The \((p, q)\)-spectral radii of \((r, s)\)-directed hypergraphs

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Abstract

An \((r, s)\)-directed hypergraph is a directed hypergraph with \(r\) vertices in tail and \(s\) vertices in head of each hyperarc. Let \(G\) be an \((r, s)\)-directed hypergraph. For any real numbers \(p, q \geq 1\), we define the \((p, q)\)-spectral radius \(\lambda_{p,q}(G)\) as

\[
\lambda_{p,q}(G) := \max_{||x||_p=||y||_q=1} \sum_{e \in E(G)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right),
\]

where \(x = (x_1, \ldots, x_m)^T\), \(y = (y_1, \ldots, y_n)^T\) are real vectors; and \(T(e), H(e)\) are the tail and head of arc \(e\), respectively. We study some properties about \(\lambda_{p,q}(G)\) including the bounds and the spectral relation between \(G\) and its components.

The \(\alpha\)-normal labeling method for uniform hypergraphs was introduced by Lu and Man in 2014. It is an effective method in studying the spectral radii of uniform hypergraphs. In this paper, we develop the \(\alpha\)-normal labeling method for calculating the \((p, q)\)-spectral radii of \((r, s)\)-directed hypergraphs. Finally, some applications of \(\alpha\)-normal labeling method are given.

Keywords: Directed hypergraph; \((p, q)\)-spectral radius; rectangular tensor; \(\alpha\)-normal labeling; weighted incidence matrix

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1. Introduction

An \((r, s)\)-directed hypergraph is a directed hypergraph with \(r\) vertices in tail and \(s\) vertices in head of each hyperarc. The purpose of this paper is to study the spectral properties of \((r, s)\)-directed hypergraphs and develop a simple method to compute the spectral radii of directed hypergraphs.

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Recall that an undirected hypergraph \( H = (V, E) \) is a pair consisting of a vertex set \( V \), and an edge set \( E \) of subsets of \( V \). A uniform hypergraph is a hypergraph in which each edge has the same size. In 2012, Cooper and Dutle [8] defined the spectra of uniform hypergraphs via eigenvalues of tensors introduced independently by Qi [28] and Lim [17]. Since then the spectral undirected hypergraph theory has been widely studied in [2, 15, 16, 24, 25, 27, 29]. For directed hypergraphs, in contrast, there are very few researches in spectral directed hypergraph theory so far. In 2016, Xie and Qi [31] investigated the spectral properties of a specific kind of directed hypergraphs via tensors. Recently, Banerjee et al. [3] represent a general directed hypergraph by different connectivity tensors and study their spectral properties.

In this paper, we introduce a parameter \( \lambda_{p,q}(G) \) for an \((r, s)\)-directed hypergraph \( G \) and real numbers \( p, q \geq 1 \) via multilinear function (see more details in Section 2), which called the \((p, q)\)-spectral radius of \( G \). Also, we give some properties about \( \lambda_{p,q}(G) \).

In [22], Lu and Man discovered a novel method for computing the spectral radii of uniform hypergraphs by introducing an \( \alpha \)-normal labeling method, which labels each corner of an edge by a positive number so that the sum of the corner labels at any vertex is 1 while the product of all corner labels at any edge is \( \alpha \). This method has been proved by many researches [13, 20, 1, 26, 33, 34, 30] to be a simple and effective method in the study of spectral radii of uniform hypergraphs. Recently, Liu and Lu [21] extend the \( \alpha \)-normal labeling method to the \( p \)-spectral radii of uniform hypergraphs. Motivated by the preceding work [22] and [21], in the present paper we develop the \( \alpha \)-normal labeling method for calculating the \((p, q)\)-spectral radii of \((r, s)\)-directed hypergraphs.

The remaining part of this paper is organized as follows. In Section 2, some preliminary definitions concerning directed hypergraphs and tensors are given. Moreover, we present the definition of \( \lambda_{p,q}(G) \) for an \((r, s)\)-directed hypergraph \( G \). Section 3 is dedicated to some basic properties of \( \lambda_{p,q}(G) \). In Section 4, we develop the \( \alpha \)-normal labeling method for calculating the \( \lambda_{p,q}(G) \) by constructing consistently \( \alpha \)-normal weighted incidence matrix for the target \((r, s)\)-directed hypergraph. Also, we present a method for comparing the \( \lambda_{p,q}(G) \) in terms of a particular value by constructing \( \alpha \)-subnormal or consistently \( \alpha \)-supernormal weighted incidence matrix. In Section 5, some applications are given.

2. Preliminaries

In this section, we will review some basic notions of tensors and directed hypergraphs, and present the definitions of \((r, s)\)-directed hypergraph and its \((p, q)\)-spectral radius. For the basics on undirected hypergraphs we follow the traditions, as in [4].

Let \( \mathbb{R} \) be the field of real numbers and \( \mathbb{R}^n \) the \( n \)-dimensional real space. Further, denote the nonnegative octant of \( \mathbb{R}^n \) by \( \mathbb{R}_+^n \). Given a vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \) and a set \( S \subseteq [n] := \{1, 2, \ldots, n\} \), write \( \mathbf{x}|_S \) for the restriction of \( \mathbf{x} \) over the set \( S \). Also, we write \( |\mathbf{x}| := (|x_1|, |x_2|, \ldots, |x_n|)^T \), and \( ||\mathbf{x}||_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p} \). For any real number \( p \geq 1 \), denote \( \mathbb{S}_p^{-1}(\mathbb{S}_{p,+}^{-1}, \mathbb{S}_{p,++}) \) the set of all (nonnegative, positive) real vectors \( \mathbf{x} \in \mathbb{R}^n \) with \( ||\mathbf{x}||_p = 1 \).
For positive integers $r$, $s$, $m$ and $n$, a real $(r,s)$-th order $(m \times n)$-dimensional rectangular tensor, or simply a real rectangular tensor, refers to a multidimensional array (also called hypermatrix) with entries $a_{i_1 \ldots i_r,j_1 \ldots j_s} \in \mathbb{R}$ for all $i_1, i_2, \ldots, i_r \in [m]$ and $j_1, j_2, \ldots, j_s \in [n]$. Recently, the (weak) Perron–Frobenius theorem for rectangular tensors were studied in [7, 18, 32, 10]. We say that $\mathcal{A} = (a_{i_1 \ldots i_r,j_1 \ldots j_s})$ is partially symmetric, if $a_{i_1 \ldots i_r,j_1 \ldots j_s}$ is invariant under any permutation of indices among $i_1, i_2, \ldots, i_r$ and any permutation of indices among $j_1, j_2, \ldots, j_s$, i.e.,

$$a_{\pi(i_1 \ldots i_r)\sigma(j_1 \ldots j_s)} = a_{i_1 \ldots i_r,j_1 \ldots j_s}, \quad \pi \in \mathfrak{S}_r, \quad \sigma \in \mathfrak{S}_s,$$

where $\mathfrak{S}_k$ is the permutation group of $k$ indices.

Let $\mathcal{A} = (a_{i_1 \ldots i_r,j_1 \ldots j_s})$ be an $(r,s)$-th order $(m \times n)$-dimensional rectangular tensor. Denote

$$\mathcal{A}\mathbf{x}^r\mathbf{y}^s := \sum_{i_1,\ldots,i_r=1}^m \sum_{j_1,\ldots,j_s=1}^n a_{i_1 \ldots i_r,j_1 \ldots j_s} x_{i_1} \cdots x_{i_r} y_{j_1} \cdots y_{j_s}. \tag{2.1}$$

A nonnegative $(r,s)$-th order $(m \times n)$-dimensional rectangular tensor $\mathcal{A} = (a_{i_1 \ldots i_r,j_1 \ldots j_s})$ is associated with an undirected bipartite graph $G(\mathcal{A}) = (V,E(\mathcal{A}))$, the bipartition of which is $V = [m] \cup [n]$, and $(i_p,j_q) \in E(\mathcal{A})$ if and only if $a_{i_1 \ldots i_r,j_1 \ldots j_s} > 0$ for some $(r+s-2)$ indices \{i_1,\ldots,i_r,j_1,\ldots,j_s\}\{i_p,j_q\}$. Following [9], the tensor $\mathcal{A}$ is called weakly irreducible if the graph $G(\mathcal{A})$ is connected.

A directed hypergraph is a pair $G = (V(G),E(G))$, where $V(G)$ is a set of vertices, and $E(G)$ is a set of hyperarcs. A hyperarc or simply arc is an ordered pair, $e = (X,Y)$, of disjoint subsets of vertices, $X$ is the tail of $e$ while $Y$ is its head. We denote the number of arcs of $G$ by $|G|$. In the following, the tail and the head of an arc $e$ will be denoted by $T(e)$ and $H(e)$, respectively. Denote

$$T(G) = \bigcup_{e \in E(G)} T(e), \quad H(G) = \bigcup_{e \in E(G)} H(e).$$

For convenience, we always assume $|T(G)| = m$ and $|H(G)| = n$ throughout this paper.

The in-degree $d^-_v$ of a vertex $v$ in directed hypergraph $G$ is the number of arcs contained $v$ in head, and the out-degree $d^+_v$ of $v$ is the number of arcs contained $v$ in tail. The degree $d_v$ of a vertex $v$ is $d^+_v + d^-_v$. The maximum in-degree and out-degree of $G$ are denoted by $\Delta^-$ and $\Delta^+$, respectively; likewise, the minimum in-degree and out-degree of $G$ are denoted by $\delta^-$ and $\delta^+$, respectively. Given two directed hypergraphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$, if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$, then $G_1$ is called the directed subhypergraph of $G_2$, denoted by $G_1 \subseteq G_2$. With any directed hypergraph $G$, we can associate an undirected hypergraph on the same vertex set simply by replacing each arc by an edge with the same vertices. This hypergraph is called the underlying hypergraph of $G$.

Now we introduce some new concepts for directed hypergraphs. In a directed hypergraph $G$, an anadiplosis walk of length $\ell$ is an alternating sequence of vertices and arcs $v_0e_1v_1e_2 \cdots v_{\ell-1}e_\ell v_{\ell}$ such that either $v_i \in T(e_i) \cap T(e_{i+1})$ or $v_i \in H(e_i) \cap H(e_{i+1})$, $i \in [\ell - 1]$. Furthermore, if $e_1, e_2, \ldots, e_{\ell}$ ($\ell \geq 2$) are all distinct arcs of $G$, and either
\(v_0 = v_\ell \in T(e_1) \cap T(e_\ell)\) or \(v_0 = v_\ell \in H(e_1) \cap H(e_\ell)\), then this anadiplosis walk is called an \textit{anadiplosis cycle}. An anadiplosis walk: \(v_0 e_1 v_1 e_2 \cdots v_{\ell-1} e_\ell v_\ell\) is called an \textit{anadiplosis semi-cycle} if \(e_1, e_2, \ldots, e_\ell\ (\ell \geq 2)\) are all distinct arcs of \(G\) and either \(v_0 = v_\ell \in T(e_1) \cap T(e_\ell)\) or \(v_0 = v_\ell \in H(e_1) \cap H(e_\ell)\). A directed hypergraph \(G\) is \textit{anadiplosis connected} if there exists a \(u-v\) anadiplosis walk for all \(u \neq v\) in \(V(G)\), and a \(u-u\) anadiplosis semi-cycle for any \(u \in T(G) \cap H(G)\). A maximal anadiplosis connected subhypergraph of \(G\) is called an \textit{anadiplosis component} of \(G\).

\textbf{Remark 2.1} In our definition above, vertex repetition is allowed in anadiplosis cycle and anadiplosis semi-cycle. In the following directed graph, \(v_0 e_1 v_1 e_2 v_2 e_3 v_3 e_4 v_4 e_5 v_5 e_6 v_0\) is an anadiplosis cycle, and \(v_2 e_6 v_0 e_1 v_1 e_2 v_2\) is an anadiplosis semi-cycle.

\textbf{Definition 2.1} A directed hypergraph \(G\) is called an \((r, s)\)-\textit{directed hypergraph} if for any arc \(e \in E(G)\), \(|T(e)| = r\) and \(|H(e)| = s\).

\textbf{Definition 2.2} Let \(G\) be an \((r, s)\)-directed hypergraph. The \textit{adjacency tensor} of \(G\) is defined as an \((r, s)\)-th order \((m \times n)\)-dimensional rectangular tensor \(A(G)\), whose \((i_1, \ldots, i_r, j_1, \ldots, j_s)\)-entry is \(\frac{1}{r! s!}\) if \(T(e) = \{i_1, i_2, \ldots, i_r\}, H(e) = \{j_1, j_2, \ldots, j_s\}\) for some \(e \in E(G)\) and 0 otherwise.

By the definition above, the adjacency tensor of an \((r, s)\)-directed hypergraph is partially symmetric. Given an \((r, s)\)-directed hypergraph \(G\), the polynomial form of \(G\) is a multilinear function \(P_G(x, y) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}\) defined for any vectors \(x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m, y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n\) as

\[
P_G(x, y) := A(G)x^Ty^T = \sum_{\substack{e \in E(G), T(e) = \{i_1, \ldots, i_r\} \\&\ H(e) = \{j_1, \ldots, j_s\}}} x_{i_1} \cdots x_{i_r} y_{j_1} \cdots y_{j_s}.
\]

We here give the definition of the \((p, q)\)-spectral radius of an \((r, s)\)-directed hypergraph.

\textbf{Definition 2.3} Let \(G\) be an \((r, s)\)-directed hypergraph. For any \(p, q \geq 1\), the \((p, q)\)-\textit{spectral radius} \(\lambda_{p,q}(G)\) of \(G\) is defined as

\[
\lambda_{p,q}(G) := \max \left\{ P_G(x, y) : x \in \mathbb{S}_p^{m-1}, y \in \mathbb{S}_q^{n-1} \right\}.
\] (2.2)
In particular, if $p = 2r$, $q = 2s$, then $\lambda_{2r, 2s}(G)$ is called the spectral radius of $G$, denoted by $\rho(G)$. That is

$$\rho(G) := \max \{ P_G(x, y) : x \in S_p^{m-1}, y \in S_q^{n-1} \}.$$  

(2.3)

If $x \in S_p^{m-1}$ and $y \in S_q^{n-1}$ are two vectors such that $\lambda_{p, q}(G) = P_G(x, y)$, then $(x, y)$ will be called an eigenpair to $\lambda_{p, q}(G)$.

