COMPARISON OF TOPOLOGIES ON ⋆-ALGEBRAS OF
LOCALLY MEASURABLE OPERATORS

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Abstract. We consider the locally measure topology \( t(M) \) on the ⋆-algebra \( \text{LS}(M) \) of all locally measurable operators affiliated with a von Neumann algebra \( M \). We prove that \( t(M) \) coincides with the \((o)\)-topology on \( \text{LS}_h(M) = \{ T \in \text{LS}(M) : T^* = T \} \) if and only if the algebra \( M \) is σ-finite and a finite algebra. We study relationships between the topology \( t(M) \) and various topologies generated by faithful normal semifinite traces on \( M \).

Introduction

The development of integration theory for a faithful normal semifinite trace \( \tau \) defined on a von Neumann algebra \( M \) has led to a need to consider the ⋆-algebra \( S(M, \tau) \) of all \( \tau \)-measurable operators affiliated with \( M \), see, e.g., [1]. This algebra is a solid ⋆-subalgebra of the ⋆-algebra \( S(M) \) of all measurable operators affiliated with \( M \). The ⋆-algebra \( S(M) \) was introduced by I. Segal [2] to describe a “noncommutative version” of the ⋆-algebra of measurable complex-valued functions. If \( M \) is a commutative von Neumann algebra, then \( M \) can be identified with the ⋆-algebra \( L_\infty(\Omega, \Sigma, \mu) \) of all essentially bounded measurable complex-valued functions defined on a measure space \( (\Omega, \Sigma, \mu) \) with a measure \( \mu \) having the direct sum property. In this case, the ⋆-algebra \( S(M) \) is identified with the ⋆-algebra \( L_0(\Omega, \Sigma, \mu) \) of all measurable complex-valued functions defined on \( (\Omega, \Sigma, \mu) \) [2].

The ⋆-algebras \( S(M, \tau) \) and \( S(M) \) are substantive examples of EW⋆-algebras \( E \) of closed linear operators, affiliated with the von Neumann algebra \( M \), which act on the same Hilbert space \( \mathcal{H} \) as \( M \) and have the bounded part \( E_b = E \cap B(\mathcal{H}) \) coinciding with \( M \) [3], where \( B(\mathcal{H}) \) is the ⋆-algebra of all bounded linear operators on \( \mathcal{H} \). A natural desire of obtaining a maximal EW⋆-algebra \( E \) with \( E_b = M \) has led to a construction of the ⋆-algebra \( \text{LS}(M) \) of all locally measurable operators affiliated with the von Neumann algebra \( M \), see, for example, [4]. It was shown in [5] that any EW⋆-algebra \( E \) satisfying \( E_b = M \) is a solid ⋆-subalgebra of \( \text{LS}(M) \).

In the case where there exists a faithful normal finite trace \( \tau \) on \( M \), all three ⋆-algebras \( \text{LS}(M), S(M), \) and \( S(M, \tau) \) coincide [6, §2.6], and a

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natural topology that endows these \(\ast\)-algebras with the structure of a topological \(\ast\)-algebra is the measure topology induced by the trace \(\tau\) [1]. If \(\tau\) is a semifinite but not a finite trace, then one can consider the \(\tau\)-locally measure topology \(t_{\tau l}\) and the weak \(\tau\)-locally measure \(t_{w\tau l}\) [7]. However, in the case where \(\mathcal{M}\) is not of finite type, the multiplication is not jointly continuous in the two variables with respect to these topologies. In this connection, it makes sense to use, for the \(\ast\)-algebra \(LS(M)\), the locally measure topology \(t(M)\), which was defined in [4] for any von Neumann algebras and which endows \(LS(M)\) with the structure of a complete topological \(\ast\)-algebra [6, §3.5].

The natural partial order on the selfadjoint part \(LS_h(M) = \{T \in LS(M) : T^* = T\}\) permits to define, on \(LS_h(M)\), an order convergence, \((o)\)-convergence, and the generated by it \((o)\)-topology \(t_0(M)\). If \(\mathcal{M}\) is a commutative von Neumann algebra, \(t(M) \leq t_0(M)\) and \(t(M) = t_0(M)\) on \(LS_h(M)\) if and only if \(\mathcal{M}\) is of \(\sigma\)-finite algebra [12, Ch. V, §6]. For non-commutative von Neumann algebras, such relations between the topologies \(t(M)\) and \(t_0(M)\) do not hold in general. For example, if \(\mathcal{M} = \mathcal{B}(\mathcal{H})\), then \(LS(M) = \mathcal{M}\) and the topology \(t(M)\) coincides with the uniform topology that is strictly stronger than the \((o)\)-topology on \(\mathcal{B}_h(\mathcal{H})\) if \(\dim(\mathcal{H}) = \infty\) [6, §3.5].

In this paper, we study relations between the topology \(t(M)\) and the topologies \(t_{\tau l}\), \(t_{w\tau l}\), and \(t_0(M)\). We find that the topologies \(t(M)\) and \(t_{\tau l}\) (resp. \(t(M)\) and \(t_{w\tau l}\)) coincide on \(S(M, \tau)\) if and only if \(\mathcal{M}\) is finite, and \(t(M) = t_0(M)\) on \(LS_h(M)\) holds if and only if \(\mathcal{M}\) is a \(\sigma\)-finite and finite. Moreover, it turns out that the topology \(t_{\tau l}\) (resp. \(t_{w\tau l}\)) coincides with the \((o)\)-topology on \(S_h(\mathcal{M}, \tau)\) only for finite traces. We give necessary and sufficient conditions for the topology \(t(M)\) to be locally convex (resp., normable). We show that \((o)\)-convergence of sequences in \(LS_h(M)\) and convergence in the topology \(t(M)\) coincide if and only if the algebra \(\mathcal{M}\) is an atomic and finite algebra.

We use the von Neumann algebra terminology, notations and results from [9, 10], and those that concern the theory of measurable and locally measurable operators from [4, 6].

1. Preliminaries

Let \(\mathcal{H}\) be a Hilbert space over the field \(\mathbb{C}\) of complex numbers, \(\mathcal{B}(\mathcal{H})\) be the \(\ast\)-algebra of all bounded linear operators on \(\mathcal{H}\), \(I\) be the identity operator on \(\mathcal{H}\), \(\mathcal{M}\) be a von Neumann subalgebra of \(\mathcal{B}(\mathcal{H})\), \(\mathcal{P}(\mathcal{M}) = \{P \in \mathcal{M} : P^2 = P = P^*\}\) be the lattice of all projections in \(\mathcal{M}\), and \(\mathcal{P}_{fin}(\mathcal{M})\) be the sublattice of its finite projections. The center of a von Neumann algebra \(\mathcal{M}\) will be denoted by \(Z(\mathcal{M})\).

A closed linear operator \(T\) affiliated with a von Neumann algebra \(\mathcal{M}\) and having everywhere dense domain \(\mathcal{D}(T) \subset \mathcal{H}\) is called measurable if there
exists a sequence \( \{P_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(\mathcal{M}) \) such that \( P_n \uparrow I \), \( P_n(\mathcal{H}) \subseteq \mathcal{D}(T) \), and \( P_n^\perp = I - P_n \in \mathcal{P}_{fin}(\mathcal{M}) \), \( n = 1, 2, \ldots \).

A set \( \mathcal{S}(M) \) of all measurable operators is a \( * \)-algebra with identity \( I \) over the field \( \mathbb{C} \). It is clear that \( \mathcal{M} \) is a \( * \)-subalgebra of \( \mathcal{S}(M) \).

A closed linear operator \( T \) affiliated with \( \mathcal{M} \) and having an everywhere dense domain \( \mathcal{D}(T) \subseteq \mathcal{H} \) is called locally measurable with respect to \( \mathcal{M} \) if there is a sequence \( \{Z_n\}_{n=1}^{\infty} \) of central projections in \( \mathcal{M} \) such that \( Z_n \uparrow I \) and \( TZ_n \in \mathcal{S}(M) \) for all \( n = 1, 2, \ldots \).

The set \( \mathcal{L}(\mathcal{M}) \) of all locally measurable operators with respect to \( \mathcal{M} \) is a \( * \)-algebra with identity \( I \) over the field \( \mathbb{C} \) with respect to the same algebraic operations as in \( \mathcal{S}(M) \). Here, \( \mathcal{S}(M) \) is a \( * \)-subalgebra of \( \mathcal{L}(\mathcal{M}) \). If \( \mathcal{M} \) is finite, or if \( \mathcal{M} \) is a factor, the algebras \( \mathcal{S}(M) \) and \( \mathcal{L}(\mathcal{M}) \) coincide.

For every \( T \in \mathcal{S}(\mathcal{Z}(\mathcal{M})) \) there exists a sequence \( \{Z_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(\mathcal{Z}(\mathcal{M})) \) such that \( Z_n \uparrow I \) and \( TZ_n \in \mathcal{M} \) for all \( n = 1, 2, \ldots \). This means that \( T \in \mathcal{L}(\mathcal{M}) \). Hence, \( \mathcal{S}(\mathcal{Z}(\mathcal{M})) \) is a \( * \)-subalgebra of \( \mathcal{L}(\mathcal{M}) \), and \( \mathcal{S}(\mathcal{Z}(\mathcal{M})) \) coincides with the center of the \( * \)-algebra \( \mathcal{L}(\mathcal{M}) \).

For every subset \( E \subseteq \mathcal{L}(\mathcal{M}) \), the sets of all selfadjoint (resp., positive) operators in \( E \) will be denoted by \( E_h \) (resp., \( E_+ \)). The partial order in \( \mathcal{L}(\mathcal{M}) \) defined by its cone \( \mathcal{L}_+(\mathcal{M}) \) is denoted by \( \leq \). For a net \( \{T_\alpha\}_{\alpha \in A} \subseteq \mathcal{L}(\mathcal{M}) \), the notation \( T_\alpha \uparrow T \) (resp., \( T_\alpha \downarrow T \)), where \( T \in \mathcal{L}(\mathcal{M}) \), means that \( T_\alpha \leq T_\beta \) (resp., \( T_\beta \leq T_\alpha \)) for \( \alpha \leq \beta \) and \( T = \sup_{\alpha \in A} T_\alpha \) (resp., \( T = \inf_{\alpha \in A} T_\alpha \)).

We say that a net \( \{T_\alpha\}_{\alpha \in A} \subseteq \mathcal{L}(\mathcal{M}) \) \( (o) \)-converges to an operator \( T \in \mathcal{L}(\mathcal{M}) \), denoted by \( T_\alpha \xrightarrow{(o)} T \), if there exist nets \( \{S_\alpha\}_{\alpha \in A} \) and \( \{R_\alpha\}_{\alpha \in A} \) in \( \mathcal{L}(\mathcal{M}) \) such that \( S_\alpha \leq T_\alpha \leq R_\alpha \) for all \( \alpha \in A \) and \( S_\alpha \uparrow T, R_\alpha \downarrow T \).

The strongest topology on \( \mathcal{L}(\mathcal{M}) \) for which \( (o) \)-convergence implies its convergence in the topology is called order topology, or the \( (o) \)-topology, and is denoted by \( t_\alpha(\mathcal{M}) \). If \( \mathcal{M} = \mathcal{L}_{\infty}(\Omega, \Sigma, \mu), \mu(\Omega) < \infty \), the \( (o) \)-convergence of sequences in \( \mathcal{L}(\mathcal{M}) \) coincides with almost everywhere convergence and convergence in the \( (o) \)-topology, \( t_\alpha(\mathcal{M}) \), with measure convergence [11, Ch. III, §9].

