Generators, Extremals and Bases of Max Cones

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Abstract

We give simple algebraic proofs of results on generators and bases of max cones, some of which are known. We show that every generating set $S$ for a cone in max algebra can be partitioned into two parts: the independent set of extremals $E$ in the cone and a set $F$ every member of which is redundant in $S$. We exploit the result that extremals are minimal elements under suitable scalings of vectors. We also give an algorithm for finding the (essentially unique) basis of a finitely generated cone.

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1 Introduction

Max algebra is the analog of linear algebra that is obtained by considering $\mathbb{R}_+$ with max times operations:

$$a \oplus b := \max(a, b),$$

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\[ a \otimes b := ab \]

and extending these in the natural way to \( \mathbb{R}^n_+ \) and \( \mathbb{R}_{nk}^+ \): If \( A = (a_{ij}) \), \( B = (b_{ij}) \) and \( C = (c_{ij}) \) are non-negative matrices of compatible sizes and \( \alpha \) is a non-negative real number, we write \( C = A \oplus B \) if \( c_{ij} = a_{ij} \oplus b_{ij} \) for all \( i, j \), \( C = A \otimes B \) if \( c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k (a_{ik}b_{kj}) \) for all \( i, j \) and \( C = \alpha \otimes A \) if \( c_{ij} = \alpha \otimes a_{ij} \) for all \( i, j \).

In this paper we define (max) cones, extremals, generating sets, independent sets, totally dependent sets, and bases in \( \mathbb{R}^n_+ \) and give simple algebraic proofs of results relating these. Some of the results are known and some have been proved in a more general setting in [W]. Here they appear as corollaries to our main result of Section 2: every generating set \( S \) for a cone in max algebra can be partitioned into two parts: the independent set of extremals \( E \) in the cone and a set \( F \) every member of which is redundant in \( S \). The main result of Section 3 is the statement that bases of max cones are the sets of minima for suitably scaled vectors. It helps to prove some topological results, and relates our problem to the classical problem of finding maxima of a set of vectors which is described in [KLP] and in [PS, Section 4.1.3]. In Section 4 we give two versions of an algorithm for finding the (essentially unique) basis of a finitely generated max cone and a MATLAB program which implements one version. This algorithm is based on the fundamental results of [CG1] about the max-linear systems of equations.

In max algebra as in linear algebra a basis is normally defined as an \textit{indexed set}, that is a \textit{sequence} if the basis is finite or countable, see [W] for a definition in max algebra or [Bou, p.10] in linear algebra. Since we wish to show the inclusion of the set of extremals (which do not have a natural order) in every generating set or basis for a cone we define the latter in term of \textit{sets} in our main sections. We thereby exclude the possibility of a repetition of elements in generating sets. But we change our point of view in a final section on algorithms for finitely generated cones since we wish to consider the generators as columns of a matrix.

The key concept of an extremal in a max cone appears in [W] under the term "irreducible element" and the key idea of Section 3 is implicit in [J] Proposition 2.9. Our topic is related to the emerging field of tropical geometry, but we do not pursue this direction further. Papers on tropical geometry relevant to max algebra are e.g. [BlYu], [DS], and [J].

Max cones have much in common with convex cones, see [R] for a general ref-
ference. This has been exploited (and generalized) in many papers including those just quoted and e.g. [CGQS] and [Z]. To this end, the basic concepts of this paper and such main results as Theorem 9, Corollary 24 and Proposition 25 have their direct analogs in convex analysis. We do not provide details, as convex geometry is also beyond our scope here.

2 Generating sets, bases and extremals

Definition 1 A subset $K$ of $\mathbb{R}_+^n$ is a max cone in $\mathbb{R}_+^n$ if it is closed under $\oplus$ and $\otimes$.

Definition 2 Let $S \subseteq \mathbb{R}_+^n$. Then $u$ is a max combination of $S$ if

$$u = \bigoplus_{x \in S} \lambda_x x, \quad \lambda_x \in \mathbb{R}_+, \quad (1)$$

where only finite number of $\lambda_x \neq 0$. The set of all max combinations will be denoted by $\text{span}(S)$. We put $\text{span}(\emptyset) = \{0\}$.

