Integrable Discrete Linear Systems
and One-Matrix Random Models

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**ABSTRACT**

In this paper we analyze one–matrix models by means of the associated discrete linear systems. We see that the consistency conditions of the discrete linear system lead to the Virasoro constraints. The linear system is endowed with gauge
invariances. We show that invariance under time–independent gauge transformations entails the integrability of the model, while the double scaling limit is connected with a time-dependent gauge transformation. We derive the continuum version of the discrete linear system, we prove that the partition function is actually the $\tau$–function of the KdV hierarchy and that the linear system completely determines the Virasoro constraints.
1. Introduction

In the study of two dimensional quantum gravity, one of the main approaches is provided by the random matrix models. In this context the most interesting and calculable model is the one Hermitian matrix model, which has been studied intensively [1] (for reviews, see [2]). At the discrete level, it is an integrable system [3], and its partition function satisfies certain constraints, which belong to the Borel subalgebra of the c=1 non-twisted Virasoro algebra[4]. After taking the double scaling limit the system is expected to become a KdV system restricted by twisted Virasoro constraints [5,6]. However, there is still a gap between the discrete level and its continuum version: first, the meaning of the Virasoro constraints is not clear; secondly, the identification of the partition function with the \( \tau \)–function as well as the appearance of the twisting has not been justified yet.

In this paper we will try to shed light upon these points by analyzing an auxiliary tool in one-matrix models: the relevant associated discrete linear systems. We will show in fact, in section 2, that a one-matrix model is equivalent to a certain discrete linear system (DLS) in which the string equation appears as a compatibility condition. In section 3 we will derive the discrete Virasoro conditions from the consistency conditions of the DLS. In section 4 we examine the “time-independent” gauge symmetry of the DLS and show that it is equivalent to the integrability of the system. In section 5 another kind of gauge symmetry is considered which includes in particular rescalings of the couplings, and leads naturally to the double scaling limit. In the last two sections we examine the continuum limit of the DLS and study its properties. In section 6 we determine the continuum analog of the DLS and find its consistency conditions. Finally, using the latter exactly as in the discrete case, in section 7 we determine the continuum Virasoro constraints. Our result is that the partition function is actually described by a KdV–\( \tau \) function, and the constraints are the twisted ones. This result had only been conjectured up to now.
2. One–Matrix Model

In this section we review the main results concerning the one–matrix model and introduce the associated discrete linear system (DLS).

The partition function of the one–matrix model is defined by [7]

\[ Z_N(t) = \int dMe^{-TrV(M)} = \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) \exp(-\sum_{i=1}^{N} V(\lambda_i)) \] (2.1)

with the potential

\[ V(M) = \sum_{r=1}^{\infty} t_r M^r \]

A powerful tool to study this model is the use of orthogonal polynomials. They are normalized as follows

\[ P_n(\lambda) = \lambda^n + \ldots \] (2.2)

and satisfy the orthogonal and recursion relations

\[ \int d\lambda e^{-V(\lambda)} P_n(\lambda)P_m(\lambda) = \delta_{n,m} h_n(t) \] (2.3)

\[ \lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda) \] (2.4)

It is convenient to introduce another set of the polynomials

\[ \xi_n(\lambda) \equiv \frac{1}{\sqrt{h_n}} e^{-\frac{1}{2}V(\lambda)} P_n(\lambda) \]

Then eqs. (2.3) and (2.4) become

\[ \int d\lambda \xi_n(t, \lambda)\xi_m(t, \lambda) = \delta_{n,m} \] (2.5a)
\[ \lambda \xi_n = \sqrt{R_{n+1}} \xi_{n+1} + S_n \xi_n + \sqrt{R_n} \xi_{n-1} \]  

(2.5b)

where the function \( R_n(t) \) are defined as the ratio

\[ R_n \equiv \frac{h_n}{h_{n-1}}, \quad n \geq 1. \]

One can show that

\[ S_n = -\frac{\partial}{\partial t_1} \ln h_n(t), \quad n \geq 0 \]  

(2.6a)

or more generally,

\[ \frac{\partial}{\partial t_r} \ln h_n(t) = -(Q^r)_{nn} \quad n \geq 0 \]  

(2.6b)

where \( Q \) is the Jacobi matrix

\[ Q_{nm} \equiv \int d\lambda \xi_m(t, \lambda) \lambda \xi_n(t, \lambda) \]  

(2.7)

\[ = \sqrt{R_{n+1}} \delta_{n,m-1} + S_n \delta_{n,m} + \sqrt{R_n} \delta_{n,m+1} \]

In the same way, we introduce the matrix \( P \)

\[ P_{nm} \equiv \int d\lambda \xi_m(\lambda) \frac{\partial}{\partial \lambda} \xi_n(\lambda) \]  

(2.8)

Using the orthogonal polynomials, we can perform the integrations in (2.1) and obtain

\[ Z_N(t) = N! h_0 h_1 h_2 \ldots h_{N-1} \]  

(2.9a)

\[ \frac{\partial}{\partial t_r} \ln Z_N(t) = -\sum_{n=0}^{N-1} Q^r_{nn} \equiv -Tr Q^r, \quad r \geq 1 \]  

(2.9b)

Next we introduce the discrete linear system alluded to in the introduction. Let us denote by \( \xi \) the column vector with components \( \xi_0, \xi_1, \xi_2, \ldots \), use the recursion
relations and differentiate the orthogonality relations with respect to $t_r$: we arrive at the following **discrete linear system** (DLS) of equations

\[
Q\xi = \lambda\xi \quad (2.10a)
\]

\[
\frac{\partial}{\partial t_r} \xi = Q^r_a \xi \quad (2.10b)
\]

\[
\frac{\partial}{\partial \lambda} \xi = P\xi \quad (2.11a)
\]

\[
P = \sum_{k=2}^{\infty} k t_k Q^{k-1}_a \quad (2.11b)
\]

where the dependence on $t$ and $\lambda$ has been understood. Here and throughout the paper we adopt the notation

\[
(Q^r_a)_{nm} \equiv \begin{cases} 
\frac{1}{2}(Q^r)_{nm}, & m < n \\
0, & m=n \\
-\frac{1}{2}(Q^r)_{nm}, & m > n
\end{cases} \quad (2.12)
\]

The product in the above equations is of course the matrix product.

