Rank-one Approximation of Joint Spectral Radius of Finite Matrix Family

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Abstract

We show that any finite set of rank-one matrices satisfies the finiteness property under the framework of (invariant) extremal norm. Formula for the computation of joint/generalized spectral radius for this class of matrix family is derived. Based on this, we further study the joint/generalized spectral radius of finite sets of general matrices through constructing rank-one approximations in terms of singular value decomposition (SVD) and a new characterization of joint/generalized spectral radius is obtained. Several benchmark examples from applications as well as their computational simulations are provided to illustrate the theoretical outcomes.

Keywords: joint/generalized spectral radius; finiteness property; rank-one; extremal norm; Barabanov norm; singular value decomposition.

1. Introduction

The joint spectral radius of a finite set of matrices plays an important role in real applications. Among its large amount of various applications we identify the following areas: wavelet theory \cite{1,2,3,4,5}, stability of switched linear systems \cite{6,7,8,9}, subdivision algorithms for generating curves \cite{10,11}, overlap-free words \cite{12}, asymptotic behavior of partition functions \cite{13}, and their references therein. Hence, the computation or approximation of joint spectral radius is of great interest in reality.

We consider a finite set $\mathcal{F} = \{A_1, A_2, \ldots, A_m\} \subset \mathbb{C}^{n \times n}$ of complex $n \times n$ matrices and define the set $\mathcal{F}_k$ of all possible products of length $k \geq 1$ with factors from $\mathcal{F}$, i.e.,

$$\mathcal{F}_k = \{A_{i_1}A_{i_2}\cdots A_{i_k} : 1 \leq i_j \leq m, \ j = 1, \ldots, k\}.$$
Let $\| \cdot \|$ be any sub-multiplicative matrix norm and $\rho(A)$ be the spectral radius of a matrix $A$. The joint spectral radius (JSR) of $\mathcal{F}$, introduced by Rota and Strang [14], is defined as

$$\hat{\rho}(\mathcal{F}) = \lim_{k \to \infty} \max_{A \in \mathcal{F}_k} \|A\|^1/k,$$

and the generalized spectral radius of $\mathcal{F}$, initiated by Daubechies and Lagarias [2], is given by

$$\bar{\rho}(\mathcal{F}) = \limsup_{k \to \infty} \max_{A \in \mathcal{F}_k} \rho(A)^1/k.$$

Since the equality $\hat{\rho}(\mathcal{F}) = \bar{\rho}(\mathcal{F})$ has been established for any finite set of matrices [15, 16, 17], we shall not distinguish between them and designate an unified notation $\rho(\mathcal{F}) (= \hat{\rho}(\mathcal{F}) = \bar{\rho}(\mathcal{F}))$ throughout the paper. Another equivalent variational way of characterizing JSR is [14]

$$\rho(\mathcal{F}) = \inf_{\| \cdot \|} \max_{A \in \mathcal{F}} \|A\|, \quad (1)$$

where the infimum is taken over the set of all sub-multiplicative matrix norms. Whenever the infimum in (1) is attained (thus a minimum), the corresponding norm $\| \cdot \|_*$ will be called an extremal norm [18]. The definition (1) is somehow attractive in the sense that its estimation of $\rho(\mathcal{F})$ avoids the computation of long matrix products if $\| \cdot \|$ is available and is computable in an efficient way.

Earlier algorithms [19, 20] for computing or approximating the joint/generalized spectral radius mostly make use of the following three important inequalities

$$\max_{A \in \mathcal{F}_k} \rho(A)^1/k \leq \rho(\mathcal{F}) \leq \max_{A \in \mathcal{F}_k} \|A\|^1/k \quad (2)$$

for any $k \geq 1$. In general, however, such a brute-force approach is far from satisfactory and the highly slow convergence in general renders this estimation impractical to many problems, in particular, for those large-scale ones. In order to obtain better approximations within current computational capacity, many numerical methods were proposed during last decade. We categorize them into two main approaches.

The first approach is to try to construct the extremal norm $\| \cdot \|_*$ or to approximate it when it exists. One necessary and sufficient condition for the existence of an extremal norm is the non-defectiveness of the corresponding normalized matrix family [21], which is not algorithmically decidable [22]. In [23], the minimization was restricted to the set of ellipsoid norms, which can be efficiently approximated by current convex optimization algorithms. This approach provides a theoretical precision estimation of $\rho(\mathcal{F})$ in limited applicable cases. In [21, 24, 25], the minimization was confined to the set of complex polytope norms. The successful construction of such a polytope norm is not guaranteed in general, and it is more suitable to be used to verify the occurrence of the finiteness property (FP) of $\mathcal{F}$ [20], that is, to check the case when there is a
positive integer \( t \) such that
\[
\rho(\mathcal{F}) = \rho(A_{i_1} A_{i_2} \cdots A_{i_t})^{1/t}
\]
for some finite product \( A_{i_1} A_{i_2} \cdots A_{i_t} \in \mathcal{F} \), and the corresponding product sequence is called an optimal sequence. Within this framework, other special extremal norms, such as Barabanov norm \([18]\), Optimal norm \([27]\), etc. are also considered. Kozyakin in \([28]\) considered an iterative algorithm which approximates \( \rho(\mathcal{F}) \) through constructing a sequence of approximated Barabanov norms under the assumption of irreducibility, however, the computational cost is very high and the issue of estimating the convergence rate remains unsolved.

The sum of squares method investigated in \([29]\) was intended to approximate the extremal norm through a multivariate polynomial with norm-like quality under which the action of matrices becomes contractive. However, to obtain an analytic extremal norm expression is quite challenging and there seem no easy solutions so far.

The second approach makes use of the cone invariance of a given matrix set \( \mathcal{F} \) for computing its JSR when such a property exists \([30]\). In \([30, 31]\), an iterative algorithm building an approximated invariant set was developed, which for a fixed dimension demonstrates polynomial time complexity with respect to \( 1/\varepsilon \), where \( \varepsilon \) is a given accuracy. In \([32]\), Blondel and Nesterov introduced a Kronecker lifting based approximation to the JSR with arbitrary accuracy under the assumption of the existence of an invariant proper cone, which can always be assured via one step of semi-definite lifting with the cost of squaring the matrix dimension. The exact nature of this cone is irrelevant to the derived accuracy of estimation. Following this methodology, a new conic programming method was offered in \([33]\), which gives an improved accuracy estimation by taking the nature of cone invariance into the consideration. In general, the existence of an invariant cone is restrictive and may exclude many interesting cases in real applications.

