PARAMETER REGULARITY OF DYNAMICAL
DETERMINANTS OF EXPANDING MAPS OF THE CIRCLE
AND AN APPLICATION TO LINEAR RESPONSE

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ABSTRACT. In order to adapt to the differentiable setting a formula for linear
response proved by Pollicott and Vytnova in the analytic setting, we show a
result of parameter regularity of dynamical determinants of expanding maps
of the circle. Linear response can then be expressed in terms of periodic points
of the perturbed dynamics.

1. Introduction. Linear response is a notion from statistical physics that denotes
the first order response of a physical system under some perturbation. The kind of
systems we are interested in here are uniformly expanding maps of the circle, and
we shall try to describe the variation of their unique invariant probability measure
absolutely continuous with respect to Lebesgue. In order to make it precise, let
$\tau \mapsto T_{\tau}$ be a curve, defined on a neighbourhood $]-\epsilon, \epsilon[$ of $0$, of $C^r$ expanding maps of
the circle $S^1$ for some $r \geq 2$ (i.e. for all $\tau \in ]-\epsilon, \epsilon[$ and $x \in S^1$ we have $|T'_{\tau}(x)| > 1$).
It is then classical (see for instance Theorems 2.1 and 2.6 in [1]) that for all $\tau \in ]-\epsilon, \epsilon[$
the map $T_{\tau}$ admits a unique invariant probability measure $\mu_{\tau} = \rho_{\tau} \, dx$ absolutely
continuous with respect to Lebesgue. Moreover, the associated density $\rho_{\tau}$ is $C^{r-1}$
and positive. The measure $\mu_{\tau}$ describes the equilibrium of our system on a physically
relevant level, it is a basic example of both physical and SRB measure (see [19] for
a discussion of this fundamental notions from smooth ergodic theory that do not
coincide in general). In this context, linear response theory adresses the question
of the regularity of the function $\tau \mapsto \mu_{\tau}$ (under some assumptions of regularity on
$\tau \mapsto T_{\tau}$) as well as the possibility to establish formulae for its derivatives when
they exist. In a more general context, where there may be no invariant measure
absolutely continuous with respect to Lebesgue, one could consider the SRB measure
instead. The case of uniformly expanding maps of the circle is presumably the
simplest one and is well-understood by now. We have for instance the following
theorem (see [2] for a proof).

Theorem 1.1. Assume that $r = 3$ and that the function $\tau \mapsto T_{\tau} \in C^3(S^1, S^1)$ is
$C^2$. Then the map $\tau \mapsto \rho_{\tau} \in C^1(S^1)$ is $C^1$.

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A convenient tool to study \( \mu_{\tau} \), and in particular to prove Theorem 1.1, is the transfer operator \( \mathcal{L}_\tau \) defined by

\[
\mathcal{L}_\tau \varphi (x) = \sum_{T^\tau y = x} \frac{1}{|T^\tau_y(y)|} \varphi (y)
\]

for \( x \in S^1 \) and \( \varphi : S^1 \to \mathbb{C} \). Indeed, \( \rho_\tau \) is an eigenvector for \( \mathcal{L}_\tau \) associated to an eigenvalue of maximal modulus, \( \mathcal{L}_\tau \rho_\tau = \rho_\tau \). Under the hypotheses of Theorem 1.1, it can be shown that the transfer operator \( \mathcal{L}_\tau \) acting on the spaces of \( C^1 \) or \( C^2 \) functions on \( S^1 \) has nice properties that allow to prove Theorem 1.1 (but also for instance exponential decay of correlation or central limit theorem for smooth observables). Let us assume for simplicity that there is a vector field \( X : S^1 \to \mathbb{R} \) (where \( S^1 = \mathbb{R}/\mathbb{Z} \) is parallelized in the usual way) such that for all \( x \in S^1 \) we have

\[
\partial_{\tau} (T^\tau_x)|_{\tau = 0} = X \circ T^\tau_x.
\]

Then we have the following formula for the derivative of \( \tau \mapsto \rho_\tau \) at 0

\[
\partial_{\tau} (\rho_\tau)|_{\tau = 0} = - (1 - \mathcal{L}_0)^{-1} \left( (X \rho_0)' \right).
\]

While 1 is an eigenvalue for \( \mathcal{L}_0 \), this expression makes sense since \( (X \rho_0)' \) belongs to the kernel of the spectral projection associated to this eigenvalue. In [11] and [8] (see also [3, Paragraph A.3]), Gouëzel, Keller and Liverani provided an abstract functional analytic framework that allows to study linear response in a more general context. The main condition to apply these results is the existence of a good scale of Banach spaces adapted to the dynamics (in particular, they can be applied to establish linear response for uniformly hyperbolic systems using anisotropic Banach spaces of distributions, see [3, 5.3]). The reader interested in linear response can refer to the survey [2].

The aim of this work is to establish a formula for linear response in terms of the periodic points of the perturbed dynamics \( T_0 \). We adapt to the linear setting a result obtained by Pollicott and Vytnova [12] in the analytic setting (based on an idea of Cvitanovic [6]). We recall the main argument: assume that \( \tau = \omega \), that is the \( T^\tau \)'s are real-analytic, that \( \tau \mapsto T^\tau \) is real-analytic, and pick \( g : S^1 \to \mathbb{R} \) a real-analytic function, then for all \( \tau \in ]-\epsilon, \epsilon[ \) and \( u \in \mathbb{R} \) introduce the formal power series

\[
z \mapsto \exp \left( - \sum_{n \geq 1} \frac{1}{n} \left( \sum_{k \geq 0} \exp \left( - u \sum_{k = 0}^{n-1} g \left( T^k_{\tau} x \right) \right) \right) z^n \right)
\]

that extends to an entire map \( z \mapsto d(z, u, \tau) \) called dynamical determinant. Moreover, the function \( (z, u, \tau) \mapsto d(z, u, \tau) \) defined this way is analytic. Exploring the properties of this dynamical determinant (in particular its relationship with thermodynamical formalism), Pollicott and Vytnova showed that for all \( \tau \in ]-\epsilon, \epsilon[ \) we have

\[
\int_{S^1} g d\mu_\tau = - \frac{\partial d}{\partial u} (1, 0, \tau),
\]

and thus

\[
\partial_{\tau} \left( \int_{S^1} g d\mu_\tau \right) \bigg|_{\tau = 0} = - \frac{\partial^2 d}{\partial u \partial (\partial (\partial z^2))} (1, 0, 0) + \frac{\partial^2 d}{\partial (\partial z^2) \partial (\partial (\partial z^2))} (1, 0, 0) \frac{\partial d}{\partial z} (1, 0, 0) \left( \frac{\partial d}{\partial z} (1, 0, 0) \right)^2.
\]
Investigating this formula, it is easy to write it in terms of the value of the derivative at $\tau = 0$ of $\tau \mapsto T_\tau$ on the periodic points of $T_0$ (see §8, in particular formula 37).

Reading the proof of Lemma 3.1 of [12], it appears that in order to adapt this result to the finitely differentiable setting one only needs to prove a result of parameter regularity for the dynamical determinant $d$ on a disk of radius strictly larger than one. This is the content of our main result, Theorem 7.1. The hypotheses of Theorem 7.1 are not the weakest possible, they have been chosen to make the exposition as simple and self-contained as possible (see Remark 1). This Theorem implies in particular that the formula 3 for the linear response holds if $\tau \mapsto T_\tau$ is regular enough (taking $t = (\tau, u)$, $T_t = T_{\tau, u} = T_\tau$ and $g_{t, u} = g_{\tau} = -ug - \log |T_\tau'|$).

More precisely, one can prove the following statement.

**Proposition 1.** Let $\tau \mapsto T_\tau$ be a $C^2$ curve, defined on a neighbourhood $]-\epsilon, \epsilon[$ of 0, of $C^7$ expanding maps of the circle. Let $g : S^1 \to \mathbb{R}$ be a $C^6$ map. Then there is some $R > 1$ such that for all $u, \tau$ sufficiently close to 0 the map defined by 1 extends on the disc of center 0 and radius $R$ to a holomorphic function $z \mapsto d(z, u, \tau)$. The function $d$ defined this way is $C^1$ and the formula 2 holds for sufficiently small $\tau$.

If $\tau \mapsto T_\tau$ is a $C^3$ curve of $C^9$ expanding maps of the circle and $g$ is $C^8$, then $d$ is $C^2$ and the formula 3 for linear response holds.

To define the dynamical determinant and prove its main properties in the real-analytic case, one may use the work of Ruelle (see [13]) : the transfer operator acting on a suitable Banach space can be shown to be nuclear in the sense of Grothendieck and the dynamical determinant is then its Fredholm determinant. To deal with the case of finite differentiability, we will use here the approach exposed in the first part of [3] (see also [4]). In particular, the main ingredient of our proof will be the decomposition of a transfer operator (in fact, of an auxiliary operator closely related, see Proposition 2, which is an adaptation of Proposition 3.15 of [3]), into a small “bounded” term and a smoothing nuclear term. The smoothing effect of the second term will be a key tool to address a common difficulty in linear response problems: if we want the dependence of $\mathcal{L}_\tau$ on $\tau$ to be regular, we need to see $\mathcal{L}_\tau$ as an operator from a space of regular functions to a space of less regular functions (for instance from $C^2(S^1)$ to $C^1(S^1)$), which could be an issue when one wants to compute determinant (since determinant are in general associated to endomorphisms).

The article is structured as follow.

In §2, we set some notations and recall some basic facts about Sobolev spaces and the Paley-Littlewood decomposition.

In §3, we use the Paley-Littlewood decomposition to state Lemma 3.1, which is a local version of the decomposition of the transfer operator mentioned above.

In §4, we define an auxiliary operator closely related to the transfer operator and use Lemma 3.1 to provide the promised decomposition of this operator (see Proposition 2).

In §5, we define a “flat trace” and a “flat determinant” for some operators and prove their main properties. These notions will be the main tool to address the lack of nuclearity of the transfer operator while considering dynamical determinant. It may be proved that the flat trace defined in §5 coincides with the flat trace defined in Section 3.2.2 of [3] in most cases.

In §6, we prove some results of regularities of the eigenelements of the transfer operator, using a method due to Gouëzel, Keller and Liverani (see [11] and [8]).
In §7, we state and prove our main result, Theorem 7.1, which is a result of parameter regularity for the dynamical determinant.

In §8, we use Theorem 7.1 to prove Proposition 1 and explain how this can be used to actually compute approximate values for the linear response, following [12].

In the analytic setting, one can prove a similar statement in the case of Anosov diffeomorphisms of the torus using the work of Rugh (see [16] and [17]). It is very likely that the method presented here adapt to the case of differentiable expanding maps in higher dimensions (see Remark 1) and to the case of Anosov diffeomorphisms (using the approach exposed in the second part of [3]).

2. The Paley–Littlewood decomposition. We start by fixing some notations and recalling some basic facts about Sobolev spaces and Paley–Littlewood decomposition needed in the following. Paley–Littlewood decomposition is a decomposition of tempered distributions with respect to dyadic ranges of frequencies. This is a common tool in the theory of functions spaces (see for instance [18]) and it has been used to study transfer operators associated to hyperbolic dynamics by Baladi and Tsujii (see for instance [4] or [3]).

We shall denote by $S$ the Schwartz class on $\mathbb{R}$ and by $S'$ the space of tempered distributions on $\mathbb{R}$. If $s \in \mathbb{R}$ write $H^s = \{ \varphi \in S' : \| \varphi \|_{H^s} < +\infty \}$ where $\| \varphi \|_{H^s}^2 = \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{\varphi}(\xi)|^2 \, d\xi$, if the Fourier transform $\hat{\varphi}$ of $\varphi$ is locally $L^2$ (otherwise set $\| \varphi \|_{H^s}^2 = +\infty$). Recall that if $s, t \in \mathbb{R}$, $\theta \in [0, 1]$ and $\varphi \in S'$, Hölder’s inequality implies that

$$\| \varphi \|_{H^{s+1-\theta}} \leq \| \varphi \|_{H^s}^{1-\theta} \| \varphi \|_{H^t}^\theta.$$  

In particular, if $B$ is some Banach space and $A : B \to S'$ is a linear operator then we have

$$\| A \|_{L(B, H^{s+1-\theta})} \leq \| A \|_{L(B, H^s)}^\theta \| A \|_{L(B, H^t)}^{1-\theta},$$  

where the space $L(B, B')$ of bounded linear operator from a Banach $B$ to another $B'$ is equipped with the operator norm $\| : L(B, B')$.

If $\psi : \mathbb{R} \to \mathbb{R}$ is some $C^\infty$ function of at most polynomial growth, denote by $\text{Op}(\psi)$ the pseudo-differential operator defined by

$$\text{Op}(\psi) \varphi (x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\xi} \psi (\xi) \hat{\varphi} (\xi) \, d\xi$$  

for $x \in \mathbb{R}$ and $\varphi \in S$. Thus $\text{Op}(\psi)$ is the operator of multiplication by $\psi$ in Fourier transform. We shall extend $\text{Op}(\psi)$ to an operator between Sobolev spaces as often as we may.

We can now present the Paley–Littlewood decomposition. Fix a $C^\infty$ even function $\chi : \mathbb{R} \to [0, 1]$ such that

$$\chi (x) = 1 \text{ for } x \leq 1 \text{ and } \chi (x) = 0 \text{ for } x \geq 2.$$  

Then for $n \in \mathbb{N}$ define $\psi_n$ by $\psi_0 = \chi$ and

$$\psi_n (x) = \chi (2^{-n} x) - \chi (2^{-n+1} x)$$  

if $n \geq 1$ and $x \in \mathbb{R}$. We shall also need the functions $\tilde{\psi}_n$ defined by $\tilde{\psi}_0 (x) = \chi \left( \frac{x}{2} \right)$ and

$$\tilde{\psi}_n (x) = \chi (2^{-n-1} x) - \chi (2^{-n+2} x)$$  

if $n \geq 1$ and $x \in \mathbb{R}$. Then the following assertions are easily proved:

$$\sum_{n \geq 1} \psi_n = 1, \quad \forall n \geq 1, x \in \mathbb{R} : \psi_n (x) = \psi_1 (2^{-n+1} x), \tilde{\psi}_n (x) = \tilde{\psi}_1 (2^{-n+1} x),$$  

for $x \in \mathbb{R}$. This decomposition is central for our study and we will use it extensively throughout the rest of the paper.
such that the following properties hold:

\[ \forall n \geq 1 : \text{supp}\, \psi_n \subseteq \{ \xi \in \mathbb{R} : 2^{n-1} \leq |\xi| \leq 2^{n+1} \}, \quad \forall n \in \mathbb{N} : \psi_n \psi_n^* = \psi_n, \]

\[ \forall n, l \in \mathbb{N} : |l - n| > 1 \Rightarrow \psi_n \psi_l = 0. \]

For all \( m \in \mathbb{N}^* \) there is a constant \( C_m \) such that for all \( n \in \mathbb{N} \)

\[ \left\| \psi_n^{(m)} \right\|_{L^\infty} \leq C_m 2^{-nm}. \quad (5) \]

Finally, using Plancherel’s formula, for all \( s \in \mathbb{R} \) there is a constant \( C > 0 \) such that for all \( \varphi \in H^s \) we have

\[ C^{-1} \| \varphi \|^2_{H^s} \leq \sum_{n \geq 0} 4^{ns} \| \text{Op} (\psi_n) \varphi \|^2_{L^2} \leq C \| \varphi \|^2_{H^s}. \quad (6) \]

In particular for all \( s, s' \in \mathbb{R} \) there is a constant \( C \) such that for all \( n \in \mathbb{N} \) we have

\[ \| \text{Op} (\psi_n) \|_{\mathcal{L}(H^{s}, H^{s'})} \leq C 2^n (s' - s). \quad (7) \]

3. Local decomposition of the transfer operator. As explained in the introduction, the strategy to prove Theorem 7.1 is based on a decomposition of the transfer operator. In order to provide this decomposition in Proposition 2, we first prove the following lemma. This lemma can be seen as a “local” version of the decomposition of the transfer operator (it takes place in \( \mathbb{R} \) and \( \mathcal{F} \) will be an inverse branch of an expanding map of the circle in the proof of Proposition 2). The proof of this lemma is an adaptation of the proof of [3, Lemma 2.21], with the necessary additions necessary to deal with the perturbation of the dynamics.

