Interior and exterior curves of finite Blaschke products

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Abstract

For a Blaschke product $B$ of degree $d$ and $\lambda$ on $\partial \mathbb{D}$, let $\ell_\lambda$ be the set of lines joining each distinct two preimages in $B^{-1}(\lambda)$. The envelope of the family of lines $\{\ell_\lambda\}_{\lambda \in \partial \mathbb{D}}$ is called the interior curve associated with $B$. In 2002, Daepp, Gorkin, and Mortini proved the interior curve associated with a Blaschke product of degree 3 forms an ellipse. While let $L_\lambda$ be the set of lines tangent to $\partial \mathbb{D}$ at the $d$ preimages $B^{-1}(\lambda)$ and the trace of the intersection points of each two elements in $L_\lambda$ as $\lambda$ ranges over the unit circle is called the exterior curve associated with $B$. In 2017, the author proved the exterior curve associated with a Blaschke product of degree 3 forms a non-degenerate conic.

In this paper, for a Blaschke product of degree $d$, we give some geometrical properties that lie between the interior curve and the exterior curve.

Keywords: Complex analysis, Blaschke product, Algebraic curve, Dual curve

MSC 30C20, 30J10

1 Introduction

A Blaschke product of degree $d$ is a rational function defined by

$$B(z) = e^{i\theta} \prod_{k=1}^{d} \frac{z - a_k}{1 - \overline{a_k} z} \quad (a_k \in \mathbb{D}, \ \theta \in \mathbb{R}). \quad (1)$$

In the case that $\theta = 0$ and $B(0) = 0$, $B$ is called canonical.

For a Blaschke product $B$ of degree $d$, set

$$f_1(z) = e^{-\frac{\theta}{d_1} z}, \quad \text{and} \quad f_2(z) = \frac{z - (-1)^d a_1 \ldots \bar{a}_d e^{i\theta}}{1 - (-1)^d a_1 \ldots \bar{a}_d e^{i\theta} z}.$$

Then, the composition $f_2 \circ B \circ f_1$ is a canonical one, and geometrical properties with respect to preimages of these two Blaschke products $B$ and $f_2 \circ B \circ f_1$ are same. Hence, we only

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need to consider a canonical Blaschke product for the following discussions. Moreover, the derivative of a Blaschke product has no zeros on \( \partial \mathbb{D} \). For instance, see [Mas13]. Hence, there are \( d \) distinct preimages of \( \lambda \in \partial \mathbb{D} \) by \( B \).

Let \( z_1, \ldots, z_d \) be the \( d \) distinct preimages of \( \lambda \in \partial \mathbb{D} \) by \( B \), and \( \ell_\lambda \) the set of lines joining \( z_j \) and \( z_k \) (\( j \neq k \)). Here, we consider the family of lines

\[
\mathcal{L}_B = \{ \ell_\lambda \}_{\lambda \in \partial \mathbb{D}},
\]

and the envelope \( I_B \) of \( \mathcal{L}_B \). We call the envelope \( I_B \) the interior curve associated with \( B \).

For a Blaschke product of degree 3, the interior curve forms an ellipse [DGM02] and corresponds to the inner ellipse of Poncelet’s theorem (cf. [Fla08]).

\[ \text{Theorem 1 (U. Daepp, P. Gorkin, and R. Mortini [DGM02])} \]

Let \( B \) be a canonical Blaschke product of degree 3 with zeros 0, \( a_1 \), and \( a_2 \). For \( \lambda \in \partial \mathbb{D} \), let \( z_1, z_2, \) and \( z_3 \) denote the points mapped to \( \lambda \) under \( B \), and write

\[
F(z) = \frac{B(z)/z}{B(z) - \lambda} = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3}.
\]

(2)

Then the lines joining \( z_1 \) and \( z_2 \) is tangent to the ellipse \( E \) with equation

\[
|z - a_1| + |z - a_2| = |1 - \overline{a_1}a_2|
\]

(3)

at the point \( \zeta_3 = \frac{m_1z_2 + m_2z_1}{m_1 + m_2} \). Conversely, every point of \( E \) is the point of tangency with \( E \) of a line that passes through two distinct points \( z_1 \) and \( z_2 \) on the unit circle for which \( B(z_1) = B(z_2) \).

