Decentralized and Distributed Temperature Control via HVAC Systems in Energy Efficient Buildings

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Abstract—In this paper, we design real-time decentralized and distributed control schemes for Heating Ventilation and Air Conditioning (HVAC) systems in energy efficient buildings. The control schemes balance user comfort and energy saving, and are implemented without measuring or predicting exogenous disturbances. Firstly, we introduce a thermal dynamic model of building systems and formulate a steady-state resource allocation problem, which aims to minimize the aggregate deviation between zone temperatures and their set points, as well as the building energy consumption, subject to practical operating constraints, by adjusting zone flow rates. Because this problem is nonconvex, we propose two methods to (approximately) solve it and to design the real-time control. In the first method, we present a convex relaxation approach to solve an approximate version of the steady-state optimization problem, where the heat transfer between neighboring zones is ignored. We prove the tightness of the relaxation and develop a real-time decentralized algorithm to regulate the zone flow rate. In the second method, we introduce a mild assumption under which the original optimization problem becomes convex, and then a real-time distributed algorithm is developed to regulate the zone flow rate. In both cases, the thermal dynamics can be driven to equilibria which are optimal solutions to those associated steady-state optimization problems. Finally, numerical examples are provided to illustrate the effectiveness of the designed control schemes.

Index Terms—Temperature control, HVAC systems, convex relaxation, distributed/decentralized control, gradient algorithms.

I. INTRODUCTION

It is reported that buildings are responsible for 40% of energy consumption, 70% of electricity consumption, and result in 30% of greenhouse gas emission [1]. Roughly speaking, Heating Ventilation and Air Conditioning (HVAC) systems in buildings account for 40% of the energy use [2]. Therefore, making HVAC systems more energy efficient is urgent for improving environmental sustainability.

To date, various control techniques have been developed for HVAC systems, including gain scheduling, optimal control, robust control, nonlinear adaptive control, Model Predictive Control (MPC), intelligent control based on artificial neural network, fuzzy logic, and genetic algorithm, and so forth [3–7]. Compared with the conventional on/off plus Proportional-Integral-Derivative (PID) control, these techniques are more robust and energy efficient. However, most of them requires centralized operation with heavy burdens of sensing, communication and computation, leading to much higher implementation cost than the conventional on/off plus PID control.

On the other hand, smart sensing, communication, computing, and actuation technologies have been stimulating the emergence of distributed/decentralized control in network systems, including the smart grid [8]–[10], smart buildings [11], [12], mobile robots [13], and intelligent transportation systems [14]. The advantages of distributed/decentralized control include: good scalability as the network grows; reduction in measurement, communication and computation compared with centralized control; privacy preserving. Thus, applying distributed/decentralized control to HVAC system control and optimization becomes an area of active research. Representative work includes, for example, distributed MPC [15]–[17], as well as mean-field game based distributed control [18]. Among these popular control mechanisms, distributed MPC is the most promising one, especially for large buildings. However, it still requires a large amount of sensing, communication and computation. And in most cases, it needs good prediction of future disturbances, i.e., outdoor temperature, sunlight, indoor occupancy, etc., which may be hard to obtain.

Contribution of this paper. This paper aims to develop real-time control schemes for HVAC systems in commercial buildings. Specifically, we aim to design decentralized and distributed algorithms to guide each thermal zone to properly adapt their supply air flow rates such that system-wide objectives are achieved under given operating conditions. The proposed control schemes (i) are scalable with respect to building structures, (ii) satisfy the system operating constraints, and (iii) ensure system efficiency, reliability and user comfort. Different from literature [15]–[18], the control schemes designed in this paper are based on solving steady-state resource allocation problems via gradient algorithms — they are dynamic feedback controllers that can be implemented without measuring or predicting disturbances, which are different from controllers based on MPC [5–7], [15]–[17].

To begin with, we provide the detailed problem setup, including an introduction of the HVAC system configuration, a commonly used thermal dynamic model of the building network, and an optimization problem that aims to minimize a weighted sum of the aggregate deviation between zone temperatures and their set points and the building energy consumption subject to practical operating constraints (Section II). Since the original optimization problem is nonconvex, two different methods are proposed to (approximately) solve it. Firstly, we present a convex relaxation approach in which the heat transfer between neighboring zones is ignored (Section III). We show
that this relaxation is always tight under a mild condition, and
develop a decentralized control scheme for real-time zone flow
rate regulation. Also, we extend the proposed method to HVAC
system management in communities/neighborhoods. Secondly,
we introduce a mild assumption under which the original
optimization problem is naturally reformulated as a convex
one (Section IV). Then a distributed algorithm is developed
for real-time zone flow rate regulation. Also, we extend the proposed method to HV AC
problems. Lastly, two numerical examples are provided to
illustrate the effectiveness of the designed control schemes
(Section V), using a building with four adjacent zones.

Notation: \( \dot{x}(t) \) denotes the derivative of a state variable \( x(t) \)
with respect to time \( t \), i.e., \( \dot{x}(t) = \frac{d}{dt} x(t) \). \( x \geq (\leq) y \) denotes
that \( x \) is much greater (less) than \( y \). The positive projection
of a function \( h(y) \) on a variable \( x \in [0, \infty) \), \( (h(y))_x^+ \) is:

\[
(h(y))_x^+ = \begin{cases} 
  h(y) & \text{if } x > 0 \\
  \max(0, h(y)) & \text{if } x = 0 
\end{cases}
\]

II. PROBLEM SETUP

A. HVAC system in buildings

The schematic of a typical HVAC system is illustrated in
Figure 1, which consists of an Air Handling Unit (AHU) for
the whole building and a set of pressure independent Variable
Air Volume (VAV) boxes for each zone (1). The AHU is
equipped with dampers, a cooling/heating coil, and a Variable
Frequency Drive (VFD) fan: the dampers mix the return air
from the building with the outside air; the cooling/heating coil
cools down/heats up the mixed air; the VFD fan adjusts its own
speed based on the total air flow rate/opening controlled by
VAV boxes to keep the duct pressure at a certain level, and
drives the cooled/heated air to each zone. Each VAV box has a
damper to control the air flow rate supplied to the zone, which
is considered as the only controllable input to the system in
this paper. Also, we will focus on optimizing HVAC operation
when the system is in either cooling or heating mode. In the
cooling mode, the AHU supply air temperature is often set to
12.8°C which generally provides the required humidity ratio
to maintain space thermal comfort (20). In the heating mode,
usually dehumidification is not necessary. Therefore, humidity
control is not considered in this paper.

