REDUCED AND NON-REDUCED LINEAR SPACES:
LINES AND POINTS

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Abstract. In this paper we consider the problem of determining
the Hilbert function of schemes $X \subset \mathbb{P}^n$ which are the generic
union of $s$ lines and one $m$-multiple point. We completely solve
this problem for any $s$ and $m$ when $n \geq 4$. When $n = 3$ we find
several defective such schemes and conjecture that they are the
only ones. We verify this conjecture in several cases.

1. Introduction

If $P$ is a point in $\mathbb{P}^n$ with corresponding ideal $I_P \subset R = k[x_0, \ldots, x_n]$
(k algebraically closed of characteristic zero), the scheme supported
on $P$ and defined by the ideal $(I_P)^m$ is called an \textit{m-multiple point}
with support $P$. In a remarkable paper \cite{AH95} J. Alexander and A.
Hirschowitz found the Hilbert function of a finite union of 2-multiple
points supported on a generic set of points in $\mathbb{P}^n$ (see also \cite{Cha01}
and \cite{BO08} for simpler proofs). This result permitted Alexander and
Hirschowitz to solve the long open problem regarding the dimensions of
the (higher) secant varieties of the Veronese varieties (see \cite{Ger96, IK99}
for an expository discussion of this important result). In a subsequent
paper \cite{CGG05} the authors showed that, in an analogous way (using
the Lemma of Terracini) one can find the dimensions of the (higher)
secant varieties to Segre embeddings of products of projective spaces,
if one could calculate the Hilbert functions of certain unions of reduced
and non-reduced schemes supported on unions of generic linear spaces
of different dimensions (for more details see Theorem 1.1. in \cite{CGG05}).
The study of such schemes is one of the principal motivations for our
work in this paper.

There is also other closely related research in the literature, e.g. some
authors have considered the problem of finding the Hilbert function of
generic $m$-multiples points in $\mathbb{P}^2$ (see the survey \cite{Min99} and \cite{CCMO03},
\cite{HR04}, \cite{Yan07}) as well as of generic $m$-multiple points in $\mathbb{P}^n$ with
$n > 2$ (see \cite{LU06}). Moreover, Hartshorne and Hirschowitz considered
the same problem for a generic union of (reduced) lines in $\mathbb{P}^n$ ($n > 2$).
In this paper we consider yet another variant of this family of problems: namely the case in which the scheme \( X \subset \mathbb{P}^n (n \geq 3) \) is composed of \( s \) generic (reduced) lines and one generic \( m \)-multiple point. A simple parameter count leads one to expect that the Hilbert function of such an \( X \), \( HF(X, \cdot) \), is

\[
HF(X, d) = \min \left\{ \binom{d + n}{n}, \binom{m + n - 1}{n} + s(d + 1) \right\}. \tag{*}
\]

If we let \( hp(X, \cdot) \) denote the Hilbert polynomial of \( X \), then (\( * \)) is really saying that

\[
HF(X, d) = \min \{ hp(\mathbb{P}^n, d), hp(X, d) \},
\]
equivalently

\[
\dim(I_X)_d = \max \left\{ \binom{d + n}{n} - \binom{m + n - 1}{n} - s(d + a), 0 \right\}.
\]

Note that in this case we say that the Hilbert function of \( X \) is bipolynomial (see also \[CCG10\] for other examples of this).

We prove (\( * \)) (see Theorem 3.2) for any \( s \) and \( m \) when \( n \geq 4 \). When \( n = 3 \), the situation is less clear. In particular, the “simple parameter count” no longer always gives the actual Hilbert function (the precise statement is given in Theorem 4.2). We conjecture that the parameter count fails (for \( n = 3 \)) if and only if \( m = d \) and \( 1 < s \leq d \). In these cases we show that \( \dim(I_X)_d = \binom{d-s+2}{2} \).

2. Basic facts and notation

Since we will make use of Castelnuovo’s inequality several times, we recall it here in a form more suited to our use (for notation and proof we refer to \[AH95\], Section 2).

**Definition 2.1.** If \( X, Y \) are closed subschemes of \( \mathbb{P}^n \), we denote by \( Res_Y X \) the scheme defined by the ideal \((I_X : I_Y)\) and we call it the residual scheme of \( X \) with respect to \( Y \), we denote by \( Tr_Y X \subset Y \) the schematic intersection \( X \cap Y \), and call it the trace of \( X \) on \( Y \). We also denote by \( X + Y \) the schematic union of \( X \) and \( Y \).

**Lemma 2.2. (Castelnuovo’s inequality):** Let \( d, \delta \in \mathbb{N}, d \geq \delta, \) let \( Y \subset \mathbb{P}^n \) be a smooth hypersurface of degree \( \delta \), and let \( X \subset \mathbb{P}^n \) be a closed subscheme. Then

\[
\dim(I_{X, \mathbb{P}^n})_d \leq \dim(I_{Res_Y X, \mathbb{P}^n})_{d-\delta} + \dim(I_{Tr_Y X, Y})_d.
\]

\[\Box\]
The following lemma gives a criterion for adding to a scheme \( X \subseteq \mathbb{P}^n \) a set of reduced points lying on a projective variety \( Y \) and imposing independent conditions to forms of a given degree in the ideal of \( X \) (see also [CCG10, Lemma 2.2]).

**Lemma 2.3.** Let \( d \in \mathbb{N} \) and let \( X \subseteq \mathbb{P}^n \) be a closed subscheme. Let \( Y \subseteq \mathbb{P}^n \) be a closed reduced irreducible subscheme, and let \( P_1, \ldots, P_s \) be generic points on \( Y \). If \( \dim(I_X)_d = s \) and \( \dim(I_{X+Y})_d = 0 \), then \( \dim(I_{X+P_1+\ldots+P_s})_d = 0 \).

