Jacobi’s bound and normal forms computations.  
A historical survey

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May 4, 2010

This work is dedicated to the memory of Giuseppa Carrà Ferro and Evgeni Vasil’evich Pankratiev.

Abstract

Jacobi is one of the most famous mathematicians of his century. His name is attached to many results in various fields of mathematics and his complete works in seven volumes have been available since the end of the xixth century and are very often quoted in many papers. It is then surprising that some of his results may have fallen into oblivion, at least in part. We will try to describe some of Jacobi’s results on ordinary differential equations and the available, published or unpublished material he left. We will then expose the selective interests of his followers and their own contributions.

There are in fact many interrelated results: a bound on the order of a differential system, a necessary and sufficient condition, given by a determinant, for the bound to be reached, an algorithm to compute the bound in polynomial time, and processes for computing normal forms using as few derivatives as possible.

We give for all of them the form under which they could have been proved or rediscovered, sometimes independently of Jacobi’s findings. In conclusion, we give the state of the art and suggest some possible applications of Jacobi’s bound to improve some algorithms in differential algebra.

1 Introduction

In two posthumous articles [17] [19], which have been recently translated [18] [20], Jacobi has introduced a bound on the order of a system of n ordinary differential equations in n unknowns. Let $A := (a_{i,j})$ be the matrix such that $a_{i,j}$ is the order of the equation $u_i$ in the unknown function $x_j$. Let $J = \max_{\sigma \in S_n} \sum_{i=1}^n a_{i,\sigma(i)}$. A sum $\sum_{i=1}^n a_{i,\sigma(i)}$ is called a transversal sum and $J$ is the maximal transversal sum. He claims that:

JACOBI’S BOUND. — The order of the system is bounded by $J$. 

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The bound is still conjectural in the general case.

**Jacobi’s Algorithm. —** Jacobi gave an algorithm to compute the bound in polynomial time, viz \( O(n^3) \) operations, instead of trying the \( n! \) permutations.

It has been forgotten and rediscovered by Kuhn in 1955 [10], using Egerváry’s results (see Schrijver’s paper [54] for historical details). The idea is to find a canon, i.e. \( \lambda \in \mathbb{N}^n \) such that, in the matrix \((a_{i,j} + \lambda_i)\) one can select maximal entries in each column that are located in different rows. Jacobi’s algorithm computes the unique canon with a minimal \( n \)-uple of integers \( \lambda \) that we denote by \( \ell \). Let \( \Lambda = \max_i \ell_i, \alpha_i = \Lambda - \ell_i \) and \( \beta_j = \max_i a_{i,j} - \alpha_i \). The truncated Jacobian matrix \( \nabla \) is the matrix \( \left( \frac{\partial u_i}{\partial x_j}^{(\alpha_i+\beta_j)} \right) \).

**The truncated determinant condition. —** Jacobi claims that the order of the system is equal to the bound \( J \) iff \( |\nabla| \neq 0 \).

This implicitly assumes the strong bound, defined with the convention \( \operatorname{ord}_x u_i = -\infty \) if \( u_i \) is free of \( x_j \) and its derivatives, but he gives no detail about what should be done in such a case.

**The shortest reduction method. —** Jacobi also asserts that it is possible to compute a normal form using only \( \ell_i \) derivatives of equation \( u_i \) and that it is impossible to compute one using a smaller number of derivatives.

This implicitly assumes \( |\nabla| \neq 0 \); if not, a greater number of derivatives may be required. This is only generically true: for some particular systems, it is possible to differentiate \( u_i \) at most \( \ell_i - \ell_{i+1} \) times, for \( i < n \), assuming \( \lambda_1 \geq \cdots \geq \lambda_n \). Jacobi also gives a bound on the order of derivation of the \( u_i \) required to compute a resolvent representation, using \( x_j \) as a differential primitive element—assuming it is one. Then, \( u_i \) must be differentiated a number of times equal to the maximal transversal sum of the matrix obtained by suppressing the row \( i \) and the column \( j \) in \( A \).

The aim of this paper is to describe the content of Jacobi’s two papers [17, 19], the genesis of these results, the history of research on the subject and the state of the art. We also describe some related documents from Jacobis Nachlaß, kept in the Archiv der Berlin-Brandenburgische Akademie der Wissenschaften and give a complete list of references.

## 2 Unpublished manuscripts

Jacobi himself is possibly the first to have forgotten his own work. According to Koenigsberger [32], his manuscripts on this subject were written around 1836 and were intended to be a part of a forsaken project of a great work on differential equations. Part of it was incorporated in his long paper on the last multiplier [21], but the bound itself was never published in his lifetime. The many versions of the text, containing numerous corrections, suggest that Jacobi was not satisfied of the reedition. However, these manuscripts were clearly intended for publication at the time he wrote them, as it is suggested by the many typographical precisions in German, written in Kurrentschrift in the margins.

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1 For the best of our knowledge... any information about material or sources not mentioned here is welcome.
In his preface to [VD], were the second article [19] was first published, Clebsch asserts that it is a little posterior to Jacobi’s lectures at Königsberg university during academic year 1842–43, when Borchardt was his student there.

On page 2231 of manuscript II/23 a), one finds a reference to a paper [22] published in 1834 and his work on normal form computation is mentioned in the second part of the last multiplier paper [21], published in 1845, but one may think that the results were obtained before the redaction of that paper; the first page of the manuscript I/58 a) shows that Jacobi began to correct it in Roma, in December 1843. Assuming that manuscripts II/13 b), II/22, II/23 a), II/23 b) where written at the same time, as many similarities of style and content suggest, it must have been between 1834 and the end of 1843, certainly not later than 1845.

These results are a by-product of his work on the isoperimetric problem: “Let $U$ be a given function of the independent variable $t$, the dependent ones $x, y, z$ etc. and their derivatives $x', x'', etc., y', y'', etc., z', z'', etc.$ etc. If we propose the problem of determining the functions $x, y, z$ in such a way that the integral

$$\int U \, dt$$

be maximal or minimal or more generally that the differential of this integral vanishes, it is known that the solution of the problem depends on the integration of the system of differential equations:

$$0 = \frac{\partial U}{\partial x} - \frac{\partial U}{\partial x'} \frac{dx'}{dt} + \frac{\partial^2 U}{\partial x'^2} \frac{dx'^2}{dt^2} - etc.,$$

$$0 = \frac{\partial U}{\partial y} - \frac{\partial U}{\partial y'} \frac{dy'}{dt} + \frac{\partial^2 U}{\partial y'^2} \frac{dy'^2}{dt^2} - etc.,$$

$$0 = \frac{\partial U}{\partial z} - \frac{\partial U}{\partial z'} \frac{dz'}{dt} + \frac{\partial^2 U}{\partial z'^2} \frac{dz'^2}{dt^2} - etc.$$

I will call these in the following isoperimetric differential equations . . . ” (see Jacobi’s last multiplier article [21], GW IV p. 495).