B. Thermal dynamic model

According to the above configuration, we model a building
as an undirected connected graph \( (N, E) \). Here \( N \) is the set
of nodes representing zones/rooms, and \( E \subseteq N \times N \) is the
set of edges. An edge \( (i, j) \in E \) means that zones \( i \) and \( j \)
are neighbors. Let \( N(i) \) denote the set of neighboring zones
of zone \( i \). The thermal dynamics for the building is described
by a reduced Resistance-Capacitance (RC) model (21) (more
discussion on the model is available in Remark 1):

\[
C_i \dot{T}_i = \frac{T^o - T_i}{R_i} + \sum_{j \in N(i)} \frac{T_j - T_i}{R_{ij}} + c_a m_i (T^s - T_i) + Q_i, \quad i \in N 
\]

where \( C_i \) is the thermal capacitance, \( T_i \) is the indoor temper-

\[ 
C_i = \frac{T^o - T_i}{R_i} + \sum_{j \in N(i)} \frac{T_j - T_i}{R_{ij}} + c_a m_i (T^s - T_i) + Q_i, \quad i \in N 
\]

where \( C_i \) is the thermal capacitance, \( T_i \) is the indoor temper-
ature which is usually a constant (17), and \( Q_i \geq 0 \) is the
heat gain from exogenous sources (e.g., user activity, solar
radiation and device operation). If \( R_i, R_{ij}, C_i \) are not known
from design specification, they can be obtained via model
identification (22), (23). In this paper, we consider the flow
rate \( m_i \) as the only control input to each zone, which is a
common practice in others’ work as well (17), (24), (25).

Proposition 1. When the HVAC system is off, i.e., \( m_i = 0 \),
the system (1) asymptotically converges to an equilibrium point
which is uniquely determined by the inputs \( T^o, Q_i \). The
HVAC system is on, the asymptotic convergence property
of system (1) remains and the equilibrium point is uniquely
determined by the inputs \( T^o, m_i, Q_i \).

This proposition can be directly derived by rearranging (1)
in state-space representation, and showing that the system
matrix is Hurwitz. Therefore, the desiderata is to design
\( m_i \) only for periods when the HVAC system is on, more
specifically, is to design the dynamics of \( m_i \) to drive (1) to
some desired state.

C. The optimization problem

In reality, each zone has a desired temperature which is the
set point determined by users. The objective of the HVAC
control considered in this paper is to regulate the temperature
to be close to the set point in each zone, and to minimize the
total energy consumption in the building. More specifically,
we consider the following steady-state optimization problem:

\[
\begin{align*}
& \min_{Z_i, m_i} \sum_{i \in N} \frac{1}{\eta} r_i (Z_i - T^o)^2 + \frac{w}{\eta} \sum_{i \in N} c_a m_i |Z_i - T^s| \\
& \quad + w s \left( \sum_{i \in N} m_i \right)^3 \\
& \text{s. t.} \quad \frac{T^o - Z_i}{R_i} + \sum_{j \in N(i)} \frac{Z_j - Z_i}{R_{ij}} + c_a m_i (T^s - Z_i) + Q_i = 0 \\
& \text{and} \quad T^o \leq Z_i \leq T^o \\
& \quad \min m_i \leq m_i \leq \max m_i 
\end{align*}
\]

Fig. 1: Schematic of a typical AHU&VAV system.
$\sum_{i \in \mathcal{N}} m_i \leq m$ (2e)

where $i \in \mathcal{N}$ in (2b)-(2d), $r_i$, $w$ are positive weight coefficients, $\eta$ is a given constant named Coefficient of Performance (COP, i.e., the ratio of the produced cold/heat to the consumed energy) of the cooling/heating coil, $s$ is a given coefficient regarding the fan power consumption, $T_i^{\text{set}}$ is the temperature set point satisfying $T_i^{\text{min}} < T_i^{\text{set}} < T_i^{\text{max}}$, $[T_i^{\text{min}}, T_i^{\text{max}}]$ is the user comfort range, $[m_i^{\text{min}}, m_i^{\text{max}}]$ is the range of $m_i$ in which $m_i^{\text{min}}$ is usually close to zero to guarantee a minimal ventilation level [16], and $m$ is the upper bound of the total flow rate in the building. Note that (i) to avoid confusion between steady-state values and temperature dynamics, we use $Z_i$ to denote the steady-state temperature value whereas $T_i$ is the temperature in the dynamic model (1), and (ii) $T^o$ and $Q_i$ are exogenous disturbances. We assume that problem (2) is feasible and satisfies Slater’s condition [26]. Moreover, we have four important remarks for problem (2).

- In (2b), the term regarding the building energy consumption contains two parts, i.e., the energy consumption of the cooling/heating coil given by $\frac{1}{\eta} \sum_{i \in \mathcal{N}} c_a m_i |Z_i - T^o|$ (weighted by $w$), and the power consumption of the supply fan which is approximately proportional to the cube of the total supply air flow [17], i.e., $s(\sum_{i \in \mathcal{N}} m_i)^3$ (weighted by $w$).
- Parameters $r_i$, $w$ are determined by users: if preferring more comfort, they can increase $r_i$ and decrease $w$, and vice versa.
- In the cooling mode, $T^o \ll Z_i$ (or $T_i$), $\forall i$; in the heating mode, $T^o \gg Z_i$ (or $T_i$), $\forall i$. This is usually true in practice. For example [17], in the cooling mode, $T^o = 12.8^\circ C$ while $T_i \geq 21.5^\circ C$. Note that once the mode is determined, the sign of $Z_i - T^o$ (or $T_i - T^o$) is determined.
- In (2d), we do not impose any lower bound constraint for the total flow rate as the lower bound usually equals $\sum_{i \in \mathcal{N}} m_i^{\text{min}}$. In addition, $m < \sum_{i \in \mathcal{N}} m_i^{\text{max}}$ holds, otherwise, this constraint would become redundant.

To conclude, the goal is to design the regulating rule for $m_i$ in a decentralized or distributed way so that system (1) reaches a steady state which is an optimal solution to problem (2).

