Formal Concept Analysis with Many-sorted Attributes

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KEYWORDS: formal context, concept lattice, network of constraints, distributed relation, satisfaction

Abstract

This paper unites two problem-solving traditions in computer science: (1) constraint-based reasoning; and (2) formal concept analysis. For basic definitions and properties of networks of constraints, we follow the foundational approach of Montanari and Rossi [3]. This paper advocates distributed relations as a more semantic version of networks of constraints. The theory developed here uses the theory of formal concept analysis, pioneered by Rudolf Wille and his colleagues [5], as a key for unlocking the hidden semantic structure within distributed relations. Conversely, this paper offers distributed relations as a seamless many-sorted extension to the formal contexts of formal concept analysis. Some of the intuitions underlying our approach were discussed in a preliminary fashion by Freuder and Wallace [1].

Introduction

The fundamental model-theoretic structures in this paper are distributed relations. Distributed relations are identified with the constraint systems of object-oriented constraint languages, which play the roles of code and data abstractions, and subsume classes, instance variables and methods [2]. Object state is defined by solving constraints. Constraints can specify various consistency requirements of objects. External hierarchical structure is definable by summation of constraints. Internal hierarchical structure is definable via formal concept analysis in terms of the concept lattice of associated formal satisfaction contexts. Formal contexts are shown to be equivalent to single-sorted distributed relations. The formal satisfaction context of any distributed relation is the inverse image of the relation along the projection domain morphism. Concept lattices form a hierarchical clustering of objects. This hierarchy represents all implications between constraints (an intensional logic).

The knowledge representation research group here at the University of Arkansas is currently using formal concept analysis in the area of natural language modeling. More specifically, we are using the conceptual hierarchies of concept lattices to investigate and represent whole language semantic space as incorporated in dictionaries and thesauri.

1 Formal Concept Analysis

Formal concept analysis is a new approach to formal logic and knowledge representation initiated by Rudolf Wille [4, 5]. Formal concept analysis starts with the primitive notion of a formal context. A (order-theoretic) formal context is a triple $C = \langle G, M, I \rangle$ consisting of two posets $G = \langle G, \leq_G \rangle$ and $M = \langle M, \leq_M \rangle$ and a binary relation $I \subseteq G \times M$ between $G$ and $M$ which respects order: $g_1 \leq_G g_2$, $g_2 I m$ imply $g_1 I m$; and $g I m_1$, $m_1 \leq_M m_2$ imply $g I m_2$. Intuitively, the elements of $G$ are thought of as entities or objects, the elements of $M$ are thought of as properties, characteristics or attributes that the entities might have, and $g I m$ asserts that “object $g$ has attribute $m$.” This definition extends the original notion of formal context which was given in a set-theoretic realm. Theoretically, there are strong reasons for enriching and extending to an order-theoretic framework in an order-theoretic setting. Practically, this enrichment offers the advantage of greater expressibility for both the system analyst and system designer, since it allows one to either describe or specify order constraints on both objects and attributes.
The collection $\text{Cxt}_G^M$ of all contexts with a fixed object set $G$ and a fixed attribute set $M$ is a poset with the subset inclusion pointwise order $I \subseteq I'$. As attribute sets $M$ are allowed to vary, we collect together all contexts with fixed object set $G$ in the poset $\text{Cxt}_G \overset{\text{df}}{=} \bigsqcup_M \text{Cxt}_G^M$. We can define direct and inverse image operators on contexts along any function $\phi: G_2 \to G_1$ between object sets as follows.

**[Direct image]** Let $(G_2, I_2, M)$ be any context. Define its direct image along $\phi$ to be the context $(G_1, \phi^\circ \circ I_2, M)$. Direct image is a monotonic function $\phi_*: \text{Cxt}_G \to \text{Cxt}_{G_1}$.

**[Inverse image]** Let $(G_1, I_1, M)$ be any context. Define its inverse image along $\phi$ to be the context $(G_2, \phi \circ \circ I_1, M)$. Inverse image is a monotonic function $\phi^*: \text{Cxt}_{G_1} \to \text{Cxt}_G$.

Contextual flow is adjoint: for any function between object sets $\phi: G_1 \to G_2$, direct and inverse image form an adjoint pair of monotonic functions

$$\phi_* \dashv \phi^*.$$

Formal concept analysis is based upon the understanding that a concept is a unit of thought consisting of two parts: its extension and its intension (Woods [6] also advocates conceptual intensions). Within the restricted scope of a formal context, the *extent* of a concept is a subset $\phi \subseteq G$ consisting of all objects belonging to the concept, whereas the *intent* of a concept is a subset $\psi \subseteq M$ which includes all attributes shared by the objects. A concept of a given context will consist of an extent/intent pair

$$(\phi, \psi).$$

Of central importance in concept construction are two *derivation operators* which define the notion of “sharing” or “commonality”. For any subset of objects $\phi \subseteq G$ the direct derivation along $I$ is $\phi^\circ \circ = \{ m \in M \mid gIm \text{ for all } g \in \phi \}$. For any subset of attributes $\psi \subseteq M$ the inverse derivation along $I$ is $\psi^\circ \circ = \{ g \in G \mid gIm \text{ for all } m \in \psi \}$. These two derivation operators form an adjointness (generalized inverse relationship) $\phi^\circ \circ \supseteq \psi \iff \phi \subseteq \psi^\circ \circ$. To demand that a concept $(\phi, \psi)$ be determined by its extent and its intent means that this adjointness should be a strict inverse relationship at the extent/intent pair $(\phi, \psi)$: the intent should contain precisely those attributes shared by all objects in the extent $\phi^\circ \circ = \psi$, and vice versa, the extent should contain precisely those objects sharing all attributes in the intent $\psi = \psi^\circ \circ$. Concept extents are identical to the closed subsets of the closure operator $\phi^\circ \circ = (\phi^\circ \circ)^\circ \circ$ defined by the adjointness of derivation, and concept intents are identical to the open subsets of the interior operator $\psi^\circ \circ = (\psi^\circ \circ)^\circ \circ$ defined by this adjointness.

