Hodge theory of holomorphic vector bundle on compact Kähler hyperbolic manifold

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Abstract

Let $E$ be a holomorphic vector bundle over a compact Kähler manifold $(X, \omega)$ with negative sectional curvature $\text{sec} \leq -K < 0$, $D_E$ be the Chern connection on $E$. In this article we show that if $C := |\Lambda, i\Theta(E)| \leq c_n K$, then $(X, E)$ satisfy a family of Chern number inequalities. The main idea in our proof is study the $L^2\bar{\partial}\tilde{E}$-harmonic forms on lifting bundle $\tilde{E}$ over the universal covering space $\tilde{X}$. We also observe that there is a closely relationship between the eigenvalue of the Laplace-Beltrami operator $\Delta_{\bar{\partial}\tilde{E}}$ and the Euler characteristic of $X$. Precisely, if there is a line bundle $L$ on $X$ such that $\chi_p(X, L^{\otimes m})$ is not constant for some integers $p \in [0, n]$, then the Euler characteristic of $X$ satisfies $(-1)^n \chi(X) \geq (n + 1) + \lfloor \frac{c_n K}{2n} \rfloor$.

Keywords. Hodge theory; Holomorphic vector bundle; Kähler hyperbolic; Chern number inequality

1 Introduction

Let us start the article by recalling a Hopf conjecture related to the negativity of Riemannian sectional curvature.

Conjecture 1.1. The Euler characteristic $\chi(X)$ of a compact $2n$-dimensional Riemannian manifold $X$ with sectional curvature $K < 0$ (resp. $K \leq 0$) satisfies $(-1)^n \chi(X) > 0$ (resp. $(-1)^n \chi(X) \geq 0$).

This is true for $n = 1$ and 2 as the Gauss–Bonnet integrands in these two low dimensional cases have the desired sign [8]. However, in higher dimensions, it is known that the sign of the sectional curvature does not determine the sign of the Gauss-Bonnet-Chern integrand [13]. The conjecture is still open in its full generality for $n \geq 3$. Therefore, Dodziuk [11] and Singer [33] suggested to use $L^2$-cohomology to approach this problem as follows: Show $\mathcal{H}_{(2)}^k (X) = \{0\}$ for $k \neq n$ which implies the $L^2$-Betti number $b_{(2)}^k (X) = 0$ for $k \neq n$ and $\mathcal{H}_{(2)}^n (X) \neq \{0\}$ which implies $b_{(2)}^n (X) \neq 0$. However, Anderson [1] constructed simply connected complete negatively curved Riemannian manifolds on which this does not
hold, thus indicating a certainly difficulty with this approach. The program outlined above was carried out by Gromov [15] when the manifold in question is Kähler and is homotopy equivalent to a closed manifold with strictly negative sectional curvatures. The main theorem in [15] states that for a Kähler hyperbolic manifold $X$, $\mathcal{H}^{p,q}_{(2)}(\tilde{X}) = \{0\}$ if and only if $p + q \neq \dim_{\mathbb{C}} X$, where $\mathcal{H}^{p,q}_{(2)}(\tilde{X})$ denotes the space of $L^2$-harmonic forms of type $(p, q)$ on $\tilde{X}$. The vanishing of $\mathcal{H}^{p,q}_{(2)}(\tilde{X})$ for $p + q \neq \dim_{\mathbb{C}} X$ is a consequence of the strong $L^2$-Lefschetz theorem. Nonvanishing for $p + q = \dim_{\mathbb{C}} X$ follows from the $L^2$-index theorem and an upper bound for the bottom of the spectrum, whose proof is based on a twisting (by tensoring $\bar{\partial} + \partial^*$ with a line bundle equipped with a connection) trick due to Vafa and Witten. By extending Gromov’s idea and notion above to the nonnegative version, Jost-Zuo [20] and Cao-Xavier [5] independently introduced the concept of Kähler parabolic and consequently settled Conjecture 1.1 in the case of $K \leq 0$ for Kähler manifolds. The study of the $L^2$-harmonic forms on a complete Riemannian manifold is a very fascinating and important subject. It also has numerous applications in the field of Mathematical Physics (see [16]). Other results on $L^2$ cohomology can be found in [2 7 12 27 28].

In this article, we consider the Hodge theory on a Hermitian vector bundle $E$ over a complete, Kähler manifold $X$, $\dim_{\mathbb{C}} X = n$, with a Kähler form $\omega$. Define a smooth Kähler metric, $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ on $X$, where $J$ is the complex structure on $X$. Let $d_A$ be a Hermitian connection on $E$. The formal adjoint operator of $d_A$ acting on $\Omega^k(X, E) := \Omega^k(X) \otimes E$ is $d_A^* = - \ast d_A \ast$, where $\Omega^k(X)$ is smooth $k$-forms on $X$ and $\ast$ is the Hodge star operator with respect to the metric $g$. We denote by $\mathcal{H}^k_{(2)}(X, E)$ the space of $L^2$ harmonic forms in $\Omega^k(X)$ with respect to the Laplace-Beltrami operator $\Delta_A := d_A d_A^* + d_A^* d_A$.

A differential form $\alpha$ on a Riemannian manifold $(X,g)$ is bounded with respect to the Riemannian metric $g$ if the $L^\infty$-norm of $\alpha$ is finite,

$$\|\alpha\|_{L^\infty(X)} := \sup_{\alpha \in X} \|\alpha(x)\|_g < \infty.$$ 

We say that $\alpha$ is $d$(bounded) if $\alpha$ is the exterior differential of a bounded form $\beta$, i.e., $\alpha = d\beta$ and $\|\beta\|_{L^\infty(X)} < \infty$.

If $\omega$ is $d$(bounded), the author in [18] extended the vanishing theorem of Gromov’s to holomorphic vector bundle case. We denote by $A^{1,1}_E$ the space of all integrable connections $d_A$, i.e., $F^{2,0}_A = F^{0,2}_A = 0$. The important observation is that if the Hermitian connection $d_A \in A^{1,1}_E$, then the operator $L^k$ could commute with $\Delta_A$ for any $k \in \mathbb{N}^+$. Following the idea in [15], the author proved a vanishing theorem on the spaces $\mathcal{H}^k_{(2)}(X, E)$. Suppose that $E$ is a holomorphic Hermitian vector bundle on $X$. We denote by $D_E = \partial_E + \bar{\partial}_E$ its Chern connection, i.e., $\bar{\partial}_E = \bar{\partial}$, by $D_E^*$ the formal adjoint of $D_E$ and by $\partial_E^*, \bar{\partial}_E^*$ the components of $D_E^*$ of type $(-1, 0)$ and $(0, -1)$. Let $\Theta(E) = \partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E$ be the curvature operator on $E$. It is clear that $\bar{\partial}^2 = 0$. Therefore, for any integer $p = 0, 1, \ldots, n$, we get a complex

$$\Omega^{p,0}(X, E) \overset{\partial}{\to} \ldots \overset{\partial}{\to} \Omega^{p,q}(X, E) \overset{\partial}{\to} \Omega^{p,q+1}(X, E) \to \ldots,$$

known as the Dolbeault complex of $(p, \bullet)$-forms with values in $E$. We can define two operators:

$$\Delta_{\bar{\partial}_E} := \bar{\partial}_E \partial^*_E + \partial^*_E \bar{\partial}_E, \quad \Delta_{\partial_E} := \partial_E \bar{\partial}^*_E + \bar{\partial}^*_E \partial_E.$$
Let us introduce, See [10, Charp V]

\[ H^{p,q}_{(2);\partial E}(X, E) := \{ \alpha \in \Omega^{p,q}_{(2)}(X, E) : \Delta\partial E\alpha = 0 \} . \]

There are many vanishing theorems for Hermitian vector bundles over a compact complex manifolds. All these theorems are based on a priori inequality for \((p, q)\)-forms with values in a vector bundle, known as the Bochner-Kodaira-Nakano inequality. This inequality naturally leads to several positivity notions for the curvature of a vector bundle ([14, 21, 22, 23, 29, 30]).

The first purpose of this paper is to study the Hodge theory of the holomorphic bundle \( E \) on the compact Kähler manifold \( X \) with negative sectional curvature. At first, we denote by

\[ C = \max_{p,q} |C_{p,q}| := ||[\Lambda, i\Theta(E)]|| \]

the operator norm of \([\Lambda, i\Theta(E)]\), where

\[ C_{p,q} := \sup_{\alpha \in \Omega^{p,q}(X,E) \setminus \{0\}} \frac{|<[\Lambda, i\Theta(E)]\alpha, \alpha>_{L^2(X)}|}{\|\alpha\|^2}. \]

We then have

**Theorem 1.2** (=Proposition 3.16 and Theorem 3.17). Let \((X, \omega)\) be a compact Kähler manifold with sectional curvature bounded from above by a negative constant, i.e.,

\[ \text{sec} \leq -K, \]

for some \( K > 0 \). Let \( E \) be a holomorphic vector bundle on \( X \), \( D_E \) be the Chern connection on \( E \). If the curvature \( \Theta(E) \) of \( D_E \) such that

\[ C := ||[\Lambda, i\Theta(E)]|| \leq c(n)K, \]

where \( c_n \) is a positive constant depends only \( n \), then for every \( p = 0, 1, \cdots, n \), the Euler characteristic

\[ \chi^p(X, E) := \int_X td(X)ch(\Omega^{p,0}(TX) \otimes E) \]

does not vanish and

\[ \text{sign} \chi^p(X, E) = (-1)^{n-p}. \]

Furthermore, for all \( 0 \leq j \leq n \), \((X, E)\) satisfy Chern number inequalities

\[ (-1)^{n+j}K_j(X, E) \geq \sum_{p=j}^{n} \binom{p}{j}. \]

**Remark 1.3.** The Chern number inequalities are always not sharp. For example, suppose that the curvature \( \Theta(E) \) of the Chern connection \( D_E \) is small enough in the sense of \( L^\infty \)-norm. Then there exists a flat connection on \( \Gamma \) on \( E \) (see [34]). Hence following Proposition 3.7 we have

\[ \chi_y(X, E) = \text{rank}(E)\chi_y(X), \]
i.e., for every \( p = 0, 1, \cdots, n \), the Euler characteristic satisfies
\[
\chi^p(X, E) = \text{rank}(E)\chi^p(X).
\]

Therefore, \( X, E \) satisfy Chern number inequalities
\[
(-1)^{n+j}K_j(X, E) = \text{rank}(E) \sum_{p=j}^{n} \binom{n}{j} h^{p,n-p}_j(X, E) \geq \text{rank}(E)(-1)^j K_j(\mathbb{CP}^n). \tag{1.1}
\]

All the equality cases in (1.1) hold if and only if \( \chi^p(X) = (-1)^{n-p}, 0 \leq p \leq n \). (see [24, Theorem 2.1]).

In [15], Gromov shown that for every \( p = 0, 1, \cdots, n \), the Euler characteristic of a compact Kähler hyperbolic manifold satisfies
\[
\text{sign}\chi^p(X) = (-1)^{n-p},
\]
as a consequence \((-1)^n\chi(X) \geq n+1\). Let \( L \) be a holomorphic line bundle on a compact Kähler manifold \( X \). We call
\[
P^{(p)}_n(m, L) := \chi^p(X, L^\otimes m)
\]
the \( p \)-Hilbert polynomial of line bundle \( L \). The second propose of this article is to show that the lower bound of the Euler characteristic \((-1)^n\chi(X)\) estimated by \( K \).