If the highest order derivative of $x_i$ in $U$ is $x_i^{(e_i)}$, the order of $x_j$ in the $i$th isoperimetric equation is $e_i + e_j$. Then, if the $e_i$ are not all equal to the maximal order $e := \max_i e_i$, we cannot compute a normal form without using auxiliary equations obtained by differentiating the $i$th isoperimetric equation $\lambda_i$ times with $\lambda_i = e - e_i$. It is also clear that $J = 2 \sum e_i$ is equal to the order of the system, provided that the Hessian matrix $(\partial^2 U/\partial x_i \partial x_j)$ has full rank. We understand how this example can have inspired the whole theory.

In a letter to his brother Moritz, on Sept. 17th 1836 [25], Jacobi writes about a huge manuscript on mechanics: “I came accross some very abstract ideas about the treatment of differential equations that appear in mechanical problems, for these differential equations, with their special form, allow some simplifications for the integration, that had not yet been remarked. These considerations will be all the more important, I think, as they extend to the differential equations that appear both in the isoperimetrical problem and the integration of partial differential equations of the first order.” It seems that the remaining manuscripts come from the time of these first investigations, so between 1836 and 1840, rather
than from some later attempt. The tables A–H given in [19] are written on page 2250 of the manuscript; on the back there is a table of the doubles of prime numbers equal to 1 modulo 8, from 2018 to 20018, a material that could reflect the strong interest of Jacobi in number theory and prime numbers, a short time before the publication of Jacobi’s *Canon arithmeticus* [24], a table of discrete logarithms, in 1839. In a letter to the Académie des Sciences de Paris, published by Liouville in 1840 [26], he evocates his work on mechanics. Liouville’s commentaries are enthousiastic and suggest that a book will appear soon. But Jacobi writes to his brother in January 1841 [25] that he is embarrassed, as he does not have enough “breath” to achieve his huge project, that should have been entitled *Phoronomie*, the study of physical bodies motion.

But in 1845, Jacobi had clearly still in mind to publish a study on normal forms computation, for he wrote: “I will expose in another paper the various ways by which this operation may be done, for this question requires many outstanding theorems that necessitate a longer exposition.” [21]

It is quite possible that this project was forgotten because of a change in Jacobi’s life—who definitely left Königsberg to Berlin after a long trip in Italy—that also opened new contacts and new scientific issues. One may also consider a possible lack of practical examples for such a general method of computing normal forms. The algorithm may have suffered the same absence of contemporary applications. Jacobi was right claiming that the problem of computing the bound was of interest by itself, but the economical questions that strongly motivated the mathematicians of the last century were not yet considered in the middle of the xixth century. (See section 7.)

3 The publication of the manuscripts

Jacobi’s widow gave the manuscripts he left to Dirichlet who began to work for their publication with his friends Borchardt and Joachimsthal. Very few documents remain from their work and the best source seems to be Königsberger [32]. The papers were in great disorder. In order to class them, they gave a number to each page. These numbers appear on the envelops were pages that seemed to form a single document or to be related were stored.

Borchardt entrusted Sigismund Cohn, who worked on the publication of some others manuscripts of Jacobi, with the documents related to the bound. Cohn indentified (see II/13 a) two sets of manuscripts suitable for publication II/13 b), II/23 b) and worked on a transcription II/13 c) of these sometimes hardly readable texts. After his death in 1861 [23], the work was continued by Borchardt who published the first paper [17] in his journal in 1865. The second [19] was published by Clebsch in the volume *Vorlesungen über Dynamik* [VD] in 1866. This one was quoted by Sofya Kovalevskaya in one of her most famous articles [39] in 1875. The fact that these papers were written in latin did not seem to have been a trouble at that time. Cohn and Borchardt could easily write themselves some paragraphs to fill gaps in the manuscripts and

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2 Almost nothing is known about him. There was a student of that name at Königsberg university in 1842-43; in 1846 some Sigismundus Cohn defended in Breslau an inaugural dissertation entitled *De medicina talmudica*. According to the *vita* that follows this work in the copy kept at the library of Alliance Israélite Universelle, he was a different man with no interest in mathematics.
Borchardt even tried to rewrite full passages in order to make them clearer. In his biography [32], published in 1904, Koenigsberger did not translate the many quotations in Latin, French and Italian.

Borchardt also wrote some kind of abstracts of the two papers, which show that he fully understood their content and that he did not consider what he published as devoid of rigor. A slightly ironical quotation of Jacobi himself “Tam quaestiones altioris indaginis poscuntur.” concludes the abstract of a part that clearly did not satisfy Borchardt’s standards and was not kept in the published version (see II/13 c).

4 Jacobi’s mathematical results

4.1 The manuscripts

The first paper [17] begins with the exposition of the bound and the truncated determinant criterion. Then, the main part of the paper is devoted to a careful explanation of the algorithm with complete proofs. To obtain this paper, Cohn put together two different texts from document II/13 b). The first 6 pages of the manuscript, reproduced in Cohn’s transcription, were not kept by Borchardt (see sec. 3). They are related to the different normal forms that a given system could have, with a quite complete description of systems of 2 equations in 2 differential unknowns.

The second [19] begins with a fast exposition of the algorithm, without any proof, followed by the example of a $10 \times 10$ matrix. Jacobi then explains how to compute a normal form, using as few derivatives of each variable as possible and how to compute a resolvent representation for some variable $x_\kappa$, using again as few derivatives of each equation as possible. This paper reproduces with very few changes a single manuscript II/23 b).

Cohn considered documents II/22, II/23 a) as unusable because they investigate how Jacobi’s last multiplier behaves when one changes the order on derivatives and computes a new normal form. Borchardt and him wanted to avoid the multiplier theory, possibly because Jacobi decided not to include this material in his paper on the subject. The § 17 in manuscript II/23 a) for 2217–2220 corresponds to the same paragraph in the last multiplier paper [21] but the manuscript considers the general situation, whereas Jacobi retreated to the linear case in the published version.

Document II/4 is not related to the bound or normal form computation, but rather to the last multiplier theory. It seems an interesting unpublished paper of Jacobi on differential equations, including results such as the linear independence criterion given in Ritt’s Differential algebra [53] p. 34.

4.2 The algorithm

A square table $(a_{i,j})$ being given, Jacobi calls transversal maxima numbers being maximal in their column, that are located in all different rows. The idea of Jacobi

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3Then these questions require further investigation.
4See Crelles 29, Heft 3 221–225 or GW IV 403–407.
for computing \( \max_{\sigma \in S_n} \sum_{i=1}^{n} a_{i, \sigma(i)} \) is to look to what he calls a canon, that is a \( n \times n \) square table where one finds a maximal set of \( n \) transversal maxima. One starts with a table \( (a_{i,j}) \) and computes a canon by adding to all the elements of row \( i \) some integer \( \ell_i \). At each step of Jacobi’s algorithm, one tries to increase the number of transversal maxima. The integers \( \ell_i \) computed by the algorithm are the smallest ones, meaning that there is no canon derived from the table \( a_{i,j} \) such that one of these integers may be smaller.

