We present $d+1$-dimensional pure magnetic Yang-Mills (YM) black strings (or 1-branes) induced by the $d$-dimensional Einstein-Yang-Mills-Dilaton (EYMD) black holes. Incorporation of non-abelian YM fields in black strings, has already been considered in the literature [1–6]. Born-Infeld (BI) version of the YM field makes our starting point which goes to the standard YM field through a limiting procedure. The lifting from black holes to black strings, (with less number of fields) is by adding an extra, compact coordinate. This amounts to the change of horizon topology from $S^{d-2}$ to a product structure. Our black string in 5-dimensions is a rather special one, with uniform Hawking temperature and non-asymptotically flat structure Local isometry in the abelian limit with the space of colliding plane waves is discussed.

I. INTRODUCTION

A black string, as a black object [1–6] is the simplest extension, of a black hole, with the addition of a compact, killing coordinate. More than one such additions makes the class of black p-branes whose first member, 1–brane is a black string. The compact coordinate may close on itself to make an $S^1$ (a black ring) so that the horizon topology of such a black string becomes that of $S^2 \times S^1$, otherwise it may simply be referred to as $S^2 \times R$. In $d$-dimensions the horizon topology becomes naturally $S^{d-2} \times S^1$. The addition of an extra, compact coordinate does affect the Komar mass and total charge of the system, since it arises in the total volume element. As the energy involves the compact dimension that means, it arises also in the thermodynamical functions, heat capacity, phase transitions and other properties. Divergence in these thermodynamical functions is known to create instability in the system. Decay of a black string into a black hole (or vice versa) is one of the possible scenarios that may take place in this context. Depending on the critical mass of a black string a possible instability was formulated through perturbation analysis by Gregory and Laflamme (GL), which is known as the GL instability [7]. It was further argued that this kind of instability threatens all black objects, not only black strings with a single compact dimension. In the presence of a magnetic charge and dilaton a black string may be stable when confined within a range of compact dimension [8]. A similar instability was demonstrated also for an electrically charged black string [9].

In this Letter our purpose is to construct generically non-asymptotically flat (NAF) and non-spherically symmetric static black strings in higher dimensions from pure magnetically charged Yang-Mills (YM) fields and dilatons. To this end we employ certain classes of Einstein-Yang-Mills-Born-Infeld-Dilaton (EYMBID) solutions that we had obtained before [10]. Black strings constructed from abelian Maxwell fields were known [11], so we wish to carry the construction to cover the non-abelian YM fields. Our method of introducing the YM fields makes use of the Wu-Yang ansatz [12] which was extended to all higher dimensions [13]. When restricted to $d = 4$, our YM field becomes almost same with the abelian Maxwell field [14], therefore in our procedure higher ($d \geq 5$) dimensions become significant. We had found also the YM version of the Born-Infeld (BI) electrodynamics which was devised originally to remove divergences in classical electromagnetism. In particular limits we can remove the BI contribution to retain the usual YM-dilaton coupling in our model. The original Kaluza-Klein (KK) theory was to employ less (or no) fields other than gravity and to derive physical fields through a dimensional reduction. Thus, one can generate dilaton and electromagnetic fields from pure gravity by proper reduction procedure. Likewise, by lifting the spacetime to higher dimensions we dissolve some of the fields and simplify our action to the extreme of pure geometrical one. Another relevant problem that we addressed before was by considering collision of pure gravitational waves in higher ($d = 5$) dimensions which correspond to the collision of Einstein-Maxwell-dilaton (EMD) fields in $d = 4$. Solving our problem leads automatically to the solution of the other. The idea can be extended to higher $p$-branes in which $p$-compact (or non-compact) dimensions are added. Local isometry between inner horizon geometry and colliding plane waves can be extended to cover black strings as we show in a particular Einstein-Maxwell example.
II. FIELD EQUATIONS AND THEIR SOLUTION IN EYMBID THEORY

