The classification of left-invariant para-Kähler structures on simply connected four-dimensional Lie groups

M. W. Mansouri and A. Oufkou

Abstract

We give a complete classification of left invariant para-Kähler structures on four-dimensional simply connected Lie groups up to an automorphism. As an application we discuss some curvatures properties of the canonical connection associated to these structures as flat, Ricci flat and existence of Ricci solitons.

Keywords: Symplectic Lie algebras, para-Kähler structures, Ricci soliton

1. Introduction and main results

An almost para-complex structure on a $2n$-dimensional manifold $M$ is a field $K$ of endomorphisms of the tangent bundle $TM$ such that $K^2 = Id_{TM}$ and the two eigendistributions $T^\pm M := \ker(Id \pm K)$ have the same rank. An almost para-complex structure $K$ is said to be integrable if the distributions $T^\pm M$ are involutive. This is equivalent to the vanishing of the Nijenhuis tensor $N_K$ defined by

$$N_K(X, Y) = [X, Y] + [KX, KY] - K[KX, Y] - K[X, KY],$$

for vector fields $X, Y$ on $M$. In such a case $K$ is called a para-complex structure. A para-Kähler structure on a manifold $M$ is a pair $(\langle \cdot, \cdot \rangle, K)$ where $\langle \cdot, \cdot \rangle$ is a pseudo-Riemannian metric and $K$ is a parallel skew-symmetric para-complex structure. If $(\langle \cdot, \cdot \rangle, K)$ is a para-Kähler structure on $M$, then $\omega = \langle \cdot, \cdot \rangle \circ K$ is a symplectic structure and the $\pm 1$–eigendistributions $T^\pm M$ of $K$ are two integrable $\omega$-Lagrangian distributions. Due to this, a para-Kähler structure can be identified with a bi-Lagrangian structure $(\omega, T^\pm M)$ where $\omega$ is a symplectic structure and $T^\pm M$ are two integrable Lagrangian distributions. Moreover the Levi-Civita connection associate to neutral metric $\langle \cdot, \cdot \rangle$ coincides with the canonical connection associate to bi-Lagrangian structure (the unique symplectic connection with parallelizes both foliations [9]). For a survey on paracomplex geometry see [6] and for background on bi-Lagrangian structures and their associated connections, the survey [7] is a good starting point and contains further references (See as well [1] and [5]).

Suppose now that $M$ is a Lie group $G$ and $\omega, \langle \cdot, \cdot \rangle$ and $K$ are left invariant. If we denote by $\mathfrak{g}$ the Lie algebras of $G$, then $(\langle \cdot, \cdot \rangle, K)$ is determined by is restrictions to the Lie algebra $\mathfrak{g}$. In this situation, $(\mathfrak{g}, \langle \cdot, \cdot \rangle, K_e)$ or $(\mathfrak{g}, \omega_e, K_e)$ is called a para-Kähler Lie algebra (e is unit of $G$), in the
rest of this paper a para-Kähler Lie algebra will be noted \((g, \omega, K)\). Recall that two para-Kähler Lie algebras \((g_1, \omega_1, K_1)\) and \((g_2, \omega_2, K_2)\) are said to be equivalent if there exists an isomorphism of Lie algebras \(T : g_1 \rightarrow g_2\) such as \(T^\ast \omega_2 = \omega_1\) and \(T^\ast K_1 = K_2\). Para-Kähler (bi-Lagrangian) structures on Lie algebras in general have been studied, for example, in \([2, 3]\) and \([4]\). In \([8]\), there is a study the existences of bi-Lagrangian structures on symplectic nilpotent Lie algebras of dimension 2, 4 and 6. A first classification of para-Kähler structures on four-dimensional Lie algebras was obtained by Calvaruso in \([6]\). Another classification based on the classification of symplectic Lie algebras is proposed by Smolentsev and Shagabudinova in \([12]\). Benayadi and Boucetta provide in \([3]\) a new characterization of para-Kähler Lie algebras using left symmetric bialgebras introduced by Bai in \([2]\). Based on this characterization we propose in this paper, the classification of para-Kähler structures on four-dimensional Lie algebras. Notice that our classification is more complete and precise than the other classifications existing in the literature.

**Notations:** For \([e_1, e_2, e_3, e_4]\) a basis of \(g\), we denote by \([e^1, e^2, e^3, e^4]\) the dual basis on \(g^\ast\) and \(e^i j\) the two-form \(e^1 \wedge e^j, e^j i\) is the symmetric two-form \(e^1 \odot e^j\) and \(E_{ij}\) is the endomorphism which sends \(e_j\) to \(e_i\) and vanishes on \(e_k\) for \(k \neq j\).

The para-Kähler Lie algebras \((g, (.,, ), K)\) is necessarily symplectic Lie algebra \((g, \omega)\). It is well known that a symplectic four-dimensional Lie algebra is necessarily solvable. The classification of symplectic four-dimensional Lie algebras \((g, \omega)\) is given by the following Table (see \([10]\)).

| Case | No vanishing brackets | \(\omega\) |
|------|-----------------------|----------|
| \(\mathfrak{r}_13\) | \([e_1, e_2] = e_3\) | \(e^{13} + e^{23}\) |
| \(\mathfrak{r}_{13,0}\) | \([e_1, e_2] = e_2\) | \(e^{23} + e^{24}\) |
| \(\mathfrak{r}_{13,1}\) | \([e_1, e_2] = e_2, [e_1, e_3] = -e_3\) | \(e^{34} + e^{23}\) |
| \(\mathfrak{g}_{2,2}\) | \([e_1, e_2] = e_2, [e_1, e_3] = e_1, [e_2, e_3] = e_2\) | \(e^{12} + e^{23}\) |
| \(\mathfrak{r}_{4,0}\) | \([e_1, e_2] = e_1, [e_3, e_4] = e_2\) | \(e^{12} + e^{23}\) |
| \(\mathfrak{r}_{4,1}\) | \([e_1, e_2] = e_1, [e_1, e_3] = e_2\) | \(e^{12} + e^{23}\) |
| \(\mathfrak{h}_{1,0}\) | \([e_1, e_2] = e_3, [e_3, e_4] = e_3, [e_4, e_1] = e_1\) | \(e^{12} - e^{24}\) |
| \(\mathfrak{h}_{1,1}\) | \([e_1, e_2] = e_3, [e_3, e_4] = e_3, [e_4, e_1] = e_1\) | \(e^{12} - e^{24}\) |
| \(\mathfrak{h}_{1,2}\) | \([e_1, e_2] = e_3, [e_4, e_1] = e_1, [e_4, e_2] = -e_2\) | \(e^{12} + e^{23}\) |
| \(\mathfrak{h}_{1,3}\) | \([e_1, e_2] = e_3, [e_3, e_4] = e_3, [e_4, e_1] = e_1, [e_4, e_2] = (1 - \lambda)e_2\) | \(e^{12} - e^{24}\) |
| \(\mathfrak{h}_{1,4}\) | \([e_1, e_2] = e_3, [e_4, e_1] = \frac{1}{2}e_1 - e_2, [e_4, e_3] = \frac{1}{2}e_1 + \frac{1}{2}e_2\) | \(\mp (e^{12} - e^{34})\) |

Table 1: Symplectic four-dimensional Lie algebras

\((\mu \geq 0, -1 \leq \beta < 1, -1 < \alpha < 0, \delta > 0)\) and \(\lambda \geq 0, \lambda \neq 1, 2)\).
Our main result is the following.

