A NEW BIJECTION BETWEEN FORESTS AND PARKING FUNCTIONS

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Abstract. In 1980, G. Kreweras [Kre80] gave a recursive bijection between forests and parking functions. In this paper we construct a nonrecursive bijection from forests onto parking functions, which answers a question raised by R. Stanley [Sta07, Exercise 6.4]. As a by-product, we obtain a bijective proof of Gessel and Seo’s formula for lucky statistic on parking functions [GS06].

1. Introduction

It is well-known (see e.g. [FR74]) that there are several bijections between forests on \( n \) vertices and parking functions with length \( n \). In 1980, G. Kreweras [Kre80] presented his work that connected recursively inversion enumerators for trees with parking functions. After that this recursive bijection was also rewritten in R. Stanley’s lecture note [Sta07] in 2004. In this book, he wrote that we need a nonrecursive bijection \( \varphi \) between the set \( F_n \) of all rooted forests on \( n \) vertices and the set \( PF_n \) of all parking functions of length \( n \) satisfying

\[
\text{inv}(F) = \binom{n+1}{2} - a_1 - \cdots - a_n
\]

where \( \varphi(F) = (a_1, \ldots, a_n) \). (See [Sta07, Exercise 6.4]) He mentioned that a “nonrecursive” bijection would be greatly preferred.

Gessel and Seo [GS06] studied the statistic lucky of parking functions. The generating function for lucky is

\[
\sum_{P \in PF_n} u^{\text{lucky}_P} = u \prod_{i=1}^{n-1} (i + (n - i + 1)u),
\]

where the sum is over all parking function \( P \) of length \( n \). This formula is proved by them, but that is not bijective. On the other side, Seo and Shin [SS07] introduced the statistic leader of forests, and whose generating function is

\[
\sum_{F \in F_n} u^{\text{lead}_F} = u \prod_{i=1}^{n-1} (i + (n - i + 1)u),
\]
where the sum is over all forest $F$ on $n$ vertices, which is proved bijectively using reverse Pr"ufer code. Since the right-hand sides of two equations (1) and (2) are same, we have to find a bijection between forests and parking functions which yields

$$ \sum_{F \in F_n} u^{\text{lead}F} = \sum_{P \in PF_n} u^{\text{lucky}P}. $$

In this paper, we construct a nonrecursive bijection $\varphi : F_n \rightarrow PF_n$ between forests and parking functions satisfying

$$ \begin{align*}
\text{inv}(F) &= \binom{n+1}{2} - |P| \\
\text{lead}(F) &= \text{lucky}(P)
\end{align*} $$

where $P = \varphi(F)$ and $|P|$ is the sum of sequences.

Moreover, reviewing the bijection $\varphi$, it has been observed that parking functions have a statistic corresponding to the statistic tree, the number of trees, in forests. When this statistic in parking functions is called critical, the bijection $\varphi$ preserves the statistics inv, lead, and tree for forests to jump, lucky, and critical for parking functions.

2. Preliminary

A graph on labeled vertices is called labeled and if a graph have one distinguished vertex, the vertex is called root and the graph is called rooted. A tree is a simple connected rooted labeled graph without cycles. A forest is a graph in which any two vertices are connected by at most one path and each connected component is a tree.

A vertex $j$ is called a descendant of a vertex $i$ if a vertex $i$ lies on the unique path from the root to a vertex $j$. This is equivalent to the statement that a vertex $i$ is a ascendant of a vertex $j$. An inversion in a rooted graph is an ordered pair $(i, j)$ such that $i > j$ and $j$ is a descendant of $i$. Let $\text{inv}(G : v)$ denote the number of ordered pairs $(v, x)$ where $v > x$ and $x$ is a descendant of $v$ in a rooted graph $G$ and $\text{inv}(G)$ the number of all inversions in a rooted graph $G$. By definition, $\text{inv}(G) = \sum_{v \in G} \text{inv}(G : v)$.

An vertex $v$ is called a leader in a rooted graph $G$ if $\text{inv}(G : v) = 0$, that is, the vertex $v$ is the smallest among its all descendants. By definition, every leaf is a leader. $\text{lead}(G)$ denotes the number of all leaders in a rooted graph $G$.

