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A NOTE ON DISK COUNTING IN TORIC ORBIFOLDS

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ABSTRACT. We compute orbi-disk invariants of compact Gorenstein semi-Fano toric orbifolds by extending the method used for toric Calabi-Yau orbifolds. As a consequence the orbi-disc potential is analytic over complex numbers.

0. INTRODUCTION

The mirror map plays a central role in the study of mirror symmetry. It provides a canonical local isomorphism between the Kähler moduli and the complex moduli of the mirror near a large complex structure limit. Such an isomorphism is crucial to counting of rational curves using mirror symmetry.

In [5] and [6], we derived an enumerative meaning of the inverse mirror maps for toric Calabi-Yau orbifolds and compact semi-Fano toric manifolds in terms of genus 0 open (orbifold) Gromov-Witten invariants (or orbi-)disk invariants. Namely, we showed that coefficients of the inverse mirror map are equal to generating functions of virtual counts of stable (orbi-)disks bounded by a regular Lagrangian moment map fiber. In particular it gives a way to effectively compute all such invariants.

In this short note we extend our method in [5] to derive an explicit formula for the orbi-disk invariants in the case of compact Gorenstein semi-Fano toric orbifolds; see Theorem 12 for the explicit formulas. This proves [4, Conjecture] for such orbifolds, generalizing [6, Theorem 1.2]:

**Theorem 1 (Open Mirror Theorem).** For a compact Gorenstein semi-Fano toric orbifold, the orbi-disk potential is equal to the (extended) Hori-Vafa superpotential via the mirror map.

We remark that the open crepant resolution conjecture [4, Conjecture 1] may be studied using this computation and techniques of analytical continuation in [5, Appendix A]:

**Corollary 2.** There exists an open neighborhood around the large volume limit where the orbi-disk potential converges.

This generalizes [6, Theorem 7.6] to the orbifold case.

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1. Preparation

1.1. Toric orbifolds.

1.1.1. Construction. A toric orbifold, as introduced in [3], is defined using certain combinatorial data called a *stacky fan* \((\Sigma, b_0, \ldots, b_{m-1})\), where \(\Sigma\) is a simplicial fan contained in the \(\mathbb{R}\)-vector space \(N_\mathbb{R} := N \otimes \mathbb{R}\) associated to a rank \(n\) lattice \(N\), and \(\{b_i \mid 0 \leq i \leq m - 1\}\) are integral generators of 1-dimensional cones (rays) in \(\Sigma\). We call \(b_i\) the *stacky vectors*.

Let \(b_{m}, \ldots, b_{m'-1} \in N\) be contained in the support of the fan \(\Sigma\) such that the vectors \(b_0, \ldots, b_{m'-1}\) generate \(N\) over \(\mathbb{Z}\). Following [24], the data
\[
(\Sigma, \{b_i\}_{i=0}^{m-1} \cup \{b_j\}_{j=m}^{m'-1})
\]
is called an *extended stacky fan*, and \(\{b_j\}_{j=m}^{m'-1}\) are called *extra vectors*. The flexibility of choosing extra vectors will be important in what follows.

Given an extended stacky fan, the *fan map*,
\[
\phi : \tilde{N} := \bigoplus_{i=0}^{m'} \mathbb{Z}e_i \to N, \quad \phi(e_i) := b_i \text{ for } i = 0, \ldots, m'-1,
\]
which is a surjective group homomorphism, gives an exact sequence (the “fan sequence”)
\[
0 \to \mathbb{L} := \text{Ker}(\phi) \xrightarrow{\psi} \tilde{N} \xrightarrow{\phi} N \to 0.
\]
Note that \(\mathbb{L} \simeq \mathbb{Z}^{m'-n}\). Tensoring with \(\mathbb{C}^\times\) gives the following exact sequence:
\[
0 \to G := \mathbb{L} \otimes \mathbb{Z} \mathbb{C}^\times \to \tilde{N} \otimes \mathbb{Z} \mathbb{C}^\times \simeq (\mathbb{C}^\times)^{m'} \xrightarrow{\phi_{\mathbb{C}^\times}} \mathbb{T} := N \otimes \mathbb{Z} \mathbb{C}^\times \to 0.
\]

Consider the set of “anti-cones”,
\[
\mathcal{A} := \left\{ I \subset \{0, 1, \ldots, m'-1\} \mid \sum_{i \notin I} \mathbb{R}_{\geq 0} b_i \text{ is a cone in } \Sigma \right\}.
\]
For \(I \in \mathcal{A}\), let \(\mathbb{C}^I \subset \mathbb{C}^{m'}\) be the subvariety defined by the ideal in \(\mathbb{C}[Z_0, \ldots, Z_{m'-1}]\) generated by \(\{Z_i \mid i \in I\}\). Define
\[
U_A := \mathbb{C}^{m'} \setminus \bigcup_{I \notin A} \mathbb{C}^I.
\]
The algebraic torus \(G\) acts on \(\mathbb{C}^{m'}\) via the map \(G \to (\mathbb{C}^\times)^{m'}\) in (1.3). Since \(N\) is torsion-free, the induced \(G\)-action on \(U_A\) is effective and has finite stabilizer groups. The global quotient stack
\[
\mathcal{X}_\Sigma := [U_A/G]
\]
is called the *toric orbifold* associated to \((\Sigma, \{b_i\}_{i=0}^{m-1} \cup \{b_j\}_{j=m}^{m'-1})\). By construction, the standard \((\mathbb{C}^\times)^{m'}\)-action on \(U_A\) induces a \(\mathbb{T}\)-action on \(\mathcal{X}_\Sigma\).

In this paper toric orbifolds \(\mathcal{X}_\Sigma\) are assumed to have semi-projective coarse moduli spaces \(X_\Sigma\). This means that \(X_\Sigma\) admits a \(\mathbb{T}\)-fixed point, and the natural map \(X_\Sigma \to \text{Spec} H^0(X_\Sigma, \mathcal{O}_{X_\Sigma})\) is
projective. More detailed discussions on semi-projective toric varieties can be found in [13, Section 7.2].

1.1.2. Twisted sectors. For a $d$-dimensional cone $\sigma \in \Sigma$ generated by $b_\sigma = (b_{i_1}, \ldots, b_{i_d})$, define

$$\Box_{b_\sigma} := \left\{ \nu \in N \mid \nu = \sum_{k=1}^{d} t_k b_{i_k}, \ t_k \in [0, 1) \cap \mathbb{Q} \right\}.$$ 

Let $N_{b_\sigma} \subset N$ be the submodule generated by $\{b_{i_1}, \ldots, b_{i_d}\}$. Then $\Box_{b_\sigma}$ is in bijection with the finite group $G_{b_\sigma} := N / N_{b_\sigma}$. It is easy to see that if $\tau$ is a subcone of $\sigma$, then $\Box_{b_\sigma} \subset \Box_{b_\tau}$. Define $\Box_{b_\tau} := \Box_{b_\sigma} \setminus \bigcup_{\tau \subset \sigma} \Box_{b_\tau}$, $\Box_\sigma := \bigcup_{\sigma \in \Sigma^{(n)}} \Box_{b_\tau}$, $\Box' \Sigma := \Box_\Sigma \setminus \{0\}$, where $\Sigma^{(n)}$ is the set of $n$-dimensional cones in $\Sigma$.