This result reminds us of the following classical result in Marden’s book [Mar66] that was proved first by Siebeck [Sie65].

\[ \text{Theorem 2 (Siebeck [Sie65])} \]

The zeros \( z'_1 \) and \( z'_2 \) of the function

\[
F(z) = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3}
\]

are the foci of the conic which touches the line segments \( z_1z_2, z_2z_3 \) and \( z_3z_1 \) in the points \( \zeta_3, \zeta_1 \), and \( \zeta_2 \) that divide these segments in the ratios \( m_1 : m_2, m_2 : m_3 \) and \( m_3 : m_1 \), respectively. If \( n = m_1 + m_2 + m_3 \neq 0 \), the conic is an ellipse or hyperbola according as \( nm_1m_2m_3 > 0 \) or \( < 0 \).

For a Blaschke product \( B \) of degree 3, let \( z_1, z_2, \) and \( z_3 \) be the preimages for some \( \lambda \in \partial \mathbb{D} \) by \( B \) and \( F \) be defined as (2), then the following folds ([DGM02 Lemma 4]),

\[
m_1 + m_2 + m_3 = 1 \quad \text{and} \quad 0 < m_j < 1 \text{ for } j = 1, 2, 3.
\]

Theorem 2 asserts the existence of “the common ellipse” for a given Blaschke product, as long as the given three points \( z_1, z_2, \) and \( z_3 \) are the preimage for some \( \lambda \) on the unit circle.

The ellipse (3) is also related to the numerical range of another specific matrix with eigenvalues \( a_1 \) and \( a_2 \) of the non-zero zero points of \( B \). Gorkin and Skubak studied such relations ([GST1]).
Moreover, for a canonical Blaschke product $B$ of degree 4 with zeros $0, a_1, a_2,$ and $a_3$, the interior curve associated with $B$ is defined by the equation of total degree 6 with respect to $z$ and $\overline{z}$. The coefficients of $z^6$ and $\overline{z}^6$ are 

$$(a_1 - a_2)^2(a_2 - a_3)^2(a_3 - a_1)^2$$

respectively, with mutually distinct $a_1, a_2,$ and $a_3$. The file size of a defining equation of this interior curve is about 200Kb as a text file. See also [Fuj13].

Thus, it is not so easy to obtain the defining equation of the interior curve by calculating the envelope of $B$ of degree greater than 4.

Next, we consider the geometrical properties of Blaschke products outside the unit disk.

Let $B$ be a canonical Blaschke product of degree $d$. For $\lambda \in \partial \mathbb{D}$, let $L_\lambda$ be the set of $d$ lines tangent to $\partial \mathbb{D}$ at the $d$ preimages of $\lambda \in \partial \mathbb{D}$ by $B$. Here, we denote by $E_B$ the trace of the intersection points of each two elements in $L_\lambda$ as $\lambda$ ranges over the unit circle. We call the trace $E_B$ the exterior curve associated with $B$.

In [Fuj17], we obtained the following.

**Theorem 3** ([Fuj17]). Let $B$ be a canonical Blaschke product of degree $d$. Then, the exterior curve $E_B$ is an algebraic curve of degree at most $d - 1$.

The proof of Theorem 3 is already described in [Fuj17], but we will give an outline proof in section 2 in order to provide the defining equation of $E_B$.

The following result comes to mind when we pay attention to the degree of $E_B$. However, we remark that the degree of the exterior curve may degenerate to less than $d - 1$ (see Remark 8).

**Theorem 4** (Siebeck [Sie65]). The zeros of the function $F(z) = \sum_{j=1}^{d} \frac{m_j}{z-z_j}$ ($m_j \in \mathbb{R}^*$), are the foci of the curve of class $d - 1$ which touches each line-segment $z_jz_k$ in a point dividing the line segment in the ratio $m_j : m_k$.

The main aim of this paper is to explore the relation between the geometrical properties of the interior curve and the exterior curve. As the main theorem in this paper, we will show that the following result in section 3.

