Consistent neutron star models with magnetic field dependent equations of state

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ABSTRACT
We present a self-consistent model for the study of the structure of a neutron star in strong magnetic fields. Starting from a microscopic Lagrangian, this model includes the effect of the magnetic field on the equation of state, the interaction of the electromagnetic field with matter (magnetisation), and anisotropies in the energy-momentum tensor, as well as general relativistic aspects. We build numerical axisymmetric stationary models and show the applicability of the approach with one example quark matter equation of state (EoS) often employed in the recent literature for studies of strongly magnetised neutron stars. For this EoS, the effect of inclusion of magnetic field dependence or the magnetisation do not increase the maximum mass significantly in contrast to what has been claimed by previous studies.

Key words: stars: neutron, magnetic fields, equation of state, methods: numerical

1 INTRODUCTION
One of the densest objects in the universe that can be observed directly, neutron stars are perfect testing grounds for theories of extreme physics. As the densities in the interior reach about $10^{15}\text{g. cm}^{-3}$, corresponding to several times normal nuclear matter saturation density, the theoretical models describing cold and dense matter, calibrated around the nuclear saturation point ($\sim 10^{14}\text{g. cm}^{-3}$) for symmetric nuclear matter (same number of protons and neutrons), must be extrapolated in density as well as asymmetry. To test these models, one can calculate the structure of neutron stars by using the energy momentum tensor and solving equations for hydrostatic equilibrium, and then compare them with astrophysical observations. One constraint that these models should satisfy is to be able to explain the highest observed neutron star mass, which is $\sim 2M_\odot$, according to the recent astrophysical reports, see Demorest et al. (2010); Antoniadis et al. (2013). Neutron stars are not only extremely dense objects, but they are known to be associated with strong magnetic fields, too. From pulsar spin-down rates, employing the simple magnetic dipole model, the estimated surface magnetic field value is typically $\sim 10^{12} - 10^{13}\text{G}$. Further, a few X-ray dim isolated neutron stars (XDINSs) and rotating radio transients (RRATs) have recently been observed with even higher magnetic fields, see Popov et al. (2006). The highest magnetic fields in neutron stars have been reported in soft gamma-ray repeaters and anomalous X-ray pulsars. The common properties of these two classes of objects have led to the proposition of a unified model of magnetars to explain the observed features. Various observations of magnetars, including the direct observation of cyclotron lines Ibrahim et al. (2004); Mereghetti (2013), indicate a magnetic field value of up to $\sim 10^{15}\text{G}$ on the surface.

As the maximum magnetic field in the interior of magnetars cannot be directly measured, it is generally estimated using the virial theorem - most estimates point towards a maximal theoretical value $\sim 10^{18}\text{G}$. Presence of a strong magnetic field can affect neutron stars in two ways. Firstly, inclusion of the magnetic field results in a modification of the energy momentum tensor, breaking the spherical symmetry of the star and resulting in an anisotropy in the latter. Secondly, it affects the Equation of State (EoS) due to Landau quantization of the constituent particles, as pointed out in Bandyopadhyay et al. (1997). One would thus expect the EoS as well as observational quantities, such as the maximum mass, to be affected by strong magnetic fields.

Previous models of magnetars, which included magnetic field effects on the EoS, computed the corresponding mass-radius relations using isotropic Tolman-Oppenheimer-Volkoff (TOV) equations, see e.g. Rabhi et al. (2000); Ferrer et al. (2010); Paulucci et al. (2011); Strickland et al. (2012); Lopes & Menezes (2012); Dexheimer et al. (2014); Casali et al. (2014). Recently Mallick & Schramm (2014) attempted to compute the structure of neutron stars in

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strong magnetic fields by a simple Taylor expansion of the energy-momentum tensor and the metric around the spherically symmetric case. However, it must be noted that at strong magnetic fields for which Landau quantization effects start to become non-negligible, the deviations from spherical symmetry are significant. There exist global models of neutron stars in which the effect of magnetic fields on the structure of neutron stars have been included in a fully general relativistic formalism, see e.g. Bocquet et al. (1993); Cardall et al. (2001); Oron (2002); Ioka & Sasaki (2004); Kinchi & Kohata (2008); Yasutake et al. (2010); Frieben & Rezzolla (2012); Yoshida et al. (2012). These simulations demonstrate that the deviation from spherical symmetry under strong magnetic fields is remarkable. However for these numerical studies the effect of magnetic field on the neutron star EoS was not taken into account.

In this paper, we present a self-consistent model to study neutron stars with strong magnetic fields, including the effect of the magnetic field on the EoS, general relativistic aspects as well as the anisotropy of the energy-momentum tensor caused by the breaking of spherical symmetry by the electromagnetic field and magnetisation. We solve Einstein’s equations in an axisymmetric metric which is determined from the axisymmetric energy momentum tensor. We investigate the effect of a strong quantizing magnetic field for the particular case of quark matter in MCFL (Magnetic Colour-Flavor-Locked) phase.

The paper is organised as follows. In Sec. 2 we construct from the Lagrangian for fermions in an (electro)-magnetic field the microscopic energy-momentum tensor. We calculate the thermodynamic average of the energy-momentum tensor and identify the contributions due to the magnetic field. In Sec. 3 we then write down the modified Maxwell equations and derive the hydrodynamic equations in presence of a magnetic field. In Sec. 4 we report our results and discuss their consequences. Finally in Sec. 5 we summarise our findings and conclusions. Some details of the derivations are provided in the Appendix. Unless otherwise stated, we work with \( c = \hbar = 1 \) and a metric signature of \((-1,1,1,1)\). The fundamental constants \( G \) and \( \mu_0 \) will be kept in the equations for better readability.

## 2 ENERGY-MOMENTUM TENSOR IN PRESENCE OF A MAGNETIC FIELD

Matter properties enter the star’s structure equations via the energy-momentum tensor as the source of the Einstein equations and for hydrodynamic equilibrium. Without the coupling to the electromagnetic field, in the general relativistic context, it is generally assumed to have the form of a perfect fluid,

\[ T^{\mu \nu} = (\varepsilon + p) u^{\mu} u^{\nu} + p g^{\mu \nu}, \tag{1} \]

where \( \varepsilon \) denotes the (matter) energy density, \( p \) the pressure, and \( u^{\mu} \) the fluid four-velocity. The EoS then relates pressure and energy density to the thermodynamically relevant parameters, chosen following the system’s equilibrium conditions. For a cold neutron star in beta equilibrium without magnetic field, the EoS is a function of only one parameter, often chosen to be baryon number density, \( n_B \), or \( \varepsilon(n_B) \). Another equivalent choice is baryon chemical potential, which is at zero temperature equal to the enthalpy per baryon \( h \), i.e. \( p(h), \varepsilon(h) \), adopted here, following Bonazzola et al. (1993); Bocquet et al. (1992).

