, \] (3.4)

where \( \partial \mathcal{T}_{h} \) denotes the set of all faces \( F \) of \( T \in \mathcal{T}_{h} \). Here, \( \ell_{F} \) denotes the distance of the vertex of \( T \) opposite to \( F \) to the face.

**Proof** We follow [10, Lemma 3.36].

Let \( \varphi_{h} \in CR_{h}^{1} \). Because \(-\Delta u = f\), we have

\[ a_{0h}(u, \varphi_{h}) - (f, \varphi_{h}) = \sum_{T \in \mathcal{T}_{h}} \int_{T} (\nabla u \cdot \nabla \varphi_{h} - f \varphi_{h}) dx \]

\[ = \sum_{T \in \mathcal{T}_{h}} \sum_{F \in \partial T_{h}} \int_{F} (n_{T} \cdot \nabla) u \varphi_{h} ds. \]

Because each face \( F \) of an element \( T \) located inside \( \Omega \) appears twice in the above sum, we have

\[ a_{0h}(u, \varphi_{h}) - (f, \varphi_{h}) = \sum_{T \in \mathcal{T}_{h}} \sum_{F \in \partial T_{h}} \int_{F} (n_{T} \cdot \nabla) u (\varphi_{h} - \bar{\varphi}_{h}) ds \]

with the mean value

\[ \bar{\varphi}_{h} := \frac{1}{|F|} \int_{F} \varphi_{h} ds. \]
Furthermore, we get

\[ a_{0h}(u, \varphi_h) - (f, \varphi_h) = \sum_{T \in \mathcal{T}_h} \sum_{F \in \partial T_h} \int_F n_T \cdot (\nabla u - \nabla u_h) (\varphi_h - \overline{\varphi}_h) \, ds \]

with the mean value

\[ n_T \cdot \nabla u := \frac{1}{|F|} \int_F (n_T \cdot \nabla) u \, ds. \]

The Cauchy–Schwarz inequality yields

\[ a_{0h}(u, \varphi_h) - (f, \varphi_h) \leq \sum_{T \in \mathcal{T}_h} \sum_{F \in \partial T_h} \| \nabla u - \nabla u_h \|_{L^2(F)} \| \varphi_h - \overline{\varphi}_h \|_{L^2(F)}. \]

For \( F \in \partial \mathcal{T}_h \), let \( \hat{T} \subset \mathbb{R}^d \) be the reference simplex and let \( \Phi_T : \hat{T} \rightarrow T \) be the corresponding affine transformation with Jacobian matrix \( A_T \). Let \( \hat{F} = \Phi_T^{-1}(F) \). Using the standard scaling argument and the trace theorem on the reference element, we have

\[ \| \varphi_h - \overline{\varphi}_h \|_{L^2(F)} \leq c \left( \frac{|F|}{|\hat{F}|} \right)^{1/2} \| \hat{\varphi}_h - \overline{\hat{\varphi}}_h \|_{H^1(\hat{T})}. \]

The Deny–Lions Lemma (see [10, Lemma B.67]) implies

\[ \| \hat{\varphi}_h - \overline{\hat{\varphi}}_h \|_{H^1(\hat{T})} \leq c |\hat{\varphi}_h|_{H^1(\hat{T})}. \]

Using the standard scaling argument again, we obtain

\[ \| \varphi_h - \overline{\varphi}_h \|_{L^2(F)} \leq c \left( \frac{|F|}{|\hat{F}|} \right)^{1/2} |\hat{\varphi}_h|_{H^1(\hat{T})} \leq c \left( \frac{|F|}{|\hat{F}|} \right)^{1/2} \| A_T \|_2 \left( \frac{\hat{T}}{T} \right)^{1/2} |\varphi_h|_{H^1(T)} \]

\[ \leq c \left( \frac{|F|}{|\hat{F}|} \right)^{1/2} h_T |\varphi_h|_{H^1(T)} = c \left( \frac{d}{\ell_F} \right)^{1/2} h_T |\varphi_h|_{H^1(T)}. \]

Here, \( \| A \|_2 \) denotes an operator norm as

\[ \| A_T \|_2 := \sup_{0 \neq x \in \mathbb{R}^d} \frac{|A_T x|}{|x|}, \]

where \( |x| := \left( \sum_{i=1}^d |x_i|^2 \right)^{1/2} \) for \( x \in \mathbb{R}^d \).

By analogous argument, we have

\[ \| \nabla u - \nabla u_h \|_{L^2(F)} \leq c \left( \frac{d}{\ell_F} \right)^{1/2} h_T |u|_{H^2(T)}. \]
We consequently get

\[ a_{hh}(u, \varphi_h) - (f, \varphi_h) \leq c \sum_{T \in \mathcal{T}_h} \sum_{F \in \partial \mathcal{T}_h} \frac{h_T^2}{\ell_F} |u|_{H^2(T)} |\varphi_h|_{H^1(T)} \]

\[ \leq c \sum_{T \in \mathcal{T}_h} \min_{F \in \partial \mathcal{T}_h} \frac{h_T^2}{\ell_F} |u|_{H^2(T)} |\varphi_h|_{H^1(T)} \]

\[ \leq c \left( \sum_{T \in \mathcal{T}_h} \frac{h_T^4}{(\min_{F \in \partial \mathcal{T}_h} \ell_F)^2} |u|_{H^2(T)}^2 \sum_{T \in \mathcal{T}_h} |\varphi_h|_{H^1(T)}^2 \right)^{1/2}, \]

which leads to (3.4).

\[ \square \]

From (3.2), (3.3) and (3.4), we have

\[ |u - u_h^{CR}|_{H^1(\mathcal{T}_h)} \leq cH|u|_{H^2(\Omega)} + c \left( \sum_{T \in \mathcal{T}_h} \frac{h_T^4}{(\min_{F \in \partial \mathcal{T}_h} \ell_F)^2} |u|_{H^2(T)}^2 \right)^{1/2}. \]

Since the order of the nonconforming term does not necessarily becomes the order \( H \), this inequality may be overestimated.

Example: Let \( 0 < h_T \leq 1 \). As examples, we consider two cases.

