Uniaxial Lifshitz Point at $O(\epsilon_L^2)$

Luiz C. de Albuquerque‡* and Marcelo M. Leite⋆†

‡ Faculdade de Tecnologia de São Paulo - FATEC/SP-CEETPS-UNESP. Praça Fernando Prestes, 30, 01124-060. São Paulo, SP, Brazil

* Departamento de Física, Instituto Tecnológico de Aeronáutica, Centro Técnico Aeroespacial, 12228-900. São José dos Campos, SP, Brazil

Abstract

The critical exponents $\nu_L^2$, $\eta_L^2$ and $\gamma_L^2$ of a uniaxial Lifshitz point are calculated at two-loop level using renormalization group and $\epsilon_L$-expansion techniques. We introduced a new constraint involving the loop momenta along the competition axis, which allows to solve the two-loop integrals. The exponent $\gamma_L^2$ obtained using our method is in good agreement with numerical estimates based on Monte Carlo simulations.

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*e-mail:claudio@fma.if.usp.br

†e-mail:leite@fma.if.usp.br
The Lifshitz point occurs in a variety of physical systems and has been extensively studied over the last twenty-five years \[1,2\]. It appears in High-$T_C$ superconductivity \[3,4\], polymer physics \[5,6\], ferroelectric liquid crystals \[7,8\], etc. In magnetic systems, the uniaxial Lifshitz critical behavior can be described by an axially next-nearest-neighbor Ising model (ANNNI). It consists of a spin-$1/2$ system on a cubic lattice \((d = 3)\) with nearest-neighbor ferromagnetic couplings and next-nearest-neighbor antiferromagnetic interactions along a single lattice axis \[11\]. The competition gives origin to a modulated phase, in addition to the ferromagnetic and paramagnetic ones. In spite of having several modulated phases, it was shown recently that around the Lifshitz critical region, a simple field-theoretic setting can be defined for this ANNNI model \[12\]. In general, the antiferromagnetic couplings can show up in \(m\) directions. In that case, the system possesses the \(m\)-fold Lifshitz critical point. Here we are going to focus our attention in the uniaxial case \((m = 1)\), since some materials present this type of critical behavior. MnP was studied both theoretically and experimentally and displays the uniaxial behavior \[13–15\].

Theoretical studies involving the uniaxial Lifshitz point have been put forth using analytical and numerical tools. Examples of the latter are high-temperature series expansion \[16\] and Monte Carlo simulations \[11,17\]. Conformal invariance calculations in \(d = 2\) (in the context of strongly anisotropic criticality) \[18\] and \(\epsilon\)-expansion techniques \[1,19,20\] have been the main analytical tools available to dealing with this kind of system.

From the field-theoretic point of view, the critical dimension of a scalar field theory describing the uniaxial Lifshitz critical behavior is found (by using the Ginzburg criterion) to be \(d_c = 4.5\). The expansion parameter is \(\epsilon_L = 4.5 - d\), where \(d\) is the space dimension of the system under consideration. (In the pure Ising model the expansion parameter is \(\epsilon = 4 - d\)). As is well known, the critical exponents for the Lifshitz point at one-loop approximation have the same dependence in \(\epsilon_L\) as those from the pure Ising model have in \(\epsilon\). Of particular importance is the effect of the mixing of the two momenta scales, i.e., along
and perpendicular to the competing axis. One can choose a convenient symmetry point to fix the external momenta scale in the quartic and quadratic directions. Then, the one-loop integral contributing to the four-point function, needed to find out the fixed point, can be performed without any approximation. The choice which simplifies the referred integral is to set the external momenta scale in the quartic direction to zero. The solution of this integral yields a leading singularity and a regular term in $\epsilon_L$. The leading singularity in $\epsilon_L$ can be chosen to be the same as that obtained in $\epsilon$ when solving the analogous one-loop integral for the Ising model (by absorbing a convenient angular factor in a redefinition of the coupling constant), even though the coefficients of the regular terms in $\epsilon_L$ and $\epsilon$ are slightly different and depend on $m$ in the Lifshitz case [20]. Thus, although the multiplicative factors to be absorbed in the coupling constant are different in the two cases, the critical exponents have the same dependence on the expansion parameter.

We can then ask ourselves if this happens to be true when we proceed to calculate the critical exponents in higher orders in the perturbative expansion. Going one step further to evaluate higher order integrals in the loop expansion would be highly desirable using the same line of reasoning. In this way, all the dependence in the external momenta is along the $(d - 1)$ directions, perpendicular to the competition axis. In the resulting $\epsilon_L$-expansion, the leading singularities can be chosen equal to those coming from the theory without competition (see below). The nontrivial new features of this expansion around the usual quadratic field theory are the coefficients of the subleading singularities and of the regular terms, which are no longer rational numbers. This approach would allow to treat this system properly in a perturbative expansion at least for correlation functions along the perpendicular directions to the competition axis. A better comprehension of this procedure might shed light in the perturbative study of higher order derivative field theories. In this sense, the Lifshitz critical behavior seems to be the natural laboratory to study higher order field theories in a perturbative framework.

