Special geometry on Calabi–Yau moduli spaces and $Q$–invariant Milnor rings

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ABSTRACT: The moduli spaces of Calabi–Yau (CY) manifolds are the special Kähler manifolds. The special Kähler geometry determines the low-energy effective theory which arises in Superstring theory after the compactification on a CY manifold. For the cases, where the CY manifold is given as a hypersurface in the weighted projective space, a new procedure for computing the Kähler potential of the moduli space has been proposed in [1–3]. The method is based on the fact that the moduli space of CY manifolds is a marginal subspace of the Frobenius manifold which arises on the deformation space of the corresponding Landau–Ginzburg superpotential. I review this approach and demonstrate its efficiency by computing the Special geometry of the 101-dimensional moduli space of the quintic threefold around the orbifold point [3].

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1 Introduction

To compute the low-energy Lagrangian of the string theory compactified on a CY manifold [4], one needs to know the Special geometry of the corresponding CY moduli space [5–8].

More precisely, the effective Lagrangian of the vector multiplets in the superspace contains $h^{2,1}$ supermultiplets. Scalars from these multiplets take value in the target space $M$, which is a moduli space of complex structures on a CY manifold and is a special Kähler manifold. Metric $G_{a\bar{b}}$ and Yukawa couplings $\kappa_{abc}$ on this space are given by the following formulae in the special coordinates $z^a$:

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K, \quad e^{-K} = -i \int_X 3 \Omega \wedge \bar{\Omega},$$
$$\kappa_{abc} = \int_X 3 \partial_a \partial_b \partial_c \Omega = \frac{\partial^3 F}{\partial z^a \partial z^b \partial z^c}, \quad (1.1)$$

where

$$z^a = \int_{A_a} \Omega, \quad \frac{\partial F}{\partial z^a} = \int_{B^a} \Omega$$

are the period integrals of the holomorphic volume form $\Omega$ on $X$. Here $A_a$ and $B^a$ form the symplectic basis in $H_3(X, \mathbb{Z})$.

We can rewrite the expression (1.1) for the Kähler potential using the periods as

$$e^{-K} = -i \Pi \Sigma \Pi^\dagger, \quad \Pi = (\partial F, z),$$

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where matrix \((\Sigma)^{-1}\) is an intersection matrix of cycles \(A_{\alpha}, B^\alpha\) equal to the symplectic unit.

The computation of periods in the symplectic basis appears to be very non-trivial. It was firstly performed for the case of the quintic CY manifold in the distinguished paper [9]. Here I present an alternative approach to the computation of Kähler potential for the case where CY manifold is given by a hypersurface \(W(x, \phi) = 0\) in a weighted projective space. The approach is based on the connection of CY manifold with a Frobenius ring which arises on the deformations of the singularity defined by the superpotential \(W_0(x)\) [10–12].

Let a CY manifold \(X\) be given as a solution of an equation

\[W(x, \phi) = W_0(x) + \sum_{s=1}^{k_{2,1}} \phi_s e_s(x) = 0\]

in some weighted projective space, where \(W_0(x)\) is a quasihomogeneous function in \(\mathbb{C}^5\) of degree \(d\) that defines an isolated singularity at \(x = 0\). The monomials \(e_s(x)\) also have degree \(d\) and are in a correspondence to deformations of the complex structure of \(X\).

Polynomial \(W_0(x)\) defines a Milnor ring \(R_0\). Inside \(R_0\) there exists a subring \(R_0^Q\) which is invariant under the action of the so-called quantum symmetry group \(Q\) that acts on \(\mathbb{C}^5\) diagonally, and preserves \(W(x, \phi)\). In many cases \(\dim R_0^Q = \dim H^3(X)\) and the ring itself has a Hodge structure \(R_0^Q = (R_0^Q)^0 \oplus (R_0^Q)^1 \oplus (R_0^Q)^2 \oplus (R_0^Q)^3\) in correspondence with the elements of degrees 0, \(d\), \(2d\), \(3d\).

Another important group is the subgroup of phase symmetries \(G\), which acts diagonally on \(\mathbb{C}^5\), commutes with the quantum symmetry \(Q\) and preserves \(W_0(x)\). It acts naturally on the invariant ring \(R_0^Q\), and this action respects the Hodge decomposition of \(R_0^Q\). This allows to choose a basis \(e_\mu(x)\) in each of the Hodge decomposition components of \(R_0^Q\) to be eigenvectors for the \(G\) group action.

On the ring \(R_0^Q\) we introduce the invariant pairing \(\eta\). The pairing turns the ring to a Frobenius algebra [13]. The pairing \(\eta\) plays an important for our construction of the explicit expression for the volume of the Calabi-Yau manifold.