Notice that $S_p^{m-1}$ and $S_q^{n-1}$ are compact sets, and $P_G(x, y)$ is continuous, thus $\lambda_{p, q}(G)$ is well defined. Clearly, equation (2.2) is equivalent to

$$\lambda_{p, q}(G) = \max_{x \neq 0, y \neq 0} \frac{P_G(x, y)}{||x||_p^p \cdot ||y||_q^q}.$$  

(2.4)

**Remark 2.2** Recall that $||x||_\infty = \max_{1 \leq i \leq m} \{|x_i|\}$ and $||y||_\infty = \max_{1 \leq j \leq n} \{|y_j|\}$. Therefore $\lim_{q, p \to \infty} \lambda_{p, q}(G) = |G|$. Denote $G_T$ the $r$-uniform hypergraph with $V(G_T) = T(G)$ and $\{i_1, i_2, \ldots, i_r\} \in E(G_T)$ if and only if $T(e) = \{i_1, i_2, \ldots, i_r\}$ for some arc $e \in E(G)$. Similarly, we can define the $s$-uniform hypergraph $G_H$. If $G_T$ has no repeated edges, then

$$\lim_{q \to \infty} \lambda_{p, q}(G) = \frac{\lambda^{(p)}(G_T)}{r},$$

where $\lambda^{(p)}(G_T)$ is the scaled $p$-spectral radius of $G_T$ by removing a constant factor $(r - 1)!$ from $[14]$. If $G_H$ has no repeated edges, we also have

$$\lim_{p \to \infty} \lambda_{p, q}(G) = \frac{\lambda^{(q)}(G_H)}{s}.$$

**Remark 2.3** If $r = s = 1$, the $(r, s)$-directed hypergraphs are exactly the directed graphs. Let $G$ be a directed graph, $A = (a_{ij})$ be a $m \times n$ matrix with row indexed by the set $T(G)$ and column indexed by the set $H(G)$, where $a_{ij} = 1$ if $(i, j)$ is an arc of $G$, and 0 otherwise. By (2.3), the spectral radius $\rho(G)$ of $G$ is exactly the largest singular value of $A$.

If $(x, y) \in S_p^{m-1} \times S_q^{n-1}$ is an eigenpair to $\lambda_{p, q}(G)$, then the vectors $x' = |x|$ and $y' = |y|$ also satisfy $||x'||_p = ||y'||_q = 1$ and so

$$\lambda_{p, q}(G) = P_G(x, y) \leq P_G(x', y') \leq \lambda_{p, q}(G),$$

which yields $\lambda_{p, q}(G) = P_G(x', y')$. Therefore, there are always nonnegative vectors $x$, $y$ such that $||x||_p = ||y||_q = 1$ and $\lambda_{p, q}(G) = P_G(x, y)$.

Let $(x, y) \in S_p^{m-1} \times S_q^{n-1}$ be an eigenpair to $\lambda_{p, q}(G)$. By Lagrange’s method, there exists a $\mu$ such that for each $i \in T(G)$ with $x_i > 0$,

$$\frac{\partial P_G(x, y)}{\partial x_i} = \sum_{e \in E(G), i \in T(e)} \left( \prod_{u \in T(e), u \neq i} x_u \right) \left( \prod_{v \in H(e)} y_v \right) = p \mu x_i^{p-1}. $$
Multiplying the $i$-th equation by $x_i$ and adding them all, we have

$$\sum_{i \in T(G)} \sum_{e \in E(G), i \in T(e)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right) = p \mu \sum_{i \in T(G)} x_i^p = p \mu.$$ 

It follows that

$$r \sum_{e \in E(G)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right) = p \mu,$$

which yields $r \lambda_{p,q}(G) = p \mu$. Therefore

$$\sum_{e \in E(G), j \in H(e)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e), v \neq j} y_v \right) = s \lambda_{p,q}(G) y_j^{q-1}.$$ 

Similarly, for each $j \in H(G)$ with $y_j > 0$, we have

$$\sum_{e \in E(G), i \in T(e)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e), v \neq j} y_v \right) = s \lambda_{p,q}(G) y_j^{q-1}.$$ 

Hence, we obtain the weak eigenequations of an $(r, s)$-directed hypergraph $G$ as follows:

\[
\begin{align*}
\left\{ \begin{array}{l}
\sum_{e \in E(G), T(e) = \{i_1, i_2, \ldots, i_r \}, H(e) = \{j_1, j_2, \ldots, j_s \}} x_{i_1}x_{i_2} \cdots x_{i_r} y_{j_1} \cdots y_{j_s} = r \lambda_{p,q}(G) x_i^p, \quad i \in T(G), \\
\sum_{e \in E(G), T(e) = \{i_1, i_2, \ldots, i_r \}, H(e) = \{j_1, j_2, \ldots, j_s \}} x_{i_1} \cdots x_{i_r} y_{j_1}y_{j_2} \cdots y_{j_s} = s \lambda_{p,q}(G) y_j^q, \quad j \in H(G).
\end{array} \right. \\
\end{align*}
\]

(2.5)

If all $x_i > 0$ and $y_j > 0$, we can cancel one factor of $x_i$ and $y_j$, and obtain the strong eigenequations of an $(r, s)$-directed hypergraph $G$ as follows:

\[
\begin{align*}
\left\{ \begin{array}{l}
\sum_{e \in E(G), T(e) = \{i_1, i_2, \ldots, i_r \}, H(e) = \{j_1, j_2, \ldots, j_s \}} x_{i_2} \cdots x_{i_i} y_{j_1} \cdots y_{j_s} = r \lambda_{p,q}(G) x_i^{p-1}, \quad i \in T(G), \\
\sum_{e \in E(G), T(e) = \{i_1, i_2, \ldots, i_r \}, H(e) = \{j_1, j_2, \ldots, j_s \}} x_{i_1} \cdots x_{i_r} y_{j_1}y_{j_2} \cdots y_{j_s} = s \lambda_{p,q}(G) y_j^{q-1}, \quad j \in H(G).
\end{array} \right. \\
\end{align*}
\]

(2.6)

Before concluding this section, we list some inequalities which will be used in the sequel (see [12]).

(1) (Generalized H"older’s inequality) Let $a_{ij} \geq 0$, $i \in [n]$, $j \in [m]$, be nonnegative real numbers, and $\alpha_1, \alpha_2, \ldots, \alpha_m$ be positive real numbers such that $\sum_{j=1}^m 1/\alpha_j = 1$. Then

$$\sum_{i=1}^n \left( \prod_{j=1}^m a_{ij} \right) \leq \prod_{j=1}^m \left( \sum_{i=1}^n a_{ij}^{\alpha_j} \right)^{1/\alpha_j}.$$ 

(2.7)

Equality holds if and only if either $x^{(j)} := (a_{1j}^{\alpha_j}, a_{2j}^{\alpha_j}, \ldots, a_{nj}^{\alpha_j})^T$, $j \in [m]$ are all proportional, or one of $x^{(j)}$ is zero vector.
Let \( a_{ij} \geq 0, i \in [n], j \in [m] \). Suppose that \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are positive real numbers such that \( \sum_{j=1}^{m} 1/\alpha_j > 1 \), then

\[
\sum_{i=1}^{n} \left( \prod_{j=1}^{m} a_{ij} \right) \leq \prod_{j=1}^{m} \left( \sum_{i=1}^{n} a_{ij}^{\alpha_j} \right)^{1/\alpha_j}.
\] (2.8)

Equality holds if and only if either one of \( x^{(j)} \) is zero vector or all but one of each vector is zero, and in the latter case, those which are positive have the same rank.

(3) **(Power Mean inequality)** Let \( a_1, a_2, \ldots, a_n \) be positive real numbers, and \( p, q \) be two nonzero real numbers such that \( p < q \). Then

\[
\left( \frac{1}{n} \sum_{i=1}^{n} a_i^p \right)^{1/p} \leq \left( \frac{1}{n} \sum_{i=1}^{n} a_i^q \right)^{1/q},
\] (2.9)

with equality if and only if \( a_1 = a_2 = \ldots = a_n \).

(4) **(Jensen’s inequality)** Let \( a_i \geq 0, i \in [n] \). If \( 0 < q < p \), then

\[
\left( \sum_{i=1}^{n} a_i^p \right)^{1/p} \leq \left( \sum_{i=1}^{n} a_i^q \right)^{1/q},
\] (2.10)

with equality holds if and only if all but one of \( a_1, a_2, \ldots, a_n \) are zero.

3. **Basic properties of \( \lambda_{p,q}(G) \)**

The first part of this section is devoted to some basic bounds about \( \lambda_{p,q}(G) \). In the second part, we give an relation of \((p, q)\)-spectral radius between \( G \) and its anadiplosis components.

Inspired from the ideas in [24], we first consider \( \lambda_{p,q}(G) \) as a function in \( p \) and \( q \) for a fixed \((r, s)\)-directed hypergraph \( G \). By changing the variables in (2.2), we obtain

\[
\lambda_{p,q}(G) = \max_{||x||=1, ||y||=1} \sum_{e \in E(G), H(e)=(i_1, \ldots, i_r)} (x_{i_1} \cdots x_{i_r})^{1/p} (y_{j_1} \cdots y_{j_s})^{1/q}.
\]

Assume \( p, q, p', q' \geq 1 \) are positive real numbers. Applying the mean value theorem, we have

\[
(x_{i_1} \cdots x_{i_r})^{1/p} (y_{j_1} \cdots y_{j_s})^{1/q} - (x_{i_1} \cdots x_{i_r})^{1/p'} (y_{j_1} \cdots y_{j_s})^{1/q'} \leq |p - p'| + |q - q'|.
\]

It follows that

\[
|\lambda_{p,q}(G) - \lambda_{p',q'}(G)| \leq |G|(|p - p'| + |q - q'|)
\leq \sqrt{2} |G| \sqrt{(p - p')^2 + (q - q')^2},
\]

which yields that \( \lambda_{p,q}(G) \) is a continuous function in \( p \) and \( q \). In Section 5 we shall return to this topic, and give more properties on the function \( \lambda_{p,q}(G) \).
3.1. Some bounds of $\lambda_{p,q}(G)$

By equation (2.2), $\lambda_{p,q}(G)$ is monotone with respect to arc addition.

**Proposition 3.1** Let $G_1$ and $G_2$ be two $(r, s)$-directed hypergraphs, and $G_1 \subseteq G_2$. Then $\lambda_{p,q}(G_1) \leq \lambda_{p,q}(G_2)$.

The following is a simple corollary of **Proposition 3.1**.

**Corollary 3.1** Let $G$ be an $(r, s)$-directed hypergraph with maximum out-degree $\Delta^+$ and maximum in-degree $\Delta^-$. Then

$$\lambda_{p,q}(G) \geq \frac{1}{r^{r/r_p s/q}} \cdot \max \left\{ (\Delta^+)^{1-(r-1)/p+s/q}, (\Delta^-)^{1-(r/p+(s-1)/q)} \right\}.$$

**Proposition 3.2** Let $G$ be an $(r, s)$-directed hypergraph with minimum out-degree $\delta^+$ and minimum in-degree $\delta^-$. Then

$$\lambda_{p,q}(G) \geq \frac{1}{r^{r/r_p s/q}} \cdot \frac{|G|^{1-(r/p+s/q)} \left( \frac{\delta^+}{r} \right)^{r/p} \left( \frac{\delta^-}{s} \right)^{s/q}}{r^{r/r_p s/q}}.$$

**Proof.** Let $x = m^{-1/p}(1, 1, \ldots, 1)^T \in \mathbb{R}^m$, $y = n^{-1/q}(1, 1, \ldots, 1)^T \in \mathbb{R}^n$. By (2.2), we have

$$\lambda_{p,q}(G) \geq P_G(x, y) = \sum_{e \in E(G)} \frac{1}{m^{r/r_p s/q}} = \frac{|G|}{m^{r/r_p s/q}} = \left( \frac{r|G|}{m} \right)^{r/p} \left( \frac{s|G|}{n} \right)^{s/q} \geq \left( \frac{r}{r^+} \right)^{r/p} \left( \frac{s}{s^-} \right)^{s/q},$$

the last inequality follows from the fact:

$$\sum_{v \in T(G)} d^+_v = r|G|, \quad \sum_{v \in H(G)} d^-_v = s|G|.$$

The proof is completed. \[\square\]

3.2. Connectedness of $(r, s)$-directed hypergraphs

We first introduce the definition of bipartite split of a directed hypergraph, which play an important role in the study of anadiplosis connectedness.

**Definition 3.1** Let $G$ be a directed hypergraph, the bipartite split $B(G)$ of $G$ is define as a bipartite directed hypergraph with the same arc set as $G$ and bipartition $V_T \cup V_H$, where $V_T$ is a copy of $T(G)$, and $V_H$ is a copy of $H(G)$.

**Example 3.1** Let $G$ be a directed graph obtained by giving an orientation to $K_4$, the bipartite split $B(G)$ of $G$ is shown as follows:
Lemma 3.1 An \((r, s)\)-directed hypergraph \(G\) is anadiplosis connected if and only if \(\overline{G}\) is connected.

Proof. \((\Rightarrow)\) Assume \(G\) is anadiplosis connected, then for any \(u \neq v\), there is a \(u-v\) anadiplosis walk: \((u = v_0)e_1v_1e_2v_2\cdots v_{\ell - 1}e_\ell(v_\ell = v)\) in \(G\). Clearly, \((u = v_0)\overline{e}_1\overline{v}_1\overline{e}_2\overline{v}_2\cdots \overline{v}_{\ell - 1}\overline{e}_\ell(\overline{v}_\ell = v)\) is a \(u-v\) walk in \(\overline{G}\). Also, for each vertex \(u \in V(\overline{G})\) with \(u \in T(G) \cap H(G)\), there exists a \(u-u\) anadiplosis semi-cycle: \((u = v_0)e_1v_1e_2v_2\cdots v_{\ell - 1}e_\ell(v_\ell = u)\) in \(G\). Therefore \((u = v_0)\overline{e}_1\overline{v}_1\overline{e}_2\overline{v}_2\cdots \overline{v}_{\ell - 1}\overline{e}_\ell(\overline{v}_\ell = u)\) is a \(u-u\) walk in \(\overline{G}\). Hence, \(\overline{G}\) is connected.

\((\Leftarrow)\) Assume \(\overline{G}\) is connected. For any \(u \neq v \in V(G)\), there is a \(u-v\) walk: \((u = v_0)\overline{e}_1\overline{v}_1\overline{e}_2\overline{v}_2\cdots \overline{v}_{\ell - 1}\overline{e}_\ell(\overline{v}_\ell = v)\) in \(\overline{G}\). Obviously, \((u = v_0)e_1v_1e_2v_2\cdots v_{\ell - 1}e_\ell(v_\ell = v)\) is an \(u-v\) anadiplosis walk in \(G\). For any \(u \in T(G) \cap H(G)\), there is a \(u-u\) walk: \((u = v_0)\overline{e}_1\overline{v}_1\overline{e}_2\overline{v}_2\cdots \overline{v}_{\ell - 1}\overline{e}_\ell(\overline{v}_\ell = v)\) in \(\overline{G}\). Then \((u = v_0)e_1v_1e_2v_2\cdots v_{\ell - 1}e_\ell(v_\ell = v)\) is a \(u-u\) anadiplosis semi-cycle in \(G\). Thus, \(G\) is anadiplosis connected. \(\square\)

Lemma 3.2 Let \(G\) be an \((r, s)\)-directed hypergraph. Then \(A(G)\) is weakly irreducible if and only if \(G\) is anadiplosis connected.

