On Lagrange duality theory for dynamics vaccination games

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Abstract

The authors study an infinite dimensional duality theory finalized to obtain the existence of a strong duality between a convex optimization problem connected with the management of vaccinations and its Lagrange dual. Specifically, the authors show the solvability of a dual problem using as basic tool an hypothesis known as Assumption S. Roughly speaking, it requires to show that a particular limit is nonnegative. This technique improves the previous strong duality results that need the nonemptiness of the interior of the convex ordering cone. The authors use the duality theory to analyze the dynamic vaccination game in order to obtain the existence of the Lagrange multipliers related to the problem and to better comprehend the meaning of the problem.

Keywords: Lagrange multipliers, Infinite dimensional duality theory, Convex problems.

1 Introduction

The main goal of the paper is to present general results for the infinite dimensional duality theory and to utilize them for analyzing dynamic vaccination games.
The duality theory (see [16, 17, 27]) firstly was defined to investigate the problem of finding, in the infinite dimensional framework, the Lagrange multipliers connected to an optimization problem or to a variational inequality subject to possibly nonlinear constraints when the interior of the ordering cone is empty. This research line, combined with a generalized constraint qualification assumption, the well known Assumption S, ensures the existence of the strong duality between a convex optimization problem and its Lagrange dual. The employment of the quasi-relative interior, firstly presented by Borwein and Lewis [9], and the concepts of tangent and normal, cone permit to go beyond the mentioned difficulty that in a lot of problems the interior of the constraint set is empty. For example, the reader can think about network equilibrium problems, the obstacle problem, the elastic-plastic torsion problem (see e.g. [4, 15, 20, 17, 27, 3, 6, 5, 19]), which all use positive cones of Lebesgue or Sobolev spaces. It is useful to point out that Assumption S is also a necessary condition to ensure the strong duality (see [10]).

In the present work, we focus our attention to archive the dual formulation of the dynamic vaccination game. In the last decade, several authors devoted their interest in the application of the game theory to vaccinating behavior under voluntary policies for human diseases [7, 8], for instance measles, mumps, pertussis and rubella [1]. In the above mentioned notes a homogeneous population where all individuals share the same perception of risk is assumed. Nevertheless, in real populations, risk perception can fluctuate a lot if we treat different social groups, or units, [21, 25]. The paper [14] deals with the dynamics of vaccinating behavior in a population divided into social units, each one having a different perceived risk of infection and vaccination, under a voluntary policy, via projected dynamical systems and variational inequalities.

In [24, 11, 8], it is underlined that, whether or not an individual decides to vaccinate, the perceived probability of their becoming infected rests on the level of disease prevalence. Disease prevalence depends on the vaccine coverage in the population (see [1]). If vaccination is voluntary, the disease prevalence is the combined effect of the vaccination decisions of other people. As a consequence, the individuals in a given population are in practice engaged in a crucial interaction (a game) with one another, utilized transmission dynamics. In particular, we stress the importance of the evolution of group’s equilibrium vaccinating strategies in a community divided into social units, each one having a special perceived risk of infection and vaccination. Let us consider an infectious disease, for which vaccination occurs shortly after birth, where parents voluntary choose to vaccinate their children, and in which individuals can be either susceptible, infectious, or immune. These are known as Susceptible-Infectious-Recovered (SIR) models, successfully tested in several epidemiology cases (see for instance [1]).

In [14] and in [12], population biology games have been firstly studied via variational inequalities. In [2] the authors analyze the evolution of the equilibrium strategies of each unit for a time interval $[0, T]$ and considered $p(t)$ a vaccination coverage function reflecting a vaccine scare happening in a population over $[0, T]$. An help to solve this argument is provided by generalized Nash games. Thanks to the theory of variational inequalities,
Barbagallo and Cojocaru present a method to obtain and compute a solution to this model. At last, in [13], hybrid system based on a continuous dynamics described by a projected system and on a finite set of exogenous event times is illustrated. Moreover, such hybrid system finds an application to track evolution strategies.

We stress that several problems arising from the theory of population dynamics and other physics phenomena are studied, not only for partial differential equations (see e.g. [28, 29]), but also by means of evolutionary variational inequalities (see for instance [22, 23]).