(I) When we use meshes including the tetrahedra \( T \) with vertices \( (0,0,0)^T \), \( (h_T,0,0)^T \), \( (0,h_T,0)^T \), and \( (0,0,h_T)^T \), we have

\[ \frac{h_T^4}{(\min_{F \in \partial \mathcal{T}_h} \ell_F)^2} |u|_{H^2(T)}^2 \leq c h_T^{2(2-\varepsilon)} |u|_{H^2(T)}^2, \]

where \( 1 < \varepsilon \leq 2 \). Since \( H = \mathcal{O}(h) \), we get

\[ |u - u_h^{CR}|_{H^1(\mathcal{T}_h)} \leq c(h + h^{2-\varepsilon}) |u|_{H^2(\Omega)}. \]

(II) When we use meshes including the tetrahedra \( T \) with vertices \( (0,0,0)^T \), \( (h_T,0,0)^T \), \( (0,h_T,0)^T \), and \( (h_T^\gamma,0,h_T^\varepsilon)^T \), we have

\[ \frac{h_T^4}{(\min_{F \in \partial \mathcal{T}_h} \ell_F)^2} |u|_{H^2(T)}^2 \leq c h_T^{2(2-\varepsilon)} |u|_{H^2(T)}^2, \]

where \( 1 < \gamma < \varepsilon \leq 1 + \gamma \) and \( 1 < \varepsilon \leq 2 \). Since \( H = \mathcal{O}(h^{1+\gamma-\varepsilon}) \), we get

\[ |u - u_h^{CR}|_{H^1(\mathcal{T}_h)} \leq c(h^{1+\gamma-\varepsilon} + h^{2-\varepsilon}) |u|_{H^2(\Omega)}. \]
3.3 Argument via the RT Interpolation Error

To overcome the difficulty, we use the relation (2.21) in Lemma 2, e.g., see also [1,20].

**Lemma 5 (Asymptotic Consistency)** We assume that $\Omega$ is convex. Let $\{T_h\}$ be a family of conformal meshes satisfying Assumption 1. Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of the homogeneous Dirichlet Poisson problem (1.1). Then, there exists $c$, independent of $H$, such that

$$\sup_{\varphi_h \in CR^1_{T_h}} \frac{a_0(u, \varphi_h) - (f, \varphi_h)}{|\varphi_h|_{H^1(T_h)}} \leq cH\|f\|. \quad (3.5)$$

**Proof** Using (2.21), we have, for any $w_h \in RT^0_h$,

$$\sup_{\varphi_h \in CR^1_{T_h}} \frac{a_0(u, \varphi_h) - (f, \varphi_h)}{|\varphi_h|_{H^1(T_h)}} = \sup_{\varphi_h \in CR^1_{T_h}} \frac{(\nabla u - w_h, \nabla \varphi_h) - (\text{div} w_h + f, \varphi_h)}{|\varphi_h|_{H^1(T_h)}}.$$

We set $w_h := I_h^{RT} \nabla u$. From Lemma 3 we get

$$\text{div}(I_h^{RT} \nabla u) = I_h^0 \text{div}(\nabla u) = -I_h^0 f.$$

Furthermore, we have, for any $\varphi_h \in CR^1_{T_h}$,

$$(-I_h^0 f + f, I_h^0 \varphi_h) = 0.$$

We thus obtain

$$(\nabla u - I_h^{RT} \nabla u, \nabla \varphi_h) - (-I_h^0 f + f, \varphi_h)$$

$$= (\nabla u - I_h^{RT} \nabla u, \nabla \varphi_h) - (-I_h^0 f + f, \varphi_h - I_h^0 \varphi_h)$$

$$\leq \|\nabla u - I_h^{RT} \nabla u\|_{L^2(\Omega)} \|
abla \varphi_h\|_{H^1(T_h)} + \|f - I_h^0 f\|\|
abla \varphi_h - I_h^0 \varphi_h\|$$

$$\leq cH|u|_{H^2(\Omega)}\|\varphi_h\|_{H^1(T_h)} + ch|f|\|
abla \varphi_h\|_{H^1(T_h)}.$$

□

We consequently obtain the error estimate of the CR finite element method on anisotropic meshes.

**Theorem 4** We assume that $\Omega$ is convex. Let $\{T_h\}$ be a family of conformal meshes satisfying Assumption 1. Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of the homogeneous Dirichlet Poisson problem (1.1) with data $f \in L^2(\Omega)$. Let $u_h^{CR} \in CR^1_{T_h}$ be the approximate solution of (3.1). Then, there exists $c$, independent of $H$, such that

$$|u - u_h^{CR}|_{H^1(T_h)} \leq cH\|f\|. \quad (3.6)$$
\[ |u - u_h^{CR}|_{H^1(\Omega)} \leq 2 \inf_{v_h \in CR_{1h}} |u - v_h|_{H^1(\Omega)} + \sup_{v_h \in CR_{1h}} \frac{a_{0h}(u, \varphi_h) - (f, \varphi_h)}{|\varphi_h|_{H^1(\Omega)}} \]

which leads to the estimate (3.6).

\( \square \)

We next give the \( L^2 \) error estimate of the CR finite element method on anisotropic meshes, see also [19][20][3].

**Theorem 5** We assume that \( \Omega \) is convex. Let \( \{T_h\} \) be a family of conformal meshes satisfying Assumption 7. Let \( u \in H^1_0(\Omega) \cap H^2(\Omega) \) be the solution of the homogeneous Dirichlet Poisson problem (3.1) with data \( f \in L^2(\Omega) \). Let \( u_h^{CR} \in CR_{1h} \) be the approximate solution of (3.1). Then, there exists \( c \), independent of \( H \), such that

\[ ||u - u_h^{CR}|| \leq cH^2||f||. \]  

**Proof** We set \( e_h := u - u_h^{CR} \). Let \( z \in H^2(\Omega) \cap H^1_0(\Omega) \) satisfy

\[ a_0(\varphi, z) = (\varphi, e_h) \quad \forall \varphi \in H^1_0(\Omega) \]  

and \( z_h^{CR} \in CR_{1h} \) satisfy

\[ a_{0h}(\varphi_h, z_h^{CR}) = (\varphi_h, e_h) \quad \forall \varphi_h \in CR_{1h}. \]