**Theorem 1.4.** Let \((X, \omega)\) be a compact Kähler manifold with sectional curvature bounded from above by a negative constant, i.e.,
\[
\text{sec} \leq -K,
\]
for some \( K > 0 \). Let \( L \) be a holomorphic line bundle on \( X \). Suppose that the \( p \)-Hilbert polynomial \( \chi^p(X, L^\otimes m) \) is not constant for some \( p \in [0, n] \). Then there exists a integer \( \tilde{m} = \tilde{m}(p) \in [-\frac{\omega(K, L)}{nC}, \frac{\omega(K, L)}{nC}] \) such that either
\[
(-1)^{n-p}\chi^p(X) \geq \left\lfloor \frac{c_nK}{nC} \right\rfloor + 1.
\]
or
\[
(-1)^{n-p}\chi^p(X, L^\otimes \tilde{m}) \geq \left\lfloor \frac{c_nK}{nC} \right\rfloor + 1,
\]
where \( C := ||\Lambda, i\Theta(L)|| \). In particular, the Euler characteristic of \( X \) satisfies
\[
(-1)^n\chi(X) \geq (n+1) + \left\lfloor \frac{c_nK}{nC} \right\rfloor.
\]

**Remark 1.5.** The conclusion of the theorem valid for all \( p = 0, \cdots, n \) if the line bundle \( L \) satisfies \( \int_X c_1^p(L) \neq 0 \). Since the canonical bundle \( K_X \) on a compact Kähler hyperbolic manifold is ample (see [6, Theorem 2.11]), then \( X \) is projective, i.e, there is an embedding \( i : X \hookrightarrow \mathbb{P}^N \). We denote by \( \mathcal{O}(1) \) the tautological line bundle on \( \mathbb{P}^N \). The pull back bundle \( i^*\mathcal{O}(1) \) of the line bundle \( \mathcal{O}(1) \) on \( X \) satisfies \( \int_X c_1^0(i^*\mathcal{O}(1)) \neq 0 \).
The lifted Kähler form $\tilde{\omega}$ on the universal covering space $\pi : (\tilde{X}, \tilde{\omega}) \to (X, \omega)$ is $d$-bounded. Set

$$Q(\omega) := \{ \theta \in \Omega^1(\tilde{X}) : \tilde{\omega} = d\theta \}.$$ 

Let $E(\theta) := \inf_{\theta \in Q(\omega)} \| \theta \|_{L^\infty(\tilde{X})}$. The eigenvalues of the Laplace-Beltrami operator $\Delta_{\overline{\partial}_E}$ on $\Omega^{p,q}_2(\tilde{X}, \tilde{E})$ ($p + q \neq n$) have a lower bounded $c_nE(\theta)^{-2} - C$. Then the Euler number of $X$ satisfies

$$( -1)^n \chi(X) \geq (n + 1) + \lfloor \frac{c_nE(\theta)^{-2}}{nC} \rfloor \geq n + \frac{c_nE(\theta)^{-2}}{nC}.$$ 

Hence, we get the following result.

**Corollary 1.6.** Let $(X, \omega)$ be a compact Kähler manifold with sectional curvature bounded from above by a negative constant, i.e.,

$$\text{sec} \leq -K,$$

for some $K > 0$. Suppose that there is a holomorphic line bundle $L$ on $X$ such that the $p$-Hilbert polynomial $\chi^p(X, L \otimes m)$ is not constant for some $p \in [0, n]$. Then

$$\sqrt{nK} - \frac{1}{2} \geq E(\theta) \geq \left[ \frac{c_n}{nC((-1)^n \chi(X) - n)} \right]^{\frac{1}{2}}.$$ 

In the Kähler surfaces case, we can get a stronger result as follows.

**Theorem 1.7.** Let $(X, \omega)$ be a compact Kähler surface with sectional curvature bounded from above by a negative constant, i.e.,

$$\text{sec} \leq -K,$$

for some $K > 0$. Suppose that there is a holomorphic line bundle $L$ on $X$ such that $\int_X c_1^2(L) \neq 0$. Then the Euler characteristic of $X$ satisfies

$$\chi(X) \geq 3 + \left( \int_X c_1^2(L) \right) \cdot \left( \left[ \frac{c_nK}{C} \right] \right)^2.$$ 

where $C := ||[\Lambda, i\Theta(L)]||$.

We denote by

$$Z_p := \{ m \in \mathbb{R} : P_n^{(p)}(m, L) = \chi^p(X) \}$$

the set of real roots of polynomial $P_n^{(p)}(m, L) - \chi^p(X)$. We denote

$$m_p(L) = \max_{m \in Z_p} |m|.$$ 

**Remark 1.8.** Following Corollary 3.19 if then Chern connection of the holomorphic line bundle on compact Kähler surface satisfies

$$C := ||[\Lambda, i\Theta(L)]|| \leq c_nk,$$

then

$$\int_X c_1(X)c_1(L) = 0.$$ 

For any $p = 0, 1, 2$, we then have (see the proof of Theorem 1.7)

$$m_p(L) = 0.$$
On higher dimensions case, we have following results.

**Theorem 1.9.** Let \((X, \omega)\) be a compact Kähler manifold with sectional curvature bounded from above by a negative constant, i.e.,

\[ \text{sec} \leq -K, \]

for some \(K > 0\). Suppose that there is a holomorphic line bundle \(L\) on \(X\) such that \(a_n := \int_X c_1^n(L) \neq 0\). Then the Euler characteristic of \(X\) satisfies

\[ (-1)^n \chi(X) \geq \max\{n + 1, n + 1 + 2|a_n| \text{sign}([c_n K - C m_p(L)])(\lfloor \frac{c_n K - C m_p(L)}{2Cn} \rfloor)^n \}, \]

where \(C := ||[\Lambda, i\Theta(L)]||\). Furthermore, if \(n\) is odd, for any \(p = 0, 1, \ldots, n\), we then have

\[ (-1)^{n-p} \chi^p(X) \geq \max\{1, 1 + 2|a_n| \text{sign}([c_n K - C m_p(L)])(\lfloor \frac{c_n K - C m_p(L)}{2Cn} \rfloor)^n \}. \]

### 2 Preliminaries

Let \(X\) be a smooth Kähler manifold with Kähler form \(\omega\) and \(E\) be a smooth vector bundle over \(X\). We denote by \(\Omega^k(X, E)\) the space of \(C^\infty\) sections of the tensor product vector bundle \(\Omega^k(X) \otimes E\) obtained from \(\Omega^k(X)\) and \(E\), i.e., \(\Omega^k(X, E) := \Gamma(\Omega^k(X) \otimes E)\). We denote by \(\Omega^{p,q}(X, E)\) the space of \(C^\infty\) sections of the bundle \(\Omega^{p,q}(X) \otimes E\). We have a direct sum decomposition

\[ \Omega^k(X, E) = \bigoplus_{p+q=k} \Omega^{p,q}(X, E). \]

For any connection \(d_A\) on \(E\), we have the covariant exterior derivatives

\[ d_A : \Omega^k(X) \otimes E \to \Omega^{k+1}(X) \otimes E. \]

Like the canonical splitting the exterior derivatives \(d = \partial + \bar{\partial}\), \(d_A\) decomposes over \(X\) into

\[ d_A = \partial_A + \bar{\partial}_A. \]

We will need some of the basic apparatus of Hermitian exterior algebra. Denote by \(L\) the operator of exterior multiplication by the Kähler form \(\omega\):

\[ L\alpha = \omega \wedge \alpha, \alpha \in \Omega^{p,q}(X, E), \]

and, as usual, let \(\Lambda\) denote its pointwise adjoint, i.e.,

\[ \langle \Lambda \alpha, \beta \rangle = \langle \alpha, L \beta \rangle. \]

Then it is well known that \(\Lambda = \ast^{-1} \circ L \circ \ast[19]\). A basic fact is
Lemma 2.1. The map $L: \Omega^{p,q}(X, E) \to \Omega^{p+1,q+1}(X, E)$ is injective for $p + q \leq n$.

The proof is then purely algebraic and can be found in standard texts on geometry. An elegant approach is through representations of $sl_2$, see [10] Chap.5, Theorem 3.12 or [10, 19].

We recall some definitions on Hermitian vector bundle [10, Charp V, Section 7]. Let $E$ be a Hermitian vector bundle of rank $r$ over a smooth Riemannian manifold $X$, $\dim \mathbb{R} X = n$. We denote respectively by $(\xi_1, \ldots, \xi_n)$ and $(e_1, \ldots, e_r)$ orthonormal frames on $TX$ and $E$ over an open subset $U \subset X$. The associated inner product of $E$ given by a positive definite Hermitian metric $h_{\lambda\mu}$ with smooth coefficients on $U$, such that $\langle e_\lambda(x), e_\mu(x) \rangle = h_{\lambda\mu}(x)$, $\forall x \in \Omega$.

When $E$ is Hermitian, one can define a natural sesquilinear map

$$\Omega^p(X, E) \times \Omega^q(X, E) \to \Omega^{p+q}(X, \mathbb{C})$$

$$(\alpha, \beta) \mapsto tr(s \wedge t)$$

combining the wedge product of forms with the Hermitian metric on $E$. If $\alpha = \sum \sigma_\lambda \otimes e_\lambda$, $\beta = \sum \tau_\mu \otimes e_\mu$, we let

$$tr(\alpha \wedge \beta) := \sum_{1 \leq \lambda, \mu \leq r} \sigma_\lambda \wedge \tau_\mu \langle e_\lambda, e_\mu \rangle.$$ 

A connection $d_A$ said to be compatible with the Hermitian structure of $E$, or briefly Hermitian, if for every $\alpha \in \Omega^p(X, E)$, $\beta \in \Omega^q(X, E)$ we have

$$dtr(\alpha \wedge \beta) = tr(d_A \alpha \wedge \beta) + (-1)^p tr(\alpha \wedge d_A \beta).$$

The inner product $\langle \cdot, \cdot \rangle$ on $\Omega^*(X, E)$ defined as, See [10] Charp VI, Section 3.1

$$\langle \alpha, \beta \rangle = \ast tr(\alpha \wedge \ast \beta), \quad \alpha, \beta \in \Omega^p(X, E).$$

We denote by $Tr$ the sesquilinear map $Tr: \Omega^p(X, EndE) \times \Omega^q(X, EndE) \to \Omega^{p+q}(X, \mathbb{C})$ induced by the map $tr: \Omega^p(X, E) \times \Omega^q(X, E) \to \Omega^{p+q}(X, \mathbb{C})$.

There are several commutation relations between the basic operators associated to a Kähler manifold $X$, all following more or less directly from the Kähler condition $d\omega = 0$; taken together, these are referred to as the Kähler identities [10, 19].

Proposition 2.2. Let $X$ be a complete Kähler manifold, $E$ a Hermitian vector bundle over $X$ and $d_A$ be a Hermitian connection on $E$. We have the following identities

(i) $[\Lambda, \bar{\partial}_A] = -\sqrt{-1}d_A^*\Lambda$, $[\Lambda, \partial_A] = \sqrt{-1}\partial_A^*\Lambda$.

(ii) $[\partial_A^*, L] = \sqrt{-1}\partial_A$, $[\bar{\partial}_A, L] = -\sqrt{-1}\bar{\partial}_A$.

Since $\omega$ is parallel, the operator $L^k: \Omega^p(X, E) \to \Omega^{p+2k}(X, E)$ defined by $L^k(\alpha) = \alpha \wedge \omega^k$ for all $p$-forms commutes with $d_A$. But the operator $L^k$ does not commute with $d_A^*$ in general, therefore the operator $L^k$ does not commute with $\Delta_A$. 

If $A$ and $B$ are operators on forms, define the (graded) commutator as

$$[A, B] = AB - (-1)^{\deg A \deg B} BA,$$

where $\deg T$ is the integer $d$ for $T: \oplus_{p+q=r} \Omega^{p,q}(X, E) \to \oplus_{p+q=r+d} \Omega^{p,q}(X, E)$. If $C$ is another endomorphism of degree $c$, the following Jacobi identity is easy to check

$$(-1)^{ca}[A, [B, C]] + (-1)^{ab}[B, [C, A]] + (-1)^{bc}[C, [A, B]] = 0.$$

At first, we observe that the operator $L^k$ commutes with $\Delta_A$ for any connection $d_A \in A^{1,1}_E$.

**Lemma 2.3.** ([18 Lemma 3.9])

$$[\Delta_A, L^k] = 2k\sqrt{-1}(F^{2,0}_A - F^{0,2}_A)L^{k-1}, \quad \forall k \in \mathbb{N}.$$

In particular, if the connection $d_A \in A^{1,1}_E$, then $\Delta_A$ commutes with $L^k$ for any $k \in \mathbb{N}$.