The algorithm may be sketched as follows. We start with the preparation process: we add suitable integers to the rows of the table, so that each row possess a maximum, i.e. an element being maximal in its column.

The second part of the algorithm starts with the prepared table the maximum of the first row being the first set of transversal maxima being considered.

A set of transversal maxima being given, we will repeatedly compute a new set containing one more element. We reorganise the rows and columns in the following way: upper rows are the rows containing the transversal maxima and lower rows the remaining ones; left columns are the columns containing the transversal maxima and right columns the remaining ones. We denote by asterisks the transversal maxima and the maxima (in their columns) that are located in the upper rows and right columns: maxima with an asterisk are the starred maxima. If there is some maximal element in some lower row and right column, we may already add it to the set of transversal maxima.

We say that there is a path (transitum dari) from row \( i \) to row \( j \) if some element of row \( j \) is equal to a starred maximum in \( i \), or if there is a path from row \( i \) to row \( i' \) and from \( i' \) to \( j \). Rows of the first class are the upper rows containing a starred element in a right column and all the rows to which there is a path from them. If there is a lower row \( j \) in the first class, there is a path from an upper row \( i_0 \) with a starred maximum \( \alpha_0 \) in a right column to an upper row \( i_1 \) possessing an element \( \alpha_1 \) equal to the transversal maxima of \( i_0 \), then a path from row \( i_1 \) to row \( i_2 \) possessing an element \( \alpha_2 \) equal to the transversal maxima of \( i_1 \), then a path from row \( i_2 \) to row \( i_3 \) and at the end a path from an upper row \( i_{r-1} \) to the lower row \( i_r = j \) containing an element \( \alpha_r \) equal to the transversal maxima in \( i_{r-1} \). We may then replace in the set of transversal maxima those located in the rows \( i_0, \ldots, i_{r-1} \) by \( \alpha_0, \ldots, \alpha_{r-1} \) and get a greater set of transversal maxima by adding \( \alpha_r \). There is no lower series of the first class iff the set of transversal maxima is maximal.

If so, we will need to increase some rows. The rows not in the first class, from which there is a path to a lower row form the third class. The second class contains the remaining rows. We will increase all the elements of the third class rows by the minimal integer such that one of them become equal to a starred element \( \beta \) in a row of the first or second class. The computations of the maxima in each column and of the partition of the rows in classes may be done in \( O(n^2) \) operations.

If the element \( \beta \) belongs to the second class, its rows goes to the third, as well as all the rows from which there is a path to it, so that the change in the classes partition may be computed in \( O(n) \). This may happen at most \( O(n) \)

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6The choice of this strange word may be related to its use in the title: *Canon arithmeticus*, that according to Schumacher came from a play on words: the computations were done by a Kanonier Unteroffizier. See Jacobi’s Briefwechsel [25] note 2 p. 62.
times before there is no more elements in the second class. If $\beta$ belongs to the first class, some lower series of the third class will go to the first, and the number of transversal maxima will be increased, as we have seen. So, we need to perform the partition in classes and exhaust the second class, with total cost $O(n^2)$, at most $n$ times before we get a maximal set of $n$ transversal maxima. The complexity of the whole algorithm is $O(n^3)$.

Basically, this algorithm is the same as Kuhn’s “Hungarian method” (see section 7), but Kuhn also adds constants to the columns, and not only to the rows. The complexity of Jacobi’s algorithm, $O(n^3)$, is the same as the complexity of the variant of Kuhn’s method given by Munkres \[41\].

Jacobi proved further that the $\ell_i$ are minimal. This result is a consequence of the two following propositions: i) there is no unchanged row of the third class, i.e. with $\ell_i = 0$; ii) the numbers added to the third class series in the algorithm are the minimal ones that may change the partitions in classes.

One may remark that this is the only place in the two posthumous papers \[17, 19\] where Jacobi provides complete proofs of his results.

### 4.3 The shortest reduction in normal form

Jacobi provides \[19\] a method to compute a normal form, using as few derivatives as possible of the given equations

$$u_1(x_1, \ldots, x_n) = 0, \ldots, u_n(x_1, \ldots, x_n) = 0. \quad (1)$$

His results are only generically true; it is easy, as we will see, to make the requested hypotheses explicit. First, he implicitly assumes that the truncated determinant does not vanish. In order to provide rigorous statements, we need to translate his findings within the framework of some formalism that did not exist at this time. We will use here an elementary approach that relies on diffiety theory \[49\].

**Definition 1.** — For short, we say that $\lambda_i$ is a canon for the order matrix $(a_{i,j})$, if $(a_{i,j} + \lambda_i)$ is a canon, i.e. a table possessing a maximal set of transversal maxima.

Let $\ell_i$ be the minimal canon of the order matrix $(a_{i,j})$, with $a_{i,j} = \ord x_j u_i$, $\Lambda = \max_i \ell_i$, $\alpha_i = \Lambda - \ell_i$ and $\beta_j = \max_i a_{i,j} - \alpha_i$. The truncated jacobian matrix $\nabla_\alpha$ is the matrix $\left( \frac{\partial u_i}{\partial x_j^{(\alpha_i + \beta_j)}} \right)$.

We will say that an ordering $< \alpha$ on derivatives is a Jacobi ordering for the system \[4\] if $k_1 - \alpha_1 < k_2 - \alpha_2$ implies $x_i^{(k_1)} < x_i^{(k_2)}$.

Jacobi calls an explicit canonical form a system

$$x_i^{(e_i)} = f_i(x), \quad 1 \leq i \leq n$$

where the functions $f_i$ only depend on derivatives of $x_j$ smaller than $e_j$.

Jacobi’s shortest reduction method may be expressed in the following way.

\[6\] Together with a short passage in the second paper \[19\] related to resolvent computation and using the same kind of combinatorial arguments. See subsection \[4.3\].

\[7\] That we may compute using Jacobi’s algorithm, see subsection \[4.2\].
Theorem 2. — i) Assume there exist some functions \( \tilde{X} : t \mapsto \tilde{x}_j(t), [a, b] \mapsto \mathbb{R} \) that form a solution of the system \( u \), such that \( |\nabla u| \neq 0 \). Then, there exists a normal form of \( u \) for a Jacobi ordering, of which \( \tilde{x} \) is solution, that may be computed using equation \( u_0 = 0 \) and its derivatives up to the order \( \ell_i \).

ii) Assume that \( \nabla u \) and all its minors of order \( n - 1 \) that do not contain row \( \mu \) have full rank, then there is no normal form of \( u \) that may be computed using only derivatives of \( u_\mu \) up to an order strictly less than \( \ell_\mu \).

