Identification of multiple cracks in an anisotropic elastic plate by boundary data

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Abstract. A linear elastic, anisotropic plate, which can contain multiple, rectilinear cracks is considered. It is assumed that the loads applied to the boundary of the plate and the displacements of the boundary are measured in a single static experiment. A method for the cracks identification using the known overdetermined boundary data is developed. The developed method is based on the reciprocity gap concept, generalized functions theory and methods of cluster analysis. Numerical examples are considered.

1. Introduction
The presence of cracks is one of the main reasons for the destruction of structural elements. In this connection, the problem of detecting and identifying cracks in elastic bodies is central among the non-destructive testing problems. Various methods have been developed for solving this problem. Among the methods for the identification of cracks based on the results of static tests, one of the most effective is the method based on the use of the reciprocity principle. The reciprocity gap concept was proposed in [1] for solving a problem of the identification of a cut inside a domain using the boundary values of harmonic function and its normal derivative. In [2] a method, based on the reciprocity principle, was applied to the problem of a plane crack identification in an isotropic linear elastic body by means of the overdetermined data on the external boundary of the body. The overdetermined data can be obtained as a result of two static experiments. The results of paper [2] were extended to the problem of identification of a plane crack in an anisotropic elastic body in [3]. The method was further developed in [4, 5], where the problems of identifying of a single ellipsoidal inclusion and, in particular, an ellipsoidal cavity and an elliptical crack, both in an isotropic and anisotropic elastic body were solved by means of the data, which can be obtained as the result of one static experiment.

The problem of identifying multiple defects has been studied to a much lesser extent. A variational method based on the use of the reciprocity functional was proposed in [6], where, as an example, the problem of identifying multiple point sources for the two-dimensional Laplace equation was considered. In [7, 8], a method for identifying several ellipsoidal inclusions both in an isotropic and anisotropic body is presented. A method for identifying multiple cracks in an isotropic plate was proposed in a recent paper [9]. The purpose of this paper is to generalize the results [9] to the case of an anisotropic plate.

2. Statement of the problem
Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with a boundary $\partial \Omega$. $\gamma_k \subset \Omega$, $k = 1, 2, \ldots, n$ are rectilinear cracks. We assume that linear elastic anisotropic body occupies the domain $S = \Omega \setminus \Gamma$, where $\Gamma$ is the boundary of the domain $S$. The cracks $\gamma_k$ are supposed to be rectilinear and the cracks located in the domain $S$. The boundary $\Gamma$ is assumed to be a part of the boundary $\partial \Omega$.
\( \Gamma = \bigcup_{k=1}^{n} \gamma_k \). We will make several assumptions concerning the set of cracks: the surfaces of cracks are free of loads; the typical lengths of the cracks have the same order (denote the typical length \( l \)); the typical distances between the cracks and the distances between the cracks and the boundary \( \partial \Omega \) exceed some value \( L \). We assume also that the lengths of cracks are small in the following sense

\[
l << L
\]

Let us introduce Cartesian coordinates \( Ox_1, x_2 \). We assume that the loads \( t^0 = (t_1^0, t_2^0) \) and displacements \( u^0 = (u_1^0, u_2^0) \) are measured on \( \partial \Omega \) in a single static test. Denote the stress-strain state in the body \( S \) as follows: \( \sigma_{\alpha\beta}(x) \) is the stress tensor, \( e_{\alpha\beta}(x) \) is the strain tensor and \( u(x) \) is the displacement vector, \( x = (x_1, x_2) \in S \). Equations of plane elasticity in the domain \( S \) have the following form:

\[
\begin{align*}
\sigma_{\alpha\beta}(x) &= \frac{1}{2} \left( u_{\alpha,\beta}(x) + u_{\beta,\alpha}(x) \right), \quad \alpha = 1, 2, \quad \beta = 1, 2 \\
e_{\alpha\beta}(x) &= C_{\alpha\beta\gamma\delta} e_{\gamma\delta}(x), \quad \gamma = 1, 2, \quad \delta = 1, 2 \\
\sigma_{\alpha\beta}(x) &= 0
\end{align*}
\]

Here \( C_{\alpha\beta\gamma\delta} \) are the elastic moduli of the anisotropic body.