Let \( T \) be a closed operator with dense domain \( \mathcal{D}(T) \subseteq \mathcal{H} \), \( T = U|T| \) the polar decomposition of the operator \( T \), where \( |T| = (T^*T)^{\frac{1}{2}} \) and \( U \) is the partial isometry in \( \mathcal{B}(\mathcal{H}) \) such that \( U^*U \) is the right support of \( T \). It is known that \( T \in \mathcal{L}(\mathcal{M}) \) if and only if \( |T| \in \mathcal{L}(\mathcal{M}) \) and \( U \in \mathcal{M} \) [6, §2.3]. If \( T \) is a self-adjoint operator affiliated with \( \mathcal{M} \), then the spectral family of projections \( \{E_\lambda(T)\}_{\lambda \in \mathbb{R}} \) for \( T \) belongs to \( \mathcal{M} \) [3, §2.1].

Let us now recall the definition of the locally measure topology. Let first \( \mathcal{M} \) be a commutative von Neumann algebra. Then \( \mathcal{M} \) is \( * \)-isomorphic to the \( * \)-algebra \( \mathcal{L}_{\infty}(\Omega, \Sigma, \mu) \) of all essentially bounded measurable complex-valued functions defined on a measure space \( (\Omega, \Sigma, \mu) \) with the measure \( \mu \) satisfying the direct sum property (we identify functions that are equal almost everywhere). The direct sum property of a measure \( \mu \) means that the Boolean algebra of all projections of the \( * \)-algebra \( \mathcal{L}_{\infty}(\Omega, \Sigma, \mu) \) is order
complete, and for any nonzero $P \in \mathcal{P}(\mathcal{M})$ there exists a nonzero projection $Q \leq P$ such that $\mu(Q) < \infty$.

Consider the $*$-algebra $LS(\mathcal{M}) = S(\mathcal{M}) = L_0(\Omega, \Sigma, \mu)$ of all measurable almost everywhere finite complex-valued functions defined on $(\Omega, \Sigma, \mu)$ (functions that are equal almost everywhere are identified). On $L_0(\Omega, \Sigma, \mu)$, define a locally measure topology $t(\mathcal{M})$, that is, the linear Hausdorff topology, whose base of neighborhoods around zero is given by

$$W(B, \varepsilon, \delta) = \{f \in L_0(\Omega, \Sigma, \mu) : \exists E \subseteq B, \mu(B \setminus E) \leq \delta, f \chi_E \in L_\infty(\Omega, \Sigma, \mu), \|f \chi_E\|_{L_\infty(\Omega, \Sigma, \mu)} \leq \varepsilon\},$$

where $\varepsilon, \delta > 0, B \in \Sigma, \mu(B) < \infty$, and $\chi(\omega) = \begin{cases} 1, & \omega \in E, \\ 0, & \omega \notin E. \end{cases}$

Convergence of a net $\{f_\alpha\}$ to $f$ in the topology $t(\mathcal{M})$, denoted by $f_\alpha \xrightarrow{t(\mathcal{M})} f$, means that $f_\alpha \chi_B \rightarrow f \chi_B$ in measure $\mu$ for any $B \in \Sigma$ with $\mu(B) < \infty$. It is clear that the topology $t(\mathcal{M})$ does not change if the measure $\mu$ is replaced with an equivalent measure. Denote by $t_h(\mathcal{M})$ the topology on $LS_h(\mathcal{M})$ induced by the topology $t(\mathcal{M})$ on $LS(\mathcal{M})$.

**Proposition 1.** If $\mathcal{M}$ is a commutative von Neumann algebra, then $t_h(\mathcal{M}) \leq t_o(\mathcal{M})$.

**Proof.** It sufficient to prove that any net $\{f_\alpha\}_{\alpha \in A} \subset LS_h(\mathcal{M})$, which $(o)$-converges to zero, also converges to zero with respect to the topology $t_h(\mathcal{M})$. Choose a net $\{g_\alpha\}_{\alpha \in A} \subset LS_h(\mathcal{M})$ such that $g_\alpha \downarrow 0$ and $-g_\alpha \leq f_\alpha \leq g_\alpha$ for all $\alpha \in A$.

Let $B \in \Sigma$ and $\mu(B) < \infty$ (we identify $\mathcal{M}$ with $L_\infty(\Omega, \Sigma, \mu)$). Then

$$-g_\alpha \chi_B \leq f_\alpha \chi_B \leq g_\alpha \chi_B, \quad \alpha \in A,$$

and, since $g_\alpha \chi_B \downarrow 0$, we have $g_\alpha \chi_B \rightarrow 0$ in measure $\mu$. Consequently, $f_\alpha \chi_B \rightarrow 0$ in measure $\mu$ and, hence, $f_\alpha \xrightarrow{t_h(\mathcal{M})} 0$. \qed

Let now $\mathcal{M}$ be an arbitrary von Neumann algebra. Identify the center $Z(\mathcal{M})$ with the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$, and $LS(Z(\mathcal{M}))$ with the $*$-algebra $L_0(\Omega, \Sigma, \mu)$. Denote by $L_+(\Omega, \Sigma, m)$ the set of all measurable real-valued functions defined on $(\Omega, \Sigma, \mu)$ and taking values in the extended half-line $[0, \infty]$ (functions that are equal almost everywhere are identified). It was shown in [2] that there exists a mapping $\mathcal{D}: \mathcal{P}(\mathcal{M}) \rightarrow L_+(\Omega, \Sigma, \mu)$ that possesses the following properties:

- (i) $\mathcal{D}(P) = 0$ if and only if $P = 0$;
- (ii) $\mathcal{D}(P) \in L_0(\Omega, \Sigma, \mu)$ if $P \in \mathcal{P}(\mathcal{M})$;
- (iii) $\mathcal{D}(P \lor Q) = \mathcal{D}(P) + \mathcal{D}(Q)$ if $PQ = 0$;
- (iv) $\mathcal{D}(U^*U) = \mathcal{D}(UU^*)$ for any partial isometry $U \in \mathcal{M}$;
- (v) $\mathcal{D}(ZP) = Z\mathcal{D}(P)$ for any $Z \in \mathcal{P}(Z(\mathcal{M}))$ and $P \in \mathcal{P}(\mathcal{M})$;
(vi) if \( \{P_\alpha\}_{\alpha \in A} \), \( P \in \mathcal{P}(\mathcal{M}) \) and \( P_\alpha \uparrow P \), then \( \mathcal{D}(P) = \sup_{\alpha \in A} \mathcal{D}(P_\alpha) \).

A mapping \( \mathcal{D} : \mathcal{P}(\mathcal{M}) \to L_+(\Omega, \Sigma, \mu) \) that satisfies properties (i)—(vi) is called a dimension function on \( \mathcal{P}(\mathcal{M}) \).

For arbitrary numbers \( \varepsilon, \delta > 0 \) and a set \( B \in \Sigma, \mu(B) < \infty \), set
\[
V(B, \varepsilon, \delta) = \{T \in \mathcal{LS}(\mathcal{M}) : \text{there exist } P \in \mathcal{P}(\mathcal{M}), Z \in \mathcal{P}(\mathcal{Z}(\mathcal{M})), \text{such that } TP \in \mathcal{M}, \|TP\|_\mathcal{M} \leq \varepsilon, Z^\perp \in W(B, \varepsilon, \delta), \mathcal{D}(ZP^\perp) \leq \varepsilon Z\},
\]
where \( \| \cdot \|_\mathcal{M} \) is the \( C^* \)-norm on \( \mathcal{M} \).

It was shown in [4] that the system of sets
\[
(1) \quad \{\{T + V(B, \varepsilon, \delta)\} : T \in \mathcal{LS}(\mathcal{M}), \varepsilon, \delta > 0, B \in \Sigma, \mu(B) < \infty\}
\]
defines a linear Hausdorff topology \( t(\mathcal{M}) \) on \( \mathcal{LS}(\mathcal{M}) \) such that sets (1) form a neighborhood base of the operator \( T \in \mathcal{LS}(\mathcal{M}) \). Here, \( (\mathcal{LS}(\mathcal{M}), t(\mathcal{M})) \) is a complete topological \( * \)-algebra, and the topology \( t(\mathcal{M}) \) does not depend on a choice of the dimension function \( \mathcal{D} \).

The topology \( t(\mathcal{M}) \) is called a locally measure topology [4].

We will need the following criterion for convergence of nets with respect to this topology.

**Proposition 2** (§3.5). (i) A net \( \{P_\alpha\}_{\alpha \in A} \subset \mathcal{P}(\mathcal{M}) \) converges to zero with respect to the topology \( t(\mathcal{M}) \) if and only if there is a net \( \{Z_\alpha\}_{\alpha \in A} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M})) \) such that \( Z_\alpha P_\alpha \in \mathcal{P}_{\text{fin}}(\mathcal{M}) \) for all \( \alpha \in A \), \( Z_\alpha t(\mathcal{Z}(\mathcal{M})) \to 0 \), and \( \mathcal{D}(Z_\alpha P_\alpha) t(\mathcal{Z}(\mathcal{M})) \to 0 \), where \( t(\mathcal{Z}(\mathcal{M})) \) is the locally measure topology on \( \mathcal{LS}(\mathcal{Z}(\mathcal{M})) \).

(ii) A net \( \{T_\alpha\}_{\alpha \in A} \subset \mathcal{LS}(\mathcal{M}) \) converges to zero with respect to the topology \( t(\mathcal{M}) \) if and only if \( E^\perp_\lambda(|T_\alpha|) t(\mathcal{M}) \to 0 \) for any \( \lambda > 0 \), where \( \{E^\perp_\lambda(|T_\alpha|)\} \) is a spectral projection family for the operator \( |T_\alpha| \).

It follows from Proposition 2 that the topology \( t(\mathcal{M}) \) induces the topology \( t(\mathcal{Z}(\mathcal{M})) \) on \( \mathcal{LS}(\mathcal{Z}(\mathcal{M})) \); hence, \( S(\mathcal{Z}(\mathcal{M})) \) is a closed \( * \)-subalgebra of \( (\mathcal{LS}(\mathcal{M}), t(\mathcal{M})) \).

It is clear that
\[
X \cdot V(B, \varepsilon, \delta) \subset V(B, \varepsilon, \delta)
\]
for any \( X \in \mathcal{M} \) with the norm \( \|X\|_\mathcal{M} \leq 1 \). Since \( V^*(B, \varepsilon, \delta) \subset V(B, 2\varepsilon, \delta) \) [6 §3.5], we have
\[
V(B, \varepsilon, \delta) \cdot Y \subset V(B, 4\varepsilon, \delta)
\]
for all \( Y \in \mathcal{M} \) satisfying \( \|Y\|_\mathcal{M} \leq 1 \). Hence,
\[
X \cdot V(B, \varepsilon, \delta) \cdot Y \subset V(B, 4\varepsilon, \delta)
\]
for any \( \varepsilon, \delta > 0 \), \( B \in \Sigma, \mu(B) < \infty \), \( X, Y \in \mathcal{M} \) with \( \|X\|_\mathcal{M} \leq 1 \), \( \|Y\|_\mathcal{M} \leq 1 \).

