WIDE SUBALGEBRAS OF SEMISIMPLE LIE ALGEBRAS

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INTRODUCTION

Let $G$ be a connected semisimple algebraic group over $\mathbb{C}$, with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. A simple finite-dimensional $\mathfrak{g}$-module $V$ is said to be $\mathfrak{h}$-indecomposable if it cannot be written as a direct sum of two proper $\mathfrak{h}$-submodules. We say that $\mathfrak{h}$ is wide, if all simple finite-dimensional $\mathfrak{g}$-modules are $\mathfrak{h}$-indecomposable. Some very special examples of indecomposable modules and wide subalgebras appeared recently in the literature, see [4, 6] and references therein. In this paper, we point out several large classes of wide subalgebras of $\mathfrak{g}$ and initiate their systematic study.

Our approach relies on the following simple observation. Suppose that $V = V_1 \oplus V_2$ is a sum of two nontrivial $\mathfrak{h}$-modules. Let $p : V \to V_1 \subset V$ be the projection along $V_2$. Then $p$ is a nontrivial idempotent in the associative algebra, $(\text{End } V)^{\mathfrak{h}}$, of $\mathfrak{h}$-invariant elements in $\text{End } V$. Consequently,

$$\{V \text{ is } \mathfrak{h}\text{-indecomposable}\} \iff \left\{(\text{End } V)^{\mathfrak{h}} \text{ does not contain } \text{non-trivial idempotents}\right\}.$$ 

The map $\text{Id}_V : V \to V$ is the unit in the associative algebra $(\text{End } V)^{\mathfrak{h}}$, and we repeatedly use the following sufficient condition for the absence of non-trivial idempotents in $(\text{End } V)^{\mathfrak{h}}$:

Suppose that $(\text{End } V)^{\mathfrak{h}} = \bigoplus_{i \in \mathbb{N}} (\text{End } V)^{\mathfrak{h}}(i)$ is graded (as associative algebra!) and $(\text{End } V)^{\mathfrak{h}}(0) = \mathbb{C} \cdot \text{Id}_V$. Then $(\text{End } V)^{\mathfrak{h}}$ does not contain non-trivial idempotents.

We prove that such a grading exists for every simple $\mathfrak{g}$-module $V$ if $\mathfrak{h}$ belongs to the following list:

(A) $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra that contains no simple ideals of $\mathfrak{g}$, and $\mathfrak{h}$ is the nilradical of $\mathfrak{p}$; in particular, if $\mathfrak{g}$ is simple, then $\mathfrak{p}$ can be any proper parabolic subalgebra (Section 2);

(B) $e \in \mathfrak{g}$ is a nilpotent element that has a non-trivial projection to any simple ideal of $\mathfrak{g}$ and $\mathfrak{h}$ is the nilradical of the centraliser of $e$; in particular, if $\mathfrak{g}$ is simple, then $e$ can be any nonzero nilpotent element (Section 3);

(C) $\mathfrak{h}$ is a certain subalgebra that consists of nilpotent elements of $\mathfrak{g}$ (= $\text{ad}$-nilpotent subalgebra) and is normalised by a Cartan subalgebra of $\mathfrak{g}$. For a sensible description, we use the standard notation on root systems, see also 1.1 below. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, $\Delta$ the root

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system of \((g, t)\), and \(g_\gamma\) the root space of \(g\) corresponding to \(\gamma \in \Delta\). If \([t, h] \subset h\) and \(h\) is ad-nilpotent, then \(h = \bigoplus_{\gamma \in \Delta_h} g_\gamma\), where \(\Delta_h\) is a closed subset of \(\Delta\) and \(\Delta_h \cap (-\Delta_h) = \emptyset\). The main result of Section 4 asserts that \(h\) is wide if and only if the closure of \(\Delta_h \cup (-\Delta_h)\) is the whole root system \(\Delta\).

\((C_1)\) A special case of this construction is a subalgebra determined by a partition of a set of simple roots \(\Pi\) in \(\Delta\). Let \(\Pi'\) be a subset of \(\Pi\). Define \(h = h(\Pi')\) to be the subalgebra of \(g\) generated by \(g_\alpha (\alpha \in \Pi')\) and \(g_{-\alpha} (\alpha \in \Pi \setminus \Pi')\). We say that \(h(\Pi')\) is a \(\Pi\)-\emph{partition subalgebra} of \(g\). Clearly, \(\dim h(\Pi') \geq \#\Pi\). It is easily seen that \(h(\Pi') \simeq h(\Pi \setminus \Pi')\) and \(h(\Pi')\) is ad-nilpotent. There is a special subset \(\tilde{\Pi} \subset \Pi\) such that \(h(\tilde{\Pi})\) is abelian and \(\dim h(\tilde{\Pi}) = \#\Pi\). Namely, \(\tilde{\Pi}\) is a set of pairwise orthogonal simple roots such that \(\Pi \setminus \tilde{\Pi}\) also consists of pairwise orthogonal roots. Since the Dynkin diagram is a tree, the partition \(\Pi = \tilde{\Pi} \cup (\Pi \setminus \tilde{\Pi})\) is unique, and in this case the vector space

\[
\bigoplus_{\alpha \in \tilde{\Pi}} g_\alpha \oplus \bigoplus_{\alpha \in \Pi \setminus \tilde{\Pi}} g_{-\alpha} \text{ or } \bigoplus_{\alpha \in \tilde{\Pi}} g_\alpha \oplus \bigoplus_{\alpha \in \Pi \setminus \tilde{\Pi}} g_{-\alpha}
\]

is already an (abelian) subalgebra of dimension \(\#\Pi\). It was proved in [4] that \(h(\tilde{\Pi})\) is wide for \(g = \mathfrak{sl}_{n+1}\). Our proof is much easier and yields a more general assertion.

\((C_2)\) Another possibility is to take \(\tilde{u} = [u^+, u^+]\), where \(u^+\) is a maximal nilpotent subalgebra of \(g\). Here \(\Delta_{\tilde{u}} = \Delta^+ \setminus \Pi\), and the closure of \(\Delta_{\tilde{u}} \cup (-\Delta_{\tilde{u}})\) equals \(\Delta\) if and only if \(g\) has no simple ideals \(\mathfrak{sl}_2\) or \(\mathfrak{sl}_3\). Invariant-theoretic properties of \(\tilde{u}\) have been studied in [11].