**Theorem 5.** Let $B$ be a canonical Blaschke product of degree $d$, and $E_B^*$ the dual curve of the homogenized exterior curve $E_B$. Then, the interior curve is given by

$$I_B : u_B^*(-z) = 0,$$

where $u_B^*(z) = 0$ is a defining equation of the affine part of $E_B^*$.

Moreover, as an application of this theorem, we construct examples of Blaschke products having two ellipses as the interior curve in section 4.

## 2 Interior and exterior curves

Although, the proof of Theorem 3 is already described in [Fuj17], in order to confirm the method of construction of the defining equation, we provide an outline proof here.
Proof of Theorem. Let $B(z) = z \prod_{k=1}^{d-1} \frac{z - a_k}{1 - a_kz}$ (where $a_k \in \mathbb{D}$) and written as follows

$$B(z) = \frac{z^d - \sigma_1z^{d-1} + \sigma_2z^{d-2} + \cdots + (-1)^{d-1}\sigma_{d-1}z + 1}{1 - \sigma_1z + \cdots + (-1)^{d-1}\sigma_{d-1}z^{d-1}},$$

where $\sigma_k$ are the elementary symmetric polynomials on variables $a_1, \cdots, a_{d-1}$ of degree $k$ ($k = 1, \cdots, d - 1$). Let $\sigma_0 = 1$ and $\sigma_d = 0$. Eliminating $\lambda$ from $B(z_1) = B(z_2) = \lambda$, we have

$$\left(\begin{array}{c} z_1^d - \sigma_1z_1^{d-1} + \cdots + (-1)^{d-1}\sigma_{d-1}z_1 \\ z_2^d - \sigma_1z_2^{d-1} + \cdots + (-1)^{d-1}\sigma_{d-1}z_2 \end{array}\right) \left(\begin{array}{c} 1 - \sigma_1z_2 + \cdots + (-1)^{d-1}\sigma_{d-1}z_2^{d-1} \\ 1 + \sigma_1z_1 + \cdots + (-1)^{d-1}\sigma_{d-1}z_1^{d-1} \end{array}\right) = - \sum_{j=1}^{d} \sum_{k=1}^{d} (-1)^{j+k}\sigma_{d-j}\sigma_{d-k}(z_1^{j}z_2^{d-j} - z_1^{d-j}z_2^{j})$$

$$= \sum_{N=1}^{d} \sum_{K=0}^{d-1} (-1)^{d-N+K}(\sigma_{d-N}\overline{\sigma}_{K} - \overline{\sigma}_N\sigma_{d-K})\left(z_1^{d}z_2^{N-K} - z_2^{d-N}\right)$$

$$= (z_1 - z_2) \sum_{N=1}^{d} \sum_{K=0}^{d-1} (-1)^{d-N+K}(\sigma_{d-N}\overline{\sigma}_{K} - \overline{\sigma}_N\sigma_{d-K})\left(z_1^2z_2^{N-2} - z_2^{d-N}\right)$$

$$\times \left((z_1 + z_2)^{2N-2} - \gamma_1z_1z_2(z_1 + z_2)^{N-3} + \cdots + \gamma_2(z_1^2z_2)^{M}(z_1 + z_2)^{R}\right) = 0,$$

where $R$ is the remainder after dividing $N - K - 1$ by 2,

$$M = \frac{N - K - 1 - R}{2}, \quad \gamma_1 = N - K - 2,$$

and $\gamma_M$ is a non-zero coefficient. The intersection point $z$ of two lines $l_1$ and $l_2$ satisfies

$$z_1z_2 = \frac{z}{\bar{z}} \quad \text{and} \quad z_1 + z_2 = \frac{2}{\bar{z}},$$

since each $l_k$ ($k = 1, 2$) is a line tangent to the unit circle at a point $z_k$. Note that the intersection point is the point at infinity if and only if $z_1 + z_2 = 0$. Hence, we have

$$\sum_{N=1}^{d} \sum_{K=0}^{d-1} (-1)^{d-N+K}(\sigma_{d-N}\overline{\sigma}_{K} - \overline{\sigma}_N\sigma_{d-K})z_1^{2N-2}z_2^{d-N}$$

$$\times \left(2^{2N-2} - 2^{N-3}\gamma_1z \bar{z} + \cdots + 2^R\gamma_Mz^M \bar{z}^M\right) = 0.$$ (5)

This equality gives a defining equation of $E_B$ with degree at most $d - 1$. \hfill \Box

When the degree is low, we can describe the exterior curve concretely, as follows.