**Proof.** By induction on \( s \).

Since \((I_{X+Y})_d = (I_X)_d \cap (I_Y)_d = (0)\) and \( \dim(I_X)_d = s > 0 \), let \( f \in (I_X)_d, f \notin (I_Y)_d \). Therefore there exists \( P \in Y, P \notin X \) such that \( f(P) \neq 0 \). It follows that \( \dim(I_{X+P})_d = s - 1 \) and thus the same holds for a generic point \( P_1 \in Y \). So we are done in case \( s = 1 \).

Let \( s > 1 \) and let \( X' = X + P_1 \). Obviously \( \dim(I_{X'+Y})_d = 0 \). Hence, by the inductive hypothesis, there exist \( s - 1 \) distinct generic points \( P_2, \ldots, P_s \) in \( Y \) such that \( \dim(I_{X'+P_2+\ldots+P_s})_d = \dim(I_{X+P_1+\ldots+P_s})_d = 0 \).

**Definition 2.4.** We say that \( C \) is a degenerate conic if \( C \) is the union of two intersecting lines \( L, M \). In this case we write \( C = L + M \).

**Definition 2.5.** Let \( L \) and \( M \) be two intersecting lines in \( \mathbb{P}^n \) (\( n \geq 3 \)), let \( P = L \cap M \), and let \( T \simeq \mathbb{P}^3 \) be a generic linear space containing the scheme \( L + M \). We call the scheme \( L + M + 2P |_T \) a degenerate conic with an embedded point or a 3-dimensional sundial (see [HH82], or [CCG10, definition 2.6 with \( m = 1 \)]).

The following lemma shows that a 3-dimensional sundial in \( \mathbb{P}^n \) is a degeneration of two generic lines in \( \mathbb{P}^n \) (see [HH82] for the case \( n = 3 \), and see [CCG10, Lemma 2.5] for the proof in a more general case).

**Lemma 2.6.** Let \( X_1 \subset \mathbb{P}^n \) (\( n \geq 3 \)) be the disconnected subscheme consisting of two skew lines \( L_1 \) and \( M \) (so the linear span of \( X_1 \) is \( < X_1 > \simeq \mathbb{P}^3 \)). Then there exists a flat family of subschemes

\[ X_\lambda \subset < X_1 > \quad (\lambda \in k) \]

whose generic fiber is the union of two skew lines and whose special fiber \( X_0 \) is the union of

- the line \( M \),
- a line \( L \) which intersects \( M \) in a point \( P \),
- the scheme \( 2P |_{< X_1 >} \), that is, the schematic intersection of the double point \( 2P \) of \( \mathbb{P}^n \) and \( < X_1 > \).
Moreover, if $H \cong \mathbb{P}^2$ is the linear span of $L$ and $M$, then $\text{Res}_H(X_0)$ is given by the (simple) point $P$.

\[ \square \]

Remark 2.7. Since it is easy to see that in $\mathbb{P}^n \ (n \geq 3)$ a 3-dimensional sundial is also a degeneration of two intersecting lines and a simple generic point which moves toward the intersection point of the two lines, by the lemma above we get that in $\mathbb{P}^n \ (n \geq 3)$ a degenerate conic with an embedded point can be viewed either as a degeneration of two generic lines, or as a degeneration of a scheme which is the union of a degenerate conic and a simple generic point.

Inasmuch as we have upper semicontinuity of the Hilbert function in a flat family, we will use the remark above several times in what follows.

Now an easy, but useful Lemma.

Lemma 2.8. Let $X \subset \mathbb{P}^n$.

(i) If $X = X_1 + \cdots + X_s$ is the union of non-intersecting closed subschemes $X_i$, if $X' = X_1 + \cdots + X_{s'} \subset X$, where $s' < s$, and if $HF(X, d) = \sum_{i=1}^{s} HF(X_i, d)$, then

\[ HF(X', d) = \sum_{i=1}^{s'} HF(X_i, d). \]

(ii) If $X = Y + mP$ is the union of a closed subscheme $Y$ and one $m$-multiple point, if $X' = Y + m'P \subset X$, where $m' < m$, and if $HF(X, d) = HF(Y, d) + \binom{m+n-1}{n}$, then

\[ HF(X', d) = HF(Y, d) + \binom{m' + n - 1}{n}. \]

(iii) If $\dim(I_X)_d = 0$, then $\dim(I_{X''})_d = 0$, for any subscheme $X'' \supset X$.

Proof. (i)

\[ HF(X, d) = \sum_{i=1}^{s} HF(X_i, d) = \sum_{i=1}^{s'} HF(X_i, d) + \sum_{i=s'+1}^{s} HF(X_i, d) \]

\[ \geq HF(X', d) + \sum_{i=s'+1}^{s} HF(X_i, d) \geq HF(X, d). \]

Hence the inequalities are equalities, and we get the conclusion.
(ii) Since \( HF(X, d) = HF(Y, d) + \binom{m+n-1}{n} \), and \( I_X = I_Y \cap I_{mP} \), from the exact sequence
\[
0 \longrightarrow R/I_Y \cap I_{mP} \longrightarrow R/I_Y \oplus R/I_{mP} \longrightarrow R/(I_Y + I_{mP}) \longrightarrow 0,
\]
we get that \( \dim(R/(I_Y + I_{mP}))_d = 0 \) and \( HF(mP, d) = \binom{m+n-1}{n} \). It follows that \( \dim(R/(I_Y + I_{m'P}))_d = 0 \) and \( HF(m'P, d) = \binom{m'+n-1}{n} \).

(iii) Obvious.

\[ \square \]

Lemma 2.9. Let \( X = X_1 + \cdots + X_s \subset \mathbb{P}^n \) be the union of non intersecting closed subschemes \( X_i \), let \( s' < s \) and \( X' = X_1 + \cdots X_{s'} \subset X \).