In numerical computation of joint/generalized spectral radius, criteria for determining if a given matrix family satisfies the FP can guide us to develop a decidable algorithm. The original finiteness conjecture \([26]\) stated that the FP is true for any finite \( n \times n \) real matrix family, which was recently proved to be false \([34, 35, 36]\). The existence of such counterexamples shows that the FP does not hold in general, however, it has been found since then that many classes of matrices possess this computationally favorable feature. In \([22]\), Blondel, Nesterov, and Theys proved the FP for the matrix family with a solvable Lie algebra and the matrix pair of form \( \{ A, A^T \} \). In \([37]\), the normal and commonly triangularizable matrix family were added into this list. Later on, a restricted version of finite conjecture claimed that the FP is true for every pair of \( n \times n \) sign-matrices \([38]\). The significance of this conjecture lies in its equivalence to the FP of all sets of rational matrices. Along this track, the case of \( 2 \times 2 \) sign-matrices pair was proved in \([39]\) with the exploration of real extremal polytope norms, but the similar conclusion for higher dimension remains unknown. Currently, the list of matrix families satisfying FP is still very short.
Rank-one matrices are the simplest class of matrices not only in theo retic analysis but also in algorithmic approximations for matrix computation since any matrix can be expressed in terms of the sum of a set of rank-one matrices, for example, in the singular value decomposition (SVD). Among all those illustrative examples appeared in existing literature related to JSR, we observed that all the cases with only rank-one matrices satisfy the finiteness conjecture. This motivates us to consider how to obtain the JSR of finite rank-one matrix family and then apply this finding to approximate the JSR of general matrix family.

The main contributions of this paper are: (i) by making use of Barabanov norm and rank-one property, we show that any finite set of rank-one matrices satisfies the finiteness property and the computation for its JSR is derived; (ii) based on the obtained result in (i), a new characterization of the JSR of general matrix family is obtained by exploiting the rank-one approximation based on SVD, which provides us a new computational scheme with some favorable numerical performance over existing algorithms.

The paper is organized as follows. In section 2 we prove that a finite set of rank-one matrices possesses the finiteness property and some important properties for the computation of its JSR are given. In section 3, we further develop this idea in the study of general matrix family. A rank-one approximation scheme based on SVD for approximating JSR is developed and a new characterization is obtained. Further discussions for non-negative matrix family are also presented in this section. Several benchmark examples from real applications as well as their numerical simulations are presented in section 4. The paper ends with concluding remarks in section 5.

2. Finite rank-one matrix family

We first give some well-known properties of rank-one matrices, which will be employed in our subsequent derivations. Given a matrix $A \in \mathbb{C}^{n \times n}$, we denote by $\text{rank}(A)$ and $\text{tr}(A)$ the rank and trace of $A$, respectively. We know from linear algebra that $\text{rank}(A) = 1$ if and only if there exist two nonzero vectors $x, y \in \mathbb{C}^n$ such that $A = xy^*$. Obviously, any rank-one matrix $A$ has at most one nonzero eigenvalue, denoted by $\lambda(A) = y^*x$. In particular, the spectral radius of a rank-one matrix $A$ is $\rho(A) = |\lambda(A)| = |\text{tr}(A)|$. For any two rank-one matrices $A_1 = x_1y_1^* \in \mathbb{C}^{n \times n}$ and $A_2 = x_2y_2^* \in \mathbb{C}^{n \times n}$, the product $A_1A_2 = x_1y_1^*x_2y_2^* = (y_1^*x_2)x_1y_2^*$ is at most rank-one. By a simple induction, arbitrary finite products of rank-one matrices remain at most rank-one.

2.1. The JSR of finite rank-one matrix family

In this subsection, we will show that any finite set $F$ of rank-one matrices possesses the finiteness property. If $\rho(F) = 0$, then by (2) it holds $\rho(A_i) = 0 = \rho(F)$ for all $1 \leq i \leq m$ and so the finiteness property is already true for $F$. Thus we will only consider $\rho(F) > 0$ case.

Recall that a general matrix family $F$ is said to be irreducible provided all the matrices in $F$ have no common non-trivial invariant linear subspaces
of \( \mathbb{C}^n \). The following lemma allows us to assume that \( \mathcal{F} \) is irreducible in the following; for otherwise, we could first reduce \( \mathcal{F} \) into several irreducible matrix families with smaller dimensions, and then carry out the same proof with each irreducible matrix family to draw the same conclusion.

**Lemma 1** ([15]). For any finite matrix family \( \mathcal{F} = \{A_1, A_2, \cdots, A_m\} \subset \mathbb{C}^{n \times n} \), there exist a nonsingular matrix \( P \in \mathbb{C}^{n \times n} \) and \( r \) positive integers \( \{n_1, n_2, \cdots, n_r\} \) with \( n_1 + n_2 + \cdots + n_r = n \) such that

\[
PA_iP^{-1} = \begin{bmatrix}
A_i^{(1)} & 0 & \cdots & 0 \\
* & A_i^{(2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & A_i^{(r)}
\end{bmatrix}
\text{ for } i = 1, 2, \cdots, m,
\]

where \( \mathcal{F}^{(j)} := \{A_1^{(j)}, A_2^{(j)}, \cdots, A_m^{(j)}\} \subset \mathbb{C}^{n_j \times n_j} \) is irreducible for \( j = 1, 2, \cdots, r \), satisfying

\[
\rho(\mathcal{F}) = \max_{1 \leq j \leq r} \rho(\mathcal{F}^{(j)}).
\]

Thus, without lose of generality, we may always assume the matrix family \( \mathcal{F} \) being irreducible. This leads to an important connection between the joint spectral radius and a special induced matrix norm, called extremal norm [37]. Due to the irreducibility of \( \mathcal{F} \), we can assume \( \rho(\mathcal{F}) = 1 \) by normalizing the matrix family \( \mathcal{F} \) by \( 1/\rho(\mathcal{F}) \), and thus it guarantees that normalized \( \mathcal{F} \) is non-defective, i.e., the semi-group of matrices generated by \( \mathcal{F} \) is bounded, and hence there exists an (invariant) extremal norm for \( \mathcal{F} \) as described in next lemma.