The collection of all concepts is ordered by generalization-specialization. One concept is more specialized (and less general) than another $(\phi, \psi) \leq_L (\phi', \psi')$ when its intent contains the other’s intent $\psi \supseteq \psi'$, or equivalently, when the opposite ordering on extents occurs $\phi \subseteq \phi'$. Concepts with this generalization-specialization ordering form a concept hierarchy for the context. The concept hierarchy is a complete lattice $\mathbf{L}(C)$ called the *concept lattice* of $C$.

The meets and joins of concepts in $\mathbf{L}(C)$ can be described in terms of unions, intersections, intent interior, and extent closure as follows:

$$\bigwedge_{k \in K} (\phi_k, \psi_k) = (\bigwedge_{k \in K} \phi_k, \bigcap_{k \in K} \psi_k)^*$$
$$\bigvee_{k \in K} (\phi_k, \psi_k) = (\bigcup_{k \in K} \phi_k, \bigcap_{k \in K} \psi_k)^*.$$

The join of a collection of concepts represents the common properties (shared characteristics) of the concepts, and the top of the concept hierarchy represents the universal concept whose extent consists of all objects. When extended to distributed relations, the meet of a collection of concepts corresponds to the natural join from relational theory, and the bottom of the concept hierarchy represents all solutions that satisfy the constraints (the solution-set concept).

In this paper we give arguments that Wille’s original notion of formal context, although quite appealing in its simplicity, now should be extended to distributed relations. Such a generalization and abstraction of formal contexts offers a powerful approach for the representation of knowledge and the reasoning about constraints.

## 2 Distributed Relations

The basic parameter in relational theory is a sorted domain. Let $N$ be a set of sorts. An $N$-sorted domain $D$ is an $N$-indexed collection of posets $D = \{ D_a \mid a \in N \}$. A single-sorted domain ($N = 1$) is just a poset. Let $\varphi N$ denote the set of all finite subsets of $N$. A finite subset of sorts $U \in \varphi N$ is called either an *arity* or an *elementary relational scheme*. Given any scheme $U = \{ a_1, ..., a_n \}$, the $U$-th power of $D$ is defined by $D^U = D_{a_1} \times \ldots \times D_{a_n}$. The empty power is $D^\emptyset = 1$, a canonical singleton poset. Any sort subset inclusion $U \subseteq V$ defines an obvious projection monotonic function $D^V \overset{\pi_U}{\to} D^U$. 
The basic variable quantity in relational theory is a relation. An \(N\)-sorted relation over domain \(D\) with scheme \(U \in \wp N\), an \((N,D)\)-relation, is a closed-below subset \(R \subseteq D^U\). We use the declaration notation \(R : U\) to denote this. For any two \((N,D)\)-relations \(S : V\) and \(R : U\), a projective containment condition exists from \(S : V\) to \(R : U\), written \(S : V \leq R : U\), when (1) \(U \subseteq V\), and (2) \(\pi_V U(S) \subseteq R\) or equivalently \(S \subseteq \pi_V^{-1}(R)\). Containment conditions (either these simple projective containment conditions or nontrivial dependency containment conditions) are examples of \textit{internal constraints} on data. Internal constraints are a particular specification form for semantics.

Let \(\text{Rel}_{N, D}\) denote the collection of all \((N,D)\)-relations. This forms a poset with projective containment order. For a single-sorted domain \(D\), \(\text{Rel}_1, D\) is the powerset of \(D\) plus the two relations \(\emptyset : \emptyset\) and \(1 : \emptyset\). The \textit{underlying scheme function} \(p_{N,D} : \text{Rel}_{N,D} \rightarrow \wp N^\text{op}\) is monotonic by the definition of projective containment.

**Fact 1** \(\text{Rel}_{N,D}\) is a complete lattice: infima are natural joins, the top is \(1 : \emptyset\), and the bottom is \(\emptyset : N\).

Given a set of sorts \(N\), a \textit{distributed relational scheme} over \(N\) is a pair \((E, \tau)\) consisting of (1) a preorder of relation (predicate) symbols or indices \(E = \langle E, \leq_E \rangle\), and (2) a monotonic function \(\tau : E \rightarrow \wp N^\text{op}\) assigning elementary relational schemes to relation indices. The triple \(\Omega = \langle E, \tau, N \rangle\) is called a \textit{first order signature} (without multiplicities). The signature is discrete when the preorder of relation symbols is the identity preorder \(E = \langle E, =_E \rangle\). In this case a signature is the same as a hypergraph. The signature is single-sorted when the set of sorts is a singleton set \(N = 1\). In this case the signature \(\Omega = \langle E, !, 1 \rangle\) is equivalent to a preorder \(E\).

Constraint-based reasoning is based upon the primitive notion of a distributed signature. Given a fixed signature \(\Omega = \langle E, \tau, N \rangle\), an \(N\)-sorted \textit{distributed relation} with scheme \(\langle E, \tau \rangle\), an \(\Omega\)-relation, is a pair \((D, R)\) consisting of (1) an \(N\)-sorted domain \(D = \{D_a \mid a \in N\}\), and (2) a monotonic function \(R : E \rightarrow \text{Rel}_{N,D}\) assigning relations to relation symbols and projective containment conditions to \(E\)-inequalities, where \(R\) is compatible with the scheme \(\langle E, \tau \rangle\) in the sense that \(R \cdot p_{N,D} = \tau\). The latter equality defines when \(R\) has distributed scheme \(\tau\).

**Facts 1**

1. Order-theoretic formal contexts \(\mathcal{C} = \langle G, M, I \rangle\) are (equivalent to) distributed relations for single-sorted signatures.

2. Networks of constraints \([3]\) \(N = \langle N, \tau, E, D, R \rangle\) are defined to be distributed relations with discrete signatures.

For formal contexts as distributed relations, (1) the signature is the preorder of attributes \(\Omega = \langle M, !, 1 \rangle\), (2) the single domain is the set of objects \(D = G\), and (3) the relation assignment \(R : M \rightarrow \text{Rel}_{1,G}\) assigns corresponding columns of the incidence relation to attributes \(R : a \rightarrow I_a \subseteq G\), so that each attribute represents a unary constraint. A network of constraints is a combinatorial construct, not a semantic algebraic construct. Distributed relations provide a semantic extension of networks of constraints by specifying projective containment conditions between constraints. Projective containment conditions help optimize the size of distributed relations.