We then have

\[ ||e_h||^2 = (e_h, e_h) = a_{0h}(u, z) - a_{0h}(u_h^{CR}, z_h^{CR}) \]

\[ = a_{0h}(u - u_h^{CR}, z - z_h^{CR}) + a_{0h}(u - u_h^{CR}, z_h^{CR}) + a_{0h}(u_h^{CR}, z - z_h^{CR}) \]

\[ + a_{0h}(u - u_h^{CR}, z_h^{CR} - I_h^CR z) + a_{0h}(u - u_h^{CR}, I_h^CR z) + a_{0h}(u_h^{CR} - I_h^CR u, z - z_h^{CR}) + a_{0h}(I_h^CR u, z - z_h^{CR}). \]  

Using Corollary 2, the first term on the right hand side of (3.11) can be estimated as

\[ a_{0h}(u - u_h^{CR}, z - z_h^{CR}) \leq |u - u_h^{CR}|_{H^1(\Omega)}|z - z_h^{CR}|_{H^1(\Omega)} \]

\[ \leq cH^2||f|| ||e_h||. \]  

For the second and fourth terms on the right hand side of (3.10), we have

\[ a_{0h}(u - u_h^{CR}, z_h^{CR} - I_h^CR z) \]

\[ = a_{0h}(u - u_h^{CR}, z_h^{CR} - z) + a_{0h}(u - u_h^{CR}, z - I_h^CR z) \]

\[ \leq |u - u_h^{CR}|_{H^1(\Omega)}(|z_h^{CR} - z|_{H^1(\Omega)} + |z - I_h^CR z|_{H^1(\Omega)}) \]

\[ \leq cH^2||f|| ||e_h||. \]
and, analogously,
\[
a_{0h}(u^{CR}_h - I^{CR}_h u, z - z^{CR}_h) \leq cH^2 \| f \| \| e_h \|. \tag{3.13}
\]
From (3.8), (3.9) and (2.21), we have
\[
a_{0h}(u - u^{CR}_h, I^{CR}_h z)
= a_{0h}(u, I^{CR}_h z) - a_{0h}(u^{CR}_h, I^{CR}_h z) = (\nabla u, \nabla h I^{CR}_h z) - (f, I^{CR}_h z)
= (\nabla u - u^{RT}_h \nabla u, \nabla h I^{CR}_h z - \nabla z) + (\nabla u, \nabla z) - (f, z)
= (\nabla u - u^{RT}_h \nabla u, \nabla h I^{CR}_h z - \nabla z) - (f + \text{div}(I^{RT}_h \nabla u), I^{CR}_h z - z).
\]
From Lemma 1 and \( \text{div}(I^{RT}_h \nabla u) = -\Pi^0_h f \), we have
\[
a_{0h}(u - u^{CR}_h, I^{CR}_h z)
= (\nabla u - u^{RT}_h \nabla u, \nabla h I^{CR}_h z - \nabla z) - (f - \Pi^0_h f, I^{CR}_h z - z)
\leq \| \nabla u - u^{RT}_h \nabla u \|_{L^2(\Omega)} \| I^{CR}_h z - z \|_{H^1(\mathcal{T}_h)} + \| f - \Pi^0_h f \| \| I^{CR}_h z - z \|
\leq cH^2 \| f \| \| e_h \|. \tag{3.14}
\]
Analogously, from \( \text{div}(I^{RT}_h \nabla z) = -\Pi^0_h e_h \), we have
\[
a_{0h}(I^{CR}_h u, z - z^{CR}_h)
= (\nabla h I^{CR}_h u - \nabla u, \nabla z - I^{RT}_h \nabla z) - (I^{CR}_h u - u, e_h + \text{div}(I^{RT}_h \nabla z))
\leq \| I^{CR}_h u - u \|_{H^1(\mathcal{T}_h)} \| \nabla z - I^{RT}_h \nabla z \|_{L^2(\Omega)} + \| I^{CR}_h u - u \| \| e_h - \Pi^0_h e_h \|
\leq cH^2 \| f \| \| e_h \|. \tag{3.15}
\]
Combining (3.10), (3.11), (3.12), (3.13), (3.14), and (3.15), we finally get
\[
\| e_h \|^2 \leq cH^2 \| f \| \| e_h \|,
\]
which leads to the target estimate. \( \Box \)

4 RT Finite Element Error Estimates

4.1 Dual mixed formulation of the Poisson problem

The Poisson equation \( -\Delta u = -\text{div} \nabla u = f \) can be written as the following system. Find \((\sigma, u) : \Omega \to \mathbb{R}^d \times \mathbb{R} \) such that
\[
\begin{align*}
\sigma &- \nabla u = 0 \quad \text{in } \Omega, \tag{4.1a} \\
\text{div} \sigma & = -f \quad \text{in } \Omega, \tag{4.1b} \\
u & = 0 \quad \text{on } \partial \Omega. \tag{4.1c}
\end{align*}
\]
We consider the following dual mixed formulation: Find \((\sigma, u) \in H(\text{div}; \Omega) \times L^2(\Omega)\) such that
\[
a(\sigma, v) + b(v, u) = 0 \quad \forall v \in H(\text{div}; \Omega), \tag{4.2a}
\]
\[
b(\sigma, q) = -(f, q) \quad \forall q \in L^2(\Omega), \tag{4.2b}
\]
where bilinear forms \(a : H(\text{div}; \Omega) \times H(\text{div}; \Omega) \to \mathbb{R}\) and \(b : H(\text{div}; \Omega) \times L^2(\Omega) \to \mathbb{R}\) are defined by
\[
a(\sigma, v) := (\sigma, v), \quad b(v, q) := (\text{div} v, q).
\]
We set \(X_0 := \{v \in H(\text{div}; \Omega); \ b(v, q) = 0 \ \forall q \in L^2(\Omega)\}\). Because there exists a constant \(c > 0\) such that
\[
a(v, v) \geq c \|v\|^2_{H(\text{div}; \Omega)} \quad \forall v \in X_0
\]
and the bilinear form \(b(. , )\) satisfies the inf–sup condition
\[
\inf_{0 \neq q \in L^2(\Omega)} \sup_{0 \neq v \in H(\text{div}; \Omega)} \frac{b(v, q)}{\|v\|_{H(\text{div}; \Omega)} \|q\|} \geq \beta_* > 0, \tag{4.3}
\]
(4.2) is uniquely solvable; e.g., see [11,6].