Proof. \((\Rightarrow)\) Assume \(A(G)\) is weakly irreducible. Then its associated bipartite graph \(G(A)\) is connected. For any \(u, v \in V(\overline{G})\), since \(G(A)\) is connected, there is a \(u-v\) walk: \((u = v_0)e_1v_1\cdots v_{\ell - 1}e_\ell(v_\ell = v)\) in \(G(A)\). By the definition of \(G(A)\), there must be \(f_i \in E(\overline{G})\) such that \(\{v_{i-1}, v_i\} \subseteq f_i\), \(i \in [\ell]\). Therefore \((u = v_0)f_1v_1\cdots v_{\ell - 1}f_\ell(v_\ell = v)\) is a \(u-v\) walk in \(\overline{G}\). By Lemma 3.1, \(G\) is anadiplosis connected.

\((\Leftarrow)\) Assume \(G\) is anadiplosis connected, according to Lemma 3.1, \(\overline{G}\) is connected. For any vertices \(u, v \in G(A)\), since \(\overline{G}\) is connected, there is a \(u-v\) walk: \((u = v_0)\overline{e}_1\overline{v}_1\cdots \overline{v}_{\ell - 1}\overline{e}_\ell(\overline{v}_\ell = v)\) in \(\overline{G}\). If there exists \(i_0 \in [\ell]\) such that \(\{v_{i_0 - 1}, v_{i_0}\} \subseteq T(e_{i_0})\) (or \(\{v_{i_0 - 1}, v_{i_0}\} \subseteq H(e_{i_0})\)), we pick any vertex \(w \in H(e_{i_0})\) (or \(w \in T(e_{i_0})\)). Then

\[(u = v_0)\overline{e}_1\overline{v}_1\cdots v_{i_0-1}\overline{v}_{i_0}u\overline{v}_{i_0}v_{i_0}\cdots v_{\ell - 1}\overline{e}_\ell(\overline{v}_\ell = v)\]
is also a $u-v$ walk in $\overline{G}$. Therefore we may assume $(u = v_0)\overline{\tau}_1v_1 \cdots v_{\ell-1}\overline{\tau}_{\ell}(v_\ell = v)$ is a $u-v$ walk in $\overline{G}$ such that $\{v_{i-1}, v_i\} \not\subseteq T(e_i)$ and $\{v_{i-1}, v_i\} \not\subseteq H(e_i)$ for any $i \in [\ell]$. By the definition of $G(A)$, $u = v_0, v_1, \ldots, v_{\ell-1}, v_\ell = v$ is a $u-v$ walk in $G(A)$, i.e., $G(A)$ is connected, which implies $A(G)$ is weakly irreducible. \hfill \Box

The following lemma establish an relation of the $(p,q)$-spectral radius between $G$ and its anadiplosis components.

**Lemma 3.3** Let $G$ be an $(r,s)$-directed hypergraph, $G_i$ be the anadiplosis components of $G$, $i = 1, 2, \ldots, k$. If $r/p + s/q \geq 1$, then

$$
\lambda_{p,q}(G) = \max_{1 \leq i \leq k} \{ \lambda_{p,q}(G_i) \}.
$$

**Proof.** For any $i \in [k]$, let $(x^{(i)}, y^{(i)})$ be an eigenpair corresponding to $\lambda_{p,q}(G_i)$. That is $\lambda_{p,q}(G_i) = P_{G_i}(x^{(i)}, y^{(i)})$. Now for any $u \in T(G), v \in H(G)$, we construct two vectors $x \in \mathbb{S}_p^{m-1}, y \in \mathbb{S}_q^{m-1}$ as follows:

$$
x_u = \begin{cases} 
(x^{(i)})_u, & \text{if } u \in T(G_i), \\
0, & \text{otherwise},
\end{cases}
$$

$$
y_v = \begin{cases} 
(y^{(i)})_v, & \text{if } v \in H(G_i), \\
0, & \text{otherwise}.
\end{cases}
$$

Therefore $\lambda_{p,q}(G) \geq P_G(x,y) = P_{G_i}(x^{(i)}, y^{(i)}) = \lambda_{p,q}(G_i)$, which yields

$$
\lambda_{p,q}(G) \geq \max_{1 \leq i \leq k} \{ \lambda_{p,q}(G_i) \}.
$$

On the other hand, let $(x,y)$ be an eigenpair corresponding to $\lambda_{p,q}(G)$. For any $u_i \in T(G_i), v_i \in H(G_i)$, by the weak eigenequations (2.5) we have

$$
\begin{align*}
\sum_{e \in E(G), u_i \in T(e)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right) &= r \lambda_{p,q}(G)x_{u_i}^p, \\
\sum_{e \in E(G), v_i \in H(e)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right) &= s \lambda_{p,q}(G)y_{v_i}^q.
\end{align*}
$$

According to Lemma 3.1, $\{ e : e \in E(G), u_i \in T(e) \} = \{ e : e \in E(G_i), u_i \in T(e) \}$. Also, we have $\{ e : e \in E(G), v_i \in H(e) \} = \{ e : e \in E(G_i), v_i \in H(e) \}$. Therefore

$$
\begin{align*}
\sum_{e \in E(G_i), u_i \in T(e)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right) &= r \lambda_{p,q}(G)x_{u_i}^p, \\
\sum_{e \in E(G_i), v_i \in H(e)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right) &= s \lambda_{p,q}(G)y_{v_i}^q.
\end{align*}
$$

(3.1)
Summing both sides on \( u_i \in T(G_i) \) and \( v_i \in H(G_i) \), respectively, we obtain

\[
\begin{align*}
\sum_{e \in E(G_i)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right) &= \lambda_{p,q}(G) \left| x_{|T(G_i)} \right|^p, \\
\sum_{e \in E(G_i)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right) &= \lambda_{p,q}(G) \left| y_{|H(G_i)} \right|^q.
\end{align*}
\] (3.2)

Hence, \( \left| x_{|T(G_i)} \right|^p = \left| y_{|H(G_i)} \right|^q \), \( i \in [k] \). Now we choose an anadiplosis component \( G_j \) such that \( x_{|T(G_j)} \neq 0 \) and \( y_{|H(G_j)} \neq 0 \). It follows from (2.4) and (3.2) that

\[
\lambda_{p,q}(G_j) \geq \frac{P_{G_j}(x_{|T(G_j)}, y_{|H(G_j)})}{\left| x_{|T(G_j)} \right|^p \cdot \left| y_{|H(G_j)} \right|^q} = \frac{\sum_{e \in E(G_j)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right)}{\left| x_{|T(G_j)} \right|^p \cdot \left| y_{|H(G_j)} \right|^q} = \frac{\lambda_{p,q}(G)}{\left| x_{|T(G_j)} \right|^p \left| y_{|H(G_j)} \right|^q} \geq \lambda_{p,q}(G).
\]

Therefore \( \max_{1 \leq i \leq k} \lambda_{p,q}(G_i) \geq \lambda_{p,q}(G) \), completing the proof. \( \square \)

However, if \( r/p + s/q < 1 \), we get a different statement as follows.

**Lemma 3.4** Let \( G \) be an \((r, s)\)-directed hypergraph, \( G_i \) be the anadiplosis components of \( G \), \( i = 1, 2, \ldots, k \). If \( r/p + s/q < 1 \), then

\[
\lambda_{p,q}(G) = \left( \sum_{i=1}^{k} \left( \lambda_{p,q}(G_i) \right)^{1/(1-(r/p+s/q))} \right)^{1-(r/p+s/q)}.
\]

**Proof.** Let \((x, y) \in S^m_{p,+} \times S^m_{q,+} \) be an eigenpair to \( \lambda_{p,q}(G) \), and let \( x^{(i)}, y^{(i)} \) be the restriction of \( x, y \) to \( T(G_i), H(G_i) \), respectively. In the light of (2.4),

\[
\lambda_{p,q}(G) = P_G(x, y) = \sum_{i=1}^{k} P_{G_i}(x^{(i)}, y^{(i)}) \leq \sum_{i=1}^{k} \lambda_{p,q}(G_i) \left| x^{(i)} \right|^p \cdot \left| y^{(i)} \right|^q.
\]

Let \( \alpha_1 = 1/(1-(r/p+s/q)) \), \( \alpha_2 = p/r \), \( \alpha_3 = q/s \), we have \( 1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 = 1 \), and applying Generalized Hölder’s inequality (2.7), we obtain

\[
\lambda_{p,q}(G) \leq \left( \sum_{i=1}^{k} \left( \lambda_{p,q}(G_i) \right)^{\alpha_1} \right)^{1/\alpha_1} \left( \sum_{i=1}^{k} \left( \left| x^{(i)} \right|^p \right)^{\alpha_2} \right)^{1/\alpha_2} \left( \sum_{i=1}^{k} \left( \left| y^{(i)} \right|^q \right)^{\alpha_3} \right)^{1/\alpha_3} \leq \left( \sum_{i=1}^{k} (\lambda_{p,q}(G_i))^{\alpha_1} \right)^{1/\alpha_1} \left( \sum_{i=1}^{k} \left| x^{(i)} \right|^p \right)^{r/p} \left( \sum_{i=1}^{k} \left| y^{(i)} \right|^q \right)^{s/q}.
\]
\[
= \left( \sum_{i=1}^{k} (\lambda_{p,q}(G_i))^{1/(1-(r/p+s/q))} \right)^{1-(r/p+s/q)}.
\]

On the other hand, let \((x^{(i)}, y^{(i)})\) be the eigenpair to \(\lambda_{p,q}(G_i)\), that is \(\lambda_{p,q}(G_i) = P_G((x^{(i)}, y^{(i)}), \ i = 1, 2, \ldots, k\). For simplicity, denote
\[
a_i = \frac{(\lambda_{p,q}(G_i))^{\alpha_1}}{\sum_{i=1}^{k} (\lambda_{p,q}(G_i))^{\alpha_1}}, \ i = 1, 2, \ldots, k.
\]

Furthermore, we let
\[
x = \left(a_1^{1/p} x^{(1)}, a_2^{1/p} x^{(2)}, \ldots, a_k^{1/p} x^{(k)}\right)^T, \\
y = \left(a_1^{1/q} y^{(1)}, a_2^{1/q} y^{(2)}, \ldots, a_k^{1/q} y^{(k)}\right)^T.
\]

Clearly, \(||x||_p^p = ||y||_q^q = 1.\) By (2.2) we see that
\[
\lambda_{p,q}(G) \geq P_G(x, y) = \sum_{i=1}^{k} P_G((a_1^{1/p} x^{(i)}, a_1^{1/q} y^{(i)})) \\
= \sum_{i=1}^{k} a_i^{r/p+s/q} \cdot P_G((x^{(i)}, y^{(i)})) \\
= \sum_{i=1}^{k} a_i^{r/p+s/q} \cdot \lambda_{p,q}(G_i) \\
= \left( \sum_{i=1}^{k} (\lambda_{p,q}(G_i))^{1/(1-(r/p+s/q))} \right)^{1-(r/p+s/q)}.
\]

The proof is completed. \(\square\)

We call the value \(e := \frac{r}{p} + \frac{s}{q}\) is the eccentricity of \(\lambda_{p,q}(G)\), and refer \(e < 1\) as the elliptical phase, \(e = 1\) as the parabolic phase, and \(e > 1\) as the hyperbolic phase. The value \(\lambda_{p,q}(G)\) behaves very different in three phases.

4. The \(\alpha\)-normal labeling methods for \((r, s)\)-directed hypergraphs

We begin this section with the following concept, which will be used frequently in the sequel.

**Definition 4.1** A weighted incidence matrix \(B = (B(v, e))\) of a (directed) hypergraph \(G\) is a \(|V| \times |E|\) matrix such that for any \(v \in V(G)\) and any \(e \in E(G)\), the entry \(B(v, e) > 0\) if \(v \in e\) and \(B(v, e) = 0\) if \(v \notin e\).

In [22], Lu and Man discovered the \(\alpha\)-normal labeling method for computing the spectral radii of uniform hypergraphs as follows.
**Theorem 4.1 ([22])** Let $H$ be a connected $k$-uniform hypergraph. Then the spectral radius of $H$ is $\rho(H)$ if and only if there is a weighted incidence matrix $B = (B(v, e))$ satisfying

1. $\sum_{e: v \in e} B(v, e) = 1$, for any $v \in V(H)$;
2. $\prod_{v \in e} B(v, e) = \alpha = (\rho(H))^{-k}$, for any $e \in E(H)$;
3. $\prod_{i=1}^{\ell} \frac{B(v_{i-1}, e_i)}{B(v_i, e_i)} = 1$, for any cycle $v_0 e_1 v_1 e_2 \cdots v_{\ell-1} e_\ell v_\ell (v_\ell = v_0)$.

In our previous paper [21], we generalized the $\alpha$-normal labeling method for computing the $p$-spectral radii of $k$-uniform hypergraphs and found a number of applications for $p > k$.

**Theorem 4.2 ([21])** Let $H$ be a $k$-uniform hypergraph and $p > k$. Then the $p$-spectral radius of $H$ is $\lambda^{(p)}(H)$ if and only if there exist a weighted incidence matrix $B = (B(v, e))$ and edge weights $\{w(e)\}$ satisfying

1. $\sum_{e \in E(G)} w(e) = 1$;
2. $\sum_{e: v \in e} B(v, e) = 1$, for any $v \in V(H)$;
3. $w(e)^{p-k} \cdot \prod_{v \in e} B(v, e) = \alpha = k^{p-k}/(\lambda^{(p)}(H))^p$, for any $e \in E(H)$;
4. For any $v \in V(H)$ and $e_i, i = 1, 2, \ldots, d$,
   $\frac{w(e_1)}{B(v, e_1)} = \frac{w(e_2)}{B(v, e_2)} = \cdots = \frac{w(e_d)}{B(v, e_d)}$.

The main focus of this section is to develop a similar method as Theorem 4.1 (and Theorem 4.2) for calculating $\lambda_{p,q}(G)$, as well as for comparing $\lambda_{p,q}(G)$ with a particular value. Before continuing, we need the following Perron–Frobenius theorem for rectangular tensors. We say an index $v$ is an isolated vertex for a rectangular tensor $A$ if the $v$-th row is zero: $a_{v_{i_2\cdots i_jj_1\cdots j_s}} \equiv 0$ or $a_{i_1\cdots i_{\ell}j_{\ell+1}\cdots j_{s}} \equiv 0$.

**Theorem 4.3 (Lu, Yang, Zhao [23])** Suppose that $A$ is an $(r,s)$-th order $(m \times n)$-dimensional nonnegative rectangular tensor with no isolated vertex.

1. If $r/p + s/q < 1$, then $A$ has a unique positive eigenvalue-eigenvectors triple.
2. If $(r - 1)/p + s/q < 1$, $r/p + (s - 1)/q < 1$, and $A$ is partially symmetric and weakly irreducible, then $A$ has a positive eigenvalue-eigenvectors triple. If further $r/p + s/q = 1$, then $A$ has a unique positive eigenvalue-eigenvectors triple.