By [3], $\Box' (\Sigma)$ is in bijection with the twisted sectors, i.e. non-trivial connected components of the inertia orbifold of $X_{\Sigma}$. For $\nu \in \Box_\Sigma$, denote by $X_\nu$ the corresponding twisted sector of $X$. Note that $X_0 = X$ as orbifolds.

The Chen-Ruan orbifold cohomology $H^*_\text{CR}(X; \mathbb{Q})$ of a toric orbifold $X$, defined in [8], is

$$H^d_{\text{CR}}(X; \mathbb{Q}) = \bigoplus_{\nu \in \Box} H^{d-\text{age}(\nu)}(X_\nu; \mathbb{Q}),$$

where $\text{age}(\nu)$ is the degree shifting number or age of the twisted sector $X_\nu$ and the cohomology groups on the right hand side are singular cohomology groups. For $\nu = \sum_{k=1}^{d} t_k b_{i_k} \in \Box_\Sigma$ where $\{b_{i_1}, \ldots, b_{i_d}\}$ generates a cone in $\Sigma$, we have

$$\text{age}(\nu) = \sum_{k=1}^{d} t_k \in \mathbb{Q}_{\geq 0}.$$ 

The $T$-action on $X$ induces $T$-actions on twisted sectors. The $T$-equivariant Chen-Ruan orbifold cohomology $H^*_\text{CR,T}(X; \mathbb{Q})$ is then defined to be

$$H^d_{\text{CR,T}}(X; \mathbb{Q}) = \bigoplus_{\nu \in \Box} H^{d-2\text{age}(\nu)}_{T}(X_\nu; \mathbb{Q}),$$

where $H^*_T(-)$ denotes $T$-equivariant cohomology. The trivial $T$-bundle over a point $pt$ defines a map $pt \to B^T$, inducing a map $H^*_T(pt, \mathbb{Q}) = H^*(B^T, \mathbb{Q}) \to H^*(pt)$. Let $Y$ be a space with a $T$-action. By construction the $T$-equivariant cohomology of $Y$ admits a map $H^*_T(pt) \to H^*_T(Y, \mathbb{Q})$. This defines a natural map

$$H^*_T(Y, \mathbb{Q}) \to H^*_T(Y, \mathbb{Q}) \otimes H^*_T(pt) H^*(pt) \simeq H^*(Y, \mathbb{Q}).$$

For a class $C \in H^*_T(Y, \mathbb{Q})$, its image under this map, which is a class in $H^*(Y, \mathbb{Q})$, is called the non-equivariant limit of $C$. 

1.1.3. Toric divisors, Kähler cones, and Mori cones. Let $\mathcal{X}$ be a toric orbifold defined by an extended stacky fan (1.1). Let $\mathcal{A}$ be the set of anticones (1.4). Applying $\text{Hom}_\mathbb{Z}(-, \mathbb{Z})$ to the fan sequence (1.2) gives the following exact sequence$^1$

$$0 \to M := N^\vee = \text{Hom}(N, \mathbb{Z}) \xrightarrow{\phi^\vee} \tilde{M} := \tilde{N}^\vee = \text{Hom}(\tilde{N}, \mathbb{Z}) \xrightarrow{\psi^\vee} L^\vee = \text{Hom}(L, \mathbb{Z}) \to 0,$$

called the “divisor sequence”. By construction, line bundles on $\mathcal{X}$ correspond to $G$-equivariant line bundles on $\mathcal{U}_A$. Because of (1.3), $\mathbb{T}$-equivariant line bundles on $\mathcal{X}$ correspond to $(\mathbb{C}^\times)^m'$-equivariant line bundles on $\mathcal{U}_A$. Because $\bigcup_{\mathcal{I} \in \mathcal{A}} \mathcal{C}^I \subset \mathbb{C}^m'$ is of codimension at least 2, we have the following descriptions of the Picard groups:

$$\text{Pic}(\mathcal{X}) \simeq \text{Hom}(G, \mathbb{C}^\times) \simeq L^\vee, \quad \text{Pic}_T(\mathcal{X}) \simeq \text{Hom}((\mathbb{C}^\times)^m', \mathbb{C}^\times) \simeq \tilde{N}^\vee = \tilde{M}.$$ 

Moreover, the natural map $\text{Pic}_T(\mathcal{X}) \to \text{Pic}(\mathcal{X})$ is identified with $\psi^\vee : \tilde{M} \to L^\vee$.

Let $\{e_i^\vee | i = 0, 1, \ldots, m' - 1\} \subset \tilde{M}$ be the basis dual to $\{e_i | i = 0, 1, \ldots, m' - 1\} \subset \tilde{N}$. For $i = 0, 1, \ldots, m' - 1$, we denote by $D_i^\vee$ the $\mathbb{T}$-equivariant line bundle on $\mathcal{X}$ corresponding to $e_i^\vee$ under the identification $\text{Pic}_T(\mathcal{X}) \simeq \tilde{M}$. Also put

$$D_i := \psi^\vee(e_i^\vee) \in L^\vee.$$ 

The collection $\{D_i \mid 0 \leq i \leq m' - 1\}$ are toric prime divisors corresponding to the generators $\{b_i \mid 0 \leq i \leq m - 1\}$ of rays in $\Sigma$, and $\{D_i^\vee \mid 0 \leq i \leq m - 1\}$ are their $\mathbb{T}$-equivariant lifts. There are natural maps

$$\tilde{M} \otimes Q \xrightarrow{\psi^\vee \otimes Q} L^\vee \otimes Q,$$

$$(\tilde{M} \otimes Q) / \left( \sum_{j=m}^{m'-1} \mathbb{Q}D_j^\vee \right) \simeq \text{H}^2T(\mathcal{X}, \mathbb{Q}) \to \text{H}^2(\mathcal{X}, \mathbb{Q}) \simeq (L^\vee \otimes \mathbb{Q}) / \left( \sum_{j=m}^{m'-1} \mathbb{Q}D_j \right).$$

Together with the natural quotient maps, they fit into a commutative diagram.

As explained in [23, Section 3.1.2], there is a canonical splitting of the quotient map $L^\vee \otimes Q \to \text{H}^2(\mathcal{X}; \mathbb{Q})$, which we now describe. For $m \leq j \leq m' - 1$, $b_j$ is contained in a cone in $\Sigma$. Let $I_j \in \mathcal{A}$ be the anticone of the cone containing $b_j$. Then we can write $b_j = \sum_{i \notin I_j} c_{ji} b_i$ for $c_{ji} \in \mathbb{Q}_{\geq 0}$.