For a fixed sorted domain \((N,D)\) the triple \(R : \langle E, \tau \rangle\) is called a \textit{distributed} \((N,D)\)-relation. For a fixed \textit{signed domain} \((\Omega, D)\), where \(\Omega = \langle E, \tau, N \rangle\) is a signature and \(D\) is an \(N\)-sorted domain, that is, for a fixed sorted domain \((N,D)\) and any relational scheme \(\langle E, \tau \rangle\), the collection \(\text{Rel}_{N,D}^{E,\tau}\) of all distributed \((N,D)\)-relations with scheme \(\langle E, \tau \rangle\) is a poset (actually, a complete Boolean algebra) with the pointwise order \(R \leq S\) when \(R_e \subseteq S_e\) for all relation indices \(e \in E\). If \(E = 1\) is a singleton, then \(\tau : 1 \rightarrow N^\text{op}\) is essentially an elementary relational scheme \(U \in \wp N^\text{op}\), and \(\text{Rel}_{N,D}^{E,\tau} = \wp D^U\) the complete Boolean algebra of (nondistributed) \((N,D)\)-relations with scheme \(U\). As the scheme \(\langle E, \tau \rangle\) varies, the posets \(\text{Rel}_{N,D}^{E,\tau}\) form our most basic semantic domains, and have syntactically specified join and cojoin transformations \(\text{Rel}_{N,D}^{E,\tau} \rightarrow \text{Rel}_{N,D}^{E,\tau}\) defined between them. As distributed schemes \(\langle E, \tau \rangle\) are allowed to vary, we collect together all distributed \((N,D)\)-relations in the poset \(\text{Rel}_{N,D} \equiv \bigsqcup_{E,\tau} \text{Rel}_{N,D}^{E,\tau}\).

3 Domain Morphisms

The process of varying (weakening or strengthening) constraint satisfaction problems [1] involves two senses (external/internal) and two options (syntactic/semantic) within each sense.

1. External sense: vary sorted domains.
   
   (a) Remove variables (domain indices): changing \(N\) to \(N'\), where \(N \supseteq N'\).
   
   (b) Enlarge variable domains: changing \(\{D_i \mid i \in N\}\) to \(\{D'_i \mid i \in N\}\), where \(D_i \subseteq D'_i\) for all \(i \in N\).
2. Internal sense: vary distributed relations.

(a) Remove constraints (relation indices): changing $E$ to $E'$, where $E \supseteq E'$.

(b) Enlarge constraint relations: changing $\{R_e \mid e \in E\}$ to $\{R'_e \mid e \in E\}$, where $R'_e \subseteq R'_e$ for all $e \in E$. As noted by [1] when $R'_e = \emptyset = D^I$ the top unconstraining relation, the constraint at $e$ has effectively been removed; so that syntactic variability can be effected to some extent by semantic variability.

The external sense for varying constraint specifications is formally defined as direct/inverse image along domain morphisms. A morphism of sorted domains $(f,\phi):(N_1,D_1) \rightarrow (N_2,D_2)$ is a pair consisting of

1. a function $f : N_1 \rightarrow N_2$ between sort sets, and

2. a $\varphi\mathcal{N}_1$-indexed collection $\phi = \{D'^U_{fU} \otimes D^U_{fU} \mid U \subseteq N_1\}$ of monotonic functions between domains, which satisfy $\phi_V \cdot \pi_{UV} = \pi_{fVU} \cdot \phi_U$ for every pair $V \supseteq U$.

Domain morphisms specify adjoint (direct/inverse image) relational flow.

**[Direct image]** Let $R_2 : (E_2,\tau_2)$ be any distributed $(N_2,D_2)$-relation. Define distributed scheme $(E_1,\tau_1)$ to be the pullback of scheme $(E_1,\tau_1)$ along the direct image sort map $\varphi : \varphi\mathcal{N}_1 \rightarrow \varphi\mathcal{N}_2$ as defined product $E_1 \overset{\text{df}}{=} \{(e_2, U) \mid e_2 \in E_2, U \subseteq N_1, \tau_2 e_2 = fU\}$ and projection function $\tau_1 : (e_2, U) \rightarrow U$. Define distributed relation $R_{1,e_2,U} \overset{\text{df}}{=} \phi_U(R_{2,e_2})$ for any $(U, e_2) \in E_1$ to be the direct image along domain function $D_2 \overset{\tau_2 e_2}{} \overset{\phi_2}{} \overset{\tau_1}{} D_1$. Finally, define the direct image distributed $(N_1,D_1)$-relation $(f,\phi)_\ast (R_2 : (E_2,\tau_2)) \overset{\text{df}}{=} R_1 : (E_1,\tau_1)$. Direct image is a monotonic function:

$(f,\phi)_\ast : \mathcal{R}_{N_2,D_2} \rightarrow \mathcal{R}_{N_1,D_1}$. 

**[Inverse image]** Let $R_1 : (E_1,\tau_1)$ be any distributed $(N_1,D_1)$-relation. Define distributed scheme $(E_2,\tau_2) \overset{\text{df}}{=} (E_1,\tau_1 \cdot \varphi f^\text{op})$. Define distributed relation $R_{2,e_1} \overset{\text{df}}{=} \phi_{f,e_1}^{-1}(R_{1,e_1})$ for any $e_1 \in E_1$ to be the inverse image along domain function $D_2 \overset{\tau_1 e_1}{} \overset{\phi_2^{-1} \tau_1}{} D_1$. Finally, define the inverse image distributed $(N_2,D_2)$-relation $(f,\phi)^\ast (R_1 : (E_1,\tau_1)) \overset{\text{df}}{=} R_2 : (E_2,\tau_2)$. Inverse image is also monotonic:

$(f,\phi)^\ast : \mathcal{R}_{N_1,D_1} \rightarrow \mathcal{R}_{N_2,D_2}$.

**Proposition 1** Relational flow is adjoint: for any morphism of sorted domains $(f,\phi):(N_1,D_1) \rightarrow (N_2,D_2)$, direct and inverse image form an adjoint pair of monotonic functions

$(f,\phi)_\ast - (f,\phi)^\ast$.

4 The Satisfaction Context

Wille’s formal contexts explicitly model relationships between objects and attributes (unary constraints). However, the more general satisfaction relationships between object tuples and constraints that we define in this section associate an order-theoretic formal context with each distributed relation $\mathcal{R} = (N,\tau,E,D,R)$ in a very natural fashion.