**Remark 1.** Though system (1) is a 1st-order RC model, using higher order RC models does not affect the formulation of (2) since it is a steady-state optimization problem. For example, for the 2nd-order model in [17, 27]

\[ C_i T_i = \frac{T^o - T_i}{R_i} + \sum_{j \in \mathcal{N}(i)} \frac{T_{ij} - T_i}{R_{ij}} + c_a m_i (T^o - T_i) + Q_i \]

\[ C_i T_{ij} = \frac{T_i - T_{ij}}{R_i} + \frac{T_j - T_{ij}}{R_{ij}} \]

where $T_{ij}$ is the temperature of the wall separating zones $i$ and $j$, and $C_i$ is the thermal capacitance of the wall, the corresponding steady-state Equation (2b) is given by

\[ \frac{T^o - Z_i}{R_i} + \sum_{j \in \mathcal{N}(i)} \frac{Z_j - Z_i}{2R_{ij}} + c_a m_i (T^o - Z_i) + Q_i = 0 \]

\[ \text{which results from the steady-state equation } Z_{ij} = (Z_i + Z_j)/2 \]

\[ (Z_{ij} \text{ is the steady-state temperature of the wall}), \text{i.e., the steady-state equation of the higher order model can be reduced to formulation (2b) by eliminating states of solids in the building envelope. Because our control design procedures proposed later are based on solving the steady-state optimization problem (2), using higher order models will not affect them.} \]

**III. METHOD I: AN APPROXIMATE SOLUTION PROCEDURE**

**A. Approximation and convex relaxation**

Problem (2) is nonconvex because the quadratic equality constraint (2b) is nonconvex in $m_i, Z_i$. In reality, due to that $Z_i, Z_j$ (or $T_i, T_j$) of neighboring zones are often very close to each other and $R_{ij}$ is not small, the total heat gain from neighboring zones is (sometimes much) less dominant compared with the heat gain from the outside plus the indoor heat gain in every zone. Therefore, in this section, we ignore the term $\sum_{j \in \mathcal{N}(i)} \frac{T_j - T_i}{R_{ij}}$ in (1) as well as $\sum_{j \in \mathcal{N}(i)} \frac{Z_j - Z_i}{R_{ij}}$ in (2b). In addition, we approximate the power consumption of the supply fan by one of its upper bound $s_0 \sum_{i \in \mathcal{N}} \frac{1}{2} m_i^2$ where $\phi$ is a given constant satisfying $\phi \geq \eta$, because using a Cauchy-Schwarz inequality, we have

\[ s_0 \sum_{i \in \mathcal{N}} \frac{1}{2} m_i^2 \geq s_0 (\sum_{i \in \mathcal{N}} m_i)^2/2|\mathcal{N}| \geq s (\sum_{i \in \mathcal{N}} m_i)^3/2|\mathcal{N}| \]

where $|\mathcal{N}|$ is the number of zones. By doing this, the objective function of the optimization problem becomes separable, which will enable us to design a decentralized controller later. To summarize, we obtain an approximate model to (1) as

\[ C_i T_i = \frac{T^o - T_i}{R_i} + c_a m_i (T^o - T_i) + Q_i \]

(3)

together with an approximate problem to (2) given by

\[ \min_{Z_i, m_i} \left\{ \frac{1}{2} r_i (Z_i - T_i)^2 + \frac{w}{\eta} c_a m_i |Z_i - T^o| + \frac{1}{2} w s_\phi m_i^2 \right\} \]

\[ \text{s. t. } \frac{T^o - Z_i}{R_i} + c_a m_i (T^o - Z_i) + Q_i = 0 \]

(4a)

\[ T_i^{\text{min}} \leq Z_i \leq T_i^{\text{max}} \]

(4b)

\[ m_i^{\text{min}} \leq m_i \leq m_i^{\text{max}} \]

(4c)

\[ \sum_{i \in \mathcal{N}} m_i \leq m \]

(4d)

where $i \in \mathcal{N}$ in (2a), (2b)-(2d). Remark that the approximate model (3) still keeps the global asymptomatic stability as in Proposition 1. We assume that problem (4) is feasible and satisfies Slater’s condition, moreover, we require $r_i > \frac{w_\phi^2}{\eta s_\phi}, i \in \mathcal{N}$ so that (4a) can be strictly convex in $Z_i, m_i$.

Although (4) is still nonconvex due to the term $m_i Z_i$ in (4b), we can actually convexify it and show that the convexification is indeed exact. From (4b), we have $m_i = \frac{T^o - Z_i + Q_i}{c_a (Z_i - T^o)}$. Define

\[ f_i(Z_i) = \frac{T^o - Z_i + Q_i}{c_a (Z_i - T^o)} = m_i \geq m_i^{\text{min}} > 0, \ i \in \mathcal{N} \]

where the physical meaning of $f_i(Z_i)$ is the approximate air flow rate for zone $i$ to stay at temperature $Z_i$. Then

\[ f_i'(Z_i) = \frac{T^o - T_i}{c_a (Z_i - T^o)^2}, \ i \in \mathcal{N} \]

\[ f_i''(Z_i) = -2 \frac{T^o - T_i}{c_a (Z_i - T^o)^3} > 0, \ i \in \mathcal{N} \]
hold, i.e., $f_i(Z_i)$ is convex in $Z_i$, because $T^s < T^o$ holds in the cooling mode, and $T^s > T^o$, $\frac{T^s-T^o}{R_i} > Q_i$ hold (as $Q_i$ is less dominant [16], [13], especially in winter) in the heating mode. Therefore, we can relax (4) as a convex optimization problem given by

$$\min_{Z_i,m_i} \sum_{i \in \mathcal{N}} \left[ \frac{1}{2} r_i(Z_i - T_i^{set})^2 + \frac{w}{\eta} c_a m_i |Z_i - T^s| + \frac{1}{2} w s \phi m_i^2 \right]$$

(5a)

s. t. \ 
$$\frac{T^s-Z_i}{c_a (Z_i-T^s)} + Q_i \leq m_i, \ i \in \mathcal{N}$$

(5b)

$$T_i^{min} \leq Z_i \leq T_i^{max}, \ i \in \mathcal{N}$$

(5c)

$$m_i^{min} \leq m_i \leq m_i^{max}, \ i \in \mathcal{N}$$

(5d)

$$\sum_{i \in \mathcal{N}} m_i \leq m$$

(5e)

Remark 2. In the cooling mode, $f_i'(Z_i) < 0, \forall i$ holds while in the heating mode, $f_i'(Z_i) > 0, \forall i$ holds, i.e., $f_i(Z_i)$ is monotonic in $Z_i, \forall i$.

B. Tightness of the convex relaxation

Before investigating the tightness of the convex relaxation, we make an assumption on the system parameters.

Assumption 1. In the cooling mode, $T_i^{set} < T^o, \forall i$ holds; in the heating mode, $T_i^{set} > T^o, \forall i$ holds. Moreover, the zone temperature set point satisfies $f_i'(T_i^{set}) \geq m_i^{min}, \forall i$.

This assumption holds in usual practice. Firstly, when the outdoor temperature is higher (in hot days)/lower (in cold days) than users’ expectation (quantified as $T_i^{set}$), the HVAC system should be turned on. So we usually have $T_i^{set} < T^o, \forall i$ for cooling and $T_i^{set} > T^o, \forall i$ for heating. Secondly, note that $m_i^{min} \approx 0, \forall i$ holds as stated earlier, and $f_i(Z_i)$ stands for the approximate flow rate for zone $i$ to stay at temperature $Z_i$. The inequality $f_i'(T_i^{set}) \geq m_i^{min}$ means that in order to make the temperature of zone $i$ be the set point, the flow rate needs to be no smaller than its minimum. Otherwise, we will have $f_i'(T_i^{set}) < m_i^{min} \approx 0$ which can lead to $T_i^{set} \geq T^o$ in the cooling mode, and $T_i^{set} \leq T^o$ in the heating mode, a contradiction to the first part of this assumption. To sum up, users can choose their set points to satisfy this assumption in reality, which is completely decentralized.