By the fan sequence (1.2) tensored with $\mathbb{Q}$, there exists a unique $D_j^\vee \in L \otimes \mathbb{Q}$ such that

$$\langle D_i, D_j^\vee \rangle = \begin{cases} 
1 & \text{if } i = j, \\
-c_{ji} & \text{if } i \notin I_j, \\
0 & \text{if } i \in I_j \setminus \{j\}.
\end{cases}$$

(1.5)

Here and henceforth $\langle - , - \rangle$ denotes the natural pairing between $L^\vee$ and $L$ (or relevant extensions of scalars). This defines a decomposition

$$L^\vee \otimes \mathbb{Q} = \text{Ker} \left( \left( D_m^\vee, \ldots, D_{m-1}^\vee \right) : L^\vee \otimes \mathbb{Q} \to \mathbb{Q}^{m'-m} \right) \oplus \bigoplus_{j=m}^{m'-1} \mathbb{Q}D_j.$$ 

Moreover, the term $\text{Ker} \left( \left( D_m^\vee, \ldots, D_{m-1}^\vee \right) : L^\vee \otimes \mathbb{Q} \to \mathbb{Q}^{m'-m} \right)$ is naturally identified with $\text{H}^2(\mathcal{X}; \mathbb{Q})$ via the quotient map $L^\vee \otimes \mathbb{Q} \to \text{H}^2(\mathcal{X}; \mathbb{Q})$, which allows us to regard $\text{H}^2(\mathcal{X}; \mathbb{Q})$ as a subspace of $L^\vee \otimes \mathbb{Q}$. 

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$^1$The map $\psi^\vee : \tilde{M} \to L^\vee$ is surjective since $N$ is torsion-free.
The extended Kähler cone of $\mathcal{X}$ is defined to be
\[
\tilde{C}_\mathcal{X} := \bigcap_{j \in A} \left( \sum_{i \in I} \mathbb{R}_{>0} D_i \right) \subset \mathbb{L}^\vee \otimes \mathbb{R}.
\]
The genuine Kähler cone $C_\mathcal{X}$ is the image of $\tilde{C}_\mathcal{X}$ under the quotient map $\mathbb{L}^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X}; \mathbb{R})$.

The splitting (1.6) of the extended Kähler cone in $\mathbb{L}^\vee \otimes \mathbb{Q}$ induces a splitting of the extended Kähler cone in $\mathbb{L}^\vee \otimes \mathbb{R}$: $\tilde{C}_\mathcal{X} = C_\mathcal{X} + \sum_{j=m}^{m'-1} \mathbb{R}_{>0} D_j$.

Recall that the rank of $\mathbb{L}^\vee$ is $r := m' - n$ while the rank of $H_2(\mathcal{X}; \mathbb{Z})$ is given by
\[
r' := r - (m' - m) = m - n.
\]
We choose an integral basis $\{p_1, \ldots, p_r\} \subset \mathbb{L}^\vee$ such that $p_a$ is in the closure of $\tilde{C}_\mathcal{X}$ for all $a$ and $p_{r+1}, \ldots, p_r \in \sum_{i=m}^{m'-1} \mathbb{R}_{>0} D_i$. Then the images $\bar{p}_1, \ldots, \bar{p}_r$ of $\{p_1, \ldots, p_r\}$ under the quotient map $\mathbb{L}^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X}; \mathbb{Q})$ gives a nef basis for $H^2(\mathcal{X}; \mathbb{Q})$ and $\bar{p}_a = 0$ for $r' + 1 \leq a \leq r$.

Choose $\{p_1^T, \ldots, p_r^T\} \subset \tilde{M} \otimes \mathbb{Q}$ such that $\bar{p}_a(p_i^T) = p_a$ for all $a$, and $\bar{p}_a^T = 0$ for $a = r' + 1, \ldots, r$.

Here, for $p \in \tilde{M} \otimes \mathbb{Q}$, denote by $\bar{p} \in H^2(\mathcal{X}; \mathbb{Q})$ the image of $p$ under the natural map $\tilde{M} \otimes \mathbb{Q} \to H^2_\mathbb{Q}(\mathcal{X}; \mathbb{Q})$. By construction, for $a = 1, \ldots, r'$, $\bar{p}_a$ is the non-equivariant limit of $\bar{p}_a^T$.

Define a matrix $(Q_{ia})$ by $D_i = \sum_{a=1}^{r'} Q_{ia} p_a$, $Q_{ia} \in \mathbb{Z}$. Denote by $\bar{D}_i$ the image of $D_i$ under $\mathbb{L}^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X}; \mathbb{Q})$. Then for $i = 0, \ldots, m - 1$, the class $\bar{D}_i$ of the toric prime divisor $D_i$ and its equivariant lift $\tilde{D}_i$ are given by
\[
\bar{D}_i = \sum_{a=1}^{r'} Q_{ia} \bar{p}_a, \quad \tilde{D}_i^T = \sum_{a=1}^{r'} Q_{ia} \bar{p}_a^T + \lambda_i, \text{ where } \lambda_i \in H^2(B\mathbb{T}; \mathbb{Q}),
\]
and for $i = m, \ldots, m' - 1$, $\bar{D}_i = 0$ in $H^2(\mathcal{X}; \mathbb{R})$, $\tilde{D}_i^T = 0$.

Let $1 \in H^0(\mathcal{X}; \mathbb{Q})$ be the fundamental class. For $\nu \in \text{Box with age}(\nu) = 1$, let $1_{\nu} \in H^0(\mathcal{X}_\nu; \mathbb{Q})$ be the fundamental class. It is then straightforward to see that
\[
H^0_{\text{CR,}\mathbb{T}}(\mathcal{X}, K_\mathcal{T}) = K_\mathbb{T}, \quad H^2_{\text{CR,}\mathbb{T}}(\mathcal{X}, K_\mathcal{T}) = \bigoplus_{\nu \in \text{Box, age}(\nu) = 1} K_\mathbb{T} 1_{\nu},
\]
where $K_\mathbb{T}$ is the field of fractions of $H^*_\mathbb{T}(\text{pt, } \mathbb{Q})$, and $H^*_{\mathbb{T}}(-, K_\mathbb{T}) := H^*_{\mathbb{T}}(-, \mathbb{Q}) \otimes_{H^*_{\mathbb{T}}(\text{pt, } \mathbb{Q})} K_\mathbb{T}$.

The dual basis of $\{p_1, \ldots, p_r\} \subset \mathbb{L}^\vee$ is given by $\{\gamma_1, \ldots, \gamma_r\} \subset \mathbb{L}$ where $\gamma_a = \sum_{i=m}^{m'-1} Q_{ia} e_i \in \tilde{N}$. Then $\{\gamma_1, \ldots, \gamma_r\}$ provides a basis of $H^2_{\text{eff}}(\mathcal{X}; \mathbb{Q})$. In particular, we have $Q_{ia} = 0$ when $m \leq i \leq m' - 1$ and $1 \leq a \leq r'$.

