MULTIDIMENSIONAL TOPOLOGICAL GALOIS THEORY

ASKOLD KHOVANSKII

Department of Mathematics, University of Toronto, Toronto, Canada

Abstract. In this preprint we present an outline of the multidimensional version of topological Galois theory. The theory studies topological obstruction to solvability of equations “in finite terms” (i.e. to their solvability by radicals, by elementary functions, by quadratures and so on). This preprint is based on the author’s book on topological Galois theory. It contains definitions, statements of results and comments to them. Basically no proofs are presented.

This preprint was written as a part of the comments to a new edition (in preparation) of the classical book “Integration in finite terms” by J.F. Ritt.

1. Introduction. In topological Galois theory for functions of one variable (see [1], [2]), it is proved that the way the Riemann surface of a function is positioned over the complex line can obstruct the representability of this function “in finite term” (i.e. its representability by radicals, by quadratures, by generalized quadratures and so on). This not only explains why many algebraic and differential equations are not solvable in finite terms, but also gives the strongest known results on their unsolvability.

In the multidimensional version of topological Galois theory analogous results are proved. But in the multidimensional case all constructions and proofs are much more complicated and involved than in the one dimensional case (see [1]).

2. Classes of functions. An equation is solvable “in finite terms” (or is solvable “explicitly”) if its solutions belong to a certain class of functions. Different classes of functions correspond to different notions of solvability in finite terms.

A class of functions can be introduced by specifying a list of basic functions and a list of admissible operations. Given the two lists, the class of functions is defined as the set of all functions that can be obtained from the basic functions by repeated application of admissible operations. Below, we define Liouvillian classes of functions in exactly this way.

Classes of functions, which appear in the problems of integrability in finite terms, contain multivalued functions. Thus the basic terminology should be made clear.

We understand operations on multivalued functions of several variables in a slightly more restrictive sense than operations on multivalued functions of single variable (one dimensional case is discussed in [1], [2]).

Key words and phrases. solvability by radicals, by elementary functions, by quadratures, by generalized quadratures.

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Fix a class of basic functions and some set of admissible operations. Can a given function (which is obtained, say, by solving a certain algebraic or a differential equation) be expressed through the basic functions by means of admissible operations? We are interested in various single valued branches of multivalued functions over various domains. Every function, even if it is multivalued, will be considered as a collection of all its single valued branches. We will only apply admissible operations (such as arithmetic operations and composition) to single valued branches of the function over various domains. Since we deal with analytic functions, it suffices only to consider small neighborhoods of points as domains.

We can now rephrase the question in the following way: can a given function germ at a given point be expressed through the germs of basic functions with the help of admissible operations? Of course, the answer depends on the choice of a point and on the choice of a single valued germ at this point belonging to the given multivalued function. It turns out, however, that for the classes of functions interesting to us the desired expression is either impossible for every germ of a given multivalued function at every point or the “same” expression serves all germs of a given multivalued function at almost every point of the space.

For functions of one variable, we use a different, extended definition of operations on multivalued functions, in which the multivalued function is viewed as a single object. This definition is essentially equivalent to including the operation of analytic continuation in the list of admissible operations on analytic germs (all details can be found in [1]). For functions of many variables, we need to adopt the more restrictive understanding of operations on multivalued functions, which is, however, no less (and perhaps even more) natural.

3. Specifics of the multidimensional case. I was always under impression that a full-fledged multidimensional version of topological Galois theory was impossible. The reason was that, to construct such a version for the case of many variables, one would need to have information on extendability of function germs not only outside their ramification sets but also along these sets. It seemed that there was nothing to extract such information from.

To illustrate the problem consider the following situation. Let $f$ be a multivalued analytic function on $\mathbb{C}^n$, whose set of singular points is an analytic set $\Sigma_f \subset \mathbb{C}^n$. Let $f_a$ be an analytic germ of $f$ at a point $a \in \mathbb{C}^n$. Let $g : (\mathbb{C}^k, b) \to (\mathbb{C}^n, a)$ be an analytic map. Consider a germ $\varphi_b$ at the point $b \in \mathbb{C}^k$ of the composition $f_a \circ g_b$. One can ask the following questions:

1) Is it true that $\varphi_b$ is a germ of a multivalued function $\varphi$ on $\mathbb{C}^k$, whose set of singular points $\Sigma_\varphi$ is contained in a proper analytic subset of $\mathbb{C}^k$?