The consistency conditions for this linear system give rise to the discrete KdV hierarchy[8]

\[
\frac{\partial Q}{\partial t_r} = [Q^r_a, Q] \quad (2.13)
\]

and to the so-called string equation

\[
[Q, P] = 1 \quad (2.14)
\]

All the integrability and criticality properties of the matrix model are encoded in the DLS. Showing this is the aim of our paper.
It is interesting to answer the question: to what extent is the correspondence between discrete linear systems and one-matrix models one to one? There certainly are linear systems that do not correspond to matrix models, however if we impose the matrix $Q$ to have the Jacobi form (2.7), the correspondence is one to one. Indeed let us start from the infinite column vector $\xi$ with orthonormalized components $\xi_0, \xi_1, \xi_2, \ldots$ as in eq.(2.5a), and write the system

$$Q\xi = \lambda \xi \quad (2.10a')$$

$$\frac{\partial}{\partial t} \xi = Q^r_a \xi \quad (2.10b')$$

Then we can reconstruct the partition function from

$$\frac{\partial}{\partial t} \ln Z_N(t) = -TrQ^r, \quad r \geq 1$$

If we define now

$$\frac{\partial}{\partial \lambda} \xi = P\xi$$

we have the consistency condition

$$\frac{\partial P}{\partial t} = [Q^r_a, P] \quad (2.11a)$$

This admit the only solution

$$P = \sum_{k=2}^{\infty} k t_k Q_a^{k-1} \quad (2.11b)$$

using simply eqs.(2.10a’) and (2.10b’) beside the orthonormality conditions.

To end this section let us pause a bit to make a comment on the DLS. As we will see, it is justified to consider eq.(2.10a) as a discrete version of the Schrödinger equation where $\lambda$ plays the role of the spectral parameter; while eq.(2.10b) shows
the KdV–type of (isospectral) deformations of the discrete Schrödinger equation we will be discussing later on. It is well known that, typically, the Liouville theory is characterized by a Schrödinger equation (actually by two of them, one for each chirality). Even though we will be using throughout the paper only the linear system (2.10-11), it is interesting to notice that we can push this analogy further by introducing a discrete version of the Drinfeld-Sokolov linear system. This is done as follows.

Introduce the two matrices $Q_1$ and $Q_2 = (Q_1)^t$, $t$ meaning transposition, by

$$
(Q_1)_{nm} = \sqrt{X_n}\delta_{n,m} + \sqrt{\frac{R_n}{X_{n-1}}}\delta_{n,m+1}
$$

and the block matrix and vector

$$
Q = \begin{pmatrix} Q_1 & -1 \\ -\lambda & Q_2 \end{pmatrix}, \quad \Xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}
$$

Then the discrete Drinfeld-Sokolov linear system is

$$
Q\Xi = 0
$$

This implies in particular $Q_2Q_1\xi_1 = \lambda\xi_1$, and we recover eq.(2.10a), provided $Q = Q_2Q_1$. The latter condition allows us to uniquely determine the $X_n$’s introduced above in terms of the $S_n$’s and $R_n$’s. One finds

$$
X_n = \frac{Y_n}{Y_{n-1}}, \quad n \geq 1
$$

where, introducing for the sake of homogeneity the notation $R_i = R_{ii-1}$,

$$
Y_0 = S_0, \quad Y_i = S_i S_{i-1} - R_0
$$

$$
Y_n = \sum_{0 \leq k \leq \frac{n}{2}} (-1)^k \sum_{i_1 < i_2 - 1 < \ldots < i_k - 1} S_{i_k+1} R_{i_k i_k-1} S_{i_k-2} \cdots S_{i_2+1} R_{i_2 i_2-1} S_{i_2-2} \cdots S_{i_1+1} R_{i_1 i_1-1} S_{i_1-2} \cdots S_0
$$

$$
(2.15)
$$

$$
(2.16)
$$

$$
(2.17)
$$

$$
(2.18)
$$

$$
(2.19)
$$
We can go further. Let us introduce the $sl_2$ generators

$$
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
$$

while $I$ will denote the identity $2 \times 2$ matrix. Next define

$$
\delta = \frac{Q_1 + Q_2}{2}, \quad \Delta = \delta I \\
\pi = \frac{Q_1 - Q_2}{2}, \quad \Pi = \pi H \\
\mathcal{E}_+ = E_+ + \lambda E_-, \quad \mathcal{A} = \Pi - \mathcal{E}_+
$$

Then the discrete Drinfeld-Sokolov linear system can be written

$$(\Delta + \mathcal{A}) \Xi = 0 \quad (2.17')$$

We notice that $\mathcal{E}_+$ is the sum of the step operators corresponding to the simple roots of the affine $sl_2$ algebra, provided we identify the spectral parameter with the loop parameter. In this sense $\mathcal{A}$ can be thought of as the discrete analogue of an $sl_2$ loop algebra connection.

3. Virasoro constraints from DLS

An important piece of information for the matrix model is contained in the so-called Virasoro constraints$^{[4,5,6]}$

$$
L_n Z_N(t) = 0, \quad n \geq -1 \quad (3.1a)
$$

where

$$
L_n = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k-1}} - 2N \frac{\partial}{\partial t_n} + \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_k \partial t_{n-k}} + N^2 \delta_{n0}, \quad (3.1b)
$$

$$
[L_n, L_m] = (n - m) L_{n+m}. \quad (3.1c)
$$

They completely determine the possible perturbations.
In this section we show that the Virasoro constraints result from the consistency conditions of the linear system (2.10-11), eqs.(2.13-14).

To this end we rewrite the string equation (2.14) in the following form

$$\sum_{k=2}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}} Q = -1,$$  \hspace{1cm} (3.2)

where we have used the KdV equations (2.13). Eq. (3.2) implies that

$$\ell S_n = -1, \quad n \geq 0; \quad \ell \equiv \sum_{k=2}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}}$$  \hspace{1cm} (3.3)

which, by (2.6a), can be re-expressed as

$$\ell \frac{\partial}{\partial t_1} \ln h_n = 1, \quad \forall n \geq 0$$

since the operator $\ell$ commutes with $\frac{\partial}{\partial t_1}$. After integrating over $t_1$, and using the formula (2.6b), we get

$$\sum_{k=1}^{\infty} k t_k (Q^{k-1})_{nn} + \alpha = 0, \quad \forall n \geq 0.$$  \hspace{1cm} (3.4)

At first sight the integration constant $\alpha$ seems to depend on $t_2, t_3, ...$, but using the discrete KdV hierarchy and the string equation one can prove that $\alpha$ is actually a constant. After summation over $n$, from (2.9b) we obtain

$$(\ell - N(t_1 - \alpha)) Z_N(t) = 0$$

We see that we can absorb $\alpha$ by a redefinition of $t_1$. So finally we get

$$\left( \sum_{k=2}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}} - N t_1 \right) Z_N(t) = 0$$  \hspace{1cm} (3.5)

which is nothing but the $L_{-1}$ constraint.
We remark that choosing even potentials would imply $S_n \equiv 0$; therefore eq. (3.3) would be meaningless and would forbid us to recover the $L_{-1}$ Virasoro condition. We will see later on that in the continuum limit this obstruction is removed.

