Topologically massive gauge theories from first order theories in arbitrary dimensions.

M. Botta Cantcheff
Centro Brasileiro de Pesquisas Fisicas (CBPF)
Departamento de Teoria de Campos e Particulas (DCP)
Rua Dr. Xavier Sigaud, 150 - Urca
22290-180 - Rio de Janeiro - RJ - Brasil.

Abstract

We thereby prove that a large class of topologically massive theories of the Cremmer-Scherk-Kalb-Ramond-type in any $d$ dimensions corresponds to gauge non-invariant first-order theories that can be interpreted as self-dual models.

The apparent clash between gauge symmetry and massive gauge bosons is avoided in the framework of topologically massive gauge theories, as it is the case for the well-known Maxwell-Chern-Simons and Cremmer-Scherk-Kalb-Ramond models (CSKR). They illustrate how Abelian gauge bosons may be attributed a physical mass without the need of bringing about Higgs scalars and spontaneous symmetry breaking. This a fundamental motivation to study this type of theories in different space-time dimensions.

This paper has a two-fold purpose: to construct first-order formulations of topologically massive theories which involve BF-terms (topological coupling between different gauge forms) in arbitrary dimensions and for all possible tensorial ranks; and afterwards, to argue that, by considering doublets of field-forms, these first order (gauge non-invariant) formulations constitute self-dual models, close in spirit to the Self-Dual system in (2 + 1)-dimensions first introduced by Townsend, Pilch and van Nieuwenhuizen. There are some recent works pointing out that Cremmer-Scherk-Kalb-Ramond models in dimension four, which include in their Lagrangian BF-terms are dual equivalent to first order ones. These authors employed the Hamiltonian embedding procedure by Batalin, Fradkin and Tyutin. Dualization of these models has also been studied by Smailagic and Spallucci, coming to results different from those found in this letter.

The parallel between these first order BF-theories, at any space-time dimension, and the Self-Dual (SD) theories in (2 + 1), exploited in this work, has recently been pointed out by Harikumar et al in the case 4-dimensional case; however, they mention a difficulty in establishing this connection as due to the the impossibility of defining self-duality in dimensions that are not of the form $d = 4k - 1$ ($k \in \mathbb{Z}_+$). Here, this objection is by-passed from the very starting point, by defining the dual operation on the space of pairs of gauge forms.

In the present approach, we proceed further and use this parallelism to define a SD model in arbitrary space-time dimension, and adapt the proof proposed by Deser and Jackiw in 2 + 1-d to manifestly show the dual correspondence between generic topologically massive models (CSKR) and the already mentioned SD theories in d-dimensions.

Finally, we shall confirm the result recently presented in ref. in four dimensions, as a particular case and generalize it to all dimensionalities.

First, let us briefly describe the well-known MCS-SD duality in (2 + 1)-dimensions. One currently

---

1e-mail: botta@cbpf.br
defines the duality operation by

\[ \star f_\mu = \frac{\chi}{\mu} \epsilon_{\mu\nu\lambda} \partial^\nu f^\lambda, \]  

(1)

where \( \mu \) is a mass parameter here introduced to render the \( \star \)-operation dimensionless. This is basically a functional curl (rotational operator).

We name self(anti-self)-duality, when the relations \( \star f = \pm f \) are (respectively) satisfied.

The so-called Self-Dual Model (Townsend, Pilch and van Nieuwenhuizen \[7\]) is given by the following action,

\[ S(f) = \int d^3x \left( \frac{\chi}{2\mu} \epsilon_{\mu\nu\lambda} f^\mu \partial^\nu f^\lambda - \frac{1}{2} f^\mu f^\mu \right). \]  

(2)

The equation of motion is the self-duality relation:

\[ f_\mu = \frac{\chi}{\mu} \epsilon_{\mu\nu\lambda} \partial^\nu f^\lambda. \]  

(3)

This model is claimed to be chiral, and the chiralities \( \chi = \pm 1 \) result defined precisely from this self-duality.