The aim of this section is to generalise the expression for the energy-momentum tensor, Eq. (1), to the case of a non-vanishing electromagnetic field taking into account the interaction of the electromagnetic field with matter. Such an expression can be obtained by taking the thermodynamic average of a microscopic energy-momentum tensor, see next subsection. Neutron star matter is essentially composed of fermions, be it hadronic (e.g. nucleons) or quark matter. Hence, for this study we will consider only the case of fermions and write down a general formalism for fermions in an electromagnetic field. For the sake of clarity, we will thereby neglect throughout the derivations any interaction apart from that with the electromagnetic field. Additional interaction terms among the particles can be added straightforwardly, see the next subsection 2.2c where we present the model used for the numerical applications.

### 2.1 Thermodynamic average of the microscopic energy-momentum tensor

Our starting point will be the microscopic energy-momentum tensor obtained from the system’s Lagrangian. The Lagrangian density of a fermion system in the presence of a magnetic field can be written as

\[ \mathcal{L} = -\bar{\psi}(x)(D_{\mu}\gamma^{\mu} + m)\psi(x) - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}. \tag{2} \]

where \( D_{\mu} = \partial_{\mu} - i q A_{\mu} \), \( q \) denotes the charge of the particles, and

\[ F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}, \tag{3} \]

is the field strength tensor of the electromagnetic field. For the scales relevant the metric can be assumed as (locally) flat, i.e. the Minkowski metric.

There are different ways to obtain the Einstein-Hilbert energy-momentum tensor appearing as source of the Einstein equations. It is defined via the requirement that the action,

\[ S = \int \mathcal{L} \sqrt{-g} d^4x \tag{4} \]

be invariant with respect to variations of the metric. This leads to

\[ T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}\mathcal{L}). \tag{5} \]

Ferrer et al. (2010) use directly this definition to derive an expression for \( T^{\mu\nu} \). Since, however, it is not obvious to define fermion fields within a non-flat metric, we will use here another strategy. In flat space, following Noether’s theorem, a divergence free energy-momentum tensor can be derived from the invariance of the Lagrangian with respect to translations in space and time. It is, however, neither symmetric nor gauge invariant, so that it is clearly not suitable as source of the Einstein equations. However, Belinfante (1940) and Rosenfeld (1940) have shown that it can be written in a symmetrised gauge-invariant form by adding the divergence free Belinfante-Rosenfeld correction term, see Weinberg (1995). In flat space, the Belinfante-Rosenfeld tensor is equivalent to
the Einstein-Hilbert energy-momentum tensor. In our case of a fermion field coupled to an electromagnetic field it is given by

\[ T^{\mu \nu} = \frac{1}{\mu_0} F^{\mu \alpha} F_{\alpha \nu} + \frac{1}{2} \bar{\psi} (\gamma^\mu D^\nu + \gamma^\nu D^\mu) \psi + g^{\mu \nu} \mathcal{L}. \]  

(6)

The first term represents the well-known contribution of the electromagnetic field and the second term, arising from the fermion field, agrees with Eq. (36) in [Ferrer et al. (2010)] showing that indeed both ways to evaluate the energy-momentum tensor are equivalent.

Since we are interested in studying the structure of a star on macroscopic length scales, we need to calculate the thermodynamic average of the microscopic energy-momentum tensor, Eq. (6). It is assumed in the following derivations that the electromagnetic fields are constant over the averaging volume. The thermal average of \( T^{\mu \nu} \) can then be written as, see [Kapusta (1994)],

\[ \langle T^{\mu \nu} \rangle = \frac{1}{\beta V Z} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp(\hat{S}) \int_0^\beta d\lambda \int d^3 x T^{\mu \nu}, \]  

(7)

where the partition function is given by

\[ Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp(\hat{S}), \]  

(8)

and the action is

\[ \hat{S} = \int_0^\beta d\lambda \int d^3 x(\mathcal{L}(\lambda, x') - \mu \bar{n}). \]  

(9)

\( \beta = 1/T \) is the inverse temperature, \( \lambda = i x^0 \), and the term \( \mu \bar{n} \) has to be introduced in grand canonical treatment to guarantee average particle number conservation. The number density operator is \( \bar{n} = -i \bar{\psi} \gamma^0 \bar{Q} \psi \), where the operator \( \bar{Q} \) associates the number density of the particle species \( a \) with its charge \( Q_a \). \( \mu \) represents the associated chemical potential.

The thermal average of the energy-momentum tensor is then given by (see Appendix A for details of the calculations)

\[ \langle T^{\mu \nu} \rangle = \frac{}{} \]  

(10)

The first two terms on the right hand side can be identified as the pure (perfect fluid) fermionic contribution, followed by the magnetisation term and finally the usual electromagnetic field contributions to the energy-momentum tensor. The magnetisation tensor \( M_{\mu \nu} \) is thereby defined as usual as the derivative of the grand canonical potential with respect to the electromagnetic field tensor, see Eq. (A.11a) and Eq. (12). The same form for the energy-momentum tensor has been given in the context of special relativistic hydrodynamics in [Huang et al. (2010)] and for the case of a perfect fluid + the electromagnetic field the above expression agrees with [Bonazzola et al. (1993)]. From now on we will drop the brackets indicating the thermal average for better readability.

In the fluid rest frame (FRF), assuming a perfect conductor, the electric field vanishes and only the magnetic field \( b_\mu \) is nonzero. The electromagnetic field tensor can then be expressed in terms of \( b_\mu \) as \( \text{Gourgoulhon} (2012) \)

\[ F_{\mu \nu} = \epsilon_{\alpha \beta \mu \nu} u^\beta b^\alpha \]  

(11)

with the Levi-Civita tensor \( \epsilon \), associated here with the Minkowski metric. The above expression, Eq. (11), is, however, more general and can be employed with any metric. If we assume in addition, that the medium is isotropic and that the magnetisation is parallel to the magnetic field, the magnetisation tensor can be written as

\[ M_{\mu \nu} = \epsilon_{\alpha \beta \mu \nu} u^\beta m^\alpha \]  

(12)

As we shall see, the dependence of the different equations on the magnetisation can now be reduced to a dependence on the scalar quantity \( x \), which can conveniently be computed in the FRF. First, the energy-momentum tensor can be rewritten in the following way

\[ T^{\mu \nu} = (\varepsilon + p) u^\mu u^\nu + p g^{\mu \nu} \]  

(13)

\[ + \frac{1}{\mu_0} \left( -b^\mu b^\nu + (b \cdot b) u^\mu u^\nu + \frac{1}{2} g^{\mu \nu} (b \cdot b) \right) + \frac{x}{\mu_0} \left( b^\mu b^\nu - (b \cdot b) (u^\mu u^\nu + g^{\mu \nu}) \right). \]

It is obvious that for a magnetic field pointing in \( z \)-direction this expression reduces to the well-known form with magnetisation, see e.g. [Ferrer et al. (2010)]. Neglecting the effect of magnetisation, i.e. taking \( x = 0 \), it agrees with the standard MHD form, see e.g. [Gourgoulhon (2012)].