Since the involution is continuous in the topology \( t(\mathcal{M}) \), the set \( \mathcal{LS}_+(\mathcal{M}) \) is closed in \( (\mathcal{LS}(\mathcal{M}), t(\mathcal{M})) \). The cone \( \mathcal{LS}_+(\mathcal{M}) \) of positive elements is also closed in \( (\mathcal{LS}(\mathcal{M}), t(\mathcal{M})) \) [4]. Hence, for every increasing (or decreasing)
net \(\{T_\alpha\}_{\alpha \in A} \subset LS_h(M)\) that converges to \(T\) in the topology \(t(M)\), we have that \(T \in LS_h(M)\) and \(T = \sup_{\alpha \in A} T_\alpha\) (resp. \(T = \inf_{\alpha \in A} T_\alpha\)) \text{[13, Ch. V, §4].}

2. Comparison of the Topologies \(t(M)\) and \(t_0(M)\)

Let \(M\) be an arbitrary von Neumann algebra, \(t_0(M)\) be the \((o)\)-topology on \(LS_h(M)\). As before, \(t_h(M)\) denotes the topology on \(LS_h(M)\) induced by the topology \(t(M)\) on \(LS(M)\).

**Theorem 1.** The following conditions are equivalent:

(i) \(t_h(M) \leq t_0(M)\); (ii) \(M\) is finite.

**Proof.** (i) \(\Rightarrow\) (ii). Suppose that \(M\) is not finite. Then there is a sequence of pairwise orthogonal and pairwise equivalent projections \(\{P_n\}_{n=1}^{\infty} \in \mathcal{P}(M)\).

Choose a partial isometry \(U_n \in M\) such that \(U_n^*U_n = P_1, U_nU_n^* = P_n, n = 1, 2, \ldots\). Set \(Q_n = \sup_{j \geq n} P_j\). Then \(Q_n \in \mathcal{P}(M)\) and \(Q_n \downarrow 0\). By condition (i) we have \(Q_n \xrightarrow{t_h(M)} 0\). Since \(P_n = P_nQ_n\), it follows from (2) that \(P_n \xrightarrow{t_h(M)} 0\).

Again using (2) we get that \(P_1 = U_1^*P_1U_1 \xrightarrow{t_h(M)} 0\) and, hence, \(P_1 = 0\), which is not true. Consequently, \(M\) is finite.

(ii) \(\Rightarrow\) (i). Let \(M\) be a finite von Neumann algebra, \(\Phi : M \mapsto \mathcal{Z}(M)\) a center-valued trace on \(M\) \text{[19, Ch. V, §2]}. The restriction \(D\) of the trace \(\Phi\) on \(\mathcal{P}(M)\) is a dimension function on \(\mathcal{P}(M)\). Let \(\{T_\alpha\}_{\alpha \in A} \subset LS_h(M)\) and \(T_\alpha \xrightarrow{(o)} 0\). Then there exists a net \(\{S_\alpha\}_{\alpha \in A} \in LS_h(M)\) such that \(S_\alpha \downarrow 0\) and \(-S_\alpha < T_\alpha \leq S_\alpha\) for all \(\alpha \in A\). Fix \(\alpha_0 \in A\) and set \(X_\alpha = XT_\alpha X, Y_\alpha = XS_\alpha X\) for \(\alpha \geq \alpha_0\), where \(X = (I + S_{\alpha_0})^{-\frac{1}{2}}\). It is clear that \(-I \leq -Y_\alpha \leq X_\alpha \leq Y_\alpha \leq I\) for \(\alpha \geq \alpha_0\) and \(Y_\alpha \downarrow 0\). Consequently, \(-I \leq -\Phi(Y_\alpha) \leq \Phi(X_\alpha) \leq \Phi(Y_\alpha) \leq I\) and \(\Phi(Y_\alpha) \downarrow 0\).

Let \(E_\lambda^+(Y_\alpha) = \{Y_\alpha > \lambda\}\) be a spectral projection for \(Y_\alpha\) corresponding to the interval \((\lambda, +\infty)\), \(\lambda > 0\). Since

\[
\mathcal{D}(E_\lambda^+(Y_\alpha)) \leq \frac{1}{\lambda} \Phi(Y_\alpha),
\]

it follows that \(\mathcal{D}(E_\lambda^+(Y_\alpha)) \xrightarrow{(o)} 0\) in \(\mathcal{Z}(M)\). By Proposition \text{[1]} we have that

\[
\mathcal{D}(E_\lambda^+(Y_\alpha)) \xrightarrow{t(\mathcal{Z}(M))} 0
\]

for all \(\lambda > 0\). Hence, Proposition \text{[2]} gives that \(Y_\alpha \xrightarrow{t(M)} 0\).

Set \(Z_\alpha = X_\alpha + Y_\alpha\). Repeating the previous reasoning and using the inequality \(0 \leq Z_\alpha \leq 2Y_\alpha\) we get that \(Z_\alpha \xrightarrow{t(M)} 0\). Consequently, \(X_\alpha = Z_\alpha - Y_\alpha \xrightarrow{t(M)} 0\) and, hence, \(T_\alpha = X^{-1}X_\alpha X^{-1} \xrightarrow{t(M)} 0\). Thus, \(t_h(M) \leq t_0(M)\). \(\square\)

**Remark 1.** In the proof of the implication (i) \(\Rightarrow\) (ii) of Theorem \text{[4]} it was shown that convergence to zero, in the topology \(t(M)\), of any sequence of projections in \(\mathcal{P}(M)\), which decreases to zero, implies that \(M\) is finite.
Let us now find conditions that would imply that the topologies \( t_h(\mathcal{M}) \) and \( t_o(\mathcal{M}) \) coincide on \( \text{LS}_h(\mathcal{M}) \). Recall that a von Neumann algebra \( \mathcal{M} \) is called \( \sigma \)-finite if any family of nonzero mutually orthogonal projections in \( \mathcal{P}(\mathcal{M}) \) is at most countable. It is known that the topology \( t(\mathcal{M}) \) on \( \text{LS}(\mathcal{M}) \) is metrizable if and only if the center \( \mathcal{Z}(\mathcal{M}) \) is \( \sigma \)-finite [4].

**Proposition 3.** If \( \mathcal{Z}(\mathcal{M}) \) is \( \sigma \)-finite, then \( t_o(\mathcal{M}) \leq t_h(\mathcal{M}) \).

**Proof.** Choose a neighborhood basis \( \{ V_k \}_{k=1}^{\infty} \) of zero in \( (\text{LS}(\mathcal{M}), t(\mathcal{M})) \) such that \( V_{k+1} + V_{k+1} \subset V_k \) for all \( k \).

Let \( \{ T_n \}_{n=1}^{\infty} \subset \text{LS}_h(\mathcal{M}) \) and \( T_n \xrightarrow{t(\mathcal{M})} 0 \). Using relation (2) and the polar decomposition \( T_n = U_n |T_n| \) we see that \( |T_n| \xrightarrow{t(\mathcal{M})} 0 \). Choose a subsequence \( \{ T_{n_k} \} \subset V_k \) and set \( S_k = \sum_{i=1}^{k} |T_{n_i}| \). It is clear that \( S_m - S_{k+1} \subset V_k \) for \( m > k \). Hence, there exists an operator \( S \in \text{LS}_h(\mathcal{M}) \) such that \( S_k \xrightarrow{t(\mathcal{M})} S \).

The sequence \( R_k = S - \sum_{i=1}^{k} |T_{n_i}| \) decreases and \( R_k \xrightarrow{t(\mathcal{M})} 0 \). Since the cone \( \text{LS}_+(\mathcal{M}) \) of positive elements is closed in \( (\text{LS}(\mathcal{M}), t(\mathcal{M})) \), we have \( R_k \downarrow 0 \) and

\[-R_{k-1} \leq |T_{n_k}| \leq T_{n_k} \leq |T_{n_k}| \leq R_{k-1}.

Consequently, \( T_{n_k} \xrightarrow{(o)} 0 \). Thus, for any sequence \( \{ T_n \}_{n=1}^{\infty} \subset \text{LS}_h(\mathcal{M}) \), which converges to \( T \in \text{LS}_h(\mathcal{M}) \) in the topology \( t(\mathcal{M}) \), there exists a subsequence \( \{ T_{n_k} \} \) such that \( \lim_{k \to \infty} T_{n_k} = T \). This means that \( t_o(\mathcal{M}) \leq t_h(\mathcal{M}) \). \( \square \)

We now describe a class of von Neumann algebras \( \mathcal{M} \) for which the topologies \( t_o(\mathcal{M}) \) and \( t_h(\mathcal{M}) \) coincide.

**Theorem 2.** The following conditions are equivalent:

(i) \( \mathcal{M} \) is finite and \( \sigma \)-finite;

(ii) \( t_o(\mathcal{M}) = t_h(\mathcal{M}) \).

**Proof.** The implication (i) \( \Rightarrow \) (ii) follows from Theorem [1] and Proposition [3].

(ii) \( \Rightarrow \) (i). If \( t_o(\mathcal{M}) = t_h(\mathcal{M}) \), then the von Neumann algebra \( \mathcal{M} \) is finite by Theorem [1].

Let \( \{ Z_j \}_{j \in \Delta} \) be a family of nonzero pairwise orthogonal projections in \( \mathcal{P}(\mathcal{Z}(\mathcal{M})) \) satisfying \( \sup_{j \in \Delta} Z_j = I \) and \( \mu(Z_j) < \infty \) (as before, we identify the commutative von Neumann algebra \( \mathcal{Z}(\mathcal{M}) \) with \( L_\infty(\Omega, \Sigma, \mu) \) and \( \text{LS}((\mathcal{Z}(\mathcal{M}))) \) with \( L_0(\Omega, \Sigma, \mu) \)). Denote by \( E \) a \( * \)-subalgebra in \( L_0(\Omega, \Sigma, \mu) \) of all functions \( f \in L_0(\Omega, \Sigma, \mu) \) satisfying \( f Z_j = \lambda_j Z_j \) for some \( \lambda_j \in \mathbb{C}, \ j \in \Delta \). It is clear that \( E \) is \( * \)-isomorphic to the \( * \)-algebra \( \mathbb{C}^\Delta = \{ \{ \lambda_j \}_{j \in \Delta} : \lambda_j \in \mathbb{C} \} \) and \( E_h \) is isomorphic to the algebra \( \mathbb{R}^\Delta = \{ \{ r_j \}_{j \in \Delta} : r_j \in \mathbb{R} \} \). Denote by \( t \) the Tychonoff topology of coordinate convergence in \( \mathbb{R}^\Delta \), and identify \( E_h \) with
If \( f_\alpha \in E_h \), then \( f_\alpha \xrightarrow{t(M)} 0 \) if and only if \( f_\alpha Z_j \xrightarrow{\mu} 0 \) for all \( j \in \Delta \). This means that the topology \( t(M) \) induces the Tychonoff topology \( t \) on \( E_h \).

Let us show that any subset \( G \subset E_h \), upper bounded in \( LS_h(M) \), is upper bounded in \( E_h \), and the least upper bounds for \( G \) in \( E_h \) and in \( S_h(M) = LS_h(M) \) are the same.

For any operator \( T \in S_+(M) \) there exists a maximal commutative \(*\)-subalgebra \( A \) of \( S(M) \) containing \( Z(M) \) and \( T \). Since \( M \) is a finite von Neumann algebra, \( N = A \cap M \) is also a finite von Neumann algebra, and \( A = S(N) \). We also have that \( Z(M) \subset N \). It is clear that \( S_h(Z(M)) \) is a regular sublattice of \( S_h(N) \), that is, the least upper bounds and the least lower bounds of bounded subsets of \( S_h(Z(M)) \) calculated in \( S_h(N) \) and in \( S_h(Z(M)) \) coincide.

