Application of the regularization method to some singularly perturbed hyperbolic partial differential equations

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Abstract. This paper considers the Cauchy problem for a hyperbolic partial differential equation containing a small parameter in the main part. When the small parameter is zero, the order of the equation does not decrease, but the original problem changes its type and becomes a Cauchy problem for a parabolic equation. The inability of a solution to the limit problem to satisfy all the conditions of the original problem leads to creation of a boundary layer region in the neighborhood of the initial conditions. The structure of an asymptotic solution to a singularly perturbed problem in the form of a regularized series makes it possible to write the solution as a single whole, without separately constructing boundary functions. In this case, the boundary effects are described by additional independent variables introduced by the spectrum of the limit operator. The principal feature of the problem considered in this paper, namely, absence of the spectrum of the limit operator, does not allow us to fully apply the method of constructing an asymptotic solution by the regularization method of S. A. Lomov. In this paper, we propose a method for solving such problems using the direct Fourier transform. An algorithm for formulating the coefficients of a regularized series is given, and zero and first terms of the asymptotic are formulated.

1. Introduction

There is a wide choice of special methods for constructing asymptotic solutions for singularly perturbed problems. This paper uses one of them, that us a regularization method developed by S.A. Lomov [1]. A great advantage of the regularization method in comparison to other methods is the fact that this method allows to write a solution as a single whole. That is, the solution is not composite, when a solution is constructed separately in the boundary zone and outside thereof. Besides, the regularized asymptotic series, which converge under certain conditions and in the usual sense, in these cases have the advantages of asymptotic series as well as the advantages of series converging in the usual sense, and their principal term in the asymptotic expansion replaces the exact solution in terms of qualitative characterization of the studied physical process [2].

2. Statement of the problem

Let us consider the problem
\[ \begin{align*} 
\triangle U - \varepsilon U_{tt} - aU_t &= 0, \\
U(x, 0, \varepsilon) &= f(x), \\
U_t(x, 0, \varepsilon) &= g(x),
\end{align*} \tag{1} \]

where \( f(x), g(x) \) are absolutely integrable functions. The equation (1) is a singularly perturbed hyperbolic equation with a small parameter and a higher derivative. If \( \varepsilon = 0 \) the problem (1)-(3) becomes a Cauchy problem for a parabolic equation. A solution to the limit problem (when \( \varepsilon = 0 \)) does not satisfy the second initial condition of the problem (3). This leads to creation of a boundary layer region in the neighborhood of \( t = 0 \).

In this paper, we will frame a solution to the problem (1)-(3) as a regularized series introduced by S.A.Lomov [1].

Following the scheme of the regularization method, along with independent variables \( x \) and \( t \), let us introduce regularizing independent variables, using the formulae:

\[ \tau_i = \frac{1}{\varepsilon} \int_0^t \lambda_i(s)ds, \quad i = 1, 2 \quad (\tau_i(0) = 0). \tag{4} \]

Pursuant to the regularization method, recalculating the problem (1)-(3), shall give a new "extended" problem regular for \( \varepsilon \) when \( \varepsilon \to 0 \). In fact, this does not happen. The reason is that the equation (1) along with a higher order derivative for variable \( t \), contains a lower order derivative for the same variable. The lower order derivative can be eliminated by introducing a new function \( w(x, t, \varepsilon) \) per formula

\[ U(x, t, \varepsilon) = w(x, t, \varepsilon) \exp \left( -\frac{t}{2\varepsilon} \right). \tag{5} \]

The problem (1)-(3) may be written as follows

\[ \begin{align*} 
\varepsilon^2 w_{xx} - \varepsilon w_{tt} + \frac{1}{4\varepsilon} w &= 0, \\
w(x, 0, \varepsilon) &= f(x), \\
w_t(x, 0, \varepsilon) &= g(x) + \frac{f(x)}{2\varepsilon}. \tag{8} 
\end{align*} \]

After solving one problem, we faced another one. When \( \varepsilon \to 0 \), the equation (6) degenerates into an algebraic equation. This means that the regularization method as described in paper [1] cannot be applied.