The paper is organized as follows. In Section 2 we recall the infinite dimensional duality theory. In Section 3 we present the dynamic vaccination game and show how Lagrange and duality theory can be applied to this model. After that, we are able to prove the existence of the Lagrange multipliers associated to the problem, which is very useful to study the behaviour of the vaccination game.

2 Lagrange duality

Let us fix $X$ a topological set, $Y$ a real normed space ordered by a convex cone $C$ with dual space $Y^*$, $Z$ a real normed space and $Z^*$ its dual space. Let us consider $S$ a convex subset of $X$, $f : S \to \mathbb{R}$ a functional, $g : S \to Y$ a mapping and $h : S \to Z$ an affine-linear mapping. Moreover, let us set

$$K = \{ x \in S : g(x) \in -C, h(x) = \theta_Z \},$$

where $\theta_Z$ is the zero element in the space $Z$.

Let us analyze the optimization problem to find $x_0 \in K$ such that

$$f(x_0) = \min_{x \in K} f(x).$$

Let us remark that its Lagrange dual problem is

$$\max_{u \in C^*, v \in Z^*} \inf_{x \in S} \{ f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle \}$$

where we indicate the usual dual cone of $C$ by

$$C^* = \{ u \in Y^* : \langle u, y \rangle \geq 0, \forall y \in C \}.$$

Let us now recall the definition of Assumption $S$. Fixed three functions $f, g, h$ as before and $K$ as in (1), we declare that Assumption $S$ is verified at a point $x_0 \in K$ if and only if

$$T_{\tilde{M}}(0, \theta_Y, \theta_Z) \cap (-\infty, 0] \times \{ \theta_Y \} \times \{ \theta_Z \} = \emptyset,$$

being

$$\tilde{M} = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)) : x \in S \setminus K, \alpha \geq 0, y \in C \}.$$
Throughout the exposition we indicate by $T_C(x)$ the tangent cone to $C$ at $x$ defined as

$$T_C(x) = \{ g \in X : g = \lim_{n \to \infty} \lambda_n (x_n - x), \lambda_n \in \mathbb{R}^+, \forall n \in N, x_n \in C, \forall n \in N, \lim_{n \to \infty} x_n = x \}.$$  

(7)

We are now enable to report the primary development on strong duality theory.

**Theorem 2.1** (see [17]) Let us assume $f : S \to \mathbb{R}$, $g : S \to Y$ two convex functions, $h : S \to Z$ an affine-linear map and Assumption S is fulfilled in the optimal solution $x_0 \in K$ to (2), thus, problem (3) is resolvable and, whenever $\bar{u} \in C^*$, $\bar{v} \in Z^*$ are the optimal solutions to (3), we purchase

$$\langle \bar{u}, g(x_0) \rangle = 0$$

(8)

and the optimal values of (3) coincides, specifically

$$f(x_0) = \max_{u \in C^*, v \in Z^*} \inf_{x \in S} (f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle).$$

(9)

Assumption S is archivable to accomplish that a suitable limit is larger or equals than zero. The present requirement is necessary in order to gain strong duality (see [10]).

It is time to formalize the coming Lagrange functional

$$\mathcal{L}(x, u, v) = f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle, \quad \forall x \in S, u \in C^*, v \in Z^*. $$

(10)

Consequently, (9) can be updated as follows

$$f(x_0) = \max_{u \in C^*, v \in Z^*} \inf_{x \in S} \mathcal{L}(x, u, v).$$

It develops, from what just considered, the below property

**Theorem 2.2** (see [16]). Let us assume that the same the assumptions of Theorem 2.1 are satisfied. Because of these, $x_0 \in K$ is an optimal solution of (2) if and only if there exist $\bar{u} \in C^*$ and $\bar{v} \in Z^*$ such that $(x_0, \bar{u}, \bar{v})$ is a saddle point of (10), equivalently

$$\mathcal{L}(x_0, u, v) \leq \mathcal{L}(x_0, \bar{u}, \bar{v}) \leq \mathcal{L}(x, \bar{u}, \bar{v}), \quad \forall x \in S, \forall u \in C^*, \forall v \in Z^*. $$

(11)

In addition

$$\langle \bar{u}, g(x_0) \rangle = 0.$$  

(12)


3 Applications to dynamic vaccination games

We introduce the dynamic vaccination game and investigate about its dual formulation, in order to show the existence of the Lagrange multipliers associated to the problem.