Using the invariant ring \(R_0^Q\) and differentials \(D_{\pm} = d \pm dW_0 \wedge\) we construct two \(Q\)-invariant cohomology groups \(H_{D_{\pm}}^5(\mathbb{C}^5)_{inv}\). These groups inherit the Hodge structure from \(R_0^Q\). We can choose in \(H_{D_{\pm}}^5(\mathbb{C}^5)_{inv}\) the eigenbases \(e_\mu(x) d^5x\) which are also invariant under the phase symmetry action.

As shown in [14], elements of these cohomology groups are in correspondence with the harmonic forms of \(H^3(X)\). This isomorphism allows to define the antilinear involution \(*\) on the invariant cohomology \(H_{D_{+}}^5(\mathbb{C}^5)_{inv}\) that corresponds to the complex conjugation on the space of the harmonic forms of \(H^3(X)\).

It turns out, that in the basis \(e_\mu(x)\) it reads

\[*e_\mu(x) d^5x = M^\nu_\mu e_\nu(x) d^5x, \quad M^\nu_\mu = \delta_{e_\mu, e_\nu} \cdot A^\mu\]

where \(e_\rho(x)\) is the unique element of degree \(3d\) in \(R_0^Q\), and \(\delta_{e_\mu, e_\nu, e_\rho}\) is 1 if \(e_\mu \cdot e_\nu = e_\rho\) and
0 otherwise.

Having \( H^\pm_5(C^5)_{\text{inv}} \) we define the relative invariant homology subgroups \( H^\pm_5, \text{inv} := H_5(C^5, W_0 = L, \text{Re}L \to \pm \infty)_{\text{inv}} \) inside the relative homology groups \( H_5(C^5, W_0 = L, \text{Re}L \to \pm \infty) \). To do this we will use the oscillatory integrals and their pairing with elements of \( H^\pm_5(C^5)_{\text{inv}} \). Using this pairing we define a cycle \( \Gamma^\pm_\mu \) in the basis of relative invariant homology to be dual to \( e^{\mu}(x) d^5x \).

At last we define periods \( \sigma^\pm_\mu(\phi) \) to be oscillatory integrals over the basis of cycles \( \Gamma^\pm_\mu \). They are equal to periods of the holomorphic volume form \( \Omega \) on \( X \) in a special basis of cycles of \( H_3(X, \mathbb{C}) \) with complex coefficients.

It follows from the phase symmetry invariance that in the chosen basis of cycles \( \Gamma^\pm_\mu \) the formula for Kähler potential has the diagonal form:

\[
e^{-K(\phi)} = \sum_\mu (-1)^{|\nu|} \sigma^+_\mu(\phi) A^\nu \sigma^-_\mu(\phi).
\]

On the other hand, as shown in [1], matrix \( A = \text{diag}\{A^\mu\} \) is equal to the product of the matrix of the invariant pairing \( \eta \) in the Frobenius algebra \( R^Q_0 \) and the real structure matrix \( M \) such that

\[
e^{-K(\phi)} = \sum_{\mu, \nu} \sigma^+_{\mu}(\phi) \eta^{\mu \lambda} M^\nu_\lambda \sigma^-_{\mu}(\phi).
\]

The real structure matrix is nothing but matrix \( M \) from (1.2). Using this we are able to explicitly compute the diagonal matrix elements \( A^\mu \) and to obtain the explicit expression for the whole \( e^{-K} \).

2 The special geometry on the CY moduli space

It was shown in in [5–8] that the moduli space \( \mathcal{M} \) of complex (or Kähler) structures of a given CY manifold is a special Kähler manifold. Namely on \( \mathcal{M} \) there exist so-called special (projective) coordinates \( z^1 \cdots z^{n+1} \) and a holomorphic homogeneous function \( F(z) \) of degree 2 in \( z \), called a prepotential, such that the Kähler potential \( K(z) \) of the moduli space metric is given by

\[
e^{-K(z)} = \int_X \Omega \wedge \bar{\Omega} = z^a \cdot \frac{\partial F}{\partial z^a} - \bar{z}^a \cdot \frac{\partial F}{\partial \bar{z}^a}.
\]

To obtain this formula, we choose Poincare dual symplectic bases \( \alpha^a, \beta^b \in H^3(X, \mathbb{Z}) \) and \( A^a, B_b \in H_3(X, \mathbb{Z}) \) and define the periods as

\[
z^a = \int_{A^a} \Omega, \quad F_b = \int_{B_b} \Omega.
\]

Then using the Kodaira Lemma

\[
\partial_a \Omega = k_a \Omega + \chi_a,
\]

we can show that

\[
F_a(z) = \frac{1}{2} \partial_a(F(z)),
\]

\[ -3 - \]
where $F(z) = 1/2z^bF_{b}(z)$.

