Tidal forces near a black hole with scalar hairy

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Abstract. We deal with static, asymptotically flat, spherically symmetric black holes supported by a minimally coupled scalar field with an arbitrary self-interaction potential. We consider a scalar black hole as a simple model of supermassive black holes in the centers of galaxies surrounded by dark matter. Both the radius of the innermost stable circular orbit and the event horizon radius of such an object are less than those of a Schwarzschild black hole with the same mass. Moreover, they can be arbitrary small, so that tidal forces, acting on a star orbiting a scalar field black hole near its photon sphere, can be extremely large and can disrupt the star. This means, in turn, that tidal effects can play an important role for the interpretation of observations in galactic astrophysics.

1. Introduction
One of the key problem in modern astrophysics is to identify with confidence the nature of strongly gravitating objects at the centers of normal galaxies [1, 2, 3, 4, 5, 6, 7, 8]. Observations of stars that are disrupted by tidal forces give us new possibilities for identification of these objects. In this paper, we compare tidal forces in the central regions of scalar field black holes of some fixed mass with those in a Schwarzschild black hole of the same mass. The motivation for this problem comes from the two following conceptions concerning supermassive black holes at the centers of galaxies. First, such black holes are surrounded by dark matter [12, 13, 14] and, thus, should not be thought of as objects located in an empty space. In our approach, dark matter in galaxies is modelled by a nonlinear self-gravitating scalar field [15, 16, 17]. Second, they are surrounded by solar-type stars which orbiting the supermassive black holes. Both these conceptions are well confirmed by modern astrophysical observations. Note, however, that there are a number of other intriguing ways for a description of dark matter in galaxies [18, 19, 20]. Our choice is supposed to give us a sufficiently wide range of possibilities to model the distribution of dark matter in the inner regions; the possibilities are provided by an arbitrary choice of a scalar field self-interaction potential.

In Section 2, we describe the mathematical background for a spherically symmetric, self-gravitating, minimally coupled scalar field with an arbitrary self-interaction potential. Section 3 is devoted to an analytical formulation and relativistic treatment of tidal forces. In Section 4, in order to clarify the difference between vacuum and scalar field black holes, we provide an illustrative example.

In this paper we adopt the metric signature \((+−−−)\) and use the geometrical system of units with \(G = 1, c = 1\). The signs of the Riemann and Ricci tensor fields are defined as \(R_{ijkl}^j = \partial_k \Gamma^{ij}_{jl} - \ldots\) and \(R_{jl} = R^j_{jl}\), respectively.
2. Spherically symmetric scalar field black holes

The action for a self-interacting real scalar field minimally coupled to gravity has the form

$$S = \frac{1}{8\pi} \int \left( -\frac{R}{2} - \langle d\phi, d\phi \rangle - 2V(\phi) \right) \sqrt{|g|} d^4 x, \quad (1)$$

where $R$ is the scalar curvature, the angle brackets denote the pointwise scalar product with respect to the metric, $\phi$ is a scalar field, and $V(\phi)$ is its self-interaction potential. For our purpose, it is convenient to write the metric of a static spherically symmetric spacetime in the area coordinates as

$$ds^2 = Adt^2 - \frac{dr^2}{f} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad A = e^{2F} F,$$

$$\quad (2)$$

where $F$ and $f$ depend only on the radial coordinate. The Einstein-Klein-Gordon field equations can be reduced to the three independent ones:

$$- \frac{f'}{r} - \frac{f - 1}{r^2} = \varepsilon \phi'^2 f + 2V, \quad (3)$$

$$\frac{f'}{r} \left( 2F' + \frac{F'}{f} \right) + \frac{f - 1}{r^2} = \varepsilon \phi'^2 f - 2V, \quad (4)$$

$$- f \phi'' - \frac{\phi'}{2} f' - \phi f \left( F' + \frac{1}{2} \frac{f'}{f} + \frac{2}{r} \right) + \varepsilon \frac{dV}{d\phi} = 0, \quad (5)$$

where a prime denotes differentiation with respect to $r$.