The Lagrangian of problem (5) is

$$L = \sum_{i \in \mathcal{N}} \left[ \frac{1}{2} r_i(Z_i - T_i^{set})^2 + \frac{w}{\eta} c_a m_i |Z_i - T^s| + \frac{1}{2} w s \phi m_i^2 \right]$$

$$+ \sum_{i \in \mathcal{N}} \left( \frac{T^s-Z_i}{c_a (Z_i-T^s)} + Q_i \right) - m_i \right] + \lambda^+ \sum_{i \in \mathcal{N}} m_i - m \right]$$

$$+ \sum_{i \in \mathcal{N}} \nu_i^+ (Z_i - T_i^{max}) + \sum_{i \in \mathcal{N}} \nu_i^- (T_i^{min} - Z_i)$$

$$+ \sum_{i \in \mathcal{N}} \mu_i^+ (m_i - m_i^{max}) + \sum_{i \in \mathcal{N}} \mu_i^- (m_i^{min} - m_i)$$

where $\zeta_i, \nu_i^+, \nu_i^-, \mu_i^+, \mu_i^-$ are the Lagrange multipliers (dual variables) for constraints (5b)-(5e). Since problem (5) is convex, feasible and satisfies Slater’s condition, the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient conditions for optimality [26], given by

$$\frac{\partial L}{\partial Z_i} = r_i(Z_i - T_i^{set}) \pm \frac{w}{\eta} c_a m_i |Z_i - T^s| + \frac{1}{2} w s \phi m_i^2$$

(6a)

$$\frac{\partial L}{\partial m_i} = w s \phi m_i \pm \frac{w}{\eta} c_a (Z_i - T^s) - \zeta_i + \mu_i^+ - \mu_i^- + \lambda^+ = 0$$

(6b)

$$\zeta_i (f_i(Z_i) - m_i) = 0, \ z_i \geq 0, f_i(Z_i) - m_i \leq 0$$

(6c)

$$\nu_i^+ (Z_i - T_i^{max}) = 0, \ \nu_i^+ \geq 0, Z_i - T_i^{max} \leq 0$$

(6d)

$$\nu_i^- (T_i^{min} - Z_i) = 0, \ \nu_i^- \geq 0, T_i^{min} - Z_i \leq 0$$

(6e)

$$\mu_i^+ (m_i - m_i^{max}) = 0, \ \mu_i^+ \geq 0, m_i - m_i^{max} \leq 0$$

(6f)

$$\mu_i^- (m_i^{min} - m_i) = 0, \ \mu_i^- \geq 0, m_i^{min} - m_i \leq 0$$

(6g)

$$\lambda^+ \left( \sum_{i \in \mathcal{N}} m_i - m \right) = 0, \ \lambda^+ \geq 0, \ \sum_{i \in \mathcal{N}} m_i - m \leq 0$$

(6h)

where $i \in \mathcal{N}$ in (6c)-(6h), and “+” takes “+” in the cooling mode and “−” in the heating mode in (6f)-(6g).

Remark 3. The convex relaxation from problem (4) to (5) is tight if and only if any solution of (5) satisfies $\zeta_i > 0$ or $\zeta_i = 0, f_i(Z_i) = m_i$, $\forall i$.

As for the tightness, we have the following theorem.

Theorem 1. Under Assumption 1, the convex relaxation from problem (4) to (5) is always tight.

Proof: Let us assume that there exists an $i$ in the solution of (6) such that $\zeta_i = 0, f_i(Z_i) < m_i$ holds. Then we can obtain $r_i(Z_i - T_i^{set}) + \frac{w}{\eta} c_a m_i = \nu_i^+ - \nu_i^-$ in the cooling mode and $r_i(Z_i - T_i^{set}) - \frac{w}{\eta} c_a m_i = \nu_i^- - \nu_i^+$ in the heating mode from (6c), and $\nu_i^+ = w s \phi m_i \pm \frac{w}{\eta} c_a (Z_i - T^s) + \mu_i^+ + \lambda^+ > 0$ from (6h). When $\nu_i^- = \nu_i^- = 0$ holds, we have $Z_i < T_i^{set}$ in the cooling mode and $Z_i > T_i^{set}$ in the heating mode. When $\nu_i^- > \nu_i^+$ holds which only happens in the cooling mode (otherwise in the heating mode, we end up with $Z_i = T_i^{max} > T_i^{set}$, a contradiction), we have $Z_i = T_i^{min} < T_i^{set}$. When $\nu_i^- > \nu_i^-$ holds which only happens in the heating mode, we have $Z_i = T_i^{max} > T_i^{set}$. These facts lead to $f_i'(T_i^{set}) < f_i(Z_i) < m_i = m_i^{min}$ by Remark 2, a contradiction to Assumption 1. Based on Remark 3, the convex relaxation from problem (4) to (5) is always tight. □

C. A decentralized algorithm

Theorem 1 indicates that solving (4) is equivalent to solving (5), while (5) can be solved in either a centralized or distributed/decentralized way. Any centralized algorithm requires to measure the outdoor temperature $T^o$ and the indoor heat gain $Q_i$ in every zone. Because these exogenous disturbances can fluctuate frequently and are not easy to obtain, the cost of centralized algorithms would be expensive. In this section, we develop a real-time decentralized algorithm that does not need measurement of these exogenous disturbances.

Since (5) is convex, feasible and satisfies Slater’s condition, we design the following algorithm to solve (5) based on a
standard primal-dual gradient method \cite{27}:

\[
\dot{Z}_i = -k_{Z_i} \left( \frac{\partial L}{\partial Z_i} \right) = k_{Z_i} \left( r_i(T_i^{\text{set}} - Z_i) - \zeta_i \frac{T_i - T_s}{c_a(Z_i - T_s)^2} - \frac{w}{\eta} c_m m_i - \nu_i^+ + \nu_i^- \right) + \frac{w}{\eta} c_m m_i - \nu_i^+ + \nu_i^- 
\]

(7a)

\[
\dot{m}_i = -k_{m_i} \left( \frac{\partial L}{\partial m_i} \right) = k_{m_i} (-w s \phi m_i + \frac{w}{\eta} c_a (Z_i - T_s) + \varsigma_i - \mu_i^+ + \mu_i^- - \lambda^+) 
\]

