SUPPLEMENTARY NOTE 1: DETAILED CALCULATIONS

In this supplement the step by step calculations of $<L_z>$ (for isotropic and anisotropic momentum distributions) and $<L_z^2>$ are shown. Starting with the calculation of $<L_z>$ in cylindrical coordinates

$$<L_z> = -i \frac{\int \mathrm{d}r \psi_i^*(r) \frac{\partial}{\partial \phi} \psi(r)}{\int \mathrm{d}r |\psi_i(r)|^2}$$

with

$$\int \mathrm{d}r |\psi_i(r)|^2 = 1 + \cos(\Delta \alpha) e^{-\frac{\sigma^2}{2}} = N$$

and

$$-i \frac{\partial}{\partial \phi} \psi_i(r) = \frac{1}{\sqrt{2}} \left[ \psi_0[k_{\perp} \rho \cos(\phi)] e^{i k_{\perp} \rho \sin(\phi)} - e^{i \Delta \alpha} k_{\perp} \rho \sin(\phi) e^{i k_{\perp} \rho \cos(\phi)} \right]$$

hence it follows

$$<L_z> = \frac{1}{2N} \int \mathrm{d}r \psi_0^2 [k_{\perp} \rho \cos(\phi) - k_{\perp} \rho \sin(\phi) + k_{\perp} \rho \cos(\phi) e^{-i \Delta \alpha} e^{i k_{\perp} \rho (\sin(\phi) - \cos(\phi))} - k_{\perp} \rho \sin(\phi) e^{i \Delta \alpha} e^{i k_{\perp} \rho (\cos(\phi) - \sin(\phi))}]$$

which, using $\int_0^{2\pi} \mathrm{d}\phi \cos(\phi) = \int_0^{2\pi} \mathrm{d}\phi \sin(\phi) = 0$, simplifies to

$$<L_z> = \frac{1}{2N} \int \mathrm{d}r k_{\perp} \rho \psi_0^2 [\cos(\phi) e^{-i \Delta \alpha} e^{i k_{\perp} \rho (\sin(\phi) - \cos(\phi))} - \sin(\phi) e^{i \Delta \alpha} e^{i k_{\perp} \rho (\cos(\phi) - \sin(\phi))}]$$

$$<L_z> = \frac{1}{2N} \int \mathrm{d}r k_{\perp} \rho \psi_0^2 [\cos(\phi) e^{-i \Delta \alpha} e^{i \sqrt{2} k_{\perp} \rho \sin(\phi - \pi/4)} - \sin(\phi) e^{i \Delta \alpha} e^{i \sqrt{2} k_{\perp} \rho \sin(\phi - \pi/4)}]$$

Then we apply the Jacobi-Anger expansion, $e^{iz \sin(\phi)} = \sum_{\ell} J_\ell(z) e^{i\ell \phi}$, and use that $\int_0^{2\pi} \mathrm{d}\phi e^{i\ell \phi} = 0$ for $\ell \neq 0$. This allows us to easily solve the azimuthal integral.

$$<L_z> = \frac{\pi}{2N} \int \mathrm{d}k_{\perp} \rho^2 \psi_0^2 [J_{-1}(\sqrt{2} k_{\perp} \rho) e^{-i \Delta \alpha} e^{i \frac{\pi}{4}} + J_1(\sqrt{2} k_{\perp} \rho) e^{-i \Delta \alpha} e^{-i \frac{\pi}{4}} -$$

$$i J_{-1}(\sqrt{2} k_{\perp} \rho) e^{i \Delta \alpha} e^{i \frac{\pi}{4}} + i J_1(\sqrt{2} k_{\perp} \rho) e^{i \Delta \alpha} e^{-i \frac{\pi}{4}}]$$

Next we use the anti-symmetry of the Bessel function of first order $J_{-1}(z) = -J_1(z)$ and begin grouping the exponential/trigonometric terms.