**Corollary 6 ([Fuj17]).** Let $B$ be a canonical Blaschke product of degree $d$ with zeros $0, a_1, \cdots, a_{d-1} \in \mathbb{D}$.

- For a canonical Blaschke product of degree 2 with zeros 0 and $a_1(\neq 0)$, the exterior curve is the line $\bar{a}_1z + a_1\bar{z} - 2 = 0$. 

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• For $d = 3$, the exterior curve is either an ellipse, a circle, a parabola, or a hyperbola.

\[
\overline{a_1}a_2 z^2 - (|a_1 a_2|^2 - |a_1 + a_2|^2 + 1) z \bar{z} + a_1 a_2 \bar{z}^2 - 2(\overline{a_1} + \overline{a_2}) z - 2(a_1 + a_2) \bar{z} + 4 = 0. \tag{6}
\]

• For $d = 4$, the defining equation of the exterior curve is written as

\[
\sigma_3 z^3 + (\sigma_1 \sigma_3 - \sigma_2 \sigma_4 - \sigma_1) z^2 \bar{z} - (\sigma_1 - \sigma_2 \sigma_1 + \sigma_3 \sigma_2) z \bar{z}^2 + \sigma_3 z^3 \\
- 2\sigma_2 z^2 - (2\sigma_1 \sigma_1 - 2\sigma_3 \sigma_3 - 4) z \bar{z} - 2\sigma_2 \bar{z}^2 + 4\sigma_1 z + 4\sigma_1 \bar{z} - 8 = 0, \tag{7}
\]

where $\sigma_k$ are the elementary symmetric polynomials on three variables $a_1, a_2, a_3$ of degree

$k (k = 1, 2, 3)$, i.e.

\[
\sigma_1 = a_1 + a_2 + a_3, \quad \sigma_2 = a_1 a_2 + a_1 a_3 + a_2 a_3 \quad \text{and} \quad \sigma_3 = a_1 a_2 a_3.
\]

Even if we use symbolic computation systems, it is hard to calculate the defining equation of the interior curve for $d = 5$. However, we can obtain the exterior curve as follows.

**Corollary 7.** For a canonical Blaschke product of degree 5, the defining equation of the exterior curve \([\overline{5}]\) is written as

\[
\sigma_4 z^4 + (\sigma_1 \sigma_3 - \sigma_2 \sigma_4 - \sigma_1) z^3 \bar{z} - (\sigma_1 \sigma_1 - \sigma_2 \sigma_2 + \sigma_3 \sigma_3 - \sigma_4 \sigma_4 - 1) z^2 \bar{z}^2 \\
+ (\sigma_3 \sigma_1 - \sigma_4 \sigma_1 - \sigma_2) z^2 \bar{z}^3 + \sigma_4 z^4 - 2\sigma_3 z^3 + 2(2\sigma_1 \sigma_1 - \sigma_1 \sigma_2 + \sigma_3 \sigma_2 + 3\sigma_4 \sigma_4) z^2 \bar{z} \\
- 2(\sigma_1 \sigma_1 - 2\sigma_1 - \sigma_4 \sigma_3) z \bar{z}^2 - 2\sigma_3 \bar{z}^3 + 4\sigma_2 \bar{z}^2 + 4(\sigma_1 \sigma_1 - \sigma_4 \sigma_4 - 3) z \bar{z} \\
+ 4\sigma_2 \bar{z}^2 - 8\sigma_1 \bar{z} - 8\sigma_1 z + 16 = 0, \tag{8}
\]

where $\sigma_k$ are the elementary symmetric polynomials on four variables $a_1, \cdots, a_4$ of degree

$k (k = 1, \cdots, 4)$.