As already pointed out e.g. by [Potekhin & Yakovlev (2012)], there has been some confusion in the literature about pressure anisotropy in the presence of a magnetic field. From the above derivations it is clear that the magnetic field does not induce any anisotropy to the matter pressure defined thermodynamically as a derivative of the partition function. It transforms as a scalar. The energy-momentum tensor, however, shows anisotropies. If the spatial elements of the FRF energy-momentum tensor are interpreted as pressures, then there is a difference induced by the orientation of the magnetic field. Often the different elements are called perpendicular and parallel pressures, but they do not correspond to the thermodynamic pressure. Let us stress that this anisotropy of the energy-momentum tensor does not arise only from the magnetic field dependence of the EoS and the magnetisation contribution, but that it is inherent already to the purely electromagnetic energy-momentum tensor.

[Blanford & Hernquist (1983)] (see [Potekhin & Yakovlev (2012)], too) claim in addition that the magnetisation contribution to the energy-momentum tensor is cancelled by the Lorentz force associated with magnetisation. We shall see in Section 3.3 that upon deriving the hydrodynamic equations of motion for the system, this is indeed the case and that the system’s equilibrium depends only on \( p, \varepsilon \) and the electromagnetic field. We prefer, however, to keep the energy-momentum tensor in its natural form, Eq. (10), including the magnetisation, and add the Lorentz force to the equilibrium equations via Maxwell equations, since we think that the physical origin of the different contributions is presented in a clearer way.

1 Note, however, the different metric convention.
2.2 Equation of state

The evaluation of the matter pressure and energy density (EoS) for different models of neutron star matter in the presence of a magnetic field can be found in many papers in the (recent) literature, see e.g. Noronha & Shovkovy (2007); Rabbi et al. (2008); Ferrer et al. (2010); Paulucci et al. (2011); Strickland et al. (2012); Sinha et al. (2013). Basically, charged particles become Landau quantized (Landau & Lifshitz (1960)) in the plane perpendicular to the magnetic field. For our numerical applications, we will employ the quark model in the MCFL phase to describe the neutron star interior. Let us now briefly summarise the main characteristics of this model.

The effect of a strong magnetic field on quark matter was extensively studied earlier by many authors, see e.g. Gatto & Ruggieri (2013); Ferrer & de la Incera (2013) and references therein. Here, we employ a simple massless three-flavor MIT bag model, supplemented with a pairing interaction of NJL-type to include the possibility of colour superconductivity in the colour-flavor locked state similar to the model in Noronha & Shovkovy (2007); Paulucci et al. (2011).

\[ \mathcal{L}_{\text{pairing}} = -\frac{G_F}{2} \sum_{n=1}^{3} (\bar{\psi} \gamma^a C \psi \gamma^a) (\gamma^T \bar{C} P_{\eta} \psi), \]

where \( C = i \gamma^5 \gamma^0 \) is the charge conjugation matrix. The quark spinors \( \psi^a \) carry colour \( a = (s, d, u) \) and flavour \( \alpha = (u, d, s) \) indices. \( \bar{P}_{\eta} = \gamma^0 P_{\eta} \gamma^0 \), and the considered pairing matrix is given by \( (P_{\eta})^{ab}_{\gamma \gamma'} = i \gamma^5 \epsilon_{\gamma \gamma'} \epsilon_{\alpha \beta} \), i.e. we only take pairing in antisymmetric channels into account. Following the same scheme as in Noronha & Shovkovy (2007), we computed the EoS of quark matter in the MCFL phase, using \( G_F = 5.15 \text{ GeV}^{-2} \), \( \Lambda = 1 \text{ GeV} \) and a bag constant \( B_{\text{bag}} = 60 \text{ MeV/fm}^3 \).

The EoS for different constant magnetic field values is displayed in Fig. 1. The effect of the magnetic field starts to become significant only at very large fields \( (B \gtrsim 10^{19} \text{ G}) \). The observed oscillations are due to the de Haas-van Alphen oscillations, pointed out already in Noronha & Shovkovy (2007).

The quantity \( \kappa \), corresponding in the FRF to the magnetisation divided by the magnetic field is shown in Fig. 2 for two different values of baryon number chemical potential. The values are in agreement with Fig. 2 of Noronha & Shovkovy (2007). It is obvious that the magnetisation in this model is too small for reasonable values of the magnetic field reachable in magnetars, to induce any considerable change in the neutron star structure. This will be confirmed by the numerical results in Section 3.

3 GLOBAL MODELS IN THE STATIONARY AND AXISYMMETRIC CASE

In order to explore the effects of the inclusion of the magnetic field onto the neutron star structure and properties, we have numerically computed, within the framework of general relativity, complete models of rotating neutron stars endowed with a magnetic field. In this section, we present the physical model built to obtain the global models and the equations that are solved. Note that, in this section, Latin letters \( i, j, \ldots \) are used for spatial indices only, whereas Greek ones \( \alpha, \mu, \ldots \) denote the spacetime indices.

Within the theory of general relativity for the gravitational field, we follow the approach by Bonazzola et al. (1993) and make the assumption of a stationary, axisymmetric spacetime, in which the matter content (the energy-momentum tensor) fulfils the circularity condition. The line element expressed in spherical-like coordinates then reads:

\[ ds^2 = -N^2 \, dt^2 + B^2 \, r^2 \sin^2 \theta (d\phi - N^{\phi} \, dt)^2 + A^2 \left( dr^2 + r^2 \, d\theta^2 \right), \]

where \( N, N^{\phi}, A \) and \( B \) are functions of coordinates \( (r, \theta) \).

3.1 Maxwell equations

In the same way as in Bocquet et al. (1993) we consider here that the electromagnetic field originates from free currents, noted hereafter simply \( j^x \), which are \( a \ priori \) independent from the movements of inert mass (with 4-velocity \( u^\alpha \)). This is a limiting assumption in our model, and one should in principle use a microscopic model to derive a distribution for the free currents, too. However, such a model would require a multi-fluid approach to model the movements of free protons and electrons, and we leave it for a future study.

Under the symmetries defined in our model (see beginning of Section 3.2) the four-potential \( A_\mu \), entering in the
definition of the electromagnetic field tensor $F^\mu\nu$ through Eq. (3), can induce either purely poloidal or purely toroidal magnetic fields [Frieben & Rezzolla (2012)]. Here, we chose a purely poloidal configuration, meaning in particular that the four-potential has vanishing components $A_\theta = A_\phi = 0$. The electric and magnetic fields measured by the Eulerian observer (whose four-velocity is $n^\mu$) are then defined as $E_\mu = F_{\mu\nu} n^\nu$ and $B_\mu = -\frac{1}{\epsilon_{\mu\nu\alpha\beta}} n^\alpha F^\nu{}_{\beta}$, with $\epsilon_{\mu\nu\alpha\beta}$ the Levi-Civita tensor associated with the metric (10). The non-zero components read:

$$E_r = \frac{1}{N} \left( \frac{\partial A_\phi}{\partial r} + N^r \frac{\partial A_\phi}{\partial r} \right)$$

(17a)

$$E_\theta = \frac{1}{N} \left( \frac{\partial A_\phi}{\partial \theta} + N^\theta \frac{\partial A_\phi}{\partial \theta} \right)$$