**Lemma 2** ([18]). For any finite irreducible matrix family \( \mathcal{F} \), there exists a vector (Barabanov) norm \( \| \cdot \|_B \) such that:

1. For all \( v \in \mathbb{C}^n \) and all \( A \in \mathcal{F} \) it holds that \( \|Av\|_B \leq \rho(\mathcal{F})\|v\|_B \).
2. For all \( v \in \mathbb{C}^n \), there exists an \( A \in \mathcal{F} \) such that \( \|Av\|_B = \rho(\mathcal{F})\|v\|_B \).

In particular, the induced matrix norm \( \| \cdot \|_B \) is an extremal norm satisfying

\[
\max_{A \in \mathcal{F}} \|A\|_B = \rho(\mathcal{F}).
\]

We are ready to prove the finiteness property of irreducible rank-one matrix family.

**Theorem 3.** Let \( \mathcal{F} = \{A_i = x_iy_i^* : i = 1, 2, \cdots, m\} \subset \mathbb{C}^{n \times n} \) be an irreducible rank-one matrix family. Then \( \mathcal{F} \) has the finiteness property and the corresponding optimal product sequence of minimal length has distinct factors.

**Proof.** We first normalize \( \mathcal{F} \) such that \( \rho(\mathcal{F}) = 1 \). By Lemma 2 choose \( v \in \mathbb{C}^n \) with \( \|v\|_B = 1 \), then for any \( k \geq 1 \) there exists a multi-index \( (i_1, i_2, \cdots, i_k) \) such that

\[
1 = \|v\|_B = \|A_{i_1}v\|_B = \|A_{i_1}A_{i_2}v\|_B = \cdots = \|A_{i_k} \cdots A_{i_2}A_{i_1}v\|_B.
\] (3)
By the pigeonhole principle, if \( k \geq (m + 1) \), then the multi-index \( (i_1, i_2, \ldots, i_k) \) has at least one repeated index. We define \( s \) to be the maximum of those \( k \)'s such that the corresponding multi-index \( (i_1, i_2, \ldots, i_k) \) satisfying (3) has no repetition. It's obvious that \( s \leq m \). Then, choosing \( k = s + 1 \) in (3) gives \( i_{s+1} = i_j \) for some unique \( 1 \leq j \leq s \), that is,

\[
1 = \|v\|_B = \cdots = \|A_{i_j} \cdots A_i v\|_B = \cdots = \|A_{i_{s+1}} A_{i_s} \cdots A_{i_j} A_{i_j} \cdots A_i v\|_B.
\]

Since \( A_{i_{s+1}} = A_{i_j} \) is rank-one matrix, its range is one-dimensional and hence

\[
A_{i_j} \cdots A_i v = \alpha z \quad \text{and} \quad A_{i_{s+1}} A_{i_s} \cdots A_{i_j} A_{i_j} \cdots A_i v = \beta z
\]

for some \( 0 \neq \alpha \in \mathbb{C}, 0 \neq \beta \in \mathbb{C} \), and \( 0 \neq z \in \mathbb{C}^n \) (we may choose \( z = x_{i_j} \) here). Then

\[
\|\alpha z\|_B = \|A_{i_j} \cdots A_i v\|_B = 1 = \|A_{i_{s+1}} A_{i_s} \cdots A_{i_j} A_{i_j} \cdots A_i v\|_B = \|\beta z\|_B,
\]

which gives \( |\alpha| = |\beta| \). Finally, we obtain

\[
\beta z = A_{i_{s+1}} A_{i_s} \cdots A_{i_j+1} (A_{i_j} \cdots A_i v) = A_{i_{s+1}} A_{i_s} \cdots A_{i_j+1} (\alpha z)
\]

and hence

\[
A_{i_{s+1}} A_{i_s} \cdots A_{i_j+1} z = \frac{\beta}{\alpha} z,
\]

where \( \frac{\beta}{\alpha} \) is an eigenvalue of \( A_{i_{s+1}} A_{i_s} \cdots A_{i_j+1} \). Therefore, by Lemma 2,

\[
1 \geq \|A_{i_{s+1}} A_{i_s} \cdots A_{i_j+1}\|_B \geq \rho(A_{i_{s+1}} A_{i_s} \cdots A_{i_j+1}) \geq \left| \frac{\beta}{\alpha} \right| = 1,
\]

which proves that \( \mathcal{F} \) has the finiteness property with

\[
\rho(\mathcal{F}) = 1 = \rho(A_{i_{s+1}} A_{i_s} \cdots A_{i_j+1})^{1/(s-j+1)},
\]

where \( 1 \leq (s - j + 1) \leq m \) and \( i_{s+1} \neq i_s \neq \cdots \neq i_{j+1} \) by the choice of \( s \).

We remark here that Theorem 5 provides us a critical structure of an optimal sequence, which will greatly improve the efficiency of specially designed search algorithms. In particular, non-repeated index indicates that the lengths of all minimal optimal sequences will not be longer than \( m \). In fact, the possible minimal optimal sequence with longest length is \( A_1 A_2 \cdots A_m \). In summary, an explicit formula for the JSR of any rank-one matrix family is given by \( \mathcal{F} = \{A_1, A_2, \cdots, A_m\} \subset \mathbb{C}^{n \times n} \) as

\[
\rho(\mathcal{F}) = \max_{1 \leq k \leq m} \left( \max_{A \in \mathcal{F}_k} \rho(A)^{1/k} \right), \tag{4}
\]

where \( \mathcal{F}_k = \{A_{i_1} A_{i_2} \cdots A_{i_k} \in \mathcal{F}_k : i_s \neq i_t \text{ when } s \neq t \} \) denotes all possible products in \( \mathcal{F}_k \) with distinct factors.
2.2. Two simple examples from current literature

In this subsection, we verify our foregoing results by two toy examples. The formula (4) provides a straightforward way to calculate the JSR for a rank-one matrix family. The search of all possible products with distinct factors of length not exceeding $m$ is sufficient to obtain the exact value of $\rho(F)$, however, most of current numerical approximation methods can only provide lower and upper bounds for JSR with no indication whether the JSR has been achieved. Moreover, our formula (4) is fully validated by the reported optimal sequences for any pair of rank-one $2 \times 2$ sign-matrices in [39].