**Remark 8.** For $d = 4$, the degree of the exterior curve \([\overline{7}]\) is not greater than 2 if and only if the Blaschke product has a double zero point at the origin and the sum of the other zero points equals to 0. While, the degree of the defining equation of the exterior curve $E_B$ is always 4 for every $B$ of degree 5.

Even though we can obtain the defining equation of the exterior curve concretely for $d \geq 6$, We abandon to describe it. Because the size of the equation is relatively large.
Figure 1: The thick curves indicate the exterior curves for canonical Blaschke products with non-zero zero points \((a_1, a_2, a_3, a_4) = (0.4 + 0.7i, 0.9i, 0.6, -0.9i)\) (upper) and \((0.4 + 0.7i, 0.9i, -0.6 + 0.6i, 0.7i)\) (lower). The envelope in each right figure is the interior curve corresponding to the left figure.

3 Proof of Theorem 5

The affine part of the projective space \(\mathbb{P}_2(\mathbb{R})\) can be identified with the complex plane \(\mathbb{C}\).

Recall that the dual curve \(C^*\) of \(C \subset \mathbb{P}_2(\mathbb{R})\) is defined by

\[ C^* = \{ L \in \mathbb{P}_2^*(\mathbb{R}) ; L \text{ is a line tangent to } C \text{ at some } p \in C \}. \]

**Proof of Theorem 5** Let \(z'\) and \(z''\) are two preimages for some \(\lambda \in \partial \mathbb{D}\), and \(\ell\) is the line joining \(z'\) and \(z''\). Let \(\zeta\) be the intersection point of two lines tangent to the unit circle at the points \(z'\) and \(z''\) (cf. Figure 2). Therefore the point \(\zeta\) is the pole and the line \(\ell\) is its polar with respect to the unit circle.

Then, the equation of \(\ell\) is written as

\[ z + z'z''\bar{z} = z' + z''. \] (9)

The intersection point \(\zeta\) satisfies

\[ z' + z'' = \frac{2}{\zeta} \quad \text{and} \quad z'z'' = \frac{\zeta}{\bar{\zeta}}. \] (10)
Figure 2: The point $\zeta$ is the pole and the line $\ell$ is its polar with respect to the unit circle.

Substituting (10) into (9), the line $\ell$ is written by the data of $\zeta$ as follows,

$$\zeta z + \zeta \bar{z} = 2.$$ 

Substituting $\zeta = \alpha + \beta i$ and $z = x + yi$ into the above equality again, the line $\ell$ is expressed as the line on the real $xy$-plane, $\alpha x + \beta y - 1 = 0$. Therefore the line $\ell \subset \mathbb{C} \subset \mathbb{P}_2(\mathbb{R})$ corresponds to the point $(-\alpha : -\beta : 1) \in \mathbb{P}_2^*(\mathbb{R})$, and this point corresponds to the point $-\zeta \in \mathbb{C}$.

Hence, the assertion is obtained from the fact that the family of all tangent lines of the interior curve $I_B$ coincides with the family of lines $L_B = \{\ell_\lambda\}_{\lambda \in D}$.

Equivalently, the converse also holds.

**Corollary 9.** Let $B$ be a canonical Blaschke product of degree $d$, and $I_B^*$ be the dual curve of the homogenized interior curve $I_B$. Then, the exterior curve is given by

$$E_B : v_B^*(-z) = 0,$$

where $v_B^*(z) = 0$ is a defining equation of the affine part of $I_B^*$.

**Remark 10.** As we mentioned in section 1, for $d = 3$, the ellipse (3) corresponds to the inner ellipse of Poncelet’s theorem. For $d \geq 4$, Theorems 2 and 5 provides the defining equation of the envelope of the family of lines $\{\ell_\lambda\}_{\lambda \in D}$, where $\ell_\lambda$ is the set of all segments joining each distinct two preimages in $B^{-1}(\lambda)$. Here, we remark that $\ell_\lambda$ includes diagonals of the $d$-sided polygon with vertices at $B^{-1}(\lambda)$. In general, the defining equation of this envelope is not always reducible, but the “outermost part” of the curve gives the so-called Poncelet curve associated with the Blaschke product. For instance, see [Mir03] and [DGSS15, Definition 5.1 and Theorem 5.2] for details about definitions and related topics of Poncelet curve.