We regard a competition being played between population aggregations, possessing the distinctness that we program to ensure a reply match in $[0, T]$. Our preamble is that at any time interval $t \in [0, T]$ there is a community of fixed proportion $N(t)$, however, we acquire $N(t) = N$ (it is comparable to affirm that communities that have the same total measurement $N$ of parents of newborns are observed in $[0, T]$). At each $t \in [0, T]$, the persons are observed with respect to their respective perceptions of the relative risks of vaccine vs. infection, scilicet $r_i$. Hence, we acquire a split of the community of parents into agglomerations of proportions $\epsilon_i(t)$, being $\sum_{i=1}^{k} \epsilon_i(t) = 1$, wherever $\epsilon_i(t) \neq 0$ or 1 requesting precisely $k$ groups or units. Unit $i$’s chance to become vaccinated at time $t$ is indicated by $P_i(t)$. Lastly, we assume that each unit’s dimension may change time to time and we formulate the proportions as functions $\epsilon_i : [0, T] \rightarrow [0, 1]$. Therefore each unit is thought to have a vaccination strategy class of functions in the followin set

$$\mathbb{K}_i(t) = \{P_i \in L^2([0, T], \mathbb{R}) : P_i(t) \in [0, 1] \text{ almost everywhere in } [0, T]\}.$$ 

Let us presume that all components of a unit $i$ have equivalent probability of significant morbidity due to vaccination, indicated by $r_{iv}(t)$, a.e. in $[0, T]$. Their chance of becoming infected, given that a proportion $p$ of the community is vaccinated, is indicated by $\pi_p^i(P(t))$, a.e. in $[0, T]$. In order to have a more general situation, we suppose that the perceived probability of becoming infected depends on the unit $i$’s probability of getting vaccinated, namely $\pi_p^i = \pi_p^i(P(t))$. Let us indicate their probability of considerable morbidity upon infection by $r_{inf}^i(t)$, a.e. in $[0, T]$. The total probability of experiencing an appreciable morbidity because of not vaccinating is thus $\pi_p^i(P(t)) r_{inf}^i(t)$, a.e. in $[0, T]$. At last, let us indicate by $r_i(t) = \frac{r_{iv}(t)}{r_{inf}^i(t)}$, a.e. in $[0, T]$, the relative risk of vaccination versus infection of unit $i$. In addition, we postulate that the population consists of parents of newborns and each group $i$ of parents is made up of individuals whose risk assessments $r_i(t)$, a.e. in $[0, T]$, are moderately close.

As a consequence, we associate to each unit a payoff $u_i : [0, T] \times L^2([0, T], \mathbb{R}) \rightarrow L^2([0, T], \mathbb{R})$ given by

$$u_i(t, P(t)) = -r_i(t)P_i(t) - \pi_p^i(P(t))(1 - P_i(t)),$$

a.e. in $[0, T]$,

whilst the constraint set $\mathbb{K}$ turns into

$$\mathbb{K} = \{P \in L^2([0, T], \mathbb{R}^k) : 0 \leq P_i(t) \leq 1, \forall i \in \{1, \ldots, k\}, \text{ a.e. in } [0, T]\}.$$
The vaccine coverage is now come to be a function \( p : [0, T] \to [0, 1] \), estimated from the model as 
\[
\sum_{j=1}^{k} \epsilon_j(t) P_j(t) = p(t),
\]
stated the action of the population units. We express a time-dependent game of vaccination approach over \( \mathbb{K} \).

**Definition 3.1** An equilibrium state of a game is a vaccination strategy vector \( P^* \in L^2([0, T], \mathbb{R}^k) \), that occurs
\[
P_i^*(t) \in \mathbb{K}_i(t) : u_i(t, P^*(t)) \geq u_i(t, P_i(t), \hat{P}_i^*(t)), \quad \forall P_i(t) \in \mathbb{K}_i(t), \quad \forall i = 1, \ldots, k, \quad (13)
\]
wherever \( \hat{P}_i^*(t) = (P_1^*(t), \ldots, P_{i-1}^*(t), P_{i+1}^*(t), \ldots, P_k^*(t)) \).