**Example 1 ([39]).** Consider the rank-one matrix pair

$$F = \left\{ A_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$  

Applying the formula (4) to obtain

$$\rho(F) = \max_{1 \leq k \leq 2} \max_{A \in F_k^*} \rho(A)^{1/k} = \max\{\rho(A_1), \rho(A_2), \rho(A_1A_2)^{1/2}\} = \sqrt{2}.$$  

While in [39] this was solved by constructing an extremal real polytope norm.

**Example 2 ([21]).** Consider the rank-one matrix family

$$F = \left\{ A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}, A_4 = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$  

Using the formula (4) to get

$$\rho(F) = \max_{1 \leq k \leq 4} \max_{A \in F_k^*} \rho(A)^{1/k} = 1.$$  

The same conclusion was derived in [21] by observing relations among all matrices, whose approach is hard to be applied to general cases.

3. Rank-one approximation of JSR

Although the JSR formula for rank-one matrix family is now available, its applicability is highly restricted since rank-one matrix family rarely occurs in practice. Therefore, in following two subsections we develop an approximation approach to expand its horizon of application. This method imitates the conventional definition of $\rho(F)$ in terms of limit superior and provides a new viewpoint on the approximation of JSR. The main idea results from the fact that the rank of any matrix products for a given set of matrices does not increase as the multiplication continues. This property provides us a feasible approach.
3.1. General matrix family

In this subsection, we will introduce a natural and insightful way of approximating $\rho(\mathcal{F})$ by utilizing previous results on rank-one matrix family. The key idea is to perform the rank-one approximation of $\mathcal{F}_k$ successively as $k$ increases. Let $A \in \mathbb{C}^{n \times n}$, from its singular value decomposition (SVD) we have the following rank-one decomposition

$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^\ast,$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the singular values, $u_i$ and $v_i$ are the $i$th left and right singular vector respectively. The customary best rank-one approximation of $A$ is trying to minimize $\|A - R\|_F$ over all rank-one matrices $R$, which is achieved by choosing $R = \sigma_1 u_1 v_1^\ast$. For our approach, we will choose a special candidate, denoted by $P(A)$, which maximizes the absolute value of its trace (or spectral radius), i.e.,

$$P(A) \equiv \sigma_i' u_i' v_i'^\ast = \arg \max_{1 \leq i \leq n} |\text{tr}(\sigma_i u_i v_i^\ast)| = \arg \max_{1 \leq i \leq n} |\rho(\sigma_i u_i v_i^\ast)|$$

as the rank-one approximation of $A$. Clearly $\sigma_i' \leq \sigma_1 = \|A\|_2$. For the convenience of further discussion, we denote the element-wise rank-one approximation of $\mathcal{F} = \{A_1, A_2, \ldots, A_m\} \subset \mathbb{C}^{n \times n}$ by

$$P(\mathcal{F}) = \{P(A_1), P(A_2), \ldots, P(A_m)\},$$

where $P(A_i)$ is the rank-one approximation of $A_i$ as defined above. Notice that $P(\mathcal{F}_k)$ is a finite rank-one matrix family, thus $\rho(P(\mathcal{F}_k))$ can be obtained by the formula (4). Our next result needs the trace characterization of JSR by Chen and Zhou [40]

$$\rho(\mathcal{F}) = \limsup_{k \to \infty} \max_{\mathcal{F}_k} \left|\text{tr}(A)\right|^{1/k}. \tag{5}$$

**Theorem 4.** For any finite matrix family $\mathcal{F} = \{A_1, A_2, \ldots, A_m\} \subset \mathbb{C}^{n \times n}$, there holds

$$\rho(\mathcal{F}) = \limsup_{k \to \infty} \rho(P(\mathcal{F}_k))^{1/k}. \tag{6}$$

**Proof.** Let $\| \cdot \|_2$ denotes the spectral matrix norm. To derive our conclusion, we will prove (6) by validating two inequalities as shown below.

Firstly, given any $k \geq 1$, for the finite rank-one matrix family $P(\mathcal{F}_k)$, we have

$$\rho(P(\mathcal{F}_k)) = \max_{1 \leq l \leq m^k} \max_{R \in P(\mathcal{F}_k)_l} \rho(R)^{1/l},$$

where $m^k$ is the cardinality of the set $P(\mathcal{F}_k)$. For any $R \in P(\mathcal{F}_k)_l$, there exist $l$
matrices $B_j (1 \leq j \leq l)$ with each $B_j \in \mathcal{F}_k$ such that

$$R = \prod_{j=1}^{l} P(B_j) = \prod_{j=1}^{l} \sigma_j u_j v_j^*,$$

where $P(B_j) = \sigma_j u_j v_j^*$ with $\sigma_j \leq \|B_j\|_2$, $u_j$ and $v_j$ are the corresponding left and right singular vectors. By Cauchy inequality there holds

$$\rho(R) \leq \prod_{j=1}^{l} \sigma_j \leq \prod_{j=1}^{l} \|B_j\|_2 \leq \left( \max_{A \in \mathcal{F}_k} \|A\|_2 \right)^l$$

for all $R \in \mathcal{P}(\mathcal{F}_k)_l$. Thus we have

$$\rho(\mathcal{P}(\mathcal{F}_k)) \leq \max_{1 \leq l \leq m_k} \left( \max_{A \in \mathcal{F}_k} \|A\|_2 \right) = \max_{A \in \mathcal{F}_k} \|A\|_2,$$

which leads to

$$\limsup_{k \to \infty} \rho(\mathcal{P}(\mathcal{F}_k))^{1/k} \leq \limsup_{k \to \infty} \left( \max_{A \in \mathcal{F}_k} \|A\|_2 \right)^{1/k} = \rho(\mathcal{F}). \quad (7)$$

Secondly, for any $k \geq 1$, let

$$B_k = \arg\max_{A \in \mathcal{F}_k} |\text{tr}(A)|.$$

Upon expressing the SVD of $B_k$ as

$$B_k = \sum_{i=1}^{n} \sigma_i u_i v_i^*,$$

the linearity of trace operator gives

$$|\text{tr}(B_k)| \leq \sum_{i=1}^{n} |\text{tr}(\sigma_i u_i v_i^*)| \leq n \left( \max_{1 \leq i \leq n} |\text{tr}(\sigma_i u_i v_i^*)| \right) = n \rho(\mathcal{P}(B_k)).$$