3 rules drawing a forests. We want that the shape of a tree is unique by drawing in only one way. When we draw a forest, we keep the following rules.

- Draw the roots at the top and all trees grow downward.
- Put the trees from left to right according to maximum label in each tree.
- Similarly, when vertices are drawn, put siblings from left to right in the order of maximum labels in their descendants.

It seems that the shape of a forest is figured as a rooted ordered forest after drawing.
Parking algorithm. Given a sequence \((p_1, p_2, \ldots, p_n)\) of length \(n\), where \(p_i\) means the favorite parking space of the \(i\)-th car, we can park \(n\) cars into parking spaces as follows:

1. Cars can be parked one by one from the first car to the last car into infinitely many parking spaces whose entrance is at the left.

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\text{Entrance} & & & & & & \\
\text{This is a one-way road}
\end{array}
\]

2. When \(i\)-th car is parked, a car has to reach at its favorite parking space \(p_i\). And then, attempt to be parked there. If the space is empty, the car is parked. Otherwise, attempt again at the next parking space without going back. Repeat this process until success to park.

3. Let \(q_i\) be the actual parking space with \(i\)-th car.

This method is called a Parking Algorithm and the notation \(PA\) is defined by

\[PA(p_1, \ldots, p_n) = (q_1, \ldots, q_n)\]

For example, given a sequence \((4, 3, 3, 1, 5)\), five cars can be parked by the Parking Algorithm as following.

| Parking Space | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---------------|---|---|---|---|---|---|---|
| Cars' Number  | 4 | ∅ | 2 | 1 | 3 | 5 | ∅ |

We get a sequence with length 5, \(PA(4, 3, 3, 1, 5) = (4, 3, 5, 1, 6)\).

If \(PA(p_1, \ldots, p_n) = (q_1, \ldots, q_n)\) and the actual parking spaces \(q_i\) is less than or equal to \(n\) for all \(i\), the sequence \((p_1, \ldots, p_n)\) is called a parking function.

A jump in a parking function is defined by the attempt to park the next space because of a non-empty parking space. Let \(\text{jump}(P : c)\) be the number of the jumps in order to park the car \(c\), that is, the difference between the favorite parking space \(p_c\) and the actual parking space \(q_c\). So we make the formula

\[\text{jump}(p_1, \ldots, p_n : c) = q_c - p_c\]

where \(PA(p_1, \ldots, p_n) = (q_1, \ldots, q_n)\). And \(\text{jump}(P)\) denotes the number of the jumps to park all cars. By definition, \(\text{jump}(P) = \sum_c \text{jump}(P : c)\). Therefore, we have

\[\text{jump}(P) = \sum_c \text{jump}(P : c) = \sum_c q_c - p_c = \left(\frac{n + 1}{2}\right) - |P|\]

where \(|P| = \sum p_i\).

A lucky in a parking function \(P\) is the car \(c\) where \(\text{jump}(P : c) = 0\), that is, the car \(c\) is parked at its favorite parking space. \(\text{lucky}(P)\) denotes the number of all lucky cars in \(P\). For example, for a given parking function \(P = (2, 4, 2, 1, 3)\), we get the sequence \(PA(P) = (2, 4, 3, 1, 5)\) by the parking algorithm.
With seeing above table, we can calculate the jump and the lucky. Actually, we have $\text{jump}(P) = 3$ since $\text{jump}(P : 3) = 1$ and $\text{jump}(P : 5) = 2$. Also, we have $\text{lucky}(P) = 3$ since lucky cars are 1, 2, and 4.

3. The Map $\varphi : F_n \to PF_n$

First of all, the map $\varphi$ is defined according to the diagram in Figure 1. Considering one forest $F \in F_{14}$ as an example, we are going to describe how to define the map $\varphi$ as follows:

(1) Draw the forest $F \in F_n$ by 3 rules in Section 2.