Therefore, according to the definition (2.1) metric $G_{a\bar{b}} = \partial_a\bar{\partial}_b K(z)$ is a special Kähler metric with prepotential $F(z)$ and with the special coordinates given by the period vector

$$
\Pi = \left(F_\alpha, z^b\right)
$$

we write the expression for the Kähler potential as

$$
e^{-K(z)} = \Pi_\mu \Sigma^{\mu\nu} \bar{\Pi}_\nu,
$$

(2.2)

where $\Sigma$ is a symplectic unit, which is an inverse intersection matrix for cycles $A^a$ and $B_b$.

Using formula (2.2), we can rewrite this expression in a basis of periods defined as integrals over arbitrary bases of cycles $q_\mu \in H_3(X, \mathbb{Z})$

$$
\omega_\mu = \int_{q_\mu} \Omega.
$$

Such that

$$
e^{-K} = \omega_\mu C^{\mu\nu} \bar{\omega}_\nu,
$$

where $C^{\mu\nu}$ is the inverse matrix of the intersection of the cycles $q_\mu$.

So to find the Kähler potential, we must compute the periods over a basis of cycles on CY manifold and find their intersection matrix.

### 3 Hodge structure on the middle cohomology of the quintic

Now let us specialize to the case where $X$ is a quintic threefold:

$$
X = \{(x_1 : \cdots : x_5) \in \mathbb{P}^4 \mid W(x, \phi) = 0\},
$$

and

$$
W(x, \phi) = W_0(x) + \sum_{t=0}^{100} \phi_t e_t(x), \quad W_0(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5
$$

(3.1)

and $e_t(x)$ are the degree 5 monomials such that each variable has the power that is a non-negative integer less than four. Let us denote monomials $e_t(x) = x_1^{t_1}x_2^{t_2}x_3^{t_3}x_4^{t_4}x_5^{t_5}$ by its degree vector $t = (t_1, \cdots, t_5)$. Then there are precisely 101 of such monomials, which can be divided into 5 sets in respect to the permutation group $S_5$: $(1, 1, 1, 1, 1), (2, 1, 1, 1, 0), (2, 2, 1, 0, 0), (3, 1, 1, 0, 0), (3, 2, 0, 0, 0)$. In these groups there are correspondingly 1, 20, 30, 30, 20 different monomials. We denote $e_0(x) := e_{(1,1,1,1,1)}(x) = x_1x_2x_3x_4x_5$ to be the so-called fundamental monomial, which will be somewhat distinguished in our picture.
For this CY dim $H_3(X) = 204$ and period integrals have the form

$$\omega_\mu(x) = \int_{q_\mu} \frac{x_5 \, dw \, dx_2 \, dx_3}{\partial W(x, \phi)/\partial x_4} = \int_{Q_\mu} \frac{dx_1 \cdots dx_5}{W(x, \phi)},$$

where $q_\mu \in H_3(X, \mathbb{Z})$ and the corresponding cycles $Q_\mu \in H_5(\mathbb{C}^5 \setminus (W(x, \phi) = 0), \mathbb{Z})$.

Cohomology groups of the Kähler manifold $X$ possess a Hodge structure $H^3(X) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)$. Period integrals measure variation of the Hodge structure on $H^3(X)$ as the complex structure on $X$ varies with $\phi$.

This Hodge structure variation is in correspondence with a Frobenius ring which we will now describe.

## 4 Hodge structure on the invariant Milnor ring

Now we will consider $W_0(x)$ as an isolated singularity in $\mathbb{C}^5$ and the associated with it Milnor ring

$$R_0 = \frac{\mathbb{C}[x_1, \ldots, x_5]}{\langle \partial W_0 \rangle}.$$ 

We can choose its elements to be unique smallest degree polynomial representatives. For the quintic threefold $X$ its Milnor ring $R_0$ is generated as a vector space by monomials where each variable has degree less than four, and dim $R_0 = 1024$.

Since the polynomial $W_0(x)$ is homogeneous one of the fifth degree it follows that $W_0(\alpha x_1, \ldots, \alpha x_5) = W_0(x_1, \ldots, x_5)$ for $\alpha^5 = 1$. This action preserves $W_0(x)$ and is trivial in the corresponding projective space and on $X$. Such a group with this action is called a quantum symmetry $Q$, in our case $Q \simeq \mathbb{Z}_5$. $Q$ obviously acts on the Milnor ring $R_0$.