Let \( G \subset E_h \) and \( S \leq T \) for all \( S \in G \). Then there exists a least upper bound \( \sup G \) in \( S_h(N) \), which, since \( S_h(Z(M)) \) is regular, belongs to \( S_h(Z(M)) \). Since \( E_h \) is a regular sublattice in \( S_h(Z(M)) \), \( \sup G \in E_h \). Consequently, any net \( \{S_\alpha\} \subset E_h \) that \((o)\)-converges to \( S \) in \( S_h(M) \) will be \((o)\)-convergent to \( S \) in \( E_h \). This means that the \((o)\)-topology \( t_o(M) \) in \( S_h(M) \) induces the \((o)\)-topology \( t_o(E_h) \) in \( E_h \). Since \( t_o(M) = t_h(M) \), the Tychonoff topology \( t \) coincides with the \((o)\)-topology in \( R^A \). Consequently, the set \( \Delta \) is at most countable [12, Ch. V, § 6], that is, \( Z(M) \) is a \( \sigma\)-finite von Neumann algebra. Since the von Neumann algebra \( M \) is finite, \( M \) is also a \( \sigma\)-finite algebra [2].

Proposition 3 and Theorems 1 and 2 give the following.

**Corollary 1.**

(i) If \( \mathcal{M} \) is a \( \sigma\)-finite von Neumann algebra but is not finite, then \( t_o(\mathcal{M}) < t_h(\mathcal{M}) \).

(ii) If \( \mathcal{M} \) is not a \( \sigma\)-finite von Neumann algebra but is finite, then \( t_h(\mathcal{M}) < t_o(\mathcal{M}) \).

Using Corollary 1 one can easily construct an example of a von Neumann algebra \( \mathcal{M} \) for which the topologies \( t_o(\mathcal{M}) \) and \( t_h(\mathcal{M}) \) are incomparable.

Let \( \mathcal{M}_1 \) be a \( \sigma\)-finite von Neumann algebra which is not finite, \( \mathcal{M}_2 \) be a not \( \sigma\)-finite von Neumann algebra which is finite, and \( \mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \). Then \( LS(\mathcal{M}) = LS(\mathcal{M}_1) \times LS(\mathcal{M}_2) \) [6, § 2.5], and the topology \( t(\mathcal{M}) \) coincides with the product of the topologies \( t(\mathcal{M}_1) \) and \( t(\mathcal{M}_2) \). Moreover, a net \( \{(T^{(1)}_\alpha, T^{(2)}_\alpha)\}_{\alpha \in A} \in LS_h(\mathcal{M}_1) \times LS_h(\mathcal{M}_2) \) is \((o)\)-convergent to an element \((T^{(1)}, T^{(2)}) \in LS_h(\mathcal{M}_1) \times LS_h(\mathcal{M}_2) \) if and only if the net \( \{T^{(k)}_\alpha\}_{\alpha \in A} \) is \((o)\)-convergent to \( T^{(k)} \), \( k = 1, 2 \). Identifying \( LS_h(\mathcal{M}_1) \) with the linear subspace \( LS_h(\mathcal{M}_1) \times \{0\} \) and \( LS_h(\mathcal{M}_2) \) with \( \{0\} \times LS_h(\mathcal{M}_2) \) we get that the \((o)\)-topology \( t_o(\mathcal{M}) \) in \( LS_h(\mathcal{M}) \) induces \((o)\)-topologies in \( LS_h(\mathcal{M}_1) \) and \( LS_h(\mathcal{M}_2) \), correspondingly. It remains to apply Corollary 1 by which the topologies \( t_o(\mathcal{M}) \) and \( t_h(\mathcal{M}) \) are incomparable.
3. **Locally measure topology on semifinite von Neumann algebras**

Let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, $\tau$ be a faithful normal semifinite trace on $\mathcal{M}$. An operator $T \in S(\mathcal{M})$ with domain $\mathcal{D}(T)$ is called $\tau$-measurable if for any $\varepsilon > 0$ there exists a projection $P \in \mathcal{P}(\mathcal{M})$ such that $P(\mathcal{H}) \subset \mathcal{D}(T)$ and $\tau(P^\perp) < \varepsilon$.

A set $S(\mathcal{M}, \tau)$ of all $\tau$-measurable operators is a $*$-subalgebra of $S(\mathcal{M})$, and $\mathcal{M} \subset S(\mathcal{M}, \tau)$. If the trace $\tau$ is finite, then $S(\mathcal{M}, \tau) = S(\mathcal{M})$.

Let $t_\tau$ be a *measure topology* $[\mathbf{1}]$ on $S(\mathcal{M}, \tau)$ whose base of neighborhoods around zero is given by

$$U(\varepsilon, \delta) = \{T \in S(\mathcal{M}, \tau) : \text{there exists a projection } P \in \mathcal{P}(\mathcal{M}),$$

such that $\tau(P^\perp) \leq \delta$, $TP \in \mathcal{M}$, $\|TP\|_\mathcal{M} \leq \varepsilon\}, \varepsilon > 0, \delta > 0$.

The pair $(S(\mathcal{M}, \tau), t_\tau)$ is a complete metrizable topological $*$-algebra. Here, the topology $t_\tau$ majorizes the topology $t(\mathcal{M})$ on $S(\mathcal{M}, \tau)$ and, if $\tau$ is a finite trace, the topologies $t_\tau$ and $t(\mathcal{M})$ coincide $[\mathbf{1}]$ $\S\S$ 3.4, 3.5. Denote by $t(\mathcal{M}, \tau)$ the topology on $S(\mathcal{M}, \tau)$ induced by the topology $t(\mathcal{M})$. It is not true in general that, if the topologies $t_\tau$ and $t(\mathcal{M}, \tau)$ are the same, then the von Neumann algebra $\mathcal{M}$ is finite. Indeed, if $\mathcal{M} = B(\mathcal{H})$, $\dim(\mathcal{H}) = \infty$, $\tau = tr$ is the canonical trace on $B(\mathcal{H})$, then $LS(\mathcal{M}) = S(\mathcal{M}) = S(\mathcal{M}, \tau) = \mathcal{M}$, and the two topologies $t_\tau$ and $t(\mathcal{M})$ coincide with the uniform topology on $B(\mathcal{H})$.

At the same time, we have the following.

**Proposition 4.** If $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal semifinite trace $\tau$ and $t_\tau = t(\mathcal{M}, \tau)$, then $\tau(I) < \infty$.

**Proof.** If $\tau(I) = \infty$, then there exists a sequence of projections $\{P_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{M})$ such that $P_n \downarrow 0$ and $\tau(P_n) = \infty$. By Theorem $[\mathbf{1}]$ $P_n \xrightarrow{t(\mathcal{M})} 0$, however, $\{P_n\}_{n=1}^\infty$ does not converge to zero in the topology $t_\tau$. 

Denote by $t_{h\tau}$ the topology on $S_h(\mathcal{M}, \tau)$ induced by the topology $t_\tau$, and by $t_{o\tau}(\mathcal{M})$ the $(o)$-topology on $S_h(\mathcal{M}, \tau)$. The topology $t_{o\tau}(\mathcal{M})$, in general, does not coincide with the topology induced by the $(o)$-topology $t_o(\mathcal{M})$ on $S_h(\mathcal{M}, \tau)$. For example, for

$$\mathcal{M} = l_\infty(\mathbb{C}) = \{\{a_n\}_{n=1}^\infty \subset \mathbb{C} : \sup_{n \geq 1} |a_n| < \infty\}$$

and

$$\tau(\{a_n\}) = \sum_{n=1}^\infty a_n, \quad a_n \geq 0,$$

we have that $LS(\mathcal{M}) = \mathbb{C}^N$ and $S(\mathcal{M}, \tau) = l_\infty(\mathbb{C})$. Here, the $(o)$-topology $t_o(\mathcal{M})$ on $LS_h(\mathcal{M}) = R^\Delta$ is the topology of the coordinatewise convergence, in particular,

$$T_n = n \cdot Z_n \xrightarrow{(o)} 0$$
in $\mathbb{R}^\Delta$, where $Z_n = \{0, \ldots, 0, 1, 0, \ldots\}$, the number 1 is at the $n$-th place. However, the sequence \( \{T_n\}_{n=1}^\infty \) does not converge in the (o)-topology $t_{or}(l_\infty(C))$, since any its subsequence is not bounded in $l_\infty(R) = (l_\infty(C))_h$ [11, Ch. VI, §3].

**Remark 2.** Since $S_\tau(M, \tau) = S(M, \tau) \cap LS_\tau(M)$ is closed in $(S(M, \tau), t_\tau)$ and $T_n \overset{t_\tau}{\to} T$ if and only if $|T_n - T| \overset{t_\tau}{\to} 0$ [8, §3.4], using metrizability of the topology $t_\tau$ and repeating the end of the proof of Proposition 3 we get that $t_{or}(M) \leq t_{h\tau}$.

**Remark 3.** Using the inclusions $U^*(\varepsilon, \delta) \subset U(\varepsilon, 2\delta)$ and $TU(\varepsilon, \delta) \subset U(\varepsilon\|T\|_M, \delta)$, where $T \in M$ we can see as in the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 4 that the equality $t_{or}(M) = t_{h\tau}$ implies that the von Neumann algebra $M$ is finite.

**Proposition 5.** Let $M$ be a semifinite von Neumann algebra, $\tau$ be a faithful normal semifinite trace on $M$. The following conditions are equivalent.

(i) Any net that is (o)-convergent in $S_h(M, \tau)$ also converges in the topology $t_{h\tau}$;

(ii) $t_{or}(M) = t_{h\tau}$;

(iii) $\tau(I) < \infty$.

**Proof.** The implication (i) $\Rightarrow$ (ii) follows from Remark 2 and definition of the topology $t_{or}(M)$.

(ii) $\Rightarrow$ (iii). By Remark 3, the von Neumann algebra $M$ is finite. Repeating the proof of Proposition 4 we see that $\tau(I) < \infty$.

(iii) $\Rightarrow$ (i). If $\tau(I) < \infty$, then $M$ is a finite von Neumann algebra, and $S(M, \tau) = LS(M)$. Hence, (i) follows from Theorem 4 \( \square \)

Together with the topology $t_\tau$ on $S(M, \tau)$, one can also consider two more Hausdorff vector topologies associated with the trace $\tau$ [7]. This are the $\tau$-locally measure topology $t_{\tau_1}$ and the weak $\tau$-locally measure topology $t_{w\tau_1}$. The sets

\[
U_\tau(\varepsilon, \delta, P) = \{T \in S(M, \tau) : \text{there exists a projection } Q \in \mathcal{P}(M) \text{ such that } Q \leq P, \tau(P - Q) \leq \delta, TQ \in M, \|TQ\|_M \leq \varepsilon\}
\]

(resp.,

\[
U_{w\tau}(\varepsilon, \delta, P) = \{T \in S(M, \tau) : \text{there exists a projection } Q \in \mathcal{P}(M) \text{ such that } Q \leq P, \tau(P - Q) \leq \delta, QTQ \in M, \|QTQ\|_M \leq \varepsilon\}),
\]

where $\varepsilon > 0$, $\delta > 0$, $P \in \mathcal{P}(M)$, $\tau(P) < \infty$, form a neighborhood base around in the topology $t_{\tau_1}$ (resp., in the topology $t_{w\tau_1}$). It is clear that $t_{w\tau_1} \leq t_{\tau_1} \leq t_\tau$, and if $\tau(I) < \infty$, all three topologies $t_{w\tau_1}$, $t_{\tau_1}$, and $t_\tau$ coincide.