2) Is it true that the monodromy group $M_\varphi$ of $\varphi$ corresponding to motions around the set $\Sigma_\varphi \subset \mathbb{C}^k$ can be estimated in terms of the monodromy group $M_f$ of $f$ corresponding to motions around the set $\Sigma_f \subset \mathbb{C}^n$? For example, if $M_f$ is a solvable group is it true that $M_\varphi$ also is a solvable group?

If the image $g(\mathbb{C}^k)$ is not contained in the singular set $\Sigma_f$ then the answers to the both questions are positive: the set $\Sigma_\varphi$ belongs to the analytic set $g^{-1}(\Sigma_f)$ and the group $M_\varphi$ is a subgroup of a certain factor group of $M_f$. These statements are not complicated and can be proved by the same arguments as in the one dimensional topological Galois theory.

Assume that the multivalued function $f$ has an analytic germ $f_a$ at a point $a$ belonging to the singular set $\Sigma_f$ (some of the germs of the multivalued function
f may appear to be nonsingular at singular points of this function). Assume now that the image \( g(C^k) \) is contained in the singular set \( \Sigma_f \) and \( a = g(b) \). It turns out that for the germ \( \varphi_b = f_a \circ g_b \) the answers to the both questions also are positive. In this situation all the proofs are more involved. They use new arguments from multidimensional complex analysis and from group theory.

It turns out that function germs can sometimes be automatically extended along their ramification sets (see [1]). That new statement from complex analysis suggests the positive answer to the first question.

To describe the connection between the monodromy group of the function \( f \) and the monodromy groups of the composition \( \varphi = f \circ g \), we introduce and develop the notion of pullback closure for groups (see [1]). The use of this operation, in turn, forces us to reconsider all arguments we used in the one dimensional version of topological Galois theory. As a result we obtain a positive answer to the second question.

4. Liouvillian classes of multivariate functions. In this section we define Liouvillian classes of functions for the case of several variables. These classes are defined in the same way as the corresponding classes for functions of one variable (see [1], [2]). The only difference is in the details.

We fix an ascending chain of standard coordinate subspaces of strictly increasing dimension: \( 0 \subset \mathbb{C}^1 \subset \cdots \subset \mathbb{C}^n \subset \cdots \) with coordinate functions \( x_1, \ldots, x_n, \ldots \) (for every \( k > 0 \), the functions \( x_1, \ldots, x_k \) are coordinate functions on \( \mathbb{C}^k \)). Below, we define Liouvillian classes of functions for each of the standard coordinate subspaces \( \mathbb{C}^k \).

To define Liouvillian classes, we will need the list of basic elementary functions and the list of classical operations.

**List of basic elementary functions.**
1. All complex constants and all coordinate functions \( x_1, \ldots, x_n \) for every standard coordinate subspace \( \mathbb{C}^n \).
2. The exponential, the logarithm and the power \( x^\alpha \), where \( \alpha \) is any complex constant.
3. Trigonometric functions: sine, cosine, tangent, cotangent.
4. Inverse trigonometric functions: arcsine, arccosine, arctangent, arccotangent.

Let us now turn to the list of classical operations on functions.

**List of classical operations.**
1. **Operation of composition** that takes a function \( f \) of \( k \) variables and functions \( g_1, \ldots, g_k \) of \( n \) variables to the function \( f(g_1, \ldots, g_k) \) of \( n \) variables.
2. **Arithmetic operations** that take functions \( f \) and \( g \) to the functions \( f + g \), \( f - g \), \( fg \) and \( f/g \).
3. **Operations of partial differentiation with respect to independent variables.** For functions of \( n \) variables, there are \( n \) such operations: the \( i \)-th operation assigns the function \( \frac{\partial f}{\partial x_i} \) to a function \( f \) of the variables \( x_1, \ldots, x_n \).
4. **Operation of integration** that takes \( k \) functions \( f_1, \ldots, f_k \) of the variables \( x_1, \ldots, x_n \), for which the differential one-form \( \alpha = f_1dx_1 + \cdots + f_kdx_k \) is closed, to the indefinite integral \( y \) of the form \( \alpha \) (i.e. to any function \( y \) such that \( dy = \alpha \)). The function \( y \) is determined by the functions \( f_1, \ldots, f_k \) up to an additive constant.
5. **Operation of solving an algebraic equation** that takes functions \( f_1, \ldots, f_n \) to the function \( y \) such that \( y^n + f_1y^{n-1} + \cdots + f_n = 0 \). The function \( y \) may not be
quite uniquely determined by the functions $f_1, \ldots, f_n$, since an algebraic equation of degree $n$ can have $n$ solutions.

