Super, Quantum and Non-Commutative Species

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Dedicated to the memory of Gian-Carlo Rota.

Abstract

We introduce an approach to the categorification of rings, via the notion of distributive categories with negative objects, and use it to lay down categorical foundations for the study of super, quantum and non-commutative combinatorics. Via the usual duality between algebra and geometry, these constructions provide categorifications for various types of affine spaces, thus our works may be regarded as a starting point towards the construction of a categorical geometry.

AMS Subject Classification: 16B50; 53D55; 81Q99.
Keywords: Deformation Quantization, Species, Feynman integrals.

1 Introduction

This work takes part in the efforts aimed to uncover the categorical foundations of quantum field theory QFT. The most developed categorical approach to QFT is via the Atiyah’s axioms for topological quantum field theory \[1, 2, 50\] and closely related axioms for other theories such as conformal field theory \[46, 47\], topological conformal field theory \[11, 35\], homotopy quantum field theory \[51, 52\], and homological quantum field theory \[13\]. These categorical approaches are most useful for the study of field theories defined over non-trivial topological spaces. In contrast the study of locally defined QFT with categorical tools remains, at large, limited and conjectural (for a promising new approach the reader may consult \[48\].)

In this work we propose an approach that, while still in its infancy, is well suited to deal with local issues in QFT. Our approach is based on a couple of elementary yet subtle observations. First observation: several statements coming from QFT may be understood in a rigorous way if regarded as taking part in formal geometry. We mention three main examples: i) Perturbative finite dimensional Feynman integrals are rigorous objects if one is willing to use fields which are formal power series, i.e. power series which may be divergent, see \[16\] and the references therein. In some cases this formal approach to Feynman integrals may also be applied in the infinite dimensional context. ii) Deformation quantization \[5\] of a finite dimensional Poisson manifold \(M\) consists in the construction of a star product \(\star\) on the space of formal power series...
in a variable $h$ with coefficients in the space of smooth functions on $M$. The main result is that the star product is essentially determined by the Poisson bracket on $M$. This fundamental fact was proved independently by Fedosov [26] and De Wilde and Lecomte [17] for symplectic Poisson manifolds; the general case was settled by Kontsevich [36]. Kontsevich’s work is done in the smooth context, however when the underlying space of the Poisson manifold is Euclidean space, his constructions applies as well in the formal context without deep changes. Notice that in both cases, smooth or formal, a formal variable $h$ has to be introduced. iii) The classification of infinite dimensional Lie groups is a notoriously difficult problem. However the classification of simple formal supersymmetries has been accomplished by Kac [32, 33].

Second observation: results in formal geometry have an underlying categorical meaning. This basic idea was introduced in combinatorics by Joyal [30, 31] via his theory of combinatorial species, which has been developed by a number of authors. The book [6] contains a comprehensive list of results and references in the theory of combinatorial species. Though not yet fully appreciated by the mathematical community at large, various constructions in the theory of combinatorial species actually has little to do with combinatorics and may be applied as well in other categorical settings. Considering the three formal constructions mentioned above one is led to the following categorical constructions: i) In Section 5 we developed a categorical version of Feynman integrals. That construction requires the extension of the notion of combinatorial species to the notion of $G$-$C$ species, where $G$ is a semisimple groupoid with finite morphisms and $C$ is a symmetric monoidal category. By definition the category of $G$-$C$ species is the category $C^G$ of functors from $G$ to $C$. ii) The categorical version of Kontsevich star product in full generality is given in Section 6. The important case of constant Poisson bracket, i.e. the categorification of the Weyl algebras, is considered in Section 7 where the notion of quantum species is introduced. The categorification of Weyl algebras allows us to look at the problem of the normal ordering of annihilation and creation operators [22, 23, 24] from a new perspective. iii) The work of Kac on the classification of formal supersymmetries opens the door for a categorical understanding of such objects; that will be the subject of our forthcoming work [20]. Pursuing this line of research will yield a plethora of examples of what might be called categorical Lie algebras.