The basic constituents in constraint-based reasoning are tuples of values representing semantic configurations of real-world objects, and ways of describing these in terms of constraining relations. The relationships between tuples and constraints is a derived relationship called satisfaction. Intuitively, for any subset $U \subseteq N$ the tuple values in $D^U$, thought of as the possible state or configuration of entities or objects, will be called object tuples. An object tuple may represent the participation of several objects or entities in a semantic whole. A tuple $x \in D^U$ is said to have tag or arity $U$ and is denoted by $x : U$. The set of all nonempty tuples of a distributed relation can be defined as the disjoint union $D^{[N]} = \bigsqcup_{U \subseteq N} D^U \overset{\text{df}}{=} \bigsqcup_{U \subseteq N} (U \times D^U)$. This is a partially ordered set with tuples in $D^{[N]}$ ordered by projection: $y : V \preceq x : U$ when $x$ is the projection of $y$; that is, when $V \supseteq U$ and $\pi_{UV}(y) = x$. This tuple projection order is an instance of meronymy, or whole-part order. The tuple $x : U$ is part of the tuple $y : V$. The empty tuple $\varepsilon : \emptyset$ is the top, smallest part in this order. A full tuple $x : N$, whose arity consists of all indices $N$, is a minimal element in this identification order, and represents a whole.

The elements of $E$ are thought of as constraints that the object tuples might satisfy. A tuple $x : U$ satisfies a constraint $e$, denoted by $x \models e$, when the tuple generalizes to, or equivalently is a specialization from, the relation of the constraint; that is, when $U \supseteq \tau_e$ and $\pi_U(\varepsilon) \in R_e$. Satisfaction is a binary relation $\models \subseteq D^{[N]} \times E$ on the set of all tuples, which respects order: $y : V \preceq x : U$, $x \models e$ imply $y \models e$; and $x \models e_1$, $e_1 \subseteq e_2$ imply $x \models e_2$. So satisfaction forms an order-theoretic formal context $\mathcal{C}_R$ called the formal satisfaction context of $\mathcal{R}$. This formal context is defined to be the triple $\mathcal{C}_R = (D^{[N]},E,\models)$, whose
contextual objects are tuples of any arity, whose contextual attributes are relation symbols (constraints) with defined scheme, and whose contextual has relationship is the satisfaction relationship. Zickwolff [7] has independently given a similar development for satisfaction. The attributes in $E$, being relation symbols with schema, are many-sorted — whence the title of this paper.

In order to define a canonical projection domain morphism from distributed relations to formal contexts, we must restrict satisfaction to full scheme tuples $\models \subseteq D^N \times E$. Let us denote this restricted context by $C^N_R = (D^N, E, \models)$. This association of the restricted formal satisfaction context with each distributed relation, is represented by the relation-to-context passage

$$\mathcal{R} \mapsto C^N_R.$$  \hspace{1cm} (1)

For any sorted domain $(N, D)$, the full $N$th power $D^N$, the unique constant function $N \downarrow 1$, and the collection of all projection functions $\pi = \{D^N \xrightarrow{\pi_N \cup} D^U \mid U \subseteq N\}$ form a canonical morphism of sorted domains called *projection*

$$\pi_{N,D} \overset{\text{def}}{=} (\downarrow, \pi): (N, D) \rightarrow (1, D^N).$$

**Theorem 1** The satisfaction context of a distributed relation is the inverse image along projection

$$C^N_R = \pi_{N,D}^\ast(R: (E, \tau)).$$

The satisfaction context of a context (regarded as a single-sorted distributed relation) is the original context. This demonstrates that distributed relations are a proper extension of formal contexts.

Passage 1 associates a lattice of semantic concepts $L(C_R)$ with each distributed relation $\mathcal{R}$, thus revealing its hidden semantic structure. The intuitions underlying this semantic structure can be expressed in terms of partial constraint satisfaction. The concept lattices of distributed relations can be directly related to the problem space interpretation of partial constraint satisfaction in [1]. According to the discussion in [1], a problem space is a set of constraint-satisfaction-problems ordered by their solution sets. In this paper we identify the appropriate problem space of a network of constraints, or more generally a distributed relation $\mathcal{R}$, to be the concept lattice of the formal satisfaction context associated with the relation $L(C_R)$. We thus identify constraint-satisfaction-problems of the problem space with formal concepts in the concept lattice — the problem intent is its set of constraints, and the problem extent is the solution set of the constraints. We refer here to the lattice elements as problem-concepts. These problem-concepts represent partial information about objects. The lattice uses the opposite sense of order than that defined in [1]. Weakening problem constraints or generalizing means moving upward in the problem-concept hierarchy. On the other hand, moving downward in the problem-concept hierarchy corresponds to the monotonic accumulation of partial information about object tuples.

Current and future work involves (1) the definition of satisfaction algorithms on the distributed relation’s concept lattice, and (2) the definition of suitably generalized similarity and distance metrics on the distributed relation’s concept lattice.

### 5 Partial Constraint Satisfaction

The concept lattices of distributed relations can be directly related to the problem space interpretation of partial constraint satisfaction in [1]. According to the discussion in [1], a problem space is a set of constraint-satisfaction-problems ordered by their solution sets. In this paper we identify the appropriate problem space of a network of constraints, or more generally a distributed relation $\mathcal{R}$, to be the concept lattice of the formal satisfaction context associated with the relation $L(C_R)$. We thus identify constraint-satisfaction-problems of the problem space with formal concepts in the concept lattice — the problem intent is its set of constraints, and the problem extent is the solution set of the constraints. We refer here to the lattice elements as problem-concepts. These problem-concepts represent partial information about objects. The lattice uses the opposite sense of order than that defined in [1]. The problem space is restricted by the strong idea of formal concept — the constraint set (intent) of each problem contains *all* constraints satisfied by the solution set (extent) of the problem. These restrictions on allowable problem-concepts make semantic sense and greatly optimize the solution search process. The semantic structure of the concept lattice specifies, through the strong idea of formal concept, how constraint-satisfaction-problems should be weakened and how allowable problems should be restricted. Weakening problem constraints or generalizing means moving upward in the problem-concept hierarchy. On the other hand, moving downward in the problem-concept hierarchy corresponds to the monotonic accumulation of partial information about object-tuples. The top of the hierarchy is unconstrained; the bottom, whose intent is all constraints, has the total solution
set of the system as its extent. Maximal constraint satisfaction corresponds to minimality in the lattice hierarchy. The join of two problem-concepts represents common constraints, and the meet corresponds to the natural join from relational theory.