Set
\[
\mathbb{K} := \{d \in \mathbb{L} \otimes \mathbb{Q} \mid \langle j, d \rangle \in \mathbb{Z} \in A \},
\]
\[
\mathbb{K}_{\text{eff}} := \{d \in \mathbb{L} \otimes \mathbb{Q} \mid \langle j, d \rangle \in \mathbb{Z}_{\geq 0} \in A \},
\]
Roughly speaking $\mathbb{K}_{\text{eff}}$ is the set of effective curve classes. In particular, the intersection $\mathbb{K}_{\text{eff}} \cap H_2(\mathcal{X}; \mathbb{R})$ consists of classes of stable maps $\mathbb{P}(1, m) \to \mathcal{X}$ for some $m \in \mathbb{Z}_{\geq 0}$. See e.g. [23] Section 3.1 for more details.

**Definition 3.** A toric orbifold $\mathcal{X}$ is called semi-Fano if $c_1(\mathcal{X}) \cdot \alpha > 0$ for every effective curve class $\alpha$, in other words, $-K_\mathcal{X}$ is nef.
For a real number \( \lambda \in \mathbb{R} \), let \( [\lambda], \lfloor \lambda \rfloor \) and \( \{\lambda\} \) denote the ceiling, floor and fractional part of \( \lambda \) respectively. Now for \( d \in \mathbb{K} \), define

\[
\nu(d) := \sum_{i=0}^{m'} [\langle D_i, d \rangle] b_i \in \mathbb{N},
\]

and let \( I_d := \{ j \in \{0, 1, \ldots, m' - 1 \} \mid \langle D_j, d \rangle \in \mathbb{Z} \} \in \mathcal{A} \). Then since we can rewrite

\[
\nu(d) = \sum_{i=0}^{m'} (\langle -\langle D_i, d \rangle \rangle + \langle D_i, d \rangle) b_i = \sum_{i=0}^{m'} \langle -\langle D_i, d \rangle \rangle b_i = \sum_{i \notin I_d} \langle -\langle D_i, d \rangle \rangle b_i,
\]

we have \( \nu(d) \in \text{Box} \), and hence \( \nu(d) \), if nonzero, corresponds to a twisted sector \( \mathcal{X}_{\nu(d)} \) of \( \mathcal{X} \).

### 1.2. Orbi-disk invariants.

We briefly review the construction of genus 0 open orbifold GW invariants of toric orbifolds following [10].

Let \( (\mathcal{X}, \omega) \) be a toric Kähler orbifold of complex dimension \( n \), equipped with the standard toric complex structure \( J_0 \) and a toric Kähler structure \( \omega \). Suppose that \( \mathcal{X} \) is associated to the stacky fan \((\Sigma, b)\), where \( b = (b_0, \ldots, b_{m-1}) \) and \( b_i = c_i v_i \). As before, \( D_i \) \( (i = 0, \ldots, m - 1) \) denotes the toric prime divisor associated to \( b_i \). Let \( L \subset \mathcal{X} \) be a Lagrangian torus fiber of the moment map \( \mu_0 : \mathcal{X} \to M_\mathbb{R} := M \otimes_{\mathbb{Z}} \mathbb{R} \), and consider a relative homotopy class \( \beta \in \pi_2(\mathcal{X}, L) = H_2(\mathcal{X}, L; \mathbb{Z}) \).

#### 1.2.1. Holomorphic orbi-disks and their moduli spaces.

A holomorphic orbi-disk in \( \mathcal{X} \) with boundary in \( L \) is a continuous map \( w : (\mathbb{D}, \partial \mathbb{D}) \to (\mathcal{X}, L) \) such that the following conditions are satisfied:

1. \((\mathbb{D}, z_1^+, \ldots, z_l^+)\) is an orbi-disk with interior orbifold marked points \( z_1^+, \ldots, z_l^+ \). Namely \( \mathbb{D} \) is analytically the disk \( D^2 \subset \mathbb{C} \), together with orbifold structure at each marked point \( z_j^+ \) for \( j = 1, \ldots, l \). For each \( j \), the orbifold structure at \( z_j^+ \) is given by a disk neighborhood of \( z_j^+ \) which is uniformized by a branched covering map \( \text{br} : z \to z^{m_j} \) for some \( m_j \in \mathbb{Z}_{>0} \).
2. For any \( z_0 \in \mathbb{D} \), there is a disk neighborhood of \( z_0 \) with a branched covering map \( \text{br} : z \to z^{m_j} \), and there is a local chart \((V_{w(z_0)}, G_{w(z_0)}, \pi_{w(z_0)})\) of \( \mathcal{X} \) at \( w(z_0) \) and a local holomorphic lifting \( \tilde{w}_{z_0} \) of \( w \) satisfying \( w \circ \text{br} = \pi_{w(z_0)} \circ \tilde{w}_{z_0} \).
3. The map \( w \) is good (in the sense of Chen-Ruan [7]) and representable. In particular, for each marked point \( z_j^+ \), the associated homomorphism

\[
(1.8) \quad h_p : \mathbb{Z}_{m_j} \to G_{w(z_j^+)}
\]

between local groups which makes \( \tilde{w}_{z_j^+} \) equivariant, is injective.

Denote by \( \nu_j \in \text{Box}(\Sigma) \) the image of the generator \( 1 \in \mathbb{Z}_{m_j} \) under \( h_j \) and let \( \mathcal{X}_{\nu_j} \) be the twisted sector of \( \mathcal{X} \) corresponding to \( \nu_j \). Such a map \( w \) is said to be of type \( \mathbf{x} := (\mathcal{X}_{\nu_1}, \ldots, \mathcal{X}_{\nu_n}) \).

There are two notions of Maslov index for an orbi-disk. The desingularized Maslov index \( \mu^{de} \) is defined by desingularizing the interior singularities (following Chen-Ruan [7]) of the pull-back bundle \( w^* T \mathcal{X} \) in [10, Section 3]. The Chern-Weil (CW) Maslov index is defined in [11] as the integral of the curvature of a unitary connection on \( w^* T \mathcal{X} \) which preserves the Lagrangian boundary condition. We will mainly use the CW Maslov index in this paper. The following lemma, which generalizes results in [9, 2, 10], can be used to compute the Maslov index of disks.

---

\(^2\)If \( m_j = 1 \), \( z_j^+ \) is a smooth interior marked point.
Lemma 4. Let \((\mathcal{X}, \omega, J)\) be a Kähler orbifold of complex dimension \(n\), equipped with a non-zero meromorphic \(n\)-form \(\Omega\) on \(\mathcal{X}\) which has at worst simple poles. Let \(D \subset \mathcal{X}\) be the pole divisor of \(\Omega\). Suppose also that the generic points of \(D\) are smooth. Then for a special Lagrangian submanifold \(L \subset \mathcal{X} \setminus D\), the CW Maslov index of a class \(\beta \in \pi_2(\mathcal{X}, L)\) is given by

\[
\mu_{\text{CW}}(\beta) = 2\beta \cdot D.
\]

Here, \(\beta \cdot D\) is defined by writing \(\beta\) as a fractional linear combination of homotopy classes of smooth disks.