**Lemma 3.1.** Let \( r \in \mathbb{N}^* \) and \( N \in \mathbb{N} \). Let \( U \) be an open set of \( \mathbb{R}^D \) for some integer \( D \). Let \( U \ni t \mapsto f_t \in C^r (\mathbb{R}) \) be a \( C^N \) family of \( C^r \) functions supported in a bounded open interval \( [a, b] \). Let \( U \ni t \mapsto F_t \in C^{r+1} (\mathbb{R}) \) be a \( C^N \) family of diffeomorphisms such that for all \( t \in U \) the derivative of \( F_t \) is bounded by \( \Lambda \) on \( \text{supp}\, f_t \). For all \( \varphi \in \mathcal{S} \) and \( t \in U \) set

\[ \mathcal{M}_t \varphi = f_t (\varphi \circ F_t). \]

Then for all \( t \in U \) the operator \( \mathcal{M}_t \varphi \) can be written as a sum

\[ \mathcal{M}_t = (\mathcal{M}_t)_b + (\mathcal{M}_t)_c \]

such that the following properties hold:

i. for all \( 0 < s < r \) there is a constant \( c > 0 \) that only depends on \( s \) and \( r \) such that for all \( t \in U \) :

\[ \left\| (\mathcal{M}_t)_b \right\|_{\mathcal{L}(H^{s}, H^{s'})} \leq c \| f_t \|_{\infty} \Lambda^s \sup_{\text{supp}\, f_t} |F_t'|^{-\frac{1}{2}}; \]

ii. for all integers \( 0 \leq k \leq N \), all \( s > k + \frac{3}{2} \) and all \( s' < r - 1 \) the map

\[ t \mapsto \mathcal{M}_t \in \mathcal{L}_{\text{nuc}} (H^{s}, H^{s'}) \]

is \( C^k \) on \( U \);

iii. for all integers \( 0 \leq k \leq N \), all \( k + \frac{3}{2} < s < r - 1 \) and all \( \epsilon > 0 \) the map

\[ t \mapsto \mathcal{M}_t \in \mathcal{L} (H^{s}, H^{s - k - \epsilon}) \]

is \( C^k \) on \( U \).
In this statement, \( \mathcal{L}_{\text{nuc}} \left( H^s, H^{s'} \right) \) denotes the space of nuclear operators from \( H^s \) to \( H^{s'} \) equipped with the nuclear operator norm (see for instance \([7, \text{Chapter V}]\) for definitions and main results).

In all this section, fix a real number \( \Lambda > 0 \) and if \( l, n \) are natural integers write
\[
l \mapsto n \quad \text{if} \quad 2^n \leq \Lambda 2^{l+6}, \quad l \not\mapsto n \quad \text{otherwise}.
\]
Then set
\[
\mathcal{A} = \{ (l, n) \in \mathbb{N}^2 : l \mapsto n \}.
\]
In order to prove Lemma 3.1 we want to set
\[
\left( \mathcal{M}_t \right)_b = \sum_{l \mapsto n} \text{Op} \left( \psi_n \right) \mathcal{M}_t \text{Op} \left( \psi_l \right) \quad \text{and} \quad \left( \mathcal{M}_t \right)_c = \sum_{l \not\mapsto n} \text{Op} \left( \psi_n \right) \mathcal{M}_t \text{Op} \left( \psi_l \right).
\]
That is the “bounded” term \( \left( \mathcal{M}_t \right)_b \) corresponds to high ranges of frequencies being moved to low range of frequencies and consequently has to be small (since we are working with positive Sobolev spaces, high frequencies are more heavily weighted than low ones). The remainder part is mainly due to the non-linearity of the map (the fact that the weight is non-constant also plays a role) and will be shown to be nuclear using oscillating integrals techniques.

First, we investigate the “bounded” term \( \left( \mathcal{M}_t \right)_b \).

**Lemma 3.2.** Let \( r \in \mathbb{N}^* \) and \( s \in \mathbb{R}_+^* \). There exists a constant \( c > 0 \) that does not depend on \( \Lambda \) such that for any \( C^{r+1} \)-diffeomorphism \( F : \mathbb{R} \rightarrow \mathbb{R} \) and any compactly supported \( C^r \) function \( f : \mathbb{R} \rightarrow \mathbb{R} \), setting for all \( \varphi \in H^s \)
\[
\mathcal{M} \varphi = f \cdot (\varphi \circ F),
\]
then the series
\[
\sum_{l \mapsto n} \text{Op} \left( \psi_n \right) \mathcal{M} \text{Op} \left( \psi_l \right)
\]
converges in \( \mathcal{L} \left( H^s, H^s \right) \) equipped with the weak operator topology to an operator with operator norm bounded by
\[
c \| f \|_{\infty} \Lambda^s \sup_{\text{supp } f} |F'|^{-\frac{1}{2}}.
\]

**Proof.** We shall prove that for any finite subset \( E \) of \( \mathcal{A} \) we have
\[
\left\| \sum_{(l, n) \in E} \text{Op} \left( \psi_n \right) \mathcal{M} \text{Op} \left( \psi_l \right) \right\|_{H^s \rightarrow H^s} \leq c \| f \|_{\infty} \Lambda^s \sup_{\text{supp } f} |F'|^{-\frac{1}{2}}
\]
The lemma follows, indeed, this implies that the net of the partial sums of the series 8 is bounded, and it has a unique accumulation point since it converges for the operator norm topology on \( \mathcal{L} \left( H^1, H^{-1} \right) \) for instance.

Let \( \varphi \in \mathcal{S} \). For all \( l \in \mathbb{N} \), set \( \varphi_l = \text{Op} \left( \psi_l \right) \varphi \). Then
\[
\left\| \sum_{(l, n) \in E} \text{Op} \left( \psi_n \right) \mathcal{M} \varphi_l \right\|_{H^s}^2 = \int_{\mathbb{R}} (1 + \xi^2)^{s} \left| \sum_{n \geq 0} \psi_n \left( \xi \right) \left( \sum_{l : (l, n) \in E} \mathcal{M} \varphi_l \left( \xi \right) \right) \right|^2 d\xi.
\]
Since for all $\xi \in \mathbb{R}$ there are at most three values of $n$ for which $\psi_n(\xi) \neq 0$, Cauchy–Schwarz implies that

\[
\left\| \sum_{(l,n) \in E} \text{Op}(\psi_n) \mathcal{M}\varphi_l \right\|_{H^s}^2 \leq 3 \sum_{n \geq 0} \int_{\mathbb{R}} (1 + \xi^2)^{s} \left| \psi_n(\xi) \left( \sum_{l: (l,n) \in E} \widehat{\mathcal{M}\varphi_l}(\xi) \right) \right|^2 d\xi \leq 3 \times 8^s \sum_{n \geq 0} \left( \sum_{l: (l,n) \in E} 2^{s_{n-l}} \right)^2 \left( \sum_{l: (l,n) \in E} 2^{s_{n-l}} \left| \psi_n(\xi) \widehat{\mathcal{M}\varphi_l}(\xi) \right|^2 \right) d\xi \leq 3 \times 8^s \sum_{n \geq 0} \left| \sum_{l: (l,n) \in E} 2^{s_{n-l}} \psi_n(\xi) \widehat{\mathcal{M}\varphi_l}(\xi) \right|^2 d\xi.
\]

Applying Cauchy–Schwarz again we get

\[
\left\| \sum_{(l,n) \in E} \text{Op}(\psi_n) \mathcal{M}\varphi_l \right\|_{H^s}^2 \leq 3 \times 8^s \int_{\mathbb{R}} \sum_{n \geq 0} \left( \sum_{l: (l,n) \in E} 2^{s_{n-l}} \right)^2 \left( \sum_{l: (l,n) \in E} 2^{s_{n-l}} \left| \psi_n(\xi) \widehat{\mathcal{M}\varphi_l}(\xi) \right|^2 \right) d\xi \leq 3A^s \frac{2^{9s}}{1 - 2^{-s}} \int_{\mathbb{R}} \sum_{(l,n) \in E} 2^{s_{n-l}} \left| \psi_n(\xi) \widehat{\mathcal{M}\varphi_l}(\xi) \right|^2 d\xi \leq 3A^s \frac{2^{9s}}{1 - 2^{-s}} \int_{\mathbb{R}} \sum_{n: (l,n) \in E} 4^{s_l} \left( \sum_{l: (l,n) \in E} 2^{s_{n-l}} \left| \psi_n(\xi) \widehat{\mathcal{M}\varphi_l}(\xi) \right|^2 \right) d\xi \leq 3A^s \frac{2^{10s}}{(1 - 2^{-s}) (2^s - 1)} \int_{\mathbb{R}} \left| \widehat{\mathcal{M}\varphi_l}(\xi) \right|^2 d\xi \leq 3A^s \frac{2^{10s}}{(1 - 2^{-s}) (2^s - 1)} \sum_{l \geq 0} 4^{s_l} \left| \mathcal{M}\varphi_l \right|^2_{L^2} \leq 3A^s \frac{2^{10s}}{(1 - 2^{-s}) (2^s - 1)} \left\| f \right\|_{L^2} \sup_{\text{supp } f} |F'|^{-1} \sum_{l \geq 0} 4^{s_l} \left| \varphi_l \right|^2_{L^2} \leq c^2 A^{2s} \left\| f \right\|_{L^2}^2 \sup_{\text{supp } f} \left| F' \right|^{-1} \left\| \varphi \right\|_{H^s}^2.
\]

We used 6 in the last line.

Now, we study the “nuclear” term $(\mathcal{M}_l)_l$ of the decomposition given by Lemma 3.1. One of its most important property is being regularizing.

**Lemma 3.3.** Let $r \in \mathbb{N}^*$ and $m \in \mathbb{N}$. Let $s, s' > 0$ with $s > m + \frac{3}{2}$ and $s' < r - 1$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a $C^r$ function compactly supported in a bounded open interval $[a, b]$ and let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a $C^{r+1}$-diffeomorphism with derivative bounded by $\Lambda$ on $\text{supp } f$. Then there is a summable sequence $(b_{n,l})_{l \geq n}$ that only depends on $a, b, s, s', m, \Lambda$, the $C^r$ norm of $f$, and the $C^{r+1}$ norm of $F$ on $[a, b]$, such that, setting for all $\varphi \in \mathcal{S}$

\[
\mathcal{M}\varphi = f \left( \varphi^{(m)} \circ F \right),
\]
for all \( l \not= n \) the operator \( \text{Op}(\tilde{\psi}_n) \mathcal{M}\text{Op}(\psi_l) \) extends to a nuclear operator from \( H^s \) to \( H^s \) with nuclear operator norm bounded by \( b_{n,l} \).

**Proof.** Since \( \varphi \mapsto \varphi^{(m)} \) is bounded from \( H^s \) to \( H^{s-m} \) and commutes with \( \text{Op}(\psi_l) \), one only has to deal with the case \( m = 0 \).

In the following, all the \( C_i \) are constants depending only on \( a, b, s, s', m, \Lambda \), the \( \mathcal{C}^r \) norm of \( f \) on \([a, b]\) and the \( \mathcal{C}^{r+1} \) norm of \( F \) on \([a, b]\).

Fix \( \epsilon > 0 \) such that \( s > \frac{3}{2} + \epsilon \) and set \( a, \eta \in \mathbb{R} \to (1 + \eta^2)^{\frac{1+\epsilon}{2}} \).

Fix \( l \not= n \) with \( l \neq 0 \). If \( \varphi \in S \) write for all \( x \in \mathbb{R} \)

\[
\text{Op}(\tilde{\psi}_n) \mathcal{M}\text{Op}(\psi_l) \varphi (x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} V_{n,l}(x, y) \text{Op}(a) \varphi (y) dy
\]  

where \( V_{n,l}(x, y) \) is defined for all \( x, y \in \mathbb{R} \) by

\[
V_{n,l}(x, y) = \int_{\mathbb{R}^3} e^{i(x-w)\xi + i(F(w)-y)\eta} f(w) \tilde{\psi}_n(\xi) \frac{\psi_l(\eta)}{(1 + \eta^2)^{\frac{1+\epsilon}{2}}} dw d\xi d\eta.
\]

If \( x \) doesn’t lie in \([a, b]\), integrating by parts \( r \) times, we get

\[
V_{n,l}(x, y) = i^r \int_{\mathbb{R}^3} e^{i(x-w)\xi + i(F(w)-y)\eta} \frac{\partial^r}{\partial w^r} \left( \frac{f(w)}{F'(w) \eta - \xi} \right) \tilde{\psi}_n(\xi) \frac{\psi_l(\eta)}{(1 + \eta^2)^{\frac{1+\epsilon}{2}}} dw d\xi d\eta.
\]

and thus, setting \( K = [a, b] \) and recalling \( 5 \),

\[
|V_{n,l}(x, y)| \leq \frac{C_1}{d(x, K)} 2^{-el} 2^{-(r-1)n}.
\]  

We notice that if \( w \in \text{supp} f \) and \( \xi, \eta \in \mathbb{R} \) then \( \tilde{\psi}_n(\xi) \psi_l(\eta) \neq 0 \) implies that \( |\xi - F'(w)\eta| \geq 2^{n-4} \). Then we write

\[
V_{n,l}(x, y) = \int_{\mathbb{R}^3} e^{i(x-w)\xi + i(F(w)-y)\eta} \frac{f(w)}{i(F'(w) \eta - \xi)} \tilde{\psi}_n(\xi) \frac{\psi_l(\eta)}{(1 + \eta^2)^{\frac{1+\epsilon}{2}}} dw d\xi d\eta
\]

Using this trick \( r \) times we write

\[
V_{n,l}(x, y) = \int_{\mathbb{R}^3} e^{i(x-w)\xi + i(F(w)-y)\eta} \frac{\partial^r}{\partial w^r} \left( \frac{f(w)}{F'(w) \eta - \xi} \right) \tilde{\psi}_n(\xi) \frac{\psi_l(\eta)}{(1 + \eta^2)^{\frac{1+\epsilon}{2}}} dw d\xi d\eta.
\]

with \( \Phi \) bounded by \( C_2 2^{-rn} \) when \( \xi \in \text{supp} (\tilde{\psi}_n) \) and \( \eta \in \text{supp} (\tilde{\psi}_l) \), and thus,

\[
|V_{n,l}(x, y)| \leq C_3 2^{-el} 2^{-(r-1)n}.
\]  

From \( 11 \) and \( 13 \), we find for all \( y \in \mathbb{R} \) :

\[
\|V_{n,l}(\cdot, y)\|_{L^2} \leq C_4 2^{-el} 2^{-(r-1)n}.
\]  

If \( y \) doesn’t lie in \( F([a, b]) \), we may improve \( 14 \) by integrating by parts twice on \( \eta \) in \( 10 \) and \( 12 \). This way we get

\[
\|V_{n,l}(\cdot, y)\|_{L^2} \leq C_4 2^{-el} 2^{-(r-1)n} h(y)
\]

where \( h \) is a positive integrable function that only depends on \( a, b \), the \( \mathcal{C}^r \) norm of \( f \) on \([a, b]\), and the \( \mathcal{C}^{r+1} \) norm of \( F \) on \([a, b]\).
Using the same kind of arguments, one can show that the function \( y \in \mathbb{R} \mapsto V_{n,t}(.,y) \in L^2 \) is continuous. Furthermore the function \( y \in \mathbb{R} \mapsto \delta_y \circ \text{Op}(a) \in (H^s)' \) is Hölder-continuous. Thus the integral
\[
\frac{1}{(2\pi)^2} \int_{\mathbb{R}} \left( \delta_y \circ \text{Op}(a) \right) \otimes V_{n,t}(.,y) \, dy \in \mathcal{L}_{\text{nuc}}(H^s, L^2)
\]
is well-defined and extends \( \text{Op}\left( \tilde{\psi}_n \right) \mathcal{M}\text{Op}(\psi_t) \) according to 9. Moreover, its nuclear operator norm is bounded by
\[
\frac{1}{(2\pi)^2} \int_{\mathbb{R}} \| \delta_y \circ \text{Op}(a) \|_{(H^s)'} \| V_{n,t}(.,y) \|_{L^2} \, dy \leq C_5 2^{-cl} 2^{-(r-1)n}.
\]
Thus, the nuclear operator from \( H^s \) to \( H^{s'} \) defined by
\[
\frac{1}{(2\pi)^2} \text{Op}(\psi_n) \circ \int_{\mathbb{R}} \left( \delta_y \circ \text{Op}(a) \right) \otimes V_{n,t}(.,y) \, dy
\]
extends the operator \( \text{Op}(\psi_n) \mathcal{M}\text{Op}(\psi_t) = \text{Op}(\tilde{\psi}_n) \mathcal{M}\text{Op}(\psi_t) \) and has nuclear operator norm bounded by (recall 7)
\[
\| \text{Op}(\psi_n) \|_{L_2 \to H^{s'}} C_5 2^{-cl} 2^{-(r-1)n} \leq C_6 2^{-cl} 2^{-(r-1-s')n} = b_{n,t}.
\]
\[\square\]

**Remark 1.** The constant \( \frac{3}{2} \) that appears in Lemma 3.3 must be understood as \( 1 + \frac{1}{2} \) and may be replaced by \( 1 + \frac{1}{p} \) by working with spaces \( H^s_p(\mathbb{R}) \) as in [3] (here, 1 is thought as the dimension and thus may be replaced by some integer \( D \) to get a more general result). This is one way of weakening the hypotheses of Theorem 7.1.