Our approach is based on a variant of the ‘restored potential method’ for constructing solutions to the Einstein-Klein-Gordon equations [21, 22, 23, 24]. Namely, we use the quadrature formulae [25]

$$F(r) = - \int_r^\infty \phi'^2 r dr, \quad \xi(r) = r + \int_r^\infty \left( 1 - e^F \right) dr,$$

$$A(r) = 2r^2 \int_r^\infty \frac{\xi - 3M}{r^4} e^F dr, \quad f(r) = e^{-2F} A,$$

$$\tilde{V}(r) = \frac{1}{2r^2} \left( 1 - 3f + r^2 \phi'^2 + 2 e^{-F} \frac{\xi - 3M}{r} \right), \quad (8)$$

which reduce equations (3)–(5) to identities. In this method, $\phi(r)$ is usually assumed to be specified as a monotonic function of class $C^2[(0, \infty)]$ with the asymptotic behaviour

$$\phi = O\left( r^{-1/2 - \alpha} \right), \quad r \to \infty \quad (\alpha > 0)$$

which, in turn, implies

$$e^F = 1 + o(r^{-1}), \quad \xi = r + o(1) \quad \text{as} \quad r \to \infty, \quad \text{and} \quad \xi(0) > 0. \quad (10)$$

Here we will use the quadratures as follows [25]. First, note that $F < 0$, $\xi > 0$, $\xi' = e^F > 0$, and $\xi'' = r \phi'^2 e^F > 0$ for all $r > 0$. Therefore for a given strictly monotonically increasing and strictly convex downwards function $\xi(r)$, satisfying the conditions (10), we can sequentially compute the functions $e^F(r), A(r), f(r)$. Second, it follows from (6) that one can find the function $\phi(r)$ by solving the problem $\phi' = \sqrt{F' / r}$, $\phi(\infty) = 0$, and then find $\tilde{V}(r)$ from (8) by direct
calculation. Third, the requirements of convexity of \( \xi(r) \) implies that \( \phi(r) \) is monotonic, and hence the self-interaction potential \( V(\phi) = \tilde{V}(r(\phi)) \) is a well-defined function.

From the quadrature (7) and the asymptotic conditions (10), we have

\[
A(r) = 1 - \frac{2M}{r} + o(1/r), \quad r \to \infty, \quad A(r) = \frac{2}{3} \frac{\xi(0) - 3M}{r} e^{F(0)} + O(1), \quad r \to 0,
\]

so that the parameter \( M \) is the Schwarzschild mass, and the spacetime under consideration is a black hole if and only if \( \xi(0) < 3M \). From (7) and the condition \( \xi(r) > r \), it can be seen that the event horizon radius \( r_h \) of a black hole of mass \( M > \xi(0)/3 \) is always less than the corresponding Schwarzschild radius \( r_s = 2M \). Moreover, \( r_h \to 0 + 0 \) if \( M \to \xi(0)/3 + 0 \). For the most part, we are interested in compact black holes whose masses (for a given function \( \xi(r) \)) are sufficiently close to the threshold value \( \xi(0)/3 \): the condition \( M \gtrsim \xi(0)/3 \) implies \( r_h \ll 2M \).

3. Tidal forces in the center-of-mass frame

We will use the orthonormal basis of vector fields, associated with the metric (2), and the dual basis of 1-forms, so that the metric components become \( (g_{ij}) = \text{diag}(1, -1, -1, -1) \). These bases are given, respectively, by

\[
e_0 = \frac{1}{e^f \sqrt{f}} \partial_t, \quad e_1 = \sqrt{f} \partial_r, \quad e_2 = \frac{1}{r} \partial_\theta, \quad e_3 = \frac{1}{r \sin \theta} \partial_\varphi,
\]

and

\[
e^0 = e^f \sqrt{f} dt, \quad e^1 = \frac{1}{\sqrt{f}} dr, \quad e^2 = r d\theta, \quad e^3 = r \sin \theta d\varphi.
\]

In an orthonormal basis, the connection 1-forms \( \omega_j^i \) are defined by \( \nabla_X e_j = \omega_j^i(X)e_i \) and obeys the relations \( \omega_0^0 = \omega_0^0, \ \omega_0^\beta = -\omega_\beta^0 \ (\alpha, \beta = 1, 2, 3) \). The connection and the curvature can be found from the Cartan structure equations. Without going into details, the algebraically independent connection 1-forms and the components of the curvature in the bases (12) and (13) are, respectively,

\[
\omega_0^1 = \sqrt{f} \left( \frac{f'}{2f} + F' \right) e^0, \quad \omega_1^0 = \sqrt{f} \left( \frac{A'}{2A} \right) e^0, \quad \omega_1^2 = \frac{\sqrt{f}}{r} e^2, \quad \omega_2^0 = \frac{\cot \theta}{r} e^3, \quad \omega_2^1 = \frac{\sqrt{f}}{r} e^3, \quad \omega_3^0 = \frac{\cot \theta}{r} e^3,
\]

and

\[
R_{0101} = -\frac{f''}{2} - \frac{3}{2} F' f' - F'' f - F'^2 f' = \frac{1}{r^2} - \phi^2 f, \quad R_{1212} = R_{1313} = \frac{f'}{2r}, \quad R_{0202} = R_{0303} = -\frac{F' f'}{r} - \frac{f'}{2r} = -\frac{f'}{2r} - \phi^2 f, \quad R_{2323} = -\frac{1 - f}{r^2}.
\]