Proof. Suppose \(\beta\) is a homotopy class of a smooth disk. Given a smooth disk representative \(u : D^2 \to \mathcal{X}\) of \(\beta\), note that the pull-back of the canonical line bundle \(u^*(K_{\mathcal{X}})\) is an honest vector bundle over \(D^2\), and hence, the proof in [2] applies to this case. Also since the CW Maslov index is topological, if \(\beta\) is given as a (fractional) linear combination, so is its CW Maslov index. Hence (1.9) for an orbi-disk class \(\beta\) also follows.

Orbi-disks in a symplectic toric orbifold have been classified [10, Theorem 6.2]. Among them, the following \textit{basic disks} corresponding to the stacky vectors and twisted sectors play an important role.

Theorem 5 ([10], Corollaries 6.3 and 6.4). Let \(\mathcal{X}\) and \(L\) be as in the beginning of this section.

1. The smooth holomorphic disks of Maslov index two (modulo \(T^n\)-action and automorphisms of the domain) are in a one-to-one correspondence with the stacky vectors \(\{b_0, \ldots, b_{m-1}\}\), whose homotopy classes are denoted as \(\beta_0, \ldots, \beta_{m-1}\).
2. The holomorphic orbi-disks with one interior orbifold marked point and desingularized Maslov index zero (modulo \(T^n\)-action and automorphisms of the domain) are in a one-to-one correspondence with the twisted sectors \(\nu \in \text{Box}'(\Sigma)\) of the toric orbifold \(\mathcal{X}\), whose homotopy classes are denoted as \(\beta_\nu\).

Lemma 6 ([10], Lemma 9.1). For \(\mathcal{X}\) and \(L\) as above, the relative homotopy group \(\pi_2(\mathcal{X}, L)\) is generated by the classes \(\beta_i\) for \(i = 0, \ldots, m-1\) together with \(\beta_\nu\) for \(\nu \in \text{Box}'(\Sigma)\).

We call these generators of \(\pi_2(\mathcal{X}, L)\) the \textit{basic disk classes}; they are the analogue of Maslov index two disk classes in toric manifolds. Basic disk classes were used in [10] to define the \textit{leading order bulk orbi-potential}, and it can be used to determine the Floer homology of torus fibers with suitable bulk deformations.

Let \(\mathcal{M}_{k+1,l}^{op,\text{main}}(\mathcal{X}, L, \beta, x)\) be the moduli space of good representable stable maps from bordered orbifold Riemann surfaces of genus zero with \(k + 1\) boundary marked points \(z_0, z_1, \ldots, z_k\) and \(l\) interior (orbi)fold marked points \(z_1^+, \ldots, z_l^+\) in the homotopy class \(\beta\) of type \(x = (\mathcal{X}_{\nu_1}, \ldots, \mathcal{X}_{\nu_l})\). Here, the superscript “\text{main}” indicates that we have chosen a connected component on which the boundary marked points respect the cyclic order of \(S^1 = \partial D^2\). By [10], \(\mathcal{M}_{k+1,l}^{op,\text{main}}(\mathcal{X}, L, \beta, x)\) has a Kuranishi structure of real virtual dimension

\[
n + \mu_{\text{CW}}(\beta) + k + 1 + 2l - 3 - 2 \sum_{j=1}^l \text{age}(\nu_j).
\]

By [10, Proposition 9.4], if \(\mathcal{M}_{1,1}^{op,\text{main}}(\mathcal{X}, L, \beta)\) is non-empty and if \(\partial \beta\) is not in the sublattice generated by \(b_0, \ldots, b_{m-1}\), then there exist \(\nu \in \text{Box}'(\Sigma), k_i \in \mathbb{N} (i = 0, \ldots, m-1)\) and \(\alpha \in \)
$H_2^{\text{eff}}(\mathcal{X})$ such that $\beta = \beta_\nu + \sum_{i=0}^{m-1} k_i \beta_i + \alpha$, where $\alpha$ is realized by a union of holomorphic (orbi-)spheres. The CW Maslov index of $\beta$ written in this way is given by $\mu_{\text{CW}}(\beta) = 2 \text{age}(\nu) + 2 \sum_{i=0}^{m-1} k_i + 2c_1(\mathcal{X}) \cdot \alpha$.

1.2.2. The invariants. Let $\mathcal{X}_{\nu_1}, \ldots, \mathcal{X}_{\nu_l}$ be twisted sectors of the toric orbifold $\mathcal{X}$. Consider the moduli space $\mathcal{M}_{1,l}^{\text{op,main}}(\mathcal{X}, L, \beta, \mathbf{x})$ of good representable stable maps from bordered orbifold Riemann surfaces of genus zero with one boundary marked point and $l$ interior orbifold marked points of type $\mathbf{x} = (\mathcal{X}_{\nu_1}, \ldots, \mathcal{X}_{\nu_l})$ representing the class $\beta \in \pi_2(\mathcal{X}, L)$. By \cite{10}, $\mathcal{M}_{1,l}^{\text{op,main}}(\mathcal{X}, L, \beta, \mathbf{x})$ carries a virtual fundamental chain, which has an expected dimension $n$ if the following equality holds:

\begin{equation}
\mu_{\text{CW}}(\beta) = 2 + \sum_{j=1}^{l} (2 \cdot \text{age}(\nu_j) - 2).
\end{equation}

For a Gorenstein orbifold, the age of every twisted sector is a non-negative integer. Now we assume that the toric orbifold $\mathcal{X}$ is semi-Fano (see Definition \ref{def:semi-fano}) and Gorenstein. Then a basic orbifold class $\beta_\nu$ has Maslov index $2 \text{age}(\nu)$, and hence every non-constant stable disk class has at least Maslov index 2.

We further restrict to the case where all the interior orbifold marked points are mapped to age-one twisted sectors, i.e. the type $\mathbf{x}$ consists of twisted sectors with age $\leq 1$. This will be enough for our purpose of constructing the mirror over $H_2^{\text{eff}}(\mathcal{X})$. In this case, the virtual fundamental chain $\mathcal{M}_{1,l}^{\text{op,main}}(\mathcal{X}, L, \beta, \mathbf{x})^{\text{vir}}$ has an expected dimension $n$ when $\mu_{\text{CW}}(\beta) = 2$, and in fact we get a virtual fundamental cycle because $\beta$ attains the minimal Maslov index and thus disk bubbling does not occur. Therefore the following definition of $\textit{genus 0 open orbifold GW invariants}$ (also termed $\textit{orbi-disk invariants}$) is independent of the choice of perturbations of the Kuranishi structures (in the general case one may restrict to torus-equivariant perturbations to make sense of the following definition following Fukaya-Oh-Ohta-Ono \cite{16,17,15}):

\begin{definition}[Orbi-disk invariants] Let $\beta \in \pi_2(\mathcal{X}, L)$ be a relative homotopy class with Maslov index given by \eqref{eq:maslov}. Suppose that the moduli space $\mathcal{M}_{1,l}^{\text{op,main}}(\mathcal{X}, L, \beta, \mathbf{x})$ has a virtual fundamental cycle of dimension $n$. Then we define $n_{1,l,\beta}^{\mathcal{X}}([\text{pt}]_L; 1_{\nu_1}, \ldots, 1_{\nu_l}) \in \mathbb{Q}$ to be the push-forward