As the reader may guess from the previous considerations a major requirement for this work is to have a solid understanding of the notion of categorification. Let us here explain informally what do we mean by such notion, and refer the reader to the body of this work for detailed definitions. The notion of categorification is under active investigation and there are various approaches to the subject. Though implicitly present in the works of the founders of category theory [40], the current activity on the subject have been greatly influenced by the works, among others, of Baez and Dolan [3, 4], Crane and Yetter [15], and Khovanov [34]. It is customary
to base the foundations of mathematics upon set theory but, as the Grothendieck’s theory of topoi has shown [29, 41], in many cases it is more enlightening to look for the categorical foundations of a given mathematical construction. The process of uncovering the categorical foundations of a set theoretical construction is named categorification. Let Cat be the category of essentially small categories; morphisms in Cat(C, D) from a category C to a category D are functors \( F : C \rightarrow D \). Let Set be the category of sets and functions as morphisms. There is a natural functor Cat \( \rightarrow \) Set called decategorification such that:

- It sends an essentially small category C to the set \( \underline{C} = \text{Ob}(C)/\text{Iso}_C \).
- It sends a functor \( F : C \rightarrow E \) into the induced map

\[
F : \underline{C} = \text{Ob}(C)/\text{Iso}_C \rightarrow \text{Ob}(E)/\text{Iso}_E = \underline{E}.
\]

Thus \( \underline{C} \) is the set of isomorphism classes of objects in C. We say that \( \underline{C} \) is the decategorification of C and also that C is a categorification of \( \underline{C} \). Notice that while a category has a unique decategorification, a set will have many categorifications. The motivating example, perhaps known implicitly to mankind since its early days, is the category \( \mathbb{B} \) whose objects are finite sets and whose morphisms are bijections between finite sets; it is easy to check that the decategorification of \( \mathbb{B} \) is the set \( \mathbb{N} \) of natural numbers.

In general we are interested in the categorification of sets provided with additional geometric or algebraic structures. For example one might try to find out what is the categorical analogue of a ring. In Section 2 we define the categorification of a ring \( R \) to be a distributive category with negative objects provided with a \( R \)-valuation. The main goal of this paper is to describe categorifications of several types of spaces, namely, noncommutative, quantum and super affine spaces. This is accomplished by identifying affine spaces with the ring of functions on them, and finding distributive categories with natural valuations on the corresponding ring of functions.

## 2 Categorification of rings

In this section we introduce the notion of categorification of rings and provide several examples. Recall that a monoidal category is a category C provided with a bifunctor \( \odot : C \times C \rightarrow C \) and natural isomorphisms \( \alpha_{x,y,z} : x \odot (y \odot z) \rightarrow (x \odot y) \odot z \) satisfying Mac Lane’s pentagon identity

\[
\alpha_{x \odot y,z,w} \circ \alpha_{x,y,z \odot w} = (\alpha_{x,y,z} \odot 1_w) \circ \alpha_{x,y,z,w}(1_x \odot \alpha_{y,z,w}).
\]

A symmetric monoidal category is a monoidal category C together with natural isomorphisms \( s_{x,y} : x \odot y \rightarrow y \odot x \) satisfying: \( s_{x,y} \circ s_{y,x} = 1_x \) and Mac Lane’s hexagon identity

\[
(s_{x,z} \odot 1_y) \circ \alpha_{x,z,y} = \alpha_{x,y,z} \circ s_{x \odot y,z}.
\]

(3)
A categorification of a ring $R$ is a triple $(C, N, | |)$ where $C$ is a distributive category, $N : C \rightarrow C$ is a functor called the negative functor, and $| | : C \rightarrow R$ is a map from the set of objects of $C$ into $R$ called the valuation map. This data should satisfy the following conditions.

1. $C$ is a distributive category, i.e. $C$ is provided with bifunctors $\oplus : C \times C \rightarrow C$ and $\otimes : C \times C \rightarrow C$ called sum and product, respectively. Functors $\oplus$ and $\otimes$ are such that:
   - There are distinguished objects 0 and 1 in $C$.
   - The triple $(C, \oplus, 0)$ is a symmetric monoidal category with unit 0.
   - The triple $(C, \otimes, 1)$ is a monoidal category with unit 1.
   - Distributivity holds. That is for objects $x, y, z$ of $C$ there are natural isomorphisms $x \otimes (y \oplus z) \simeq (x \otimes y) \oplus (x \otimes z)$ and $(x \oplus y) \otimes z \simeq (x \otimes z) \oplus (y \otimes z)$.

