QUERY COMPLEXITY OF CLUSTERING WITH SIDE INFORMATION

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Abstract. Suppose, we are given a set of $n$ elements to be clustered into $k$ (unknown) clusters, and an oracle/expert labeler that can interactively answer pair-wise queries of the form, "do two elements $u$ and $v$ belong to the same cluster?". The goal is to recover the optimum clustering by asking the minimum number of queries. In this paper, we initiate a rigorous theoretical study of this basic problem of query complexity of interactive clustering, and provide strong information theoretic lower bounds, as well as nearly matching upper bounds. Most clustering problems come with a similarity matrix, which is used by an automated process to cluster similar points together. Nevertheless, obtaining an ideal similarity function is extremely challenging due to ambiguity in data representation, poor data quality etc., and this is one of the primary reasons that makes clustering hard. To improve accuracy of clustering, a fruitful approach in recent years has been to ask a domain expert or crowd to obtain labeled data interactively. Many heuristics have been proposed, and all of these use a similarity function to come up with a querying strategy. However, there is no systematic theoretical study.

Our main contribution in this paper is to show the dramatic power of side information aka similarity matrix on reducing the query complexity of clustering. A similarity matrix represents noisy pair-wise relationships such as one computed by some function on attributes of the elements. A natural noisy model is where similarity values are drawn independently from some arbitrary probability distribution $f_+$ when the underlying pair of elements belong to the same cluster, and from some $f_-$ otherwise. We show that given such a similarity matrix, the query complexity reduces drastically from $\Theta(nk)$ (no similarity matrix) to $O(k^2 \log n H^2(f_+ \| f_-))$ where $H^2$ denotes the squared Hellinger divergence. Moreover, this is also information-theoretic optimal within an $O(\log n)$ factor. Our algorithms are all efficient, and parameter free, i.e., they work without any knowledge of $k$, $f_+$ and $f_-$, and only depend logarithmically with $n$. Our lower bounds could be of independent interest, and provide a general framework for proving lower bounds for classification problems in the interactive setting. Along the way, our work also reveals intriguing connection to popular community detection models such as the stochastic block model, significantly generalizes them, and opens up many venues for interesting future research.

1. Introduction

Clustering is one of the most fundamental and popular methods for data classification. In this paper we initiate a rigorous theoretical study of clustering with the help of an oracle, a model that saw a recent surge of popular heuristic algorithms.

Suppose we are given a set of $n$ points, that need to be clustered into $k$ clusters where $k$ is unknown to us. Suppose there is an oracle that either knows the true underlying clustering or can compute the best clustering under some optimization constraints. We are allowed to query the oracle whether any two points belong to the same cluster or not. How many such queries are needed to be asked at minimum to perform the clustering exactly? The motivation to this problem lies at the heart of modern machine learning applications where the goal is to facilitate more accurate learning from less data by interactively asking for labeled data, e.g., active learning and crowdsourcing. Specifically, automated clustering algorithms that rely just on a similarity matrix often return inaccurate results. Whereas, obtaining few labeled data adaptively can help in significantly improving its accuracy. Coupled with this observation, clustering with an oracle
has generated tremendous interest in the last few years with increasing number of heuristics developed for this purpose \cite{21, 37, 13, 39, 40, 18, 36, 12, 20, 28}. The number of queries is a natural measure of “efficiency” here, as it directly relates to the amount of labeled data or the cost of using crowd workers—however, theoretical guarantees on query complexity is lacking in the literature.

On the theoretical side, query complexity or the decision tree complexity is a classical model of computation that has been extensively studied for different problems \cite{16, 4, 8}. For the clustering problem, one can obtain an upper bound of $O(nk)$ on the query complexity easily and it is achievable even when $k$ is unknown \cite{37, 13}: to cluster an element at any stage of the algorithm, ask one query per existing cluster with this element (this is sufficient due to transitivity), and start a new cluster if all queries are negative. It turns out that $\Omega(nk)$ is also a lower bound, even for randomized algorithms (see, e.g., \cite{13}). In contrast, the heuristics developed in practice often ask significantly less queries than $nk$. What could be a possible reason for this deviation between the theory and practice?

Before delving into this question, let us look at a motivating application that drives this work.

A Motivating Application: Entity Resolution. Entity resolution (ER, also known as record linkage) is a fundamental problem in data mining and has been studied since 1969 \cite{17}. The goal of ER is to identify and link/group different manifestations of the same real world object, e.g., different ways of addressing (names, email address, Facebook accounts) the same person, Web pages with different descriptions of the same business, different photos of the same object etc. (see the excellent survey by Getoor and Machanavajjhala \cite{19}). However, lack of an ideal similarity function to compare objects makes ER an extremely challenging task. For example, DBLP, the popular computer science bibliography dataset is filled with ER errors \cite{29}. It is common for DBLP to merge publication records of different persons if they share similar attributes (e.g. same name), or split the publication record of a single person due to slight difference in representation (e.g. Marcus Weldon vs Marcus K. Weldon). In recent years, a popular trend to improve ER accuracy has been to incorporate human wisdom. The works of \cite{39, 40, 37} (and many subsequent works) use a computer-generated similarity matrix to come up with a collection of pair-wise questions that are asked interactively to a crowd. The goal is to minimize the number of queries to the crowd while maximizing the accuracy. This is analogous to our interactive clustering framework. But intriguingly, as shown by extensive experiments on various real datasets, these heuristics use far less queries than $nk$ \cite{39, 40, 37}—barring the $\Omega(nk)$ theoretical lower bound. On a close scrutiny, we find that all of these heuristics use some computer generated similarity matrix to guide in selecting the queries. Could these similarity matrices, aka side information, be the reason behind the deviation and significant reduction in query complexity?

Let us call this clustering using side information, where the clustering algorithm has access to a similarity matrix. This can be generated directly from the raw data (e.g., by applying Jaccard similarity on the attributes), or using a crude classifier which is trained on a very small set of labelled samples. Let us assume the following generative model of side information: a noisy weighted upper-triangular similarity matrix $W = \{w_{i,j}\}$, $1 \leq i < j \leq n$, where $w_{i,j}$ is drawn from a probability distribution $f_+$ if $i, j, i \neq j$, belong to the same cluster, and else from $f_-$. However, the algorithm designer is given only the similarity matrix without any information on $f_+$ and $f_-$. In this work, one of our major contributions is to show the separation in query complexity of clustering with and without such side information. Indeed the recent works of \cite{18, 32} analyze popular heuristic algorithms of \cite{37, 40} where the probability distributions are obtained from real datasets which show that these heuristics are significantly suboptimal even for very simple distributions. To the best of our knowledge, before this work, there existed no algorithm that works for arbitrary unknown distributions $f_+$ and $f_-$ with near-optimal performances. We develop a generic framework for proving information theoretic lower bounds for interactive clustering using side information, and design efficient algorithms for arbitrary $f_+$ and $f_-$ that nearly match the lower bound. Moreover, our algorithms are parameter free, that is they work without any knowledge of $f_+, f_-$ or $k$. 
Connection to popular community detection models. The model of side information considered in this paper is a direct and significant generalization of the planted partition model, also known as the stochastic block model (SBM) \cite{27 15 4 1 24 23 11 33}. The stochastic block model is an extremely well-studied model of random graphs which is used for modeling communities in real world, and is a special case of a similarity matrix we consider. In SBM, two vertices within the same community share an edge with probability \( p \), and two vertices in different communities share an edge with probability \( q \). It is often assumed that \( k \), the number of communities, is a constant (e.g. \( k = 2 \) is known as the planted bisection model and is studied extensively \cite{11 33 15} or a slowly growing function of \( n \) (e.g. \( k = o(\log n) \)). The points are assigned to clusters according to a probability distribution indicating the relative sizes of the clusters. In contrast, not only in our model \( f_+ \) and \( f_- \) can be arbitrary probability mass functions (pmfs), we do not have to make any assumption on \( k \) or the cluster size distribution, and can allow for any partitioning of the set of elements (i.e., adversarial setting). Moreover, \( f_+ \) and \( f_- \) are unknown. For SBM, parameter free algorithms are known relatively recently for constant number of linear sized clusters \cite{3 23}.

There are extensive literature that characterize the threshold phenomenon in SBM in terms of \( p \) and \( q \) for exact and approximate recovery of clusters when relative cluster sizes are known and nearly balanced (e.g., see \cite{2} and therein for many references). For \( k = 2 \) and equal sized clusters, sharp thresholds are derived in \cite{1 33} for a specific sparse region of \( p \) and \( q \). In a more general setting, the vertices in the \( i \)th and the \( j \)th communities are connected with probability \( q_{ij} \) and threshold results for the sparse region has been derived in \cite{2} - our model can be allowed to have this as a special case when we have pmfs \( f_{i,j} \)s denoting the distributions of the corresponding random variables. If an oracle gives us some of the pairwise binary relations between elements (whether they belong to the same cluster or not), the threshold of SBM must also change. But by what amount? This connection to SBM could be of independent interest to study query complexity of interactive clustering with side information, and our work opens up many possibilities for future direction.

Developing lower bounds in the interactive setting appears to be significantly challenging, as algorithms may choose to get any deterministic information adaptively by querying, and standard lower bounding techniques based on Fano-type inequalities \cite{9 30} do not apply. One of our major contributions in this paper is to provide a general framework for proving information-theoretic lower bound for interactive clustering algorithms which holds even for randomized algorithms, and even with the full knowledge of \( f_+ \), \( f_- \) and \( k \). In contrast, our algorithms are computationally efficient and are parameter free (works without knowing \( f_+ \), \( f_- \) and \( k \)). The technique that we introduce for our upper bounds could be useful for designing further parameter free algorithms which are extremely important in practice.