We define a subring $R_0^Q$ to be a $Q$-invariant part of the Milnor ring

$$R_0^Q := \{ e_\mu(x) \in R_0 \mid e_\mu(\alpha x) = e_\mu(x) \}, \alpha^5 = 1.$$ 

$R_0^Q$ is multiplicatively generated by 101 fifth-degree monomials $e_l(x)$ from (3.1) and consists of elements of degree 0, 5, 10 and 15. The dimensions of the corresponding subspaces are 1, 101, 101 and 1.

This degree filtration defines a Hodge structure on $R_0^Q$. Actually, $R_0^Q$ is isomorphic to $H^3(X)$ and this isomorphism sends the degree filtration on $R_0^Q$ to the Hodge filtration on $H^3(X)$ [14].

Let us denote $\chi^i_j = g^i_k \chi^j_k$ as an extrinsic curvature tensor and $g_{ik}$ is a metric for the hypersurface $W(x, \phi) = 0$ in $\mathbb{P}^4$. Then the isomorphism above can be written as a map from $R_0^Q$ to closed differential forms in $H^3(X)$:

$$1 \rightarrow \Omega_{ijk} \in H^{3,0}(X),$$

$$\chi^1_0(x) \rightarrow e_\mu(x) \chi^1_0 \Omega_{ijk} \in H^{2,1}(X) \text{ if } |\mu| = 5,$$

$$e_\mu(x) \rightarrow e_\mu(x) \chi^1_0 \chi^m_0 \Omega_{lmk} \in H^{1,2}(X) \text{ if } |\mu| = 10,$$

$$e_\mu(x) = x_1^3 x_2^3 x_3^3 x_4^3 x_5^3 \rightarrow \chi^1_0 \chi^m_0 \chi^p_0 \Omega_{lmk} = \kappa \Omega \in H^{0,3}(X).$$ (4.1)
The details of this map can be found in [14]. We also introduce the notation \( e_\mu(x) \) for elements of the monomial basis of \( R_0^Q \), where \( \mu = (\mu_1, \cdots, \mu_5) \), \( \mu_i \in \mathbb{Z}_+ \), \( e_\mu(x) = \prod_i x_i^{\mu_i} \) and the degree of \( e_\mu(x) \) \( \mu = \sum \mu_i \) is equal to zero module 5. In particular, \( \rho = (3, 3, 3, 3, 3) \), that is \( e_\rho(x) \) is the unique degree 15 element of \( R_0^Q \).

The phase symmetry group \( \mathbb{Z}_5^3 \) acts diagonally on \( \mathbb{C}^5 \): \( \alpha \cdot (x_1, \cdots, x_5) = (\alpha_1 x_1, \cdots, \alpha_5 x_5) \), \( \alpha_i^5 = 1 \). This action preserves \( W_0 = \sum_i x_i^5 \). The mentioned above quantum symmetry \( Q \) is a diagonal subgroup of the phase symmetries. Basis \( \{e_\mu(x)\} \) consists of the eigenvectors of the phase symmetry and each \( e_\mu(x) \) has a unique weight. Note that the action of the phase symmetry preserves the Hodge decomposition.

Another important fact is that on the invariant ring \( R_0^Q \) there exists a natural invariant pairing turning it into a Frobenius algebra [13]:

\[
\eta_{\mu \nu} = \text{Res} \frac{e_\mu(x) e_\nu(x)}{\prod_i \partial_i W_0(x)}.
\]

Up to an irrelevant constant for the monomial basis it is \( \eta_{\mu \nu} = \delta_{\mu+\nu, \rho} \). This pairing plays a crucial role in our construction.

Let us introduce a couple of Saito differentials as in [1] on differential forms on \( \mathbb{C}^5 : D_\pm = d \pm dW_0(x) \wedge \). They define two cohomology groups \( H^*_D^+ (\mathbb{C}^5) \). The cohomologies are only nontrivial in the top dimension \( H^5_D^+ (\mathbb{C}^5) \simeq R_0 \). The isomorphism \( J \) has an explicit description

\[
J(e_\mu(x)) = e_\mu(x) \d^5 x, \quad e_\mu(x) \in R_0.
\]

We see, that \( Q = \mathbb{Z}_5 \) naturally acts on \( H_5^D (\mathbb{C}^5) \) and \( J \) sends the elements of \( Q \)-invariant ring \( R_0^Q \) to \( Q \)-invariant subspace \( H_5^D (\mathbb{C}^5)_{\text{inv}} \). Therefore, the latter space obtains the Hodge structure as well. Actually, this Hodge structure naturally corresponds to the Hodge structure on \( H^3(X) \).