We report on what we believe to be the first study of critical exponents at two-loop order for the uniaxial Lifshitz point. Using $\lambda \phi^4$ field theory and the expansion in powers of
$\epsilon_L = 4.5 - d$ in the Lifshitz critical point, we give a detailed account of the calculation of the exponents $\nu_{L2}$ and $\eta_{L2}$, which by now can be viewed as a worked out example of a recent generalization obtained for anisotropic behaviors \cite{20}. In order to solve higher-loop integrals we introduce a constraint relating the loop momenta in internal and external subdiagrams along the competing axis. The results for these integrals are consistent with homogeneity of the Feynman integrals in the external quadratic momenta scale. The exponents $\nu_{L2}$ and $\eta_{L2}$ are associated with the directions perpendicular to the competition axis. (The exponents $\nu_{L4}$ and $\eta_{L4}$ associated with the competition axis are not going to be considered here and we shall analyse them in another work.) We then obtain the exponent $\gamma_{L2}$ via scaling relations.

The paper is organized as follows. In section 2 we introduce the notation and calculate the relevant integrals for the determination of the critical exponents at two-loop level. We present the critical exponents $\eta_{L2}$, $\nu_{L2}$ and $\gamma_{L2}$ in section 3. In section 4 we discuss our results and compare the $\gamma_{L2}$ exponent with numerical estimates based on Monte Carlo methods and high-temperature series.

**II. CALCULATION OF HIGHER LOOP INTEGRALS**

The Lifshitz critical behavior can be described using a modified $\lambda\phi^4$ field theory. The bare Lagrangian associated with the uniaxial critical behavior is given by:

\[
L = \frac{1}{2} |\nabla_1^2 \phi_0|^2 + \frac{1}{2} |\nabla_{(d-1)} \phi_0|^2 + \delta_0 \frac{1}{2} |\nabla_1 \phi_0|^2 + \frac{1}{2} t_0 \phi_0^2 + \frac{1}{4!} \lambda_0 \phi_0^4.
\]  

(1)

The competition is responsible for the appearance of the quartic term in the free propagator. The Lifshitz critical point is characterized by the values $t_0 = \delta_0 = 0$. From now on, this is the case which interests us in this work. First, we are going to compute the renormalized coupling constant at the fixed point. In order to find out the factor that shall be absorbed in the coupling constant, we quickly review the one-loop contribution to the four point function \cite{12}. The relevant integral is:

\[
I_2 = \int \frac{d^{d-1}qdk}{((k+k')^4 + (q+p)^2)(k^4 + q^2)}.
\]

(2)
The external momenta are \( k' \) along the quartic (competing) direction and \( \vec{p} \) along the \((d - 1)\) quadratic directions. We then choose a symmetry point which simplifies the integral at external momenta \( k' = 0, p^2 = 1 \). Using Schwinger’s parameterization we get

\[
\int \frac{d^{d-1}q dk}{(k^4 + (q + p)^2)(k^4 + q^2)} = \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \left( 2 \int_0^\infty dk \exp(-(\alpha_1 + \alpha_2)k^4) \right) \\
\times \int d^{d-1} q \exp(-(\alpha_1 + \alpha_2)q^2 - 2\alpha_2 q.p - \alpha_2 p^2). \tag{3}
\]

The \( \vec{q} \) integral is straightforward. It can be performed to give

\[
\int d^{d-1} q \exp(-(\alpha_1 + \alpha_2)q^2 - 2\alpha_2 q.p - \alpha_2 p^2) = \frac{1}{2} S_{d-1} \Gamma\left( \frac{d - 1}{2} \right) (\alpha_1 + \alpha_2)^{-\frac{d-1}{2}} \exp\left( -\frac{\alpha_1 \alpha_2 p^2}{\alpha_1 + \alpha_2} \right). \tag{4}
\]

The next step is to compute the \( k \) integration, which is:

\[
2 \int_0^\infty dk \exp(-(\alpha_1 + \alpha_2)k^4) = \frac{1}{2} (\alpha_1 + \alpha_2)^{-\frac{d-1}{4}} \Gamma\left( \frac{1}{4} \right). \tag{5}
\]

Replacing equations (4), (5) into equation (3) together with the value \( p^2 = 1 \), one finds

\[
\left( \int \frac{d^{d-1}q dk}{(k^4 + (q + p)^2)(k^4 + q^2)} \right)_{p^2=1} = \frac{1}{4} S_{d-1} \Gamma\left( \frac{d - 1}{2} \right) \Gamma\left( \frac{1}{4} \right) \\
\times \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \exp\left( -\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \right) (\alpha_1 + \alpha_2)^{-\frac{(d-1)}{2} - \frac{d}{4}}. \tag{6}
\]