Other Related works. The interactive framework of clustering model has been studied before where the oracle is given the entire clustering and the oracle can answer whether a cluster needs to be split or two clusters must be merged \cite{7 6}. Here we contain our attention to pair-wise queries, as in all practical applications that motivate this work \cite{39 10 21 37}. In most cases, an expert human or crowd serves as an oracle. Due to the scale of the data, it is often not possible for such an oracle to answer queries on large number of input data. Only recently, some heuristic algorithms with \( k \)-wise queries for small values of \( k \) but \( k > 2 \) have been proposed in \cite{36}, and a non-interactive algorithm that selects random triangle queries have been analyzed in \cite{38}. Perhaps conceptually closest to us is a recent work by Asthiani et al. \cite{5}, that was done independently of ours and appeared subsequent to a previous version of this work \cite{31}. In \cite{5}, pair-wise queries for clustering is considered. However, their setting is very different. They consider the specific NP-hard \( k \)-means objective with distance matrix which must be a metric and must satisfy a deterministic separation property. Their lower bounds are computational and not information theoretic; moreover their algorithm must know the parameters. There exists a significant gap between their lower and upper bounds: \( \sim \log k \) vs \( k^2 \), and it would be interesting if our techniques can be applied to improve this.

Most recent works consider the region of interest as \( p = 1 - \frac{2 \log n}{n} \) and \( q = \frac{b \log n}{n} \) for some \( a > b > 0 \).
Here we have assumed the oracle always returns the correct answer. To deal with the possibility that the crowdsourced oracle may give wrong answers, there are simple majority voting mechanisms or more complicated techniques [30, 12, 20, 28, 10, 38] to handle such errors. If we assume the errors are independent—since answers are collected from independent crowdsworkers, then we can simply ask each query \( O(\log n) \) times and take the majority vote as the correct answer according to the Chernoff bound. Here our main objective is to study the power of side information, and we do not consider the more complex scenarios of handling erroneous oracle answers.

**Contributions.** Formally the problem we study in this paper can be described as follows.

**Problem 1 (Query-Cluster with an Oracle).** Consider a set of elements \( V \equiv [n] \) with \( k \) latent clusters \( V_i, i = 1, \ldots, k \), where \( k \) is unknown. There is an oracle \( O : V \times V \rightarrow \{\pm 1\} \), that when queried with a pair of elements \( u, v \in V \times V \), returns +1 if \( u \) and \( v \) belong to the same cluster, and −1 if \( u \) and \( v \) belong to different clusters. The queries \( Q \subseteq V \times V \) can be done adaptively. Consider the side information \( W = \{w_{u,v} : 1 \leq u < v \leq n\} \), where the \((u, v)\)th entry of \( W \), \( w_{u,v} \), is a random variable drawn from a discrete probability distribution \( f_+ \) if \( u, v \) belong to the same cluster, and is drawn from a discrete probability distribution \( f_- \) if \( u, v \) belong to different clusters. The parameters \( k, f_+ \) and \( f_- \) are unknown. Given \( V \) and \( W \), find \( Q \subseteq V \times V \) such that \(|Q|\) is minimum, and from the oracle answers and \( W \) it is possible to recover \( V_i, i = 1, 2, \ldots, k \).

Without side information, as noted earlier, it is easy to see an algorithm with query complexity \( O(nk) \) for Query-Cluster. When no side information is available, it is also not difficult to have a lower bound of \( \Omega(nk) \) on the query complexity. Our main contributions are to develop strong information theoretic lower bounds as well as nearly matching upper bounds when side information is available, and characterize the effect of side information on query complexity precisely.

**Upper Bound (Algorithms).** We show that with side information \( W \), a drastic reduction in query complexity of clustering is possible, even with unknown parameters \( f_+, f_- \), and \( k \). We propose a Monte Carlo randomized algorithm that reduces the number of queries from \( O(nk) \) to \( O(nk^2 \log n / \mathcal{H}(f||g)) \), where \( \mathcal{H}(f||g) \) is the Hellinger divergence between the probability distributions \( f \) and \( g \), and recovers the clusters accurately with high probability (with success probability \( 1 - \frac{1}{n} \)) without knowing \( f_+ \), \( f_- \) or \( k \) (see, Theorem 1). Depending on the value of \( k \), this could be highly sublinear in \( n \). Note that, the squared Hellinger divergence between two pnfs \( f \) and \( g \) is defined to be,

\[
\mathcal{H}^2(f||g) = \frac{1}{2} \sum_i \left( \sqrt{f(i)} - \sqrt{g(i)} \right)^2.
\]

We also develop a Las Vegas algorithm, that is one which recovers the clusters with probability 1 (and not just with high probability), with query complexity \( O(n \log n + k^2 \log n / \mathcal{H}^2(f_+||f_-)) \). Since \( f_+ \) and \( f_- \) can be arbitrary, not knowing the distributions provides a major challenge, and we believe, our recipe could be fruitful for designing further parameter-free algorithms. We note that all our algorithms are computationally efficient - in fact, the time required is bounded by the size of the side information matrix, i.e., \( O(n^2) \).

**Theorem 1.** Let, the number of clusters \( k \) be unknown and \( f_+ \) and \( f_- \) be unknown discrete distributions with fixed cardinality of support. There exists an efficient (polynomial-time) Monte Carlo algorithm for Query-Cluster that has query complexity \( O(\min(nk, \frac{k^2 \log n}{\mathcal{H}^2(f_+||f_-)}) \) and recovers all the clusters accurately with probability \( 1 - o(\frac{1}{n}) \). Plus there exists an efficient Las Vegas algorithm that with probability \( 1 - o(\frac{1}{n}) \) has query complexity \( O(n \log n + \min(nk, \frac{k^2 \log n}{\mathcal{H}^2(f_+||f_-)}) \).

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our lower bound holds for continuous distributions as well.

for simplicity of expression, we treat the sample space to be of constant size. However, all our results extend to any finite sample space scaling linearly with its size.
Lower Bound. Our main lower bound result is information theoretic, and can be summarized in the following theorem. Note especially that, for lower bound we can assume the knowledge of $k, f_+, f_-$ in contrast to upper bounds, which makes the results stronger. In addition, $f_+$ and $f_-$ can be discrete or continuous distributions. Note that, when $\mathcal{H}^2(f_+\|f_-)$ is close to 1, e.g., when the side information is perfect, no queries are required. However, that is not the case in practice, and we are interested in the region where $f_+$ and $f_-$ are “close”, that is $\mathcal{H}^2(f_+\|f_-)$ is small.

**Theorem 2.** Assume $\mathcal{H}^2(f_+\|f_-) \leq \frac{1}{18}$. Any (possibly randomized) algorithm with the knowledge of $f_+, f_-$, and the number of clusters $k$, that does not perform $\Omega \left( \min \{ nk, \frac{k^2}{\mathcal{H}^2(f_+\|f_-)} \} \right)$ expected number of queries, will be unable to return the correct clustering with probability at least $\frac{1}{2}$. And to recover the clusters with probability 1, the number of queries must be $\Omega \left( n + \min \{ nk, \frac{k^2}{\mathcal{H}^2(f_+\|f_-)} \} \right)$.

The lower bound therefore matches the query complexity upper bound within a logarithmic factor.

Note that, when no querying is allowed, this turns out exactly to be the setting of stochastic block model though with much general distributions. We have analyzed this case in Appendix A. To see how the probability of error must scale, we have used a generalized version of Fano’s inequality (e.g., [22]). However, when the number of queries is greater than zero, plus when queries can be adaptive, any such standard technique fails. Hence, significant effort has to be put forth to construct a setting where information theoretic minimax bounds can be applied. This lower bound could be of independent interest, and provides a general framework for deriving lower bounds for fundamental problems of classification, hypothesis testing, distribution testing etc. in the interactive learning setting. They may also lead to new lower bound proving techniques in the related multi-round communication complexity model where information again gets revealed adaptively.

**Organization.** The proof of the lower bound is provided in Section 2 and the algorithms are presented in Section 3. Section 3.1 contains the Monte Carlo algorithm. The Las Vegas algorithm is presented in 3.3. Generalization of the stochastic block model, as well as exciting future directions are discussed in Appendix A and B.

## 2. Lower Bound (Proof of Theorem 2)

In this section, we develop our information theoretic lower bounds. We prove a more general result from which Theorem 2 follows easily.

**Lemma 1.** Consider the case when we have $k$ equally sized clusters of size $a$ each (that is total number of elements is $n = ka$). Suppose we are allowed to make at most $Q$ adaptive queries to the oracle. The probability of error for any algorithm for Query-Cluster is at least,

$$1 - \frac{2}{k} \left( 1 + \sqrt{\frac{4Q}{ak}} \right)^2 - \frac{4Q}{ak(k - 1)} - 2\sqrt{\mathcal{H}(f_+\|f_-)}.$$ 

The main high-level technique to prove Lemma 1 is the following. Suppose, a node is to be assigned to a cluster. This situation is obviously akin to a $k$-hypothesis testing problem, and we want to use a lower bound on the probability of error. The side information and the query answers constitute a random vector whose distributions (among the $k$ possible) must be far apart for us to successfully identify the clustering. But the main challenge comes from the interactive nature of the algorithm since it reveals deterministic information and into characterizing the set of elements that are not queried much by the algorithm.

**Proof of Lemma 1.** Since the total number of queries is $Q$, the average number of queries per element is at most $\frac{4Q}{ak}$. Therefore there exist at least $\frac{4Q}{ak}$ elements that are queried at most $T < \frac{4Q}{ak}$ times. Let $x$ be one such element. We just consider the problem of assignment of $x$ to a cluster (assume, otherwise the clustering is done), and show that any algorithm will make wrong assignment with positive probability.