(i) If \( \dim(I_X)_d = \binom{d+n}{n} - \sum_{i=1}^s HF(X_i, d) \) (the expected value), then also \( \dim(I_{X'})_d \) is as expected, that is
\[
\dim(I_{X'})_d = \binom{d+n}{n} - \sum_{i=1}^{s'} HF(X_i, d).
\]

(ii) If \( \dim(I_X)_d = 0 \), then \( \dim(I_{X''})_d = 0 \), for any subscheme \( X'' \supset X' \).

We now recall the basic theorem of Hartshorne and Hirschowitz about the Hilbert function of generic lines.

Theorem 2.10. \cite{HH82} Theorem 0.1] Let \( n, d \in \mathbb{N} \). For \( n \geq 3 \), the ideal of the scheme \( X \subset \mathbb{P}^n \) consisting of \( s \) generic lines has the expected dimension, that is,
\[
\dim(I_X)_d = \max \left\{ \binom{d+n}{n} - s(d+1); 0 \right\},
\]
or equivalently
\[
HF(X, d) = \min \left\{ hp(\mathbb{P}^n, d) = \binom{d+n}{n}, hp(X, d) = s(d+1) \right\},
\]
that is, \( X \) has bipolynomial Hilbert function.

\[ \square \]

To be more precise the following equivalent statement is the actual theorem proved in \cite{HH82}:
Theorem 2.11. [HH82 Theorem 0.2] Let \( n, d \in \mathbb{N} \). Let
\[
t = \left\lfloor \frac{(d+n)}{d+1} \right\rfloor; \quad r = \binom{d+n}{n} - t(d+1),
\]
and let \( L_1, \ldots, L_{t+1} \) be \( t+1 \) generic lines in \( \mathbb{P}^n \). For \( n \geq 3 \), the ideal of the scheme \( X \subset \mathbb{P}^n \) consisting of the \( t \) lines \( L_1, \ldots, L_t \) and \( r \) generic points lying on \( L_{t+1} \) has the expected dimension, that is,
\[
\dim(I_X)_d = 0.
\]

\( \square \)

Remark 2.12. By Lemma 2.8, the statement of Theorem 2.11 easily implies the one of Theorem 2.10; moreover, by Lemma 2.3, it is easy to prove that also the converse holds.

We now recall the following technical result, we refer the reader to [CCG11] for a proof.

Theorem 2.13. Let \( n \geq 3 \) and let \( X \subset \mathbb{P}^n \) be the union of \( s \) generic 3-dimensional sundials and \( l \) generic lines. Then \( X \) has bipolynomial Hilbert function, that is,
\[
HF(X, d) = \min \left\{ \left( \binom{d+n}{n}; (d+1)(2s+l) \right) \right\}.
\]

Equivalently, the following schemes have the expected Hilbert Function in degree \( d \):
\[
W = \begin{cases} 
\hat{C}_1 + \cdots + \hat{C}_s + P_1 + \cdots + P_r & \text{for } t \text{ even} \\
\hat{C}_1 + \cdots + \hat{C}_s + M + P_1 + \cdots P_r & \text{for } t \text{ odd}
\end{cases}
\]
\[
T = \begin{cases} 
\hat{C}_1 + \cdots + \hat{C}_s + M & \text{for } t \text{ even and } r > 0 \\
\hat{C}_1 + \cdots + \hat{C}_{s+1} & \text{for } t \text{ odd and } r > 0
\end{cases}
\]

where
\[
t = \left\lfloor \frac{(d+n)}{d+1} \right\rfloor, \quad r = \binom{d+n}{n} - t(d+1), \quad s = \left\lfloor \frac{t}{2} \right\rfloor,
\]

where the \( \hat{C}_i \) are degenerate conics with an embedded point, that is 3-dimensional sundials, the \( P_i \) are generic points and \( M \) is a generic line, that is,
\[
\dim(I_W)_d = \exp \dim(I_W)_d = \binom{d+n}{n} - t(d+1) - r = 0;
\]
\[
\dim(I_T)_d = \exp \dim(I_W)_d = \max \left\{ \binom{d+n}{n} - (t+1)(d+1) ; 0 \right\} = 0.
\]
3. The main theorem in $\mathbb{P}^n$, for $n \geq 4$

In this section we will prove (see Theorem 3.2) that for $n \geq 4$, the ideal of the scheme $X \subset \mathbb{P}^n$ consisting of $s$ generic lines and a generic point of multiplicity $m$ has the expected dimension. We start with the following proposition, which, for $m \leq d$, is equivalent to Theorem 3.2 (see Remark 2.12 for an analogous situation).

**Proposition 3.1.** Let $n, d, m \in \mathbb{N}$, $n \geq 4$, $m \leq d$. Let

$$e = \left\lfloor \frac{(d+n) - \binom{m+n-1}{n}}{d+1} \right\rfloor; \quad r = \binom{d+n}{n} - \binom{m+n-1}{n} - e(d+1).$$

The ideal of the scheme $X \subset \mathbb{P}^n$ consisting of $e$ generic lines $L_1, \ldots, L_e$, $r$ generic points $P_1, \ldots, P_r$ lying on a generic line $L$ and a generic point $P$ of multiplicity $m$ has the expected dimension, that is,

$$\dim(I_X)_d = \binom{d+n}{n} - \binom{m+n-1}{n} - e(d+1) - r = 0.$$

**Proof.** We will prove the theorem by induction on $d - m$.

Let $d = m$. Since for $d = m$ any form of degree $d$ in $I_X$ represents a cone with $P$ as vertex, it follows that

$$\dim(I_X)_d = \dim(I_W)_d,$$

where $W \subset \mathbb{P}^{n-1}$ consists of $e$ generic lines and $r$ generic points lying on a line. Since for $d = m$ we have $\binom{d+n}{n} - \binom{m+n-1}{n} = \binom{d+n-1}{n-1}$, we get

$$e = \left\lfloor \frac{(d+n-1)}{d+1} \right\rfloor; \quad r = \binom{d+n-1}{n-1} - e(d+1).$$

So by Theorem 2.11 we get

$$\dim(I_W)_d = \binom{d+n-1}{n-1} - e(d+1) - r = 0,$$

and we are done for $m = d$.