Therefore, for $k \geq 1$ we have

$$\rho(\mathcal{P}(\mathcal{F}_k))^{1/k} \geq \rho(\mathcal{P}(B_k))^{1/k} \geq (n^{-1} |\text{tr}(B_k)|)^{1/k} = n^{-1/k} \left( \max_{A \in \mathcal{F}_k} |\text{tr}(A)| \right)^{1/k},$$

which in together with the equality (5) gives

$$\limsup_{k \to \infty} \rho(\mathcal{P}(\mathcal{F}_k))^{1/k} \geq \limsup_{k \to \infty} n^{-1/k} \left( \max_{A \in \mathcal{F}_k} |\text{tr}(A)| \right)^{1/k} = \rho(\mathcal{F}). \quad (8)$$

Finally, combining (7) and (8) the proof is completed. \qed

The above result reveals that the rank-one approximation of a matrix family
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will approximate its JSR in sense of limit superior. Observing the left and right singular vector pair of a normal matrix are orthonormal, the following result generalizes the one given by [37] on normal matrix family.

**Corollary 1.** For any finite matrix family \( \mathcal{F} = \{A_1, A_2, \ldots, A_m\} \subset \mathbb{C}^{n \times n} \), if there exists a sequence \( A_1, A_2 \cdots A_j \in \mathcal{F} \) is normal, then \( \rho(\mathcal{F}) \geq \|A_1 A_2 \cdots A_j\|^{1/j}_2 \).

In particular, if

\[
\|A_1, A_2 \cdots A_j\|^{1/j}_2 = \max_{A \in \mathcal{F}_j} \|A\|^{1/j}_2
\]

then we have

\[
\rho(\mathcal{F}) = \|A_1 A_2 \cdots A_j\|^{1/j}_2.
\]

**Proof.** We conduct the SVD for the matrix \( A = A_1 A_2 \cdots A_j \). This gives \( A = U D U^* \), where \( U \) is an unitary matrix, and \( D \) is diagonal. Now consider the sequence \( \{A^\ell\}_{\ell=1}^\infty \). Notice that \( A^\ell = U D^\ell U^* \) and \( \rho(P(A^\ell)) = \|A\|^2_\ell \). Thus

\[
\lim_{k \to \infty} \rho(P(F_k))^{1/k} \geq \lim_{\ell \to \infty} \rho(P(A^\ell))^{1/\ell j} = \|A\|^{1/j}_2.
\]

According to Theorem 3, the conclusion thus follows.

In order to demonstrate the effectiveness of our approach, let’s consider the following well-known 2 \( \times \) 2 irreducible matrix pair

\[
\mathcal{F} = \left\{ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = b \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}
\]

with \( b > 0 \), which was recently employed to disprove the finiteness conjecture [35]. The authors in [24] show that constructing an exact real polytope extremal norm along with computational investigation that \( \rho(\mathcal{F}) = \sigma_1 \sqrt{b} \) when \( b \in [\frac{4}{5}, 1] \) and the minimal optimal sequence is \( A_1 A_2 \).

Note that the SVD of \( A_1 \) (calculated by Mathematica 8) is given by

\[
A_1 = \begin{bmatrix} u_1 & u_2 \\ \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^*
\]

\[
= \begin{bmatrix} \frac{1+\sqrt{5}}{\sqrt{2}(5+\sqrt{5})} & \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}} \\ \frac{\sqrt{5}+1}{2} & 0 \\ \frac{\sqrt{5}-1}{2} & \sqrt{10-2\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}+1}{2} & 0 \\ \frac{\sqrt{5}-1}{2} & \sqrt{10-2\sqrt{5}} \end{bmatrix}^*,
\]

and thus the SVD of \( A_2 \) is

\[
A_2 = b A_1^* = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} b\sigma_1 & 0 \\ 0 & b\sigma_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix}^*.
\]

Direct calculation yields \( u_1^* u_1 = v_1^* v_1 = 1 \) and \( v_1^* u_1 = u_1^* v_1 = \frac{2}{\sqrt{5}} \). Hence, the rank-one approximation of \( \mathcal{F} \) based on SVD is

\[
P(\mathcal{F}_1) = P(\mathcal{F}) = \{\sigma_1 u_1 v_1^*, b\sigma_1 v_1 u_1^*\}.
\]
According to (4), if $b \in [\frac{4}{5}, 1]$ then there holds
\[
\rho(P(F_1)) = \max \left\{ \sigma_1 v_1^* u_1, b \sigma_1 u_1^* v_1, \sigma_1 \sqrt{b} \sqrt{u_1^* u_1} \cdot v_1^* v_1 \right\}
\]
\[
= \sigma_1 \cdot \max \left\{ \frac{2}{\sqrt{5}} b, \frac{2}{\sqrt{5}} \sqrt{b} \right\} = \sigma_1 \sqrt{b} = \rho(F).
\]
Here only one step of rank-one approximation gives $\rho(F)$.

3.2. Nonnegative matrix family

In numerical implementation, it would be more favorable to have a limit rather than a limit superior in (6) because the former can be evidently observed within sufficient steps of approximation. In this subsection, we are going to further develop the limit property for nonnegative matrix family.

In this paper by the notation $A \geq 0$ and $A > 0$ we mean the matrix $A$ is nonnegative and positive in entry-wise sense, respectively. Suppose the considering matrix family $\mathcal{F} = \{A_1, A_2, \ldots, A_m\} \subset \mathbb{R}^{n \times n}$ is nonnegative, i.e., $A_i \geq 0$, there is an elegant limit expression of JSR established by Blondel and Nesterov [32]
\[
\rho(\mathcal{F}) = \lim_{k \to \infty} \rho^{1/k}(A_1^\otimes k + \ldots + A_m^\otimes k),
\]
where $A_i^\otimes k$ represents the $k$-th Kronecker power of $A_i$. This expression will play an important role in our following approach. For square matrices of the same size, the following properties for Kronecker product can be found in a standard matrix analysis textbook:

(i) $(A_i \otimes A_j)(A_s \otimes A_t) = (A_i A_s) \otimes (A_j A_t)$;

(ii) $\text{tr} (A_{i_1} A_{i_2} \ldots A_{i_\ell})^\otimes k = \text{tr}^k (A_{i_1} A_{i_2} \ldots A_{i_\ell})$ for any positive integer $k$;

(iii) $(A^\otimes k)^\ell = (A^\ell)^\otimes k$ for any positive integers $k, \ell$.