From Theorem 4.3 and Lemma 3.2, we have the following statement.

**Theorem 4.4** Suppose that $G$ is an $(r,s)$-directed hypergraphs with no isolated vertex.

1. If $r/p + s/q < 1$, then $G$ has a unique positive eigenpair to $\lambda_{p,q}(G)$. 

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(2) If \( r/p + s/q = 1 \), and \( G \) is anadiplosis connected, then \( G \) has a unique positive eigenpair to \( \lambda_{p,q}(G) \).

We say that vertices \( u \) and \( v \) are equivalent in \( G \), in writing \( u \sim v \), if there exists an automorphism \( \pi \) of \( G \) such that \( \pi(u) = v \). The following is a direct corollary of Theorem 4.4.

**Corollary 4.1** Let \( G \) be an \((r, s)\)-directed hypergraph, and let \( u, v \in T(G) \) (or \( u, v \in H(G) \)), \( u \sim v \). Suppose that \((x, y) \in \mathbb{S}^{m-1}_{p,+} \times \mathbb{S}^{n-1}_{q,+} \) is an eigenpair to \( \lambda_{p,q}(G) \). Then \( x_u = x_v \) (or \( y_u = y_v \)) if one of the following holds:

1. \( r/p + s/q < 1 \);
2. \( r/p + s/q = 1 \) and \( G \) is anadiplosis connected.

### 4.1. Parabolic phase: \( \frac{r}{p} + \frac{s}{q} = 1 \)

**Definition 4.2** An \((r, s)\)-directed hypergraph \( G \) is called parabolic \( \alpha \)-normal if \( r/p + s/q = 1 \) and there exists a weighted incidence matrix \( B \) satisfying

\[
\sum_{e: u \in T(e)} B(u, e) = \sum_{e: v \in H(e)} B(v, e) = 1, \text{ for any } u \in T(G), \ v \in H(G);
\]

\[
\prod_{u \in T(e)} (B(u, e))^{1/p} \cdot \prod_{v \in H(e)} (B(v, e))^{1/q} = \alpha, \text{ for any } e \in E(G).
\]

Moreover, the weighted incidence matrix \( B \) is called parabolic consistent if for any anadiplosis cycle \( v_0e_1v_1e_2 \cdots v_{\ell-1}e_\ell v_\ell \) (\( v_\ell = v_0 \)),

\[
\prod_{i=1}^{\ell} \frac{B(v_{i-1}, e_i)}{B(v_i, e_i)} = 1.
\]

**Example 4.1** Consider the following \((2, 1)\)-directed hypergraph \( C^{(2,1)}_\ell \) with \( \ell \) arcs (\( \ell \) is even). Clearly, \( C^{(2,1)}_\ell \) is an anadiplosis cycle. Here, black (white) node represents the vertex in tail (head).
We label the value $B(v,e)$ at vertex $v$ near the side of arc $e$. If $d_v = 1$, then it has the trivial value 1, and therefore we omit its labeling. If $d_v = 2$, we let $B(v,e) = 1/2$. It can be checked that $G^{(2,1)}_e$ is parabolic consistently $2^{-(1/p+1/q)}$-normal.

**Lemma 4.1** Let $G$ be an $(r,s)$-directed hypergraph with $r/p + s/q = 1$. If $G$ is anadiplosis connected, then the $(p,q)$-spectral radius of $G$ is $\lambda_{p,q}(G)$ if and only if $G$ is parabolic consistently $\alpha$-normal with

$$\alpha = \frac{1}{r^{r/p} s^{s/q} \lambda_{p,q}(G)}.$$

**Proof.** We first show that it is necessary. By Theorem 4.4, let $x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{S}_{p,++}^{m-1}$ and $y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{S}_{q,++}^{n-1}$ be an eigenpair to $\lambda_{p,q}(G)$. Define a weighted incidence matrix $B$ of $G$ as follows:

$$B(v,e) = \begin{cases} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{u \in H(e)} y_u \right), & \text{if } v \in T(e), \\
\frac{\prod_{u \in T(e)} x_u}{r \lambda_{p,q}(G)} y_v, & \text{if } v \in H(e), \\
0, & \text{otherwise.} \end{cases} \quad (4.1)$$

For any $u \in T(G)$, $v \in H(G)$, by equation (2.6) we have

$$\sum_{e: u \in T(e)} B(u,e) = \sum_{e: v \in H(e)} B(v,e) = 1$$

and

$$\sum_{e: u \in T(e)} B(u,e) = \sum_{e: v \in H(e)} B(v,e) = 1.$$ 

Also, for any $e \in E(G)$, it can be checked that

$$\prod_{u \in T(e)} \left( B(u,e) \right)^{1/p} = \left( \prod_{u \in T(e)} x_u \prod_{v \in H(e)} y_v \right)^{r/p} \left( r \lambda_{p,q}(G) \right)^{r/p} \prod_{v \in T(e)} x_v$$

and

$$\prod_{v \in H(e)} \left( B(v,e) \right)^{1/q} = \left( \prod_{u \in T(e)} x_u \prod_{u \in H(e)} y_u \right)^{s/q} \left( s \lambda_{p,q}(G) \right)^{s/q} \prod_{u \in H(e)} y_u.$$ 

It follows from $r/p + s/q = 1$ that

$$\prod_{u \in T(e)} \left( B(u,e) \right)^{1/p} \cdot \prod_{v \in H(e)} \left( B(v,e) \right)^{1/q} = \frac{1}{r^{r/p} s^{s/q} \lambda_{p,q}(G)} = \alpha.$$ 

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To show that $B$ is parabolic consistent, for any anadiplosis cycle $v_0 e_1 v_1 e_2 \cdots v_{\ell - 1} e_{\ell} v_{\ell} (v_{\ell} = v_0)$, by the definition of anadiplosis cycle and (4.1) we conclude that

$$B(v_i, e_{i+1}) = \frac{\prod_{u \in T(e_{i+1})} x_u}{\prod_{u \in T(e_i)} x_u} \frac{\prod_{v \in H(e_{i+1})} y_v}{\prod_{v \in H(e_i)} y_v}, \quad i \in [\ell - 1].$$

For short, we denote $Z(e) := \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{u \in H(e)} y_u \right)$ for any $e \in E(G)$. Therefore

$$\prod_{i=1}^{\ell} B(v_{i-1}, e_i) = B(v_0, e_1) \cdot \frac{\prod_{i=1}^{\ell-1} B(v_i, e_{i+1})}{B(v_{\ell}, e_{\ell})} = \frac{Z(e_1)}{Z(e_{\ell})} \cdot \frac{\prod_{i=1}^{\ell-1} Z(e_{i+1})}{Z(e_i)} = 1.$$

Now we show that it is also sufficient. Assume that $B$ is a parabolic consistent $\alpha$-normal weighted incident matrix of $G$. For any nonnegative vectors $x = (x_1, x_2, \ldots, x_m)^T \in S_{p,+}^n$ and $y = (y_1, y_2, \ldots, y_m)^T \in S_{q,+}^n$, by H"{o}lder's inequality and AM–GM inequality we have

$$P_G(x, y) = \sum_{e \in E(G)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right)$$

$$= \frac{1}{\alpha} \sum_{e \in E(G)} \left( \prod_{u \in T(e)} (B(u, e))^{1/p} x_u \right) \left( \prod_{v \in H(e)} (B(v, e))^{1/q} y_v \right)$$

$$\leq \frac{1}{\alpha} \left( \sum_{e \in E(G)} \prod_{u \in T(e)} (B(u, e))^{1/r} x_u^{r/p} \right) \left( \sum_{e \in E(G)} \prod_{v \in H(e)} (B(v, e))^{1/s} y_v^{q/s} \right)^{s/q}$$

$$\leq \frac{1}{r^{r/p s/q} \alpha} \left( \sum_{e \in E(G)} \sum_{u \in T(e)} B(u, e) x_u^p \right)^{r/p} \left( \sum_{e \in E(G)} \sum_{v \in H(e)} B(v, e) y_v^q \right)^{s/q}$$

$$= \frac{1}{r^{r/p s/q} \alpha} \cdot ||x||_p^r \cdot ||y||_q^s = \frac{1}{r^{r/p s/q} \alpha}. $$

This inequality implies

$$\lambda_{p,q}(G) \leq \frac{1}{r^{r/p s/q} \alpha}. \quad (4.2)$$

The equality holds if $G$ is parabolic $\alpha$-normal and there is a nonzero solution $(x, y)$ to the following equations:

$$r B(i_1, e) x_{i_1}^p = \cdots = r B(i_r, e) x_{i_r}^p = s B(j_1, e) y_{j_1}^q = \cdots = s B(j_s, e) y_{j_s}^q \quad (4.3)$$

for any $e \in E(G)$, $T(e) = \{i_1, i_2, \ldots, i_r\}$ and $H(e) = \{j_1, j_2, \ldots, j_s\}$. Fix a vertex $u_0 \in T(G)$, now we consider any vertex $u \in T(G)$. Since $G$ is anadiplosis connected, there exists a $u_0 - u$ anadiplosis walk: $u_0 e_1 u_1 e_2 \cdots u_{\ell-1} e_{\ell} u_{\ell} (u_{\ell} = u)$ in $G$. Define

$$x_u^* = \left( \prod_{i=1}^{\ell} \frac{B(u_{i-1}, e_i)}{B(u_i, e_i)} \right)^{1/p} x_{u_0}^*$$

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where \( x^*_{u_0} \) is determined by the condition \( ||x^*||_p = 1 \). Similarly, for any \( v \in H(G) \), there is a \( u_0-v \) anadiplosis walk: \( u_0f_1v_1f_2 \cdots v_{e-1}fv'v'(v_e = v) \) in \( G \). Define

\[
y^*_v = \left( \frac{r}{s} \prod_{i=1}^{e'} \frac{B(v_{i-1}, f_i)}{B(v_i, f_i)} \right)^{1/q} \left( x^*_{u_0} \right)^{p/q}.
\]

The consistent condition guarantees that \( x^*_v \) and \( y^*_v \) are independent of the choice of the anadiplosis walk. It is easy to check that \((x^*, y^*)\) is a solution of \((4.3)\), and

\[
\sum_{v \in H(G)} (y^*_v)^q = \sum_{e \in E(G)} \sum_{v \in H(e)} B(v, e)(y^*_v)^q
= \sum_{e \in E(G)} \sum_{u \in T(e)} B(u, e)(x^*_u)^p
= \sum_{u \in T(G)} (x^*_u)^p = 1.
\]

Therefore \( \lambda_{p,q}(G) = r^{-r/p} s^{-s/q} \alpha^{-1} \). The proof is completed. \( \square \)

**Remark 4.1** According to the proof of Lemma 4.1, equation \((4.3)\) is a sufficient condition for the equality holding in \((4.2)\). We remark that it is also a necessary condition. That is, if \( B \) is a parabolic consistently normal labeling of \( G \), and \((x, y) \in S_{p,++}^{m-1} \times S_{q,++}^{n-1} \) is an eigenpair to \( \lambda_{p,q}(G) \), then \((4.3)\) holds. Indeed, we assume

\[
B(i_1, e)x^p_{i_1} = \cdots = B(i_r, e)x^p_{i_r} = cB(j_1, e)y^q_{j_1} = \cdots = cB(j_s, e)y^q_{j_s}.
\]

By \( ||x||_p = ||y||_q = 1 \), we have

\[
1 = \sum_{u \in T(G)} x^p_u = \sum_{e \in E(G)} \sum_{u \in T(e)} B(u, e)x^p_u
= \sum_{e \in E(G)} \sum_{v \in H(e)} \frac{rc}{s} B(v, e)y^q_v
= \frac{rc}{s} \sum_{v \in H(G)} y^q_v = \frac{rc}{s},
\]

which yields that \( rc = s \).

In what follows, we give a method for comparing the \((p, q)\)-spectral radius with a particular value. It is convenient to introduce the following concepts.

**Definition 4.3** An \((r, s)\)-directed hypergraph \( G \) is called **parabolic \( \alpha \)-subnormal** if \( r/p + s/q = 1 \) and there exists a weighted incidence matrix \( B \) satisfying

\[
(1) \sum_{e : u \in T(e)} B(u, e) \leq 1, \quad \sum_{e : v \in H(e)} B(v, e) \leq 1, \text{ for any } u \in T(G), v \in H(G);
\]

\[
(2) \sum_{e : v \in H(e)} B(u, e) \leq 1, \quad \sum_{e : u \in T(e)} B(v, e) \leq 1, \text{ for any } v \in H(G), u \in T(G);
\]

\[
(3) \sum_{e : u \in T(e)} B(v, e) \leq 1, \quad \sum_{e : v \in H(e)} B(u, e) \leq 1, \text{ for any } u \in T(G), v \in H(G);
\]

\[
(4) \sum_{e : v \in H(e)} B(v, e) \leq 1, \quad \sum_{e : u \in T(e)} B(u, e) \leq 1, \text{ for any } v \in H(G), u \in T(G).
\]

In what follows, we provide some examples of \((r, s)\)-directed hypergraphs that satisfy the above conditions.
Moreover, $G$ is called \textit{strictly parabolic $\alpha$-subnormal} if it is parabolic $\alpha$-subnormal but not parabolic $\alpha$-normal.

Here is an example of parabolic $\alpha$-subnormal directed hypergraph.

\textbf{Example 4.2} Consider the following $(2,1)$-directed hypergraph $P_{(2,1)}^\ell$. By labeling $P_{(2,1)}^\ell$ as follows:

\[ \ldots \quad \begin{array}{c}
\begin{array}{cccccccc}
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1
\end{array}
\end{array} \]

We can check that $P_{(2,1)}^\ell$ is parabolic $2^{-(1/p+1/q)}$-subnormal.

