CASTELNUOVO REGULARITY FOR SMOOTH SUBVARIETIES OF DIMENSIONS 3 AND 4

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§0. Introduction.

Let $X$ be a smooth, non-degenerate projective variety of dimension $n$ and of degree $d$ in $\mathbb{P}^r$. We say that $X$ is $k$–normal if the homomorphism

$$H^0(\mathbb{P}^r, \mathcal{O}(k)) \rightarrow H^0(X, \mathcal{O}_X(k))$$

is surjective, i.e., hypersurfaces of degree $k$ cut out a complete linear system on $X$. We say that $X$ is $k$–regular if $H^i(\mathbb{P}^r, \mathcal{I}_X(k - i)) = 0$ for all $i \geq 1$, where $\mathcal{I}_X$ is the sheaf of ideals of $X$ in $\mathcal{O}_{\mathbb{P}^r}$. It is easy to see that $X$ is $(k+1)$–regular if and only if $X$ is $k$–normal and $H^i(X, \mathcal{O}_X(k - i)) = 0$ for all $i > 0$. Let $\text{reg}(X) = \min\{k \in \mathbb{Z}: X \text{ is } k\text{-regular}\}$.

The importance of $k$–regularity stems from the following well-known results ([Mu1], lecture 14): if $X$ is $k$–regular, then the saturated ideal $\mathcal{I}_X$ is generated by homogeneous polynomials of degree at most $k$ and hence there is no $(k+1)$-secant line to $X$. Furthermore, the Hilbert polynomial and the Hilbert function of $X$ have the same values for all integers $m \geq k - 1$.

There is a well-known conjecture concerning the $k$–normality and $k$–regularity of $X$:

**Regularity conjecture** [EG], [GLP].

1. $X$ is $k$–normal for all $k \geq \deg(X) - \text{codim}(X)$.
2. $X$ is $k$–regular for all $k \geq \deg(X) - \text{codim}(X) + 1$,
   i.e., $\text{reg}(X) \leq d - (r - n) + 1$.

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As F. Zak has pointed out, the generalized Bertini theorem shows that a dimension $n$, degree $d$, non-degenerate variety $X$ in $\mathbb{P}^r$ can have at most a $(d - (r - n) + 1)$ secant line, thus yielding further credence to the conjecture. The conjecture is also sharp in the sense that for all $n$, there exist varieties of dimension $n$ with regularity $(d - (r - n) + 1)$.

It would already be interesting to establish the weaker bound $\text{reg}(X) \leq \deg(X)$ for an arbitrary dimensional, smooth variety $X$, since by projection it is possible to produce many equations of degree $d$ vanishing on $X$ – cones of projections – in fact, enough to cut out $X$ locally ideal-theoretically. This is a classical result, which is the foundation of the theory of Chow coordinates. For a modern proof, see [Mu2]. It is worth noting, however, that unpublished examples of Bayer-Pinkham show that the ideal generated by the cones of projection can be strictly smaller, in degree $d$, than the saturated ideal, e.g., for elliptic curves.

The purpose of this paper is to give new bounds for regularity in dimensions 3 and 4 which are only slightly worse than the optimal ones suggested by the conjecture. Our method will yield new bounds up to dimension 14, but as they get progressively worse as the dimension goes up, we have not written them down here.

**Theorem.** Let $X$ be a smooth variety of dimension $n$ and of degree $d$ in $\mathbb{P}^r$

(i) If $n = 3$, then $\text{reg}(X) \leq d - \text{codim}(X) + 2$.

(ii) If $n = 4$, then $\text{reg}(X) \leq d - \text{codim}(X) + 5$.

The main body of the proof is given in §3. Note that for $n = 3$ (resp. 4) we are off by 1 (resp. 4) from the conjectured bound. In the case $n = 4$, our bound is precisely $d$ in the “standard” case $r = 2n + 1$.

Here is a sketch of the history of the regularity conjecture.

0.1 Curves: The case of curves in $\mathbb{P}^3$ was settled by Castelnuovo in 1893 ([C]); that of curves in $\mathbb{P}^r$, $r \geq 3$, by Gruson, Lazarsfeld, and Peskine in 1983 ([GLP]).

0.2 Surfaces: The first bound for smooth surfaces was obtained by Pinkham([P]) using geometric methods. Shortly afterward Lazarsfeld([L]) obtained the full conjecture using more cohomological techniques. It should be noted that Pinkham’s proof relied on the classically established “fact” that the 3 inverse images above a triple point of a generic projection of a smooth surface in $\mathbb{P}^5$ to $\mathbb{P}^3$ lie in general linear position in their fiber (which is a $\mathbb{P}^2$). Vladimir Greenberg ([L], p.425) noticed that the classic proofs of this fact rely on an unproven general position assertion. More recently, it has become clear that the required “fact” is false (despite the recent claim to the contrary by Ran [Ran1], 2.3). This issue will be discussed in detail in the forthcoming Columbia Ph.D. thesis of Dobler ([D]).

