Supersolvable Frobenius groups with nilpotent centralizers

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Abstract. Let $FH$ be a supersolvable Frobenius group with kernel $F$ and complement $H$. Suppose that a finite group $G$ admits $FH$ as a group of automorphisms in such a manner that $C_G(F) = 1$ and $C_G(H)$ is nilpotent of class $c$. We show that $G$ is nilpotent of $(c, |FH|)$-bounded class.

1. Introduction

Let a group $A$ act by automorphisms on a group $G$. We denote by $C_G(A)$ the set $C_G(A) = \{x \in G; \ x^a = x \text{ for all } a \in A\}$, the centralizer of $A$ in $G$ (the fixed-point subgroup). Very often the structure of $C_G(A)$ has strong influence over the structure of $G$. Recently, prompted by Mazurov’s problem 17.72 in the Kourovka Notebook [8], some attention was given to the situation where a Frobenius group $FH$ acts by automorphisms on a finite group $G$. Recall that a Frobenius group $FH$ with kernel $F$ and complement $H$ can be characterized as a finite group that is a semidirect product of a normal subgroup $F$ by $H$ such that $C_F(h) = 1$ for every $h \in H \setminus \{1\}$. By Thompson’s theorem [11] the kernel $F$ is nilpotent, and by Higman’s theorem [4] the nilpotency class of $F$ is bounded in terms of the least prime divisor of $|H|$.

In the case where the kernel $F$ acts fixed-point-freely on $G$, some results on the structure of $G$ were obtained by Khukhro, Makarenko and Shumyatsky in [7]. The authors prove that various properties of $G$ are in a certain sense close to the corresponding properties of its subgroup $C_G(H)$, possibly also depending on $H$. In particular, they proved the following result.

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Theorem 1.1 ([7], Theorem 2.7 (c)). Suppose that a finite group $G$ admits a Frobenius group of automorphisms $FH$ with kernel $F$ and complement $H$ such that $C_G(F) = 1$ and $C_G(H)$ is nilpotent. Then $G$ is nilpotent.

Under the additional assumption that the Frobenius group of automorphisms $FH$ is metacyclic, that is, supposing that the kernel $F$ is cyclic, the authors use some Lie ring methods to obtain upper bounds for the nilpotency class of $G$. They proved that if $C_G(H)$ is nilpotent of class $c$, then the nilpotency class of $G$ is bounded in terms of $c$ and $|H|$. In the case when $GF$ is also a Frobenius group with kernel $G$ and complement $F$ (so that $GFH$ is a double Frobenius group) the latter result was obtained earlier in [10]. Examples in [7] show that the result on the nilpotency class of $G$ is no longer true in the case of non-metacyclic Frobenius groups.

Recall that a group $N$ is supersolvable if $N$ possesses a normal series with cyclic factors such that each term is normal in $N$. It is easy to see that a supersolvable group possesses a chief series whose factors have prime order.

Throughout the paper we use the expression “$(a, b, \ldots)$-bounded” to mean “bounded from above in terms of $a, b, \ldots$ only”. In the present paper we consider the situation in which a not necessarily metacyclic Frobenius group acts by automorphisms on a finite group. More precisely, we prove the following theorem.

Theorem 1.2. Let $FH$ be a supersolvable Frobenius group with kernel $F$ and complement $H$. Suppose that $FH$ acts on a finite group $G$ in such a way that $C_G(F) = 1$ and $C_G(H)$ is nilpotent of class $c$. Then $G$ is nilpotent of $(c, |FH|)$-bounded class.

Note that $G$ in Theorem 1.2 is nilpotent by Theorem 1.1. We wish to show that the nilpotency class of $G$ is $(c, |FH|)$-bounded. It should be mentioned that in the case of metacyclic $FH$, Khukhro, Makarenko and Shumyatsky in [7, Theorem 5.1] give a bound independent of the order $|F|$. At present it is unclear if in Theorem 1.2 the bound can be made independent of $|F|$. The proof is based on an analogous result on Lie algebras.