(2) Change the forest $F$ to the tree $T$, adding the vertex $n+1$ at the top and connecting new vertex to each root of trees in $F$. 

\[
\begin{array}{c|cccc}
q_c & 1 & 2 & 3 & 4 & 5 \\
\hline
\text{c} & 4 & 1 & 3 & 2 & 5 \\
\text{p_c} & 1 & 2 & 2 & 4 & 3 \\
q_c - p_c & 0 & 0 & 1 & 0 & 2 \\
\end{array}
\]
(3) Rearrange the label on vertices by the following pseudo-code:
   for all $v \in V$ do
   i) find the maximum label $m$ on descendants of $v$.
   ii) label $m$ on $v$.
   iii) rearrange the other labels in descendants of $v$ by order-preserving.
   end do
For example, after rearranging labels on descendants of $v$ labeled by 8 in the tree $T$, we label 13, the maximum of descendants on $v$, on the vertex $v$.

This is well-defined, that is independent to an order of choosing vertices $v \in V$.

(4) The decreasing tree $D$ is made after acting above process on all vertices. The map $\theta_F$ is induced by the correspondence of labels in a tree after relabeling. For example, $\theta_F(10) = 9$. 
(5) Because we cannot remake the original tree $T$ from the tree $D$ alone, we need another tree induced from the unused information of $T$, that is, $\text{inv}(T : v)$. So we make a new tree $I$ such that each vertex $v$ is labeled with $\text{inv}(T : v)$. In order to distinguish it from other labels, we use the box.

Note that we can produce the original tree $T$ from only two trees $D$ and $I$.

(6) Additionally, label the vertices indexed by post-order which is indicated by circle. The tree $C$ is only dependent to the underlying graph of $T$, that is, its tree structure. This is the reason why we define the method to redraw the tree in the inverse map $\varphi^{-1}$.

(7) Finally, we make the tree $D \times (C - I)$, in which the plain labels are induced by $D$ and the circled labels are induced by $C$ subtracted by $I$. 
\[ \varphi^{-1} : P \xrightarrow{PA} PA(P) \xrightarrow{D} T \xrightarrow{F} C \]

\[ \xrightarrow{I} \]

**Figure 2.** Diagram of \( \varphi^{-1} \)

(8) In the sequel, we delete the tree-structure from \( D \times (C - I) \) and write the circle number by the order of plain number. Since the last is always \( 1 \), we delete it and the rest becomes a parking function. Although all labels of \( I \) are zeros in the worst case, the set of circle labels of \( C - I \) becomes \([n]\). Since every permutation is a parking function, it becomes a parking function in the worst case. For continuing example, we get the sequences

\[
\begin{align*}
1 & \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \\
\text{Cars' Number}, & \ 6 \ 2 \ 10 \ 9 \ 4 \ 3 \ 5 \ 14 \ 12 \ 1 \ 8 \ 13 \ 7 \ 11 \ 15
\end{align*}
\]

Below the plain label 15, there is always circled label 1. So we can omit it, and then the second row (circled label) becomes a parking function \( P \) of length 14.

\[ P = 10 \ 2 \ 6 \ 5 \ 7 \ 1 \ 13 \ 1 \ 4 \ 14 \ 9 \ 11 \ 5 \]

4. **The Inverse Map** \( \varphi^{-1} : PF_n \rightarrow F_n \)

In this section, we construct the inverse map \( \varphi^{-1} \) from parking functions to forests as Figure 2. We start from the previous example \( P \in PF_{14} \),

\[ P = 10 \ 2 \ 6 \ 5 \ 7 \ 1 \ 13 \ 1 \ 4 \ 14 \ 9 \ 11 \ 5 \]

After adding the 1 at the end of \( P \), 15 cars are parked by the parking algorithm as follows:

| Parking Space | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---------------|---|---|---|---|---|---|---|---|---|-----|----|----|----|----|----|
| Cars' Number, c | 6 | 2 | 10 | 9 | 4 | 3 | 5 | 14 | 12 | 1 | 8 | 13 | 7 | 11 | 15 |
At this time, we record the jump for every car in third row. And then, we draw an edge between the car $c$ and the closest car on its right which is larger than $c$.

| Parking Space | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| Cars’ Number, $c$ | 6 | 2 | 10 | 9 | 4 | 3 | 5 | 14 | 12 | 1 | 8 | 13 | 7 | 11 | 15 |
| jump($P : c$) | 0 | 0 | 2 | 0 | 0 | 0 | 3 | 0 | 0 | 1 | 1 | 0 | 0 | 14 |

We get the tree-structure on the cars as vertices. If we consider 15 as a root, we can rebuild three trees $C$, $D$, and $I$. The parking space in the first row becomes the tree $C$, the cars’ number in the second row becomes the tree $D$, and the jump in the third row becomes the tree $I$.