We immediately deduce the following result from Lemmas 3.2 and 3.3.

**Corollary 1.** Let \( r \in \mathbb{N}^+ \) and \( m \in \mathbb{N} \). Let \( s \in \mathbb{R} \) with \( m + \frac{3}{2} < s < m + r - 1 \). Let \( f : \mathbb{R} \to \mathbb{R} \) be a \( C^r \) function compactly supported in a bounded open interval \( [a,b] \) and \( F : \mathbb{R} \to \mathbb{R} \) be a \( C^{r+1} \)-diffeomorphism. Then setting for all \( \varphi \in \mathcal{S} \)
\[
\mathcal{M}\varphi = f\left( \varphi^{(m)} \circ F \right),
\]
the operator \( \mathcal{M} \) is bounded from \( H^s \) to \( H^{s-m} \). Moreover its operator norm between these spaces is bounded by a constant that only depends on \( a, b, s, m, \Lambda \), the \( C^r \) norm of \( f \), and the \( C^{r+1} \) norm of \( F \) on \( [a,b] \).

Finally, we investigate the regularity of the dependence in \( t \) of the operator \( \mathcal{M}_t \).

**Lemma 3.4.** Let \( r \in \mathbb{N}^+ \) and \( m \in \mathbb{N} \). Let \( s \in \mathbb{R} \) with \( m + \frac{3}{2} < s < m + r - 1 \). Let \( U \) be an open set of \( \mathbb{R}^D \) for some integer \( D \). Let \( U \ni t \mapsto f_t \in C^r(\mathbb{R}) \) be a continuous family of \( C^r \) functions supported in \( [a,b] \). Let \( U \ni t \mapsto F_t \in C^{r+1}(\mathbb{R}) \) be a continuous family of diffeomorphisms of \( \mathbb{R} \) into itself. For all \( \varphi \in \mathcal{S} \) and \( t \in U \) set
\[
\mathcal{M}_t \varphi = f_t\left( \varphi^{(m)} \circ F_t \right).
\]
Then for all \( \epsilon > 0 \) the function \( t \mapsto \mathcal{M}_t \in \mathcal{L}(H^s, H^{s-m-\epsilon}) \) is continuous on \( U \).

**Proof.** As previously, we may and will suppose that \( m = 0 \). If \( V \) is an open set of \( \mathbb{R} \), we denote by \( H^s_0(V) \) the closure in \( H^s \) of the set of \( C^\infty \) functions compactly supported in \( V \). Then, we can write
\[
\mathcal{M}_t = i \circ \mathcal{M}_t \circ \theta
\]
where in the left-hand side $M_t$ is seen as an operator from $H^s$ to $H^{s-r}$ and in the right-hand side $M_t$ is seen as an operator from $H^s_0([a,b])$ to $H^s_0(F^{-1}[a,b])$. $i$ is the (compact) embedding of $H^s_0(F^{-1}[a,b])$ in $H^{s-r}$ and $\theta : H^s \to H^s_0([a,b])$ is the operator of multiplication by a $C^\infty$ function supported in $[a,b]$ with value 1 on the support of the $f_t$. Thanks to the compactness of $i$, we only need to show that $t \mapsto M_t \in \mathcal{L}(H^s_0([a,b]), H^s_0(F^{-1}[a,b]))$ is continuous for the strong topology, that is for all $\varphi \in H^s_0([a,b])$ the function $t \mapsto M_{t,\varphi} \in H^s_0(F^{-1}[a,b])$ is continuous. Thanks to Corollary 1, we only need to prove this when $\varphi$ is $C^\infty$ function supported in $[a,b]$. But for such a function, the continuity is obvious (it even holds in $C^0_0(F^{-1}[a,b])$).

Lemma 3.5. Let $r \in \mathbb{N}^*, m \in \mathbb{N}$ and $N \in \mathbb{N}$. Let $s \in \mathbb{R}$ with $N + m + \frac{3}{2} < s < m + r - 1$. Let $U$ be an open set of $\mathbb{R}^D$ for some integer $D$. Let $U \ni t \mapsto F_t \in C^{r+1}(\mathbb{R})$ be a $C^N$ family of diffeomorphisms. Let $U \ni t \mapsto f_t \in C^r(\mathbb{R})$ be a $C^N$ family of $C^r$ functions supported in $[a,b]$. For all $\varphi \in \mathcal{S}$ and $t \in U$ set

$$M_{t,\varphi} = f_t \left( \varphi^{(m)} \circ F_t \right).$$

Then for all $\epsilon > 0$ the function $U \ni t \mapsto M_t \in \mathcal{L}(H^s, H^{s-m-N-\epsilon})$ is $C^N$.

Proof. The proof is an induction on $U$. The case $N = 0$ has been dealt with in Lemma 3.4. Let $N \geq 1$. As usual, we may suppose $m = 0$. Furthermore, we only need to deal with the case $\epsilon < s - N - \frac{1}{2}$. Let $i \in \{1, \ldots, D\}$ and for all $t = (t_1, \ldots, t_D) \in U$ and $\varphi \in H^s$, write

$$A_i \varphi = \frac{\partial f_t}{\partial t_i} \cdot (\varphi \circ F_t) + f_t \frac{\partial F_t}{\partial t_i} \left( \varphi' \circ F_t \right).$$

By induction hypothesis, $t \mapsto A_i \in \mathcal{L}(H^s, H^{s-m-N-\epsilon})$ is $C^{N-1}$. Consequently, we only need to check that $A_i$ is the partial derivative of $t \mapsto M_t$ with respect to $t_i$. Let $T = (T_1, \ldots, T_d) \in U$ and set for $t_i$ sufficiently close to $T_i :$

$$F(t_i) = M_T + \int_{T_i}^{t_i} A_{1, \ldots, \tau_1, \ldots, t_d} \, d\tau_i \in \mathcal{L}(H^s, H^{s-m-N-\epsilon}).$$

Now, if $\varphi \in \mathcal{S}$ and $y \in \mathbb{R}$, since $\delta_y$ is continuous on $H^{s-N-\epsilon}$, we have

$$F(t_i) \varphi (y) = M_T \varphi (y) + \int_{T_i}^{t_i} A_{1, \ldots, \tau_1, \ldots, t_d} \varphi (y) \, d\tau_i = M_{T_1, \ldots, T_i, \ldots, T_d} \varphi (y).$$

Thus $F(t_i) \varphi = M_{T_1, \ldots, t_i, \ldots, T_d} \varphi$ and finally $F(t_i) = M_{T_1, \ldots, t_i, \ldots, T_d}$. Consequently, $A_i$ is the derivative of $t \mapsto M_t$ with respect to $t_i$.

Now, we can prove our “local decomposition” lemma.

Proof of Lemma 3.1. Set

$$(M_t)_k = \sum_{l \to n} \text{Op} (\psi_n) M_t \text{Op} (\psi_l) \text{ and } (M_t)_c = \sum_{l \neq n} \text{Op} (\psi_n) M_t \text{Op} (\psi_l).$$

Then the first point is a consequence of Lemma 3.2. The second point is deduced from Lemma 3.5 by a standard argument of dominated convergence (the domination being a consequence of Lemma 3.3). Finally, the third point is an immediate consequence of the second point and Lemma 3.5.
4. Decomposition of the transfer operator. We want now to use Lemma 3.1 in order to get the announced decomposition of the transfer operator. For technical reasons, it will be easier to study an auxiliary operator $\mathcal{K}_t$ obtained by cutting the circle into pieces in order to reduce to the local picture studied in §3. The main result of this section is Proposition 2 which describes this decomposition of the auxiliary operator $\mathcal{K}_t$ and will be a key tool in the proof of Theorem 7.1.

We start by fixing some notations. Let $U$ be an open set of $\mathbb{R}^D$. Let $r \in \mathbb{N}^*$, $N \in \mathbb{N}$. Let $0 < \lambda < 1$. Let $t \mapsto T_t \in C^{r+1}(S^1, S^1)$ be a $C^N$ function on $U$ whose values are expanding maps of the circle with expansion constant $\lambda$ and $\lambda^r$. For technical reasons, it will be easier to study an auxiliary operator $\mathcal{L}_t$ in order to get the announced decomposition of the transfer operator. For technical reasons, we shall associate to $L$ an operator $\mathcal{K}_t$ with similar properties and then get a decomposition for $\mathcal{K}_t$.

We need further notation to do so. Let $K$ be a compact subset of $U$ and $\tilde{K}$ a compact neighbourhood of $K$ in $U$. We may choose a finite cover $\alpha = (V\omega)_{\omega \in \Omega}$ of $S^1$ by open intervals with the following properties:

1. for all $\omega \in \Omega$, the canonical projection $\pi : \mathbb{R} \to S^1$ has a $C^\infty$ local inverse $\kappa_\omega$ defined on $V\omega$;
2. for all $t \in \tilde{K}$ and all $\omega \in \Omega$, the map $T_t$ induces a diffeomorphism from a neighbourhood of $V\omega$ to a neighbourhood of $T_t(V\omega)$;
3. for all $t \in \tilde{K}$

$$\sup \left\{ \text{diam} \ V : V \in \bigwedge_{i=0}^{m-1} T_t^{-i} \alpha \right\} \rightarrow 0;$$

4. for all $t \in \tilde{K}$ and $m \in \mathbb{N}^*$ the elements of $\bigwedge_{i=0}^{m-1} T_t^{-i} \alpha$ are open intervals;
5. denoting for all $\omega \in \Omega$ by $W_\omega$ the open interval of $S^1$ with same center as $V_\omega$ but three times as long, the cover $\tilde{\alpha} = (W_\omega)_{\omega \in \Omega}$ also satisfies the four properties above.

Indeed, these properties hold as soon as the diameter of the elements of $\alpha$ is small enough, “small enough” being uniform in $t$ thanks to the compactness of $\tilde{K}$.

For all $m \in \mathbb{N}^*$, $t \in U$ and $\vec{j} = (\omega_0, \ldots, \omega_{m-1}) \in \Omega^m$ let us write $V_{\vec{j},t} = \bigcap_{i=0}^{m-1} T_t^{-i} V_{\omega_i}$ and $\alpha_{m,t} = \bigwedge_{i=0}^{m-1} T_t^{-i} \alpha = \{ V_{\vec{j},t} : \vec{j} \in \Omega^m \}$. Replacing, $V_\omega$ by $W_\omega$, we define in the same way $W_{\vec{j},t}$ and $\tilde{\alpha}_{m,t}$. For all $m \in \mathbb{N}^*$, $t \in U$ and $x \in S^1$ write

$$g_{m,t}(x) = \sum_{i=0}^{m-1} g_t \left( T_t^{-i}(x) \right). \quad (16)$$

By a standard bounded distortion argument, there is a constant $M > 0$ such that for all $t \in \tilde{K}$, $m \in \mathbb{N}^*$ and $V \in \tilde{\alpha}_{m,t}$, if $x, y \in V$ then

$$|g_{m,t}(x) - g_{m,t}(y)| \leq M \text{ and } \left| (T_t^m)'(x) \right| \leq M \left| (T_t^m)'(y) \right|.$$

Now, choose a $C^\infty$ partition of unity $(\theta_\omega)_{\omega \in \Omega}$ adapted to the cover $\alpha$. For all $\omega \in \Omega$, choose a $C^\infty$ function $h_\omega$ compactly supported in $\kappa_\omega(V_\omega)$ with $h_\omega = 1$ on
\( \kappa_\omega (\supp \theta_\omega) \). For all \( s \in \mathbb{R} \) set

\[
\mathcal{B}^s = \bigoplus_{\omega \in \Omega} H^s
\]
equipped with the norm

\[
\| \cdot \|_{\mathcal{B}^s} : (\varphi_\omega)_{\omega \in \Omega} \mapsto \sqrt{\sum_{\omega \in \Omega} \| \varphi_\omega \|^2_{H^s}},
\]
which ensures that \( \mathcal{B}^s \) is a Hilbert space.

Define

\[
S : (\varphi_\omega)_{\omega \in \Omega} \in (\mathbb{C}^\mathbb{R})^\Omega \mapsto \sum_{\omega \in \Omega} \theta_\omega (\varphi_\omega \circ \kappa_\omega) \in \mathbb{C}^{S^1}
\]
and

\[
P : \varphi \in \mathbb{C}^{S^1} \mapsto (h_\omega \varphi \circ \pi)_{\omega \in \Omega} \in (\mathbb{C}^\mathbb{R})^\Omega,
\]
and notice that \( S \circ P = Id \).

For all \( t \in U \) define \( K_t \) on \( \bigoplus_{\omega \in \Omega} S \) by

\[
K_t \varphi = (P \circ L_t \circ S) \varphi.
\]

We are now ready to state and prove our key decomposition of the auxiliary operator \( K_t \). As in the local case, this is a decomposition as a sum of a small “bounded” term \( (K_t)_b \) (the infimum that appears in the right hand-side of 19 has a thermodynamical interpretation that will be made clear in Lemma 6.2) and a “nuclear” smoothing term \( (K_t)_c \). While this decomposition is a fundamental tool in [3] (see [3, Proposition 3.15]), we need here to make explicit the dependence on the parameter \( t \) of the terms of this decomposition.

**Proposition 2.** For all \( t \in K \) and all \( \frac{r}{2} < s < r - 1 \) the operator \( K_t \) extends to a bounded operator from \( \mathcal{B}^s \) to itself. Moreover, for all \( m \in \mathbb{N}^* \) the operator \( K_t^m \) can be written as a sum

\[
K_t^m = (K_t^m)_b + (K_t^m)_c
\]
such that the following properties hold:

i. for all \( \frac{r}{2} < s < r \), there exists a constant \( c > 0 \) such that for all \( m \in \mathbb{N}^* \) and all \( t \in K \):

\[
\|(K_t^m)_b\|_{\mathcal{L}(\mathcal{B}^s, \mathcal{B}^s)} \leq c \lambda^m(s - \frac{r}{2}) \inf_{\beta} \sum_{V \in \beta} \exp \left( \sup_{V \in \beta} g_{m,t} \right);
\]

ii. for all integers \( 0 \leq k \leq N \), all \( \frac{r}{2} + k < s < r - 1 \), all \( m \in \mathbb{N}^* \) and all \( \epsilon > 0 \) the map

\[
t \mapsto (K_t^m)_b \in \mathcal{L}(\mathcal{B}^s, \mathcal{B}^{s - k - \epsilon})
\]
extends to a \( C^k \) function on a neighbourhood of \( K \);

iii. for all integers \( 0 \leq k \leq N \), all \( s > k + \frac{r}{2} \), all \( s' < r - 1 \) and all \( m \in \mathbb{N}^* \) the map

\[
t \mapsto (K_t^m)_c \in \mathcal{L}_{nuc}(\mathcal{B}^s, \mathcal{B}^{s'})
\]
extends to a \( C^k \) function on a neighbourhood of \( K \).