**Proof.** The case of $k = 1$: the operators $d_A, d^*_A$ and $L$ satisfy the following Jacobi identity:

$$-[L, [d_A, d^*_A]] + [d^*_A, [L, d_A]] + [d_A, [d^*_A, L]] = 0.$$

Then we have

$$[L, \Delta_A] = [d_A, [d^*_A, L]] = [\partial_A + \bar{\partial}_A, \sqrt{-1}(\partial_A - \bar{\partial}_A)]$$

$$= [\sqrt{-1}\partial_A, \partial_A] - [\sqrt{-1}\bar{\partial}_A, \bar{\partial}_A]$$

$$= 2\sqrt{-1}(F^{2,0}_A - F^{0,2}_A).$$

We suppose that the case of $p = k - 1$ is true, i.e.,

$$[\Delta_A, L^{k-1}] = 2(k-1)\sqrt{-1}(F^{2,0}_A - F^{0,2}_A)L^{k-2}.$$

Thus if $p = k$, we have

$$[\Delta_A, L^k] = [\Delta_A, L]L^{k-1} + L[\Delta_A, L^{k-1}]$$

$$= 2\sqrt{-1}(F^{0,2}_A - F^{2,0}_A)L^{k-1} + 2(k-1)\sqrt{-1}L(F^{2,0}_A - F^{0,2}_A)L^{k-2}$$

$$= 2k\sqrt{-1}(F^{0,2}_A - F^{2,0}_A)L^{k-1}.$$

If $d_A \in A^{1,1}_E$, then $[\Delta_A, L^k] = 0$. \qed

### 3 Harmonic forms on vector bundle $E$

As we derive estimates in this section (and also following sections), there will be many constants which appear. To simplify notation we shall write $a \lesssim b$, for $a \leq \text{const}_n b$, and $a \approx b$, for $b \lesssim a \lesssim b$. 


3.1 Uniform positive lower bounds for the least eigenvalue of $\Delta_{\partial E}$

Let $(X, g)$ be an oriented, smooth, Riemannian manifold, $\dim_{\mathbb{R}} X = n$, and $E$ be a Hermitian vector bundle over $X$. Assume now that $d_A$ is a Hermitian connection on $E$. The formal adjoint operator of $d_A$ acting on $\Omega^p(X, E)$ is $d_A^* = (-1)^{np+1} * d_A*$, where the operator $*: \Omega^p(X, E) \to \Omega^{n-p}(X, E)$ induced by the Hodge-Poincaré-de Rahm operator $*$. Indeed, if $\alpha \in \Omega^p(X, E)$, $\beta \in \Omega^{p+1}(X, E)$ have compact support, we get

$$\int_X \langle d_A \alpha, \beta \rangle = \int_X \langle \alpha, d_A^* \beta \rangle.$$ 

The Laplace-Beltrami operator associated to $d_A$ is the second order operator $\Delta_A = d_A d_A^* + d_A^* d_A$. The space of $L^2$-harmonic forms of degree of $k$ respect to the Laplace-Beltrami operator $\Delta_A$ is defined by

$$\mathcal{H}^k_{(2)}(X, E) = \{ \alpha \in \Omega^k_{(2)}(X, E) : \Delta_A \alpha = 0 \}.$$ 

Define the $\delta$-Laplacian by setting $\Delta_\delta := [\delta, \delta^*]$. For all $(p, q)$, we denote by

$$\mathcal{H}^{p,q}_{(2),\delta}(X, E) := \ker(\Delta_\delta) \cap \Omega^{p,q}_{(2)}(X, E)$$

the space of $L^2$-$\delta$-harmonic forms in bidegree $(p, q)$. We have an useful lemma as follows.

**Lemma 3.1.** ([18 Lemma 3.2]) Let $X$ be a complete Riemannian manifold $X$, $E$ a Hermitian vector bundle over $X$. Then

$$\mathcal{H}^k_{(2)}(X, E) = \ker d_A \cap \ker d_A^* \cap \Omega^k_{(2)}(X, E),$$

$$\mathcal{H}^{p,q}_{(2),\delta}(X, E) = \ker \delta \cap \ker \delta^* \cap \Omega^{p,q}_{(2)}(X, E),$$

where $\delta = \bar{\partial}_A$ or $\partial_A$.

**Theorem 3.2.** Let $(X, \omega)$ be a complete, Kähler manifold, $\dim_{\mathbb{C}} X = n$, with a $d$-(bounded) Kähler form $\omega$, i.e., there is a bounded 1-form $\theta$ such that $\omega = d\theta$, $d_A \in \mathcal{A}^{1,1}_E$ be a smooth Hermitian integrable connection on a Hermitian vector bundle $E$ over $X$. Then

$$\mathcal{H}^k_{(2)}(X, E) = \{ 0 \}, \forall k \neq n.$$ 

**Proof.** Let $k < n$. For every $d_A$-closed $L^2$ $k$-form $\alpha$, the form

$$L^{n-k} \alpha = \omega^{n-k} \wedge \alpha = d_A(\omega^{n-k-1} \wedge \theta \wedge \alpha)$$

is $L^2$. We denote $\beta = \omega^{n-k-1} \wedge \theta \wedge \alpha$. One can see that $\beta$ is $L^2$. In particular, if $\alpha$ is $\Delta_A$-harmonic, then $L^k \alpha = 0$ is also $\Delta_A$-harmonic. Following Proposition [18 Proposition 3.7], we get $L^{n-k} \alpha = 0$. This implies, by Lemma [241] that $\alpha = 0$. The case $k > n$ follows by $E^*$ is a holomorphic vector bundle on $X$ and $\mathcal{H}^k_{(2)}(X, E) \approx \mathcal{H}^{2n-k}_{(2)}(X, E^*) = \{ 0 \}$. \qed
We want to sharpen the Lefschetz vanishing theorem\[3.2\] by giving a lower bound on the spectrum of the Laplace operator $\Delta_A$ on $L^2$-forms $\Omega^k(X, E)$ for $k \neq n$.

**Theorem 3.3.** ([[18 Theorem 1.3]]) Let $(X, \omega)$ be a complete, Kähler manifold, $\dim_{\mathbb{C}} X = n$, with a $d$-bounded Kähler form $\omega$, i.e., there is a bounded 1-form $\theta$ such that $\omega = d\theta$. Let $A^{11}_E \in \mathcal{A}_{E}^{11}$ be a smooth Hermitian integrable connection on a Hermitian vector bundle $E$ over $X$. If $\alpha \in \Omega^k(X, E)$ such that $\Delta_A \alpha \in L^2$, $(k \neq n)$, then we have the inequality

$$
c_{n,k} \|\theta\|^2_{L^\infty(X)} \|\alpha\|^2_{L^2(X)} \leq \langle \alpha, \Delta_A \alpha \rangle_{L^2(X)},
$$

where $c_{n,k} > 0$ is a constant which depends only on $n, k$.

**Proof.** Let $\alpha$ be a $p$-form on vector bundle $(p < n)$, we denote $\beta = L^k\alpha = \omega^k \wedge \alpha$. We recall the operator $L^k : \Omega^p(X, E) \rightarrow \Omega^{p+2k}(X, E)$ for a given $p < n$ and $p + k = n$. Since the Lefschetz theorem $L^k$ is a bijective quasi-isometry,

$$
\|\alpha\|_{L^2(X)} \approx \|\beta\|_{L^2(X)}.
$$

If $\alpha$ is in $L^2$, $\beta$ is also in $L^2$. Since $I_A^{0,2} = 0$, following Lemma\[2,3\] $[L^k, \Delta_A] = 0$. Then we have

$$
\langle \Delta_A \beta, \beta \rangle_{L^2(X)} = \langle L^k(\Delta_A \alpha) L^k \alpha \rangle_{L^2(X)} \approx \langle \Delta_A \alpha, \alpha \rangle_{L^2(X)}.
$$

We write $\beta = d_A\eta - \tilde{\alpha}$, for $\eta = \alpha \wedge \omega^{k-1} \wedge \theta$ and $\tilde{\alpha} = d_A\alpha \wedge \omega^{k-1} \wedge \theta$. Observe that

$$
\|\eta\|_{L^2(X)} \lesssim \|\theta\|_{L^\infty(X)} \|\alpha\|_{L^2(X)} \lesssim \|\theta\|_{L^\infty(X)} \|\beta\|_{L^2(X)},
$$

and

$$
\|\tilde{\alpha}\|_{L^2(X)} \lesssim \|d_A\alpha\|_{L^2(X)} \|\theta\|_{L^\infty(X)} \lesssim \langle \Delta_A \alpha, \alpha \rangle_{L^2(X)}^{1/2} \|\theta\|_{L^\infty(X)}.
$$

We then have

$$
\|\beta\|^2_{L^2(X)} \leq \|\beta, d_A\eta\|_{L^2(X)}^2 + \|\beta, \tilde{\alpha}\|_{L^2(X)}^2 \\
\leq \|d_A'\beta, \eta\|_{L^2(X)} + \|\beta, \tilde{\alpha}\|_{L^2(X)}^2 \\
\lesssim \langle \Delta_A \beta, \beta \rangle_{L^2(X)} \|\theta\|_{L^\infty(X)} \|\beta\|_{L^2(X)} + \|\beta\|_{L^2(X)} \|d_A\alpha\|_{L^2(X)} \|\theta\|_{L^\infty(X)} \\
\lesssim \langle \Delta_A \alpha, \alpha \rangle_{L^2(X)}^{1/2} \|\theta\|_{L^\infty(X)} \|\beta\|_{L^2(X)}.
$$

This yields the desired estimation

$$
\|\alpha\|^2_{L^2(X)} \lesssim \|\beta\|^2_{L^2(X)} \lesssim \|\theta\|^2_{L^\infty(X)} \langle \Delta_A \alpha, \alpha \rangle_{L^2(X)}.
$$

The case $p > n$ follows by $E^*$ is a holomorphic vector bundle on $X$, the Poincaré duality as the operator $*_{E} : \Omega^p(X, E) \rightarrow \Omega^{2n-p}(X, E^*)$ commutes with $\Delta_A$ and is isometric for the $L^2$-norms. \hfill \Box
Lemma 3.4. If $d_A \in \mathcal{A}^{1,1}_E$, then
$$\Delta_A = \Delta_{\partial E} + \Delta_{\bar{\partial} E}.$$

Proof. Following the definitions of $\Delta_A$, $\Delta_{\partial E}$ and $\Delta_{\bar{\partial} E}$, we have
$$\Delta_A = [\partial E, \partial^*_E] + [\bar{\partial} E, \partial^*_E] + [\partial E, \bar{\partial} E] = \Delta_{\partial E} + \Delta_{\bar{\partial} E} + [\partial E, \partial^*_E].$$

Following identities on Proposition 2.2, we have
$$[\partial E, \partial^*_E] = i[\Theta(E), \Lambda] = i[\partial E, [\partial E, \Lambda]] - i[\bar{\partial} E, [\partial E, \Lambda]].$$

Therefore,
$$[\partial E, \partial^*_E] = 0.$$

By the similar way, we also get
$$[\bar{\partial} E, \partial^*_E] = 0.$$

Therefore, we have
$$\Delta_A = \Delta_{\partial E} + \Delta_{\bar{\partial} E}.$$

Proposition 3.5. Let $(X, \omega)$ be a complete, Kähler manifold, $\dim_{\mathbb{C}} X = n$, with a $d$-bounded Kähler form $\omega$, i.e., there is a bounded 1-form $\theta$ such that $\omega = d\theta$, $D_E$ be the Chern connection on a holomorphic Hermitian vector bundle $E$ over $X$. Then for any $\alpha \in \Omega^{p,q}(X, E)$, $(k := p+q \neq n)$, such that $\Delta_{\bar{\partial} E} \alpha \in L^2$, which satisfies the inequality
$$(c(n, k)\|\theta\|_{L^\infty(X)} - \|[i\Theta(E), \Lambda]\|_{L^2(X)})\|\alpha\|_{L^2(X)} \leq \langle \Delta_{\bar{\partial} E} \alpha, \alpha \rangle_{L^2(X)}.$$

where $c_{n,k} > 0$ is a constant which depends only on $n, k$. Furthermore, if
$$\|[i\Theta(E), \Lambda]\| \leq c_n\|\theta\|_{L^\infty(X)}^2,$$

where $c_n$ is a uniformly positive constant only depends on $n$ which satisfies $c_n < \inf c_{n,k}$, we then have
$$\mathcal{H}^{p,q}_{(2),\partial E}(X, E) = 0, \forall p + q \neq n.$$