**Step 1: Setting up the hypotheses.** Note that, the side information matrix $W = (w_{i,j})$ is provided where the $w_{i,j}$s are independent random variables. Now assume the scenario when we
use an algorithm ALG to assign \( x \) to one of the \( k \) clusters, \( V_u, u = 1, \ldots, k \). Therefore, given \( x \), ALG takes as input the random variables \( w_{i,x} \) where \( i \in \mathcal{U}_V \), makes some queries involving \( x \) and outputs a cluster index, which is an assignment for \( x \). Based on the observations \( w_{i,x} \), the task of ALG is thus a multi-hypothesis testing among \( k \) hypotheses. Let \( H_u, u = 1, \ldots, k \) denote the \( k \) different hypotheses \( H_u : x \in V_u \). And let \( P_u, u = 1, \ldots, k \) denote the joint probability distributions of the random matrix \( W \) when \( x \in V_u \). In short, for any event \( A \), \( P_u(A) = \Pr(A|H_u) \). Going forward, the subscript of probabilities or expectations will denote the appropriate conditional distribution.

**Step 2: Finding “weak” clusters.** There must exist \( t \in \{1, \ldots, k\} \) such that,

\[
\sum_{i=1}^{k} P_i \{ \text{a query made by ALG involving cluster } V_u \} \leq \mathbb{E}_t \{ \text{Number of queries made by ALG} \} \leq T.
\]

We now find a subset of clusters, that are “weak,” i.e., not queried enough if \( H_t \) were true. Consider the set \( J' = \{ v \in \{1, \ldots, k\} : P_t \{ \text{a query made by ALG involving cluster } V_v \} \leq \frac{2T}{k(1-\beta)} \} \), where \( \beta = \frac{1}{1+\sqrt{\frac{2T}{k}}} \). We must have, \((k-|J'|) \cdot \frac{2T}{k(1-\beta)} \leq T \), which implies, \(|J'| \geq \frac{(1+\beta)k}{2}\).

Now, to output a cluster without using the side information, ALG has to either make a query to the actual cluster the element is from, or query at least \( k-1 \) times. In any other case, ALG must use the side information (in addition to using queries) to output a cluster. Let \( E' = \{ u \in \{1, \ldots, k\} : P_t \{E'\} \leq \frac{2}{3k} \} \). Since, \( \sum_{u=1}^{k} P_t \{E'\} \leq 1 \), we must have, \((k-|J'|) \cdot \frac{2}{3k} \leq 1 \), or \(|J'| > k - \frac{4k}{3} = \frac{(2-\beta)k}{3} \). We have, \(|J' \cap J''| > \frac{(1+\beta)k}{2} + \frac{(2-\beta)k}{3} - k = \frac{k}{2} \). This means, \( \{V_u : u \in J' \cap J''\} \) contains more than \( \frac{ak}{2} \) elements. Since there are \( \frac{ak}{2} \) elements that are queried at most \( T \) times, these two sets must have nonzero intersection. Hence, we can assume that, \( x \in V_t \) for some \( \ell \in J' \cap J'' \), i.e., let \( H_t \) be the true hypothesis. Now we characterize the error events of the algorithm ALG in assignment of \( x \).

**Step 3: Characterizing error events for “x”**. We now consider the following two events.

\( E_1 = \{ \text{a query made by ALG involving cluster } V_t \} \); \( E_2 = \{ k-1 \text{ or more queries were made by ALG} \} \).

Note that, if the algorithm ALG can correctly assign \( x \) to a cluster without using the side information then either of \( E_1 \) or \( E_2 \) must have to happen. Recall, \( \mathcal{E} \) denotes the event that ALG outputs cluster \( V_t \) using the side information. Now consider the event \( \mathcal{E} = \mathcal{E}' \cup \mathcal{E}_1 \cup \mathcal{E}_2 \). The probability of correct assignment is at most \( P_t(\mathcal{E}) \). We now bound this probability of correct recovery from above.

**Step 4: Bounding probability of correct recovery via Hellinger distance.** We have,

\[
P_t(\mathcal{E}) \leq P_t(\mathcal{E}) + |P_t(\mathcal{E}) - P_t(\mathcal{E})| \leq P_t(\mathcal{E}) + \|P_t - P_t\|_{TV} \leq P_t(\mathcal{E}) + \sqrt{2H(P_t||P_t)},
\]

where, \( \|P - Q\|_{TV} = \sup_A \{P(A) - Q(A)\} \) denotes the total variation distance between two probability distributions \( P \) and \( Q \) and in the last step we have used the relationship between total variation distance and the Hellinger divergence (see, for example, [35, Eq. (3)]). Now, recall that \( P_t \) and \( P_t \) are the joint distributions of the independent random variables \( w_{i,x} \) \( i \in \mathcal{U}_V \). Now, we use the fact that squared Hellinger divergence between product distribution of independent random variables are less than the sum of the squared Hellinger divergence between the individual distribution. We also note that the divergence between identical random variables are 0. We obtain

\[
\sqrt{2H(P_t||P_t)} \leq \sqrt{2 \cdot 2aH^2(f_+||f_-) = 2\sqrt{aH}(f_+||f_-)}.
\]

This is true because the only times when \( w_{i,x} \) differs under \( P_t \) and under \( P_t \) is when \( x \in V_t \) or \( x \in V_t \). As a result we have, \( P_t(\mathcal{E}) \leq P_t(\mathcal{E}) + 2\sqrt{aH}(f_+||f_-) \). Now, using Markov inequality

\[
P_t(\mathcal{E}_2) \leq \frac{T}{k-1} \leq \frac{4Q}{ak(k-1)}.
\]

Therefore,

\[
P_t(\mathcal{E}) \leq P_t(\mathcal{E}') + P_t(\mathcal{E}_1) + P_t(\mathcal{E}_2) \leq \frac{2}{\beta k} + \frac{8Q}{ak^2(1-\beta)} + \frac{4Q}{ak(k-1)}.
\]
Therefore, putting the value of $\beta$ we get, $P_{\tau}(\mathcal{E}) \leq \frac{2}{n} \left( 1 + \sqrt{\frac{4Q}{ak}} \right)^2 + \frac{4Q}{ak(n-1)} + 2\sqrt{a}H(f_+ \parallel f_-)$, which proves the lemma. □

Proof of Theorem 3. Suppose, $a = \lfloor \frac{nk}{H^2(f_+ \parallel f_-)} \rfloor$. Then $a \geq 2$, since $H^2(f_+ \parallel f_-) \leq \frac{1}{18}$. Also, we can take $nk \geq k^2a$, since otherwise the theorem is already proved from the $nk$ lower bound. Consider the situation when we are already given a complete cluster $V_k$ with $n - (k-1)a$ elements, remaining $(k-1)$ clusters each has 1 element, and the rest $(a-1)(k-1)$ elements are evenly distributed (but yet to be assigned) to the $k-1$ clusters. Now we are exactly in the situation of Lemma 4 with $k-1$ playing the role of $k$. If we have $Q < \frac{k^2}{27}$, The probability of error is at least $1 - o_k(1) - \frac{1}{6} - \frac{2}{3} = 1 - o_k(1)$, where $o_k(1)$ is a term that goes to 0 with $k$. Therefore $Q$ must be $\Omega(\frac{k^2}{H^2(f_+ \parallel f_-)})$. Note that, in this proof we have not in particular tried to optimize the constants.

If we want to recover the clusters with probability 1, then $\Omega(n)$ is a trivial lower bound. Hence, coupled with the above we get a lower bound of $\Omega(n + \min \{nk, \frac{k^2}{H^2(f_+ \parallel f_-)} \})$ in that case. □

3. Algorithms

We propose two algorithms (Monte Carlo and Las Vegas) both of which are completely parameter free that is they work without any knowledge of $k$, $f_+$ and $f_-$, and meet the respective lower bounds within an $O(\log n)$ factor. We first present the Monte Carlo algorithm which drastically reduces the number of queries from $O(nk)$ (no side information) to $O(\frac{k^2 \log n}{H^2(f_+ \parallel f_-)})$, and recovers the clusters exactly with probability at least $1 - o_k(1)$. Next, we present our Las Vegas algorithm.

Our algorithm uses a subroutine called Membership that takes as input an element $v \in V$ and a subset of elements $C \subseteq V \setminus \{v\}$, Assume that $f_+, f_-$ are discrete distributions over fixed set of $q$ points $a_1, a_2, \ldots, a_q$; that is $w_{i,j}$ takes value in the set $\{a_1, a_2, \ldots, a_q\}$. Define the empirical “inter” distribution $p_{v,C}$ for $i = 1, \ldots, q$, $p_{v,C}(i) = \frac{1}{|C|} \sum_{u \in C} w_{i,u}$. Also compute the “intra” distribution $p_C$ for $i = 1, \ldots, q$, $p_C(i) = \frac{1}{|C|(|C|-1)} \sum_{u \not= v} w_{i,u}$. Then we use $\text{Membership}(v, C) = -H^2(p_{v,C} \parallel p_C)$ as affinity of vertex $v$ to $C$, where $H(p_{v,C} \parallel p_C)$ denotes the Hellinger divergence between distributions. Note that, since the membership is always negative, a higher membership implies that the ‘inter’ and ‘intra’ distributions are closer in terms of Hellinger distance.

Designing a parameter free Monte Carlo algorithm seems to be highly challenging as here, the number of queries depends only logarithmically with $n$. Intuitively, if an element $v$ has the highest membership in some cluster $C$, then $v$ should be queried with $C$ first. Also an estimation from side information is reliable when the cluster already has enough members. Unfortunately, we know neither whether the current cluster size is reliable, nor we are allowed to make even one query per element.