We can perform one of the integrals in the Schwinger parameters using a change of variables. Set \( v = \frac{\alpha_2}{\alpha_1 + \alpha_2} \), and \( u = \alpha_1 v \). The integral over \( u \) can be done, and we are left with

\[
\int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \exp\left( -\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \right) (\alpha_1 + \alpha_2)^{-\frac{(d-1)}{2} - \frac{d}{4}} \\
= \Gamma\left( 2 - \left( \frac{d - 1}{2} + \frac{1}{4} \right) \right) \int_0^1 dv (v(1 - v))^{\frac{d-1}{2} + \frac{1}{4} - 2}. \tag{7}
\]

Now we make the continuation \( d = 4.5 - \epsilon_L \). One obtains a result in terms of Gamma functions with non integer arguments. A useful identity involving the expansion of Gamma functions around a small number is given by:

\[
\Gamma(a + bx) = \Gamma(a) \left[ 1 + b x \psi(a) + O(x^2) \right], \tag{8}
\]
where $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$. This allows one to obtain the $\varepsilon_L$-expansion when the Gamma functions have non integer arguments. Replacing Eqs. (7), (8) into Eq. (6), we obtain:

$$I_2 = \frac{1}{2} \Gamma(7/4) \Gamma(1/4) S_{d-1} \frac{1}{\varepsilon_L} (1 + i_2 \varepsilon_L),$$

where $i_2 = 1 + \frac{1}{2} (\psi(1) - \psi(\frac{7}{4}))$. We absorb in the coupling constant a geometric angular factor, which is $\frac{1}{2} \Gamma(7/4) \Gamma(1/4) S_d$, where $S_d = [2^{d-1} \pi^2 \Gamma(\frac{d}{2})]^{-1}$. Then the redefined integral is:

$$\tilde{I}_2 = \frac{I_2}{\frac{1}{2} \Gamma(7/4) \Gamma(1/4) S_{d-1}},$$

or

$$\tilde{I}_2 = \frac{1}{\varepsilon_L} (1 + i_2 \varepsilon_L).$$

We suppress the tilde hereafter to simplify the notation. We have to keep in mind that we should divide out this factor for each loop integration. We now turn our attention to higher-loop integrals. In practice, we have to calculate the two-loop integrals $I_{4SP} \equiv I_4$, $\frac{\partial}{\partial p^2} I_3|_{SP} \equiv I'_3$ and the three-loop integral $\frac{\partial}{\partial p^2} I_5|_{SP} \equiv I'_5$, in order to find the fixed point at two-loop level and the critical exponents. The subscript $SP$ is used to denote our choice of the subtraction point. They are given by (see Figure 1):

$$I_3 = \int \frac{d^{d-1}q_1 d^{d-1}q_2 dk_1 dk_2}{(q_1^2 + k_1^4)(q_2^2 + k_2^4)((q_1 + q_2 + p)^2 + (k_1 + k_2 + k')^4)},$$

$$I_5 = \int \frac{d^{d-1}q_1 d^{d-1}q_2 d^{d-1}q_3 dk_1 dk_2 dk_3}{(q_1^2 + k_1^4)(q_2^2 + k_2^4)(q_3^2 + k_3^4)((q_1 + q_2 + p)^2 + (k_1 + k_2 + k')^4)} \times \frac{1}{(q_1 + q_3 - p)^2 + (k_1 + k_3 - k')^4}.$$

$$I_4 = \int \frac{d^{d-1}q_1 d^{d-1}q_2 dk_1 dk_2}{(q_1^2 + k_1^4)((P - q_1)^2 + (K' - k_1)^4)(q_2^2 + k_2^4)} \times \frac{1}{(q_1 - q_2 + p_3)^2 + (k_1 - k_2 + k_3)^4}.$$


In the first two integrals, $\vec{p}$ is the external momentum (associated with the two-point vertex) along $(d-1)$ directions, whereas $k'$ is the external momentum along the competition axis. Inside the integral $I_4$, $P = p_1 + p_2$, with $p_1, p_2, p_3$ being the external momenta (associated with the four-point vertex) along the quadratic directions, and $K' = k_1' + k_2'$, with $k_1', k_2', k_3'$ the external momenta along the quartic direction. The symmetry point is chosen as follows. We set all the external momenta at the competition axis equal to zero. For the four-point vertex, the external momenta along the quadratic directions are chosen as $p_i p_j = \frac{\kappa^2}{4}(4\delta_{ij} - 1)$. We fix the momentum scale of the two-point function through $p^2 = \kappa^2 = 1$.