(7b)

\[
\dot{\zeta}_i = k_{\zeta_i} \left( \frac{\partial L}{\partial \zeta_i} \right) = k_{\zeta_i} \left( \frac{T_i - Z_i}{c_a(Z_i - T_s)} - m_i \right)_\varsigma 
\]

(7c)

\[
\dot{\nu}_i^+ = k_{\nu_i^+} \left( \frac{\partial L}{\partial \nu_i^+} \right)_\nu^+ = k_{\nu_i^+} (Z_i - T_i^{\text{max}})_\nu^+ 
\]

(7d)

\[
\dot{\nu}_i^- = k_{\nu_i^-} \left( \frac{\partial L}{\partial \nu_i^-} \right)_\nu^- = k_{\nu_i^-} (Z_i - T_i^{\text{min}})_\nu^- 
\]

(7e)

\[
\dot{\mu}_i^+ = k_{\mu_i^+} \left( \frac{\partial L}{\partial \mu_i^+} \right)_\mu^+ = k_{\mu_i^+} (m_i - m_i^{\text{max}})^+ \mu_i^+ 
\]

(7f)

\[
\dot{\mu}_i^- = k_{\mu_i^-} \left( \frac{\partial L}{\partial \mu_i^-} \right)_\mu^- = k_{\mu_i^-} (m_i^{\text{min}} - m_i^+)^- \mu_i^- 
\]

(7g)

\[
\dot{\lambda}^+ = k_{\lambda^+} \left( \frac{\partial L}{\partial \lambda^+} \right)_\lambda^+ = k_{\lambda^+} \left( \sum_{i \in N} m_i - \overline{m} \right)_\lambda^+ 
\]

(7h)

where \( i \in N \) in \((7a)-(7h))\), \( k_{Z_i}, k_{m_i}, k_{\zeta_i}, k_{\nu_i^+}, k_{\nu_i^-}, k_{\mu_i^+}, k_{\mu_i^-}, k_{\lambda^+} \) are positive scalars representing the controller gains, and "+" takes "-" in the cooling mode and "-" in the heating mode in \((7a)-(7h))\).  Note that \( T_i \) has its own dynamics given by \((7a)-(7h))\) and thus can not be designed, which is why we replace \( T_i \) with \( Z_i \) initially, i.e., \( Z_i, i \in N \) are ancillary state variables. According to \cite{27, 28}, it is true that \((7a)-(7h))\) asymptotically converges to an equilibrium point which is the unique optimal solution of \((5))\), since the optimization problem is convex and the objective function is strictly convex in the decision variables. Now using \( m_i \) in \((7a)-(7h))\) as the input to system \((3))\), we can naturally obtain a real-time decentralized controller to regulate \((5))\) to a steady state which solves \((4))\).

**Theorem 2.** Given constant/step change/slow-varying \( T_i^0, Q_i \), under Assumption \(7), the equilibrium point of the overall system \((3))\) and \((4))\) is unique, asymptotically stable and \( T_i, m_i \) of the equilibrium point is the optimal solution of \((4))\).

**Proof:** For constant/step change/slow-varying disturbances \( T_i^0, Q_i \), since \((5))\) is convex and the objective function is strictly convex in \( Z_i, m_i \), its optimal solution is unique \cite{26}. Therefore, the equilibrium point of \((7a)-(7h))\) is unique, asymptotically stable \cite{27, 28} and is the optimal solution of \((5))\) by Theorem \(1).\) So the equilibrium point of \((3))\) and \((4))\) is also asymptotically stable, due to the cascade nature, i.e., \((7a)-(7h))\)\). Under the fact that the equilibrium point of \((3))\) is uniquely determined by the inputs \( T_i^0, m_i, Q_i \), where \( m_i \) is given by \((7a))\), we have \( T_i = Z_i, (T_i \) is the state given by \((3))\) when the overall system reaches steady state, which completes the proof. \( \square \)

![Fig. 2: Information exchange of the decentralized controller.](image)

This theorem requires \( T_i^0, Q_i \) to be either constant, step change, or slow-varying, which holds in practice as these disturbances vary at a time-scale of minutes. Remark that our controller operates in real-time, i.e., at a time-scale of seconds.

In \((7a)-(7h))\), the disturbances \( T_i^0, Q_i \) appear. Motivated by \cite{29}, to make the algorithm implementable without measuring these terms, we substitute \((3))\) into \((7a)-(7h))\) to get

\[
\dot{Z}_i = k_{Z_i} \left( r_i(T_i^{\text{set}} - Z_i) + \frac{w}{\eta} c_m m_i - \nu_i^+ + \nu_i^- \right) 
\]

(8a)

\[
\dot{\zeta}_i = k_{\zeta_i} \left( \frac{T_i - Z_i}{c_a(Z_i - T_s)} + C_i T_i - c_a m_i (T_i - T_s)^2 \right) - \varsigma_i 
\]

(8b)

in which the derivative action \( T_i \) can be implemented by using a differentiator with some form of filtering \cite{30}.\)

**Implementation.** The proposed control scheme \((7a)-(7h))\) \((5))\) and \((6))\) is completely decentralized as shown in Figure \(2\) and can be implemented as follow. Given \( T_i^0, R_i, C_i, r_i, w, \phi, s, \eta, m_i^{\text{min}}, m_i^{\text{max}}, \) each zone in the building collects \( T_i^{\text{set}} \), \( [T_i^{\text{min}}, T_i^{\text{max}}] \) from users, locally measures its indoor temperature \( T_i \), receives the feedback signal \( \lambda^+ \) from the supply fan/duct, and then uses this information to update \( Z_i, m_i, \zeta_i, \nu_i^+, \nu_i^-, \mu_i^+, \mu_i^- \) based on \((7a)-(7b)), \((7d)-(7g))\). On the other hand, given \( \overline{m} \), the supply fan/duct locally measures the total flow rate \( \sum_{i \in N} m_i \) which is proportional to the fan speed \cite{24} and uses \((7h))\) to update \( \lambda^+ \), then broadcasts it. Here \( T_i^0, R_i, C_i, s, \eta, m_i^{\text{min}}, m_i^{\text{max}}, \overline{m} \) are building parameters and \( T_i^{\text{set}}, [T_i^{\text{min}}, T_i^{\text{max}}], r_i, w, \phi \) are parameters specified by users.

