Magneto-static vortices in two dimensional
Abelian gauge theories

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Abstract

We study the existence of vortices of the Klein-Gordon-Maxwell equations in the two dimensional case. In particular we find sufficient conditions for the existence of vortices in the magneto-static case, i.e. when the electric potential $\phi = 0$. This result, due to the lack of suitable embedding theorems for the vector potential $A$ is achieved with the help of a penalization method.

1 Introduction

In the Abelian gauge theory the interaction between a matter field $\psi$ obeying the nonlinear Klein-Gordon equation and the electromagnetic field represented by the gauge potentials $(A, \phi)$ is described by considering the Lagrangian density (see e.g. [13], [14])

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 - W(|\psi|)$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$ and

$$\mathcal{L}_0 = \frac{1}{2} \left[ |(\partial_t + i\phi)\psi|^2 - |(\nabla - iA)\psi|^2 \right]$$

$$\mathcal{L}_1 = \frac{1}{2} |\partial_t A + \nabla \phi|^2 - \frac{1}{2} |\nabla \times A|^2$$

being $W$ is a suitable nonlinear term $W : \mathbb{R}^+ \rightarrow \mathbb{R}$. Making the variation of the total action

$$S(\psi, \phi, A) = \int (\mathcal{L}_0 + \mathcal{L}_1 - W(|\psi|)) \, dxdt$$

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with respect to $\psi, \phi, A$ we have the following set of equations

$$
(\partial_t + i\phi)^2 \psi - (\nabla - iA)^2 \psi + W'(|\psi|)\frac{\psi}{|\psi|} = 0 \quad (1.3)
$$

$$
\nabla \cdot (\partial_t A + \nabla \phi) = \left( \text{Im} \frac{\partial \psi}{\psi} + \phi \right) |\psi|^2 \quad (1.4)
$$

$$
\nabla \times (\nabla \times A) + \partial_t (\partial_t A + \nabla \phi) = \left( \text{Im} \frac{\nabla \psi}{\psi} - A \right) |\psi|^2 \quad (1.5)
$$

which correspond to the Euler-Lagrange equations for (1.1). We refer to these equations as the Klein-Gordon-Maxwell (KGM) equations.

Many papers are concerned with the existence of stationary solutions of (1.3)-(1.5) in the static situation, i.e functions of the following form

$$
\psi(x, t) = u(x) e^{i(S(x) - \omega t)}, \ u \in \mathbb{R}^+, \ \omega \in \mathbb{R}, \ S \in \mathbb{R}/2\pi\mathbb{Z}. \quad (1.6)
$$

with the electromagnetic potentials satisfying

$$
\partial_t A = 0 \quad \text{and} \quad \partial_t \phi = 0
$$

In this case equations (1.3)-(1.5) become

$$
-\Delta u + \left[ |\nabla S - A|^2 - (\phi - \omega)^2 \right] u + W'(u) = 0 \quad (1.7)
$$

$$
-\nabla \cdot [(\nabla S - A)u^2] = 0 \quad (1.8)
$$

$$
-\Delta \phi = (\omega - \phi)u^2 \quad (1.9)
$$

$$
\nabla \times (\nabla \times A) = (\nabla S - A)u^2. \quad (1.10)
$$

It is possible to have three types of stationary non-trivial solutions

- electro-static solutions: $A = 0, \phi \neq 0$
- magneto-static solutions: $A \neq 0, \phi = 0$
- electromagneto-static solutions: $A \neq 0, \phi \neq 0$

under suitable assumptions on the nonlinear term $W$.

If the stationary solution $\psi(x, t) = u(x)e^{i(S(x) - \omega t)}$ admits a phase that depends only on time, i.e $S(x) = 0$, we call this solution a standing wave solution, whereas if $S(x) \neq 0$ we call this solution a vortex.

In the literature there exist results both for standing waves and vortices in the electro, magneto and electromagneto-static case, see for instance the books [8], [12] and the more recent papers [6], [4], [5] and [7]. In particular, for what concerns the existence of vortices, the classical results of [1] and [10] are obtained in the two dimensional case with a double-well shaped function $W$ of the type $W(s) = (1 - s^2)^2$, whereas in [6] three dimensional vortices are studied with $W(s) = \frac{1}{2} s^2 - \frac{s^p}{p}$ with $2 < p < 6$. 

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In this paper we study two dimensional vortices in the magneto-static case, i.e for $\phi = 0$. This problem has a physical relevance due to the fact that two dimensional magneto-static vortices arise in superconductivity, see for instance [9]. The assumption $\phi = 0$ readily implies $\omega = 0$, hence stationary solutions do not depend on time and have null angular momentum although they have non-vanishing magnetic momentum.

We consider solutions $\psi$ of equations (1.7)-(1.10) of the form (1.6) with $\omega = 0$ and $S(x) = k\theta(x)$ where $\theta$ is the angular function

$$
\theta(x) = \text{Im} \log(x_1 + ix_2), \ x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}
$$

and $k \in \mathbb{Z} \setminus \{0\}$ is a constant. A solution $\psi$ with this choice of $S$ is a vortex and the constant $k$ is called the vorticity. Notice that the function $\theta$ and its gradient $\nabla \theta(x) = (\frac{x_2}{r^2}, -\frac{x_1}{r^2}, 0)$ are $C^\infty$ in $\mathbb{R}^2 \setminus \{0\}$ and $|\nabla \theta| = 1/r$.

With this ansatz equations (1.7)-(1.10) reduce to

$$
-\Delta u + |k\nabla \theta - A|^2 u + W'(u) = 0 \quad (1.11)
$$

$$
\nabla \times (\nabla \times A) = (k\nabla \theta - A) u^2. \quad (1.12)
$$

The existence of non-trivial solutions of (1.11) and (1.12) depends on the assumptions on the nonlinear term $W$. For example if $W'(s)s \geq 0$ then one can prove that any solution $(u, A)$ has necessarily $u \equiv 0$. We prove that under the following assumptions on $W$ there exists a solution with nontrivial $u$.

We take the potential $W$ of the following type

$$
W(s) = \frac{1}{2}s^2 - R(s)
$$

where $R: \mathbb{R}^+ \to \mathbb{R}$ satisfies:

- $R(0) = R'(0) = 0$,
- $\exists c > 0$ and $p > 2$ such that $|R(s)| \leq cs^p$,
- $sR'(s) \geq pR(s) > 0$ for $s > 0$.

Before stating our main result, we make a short remark on notation. Our problem is defined in $\mathbb{R}^2$, however, to give sense to expressions like $\nabla \times A$, vectors will be thought of as three-vectors with null third component and depending only on two variables $(x_1, x_2)$. In particular hereafter we use the notation $|\nabla A|^2 = \sum_{i,j=1}^3 (\partial_i A_j)^2$.

The main result of the paper is the following.

**Theorem 1.1.** Under the above conditions on the potential $W$ there exists a (non-trivial) solution $(u_0, A_0)$ of (1.11) and (1.12) in the sense of distributions, where
• \( u_0 \) is positive, radial and satisfies

\[
\int |\nabla u_0|^2 \, dx + \int \left( 1 + \frac{1}{r^2} \right) u_0^2 \, dx < +\infty, \quad r^2 = x_1^2 + x_2^2;
\]

• \( A_0 \) is divergence free and \( \int |\nabla A_0|^2 \, dx < +\infty \).