Much of Lazarsfeld’s argument for surfaces generalizes immediately to arbitrary dimension. The only missing ingredient is information on the length and position of the fibers of a general linear projection $\pi_\Lambda : X^n \to \mathbb{P}^{n+1}$ in order to separate
points in the fiber by homogeneous polynomials of some given degree. Two results give partial results in this direction: first, the theorem of Mather ([Ma1]) showing that for \( n \leq 14 \), generic linear projections of \( X \) to \( \mathbb{P}^{n+1} \) are stable enables us, in small dimension, to use his results on stable mappings([Ma2]). Note however that Mather’s results give no information on the position of the points in the fiber, e.g., are they in general linear position? A partial answer (in all dimensions) to the position question is given by Ran’s (dimension +2)-secant lemma ([Ran2]): the family of all \((n+2)\)-secant lines to a \( n \)-dimensional smooth projective variety \( X \) is at most \((n+1)\)-dimensional.

0.3 Threefolds: In 1987, V. Greenberg([G]) modified Lazarsfeld’s argument to give the same bound as ours for threefolds. Unfortunately, at one step in the proof, there is an unsubstantiated claim which leaves the impression that the proof is incomplete. In this note we follow Greenberg’s proof closely, and justify the unsubstantiated claim. It should be noted that the bound we obtain in dimension 4 is stronger than Greenberg’s.

In 1989, Z. Ran([Ran1]) claimed the regularity conjecture for smooth threefolds \( X \) in \( \mathbb{P}^r, r \geq 9 \). The proof relies on differential geometric techniques to show that the family of all 4-secant lines to \( X \) is at most 4-dimensional. The method of proof definitely fails for embeddings in \( \mathbb{P}^7 \) (the most important case) and \( \mathbb{P}^8 \), which our method covers. For a further discussion of Ran’s result, see [D].

0.4 Above dimension 3 only weaker bounds were known: Mumford showed in 1984 that \( \text{reg}(X) \leq (n+1)(d-1)-n+1 \), where \( \text{dim}(X) = n \) and \( \text{deg}(X) = d \) ([BM]) which is improved to \( \text{reg}(X) \leq \min\{e, n+1\}(d-1)-n+1 \), \( e = \text{codim}X \) in [BEL].

0.5 For a discussion of what is known in the small codimension cases that we do not treat here, we refer to [Kw].

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§1. Basic background. In this section, we recall the definitions and well-known results which are used in the following sections.

**Definition 1.1.** For a coherent sheaf \( \mathcal{F} \) on \( \mathbb{P}^n_k \), \( \mathcal{F} \) is \( m \)-regular if \( H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0 \) for all \( i > 0 \) and \( \text{reg}(\mathcal{F}) \) is defined by \( \inf \{ m \in \mathbb{Z} : \mathcal{F} \text{ is } m\text{-regular} \} \).
Proposition 1.2. Let $F$ be a coherent sheaf on $\mathbb{P}^n$. If $F$ is $m$–regular, then

(a) $F(m)$ is generated by its sections.
(b) The multiplication map $H^0(F(m)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(F(m+t))$ is surjective for $t \geq 0$.
(c) $F$ is $(m+t)$–regular for $t \geq 0$.

Proof. See [Mu1], pg. 100. □

Proposition 1.3.

(a) Let $F$ be a $p$–regular vector bundle and $G$ be a $q$–regular vector bundle on $\mathbb{P}^n$ which is defined over an algebraically closed field of characteristic zero. Then $F \otimes G$ is $(p+q)$–regular and $S^k(F)$ and $\Lambda^k(F)$ are $(kp)$–regular.
(b) Let $F$ be a coherent sheaf on $\mathbb{P}^n$ and $\cdots \to F_i \to \cdots \to F_0 \to F \to 0$ be an exact sequence of coherent sheaves on $\mathbb{P}^n$ such that $F_i$ is $(p+i)$–regular. Then $F$ is $p$–regular.

Proof. See [L], pg. 428. □

Theorem 1.4. Let $X \subset \mathbb{P}^r$ be a smooth $n$-dimensional subvariety and $\pi_\Lambda : X^n \to \overline{X} \subset \mathbb{P}^{n+1}$ a generic linear projection to a hypersurface. Let $\overline{X}_k$ be the closed set of points $\overline{x} \in \overline{X}$ such that the scheme-theoretic length of the fiber $\pi_\Lambda^{-1}(\overline{x})$ is at least $k$, and $X_k$ the inverse image of $\overline{X}_k$, so $X_{k+1} \subset X_k$ for all $k$. Assume that $n \leq 14$, so that we are in Mather’s “nice” range. Then $X_{n+2}$ is empty, and $X_k$ has dimension at most $n+1-k$. If $X_k$ has the expected dimension $n+1-k$, then outside of a closed subset of smaller dimension each fiber above $\overline{X}_k$ is a set of $k$ reduced points.