\textbf{Lemma 4.2} Let $G$ be an $(r,s)$-directed hypergraph. If $G$ is parabolic $\alpha$-subnormal, then the $(p,q)$-spectral radius of $G$ satisfies

\[ \lambda_{p,q}(G) \leq \frac{1}{r^r/p^{s}q^{s}/\alpha}. \]

\textbf{Proof.} For any nonnegative vectors $x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{S}_{p,+}^m$ and $y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{S}_{q,+}^n$, by Hölder’s inequality and AM–GM inequality, we deduce that

\[
P_G(x,y) \leq \frac{1}{\alpha} \sum_{e \in E(G)} \left( \prod_{u \in T(e)} (B(u,e))^{1/p} x_u \right) \left( \prod_{v \in H(e)} (B(v,e))^{1/q} y_v \right)
\]

\[
\leq \frac{1}{\alpha} \left( \sum_{e \in E(G)} \prod_{u \in T(e)} (B(u,e))^{1/r} x_u^{p/r} \right) \left( \sum_{e \in E(G)} \prod_{v \in H(e)} (B(v,e))^{1/s} y_v^{q/s} \right)
\]

\[
\leq \frac{1}{r^r/p^{s}q^{s}/\alpha} \left( \sum_{e \in E(G)} \sum_{u \in T(e)} B(u,e) x_u^{p/r} \right) \left( \sum_{e \in E(G)} \sum_{v \in H(e)} B(v,e) y_v^{q/s} \right)
\]

\[
\leq \frac{1}{r^r/p^{s}q^{s}/\alpha} \cdot ||x||_p \cdot ||y||_q = \frac{1}{r^r/p^{s}q^{s}/\alpha},
\]

which implies $\lambda_{p,q}(G) \leq r^{-r/p}s^{-s/q}\alpha^{-1}$. When $G$ is strictly parabolic $\alpha$-subnormal, this inequality is strict, and therefore $\lambda_{p,q}(G) < r^{-r/p}s^{-s/q}\alpha^{-1}$. \hfill \Box

\textbf{Definition 4.4} An $(r,s)$-directed hypergraph $G$ is called \textit{parabolic $\alpha$-supernormal} if $r/p + s/q = 1$ and there exists a weighted incidence matrix $B$ satisfying

\[ (1) \quad \sum_{e: u \in T(e)} B(u,e) \geq 1, \quad \sum_{e: v \in H(e)} B(v,e) \geq 1, \text{ for any } u \in T(G), v \in H(G); \]
Moreover, $G$ is called \textit{strictly parabolic $\alpha$-supernormal} if it is parabolic $\alpha$-supernormal but not parabolic $\alpha$-normal.

**Lemma 4.3** Let $G$ be an $(r, s)$-directed hypergraph. If $G$ is parabolic consistently $\alpha$-supernormal, then the $(p, q)$-spectral radius of $G$ satisfies

$$
\lambda_{p, q}(G) \geq \frac{1}{r^{r/p} s^{s/q}}.
$$

**Proof.** The parabolic consistent condition implies that there exist nonnegative vectors $x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}_{p,+}^n$ and $y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}_{q,+}^n$ satisfying (4.3). Therefore

$$
P_G(x, y) \geq \frac{1}{\alpha} \sum_{e \in E(G)} \left( \prod_{u \in T(e)} (B(u, e))^{1/p} x_u \right) \left( \prod_{v \in H(e)} (B(v, e))^{1/q} y_v \right)
$$

$$
= \frac{1}{\alpha} \left( \sum_{e \in E(G)} \prod_{u \in T(e)} (B(u, e))^{1/r} x_u^{p/r} \right)^{r/p} \left( \sum_{e \in E(G)} \prod_{v \in H(e)} (B(v, e))^{1/s} y_v^{q/s} \right)^{s/q}
$$

$$
\geq \frac{1}{r^{r/p} s^{s/q}} \cdot \|x\|^r \cdot \|y\|^s = \frac{1}{r^{r/p} s^{s/q}}
$$

which implies $\lambda_{p, q}(G) \geq r^{-r/p} s^{-s/q}$. When $G$ is strictly parabolic $\alpha$-supernormal, this inequality is strict, and therefore $\lambda_{p, q}(G) > r^{-r/p} s^{-s/q}$. \hfill \Box

### 4.2. Elliptic phase: $\frac{r}{p} + \frac{s}{q} < 1$

Given an $(r, s)$-directed hypergraph $G$, for each arc $e \in E(G)$, we put a weight $w(e) > 0$ on $e$. We now introduce the following concepts.

**Definition 4.5** An $(r, s)$-directed hypergraph $G$ is called \textit{elliptic $\alpha$-normal} if $r/p + s/q < 1$ and there exist a weighted incidence matrix $B$ and $\{w(e)\}$ satisfying

1. \( \sum_{e \in E(G)} w(e) = 1; \)
2. \( \sum_{e: u \in T(e)} B(u, e) = \sum_{e: v \in H(e)} B(v, e) = 1, \) for any $u \in T(G)$, $v \in H(G);$
3. \( w(e)^{1-(r/p+s/q)} \prod_{u \in T(e)} (B(u, e))^{1/p} \prod_{v \in H(e)} (B(v, e))^{1/q} = \alpha, \) for any $e \in E(G).$
Moreover, the weighted incidence matrix $B$ and $\{w(e)\}$ are called *elliptic consistent* if for any $u \in T(G)$, $v \in H(G)$, $e_1, \ldots, e_d$ and $f_1, \ldots, f_h$ are arcs contained $u$ and $v$ in tail and head, respectively,

$$\frac{w(e_1)}{B(u, e_1)} = \cdots = \frac{w(e_d)}{B(u, e_d)}, \quad \frac{w(f_1)}{B(v, f_1)} = \cdots = \frac{w(f_h)}{B(v, f_h)}.$$

**Lemma 4.4** Let $G$ be an $(r, s)$-directed hypergraph with $r/p + s/q < 1$. Then the $(p, q)$-spectral radius of $G$ is $\lambda_{p,q}(G)$ if and only if $G$ is elliptic consistently $\alpha$-normal with

$$\alpha = \frac{1}{r^{r/p}s^{s/q}\lambda_{p,q}(G)}.$$

**Proof.** We first show that it is necessary. By Theorem 4.4, let $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T \in S_{p,+}^{m-1}$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T \in S_{q,+}^{n-1}$ be the eigenpair to $\lambda_{p,q}(G)$. Define a weighted incidence matrix $B = (B(v, e))$ and $\{w(e)\}$ as follows:

$$B(v, e) = \begin{cases} \prod_{u \in T(e)} x_u \prod_{u \in H(e)} y_u, & \text{if } v \in T(e), \\ \prod_{u \in T(e)} x_u \prod_{u \in H(e)} y_u, & \text{if } v \in H(e), \\ 0, & \text{otherwise.} \end{cases} \quad (4.4)$$

$$w(e) = \frac{\prod_{u \in T(e)} x_u \prod_{u \in H(e)} y_u}{\lambda_{p,q}(G)}. \quad (4.5)$$

For any $u \in T(G)$, $v \in H(G)$, by (2.6) we see that

$$\sum_{e: u \in T(e)} B(u, e) = \frac{\sum_{e: v \in T(e)} \prod_{v \in T(e)} x_v \prod_{v \in H(e)} y_v}{r\lambda_{p,q}(G)x_v} = 1$$

and

$$\sum_{e: v \in H(e)} B(v, e) = \frac{\sum_{e: v \in H(e)} \prod_{u \in T(e)} x_u \prod_{u \in H(e)} y_u}{s\lambda_{p,q}(G)y_v} = 1.$$
To show that \( B \) and \( \{ w(e) \} \) are consistent, for any \( u \in T(G) \), \( v \in H(G) \), according to (4.4) and (4.5) we see that

\[
\frac{w(e_1)}{B(u, e_1)} = \cdots = \frac{w(e_d)}{B(u, e_d)} = r x^p, \quad \frac{w(f_1)}{B(v, f_1)} = \cdots = \frac{w(f_h)}{B(v, f_h)} = s y^q_v.
\]

Now we show that it is also sufficient. Assume that \( G \) is elliptic consistently \( \alpha \)-normal with weighted incident matrix \( B \) and \( \{ w(e) \} \). Denote

\[
\alpha_1 = \frac{1}{1 - (r/p + s/q)}, \quad \alpha_2 = \frac{p}{r}, \quad \alpha_3 = \frac{q}{s}.
\]

Clearly, \( 1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 = 1 \). For any nonnegative vectors \( x = (x_1, x_2, \ldots, x_m)^T \in S_{p,+}^{m-1} \) and \( y = (y_1, y_2, \ldots, y_n)^T \in S_{q,+}^{n-1} \), by Generalized Hölder’s inequality (2.7) and AM–GM inequality, we have

\[
P_G(x, y) = \sum_{e \in E(G)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right)
= \frac{1}{\alpha} \sum_{e \in E(G)} \left( w(e)^{1-(r/p+s/q)} \prod_{u \in T(e)} (B(u, e))^{1/p} x_u \prod_{v \in H(e)} (B(v, e))^{1/q} y_v \right)
\leq \frac{1}{\alpha} \left( \sum_{e \in E(G)} w(e)^{1-(r/p+s/q)\alpha_1} \right)^{1/\alpha_1} \left( \sum_{e \in E(G)} \prod_{u \in T(e)} (B(u, e))^{\alpha_2/p} x_u^{\alpha_2} \right)^{1/\alpha_2}
\times \left( \sum_{e \in E(G)} \prod_{v \in H(e)} (B(v, e))^{\alpha_3/q} y_v^{\alpha_3} \right)^{1/\alpha_3}
= \frac{1}{\alpha} \left( \sum_{e \in E(G)} \prod_{u \in T(e)} (B(u, e))^{1/r} x_u^{r/p} \right)^{r/p} \left( \sum_{e \in E(G)} \prod_{v \in H(e)} (B(v, e))^{1/s} y_v^{s/q} \right)^{s/q}
\leq \frac{1}{r^{p/s} q^{s/q} \alpha} \left( \sum_{e \in E(G)} \sum_{u \in T(e)} B(u, e) x_u^{p} \right)^{r/p} \left( \sum_{e \in E(G)} \sum_{v \in H(e)} B(v, e) y_v^{q} \right)^{s/q}
= \frac{1}{r^{p/s} q^{s/q} \alpha} \cdot \|x\|_{p}^{r/p} \cdot \|y\|_{q}^{s/q} = \frac{1}{r^{r/p} s^{s/q} \alpha}.
\]

This inequality implies \( \lambda_{p,q}(G) \leq r^{-r/p} s^{-s/q} \alpha^{-1} \).

The equality holds if \( G \) is elliptic \( \alpha \)-normal and there is a nonzero solution \((x, y)\) to the following equations:

\[
r B(i_1, e)x_{i_1}^{p} = \cdots = r B(i_r, e)x_{i_r}^{p} = s B(j_1, e)y_{j_1}^{q} = \cdots = s B(j_s, e)y_{j_s}^{q} = w(e) \tag{4.6}
\]

for any \( e \in E(G) \), \( T(e) = \{i_1, i_2, \ldots, i_r\} \) and \( H(e) = \{j_1, j_2, \ldots, j_s\} \). Assume \( u \in T(G) \), \( v \in H(G) \), and \( u \in T(e) \), \( v \in H(f) \) for some arcs \( e, f \in E(G) \). Define

\[
x_u^* = \left( \frac{w(e)}{r B(u, e)} \right)^{1/p} \tag{4.7}
\]
and

\[ y_w^* = \left( \frac{w(f)}{sB(v,f)} \right)^{1/q}. \quad (4.8) \]

The consistent conditions guarantee that \( x_u^* \) and \( y_v^* \) are independent of the choice of the arcs \( e \) and \( f \). It is easy to check that \((x^*, y^*)\) is a solution of (4.6). Equations (4.7) and (4.8) also imply that

\[
\begin{aligned}
    rB(u,e)(x_u^*)^p &= w(e), & & \text{if } u \in T(e), \\
    sB(v,f)(y_v^*)^q &= w(f), & & \text{if } v \in H(f),
\end{aligned}
\]

from which it follows that

\[
\begin{aligned}
    ||x^*||_p^p &= \sum_{u \in T(G)} (x_u^*)^p = \sum_{e \in E(G)} \sum_{u \in T(e)} B(u,e)(x_u^*)^p = \sum_{e \in E(G)} w(e) = 1 \\
    ||y^*||_q^q &= \sum_{v \in H(G)} (y_v^*)^q = \sum_{f \in E(G)} \sum_{v \in H(f)} B(v,f)(y_v^*)^q = \sum_{f \in E(G)} w(f) = 1.
\end{aligned}
\]

Therefore \( \lambda_{p,q}(G) = r^{-r/p} s^{-s/q} \alpha^{-1} \), completing the proof. \( \square \)

An \((r, s)\)-directed hypergraph \( G \) is called an out-hyperstar (or in-hyperstar) if each two arcs of \( G \) share the same vertex in the tail (or head) of each arc. The same vertex is called the center of \( G \).

**Example 4.3** Let \( G \) be an out-hyperstar with \( k \) arcs and \( r/p + s/q < 1 \). We define a weighted incidence matrix \( B \) and \( \{w(e)\} \) for \( G \) as follows:

\[
B(v,e) = \begin{cases} 
1/k, & \text{if } v \text{ is the center}, \\
1, & \text{else if } v \in e, \\
0, & \text{otherwise},
\end{cases}
\]

\[
w(e) = 1/k.
\]

It can be checked that \( G \) is elliptic consistently \( \alpha \)-normal with \( \alpha = k^{(r-1)/p + s/q - 1} \). Therefore

\[
\lambda_{p,q}(G) = k^{1-(r-1)/p + s/q - 1}. 
\]

In particular, if \( r/p + s/q = 1 \), we have

\[
\lambda_{p,q}(G) = \frac{k^{1/p}}{r^{r/p} s^{s/q}} 
\]

by taking \( r/p + s/q \to 1 \). Similarly, we can prove that if \( G \) is an in-hyperstar with \( k \) arcs, then

\[
\lambda_{p,q}(G) = \begin{cases}
\frac{k^{1-(r/p + (s-1)/q)}}{r^{r/p} s^{s/q}}, & \text{if } r/p + s/q < 1, \\
\frac{k^{1/q}}{r^{r/p} s^{s/q}}, & \text{if } r/p + s/q = 1.
\end{cases}
\]

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**Definition 4.6** An \((r, s)\)-directed hypergraph \(G\) is called elliptic \(\alpha\)-subnormal if \(r/p + s/q < 1\) and there exist a weighted incidence matrix \(B\) and \(\{w(e)\}\) satisfying

1. \(\sum_{e \in E(G)} w(e) \leq 1\);
2. \(\sum_{e: u \in T(e)} B(u, e) \leq 1\), \(\sum_{e: v \in H(e)} B(v, e) \leq 1\), for any \(u \in T(G), v \in H(G)\);
3. \(w(e)^{1-(r/p+s/q)} \prod_{u \in T(e)} (B(u, e))^{1/p} \cdot \prod_{v \in H(e)} (B(v, e))^{1/q} \geq \alpha\), for any \(e \in E(G)\).

Moreover, \(G\) is called strictly elliptic \(\alpha\)-subnormal if it is elliptic \(\alpha\)-subnormal but not elliptic \(\alpha\)-normal.

