Born’s Rule Is Insufficient in a Large Universe *

Don N. Page †
Theoretical Physics Institute
Department of Physics, University of Alberta
Room 238 CEB, 11322 – 89 Avenue
Edmonton, Alberta, Canada T6G 2G7

2010 March 4

Abstract

Probabilities in quantum theory are traditionally given by Born’s rule as the expectation values of projection operators. Here it is shown that Born’s rule is insufficient in universes so large that they contain identical multiple copies of observers, because one does not have definite projection operators to apply. Possible replacements for Born’s rule include using the expectation value of various operators that are not projection operators, or using various options for the average density matrix of a region with an observation. The question of what replacement to use is part of the measure problem in cosmology.

*Alberta-Thy-03-10, arXiv:yymm.nnnn [hep-th]
†Internet address: don@phys.ualberta.ca
Probabilities in quantum theory are traditionally given by Born’s rule [1]. This rule says that probabilities are the absolute squares of quantum amplitudes. More precisely, Born’s rule gives probabilities of measurement or observation results as the expectation values of a complete orthogonal set of projection operators. This rule seems to work well for ordinary laboratory settings, where one is considering the observations of a specific observer and knows where he or she is within the quantum state. However, the universe may be so large that it contains identical multiple copies of the observer and the measurement situation, so that the observer does not know which copy he or she is. Here I show that Born’s rule is insufficient in such cases. Normalized probabilities for the outcomes that can be distinguished by a local observer cannot be given by the expectation values of any projection operators in a global quantum state of the universe. There are several possible replacements for Born’s rule, such as using the expectation values of various operators that are not projection operators. The measure problem in cosmology [2, 3, 4] is a reflection of the uncertainty of what the correct rule is.

Traditional quantum theory uses Born’s rule for the probability of an observation $O_j$ (the result of an observation) as $P_j \equiv P(O_j) = \langle P_j \rangle$ where $P_j$ is the projection operator onto the observational result $O_j$, and where $\langle \ldots \rangle$ denotes the quantum expectation value of whatever operator replaces the $\ldots$ inside the angular brackets. Born’s rule works when one knows where the observer is within the quantum state (e.g., in the quantum state of a single laboratory rather than of the universe), so that one has definite orthonormal projection operators. However, Born’s rule does not work in a universe large enough that there may be identical copies of the observer at different locations, since then one does not know uniquely where the observer is or what the projection operators are.
For example, suppose there are two copies of the observer, at locations $A$ and $B$. The two copies are assumed to be identical, by which I mean all local observations the observer might make cannot distinguish them. The two copies may be distinguished globally by their different locations, but that information is not available to the copies of the local observer themselves. Suppose, for simplicity, that each copy of the observer makes an observation that can give either the result 1 or 2, with no other possibilities. One would like a theory of the universe (including a specification of its quantum state) that would give normalized probabilities of getting the results 1 and 2, say $P_1$ and $P_2$ respectively, without having to specify the inaccessible information of what the location is.

Born’s rule would give the probabilities $P_1^A = \langle P_1^A \rangle$ and $P_2^A = \langle P_2^A \rangle$ if the observer knew that it were at location $A$ with the projection operators there being $P_1^A$ and $P_2^A$. Similarly, it would give the probabilities $P_1^B = \langle P_1^B \rangle$ and $P_2^B = \langle P_2^B \rangle$ if the observer knew that it were at location $B$. (All these projection operators act on the full quantum state, but they act nontrivially only at their respective locations $A$ and $B$. For simplicity we shall assume that the two locations are at spacelike separations, so that the two sets of projection operators commute with each other.)

However, if the observer is not certain to be at either $A$ or $B$, and if $P_1^A \neq P_1^B$, then neither $P_1^A$ nor $P_1^B$ would be the probability $P_1$ of simply getting the observational result 1. I shall assume that $P_1$ must be a weighted mean of $P_1^A$ and $P_1^B$ with both weights positive, and so be strictly between $P_1^A$ and $P_1^B$. However, there is no state-independent projection operator that gives an expectation value with this property for all possible quantum states, as I shall now prove. (If one were allowed to choose the projection operator to depend on the quantum state, then one could get any expectation value one wanted from any quantum state, so I shall exclude that possibility.)
Consider normalized pure quantum states of the form

\[ |\psi\rangle = b_{12} |12\rangle + b_{21} |21\rangle, \]  

(1)

with arbitrary normalized complex amplitudes \( b_{12} \) and \( b_{21} \). The component \(|12\rangle\) represents the observation 1 in the region A and the observation 2 in the region B; the component \(|21\rangle\) represents the observation 2 in the region A and 1 in the region B. Therefore, \( P_A^A = P_B^B = |b_{12}|^2 \), and \( P_A^B = P_B^A = |b_{21}|^2 \).

For Born’s rule to give the possibility of both observational probabilities’ being nonzero in the two-dimensional quantum state space being considered, the orthonormal projection operators should each be of rank one, of the form

\[ P_1 = |\psi_1\rangle\langle \psi_1|, \quad P_2 = |\psi_2\rangle\langle \psi_2|, \]  

(2)

where \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are two orthonormal pure states.