$$<L_z> = -\frac{\pi}{N} \int \mathrm{d}k_{\perp} \rho^2 \psi_0^2 [J_1(\sqrt{2} k_{\perp} \rho) e^{-i \Delta \alpha} \sin(\frac{\pi}{4}) - e^{i \Delta \alpha} \cos(\frac{\pi}{4})]$$

$$<L_z> = 2\pi \frac{\sin(\Delta \alpha)}{\sqrt{2N}} \int \mathrm{d}k_{\perp} \rho^2 \psi_0^2 J_1(\sqrt{2} k_{\perp} \rho)$$
Which can be rewritten into the form of a standard Hankel transform of first order with known result. This brings us to the equation (8) seen in the main text

$$\langle L_z \rangle = \sin(\Delta \alpha) \frac{k^2 \sigma^2}{4N} e^{-\frac{\sigma^2 \Delta^2}{4}}$$

Next we examine the generalized case where the momentum distribution is anisotropic (see equation (13) in the main text). This is best done in Cartesian coordinates:

$$L_z \psi_t = -i(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}) \psi_t$$

$$= -\frac{i}{\sqrt{\pi \sigma_x \sigma_y}} e^{\frac{-x^2}{\sigma_x^2} - \frac{y^2}{\sigma_y^2}} (2xy(e^{ik \perp y} + e^{i\Delta \alpha} e^{ik \perp x})[\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}] + ik \perp xe^{ik \perp y} - i k \perp ye^{i\Delta \alpha} e^{ik \perp x})$$

Hence it follows

$$\langle L_z \rangle = \frac{k \perp}{\pi \sigma_x \sigma_y N} \int dx dy e^{\frac{-2x^2}{\sigma_x^2} - \frac{2y^2}{\sigma_y^2}} [xe^{ik \perp (x-y) - i\Delta \alpha} - ye^{ik \perp (x-y) + i\Delta \alpha}]$$

Where all odd terms have been dropped since their integral is zero. To proceed we use $i \frac{\partial}{\partial k \perp} e^{-ik(a+b)} = (a+b)e^{-ik(a+b)}$ to get

$$\langle L_z \rangle = \frac{k \perp}{\pi \sigma_x \sigma_y N} \int dx dy e^{\frac{-2x^2}{\sigma_x^2} - \frac{2y^2}{\sigma_y^2}} [\frac{1}{\sigma_x^2} \frac{\partial}{\partial k \perp} e^{ik \perp (x-y) - i\Delta \alpha} = \frac{i}{\sigma_x^2} \frac{\partial}{\partial k} e^{ik \perp (x-y) - i\Delta \alpha} - e^{ik \perp (x-y) + i\Delta \alpha}]$$

Note the final term is minus the complex conjugate of our previous expression for $\langle L_z \rangle$ hence it follows

$$\langle L_z \rangle = \frac{k \perp}{\pi \sigma_x \sigma_y N} \int dx dy e^{\frac{-2x^2}{\sigma_x^2} - \frac{2y^2}{\sigma_y^2}} \frac{\partial}{\partial k \perp} [e^{ik \perp (x-y) - i\Delta \alpha} - e^{ik \perp (x-y) + i\Delta \alpha}]$$

and since expectation values must be real we can conclude

$$\langle L_z \rangle = \frac{ik \perp}{2\pi \sigma_x \sigma_y N} \int dx dy e^{\frac{-2x^2}{\sigma_x^2} - \frac{2y^2}{\sigma_y^2}} \frac{\partial}{\partial k \perp} [e^{ik \perp (x-y) - i\Delta \alpha} - e^{ik \perp (x-y) + i\Delta \alpha}]$$

We may now swap integration and differentiation and realize that we are left with a standard Fourier transform

$$\langle L_z \rangle = \frac{ik \perp}{2\pi \sigma_x \sigma_y N} \frac{\partial}{\partial k \perp} \int dx dy e^{\frac{-2x^2}{\sigma_x^2} - \frac{2y^2}{\sigma_y^2}} [e^{ik \perp (x-y) - i\Delta \alpha} - e^{ik \perp (x-y) + i\Delta \alpha}]$$