**Theorem 1.1.** Let \((g, \omega, K)\) be a four-dimensional para-Kähler Lie algebra. Then \((g, \omega, K)\) is isomorphic to one of the following Lie algebras with the given para-Kähler structures:

**Lie algebra** \(\mathfrak{r}^3\)

*For \(\omega = e^{14} + e^{23}\)*

\[k_1 = E_{11} - E_{21} - E_{22} + E_{33} - E_{43} - E_{44}\]
\[k_2 = E_{11} - E_{22} + E_{33} - E_{44}\]
\[k_3 = -E_{11} + E_{22} - E_{33} + E_{44}\]

**Lie algebra** \(\mathfrak{r}^{3,0}\)

*For \(\omega = e^{12} + e^{34}\)*

\[k_1 = -E_{11} + E_{22} - E_{33} + E_{44}\]
\[k_2 = E_{11} + xE_{12} - E_{22} + E_{33} - E_{44}\]

**Lie algebra** \(\mathfrak{r}^{3,-1}\)

*For \(\omega = e^{14} + e^{23}\)*

\[k_1 = E_{11} + E_{22} \pm E_{23} - E_{33} - E_{44}\]
\[k_2 = -E_{11} - E_{22} \pm E_{23} + E_{33} + E_{44}\]
\[k_3 = E_{11} + E_{22} + xE_{14} - E_{33} - E_{44}\]

**Lie algebra** \(\mathfrak{r}^{1,\gamma_2}\)

*For \(\omega = e^{12} + \mu e^{13} + e^{34}, (\mu > 0)\)*

\[k_1 = -E_{11} + 2E_{13} + E_{22} + E_{33} - 2E_{42} - E_{44}\]
\[k_2 = -E_{11} + E_{22} + E_{33} - E_{44}\]
\[k_3 = E_{11} - 2E_{13} - E_{22} - E_{33} + 2E_{42} + E_{44}\]

*For \(\omega = e^{12} + e^{34}\)*

\[k_1 = E_{11} - E_{22} - 2E_{24} + 2E_{31} - E_{33} + E_{44}\]
\[k_2 = -E_{11} + xE_{12} + E_{13} + xE_{14} + E_{22} + xE_{32} + xE_{34} - E_{42} + \frac{4}{5}E_{43}\]
\[k_3 = -E_{11} + E_{22} - E_{33} + E_{44}\]
\[k_4 = -E_{11} + E_{22} + E_{33} + xE_{43} - E_{44}\]
\[k_5 = E_{11} - 2E_{13} - E_{22} - E_{33} + 2E_{42} + E_{44}\]
\[k_6 = E_{11} + xE_{12} - E_{22} + E_{33} + yE_{34} - E_{44}\]
\[k_7 = E_{11} + xE_{12} + xE_{14} - E_{22} + xE_{32} + E_{33} + xE_{34} - E_{44}\]

**Lie algebra** \(\mathfrak{r}^{1,\gamma_2'}\)

*For \(\omega = e^{14} + e^{23}\)*

\[k_1 = E_{11} + xE_{14} - E_{22} - \frac{4}{5}E_{32} + E_{33} - E_{44}\]
\[k_2 = -E_{11} - E_{22} + E_{33} + E_{44}\]
\[
\begin{align*}
K_3 &= xE_{11} + 2yE_{12} + (1 - x)E_{13} - 2yE_{14} - 2yE_{21} + xE_{22} + 2yE_{23} + (1 + x)E_{31} + 2yE_{32} - xE_{33} - 2yE_{34} - 2yE_{41} + (1 + x)E_{42} + 2yE_{43} - xE_{44} \\
K_4 &= E_{11} + xE_{14} + E_{22} - E_{33} - E_{44}
\end{align*}
\]

**Lie algebra** \(t_{4,0}\)

For \(\omega = e^{14} \pm e^{23}\)
\[
\begin{align*}
K_1 &= -E_{11} + E_{22} - E_{33} + E_{44} \\
K_2 &= E_{11} - E_{22} + E_{33} - E_{44}
\end{align*}
\]

**Lie algebra** \(t_{4,-1}\)

For \(\omega = e^{13} + e^{24}\)
\[
\begin{align*}
K_1 &= E_{11} + xE_{14} - E_{22} - E_{33} + E_{44} \\
K_2 &= -E_{11} + E_{22} + E_{33} - E_{44}
\end{align*}
\]

**Lie algebra** \(t_{4,-1,\beta} (-1 < \beta < 1)\)

For \(\omega = e^{12} + e^{34}\)
\[
\begin{align*}
K_1 &= E_{11} + E_{12} - E_{22} - E_{33} + E_{44} \\
K_2 &= E_{11} - E_{12} - E_{22} - E_{33} + E_{44} \\
K_3 &= E_{11} - E_{22} - E_{33} + E_{44} \\
K_4 &= E_{11} - E_{22} + xE_{34} + \frac{1}{x}E_{43} \\
K_5 &= E_{11} - E_{22} + E_{33} - E_{44} \\
K_6 &= -E_{11} + E_{12} + E_{22} + E_{33} - E_{44} \\
K_7 &= -E_{11} - E_{12} + E_{22} + E_{33} - E_{44}
\end{align*}
\]

**Lie algebra** \(t_{4,-1,-1}\)

For \(\omega = e^{12} + e^{34}\)
\[
\begin{align*}
K_1 &= -E_{11} + xE_{21} + E_{22} + E_{23} - E_{33} + E_{44} \\
K_2 &= -E_{11} + E_{21} + E_{22} - E_{33} + E_{44} \\
K_3 &= -E_{11} - E_{21} + E_{22} - E_{33} + E_{44} \\
K_4 &= -E_{11} + E_{22} - E_{33} + E_{44} \\
K_5 &= E_{11} - E_{12} - E_{22} - E_{33} + E_{44} \\
K_6 &= E_{11} + E_{12} - E_{22} - E_{33} + E_{44} \\
K_7 &= E_{11} - E_{22} - E_{33} + xE_{43} + E_{44} \\
K_8 &= -E_{11} + E_{22} + E_{33} - E_{44} \\
K_9 &= E_{11} + E_{21} - E_{22} + E_{33} - E_{44} \\
K_{10} &= E_{11} - E_{21} - E_{22} + E_{33} - E_{44}
\end{align*}
\]

**Lie algebra** \(t_{4,\alpha,-\omega} (-1 < \alpha < 0)\)

For \(\omega = e^{14} + e^{23}\)
\[
\begin{align*}
K_1 &= -E_{11} + E_{22} + E_{23} - E_{33} + E_{44}
\end{align*}
\]
\[ K_2 = -E_{11} + E_{22} - E_{23} - E_{33} + E_{44} \]
\[ K_3 = 2(-E_{11} + E_{22} - E_{33} + xE_{41} + E_{44}) \]
\[ K_4 = E_{11} + E_{22} - E_{33} - E_{44} \]
\[ K_5 = E_{11} + E_{22} + E_{32} - E_{33} - E_{44} \]
\[ K_6 = -E_{11} - E_{22} + E_{32} + E_{33} + E_{44} \]
\[ K_7 = E_{11} + E_{22} - E_{32} - E_{33} - E_{44} \]
\[ K_8 = E_{11} - E_{22} + E_{23} + E_{33} - E_{44} \]
\[ K_9 = E_{11} - E_{22} + E_{23} + E_{33} - E_{44} \]
\[ K_{10} = -E_{11} - E_{22} - E_{32} + E_{33} + E_{44} \]
\[ K_{11} = -E_{11} - E_{22} + E_{33} + E_{44} \]

**Lie algebra \( k_{4,1} \)**

*For \( \omega = e^{12} - e^{34} \)*
\[ K_1 = E_{11} + E_{12} - E_{22} - E_{33} + E_{44} \]
\[ K_2 = E_{11} + E_{12} - E_{22} - E_{33} + E_{44} \]
\[ K_3 = E_{11} - E_{22} - E_{33} + xE_{43} + E_{44} \]
\[ K_4 = E_{11} - E_{22} + E_{33} - E_{44} \]
\[ K_5 = -E_{11} + E_{12} + E_{22} + E_{33} - E_{44} \]
\[ K_6 = -E_{11} - E_{12} + E_{22} + E_{33} - E_{44} \]
\[ K_7 = E_{11} + E_{21} - E_{22} + xE_{23} + E_{33} - xE_{41} - E_{44} \]
\[ K_8 = -E_{11} + E_{21} + E_{22} - E_{33} + E_{44} \]
\[ K_9 = -E_{11} - E_{21} + E_{22} - E_{33} + E_{44} \]
\[ K_{10} = -E_{11} + E_{22} - E_{33} + xE_{43} + E_{44} \]

*For \( \omega = e^{12} - e^{34} + e^{24} \)*
\[ K_1 = E_{11} + xE_{12} - E_{22} - E_{33} + E_{44} \]
\[ K_2 = -E_{11} + xE_{12} + E_{22} + E_{33} - E_{44} \]

**Lie algebra \( k_{4,2} \)**

*For \( \omega = e^{12} - e^{34} \)*
\[ K_1 = E_{11} + E_{12} - E_{22} - E_{33} + E_{44} \]
\[ K_2 = E_{11} - E_{12} - E_{22} - E_{33} + E_{44} \]
\[ K_3 = E_{11} - E_{22} - E_{33} + E_{44} \]
\[ K_4 = E_{11} - E_{22} + E_{33} + xE_{43} - E_{44} \]