4.2 RT Approximate Problem

We consider the following RT approximate problem. Find \((\sigma_h^{RT}, u_h^{RT}) \in RT_0^0 \times M_0^0\) such that
\[
a(\sigma_h^{RT}, v_h) + b(v_h, u_h^{RT}) = 0, \quad \forall v_h \in RT_0^0, \tag{4.4a}
\]
\[
b(\sigma_h^{RT}, q_h) = -(f, q_h), \quad \forall q_h \in M_0^0. \tag{4.4b}
\]
This setting is conforming because \(RT_0^0 \times M_0^0 \subset H(\text{div}; \Omega) \times L^2(\Omega)\). It is given later that the discrete inf–sup condition
\[
\inf_{q_h \in M_0^0} \sup_{v_h \in RT_0^0} \frac{b(v_h, q_h)}{\|v_h\|_{H(\text{div}; \Omega)} \|q_h\|} \geq c_* > 0
\]
holds, where \(c_*\) is a constant independent of \(h\).

4.3 Error Estimates of the RT Finite Element Approximation

This section gives error estimates of the mixed finite element approximation [14]. We emphasise that we do not impose the shape regularity condition and the maximum-angle condition for the mesh partition. That is, we assume that \(\{T_h\}\) is a family of conformal meshes satisfying Assumption [1]
Lemma 6 Let $D \subset \mathbb{R}^d$ be a bounded domain. For any $g \in L^2(D)$, there exists $v \in H^1(D)^d$ such that
\[
\text{div } v = g \quad \text{in } D \tag{4.5}
\]
and
\[
|v|_{H^1(D)^d} \leq \|g\|_{L^2(D)}, \quad \|v\|_{L^2(D)^d} \leq C_P(D)\|g\|_{L^2(D)}, \tag{4.6}
\]
where $C_P(D)$ is the Poincaré constant.

Proof The proof can be found in [3, Lemma 2.2]. □

We next give the discrete inf–sup condition.

Lemma 7 (Discrete inf–sup condition) If $C_{\text{RT}}^2 H \leq 1$, there exists a constant $c_*$, depending only on the Poincaré constant, such that
\[
\inf_{q_h \in M_0^h} \sup_{v_h \in RT^0} \frac{b(v_h, q_h)}{\|v_h\|_{H(div; \Omega)} \|q_h\|} \geq c_*>0, \tag{4.7}
\]
where $C_{\text{RT}}^g$ is the constant appearing in Corollary [1].

Proof Let $q_h \in M_0^h$. From Lemma [6] there exists $v \in H^1(\Omega)^d$ such that $\text{div } v = q_h$ in $\Omega$, $|v|_{H^1(\Omega)^d} \leq \|q_h\|$, and $\|v\|_{L^2(\Omega)^d} \leq C_P(\Omega)\|q_h\|$.

By the Gauss theorem, we have
\[
\sum_{T \in \mathcal{T}_h} \int_{\partial T} v \cdot n_T ds = \sum_{T \in \mathcal{T}_h} \int_T \text{div } v dx = \int_{\Omega} q_h dx.
\]
From the definition of the Raviart–Thomas interpolation, we conclude that
\[
\int_{\Omega} \text{div}(I^R_T v)p_h dx = \sum_{T \in \mathcal{T}_h} p_h \int_T \text{div}(I^R_T v) dx = \sum_{T \in \mathcal{T}_h} p_h \int_{\partial T} n_T \cdot (I^R_T v) ds
\]
\[
= \sum_{T \in \mathcal{T}_h} p_h \int_{\partial T} v \cdot n_T ds = \int_{\Omega} q_h p_h dx \quad \forall p_h \in M_0^h.
\]
Therefore, it follows that $\text{div}(I^R_T v) = q_h$.

From the definitions, we have
\[
\|I^R_T v\|^2_{H(div; \Omega)} = \|I^R_T v\|^2_{L^2(\Omega)^d} + \|\text{div}(I^R_T v)\|^2
\]
\[
\leq 2\|I^R_T v - v\|^2_{L^2(\Omega)^d} + 2\|v\|^2_{L^2(\Omega)^d} + \|q_h\|^2
\]
\[
\leq 2(C_{\text{RT}}^2 H)^2 \|v\|^2_{H^1(\Omega)^d} + 2C_P(\Omega)^2 \|q_h\|^2 + \|q_h\|^2
\]
\[
\leq (3 + 2C_P(\Omega)^2) \|q_h\|^2.
\]
We thus have
\[
\sup_{v_h \in RT^0_h} \frac{b(v_h, q_h)}{\|v_h\|_{H(div; \Omega)}} \geq \frac{b(I^R_T v, q_h)}{\|I^R_T v\|_{H(div; \Omega)}} \geq \frac{1}{(3 + 2C_P(\Omega)^2)^{1/2}} \frac{(q_h, q_h)}{\|q_h\|},
\]
and the proof of (4.7) is completed with $c_* := (3 + 2C_P(\Omega)^2)^{-1/2}$. □
From the discrete equations (4.4) and their continuous counterpart (4.2), we obtain the Galerkin orthogonality

\[ a(\sigma - \sigma_h^{RT}, v_h) + b(v_h, u - u_h^{RT}) = 0 \quad \forall v_h \in RT_h^0, \quad (4.8a) \]
\[ b(\sigma - \sigma_h^{RT}, q_h) = 0 \quad \forall q_h \in M_h^0. \quad (4.8b) \]

We then get the following Céa-lemma-type estimates with the help of (4.8) and the inf–sup condition (4.7).