The overdetermined data on the external boundary of the body are of the form:

\[
\sigma_{\alpha\beta}(x') n_{\beta}(x') = t^0_{\alpha}(x'), \quad u_{\alpha}(x') = u^0_{\alpha}(x'), \quad x' = (x'_1, x'_2) \in \partial \Omega
\]

Here \( n(x') = (n_1(x'), n_2(x')) \) is a unit outward normal to the boundary \( \partial \Omega \) at the point \( x' \) and convention of summation for repeated indices is used.

The conditions on the cracks surfaces have the form:

\[
\sigma_{\alpha\beta}(x^*) n_{\beta}(x^*) = 0, \quad x^* \in \gamma_k
\]

where \( (N_1^k, N_2^k) \) is a normal to \( \gamma_k \).

The problem is to reconstruct the cracks \( \gamma_k \), using the overdetermined boundary data \( t^0(x') \) and \( u^0(x') \) on the external boundary \( \partial \Omega \).

3. A method for identification of the cracks

An arbitrary elastic field in the domain \( \Omega \) without cracks we will call by a regular elastic field. The regular elastic fields we will mark with a superscript \( r \). Introduce a reciprocity gap functional depending on a regular elastic field \( u^r \)

\[
RG(u^r) = \int_{\partial \Omega} \left( t^0_{\alpha}(x') u^r_{\alpha}(x') - t^0_{\alpha}(x') u^0_{\alpha}(x') \right) \, dl, \quad t^0_{\alpha}(x') = \sigma_{\alpha\beta}(x') n_{\beta}(x')
\]

Because the functions \( t^0_{\alpha}(x') \) and \( u^0_{\alpha}(x') \) are supposed known on the boundary \( \partial \Omega \), the values of the reciprocity gap functional can be calculated for all regular elastic fields. Below, we will solve the considered problem by expressing the cracks parameters by means of the values of the reciprocity gap functional. The right hand side of the equation (5) can be reduced to the sum of integrals over the cracks, using the reciprocity principle.
Here square brackets denote jumps of displacements on cracks.

We construct regular elastic fields that will be used for the identification of cracks. Equations (2) can be rewritten in the form:

\[ C_{\alpha\beta\gamma\delta} u_{x}^{\alpha\beta \gamma \delta} (x) = 0 \]  

(7)

We will search for the solution \( u_{x}^{\alpha \beta} (x_1, x_2) \) of equation (7) in the form

\[ u_{x}^{\alpha \beta} (x_1, x_2) = f_{\gamma} (x_1 + sx_2) \]  

(8)

It follows from equations (7) and (8)

\[ m_{\alpha\beta} (s) f_{\gamma}^* = 0, \quad m_{\alpha\beta} (s) = C_{\alpha\gamma\gamma} + \left( C_{\alpha\gamma\beta} + C_{\alpha\beta\gamma} \right) s + C_{\alpha\gamma\gamma} s^2, \quad \alpha = 1, 2, \quad \gamma = 1, 2 \]  

(9)

Consider a matrix \( \mathbf{M}(s) = \left( m_{\alpha\beta} (s) \right) \). Equations (9) have a nonzero solution if and only if the following equality holds

\[ \det(\mathbf{M}(s)) = 0 \]  

(10)

Equation (10) is an algebraic equation of fourth order in \( s \). For simplicity, we assume that equation (10) does not have multiple roots. According to [10], the imaginary parts of the roots are not zero. Thus, the roots are of the form:

\[ s_1 = \alpha_1 + i\beta_1, \quad s_2 = \alpha_2 + i\beta_2, \quad s_3 = \overline{\alpha_1}, \quad s_4 = \overline{\alpha_2}, \quad \beta_\gamma > 0, \quad \gamma = 1, 2 \]  