In Section 5, we gather simple general properties of wide subalgebras and discuss a relationship between wide subalgebras and epimorphic subgroups. A subgroup of \(H \subset G\) is \emph{epimorphic} if the following condition holds: \(\text{If } \mathbb{V} \text{ is a finite-dimensional rational } G\text{-module and } \mathbb{V} = V_1 \oplus V_2 \text{ is a direct sum of } H\text{-modules, then the subspaces } V_1, V_2 \text{ are actually } G\text{-invariant} \) (see [1]). For a simple \(G\)-module \(\mathbb{V}\), this is just the \(H\)-indecomposability condition. Therefore, if \(H\) is epimorphic, then \(\text{Lie}(H)\) is wide. However, our work shows that there are much more wide subalgebras than Lie algebras of epimorphic subgroups. Indeed, epimorphic subgroups are also characterised by the property that \(\mathbb{C}[G]^H = \mathbb{C}\), hence they cannot be unipotent, whereas all wide subalgebras described in (A), (B), and (C) are ad-nilpotent. We also give an example of a two-dimensional wide subalgebra of \(g\) and provide a quick derivation (and generalisation) for the results of [6].

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1. Notation and Other Preliminaries

1.1. **Notation.** We fix a triangular decomposition \(g = u^+ \oplus t \oplus u^-\) and various objects associated with the root system \(\Delta = \Delta(g, t)\). Specifically,
\[ \Delta^+ \text{ is the set of positive roots (} = \text{ the roots of } u^+); \]
\[ \Pi = \{\alpha_1, \ldots, \alpha_n\} \text{ is the set of simple roots in } \Delta^+; \]
\[ \{\varphi_\alpha \mid \alpha \in \Pi\} \text{ are the fundamental weights and } \mathcal{X}_+ \text{ is the set of dominant weights} \]
\[ \text{corresponding to } \Pi; \]
\[ \mathcal{Q} = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i \text{ is the root lattice}, \mathcal{E} = \mathcal{Q} \otimes_{\mathbb{Z}} \mathbb{R}, \text{ and } \mathcal{P} \text{ is the weight lattice in } \mathcal{E}. \]
\[ (\ , \ ) \text{ is a Weyl group invariant inner product in } t. \] Using this inner product, we identify \( t \) and \( t^* \), and regard \( \mathcal{E} \) as a real form of \( t. \)

For any \( \gamma \in \Delta, \) let \( g_\gamma \) denote the corresponding root space. We also fix a nonzero element \( e_\gamma \in g_\gamma. \) All \( g \)-modules are assumed to be finite-dimensional. Write \( \gamma_g(M) \) or \( g^M \) for the centraliser of a subset \( M \subset g. \)

1.2. Rational semisimple elements and gradings. Let \( h \in g \) be a rational semisimple element, i.e., the eigenvalues of \( h \) in \( g \) are rational. Then \( h \) has rational eigenvalues in any finite-dimensional \( g \)-module \( V. \) Therefore,

\[ (1.1) \quad V = \bigoplus_{i \in \mathcal{Q}} V_h(i), \]

where \( V_h(i) = \{v \in V \mid \rho_V(h)v = iv \} \) and \( \rho_V : g \to \text{End } V = \text{gl}(V) \) is the representation. We also say that \( (1.1) \) is the \( h \)-grading of \( V. \) Each subspace \( V_h(i) \) is \( g^h \)-stable.

**Lemma 1.1.** Let \( h \in g \) be a rational semisimple element. Given a \( g \)-module \( V, \) consider the \( h \)-grading of the \( g \)-module \( \text{End } V, \) \( \text{End } V = \bigoplus_{i \in \mathcal{Q}} (\text{End } V)_h(i). \) Then

1. (i) this is an associative algebra grading;
2. (ii) if \( h \) is a subalgebra of \( g, \) then \( (\text{End } V)^h \) is an associative subalgebra of \( \text{End } V. \) Moreover, if \( [h, h] \subset h, \) then \( (\text{End } V)^h \) inherits the \( h \)-grading.

**Proof.** (i) The \( g \)-module structure in \( \text{End } V \) is given by

\[ (x, A) \mapsto \rho_V(x)A - A\rho_V(x) = [\rho_V(x), A], \quad x \in g, A \in \text{End } V. \]

If \( [\rho_V(h), A] = iA \) and \( [\rho_V(h), B] = jB \) with \( i, j \in \mathcal{Q}, \) then

\[ [\rho_V(h), AB] = \rho_V(h)AB - AB\rho_V(h) = (iA + A\rho_V(h))B - A(\rho_V(h)B - jB) = (i + j)AB. \]

(ii) Similarly. \( \square \)

**Lemma 1.2.** Let \( A \) be a finite-dimensional \( \mathbb{N} \)-graded unital associative algebra, \( A = \bigoplus_{i \in \mathbb{N}} A(i). \) Suppose that \( A(0) = \mathbb{C}1, \) where \( 1 \) is the unit. Then \( 1 \) is the only idempotent of \( A. \)

**Proof.** Any \( p \in A \) can be written as \( p = cI + q, \) where \( c \in \mathbb{C} \) and \( q \in \bigoplus_{i \geq 1} A(i). \) If \( p^2 = p, \) then \( c = 1 \) and \( q^2 + q = 0. \) As \( q^n = 0 \) for \( n > 0, \) \( 1 + q \) is invertible and \( q = 0. \) \( \square \)
**Warning.** If \( \dim \mathcal{A}(0) \geq 2 \), then \( \mathcal{A} \) may have non-trivial idempotents that are not contained in \( \mathcal{A}(0) \).

We also need a slightly different version that concerns the case in which \((\text{End } V)^\mathcal{h}_\nu \) is positively multigraded. If \([t, \mathfrak{h}] \subset \mathfrak{h} \), then the associative algebra \((\text{End } V)^\mathcal{h}_\nu \) is being decomposed in a finite sum of \(t\)-weight spaces,

\[
(\text{End } V)^\mathcal{h}_\nu = \bigoplus_{\nu \in \mathcal{P}} (\text{End } V)^\mathcal{h}_{\nu}.
\]

**Lemma 1.3.** Suppose that the set \( \mathcal{P}(V, \mathfrak{h}) = \{ \nu \in \mathcal{P} \mid (\text{End } V)^\mathcal{h}_{\nu} \neq 0 \} \) is contained in a closed strictly convex cone \( \mathcal{C} \subset \mathcal{E} \) and \((\text{End } V)^\mathcal{h}_0 = \mathbb{C} \cdot \text{id}_V \). Then \( \text{id}_V \) is the only idempotent in \((\text{End } V)^\mathcal{h}_\nu \) and \( V \) is \( \mathfrak{h} \)-indecomposable.

**Proof.** Let \( h \in t \) be a rational element such that \( \mu(h) > 0 \) for all \( \mu \in \mathcal{C} \setminus \{0\} \) and \( \mu(h) \in \mathbb{Z} \) for all \( \mu \in \mathcal{P}(V, \mathfrak{h}) \). Then (1.2) can be specialised to the \( h \)-grading, where Lemmas 1.1 and 1.2 apply. Alternatively, one can directly prove that (1.2) is an associative algebra grading and the argument of Lemma 1.2 goes through for positive multigradings. \( \square \)

**Remark 1.4.** For future use, we recall the standard fact that if \( V \) is a simple \( \mathfrak{g} \)-module, then all \( t \)-weights of the \( \mathfrak{g} \)-module \( \text{End } V \) belong to the root lattice \( \mathfrak{Q} \).