**Lemma 4.5** Let \(G\) be an \((r, s)\)-directed hypergraph. If \(G\) is elliptic \(\alpha\)-subnormal, then the \((p, q)\)-spectral radius of \(G\) satisfies

\[
\lambda_{p,q}(G) \leq \frac{1}{r^{1/p} s^{1/q} \alpha}.
\]

**Proof.** For any nonnegative vectors \(x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{S}_p^{m-1}\) and \(y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{S}_q^{n-1}\), by Generalized Hölder’s inequality and AM–GM inequality, we deduce that

\[
P_G(x, y) = \sum_{e \in E(G)} \left( \prod_{u \in T(e)} x_u \right) \left( \prod_{v \in H(e)} y_v \right)
\]

\[
\leq \frac{1}{\alpha} \sum_{e \in E(G)} \left( w(e)^{1-(r/p+s/q)} \cdot \prod_{u \in T(e)} (B(u, e))^{1/p} x_u \cdot \prod_{v \in H(e)} (B(v, e))^{1/q} y_v \right)
\]

\[
\leq \frac{1}{\alpha} \left( \sum_{e \in E(G)} \prod_{u \in T(e)} (B(u, e))^{1/r} x_u^{p/r} \right)^{r/p} \left( \sum_{e \in E(G)} \prod_{v \in H(e)} (B(v, e))^{1/s} y_v^{q/s} \right)^{s/q}
\]

\[
\leq \frac{1}{r^{1/p} s^{1/q} \alpha} \left( \sum_{e \in E(G)} \sum_{u \in T(e)} B(u, e) x_u^{p/r} \right)^{r/p} \left( \sum_{e \in E(G)} \sum_{v \in H(e)} B(v, e) y_v^{q/s} \right)^{s/q}
\]

\[
\leq \frac{1}{r^{1/p} s^{1/q} \alpha} \cdot ||x||_p \cdot ||y||_q = \frac{1}{r^{1/p} s^{1/q} \alpha},
\]

yielding \(\lambda_{p,q}(G) \leq r^{-r/p} s^{-s/q} \alpha^{-1}\). When \(G\) is strictly elliptic \(\alpha\)-subnormal, this inequality is strict, and therefore \(\lambda_{p,q}(G) < r^{-r/p} s^{-s/q} \alpha^{-1}\). \(\square\)

**Definition 4.7** An \((r, s)\)-directed hypergraph \(G\) is called elliptic \(\alpha\)-supernormal if \(r/p + s/q < 1\) and there exist a weighted incidence matrix \(B\) and \(\{w(e)\}\) satisfying

1. \(\sum_{e \in E(G)} w(e) \geq 1\);
\[
(2) \sum_{e: u \in T(e)} B(u, e) \geq 1, \quad \sum_{e: v \in H(e)} B(v, e) \geq 1, \text{ for any } u \in T(G), v \in H(G);
\]

\[
(3) w(e)^{1 - (r/p + s/q)} \prod_{u \in T(e)} (B(u, e))^{1/p} \prod_{v \in H(e)} (B(v, e))^{1/q} \leq \alpha, \text{ for any } e \in E(G).
\]

Moreover, \( G \) is called \textit{strictly elliptic} \( \alpha \)-\textit{supernormal} if it is elliptic \( \alpha \)-supernormal but not elliptic \( \alpha \)-normal.

\section*{Lemma 4.6}

\textit{Let} \( G \) \textit{be an} \( (r, s) \)-\textit{directed hypergraph. If} \( G \) \textit{is elliptic consistently} \( \alpha \)-\textit{supernormal, then the} \( (p, q) \)-\textit{spectral radius of} \( G \) \textit{satisfies}

\[
\lambda_{p,q}(G) \geq \frac{1}{r^{r/p} s^{s/q} \alpha}.
\]

\textit{Proof.} Define vectors \( x \in \mathbb{S}^{m-1}_{p,+} \) and \( y \in \mathbb{S}^{n-1}_{q,+} \) as follows:

\[
x_u = \left( \frac{w(e)}{r B(u, e)} \right)^{1/p}, \quad u \in T(e); \quad y_v = \left( \frac{w(f)}{s B(v, f)} \right)^{1/q}, \quad v \in H(f).
\]

The consistent conditions guarantee that \( x_u \) and \( y_v \) are independent of the choice of the arcs \( e \) and \( f \). Hence, we have

\[
\lambda_{p,q}(G) \geq P_G(x, y)
\]

\[
= \frac{1}{r^{r/p} s^{s/q} \alpha} \sum_{e \in E(G)} \prod_{u \in T(e)} (B(u, e))^{1/p} \prod_{v \in H(e)} (B(v, e))^{1/q}
\]

\[
\geq \frac{1}{r^{r/p} s^{s/q} \alpha \alpha} \sum_{e \in E(G)} w(e)
\]

\[
\geq \frac{1}{r^{r/p} s^{s/q} \alpha \alpha}.
\]

When \( G \) is strictly elliptic \( \alpha \)-supernormal, this inequality is strict, and therefore \( \lambda_{p,q}(G) > \frac{1}{r^{r/p} s^{s/q} \alpha - 1} \).

\section*{4.3. Hyperbolic phase: \( \frac{r}{p} + \frac{s}{q} > 1 \)}

Due to the fact that the Perron–Frobenius Theorem fails for general \( (r, s) \)-directed hypergraph \( G \) when \( r/p + s/q > 1 \), the theory is less effective than the case \( r/p + s/q \leq 1 \). However, we can still define the hyperbolic \( \alpha \)-normal for \( r/p + s/q > 1 \) as Definition 4.5, and prove the following result.

\textbf{Theorem 4.5} \textit{For} \( r/p + s/q > 1 \), \textit{and any} \( (r, s) \)-\textit{directed hypergraph} \( G \) \textit{with} \( (p, q) \)-\textit{spectral radius} \( \lambda_{p,q}(G) \), \textit{there exists an induced sub-dirhypergraph} \( G' \) \textit{of} \( B(G) \) \textit{such that} \( G' \) \textit{is hyperbolic consistently} \( \alpha \)-\textit{normal with} \( \alpha = (r^{r/p} s^{s/q} \lambda_{p,q}(G))^{-1} \).

Conversely, we have

\[
\lambda_{p,q}(G) = \frac{1}{r^{r/p} s^{s/q} \alpha} \max_i \{ \alpha_i^{-1} \},
\]

where the maximum is taken over all \( \alpha_i \) such that there is a hyperbolic consistent \( \alpha_i \)-normal labeling on some induced sub-dirhypergraph of \( B(G) \).
Proof. For short, denote $B := B(G)$. Assume that $(x, y) \in S_{p}^{m-1} \times S_{q}^{n-1}$ is an eigenpair corresponding to $\lambda_{p,q}(G)$. Let $S_1 := \{u \in T(G) : x_u > 0\}, S_2 := \{v \in H(G) : y_v > 0\}$. Consider the induced dirhypergraph $B[S_1 \cup S_2]$. By Lemma 3.3,

$$\lambda_{p,q}(G) = \lambda_{p,q}(B) = \lambda_{p,q}(B[S_1 \cup S_2]).$$

It can be proved that $G' = B[S_1 \cup S_2]$ is the desired induced sub-dirhypergraph. The proof is similar to Lemma 4.4.

Conversely, assume that $G_i$ is an induced sub-dirhypergraph of $B$, and $\{B(v, e)\}$ and $\{w(e)\}$ are hyperbolic consistent $\alpha$-normal labeling of $G_i$. Define vectors $x = (x_1, \ldots, x_m)^T \in S_{p}^{m-1}, y = (y_1, \ldots, y_n)^T \in S_{q}^{n-1}$ for $G$ as follows:

$$x_u = \begin{cases} \left( \frac{w(e)}{rB(u, e)} \right)^{1/p}, & \text{if } u \in T(e), e \in E(G_i), \\ 0, & \text{otherwise}, \end{cases}$$

$$y_v = \begin{cases} \left( \frac{w(f)}{sB(v, f)} \right)^{1/q}, & \text{if } v \in H(f), f \in E(G_i), \\ 0, & \text{otherwise}. \end{cases}$$

The consistent conditions guarantee that $x_u$ and $y_v$ are independent of the choice of the arcs $e$ and $f$. It follows that

$$\lambda_{p,q}(G) \geq P_G(x, y) = \frac{1}{r^{r/p}s^{s/q}} \sum_{e \in E(G)} \frac{w(e)^{r/p+s/q}}{\prod_{u \in T(e)} (B(u, e))^{1/p} \prod_{v \in H(e)} (B(v, e))^{1/q}} = \frac{1}{r^{r/p}s^{s/q}q\alpha_i} \sum_{e \in E(G)} w(e) = \frac{1}{r^{r/p}s^{s/q}q\alpha_i}.$$

Combining with the first part of this theorem, we have

$$\lambda_{p,q}(G) = \frac{1}{r^{r/p}s^{s/q}q\alpha_i} \max_i \{\alpha_i^{-1}\}.$$

The proof is completed. □

We also can define the hyperbolic $\alpha$-subnormal for $r/p + s/q > 1$ as Definition 4.3. According to the proof of Lemma 4.2 and (2.8), we still have the following result.

**Theorem 4.6** Let $G$ be an $(r, s)$-directed hypergraph. If $G$ is hyperbolic $\alpha$-subnormal, then the $(p, q)$-spectral radius of $G$ satisfies

$$\lambda_{p,q}(G) \leq \frac{1}{r^{r/p}s^{s/q}q\alpha_i}.$$
5. Applications

In this section, we shall give some applications of the $\alpha$-normal labeling method in the study of $(p, q)$-spectral radius. For short, we denote $\gamma(p, q) := 1 - (r/p + s/q)$ in this section.

5.1. Some degree based bounds

**Proposition 5.1** Let $G$ be an $(r, s)$-directed hypergraph with maximum out-degree $\Delta^+$ and maximum in-degree $\Delta^-$. 

(1) If $r/p + s/q \geq 1$, then

$$\lambda_{p,q}(G) \leq \left(\frac{\Delta^+}{r}\right)^{r/p} \left(\frac{\Delta^-}{s}\right)^{s/q}.$$ 

(2) If $r/p + s/q < 1$, then

$$\lambda_{p,q}(G) \leq |G|^{1-(r/p+s/q)} \left(\frac{\Delta^+}{r}\right)^{r/p} \left(\frac{\Delta^-}{s}\right)^{s/q}.$$ 

**Proof.** (1). Assume $r/p + s/q \geq 1$. Without loss of generality, we can assume $G$ is anadiplosis connected. Otherwise, we consider an anadiplosis connected component instead. Construct a weighted incidence matrix $B = (B(v, e))$ for $G$ as follows:

$$B(v, e) = \begin{cases} 
1/\Delta^+, & \text{if } v \in T(e), \\
1/\Delta^-, & \text{if } v \in H(e), \\
0, & \text{otherwise.}
\end{cases}$$

For any $u \in T(G), v \in H(G)$, we see that

$$\sum_{e: u \in T(e)} B(u, e) \leq 1, \quad \sum_{e: v \in H(e)} B(v, e) \leq 1.$$ 

For any arc $e \in E(G)$,

$$\prod_{u \in T(e)} (B(u, e))^{1/p} \cdot \prod_{v \in H(e)} (B(v, e))^{1/q} \geq \frac{1}{(\Delta^+)^{r/p}(\Delta^-)^{s/q}}.$$ 

Using Lemma 4.2 and Theorem 4.6 gives

$$\lambda_{p,q}(G) \leq \frac{1}{r^{r/p}s^{s/q}\gamma} = \left(\frac{\Delta^+}{r}\right)^{r/p} \left(\frac{\Delta^-}{s}\right)^{s/q}.$$ 

(2). When $r/p + s/q < 1$, we define a weighted incidence matrix $B = (B(v, e))$ and $\{w(e)\}$ for $G$ as follows:

$$B(v, e) = \begin{cases} 
1/\Delta^+, & \text{if } v \in T(e), \\
1/\Delta^-, & \text{if } v \in H(e), \\
0, & \text{otherwise.}
\end{cases}$$
\[ w(e) = 1/|G|. \]

It can be checked that \( G \) is elliptic \( \alpha \)-subnormal with
\[ \alpha = \frac{1}{|G|^\gamma(p,q)(\Delta^+)^{r/p}(\Delta^-)^{s/q}}. \]

According to Lemma 4.5, we have
\[ \lambda_{p,q}(G) \leq \frac{1}{r^{r/p}s^{s/q}\alpha} = |G|^\gamma(p,q) \left( \frac{\Delta^+}{r} \right)^{r/p} \left( \frac{\Delta^-}{s} \right)^{s/q}. \]

The proof is completed. \( \square \)

**Proposition 5.2** Let \( G \) be an \((r, s)\)-directed hypergraph.

1. If \( r/p + s/q \geq 1 \), then
\[ \lambda_{p,q}(G) \leq \frac{1}{r^{r/p}s^{s/q}} \max_{e \in E(G)} \left\{ \prod_{u \in T(e)} \left( d_u^+ \right)^{1/p} \prod_{v \in H(e)} \left( d_v^- \right)^{1/q} \right\}. \]

2. If \( r/p + s/q < 1 \), then
\[ \lambda_{p,q}(G) \leq \frac{|G|^{1-(r/p+s/q)}}{r^{r/p}s^{s/q}} \max_{e \in E(G)} \left\{ \prod_{u \in T(e)} \left( d_u^+ \right)^{1/p} \prod_{v \in H(e)} \left( d_v^- \right)^{1/q} \right\}. \]

**Proof.** (1). Assume \( r/p + s/q \geq 1 \). Without loss of generality, we can assume \( G \) is anadiplosis connected. Otherwise, we consider an anadiplosis connected component instead. We construct a weighted incidence matrix \( B = (B(v, e)) \) for \( G \) as follows:
\[ B(v, e) = \begin{cases} 1/d_v^+, & \text{if } v \in T(e), \\ 1/d_v^-, & \text{if } v \in H(e), \\ 0, & \text{otherwise.} \end{cases} \]

Clearly, for any \( u \in T(G), v \in H(G) \), we see that
\[ \sum_{e: u \in T(e)} B(u, e) = \sum_{e: v \in H(e)} B(v, e) = 1. \]

For any arc \( e \in E(G) \),
\[ \prod_{u \in T(e)} (B(u, e))^{1/p} \cdot \prod_{v \in H(e)} (B(v, e))^{1/q} \geq \frac{1}{\max_{e \in E(G)} \left\{ \prod_{u \in T(e)} \left( d_u^+ \right)^{1/p} \prod_{v \in H(e)} \left( d_v^- \right)^{1/q} \right\}}. \]

By Lemma 4.2 and Theorem 4.6, we have
\[ \lambda_{p,q}(G) \leq \frac{1}{r^{r/p}s^{s/q}\alpha} = \frac{1}{r^{r/p}s^{s/q}} \max_{e \in E(G)} \left\{ \prod_{u \in T(e)} \left( d_u^+ \right)^{1/p} \prod_{v \in H(e)} \left( d_v^- \right)^{1/q} \right\}. \]
(2). Assume \( r/p + s/q < 1 \), we define a weighted incidence matrix \( B = (B(v, e)) \) and \( \{w(e)\} \) for \( G \) as follows:

\[
B(v, e) = \begin{cases} 
1/d_v^+, & \text{if } v \in T(e), \\
1/d_v^-, & \text{if } v \in H(e), \\
0, & \text{otherwise.}
\end{cases}
\]

\[ w(e) = 1/|G|. \]

It can be checked that \( G \) is elliptic \( \alpha \)-subnormal with

\[
\alpha = \frac{1}{|G|^\gamma(p,q)} \max_{e \in E(G)} \left\{ \prod_{u \in T(e)} (d_u^+)^{1/p} \prod_{v \in H(e)} (d_v^-)^{1/q} \right\}.
\]

By Lemma 4.5 we have

\[
\lambda_{p,q}(G) \leq \frac{1}{r^r/p^s/q^{\alpha}} |G|^\gamma(p,q) \max_{e \in E(G)} \left\{ \prod_{u \in T(e)} (d_u^+)^{1/p} \prod_{v \in H(e)} (d_v^-)^{1/q} \right\}.
\]

The proof is completed.

\[ \square \]

5.2. Monotonicity and convexity of \( \lambda_{p,q}(G) \)

In this subsection, we consider \( \lambda_{p,q}(G) \) as a function of \( p, q \) for a fixed \((r, s)\)-directed hypergraph \( G \), and study some properties of the function \( \lambda_{p,q}(G) \).