The complex conjugation acts on \( H^3(X) \) so that \( \overline{H^{p,q}(X)} = H^{q,p}(X) \), in particular \( \overline{H^{2,1}(X)} = H^{1,2}(X) \). Through the isomorphism between \( R_0^Q \) and \( H^3(X) \) the complex conjugation acts also on the elements of the ring \( R_0^Q \) as \( * e_\mu(x) = p_\mu e_{\rho-\mu}(x) \), where \( p_\mu p_{\rho-\mu} = 1 \) and \( p_\mu \) is a constant to be determined. In particular, differential form built from the linear combinations \( e_\mu(x) + p_\mu e_{\rho-\mu}(x) \in H^3(X, \mathbb{R}) \) is real.

## 5 Oscillatory representation and computation periods \( \sigma_\mu(\phi) \)

Relative homology groups \( H_5(\mathbb{C}^5, W_0 = L, \text{Re} L \to \pm \infty) \) have a natural pairing with \( Q \)-invariant cohomology groups \( H_5^D (\mathbb{C}^5)_{\text{inv}} \) defined as

\[
\langle e_\mu(x) d^5 x, \Gamma^\pm \rangle = \int_{\Gamma^\pm} e_\mu(x) e^{\mp W_0(x)} d^5 x, \quad H_5(\mathbb{C}^5, W_0 = L, \text{Re} L \to \pm \infty).
\]

Using this we introduce two \( Q \)-invariant homology groups\(^1\) \( H_5^{\pm, \text{inv}} \) as quotient of \( H_5(\mathbb{C}^5, W_0 = L, \text{Re} L \to \pm \infty) \) with respect to the subgroups orthogonal to \( H_5^D (\mathbb{C}^5)_{\text{inv}} \).

\(^1\)We are grateful to V. Vasiliev for explaining to us the details about these homology groups and their connection with the middle homology of \( X \).

- 6 -
Now we introduce bases $\Gamma^{\pm}_\mu$ in the homology groups $H^{\pm}_{D_5}(\mathbb{C}^5)_{inv}$ using the duality with the bases in $H^5_{D_5}(\mathbb{C}^5)_{inv}$:

$$\int_{\Gamma^{\pm}_\mu} e_\nu(x) e^{\mp W_0(x)} d^5 x = \delta_{\mu \nu},$$

and the corresponding periods

$$\sigma^{\pm}_{\alpha \mu}(\phi) := \int_{\Gamma^{\pm}_\mu} e_\alpha(x) e^{\mp W(x, \phi)} d^5 x,$$

$$\sigma^{\mp}_{\mu}(\phi) := \sigma^{\pm}_{0 \mu}(\phi)$$

which are understood as series expansions in $\phi$ around zero.

The periods $\sigma^{\pm}_{\mu}(\phi)$ satisfy the same differential equation as periods $\omega_{\mu}(\phi)$ of the holomorphic volume form on $X$. Moreover, these sets of periods span same subspaces as functions of $\phi$. Therefore we can define cycles $Q^{\pm}_\mu \in H^{\pm}_{D_5}(\mathbb{C}^5)_{inv}$ such that

$$\int_{Q^{\pm}_\mu} e^{\mp W(x, \phi)} d^5 x = \int_{q_\mu} \Omega = \int_{Q^{\pm}_\mu} \frac{d^5 x}{W(x, \phi)}.$$  \hfill (5.2)

So the periods $\omega^{\pm}_{\alpha \mu}(\phi)$ are given by the integrals over these cycles analogous to (5.1).

With these notations the idea of computation of periods [15]

$$\sigma^{\pm}_{\mu}(\phi) = \int_{\Gamma^{\pm}_\mu} e^{\mp W(x, \phi)} d^5 x$$  \hfill (5.3)

can be stated as follows.

To explicitly compute $\sigma^{\pm}_{\mu}(\phi)$, first we expand the exponent in the integral (5.3) in $\phi$ representing $W(x, \phi) = W_0(x) + \sum_s \phi_s e_s(x)$

$$\sigma^{\pm}_{\mu}(\phi) = \sum_m \left( \prod_s \frac{(\pm \phi_s)^{m_s}}{m_s!} \right) \int_{\Gamma^{\pm}_\mu} \prod_s e_s(x)^{m_s} e^{\mp W_0(x)} d^5 x.$$  \hfill (5.4)

We note, that $\sigma^-_{\mu}(\phi) = (-1)^{|\mu|} \sigma^+_{\mu}(\phi)$, so we focus on $\sigma_{\mu}(\phi) := \sigma^+_{\mu}(\phi)$.