To overcome this bottleneck, we propose an iterative-update algorithm which we believe will find more uses in developing parameter free algorithms. We start by querying a few points so that there is at least one cluster with $\Theta(\log n)$ points. Now based on these queried memberships, we learn two empirical distributions $p^1_+$ from intra-cluster similarity values, and $p^1_-$ from inter-cluster similarity values. Given an element $v$ which has not been clustered yet, and a cluster $C$ with the highest number of current members, we would like to consider the submatrix of side information pertaining to $v$ and all $u \in C$ and determine whether that side information is generated from $f_+$ or $f_-$. We know if the statistical distance between $f_+$ and $f_-$ is small, then we would need more members in $C$ to successfully do this test. Since, we do not know $f_+$ and $f_-$, we compute the squared Hellinger divergence between $p^1_+$ and $p^1_-$, and use that to compute a threshold $\tau_1$ on the size of $C$. If $C$ crosses this size threshold, we just use the side information to determine if $v$ should belong to $C$. Otherwise, we query further until there is one cluster with size $\tau_1$, and re-estimate the empirical distributions $p^2_+$ and $p^2_-$. Again, we recompute a threshold $\tau_2$, and stop if the cluster under consideration crosses this new threshold. If not we continue. Interestingly, we can show when the process converges, we have a very good estimate of $H(f_+ \parallel f_-)$ and, moreover it converges fast.
3.1. Monte Carlo Algorithm. The algorithm has several phases.

Phase 1. Initialization. We initialize the algorithm by selecting any vertex \( v \) and creating a singleton cluster \( \{ v \} \). We then keep selecting new vertices randomly and uniformly that have not yet been clustered, and query the oracle with it by choosing exactly one vertex from each of the clusters formed so far. If the oracle returns +1 to any of these queries then we include the vertex in the corresponding cluster, else we create a new singleton cluster with it. We continue this process until at least one cluster has grown to a size of \( \lceil C \log n \rceil \), where \( C \) is an appropriately chosen constant depending on \( q \).

Observation 3. The number of queries made in Phase 1 is at most \( O(k^2 \log n) \).

Proof. We stop the process as soon as a cluster has grown to size of \( \lceil C \log n \rceil \). Therefore, we may have clustered at most \( k \ast \lceil C \log n \rceil \) vertices at this stage, each of which may have required \( k \) queries to the oracle, one for every cluster. \( \square \)

Phase 2. Iterative Update. Let \( C_1, C_2, \ldots, C_{l_x} \) be the set of clusters formed after the \( x \)-th iteration for some \( l_x \leq k \), where we consider Phase 1 as the 0-th iteration. We estimate

\[
p_{+,x} = \frac{1}{\sum_{i=1}^{l_x} \binom{|C_i|}{2}} \cdot |\{ u, v \in C_i : w_{u,v} = a_i \}|, \quad \text{and}
\]

\[
p_{-,x} = \frac{1}{\sum_{i=1}^{l_x} \sum_{j<i} |C_i||C_j|} \cdot |\{ u \in C_i, v \in C_j, i < j, i, j \in [1, l_x] : w_{u,v} = a_i \}|
\]

Define

\[
M_x^E = \frac{C \log n}{\mathcal{H}(p_{+,x}||p_{-,x})^2}.
\]

If there is no cluster of size at least \( M_x^E \) formed so far, we select a new vertex yet to be clustered and query it exactly once with the existing clusters (that is by selecting one arbitrary point from every cluster and querying the oracle with the new vertex and the selected one), and include it in an existing cluster or create a new cluster with it based on the query answer. We then set \( x = x + 1 \) and move to the next iteration to get updated estimates of \( p_{+,x}, p_{-,x}, M_x^E \) and \( l_x \).

Else if there is a cluster of size at least \( M_x^E \), we stop and move to the next phase.

Phase 3. Processing the grown clusters. Once Phase 2 has converged, let \( p_+, p_-, \mathcal{H}(p_+||p_-), M^E \) and \( l \) be the final estimates. For every cluster \( C \) of size \( |C| \geq M^E \), we call it grown and we do the following.

(3A.) For every unclustered vertex \( v \), if \( \text{Membership}(v, C) \geq -\left( \frac{4\mathcal{H}(p_+||p_-)}{C} - \frac{2\mathcal{H}(p_+||p_-)^2}{C \sqrt{\log n}} \right) \), then we include \( v \) in \( C \) without querying.

(3B.) We create a new list \( \text{Waiting}(C) \), initially empty. If

\[
-\left( \frac{4\mathcal{H}(p_+||p_-)}{C} - \frac{2\mathcal{H}(p_+||p_-)^2}{C \sqrt{\log n}} \right) > \text{Membership}(v, C) \geq -\left( \frac{4\mathcal{H}(p_+||p_-)}{C} + \frac{2\mathcal{H}(p_+||p_-)^2}{C \sqrt{\log n}} \right),
\]

then we include \( v \) in \( \text{Waiting}(C) \). For every vertex in \( \text{Waiting}(C) \), we query the oracle with it by choosing exactly one vertex from each of the clusters formed so far starting with \( C \). If oracle returns answer “yes” to any of these queries then we include the vertex in that cluster, else we create a new singleton cluster with it. We continue this until \( \text{Waiting}(C) \) is exhausted.

We then call \( C \) completely grown, remove it from further consideration, and move to the next grown cluster. if there is no other grown cluster, then we move back to Phase 2.

3.2. Analysis. One of the important tools that will be used in this section is Sanov’s theorem from the large-deviation theory.

the precise value of \( C \) can be deduced from the proof given \( q \)
Lemma 2 (Sanov’s theorem). Let \(X_1, \ldots, X_n\) are iid random variables with a finite sample space \(X\) and distribution \(P\). Let \(P^n\) denote their joint distribution. Let \(E\) be a set of probability distributions on \(X\). The empirical distribution \(P_n\) gives probability \(P_n(A) = \frac{1}{n} \sum_{i=1}^{n} 1_{X_i \in A}\) to any event \(A\). Then,
\[
P^n(\{x_1, \ldots, x_n\} : \tilde{P}_n \in E) \leq (n + 1)^{|X|} \exp(-n \min_{P^* \in E} D(P^* \| P)).
\]

A continuous version of Sanov’s theorem is also possible, especially when the set \(E\) is convex (as a matter of fact the polynomial term in front of the right hand side can be omitted in certain cases), but we omit here for clarity. The Sanov’s theorem states, if we have an empirical distribution \(P^n\) and a set of all distributions satisfying certain property \(E\), then the probability \(P^n \in E\) decreases exponentially with the minimum KL divergence of \(P^n\) with any distribution in \(E\). Note that, the KL divergence in the exponent of the Sanov’s theorem naturally indicates an upper bound in terms of KL divergence. However, a major difficulty in dealing with KL divergence is that it is not a distance and does not satisfy triangle inequality. We overcome that by dealing with Hellinger distance instead.

There are two parts to the analysis, showing the clusters are correct with high probability and determining the query complexity.

Lemma 3. With probability at least \(1 - \frac{6}{n^2}\) all of the following holds for an appropriately chosen constant \(B\)

(a) \(\mathcal{H}(p_+ \| f_+) \leq \frac{2\mathcal{H}(p_+ \| p_-)^2}{B \sqrt{\log n}}\)

(b) \(\mathcal{H}(p_- \| f_-) \leq \frac{2\mathcal{H}(p_+ \| p_-)^2}{B \sqrt{\log n}}\)

(c) \(\mathcal{H}(p_+ \| p_-) \left(1 + \frac{4\mathcal{H}(p_+ \| p_-)}{B \sqrt{\log n}}\right) \geq \mathcal{H}(f_+ \| f_-) \geq \mathcal{H}(p_+ \| p_-) \left(1 - \frac{4\mathcal{H}(p_+ \| p_-)}{B \sqrt{\log n}}\right)\)

Proof. Let \(C\) be a cluster which according to the updated estimates of \(p_+\) and \(p_-\) has crossed the updated \(M^E\) threshold. Since \(|C| \geq M^E\), \(p_+\) is estimated based on at least \((\frac{M^E}{2})^2\) edges. We assume the largest cluster size in the input instance is at most \(\frac{n}{2}\). Suppose the total number of vertices selected in Phase 1 and Phase 2 before \(C\) grew to \(M^E\) is strictly less than \(\frac{3M^E}{2}\). Then the expected number of vertices selected from \(C\) is at most \(\frac{3M^E}{4}\). Then, by the Chernoff bound, the probability that the number of vertices selected from \(C\) is \(M^E\) is at most \(e^{-\frac{M^E}{3n}}\). Taking \(C \geq 118\), we get with probability at least \(1 - \frac{1}{n^2}\), the number of vertices chosen from outside \(C\) is at least \(\frac{M^E}{2}\). Thus, \(p_-\) is estimated based on at least \((\frac{M^E}{2})^2\) edges.

Here, we use the following version of the Chernoff bound.