Let us suppose that the following assumptions on the payoff functions \( u_i \) and on \( \nabla u \) are satisfied:

(i) \( u_i(t, P(t)) \) is continuously differentiable a.e. in \( [0, T] \),

(ii) \( \nabla u \) is a Carathéodory function such that
\[
\exists h \in L^2([0, T]) : \| \nabla u(t, P(t)) \|_{mn} \leq h(t) \| P(t) \|_{mn}, \quad \text{a.e. in } [0, T], \quad \forall P \in L^2([0, T], \mathbb{R}^k), \quad (14)
\]

(iii) \( u_i(t, P(t)) \) is pseudoconcave with respect to \( P_i \), \( i = 1, \ldots, k \), that is, for a.e. in \( [0, T] \):
\[
\left\langle \frac{\partial u_i}{\partial P_i} (t, P_1, \ldots, P_i, \ldots P_k), P_i - Q_i \right\rangle \geq 0
\]
\[
\implies u_i(t, P_1, \ldots, P_i, \ldots P_k) \geq u(t, P_1, \ldots, Q_i, \ldots P_k).
\]

We underline that the time-dependent vaccination equilibrium is characterized by means of an evolutionary variational inequality, as we put in evidence in the successive result (see [2], Theorem 3.2):

**Theorem 3.1** Let us assume that assumptions (i), (ii) and (iii) are satisfied. A time-dependent vaccination equilibrium is reached, in accord with Definition 3.1, if and only if it satisfies the evolutionary variational inequality
\[
\int_0^T \langle -\nabla u(t, Q(t)), P(t) - Q(t) \rangle dt \geq 0, \quad \forall P \in K. \quad (15)
\]

We keep in mind that, in the Hilbert space \( L^2([0, T], \mathbb{R}^k) \),
\[
\ll \phi, y \gg = \int_0^T \langle \phi(t), y(t) \rangle dt
\]
is the duality mapping, being \( \phi \in (L^2([0, T], \mathbb{R}^k))^* = L^2([0, T], \mathbb{R}^k) \) and \( y \in L^2([0, T], \mathbb{R}^k) \).

Our purpose is to acquire the existence of solution to (15). To this aim we take into account some preliminaries (see [26]).
Definition 3.2 If \( \nabla u = \left( \frac{\partial u}{\partial P_1}, \ldots, \frac{\partial u}{\partial P_k} \right) \) is as before. We say that

- the mapping \( -\nabla u \) is pseudomonotone in the sense of Karamardian (briefly K-pseudomonotone) iff for every \( P, Q \in \mathbb{K} \)
  \[ \ll -\nabla u(Q), P - Q \gg \geq 0 \Rightarrow \ll -\nabla u(P), P - Q \gg \geq 0; \]

- the mapping \( -\nabla u \) is pseudomonotone in the sense of Brezis (briefly B-pseudomonotone) iff
  1. for each sequence \( \{P_n\} \) weakly converging to \( u \) (in short \( P_n \rightharpoonup P \) in \( \mathbb{K} \)) and such that \( \limsup_n \ll -\nabla u(P_n), P_n - Q \gg \leq 0 \), it ensures that
     \[ \liminf_{n \to +\infty} \ll -\nabla u(P_n), P_n - Q \gg \geq \ll -\nabla u(P), P - Q \gg, \quad \forall Q \in \mathbb{K}; \]
  2. for each \( Q \in \mathbb{K} \) the function \( P \mapsto \ll -\nabla u(P), P - Q \gg \) is lower bounded on the bounded subset of \( \mathbb{K} \);

- the mapping \( -\nabla u \) is lower hemicontinuous along line segments, iff the function \( \xi \mapsto \ll -\nabla u(\xi), P - Q \gg \) is lower semicontinuous for every \( P, Q \in \mathbb{K} \) on the line segments \([P, Q]\);

- the mapping \( -\nabla u \) is hemicontinuous in the sense of Fan (briefly F-hemicontinuous) iff the function \( P \mapsto \ll -\nabla u(P), P - Q \gg \) is weakly lower semicontinuous on \( \mathbb{K} \), for every \( Q \in \mathbb{K} \).

Finally, we can explain the existence result that can be achieved making use of some arguments contained in \([26]\).