3. Regularization of singularities

To solve these problems, let us apply Fourier transform [3] for variable \( x \) to the problem (6)-(8).

This gives us

\[ \begin{align*} 
4\varepsilon^2 W'' + (4\varepsilon^2 \sigma^2 - 1)W &= 0, \\
W(\sigma, 0, \varepsilon) &= F(\sigma), \\
W'(\sigma, 0\varepsilon) &= G(\sigma) + \frac{F(\sigma)}{2\varepsilon}, \tag{11} 
\end{align*} \]
where
\[ w(x, 0, \varepsilon) \leftrightarrow W(\sigma, 0, \varepsilon), \]  
(12)  
\[ f(x) \leftrightarrow F(\sigma), \]  
(13)  
\[ g(x) \leftrightarrow G(\sigma). \]  
(14)
By substitution, we convert the problem (9)-(11) to
\[ 2\varepsilon W' = Z \]  
(15)
or
\[ 2\varepsilon \begin{pmatrix} W' \\ Z' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 - 4\sigma^2 \varepsilon & 0 \end{pmatrix} \begin{pmatrix} W \\ Z \end{pmatrix}. \]  
(16)
The initial conditions are as follows
\[ \begin{pmatrix} W(\sigma, 0, \varepsilon) \\ Z(\sigma, 0, \varepsilon) \end{pmatrix} = \begin{pmatrix} F(\sigma) \\ F(\sigma) \end{pmatrix} + 2\varepsilon \begin{pmatrix} 0 \\ G(\sigma) \end{pmatrix}. \]  
(17)
Let us set a limit operator matrix
\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]  
(18)
From the system
\[ A\varphi(t) = \lambda_i \varphi(t) \]  
(19)
let us find an eigenvector of the matrix.  
Let us introduce regularizing functions per formulae
\[ \tau_1 = \frac{t}{2\varepsilon} \equiv g_1(t, \varepsilon), \tau_2 = -\frac{t}{2\varepsilon} \equiv g_2(t, \varepsilon). \]  
(20)
Instead of the desired solution
\[ \begin{pmatrix} W(\sigma, t, \varepsilon) \\ Z(\sigma, t, \varepsilon) \end{pmatrix} \]  
(21)
to the problem (16)-(17) we get an “extended” function
\[ \begin{pmatrix} \bar{W}(\sigma, t, \tau_1, \tau_2, \varepsilon) \\ \bar{Z}(\sigma, t, \tau_1, \tau_2, \varepsilon) \end{pmatrix} = \begin{pmatrix} W(\sigma, t, \tau, \varepsilon) \\ Z(\sigma, t, \tau, \varepsilon) \end{pmatrix}. \]  
(22)
For the “extended” function we get the following problem
\[ 2\varepsilon \frac{\partial}{\partial t} \begin{pmatrix} \bar{W} \\ \bar{Z} \end{pmatrix} + \frac{\partial}{\partial \tau_1} \begin{pmatrix} \bar{W} \\ \bar{Z} \end{pmatrix} - \frac{\partial}{\partial \tau_2} \begin{pmatrix} \bar{W} \\ \bar{Z} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 - 4\sigma^2 \varepsilon & 0 \end{pmatrix} \begin{pmatrix} \bar{W} \\ \bar{Z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]  
(23)
\[ \begin{pmatrix} \bar{W}(\sigma, 0, 0, \varepsilon) \\ \bar{Z}(\sigma, 0, 0, \varepsilon) \end{pmatrix} = \begin{pmatrix} F(\sigma) \\ F(\sigma) \end{pmatrix} + 2\varepsilon \begin{pmatrix} 0 \\ G(\sigma) \end{pmatrix}. \]  
(24)
The link between the problem (23)-(24) and the problem (16)-(17) is that if (22) is a solution to the problem (23)-(24), then its "restriction" will be an exact solution to the problem (16)-(17), i.e.

\[
\begin{pmatrix}
\tilde{W}(\sigma,t,\tau,\varepsilon) \\
\tilde{Z}(\sigma,t,\tau,\varepsilon)
\end{pmatrix}
\bigg|_{\tau_1=g_1(t,\varepsilon),\tau_2=g_2(t,\varepsilon)}
= \begin{pmatrix}
W(\sigma,t,\varepsilon) \\
Z(\sigma,t,\varepsilon)
\end{pmatrix}.
\]

(25)

This is a required condition for regularization.