(17b)

$$B_r = \frac{1}{Br^2 \sin \theta} \frac{\partial A_r}{\partial \theta}$$

(17c)

$$B_\theta = -\frac{1}{B \sin \theta} \frac{\partial A_\phi}{\partial r}$$

(17d)

The homogeneous Maxwell equation $F_{[\mu\nu]} = 0$ (Faraday-Gauss) is automatically fulfilled, when taking the form in Eq. (3) for the tensor $F^\mu\nu$. The inhomogeneous Maxwell equation (Gauss-Ampère) in presence of external magnetic field ($\nabla_u$ is the covariant derivative associated with the metric (10)),

$$\frac{1}{\mu_0} \nabla_u F^{\mu\nu} = j_r^{\text{free}} + \nabla_u M^{\mu\nu} ,$$

(18)

can then be transformed to give

$$\nabla_u F^{\mu\nu} = \frac{1 - x}{x} (\mu_0)^2 j_r^{\text{free}} + F^{\mu\nu} \nabla_u x .$$

(19)

This equation can be expressed in terms of the two non-vanishing components of $A^\mu$, with the Maxwell-Gauss equation

$$\Delta_3 A_\theta = \frac{1}{x - 1} \times [\mu_0 A^2 (g_{\mu\nu} j_r^{\text{free}} + g_{\mu\nu} j_r^{\text{free}}) + \partial A_r \partial x]$$

$$- \frac{B^2}{N^2} N^r r^2 \sin^2 \theta \partial A_\phi \partial N^r$$

$$- \left( 1 + \frac{B^2}{N^2} r \sin^2 \theta \right) (\partial A_\phi \partial N^r)$$

$$- \partial A_r + 2 N^r \partial A_\phi \right) \partial (\beta - \nu)$$

$$- \frac{2 N^r}{r} \left( \frac{\partial A_\phi}{\partial r} + \frac{1}{r \tan \theta} \frac{\partial A_\phi}{\partial \theta} \right) ,$$

(20)

and the Maxwell-Ampère equation

$$\Delta_3 \left( \frac{A_\phi}{r \sin \theta} \right) = \frac{1}{x - 1} \times [\mu_0 A^2 B^2 (j_r^{\text{free}} - N^r j_r^{\text{free}}) r \sin \theta]$$

$$+ \frac{1}{r \sin \theta} \partial A_\phi \partial x$$

$$+ \frac{B^2}{N^2} \sin \theta \partial N^r \left( \partial A_r + N^r \partial A_\phi \right)$$

$$+ \frac{1}{r \sin \theta} \partial A_\phi \partial (\beta - \nu) ,$$

(21)

with the following notations:

$$\nu = \ln N, \quad \alpha = \ln A, \quad \beta = \ln B,$$

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial}{\partial \theta}$$

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$$\Delta_3 = \Delta_3 - \frac{1}{r^2 \sin^2 \theta}$$

$$\partial a \partial b = \frac{\partial a}{\partial r} \frac{\partial b}{\partial r} + \frac{1}{r^2} \frac{\partial a}{\partial \theta} \frac{\partial b}{\partial \theta} .$$

In the case without magnetisation, $(x = 0)$, Eqs. (6) and (7) of [Bocquet et al. (1995)] are recovered.

### 3.2 Einstein equations and energy-momentum tensor

Under the present assumptions of a stationary, axisymmetric spacetime, the Einstein equations result in a set of four elliptic partial differential equations for the metric potentials defined in Eq. (10):

$$\Delta_3 \nu = 4 \pi G A^2 (E + S^i) + \frac{B^2}{2 N^2} \left( \partial N^r \right)^2$$

$$- \partial \nu \partial (\nu + \beta)$$

(22a)

$$\Delta_3 (N^r r \sin \theta) = -16 \pi G A^2 (E + S^i) + \frac{B^2}{2 N^2} \left( \partial N^r \right)^2$$

$$- r \sin \theta \partial N^r \partial (3 \beta - \nu)$$

(22b)

$$\Delta_2 [(NB - 1) r \sin \theta] = 8 \pi G A^2 B r \sin \theta \left( S_{r r} + S_{\theta \theta} \right)$$

(22c)

$$\Delta_2 (\nu + \alpha) = 8 \pi G A^2 S_{\nu} + \frac{3 B^2}{2 N^2} \left( \partial N^r \right)^2$$

$$(\partial \nu)^2 ,$$

(22d)

with the same notations as those introduced in Eqs. (20-21).

Finally, $E_i, J_i, S^i_j$ are quantities obtained from the so-called 3+1 decomposition of the energy-momentum tensor (for definitions, see e.g. [Gourgoulhon (2012)]). In our case of Eq. (10) describing a perfect fluid endowed with a magnetic field, including magnetisation effects, they can be written in axisymmetric stationary symmetries as:

$$E = \Gamma^2 (\varepsilon + p) - \frac{1}{2 \mu_0} \left[ (1 + 2)x E^i E_i + B^i B_i \right] ,$$

(23a)

$$J_\nu = \Gamma^2 (\varepsilon + p) U$$

$$+ \frac{1}{\mu_0} \left[ A^2 (B^i E^i - E^i B^i) + x B^i B_i U \right] ,$$

(23b)

$$S^i_{r r} = p + \frac{1}{2 \mu_0} \left( E^i E_i - E^i E_i + B^i B_i - B^i B_i \right)$$

$$+ \frac{2 x}{1 + \Gamma^2} B^i B_i ,$$

(23c)

$$S^i_{\theta \theta} = p + \frac{1}{2 \mu_0} \left( E^i E_i - E^i E_i + B^i B_i - B^i B_i \right)$$

$$+ \frac{2 x}{1 + \Gamma^2} B^i B_i ,$$

(23d)

$$S^i_{r \theta} = p + \Gamma^2 (\varepsilon + p) U^2 + \frac{1}{2 \mu_0} \left[ E^i E_i + B^i B_i \right]$$

$$+ \frac{2 x}{1 + \Gamma^2} (1 + \Gamma^2 U^2) B^i B_i ,$$

(23e)

all other components of $J_i$ and $S^i_j$ being zero. $x$ is the magnetisation, defined by Eq. (13) and $U$ is the physical fluid velocity in the $\varphi$ direction, as measured by the Eulerian observer; it is given by

$$U = \frac{B r \sin \theta}{N} (\Omega - N^r) ,$$

(24)
with \( \Omega = u^\phi / \mu^\phi \) being the fluid coordinate angular velocity (gauge independent). The electric \((E^i)\) and magnetic \((B^i)\) fields have been defined in Section 3.1.