In [11, 6] we can find some other results bounding the nilpotency class for groups acted on by Frobenius groups of automorphisms with non-cyclic kernels.
2. Some results on graded Lie algebras

Let $A$ be an additively written abelian group and $L$ an $A$-graded Lie algebra:

$$L = \bigoplus_{a \in A} L_a, \quad [L_a, L_b] \subseteq L_{a+b}.$$ 

The particular case of $\mathbb{Z}/n\mathbb{Z}$-graded Lie algebras arises naturally in the study of Lie algebras admitting an automorphism $\varphi$ of order $n$. This is due to the fact that, after the ground field is extended by a primitive $n$th root of unity $\omega$, the eigenspaces $L_i = \{x \in L \mid x^\varphi = \omega^ix\}$ behave like the components of a $\mathbb{Z}/n\mathbb{Z}$-grading. For example, the proof of a well-known theorem of Kreknin [9] stating that a Lie ring $L$ admitting a fixed-point-free automorphism of finite order $n$ is soluble with derived length bounded by a function of $n$, is reduced to proving the solvability of a $\mathbb{Z}/n\mathbb{Z}$-graded Lie ring with $L_0 = 0$.

**Theorem 2.1.** Let $n$ be a positive integer and $L$ be a $\mathbb{Z}/n\mathbb{Z}$-graded Lie algebra. If $L_0 = 0$, then $L$ is solvable of $n$-bounded derived length.

In the special case where $n$ is a prime we have the following well-known result proved by Higman.

**Theorem 2.2.** [4] Let $p$ be a prime number and $L$ be a $\mathbb{Z}/p\mathbb{Z}$-graded Lie algebra. If $L_0 = 0$, then $L$ is nilpotent of $p$-bounded class.

If $J, Y, J_1, \ldots, J_s$ are subsets of $L$ we use $[J, Y]$ to denote the subspace of $L$ spanned by the set $\{[j, y] \mid j \in J, y \in Y\}$ and for $i \geq 2$ we write $[J_1, \ldots, J_i]$ for $[[J_1, \ldots, J_{i-1}], J_i]$.

The next two results are also criteria for solvability and nilpotency of graded Lie algebra, respectively.

**Theorem 2.3.** [5] Theorem 1] Let $n$ be a positive integer and $L$ be a $\mathbb{Z}/n\mathbb{Z}$-graded Lie algebra. Suppose that $[L, L_0, \ldots, L_0]_m = 0$. Then, $L$ is solvable of $(n,m)$-bounded derived length.

**Theorem 2.4.** [6] Proposition 2] Let $A$ be an additively written abelian group and $L$ an $A$-graded Lie algebra. Suppose that there are at most $d$ nontrivial grading components among the $L_a$. If $[L, L_a, \ldots, L_a]_m = 0$ for all $a \in A$, then $L$ is nilpotent of $(d,m)$-bounded class.
Now, let \( p \) be a prime number and \( L \) be a \( \mathbb{Z}/p\mathbb{Z} \)-graded Lie algebra. Assume that there exist non-negative integers \( u \) and \( v \) such that
\[
(1) \quad [L, L_0, \ldots, L_0] = 0
\]
and
\[
(2) \quad [[L, L] \cap L_0, L_a, \ldots, L_a] = 0, \quad \text{for all } a \in \mathbb{Z}/p\mathbb{Z}.
\]

We finish this section showing that conditions (1) and (2) together are sufficient to conclude that \( L \) is nilpotent of \((p, u, v)\)-bounded class.

By Theorem 2.3, condition (1) implies that \( L \) is solvable with \((p, u)\)-bounded derived length. Thus, we can use induction on the derived length of \( L \). If \( L \) is abelian, there is nothing to prove. Assume that \( L \) is metabelian. In this case, \([x, y, z] = [x, z, y] \), for every \( x \in [L, L] \) and \( y, z \in L \).

For each \( b \in \mathbb{Z}/p\mathbb{Z} \) we denote \([L, L] \cap L_0, L_a, \ldots, L_a\) by \( L'_b \). By Theorem 2.3 it is sufficient to prove that there exists a \((p, u, v)\)-bounded number \( t \) such that
\[
[L'_b, L_a, \ldots, L_a] = 0, \quad \text{for all } a, b \in \mathbb{Z}/p\mathbb{Z}.
\]

If \( b = 0 \), this follows from (2) with \( t = v \). Suppose that \( b \neq 0 \). If \( a = 0 \), the commutator is zero from (1) with \( t = u \). In the case where \( a \neq 0 \) we can find a positive integer \( s < p \) such that \( b + sa = 0 \) (mod \( p \)). Therefore, we have \([L'_b, L_a, \ldots, L_a] \subseteq [L'_0, L_a, \ldots, L_a] \). We know that the latter commutator is 0 by (2).