Lemma 4 (The Chernoff Bound). Let \(X_1, X_2, \ldots, X_n\) be independent random variable taking values in \([0, 1]\) with \(E[X_i] = \mu_i\). Let \(X = \sum_{i=1}^{n} X_i\), and \(\mu = E[X]\). Then the following holds

1. For \(0 < \delta \leq 1\), \(Pr[X \leq (1 - \delta)\mu] \leq e^{-\mu\delta^2/2}\)
2. For \(0 < \delta \leq 1\), \(Pr[X \geq (1 + \delta)\mu] \leq e^{-\mu\delta^2/3}\)

(a) \(\text{Let } M = \left(\frac{M^E}{2}\right)^2 \geq \left(\frac{M^E}{3}\right)^2\). Now, select \(\delta = \sqrt{\frac{C' \log n}{M}}\), where \(C'\) is a constant that ensures \(n^{2C'} \geq \frac{M^E}{27\sqrt{3}}\). Thus, \(\delta \geq \frac{1}{2}\).\n
\[
\Pr \left( \mathcal{H}(p_+ \| f_+) \geq \delta \right) = f_+ \left( \{ p_+ : \mathcal{H}(p_+ \| f_+) \geq \delta \} \right) = (M + 1)^q \exp(-M \min_{p : \mathcal{H}(p \| f_+) \geq \delta} D(p \| f_+)),
\]

Here in the last step we have used Sanov’s theorem (see, Lemma 2). Using the relationship between KL-divergence and Hellinger distance, we get

\[
D(p \| f_+) \geq 2\mathcal{H}^2(p \| f_+) \geq 2\delta^2
\]

We could have also assumed the largest cluster size is at most \(n(1-\epsilon)\) for some constant \(\epsilon > 0\) and adjust the constants appropriately.

note that the version of the Chernoff bound also holds for sampling without replacement, which is the case here [26].
where in the last step we used the optimization condition under the Sanov’s theorem. Setting 
\( \delta = \sqrt{\frac{C^2 \log n}{M^{E}}}, M \geq \frac{(M^E)^2}{8} = \frac{C^2 \log^2 n}{3 \mathcal{H}(p_+ \Vert p_-)} \), we get \( \delta = \frac{\sqrt{C^2} \mathcal{H}(p_+ \Vert p_-)}{\sqrt{C^{2}}} \). Let us take \( B' = \frac{C}{\sqrt{C^{2}}} \), and \( B = \sqrt{\frac{C}{C^{2}}} \), we have \( B \leq B' \) and we get
\[
\Pr \left( \mathcal{H}(p_+ \| f_+) \geq \frac{2 \mathcal{H}(p_+ \| p_-)^2}{B' \sqrt{\log n}} \right) \leq \frac{1}{n^3}
\]
Hence,
\[
\Pr \left( \mathcal{H}(p_+ \| f_+) \geq \frac{2 \mathcal{H}(p_+ \| p_-)^2}{B \sqrt{\log n}} \right) \leq \frac{1}{n^3}
\]
(b) Following a similar argument as above, we get
\[
\Pr \left( \mathcal{H}(p_- \| f_-) \geq \frac{2 \mathcal{H}(p_+ \| p_-)^2}{B \sqrt{\log n}} \right) \leq \frac{1}{n^3}
\]
(c) Now
\[
\mathcal{H}(f_+ \| f_-) \geq \mathcal{H}(p_+ \| p_-) - \mathcal{H}(p_+ \| f_+) - \mathcal{H}(p_- \| f_-)
\]
by applying triangle inequality
\[
\geq \mathcal{H}(p_+ \| p_-) - \frac{4 \mathcal{H}(p_+ \| p_-)^2}{B \sqrt{\log n}}
\]
from (a) and (b) with probability at least \( 1 - \frac{2}{n^3} \)
\[
= \mathcal{H}(p_+ \| p_-) \left( 1 - \frac{4 \mathcal{H}(p_+ \| p_-)^2}{B \sqrt{\log n}} \right)
\]
Similarly,
\[
\mathcal{H}(p_+ \| p_-) \geq \mathcal{H}(f_+ \| f_-) - \mathcal{H}(p_+ \| f_+) - \mathcal{H}(p_- \| f_-)
\]
by triangle inequality
\[
\geq \mathcal{H}(f_+ \| f_-) - \frac{4 \mathcal{H}(p_+ \| p_-)^2}{B \sqrt{\log n}}
\]
from (a) and (b) with probability at least \( 1 - \frac{2}{n^3} \)
Hence, by union bound all of (a), (b) and (c) hold with probability at least \( 1 - \frac{6}{n^3} \).

**Lemma 5.** Let \( C \) be a cluster considered in Phase 3 of size at least \( M^E \) then the following holds with probability at least \( 1 - o_n(1) \).

(a) If \( \text{Membership}(v, C) > \left( \frac{\mathcal{H}(p_+ \| p_-)}{B} - \frac{2 \mathcal{H}(p_+ \| p_-)^2}{B \sqrt{\log n}} \right) \) then \( v \) is in \( C \)

(b) If \( v \in C \) then \( \text{Membership}(v, C) \geq \left( \frac{\mathcal{H}(p_+ \| p_-)}{B} + \frac{2 \mathcal{H}(p_+ \| p_-)^2}{B \sqrt{\log n}} \right) \)

**Proof.** Suppose \( v \in C \). Then for any \( \delta > 0 \), we have
\[
\Pr \left( \mathcal{H}(p_{v,C} \| f_+) > \delta \mid v \in C \right) = f_+ \left( \mathcal{H}(p_{v,C} \| f_+) > \delta \right)
\]
\[
\leq (M^E + 1)^q \exp(-M^E \min_{p: \mathcal{H}(p \| f_+)} D(p \| f_+)) \quad \text{(by Sanov’s theorem)}
\]
\[
\leq (M^E + 1)^q \exp(-M^E \min_{p: \mathcal{H}(p \| f_+)} 2 \mathcal{H}^2(p \| f_+))
\]
(notating the relationship between KL-divergence and Hellinger distance)
\[
\leq (M^E + 1)^q \exp(-2M^E \delta^2)
\]
Setting \( M^E \delta^2 = C' \log n \), we get \( \delta = \sqrt{\frac{C' \log n}{M^E}} = \sqrt{\frac{C'}{C} \mathcal{H}(p_+ \| p_-)} = \frac{\mathcal{H}(p_+ \| p_-)}{B} \) (by noting the value of \( B \)), we get
\[
\Pr \left( \mathcal{H}(p_{v,C} \| f_+) > \frac{\mathcal{H}(p_+ \| p_-)}{B} \mid v \in C \right) \leq \frac{1}{n^3} \quad \text{(by noting the value of } C')
\]
Similarly,
\[
\Pr \left( \mathcal{H}(p_{v,C} \| f_-) > \frac{\mathcal{H}(p_+ \| p_-)}{B} \mid v \in C \right) \leq \frac{1}{n^3}
\]

Similarly,
Therefore, with at least $1 - \frac{2}{n^2}$ probability (by applying union bound over all $v$ the following hold. (i) If $v \in C$ then $H(p_{v, C} \| f_+^v) < \frac{H(p_+ \| p_-)}{B}$ and (ii) If $v \notin C$ then $H(p_{v, C} \| f_-^v) < \frac{H(p_+ \| p_-)}{B} - 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}}$. Suppose if possible $v \notin C$. Then, we have

$$H(p_{v, C} \| f_+^v) \leq H(p_{v, C} \| p_+^v) + H(p_+ \| f_+^v)$$

by triangle inequality

$$< \frac{H(p_+ \| p_-)}{B} - 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}} + H(p_+ \| f_+^v)$$

applying condition on $\text{Membership}(v, C)$ from Lemma 3 (a) with probability at least $1 - \frac{1}{n^3}$

Then we have,

$$H(p_{v, C} \| f_-^v) \geq H(f_- \| f_-^v) - H(p_{v, C} \| f_+^v)$$

by triangle inequality

$$\geq H(p_+ \| p_-) - 4\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}} - H(p_{v, C} \| f_+^v)$$

from Lemma 3 (c) with probability at least $1 - \frac{2}{n^2}$

$$\geq \left(1 - \frac{1}{B}\right)H(p_+ \| p_-) - 4\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}}$$

with probability at least $1 - \frac{3}{n^3}$

$$\geq \left(1 - \frac{1}{B} - 4\frac{B \sqrt{\log n}}{B \sqrt{\log n}}\right)H(p_+ \| p_-)$$

since $H(p_+ \| p_-) \leq 1$

$$> \frac{H(p_+ \| p_-)}{B}$$

by taking $B > 6$, or $C \geq 36C'$. This contradicts that $v \notin C$.

(b) Now assume $v \in C$ but $\text{Membership}(v, C) \geq \left(\frac{H(p_+ \| p_-)}{B} + 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}}\right)$, that is $H(p_{v, C} \| p_+) \geq \frac{H(p_+ \| p_-)}{B} + 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}}$. We have

$$H(p_{v, C} \| f_+^v) \geq H(p_{v, C} \| p_+) - H(f_+ \| p_+)$$

$$\geq \frac{H(p_+ \| p_-)}{B} + 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}} - H(f_+ \| p_+)$$

applying condition on $\text{Membership}(v, C)$

$$\geq \frac{H(p_+ \| p_-)}{B}$$

from Lemma 3 (a) with probability at least $1 - \frac{1}{n^3}$

This contradicts the fact that $v \in C$.

\[\square\]

**Corollary 1.** Let $C$ be a cluster considered in Phase 3 of size at least $M^E$ then the following hold with probability at least $1 - \frac{2}{n^2}$.

(a) Vertices that are included in $C$ in Phase (3A) truly belong to $C$.

(b) Vertices that are not in $\text{Waiting}(C)$ can not be in $C$.

**Proof.** Follows from Lemma 5 (a) and (b) respectively. \[\square\]

**Lemma 6.** Let $C$ be a cluster considered in Phase 3 of size at least $M^E$ and $\hat{C}$ denotes the true cluster with $C \subseteq \hat{C}$. Then after Phase (3A), $|\hat{C} \setminus C| = o(1)$ with probability at least $1 - \frac{1}{n^7}$.

**Proof.** We have from Lemma 5 that for $v$ to belong to $\hat{C}$, it must satisfy $\text{Membership}(v, C) \geq \left(\frac{H(p_+ \| p_-)}{B} + 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}}\right)$. On the other hand, if $v$ has $\text{Membership}(v, C) > \left(\frac{H(p_+ \| p_-)}{B} - 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}}\right)$ then $v$ has already been included in $C$. Therefore, the grey region of $\text{Membership}(v, C)$ values for which we cannot decide on whether or not to include $v$ to $C$ is when $\text{Membership}(v, C) \in \left[\frac{H(p_+ \| p_-)}{B} \pm 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}}\right]$, that is $H(p_{v, \hat{C}} \| p_+) \leq \frac{H(p_+ \| p_-)}{B} \pm 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}}$.

