Quantum Computation Based Probability Density Function Estimation

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Signal processing techniques will lean on blind methods in the near future, where no redundant, resource allocating information will be transmitted through the channel. To achieve a proper decision, however, it is essential to know at least the probability density function (pdf), which to estimate is classically a time consuming and/or less accurate hard task, that may make decisions to fail. This paper describes the design of a quantum assisted pdf estimation method also by an example, which promises to achieve the exact pdf by proper setting of parameters in a very fast way.

Keywords: PDF estimation, decision, database searching

1. Introduction

The latest prognoses forecast a transistor size shrinkage to its 1 nm limit up to the year 2010, where the impacts of the quantum effects rise remarkably. Several new methods were published to replace the silicon based technology, one of them exploits the power of quantum computation. Unfortunately, it does not exists quantum computer solution with proper size, so far. In addition several research papers showed the constraints of the application range of quantum computers, that would not relegate classical computation which shows the demand on quantum assisted computation, which could amplify computational power enormously in the next future. This research work focuses on the exploitation of the strengths of quantum computation in classically difficult problems.

In every communication chain a decision has to be made at least in the receiver side, often without any a-priori known information, e.g. a-posteriori probability or the probability density function (pdf) of the source.

The estimation of the probability density function of observed signals shows a significant interest in many signal processing methods, such as pattern recognition, independent component analysis or detection. It exists a lot of well defined estimation techniques based on histogram or kernel method, which requires a large number of samples n to estimate the pdf almost sure as described for the $L_1$ case
by
\[
\lim_{n \to \infty} \|f - f_n\| = \lim_{n \to \infty} \int |f(x) - f_n(x)| \, dx \to 0.
\]
(1)

A more tight lower bound for kernel estimator can be found, e.g. in\(^3\). In case of decision problems, however, the knowledge of the whole pdf is often not needed, rather the pdf at a single point is of interest.

The rest of the paper is organized as follows: the overall system model is introduced in Section 2. In Section 3 the probability density function estimation is showed, and Section 4 concludes the paper.

2. System Model

By help of \textit{a-priori} knowledge of the observed system\(^4\), a quantum register
\[
|\varphi\rangle = \sum_{x=0}^{N-1} \varphi_x |x\rangle; \quad \varphi_x \in \mathbb{C},
\]
(2)
as a database is used to store all the raw quantized parameters, e.g. delay, heat, velocity, etc. values\(^5\). The database (2) have to be set up in the hardware only once for the whole estimation process. To handle the large amount of data a virtual database \(y = g(s, x)\), should be introduced\(^7\), where \(s\) describes the behavior of the system and \(x\) denotes the index of the qregister \(|\varphi\rangle\), respectively. The function \(y_i = g(s, x_i)\) points to an record in the virtual database\(^6\).

2.1. Properties of the Virtual Database Generating Function

The function \(g(s, x)\) is not obligingly mutual unambiguous consequently, i.e. it is not reversible, except for several special cases, when the virtual database contains \(\hat{r} = g(s, x)\) only once. In this case the parameter settings of the system are easy to determine. Henceforth the fact should be kept in mind that \(g(s, x)\) is in almost every case a so called one way function which is easy to evaluate in one direction, but to estimate the inverse is rather hard.

The function \(g(.)\) generates all the possible disturbances additional to the considered input value. This is of course a large amount of information, \(2N = 2^{n+1}\), where \(n\) is the length of the qregister \(|\varphi\rangle\). For an example let us assume a 15-qbit qregister. The function \(g(.)\) in (2) generates \(2^{15} = 32,768\) output values. Taking into account the large number of possible points in the set surface the optimal classification in a classical way becomes difficult. At the first glance this problem looks more difficult to solve, however, using the Deutsch-Jozsa\(^8\) quantum parallelization algorithm, an arbitrary unitary operation can be executed on all the prepared states contemporaneously.