**D. an extension.**

In this section, we provide an alternative scenario which fits the above solution procedure. Consider a community or a neighborhood consisting of a number of separated houses. Each house is equipped with an HVAC system. Since these houses are separated (i.e., \( R_{ij} = \infty \)), we can naturally use system \((5))\) to model their temperature dynamics. The objective is to regulate the temperature in each house to be close to its set point, meanwhile, to minimize the total energy consumption of the community. As a result, we end up with a steady-
state optimization problem given by (2). Note that (i) each entire house is modeled by a RC model (we do not specify zones inside the house) and has a unique temperature set point determined by users; (ii) \( m_i^{\text{min}} = 0 \) holds in (4) because we now allow houses to turn off their HVAC systems; (iii) \( \bar{m} \) in (3) is an indicator of the upper bound of the total energy consumption in the community; (iv) the supply air temperature can be different in houses, i.e., \( T^s \) can be replaced by \( T^s_i \) in (3) and (4e)-(4f) (the mixed mode case, i.e., some of HVAC systems are in the cooling mode while the others are in the heating mode, is naturally included), and similar for \( w, \eta, s, \phi \).

Correspondingly, we modify Assumption 1 a bit as follow.

**Assumption 2.** For any house \( i \), when the HVAC system is on, the house temperature set point satisfies \( f_i(T_i^{\text{ext}}) > 0, \forall i \). Otherwise, the HVAC in house \( i \) is turned off and all terms and constraints relating to house \( i \) in problem (2) are excluded.

As before, users can choose their set points to satisfy this assumption, which is decentralized. Under this assumption and following the proof of Theorem 1, we can still ensure the tightness of the convex relaxation procedure in Section III-A. Moreover, we can still derive a decentralized algorithm similar to (7) for HVAC system management in the community. What needs to be emphasized is that now (7) is updated in a control center of the community, which receives \( m_i \) from each house and sends \( \lambda^+ \) to them (when \( m_i = 0 \), the control center can know that house \( i \) has turned off its HVAC system).

**IV. Method II: The General Case**

In this section, we focus on solving problem (2) directly. The key step is to reformulate (2) as an optimization problem without using the decision variable \( m_i \) by substituting (2b) into (2a) and (2d)-(2e) to eliminate \( m_i \). Before showing the details, let us first consider a simple scenario, i.e., a building with two adjacent zones, to gain an insight into this method.

**A. An illustrating example**

Let the constraint set of problem (2) be

\[
\begin{align*}
30 - Z_1 \leq 15 & \quad Z_2 - Z_1 \leq 15 \\
30 - Z_2 \leq 16 & \quad Z_1 - Z_2 \leq 18 \\
0.01 \leq m_1 \leq 0.5 & \quad 0.01 \leq m_2 \leq 0.5 \\
0.1 \leq m_1 + m_2 \leq 0.7
\end{align*}
\]

where we do not add user comfort constraint (2) (the reason will become clear later). The domain for \( m_1, m_2, Z_1, Z_2 \) is shown in Figure 3(a)-(b), which are nonconvex surfaces. However, by rewriting \( m_1, m_2 \) as functions of \( Z_1, Z_2 \)

\[
m_1 = \frac{30 - Z_1}{15} + \frac{Z_2 - Z_1}{18} + 0.1 \frac{1.012(Z_1 - 12.8)}{15} + 0.2 \quad m_2 = \frac{30 - Z_2}{16} + \frac{Z_1 - Z_2}{18} + 0.2 \frac{1.012(Z_2 - 12.8)}{15}
\]

and substituting them into all three inequality constraints, we can see that the constraint set for \( Z_1, Z_2 \) is actually convex as shown in Figure 4, which is also the two-dimensional view of Figure 3(a)-(b). The reason is as follows. Since the cooling mode is considered (i.e., \( Z_1, Z_2 \ll T^s = 12.8 \)), the constraints 0.01 \leq m_1 \leq 0.5 and 0.01 \leq m_2 \leq 0.5 are equivalent to

\[
0.01 \times 1.012(Z_1 - 12.8) \leq \frac{30 - Z_1}{15} + \frac{Z_2 - Z_1}{18} + 0.1 \leq 0.5 \times 1.012(Z_1 - 12.8)
\]

\[
0.01 \times 1.012(Z_2 - 12.8) \leq \frac{30 - Z_2}{16} + \frac{Z_1 - Z_2}{18} + 0.2 \leq 0.5 \times 1.012(Z_2 - 12.8)
\]

which are linear, and therefore, convex. So the convexity of the constraint set only depends on the convexity of the function \( m_1 + m_2 \). Then we calculate the Hessian matrix of

\[
m_1 + m_2 = \frac{30 - Z_1}{15} + \frac{Z_2 - Z_1}{18} + 0.1 \frac{1.012(Z_1 - 12.8)}{15} + 0.2 \frac{1.012(Z_2 - 12.8)}{15}
\]
with respect to \( Z_1, Z_2 \). Any point in the domain in Figure 3(c) has a positive semi-definite Hessian matrix of \( m_1 + m_2 \).

Next, we modify the upper bound of \( m_1 + m_2 \) to be 0.5 and 0.3, and draw the constraint sets of \( Z_1, Z_2 \), as illustrated in Figure 4. It can be seen that these constraint sets are also convex, because any point in the domains displayed in Figure 4 has a positive semi-definite Hessian matrix of \( m_1 + m_2 \). Note that including user comfort constraint (2c) will not affect the above analysis since it does not affect the convexity of the constraint set of \( Z_1, Z_2 \) but only makes the set smaller, i.e., the above analysis is actually less conservative than that with constraint (2b). These numerical results indicate that once we exclude \( m_i \) from problem (2), convexity can be obtained.

### B. Solution procedure

Now we provide the solution procedure for (2). Define

\[
h(Z) = \sum_{i \in N} m_i = \sum_{i \in N} \left( \frac{T^o - Z_i}{R_i} + \sum_{j \in N(i)} \frac{Z_j - Z_i}{R_{ij}} + Q_i \right),
\]

where \( Z \) is a collection of \( Z_i, i \in N \). By substituting (2b) into (2a) and (2) into (2c) to eliminate \( m_i \), problem (2) becomes

\[
\min_{Z_i} \sum_{i \in N} \frac{1}{2} r_i (Z_i - T^o) \leq 0, \quad \text{s. t. } T^o - Z_i \\
\sum_{i \in N} m_i \leq m_i \max \leq m_i \max \\
\frac{T^o - Z_i}{R_i} + \sum_{j \in N(i)} \frac{Z_j - Z_i}{R_{ij}} + Q_i \leq m_i \max \\
h(Z) \leq m_i.
\]

where \( i \in N \) in (9b)-(9d). Constraint (9d) can actually become linear once the cooling/heating mode (the sign of \( m_i \)) is determined (also, the absolute value sign in (9d) can be removed). The convexity of the constraint set only depends on constraint (9d). Now we claim that the HVAC system satisfies the following assumption under normal operating conditions.