Proof. Following the Bochner-Kodaira-Nakano formula [10 Chapter VII., Corollary 1.3]
$$\Delta_{\bar{\partial} E} = \Delta_{\partial E} + [i\Theta(E), \Lambda],$$

where $\mathcal{H}^{p,q}_{(2),\partial E}(X, E)$ is the $(2, p+q)$-cohomology group with respect to the Chern connection $D_E$.
we have
\[ \Delta_E = \Delta_{\bar{\partial}_E} + \Delta_{\partial_E} = 2\Delta_{\bar{\partial}_E} - [i\Theta(E), \Lambda], \]
where \( \Delta_E := D_ED_E^* + D_E^*D_E \). Then for any \( \alpha \in \Omega^{p,q}_{(2)}(X, E) \), \((p + q \neq n)\), we have
\[
\langle \Delta_E \alpha, \alpha \rangle_{L^2(X)} \leq 2\langle \Delta_{\bar{\partial}_E} \alpha, \alpha \rangle_{L^2(X)} + |[i\Theta(E), \Lambda]\alpha, \alpha \rangle_{L^2(X)} |
\]
\[
\leq 2\langle \Delta_{\bar{\partial}_E} \alpha, \alpha \rangle_{L^2(X)} + |[i\Theta(E), \Lambda]| \cdot \|\alpha\|_{L^2(X)}. \]
We then have
\[
\langle \Delta_{\bar{\partial}_E} \alpha, \alpha \rangle_{L^2(X)} \geq (c(n, k)\|\theta\|_{L^\infty(X)}^2 - |[i\Theta(E), \Lambda]|)\|\alpha\|_{L^2(X)}, \]
where \( c(n, k) \) is a uniformly positive constant.

For any \( k \neq n \), if
\[
|\langle \Lambda, i\Theta(E) \rangle | \leq c_n\|\theta\|_{L^\infty(X)}^2 < c_{n,k}\|\theta\|_{L^2(X)}^2, \]
then every \( \alpha \in \mathcal{H}^{p,q}_{(2),\bar{\partial}_E}(X, E) \), we get
\[
0 \leq (c(n, k)\|\theta\|_{L^\infty(X)}^2 - |[i\Theta(E), \Lambda]|)\|\alpha\|_{L^2(X)} \leq 0, \]
i.e., \( \alpha = 0 \). We complete the proof of this theorem.

A compact Kähler manifold \((X, J, \omega)\) with sectional curvature bounded form above by a negative constant, i.e., \( \text{sec} \leq -K \) for some \( K > 0 \). We denote by \((\tilde{X}, \tilde{J}, \tilde{\omega})\) the universal covering space of \((X, J, \omega)\). Since \( \pi \) is local isometry, the sectional curvature of \( \tilde{X} \) also bounded form above by the negative constant \( K \). By [6, Lemma 3.2], there exists 1-form \( \theta \) on \( \tilde{X} \) such that
\[ \tilde{\omega} = d\theta \]
and
\[ \|\theta\|_{L^\infty(\tilde{X})} \leq \sqrt{n}K^{-\frac{1}{2}}. \]

**Corollary 3.6.** Let \((X, \omega)\) be a compact Kähler manifold with sectional curvature bounded from above by a negative constant, i.e.,
\[ \text{sec} \leq -K, \]
for some \( K > 0 \). Let \( E \) be a holomorphic vector bundle on \( X \), \( D_E \) be the Chern connection on \( E \). Let \( \pi : (\tilde{X}, \tilde{g}) \to (X, g) \) be the universal covering with \( \tilde{g} = \pi^*g, \tilde{E} = \pi^*E \) the pull back bundle over \( \tilde{X} \). Then for any \( \alpha \in \Omega^{p,q}_{(2)}(\tilde{X}, \tilde{E}) \) such that \( \Delta_{\bar{\partial}_E} \alpha \in L^2 \), which satisfies the inequality
\[
(c(n, k)K/n - |\langle \Lambda, i\Theta(E) \rangle |)\|\alpha\|_{L^2(X)} \leq \langle \Delta_{\bar{\partial}_E} \alpha, \alpha \rangle_{L^2(\tilde{X})}, \]
where \( c_{n,k} > 0 \) is a constant which depends only on \( n, k \). Furthermore, if
\[ |\langle \Lambda, i\Theta(E) \rangle | \leq c_nK, \]
where \( c_n \) is a positive constant only depends on \( n \) which satisfies \( c_n < \inf c_{n,k}/n \), we then have
\[ \mathcal{H}^{p,q}_{(2),\bar{\partial}_E}(\tilde{X}, \tilde{E}) = 0, \forall p + q \neq n. \]
3.2 Nonvanishing results

In [15], Gromov proved a nonvanishing for \( p + q = \text{dim}_\mathbb{C} X \) follows from the \( L^2 \)-index theorem and an upper bound for the bottom of the spectrum [15 Main Theorem]. A special case of a conjecture of Hopf follows from the main theorem. Namely, the Euler characteristic \( \chi(X) \) of a compact, negatively curved Kähler manifold \( X \) of complex dimension \( n \) satisfies \( \text{sign} \chi(X) = (-1)^n \). Let \( E \) be a holomorphic vector bundle equipped with a Hermitian metric and Hermitian connection \( d_A \) over a compact Kähler manifold. Suppose \( X \) is a compact Kähler manifold with underlying Riemann metric \( g \). We denote by \( \nabla^g \) the Hermitian connections induced by the Levi-Civita connection on \( \Omega^{\bullet, \bullet} \), \( \cdot \), \( T X \). Let \( D_E \) be the Chern connection on \( E \). Thus the twist bundle \( \Omega^{p,0} T X \otimes E \) is also a holomorphic vector bundle on \( X \). We denote by \( \chi^p(X, E) \) the index of the operator

\[
D_p = \bar{\partial}_E + \partial_E^* : \Omega^{p, *}(X, E) \to \Omega^{p, * \pm 1}(X, E).
\]

By definition

\[
\chi^p(X, E) = \text{Index}(D_p)
= \text{dim}_\mathbb{C}(\ker D_p) - \text{dim}_\mathbb{C}(\text{coker} D_p)
= \text{dim}_\mathbb{C} \bigoplus_q \text{even} \mathcal{H}_{\bar{\partial}_E}^{p,q} - \text{dim}_\mathbb{C} \bigoplus_q \text{odd} \mathcal{H}_{\partial_E}^{p,q}
= \sum_{q=0}^n (-1)^q h^{p,q}(X, E),
\]

where

\[
\mathcal{H}_{\bar{\partial}_E}^{p,q} = \{ \alpha \in \Omega^{p,q}(X, E) : D_p \alpha = 0 \}
\]

are the spaces of \( \bar{\partial}_E \)-harmonic forms and \( h^{p,q}(X, E) := \dim \mathcal{H}_{\bar{\partial}_E}^{p,q} \) the Hodge numbers of \( (X, E) \). In particular, \( \chi^0(X, E) \) called the *Euler-Poincaré* characteristic [19 Section 5]. The Hirzebruch-Riemann-Roch theorem gives

\[
\chi^p(X, E) = \int_X t^d(X) \text{ch}(\Omega^{p,0} T X \otimes E) = \int_X t^d(X) \text{ch}(\Omega^{p,0} T X) \text{ch}(E).
\]

Given a compact \( n \)-dimensional manifold \( X \), one can associate polynomial \( \chi_y(X) \), called the Hirzebruch \( \chi_y \)-genus, in terms of their Hodge number

\[
h^{p,q}(X) := \dim \mathcal{H}_{\bar{\partial}}^{p,q}(X)
\]

as follows:

\[
\chi_y(X) := \sum_{p=0}^n \chi^p(X) \cdot y^p := \sum_{p=0}^n [\sum_{q=0}^n (-1)^q h^{p,q}(X)] y^p.
\]

where \( \chi^p(X) := \sum_{q=0}^n (-1)^q h^{p,q}(X) \) \((0 \leq p \leq n)\). The \( \chi^p(X) \)-genus was first introduced by Hirzebruch [17]. On a holomorphic bundle over compact complex manifold, we also define a polynomial as follows:

\[
\chi_y(X, E) := \sum_{p=0}^n \chi^p(X, E) \cdot y^p = \sum_{p=0}^n [\sum_{q=0}^n (-1)^q h^{p,q}(X, E)] y^p.
\]
The general form of the Hirzebruch-Riemann-Roch theorem, which is a corollary of the Atiyah-Singer index theorem, allows us to compute \( \chi_y(X, E) \) in terms of the Chern numbers of \( X, E \) as follows:

\[
\chi_y(X, E) = \int_X td(X)ch(\bigoplus_{p=0}^{n}\Omega^{p,0}(TX)y^p)ch(E).
\]

Let \( \gamma_i \) denote the formal Chern roots of \( TX \) (see [19, Corollary 5.14]), i.e., \( i \)-th elementary symmetric polynomial of \( \gamma_1, \cdots, \gamma_n \) represents the \( i \)-th Chern class of \((X, J)\):

\[
c_1 = \gamma_1 + \cdots + \gamma_n, \quad c_2 = \sum_{1 \leq i < j \leq n} \gamma_i\gamma_j, \cdots, \quad c_n = \gamma_1 \cdots \gamma_n.
\]

Then

\[
\int_X td(X) = \prod_{i=1}^{n} \frac{\gamma_i}{1 - e^{-\gamma_i}},
\]

and

\[
ch(\bigoplus_{p=0}^{n}\Omega^{p,0}(TX)y^p) = \prod_{i=1}^{n} (1 + ye^{-\gamma_i}).
\]

**Proposition 3.7.**

\[
\chi_y(X, E) = \int_X ch(E) \prod_{i=1}^{n} (1 + ye^{-\gamma_i}) \frac{\gamma_i}{1 - e^{-\gamma_i}}.
\]

In particular, If \( E \) is a flat bundle, then

\[
\chi_y(X, E) = rank(E) \int_X \prod_{i=1}^{n} (1 + ye^{-\gamma_i}) \frac{\gamma_i}{1 - e^{-\gamma_i}} = rank(E)\chi_y(X).
\]

**Proof.** If \( E \) is a flat bundle, there exists a flat connection \( d_\Gamma \) on the Hermitian vector bundle \( E \). We can write the connection \( d_A = d_\Gamma + a \), where \( a \) is a 1-form take value in \( End(E) \). Therefore, \( F_A = d_\Gamma a + a \wedge a \). Then, \([ch(E)] = [Tr(exp F_A)] = rank(E)\), i.e, there exists a differential form \( \eta \) such

\[
ch(E) = rank(E) + d\eta.
\]

Noting that \( d(td(X)ch(\Omega^{p,0}(X))) = 0 \). We then have,

\[
\chi_y(X, E) = \int_X td(X)ch(\bigoplus_{p=0}^{n}\Omega^{p,0}(TX)y^p)(rank(E) + d\eta) = rank(E)\int_X td(X)ch(\bigoplus_{p=0}^{n}\Omega^{p,0}(TX)y^p) + \int_X d(td(X)ch(\bigoplus_{p=0}^{n}\Omega^{p,0}(TX)y^p) \wedge \eta) = rank(E)\chi_y(X).
\]
Remark 3.8. The $\chi_y$-genus famously satisfies

$$\chi_y(X) = (-y)^n \cdot \chi_{y-1}(X),$$

which are equivalent to the relations $\chi^p(X) = (-1)^n \chi^{n-p}(X)$ and can be derived from the Serre duality for the Hodge number:

$$\chi^p(X) = \sum_{q=0}^{n} (-1)^q h^{p,q}(X) = \sum_{q=0}^{n} (-1)^q h^{n-p,n-q}(X)$$

$$= (-1)^n \sum_{q=0}^{n} (-1)^q h^{n-p,q}(X) = (-1)^n \chi^{n-p}(X).$$

But for any holomorphic vector bundle $E$ over a compact complex manifold $X$ there exists $\mathbb{C}$-linear isomorphisms (Serre duality [19, Corollary 4.1.16]):

$$\mathcal{H}^{p,q}_{\overline{\partial} E}(X, E) \cong \mathcal{H}^{n-p,n-q}_{\overline{\partial} E}(X, E^*),$$

so $\chi_y(X, E)$ always cannot satisfies $\chi_y(X, E) = (-y)^n \cdot \chi_{y-1}(X, E)$.