Assume $m < d$. Let

$$e' = \left\lfloor \frac{(d-1+n) - \binom{m+n-1}{n} - r}{d} \right\rfloor;$$

$$r' = \binom{d-1+n}{n} - \binom{m+n-1}{n} - r - e'd.$$

Since $(d-1+n) - \binom{m+n-1}{n} - r \geq 0$, we have $e' \geq 0$ (see the Appendix, Lemma 5.1 (i)).
Notice that $e - e' - 2r' \geq 0$ (this inequality is treated in the Appendix, Lemma 5.1 (ii)). Using this inequality we construct a scheme $Y$ obtained from $X$ by specializing some lines and by degenerating other pairs of lines into a hyperplane $H \simeq \mathbb{P}^{n-1}$.

More precisely, we specialize $e-e'-2r'$ lines into $H$ and we degenerate $r'$ pairs of lines in order to obtain the following specialization of $X$:

$$Y = \tilde{C}_1 + \cdots + \tilde{C}_{e'} + M_1 + \cdots + M_{e-e'-2r'} + L_1 + \cdots + L_{e'} + mP + P_1 + \cdots + P_r,$$

where the $M_i \subset H$ are generic lines and the $\tilde{C}_i \subset H_i \simeq \mathbb{P}^3$ are 3-dimensional sundials such that $\tilde{C}_i$ is the union of a degenerate conic $C_i$ lying on $H$ and a double point $2Q_i|H \ns \not\subset H$.

So we have

$$\text{Res}_H Y = Q_1 + \cdots + Q_{e'} + L_1 + \cdots + L_{e'} + mP + P_1 + \cdots + P_r \subset \mathbb{P}^n,$$

$$\text{Tr}_H Y = C_1 + \cdots + C_{e'} + M_1 + \cdots + M_{e-e'-2r'} + T_1 + \cdots + T_{e'} \subset H \simeq \mathbb{P}^{n-1},$$

where $T_i = L_i \cap H$ and the $T_i$ are generic points.

Since $e' \geq r'$ (this inequality is proved in the Appendix, Lemma 5.1 (iii)) and $r' \leq d-1$, by Remark 2.7, Lemma 2.8 and Theorem 2.13 we get that the dimension of $\dim(I_{\text{Tr}_H Y})$ is as expected, that is,

$$\dim(I_{\text{Tr}_H Y}) = \left(\frac{d + n - 1}{n - 1}\right) - r'(2d + 1) - (e - e' - 2r')(d + 1) - e' = \left(\frac{d + n - 1}{n - 1}\right) - (e - e')(d + 1) + r' - e' = \left(\frac{d + n - 1}{n - 1}\right) - e(d + 1) + \left(\frac{d - 1 + n}{n}\right) - \left(\frac{m + n - 1}{n}\right) - r = \left(\frac{d + n - 1}{n - 1}\right) - e(d + 1) + \left(\frac{d - 1 + n}{n}\right) - \left(\frac{d + n}{n}\right) + e(d + 1) = 0.$$
and
\[ \dim(I_{Y_1 + L})_{d-1} = \max\{r - d; 0\} = 0. \]

Hence, by Lemma 2.3 we get
\[ \dim(I_{\text{Res}_L Y})_{d-1} = 0. \]

Now, since \( \dim(I_{\text{Tr}_Y H}) = \dim(I_{\text{Res}_L Y})_{d-1} = 0 \), by Castelnuovo’s Inequality (see Lemma 2.2) the conclusion follows.

\[ \square \]

**Theorem 3.2.** Let \( n, d, s, m \in \mathbb{N} \). For \( n \geq 4 \), the ideal of the scheme \( X \subset \mathbb{P}^n \) consisting of \( s \) generic lines and a generic point \( P \) of multiplicity \( m \) has the expected dimension, that is,
\[ \dim(I_X)_d = \max \left\{ \left( d + \frac{n}{n} \right) - \left( \frac{m + n - 1}{n} \right) - s(d + 1), 0 \right\}. \]

**Proof.** Obvious for \( m > d \). For \( m \leq d \) the conclusion follows from Proposition 3.1 and Lemma 2.8.

\[ \square \]

4. The main theorem in \( \mathbb{P}^3 \)

**Proposition 4.1.** Let \( d \in \mathbb{N}, d \geq 3 \). Let
\[ e = \left\lfloor \left( \frac{d+3}{3} \right) - 4 \right\rfloor; \quad r = \left( \frac{d+3}{3} \right) - 4 - e(d + 1). \]

The ideal of the scheme \( X \subset \mathbb{P}^3 \) consisting of \( e \) generic lines \( L_1, \ldots, L_e \), \( r \) generic points \( P_1, \ldots, P_r \), and a generic double point supported on \( P \) has the expected dimension, that is,
\[ \dim(I_X)_d = \left( \frac{d+3}{3} \right) - 4 - e(d + 1) - r = 0. \]

**Proof.** We will prove the theorem by induction on \( d \).

For \( d = 3 \) we have \( e = 4, r = 0 \) so
\[ X = 2P + L_1 + \cdots + L_4. \]

Since the trace of \( X \) on the plane \( < P, L_i > \) is formed by the line \( L_i \), one double point and three simple points, then the surfaces defined by the forms of degree 3 in \( I_X \) have the plane \( < P, L_i > \) as a fixed component. But the four planes \( < P, L_i > \) cannot be fixed components for a surface of degree 3. It follows that \( \dim(I_X)_3 = 0. \)
For $d = 4$ we have $e = 6$, $r = 1$ so
$$X = 2P + L_1 + \cdots + L_6 + P_1.$$ 
Now we degenerate the scheme $X$: first we degenerate the lines $L_1$ and $L_2$, so that they become a 3-dimensional sundial $\hat{C}$, then we specialize the line $L_3$ on the plane $H = \langle P, R, P_1 \rangle$, where $R$ is the double point of $\hat{C}$. Let
$$\tilde{X} = 2P + \hat{C} + L_3 + \cdots + L_6 + P_1$$ 
be the degenerate scheme.