This work has been inspired by the recent work by Benci and Fortunato [6], in which the existence of three dimensional vortices for KGM in the electro, magneto and electromagneto-static case is proved under the same assumptions on \( W \), by using a mountain pass argument in a suitable functional space.

The two dimensional case, due to the lack of suitable embedding theorems concerning the vector potentials \( A \), shall be treated however with a different approach. We cannot barely apply the same ideas of [6] due to the fact that the same mountain pass argument cannot be used. In this paper we follow a penalization argument, finding solutions of the “perturbed problem”

\[
\begin{aligned}
\begin{cases}
-\Delta u + |k\nabla \theta - A|^2 u + W'(u) = 0 \\
\nabla \times (\nabla \times A) + \varepsilon A = (k\nabla \theta - A) u^2
\end{cases}
(P_{\varepsilon})
\end{aligned}
\]

for \( \varepsilon \in (0, 1) \). A solution of the initial problem (1.11) and (1.12) will then be obtained by taking the limit for \( \varepsilon \to 0 \) of the solutions \((u_{\varepsilon}, A_{\varepsilon})\) of \((P_{\varepsilon})\).

One of the advantages of the perturbed problem is that the space of vector potentials \( A \) can be chosen such that a mountain pass theorem can be applied to find “weak” solutions of the problem.

The paper is organized as follows: in Section 2 we introduce all the functional spaces that will be used, and in Section 3 we introduce a natural constraint for the functional associated to \((P_{\varepsilon})\), that is a manifold on which the problem is more tractable. Finally, in Section 4 we prove the main theorem.

2 Functional framework

In the following, unlike otherwise specified, all the integrals, norms and functional spaces are intended on \( \mathbb{R}^2 \).

We denote by \( \| \cdot \|_p \) the \( L^p \) norm, \( H^1 \) is the usual Sobolev space with norm

\[
\| u \|_{H^1}^2 = \int (|\nabla u|^2 + u^2) \, dx
\]

and \( \tilde{H}^1 \) is the weighted Sobolev space endowed with norm

\[
\| u \|_{\tilde{H}^1}^2 = \| u \|_{H^1}^2 + \int \frac{u^2}{r^2} \, dx, \quad r^2 = x_1^2 + x_2^2.
\]
We denote the $L^p$ norm of a vector $\mathbf{X}$ as
\[
\|\mathbf{X}\|_p := \left\| (\mathbf{X} \cdot \mathbf{X})^{\frac{1}{2}} \right\|_p
\]
where no symbol is used for the inner product between vectors. Using this notation, $\|A\|_{H^1}^2 = \|\nabla A\|_2^2 + \|A\|_2^2$ where $\|\nabla A\|_2^2 = \sum_j \|\nabla A_j\|_2^2$.

Let us define the space $H = \tilde{H}^1 \times (H^1)^3$ with norm $\|(u, A)\|_H^2 = \|u\|_{H^1}^2 + \|A\|_{H^1}^2$, and the functional on $H$
\[
J_\varepsilon(u, A) = \frac{1}{2} \int (|\nabla u|^2 + |\nabla \times A|^2) \, dx + \frac{1}{2} \int |k\nabla \theta - A|^2 u^2 \, dx
+ \frac{\varepsilon}{2} \int |A|^2 \, dx + \int W(u) \, dx.
\]

Straightforward computations show that $J_\varepsilon$ is well defined and $C^1$ on $H$ thanks to the growth conditions on $W$, and its Euler-Lagrange equations are (2.1). Hence a critical point $(u, A)$ of $J_\varepsilon$ in $H$ is a weak solutions of (2.1), that is
\[
\int (\nabla u \nabla v + |k\nabla \theta - A|^2 u^2 + W'(u)v) \, dx = 0, \forall v \in \tilde{H}^1 \tag{2.1}
\]
\[
\int ((\nabla \times A)(\nabla \times V) + \varepsilon AV + (A - k\nabla \theta) V u^2) \, dx = 0, \forall V \in (H^1)^3. \tag{2.2}
\]

For details we refer to [6] where the case $N = 3$ is treated.

**Remark 2.1.** We can extend the potential $W$ to be defined on $\mathbb{R}$ by letting $R(s) = 0$ for $s \leq 0$. Using this extension one proves that if the couple $(u, A)$ satisfies (2.1) and (2.2), then $u \geq 0$ a.e., hence $u$ has some physical consistency. Indeed, denoting with $u^-(x) = \min\{u(x), 0\}$ and taking $v = u^-$ in (2.1), we have
\[
\int \left[ |\nabla u^-|^2 + |k\nabla \theta - A|^2 (u^-)^2 + W'(u^-)u^- \right] \, dx = 0.
\]

Since $W'(s) = s - R'(s) = s$ for $s \leq 0$ we have
\[
\int \left[ |\nabla u^-|^2 + |k\nabla \theta - A|^2 (u^-)^2 + (u^-)^2 \right] \, dx = 0
\]
and then $u^- = 0$ a.e.
A difficulty that arises looking at vortices of KGM is that the space $\mathcal{D}$ of test functions is not contained in $\hat{H}^1$. Hence a weak solution of $\langle P_\varepsilon \rangle$, a priori, does not satisfy it in the sense of distributions, specifically $(2.1)$ for $v \in \mathcal{D}$. Fortunately this circumstance does not happen. In Proposition 4.9 we show that a weak solution satisfying $(2.1)$ and $(2.2)$ turns out to be a solution in the sense of distributions. Hence we first find a weak solution of $\langle P_\varepsilon \rangle$ and then obtain also a solution in the sense of distributions.

3 A natural constraint for $J_\varepsilon$

To study the existence of critical points of the functional $J_\varepsilon$, we restrict ourselves to a submanifold of the space $H$. This is due to some difficulties. Although the introduction of the parameter $\varepsilon$ helps us to work in the familiar space $(H^1)^3$, the functional $J_\varepsilon$, contains a term, $\int |\nabla \times A|^2 \, dx$, which is not a Sobolev norm.

To overcome this problem, looking at the identity
\[
\int \left( |\nabla \times X|^2 + (\nabla \cdot X)^2 \right) = \int |\nabla X|^2 \quad (3.1)
\]
for regular vectors $X$ with compact support, it seems natural, if we want to deal with $|\nabla A|^2$ in place of $|\nabla \times A|^2$, to take the manifold of divergence free vector fields.

Moreover, by classical results on symmetric solutions of elliptic problems, we are naturally led to introduce a constraint also on $u$, considering only radial functions.

Hence we introduce a manifold $V \subset H$ such that

(a) it is a “natural constraint” for $J_\varepsilon$, namely its constrained critical points on $V$ are critical points on $H$;

(b) any $A \in V$ is divergence free;

(c) any $u$ in $V$ is radially symmetric.

To be more precise, define
\[
\mathcal{A}_0 = \{ b \nabla \theta : b \in C^\infty_0(\mathbb{R}^2 \setminus \{0\}) \text{ and radial} \}
\]
and let
\[
\mathcal{A} = \text{the closure of } \mathcal{A}_0 \text{ in the } (H^1)^3 \text{ norm.}
\]

Since we consider only radial functions $b(x)$, they only depend on $r = |x| = (x_1^2 + x_2^2)^{1/2}$. Hence we simply write $b(r)$. Moreover, notice that if $X$ is the closure in the norm
\[
\|f\|_2^2 = \int_0^{+\infty} \frac{f^2(r)}{r} \, dr + \int_0^{+\infty} \frac{(f'(r))^2}{r} \, dr
\]
of \( C_0^\infty(0, +\infty) \), then
\[
\mathcal{A} = \{ b\nabla \theta : b \in \mathcal{X} \}.
\]

Moreover any \( b \in \mathcal{X} \) can be continuously extended to 0 by setting \( b(0) = 0 \) and it results \( b(r) = \int_0^r b'(t) \, dt \).