*For \( \omega = e^{14} - e^{23} \)*
\[ K_1 = -E_{11} - E_{22} + \frac{1}{2}x^2 + E_{32} + E_{33} - 2xE_{14} + E_{44} \]
\[ K_2 = -E_{11} - E_{22} - 2E_{31} + xE_{32} + E_{33} + 2E_{24} + E_{44} \]
\[ K_3 = E_{11} + E_{22} - 2E_{31} + xE_{32} - E_{33} + 2E_{24} - E_{44} \]
\[ K_4 = E_{11} - E_{22} + xE_{12} + xE_{32} + E_{33} + xE_{14} + xE_{34} - E_{44} \]
\[ K_5 = -E_{11} + E_{22} - E_{33} + xE_{41} + E_{44} \]
\[ K_6 = E_{11} - E_{22} - 2xE_{23} + E_{33} + xE_{41} - E_{44} \]
\[ K_7 = -E_{11} + 2xE_{21} + E_{22} - 2xE_{23} - E_{33} - 2xE_{41} - 2xE_{43} + E_{44} \]
\[ K_5 = -E_{11} + E_{22} - 2xE_{23} + E_{33} + xE_{41} + E_{44} \]
\[ K_6 = E_{11} - E_{22} - 2xE_{23} + E_{33} + xE_{41} - E_{44} \]
\[ K_7 = -E_{11} + 2xE_{21} + E_{22} - 2xE_{23} - E_{33} - 2xE_{41} - 2xE_{43} + E_{44} \]

For \( \omega = e^{14} + e^{23} \)
\[ K_1 = -E_{11} - E_{22} + \frac{1}{4}xE_{32} + E_{33} - 2xE_{14} + E_{44} \]
\[ K_2 = -E_{11} + E_{22} - E_{33} + xE_{41} + E_{44} \]
\[ K_3 = E_{11} - E_{22} + xE_{23} + E_{33} - \frac{1}{4}xE_{14} - E_{44} \]

**Lie algebra \( \mathfrak{d}_{4,1} \)**

For \( \omega = e^{12} - e^{34} \)
\[ K_1 = E_{11} - E_{22} - E_{33} + xE_{43} + E_{44} \]
\[ K_2 = E_{11} - E_{22} + E_{33} - E_{44} \]

**Lie algebra \( \mathfrak{d}_{4,1} (\lambda > \frac{1}{2}, \lambda \neq 1, 2) \)**

For \( \omega = e^{12} - e^{34} \)
\[ K_1 = E_{11} + E_{12} - E_{22} - E_{33} + E_{44} \]
\[ K_2 = E_{11} - E_{12} - E_{22} - E_{33} + E_{44} \]
\[ K_3 = -E_{11} + E_{21} + E_{22} - E_{33} + E_{44} \]
\[ K_4 = -E_{11} - E_{21} + E_{22} - E_{33} + E_{44} \]
\[ K_5 = E_{11} - E_{22} - E_{33} + xE_{43} + E_{44} \]
\[ K_6 = -E_{11} + E_{22} - E_{33} + xE_{43} + E_{44} \]
\[ K_7 = -E_{11} + xE_{12} + E_{22} + E_{33} - E_{44} \]
\[ K_8 = E_{11} + xE_{21} - E_{22} + E_{33} - E_{44} \]

**Lie algebra \( \mathfrak{d}_4 \)**

For \( \omega = \pm(e^{12} - e^{34}) \)
\[ K_1 = E_{11} - E_{22} - E_{33} + E_{44} \]
\[ K_2 = -E_{11} + E_{22} + E_{33} - E_{44} \]

**Corollary 1.1.** The symplectic Lie algebras \( \mathfrak{h}_{3,0}, \mathfrak{n}_4, \mathfrak{n}_r'_{4,0}, \mathfrak{n}_{4,4} \) and \( \mathfrak{b}'_{4,4} \) does not admit a para-Kähler structure.

The paper is organized as follows. Section 2 contains the basic results which are essential to the classification of four-dimensional para-Kähler Lie algebras (proof of the Theorem 1.1). Theorem 2.1 and Theorem 2.2 are the key steps in this proof. Section 3 is devoted to some curvature properties of four-dimensional para-Kähler metrics. Section 4 contains the tables of Theorems 2.2 and the isomorphisms tables used in the proof of Theorem 1.1. The software Maple 18® has been used to check all needed calculations.
2. Proof of Theorem 1.1

In this section we begin with a reminder of the new approach introduced by Benayadi and Boucetta in \[3\], which characterizes the para-Kähler Lie algebras. Recall that, a para-Kähler Lie algebra \((g, \langle \cdot , \cdot \rangle, K)\) is carries a Levi-Civita product, the product characterized by Koszul’s formula:

\[ 2\langle u,v,w \rangle = \langle [u,v], w \rangle + \langle [v,w], u \rangle + \langle [w,u], v \rangle. \]

The subalgebras \(g^1 = \ker(K - Id_g)\) and \(g^{-1} = \ker(K + Id_g)\) have the following properties, \(g^1\) and \(g^{-1}\) are isotropic with respect to \(\langle \cdot , \cdot \rangle\), Lagrangian with respect to \(\omega\) and checking that \(g = g^1 \oplus g^{-1}\), moreover the restriction of the Levi-Civita product on \(g^1\) and \(g^{-1}\) induces a left symmetric structures. i.e. for any \(u, v, w \in g^1\) (resp. \(g^{-1}\)),

\[ \text{ass}(u,v,w) = \text{ass}(v,u,w) \]

where \(\text{ass}(u,v,w) = (u.v).w - u.(v.w)\). In particular, \(g^1\) and \(g^{-1}\) are left symmetric algebras.

For any \(u \in g^{-1}\), let \(u^*\) denote the element of \((g^1)^*\) given by \(u^*(v) = \langle u, v \rangle\). The map \(u \mapsto u^*\) realizes an isomorphism between \(g^{-1}\) and \((g^1)^*\). Thus, we can identify \((g, \langle \cdot , \cdot \rangle, K)\) relative to the phase space \((g^1 \oplus (g^1)^*), \langle \cdot , \cdot \rangle_0, K_0)\), where \(\langle \cdot , \cdot \rangle_0\) and \(K_0\) are given by:

\[ \langle u + a, v + \beta \rangle_0 = \alpha(u) - \beta(v) \quad \text{and} \quad K_0(u + a) = u - a. \]

Both \(g^1\) and \((g^1)^*\) carry a left symmetric algebra structure. For any \(u \in g^1\) and for any \(\alpha \in (g^1)^*\), we denote \(L_\alpha : g^1 \to g^1\) and \(L_\alpha : (g^1)^* \to (g^1)^*\) as the left multiplication by \(u\) and \(\alpha\), respectively, i.e., for any \(v \in g^1\) and any \(\beta \in (g^1)^*\),

\[ L_\alpha v = u.v \quad \text{and} \quad L_\alpha \beta = \alpha.\beta. \]

The Levi-Civita product (and the Lie bracket) on \(g\) is determined entirely by their restrictions to \((g^1)^*\) and \(g^1\): For any \(u \in g^1\) and for any \(\alpha \in (g^1)^*\),

\[ u.\alpha = L_\alpha u \quad \text{and} \quad \alpha.u = -L_\alpha X. \]

Conversely, let \(U\) be a finite dimensional vector space and \(U^*\) is its dual space. We suppose that both \(U\) and \(U^*\) have the structure of a left symmetric algebra. We extend the products on \(U\) and \(U^*\) to \(U \oplus U^*\) for any \(X, Y \in U\) and for any \(\alpha, \beta \in U^*\), by putting

\[ (X + a).(Y + \beta) = X.Y - L_{\alpha} X - L_{\beta} Y + \alpha.\beta. \tag{1} \]

We say that two left symmetric products on \(U\) and \(U^*\) is Lie-extendible if the commutator of the product on \(U \oplus U^*\) given by (1) is a Lie bracket. In this case we have the following theorem:

**Theorem 2.1.** \[3\] Let \((U, \cdot, \cdot)\) and \((U^*, \cdot, \cdot)\) be two Lie-extendible left symmetric products. Then, \((U \oplus U^*, \langle \cdot , \cdot \rangle_0, K_0)\), endowed with the Lie algebra bracket associated with the product given by (1) is a para-Kähler Lie algebra. Where \(\omega_0, \langle \cdot , \cdot \rangle_0\) and \(K_0\) are given by:

\[ \omega_0(u + a, v + \beta) = \beta(u) - \alpha(v), \quad \langle u + a, v + \beta \rangle_0 = \alpha(u) + \beta(v) \quad \text{and} \quad K_0(u + a) = u - a. \]

Moreover, all para-Kähler Lie algebras are obtained in this manner.

