Some recent advances in the understanding of $\mathcal{N} = 4$ supersymmetric Yang-Mills thermodynamics

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The interest in the thermodynamics of supersymmetric Yang-Mills started after Maldacena proposed the duality between string theory on AdS backgrounds and the large-$N$ limit of SYM theories. One of the motivations to study the thermal properties of $\mathcal{N} = 4$ supersymmetric Yang-Mills in four dimensions (SYM$_{4,4}$) is that at high temperatures, the weak-coupling limit of this theory has many similarities with high temperature quantum chromodynamics (QCD). In this proceedings contribution, we review recent calculations of the resummed perturbative free energy of $\mathcal{N} = 4$ supersymmetric Yang-Mills in four spacetime dimensions through second order in the ‘t Hooft coupling $\lambda$ at finite temperature and zero chemical potential. We compare our final result with prior results obtained in the weak and strong-coupling limits and construct a generalized Padé approximant that interpolates between the weak-coupling result and the large-$N_c$ strong-coupling result.
1. Introduction

The perturbative expansion of the free energy of hot non-Abelian gauge theory and in our case $\mathcal{N} = 4$ supersymmetric Yang-Mills in four dimensions with $N_c$ colors and gauge coupling $g$ can be written in the form

$$\lim_{\lambda \to 0} F \sim T^4 \left[ a_0 + a_2 \lambda + a_3 \lambda^{3/2} + (a_4 + a'_4 \log \lambda) \lambda^2 + O(\lambda^{5/2}) \right], \quad (1)$$

where $\lambda = g^2 N_c$ is the 't Hooft coupling. The leading term in this expression is the free energy of an ideal plasma and the $O(\lambda)$ correction can be obtained by computing two-loop Feynman diagrams. The next contribution is $O(\lambda^2)$ and comes from three-loop contributions. However, a problem emerges because one finds uncanceled infrared divergences at the three-loop level if one uses bare propagators. This happens in QCD as well and there the infrared divergences can be eliminated by summing over the so-called ring diagrams [1]. The solution is similar in SYM$_{4,4}$ and the only difference with QCD is the number and types of degrees of freedom. In the weak-coupling limit, the free energy of SYM$_{4,4}$ has been calculated through order $\lambda^3$ in [2] and in the opposite limit of strong coupling, the behavior of the SYM$_{4,4}$ free energy was studied using the AdS/CFT correspondence in [3].

2. $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 4-dimensions (SYM$_{4,4}$)

The SYM$_{4,4}$ theory can be obtained by dimensional reduction of SYM$_{1,D}$ in $D = D_{\text{max}} = 10$ with all fields being in the adjoint representation of $SU(N_c)$. The Lagrangian that generates the perturbative expansion for SYM$_{4,4}$ in Minkowski-space can be expressed as

$$\mathcal{L}_{\text{SYM}_{4,4}} = \text{Tr} \left[ -\frac{1}{2} G_{\mu \nu}^2 + (D_{\mu} \Phi_A)^2 + i \bar{\psi}_i D\psi_i - \frac{1}{2} g^2 (i[\Phi_A, \Phi_B])^2 
- ig \bar{\psi}_i [\alpha^p_{ij} X_p + i \beta^q_{ij} \gamma_5 Y_q, \psi_j] \right] + \mathcal{L}_{gf} + \mathcal{L}_{gh} + \Delta \mathcal{L}_{\text{SYM}}, \quad (2)$$

with $\Phi_A \in (X_1, Y_1, X_2, Y_2, X_3, Y_3)$ and $X_p$ and $Y_q$ denote scalars and pseudoscalar fields, respectively.

We are in general interested in supersymmetric field theories with supercharges in dimensions $D \leq D_{\text{max}}$, with $D$ being an integer. The evaluation of Feynman diagrams for theories that are obtained by dimensional reduction of SYM$_{1,D}$ can be carried out in a simple way that preserves the supersymmetry by taking all fields to be $D$-dimensional tensors or spinors and all momentum to be $d = D - 2e$ vectors. This scheme was introduced by W. Siegel and is called regularization by dimension reduction (RDR) [4].

3. Resummation in SYM$_{4,4}$

Since we want to obtain the thermodynamic functions up to $O(\lambda^2)$, we need to calculate Feynman diagrams through three loop order. However, at three loop level in QCD [1], infrared divergences appear that need to be canceled by summing over the ring diagrams appearing in the
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thermal mass counterterm. As detailed in ref. [1], in order to systematically resum the necessary diagrams, we need to modify the static bosonic propagators by incorporating gluon and scalar thermal masses, $m_D$ and $M$, respectively.

Following Arnold and Zhai, we introduce thermal masses, $m_D$ and $M$, only for the zero Matsubara modes of the gluon and scalar fields. The resulting reorganized Lagrangian density in frequency space can be rewritten as

$$L_{\text{SYM}_{4}}^{\text{resum}} = \{ L_{\text{SYM}_{4}} + \text{Tr} [m_D^2 F_0^2 \delta_p \delta_{p_0} - M^2 \Phi^2 \delta_{p_0}] \} - \text{Tr} [m_D^2 F_0^2 \delta_p \delta_{p_0} - M^2 \Phi^2 \delta_{p_0}] \ , \quad (3)$$

Then we absorb the two $A_0^2$ and $\Phi^2$ terms in the curly brackets into our unperturbed Lagrangian $L_0$, and treat the two terms outside the curly brackets as a perturbation.

![Feynman diagrams up to 3-loop order](https://via.placeholder.com/150)

Figure 1: Feynman diagrams up to 3-loop order. The dashed lines indicate a scalar field and dotted lines indicate a ghost field. The crosses are the thermal counter-terms.