Define also \( \mathcal{D}_r = \{ u \in \mathcal{D} : u = u(r) \} \) and
\[
\hat{H}_1^r = \text{the closure of } \mathcal{D}_r \text{ in the } \hat{H}_1 \text{ norm}.
\]

The natural manifold we consider is then defined by
\[
V = \hat{H}_1^r \times \mathcal{A} \tag{3.2}
\]
with norm \( \|(u, A)\|_V = \|(u, A)\|_H \) (see Section 2).

**Remark 3.1.** The manifold \( V \) is closed and convex, hence it is weakly closed in \( H \). This will be used in the next section.

We summarise the main properties of \( \mathcal{A} \) and the advantages to consider \( V \). First, since we are now dealing with radial functions \( u \), we recall the following result which is used in the computations.

**Theorem 3.2 (3).** The space \( H^1_r(\mathbb{R}^2, \mathbb{R}) \) is compactly embedded in \( L^s(\mathbb{R}^2, \mathbb{R}) \) for \( s \in (2, +\infty) \).

For what concerns the vectors \( \mathcal{A} \), the identity [3.1] and vector calculus imply that

**Lemma 3.3.** For \( A \in \mathcal{A} \) we have
1) \( \int |\nabla \times A|^2 \, dx = \int |\nabla A|^2 \, dx \);
2) \( \nabla \times (\nabla \times A) = -\Delta A \).

On \( V \) the functional \( J_\varepsilon \) has the following form to which we refer hereafter
\[
J_\varepsilon(u, A) = \frac{1}{2} \int (|\nabla u|^2 + |\nabla A|^2) \, dx + \frac{1}{2} \int |k\nabla \theta - A|^2 u^2 \, dx \\
+ \frac{\varepsilon}{2} \int |A|^2 \, dx + \int W(u) \, dx \tag{3.3}
\]

A critical point \( (u_0, A_0) \) of \( J \) on \( V \) satisfies:
\[
\int (\nabla u_0 \nabla v + |k\nabla \theta - A_0|^2 u_0 v + W'(u_0)v) \, dx = 0, \forall v \in \hat{H}_1^r \tag{3.4}
\]
\[
\int (\nabla A_0 \cdot \nabla V + \varepsilon A_0 V + (A_0 - k\nabla \theta) V u_0^2) \, dx = 0, \forall V \in \mathcal{A}. \tag{3.5}
\]
i.e. it is a weak solution in $V$ of
\[
\begin{aligned}
-\Delta u + |k \nabla \theta - A|^2 u + W'(u) &= 0 \\
-\Delta A + \epsilon A &= (k \nabla \theta - A) u^2
\end{aligned}
\] (P$_\varepsilon$)

The manifold $V$ defined in (3.2) is a natural constraint according to the following theorem.

**Theorem 3.4.** Assume that $(u_0, A_0)$ is a critical point of $J_\varepsilon$ in $V$, i.e.
\[
dJ_\varepsilon(u_0, A_0)[v, V] = 0 \quad \forall (v, V) \in V.
\]
Then
\[
dJ_\varepsilon(u_n, A_n)[v, V] = 0 \quad \forall (v, V) \in \tilde{H}^1 \times (H^1)^3.
\] (3.6)

**Proof.** The result will be obtained making use of the Palais Principle of Symmetric Criticality [11].

Let us first observe that $J_\varepsilon$ is invariant under the group action
\[
T_g : (u, A) \mapsto (u \circ g, g^{-1} \circ V \circ g)
\]
where $g \in O(2)$ is a rotation in $\mathbb{R}^2$. We compute the set of fixed points for this action. Clearly in the first variable, $u$, this set is nothing but $\tilde{H}^1$ and
\[
\partial_u J_\varepsilon(u_0, A_0)[v] = 0 \quad \text{for any } v \in \tilde{H}^1.
\] (3.7)

Moreover writing a generic vector $V(x, y)$ as
\[
V(x, y) = a(x_1, x_2)t + b(x_1, x_2)r
\]
where $t = (x_2/r, -x_1/r)$ and $r = (x_1/r, x_2/r)$, being as usual $r^2 = x_1^2 + x_2^2$, the requirement that $g^{-1} \circ V \circ g = V$ implies that the coefficients $a$ and $b$ are radial. Hence vectors of type $a(r)t + b(r)r$ are fixed by the action of $T_g$ on the second variable.

We claim that
\[
\partial_A J_\varepsilon(u_0, A_0)[a(r)t + b(r)r] = 0.
\] (3.8)

Indeed, by assumption,
\[
A_0 = b_0 \nabla \theta = \frac{b_0(r)}{r} t
\]
and $\partial_A J_\varepsilon(u_0, A_0)[a(r)t] = 0$. In order to prove (3.8) we have only to show that
\[
\partial_A J_\varepsilon(u_0, A_0)[b(r)r] = 0.
\]
Since $\nabla \times (b(r)r) = 0$ and the vectors $t$ and $r$ are orthogonal, we have

$$
\partial A J_\varepsilon(u_0, A_0)[b(r)r] = \int (\nabla \times A_0) (\nabla \times (b(r)r)) \, dx
+ \int u^2(A_0 - k \nabla \theta) b(r) dx + \varepsilon \int A_0 b(r) r \, dx = 0
$$

which proves the claim.

We conclude, by (3.7) and (3.8), that the couple $(u_0, A_0)$ is a critical point of $J_\varepsilon$ on the set of fixed points for the action of $T_g$ on $\tilde{H}^1 \times (H^1)^3$. Hence the Palais Principle applies and we get (3.6). □

## 4 Proof of Theorem 1.1

By the previous results, we are reduced to study the functional $J_\varepsilon$ defined in (3.3) on $V = \tilde{H}_1^1 \times A$.

### 4.1 Solution of the perturbed problem

**Proposition 4.1.** The functional $J_\varepsilon$ is weakly lower semicontinuous on $V$.

**Proof.** Using (3.3), we can write

$$
J_\varepsilon(u, A) = \frac{1}{2} \|u\|_{\tilde{H}_1^1}^2 + \frac{1}{2} \|\nabla A\|_2^2 + \frac{\varepsilon}{2} \|A\|_2^2 - \int R(u) \, dx
+ \frac{1}{2} \|A\|_2^2 u^2 \, dx - k \int \nabla \theta A u^2 \, dx,
$$

hence it is sufficient to show that the last two terms are weakly continuous.