**Assumption 3.** The Hessian matrix of \( h(Z) \), i.e.,

\[
\frac{\partial^2 h(Z)}{\partial Z^2} = \begin{bmatrix}
\frac{\partial^2 h(Z)}{\partial Z_i^2} & \frac{\partial^2 h(Z)}{\partial Z_i \partial Z_j} \\
\frac{\partial^2 h(Z)}{\partial Z_j \partial Z_i} & \frac{\partial^2 h(Z)}{\partial Z_j^2}
\end{bmatrix}
\]

\[
\frac{\partial^2 h(Z)}{\partial Z^2} = -2 \left( \frac{T^o - T^o}{R_i} + \sum_{j \in N(i)} \frac{T^o - Z_i}{R_{ij}} + Q_i \right) < 0
\]

\[
\frac{\partial h(Z)}{\partial Z_i} = -m_i \max - \frac{1}{R_{ij} c_a (Z_i - T^o)^2} < 0
\]

is positive semi-definite.

Assumption 3 is not conservative and it usually holds in practice, because \( \frac{\partial^2 h(Z)}{\partial Z^2} \) is usually diagonal dominant due to that \( Z_i, Z_j \) of neighboring zones do not deviate too much from each other (this can also be guaranteed by properly adjusting user comfort range), i.e.,

\[
\frac{|\partial^2 h(Z)|}{\partial Z^2} \leq \sum_{j \in N(i)} \left| \frac{\partial^2 h(Z)}{\partial Z_i \partial Z_j} \right| \approx -2 \left( \frac{T^o - T^o}{R_i} + \sum_{j \in N(i)} \frac{T^o - Z_i}{R_{ij}} + Q_i \right) c_a (Z_i - T^o)^2 > 0.
\]

This fact is demonstrated in the previous example: even though \( Z_1, Z_2 \) deviate about 8°C from each other, the constraint set remains convex as shown in Figures 3(c)-4.

Under Assumption 3 and the sign of \( Z_i - T^o \) is determined, problem (2) is convex since the objective function (9) is strictly convex in \( Z_i \) as convexity is preserved from \( h(Z) \) to \( (h(Z))^2 \). Motivated by a standard primal-dual gradient method [27], we design the following algorithm to solve (2) (assuming the cooling mode here, i.e., \( Z_i > T^o \), the heating mode case is similar):

\[
\dot{Z}_i = k_{Z_i} \left( r_i (T^o - Z_i) - \mu_i^+ \left( \frac{1}{R_i} + \sum_{j \in N(i)} \frac{1}{R_{ij}} + m_i \max c_a \right) \right)
\]

\[
- \sum_{j \in N(i)} \frac{\mu_i^+}{R_{ij}} - \mu_i^- \left( \frac{1}{R_i} + \sum_{j \in N(i)} \frac{1}{R_{ij}} + m_i \max c_a \right) - \nu_i^+ \left( \frac{T^o - T^o}{R_i} + \sum_{j \in N(i)} \frac{T^o - Z_i}{R_{ij}} - Q_i \right)
\]

\[
+ \sum_{j \in N(i)} \frac{1}{R_{ij} c_a (Z_j - T^o)^2} \right) (wsh^2(Z) + \lambda^+) + \frac{\nu_i^-}{\eta R_i}
\]

(10a)

(10b)

(10c)

(10d)

(10e)

(10f)

where \( \nu_i^+, \nu_i^-, \mu_i^+, \mu_i^- \), \( \lambda^+ \) are the Lagrange multipliers (dual variables) for constraints (2b)-(2d), \( k_i, k_i, k_{\mu_i^+}^+, k_{\mu_i^-}^-, k_i, k_i, k_{\mu_i^-}^-, \lambda^+ \) are positive scalars representing the controller gains, and \( i \in N \) in (10b)-(10e). In addition, we adopt the following low pass dynamics for each \( m_i \) as the control input to system (1):

\[
m_i = k_m \left( \frac{T^o - Z_i}{R_i} + \sum_{j \in N(i)} \frac{Z_j - Z_i}{R_{ij}} + Q_i \right)
\]

(11)

where \( k_m \) is the homogeneous controller gain. The reason for using the low pass dynamics is that it can attenuate high frequency noises to help improve system performance.

**Theorem 3.** Given constant/step change/slow-varying \( T^o, Q_i \) (remark that they vary at a time-scale of minutes), under Assumption 3 any trajectory of system (1), (10), and (11) asymptotically converges to a unique equilibrium point at which
A differentiator with some form of filtering \[29\], to make the algorithm implementable without measuring these terms, we substitute \[1\] into \([10]-[11]\) to get
\[
\dot{Z}_i = k_Z \left( r_i(T_i^{\text{set}} - Z_i) + \frac{w}{\eta R_i} - \nu_i^+ + \nu_i^- \right) \\
+ \mu_i^+ \left( \frac{1}{R_i} + \sum_{j \in N(i)} \frac{1}{R_{ij}} + m_i^{\max} c_a \right) - \sum_{j \in N(i)} \mu_j^+ \frac{1}{R_{ij}} \\
- \mu_i^- \left( \frac{1}{R_i} + \sum_{j \in N(i)} \frac{1}{R_{ij}} + m_i^{\min} c_a \right) + \sum_{j \in N(i)} \mu_j^- \frac{1}{R_{ij}} \\
- (3wsh^2(Z) + \lambda^+) \left( \sum_{j \in N(i)} \frac{1}{R_{ij}} c_a(Z_j - T^*) \right) + \\
\frac{T^* - Z_i - C_i \dot{T}_i + \sum_{j \in N(i)} \frac{T^* - Z_j + T_j - T_i}{R_{ij}} - c_a m_i(T_i - T^*)}{c_a(Z_i - T^*)^2}
\]
(12a)
\[
\dot{\mu}_i^+ = k_{\mu_i} \left( \frac{T_i - Z_i}{R_i} + C_i \dot{T}_i + \sum_{j \in N(i)} \frac{Z_j - Z_i - T_j + T_i}{R_{ij}} \right) + \frac{c_a m_i(T_i - T^*) - m_i^{\max} c_a(Z_i - T^*)}{\mu_i^+} \\
+ \sum_{j \in N(i)} \frac{Z_j - Z_i - T_j + T_i}{R_{ij}} - c_a m_i(T_i - T^*)
\]
(12b)
\[
\dot{\mu}_i^- = k_{\mu_i} \left( m_i^{\min} c_a(Z_i - T^*) - \frac{T_i - Z_i}{R_i} - C_i \dot{T}_i \right) + \frac{\sum_{j \in N(i)} \frac{Z_j - Z_i - T_j + T_i}{R_{ij}} - c_a m_i(T_i - T^*)}{\mu_i^-}
\]
(12c)
\[
\dot{\lambda}^+ = k_{\lambda} (h(Z) - \bar{m}) \lambda^+ \\
h(Z) = \frac{1}{k_m} \sum_{i \in N} m_i + \sum_{i \in N} m_i.
\]
(13a)
(13b)

In which the derivative action \( \dot{T}_i \) can be implemented by using a differentiator with some form of filtering \[30\].