Current and future work involves the definition of satisfaction algorithms on the distributed relation’s concept lattice (including generalization of analogs to Freuder and Wallace’s satisfaction algorithms from maximal satisfaction, which is only one part of the distributed relation’s concept lattice, to the entire lattice), and the definition of suitably generalized similarity and distance metrics on the distributed relation’s concept lattice.

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Current and future work involves the definition of satisfaction algorithms on the distributed relation’s concept lattice (including generalization of analogs to Freuder and Wallace’s satisfaction algorithms from maximal satisfaction, which is only one part of the distributed relation’s concept lattice, to the entire lattice), and the definition of suitably generalized similarity and distance metrics on the distributed relation’s concept lattice.

7 Relational Interior

Any distributed relation contains within it another distributed relation called its interior which satisfies all possible projective containment conditions. Interior is derived from an interesting adjointness between projection and solution.

On the one hand, the solution set of a distributed relation $R \in \text{Rel}_{N,D}^E$ is defined by $\Pi R \overset{df}{=} \{ x \in D^N \mid x\vDash e \text{ for all } e \in E \}$. Two distributed relations $R,S \in \text{Rel}_{N,D}^E$ are equivalent when they have the same solution set $\Pi R = \Pi S$. Define $\Pi^{-1}P \overset{df}{=} \{ R \mid \Pi R = P \}$ to be the (possibly empty) collection of distributed relations with solution set $P$. By definition of the solution set $\Pi R$, the projection $\pi_{N,e}(x)$ of every tuple $x \in \Pi R$ is contained in the $e$-th constraint relation $\pi_{N,e}(x) \in R_e$. This fact is expressed by the “counit inequality” $\pi(\Pi R) \subseteq R$.

On the other hand, let $P \subseteq D^N$ be any subset of full object tuples. We regard $P$ as a potential solution set. The projection of $P$ is the $E$-indexed collection of relations $\pi P \overset{df}{=} \{ \pi_{N,e}(P) \mid e \in E \}$. The projection of $P$ clearly satisfies the projective containment conditions $\pi_{N,e}(\pi_{N,e}(P)) \subseteq \pi_{N,e}(P)$, and hence defines a distributed relation $(N,\tau,E,\tau,D,\pi P)$. The fact that every tuple in $P$ satisfies every constraint in $E$ is expressed by the “unit inequality” $P \subseteq \Pi(\pi P)$.

There are two ways to define a context on full tuples.

1. For any subset of full tuples $P \subseteq D^N$, define an associated order-theoretic formal context $C_P = \langle G,M,I_P \rangle$ by $xI_P e \iff x \in P$.

Since the projection $\pi$ and product $\Pi$ operators are monotonic functions, we have the following proposition.
Proposition 2. The projection operator $\pi$ and the solution operator $\Pi$ form an adjointness or Galois connection

$$\pi \dashv \Pi$$

between distributed relations $R$ and potential solution sets $P$, expressed by the logical equivalence: $\pi P \subseteq R$ if and only if $P \subseteq \Pi R$.

By the above adjointness, projection preserves joins

$$\Pi(\sqcap) \dashv$$

and connection $\Pi\circ$. By the above adjointness, projection preserves joins $\Pi(\sqcap) \dashv$ and connection $\Pi\circ$. For any distributed relation $R \in \text{Rel}_{N,D}$ with distributed scheme $\langle E, \tau \rangle$, the projection of the solution $R^\circ \equiv \Pi(\sigma(R)) = \pi(\Pi(R))$ is called the interior of $R$. The interior is a distributed relation which satisfies all possible projective containment conditions. The above adjointness implies the existence of minimal equivalent networks of constraints [3]. The interior $R^\circ$ is optimal w.r.t. satisfaction — it is the smallest distributed relation having solution set $\Pi R$:

1. $\Pi(R^\circ) = \Pi R$; and
2. if $S$ is an $E$-indexed distributed relations having solution set $\Pi S = \Pi R$, then $R^\circ \subseteq S$.

In particular, interior is the minimal distributed relation $R^\circ = \bigwedge \Pi^{-1}\Pi R$ equivalent to $R$. These results were stated and proved in [3], but the proofs were not expressed by use of the simplifying notion of an adjointness.

8 Relational Participation

The previous discussion leads one to ask why certain tuples are not included in the interior of a distributed relation $R$. For any relational constraint index $e \in E$ a tuple $x \cdot \tau_e$ is in the difference $R_e \setminus R^\circ_e$ when it does not participate in a combining relationship with other tuples to form a full object tuple that respects all constraints in $E$. In other words, it is isolated with respect to $R$. In this section we will relativize this idea of relational participation and make it more explicit.

We can emphasize the importance of certain concepts in a concept lattice $L(\mathcal{C})$ for a formal context $\mathcal{C} = \langle G, M, I \rangle$ by specifying this collection of concepts as a suborder $P \subseteq L(\mathcal{C})$. More precisely, suppose that $P \subseteq L(\mathcal{C})$ is an injective monotonic function which is left adjoint to $L(\mathcal{C}) \rightarrow P$. The trivial example of such a suborder is the principal ideal $\downarrow c$ of any concept $c \in L(\mathcal{C})$, where $\iota$ is subset inclusion $\iota(c') = c'$ for all $c' \leq_L c$ and $\iota^\circ$ is meet $\iota^\circ(d) \equiv c \wedge_L d$ for all $d \in L(\mathcal{C})$. As a special case, relevant for relational interior, is $c = \downarrow 1$. Define an associated formal context $C_P = \langle G, M, I_P \rangle$ by

$$g | P \iff \exists p \in P \gamma(g) \leq_L \iota(p) \leq_L \mu(m).$$

9 Domain Morphisms

In a global sense, following the discussion in [1], the process of varying (weakening or strengthening) constraint satisfaction problems involves two senses with 2 options each.

1. External sense: vary sorted domains.
   (a) Remove variables (domain indices): changing $N$ to $N'$, where $N \supseteq N'$.
   (b) Enlarge variable domains: changing $\{D_i \mid i \in N\}$ to $\{D'_i \mid i \in N\}$, where $D_i \subseteq D'_i$ for all $i \in N$.