Theorem 3.2 Let us assume that assumptions (i), (ii) and (iii) are satisfied. If \( -\nabla u \) is B-pseudomonotone, or F-hemicontinuous or \( -\nabla u \) is K-pseudomonotone, then (15) admits a solution.

Let us note that (14) of (ii) guarantees the lower semicontinuity along line segments of \( -\nabla u \). As a consequence, assuming that \( -\nabla u \) is a K-pseudomonotone operator, then the existence of a solution is ensured without entering supplementary assumptions besides (14).

Now we apply the infinite dimensional duality theory to the dynamic vaccination game. For this intent, let \( Q \in \mathbb{K} \) be a solution of (15) and let us set up
\[
\psi(P) = \ll -\nabla u(Q), P - Q \gg, \quad \forall P \in \mathbb{K},
\]
that is
\[
\psi(P) = \int_0^1 - \sum_{i=1}^k \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i(t) - Q_i(t)) dt, \quad \forall P \in \mathbb{K}.
\]
Moreover, let us note
\[
\psi(P) \geq 0 \quad \forall P \in \mathbb{K}
\]
and
\[
\min_{P \in \mathbb{K}} \psi(P) = \psi(Q) = 0. \quad (16)
\]
We combine (15) and the subsequent Lagrange functional:
\[
\mathcal{L}(P, \alpha, \beta) = \psi(P) - \ll \alpha, P \gg + \ll \beta, P - 1 \gg,
\]
for every \( P \in L^2([0, T], \mathbb{R}^k) \), \( (\alpha, \beta) \in C^* \), being
\[
C^* = \left\{ (\alpha, \beta) \in L^2([0, T], \mathbb{R}^k) \times L^2([0, T], \mathbb{R}^k) : \alpha(t) \geq 0, \beta(t) \geq 0, \right. \\
\left. \text{a.e. in } [0, T] \right\}
\]
the dual cone of the ordering cone \( C \) of \( L^2([0, T], \mathbb{R}^k) \times L^2([0, T], \mathbb{R}^k) \). We point out that, in our case, \( C = C^* \). We are ready to demonstrate that our problem complies Assumption S. To purchase this goal, we need to state a preparatory lemma.

**Lemma 3.1** Let us assume that assumptions (i), (ii) and (iii) are satisfied. Let \( P \in \mathbb{K} \) be a solution of (15) and fix, for \( i = 1, 2, \ldots, m \),
\[
E_i^- = \{ t \in [0, T] : Q_i(t) = 0 \}, \quad E_i^0 = \{ t \in [0, T] : 0 < Q_i(t) < 1 \}, \\
E_i^+ = \{ t \in [0, T] : Q_i(t) = 1 \}.
\]
Then, we reach
\[
\frac{\partial u_i(t, Q(t))}{\partial P_i} \leq 0, \ \text{a.e. in } E_i^-, \quad \frac{\partial u_i(t, Q(t))}{\partial P_i} = 0, \ \text{a.e. in } E_0^i, \\
\frac{\partial u_i(t, Q(t))}{\partial P_i} \geq 0, \ \text{a.e. in } E_i^+.
\]

**Proof** In order to advantage the reader, we report the used technique and some details
(see also [4], Lemma 4.7). We emphasize that

\[ \ll -\nabla u(Q), P - Q \gg = - \int_0^T \sum_{i=1}^k \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i(t) - Q_i(t)) dt \]

\[ = - \sum_{i=1}^k \int_{E^-_i} \frac{\partial u_i(t, Q(t))}{\partial P_i} P_i(t) dt \]

\[ - \sum_{i=1}^k \int_{E^0_i} \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i(t) - Q_i(t)) dt \]

\[ - \sum_{i=1}^k \int_{E^+_i} \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i(t) - 1) dt \geq 0. \]

Let us presuppose that \( P_l(t) = Q_l(t) \) for \( l \neq i \), we get, for every \( 0 \leq P_i(t) \leq 1 \) and every \( i = 1, 2, \ldots, m \),

\[ \ll -\nabla u(Q), P - Q \gg = - \int_{E^-_i} \frac{\partial u_i(t, Q(t))}{\partial P_i} P_i(t) dt \]

\[ - \int_{E^0_i} \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i(t) - Q_i(t)) dt \]

\[ - \int_{E^+_i} \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i(t) - 1) dt \geq 0. \]