(11)

Since, by assumption, the roots of equation (10) are simple, the solution space of equations \( m_{\alpha\beta} (s) f_{\gamma}^* = 0 \) for \( \gamma = 1, 2 \) is one-dimensional. Denote \( \lambda_{\gamma} = \left( \lambda_{\gamma1}, \lambda_{\gamma2} \right)^T \) the normalized solution of the equations. Here the superscript \( T \) denotes transpose of the matrix. Let \( g(x_1 + sx_2) \) be an arbitrary smooth function. It follows from the equations (9) – (11) that the vector-functions \( u^{\gamma} = \text{Re} \left( \lambda_{\gamma} g(z_p) \right) \) and \( u^{\gamma} = \text{Im} \left( \lambda_{\gamma} g(z_p) \right) \) are regular elastic fields. Here \( z_p = x_1 + s_p x_2 \). Let us note that here and below there is no summation over the repeating index \( p \).

Choose any of the two roots of \( s_p, \quad p = 1, 2 \) . It follows from equation (6) and the definition of the regular elastic fields \( u^{\gamma} \) and \( u^{\gamma} \)

\[ \mathbf{R}_G (u^{\gamma}) + i\mathbf{R}_G (u^{\gamma}) = -\sum_{k=1}^{n} \left( \left( C_{\alpha\beta\gamma} + s_p C_{\alpha\beta\gamma} \right) \lambda_{\gamma1} + \left( C_{\alpha\beta\gamma} + s_p C_{\alpha\beta\gamma} \right) \lambda_{\gamma2} \right) N_{\beta} \left[ u_{x}^{\alpha \beta} (x) \right] H(z_p) dl \]  

(12)

Here \( H(z_p) = g'(z_p) \).

Introduce a functional space \( K \) that consists of functions having the form \( H(x_1 + s_p x_2) \) that are defined in \( \overline{\Omega} \) and holomorphic in variable \( z_p = x_1 + s_p x_2 \). \( \overline{\Omega} \) is a closure of \( \Omega \). The topology in \( K \) is determined by the topology of uniform convergence in \( \overline{\Omega} \). The space of linear continuous functionals over \( K \) is denoted by \( K' \). Consider a distribution (generalized function) \( f \in K' \).
\[ F = -\sum_{k=1}^{N} \left( \left( C_{opp1} + s_{k}C_{opp2} \right) \delta_{1} + \left( C_{opp2} + s_{k}C_{opp1} \right) \delta_{2} \right) \left[ \begin{array}{c} N_{1} \n \end{array} \right] \left[ H_{n} \left( x \right) \right] \delta_{y} \] (13)

Here \( \delta_{y} \) – дельта-функция с носителем на трещине \( y_{k} \).

It follows from equations (12) and (13)

\[ RG\left( u^{\&} \right) + iRG\left( u^{\&} \right) = \left( F, H \left( z_{p} \right) \right) \] (14)

We will approximate the distribution \( F \) by means of a set of distributions \( F_{q}, N = 1,2,\cdots \). The distributions \( F_{q} \) are constructed in the form

\[ F_{N} = \sum_{q=1}^{N} A_{qN} \delta \left( x - x_{N}^{q} \right), \quad x = \left( x_{1}, x_{2} \right), \quad x_{N}^{q} = \left( x_{1}^{q}, x_{2}^{q} \right) \] (15)

Here \( A_{qN} \) – constants. The values \( A_{qN} \) and coordinates \( x_{N}^{q} \) are unknown and should be determined. To determine the values the following procedure is used.