### 3.3 Magnetostatic equilibrium

The equations for magnetostatic equilibrium can be derived from the conservation of energy and momentum, expressed as vanishing divergence of the energy-momentum tensor:

\[
\nabla_\mu T^{\mu\nu} = 0.
\]

This can be detailed as:

\[
\nabla_\mu T^{\alpha\beta} = \nabla_\alpha T^{\beta\mu} - F^{\alpha\beta} j^{\mu\text{free}} - \frac{x}{2\mu_0} F^{\sigma\tau} \nabla_\beta F^{\sigma\tau},
\]

where \(T^{\alpha\beta}\) represents the perfect-fluid contribution to the energy-momentum tensor; one can recognize the usual Lorentz force term, too, arising from free currents. In the absence of magnetisation, the expression is the same as in Bonazzola et al. (1993).

As in Bocquet et al. (1993), in the case of rigid rotation (\(\Omega\) constant across the star), a first integral of the following expression is sought:

\[
(\varepsilon + p) \left( \frac{1}{\varepsilon + p} \frac{\partial p}{\partial x^i} + \frac{\partial \ln h}{\partial x^i} \right) - F_{\mu\nu} j^{\mu\text{free}} - \frac{x}{2\mu_0} F^{\mu\nu} \nabla_i F^{\mu\nu} = 0 \quad (27)
\]

In order to obtain this first integral, one introduces the enthalpy per baryon and its derivatives. It can be shown that, even in the presence of the magnetic field, the logarithm of the enthalpy per baryon represents again a first integral of the fluid equations. To that end, let us first note that for the neutron star case with a magnetic field in beta-equilibrium and at zero temperature, the enthalpy is a function of both baryon density and magnetic field:

\[
h = h(n_b, b) = \frac{\varepsilon + p}{n_b} = \mu_b. \quad (28)
\]

Hence we have

\[
\frac{\partial \ln h}{\partial x^i} = \frac{1}{h} \left( \frac{\partial h}{\partial n_b} \frac{\partial n_b}{\partial x^i} + \frac{\partial h}{\partial b} \frac{\partial b}{\partial x^i} \right). \quad (29)
\]

In addition, the following thermodynamic relations are valid under the present assumptions:

\[
\frac{\partial h}{\partial n_b} |_{b} = \frac{\partial p}{\partial n_b} |_{b}, \quad (30)
\]

\[
\frac{\partial p}{\partial b} |_{n_b} = m(\varepsilon - \mu_b) = -\frac{\partial \varepsilon}{\partial b} |_{n_b}. \quad (31)
\]

And we obtain for the derivative of the logarithm of the enthalpy:

\[
\frac{\partial \ln h}{\partial x^i} = \frac{1}{\varepsilon + p} \left[ \frac{\partial p}{\partial n_b} |_{b} \frac{\partial n_b}{\partial x^i} + \left( \frac{\partial h}{\partial n_b} |_{n_b} - m \right) \frac{\partial p}{\partial x^i} \right] - \frac{m}{\varepsilon + p} \frac{\partial \varepsilon}{\partial x^i}.
\]

\[
= \frac{1}{\varepsilon + p} \left[ \frac{\partial p}{\partial x^i} - m \frac{\partial b}{\partial x^i} \right]. \quad (32)
\]

The second term in Eq. (27) \(- F_{\mu\nu} j^{\mu\text{free}}\) is treated as in Bonazzola et al. (1993) and we assume that i) matter is a perfect conductor \((\alpha_i = -\Omega A_i \text{ inside the star}); ii) it is possible to relate the components of the electric current density to the electromagnetic potential \(A_i\), through an arbitrary function \(f\), called the current function:

\[
j^\phi - \Omega j^\rho = (\varepsilon + p) f (A_i). \quad (33)
\]

Under these assumptions, the Lorentz force term becomes:

\[
F_{\mu\nu} j^{\mu\text{free}} = (j^\rho - \Omega j^\phi) \frac{\partial A_i}{\partial x^i} = (\varepsilon + p) \frac{\partial M}{\partial x^i}, \quad (34)
\]

with

\[
M(r, \theta) = - \int_0^{A_i(r, \theta)} f(x) dx. \quad (35)
\]

The last term can be written in terms of the magnetic field \(b^i\) in the FRF as (with \(b^2 = b^ib_i\)):

\[
\frac{x}{2\mu_0} F^{\mu\nu} \nabla_i F^{\mu\nu} = \frac{x}{\mu_0} (b^j \nabla_i b^j - b^i \nabla^j u^j) = b \nabla_i b = m \frac{\partial b_i}{\partial x^i}, \quad (36)
\]

from the expression (31), and the definition (13).

Thus, this last term cancels with its counterpart in Eq. (22) and the first integral (27) keeps exactly the same form as without magnetisation:

\[
\ln h(r, \theta) + \nu(r, \theta) - \ln \Gamma(r, \theta) + M(r, \theta) = \text{const.} \quad (37)
\]

### 3.4 Numerical resolution

The equations have been solved with the library \textsc{Loren}, using spectral methods to solve Poisson-like partial differential equations appearing in the Einstein-Maxwell system (20) and (21). For more details about these methods, see e.g. Grandclement & Novak (2000). The code follows the algorithm presented by Bocquet et al. (1993), but with the modification of the inclusion of new magnetisation terms, i.e. depending on the magnetisation \(r\) defined in Eq. (13), in these partial differential equations. However, as it has been shown in Eq. (37) the expression for the equilibrium of the fluid in the gravitational and magnetic fields does not change.

The most important difference with Bocquet et al. (1993) comes from the use of an EoS which gives all the needed variables: \(p, \varepsilon, n_b, x\); depending on two parameters (instead of one): the enthalpy \(h\) (28) and the magnetic field amplitude in the FRF \(b = \sqrt{\mu_0 B^i B_i}\). These quantities are first computed and stored on a table once for all. This is then read by the code computing the equilibrium global models, and a bi-dimensional interpolation using Hermite polynomials is used, following the method described by Sweezey (1990), to ensure thermodynamic consistency of the interpolated quantities \((p(h, b), \varepsilon(h, b), n_b(h, b)\) and \(x(h, b)\)).

The free physical parameters entering our model are: the EoS, the current function \(f\), the rotation frequency \(\Omega\) and the logarithm of the central enthalpy \(H_c = \ln(h(r = 0))\). Once the equilibrium configuration has been computed, global quantities are obtained either from integration over the star’s volume (e.g. baryonic mass \(M_b\)) or from the asymptotic behaviour of the gravitational field (e.g. gravitational mass \(M_G\)) and of the electromagnetic field (e.g. magnetic moment \(M\)). Detailed definitions and formulae can be found in Bonazzola et al. (1993) and Bocquet et al. (1993).
4 RESULTS AND DISCUSSION

We computed models of fully relativistic neutron stars with a poloidal magnetic field, employing the EoS described in Sec. 2.2, and a constant current function \( f(x) = f_0 \). As shown in Bocquet et al. (1995), the choice of other current functions for \( f \) would not alter the conclusions. Varying \( f_0 \) allowed us to vary the intensity of the magnetic field, as measured for instance by the value of the radial component at the star’s pole (polar magnetic field), or by the magnetic moment. The variation of the central enthalpy has a direct influence on the star’s masses (\( M_B \) and \( M_G \)), although they depend on the rotation frequency and magnetic field strength, too. To demonstrate pure magnetic field effects on the neutron star configurations, we first computed static neutron stars.