Evidently, $\text{span}(S)$ is a cone. If $\text{span}(S) = K$, we call $S$ a set of generators for $K$.

Definition 3 An element $u \in K$ is an extremal in $K$ if

$$u = v \oplus w, \quad v, w \in K \implies u = v \text{ or } u = w. \quad (2)$$

If $u$ is an extremal in $K$ and $\lambda > 0$ then $\lambda u$ is also an extremal in $K$. Extremals are defined in [W] under the name "irreducible elements".

Definition 4 An element $x \in \mathbb{R}_+^n$ is scaled if $||x|| = 1$.

In Section 2, $||x||$ may be any norm in $\mathbb{R}^n$ (they are all equivalent). However, in Section 3 we specialize to the max norm, $||x|| = \max x_i$ in order to exploit the property that it is max linear on $\mathbb{R}_+^n$. If $S \subseteq \mathbb{R}_+^n$ we may call $S$ scaled to indicate that it consists of scaled elements.

Definition 5 Let $S$ be a set of vectors in $\mathbb{R}_+^n$.

1. The set $S$ is dependent if, for some $x \in S$, $x$ is a max combination of $S \setminus \{x\}$. Otherwise, $S$ is independent.
2. The set $S$ is totally dependent if every $x \in S$ is a max combination of $S \setminus \{x\}$.

Thus the empty set of vectors is both independent and totally dependent. Since $\text{span}(\emptyset) = \{0\}$, the set $\{0\}$ is totally dependent.

**Definition 6** Let $K$ be a cone in $\mathbb{R}^n_+$. A set $S$ of vectors in $\mathbb{R}^n_+$ is a basis for $K$ if it is an independent set of generators for $K$.

**Lemma 7** Let $S$ be a set of scaled generators for the cone $K$ in $\mathbb{R}^n_+$ and let $u$ be a scaled extremal in $K$. Then $u \in S$.

**Proof.** Suppose $u$ is given by the max combination (1). Since the number of nonzero $\lambda_x$ is finite, we may use Definition 3 and induction to show that $u = \lambda_x x$ for some $x$. But $u$ and $x$ are both scaled, hence $u = x$ and $u \in S$. □

**Lemma 8** The set of scaled extremals of a cone is independent.

**Proof.** If the set $E$ of scaled extremals is nonempty let $u$ be a scaled extremal in $K$ and apply Lemma 7 to the cone $K_1 := \text{span}(E \setminus \{u\})$. This shows $u \not\in K_1$ and the result is proved. □

Below we use subscripts for elements of vectors in $\mathbb{R}^n_+$ and superscripts to label vectors. We now state and prove the main result of our paper.

**Theorem 9** Let $S$ be a set of scaled generators for a cone $K$ in $\mathbb{R}^n_+$ and let $E$ be the set of scaled extremals in $K$. Then

1. $E \subseteq S$.
2. Let $F = S \setminus E$. Then for any $u \in F$, the set $S' := S \setminus \{u\}$ is a set of generators for $K$.

**Proof.** Assertion 1 repeats Lemma 7.

To prove Assertion 2, let $u \in F$. Since $u$ is not an extremal, we have $u = v \oplus w$, $v, w \in K$ and both $u \neq v$ and $u \neq w$. Since $S$ generates $K$, we have

\begin{align*}
v &= \lambda_u u \oplus \bigoplus_{x \in S'} \lambda_x x, \\w &= \mu_u u \oplus \bigoplus_{x \in S'} \mu_x x.
\end{align*}

(3)

(4)
If \( \lambda_u \geq 1 \) then \( v \geq u = v \oplus w \geq v \) and hence \( v = u \), contrary to assumption. It follows that \( \lambda_u < 1 \) and similarly \( \mu_u < 1 \). Thus

\[
u = v \oplus w = \gamma_u u \oplus z
\] (5)

where

\[
z = \bigoplus_{x \in S'} \gamma_x x
\] (6)

and

\[
\gamma_x = \max(\lambda_x, \mu_x), \ x \in S.
\] (7)