Now we can study the solution of the higher-loop integrals shown above. Consider the integral $I_3$. With our choice for the quartic external momenta it is given by:

$$I_3 = \int \frac{d^{d-1}q_1}{(q_1^2 + k_1^4)} \int \frac{d^{d-1}q_2}{(q_2^2 + k_2^4)} \int \frac{d^{d-1}q_3}{((q_1 + q_2 + p)^2 + (k_1 + k_2)^4)}. \quad (15)$$

First, we perform the integral over the internal bubble, i.e., we integrate over $q_2$ and $k_2$. In order to solve the internal bubble we demand that the loop momenta $k_1$ should be related to $k_2$. Note that we could have chosen the other way around, since the integral is symmetric under the exchange $k_1 \leftrightarrow k_2, q_1 \leftrightarrow q_2$. There are two issues which need to be emphasized here. First, the Lifshitz point condition eliminates the quadratic part of the momenta along the competition axis, for $\delta_0 = 0$. Second, the remaining quartic part of the integral mixes the two loop momenta $(k_1, k_2)$ in different subdiagrams, yielding crossed terms which are difficult to integrate. Indeed, using Schwinger parameters and carrying out the integration over $q_2$ first, we obtain

$$I_3(p, 0) = \frac{1}{2} \mathcal{S}_{d-m} \Gamma\left(\frac{d-1}{2}\right) \int \frac{d^{d-1}q_1}{q_1^2 + (k_1^4)^2} \times \int_0^{\infty} d\alpha_1 d\alpha_2 (\alpha_1 + \alpha_2)^{-\frac{d-1}{2}} e^{\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} (q_1 + p)^2} \int dk_2 e^{-\alpha_1 k_2^4} e^{-\alpha_2 (k_1 + k_2)^4}. \quad (16)$$

In order to integrater over $k_2$, we have to expand the argument of the last exponential. This results in a complicated function of $\alpha_1, \alpha_2, k_1$ and $k_2$, which has no elementary primitive.
Considering the remaining terms as a damping factor to the integrand, the maximum of the integrand will be either at $k_1 = 0$ or at $k_1 = -2k_2$. (The most general choice $k_1 = -\alpha k_2$ yields a hypergeometric function.) The constraint eliminates the crossed terms and is the simplest way to disentangle the two quartic integrals in the loop momenta. The choice $k_1 = -2k_2$ implies that $k_1$ varies in the internal bubble, but not in an arbitrary manner. Its variation is dominated by $k_2$ through this constraint, which eliminates the dependence on $k_1$ in the internal subdiagram. Integrating over $k_2$ yields a simple factor to the remaining parametric integrals (over the two Schwinger parameters) proportional to $(\alpha_1 + \alpha_2)^{-1/2}$. After solving the parametric integrals we find:

$$I_3 = \int \frac{d^{d-1}q_1dk_1}{(q_1^2 + k_1^4)[(q_1 + p)^2]^{\frac{d-4}{2}}} I_2. \quad (17)$$

We use Schwinger’s parameterization again to solve this integral along the quartic direction. We obtain:

$$I_3 = I_2 \int \frac{d^{d-1}q_1}{[q_1^2]^{\frac{d}{2}}[(q_1 + p)^2]^{\frac{d-4}{2}}} . \quad (18)$$

The difference with respect to the pure $\phi^4$ theory is that after performing the quartic integral, we get an exponent for the quadratic part of the momenta which is not an integer. At this point, one can use Feynman parameters to solve the momentum integrals. The dependence of the integral in the external momenta is proportional to $(p^2)^{1-\epsilon_L}$, in conformity with the homogeneity of the Feynman integrals in the external momenta scale. Deriving with respect to $p^2$ and setting $p^2 = 1$, we find:

$$I'_3 = -\frac{1}{7\epsilon_L} \left[ 1 + \left( i_2 + \frac{6}{7} \right) \epsilon_L \right]. \quad (19)$$

The integral $I_4$ can be calculated in a similar fashion. First, we set the external momenta along the competing direction equal to zero. Therefore

$$I_4 = \int \frac{d^{d-1}q_1dk_1}{(q_1^2 + k_1^4)((P - q_1)^2 + k_1^4)} \times \int \frac{d^{d-1}q_2dk_2}{(q_2^2 + k_2^4)[(q_1 - q_2 + p_3)^2 + (k_1 + k_2)^4]} , \quad (20)$$
where we changed variables from \( k_2 \rightarrow -k_2 \). We set \( k_1 = -2k_2 \) in the internal bubble \( q_2, k_2 \) (as we did for \( I_3 \)), and integrate over \( q_2, k_2 \). We then have

\[
I_4 = I_2 \int \frac{d^{d-1} q_1 dk_1}{(q_1^2 + k_1^2)((P - q_1)^2 + k_1^2) \left[ (q_1 + p_3)^2 \right]^{\frac{d}{2}}}.
\]  
(21)