In order to derive the other Virasoro constraints, we introduce the quantities

$$B_n^{(r)} \equiv \sqrt{R_{n-1} \ldots R_{n-r}} P_{n,n-r}, \quad r \geq 0. \quad (3.6)$$

Due to the string equation (2.14), one finds for the above symbols the recursion relations

$$B_{n+1}^{(r+1)} - B_n^{(r+1)} = (S_{n-r} - S_n)B_n^{(r)} + R_{n-r}B_n^{(r-1)}$$

$$- R_{n-1}B_{n-1}^{(r-1)} + \delta_{r,0}$$

The first few ones are as follows

$$B_n^{(0)} = 0, \quad B_n^{(1)} = n$$

$$B_n^{(2)} = S_0 + S_1 + \ldots + S_{n-2} - (n-1)S_{n-1}$$

$$B_n^{(3)} = \sum_{i=0}^{n-3} S_i^2 + \sum_{i=0}^{n-4} S_i S_{i+1} - \sum_{i=0}^{n-4} S_i (S_{n-2} + S_{n-1})$$

$$- S_{n-3}S_{n-1} + 2 \sum_{i=0}^{n-3} R_i - (n-2)R_{n-2} \quad \ldots$$

(3.7)

On the other hand, from the KdV equations, it is easy to see that

$$\frac{\partial}{\partial t_1} TrQ = \frac{\partial}{\partial t_1} \sum_{i=0}^{N-1} S_i = -R_{N-1}. \quad (3.8)$$

Furthermore, since $Q^r$ is a symmetric matrix, from the eqs. (2.11), (3.4) with $\alpha = 0$,
one finds that
\[
\sum_{k=1}^{\infty} kt_k Q_{nl}^{(k-1)} = \begin{cases} 
  P_{nl}, & n > l \\
  0, & n = l \\
  -P_{nl}, & n < l 
\end{cases}.
\]
(3.9)

Therefore, for any positive integer \(r\), one obtains
\[
\sum_{k=1}^{\infty} kt_k Q_{nn}^{(k+r)} = \sum_{k=1}^{\infty} \sum_{l=n-r-1}^{n+r+1} (kt_k Q_{nl}^{(k-1)})_{nl} (Q^{r+1})_{ln} 
\]
\[= B_{n+r+1}^{(r+1)} + B_n^{(r+1)} + \ldots
\]
(3.10)

Then, after summation over \(n\), the above equations give the \(L_r\)–constraint. For instance, for the first three cases, it is easy to check that
\[
\sum_{k=1}^{\infty} kt_k Q_{nn}^k = \sum_{k=1}^{\infty} kt_k Q_{n,n-1}^k Q_{n-1,n} + \sum_{k=1}^{\infty} kt_k Q_{nn}^{k-1} Q_{nn} + \sum_{k=1}^{\infty} kt_k Q_{n,n+1}^{k-1} Q_{n+1,n}
\]
\[= B_n^{(1)} + B_{n+1}^{(1)} = 2n + 1
\]
\[
\sum_{k=1}^{\infty} kt_k Q_{nn}^{k+1} = B_{n+1}^{(1)} (S_n + S_{n+1}) + B_n^{(1)} (S_n + S_{n-1}) + B_n^{(2)} + B_{n+2}^{(2)}
\]
\[= 2[S_0 + S_1 + \ldots + S_{n-1} + (n+1)S_n]
\]
\[
\sum_{k=1}^{\infty} kt_k Q_{nn}^{k+2} = B_{n+3}^{(3)} + B_n^{(3)} + B_n^{(2)} (S_n + S_{n-1} + S_{n-2}) + B_{n+2}^{(2)} (S_n + S_{n+1} + S_{n+2})
\]
\[+ B_n^{(1)} (R_n + R_{n-1} + R_{n-2} + S_{n-1}^2 + S_n S_{n-1} + S_n^2)
\]
\[+ B_{n+1}^{(1)} (R_n + R_{n-1} + R_{n+1} + S_{n+1}^2 + S_n S_{n+1} + S_n^2)
\]
which after summing over \(n\), and making use of (2.9b), (3.7) and (3.8), become the
Virasoro constraints

\[ L_n Z_N(t) = 0, \quad n = 0, 1, 2 \quad (3.11a) \]

\[ L_n = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} - 2N \frac{\partial}{\partial t_n} + \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_k \partial t_{n-k}} + N^2 \delta_{n,0}. \quad (3.11b) \]

The Virasoro algebraic structure (3.1c) ensures that the higher order constraints are also true.

4. Gauge Symmetry and Integrability

As we anticipated in the introduction the discrete linear system (2.10) is characterized by gauge symmetries which, on the one hand, ensure integrability and, on the other hand, allow us to envisage the double scaling limit as a singular gauge transformation. This section and the following one are devoted to studying the properties of these gauge transformations.

Let us consider the following transformation (at fixed \( t_k \)'s)

\[
\begin{align*}
\xi &\rightarrow \hat{\xi} = G^{-1} \xi, \\
Q &\rightarrow \hat{Q} = G^{-1} Q G
\end{align*}
\]

\( (4.1) \)

where \( G \) is a unitary matrix. If the transformed Jacobi matrix \( \hat{Q} \) has the same structure as \( Q \), i.e. only the diagonal line and the first two off–diagonal lines are non–zero, and, moreover, if

\[
\begin{align*}
\hat{Q} \hat{\xi} &= \lambda \hat{\xi} \\
\frac{\partial}{\partial t_r} \hat{\xi} &= \hat{Q}^\ast_{rr} \hat{\xi}
\end{align*}
\]

\( (4.2) \)

then, we say that our linear system is gauge invariant.
Let us examine these transformations more closely by considering the infinitesimal transformation

\[ G = 1 + \varepsilon g. \]

Then, the invariance requires the matrix \( g \) to satisfy the equations

\[ \hat{Q} = Q + \varepsilon [Q, g] \] (4.3a)

\[ \frac{\partial}{\partial t} g = \left[ Q^r, g \right] - \left[ Q^r, g \right]_a. \] (4.3b)

A non-trivial solution is

\[ g = \sum_k b_k Q^k_a, \] (4.4)

where \( b_k \)'s are time-independent constants. By abuse of language we will call this a “time-independent gauge transformation”.

Let us consider the case when only \( b_k \) is nonzero. Then

\[ \delta Q = \hat{Q} - Q = \varepsilon b_k [Q, Q^k_a] = -\varepsilon b_k \frac{\partial}{\partial t_k} Q. \] (4.5)

This corresponds to the transformation \( t_k \rightarrow t_k - \varepsilon b_k \), which can be rephrased by saying that the tuning of the time parameters is realized by means of the gauge transformation (4.4).