The problem (23)-(24) is already regular on $\varepsilon$, therefore we solve it as a series from a classic perturbation theory.

\[
\begin{pmatrix}
\tilde{W}(\sigma,t,\tau,\varepsilon) \\
\tilde{Z}(\sigma,t,\tau,\varepsilon)
\end{pmatrix}
= \sum_{k=0}^{\infty} \varepsilon^k
\begin{pmatrix}
\tilde{W}_k(\sigma,t,\tau) \\
\tilde{Z}_k(\sigma,t,\tau)
\end{pmatrix}.
\]

(26)

4. Iteration problems

Plugging the series (26) in the problem (23)-(24) and equating the coefficients of the same powers $\varepsilon$, we get problems for determination of coefficients of the series (26)

\[
\begin{pmatrix}
\partial \frac{\partial}{\partial \tau_1} - \partial \frac{\partial}{\partial \tau_2}
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_0 \\
\tilde{Z}_0
\end{pmatrix}
- \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_0 \\
\tilde{Z}_0
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

(27)

\[
\begin{pmatrix}
\tilde{W}_0(\sigma,0,0) \\
\tilde{Z}_0(\sigma,0,0)
\end{pmatrix}
= \begin{pmatrix}
F(\sigma) \\
F(\sigma)
\end{pmatrix},
\]

(28)

\[
\begin{pmatrix}
\partial \frac{\partial}{\partial \tau_1} - \partial \frac{\partial}{\partial \tau_2}
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_1 \\
\tilde{Z}_1
\end{pmatrix}
- \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_1 \\
\tilde{Z}_1
\end{pmatrix}
= \begin{pmatrix}
-2 \partial \frac{\partial}{\partial t}
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_0 \\
\tilde{Z}_0
\end{pmatrix}
- \begin{pmatrix}
0 & 0 \\
4\sigma^2 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_0 \\
\tilde{Z}_0
\end{pmatrix},
\]

(29)

\[
\begin{pmatrix}
\tilde{W}_1(\sigma,0,0) \\
\tilde{Z}_1(\sigma,0,0)
\end{pmatrix}
= \begin{pmatrix}
0 \\
2G(\sigma)
\end{pmatrix},
\]

(30)

\[
\begin{pmatrix}
\partial \frac{\partial}{\partial \tau_1} - \partial \frac{\partial}{\partial \tau_2}
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_k \\
\tilde{Z}_k
\end{pmatrix}
- \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_k \\
\tilde{Z}_k
\end{pmatrix}
= \begin{pmatrix}
-2 \partial \frac{\partial}{\partial t}
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_{k-1} \\
\tilde{Z}_{k-1}
\end{pmatrix}
- \begin{pmatrix}
0 & 0 \\
4\sigma^2 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{W}_{k-1} \\
\tilde{Z}_{k-1}
\end{pmatrix},
\]

(31)

\[
\begin{pmatrix}
\tilde{W}_k(\sigma,0,0) \\
\tilde{Z}_k(\sigma,0,0)
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

(32)

We construct solutions to the problems (27)-(32) in a space of non-resonant solutions $E$, described in the paper [1],

\[
\begin{pmatrix}
\tilde{W}_k \\
\tilde{Z}_k
\end{pmatrix}
= \alpha_{k1}(t)\varphi_1 e^{\tau_1} + \alpha_{k2}(t)\varphi_2 e^{\tau_2},
\]

(33)

Where $\alpha_{ki}(t) \in C^\infty[0,T]$ - arbitrary scalar functions, $\varphi_i$ - eigenvectors of matrix of limit operator $i = 1, 2$. 