We observe that \( \gamma_u = \max(\lambda_u, \mu_u) < 1 \). It follows that \( \gamma_u u_i < u_i \) if \( u_i > 0, i = 1, \ldots, n \). We conclude by (5) that \( u = z \). Thus, by (6), in any max combination involving \( u \), this vector can be replaced by a max combination of vectors in \( S' \) and the theorem is proved. \( \Box \)

Thus we have shown that every set of generators for a cone \( K \) must contain all the extremals of \( K \). The following example shows that the set \( F \) of Theorem 9 need not be totally dependent.

**Example 10** Let \( K \) be the cone in \( \mathbb{R}^2_+ \) generated by \( u_r = [1, 1/r]^T, r = 1, \ldots \). Then \( u^1 \) is the unique scaled extremal in \( K \). But the set \( F = \{u_r : r = 2, \ldots\} \) is not totally dependent since \( u^2 \) is an extremal in \( \text{span}(F) \).

In a very general context, Wagneur [W] showed that if a basis exists for a cone \( K \) then it is essentially unique, see also [Se] for further references. The following result shows somewhat more.

**Theorem 11** Let \( E \) be the set of scaled extremals in a max cone \( K \). Let \( S \subseteq K \) consist of scaled elements. Then the following are equivalent:

1. The set \( S \) is a minimal set of generators for \( K \).
2. \( S = E \) and \( S \) generates \( K \).
3. The set \( S \) is a basis for \( K \).

**Proof.**
1. \( \implies \) 2. By Theorem 9 we have \( S = E \cup F \) where every element of \( F \) is redundant in \( S \). But since \( S \) is a minimal set of generators, we must have \( F = \emptyset \). Hence \( S = E \).
2. \( \implies \) 3. The set \( E \) is independent and a generating set.
3. \[\implies 1.\] By independence of \( S \) the span of a proper subset of \( S \) is strictly contained in \( \text{span}(S) \). □

Thus we have shown that if a cone has a (scaled) basis then it must be its set of (scaled) extremals. We note that a maximal independent set in a cone \( K \) need not be a basis for \( K \) as is shown by the following example.

**Example 12** Let \( K \subseteq \mathbb{R}^2_+ \) consist of all \( [x_1, \ x_2]^T \) with \( x_1 \geq x_2 > 0 \). If \( 1 > a > b > 0 \), then \( \{[1, \ a]^T, \ [1, \ b]^T\} \) is a maximal independent set in \( K \) which does not generate \( K \).

**Corollary 13** If \( K \) is a finitely generated cone, then its set of scaled extremals is the unique scaled basis for \( K \).

*Proof.* Since \( K \) is finitely generated, there exists a minimal set of generators \( S \). By Theorem 11 \( S = E \) and \( S \) is a basis. □

**Corollary 14** If \( S \) is a nonempty scaled totally dependent set in \( \mathbb{R}^n_+ \) then \( S \) is infinite.

*Proof.* Suppose that \( S \) is finite and let \( K = \text{span}(S) \). By Corollary 13 \( K \) contains scaled extremals which, by Theorem 9, must be contained in \( S \) given that \( K = \text{span}(S) \). But then \( S \) is not totally dependent. This contradiction proves the result. □

**Corollary 15** Let \( K \) be a cone in \( \mathbb{R}^n_+ \). The following are equivalent:

1. There is no extremal in \( K \).
2. There exists a totally dependent set of generators for \( K \).
3. Every set of generators for \( K \) is totally dependent.

*Proof.* Since there always exists a set of generators for \( K \) (e.g \( K \) itself), each of the Conditions 2. and 3. is equivalent to Condition 1. by Theorem 9. □
3 Topology and order

We now consider \( \mathbb{R}^n_+ \) in the topology induced by the Euclidean topology of \( \mathbb{R}^n \). That is, a set in \( \mathbb{R}^n_+ \) will be called open if and only if it is the intersection of an open subset of \( \mathbb{R}^n \) with \( \mathbb{R}^n_+ \). A cone \( K \) is called open if \( K \setminus \{0\} \) is open, and it is called closed if it is closed as a subset of \( \mathbb{R}^n_+ \), or equivalently of \( \mathbb{R}^n \).