   See Laplaza’s works [38, 39] for a complete definition, including coherence theorems, of a category with two monoidal structures satisfying the distributive property.

2. The functor $N : C \rightarrow C$ must be such that for $x, y \in C$ the following properties hold: $N(x \oplus y) \simeq N(x) \oplus N(y)$, $N(0) = 0$, and $N^2$ is the identity functor.

3. The map $| | : C \rightarrow R$ is such that for $x, y \in C$ we have:
   - $|x| = |y|$ if $x$ and $y$ are isomorphic.
   - $|x \oplus y| = |x| + |y|$, $|x \otimes y| = |x||y|$, $|1| = 1$, and $|0| = 0$.
   - $|N(x)| = -|x|$.

   We make a few remarks regarding the notion of categorification of rings. If $R$ is a semi-ring then a categorification of $R$ is defined as above omitting the existence of the functor $N$. A categorification is said to be surjective if the valuation map is surjective. Notice that we do not require that $\oplus$ and $\otimes$ be the coproduct and product of $C$, although they could be. We stress that our definition only demands that $|a \oplus N(a)| = 0$. Demanding that $a \oplus N(a)$ be isomorphic to 0 would reduce drastically the scope of our definition. In practice we prefer to write $-a$ instead of $N(a)$.

A ring $R$ is a categorification of itself, since one may consider $R$ as the category whose object set is $R$ and with identities as the only morphisms. The valuation map is the identity map and the negative of $r \in R$ is $-r$. Thus, there is not existence problem attached to the notion of categorification: all rings admit a categorification. It will become clear from the examples given below that one should not expect the categorification of a ring to be unique. Quite the contrary, the philosophy behind the notion of categorification is that valuable information about a ring can be obtained by looking at its various categorifications, just like we can learn valuable
information about a group by looking at its various representations.

A functor $\varphi : C \to D$ between distributive categories is a functor that is monoidal with respect to $\oplus$ and $\otimes$. If both $C$ and $D$ have negative objects, then we demand in addition that the functor $\varphi$ respects the negative functors on $C$ and $D$. Notice that $\otimes$ is not required to be symmetric; if it is symmetric then we say that $C$ is a symmetric distributive category.

**Lemma 1.** For each distributive category $C$, there exists a distributive category with negative objects $\mathbb{Z}_2$-$C$, and an inclusion functor $i : C \to \mathbb{Z}_2$-$C$ such that for any given distributive category with negative objects $D$ and any given functor $\varphi : C \to D$, there is a unique functor $\psi : \mathbb{Z}_2$-$C \to D$ such that $\psi \circ i = \varphi$, i.e. the following diagram commutes

$$
\begin{array}{ccc}
C & \xrightarrow{i} & \mathbb{Z}_2$-$C \\
\downarrow{\varphi} & & \downarrow{\psi} \\
D & & 
\end{array}
$$

**Proof.** First define $\mathbb{Z}_2$-$C$ as the category $\mathbb{Z}_2$-$C = C \times C$ with sums and products given by

$$(a_1, a_2) \oplus (b_1, b_2) = (a_1 \oplus b_1, a_2 \oplus b_2),$$

$$(a_1, a_2) \otimes (b_1, b_2) = (a_1 \otimes b_1 \oplus a_2 \otimes b_2, a_1 \otimes b_2 \oplus a_2 \otimes b_1).$$

The negative functor $N$ is given by $N(a, b) = (b, a)$. The inclusion functor $i : C \to \mathbb{Z}_2$-$C$ is given by $i(a) = (a, 0)$. Given $\varphi : C \to D$, then the functor $\psi : \mathbb{Z}_2$-$C \to D$ is given by

$$\psi(a_1, a_2) = a_1 \oplus N(a_2).$$

**Lemma 2.** Let $| \cdot | : C \to R$ be a valuation on a distributive category $C$. There is a natural valuation $| \cdot | : \mathbb{Z}_2$-$C \to R$ on $\mathbb{Z}_2$-$C$ such that $|i(x)| = |x|$ for $x \in C$.