Now,

$$\Pr\left(H(p_{v, C} \| p_+) \leq \frac{H(p_+ \| p_-)}{B} + 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}}\right) \leq \Pr\left(H(p_{v, \hat{C}} \| p_+) \leq \frac{H(p_+ \| p_-)}{B} + 2\frac{H(p_+ \| p_-)^2}{B \sqrt{\log n}}\right)$$


\[(M^E + 1)^9 \exp \left( -M^E \right) \min_{p: H(p||p_+)} \frac{H(p||p_+)}{2H(p||p_-) + 2H(p_+||p_-) - 2H(p||p_-)} \text{ by Sanov’s theorem} \]

Now,
\[D(p||f_+) \geq 2H(p||f_+)^2 \geq 2 \left( H(p||p_+) - H(p_+||f_+) \right)^2 \text{ by triangle inequality} \]
\[\geq 2 \left( \frac{H(p_+||p_-)}{B} - \frac{2H(p_+||p_-)^2}{B^2 \log n} - H(p_+||f_+) \right)^2 \text{ from the optimization condition} \]
\[\geq 2 \left( \frac{H(p_+||p_-)}{B} - \frac{4H(p_+||p_-)^2}{B^2 \log n} \right)^2 \text{ from Lemma 3(a) with probability at least } 1 - \frac{1}{n^3} \]
\[= \frac{2H^2(p_+||p_-)}{B^2} \left( 1 - \frac{4H(p_+||p_-)}{B^2 - 1} \right)^2 \geq \frac{2H^2(p_+||p_-)}{B^2} \left( 1 - \frac{4B}{B} \right)^2 \]
\[\geq \frac{2H^2(p_+||p_-)}{27B} \text{ by inserting the minimum value for } \frac{1}{B} \left( 1 - \frac{4B}{B} \right)^2 \]

Now \(M^E \geq \frac{C \log n}{H(p||p_-)}\). Hence,
\[\Pr \left( H(p_+||p_-) \in \frac{H(p_+||p_-)}{B} \pm \frac{2H(p_+||p_-)^2}{B^2 \log n} \right) \leq (M^E + 1)^9 \exp \left( -\frac{2C}{27B} \log n \right) + \frac{1}{n^3} = (M^E + 1)^9 \exp \left( -\frac{4\sqrt{C^2}}{27B^2} \log n \right) + \frac{1}{n^3} \leq \frac{2}{n^3} \]

Hence the expected number of vertices \(v \in C\) in the grey region is \(\leq \frac{2}{n^3}\). Thus by simple Markov inequality, after Phase (3A), the probability that \(|\hat{C} \setminus C| \geq 4\) is at most \(\frac{1}{2n^2}\). Hence, with probability at least \(1 - \frac{1}{2n^2}\), the size is bounded by 4.

**Lemma 7.** The algorithm asks at most \(O\left(\frac{k^2 \log n}{n\log |f_+||f_-|} \right)\) queries over the three phases with probability \(1 - o_n(1)\).

**Proof.** In Phase 1, as seen from Observation 3, the number of queries is \(O(k^2 \log n) \leq O\left(\frac{k^2 \log n}{n\log |f_+||f_-|} \right)\), as \(0 \leq H(f_+||f_-)^2 \leq 1\).

In Phase 2, from Lemma 3, at any time when we have a grown cluster
\[H(p_+||p_-) \geq H(f_+||f_-) - H(p_+||f_+) - H(p_+||f_-) \text{ by triangle inequality} \]
\[\geq H(f_+||f_-) - \frac{4H(p_+||p_-)^2}{B^2 \log n} \text{ from Lemma 3} \]

Therefore,
\[H(p_+||p_-) \geq \frac{H(f_+||f_-)}{1 + \frac{4H(p_+||p_-)^2}{B^2 \log n}} \geq \frac{H(f_+||f_-)}{2} \]

This also shows whenever one cluster has grown to a size of \(\frac{4C \log n}{n\log |f_+||f_-|}\), then \(M^E\) must cross the threshold based on the newest estimate of \(p_+\) and \(p_-\). Hence, Phase 2 never grows a cluster beyond a size of \(O\left(\frac{\log n}{n\log |f_+||f_-|} \right)\) with probability \(1 - \frac{1}{n^3}\). Hence, in Phase 2, the total number of queries can be at most \(O\left(\frac{k^2 \log n}{n\log |f_+||f_-|} \right)\).

In Phase 3, the total number of queries made is at most \(O(k^2)\) with probability at least \(1 - \frac{1}{2n}\) due to Lemma 6 and applying union bound over all the clusters.

Thus, we get the overall query complexity is \(O\left(\frac{k^2 \log n}{n\log |f_+||f_-|} \right)\) with probability \(1 - o_n(1)\), where \(o_n(1)\) denotes a function of \(n\) that goes to 0 with \(n\). \(\square\)

Putting together all the lemmas, we arrive at the statement of Theorem 1.
3.3. A Las Vegas Algorithm for Query-Cluster with an Oracle. In this section, we design a Las Vegas algorithm for clustering with oracle.

Recall that, our algorithm uses a subroutine called Membership that takes as input an element $v \in V$ and a subset of elements (cluster) $C \subseteq V \setminus \{v\}$. Assume that $f_+, f_-$ are discrete distributions over $q$ points $a_1, a_2, \ldots, a_q$; that is $w_{i,j}$ takes value in the set $\{a_1, a_2, \ldots, a_q\}$. We defined the empirical “inter” distribution $p_{v,C}$ for $i = 1, \ldots, q$, $p_{v,C}(i) = \frac{1}{|C|} \cdot |\{u \in C : w_{u,v} = a_i\}|$. Also compute the “intra” distribution $p_{C}$ for $i = 1, \ldots, q$, $p_{C}(i) = \frac{1}{|C|} \cdot |\{u \in C : u \neq v, w_{u,v} = a_i\}|$. Then we use Membership$(v,C) = -\mathcal{H}(p_{v,C}||p_{C})$ as affinity of vertex $v$ to cluster $C$, where $\mathcal{H}(p_{v,C}||p_{C})$ denotes the Hellinger divergence between distributions. Note that, since the membership is always negative, a higher membership implies that the ‘inter’ and ‘intra’ distributions are closer in terms of Hellinger distance.

The algorithm works as follows. Let $C_1, C_2, \ldots, C_l$ be the current clusters in nonincreasing order of size. We find the minimum index $j \in [1,l]$ such that there exists a vertex $v$ not yet clustered, with the highest average membership to $C_j$, that is $\text{Membership}(v,C_j) \geq \text{Membership}(v,C_{j'})$, $\forall j' \neq j$, and $j$ is the smallest index for which such a $v$ exists. We first check if $v \in C_j$ by querying $v$ with any current member of $C_j$. If not, then we group the clusters $C_1, C_2, \ldots, C_{l-1}$ in at most $[\log n]$ groups such that clusters in group $i$ have size in the range $\left]\frac{|C_1|}{2^i}, \frac{|C_1|}{2^{i+1}}\right]$. For each group, we pick the cluster which has the highest average membership with respect to $v$, and check by querying whether $v$ belongs to that cluster. Even after this, if the membership of $v$ is not resolved, then we query $v$ with one member of each of the clusters that we have not checked with previously. If $v$ is still not clustered, then we create a new singleton cluster with $v$ as its sole member.

The pseudocode of the algorithm is given in Figure 1. We now give a proof of the Las Vegas part of Theorem 1 here using Algorithm 1. We crucially use the following lemma which proves a strong concentration inequality adapting the Sanov’s Theorem (see Lemma 2) of information theory.

**Lemma 8.** Suppose, $C, C' \subseteq V$, $C \cap C' = \emptyset$ and $|C| \geq M, |C'| \geq M = \frac{32 \log n}{\mathcal{H}(f_+||f_-)}$. Then,

$$\Pr\left(\text{Membership}(v,C') \geq \text{Membership}(v,C) \mid v \in C\right) \leq \frac{2}{n^\beta}.$$  

**Proof.** Let $\beta = \frac{\mathcal{H}(f_+||f_-)}{2}$. If Membership$(v,C') \geq$ Membership$(v,C)$ then we must have, $\mathcal{H}(p_{v,C'}||p_{C'}) \leq \mathcal{H}(p_{v,C}||p_{C})$. This means, either $\mathcal{H}(p_{v,C'}||p_{C'}) \leq \frac{\beta}{2}$ or $\mathcal{H}(p_{v,C}||p_{C}) \geq \frac{\beta}{2}$. Now, using triangle inequality,

$$\Pr\left(\mathcal{H}(p_{v,C'}||p_{C'}) \leq \frac{\beta}{2}\right) \leq \Pr\left(\mathcal{H}(p_{v,C'}||f_+) - \mathcal{H}(p_{C}||f_+) \leq \frac{\beta}{2}\right) \leq \Pr\left(\mathcal{H}(p_{v,C'}||f_+) \leq \beta \text{ or } \mathcal{H}(p_{C}||f_+) \geq \frac{\beta}{2}\right) \leq \Pr\left(\mathcal{H}(p_{v,C'}||f_+) \leq \beta\right) + \Pr\left(\mathcal{H}(p_{C}||f_+) \geq \frac{\beta}{2}\right).$$

Similarly,

$$\Pr\left(\mathcal{H}(p_{v,C}||p_{C}) \geq \frac{\beta}{2}\right) \leq \Pr\left(\mathcal{H}(p_{v,C}||f_+) + \mathcal{H}(p_{C}||f_+) \geq \frac{\beta}{2}\right) \leq \Pr\left(\mathcal{H}(p_{v,C}||f_+) \geq \frac{\beta}{4}\right) \leq \frac{\beta}{4} + \Pr\left(\mathcal{H}(p_{C}||f_+) \geq \frac{\beta}{4}\right).$$

Now, using Sanov’s theorem (Lemma 2), we have,

$$\Pr\left(\mathcal{H}(p_{v,C'}||f_+) \leq \beta\right) \leq (M + 1)^9 \exp\left(-M \min_{p: \mathcal{H}(p||f_+)} D(p||f_-)\right).$$

At the optimizing $p$ of the exponent,

$$D(p||f_-) \geq 2\mathcal{H}(p||f_-) \geq 2(\mathcal{H}(f_+||f_-) - \mathcal{H}(p||f_+))^2 \geq 2(2\beta - \beta)^2$$

from using triangle inequality from noting the value of $\beta$. 

\[ \text{relation between Hellinger and KL}\]
Algorithm 1 Query-Cluster with Side Information. Input: \{V, W\} (Note: O is the perfect oracle.