The first point to emphasize is that, as it has already been illustrated, e.g. in Bocquet et al. (1995); Cardall et al. (2001), the stellar configurations can strongly deviate from spherical symmetry due to the anisotropy of the energy-momentum tensor in presence of a non-vanishing electromagnetic field. As an example we show in Fig. 3 the magnetic field lines and the enthalpy profile in the \((r, \theta)\)-plane for a configuration with a magnetic moment of \( 3.25 \times 10^{32} \) A m\(^2\) and a baryon mass of \( 2.56 M_\odot \). These values correspond to a polar magnetic field of \( 8.16 \times 10^{17} \) G and a gravitational mass of \( 2.22 M_\odot \). The asymmetric shape of the star due to the Lorentz forces exerted by the electromagnetic field on the fluid is evident from the figures. Upon increasing the magnetic field strength the star’s shape becomes more and more elongated, finally reaching a toroidal shape, see Cardall et al. (2001). However, our code is not able to treat this change of topology and the configuration shown in Figs. 3 represents the limit in terms of magnetic field strength, that can be computed within our numerical framework. Therefore for this study, we compute stellar configurations within this maximum limit. Nevertheless, note that the polar magnetic field value is well above any observed magnetic field in magnetars.

The determination of the maximum gravitational mass is usually performed considering sequences of constant magnetic moment \( M \) and increasing central enthalpy \( H_c \) (see Bocquet et al. (1993)). To be able to relate better with astrophysical observations of magnetars, in Fig. 4 we plot the value of the polar magnetic field corresponding to the values of the magnetic moment for a neutron star having baryonic mass \( 1.6 M_\odot \). In this figure, three curves have been plotted, corresponding to three types of configurations:

(i) A full model as described in Sect. 3 denoted by EoS(B), \( M \);
(ii) A model with magnetic field dependence of the EoS, but no inclusion of the magnetisation terms \( x \) in the energy-momentum tensor – setting \( x = 0 \) in Eqs. (19) and (23) – denoted by EoS(B), no \( M \);
(iii) A bare model where both these effects are excluded (no EoS(B), no \( M \)), which is a case comparable with the study by Bocquet et al. (1993).

These settings shall be used later in this work, too. The polar magnetic field increases linearly with the magnetic moment, and is indistinguishable between the three cases discussed above, i.e. with and without inclusion of magnetic field dependence of the EoS and magnetisation. The relation between the magnetic moment and the polar magnetic field changes only slightly depending on the baryon mass of the star, the present figure can therefore be used as a guideline for all the configurations shown within this work.

We further studied the influence of using magnetic field dependent EoS on the neutron star maximal mass, and we
have retrieved this result in the left panel of Fig. 6. It is evident that there is very little difference on inclusion of the full model, with respect to the one by Bocquet et al. (1995), but in order to be more precise, we plotted in the right panel of Fig. 6 the relative differences in the maximum gravitational masses as functions of the magnetic moment, with and without the inclusion of the above magnetic field effects compared to the case excluding these effects. In this right panel, we see that even for very high magnetic moments, corresponding to polar magnetic field much higher than those observed in magnetars (see Fig. 4 for correspondence), the relative difference in the maximal mass of magnetised neutron stars is at most of the order $10^{-3}$ and therefore negligible compared with uncertainties existing in the EoS models.

Another neutron star parameter of astrophysical interest is the compactness $\mathcal{C}$, which is the dimensionless ratio of the gravitational mass and radius

$$\mathcal{C} = \frac{GM}{R_{\text{circ}}} c^2,$$  

(38)

where $R_{\text{circ}}$ is the circumferential equatorial radius (see Bonazzola et al. (1993)). We studied the behaviour of the compactness of a neutron star of baryon mass $1.6 M_\odot$ with magnetic moment, as illustrated in the Fig. 4. The compactness was found to decrease with increase in magnetic moment. This is understandable from the centrifugal forces exerted by the Lorentz force on matter at the centre, increasing with increasing magnetic moment, i.e. magnetic field, see e.g. the discussion in Cardall et al. (2001). Again the lines corresponding to the cases with and without magnetisation or magnetic field effects in the EoS are almost indistinguishable and the main effect arises from the purely electromagnetic part already included in Bocquet et al. (1993).

As neutron stars are expected to be good sources of gravitational radiation, we have looked at the influence of magnetisation and magnetic field-dependent EoS on this mechanism. The scenario we studied was the emission of gravitational waves when the magnetic dipole is not aligned with the rotation axis, and magnetic field deformation induces a time variation of the quadrupole moment. This setting has previously been addressed by Bonazzola & Gourgoulhon (1996), who have shown that, using the standard quadrupole formula to compute the characteristics of gravitational waves, it was possible to split the quadrupole moment into two parts: a first depending on the rotation, and a second on the distortion of the star. Thus, it is possible to study the amplitude of gravitational waves by looking at the deformation of static (non-rotating) stars. We therefore considered a sequence of non-rotating stars, with fixed baryon mass $M_B = 2M_\odot$ and increasing magnetic moment. For this sequence, we looked at the quadrupole moment $Q$, as defined in Eq. (7) of Salgado et al. (1994), as a function of the magnetic moment $\mathcal{M}$. First, we recovered the behaviour $Q \sim \mathcal{M}^2$, given in Eq. (32) of Bonazzola & Gourgoulhon (1996). Then, we looked again at the relative difference in $Q$, between a sequence using the full approach devised here, and a simpler one with the EoS not depending on the magnetic field and no magnetisation term. Contrary to the maximum mass case, this difference remains almost independent of the magnetic moment, with a value $\sim 10^{-3}$. This shows that the gravitational wave
emission properties are very little sensitive to the use of a magnetic-field dependent EoS.

Finally, we computed rotating configurations along a sequence of constant magnetic dipole moments. For the moment the observed magnetars all rotate very slowly with periods of the order of seconds, see Mereghetti (2013), mainly because the strong magnetic fields induce a very rapid spin-down. This means that the fast rotating configurations do not have any realistic observed counterpart for the moment, and we perform this investigation mainly for curiosity. As obtained in the static case, the maximum gravitational mass was found to increase with the magnetic dipole moment $M$.

In particular, we chose a sequence of neutron stars rotating at 700 Hz, close to the frequency of the fastest known rotating pulsar, which rotates at 716 Hz (Hessels et al. (2006)). In Fig. 8 we see the same behaviour for both cases: the maximal mass increases with the magnetic field and, although it is not shown in the figure, the effects of magnetisation or inclusion of the magnetic field are very small, as in the non-rotating case.

