Single Polygon Counting for \( m \) Fixed Nodes in Cayley Tree: Two Extremal Cases

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Abstract

We denote a polygon as a connected component in Cayley tree of order 2 containing certain number of fix vertices. We found an exact formula for a polygon counting problem for two cases, in which, for the first case the polygon contain a full connected component of a Cayley tree and for the second case the polygon contain two fixed vertices. From these formulas, which is in the form of finite linear combination of Catalan numbers, one can find the asymptotic estimation for a counting problem.

Keyword: Cayley tree, connected component, Catalan number.
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1 Introduction

In the study of computer science, networks appears very often and attracted a lot of attention of researchers\(^{[1,2]}\). In a generic undirected network particle can enter or exit at any arbitrary sites. Again, the presence of loops in the generic networks, also make the study of particle transport difficult. A prototype network is a Cayley tree\(^{[3]}\), where the direction of transport, the entry and exit points are well defined. Absence of loops make the study relatively simpler. Again, several physical systems like, water transport in trees, transport of antibody in idiotypic networks in immune system\(^{[4]}\), and air circulation in lung\(^{[5]}\) are strikingly similar to this model system of Cayley trees.

There are extensively many combinatorial problems on Cayley tree e.g. the connection between prefix ordered sequences and rooted labelled trees\(^{[6,7,8,9]}\) and Dyck path\(^{[10]}\). It is not surprise that most of them are related to the well known Catalan numbers\(^{[11]}\). Catalan numbers is one of the most frequently encounter integer sequence in counting

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problem [8]. Nowadays, there are many applications as well as generalizations of such numbers [12, 13, 14].

In this paper, we will consider the following problem: to find the number of all different connected components of a Cayley tree with \( n \) number of vertices, containing the given \( m \) number of vertices (where \( n \geq m \)). We borrow the term using in integer lattice, i.e. polygon, for this connected component. In computer science, one can simply visualize this scenario as we are setting up \( m \) routers and each router is expands in a rooted tree, how many different way we can arrange the network with given \( n \) (nodes). As pointed out in [2], a giant connected component is analogous to the percolation cluster in condensed matter. The size distribution of these finite connected components also describe the topology of a random network. So, we see this research not merely as a mathematical exercise but it does provide some applications in network theory as well.

Note that in [15], the posted problem was solved for the case \( m = 1 \), namely, it was shown that the number of different connected component containing a fixed root \( x_0 \in V \) in a semi-infinite Cayley tree for a given \( n \) number of vertices is exactly the Catalan number

\[
C_n = \frac{1}{n+1} \binom{2n}{n}, \quad n \in \mathbb{N}.
\]  

In this paper, first we give a topological structure of the connected component of the Cayley tree. Using these structures, we consider two extremal cases for single polygon counting problem. A recurrent formula for the said problem is derived combinatorically, then an explicit form of single polygon counting is found in term of linear combination of Catalan numbers where the coefficients do not depend on \( n \). The linear combination formulas are derived using generating vectors, analogues to generating function, which will be defined later. From these formulas, one can easily find the asymptotic behavior of the derived numbers.

### 2 A topological structure of the connected component of the Cayley tree

Recall that a Cayley tree of order \( k \) [3], denoted as \( \Gamma^k \), is a graph with no cycles, each vertex emanates \( k+1 \) edges. We denote the set of all vertices as \( V \) and the set of all edges as \( E \), i.e. \( \Gamma^k = (V, E) \). In this paper we restrict ourselves to the Cayley tree of order 2, i.e. \( \Gamma^2 \). Two vertices \( x, y \in V \) are called nearest neighbors if \( \langle x, y \rangle \in E \). Let \( K = (V_K, E_K) \) be a finite connected component of a Cayley tree, where \( V_K \) contains at least two vertices of the tree.

For a given \( x \in V_K \) we put

\[
\mathcal{N}_K(x) = \{ y \in V_K : \langle x, y \rangle \in E_K \}.
\]

The elements of \( \mathcal{N}_K(x) \) are called the nearest neighbors of \( x \) in \( K \). We stand \( |\mathcal{N}_K(x)| \) for the number of the elements of \( \mathcal{N}_K(x) \).

**Definition 2.1** A vertex \( x \in V_K \) is called a boundary of \( K \), if \( |\mathcal{N}_K(x)| = 1 \) i.e. the nearest neighbor of \( x \) in \( K \) is one. Otherwise, it is called an interior vertex of \( K \).

By \( \partial K \) and \( \text{int} K \) we denote the set all of boundary and interior vertices of \( K \), respectively.
Remark 1. Note that the given definition of the boundary and interior is totally different from the ordinary definitions of boundary and interior points of a set, which is used in topological spaces or in the graph theory.

So, by definition, we have

\[ \partial K = \{ x \in V_K : |N_K(x)| = 1 \}, \]
\[ int K = \{ x \in V_K : |N_K(x)| \geq 2 \}, \]

and

\[ \partial K \cap int K = \emptyset, \quad \partial K \cup int K = V_K. \]

**Definition 2.2** A finite connected component \( K = (V_K, E_K) \) is said to be full in a Cayley tree, if \( |N_K(x)| = 3 \) for any \( x \in int K \).

**Remark 2.** If \( int K = \emptyset \) then by definition we say that \( K \) is full.

**Proposition 2.1** For any finite connected component \( K = (V_K, E_K) \) there exists one and only one full connected component \( \overline{K} = (\overline{V}_K, \overline{E}_K) \), containing \( K \) and contained in an arbitrary full connected component \( K' = (V_{K'}, E_{K'}) \) which contains \( K \).

**Proof.** Assume that \( int K \neq \emptyset \), otherwise nothing to prove. Let us construct the following connected component \( \overline{K} = (\overline{V}_K, \overline{E}_K) \) of the tree:

\[ V_\overline{K} = \{ \overline{x} \in V : \exists x \in int K, (x, \overline{x}) \in E \} = \bigcup_{x \in int K} N_E(x), \]

where \( N_E(x) \) is a set all the nearest neighbors of \( x \) in the tree.

From this construction one can see that

\[ int \overline{K} = int K, \quad \partial \overline{K} \supset \partial K, \quad |N_\overline{K}(x)| = 3, \quad x \in int \overline{K}, \]

which means \( \overline{K} = (V_\overline{K}, E_\overline{K}) \) is a full connected component containing \( K \). It follows from this construction that \( \overline{K} = (V_\overline{K}, E_\overline{K}) \) contained any full connected component \( K' = (V_{K'}, E_{K'}) \) which contains \( K \).