The trace of $\tilde{X}$ on the plane $H$ is
$$Tr_H \tilde{X} = 2P|_H + 2R|_H + L_3 + P_1 + (L_4 + L_5 + L_6) \cap H \subset H \simeq \mathbb{P}^2,$$

hence
$$\dim(I_{Tr_H \tilde{X}})_4 = \dim(I_{Tr_H \tilde{X} - L_3})_3.$$ 
Since $(Tr_H \tilde{X} - L_3)$ is the union of two double points and four simple points, it follows that $\dim(I_{Tr_H \tilde{X}})_4 = 0$. So $H$ is a fixed component for the forms of $(I_{\tilde{X}})_4$, and we have
$$\dim(I_{\tilde{X}})_4 = \dim(I_{Res_H \tilde{X}})_3,$$
where $Res_H \tilde{X}$ is the union of three lines, a point and a degenerate conic $C$, say
$$Res_H \tilde{X} = P + C + L_4 + L_5 + L_6.$$ 

Now, if we degenerate $P$ and $C$, we obtain again the sundial $\hat{C}$, so, by Theorem 2.13 we have
$$\dim(I_{Res_H \tilde{X}})_3 = 0,$$
and from here we get $\dim(I_{X})_4 = 0$.

Now let $d \geq 5$. Let $Q$ be a smooth quadric: we will specialize some of the lines of the scheme $X$ on $Q$. We consider three cases.

**Case 1:** $d \equiv 0 \mod 3$.
Let $d = 3h$. Note that, since $d \geq 5$, then $h \geq 2$. We have:
$$e = \frac{(h + 1)(3h + 2)}{2} - 1, \quad r = 3(h - 1) \geq 3,$$
$$X = 2P + L_1 + \cdots + L_e + P_1 + \cdots + P_r.$$ 
Let $\tilde{X}$ be the scheme obtained from $X$ by specializing $2h + 1$ lines in such a way that the lines $L_1, \ldots, L_{2h+1}$ become lines of the same ruling on $Q$, (the lines $L_{2h+2}, \ldots, L_e$ remain generic lines, not lying on $Q$),
and by specializing on $Q$ the points $P_1$ and $P_2$. We have

$$\text{Res}_Q \tilde{X} = 2P + L_{2h+2} + \ldots + L_e + P_3 + \ldots + P_r.$$ 

By the inductive hypothesis we have:

$$\dim(I_{\text{Res}_Q \tilde{X}})_{d-2} = \binom{3h+1}{3} - 4 - (e-2h-1)(3h-1) - (r-2)$$

$$= \frac{h(3h+1)(3h-1)}{2} - 4 - \frac{(h+1)(3h-2)}{2}(3h-1) - (3h-5) = 0.$$ 

Now

$$\text{Tr}_Q \tilde{X} = L_1 + \ldots + L_{2h+1} + \text{Tr}_Q(L_{2h+2} + \ldots + L_e) + P_1 + P_2.$$ 

Since the trace on $Q$ of the $(e-2h-1)$ lines $L_{2h+2}, \ldots, L_e$ consists of $2(e-2h-1)$ generic points, we have that $\text{Tr}_Q \tilde{X}$ consists of $(2h+1)$ lines of the same ruling, and $(2e-4h)$ generic points. Thinking of $Q$ as $\mathbb{P}^1 \times \mathbb{P}^1$, we see that the forms of degree $3h$ in the ideal of $\text{Tr}_Q \tilde{X}$ are curves of type $(3h-(2h+1),3h) = (h-1,3h)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ passing through $(2e-4h)$ generic points. Hence

$$\dim(I_{\text{Tr}_Q \tilde{X}})_{3h} = h(3h+1) - 2e + 4h = 0.$$ 

So by Lemma 2.2 and by the semicontinuity of the Hilbert function we get $\dim(I_X)_{3h} = 0$.

**Case 2: $d \equiv 2 \mod 3$.**

For computation of this case, recall that we will think of $Q$ as $\mathbb{P}^1 \times \mathbb{P}^1$ and that (see, for instance, [CGG05, Section 2]) in the case we are treating each of the double points on $Q$ will give three independent condition to our forms.

Let $d = 3h + 2$. We have:

- For $h = 1$: $d = 5$; $e = 8$; $r = 4$;
- For $h = 2$: $d = 8$; $e = 17$; $r = 8$;
- For $h \geq 3$: $d = 3h + 2$; $e = \frac{3(h+1)(h+2)}{2}$; $r = h - 3$.