Let us remark that if $M = B(H)$ and $\tau = tr$, the topology $t_{\tau_1}$ coincides with the strong operator topology, and the topology $t_{w\tau_1}$ with the weak
operator topology, that is, if \( \dim(\mathcal{H}) = \infty \), we have \( t_{\text{wrt}} < t_{\tau} < t_\tau \) in this case.

The following criterion for convergence of nets in the topologies \( t_{\text{wrt}} \) and \( t_{\tau} \) can be obtained directly from the definition.

**Proposition 6 (\( \mathbb{I} \)).** If \( \{T_\alpha\}_{\alpha \in A}, T \subset S(\mathcal{M}, \tau) \), then \( T_\alpha \xrightarrow{t_{\tau}} T \) (resp., \( T_\alpha \xrightarrow{t_{\text{wrt}}} T \)) if and only if \( T_\alpha P \xrightarrow{t_\tau} TP \) (resp., \( PT_\alpha P \xrightarrow{t_\tau} PTP \)) for all \( P \in \mathcal{P}(\mathcal{M}) \) satisfying \( \tau(P) < \infty \).

Let us also list the following useful properties of the topologies \( t_{\tau} \) and \( t_{\text{wrt}} \).

**Proposition 7 (\( \mathbb{I} \)).** Let \( T_\alpha, S_\alpha \in S(\mathcal{M}, \tau) \). Then

(i) \( T_\alpha \xrightarrow{t_{\tau}} \iff |T_\alpha| \xrightarrow{t_{\tau}} 0 \iff |T_\alpha|^2 \xrightarrow{t_{\text{wrt}}} 0 \);

(ii) if \( T_\alpha \xrightarrow{t_{\tau}} T, S_\alpha \xrightarrow{t_{\tau}} S \), and the net \( \{S_\alpha\} \) is \( t_{\tau} \)-bounded, then \( T_\alpha S_\alpha \xrightarrow{t_{\tau}} TS \);

(iii) if \( 0 \leq S_\alpha \leq T_\alpha \) and \( T_\alpha \xrightarrow{t_{\text{wrt}}} 0 \), then \( S_\alpha \xrightarrow{t_{\text{wrt}}} 0 \).

To compare the topologies \( t_{\text{wrt}} \) and \( t_{\tau} \) with the topology \( t(\mathcal{M}) \), we will need the following property of the topology \( t(\mathcal{M}) \).

**Proposition 8.** The topology \( t(\mathcal{M}) \) induced the topology \( t(PMP) \) on \( \text{LS}(PMP) \), where \( 0 \neq P \in \mathcal{P}(\mathcal{M}) \).

**Proof.** Let \( \{Q_\alpha\}_{\alpha \in A} \subset \mathcal{P}(PMP) \) and \( Q_\alpha \xrightarrow{t(\mathcal{M})} 0 \). By Proposition \( \mathbb{I} i) \) there exists a net \( \{Z_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M})) \) such that \( Z_\alpha Q_\alpha \in \mathcal{P}(\mathcal{M}) \) for any \( \alpha \in A, Z_\alpha \xrightarrow{t(\mathcal{Z}(\mathcal{M}))} 0 \), and \( \mathcal{D}(Z_\alpha Q_\alpha) \xrightarrow{t(\mathcal{Z}(\mathcal{M}))} 0 \). The projection \( R_\alpha = PZ_\alpha \) is in the center \( \mathcal{Z}(PMP) \) of the von Neumann algebra \( PMP \), and \( R_\alpha Q_\alpha = Z_\alpha Q_\alpha \) is a finite projection in \( PMP \). Denote by \( Z(P) \) the central support of the projection \( P \). The mapping \( \psi : PZ(\mathcal{M}) \rightarrow Z(P)Z(\mathcal{M}) \) given by

\[
\psi(PZ) = Z(P)Z, \quad Z \in \mathcal{Z}(\mathcal{M}),
\]

is a \(*\)-isomorphism from \( PZ(\mathcal{M}) \) onto \( Z(P)Z(\mathcal{M}) \). Since \( Z(PMP) = PZ(\mathcal{M}) \) \([\text{9]} \) Sect. 3.1.5], the \(*\)-algebras \( Z(PMP) \) and \( Z(P)Z(\mathcal{M}) \) are \(*\)-isomorphic.

It is clear that

\[
\mathcal{D}_P(Q) := Z(P)\mathcal{D}(Q), \quad Q \in \mathcal{P}(PMP),
\]

is a dimension function on \( \mathcal{P}(PMP) \), where \( \mathcal{D} \) is the initial dimension function on \( \mathcal{P}(\mathcal{M}) \). We have that

\[
\mathcal{D}_P(R_\alpha Q_\alpha) = \mathcal{D}_P(Z_\alpha Q_\alpha) = Z(P)\mathcal{D}(Z_\alpha Q_\alpha) \xrightarrow{t(Z(P)Z(\mathcal{M}))} 0.
\]

Moreover,

\[
P - R_\alpha = P(I - Z_\alpha) \xrightarrow{\psi} Z(P)Z^\perp_\alpha \xrightarrow{t(Z(P)Z(\mathcal{M}))} 0.
\]
Hence, by Proposition 2(i) we get that $Q_\alpha \xrightarrow{t(M')} 0$.

In the same way we can prove that $Q_\alpha \xrightarrow{t(M')} 0$, $\{Q_\alpha\} \subset P(MMP)$, implies that $Q_\alpha \xrightarrow{t(M)} 0$.

Let now $\{T_\alpha\} \subset LS(MMP)$ and $T_\alpha \xrightarrow{t(M)} 0$. By Proposition 2(ii), we have that $E^\alpha_\lambda(|T_\alpha|) \xrightarrow{t(M')} 0$ for any $\lambda > 0$, where $\{E^\alpha_\lambda(|T_\alpha|)\}$ is a family of spectral projections for $|T_\alpha|$. Denote by $\{E^P_\lambda(|T_\alpha|)\}$ the family of spectral projections for $|T_\alpha|$ in $L \times S(MMP)$, $\lambda > 0$. It is clear that $E^\lambda_\alpha(|T_\alpha|) = P^\perp + E^P_\lambda(|T_\alpha|)$ and $E^P_\lambda(|T_\alpha|) = P - E^P_\lambda(|T_\alpha|)$ for all $\lambda > 0$. It follows from above that $P - E^P_\lambda(|T_\alpha|) \xrightarrow{t(M')} 0$ for all $\lambda > 0$. Hence, by Proposition 2(ii), it follows that $T_\alpha \xrightarrow{t(M')} 0$.

One can similarly prove that the convergence $T_\alpha \xrightarrow{t(M')} 0$ implies the convergence $T_\alpha \xrightarrow{t(M)} 0$.

**Theorem 3.** $t_{\tau \ell} \leq t(M, \tau)$.

**Proof.** If $\{T_\alpha\} \subset S(M, \tau)$ and $T_\alpha \xrightarrow{t(M)} 0$, then $|T_\alpha|^2 \xrightarrow{t(M)} 0$. Let $P \in P(M)$ and $\tau(P) < \infty$. By Proposition 8 we have that $P|T_\alpha|^2 P \xrightarrow{t(M')} 0$. Since $\tau(P) < \infty$, it follows that $LS(MMP) = S(MMP, \tau)$ and the topology $t(MMP)$ coincides with the measure topology $t_\tau$, that is, $P|T_\alpha|^2 P \xrightarrow{t_\tau} 0$. By Proposition 4 we get that $|T_\alpha|^2 \xrightarrow{t_{\tau \ell}} 0$. Hence, it follows from Proposition 7(i) that $T_\alpha \xrightarrow{t_{\tau \ell}} 0$. $\square$

**Remark 4.** It follows from Theorem 3 that the inequalities

$$t_{\tau \ell} \leq t_{\tau} \leq t(M, \tau) \leq t_\tau$$

always hold. If $M = B(H) \times L_\infty(0, \infty)$, $\tau((T, f)) = \text{tr} T + \int_0^\infty f d\mu$, where $T \in B_+(H)$, $0 \leq f \in L_\infty(0, \infty)$, $\mu$ is the linear Lebesgue measure on $[0, \infty)$, dim $H = \infty$, then $S(M, \tau) = B(H) \times S(L_\infty[0, \infty), \mu)$ and, in this case, the following strict inequalities hold:

$$t_{\tau \ell} < t_{\tau} < t(M, \tau) < t_\tau.$$
\[ (ii) \quad \Phi(|T_\alpha|) \xrightarrow{t(Z(M))} 0; \]
\[ (iii) \quad T_\alpha \xrightarrow{t(M)} 0. \]

**Proof.** 1). If \( \tau(|T_\alpha|) \to 0 \), then it follows at once from the inequality \( \tau(|\alpha| > \lambda) \leq \frac{1}{\lambda} \tau(|T_\alpha|) \), \( \lambda > 0 \), that \( T_\alpha \xrightarrow{\tau} 0 \) (here we do not need the condition that \( \sup_{\alpha \in A} \|T_\alpha\|_M \leq 1 \)). Conversely, let \( T_\alpha \xrightarrow{\tau} 0 \). Then for every \( \varepsilon > 0 \) there exist \( \alpha(\varepsilon) \in A \) and \( P_\alpha \in \mathcal{P}(M) \), \( \alpha \geq \alpha(\varepsilon) \), such that

\[
\tau(P_\alpha^1) \leq \varepsilon, \quad T_\alpha P_\alpha \in M, \quad \|T_\alpha P_\alpha\|_M \leq \varepsilon.
\]

Consequently, \( \|T_\alpha|P_\alpha\|_M \leq \varepsilon \) and \( \tau(|T_\alpha|P_\alpha) \leq \varepsilon \tau(I) \). Whence,

\[
\tau(|T_\alpha|) \leq \varepsilon \tau(I) + \tau(|T_\alpha|P_\alpha^1) \leq \varepsilon \tau(I) + \varepsilon \sup_{\alpha \in A} \|T_\alpha\|_M
\]

for all \( \alpha \geq \alpha(\varepsilon) \), that is, \( \tau(|T_\alpha|) \to 0 \).

2). \( (i) \Rightarrow (ii) \). If \( T_\alpha \xrightarrow{\tau} 0 \), then \( |T_\alpha| \xrightarrow{\tau} 0 \) (Proposition 7) and thus \( |T_\alpha| \xrightarrow{\text{wrt}} 0 \). Let \( \mathcal{P}_\tau(M) = \{P \in \mathcal{P}(M) : \tau(P) < \infty \} \). For every finite subset \( \beta = \{P_1, P_2, \ldots, P_n\} \subset \mathcal{P}_\tau(M) \), let \( Q_\beta = \sup_{1 \leq i \leq n} P_i \). Denote by \( B = \{\beta\} \) the directed set of all finite subsets of \( \mathcal{P}_\tau(M) \), ordered by inclusion. It is clear that \( Q_\beta \uparrow I \) and \( Q_B \in \mathcal{P}_\tau(M) \) for all \( \beta \in B \).