Let us prove the uniqueness. Suppose the contrary, i.e. there exists \( \tilde{K} = (V_{\tilde{K}}, E_{\tilde{K}}) \) which satisfies the assertion of the proposition. Then we have

\[ int K \subset int \tilde{K} \subset int \overline{K} \subset int K. \]

Therefore \( int \tilde{K} = int \overline{K} \). Since \( \tilde{K} \) and \( \overline{K} \) are full, then we have \( \partial \tilde{K} = \partial \overline{K} \). This means \( \tilde{K} = \overline{K} \), which completes the proof.

A full connected component \( \overline{K} = (V_\overline{K}, E_\overline{K}) \) which satisfies the assertion of Proposition 2.1 is called the **minimal full component over** \( K = (V_K, E_K) \).

**Theorem 2.1** A finite connected component \( K = (V_K, E_K) \) is full if and only if \( |\partial K| - |int K| = 2 \).
Proof. Only if Part. Let $K = (V_K, E_K)$ be a full connected component. Then we will prove that $|\partial K| - |\text{int} K| = 2$. We use mathematical induction w.r.t. the number $|\text{int} K|$ of interior vertices of $K$.

Let $\text{int} K = \emptyset$. Then $K = \langle x, y \rangle \in E$ and $\partial K = \{x, y\}$. Therefore we have $|\partial K| - |\text{int} K| = 2 - 0 = 2$. We suppose that the assertion of Theorem is true for any full connected component $K$ with $|\text{int} K| \leq k$, i.e. $|\partial K| - |\text{int} K| = 2$.

Now assume that $|\text{int} K| = k + 1$. Let us consider a full connected subcomponent $\tilde{K} = (V_{\tilde{K}}, E_{\tilde{K}})$ of $K$ such that:

(i) $V_{\tilde{K}} \subset V_K$, $E_{\tilde{K}} \subset E_K$;
(ii) $\text{int} \tilde{K} \subset \text{int} K$ and $|\text{int} \tilde{K}| = k$ i.e. $\text{int} \tilde{K} = \{x_1, x_2, \ldots, x_k\}$;
(iii) If $\text{int} K \setminus \text{int} \tilde{K} = \{x_{k+1}\}$ then we have $\text{int} \tilde{K} \cap N_K(x_{k+1}) = 1$.

From the construction one can see that

$$x_{k+1} \in \partial \tilde{K}, \quad \partial \tilde{K} \setminus \{x_{k+1}\} \subset \partial K, \quad |\partial K \setminus \partial \tilde{K}| = 2.$$

Then we have $|\partial K| = |\partial \tilde{K}| + 1$ and $|\text{int} K| = |\text{int} \tilde{K}| + 1$. Since $\tilde{K}$ is a full connected component with $|\text{int} \tilde{K}| = k$, so according to the assumption of induction we have $|\partial \tilde{K}| - |\text{int} \tilde{K}| = 2$. Consequently, one finds

$$|\partial K| - |\text{int} K| = |\partial K| - |\partial \tilde{K}| + 1 - |\text{int} \tilde{K}| - 1 = |\partial K| - |\text{int} \tilde{K}| = 2.$$

If part. Let $K = (V_K, E_K)$ be a connected component with $|\partial K| - |\text{int} K| = 2$. We shall show that $K$ is full. To do it, let us consider the minimal full connected component $\overline{K} = (V_{\overline{K}}, E_{\overline{K}})$ containing $K = (V_K, E_K)$. We then have $\text{int} K = \text{int} \overline{K}$ and $\partial K \subset \partial \overline{K}$ (see Proposition 2.1). Since $\overline{K}$ is full, it then follows from only if part of this Theorem that $|\partial \overline{K}| - |\text{int} \overline{K}| = 2$. Therefore,

$$|\partial \overline{K}| = |\text{int} \overline{K}| + 2 = |\text{int} K| + 2 = |\partial K|.$$

Finiteness of $\partial \overline{K}$ and $\partial K \subset \partial \overline{K}$ imply $\partial K = \partial \overline{K}$. This means $K = \overline{K}$, which completes the proof.

Corollary 2.1 For any connected component $K = (V_K, E_K)$ of a Cayley tree we have

$$2 \leq |\partial K| \leq |\text{int} K| + 2. \quad (2.1)$$

3 Polygon counting problem for $m$ vertices

Suppose that an arbitrary $m$ number of vertices of a Cayley tree $\Gamma^2 = (V, E)$ be given. Namely $V_m = \{x_1, x_2, \ldots, x_m\}$. Let us state our problem:

Problem 3.1 Find the number of all connected components of a Cayley tree with $n$ number of vertices containing the given $m$ number of vertices, where $n \geq m$.

Let

$$G_{n,V_m}^m = \{K = (V_K, E_K): |V_K| = n, \ V_K \supset V_m\}$$

be the set of all connected components containing the given $m$ vertices $V_m$. Our main task is to evaluate the number $|G_{n,V_m}^m|$ of elements of $G_{n,V_m}^m$. Since there always exists a
connected component containing the given \( m \) vertices in a Cayley tree. Among such kind of connected components we take minimal one, i.e. by a minimal connected component containing vertices \( V_m \) we mean a connected component \( K(V_m) = (V_K, E_K) \) such that \( V_m \subset V_K \) and if \( V_m \) is contained in another connected component \( K' = (V', E') \) then one has \( V_K \subset V' \) and \( E_K \subset E' \). One can see that if \( K(x_i, x_j) \) is a shortest path connecting two vertices \( x_i \) and \( x_j \) then

\[
K(V_m) = \bigcup_{i=1}^{m-1} K(x_i, x_{i+1}).
\]

In this case, we can reformulate our problem as follows:

**Problem 3.2** Find the number of all connected components of a Cayley tree with \( n \) number of vertices, containing the given connected component \( K(V_m) \).