This transformation has remarkable properties. On the one hand it can be considered as the discrete version of the conformal transformations, on the other hand it leads to the integrability of the linear system (and consequently to that of the one-matrix model).
Let us consider in detail the latter claim. We can think of $\delta Q$ given by eq.(4.5) as originating from a Poisson bracket in the following sense:

$$\delta Q \equiv \varepsilon \{ A_r, Q \}$$

(4.6)

That is

$$\{ A_r, Q \} \equiv [Q^r_a, Q].$$

(4.7)

where $A_r$ represents a Hamiltonian to be determined. For the Hamiltonians we make the ansatz

$$H_r \equiv \frac{1}{r} \sum_{n=0}^{\infty} Q^r_{nn} \quad r = 1, 2, \ldots$$

(4.8)

and corresponding to the choice $A_r = H_r, H_{r-1}, H_{r-2}, \ldots$, we obtain different Poisson brackets. More explicitly we write

$$\{ H_{r-k+1}, Q \}_k = [Q^r_0, Q],$$

(4.9)

In the following we will study in detail only the cases $k = 1, 2, 3$. Comparing the LHS with the RHS which can be explicitly calculated, we can obtain the Poisson brackets for $R_i$ and $S_i$. Of course we have no a priori guarantee that the brackets so obtained satisfy the Jacobi identity. This has to be checked a posteriori.

While explicitly working out the Poisson brackets one realizes that there are two distinct regimes according to whether $S_i = 0$ or $\neq 0$.

i) First regime, i.e. $S_i = 0$. We find two meaningful Poisson brackets:

$$\{ R_i, R_j \}_1 = R_i R_j (\delta_{i,i+1} - \delta_{i,j+1})$$

(4.10)

and

$$\{ R_i, R_j \}_3 = R_i R_j (R_i + R_j)(\delta_{i,i+1} - \delta_{i,j+1})$$
\[ + R_j R_{j-1} R_{j-2} \delta_{j,i+2} - R_i R_{i-1} R_{i-2} \delta_{i,j+2} \]  

while

\[ \{ R_i, R_j \}_2 \equiv 0 \]

ii) *Second regime*, i.e. \( S_i \neq 0 \). We have two Poisson brackets:

\[ \{ R_i, R_j \}_1 = R_i R_j (\delta_{j,i+1} - \delta_{i,j+1}) \]  

\[ \{ R_i, S_j \}_1 = R_i S_j (\delta_{j,i+1} - \delta_{i,j}) \]  

\[ \{ S_i, S_j \}_1 = R_i \delta_{j,i+1} - R_j \delta_{i,j+1}. \]

and

\[ \{ R_i, R_j \}_2 = 2R_i R_j (S_i \delta_{i,j-1} - S_j \delta_{i,j+1}) \]

\[ \{ R_i, S_j \}_2 = R_i R_j (\delta_{i,j-1} + \delta_{i,j}) - R_i R_{j+1} (\delta_{i,j+1} + \delta_{i,j+2}) \]

\[ + R_i S_j (\delta_{i,j} - \delta_{i,j+1}) \]

\[ \{ S_i, S_j \}_2 = (S_i + S_j) (R_j \delta_{i,j-1} - R_i \delta_{i,j+1}) \]

For \( k = 3 \) eq.(4.9) does not define a consistent bracket.

Let us conclude this section with a few remarks. Due to eq.(4.7) one would expect all the Hamiltonians to commute

\[ \{ H_n, H_m \} = 0. \]

for any meaningful Poisson structure. However due to the subtleties connected with traces of infinite matrices, one should verify this property starting from the Poisson brackets (4.9) and (4.10). We have done it for the first few cases.
Moreover, from the definition of the Poisson brackets, we know that

$$\{H_{r+2}, Q\}_1 = \{H_r, Q\}_3$$

in the first regime and

$$\{H_{r+1}, Q\}_1 = \{H_r, Q\}_2$$

in the second, namely the two Poisson structures are compatible with each other. That is to say, in both regimes the linear system has infinitely many conserved quantities and possesses a bi–Hamiltonian structure.

Finally the Poisson brackets (4.10) and (4.11) are the same we come across in the lattice version of the Liouville model [9]. In that case $R_i$ plays the role of a lattice deformation of the classical Virasoro generators. However one should remember that in the first regime we cannot impose the string equation (see remark after eq.(3.5)). So we find a remarkable similarity of structures in the Liouville model on the lattice and in the one-matrix model unconstrained by the string equation. Strictly speaking, the Poisson structures of the one-matrix model are those of the second regime. It is an interesting open problem what field theory on the lattice they correspond to.

5. Reparametrization and Time-Dependent Gauge Transformations

The DLS (2.10) is form invariant under reparametrization of the $t_k$ couplings. That is

$$Q(\tilde{t})\xi(\tilde{t}) = \lambda\xi(\tilde{t}), \quad \frac{\partial}{\partial t_r}\xi(\tilde{t}) = \left(Q_a^r(\tilde{t})\right)\xi(\tilde{t})$$

where $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, ...)$ and $\tilde{t}_k$ is a smooth functions of the $t_k$’s.

This invariance is a formal one. However, by combining gauge and reparametrization transformations, we can obtain significant symmetries of the system. Let us
consider the transformations

\[
\begin{align*}
\tilde{t}_k &= t_k + \varepsilon (k - n) t_{k-n}, \quad \forall k > n, \quad n \geq -1 \\
\tilde{t}_n &= t_n - 2N\varepsilon, \quad n \geq 1 \\
\hat{\xi}(\tilde{t}) &= G^{-1}(\tilde{t})\xi(\tilde{t}), \\
\hat{Q}(\tilde{t}) &= G^{-1}(\tilde{t})Q(\tilde{t})G(\tilde{t})
\end{align*}
\]

(5.1)

In this kind of setup it is possible to find \( G = 1 + \varepsilon g \) so that

\[
\hat{Q}(\tilde{t}) = Q(t)
\]

(5.2)

and the linear system becomes

\[
\begin{align*}
Q(t)\hat{\xi}(\tilde{t}) &= \lambda \hat{\xi}(\tilde{t}) \\
\frac{\partial}{\partial t} \hat{\xi}(\tilde{t}) &= Q^a(t)\hat{\xi}(\tilde{t})
\end{align*}
\]

(5.3)

We refer to these as time-dependent gauge transformations.

Let us consider two examples

\[
\begin{align*}
g &= P, \quad n = -1 \\
g &= QP, \quad n = 0
\end{align*}
\]

(5.4)

In these two cases eq.(5.2) is satisfied (the linear system is invariant but remark that \( \xi \) is not invariant for \( n = 0 \)) and this fact leads straightforwardly to the \( L_{-1} \) and \( L_0 \) Virasoro constraints. We could as well obtain the other Virasoro constraints, but in these cases the matrix \( g \) has a complicated form and will not be written down here.
6. THE DOUBLE SCALING LIMIT

The purpose of this section is to recover a continuum version of the DLS (2.10) in the double scaling limit. In general we will exploit the idea that the double scaling limit is mimicked by a finite version of the transformation (5.1) above when \( n = 0 \):

\[ t_r \to \gamma r t_r, \quad \forall r \]  

(6.1)

where \( \gamma \) is a finite constant. We recall that under this transformation \( Q \) remains invariant. In other words, the double scaling limit is connected with a singular case of a symmetry operation on our DLS.