4
5. Formation of regularized series

Let us formulate the principal term of the asymptotic of the problem (23)-(24). To do this, let us solve the problem (27)-(28). We look for a solution to this problem in the form of

\[
\begin{pmatrix}
\tilde{W}_0(\sigma, t, \tau) \\
\tilde{Z}_0(\sigma, t, \tau)
\end{pmatrix} = \alpha_{01}(t)\varphi_1 e^{\tau_1} + \alpha_{02}(t)\varphi_2 e^{\tau_2}.
\]

(34)

We define the initial conditions for functions \(\alpha_{01}(t)\) and \(\alpha_{02}(t)\) from (28). To do this, we plug (34) in (28). We get

\[
\alpha_{01}(0) = F(\sigma), \quad \alpha_{02}(0) = 0.
\]

(35)

To determine equations for defining the functions \(\alpha_{01}(t)\) and \(\alpha_{02}(t)\), we use the theorem on solvability of non-resonant solutions in the space [1] for the equation

\[
T_0 z(t, \tau) \equiv \lambda_1(t) \frac{\partial z}{\partial \tau_1} + \lambda_2(t) \frac{\partial z}{\partial \tau_2} - A(t) z = f(t, \tau).
\]

(36)

To do this, let us return to the first part of the system (29)-(30)

\[
f_1(t, \tau) \equiv e^{\tau_1} \begin{pmatrix}
-2\dot{\alpha}_{01}(t) \\
-2\dot{\alpha}_{01}(t) - 4\sigma^2\alpha_{01}(t)
\end{pmatrix} + e^{\tau_2} \begin{pmatrix}
2\dot{\alpha}_{02}(t) \\
-2\dot{\alpha}_{02}(t) + 4\sigma^2\alpha_{02}(t)
\end{pmatrix}.
\]

(37)

For the system (29)-(30) to be solvable, it is necessary and sufficient that

\[
\begin{pmatrix}
-2\dot{\alpha}_{01}(t) \\
-2\dot{\alpha}_{01}(t) - 4\sigma^2\alpha_{01}(t)
\end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv 0,
\]

(38)

\[
\begin{pmatrix}
2\dot{\alpha}_{02}(t) \\
-2\dot{\alpha}_{02}(t) + 4\sigma^2\alpha_{02}(t)
\end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \equiv 0,
\]

(39)

where \(\psi_1 = (1, 1)\), \(\psi_2 = (1, -1)\) are eigenvectors of adjoint matrices matrix in the equation (27).

By expanding the dot products, we get

\[
4\dot{\alpha}_{01}(t) + 4\sigma^2\alpha_{01}(t) = 0,
\]

(40)

\[
4\dot{\alpha}_{02}(t) - 4\sigma^2\alpha_{02}(t) = 0.
\]

(41)

By adding to the equations (40), (41), the corresponding conditions (35), we get problems for definition of functions \(\alpha_{0i}(t)\).

By solving these problems, we find

\[
\alpha_{01}(t) = F(\sigma)e^{-\sigma^2 t}, \quad \alpha_{02}(t) = 0.
\]

(42)

Therefore, a solution to the problem (27)-(28) is defined uniquely and is as follows

\[
\begin{pmatrix}
\tilde{W}_0(\sigma, t, \tau) \\
\tilde{Z}_0(\sigma, t, \tau)
\end{pmatrix} = F(\sigma)e^{-\sigma^2 t}\varphi_1 e^{\tau_1}.
\]

(43)

By “restricting” the equation (43) for \(\tau_1 = \frac{t}{2\sigma}\), we get the principal term of the asymptotic of the problem (16)-(17)

\[
\begin{pmatrix}
\tilde{W}_0(\sigma, t, \tau) \\
\tilde{Z}_0(\sigma, t, \tau)
\end{pmatrix} = F(\sigma)e^{-t(\sigma^2 - \frac{1}{4\sigma})} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