**Implementation.** The proposed controller \([10]-[12]\) and \([12]\) is completely distributed as shown in Figure 5 and can be implemented as follow. Given \( T^*, R_i, R_{ij}, C_i, r_i, w, s, \eta, m_i^{\min}, m_i^{\max}, \) each zone in the building collects \( T_i^{\text{set}}, T_i^*, T_i^{\text{min}}, T_i^{\text{max}} \) from users, locally measures its indoor temperature \( T_i \), receives the feedback signals \( 3wsh^2(Z) + \lambda^+ \) from the supply fan/duct and \( T_j, Z_j, \mu_j^+, \mu_j^- \) from its neighboring zones, and then uses these information to update \( Z_i, \nu_i^+, \nu_i^-, \mu_i^+, \mu_i^- \) based on \([10b]-[10c]\), \([12a]-[12b]\) and \([12c]\). On the other hand, given \( m \), the supply fan/duct locally measures the total flow rate \( \sum_{i \in N} m_i \), which is proportional to the fan speed \[24\], and uses \([12d]\) to update \( \lambda^+ \) and then broadcasts \( 3wsh^2(Z) + \lambda^+ \). Note that the supply fan/duct can compute \( \lambda^+ \) and \( h(Z) \) via
\[
\dot{\lambda}^+ = k_{\lambda} (h(Z) - \bar{m}) \lambda^+ \\
h(Z) = \frac{1}{k_m} \sum_{i \in N} m_i + \sum_{i \in N} m_i.
\]
(13a)
(13b)

Also, \( T^*, R_i, R_{ij}, C_i, r_i, w, s, \eta, m_i^{\min}, m_i^{\max}, \eta \) are building parameters and \( T_i^{\text{set}}, T_i^*, T_i^{\text{min}}, T_i^{\text{max}} \), \( r_i, w \) are given by users.

### V. Numerical Investigations

In this section, we present two numerical examples for scenarios described in Sections [III]-[IV] respectively, using a house with four adjacent zones as illustrated in Figure 1. Only the cooling case is presented in the following, while the heating case is similar under the proposed control schemes.

The parameters of the simulations are from \([31], [22]\) all \( C_i = 20kJ/\degree C, R_i = 15^\circ C/kW, R_{ij} = 23^\circ C/kW, c_a = 0.012kJ/kg/\degree C, T^* = 12.8^\circ C \) (cooling), \( s = 2kW/(kg/s)^3, \eta = 2.9 \), all comfort ranges are within \( \pm 1.5^\degree C \) of their set points, all \( m_i^{\min}, m_i^{\max} = [0.01, 0.45]kg/s, \eta = 0.5kg/s, \phi = 1kg/s, r_i = 0.1p.u., w = 1p.u., all \( k_{\lambda} = 0.067p.u., all \( k_{\mu_i} = k_{\lambda_i} = k_{\phi_i} = k_{\phi_i} = k_{\lambda_i} = k_{\eta_i} = 1p.u. \) (i.e., per unit), and the disturbance injection is shown in Figure 6.

The simulation result of the first scenario is illustrated in Figures \([7]-[9]\) in which we changed \( w = 1p.u. \) to \( w = 0.1p.u. \) at 12h. The curves labelled with “app” indicate the case of using \([7b]-[7d]\) for the approximate model \[4\] while the curves labelled without “app” are for the accurate
Temperature (c−degree)

and energy saving when using the proposed controller. All

= 0

not saturate in other periods. Interestingly, there are always

starting from

the weight coefficient

deviations with respect to their set points before decreasing

Fig. 8 and explained in Section III as well. The temperature

from the other (outside and indoor) sources, as shown in

two cases is very small due to that the heat transfer between

model (1). It can be seen that the difference between these
two cases is very small due to that the heat transfer between
neighbor zones is negligible compared with the heat gain from
the other (outside and indoor) sources, as shown in Figure 8
and explained in Section III as well. The temperature
deviations with respect to their set points before decreasing
since starting from 12h, user comfort becomes more important
while energy consumption saving becomes less. The total
flow rate reaches its maximum from 12h to 16h, while does
not saturate in other periods. Interestingly, there are always
deviations from the real temperatures to their set points (unless
w = 0), because there is a tradeoff between user comfort
and energy saving when using the proposed controller. All
ζi are always positive, indicating that the convex relaxation
procedure proposed in Section III always ensures tightness.

In the second scenario, we modify the parameters as fol-

ows: all comfort ranges are within ±1.8°C of their set points,

min = [0.01, 0.15]kg/s, \( \bar{m} = 0.5 \text{kg/s} \) before 16h

while \( \bar{m} \) decreases to 0.4kg/s after 16h, \( w = 0 \) before 8h

and \( w = 1 \text{p.u.} \) after 8h (the other parameters remain the same).
The simulation result is illustrated in Figures 10-11. We can
see that the curves of the ancillary states \( Z_i \) almost coincide
with the temperature curves \( T_i \) as expected. Before 8h, the
temperatures are exactly the same as their set points since
there is no consumption reduction purpose and the total flow
rate dose not saturate during this period. After 8h, deviations
appear due to the consideration of energy saving. The total
flow rate reaches its maximum around 15h, and from 16h to
18h, thus, the zone temperatures further deviate from their set
points (due to the lack of capacity). All these two scenarios
inspire us that tuning the weight coefficient \( w \) (or equivalently
\( r_i \), as the optimal solution of (2)/(4) depends on the ratio
\( r_i / w, i \in \mathbb{N} \) can balance user comfort and HVAC system
energy consumption – there always exists a tradeoff between
user comfort and energy saving.

VI. CONCLUSION AND FUTURE WORK

This paper presents decentralized/distributed control frameworks
on real-time temperature regulation for HVAC systems in
energy efficient buildings. The proposed controllers adjust
air flow rates in each zone, which balances user comfort
and energy saving. Moreover, they can automatically adapt
to changes of disturbances such as the outdoor temperature
and indoor heat gains, without measuring or predicting those
values. Also, the implementation of the controllers are simple.
Future work includes: investigating, e.g., the \( H_2 \) and \( H_\infty \)
performances of the controlled systems; considering the fact of
reheating by each VAV box, and natural ventilation scheduled
by users (these will be characterized as extra exogenous
inputs to the thermal dynamics); studying the interaction
between the controlled building HVAC systems and power
grids/microgrids; last but not least, extending the control
design frameworks to other heating and cooling systems, for
instance, ground source heat pump systems.
Fig. 10: Temperatures in each zone under (10b)-(10c) and (12).

Fig. 11: Air flow rates, the total consumption and ancillary variables under (10b)-(10c) and (12).

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