These previous ideas were applied in a similar way in [3] and [2].

3. Bounding nilpotency class of Lie algebras

In this section \( FH \) denotes a finite supersolvable Frobenius group with kernel \( F \) and complement \( H \).

Let \( R \) be an associative ring with unity. Assume that the characteristic of \( R \) is coprime with \( |F| \) and the additive group of \( R \) is finite (or locally finite). Let \( L \) be a Lie algebra over \( R \). The main goal of this section is to prove the following theorem.

**Theorem 3.1.** Suppose that \( FH \) acts by automorphisms on \( L \) in such a way that \( C_L(F) = 0 \) and \( C_L(H) \) is nilpotent of class \( c \). Then \( L \) is nilpotent of \((c, |FH|)\)-bounded class.
If $F$ is cyclic, then $L$ is nilpotent of $(c, |H|)$-bounded class by [7] Theorem 5.1. Therefore without loss of generality we may assume that $F$ is not a cyclic group.

Let $Z$ be a subgroup of prime order $p$ of $Z(F)$ such that $Z \triangleleft FH$ and let $\varphi$ be a generator of $Z$.

Now, let $\omega$ be a primitive $p$th root of unity. We extend the ground ring of $L$ by $\omega$ and denote by $\tilde{L}$ the algebra $L \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. The action of $FH$ on $L$ extends naturally to $\tilde{L}$. Note that $C_{\tilde{L}}(F) = 0$ and $C_{\tilde{L}}(H)$ is nilpotent of class $c$. Since $L$ and $\tilde{L}$ have the same nilpotency class, it is sufficient to bound the class of $\tilde{L}$. Hence, without loss of generality it will be assumed that the ground ring contains $\omega$ so that we will work with $L$ rather than $\tilde{L}$.

For each $i = 0, \ldots, p - 1$ we denote by $L_i$ the $\varphi$-eigenspace corresponding to eigenvalue $\omega^i$, that is, $L_i = \{x \in L ; x^{\varphi} = \omega^ix\}$. We have

$$L = \bigoplus_{i=0}^{p-1} L_i \quad \text{and} \quad [L_i, L_j] \subseteq L_{i+j \pmod{p}}.$$  

Note that $L_0 = C_L(Z)$. It is easy to see that $FH/Z$ acts on $L_0$ in such a manner that $C_{L_0}(F/Z) = 0$. Thus, by induction on $|FH|$ we conclude that $L_0$ is nilpotent of $(c, |FH|)$-bounded class.

A proof of the next lemma can be found in [7] Lemma 2.4. It will be useful to decompose the subalgebra $L_0$ into the fixed-points of the subgroups $H^f$, $f \in F$.

**Lemma 3.2.** Suppose that a finite group $N$ admits a Frobenius group of automorphisms $KB$ with kernel $K$ and complement $B$ such that $C_N(K) = 1$. Then $N = \langle C_N(B^y); y \in K \rangle$.

In the above lemma we can write $N = \langle C_N(B^y); y \in K \rangle$, since $C_N(B^y) = C_N(B^y)$.

For any $f \in F$ we denote by $V_f$ the subalgebra $C_L(ZH^f) \subseteq L_0$. Note that $ZH^f$ is also a Frobenius group and that $ZH^f = ZH$ whenever $f \in Z$.

**Lemma 3.3.** We have $L_0 = \sum_{f \in F} V_f$.

**Proof.** The subalgebra $L_0$ is $FH$-invariant and so it admits the natural action by the group $FH/Z$. Moreover, $F/Z$ acts fixed-point-freely on $L_0$. Since the additive group of $L$ is finite, it is immediate from Lemma 3.2 that $L_0 = \langle C_{L_0}(H^f); \bar{f} \in F/Z \rangle$. As a consequence we have that $L_0 = \langle C_L(ZH^f); f \in F \rangle$. 