$$n_{1,l,\beta}^{\mathcal{X}}([\text{pt}]_L; 1_{\nu_1}, \ldots, 1_{\nu_l}) := ev_0 (\mathcal{M}_{1,l}^{\text{op,main}}(\mathcal{X}, L, \beta, \mathbf{x})^{\text{vir}}) \in H_n(L; \mathbb{Q}) \cong \mathbb{Q},$$

where $ev_0 : \mathcal{M}_{1,l}^{\text{op,main}}(\mathcal{X}, L, \beta, \mathbf{x}) \to L$ is evaluation at the boundary marked point, $[\text{pt}]_L \in H^n(L; \mathbb{Q})$ is the point class of the Lagrangian torus fiber $L$, and $1_{\nu_j} \in H^0(\mathcal{X}_{\nu_j}; \mathbb{Q}) \subset H_{\text{CR}}^{2 \text{age}(\nu_j)}(\mathcal{X}; \mathbb{Q})$ is the fundamental class of the twisted sector $\mathcal{X}_{\nu_j}$.
\end{definition}

2. Geometric constructions

Let $\beta \in \pi_2(\mathcal{X}, L)$ be a disk class with $\mu_{\text{CW}}(\beta) = 2$. By the discussion in Section \ref{sec:toric-orbifolds} we can write

$$\beta = \beta_d + \alpha$$

with $\alpha \in H_2(\mathcal{X}, \mathbb{Z})$, $c_1(\mathcal{X}) \cdot \alpha = 0$ and either $\beta_d \in \{\beta_0, \ldots, \beta_{m-1}\}$ or $\beta_d \in \text{Box}'(\mathcal{X})$ is of age 1. Denote by $b_{\alpha} \in \mathbb{N}$ the element corresponding to $\beta_d$. 

Recall that the fan polytope $\mathcal{P} \subset \mathbb{R}_+$ is the convex hull of the vectors $b_0, \ldots, b_{m-1}$. Note that $b_d \in \mathcal{P}$. Denote by $F(b_d)$ the minimal face of the fan polytope $\mathcal{P}$ that contains the vector $b_d$. Let $F$ be a top-dimensional face of $\mathcal{P}$ that contains $F(b_d)$. Let $\Sigma_{\beta_d} \subset \Sigma$ be the minimal convex subfan containing all $\{b_0, \ldots, b_{m-1}\} \cap F$. The vectors
\[
\{b_0, \ldots, b_{m'-1}\} \cap \sum_{b_j \in \{b_0, \ldots, b_{m-1}\} \cap F} \mathbb{Q}_{\geq 0} b_j
\]
determine a fan map. Let
\[
\mathcal{X}_{\beta_d} \subset \mathcal{X}
\]
be the associated toric suborbifold (of the same dimension). Since $\mathcal{X}$ is Gorenstein, we have

**Lemma 8.** $\mathcal{X}_{\beta_d}$ is a toric Calabi-Yau orbifold.

Note that $\mathcal{X}_{\beta_d}$ depends on the choice of the face $F$, not just $\beta_d$. We use $\mathcal{X}_{\beta_d}$ to compute open Gromov-Witten invariants of $\mathcal{X}$ in class $\beta = \beta_d + \alpha$.

In what follows we show that $\mathcal{X}_{\beta_d} \subset \mathcal{X}$ contains all stable orbi-disks of $\mathcal{X}$ of class $\beta$. First, we have the following analogue of [6, Proposition 5.6].

**Lemma 9.** Let $f : \mathcal{D} \cup \mathcal{C} \to \mathcal{X}$ be a stable orbi-disk map in the class $\beta = \beta_d + \alpha$, where $\mathcal{D}$ is a (possibly orbifold) disk and $\mathcal{C}$ is a (possibly orbifold) rational curve such that $f_*[\mathcal{D}] = \beta_d$ and $f_*[\mathcal{C}] = \alpha$ with $c_1(\alpha) = 0$. Then we have
\[
f(\mathcal{C}) \subset \bigcup_{b_j \in F(b_d)} D_j,
\]
and $[f(\mathcal{C})] \cdot D_j = 0$ whenever $b_j \notin F(b_d)$.

**Proof.** Since $c_1(\alpha) = 0$, $\mathcal{C}$ should lie in toric divisors of $\mathcal{X}$. Suppose $\beta_d$ is a smooth disk class. Then the sphere component $\mathcal{C}_0$ meeting the disk component $\mathcal{D}$ maps into the divisor $D_d$ and it should have non-negative intersection with other toric divisors. By [21, Lemma 4.5] which easily extends to the simplicial setting, we have desired statement for $f(\mathcal{C}_0)$.

If $\beta_d$ is an orbi-disk class, then we can write the corresponding $b_d \in N$ as $b_d = \sum_{b_i \in \sigma} c_i b_i$, with $\sum c_i = 1, c_i \in [0,1) \cap \mathbb{Q}$. For the sphere component $\mathcal{C}_0$ meeting the disk component $\mathcal{D}_{\sigma}$, we have $f(\mathcal{C}_0) \subset \bigcup_{b_i \in \sigma} D_i$ and each $b_i \in \sigma$ satisfies $b_i \in F(b_d)$. Hence $f(\mathcal{C}_0) \subset \bigcup_{b_i \in F(b_d)} D_i$ and $f(\mathcal{C}_0) \cdot D_j = 0$ for $b_j \notin F(b_d)$.

Let $\mathcal{C}_1 \subset \mathcal{C}$ be a sphere component meeting $\mathcal{C}_0$, then we have $f(\mathcal{C}_1) \subset F(b_j)$ for some $b_j \in F(b_d)$ by the intersection condition. Now, we can follow the proof of [6, Proposition 5.6] shows that $f(\mathcal{C}_1) \subset \bigcup_{b_i \in F(b_d)} D_i$. The result follows by repeating this argument for one sphere component at a time. \qed

Partition $\{b_0, \ldots, b_{m-1}\} \cap F(b_d)$ into the disjoint union of two subsets,
\[
\{b_0, \ldots, b_{m-1}\} \cap F(b_d) = F(b_d)^c \bigsqcup F(b_d)^{\text{nc}},
\]
where $b_i \in F(b_d)^c$ if $D_i \subset \mathcal{X}_{\beta_d}$ and $b_i \in F(b_d)^{\text{nc}}$ if $D_i \not\subset \mathcal{X}_{\beta_d}$.

**Lemma 10.** Let $f : \mathcal{D} \cup \mathcal{C} \to \mathcal{X}$ be as in Lemma 9 Then we have $f(\mathcal{D} \cup \mathcal{C}) \subset \mathcal{X}_{\beta_d}$. 

Proof. Certainly \( f(D) \subset X_{\beta_d} \). We claim that

\[
(2.1) \quad f(C) \subset \bigcup_{b_j \in F(b_d)^c} D_j,
\]

from which the lemma follows.