**Corollary 16** If \( K \) is an open cone in \( \mathbb{R}^n_+ \) that does not contain unit vectors, then every generating set for \( K \) is totally dependent.

**Proof.** It is enough to show that there is no extremal in \( K \), for then the result follows by Theorem 9. Let \( u \in K \). Since \( u \) is not a unit vector, there are at least two indices \( k, l \in \text{supp}(u) \). Since \( K \) is open, we have \( w^p = u - \varepsilon e^p \in K \), \( p = k, l \) for sufficiently small \( \varepsilon \) and \( u = w^k \oplus w^l \). None of \( w^p \), \( p = k, l \) is equal to \( u \), hence \( u \) is not an extremal, and the corollary follows. \( \square \)

An example of an open cone in \( \mathbb{R}^n_+ \) is furnished by the cone \( K \) of all positive vectors in \( \mathbb{R}^n_+ \). We note that, for this particular case, Corollary 16 was shown in [CGB]. Another example of an open cone consists of all vectors \([a, b]^T \) in \( \mathbb{R}^2_+ \) with \( a > b > 0 \).

**Definition 17** Let \( v \in \mathbb{R}^n_+ \). Then the *support* of \( v \) is defined by

\[
\text{supp}(v) = \{ j \in \{1, \ldots, n\} : v_j > 0 \}.
\]

The cardinality of \( \text{supp}(u) \) will be written as \( |\text{supp}(u)| \).

In order to relate the natural partial order on \( \mathbb{R}^n_+ \) to results on extremals of cones we introduce a scaling of vectors in \( \mathbb{R}^n_+ \) for each \( j \in \{1, \ldots, n\} \) such that for each scaled vector \( v_j = 1 \).

**Definition 18**

1. Let \( u \in \mathbb{R}^n_+ \) and suppose \( j \in \text{supp}(u) \). Then we define

\[
u(j) = u/u_j.
\]

2. Let \( S \subseteq \mathbb{R}^n_+ \). We define \( S(j) = \{ u(j) : u \in S \text{ and } j \in \text{supp}(u) \} \) for all \( j = 1, \ldots, n \).

3. Let \( S \subseteq \mathbb{R}^n_+ \). An element \( u \in S \) is called *minimal* in \( S \), if \( v \leq u \) and \( v \in S \) implies that \( v = u \).
4. Let $K$ be a cone in $\mathbb{R}_n^+$, let $u \in K$, and let $j \in \text{supp}(u)$. We define

$$D_j(u) = \{ v \in K(j) : v \leq u(j) \}.$$ 

**Proposition 19** Let $S \subseteq \mathbb{R}_n^+$. Then the following are equivalent:

1. $u \in \text{span}(S)$.

2. For each $j \in \text{supp}(u)$ there is an $x^j \in S$ such that $j \in \text{supp}(x^j)$ and $x^j(j) \in D_j(u)$.

**Proof.** $2. \implies 1$: If 2. holds, then $u = \bigoplus_{j \in \text{supp}(u)} \lambda_j x^j$ where $\lambda_j = u_j / x^j_j$.

1. $\implies 2$. Conversely if 1. holds, then it follows immediately from (1) that for each $j \in \text{supp}(u)$ there is an $x^j \in S$ with $\lambda_j x^j \leq u$ and $(\lambda_j x^j)_j = u_j$. Clearly, $\lambda_j = u_j / x^j_j$ which yields 2. 

The following immediate corollary to Proposition 19 is essentially found as [DS, Proposition 5] and called "Tropical Carathéodory Theorem".

**Corollary 20** Let $S \subseteq \mathbb{R}_n^+$. Then $u \in \text{span}(S)$ if and only if there are $k$ vectors $x^1, \ldots, x^k \in S$, where $k \leq |\text{supp}(u)|$, such that $u \in \text{span}\{x^1, \ldots, x^k\}$. 