We use Schwinger’s parameterization to get rid of the \( k_1 \) integral. After performing the change of variables used to calculate the one-loop integral and a re-scaling, we can solve one of the parametric integrals to get

\[
I_4 = I_2 \int_0^1 dz \int \frac{d^{d-1} q_1}{(q_1^2 - 2z P.q_1 + z P^2)^{\frac{d}{2}} \left[ (q_1 + p_3)^2 \right]^{\frac{d}{2}}}.
\]  
(22)

In order to perform the integral over \( q_1 \), we make use of a Feynman parameter obtaining

\[
I_4 = \frac{1}{2} I_2 \left( 1 - \frac{\epsilon_L}{2} \psi \left( \frac{7}{4} \right) \frac{\Gamma(\epsilon_L)}{\Gamma \left( \frac{\epsilon_L}{2} \right)} \right) \int_0^1 dy \ y^{\frac{d}{2}} (1 - y)^{\frac{1}{2} + \epsilon_L - 1} \times \int_0^1 dz \ [yz(1 - yz)P^2 + y(1 - y)p_3^2 - 2yz(1 - y)p_3 P]^{-\epsilon_L}.
\]  
(23)

There is a subtlety that needs to be analyzed with care. Here we proceed in complete analogy to the pure \( \phi^4 \) theory [21]. The integral over \( y \) is singular at \( y = 1 \) when \( \epsilon_L = 0 \).

We add and subtract the value of the integrand in the last integral at the point \( y = 1 \)

\[
[yz(1 - yz)P^2 + y(1 - y)p_3^2 - 2yz(1 - y)p_3 P]^{-\epsilon_L} = [z(1 - z)P^2]^{-\epsilon_L} - \epsilon_L \ln \left\{ \frac{[yz(1 - yz)P^2 + y(1 - y)p_3^2 - 2yz(1 - y)p_3 P]}{z(1 - z)P^2} \right\} + O(\epsilon_L^2).
\]  
(24)

As \( y \rightarrow 1 \) the logarithm is zero when \( \epsilon_L = 0 \), leading to a well defined result for the integral over \( y \). The coefficient of the integral is proportional to \( \frac{1}{\epsilon_L} \), which cancels the \( \epsilon_L \) in front of the logarithm. The logarithm contributes only to the order \( \epsilon_L^0 \) and can be neglected. We then find

\[
I_4 = \frac{1}{2 \epsilon_L^2} [1 + 3 i \epsilon_L] \text{.}
\]  
(25)

Finally, let us describe the computation of the three-loop integral \( I_5 \). At zero external momenta along the competition axis, this integral reads:
\[ I_5 = \int \frac{d^{d-1}q_1dk_1}{(q_1^2 + k_1^4)} \int \frac{d^{d-1}q_2dk_2}{(q_2^2 + k_2^4)((q_1 + q_2 - p)^2 + (k_1 + k_2)^4)} \]

\[ \times \int \frac{d^{d-1}q_3dk_3}{(q_3^2 + k_3^4)((q_1 + q_3 - p)^2 + (k_1 + k_3)^4)}. \]  \hspace{1cm} (26)

The integral is symmetric in \( q_2 \leftrightarrow q_3, k_2 \leftrightarrow k_3 \). As the loop momenta are dummy variables, the two internal bubbles are really the same. That is why we do not need more than one relation among the loop momenta in the external and internal bubbles, even though this is a three-loop diagram (with two internal bubbles). As a matter of fact, we can solve the two integrals independently in the following way. In the internal bubble \( q_3, k_3 \) we set \( k_1 = -2k_3 \), as well as \( k_1 = -2k_2 \) over the other internal bubble \( q_2, k_2 \). Apparently we have two different relations among the loop momenta, but one of them is artificial. This means that the two internal bubbles give the same contribution, i.e. the integration over the internal bubbles is proportional to \( I_2^2 \). Thus

\[ I_5 = I_2^2 \int \frac{d^{d-1}q_1dk_1}{(q_1^2 + k_1^4)((q_1 + p)^2 + (k_1 + p)^4)} \epsilon_L. \]  \hspace{1cm} (27)

We integrate first over \( k_1 \) and proceed analogously as before, to find the following result:

\[ I'_5 = -\frac{4}{21\epsilon_L^2} \left[ 1 + \left( \frac{2i_2 + 8}{7} \right) \epsilon_L \right]. \]  \hspace{1cm} (28)

We stress once again that the constraint preserves the physical principle of homogeneity in all the higher-loop Feynman integrals in the external momenta scale, which is consistent with scaling theory. With the integrals calculated in this way, we can find out the exponents as is going to be shown in the next section.

\[ \text{III. CRITICAL EXPONENTS} \]