Let $(u_n, A_n) \rightharpoonup (u, A)$ in $V$ for a given $(u, A) \in V$. Then the norms $\|(u_n, A_n)\|_H$ are bounded. We prove that

$$
\int |A_n|^2 u^2_n \, dx \to \int |A|^2 u^2 \, dx \quad \nabla \theta A_n u^2_n \, dx \to \nabla \theta A u^2 \, dx.
$$

To prove the first convergence, we write

$$
\left| \int (|A_n|^2 u_n^2 - |A|^2 u^2) \, dx \right| \leq a_n + b_n
$$

with

$$
a_n = \int |A_n|^2 |u_n^2 - u^2| \, dx \leq \|A_n\|^2_4 \left( \int |u_n^2 - u^2|^2 \, dx \right)^{1/2}
$$

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$$b_n = \left| \int u^2 (|A_n|^2 - |A|^2) \, dx \right|$$

By the compactness result of Theorem 3.2 up to a sub-sequence, we can assume that

$$u_n^2 \to u^2 \text{ a.e. and } \|u_n^2\|_2 \to \|u^2\|_2,$$

and by the classical Sobolev embedding $H^1(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$ for all $q \in [2, +\infty)$, the norm $\|A_n\|_4$ is bounded. Hence it follows that $a_n \to 0$.

For what concerns $b_n$, by applying again the Sobolev embedding it follows that the functions $|A_n|^2$ are in $L^p(\mathbb{R}^2)$ for all $p \in [1, +\infty)$ and with bounded norms. Hence in particular $|A_n|^2$ are bounded in $L^2$. Hence, up to a sub-sequence they converge weakly in $L^2$. Again by Theorem 3.2, the function $u^2$ is in $L^2$, hence $b_n \to 0$.

Analogously, to prove the second convergence in (4.1)

$$\left| \int (\nabla \theta (A u^2 - A_n u_n^2)) \, dx \right| \leq a_n' + b_n'$$

with

$$a_n' = \int \frac{1}{r} |A_n| |u^2 - u_n^2| \, dx$$

$$b_n' = \int \frac{1}{r} u^2 |A - A_n| \, dx.$$

Using the Schwarz inequality

$$b_n' \leq \left( \int |A - A_n|^2 u^2 \, dx \right)^{1/2} \left( \int \frac{u^2}{r^2} \, dx \right)^{1/2} \leq \|u\|_{\bar{H}_r^1} \left( \int |A - A_n|^2 u^2 \, dx \right)^{1/2}$$

and we can apply the same argument as before to the functions $|A - A_n|^2$ to obtain, up to a sub-sequence, the weak convergence in $L^2$. Hence $b_n'$ is vanishing.

It remains to prove that $a_n' \to 0$. From this the second convergence in (4.1) follows and the proof is completed.

It results $a_n' \leq c_n' + d_n'$ where

$$c_n' = \int \frac{1}{r} |A_n| |u - u_n| \, dx \quad d_n' = \int \frac{1}{r} |A_n| |u_n| |u - u_n| \, dx.$$
Now we have
\[
c_n' \leq \left( \int \left( \frac{|A_n| |u|}{r} \right)^{3/2} \frac{dx}{r^{1/2}} \right)^{2/3} \left( \int |u_n - u|^3 \frac{dx}{r^2} \right)^{1/3}
\]
\[
\leq \left( \int \left( \frac{|A_n|^{3/2} |u|^{1/2}}{r^{1/2}} \right)^2 dx \right)^{1/2} \left( \int \frac{u^2}{r^2} \frac{dx}{r} \right)^{1/2} \|u_n - u\|_3
\]
\[
\leq \|A_n\|_6 \|u\|_{H^1_r}^{1/3} \|u\|_{H^1_r}^{2/3} \|u_n - u\|_3
\]
\[
= \|A_n\|_6 \|u\|_{H^1_r} \|u_n - u\|_3 \rightarrow 0
\]
by Theorem 3.2 and because the norms \(\|A_n\|_{H^1}\) are bounded. Similarly
\[
d_n' \leq \|A_n\|_6 \|u\|_{H^1_r} \|u_n - u\|_3 \rightarrow 0
\]
which proves that \(a_n' \rightarrow 0\).

The next proposition establishes a geometrical property of \(J_\varepsilon\) which enables us to deduce a sequence of “quasi-solutions” i.e. a Palais-Smale sequence (PS for short).

**Proposition 4.2.** The functional \(J_\varepsilon\) has the Mountain Pass geometry on \(V\).

**Proof.** By Mountain Pass geometry we mean that there exist two constants \(\rho, \alpha > 0\) and a point \((\bar{u}, \bar{A})\) with \(\|(\bar{u}, \bar{A})\|_V > \rho\) such that
\[
J_\varepsilon(0, 0) = 0
\]
\[
J_\varepsilon(u, A) \geq \alpha \quad \text{for} \quad \|(u, A)\|_V = \rho, \quad (4.2)
\]
\[
J_\varepsilon(\bar{u}, \bar{A}) \leq 0, \quad (4.3)
\]
see [2]. It is worth noticing that \(\bar{u}\) can be chosen independently on \(\varepsilon\).
Let us first compute
\[
\int |k\nabla \theta - A|^2 u^2 \, dx \geq \int \left( |A|^2 - \frac{2|kA|}{r} + \frac{1}{r^2} \right) u^2 \, dx
\]
\[
= \int \left[ |A|^2 - 2\left( |kA|\sqrt{\frac{1}{r^2}} + \frac{1}{r} \right) \right] u^2 \, dx
\]
\[
\geq \int \left[ |A|^2 - 2|kA|^2 - \frac{1}{2r^2} + \frac{1}{r^2} \right] u^2 \, dx
\]
\[
= (1 - 2k^2) \int |A|^2 u^2 \, dx + \frac{1}{2} \int \frac{u^2}{r^2} \, dx
\]
\[
\geq \frac{1}{2} \int \frac{u^2}{r^2} \, dx + (1 - 2k^2)\|A\|_6^2 \|u\|_3^2
\]
\[
\geq \frac{1}{2} \int \frac{u^2}{r^2} \, dx - \frac{2k^2 - 1}{2}\|A\|_6^4 - \frac{2k^2 - 1}{2} \|u\|_3^4
\]
\[
\geq \frac{1}{2} \int \frac{u^2}{r^2} \, dx - c_1\|A\|_{H^1}^4 - c_2\|u\|_{H^1}^4.
\]

So we have
\[
J_\varepsilon(u, A) \geq \frac{1}{2} \int |\nabla u|^2 \, dx + \frac{1}{2} \int \frac{u^2}{r^2} \, dx - c_1\|A\|_{H^1}^4 - c_2\|u\|_{H^1}^4
\]
\[
+ \frac{1}{2}\|\nabla A\|_2^2 + \frac{\varepsilon}{2} \int |A|^2 \, dx + \int W(u) \, dx
\]
\[
\geq \|u\|_{H^1}^2 \left( \frac{1}{2} - c_2\|u\|_{H^1}^2 \right) + \|A\|_{H^1}^2 \left( \frac{\varepsilon}{2} - c_1\|A\|_{H^1}^2 \right)
\]
\[
- \int R(u) \, dx.
\]

By the assumptions on \( R(s) \)
\[
\int |R(u)| \, dx \leq c\|u\|_p^p \leq c'\|u\|_{H^1}^p,
\]
and hence \( J_\varepsilon \) has a strict local minimum in \((0, 0)\) and (4.2) is satisfied.