(44)
Let us apply Fourier inversion [3] for $W_0(\sigma, t, \varepsilon)$
\[
w_0(x, t, \varepsilon) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\sigma) \exp\left(-t(\sigma^2 - \frac{1}{2\varepsilon})\right) \exp(i\sigma x) d\sigma =
\]
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(f(x)) \Phi\left(\frac{1}{2\sqrt{\pi} t} \exp\left(-\frac{x^2}{4t}\right)\right) \exp\left(\frac{t}{2\varepsilon}\right) \exp(i\sigma x) d\sigma =
\]
\[
\frac{1}{\sqrt{2\pi}} \exp\left(\frac{t}{2\varepsilon}\right) \int_{-\infty}^{\infty} \Phi(f(x)) \Phi\left(\frac{1}{2\sqrt{\pi} t} \exp\left(-\frac{x^2}{4t}\right)\right) \exp(i\sigma x) d\sigma =
\]
\[
\exp\left(\frac{t}{2\varepsilon}\right) f(x) * \frac{1}{\sqrt{2\pi} t} \exp\left(-\frac{x^2}{4t}\right) = \frac{1}{\sqrt{2\pi} t} \exp\left(\frac{t}{2\varepsilon}\right) \int_{-\infty}^{\infty} f(\zeta) \exp\left(-\frac{(x - \zeta)^2}{4t}\right) d\zeta.
\]

We used the known fact [3], that
\[
\exp(-\sigma^2 t) = \Phi\left(\frac{1}{2\sqrt{\pi} t} \exp\left(-\frac{x^2}{4t}\right)\right)
\]
and the property of the Fourier transform for the convolution. Then, given the substitution (5), we get from (40) the principal term of the asymptotic of the problem (1)-(3)
\[
U_0(x, t, \varepsilon) = \frac{1}{2\sqrt{\pi} t} \int_{-\infty}^{\infty} f(\zeta) \exp\left(-\frac{(x - \zeta)^2}{4t}\right) d\zeta.
\]

By proceeding similarly, we can get the following $U_1(x, t, \varepsilon)$ approximation in the problem (1)-(3).

We look for a solution to the problem (29) to be as follows
\[
\begin{bmatrix} \tilde{W}_1(\sigma, t, \tau) \\ \tilde{Z}_1(\sigma, t, \tau) \end{bmatrix} = \alpha_{11}(t)\varphi_1 e^{\tau_1} + \alpha_{12}(t)\varphi_2 e^{\tau_2}.
\]

Let us determine initial conditions for functions $\alpha_{11}(t)$, $\alpha_{12}(t)$. To do this, we plug (47) in the condition (30). This gives us
\[
\alpha_{11}(0) = G(\sigma), \quad \alpha_{12}(0) = G(\sigma).
\]

To find equations for the functions $\alpha_{11}(t)$, $\alpha_{12}(t)$ let us move to the system (31). Given (47), the right hand part of this system is as follows
\[
f_2(t, \tau) = e^{\tau_1} \begin{bmatrix} -2\dot{\alpha}_{11}(t) \\ -2\dot{\alpha}_{11}(t) - 4\sigma^2\alpha_{11}(t) \end{bmatrix} + e^{\tau_2} \begin{bmatrix} 2\dot{\alpha}_{12}(t) \\ -2\dot{\alpha}_{12}(t) + 4\sigma^2\alpha_{12}(t) \end{bmatrix}.
\]

Then the solvability conditions shall be expressed as follows
\[
\left\langle \begin{bmatrix} -2\dot{\alpha}_{11}(t) \\ -2\dot{\alpha}_{11}(t) - 4\sigma^2\alpha_{11}(t) \end{bmatrix}, \varphi_1 \right\rangle = 0,
\]
\[
\left\langle \begin{pmatrix} 2\dot{\alpha}_{12}(t) \\ -2\ddot{\alpha}_{12}(t) + 4\sigma^2\alpha_{12}(t) \end{pmatrix}, \varphi_2 \right\rangle = 0. \tag{51}
\]

By expanding the dot products (50), (51), we get equations for determination of the functions \(\alpha_{11}(t), \alpha_{12}(t)\)
\[-4\dot{\alpha}_{11}(t) - 4\sigma^2\alpha_{11}(t) = 0, \tag{52}\]
\[-4\dot{\alpha}_{12}(t) + 4\sigma^2\alpha_{12}(t) = 0. \tag{53}\]