Let now \(U\) be a 2-dimensional vector space and \(U^*\) its dual space and let \(\{e_1, e_2\}, \{e_3, e_4\}\) be a basis of \(U\) and \(U^*\). We base on the previous theorem and the classification of real left-symmetric algebras in dimension 2 listed below (see Theorem 1.2. of \[11\]),
we get $\omega = e^{13} + e^{24}$, $(., .) = e^{13} + e^{24}$ and $K = E_{11} + E_{22} - E_{33} - E_{44}$.

Remark 1. $b$ stands for algebras with non-commutative associated Lie algebra and $c$ stands for algebras with commutative associated Lie algebra.

Theorem 2.2. Let $(\mathfrak{g}, (., .), K)$ be a four-dimensional para-Kähler Lie algebra. Then there exists a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that

\[
\begin{align*}
\omega &= e^{13} + e^{24}, \quad (., .) = e^{13} + e^{24} \quad \text{and} \quad K = E_{11} + E_{22} - E_{33} - E_{44}.
\end{align*}
\]

and the non vanishing Lie brackets as listed in the Table 2 and 3.

Proof. We will give the proof in the case $B_2$ since all cases should be handled in a similar way. In that case the left-symmetric product in $U$ is given by $e_2. e_1 = e_1, e_2. e_2 = e_1 + e_2$ and let

\[
\begin{align*}
e_3. e_3 &= a_{33} e_3 + b_{33} e_4 \\
e_3. e_4 &= a_{34} e_3 + b_{34} e_4 \\
e_4. e_3 &= a_{43} e_3 + b_{43} e_4 \\
e_4. e_4 &= a_{44} e_3 + b_{44} e_4
\end{align*}
\]

be an arbitrary product in $U^*$, let’s look for products in $U^*$ which satisfy the Jacobi identity $\hat{f}[[e_1, e_j], e_k] = 0$ with $1 \leq i < j < k \leq 4$, where $\hat{f}$ denotes the cyclic sum.

The identity $\hat{f}[[e_1, e_2], e_3] = 0$ and $\hat{f}[[e_1, e_2], e_4] = 0$ is equivalent to

\[
\begin{align*}
b_{34} + a_{33} + a_{43} &= 0 \\
a_{44} &= 0 \\
b_{34} + a_{43} &= 0
\end{align*}
\]

suppose that $a_{44} = 0$, $b_{34} + a_{43} = 0$ and $b_{34} = -a_{33} - a_{43}$. The identity $\hat{f}[[e_1, e_3], e_4] = 0$ and $\hat{f}[[e_2, e_3], e_4] = 0$ is equivalent to

\[
\begin{align*}
a_{33} a_{34} - 2a_{33} a_{43} - a_{34} d_{43} - a_{43} d_{43} &= 0 \\
a_{34} (a_{33} + a_{43}) &= 0 \\
a_{34} &= 0
\end{align*}
\]

we get $a_{34} = 0$, $a_{43} = 0$, $a_{33} = 0$, and $d_{43} = 0$. Then the product in $U^*$ is given by $e_3. e_4 = e_4$ (who is indeed a left-symmetric product) and the Lie bracket in $U \oplus U^*$ is given by

\[
[e_1, e_2] = -e_1, \quad [e_2, e_3] = xe_1 - e_3 - e_4, \quad [e_2, e_4] = -e_4.
\]
Proof. of the Theorem

The Theorem 2.2 confirms that for each Lie algebra $\mathfrak{g}$ of the tables 4 and 5 there exist a base $B_0 = \langle e_1, e_2, e_3, e_4 \rangle$ such that the para-Kähler structure is given by

$$\omega = e^{13} + e^{24} \quad \text{and} \quad K = E_{11} + E_{22} - E_{33} - E_{44}$$

and the Lie brackets depend on some parameters. In Tables 6 and 7 we build a family of isomorphisms (depending on the values of parameters) from $\mathfrak{g}$ (or $\mathfrak{C}_{1,2,3}$) onto a four-dimensional Lie algebra, (say $A$) of the Table 1. Each isomorphism is given by the passage matrix $P$ from $B_0$ to $B = \langle f_1, f_2, f_3, f_4 \rangle$. The image by $P$ of the para-Kähler structure $(\omega, K)$ is given by the matrices of its component in the bases $B$ and $B'$ by

$$(P \circ \omega \circ P = \omega_i \quad \text{and} \quad P^{-1} \circ K \circ P = K_i.$$

In this way we collect all the possible para-Kähler structures $(\omega_i, K_i)$ on $A$. Thereafter, we proceed to the classification in $A$ (up to automorphism).

We will give the proof in the case $\mathfrak{r}_{3,0}$ since all cases should be handled in a similar way. We will show that the Lie algebra $\mathfrak{r}_{3,0}$ admits two non-equivalent para-Kähler structures. Note that in this case the non vanishing Lie bracket is

$$[f_1, f_2] = f_3$$

the symplectic form is $\omega_0 = f^{12} + f^{34}$ and the automorphisms is

$$T = \begin{pmatrix}
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & a_{2,1} & 0 & 0 \\
0 & 0 & a_{3,3} & a_{3,4} \\
0 & a_{4,1} & 0 & a_{4,4}
\end{array}
\end{pmatrix}.$$ 

The groups of automorphisms of four dimensional Lie algebras were given in 10. From Table 6 and Table 7 $\mathfrak{r}_{3,0}$ is obtained four times.

1. The transformation: $f_1 = -e_4, f_2 = e_2, f_3 = e_3, f_4 = e_1$ gives an isomorphism from $\mathfrak{C}_{1,6}$ to $\mathfrak{r}_{3,0}$ and the para-Kähler structure obtained on $\mathfrak{r}_{3,0}$ is

$$\omega_1 = f^{12} - f^{34} \quad \text{and} \quad K_1 = -E_{11} + E_{22} - E_{33} + E_{44}.$$ 

2. The transformation: $f_1 = -e_2, f_2 = -ye_2 + e_4, f_3 = e_1, f_4 = e_3$ gives an isomorphism from $\mathfrak{C}_{2,1}$ with $x = 0$ to $\mathfrak{r}_{3,0}$ and the para-Kähler structure obtained on $\mathfrak{r}_{3,0}$ is

$$\omega_2 = -f^{12} + f^{34} \quad \text{and} \quad K_2 = E_{11} - 2yE_{12} - E_{22} + E_{33} - E_{44}.$$ 

3. The transformation: $f_1 = -e_2, f_2 = e_4, f_3 = e_1, f_4 = e_3$ gives an isomorphism from $\mathfrak{C}_{2,2}$ with $x = 0, y = 0$ to $\mathfrak{r}_{3,0}$ and the para-Kähler structure obtained on $\mathfrak{r}_{3,0}$ is

$$\omega_3 = -f^{12} + f^{34} \quad \text{and} \quad K_3 = E_{11} - E_{22} + E_{33} - E_{44}.$$ 

4. The transformation: $f_1 = e_1 - e_2, f_2 = e_4, f_3 = e_1, f_4 = e_3$ gives an isomorphism from $\mathfrak{C}_{2,3}$ with $x = 0, y = 0$ to $\mathfrak{r}_{3,0}$ and the para-Kähler structure obtained on $\mathfrak{r}_{3,0}$ is

$$\omega_4 = -f^{12} + f^{24} + f^{34} \quad \text{and} \quad K_4 = E_{11} - E_{22} + E_{33} - E_{44}.$$
the algebra $\mathfrak{rr}_{3,0}$ support $\omega_0$ as a unique symplectic structure (up to automorphism), therefore there are four families of automorphisms $T_i$, $i \in \{1, \ldots, 4\}$ such that $T_i^* \omega_0 = \omega_0$ for $i \in \{1, \ldots, 4\}$, a direct calculation gives us

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & 1 & 0 & 0 \\ 0 & 0 & a_{3,4} & a_{4,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & -1 & 0 & 0 \\ 0 & 0 & a_{3,4} & a_{4,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & 1 & 0 & 0 \\ 0 & 0 & a_{3,4} & a_{4,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & -1 & 0 & 0 \\ 0 & 0 & a_{3,4} & a_{4,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix}.$$ 