It is clear that any conjugated $H^f$ of $H$ can be considered as a Frobenius complement of $FH$. Now we describe the action $H^f$ on
the homogeneous components \( L_i \). Since \( H \) is cyclic, we can choose a generator \( h \) of \( H \) and \( r \in \{1, 2, \ldots, p - 1\} \) such that \( \varphi^{h^{-1}} = \varphi^r \). Then \( r \) is a primitive \( q \)-th root of 1 in \( \mathbb{Z}/p\mathbb{Z} \). The group \( H \) permutes the homogeneous components \( L_i \) as follows: \( L_i^h = L_{ri} \) for all \( i \in \mathbb{Z}/p\mathbb{Z} \). Indeed, if \( x_i \in L_i \), then \( (x_i^h)^\varphi = x_i^{h \varphi^{h^{-1}} h} = (x_i^\varphi)^h = \omega^r x_i^h \). On the other hand, since \( F \) commutes with \( \varphi \), we also have \( L_i^h = L_{ri} \) for all \( i \in \mathbb{Z}/p\mathbb{Z} \) (because if \( x_i \in L_i \), then \( (x_i^h)^\varphi = \omega^r x_i^h \)). Thus, the action of \( H^f \) on the components \( L_i \) coincides with the action of \( H \).

To lighten the notation we establish the following convention.

**Remark 3.4 (Index Convention).** In what follows, for a given \( u_s \in L_s \) we denote both \( u_s^h \) and \( u_s^{(h_f)^r} \) by \( u_{r_i,s} \), since \( L_s^h = L_s^{(h_f)^r} = L_{ri,s} \). Therefore, we can write \( u_s + u_{r_1,s} + \cdots + u_{r_{q-1},s} \) to mean a fixed-point of \( H^f \) for any \( f \in F \).

**Lemma 3.5.** There exists a \((c, |FH|)\)-bounded number \( u \) such that \( \left[ \underbrace{L, L_0, \ldots, L_0} \right] = 0 \).

**Proof.** It suffices to prove that \( \left[ \underbrace{L_b, L_0, \ldots, L_0} \right] = 0 \) for any \( b \in \{0, 1, \ldots, p - 1\} \).

Taking into account that \( L_0 \) is nilpotent of \((c, |FH|)\)-bounded class, we may assume \( b \neq 0 \).

By Lemma 3.3 \( L_0 = \sum_f V_f \). Let \( u_b \in L_b \). Since \( u_b + u_{rb} + \cdots + u_{r^{q-1}b} \in C_L(H^f) \) and \( V_f \subseteq C_L(H^f) \), we have
\[
\left[ u_b + u_{rb} + \cdots + u_{r^{q-1}b}, V_f, \ldots, V_f \right] = 0.
\]
Thus, \( \sum_{i=0}^{q-1} [u_{r^ib}, V_f, \ldots, V_f] = 0 \). On the other hand, \( [u_{r^ib}, \underbrace{V_f, \ldots, V_f}_c] \in L_{r^ib} \) and \( L_{r^ib} \neq L_{r^jb} \) whenever \( i \neq j \). Therefore, we obtain \( \left[ \underbrace{L_b, V_f, \ldots, V_f}_c \right] = 0 \).

Let \( S \) be an \( FH \)-invariant subalgebra of \( L_0 \) and let \( S_f = S \cap V_f \) for \( f \in F \). It follows that \( S = \sum_f S_f \). Now, we will show that there exists a \((c, |FH|)\)-bounded number \( u \) such that \( \left[ \underbrace{L_b, S, \ldots, S}_u \right] = 0 \) using induction on the nilpotency class of \( S \). Since \( [S, S]_u \) is nilpotent of smaller class, there exists a \((c, |FH|)\)-bounded number \( u_1 \) such that \( \left[ \underbrace{L_b, [S, S], \ldots, [S, S]}_{u_1} \right] = 0 \).
Now, set \( l = (c-1)|F|+1 \) and \( W = [L_b, S_{i_1}, \ldots, S_{i_l}] \) for some choice of \( S_{i_j} \) in \( \{S_f; f \in F\} \). It is clear that for any permutation \( \pi \) of the symbols \( i_1, \ldots, i_l \) we have \( W \leq [L_b, S_{\pi(i_1)}, \ldots, S_{\pi(i_l)}] + [L_b, [S, S]] \). Also, note that the number \( l \) is big enough to ensure that some \( S_{i_j} \) occurs in the list \( S_{i_1}, \ldots, S_{i_l} \) at least \( c \) times. Thus, we deduce that \( W \leq \sum_{c}^{l} [L_b, S_{i_j}, \ldots, S_{i_l}], \ldots, S_{i_l}] + [L_b, [S, S]] \), where the asterisks denote some of the subalgebras \( S_j \) which, in view of the fact that \( [L_b, S_{i_1}, \ldots, S_{i_l}] = 0 \), are of no consequence. Hence, \( W \leq [L_b, [S, S]] \).