Because the label $I(v)$ of vertex $v$ in the tree $I$ stands for $\text{inv}(T : v)$, we can make the tree $T$ from the decreasing tree $D$ as follows:

for all vertex $v$ in the tree $D$ do
  i) find $(I(v) + 1)$-th smallest label $m$ on descendants of $v$.
  ii) label $m$ on $v$.
  iii) rearrange the other labels in descendants of $v$ by order-preserving.
end do

The pseudo-code above is an inverse map of $\theta_F$. After we make the tree $T$, we can get the forest $F$ from $T$ by deleting the maximum vertex of $T$.

**Theorem 1.** The above algorithm from a parking function to a forest is the inverse map of $\varphi$.

**Proof.** It is enough to show that the tree-structure deleted in the map $\varphi$ and the tree-structure made in the inverse map $\varphi^{-1}$ are the same. If all labels of the tree $I$ are zeros, the circled labels in the tree $D \times (C - I)$ are distinct. If so, a parking function $P$ is a permutation, that is, all cars are lucky. If $P$ is a permutation, $PA(P) = P$. A parent of a car $c$ is larger than $c$ since the tree $D$ is decreasing and it is on the right of $c$ after parking algorithm because of a post-order. In this time, we can make the tree $D$ from a permutation $P^{-1}$.

Since the tree $D$ is decreasing, all cars corresponding to descendants of $v$ already parked when a car corresponding to $v$ is parking. Using labels of the tree $C - I$ instead of the tree $C$, a favorite parking space of car $c$ corresponding to a vertex $v$ decreases by $\text{inv}(T : v)$ but parking space at which car $c$ parks actually is not changed. Hence we can make the tree $D$ from $PA(P)$.

5. **Statistics**

After we observe the map $\varphi$, we can get Lemma 2:

**Lemma 2.** The map $\varphi$ has two following properties.

- $\text{inv}(F : v) = \text{jump}(\varphi(F) : \theta_F(v))$
- If $v$ is a root of a tree in $F$, then $\theta_F(v)$ is a right-to-left maximum in $PA(\varphi(F))^{-1}$. 
Proof. If we use the labels of the tree $C$ instead of the tree $C - I$, all cars are lucky. Using labels of the tree $C - I$ instead of the tree $C$, $\text{jump}(P : c)$ increases by $\text{inv}(T : v)$. Thus $\text{inv}(F : v) = \text{jump}(\varphi(F) : \theta_F(v))$.

If a vertex $v$ is a root of a tree in $F$, a parent of $v$ is the root of $T$. So there is no car larger than the car $\theta_F(v)$ on its right. Hence the car $\theta_F(v)$ is a right-to-left maximum in $PA(\varphi(F))^{-1}$. □

Let $\text{inv}(F)$ be a type of inversions of $F$ and $\text{jump}(P)$ be a type of jumps of $P$ defined by

\[
\begin{align*}
\text{inv}(F) &= (\text{lead}_0(F), \ldots, \text{lead}_n(F)) \\
\text{jump}(P) &= (\text{lucky}_0(P), \ldots, \text{lucky}_n(P))
\end{align*}
\]

where $\text{lead}_i(F)$ is the number of vertices $v$ such that $\text{inv}(F : v) = i$ and $\text{lucky}_i(F)$ is the number of cars $c$ such that $\text{jump}(P : c) = i$.

The car $c$ is called critical if there is no empty parking space on the right of the car $c$ after it is parked. Let $\text{tree}(F)$ be the number of trees in a forest $F$ and $\text{critical}(P)$ be the number of critical cars in a parking function $P$. Note that any critical car becomes a right-to-left maximum in $PA(\varphi(F))^{-1}$ and its converse is also true.