In other words, if

\[
G_{n,K}^m(V_m) = \{ K = (V_K, E_K) : K \supset K(V_m), \ |V_K| = n \}
\]

is the set all of the connected components containing the given connected component \( K(V_m) \) then we have \( \left| G_{n,K}^m(V_m) \right| = \left| G_{n,K}^m \right| \). It is easy to check that if

\[
G_{n,K}^{\partial K(V_m)} = \{ K = (V_K, E_K) : V_K \supset \partial K(V_m), \ |V_K| = n \}
\]

is the set all of the connected components containing the given number \( |\partial K(V_m)| \) of vertices \( \partial K(V_m) \), then we have

\[
\left| G_{n,K}^{\partial K(V_m)} \right| = \left| G_{n,K}^m \right| = \left| G_{n,V_m}^m \right|.
\]

Therefore, in what follows, we shall calculate the number of elements of \( G_{n,K}^{\partial K} \) which is the set all of the connected components containing the given number \( |\partial K| \) of vertices \( \partial K \). From the inequality (2.1) we can get estimation to the number \( |\partial K| \) as follows

\[
2 \leq |\partial K| \leq |interrupted K| + 2.
\]

In next sections, we will solve the polygon calculating problem for two extremal cases of the number \( |\partial K| \), namely we shall consider cases: \( |\partial K| = |interrupted K| + 2 \) and \( |\partial K| = 2 \). In other words, in the first case \( K \) is a full connected component and in the second case \( K \) is a shortest path which connects two given vertices.

## 4 Polygon counting problem for a full connected component

Let \( K = (V_K, E_K) \) be a given full connected component of a Cayley tree with \( |\partial K| = m \), where \( m \geq 2 \). We then know that \( |interrupted K| = m - 2 \). By \( E_{n,m-2}^{(m)} \) we denote the number of elements of \( G_{n,K}^{\partial K} \).

**Proposition 4.1** Let \( C_n \) be Catalan numbers, where \( n \in \mathbb{N} \) (see (1.1)) then we have

\[
E_{n,m-2}^{(m)} = \sum_{r_1 + \cdots + r_m = n-m+2, r_1, \ldots, r_m \geq 1} C_{r_1} \cdots C_{r_m}. \quad (4.1)
\]
Proof.

Since there exists only unique full connected component with boundary $\partial K$ which connects all $x_i$'s, (see Figure 1) the number of vertices in the full connected component $m - 2$ is always counted, left only $n - m + 2$ vertices for different configuration.

Figure 1: A connected component containing vertices $x_1, x_2, \ldots, x_m$, connected component which emanate out from $x_i$ is not shown.

For each vertex $x_i \in \partial K$, let $r_i$ denote the number of vertices emanate out from $x_i$, but not back to full connected component. It is known from [15], there allow $C_{r_i}$ number of different connected component in each box for given $r_i$ vertices (See Figure 2). The number of combination for all fixed $r_i$ over $\partial K$ is the product of $C_{r_1}C_{r_2} \cdots C_{r_{m-1}}C_{r_m}$ (For illustration, see Figure 3). The total number is then the products sum over different combination of $r_1 + r_2 + \cdots + r_{m-1} + r_m + m - 2 = n$ i.e. the sum of all vertices equals to $n$. Note that the vertices $x_i$ in the full connected component are always occupied, therefore $r_i$ is always at least 1. Theorem is proved.

Let us calculate $F_{n,m-2}^{(m)}$ for small $m$. For this we will use the following recurrence
Let us list the first few terms of the formula for the Catalan number

\[ C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \]

here \( C_0 := 1 \).

1°. Let \( m = 2 \). We then have

\[ F_{n,0}^{(2)} = \sum_{r_1+r_2=n} C_{r_1} C_{r_2} = \sum_{r_1=1}^{n-1} C_{r_1} \sum_{r_2=1}^{n-1-r_1} C_{r_2} = C_{n+1} - 2C_0 C_n \]

\[ = C_{n+1} - 2C_n. \quad (4.2) \]

Let us list the first few terms of \( F_{n,0}^{(2)} \) (see Sloane [16] A002057):

1, 4, 14, 48, 165, 572, 2002, 7072, 25194, ...

2°. Let \( m = 3 \). We then get

\[ F_{n,1}^{(3)} = \sum_{r_1+r_2+r_3=n-1} C_{r_1} C_{r_2} C_{r_3} = \sum_{r_1=1}^{n-3} C_{r_1} \sum_{r_2=1}^{n-1-r_1} C_{r_2} C_{r_3} \]

\[ = \sum_{r_1=1}^{n-3} C_{r_1} F_{n-1-r_1,0}^{(2)} = \sum_{r_1=1}^{n-3} C_{r_1} (C_{n-r_1} - 2C_{n-1-r_1}) = \]

\[ = \sum_{r_1=1}^{n-1} C_{r_1} C_{n-r_1} - C_1 C_{n-1} - C_2 C_{n-2} - 2 \left( \sum_{r_1=1}^{n-2} C_{r_1} C_{n-1-r_1} - C_1 C_{n-2} \right) \]

\[ = \sum_{r_1=1}^{n-1} C_{r_1} C_{r_2} - 2 \sum_{r_1=1}^{n-1} C_{r_1} C_{r_2} - C_1 C_{n-1} = F_{n,0}^{(2)} - 2F_{n-1,0}^{(2)} - C_{n-1} \]

\[ = C_{n+1} - 4C_n + 3C_{n-1}. \quad (4.3) \]

The first few terms of \( F_{n,1}^{(3)} \) (see Sloane [16] A003517) are as follows:

1, 6, 27, 110, 429, 1638, 6188, 23256, 87210, ...

3°. Let \( m = 4 \). By means the formulas for \( F_{n,1}^{(3)}, F_{n,0}^{(2)} \) we could get the following formula for \( F_{n,2}^{(4)} \) after algebraic manipulations

\[ F_{n,2}^{(4)} = \sum_{r_1+\cdots+r_4=n-2} C_{r_1} \cdots C_{r_4} = \sum_{r_1=1}^{n-5} C_{r_1} \sum_{r_2+\cdots+r_4=n-2-r_1} C_{r_2} C_{r_3} C_{r_4} \]

\[ = \sum_{r_1=1}^{n-5} C_{r_1} F_{n-2-r_1,1}^{(3)} = \cdots = C_{n+1} - 6C_n + 10C_{n-1} - 4C_{n-2}. \quad (4.4) \]
The first few terms of $F_{n,2}^{(4)}$ (see Sloane [16] A003518) are as follows:

\[ 1, 8, 44, 208, 910, 3808, 15504, 62016, 245157, \ldots \]

These calculations lead us to the following conjecture: the number $F_{n,m-2}^{(m)}$ is represented as a linear combination of the Catalan numbers.

Note that such a representation allows us to find asymptotical estimation for $F_{n,m-2}^{(m)}$.

Below we want to realize the stated conjecture.

In the sequel, for any given vector $a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ we put

\[ a_\uparrow = (a_1, a_2, \ldots, a_m, 0), \]

clearly $a_\uparrow \in \mathbb{R}^{m+1}$.