---

\( ^1 \)This decomposition does not depend on \( s \) in the following sense: the operators \( (K_t^m)_b \) and \( (K_t^m)_c \) commute with the natural injections between spaces \( \mathcal{B}^s \) for different values of \( s \).
Proof. The boundedness of $\mathcal{K}_t$ on $\mathcal{B}^n$ will be a consequence of the decomposition \ref{18} for $m = 1$. Fix an integer $m \in \mathbb{N}^*$.

If $t \in K$ choose a subset $I = I_{t,m}$ of $\Omega^m$ that reaches the infimum

$$\sum_{\mathcal{J} \in I} \exp \left( \sup_{V_{\mathcal{J},t}} g_{m,t} \right) = \inf_{J \subseteq \Omega^m} \inf_{S^1 = \bigcup_{\mathcal{J} \in J} W_{\mathcal{J},t}} \sum_{\mathcal{J} \in J} \exp \left( \sup_{V_{\mathcal{J},t}} g_{m,t} \right). \quad (20)$$

Then find a neighbourhood $U_{t,m} \subseteq K$ of $t$ in $U$ with the following properties:

1. for all $t' \in U_{t,m}$
   $$S^1 = \bigcup_{\mathcal{J} \in I} W_{\mathcal{J},t'}; \quad (21)$$

2. for all $t' \in U_{t,m}$
   $$\sum_{\mathcal{J} \in I} \exp \left( \sup_{V_{\mathcal{J},t'}} g_{m,t'} \right) \leq 2 \inf_{J \subseteq \Omega^m} \inf_{S^1 = \bigcup_{\mathcal{J} \in J} V_{\mathcal{J},t'}} \sum_{\mathcal{J} \in J} \exp \left( \sup_{V_{\mathcal{J},t'}} g_{m,t'} \right); \quad (22)$$

3. for all $t' \in U_{t,m}$ and $\mathcal{J} \in I$
   $$\frac{1}{2} \inf_{W_{\mathcal{J},t}} |(T^m_t)| \leq \inf_{W_{\mathcal{J},t'}} |(T^m_{t'})| \leq 2 \inf_{W_{\mathcal{J},t}} |(T^m_t)|. \quad (23)$$

We explain briefly how to find such a neighbourhood. The first point is easy, one only needs to notice that \ref{21} is equivalent to

$$\forall x \in S^1 : \sum_{(\omega_0, \ldots, \omega_{m-1}) \in I} \prod_{i=0}^{m-1} d \left( T^i_t (x), S^1 \setminus W_{\omega_i} \right) > 0.$$  

The third point is an argument of continuity, using the fact that for all $t' \in K$ the set $W_{\mathcal{J},t'}$ is an interval. The second point is more complicated. If $J \subseteq \Omega^m$ is such that $(W_{\mathcal{J},t})_{\mathcal{J} \in J}$ doesn’t cover $S^1$ then for $t'$ sufficiently close to $t$, $(V_{\mathcal{J},t'})_{\mathcal{J} \in J}$ doesn’t cover $S^1$. Consequently, for $t'$ sufficiently close to $t$ we have

$$\inf_{J \subseteq \Omega^m} \inf_{S^1 = \bigcup_{\mathcal{J} \in J} W_{\mathcal{J},t}} \sum_{\mathcal{J} \in J} \exp \left( \sup_{V_{\mathcal{J},t'}} g_{m,t'} \right) \leq \inf_{J \subseteq \Omega^m} \inf_{S^1 = \bigcup_{\mathcal{J} \in J} V_{\mathcal{J},t'}} \sum_{\mathcal{J} \in J} \exp \left( \sup_{V_{\mathcal{J},t'}} g_{m,t'} \right).$$

But the infimum on the left-hand side of this inequality is taken on a set that does not depend of $t'$, so we can use the same kind of argument as for the third point, recalling \ref{20}.

Now, $(U_{t,m})_{t \in K}$ is an open cover of $K$ and consequently, one only needs to get the decomposition \ref{18} on each of its elements separately (then glue the different decompositions using a partition of unity). So fix $t_0 \in K$ and write $I = I_{t_0,m}$ the subset of $\Omega^m$ that appears in the definition of $U_{t_0,m}$. For all $\omega \in \Omega$ choose a $C^\infty$ function $\tilde{\chi}_\omega : S^1 \to \mathbb{R}$ such that $0 \leq \tilde{\chi}_\omega \leq 1$ and $\tilde{\chi}_\omega (x) > 0$ if and only if $x \in W_\omega$. Then for all $m \in \mathbb{N}^*$, $t \in U_{t_0,m}$ and $\mathcal{J} = (\omega_0, \ldots, \omega_{m-1}) \in I$, set:

$$\tilde{\chi}_{\mathcal{J},t} : x \in S^1 \mapsto \prod_{i=0}^{m-1} \tilde{\chi}_{\omega_i} (T^i_t (x)).$$
and
\[ \chi_{\omega, t} : x \in S^1 \mapsto \frac{\tilde{\chi}_{\omega,t}(x)}{\sum_{\omega' \in I} \tilde{\chi}_{\omega',t}(x)}, \]
which is well-defined thanks to 21. Thus we have for all \( t \in U_{t_{0}, t} \):
\[
\sum_{\omega \in I} \chi_{\omega, t} = 1 \quad \text{and} \quad \forall \omega \in I : \forall x \in S^1 : \chi_{\omega, t}(x) > 0 \iff x \in W_{\omega,t}.
\]
Then for all \( \omega \in I \), \( t \in U_{t_{0}, t} \) and \( \varphi \in C^\infty(S^1) \), define
\[
L^m_{\omega,t} \varphi : x \in S^1 \mapsto \sum_{T^m_{t}(y)=x} \chi_{\omega, t}(y) e^{g_{m,t}(y)} \varphi(y) = \left( \chi_{\omega, t} e^{g_{m,t}} \varphi \right) \circ \left( T^m_{t} \big|_{W_{\omega,t}} \right)^{-1}(x)
\]
and then
\[
K^m_{\omega, t} = D \circ L^m_{\omega,t} \circ S.
\]
These definitions immediately imply for all \( t \in U_{t_{0}, t} \)
\[
L^m_t = \sum_{\omega \in I} L^m_{\omega,t} \quad \text{and} \quad K^m_t = \sum_{\omega \in I} K^m_{\omega,t}.
\]
Now, fix \( \omega \in I \) and write \( K^m_{\omega, t} \) as a matrix of operators \( (A_{\omega, \omega', t})_{\omega, \omega' \in \Omega} \) that is, for all \( \varphi = (\varphi_\omega)_{\omega \in \Omega} \in \bigoplus_{\omega \in \Omega} S \), we have
\[
K^m_{\omega, t} \varphi = \left( \sum_{\omega' \in \Omega} A_{\omega, \omega', t} \varphi_\omega \right)_{\omega \in \Omega}.
\]
If \( \omega, \omega' \in \Omega \) and \( \varphi \in S \), we have
\[
A_{\omega, \omega', t} \varphi = h_\omega \left( \chi_{\omega', t} e^{g_{m,t} \cdot \theta_{\omega'}} \circ \left( T^m_{t} \big|_{W_{\omega',t}} \right)^{-1} \circ \pi \right) \left( \varphi \circ \kappa_{\omega'} \circ \left( T^m_{t} \big|_{W_{\omega',t}} \right)^{-1} \circ \pi \right).
\]
The map \( \kappa_{\omega'} \circ \left( T^m_{t} \big|_{W_{\omega',t}} \right)^{-1} \circ \pi \) may be extended in a \( C^\infty \)-diffeomorphism \( F_\omega \) of \( \mathbb{R} \) in a consistent way: take the inverse of a lift of \( T^m_{t} \) that extends \( \kappa_\omega \circ \left( T^m_{t} \big|_{W_{\omega',t}} \right)^{-1} \circ \pi \big|_{\kappa_{\omega'}(V_{\omega'})} \). Now apply Lemma 3.1 with this \( F_\omega \), \( f_\omega = h_\omega \left( \chi_{\omega', t} e^{g_{m,t} \cdot \theta_{\omega'}} \circ \pi \circ F_\omega \right) \) and \( \Lambda = 2 \left( \inf_{W_{\omega', t_{0}}} \left| (T^m_{t})' \right| \right)^{-1} \), recalling 23. Thus, using Lemma 3.1, we get a decomposition of \( A_{\omega, \omega', t} \) in
\[
A_{\omega, \omega', t} = (A_{\omega, \omega', t})_b + (A_{\omega, \omega', t})_c
\]
with the expected regularity and, for all \( \frac{1}{2} < s < r \), a constant \( c_s \), that only depends of \( s \), such that

\[
\| (A_{\omega, \omega', t})_b \|_{H^s \rightarrow H^{r-s}} \leq c_s \exp \left( \sup_{W_{\omega', t}} g_{m,t} \right) \Lambda^s \sup_{W_{\omega', t}} \left| (T^m_{t})' \right|^\frac{s}{2} \leq 4^s c_s e^M \exp \left( \sup_{V_{\omega', t}} g_{m,t} \right) \left( \inf_{W_{\omega', t}} \left| (T^m_{t})' \right| \right)^{-s} \sup_{W_{\omega', t}} \left| (T^m_{t})' \right|^\frac{s}{2}.
\]
\[ \leq 4^s c_s e^M \sqrt{M} \exp \left( \sup_{V \in \Omega} g_{m,t} \right) \left( \inf_{W \in \Omega} \left| (T^m_t)' \right| \right)^{-s + \frac{1}{2}} \]

\[ \leq 4^s c_s e^M \sqrt{M} \exp \left( \sup_{V \in \Omega} g_{m,t} \right) \lambda^m(s - \frac{1}{2}), \]

where \( M \) has been introduced in 17 (that we used on second and third line). We also used 23 on second line and 15 on the last line. From this, we deduce a decomposition of \( K_{m,t}^{\Omega} \) with the expected regularity and the same estimate of the operator norm up to a multiplicative factor \((\#\Omega)^2\). Summing over \( \omega \in I \), we get the decomposition 18 with the expected regularity and

\[ \| (K^m_t)_{\#} \|_{H^{s} \rightarrow H^{s}} \leq 2 \times 4^s c_s e^M (\#\Omega)^2 \lambda^m(s - \frac{1}{2}) \inf_{\beta \text{ subcover of } \alpha_{m,t}} \sum_{V \in \beta} \exp \left( \sup_{V} g_{m,t} \right) \]

thanks to 22.

\[ \square \]

**Remark 2.** As a consequence of Proposition 2 for all integers \( 0 \leq k \leq N \), for all \( \frac{3}{2} < s < r - 1 \), all \( m \in \mathbb{N}^* \) and all \( \epsilon \) the map

\[ t \mapsto K_t \in \mathcal{L} (\mathcal{B}^s, \mathcal{B}^{s-k-\epsilon}) \]

is \( C^k \) on a neighborhood of \( K \).

5. **Flat trace of the transfer operator.** In order to use Proposition 2 to prove Theorem 7.1, we need to explain the link between the dynamical determinant from Theorem 7.1 (which is given by 1 in the application) and the transfer operator \( \mathcal{L}_t \) (in fact, we shall use the auxiliary operator \( K_t \) from §4 instead). In the analytic setting, the transfer operator \( \mathcal{L}_t \) acting on some space of holomorphic functions can be shown to be nuclear and the dynamical determinant is then its Fredholm determinant (see for instance [5]). This is *a priori* not true in our setting. Consequently, this section is dedicated to the definitions and basic properties of the “flat trace” and “flat determinant”, which are key tools in the study of dynamical determinants in [3] or [4], and will allow us to bypass this difficulty. Although we will not define these objects in the same way as in [3] or [4], it could be shown that both definitions agree in most cases (using for instance the third point of Lemma 5.1 and [3, Proposition 3.13]).

For all \( \epsilon > 0 \) we set for all \( x \in \mathbb{R} \):

\[ \rho_\epsilon (x) = \frac{1}{\epsilon} \rho \left( \frac{x}{\epsilon} \right) \quad \text{and} \quad \chi_\epsilon = F^{-1} (\rho_\epsilon), \]

where \( F^{-1} \) is the inverse of the Fourier transform and \( \rho : \mathbb{R} \rightarrow \mathbb{R} \) is a \( C^\infty \) function, taking values in \([0, 1]\), compactly supported, of integral 1 and identically equal to 1 one a neighbourhood of 0.

For all \( \varphi \in \mathcal{S}' \) we write

\[ J_\epsilon \varphi = \rho_\epsilon * (\chi_\epsilon \varphi) \]

and then for all \( \varphi = (\varphi_\omega)_{\omega \in \Omega} \in \bigoplus_{\omega \in \Omega} \mathcal{S}' \)

\[ I_\epsilon \varphi = (J_\epsilon \varphi_\omega)_{\omega \in \Omega}. \]
The basic properties of these operators are listed in the following lemma.

**Lemma 5.1.** For all \( s, s' > \frac{1}{2} \) the following properties hold:

i. for all \( \epsilon > 0 \), \( J_\epsilon \) (resp. \( I_\epsilon \)) defines a nuclear operator of order 0 from \( H^s \) to \( H^{s'} \) (resp. from \( B^s \) to \( B^{s'} \));

ii. there is a constant \( C \) such that for all \( \epsilon \in [0, 1] \) we have \( \|J_\epsilon\|_{H^s \rightarrow H^{s'}} \leq C \) (resp. \( \|I_\epsilon\|_{B^s \rightarrow B^{s'}} \leq C \));

iii. for all \( \varphi \in H^s \) (resp. \( B^s \) \( ) \), \( J_\epsilon \varphi \) (resp. \( I_\epsilon \varphi \)) tends to \( \varphi \) in \( H^s \) (resp. \( B^s \)) as \( \epsilon \) tends to 0.

**Proof.** The first point is immediate since \( J_\epsilon \) factorizes through \( S \) which is a nuclear space (see for instance Remark 6 p.35 and Theorem 10 p.55 of the second part of [9]).

The norm in \( L(H^s, H^{s'}) \) of \( \varphi \in H^s \rightarrow \rho_\epsilon \ast \varphi \in H^{s'} \) is bounded by \( \|\rho_\epsilon\|_{L^\infty} \leq \|\rho_\epsilon\|_{L^1} = 1 \). We get a uniform bound on the norm in \( L(H^s, H^{s'}) \) of \( \varphi \in H^s \rightarrow \chi_\epsilon \varphi \) by a classical Leibniz inequality (for instance Corollary 4.2.2 p.205 of [18]).

The third point is a consequence of the second one and the fact that the convergence holds for functions in \( S \) by an argument of density.

Thus if \( s \in \mathbb{R} \) and \( A \) is a bounded operator from \( H^s \) to itself, \( A \circ J_\epsilon \) is a nuclear operator and since the Hilbert space \( H^s \) has the approximation property, we can set

\[
\text{tr}_\epsilon A = \text{tr} (A \circ J_\epsilon),
\]

and then the “flat trace” of \( A \) is defined as

\[
\text{tr}^\flat A = \lim_{\epsilon \to 0} \text{tr}_\epsilon A,
\]

provided the limit exists. Replacing \( J_\epsilon \) by \( I_\epsilon \) we get similar definitions for operators from \( B^s \) to itself. If we write a bounded operator \( A \) on \( B^s \) as a matrix \((A_{\omega,\omega'})_{\omega,\omega' \in \Omega}\) of bounded operators on \( H^s \), it can easily be shown that

\[
\text{tr}_\epsilon A = \sum_{\omega \in \Omega} \text{tr}_\epsilon A_{\omega,\omega}.
\]

Thus the flat trace of \( A \) is defined if the flat trace of \( A_{\omega,\omega} \) is defined for all \( \omega \in \Omega \), if so

\[
\text{tr}^\flat A = \sum_{\omega \in \Omega} \text{tr}^\flat A_{\omega,\omega}.
\]

From the third point of Lemma 5.1, the “flat trace” coincides with the usual trace for nuclear operators.