For $h = 1$, we have

$$X = 2P + L_1 + \ldots + L_8 + P_1 + \ldots + P_4.$$ 

Specialize the scheme $X$ in such a way that the lines $L_1, \ldots, L_4$ become lines of the same ruling on $Q$, and the points $P$ and $P_1$ become points on $Q$. We get

$$\text{Res}_Q \tilde{X} = P + L_5 + \ldots + L_8 + P_2 + \ldots + P_4,$$

$$\text{Tr}_Q \tilde{X} = 2P|_Q + L_1 + \ldots + L_4 + \text{Tr}_Q(L_5 + \ldots + L_8) + P_1.$$
and we easily get
\[ \dim(I_{\text{Res}_Q \tilde{X}})_3 = 20 - 1 - 16 - 3 = 0, \]
\[ \dim(I_{\text{Tr}_Q \tilde{X}})_5 = 12 - 3 - 8 - 1 = 0. \]
For \( h = 2 \), we have
\[ X = 2P + L_1 + \cdots + L_{17} + P_1 + \cdots + P_8. \]
Specialize the scheme \( X \) so that the lines \( L_1, \ldots, L_6 \) become lines of the same ruling on \( Q \), and the points \( P, P_1 \) and \( P_2 \) become points on \( Q \). We get
\[ \text{Res}_Q \tilde{X} = P + L_7 + \cdots + L_{17} + P_3 + \cdots + P_8, \]
\[ \text{Tr}_Q \tilde{X} = 2P|_Q + L_1 + \cdots + L_6 + \text{Tr}_Q(L_7 + \cdots + L_{17}) + P_1 + P_2, \]
and we have
\[ \dim(I_{\text{Res}_Q \tilde{X}})_6 = 84 - 1 - 77 - 6 = 0, \]
\[ \dim(I_{\text{Tr}_Q \tilde{X}})_8 = 27 - 3 - 22 - 2 = 0. \]
For \( h \geq 3 \), we have
\[ X = 2P + L_1 + \cdots + L_e + P_1 + \cdots + P_{h-3}. \]
Now we degenerate the lines \( L_1 \) and \( L_2 \), so that they become a 3-dimensional sundial \( \tilde{C} = C + 2R \), where \( C \) is a degenerate conic and \( 2R \) is a double point, then we specialize the points \( R, P, P_1 \ldots P_{h-3} \) so that they become points on \( Q \), and the lines \( L_3, \ldots, L_{2h+4} \) so that they become lines of the same ruling on \( Q \). Let \( \tilde{X} \) be the specialized scheme. We have
\[ \text{Res}_Q \tilde{X} = P + C + L_{2h+5} + \cdots + L_e, \]
and, by Remark 2.7 we get
\[ \dim(I_{\text{Res}_Q \tilde{X}})_3h = \binom{3h + 3}{3} - 2(3h + 1) - (e - 2h - 4)(3h + 1) = 0. \]
Moreover
\[ \text{Tr}_Q \tilde{X} = 2P|_Q + 2R|_Q + L_3 + \cdots + L_{2h+4} + \text{Tr}_Q(L_{2h+5} + \cdots + L_e) + P_1 + \cdots + P_{h-3}, \]
and we get
\[ \dim(I_{\text{Tr}_Q \tilde{X}})_{3h+2} = (h+1)(3h+3) - 3 - 3 - 2 - 2(e - 2h - 5 + 1) - (h - 3) = 0. \]
So by Lemma 2.2 and by the semicontinuity of the Hilbert function we get \( \dim(I_X)_{3h+2} = 0. \)
Case 3: \( d \equiv 1 \mod 3 \).

Let \( d = 3h + 1 \). Note that \( h \geq 2 \). We have:

\[
e = \frac{(h+1)(3h+4)}{2} - 1; \quad r = 3h - 2.
\]

Specialize the scheme \( X \) in such a way that the lines \( L_1, \ldots, L_{2h+1} \) become lines of the same ruling on \( Q \), and the points \( P \) and \( P_1, \ldots, P_{2h-1} \) become points on \( Q \). Let \( \tilde{X} \) be the specialized scheme.

So

\[
\text{Res}_Q \tilde{X} = P + L_{2h+2} + \ldots + L_e + P_{2h} + \ldots + P_{3h-2};
\]

and by Theorem \( 2.10 \) we have

\[
\dim(I_{\text{Res}_Q \tilde{X}})_{3h-1} = \left( \frac{3h+2}{3} \right) - 1 - 3h(e - 2h - 1) - (h - 1)
\]

\[
= \frac{h(3h+2)(3h+1)}{2} - 1 - \frac{9h^2(h+1)}{2} - h + 1 = 0.
\]

The trace of \( \tilde{X} \) on \( Q \) consists of the \( (2h+1) \) lines of the same ruling \( L_1, \ldots, L_{2h+1} \), the double point \( P \), the simple points \( P_1, \ldots, P_{2h-1} \), and the trace of the lines \( L_{2h+2}, \ldots, L_e \). As usual, thinking of \( Q \) as \( \mathbb{P}^1 \times \mathbb{P}^1 \), we see that the forms of degree \( 3h+1 \) in the ideal of \( T_{\text{Res}_Q \tilde{X}} \) are curves of type \( ((3h+1) - (2h+1), 3h+1) = (h, 3h+1) \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). Hence, since it is easy to prove that the double point \( P \) gives 3 independent conditions to our forms (see, for instance, [CGG05, Section 2]), we have

\[
\dim(I_{T_{\text{Res}_Q \tilde{X}}}x)_{3h+1} = (h+1)(3h+2) - 3 - (2h-1) - 2(e - 2h - 1) = 0.
\]

So also in this case, by Lemma \( 2.2 \) and by the semicontinuity of the Hilbert function, we get \( \dim(I_X)_{3h+1} = 0 \).

\hfill \Box

**Theorem 4.2.** Let \( d, s, m \in \mathbb{N}, \ d \geq 1 \). Let \( X \subset \mathbb{P}^3 \) be the scheme consisting of \( s \geq 1 \) generic lines and a generic point \( P \) of multiplicity \( m \geq 1 \).

(i) The ideal of \( X \subset \mathbb{P}^3 \) has the expected dimension, that is,

\[
\dim(I_X)_d = \exp\dim(I_X)_d = \max \left\{ \left( \frac{d+3}{3} \right) - \left( \frac{m+2}{3} \right) - s(d+1), 0 \right\},
\]

(a) for \( m > d \);
(b) for \( m = d \) and \( s > d \), or for \( m = d \) and \( s = 1 \);
(c) for \( m = d - 1 \);
(d) for \( m < d - 1 \) and \( 1 \leq s \leq m + 2 \);
(e) for \( m = 2 \), and \( d \geq 3 \);
(f) for $m = 1$.