Let us consider a \( k \)-th order critical point and define, as usual, the continuum variables

\[ x \equiv \frac{n}{\beta}, \quad R(x) \equiv R_n, \quad \xi(x) \equiv \xi_n. \]

Moreover we set

\[ \epsilon \equiv \left( \frac{1}{\beta} \right)^{\frac{k+1}{2k+1}}, \quad \tilde{t}_0 = (1 - \frac{n}{\beta}) \beta^{\frac{2k}{2k+1}}, \quad \partial \equiv \frac{\partial}{\partial t_0}, \]  

(6.2)

The double scaling limit corresponds to

\[ \beta \to \infty, \quad N \to \infty, \quad \tilde{t}_0 \text{ fixed} \]

For large \( n \sim N \) one has

\[ x = 1 - \epsilon^{2k} \tilde{t}_0, \quad R(x) = 1 + \epsilon^2 u(\tilde{t}_0) \]  

(6.3)

where \( u(\tilde{t}_0) \) is the specific heat.
The latter ansatz requires a comment. Let us consider the string equation (2.14), suitably rescaled

\[
[Q, \bar{P}] = \frac{1}{\beta}, \quad \bar{P} = \sum_{r=2}^{\infty} r \tilde{t}_r Q_a^{r-1}, \quad t_r = \beta \tilde{t}_r \quad (6.4)
\]

where \( \tilde{t}_r \) are renormalized coupling constants (see below). This equation establishes strong restrictions between the limiting expressions of \( Q \) and \( \bar{P} \). With the simplifying assumption \( t_{2r+1} = 0 \quad \forall r \), the above ansatz is correct, as is well known; if we switch on the odd interactions the analysis is more complicated, the above ansatz is still correct but we have not been able to exclude other solutions. In the following we will stick to the case of even potentials and to (6.3).

Let us now write down a few expansions which will be useful in the following. It is easy to see that

\[
R_{n+1} = R(x \mp \frac{1}{\beta}) = 1 + \epsilon^2 u(\tilde{t}_0 \pm \epsilon)
\]

\[
\xi_{n+1} = \xi(x \mp \frac{1}{\beta}) = e^{\pm \epsilon \partial} \xi(\tilde{t}_0).
\]

Then, using Taylor expansion, we have the following formulas

\[
R_{n+b} = 1 + \epsilon^2 u - b \epsilon^3 u' + \frac{b^2}{2} \epsilon^4 u'' - \frac{b^3}{6} \epsilon^5 u''' + \ldots \quad (6.5a)
\]

\[
\prod_{i=0}^{q} R_{n+b+i} = 1 + (q+1)\epsilon^2 u - \frac{1}{2}(q+1)(2b+q)\epsilon^3 u' + \epsilon^4 \left( \sum_{i=0}^{q} \frac{(b+i)^2}{2} u'' + \frac{(q+1)q}{2} u^2 \right) \quad (6.5b)
\]

\[
- \epsilon^5 \left( \sum_{i=0}^{q} \frac{(b+i)^3}{6} u''' + \frac{(2b+q)(q+1)q}{2} uu' \right) + \ldots
\]
\[ \prod_{i=0}^{q} \sqrt{R_{n+b+i}} = 1 + \frac{q + 1}{2} \epsilon^2 u - \frac{(q + 1)(2b + q)}{4} \epsilon^3 u' \]

\[ + \epsilon^4 \left( \sum_{i=0}^{q} \frac{(b + i)^2}{4} u'' + \frac{(q + 1)(q - 1)}{8} u^2 \right) \]  \hspace{1cm} (6.5c)

\[ - \epsilon^5 \left( \sum_{i=0}^{q} \frac{(b + i)^3}{12} u''' + \frac{(2b + q)(q + 1)(q - 1)}{8} uu' \right) + \ldots \]

Using these formulas one can derive

\[ Q = 2 + \epsilon^2 (\partial^2 + u) + O(\epsilon^3), \]  \hspace{1cm} (6.6a)

\[ -Q_a = \epsilon \partial + \frac{1}{6} \epsilon^3 (\partial^3 + 3u \partial + \frac{3}{4} u') + \frac{1}{8} \epsilon^4 (2u' \partial + u'') \]

\[ + \frac{1}{8} \epsilon^5 \left( \frac{1}{15} \partial^5 + \frac{2}{3} u \partial^3 + u \partial^2 + u'' \partial - u^2 \partial - uu' + \frac{1}{3} u''' \right) + \ldots, \]  \hspace{1cm} (6.6b)

\[ -Q_a^2 = 2 \epsilon \partial + \frac{4}{3} \epsilon^3 (\partial^3 + \frac{3}{2} u \partial + \frac{3}{4} u') + \frac{1}{2} \epsilon^4 (2u' \partial + u'') \]

\[ + \epsilon^5 \left( \frac{4}{15} \partial^5 + \frac{4}{3} u \partial^3 + 2u' \partial^2 + \frac{3}{2} u'' \partial + \frac{5}{12} u''' \right) + \ldots, \]  \hspace{1cm} (6.6c)

\[ -Q_a^3 = 6 \epsilon \partial + \epsilon^3 (5 \partial^3 + 9u \partial + \frac{9}{2} u') + \frac{9}{2} \epsilon^4 (2u' \partial + u'') + \frac{3}{2} \epsilon^5 \left( \frac{41}{10} \partial^5 \right. \]

\[ + 15u \partial^3 + \frac{45}{2} u' \partial^2 + \frac{37}{2} u'' \partial + \frac{9}{2} u^2 \partial + \frac{9}{2} uu' + \frac{11}{2} u''' \Bigg) + \ldots, \]  \hspace{1cm} (6.6d)

\[ -Q_a^4 = 12 \epsilon \partial + 16 \epsilon^3 (\partial^3 + \frac{3}{2} u \partial + \frac{3}{4} u') + 6 \epsilon^4 (2u' \partial + u'') + 2 \epsilon^5 \left( \frac{24}{5} \partial^5 \right. \]

\[ + 16u \partial^3 + 24u' \partial^2 + 19u'' \partial + 6u^2 \partial + 6uu' + \frac{11}{2} u''' \Bigg) + \ldots, \]  \hspace{1cm} (6.6e)

\[ -Q_a^5 = 30 \epsilon \partial + \epsilon^3 (45 \partial^3 + 75u \partial + \frac{75}{2} u') + 75 \epsilon^4 (2u' \partial + u'') + \frac{5}{4} \epsilon^5 (29 \partial^5) \]
\[ + 90u \partial^3 + 135u' \partial^2 + 111u'' \partial + 45u'^2 \partial + 45uu' + 33u'' \] + \ldots \quad (6.6f)

etc.