Let \( V, U \) be neighborhoods in \( (S(Z(M)), t(Z(M))) \) of zero such that \( V + V \subset U \) and \( XV \subset V \) for any \( X \in Z(M) \) with \( \|X\|_M \leq 1 \). Since \( \Phi(Q_\beta^1) \downarrow 0 \), there exists \( \beta_0 \in B \) such that \( \Phi(Q_{\beta_0}^1) \in V \). Since

\[ 0 \leq \Phi(Q_{\beta_0}^1|T_\alpha|Q_{\beta_0}^1) = \sup_{\alpha \in A} \|T_\alpha\|_M \Phi(Q_{\beta_0}^1) \leq \Phi(Q_{\beta_0}^1), \]

we have that \( \Phi(Q_{\beta_0}^1|T_\alpha|Q_{\beta_0}^1) \in V \) for all \( \alpha \in A \). Identify the center \( Z(M) \) with \( L_\infty(\Omega, \Sigma, \mu) \) and, for \( E \in \Sigma, \mu(E) < \infty \), consider a faithful normal finite trace \( \nu_E \) on \( \chi E M \) defined by

\[
\nu_E(X) = \int_X \Phi(X) d\mu.
\]

Since \( |T_\alpha| \xrightarrow{\text{wrt}} 0 \), we have that \( X_\alpha = Q_{\beta_0}|T_\alpha|Q_{\beta_0} \xrightarrow{\tau} 0 \). Consequently, \( \chi E X_\alpha \xrightarrow{\tau} 0 \) and, hence, \( \chi E X_\alpha \xrightarrow{\tau E} 0 \). Using item 1) we see that

\[
\int_E \Phi(X_\alpha) d\mu = \nu(\chi E X_\alpha) \longrightarrow 0.
\]

Consequently, \( \Phi(X_\alpha) \xrightarrow{t(Z(M))} 0 \). Hence, there exists \( \alpha(V) \in A \) such that \( \Phi(X_\alpha) \in V \) for all \( \alpha \geq \alpha(V) \). Using that \( \Phi(XY) = \Phi(YX) \) for \( X, Y \in M \) [9 Sect. 7.11] we get that

\[
\Phi(|T_\alpha|) = \Phi(Q_{\beta_0}|T_\alpha|Q_{\beta_0}) + \Phi(Q_{\beta_0}^1|T_\alpha|Q_{\beta_0}^1) \in V + V \subset U
\]

for \( \alpha \geq \alpha(V) \), which implies the convergence \( \Phi(|T_\alpha|) \xrightarrow{t(Z(M))} 0 \).
(ii) ⇒ (iii). If $\Phi(|T_\alpha|) \xrightarrow{t(M)} 0$, then it follows from $\Phi(E^+_\lambda(|T_\alpha|)) \leq 1 \Phi(|T_\alpha|)$ that $E^+_\lambda(|T_\alpha|) \xrightarrow{t(M)} 0$ for all $\lambda > 0$.

Setting $Z_\alpha = I$ and using Proposition [2]i we get that $E^+_\lambda(|T_\alpha|) \xrightarrow{t(M)} 0$ for all $\lambda > 0$ and, hence, $T_\alpha \xrightarrow{t(M)} 0$, see Proposition [2]ii.

The implication (iii) ⇒ (i) follows from Theorem [3] □

**Theorem 4.** Let $\mathcal{M}$ be a semifinite von Neumann algebra, $\tau$ be a faithful normal semifinite trace on $\mathcal{M}$. The following condition are equivalent:

(i) $t_{wrl} = t(\mathcal{M}, \tau)$;
(ii) $t_{rl} = t(\mathcal{M}, \tau)$;
(iii) $\mathcal{M}$ is finite.

**Proof.** (i) ⇒ (ii). If $t_{wrl} = t(\mathcal{M}, \tau)$, then the operation of multiplication in $(S(\mathcal{M}, \tau), t_{wrl})$ is jointly continuous. In this case, as was shown in [7] Theorem 4.1, $t_{rl} = t_{wrl}$ and $\mathcal{M}$ is of finite type. The implication (ii) ⇒ (iii) is proved similarly.

(iii) ⇒ (i). Let $\mathcal{M}$ be a finite von Neumann algebra. Then $t_{wrl} = t_{rl}$ [7] Theorem 4.1. Let $\{T_\alpha\} \subset S(\mathcal{M}, \tau)$ and $T_\alpha \xrightarrow{t_{wrl}} 0$. It follows from the identity $t_{wrl} = t_{rl}$ and Proposition [7] that $|T_\alpha| \xrightarrow{t_{wrl}} 0$.

If $\lambda \geq 1$, we have that $0 \leq E^+_\lambda(|T_\alpha|) \leq |T_\alpha|$, hence $E^+_\lambda(|T_\alpha|) \xrightarrow{t_{wrl}} 0$ by Proposition [7]. Using Proposition [9] item 2, we get that $E^+_\lambda(|T_\alpha|) \xrightarrow{t(M)} 0$ for all $\lambda \geq 1$. Since

$$E^+_\lambda(|T_\alpha|E^+_1(|T_\alpha|)) = \begin{cases} E^+_1(|T_\alpha|), & 0 < \lambda < 1, \\ E^+_\lambda(|T_\alpha|), & \lambda \geq 1, \end{cases}$$

by Proposition [2](ii) we get that $|T_\alpha|E^+_1(|T_\alpha|) \xrightarrow{t(M)} 0$. Now, it follows from the inequality $|T_\alpha|E_1(|T_\alpha|) \leq |T_\alpha|$ that $|T_\alpha|E_1(|T_\alpha|) \xrightarrow{t_{wrl}} 0$. Since $t_{wrl} = t_{rl}$ and $||T_\alpha|E_1(|T_\alpha|)||_M \leq 1$, we have that $|T_\alpha|E_1(|T_\alpha|) \xrightarrow{t(M)} 0$ by Proposition [3] item 2.

Hence, $|T_\alpha| = |T_\alpha|E_1(|T_\alpha|) + |T_\alpha|E^+_1(|T_\alpha|) \xrightarrow{t(M)} 0$, and so $T_\alpha \xrightarrow{t(M)} 0$. With a use of Theorem [3] this shows that $t_{wrl} = t(\mathcal{M}, \tau)$. □

4. Comparison of the topologies $t_{rl}$ and $t_{wrl}$ with the (o)-topology on $S_h(\mathcal{M}, \tau)$

Let us denote by $t_{hrl}$ (resp., $t_{h_{wrl}}$) the topology on $S_h(\mathcal{M}, \tau)$ induced by the topology $t_{rl}$ (resp., $t_{wrl}$), and find a connection between these topologies and the (o)-topology $t_{or}(\mathcal{M})$.

**Proposition 10.** $t_{h_{wrl}} \leq t_{hrl} \leq t_{or}(\mathcal{M})$.

**Proof.** Let $\{T_\alpha\}_{\alpha \in A} \subset S_h(\mathcal{M}, \tau)$, $T_\alpha \downarrow 0$, $P \in P(\mathcal{M})$, $\tau(P) < \infty$. Since $(PT_\alpha P) \downarrow 0$, we have that $PT_\alpha P \xrightarrow{t_{hrl}} 0$ by Proposition [5]. Consequently,
$T_\alpha \overset{t_{wrl}}{\longrightarrow} 0$ by Proposition \[5\]. Let $0 \leq S_\alpha \leq T_\alpha$, $S_\alpha \in S_h(M, \tau)$. By Proposition \[7\](iii), $S_\alpha \overset{t_{wrl}}{\longrightarrow} 0$ and, hence, $\sqrt{S_\alpha} \overset{t_{rl}}{\longrightarrow} 0$ by Proposition \[7\](i). Let us show that $S_\alpha \overset{t_{rl}}{\rightarrow} 0$.

Let $\mu_t(T) = \inf\{\|TP\|_M : P \in P(M), \tau(P^\perp) \leq t\}$, $t > 0$, be a non-increasing rearrangement of the operator $T$. Fix $\alpha_0 \in A$. For every $\alpha \geq \alpha_0$, we have

$$\mu_t(\sqrt{S_\alpha}) = \sqrt{\mu_t(S_\alpha)} \leq \sqrt{\mu_t(T_\alpha)} \leq \sqrt{\mu_t(T_{\alpha_0})},$$

in particular,

$$\sup_{\alpha \geq \alpha_0} \mu_t(\sqrt{S_\alpha}) \leq \sqrt{\mu_t(T_{\alpha_0})} < \infty$$

for all $t > 0$. Consequently, the net $\{\sqrt{S_\alpha}\}_{\alpha \geq \alpha_0}$ is $t_{\tau}$-bounded \[7\] Lemma 1.2 and, hence, $S_\alpha \overset{t_{rl}}{\longrightarrow} 0$ by Proposition \[7\](ii). Repeating now the proof of the implication $(ii) \Rightarrow (i)$ in Theorem \[4\] we get that $t_{hrl} \leq t_{or}(M)$.

The inequality $t_{hwr}l \leq t_{hrl}$ follows from the inequality $t_{wrl} \leq t_{rl}$. \[□\]

**Corollary 2.** (i) If $t_{wrl} = t_\tau$ (resp., $t_{rl} = t_\tau$), then $\tau(I) < \infty$.
(ii) If $t_{hwr}l = t_{hrl}$, then the algebra $M$ is finite.
(iii) If $t_{hwr}l = t_{h\tau}$ (resp., $t_{hrl} = t_{h\tau}$), then $\tau(I) < \infty$.

**Proof.** (i) It follows from $t_{wrl} = t_\tau$ that $t_{rl} = t_\tau$. Consequently, $t_{h\tau} \leq t_{or}(M)$ and, hence, $\tau(I) < \infty$ by Proposition \[5\]

(ii) If $T_\alpha \overset{t_{wrl}}{\longrightarrow} 0$, then $T_\alpha \overset{t_{wrl}}{\longrightarrow} 0$ and, hence,

$$Re T_\alpha = \frac{1}{2}(T_\alpha + T_\alpha^*) \overset{t_{wrl}}{\longrightarrow} 0, \quad Im T_\alpha = \frac{1}{2i}(T_\alpha - T_\alpha^*) \overset{t_{wrl}}{\longrightarrow} 0.$$

Since $t_{hwr}l = t_{hrl}$, we get that $T_\alpha \overset{t_{rl}}{\longrightarrow} 0$. Hence, $t_{wrl} = t_{rl}$, which implies that $M$ is finite \[7\].

Item $(iii)$ follows from Propositions \[5\] and \[10\]. \[□\]

**Corollary 3.** The following conditions are equivalent:

(i) $t_{wrl} = t_\tau$;
(ii) $t_{rl} = t_\tau$;
(iii) $\tau(I) < \infty$.

**Proof.** The implication $(i) \Rightarrow (ii)$ follows from the inequalities $t_{wrl} \leq t_{rl} \leq t_\tau$, $(ii) \Rightarrow (iii)$ from Propositions \[5\] and \[10\] and the implication $(iii) \Rightarrow (i)$ is clear. \[□\]

By Proposition \[5\] if $\tau(I) < \infty$, we have the following:

$$t_{hwr}l = t_{hrl} = t_{or}(M) = t_{h\tau}.$$

The following theorem permits to construct examples of von Neumann algebras $M$ for which

$$t_{hwr}l < t_{hrl} < t_{or}(M) < t_{h\tau}.$$

**Theorem 5.** If $t_{hrl} = t_{or}(M)$, then $\tau(I) < \infty$. 


Proof. Assume that $\tau(I) = +\infty$ and first consider the $\sigma$-finite von Neumann algebra $\mathcal{M}$. In this case, there is a faithful normal positive linear functional $\varphi$ on $\mathcal{M}$ [10, Ch. II, §3]. If we have a net $\{T_\alpha\} \subset \mathcal{M}_+$ and $T_\alpha \downarrow 0$, then $\varphi(T_\alpha) \downarrow 0$ and, hence, there is a sequence of indices $\alpha_1 \leq \alpha_2 \leq \ldots$ such that $\varphi(T_{\alpha_n}) \downarrow 0$, which implies $T_{\alpha_n} \downarrow 0$.