Proof. This follows from the main theorem of [Ma1] and the discussion in §5 of [Ma2]. A key ingredient is the inequality, for any $\overline{x} \in \overline{X}$

$$\sum_{x \in \pi^{-1}(\overline{x})} (\delta_x + \gamma_x) \leq n + 1$$

where $\delta_x$ is the length of the fiber at $x$ and $\gamma_x$ is another non-negative invariant defined (and computed) for all stable germs in the “nice” range in [Ma2]. It is also known that $\delta_x = m_x$, where $m_x$ is the maximum number of points in the fiber of the germ map in the neighborhood of $x$ ([DG]). □

In the classical literature (for example in Bertini’s book [B], chapter 9, no.9, page 195 or more recently in [Llu]) this result is claimed in all dimensions. The proof is incorrect, as pointed out in [G], pg. 6: assuming that $X$ is $n$–dimensional and non–degenerate in $\mathbb{P}^{2n+1}$, the authors implicitly assume that the family of $n$–dimensional linear subspaces of $\mathbb{P}^{2n+1}$ intersecting $X$ in at least $n+1$ points is an irreducible variety whose generic point is a linear span of $n+1$ points of
X in general position. It is easy to give counterexamples to this assertion: the following one, due to Lazarsfeld, appears in [G]: let \( E \) be a plane elliptic curve, and embed \( X = \mathbb{P}^1 \times E \subset \mathbb{P}^1 \times \mathbb{P}^2 \) by the Segre embedding to \( \mathbb{P}^5 \). Then the family of trisecant planes has at least two components. The generic element of the first is the linear span of 3 points of \( X \) in general position, while the generic element of the second is the generic plane passing through the generic line in the Segre plane \( x_0 \times \mathbb{P}^2 \). (After this paper was prepared, Joel Roberts kindly brought to our attention his unpublished 1969 Harvard Ph. D thesis, "Ordinary Singularities of Projective Varieties", where the problem with the classical method of proof is discussed in section 3.5.)

It would be very interesting to know what actually happens outside the “nice” range of Mather. See [MP], pg. 111 for a related conjecture. In this direction we have a result of Ran, which we quote but will not use in this paper:

**Theorem 1.5 (The dimension+2-secant lemma).** Let \( X \subset \mathbb{P}^r \) be a smooth \( n \)-dimensional subvariety and let \( Y \) be an irreducible variety parametrizing a family of lines in \( \mathbb{P}^r \). Assume that, for a general \( L_y \), the length of the scheme-theoretic intersection \( L_y \cap X \) is at least \( n + 2 \). Then we have

\[
\dim(\bigcup_{y \in Y} L_y) \leq n + 1
\]

**Proof.** See [Ran2]. □

The importance of the result is that it is true in all dimensions. In the nice range it is an immediate corollary of Theorem 1.4.

In this note, where we only treat the case \( n \leq 4 \), we do not use the full strength of Theorem 1.4. Indeed if \( n \leq 5 \), all fibers of \( \pi_\Lambda : X^n \to \overline{X} \subset \mathbb{P}^{n+1} \) as above are curvilinear, by an easy dimension count. This implies that the result we need follows from [PR], [MP] or Theorem 1.5 above. Note that “curvilinear” simply means that the finite algebra (the algebra \( Q(f_S) \) associated to the multigerm, defined in [Ma2], pg.188) appearing in the fiber of the projection has rank at most 1, in the language of [Ma2]. In other words, it is a product of rings of the form \( \mathbb{C}[x]/(x^k) \). Some authors ([MM], [MP]) say that \( \pi_\Lambda \) is of corank 1.

§2. A Construction of Lazarsfeld [L].

We first establish some notation that we will use throughout the rest of this note.