(ii) For $m = d \geq 2$ and $2 \leq s \leq d$, the dimension of $(I_X)_d$ is

$$\dim(I_X)_d = \binom{d-s+2}{2} \neq \exp \dim(I_X)_d,$$

and the defect is:

$$\delta = \begin{cases} \binom{s}{2} & \text{for } s \leq \frac{d+2}{2}, \\ \binom{d-s+2}{2} & \text{for } \frac{d+2}{2} \leq s \leq d \end{cases}.$$

Proof. (i) (a) Obvious. We have $\dim(I_X)_d = \exp \dim(I_X)_d = 0$.

(i) (b) and (ii). If $m = d$ any form of degree $d$ in $I_X$ represents a cone whose vertex contains $P$. Hence

$$\dim(I_X)_d = \dim(I_X')_d,$$

where $X' \subset \mathbb{P}^2$ is the projection of $X$ from $P$ in a $\mathbb{P}^2$ and it is a scheme consisting of $s$ generic lines. Hence, for $s > d$, we immediately get $\dim(I_X)_d = 0$.

For $s \leq d$ we have

$$\dim(I_X)_d = \binom{d-s+2}{2}.$$

Since in this case the expected dimension of $(I_X)_d$ is

$$\exp \dim(I_X)_d = \max \left\{ \left( \frac{d+3}{3} \right) - \left( \frac{d+2}{3} \right) - s(d+1), 0 \right\}$$

$$= \begin{cases} \binom{d+2}{2} - s(d+1) & \text{for } s \leq \frac{d+2}{2} \\ 0 & \text{for } s \geq \frac{d+2}{2} \end{cases},$$

then for $s = 1$ we have $\dim(I_X)_d = \exp \dim(I_X)_d$, and so we are done with (i)(b).

For $2 \leq s \leq d$ the defect is

$$\dim(I_X)_d - \exp \dim(I_X)_d = \begin{cases} \binom{s}{2} & \text{for } s \leq \frac{d+2}{2} \\ \binom{d-s+2}{2} & \text{for } s \geq \frac{d+2}{2} \end{cases},$$

so we have proved (ii).

(i) (c). By induction on $d$. Obvious for $d = 1$, let $d > 1$.

Let

$$X = L_1 + \cdots + L_s + mP$$
be our scheme, where the $L_i$ are generic lines. Since $d = m + 1$, we have that
\[
exp \dim(I_X)_d = \max \left\{ \left( \frac{d+3}{3} \right) - \left( \frac{d+1}{3} \right) - s(d+1); 0 \right\}
\]
\[
= \max \left\{ (d+1)^2 - s(d+1); 0 \right\},
\]
hence it is enough to prove that $(I_X)_d$ has the expected dimension for $s = d + 1$, and the conclusion will follow from Lemma 2.8.

Let $H \cong \mathbb{P}^2$ be the plane though $P$ and $L_1$. The trace of $X$ on $H$ is
\[
Tr_H X = mP|_H + L_1 + R_2 + \cdots + R_{d+1},
\]
where $R_i = L_i \cap H$, and the $R_i$ are $d$ generic points on $H$.

Since $L_1$ is a fixed component for the curves defined by the forms of $I_{Tr_H X}$, we have
\[
\dim(I_{Tr_H X})_d = \dim(I_{Tr_H X - L_1})_{d-1} = \left( \frac{d+1}{2} \right) - \left( \frac{m+1}{2} \right) - d
\]
\[
= \left( \frac{d+1}{2} \right) - \left( \frac{d}{2} \right) - d = 0.
\]
It follow that $H$ is a fixed component for the forms of $(I_X)_d$, so
\[
\dim(I_X)_d = \dim(I_{Res_H X})_{d-1}
\]
where
\[
Res_H X = (m-1)P + L_2 + \cdots + L_s = (d-2)P + L_2 + \cdots + L_{d+1}.
\]
By the inductive hypothesis we get
\[
\dim(I_{Res_H X})_{d-1} = \left( \frac{d+2}{3} \right) - \left( \frac{d}{3} \right) - d^2 = 0,
\]
and we are done with (i) (c).

(i) (d). Since for $m = d - 1$, and $s = m + 2$ by (i) (c) we have
\[
\dim(I_X)_d = \left( \frac{d+3}{3} \right) - \left( \frac{m+2}{3} \right) - s(d+1),
\]
by Lemma 2.8 (i) and (ii) we get the conclusion.

(i)(e). Let $m = 2$ and $d \geq 3$. We have to prove that
\[
\dim(I_X)_d = \exp \dim(I_X)_d = \max \left\{ \left( \frac{d+3}{3} \right) - 4 - s(d+1), 0 \right\}.
\]
If \((d+3)/3 - 4 - s(d+1) \geq 0\), let
\[
e = \left\lfloor \frac{(d+3) - 4}{d+1} \right\rfloor; \quad r = \left(\frac{d+3}{3}\right) - 4 - e(d+1),
\]
and let \(P_1, \ldots, P_r\) be generic points.

By Proposition 4.1 we know that for \(s = e\)
\[
\dim(I_{X+P_1+\cdots+P_r})_d = 0,
\]
hence for \(s = e\) we have
\[
\dim(I_X)_d = r = \exp \dim(I_X)_d
\]
and now the conclusion follows from Lemma 2.8 (i).