The $d$-dimensional action in EYMBID theory is given by ($G(d) = 1$)

$$ I = \frac{1}{16\pi} \int d^d x \sqrt{-g} \left( -\bar{R} + \frac{4}{d-2} (\nabla \psi)^2 - L(\bar{F}, \psi) \right), $$

where $\bar{R}$ is the Ricci scalar, $\psi$ is the dilaton and YM 2-form field is defined by

$$ \bar{F}^{(a)} = d\bar{A}^{(a)} + \frac{1}{2\sigma} C_{(b)(c)}^{(a)} \bar{A}^{(b)} \wedge \bar{A}^{(c)} $$

with structure constants $C_{(b)(c)}^{(a)}$. Our choice of YM potential $\bar{A}^{(a)}$ follows from the higher dimensional Wu-Yang ansatz [12, 13] and the coupling constant $\sigma$ is expressed in terms of the YM charge. The Lagrangian $L(\bar{F}, \psi)$ is chosen as

$$ L(\bar{F}, \psi) = 4\beta^2 e^{4\alpha \psi/(d-2)} \left( 1 - \sqrt{1 + \frac{\text{Tr}(\bar{F}^{(a)} \bar{F}^{(a)} \lambda \sigma) e^{-8 \alpha \psi/(d-2)}}{2\beta^2}} \right) $$

in which $\alpha$ and $\beta$ are the dilaton and BI parameters, respectively. For brevity, let us introduce the following abbreviations

$$ \mathcal{L}(X) = 1 - \sqrt{1 + X}, \quad X = \frac{\text{Tr}(\bar{F}^{(a)} \bar{F}^{(a)} \lambda \sigma) e^{-8 \alpha \psi/(d-2)}}{2\beta^2}, \quad \text{Tr}(\cdot) = \sum_{a=1}^{(d-1)(d-2)/2} \cdot $$

Our choice for the metric ansatz is

$$ d\bar{s}^2 = -\bar{f}(r) \, dt^2 + \frac{dr^2}{\bar{f}(r)} + \bar{h}(r)^2 \, d\Omega^2_{(d-2)}, $$

in which $\bar{f}(r)$ and $\bar{h}(r)$ stand for functions of $r$ and $d\Omega^2_{(d-2)}$ is the $d$-2-dimensional unit spherical line element. Variation of (1) with respect to $\bar{g}_{\mu\nu}$ and $\psi$ yields

$$ \bar{R}_{\mu\nu} = \frac{4}{d-2} \bar{\nabla}_\mu \psi \bar{\nabla}_\nu \psi - 4e^{-4\alpha \psi/(d-2)} \left( \text{Tr} \left( \bar{F}^{(a)} \bar{F}^{(a)} \lambda \right) \partial_X \mathcal{L}(X) \right) + \frac{4\beta^2}{d-2} e^{4\alpha \psi/(d-2)} \mathcal{K}(X) \bar{g}_{\mu\nu}, $$

$$ \bar{\nabla}^2 \psi = 2\alpha \beta^2 e^{4\alpha \psi/(d-2)} \mathcal{K}(X). $$

in which we have further abbreviated

$$ \mathcal{K}(X) = 2X \partial_X \mathcal{L}(X) - \mathcal{L}(X) \quad \left( \partial_X \mathcal{L}(X) = \frac{\partial \mathcal{L}(X)}{\partial X} \right). $$

The YM-equations take the form

$$ d \left( e^{-4\alpha \psi/(d-2)} \bar{F}^{(a)} \right) + \frac{1}{\sigma} C_{(b)(c)}^{(a)} e^{-4\alpha \psi/(d-2)} \bar{A}^{(b)} \wedge \bar{F}^{(c)} = 0 $$

where the hodge star $\ast$ implies duality. It can readily be seen that for $\beta \to 0$ (i.e. no YM-field) Eq.s (6)-(7) reduce to

$$ \bar{R}_{\mu\nu} = \frac{4}{d-2} \bar{\nabla}_\mu \psi \bar{\nabla}_\nu \psi, $$

$$ \bar{\nabla}^2 \psi = 0 $$

which refer simply to the gravity coupled with a massless scalar field. The limit $\beta \to \infty$ removes BI theory and we obtain simply EYMD theory, which will be the subject of Sec. III.
Exact solution for the metric (5) and dilaton are given by \[10\]
\[
\bar{f}(r) = \Xi \left(1 - \left(\frac{r_+}{r}\right)^{\frac{(d-3)\alpha^2+1}{2\alpha^2+1}}\right) r^{\frac{\alpha^2+1}{\alpha^2+1}}, \\
\bar{h}(r) = \xi e^{-2\alpha\psi/(d-2)}, \quad (\xi = \text{constant}) \\
\psi = -\frac{(d-2)}{2} \frac{\alpha \ln r}{\alpha^2 + 1},
\]
in which

\[
\Xi = -\frac{4\beta^2 (\alpha^2 + 1)^2 K(X)}{(d-2)((d-3)\alpha^2 + 1)} \tag{12}
\]
and

\[
r_+ = \left(\frac{4 (\alpha^2 + 1) M_{QL}}{(d-2) \Xi \alpha^2 A^{n-1}}\right). \tag{13}
\]