To compute the critical exponents associated to the ferromagnetic planes at the Lifshitz critical point, one may use the standard field-theoretic approach \[21\]. This is possible since
no new renormalization constants need to be introduced in this case\footnote{This is not valid in the calculation of the critical exponents along the competition axis.}. From the results for $I'_3$, Eq. (19), and $I'_5$, Eq. (28), we note that these integrals do not have the same leading singularities as in pure $\phi^4$ theory. As an approximation, we introduce a weight factor for the two point vertex function, in order to identify the leading singularities with the ones appearing in the pure $\phi^4$ field theory. This factor is $7/8$ for the integrals above. This approximation is suitable when one considers the generalization for the $m$-fold case with $m \neq 8$. In this way, we have a smooth transition from the Isinglike case ($m = 0$) to the general Lifshitz anisotropic critical behavior ($m \neq 8$) \cite{20}. The bare and renormalized quantities are related through $\phi_0 = Z^{1/2}_\phi \phi$ and $u_0 = Z^{-2}_u u$, where $\phi_0$ and $\lambda_0 \equiv \kappa^r \cdot u_0$ are the bare parameters in Eq. (1). As usual, $Z_\phi \equiv 1 + \delta_\phi$ and $Z_u \equiv 1 + \delta_u$ are the wave-function and coupling constant renormalization constants, respectively. In addition, we introduce the composite field renormalization constant $Z_{\phi^2}$, and also $\bar{Z}_{\phi^2} = Z_{\phi^2} Z_\phi \equiv 1 + \bar{\delta}_{\phi^2}$. We have

$$
\beta_u = -\epsilon L \left( \frac{\partial \ln u_0}{\partial u} \right)
$$

$$
\gamma_\phi = \beta_u \frac{\partial \ln Z_\phi}{\partial u}
$$

$$
\bar{\gamma}_{\phi^2} = -\beta_u \frac{\partial \ln \bar{Z}_{\phi^2}}{\partial u}
$$

The fixed point $u^*$ is given by the solution of the equation $\beta_u(u^*) = 0$, and the critical exponents by the relations $\eta_{L2} = \gamma_\phi(u^*)$ and $\nu_{L2}^{-1} = 2 - \eta_{L2} - \bar{\gamma}_{\phi^2}(u^*)$. In case the order parameter has a $O(N)$ symmetry, the formulas relating the integrals computed in section 2 and the renormalization constants defined above are:

$$
\delta_\phi = B_2 u^2 I'_3 + \left( 2B_3 I'_2 I'_3 - B_3 I'_5 \right) u^3 + O(u^4)
$$

$$
\delta_u = 3A_1 u I_2 + 3 \left( 6A_1^2 I_2^2 - A_2^{(1)} I_2^2 - 2A_2^{(2)} I_4 \right) u^2 + O(u^3)
$$

$$
\bar{\delta}_{\phi^2} = C_1 u I_2 + (C_2 I_2^2 - C_1 I_4) u^2 + O(u^3)
$$
where $A_1 = (N + 8)/18$, $A_1^{(1)} = (N^2 + 6N + 20)/108$, $A_1^{(2)} = (5N + 22)/54$, $B_2 = (N + 2)/18$, $B_3 = (N + 2)(N + 8)/108$, $C_1 = (N + 2)/6$, and $C_2 = (N + 2)(N + 8)/36$.

With this information we compute the fixed point at two-loop level. We expand the dimensionless bare coupling constant $u_0$ in terms of the renormalized coupling $u$ and the $\epsilon_L$ parameter. Using Eqs. (29) and (30), we find the fixed point for the $O(N)$ symmetric theory in the following form:

$$u^* = \frac{6}{8 + N} \epsilon_L \left\{ 1 + \epsilon_L \left[ \left( \frac{4(5N + 22)}{(8 + N)^2} - 1 \right) i_2 - \frac{(2 + N)}{(8 + N)^2} \right] \right\}.$$

(31)

Therefore, the exponents $\eta_{L2}$ and $\nu_{L2}$ are given by:

$$\eta_{L2} = \frac{1}{2} \frac{\epsilon_L^2}{(8 + N)^2} \left[ \epsilon_L^3 \left( \frac{2 + N}{(8 + N)^2} \right) \left\{ \left( \frac{4(5N + 22)}{(8 + N)^2} - \frac{1}{2} \right) i_2 + \frac{1}{7} - \frac{(2 + N)}{(8 + N)^2} \right\} \right].$$

(33)

$$\nu_{L2} = \frac{1}{4} + \frac{1}{8 + N} \left[ 2(14N + 40) i_2 - 2(2 + N) + (8 + N)(3 + N) \right] \epsilon_L^2.$$

(35)

Now using Fisher's law along directions perpendicular to the competing axis, namely $\gamma_{L2} = \nu_{L2}(2 - \eta_{L2})$, the exponent $\gamma_{L2}$ can be written as:

$$\gamma_{L2} = 1 + \frac{1}{2} \frac{\epsilon_L^2}{8 + N} \left[ 4(8 + N)^3 \left\{ 12 + 8N + N^2 + 4 i_2 (20 + 7N) \right\} \epsilon_L^2. \right]$$