Conducting the transform and grouping the exponential/trigonometric terms leads to

$$\langle L_z \rangle = \frac{k \perp}{2N} \frac{\partial}{\partial k \perp} e^{-\frac{(\sigma_x^2 + \sigma_y^2) \Delta^2}{8}} \sin(\Delta \alpha)$$

Finally carrying out the differentiation leads to the result shown in equation (14) of the main text

$$\langle L_z \rangle = \sin(\Delta \alpha) \frac{k^2 \sigma_x^2 + \sigma_y^2}{8N} e^{-\frac{k^2 (\sigma_x^2 + \sigma_y^2)}{8}}$$

Finally we calculate the second moment of the OAM distribution, $\langle L_z^2 \rangle$

$$\langle L_z^2 \rangle = -\frac{\int dr \psi_t^*(r) \frac{\partial^2}{\partial \sigma^2} \psi_t(r)}{N}$$
\[
\frac{\partial^2}{\partial \phi^2} \psi(r) = -\frac{1}{\sqrt{2}} \psi_0[(k_{\perp}^2 \rho^2 \sin^2(\phi) + i k_{\perp} \rho \cos(\phi)) e^{i \Delta \alpha} e^{i k_{\perp} \rho \cos(\phi)} + (k_{\perp}^2 \rho^2 \cos^2(\phi) + i k_{\perp} \rho \sin(\phi)) e^{i k_{\perp} \rho \sin(\phi)}]
\]

Therefore

\[
< L_z^2 > = \frac{1}{2N} \int d\phi |\psi_0|^2 [(k_{\perp}^2 \rho^2 \sin^2(\phi) + i k_{\perp} \rho \cos(\phi)) + (k_{\perp}^2 \rho^2 \cos^2(\phi) + i k_{\perp} \rho \sin(\phi)) + (k_{\perp}^2 \rho^2 \sin^2(\phi) + i k_{\perp} \rho \cos(\phi)) e^{i \Delta \alpha} e^{i k_{\perp} \rho \cos(\phi)} - \sin(\phi)]
\]

First we use \(\cos(\phi) - \sin(\phi) = -\sqrt{2} \sin(\phi - \frac{\pi}{4})\)

\[
< L_z^2 > = \frac{1}{2N} \int d\phi |\psi_0|^2 [k_{\perp}^2 \rho^2 (\sin^2(\phi) + k_{\perp} \rho \cos(\phi)) + (k_{\perp}^2 \rho^2 \cos^2(\phi) + i k_{\perp} \rho \sin(\phi)) e^{i \Delta \alpha} e^{i \sqrt{2} k_{\perp} \rho \sin(\phi - \frac{\pi}{4})}]
\]

We simplify the expression by using the identity \(\cos^2(\phi) + \sin^2(\phi) = 1\)

\[
< L_z^2 > = \frac{1}{2N} \int d\rho |\psi_0|^2 [k_{\perp}^2 \rho^2 + (k_{\perp}^2 \rho^2 \sin^2(\phi) + i k_{\perp} \rho \cos(\phi)) e^{i \Delta \alpha} e^{-i \sqrt{2} k_{\perp} \rho \sin(\phi - \frac{\pi}{4})}]
\]

We solve the azimuthal integral by using the Jacobi-Anger expansion again, \(e^{i \alpha \sin(\phi)} = \sum_{\ell} J_{\ell}(z) e^{i \ell \phi}\), and again use that \(\int_0^{2\pi} d\phi e^{i \ell \phi} = 0\) for \(\ell \neq 0\). Note that the latter identity paired with the trigonometric terms in the previous line filter out all but the \(\ell = 0\) and \(\ell = \pm 1\) terms of the Jacobi-Anger expansion.