Thus we obtain four para-Kähler structures on $\mathfrak{rr}_{3,0}$ given by $(\omega_0, K_{0i})$, $i \in \{1, \ldots, 4\}$ with $K_{0i} = T_i^{-1} \circ K_i \circ T_i$ a direct calculation gives us

$$K_{01} = -E_{11} + E_{22} - E_{33} + E_{44}$$
$$K_{02} = E_{11} + 2yE_{12} - E_{22} + E_{33} - E_{44}$$
$$K_{03} = E_{11} - E_{22} + E_{33} - E_{44}$$
$$K_{04} = -E_{11} + E_{22} + E_{33} - E_{44}$$

Noticing that $K_{03}$ is a sub-case of $K_{02}$ and that $(\omega_0, K_{04})$ is isomorphic to $(\omega_0, K_{01})$. Indeed we have $L^* \omega_0 = \omega_0$ and $L_i^{-1} \circ K_{04} \circ L_i = K_{01}$ with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$ 

We complete the proof by showing that $(\omega_0, K_{01})$ is not isomorphic to $(\omega_0, K_{02})$. Indeed, the symplectomorphism group of $\omega_0$ is generated by

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & 1 & 0 & 0 \\ 0 & 0 & a_{3,4} & a_{4,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & 1 & 0 & 0 \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & -a_{3,4}^{-1} & 0 \end{pmatrix}$$

a simple calculation gives us

$$f^2((L_1^{-1} \circ K_{01} \circ L_1 - K_{02})(f_1)) = 2 \quad \text{and} \quad f^1((L_2^{-1} \circ K_{01} \circ L_2 - K_{02})(f_1)) = -2$$

so $L_1^{-1} \circ K_{01} \circ L_1 \neq K_{02}$ and $L_2^{-1} \circ K_{01} \circ L_2 \neq K_{02}$.
3. Application: Curvature properties of four-dimensional para-Kähler Lie algebras

Let now \((q, \omega, K)\) denote a four-dimensional para-Kähler Lie algebra. Let \(\nabla: g \times \mathfrak{g} \rightarrow g\) be the Levi-Civita product associated to a left-invariant pseudo-Riemannian metric \(h(X, Y) = \omega(KX, Y)\). The connection \(\nabla\) is also called Hess connection. The curvature tensor is then described in terms of the map

\[
R : \quad g \times g \rightarrow g \lceil g \lceil \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \right. \quad (2)
\]

The Ricci tensor is the symmetric tensor \(\text{ric}\) given by \(\text{ric}(X, Y) = \text{tr}(Z \mapsto R(X, Z)Y)\) and the Ricci operator \(\text{Ric} : g \rightarrow g\) is given by the relation \(\langle \text{Ric}(X), Y \rangle = \text{ric}(X, Y)\). The scalar curvature is defined in the standard way by \(s = \text{tr}(\text{Ric})\).

Recall that \((g, h)\) is called flat if \(R = 0\), Ricci flat if \(\text{Ric} = 0\) and Ricci soliton if

\[
\mathcal{L}_X h + \text{ric} = \lambda h, \quad (3)
\]

where \(X = \sum x_i e_i + \sum y_i e_i\) is a vector field and \(\lambda\) is a real constant, in that case if \(X = 0\) then \(h\) is called Einstein metric and if \(\lambda\) is positive, zero, or negative then \(h\) is called a shrinking, steady, or expanding Ricci soliton, respectively. We give in the following theorem some geometrical situations for the left invariant four-dimensional dimensional para-Kähler Lie groups.

**Theorem 3.1.** Let \((g, \omega, K)\) be a class of para-Kähler Lie algebras obtained in Theorem 3.1. The associated para-Kähler metric and some of his properties are given in the following tables.
| $r_4$  | $e^{12} = e^{14} + e^{23}, x \neq 0$ | $e^{12} = e^{14} + e^{23}, x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ |
|--------|----------------------------------|----------------------------------|----------------------------------|
| $r_4$  | $\pm(e^{14} + e^{23}), \beta \neq 0$ | $\pm(e^{14} + e^{23}), \beta \neq 0$ | $\pm(e^{14} + e^{23}), \beta \neq 0$ |
| $r_{4,0}$ | $\pm(e^{14} + e^{23}), \beta \neq 0$ | $\pm(e^{14} + e^{23}), \beta \neq 0$ | $\pm(e^{14} + e^{23}), \beta \neq 0$ |
| $r_{4,1}$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ |
| $r_{4,1,0}$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ |
| $r_{4,1,1}$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ |
| $d_{4,1}$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ |
| $d_{4,2}$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ |
| $d_{4,3}$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ |
| $d_{4,4}$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ |
| $d_{4,5}$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ | $\pm(e^{14} + e^{23}), x \neq 0$ |
Table 3: Curvature properties of four-dimensional para-Kähler Lie algebras

| b₄ | ±(e₁² - e₄⁴) | No | Yes | 0 | (0, 0, 0, 0) |
|----|--------------|----|-----|---|----------------|
| Hₚ | e₁⁺ ∈ ℝ | No | No | 3/2 | (0, 0, 0, 0) |
| H₁ | x^n | No | Yes | 0 | (0, 0, 0, 0) |
| H₂ | x^n | Yes | Yes | -x₄ | (0, 0, 0, x₄) |

\[ h₁ = y(e₁¹ - e₁³ - e₁² + e₄² + e₃³ - e₄⁴) - (2 + x)e₁² + (x + 1)(e₄¹ + e₄³) - xe₃⁴ \]
\[ h₂ = y(e₁¹ - e₁³ - e₁² + e₄² + e₃³ - e₄⁴) - 2e₁² + e₄² + e₃⁴ \]

**Proof.** We report below the details for the case of \( b₄ \); the other cases are treated in the same way. Let \( \{e₁, e₂, e₃, e₄\} \) denotes the basis used in theorem 1.1 for \( b₄ \). The non-isomorphic para-Kähler structures in \( b₄ \) are \((ω, K₁)\) and \((ω, K₂)\) with \( ω = e₁² - e₄⁴ \), \( K₁ = E₁₁ - E₂₂ - E₃₃ + xe₄₄ + E₄₄ \) and \( K₂ = E₁₁ - E₂₂ + E₃₃ - E₄₄ \).

The corresponding compatible metric to \((ω, K₁)\) is uniquely determined by \( h₁(X, Y) = (K₁X, Y) \).

Hence, para-Kähler metrics in \( b₄ \) are of the form

\[
h₁ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & x & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad h₂ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
\]

For \( h₁, x \neq 0 \), using the Koszul formula, the Levi-Civita connection is described

\[
\nabla e₁ = \begin{pmatrix} 0 & 0 & -\frac{i}{2}x & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{i}{2}x & 0 & 0 \end{pmatrix}, \quad \nabla e₂ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2}x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\]

Then we calculate the curvature matrices \( R(eᵢ, eⱼ) \) (for \( 1 \leq i < j \leq 4 \)) and we find

\[
R(e₁, e₂) = \begin{pmatrix} -x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -\frac{i}{2}x & 0 \\ 0 & 0 & -\frac{i}{2}x & -x \end{pmatrix}, \quad R(e₁, e₃) = \begin{pmatrix} 0 & 0 & -\frac{i}{2}x & -\frac{x}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R(e₁, e₄) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{x}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{x}{2} & 0 & 0 & 0 \end{pmatrix}.
\]

\[
R(e₂, e₃) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2}x & 0 \\ 0 & 0 & 0 & 0 \\ \frac{i}{2}x & 0 & 0 & 0 \end{pmatrix}, \quad R(e₂, e₄) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

13
Then, solving equation $L\varepsilon = \lambda e$, for $x \neq 0$, we obtain

$$\lambda = \frac{3}{2}x \quad \text{and} \quad X = 0.$$  

Notice that, in this case, the para-Kähler metric is a Einstein metric not Ricci flat. For $h_1$ with $x = 0$ and $h_2$

$$\nabla_{e_1} e_2 = e_3, \quad \nabla_{e_1} e_4 = -e_1, \quad \nabla_{e_1} e_1 = -\frac{1}{2}e_1, \quad \nabla_{e_1} e_2 = \frac{1}{2}e_2, \quad \nabla_{e_2} e_3 = e_3, \quad \nabla_{e_2} e_4 = -e_4$$

This para-Kähler structure is flat ($R(e_i, e_j)$ = 0 for 1 $\leq$ i $<$ j $\leq$ 4). The Lie derivative $L\varepsilon h_1$ of the metric $h_1$, is given by

$$L\varepsilon h_1 = \begin{pmatrix} 0 & -x_4 & 0 & \frac{1}{2}x_2 \\ -x_4 & 0 & 0 & -\frac{1}{2}x_1 \\ 0 & 0 & 0 & -x_4 \\ \frac{1}{2}x_2 & -\frac{1}{2}x_1 & -x_4 & 2x_3 \end{pmatrix}.$$  