To see (2.1), we write \( C = C_c \cup C_{nc} \) where \( C_c \) consists of components of \( C \) which lie in \( \bigcup_{b_j \in F(b_d)^c} D_j \), and \( C_{nc} \) consists of the remaining components. Set \( A := f_*[C_c] \) and \( B := f_*[C_{nc}] \). Then \( \alpha = A + B \).

Since \( -K_X \) is nef and \( -K_X \cdot \alpha = 0 \), we have \( -K_X \cdot A = 0 = -K_X \cdot B \). Write \( B = \sum k_b B_k \) as an effective linear combination of the classes \( B_k \) of irreducible 1-dimensional torus-invariant orbits in \( X \). Again because \( -K_X \) is nef, we have \( -K_X \cdot B_k = 0 \) for all \( k \). Each \( B_k \) corresponds to an \((n - 1)\)-dimensional cone \( \sigma_k \in \Sigma \). In the expression \( B = \sum k_b B_k \), there is at least one (non-zero) \( B_k \) which is not contained in \( \bigcup_{b_j \in F(b_d)^c} D_j \). As a consequence, either \( \sigma_k \) contains a ray \( \mathbb{R}_{\geq 0} b_j \) with \( b_j \notin F(b_d) \), or there exists a \( b_j \notin F(b_d) \) such that \( \sigma_k \) and \( b_j \) span an \( n \)-dimensional cone in \( \Sigma \).

Since \( B_k \) is not contained in \( \bigcup_{b_j \in F(b_d)^c} D_j \), we see that if \( b_i \in F(b_d)^c \) then \( b_i \notin \sigma_k \). Also, \( D \cdot B_k \geq 0 \) for every toric prime divisor \( D \subset X \) not corresponding to a ray in \( \sigma_k \).

By Lemma 4.5] (which easily extends to the simplicial setting), we have \( D \cdot B_k = 0 \) for every toric prime divisor \( D \subset X \) corresponding to an element in \( \{b_1, \ldots, b_m\} \setminus F(\sigma_k) \), where \( F(\sigma_k) \subset \mathcal{P} \) is the minimal face of \( \mathcal{P} \) containing rays in \( \sigma_k \). Since the divisors \( D \subset X \) corresponding to \( \{b_1, \ldots, b_m\} \setminus F(\sigma_k) \) span \( H^2(X) \), we have \( B_k = 0 \), a contradiction.

Let \( x = (\mathcal{X}_1, \ldots, \mathcal{X}_m) \) be an \( l \)-tuple of twisted sectors of \( X_{\beta_d} \). Then Lemma 10 implies that the natural inclusion \( \mathcal{M}_{1,l,\beta, x}^{op,main}(X_{\beta_d}, L, \beta, x) \hookrightarrow \mathcal{M}_{1,l,\beta, x}^{op,main}(X, L, \beta, x) \) is a bijection. Since \( X_{\beta_d} \subset X \) is open, the local deformations and obstructions of stable discs in \( X_{\beta_d} \) and their inclusion in \( X \) are isomorphic. It follows that

**Proposition 11.** The moduli spaces \( \mathcal{M}_{1,l,\beta, x}^{op,main}(X, L, \beta, x) \) of disks in \( X \) is isomorphic as Kuranishi spaces to the moduli spaces \( \mathcal{M}_{1,l,\beta, x}^{op,main}(X_{\beta_d}, L, \beta, x) \) of disks in \( X_{\beta_d} \). Consequently

\[
n_{\mathcal{X}_1, \mathcal{X}_2}^X([\text{pt}]_L; 1_{\nu_1}, \ldots, 1_{\nu_l}) = n_{1,1,\beta}^X([\text{pt}]_L; 1_{\nu_1}, \ldots, 1_{\nu_l}).
\]

Since \( X_{\beta_d} \) is a toric Calabi-Yau orbifold, the open Gromov-Witten invariants \( n_{1,1,\beta}^X([\text{pt}]_L; 1_{\nu_1}, \ldots, 1_{\nu_l}) \) have been computed in [5]. By Proposition 11 this gives open Gromov-Witten invariants of \( X \). Explicitly they are given as follows.

Using the toric data of \( X_{\beta_d} \), we define

\[
\Omega_j^{X_{\beta_d}} := \{ d \in \mathbb{K}_{\text{eff}} \mid \nu(d) = 0, \langle D_j, d \rangle \in \mathbb{Z}_{<0} \text{ and } \langle D_i, d \rangle \in \mathbb{Z}_{\geq 0} \forall i \neq j \}, \quad j = 0, 1, \ldots, m - 1,
\]

\[
\Omega_j^{X_{\beta_d}} := \{ d \in \mathbb{K}_{\text{eff}} \mid \nu(d) = b_j \text{ and } \langle D_i, d \rangle \notin \mathbb{Z}_{<0} \forall i \}, \quad j = m, m + 1, \ldots, m' - 1.
\]

\[
(2.2) \quad A_j^{X_{\beta_d}}(y) := \sum_{d \in \Omega_j^{X_{\beta_d}}} y^d \frac{(1 - \langle D_j, d \rangle - 1)(\langle D_j, d \rangle - 1)!}{\prod_{i \neq j} \langle D_i, d \rangle!}, \quad j = 0, 1, \ldots, m - 1,
\]
Theorem 12. If \( \beta_{i_0} \) is a basic smooth disk class corresponding to the ray generated by \( b_{i_0} \) for some \( i_0 \in \{0, 1, \ldots, m - 1\} \), then we have

\[
\sum_{\alpha \in H_2^{\text{eff}}(X)} \sum_{l \geq 0} \sum_{v \in \Box'(\Sigma_{b_{i_0}})_{\alpha} = 1} \prod_{l=1}^{\ell} \frac{\tau_{v_{l}}}{l!} n_{1, l, \beta_{i_0} + \alpha}([pt]_L; \prod_{i=1}^{l} 1_{v_{i}}) q^{\alpha} = \exp \left( -A_{i_0}^{X,b_{i_0}}(y(q, \tau)) \right)
\]

via the inverse \( y = y(q, \tau) \) of the toric mirror map (2.4).

If \( \beta_{j_0} \) is a basic orbi-disk class corresponding to \( \nu_{j_0} \in \Box'(\Sigma)_{\alpha = 1} \) for some \( j_0 \in \{m, m + 1, \ldots, m' - 1\} \), then we have

\[
\sum_{\alpha \in H_2^{\text{eff}}(X)} \sum_{l \geq 0} \sum_{v \in \Box'(\Sigma_{b_{j_0}})_{\alpha = 1}} \prod_{l=1}^{\ell} \frac{\tau_{v_{l}}}{l!} n_{1, l, \nu_{j_0} + \alpha}([pt]_L; \prod_{i=1}^{l} 1_{v_{i}}) q^{\alpha}
\]

\[
= y^{D_{j_0}} \exp \left( -\sum_{i \notin I_{j_0}} c_{j_0i} A_{i}^{X,b_{j_0}}(y(q, \tau)) \right),
\]

via the inverse \( y = y(q, \tau) \) of the toric mirror map (2.4), where \( D_{j_0} \in \mathbb{K}_{\text{eff}} \) is the class defined in (1.5), \( I_{j_0} \in \mathcal{A} \) is the anticone of the minimal cone containing \( b_{j_0} = \nu_{j_0} \) and \( c_{j_0i} \in \mathbb{Q} \cap [0, 1) \) are rational numbers such that \( b_{j_0} = \sum_{i \notin I_{j_0}} c_{j_0i} b_{i} \).