Theorem 9 shows that in any set of generators of a cone an element which is not an extremal is redundant. Hence any finite number of such generators are redundant. The following corollary to Proposition 19 may be combined with Theorem 22 to yield conditions for an infinite set of generators to be redundant.

**Corollary 21** Let $K$ be a cone in $\mathbb{R}_n^+$ and let $T$ be a set of generators for $K$. Let $U \subseteq T$ and let $S = T \setminus U$. Then $S$ generates $K$ if and only if each $u \in T$ satisfies condition 2. of Proposition 19. 

**Theorem 22** Let $S \subseteq \mathbb{R}_n^+$ be generated by the cone $K$ and let $u \in S$, $u \neq 0$. Then the following are equivalent:

1. $u$ is an extremal in $K$.

2. For some $j \in \text{supp}(u)$, $u(j)$ is minimal in $K(j)$.

3. For some $j \in \text{supp}(u)$, $u(j)$ is minimal in $S(j)$. 

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Proof. 1.  $\Rightarrow$ 3. If $|\text{supp}(u)| = 1$ then $u(j)$ is minimal in $S(j)$. So suppose that $|\text{supp}(u)| > 1$ and that $u(j)$ is not minimal in $S(j)$ for any $j \in \text{supp}(u)$. Then for each $j \in \text{supp}(u)$ there exists $x^j \in S(j)$ such that $x^j \leq u(j)$, $x^j \neq u(j)$. Therefore $u = \bigoplus_{j \in \text{supp}(u)} u_j x^j$, and $u$ is proportional with none of $x^j$. Hence $u$ is not an extremal in $K$.

3. $\Rightarrow$ 2. Let $v \in K$ and assume that $j \in \text{supp}(v)$ and $v(j) \leq u(j)$. We need to show that $v(j) = u(j)$. By Proposition 19, there is a $w \in S$ such that $w(j) \leq v(j)$. Thus $w(j) \leq v(j) \leq u(j)$ and by Assumption 3. it follows that $w(j) = v(j) = u(j)$.

2. $\Rightarrow$ 1. Let $u(j)$ be minimal in $K(j)$ for some $j \in \text{supp}(u)$ and suppose that $u = v \oplus w$, $v, w \in K$. Then both $v \leq u$ and $w \leq u$ and either $v_j = u_j$ or $w_j = u_j$, say (without loss of generality) that $v_j = u_j$. Hence $v \in K(j)$ and it follows from 2. that $v(j) = u(j)$. Hence also $v = u$ which proves 1.

Note that in Theorem 22 we can of course have $S = K$.

**Corollary 23** Let $K$ be a cone in $\mathbb{R}^n_+$. Then $K$ is generated by its extremals (and hence has an essentially unique basis) if and only if $D_j(u)$ has a minimal element for each $u \in K$ and each $j \in \text{supp}(u)$.

**Proof.** Suppose that $x^j$ is a minimal element of $D_j(u)$. Since, for $v \in K(j)$, $v \leq x^j$ implies that $v \in D_j(u)$, $x^j$ is also a minimal element of $K(j)$. We now obtain the Corollary by combining Proposition 19 and Theorem 22.

**Corollary 24** Let $K$ be a closed cone in $\mathbb{R}^n_+$. Then $K$ is generated by its set of extremals.

**Proof.** Let $u \in K$ and let $j \in \text{supp}(u)$. It is easily shown that $D_j(u)$ is compact since $K$ is closed. Hence $D_j(u)$ contains a minimal element $x^j$. The result now follows by Corollary 23.

The max norm is max linear on $\mathbb{R}^n_+$:

$$||\lambda u \oplus \mu v|| = \lambda ||u|| \oplus \mu ||v||.$$  \hspace{1cm} (8)

This is exploited in the following proposition.