## 2. The nilradical of a proper parabolic subalgebra is wide

Let \( \Pi' \) be an arbitrary subset of \( \Pi \). If \( \gamma = \sum_{\alpha \in \Pi} a_\alpha \alpha \in \Delta \), then \( \text{ht}_{\Pi'}(\gamma) = \sum_{\alpha \in \Pi'} a_\alpha \) is called the \( \Pi' \)-height of \( \gamma \). For \( \Pi' = \Pi \), one obtains the usual notion of the height.

Let \( \mathfrak{p} \) be the standard parabolic subalgebra of \( \mathfrak{g} \) determined by \( \Pi' \subset \Pi \). That is,

\[
\mathfrak{p} = t \oplus \left( \bigoplus_{\gamma : \text{ht}_{\Pi'}(\gamma) > 0} \mathfrak{g}_\gamma \right).
\]

Then \( \mathfrak{p}_{\text{nil}} = n = \bigoplus_{\gamma : \text{ht}_{\Pi'}(\gamma) > 0} \mathfrak{g}_\gamma \) is the nilpotent radical of \( \mathfrak{p} \), and

\[
\mathfrak{l} = t \oplus \left( \bigoplus_{\gamma : \text{ht}_{\Pi'}(\gamma) = 0} \mathfrak{g}_\gamma \right)
\]

is the standard Levi subalgebra of \( \mathfrak{p} \).

**Lemma 2.1.** If \( V \) is a simple \( \mathfrak{g} \)-module, then \( V^n \) is a simple \( t \)-module.

**Proof.** If \( \mathfrak{u}(\mathfrak{l}) \) is an arbitrary maximal nilpotent subalgebra of \( \mathfrak{l} \), then \( \mathfrak{u}(\mathfrak{l}) \oplus n \) is a maximal nilpotent subalgebra of \( \mathfrak{g} \). Therefore, \( \dim (V^n)^{\mathfrak{u}(\mathfrak{l})} = \dim V^{\mathfrak{u}(\mathfrak{l}) \oplus n} = 1 \). \( \square \)

We extend the \( \Pi' \)-height to the whole of \( \mathcal{P} \), using the same formulae as above. That is, if \( \nu = \sum_{\alpha \in \Pi} b_\alpha \alpha \in \mathcal{P} \), then \( \text{ht}_{\Pi'}(\nu) = \sum_{\alpha \in \Pi'} b_\alpha \). The coefficients \( b_\alpha \) and hence \( \text{ht}_{\Pi'}(\nu) \) can
be rational. More precisely, \( b_\alpha \in \mathbb{1}/\mathbb{Z} \), where \( f = [\mathcal{P} : \mathcal{Q}] \) is the index of connection of \( \Delta \). In this way, one obtains the canonical grading of type \( \Pi' \) in any \( g \)-module \( V \). Namely,

\[
V = \bigoplus_{i \in \mathbb{Z}} V(i),
\]

where \( V(i) \) is the sum of weight spaces of \( W \) corresponding to the weights of \( \Pi' \)-height \( i \). Obviously,

\[
ht_{\Pi'}(\nu) = (\sum_{\alpha \in \Pi'} \varphi_{\alpha}^\vee, \nu),
\]

where \( \varphi_{\alpha}^\vee = 2\varphi_{\alpha}/(\alpha, \alpha) \). Therefore, the grading of type \( \Pi' \) is nothing but the \( h \)-grading in the sense of Subsection 1.2, with \( h = \sum_{\alpha \in \Pi'} \varphi_{\alpha}^\vee \in t^* \approx t \).

Clearly, each \( V(i) \) is an \( l \)-module and \( g_\alpha V(i) \subset V(i+1) \) if \( \alpha \in \Pi' \), i.e., \( ht_{\Pi'}(\alpha) = 1 \). If \( V \) is a simple \( g \)-module, then all \( i \in \mathbb{Q} \) such that \( V(i) \neq 0 \) give rise to one and the same element in \( \mathbb{Q}/\mathbb{Z} \). Moreover, if all the weights of \( V \) belong to \( \mathbb{Q} \), then the grading of type \( \Pi' \) is a \( \mathbb{Z} \)-grading on \( V \).

To avoid a cumbersome notation, we assume below that \( g \) is simple (see also Remark 2.5). Let \( p \) be a proper parabolic subalgebra, i.e., \( \Pi' \neq \emptyset \).

**Lemma 2.2.** Let \( V \) be a simple \( g \)-module equipped with the canonical grading of type \( \Pi' \) (2.1). Set \( m = \max\{i \mid V(i) \neq 0\} \). Then (i) \( V^n = V(m) \) and \( m \geq 0 \); (ii) \( m = 0 \) if and only if \( V \) is a trivial one-dimensional module.

**Proof.** Let \( \lambda \in \mathfrak{X}_+ \) be the highest weight of \( V \), and \( \lambda = \sum_{\alpha \in \Pi} c_\alpha \alpha \). Clearly, \( m = ht_{\Pi'}(\lambda) = \sum_{\alpha \in \Pi'} c_\alpha \). Here all the coefficients \( c_\alpha \) are strictly positive if \( \lambda \neq 0 \). (This follows from the fact that all the entries of the inverse of the Cartan matrix of \( \Delta \) are strictly positive [10].) Hence if \( \lambda \neq 0 \), then \( ht_{\Pi'}(\lambda) > 0 \) for any non-empty \( \Pi' \).

Since \( g_\gamma V(i) \subset V(i+ht_{\Pi'}(\gamma)) \) for any \( \gamma \in \Delta^+ \), we have \( V(m) \subset V^n \). On the other hand, \( V^n \) is a simple \( l \)-module (Lemma 2.1), hence \( V(m) = V^n \). \( \square \)

**Theorem 2.3.** For any nonempty subset \( \Pi' \subset \Pi \) and any simple finite-dimensional \( g \)-module \( V \),

(i) the grading of type \( \Pi' \) on \( (\text{End } V)^n \) is actually an \( \mathbb{N} \)-grading and \( (\text{End } V)^n(0) = \mathbb{C} \cdot \text{Id}_V \).

(ii) \( (\text{End } V)^n \) contains no non-trivial idempotents and, therefore, \( n \) is a wide subalgebra of \( g \).

**Proof.** (i) Since the weights of the \( g \)-module \( \text{End } V \) belong to the root lattice \( \mathcal{Q} \), the grading of type \( \Pi' \) on \( \text{End } V \) is actually a \( \mathbb{Z} \)-grading. We have \( \text{End } V \simeq V \otimes V^* = \bigoplus_{j=1}^k V_j \), where \( V_j \) are certain simple \( g \)-modules. If \( V_j = \bigoplus_{i \in \mathbb{Z}} V_j(i) \) be the \( \mathbb{Z} \)-grading of type \( \Pi' \) and \( m_j := \max\{i \mid V_j(i) \neq 0\} \), then

\[
(\text{End } V)^n = \bigoplus_{j=1}^k V_j(m_j)
\]
is the direct sum of simple \( l \)-modules and also a (refinement of) \( \mathbb{N} \)-grading. By the Schur lemma, \( V \otimes V^* \) contains a unique trivial one-dimensional \( g \)-module, and this unique trivial module is the line through \( l d_V : V \to V \). In view of Lemma 2.2, we may assume that \( m_1 = 0 \) and \( m_j > 0 \) for \( j \geq 2 \).