Let \( X \) be a smooth non–degenerate complex projective variety of degree \( d \) and of dimension \( n \) in \( \mathbb{P}^r \). Let \( \Lambda = \mathbb{P}^{r-n-2} \) be a general linear space of dimension \( (r-n-2) \), disjoint from \( X \), and let \( p_1 : Bl_\Lambda \mathbb{P}^r \to \mathbb{P}^r \) be the blow–up of \( \mathbb{P}^r \) along \( \Lambda \). Let \( p_2 : Bl_\Lambda \mathbb{P}^r \to \mathbb{P}^{n+1} \) be the projection associated to \( \Lambda \). We may realize the projection by choosing any linear subspace \( \Psi \subset \mathbb{P}^r \) of dimension \( n + 1 \) not intersecting \( \Lambda \). In the intrinsic realization of a projection, a point in the projected
space corresponds to a linear subspace $H$ of $\mathbb{P}^r$, of dimension $r - n - 1$, containing $\Lambda$. After choosing a $\Psi$, we can parametrize the projection by the (unique) point of intersection $q = \Psi \cap H$. Going the other way, we can reconstruct $H$ from $q \in \Psi$ by taking $H = \langle \Lambda, q \rangle$. Obviously the projection does not depend on the choice of $\Psi$. In the next section we will vary $\Psi$ without changing $\Lambda$, in order to ensure that $\Psi$ does not vanish at a given finite set of points. This is obviously always possible, and amounts to a change of coordinates. We choose homogeneous coordinates $T_0, T_1, \ldots, T_r$ on $\mathbb{P}^r$ so that $\Lambda = Z(T_0, T_1, \ldots, T_{n+1})$, the locus where $T_0, T_1, \ldots, T_{n+1}$ all vanish, and $\Psi = Z(T_{n+2}, \ldots, T_r)$. Setting $V = \mathbb{C} \cdot T_{n+2} \oplus \mathbb{C} \cdot T_{n+3} \oplus \cdots \oplus \mathbb{C} \cdot T_r$, and $W = \mathbb{C} \cdot T_0 \oplus \mathbb{C} \cdot T_1 \oplus \cdots \oplus \mathbb{C} \cdot T_{n+1}$, we also write $\Lambda = \mathbb{P}(V)$ and $\Psi = \mathbb{P}(W)$. Note that $\text{Bl}_\Lambda \mathbb{P}^r = \mathbb{P}(\mathcal{O}_\Psi(1) \oplus (V \otimes \mathcal{O}_\Psi)) = \{ (x, q) : x \in L_q = \langle \Lambda, q \rangle, q \in \Psi \}$. Consider the diagram

$$
\begin{align*}
\text{Bl}_\Lambda \mathbb{P}^r &= \mathbb{P}(\mathcal{O}_\Psi(1) \oplus (V \otimes \mathcal{O}_\Psi)) \\
&\xrightarrow{p_2} \Psi = \mathbb{P}^{n+1} \\
&\xrightarrow{p_1} X \subset \mathbb{P}^r
\end{align*}
$$

(2.1)

Let $\pi_\Lambda$ be the restriction of $p_2 \circ p_1^{-1}$ to $X$. By the choice of $\Lambda$, $\pi_\Lambda$ is a (finite) morphism.

From the exact sequence $0 \to \mathcal{I}_{X/\mathbb{P}^r}(k) \to \mathcal{O}_{\mathbb{P}^r}(k) \to \mathcal{O}_X(k) \to 0$ for each $k \in \mathbb{Z}$, we have the following sequence

$$0 \to p_2^*(p_1^*\mathcal{I}_{X/\mathbb{P}^r}(k)) \to p_2^*(p_1^*\mathcal{O}_{\mathbb{P}^r}(k)) \xrightarrow{\omega_{n,k}} p_2^*(p_1^*\mathcal{O}_X(k)) \to R^1p_2^*(p_1^*\mathcal{I}_{X/\mathbb{P}^r}(k)) \to 0$$

because $R^1p_2^*(p_1^*\mathcal{O}_{\mathbb{P}^r}(k)) = R^1p_2^*(\mathcal{O}_{\mathbb{P}(E)}(k)) = 0$, where $E = \mathcal{O}_\Psi(1) \oplus (V \otimes \mathcal{O}_\Psi)$. Note that $p_2^*(p_1^*\mathcal{O}_X(k)) = \pi_\Lambda^*(\mathcal{O}_X(k)$ and $p_1^*\mathcal{O}_{\mathbb{P}^r}(1) = \mathcal{O}_{\mathbb{P}(E)}(1)$ which is the tautological line bundle.

The main issue is to prove the surjectivity of $\omega_{n,k}$ for some $n$ and $k$, where $\omega_{n,k}$ is the map in the above exact sequence. By Nakayama’s lemma, it suffices to show that for all $q \in \Psi$,

$$
\begin{align*}
p_2^*(p_1^*\mathcal{O}_{\mathbb{P}^r}(k)) \otimes \mathbb{C}(q) &\xrightarrow{\omega_{n,k} \otimes \mathbb{C}(q)} p_2^*(p_1^*\mathcal{O}_X(k)) \otimes \mathbb{C}(q) \\
&\cong H^0(L_q, \mathcal{O}_{L_q}(k)) \xrightarrow{\cong} H^0(L_q, \mathcal{O}_{\pi_\Lambda^{-1}(q)}(k))
\end{align*}
$$