On the other hand, the gauge-invariant combination of a Chern-Simons term with a Maxwell action,

\[ S_{MCS}[A] = \int d^3x \left( \frac{1}{4\mu^2} F_{\mu\nu} F_{\mu\nu} - \frac{\chi}{2\mu} \epsilon_{\mu\nu\lambda} A_\mu \partial^\nu A^\lambda \right), \]  

(4)

is the topologically massive theory, which is known to be equivalent \[13\] to the self-dual model (2). \( F_{\mu\nu} \) is the usual Maxwell field strength,

\[ F_{\mu\nu}[A] \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = 2\partial_{[\mu}A_{\nu]}. \]  

(5)

This equivalence has been verified with the Parent Action Approach \[14\]. We write down the general parent action proposed by Deser and Jackiw in \[13\], which proves this equivalence:

\[ S_{Pare\text{nt}}[A, f] = \chi S_{CS}[A] - \int d^3x [\epsilon_{\mu\nu\lambda} F_{\nu\lambda}[A] f_\mu + \mu f_\mu f^\mu], \]  

(6)

where

\[ S_{CS}[A] \equiv \int d^3x \epsilon_{\mu\nu\lambda} (A_\mu \partial_\nu A_\lambda), \]  

(7)

is the Chern-Simons action \[3\].

For general dimensions, it is possible to define self (and anti-self)-duality for pairs (doublets) of form-fields with different ranks \[6\]; so, a parallel of this structure with the one in \( d \) dimensions will be observed.

The problem of defining the Hodge duality for all dimensions is well-known; for instance, in Lorentzian four-dimensional spacetime, the main obstruction to self-duality comes from the relation of double-dualization \[3\] for a rank-two tensor:

\[ **F = (-1)^q F, \]  

(9)

\(^2\text{For a generic } q\text{-form, } A, \text{ the Hodge dual is defined by}
\[ (**A)_{\mu_1 \cdots \mu_q} \equiv \frac{1}{q!} \epsilon^{\mu_1 \cdots \mu_q \nu_1 \cdots \nu_d} A_{\nu_1 \cdots \nu_d}. \]  

(8)
where \( s \) is the signature of space time \( [3] \). For the case of the Lorentzian metric, where \( s \) is an odd number, the self-duality concept seems inconsistent with the double dualization operation due to the minus sign in \( (9) \). This problem remains for dimensionality \( d = 4m \) (\( m \in \mathbb{Z}_+ \) \( [15] \)); in contrast, it is absent for \( d = 4m - 2 \). Thus, self-duality is claimed to be well-defined (only) in such a dimensionality. First, let us recall that \( (9) \) has led to the prejudice that the (Abelian) Maxwell theory would not possess manifest self-duality solutions.

The resolution of this obstruction came with the recognition of an internal two-dimensional structure hidden in the space of fields. Transformations in this internal duality space extends the self-duality concept to this case and is currently known under the names of Schwarz and Sen \( [16] \), but this deep unifying concept has also been appreciated by others \( [17] \). The actions worked out correspond to self-dual and anti-self-dual representation of a given theory and make use of the internal space concept. The duality operation is now defined to include the internal (two dimensional) index \((i, j)\) in the fashion

\[
\hat{F}^i = e^{ij} * F^j
\]

where the \( 2 \times 2 \) matrix, \( e \), depends on the signature and dimension of the spacetime in the form:

\[
e^{\alpha\beta} = \begin{cases} 
\sigma^{\alpha\beta}, & \text{if } d = 4m - 2 \\
\epsilon^{\alpha\beta}, & \text{if } d = 4m ,
\end{cases}
\]

with \( \sigma^{\alpha\beta} \) being the first of the Pauli matrices and \( \epsilon^{\alpha\beta} \) is the totally antisymmetric \( 2 \times 2 \) matrix with \( \epsilon^{1,2} = 1 \).