(ii) The grading of type \( \Pi' \) in \( \text{End} \, V \) is also the \( (\sum_{\alpha \in \Pi'} \varphi_\alpha^\vee) \)-grading (see Eq. (2.2)). Then Lemma 1.1(i) guarantees us that this is an associative algebra grading. Furthermore, \( t \) normalises \( n \) and \( \sum_{\alpha \in \Pi'} \varphi_\alpha^\vee \) is identified with an element of \( t \). Therefore, (2.3) is also an associative algebra grading and, by Lemma 1.2, \( l d_V \) is the only idempotent in \((\text{End} \, V)^n\).

**Remark 2.4.** It is known that \( \dim n = \dim G/P \geq \text{rk}(g) \), and the equality only occurs for the maximal parabolic subalgebra of \( sl_{n+1} \) such that \( \Pi' = \{ \alpha \} \) and \( \alpha \) is an extreme root in the Dynkin diagram, see [14]. In particular, if \( \dim n = \text{rk}(g) \), then \( n \) is abelian.

**Remark 2.5.** If \( g \) is semisimple but not simple, then \( g = \prod_j g^{(j)} \) is the product of simple ideals and \( \Pi = \bigcup_j \Pi^{(j)} \). It is then easily seen that Lemma 2.2 and Theorem 2.3 remain true if \( \Pi' \cap \Pi^{(j)} \neq \emptyset \) for all \( j \), i.e., if \( p \) does not contain simple ideals of \( g \).

3. **THE NILRADICAL OF THE CENTRALISER OF A NON-DEGENERATE NILPOTENT ELEMENT IS WIDE**

Let \( \mathcal{N} \) be the set of all nilpotent elements of \( g \). Throughout this section, we assume that \( e \in \mathcal{N} \) is nonzero. To present a (well-known) description of the nilpotent radical of \( g^e \), we need the machinery of \( sl_2 \)-triples and respective \( \mathbb{Z} \)-gradings of \( g \). By the Morozov-Jacobiobson theorem, any nonzero \( e \in \mathcal{N} \) can be embedded into an \( sl_2 \)-triple \( \{ e, h, f \} \) (i.e., \( [h, e] = 2e, [h, f] = -2f, [e, f] = h \)) [5, 3.3]. The eigenvalues of \( h \) in any \( g \)-module are integral, hence the \( h \)-grading in any simple \( g \)-module is actually a \( \mathbb{Z} \)-grading.

As in Subsection 1.2, the semisimple element \( h \) determines the \( h \)-grading of \( g \):

\[
g = \bigoplus_{i \in \mathbb{Z}} g_h(i),
\]

where \( g_h(i) = \{ x \in g \mid [h, x] = ix \} \). Then \( e \in g_h(2) \) and \( f \in g_h(-2) \).

The following facts on the structure of this grading and the centraliser \( g^e \) are standard, see [13, ch. III, § 4] or [5, Ch. 3].

**Proposition 3.1.** Let \( \{ e, h, f \} \) be an \( sl_2 \)-triple. Then

(i) the Lie algebra \( g^e \) is non-negatively graded: \( g^e = \bigoplus_{i \geq 0} g_h^e(i) \), where \( g_h^e(i) = g^e \cap g_h(i) \). Here \( g^e_{nil} := \bigoplus_{i \geq 1} g_h^e(i) \) is the nilpotent radical and \( g^e_{red} := g^e_h(0) \) is a Levi subalgebra of \( g^e \); actually, \( g^e(h) = g_{(e, h, f)}^e \).

(ii) \( \text{ad} \, e : g_h(i - 2) \to g_h(i) \) is injective for \( i \leq 1 \) and surjective for \( i \geq 1 \);

(iii) \( \dim g^e = \dim g_h(0) + \dim g_h(1) \) and \( \dim g^e_{nil} = \dim g_h(1) + \dim g_h(2) \).
This provides a rather good understanding of the nilpotent radical \( \mathfrak{g}_{\text{nil}} \). Recall that \( e \) is said to be \textit{principal}, if \( \dim \mathfrak{g}^e = \text{rk}(\mathfrak{g}) \), and then \( \mathfrak{g}^e = \mathfrak{g}_{\text{nil}}^e \). In this case we also say that \( \langle e, h, f \rangle \) is a principal \( \mathfrak{sl}_2 \)-subalgebra.

If \( \mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \) is a sum of two ideals and \( e = e_1 + e_2 \), with \( e_i \in \mathfrak{g}^{(i)} \), then \( \mathfrak{g}^e = (\mathfrak{g}^{(1)})^{e_1} \oplus (\mathfrak{g}^{(2)})^{e_2} \). Therefore, \( (\mathfrak{g}^{(1)})^{e_1}_{\text{nil}} = 0 \) if and only if \( e_1 = 0 \). We say that \( e \in \mathfrak{N} \) is \textit{non-degenerate}, if \( e \) has a non-trivial projection to every simple ideal of \( \mathfrak{g} \). If \( \mathfrak{g} \) is simple, then any nonzero \( e \in \mathfrak{N} \) is non-degenerate.

**Lemma 3.2.** Suppose that \( e \in \mathfrak{N} \) is non-degenerate. Then the subalgebra generated by \( \mathfrak{g}_{\text{nil}}^e \) and \( f \) is the whole of \( \mathfrak{g} \).

**Proof.** Set \( \mathfrak{s} = \langle e, h, f \rangle \). It is a three-dimensional simple subalgebra of \( \mathfrak{g} \). Consider \( \mathfrak{g} \) as \( \mathfrak{s} \)-module. By Proposition 3.1(i), \( \mathfrak{g}_{\text{nil}}^e \) is the linear span of the highest vectors of all nontrivial simple \( \mathfrak{s} \)-modules in \( \mathfrak{g} \). Therefore, the minimal \( \text{(ad} f) \)-stable subspace containing \( \mathfrak{g}_{\text{nil}}^e \), say \( \mathbb{U} \), is the sum of all nontrivial \( \mathfrak{s} \)-submodules in \( \mathfrak{g} \), and the subalgebra generated by \( \mathfrak{g}_{\text{nil}}^e \) and \( f \) coincides with the subalgebra generated by \( \mathbb{U} \). The reductive algebra \( \mathfrak{g}^\mathfrak{s}(0) = \mathfrak{g}^\mathfrak{s} \) is the sum of all trivial \( \mathfrak{s} \)-modules. Hence \( \mathbb{U} \oplus \mathfrak{g}^\mathfrak{s} = \mathfrak{g} \) and \( [\mathfrak{g}^\mathfrak{s}, \mathbb{U}] \subset \mathbb{U} \). Let \( \langle \mathbb{U} \rangle \) be the subalgebra generated by \( \mathbb{U} \). Then \( [\mathfrak{g}^\mathfrak{s}, \langle \mathbb{U} \rangle] \subset \langle \mathbb{U} \rangle \) and \([\mathbb{U}, \langle \mathbb{U} \rangle] \subset \langle \mathbb{U} \rangle \). Hence \( \langle \mathbb{U} \rangle \) is an ideal of \( \mathfrak{g} \). By the assumption, \( \langle \mathbb{U} \rangle \) has non-trivial projections to all simple ideals of \( \mathfrak{g} \). Hence \( \langle \mathbb{U} \rangle = \mathfrak{g} \) (cf. also [9, Lemma 4.1]).