**Proof.** Define $| \cdot | : \mathbb{Z}_2$-$C \to R$ by $|(a, b)| = |a| - |b|$.

Lemma 2 allows us to define valuations with rings as codomain from valuations with semirings as codomain. Next paragraphs introduce a list of examples of distributive categories provided with valuations.

Let $\text{set}$ be the category of finite sets and maps as morphisms. The distributive structure on $\text{set}$ is given by disjoint union $x \sqcup y$ and Cartesian product $x \times y$. The map $| \cdot | : \text{set} \to \mathbb{N}$ sending $x$ into its cardinality $|x|$ defines a valuation on $\text{set}$.
Let $vect$ be the category of finite dimensional vector spaces. It is a distributive category with $\oplus$ and $\otimes$ defined as the direct sum and tensor product of vector spaces. The map $|\cdot| : vect \rightarrow \mathbb{N}$ given by $|V| = \dim(V)$ defines a valuation on $vect$.

Let $\mathbb{Z}_2$-vect be the category of finite dimensional $\mathbb{Z}_2$-graded vector space. Let $V, W \in \mathbb{Z}_2$-vect be given by $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$. Direct sum and tensor product on $\mathbb{Z}_2$-vect are given, respectively, by $V \oplus W = (V_0 \oplus W_0) \oplus (V_1 \oplus W_1)$ and $V \otimes W = [(V_0 \otimes W_0) \oplus (V_1 \otimes W_1)] \oplus [(V_0 \otimes W_1) \oplus (V_1 \otimes W_0)]$

The map $|\cdot| : \mathbb{Z}_2$-vect $\rightarrow \mathbb{Z}$ given by $|V_0 \oplus V_1| = \dim(V_0) - \dim(V_1)$ is a valuation on $\mathbb{Z}_2$-vect.

Recall [15] that the Möbius function $\mu : x \times x \rightarrow x$ of a finite partially ordered set $(x, \leq)$ is defined as follows: for $i, k$ incomparable elements of $x$ we set $\mu(i, k) = 0$. For $i \leq k$ the Möbius function satisfies the recursive relation:

$$\sum_{i \leq j \leq k} \mu(i, j) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Let $mposet$ be the full subcategory of the category of posets (partially ordered sets) whose objects $(x, \leq)$ are such that each equivalence class $c$, under the equivalence relation on $x$ generated by $\leq$, has a minimum $m_c$ and a maximum $M_c$. The sum functor is disjoint union of posets and the product functor is the Cartesian product of posets. Consider the map $|\cdot| : mposet \rightarrow \mathbb{Z}$ given by $|(x, \leq)| = \sum_c \mu(m_c, M_c)$, where the sum runs over the set $\{c\}$ of equivalence classes on $x$. The map $|\cdot|$ defines a valuation on the category $mposet$.

Let $vman$ be the category of pairs $(M, v)$, where $M$ is a finite disjoint union of finite dimensional oriented smooth manifolds, and $v$ is a map that sends each connected component $c$ of $M$ into a top differential form $v(c) \in \Omega^{\dim(c)}(c)$. Morphisms in $vman((M, v_M), (N, v_N))$ are smooth maps $f : M \rightarrow N$ such that $f^*v_N = v_M$. The sum functor is disjoint union and the product functor is Cartesian product. The map $v_{M \times N}$ sends the connected component $c \times d$ to the differential form $v_{M \times N}(c \times d) = \pi_M^*v_M(c) \wedge \pi_N^*v_N(d)$, where $\pi_M$ and $\pi_N$ are the projections from $M \times N$ onto $M$ and $N$, respectively. The map $|\cdot| : vman \rightarrow \mathbb{R}$ given by $|(M, v)| = \sum_{c \in \pi_0(M)} \int_c v(c)$.
defines a valuation on top by Fubini’s theorem.