5 CONCLUSIONS

In this work, we developed a self-consistent approach to determine the structure of neutron stars in strong magnetic fields, relevant for the study of magnetars. Starting from the microscopic Lagrangian for fermions in a magnetic field, we derived a general expression for the energy-momentum tensor of one fluid in presence of a non-vanishing electromagnetic field. Due to the perfect conductor assumption, the electric field vanishes in the fluid rest frame, and therefore only magnetisation and the magnetic field dependence of the equation of state enter the final results. Equations for the star’s equilibrium are obtained as usual from the conservation of the energy-momentum tensor coupled to Maxwell and Einstein equations. This consistent derivation shows in particular that, as claimed by Blandford & Hernquist (1982), the equilibrium only depends on the thermodynamic equation of state and magnetisation explicitly only enters Maxwell and Einstein equations. This should answer some discussion in the recent literature.

Figure 6. Neutron star maximal mass (left panel) and relative difference in this mass among three models, as a function of magnetic moment. The three models correspond to the possibility or not of including of magnetisation term $x$ (“M” or “no M”), and to the magnetic field dependence or not of the EoS (“EoS(B)” or “no EoS(B)”).

Figure 7. Compactness as a function of magnetic moment for neutron star with baryon mass 1.6 $M_{\odot}$ with and without magnetic field dependence and magnetisation (see Fig.6).
on the role of magnetisation, see e.g. Dexheimer et al. (2014) and Potekhin & Yakovlev (2012).

We have extended an existing axisymmetric numerical code to solve these coupled equations and to obtain static and rotating neutron star configurations in general relativity. Taking as an example the equation of state of quark matter in the MCFL phase, we then investigated the effect of inclusion of the dependence of the EoS on the magnetic field as well as the magnetisation on the structure of the neutron star. In contrast to the results obtained previously by other authors, see Paulucci et al. (2011); Dexheimer et al. (2014); Sinha et al. (2013), we found that the effect of inclusion of the magnetic field dependence on the EoS does not change significantly the star’s structure. In particular, we have shown that the maximum mass of the neutron star is only slightly modified even for the strongest magnetic fields considered, well above those values which we could consider as realistic from present magnetar observations. The difference to previous results arises due to the fact that in these works the isotropic TOV equations were used to solve for the star’s structure, whereas the magnetic field causes the star to deviate from spherical symmetry considerably. We hereby confirm, within our fully consistent model, the conjecture by Broderick et al. (2002), who argued that the influence of the magnetic field on the EoS starts to become important only for values where the star has already taken a toroidal shape, comparing the model by Cardall et al. (2001) with calculations of the magnetic field dependence of several realistic EoS.

An obvious question is of course to which extent we could expect higher field values with stronger effects induced via the magnetic field dependence of the equation of state. It is the surface poloidal magnetic field which is estimated from the observation of spin-down of pulsars via electromagnetic radiation. In this work, we have studied magnetised neutron star models with a purely poloidal magnetic configuration, taking maximum field values well above those observed. It has, however, been argued that differential rotation in neutron stars could amplify the seed poloidal field resulting in the generation of a toroidal field with higher absolute value than the poloidal one. The effect of purely toroidal fields on the structure of neutron stars has already been elaborately studied in Kiuchi & Kotake (2008); Yasutake et al. (2010); Frieben & Rezzolla (2012); Yoshida et al. (2012). The virial theorem allows to estimate a maximum limit to the magnetic field in the interior, irrespective of the configuration, from the observed poloidal surface fields, indicating that the maximum values considered within this work are probably not exceeded in realistic situations. Thus, although the study of magnetised neutron star models with purely poloidal magnetic field is not the most general one, it gives us a fairly good idea about the effect of the maximum field on the stellar structure. Our formalism can of course be extended to any magnetic field configuration and in a future work this statement should be tested quantitatively.

In addition, as the main aim of the present work was to build consistent models and to show the numerical applicability of the formalism, we have tested it and computed neutron star models only with one particular equation of state. In principle, it could be that another model shows stronger effects. For hadronic models, as shown in Broderick et al. (2001), the effect of Landau quantization on the proton Fermi energy starts to become considerable for field values of the order $5\times 10^{15}$ G, thus too high to influence the magnetar structure. Another point is that we have not taken into account the contribution of anomalous magnetic moments. A simple estimate of energetics shows, however, that the magnetic interaction energy for (vacuum) nucleonic magnetic moments is of the order $0.006 B_{10^{15} G}$ MeV, thus of the order of several MeV for fields of $10^{18}$ G. This is still very small compared with the energy scale of a high density EoS, we therefore do not expect a very strong effect on the global structure, either. To be more quantitative, in a typical hadronic RMF model, with Landau quantization and anomalous magnetic moments, the magnetisation is about ten times larger than in our quark matter example, see Rabhi et al. (2014), and the effects should thus be only slightly larger.

The possibility of a ferromagnetic instability in nuclear matter has been discussed, too, see e.g. Vidaurre et al. (1984); Maruyama & Tatsumi (2001). The general opinion was that this instability is unphysical since no such behaviour is observed for nuclear matter in heavy ion collisions nor in microscopic nuclear matter calculations starting from the fundamental nucleon-nucleon interaction. Ferromagnetic behaviour in neutron star matter – quark or hadronic – is, however, not unambiguously refuted, see for instance the recent works of Diener & Scholz (2013) and Tsue et al. (2013). In the context of strong magnetic fields inside a neutron star, ferromagnetism could considerably enhance the effect of the magnetic field on the equation of state and the corresponding neutron star structure. The magnetic catalysis effect, modifying hadronic masses in a strong magnetic field, see Haber et al. (2013), could additionally influence the equation of state, too.

To close, let us mention that our code is part of LORENE.
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APPENDIX A: DERIVATION OF THERMALLY AVERAGED ENERGY-MOMENTUM TENSOR

The following method follows closely that employed in Ferrer et al. (2010). For the further developments, we will
distinguish between the fermionic contribution to $T_{\mu\nu}^f$.

\[
T_{\mu\nu}^f = \frac{1}{2} \bar{\psi}(\gamma^\mu \partial^\nu + \gamma^\nu \partial^\mu) \psi - \frac{1}{2} (\partial^\mu A^\nu + \partial^\nu A^\mu) - g^\mu\nu \bar{\psi}(x)(D_\mu \gamma^\nu + m)\psi(x),
\]

and the purely electromagnetic one,

\[
T_{\mu\nu}^{EM} = -\frac{1}{\mu_0} \left( F^{\mu\alpha} F_{\alpha}^\nu + g^{\mu\nu} \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right).
\]

Within the former expression we have introduced the electromagnetic current,

\[
j^\mu = \frac{\partial \mathcal{L}}{\partial \partial^\mu A_\mu} = i q \bar{\psi} \gamma^\mu \psi.
\]

The electromagnetic part is treated as completely classical here and we will assume in addition that the electromagnetic fields are constant over the averaging volume. It is then trivial to show that for the purely electromagnetic part we obtain simply

\[
<T_{\mu\nu}^{EM}> = -\frac{1}{\mu_0} \left( F^{\mu\alpha} F_{\alpha}^\nu + g^{\mu\nu} \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right).
\]