is surjective. Equivalently, using the bottom horizontal arrow, it is enough to show that the finite scheme $\pi_\Lambda^{-1}(q)$ in $L_q = \langle \Lambda, q \rangle$ is $k$–normal for all $q \in \Psi$. Therefore, we need some information on the finite schemes $\pi_\Lambda^{-1}(q)$ appearing in the fiber of
a generic projection $\pi_X : X \to \Psi$. Note that a finite scheme of length $k$ is $(k-1)$-normal and it fails to be $(k-2)$-normal if and only if it is contained in a line. When the fiber is reduced, to prove surjectivity in degree $k$ it is enough, for each point $p$ in the fiber, to find a polynomial of degree $k$ vanishing at all the points in the fiber other than $p$, and not vanishing at $p$.

Suppose that $\omega_{n,k}$ is surjective for some $k > 0$. Recall that

\[ p_2^*(p_1^*O_{\mathbb{P}^r}(k)) = \text{Sym}^k(\mathcal{O}_\Psi(1) \oplus (V \otimes \mathcal{O}_\Psi)) \]

\[ = \mathcal{O}_\Psi(k) \oplus (V \otimes \mathcal{O}_\Psi(k-1)) \oplus (S^2(V) \otimes \mathcal{O}_\Psi(k-2)) \oplus \cdots \oplus (S^k(V) \otimes \mathcal{O}_\Psi), \]

where $S^i(V)$ is the $i$-th symmetric power of $V$. After twisting by $(-k)$, we get the exact sequence

\[ 0 \to E_{n,k} \to (S^k(V) \otimes \mathcal{O}_\Psi(-k)) \oplus \cdots \oplus (V \otimes \mathcal{O}_\Psi(-1)) \oplus \mathcal{O}_\Psi \xrightarrow{\tilde{\omega}_{n,k}} \pi_{\Lambda_*} \mathcal{O}_X \to 0, \]

where $E_{n,k} = \text{Ker}(\tilde{\omega}_{n,k})$. Since $\pi_{\Lambda_*} \mathcal{O}_X$ is a locally Cohen-Macaulay module of codimension $1$ for $\mathcal{O}_\Psi$, $E_{n,k}$ is locally free.

Now, following Greenberg ([G]), we generalize this construction. For each $j \geq 1$ we choose a subspace $V_j$ of $S^j(V)$ such that the map $\tilde{\omega}_{n,k,j}$ obtained by restriction,

\[ (2.3) \quad (V_k \otimes \mathcal{O}_\Psi(-k)) \oplus \cdots \oplus (V_1 \otimes \mathcal{O}_\Psi(-1)) \oplus \mathcal{O}_\Psi \xrightarrow{\tilde{\omega}_{n,k,j}} \pi_{\Lambda_*} \mathcal{O}_X \]

is still surjective. We write $\mathcal{V}$ for the collection of $V_j$. We will always take $V_1 = V$ and $V_2 = S^2(V)$. As before, the kernel $E_{n,k,j}$ of $\tilde{\omega}_{n,k,j}$ is locally free.

**Lemma 2.1.** If $\tilde{\omega}_{n,k,j}$ is surjective, then:

1. If $H^1(E_{n,k,j}(m)) = 0$, then $X$ is $m$-normal.
2. $\text{reg}(X) \leq \text{reg}(E_{n,k,j}) = \text{reg}(\Lambda_{\text{rank}(E_{n,k,j})-1}(E_{n,k,j}^* \otimes \text{det}E_{n,k,j}))$

   \[ \leq (\text{rank}(E_{n,k,j}) - 1)\text{reg}(E_{n,k,j}) - c_1(E_{n,k,j}). \]

**Proof.** (1) is a straightforward extension of the proof of lemma 1.5 of [L]. For (2), first check that $\text{reg}(X) \leq \text{reg}(E_{n,k,j})$. Suppose that $E_{n,k,j}$ is $(m + 1)$-regular, in other words, $h^i(\Psi, E_{n,k,j}(m + 1 - i)) = 0$, $i > 0$. We can show that $X$ is $(m + 1)$-regular, equivalently, $X$ is $m$-normal and $h^i(\mathbb{P}^r, \mathcal{O}_X(m - i)) = 0$, $i > 0$. From the exact sequence induced from (2.3),

\[ 0 \to E_{n,k,j} \to V_k \otimes \mathcal{O}_\Psi(-k) \oplus \cdots \oplus V_1 \otimes \mathcal{O}_\Psi(-1) \oplus \mathcal{O}_\Psi \xrightarrow{\tilde{\omega}_{n,k,j}} \pi_{\Lambda_*} \mathcal{O}_X \to 0 \]

it is easy to check that $H^i(\Psi, E_{n,k,j}(m + 1 - i)) = 0$, $i > 0$ implies $H^i(\mathbb{P}^r, \mathcal{O}_X(m - i)) = 0$ for $i > 0$. Therefore, $X$ is $(m + 1)$-regular and $\text{reg}(X) \leq \text{reg}(E_{n,k,j})$. The second inequality follows from the isomorphism $E_{n,k,j} \simeq \Lambda_{\text{rank}(E_{n,k,j})-1}(E_{n,k,j}^* \otimes \text{det}E_{n,k,j})$ and Proposition 1.3 (a). □
Lemma 2.2. \( \text{reg} (E_{n,k,V}^*) \leq (-2) \)