A simple \( \mathfrak{g} \)-module with highest weight \( \lambda \in \mathfrak{X}_+ \) is denoted by \( \mathcal{R}(\lambda) \), and \( \rho_\lambda \) is the corresponding representation of \( \mathfrak{g} \). If \( \mathfrak{a} \) is any subset of \( \mathfrak{g} \), then

\[
\mathcal{R}(\lambda)^\mathfrak{a} = \{ v \in \mathcal{R}(\lambda) \mid \rho_\lambda(x)v = 0 \ \forall x \in \mathfrak{a} \}.
\]

In particular, \( \mathcal{R}(\lambda)^h \) is the zero weight space of the \( h \)-grading of \( \mathcal{R}(\lambda) \).

**Proposition 3.3.** Let \( \{ e, h, f \} \) be an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \). If \( e \) is non-degenerate and \( \lambda \neq 0 \), then the \( h \)-eigenvalues in \( \mathcal{R}(\lambda)^{\mathfrak{g}_{\text{nil}}^e} \) are strictly positive.

**Proof.** It follows from the theory of \( \mathfrak{sl}_2 \)-representations that the \( h \)-eigenvalues in \( \mathcal{R}(\lambda)^e \) are nonnegative. Hence the same is true for \( \mathcal{R}(\lambda)^{\mathfrak{g}_{\text{nil}}^e} \subset \mathcal{R}(\lambda)^e \), and our goal is to prove that \( 0 \) does not occur as an \( h \)-eigenvalue in \( \mathcal{R}(\lambda)^{\mathfrak{g}_{\text{nil}}^e} \). If \( v \in \mathcal{R}(\lambda)^h \cap \mathcal{R}(\lambda)^{\mathfrak{g}_{\text{nil}}^e} \), then \( v \) is killed by both \( h \) and \( e \). Therefore, \( \rho_\lambda(f)(v) = 0 \). Thus, \( v \) is killed by \( f \) and \( \mathfrak{g}_{\text{nil}}^e \). By Lemma 3.2, the subalgebra generated by \( f \) and \( \mathfrak{g}_{\text{nil}}^e \) is \( \mathfrak{g} \). Hence \( v \in \mathcal{R}(\lambda)^{\mathfrak{a}} = \{ 0 \} \).

Now, we are ready to prove the main result of this section.

**Theorem 3.4.** For any non-degenerate \( e \in \mathfrak{N} \) and any simple finite-dimensional \( \mathfrak{g} \)-module \( \mathcal{R}(\lambda) \), we have

(i) the \( h \)-grading on \((\text{End } \mathcal{R}(\lambda))^{\mathfrak{g}_{\text{nil}}^e}\) is an \( \mathbb{N} \)-grading and \((\text{End } \mathcal{R}(\lambda))^{\mathfrak{g}_{\text{nil}}^e}(0) = \mathbb{C} \cdot \text{Id}_{\mathcal{R}(\lambda)}\).
4. Some regular ad-nilpotent subalgebras are wide

A subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) is said to be regular, if it is normalised by a Cartan subalgebra of \( \mathfrak{g} \). Without loss of generality, one may only consider regular subalgebras such that \([t, \mathfrak{h}] \subset \mathfrak{h}\) for our fixed \( t \). We additionally assume below that \( \mathfrak{h} \) is ad-nilpotent. Then \( \mathfrak{h} = \bigoplus_{\lambda \in \Delta_\mathfrak{h}} \mathfrak{g}_\lambda \), where \( \Delta_\mathfrak{h} \) is a closed subset of \( \Delta \) and \( \Delta_\mathfrak{h} \cap (-\Delta_\mathfrak{h}) = \emptyset \). (Note that we do not assume here that \( \Delta_\mathfrak{h} \subset \Delta^+ \).) Recall that a subset \( \Gamma \subset \Delta \) is closed if whenever \( \gamma_1, \gamma_2 \in \Gamma \) and \( \gamma_1 + \gamma_2 \in \Delta \), then \( \gamma_1 + \gamma_2 \in \Gamma \); the closure of \( \Gamma \) is the smallest closed subset of \( \Delta \) containing \( \Gamma \). Write \( \mathcal{R}(\lambda)_\mu \) for the \( \mu \)-weight space of \( \mathcal{R}(\lambda) \). As is well-known [3, Ch. VIII, §7],

\[
\mathcal{R}(\lambda)_0 \neq \{0\} \iff \lambda \in \Omega \iff \text{all the weights of } \mathcal{R}(\lambda) \text{ belong to } \Omega.
\]

Lemma 4.1. Let \( \mathfrak{h} \) be as above. Then

(i) for any \( \lambda \in \mathcal{X}_+ \), we have \( \mathcal{R}(\lambda)^h \subset \bigoplus_{\mu \in \mathcal{E}} \mathcal{R}(\lambda)_\mu \), where \( \mathcal{C}(\mathfrak{h}) = \{ \mu \in \mathcal{E} \mid (\mu, \gamma) \geq 0 \ \forall \gamma \in \Delta_\mathfrak{h} \} \) is a closed cone in \( \mathcal{E} \), which does not depend on \( \lambda \);

(ii) suppose that the closure of \( \Delta_\mathfrak{h} \cup (-\Delta_\mathfrak{h}) \) equals \( \Delta \). Then \( \mathcal{C}(\mathfrak{h}) \) is a strictly convex cone and \( \mathcal{R}(\lambda)^h \subset \bigoplus_{\mu \in \mathcal{C}(\mathfrak{h}) \setminus \{0\}} \mathcal{R}(\lambda)_\mu \) for any nonzero \( \lambda \in \mathcal{X}_+ \).
Proof. (i) If \( \gamma \in \Delta \), then \( \mathcal{R}(\lambda)^0 \subset \bigoplus_{\mu: (\mu, \gamma) \geq 0} \mathcal{R}(\lambda)_\mu \), see [3, Ch. VIII, \S 7]. Therefore,

\[
\mathcal{R}(\lambda)^h \subset \bigcap_{\gamma \in \Delta} \mathcal{R}(\lambda)^{\gamma} \subset \bigoplus_{\mu \in \mathcal{C}(h)} \mathcal{R}(\lambda)_\mu, 
\]