Finally we notice that, by using again the assumptions on \( R \), for any \( u_0 \in \tilde{H}^1_r \) it holds
\[
\lim_{t \to +\infty} J_\varepsilon(tu_0, 0) = -\infty.
\]

Concluding, there exists a point \((\bar{u}, 0)\) such that \( J_\varepsilon(\bar{u}, 0) < 0 \), hence (4.3). \( \square \)

By the \( C^1 \) regularity of the functional \( J_\varepsilon \) and Proposition 4.2, applying a weak form of the Mountain Pass Theorem we deduce the
existence of a PS sequence for $J_\varepsilon$ at some level $c_\varepsilon > 0$. That is there exists a sequence $(u_n, A_n) \subset V$ such that

$$J_\varepsilon(u_n, A_n) \to c_\varepsilon \quad \text{and} \quad dJ_\varepsilon(u_n, A_n) \to 0 \quad \text{in} \quad V'.$$

It is understood that the sequence $(u_n, A_n)$ also depend on $\varepsilon$, but for simplicity we omit this dependence here and in the next two results.

The following lemma is fundamental.

**Lemma 4.3.** Let $(u_n, A_n) \subset V$ be a PS sequence for the functional $J_\varepsilon$ at level $c_\varepsilon$. Then it is bounded.

**Proof.** If $(u_n, A_n) \subset V$ is a PS sequence, by definition

$$\partial_u J_\varepsilon(u_n, A_n)[u_n] = \lambda_n[u_n] \quad (4.4)$$

$$J_\varepsilon(u_n, A_n) = c_{\varepsilon,n} \to c_\varepsilon \quad (4.5)$$

where $\lambda_n \to 0$ in $(\hat{H}^1)'$.

Evaluating

$$J_\varepsilon(u_n, A_n) - \frac{1}{p} \partial_u J_\varepsilon(u_n, A_n)[u_n] = c_{\varepsilon,n} - \frac{1}{p} \lambda_n[u_n]$$

we find

$$\frac{p - 2}{2p} \int |\nabla u_n|^2 \, dx + \frac{p - 2}{2p} \int |k\nabla \theta - A_n|^2 u_n^2 \, dx + \frac{1}{2} \int |\nabla A_n|^2 \, dx$$

$$+ \frac{\varepsilon}{2} \int |A_n|^2 \, dx + \int \left( W(u_n) - \frac{1}{p} W'(u_n) u_n \right) \, dx = c_{\varepsilon,n} - \frac{1}{p} \lambda_n[u_n].$$

(4.6)

Recalling the assumptions on $W$, we get

$$W(u_n) - \frac{1}{p} W'(u_n) u_n = \frac{p - 2}{2p} u_n^2 + \frac{1}{p} R'(u_n) u_n - R(u_n) \geq \frac{p - 2}{2p} u_n^2$$

hence (4.6) implies

$$\frac{p - 2}{2p} \int |\nabla u_n|^2 \, dx + \frac{p - 2}{2p} \int |k\nabla \theta - A_n|^2 u_n^2 \, dx + \frac{1}{2} \int |\nabla A_n|^2 \, dx$$

$$+ \frac{\varepsilon}{2} \int |A_n|^2 \, dx + \frac{p - 2}{2p} \int u_n^2 \, dx \leq c_{\varepsilon,n} - \frac{1}{p} \lambda_n[u_n]$$

hence

$$\frac{p - 2}{2p} \| u_n \|^2_{H^1} + \frac{1}{2} \| \nabla A_n \|^2_2 + \frac{\varepsilon}{2} \| A_n \|^2_2 \leq c_{\varepsilon,n} + \frac{1}{p} \| \lambda_n \|(\hat{H}^1)', \| u_n \|_{H^1}. \quad (4.7)$$

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By (4.7) we deduce that \( \{\|u_n\|_{H^1}\} \) and \( \{\|A_n\|_{H^1}\} \) are bounded. In particular there exists a constant \( C > 0 \) such that \( \int R(u_n) \, dx \leq C \).

Finally we have

\[
M \geq \frac{1}{2} \|u_n\|^2_{H^1} - k \int |u_n|^2 \, dx - \int R(u_n) \, dx
\]

\[
\geq \frac{1}{2} \|u_n\|^2_{H^1} - \frac{|k|}{2} \int \left( 4|A_n|^2 + \frac{1}{4r^2} \right) u_n^2 \, dx - C
\]

\[
\geq \frac{1}{2} \|u_n\|^2_{H^1} - 2|k| \int |A_n|^2 u_n^2 \, dx - \frac{1}{8} \int u_n^2 \, dx - C
\]

\[
\geq \frac{1}{2} \|u_n\|^2_{H^1} - 2|k| \|A_n\|^2 \|u_n\|^2_{L^2} - \frac{1}{8} \|u_n\|^2_{H^1} - C
\]

\[
= \frac{3}{8} \|u_n\|^2_{H^1} - 2|k| \|A_n\|^2 \|u_n\|^2_{L^2} - \frac{1}{8} \|u_n\|^2_{H^1} - C
\]

which shows that \( \{u_n\} \) is bounded in \( H^1 \) since \( \{\|A_n\|_{H^1}\} \) and \( \{\|u_n\|_{H^1}\} \) are bounded by Sobolev embedding theorems.

The next step is to prove that any PS sequence is bounded away from zero.

**Proposition 4.4.** If \((u_n, A_n)\) is a PS sequence for \( J_\varepsilon \) at level \( c_\varepsilon > 0 \) then for some \( c > 0 \)

\[
\|u_n\|^p_p \geq c > 0.
\]

**Proof.** Let \( \{(u_n, A_n)\} \) be a bounded PS sequence satisfying (4.4) and (4.5). Since \( \{\|u_n\|_{H^1}\} \) is bounded,

\[
\| \nabla u_n \|^2_2 + \int |k\nabla \theta - A_n|^2 u_n^2 \, dx + \int W'(u_n) u_n \, dx = \lambda_n [u_n] \to 0. \tag{4.8}
\]

Hence

\[
\| \nabla u_n \|^2_2 + \int |k\nabla \theta - A_n|^2 u_n^2 \, dx + \frac{1}{2} \int u_n^2 \, dx = \lambda_n [u_n] + \int R'(u_n) u_n \, dx \leq \lambda_n [u_n] + \|u_n\|^p_p
\]

(4.9)

from which it follows

\[
\|u_n\|^2_{H^1} \leq \lambda_n [u_n] + \|u_n\|^p_p. \tag{4.10}
\]

We argue by contradiction. If \( \|u_n\|^p \to 0 \), using (4.8) and (4.10) we obtain

\[
\|u_n\|^2_{H^1} \to 0 \quad \text{and} \quad \int R(u_n) \, dx \to 0 \quad \text{(4.11)}
\]
and coming back to (4.9)
\[
\int |k \nabla \theta - A_n|^2 u_n^2 \, dx \to 0. \tag{4.12}
\]

On the other hand since \( \partial_A J_\varepsilon(u_n, A_n) \to 0 \) it holds
\[
-\Delta A_n + \varepsilon A_n - (k \nabla \theta - A_n) u_n^2 = \delta_n \to 0 \quad \text{in } ((H^1)^3)',
\]
and, since \( \{\|A_n\|_{H^1}\} \) is bounded,
\[
\|\nabla A_n\|^2_2 + \varepsilon \|A_n\|^2_2 - \int (k \nabla \theta - A_n) A_n u_n^2 \, dx = \delta_n[A_n] \to 0. \tag{4.13}
\]