\(^{6}\)This is not an \textit{a-priori} information about the sequence addressed above.
2.2. The Estimator

Roughly speaking the task is to find the entry (entries) in the virtual databases which is (are) equal to the observed data \( r \). To accomplish the database search the Grover database search algorithm should be invoked\(^7\), where we feed the received signal \( r(t) \) and \( g(s, x) \) to the oracle \( O \). Because of the fact of tight bound, in real application less iterations would be also appropriate\(^10\). Employing the Grover database search algorithm we are able to find the entries in the virtual databases, however, it is not needed to perform a complete search because the search result—the exact index (indices) of the searched item(s) is (are) not interesting but the number how often a given configuration is involved in \( g(s, x) \) or not. For that purpose a new function

\[
 f(r|s) = \frac{\#(x : r = g(s, x))}{\#(x)}, \quad (3)
\]

is defined\(^7\), which counts the number of similar entries in the virtual database, which corresponds to the conditional probability density function \( r \) to be in the set \( s \). For that reason it is worth stepping forward to quantum counting\(^11\) based on Grover iteration (Fig. 1).

3. Estimation Method

In this section we deal with the estimation of the probability density function at a single input point and with the estimation of the whole pdf by quantum assisted way.

3.1. Decision in One Point

Based on the proposed pdf estimation method a decision is possible at a single input point. As an example a decision has to be made in a telecommunication receiver. A received signal \( r \) is given, which is the Bernoulli-distributed source signal \( a \in \mathbb{R}^{[0,1]} \) disturbed by the communication channel. For the sake of simplicity only an Additive White Gaussian Noise (AWGN) channel is assumed, \( r(t) = a(t) + n(t) \), where \( n \) is the Gaussian distributed noise sample. In case of no known a-priori probability a decision in Maximum Likelihood (ML) sense seems to be appropriate, which maximize the likelihood function based on the (estimated) pdf. The value \( r = -0.8 \)
was received in the detector, the task is to decide whether \( a = -1 \) with noise vector \( n = 0.2 \) or \( a = 1 \) with \( n = -1.8 \) was detected as shown in Fig 3. Employing the pdf estimator and the decision device (a simple comparator) as shown in Fig. 2 the number of the overall accuracies of the given \( r \) in the system is calculated, which is slightly more than 1400 for the \( a = -1 \) belonging to the solid lined pdf, and less than 200 for the other one. In this way the likelihood functions are well calculated and a fast decision is possible. You may find another example in\(^7\).

### 3.2. The Whole Pdf Estimation

In many cases, however, the knowledge of the whole pdf is required, e.g. for independence testing of input sequences by KL divergence

\[
KL(X \parallel Y) = P \left( X \log \left( \frac{X}{Y} \right) \right),
\]

where \( X = \prod_i Y_i \) have to be fulfilled for the independence. To get know the pdf in quantum assisted way a estimation method introduced in (3.1) should be invoked for a large number of input values \( (r) \) to get a good approximation

\[
\lim_{j \to J} \| f(x_j|s)|_{x=r} - f(x) \| \to 0,
\]

in \( L_1 \) sense. Should be \( J \) chosen to \( \infty \) the method may converge to continuous pdf estimation, which is impossible because of time consumption and quantized behavior of the element in the database (2). The problem is quite similar to the problem of the number of bins in classical histogram method\(^12\).

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**Fig. 2.** Quantum Assisted Detector

**Fig. 3.** Calculated appearance of possible input signals in Gaussian distributed noisy environment
4. Concluding Remarks

This paper provides a brief introduction to quantum assisted probability density function estimation. The inaccuracy of the pdf estimation may result in wrong detection, however, the quantum assisted estimator is able to achieve the exact value of the pdf at a single point (received signal) that could make a decision more accurate. The proposed qregister $|\psi\rangle$ have to be set up only once before the estimation. The virtual databases are generated once and directly leaded to the Grover block in the quantum counting circuit, which reduce the computational complexity, substantially. The paper should be regarded as a starting point to further analyze the properties of the quantum assisted pdf estimator.

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