For each of the summands in (5.4) the form $\prod_s e_s(x)^{m_s} d^5 x$ belongs to $H^5_{D_5}(\mathbb{C}^5)_{inv}$, because it is $Q$–invariant. Therefore, we can expand it in the basis $e_{\mu}(x) d^5 x \in H^5_{D_5}(\mathbb{C}^5)_{inv}$. Namely we can find such a polynomial 4–form $U$, that

$$\prod_s e_s(x)^{m_s} d^5 x = \sum_{\nu} C_{\nu}(m) e_{\nu}(x) d^5 x + D_{+}U.$$  

In result we obtain for the integral in (5.4)

$$\int_{\Gamma^{\pm}_\mu} \prod_s e_s(x)^{m_s} e^{\mp W_0(x)} d^5 x = C_{\mu}(m).$$
So from (5.4) we have

$$\sigma_\mu(\phi) = \sum_m \left( \prod_s \frac{\phi^{m_s}_s}{m_s!} \right) \int_{\Gamma_{\mu}} \prod_{s,i} x_i^{\sum_s m_s s_i} e^{-W_\mu(x)} d^5 x. \quad (5.5)$$

We can rewrite the sum in the exponent of $x_i$ as $\sum_s m_s s_i = 5n_i + \nu_i$, $\nu_i < 5$. Therefore we need to compute the coefficients $c^m_\nu$ in the equations

$$\prod_i x_i^{5n_i + \nu_i} d^5 x = \sum_\nu c^m_\nu e_\nu(x) d^5 x + D_+ U.$$ 

Note that

$$D_+ \left( \frac{1}{5} x_1^{5n+k-4} f(x_2, \cdots, x_5) dx_2 \wedge \cdots \wedge dx_5 \right) =$$

$$= \left[ x_1^{5n+k} + \left( n + \frac{k - 4}{5} \right) x_1^{5(n-1)+k} \right] f(x_2, \cdots, x_5) d^5 x \quad (5.6)$$

Therefore in $D_+$ cohomology we have

$$\prod_i x_i^{5n_i + \nu_i} d^5 x = - \left( n_1 + \frac{\nu_1 - 4}{5} \right) x_1^{5(n-1)+\nu_i} \prod_{i=2}^5 x_i^{5n_i + \nu_i} d^5 x, \nu_i < 5. \quad (5.7)$$

By induction we obtain

$$\prod_i x_i^{5n_i + \nu_i} d^5 x = (-1)^{\Sigma_i \nu_i} \prod_i \left( \frac{\nu_i + 1}{5} \right) n_i \prod_i x_i^{\nu_i} d^5 x, \nu_i < 5. \quad (5.8)$$

where $(a)_n = \Gamma(a + n)/\Gamma(a)$.

Using (5.6) once again, we see that if any $\nu_i = 4$ then the differential form is trivial and the integral is zero. Hence, rhs of (5.8) is proportional to $e_\nu(x)$ and gives the desired expression. Plugging (5.8) into (5.5) and integrating over $\Gamma^+_\mu$ we obtain the answer

$$\sigma_\mu(\phi) = \sigma^+_\mu(\phi) = \sum_{n_i \geq 0} \prod_i \left( \frac{\mu_i + 1}{5} \right) \sum_{n_i \in \Sigma_n} \prod_s \frac{\phi^{m_s}_s}{m_s!},$$

where

$$\Sigma_n = \{ m \mid \sum_s m_s s_i = 5n_i + \mu_i \}.$$ 

Further we will also use the periods with slightly different normalization, which turn out to be convenient

$$\sigma_\mu^*(\phi) = \prod_i \Gamma \left( \frac{\mu_i + 1}{5} \right) \sigma_\mu(\phi) = \sum_{n_i \geq 0} \prod_i \Gamma \left( n_i + \frac{\mu_i + 1}{5} \right) \sum_{m \in \Sigma_n} \prod_s \frac{\phi^{m_s}_s}{m_s!}. \quad (5.9)$$
6 Computation of the Kähler potential

Pick any basis $Q^\pm_\mu$ of cycles with integer or real coefficients as in (5.2). Then for the Kähler potential we have the formula

$$e^{-K} = \omega_\mu^+(\phi) C^{\mu\nu} \omega_\nu^-(\phi)$$

(6.1)

in which the matrix $C^{\mu\nu}$ is related with the Frobenius pairing $\eta$ as

$$\eta_{\alpha\beta} = \omega^+_{\alpha\mu}(0) C^{\mu\nu} \omega^-_{\beta\nu}(0).$$

(6.2)

The derivation of the last relation is given in [16, 17].

Let also $T^\pm_\nu$ be the matrix that connects the cycles $Q^\pm_\mu$ and $\Gamma^\pm_\nu$.

That is

$$Q^\pm_\mu = (T^\pm_\nu)^\mu \Gamma^\pm_\nu.$$

Then $M = (T^-)^\mu_\nu \Gamma^\pm_\nu$ is a real structure matrix, that is $M \bar{M} = 1$ and by construction $M$ doesn’t depend on the choice of basis $Q^\pm_\mu$. $M$ is only defined by our choice of $\Gamma^\pm_\mu$.