We pick out that

\[ P_i \begin{cases} 
\geq 0 & \text{in } E^-_i \\
= Q_i & \text{in } E^0_i \\
= Q_i & \text{in } E^+_i 
\end{cases} \]

so that (17) could be written as follows:

\[ - \int_{E^-_i} \frac{\partial u_i(t, Q(t))}{\partial P_i} P_i(t) dt \geq 0. \] (18)

Our “finishing line” is to show that \( \frac{\partial u_i(t, Q(t))}{\partial P_i} \leq 0 \), a.e. in \( E^-_i \). To this purpose, if there exists a subset \( F \) of \( E^-_i \), with \( m(F) > 0 \), such that \( -\frac{\partial u_i(t, Q(t))}{\partial P_i} < 0 \) in \( F \), picking

\[ P_i \begin{cases} 
= 0 & \text{in } E^-_i \setminus F \\
> 0 & \text{in } F 
\end{cases} \]
it results
\[- \int_F \frac{\partial u_i(t, Q(t))}{\partial P_i} P_i(t) dt < 0,\]
that contradicts (18). The product effect is that
\[\frac{\partial u_i(t, Q(t))}{\partial P_i} \leq 0, \text{ in } E_i^+.\]
Using similar arguments, we reach
\[\frac{\partial u_i(t, Q(t))}{\partial P_i} = 0, \text{ in } E_0^i.\]
In \(E_+^i\), selecting
\[P_i \begin{cases} = 0 & \text{ in } E_-^i \\ = Q_i & \text{ in } E_0^i \\ \leq 1 & \text{ in } E_+^i \end{cases},\]
we gain
\[- \int_{E_+^i} \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i(t) - 1) dt \geq 0.\]
When \(P_i - 1 \leq 0\) in \(E_+^i\) then we prove that \(\frac{\partial u_i(t, Q(t))}{\partial P_i} \geq 0\). As a matter of fact, if there exists \(F \subseteq E_+^i\), with \(m(F) > 0\), such that \(\frac{\partial u_i(t, Q(t))}{\partial P_i} < 0\) in \(F\), considering
\[P_i \begin{cases} = 1 & \text{ in } E_+^i \setminus F \\ < 1 & \text{ in } F \end{cases},\]
it follows
\[- \int_F \frac{\partial u_i(t, Q(t))}{\partial P_i} P_i(t) dt < 0.\]
As an effect,
\[\frac{\partial u_i(t, Q(t))}{\partial P_i} \geq 0, \text{ in } E_+^i.\]
\(\Box\)

Now we are ready to announce the main result of the article.

**Theorem 3.3** Let us assume that assumptions (i). (ii) and (iii) are satisfied. Problem (16) satisfies Assumption S at the minimal point \(P \in K\).
**Proof** We remember that

\[ \psi(P) = \langle -\nabla u(Q), P - Q \rangle, \quad \forall P \in \mathbb{K}. \]

To reach *Assumption S*, we demonstrate that if \( (l, \theta_{L^2([0,T],\mathbb{R}^k)}, \theta_{L^2([0,T],\mathbb{R}^k)}) \) belongs to \( T_M(0, \theta_{L^2([0,T],\mathbb{R}^k)}, \theta_{L^2([0,T],\mathbb{R}^k)}) \), specifically if

\[
\begin{align*}
 l &= \lim_{n \to +\infty} \lambda_n(\psi(P_n) - \psi(Q) + \alpha_n), \\
\theta_{L^2([0,T],\mathbb{R}^k)} &= \lim_{n \to +\infty} \lambda_n(-P_n + R_n), \\
\theta_{L^2([0,T],\mathbb{R}^k)} &= \lim_{n \to +\infty} \lambda_n(P_n - 1 + S_n),
\end{align*}
\]

with \( \lambda_n > 0, \ 0 = \lim_{n \to +\infty} (\psi(P_n) - \psi(Q) + \alpha_n), \ \theta_{L^2([0,T],\mathbb{R}^k)} = \lim_{n \to +\infty} (-P_n + R_n), \ \theta_{L^2([0,T],\mathbb{R}^k)} = \lim_{n \to +\infty} (P_n(t) - 1 + S_n), \ P_n \in L^2([0,T],\mathbb{R}^k) \setminus \mathbb{K}, \ \alpha_n \geq 0, \ R_n \in C, \) then \( l \geq 0. \)