We will consider a fluid star moving along a geodesic in a scalar field black hole spacetime with the metric (2). More precisely, one should regard the center of mass as moving along the geodesic while other parts of the star deviate from geodesic motion. We will assume that the geodesic is located in the equatorial plane, and that the mass and size of the star are much smaller than the black hole mass \( M \) and the horizon radius \( r_h \), respectively. Let \( U = U^i e_i \) be the four-velocity of the center of mass, such that \( U^2 = 0 \). The integrals of motion can be written as

\[
E = \sqrt{A} U^0, \quad J = r U^3, \quad (U^1)^2 = \frac{1}{f} \left( \frac{dr}{ds} \right)^2 = \frac{1}{A} \left( E^2 - V_{eff} \right), \quad V_{eff} = A \left( 1 + \frac{J^2}{r^2} \right),
\]

where \( E \) is the specific energy and \( J \) the specific angular momentum.
For the sake of simplicity and analytical clarity, we will consider below circular geodesics in which \( U^1 = 0 \). In a circular orbit, the geodesic equation for the component \( U^1 \) and the relations (17) give

\[
\frac{rA'}{2A} (U^0)^2 = (U^3)^2, \quad E^2 = A \left( 1 - \frac{rA'}{2A} \right)^{-1}, \quad J^2 = \frac{r^3A'}{2A} \left( 1 - \frac{rA'}{2A} \right)^{-1}. \tag{18}
\]

A correct description of tidal effects must be based on an ‘instantaneous local inertial frame’ in which the center of mass of the star is at rest at the time of observation. In other words, in the corresponding tangent spaces along the geodesic, we choose the dual orthonormal bases (of the obvious forms)

\[
\varepsilon_0 = U = U^0 e_0 + U^3 e_3, \quad \varepsilon_1 = \cos(\sigma s) e_1 - \sin(\sigma s) e_3^*, \quad \varepsilon_2 = e_2, \quad \varepsilon_3 = \sin(\sigma s) e_1 + \cos(\sigma s) e_3^*, \tag{19}
\]

\[
\varepsilon^0 = U^0 e_0 - U^3 e_3^*, \quad \varepsilon^1 = \cos(\sigma s) e_1^* - \sin(\sigma s) e_3^*, \quad \varepsilon^2 = e_2^*, \quad \varepsilon^3 = \sin(\sigma s) e_1^* + \cos(\sigma s) e_3^*, \tag{20}
\]

where

\[
e_3^* = U^3 e_0 + U^0 e_3, \quad e_3^* = -U^3 e_0 + U^0 e_3, \quad \sigma = \frac{\sigma_3(U)}{U^3} = \frac{\sigma_3(U)}{U^0} = \frac{\sqrt{J} U^3}{r^2 U^0}. \]

The ‘frequency’ \( \sigma \) is obtained from the conditions of parallel transport of the spatial basis vectors: \( \nabla_\xi \varepsilon_\alpha = 0, \ \alpha = 1, 2, 3; \) one should take into account expressions (14) and the first equality (18) which holds in circular orbits.

The components of specific tidal force (acceleration) are given by the expression

\[
F_i^j \equiv \frac{D^2 \eta^i}{ds^2} = R^i_{jk} U^k U^j \eta^l = P^i_{jl} \eta^l,
\]

where \( \eta^l \) is a vector ‘connecting’ the center of mass with some other point of the star, and \( P = R(\cdot, U, U, \cdot) \) is the tidal tensor. For simplicity of computations (and without loss of generality), one can set \( s = 0 \) in (19) and (20), then it follows from (15) – (18) that

\[
P = \left( 1 - \frac{rA'}{2A} \right)^{-1} \left[ u(r) \varepsilon_1 \otimes \varepsilon^1 + v(r) \varepsilon_2 \otimes \varepsilon^2 + w(r) \varepsilon_3 \otimes \varepsilon^3 \right]. \tag{21}
\]

where

\[
u(r) = \frac{1 - f}{r^2} + \phi^2 f + \frac{f'A'}{4A}, \quad v(r) = -\frac{f'}{2r} - \phi'^2 f - \frac{A'}{2A} \frac{1 - f}{r}, \quad w(r) = -\frac{f'}{2r} - \phi'^2 f.
\]