Consider the functions \( H_{m} \left( z_{p} \right) = \left( z_{p} / L \right) \in K \). Construct regular elastic fields \( r_{m} \) and \( \rho_{m} \) using the holomorphic function \( H_{m} \left( z_{p} \right) \). Denote

\[ RG\left( u^{\&} \right) + iRG\left( u^{\&} \right) = \left( F, H_{m} \left( z_{p} \right) \right) = b_{m} \] (16)

Note that the left hand side of the equation (16) and consequently the values \( b_{m} \) can be calculated by means of input data. Consider a system of equations

\[ \left( F_{q}, H_{m} \left( z \right) \right) = b_{m}, \quad m = 0,1,\cdots,2N-1 \] (17)

It follows from the equations (15) and (17)

\[ \sum_{q=1}^{N} A_{qN} w_{q}^{m} = b_{m}, \quad z_{qN} = x_{N}^{q} + s_{p} x_{2}^{q}, \quad w_{q}^{m} = z_{qN} / L, \quad m = 0,1,\cdots,2N-1 \] (18)

The equations having the form of equation (18), are encountered in a number of problems. The methods of solving such equations are well developed. Here we use the method outlined in [11]. Construct the Hankel matrices

\[ H_{0} = \begin{pmatrix} b_{0} & b_{1} & \cdots & b_{N-1} \\ b_{1} & b_{2} & \cdots & b_{N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N-1} & b_{N-2} & \cdots & b_{2N-2} \end{pmatrix}, \quad H_{1} = \begin{pmatrix} b_{0} & b_{2} & \cdots & b_{N} \\ b_{1} & b_{3} & \cdots & b_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N} & b_{N+2} & \cdots & b_{2N-1} \end{pmatrix} \] (19)

As it was shown in [11], the values \( w_{qN}, q = 1,2,\cdots,N \) are the eigenvalues of the generalized eigenvalue problem

\[ H_{1}a = wH_{0}a \] (20)

where \( a \) is the eigenvector.

After determination of the values \( w_{qN} \) we solve a system of linear algebraic equations (18) with respect to \( A_{qN} \), where \( m = 0,1,2,\cdots,N-1 \).
Thus, we construct a sequence of distributions $F_N$, $N = 1, 2, \ldots$. In case $N > n$ we can expect that the supports of the generalized function $F_N$ are located near the cracks. At the same time, due to numerical errors, some of the eigenvalues $w_{qN}$ can be spurious and correspond to points located away of the cracks. The spurious solutions are excluded in several steps. At the first step we exclude the eigenvalues $w_{qN}$ that satisfy at least one of the following two criteria:

1. The eigenvalue $w_{qN} = \left( x_1^{qN} + x_2^{qN} \right) / L$ is excluded if the point $\left( x_1^{qN}, x_2^{qN} \right)$ is located outside the domain $\Omega$.

2. Denote $A_{N_{\text{max}}} = \max_{q=1,\ldots,N} \left| A_{qN} \right|$. The eigenvalue $w_{qN}$ is excluded if the following inequality is valid $\left| A_{qN} \right| / A_{N_{\text{max}}} < \epsilon_{cr}$. Here $\epsilon_{cr}$ is a small threshold value. In the examples considered below we took $\epsilon_{cr} = 0.01$.

If for a considered $N$ some of the eigenvalues $w_{qN}$ are excluded and only eigenvalues $w_{qN}$, $q = 1, 2, \ldots, N_N$, $N_N < N$ are remained, then the coefficients $A_{qN}$ should be recalculated by means of the equations (18) with $m = 0, 1, \ldots, N_N - 1$.