4 Examples

For a Blaschke product $B$ of degree 4, the interior curve $I_B$ is an ellipse if and only if $B$ is a composition of two Blaschke products of degree 2. See [Fuj17], and also see [GW17] for the relationship between this result and the numerical range of shift operators.

Here, as an application of Theorem 5, we construct a Blaschke product of degree 5 whose interior curve is a union of two ellipses.
Example 11. Let
\[ B_{a,b}(z) = \frac{z^2 - a}{1 - a^2} \frac{z^2 - b}{1 - b^2} \quad (0 < a, b < 1), \]
where \( a, b \) satisfy the equality \( a^3b^3 - 2a^2b^2 - (b^2 + a^2) + 3ab = 0 \).
In this case, \( E_{B_{a,b}} \) is given as follows,
\[ E_{B_{a,b}} : (a(b + 1)^2x^2 + a(b - 1)^2y^2 - 4b) \]
\[ ((a^2b^3 - ab^2 + 2b^2 + 3b - a)x^2 + (a^2b^3 - ab^2 - 2b^2 + 3b - a)y^2 - 4b) = 0, \quad (11) \]
where \( z = x + iy \).
Therefore, the exterior curve is a union of two ellipses for every \( B_{a,b} \). Then, the interior curve is also a union of two ellipses because the dual curve of an irreducible conic is also an irreducible conic and the interior curve is a compact curve in \( \mathbb{D} \). See Figure 3.

\[ \begin{align*}
\text{Figure 3: The thick curve indicates the exterior curve } & E_{B_{a,b}}, \text{ where } (a, b) = (0.16, 0.0616). \\
\text{The envelope in the right figure is the interior curves correspond to the left figure.}
\end{align*} \]

In fact, \( I_{B_{a,b}} \) is given by,
\[ I_{B_{a,b}} : \left( \frac{4b}{a(b + 1)^2}x^2 + \frac{4b}{a(b - 1)^2}y^2 - 1 \right) \left( \frac{4a}{b(a + 1)^2}x^2 + \frac{4a}{b(a - 1)^2}y^2 - 1 \right) = 0. \]
The two foci are \( \pm \sqrt{a} \) (the first factor) and \( \pm \sqrt{b} \) (the second factor).

Example 12. Let
\[ B_c(z) = z \left( \frac{z - \frac{1}{c}}{1 - \frac{1}{cz}} \right)^2 \left( \frac{z - c}{1 - cz} \right)^2 \quad (0 < c < 1), \]
where \( c \) is a solution of \( c^3 - 72c^2 + 48c - 4 = 0 \). There are two possibilities of \( c \),
\[ c \approx 0.0976036, \quad \text{or} \quad c \approx 0.5745591. \]
In this case, $E_{Bc}$ is given as follows,

$$E_{Bc} : \left( 4z^2 + (−225zc + 8\bar{z} - 64)z + 4\bar{z}^2 - 64\bar{z} + 256 \right)$$

$$\left( 16c^2z^2 + (−257zc^2 + (272\bar{z} - 64)c - 64\bar{z})z + 16\bar{z}^2c^2 - 64\bar{z}c + 64 \right) = 0. \quad (12)$$

In the case of $c \approx 0.0976036$, (12) is a union of two ellipses. See Figure 4.

![Figure 4: The thick curve indicates the exterior curve for $B_c$ with $c \approx 0.0976036$. The envelope in the right figure is the interior curve.](image)

The other case, (12) is a union of an ellipse and a hyperbola. See Figure 5. In any case, the interior curve should be a union of two ellipses.

In fact, the interior curve is the union of the following two circles,

$$|z - \frac{1}{4}| = \frac{15}{16}\sqrt{c}, \quad \text{and} \quad |z - c| = \frac{1}{8}(17c - 8).$$

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