Now let
\[
\left(\frac{d+3}{3}\right) - 4 - s(d+1) < 0.
\]

In this case we have
\[
s > \frac{(d+3) - 4}{(d+1)} = \begin{cases} 
\frac{(h+1)(3h+2)}{2} - \frac{4}{3h+1} & \text{for} \quad d = 3h \\
\frac{(h+1)(3h+4)}{2} - \frac{4}{3h+2} & \text{for} \quad d = 3h + 1 \\
\frac{3(h+1)(h+2)}{2} + \frac{h-3}{3h+3} & \text{for} \quad d = 3h + 2
\end{cases}
\]
that is,
\[
s \geq \begin{cases} 
\frac{(h+1)(3h+2)}{2} & \text{for} \quad d = 3h; \\
\frac{(h+1)(3h+4)}{2} & \text{for} \quad d = 3h + 1 \\
9 & \text{for} \quad d = 5 \\
18 & \text{for} \quad d = 8 \\
\frac{3(h+1)(h+2)}{2} + 1 & \text{for} \quad d = 3h + 2, \quad h \geq 3
\end{cases}
\]
Since


\[ t = \left\lceil \frac{(d+3)}{3(d+1)} \right\rceil = \begin{cases} 
\frac{(h+1)(3h+2)}{2} & \text{for } d = 3h; \\
\frac{(h+1)(3h+4)}{2} & \text{for } d = 3h + 1 \\
10 & \text{for } d = 5 \\
19 & \text{for } d = 8 \\
\frac{3(h+1)(h+2)}{2} + 1 & \text{for } d = 3h + 2, \ h \geq 3
\end{cases} \]

then, except for \( d = 5 \) and \( d = 8 \), by Theorem 2.10 we immediately get \( \dim(I_X)_d = 0 \).

We remain with the cases \( d = 5; \ s = 9 \) and \( d = 8; \ s = 18 \). We omit the proves of these cases.

(i) (f) immediately follows from Theorem 2.10. \( \square \)

5. Appendix

Lemma 5.1. Let \( n \geq 4, \ m < d \) and

\[ e = \left\lceil \frac{(d+n)-(m+n-1)}{d+1} \right\rceil; \quad r = \binom{d+n}{n} - \binom{m+n-1}{n} - e(d+1); \]

\[ e' = \left\lceil \frac{(d-1+n)-(m+n-1)-r}{d} \right\rceil; \]

\[ r' = \binom{d-1+n}{n} - \binom{m+n-1}{n} - r - e'd. \]

Then:

(i) \( e' \geq 0 \);
(ii) \( e - e' - 2r' \geq 0 \);
(iii) \( e' \geq r' \).

Proof. (i) Since \( n \geq 4 \) and \( r \leq d \), we have

\[ e' = \binom{d-1+n}{n} - \binom{m+n-1}{n} - r \geq \binom{d-1+n}{n} - \binom{d-1+n-1}{n} - d \]

\[ = \binom{d+n-2}{d-1} - d \geq \binom{d+2}{d-1} - d \geq 0. \]
(ii) Since $e' + 2r'$ is an integer, then the inequality $e \geq e' + 2r'$ is equivalent to $\binom{d+n}{n} - \binom{m+n-1}{n} \geq e' + 2r'$. Hence, if we prove that

$$
\binom{d+n}{n} - \binom{m+n-1}{n} - (d+1)e' - 2(d+1)r' \geq 0
$$

we are done.

Now

$$
\binom{d+n}{n} - \binom{m+n-1}{n} - (d+1)e' - 2(d+1)r' = \binom{d+n}{n} + (2d+1)\binom{m+n-1}{n} +
$$

$$(d+1)(2d-1)e' - 2\binom{d-1+n}{n}(d+1) + 2r(d+1)
$$

$$
\geq \binom{d+n}{n} + (2d+1)\binom{m+n-1}{n} +
$$

$$(d+1)(2d-1)\left(\frac{\binom{d-1+n}{n} - \binom{m+n-1}{n} - r}{d} - 1\right) - 2\binom{d-1+n}{n}(d+1) + 2r(d+1)
$$

$$
= \frac{1}{d}\left(\binom{d+n}{n} - (d+1)\binom{d-1+n}{n} + \binom{m+n-1}{n} + r(d+1) - d(2d^2 + d - 1)\right)
$$

$$
= \frac{1}{d}\left((n-1)\binom{d+n-1}{d-1} + \binom{m+n-1}{n} + r(d+1) - d(2d^2 + d - 1)\right)
$$

$$
\geq \frac{1}{d}\left((n-1)\binom{d+n-1}{d-1} - d(2d^2 + d - 1)\right).
$$

For $n \geq 5$ we have

$$(n-1)\binom{d+n-1}{d-1} - d(2d^2+d-1) \geq \frac{1}{30}d(d+1)((d+2)(d+3)(d+4)-60d+30) \geq 0$$

for any $d \geq 1$.

For $n = 4$, we have

$$(n-1)\binom{d+n-1}{d-1} - d(2d^2+d-1) = \frac{1}{8}d(d+1)(d^2 - 11d + 14),$$

and this is positive for $d = 1$ and $d \geq 10$. Hence, except for $n = 4$ and $2 \leq d \leq 9$, we have proved that $e - e' - 2r' \geq 0$. For $n = 4$ by direct computation we find:
It follows that also in these cases we have \( e - e' - 2r' \geq 0 \), and this completes the proof.

(iii) \( r' \) is an integer, hence it suffices to prove that
\[
\begin{align*}
\binom{d-1+n}{n} - \binom{m+n-1}{n} - r - r' & \geq 0.
\end{align*}
\]

Since \(m \leq d - 1\), \(r \leq d\) and \(r' \leq d - 1\), \(n \geq 4\) we have
\[
\begin{align*}
\binom{d-1+n}{n} - \binom{m+n-1}{n} - r - r'd & \geq \binom{d-1+n}{n} - \binom{d-1+n-1}{n} - d - (d-1)d \\
= \binom{d+n-2}{d-1} - d^2 & \geq \binom{d+2}{3} - d^2 = \binom{d}{3} \geq 0,
\end{align*}
\]
and the conclusion follows. 

\[\square\]

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