Classical estimates give
\[
\left| \int (k \nabla \theta - A_n) A_n u_n^2 \, dx \right| \leq \left( \int |k \nabla \theta - A_n|^2 u_n^2 \, dx \right)^{1/2} \left( \int |A_n|^2 u_n^2 \, dx \right)^{1/2} \to 0
\]
by (4.12) and since \( \int |A_n|^2 u_n^2 \, dx \) is bounded by the Schwartz inequality. Therefore by (4.13) we get
\[
\|\nabla A_n\|^2_2 + \varepsilon \|A_n\|^2_2 \to 0. \tag{4.14}
\]

Finally, by (4.11), (4.12) and (4.14)
\[
J_\varepsilon(u_n, A_n) = \frac{1}{2} \|u_n\|^2_{H^1} + \frac{1}{2} \int |k \nabla \theta - A_n|^2 u_n^2 \, dx \\
+ \frac{1}{2} \|\nabla A_n\|^2_2 + \varepsilon \|A_n\|^2_2 - \int R(u_n) \, dx \to 0.
\]

This is a contradiction since \( J_\varepsilon(u_n, A_n) \to c_\varepsilon > 0. \)

By the previous results, for every \( \varepsilon \in (0, 1) \) there exists \( (u_{n,\varepsilon}, A_{n,\varepsilon}) \), a bounded PS sequence for \( J_\varepsilon \) at level \( c_\varepsilon \). So we can extract a weakly convergent sub-sequence, denoted again with \( (u_{n,\varepsilon}, A_{n,\varepsilon}) \), to a certain \( (u_\varepsilon, A_\varepsilon) \in V \). We know that \( u_\varepsilon \neq 0 \) (Proposition 4.4) and \( J_\varepsilon(u_\varepsilon, A_\varepsilon) \leq c_\varepsilon \) (Proposition 4.1). We have proved that
\[
\begin{align*}
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \件
Proof. Let $\varepsilon$ be fixed. Since $(u_{n,\varepsilon}, A_{n,\varepsilon})$ is a PS sequence for $J_\varepsilon$ we have

$$-\Delta u_{n,\varepsilon} + |k\nabla \theta - A_{n,\varepsilon}|^2 u_{n,\varepsilon} + W'(u_{n,\varepsilon}) = \lambda_{n,\varepsilon} \to 0 \quad \text{in} \quad (\dot{H}^1_\varepsilon)'$$

which evaluated on $v \in \dot{H}^1_\varepsilon$ gives

$$\int \nabla u_{n,\varepsilon} \nabla v \, dx + k^2 \int \frac{u_{n,\varepsilon} v}{r^2} \, dx + \int u_{n,\varepsilon} v \, dx$$

$$- 2k \int \nabla \theta A_{n,\varepsilon} v \, dx - \int R(u_{n,\varepsilon}) v \, dx = \lambda_{n,\varepsilon} [v].$$

Applying the same arguments of the proof of Proposition 4.1, letting $n \to \infty$ we find that $(u_{\varepsilon}, A_{\varepsilon})$ is a solution of the first equation in $(P_\varepsilon)$, i.e. satisfies (3.4) with $v \in \dot{H}^1_\varepsilon$. Analogously

$$-\Delta A_{n,\varepsilon} + \varepsilon A_{n,\varepsilon} - (k\nabla \theta - A_{n,\varepsilon}) u_{n,\varepsilon}^2 = \delta_{n,\varepsilon} \to 0 \quad \text{in} \quad ((H^1)^3)'$$

which evaluated on $V \in \mathcal{A}$ and passing to the limit in $n$ gives

$$\int \nabla A_{\varepsilon} \cdot \nabla V \, dx + \varepsilon \int A_{\varepsilon} V \, dx = \int (k\nabla \theta - A_{\varepsilon}) Vu_{\varepsilon}^2 \, dx$$

so that $(u_{\varepsilon}, A_{\varepsilon})$ solves (3.5) with $V \in \mathcal{A}$.

Remark 4.6. By Theorem 3.4 $(u_{\varepsilon}, A_{\varepsilon})$ satisfies also (2.1) and (2.2).

4.2 ...and now $\varepsilon \to 0$

In this section all the limits are taken for $\varepsilon$ which tends to $0^+$.

As we have seen, for any $\varepsilon \in (0, 1)$, $u_{\varepsilon} \neq 0$. Actually we have the following

Lemma 4.7. There exists a positive constant, $C$ such that for every $\varepsilon \in (0, 1)$

$$0 < C \leq \|u_{\varepsilon}\|_{H^1}.$$  

Proof. Since every $(u_{\varepsilon}, A_{\varepsilon})$ satisfies $[P_\varepsilon]$, we have

$$\|u_{\varepsilon}\|_{H^1}^2 + \int |k\nabla \theta - A_{\varepsilon}|^2 u_{\varepsilon}^2 \, dx - \int R'(u_{\varepsilon}) u_{\varepsilon} \, dx = 0$$

hence

$$\|u_{\varepsilon}\|_{H^1}^2 \leq \int R'(u_{\varepsilon}) u_{\varepsilon} \, dx \leq c \|u_{\varepsilon}\|_{H^1}^p,$$

which shows that $\{u_{\varepsilon}\}$ is bounded away from zero. \qed
We also need to know that the sequence \( \{u_\varepsilon\} \) is bounded in \( \hat{H}_1^r \). This is stated in the next Lemma. We first give some preliminary remarks.

Recalling the definition of the mountain pass level

\[
c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J_\varepsilon(\gamma(t)),
\]

where \( \Gamma = \{ \gamma \in C([0,1], V) : \gamma(0) = (0,0), \gamma(1) = (\bar{u},0) \} \) and \( J_\varepsilon(\bar{u},0) \leq 0 \) (see Proposition 4.2), consider the path

\[
\gamma_0 : t \in [0,1] \mapsto (t\bar{u},0) \in V.
\]

Then

\[
c_\varepsilon \leq \max_{0 \leq t \leq 1} J_\varepsilon(\gamma_0(t)) = \max_{0 \leq t \leq 1} J_0(t\bar{u},0)
\]

which says that \( \{c_\varepsilon\} \) is bounded by some positive constant \( K \) which does not depend on \( \varepsilon \).

Also we know that

\[
\partial_u J(u_\varepsilon, A_\varepsilon) = 0,
\]

\[
J(u_\varepsilon, A_\varepsilon) \leq c_\varepsilon.
\]

Again evaluating \( J(u_\varepsilon, A_\varepsilon) - \frac{1}{p} \partial_u J(u_\varepsilon, A_\varepsilon)[u_\varepsilon] \leq c_\varepsilon \) we find

\[
\frac{p-2}{2p} \|\nabla u_\varepsilon\|_2^2 + \frac{p-2}{2p} \int |k\nabla \theta - A_\varepsilon|^2 u_\varepsilon^2 \, dx + \frac{1}{2} \|\nabla A_\varepsilon\|^2_2
\]

\[
+ \frac{\varepsilon}{2} \int |A_\varepsilon|^2 \, dx + \int \left( W(u_\varepsilon) - \frac{1}{p} W'(u_\varepsilon) u_\varepsilon \right) \, dx \leq c_\varepsilon \leq K.
\]

Since by the assumptions

\[
W(u_\varepsilon) - \frac{1}{p} W'(u_\varepsilon) u_\varepsilon \geq \frac{p-2}{2p} u_\varepsilon^2
\]

we find

\[
\frac{p-2}{2p} \left( \|u_\varepsilon\|^2_{H_1^r} + \int |k\nabla \theta - A_\varepsilon|^2 u_\varepsilon^2 \, dx \right) + \frac{1}{2} \|\nabla A_\varepsilon\|^2_2 + \frac{\varepsilon}{2} \int |A_\varepsilon|^2 \, dx \leq K.
\]

so that \( \{u_\varepsilon\} \) is bounded in \( H_1^r \).