(ii) If the closure of \( \Delta_h \cup (-\Delta_h) \) equals \( \Delta \), then \( \Delta_h \) contains a basis for \( \mathcal{E} \) and hence \( \mathcal{E}(h) \) is strictly convex. Therefore, it remains to prove that even if \( \mathcal{R}(\lambda)_0 \neq 0 \) (i.e., \( \lambda \in \Omega \)), then still \( \mathcal{R}(\lambda)^h_0 = 0 \). Indeed,

\[
\mathcal{R}(\lambda)^h_0 = \bigcap_{\gamma \in \Delta} \text{Ker} (\text{ad}_{\gamma} : \mathcal{R}(\lambda)_0 \rightarrow \mathcal{R}(\lambda)_\gamma).
\]

But it follows from the \( \mathfrak{sl}_2 \)-theory applied to the subalgebra generated by \( g, \gamma \) and \( g_{-\gamma} \) that

\[
\text{Ker} (\text{ad}_{\gamma} : \mathcal{R}(\lambda)_0 \rightarrow \mathcal{R}(\lambda)_\gamma) = \text{Ker} (\text{ad}_{-\gamma} : \mathcal{R}(\lambda)_0 \rightarrow \mathcal{R}(\lambda)_{-\gamma}).
\]

Therefore, \( \mathcal{R}(\lambda)^h_0 \) is also a fixed point subspace of the subalgebra generated by \( t \) and all \( g, \gamma, g_{-\gamma} \) with \( \gamma \in \Delta_h \). The hypothesis on the closure implies that this subalgebra equals \( g \). Hence \( \mathcal{R}(\lambda)^h_0 = \mathcal{R}(\lambda)^g = \{0\} \) if \( \lambda \neq 0 \). \( \square \)

**Theorem 4.2.** Let \( \mathfrak{h} \) be an ad-nilpotent subalgebra of \( \mathfrak{g} \) normalised by \( t \) and \( \Delta_h \) the corresponding set of roots.

(i) Suppose that the closure of \( \Delta_h \cup (-\Delta_h) \) equals \( \Delta \). Then, for any \( \lambda \in \mathfrak{X}_+ \), the associative algebra \( (\text{End} \mathcal{R}(\lambda))^h \) does not contain non-trivial idempotents; hence \( \mathcal{R}(\lambda) \) is \( \mathfrak{g} \)-indecomposable and thereby \( \mathfrak{h} \) is wide.

(ii) Conversely, if \( \mathfrak{h} \) is wide, then the closure of \( \Delta_h \cup (-\Delta_h) \) equals \( \Delta \).

Proof. (i) As before, \( \text{End} \mathcal{R}(\lambda) \simeq \mathcal{R}(\lambda) \otimes \mathcal{R}(\lambda)^* = \bigoplus_{i=1}^{k} \mathcal{R}(\lambda_i) \), where all \( \lambda_i \in \Omega \cap \mathfrak{X}_+ \), and we may assume that \( \lambda_1 = 0 \), while \( \lambda_i \neq 0 \) for \( i \geq 2 \). Then

\[
(\text{End} \mathcal{R}(\lambda))^h = \mathcal{R}(0) \oplus \bigoplus_{i=2}^{k} \mathcal{R}(\lambda_i)^h.
\]

By Lemma 4.1, each \( \mathcal{R}(\lambda_i)^h \) is \( \mathcal{C}(h) \)-graded and the component of grade 0 is just \( \mathcal{R}(0) = \mathbb{C} \cdot \text{Id}_{\mathcal{R}(\lambda)} \). Because this grading of \( (\text{End} \mathcal{R}(\lambda))^h \) is determined by weights of \( t \) and these weights are contained in the strictly convex cone \( \mathcal{C}(h) \), it is an associative algebra grading and the only idempotent sitting in \( (\text{End} \mathcal{R}(\lambda))^h_{\text{I}^{\text{II}}} \) is \( \text{Id}_{\mathcal{R}(\lambda)} \) (see Lemma 1.3). Thus, \( \mathcal{R}(\lambda) \) is \( \mathfrak{h} \)-indecomposable, and we are done.

(ii) Let \( \hat{\Delta} \) be the closure of \( \Delta_h \cup (-\Delta_h) \). Assume that \( \hat{\Delta} \neq \Delta \). Then \( \hat{\mathfrak{g}} := t \oplus (\bigoplus_{\gamma \in \hat{\Delta}} \mathfrak{g}_\gamma) \) is a proper reductive subalgebra of \( \mathfrak{g} \) and \( \mathfrak{h} \subset \hat{\mathfrak{g}} \). Hence the simple \( \mathfrak{g} \)-module \( \mathfrak{g} \) is decomposable as \( \hat{\mathfrak{g}} \)- and \( \mathfrak{h} \)-module. \( \square \)

In the rest of the section, we consider important examples illustrating Theorem 4.2.
Example 4.3 (Parabolic subalgebras). Let \( n \) be the nilradical of a standard parabolic subalgebra \( p \). It is easily seen that if \( p \) contains no simple ideals of \( g \), then the closure of \( \Delta_n \cup (-\Delta_n) \) equals \( \Delta \). Therefore, Theorem 2.3 follows from Theorem 4.2(i). But we include a separate treatment for the nilpotent radicals, because it does not require multigradings and yields a more complete information.

Example 4.4 (The derived algebra of \( u^+ \)). For the ad-nilpotent subalgebra \( \tilde{u} := [u^+, u^+] \), we have \( \Delta_{\tilde{u}} = \Delta^+ \setminus \Pi \). If \( g \) has no simple ideals \( sl_2 \) or \( sl_3 \), then the closure of \( (\Delta^+ \setminus \Pi) \cup (-\Delta^+ \setminus \Pi) \) is \( \Delta \). Hence \( \tilde{u} \) is wide in all these cases.

By [11, Sect. 4], the cone \( C(\tilde{u}) \) is generated by the weights \( \varphi_\alpha, \varphi_\alpha - \alpha (\alpha \in \Pi) \); and it also follows from [11, Sect. 1] that \( R(\lambda)\tilde{u} \) is positively \( \rho^\vee \)-graded, where \( \rho^\vee = \frac{1}{2} \sum_{\gamma \in \Delta^+} \gamma^\vee \).

Example 4.5 (\( \Pi \)-partition subalgebras). Let \( \Pi' \) be a subset of \( \Pi \). As the following exposition is symmetric with respect to \( \Pi' \) and \( \Pi'' = \Pi \setminus \Pi' \), it is convenient to think of it as a partition \( \Pi = \Pi' \sqcup \Pi'' \).