\[
< L_z^2 > = \frac{1}{2N} \int d\rho |\psi_0|^2 [2\pi k_{\perp}^2 \rho^2 + (\pi k_{\perp}^2 \rho^2 J_0(\sqrt{2} k_{\perp} \rho) - \sqrt{2} \pi k_{\perp} \rho J_1(\sqrt{2} k_{\perp} \rho)) e^{i \Delta \alpha} + (\pi k_{\perp}^2 \rho^2 J_0(\sqrt{2} k_{\perp} \rho) - \sqrt{2} \pi k_{\perp} \rho J_1(\sqrt{2} k_{\perp} \rho)) e^{-i \Delta \alpha}]
\]

Here we have once again used the asymmetry of the first order Bessel function. Next we group together the trigonometric terms

\[
< L_z^2 > = \frac{1}{2N} \int d\rho |\psi_0|^2 [2\pi k_{\perp}^2 \rho^2 + \cos(\Delta \alpha)(2\pi k_{\perp}^2 \rho^2 J_0(\sqrt{2} k_{\perp} \rho) - \sqrt{8} \pi k_{\perp} \rho J_1(\sqrt{2} k_{\perp} \rho))]
\]

Now we attempt to solve the radial integrals

\[
< L_z^2 > = \frac{1}{\pi \sigma^2 N} \int d\rho \rho e^{-2 \sigma^2 \rho} [2\pi k_{\perp}^2 \rho^2 + \cos(\Delta \alpha)(2\pi k_{\perp}^2 \rho^2 J_0(\sqrt{2} k_{\perp} \rho) - \sqrt{8} \pi k_{\perp} \rho J_1(\sqrt{2} k_{\perp} \rho))]
\]

The first integral seen above:

\[
\int_0^\infty d\rho 2\pi k_{\perp}^2 \rho^3 e^{-2 \sigma^2 \rho} = \left[ -\pi k_{\perp}^2 \frac{\sigma^2}{2} e^{-\frac{\sigma^2 \rho}{2}} \right]_0^\infty + \int_0^\infty d\rho \pi k_{\perp}^2 \sigma^2 e^{-2 \frac{\sigma^2 \rho}{4}} = \frac{\pi k_{\perp}^2 \sigma^4}{4}
\]
The next radial integral in $< L_z^2 >$ is a Hankel transform with a known result:

$$2\pi \cos(\Delta \alpha) \int_0^\infty d\rho k_{\perp}^2 \rho^3 e^{-2\sigma^2 \rho^2} J_0(\sqrt{2}k_{\perp}\rho) = \frac{\pi k_{\perp}^2 \sigma^4}{4} \cos(\Delta \alpha) e^{-\frac{\sigma^2 \rho^2}{4}} (1 - \frac{k_{\perp}^2 \sigma^2}{4})$$

The final integral is the same Hankel transform as for the first moment

$$-\sqrt{8}\pi \cos(\Delta \alpha) \int d\rho e^{-2\sigma^2 \rho^2} J_1(\sqrt{2}k_{\perp}\rho) = -\cos(\Delta \alpha) \frac{\pi k_{\perp}^2 \sigma^4}{4} e^{-\frac{k_{\perp}^2 \sigma^2}{4}}$$

Hence we find

$$< L_z^2 > = \frac{k_{\perp}^2 \sigma^2}{4N} + \cos(\alpha) \frac{k_{\perp}^2 \sigma^2}{4N} e^{-\frac{\sigma^2 \rho^2}{4}} (1 - \frac{k_{\perp}^2 \sigma^2}{4}) - \cos(\alpha) \frac{k_{\perp}^2 \sigma^2}{4N} e^{-\frac{k_{\perp}^2 \sigma^2}{4}}$$

$$< L_z^2 > = \frac{k_{\perp}^2 \sigma^2}{4N} - \cos(\alpha) \frac{k_{\perp}^2 \sigma^2}{16N} e^{-\frac{\sigma^2 \rho^2}{4}}$$