It can be directly verified by direct differentiation in (7) that

\[
1 - \frac{rA'}{2A} = \frac{\xi - 3M}{rA} e^F > 0 \tag{22}
\]

in the region in which \( \xi > 3M \). Note first that the equality \( \xi = 3M \) determines the photon orbit of some radius \( r_{ph} \); and second, for scalar field black holes, the various numerical simulations show that (analogously to Schwarzschild black holes) \( r_{ph} \approx (3/2) r_h \) and \( r_{ISCO} \approx 3r_h \).
4. An illustrative example

As it is usually done in gravitational physics, we will use the mass $M$ as the unit of length, that is, will set $M = 1$ from now on. The strictly monotonic piecewise analytic function

$$\xi = a + (1 - a/6)r + (a/108)r^2, \quad 0 < r \leq 6, \quad \text{and} \quad \xi = r + 2a/r, \quad 6 < r < \infty$$  \hspace{1cm} (23)

has continuous derivatives up to the second order at the point $r = 6$ and determines a scalar field black hole for $0 < a < 3$, where $a$ is the parameter of 'intensity' of the scalar field. The explicit expression for the corresponding metric and field functions are too cumbersome to be presented here. The metric functions $A$ and $f$ are plotted in Figure 1 (left panel) in comparison to the Schwarzschild black hole of the same mass $M$. In the former case, $r_h \simeq 0.242$, $r_{ph} \simeq 0.368$, and $r_{ISCO} \simeq 0.821$, while in the latter case, $r_{St} = 2$, $r_{ph} = 3$, and $r_{ISCO} = 6$; note that the radius $r$ is measured in units of $M$. One of the important consequences of this difference consist in the following: for a deviation vector $\eta$, the tidal forces at the radius $r_{ISCO}$ are typically of orders $|\eta|$ and $10^{-3}|\eta|$ for the scalar field black hole and the Schwarzschild black hole, respectively.

For definiteness, we consider a solar-type star orbiting a supermassive black hole near (in the pericenter) the innermost stable circular orbit. The star undergoes the outflow of matter or disruption if the attractive gravitational force $F_g$ at the star surface equals or, respectively, is much less than the tidal force $F_t$ at the surface. For the supermassive black hole Sgr A* ($M \approx 4 \cdot 10^6 M_\odot$) in our Galaxy, a rough estimation of this situation can be obtain as follows. Let $M_\star$ and $R_\star$ be the mass and the radius of the star, respectively. In units of $M$, one has $M_\star \approx 2.5 \cdot 10^{-7}$ and $R_\star \approx 7 \cdot 10^{-2}$, so that $F_g \sim M_\star/R_\star^2 \sim 5 \cdot 10^{-5}$ while $F_t^{(v)} \sim 7 \cdot 10^{-5}$ if the black hole is in vacuum, and $F_t^{(sf)} \sim 7 \cdot 10^{-2}$ if it is surrounded by the scalar field. In this example, the force $F_t^{(v)}$ leads to the outflow of matter from the star surface, while the force $F_t^{(sf)}$ leads to the disruption of the star.

![Figure 1](image_url)

**Figure 1.** The (a) panel shows the key metric functions determined by (23) in comparison to the Schwarzschild solution with the same mass: $r_h \simeq 0.242$, $r_{ph} \simeq 0.368$, and $r_{ISCO} \simeq 0.821$. The radius $r$ is measured in units of mass. In the (b) panel are plotted the components of the tidal tensor $P = P^1_1 \varepsilon_1 \otimes \varepsilon^1 + P^2_2 \varepsilon_2 \otimes \varepsilon^2 + P^3_3 \varepsilon_3 \otimes \varepsilon^3$ given by the expression (21).
5. Discussion
We consider the results of this paper as a 'first approximation' to a future realistic theoretical model of deformations and disruptions of stars in the tidal field of a supermassive black hole located at the center of a normal galaxy, and, consequently, surrounded by dark matter which, in turn, is concentrated in a region near the horizon. In our model, in which dark matter is regarded as a self-gravitating scalar field, tidal forces increase to infinity as the orbital radius approaches to that of the photon orbit; it is seen directly from expressions (21) and (22). The illustrative example also shows that a solar-type star will be disrupted by tidal forces in our model. However, if Sgr A* is a vacuum black hole, then the star undergoes only the outflow of its matter. In the latter case, white dwarfs and neutron stars will not undergo the appreciable tidal effects in the region $r > r_h$; this means that these effects become completely unobservable.

In spite of the restriction of our consideration to the case of circular orbits, numerical simulations of the problem show that the results obtained in this way have a more general sense: we can be confident that the radial and azimuthal tidal forces in the pericenter are not less in magnitude than those in the circular orbit of the same radius. One can hope that future precise astronomical instruments will increase the possibilities to discriminate black holes in the centers of galaxies from other strongly gravitating objects using, among other things, tidal effects.

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