Proof. By Proposition 11 \( n_{1, l, \beta}([pt]_L; 1_{v_{1}}, \ldots, 1_{v_{l}}) = n_{1, l, \beta}([pt]_L; 1_{v_{1}}, \ldots, 1_{v_{l}}) \), and so the LHS of (2.5) is equal to

\[
\sum_{\alpha \in H_2^{\text{eff}}(X_{b_{i_0}})} \sum_{l \geq 0} \sum_{v \in \Box'(\Sigma_{b_{i_0}})_{\alpha = 1}} \prod_{l=1}^{\ell} \frac{\tau_{v_{l}}}{l!} n_{1, l, \beta_{i_0} + \alpha}([pt]_L; \prod_{i=1}^{l} 1_{v_{i}}) q^{\alpha}
\]

which in turn is equal to \( \exp \left( -A_{i_0}^{X,b_{i_0}}(y(q, \tau)) \right) \) by [5] Theorem 1.4. The deduction for (2.6) is similar. \( \square \)

Example 2.1. \( \mathbb{P}^2 / \mathbb{Z}_3 \) is a Gorenstein Fano toric orbifold. Its fan and polytope pictures are shown in Figure 7. It has three toric divisors \( D_1, D_2, D_3 \) corresponding to the rays generated by \( v_1 = (-1, -1), v_2 = (2, -1), v_3 = (-1, 2) \). By pairing with the dual vectors \((1, 0)\) and \((0, 1)\), the linear equivalence relations are \( 2D_2 - D_3 - D_1 \sim 0 \) and \( 2D_3 - D_2 - D_1 \sim 0 \), and so \( D_1 \sim D_2 \sim D_3 \). It has three orbifold points corresponding to the three vertices in the polytope picture. Locally it is \( \mathbb{C}^2 / \mathbb{Z}_3 \) around each orbifold point.
Fix a Lagrangian torus fiber. \( \mathbb{P}^2/\mathbb{Z}_3 \) has nine basic orbi-disk classes corresponding to the nine lattice points on the boundary of the fan polytope. Three of them are smooth disk classes and denote them by \( \beta_1, \beta_2, \beta_3 \). The basic orbi-disk classes corresponding to the two lattice points \((2v_1 + v_2)/3\) and \((v_1 + 2v_2)/3\) are denoted by \( \beta_{112} \) and \( \beta_{122} \), which pass through the twisted sectors \( \nu_{112} \) and \( \nu_{122} \) respectively. Then \( 2\beta_1 + \beta_2 - 3\beta_{112} \) (or \( 2\beta_2 + \beta_1 - 3\beta_{122} \)) is the class of a constant orbi-sphere passing through the twisted sector \( \nu_{112} \) (or \( \nu_{122} \) resp.). In particular the area of \( \beta_{112} \) equals to \((2\beta_1 + \beta_2)/3\). Other basic orbi-disk classes have similar notations.

Theorem 12 provides a formula for the open GW invariants \( n_{1, L, \beta_{112}}(\text{pt}; l(\prod_{i=1}^t 1_{\nu_i})) \) where \( \nu_i \) is either \( \nu_{112} \) or \( \nu_{122} \) for each \( i \). To write down the invariants more systematically, we consider the open GW potential as follows.

Let \( q \) be the Kähler parameter of the smooth sphere class \( \beta_1 + \beta_2 + \beta_3 \in H_2(\mathbb{P}^2/\mathbb{Z}_3) \). The basic orbi-disk classes correspond to monomials in the disk potential \( q^3 z^\tau \), where \( q^3 = q^{3z} = q^{3\beta_{112}} = q^{3\beta_{122}} = 1 \), \( q^3 = q^{3\beta_1 + \beta_2 + \beta_3} = q \), \( q^{3\beta_{223}} = q^{(2\beta_2 + \beta_3)/3} = q^{3\beta_{33}^3} = q^{1/3} \), and similar for other basic orbi-disk classes. The Kähler parameters corresponding to the twisted sectors \( \nu_{112}, \nu_{122} \) are denoted as \( \tau_{112}, \tau_{122} \) (and similar for other twisted sectors).

By [5] Example 1, Section 6.5, the open GW potential for \( \mathbb{C}^2/\mathbb{Z}_3 \) is given by \( w(z - \kappa_0(\tau_{112}, \tau_{122}))(z - \kappa_1(\tau_{112}, \tau_{122}))(z - \kappa_2(\tau_{112}, \tau_{122})) \) where

\[
(2.7) \quad \kappa_k(\tau_1, \tau_2) = \zeta^{2k+1} \prod_{r=1}^{2} \exp \left( \frac{1}{3} \zeta^{(2k+1)r} \tau_r \right), \quad \zeta := \exp(\pi \sqrt{-1}/3).
\]

By Proposition 14, the disk invariants of \( \mathbb{P}^2/\mathbb{Z}_3 \) equal to those of \( \mathbb{C}^2/\mathbb{Z}_3 \). Thus the open GW potential of \( \mathbb{P}^2/\mathbb{Z}_3 \) is given by

\[
W = z^{-1} w^{-1}(z - \kappa_0(\tau_{112}, \tau_{122}))(z - \kappa_1(\tau_{112}, \tau_{122}))(z - \kappa_2(\tau_{112}, \tau_{122}))
\]

\[
+ z^{-1} w^{-1} (q^{1/3} z^{-1} w - \kappa_0(\tau_{113}, \tau_{133}))(q^{1/3} z^{-1} w - \kappa_1(\tau_{113}, \tau_{133}))(q^{1/3} z^{-1} w - \kappa_2(\tau_{113}, \tau_{133}))
\]

\[
+ z^{-2} w^{-1} (q^{1/3} z^{-1} w - \kappa_0(\tau_{223}, \tau_{233}))(q^{1/3} z^{-1} w - \kappa_1(\tau_{223}, \tau_{233}))(q^{1/3} z^{-1} w - \kappa_2(\tau_{223}, \tau_{233}))
\]

\[
- z^{-1} w^{-1} - z^{-2} w^{-1} - q z^{-1} w^2.
\]

\[
\text{Figure 1. The fan and polytope picture for } \mathbb{P}^2/\mathbb{Z}_3.\]
Then generating functions of open orbifold GW for $\beta_{112}$ and $\beta_{122}$ are given by the coefficients of $w^{-1}$ and $zw^{-1}$ in $W$ respectively.

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