2. Internal sense: vary distributed relations.
   (a) Remove constraints (relation indices): changing $E$ to $E'$, where $E \supseteq E'$.
   (b) Enlarge constraint relations: changing $\{R_e \mid e \in E\}$ to $\{R'_e \mid e \in E\}$, where $R_e \subseteq R'_e$ for all $e \in E$. As noted by [1] when $R'_e = \top = D^\iota$ the top unconstraining relation, the constraint at $e$ has effectively been removed; so that syntactic variability can be effected to some extent by semantic variability.

The external sense for varying constraint specifications is formally defined as direct/inverse image along domain morphisms. A morphism of sorted domains $(f, \phi) : \langle N_1, D_1 \rangle \rightarrow \langle N_2, D_2 \rangle$ is a pair consisting of

1. a function $f : N_1 \rightarrow N_2$ between sort sets, and
2. a $\varphi N_1$-indexed collection $\phi = \{E^U \phi_{\varphi(U)} \subseteq D^U \mid U \subseteq N_1\}$ of functions between domains, which satisfy $\phi_V \cdot \pi_{V,U} = \pi_{fV,fU} \cdot \phi_U$ for every pair $U \subseteq V$ of subsets of $N_1$.

Given any sorted domain $\langle N, D \rangle$, the full $N$th power $D^N$, the unique constant function $N \rightarrow 1$, and the collection of all projection functions $\pi = \{D^N \rightarrow D^U \mid U \subseteq N\}$ form a canonical morphism of sorted domains called projection $\pi_N \equiv (\iota, \mu) : \langle N, D \rangle \rightarrow \langle 1, D^N \rangle$. 
Morphisms of sorted domains specify adjoint (direct/inverse) image relational flow. Let \( (f, \phi) : (N_1, D_1) \to (N_2, D_2) \) be any morphism of sorted domains. Define direct and inverse image operators as follows.

**[Direct image]** Let \( R_2 : (E_2, \tau_2) \) be any distributed \( (N_2, D_2) \)-relation. Define scheme \( (E_1, \tau_1) \) to be the pullback of scheme \( (E_1, \tau_1) \) along the direct image sort map \( \phi f : \phi N_1 \to \phi N_2 \) with fibered product \( E_1 \overset{df}{=} \{ (U, e_2) \mid U \subseteq N_1, e_2 \in E_2, fU = \tau_2 e_2 \} \) and projection function \( \tau_1 : (U, e_2) \mapsto U \). Define distributed relation \( R_{1,(U,e_2)} \overset{df}{=} \phi U (R_2, e_2) \) for any \((U,e_2) \in E_1\) to be the direct image along domain function \( D_2^{\tau_2,e_2} = D_1^{fU} \phi U D_1^{e_2} \). Finally, define the direct image distributed \( (N_1, D_1) \)-relation \( (f, \phi)_* (R_2 : (E_2, \tau_2)) \overset{df}{=} R_1 : (E_1, \tau_1) \). Direct image is a monotonic function

\[
(f, \phi)_* : \text{Rel}_{N_2, D_2} \to \text{Rel}_{N_1, D_1}.
\]

**[Inverse image]** Let \( R_1 : (E_1, \tau_1) \) be any distributed \( (N_1, D_1) \)-relation. Define distributed relation \( R_{2,e_1} \overset{df}{=} \phi^{-1}_U (R_{1,e_1}) \) for any \( e_1 \in E_1 \) to be the inverse image along domain function \( D_2^{\tau_1,e_1} \phi^{-1}_U D_1^{e_1} \). Finally, define the inverse image distributed \( (N_2, D_2) \)-relation \( (f, \phi)^* (R_1 : (E_1, \tau_1)) \overset{df}{=} R_2 : (E_2, \tau_2) \). Inverse image is a monotonic function

\[
(f, \phi)^* : \text{Rel}_{N_1, D_1} \to \text{Rel}_{N_2, D_2}.
\]

**Proposition 3** Relational flow is adjoint: for any morphism of sorted domains \( (f, \phi) : (N_1, D_1) \to (N_2, D_2) \), direct and inverse image form an adjoint pair of monotonic functions

\[
(f, \phi)_* \dashv (f, \phi)^*.
\]

**Proposition 4** The satisfaction context of a distributed relation is the inverse image along projection

\[
C_R = \pi_{N,D}^\ast (R : (E, \tau)).
\]

The satisfaction context of a context (regarded as a single-sorted distributed relation) is the original context. This demonstrates that distributed relations are a proper extension of formal contexts.

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### A Networks of Constraints

The basic constituents in constraint-based reasoning are tuples of values representing real-world objects, and ways of describing these in terms of constraining relations. The relationship between tuples and constraints is a derived relationship called satisfaction. Constraint-based reasoning is based upon the primitive notion of a network of constraints. A network of constraints [3] is a quintuple \( N = \langle N, \tau, E, D, R \rangle \) consisting of the following.

1. A specification or syntactic part, represented by a hypergraph \( \Omega = \langle N, \tau, E \rangle \), with a set \( N \) of nodes called domain indices, sorts or variables; and a set \( E \) of hyper-edges called constraint indices or just constraints, such that each edge \( e \in E \) is typed \( e : I \) by a subset of domain indices \( \tau_e \overset{df}{=} I \subseteq N \) called its connection or scheme. The scheme is a subset function \( \tau : E \to \phi N \) where
\(N\) is the power-set of \(N\); or equivalently, a relation \(\tau \subseteq N \times E\) with \(i\)th when \(i\) indexes a domain of the constraint \(e\). (So the triple \(\langle N, E, \tau \rangle\) is a formal context as defined below, representing the "syntactic concepts" of the constraint network \(N\).

These formal syntactic concepts consist of the intersections of schemes as intent and projectable constraints as extent.) In database theory elements of \(N\) would be called attributes; but this terminology would be confusing here since formal concept analysis has attributes which correspond here to elements of \(E\). Sorts are identified with variables here. In order to separate these, the hypergraph \(\Omega = \langle N, \tau, E \rangle\) must be generalized to a multisorted first order signature.