Consider a sequence of generalized functions $F_N$. Since we can use only a finite number of generalized functions, we will assume that $N \leq N_{\text{max}}$. We assume also that the value $N_{\text{max}}$ is much greater than the number of cracks $n$. In the examples considered below it is taken $N_{\text{max}} = 20$. After excluding the spurious solutions by means of the criteria 1 and 2, pointed out above, the remaining points $\left( x_1^{qN}, x_2^{qN} \right)$ corresponding the eigenvalue $w_{qN}$ are concentrated mainly near the cracks. Thus, the points $\left( x_1^{qN}, x_2^{qN} \right)$ form clusters corresponding to the cracks. The number of clusters can be found both as by visual analysis of the distribution of the remaining points and formally by the use a clustering algorithm. For correct determination of the clusters, it is necessary to use the clustering algorithm, which takes into account the properties of the clusters. The clusters corresponding to cracks should be well separated and elongated. In a numerical analysis of the examples considered below, we compared several clustering algorithms built into the software package Mathematica. The results presented in the next Section were obtained by means of the use of the function FindCluster with the option Optimize and CosineDistance as an appropriate distance function for elongated clusters.

Because the clusters are located near rectilinear cracks it is possible to use linear regression model for approximate determination of the lines containing cracks. The clusters obtained by means of the clustering algorithm still can contain spurious points due to supports of generalized functions $F_N$ when $N < n$. These points can be detected by means of a standard procedure of the detection of outliers [12]. The algorithms of linear regression and detection of outliers are considered in details in [9]. So we will not repeat them here.

Each crack $\gamma_k$ is completely determined by its length $l_k$, the coordinates of the center $\left( x_1^0, x_2^0 \right)$ and the angle of inclination $\phi_k$ to the $x_1$ axis. The angle $\phi_k$ is determined by the linear regression. We determine now other parameters of the crack $\gamma_k$. The main problem arising in application of the reciprocity gap functional method to a problem of identification of multiple defects is that all defects contribute in the value of the functional. It is difficult to separate the contribution of each defect in the value of the reciprocity gap functional but using the results of clustering the problem can be solved. The cluster, cleared from the outliers, that corresponds to the crack $\gamma_k$ we denote $O_k$. Consider generalized function $F_N$. From equations (17) and (18) it follows that the contribution to the value of $b_m$, made by a crack (cluster) $\gamma_k$, is equal to $b_{mk}$, where

$$b_{mk} = \int_{O_k} \mathcal{T}(\mathbf{x}, \mathbf{x}^*) \, d\mathbf{x}$$
\[
\sum_{x \in \partial \Omega} A_{\alpha \nu} \nu_{\alpha \nu} = b_{\text{in}}
\]  

We assume that each crack is well-separated from the boundary of the body and other cracks. From this assumption it follows that the crack opening displacement has approximately the same form as the crack opening displacement of a single crack in an infinite solid subjected to constant stresses at the infinity. Thus, the crack opening displacement has a form, see [13]

\[
\left[ u_{\alpha}(x) \right] = C_{\alpha} \left( 1 - \frac{4(x_i - x_{i0})^2}{l_i^2 \cos^2 \varphi_k} \right)^{1/2}, \quad C_{\alpha} = \text{const}, \quad x = (x_i, x_2) \in \gamma_k
\]  

It follows from (12) and (22)

\[
C \int_{\gamma_k} \left( \frac{z_k}{L} \right) \left( 1 - \frac{4(x_i - x_{i0})^2}{l_i^2 \cos^2 \varphi_k} \right)^{1/2} dl = b_{\text{in}}, \quad C = \text{const}
\]

From equation (23) we obtain the following equalities by means of direct calculations

\[
b_{k1} = z_{k0}^o b_{\nu o} L^1, \quad z_{k0}^o = x_{i0}^o + s_p x_{2k}^o
\]

\[
b_{k2} = \left[ \frac{l_k^2}{16} \left( \cos \varphi_k + s_p \sin \varphi_k \right)^2 + \left( z_{k0}^o \right)^2 \right] \frac{b_{\nu o}}{L^2}
\]