Then, solving equation $L\varepsilon = \lambda h$, for $x = 0$ (or for $h_2$) we obtain

$$\lambda = -x_4 \quad \text{and} \quad X = x_4 e_4.$$
### 4. Tables

| Lie algebra | No zero brackets |
|-------------|------------------|
| $B_{1,0}^{3}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_4] = -x e_4$ |
| $B_{1,0}^{2}$ | $[e_1, e_2] = -e_1, [e_1, e_4] = -\frac{x}{2} e_1, [e_2, e_3] = -e_3, [e_2, e_4] = x e_2 - a e_4, [e_3, e_4] = \frac{4}{3} e_3$ |
| $B_{1,2}^{3}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_4] = 2 x e_2 + 2 e_4, [e_3, e_4] = -x e_3$ |
| $B_{1,2}^{2}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_2 - e_3, [e_2, e_4] = x e_2 + 2 e_4, [e_3, e_4] = -2 x e_4$ |
| $B_{1,1}^{3}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = -x e_1, [e_2, e_4] = x e_2 + e_4, [e_3, e_4] = -x e_3$ |
| $B_{1,1}^{2}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_4] = x e_2 - e_4, [e_3, e_4] = -x e_3$ |
| $B_{1,1}^{1}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_4] = x e_2 - e_4, [e_3, e_4] = -x e_3$ |
| $B_{1,0}^{1}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_4] = x e_2 - e_4, [e_3, e_4] = -x e_3$ |
| $B_{2,1}^{1}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = -\frac{e_1}{2}, [e_1, e_4] = -x e_1, [e_2, e_3] = \frac{1}{3} e_1 + \frac{1}{2} e_2 - e_3$ |
| $B_{2,1}^{2}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_1 + \frac{1}{3} e_1 - e_3$ |
| $B_{2,2}^{1}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_1 + \frac{1}{3} e_1 - e_3$ |
| $B_{2,2}^{2}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_1 + \frac{1}{3} e_1 - e_3$ |
| $B_{2,3}^{1}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_1 + \frac{1}{3} e_1 - e_3$ |
| $B_{2,3}^{2}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_1 + \frac{1}{3} e_1 - e_3$ |
| $B_{2,4}^{1}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_1 + \frac{1}{3} e_1 - e_3$ |
| $B_{2,4}^{2}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_1 + \frac{1}{3} e_1 - e_3$ |
| $B_{2,5}^{1}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_1 + \frac{1}{3} e_1 - e_3$ |
| $B_{2,5}^{2}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_1 + \frac{1}{3} e_1 - e_3$ |
| $B_{2,5}^{3}$ | $[e_1, e_2] = -e_1, [e_1, e_3] = x e_1 - e_3, [e_2, e_3] = x e_1 + \frac{1}{3} e_1 - e_3$ |

Table 4: Four dimensional Para-Kähler Lie algebras coming from b
Table 5: Four dimensional Para-Kähler Lie algebras coming from $\alpha$

| Source | Isomorphism | Target |
|--------|-------------|--------|
| $S^1_{1,\alpha}$ | $|\alpha| < 1, \alpha \neq 0$ | $f_1 = e_1, f_2 = -\frac{c}{d} e_1 + e_3, f_3 = e_4, f_4 = e_2$ | $\mathfrak{t}_{4, -1, -\alpha}$ |
| $S^1_{-1,\alpha}$ | $|\alpha| > 1, \alpha \neq -2$ | $f_1 = e_4, f_2 = e_1, f_3 = -\frac{c}{d} e_1 + e_3, f_4 = -\frac{c}{d} e_2$ | $\mathfrak{t}_{4, -1, \alpha}$ |
| $S^2_{1,\alpha}$ | $|\alpha| < 1, \alpha \neq 0$ | $f_1 = e_1, f_2 = e_3, f_3 = -\frac{c}{d} e_2 + e_4, f_4 = -\frac{c}{d} e_2 + e_4$ | $\mathfrak{t}_{4, -1, -\alpha}$ |
| $S^2_{-1,\alpha}$ | $|\alpha| > 1, \alpha \neq -2$ | $f_1 = -\frac{c}{|\alpha|} e_2 + e_4, f_2 = e_1, f_3 = e_3, f_4 = -\frac{c}{|\alpha|} e_2 + e_4$ | $\mathfrak{t}_{4, -1, \alpha}$ |