We now resume defining Liouvillian classes of functions.

**Functions of $n$ variables representable by radicals.** List of basic functions: all complex constants and all coordinate functions. List of admissible operations: composition, arithmetic operations and the operation of taking the $m$-th root $f^{1/m}$, $m = 2, 3, \ldots$, of a given function $f$.

**Functions of $n$ variables representable by $k$-radicals.** This class of functions is defined in the same way as the class of functions representable by radicals. We only need to add the operation of solving algebraic equations of degree $\leq k$ to the list of admissible operations.

**Elementary functions of $n$ variables.** List of basic functions: elementary functions. List of admissible operations: composition, arithmetic operations, differentiation.

**Generalized elementary functions of $n$ variables.** This class of functions is defined in the same way as the class of elementary functions. We only need to add the operation of solving algebraic equations to the list of admissible operations.

**Functions of $n$ variables representable by quadratures.** List of basic functions: basic elementary functions. List of admissible operations: composition, arithmetic operations, differentiation, integration.

**Functions of $n$ variables representable by $k$-quadratures.** This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations of degree at most $k$ to the list of admissible operations.

**Functions $n$ variables representable by generalized quadratures.** This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations to the list of admissible operations.

5. **Strong non representability in finite terms.** Topological obstructions to the representability of functions in finite terms relate to branching. It turns out that if a function does not belong to a certain Liouvillian class by topological reasons then it automatically does not belong to a much wider extended Liouvillian class of functions.

Such an extended Liouvillian class is defined as follows: its list of admissible operations is the same as in the original Liouvillian class and its list of basic functions is the list of basic function in the original class extended by all single valued functions of any number of variables having proper analytic set of singular points.

**Definition.** A germ $f$ is a germ of function belonging to the extended class of functions representable by by quadratures if it can be represented by germs of basic elementary functions and by germs of single valued functions, whose set of singular points is a proper analytic set, by means of composition, integration, arithmetic operations and differentiation.
Definition. A germ $f$ is strongly non representable by quadratures if it is not a germ of function from the extended class of functions representable by quadratures.

The definition of strong non representability of a germ $f$ by radicals, by $k$-radical, by elementary functions, by generalized elementary functions, by $k$-quadratures and by generalized quadratures is similar to the above definition.

6. Holonomic systems of linear differential equations. Consider a system of $N$ linear differential equations $L_j(y) = 0, j = 1, \ldots, N,$

$$L_j(y) = \sum a_{i_1, \ldots, i_n} \frac{\partial^{i_1+\cdots+i_n} y}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = 0,$$

on an unknown function $y$, whose coefficients $a_{i_1, \ldots, i_n}$ are analytic functions in a domain $U \subset \mathbb{C}^n$.

The system (1) is holonomic if at every point $a \in U$ the $\mathbb{C}$-linear space $V_a$ of germs $y_a$ satisfying the system (1) has finite dimension, $\dim_{\mathbb{C}} V_a = d(a) < \infty$. Holonomic systems can be considered as a multidimensional generalization of linear differential equation on one unknown function of a single variable. Kolchin obtained a generalization of the Picard–Vessiot theory (Galois theory for linear differential equations) to the case of holonomic systems of differential equations [3].

Holonomic system (1) has the following properties:

1) There exists an analytic singular hypersurface $\Sigma \subset U$ such that the dimension $d(a) = \dim_{\mathbb{C}} V_a$ is constant $d(a) \equiv d$ on $U \setminus \Sigma$.

2) Let $\gamma : I \to U \setminus \Sigma$ be a continuous map, where $I$ is the unit segment $0 \leq t \leq 1$ and $\gamma(0) = a, \gamma(1) = b$. Then the space $V_a$ of solutions of (1) at the point $a$ admits analytic continuation along $\gamma$ and the space obtained by the continuation at the point $b$ is the space $V_b$ of solutions of (1) at the point $b$.

3) If all equations of the system (1) admit analytic continuation to some domain $W$, then the system obtained by such a continuation is a holonomic system in the domain $W$.