Recall that a square matrix $A \geq 0$ is called to be primitive if $A^\ell > 0$ for some integer $\ell \geq 1$. It is easy to see that if $A$ is primitive, then $A^\otimes k$ is also primitive according to (iii). An important property for a primitive matrix is that its spectral radius can be expressed as
\[
\rho(A) = \lim_{k \to \infty} \text{tr}^{1/k}(A^k),
\]
instead of limit superior in general cases.

**Lemma 5.** Let $\mathcal{F} = \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n \times n}$ be a family of nonnegative matrices. If there exists an integer $j \geq 1$ such that $A_{i_1} A_{i_2} \cdots A_{i_j} \in \mathcal{F}_j$ is primitive, then for any $k \geq 1$ the matrix $A_1^\otimes k + \ldots + A_m^\otimes k$ is primitive.
Proof. Since $A_{i_1}A_{i_2}\cdots A_{i_j}$ is primitive, there exists a positive integer $\ell$ such that
\[ (A_{i_1}A_{i_2}\cdots A_{i_j})^\ell > 0. \]
The conclusion follows from the observation
\[ \left[ (A\otimes k_1 + \cdots + A\otimes k_m)^j \right]^\ell \geq \left[ (A_{i_1}A_{i_2}\cdots A_{i_j})^\otimes k \right]^\ell = \left[ (A_{i_1}A_{i_2}\cdots A_{i_j})^{\otimes k} \right] > 0. \]

The following lemma generalizes a recent result given by Xu [41].

Lemma 6. Let $F = \{ A_1, \ldots, A_m \} \subset \mathbb{R}^{n \times n}$ be a family of nonnegative matrices. If there exists an integer $j \geq 1$ such that $A_{i_1}A_{i_2}\cdots A_{i_j} \in F_j$ is primitive, then we have
\[ \rho(F) = \lim_{l \to \infty} \max_{A \in F_l} \text{tr}^{1/l}(A). \]

Proof. For any integers $k, l \geq 1$, there holds
\[ \text{tr} \left( A_1^{\otimes k} + \cdots + A_m^{\otimes k} \right)^l = \text{tr} \left[ \sum_{1 \leq i_1, \ldots, i_l \leq m} (A_{i_1}\cdots A_{i_l})^{\otimes k} \right] = \sum_{1 \leq i_1, \ldots, i_l \leq m} \text{tr} (A_{i_1}\cdots A_{i_l})^{\otimes k} = \sum_{1 \leq i_1, \ldots, i_l \leq m} \text{tr}^k (A_{i_1}\cdots A_{i_l}), \]
i.e.
\[ \text{tr} \left( A_1^{\otimes k} + \cdots + A_m^{\otimes k} \right)^l = \sum_{A \in F_l} \text{tr}^k (A), \]
where the cardinality of $F_l$ is $m^l$. This leads to
\[ \left[ \text{tr}^{1/l} \left( A_1^{\otimes k} + \cdots + A_m^{\otimes k} \right) \right]^{1/k} \leq m^{1/k} \left( \max_{A \in F_l} \text{tr}^{1/l}(A) \right). \]
By assumption that $A_{i_1}A_{i_2}\cdots A_{i_j} \in F_j$ is primitive for some $j \geq 1$, we know from Lemma 5 that $A_1^{\otimes k} + \cdots + A_m^{\otimes k}$ is primitive for all positive integer $k$. For a fixed $k \geq 1$, by taking the limit inferior over $l$ on both sides and noting (10), we have
\[ \rho^{1/k}(A_1^{\otimes k} + \cdots + A_m^{\otimes k}) \leq m^{1/k} \liminf_{l \to \infty} \max_{A \in F_l} \text{tr}^{1/l}(A). \]
Now by letting $k \to \infty$ and utilizing (9) we obtain
\[ \rho(F) \leq \liminf_{l \to \infty} \max_{A \in F_l} \text{tr}^{1/l}(A). \]
By combining this with the known equality

$$\rho(\mathcal{F}) = \limsup_{l \to \infty} \max_{A \in \mathcal{F}_l} \frac{1}{l} \text{tr}(A)$$

we derive

$$\limsup_{l \to \infty} \max_{A \in \mathcal{F}_l} \frac{1}{l} \text{tr}(A) = \rho(\mathcal{F}) \leq \liminf_{l \to \infty} \max_{A \in \mathcal{F}_l} \frac{1}{l} \text{tr}(A),$$

which leads to the conclusion.

**Corollary 2.** Let $\mathcal{F} = \{A_1, \cdots, A_m\} \subset \mathbb{R}^{n \times n}$ be a family of nonnegative matrices. If there exists an integer $j \geq 1$ such that $A_{i_1} A_{i_2} \cdots A_{i_j} \in \mathcal{F}_j$ is primitive, then there holds

$$\rho(\mathcal{F}) = \lim_{k \to \infty} \rho(\mathcal{P}(\mathcal{F}_k))^{1/k}. \quad (11)$$

**Proof.** Similar to the proof of Theorem [4] by repeating the argument in the first part it follows the inequality

$$\limsup_{k \to \infty} \rho(\mathcal{P}(\mathcal{F}_k))^{1/k} \leq \rho(\mathcal{F}). \quad (12)$$

While in the second part, we have

$$\rho(\mathcal{P}(\mathcal{F}_k))^{1/k} \geq n^{-1/k} \left( \max_{A \in \mathcal{F}_k} |\text{tr}(A)| \right)^{1/k}$$

for all $k \geq 1$, taking the limit inferior on both sides and employing Lemma [10] gives

$$\liminf_{k \to \infty} \rho(\mathcal{P}(\mathcal{F}_k))^{1/k} \geq \lim_{k \to \infty} n^{-1/k} \left( \max_{A \in \mathcal{F}_k} |\text{tr}(A)| \right)^{1/k} = \rho(\mathcal{F}). \quad (13)$$