> Initialization.
1: Pick an arbitrary element v and create a new cluster \{v\}. Set V = V \ v
2: while V ≠ \emptyset do
   > Let the number of current clusters be \(l \geq 1\)
3:   Order the existing clusters in nonincreasing size.
4:   Let \(|C_1| \geq |C_2| \geq \ldots \geq |C_l|\) be the ordering (w.l.o.g).
5:   for j = 1 to l do
6:      If \(\exists v \in V\) such that \(j = \max_{i \in [1,l]} \text{Membership}(v, C_i)\), then select v and Break;
7:      end for
8:   if \(O(v, u) == " + 1"\) then
9:      Include v in \(C_j\). V = V \ v
10:  end if
11: Group \(C_1, C_2, \ldots, C_{l-1}\) into s consecutive classes \(H_1, H_2, \ldots, H_s\) such that the clusters in group \(H_i\) have their current sizes in the range \([|C_1|, |C_{l-1}|]\)
12:   for i = 1 to s do
13:      \(j = \max_{a \in H_i} \text{Membership}(v, C_a)\)
14:      \(O(v, u)\) where \(u \in C_j\).
15:      if \(O(v, u) == " + 1"\) then
16:         Include v in \(C_j\). V = V \ v. Break.
17:      end if
18:   end for
19:   if v \in V then
20:      for i = 1 to l + 1 do
21:         if i = l + 1 then \(\triangleright v\) does not belong to any of the existing clusters
22:            Create a new cluster \{v\}. Set V = V \ v
23:         else
24:            if \(\exists u \in C_i\) such that \((u, v)\) has already been queried then
25:               \(O(v, u)\)
26:            endif
27:            if \(O(v, u) == " + 1"\) then
28:               Include v in \(C_j\). V = V \ v. Break.
29:            end if
30:         end if
31:      end for
32:   end if
33: end if
34: end while

\[ = \frac{\mathcal{H}^2(f_+ \| f_-)}{2}. \]

Again, using Sanov’s theorem (Lemma 2), we have,

\[ \Pr \left( \mathcal{H}(p_C || f_+) \geq \frac{\beta}{2} \right) \leq (M + 1)^q \exp(-M \min_{p: \mathcal{H}(p || f_+) \geq \frac{\beta}{2}} D(p || f_+)). \]

At the optimizing \(p\) of the exponent,

\[ D(p || f_+) \geq 2\mathcal{H}^2(p || f_+) \]

relation between Hellinger and KL divergences [35].
Now substituting this in the exponent, using the value of $M$ and doing the same exercise for the other two probabilities we get the claim of the lemma.

**(Proof of Theorem 1, Las Vegas Algorithm.** First, The algorithm never includes a vertex in a cluster without querying it with at least one member of that cluster. Therefore, the clusters constructed by our algorithm are always proper subsets of the original clusters. Moreover, the algorithm never creates a new cluster with a vertex $v$ before first querying it with all the existing clusters. Hence, it is not possible that two clusters produced by our algorithm can be merged.

Let $C_1, C_2, ..., C_l$ be the current non-empty clusters that are formed by Algorithm 1, for some $l \leq k$. Note that Algorithm 1 does not know $k$. Let without loss of generality $|C_1| \geq |C_2| \geq ... \geq |C_l|$. Let there exists an index $i \leq l$ such that $|C_1| \geq |C_2| \geq ... \geq |C_i| \geq M$, where $M = \frac{32 \log n}{f_+(1)}$. Of course, the algorithm does not know either $i$ or $M$. If even $|C_1| < M$, then $i = 0$. Suppose $j'$ is the minimum index such that there exists a vertex $v$ with highest average membership in $C_{j'}$. There are few cases to consider based on $j' \leq i$, or $j' > i$ and the cluster that truly contains $v$.

**Case 1.** $v$ truly belongs to $C_{j'}$. In that case, we just make one query between $v$ and an existing member of $C_{j'}$ and the first query is successful.

**Case 2.** $j' \leq i$ and $v$ belongs to $C_j, j \neq j'$ for some $j \in \{1, \ldots, i\}$. Here we have Membership($v, C_{j'}$) ≥ Membership($v, C_j$). Since both $C_j$ and $C_{j'}$ have at least $M$ current members, then using Lemma 8, this happens with probability at most $\frac{1}{n^2}$. Therefore, the number of queries involving $v$ before its membership gets determined is $\leq 1$ with probability at least $1 - \frac{2k}{n^2}$.

**Case 4.** $v$ belongs to $C_j, j \neq j'$ for some $j > i$. In this case, the algorithm may make $k$ queries involving $v$ before its membership gets determined.

**Case 5.** $j' > i$, and $v$ belongs to $C_j$ for some $j \leq i$. In this case, there exists no $v$ with its highest membership in $C_1, C_2, ..., C_t$.

Suppose $C_1, C_2, ..., C_{j'}$ are contained in groups $H_1, H_2, ..., H_s$ where $s \leq \lceil \log n \rceil$. Let $C_j \in H_t, t \in [1, s]$. Therefore, $|C_j| \in \left[ \frac{|C_1|}{n^2}, \frac{|C_1|}{2n^2} \right]$. If $|C_j| \geq 2M$, then all the clusters in group $H_t$ have size at least $M$. Now with probability at least $1 - \frac{2}{n^2}$, Membership($v, C_j$) ≥ Membership($v, C_{j'}$) for every cluster $C_{j'} \in H_t$. In that case, the membership of $v$ is determined within at most $\lceil \log n \rceil$ queries. Else, with probability at most $\frac{2}{n^2}$, there may be $k$ queries to determine the membership of $v$.

Therefore, once a cluster has grown to size $2M$, the number of queries to resolve the membership of any vertex in those clusters is at most $\lceil \log n \rceil$ with probability at least $1 - \frac{2}{n^2}$. Hence, for at most $2kM$ elements, the number of queries made to resolve their membership can be $k$. Thus the total number of queries made by Algorithm 1 is $O(n \log n + Mk^2) = O(n \log n + \frac{k^2 \log n}{f_+(1)})$ with probability $1 - o_n(1)$.

**Remark 1.** While, for the more general setting with unknown $f_{i,j}$s (distribution referring to similarity of cluster $i$ and $j$), we do not know how to extend this algorithm yet, if the parameters were known it is possible to extend our algorithm to such setting. We can calculate $M_i = O(\frac{\log n}{f_+(1)}), H^2(f_+, f_{i,i})$, and thus whenever the $i$ th clusters grows to size $M_i$, remainder of its members can be inferred.

Since, we handle very generic distributions, our upper bounds are off by a factor of $O(\log n)$ from the lower bound. Tightening this bound, e.g. for sparse SBM to match the conjectured trade-off between queries and threshold remains an important open question.
Discussion. This is the first rigorous theoretical study of interactive clustering with side information, and it unveils many interesting directions for future study of both theoretical and practical significance (see Appendix B for more details). Having arbitrary $f_+, f_−$ significantly generalizes SBM. Also it raises an important question about how SBM recovery threshold changes with queries. For sparse region of SBM, where $f_+$ is Bernoulli($\frac{a'\log n}{n}$) and $f_−$ is Bernoulli($\frac{b' \log n}{n}$), $a' > b'$, Lemma 1 is not tight yet. However, it shows the following trend. By setting $a = 1 - \frac{QQ}{nk}$ and ignoring the lower order terms and a $\sqrt{\log n}$ factor, recovery error becomes $\approx (1 - \frac{Q}{nk}) - \frac{H(a'[b'])}{\sqrt{k}}$. We conjecture with $Q$ queries, the sharp recovery threshold of sparse SBM changes from $(\sqrt{a'} - \sqrt{b'}) \geq \sqrt{k}$ to $(\sqrt{a'} - \sqrt{b'}) \geq \sqrt{k} (1 - \frac{Q}{nk})$. Proving this bound remains an exciting open question.

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Consider the case when we allow zero query to the oracle. The clustering has to be done just by using the side information matrix. This is a direct generalization to the well-known stochastic block model. Indeed, if $f_+$ is Bernoulli$(p)$ and $f_-$ is Bernoulli$(q)$, then the side information matrix is a binary matrix, as in the case of stochastic block model \cite{Hofding, Hofding2, Polyanskiy, Polyanskiy2}. It is clear that if the clustering input instance is adversarial, then it is impossible to recover the clusters with high probability. For example, think of the situation that $k - 1$ clusters are of size 1 each. In that case, one of these $k - 1$ small cluster points cannot be assigned to the correct cluster without querying, with a positive probability. Note that, we will not be able to have such an argument later when querying is allowed, which makes that case significantly difficult.