Let $a \notin \Sigma$ be a point not belonging to the hypersurface $\Sigma$. Take an arbitrary path $\gamma(t)$ in the domain $U$ originating and terminating at $a$ and avoiding the hypersurface $\Sigma$. Solutions of this system admit analytic continuations along the path $\gamma$, which are also solutions of the system. Therefore, every such path $\gamma$ gives rise to a linear map $M_{\gamma}$ of the solution space $V_a$ to itself. The collection of linear transformations $M_{\gamma}$ corresponding to all paths $\gamma$ form a group, which is called the monodromy group of the holonomic system.

7. $\mathcal{S}\mathcal{C}$-germs. There is a wide class of $\mathcal{S}$-functions in one variable containing all Liouvilian functions and stable under classical operations, for which the monodromy group is defined. The class of $\mathcal{S}$-functions plays an important role in the one dimensional version of topological Galois theory (see [1], [2]). Is there a sufficiently wide class of multivariate function germs with similar properties?

For a long time, I thought that the answer to this question was negative. In this section the class of $\mathcal{S}\mathcal{C}$-germs is defined. This provides an affirmative answer to this question.

A subset $A$ in a connected $k$-dimensional analytic manifold $Y$ is called meager if there exists a countable set of open domains $U_i \subset M$ and a countable collection of proper analytic subsets $A_i \subset U_i$ in these domains such that $A \subset \bigcup A_i$.

The following definition plays a key role in what follows.
**Definition.** A germ $f_a$ of an analytic function at a point $a \in \mathbb{C}^n$ is an $\text{SC}$-germ if the following condition is fulfilled. For every connected complex analytic manifold $Y$, every analytic map $G : Y \to \mathbb{C}^n$ and every preimage $b$ of the point $a$, $G(b) = a$, there exists a meager set $A \subset Y$ such that, for every path $\gamma : [0, 1] \to Y$ originating at the point $b$, $\gamma(0) = b$ and intersecting the set $A$ at most at the initial moment, $\gamma(t) \notin A$ for $t > 0$, the germ $f_a$ admits an analytic continuation along the path $G \circ \gamma : [0, 1] \to \mathbb{C}^n$.

The following lemma is obvious.

**Lemma 1.** The class of $\text{SC}$-germs contains all germs of analytic functions on $\mathbb{C}^N \setminus \Sigma$ where $\Sigma$ is an analytic subset in $\mathbb{C}^N$ where $N$ is a natural number. In particular the class contains all analytic germs of $\text{S}$-functions of one variable and all germs of meromorphic functions of many variables.

The proof of the following Theorem 2 uses the results on extendability of multi-valued analytic functions along their singular point sets (see [1]).

**Theorem 2 (on stability of the class of $\text{SC}$-germs).** The class of $\text{SC}$-germs on $\mathbb{C}^n$ is stable under the operation of taking the composition with $\text{SC}$-germs of $m$-variable functions, the operation of differentiation and integration. It is stable under solving algebraic equations whose coefficients are $\text{SC}$-germs and under solving holonomic systems of linear differential equations whose coefficients are $\text{SC}$-germs.

Theorem 2 implies the following corollary.

**Corollary 3.** If a germ $f$ is not an $\text{SC}$-germ then $f$ is strongly non-representable by generalized quadratures. In particular it cannot be a germ of a function belonging to a certain Liouvilian class.

**8. Monodromy group of a $\text{SC}$-germs.** The monodromy group and the monodromy pair of a $\text{SC}$-germ $f_a$ can be defined in the same way as for $\text{S}$-functions of one variable. By definition the set $\Sigma \subset \mathbb{C}^n$ of singular points of $f_a$ is meager set. Take any point $x_0 \in \mathbb{C}^n \setminus \Sigma$ and consider the action of the fundamental group $\pi_1(\mathbb{C}^n \setminus \Sigma, x_0)$ on the set $F_{x_0}$ of all germs equivalent to the germ $f_a$. The monodromy group of $f_a$ is the image of the fundamental group under this action. The monodromy pair of $f_a$ is the pair $[\Gamma, \Gamma_0]$ where $\Gamma$ is the monodromy group and $\Gamma_0$ is the stationary subgroup of a germ $f \in F_{x_0}$. Up to an isomorphism the monodromy group and the monodromy pair are independent of a choice of the point $x_0$ and the germ $f$.

