ON THE COEFFICIENTS OF DIVISORS OF $x^n - 1$

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Abstract. Let $a(r, n)$ be $r$th coefficient of $n$th cyclotomic polynomial. Suzuki proved that $\{a(r, n) | r \geq 1, n \geq 1\} = \mathbb{Z}$. If $m$ and $n$ are two natural numbers we prove an analogue of Suzuki's theorem for divisors of $x^n - 1$ with exactly $m$ irreducible factors. We prove that for every finite sequence of integers $n_1, \ldots, n_r$ there exists a divisor $f(x) = \sum_{i=0}^{\deg(f)} c_i x^i$ of $x^n - 1$ for some $n \in \mathbb{N}$ such that $c_i = n_i$ for $1 \leq i \leq r$. Let $H(r, n)$ denote the maximum absolute value of $r$th coefficient of divisors of $x^n - 1$. In the last section of the paper we give tight bounds for $H(r, n)$.

1. Introduction

The $n$th cyclotomic polynomial $\phi_n(x)$ is given by

$$\phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(d/n)},$$

where $\mu(n)$ is the Mobius function. $\phi_n(x)$ is an irreducible polynomial of degree $\phi(n)$. $x^n - 1$ can be factored in the following way

$$x^n - 1 = \prod_{d|n} \phi_d(x).$$

Let $A(n)$ denote the largest coefficient of $\phi_n(x)$. Bateman in [2] proved the following inequality

$$A(n) \leq n^{2k-1},$$

where $k$ is the number of distinct odd prime factors. $A(n)$ has been investigated in papers [10], [3] and [6]. For a polynomial $f \in \mathbb{Z}[x]$, let $H(f)$ denote the absolute value of largest coefficient of $f$. Pomerance and Ryan in [7] introduced the function

$$B(n) := \max \{H(f) : f|(x^n - 1)\}.$$ 

They obtained a tight estimate for $B(n)$ and proved that

$$\limsup_{n \to \infty} \frac{\log \log B(n)}{\log n / \log \log n} = 3.$$ 

Let $d(n)$ denote number of divisors of $n$. Thompson in Theorem 1.2 of [12] has proved that for any function $\psi(n)$ defined on natural numbers such that $\psi(n) \to \infty$ as $n \to \infty$ we have

$$B(n) \leq n^{d(n)\psi(n)}$$

for a set of natural numbers of density 1.

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The function $B(n)$ has been the subject of the papers [5] and [9]. If $\phi_n(x) = \sum_{r=0}^{\phi(n)} a(r, n)x^r$, Suzuki in [11] proves that $\{a(r, n)|r \geq 1, n \geq 1\} = \mathbb{Z}$. We prove an analogue for divisors of $x^n - 1$ in the second section. In the third section we prove that for every finite sequence of integers $n_1, \ldots, n_r$, there exists a divisor $f(x) = \sum_{i=0}^{\deg(f)} c_i x^i$ of $x^n - 1$ for some $n \in \mathbb{N}$ such that $c_i = n_i$ for $1 \leq i \leq r$.

For $f \in \mathbb{Z}[x]$, let $(f)_r$ denote the $r$th coefficient of $f$. Then $H(r, n)$ is defined in the following way

$$H(r, n) = \max\{|(g)_r| : g|(x^n - 1)|$$

In the last section of the paper we give an upper bound for $H(r, n)$ in terms of number of divisors of $n$ and then we show that the upper bound cannot be improved significantly.

2. An analogue of Suzuki’s Theorem

If $m$ and $n$ are two natural numbers then there exists a divisor $f(x)$ of $x^l - 1$ for some $l \in \mathbb{N}$ with exactly $m$ irreducible factors such that $(f)_r = n$ and $(f)_s = -n$ for some $r, s \in \mathbb{N}$. We prove the following proposition in this section.

**Proposition 2.1.** Let $n$ and $m$ be two natural numbers then there exists a divisor $f(x) = \sum_{i=0}^{\deg(f)} a_i x^i$ of $x^l - 1$ for some $l \in \mathbb{N}$ with exactly $m$ irreducible divisors such that $\{-n, \cdots, 0, \cdots, n\} \subset \{a_1, \cdots, a_{\deg(f)}\}$.