In [1] we deduced from (6.1) and (6.2) the formula

$$e^{-K(\phi)} = \sigma^+_{\mu}(\phi) \eta^{\mu\lambda} M^{\nu}_{\lambda} \sigma^-_{\nu}(\phi) = \sigma_{\mu} A^{\mu\nu} \sigma^-_{\nu},$$

(6.3)

where $\eta^{\mu\nu} = \eta_{\mu\nu} = \delta_{\mu,\nu} \delta_{\mu,\nu}$.

Now we show that the matrix $A^{\mu\nu}$ in (6.3) is diagonal. To do this we extend the action of the phase symmetry group to the action $\mathcal{A}$ on the parameter space $\{\phi_s\}$ such that $W = W_0 + \sum_s \phi_s e_s(x)$ is invariant under this new action. It easy to see that each $e_s(x)$ has an unique weight under this group action. Action $\mathcal{A}$ can be compensated using the coordinate tranformation and therefore is trivial on the moduli space of the quintic (implying that point $W = W_0$ is an orbifold point of the moduli space).

In particular, $e^{-K} = \int_X \Omega \wedge \bar{\Omega}$ is $\mathcal{A}$ invariant. Consider

$$e^{-K} = \sigma_{\mu} A^{\mu\nu} \sigma^-_{\nu}$$

as a series in $\phi_s$ and $\bar{\phi}_t$. Each monomial has a certain weight under $\mathcal{A}$, For the series to be invariant, each monomial must have weight 0. But weight of $\sigma_{\mu} \bar{\sigma}_{\nu}$ equals to $\mu - \nu$ and due to non-degeneracy of weights of $\sigma_{\mu}$ only the ones with $\mu = \nu$ have weight zero. Thus, (6.3) becomes

$$e^{-K} = \sum_{\mu} A^\mu |\sigma_{\mu}(\phi)|^2.$$

Moreover, the matrix $A$ should be real and, because $A = \eta \cdot M, \ M \bar{M} = 1$ and $\eta_{\mu\nu} = \delta_{\mu+\nu,\mu}$, we have

$$A^\mu A^{\rho-\mu} = 1.$$

(6.4)

Monodromy considerations To fix finally the real numbers $A^\mu$ we use monodromy invariance of $e^{-K}$ around $\phi_0 = \infty$. Pick some $t = (t_1, t_2, t_3, t_4, t_5)$ with $|t| = 5$ and let $\phi_s|_{s \neq t, 0} = 0$. We will consider only the first order in $\phi_t$. 

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Then the condition that period $\sigma_\mu(\phi)$ contains only non-zero summands of the form $\phi_0^{n_\mu} \phi_t$ implies that $\mu = t + \text{const} \cdot (1, 1, 1, 1, 1) \mod 5$. For each $t$ from the table below the only such possibilities are $\mu = t$ and $\mu = \rho - \rho' = (3, 3, 3, 3, 3) - t'$, where $t'$ denotes a vector obtained from $t$ by permutation (written explicitly in the table below) of its coordinates.

Therefore, in this setting (6.3) becomes

$$e^{-K} = \sum_{k=0}^{3} a_k |\hat{\sigma}_{(k,k,k,k,k)}|^2 + a_t |\hat{\sigma}_t|^2 + a_{\rho-\rho'} |\hat{\sigma}_{\rho-\rho'}|^2 + O(\phi_5^2),$$

here we use periods $\hat{\sigma}$ from (5.9) and denote $a_t = A^t / \prod_i \Gamma((t_i+1)/5)^2$. And the coefficients $a_k, k = 0, 1, 2, 3$ are already known from [9]. This expression has to be monodromy invariant under the transport of $\phi_0$ around $\infty$. From the formula (5.9) we have

$$F_1 = \hat{\sigma}_t(\phi_t, \phi_0) = g_t \phi_k F(a, b; a + b | (\phi_0/5)^3) + O(\phi_5^6),$$

$$F_2 = \hat{\sigma}_{\rho-\rho'}(\phi_t, \phi_0) = g_{\rho-\rho'} \phi_t \phi_0^{1-a-b} F(1-a, 1-b; 2-a-b | (\phi_0/5)^3) + O(\phi_5^6),$$

where $g_t, g_{\rho-\rho'}$ are some constants. Explicitly for all different labels $t$

| $t$   | $\rho - \rho'$ | (a, b) |
|-------|----------------|--------|
| (2,1,1,1,0) | (3,2,2,2,1)  | (2,5,2,5) |
| (2,2,1,0,0) | (3,3,2,1,1)  | (1,5,3,5) |
| (3,1,1,0,0) | (0,3,3,2,2)  | (1,5,2,5) |
| (3,2,0,0,0) | (1,0,3,3,3)  | (1,5,1,5) |