**Remark.** If a $\text{SC}$-germ $f_a$ is defined at a singular point $a \in \Sigma$ then the monodromy group of $f_a$ along $\Sigma$ is defined: one can consider continuations of $f_a$ along curves $\gamma$ belonging to $\Sigma$ and define a singular set $\Sigma_1 \subset \Sigma$ for $f_a$ along $\Sigma$. The monodromy group of $f_a$ along $\Sigma$ corresponds to the action the fundamental group of $\pi_1(\Sigma \setminus \Sigma_1, x_1)$ on the set of germs at $x_1 \in \Sigma \setminus \Sigma_1$ obtained by continuation of $f_a$ along $\Sigma$. If the point $a$ belongs to $\Sigma_1$ then one can define also a monodromy group of $f_a$ along $\Sigma_1$ and so on. Thus in the multidimensional case one can associate to an $\text{SC}$-germ an an hierarchy of monodromy groups. All these monodromy groups (and corresponding monodromy pairs) appear in multidimensional topological Galois theory. But the monodromy group and the monodromy pair we discuss above are most important for our purposes.
9. Stability of certain classes of $\mathcal{S}\mathcal{C}$-germs. One can prove the following theorems.

**Theorem 4 (see [1]).** The class of all $\mathcal{S}\mathcal{C}$-germs, having a solvable monodromy is stable under composition, arithmetic operations, integration and differentiation. This class contains all germs of basic elementary functions and all germs of single valued functions whose set of singular points is a proper analytic set.

**Theorem 5 (see [1]).** The class of all $\mathcal{S}\mathcal{C}$-germs, having a $k$-solvable monodromy pair (see [1], [2]) is stable under composition, arithmetic operations, integration, differentiation and solution of algebraic equations of degree at most $k$. This class contains all germs of basic elementary functions and all germs of single valued functions whose set of singular points is a proper analytic set.

**Theorem 6 (see [1]).** The class of all $\mathcal{S}\mathcal{C}$-germs, having an almost solvable monodromy pair (see [1], [2]) is stable under composition, arithmetic operations, integration, differentiation and solution of algebraic equations. This class contains all germs of basic elementary functions and all germs of single valued functions whose set of singular points is a proper analytic set.

Theorems 4–6 imply the following corollaries.

**Result on quadratures.** If the monodromy group of a $\mathcal{S}\mathcal{C}$-germ $f$ is not solvable, then $f$ is strongly non representable by quadratures.

**Result on $k$-quadratures.** If the monodromy pair of a $\mathcal{S}\mathcal{C}$-germ $f$ is not $k$-solvable, then $f$ is strongly non representable by $k$-quadratures.

**Result on generalized quadratures.** If the monodromy pair of a $\mathcal{S}\mathcal{C}$-germ $f$ is not almost solvable, then $f$ is strongly non representable by generalized quadratures.

10. Solvability and non solvability of algebraic equation. Consider an irreducible algebraic equation

$$P_n y^n + P_{n-1} y^{n-1} + \cdots + P_0 = 0$$

whose coefficients $P_n, \ldots, P_0$ are polynomials of $N$ complex variables $x_1, \ldots, x_N$. Let $\Sigma \subset \mathbb{C}^N$ be the singular set of the equation (2) defined by the equation $P_n J = 0$ where $J$ is the discriminant of the polynomial (2).

**Theorem 7 (see [1], [2], [4]).** Let $y_{x_0}$ be a germ of analytic function at a point $x_0 \in \mathbb{C}^N \setminus \Sigma$ satisfying the equation (2). If the monodromy group of the equation (2) is solvable (is $k$-solvable) then the germ $y_{x_0}$ is representable by radicals (is representable by $k$-radicals).

According to Camille Jordan’s theorem (see [4]) the Galois group of the equation (2) over the field $\mathcal{R}$ of rational functions of $x_1, \ldots, x_N$ it is isomorphic to the monodromy group of this equation (2). Thus Theorem 7 follows from Galois theory (see [1], [4]).

**Theorem 8 (see [1]).** Let $y_{x_0}$ be a germ of analytic function at a point $x_0 \in \mathbb{C}^N$ satisfying the equation (2). If the monodromy group of the equation is not solvable (is not $k$-solvable) then the germ $y_{x_0}$ is strongly non representable by quadratures (is strongly non representable by $k$-quadratures).
Theorem 8 follows from the results on quadratures and on $k$-quadratures from the previous section.