We have

\[
\begin{align*}
l &= \lim_{n \to +\infty} \lambda_n \left( -\sum_{i=1}^{k} \int_{0}^{T} \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i^n(t) - Q_i(t)) dt + \alpha_n \right) \\
&\geq \lim_{n \to +\infty} \lambda_n \left( -\sum_{i=1}^{k} \int_{E_i^+} \frac{\partial u_i(t, Q(t))}{\partial P_i} P_i^n(t) dt \\
&- \sum_{i=1}^{k} \int_{E_i^0} \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i^n(t) - Q_i(t)) dt \\
&- \sum_{i=1}^{k} \int_{E_i^-} \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i^n(t) - 1) dt \right).
\end{align*}
\]

Furthermore we notice that

\[
\begin{align*}
&\lim_{n \to +\infty} \lambda_n \left[ -\sum_{i=1}^{k} \int_{E_i^-} \frac{\partial u_i(t, Q(t))}{\partial P_i} P_i^n(t) dt \right] \\
&= \lim_{n \to +\infty} \lambda_n \left[ \sum_{i=1}^{k} \left( -\int_{E_i^-} \frac{\partial u_i(t, Q(t))}{\partial P_i} (P_i^n(t) - R_i^n(t)) dt \\
&- \int_{E_i^+} \frac{\partial u_i(t, Q(t))}{\partial P_i} R_i^n(t) dt \right) \right] \geq 0,
\end{align*}
\]

because of \( \lim_{n \to +\infty} \lambda_n (P_i^n - R_i^n) = \theta_{L^2([0,T],\mathbb{R})}, \) in \( L^2([0,T],\mathbb{R}), \ R_i^n \geq 0, \ \lambda_n \geq 0, \)

\[
\frac{\partial u_i(t, Q(t))}{\partial P_i} \leq 0 \text{ in } E_i^+.
\]

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In addition, we observe that
\[
\lim_{n \to +\infty} \lambda_n \sum_{i=1}^{k} \int_{E^i_0} \frac{\partial u_i(t, Q(t))}{\partial P_i} (P^n_i(t) - Q_i(t)) dt = 0,
\]
being \( \frac{\partial u_i(t, Q(t))}{\partial P_i} = 0 \) in \( E^i_0 \).

Since \( \lim_{n \to +\infty} \lambda_n (Q^n_i + S^n_i - 1) = 0 \), in \( L^2([0, T], \mathbb{R}) \), \( \lambda_n \geq 0 \), \( S^n_i \geq 0 \), \( \frac{\partial u_i(t, Q(t))}{\partial P_i} \geq 0 \), in \( E^+_i \), it results
\[
\lim_{n \to +\infty} \lambda_n \sum_{i=1}^{k} \int_{E^+_i} -\frac{\partial u_i(t, Q(t))}{\partial P_i} (P^n_i(t) - 1) dt = 0.
\]
Hence, we have
\[
\lim_{n \to +\infty} \lambda_n \sum_{i=1}^{k} \int_{E^+_i} -\frac{\partial u_i(t, Q(t))}{\partial P_i} (P^n_i(t) - S^n_i(t)) dt = 0.
\]

Hence, the claim is completely achieved. \( \square \)

Let us now prove a necessary and sufficient condition concerning solutions to (15).

**Theorem 3.4** Let us assume that assumptions (i), (ii) and (iii) are satisfied. \( P \in \mathbb{K} \) is a solution to (15) if and only if there exist \( \alpha^*, \beta^* \in L^2([0, T], \mathbb{R}^k) \) such that:

(I) \( \alpha^*(t), \beta^*(t) \geq 0 \), a.e. in \([0, T]\);

(II) \( \alpha^*(t) Q(t) = 0 \), a.e. in \([0, T]\),
\( \beta^*(t)(Q(t) - 1) = 0 \), a.e. in \([0, T]\);

(III) \( -\nabla u(t, Q(t)) + \beta^*(t) = \alpha^*(t) \), a.e. in \([0, T]\).