**Theorem 3 (Main Theorem).** There is a nonrecursive bijection $\varphi : F_n \to PF_n$ between forests and parking functions satisfying

\[
(\text{inv}, \text{tree})(F) = (\text{jump}, \text{critical})(\varphi(F))
\]

**Proof.** By Lemma 2 there is the correspondence $\theta_F$ between all vertices $v$ in the forest $F$ and all cars $c$ in the parking function $\varphi(F)$ such that $\text{inv}(F : v) = \text{jump}(\varphi(F) : \theta_F(v))$. So we have $\text{lead}_i(F) = \text{lucky}_i(\varphi(F))$ for all $i = 0, \ldots, n$ and

\[
\text{inv}(F) = \text{jump}(\varphi(F)).
\]

By the map $\theta_F$, each root of trees in $F$ corresponds to each of right-to-left maximums in $PA(\varphi(F))^{-1}$. Hence we have

\[
\text{tree}(F) = \text{critical}(\varphi(F)).
\]

□

Let $I_n$ and $J_n$ be homogeneous polynomials of degree $n$,

\[
\begin{align*}
I_n(q; c) &= \sum_{F \in F_n} q^{\text{inv}(F)}c^{\text{tree}(F)} \\
J_n(q; c) &= \sum_{P \in PF_n} q^{\text{jump}(P)}c^{\text{critical}(P)}
\end{align*}
\]

where $q^v = q_0^{v_0}q_1^{v_1} \cdots q_n^{v_n}$.

**Theorem 4.** For a nonnegative integer $n$, we have

\[
I_n(q; c) = J_n(q; c)
\]
Proof. By Theorem 3, there exists the bijection $\varphi : F_n \rightarrow PF_n$ such that

$$q^{\text{inv}(F)}c^{\text{tree}(F)} = q^{\text{jump}(\varphi(F))}c^{\text{critical}(\varphi(F))}.$$

So we have $I_n(q; c) = J_n(q; c)$. \qed

Corollary 5. We have

$$\sum_{F \in F_n} q^{\text{inv}(F)}u^{\text{lead}(F)}c^{\text{tree}(F)} = \sum_{P \in PF_n} q^{\text{jump}(P)}u^{\text{lucky}(P)}c^{\text{critical}(P)}$$

Proof. By Theorem 4, $I_n(u, q, q^2, \ldots ; c) = J_n(u, q, q^2, \ldots ; c)$. Simplifying it by

$$\text{inv}(F) = \sum_i i \cdot \text{lead}_i(F),$$
$$\text{lead}(F) = \text{lead}_0(F),$$
$$\text{jump}(P) = \sum_i i \cdot \text{lucky}_i(P) = \binom{n+1}{2} - |P|, \text{ and}$$
$$\text{lucky}(P) = \text{lucky}_0(P),$$

we are done. \qed

Corollary 6. We have

$$\sum_{P \in PF_n} c^{\text{critical}(P)}u^{\text{lucky}(P)} = cu^{n-1} \prod_{i=1}^{n-1} (i + (n-i)u + cu),$$

which have a bijective proof.

Proof. By Theorem 4, $I_n(u, 1, 1, \ldots ; c) = J_n(u, 1, 1, \ldots ; c)$. We get

$$\sum_{F \in F_n} u^{\text{lead}(F)}c^{\text{tree}(F)} = \sum_{P \in PF_n} u^{\text{lucky}(P)}c^{\text{critical}(P)}.$$

Recall the formula in [SS07, Eq.(1)],

$$\sum_{F \in F_n} u^{\text{lead}(F)}c^{\text{tree}(F)} = P_n(1, u, cu)$$

where $P_n(a, b, c) = c \prod_{i=1}^{n-1} (ia + (n-i)b + c)$. Combining above two formulae, we are done. \qed

Forests and parking functions have not only the same cardinality, but also many equinumerous statistics. The map $\varphi$ corresponds simultaneously between statistics inv, lead, and tree in forests and statistics jump, lucky, and critical in parking functions. Also, while the $\varphi$ makes a correspondence between combinatorial objects, the $\theta$ makes a correspondence between vertices in forests and cars in parking functions in detail.
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