We also observe that

$$\chi_y(X, E)|_{y=0} = \chi^0(X, E) = \int_X td(X) ch(E)$$

and

$$\chi_y(X, E)|_{y=-1} = \int_X ch(E) \prod_{i=1}^{n} \gamma_i = \text{rank}(E) \chi(X).$$

Let $X$ be a Riemannian manifold and $\Gamma$ a discrete group of isometrics of $X$, such that the differential operator $\mathcal{D}$ commutes with the action of $\Gamma$. This presupposes that the action of $\Gamma$ lifts to the pertinent bundles $E$ and $E'$, and then the commutation between the actions of $\Gamma$ on sections of $E$ and $E'$ and $\mathcal{D} : C^\infty(E) \to C^\infty(E')$ makes sense. A trivial example is that of Galois action for a covering map $X \to X_0$, where $\mathcal{D}$ is the pull back from an operator on $X_0$. We consider a $\Gamma$-invariant Hermitian line bundle $(L, \nabla)$ on $X$ we assume $X/\Gamma$ is compact, and we state Atiyah’s $L^2$-index theorem for $\mathcal{D} \otimes \nabla$.

Theorem 3.9. [15] Theorem 2.3.A/ Let $\mathcal{D}$ be a first-order elliptic operator. Then there exists a closed nonhomogeneous form

$$I_\mathcal{D} = I^0 + I^1 + \cdots + I^n \in \Omega^*(X) = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^n$$

invariant under $\Gamma$, such that the $L^2$-index of the twisted operator $\mathcal{D} \otimes \nabla$ satisfies

$$L^2 \text{Index}_\Gamma(\mathcal{D} \otimes \nabla) = \int_{X/\Gamma} I_\mathcal{D} \wedge \exp [\omega],$$

where $[\omega]$ is the Chern form of $\nabla$, and

$$\exp [\omega] = 1 + [\omega] + \frac{[\omega] \wedge [\omega]}{2!} + \frac{[\omega] \wedge [\omega] \wedge [\omega]}{3!} + \cdots.$$
Remark 3.10. (1) $L^2\text{Index}_\Gamma(D \otimes \nabla) \neq 0$ implies that either $D \otimes \nabla$ or its adjoint has a non-trivial $L^2$-kernel.
(2) The operator $D$ used in the present paper is the operator $\bar{\partial}_E + \bar{\partial}_E^*$. In this case the $I_0$-component of $I_D$ is non-zero. Hence $\int_{X/\Gamma} I_D \wedge \exp \alpha[\omega] \neq 0$, for almost all $\alpha$, provided the curvature form $[\omega]$ is “homologically nonsingular” $\int_{X/\Gamma}[\omega]^n \neq 0$.

We may start with $\Gamma$ acting on $(L, \nabla)$ and then pass (if the topology allows) to the $k$-th root $(L, \nabla)^\frac{k}{k}$ of $(L, \nabla)$ for some $k > 2$. Since the bundle $(L, \nabla)^\frac{k}{k}$ is only defined up to an isomorphism, the action of $\Gamma$ does not necessarily lift to $L$. Yet there is a larger group $\Gamma_k$ acting on $(L, \Gamma)$, where $0 \to \mathbb{Z}/k\mathbb{Z} \to \Gamma_k \to \Gamma \to 1$. In the general case where $\omega(\nabla)$ is $\Gamma$-equivariant, the action of $\Gamma$ on $(L, \nabla)$ is defined up to the automorphism group of $(L, \nabla)$ which is the circle group $S^1 = \mathbb{R}/\mathbb{Z}$ as we assume $X$ is connected. Thus we have a non-discrete group, say $\Gamma$, such that $1 \to S^1 \to \Gamma \to \Gamma \to 1$, and such that the action of $\Gamma$ on $X$ lifts to that of $\Gamma$ on $(L, \nabla)$. This gives us the action of $\Gamma$ on the spaces of sections of $E \otimes L$ and $E' \otimes L$, and we can speak of the $\Gamma$-dimension of $\ker(D \otimes \nabla)$ and $\coker(D \otimes \nabla)$. The proof by Atiyah of the $L^2$-index theorem does not change a bit, and the formula (3.4) remains valid with $\Gamma$ in place of $\Gamma$,

$$L^2\text{Index}_\Gamma(D \otimes \nabla) = \int_{X/\Gamma} I_D \wedge \exp[\omega].$$

Here again, the relevant fact is the implication

$$\int_{X/\Gamma} I_D \exp[\omega] > 0 \Rightarrow \ker D \otimes \nabla \neq 0.$$

Gromov defined the lower spectral bound $\lambda_0 = \lambda_0(D) \geq 0$ as the upper bounded of the negative numbers $\lambda$, such that $\|De\|_{L^2} \geq \lambda\|e\|_{L^2}$ for those sections $e$ of $E$ where $De$ in $L^2$. Let $D$ be a $\Gamma$-invariant elliptic operator on $X$ of the first order, and let $I_D = I_0 + I_1 + \cdots + I^n \in \Omega^*(X)$ be the corresponding index form on $X$. Let $\omega$ be a closed $\Gamma$-invariant 2-form on $X$ and denote by $I_\alpha^n$ the top component of product $I_D \wedge \exp \alpha \omega$, for $\alpha \in \mathbb{R}$. Hence $I_\alpha^n$ is an $\Gamma$-invariant $n$-form on $X$, $\dim X = n$ depending on parameter $\alpha$.

Theorem 3.11. ([15] 2.4.A. Theorem) Let $H^1_{dR}(X) = 0$ and let $X/\Gamma$ be compact and $\int_{X/\Gamma} I_\alpha^n \neq 0$, for some $\alpha \in \mathbb{R}$. If the form $\omega$ is $d$-bounded, then either $\lambda_0(D) = 0$ or $\lambda_0(D^*) = 0$, where $D^*$ is the adjoint operator.

We then have

Theorem 3.12. Let $(X, \omega)$ be a compact Kähler manifold with sectional curvature bounded from above by a negative constant, i.e.,

$$\sec \leq -K,$$

for some $K > 0$. Let $E$ be a holomorphic vector bundle on $X$, $D_E$ be the Chern connection on $E$. If the curvature $\Theta(E)$ of $D_E$ such that

$$\|[\Lambda, i\Theta(E)]\| \leq c(n)K,$$
then the spaces of $L^2 \Delta_{\partial_E}$-harmonic $(p, q)$-forms on the lifting bundle $\tilde{E}$ satisfy

$$
\begin{aligned}
\mathcal{H}^{p,q}_{(2),\partial E}(\tilde{X}, \tilde{E}) &= \{0\}, \ p + q \neq n \\
\mathcal{H}^{p,q}_{(2),\partial E}(\tilde{X}, \tilde{E}) &\neq \{0\}, \ p + q = n
\end{aligned}
$$

Proof. Since $\pi$ is a local isometry, the Chern curvature $\Theta(\tilde{E})$ also satisfies

$$
||[\Lambda, i\Theta(\tilde{E})]| \leq c(n)K.
$$

We denote by $\tilde{D}_p$ the lifted of $D_p$ for $p \geq 0$. Following Proposition [3.5] we get $\mathcal{H}^{p,q}_{(2),\partial E}(\tilde{X}, \tilde{E}) = \{0\}$ for any $p + q \neq n$. Since $\int_X [\omega]^n \neq 0$, by Theorem [3.11] we obtain that either

$$
\ker \tilde{D}_p \cap \bigoplus \Omega^{p,+}_{(2)}(X, E) = \oplus_q \text{even} \mathcal{H}^{p,q}_{(2),\partial E}(\tilde{X}, \tilde{E}) \neq 0
$$

or

$$
\ker \tilde{D}_p \cap \bigoplus \Omega^{p,-}_{(2)}(X, E) = \oplus_q \text{odd} \mathcal{H}^{p,q}_{(2),\partial E}(\tilde{X}, \tilde{E}) \neq 0.
$$

Therefore, for any $p + q = n$, we have

$$
\mathcal{H}^{p,q}_{(2),\partial E}(\tilde{X}, \tilde{E}) \neq 0.
$$

\[ \square \]

3.3 $L^2$-Hodge numbers on vector bundle $E$

We assume throughout this subsection that $(X, g, J)$ is a compact complex $n$-dimensional manifold with a Hermitian metric $g$, and $\pi : (\tilde{X}, \tilde{g}, \tilde{J}) \to (X, g, J)$ its universal covering with $\Gamma = \pi_1(X)$ as an isometric group of deck transformations. Let $E \to X$ be a holomorphic bundle on $X$. We denote by $\tilde{g} := \pi^*g$ the pull-back metric on $\tilde{X}$ and $\tilde{E} := \pi^*E$ the pull-back bundle on $\tilde{X}$. We call an open set $U \subset \tilde{X}$ a fundamental domain of the action of $\tilde{\Gamma}$ on $\tilde{X}$ if the following conditions are satisfied:

1. $\tilde{X} = \bigcup_{\gamma \in \Gamma} \gamma(U)$,
2. $\gamma_1(U) \cap \gamma_2(U) = \emptyset$ for $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$ and
3. $\tilde{U} \setminus U$ has zero measure.

We then have (see [4] or [26] Section 3.6.1)

$$
\Omega^{p,q}_{(2)}(\tilde{X}, \tilde{E}) \cong L^2(\Gamma) \otimes \Omega^{p,q}_{(2)}(U, \tilde{E}|_U) \cong L^2(\Gamma) \otimes \Omega^{p,q}(X, E),
$$

where a basis for $L^2(\Gamma)$ is given by the function $\delta_\gamma$ with $\gamma \in \Gamma$ defined by $\delta_\gamma(\gamma') = 1$ if $\gamma = \gamma'$ and $\delta_\gamma(\gamma') = 0$ if $\gamma \neq \gamma'$. Consider now a $\Gamma$-module $V \subset \Omega^{p,q}_{(2)}(\tilde{X}, \tilde{E})$, that is a closed subspace of $\Omega^{p,q}_{(2)}(\tilde{X}, \tilde{E})$ which is invariant under the action of $\Gamma$. If $\{\eta_i\}_{i \in \mathbb{N}}$ is an orthonormal basis for $V$ then the following quantity is finite:

$$
\sum_{i \in \mathbb{N}} \int_U |\eta_i|^2 dvol_{\tilde{g}|_U}
$$
and does not depend either on the choice of the orthonormal basis of $V$ or on the choice of the fundamental domain of the action of $\Gamma$ on $\tilde{X}$. The von Neumann dimension of a $\Gamma$-module $V$ is therefore defined as

$$\dim_\Gamma(V) = \sum_{i \in \mathbb{N}} \int_U |\eta_i|^2 dvol_{\tilde{g}}|_U,$$

where $\{\eta_i\}_{i \in \mathbb{N}}$ is any orthonormal basis for $V$ and $U$ is any fundamental domain of the action of $\Gamma$ on $\tilde{X}$. Since the Laplacian $\Delta_{\tilde{\partial}_E}$ commutes with the action of $\Gamma$, a natural and important example of $\Gamma$-module is provided by the space of $L^2(\tilde{\partial}_E)$-harmonic forms of bidegree $(i, j)$, $H^\Gamma_{(2);\tilde{\partial}_E}^{p,q}(\tilde{X}, \tilde{E})$ for each $i, j = 0, \ldots, n$ (see [26, Section 3.6.2]). We denote by $\dim_\Gamma H^\Gamma_{(2);\tilde{\partial}_E}^{p,q}(\tilde{X}, \tilde{E})$ the Von Neumann dimension of $H^\Gamma_{(2);\tilde{\partial}_E}^{p,q}(\tilde{X}, \tilde{E})$ with respect to $\Gamma$, which is a nonnegative real number. We have the following two basic facts.