We follow the proof of Suzuki [11]. We require the following lemmas for proving Theorem 2.1.

**Lemma 2.2.** Let $t$ be any integer greater than 2. Then there exist $t$ distinct primes $p_1 < p_2 < \cdots < p_t$ such that $p_1 + p_2 > p_t$.

**Proof.** Let $\pi(n)$ be prime counting function. From the prime number therem one can see that $\lim_{n\to\infty}(\pi(2n) - \pi(n)) = \infty$. Hence for a large $n$ we can choose $t$ distinct primes between $(n, 2n]$ and the $t$ primes picked from the set $(n, 2n]$ clearly satisfy the hypothesis. \(\square\)

**Lemma 2.3.** If $p$ is a prime.
If $p|n$ then $\phi_{np}(x) = \phi_n(x^p)$.
If $p \nmid n$ then $\phi_{np}(x) = \frac{\phi_n(x^p)}{\phi_n(x)}$.
If $n > 1$ is odd then $\phi_{2n}(x) = \phi_n(-x)$.

**Proof.** All the three statements can be proved from [11]. \(\square\)

**Proof of Proposition 2.1**

Choose an odd $t$ strictly greater than $n + 1$ and from Lemma 2.2, there exist $t$ primes $p_1 < \cdots < p_t$ such that $p_1 + p_2 > p_t$. Let $p = p_t$ and $N = \prod_{i=1}^{t} p_i$. 
From (1), we have
\[ \phi_N(x) = \prod_{d|N} (1 - x^d)^{\mu\left(\frac{N}{d}\right)} \]
\[ \equiv (1 - x)^{-1} \prod_{i=1}^{t} (1 - x^{p_i})( \mod x^{p+1}) \]
\[ \equiv (1 + x + \cdots + x^p) (1 - x^{p_1} \cdots - x^{p_t}) ( \mod x^{p+1}) \]
\[ \equiv \sum_{i=0}^{p} c_i x^i ( \mod x^{p+1}), \]
where \( c_i = \begin{cases} 
1 & 0 \leq i < p_1 \\
1 - k & p_k \leq i < p_{k+1} \\
1 - t & i = p.
\end{cases} \)

From Lemma 2.3 we have
\[ \phi_{2N}(x) = \phi_N(-x) \]
\[ \equiv \sum_{i=0}^{p} c_i (-1)^i x^i ( \mod x^{p+1}). \]

We can see that \( \{-n, \cdots, 0, \cdots, n\} \subset \{-c_1, \cdots, c_p(-1)^p\} \). Hence Proposition 2.1 is true for \( m = 1 \). Choose a prime \( p' \) greater than \( p \), from Lemma 2.3 we have
\[ \phi_{2Np'}(x) \phi_{p'}(x) = \phi_{2N}(x^p). \]

Hence the set \( \{-n, \cdots, n\} \) is a subset of set of coefficients of \( \phi_{2Np'}(x) \phi_{p'}(x) \). Hence the proposition is true for \( m = 2 \). Choose square free numbers \( n_1 < n_2 < \cdots < n_k \) and \( n'_1 < \cdots < n'_k \) such that each \( n_i \) has exactly two distinct prime factors and each \( n'_i \) are prime and every prime factor of \( n_i \) and \( n'_i \) is greater than \( 2Np' \). From (1), one can see that
\[ \phi_{2N}(x) \prod_{i=1}^{k} \phi_{n_i}(x) \prod_{i=1}^{k} \phi_{n'_i}(x) \equiv \phi_{2N}(x) \prod_{i=1}^{k} (1 - x) \prod_{i=1}^{k} (1 - x)^{-1} \]
\[ \equiv \phi_{2N}(x) ( \mod x^{2N}). \]

Hence the proposition is true for \( m = 2k + 1 \). For \( m = 2k \) consider the product
\[ \phi_{2Np'}(x) \phi_{p'}(x) \prod_{i=1}^{k-1} \phi_{n_i}(x) \prod_{i=1}^{k-1} \phi_{n'_i}(x) \equiv \phi_{2Np'}(x) \phi_{p'}(x) ( \mod x^{2Np'}). \]

Hence the proposition is true for \( m = 2k \) for \( k \geq 2 \) which completes the proof of the proposition. \( \square \)

For every natural number \( r \) the set \( \{|a(r, n) : n \in N\} \) remains bounded and let \( C(r) \) denote the maximum value of the set. \( C(r) \) has been studied by Erdős in [4]. Bachman in [11] has obtained the following asymptotic formula
\[ \log C(r) = C_o \frac{\sqrt{r}}{(\log r)^4} \left( 1 + O \left( \frac{\log \log r}{\sqrt{\log r}} \right) \right). \]
However, this is not true for rth coefficients of divisors of $x^n - 1$. The rth coefficient of divisors of $x^n - 1$ are unbounded. In fact more can be said about the first r coefficients of divisors of $x^n - 1$.