Proof. — i) We may reorder the unknowns \( x_j \) and the equations \( u_i \), so that the sequence \( \alpha_i \) of definition 1 is increasing and the \( n \) principal minors of \( \nabla u \) have non vanishing determinants. Let \( D_k \) be the determinant of the \( k^{th} \) minor. Consider the jacobian matrix of the system \( \{ x^{(\alpha_i, + \beta_i, + \kappa)} | 1 \leq i \leq n, 0 \leq k \leq \ell_i \} \), with respect to the set of derivatives \( E := \{ x^{(\alpha_i, + \beta_i, + \kappa)} | 1 \leq i \leq n, 0 \leq k \leq \ell_i \} \). Its determinant is a product of powers of the \( D_i \) and so is not vanishing. We can then use the implicit functions theorem and find, on a suitable open set containing \( (\tilde{x}_1(0), \ldots, \tilde{x}_1(0), \ldots, x^{(\alpha_i, + \beta_i, + \ell_i)}) \), \( x^{(\alpha_i, + \beta_i, + \ell_i)} \), functions expressing the derivatives of \( E \), depending on the derivatives of the set \( S := \{ x^{(\alpha_i, + \beta_i)} | 1 \leq i \leq n, 0 \leq k < \alpha_i + \beta_i \} \). Hence we get a normal form:

\[
x^{(\alpha_i, + \beta_i)} = f_i(x),
\]

of which \( \tilde{X} \) is a solution. The order associated to this normal form is a Jacobi ordering.

ii) Assume that there exists a normal form that may be computed using derivatives of \( u_\mu \) up to order \( \lambda_\mu \), and that \( \lambda_\mu < \ell_\mu \). Let \( \lambda_\mu :\max_{i=1}^n \lambda_i - \ell_i \). As \( \nabla u \neq 0 \), we see that there is an element in the normal form depending of some \( u^{(\lambda_0)}_i \) with \( \lambda_0 - \ell_0 = s \), and of the form \( x^{(\beta_0 + \Lambda + s)} = g(x) \), where \( \Lambda = \max \ell_i \). Remaining elements in the normal form have left side derivatives \( x_j^{(\beta_j + \Lambda + s')} \), with \( s' \leq s \), so \( g \) does not depend on derivatives \( x_j^{(\beta_j + \Lambda + s')} \); this implies that the minor of \( \nabla u \) obtained by supressing row \( i_0 \) and column \( j_0 \) is not of full rank, a contradiction.

Jacobi gave no proof for these statements; the style of the article [10] and of most of his manuscripts on the subject is that of a mathematical cook-book: in the best case, proofs are reduced to a short sketch. Ritt surmised [52] that the bound was suggested to Jacobi by such considerations on normal form computation. This natural assumption is confirmed by Jacobi’s claim [17] that his normal form reduction method provides an alternative proof of the bound. We easily see, according to the shape of the normal form above, that the order of \( u \) is \( \sum_{i=1}^n \alpha_i + \beta_i = J \).

Jacobi also claims that there are as many normal forms of this kind as there are permutations \( \sigma \) such that \( \sum_{i=1}^n a_{i, \sigma(i)} = J \). This is perhaps his single claim that does not stand, even under suitable genericity hypotheses, as shown by the example \( x_1'' + x_2'' + x_3'' = 0, x_2, x_3 = 0 \) and \( x_1 = -x_2 - x_3 \): we only have one possible permutation, but two possible normal forms for shortest reductions: \( x_1'' = -x_2'' - x_3'' \), \( x_2 = 0 \) and \( x_1'' = -x_2'' - x_3'' \), \( x_3 = 0 \). If we want to compute a normal form using a different kind of orderings, we may need to differentiate the defining equations a greater number of times. Jacobi provided bounds for the computation of resolvents.
4.4 Resolvent computation

In § 4 of the second article [19], Jacobi investigates the computation of resolvent, i.e. of normal forms such that one equation of order $J$, $x_i^{(J)} = f_i(x_{i_0})$, only depends on a single indeterminate $x_{i_0}$ and the remaining ones, $x_i = f_i(x_{i_0})$, $i \neq i_0$, express the other variables as functions of $x_{i_0}$ and its derivatives of order lower than $J$. He writes: “As mathematicians use to consider such kind of normal forms before others, I will indicate how many times each of the proposed differential equations $u_1 = 0$, $u_2 = 0$, ..., $u_n = 0$ are to be differentiated in order to make appear auxiliary equations necessary for that reduction.” Jacobi’s presentation requires the implicit hypothesis $|\nabla u| \neq 0$, but his results stand in the more general situation of quasi-regular systems[50].

Let $A := (a_{i,j})$ be defined as in the introduction, Jacobi starts with the new matrix $A'$ defined by adding to each row of $A$ the corresponding numbers $\ell_i$ defined above. He assumes that a set of transversal maxima is chosen in the canons $A'$; he denotes transversal maxima with asterisks and calls them starred terms, and he underlines the terms being equal to the stared maximum located in their column. He says that row $i_1$ is attached (annexa) to row $i_0$ if it contains a stared term equal to some underlined term in row $i_0$ or in some row attached to row $i_0$. The row $i_0$ is implicitly assumed to be attached to itself. If not all rows are attached to row $i_0$, he increases all the rows attached to it of the same minimal number that makes new underlined elements to appear and new rows to be attached to row $i_0$. The process is to be continued until all rows are attached to it. We get a new matrix $A''$; the last step is to increase all its elements of a same number that makes the stared term in row $i_0$ become equal to the order $J$ of the system $u$; we get a new matrix $A''', \text{ which is obtained by increasing the rows of $A$ by a sequence of numbers $h_i$, } 1 \leq i \leq n, \text{ that will be the researched orders of derivation.}$

**Example 3. —** We consider a simple system $x_1'' - x_2' = 0$, $x_2'' - x_3 = 0$ and $x_3' - x_2 = 0$. For this system, the order matrix

$$A = A' = \begin{pmatrix} 2^* & 1 & -\infty \\ -\infty & 2^* & 0 \\ -\infty & 0 & 1^* \end{pmatrix}$$

is already a canon. It is not possible to construct a resolvent representation using $x_2$ or $x_3$, but it is possible with $x_1$. There is no row attached to the first one. Increasing it by 1, row 2 becomes attached to row 1. Then, increasing rows 1 and 2 by 1, row 3 becomes attached to row 1. So, we get the matrix

$$A'' = \begin{pmatrix} 4^* & 3 & -\infty \\ -\infty & 3^* & 1^* \\ -\infty & 0 & 1^* \end{pmatrix}.$$ 

The order $J$ of the system is 5 and we need to increase all terms by 1, so that the stared term of row 1 be made equal to $J$. We get the new matrix

$$A''' = \begin{pmatrix} 5^* & 4 & -\infty \\ -\infty & 4^* & 2^* \\ -\infty & 1 & 2^* \end{pmatrix}.$$
To obtain $A''$, one has increased row 1 of $A$ by 3, row 2 by 2 and row 3 by 1, so one needs to differentiate the first equation 3 times, the second 2 times and the third 1 time to compute a resolvent representation for $x_1$: $x_1^{(5)} = x_1''$, $x_2 = x_1^{(4)}$, $x_3 = x_1''$.

