Toward a Unit Distance Embedding for the Heawood graph

Mitchell A. Harris
Harvard Medical School
Department of Radiology
Boston, MA 02114, USA
harris.mitchell@mgh.harvard.edu

Abstract. The unit distance embeddability of a graph, like planarity, involves a mix of constraints that are combinatorial and geometric. We construct a unit distance embedding for $H - e$ in the hope that it will lead to an embedding for $H$. We then investigate analytical methods for a general decision procedure for testing unit distance embeddability.

1 Introduction

Unit distance embedding of a graph is an assignment of coordinates in the plane to the vertices of a graph such that if there is an edge between two vertices, then their respective coordinates are exactly distance 1 apart. To bar trivial embeddings, such as for bipartite graphs having all nodes in one part located at $(0,0)$, for the other part at $(0,1)$, it is also required that the embedded points be distinct. There is no restriction on edge crossing. A graph is said to be unit distance embeddable if there exists such an embedding (with the obvious abbreviations employed, such as UD embeddable, a UDG, etc).

For some well-known examples, $K_4$ is not UDG but $K_4 - e$ is. The Moser spindle is UDG, and is also 4 colorable [5], giving the largest currently known lower bound to the Erdős colorability of the plane [3]. The graph $K_{2,3}$ is not because from two given points in the plane there are exactly two points of distance 1, but the graph wants three; removing any edge allows a UD embedding.

The property UD is hereditary so by the Graph Minor theorem, the property has a finite number of forbidden minors, and by the previous paragraph, $K_4$ and $K_{2,3}$ are two of them.

The Petersen graph $P$ is a UDG, as well as $K_2^n$. 
A UDG is *rigid* if there are only a finite number of UD embeddings, that is, one embedding cannot be transformed continuously to any other.

Now consider the Heawood graph $H$, also known as the point-line graph of the Fano plane, the (3,6) cage, or the smallest cubic graph of girth 6, and specified by LCF notation as $(5, -5)^7$ or the difference set $\{1, 2, 4\}$ mod 14. It has the following non-UD embedding:

2 The Construction

Consider first the Heawood graph with two adjacent vertices removed. We will show that this graph is UDE and then add back in the two vertices (but not the mutual incident edge). The graph $H - \{1, a\}$ ((a) in the figure) is isomorphic to the Möbius ladder $M_4$ with a vertex inserted on each 'rung' (b):
The difficulty with UD embedding this modified ladder is that opposing vertices are mutually more than distance 2 apart. So we transform a smaller graph and then build back up. First, it is easy to UD embed the nine points that make up the 8-cycle and one rung on a square. 'Folding' over the rung puts the opposite corners closer together, and perturbing $f$, $d$, and $4$ a little preserves distinctness (a). The rest of the rungs can then be added since the end vertices are now all within distance 2 (b). The last two points of $H$ can now be added in (c) giving a UDE of $H$–.

The cycle and one rung, plus the other three rungs, plus the last two points

Though this is essentially a proof by picture, necessitating all the usual Euclidean caveats about inferring from idiosyncrasies of the specific diagram, it still follow. Each of the transformations can be seen to preserve or enforce unit distance and preserve distinctness. When a vertex and two edges are added, the UD condition can be satisfied in exactly two ways. The only real choice made here is for vertices 2, 6, $a$ and 1, and we make those choices (exercise for the reader) such that things work.

In order to get a UDE of the full graph $H$, the last item to take care of is the edge between 1 and $a$. Given that the suggested embedding is highly constrained by the strict placement of the six initial vertices, and the freedoms
in the perturbations and binary choices for the rest, can things be modified slightly enough so that 1 and a are a unit apart and vertex embeddings are kept distinct?

If one could show that there is a configuration where $\overline{1a} < 1$, a configuration another where $\overline{1a} > 1$, and a continuous transformation between the two, then we’d have a proof of the existence of a unit embedding. This is not exactly a constructive embedding but a proof nonetheless, from which a numerically accurate embedding can be approximated.

### 3 Analytic and Automatic Solutions

For some graphs there are obvious ‘by-hand’ proofs or disproofs of embeddability or the lack thereof. But we also seek a general algorithm to determine UDE.

An unit distance embedding graph can be modeled by a set of multinomial equations that express the coordinates of the vertices of an edge in a distance constraint. For example, if $x_a, y_a$ and $x_b, y_b$ are the coordinates of an edge between $a$ and $b$, then by the Euclidean distance formula:

\[(x_a - x_b)^2 + (y_a - y_b)^2 = 1,\]

and each edge of a graph produces another such equality constraint.