Let top the category of topological spaces with finite dimensional \( \mathbb{C} \)-cohomology groups. The sum functor is disjoint union and the product functor is Cartesian product of topological spaces. By the Künneth’s formula the map \(| | : top \to \mathbb{C}[[t]]\) given by

\[
|X| = \sum_{i=0}^{\infty} \dim_{\mathbb{C}}(H^i(X)) t^i
\]
defines a valuation on top.

Let symp be the category whose objects are finite dimensional symplectic manifolds. We allow disconnected manifolds with components of various dimension. Morphisms in symp from \((M, \omega_M)\) to \((N, \omega_N)\) are smooth maps \(f : M \to N\) such that \(f^*\omega_N = \omega_M\). The distributive structure on symp is given by disjoint union and Cartesian product, where the symplectic structure on \(M \times N\) is given by \(\omega_{M \times N} = \pi_M^*\omega_M + \pi_N^*\omega_N\). The valuation map \(| | : symp \to \mathbb{R}\) is given by

\[
|(M, \omega_M)| = \sum_{c \in \pi_0(M)} \int_c \frac{\dim_{\mathbb{C}}}{c^2} \omega_M^2.
\]

Let \(C\) be a distributive category provided with a valuation map \(| | : C \to \mathbb{R}\). Let \(C^\mathbb{N}\) be the category of \(\mathbb{N}\)-graded \(C\)-objects, i.e. the category of functors \(\mathbb{N} \to C\), where \(\mathbb{N}\) is the category whose objects are the natural numbers and with identities morphisms only. The sum and product functors are given by

\[
(F \oplus G)(k) = F(k) \oplus G(k),
\]
\[
(F \otimes G)(k) = \bigoplus_{i+j=k} F(i) \otimes G(j).
\]
The map \(| | : C^\mathbb{N} \to \mathbb{R}[[t]]\) given by \(|F| = \sum_{k \in \mathbb{N}} |F(k)| t^k\) defines a valuation on \(C^\mathbb{N}\).

Let gpd be the category of finite groupoids. Recall that a finite groupoid \(G\) is a category such that the objects of \(G\) form a finite set, \(G(x, y)\) is a finite set for all \(x, y \in G\), and all morphisms in \(G\) are invertible. The sum and product functors on gpd are, respectively, disjoint union and Cartesian product of categories. The valuation map \(| | : gpd \to \mathbb{Q}\) is given by

\[
|G| = \sum_{x \in G} \frac{1}{|G(x, x)|}.
\]
This example has been exploited by Díaz and Blandín in order to propose a model for the study of the combinatorics of rational numbers.
Perhaps the best known example of categorification is the following. Let $M$ be a compact topological space and $\text{vect}_M$ be the category of finite rank $\mathbb{C}$-vector bundles on $M$. The canonical map $\pi : \text{vect}_M \rightarrow K_0(M)$, where $K_0(M)$ is the degree zero $K$-theory group of $M$, is a valuation map.

From now on we will make the following assumption. Let $x$ be a set of cardinality $n$, $C$ a symmetric monoidal category with product $\odot$, and $f : x \rightarrow C$ a map. Consider the category $\mathbb{L}(x)$ of linear orderings on $x$. Objects in $\mathbb{L}(x)$ are bijections $\alpha : [n] \rightarrow x$ where $[n] = \{1, 2, ..., n\}$. Morphisms in $\mathbb{L}(x)$ from $\alpha$ to $\beta$ are given by:

$$\mathbb{L}(x)(\alpha, \beta) = \{\sigma : [n] \rightarrow [n] \mid \beta \sigma = \alpha \text{ and } \sigma \text{ is a bijection}\}.$$ 

Consider the functor $\hat{f} : \mathbb{L}(x) \rightarrow C$ given by

$$\hat{f}(\alpha) = \odot_{i=1}^{n} f(\alpha(i)).$$

The image $\hat{f}(\sigma)$ of a morphism $\sigma$ in $\mathbb{L}(x)$ is obtained using the symmetry map of $C$. With this notation we define the $\odot$-product of objects in $C$ indexed by an unordered set $x$ as follows:

$$\bigodot_{i \in x} f(i) = \text{colim}(\hat{f}).$$

From now on we assume that our distributive categories are such that for each map $f : x \rightarrow C$ the colimit of the associated functor $\hat{f}$ exists for $\odot = \oplus$. Moreover, if $C$ happens to be a symmetric distributive category, then we also assume that the colimit above exist for $\odot = \otimes$.