3.1 The resummed one-loop free energy

The resummed one-loop free energy can be written as

$$F_{1\text{-loop}}^{\text{resum}} = d_A F_{0a} + d_F F_{0b} + d_S F_{0c} + d_A F_{0d} , \quad (4)$$

with $d_F = 4d_A$ and $d_S = 6d_A$. By using resummed gluonic and scalar propagators, imposing $D = 4$, $m_D^2 = 2\lambda T^2$, $M^2 = \lambda T^2$, and truncating at $O(\epsilon^0)$ one obtains

$$F_{1\text{-loop}}^{\text{resum}} = -d_A \left( \frac{\pi^2 T^4}{6} \right) \left[ 1 + \frac{3 + \sqrt{2}}{\pi^3} \lambda^{3/2} \right] . \quad (5)$$
3.2 The resummed two-loop free energy

The SYM\(_{4,4}\) two-loop free energy can be written as

\[
F_{\text{resum}}^{2\text{-loop}} = d_A \left\{ \lambda \left[ F_{1a} + F_{1b} + F_{1c} + F_{1d} + F_{1e} + F_{1f} + F_{1g} + F_{1h} \right] + F_{1i} + F_{1j} \right\} .
\] (6)

By using resummed gluonic and scalar propagators one obtains

\[
F_{\text{resum}}^{2\text{-loop}} = -d_A \left( \frac{\pi^2 T^4}{6} \right) \left[ -\frac{3}{2\pi^2} \lambda - \frac{3}{2\pi^4} \left( \frac{23}{8} + \frac{3\sqrt{2}}{4} + \frac{15\log 2}{4} - \log \lambda \right) \right] .
\] (7)

3.3 The resummed three-loop free energy

The calculation of the massless three-loop vacuum Feynman diagrams in SYM\(_{4,4}\) can be accomplished more simply in the corresponding SYM\(_{1,10}\) theory. As a result of this equivalence, one can consider the much smaller set of SYM\(_{1,10}\) graphs presented in fig. 1(d), which are topologically equivalent to three-loop QCD vacuum graphs. The three-loop results in SYM\(_{4,4}\) can be obtained by imposing \(D = D_{\text{max}} = 10, d = 4 - 2\epsilon\) in the SYM\(_{1,10}\) theory.

\[
F_{\text{vacuum}}^{3\text{-loop}} = d_A \lambda^2 \left[ F_{2a} + F_{2b} + F_{2c} + F_{2d} + F_{2e} + F_{2f} + F_{2g} + F_{2h} + F_{2i} + F_{2j} \right] \left| \frac{D}{d_{\text{max}} = 10} = 10 \right| \left( d = 4 - 2\epsilon \right) .
\] (8)

Infrared divergences are generated in eq. (8) due to 3-momentum integrations. These divergences are canceled by thermal mass counterterm diagrams in fig. 1(c).

\[
F_{\text{resum}}^{3\text{-loop}} = F_{\text{vacuum}}^{3\text{-loop}} + F_{\text{act}}^{3\text{-loop}} + F_{\text{bct}}^{3\text{-loop}}
\]

\[
= -d_A \left( \frac{\pi^2 T^4}{6} \right) \left( \frac{27}{8} + 3y + 3 \frac{\zeta'(1)}{\zeta(-1)} + 5 \log 2 - 6 \log \pi \right) .
\] (9)

4. SYM\(_{4,4}\) thermodynamic functions to \(O(\lambda^2)\)

Combining eqs. (5), (7), and (9), we obtain our final result for the resummed free energy in the RDR scheme through \(O(\lambda^2)\).

\[
F = -d_A \left( \frac{\pi^2 T^4}{6} \right) \left[ 1 - \frac{3}{2\pi^2} \lambda + \left( 3 + \sqrt{2} \right) \left( \frac{\lambda}{\pi^2} \right)^{3/2} \right]
\]

\[
+ \left[ -\frac{21}{8} - \frac{9\sqrt{2}}{8} + \frac{3}{2} \zeta(1) - \frac{25}{8} \log 2 + \frac{3}{2} \log \frac{\lambda}{\pi^2} \right] \left( \frac{\lambda}{\pi^2} \right) .
\] (10)

5. Conclusions and Outlook

In this work, we reviewed the computation of the thermodynamic function of SYM\(_{4,4}\) to \(O(\lambda^2)\). The final result, presented in eq. (10), extends our knowledge of weak-coupling SYM\(_{4,4}\) thermodynamics to include terms at \(O(\lambda^2)\) and \(O(\lambda^2 \log \lambda)\). With the \(O(\lambda^2)\) and \(O(\lambda^2 \log \lambda)\) coefficients in the SYM\(_{4,4}\) free energy, we then constructed a large-\(N_c\) Padé approximant that interpolates between the weak- and strong-coupling limits. Figure 2 summarizes our findings.

\[\text{We have noticed a small error in Ref. [5], recently. The first term on the second line of (10) should be } -\frac{21}{8} \text{ instead of } -\frac{45}{16} \text{ as obtained in Ref. [5]. Equation (10) already includes this correction.}\]
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Figure 2: The entropy density $S$ normalized by the $S_{\text{ideal}}$ in SYM$_{4,4}$ as a function of the 't Hooft coupling $\lambda$.

We have recently rederived the final result (10) using effective field theory techniques [6]. We are also working on computing the coefficient of $\lambda^{5/2}$ in the SYM$_{4,4}$ free energy using effective field theory methods. Finally, we also plan to pursue a three-loop HTLpt calculation of SYM$_{4,4}$ thermodynamics, extending our prior two-loop HTLpt results [7, 8].

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