Proof. See [L], Lemma 2.1, where \( A^* \) is \((-1)\)-regular and \( B^* \) is \((-2)\)-regular. From the sequence \( 0 \to A^* \to B^* \to E_{n,k,V}^* \to 0 \) in the same lemma of [L] and Proposition 1.3 (b), \( E_{n,k,V}^* \) is \((-2)\)-regular. \( \square \)

Proposition 2.3 (Greenberg). Assume \( \tilde{\omega}_{n,k,V} \) is surjective. Then

\[
\text{reg}(X) \leq \deg(X) - \text{codim}(X) + 1 + \sum_{j=3}^{k} (j-2) \dim V_j
\]

Proof. This is an easy computation obtained from Lemma 2.1,(2). \( \square \)

The goal is therefore to get the \( V_j \) as small as possible. Curiously, the quadratic polynomials \( V_2 \) do not contribute to the final result. To obtain the regularity conjecture one must have \( V_j = 0 \) for all \( j \geq 3 \). This is what happens for smooth curves and surfaces, see [L] pg. 425.

§3. Castelnuovo regularity for smooth subvarieties of dimensions 3, 4.

We first deal with the case \( n = 3 \). By Mather’s Theorem 1.4, we see that outside of the finite set of points \( \overline{X}_4 \) (using the notation of that theorem) the fibers of the general projection have length at most three, and so can be separated by quadratic polynomials. Above the points \( \overline{X}_4 \), the fiber consists of 4 reduced points and this too can be separated by polynomials of degree 2 unless the four points are aligned. In the next theorem, we show that this can be handled by a subspace \( V_3 \) of dimension 1.

Theorem 3.1. Let \( X \) be a smooth 3-fold of degree \( d \) in \( \mathbb{P}^r \). Then we have \( \text{reg}(X) \leq \deg(X) - \text{codim}(X) + 2 \)

Proof. We use the notation of §2, simply setting \( n = 3 \).

By Theorem 1.4, the finite scheme \( \pi^{-1}_{\Lambda}(q) \) in \( L_q = \langle \Lambda, q \rangle \) has length at most 4 for all \( q \in \Psi \). As in Theorem 1.4, let \( \overline{X}_k = \{ q \in \overline{X} \mid \text{the length of } \pi^{-1}_{\Lambda}(q) \geq k \} \), and let \( Y_4 \) be the subset of \( \overline{X}_4 \) above which the fiber consists of 4 distinct, collinear points. Then the commutative diagram (2.2) is surjective for all \( q \notin Y_4 \) and \( k = 2 \). In other words, every fiber \( \pi^{-1}_{\Lambda}(q), q \notin Y_4 \) is 2-normal in \( L_q = \langle \Lambda, q \rangle \). Therefore, the morphism \( \omega_{3,2} : O_{\Psi}(2) \oplus (V \otimes O_{\Psi}(1)) \oplus (S^2(V) \otimes O_{\Psi}) \to \pi_{\Lambda} \otimes O_X(2) \) is surjective for all \( y \notin Y_4 \). Thus the morphism

\[
\omega_{3,2} \otimes O_{\Psi}(1) : O_{\Psi}(3) \oplus (V \otimes O_{\Psi}(2)) \oplus (S^2(V) \otimes O_{\Psi}(1)) \to \pi_{\Lambda} \otimes O_X(3)
\]

is also surjective for all \( y \notin Y_4 \). Recall that \( \overline{X}_4 \), and therefore its subset \( Y_4 \) is (at most) a finite set by Theorem 1.4. Write \( Y_4 = \{ q_1, \ldots, q_t \} \), so the points
\( \pi^{-1}_A(q_i) \) are collinear. Denote by \( \ell_{q_i} \) the line supporting the fiber above \( q_i \), i.e., \( \pi^{-1}_A(q_i) \subset \ell_{q_i} \subset L_{q_i} \) for all \( q_i \in Y_4 \). Let \( p_i \) be the point of intersection of the line \( \ell_{q_i} \) with \( \Lambda \). Now, choose a linear form \( H(T_5, \ldots, T_r) \) on \( \Lambda \) that does not vanish on \( \{ p_1, p_2, \ldots, p_t \} \). Viewing, in the obvious way, \( H \) as a form on \( \mathbb{P}^r \), we can restrict \( H \) to \( \ell_{q_i} \) and by construction, \( H \) vanishes only at the point \( q_i \). Let \( U(T_0, \ldots, T_4) \) be a linear form on \( \Psi \) that does not vanish on \( \{ q_1, q_2, \ldots, q_t \} \). Then \( U \) restricted to the line \( \ell_{q_i} \) only vanishes at \( p_i \). Clearly \( \{ H, U \} \), when restricted to \( \ell_{q_i} \), is a homogeneous coordinate system for all \( i = 1, 2, \ldots, t \), and \( \pi^{-1}_A(q_i) \subset \ell_{q_i} \), for all \( q_i \in Y_4 \), can be separated by cubic polynomials in \( \{ H, U \} \). Therefore, the natural morphism