Let us see now some consequences of the above expansions. First of all let us notice that in the continuum limit the reduction to even potentials does not contradict the string equation as in the discrete case. We should remember that the contradiction is exposed in eq. (3.3). From the above expansions it is not difficult to see that in the continuum limit it does not make sense to single out an equation like (3.3), while the LHS of the string equation is replaced by a differential operator even if \( t_{2r+1} = 0 \) \( \forall r \). So in the continuum limit one can safely choose an even potential.

Next we consider the continuum limit of (2.10a). From eq. (6.6a) we see that in a neighbourhood of the critical point \( \lambda \sim 2 \). Therefore we introduce the renormalized quantities

\[ \tilde{\lambda} \equiv \epsilon^{-2}(\lambda - 2), \quad \tilde{Q} \equiv \partial^2 + u \quad (6.7a) \]

\[ \frac{\partial}{\partial \tilde{\lambda}} \xi = \tilde{P} \xi, \quad \tilde{P} = \epsilon^2 P = \epsilon^{1-2k} \hat{P}. \quad (6.7b) \]

Then, the discrete Schrödinger equation goes over to its continuum version

\[ (\partial^2 + u)\tilde{\xi}(\tilde{t}_0) = \tilde{\lambda} \tilde{\xi}(\tilde{t}_0) \quad (6.8) \]

Similarly the string equation becomes

\[ [\tilde{Q}, \tilde{P}] = 1. \quad (6.9) \]

We are now in a condition to explicitly determine critical points. From eq. (2.11b) and (6.7b) (recall that we are working with the simplifying assumption of even po-
tential) we have

\[ \tilde{P} = \epsilon^2 (2t_2 Q_a + 4t_4 Q_a^3 + 6t_6 Q_a^5 + \ldots) \]
\[ = \epsilon^{1-2k} (2\tilde{t}_2 Q_a + 4\tilde{t}_4 Q_a^3 + 6\tilde{t}_6 Q_a^5 + \ldots) \]  

(6.10)

We remarked above that the string equation (6.9) puts severe restrictions not only in the discrete case but also in the double scaling limit. From eq.(6.6) and (6.7), we see in particular that all the operators \( Q'_a \)'s are vanishing in the limit \( \epsilon \to 0 \), so that if one wants the string equation to be satisfied, one must let a certain subset of bare coupling constants in \( P \) go to infinity (DSL). The practical recipe is to look for combinations of \( \tilde{t}_2, \tilde{t}_4 \)'s such that all the singular terms in the second expression of (6.10) vanish. Let us see a few significant examples.

i) \( k=2 \). In this case only \( \tilde{t}_2 \) and \( \tilde{t}_4 \) are nonzero, and \( \beta = \epsilon^{-5} \). Then, from (6.6b,d) we see that only if we set

\[
\begin{align*}
  t_2 &= \frac{15}{8} \epsilon^{-5} \tilde{t}_2 = \epsilon^{-5} \tilde{t}_2, \\
  t_4 &= -\frac{5}{32} \epsilon^{-5} \tilde{t}_2 = \epsilon^{-5} \tilde{t}_4,
\end{align*}
\]

are we able to eliminate the \( \epsilon^{-1} \partial \) term in \( \tilde{P} \), and we get the known operator

\[ \tilde{P} = \frac{5}{2} \tilde{t}_2 (\partial^2 + u)^{\frac{3}{2}} + \mathcal{O}(\epsilon) \]

The string equation becomes

\[ -\frac{5}{2} \tilde{t}_2 R'_2[u] = 1 \]

where we have introduced the Gelfand–Dickii polynomials

\[ R'_k[u] = [(\partial^2 + u)^{k-\frac{1}{2}} \partial^2 + u]. \]  

(6.11)

As is well-known at the critical point \( \tilde{t}_2^c = \frac{8}{15} \), the above string equation is the
Painlevé equation of first kind

\[ \tilde{t}_0 = \frac{1}{3} u'' + u^2. \]

ii) \( k=3 \). Only \( t_2, t_4 \) and \( t_6 \) are non-vanishing, \( \beta = \epsilon^{-7} \). According to the above recipe we kill all the negative powers of \( \epsilon \) in \( \tilde{P} \) if we put

\[
\begin{align*}
&\begin{cases}
t_2 = -\frac{105}{32} \epsilon^{-7} \tilde{t}_3 = \epsilon^{-7} \tilde{t}_2, \\
t_4 = \frac{35}{64} \epsilon^{-7} \tilde{t}_3 = \epsilon^{-7} \tilde{t}_4, \\
t_6 = -\frac{7}{192} \epsilon^{-7} \tilde{t}_3 = \epsilon^{-7} \tilde{t}_6, 
\end{cases}
\end{align*}
\]

It is straightforward to show that

\[ \tilde{P} = \frac{7}{2} \tilde{t}_3 (\partial^2 + u) + \mathcal{O}(\epsilon) \]

and the string equation becomes

\[ -\frac{7}{2} \tilde{t}_3 R'_3[u] = 1. \]

The third critical point is at \( \tilde{t}_3^c = -\frac{16}{35} \) and the differential equation is

\[ \tilde{t}_0 = -(u^3 - \frac{1}{2} u^2 - uu'' + \frac{1}{10} u^{(4)}). \]

One can proceed in this way and determine higher order critical points. As is well-known, on a very general ground the form of the operator \( \tilde{P} \) must be as follows

\[
\tilde{P} = \sum_{n=1}^{\infty} (n + \frac{1}{2}) \tilde{t}_n C_n \epsilon^{n+\frac{1}{2}} + \mathcal{O}(\epsilon). \quad (6.12)
\]

We conjecture that this form is induced by the following coupling redefinitions:

\[
t_{2r} = - \sum_{n=r}^{\infty} (n + \frac{1}{2}) \tilde{t}_n C_n \epsilon^{-(2n+1)} = \sum_{n=r}^{\infty} \Gamma_n \Gamma_n \tilde{t}_n \quad (6.13)
\]
where
\[ a_r^{(n)} = (-1)^{r+1} \frac{n!(r-1)!}{(n-r)!(2r)!} \]

We checked eq.(6.13) for the first few cases and found
\[ C_1 = 1, \quad C_2 = -\frac{3}{4}, \quad C_3 = \frac{5}{8}, \quad C_4 = -\frac{35}{64}, \ldots \]

In general one has
\[ C_n = (-1)^{n+1} \frac{(2n-1)!!}{2^{n-1}n!}. \]

These factors are determined in such a way as to reproduce the standard KdV hierarchy.