Let a net $\{T_\alpha\} \subset S_h(\mathcal{M}, \tau)$ $(o)$-converge to zero in $S_h(\mathcal{M}, \tau)$. Choose $S_\alpha \in S_+(\mathcal{M}, \tau)$ such that $-S_\alpha \leq T_\alpha \leq S_\alpha$ and $S_\alpha \downarrow 0$. Fix $\alpha_0$ and set $X = (I+S_\alpha)^{1/2}$, $Y_\alpha = XS_\alpha X$, $\alpha \geq \alpha_0$. Then $Y_\alpha \in \mathcal{M}_+$, $Y_\alpha \downarrow 0$, and hence there is a sequence $\alpha_1 \leq \alpha_2 \leq \ldots$ such that $Y_{\alpha_n} \downarrow 0$. Consequently, $S_{\alpha_n} \downarrow 0$ and $T_{\alpha_n} \frac{(o)}{t} \rightarrow 0$.

Hence, the subset $F \subset S_h(\mathcal{M}, \tau)$ is closed in the $(o)$-topology $t_oT(\mathcal{M})$ if and only if $F$ contains $(o)$-limits of all $(o)$-convergent sequences of elements in $F$.

Choose a sequence $\{P_n\}$ of nonzero pairwise orthogonal projections in $\mathcal{P}(\mathcal{M})$ satisfying $1 \leq \tau(P_n) < \infty$ and show that $F = \{\sqrt{n}P_n\}_{n=1}^\infty$ is closed in the $(o)$-topology $t_oT(\mathcal{M})$.

If $\{T_k\}_{k=1}^\infty \subset F$ is an $(o)$-convergent sequence of pairwise distinct elements, then $T_k = \sqrt{n_k}P_{n_k} \leq S$, $k = 1, 2, \ldots$, for some $S \in S_+(\mathcal{M}, \tau)$ and, hence, $0 \leq P_{n_k} \leq \frac{1}{\sqrt{n_k}} S \frac{t}{T} \rightarrow 0$. Consequently, $\tau(P_{n_k}) \rightarrow 0$, which contradicts the inequality $\tau(P_{n_k}) \geq 1$, $k = 1, 2, \ldots$.

Hence, the set $F$ is closed in the $(o)$-topology $t_oT(\mathcal{M})$.

It remains to show that this set $F = \{\sqrt{n}P_n\}_{n=1}^\infty$ is not closed in the topology $t_wT$ (here we do not use that the algebra $\mathcal{M}$ is $\sigma$-finite). Denote by $\mathcal{M}^+_\sigma$ the set of all positive normal linear functionals on $\mathcal{M}$, and let $t_\sigma$ be the $\sigma$-strong topology on $\mathcal{M}$ generated by the family of seminorms $p_\psi(T) = \psi(T^*T)^{1/2}$, $\psi \in \mathcal{M}^+_\sigma$, $T \in \mathcal{M}$ [10, Ch. II, §2]. It is clear that the linear functional $\varphi_Q(T) = \tau(QTQ)$ belongs to $\mathcal{M}^+_\sigma$ for all $Q \in P_\tau(\mathcal{M})$ such that $\tau(Q) < \infty$. Thus, the convergence $T_{\alpha} \frac{t}{T} \rightarrow 0$ implies that $\tau(QT_{\alpha}^*T_{\alpha}Q) = \varphi_Q(T_{\alpha}^*T_{\alpha}) = p^{2}_{\varphi_Q}(T_{\alpha}) \rightarrow 0$, that is, $T_{\alpha}Q = QT_{\alpha} \frac{t}{T} \rightarrow 0$. By [14] we have that $|T_{\alpha}Q| \frac{t}{T} \rightarrow 0$ and, hence, $T_{\alpha}Q = QT_{\alpha} \frac{t}{T} \rightarrow 0$, that is, $T_{\alpha} \frac{t}{T} \rightarrow 0$ by Proposition 6 Consequently, the topology $t_\sigma$ majorizes the topology $t_wT$ on $\mathcal{M}$.

Let us now show that $T = 0$ belongs to the closure of the set $F$ in the topology $t_\sigma$. The sets

$$V(\varphi_1, \ldots, \varphi_n, \varepsilon) = \{T \in \mathcal{M} : p_{\varphi_i}(T) \leq \varepsilon, i = 1, 2, \ldots, n\}$$

form a neighborhood base around zero in the topology $t_\sigma$, where $\{\varphi_i\}_{i=1}^n \subset \mathcal{M}^+_\sigma$, $\varepsilon > 0$, $n \in \mathbb{N}$. If $\varphi = \sum_{i=1}^n \varphi_i$, then $\varphi \in \mathcal{M}^+_\sigma$ and $p_{\varphi_i}(T) \leq p_{\varphi}(T)$, $i = 1, 2, \ldots, n$. Hence, the system of subsets $\{V(\varphi, \varepsilon) : \varphi \in \mathcal{M}^+_\sigma, \varepsilon > 0\}$ is a neighborhood base around zero in the topology $t_\sigma$. If $V(\varphi, \varepsilon) \cap F = \emptyset$, then
\[ \varepsilon < p_{t^*}(\sqrt{n}P_n) = \sqrt{n}\varphi(P_n)^{\frac{1}{2}} \] and, hence, \( \varphi(P_n) > \frac{2}{n} \) for all \( n = 1, 2, \ldots, \) which is impossible, since \( \sum_{n=1}^{\infty} \varphi(P_n) = \varphi(\sup P_n) < +\infty. \)

Consequently, \( V(\varphi, \varepsilon) \cap F \neq \emptyset \) for all \( \varphi \in \mathcal{M}_+^* \), \( \varepsilon > 0 \). This means that \( T = 0 \) belongs to the closure of the set \( F \) in the topology \( t_\sigma \). Since \( t_\sigma \) majorizes the topology \( t_{rT} \) on \( \mathcal{M} \), zero belongs to the closure of the set \( F \) in the topology \( t_{rT} \). Consequently, the set \( F \) is not closed in \( (\mathcal{S}_h(\mathcal{M}, \tau), t_{hrT}) \) and, hence, \( t_{hrT} < t_{or}(\mathcal{M}). \)

Let now \( \mathcal{M} \) be a not \( \sigma \)-finite von Neumann algebra, \( \{P_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{M}) \) be as before, \( P = \sup_{n} P_n \), and \( \mathcal{A} = PMP \). It is clear that \( \varphi(T) = \sum_{n=1}^{\infty} \frac{\tau(T_n TP_n)}{2^n \tau(T_n)} \), \( T \in \mathcal{A} \), is a faithful normal linear functional on \( \mathcal{A} \), and thus the algebra \( \mathcal{A} \) is \( \sigma \)-finite \([10] \) Ch. II, \( §3 \).

Let \( \{T_\alpha\}_{\alpha \in A} \subset \mathcal{S}_h(\mathcal{A}, \tau) \subset \mathcal{S}_h(\mathcal{M}, \tau) \), and \( T_\alpha \overset{(o)}{\to} T \) in \( \mathcal{S}_h(\mathcal{M}, \tau) \), that is, there is a net \( \{S_\alpha\}_{\alpha \in A} \subset \mathcal{S}_+(\mathcal{M}, \tau) \) such that \( -S_\alpha \leq T_\alpha - T \leq S_\alpha \) and \( S_\alpha \downarrow 0 \). Then

\[ -PS_\alpha P \leq P(T_\alpha - T)P = T_\alpha - PTP \leq PS_\alpha P \]

and \( PS_\alpha P \downarrow 0 \), that is, \( T_\alpha \overset{(o)}{\to} PTP \) in \( \mathcal{S}_h(\mathcal{A}, \tau) \) and in \( \mathcal{S}_h(\mathcal{M}, \tau) \). Consequently, \( T = PTP \) so that \( T \in \mathcal{S}_h(\mathcal{A}, \tau) \). This means that \( \mathcal{S}_h(\mathcal{A}, \tau) \) is closed in \( (\mathcal{S}_h(\mathcal{M}, \tau), t_{or}(\mathcal{M})) \), and the \( (o) \)-topology \( t_{or}(\mathcal{M}) \) induces the \( (o) \)-topology \( t_{or}(\mathcal{A}) \) on \( \mathcal{S}_h(\mathcal{A}, \tau) \). In particular, the set \( F = \{\sqrt{n}P_n\}_{n=1}^{\infty} \) is closed in \( (\mathcal{S}_h(\mathcal{M}, \tau), t_{or}(\mathcal{M})) \), although it is not closed in the topology \( t_{hrT} \).

Proposition \([10] \) and Theorem \([5] \) immediately give the following.

**Corollary 4.**

(i) If \( t_{hrT} = t_{or}(\mathcal{M}) \), then \( \tau(I) < \infty \).

(ii) If \( \mathcal{M} \) is not finite, then \( t_{hrT} < t_{or}(\mathcal{M}) < t_{hr} \).

(iii) If \( \mathcal{M} \) is finite and \( \tau(I) = +\infty \), then \( t_{hrT} = t_{hrT} < t_{or}(\mathcal{M}) < t_{hr} \).

5. **The locally measure topology on atomic algebras**

Necessary and sufficient conditions on the algebra \( \mathcal{M} \) so that the topology \( t_{rT} \) would be locally convex (resp., normable) were given in the paper of A. M. Bikchentaev \([8] \). Let us give a similar criterion for the topology \( t(\mathcal{M}). \)

A nonzero projection \( P \in \mathcal{P}(\mathcal{M}) \) is called an atom if \( 0 \neq Q \leq P, \ Q \in \mathcal{P}(\mathcal{M}) \), implies that \( Q = P \).

A von Neumann algebra \( \mathcal{M} \) is atomic if every nonzero projection in \( \mathcal{M} \) majorizes some atom. Any atomic von Neumann algebra \( \mathcal{M} \) is \( * \)-isomorphic to the \( C^* \)-product

\[ C^* - \prod_{j \in J} \mathcal{M}_j = \{\{T_j\}_{j \in J} : T_j \in \mathcal{M}_j, \sup_{j \in J} \|T_j\|_{\mathcal{M}_j} < +\infty\} \]

where \( \mathcal{M}_j = \mathcal{B}(\mathcal{H}_j), \ j \in J. \) Since \( LS(\mathcal{B}(\mathcal{H}_j)) = \mathcal{B}(\mathcal{H}_j) \) and \( LS(C^* - \prod_{j \in J} \mathcal{M}_j) = \prod_{j \in J} LS(\mathcal{M}_j) \) \([6] \) Ch. II, \( §3 \), we have that, for an atomic von
Neumann algebra $\mathcal{M}$, the $*$-algebra $LS(\mathcal{M})$ is $*$-isomorphic to the direct product $\prod_{j \in J} \mathcal{B}(\mathcal{H}_j)$ of the algebras $\mathcal{B}(\mathcal{H}_j)$. By Proposition 2 the topology $t(\mathcal{M})$ coincides with the Tychonoff product of the topologies $t(\mathcal{B}(\mathcal{H}_j))$. Since $t(\mathcal{B}(\mathcal{H}_j))$ is a uniform topology $t_{\mathcal{B}(\mathcal{H}_j)}$ on $\mathcal{B}(\mathcal{H}_j)$ generated by the norm $\| \cdot \|_{\mathcal{B}(\mathcal{H}_j)}$, the topology $t(\mathcal{M})$ is locally convex. For every $0 \leq \{T_j\}_{j \in J} \subseteq C^* - \prod_{j \in J} \mathcal{B}(\mathcal{H}_j)$, set $\tau(\{T_j\}_{j \in J}) = \sum_{j \in J} tr_j(T_j)$, where $tr_j$ is the canonical trace on $\mathcal{B}(\mathcal{H}_j)$. It is clear that $\tau$ is a faithful normal semifinite trace on the atomic von Neumann algebra $\mathcal{M} = C^* - \prod_{j \in J} \mathcal{M}_j$, and the topology $t_{\mathcal{M}_j}$ is also locally convex [8], however, $t_{\mathcal{M}_j} \neq t(\mathcal{M})$ if $\dim \mathcal{H}_j = \infty$ for at least one index $j \in J$.