Let

\[ A_m = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -2C_0 & 1 & 0 & \cdots & 0 & 0 \\ -C_1 & -2C_0 & 1 & \cdots & 0 & 0 \\ -C_2 & -C_1 & -2C_0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -C_{m-3} & -C_{m-4} & -C_{m-5} & \cdots & 1 & 0 \\ -C_{m-2} & -C_{m-3} & -C_{m-4} & \cdots & -2C_0 & 1 \end{pmatrix} \] (4.5)

and

\[ B_{[m-1 \times m]} = \begin{pmatrix} -C_m & -C_{m-1} & -C_{m-2} & \cdots & -C_3 & -C_2 & -C_1 \\ -C_{m+1} & -C_m & -C_{m-1} & \cdots & -C_4 & -C_3 & -C_2 \\ -C_{m+2} & -C_{m+1} & -C_m & \cdots & -C_5 & -C_4 & -C_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -C_{2m-2} & -C_{2m-3} & -C_{2m-4} & \cdots & -C_{m+1} & -C_m & -C_{m-1} \end{pmatrix} \] (4.6)

be $m \times m$ and $(m-1) \times m$ matrices, respectively (where $m \geq 2$).

Let us consider the following vectors

\[ a^{(1)} = 1, \quad a^{(2)} = A_2 a_\uparrow^{(1)}, \quad a^{(3)} = A_3 a_\uparrow^{(2)}, \quad \ldots, \quad a^{(m+1)} = A_{m+1} a_\uparrow^{(m)}. \]

**Theorem 4.1** Let $A_m$ and $B_{[m-1 \times m]}$ be the matrices given by (4.5) and (4.6), here $m \geq 2$. If $a^{(1)} = 1$, $a^{(m+1)} = A_{m+1} a_\uparrow^{(m)}$ and $B_{[m-1 \times m]} a^{(m)} = \theta_{m-1}$ then $B_{[m \times m+1]} a^{(m+1)} = \theta_m$, here $\theta_m = (0, 0, \ldots, 0)$ is zero vector of $\mathbb{R}^m$.

**Proof.** We use mathematical induction w.r.t. $m \geq 2$. Let $m = 2$. Since $a^{(1)} = 1$ and

\[ a^{(2)} = A_2 a_\uparrow^{(1)} = \begin{pmatrix} 1 \\ -2C_0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1, -2C_0), \]

\[ B_{[1 \times 2]} a^{(2)} = (-C_2, -C_1) \begin{pmatrix} 1 \\ -2C_0 \end{pmatrix} = 0 = \theta_1, \]

we obtain

\[ a^{(3)} = A_3 a_\uparrow^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ -2C_0 & 1 & 0 \\ -C_1 & -2C_0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2C_0 \end{pmatrix} = (1, -4C_0, -C_1 + 4C_0^2) \]

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\[ B_{[2 \times 3]} a^{(3)} = \begin{pmatrix} -C_3 & -C_2 & -C_1 \\ -C_4 & -C_3 & -C_2 \end{pmatrix} \begin{pmatrix} 1 \\ -4C_0 \\ -C_1 + 4C_0^2 \end{pmatrix} = (0, 0) = \theta_2. \]

We assume that for \( m = k - 1 \) the statement of the Theorem is true, i.e., \( B_{[k-1 \times k]} a^{(k)} = \theta_{k-1} \) which is equivalent

\[- \sum_{j=1}^{k} C_{k+i-j} a^{(k)}_j = 0 \tag{4.7} \]

where \( i = 1, k-1, j = 1, k \) and \( a^{(k)} = (a^{(k)}_1, \ldots, a^{(k)}_k) \).

We will prove the statement of Theorem for \( m = k \). Let \( D_{[k \times k+1]} = B_{[k \times k+1]} A_{k+1} \). We then have

\[ B_{[k \times k+1]} a^{(k+1)} = B_{[k \times k+1]} A_{k+1} a^{(k)} = D_{[k \times k+1]} a^{(k)} \]

Let us evaluate the matrix \( D_{[k \times k+1]} \). We know that

\[ A_{k+1} = (a_{ij})_{i,j=1}^{k+1}, \quad a_{ij} = \begin{cases} 0, & i < j \\ 1, & i = j \\ -2C_0, & i = j + 1 \\ -C_{i-j-1}, & i > j + 1 \end{cases} \]

and

\[ B_{[k \times k+1]} = (b_{ij})_{i,j=1}^{k+k+1} = (-C_{k+1-i-j})_{i,j=1}^{k,k+1} \]

Hence, we get

\[ d_{ij} = \sum_{l=1}^{k+1} b_{il} a_{lj} = \sum_{l=1}^{k+1} b_{il} a_{lj} \\
= -C_{k+1-l-j} + 2C_0 C_{k+1-i-j} + C_I C_{k+i-j-1} + C_{k+i-j-2} + \cdots + C_{k-j} C_i \\
= - \sum_{l=0}^{k-i-j} C_I C_{k+i-j-l} + \sum_{l=0}^{k-i-j} C_I C_{k+i-j-l} + C_{k+i-j} C_0 \\
= - \sum_{l=k-j+1}^{k-i-j} C_I C_{k+i-j-l} + C_{k+i-j} C_0 \]

If we set \( r = l - k + j \) then one can have

\[ d_{ij} = - \sum_{r=1}^{i} C_{k-j+r} C_{i-r} + C_{k+i-j} C_0. \tag{4.8} \]

where \( i = 1, k \) and \( j = 1, k+1 \). Therefore, it follows from (4.8) that

\[ \left( D_{[k \times k+1]} a^{(k)} \right)_{i} = \sum_{j=1}^{k+1} d_{ij} a^{(k)}_j = \sum_{j=1}^{k} d_{ij} a^{(k)}_j \\
= \sum_{j=1}^{k} \left( - \sum_{r=1}^{i} C_{k-j+r} C_{i-r} + C_{k+i-j} C_0 \right) a^{(k)}_j \\
= \sum_{r=1}^{i} C_{i-r} \left( - \sum_{j=1}^{k} C_{k+r-j} a^{(k)}_j \right) C_0 \sum_{j=1}^{k} C_{k+i-j} a^{(k)}_j \tag{4.9} \]
where \( i = \frac{1}{k} \). Consequently, from (4.9) and (4.7) one finds:

(i) If \( i = 1 \) then

\[
\left( D_{[k \times k+1]} \right)_{1,1}^{(k)} = -C_0 \sum_{j=1}^{k} C_{k+1-j} a_{j}^{(k)} + C_0 \sum_{j=1}^{k} C_{k+1-j} a_{j}^{(k)} = 0;
\]

(ii) If \( 1 < i \leq k \) then

\[
\left( D_{[k \times k+1]} \right)_{i,i}^{(k)} = \sum_{r=1}^{i-1} C_i \left( -\sum_{j=1}^{k} C_{k+r-j} a_{j}^{(k)} \right) = 0.
\]

This means that \( D_{[k \times k+1]} a_{1}^{(k)} = \theta_k \) and completes the proof.