Consider a generalized function \( F_{\nu} \). Using equation (21) we can calculate the values \( b_{\nu o}, b_{\nu 1}, \) and \( b_{\nu 2} \). The angle \( \varphi_k \) we calculate as a result of cluster analysis. After that the value \( z_{k0}^o \) and consequently the center of the crack \( \gamma_k \) is determined using equation (24). Finally, the length \( l_k \) of the crack \( \gamma_k \) is determined from the equation (25). Thus, all parameters of the crack are reconstructed. Note that according to equation (21), to calculate the values \( b_{\text{in}} \) it is possible using various generalized functions \( F_{\nu} \). It is necessary only that at least two delta-functions in the generalized function \( F_{\nu} \) were concentrated at the points belonging to the cluster \( O_k \) corresponding to the crack \( \gamma_k \), and there would be no outliers among them. The examples considered below show that the results weakly depend on the choice of a generalized function of \( F_{\nu} \) satisfying the above conditions.

4. Numerical examples

Consider several examples. In all examples \( \Omega \) is a square plate \( \Omega = \{ x : |x_i| \leq 10, \alpha = 1, 2 \} \), containing three cracks. Parameters of the cracks are: \( l_1 = 1, \quad (x_{11}^0, x_{21}^0) = (1, 7), \quad \varphi_1 = 0, \quad l_2 = 0.8, \quad (x_{12}^0, x_{22}^0) = (-6, 3), \quad \varphi_2 = 45^\circ, \quad l_3 = 1.2, \quad (x_{13}^0, x_{23}^0) = (4, -2), \quad \varphi_3 = -45^\circ. \) Here all lengths and coordinates are given in meters. The material of the plate is of glass-epoxy. The elastic moduli of the material are taken from the paper [14]. In the coordinate system \( Oy_1 y_2 \) with axes directed along the symmetry axes of the material the relations between the stresses and strains are of the form:

\[
\begin{align*}
 e_{11} &= \frac{1}{E_1} \left( \sigma_{11} - \nu_{12} \sigma_{22} \right), \\
 e_{12} &= \frac{1}{2\sigma_{12}} \sigma_{12}, \\
 e_{22} &= \frac{1}{E_2} \left( \sigma_{22} - \nu_{21} \sigma_{11} \right)
\end{align*}
\]

where the elastic moduli satisfy to the equality \( \nu_{12} / E_1 = \nu_{21} / E_2. \)
The elastic moduli for the considered material are: \( E_1 = 48.26 \) GPa, \( E_2 = 17.24 \) GPa, \( \nu_{12} = 0.29 \), \( G_{12} = 6.89 \) GPa. The considered applied loads correspond to uniaxial tension in the direction of the axis \( x_2 \), see figure 1.

**Example 1.** First, we consider the case when the coordinates \( Oy_1, y_2 \) coincide with the coordinates \( Ox_1, x_2 \). In this case the roots of equation (10) are equal \( s_1 = 0.68567i \), \( s_2 = 2.44013i \). Location of the supports of the generalized functions \( F_N \), corresponding to the root \( s_1 \), is shown in the left part of the figure 2. Location of the supports of the generalized functions, remaining after removing the spurious solutions by means of the mentioned above criteria 1 and 2, is given in the right part of the figure 2.

**Figure 1.** The body containing three cracks and subjected to uniaxial tension.

**Figure 2.** Location of the supports of generalized functions before and after removing of spurious solutions. \( s_p = s_1 \).

It is possible to determine visually from the right part of the figure 2 that the points form three clusters. The same result is obtained automatically by the use the function FindCluster with the pointed out above options. The results of clustering are presented in the left part of the figure 3. In the right part of the figure 3 the clusters after removing the outliers are shown.
As we mentioned above, the angles of inclination of the cracks to the axis $x_1$ are determined by applying the linear regression model to the clusters. The obtained results are: $\varphi_1 = -0.05^\circ$, $\varphi_2 = 42.58^\circ$, $\varphi_3 = -44.90^\circ$. The results of reconstruction of the lengths and centers of the cracks are presented in Table 1.