**Theorem 2.4.** For a given finite sequence of integers $\{n_i\}_{i=1}^l$ there exists a divisor

$$f(x) = \sum_{i=0}^{d_{\deg(f)}} a_i x^i$$

of $x^l - 1$ for some l such that $a_i = n_i$ for $1 \leq i \leq r$.

3. **Proof of Theorem 2.4**

The following Lemma is needed for proving Theorem 2.4.

**Lemma 3.1.** For every $n \in \mathbb{N}$, there exist two sequences of polynomials $\{d_m^{(n)}(x)\}_{m=1}^\infty$ and $\{d_m^{(n)}(x)\}_{m=1}^\infty$ where $d_m^{(n)}(x)$ and $d_m^{(n)}(x)$ are divisors of $x^{l_m} - 1$ and $x^{l_m} - 1$ respectively for some $l_m, l_m' \in \mathbb{N}$ and have the following properties.

1. $\gcd(d_m^{(n)}, d_m^{(n)}) = 1$ for $m_1 \neq m_2$.
2. $\gcd(d_m^{(n)}, d_m^{(n)}) = 1$ for $m_1 \neq m_2$.
3. $\gcd(d_m^{(n)}, d_m^{(n)}) = 1$ for all $m_1$ and $m_2$.
4. $d_m^{(n)}(x) \equiv 1 - x^n \pmod{x^{n+1}}$.
5. $d_m^{(n)}(x) \equiv 1 + x^n \pmod{x^{n+1}}$.

**Proof.** From (2)

$$1 - x^n = - \prod_{d\mid n} \phi_d(x).$$

Let $\{n_i\}_{i=1}^\infty$ be a strictly increasing sequence of all square free numbers with exactly two prime factors such that each of the prime divisor is strictly greater than n. From (1), if $1 < d \leq n$ then

$$\phi_{n_i, d}(x) \equiv \phi_d(x) \pmod{x^{n+1}}.$$

If $d = 1$ then $\phi_{n_i, d}(x) \equiv -\phi_1(x) \pmod{x^{n+1}}$. Hence for every $m$

$$\prod_{d\mid n} \phi_{n_m, d}(x) \equiv 1 - x^n \pmod{x^{n+1}}.$$  

From (2), we have

$$\frac{x^{2n} - 1}{x^n - 1} = 1 + x^n = \prod_{d\mid 2n, d\mid n} \phi_d(x).$$

Thus, we have

$$\prod_{d\mid 2n, d\mid n} \phi_{n_m, d}(x) \equiv 1 + x^n \pmod{x^{n+1}}.$$  

Define

$$d_m^{(n)}(x) := \prod_{d\mid n} \phi_{n_m, d}(x)$$

and

$$d_m^{(n)}(x) := \prod_{d\mid 2n, d\mid n} \phi_{n_m, d}(x).$$

Clearly, $\{d_m^{(n)}(x)\}_{m=1}^\infty$ and $\{d_m^{(n)}(x)\}_{m=1}^\infty$ satisfy properties (1) to (5). $\square$
Proof of Theorem 2.4

Proof. The proof is by induction on \( r \). For \( r = 1 \) and for some \( n_1 \in \mathbb{Z} \). If \( n_1 = 0 \), let \( f(x) = d_1^{(2)}(x) \equiv 1 + n_1 x \pmod{x^2} \), if \( n_1 > 0 \) let \( f(x) = \prod_{i=1}^{n_1} d_i^{(1)}(x) \equiv 1 + n_1 x \pmod{x^2} \) and if \( n_1 < 0 \) let \( f(x) = \prod_{i=1}^{-n_1} d_i^{(1)}(x) \equiv 1 + n_1 x \pmod{x^2} \). Hence the theorem is true for \( r = 1 \).