The double dualisation operation,

\[
(\hat{F})^i = F^i
\]

generalizes \( (9) \) to allow consistency with self-duality. It has been shown that this prescription works in the construction of self-dual Maxwell actions \( [18] \).

This structure has always been considered in the literature only for tensorial objects where the field has the same tensorial rank that its corresponding dual. However, we may generalize further these ideas, introducing more general doublets \( [6] \).

Let a \( d \)-dimensional space-time with signature \( s \), and a generic element \( \Phi \equiv (a, b) \) in the space \( \Lambda_p \times \Lambda_{d-p} \). i.e, \( a \) , \( b \) are either a \( p \)-form and a \( (d-p) \)-form respectively. Thus, one may define a Hodge-type operation for these objects \( [9] \) by means of

\[
*\Phi \equiv (b, S_p * a),
\]

where \( S_q \) is a number defined by the double dualisation operation, for a generic \( q \)-form \( A \): \( * (\star A) = S_q A \). This depends on the signature \( s \) and dimension of the spacetime in the form \( S_q = (-1)^{(s+q)(d-q)} \).

Notice that \( * \) applied to doublets is defined such that its components are interchanged with a supplementary change of sign for the second component.

Notice that this Hodge-type self (anti-self)-duality is well-defined, since

\[
*\Phi = \pm \Phi,
\]

is consistent with the double dualization requirement, \( * (\star \Phi) = \Phi \).

For our purpose in this paper, we are more interested in proposing and working with another type of dual-operation of a similar nature to the duality we describe above for the case of \( 2 + 1 \)-dimensions.

\(^3\)i.e, this is the number of minuses occurring in the metric.

\(^4\)Which clearly includes the case \( p = d/2 \) described above.
Let a $d$-dimensional space-time with signature $s$: we consider the tensor doublet,

$$\mathcal{F} := (f_{\mu_1\cdots\mu_p}, g_{\mu_1\cdots\mu_{d-p-1}}),$$

where $f$ is a $p(<d)$-form (a totally antisymmetric tensor type $(0;p)$), and $g$ is a $(d-p-1)$-form. $\mathcal{F}$ is an element of the space $\Delta_p \equiv \Lambda_p \times \Lambda_{d-[p+1]}$.

There is also a well defined notion of self (and anti-self)-duality for the objects in this space. Consider the doublet of gauge fields $A \equiv (a_{\mu_1\cdots\mu_p}, b_{\mu_1\cdots\mu_{d-p-1}})$ in addition to $\mathcal{F} = (f_{\mu_1\cdots\mu_p}, g_{\mu_1\cdots\mu_{d-p-1}})$; the parent action proposed is:

$$S_P[A, \mathcal{F}] = S_{BF}[A] - \int dx^d \epsilon^{\mu_1\cdots\mu_d} \left[ b_{\mu_1\cdots\mu_{d-p-1}} \partial_{\mu_{d-p}} f_{\mu_{d-p+1}\cdots\mu_d} + g_{\mu_1\cdots\mu_{d-p-1}} \partial_{\mu_{d-p}} a_{\mu_{d-p+1}\cdots\mu_d} \right] + \int dx^d \frac{m}{2} \left[ [p+1]! g_{\mu_1\cdots\mu_{d-p-1}} f_{\mu_{d-p+1}\cdots\mu_d} + (-1)^s[d-p-1]! f_{\mu_1\cdots\mu_p}^p \right],$$

where

$$S_{BF}[A] = \int dx^d \left[ -b_{\mu_1\cdots\mu_{d-p-1}} \epsilon^{\mu_1\cdots\mu_d} \partial_{\mu_{d-p}} a_{\mu_{d-p+1}\cdots\mu_d} \right]$$

may be recognized as a BF-action.