Here $M_{QL}$ stands for the Quasi-local mass \[15\] and the constants satisfy the constraint condition

\[
4K(X) \beta^2 A^4 (\alpha^2 - 1) + (d-2) (d-3) (4Q^2 \partial_X \mathcal{L} + A^2) = 0, \tag{14}
\]
which makes $A$, proportional to the YM charge $Q$,

\[
A = \pm \sqrt{2}Q. \tag{15}
\]

We recall that from the higher dimensional Wu-Yang ansatz the YM potential is given by \[12\]
\[
\bar{A}^{(a)} = \frac{Q}{r^2} C_{(b)(c)}^{(a)} x^b dx^c, \quad Q = \text{YM magnetic charge}, \quad r^2 = \sum_{i=1}^{d-1} x_i^2, \tag{16}
\]
in which $Q$ is the only non-zero YM charge.

### III. BLACK STRINGS IN EYMBID THEORY

Now let’s consider a $(d + 1)$—dimensional space time as

\[
ds^2 = e^{-b\psi} ds^2 + e^{b(d-2)\psi} dz^2 \tag{17}
\]
in which $b =$constant, $ds^2$ is our line element (5) and $z$ is an additional, compact Killing coordinate ($0 < z \leq L$). Our new action modifies into

\[
I = \frac{1}{16\pi G_{(d+1)}} \int_0^L dz \int d^d x \sqrt{-g} \left(R + 4\beta^2 \left(1 - \sqrt{1 + \frac{\text{Tr}(F^{(a)}_{\lambda\sigma} F^{(a)\lambda\sigma})}{2\beta^2}}\right)\right), \tag{18}
\]
where the $d + 1$ and $d$—dimensional quantities are related as follow

\[
\sqrt{-g} = e^{-b\psi} \sqrt{-\bar{g}}, \quad R = e^{b\psi} \left[\bar{R} + b \left\{\nabla^2 \psi - \frac{(d-1)(d-2)}{4} b (\nabla \psi)^2\right\}\right], \\
A^{(a)} = \bar{A}^{(a)} \\
\mathcal{F} = e^{2b\psi} \bar{\mathcal{F}}, \\
(\mathcal{F} = F^{(a)}_{\lambda\sigma} F^{(a)\lambda\sigma}, \quad \bar{\mathcal{F}} = \bar{F}^{(a)}_{\lambda\sigma} \bar{F}^{(a)\lambda\sigma}). \tag{19}
\]
It is a straightforward calculation to show that while $\tilde{F}^{(a)}$ satisfies the YM equations in $d$-dimensions $F^{(a)} = \tilde{F}^{(a)}$ satisfies the YM equations in $d + 1-$dimensions. By substitutions of (19) directly into (18) and integrating over the compact coordinate (i.e. $\int_0^L dz = L$) we arrive at the action

$$I = \frac{L}{16\pi G_d} \int d^d x \sqrt{-g} \left( \tilde{R} - \frac{4}{d - 2} \left( \nabla \psi \right)^2 + 4 \beta^2 e^{\frac{4(\alpha - 2)\sqrt{-1}}{2\beta^2}} \left( 1 - \sqrt{1 + \frac{\tilde{F} e^{-\frac{4\alpha \sqrt{-1}}{2\beta^2}} \nabla \psi}{2\beta^2}} \right) \right),$$

(20)

in which the Newton’s gravitational constant $G_{(d)}$ is related to its 4d version by $G_{(d)} = G_{(4)} L^{d-4}$. This is nothing but the $d$-dimensional action for the choice $\alpha = \frac{1}{\sqrt{d-1}}$ in the EYMBID theory. Now, by expressing the $d+1$-dimensional black string metric in the form

$$ds^2 = -f_1 (r) dt^2 + \frac{dr^2}{f_2 (r)} + f_3 (r) dz^2 + f_4 (r) d\Omega_{(d-2)}^2$$

(21)

in which $f_i (r)$ are functions of $r$ only serves to construct the latter from the known solution of EYMBID black hole solution in $d$-dimensions. The routine identification of the metric functions go as follows

$$f_1 (r) = e^{-b \psi} \bar{f} (r); \quad f_2 (r) = e^{b \psi} \bar{f} (r); \quad f_3 (r) = e^{b(d-2) \psi}; \quad f_4 (r) = e^{-b \psi} \bar{h} (r)^2$$

(22)

where $\psi, \bar{f} (r)$ and $\bar{h} (r)$ are given in (11).