According to (12) and (13), we thus have

$$\limsup_{k \to \infty} \rho(\mathcal{P}(\mathcal{F}_k))^{1/k} \leq \rho(\mathcal{F}) \leq \liminf_{k \to \infty} \rho(\mathcal{P}(\mathcal{F}_k))^{1/k},$$

which implies the conclusion. \qed

The primitive condition in Corollary 2 is verifiable in numerical computations since a nonnegative matrix $A \in \mathbb{R}^{n \times n}$ is primitive if and only if $A^{n^2 - 2n + 2}$ is positive [42]. Moreover, such an approximation to JSR is earlier to be observed than the limit superior as $k$ increases, which will be illustrated in following numerical examples.
4. Numerical examples

To demonstrate the effectiveness of our proposed approach, we next present numerical simulations on several important examples from current literature. All experiments are performed on MATLAB 7.10 with a machine precision of $10^{-16}$. For $k \geq 1$, we denote
\[
\hat{\rho}_k(F) = \max_{A \in F_k} \| A \|_2^{1/k}, \quad \bar{\rho}_k(F) = \max_{A \in F_k} \rho(A)^{1/k}, \quad \text{and} \quad \tilde{\rho}_k(F) = [\rho(P(F_k))]^{1/k}.
\]

4.1. Generalized partition function

In number theory, a long-lasting problem is to estimate the asymptotic growth of the generalized partition function $f_{p,c}(t)$ defined as the total number of different $p$-adic expansions $t = \sum_{j=0}^{\infty} c_j p^j$ with $c_j \in \{0, 1, \ldots, c - 1\}$. It has been shown that for given positive integers $p$ and $c$ there exist positive constants $C$ and $\gamma$ such that $f_{p,c}(t) \geq C t^\gamma$ as $t \to \infty$. Moreover, there is a procedure [13] to construct a family of binary matrices $F$ dependent on $p$ and $c$ with the relation $\rho(F) = p^\gamma$.

In [33], a conic programming approach was proposed to approximate the JSR of this matrix family $F$ of dimension $7 \times 7$ with $(p, c) = (3, 14)$. In this case the matrix family is given by
\[
F = \left\{ A_1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix},
A_2 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix},
A_3 = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}\right\}.
\]

According to [33] the estimated interval of $\rho(F)$ is given by $[4.72, 4.8]$, where the lower bound is attained by using the sequence $A_1A_2$ and the upper bound is searched through the conic algorithm among all possible matrix products within length $k \leq 9$. There is no indication what is the value of $\rho(F)$ in their approach.

The first step rank-one approximation $P(F)$ by formula (4) gives
\[
\rho(P(F)) \approx 4.7915415825,
\]
which falls into the interval $[4.72, 4.8]$. The numerical values of $\hat{\rho}_k(F)$, $\bar{\rho}_k(F)$, and $\tilde{\rho}_k(F)$ for $k \geq 1$ are plotted and shown in Fig. and Table respectively.
Table 1: The values of $\bar{\rho}_k(F), \hat{\rho}_k(F), \tilde{\rho}_k(F)$ with respect to $k$, $(p, c) = (3, 14)$.

| $k$ | $\bar{\rho}_k(F)$ | $\hat{\rho}_k(F)$ | $\tilde{\rho}_k(F)$ | $|\tilde{\rho}_k(F) - \rho^{1/2}(A_1A_2)|$ |
|-----|-----------------|-----------------|-----------------|-------------------------------|
| 1   | 5.262878        | 4.6690790883    | 4.7915415825    | 6.95e-02                      |
| 2   | 5.046134        | 4.7220451340    | 4.7208642368    | 1.18e-03                      |
| 3   | 4.936157        | 4.7122439907    | 4.7216905518    | 3.55e-04                      |
| 4   | 4.881518        | 4.7220451340    | 4.7220575153    | 1.24e-05                      |
| 5   | 4.849140        | 4.7164125255    | 4.7220470073    | 1.87e-06                      |
| 6   | 4.827731        | 4.7220451340    | 4.7220461006    | 9.67e-07                      |
| 7   | 4.812488        | 4.7180343424    | 4.7220452529    | 1.19e-07                      |
| 8   | 4.801089        | 4.7220451340    | 4.7220451879    | 5.39e-08                      |

Figure 1: The values of $\bar{\rho}_k(F), \hat{\rho}_k(F), \tilde{\rho}_k(F)$ with respect to $k$, $(p, c) = (3, 14)$.

4.2. Asymptotics of Overlap-free words

A word on the binary alphabet $\{a, b\}$ is said to be overlap-free if it has no subwords (or factors) of the form $xwxwx$, where $x \in \{a, b\}$ and $w$ could be a word or empty. For instance, the word $baabaab$ is overlap-free, but $baababab$ is not. The asymptotic growth of the number $t_l$ of binary overlap-free words of length $l$ could be expressed in terms of the JSR of a matrix pair $F$ as

$$\limsup_{l \to \infty} \frac{\ln t_l}{\ln l} = \log_2 \rho(F),$$

Notice that the product $A_1A_2A_3$ is positive, according to Corollary 2, we know there holds $\lim_{k \to \infty} \tilde{\rho}_k(F) = \rho(F)$. Therefore, we also compute the absolute difference between $\tilde{\rho}_k(F)$ and $\rho^{1/2}(A_1A_2)$, which provides strongly numerical evidence that $A_1A_2$ is the optimal sequence of $\rho(F)$.
The set $\mathcal{F} = \{A_1, A_2\} \subset \{0, 1, 2, 4\}^{20 \times 20}$ is given by

$$A_1 = \begin{bmatrix} C & 0 \\ D & B \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} D & B \\ 0 & C \end{bmatrix}$$

with sub-matrices

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
presented in Fig. 2 and Table 2 respectively. By observing that

$$|\tilde{\rho}_k(F) - \rho(A_1A_2)^{1/2}| \leq 5 \times 10^{-4}$$

when \( k \geq 9 \), it is reasonable to believe that their conjecture is true.