**Theorem 5.1** Let \( G \) be an \((r, s)\)-directed hypergraph with \( r/p + s/q < 1 \). Then the function \((r|G|)^{r/p}|G|^s/q\lambda_{p,q}(G)\) is non-increasing in both \( p \) and \( q \).

**Proof.** Assume that \( G \) is elliptic consistently \( \alpha \)-normal with weighted incidence matrix \( B \) and weights \( \{w(e)\} \) for \( \lambda_{p,q}(G) \). Let \( p < p' \). We define a weighted incidence matrix \( B' \) and \( \{w'(e)\} \) for \( \lambda_{p',q}(G) \) as follows:

\[
B'(v, e) = \begin{cases} 
(B(v, e))^{p'/p}, & \text{if } v \in T(e), \\
B(v, e), & \text{if } v \in H(e),
\end{cases}
\]

\[ w'(e) = \frac{w(e)^\gamma(p,q)/\gamma(p',q)}{|G|^{r/p^r/q^{p'/q}}}. \]

In what follows, we shall prove that \( \{B'(v, e)\} \) and \( \{w'(e)\} \) are elliptic \( \alpha' \)-subnormal labeling for \( \lambda_{p',q}(G) \) with \( \alpha' = \alpha |G|^{r/p' - r/p}. \)

(i). Using Hölder’s inequality gives

\[
\sum_{e \in E(G)} w'(e) = |G|^{-r/p + s/q} \sum_{e \in E(G)} w(e)^{\gamma(p,q)/\gamma(p',q)} \leq 1.
\]
(ii) For any $u \in T(G)$ and $v \in H(G)$, we have
\[
\sum_{e: u \in T(e)} B'(u, e) = \sum_{e: u \in T(e)} (B(u, e))^{p'/p} \leq \sum_{e: u \in T(e)} B(u, e) = 1
\]
and
\[
\sum_{e: v \in H(e)} B'(v, e) = \sum_{e: v \in H(e)} B(v, e) = 1.
\]

(iii) For each arc $e$, we have
\[
w'(e)^{\gamma(p') \prod_{u \in T(e)} (B'(u, e))^{1/p'} \prod_{v \in H(e)} (B'(v, e))^{1/q}} = \frac{w(e)^{\gamma(p) \prod_{u \in T(e)} (B(u, e))^{1/p} \prod_{v \in H(e)} (B(v, e))^{1/q}}}{|G|^{r/p-r/p'}} = \frac{\alpha}{|G|^{r/p-r/p'}}.
\]
Hence, $G$ is elliptic $\alpha'$-subnormal for $\lambda_{p', q}(G)$ with $\alpha' = \alpha|G|^{r/p-r/p'}$. It follows from Lemma 4.5 that
\[
\lambda_{p', q}(G) \leq \frac{1}{r^{r/p}s/q\alpha'} = (r|G|)^{r/p-r/p'} \lambda_{p, q}.
\]
Therefore, we obtain
\[
(r|G|)^{r/p'} \lambda_{p', q}(G) \leq (r|G|)^{r/p} \lambda_{p, q}(G).
\]
Similarly, for $q' > q$, we can prove that
\[
(s|G|)^{s/q} \lambda_{p', q}(G) \leq (s|G|)^{s/q} \lambda_{p, q}(G).
\]
Thus, for any $p' > q$ and $q' > q$, we have
\[
(r|G|)^{r/p'} (s|G|)^{s/q} \lambda_{p', q}(G) \leq (r|G|)^{r/p'} (s|G|)^{s/q} \lambda_{p', q}(G) \leq (r|G|)^{r/p} (s|G|)^{s/q} \lambda_{p, q}(G).
\]
The proof is completed. \hfill \Box

**Lemma 5.1** Let $G$ be an $(r, s)$-directed hypergraph with $r/p + s/q < 1$. Suppose that $G$ is elliptic consistently $\alpha$-normal with weights $\{w(e)\}$. Then
\[
[\alpha(\delta^+)^{r/p}(\delta^-)^{s/q}]^{1/[1-(r/p+s/q)]} \leq w(e) \leq [\alpha(\Delta^+)^{r/p}(\Delta^-)^{s/q}]^{1/[1-(r/p+s/q)]}.
\]

**Proof.** The consistent conditions in Definition 4.5 imply that
\[
B(u, e) = \begin{cases} \frac{w(e)}{\sum_{f: u \in T(f)} w(f)}, & \text{if } u \in T(e), \\ \frac{w(e)}{\sum_{f: u \in H(f)} w(f)}, & \text{if } u \in H(e). \end{cases}
\]
By item (3) in Definition 4.5, we obtain

\[ w(e) = \alpha \prod_{u \in T(e)} \left( \sum_{f : u \in T(f)} w(f) \right)^{1/p} \cdot \prod_{v \in H(e)} \left( \sum_{f : v \in H(f)} w(f) \right)^{1/q} \tag{5.1} \]

Without loss of generality, assume \( w(e_1) = \min \{ w(e) : e \in E(G) \} \), and \( w(e_2) = \max \{ w(e) : e \in E(G) \} \). Using equation (5.1) gives

\[
\begin{align*}
  w(e_1) & \geq \alpha[(\delta^+)w(e_1)]^{r/p} \cdot [(\delta^-)w(e_1)]^{s/q} \\
  & = \alpha(\delta^+)^{r/p}(\delta^-)^{s/q} \cdot w(e_1)^{r+s/q},
\end{align*}
\]

which follows that

\[ w(e_1) \geq [\alpha(\delta^+)^{r/p}(\delta^-)^{s/q}]^{1/(p,q)}. \]

Similarly, we can prove the right side. \( \Box \)

**Theorem 5.2** Suppose that \( G \) is an \((r,s)\)-directed hypergraph with \( r/p + s/q < 1 \). Let

\[
f_G(x) := \left( \frac{r}{\Delta^+} \right)^{r/(px)} \left( \frac{s}{\Delta^-} \right)^{s/(qx)} \lambda_{px,qx}(G) \frac{1}{1-(r/(px)+s/(qx))},
\]

and

\[
g_G(x) := \left( \frac{r}{\delta^+} \right)^{r/(px)} \left( \frac{s}{\delta^-} \right)^{s/(qx)} \lambda_{px,qx}(G) \frac{1}{1-(r/(px)+s/(qx))},
\]

then \( f_G(x) \) is non-decreasing on \((r/p + s/q, \infty)\) while \( g_G(x) \) is non-increasing on \((r/p + s/q, \infty)\).

**Proof.** For any \( x_1 > r/p + s/q \), let \( G \) be elliptic consistently \( \alpha_1 \)-normal with weighted incidence matrix \( B_1 \) and weights \( \{w_1(e)\} \) for \( \lambda_{px_1,qx_1}(G) \). Therefore

\[
\begin{align*}
  \sum_{e \in E(G)} w_1(e) &= 1, \\
  \sum_{e : u \in T(e)} B_1(u,e) &= \sum_{e : v \in H(e)} B_1(v,e) = 1, \quad u \in T(G), \ v \in H(G), \quad \sum_{e \in E(G)} \left( B_1(u,e) \right)^{1/(px_1)} \cdot \prod_{v \in H(e)} \left( B_1(v,e) \right)^{1/(qx_1)} = \alpha_1.
\end{align*}
\]

Let \( x_2 > x_1 \). We now define a weighted incidence matrix \( B_2 \) and \( \{w_2(e)\} \) for \( \lambda_{px_2,qx_2}(G) \) as follows:

\[
B_2(v,e) = B_1(v,e), \quad w_2(e) = w_1(e).
\]

It is clear that

\[
\sum_{e \in E(G)} w_2(e) = \sum_{e \in E(G)} w_1(e) = 1.
\]

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We also have
\[ \sum_{e: u \in T(e)} B_2(u, e) = 1, \quad \sum_{e: v \in H(e)} B_2(v, e) = 1. \]

Using Lemma 4.4 gives
\[ w_2(e)^{\gamma(p_{x_2,q_{x_2}})} \prod_{u \in T(e)} (B_2(u, e))^{1/(p_{x_2})} \prod_{v \in H(e)} (B_2(v, e))^{1/(q_{x_2})} \]
\[ = w_1(e)^{\gamma(p_{x_2,q_{x_2}})} \prod_{u \in T(e)} (B_1(u, e))^{1/(p_{x_2})} \prod_{v \in H(e)} (B_1(v, e))^{1/(q_{x_2})} \]
\[ = w_1(e)^{1-x_1/x_2} \cdot \alpha_1^{x_1/x_2} \]
\[ \leq \alpha_1^{(p_{x_2,q_{x_2}})/(p_{x_1,q_{x_1}})} \left( (\Delta^+)^{r/(p_{x_1})} (\Delta^-)^{s/(q_{x_1})} \right)^{1-x_1/x_2}. \]

Therefore, \( G \) is elliptic consistently \( \alpha_2 \)-supernormal for \( \lambda_{p_{x_2,q_{x_2}}}(G) \) with
\[ \alpha_2 = \alpha_1^{(p_{x_2,q_{x_2}})/(p_{x_1,q_{x_1}})} \left( (\Delta^+)^{r/(p_{x_1})} (\Delta^-)^{s/(q_{x_1})} \right)^{1-x_1/x_2}. \]

According to Lemma 4.6 and \( (\alpha_1)^{-1} = r^{r/(p_{x_1})} s^{s/(q_{x_1})} \lambda_{p_{x_1,q_{x_1}}}(G) \), we see
\[ \lambda_{p_{x_2,q_{x_2}}}(G) \geq \frac{1}{r^{r/(p_{x_2})} s^{s/(q_{x_2})} \alpha_2} \]
\[ = \frac{r^{r/(p_{x_1})} s^{s/(q_{x_1})} \lambda_{p_{x_1,q_{x_1}}}(G)}{r^{r/(p_{x_2})} s^{s/(q_{x_2})} (\Delta^+)^{r/(p_{x_1})} (\Delta^-)^{s/(q_{x_1})}} \]
\[ \leq \left[ \left( \frac{r}{\Delta^+} \right)^{r/(p_{x_1})} \left( \frac{s}{\Delta^-} \right)^{s/(q_{x_1})} \right] \left( \frac{p_{x_2,q_{x_2}}}{p_{x_1,q_{x_1}}} \right)^{1-x_1/x_2}, \]
which implies that \( f_G(x) \) is non-decreasing in \( x \). Similarly, we can prove that \( g_G(x) \) is non-increasing on \( (r/p + s/q, \infty) \). \( \square \)

Theorem 5.3 For any \((r,s)\)-directed hypergraph \( G \) with \( r/p + s/q < 1 \), the function \( pq \log(\lambda_{p,q}(G)) \) is concave upward in \( p \) (and in \( q \)).

Proof. For any \( p_1 < p < p_2 \), write \( p = \mu p_1 + (1 - \mu) p_2 \), where \( \mu = (p_2 - p)/(p_2 - p_1) \).

Let \( G \) be elliptic consistently \( \alpha_i \)-normal with weighted incident matrix \( B_i \) and \( \{ w_i(e) \} \) for \( \lambda_{p_i,q}(G) \), \( i = 1, 2 \).

We define a weighted incidence matrix \( B \) and \( \{ w(e) \} \) for \( \lambda_{p,q}(G) \) as follows:
\[ B(u,e) = \mu B_1(u,e) + (1 - \mu) B_2(u,e), \text{ if } u \in T(e), \]
\[ B(v,e) = \eta B_1(v,e) + (1 - \eta) B_2(v,e), \text{ if } v \in H(e), \]
\[ w(e) = \xi w_1(e) + (1 - \xi) w_2(e), \]
where
\[ \eta = \frac{p_1}{p} \mu, \quad \xi = \frac{p_1 q - (rq + sp_1)}{pq - (rq + sp)} \mu. \]
For any vertices $u \in T(G), v \in H(G)$, we have
\[
\sum_{e: u \in T(e)} B(u, e) = \mu \sum_{e: u \in T(e)} B_1(u, e) + (1 - \mu) \sum_{e: u \in T(e)} B_2(u, e)
= \mu + (1 - \mu) = 1.
\]

Also, we have
\[
\sum_{e: v \in H(e)} B(v, e) = \eta \sum_{e: v \in H(e)} B_1(u, e) + (1 - \eta) \sum_{e: v \in H(e)} B_2(v, e)
= \eta + (1 - \eta) = 1.
\]

For each arc $e \in E(G)$, it follows from Young’s inequality that
\[
\left( \frac{w(e)^{1-(r/p+s/q)}}{w(e)^{1/p}} \prod_{u \in T(e)} \left( B(u, e) \right)^{1/p} \prod_{v \in H(e)} \left( B(v, e) \right)^{1/q} \right)^{pq}
\geq w_1(e)^{[pq-(rq+sp)]} \prod_{u \in T(e)} \left( B_1(u, e) \right)^{\mu q} \prod_{v \in H(e)} \left( B_1(v, e) \right)^{\eta p}
\times w_2(e)^{(1-\xi)[pq-(rq+sp)]} \prod_{u \in T(e)} \left( B_2(u, e) \right)^{(1-\mu)q} \prod_{v \in H(e)} \left( B_2(v, e) \right)^{(1-\eta)p}
\]
\[
= (\alpha_1)^{p_1 q \mu} (\alpha_2)^{p_2 q (1-\mu)}.
\]

Hence, $G$ is elliptic $\alpha$-subnormal for $\lambda_{p,q}(G)$ with $\alpha^{pq} = (\alpha_1)^{p_1 q \mu} (\alpha_2)^{p_2 q (1-\mu)}$. Using Lemma 4.5 gives
\[
pq \log (\lambda_{p,q}(G)) \leq - \log (r^{rq} s^{sp} \alpha^{pq})
= - \log (r^{rq} s^{sp} \alpha_1^{p_1 q \mu} \alpha_2^{p_2 q (1-\mu)})
= - \log (r^{rq} s^{sp} - p_1 q \mu \log \alpha_1 - p_2 q (1-\mu) \log \alpha_2)
= \mu p_1 q \log (\lambda_{p_1,q}(G)) + (1 - \mu) p_2 q \log (\lambda_{p_2,q}(G))
\]
which implies that the function $pq \log (\lambda_{p,q}(G))$ is concave upward in $p$. Similarly, we can prove that $pq \log (\lambda_{p,q}(G))$ is also concave upward in $q$. \hfill \square

**Theorem 5.4** For any $(r, s)$-directed hypergraph $G$ with $r/p + s/q < 1$, the function
\[
h_G(1/p, 1/q) := \log (\lambda_{p,q}(G))
\]
is concave upward in $1/p$ and $1/q$.