**Theorem 6** Let \( \sigma \in H^1(\Omega)^d \) and \( \sigma_h^{RT} \in RT_h^0 \) be the solutions of (4.1) and (4.4), respectively. We then have

\[ \| \sigma - \sigma_h^{RT} \|_{L^2(\Omega)^d} \leq \| \sigma - I_h^{RT} \sigma \|_{L^2(\Omega)^d}. \quad (4.9) \]

Furthermore, let \((\sigma, u) \in H^1(\Omega)^d \times L^2(\Omega)\) and \((\sigma_h^{RT}, u_h^{RT}) \in RT_h^0 \times M_h^0\) be the solutions of (4.1) and (4.4), respectively. Then, if \( C_{RT}^g H \leq 1 \), it holds that

\[ \| u - u_h^{RT} \| \leq \| u - I_h^0 u \| + c_1^{-1} \| \sigma - \sigma_h^{RT} \|_{L^2(\Omega)^d}. \quad (4.10) \]

Here, \( C_{RT}^g \) and \( c_1 \) are respectively the constants appearing in Corollary 1 and Lemma 7.

**Proof** The proof can be found in [5, Lemma 3.7, Lemma 3.9]. \( \square \)

Using Theorem 6 and the interpolation error estimates of Corollary 1 and 3, we thus have the error estimates of the mixed finite element approximation (4.4) on anisotropic meshes violating the maximum-angle condition.

**Theorem 7** Let \((\sigma, u) \in H^1(\Omega)^d \times H^1(\Omega)\) and \((\sigma_h^{RT}, u_h^{RT}) \in RT_h^0 \times M_h^0\) be the solutions of (4.1) and (4.4), respectively. Then, there exists a constant \( c_1 > 0 \), independent of \( \sigma, \) \( u, \) \( h, \) \( H, \) and the geometric properties of \( \mathcal{T}_h, \) such that

\[ \| \sigma - \sigma_h^{RT} \|_{L^2(\Omega)^d} \leq c_1 H |\sigma|_{H^1(\Omega)^d}. \quad (4.11) \]

Furthermore, if \( C_{RT}^g H \leq 1 \), there exists a constant \( c_2 > 0 \), depending on the discrete inf–sup condition but independent of \( \sigma, u, h, H, \) and the geometric properties of \( \mathcal{T}_h \)

\[ \| u - u_h^{RT} \| \leq c_2 \left( h |u|_{H^1(\Omega)} + H |\sigma|_{H^1(\Omega)^d} \right). \quad (4.12) \]

Here, \( C_{RT}^g \) is the constant appearing in Corollary 1.
5 Relationship between the RT and CR Finite Element Approximation

This section shows the relationship between the RT and CR problems. Find \((\bar{\sigma}_h^{RT}, \bar{u}_h^{RT}) \in RT_h^0 \times M_h^0\) such that
\[
a(\bar{\sigma}_h^{RT}, v_h) + b(v_h, \bar{u}_h^{RT}) = 0 \quad \forall v_h \in RT_h^0, \quad \tag{5.1a}
\]

\[
b(\bar{\sigma}_h^{RT}, q_h) = - (\Pi_h^0 f, q_h) \quad \forall q_h \in M_h^0 \quad \tag{5.1b}
\]

and find \(\bar{u}_h^{CR} \in CR_{h0}\) such that
\[
a_{0h}(\bar{u}_h^{CR}, \varphi_h) = (\Pi_h^0 f, \varphi_h) \quad \forall \varphi_h \in CR_{h0}^1. \quad \tag{5.2}
\]

Here, \(5.2\) is the CR approximation of the Poisson equation
\[
-\Delta \bar{u} = \Pi_h^0 f \quad \text{in } \Omega, \quad \bar{u} = 0 \quad \text{on } \partial \Omega. \quad \tag{5.3}
\]

In the case of \(d = 2\), it is well known that there exists a relationship between \((\bar{\sigma}_h^{RT}, \bar{u}_h^{RT})\) and \(\bar{u}_h^{CR}\) introduced by Marini; for example, [21]. See also [19,17,20]. We here show the relation in the three dimensional case.

Let us consider a tetrahedron \(T \subset \mathbb{R}^3\) such as that in Figure 1. Let \(x_i\) \((i = 1, 2, 3, 4)\) be the vertices and \(m_{i,j}\) the midpoints of edges of the tetrahedron; that is, \(m_{i,j} := \frac{1}{2}(x_i + x_j)\). Furthermore, for \(1 \leq i \leq 4\), let \(F_i\) be the face of the tetrahedron opposite \(x_i\). Then, by simple calculation, we find the equality
\[
L := \sum_{i=1}^4 |x_i - x_T|^2 = |m_{1,4} - m_{2,3}|^2 + |m_{1,4} - m_{2,4}|^2 + |m_{1,2} - m_{3,4}|^2,
\]

holds, where \(x_T\) is the barycentre of \(T\) such that \(x_T := \frac{1}{4} \sum_{i=1}^4 x_i\).

We present a quadrature scheme over a simplex \(T \subset \mathbb{R}^3\) (e.g., [22, p.307]) that is easily conformed.

---

Fig. 1 Tetrahedron
Lemma 8 For any \( f \in C^0(T) \), the quadrature scheme

\[
\int_T f(x) dx \sim -\frac{|T|}{20} \sum_{i=1}^{4} f(x_i) + \frac{|T|}{5} \sum_{1 \leq i < j \leq 4} f(m_{i,j})
\]

is exact for polynomials lower than degree 2;

\[
\int_T f(x) dx + \frac{|T|}{20} \sum_{i=1}^{4} f(x_i) - \frac{|T|}{5} \sum_{1 \leq i < j \leq 4} f(m_{i,j}) = 0 \quad \forall f \in \mathcal{P}^2(T). \tag{5.4}
\]

Define the function \( \varphi_T \) by

\[
\varphi_T(x) := \begin{cases} 
L - 12|x - x_T|^2, & \text{on } T, \\
0, & \text{otherwise.}
\end{cases}
\tag{5.5}
\]

We then have the following lemma.