**Lemma 3.13.**

$$\dim_\Gamma H^\Gamma_{(2)}(M) = 0 \iff H^\Gamma_{(2)}(M) = \{0\},$$

and $\dim_\Gamma H$ is additive: Given

$$0 \to H_1 \to H_2 \to H_3 \to 0,$$

one have

$$\dim_\Gamma H_2 = \dim_\Gamma H_1 + \dim_\Gamma H_3.$$

Then the $L^2$-Hodge numbers of $(X, E)$, denote by $h^{p,q}_{(2)}(X, E)$, are defined to be

$$h^{p,q}_{(2)}(X, E) = \dim_\Gamma H^\Gamma_{(2);\tilde{\partial}_E}^{p,q}(\tilde{X}, \tilde{E}) \in \mathbb{R}_{\geq 0}, \ (0 \leq p, q \leq n).$$

The Dolbeault-type operators $\mathcal{D}_p$ can be lifted to $(\tilde{X}, \tilde{E})$:

$$\tilde{\mathcal{D}}_p : \Omega^{p,\ast+1}_{(2)}(\tilde{X}, \tilde{E}) \to \Omega^{p,\ast}_{(2)}(\tilde{X}, \tilde{E}),$$

and one can define the $L^2$-index of the lifted operators $\tilde{\mathcal{D}}_p$ by

$$L^2\text{Index}_\Gamma(\tilde{\mathcal{D}}_p) = \dim_\Gamma(\ker \tilde{\mathcal{D}}_p) - \dim_\Gamma(\coker \tilde{\mathcal{D}}_p)$$

$$= \dim_\Gamma(\bigoplus_{q \text{ even}} H^\Gamma_{(2);\tilde{\partial}_E}^{p,q}(\tilde{X}, \tilde{E})) - \dim_\Gamma(\bigoplus_{q \text{ odd}} H^\Gamma_{(2);\tilde{\partial}_E}^{p,q}(\tilde{X}, \tilde{E}))$$

$$= \sum_{q \text{ even}} \dim_\Gamma H^\Gamma_{(2);\tilde{\partial}_E}^{p,q}(\tilde{X}, \tilde{E}) - \sum_{q \text{ odd}} \dim_\Gamma H^\Gamma_{(2);\tilde{\partial}_E}^{p,q}(\tilde{X}, \tilde{E})$$

$$= \sum_{q=0}^n (-1)^q h^{p,q}_{(2)}(X, E).$$

We recall the Atiyah’s $L^2$-index theorem [3, 31].
**Theorem 3.14.** [31, Theorem 6.1] Let $X$ be closed Riemannian manifold, $P$ a determined elliptic operator on sections of certain bundles over $X$. Denote by $\tilde{P}$ its lift of $P$ to the universal covering space $\tilde{X}$. Let $\Gamma = \pi_1(M)$. Then the $L^2$ kernel of $\tilde{P}$ has a finite $\Gamma$-dimension and

$$L^2\text{Index}_\Gamma(\tilde{P}) = \text{Index}(P).$$

We define $L^2$- Euler characteristics

$$\chi^p_{(2)}(X, E) = \sum_{q=0}^{n} (-1)^q h^{p,q}_{(2)}(X, E)$$

on the holomorphic bundle $\Omega^{p,0}(X) \otimes E$ over a compact Kähler manifold. The celebrated $L^2$-index theorem of Atiyah [3] asserts that

$$\text{Index}(D_p) = L^2\text{Index}_\Gamma(\tilde{D}_p)$$

so we have the following crucial identities between $\chi^p(X, E)$ and the $L^2$-Hodge numbers $h^{p,q}_{(2)}(X, E)$:

$$\chi^p(X, E) = \chi^p_{(2)}(X, E) = \sum_{q=0}^{n} (-1)^q h^{p,q}_{(2)}(X, E).$$

**Theorem 3.15.** Let $(X, \omega)$ be a compact Kähler manifold, $E$ a holomorphic bundle on $X$. Then for any $p = 0, \ldots, n$,

$$\chi^p(X, E) = \chi^p_{(2)}(X, E) = \sum_{q=0}^{n} (-1)^q h^{p,q}_{(2)}(X, E).$$

In [15], Gromov proved that if $X$ is Kähler hyperbolic, $\dim \mathbb{C} X = n$, then for every $p = 0, 1, \ldots, n$, the Euler characteristic

$$\chi^p(X) = \int_X td(X) \text{ch}(\Omega^{p,0}(TX))$$

does not vanish and $\text{sign} \chi^p = (-1)^{n-p}$. We will extend the result to holomorphic vector bundle case.

**Proposition 3.16.** Let $(X, \omega)$ be a compact Kähler manifold with sectional curvature bounded from above by a negative constant, i.e.,

$$\text{sec} \leq -K,$$

for some $K > 0$. Let $E$ be a holomorphic vector bundle on $X$, $D_E$ be the Chern connection on $E$. If the curvature $\Theta(E)$ of $D_E$ such that

$$|[\Lambda, i\Theta(E)]| \leq c(n)K,$$

then

$$\begin{cases} h^{p,q}_{(2)}(X, E) = 0, p + q \neq n \\ h^{p,q}_{(2)}(X, E) \geq 1, p + q = n \end{cases}$$

In particular, for every $p = 0, 1, \ldots, n$, the Euler characteristic

$$(-1)^{n-p} \chi^p(X, E) \geq 1.$$
Proof. Following Theorems 3.12 and 3.15, we have
\[(−1)^{n−p}χ^p(X, E) = h^{p,n−p}_{(2)}(X, E) ≥ 1.\]

If we denote by \(K_j(M, E)\) (\(0 ≤ j ≤ n\)) the coefficients in the Taylor expansion of \(χ_y(X, E)\) at \(y = −1\), i.e.,
\[χ_y(X, E) := \sum_{j=0}^n K_j(X, E) \cdot (y + 1)^j.\]
Following the idea of Li in \[24\], we then have

**Theorem 3.17.** Let \((X, ω)\) be a compact Kähler manifold with sectional curvature bounded from above by a negative constant, i.e.,

\[\text{sec} ≤ −K,\]

for some \(K > 0\). Let \(E\) be a holomorphic vector bundle on \(X\), \(D_E\) be the Chern connection on \(E\). If the curvature \(Θ(E)\) of \(D_E\) such that
\[C := |[Λ, iΘ(E)]| ≤ c(n)K,\]
then for all \(0 ≤ j ≤ n\), \((X, E)\) satisfy Chern number inequalities
\[(−1)^{n+j}K_j(X, E) ≥ \sum_{p=j}^n \binom{n}{p}.\]

**Proof.** Following Proposition 3.16 we get
\[χ^p(X, E) = \sum_{q=0}^n (−1)^q h^{p,q}(X, E) = (−1)^{n−p}h^{p,n−p}_{(2)}(X, E).\]

We have
\[(-1)^n \sum_{j=0}^n K_j(X, E) \cdot (y + 1)^n = (-1)^n χ_y(X, E)\]
\[= (-1)^n \sum_{p=0}^n χ^p(X, E) \cdot y^p\]
\[= \sum_{p=0}^n h^{p,n−p}(X, E) \cdot (−y)^p.\]
Now comparing the coefficients of the Taylor expansion at \(y = −1\) on both sides of (3.5) yields
\[(-1)^nK_j(X, E) = \frac{1}{j!} \left[ \sum_{p=0}^n h^{p,n−p}_{(2)}(X, E) \cdot (−y)^p \right]^{(j)} |_{y = −1}\]
\[= (-1)^j \sum_{p=j}^n \binom{n}{p} h^{p,n−p}_{(2)}(X, E).\]
This implies that

$$(-1)^{n+j} K_j(X, E) \geq \sum_{p=j}^{n} \binom{p}{j}.$$ 

\[ \square \]

**Remark 3.18.** If $E$ is flat bundle, then

$$\chi_y(X, E) = \text{rank}(E) \chi_y(X) = \sum_{j=0}^{n} K_j(X) \cdot (y + 1)^j.$$ 

Therefore,

$$K_j(X, E) = \text{rank}(E) K_j(X).$$ 

The first few terms are given by

$$K_0(X) = c_n[X], \quad K_1(X) = -\frac{1}{2} nc_n[X], \ldots.$$ 

A recursive algorithm for calculating $K_j$ was described in [25, p. 144]. The formulas $K_j$ for $j \leq 6$ are presented, respectively, in [25, pp. 141–143], [32, p. 145].

If $E$ is a holomorphic bundle, then the first few terms of $K_j(X, E)$ are given by

$$K_0(X, E) = \chi_y(X, E)|_{y=-1} = \text{rank}(E)c_n[X],$$

$$K_1(X, E) = \frac{d}{dy} \chi_y(X, E)|_{y=-1}$$

\[= \frac{d}{dy} \int_X \text{ch}(E) \prod_{i=1}^{n} (1 + ye^{-\gamma_i}) \frac{\gamma_i}{1 - e^{-\gamma_i}}|_{y=-1} \]\n
\[= \int_X \text{ch}(E) t d(X) \sum_{i=1}^{n} \frac{e^{-\gamma_i}}{1 - e^{-\gamma_i}} \prod_{i=1}^{n} (1 - e^{-\gamma_i}) \]\n
\[= \int_X \text{ch}(E) \prod_{i=1}^{n} \gamma_i \sum_{i=1}^{n} \frac{e^{-\gamma_i}}{1 - e^{-\gamma_i}} \]\n
\[= \int_X \text{ch}(E) \prod_{i=1}^{n} \gamma_i \sum_{i=1}^{n} \left( -1 + \frac{1}{1 - e^{-\gamma_i}} \right) \]\n
\[= \int_X \text{ch}(E) \sum_{i=1}^{n} ((\prod_{j \neq i} \gamma_j)(\frac{\gamma_i}{1 - e^{-\gamma_i} - \gamma_i})) \]\n
\[= \int_X \text{ch}(E) \sum_{i=1}^{n} ((\prod_{j \neq i} \gamma_j)(1 - \frac{\gamma_i}{2})) \]\n
\[= \int_X (\text{rank}(E) + c_1(E))(c_{n-1}(X) - \frac{n}{2} c_n(X)) \]\n
\[= -\frac{\text{rank}(E)}{2} nc_n[X] + \langle c_{n-1}(X)c_1(E), X \rangle. \]
Here we use the fact
\[ c_{n-1}(X) = \sum_{i=1}^{n} (\prod_{j \neq i} \gamma_j), \quad c_n(X) = \prod_{i=1}^{n} \gamma_i. \]

**Corollary 3.19.** Let \((X, \omega)\) be a compact Kähler surface with sectional curvature bounded from above by a negative constant, i.e.,
\[ \text{sec} \leq -K, \]
for some \(K > 0\). Let \(E\) be a holomorphic vector bundle on \(X\), \(D_E\) be the Chern connection on \(E\). If the curvature \(\Theta(E)\) of \(D_E\) such that
\[ C := |[\Lambda, i\Theta(E)]| \leq c(n)K, \]
we then have
\[ \int_X c_1(X)c_1(E) = 0. \]
Furthermore, if \(\text{ch}_2(E) = 0\), then
\[ \chi_y(X, E) = \text{rank}(E)\chi_y(X). \]