**Proposition 25** If $S \subset \mathbb{R}^n_+$ is compact and $0 \notin S$, then the cone $K = \text{span}(S)$ is closed.
Proof. Consider a sequence \( u^i \in K \) converging to \( v \). Then, by Corollary 20 we have
\[
u^i = \bigoplus_{s=1}^{n} \lambda_{is} w^{is},
\]
where \( w^{is} \in S \) and \( \lambda_{is} \in \mathbb{R}_+ \). By (8)
\[
||v^i|| = \bigoplus_{s=1}^{n} \lambda_{is} ||w^{is}||. \tag{9}
\]
Since the sequence \( u^i \) converges (to \( v \)), the norms \( ||v^i|| \) are bounded from above by some \( M_1 > 0 \). On the other hand, we have \( ||w^{is}|| \geq M_2 \) for some \( M_2 > 0 \), since \( S \) is closed and does not contain 0. Then by (9) \( \lambda_{is} ||w^{is}|| \leq M_1 \) for all \( i \) and \( s \), and \( \lambda_{is} \leq M_1 M_2^{-1} \) for all \( i \) and \( s \). Thus \( \lambda_{is} \) are bounded from above. But \( ||w^{is}|| \) are also bounded from above, since \( S \) is compact. This implies that there is a subsequence \( u^{j(i)} \) such that for all \( s = 1, \ldots, n \) the sequences \( w^{j(i)s} \) and \( \lambda_{j(i)s} \) converge. Denote their limits by \( \bar{w}^s \) and \( \bar{\lambda}_s \), respectively, then \( \bar{w}^s \in S \) and \( \bar{\lambda}_s \leq M_1 M_2^{-1} \). By continuity of \( \oplus \) and \( \otimes \) we obtain that
\[
v = \bigoplus_{s=1}^{n} \bar{\lambda}_s \bar{w}^s. \tag{10}
\]
Thus \( v \in K \). \( \square \)

Corollary 26 If the set of scaled extremals of a max cone \( K \) is closed and generates \( K \), then \( K \) is closed.

The converses of both Corollary 24 and Corollary 26 are false as is shown by the following example.

Example 27 1. In \( \mathbb{R}^3_+ \) let \( S \) consist of all vectors \( [x_1, x_2, 1]^T \), \( 0 \leq x_1 < 1/2 \) such that \( x_1 + x_2 = 1 \) and let \( K = \text{span}(S) \). Then the section of \( K \) given by \( x_3 = 1 \) consists of all vectors \( [x_1, x_2, 1]^T \), \( 0 \leq x_1 < 1/2 \), \( 0 \leq x_2 \leq 1 \) such that \( x_1 + x_2 \geq 1 \). Thus \( K \) is not closed. The set of scaled extremals of \( K \) is \( S \).

2. Now let \( S' = S \cup \{u\} \), where \( u = [1/2, 0, 1]^T \) and let \( K' = \text{span}(S') \). Then the section of \( K' \) given by \( x_3 = 1 \) consists of \( K \) together with the line segment whose end points are \( u \) and \( [1/2, 1, 1]^T \). Thus \( K' \) is closed. The set of scaled extremals of \( K' \) is \( S' \) which is not closed.
The cross sections of $K$ and $K'$ by $x_3 = 1$ are shown on Figure 1, together with the generating sets $S$ and $S' = S \cup \{u\}$. We also have the following extension of Corollary 13.

**Corollary 28** Any finitely generated max cone $K$ is closed and its set of scaled extremals is the unique scaled basis of $K$.

4 Algorithmic considerations

As explained in the introduction we redefine our basic concepts for this section which is concerned with finitely generated cones. We also restate a suitable adaptation of Corollary 13.

**Definition 29** Let $V \in \mathbb{R}^{nk}_+$ and let $V^i$ be the matrix obtained from $V$ by deleting column $i$, $i = 1, \ldots, k$. Then the cone $K$ generated by the columns $(v^1, \ldots, v^k)$ of $V$ consists of all vectors of form $V \otimes x$, $x \in \mathbb{R}^k$. Further, the columns of $V$ form a basis for $K$ if, for $i$, $i = 1, \ldots, k$, there is no $x \in \mathbb{R}^{k-1}$ such that $V_i \otimes x = v^i$.