Now, if \( A : B^s \to B^s \) is such that for all \( m \in \mathbb{N}^* \) the flat trace of \( A^m \) is defined, set

\[
\det^\flat (I - zA) = \exp \left\{ -\sum_{n \geq 1} \frac{\tr^\flat (A^n)}{n} z^n \right\} = \sum_{n \geq 0} a_n z^n \in \mathbb{C}[[z]] \quad (25)
\]

that is \( \sum_{n \geq 0} a_n z^n \) is the formal power series recursively defined by

\[
a_0 = 1 \text{ and } a_n = -\frac{1}{n} \sum_{k=0}^{n-1} a_k \tr^\flat (A^{n-k}) \text{ for } n \geq 1. \quad (26)
\]

Thus if \( A \) and \( B \) are operators such that \( AB = BA = 0 \) and for all \( m \in \mathbb{N}^* \) the flat traces of \( A^m \) and \( B^m \) are defined, we have

\[
\det^\flat (I - z(A + B)) = \det^\flat (I - zA) \det^\flat (I - zB).
\]
First, we show that all the powers of the transfer operator \( \mathcal{K}_t \) have a flat trace and we compute it.

**Lemma 5.2.** For all \( m \in \mathbb{N}^* \), all \( t \in U \) and all \( \frac{3}{2} < s < r - 1 \) the flat trace of the operator \( \mathcal{K}_t^m : \mathcal{B}^s \to \mathcal{B}^s \) is defined and

\[
\text{tr}^\flat (\mathcal{K}_t^m) = \sum_{T_t^m x = x} \exp \left( g_{m,t}(x) \right) \frac{1}{1 - \left( (T_t^m)'(x) \right)}.
\]  

(27)

**Proof.** As in the proof of Proposition 2, write

\[
\mathcal{K}_t^m = \sum_{\omega \in \mathcal{I}} \mathcal{K}_{t,\omega}^m.
\]

Choose \( \mathcal{I} \in I \) and write \( \mathcal{K}_{t,\omega}^m \) as a matrix of operators \((A_{\omega,\omega',t})_{\omega,\omega' \in \Omega} \). For all \( \omega \in \Omega, \varphi \in H^s \) and \( x \in \mathbb{R} \) we can write thanks to 24

\[
(A_{\omega,\omega,t} \circ J_x) \varphi(x) = \int_{\mathbb{R}} h_\omega(x) \left( \chi_\omega,t e^{g_{m,t} \theta_\omega} \circ \pi \circ F_t(x) \rho_e (F_t(x) - y) \chi_e(y) \varphi(y) \right) dy,
\]

which expresses \( A_{\omega,\omega,t} \) as an integral of nuclear operators. Thus

\[
\text{tr}_e A_{\omega,\omega,t} = \int_{\mathbb{R}} h_\omega(x) \left( \chi_\omega,t e^{g_{m,t} \theta_\omega} \circ \pi \circ F_t(x) \rho_e (F_t(x) - y) \chi_e(y) \right) dx.
\]

Since \( F_t \) has its derivative bounded by \( \lambda^m < 1 \), the map \( x \mapsto F_t(x) - x \) is a diffeomorphism from \( \mathbb{R} \) to itself, let \( G \) be its inverse. Denote by \( x^* \) the unique fixed point of \( F_t \) and perform the change of variables \( u = x^* - F_t(x) + x \) to get

\[
\text{tr}_e A_{\omega,\omega,t} = \lim_{\epsilon \to 0} \int_{\mathbb{R}} h_\omega(x^*) \left( \chi_\omega,t e^{g_{m,t} \theta_\omega} \circ \pi (x^*) \right) dx^*.
\]

Recall that \( F_t \) is the inverse of a lift of \( T_t^m \) that extends \( \kappa_\omega \circ (T_t^m|_{W_{\omega,t}}) \circ \pi|_{\kappa_\omega(W_{\omega,t})} \), thus \( \pi(x^*) \) is a fixed point of \( T_t^m \). Since \( T_t^m \) induces an expanding diffeomorphism from \( W_{\omega,t} \) to \( T_t^m(W_{\omega,t}) \), it has at most one fixed point in \( W_{\omega,t} \). If \( y \) is such a fixed point and \( y \in V_\omega \) then \( \kappa_\omega(y) \) is a fixed point of \( F_t \), thus \( y = \pi(x^*) \) and

\[
\text{tr}^\flat A_{\omega,\omega,t} = \frac{\chi_\omega,t(y) e^{g_{m,t}(y) \theta_\omega(y)}}{1 - (T_t^m)'(y)}.
\]

Otherwise, \( h_\omega(x^*) = 0 \) or \( \chi_\omega,t \circ \pi(x^*) = 0 \) and \( \text{tr}^\flat A_{\omega,\omega,t} = 0 \). Finally, we always have

\[
\text{tr}^\flat A_{\omega,\omega,t} = \sum_{T_t^m(x) = x} \frac{\chi_\omega,t(x) e^{g_{m,t}(x) \theta_\omega(x)}}{1 - (T_t^m)'(x)}.
\]

Summing over \( \omega \in \Omega \) and then \( \mathcal{I} \in I \), we get

\[
\text{tr}^\flat (\mathcal{K}_t^m) = \sum_{T_t^m x = x} \frac{\exp \left( g_{m,t}(x) \right)}{1 - \left( (T_t^m)'(x) \right)}.
\]
Formula 27 of Lemma 5.2 implies that the flat determinant of $K_t$ is the dynamical determinant of Theorem 7.1 (and is given by 1 in the application):

$$\text{det}^b (I - zK_t) = \exp \left( - \sum_{n \geq 1} \frac{1}{n} \sum_{T^n x = x} \exp \left( g_{n,t} (x) \right) \left( (T^n_t)^\prime (x) \right)^{-1} z^n \right). \quad (28)$$

We want now to show that the product of “bounded” terms $(K_m^m)_k$ of the decompositions 18 of large enough powers of $K_t$ have no trace. That’s the point of Lemma 5.4 and will be essential in the proof of Theorem 7.1: we will only have to deal with traces of nuclear operators. To do so, we need first to state an abstract property of the flat trace. Notice that the convergence in weak operator topology is the convergence that appears in Lemma 3.2.

**Lemma 5.3.** If $s \in \mathbb{R}$ and $(u_k)_{k \geq 0}$ is a sequence of bounded operators on $H^s$ such that the series $\sum_{k \geq 0} u_k$ converges in the weak operator topology, then for all $\epsilon > 0$ we have

$$\text{tr}_\epsilon \left( \sum_{k=0}^{+\infty} u_k \right) = \sum_{k=0}^{+\infty} \text{tr}_\epsilon u_k.$$

Furthermore, the same is true replacing $H^s$ by $B^s$.

**Proof.** First, using Banach–Steinhauss Theorem twice, we find that there exists $M$ such that for all $n \in \mathbb{N}$, we have

$$\left\| \sum_{k=0}^{n} u_k \right\|_{H^s \to H^s} \leq M.$$

Then, write $J_\epsilon$ as a sum of rank one operators

$$J_\epsilon = \sum_{m \geq 0} l_m \otimes x_m$$

with $l_m \in (H^s)^\prime$, $x_m \in H^s$ and

$$\sum_{m \geq 0} \| l_m \|_{(H^s)^\prime} \| x_m \|_{H^s} < +\infty.$$

Thus for all $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n} \text{tr}_\epsilon u_k = \sum_{k=0}^{n} \sum_{m \geq 0} l_m (u_k (x_m)) = \sum_{m \geq 0} l_m \left( \left( \sum_{k=0}^{n} u_k \right) (x_m) \right).$$

For all $m \in \mathbb{N}$ and $n \in \mathbb{N}$ we have

$$\left| l_m \left( \left( \sum_{k=0}^{n} u_k \right) (x_m) \right) \right| \leq M \| l_m \|_{(H^s)^\prime} \| x_m \|_{H^s}.$$

Thus by dominated convergence and convergence in the weak operator topology:

$$\sum_{k=0}^{+\infty} \text{tr}_\epsilon u_k = \sum_{m \geq 0} \lim_{n \to +\infty} l_m \left( \left( \sum_{k=0}^{n} u_k \right) (x_m) \right).$$
\[
= \sum_{m \geq 0} l_m \left( \left( \sum_{k=0}^{+\infty} u_k \right) (x_m) \right) = \text{tr}_\epsilon \left( \sum_{k=0}^{+\infty} u_k \right).
\]

Lemma 5.4. There is an integer \( L \) such that if \( m_1, \ldots, m_J \) are integers greater than \( L \) then for all \( t \in K \)
\[
\text{tr}^b \left( \prod_{j=1}^J (K_t^m) \right) = 0.
\]

Proof. Write \( \prod_{j=1}^J (K_t^m) \) as a matrix \( (B_{\omega, \omega', t})_{\omega, \omega' \in \Omega} \). From the construction of the \( (K_t^m) \), it comes that, for all \( \omega \in \Omega \), the operator \( B_{\omega, \omega, t} \) can be written as a sum (in weak operator topology) of terms of the form
\[
\text{Op} \left( \psi_n \right) \mathcal{M}_1 \text{Op} \left( \psi_1 \right) \ldots \text{Op} \left( \psi_{n_J} \right) \mathcal{M}_J \text{Op} \left( \psi_J \right)
\]
(29)
with the \( \mathcal{M}_j \) as in §3 and
\[
2^{n_j} \leq \lambda^m \cdot 2^{l_j+6} \leq \lambda^L \cdot 2^{l_j+6}
\]
(30)
for all \( j \in \{1, \ldots, J\} \), which implies
\[
n_j \leq L \log_2 \lambda + l_j + 6.
\]

We shall show that, provided \( \epsilon \) is small enough and \( L \) large enough, the “epsilon trace” of all these operators is zero, which ends the proof with Lemma 5.3.

Let \( u \) be the operator defined by 29 composed by \( J_\epsilon \). If there is \( j \in \{1, \ldots, J-1\} \) such that \( \psi_1 \psi_{n_J} = 0 \) then \( u \) is zero and so is its trace. Otherwise, for all \( j \in \{1, \ldots, J-1\} \) we have
\[
l_j \leq n_{j+1} + 1,
\]
which leads with 30 to
\[
n_1 \leq J (L \log_2 \lambda + 1) + l_J + 6.
\]
Thus, provided \( L \) is large enough
\[
n_1 + 1 \leq l_J - 1.
\]
(31)

Suppose that \( \varphi \) is an eigenvector of \( u \) corresponding to a non-zero eigenvalue. Then \( \hat{\varphi} \) is supported in \( \text{supp} \psi_{n_1} \). But, provided \( \epsilon \) is small enough, \( \rho_\epsilon \) is supported in \([-1,1]\) and thus \( \overline{J_\epsilon} \varphi = \hat{\rho}_\epsilon (\rho_\epsilon \ast \hat{\varphi}) \) is supported in \([-2^{n_1} - 1; 2^{n_1} + 1]\) that doesn’t intersect \( \text{supp} \psi_j \) according to 31. Consequently, \( \varphi \) must be zero, which is absurd, and thus the spectrum of \( u \) is \( \{0\} \). Since \( u \) is nuclear of order 0 \( (J_\epsilon \) is), its trace is zero (see for example Corollary 4 page 18 of the second part of [9]).

6. Linear response a priori. In this section, we prove that the greatest eigenvalue of \( K_t \), the associated eigenprojection and the resolvant \( (z - K_t)^{-1} \) for \( z \) far enough from the spectrum have a regular dependence in \( t \). To do so we use methods introduced by Gouëzel, Keller and Liverani in [11] and [8] (see also Paragraph A.3 in [3] for a sum-up). Notice that the spaces \( \mathcal{B}^s \) are not the same as those used, for instance, in [3]. In particular, there is no compact injection of \( \mathcal{B}^{s'} \) in \( \mathcal{B}^s \) when \( s' < s \). However, the results of [11] and [8] do not require compact injection and thus we can prove that the spectrum of the transfer operator has the expected behaviour on the Banach spaces \( \mathcal{B}^s \).
First, we recall that the transfer operator has a spectral gap. We shall denote by $P_t$ the topological pressure associated with the dynamics $T_t$ (see §3.4 of [14] for a definition).

**Lemma 6.1.** For all $t \in K$ and all $\frac{3}{2} < s < r - 1$, the real number $e^{P_t(g_t)}$ is an eigenvalue of $K_t : H^s \to H^s$. Moreover, this eigenvalue is simple and there is $\eta < e^{P_t(g_t)}$ that does not depend of $s$ (but may depend of $t$) such that $e^{P_t(g_t)}$ is the only element of the spectrum of $K_t$ acting on $B^s$ of modulus larger than $\eta$.

*Proof.* From Proposition 2 and [10, Corollaire 1], we deduce that the essential spectral radius of $K_t$ acting on $B^s$ for any $\frac{3}{2} < s < r - 1$ is smaller than $\lambda e^{P_t(g_t)}$, that is the intersection of the spectrum of $K_t$ acting on $B^s$ with $\{ z \in \mathbb{C} : |z| > \lambda e^{P_t(g_t)} \}$ is made of eigenvalues of finite multiplicity. Next, note that the lemma is true while replacing $K_t$ by $\mathcal{L}_t$ and $B^s$ by $C^2 (\mathbb{R}, \mathbb{C})$ or $C^r (S^1, \mathbb{C})$ (see for instance Theorem 3.6 of [15]).

Now, if $\mu$ is a complex number of modulus strictly greater than $\lambda e^{P_t(g_t)}$, the map $S$ sends the generalized eigenspace of $K_t$ acting on $B^s$ associated to $\mu$ into the generalized eigenspace of $\mathcal{L}_t$ acting on $C^2 (S^1, \mathbb{C})$ associated to $\mu$. Moreover, the induced map is injective, indeed if $\varphi$ is a generalized eigenvector such that $S\varphi = 0$, since $K_t = P_t \mathcal{L}_t S$ we have $K_t \varphi = 0$. But since $\varphi$ is a generalized eigenvector there is $m \in \mathbb{N}$ such that $\mu^m \varphi = (\mu - \lambda e^{P_t(g_t)})^m \varphi = 0$ and thus $\varphi = 0$. Thus the dimensions of the generalized eigenspaces for $K_t$ are smaller than the dimensions of the corresponding generalized eigenspaces for $\mathcal{L}_t$ acting on $C^2 (S^1, \mathbb{C})$, which proves the second part of the lemma.

Reciprocally, we have to prove that $e^{P_t(g_t)}$ is indeed an eigenvalue for $K_t$. Let $\psi$ be an eigenvector for $\mathcal{L}_t$ acting on $C^r (S^1, \mathbb{C})$ associated to the eigenvalue $e^{P_t(g_t)}$. Then the coordinates of $P_t \psi$ are $C^r$ and compactly supported, thus $P_t \psi \in B^s$, and $P_t \psi$ is an eigenvector of $K_t$ for the eigenvalue $e^{P_t(g_t)}$ (the map $P$ is injective).

We need some technical estimates to apply the results from [11] and [8].