A \( \Pi \)-\textit{partition subalgebra} of \( g \) is the Lie algebra generated by the root spaces \( g_\alpha (\alpha \in \Pi') \) and \( g_{-\alpha} (\alpha \in \Pi'') \). Write \( h(\Pi') \) for this subalgebra.

Here are some simple observations related to these subalgebras:

- \( \dim h(\Pi') \geq rk(g) \) and \( t \) normalises \( h(\Pi') \);
- \( h(\Pi') \simeq h(\Pi'') \) (use the Weyl involution of \( g \));
- \( h(\Pi) = u^+ \) and \( h(\varnothing) = u^- \);
- The weights \( \Pi' \cup (-\Pi'') \) are contained in an open half-space of \( \mathcal{E} \).

The last property implies that \( h(\Pi') \) is contained in a maximal nilpotent subalgebra of \( g \). Hence \( h(\Pi') \) consists of nilpotent elements and \( \Delta_{h(\Pi')} \cap (-\Delta_{h(\Pi')}) = \emptyset \).

Since \( \Delta_{h(\Pi')} \cup (-\Delta_{h(\Pi')}) \supset \Pi \cup (-\Pi) \), the closure of \( \Delta_{h(\Pi')} \cup (-\Delta_{h(\Pi')}) \) is \( \Delta \). Hence Theorem 4.2(i) applies here and all \( \Pi \)-partition subalgebras are wide.

The most interesting \( \Pi \)-partition subalgebra occurs if the roots in \( \Pi' \) are pairwise orthogonal (= disjoint on the Dynkin diagram) and the same property also holds for \( \Pi'' \). Since the Dynkin diagram is a tree, such a partition of \( \Pi \) is unique, so there are two (isomorphic) respective subalgebras of \( g \). This partition of \( \Pi \) is said to be disjoint and its parts are denoted by \( \{ \tilde{\Pi}, \tilde{\Pi}' \} \). This discussion yields the following simple but useful assertion.

**Proposition 4.6.** For a partition \( \Pi = \Pi' \sqcup \Pi'' \), the following conditions are equivalent:

1) this partition is disjoint, i.e., \( \Pi' = \tilde{\Pi} \) or \( \tilde{\Pi}' \);
2) \( \dim h(\Pi') = rk(g) \);
3) \( h(\Pi') \) is abelian;
4) \( \Delta_{h(\Pi')} = \Pi' \cup (-\Pi'') \).
In [4], it is proved that $h(\Pi)$ is wide for $g = sl_{n+1}$. But that proof is rather technical and exploits Littelmann’s theory of standard bases for the $sl_{n+1}$-representations. Our approach provides a much simpler proof for a much stronger result (Theorem 4.2).

5. On a General Approach to Wide Subalgebras and Indecomposable Representations

5.1. Simple properties. Here we discuss some general properties of wide subalgebras of $g$ and related problems.

Lemma 5.1.

(i) If $a_1 \subset a_2$ and $a_1$ is wide, then so is $a_2$;
(ii) If $a \subset s \subset g$ and $s$ is reductive, then $a$ is not wide;
(iii) If $a$ is wide, then $z_g(a)$ is an $ad$-nilpotent subalgebra.

Proof. (i) Obvious.
(ii) The simple $g$-module $g$ is decomposable as $s$-module.
(iii) If $s \in z_g(a)$ is semisimple, then $a$ is contained in the reductive subalgebra $z_g(s)$, hence $a$ is not wide. That is, $z_g(a)$ does not contain semisimple elements. As $z_g(a)$ is an algebraic Lie algebra [2, 7.4], it contains the semisimple part of every element. Therefore, $z_g(a)$ must contain only nilpotent elements. □

All wide subalgebras occurring in Section 4 are of dimension at least $rk(g)$ (see also Remark 2.4), and the same is true for the nilpotent radicals of centralisers of non-degenerate nilpotent elements, see Section 3. Moreover, in both cases, the subalgebras of dimension $rk(g)$ are necessarily abelian. A partial explanation is given by

Lemma 5.2. Suppose that $a$ is wide and regular. Then $\dim a \geq rk(g)$ and if $\dim a = rk(g)$, then $a$ is $ad$-nilpotent and abelian.

Proof. If $[t, a] \subset a$, then $a = \tilde{t} \oplus (\bigoplus_{\gamma \in \Delta_a} g_\gamma)$, where $\tilde{t}$ is a subspace of $t$ such that $\tilde{t} \supset \Delta_a \cap (-\Delta_a)$ (in the last embedding we identify $t$ and $t^*$). If $\dim a < rk(g)$, then $\# \Delta_a < rk(g)$ and $z_g(a)$ certainly contains a nonzero element of $t$, i.e., $a$ cannot be wide. If $\dim a = rk(g)$, then a similar argument shows that we must have $\# \Delta_a = rk(g)$, the elements of $\Delta_a$ are linearly independent and $\tilde{t} = 0$. Moreover, since $\Delta_a$ is linearly independent and closed, $a$ is abelian. □

However, $rk(g)$ provides the strict lower bound only for the dimension of regular wide subalgebras of $g$. We prove below that every simple Lie algebra has a wide commutative subalgebra of dimension 2. Note also that it may happen that $a$ is not wide, but there still exist families of $a$-indecomposable simple $g$-modules. Here is a sample reason for such phenomenon.
Lemma 5.3. Let $\tilde{g} \subset g$ be a proper semisimple subalgebra and $a$ is wide in $\tilde{g}$. Suppose that a simple $g$-module $R(\lambda)$ remains simple as $\tilde{g}$-module. Then $R(\lambda)$ is $a$-indecomposable.

5.2. Wide subalgebras and epimorphic subgroups. A subgroup $H \subset G$ is said to be epimorphic, if $C[G]^H = C$. Equivalently, $H$ is epimorphic if $\mathcal{R}(\lambda)^H = \{0\}$ unless $\lambda = 0 [1]$. One easily proves that $H$ is epimorphic if and only if the identity component of $H$ is. Therefore, we may say that a subalgebra $h$ is epimorphic if $h = \text{Lie}(H)$ and $H$ is epimorphic in the above sense.

By [1, Theorem 1], $h$ is epimorphic if and only if the following condition holds: If $\forall$ is a $g$-module and $V = V_1 \oplus V_2$ is a sum of $h$-modules, then $V_1$ and $V_2$ are actually $g$-invariant. Compare this with the definition of a wide subalgebra, which requires indecomposability only for the simple $g$-modules!

This implies that any epimorphic subalgebra is wide. Alternatively, one may notice that if $h$ is epimorphic, then $(\text{End} R(\lambda))^h = C \cdot \text{Id}_{R(\lambda)}$ for all $\lambda \in X_+$ and hence $h$ is wide. There is a close relationship between regular wide and epimorphic subalgebras.

Proposition 5.4. Suppose that the subalgebra $h \subset g$ is ad-nilpotent and $[t, h] \subset h$. Then $h \oplus t$ is epimorphic if and only if $h$ is wide.