Moreover by \( (4.15) \) other information can be deduced.

**Lemma 4.8.** The following facts hold:

1. \( \{u_\varepsilon\} \) is bounded in \( \hat{H}_1^r \),
2. \( \{A_\varepsilon\} \) is bounded in \( H_1^{loc} \),

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3. \( \lim_{\varepsilon \to 0} \int A_\varepsilon \mathbf{V} \, dx = 0 \) for any \( \mathbf{V} \in (L^2)^3 \).

**Proof.** 1. Since we have already proved that \( \{ u_\varepsilon \} \) is bounded in \( H^1_r \), it remains to prove the boundedness of \( \int \frac{u_\varepsilon^2}{r^2} \, dx \). We write
\[
\int \frac{u_\varepsilon^2}{r^2} \, dx = \int_0^1 \frac{u_\varepsilon^2}{r} \, dr + \int_1^{+\infty} \frac{u_\varepsilon^2}{r} \, dr
\]
and both integrals in the right hand side are uniformly bounded in \( \varepsilon \), indeed
\[
\int_1^{+\infty} \frac{u_\varepsilon^2}{r} \, dr \leq \int_1^{+\infty} ru_\varepsilon^2 \, dr \leq \int u_\varepsilon^2 \, dx \leq \| u_\varepsilon \|_{H^1_r}^2 \leq K,
\]
and by (4.15)
\[
\frac{2p}{p-2} K \geq \int |k \nabla \theta - A_\varepsilon|^2 u_\varepsilon^2 \, dx = \int |k - b_\varepsilon|^2 \frac{u_\varepsilon^2}{r^2} \, dx
\]
\[
\geq \int_0^1 |k - b_\varepsilon|^2 \frac{u_\varepsilon^2}{r} \, dr \geq c \int_0^1 \frac{u_\varepsilon^2}{r} \, dr
\]
where the constant \( c \) can be chosen independently on \( \varepsilon \) since \( k \neq 0 \) and \( b_\varepsilon(0) = 0 \).

2. We have only to show that for any \( \rho > 0 \)
\[
\int_{B_\rho} |A_\varepsilon|^2 \, dx \quad \text{is bounded independently on } \varepsilon
\]
where \( B_\rho \) is the ball in \( \mathbb{R}^2 \) centered in 0 and with radius \( \rho \).

We have
\[
K \geq \int |\nabla A_\varepsilon|^2 \, dx = \int |\nabla \times A_\varepsilon|^2 \, dx
\]
\[
= \int |\nabla \times (b_\varepsilon \nabla \theta)|^2 \, dx = \int \frac{(b'_\varepsilon)^2}{r^2} \, dx
\]
\[
= \int_0^{+\infty} \frac{(b'_\varepsilon)^2}{r} \, dr. \quad (4.16)
\]
Let us fix \( \rho > 0 \). For \( r \leq \rho \), by the elementary inequality \( b_\varepsilon(r) = \int_0^r b'_\varepsilon(t) \, dt \leq \sqrt{r} (\int_0^r (b'_\varepsilon(t))^2 \, dt)^{1/2} \) and (4.16) we find
\[
(b_\varepsilon(r))^2 \leq r \int_0^r (b'_\varepsilon(t))^2 \, dt
\]
\[
\leq r^2 \int_0^r \frac{(b'_\varepsilon(t))^2}{t} \, dt
\]
\[
\leq r^2 K. \quad (4.17)
\]
Now we can evaluate $\int_{B_\rho} |A_\varepsilon|^2\,dx$. It results
\[
\int_{B_\rho} |A_\varepsilon|^2\,dx = \int_0^\rho \frac{(b_\varepsilon)^2}{r}\,dr \leq K \int_0^\rho \,dr = \frac{1}{2} \rho^2 K
\]
in virtue of (4.17).

3. By (4.15) we deduce that $\{\sqrt{\varepsilon}A_\varepsilon\}_{\varepsilon \in (0,1)}$ is bounded in $L^2$, so up to a sub-sequence, it weakly converges to a certain $X$ in $(L^2)^3$, that is
\[
\int \sqrt{\varepsilon}A_\varepsilon V\,dx \to \int X V\,dx \quad \forall V \in (L^2)^3
\]
and hence the conclusion follows. $\square$

As a consequence of Lemma 4.8 we infer that there exists $(u_0, A_0) \in \hat{H}_r^1 \times A = V$, such that as $\varepsilon \to 0$
\[
u = u_0 \quad \text{in} \quad \hat{H}_r^1, \quad (4.18)
\]
\[
A = A_0 \quad \text{in} \quad H_{loc}^1, \quad (4.19)
\]
and so by Theorem 3.2 and usual Sobolev embedding theorems
\[
\begin{align*}
\|u_\varepsilon\|_{L^p} &\to \|u_0\|_{L^p} \quad \text{for} \quad 2 < p < +\infty, \quad (4.20) \\
\|A_\varepsilon\|_{L^p_{loc}} &\to \|A_0\|_{L^p_{loc}} \quad \text{for} \quad 1 \leq p < +\infty, \quad (4.21) \\
A_\varepsilon &\to A_0 \quad \text{a.e. in} \quad \mathbb{R}^2. \quad (4.22)
\end{align*}
\]