![Figure 2: The values of \( \tilde{\rho}_k(F), \hat{\rho}_k(F), \check{\rho}_k(F) \) with respect to \( k \).](image)

| \( k \) | \( \tilde{\rho}_k(F) \) | \( \hat{\rho}_k(F) \) | \( \check{\rho}_k(F) \) | \( |\tilde{\rho}_k(F) - \rho(A_1A_2)^{1/2}| \) |
|---|---|---|---|---|
| 1 | 5.896964 | 2.4206250653 | 1.0000000000 | 1.52e+00 |
| 2 | 3.309093 | 2.5179340409 | 2.4799585961 | 3.80e-02 |
| 3 | 3.029307 | 2.5048603453 | 2.527922425 | 1.00e-02 |
| 4 | 2.924657 | 2.5179340409 | 2.549319895 | 2.80e-02 |
| 5 | 2.841023 | 2.5080155383 | 2.521385520 | 2.20e-03 |
| 6 | 2.778162 | 2.5179340409 | 2.5268628549 | 8.93e-03 |
| 7 | 2.736156 | 2.5099337275 | 2.5190358732 | 1.10e-03 |
| 8 | 2.705763 | 2.5179340409 | 2.5199752844 | 2.04e-03 |
| 9 | 2.687999 | 2.518420373 | 2.5180910647 | 1.57e-04 |
| 10 | 2.669268 | 2.5179340409 | 2.5184122994 | 4.78e-04 |
| 11 | 2.654756 | 2.5129654473 | 2.5179476144 | 1.36e-05 |
| 12 | 2.642173 | 2.5179340409 | 2.5180554550 | 1.21e-04 |
| 13 | 2.632798 | 2.5137397302 | 2.517939058 | 5.86e-06 |
4.3. Matrix pair with a rank-one member

Consider the matrix pair

\[ \mathcal{F} = \left\{ A_1 = \begin{bmatrix} 1 & \frac{1}{\sqrt{7}} \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \right\}. \]

It is easy to show that

\[ \rho(\mathcal{F}) = \max_{\ell \geq 1} \left( \frac{\ell}{\sqrt{17}} \right)^{\frac{\ell}{\sqrt{7}}} = \left( \frac{8}{\sqrt{7}} \right)^{1/9} \approx 1.130819895422034 \]

with the optimal sequence \( A_1^8A_2 \). The numerical values of \( \hat{\rho}_k(\mathcal{F}), \tilde{\rho}_k(\mathcal{F}), \) and \( \tilde{\rho}_k(\mathcal{F}) \) are plotted in Fig. 3 and reported in Table 3, respectively. One can see that for small \( k \), \( \tilde{\rho}_k(\mathcal{F}) \) approaches \( \rho(\mathcal{F}) \) much better than \( \hat{\rho}_k(\mathcal{F}) \) and \( \bar{\rho}_k(\mathcal{F}) \). Moreover, the approximation \( \tilde{\rho}_9(\mathcal{F}) \) does provide the exact value of \( \rho(\mathcal{F}) \), because \( A_1^8A_2 \) is rank-one.

Figure 3: The values of \( \hat{\rho}_k(\mathcal{F}), \tilde{\rho}_k(\mathcal{F}), \bar{\rho}_k(\mathcal{F}) \) with respect to \( k \).

4.4. An example with oscillated approximation

Consider the matrix pair

\[ \mathcal{F} = \left\{ A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \right\}. \]

The numerical values of \( \hat{\rho}_k(\mathcal{F}), \tilde{\rho}_k(\mathcal{F}), \) and \( \bar{\rho}_k(\mathcal{F}) \) are plotted and reported in Fig. 4 and Table 4, respectively. Although this matrix family does not satisfy the assumption of Corollary 2, one can easily show that \( \rho(\mathcal{F}) = \lim_{k \to \infty} \hat{\rho}_k(\mathcal{F}) \).
This example illustrates that the assumption of Corollary 2 is sufficient but not necessary.

5. Concluding Remarks

The computation of joint/generalized spectral radius has been proven to be challenging and difficult in current literature since both of them are the characteristics of the worst-case operation count which usually grows faster than
Table 4: The values of $\hat{\rho}_k(F)$, $\tilde{\rho}_k(F)$, $\tilde{\rho}_k(F)$ with respect to $k$.

| $k$ | $\hat{\rho}_k(F)$ | $\tilde{\rho}_k(F)$ | $\hat{\tilde{\rho}}_k(F)$ | $|\hat{\rho}_k(F) - \rho(F)|$ |
|-----|-------------------|-------------------|------------------|------------------|
| 1   | 1.000000          | 1.000000000000    | 0.500000000000   | 5.00e-01         |
| 2   | 1.000000          | 1.000000000000    | 1.000000000000   | 0.00e+00         |
| 3   | 1.000000          | 1.000000000000    | 0.7937005260     | 2.06e-01         |
| 4   | 1.000000          | 1.000000000000    | 1.000000000000   | 0.00e+00         |
| 5   | 1.000000          | 1.000000000000    | 0.8705506533     | 1.29e-01         |
| 6   | 1.000000          | 1.000000000000    | 1.000000000000   | 0.00e+00         |
| 7   | 1.000000          | 1.000000000000    | 0.9057236643     | 9.43e-02         |
| 8   | 1.000000          | 1.000000000000    | 1.000000000000   | 0.00e+00         |
| 9   | 1.000000          | 1.000000000000    | 0.9258747123     | 7.41e-02         |
| 10  | 1.000000          | 1.000000000000    | 1.000000000000   | 0.00e+00         |
| 11  | 1.000000          | 1.000000000000    | 0.9389309107     | 6.11e-02         |
| 12  | 1.000000          | 1.000000000000    | 1.000000000000   | 0.00e+00         |
| 13  | 1.000000          | 1.000000000000    | 0.9480775143     | 5.19e-02         |

any polynomial in terms of matrix sizes. To identify which class of matrix families has finiteness property may lead to various efficient algorithms which can reduce the computational cost in a significant way.

In this paper, we show that any family of rank-one matrices possesses the finiteness property and an explicit formula of its joint/generalized spectral radius is obtained. Furthermore, by making use the rank-one approximation based on the singular value decomposition, a new characterization of joint/generalized spectral radius is derived. Numerical simulations on several benchmark examples from applications show great promise for our proposed approach as an alternative approximation to joint/generalized spectral radius. However, finding the theoretical convergence rate of our approach is still open.

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