It is worth noticing that 1) in eq.(6.12) and (6.13) we are considering all the critical points at a time, and 2) the time transformation (6.13) is made of a reparametrization plus a scale transformation of the type (6.1).

What is left for us to do is to analyze the continuum limit of the KdV hierarchy. On the basis of eq.(6.13) one naively has

\[ \frac{\partial}{\partial t_n} = - \sum_{r=1}^{n} \left( n + \frac{1}{2} \right) C_n \epsilon^{-(2n+1)} a_r^{(n)} \frac{\partial}{\partial t_{2r}}. \tag{6.14} \]

So, in particular, on the basis of (2.10) and (6.6) we must have

\[ \frac{\partial}{\partial t_1} \xi = \left( \frac{3}{2} \epsilon^{-2} \partial + (\partial^2 + u)^{\frac{3}{2}} + \mathcal{O}(\epsilon) \right) \xi \]

\[ \frac{\partial}{\partial t_2} \xi = \left( -\frac{15}{8} \epsilon^{-4} \partial + (\partial^2 + u)^{\frac{5}{2}} + \mathcal{O}(\epsilon) \right) \xi \]

\[ \frac{\partial}{\partial t_3} \xi = \left( \frac{35}{32} \epsilon^{-6} \partial + (\partial^2 + u)^{\frac{7}{2}} + \mathcal{O}(\epsilon) \right) \xi \]

etc. These are however naive formulae since \( \{ \tilde{t}_n, \ n \geq 1 \} \) is not a complete set of parameters after we take the continuum limit. To correct this we have to allow
also for a \( \partial \)-dependent term in the RHS of (6.14). This additional term takes care exactly of the divergent terms (in the \( \epsilon \to 0 \) limit) in the RHS of the above equations.

Finally the evolution equations become

\[
\frac{\partial}{\partial t_n} \tilde{\xi} = (\partial^2 + u)^{n+\frac{1}{2}} \tilde{\xi}, \quad n \geq 0
\]  

which result in the standard KdV flow

\[
\frac{\partial}{\partial t_n} u = [(\partial^2 + u)^{n+\frac{1}{2}}, \partial^2 + u], \quad n \geq 0
\]  

In eq. (6.15) \( \tilde{\xi} \) is the limit of \( \xi \) possibly multiplied by an \( \epsilon \)-dependent factor which may be necessary in order to obtain a finite result.

7. The Virasoro Constraints in the Continuum Limit

In the previous section starting from the DLS we have obtained, near criticality, a continuous linear system

\[
(\partial^2 + u)\tilde{\xi} = \tilde{\lambda} \tilde{\xi}
\]  

\[
\frac{\partial}{\partial t_n} \tilde{\xi} = (\partial^2 + u)^{n+\frac{1}{2}} \tilde{\xi}
\]  

\[
\tilde{P} = \sum_{n=1}^{\infty} (n + \frac{1}{2}) \tilde{t}_n \tilde{Q}_{n+\frac{1}{2}}
\]  

whose consistency conditions are

\[
[\tilde{Q}, \tilde{P}] = 1
\]  

\[
\frac{\partial}{\partial t_n} u = [(\partial^2 + u)^{n+\frac{1}{2}}, \partial^2 + u]
\]  

\[
\frac{\partial}{\partial t_n} \tilde{P} = [(\partial^2 + u)^{n+\frac{1}{2}}, \tilde{P}].
\]  

We want now to recover the Virasoro constraints in this continuous system. The strategy is the same as for the discrete case. We use essentially the string
equation (7.2a). First of all, as is well known, in the continuum limit the partition function behaves like

\[
\ln Z = \sum_{k=1}^{N-1} (N-k) \ln R_k + \text{constant terms}
\]

\[
\to \int_0 \tilde{t}_0 (\tilde{t}_0 - \tilde{t}_0') u(\tilde{t}_0') + \mathcal{O}(\epsilon) + \text{regular terms}
\]

\[
\Rightarrow \partial^2 \ln \tilde{Z} = u(\tilde{t}_0)
\]

where, in taking the continuum limit, we passed to the normalized partition function

\[
\tilde{Z} \equiv \frac{Z(\tilde{t})}{Z(T)}
\]

the parameter \( T \) being a reference point connected with the extremum of integration \( \tilde{t}_0 = 0 \).

Now, using (7.1c), eq.(7.2a) can be written

\[
-k + \frac{1}{2} i_k' \mathcal{R}_k'[u] = 1.
\]

Integrating once with respect to \( t_0 \), we obtain

\[
\sum_{k=1}^{\infty} (k + \frac{1}{2}) \tilde{t}_k \left( \mathcal{R}_k[u] - \mathcal{R}_k[0] \right) + \tilde{t}_0 = 0
\]

where \( \mathcal{R}_k[0] \) is \( \mathcal{R}_k[u] \) computed at \( \tilde{t}_0 = 0 \). If one directly considers the continuum limit of the discrete Virasoro constraints, it is not this actually the case...

This is actually the case...
In order to simplify the next formulas let us introduce the recursion operator

\[ \hat{\phi} \equiv \frac{1}{4} \partial^2 + u + \frac{1}{2} u' \partial^{-1} \]

and define the recursion relation for the Gelfand–Dikii polynomials

\[ \mathcal{R}'_{n+1} = \hat{\phi} \mathcal{R}'_n = \hat{\phi}^n \partial u. \]  

(7.7)

Remembering that, on the basis of our conventions, we have

\[ \partial^{-1} \mathcal{R}'_k[u] = \mathcal{R}_k[u] - \mathcal{R}_k[0] \]

eq.(7.6) can be rewritten

\[ F \equiv \tilde{t}_0 + \sum_{k=1}^{\infty} (k + \frac{1}{2}) \tilde{t}_k \partial^{-1} \hat{\phi}^{k-1} \partial u = 0. \]  

(7.8a)

On the same basis we can write

\[ \partial^{-1} \hat{\phi}^{n+1} \partial F = 0, \quad n \geq -1 \]  

(7.8b)

To obtain these equations we have used only the string equation. We can as well envisage eqs.(7.8) as a consequence of a symmetry of the system, precisely as a consequence of the fact that \( u(\tilde{t}_0) \) and the KdV hierarchy are invariant under the transformations

\[
\begin{align*}
\tilde{t}_k &\rightarrow \tilde{t}_k = \tilde{t}_k + \epsilon (k - n + \frac{1}{2})\tilde{t}_{k-n}, \\
u(\tilde{t}) &\rightarrow \hat{\phi}^{n+1} \cdot 1 + u(\tilde{t}_k)
\end{align*}
\]

(7.9)
The generators associated with (7.9) are

\[ L_{-1} = \sum_{k=1}^{\infty} \left( k + \frac{1}{2} \right) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k-1}} + \frac{1}{4\rho} \tilde{t}_0^2, \]

\[ L_0 = \sum_{k=0}^{\infty} \left( k + \frac{1}{2} \right) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_k} + \frac{1}{16}, \quad (7.10) \]

\[ L_n = \sum_{k=0}^{\infty} \left( k + \frac{1}{2} \right) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k+n}} + \frac{\rho}{4} \sum_{k=1}^{n} \frac{\partial^2}{\partial \tilde{t}_{k-1} \partial \tilde{t}_{n-k}}, \quad n \geq 1. \]

Here, for later purposes, we have introduced a constant \( \rho \) (for example, by rescaling all the \( \tilde{t} \)'s).