**Theorem 4.2** Let \( A_m \) be the matrix given by (4.5) and \( a^{(1)} = 1 \), \( a^{(m+1)} = A_{m+1} a^{(m)} \). Let \( C_n \) be the Catalan number. Then

\[
F_{m,n-2}^m = a_1^{(m)} C_{n+1} + a_2^{(m)} C_n + \cdots + a_m^{(m)} C_{n-m+2},
\]

where \( a^{(m)} = (a_1^{(m)}, a_2^{(m)}, \ldots, a_m^{(m)}) \).

**Proof.** We use mathematical induction w.r.t. \( m \geq 2 \). Let \( m = 2 \). Since \( a^{(1)} = 1 \) and

\[
a^{(2)} = A_2 a^{(1)} = \begin{pmatrix} 1 & 0 \\ -2C_0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1, -2C_0),
\]

from (4.2) we obtain that

\[
F_{n,0}^{(m)} = C_{n+1} - 2C_n = 1 \cdot C_{n+1} - 2C_0 \cdot C_n = a_1^{(2)} C_{n+1} + a_2^{(2)} C_n.
\]

We suppose that the statement of Theorem is true for \( m = k \), i.e.,

\[
F_{m,n-2}^{(k)} = a_1^{(k)} C_{n+1} + a_2^{(k)} C_n + \cdots + a_k^{(k)} C_{n-k+2}, \tag{4.10}
\]

and we prove for \( m = k + 1 \). Then, it follows from (4.1) that

\[
F_{n,k-1}^{(k+1)} = \sum_{r_1 + \cdots + r_{k+1} = n-k+1 \atop r_1, \ldots, r_{k+1} \geq 1} C_{r_1} \cdots C_{r_{k+1}}
\]

\[
= \sum_{r_1=1}^{n-2k+1} C_{r_1} \sum_{r_2+\cdots+r_{k+1}=n-k+1-r_1} C_{r_2} \cdots C_{r_{k+1}} = \sum_{r_1=1}^{n-2k+1} C_{r_1} F_{n-1-r_1,k-2}^{(k)}
\]

Using (4.10), from the last equality we get

\[
F_{n,k-1}^{(k+1)} = \sum_{r_1=1}^{n-2k+1} C_{r_1} \sum_{j=1}^{k} a_j^{(k)} C_{n+1-r_1-j} = \sum_{r_1=1}^{n-2k+1} a_j^{(k)} \sum_{r_1=1}^{k} C_{r_1} C_{n+1-r_1-j}
\]

\[
= \sum_{j=1}^{k} a_j^{(k)} \left( \sum_{r_1=1}^{n-j} C_{r_1} C_{n+1-r_1-j} - \sum_{r_1=n-2k+2}^{n-j} C_{r_1} C_{n+1-r_1-j} \right)
\]

\[
= \sum_{j=1}^{k} a_j^{(k)} \left( F_{n+1-j,0}^{(2)} - \sum_{r_1=n-2k+2}^{n-j} C_{r_1} C_{n+1-r_1-j} \right)
\]
If we set $i = n + 1 - j - r_1$ then using (4.12) one gets

\[
F^{(k+1)}_{n,k-1} = \sum_{j=1}^{k} a^{(k)}_j \left( F^{(2)}_{n+1-j,0} - \sum_{i=1}^{2k-1-j} C_i C_{n+1-j-i} \right)
\]

\[
= \sum_{j=1}^{k} a^{(k)}_j \left( C_{n+2-j} - 2C_0 C_{n+1-j} - \sum_{i=1}^{2k-1-j} C_i C_{n+1-j-i} \right)
\]

\[
= a^{(k)}_1 \left( C_{n+1} - 2C_0 C_n - \sum_{i=1}^{2k-2} C_i C_{n-i} \right)
\]

\[
+ a^{(k)}_2 \left( C_n - 2C_0 C_{n-1} - \sum_{i=1}^{2k-3} C_i C_{n-1-i} \right) + \ldots
\]

\[
+ a^{(k)}_l \left( C_{n+2-l} - 2C_0 C_{n+1-l} - \sum_{i=1}^{2k-1-l} C_i C_{n+1-l-i} \right) + \ldots
\]

\[
+ a^{(k)}_k \left( C_{n+2-k} - 2C_0 C_{n+1-k} - \sum_{i=1}^{k-1} C_i C_{n+1-k-i} \right)
\]

follow by

\[
F^{(k+1)}_{n,k-1} = a^{(k)}_1 C_{n+1} + \left( a^{(k)}_2 - 2C_0 a^{(k)}_1 \right) C_n + \left( a^{(k)}_3 - 2C_0 a^{(k)}_2 - C_1 a^{(k)}_1 \right) C_{n-1}
\]

\[
+ \left( a^{(k)}_4 - 2C_0 a^{(k)}_3 - C_1 a^{(k)}_2 - C_2 a^{(k)}_1 \right) C_{n-2} + \ldots
\]

\[
+ \left( a^{(k)}_l - 2C_0 a^{(k)}_{l-1} - C_1 a^{(k)}_{l-2} - \ldots - C_{l-2} a^{(k)}_1 \right) C_{n-l+2} + \ldots
\]

\[
+ \left( a^{(k)}_k - 2C_0 a^{(k)}_{k-2} - \ldots - C_{k-2} a^{(k)}_1 \right) C_{n-k+2}
\]

\[
+ \left( -2C_0 a^{(k)}_k - C_1 a^{(k)}_{k-1} - C_2 a^{(k)}_{k-2} - \ldots - C_{k-1} a^{(k)}_1 \right) C_{n-k+1}
\]

\[
+ \left( -C_1 a^{(k)}_k - C_2 a^{(k)}_{k-1} - C_3 a^{(k)}_{k-2} - \ldots - C_k a^{(k)}_1 \right) C_{n-k}
\]

\[
+ \left( -C_{k-1} a^{(k)}_k - C_k a^{(k)}_{k-1} - C_{k+1} a^{(k)}_{k-2} - \ldots - C_{2k-2} a^{(k)}_1 \right) C_{n-2k+2}
\]

\[
= \left( A_{k+1} a^{(k)} \right)_{1} C_{n+1} + \ldots + \left( A_{k+1} a^{(k)} \right)_{k+1} C_{n-k+1}
\]

\[
+ \left( B_{[k-1×k]} a^{(k)} \right)_{1} C_{n-k} + \ldots + \left( B_{[k-1×k]} a^{(k)} \right)_{k-1} C_{n-2k+2}
\]

Taking into account

\[
A_{k+1} a^{(k)} = a^{(k+1)}, \quad B_{[k-1×k]} a^{(k)} = \theta_{k-1}.
\]

we obtain

\[
F^{(k+1)}_{n,k-1} = a^{(k+1)}_1 C_{n+1} + a^{(k+1)}_2 C_n + \ldots + a^{(k+1)}_{k+1} C_{n-k+1},
\]

and this completes the proof.