**Proof.** According to Lemma 4.4, let $G$ be elliptic consistently $\alpha_i$-normal with weighted incident matrix $B_i$ and $\{w_i(e)\}$ for $\lambda_{p_i,q_i}(G)$, where $(\alpha_i)^{-1} = r^{rq} s^{sp} \lambda_{p_i,q_i}(G)$, $i = 1, 2$.

For any $(1/p, 1/q)$, write
\[
\left( \frac{1}{p}, \frac{1}{q} \right) = \mu \left( \frac{1}{p_1}, \frac{1}{q_1} \right) + (1 - \mu) \left( \frac{1}{p_2}, \frac{1}{q_2} \right),
\]

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where
\[ \mu = \frac{p_1(p_2 - p)}{p(p_2 - p_1)} = \frac{q_1(q_2 - q)}{q(q_2 - q_1)}. \]
Furthermore, let
\[ \mu_1 = \frac{p}{p_1}, \mu_2 = \frac{q}{q_1}, \xi = \gamma \left( \frac{p_1}{p}, \frac{q_1}{q} \right) \mu. \]
We define a weighted incidence matrix \( B \) and \( \{ w(e) \} \) for \( \lambda_{p,q}(G) \) as follows:
\[
B(v, e) = \begin{cases} 
\mu_1B_1(v, e) + (1 - \mu_1)B_2(v, e), & \text{if } v \in T(e), \\
\mu_2B_1(v, e) + (1 - \mu_2)B_2(v, e), & \text{if } v \in H(e), \\
\end{cases}
\]
\[
w(e) = \xi w_1(e) + (1 - \xi)w_2(e).
\]
It can be checked that \( G \) is elliptic \( \alpha \)-subnormal for \( \lambda_{p,q}(G) \) with \( \alpha = (\alpha_1)^{\mu}(\alpha_2)^{1-\mu} \). By Lemma 4.5, we have
\[
\log \left( \lambda_{p,q}(G) \right) = \log \left( r^{r/p} s^{s/q} \lambda_{p,q}(G) \right) - \frac{r \log r}{p} - \frac{s \log s}{q} \\
\leq - \log \alpha - \frac{r \log r}{p} - \frac{s \log s}{q} \\
= - (\mu \log \alpha_1 + (1 - \mu) \log \alpha_2) - \frac{r \log r}{p} - \frac{s \log s}{q} \\
= \mu \log \left( r^{r/p_1} s^{s/q_1} \lambda_{p_1,q_1}(G) \right) + (1 - \mu) \log \left( r^{r/p_2} s^{s/q_2} \lambda_{p_2,q_2}(G) \right) \\
- \frac{r \log r}{p} - \frac{s \log s}{q} \\
= \mu \log \left( \lambda_{p_1,q_1}(G) \right) + (1 - \mu) \log \left( \lambda_{p_2,q_2}(G) \right).
\]
Thus the function \( h_G(1/p, 1/q) \) is concave upward in \( 1/p \) and \( 1/q \).
\[ \square \]
Corollary 5.1 For any \( (r, s) \)-directed hypergraph \( G \) with \( r/p + s/q < 1 \), the function \( \log \left( \lambda_{px,qx}(G) \right) \) is concave upward in \( 1/x \) on the interval \( (r/p + s/q, \infty) \).

Theorem 5.5 Let \( G \) be an \( (r, s) \)-directed hypergraph and \( r/p + s/q < 1 \). Then the function \( x \log \left( \lambda_{px,qx}(G) \right) \) is concave upward in \( x \) on the interval \( (r/p + s/q, \infty) \).

Proof. For any \( x_2 > x_1 > r/p + s/q \), let \( G \) be elliptic consistently \( \alpha_i \)-normal with weighted incident matrix \( B_i \) and \( \{ w_i(e) \} \) for \( \lambda_{px_i,qx_i}(G), i = 1, 2 \).

For \( x > r/p + s/q \), write \( x = \mu x_1 + (1 - \mu)x_2 \), where \( \mu = (x_2 - x)/(x_2 - x_1) \). We define a weighted incidence matrix \( B \) and \( \{ w(e) \} \) for \( \lambda_{px,qx}(G) \) as follows:
\[
B(v, e) = \mu B_1(v, e) + (1 - \mu)B_2(v, e), \ w(e) = \xi w_1(e) + (1 - \xi)w_2(e),
\]
where
\[
\xi = \frac{\mu x_1 \gamma(px_1, qx_1)}{x - (r/p + s/q)}.
\]
By some simple computation, we have
\[
\left[ w(e)^{\gamma(px,qx)} \prod_{u \in T(e)} (B(u,e))^{1/p} \prod_{v \in H(e)} (B(v,e))^{1/q} \right]^x
= w(e)^{x-(r/p+s/q)} \prod_{u \in T(e)} (B(u,e))^{1/p} \prod_{v \in H(e)} (B(v,e))^{1/q} \\
\geq w_1(e)^{x-(r/p+s/q)} \prod_{u \in T(e)} (B_1(u,e))^{\mu/p} \prod_{v \in H(e)} (B_1(v,e))^{\mu/q} \\
\times w_2(e)^{(1-x)(r+s/p+q)} \prod_{u \in T(e)} (B_2(u,e))^{(1-\mu)/p} \prod_{v \in H(e)} (B_2(v,e))^{(1-\mu)/q} \\
= (\alpha_1)^{\mu x_1} (\alpha_2)^{(1-\mu)x_2}.
\]

Hence, $G$ is elliptic $\alpha$-subnormal for $\lambda_{px,qx}(G)$ with $\alpha^x = (\alpha_1)^{\mu x_1} (\alpha_2)^{(1-\mu)x_2}$. It follows from Lemma 4.5 that
\[
x \log (\lambda_{px,qx}(G)) \leq \mu x_1 \log (\lambda_{px1,qx1}(G)) + (1 - \mu)x_2 \log (\lambda_{px2,qx2}(G)).
\]
The proof is completed. \hfill \Box

\section{5.3. Miscellaneous results}

The following theorem establish an relation of spectral radius between $G$ and the underlying of $B(G)$.

\textbf{Theorem 5.6} Let $G$ be an $(r,s)$-directed hypergraph with $r/p + s/q = 1$. Suppose that $G$ is the underlying hypergraph of $B(G)$, and $\rho(G)$ is the spectral radius of $G$.

1. If $p \leq q$, then
   \[
   \lambda_{p,q}(G) \leq \frac{1}{r^q/p^s} (\rho(G))^{(r+s)/p};
   \]
2. If $p > q$, then
   \[
   \lambda_{p,q}(G) \leq \frac{1}{r^q/p^s} (\rho(G))^{(r+s)/q};
   \]
3. If $p = q = r + s$, then
   \[
   \lambda_{p,q}(G) = \frac{1}{r^q/r^s} \rho(G).
   \]

\textbf{Proof.} By Lemma 3.3, we may assume that $G$ is anadiplosis connected. According to Theorem 4.1, let $B = (B(v,\overline{v}))$ be the weighted incidence matrix of $G$ satisfying
1. $\sum_{v \in V} B(v,\overline{v}) = 1$, for any $v \in V(G)$;
2. $\prod_{v \in V} B(v,\overline{v}) = \overline{\alpha} = (\rho(G))^{-(r+s)}$, for any $\overline{v} \in E(G)$;
3. $\prod_{i=1}^{\ell} B(v_{i-1},e_i) = 1$, for any cycle $v_0 e_1 v_1 e_2 \cdots v_{\ell - 1} e_{\ell} (v_\ell = v_0)$.

Now we define a weighted incidence matrix $B = (B(v,e))$ for $G$ as
\[
B(v,e) = B(v,e), \text{ for any } e \in E(G).
\]
Clearly, for any \( u \in T(G), \ v \in H(G) \),
\[
\sum_{e: u \in T(e)} B(u, e) = \sum_{v \in H(e)} B(v, e) = 1, \quad \sum_{e: v \in H(e)} B(v, e) = \sum_{v \in H(e)} B(v, e) = 1.
\]

Also, for any \( e \in E(G) \),
\[
\prod_{u \in T(e)} (B(u, e))^{1/p} \cdot \prod_{v \notin H(e)} (B(v, e))^{1/q} \geq \begin{cases} 
(\overline{G})^{1/p}, & \text{if } p \leq q, \\
(\overline{G})^{1/q}, & \text{if } p > q.
\end{cases}
\]

If \( p \leq q \), \( G \) is parabolic \((\overline{G})^{1/p}\)-subnormal. By Lemma 4.2 we have
\[
\lambda_{p,q}(G) \leq \frac{1}{r^s p G(\overline{G})^{1/p}} = \frac{1}{r^s p G(\overline{G})^{1/q}}.
\]

If \( p > q \), \( G \) is parabolic \((\overline{G})^{1/q}\)-subnormal. By Lemma 4.2 we conclude that
\[
\lambda_{p,q}(G) \leq \frac{1}{r^s p G(\overline{G})^{1/q}} = \frac{1}{r^s p G(\overline{G})^{1/q}}.
\]

Let \( p = q = r + s \). By Definition 4.2, equation (5.2) is a parabolic consistent \((\overline{G})^{1/(r+s)}\)-normal labeling of \( G \). Therefore \( \lambda_{p,q}(G) = (r^s s)^{-1/(r+s)} \rho(G) \). \( \square \)

Let \( G = (V, E) \) be an \((r, s)\)-directed hypergraph. For each \( u \in V \) (and \( e \in E \)), let \( V_u \) (and \( T_e, H_e \)) be a new vertex set with \( k \) (and \( a, b \)) elements such that all these new sets are pairwise disjoint. Then the \emph{power} of \( G \), denoted by \( G(k; a, b) \), is defined as the \((kr + a, ks + b)\)-directed hypergraph with the vertex set
\[
V(G(k; a, b)) = \left( \bigcup_{u \in V} V_u \right) \bigcup_{e \in E} \left( \bigcup_{u \in T(e)} V_u \bigcup T_e \right) \bigcup_{u \in H(e)} \left( V_u \bigcup H_e \right).
\]

and arc set
\[
E(G(k; a, b)) = \left\{ \overline{e} = \left( \bigcup_{u \in T(e)} V_u \bigcup T_e, \bigcup_{u \in H(e)} V_u \bigcup H_e \right) : e \in E(G) \right\}.
\]

**Theorem 5.7** Let \( G \) be an \((r, s)\)-directed hypergraph, and \( G(k; a, b) \) be the power of \( G \) with \( as = br \). Then
\[
\rho(G(k; a, b)) = \frac{(\sqrt{k} s \rho(G) (kr + a))}{\sqrt{(kr + a)(ks + b)}}.
\]

**Proof.** Assume that \( B = (B(u, e)) \) is the parabolic consistent \( \alpha \)-normal labeling of \( G \). Now define a weighted incidence matrix \( B' \) for \( G(k; a, b) \) as follows:
\[
B'(v, e) = \begin{cases} 
B(u, e), & \text{if } v \in V_u \text{ for some } u \in e, \\
1, & \text{if } v \in T_e \bigcup H_e, \\
0, & \text{otherwise.}
\end{cases}
\]

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Clearly, for any vertex \( v \in V(G(k; a, b)) \),
\[
\sum_{\tilde{e}: v \in T(\tilde{e})} B'(v, \tilde{e}) = \sum_{\tilde{e}: v \in H(\tilde{e})} B(v, \tilde{e}) = 1.
\]

Notice that \( as = br \). Therefore, \( G(k; a, b) \) is parabolic consistently \( \alpha' \)-normal with
\[
\alpha' = \prod_{v \in T(\tilde{e})} (B'(v, \tilde{e}))^{1/(kr+a)} \cdot \prod_{v \in H(\tilde{e})} (B'(v, \tilde{e}))^{1/(ks+b)} = \alpha^{kr/(kr+a)}.
\]

It follows from Lemma 4.1 that
\[
\rho(G(k; a, b)) = \frac{1}{\sqrt{(kr+a)(ks+b)} \alpha'} = \frac{1}{\sqrt{(kr+a)(ks+b)} \alpha^{kr/(kr+a)}} = \frac{(\sqrt{rs} \rho(G))^{kr/(kr+a)}}{\sqrt{(kr+a)(ks+b)}}.
\]

The proof is completed. \( \Box \)

6. Concluding remarks

In this paper, we establish an initial spectral theory of directed hypergraphs by introducing the \((p, q)\)-spectral radius \( \lambda_{p,q}(G) \) for an \((r, s)\)-directed hypergraph \( G \). More precisely, we present some properties of \( \lambda_{p,q}(G) \), and develop a simple method for calculating \( \lambda_{p,q}(G) \) via weighted incident matrix, as well as for comparing the \( \lambda_{p,q}(G) \) with a particular value. The main results of this paper are focus on general \( p, q \geq 1 \). It is interesting to consider the case \( p = 2r, q = 2s \), in which case the statements are concise and nontrivial. That would be our next topic to investigate.

For directed graphs, it is known that there are several different matrices associated to a directed graph \( G \) to capture the adjacency of the directed graph. One candidate is the adjacency matrix \( A(G) \), which is not symmetric. The \((i,j)\)-entry of \( A(G) \) is 1 if there is an arc from the vertex \( i \) to \( j \), and 0 otherwise (see more in [5]). Another candidate is the skew-symmetric adjacency matrix, where the \((i,j)\)-entry is 1 if there is an arc from \( i \) to \( j \), and \(-1\) if there is an arc from \( j \) to \( i \) (and 0 otherwise) [6]. Recently, the Hermitian adjacency matrix \( H(G) \) is introduced by Guo and Mohar [11], and independently by Liu and Li [19]. The \((i,j)\)-entry \( h_{ij} \) of \( H(G) \) is given by

\[
h_{ij} = \begin{cases} 
1, & \text{if } (i,j) \in E(G) \text{ and } (j,i) \in E(G), \\
i, & \text{if } (i,j) \in E(G) \text{ and } (j,i) \notin E(G), \\
-i, & \text{if } (i,j) \notin E(G) \text{ and } (j,i) \in E(G), \\
0, & \text{otherwise},
\end{cases}
\]
where $i$ is the imaginary unit. This paper provides a new direction to study the spectral properties of directed graphs, which have a great relationship with the anadiplosis connectedness of directed graphs. It would be an interesting topic to study the spectrum of a directed graph via the singular values of its adjacency matrix $A$ in Definition 2.2 or equivalently the nonnegative eigenvalues of the following block matrix

$$
\begin{pmatrix}
0 & A \\
A^T & 0
\end{pmatrix}.
$$

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