Consider the universal degree $n$ algebraic function $y(a_n, \ldots, a_0)$ defined by the equation

$$a_n y^n + \cdots + a_0 = 0.$$  

It is easy to see that the monodromy group of the equation (3) is isomorphic to the group $S_n$ of all permutations of $n$ element. For $n \geq 5$ the group $S_n$ is unsolvable and it is not $k$-solvable group for $k < n$. Thus Theorem 8 implies the following strongest known version of the Abel-Ruffini Theorem.

**Theorem 9 (a version of the Abel–Ruffini Theorem).** Let $y_a$ be a germ of analytic function at a point $a$ satisfying the universal degree $n \geq 5$ algebraic equation. If $n \geq 5$ then the germ $y_a$ is strongly non representable by $(n - 1)$ quadratures. In particular the germ $y_a$ is strongly non representable by quadratures.

### 10. Solvability and non solvability of holonomic systems of linear differential equations.

Consider a system of $N$ linear differential equations $L_j(y) = 0$, $j = 1, \ldots, N$,

$$L_j(y) = \sum a_{i_1, \ldots, i_n} \frac{\partial^{i_1+\cdots+i_n} y}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = 0,$$

on an unknown function $y$, whose coefficients $a_{i_1, \ldots, i_n}$ are rational functions of $n$ complex variables $x_1, \ldots, x_n$. Assume that the system (4) is holonomic in $\mathbb{C}^n \setminus \Sigma_1$ where $\Sigma_1$ is the union of poles of the coefficients $a_{i_1, \ldots, i_n}$. Let $\Sigma_2 \subset \mathbb{C}^n \setminus \Sigma_1$ be the singular hypersurface of a holonomic system (4).

Every germ $y_a$ of a solution of the system at a point $a \in \mathbb{C}^n \Sigma$ where $\Sigma = \Sigma_1 \cup \Sigma_2$ admits an analytic continuation along every path avoiding the hypersurface $\Sigma$ so the monodromy group of the system (4) is well-defined.

**Theorem 10 (see [1]).** If the monodromy group of the holonomic system (4) is not solvable (not $k$-solvable, not almost solvable), then a germ $y_a$ of almost every solution at a point $a \in \mathbb{C}^n \setminus \Sigma$ is strongly non representable by quadratures (is strongly non representable by $k$-quadratures, is strongly non representable by generalized quadratures).

Theorem 10 follows from the results on quadratures, on $k$-quadratures and on generalized quadratures from section 9.

A holonomic system is said to be regular, if near the singular set $\Sigma$ and near infinity the solutions of the system grow at most polynomially.

**Theorem 11 (see [1]).** If the monodromy group of a regular holonomic system is solvable (is $k$-solvable, is almost solvable), then a germ $y_a$ of almost every solution at a point $a \in \mathbb{C}^n \setminus \Sigma$ is representable by quadratures (is representable by $k$-quadratures, is representable by generalized quadratures).

### 11. Completely integrable systems of linear differential equations with small coefficients.

Consider a completely integrable system of linear differential equations of the following form

$$dy = Ay,$$
where \( y = y_1, \ldots, y_N \) is an unknown vector-function, and \( A \) is a \((N \times N)\)-matrix consisting of differential one-forms with rational coefficients on the space \( \mathbb{C}^n \) satisfying the condition of complete integrability \( dA + A \wedge A = 0 \) and having the following form:

\[
A = \sum_{i=1}^{k} A_i \frac{dl_i}{l_i},
\]

where \( A_i \) are constant matrices, and \( l_i \) are linear (not necessarily homogeneous) functions on \( \mathbb{C}^n \).

If the matrices \( A_i \) can be simultaneously reduced to the triangular form, then system (5), as any completely integrable triangular system, is solvable by quadratures. Of course, there exist solvable systems that are not triangular. However, if the matrices \( A_i \) are sufficiently small, then there are no such systems. Namely, the following theorem holds.

**Theorem 12 (see [1]).** A system (5) that does not reduce to the triangular form and such that the matrices \( A_i \) have sufficiently small norms is unsolvable by generalized quadratures in the following strong sense. At every point \( a \in \mathbb{C}^n \) where the matrix \( A \) is regular, and for almost any germ \( y_a = (y_1, \ldots, y_N)_a \) of a vector-function satisfying the system (5), there is a component \( (y_i)_a \) which is strongly non representable by generalized quadratures.

Multidimensional Theorem 12 is similar to the one dimensional Corollary 20 from [2]. Their proofs (see [1]) are also similar. We only need to replace the reference to the (one-dimensional) Lappo-Danilevsky theory with the reference to the multidimensional version of it from [5].

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