Remark 3. Since the vector $a^{(m)}$ does not depend on $n$ and all coordinates of the vector $a^{(m)}$ are nonzero, one can find an estimate easily through the following well-known estimation of Catalan numbers

\[
C_n = O(1) \frac{4^n}{n^{3/2}} \quad (4.11)
\]

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Corollary 4.1 The asymptotic estimation of $F_{n,m-2}^m$ is given by

$$F_{n,m-2}^m = O(1)C_{n-m+2} = O(1)\frac{4^{n-m+2}}{(n-m+2)^{3/2}}.$$ 

5 Polygon counting problem for two vertices

Let $x_1$ and $x_m$ be two given vertices and $K(x_1, x_m) = (V_m, E_m)$ be the shortest path connecting the given vertices $x_1$ and $x_m$. We then know that $|\partial K| = \{x_1, x_m\}$. We suppose that $|V_m| = m$ and $m \geq 2$. Then it is clear that $|\text{int} K(x_1, x_m)| = m - 2$. By $T_{n,m}^{(2)}$ we denote the number of elements of $G_{n,K(x_1, x_m)}^{\partial K(x_1, x_m)}$ which is the set all of the connected component, with $n$ number of vertices, containing two given vertices $x_1$ and $x_m$.

Proposition 5.1 Let $C_n$ be Catalan numbers, where $n \in \mathbb{N}$, then we have

$$T_{n,m}^{(2)} = \sum_{\substack{r_1 + \cdots + r_m = n-m+2 \\ r_1, r_m \geq 1, r_2, \ldots, r_{m-1} \geq 0}} C_{r_1} \cdots C_{r_m}. \tag{5.1}$$

Proof. Since vertices $x_1$ and $x_m$ always present (see Figure 3) and connected by a unique shortest path with $m$ vertices, we denote this set of vertices by $V_m$, the collection of $x_1$, $x_m$ and vertices along shortest path. For each vertex $x_i \in V_m$, there allow $C_{r_i}$ number of different connected component, which the connected component not in the $V_m$, emanate from $x_i$ with $r_i$ number of vertices (see Figure 4).

![Figure 4: A connected component containing vertices $x_1$ and $x_m$ only, connected component which emanate out from them is not shown.](image)

The number of combination for each fix $x_i$ over $V_m$ is the product of $C_{r_1} \cdot C_{r_2} \cdots C_{r_m}$. The total number is then sum over $r_1 + r_2 + \cdots + r_{m-1} + r_m + m - 2 = n$ which excluding the $m - 2$ vertices. Since $x_1$ and $x_m$ always present, $r_1$ and $r_m$ is always at least one. Theorem proved.
Figure 5: There are \( m - 2 \) vertices along short path, excluding \( x_1 \) and \( x_2 \). From each \( x_i \) emanate another single connected component with \( r_i \) vertices.

Let us calculate \( T_{n,m}^{(2)} \) for small \( m \).

1°. Let \( m = 2 \). We then have

\[
T_{n,2}^{(2)} = \sum_{r_1 + r_2 = n} C_{r_1} C_{r_2} = C_{n+1} - 2 C_n. \tag{5.2}
\]

2°. Let \( m = 3 \). We then get

\[
T_{n,3}^{(2)} = \sum_{r_1 + r_2 + r_3 = n-1} C_{r_1} C_{r_2} C_{r_3} = \sum_{r_2=0}^{n-3} C_{r_2} T_{n-1-r_2,2}^{(2)} = \sum_{r_2=0}^{n-3} C_{r_2} (C_{n-r_2} - 2 C_{n-1-r_2})
= \sum_{r_2=0}^{n} C_{r_2} C_{n-r_2} - C_0 C_n - C_1 C_{n-1} - C_2 C_{n-2} - 2 \left( \sum_{r_2=0}^{n-1} C_{r_2} C_{n-1-r_2} - C_0 C_{n-1} - C_1 C_{n-2} \right)
= C_{n+1} - 3 C_n + C_{n-1}. \tag{5.3}
\]

3°. Let \( m = 4 \). After a little bit algebraic manipulation we could get the following formula for \( T_{n,4}^{(2)} \)

\[
T_{n,4}^{(2)} = \sum_{r_1 + \cdots + r_4 = n-2} C_{r_1} \cdots C_{r_4} = \sum_{r_2=0}^{n-4} C_{r_2} T_{n-1-r_2,3}^{(2)} = \cdots = C_{n+1} - 4 C_n + 3 C_{n-1}. \tag{5.4}
\]

If we denote by \( T_{n,1}^{(2)} := C_{n+1} - C_n \) for the sake of the beauty of the formula, then one can observe the following recurrence formula

\[
T_{n,3}^{(2)} = T_{n,2}^{(2)} - T_{n-1,1}^{(2)} \tag{5.5}
\]

\[
T_{n,4}^{(2)} = T_{n,3}^{(2)} - T_{n-1,2}^{(2)} \tag{5.6}
\]
Remark 4. It is clear that $T_{m,m}^{(2)} = 1$ for any $m \geq 2$.
Let us show this recurrence formula for $T_{n,m}^{(2)}$ in general case.

**Theorem 5.1** We have the following recurrence formula for $T_{n,m}^{(2)}$

$$T_{n,m}^{(2)} = T_{n,m-1}^{(2)} - T_{n-1,m-2}^{(2)}$$

(5.7)

where $n > m$ and $m \geq 3$.