Varying $S_P$ with respect to $\mathcal{F}$, we obtain

$$\mathcal{F} = -\frac{1}{m} * dA;$$
plugging this back into \( (19) \), we recover the topologically massive gauge action (CSKR):

\[
S_{CSKR}[\mathcal{A}] = S_{BF}[\mathcal{A}] - \int \frac{d^dx}{2m} \left( (-1)^s [d-p-1]! (\partial_{\mu_1} \cdots \partial_{\mu_{d-p}})^2 + [p+1]! (\partial_{\mu_1} \cdots \partial_{d-p-1})^2 \right)
\]  

(22)

We shall observe that this is invariant under the gauge transformations; \( \mathcal{A} \rightarrow \mathcal{A} + dD \), where \( dD \) is a pure gauge doublet, i.e., it is a pair of exact differentials of \((p-1, d-p-2)\)-forms.

Now, we vary \( S_P \) with respect to \( \mathcal{A} \) and obtain:

\[
* d(\mathcal{A} - F) = 0; \quad (23)
\]

or in components,

\[
* d(a - f) = 0 \quad * d(b - g) = 0. \quad (24)
\]

This implies that the differences \( a - f \) and \( b - g \) may locally be written as exact forms; therefore, one it is possible to express the solution to these equations as

\[
\mathcal{A} = F + dD. \quad (25)
\]

Putting this back into the action \( (19) \), we recover the SD theory \( (16) \) up to topological terms.

This completes the proof of our main statement.

As an example, one can particularize this result for the special dimensionality, \( d = 3 + 1 \). In this case, only two tensor doublets may be chosen: \( \mathcal{G} = (A_\mu, B_{\nu\rho}) \) and \( \mathcal{H} = (\phi, F_{\nu\rho\alpha}) \). The first one describes a Cremmer-Scherk-Kalb-Ramond massive spin-one particle and, by virtue of the general result proven before, its dynamics may alternatively be described by either, the Cremmer-Scherk-Kalb-Ramond theory

\[
S_{CSKR}(\mathcal{G}) = \int d^4x \left( \frac{1}{2m} \partial_{[\mu} A_{\nu]} \partial^{[\rho} A^{\nu]} - \frac{1}{2m} \partial_{[\mu} B_{\nu\rho]} \partial^{[\rho} B^{\nu]} + B_{\mu\nu} \varepsilon^{\mu\nu\rho\sigma} \partial_{[\rho} A_{\sigma]} \right),
\]

(26)

or the first-order SD model:

\[
S_{DSD}(\tilde{\mathcal{G}}) = \int d^4x \left( - \tilde{A}_\sigma \tilde{A}^\sigma + \tilde{B}_{\mu\nu} \tilde{B}^{\mu\nu} + \frac{1}{m} \tilde{A}_\sigma \varepsilon^{\mu\nu\rho\sigma} \partial_{[\rho} \tilde{B}_{\mu]} \right),
\]

(27)

which is gauge non-invariant. This confirm the result recently presented in ref. [8].

The second possible doublet in four dimensions describes a scalar (spin-zero) massive particle whose dynamics may be given by a topologically massive action,

\[
S_{TM}(\mathcal{H}) = \int d^4x \left( \frac{1}{2m} \partial_{[\mu} F_{\nu\rho\alpha]} \partial^{[\mu} F^{\nu\rho\alpha]} - 3! \partial_{[\sigma} \phi \partial^{[\sigma} \phi + \phi \varepsilon^{\mu\nu\rho\sigma} \partial_{[\rho} F_{\nu\sigma]} \right),
\]

(28)

or alternatively, by a first order (SD) model:

\[
S_{DSD}(\tilde{\mathcal{H}}) = \int d^4x \left( \tilde{\phi}^2 - \frac{3!}{m} \tilde{F}_{\mu\rho\sigma} \tilde{F}^{\mu\rho\sigma} + \frac{2}{m} \phi \varepsilon^{\mu\nu\rho\sigma} \partial_{[\rho} \tilde{F}_{\mu\sigma]} \right).
\]

(29)

Doublet Hodge Duality has been defined in a similar sense to the duality in 3d \[3\]. This suggests a list of formal correspondences between theories in 3d which involve self-duality and similar models in other

\[6\] However, in ref. [3], this duality is shown by using the Batalin, Fradkin and Tyutin embedding technique \[1\].
dimensions. This constitutes by itself a very important application of this formalism since one can, in principle, translate the constructions of 3d to arbitrary dimensions.