 références
| $B_{1,-2}$ | $x, y \neq 0$ | $f_1 = -ye_1 + e_2 + \frac{1}{3}e_4, f_2 = -\frac{1}{3}ye_1 + e_3, f_3 = ye_1, f_4 = e_2 + e_3$ | $d_{a,2}$ |
| $B_{1,-2}$ | $x \neq 0, y = 0$ | $f_1 = e_2 + \frac{4}{3}e_4, f_2 = e_3, f_3 = e_1, f_4 = e_2 + \frac{1}{3}e_4$ | $r_{4,-1,-1}$ |
| $B_{1,-2}$ | $x = 0$ | $f_1 = e_4, f_2 = -\frac{1}{2}e_1 + e_3, f_3 = -xe_1, f_4 = -\frac{1}{2}e_3 + \left(\frac{4}{3}\right)e_4$ | $d_{a,2}$ |
| $B_{1,-2}$ | $x \neq 0$ | $f_1 = e_4, f_2 = \frac{1}{2}e_1 + e_2 + \frac{1}{3}e_3, f_3 = -xe_1, f_4 = -\frac{1}{2}e_3 + \left(\frac{4}{3}\right)e_4$ | $d_{a,2}$ |
| $B_{1,-1}$ | $x = 0$ | $f_1 = e_4, f_2 = -\frac{1}{2}e_1 + e_3, f_3 = e_1, f_4 = \frac{1}{2}e_2$ | $r_{4,-1,-1}$ |
| $B_{1,0}$ | | $f_1 = e_2, f_2 = e_1, f_3 = -\frac{1}{2}e_1 + e_3, f_4 = e_4$ | $r_{3,-1}$ |
| $B_{2,0}$ | | $f_1 = e_2, f_2 = e_1, f_3 = xe_1 + e_4$ | $r_{3,-1}$ |
| $B_{1,1}$ | $x \neq 0$ | $f_1 = e_1, f_2 = xe_3 - \frac{1}{3}e_4, f_3 = \frac{1}{2}e_1 + xe_2 - e_4, f_4 = e_2$ | $r_{4,-1,-1}$ |
| $B_{1,1}$ | | $f_1 = e_1, f_2 = -\frac{1}{2}e_1 + e_3, f_3 = e_4, f_4 = e_2$ | $r_{4,-1,-1}$ |
| $B_1$ | | $f_1 = e_1, f_2 = -e_4, f_3 = -\frac{1}{2}e_1 + e_3, f_4 = e_2$ | $r_{4,-1,-1}$ |
| $B_{1,y}$ | $\frac{1}{3} > \frac{1}{2}$ | $f_1 = e_1, f_2 = xe_1 + \frac{e_2}{e_3}, f_3 = -\frac{1}{2}e_1 + e_3, f_4 = -e_2$ | $d_{a,1}$ |
| $B_{1,y}$ | $\frac{1}{3} < \frac{1}{2}$ | $f_1 = -xe_1 + \frac{e_2}{e_3}, f_2 = e_1, f_3 = e_1 + e_4, f_4 = -e_2$ | $d_{a,1}$ |
| $B_{2,y}$ | $x = 0$ | $f_1 = e_1, f_2 = e_1, f_3 = e_4, f_4 = e_2$ | $d_{a,1}$ |
| $B_{2,y}$ | $\frac{1}{3} > \frac{1}{2}$ | $f_1 = e_1 + xae_2 - e_4, f_2 = e_3, f_3 = xae_2 - e_4, f_4 = -e_2 + \frac{1}{3}e_3$ | $d_{a,1}$ |
| $B_{2,y}$ | $\frac{1}{3} < \frac{1}{2}$ | $f_1 = -xae_2 + e_4, f_2 = e_1, f_3 = -xae_2 + e_4, f_4 = \frac{1}{3}e_1 - e_2$ | $d_{a,1}$ |
| $B_{2,y}$ | | $f_1 = e_3, f_2 = e_1, f_3 = -2xe_2 + e_4, f_4 = -e_2$ | $d_{a,1}$ |
| $B_{3,y}$ | $\frac{1}{3} > \frac{1}{2}$ | $f_1 = e_1, f_2 = \frac{1}{3}xe_1 - \frac{1}{3}xe_2 + e_3, f_3 = -e_4, f_4 = e_2$ | $d_{a,2}$ |
| $B_{3,y}$ | $\frac{1}{3} < \frac{1}{2}$ | $f_1 = e_1, f_2 = \frac{1}{3}xe_1 - \frac{1}{3}xe_2 + e_3, f_3 = -e_4, f_4 = e_2$ | $d_{a,2}$ |
| $B_{3,y}$ | $y \neq 0$ | $f_1 = e_1, f_2 = -e_3 - \frac{1}{3}e_4, f_3 = xe_1 - \frac{1}{3}ye_2 + e_4, f_4 = \frac{1}{2}ye_4$ | $d_{a,2}$ |
| $B_{3,y}$ | $x = 0, y = 0$ | $f_1 = e_1, f_2 = xe_1 + e_3, f_3 = -e_4, f_4 = -e_2$ | $d_{a,1}$ |
| $B_{3,y}$ | $x \neq 0, y = 0$ | $f_1 = \frac{1}{3}xe_1 - \frac{1}{3}xe_2 + e_3, f_2 = e_4, f_3 = -\frac{1}{2}e_1 - \frac{1}{3}e_3, f_4 = e_1 - \frac{1}{3}e_2$ | $r_{3,2}$ |
| $B_{3,y}$ | $x = 0, y \neq 0, z \neq 0$ | $f_1 = xe_1 - xe_2 - \frac{1}{2}e_3, f_2 = -xe_1 - ye_2 + e_4, f_3 = xe_1 - ye_2 + e_4$ | $r_{3,2}$ |
| $B_{3,y}$ | $x = 0, y \neq 0, z = 0$ | $f_1 = e_1, f_2 = e_3, f_3 = ye_2 - e_4, f_4 = -e_2$ | $d_{a,1}$ |
| $B_{3,y}$ | $xy \neq 0, x^2 + 4yz = 0$ | $f_1 = ye_2 + e_4, f_2 = -\frac{1}{2}e_1 - \frac{1}{3}e_2 - e_3, f_3 = \frac{1}{2}e_1 + \frac{1}{3}e_2 - \frac{1}{2}e_4, f_4 = -e_2$ | $d_{a,1}$ |
| $B_{3,y}$ | $xy \neq 0, x^2 + 4yz > 0$ | $f_1 = \frac{\sqrt{x^2 + 4yz}}{\sqrt{x^2 + 4yz}} e_1 - \frac{1}{\sqrt{x^2 + 4yz}} e_2 + \frac{1}{\sqrt{x^2 + 4yz}} e_3, f_2 = (x + \sqrt{x^2 + 4yz}) e_1 - 2ye_2 + 2e_4, f_3 = -\frac{1}{\sqrt{x^2 + 4yz}} e_1 + \frac{1}{\sqrt{x^2 + 4yz}} e_2 - \frac{1}{\sqrt{x^2 + 4yz}} e_3, f_4 = -\frac{1}{\sqrt{x^2 + 4yz}} e_1 - ye_2 + e_4$ | $r_{3,2}$ |
| Source | Isomorphism | Target |
|--------|-------------|--------|
| $\mathcal{B}_{3,1}$ | $xy \neq 0, x^2 + 4yz < 0$ | $f_1 = \frac{\sqrt{\alpha^2 - 4yz}}{2}e_1 - e_2, f_2 = -\frac{\alpha}{\sqrt{\alpha^2 - 4yz}}e_1 - \frac{y\sqrt{4yz - \alpha^2}}{2\sqrt{\alpha^2 - 4yz}}e_2 + \frac{2}{\sqrt{\alpha^2 - 4yz}}e_3 + e_4, f_3 = \frac{\sqrt{\alpha^2 - 4yz}}{2}e_1 - e_2 + \frac{\alpha}{\sqrt{\alpha^2 - 4yz}}e_3, f_4 = -\frac{\alpha}{\sqrt{\alpha^2 - 4yz}}e_1 - ye_2 + e_4$ | $\mathfrak{b}_{4,1}$ |
| $B_4$ | $x = 0$ | $f_1 = -e_1, f_2 = ye_1 + e_3 - e_4, f_3 = e_4, f_4 = -e_2$ | $\mathfrak{b}_{4,1}$ |
| $B_4$ | $x \neq 0$ | $f_1 = \frac{\alpha}{\sqrt{\alpha^2 - 4yz}}e_1 - \frac{1}{\sqrt{\alpha^2 - 4yz}}e_3, f_2 = -xe_1 + e_4, f_3 = \frac{\alpha}{\sqrt{\alpha^2 - 4yz}}e_1 - e_2 + \frac{\alpha}{\sqrt{\alpha^2 - 4yz}}e_3, f_4 = e_4$ | $\mathfrak{b}_{4,1}$ |
| $B_{5,1}$ | | $f_1 = -xe_2 - e_3 + e_4, f_2 = e_1, f_3 = e_3, f_4 = e_1 + e_2$ | $\mathfrak{b}_{4,2}$ |
| $B_{5,2}$ | $x \neq 0$ | $f_1 = \frac{1}{\alpha}e_2 - \frac{1}{\alpha}e_4, f_2 = e_3, f_3 = \frac{\alpha}{\sqrt{\alpha^2 - 4yz}}e_1 + e_3, f_4 = \frac{\alpha}{\sqrt{\alpha^2 - 4yz}}e_4$ | $\mathfrak{b}_{4,2}$ |
| $B_{5,2}$ | $x = 0$ | $f_1 = e_3 + e_4, f_2 = e_1, f_3 = e_3, f_4 = -e_1 + e_2$ | $\mathfrak{b}_{4,2}$ |
| $B_{5,3}$ | $x \neq 0$ | $f_1 = -e_3 - e_4, f_2 = e_1, f_3 = xe_1 + xe_2 + e_3, f_4 = -\frac{1}{4}e_3$ | $\mathfrak{b}_{4,2}$ |
| $B_{5,3}$ | $x = 0$ | $f_1 = e_4, f_2 = e_1, f_3 = e_4, f_4 = e_2$ | $\mathfrak{b}_{4,2}$ |
| $B_{5,4}$ | $x \neq 0$ | $f_1 = e_3 + e_4, f_2 = e_1, f_3 = -xe_1 + xe_2 + e_3, f_4 = -e_1 + e_2$ | $\mathfrak{b}_{4,2}$ |
| $B_{5,4}$ | $x = 0$ | $f_1 = -e_3 + e_4, f_2 = e_1, f_3 = e_3, f_4 = e_1 + e_2$ | $\mathfrak{b}_{4,2}$ |
| $B_{5,1}$ | $x \neq 0$ | $f_1 = -xe_2 + e_4, f_2 = e_1, f_3 = -e_3, f_4 = e_2$ | $\mathfrak{b}_{4,2}$ |
| $B_{5,2}$ | $x = 0$ | $f_1 = e_2 + \frac{1}{\alpha}e_4, f_2 = e_1, f_3 = xe_1 + 2e_3, f_4 = -\frac{1}{\alpha}e_4$ | $\mathfrak{b}_{4,2}$ |