Table 1. Results of reconstruction of the cracks parameters. $s_p = s_1$.

| Names of parameters | Exact data | $N = 6$ | $N = 7$ | $N = 8$ |
|---------------------|------------|---------|---------|---------|
| $(x_{11}, x_{21})$ (m) | (1, 7) | (1.000, 6.999) | (0.999, 7.000) | (0.991, 7.001) |
| $l_1$ (m) | 1 | 0.986 | 1.011 | 1.009 |
| | | | | |
| $(x_{12}, x_{22})$ (m) | (-6, 3) | (-5.998, 3.000) | (-5.999, 3.000) | (-5.999, 2.999) |
| $l_2$ (m) | 0.8 | 0.802 | 0.803 | 0.786 |
| | | | | |
| $(x_{13}, x_{23})$ (m) | (4, -2) | (4.001, -2.000) | (4.000, -2.001) | (4.004, -2.001) |
| $l_3$ (m) | 1.2 | 1.184 | 1.183 | 1.184 |

The presented results show that the cracks parameters are reconstructed with a good accuracy independently on the used generalized function $F_N$.

The results presented in figures 2, 3 and table 1 are obtained using the root $s_1$. Below we show that similar results can be obtained using the root $s_2$. Location of supports of generalized functions before and after removing of spurious solutions is presented in figure 4. Results of clustering before and after removing the outliers are presented in figure 5.
Figure 4. Location of the supports of generalized functions before and after removing of spurious solutions. $s_p = s_z$.

Figure 5. Results of clustering before and after removing the outliers. $s_p = s_z$.

The angles obtained by the linear regression are: $\varphi_1 = 0.05^\circ$, $\varphi_2 = 42.88^\circ$, $\varphi_3 = -42.03^\circ$. The results of reconstruction of the lengths and centers of the cracks are presented in Table 2.

Table 2. Results of reconstruction of the cracks parameters. $s_p = s_z$.

| Names of parameters | Exact data | $N = 8$ | $N = 9$ | $N = 10$ | $N = 9$ | $N = 10$ | $N = 11$ | $N = 7$ | $N = 8$ |
|---------------------|------------|---------|---------|---------|---------|---------|---------|---------|---------|
| $(x_{11}^0, x_{21}^0)$ (m) | (1, 7) | (1.000, 7.000) | (1.000, 7.000) | (1.000, 7.000) | (1.000, 7.000) | (1.000, 7.000) | (1.000, 7.000) | (1.000, 7.000) | (1.000, 7.000) |
| $l_1$ (m) | 1 | 1.003 | 1.007 | 1.006 | 1.007 | 1.006 | 1.007 | 1.006 | 1.006 |
| $(x_{12}^0, x_{22}^0)$ (m) | (-6, 3) | (-5.999, 3.000) | (-6.000, 3.000) | (-6.000, 3.000) | (-6.000, 3.000) | (-6.000, 3.000) | (-6.000, 3.000) | (-6.000, 3.000) | (-6.000, 3.000) |
| $l_z$ (m) | 0.8 | 0.809 | 0.815 | 0.816 | 0.815 | 0.816 | 0.815 | 0.816 | 0.816 |
| $(x_{13}^0, x_{23}^0)$ (m) | (4, -2) | (4.005, -2.001) | (3.999, -2.002) | (4.000, -2.000) | (3.999, -2.002) | (4.000, -2.000) | (3.999, -2.002) | (4.000, -2.000) | (3.999, -2.002) |
| $l_2$ (m) | 1.2 | 1.215 | 1.204 | 1.237 | 1.204 | 1.237 | 1.204 | 1.237 | 1.237 |
The accuracy of reconstruction of the cracks parameters by means of the roots $s_1$ and $s_2$ is approximately the same.