**Lemma 6.2.** Let $e^{P_0(m)} < R$. There is a constant $C > 0$ such that for all $t \in K$ sufficiently close to 0 and all $m \in \mathbb{N}^*$ we have

$$Z(t, m) = \inf_{\beta \text{ subcover of } \alpha} \sum_{\substack{V \in \beta \\text{ s.t.}}} \exp \left( \sup_{\mathbf{V} \in \beta} g_{m, t} \right) \leq C R^m.$$ 

*Proof.* We know that $e^{P_0(g_0)} = \inf_{m \in \mathbb{N}^*} Z(0, m)^{\frac{1}{m}}$. Thus there exists $m_0 \in \mathbb{N}^*$ such that $Z(0, m_0) < R^{m_0}$. Since $Z(\cdot, m_0)$ is upper semi-continuous, this inequality still holds replacing 0 by $t$ sufficiently close. For such a $t$, we can write for $m = q m_0 + r$:

$$Z(t, m) \leq Z(t, m_0)^q Z(t, r) \leq R^{q m_0} \sup_{k=0, \ldots, q-1} \| Z(\cdot, k) \|_{\infty, K} \leq C R^m.$$

**Lemma 6.3.** For each $\rho > \lambda e^{P_0(g_0)}$ there exists a neighbourhood $W$ of 0 in $K$ such that for all $s, s' \in \left[ \frac{3}{2}, r - 1 \right]$, there are some constants $C_1, C_2, M$ such that for all $m \in \mathbb{N}^*$, $t \in W$ and $\varphi \in B^s$ we have

$$\| K_t^m \varphi \|_{B^{s'}} \leq C_1 \rho^m \| \varphi \|_{B^s} + C_2 M^m \| \varphi \|_{B^{s'}}.$$

*Proof.* Choose $\rho' \in \left[ \lambda e^{P_0(g_0)}, \rho \right[. $ Applying Lemma 6.2 and recalling Proposition 2, we have for $t$ sufficiently close to 0, all $\sigma \in \{ s, s' \}$ and all $m \in \mathbb{N}^*$

$$\| (K_t^m)_{\sigma} \|_{\mathcal{L}(B^s, B^{s'})} \leq C_1 (\rho')^m.$$
for some constant $C_1$. Choose $L$ large enough so that $C_1 (\rho^L)^L \leq \rho^L$ and then write for $m = qL + r \in \mathbb{N}^*$ (with $0 \leq r < L$)
\[
K_m^t = ((\mathcal{K}_L^t)^q)_b \circ (\mathcal{K}_L^t)_c + \sum_{k=0}^{q-1} \mathcal{K}_L^t \circ (\mathcal{K}_L^t)_c \circ ((\mathcal{K}_L^t)_b)^{q-k-1} \circ (\mathcal{K}_L^t)_b .
\]
Thus if $\varphi \in B^s$ and $t$ is sufficiently close to 0, we have
\[
\|K_m^t \varphi\|_{B^r} \leq C \rho^m \|\varphi\|_{B^r} + \left[ \|K_t\|_{\mathcal{L}(B^r, B^s)} \|\mathcal{K}_t\|_{\mathcal{L}(B^s, B^r)} \right. \\
+ \sum_{k=0}^{q-1} \|K_t\|_{\mathcal{L}(B^r, B^s)} \|\mathcal{K}_t\|_{\mathcal{L}(B^s, B^r)} C_1 \rho^{(q-k-1)L+r} \|\varphi\|_{B^r}' .
\]
From which we get
\[
\|K_m^t \varphi\|_{B^r} \leq C_1 \rho^m \|\varphi\|_{B^r} + C_2 M^m \|\varphi\|_{B^r'}
\]
for some constants $C_2, M$.

From now on, we suppose that $K$ is a rectangle. Fix $\eta \in \] \lambda e^{p_b(0)} , e^{p_b(0)} [$ as in Lemma 6.1 for $t = 0$ and $0 < \delta < e^{p_b(0)} - \eta$. Set
\[
V_{\delta, \eta} = \left\{ z \in \mathbb{C} : |z| \leq \eta \text{ or } \left| z - e^{p_b(0)} \right| \leq \delta \right\}.
\]
We state a result of continuity and then a result of differentiability.

**Lemma 6.4.** If $N \geq 1$ then for all $\frac{5}{2} < s < r - 1$ and all $\epsilon > 0$, there is a neighbourhood $W$ of 0 in $K$ such that for all $t \in W$, the spectrum of $K_t$ acting on $B^s$ is contained in $V_{\delta, \eta}$. Moreover there exists a constant $C$ such that for all $t \in W$ and $z \in \mathbb{C} \setminus V_{\delta, \eta}$ we have
\[
\left\| (z - K_t)^{-1} \right\|_{\mathcal{L}(B^r, B^s)} \leq C
\]
and the map
\[
t \mapsto (z - K_t)^{-1} \in \mathcal{L}(B^s, B^{s-\epsilon})
\]
is continuous on $W$.

**Proof.** We want to apply Theorem 1 of [11] twice with $\| . \| = \| . \|_{B^r}$ and $\| . \| = \| . \|_{B^{s-\epsilon}}$ and $\delta$ replaced by $\frac{\delta}{2}$. This theorem is stated in a one-dimensional setting but in view of the dependences in the data of the constants appearing in the results (that are explicitly given), it may be applied here. Thanks to Proposition 2 and Lemma 6.3, there is a neighbourhood $W'$ of 0 in $K$ such that the conditions (2) and (3) of [11] are fulfilled by the family $(K_t)_{t \in W'}$ (with $\alpha = \rho < \eta$). Since $t \mapsto K_t \in \mathcal{L}(B^s, B^{s-1-\epsilon})$ is $C^1$ on a neighbourhood of $K$, we get by interpolating between $B^s$ and $B^{s-1-\epsilon}$ (that is applying the inequality 4)
\[
\|K_{t_1} - K_{t_0}\|_{\mathcal{L}(B^s, B^{s-\epsilon})} \leq C |t_1 - t_0| \beta
\]
for all $t_0, t_1 \in K$ and some constants $C$ and $\beta > 0$. Thus the condition (5) of [11] is fulfilled on $W'$. As pointed out in Remark 6 of [11], the condition (4) is unnecessary here since $\mathbb{C} \setminus V_{\frac{\delta}{2}, \eta}$ is connected.

Thus applying this theorem and the remark, we find a neighbourhood $W$ of 0 in $t$ such that for all $t \in W$ the spectrum of $K_t$ acting on $B^s$ is contained in $V_{\frac{\delta}{2}, \eta}$. Now, we may apply the theorem from [11] again, taking each point of $W$ as the origin, to end the proof of the lemma.
Lemma 6.5. For all integers \( 0 \leq k \leq N - 1 \), all real \( \frac{r}{2} + k < s < r - 1 \), and all \( \epsilon > 0 \), there is a neighbourhood \( W \) of 0 in \( K \) such that for all \( z \in \mathbb{C} \setminus V_{\delta, \eta} \) the map

\[
t \mapsto (z - K_t)^{-1} \in \mathcal{L}(B^s, B^{s-k-\epsilon})
\]

is \( C^k \) on \( W \). Moreover, for all multi-indices \( \alpha \) with \( |\alpha| \leq k \) we have

\[
\frac{\partial^\alpha}{\partial t^\alpha} (z - K_t)^{-1} = \sum_{\alpha_1, \ldots, \alpha_j \neq 0} (z - K_t)^{-1} \left( \frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} K_t \right) (z - K_t)^{-1} \ldots \left( \frac{\partial^{\alpha_j}}{\partial t^{\alpha_j}} K_t \right) (z - K_t)^{-1}.
\]

Proof. The case \( k = 0 \) has been dealt with in Lemma 6.4.

The case \( k = 1 \) is a consequence of Theorem A.4 of [3] with the spaces \( B^0, B^1, B^2 \) of [3] being here respectively \( B^s, B^{s-1-\epsilon}, B^{s-2-\epsilon} \) (for sufficiently small \( \epsilon \)). There is a neighborhood \( W \) of 0 on which the hypotheses (A.2)-(A.7) are fulfilled (even when replacing 0 by another element \( t_0 \) of \( W \)): (A.2) and (A.3) are consequences of Proposition 2 and Lemma 6.3, (A.4) is contained in Lemma 6.3, (A.5) is implied by Taylor formula since \( t \mapsto K_t \in \mathcal{L}(B^s, B^{s-1-\epsilon}) \) is \( C^1 \) (as pointed out in Remark 2), (A.6) and (A.7) are a consequence of the fact that \( t \mapsto K_t \in \mathcal{L}(B^s, B^{s-2-\epsilon}) \) is \( C^2 \). Applying this theorem in each direction and interpolating between \( B^{s-1-\epsilon} \) and \( B^{s-2-\epsilon} \), it comes that the map defined by (33) admits partial derivatives given by (34) on a neighborhood of 0. These partial derivatives are continuous on a neighborhood of 0 as a consequence of Lemma 6.4.

The end of the proof is an induction. We show how to get \( k = 2 \) from \( k = 1 \). Notice that we cannot calculate the second partial derivatives by differentiating the partial derivatives as products. However, the expected formula will stand. Indeed, fix \( i, j \in \{1, \ldots, d\} \) and, setting \( A_t = (z - K_t)^{-1} \), write the growth rate

\[
\frac{A_{t+h^e_i} \partial}{\partial t_j} K_{t+h^e_j} A_{t+h^e_j} - A_t \partial K_t A_t
\]

\[
- \left( \frac{\partial A_t}{\partial t_j} A_t \partial K_t A_t + A_t \frac{\partial^2}{\partial t_i \partial t_j} K_t A_t + A_t \partial K_t \partial A_t \right),
\]

where \( e_j \) is the \( j \)-th vector of the canonical basis of \( \mathbb{R}^d \), as

\[
\left( \frac{A_{t+h^e_i} - A_t}{h} - \frac{\partial A_t}{\partial t_j} A_t \right) \partial K_{t+h^e_j} A_{t+h^e_j}
\]

\[
+ A_t \left( \frac{\partial}{\partial t_j} K_{t+h^e_j} - \frac{\partial}{\partial t_j} K_t \right) A_{t+h^e_j}
\]

\[
+ A_t \frac{\partial}{\partial t_j} K_t \left( \frac{A_{t+h^e_j} - A_t}{h} - \frac{\partial A_t}{\partial t_j} A_t \right) + \frac{\partial}{\partial t_j} A_t \left( \frac{\partial}{\partial t_i} K_{t+h^e_j} A_{t+h^e_j} - \frac{\partial}{\partial t_i} K_t A_t \right)
\]

\[
+ A_t \frac{\partial^2}{\partial t_i \partial t_j} K_t (A_t - A_{t+h^e_j}).
\]

The norm of this expression as an operator from \( B^s \) to \( B^{s-2-\epsilon} \) is smaller than

\[
\left\| \frac{A_{t+h^e_j} - A_t}{h} - \frac{\partial}{\partial t_j} A_t \right\|_{\mathcal{L}(B^s, B^{s-2-\epsilon})} \left\| \frac{\partial}{\partial t_i} K_{t+h^e_j} \right\|_{\mathcal{L}(B^s, B^{s-1})} \left\| A_{t+h^e_j} \right\|_{\mathcal{L}(B^s, B^s)}
\]
+ \| A_t \|_{\mathcal{L}(B^{s'},B^{s'})} \left\| \frac{\partial}{\partial t} K_{t+h\epsilon_j} - \frac{\partial}{\partial t} K_t \right\|_h \left\| A_t + h\epsilon_j \right\|_{\mathcal{L}(B^s,B^{s'})} \\
+ \| A_t \|_{\mathcal{L}(B^{s'},B^{s'})} \left\| \frac{\partial}{\partial t} K_t \right\|_{\mathcal{L}(B^s,B^{s'})} \left\| \frac{A_t + h\epsilon_j - A_t}{h} \right\|_{\mathcal{L}(B^s,B^{s'})} - \left\| \frac{\partial}{\partial t} A_t \right\|_{\mathcal{L}(B^s,B^{s'})} \\
+ \left\| \frac{\partial}{\partial t} A_t \right\|_{\mathcal{L}(B^s,B^{s'})} \left\| \frac{\partial}{\partial t} K_{t+h\epsilon_j} \right\|_{\mathcal{L}(B^s,B^{s'})} \left\| A_t + h\epsilon_j - \frac{\partial}{\partial t} A_t \right\|_{\mathcal{L}(B^s,B^{s'})} \\
+ \left\| A_t \frac{\partial^2}{\partial t \partial t_j} K_t \right\|_{\mathcal{L}(B^s,B^{s'})} \left\| A_t - A_t + h\epsilon_j \right\|_{\mathcal{L}(B^{s'},B^{s'})},
\]

where \( s_1 = s - 1 - \frac{\epsilon}{2}, \) \( s_2 = s - 2 - \epsilon \) and \( s' = s - \frac{\epsilon}{2}. \) From the previous cases, we get that, provided \( t \) is in some neighbourhood of 0, this expression tends to 0 as \( h \) tends to 0. Consequently, \( A_t \) has the second partial derivatives announced. Since they are continuous, the result is proved for \( k = 2. \) The idea of the proof in the general case is the same, up to more notational issues. \( \square \)

We also need some information about the greatest eigenvalue of \( K_t \) and the associated spectral projection.

**Lemma 6.6.** For all \( \frac{5}{2} < s < r - 1, \) there is a neighbourhood \( W \) of 0 in \( K \) such that for all \( t \in W, e^{\rho_t(g_t)} \in \mathbb{D}(e^{\rho_0(g_0)}, \delta) \) and \( e^{\rho_t(g_t)} \) is the only element of the spectrum of \( K_t \) that has modulus greater than \( \eta. \)

**Proof.** Apply Corollary 1 of [11] together with Lemma 6.1. \( \square \)

**Lemma 6.7.** For all \( 0 \leq k \leq N - 1, \) all \( \frac{5}{2} + k < s < r - 1 \) and all \( \frac{5}{2} < s' < r - 1 - k, \) there is a neighbourhood \( W \) of 0 in \( K \) such that for all \( t \in W \) the spectrum of \( K_t \) is contained in \( V_{\delta,0} \) and setting for all \( t \in W \)

\[ \Pi_t = \frac{1}{2\pi i} \int_{\Gamma} (z - K_t)^{-1} \mathrm{d}z, \]

where \( \Gamma \) is a circle of center \( e^{\rho_0(g_0)} \) and of radius slightly larger than \( \delta, \) the map

\[ t \mapsto \Pi_t \in \mathcal{L}_{\text{nuc}} \left( B^s, B^{s'} \right) \]

is \( C^k \) on \( W. \)

**Proof.** The first point is only a reminder of Lemma 6.4. Recall that \( \Pi_t \) is a spectral projection. Choose \( \epsilon > 0 \) sufficiently small and \( \varphi_0 \in B^{s'+k+\epsilon} \cap B^s \) such that \( \Pi_0 \varphi_0 \neq 0 \) (for instance take an eigenvector of \( K_0 \) for the eigenvalue \( e^{\rho_0(g_0)} \)). Then choose a linear form \( l \) on \( B^s \) such that \( l(\Pi_0 \varphi_0) \neq 0. \) Set \( \rho_t = \Pi_t \varphi_0 \) and notice that Lemma 6.5 and dominated convergence imply that

\[ t \mapsto \rho_t \in B^{s'} \]

is \( C^k \) on a neighbourhood of 0. In particular \( \rho_t \neq 0 \) for \( t \) close enough of 0. Applying Lemmas 6.1 and 6.6, the spectral projector \( \Pi_t \) has rank one when acting on \( B^s, \) providing \( t \) is small enough, and can thus be written

\[ \Pi_t = m_t \otimes \rho_t \]

with \( m_t \in (B^s)' \). But for \( t \) sufficiently close to 0 so that \( l(\rho_t) \neq 0, \) \( m_t \) may be written as

\[ m_t = \frac{l \circ \Pi_t}{l(\rho_t)} \]

(35)
in which \( \Pi_t \) is seen as an operator from \( B^s \) to \( B^0 \), and thus has a \( C^k \) dependence in \( t \). Consequently, \( m_t \) has a \( C_k \) dependence in \( t \) on a neighbourhood of 0. Hence, the formula 35 ends the proof. \( \square \)

**Remark 3.** The fact that \( s' \) may be chosen greater than \( s \) (for appropriate values of \( k \) and \( r \)) in Lemma 6.7 will be crucial in the proof of Theorem 7.1. However, the proof of this fact heavily relies on the fact that \( \Pi_t \) has rank one. Consequently, it seems that the method presented here does not allow to make \( R \) as large as we could expect (considering for instance [3, Theorem 3.5]), because of the eventual presence of non-simple eigenvalues.

**Lemma 6.8.** If \( r \geq 4 \) and \( k = \min (N-1, r-4) \), then the map \( t \mapsto e^{P_t(g_t)} \) is \( C^k \) on \( U \).

**Proof.** Choose \( s \in ]\frac{1}{2} + k, r-1[ \), \( \varphi \in B^s \) and \( l \in (B^0)' \) such that \( l(\Pi_0 \varphi) \neq 0 \), write

\[
e^{P_t(g_t)} = \frac{l(\Pi_t \varphi)}{l(\Pi_0 \varphi)} = \frac{l \left( \int_{t} z (z - \kappa_t)^{-1} \varphi dz \right)}{l \left( \int_{t} (z - \kappa_t)^{-1} \varphi dz \right)}
\]

and apply Lemma 6.5 (of course, one can work the same way on the neighbourhood of any point of \( U \)). \( \square \)

**7. Regularity of dynamical determinants.** We are now ready to prove our main result, which states roughly that the dynamical determinant 28 depends regularly on the parameter \( t \). The idea behind the proof is simple. We rely on a proof of the holomorphic extension of the dynamical determinant to a disc of radius \( R > e^{-P_0(g_0)} \) (the first proof given in the paragraph 3.3.1 of [3]). The main issue is that differentiating the “bounded” term in the decomposition of \( \kappa_t \) given in Proposition 2 imposes to lose regularity on the level of the Sobolev spaces we are working with. This problem is dealt with by compensating this loss of regularity by the smoothing effect of the second term of this decomposition.