**Theorem 4.** — Assume that $|\nabla u| \neq 0$ and that $x_j$ is a differential primitive element, then a resolvent representation of $u$ using $x_j$, may be computed using derivatives of equation $u_i$ up to order $h_i$.

Jaccobi gave no proof for this result; we propose for the reader’s convenience the following elementary one, that follows Jacobi’s presentation.

**Proof.** — Using theorem 2 we may assume that the system admits a normal form of the shape $x_i^{(a_{i,j})} = f_j(x)$, that can be obtained using derivatives of $u_i$ up to order $\ell_i$ at most. Let us denote by $w_i$ the $i^{th}$ equation of this normal form. We may make a more precise evaluation of the order of derivation requested to compute $w_{i_0}$. If $x_i^{(a_{i,j}+j)}$ appears in $u_{i_0}$, then we will need the $(\ell_i - \ell_{i_0})^{th}$ derivative of $w_i$ to compute $w_{i_0}$. Then, the row $i$ is attached to the row $i_0$ in $A'$. We may recursively prove that, if the $(\ell_i - \ell_{i_0})^{th}$ derivative of $u_i$ is needed, then the row $i$ is attached to the row $i_0$ in $A'$. More precisely, if we need to increase the row $i_0$ of $A'$ by $s \leq \ell_i - \ell_{i_0}$, so that row $i$ becomes attached to it, we only need to differentiate $u_i$ up to order $\ell_i - \ell_{i_0} - s$ in order to compute $w_{i_0}$.

As $|\nabla u| \neq 0$, the order of the system is $J$ and for computing a resolvent, we need the first $J$ derivatives of $x_{i_0}$; we shall differentiate equation $w_{i_0}$, which is of order $a_{i_0,i_0}$ in $x_{i_0}$, up to order $J - a_{i_0,i_0}$. At the beginning of the process and after each step of differentiation, if some $x_j^{(a_{j,j})}$, $j \neq i_0$ appears in the right side, we may substitute to it the expression $f_j(x)$, and repeat the process until no such derivative appears. If the row $i$ of $A'$ is attached to row $i_0$, then we may compute the derivative of order $\ell_{i_0}$ of $w_i$ using the derivatives of $u_i$ of order $\ell_i$ at most, and so we may compute a resolvent using the derivatives of $u_i$ of order at most $J - (a_{i_0,i_0} + \ell_i) + \ell_{i_0} = h_i$. If the row $i$ in $A'$ becomes attached to row $i_0$ after this last row has been increased of $e$, then we may compute the derivative of $w_{i_0}$ of order $e + \ell_{i_0}$ using the derivatives of $u_i$ up to order at most $\ell_i$. So, we may get the derivative of order $J - a_{i_0,i_0} + \ell_{i_0}$, requested to get the resolvent, using derivatives of $u_i$ up to order at most $J - a_{i_0,i_0} - (e + \ell_{i_0}) + \ell_{i_0} = h_i$, hence the result.

**Remark 5.** — It is easily seen that the maximal possible value for $h_i$ is $J - a_{i_0,i_0} + \ell_i - \ell_{i_0} = J - (\beta_{i_0} + \gamma) + \ell_i$. If the equation $u_i$ has order $e_i$, the sum of the $h_i$, that is the number of equations in the system one needs to solve to compute a resolvent, is maximal when $e_i$ is the order of $u_i$ in all the variables: $h_i$ is at most $\sum_{i \neq j} e_i$, and $\sum_{i=1}^{n} h_i \leq (n-1) \sum_{i=1}^{n} e_i$. On the other hand, assume that, after some reordering, the rows are listed by successive order of “attachment” to row $i_0 = 1$. We have $h_1 = \sum_{i \geq 1} a_{i,i}$ and for $i > 1$, $h_i \geq \sum_{k>i} a_{k,k}$, so that $\sum_{i=1}^{n} h_i \geq \sum_{i=1}^{n} (i-1) a_{i,i}$.

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9This means that a resolvent exists for that element, see Cluzeau and Hubert [2] for details.

10This idea appears in some unpublished manuscripts of Jaccobi, e. g. in III/23 a) p. 2217 a: "et simulac in dextra parte obvertit variabulus $x_{i,j}$ differentialem $f_{i,j,i}^{(m)}$, eius e $f_{i,j}$ substitu tuo valorem $f_{i,j}$." (Mathematical notations between square brackets have been changed to correspond with those used in our proof.) See our translation [2] p. 37.
Remark 6. — In more general situations, that is when $|\nabla u| = 0$, one may
need to differentiate the defining equations a greater number of times, that
may also depend on the degree of the equations. But for quasi-regular systems
\cite{37,49}, Jacobi’s bound for the $h_i$ still stands\cite{50}.

Jacobi did not stop at this stage; he has also provided the following elegant
version of his result.

Theorem 7. — The order $h_i$ up to which one needs to differentiate equation
$u_i$ in order to compute a resolvent representation for $u$ using $x_{i0}$ as a primitive
element, is equal to the maximal transversal sum of the $(n-1) \times (n-1)$ matrix
obtained by removing from $A$ line $i$ and column $i0$.

Jacobi gives a complete proof of the equality between the order $h_i$ defined by
his process and the maximal transversal sum of the above theorem. It relies on
the same kind of argument as the proof of his algorithm. We refer to his paper
\cite{19} for details\footnote{See our translation \cite{20} p. 62–63.}.

We may remark that a naïve use of this theorem requires $O(n^4)$
operations to compute the $h_i$ using Jacobi’s or Munkres’ algorithm \cite{41}. But Jacobi’s
process, described above, only requires $O(n^3)$ operations, once the $\ell_i$ have been
computed with the same complexity. Jacobi states first an efficient algorithmic
version of his result, before giving an elegant—but less efficient—mathematical
version.

These sharp bounds provided by Jacobi for computing normal forms can be
used in a straightforward way to improve many algorithms developed in recent
years. His results on normal form are especially important for new methods of
resolution for algebraic systems relying on the representation of polynomials as
\textit{Straight-Line Programs} \cite{12} that begin to be extended to differential systems
\cite{7,8,9,10}.

4.5 The bound

The work of Jacobi suffers from the lack of a rigorous theory allowing precise
definitions of the mathematical objects he considers. However, if one has in mind
that his goal is to consider physical situations, that imply implicit hypotheses,
his proof of the bound is not so weak as one may think at first sight.

Proof of the bound. — The proof relies on three successive steps. The
first is to claim that one can reduce to linear equations. This, of course, will not
stand for all systems, but for those expressing the laws of mechanics or other
problems of physical interest, we can take this for granted.