The proof of Theorem 1.1 is finished once we prove the following

**Proposition 4.9.** The couple $(u_0, A_0)$ is a solution of (1.11) and (1.12) in the sense of distributions.

**Proof.** By Proposition 4.5, $(u_\varepsilon, A_\varepsilon)$ are weak solutions of
\[
-\Delta u_\varepsilon + |k\nabla \theta - A_\varepsilon|^2 u_\varepsilon + W'(u_\varepsilon) = 0,
\]
\[
-\Delta A_\varepsilon + \varepsilon A_\varepsilon = (k\nabla \theta - A_\varepsilon)u_\varepsilon^2,
\]
i.e. satisfy (3.4) and (3.5). Moreover by Theorem 3.4 they satisfies (3.4) and (3.5) also with $v \in \hat{H}^1$ and $V \in (H^1)^3$. In particular, for $v \in \mathcal{D} \left( \mathbb{R}^2 \setminus \{0\} \right)$, (3.4) reads
\[
\int \nabla u_\varepsilon \nabla v\,dx + k^2 \int \frac{u_\varepsilon v}{r^2}\,dx + \int u_\varepsilon v\,dx + \int A_\varepsilon^2 u_\varepsilon v\,dx \\
- 2k \int \nabla \theta A_\varepsilon u_\varepsilon v\,dx - \int R'(u_\varepsilon)v\,dx = 0.
\]
By (4.18) we have
\[ \int \nabla u_{\varepsilon} \nabla v \, dx + k^2 \int \frac{u_{\varepsilon}^2}{r^2} \, dx + \int u_{\varepsilon} v \, dx \to \int \nabla u_0 \nabla v \, dx + k^2 \int \frac{u_0^2}{r^2} \, dx + \int u_0 v \, dx \]  
(4.23)
and it is clear that
\[ \int R'(u_{\varepsilon}) v \, dx \to \int R'(u_0) v \, dx. \]  
(4.24)
Moreover, if \( B \) is a ball containing the support of \( v \) we have
\[ \left| \int |A_{\varepsilon}|^2 u_{\varepsilon} v \, dx - \int |A_0|^2 u_0 v \, dx \right| \leq \int_B |v||A_{\varepsilon}|^2 |u_{\varepsilon} - u| \, dx + \int_B |v||A_{\varepsilon}|^2 - |A_0|^2 |u| \, dx \]
and by (4.20), (4.21) and (4.22)
\[ \int_B |v||A_{\varepsilon}|^2 |u_{\varepsilon} - u| \, dx \leq c |A_{\varepsilon}|^2 \|L^3(B)\| |u_{\varepsilon} - u|_3 \to 0 \]
\[ \int_B |v||A_{\varepsilon}|^2 - |A_0|^2 |u| \, dx \leq c \||A_{\varepsilon}|^2 - |A_0|^2\|_L^2(B) \|u\|_2 \to 0. \]
This shows that
\[ \int |A_{\varepsilon}|^2 u_{\varepsilon} v \, dx \to \int |A_0|^2 u_0 v \, dx. \]  
(4.25)
Similarly,
\[ \left| \int \nabla \theta A_{\varepsilon} u_{\varepsilon} v \, dx - \int \nabla \theta A_0 u_0 v \, dx \right| \leq \int_B |\nabla \theta A_{\varepsilon} u_{\varepsilon} v - \nabla \theta A_0 u_0 v| \, dx + \int_B |\nabla \theta A_{\varepsilon} u_0 v - \nabla \theta A_0 u_0 v| \, dx \]
and it holds
\[ \int_B \frac{|v|}{r} |A_{\varepsilon}| |u_{\varepsilon} - u_0| \, dx \leq c \|A_{\varepsilon}\|_{L^{3/2}(B)} \|u_{\varepsilon} - u_0\|_{L^3(B)} \to 0, \]
\[ \int_B \frac{|v|}{r} |u_0| |A_{\varepsilon} - A_0| \, dx \leq c \|u_0\|_2 \|A_{\varepsilon} - A_0\|_{L^2(B)} \to 0 \]
(notice that the function \( v/r \) still belongs to \( D (\mathbb{R}^2 \setminus \{0\}) \)). In other words
\[ \int \nabla \theta A_{\varepsilon} u_{\varepsilon} v \, dx \to \int \nabla \theta A_0 u_0 v \, dx. \]  
(4.26)
Putting together (4.23), (4.24), (4.25) and (4.26) we infer that for any \( v \in D(R^2 \setminus \{0\}) \)

\[
\int \nabla u_0 \nabla v \, dx + \int |k \nabla \theta - A_0|^2 u_0 v \, dx + \int W(u_0) v \, dx = 0. \tag{4.27}
\]

To conclude that \((u_0, A_0)\) is a solution of (1.11) in the sense of distributions we need to show that (4.27) is still true for \( v \in D \). This is done by following an argument of [6] to which the reader is referred, here we sketch the main steps.

**Step 1** First, one defines a family of smooth and radial functions on \( R^2 \)

- \( \chi_n(r) = 1 \) for \( r \geq 2/n \),
- \( \chi_n(r) = 0 \) for \( r \leq 1/n \),
- \( |\nabla \chi_n(r)| \leq 1 \),
- \( |\nabla \chi_n(r)| \leq 2n \),
- \( \chi_{n+1}(r) \geq \chi_n(r) \).

It is not difficult to prove that if \( \varphi \in H^1 \cap L^\infty \) has bounded support

\[
\varphi_n := \varphi \chi_n \rightharpoonup \varphi \quad \text{in} \quad H^1.
\tag{4.28}
\]

Thank to these cut-off functions can be proved that \((u_0, A_0)\) is a solution of (2.1) in the sense of distributions.

**Step 2** Now take \( v \in D \), and choose \( \varphi_n = v^+ \chi_n \in \dot{H}^1 \) as test functions in (2.1). Observe that there exists a ball \( B \) such that all the functions \( \varphi_n \) have support in \( B \). Then the proof of Theorem 8 of [6] can be adapted here. Hence, taking the limit in \( n \) and making use of (4.28) (that in this case means \( \varphi_n \rightharpoonup v^+ \) in \( H^1 \)) we find that (2.1) is satisfied with \( v^+ \) as test functions. Since the same is true for \( v^- \), this yields that \((u_0, A_0)\) solves in the sense of distributions (2.1), or equivalently (1.11).

We now prove that \((u_0, A_0)\) is a solution of (1.12) in the sense of distributions. Certainly \((u_0, A_0)\) satisfies also (2.2) with \( V \in (D)^3 \). We have to prove that \((u_0, A_0)\) solves equation (1.12) in the sense of distributions, or equivalently, since we are in the natural constraint, the equation

\[
-\Delta A = (k \nabla \theta - A) u^2 \tag{4.29}
\]

in the sense of distributions.
Therefore take $V \in (D)^3$ and let $B$ be a ball containing the support of $V$. We know that
\[ \int \nabla A_\varepsilon \cdot \nabla V \, dx + \varepsilon \int A_\varepsilon V \, dx = \int (k\nabla \theta - A_\varepsilon) V u_\varepsilon^2 \, dx \]
and we want to pass to the limit for $\varepsilon \to 0$.

We have
\[ \left| \int (k\nabla \theta - A_\varepsilon) V u_\varepsilon^2 \, dx - \int (k\nabla \theta - A_0) V u_0^2 \, dx \right| \leq \int_B u_\varepsilon^2 |V||A_0 - A_\varepsilon| \, dx + \int_B |V||k\nabla \theta - A_0||u_\varepsilon^2 - u_0^2| \, dx. \]

Now, again using (4.20) and (4.21)
\[ \int_B u_\varepsilon^2 |V||A_0 - A_\varepsilon| \, dx \leq \max|V| \|u_\varepsilon\|^2_4 \|A_0 - A_\varepsilon\|_{L^2(B)} \to 0 \]
\[ \int_B |V||k\nabla \theta - A_0||u_\varepsilon^2 - u_0^2| \, dx \leq \max|V| \|k\nabla \theta - A_0\|_{L^2(B)} \|u_\varepsilon^2 - u_0^2\|_3 \to 0 \]
so that
\[ \int (k\nabla \theta - A_\varepsilon) V u_\varepsilon^2 \, dx \to \int (k\nabla \theta - A_0) V u_0^2 \, dx. \] (4.30)

Moreover by (4.15) there exists $B \in L^2$ such that $\nabla A_\varepsilon \rightharpoonup B$ in $L^2$ and therefore the convergence is in the sense of distributions, that is
\[ \nabla A_\varepsilon \to B \quad \text{in} \quad D'. \]

On the other hand (4.19) implies $A_\varepsilon \to A$ in $D'$ and then
\[ \nabla A_\varepsilon \to \nabla A_0 \quad \text{in} \quad D' \]
so necessarily $\nabla A_0 = B \in L^2$. Finally, by virtue of (3) of Lemma 4.8
\[ \int \nabla A_\varepsilon \cdot \nabla V \, dx + \varepsilon \int A_\varepsilon V \, dx \to \int \nabla A_0 \cdot \nabla V \, dx. \] (4.31)

By (4.30) and (4.31) equation (4.29) is satisfied in the sense of distributions. Hence $(u_0, A_0)$ satisfies (1.12) in the sense of distributions. \qed

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