Let us assume the theorem is true for \( r = k \). For a sequence \( \{n_i\}_{i=1}^{r+1} \) from our assumption there exists a divisor \( f'(x) = \sum_{i=0}^{\deg(f')} a_i x^i \) of \( x^{l'} - 1 \) for some \( l' \) such that \( a_i = n_i \) for \( 1 \leq i \leq r \).

If \( a_{r+1} = n_{r+1} \) set \( f(x) = f'(x) \).

If \( a_{r+1} > n_{r+1} \) set

\[
f(x) = f'(x) \prod_{i=1}^{a_{r+1}-n_{r+1}} d_{n_{j_i}}^{(r+1)}(x) \equiv a_0 + \sum_{i=1}^{r+1} n_i x^i \pmod{x^{r+2}},
\]

where \( d_{n_{j_i}}(x) \) are chosen such that they are relatively prime to \( f'(x) \).

If \( a_{r+1} < n_{r+1} \) set

\[
f(x) = f'(x) \prod_{i=1}^{-a_{r+1}+n_{r+1}} d_{n_{j_i}}^{(r+1)}(x) \equiv 1 + \sum_{i=1}^{r+1} n_i x^i \pmod{x^{r+2}},
\]

where \( d_{n_{j_i}}(x) \) are chosen such that they are relatively prime to \( f'(x) \). Since the divisors are relatively prime to \( f'(x) \), \( f(x) \) is a divisor of \( x^l - 1 \) for some \( l \). Hence the theorem is true for \( r = k + 1 \) which completes the proof of the theorem. \( \square \)

4. Upper and Lower Bounds on \( H(r, n) \)

We give an upper bound for \( H(r, n) \) in terms of number of divisors of \( n \).

**Theorem 4.1.** For a given natural number \( n \) there exists a constant \( c(r) \) only depending on \( r \) such that

\[
H(r, n) \leq \frac{1}{2^{r+1}} d(n)^r + c(r) d(n)^{r-1},
\]

where \( d(n) \) is number of divisors of \( n \).

**Proof.** Let \( n \) be a natural number. From (2)

\[
x^n - 1 = \prod_{d|n} \phi_d(x).
\]

Any divisor \( f(x) \in \mathbb{Z}[x] \) will be of the form

\[
f(x) = \prod_{m \in S} \phi_m(x),
\]

where \( S \) is a set of divisors of \( n \).
where $S$ is a subset of set of all divisors of $n$. From (1)

$$f(x) = \prod_{m \in S} \prod_{d|m} (x^d - 1)^{\mu \left( \frac{m}{d} \right)}$$

$$= \prod_{d|n} (x^d - 1)^{s_1(d) - s_2(d)}$$

$$= \prod_{d|n, d \leq r} (x^d - 1)^{s_1(d) - s_2(d)} \pmod{x^{r+1}},$$

where

$$s_1(d) := |\{m \in S : d|m, \mu \left( \frac{m}{d} \right) = 1\}|,$$

$$s_2(d) := |\{m \in S : d|m, \mu \left( \frac{m}{d} \right) = -1\}|.$$

Since $|s_1(d) - s_2(d)| \leq \frac{d(n)}{2}$, the coefficients of $f(x)$ mod $x^{r+1}$ are dominated by

$$g(x) = \left( \prod_{i=1}^{r} (1 - x^i)^{-\frac{1}{2}} \right)^{d(n)}$$

$$= \sum_{i=0}^{r} c_i x^i \pmod{x^{r+1}},$$

where $c_0 = 1$ and $c_1 = \frac{1}{2}$. When $(\sum_{i=0}^{r} c_i x^i)^{d(n)}$ is expanded using multinomial theorem, coefficient of $x^r$ will be

$$\left( \sum_{i=0}^{r} c_i x^i \right)^{d(n)} \equiv \sum_{i_0 + \cdots + i_r = d(n)} \frac{d(n)! c_1^{i_1} c_2^{i_2} \cdots c_r^{i_r}}{(d(n) - i_1 - i_2 - \cdots - i_r)! i_1! \cdots i_r!}$$

$$= \frac{1}{2^r r!} d(n)^r + O(d(n)^{r-1}).$$

Hence

$$(f(x))_r \leq \frac{1}{2^r r!} d(n)^r + c(r)d(n)^{r-1},$$

for some constant $c(r) > 0$. Thus, $H(r, n) \leq \frac{r}{2^r r!} d(n)^r + c(r)d(n)^{r-1}$ for all $n \in \mathbb{N}$. □

An immediate consequence of this theorem is the following corollary.