**Proof** Making use of Theorem 2.2 there exists \( (\alpha^*, \beta^*) \in C^* \) such that \( (Q, \alpha^*, \beta^*) \) is a saddle point of the functional \( \mathcal{L} \), namely
\[
\mathcal{L}(Q, \alpha, \beta) \leq \mathcal{L}(Q, \alpha^*, \beta^*) \leq \mathcal{L}(P, \alpha^*, \beta^*), \quad \forall (\alpha, \beta) \in C^*, \ P \in L^2([0, T], \mathbb{R}^k),
\]
and, in addition,
\[
\ll \alpha^*, Q \rr = 0, \quad (20)
\]
\[
\ll \beta^*, Q - 1 \rr = 0. \quad (21)
\]
Since $\alpha, \beta \geq 0$, $Q \geq 0$, $Q - 1 \leq 0$, a.e. in $[0, T]$, (20) implies
$$\alpha^*(t)Q(t) = 0, \quad \text{a.e. in } [0, T],$$
$$\beta^*(t)(Q(t) - 1) = 0, \quad \text{a.e. in } [0, T].$$
Thanks to (19) we obtain, for every $P \in L^2([0, T], \mathbb{R}^k)$,
$$\mathcal{L}(P, \alpha^*, \beta^*) = \ll -\nabla u(Q), P - Q \gg - \ll \alpha^*, P \gg + \ll \beta^*, P - 1 \gg \geq 0 = \mathcal{L}(Q, \alpha^*, \beta^*).$$
Bearing in mind (20), we reach
$$\ll -\nabla u(Q) - \alpha^* + \beta^*, P - Q \gg \geq 0, \quad \forall P \in L^2([0, T], \mathbb{R}^k).$$
Let us fix
$$P^1 = Q + \varepsilon, \quad P^2 = Q - \varepsilon, \quad \forall \varepsilon \in C^\infty_0([0, T], \mathbb{R}^k),$$
we acquire, for every $\varepsilon \in C^\infty_0([0, T], \mathbb{R}^k),$
$$\mathcal{L}(P^1, \alpha^*, \beta^*) = - \ll -\nabla u(Q) - \alpha^* + \beta^*, \varepsilon \gg \geq 0, \quad (22)$$
$$\mathcal{L}(P^2, \alpha^*, \beta^*) = \ll -\nabla u(Q) - \alpha^* + \beta^*, \varepsilon \gg \geq 0. \quad (23)$$
As a consequence, by (22), we derive, for every $\varepsilon \in C^\infty_0([0, T]):$
$$\ll -\nabla u(Q) - \alpha^* + \beta^*, \varepsilon \gg = 0,$$
precisely, we gain
$$-\nabla u(t, Q(t)) - \alpha^*(t) + \beta^*(t) = 0, \quad \text{a.e. in } [0, T]. \quad (24)$$
Vice versa, if there exists $Q \in K$, $\alpha^* \in L^2([0, T], \mathbb{R}^k)$ and $\beta^* \in L^2([0, T], \mathbb{R})$ that satisfy assumptions (I), (II) and (III), it arises that $(Q, \alpha^*, \beta^*)$ is a saddle point of $\mathcal{L}$. Then, reminding Theorem 2.2, we affirm that $Q$ is a solution to (15). □
Recalling the above proved Theorem 3.4, we can emphasize the importance of the Lagrange multipliers $\alpha^*$ and $\beta^*$ on the understanding and the management of the vaccination game problem. Indeed, at a fixed time $t \in [0, T]$, it emerges that
(a) if $\alpha^*(t) > 0$ then, using (II), we have $Q(t) = 0$, namely the group $i$’s probability of getting vaccinated is null;
(b) if $Q(t) > 0$ then, bearing in mind (II), $\alpha^*(t) = 0$ and, making use of (III), it results
$$\nabla u_i(t, Q(t)) = \beta^*(t),$$
namely $\beta^*(t)$ is equal to the marginal payoff;
(c) if $\beta^*(t) > 0$ then, from (II), we obtain $Q(t) = 1$, namely the group $i$’s probability of getting vaccinated is maximum;
(d) if $Q(t) < 1$ then, from (II), $\beta^*(t) = 0$ and, taking into account (III), we obtain
$$-\nabla u_i(t, Q(t)) = \alpha^*(t),$$
namely $\beta^*(t)$ is equal to the opposite of the marginal payoff.
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