*Proof.* We use a mathematical induction w.r.t. $m$. Let $m = 3$. In (5.5) it is shown that the recurrence formula (5.7) is true. We suppose that the recurrence formula (5.7) is true for any $m \leq k$ and we prove for $m = k + 1$. We know that

$$T_{n,k+1}^{(2)} = \sum_{r_2=0}^{n-k-1} C_{r_2} \left( T_{n-1-r_2,k-1}^{(2)} - T_{n-2-r_2,k-2}^{(2)} \right)$$

\[= \sum_{r_2=0}^{n-k-1} C_{r_2} T_{n-1-r_2,k-1}^{(2)} - \sum_{r_2=0}^{n-k-1} C_{r_2} T_{n-2-r_2,k-2}^{(2)}\]

\[= \sum_{r_2=0}^{n-k} C_{r_2} T_{n-1-r_2,k-1}^{(2)} - C_{n-k} T_{k-1,k-1}^{(2)} - \sum_{r_2=0}^{n-k} C_{r_2} T_{n-2-r_2,k-2}^{(2)} + C_{n-k} T_{k-2,k-2}^{(2)}\]

\[= T_{n,k}^{(2)} - T_{n-1,k-1}^{(2)} .\]

This completes the proof.

Let us try to find an analytic formula for calculating $T_{n,m}^{(2)}$.

**Theorem 5.2** There exist vectors

$$a^{[m]} = \left( a_1^{[m]}, -a_2^{[m]}, \ldots, (-1)^{l-1} a_l^{[m]}, \ldots, (-1)^{[m]-1} a_{[m]}^{[m]} \right)$$

such that $a_l^{[m]} > 0$ for any $l = 1, [m]$ and

$$T_{n,m}^{(2)} = a_1^{[m]} C_{n+1} - a_2^{[m]} C_n + \cdots + (-1)^{l-1} a_l^{[m]} C_{n+2-l} + \cdots + (-1)^{[m]-1} a_{[m]}^{[m]} C_{n+2-[m]} ,$$

here $[m] := \left[ \frac{m-1}{2} \right] + 2$.

*Proof.* We use a mathematical induction w.r.t. $m$. In (5.2), (5.3), (5.4) for $m = 2, 3, 4$, we have already found the vectors $a^{[2]}, a^{[3]}, a^{[4]}$ satisfying the assertion of Theorem.
We suppose that the assertion of Theorem is true for \( m \leq k \) and we prove for \( m = k+1 \). Due to assumption of induction we have

\[
T_{n,k}^{(2)} = a_{[k]}^{[k]}C_{n+1} - a_2^{[k]}C_n + \cdots + (-1)^{[k]}a_{[k]}^{[k]}C_{n+[k]},
\]

\[
T_{n-1,k-1}^{(2)} = a_{[k-1]}^{[k-1]}C_n - a_2^{[k-1]}C_{n-1} + \cdots + (-1)^{[k-1]}a_{[k-1]}^{[k-1]}C_{n+1+[k-1]}.
\]

We know that if \( k = 2t \) then \([k+1] = t + 2\), \([k] = [k-1] = t + 1\) and

\[
n + 1 - [k-1] = n + 2 - [k+1] = n - t,
\]

and if \( k = 2t + 1 \) then \([k+1] = [k] = [k-1] = t + 1\) and

\[
n + 1 - [k-1] = n + 2 - [k] = n + 2 - [k+1] = n - t.
\]

Then, it follows from (5.7) that

\[
T_{n,k+1}^{(2)} = T_{n,k}^{(2)} - T_{n-1,k-1}^{(2)}
\]

\[
= a_{[k+1]}^{[k+1]}C_{n+1} - a_2^{[k+1]}C_n + \cdots + (-1)^{[k]}a_{[k]}^{[k]}C_{n+[k]} + \cdots + \]

\[
+ \cdots + (-1)^{[k+1]}a_{[k+1]}^{[k+1]}C_{n+2-[k+1]},
\]

where

\[
a_{[k]}^{[k+1]} = a_{[k]}^{[k]},
\]

\[
a_{2}^{[k+1]} = a_{2}^{[k]} + a_{1}^{[k-1]},
\]

\[\cdots\]

\[
a_{i}^{[k+1]} = a_{i}^{[k]} + a_{i-1}^{[k-1]},
\]

\[\cdots\]

\[
a_{[k+1]}^{[k+1]} = \begin{cases} a_{[k]}^{[k]} + a_{[k]}^{[k-1]}, & k = 2t + 1 \\ a_{[k-1]}^{[k-1]}, & k = 2t. \end{cases}
\]

Since \( a_{i}^{[k]} > 0 \) and \( a_{j}^{[k-1]} > 0 \) for any \( i = 1, [k] \) and \( j = 1, [k-1] \), we have \( a_{i}^{[k+1]} > 0 \) for all \( l = 1, [k+1] \). This completes the proof of Theorem.

Right now, we will try to find an explicit form of the vector

\[
a_{[m]} = \left( a_{1}^{[m]}, -a_{2}^{[m]}, \ldots, (-1)^{l-1}a_{l}^{[m]}, \ldots, (-1)^{[m]-1}a_{[m]}^{[m]} \right)
\]

for any \( m \geq 3 \). We will call this vector as a generating vector of \( T_{n,m}^{(2)} \).

Let us introduce the following notations for a given vector \( a^{(m)} = (a_1, \ldots, a_m) \in \mathbb{R}^{m-1} \)

\[
a_{k}^{(m)} = (0, a^{(m)}) = (0, a_{1}^{(m)}, \ldots, a_{m}^{(m)}),
\]

\[
a_{m}^{(m)} = \begin{cases} a^{(m)}, & m + 1 > [m] \\ a^{(m)}, & m + 1 = [m] \end{cases}
\]

here \( (a^{(m)}, 0) = (a_{1}^{(m)}, \ldots, a_{m}^{(m)}, 0) \) and \( [m] \leq \frac{m-1}{2} + 2 \).
Let 

$$\mathcal{T}_m = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-C_0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-C_1 & -C_0 & 1 & 0 & \cdots & 0 & 0 \\
-C_2 & -C_1 & -C_0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-C_{m-3} & -C_{m-4} & -C_{m-5} & -C_{m-6} & \cdots & 1 & 0 \\
-C_{m-2} & -C_{m-3} & -C_{m-4} & -C_{m-5} & \cdots & -C_0 & 1
\end{pmatrix}$$

be $m \times m$ matrix.