However, once the state-independent projection operators are fixed, then if the quantum state is \( |\psi\rangle = |\psi_1\rangle \), the expectation values of the two projection operators are \( \langle P_1 \rangle \equiv \langle \psi|P_1|\psi \rangle = \langle \psi_1|\psi_1\rangle \langle \psi_1|\psi_1 \rangle = 1 \) and \( \langle P_2 \rangle \equiv \langle \psi|P_2|\psi \rangle = \langle \psi_1|\psi_2\rangle \langle \psi_2|\psi_1 \rangle = 0 \). These extreme values of 1 and 0 are not positively weighted means of \( P_A^A \) and \( P_B^B \) and of \( P_A^B \) and \( P_A^A \) for any choice of \( |\psi_1\rangle \) and \( |\psi_2\rangle \) and any normalized choice of positive weights. Therefore, no matter what the orthonormal projection operators \( P_1 \) and \( P_2 \) are, there is at least one quantum state (and actually an open set of states) that gives expectation values that are not positively weighted means of the observational probabilities at the two locations. Thus Born’s rule fails. This proof is simpler and uses much weaker assumptions than my previous arguments for the failure of Born’s rule [4, 5, 6, 7].

There are many logically possible replacements of Born’s rule [4, 5, 6, 7]. Here I shall describe only three of them. It is simplest to start with definitions of unnormalized relative probabilities (nonnegative, but not necessarily summing to unity).
and then to define the normalized first-person probabilities $P_j = P(O_j)$ to be these relative probabilities divided by their sum over all possible observations.

One choice (theory $T_3$ in [6]) would be to take the relative probabilities to be the expectation values of the numbers of occurrence of the observation. This rule was called *volume weighting*.

A second choice (theory $T_4$ in [6]) would be for the relative probabilities to be the expectation values of the fraction of the number of locations in which the observation occurs. This rule was called *volume averaging*. It differs from volume weighting when the quantum state is a superposition of different numbers of locations (e.g., different sizes for the universe). Volume weighting gives more weight to components of the quantum state in which there are more locations for observers. On the other hand, for volume averaging, it does not matter how many locations there are in a quantum component, but only the fraction of the number of locations where it occurs (for fixed quantum amplitudes for the components).

A third choice (theory $T_5$ in [6]) for each relative probability would be the expectation value of the fraction of all observations that are the one in question. This rule was called *observational averaging*. It would seem to be the most natural rule to use if one assumed wavefunction collapse [8, 9] and multiplied the quantum probability for a particular quantum component with the probability of randomly choosing a particular observation out of all the observations in that quantum component.

All three of these rules may be interpreted as replacing Born’s rule of the probabilities as expectation values of projection operators $P_j$ with rules for giving the probabilities as expectation values of other operators $Q_j$, which might be called *observation operators*. That is, these rules are linear in the quantum state (an entity that assigns expectation values to operators). It is logically possible that the rules for extracting observational probabilities from the quantum state are instead non-linear [6], though the examples proposed so far for this seem rather contrived and
more complicated than the linear rules. It would perhaps be most conservative first to explore the various linear rules.

The ambiguity of what to use to replace Born’s rule is reflected in the measure problem in cosmology \[2, 3, 4\]. The usual focus of the measure problem is how to regulate the infinities that occur when one has an infinite universe. However, the need to replace Born’s rule shows that there is an ambiguity even for finite but large universes. It would not be enough to have the actual quantum state of the universe; one would also need the rules for extracting the first-person probabilities of observations from the quantum state. Here I have shown that Born’s rule is insufficient for getting reasonable first-person probabilities in a universe large enough for many identical copies of the observer.

I am grateful for discussions with Andreas Albrecht, Tom Banks, Raphael Bousso, Sean Carroll, Brandon Carter, Ben Freivogel, Alan Guth, Daniel Harlow, James Hartle, Thomas Hertog, Gary Horowitz, Matthew Kleban, Andrei Linde, Seth Lloyd, Juan Maldacena, Donald Marolf, Mahdiyar Noorbala, Daniel Phillips, Joe Polchinski, Stephen Shenker, Eva Silverstein, Mark Srednicki, Leonard Susskind, Herman Verlinde, Alex Vilenkin, Alexander Westphal, and others. This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.
References

[1] M. Born, Z. Phys. 37, 863-867 (1926).

[2] J. Garcia-Bellido, A. D. Linde and D. A. Linde, Phys. Rev. D 50, 730 (1994) [arXiv:astro-ph/9312039].

[3] A. Vilenkin, Phys. Rev. Lett. 74, 846-849 (1995) [arXiv:gr-qc/9406010].

[4] D. N. Page, J. Cosmol. Astropart. Phys. 0810, 025 (2008) [arXiv:0808.0351].

[5] D. N. Page, Phys. Lett. B 678, 41-44 (2009) [arXiv:0808.0722].

[6] D. N. Page, J. Cosmol. Astropart. Phys. 0708, 008 (2009) [arXiv:0903.4888].

[7] D. N. Page, “Born Again,” arXiv:0907.4152.

[8] D. N. Page, “Observational Consequences of Many-Worlds Quantum Theory,” arXiv:quant-ph/9904004.

[9] D. N. Page, in General Relativity and Relativistic Astrophysics, Eighth Canadian Conference, Montreal, Quebec, 1999, edited by C. P. Burgess and R. C. Myers (American Institute of Physics, Melville, New York, 1999), pp. 225-232 [arXiv:gr-qc/0001001].