\[
[\mathcal{O}_X(3) \oplus (V \otimes \mathcal{O}_X(2)) \oplus (S^2(V) \otimes \mathcal{O}_X(1)) \oplus (H^3 \otimes \mathcal{O}_X)] \otimes \mathcal{C}(q) \rightarrow \pi_* \mathcal{O}_X(3) \otimes \mathcal{C}(q)
\]

is surjective for all \( q \in Y_4 \) because the left-side of the morphism generates all cubic polynomials in the variables \( \{ H, U \} \) on the line. In other words, taking \( V_3 \) to be the one-dimensional space generated by \( H^3, \omega_{3,3,1} \) is surjective so by Proposition 2.3, the Theorem is proved. \( \square \)

We now turn to the case of dimension 4.

**Theorem 3.2.** Let \( X \) be a smooth, non-degenerate subvariety of degree \( d \) and of dimension 4 in \( \mathbb{P}^r \). Then we have reg\((X) \leq d - \text{codim} X + 5 \)

We will show that we can take \( U_3 \) of dimension 2, \( U_4 \) of dimension 1, and all higher \( U \) of dimension 0. The codimension 2 case is dealt with in [Kw], where we get reg\((X) \leq d – 1 \) for smooth threefold or fourfold \( X \) and will not be discussed here, so we assume \( n \geq 7 \).

**Proof.** As in §2, take a generic projection \( \pi_A : X^4 \rightarrow \mathcal{X} \subset \Psi = \mathbb{P}^5 \).

Let \( \mathcal{X}_k \) and \( \mathcal{X}_k \) have the same meaning as in Theorem 1.4, and let \( Y_4 \) be the subset of \( \mathcal{X}_4 \) where the fiber contains 4 distinct collinear points. Let \( Y_5 = Y_4 \cap \mathcal{X}_5 \). By Theorem 1.4, \( \dim Y_k \leq 5 - k \). So, \( Y_4 \) is (at most) a 1-dimensional subvariety in \( \pi_A(X) \subset \Psi \) and \( Y_5 = \{ q_1, \ldots, q_t \} \) is a finite set of \( t \) points. For each point \( q \in Y_4 \), there is a unique line \( \ell_q \) which is the support of the collinear points in the fiber above \( q \). As \( q \) varies in \( Y_4 \), the lines \( \ell_q \) intersect \( \Lambda \) in a variety \( Z \) which is at most 1-dimensional. As \( q_i \) varies in \( Y_5 \), the lines \( \ell_{q_i} \) intersect \( \Lambda \) in points \( p_i \). Choose linear forms \( H_1, H_2 \) on \( \Lambda \) such that \( H_1 \) is nonzero at \( \{ p_1, p_2, \ldots, p_t \} \) and \( H_2 \) is nonzero at \( H_1 \cap Z = \{ t_1, \ldots, t_m \} \subset \Lambda \). The morphism in the diagram (2.2) is surjective for all \( q \notin Y_4 \). Therefore, the morphism \( \omega_{4,2} : \mathcal{O}_X(2) \oplus V \otimes \mathcal{O}_X(1) \oplus S^2(V) \otimes \mathcal{O}_X \rightarrow \pi_* \mathcal{O}_X(2) \) is surjective for all \( q \notin Y_4 \). Note then that the morphism

\[
\omega_{4,2} \otimes \mathcal{O}_X(1) : \mathcal{O}_X(3) \oplus (V \otimes \mathcal{O}_X(2)) \oplus (S^2(V) \otimes \mathcal{O}_X(1)) \rightarrow \pi_* \mathcal{O}_X(3)
\]

is also surjective for all \( q \notin Y_4 \). We first treat the fibers above \( Y_4 \setminus Y_5 \). By construction, one of \( H_1 \) and \( H_2 \) acts as the second homogeneous coordinate on the
support of the fiber, which is a line. So if we let $V_3$ be the 2–dimensional subspace of $S^3(V)$ generated by $H_1^3$ and $H_2^3$, the corresponding map $\omega_{4,3,U}$ is surjective everywhere except possibly above $Y_5$.