**Theorem 5.3** Let $a^{[m+1]}$ and $a^{[m]}$ be the generating vectors of $T^{(2)}_{n,m+1}$ and $T^{(2)}_{n,m}$, respectively. Then we have the following formula

$$a^{[m+1]} = \mathcal{T}_{m+1} a^{[m]}$$

(5.9)

where $[m] = \left\lfloor \frac{m-1}{2} \right\rfloor + 2$ and $m \geq 2$.

**Proof.** We know that if $a^{[m+1]}$, $a^{[m]}$, $a^{[m-1]}$ are the generating vectors of $T^{(2)}_{n,m+1}$, $T^{(2)}_{n,m}$, $T^{(2)}_{n,m-1}$, respectively then it follows from (5.8) that

$$a^{[m+1]} = a^{[m]} - a^{[m-1]}$$

(5.10)

We use mathematical induction w.r.t. $m \geq 2$ to prove the formula (5.9).

For $m = 2, 3$, the formula (5.9) is true. We suppose that the formula (5.9) is true for $m \leq k$ and we prove for $m = k + 1$. Here, it can happen two case either $[k+1] = [k]$ or $[k+1] > [k]$. We will consider the first case, i.e. $[k+1] = [k]$. Analogously, one can show for the second case.

Since $[k+1] = [k]$ due to (5.10) we have

$$a^{[k+1]} = a^{[k]} - a^{[k-1]} = a^{[k]} - a^{[k-1]}.$$  \hspace{1cm} (5.11)

It follows from (5.11) and the assumption of induction that

$$a^{[k+1]} = \mathcal{T}_{[k]} a^{[k-1]} - a^{[k-1]}$$

$$= \mathcal{T}_{[k]} a^{[k]} + \mathcal{T}_{[k]} a^{[k-1]} - \mathcal{T}_{[k]} a^{[k]} - a^{[k-1]}$$

$$= \mathcal{T}_{[k]} a^{[k]} + \mathcal{T}_{[k]} (a^{[k-1]} - a^{[k]}) - a^{[k-1]}.$$  \hspace{1cm} (5.11)

Since $[k+1] = [k]$ we have $\mathcal{T}_{[k+1]} = \mathcal{T}_{[k]}$ and

$$a^{[k+1]} = \mathcal{T}_{[k+1]} a^{[k]} + \mathcal{T}_{[k]} (a^{[k-1]} - a^{[k]}) - a^{[k-1]}.$$  \hspace{1cm} (5.11)

Right now we want to show that

$$\mathcal{T}_{[k]} (a^{[k-1]} - a^{[k]}) = a^{[k-1]}$$

We know that $[k+1] = [k]$. Then $a^{[k]} = a^{[k]}$ and

$$a^{[k-1]} - a^{[k]} = a^{[k-1]} - a^{[k]} = a^{[k-2]}.$$  \hspace{1cm} (2)
Since \( [k - 1] = [k - 2] \) due to assumption of induction we have
\[
a^{[k-1]} = T_{[k-1]}a^{[k-2]} = T_{[k-1]}a^{[k-2]}.
\]
From the construction of \( T_{[k]} \) it follows immediately that
\[
T_{[k]} \left( a^{[k-1]} - a^{[k]} \right) = T_{[k]}a^{[k-2]} = \left( 0, T_{[k-1]}a^{[k-2]} \right) = a^{[k-1]}.
\]
This completes the proof.

Analogously, using the estimation (4.11) of Catalan number \( s \), one finds the following asymptotical estimation for \( T_{n,m}^{(2)} \).

**Corollary 5.1** The asymptotic estimation of \( T_{n,m}^{(2)} \) is given by
\[
T_{n,m}^{(2)} = O(1)C_{n-[m]+2} = O(1) \frac{4^n-[m]+2}{(n-[m]+2)^{3/2}},
\]
where \( [m] = \left\lfloor \frac{m-1}{2} \right\rfloor + 2 \).

### 6 Conclusion

We obtained an exact formula for \( F_{n,m}^{m-2} \) and \( T_{n,m}^{(2)} \) as a linear combination of Catalan numbers, where the coefficient does not depend on \( n \). An estimate is then derived.

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### References

[1] Miguens, J. I. L. and Mendes, J. F. F.: Travel and tourism: Into a complex network, Physica A, Volume 387, Issue 12, 1, (2008)

[2] Goltsev, A. V., Dorogovtsev, S. N., Mendes, J. F. F., Critical phenomena in complex networks, Rev. Mod. Phys. 80, 1275 (2008).

[3] Baxter R. J.: *Exactly Solved Models in Statistical Mechanics*, Academic Press, London/New York, (1982).

[4] Anderson R.W., Neumann A.U., Perelson A.S., A Cayley tree immune network model with antibody dynamics, Bull. Math. Bio., 55 (1993), 1091–1131.

[5] Barabasi A., Buldyrev S.V., Stanley H.E., Suki B., Avalanches in the lung: A statistical mechanical model. Phys. Rev. Lett 76(1996), 2192–2195.

[6] Sunik, Z., Self describing sequences and the Catalan family tree, Elect. J. Combin., 10 (No. 1, 2003).

[7] Klarner, D. A. "Correspondences Between Plane Trees and Binary Sequences." J. Comb. Th. 9, 401-411, 1970.
[8] Hilton, P. and Pedersen, J. "Catalan Numbers, Their Generalization, and Their Uses." Math. Int. 13, 64-75, 1991.

[9] Solomon, N. and Solomon, S., A Natural Extension of Catalan Numbers, Journal of Integer Sequences, Vol. 11 (2008).

[10] Barcucci, E., Lungo, A. D., Pergola, E., and Pinzani, R., Permutations avoiding an increasing number of length-increasing forbidden subsequences, Discrete Mathematics & Theoretical Computer Science, Volume 4, 1 (2000), pp. 31-44

[11] Stanley, R.P.: Catalan addendum to Enumerative Combinatorics, Volume 2.

[12] Aigner, M. Catalan-like Numbers and Determinants, J. Combinatorial Theory. Series A, 87 (1999) 33–51.

[13] Heubach, S., Li, N.Y., Mansour, T, Staircase tilings and k-Catalan structures, Discrete Mathematics, 308 (2008), 5954–5964

[14] Mansour, T., Sun, Y, Identities involving Narayana polynomials and Catalan numbers, Discrete Mathematics, 309 (2009), 4079–4088

[15] Pah, C. H.: An application of Catalan number on Cayley tree of order 2: Single polygon counting, Bull. Malays. Math. Sci. Soc. (2) 31 (2), 175-183, (2008).

[16] Neil J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, 2010, published electronically at http://www.research.att.com/njas/sequences/