**Theorem 7.1.** Let \( U \) be an open set of \( \mathbb{R}^D \) with \( 0 \in U \). Let \( r \geq 4 \) be an integer and \( N \in \mathbb{N}^* \). Let \( t \mapsto T_t \in C^{r+1}(S^1, S^1) \) be a \( C^N \) function on \( U \) whose values are expanding maps with uniform expansion constant \( \lambda^{-1} > 1 \) (that is, 15 holds). Let \( t \mapsto g_t \in C^r(S^1, \mathbb{R}) \) be a \( C^N \) function on \( U \). Set \( k = \min \left( \left[ \frac{r}{2} - \frac{7}{4} \right], r-4, N-1 \right) \). Then there are \( R > e^{-P_0(g_0)} \) and a neighbourhood \( W \) of 0 in \( U \) such that:

i. for all \( t \in W \), the power series

\[
\sum_{n \geq 1} \frac{1}{n} \sum_{T_t^n x = x} \exp \left( g_{n,t} (x) \right) \frac{z^n}{1 - \left( (T_t^n)' (x) \right)^{-1} z^n},
\]

where \( g_{n,t} \) is defined by 16, has a non-zero convergence radius;

ii. for all \( t \in W \) the map

\[
z \mapsto \exp \left( - \sum_{n \geq 1} \frac{1}{n} \sum_{T_t^n x = x} \frac{\exp \left( g_{n,t} (x) \right)}{1 - \left( (T_t^n)' (x) \right)^{-1} z^n} \right)
\]

extends on \( \mathbb{D} (0, R) \) to a holomorphic function \( z \mapsto d(z, t) \);

iii. for all \( t \in W \), the real number \( e^{-P_t(g_t)} \) is the unique zero of \( d(\cdot, t) \) and it is simple;

iv. the map \( (z, t) \mapsto d(z, t) \) is \( C^k \) on \( \mathbb{D} (0, R) \times W \).
Moreover, since $\rho = 1$, there is for all $t$ we may choose a neighbourhood $W$ of 0 in $K$ such that there is a constant $C$ such that

1. for all $t \in W$, the spectrum of $K_t$ acting on $\mathcal{B}^{s_0}$, on $\mathcal{B}^{s_1}$ and on $\mathcal{B}^{s_0+\alpha}$ is contained in $V_{\delta,\eta}$ (see Lemma 6.4 and notice that $s_0, s_1$ and $s_0 + \alpha$ are between $\frac{5}{2}$ and $r - 1 = 5$);
2. for all $t \in W$ and $z \in \mathbb{C} \setminus V_{\delta,\eta}$:
   \[ \| (z - K_t)^{-1} \|_{\mathcal{B}^{s_0} \to \mathcal{B}^{s_0}} \leq C \]
   (see Lemma 6.4)
3. for all $z \in \mathbb{C} \setminus V_{\delta,\eta}$ the map
   \[ t \mapsto (z - K_t)^{-1} \in \mathcal{L}(\mathcal{B}^{s_0+\alpha}, \mathcal{B}^{s_0}) \]
   is continuous on $W$ with $C^0$ norm bounded by $C$ (see Lemma 6.4);
4. for all $z \in \mathbb{C} \setminus V_{\delta,\eta}$ the map
   \[ t \mapsto (z - K_t)^{-1} \in \mathcal{L}(\mathcal{B}^{s_1}, \mathcal{B}^{s_0}) \]
   is $C^1$ on $W$ with $C^1$ norm bounded by $C$ (see Lemma 6.5 and notice that $s_1 - s_0 > k = 1$ and $s_1 > \frac{5}{2} + k = \frac{7}{2}$);
5. for all $t \in W$ the spectral projection $\Pi_t$ has rank one (see Lemma 6.1 and 6.6);
6. the map
   \[ t \mapsto \Pi_t \in \mathcal{L}_{\text{nuc}}(\mathcal{B}^{s_1-1}, \mathcal{B}^{s_1}) \]
   is continuous on $W$ with $C^0$ norm bounded by $C$ (see Lemma 6.7 and notice that $\frac{5}{2} < s_1 < s_1 < r - 1 = 5$);
7. the map
   \[ t \mapsto \Pi_t \in \mathcal{L}_{\text{nuc}}(\mathcal{B}^{s_0-\alpha}, \mathcal{B}^{s_1+\alpha}) \]
   is $C^1$ on $W$ with $C^1$ norm bounded by $C$ (see Lemma 6.7 and notice that $\frac{5}{2} + 1 = \frac{7}{2} < s_0 - \alpha < s_0 + \alpha < r - 1 - 1 = 4$);
8. there is $\rho_0 \in \left\{ \lambda : P_{1}(g_0), e^{P_{1}(g_0)} \right\}$ such that for all $t \in W$, $m \in \mathbb{N}^*$ and $l \in \{-1, 0\}$
   \[ \| (K^m_t)_{b} \|_{\mathcal{L}(\mathcal{B}^{s_0}, \mathcal{B}^{s_0})} \leq C \rho_0^m \]
   (see Proposition 2 and Lemma 6.2 and notice that $\frac{1}{2} < s_{-1} < s_0 < r = 6$).

Moreover, since $W \subseteq K$, Proposition 2 implies that the following properties hold (notice in particular that in Proposition 2 the same $K$ can be used for different values of $s$ and $m$):
9. for all $0 < \epsilon \leq \alpha$ and $m \in \mathbb{N}^*$ the maps
   \[ t \mapsto (K^m_t)_{b} \in \mathcal{L}(\mathcal{B}^{s_0}, \mathcal{B}^{s_0-\epsilon}) \]
   and
   \[ t \mapsto (K^m_t)_{b} \in \mathcal{L}(\mathcal{B}^{s_1+\epsilon}, \mathcal{B}^{s_1-1}) \]
   are continuous on $W$ (notice that $s_{-1} > \frac{3}{2}$);
10. for all $m \in \mathbb{N}^*$ the map
    \[ t \mapsto (K^m_t)_{b} \in \mathcal{L}(\mathcal{B}^{s_0-\alpha}, \mathcal{B}^{s_1+\alpha}) \]
    is $C^1$ on $W$ (notice that $s_0 - \alpha > \frac{5}{2}$);
11. for all $m \in \mathbb{N}^*$ the map

$$t \mapsto (\mathcal{K}^m_t)_{cc} \in \mathcal{L}_{nuc}(\mathcal{B}^{s-1}, \mathcal{B}^{s_1})$$

is continuous on $W$ (notice that $\frac{1}{2} < s_0$ and $s_1 < r - 1$);

12. for all $m \in \mathbb{N}^*$ the map

$$t \mapsto (\mathcal{K}^m_t)_{cc} \in \mathcal{L}_{nuc}(\mathcal{B}^{s_0-\alpha}, \mathcal{B}^{s_0+\alpha})$$

is $C^1$ on $W$ (notice that $s_0 - \alpha > \frac{s}{2}$ and $s_0 + \alpha < r - 2$).

Then for all $t \in W$ write $\mathcal{K}_{0,t} = \Pi_t \mathcal{K}_t = e^{P_t(\mathcal{g}_t)} \Pi_t$ and $\mathcal{K}_{1,t} = (Id - \Pi_t) \mathcal{K}_t$, where $\mathcal{K}_t$ is seen as acting on $\mathcal{B}^{s_0}$, then we have

$$\mathcal{K}_t = \mathcal{K}_{0,t} + \mathcal{K}_{1,t}$$

and for all $m \in \mathbb{N}^*$ the flat traces of $\mathcal{K}^m_t$ and of $\mathcal{K}^m_{0,t}$ (which has rank one) are well-defined and thus the flat trace of $\mathcal{K}^m_t$ too. Consequently, we have

$$d(z, t) = \det^b(I - z \mathcal{K}_t) = \det^b(I - z \mathcal{K}_{0,t}) \det^b(I - z \mathcal{K}_{1,t})$$

$$= (1 - z e^{P_t(\mathcal{g}_t)}) \det^b(I - z \mathcal{K}_{1,t})$$

as formal power series. Then set, for all $m \in \mathbb{N}^*$ and $t \in W$, $h_m(t) = \text{tr}^b(\mathcal{K}^m_t)$. We want to show that the $C^1$ norm of $h_m$ is bounded by $C' \rho^m$ for some constants $C' > 0$ and $\rho \leq e^{P_t(\mathcal{g}_t)}$. To do that, write for all $m \in \mathbb{N}^*$ and all $t \in W$

$$\mathcal{K}^m_{1,t} = (\mathcal{K}^m_t)_{cc} + (\mathcal{K}^m_{1,t})_{cc}$$

where $k = \mathcal{K}^m_{1,t} - (\mathcal{K}^m_t)_{cc} = \mathcal{K}^m_{1,t}-(\mathcal{K}^m_t)_{cc}$.

Notice that properties 6, 7, 11 and 12 together with Lemma 6.8 imply that

13. for all $m \in \mathbb{N}^*$ the maps

$$t \mapsto (\mathcal{K}^m_t)_{cc} \in \mathcal{L}_{nuc}(\mathcal{B}^{s-1}, \mathcal{B}^{s_1})$$

and

$$t \mapsto (\mathcal{K}^m_{1,t})_{cc} \in \mathcal{L}_{nuc}(\mathcal{B}^{s_0-\alpha}, \mathcal{B}^{s_0+\alpha})$$

are respectively continuous and $C^1$ on $W$.

Now choose $\rho_1 \in [\rho_0, e^{P_t(\mathcal{g}_t)}]$ and an integer $L$ large enough so that $C' \rho_1^L \leq \rho_2^L$ and Lemma 5.4 holds for $s = s_0$. Then if $m \geq L$ write $m = qL + r$ with $L \leq r < 2L$ and then

$$\mathcal{K}^m_{1,t} = (\mathcal{K}^L_{1,t})_{cc} + (\mathcal{K}^L_{1,t})_{cc} + \sum_{k=0}^{q-1} \mathcal{K}^{kL}_{1,t} (\mathcal{K}^L_{1,t})_{cc} (\mathcal{K}^L_{1,t})_{cc} (\mathcal{K}^L_{1,t})_{cc} (\mathcal{K}^L_{1,t})_{cc}.$$

This implies with Lemma 5.4

$$h_m(t) = \text{tr}(\mathcal{K}^{qL}_{1,t} (\mathcal{K}^L_{1,t})_{cc}) + \sum_{k=0}^{q-1} \text{tr}(\mathcal{K}^{kL}_{1,t} (\mathcal{K}^L_{1,t})_{cc} (\mathcal{K}^L_{1,t})_{cc} (\mathcal{K}^L_{1,t})_{cc} (\mathcal{K}^L_{1,t})_{cc}).$$

As in the proof of Lemma 6.5, one may prove using in particular properties 3, 4, 9, 10 and 13 above that the nuclear operator from $\mathcal{B}^{s_0}$ to itself $\mathcal{K}^{qL}_{1,t} (\mathcal{K}^L_{1,t})_{cc}$ has a $C^1$ dependence in $t$ (the constant $\alpha$ has only been introduced to do so by making certain operators continuous functions of $t$ by losing some regularity, it will not appear in the remainder of the proof) with for $i \in \{1, \ldots, d\}$:

$$\left\| \frac{\partial}{\partial t_i} \left( \mathcal{K}^{qL}_{1,t} (\mathcal{K}^L_{1,t})_{cc} \right) \right\|_{\mathcal{L}_{nuc}(\mathcal{B}^{s_0}, \mathcal{B}^{s_0})}.$$
\[ \left\| \frac{\partial}{\partial t_i} \left( K_{1,t}^{qL} \right) (K_{t,t}^{r})_{cc} + K_{1,t}^{qL} \frac{\partial}{\partial t_i} \left( (K_{t,t}^{r})_{cc} \right) \right\|_{\mathcal{L}_{nu,c}(\mathcal{B}^{(0,0)})} \]
\[ \leq \frac{1}{2\pi} \left( \int_{\gamma} z^{qL} \frac{\partial}{\partial t_i} \left( (z - K_t)^{-1} \right) dz \right) \left\| (K_{t,t}^{r})_{cc} \right\|_{\mathcal{L}_{nu,c}(\mathcal{B}^{(0,0)})} \]
\[ + \frac{1}{2\pi} \left( \int_{\gamma} z^{qL} (z - K_t)^{-1} dz \right) \frac{\partial}{\partial t_i} \left( (K_{t,t}^{r})_{cc} \right) \left\| (K_{t,t}^{r})_{cc} \right\|_{\mathcal{L}_{nu,c}(\mathcal{B}^{(0,0)})} \]
\[ \leq C' \rho_2^m, \]
where \( \gamma \) is a circle of center 0 and of radius \( \rho_2 \) slightly greater than \( \eta \) and \( C' \) is some constant. If \( k \in \{0, \ldots, q - 1\} \) we can write \( \frac{\partial}{\partial t_i} \left( K_{1,t}^{kL} \right) (K_{t,t}^{L})_{cc} (K_{t,t}^{L})_{q-k-1} (K_{t,t}^{r})_{b} \) in the same way as
\[ \frac{1}{2\pi} \left( \int_{\gamma} z^{kL} \frac{\partial}{\partial t_i} \left( (z - K_t)^{-1} \right) dz \right) \left( K_{t,t}^{L} \right)_{cc} (K_{t,t}^{L})_{q-k-1} (K_{t,t}^{r})_{b} \]
\[ + \frac{1}{2\pi} \left( \int_{\gamma} z^{kL} (z - K_t)^{-1} dz \right) \frac{\partial}{\partial t_i} \left( (K_{t,t}^{L})_{cc} \right) (K_{t,t}^{L})_{q-k-1} (K_{t,t}^{r})_{b} \]
\[ + \sum_{j=0}^{q-k-1} \frac{1}{2\pi} \left( \int_{\gamma} z^{kL} (z - K_t)^{-1} dz \right) \left( K_{t,t}^{L} \right)_{cc} (K_{t,t}^{L})_{j} \frac{\partial}{\partial t_i} \left( (K_{t,t}^{L})_{b} \right) (K_{t,t}^{L})_{q-k-j-2} (K_{t,t}^{r})_{b} \]
\[ + K_{1,t}^{kL} \left( (K_{t,t}^{L})_{cc} (K_{t,t}^{L})_{q-k-1} \right) \frac{\partial}{\partial t_i} \left( (K_{t,t}^{r})_{b} \right) \]
and thus
\[ \left\| \frac{\partial}{\partial t_i} \left( K_{1,t}^{kL} \right) (K_{t,t}^{L})_{cc} (K_{t,t}^{L})_{q-k-1} (K_{t,t}^{r})_{b} \right\|_{\mathcal{L}_{nu,c}(\mathcal{B}^{(0,0)})} \]
is smaller than
\[ \frac{1}{2\pi} \left( \int_{\gamma} z^{kL} \frac{\partial}{\partial t_i} \left( (z - K_t)^{-1} \right) dz \right) \left( K_{t,t}^{L} \right)_{cc} \left( K_{t,t}^{L} \right)_{q-k-1} \left( K_{t,t}^{r} \right)_b \]
\[ \times \left\| (K_{t,t}^{L})_b \right\|_{\mathcal{L}(\mathcal{B}^{(0,0)}, \mathcal{B}^{(0,0)})} \left\| (K_{t,t}^{r})_b \right\|_{\mathcal{L}(\mathcal{B}^{(0,0)}, \mathcal{B}^{(0,0)})} \]
\[ + \frac{1}{2\pi} \left( \int_{\gamma} z^{kL} (z - K_t)^{-1} dz \right) \frac{\partial}{\partial t_i} \left( (K_{t,t}^{L})_{cc} \right) \left( K_{t,t}^{L} \right)_{q-k-1} \left( K_{t,t}^{r} \right)_b \]
\[ \times \left\| (K_{t,t}^{L})_b \right\|_{\mathcal{L}(\mathcal{B}^{(0,0)}, \mathcal{B}^{(0,0)})} \left\| (K_{t,t}^{r})_b \right\|_{\mathcal{L}(\mathcal{B}^{(0,0)}, \mathcal{B}^{(0,0)})} \]
\[ + \sum_{j=0}^{q-k-1} \frac{1}{2\pi} \left( \int_{\gamma} z^{kL} (z - K_t)^{-1} dz \right) \left( K_{t,t}^{L} \right)_{cc} \left( K_{t,t}^{L} \right)_{q-k-1} \left( K_{t,t}^{r} \right)_b \]
\[ \times \left\| (K_{t,t}^{L})_b \right\|_{\mathcal{L}(\mathcal{B}^{(0,0)}, \mathcal{B}^{(0,0)})} \left( K_{t,t}^{L} \right)_b \left( K_{t,t}^{r} \right)_b \]
\[ \times \left( K_{t,t}^{r} \right)_b \left( K_{t,t}^{r} \right)_b \]
\[ + \frac{1}{2\pi} \left( \int_{\gamma} z^{kL} (z - K_t)^{-1} dz \right) \left( K_{t,t}^{L} \right)_{cc} \left( K_{t,t}^{L} \right)_{q-k-1} \left( K_{t,t}^{r} \right)_b \]
\[ \times \left\| (K_{t,t}^{L})_b \right\|_{\mathcal{L}(\mathcal{B}^{(0,0)}, \mathcal{B}^{(0,0)})} \left( K_{t,t}^{L} \right)_b \left( K_{t,t}^{r} \right)_b \]
\[ \times \left( K_{t,t}^{r} \right)_b \left( K_{t,t}^{r} \right)_b \]
\[ + \frac{1}{2\pi} \left( \int_{\gamma} z^{kL} (z - K_t)^{-1} dz \right) \left( K_{t,t}^{L} \right)_{cc} \left( K_{t,t}^{L} \right)_{q-k-1} \left( K_{t,t}^{r} \right)_b \]
5. for all $t \leq 6$. for all 1
6. the map $z$
7. for all $r > 7$
t for $54$ MALO JÉZÉQUEL
8. there is $\rho \in \{ -k, \ldots, k \}$
9. Let us give a hint on the way to write the proof for general $C$
is $C$ is constant on $W$
is $C$
are between 5 and $C$
is $C$
are small enough;
10. for all $t \in W$ and $z \in \mathbb{C} \setminus V_{\delta, \eta}$
\[ \left\| (z - K_\delta)^{-1} \right\|_{B^s \to B^0} \leq C \]
(see Lemma 6.4 and notice that $\frac{k}{2} < s_0 < r - 1$ provided $\delta$ is small enough)
3. for all $z \in \mathbb{C} \setminus V_{\delta, \eta}$ the map
\[ t \mapsto (z - K_\delta)^{-1} \in \mathcal{L}(B^{s_0 + \alpha}, B^{s_0}) \]
is continuous on $W$ with $C^0$ norm bounded by $C$ (see Lemma 6.4);
4. for all $z \in \mathbb{C} \setminus V_{\delta, \eta}$ and $1 \leq l \leq k$ the map
\[ t \mapsto (z - K_\delta)^{-1} \in \mathcal{L}(B^{s_l}, B^{s_0}) \]
is $C^l$ on $W$ with $C^l$ norm bounded by $C$ (see Lemma 6.5 and notice that $s_l - s_0 > l$,
$s_l > \frac{5}{2} + l$ and $s_l \leq s_k = \frac{5}{2} + 2k + \delta + k\beta < r - 1$ provided $\delta$ and $\beta$ are small enough);
5. for all $t \in W$ the spectral projection $\Pi_t$ has rank one (see Lemma 6.1 and 6.6);
6. for all $1 \leq l \leq k$ the map
\[ t \mapsto \Pi_t \in \mathcal{L}_{\text{nuc}}(B^{s-l}, B^{s_l}) \]
is $C^{k-l}$ on $W$ with $C^{k-l}$ norm bounded by $C$ (see Lemma 6.7 and notice that $\frac{5}{2} + k - l < s_{-l} < s_l < r - 1 - (k - l) = 5$ provided $\beta < \frac{1}{k}$ and $\delta$ is small enough);
7. the map
\[ t \mapsto \Pi_t \in \mathcal{L}_{\text{nuc}}(B^{s_0 - \alpha}, B^{s_0 + \alpha}) \]
is $C^k$ on $W$ with $C^k$ norm bounded by $C$ (see Lemma 6.7 and notice that $\frac{5}{2} + k < s_0 - \alpha < s_0 + \alpha < r - 1 - k$ provided $\alpha$ and $\delta$ are small enough);
8. there is $\rho_0 \in \{ \lambda e^{P_0(\varphi_0)}, e^{P_0(\varphi_0)} \}$ such that for all $t \in W$, $m \in \mathbb{N}^*$ and $l \in \{ -k, \ldots, 0 \}$
\[ \|(K_\delta^{-1})_t\|_{\mathcal{L}(B^{s_l}, B^{s_l})} \leq C\rho_0^m \]
(see Proposition 2 and Lemma 6.2 and notice that $\frac{1}{2} < s_{-k} < s_0 < r)$;
9. for all \( 0 < \epsilon \leq \alpha, m \in \mathbb{N}^* \) and \( l \in \{-k+1, \ldots, 0\} \) the maps
\[
t \mapsto (K^m_t)^b \in \mathcal{L}(B^{s_l}, B^{s_l-\epsilon})
\]
and
\[
t \mapsto (K^m_t)^b \in \mathcal{L}(B^{s_{l+1}}, B^{s_l})
\]
are continuous on \( W \) (see Proposition 2 and notice that \( s_{-k} > \frac{3}{2} \));
10. for all \( m \in \mathbb{N}^*, l \in \{-k+1, \ldots, 0\} \) and \( 1 \leq u \leq k + l \) the map
\[
t \mapsto (K^m_t)^b \in \mathcal{L}(B^{s_l-\alpha}, B^{s_l-\alpha+\epsilon})
\]
is \( C^u \) on \( W \) (see Proposition 2 and notice that \( \frac{3}{2} + k - l < s_{-l} \) and \( s_l < r - 1 - (k - l) \) provided \( \delta \) and \( \epsilon \) are small enough);
11. for all \( 1 \leq l \leq k \) and \( m \in \mathbb{N}^* \) the map
\[
t \mapsto (K^m_t)^b \in \mathcal{L}_{nuc}(B^{s_l-\epsilon}, B^{s_l})
\]
is \( C^{k-l} \) on \( W \) (see Proposition 2 and notice that \( s_0 - \alpha > \frac{3}{2} + k \) and \( s_0 + \alpha < r - 1 - k \) provided \( \delta \) is small enough).

The strategy to end the proof is then the same as in the case \( k = 1 \): we prove as in Lemma 6.5 that the nuclear terms that appear in the decomposition 36 of \( K^m_{t,t} \) have a \( C^k \) dependence on \( t \) with the expected derivatives (including higher derivatives). As in the case \( k = 1 \), this is the only part of the proof in which the parameter \( \alpha \) is needed. The main idea is the same as before: the loss of regularity due to the differentiation of the transfer operator with respect to \( t \) is compensated by the smoothing effect of the nuclear operator \( (K^m_{t,t})_{cc} \), and the properties above are sufficient to make this happen (and explain why \( r \) grows like \( 2k \) instead of \( k \)). We can then get a geometric higher bounds for the \( C^k \) norm of \( h_m \), as we did in the case \( k = 1 \), which ends the proof.

8. Application : Linear response and periodic points in finite differentiability. Finally, we briefly recall the proof of Lemma 3.1 of [12] to get formulae

\[ 2 \] and \( 3 \) in a differentiable setting.

**Corollary 2.** Let \( \tau \mapsto T_\tau \) be a \( C^2 \) curve, defined on a neighbourhood \( [-\epsilon, \epsilon] \) of \( 0 \), of \( C^7 \) expanding maps of the circle. Let \( g : S^1 \to \mathbb{R} \) be a \( C^0 \) map. Then there is some \( R > 1 \) such that for all \( u, \tau \) sufficiently close to \( 0 \) the map defined by \( 1 \) extends on the disc of center \( 0 \) and radius \( R \) to a holomorphic function \( z \mapsto d(z, u, \tau) \). The function \( d \) defined this way is \( C^1 \) and the formula 2 holds for sufficiently small \( \tau \).

If \( \tau \mapsto T_\tau \) is a \( C^3 \) curve of \( C^0 \) expanding maps of the circle and \( g \) is \( C^8 \), then \( d \) is \( C^2 \) and the formula 3 for linear response holds.

**Proof.** The definition and regularity of the function \( d \) are immediate consequences of Theorem 7.1 taking \( U = [-\epsilon, \epsilon] \times \mathbb{R}, \ t = (\tau, u), g_{\tau,u} = -ug - \log |T_\tau| \) and \( T_{\tau,u} = T_\tau \).

The end of the proof is just a reminder of the proof of Lemma 3.1 in [12].

Write \( z(u, \tau) = \exp(-P_\tau (-ug - \log |T_\tau|)) \) (with \( P_\tau = P_t \) the topological pressure of the dynamics \( T_\tau \)). For \( u, \tau \) sufficiently small, \( z(u, \tau) \) is in the disc of radius \( R \) and center \( 0 \). Notice that the regularity of the function \( u, \tau \mapsto z(u, \tau) \) can be seen
as a consequence of Lemma 6.8 or of the implicit function theorem. Differentiating the equation \( d(z(u, \tau), u, \tau) = 0 \) we get 2 recalling that

\[
\frac{\partial}{\partial u} \left( P_\tau (-u g - \log |T'_\tau|) \right) \bigg|_{u=0} = - \int_{S^1} g \, d\mu_\tau.
\]

See for example §7.28 and §7.12 in [14] for a proof of this formula. The formula 3 for linear response is obtained by differentiating 2.

Suppose that the hypotheses of Proposition 1 hold. Let \( d \) be the function introduced in Proposition 1. Write it as the sum of a power series

\[
d(z(u, \tau), u, \tau) = \sum_{n \geq 0} a_n(u, \tau) z^n.
\]

Notice that this sum converges as soon as \( |z| < R \). Then the formula 3 may be written

\[
\frac{\partial}{\partial \tau} \left( \int_{S^1} g \, d\mu_\tau \right) \bigg|_{\tau=0} = - \frac{\sum_{n \geq 1} \partial^2 a_n}{\sum_{n \geq 1} na_n}(0, 0) \left( \frac{\sum_{n \geq 1} \partial a_n}{\partial u}(0, 0) \right)^2.
\]

(37)

We recall now the method proposed in the paragraph 4.1 of [12] to explicitly calculate the terms of the series that appear in 37 in terms of the periodic points of \( T_0 \) and of the derivatives of \( \tau \mapsto T_\tau \) at 0. To make the calculation easier we shall suppose that the \( T_\tau \) are orientation-preserving (we get the orientation-reversing case by some sign changes). Then, setting

\[
b_n(u, \tau) = \sum_{T^k_{\tau} x = x} \exp \left( -u \sum_{k=0}^{n-1} g(T^k_{\tau} x) \right) = \sum_{T^k_{\tau} x = x} \frac{\exp \left( -u \sum_{k=0}^{n-1} g(T^k_{\tau} x) \right)}{|(T^k_{\tau})'(x) - 1|},
\]

we find, using formulae 25, 26 and 27, that the \( a_n \) are recursively defined by

\[
a_0 = 1 \quad \text{and} \quad a_n = -\frac{1}{n} \sum_{j=0}^{n-1} a_j b_{n-j}.
\]

We immediately deduce similar formulae for the derivatives of the \( a_n \):

\[
\frac{\partial a_0}{\partial u} = \frac{\partial a_0}{\partial \tau} = \frac{\partial^2 a_0}{\partial u \partial \tau} = 0,
\]

\[
\frac{\partial a_n}{\partial u} = -\frac{1}{n} \sum_{j=0}^{n-1} \left( \frac{\partial a_j}{\partial u} b_{n-j} + a_j \frac{\partial b_{n-j}}{\partial u} \right),
\]

\[
\frac{\partial a_n}{\partial \tau} = -\frac{1}{n} \sum_{j=0}^{n-1} \left( \frac{\partial a_j}{\partial \tau} b_{n-j} + a_j \frac{\partial b_{n-j}}{\partial \tau} \right),
\]

and

\[
\frac{\partial^2 a_n}{\partial u \partial \tau} = -\frac{1}{n} \sum_{j=0}^{n-1} \left( \frac{\partial^2 a_j}{\partial u \partial \tau} b_{n-j} + \frac{\partial a_j}{\partial u} \frac{\partial b_{n-j}}{\partial \tau} + \frac{\partial a_j}{\partial \tau} \frac{\partial b_{n-j}}{\partial u} + a_j \frac{\partial^2 b_{n-j}}{\partial u \partial \tau} \right).
\]
In order to apply these formulae, we need to calculate the derivatives of $b_n$. To do so we set: $X_n = \frac{\partial}{\partial \tau} (T^n_{\tau}) |_{\tau=0}$ for all non-zero integer $n$ and $X = X_1$. Then we have

$$X_n = \sum_{k=0}^{n-1} X \circ T_0^k (T_0^{n-1-k})' \circ T_0^{k+1},$$

and, for all $\tau \in [-\epsilon, \epsilon]$,

$$\frac{\partial b_n}{\partial u} (0, \tau) = - \sum_{T_0^n x = x} \frac{\sum_{k=0}^{n-1} g (T_0^k x)}{(T_0^n)'(x) - 1}. \tag{38}$$

Now, if $x = x_0$ is such that $T_0^n x_0 = x_0$, we may “follow” $x$ on small time by the implicit function theorem, there is a smooth map $\tau \mapsto x_{\tau}$ defined on a neighborhood of zero, such that $T_{\tau} x_{\tau} = x_{\tau}$ whenever it is defined. Moreover, we have

$$\frac{\partial x_{\tau}}{\partial \tau} |_{\tau=0} = \frac{X_n (x_0)}{1 - (T_0^n)'(x_0)}. \tag{39}$$

Since all the periodic points of period $n$ of $T_{\tau}$ are obtained this way, this leads to

$$\frac{\partial b_n}{\partial \tau} (0, 0) = - \sum_{T_0^n x = x} \frac{X_n' (x) + (T_0^n)'(x) \frac{X_n (x)}{1 - (T_0^n)'(x)}}{((T_0^n)'(x) - 1)^2}. \tag{40}$$

From 39, we also deduce that

$$\frac{\partial}{\partial \tau} \left( \frac{1}{(T_0^n)'(x_{\tau}) - 1} \right) |_{\tau=0} = - \frac{1}{((T_0^n)'(x_0) - 1)^2} \left( X_n' (x_0) + (T_0^n)'(x_0) \frac{X_n (x_0)}{1 - (T_0^n)'(x_0)} \right). \tag{41}$$

Since for all $0 \leq k \leq n - 1$, the point $T_{\tau}^k x_{\tau}$ is also a periodic point of period $n$ for $T_{\tau}$, we have

$$\frac{\partial T_{\tau}^k x_{\tau}}{\partial \tau} |_{\tau=0} = \frac{X_n (T_0^k x_0)}{1 - (T_0^n)'(T_0^k x_0)}$$

and thus

$$\frac{\partial g (T_{\tau}^k x_{\tau})}{\partial \tau} |_{\tau=0} = \frac{g' (T_0^k x_0)}{1 - (T_0^n)'(T_0^k x_0)} X_n (T_0^k x_0). \tag{41}$$

Using 40 and 41 to differentiate respectively the denominator and the numerator in 38, we get

$$\frac{\partial^2 b_n}{\partial u \partial \tau} (0, 0) = \sum_{T_0^n x = x} \left[ \frac{\sum_{k=0}^{n-1} g \circ T_0^k}{((T_0^n)' - 1)^2} \left( (T_0^n)'' \frac{X_n}{1 - (T_0^n)'} + X_n' \right) + \frac{1}{(T_0^n)' - 1} \sum_{k=0}^{n-1} \frac{X_n \circ T_0^k g' \circ T_0^k}{(T_0^n)' \circ T_0^k - 1} \right] (x).$$

Finally using Cauchy’s formula and differentiation under the integral, the partial derivatives of $d$ are holomorphic on the disc of center and radius $R$. Thus the power series that appear in formula 37 converge exponentially fast (we get a much faster convergence in the analytic setting, see [12]).
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