From a mathematical standpoint, the most general condition under which
this may be done is the one given by Johnson \cite{29}, expressing that a differential
system is in some way “regular”\footnote{This was to be developed later by Kondratieva \textit{et al.} \cite{35,37,49}.}. Under such hypotheses, one may from a
system $u_i = 0$ build a new system $\delta u_i = 0$, which is nothing else than what
is described by Johnson as “Kähler differentials” \cite{28}. Let $\tilde{X} : t \mapsto \tilde{x}(t)$ be a
solution of $u$. It is a quasi-regular solution of $u$ at $t_0$ if for all $r \in \mathbb{N}$, when
substituting to any derivative $x^{(k)}_j$ the value $\tilde{x}^{(k)}_j(t_0)$, the jacobian matrix of the
system $u, \ldots, u^{(r)}$ has full rank $n(r+1)$. Then the jet of $\tilde{X}$ at $t_0$ is a regular
point of the subspace $V$ of $J^\infty(\mathbb{R}, \mathbb{R}^n)$ defined by $u$ and the order of $u$, that is also the dimension of $V$, is equal to the dimension of its tangent space, defined by $\delta u$.

It is easily seen that, if $\nabla u \neq 0$ for $\tilde{X}$, then $\tilde{X}$ is quasi-regular. We say “quasi-regular”, for some “singular solutions”, according to Ritt’s terminology, such as $x = 0$ for the equation $(x')^2 - 4x = 0$, may satisfy the Johnson condition.

Jacobi uses in fact a stronger result, but without proof, claiming that if the system $u$ has order $e$ and if its general solution depends on $e$ arbitrary constants $a_1, \ldots, a_e$, then the set $\partial x/\partial a_i$ is a basis of solutions of the linear system $\delta u$. I don’t know if it was “well known” at that time.

The next step is more surprising, for Jacobi claims, with no justification, that it is enough to consider a linear system with constant coefficients. There is a beginning of proof, stricken out by Jacobi on page 2203a of manuscript II/13b: “In exploring the order of a system, as one considers only the highest derivatives in the linear differential equations to which the proposed ones have been reduced, one may assume that the coefficients are constants. For, having differentiated the equations $[\delta u = 0]$ many times, in order to obtain new equations”… This interrupted sentence suggests that the shortest reduction process used to compute normal forms was the basic idea.

In fact, provided that $|\nabla u|$ does not vanish, we only need to consider the coefficients of the leading derivatives when computing the order: derivatives of the coefficients will only affect smaller derivatives of the variables. If $|\nabla u|$ vanishes, assume that the equations $u_i$ are sorted by increasing $\alpha_i$ and that $i_0$ is the smallest integer such that the first $i_0$ rows of $\nabla u$ are linearly dependent, satisfying $\sum_{i=1}^{i_0} c_i l_i = 0$, then we may in $\delta u$ replace $\delta u_{i_0}$ by $\sum_{i=1}^{i_0-1} c_i (\delta u_i)(\ell_i - \ell_{i_0})$, as $c_{i_0} \neq 0$, this new system is equivalent to $\delta u$ and the leading derivatives in $\delta u_{i_0}$ have been removed: so the new system has a strictly smaller Jacobi number $J$. This gives an easy proof by induction of the result.

We could conclude from such considerations on normal forms of linear systems, but Jacobi uses a different kind of argument for the last step of his proof. Having reduced his investigations to the case of a linear system with constant coefficients, he looks for solutions of the form $x_j = c_j e^{\lambda t}$. Substituting such an expression in his linear equations, he gets a system of the form $\sum_{j=1}^n P_{i,j}(\lambda) c_j = 0$, $1 \leq i \leq n$ where $\deg \lambda P_{i,j} = a_{i,j}$. The number of possible values for $\lambda$ is the degree of the determinant $|P|$, which is at most $\max_{\sigma \in S_n} \sum_{i=1}^n a_{i,\sigma(i)} = J$, with equality if $|\nabla u| \neq 0$. Jacobi does not consider the case of multiple solutions, etc. but there is no difficulties.

We cannot know precisely how Jacobi could have detailed his demonstration. However, we have shown that we can design a complete proof, using elementary arguments, following the indications he left in his manuscripts, provided that we retreat to the safe ground of regular systems. For an account of the research on singular solutions during the first half of the $19^{th}$ century, see the work of Houtain [27].

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13 We call here a leading derivative of an equation, a derivative $x_j^{(k)}$ such that $k - \beta_j$ is maximal, i.e. with a maximal Jacobi order.
5 The second part of the xixth century

The first publication of these two papers[17, 19] in 1865 and 1866 and a new publication in 1890 in the volume V of Jacobi’s complete works did not stimulate further research on the subject in Germany. The works of his continuators during the xixth century are very superficial. They did not seem to have considered the subject could deserve a real mathematical effort.

Nanson in 1876 [38], considers the linear case with constant coefficients, in 2 and 3 variables. He rejects the idea of substituting \( x_i = c_i e^{\lambda t} \) in the equation, in order to obtain the order, claiming that one should first compute the order in order to be sure that a complete solution of that kind could be obtained. He proceeds heuristically, eliminating one variable after the other and bounding at each step the orders in each variable of the equations he gets.

Jordan in 1883 [30] considered the non linear situation with 4 variables. He tried to eliminate \( x_2, x_3, x_4 \) in order to compute a resolvent for \( x_1 \), using arguments that only work in the most general situation. From heuristic considerations on the number of derivatives to eliminate, he established theorem \( 7 \) in four variables: one needs to differentiate \( u_i \) a number of times equal to \( h_i \), the maximal transversal sum of the matrix obtained by removing from the order matrix \( A \) row \( i \) and column 1. So, the order of the resolvent will be \( \max_{i=1}^4 a_{i,1} + h_i = J \).

Nanson, who referred to Boole [11] for systems of order 1 and Cournot [6] for systems of two equations, did not quote Jacobi. Jordan—who did not quote Nanson—did not have a full view of Jacobi’s work. He claimed that Jacobi had given an “indirect” proof and that he would give a “direct” one.

The work of Chrystal in 1895 [3] was rigorous, but he only considered the easy linear case with constant coefficients—Jacobi’s arguments only worked for all different eigenvalues. Ritt, who gave these references [52] also referred to a paper by Sarinski (Communications of the University of Warsaw, 1902) that I was unable to find.

6 Ritt’s work

Ritt, who was known to be fluent in many languages, certainly had a better view of Jacobi’s results. However, a century after Jacobi wrote them, the style and spirit of mathematics did change. One expects rigorous proofs, but also intrinsic results, attached to geometrical objects. One thinks of varieties (or “components”) and not of systems: more precisely, in his article [52], published in 1935, a system means a component. It is remarkable that this change also concerned a mathematician like Ritt, who knew well some very applied style of mathematics and whose activity was dominated by the spirit of “classicism” [15].