(37)

Previous results in the literature only yielded the exponent $\eta_{L2}$ at $O(\epsilon_L^2)$ and the exponent $\nu_{L2}$ at $O(\epsilon_L)$. For this uniaxial case, our results express the critical exponents in a higher order in $\epsilon_L$ compared to earlier investigations. A detailed comparison with other methods is provided in the next section.
IV. DISCUSSION

First of all, our result for the exponent $\eta_{L2}$ is in agreement with Mukamel’s calculation \[22\] at $O(\epsilon_L^2)$. Therefore, our method is equivalent to integrating over the momentum shell as was done in his work using the Landau-Ginzburg-Wilson Hamiltonian approach.

For the ANNNI model, $\gamma_{L2} = 1.4 \pm 0.06$ is the former Monte Carlo output \[11\], whereas the best estimates from the high-temperature series is $\gamma_{L2} = 1.62 \pm 0.12 \[16\]$. Note that we use the subscript $\gamma_{L2}$ instead of $\gamma_L$, since it was shown recently that the exponents parallel and perpendicular to the competition axis obey independent scaling laws \[23\]. Our two-loop calculation obtained from the $\epsilon_L$-expansion via the scaling law (when neglecting $O(\epsilon_L^3)$ terms) in three dimensions yields $\gamma_{L2} = 1.45$. This agrees (within the error bar) with the former Monte Carlo result, the difference being of order $10^{-2}$.

Nevertheless, the most recent high-precision numerical Monte Carlo estimate for the ANNNI model yielded $\gamma_{L2} = 1.36 \pm 0.03 \[17\]$. In order to figure out how to extract the best numerical results from the $\epsilon_L$-expansion when the $\epsilon_L$ parameter is not small (which is the case for $d = 3$), a comparison with the Ising model is worthwhile. For the exponent $\gamma$ in three dimensions, the $\epsilon$-expansion gives a contribution of 0.167 at $O(\epsilon)$ and 0.077 at $O(\epsilon^2) \[21\]$. The $O(\epsilon)$ contributes with 13% and the order $O(\epsilon^2)$ with 6% to the value of $\gamma$ (1.24), respectively. For the uniaxial Lifshitz case, the contributions for the $\gamma_{L2}$ index are 0.25 (17%) and 0.196 (14%). The very close values of the contributions of first and second order to $\gamma_{L2}$ (as the $\epsilon_L$ parameter is 1.5 not being a small number), indicates that neglecting $O(\epsilon_L^3)$ could be a dangerous step in obtaining the exponent $\gamma_{L2}$ via scaling relations in a more accurate way. Indeed, had we replaced the numerical values obtained for $\nu_{L2} = 0.73$, $\eta_{L2} = 0.04$ and $d = 3$ directly into the scaling law, we would have obtained $\gamma_{L2} = 1.43$. As argued in \[23\] for the other critical exponents $\alpha_{L2}$ and $\beta_{L2}$, whenever $\epsilon_L > 1$ one should use the numerical values of $\nu_{L2}$, $\eta_{L2}$ obtained from the $\epsilon_L$-expansion for fixed values of $(N, d, m)$ in order to obtain the numerical values of the other exponents via scaling laws. Therefore, provided we give this new interpretation to the numerical output of the $\epsilon_L$-expansion when
\[ \epsilon_L > 1, \] we consider that the agreement between the numerical (Monte Carlo) and analytical (\( \epsilon_L \)-expansion) results is remarkable for \( d = 3 \). The numerical value obtained here for the correlation length exponent is \( \nu_{L2} = 0.73 \). The experimental value of this critical index is still lacking. We hope our result sheds some light towards its experimental determination.

The extension of the present method to the calculation of critical exponents for the \( m \)-fold \( (m \neq 8) \) case reduces to the Ising-like case when \( m = 0 \) and to the present case when \( m = 1 \) \[20\]. An interesting open question is the calculation of the critical exponents \( \nu_{LA} \) and \( \eta_{LA} \) using the \( \epsilon_L \)-expansion at two-loop level. The approach followed here is not suitable to computing these critical exponents (parallel to the competition axis), since our choice of the symmetry point prevents a proper treatment in this direction. The possibility of devising another symmetry point to deal with these exponents seems to be feasible, and will be reported elsewhere.