An interesting possibility that we open up is the study of bosonization in arbitrary dimensions, mainly in higher dimensions. This is not a trivial matter\cite{19, 20, 21}, but with the help of the technique suggested here, \( d \geq 4 \) bosonization comes out in connection with a topologically massive model that mixes different gauge forms. Results on this issue shall soon be reported elsewhere\cite{22}.

Acknowledgements: The author is indebted to J. A. Helayel-Neto for invaluable discussions and pertinent corrections on the manuscript. Thanks are due to the GFT-UCP for the kind hospitality. CNPq is also acknowledged for the invaluable financial help.

References

[1] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. 140 (1982) 372.
[2] E. Cremer and J. Scherk, Nucl. Phys. B72 (1974) 117.
[3] M. Kalb and P. Ramond, Phys. Rev. D9 (1974) 2273.
[4] T.J. Allen, M.J. Bowick, and A. Lahiri, Mod. Phys. Lett. A6 (1991) 559; R. Amorim and J. Barcelos-Neto, Mod. Phys. Lett. A10 (1995) 917.
[5] A. Aurulia and Y. Takahashi, Prog. Theor. Phys. 66 (1981) 693; Phys. Rev. D23 (1981) 752; T. J. Allen, M. J. Bowick and A. Lahiri, Mod. Phys. Lett. A6 (1991) 559.
[6] M. Botta Cantcheff, "Hodge-type self(antiself)-duality for general p-form fields in arbitrary dimensions ", hep-th/0107123.
[7] P. K. Townsend, K. Pilch and P. van Nieuwenhuizen, Phys. Lett. B 136 (1984) 38.
[8] E. Harikumar, M. Sivakumar, Nucl Phys-B 565 (2000) 385, and references therein.
[9] E. Harikumar, M. Sivakumar, Mod.Phys Lett A 15 (2000) 121.
[10] E. Harikumar, M. Sivakumar, "Hamiltonian vs Lagrangian embedding of massive spin one theory involving two form field." hep-th/0104107.
[11] I. A. Batalin and E. S. Fradkin, Nucl. Phys. B279 (1987) 514; I. A. Batalin and I. V. Tyutin, Int. J. Mod. Phys. A6 (1991) 3255.
[12] E.Smailagic, E. Spallucci, Phys. Rev. D61 (2000) 067701.
[13] S. Deser and R. Jackiw, Phys. Lett. B 139 (1984) 2366.
[14] For a review in the use of the master action in proving duality in diverse areas see: S. E. Hjelmeland, U. Lindström, UIO-PHYS-97-03, May 1997. e-Print Archive: hep-th/9705123.
[15] S. Deser, A. Gomberoff, M Henneaux and C. Teitelboim, Duality, Self-Duality, Sources and Charge Quantization in Abelian N-Form Theories, hep-th/9702184; C. Wotzasek, Phys.Rev.D58:125026,1998; R. Banerjee, C. Wotzasek, Phys.Rev.D63:045005,2001
[16] J. Schwarz and A. Sen, Nucl. Phys. B411 (1994) 35
[17] D. Zwanziger, Phys. Rev. D3 (1971) 880; S. Deser and C. Teitelboim, Phys. Rev. D13 (1976) 1592.
[18] R.Banerjee and C.Wotzasek, Nucl.Phys B527(1998) 402.
[19] M.Lüscher, Nucl. Phys. B326 (1989) 557.
[20] E. C. Marino, Phys. Lett. B263 (1991) 63.

[21] R. Banerjee, C. Marino, Phys. Rev. D56 (1997) 3763.

[22] M. Botta Cantcheff, J. A. Helayel-Neto, work in progress.