**Proof.** For any \(\alpha \in \Omega^{2,0}(X, E)\) or \(\alpha \in \Omega^{0,2}(X, E)\), we observe that
\[ [\Lambda, i\Theta(E)]\alpha = 0. \]
Therefore,
\[ \Delta_E\alpha = 2\Delta_{\partial_E}\alpha = 2\Delta_{\bar{\partial}_E}\alpha. \]
Hence
\[ \ker \Delta_{\partial_E} \cap \Omega^{0,2}(X, E) = \ker \Delta_E \cap \Omega^{0,2}(X, E) \]
and
\[ \ker \Delta_{\bar{\partial}_E} \cap \Omega^{2,0}(X, E) = \ker \Delta_E \cap \Omega^{2,0}(X, E). \]
Noticing that \(\Omega^{0,2}(X, E) \cong \Omega^{2,0}(X, E)\). We then have
\[ \ker \Delta_{\partial_E} \cap \Omega^{2,0}(X, E) \cong \ker \Delta_{\bar{\partial}_E} \cap \Omega^{0,2}(X, E). \]
One can see that
\[ h^{2,0}_{(2)}(X, E) = h^{0,2}_{(2)}(X, E). \]
Noticing that
\[ \chi_y(X, E)|_{y=-1} = h^{0,2}_{(2)}(X, E) + h^{1,1}_{(2)}(X, E) + h^{2,0}_{(2)}(X, E). \]
We then have
\[ -K_1(X, E) = \text{rank}(E) \int_X c_2(X) - \int_X c_1(X)c_1(E) \]
\[ = h^{1,1}_{(2)}(X, E) + 2h^{2,0}_{(2)}(X, E) \]
\[ = h^{0,2}_{(2)}(X, E) + h^{1,1}_{(2)}(X, E) + h^{2,0}_{(2)}(X, E) \]
\[ = \chi_y(X, E)|_{y=-1} \]
\[ = \text{rank}(E)\chi(X). \]
Therefore, we get
\[ \int_X c_1(X)c_1(E) = 0. \]

Noticing that
\[ td(X) = 1 + \frac{c_1(X)}{2} + \frac{c_1^2(X) + c_2(X)}{12}, \quad ch(E) = rank(E) + c_1(E) + \frac{c_2(E) - 2c_2(E)}{2}. \]

By the definition of \( K_2(X, E) \), we then have
\[
K_2(X, E) = \frac{1}{2} \frac{d^2}{dy^2} \chi_y(X, E) |_{y=-1}
\]
\[
= \frac{1}{2} \frac{d^2}{dy^2} \int_X td(X)ch(E) \prod_{i=1}^2 (1 + ye^{-\gamma_i}) |_{y=-1}
\]
\[
= \frac{1}{2} \frac{d^2}{dy^2} \int_X td(X)ch(E)e^{-\gamma_1 - \gamma_2}y^2 |_{y=-1}
\]
\[
= \int_X td(X)ch(E)e^{-c_1(X)}
\]
\[
= \int_X (1 + \frac{c_1(X)}{2} + \frac{c_1^2(X) + c_2(X)}{12})(1 - c_1(X) + \frac{c_1^2(X)}{2})(\text{rank}(E) + c_1(E) + \frac{c_2(E) - 2c_2(E)}{2})
\]
\[
= \int_X (\frac{c_1^2(E) - 2c_2(E)}{2} - c_1(X)c_1(E) + \text{rank}(E)\frac{c_1^2(X)}{2} - \text{rank}(E)\frac{c_1^2(X)}{2} + \frac{c_1(X)}{2}c_1(E)
\]
\[
+ \text{rank}(E)\frac{c_1^2(X) + c_2(X)}{12})
\]
\[
= \text{rank}(E)K_2(X) - \langle \frac{c_1(X)c_1(E)}{2}, X \rangle + \langle \frac{c_1^2(E) - 2c_2(E)}{2}, X \rangle.
\]

We also have
\[ K_2(X, E) = h_{(2)}^{2,0}(X, E), \quad K_2(X) = h_{(2)}^{2,0}(X). \]

Therefore, we get
\[ \int_X \frac{c_1^2(E) - 2c_2(E)}{2} = h_{(2)}^{2,0}(X, E) - \text{rank}(E)h_{(2)}^{2,0}(X). \]

If \( ch_2(E) = 0 \), then
\[ K_2(X, E) = \text{rank}(E)K_2(X). \]

Notice that \( K_0(X, E) = \text{rank}(E)K_0(X) \) and \( K_1(X, E) = \text{rank}(E)K_1(X). \) Therefore,
\[ \chi_y(X, E) = \text{rank}(E)\chi_y(X). \]

\[ \square \]

**Proof of Theorem 1.7** If \( C := ||\Lambda, i\Theta(L)|| > c(n)K \), i.e., \( \left| \frac{c_nK}{c} \right| = 0 \), then \( \chi(X) \geq 3. \)

If \( C \leq c_nK \), i.e., there is a positive integer \( N \) such that
\[ C(N + 1) > c_nK \geq CN = ||\Lambda, i\Theta(L^{\otimes N})||, \]
then for any $|m| \leq N$, following Corollary 3.19, we get

$$\langle c_1(L^\otimes m)c_1(X), X \rangle = 0.$$ 

Therefore,

$$K_1(X, L^\otimes m) = K_1(X) = -c_2[X].$$

By the definition of $\chi_y(X, E)$, we get

$$\chi_y(X, L^\otimes m) = \chi^0(X, L^\otimes m) + \chi^1(X, L^\otimes m)y + \chi^2(X, L^\otimes m)y^2 = K_0(X, L^\otimes m) + K_1(X, L^\otimes m) \cdot (y + 1) + K_2(X, L^\otimes m) \cdot (y + 1)^2.$$ 

Therefore, for any $|m| \leq N$, we have

$$\chi^0(X, L^\otimes m) = K_0(X, L^\otimes m) + K_1(X, L^\otimes m) + K_2(X, L^\otimes m) = K_0(X) + K_1(X) + K_2(X) + \frac{m^2}{2} \int_X c_1^2(L) = \chi^0(X) + \frac{m^2}{2} \int_X c_1^2(L) \geq 1,$$

$$\chi^1(X, L^\otimes m) = K_1(X, L^\otimes m) + 2K_2(X, L^\otimes m) = K_1(X) + 2K_2(X) + m^2 \int_X c_1^2(L) = \chi^1(X) + m^2 \int_X c_1^2(L) \leq -1,$$

$$\chi^2(X, L^\otimes m) = K_2(X, L^\otimes m) = K_2(X) + \frac{m^2}{2} \int_X c_1^2(L) = \chi^2(X) + \frac{m^2}{2} \int_X c_1^2(L) \geq 1.$$

Following [15, 0.4.A. Theorem], we also have $(-1)^p \chi^p(X) \geq 1$, $p = 0, 1, 2$.

When $\int_X c_1^2(L) > 0$, for any $|m| \leq N$, we obtain that

$$\chi^0(X, L^\otimes m) \geq 1 + \frac{m^2}{2} \int_X c_1^2(L),$$

$$- \chi^1(X, L^\otimes m) \geq 1,$$

$$\chi^2(X, L^\otimes m) \geq 1 + \frac{m^2}{2} \int_X c_1^2(L).$$

Therefore,

$$\chi(X) = \chi_y(X, L^\otimes m)|_{y=-1} = \sum_{i=0}^2 (-1)^i \chi^i(X, L^\otimes m) \geq 3 + m^2 \int_X c_1^2(L).$$
When $\int_X c_1^2(L) < 0$, for any $|m| \leq N$, we obtain that

$$
\chi^0(X, L^\otimes m) \geq 1,
$$

$$
- \chi^1(X, L^\otimes m) \geq 1 - m^2 \int_X c_1^2(L),
$$

$$
\chi^2(X, L^\otimes m) \geq 1.
$$

Therefore,

$$
\chi(X) = \chi_y(X, L^\otimes m)|_{y=-1}
$$

$$
= \sum_{i=0}^2 (-1)^i \chi^i(X, L^\otimes m)
$$

$$
\geq 3 - m^2 \int_X c_1^2(L).
$$

Therefore, for all cases, we get

$$
\chi(X) \geq 3 + \left| \int_X c_1^2(L) \cdot \left( \frac{c_nK}{C^*} \right)^2 \right|.
$$

\[\square\]

## 4 Eigenvalue and Euler characteristic

### 4.1 Hilbert polynomial of line bundle

The Hilbert polynomial of polarized manifold $(X, L)$, i.e., $L$ is an ample line bundle on a compact Kähler manifold $X$, is defined as the functional $P_{(X,L)}(m) := \chi(X, L^\otimes L)$. Indeed, $P_{(X,L)}(m)$ is a polynomial in $n$. Following Kodaira vanishing theorem (see [19, Proposition 5.27]), we also have $P_{(X,L)}(m) = h^0(X, L^\otimes m)$ for $m \gg 1$. Notice that

$$
[t(X) ch(\Omega^{p,0}(TX)) ch(L^\otimes m)]_{2n} = \sum_{i=0}^n [t(X) ch(\Omega^{p,0}(TX))]_{2n-2i} [ch^m(L)]_{2i},
$$

$$
= \sum_{i=0}^n [t(X) ch(\Omega^{p,0}(TX))]_{2n-2i} \frac{(mc_1(L))^i}{i!},
$$

where $[t(X) ch(\Omega^{p,0}(TX))]_{2n-2i}$ is the part of $2n - 2i$-form of $t(X) ch(\Omega^{p,0}(TX))$. We denote

$$
a_i := \int_X [t(X) ch(\Omega^{p,0}(TX))]_{2n-2i} \wedge \frac{c_1(L)}{i!}.
$$

Therefore,

$$
P^{(p)}_n(m, L) := \sum_{i=0}^n a_i m^i = \int_X t(X) ch(\Omega^{p,0}(TX)) ch(L^\otimes m).
$$

$P^{(p)}_n(m, L)$ is a polynomial of degree $n$ if only if $\int_X c_1^n(L) \neq 0$. We can introduce the following definition.
Definition 4.1. Let \( L \) be a holomorphic line bundle on a compact Kähler manifold \( X \). We call
\[
P^{(p)}_n(m, L) := \chi_p(X, L^\otimes m)
\]
the \( p \)-Hilbert polynomial of line bundle \( L \).

Lemma 4.2. Let \( P_n(m) \) be a numerical polynomial of degree \( n \geq 1 \). Suppose that \( P_n(m) \) is not constant. If \( N = \{i_1, \cdots, i_{2nL+1}\} \), where the integers \( \{i_j\}_{j=1, \cdots, 2nL+1} \) that are not equal to each other, then there exists integer \( \tilde{i} \) such that
\[
|P_n(\tilde{i})| \geq L.
\]

Proof. Since \( P_n(m) \) is not constant, there is an coefficient \( a_i \neq 0 \). Notice that for any integer \( i \), the equation \( P_n(x) = i \) has at most \( n \) solutions. Therefore, \( 0 \leq |P_n(\tilde{i})| \leq L - 1 \) has at most \( 2nL \) solutions. Hence there exists an integer \( \tilde{i} \in N \) such that \( |P_n(\tilde{i})| \geq L \).

Assume that \( X \) has a Kähler metric \( \omega \). Let
\[
\gamma_1(x) \leq \cdots \gamma_n(x)
\]
be the eigenvalues of \( i\Theta(L)_x \) with respect to \( \omega_x \) at each point \( x \in X \), and let
\[
i\Theta(L) = i \sum_{1 \leq j \leq n} \gamma_i \xi_j \wedge \bar{\xi}_j, \quad \xi_j \in T^*_x X
\]
be a diagonalization of \( i\Theta(L)_x \). We then have
\[
\langle [i\Theta(L), \Lambda]u, u \rangle = \sum_{J,K} (\sum_{j \in J} \gamma_j + \sum_{j \in K} \gamma_j - \sum_{1 \leq j \leq n} \gamma_j) |u_{J,K}|^2.
\]

Lemma 4.3. Let \( L \) be a line on a compact Kähler manifold \((X, \omega)\). Then \( C = 0 \) if only if \( \Theta(L) = 0 \). Furthermore, if \( L \) is not flat, then there is an uniform positive constant \( \varepsilon \in (0, 1) \) such that
\[
C > \varepsilon.
\]

Proof. If \( C = 0 \), then for any \( J, K \), we have
\[
\sum_{J,K} (\sum_{j \in J} \gamma_j + \sum_{j \in K} \gamma_j - \sum_{1 \leq j \leq n} \gamma_j) = 0.
\]
We take \( J = \{1, \cdots, n\} \) and \( K = j \), then we get
\[
\gamma_j = 0, \forall 0 \leq j \leq n,
\]
i.e., \( \Theta(L) = 0 \).