Let us look at the scenario, when there are $k$ clusters of size $\frac{n}{k}$ each. Suppose $V = \bigcup_{i=1}^{k} V_i$ is the correct clustering. Consider the different clustering instances, that can be derived from the correct clustering, by swapping any two points $a \in V_i$ and $b \in V_j$, $i \neq j$. There are $K = \binom{k}{2} \frac{n^2}{k^2} = \frac{n^2}{2} (1 - 1/k)$ such different clusterings (partitions) possible. Let us consider these $K$ different cases as $K$ hypotheses, and try to identify which one of them is true based on the side information matrix.

Let $Q_t, t = 1, \ldots, K$ be the joint probability distributions of the side information matrix under hypothesis $t$, $t = 1, \ldots, K$. Also, let the correct clustering be the zeroth hypothesis and induces a joint probability distribution $Q_0$.

In this type of multi-hypothesis testing problem, a standard tool to lower bound probability of error is Fano’s inequality. However, Fano’s inequality in its usual form in hypothesis testing (see, \cite{Hofding} Thm. 7) does not give the tightest possible result in our case. We instead use another form of Fano’s inequality from \cite{Hofding2} Thm. II.1 Eq. (5) - therein taking $Q = Q_0$ and taking $f(x) = x \log x$, we have, the probability of error $P_c$ of this hypothesis testing problem (to identify
between the $K$ hypotheses) given by,

$$\frac{1}{K} \sum_i D(Q_i||Q_0) \geq (1 - P_e) \log(K(1 - P_e)) + P_e \log(KP_e/(K - 1))$$

where $D(f||g)$ is the Kullback-Leibler (KL) divergence. The KL divergence between joint distribution of independent random variables is sum of the KL divergence of the marginals, and the only times when the distributions of $w_i$ differs under $Q_i$ and under $Q_0$ is when $i$ or $j$ belong to the two clusters where elements were swapped. There are $\frac{4n^2}{K}$ such instances, among them $\frac{2n}{K}$ contributes $D(f_+||f_-)$ to the sum and $\frac{2n}{K}$ contributes $D(f_-||f_+)$ to the sum. Therefore we obtain,

$$P_e \geq 1 - \frac{1}{K} \sum_i D(Q_i||Q_0) + \log 2 \geq 1 - \frac{2n}{K} \Delta(f_+||f_1) \log \frac{n^2}{K(1 - 1/k)} \approx 1 - n\Delta(f_+||f_1)/k \log n,$$

where $\Delta(f||g) = D(f||g) + D(f||g)$.

One particular regime of interest in the literature of stochastic block model appear (see, [2,33]) when, $f_+ \sim \text{Bernoulli}\left(\frac{a \log n}{n}\right)$ and $f_- \sim \text{Bernoulli}\left(\frac{b \log n}{n}\right)$. Then $D(f_+||f_-) = \frac{a \log n}{n} \log \frac{a}{b} + \left(1 - \frac{a \log n}{n}\right) \log \frac{1 - a \log n}{b \log n}$ and $\Delta(f_+||f_-) = (a - b) \log \frac{a}{b} - \log \frac{a}{b} - \frac{1}{2} \log \frac{a \log n}{b \log n} \approx \log \frac{n}{a \log n} \cdot (a - b) \log \frac{a}{b}$. In this case, $P_e \geq 1 - \frac{a \log n}{n} \log \frac{a}{b}$, and, $P_e > 0$ as long as $(a - b) \log \frac{a}{b} < k$. This lower bound can be improved by considering generalized versions of Fano’s inequality involving Hellinger divergence.

In particular, by constructing a different hypothesis testing scenario and using a generalized version of Fano’s inequality we can obtain the following result on probability $P_e$ of erroneous clustering. In particular, we can use a generalized version of Fano’s inequality due to Polyanskiy and Verdú [34, Thm. 4], says that the probability of error $P_e$ is given by (considering Renyi divergence of order $\frac{1}{2}$),

$$-2 \log \left(\sqrt{\frac{1 - P_e}{K}} + \sqrt{P_e(1 - \frac{1}{K})}\right) \leq - \log \sum_y \frac{1}{K} \sum_{j=1}^K \sqrt{Q_j(y)}^2$$

which implies for us,

$$\left(\sqrt{\frac{1 - P_e}{K}} + \sqrt{P_e(1 - \frac{1}{K})}\right)^2 \geq \frac{1}{K^2} \sum_j \sum_i \sum_y \sqrt{Q_j(y)Q_i(y)} = \frac{1}{K^2} \sum_j \sum_i \left(1 - H^2(Q_i||Q_j)\right)$$

$$= 1 - H^2(Q_i||Q_j) = 1 - \left(1 - (1 - H^2(f_+||f_-))^{\frac{1}{4}}\right) = (1 - H^2(f_+||f_-))^{\frac{4}{1}}\frac{K}{K},$$

where we had to crucially used the following fact: if $P_i^m$ and $Q_i^m$ denote joint distributions of $m$ of independent $P_i$ and independent $Q_i$, $i = 1,\ldots, n$, random variables, then,

$$H^2(P_i^m||Q_i^m) = 1 - \int x_{x_1,x_m} P_i^m(x_1,\ldots,x_m)Q_i^m(x_1,\ldots,x_m)dx_{x_1,\ldots,dx_m}$$

$$= 1 - \lim_{m \to \infty} \int x P_i(x)Q_i(x) dx$$

using Tonelli’s theorem

$$= 1 - \lim_{m \to \infty} (1 - H^2(P_i||Q_i)) \leq \sum_{i=1}^m H^2(P_i||Q_i).$$
Again, we assume \( f_+ \sim \text{Bernoulli}(\frac{a \log n}{n}) \) and \( f_- \sim \text{Bernoulli}(\frac{b \log n}{n}) \). In this case,

\[
\sqrt{k + \frac{\sqrt{e}}{n}} \geq \left(\sqrt{ab} \frac{\log n}{n} + \sqrt{2 - ab} \frac{\log n}{n}\right)^\frac{2}{3} = \left(1 - \left(\sqrt{ab} - \frac{ab \log n}{n}\right) \frac{\log n}{n}\right)^\frac{2}{3}.
\]

This implies, \( \sqrt{P_e} \geq n \left(\frac{a + b - \sqrt{ab}}{2}\right)^\frac{2}{3} - \sqrt{k} n^{-1/2} \). In particular, if \( \left(\frac{a + b - \sqrt{ab}}{2}\right)^\frac{2}{3} < \frac{1}{2} \), then \( P_e > 0 \). Hence, \( P_e > 0 \) if

\[
\sqrt{a} - \sqrt{b} < \sqrt{k}.
\]

While in this regime, this result is slightly suboptimal compared to the lower bound of [2], where the corresponding bound was \( \sqrt{a} - \sqrt{b} < \sqrt{k} \), note that our bound works for arbitrary \( f_+, f_- \) and across all regimes; moreover we have not tried to optimize the constants here.

**Appendix B. Connections & Future Direction**

This is the first work that rigorously study the query complexity of clustering with side information. We introduce new general information theoretic methods; as well as use, information theoretic inequalities to design efficient algorithms for clustering with near-optimal complexity. Our algorithms are entirely parameter free, and are computationally efficient. This work reveals interesting connection to the well-studied model of the stochastic block model and, generalize them in a significant way by considering arbitrary distribution for noise opposed to only Bernoulli noise, and opens up new direction of study in the general area of clustering and community detection.

Even for the zero-query case, using generalized Fano’s inequality in multiple hypothesis testing, we can derive simple lower bounds for SBM with arbitrary \( f_+, f_- \) and cluster size distribution, matching closely the bounds for the sparse region \( f_+ \sim \text{Bernoulli}(\frac{a \log n}{n}) \) and \( f_- \sim \text{Bernoulli}(\frac{b \log n}{n}) \) and cluster size \( \sim \frac{2}{k} \). Extending this lower bound to consider adaptive querying comes as a major challenge, as querying may reveal different deterministic information under different hypothesis. We propose a general framework for deriving such lower bounds, and in the process it reveals an interesting trend on how the threshold of recovery should change with querying: from \( \sqrt{a} - \sqrt{b} \geq \sqrt{k} \) to \( \sqrt{a} - \sqrt{b} \geq \sqrt{k} \left(1 - \frac{n}{2k}\right) \) (see Lemma [1]). That is querying can help reduce the threshold when \( O(n) \) edges have been queried as \( k \) is a constant. Currently, there is a \( \sqrt{\log n} \) gap to achieve this bound as our lower bounds deal with very generic distributions and cluster sizes. Closing this gap for the stochastic block model with querying remains an interesting open question.

We propose two computationally efficient algorithms that match the query complexity lower bound within \( \log n \) factor and are completely parameter free. In particular, our iterative-update method to design Monte-Carlo algorithm provides a general recipe to develop any parameter-free algorithm, which are of extreme practical importance. The convergence result is established by extending Sanov’s theorem from the large deviation theory which gives bound only in terms of KL-divergence. Due to the generality of the distributions, the only tool we could use is Sanov’s theorem. However, Hellinger distance comes out to be the right measure both for lower and upper bounds. If \( f_+ \) and \( f_- \) are common distributions like Gaussian, Bernoulli etc., then other concentration results stronger than Sanov may be applied to improve the constants and a logarithm factor to show the trade-off between queries and thresholds as in sparse SBM. While some of our results apply to general \( f_{i,j} \)’s, a full picture with arbitrary \( f_{i,j} \)’s and closing the gap of \( \log n \) between the lower and upper bound remain an important future direction.

There is also a very recent result by [5] that studies the specific \( k \)-means clustering problem with a different side information model. While the setting is quite different, we believe their results can be significantly improved (for example, they only show a lower bound of \( \Omega(\log k + \log n) \).
to overcome NP-hardness of the problem) using our general methods - which promises to be an interesting future work.