Above $Y_5$, we have two kinds of fibers:

(1) 5 reduced collinear points (this is 4–normal)
(2) 4 reduced collinear points plus one extra point off the line in $L_q$.

We note that there is no fiber which contains four aligned points, and one of the points in the fiber is nonreduced and has multiplicity 2. This case is ruled out by J.Mather’s inequality that we quote on page 4 (as in the case of surfaces).

In the first case, it is necessary to include the element $H_4^1$ in $V_4$ to obtain surjectivity of the corresponding $\omega_{4,4,U}$. Note that by construction $H_1$ only vanishes at the point $q_i$ on the line $l_{q_i}$. The argument is the same as in dim($X$) = 3.

The second case is more delicate. To prove the Theorem, we must avoid adding any extra global sections to $V_3$ or $V_4$. Since we can work one fiber at a time, we drop the $i$ index. The fiber above $q \in Y_5 \subset \Psi$ spans a 2-dimensional linear space in $L_q$ which meets $\Lambda$ in a line $N_q$ containing the point $p \in \Lambda$. Since we have only a finite number of $N_q$, we may of course assume that neither $H_1$ nor $H_2$ vanishes identically on $N_q$. We may also assume that $q \notin X \cap \Psi$, by moving $\Psi$ as explained in §2, since $Y_5$ is a finite set. Thus the fiber $\mathbb{P}^2$ has homogeneous coordinates $(U, H_1, H_2)$, where the line $U = 0$ is the intersection of the fiber with $\Lambda$. Thus we can assume that the four aligned points have coordinates $(u_i, a, b)$ and the fifth point has coordinates $(u, c, d)$. By construction, none of the $u_i$ or $u$ are equal to 0. We may choose the $H_j$ so that the same is true for $a, b, c, d$. We also know that none of the points is the point $q = (1, 0, 0)$.

We need to prove the following easy lemma, where for simplicity of notation we set $H_1 = X$ and $H_2 = Y$.

**Lemma 3.3.** Let $U, X, Y$ be homogeneous coordinates on $\mathbb{P}^2$, and suppose given 5 points, $p_i = (u_i, a, b)$, $1 \leq i \leq 4$ and $p_5 = (u, c, d)$. Assume that none of the $u_i, u, a, b, c, d$ are 0. Then the points can be separated using only cubics whose equation only contain the monomials

$$U^3, U^2X, U^2Y, UX^2, UXY, UY^2, X^3, Y^3,$$

**Proof.** This is easy to check by elementary linear algebra. By symmetry it is enough to construct a cubic that vanishes at $p_1, p_2, p_3,$ and $p_5$. (Note that it is trivial to construct a linear form vanishing at the four aligned points, but not at $p_5$.) Consider the cubic polynomial

$$U^3+(a_{1,0}U^2X+a_{0,1}U^2Y)+(a_{2,0}UX^2+a_{1,1}UXY+a_{0,2}UY^2)+(a_{3,0}X^3+a_{0,3}Y^3) = 0$$
First, note that this cubic doesn’t vanish identically on the line $\ell = \{aY - bX = 0\}$ containing the 4-collinear points because it doesn’t vanish at the point $(1, 0, 0)$ which is on $\ell$. Evaluating $X = a$, $Y = b$, we must have the cubic polynomial in $U$:

$$U^3 + (a_{1,0}a + a_{0,1}b)U^2 + (a_{2,0}a^2 + a_{1,1}ab + a_{0,2}b^2)U^2 + (a_{3,0}a^3 + a_{0,3}b^3)$$

This must be equal to $(U - u_1)(U - u_2)(U - u_3)$ if the polynomial is to vanish at $p_1$, $p_2$, $p_3$. For this to happen, equating coefficients, we must solve a system of three linear equations in the seven unknowns $a_{1,0}, a_{0,1}, a_{2,0}, a_{1,1}, a_{0,2}, a_{3,0}, a_{0,3}$. This yields a 4-dimensional family of solutions. Finally we must force the solution to pass through $p_5$. This is one extra linear condition. So we have found a cubic (in fact a 3-dimensional family) passing through $p_1$, $p_2$, $p_3$ and $p_5$. Since it does not contain the line $\ell$, and since we have all its intersections with $\ell$, it does not contain $p_4$. Therefore, the point $p_4$ can be separated from the others. □

We have shown that taking $V_3 = \langle H_1^3, H_2^3 \rangle$ of dimension 2, $V_4 = \langle H_2^3 \rangle$ of dimension 1, $\tilde{\omega}_{4,4,Y}$ is surjective, and therefore, by Proposition 2.3, the theorem is proved. □

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