Lemma 9 It holds that

\[
\frac{1}{|F_i|} \int_{F_i} \varphi_T(x) ds = 0, \quad i = 1, 2, 3, 4, \tag{5.6}
\]

\[
\frac{1}{|T|} \int_T \varphi_T(x) dx = \frac{2}{5} L, \tag{5.7}
\]

\[
\frac{1}{|T|} \int_T |\nabla \varphi_T(x)|^2 dx = \frac{144}{5} L. \tag{5.8}
\]

Proof From second-order three-point numerical integration over \( F_1 \),

\[
\int_{F_1} f(x) ds = \frac{|F_1|}{3} (f(m_{2,3}) + f(m_{3,4}) + f(m_{2,4})) \quad \forall f \in \mathcal{P}^2(T),
\]

we have

\[
\frac{1}{|F_1|} \int_{F_1} \varphi_T(x) ds
= \frac{1}{3} (\varphi_T(m_{2,3}) + \varphi_T(m_{3,4}) + \varphi_T(m_{2,4}))
= \frac{1}{3} (3L - 12 \left( |m_{2,3} - x_T|^2 + |m_{3,4} - x_T|^2 + |m_{2,4} - x_T|^2 \right))
= \frac{1}{3} \left( 3L - \frac{12}{4} \left( |m_{2,3} - m_{1,4}|^2 + |m_{3,4} - m_{1,2}|^2 + |m_{2,4} - m_{1,3}|^2 \right) \right) = 0,
\]

which leads to (5.6).
Next, using (5.4), we have
\[
\frac{1}{|T|} \int_T \varphi_T(x)dx = -\frac{1}{20} \sum_{i=1}^{4} \varphi_T(x_i) + \frac{1}{5} \sum_{1 \leq i < j \leq 4} \varphi_T(m_{i,j})
\]
\[
= -\frac{1}{20} \left( 4L - 12 \sum_{i=1}^{4} |x_i - x_T|^2 \right) + \frac{1}{5} \left( 6L - 12 \sum_{1 \leq i < j \leq 4} |m_{i,j} - x_T|^2 \right)
\]
\[
= \frac{2}{5} L,
\]
which leads to (5.7). We here used
\[
\sum_{1 \leq i < j \leq 4} |m_{i,j} - x_T|^2 = |m_{1,2} - x_T|^2 + |m_{1,3} - x_T|^2 + |m_{1,4} - x_T|^2
\]
\[
+ |m_{2,3} - x_T|^2 + |m_{2,4} - x_T|^2 + |m_{3,4} - x_T|^2
\]
\[
= \frac{1}{4} \left( 2|m_{1,2} - m_{3,4}|^2 + 2|m_{1,3} - m_{2,4}|^2 + 2|m_{1,4} - m_{2,3}|^2 \right) = \frac{L}{2}.
\]

We similarly obtain
\[
\frac{1}{|T|} \int_T |\nabla \varphi_T(x)|^2dx = \frac{24^2}{|T|} \int_T |x - x_T|^2dx
\]
\[
= -\frac{24^2}{20} \sum_{i=1}^{4} |x_i - x_T|^2 + \frac{24^2}{5} \sum_{1 \leq i < j \leq 4} |m_{i,j} - x_T|^2 = \frac{144}{5} L,
\]
which leads to (5.8). \(\square\)

We set the bubble space \(B_h\) by
\[
B_h := \{ b_h \in L^2(\Omega); \ b_h|_T \in \text{span}\{\varphi_T\}, \ \forall T \in T_h \}.
\]
Then, for any \(\psi_h \in CR^{1}_{h0}\) and \(b_h \in B_h\), it holds that
\[
(\nabla_h \psi_h, \nabla_h b_h) = \sum_{T \in T_h} \int_T \nabla \psi_h \cdot \nabla b_h dx
\]
\[
= \sum_{T \in T_h} \left\{ \sum_{F \subset \partial T} \int_F (n_F \cdot \nabla \psi_h) b_h ds - \int_T \Delta \psi_h b_h dx \right\} = 0.
\]
We here used the facts that \(n_F \cdot \nabla \psi_h\) is constant on \(F\), (5.6), and \(\Delta \psi_h = 0\) on \(T\). That is to say, two finite element spaces \(CR^{1}_{h0}\) and \(B_h\) are orthogonal to each other.
Furthermore, we define the finite element space \(X_{bCR}^h\) by
\[
X_{bCR}^h := CR_0^1 + B_h = \{\psi_h + b_h; \; \psi_h \in CR_0^1, \; b_h \in B_h\}. \tag{5.10}
\]
We consider the following finite element problem. Find \(u_{bCR}^h \in X_{bCR}^h\) such that
\[
a_{0h}(u_{bCR}^h, \varphi_h) = (\nabla_h u_{bCR}^h, \nabla_h \varphi_h) = (\Pi_0^h f, \varphi_h) \quad \forall \varphi_h \in X_{bCR}^h. \tag{5.11}
\]
The solution \(u_{bCR}^h \in X_{bCR}^h\) is then decomposed as \(u_{bCR}^h = \bar{u}_{CR}^h + b_h\) with \(\bar{u}_{CR}^h \in CR_0^1\) and \(b_h \in B_h\). Note that \(\bar{u}_{CR}^h\) and \(b_h\) respectively satisfy (5.2) and the equation
\[
a_{0h}(b_h, c_h) = (\nabla_h b_h, \nabla_h c_h) = (\Pi_0^h f, c_h) \quad \forall c_h \in B_h. \tag{5.12}
\]
On each element \(T \in \mathcal{T}_h\), (5.12) has the form
\[
\gamma_T \int_T \nabla \varphi_T \cdot \nabla \varphi_T dx = \int_T \Pi_0^h f \varphi_T dx, \quad \gamma_T \in \mathbb{R}. \tag{5.13}
\]
From (5.7) and (5.8), we have
\[
\gamma_T = \frac{1}{l^2} \Pi_0^h f \quad \forall T \in \mathcal{T}_h.
\]

**Theorem 8** Let \(u_{bCR}^h \in X_{bCR}^h\) be the solution of (5.11) and \((\sigma_{RT}^h, \bar{u}_{RT}^h) \in RT_0^0 \times M_0^0\) the solution of (5.1). We then have \(\nabla_h u_{bCR}^h \in RT_0^0\) and
\[
\sigma_{RT}^h = \nabla u_{bCR}^h \quad \forall T \in \mathcal{T}_h, \tag{5.14}
\]
\[
\bar{u}_{RT}^h = \Pi_0^h \bar{u}_{CR}^h \quad \forall T \in \mathcal{T}_h. \tag{5.15}
\]

**Proof** The proof can be found in [14]. \(\square\)

From Theorem 8 for \(d = 3\), the following lemma holds.