Further, for any choice of \( S_{i_1}, \ldots, S_{i_l} \in \{S_f; f \in F\} \) the same argument shows that
\[
[W, S_{i_1}, \ldots, S_{i_l}] \leq [W, [S, S]] \leq [L_b, [S, S], [S, S]].
\]

More generally, for any \( m \) and any \( S_{i_1}, \ldots, S_{i_{ml}} \in \{S_f; f \in F\} \) we have
\[
[L_b, S_{i_1}, \ldots, S_{i_{ml}}] \leq [L_b, [S, S], \ldots, [S, S]].
\]

Put \( u = u_{i_1} \). The above shows that
\[
[L_b, S_{i_1}, \ldots, S_{i_{iu}}] \leq [L_b, [S, S], \ldots, [S, S]] = 0.
\]

Of course, this implies that \( [L_b, S_{i_1}, \ldots, S_{i_{iu}}] = 0 \). The lemma is now straightforward from the case where \( \bar{S} = L_0 \). \( \square \)

**Proof of Theorem 3.1**

In view of Lemma 3.5, applying the arguments of Section 2, Theorem 3.1 holds if we show the following:

**Lemma 3.6.** Let \( L \) be metabelian. There exists a \((c, |FH|)-bounded\) number \( v \) such that \([L, L] \cap L_0, L_{a_1}, \ldots, L_{a_v} = 0\), for all \( a \in \mathbb{Z}/p\mathbb{Z} \).

**Proof.** Recall that \([L, L] \cap L_0 = \sum_f [L, L] \cap V_f \) where \( V_f = C_L(ZH^f) \). Set \( V = C_L(ZH) \) and \( V' = [L, L] \cap V \). First we prove that \([V', L_{a_1}, \ldots, L_{a_v} = 0\) for any \( a \in \mathbb{Z}/p\mathbb{Z} \).

For any \( u_a \in L_a \) we have \( u_a + u_{ra} + \cdots + u_{r^{a-1}} \in C_L(H) \) (under Index Convention). Thus,
\[
[V', u_a + u_{ra} + \cdots + u_{r^{a-1}}, \ldots, v_a + v_{ra} + \cdots + v_{r^{a-1}}] = 0
\]
for any $c$ elements $u_a, \ldots, v_a \in L_a$.

Let $T$ denote the span of all the sums $x_a + x_{ra} + \cdots + x_{r^{q-1}a}$ over $x_a \in L_a$ (in fact, $T$ is the fixed-point subspace of $H$ on $\bigoplus_{i=0}^{q-1} L_{r^i a}$). Then the latter equality means that

$$[V', \overbrace{T, \ldots, T}^c] = 0.$$ 

Applying $\varphi^j$ we also obtain

$$[V', \overbrace{T\varphi^j, \ldots, T\varphi^j}^c] = [(V')\varphi^j, \overbrace{T\varphi^j, \ldots, T\varphi^j}^c] = 0.$$ 

A Vandermonde-type linear algebra argument shows that $L_a \subseteq \sum_{j=0}^{q-1} T\varphi^j$. Actually this fact is a consequence of the following result:

**Lemma 3.7. [7] Lemma 5.3** Let $\langle \alpha \rangle$ be a cyclic group of order $n$, and $\omega$ a primitive $n$th root of unity. Suppose that $M$ is a $\mathbb{Z}[\omega] \langle \alpha \rangle$-module such that $M = \sum_{i=1}^m M_{t_i}$, where $x\alpha = \omega^{t_i} x$ for $x \in M_{t_i}$ and $0 \leq t_1 < t_2 < \cdots < t_m < n$. If $z = y_{t_1} + y_{t_2} + \cdots + y_{t_m}$ for $y_{t_i} \in M_{t_i}$, then for some $m$-bounded number $d_0$ every element $n^{d_0} y_{t_s}$ is a $\mathbb{Z}[\omega]$-linear combination of the elements $z, z\alpha, \ldots, z\alpha^{m-1}$.