**IV. THE $\beta \to \infty$ LIMIT OF EYMBID THEORY**

From the action (1) it can be shown easily that

$$\lim_{\beta \to \infty} L (\bar{F}, \psi) = -e^{-4\alpha \psi/(d-2)} \bar{F}$$

(23)

which is the dilaton-YM coupling Lagrangian and accordingly the resulting field equations reduce to

$$\tilde{R}^{\nu}_{\mu} = \frac{4}{d-2} \delta^{\nu}_{\mu} \left( \nabla \psi \right)^2 + 2e^{-4\alpha \psi/(d-2)} \left[ \text{Tr} \left( \bar{F}^{(a)} \bar{F}^{(a)\nu\lambda} \right) - \frac{1}{2(d-2)} \bar{F}^2 \right],$$

$$\nabla^2 \psi = -\frac{1}{2} e^{-4\alpha \psi/(d-2)} \bar{F}.$$  

(24)

(25)

With the invariant YM equation and same ansatz metric as (5) we obtain the solution

$$\psi = -\frac{(d-2)}{2} \frac{\alpha \ln r}{\alpha^2 + 1}.$$  

$$\bar{f} (r) = \Xi \left( 1 - \left( \frac{r}{r_c} \right)^{(d-2)\alpha^2 + 1} \right) \left( \frac{r_c}{r} \right)^{\frac{d-1}{(d-2)\alpha^2 + 1}}.$$  

$$\bar{h} (r) = \xi e^{-2\alpha \psi/(d-2)} \quad (\xi = \text{constant}).$$  

$$\Xi = \frac{(d-3)}{(d-3) \alpha^2 + 1} Q^2,$$  

$$\xi^2 = Q^2 \left( \alpha^2 + 1 \right).$$  

(26)

The corresponding black string metric is same as (21), and upon setting $\alpha = \frac{1}{\sqrt{d-1}}$ and $b = -\frac{4}{(d-2)\sqrt{d-1}}$ the $d-$dimensional action casts into

$$I = \frac{1}{16\pi G_{(d)}} \int_{\mathcal{M}} d^d x \sqrt{-g} \left( \bar{R} - \frac{4}{d-2} \left( \nabla \psi \right)^2 - e^{-4\alpha \psi/(d-2)} \bar{F} \right).$$

(27)

By choosing the $(d + 1)$-line element as in (22) we identify the black string metric functions easily.
As an example to the foregoing analysis we obtain a particular 5d–black string as follows. We recall that in \(d = 4\), EYMD black hole solution coincides with the NAF black hole solution in EMD gravity. Thus, for \(d = 4\) we obtain

\[
\psi = -\frac{\alpha \ln r}{\alpha^2 + 1}, \quad \tilde{f}(r) = \Xi \left(1 - \frac{r_+}{r}\right) r^2, \quad \Xi = \frac{3}{4Q^2}, \quad \xi^2 = Q^2 \left(\frac{4}{3}\right),
\]

(28)

\[
\bar{h}(r) = \xi r^{1/4}, \quad \alpha = \frac{1}{\sqrt{3}}, \quad b = -\frac{2}{\sqrt{3}}, \quad e^{-b \psi} = r^{-1/2}.
\]

Now from the identification (28) the 5D–black string metric follows

\[
ds^2 = -\frac{3(r - r_+)}{4Q^2} dt^2 + \frac{4Q^2}{3r (r - r_+)} dr^2 + r dz^2 + \frac{4Q^2}{3} d\Omega_2^2.
\]

(29)

The resulting action is expressed by

\[
I = \frac{1}{16\pi G_5} \int_0^L dz \int d^4x \sqrt{-g} (R - F)
\]

(30)

in which \(F\) is the Maxwell invariant without dilaton. Here we also calculate the Hawking temperature

\[
T_H = \frac{1}{4\pi} f'_1 (r_+) = \frac{3}{16\pi Q^2}.
\]

(31)

In the following section we shall concentrate on this particular case and show that the line element (29) can be interpreted as a space time of colliding plane gravitational waves coupled with the abelian Maxwell fields.

V. LOCAL EQUIVALENCE WITH COLLIDING PLANE WAVES

It is well known that inner horizon region of a 4-dimensional Schwarzschild black hole (SBH) even a Kerr black hole is dynamic, admitting two space-like Killing vectors [16]. This turns out to be equivalent to the region of colliding waves which also admits two space like Killing vectors. For the Reissner-Nordström (RN) and Kerr black holes we have two horizons, outer = event, and inner. In between the two horizons again we face a similar equivalence with colliding waves. For the black string metric (29) we shall show now that the same rule also remains intact, with the difference that an additional space like Killing vector arise.