One interest of Ritt in this subject was to secure results that could be applied to components intersection and it is not the case for Jacobi’s bound (see

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14He worked performing computations in the Naval Observatory in Washington during his studies and helped to organise a computation group working for the US artillery during World War I [15].

15In Ritt’s obituary [44] p. 310, E.R. Lorch writes “His media are complex function theory of the nineteenth century and differential equations. Much of his work could have been written a half century earlier.”
Having developed a theory that allows to characterize “singular components”, he wonders if the bound stands for all components, including those that do not satisfy natural hypotheses of regularity: a difficult question that was certainly not considered by Jacobi.

The less convincing part of Jacobi’s “proof” is to go from time varying systems to constant coefficients. Ritt’s proof in the linear case \[52\] solves the problem. But Ritt, who only considers \[52, 53\] elimination orderings does not prove the necessary and sufficient condition for the bound to be reached, given by the non vanishing of the truncated Jacobian. Such a condition is, as we have seen in \[4.5\] more easily proved with an orderly ordering for an adapted order defined in this way: \[\tilde{\text{ord}}_{x_{i}} u_{j} := \text{ord}_{x_{j}} u_{i} - \beta_{j},\] with \(\beta_{j}\) defined as in def. \[1\]. Ritt’s proof relies on some simplified method for computing a characteristic set in the linear case. He proves the strong bound, using the convention \[\text{ord}_{x_{j}} u_{i} = -\infty\] if \(x_{j}\) and its derivatives do not appear in \(u_{i}\).

His 1935 paper \[52\] concludes with a proof of the bound for any component of dimension zero of a system of two polynomial equations \(A\) and \(B\) in two variables, that is reproduced with a few modifications in \textit{Diff. alg.} \[53\] p. 136–138. One may remark that in the 1935 article \[52\], a footnote precises that if one of the variables does not appear in \(A\), the order of \(A\) in this variable is 0. In \[52\], a new footnote claims that we can in fact prove the strong bound. The requested modifications in the proof seem easy, but are not given explicitly. In 1935 \[52\], Ritt refers to Gourin \[13\] for the following result: if a zero dimensional differential ideal \(I\) contains a zero dimensional differential ideal \(J\), then \(\text{ord} I < \text{ord} J\).

An important argument is not explicitly stated in the proof. Ritt reduces the situation to the case of two polynomials, \(A\) and \(E\), were \(E\) depends only on \(x_{1}\). He claims that \(E\) must effectively depend on that variable. For this, he needs to use the fact that a single equation in two variables cannot define a component of dimension 0, which is true by \textit{Diff. alg.} \[53\] chap. III § 1 p. 57.

In \textit{Diff. alg.} \[53\] p. 139–144, Ritt also proved an important result. He considers irreducible differential polynomials in two variables, and investigates the order of the intersection of their general solutions. He first proves that Jacobi’s bound stands if \(A\) and \(B\) have order not greater than unity. Then, he exhibits a family of polynomials of order \(r > 3\) in \(x_{1}\) and \(x_{2}\), the general solution of which intersects the manifold of \(x_{1}\) in an irreducible manifold of order \(2r - 3\). So, the bound cannot stand for manifolds, but just for systems.

7 The assignment problem

In 1944, the R.A.F. tried to optimize the reaffectation of soldiers of disbanded units \[54\]. No practical solution could be used before the end of World War II, but this initiated the first research on the problem. It was then considered to optimize the affectation of \(n\) workers to \(n\) tasks, \(a_{i,j}\) representing the productivity of worker \(i\) if affected to task \(j\). One looks for a maximum, with the constraint that two different workers must be given two different tasks. The Monge problem \[47\] may be considered as a first, continuous, example of such problems (how to transport earth from a given area to some other with the least amount of carriage).
It is not the place to give much details about the discovery of the Hungarian method by Harold Kuhn in 1955 [54]. Anyway, it may be of interest to consider the situation from the standpoint of the transmission of mathematical results. Jacobi’s algorithm was sleeping in papers written in a dead language, with titles that cannot be related to the assignment problem. It also seems that the mathematical community was not always of a great help for the practitioner who wanted to solve his optimization problem in a short time. It is amazing that trying \( n! \) possibilities may have been considered as an acceptable solution, provided that their number is finite, in the middle of the \( xx^{th} \) century (see Schrijver [54] p. 8). For Jacobi, trying \( n! \) solution was not a solution at all. He claimed indeed to look for a solution, whereas we would rather say that we are looking for an efficient one. The efficiency issue was at that time—very strongly—implicit.

One may also notice that rediscovering Jacobi’s method took more than 10 years, from 1944 to 1955, and that prominent mathematicians such as John von Neuman considered the problem. It could have been much longer if Kuhn did not translate from Hungarian Egerváry’s paper [11] that allowed him to conclude. Inspired by Kőnig, a pioneer of graph theory [38], Egerváry considered the problem as a weighted variant of the maximal matching problem, but he did not give a polynomial time algorithm [31]. Possibly, Egerváry could have contributed to the question himself, but it seems that the research was concentrated in the eastern part of the world, mostly in the United States. It was also strongly motivated by economical and organizational issues and one may guess that it did not facilitate collaborations with eastern scientists during the cold war. Egerváry heard of Kuhn’s algorithm as late as in 1957 and went back to such matters with two papers on the transportation problem. Tragic circumstances interrupted his research. If, after the war, the Hungarian Academy of Sciences provided him good working conditions that stimulated his work, he killed himself in 1958, persecuted by bureaucrats [16].

Richard Cohn is the first, for the best of my knowledge, to have made a link between Jacobi’s work and the assignment problem [5], but the information did not spread into the optimization community before 2005.

### 8 The second part of the \( xx^{th} \) century

The second part of the \( xx^{th} \) century is dominated by the work of Richard Cohn and his students.

#### 8.1 Greenspan’s bound

Greenspan proved a different bound, in the framework of difference algebra [14] in 1959. It is easily translated in differential algebra: let \( r_j = \max_{i=1}^n \ord_{x_j} u_i \) and \( \eta_j \) be the greatest integer such that \( \ord_{x_j} u_i(\eta_j) \leq r_j \), \( 1 \leq j \leq n \), Greenspan’s bound is \( G := \sum_{j=1}^n r_j - \max_{i=1}^n \eta_j \). It was proved by Cohn in 1980 [4] that the order of any zero dimensional component of an arbitrary differential system is bounded by \( G \). One may remark that this result implies Jacobi’s bound in two variables, and that it may be proved using an adapted version of Ritt’s proof.