2. An interpretative or semantic part, which interprets the hypergraph, as an \(N\)-indexed collection \(\{D_i \mid i \in N\}\) of sets of values called domains, or an \(N\)-sorted domain; and an \(E\)-indexed collection \(\{R_e \mid e \in E, I = \tau_e\}\) of constraint relations, where \(D = \prod_{i \in I} D_i\) is the Cartesian product of all domains with indices in \(I\).

### Table 1: Network of Constraints \(N\)

| \(\tau_1\) | \(\tau_2\) | \(\tau_3\) | \(\tau_4\) | \(\tau_5\) |
|{\(a_1\)}|{\(a_1, a_2, a_3\)}|{\(a_1, a_4\)}|{\(a_2, a_5\)}|{\(a_3, a_4, a_5\)}|

### Table 2: Satisfaction Context \(C_N\)

| \(\tau_1\) | \(\tau_2\) | \(\tau_3\) | \(\tau_4\) | \(\tau_5\) |
|{\(a_1\)}|{\(a_1, a_2, a_3\)}|{\(a_1, a_4\)}|{\(a_2, a_5\)}|{\(a_3, a_4, a_5\)}|

### Table 3: Interior System \(N^o\)

\(\Pi R = \{(f, t, t, t, t), (f, t, t, t, t)\}\)

Table 3 gives the interior within the network of constraints presented in Table 1. This was erroneously described in [3]. The error may have caused Montanari and Rossi to overlook the important projective containment conditions for systems of constraints, which lead to the definition of distributed relations. In this example we see, as [3] have pointed out, why distributed relations are more optimal w.r.t. storage requirements than networks: although \(\Pi R\) is equivalent to \(R\), it has only half as many tuples. In addition, there are only half as many concepts in the concept lattice of the interior.

This paper takes the position that the hidden semantic structure of the system of constraints inside any distributed relation \(R\) (and in particular, any network of constraints \(N\)) is revealed by the concept lattice \(L(C_R)\) of the formal satisfaction context associated with the system. For any collection of constraint
indices \( \psi \subseteq E \), the inverse derivation

\[
\psi_e^{\sigma} = \{ x \in D^N \mid x \sigma e \text{ for all constraints } e \in \psi \}
\]

\[
= \{ x \in D^N \mid \pi_e(x) \in R_e \text{ for all constraints } e \in \psi \}
\]
corresponds to the natural database join \( \land \{ R_e \mid e \in \psi \} \) of the constraint relations indexed in the set \( \psi \). The most efficient way to describe concepts in any concept lattice is by means of generators and successors: the extent of a concept is the set of tuple generators of all preceding concepts (concepts below it); and the intent of a concept is the set of constraint generators of all succeeding concepts (concepts above it). The 23 concepts for the satisfaction context of the network of constraints presented in Table 1, are described in Table 4 and Table 5 by listing their generators and successors. Concepts are indexed in lexic (reverse numeric) order of their satisfaction bit vector for constraint indices. The top of the lattice is \( C_1 = (D^N, \emptyset) \) and is labeled by the single tuple \((t, f, t, f, t)\) which satisfies no constraint. The bottom of the lattice is \( C_{23} = (\Pi R, E) \) and is labeled by the solution set \( \{ (f, f, t, t, f), (f, f, t, t, f), (t, t, t, f, t) \} \), the two tuples which satisfy all constraints. From Table 4 and Table 5 we can compute extents and intents of concepts: for example, the concept \( C_{10} \) has extent \( \{ (t, f, f, f, f), (t, f, f, f, f), (t, f, f, f, f), (t, f, f, f, f), (t, f, f, f, f) \} \) and intent \( \{ e_2, e_5 \} \). Here, as for all concepts, the extent is the solution set for the constraints in the intent, and the intent is all constraints satisfied by this solution set — they determine each other. The extent is also the natural database join \( R_{e_2} \bowtie R_{e_5} \) of the constraint relations indexed in the intent.

Table 6 represents the order relation for the concept lattice. The matrix is lower triangular because of two facts: (1) the concepts are listed in lexic (reverse numeric) order; and (2) the concept lattice partial order is a suborder of the lexic total order.

### C Relational Interior

Any distributed relation contains within it another distributed relation called its interior which satisfies all possible projective containment conditions. Interior is derived from an interesting adjointness between projection and solution.

On the one hand, the solution set of a distributed relation \( R \in \text{Rel}_{N,D}^{E,\tau} \) is defined by \( \Pi R \overset{df}{=} \{ x \in D^N \mid x \tau e \text{ for all } e \in E \} \). Two distributed relations \( R, S \in \text{Rel}_{N,D}^{E,\tau} \) are equivalent when they have the same solution set \( \Pi R = \Pi S \). Define \( \Pi^{-1} P \overset{df}{=} \{ R \mid \Pi R = P \} \) to be the (possibly empty) collection of distributed

---

**Table 4: Concept Generators**

| concept | generators | constraints |
|---------|------------|-------------|
| \( C_1 \) | \( \emptyset \) | \( e_5 \) |
| \( C_2 \) | \( (t, f, t, t) \) | \( e_4 \) |
| \( C_3 \) | \( (t, f, t, f), (t, t, f, t) \) | \( e_3 \) |
| \( C_4 \) | \( (t, t, f, f) \) | \( e_2 \) |
| \( C_5 \) | \( (t, f, f, f) \) | \( e_1 \) |
| \( C_6 \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |
| \( C_7 \) | \( (t, t, t, t) \) | \( e_5 \) |
| \( C_8 \) | \( (t, f, t, t) \) | \( e_5 \) |
| \( C_9 \) | \( (t, f, f, f) \) | \( e_5 \) |
| \( C_{10} \) | \( (t, f, f, f) \) | \( e_5 \) |
| \( C_{11} \) | \( (t, f, t, f), (t, t, f, t) \) | \( e_5 \) |
| \( C_{12} \) | \( (t, f, f, f) \) | \( e_5 \) |
| \( C_{13} \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |
| \( C_{14} \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |
| \( C_{15} \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |
| \( C_{16} \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |
| \( C_{17} \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |
| \( C_{18} \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |
| \( C_{19} \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |
| \( C_{20} \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |
| \( C_{21} \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |
| \( C_{22} \) | \( (t, f, f, f), (t, f, f, f) \) | \( e_5 \) |