**Corollary 4.2.**

$$H(r, n) \leq (1 + o(1)) n^{r \left( \log 2 + o(1) \right) \frac{\log \log n}{\log n}}.$$

**Proof.** This follows from the theorem of Ramanujan [8] that

$$d(n) \leq n^{\left( \frac{\log 2 + o(1)}{\log \log n} \right)}.$$

□

Now we show that the inequality can be reversed for infinitely many $n$. 
**Theorem 4.3.** There exists a sequence of natural numbers \( \{n_k\}_{k=1}^{\infty} \) such that

\[
H(r, n_k) \geq \frac{1}{2^r r!} d(n_k)^r - c_1(r)d(n_k)^{r-1}
\]

and \( d(n_k) \to \infty \) as \( k \to \infty \) where \( c_1(r) \) is a constant only depending on \( r \).

**Proof.** Let \( n_k \) be the product of first \( k \) primes with \( k \geq r \). Let

\[
f_k(x) = \prod_{m|n_k \atop \mu(m)=1} \phi_m(x)
\]

\[
= \prod_{m|n_k \atop \mu(m)=1} \prod_{d|m} (x^d - 1)^{\mu(d)}
\]

\[
= \prod_{m|n_k \atop \mu(m)=1} \prod_{d|m} (x^d - 1)^{\mu(d)}
\]

\[
= \prod_{d|n_k \atop \mu(m)=1} \prod_{m \equiv 0 \pmod d} (x^d - 1)^{\mu(d)}
\]

Since \(|\{m|n_k : m \equiv 0 \pmod d, \mu(m) = 1\}| = 2^{k-v(d)-1}\) for \( d|n_k \) and \( d \neq n_k \), where \( v(d) \) denotes number of distinct prime factors of \( d \), we have

\[
f_k(x) \equiv \prod_{d \leq r} (x^d - 1)^{\mu(d)2^{k-v(d)-1}} \pmod{x^{r+1}}
\]

\[
\equiv \left( \prod_{d \leq r} (1 - x^d \mu(d)2^{-v(d) - 1}) \right)^{2^k} \pmod{x^{r+1}}
\]

\[
\equiv \sum_{i=0}^{r} c_i x^i d(n_k) \pmod{x^{r+1}},
\]

where \( c_0 = 1 \) and \( c_1 = -\frac{1}{2^r} \). Expanding \( f_k(x) \) using multinomial theorem we can arrive at

\[
((\sum_{i=0}^{r} c_i x^i d(n))_r = \sum_{i_0+\ldots+i_r=d(n_k)} \frac{d(n_k)!c_1^{i_1}c_2^{i_2}\ldots c_r^{i_r}}{(d(n) - i_1 - i_2 - \ldots - i_r)!i_1!\ldots i_r!}
\]

\[
= (-1)^r \frac{d(n_k)^r}{2^r r!} + O(d(n_k)^{r-1}).
\]

Therefore there exists a constant \( c_1(r) \) independent of \( k \) such that

\[
|(f_k(x))_r| \geq \frac{1}{2^r r!} d(n_k)^r - c_1(r)d(n_k)^{r-1}.
\]

Thus,

\[
H(r, n_k) \geq \frac{1}{2^r r!} d(n_k)^r - c_1(r)d(n_k)^{r-1}.
\]

\[\square\]

Since \( n_k \) is product of first \( k \) primes. We have \( d(n_k) = n_k^{\frac{\log 2+ \epsilon_k}{\log \log n_k}} \) for some \( \epsilon_k \) and \( \epsilon_k \to 0 \) as \( k \to \infty \). We have the following corollary.
Corollary 4.4. For every $r$, there exists a sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$H(r, n_k) \geq (1 + \epsilon_k)n_k^{r \log 2 + \epsilon_k \log n_k},$$

as $k \to \infty$, $\epsilon_k \to 0$ and $\epsilon'_k \to 0$.

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