Now we can apply Lemma 3.7 with $\alpha = \varphi$, $M = L_a + L_{ra} + \cdots + L_{r^{q-1}a}$ and $m = q$ to $w = u_a + u_{ra} + \cdots + u_{r^{q-1}a} \in T$ for any $u_a \in L_a$, because here the indices $r^ia$ can be regarded as pairwise distinct residues modulo $p$ ($r$ is a primitive $q$th root of 1 modulo $p$). Since $p$ is invertible in our ground ring, follows that $L_a \subseteq \sum_{j=0}^{q-1} T\varphi^j$.

Set $v = (c-1)q+1$. We now claim that $[V', \overbrace{L_a, \ldots, L_a}^v] = 0$. Indeed, after replacing $L_a$ with $\sum_{j=0}^{q-1} T\varphi^j$ and expanding the sums, in each commutator of the resulting linear combination we can freely permute the entries $T\varphi^j$, since $L$ is metabelian. Since there are sufficiently many of them, we can place at least $c$ of the same $T\varphi^j$ for some $j_0$ right after $V'$ at the beginning, which gives 0.

Note that in the case where $f \in F \setminus Z$ we can consider the Frobenius group $ZH^f$ and conclude in a similar way that

$$[[L, L] \cap V_f, \overbrace{u_a, \ldots, v_a}^v] = 0$$

for any $v$ elements $u_a, \ldots, v_a \in L_a$. □
4. Proof of Theorem 1.2

We will use the following results.

**Lemma 4.1.** Let $G$ be a finite $p$-group admitting a nilpotent group of automorphisms $F$ such that $C_G(F) = 1$. Let $F_{p'}$ be the Hall $p'$-subgroup of $F$. Then $C_G(F_{p'}) = 1$.

**Proof.** The subgroup $C_G(F_{p'})$ is $F$-invariant, and so, it admits the natural action by the $p$-group $F/F_{p'}$. Since a finite $p$-group cannot act without nontrivial fixed points on another $p$-group, we must have $C_G(F_{p'}) = 1$. □

**Lemma 4.2.** [7, Lemma 2.2] Let $G$ be a finite group admitting a nilpotent group of automorphisms $F$ such that $C_G(F) = 1$. If $N$ is an $F$-invariant normal subgroup of $G$, then $C_{G/N}(F) = 1$.

**Lemma 4.3.** [7, Theorem 2.3] Suppose that a finite group $G$ admits a Frobenius group of automorphisms $FH$ with kernel $F$ and complement $H$. If $N$ is an $FH$-invariant normal subgroup of $G$ such that $C_N(F) = 1$, then $C_{G/N}(H) = C_G(H)N/N$.

We know that $G$ in Theorem 1.2 is nilpotent. We wish to show that the nilpotency class of $G$ is $(c, |FH|)$-bounded. It is easy to see that without loss of generality we may assume that $G$ is a $p$-group. Moreover, by Lemma 4.1 we also may assume that $(|G|, |F|) = 1$.

Consider the associated Lie ring of the group $G$

$$L(G) = \bigoplus_{i=1}^{n} \gamma_i/\gamma_{i+1},$$

where $n$ is the nilpotency class of $G$ and $\gamma_i$ are the terms of the lower central series of $G$. The nilpotency class of $G$ coincides with the nilpotency class of $L(G)$. The action of the group $FH$ on $G$ induces naturally an action of $FH$ on $L(G)$. Since $F$ acts fixed-point-freely on every quotient $\gamma_i/\gamma_{i+1}$, it follows by Lemma 4.2 that $C_{L(G)}(F) = 0$. We observe that the subring $C_{L(G)}(H)$ is nilpotent of class at most $c$ by Lemma 4.3. Theorem 3.1 now tells us that $L(G)$ is nilpotent of $(c, |FH|)$-bounded class. The proof is complete.

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