**Table 5: Concept Successors**

| concept | successor_concepts |
|---------|-------------------|
| \( C_1 \) | \( \emptyset \) |
| \( C_2 \) | \( [C_1] \) |
| \( C_3 \) | \( [C_1] \) |
| \( C_4 \) | \( [C_2, C_3] \) |
| \( C_5 \) | \( [C_1] \) |
| \( C_6 \) | \( [C_2, C_6] \) |
| \( C_7 \) | \( [C_1] \) |
| \( C_8 \) | \( [C_4, C_6, C_7] \) |
| \( C_9 \) | \( [C_1, C_8, C_9] \) |
| \( C_{10} \) | \( [C_2, C_9] \) |
| \( C_{11} \) | \( [C_1, C_9] \) |
| \( C_{12} \) | \( [C_4, C_{10}, C_{11}] \) |
| \( C_{13} \) | \( [C_5, C_{10}] \) |
| \( C_{14} \) | \( [C_6, C_{10}, C_{13}] \) |
| \( C_{15} \) | \( [C_7, C_{11}, C_{13}] \) |
| \( C_{16} \) | \( [C_8, C_{12}, C_{14}, C_{15}] \) |
| \( C_{17} \) | \( [C_9] \) |
| \( C_{18} \) | \( [C_7, C_{17}] \) |
| \( C_{19} \) | \( [C_7, C_{17}] \) |
| \( C_{20} \) | \( [C_8, C_{18}, C_{19}] \) |
| \( C_{21} \) | \( [C_{13}, C_{17}] \) |
| \( C_{22} \) | \( [C_{15}, C_{18}, C_{21}] \) |
| \( C_{23} \) | \( [C_{16}, C_{20}, C_{22}] \) |
relations with solution set \( P \). By definition of the solution set \( \Pi R \), the projection \( \pi_{Nτ_e}(x) \) of every tuple \( x ∈ Π R \) is contained in the \( e \)-th constraint relation \( \pi_{Nτ_e}(x) ∈ R_e \). This fact is expressed by the “count inequality” \( \pi(Π R) ⊆ R \).

On the other hand, let \( P ⊆ D^N \) be any subset of full object tuples. We regard \( P \) as a potential solution set. The projection of \( P \) is the \( E \)-indexed collection of relations \( \pi P \overset{df}{=} \{ \pi_{Nτ_e}(P) \mid e ∈ E \} \). The projection of \( P \) clearly satisfies the projective containment conditions \( \pi_{τ_e,τ}(\pi_{Nτ_e}(P)) ⊆ \pi_{Nτ}(P) \), and hence defines a distributed relation \( (N, τ, E, D, πP) \). The fact that every tuple in \( P \) satisfies every constraint in \( E \) is expressed by the “unit inequality” \( P ⊆ Π(πP) \).

Table 6: Order Relation for Concept Lattice \( L(C_N) \)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1 |  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 2 |  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 3 |  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 4 |  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 5 |  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 6 |  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 7 |  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 8 |  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 9 |  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 10|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 11|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 12|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 13|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 14|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 15|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 16|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 17|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 18|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 19|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 20|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 21|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 22|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 23|  |  |  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

There are two ways to define a context on full tuples.

1. For any subset of full tuples \( P ⊆ D^N \), define an associated order-theoretic formal context \( C_P = \langle G, M, I_P \rangle \) by

\[
x I_P e \text{ if and only if } x ∈ P.
\]

Since the projection \( π \) and product \( Π \) operators are monotonic functions, we have the following proposition.

**Proposition 5** The projection operator \( π \) and the solution operator \( Π \) form an adjointness or Galois connection

\[
π \dashv Π
\]

between distributed relations \( R \) and potential solution sets \( P \), expressed by the logical equivalence: \( πP ⊆ R \) \( iff \ P ⊆ ΠR \).

By the above adjointness, projection preserves joins \( π(∪_k P_k) = ∪_k π(P_k) \), and solution preserves meets \( Π(∩_k R_k) = ∩_k ΠR_k \).

For any distributed relation \( R ∈ Rel_{N,D} \) with distributed scheme \( (E, τ) \), the projection of the solution \( R_e \overset{df}{=} Ππ(R) = Π(Π(R)) \) is called the interior of \( R \). The interior is a distributed relation which satisfies all possible projective containment conditions. The above adjointness implies the existence of minimal equivalent networks of constraints [3]. The interior \( R_e \) is optimal w.r.t. satisfaction — it is the smallest distributed relation having solution set \( ΠR \):

1. \( Π(R^e) = ΠR \); and
2. if \( S \) is an \( E \)-indexed distributed relations having solution set \( ΠS = ΠR \), then \( R^e ⊆ S \).

In particular, interior is the minimal distributed relation \( R^e = \bigwedge Π^{-1}ΠR \) equivalent to \( R \). These results were stated and proved in [3], but the proofs were not expressed by use of the simplifying notion of an adjointness.

### D Relational Participation

The previous discussion leads one to ask why certain tuples are not included in the interior of a distributed relation \( R \). For any relational constraint index \( e ∈ E \) a tuple \( x : τ_e \) is in the difference \( R_e \setminus R_e' \) when it does not participate in a combining relationship with other tuples to form a full object tuple that respects all constraints in \( E \). In other words, it is isolated with respect to \( R \). In this section we will relativize this idea of relational participation and make it more explicit.

We can emphasize the importance of certain concepts in a concept lattice \( L(C) \) for a formal context \( C = \langle G, M, I \rangle \) by specifying this collection of concepts as a suborder \( P ⊆ L(C) \). More precisely, suppose that \( P ⊆ L(C) \) is an injective monotonic function which is left adjoint to \( L(C) \xrightarrow{ι} P \). The trivial example of such a suborder is the full lattice, where \( ι \) and \( ι^\text{−1} \) are both identity. The simplest nontrivial example of such a suborder is the principal ideal \( ↓_C c \) of any concept \( c ∈ L(C) \), where \( c \) is the subset inclusion \( ι(c') = c' \) for all \( c' ≤_L c \) and \( ι^\text{−1} \) meet \( ι^\text{−1}(d) = c^\text{−1} \wedge_L d \) for all \( d ∈ L(C) \). As a special case, relevant for relational interior, \( c = ↓_M \). Define an associated formal context \( C_P = \langle G, M, I_P \rangle \) by

\[
γl_{p,m} \text{ if } \exists x ∈ P \text{ such that } \gamma(y) ≤_L ι(p) ≤_L ι(μ(m)).
\]