For a set of non-linear multinomial equations, there is a decision procedure that, though it doesn’t necessarily produce closed-form coordinates, it will give a yes-no answer to whether the set of constraints has a solution. Gröbner basis completion takes a set of multinomials and reduces it to a ‘minimal’ set, such that the minimal set has the multinomial ‘1’ as its sole member if and only if there is no solution. If this minimal set, called the Gröbner basis, is not ‘1’, then it is a set of multinomials, from which one attempt to numerically extract coordinates (variations on multivariate Newton-Raphson with all their attendant problems of convergence), or using other methods, attempt to symbolically extract coordinates (polynomial factoring, root extraction, etc).

For example, $K_4 - e$, with edges (1, 2), (1, 3), (2, 3), (2, 4), (3, 4), has the system:

\[
\begin{align*}
(x_1 - x_2)^2 + (y_1 - y_2)^2 &= 1 \\
(x_1 - x_3)^2 + (y_1 - y_3)^2 &= 1 \\
(x_2 - x_3)^2 + (y_2 - y_3)^2 &= 1 \\
(x_2 - x_4)^2 + (y_2 - y_4)^2 &= 1 \\
(x_3 - x_4)^2 + (y_3 - y_4)^2 &= 1
\end{align*}
\]

To oversimplify, the completion algorithm will do a generalization of Gaussian elimination, eliminating largest common terms between two (expanded) multinomial equations. For the above system, setting $(x_1, y_1)$ to $(0, 0)$ and $(x_2, y_2)$ to $(1, 0)$ to reduce some processing, we get the following reduced system:
\[4y_4^3 = 3y_4\]
\[x_4 = 2y_4^2\]
\[y_3y_4 = y_4^2\]
\[4y_3^2 = 3\]
\[2x_3 = 1\]
\[y_2 = 0\]
\[x_2 = 1\]
\[y_1 = 0\]
\[x_1 = 0\]

where, in this instance, it is easy to extract the values of the coordinates by back-substitution to get:

\[(x_1, y_1) = (0, 0)\]
\[(x_2, y_2) = (1, 0)\]
\[(x_3, y_3) = (\frac{1}{2}, \pm \frac{\sqrt{3}}{2})\]
\[(x_4, y_4) = (0, 0) \text{ or } (\frac{3}{2}, \pm \frac{\sqrt{3}}{2})\]

This answer has to be checked for duplicate vertex embeddings, but when \((x_4, y_4) = (0, 0)\) is removed, there is still a legal embedding left.

For the example of \(K_{2,3}\), where there are many (continuous) solutions to the system. Computing by mechanically using a symbolic algebra package, we get:

\[x_5^2 + y_5^2 = 1\]
\[x_3^2 + y_3^2 = 1\]
\[x_2x_5 + y_2y_5 = x_2\]
\[x_2x_4 + y_2y_4 = x_2\]
\[x_2^2 + y_2^2 = 2x_2\]
\[y_2y_5 = y_2y_5^2\]
\[y_2y_5 + x_5y_2y_5 = x_2y_5^2\]
\[x_2y_2y_5 = x_2y_5^2\]
\[y_2^2y_4 = y_2y_4^2\]
\[x_5y_2y_4 + y_2y_5 = x_4y_2y_5 + y_2y_4\]
\[y_2y_4 + x_4y_2y_4 = x_2y_4^2\]
\[x_2y_2y_4 = x_2y_4^2\]
\[x_5y_2^2 + 2y_2y_5 = x_2y_5^2 + y_2^2\]
\[ x_4 y_2^2 + 2 y_2 y_4 = x_2 y_4^2 + y_2^2 \]
\[ 2 y_2 y_4 y_5 + x_4 y_2 y_5^2 = y_2 y_5^2 + x_2 y_4 y_5^2 \]
\[ y_2^2 y_5^2 = y_2 y_4 y_5^2 \]
\[ x_2 y_4 y_5^2 = x_2 y_4 y_5^2, \]

which is a formidable system to digest by hand, especially when you realize that it really does boil down to the three conceptual cases of 1 and 2 embedded at the same point (the other three placed freely distance 1 around them), or 3,4,5 at the same point (with 1 and 2 free), or ***. In any case, checking for duplicates, once a reduced Gröbner basis is computed, is still a nontrivial task.

### 4 Comments

Chvátal et al. [2, problem 21] posed the question in terms of bounds on the number of vertices in a UDG, noting the lack of an extant embedding for \( H \) (in terms of projective planes). Hochberg [1] describes an algorithm for showing the impossibility of an embedding for a given graph, unfortunately by experience not tractable on \( H \). Gerbracht [4] found an analytic embedding for the Harborth graph, the smallest known non-crossing UDG or matchstick graph. He found a polynomial in one variable that determines the finite set of possibilities for one point of the embedding, from which all the rest are determined.

### References

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