By adding the corresponding initial conditions (48) to the latter equations, we uniquely define functions \(\alpha_{11}(t), \alpha_{12}(t)\)
\[\alpha_{11}(t) = e^{-\sigma^2 t} G(\sigma), \]
\[\alpha_{12}(t) = e^{\sigma^2 t} G(\sigma). \]

Given the latter equalities, the equation (47) will be as follows
\[
\begin{pmatrix} \bar{W}_1(\sigma, t, \tau) \\ \bar{Z}_1(\sigma, t, \tau) \end{pmatrix} = e^{-\sigma^2 t} G(\sigma) \varphi_1 e^{\tau_1} + e^{\sigma^2 t} G(\sigma) \varphi_2 e^{\tau_2}. \tag{54}\]

By ”restricting” \((\tau_1 = \frac{t}{2\varepsilon}, \tau_2 = -\frac{t}{2\varepsilon})\) in (47), we get another approximation of solution to the problems (16)-(17)
\[
\begin{pmatrix} W_1(\sigma, t, \tau) \\ Z_1(\sigma, t, \tau) \end{pmatrix} = e^{-t(\sigma^2 - \frac{1}{4\varepsilon})} G(\sigma) \varphi_1 + e^{-t(-\sigma^2 + \frac{1}{4\varepsilon})} G(\sigma) \varphi_2. \tag{55}\]

Let us apply Fourier inversion for \(W_1(\sigma, t, \varepsilon)\)
\[w_1(x, t, \varepsilon) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\sigma) \exp \left(-t \left(\sigma^2 - \frac{1}{2\varepsilon}\right)\right) \exp(i\sigma x) d\sigma - \]
\[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\sigma) \exp \left(-t \left(-\sigma^2 + \frac{1}{2\varepsilon}\right)\right) \exp(i\sigma x) d\sigma = \]
\[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(g(x)) \Phi \left(\frac{1}{2\sqrt{\pi} t} \exp \left(-\frac{x^2}{4t}\right)\right) \exp \left(\frac{t}{2\varepsilon}\right) \exp(i\sigma x) d\sigma - \]
\[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(g(x)) \Phi \left(\frac{1}{2\sqrt{\pi} t} \exp \left(\frac{x^2}{4t}\right)\right) \exp \left(-\frac{t}{2\varepsilon}\right) \exp(i\sigma x) d\sigma = \]
\[\exp \left(\frac{t}{2\varepsilon}\right) g(x) * \frac{1}{2\sqrt{\pi} t} \exp \left(-\frac{x^2}{4t}\right) - \exp \left(-\frac{t}{2\varepsilon}\right) g(x) * \frac{1}{2\sqrt{\pi} t} \exp \left(\frac{x^2}{4t}\right) = \tag{56}\]
Then, taking into account the substitution (5), we get

\[
U_1(x, t, \varepsilon) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\zeta) \exp \left( -\frac{(x - \zeta)^2}{4t} \right) d\zeta - \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\zeta) \exp \left( \frac{(x - \zeta)^2}{4t} \right) d\zeta.
\]  

As a result, we will get

\[
U(x, t, \varepsilon) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\zeta) \exp \left( -\frac{(x - \zeta)^2}{4t} \right) d\zeta +
\]

\[
\frac{\varepsilon}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\zeta) \exp \left( -\frac{(x - \zeta)^2}{4t} \right) d\zeta - \frac{\varepsilon}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\zeta) \exp \left( \frac{(x - \zeta)^2}{4t} \right) d\zeta.
\]  

All further approximations equal zero due to the fact that they are solutions to homogeneous equations with zero initial conditions.

References
[1] Lomov S A 1981 Introduction to the General Theory of Singular Perturbation (Moscow: Science)
[2] Lomov S A and Lomov I S 2011 Foundations of Mathematical Theory of Boundary Layer (Moscow: Moscow University Publisher’s House)
[3] Tikhonov A N and Samarsky A A 2004 Mathematical Physics Equations (Moscow: Moscow University Publisher’s House)