**Proposition 30** Let $V \in \mathbb{R}^{nk}_+$. Then there exists a unique submatrix $U \in \mathbb{R}^{np}_+, 0 \leq p \leq k$ whose columns form a basis for the cone generated by the columns of $V$ (and every other basis is of form $UPD$, where $P$ is a permutation matrix and $D$ is a diagonal matrix with nonzero diagonal elements).
We shall apply the following proposition. Note that all statements in this proposition have been proved in a more general setting in [CG1, Theorem 14.3]. See also [Bu] and [CG2, Chapter III].

**Proposition 31** Let $U \in \mathbb{R}^{nk}_{+}$ with all columns nonzero and let $v \in \mathbb{R}^n_{+}$. Let $x \in \mathbb{R}^k_{+}$ be defined by

$$x_i = \min \{v_j / u^i_j : u^i_j \neq 0, j = 1, \ldots, n\} \quad (11)$$

for $i = 1, \ldots, k$. Then

$$U \otimes x \leq v, \quad (12)$$

$$x = \max \{z \in \mathbb{R}^k_{+} : U \otimes z \leq v\}, \quad (13)$$

$$U \otimes x = \max \{U \otimes z : z \in \mathbb{R}^k_{+}, U \otimes z \leq v\}. \quad (14)$$

Further, there exists $z \in \mathbb{R}^k_{+}$ such that $U \otimes z = v$ if and only if $U \otimes x = v$.

**Proof.** Assertion (12) follows from the observation that $U \otimes z \leq v$ if and only if $z_i \leq v_j / u^i_j$ if $j \in \supp(u^i)$, $i = 1, \ldots, k$. Note that $x \in \mathbb{R}^k_{+}$ since no column of $U$ is zero. Since $\otimes$ is isotone (that is, $x \leq y$ implies $A \otimes x \leq A \otimes y$), assertions (13) and (14) follow immediately. For the final statement assume that $U \otimes z = v$ for some $z$. By (12) and (14) we have $v = U \otimes z \leq U \otimes x \leq v$, and the statement follows. The converse is trivial. \qed

**Algorithm 32** Input: $V \in \mathbb{R}^{nk}_{+}$.
Output: An $n \times p$ submatrix $U$ of $V$ whose columns form the essentially unique basis for the cone generated by the columns of $V$.

**Step 1.** Initialize $U = V$.
**Step 2.** For each $j = 1, \ldots, k$ if $u^j \neq 0$ set $v = u^j$, and for each $i \neq j$ compute $x_i$ by (11), if $u^i \neq 0$, and set $x_i = 0$ otherwise. If $U \otimes x = v$, set $u^j = 0$.
**Step 3.** Delete the zero columns of $U$. The remaining columns of $U$ are the basis we seek.

**Remark 33** The restriction in Proposition 31 that each column $U \in \mathbb{R}^{nk}_{+}$ must have a positive element was imposed to avoid definitions for $a/0$, $a > 0$, or $0/0$. The restriction is inessential in the sense that for general $U \in \mathbb{R}^{nk}_{+}$ we may define $x_i$ by (11) whenever $u^i \neq 0$ and choose $x_i$ arbitrarily in $\mathbb{R}^k_{+}$ whenever $u^i = 0$. Then all assertions of the Proposition still hold, with exception of (13). It is possible to extend $\mathbb{R}^n_{+}$ by adding a maximal element $\infty$ so that (13) still holds.
We omit details and present the MATLAB program maxbas that implements Algorithm 32 but employs such an extension. We also give an example produced by MATLAB’s rand function with some elements put equal to 0. Note that in [CG1, Theorem 16.2] a similar algorithm called \emph{A-test} has been presented. It enables to identify columns that are dependent on other columns of an \( n \times k \) matrix in \( O(nk^2) \) time. However, there is no discussion of bases in connection with this method in [CG1].