When $\phi_0$ goes around infinity

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = B \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},$$

where

$$B = \frac{1}{|s(a + b)|} \left( \frac{c(a-b) - e^{i\pi(a+b)}}{2e^{2\pi i(a+b)} s(a) s(b) e^{\pi i(a+b)} [e^{2\pi i a} + e^{2\pi i b} - 2]/2} \right).$$

Here $c(x) = \cos(\pi x)$, $s(x) = \sin(\pi x)$. It is straightforward to show the following

**Proposition 1.**

$$a_t |\hat{\sigma}_t|^2 + a_{\rho-\rho'} |\hat{\sigma}_{\rho-\rho'}|^2 = a_t \prod_i \Gamma \left( \frac{t_i + 1}{5} \right)^2 |\sigma_t|^2 + a_{\rho-\rho'} \prod_i \Gamma \left( \frac{4 - t_i}{5} \right)^2 |\sigma_{\rho-\rho'}|^2$$

is $B$-invariant iff $a_t = -a_{\rho-\rho'}$.

Due to symmetry we have $a_{\rho-\rho'} = a_{\rho-t}$ in each case. From (6.4) it follows that the product of the coefficients at $|\sigma_\mu|^2$ and $|\sigma_{\rho-\mu}|^2$ in the expression for $e^{-K}$ should be 1:

$$A^\rho \cdot A^t = a_{\rho-\rho'} \cdot a_t \prod_i \Gamma \left( \frac{t_i + 1}{5} \right)^2 \Gamma \left( \frac{4 - t_i}{5} \right)^2 = 1.$$
Due to reflection formula $a_t = \pm \prod_i \sin(\pi(t_i + 1)/5)$ up to a common factor of $\pi$. The sign turns out to be minus for Kähler metric to be positive definite in the origin. Therefore

$$A^\mu = (-1)^{\text{deg}(\mu)/5} \prod_i \gamma \left( \frac{\mu_i + 1}{5} \right).$$

Finally the Kähler potential becomes

$$e^{-K(\phi)} = \sum_{\mu=0}^{203} (-1)^{\text{deg}(\mu)/5} \prod \gamma \left( \frac{\mu_i + 1}{5} \right) |\sigma_\mu(\phi)|^2,$$

(6.5)

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$.

7 Real structure on the cycles $\Gamma^\pm_\mu$

Let cycles $\gamma_\mu \in H_3(X)$ be the images of cycles $\Gamma^+_\mu$ under the isomorphism $\mathcal{H}^+_3,^{\text{inv}} \simeq H_3(X)$.

Complex conjugation sends $(2, 1)$-forms to $(1, 2)$-forms. Similarly it extends to a mapping on the dual homology cycles $\gamma_\mu$.

**Lemma 1.** Conjugation of homology classes has the following form: $*\gamma_\mu = p_\mu \gamma_{\rho - \mu}$, where $\rho = (3, 3, 3, 3, 3)$ is a unique maximal degree element in the Milnor ring.

**Proof.** We perform a proof for the cohomology classes represented by differential forms. For one-dimensional $H^{3,0}(X)$ and $H^{0,3}(X)$ it is obvious. Let

$$\Omega_{2,1} := e_1(x) \chi^1_\mu \Omega_{ijk} \in H^{2,1}(X).$$

Any element from $H^{1,2}(X)$ is representable by a degree 10 polynomial $P(x)$ as follows from (4.1) as

$$\overline{\Omega_{2,1}} = \Omega_{1,2} := P(x) \chi^1_\mu \chi^m_\gamma \Omega_{lmk} \in H^{1,2}(X).$$

The group of phase symmetries modulo common factor acts by isomorphisms on $X$. Therefore, it also acts on the differential forms. Lhs and rhs of the previous equation should have the same weight under this action, and weight of the lhs is equal $-t$ modulo $(1, 1, 1, 1, 1)$. It follows that $P(x) = p_t e_{\rho - t}(x)$ with some constant $p_t$. \qed

Using this lemma and applying the complex conjugation of cycles to the formula (6.3) to obtain

$$e^{-K} = \sum_{\mu} A^\mu |\sigma_\mu|^2 = \sum_{\mu} p_\mu^2 A^\mu |\sigma_{\rho - \mu}|^2;$$

it follows that $A^\mu = \pm 1/p_\mu$. Now formula (6.5) implies

$$p_\mu = \prod \gamma \left( \frac{4 - \mu_i}{5} \right).$$
8 Conclusion

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