In recent articles, some authors \[24\] studied an alternative field-theoretic approach based on coordinate space calculations. In the first paper they recovered the results of reference \[19\] for the cases \( m = 2, 6 \) analytically, but only could get the exponents numerically for the \( m = 1 \) case, working entirely in coordinate space. It is worth emphasizing that they computed the fixed point only at one-loop order (see equation (82) in the mentioned paper). In the second paper, they computed the critical exponents at second order in perturbation theory by making use of a hybrid mechanism, going to coordinate or momentum space according to the necessity through a scaling function related to the free propagator in coordinate space. They obtained the exponents, whose coefficients of each power of \( \epsilon_L \) are integrals to be performed numerically. The very similar values obtained for the exponents using their method or ours confirms that momentum and coordinate space calculations should give the same results, since either our approximation or the numerical approximation made by them \[24\] is responsible for a rather small deviation in the two results when compared to the above numerical values.

In conclusion, we have found a way to perform two- and three-loop integrals for the uniaxial Lifshitz point, needed to calculate universal properties at directions perpendicular
to the competition axis. The constraint in the loop momenta at the competition direction is the key ingredient to carry out the calculations. In this approximation, the loop momenta along the competition axis are not conserved when one uses this constraint. However the momenta along the \((d - 1)\) directions are conserved. Momentum non-conservation along the competing direction as a higher-order effect does not seem to affect the critical exponents considered here in a significant way, as indicated by the comparison of our study with the available numerical data for the \(d = 3\) case.

It might be interesting to study general field theories including higher order derivative terms in this new framework. Topics including the extension of the present method to the region out of the Lifshitz point \((\delta_0 \neq 0)\) and two-loop calculations using a modified symmetry point along the competition axis are in development.

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REFERENCES

[1] Hornreich R M, Luban M and Shtrikman S 1975 Phys. Rev. Lett. 35 1678

[2] Hornreich R M 1980 Journ. Magn. Magn. Mat. 15-18 387

[3] Hayden S M et al. 1991 Phys. Rev. Lett. 66 821

[4] Keimer B et al. 1991 Phys. Rev. Lett. 67 1930

[5] Sachdev S and Ye J 1992 Phys. Rev. Lett. 69 2411

[6] Fredrickson G H and Milner S T 1991 Phys. Rev. Lett. 67 835

[7] Bates F S et al 1995 Phys. Rev. Lett. 75 4429

Bates F S et al 1997 ibid. 79 849

[8] Netz P R and Schick M 1996 Phys. Rev. Lett. 77 302

[9] Rananavare S B, Pisipati V G K M and Wong E W 1994 Phys. Rev. Lett. 72 3558

[10] Zalar B et al 1998 Phys. Rev. Lett. 80 4458

[11] Selke W 1988 Phys. Rep. 170 213

Selke W 1998 Phase Transitions and Critical Phenomena, edited by C. Domb and J. Lebowitz (Academic Press, London), vol.15

[12] Leite M M 2000 Phys. Rev. B 61 14691

[13] Yokoi C S O , Coutinho-Filho M D and Salinas S R 1981 Phys. Rev. B 24 5430

Yokoi C S O , Coutinho-Filho M D and Salinas S R 1984 ibid. 29 6341

[14] Shapira Y 1982 J. Appl. Phys. 53 1914

[15] Bindilatti V, Becerra C C and Oliveira N 1989 Phys. Rev. B 40 9412

[16] Mo Z and Ferer M 1991 Phys. Rev. B 43 10890

[17] Pleimling M and Henkel M 2001 Phys. Rev. Lett. 87 125702
[18] Henkel M 1997 Phys. Rev. Lett. 78 1940

[19] Mergulhão Jr C and Carneiro C E I 1998 Phys. Rev. B 58 6047
    Mergulhão Jr C and Carneiro C E I 1999 ibid. 59 13954

[20] de Albuquerque L C and Leite M M 2001 J. Phys. A 34 L327

[21] Amit D J 1984 Field Theory, the Renormalization Group and Critical Phenomena
    (World Scientific, Singapore)

[22] Mukamel D 1979 J. Phys. A 10 L249

[23] Leite M M 2001 Preprint hep-th/0109037

[24] Diehl H W and Shpot M 2000 Phys. Rev. B 62 12338
    Shpot M and Diehl H W 2001 Preprint cond-mat/0106105 Nucl. Phys. B (in press)
Figure captions

- Figure 1. Feynman graphs corresponding to the integrals: (a) $I_2$; (b) $I_3$; (c) $I_4$; (d) $I_5$. The broken lines in the graphs (b), (c), and (d) define the "internal" bubbles in each case. The momenta $q_i, k_i$ refer to the loop momenta in the quadratic and quartic directions, respectively. The labels $p_i, k_i'$ denote the external momenta in the quadratic and quartic directions, respectively.
q_1,k_1, q_2,k_2 \quad q_1-P,k_1-K' \quad q_2-P,k_2-P_1-k_1-k_3'
\(q_1, k_1\)

\(q_2, k_2\) \(\rightarrow\) \(q_1 + q_2 - p, k_1 + k_2 - k'\)

\(q_3, k_3\) \(\rightarrow\) \(q_1 + q_3 - p, k_1 + k_3 - k'\)

\(d\)