\footnotesize

\begin{verbatim}
function [B,f] = maxbas(A),
%     B = the unique max times basis for the max col space of A
%     f = indices of columns of B in A
%     calls maxpr, max multiplication of matrices

[m,n] = size(A); B = A; t = max(max(A));
for j = 1:n
    v = compl(j,n);
    c = B(:,j); BB = B(:,v);
    warning('off'),
    e = ones(1,n-1); C = c*e; x = min(C./BB)';
    z = maxpr(BB,x);
    if abs(c-z) < t*eps, B(:,j) = 0; end,
end u = max(B); f = find(u > t*eps); B = B(:,f);

A =
Columns 1 through 5
0.9049  0.5612  0.0069  0.2544  0.0136
0.2822  0.0  0  0.8030  0.5616
0.0650  0.7727  0.1957  0.6678  0.4546

>> [B,f] = maxbas(A)
B =
    0.9049  0.5612  0.0069  0.0136
    0.2822  0.0  0  0.5616
    0.0650  0.7727  0.1957  0.4546
f =
    1  2  3  5
\end{verbatim}

We note that a second form of the algorithm may be based on set covering condition (15) below, which appears in [CG1, Theorem 15.6]. It can also be found in [Bu] Section 2 but only in the case when all vectors are positive. See also [AGK, Theorem 3.5] for an interesting functional generalization of this condition (and more). With \( v \) and \( U \) as in Step 2, denote by \( N_i \) the set \( \{ j : v(j) \geq u^i(j) \} \). By Proposition 19, \( v \in \text{span}(u^1, \ldots, u^m) \) if and only if
\[
\bigcup_{i=1}^{m} N_i = \text{supp}(v). \tag{15}
\]

With \( x \) given by (11), we note that
\[
N_i = \begin{cases} 
\{ j \in \text{supp}(u^i) : v_j/u^i_j = x_i \} & \text{if } x_i \neq 0, \\
\emptyset & \text{if } x_i = 0.
\end{cases} \tag{16}
\]
Thus Step 2 in Algorithm 32 may be replaced by

**Step 2’**: For each \( j = 1, \ldots, k \) such that \( u^j \neq 0 \): set \( v = u^j \) and for each \( i \neq j \) compute \( N_i = \{ j : v(j) \geq u^i(j) \} \) according to (16), if \( u^i \neq 0 \), and set \( N_i = \emptyset \) otherwise. If \( \bigcup_{i \neq j} N_i = \text{supp}(v) \), set \( u^j = 0 \).

Our algorithms are of complexity \( O(nk^2) \).

If \( S \) is the set of columns of the matrix \( U \), then it follows from Theorem 22 that a basis for the cone generated by \( S \) consists of the union of the \( n \) sets \( M(j), j = 1, \ldots, n \), where \( M(j) \) consist of the vectors minimal in \( S(j) \). The problem of finding all maxima (or minima) of \( k \) vectors in \( \mathbb{R}^n \) is considered in [KLP], and also in [PS, Section 4.1.3], where it is dubbed the problem of Erehwon Kings. The computational complexity of methods developed in [KLP] and [PS] is bounded from above by \( O(n^2 k (\log_2 k)^{n-2}) + O(k \log_2 k) \), \( n \geq 2 \), see [PS, Theorem 4.9] and [KLP, Theorem 5.2].

To solve our problem we can apply these methods to each \( S(j), j = 1, \ldots, n \) separately. Taking into account that for each \( j \) we need \( O(nk) \) operations to find the coordinates of essentially \( (n - 1) \)-dimensional vectors in \( S(j) \), this yields an alternative method with complexity not smaller than \( O(n^2 k) \) and not greater than \( O(n^3 k (\log_2 k)^{n-3}) + O(k \log_2 k) \), \( n \geq 3 \). This method may be preferred if \( \log_2 k \) is substantially larger than \( n \).

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\[\text{That is, the complexity is not greater than } O(k \log_2 k) \text{ if } n = 2 \text{ and not greater than } O(n^2 k (\log_2 k)^{n-2}) \text{ if } n \geq 3.\]
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