Table 6: Isomorphisms from the Lie algebras obtained in Table 4 onto the Lie algebras in Table 1.
| \(C_{1.8}\) | \(f_1 = e_1, f_2 = e_3, f_3 = e_2, f_4 = -e_4\) | \(d_{4.1}\) |
| \(C_{1.9}\) | \(f_1 = \frac{3}{4}e_3 - \frac{1}{4}e_4, f_2 = e_1 - e_2, f_3 = -\frac{1}{2}e_3 + \frac{1}{4}e_4, f_4 = e_1 + e_2\) | \(r_{3.2}\) |
| \(C_{1.10}\) | \(f_1 = -e_4, f_2 = e_3, f_3 = e_1, f_4 = -e_2\) | \(r_2'\) |
| \(C_{2.1}\)  | \(x = 0\) | \(f_1 = -e_2, f_2 = -ye_2 + e_4, f_3 = e_1, f_4 = e_3\) | \(r_{5.0}\) |
| \(C_{2.2}\)  | \(x \neq 0\) | \(f_1 = \frac{1}{2}e_3, f_2 = e_1, f_3 = -\frac{1}{2}e_1 - e_2, f_4 = e_4\) | \(r_{3.2}\) |
| \(C_{2.3}\)  | \(x = 0, y = 0\) | \(f_1 = -e_4, f_2 = -ye_1, f_3 = e_3, f_4 = -e_2\) | \(r_{4.0}\) |
| \(C_{2.4}\)  | \(x = 0, y = 0\) | \(f_1 = -e_2, f_2 = e_4, f_3 = e_1, f_4 = e_3\) | \(r_{5.0}\) |
| \(C_{2.5}\)  | \(x \neq 0\) | \(f_1 = e_1 - e_2, f_2 = -xe_1 + e_4, f_3 = e_2 - \frac{1}{2}e_1, f_4 = e_1\) | \(r_{3.2}\) |
| \(C_{2.6}\)  | \(x = 0, y \neq 0\) | \(f_1 = e_4, f_2 = -ye_1, f_3 = e_3, f_4 = -e_2\) | \(r_{4.0}\) |
| \(C_{3.1}\)  | \(z = 0, x \neq 0\) | \(f_1 = e_1, f_2 = -\frac{1}{2}e_4, f_3 = -\frac{1}{2}e_1 + e_2, f_4 = -\frac{1}{2}e_3\) | \(r_{4.0}\) |
| \(C_{3.2}\)  | \(z = 0, x = 0\) | \(f_1 = -e_2, f_2 = -e_3, f_3 = ye_1 - e_4, f_4 = e_1\) | \(r_{3.3}\) |
| \(C_{3.3}\)  | \(z \neq 0, x = 0\) | \(f_1 = \frac{1}{2}e_4, f_2 = e_1, f_3 = -2ye_1 - 2xe_2 + e_4, f_4 = e_1\) | \(r_{3.3}\) |
| \(C_{3.4}\)  | \(z \neq 0, -1 \leq \frac{x}{z} < 1\) | \(f_1 = e_4, f_2 = \frac{2e_1}{x-z}e_1 - 2xe_2 + e_4, f_3 = e_1, f_4 = -\frac{1}{2}e_3\) | \(r_{4,-1,4}\) |
| \(C_{3.5}\)  | \(z \neq 0, -1 > \frac{x}{z} > 1\) | \(f_1 = e_1, f_2 = \frac{2z}{x-z}e_1 - 2xe_2 + e_4, f_3 = e_4, f_4 = -\frac{1}{2}e_3\) | \(r_{4,-1,4}\) |
| \(C_{3.6}\)  | \(z \neq 0, -1 > \frac{x}{z} > 1\) | \(f_1 = e_1, f_2 = e_4, f_3 = \frac{2z}{x-z}e_1 - 2xe_2 + e_4, f_4 = -\frac{1}{2}e_3\) | \(r_{4,-1,4}\) |
| \(C_{3.7}\)  | \(z \neq 0, \frac{x}{z} = 1\) | \(f_1 = -\frac{1}{2}e_1 + e_2 + \frac{1}{2}e_4, f_2 = e_4, f_3 = e_1, f_4 = \frac{1}{2}e_3\) | \(r_{4,-1,4}\) |
| \(C_{3.8}\)  | \(z = 0, x \neq 0\) | \(f_1 = -e_2, f_2 = e_3 - e_1, f_3 = ye_1 - e_4, f_4 = e_1\) | \(r_{3.3}\) |
| \(C_{3.9}\)  | \(z = 0, x \neq 0\) | \(f_1 = \frac{1}{2}e_4, f_2 = -\frac{1}{2}e_1 + e_2, f_3 = -xe_1 - e_2, f_4 = \frac{1}{2}e_3\) | \(d_{4.1}\) |
| \(C_{3.10}\) | \(z \neq 0, x \neq 0, \frac{x}{z} = \frac{1}{2}\) | \(f_1 = \frac{1}{2}e_4, f_2 = \frac{x}{z}e_1 + e_2, f_3 = (x - 2z)e_2 + e_4, f_3 = x(x - 2z)e_1, f_4 = -\frac{2z(x - 2z)}{z}e_1 - \frac{1}{4}e_3\) | \(d_{4,\frac{1}{2}}\) |
| \(C_{3.11}\) | \(z \neq 0, x \neq 0, \frac{x}{z} > \frac{1}{2}\) | \(f_1 = (x - 2z)e_2 + e_4, f_2 = \frac{2z(x - 2z)}{x-z}e_1 + e_4, f_3 = x(x - 2z)e_1, f_4 = \frac{-2z(x - 2z)}{x-z}e_1 - \frac{1}{4}e_3\) | \(d_{4,\frac{1}{2}}\) |
| \(C_{4.1}\)  | \(f_1 = -ye_1 - xe_2 + e_3, f_2 = e_1, f_3 = -xe_4 + e_4, f_4 = -xe_4 - e_2 + e_4\) | \(d_{4.1}\) |
| \(C_{4.2}\)  | \(f_1 = e_3, f_2 = e_1, f_3 = -xe_2 + e_4, f_4 = -e_2\) | \(d_{4.1}\) |
| \(C_{5.1}\)  | \(f_1 = \frac{1}{2}e_1 - \frac{1}{2}e_2, f_2 = (x+y)e_1 + (x-y)e_2 - e_3 + e_4, f_3 = -(\frac{1}{2} + x + y)e_1 - \frac{1}{4} + x + y)e_2 + e_3 + e_4, f_4 = -(x + y)e_1 - (x + y)e_2 + e_3 + e_4\) | \(r_{3.2}\) |
Table 7: Isomorphisms from the Lie algebras obtained in Table 5 onto the Lie algebras in Table 4

| $C^+_{5,2}$ | $f_1 = \frac{1}{2}e_1 - \frac{1}{2}e_2$, $f_2 = -xe_2 - e_3 + e_4$, $f_3 = -\frac{1}{2}e_1 - \frac{1}{2}e_2$, $f_4 = -xe_2 + e_3 + e_4$ | $r_2^e^2$ |
| $C^-_{5,1}$ | $f_1 = ye_1 - (x + 1)e_2 + e_3$, $f_2 = -(x+1)e_1 - ye_2 + e_4$, $f_3 = ye_1 - xe_2 + e_3$, $f_4 = -xe_2 + e_3 + e_4$ | $r'_2$ |
| $C^-_{5,2}$ | $f_1 = -e_2 - e_3$, $f_2 = e_1 - xe_2 + e_4$, $f_3 = -e_3$, $f_4 = -xe_2 + e_4$ | $r'_2$ |

Acknowledgments:

The authors would like to thank sincerely Professor Mohamed Boucetta for his many suggestions which were of great help to improve our work.

References

[1] D. V. Alekseevsky, C. Medori and A. Tomassini, *Homogeneous para-Kähler Einstein manifolds*, Russian Mathematical Surveys, Volume 64, Number 1

[2] C. Bai, *Left-symmetric bialgebras and an analogue of the classical Yang-Baxter equation*, Commun. Contemp. Math. 10 (2008), no. 2, 221-260.

[3] S. Benayadi, M. Boucetta. *On para-Kähler and hyper-para-Kähler Lie algebras*, J. Algebra 436, (2015),61-101.

[4] O. Bouzour and M. W. Mansouri, *Bi-Lagrangian Structure on the Symplectic Affine Lie Algebra af(2, R)* Journal of Geometry and Symmetry in Physics-56, (2020), 45-57.

[5] N. B. Boyom, *Métriques Kähleriennes affinement plates de certaines variétés symplectiques I* Proc. London Math. Soc. (1993), 358-80.

[6] G. Calvaruso, *A complete classification of four-dimensional para-Kähler Lie algebras*, Complex Manifolds (2), (2015), 1-10.

[7] F. Etayo, R. Santamaria and U. R. Trias *The geometry of a bi-Lagrangian manifold*, Diff Geo and its App (24), (1), (2006), 33-59.

[8] M. J. D. Hamilton, *Bi-Lagrangian structures on nilmanifolds*, Jour of Geom and Phys (140), (2019), 10-25. arXiv:1810.06518

[9] H. Hess, *Connections on symplectic manifolds and geometric quantization*, Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca) (1979), pp 153-166.

[10] G. Ovando *Four Dimensional Symplectic Lie Algebras*, Contributions to Algebra and Geometry, 47 (2006), No. 2, 419-434.

[11] A. Konyaev, *Nijenhuis geometry II: Left-symmetric algebras and linearization problem for Nijenhuis operators*, arXiv:1903.06411

[12] N. K. Smolentsev and I. Y. Shagabutinova, *On the classification of left-invariant para-Kähler structures on four-dimensional Lie groups* arXiv:2008.05664

20