For this purpose we cast the metric (29) into the form with three space like Killing vectors. Without loss of generality we choose \(r_+ = 2\) and \(Q^2 = 3/4\) and apply the following coordinate transformation

\[
r = 1 + \sin (au + bv)
\]

\[
\cos \theta = \sin (au - bv)
\]

\[
\phi = y
\]

\[
t = x
\]

\[
z = z
\]

(32)

in which \((a, b)\) are constants representing the strength (energy) of the waves and \((u, v)\) are the null coordinates. (One can make a further choice of \(a = 1 = b\), which makes the waves with unit energies.) Upon substitution of (32) into (29) and making appropriate scaling for both coordinates and \(ds^2\) one arrives at the line element

\[
ds_5^2 = -2dudv + (1 - \sin (au + bv)) dx^2 + \cos^2 (au - bv) dy^2 + (1 + \sin (au + bv)) dz^2.
\]

(33)

Now, to interpret this line element as a colliding plane wave we introduce Heaviside step (i.e. \(\theta (u) = +1\) for \(u > 0\) and \(\theta (u) = 0\) for \(u < 0\)) function in accordance with

\[
u \rightarrow u\theta (u),
\]

\[
v \rightarrow v\theta (v),
\]

(34)

to determine the global region metric

\[
ds_5^2 = -2dudv + (1 - \sin [au\theta (u) + bv\theta (v)]) dx^2 + \cos^2 (au\theta (u) - bv\theta (v)) dy^2 + (1 + \sin [au\theta (u) + bv\theta (v)]) dz^2.
\]

(35)
In doing this, of course it is checked that no extra currents arise at the boundaries. The Ricci scalar for this metric is given by 

$$R = ab\theta (u) \theta (v),$$

which implies that we have a non-singular colliding wave metric [17]. It should be remarked that there are null singularities at $au = \frac{\pi}{2}$, $v = 0$ ($av = \frac{\pi}{2}$, $u = 0$) on the null boundaries. The surface $au + bv = \frac{\pi}{2}$, on the other hand represents the horizon, not singularity. The incoming regions are obtained by setting $(u < 0, v > 0)$ and $(u > 0, v < 0)$. No wave region $(v < 0, u < 0)$ is given by the flat metric

$$ds^2 = -2dudv + dx^2 + dy^2 + dz^2.$$ (36)

Together with the colliding wave metric (35) the colliding electromagnetic vector potential is given by

$$A_\mu = \frac{\sqrt{3}}{2} \delta_\mu^y \sin (au\theta (u) - bv\theta (v)).$$ (37)

We can draw the conclusion from this result, with reference to the 5d black string and 4d black hole identification above, that colliding EM plane waves in 5d, is completely equivalent to colliding EMD plane waves in 4d. Such an equivalence (or isometry), however, for the non-Abelian YM fields, remains to be established. In other words, dilaton emerges as a result of reduction procedure of dimensionality in the collision problem as well. This technique works in general for any higher dimensional p-branes where the dimensionality is changed [18]. It is interesting to note also that any additional branes in the form of

$$ds^2_{5+k} = ds^2_5 + \sum_{i=1}^{k} (dw_i)^2$$ (38)

where $ds^2_5$ is given by (42) and $w_i$ are k-extra coordinates, doesn’t change the nature of the solution. It is needless to add finally that the aforesaid rule applies at best to the vacuum solutions such as Khan-Penrose [19] and its double polarized extension Nutku-Halil [20] solutions.

VI. CONCLUSION

Black string in $(d + 1)$–dimensions are constructed from d-dimensional black hole in the presence of non-abelian YM fields. Particular black string are obtained from the EYMD black holes. Dilatonic coupling is strong enough to convert spacetime into a non-asymptotically flat (NAF) one. We go one more step ahead to consider Born-Infeld (BI) coupling out of the YM field which makes the overall model further non linear. In the limit ($\beta \to \infty$) of BI we recover the standard YM theory coupled with the dilaton. We didn’t investigate the Gregory-Laflame (GL) type instability for our black string, but it is our belief that it remains intact. Our black string has a unique structure: it admits a uniform Hawking temperature $T_H$, irrespective of both the horizon radius and quasi local mass. This implies that any thermodynamical stability analogies faced on $T_H$ fails to work. As a final contribution, in the abelian Maxwell limit we establish local isometry between black string and colliding plane wave spacetimes which was well-known in the realm of black holes.

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