The fermionic part is a little bit more complicated to evaluate. Let us first compute some derivatives of the partition function related to thermodynamic quantities. We follow here the standard formalism, as employed e.g. in Ferrer et al. (2014). We will thereby factorise the partition function as $Z = Z_{BD} Z_f$ which is possible since we treat the electromagnetic part as classical and constant. It can thus be taken out of the integral. The first derivative will be $\partial Z_f/\partial \beta$. To that end we perform the variable transformation $\lambda \rightarrow \chi \beta$. The partition function then reads

\[
Z_f = \int D\psi D\bar{\psi} \exp\left\{-\beta \int d\lambda \int d^3 x \lambda \left[ \frac{1}{2} \partial^\mu \psi \partial_\mu \psi + \frac{i}{\hbar} \partial^\mu \bar{\psi} \partial_\mu \psi \right] \right\}.
\]

Reintroducing the electromagnetic part of the partition function we can rewrite the above expression as a thermal expectation value

\[
\frac{1}{\beta V} \frac{\partial \ln Z_f}{\partial \beta} = \frac{1}{\beta V} \int D\psi D\bar{\psi} \exp\left(\tilde{S}_f\right) \times \int_0^\beta d\lambda \int d^3 x (\mathcal{L}_f - \mu \hat{n} - \bar{\psi} \partial_\lambda \gamma^0 \psi) . \tag{A3}
\]

In the above expression we recognise the grand canonical potential,

\[
\Phi_f = -\frac{1}{\beta V} \ln Z_f , \tag{A5}
\]

with the following well-known thermodynamic relations

\[
\frac{1}{\beta V} \frac{\partial \ln Z_f}{\partial \beta} = -\Phi_f + T \frac{\partial \Phi_f}{\partial T} = -\Phi_f - Ts = -\varepsilon + \mu n . \tag{A6b}
\]

\[
\frac{1}{\beta V} V \frac{\partial \ln Z_f}{\partial V} = -\frac{\partial \Phi_f}{\partial V} = p . \tag{A6c}
\]

We have introduced here the number density $n$, the entropy density $s$, pressure $p$ and the energy density $\varepsilon$.

Following the same procedure with the variable transform $x' \rightarrow L_1 x'$ and the volume given by $V = L_1 L_2 L_3$ we can show that

\[
\frac{1}{\beta V} V \frac{\partial \ln Z_f}{\partial V} = (\bar{\psi} \gamma^\mu \partial^\mu \psi + \mathcal{L}_f) . \tag{A7}
\]

The next step will be the derivative with respect to the electromagnetic field strength tensor. Assuming a perfect conductor, the electric field vanishes in the FRF and only the magnetic field $b_\mu$ is nonzero. In order to be precise we will, under the assumption of constant electromagnetic fields over the averaging volume, take

\[
A_\mu = -\frac{1}{2} F_{\mu\nu}(y)(x - y)^\nu \tag{A8}
\]

at some given point $y$, where $F_{\mu\nu}$ is given by Eq. [11].

It can be shown that

\[
\frac{\partial \Phi}{\partial \tau} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} u^\alpha (x-y)^\nu j^\mu . \tag{A9}
\]

Hence,

\[
\frac{1}{\beta V} \frac{\partial \ln Z_f}{\partial \tau} = \frac{1}{\beta V} \int D\psi D\bar{\psi} \exp(\tilde{S}_f) \times \int_0^\beta d\lambda \int d^3 x (\mathcal{L}_f - \mu \hat{n} - \bar{\psi} \partial_\lambda \gamma^0 \psi) = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} u^\alpha (x-y)^\nu j^\mu . \tag{A10}\]

The above equation gives the magnetisation vector,

\[
m_\tau = -\frac{\partial \Phi_f}{\partial \tau} = \frac{m}{b} n \tag{A11a}
\]

\[
= -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} u^\alpha (x-y)^\nu j^\mu , \tag{A11b}
\]

As can be seen here, in isotropic matter, the magnetisation is aligned with the magnetic field, and the polarisation components vanish due to the perfect conductor assumption in the FRF. From Eq. [A11a] the magnetisation tensor, $M_{\mu\nu}$, can be calculated via Eq. [12]. It is obviously antisymmetric, $M_{\mu\nu} = -M_{\nu\mu}$, as it should be and fulfills the requirement that its divergence gives the expectation value of the current, too.

Using Eq. [A8], we can write in addition

\[
\langle A^\mu j^\nu \rangle = -\frac{1}{2} \epsilon_{\mu\rho\sigma\nu} b_\sigma (x-y)_\rho j^\mu . \tag{A12}
\]

Inserting Eq. [A11a] into Eq. [12] and using the antisymmetry of $F_{\mu\nu} = -F_{\nu\mu}$ we obtain finally

\[
-\frac{1}{2} \langle A_\mu j^\nu + A_\nu j_\mu \rangle = \frac{1}{2} (F_\mu^\rho M_{\rho\nu} + F_\nu^\rho M_{\rho\mu}) . \tag{A13}\]

Let us now come back to the energy-momentum tensor. Neglecting for the moment the coupling to the electromagnetic field, we can show that, in the FRF, the non-diagonal elements of the energy-momentum tensor in homogeneous isotropic matter have a vanishing expectation value. This can be seen by writing $\psi(x)$ in a Fourier basis with respect to momentum $p^0$. The derivative $i\partial_\lambda$ adds a factor $p^\lambda$ and the non-diagonal elements become the product of an even
and an odd function in $p^\mu$. This product vanishes upon integration over $d^4p$. Formulated in another way we use here the fact that the movement of one homogeneous isotropic fluid is described by one vector, its four-velocity $u^\mu$. As a consequence the energy-momentum tensor necessarily has the following structure

$$
\langle T_{\mu\nu}^f \rangle = a_1 u^\mu u^\nu + a_2 g^{\mu\nu},
$$

(A14)

with two scalar functions $a_{1,2}$ which can be evaluated in the FRF as

$$a_2 = -(u \cdot u) \langle T^{ii}_f \rangle = -(u \cdot u) \frac{\partial \ln Z}{\beta \partial V} = -(u \cdot u)p \quad \text{(A15)}$$

$$a_1 = \langle T^{00}_f \rangle + \langle T^{ii}_f \rangle = \varepsilon + p. \quad \text{(A16)}$$

$p$ and $\varepsilon$ are the pressure and energy density of the system, respectively.

In the presence of an electromagnetic field, the eigenstates of charged fermions are no longer momentum eigenstates. They become quantized in the plane perpendicular to the direction of the magnetic field $b^\mu$, see Landau & Lifshitz (1960). It can again be shown, e.g. with the help of an explicit representation of the spinor fields in presence of a magnetic field, that in the FRF the thermal average of the non-diagonal elements of $T_{\mu\nu}^f$ vanish except those involving explicitly the magnetic field. We thus obtain formally the same result for the fermionic part of the energy-momentum tensor,

$$
\langle T_{\mu\nu}^f \rangle = (\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu} + \frac{1}{2} (F^\nu_{\mu} M^{\tau\mu} + F^\mu_{\nu} M^{\tau\nu}).
$$

(A17)

Pressure and energy density are defined as derivatives of the fermionic partition function in the FRF as before. Putting everything together we obtain the final result, see Eq. (10).