Proof. Here $h$ is the nilpotent radical of the regular solvable subalgebra $\tilde{h} = h \oplus t$.

By [12, Korollar 3.6], if $\tilde{h}$ is epimorphic, then the closure of $\Delta_h \cup (-\Delta_h)$ is $\Delta$. Hence $h$ is wide in view of Theorem 4.2(ii).

Conversely, if $h$ is wide, then the closure of $\Delta_h \cup (-\Delta_h)$ is $\Delta$ according to Theorem 4.2(ii), and again [12, Korollar 3.6] shows that $h \oplus t$ is epimorphic.

Any simple Lie algebra contains a three-dimensional solvable epimorphic subalgebra (see [1, n. 5(b)]), but this subalgebra is neither regular nor ad-nilpotent. Below, we recall the construction and show that the two-dimensional nilradical of that subalgebra is wide.

Example 5.5. Let $s := \langle e, h, f \rangle$ be a principal $sl_2$-subalgebra of a simple Lie algebra $g$. Then $s$ is not contained in a proper regular semisimple subalgebra of $g$ [7, Theorem 9.1]. Actually, $s$ is either a maximal semisimple subalgebra, or is contained in a unique maximal proper semisimple subalgebra $\tilde{g}$ of $g$, see [8] for the classical Lie algebras and [7] for the exceptional algebras. For instance, if $g$ is of type $E_6$, then $\tilde{g}$ is of type $F_4$, whereas for all other exceptional algebras, one has $s = \tilde{g}$ [7, Theorem 15.2]. By Lemma 3.2, $\tilde{g}$ cannot contain the whole of $g^c = g^c_{\text{nil}}$. Therefore, one can pick an $h$-eigenvector $\tilde{e} \in g^c$ such that $s$ and $\tilde{e}$ generate the whole of $\tilde{g}$. In other words, $f$ and the commutative subalgebra $a = \langle e, \tilde{e} \rangle$ generate the whole of $\tilde{g}$. Applying then Proposition 3.3 and Theorem 3.4 to $a$ (in place of $g^c_{\text{nil}}$), we conclude that $a$ is wide. Here $\langle h, e, \tilde{e} \rangle$ is an epimorphic subalgebra of $g$ described in [1], and $a$ is its nilradical.

This prompts the following
**Question 1.** Let \( a \) be an epimorphic algebraic subalgebra of \( g \). Is it true that the nilpotent radical \( a_{\text{nil}} \) is wide?

### 5.3. Example: the euclidean Lie algebra \( e_3 \).

Following \([6]\), we denote by \( e_3 \) the semi-direct product \( so_3 \ltimes \mathbb{C}^3 \cong sl_2 \ltimes sl_2 \). It is proved in \([6]\) that, for a certain embedding \( e_3 \subset sl_4 \), the simple \( sl_4 \)-modules \( R(m\varphi_1) \) and \( R(m\varphi_3) \) are \( e_3 \)-indecomposable for all \( m \in \mathbb{N} \), whereas \( R(\varphi_2) \) and \( R(2\varphi_2) \) are not. [We use the obvious numbering of the fundamental weights of \( sl_4 \).] To illustrate the usefulness of our methods, we provide a simpler derivation (and a generalisation) of those results.

The embedding \( e_3 \subset sl_4 \) is given by Equations (3.1) and (4.1) in \([6]\). Making a suitable permutation of the corresponding basis vectors of \( \mathbb{C}^4 \), one easily finds that \( e_3 \) can be regarded as the subalgebra

\[
(5.1) \quad \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \mid A, B \in sl_2 \right\} \subset sl_4.
\]

Let \( \psi \) be the skew-symmetric bilinear form on \( \mathbb{C}^4 \) with the matrix

\[
\Psi = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\]

and let \( sp_4 \) denote the stabiliser of the form \( \psi \). That is, \( sp_4 = \{ g \in sl_4 \mid \Psi g + g^t \Psi = 0 \} \). Then one readily verifies that \( e_3 \subset sp_4 \). Hence \( e_3 \) is not wide in \( sl_4 \), in view of Lemma 5.1(ii). However,

**Lemma 5.6.** \( e_3 \) is wide in \( sp_4 \).

**Proof.** Let \( \tilde{\alpha}_1, \tilde{\alpha}_2 \) be the simple roots of \( sp_4 \) (\( \tilde{\alpha}_1 \) is short) and \( p(2) \) the parabolic subalgebra of \( sp_4 \) corresponding to \( \Pi' = \{ \tilde{\alpha}_2 \} \) (see notation of Section 2). Then \( e_3 \) is of codimension 1 in \( p(2) \) and \( e_3 \supseteq p(2)_{\text{nil}} \). In the above Eq. (5.1), \( p(2)_{\text{nil}} \) is the set of matrices with \( A = 0 \). By Theorem 2.3 and Lemma 5.1(i), we conclude that \( e_3 \) is wide in \( sp_4 \). \( \square \)

**Theorem 5.7.** The simple \( sl_4 \)-module \( R(\lambda) \) is \( e_3 \)-indecomposable if and only if \( \lambda \in \{ m\varphi_1, m\varphi_3 \} \) with any \( m \in \mathbb{N} \).

**Proof.** As is well-known, the simple \( sl_4 \)-modules \( R(m\varphi_1) \) and \( R(m\varphi_3) \) remain simple as \( sp_4 \)-modules. Hence they are \( e_3 \)-indecomposable. On the other hand, all other simple \( sl_4 \)-modules are decomposable as \( sp_4 \)-modules. This can be verified using Weyl’s dimension formula \([3, \text{Ch. VIII, \S 9.2}]\). Namely, if \( \lambda = a_1 \varphi_1 + a_2 \varphi_2 + a_3 \varphi_3 \), then \( R(\lambda)|_{sp_4} \) contains the simple \( sp_4 \)-module with highest weight \( \tilde{\lambda} = (a_1 + a_3)\tilde{\varphi}_1 + a_2\tilde{\varphi}_2 \). Then Weyl’s formula shows that \( \dim R(\lambda) > R(\tilde{\lambda}) \) if \( \lambda \neq m\varphi_1, m\varphi_3 \).
Alternatively, one can refer to the seminal work of E.B. Dynkin on maximal subgroups. Specifically, in [8, Theorem 4.1], Dynkin describes all irreducible representations of $\mathfrak{sl}_n$ that remain irreducible upon the restriction to a semisimple subalgebra.

**Remark.** The subalgebra $\mathfrak{sp}_4$ is symmetric in $\mathfrak{sl}_4$, and it is known that $\mathcal{R}(\lambda)^{\mathfrak{sp}_4} \neq 0$ if and only if $\lambda = m\varphi_2$. This again shows that $\mathcal{R}(m\varphi_2)$ is decomposable as $\mathfrak{sp}_4$-module for all $m \in \mathbb{N}$.

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