Even though we will not use it in the following, it is worth mentioning that there is a larger symmetry of the system: the latter is also invariant under the transformations

\[ \tilde{t}_k \rightarrow \tilde{t}_k + \epsilon \quad (7.11a) \]

whose generators are given by

\[ V_k = \frac{\partial}{\partial \tilde{t}_k}, \quad k \geq 0. \quad (7.11b) \]

The generators \( V_k \)'s and \( L_n \)'s characterize the master symmetry of the KdV hierarchy [9,11]. The corresponding algebra is

\[ [V_k, V_l] = 0, \quad k, l \geq 0, \quad (7.12a) \]

\[ [V_k, L_n] = (k + \frac{1}{2})V_{k+n}, \quad k \geq 0, k + n \geq 0, \quad (7.12b) \]

\[ [V_0, L_{-1}] = \frac{1}{2\rho} \tilde{t}_0 \quad (7.12c) \]

\[ [L_n, L_m] = (n - m)L_{n+m}, \quad n, m \geq -1. \quad (7.12d) \]
But let us return to the derivation of the Virasoro constraint. Representing
now (7.8) in terms of the partition function, we get for \( n = -1, 0 \)
\[
\partial \left( \sum_{n=1}^{\infty} \left( n + \frac{1}{2} \right) \tilde{t}_n \frac{\partial}{\partial \tilde{t}_{n-1}} \ln \tilde{Z} + \frac{1}{2 \rho} \tilde{t}_0^2 \right) = 0
\]
\[
\partial \left( \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) \tilde{t}_n \frac{\partial}{\partial \tilde{t}_n} \ln \tilde{Z} \right) = 0
\]
or, in general,
\[
\partial \left( \frac{l_n \sqrt{\tilde{Z}}}{\sqrt{\tilde{Z}}} \right) = 0, \quad n \geq -1. \tag{7.13}
\]
where, by definition,
\[
l_0 = L_0 - \frac{1}{16}, \quad l_n = L_n, \quad n \neq 0.
\]
The most general solution of (4.28) has the form
\[
l_n \sqrt{\tilde{Z}} = b_n \sqrt{\tilde{Z}} \tag{7.14}
\]
where the \( b_n \)'s are \( \tilde{t}_0 \)-independent but arbitrary functions of the other parameters.
In order to determine them we remark that they must be compatible with the
algebra (7.12). In particular they must satisfy
\[
[l_n, b_m] - [l_m, b_n] = (n - m) b_{n+m} + \frac{1}{8} n \delta_{n+m,0}. \quad (7.15)
\]
Moreover we remark that (7.12) is a graded algebra provided we define the degree
as follows
\[
\deg[\tilde{t}_k] \equiv -k, \quad \deg[\frac{\partial}{\partial \tilde{t}_l}] \equiv l, \quad \deg[\rho] \equiv 1
\]
Therefore, \( \deg[L_n] = n \) and, from (7.14),
\[
\deg[b_n] = n, \quad n \geq -1 \quad (7.16)
\]
The general form of $b_n$ will be a sum of monomials of the following type

$$\mathcal{M}_n(p, q_1, ..., q_a) = \text{const} \, \rho^p \, \tilde{t}_{n_1}^{q_1} \cdots \tilde{t}_{n_a}^{q_a}$$

where $p$ is a real number and $q_1, q_2, ...$ are nonnegative real numbers (we exclude negative exponents as it is natural to require a smooth limit of $b_n$ as any one of the couplings vanishes). Next we remember that the parameter $\rho$ appeared on the scene because of a rescaling $\tilde{t}_n \to \sqrt{\rho} \tilde{t}_n$. Therefore if we perform the opposite rescaling the dependence of $b_n$ on $\rho$ must disappear. This implies the condition $p = \frac{1}{2}(q_1 + ... + q_a)$. So the degree of the above monomial would be

$$\text{deg}[\mathcal{M}_n(p, q_1, ..., q_a)] = \sum_{i=1}^{a} \left( \frac{1}{2} - n_i \right) q_i$$  \hspace{1cm} (7.17)

Comparing eq.(7.16) with (7.17), we see all the $b_n$'s are zero except perhaps for $b_0$ which must be a constant, and $b_{-1}$, which could depend on $\tilde{t}_1$ and $\rho$. We now use the consistency conditon (7.15). For $n = 1$ and $m = -1$ it tells us that

$$\frac{5}{2} \frac{\partial b_{-1}}{\partial \tilde{t}_1} = 2b_0 + \frac{1}{8}$$

This allows us to conclude that

$$b_0 = -\frac{1}{16}.$$

and $b_{-1}$ does not depend on $\tilde{t}_1$. Next we use again (7.15) for $n = 0$ and $m = -1$ and conclude that

$$b_{-1} = 0$$

Collecting the above results we obtain the Virasoro constraints

$$L_n \sqrt{Z} = 0, \hspace{1cm} n \geq -1$$  \hspace{1cm} (7.18)

with $L_n$ given by (7.10). Another proof of the same result is given in Appendix.
Finally we recall that due to (7.2b) and (7.3), $\sqrt{Z}$ is a $\tau$-function of the KdV hierarchy.

Appendix

This Appendix is devoted to another proof of eq.(7.18). We start from eq.(7.15) for $m = -1$ and $m = 0$ and analyze the $\tilde{t}_0$ dependence. Since all the $b_n$ are $\tilde{t}_0$-independent we get, in the first case ($m = -1$),

$$\frac{\partial b_{-1}}{\partial \tilde{t}_n} = 0, \quad n \geq 1$$

and, in the second case ($m = 0$),

$$\frac{\partial b_0}{\partial \tilde{t}_n} = 0, \quad n \geq 1$$

In conclusion both $b_0$ and $b_{-1}$ are constant. Applying now again eq.(7.15) for $n = 0$ and $m = -1$, we obtain

$$b_{-1} = 0$$

i.e. eq.(7.18) is true for $n = -1$. Since we have shown above that $\sqrt{Z}$ is a $\tau$-function of the KdV hierarchy, we can now apply a theorem of ref.[11] which asserts that if $L_{-1} \tau = 0$, then $L_n \tau = 0$, $\forall n \geq -1$. Consequently eq.(7.18) is proven.

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