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16Kőnig committed suicide in October 1944, a few days before Budapest Jews were forced into the ghetto and their deportation began.
8.2 Lando’s bound

Barbara Lando proved in 1970 [42] the “weak bound” for order one differential systems. The “weak bound” means that if \( x_j \) and its derivatives do not appear in \( u_i \), we use the convention \( \text{ord}_{x_j} u_i = 0 \). This result was translated in difference algebra [43] in 1972. Proving the strong bound for order one systems would imply the strong bound for any system, for the strong bound is compatible with the classical reduction of a system to order one equations. Lando’s proof uses results on matrices of zeros and ones that are very close to some theorems of König and Egerváry.

B. Lando also proved that any order matrix \( A \) is the order matrix of a system for which the bound is reached.

8.3 Tomásovíc and PDE systems

There was an attempt to generalize the bound to partial differential systems [55], due to Tomásovíc in 1976. Consider a system of \( n \) partial differential equations in \( n \) variables and \( m \) derivatives. Let \( \mathcal{P} \) be a component of \( \{u\} \) and

\[
\omega_\mathcal{P}(r) = \sum_{i=0}^{m} a_i \binom{r+i}{i}
\]

be the Hilbert polynomial of \( \mathcal{P} \). If the dimension of \( \mathcal{P} \) is 0, then \( a_m = 0 \). The Jacobi conjecture of Tomásovíc states that \( a_{m-1} \leq J \), where \( J \) is defined as above, according to the order matrix of the PDE system. This conjecture had already been presented by his thesis adviser, Kolchin, in 1966 [33]. Tomásovíc proved it for linear systems and for \( n \leq 2 \). A proof in the linear case was also given by Kondratieva et al. [36].

Tomásovíc’s results remained unpublished, due to his untimely death. His dissertation contains interesting material that requires further examination.

8.4 Order and dimension

In 1983, Cohn proved that the bound would imply the “dimension conjecture”: every component defined by a system of \( r \) equations has differential codimension at most \( r \). Tomásovíc proved in the chap. 6 of his thesis [55] that the dimension conjecture is equivalent to this one: If a system has a component of differential dimension 0, then its Jacobi number \( J \) is not \( -\infty \).

We have seen in section 6 that Ritt’s proof in two variables requires the dimension conjecture for \( r = 1 \). More precisely, Cohn’s proof shows that proving the bound for a system of \( n \) equations implies the dimension conjecture for a system of equations in \( r < n \) variables. Furthermore, Cohn proved that the weak bound and the dual of “Bézout’s bound” [47] also imply the dimension conjecture.

It is known that the intersection of two manifolds of differential codimensions \( r \) and \( s \) may contain components of codimension greater than \( r + s \) (see Diff. alg. [53] p. 133) and, as we have already seen above in section 6, Jacobi’s bound may only be expected to stand for systems and not for manifolds. If the examples

\[^{17}\text{It is defined as } \sum_{i=1}^{n} \max_{j} \text{ord}_{x_j} u_i.\]
given by Ritt in these two cases are clearly distinct, we may remark that they are both closely related to the structure of the singular place of the manifold. We still need to understand better such paradoxical behaviours and their possible connections.

8.5 Kondratieva’s proof

As we have already seen in subsection 4.5, the case where Jacobi’s linearization argument works corresponds to the regularity hypothesis defined by Johnson in order to prove Janet’s conjecture [29] in 1978: the differentials \( du^{(k)}_i \), \( 1 \leq i \leq n, k \in \mathbb{N} \), are linearly independent. Kondratieva et al. call such systems independent systems; they were able in 1982 [35] to prove the strong bound, using first linearization as in Jacobi’s approach [17], then Ritt’s proof for the linearized system \( du \). However, this result received little attention, the paper being written in Russian and difficult to find. A new proof, also valid for independent partial differential systems has been given in 2008 by the same authors [37].

8.6 Other works

In 1960, Jacobi’s strong bound was rediscovered independently by Volevich [56] for arbitrary linear systems, assuming that the truncated determinant does not vanish. One may also mention the works of Magnus, [45, 46] who refers to Chrystal and Jacobi.

9 Beginning of the xxI\textsuperscript{th} century

Hrushovski in 2004 [16] proposed a proof for the bound in difference algebra. His method is completely different from those used so far in this field, but it does not seem possible, or at least not easy, to deduce from this result a proof in the differential case.

See our article [49] for a proof of the truncated jacobian condition in the framework of diffiety theory, assuming the regularity condition of Johnson. A generalization to underdetermined systems is also considered. Kondratieva et al. [37] provided a proof of Jacobi’s bound for independent partial differential systems.

In 2001, Pryce [51] rediscovered Jacobi’s shortest reduction method in order to provide a efficient method of computing power series solution of implicit differential algebraic systems.

10 Conclusion

One says that Jacobi once told to a student who wanted to read all the mathematical literature before starting his research: “Where would you be if your father before marrying your mother had wanted to see all the girls of the world?” So, we may hope that he would have forgiven us for having forgotten some of his results.

We see nevertheless that a closer look to the past may be fruitfull. Contemporary mathematicians who turn to computer algebra will find some common
spirit with the approach of these times, when a great familiarity with hand computation produced a still unformal but deep attention to efficiency.

Some of the results presented here, beyond their intrinsic mathematical interest, could help producing improved bounds of complexity or designing new algorithms. E.g., the shortest reduction leads to the choice of an ordering for which the computation of a characteristic set may be easier.

Thanks

I express my gratitude to the late Evgeny Pankratiev, Marina Kondratieva, Alexandr Mihailev, Brahim sadik and the referees for their careful rereading and many corrections.

Thanks to Richard Cohn, Marina Kondratieva, Harold Kuhn, William Sit for scientific comments and historical precisions. I also express my thanks to Dr Wolfgang Knobloch and Dr. Vera Enke (Archiv der BBAW) for their precious help in my search for Jacobi’s manuscripts, to Bernd Bank for achieving the deciphering of Cohn’s letter II/13 a), to mgr Ivo Laborer (Archivum Państwowe we Wrocławiu) and mgr Marlena Koter (Archivum Państwowe w Olsztynie), to Jean-Marie Strelcyn for his kind providing translations of letters from the polish archives, to Bärbel Mund (Niedersächsische Staats- und Universitätsbibliothek Göttingen), Mikael Ragstedt (library of the Mittag-Leffler Institute).

Last but not least, I express my gratitude to all the staff of the Archives and Bibliothèque Centrale de l’École polytechnique.

The “Groupe Aleph et GÉode” provided financial support for paying copies of the documents.

Warning

The bibliography is divided in four parts. The first, Manuscripts contains primary material, mostly Jacobi’s manuscript, comming from Jacobis Nachlaß, Archiv der Berlin-Brandenburgische Akademie der Wissenschaften. Documents are denoted by their archive index, e.g. [I/58 a]]. The second, Complete works contains the book were Jacobi’s work were published. They are denoted by [GW . . . ], [VD], . . . The third Crelle Journal contains the issues were the quoted papers of Jacobi were published. Their are denoted by [Crelle. . . ] The fourth and last contains the remaining material. They are denoted by numbers.

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Manuscripts

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