**Example 2.** In the example considered above the axes of symmetry of the material were parallel to the sides of the square. Here we consider the case when the coordinates $O_{y_1,y_2}$ are rotated with respect to coordinates $O_{x_1,x_2}$ on the angle $\pi/3$. In this case we denote the roots of the equation (10) by $s'_1$ and $s'_2$. These roots are connected with the roots $s_1$ and $s_2$ in the coordinates $O_{y_1,y_2}$ by the following equations [10]:

$$
\psi = -\pi/3.
$$

In the considered case $\psi = -\pi/3$. Location of supports of generalized functions before and after removing of spurious solutions is presented in figure 6. Results of clustering before and after removing the outliers are presented in figure 7. The presented results are obtained for the root $s'_1$.

**Figure 6.** Location of the supports of generalized functions before and after removing of spurious solutions. $\psi = -\pi/3$.

**Figure 7.** Results of clustering before and after removing the outliers. $\psi = -\pi/3$. 
The angles obtained by the linear regression are: \( \varphi_1 = 0.57^\circ, \varphi_2 = 41.26^\circ, \varphi_3 = -45.02^\circ \). The results of reconstruction of the lengths and centers of the cracks are presented in table 3.

The presented results show that the accuracy of determining the parameters of cracks does not decrease when the symmetry axes of the material are rotated with respect to the sides of the square. We do not present here the results of the reconstruction of the parameters of cracks using the root \( s'_2 \), since they are quite analogous.

Table 3. Results of reconstruction of the cracks parameters. \( \psi = -\pi / 3 \).

| Names of parameters | Exact data | \( N = 6 \)       | \( N = 7 \)       | \( N = 8 \)       |
|---------------------|------------|-------------------|-------------------|-------------------|
| \((x_{11}^0, x_{21}^0)\) (m) | (1,7)      | (1.000,7.000)     | (1.000,7.000)     | (0.999,7.000)     |
| \(l_1\) (m)         | 1          | 1.005             | 1.003             | 1.006             |
|                     |            | \( N = 6 \)       | \( N = 7 \)       | \( N = 8 \)       |
| \((x_{12}^0, x_{22}^0)\) (m) | (–6,3)     | (–5.999,3.000)    | (–5.999,3.000)    | (–5.999,3.000)    |
| \(l_2\) (m)         | 0.8        | 0.806             | 0.810             | 0.807             |
|                     |            | \( N = 6 \)       | \( N = 7 \)       | \( N = 8 \)       |
| \((x_{13}^0, x_{23}^0)\) (m) | (4,–2)     | (4.001,–2.000)    | (4.001,–2.000)    | (4.001,–2.000)    |
| \(l_3\) (m)         | 1.2        | 1.186             | 1.188             | 1.183             |

5. Conclusions
A method for identification of multiple, well-separated, rectilinear cracks in an anisotropic plate is presented. The method uses for the cracks reconstruction completely overdetermined data on the boundary of the body. The method is a generalization to the case of anisotropic body of the results obtained by the authors recently for the case of isotropic bodies. The presented numerical results show that the method enables to reconstruct the cracks parameters with sufficiently high accuracy for various types of anisotropy. In the proposed method we obtain simultaneously a set of approximations of the cracks parameters by means of using different introduced generalized functions. Numerical results show that they weakly depend on the choice of the generalized function used for identification. The proposed method contains two parameters: the threshold value for removing of spurious solutions \( \varepsilon_{cr} \) and maximal number of delta-functions used for approximation \( N_{max} \). In the presented results we took \( \varepsilon_{cr} = 0.01 \) and \( N_{max} = 20 \), but we carried out calculations for various values of the specified parameters. These calculations showed that the parameters can vary within fairly wide limits. For example the influence of the choice of the value \( \varepsilon_{cr} \) in the interval \( 1 \cdot 10^{-3} \leq \varepsilon_{cr} \leq 1 \cdot 10^{-2} \), on the results is insignificant. The possibilities of increasing of the value \( N_{max} \) are limited only by the accuracy of the calculations. The calculations were carried out up to \( N_{max} = 50 \).

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