We suppose that the constant \( \varepsilon \) does not exist. We may then choose a sequence non-flat bundles \( \{L_i\}_{i \in \mathbb{N}} \) such that \( C_i \to 0 \) as \( i \to \infty \). We have
\[
|\Theta(L_i)| \leq \sum_{j=1}^n |\gamma_j| \leq nC_i.
\]
If \( i \) large enough, then there exists a flat connection on \( L_i \) (see [34]), contradicting our initial assumption regarding the sequence \( \{L_i\}_{i \in \mathbb{N}} \).
Proof of Theorem \ref{thm:vanishing} Since \( \chi^p(X, L^\otimes m) \) is not constant for some \( p \in [0, n] \), we obtain that \( L \) is not flat, i.e.,

\[ C := ||i\Theta(L), \Lambda|| > 0. \]

For any \( K > 0 \), there is an integer \( N \) such that

\[ C(nN + 1) > c_KK \geq C(nN) = ||i\Theta(L^\otimes(nN)), \Lambda||. \]

Notice that \( P_n^{(p)}(m, L) \) is integer for any \( |m| \leq nN \). Following Lemma \ref{lem:integer}, then there is a integer \( \tilde{m} = \tilde{m}(p) \) such that

\[ |P_n^{(p)}(\tilde{m}, L) - \chi^p(X)| \geq N. \]

We then have either

\[ (-1)^{n-p+1}(P_n^{(p)}(\tilde{m}, L) - \chi^p(X)) \geq N \]

or

\[ (-1)^{n-p+1}(P_n^{(p)}(\tilde{m}, L) - \chi^p(X)) \leq -N. \]

If \( (-1)^{n-p+1}(P_n^{(p)}(\tilde{m}, L) - \chi^p(X)) \leq -N \), then

\[ (-1)^{n-p}\chi^p(X, L^\otimes \tilde{m}) = (-1)^{n-p}\chi^p(X) + (-1)^{n-p}(P_n^{(p)}(\tilde{m}, L) - \chi^p(X)) \geq 1 + N. \]

Following Theorem \ref{thm:vanishing} for any \( |m| \leq nN \), we get

\[ (-1)^{n-p}\chi^p(X, L^\otimes m) = (-1)^{n-p}\chi^p(X) + (-1)^{n-p}(P_n^{(p)}(\tilde{m}, L) - \chi^p(X)) \geq 1. \]

If \( (-1)^{n-p+1}(P_n^{(p)}(\tilde{m}, L) - \chi^p(X)) \geq N \), then there is a integer \( \tilde{m} \) such that

\[ (-1)^{n-p}\chi^p(X) = (-1)^{n-p}\chi^p(X, L^\otimes \tilde{m}) + (-1)^{n-p+1}(P_n^{(p)}(\tilde{m}, L) - \chi^p(X)) \geq 1 + N. \]

Following the vanishing theorem and the Atiyah’s \( L^2 \)-index theorem, we obtain that either

\[ (-1)^{n-p}\chi^p(X) = (-1)^{n-p}\chi_{(2)}^p(X) = h_{(2)}^{n-p,p}(X) \geq 1 + N \]

or

\[ (-1)^{n-p}\chi^p(X, L^\otimes \tilde{m}) = (-1)^{n-p}\chi_{(2)}^p(X, L^\otimes \tilde{m}) = h_{(2)}^{n-p,p}(X, L^\otimes \tilde{m}) \geq 1 + N. \]

Therefore, we get

\[ (-1)^n\chi(X) = (-1)^n\chi_{(2)}(X) = \sum_{p=0}^n h_{(2)}^{n-p,p}(X) \geq (n + 1) + N \]

or

\[ (-1)^n\chi(X, L^\otimes \tilde{m}) = (-1)^n\chi_{(2)}^p(X, L^\otimes \tilde{m}) = \sum_{p=0}^n h_{(2)}^{n-p,p}(X, L^\otimes \tilde{m}) \geq (n + 1) + N. \]

We observe that for any \( m \in \mathbb{Z} \),

\[ \chi_g(X, L^\otimes m)|_{y=-1} = \chi(X). \]

Hence, the Euler number of \( X \) must satisfy

\[ (-1)^n\chi(X) = (-1)^n\chi(X, L^\otimes \tilde{m}) \geq (n + 1) + N. \]

\[ \square \]
4.2 The roots of Hilbert polynomial

We denote by
\[ Z^\pm = \{ m \in \mathbb{R}^\pm : P_n^{(p)}(m, L) = \chi^p(X) \} \]
the set of positive (resp. negative) roots of \( (P_n^{(p)}(m, L) - \chi^p(X)) \). We denote
\[ C^\pm := \max_{m \in Z^\pm} |m|, \]
when \( Z^\pm = \emptyset \), we denote \( C^\pm = 0 \). It’s easy to see that \( C^\pm \) depends on \( K \) and \( c_1(L) \).

Lemma 4.4. ([9, Lemma 16.3]) Let \( P_n(m) \) be a numerical polynomial of degree \( n \geq 1 \) and with leading coefficient \( \frac{1}{n!} a_n \in \mathbb{Z}, a_n > 0 \). We assume that \( P_n(m) \geq 0 \) for all \( m \geq m_0 \). Then for any \( k \in \mathbb{N} \), there exists \( m \in [m_0, m_0 + kn] \) such that
\[ P_n(m) \geq \frac{a_n k^n}{2n-1}. \]

Proof. By virtue of Newton’s formula for the iterated differentials \( \Delta P_n(m) = P_n(m + 1) - P_n(m) \), we obtain
\[ \Delta^n P_n(m) = \sum_{1 \leq j \leq n} (-1)^j \binom{n}{j} P_n(m + N - j) = a_n. \]
Consequently, if \( j \in \{0, 2, 4, 2\lfloor n/2 \rfloor \} \subset [0, n] \) is the even integer realizing the maximum of \( P(m_0 + n - j) \) on this set, we obtain
\[ 2^{n-1} P(m_0 + n - j) \geq (\binom{n}{0} + \binom{n}{2} + \cdots) P(m_0 + n - j) \geq a_n, \]
whereby we obtain the existence of an integer \( m \in [m_0, m_0 + n] \) with \( P_n(m) \geq \frac{a_n k^n}{2n-1} \). The result is therefore prove for \( k = 1 \). In general case, we apply this particular result to the polynomial \( Q_n(m) = P_n(km - (k-1)m_0) \), for which the leading coefficient is \( \frac{1}{n!} a_n k^n \).

Proposition 4.5. Let \((X, \omega)\) be a compact Kähler manifold with sectional curvature bounded from above by a negative constant, i.e.,
\[ \sec \leq -K, \]
for some \( K > 0 \). Suppose that there is a holomorphic line bundle \( L \) on \( X \) such that \( a_n := \int_X c_1^n(L) \neq 0 \). If \( c_n K \geq CC^\pm \), then there exists an integer \( \tilde{m} \in [-\frac{c_n K - CC^\pm}{nC}, \frac{c_n K - CC^\pm}{nC}] \) such that either
\[ (-1)^{n-p} \chi^p(X, L^{\otimes \tilde{m}}) \geq 2|a_n|\left(\left|\frac{c_n K - CC^\pm}{2Cn}\right|\right)^n + 1 \]
or
\[ (-1)^{n-p} \chi^p(X) \geq 2|a_n|\left(\left|\frac{c_n K - CC^\pm}{2Cn}\right|\right)^n + 1. \]
Proof. For any $K > 0$, there are integers $N^+, N$ such that

$$Cn(N + 1) > c_nK - CC^\pm \geq C(nN).$$

and

$$N^+ + 1 > C^\pm \geq N^+.$$

Since $c_nK \geq CC^\pm$, $N \geq 0$. Therefore, we have

$$c_nK \geq C(nN + N^+) \geq |\langle i\Theta(L^{\otimes nN+\pm}), \Lambda \rangle|.$$

Notice that $\text{sign}(a_n)(P_n^{(p)}(m, L) - \chi^p(X))$ is positive integer for any $m > N^+$ and $\text{sign}(a_n)(P_n^{(p)}(m, L) - \chi^p(X))$ is negative (resp. positive) integer for any $m < -N^-$ when $n$ is odd (resp. even). Following the way in Lemma 4.4, then there is a integer $N^+ \leq \tilde{m} \leq nN + N^+$ (resp. $-nN - N^- \leq \tilde{m} \leq -N^-$) such that

$$\text{sign}(a_n)(P_n^{(p)}(\tilde{m}, L) - \chi^p(X)) \geq |a_n|n^{\n/2^{n-1}} (\text{resp.} \ (-1)^n\text{sign}(a_n)P_n(\tilde{m}) \geq |a_n|n^{\n/2^{n-1}}).$$

Following Theorem 1.2 for any $|m| \leq (nL + N^\pm)$, we get

$$(-1)^{n-p}\chi^p(X, L^{\otimes m}) = (-1)^{n-p}\chi^p(X) + (-1)^{n-p}(P_n^{(p)}(m, L) - \chi^p(X)) \geq 1$$

and

$$(-1)^{n-p}\chi^p(X) = (-1)^{n-p}\chi^p(X, L^{\otimes m}) + (-1)^{n-p+1}(P_n^{(p)}(m, L) - \chi^p(X)) \geq 1.$$}

We then have

$$\max\{|(-1)^{n-p}\chi^p(X), (-1)^{n-p}\chi^p(X, L^{\otimes \tilde{m}})\} \geq 1 + |P_n^{(p)}(\tilde{m}, L) - \chi^p(X)|$$

$$\geq 1 + |a_n|n^{\n/2^{n-1}}$$

$$\geq 2|a_n|(|\frac{c_nK - CC^\pm}{2Cn}|)^n + 1.$$

We denote by

$$Z_p := \{m \in \mathbb{R} : P_n^{(p)}(m, L) = \chi^p(X)\}$$

the set of real roots of polynomial $P_n^{(p)}(m, L) - \chi^p(X)$. We denote

$$m_p(L) = \max_{m \in Z_p} |m|,$$

(when $Z = \emptyset$, we denote $m_p(L) = 0$). It’s also easy to see $m_p(L)$ depends on $K$ and $c_1(L)$. 


Proof of Theorem 1.9 There exists an integer $N$ such that

$$Cn(N + 1) > c_n K - Cm_p(L) \geq CnN.$$  

If $N \leq 0$, then $(-1)^{n-p} \chi^p(X) \geq 1$.

If $N > 0$, we then have

$$c_n K \geq C(nN + m_p(L)) \geq \left| [i\Theta(L^\otimes(nN + |m_p(L)|)), \Lambda] \right|.$$  

Following Theorem 1.2 for any $|m| \leq (nN + |m_p(L)|)$, we get

$$(-1)^{n-p} \chi^p(X, L^\otimes m) \geq 1,$$

i.e.,

$$(-1)^{n-p} \chi^p(X) \geq (-1)^{n-p+1}(P^{(p)}_n(m, L) - \chi^p(X)) + 1.$$  

When $n$ is odd, we get $\text{sign}(a_n)(P^{(p)}_n(m, L) - \chi^p(X))$ is positive integer for any $m > m_p(L)$ and $\text{sign}(a_n)(P^{(p)}_n(m, L) - \chi^p(X))$ is negative integer for any $m < -m_p(L)$. Therefore, following Lemma 4.4 we get

$$(-1)^{n-p} \chi^p(X) \geq \max_{nN + |m_p(L)| \geq |m| > m_p(L)} (-1)^{n-p+1}(P^{(p)}_n(m, L) - \chi^p(X)) + 1 \geq 1 + |a_n|N^n/2^{n-1}.$$  

When $n$ is even, we get $\text{sign}(a_n)(P^{(p)}_n(m, L) - \chi^p(X))$ is positive integer for any $|m| > m_p(L)$. Following Proposition 4.5 there exists an integer $\tilde{m} \in [-nN - |m_p(L)|, nN + |m_p(L)|]$ such that either

$$(-1)^{n-p} \chi^p(X, L^\otimes \tilde{m}) \geq 1 + |a_n|N^n/2^{n-1}$$

or

$$(-1)^{n-p} \chi^p(X) \geq 1 + |a_n|N^n/2^{n-1}.$$  

Therefore, for all cases, we get

$$(-1)^n \chi(X) \geq \max\{n + 1, n + 1 + 2|a_n|\text{sign}(\lfloor c_n K - Cm_p(L) \rfloor) |\lfloor \frac{c_n K - Cm_p(L)}{2Cn} \rfloor|^n\}.$$  

\[\square\]

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