**Proposition 11.** The topology $t(\mathcal{M})$ is locally convex if and only if $\mathcal{M}$ is $*$-isomorphic to the $C^*$-product $C^* - \prod_{j \in J} \mathcal{M}_j$, where $\mathcal{M}_j$ are factors of type I or type III.

**Proof.** Let $t(\mathcal{M})$ be a locally convex topology on $LS(\mathcal{M})$. Since $t(\mathcal{M})$ induces the topology $t(\mathcal{Z}(\mathcal{M}))$ on $\mathcal{Z}(\mathcal{M})$, we have that $(S(\mathcal{Z}(\mathcal{M})), t(\mathcal{Z}(\mathcal{M})))$ is a locally convex space. It follows from [12, Ch. V, §3] that $\mathcal{Z}(\mathcal{M})$ is an atomic von Neumann algebra. Hence, the algebra $\mathcal{M}$ is $*$-isomorphic to the $C^*$-product $C^* - \prod_{j \in J} \mathcal{M}_j$, where $\mathcal{M}_j$ are factors for all $j \in J$. Let $\mathcal{M}_{j_0}$ be of type II-factor. Then there exists a nonzero finite projection $P \in \mathcal{P}(\mathcal{M})$ such that $PMP$ is of type II_1. It follows from [12, Ch. V §3] that $S(PMP, t(PMP))$ has not nonzero continuous linear functional and, hence, the topology $t(PMP)$ can not be locally convex. By Proposition 8 the topology $t(\mathcal{M})$ can not be locally convex too. Consequently, $\mathcal{M}_j$ are either of type I or type III factors for all $j \in J$.

Conversely, let $\mathcal{M} = C^* - \prod_{j \in J} \mathcal{M}_j$, where $\mathcal{M}_j$ are of type I or type III factors. Then $LS(\mathcal{M}_j) = \mathcal{M}_j$, $t(\mathcal{M}_j) = t_{\mathcal{M}_j}$, $LS(\mathcal{M}) = \prod_{j \in J} \mathcal{M}_j$ and, hence, the topology $t(\mathcal{M})$ is a Tychonoff product of the normed topologies $t(\mathcal{M}_j)$, that is, $t(\mathcal{M})$ is a locally convex topology. \qed

**Corollary 5.** The topology $t(\mathcal{M})$ can be normed if and only if $\mathcal{M} = \prod_{j = 1}^n \mathcal{M}_j$, where $\mathcal{M}_j$ are of type I or type III factors, $j = 1, 2, ..., n$, and $n$ is a positive integer.

**Proof.** If the topology $t(\mathcal{M})$ is normable, then $(S(\mathcal{Z}(\mathcal{M})), t(\mathcal{Z}(\mathcal{M})))$ is a normable vector space. It follows from [12, Ch. V, §3] that $\mathcal{Z}(\mathcal{M})$ is a finite dimensional algebra, which implies that $\mathcal{M} = \prod_{j = 1}^n \mathcal{M}_j$, where $\mathcal{M}_j$ are factors, $j = 1, 2, ..., n$. By Proposition 11 the factors $\mathcal{M}_j$ are either of type I or type III for all $j = 1, 2, ..., n$. 
Let us also mention one more useful property of the topologies \( t(\mathcal{M}) \) if \( \mathcal{M} \) is an atomic finite von Neumann algebra.

**Proposition 12.** The following conditions are equivalent:

(i) \( \mathcal{M} \) is an atomic finite von Neumann algebra;

(ii) if \( \{T_n\}_{n=1}^{\infty} \subset LS_h(\mathcal{M}) \), then \( T_n \xrightarrow{t(\mathcal{M})} 0 \) if and only if \( T_n \xrightarrow{(o)} 0 \).

**Proof.** (i) \( \Rightarrow \) (ii). Since \( \mathcal{M} \) is a finite von Neumann algebra, it follows from \( T_n \xrightarrow{(o)} 0 \) that \( T_n \xrightarrow{t(\mathcal{M})} 0 \) by Theorem 1. Since \( \mathcal{M} \) is atomic, we have \( \mathcal{M} = C^* - \prod_{j \in J} B(\mathcal{H}_j) \). If \( T_n = \{T_n^{(j)}\}_{j \in J}, T_n^{(j)} \in B(\mathcal{H}_j), \) and \( T_n \xrightarrow{t(\mathcal{M})} 0 \), then 
\[
\|T_n^{(j)}\|_{B(\mathcal{H}_j)} \to 0 \text{ as } n \to \infty \text{ for all } j \in J.
\]
Since \( T_n^{(j)} \leq \|T_n^{(j)}\|_{B(\mathcal{H}_j)} \cdot I_{B(\mathcal{H}_j)} \), it follows that \( \{T_n^{(j)}\}_{j=1}^{\infty} \) converges to zero in \( B(\mathcal{H}_j) \) and, hence, \( T_n \xrightarrow{(o)} 0 \).

(ii) \( \Rightarrow \) (i). It follows from Remark that \( \mathcal{M} \) is finite. Identify the center \( Z(\mathcal{M}) \) with \( L_\infty(\Omega, \Sigma, \mu) \). By condition (ii), any sequence in \( L_0(\Omega, \Sigma, \mu) \) that \( \mu \)-almost everywhere converges is convergent in the topology \( t(\mathcal{M}) \).

Consequently, \( Z(\mathcal{M}) \) is an atomic von Neumann algebra and, hence, \( \mathcal{M} = C^* - \prod_{j \in J} M_j \), where \( M_j \) are finite factors of types \( I \) or \( II \). If there is an index \( j_0 \in J \) for which \( M_{j_0} \) is of type \( II \), then there exists a nonzero projection \( P \in \mathcal{P}(\mathcal{M}) \), \( \tau(P) = 1 \), such that \( PMP \) is of type \( I_1 \).

Let \( \mathcal{A} \) be a maximal commutative \(*\)-subalgebra of \( PMP \). Then \( \mathcal{A} \) has no atoms and there is a collection \( Q_n^{(k)} \), \( 1 \leq k \leq n \), of pairwise orthogonal projections in \( \mathcal{P}(\mathcal{A}) \) such that \( \sup_{1 \leq k \leq n} Q_n^{(k)} = P \) and \( \tau(Q_n^{(k)}) = \frac{1}{n}, k = 1, 2, \ldots, n \).

Set \( X_n^{(k)} = nQ_n^{(k)}, k = 1, 2, \ldots, n \), and index the operators \( X_n^{(k)} \) by setting \( T_1 = X_1^{(1)}, T_2 = X_2^{(1)}, T_3 = X_2^{(2)}, \ldots \). It is clear that \( T_n \xrightarrow{t} 0 \) and, by condition (ii), \( T_n \xrightarrow{(o)} 0 \) in \( LS_h(PMP) \), that is, there exists a sequence \( \{S_n\}_{n=1}^{\infty} \subset LS_+(PMP) \) such that \( S_n \downarrow 0 \) and \( 0 \leq T_n \leq S_n, n = 1, 2, \ldots \).

Since \( E_n^{\perp}(S_1) \in PMP \) and \( E_n^{\perp}(S_1) \downarrow 0 \), there is an index \( n_0 \) such that \( \tau(E_n^{\perp}(S_1)) < \frac{1}{2} \). Set \( E = PE_n(S_1) \) and \( L_n = ES_n E \). It is clear that \( \frac{1}{2} < \tau(E) \leq 1, 0 \leq L_n \leq n_0 E, L_n \downarrow 0, \) and \( 0 \leq ET_n E \leq L_n, n = 1, 2, \ldots \).

Since \( \tau(L_n) \downarrow 0 \), the inequality \( \tau(E^{\perp}_n(L_n)) \leq \frac{1}{2} \tau(L_n) \) implies that \( \tau(E^{\perp}_n(L_n)) \to 0 \) for all \( \varepsilon > 0 \).

Fix \( \varepsilon \in (0, 1) \) and choose an index \( n_1 \) such that \( \tau(E_n^{\perp}(L_{n_1})) < \frac{1}{2} \). For the projection \( G = PE_n(L_{n_1}) \), we have that \( GET_n E G \leq GL_n G \leq GL_n G \leq \varepsilon G \) for all \( n \geq n_1 \), that is, \( \|Q_n^{(k)} EG\|_{\mathcal{M}} \leq \sqrt{\varepsilon} \) for any \( n \geq n_1, k = 1, 2, \ldots, n \).

If \( E \wedge G = 0 \), then \( 1 = \tau(P) \geq \tau(E \vee G) = \tau(E) + \tau(G) > 1 \). Consequently, \( E \wedge G \neq 0 \), so that there exists a vector \( \xi \in (E \wedge G)(\mathcal{H}) \subset P(\mathcal{H}) \) with \( \|\xi\|_{\mathcal{H}} = 1 \), where \( \|\cdot\|_{\mathcal{H}} \) is the norm on the Hilbert space \( \mathcal{H} \) on which the von
Neumann algebra $M$ acts. For each $n \geq n_1$, we have that
\[
1 = \|P\xi\|_H^2 = \sum_{k=1}^{n} \|Q_n^{(k)}\xi\|_H^2 = \sum_{k=1}^{n} \|(Q_n^{(k)}EG)\xi\|_H^2 \leq \frac{\varepsilon}{n} + \frac{\varepsilon}{n} + \ldots + \frac{\varepsilon}{n} = \varepsilon < 1.
\]
This contradiction shows that the sequence $\{T_n\}_{n=1}^\infty$ can not be $(o)$-convergent to zero in $LS_h(PMP)$. Hence, all the factors $M_j$, $j \in J$, are of type $I$, that is, $M$ is an atomic von Neumann algebra. \[\square\]

**Corollary 6.** The following conditions are equivalent.

(i) Any $t_{w\tau l}$-convergent sequence in $S_h(M, \tau)$ is $(o)$-convergent.
(ii) Any $t_{\tau l}$-convergent sequence in $S_h(M, \tau)$ is $(o)$-convergent.
(iii) Any $t_\tau$-convergent sequence in $S_h(M, \tau)$ is $(o)$-convergent.
(iv) $M$ is an atomic von Neumann algebra and $\tau(I) < \infty$.

**Proof.** The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow from the inequalities $t_{w\tau l} \leq t_{\tau l} \leq t_\tau$.

(iii) $\Rightarrow$ (iv). Since the topology $t_\tau$ is metrizable, it follows from Remark 2 that $t_{h\tau} = t_{o\tau}$ and, hence, $\tau(I) < \infty$ by Proposition 5 in particular, $t_\tau = t(M)$. It remains to apply Proposition 12.

The implication (iv) $\Rightarrow$ (i) follows from Theorem 4 and Proposition 12. \[\square\]

**Remark 5.** It was shown in the proof of the implication (ii) $\Rightarrow$ (i) in Proposition 12 that for a non-atomic von Neumann algebra $M$ with a faithful normal trace $\tau$ there always exists a sequence $\{E_n\}_{n=1}^\infty$ of pairwise commuting projections in $\mathcal{P}(M)$ such that $E_n \xrightarrow{t_\tau} 0$, however, $\{E_n\}_{n=1}^\infty$ does not $(o)$-converge in $S_h(M, \tau)$.

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