**Lemma 10** Let \(u_{CR}^h \in CR_0^1\) be the solution of (5.2) and \((\sigma_{RT}^h, \bar{u}_{RT}^h) \in RT_0^0 \times M_0^0\) the solution of (5.1). We then have the relationships
\[
\sigma_{RT}^h|_T = \nabla u_{CR}^h - \frac{1}{3} \Pi_0^h f(x - x_T) \quad \forall T \in \mathcal{T}_h, \tag{5.16}
\]
\[
\bar{u}_{RT}^h|_T = \Pi_0^h \bar{u}_{CR}^h + \frac{1}{180} \Pi_0^h f \sum_{i=1}^4 |x_i - x_T|^2 \quad \forall T \in \mathcal{T}_h. \tag{5.17}
\]

Using relationship between the RT and CR finite element methods, we have the error estimate of the CR finite element approximation with the bubble function.
Lemma 11 We assume that $\Omega$ is convex. Let $\{T_h\}$ be a family of conformal meshes satisfying Assumption 7. Let $\bar{u} \in H^1(\Omega) \cap H^2(\Omega)$ be the solution of (5.3) and $u^{CR}_h \in X^{CR}_h$ be the solution of the Crouzeix–Raviart problem (5.11). There then exists a constant $c > 0$ independent of $\bar{u}$, $h$, $H$ and the geometric properties of $T_h$ such that

$$|\bar{u} - u^{CR}_h|_{H^1(\gamma_h)} \leq cH\|\Pi^0_h f\|. \tag{5.18}$$

Proof Let $(\bar{\sigma}^{RT}_h, \bar{u}^{RT}_h) \in R^0_T \times M^0_T$ be the solution of (5.1). From Theorem 8, it holds that $\nabla_h u^{CR}_h \in RT^0_h$ and $\bar{\sigma}^{RT}_h = \nabla_h u^{CR}_h$. Setting $\bar{\sigma} := \nabla \bar{u} \in H^1(\Omega)^d$, we then have, using inequality (4.11), that

$$|\bar{u} - u^{CR}_h|_{H^1(\gamma_h)} = \left(\sum_{T \in \Gamma_h} \|\bar{\sigma} - \bar{\sigma}^{RT}_h\|^2_{L^2(T)}\right)^{1/2} \leq cH|\bar{\sigma}|_{H^1(\Omega)^d} = cH|\bar{u}|_{H^2(\Omega)} \leq cH\|\Pi^0_h f\|.$$

$\square$

6 Numerical Results

This section presents results of numerical examples. Let $\Omega := (0, 1)^3$. Let $u^L_h$ and $u^{CR}_h$ be the $P_1$-Lagrange and $P_1$-CR finite element solutions, respectively, for the model problem

$$-\Delta u = 2y(1-y)z(1-z) + 2x(1-x)z(1-z) + 2x(1-x)y(1-y) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

which is the exact solution $u = x(1-x)y(1-y)z(1-z)$.

Let $M$ be the division number of each side of the bottom face and $N$ the division number of the height of $\Omega$ with $N \sim M^\gamma$ (see Fig. [3]). There are two elements as shown in Fig. [3].

If an exact solution $u$ is known, the error $e_h := u - u_h$ and $e_{h/2} := u - u_{h/2}$ are computed numerically for two mesh sizes $h$ and $h/2$. The convergence indicator $r$ is defined by

$$r = \frac{1}{\log(2)} \log \left( \frac{\|e_h\|_X}{\|e_{h/2}\|_X} \right).$$

We set $h := \frac{1}{4M}$. The parameter $H$ is then $H = O(h^{2-\gamma})$. We compute the convergence order with respect to $H^1_0$ and $L^2$ norms defined by

$$\text{Err}^L_h(H^1_0) := \frac{|u - u^L_h|_{H^1(\Omega)}}{\|\Delta u\|}, \quad \text{Err}^L_h(L^2) := \frac{\|u - u^L_h\|}{\|\Delta u\|},$$

$$\text{Err}^{CR}_h(H^1_0) := \frac{|u - u^{CR}_h|_{H^1(\gamma_h)}}{\|\Delta u\|}, \quad \text{Err}^{CR}_h(L^2) := \frac{\|u - u^{CR}_h\|}{\|\Delta u\|}.$$
for three cases: $\gamma = 1.5$, $\gamma = 1.9$ and $\gamma = 2.0$. In order to compute the above norms, we use the five-order fifteen-point numerical integration introduced in [16]. The results are give in Table 1 when $\gamma = 1.5$, Table 2 when $\gamma = 1.9$, and Table 3 when $\gamma = 2.0$. Further, $N^L_p$ and $N^{CR}_p$ denote respectively the degrees of freedom for the $P^1$-Lagrange finite element and the $P^1$-CR finite element.

Table 1 Error of the $P^1$-Lagrange finite element solution ($\gamma = 1.5$)

| $M$ | $N$ | $h$   | $H$   | $N^L_p$ | $\text{Err}_{L^2}(H^1)$ | $r$     | $\text{Err}_{L^2}(L^2)$ | $r$     |
|-----|-----|-------|-------|---------|--------------------------|--------|--------------------------|--------|
| 4   | 8   | 2.50e-01 | 5.00e-01 | 225     | 1.2043e-01               | 0.78   | 9.5321e-03               | 1.59   |
| 8   | 22  | 1.25e-01 | 3.54e-01 | 1,863   | 7.0318e-02               | 0.78   | 3.1646e-03               | 1.33   |
| 16  | 64  | 6.25e-02 | 2.50e-01 | 18,785  | 4.4662e-02               | 0.65   | 1.2570e-03               | 1.33   |
| 32  | 182 | 3.13e-02 | 1.77e-01 | 199,287 | 2.9476e-02               | 0.60   | 5.4477e-04               | 1.21   |

Observing the numerical results, the convergence indicators $r$ in each norms are respectively

$$ |u - u^L_h|_{H^1(\Omega)} = \mathcal{O}(H), \quad \|u - u^L_h\| = \mathcal{O}(H^2), $$

$$ |u - u^{CR}_h|_{H^1(\gamma_a)} = \mathcal{O}(h), \quad \|u - u^{CR}_h\| = \mathcal{O}(h^2), $$
Table 2 Error of the $P^1$-CR finite element solution ($\gamma = 1.5$)

| $M$ | $N$ | $h$ | $H$ | $N^p_{\text{CR}}$ | $\text{Err}^h_{\text{CR}}(H^1_0)$ | $r$ | $\text{Err}^h_{\text{CR}}(L^2)$ | $r$ |
|-----|-----|-----|-----|-----------------|---------------------------------|-----|-----------------|-----|
| 4   | 8   | 5.00e-01 | 1.440 | 8.2509e-02 | 4.8242e-03 |
| 8   | 22  | 1.25e-01 | 4.63e-02 | 4.0629e-02 | 8.8556e-04 | 2.11 |
| 16  | 64  | 6.25e-02 | 2.00e-02 | 2.0042e-02 | 2.0854e-04 | 2.11 |
| 32  | 182 | 3.13e-02 | 9.05e-03 | 9.0579e-03 | 4.8960e-05 | 2.07 |

Table 3 Error of the $P^1$-Lagrange finite element solution ($\gamma = 1.9$)

| $M$ | $N$ | $h$ | $H$ | $N^p_{L}$ | $\text{Err}^h_{L}(H^1_0)$ | $r$ | $\text{Err}^h_{L}(L^2)$ | $r$ |
|-----|-----|-----|-----|-----------|---------------------|-----|---------------------|-----|
| 4   | 14  | 2.50e-01 | 8.71e-01 | 345 | 1.4873e-01 | 1.4032e-02 |
| 8   | 52  | 1.25e-01 | 8.12e-01 | 4.293 | 1.2167e-01 | 0.29 | 9.3061e-03 | 0.59 |
| 16  | 194 | 6.25e-02 | 7.58e-01 | 509 | 1.0919e-01 | 0.16 | 7.4989e-03 | 0.31 |
| 32  | 724 | 3.13e-02 | 7.07e-01 | 7,508,480 | 1.0128e-01 | 0.11 | 6.4558e-03 | 0.22 |

Table 4 Error of the $P^1$-CR finite element solution ($\gamma = 1.9$)

| $M$ | $N$ | $h$ | $H$ | $N^p_{\text{CR}}$ | $\text{Err}^h_{\text{CR}}(H^1_0)$ | $r$ | $\text{Err}^h_{\text{CR}}(L^2)$ | $r$ |
|-----|-----|-----|-----|-----------------|---------------------------------|-----|-----------------|-----|
| 4   | 14  | 2.50e-01 | 8.71e-01 | 2,496 | 7.9756e-02 | 3.2993e-03 |
| 8   | 52  | 1.25e-01 | 8.12e-01 | 35,072 | 3.9708e-02 | 1.01 | 7.7177e-04 | 2.10 |
| 16  | 194 | 6.25e-02 | 7.58e-01 | 509,568 | 1.9814e-02 | 0.16 | 1.8781e-04 | 2.04 |
| 32  | 724 | 3.13e-02 | 7.07e-01 | 7,508,480 | 9.9003e-03 | 0.11 | 4.6546e-05 | 2.01 |

Table 5 Error of the $P^1$-Lagrange finite element solution ($\gamma = 2.0$)

| $M$ | $N$ | $h$ | $H$ | $N^p_{L}$ | $\text{Err}^h_{L}(H^1_0)$ | $r$ | $\text{Err}^h_{L}(L^2)$ | $r$ |
|-----|-----|-----|-----|-----------|---------------------|-----|---------------------|-----|
| 4   | 16  | 2.50e-01 | 1.00 | 425 | 1.5862e-01 | 1.5909e-02 |
| 8   | 64  | 1.25e-01 | 1.00 | 1,024 | 1.1162e-01 | 1.1144e-02 | 0.03 |
| 16  | 256 | 6.25e-02 | 1.00 | 1,024 | 1.1162e-01 | 1.1144e-02 | 0.03 |
| 32  | 1,024 | 3.13e-02 | 1.00 | 1,024 | 1.1162e-01 | 1.1144e-02 | 0.03 |

where $H = O(h^{2-\gamma})$. Meanwhile, the theoretical results are as follows:

$$|u - u^L|_{H^1(\Omega)} = O(H), \quad \|u - u^L\| = O(H^2),$$

$$|u - u^CR|_{H^1(\Omega)} = O(H), \quad \|u - u^CR\| = O(H^2),$$

if $\Omega$ is convex and $u \in H^2(\Omega) \cap H^1_0(\Omega)$. In this numerical examples, the CR finite element approximation is superior to the Lagrange finite element approximation on this anisotropic meshes. The theoretical explanation of this point is still open.

Table 6 Error of the $P^1$-CR finite element solution ($\gamma = 2.0$)

| $M$ | $N$ | $h$ | $H$ | $N^p_{CR}$ | $\text{Err}^h_{CR}(H^1_0)$ | $r$ | $\text{Err}^h_{CR}(L^2)$ | $r$ |
|-----|-----|-----|-----|-----------|---------------------|-----|---------------------|-----|
| 4   | 16  | 2.50e-01 | 1.00 | 2,848 | 7.9473e-02 | 3.2264e-03 |
| 8   | 64  | 1.25e-01 | 1.00 | 1,024 | 1.9830e-02 | 1.00 | 1.8680e-04 | 2.03 |
| 16  | 256 | 6.25e-02 | 1.00 | 1,024 | 1.9830e-02 | 1.00 | 1.8680e-04 | 2.03 |
| 32  | 1,024 | 3.13e-02 | 1.00 | 1,024 | 1.9830e-02 | 1.00 | 1.8680e-04 | 2.03 |
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