### 3 Categorification of non-commutative affine space

In this section we begin the study of the main topic of this work, namely, the categorification of certain affine spaces. In the previous section we gave a precise definition of the notion of categorification of rings, and constructed various examples. To categorify spaces we recall the duality

$$\text{geometry} \quad \longleftrightarrow \quad \text{algebra}$$

between geometry and algebra which assigns – in its simplest version – to each space its corresponding ring of functions. For example if our space is a topological space, then we consider the ring of continuous functions on it. If instead, it is a smooth manifold, then one considers the ring of smooth functions on it. If it is an affine variety we consider the ring of polynomials functions on it, and so on. This duality has been of great use in functional analysis, algebraic geometry, non-commutative geometry, and further applications are to be expected. The key point to keep in mind is that once the appropriated ring of functions for a given space have
been determined, then the geometric properties of that space will be encoded in the algebraic properties of the corresponding ring.

Let $R$ be a commutative ring. The non-commutative formal $d$-dimensional affine space over $R$ is the space whose associated ring of functions is $R\langle\langle x_1,\ldots,x_d\rangle\rangle$, the ring of formal power series with coefficients in $R$ in the non-commutative variables $x_1,\ldots,x_d$. We find a categorification of non-commutative affine $d$-space as follows: we are going to define a distributive category $L^d$ such that any symmetric distributive category $C$ provided with a valuation map $|\ |: C \rightarrow R$ induces a valuation map $|\ |: C^{L^d} \rightarrow R\langle\langle x_1,\ldots,x_d\rangle\rangle$.

**Definition 3.** For $d \in \mathbb{N}^+$ we let $L^d$ be the category such that:

1. Objects of $L^d$ are triples $(x,\leq,f)$ where $x$ is a finite set, $\leq$ is a linear order on $x$, and $f : x \rightarrow [d]$ is a map.

2. The set of morphisms $L^d((x,\leq,f),(y,\leq,g))$ is given by

   $$\{ \varphi : x \rightarrow y \mid g \circ \varphi = f, \varphi \text{ is a bijection and } \varphi(i) < \varphi(j) \text{ for } i < j \}.$$ 

Note that $L^d$ is a groupoid and that there is at most one morphism between any pair of objects in $L^d$. Given essentially small categories $C$ and $D$, we let $D^C$ be the category of functors from $C$ to $D$. Morphisms in $D^C(F,G)$ are natural transformations $F \rightarrow G$.

**Definition 4.** Let $C$ be a symmetric distributive category. The category $C^{L^d}$ of functors from $L^d$ to $C$ will be called the category of non-commutative $C$-species of type $d$. We denote by $C^{L^d}_+$ the full subcategory of $C^{L^d}$ whose objects are functors $F \in C^{L^d}$ such that $F(\emptyset) = 0$.

The category $C^{L^d}_+$ is most useful when $C$ does not admit arbitrary sums $\bigoplus_{i \in I} a_i$. Recall that the ordered disjoint union of posets is given by $(x,\leq) \sqcup (y,\leq) = (x \sqcup y,\leq)$, where $\leq$ on $x \sqcup y$ is such that its restriction to $x$ agrees with the order on $x$, its restriction to $y$ agrees with the order on $y$, and $i \leq j$ for $i \in x$, $j \in y$. An ordered partition into $n$-pieces of a poset $(x,\leq)$ is a $n$-tuple of non-empty posets $(x_1,\leq),\ldots,(x_n,\leq)$ such that:

$$(x_1,\leq) \sqcup \cdots \sqcup (x_n,\leq) = (x,\leq).$$

We let $\text{opar}(x,\leq)$ be the set of all ordered partitions of $(x,\leq)$. To simplify notation we denote the restrictions of $\leq$ to the various subsets of $x$ by the same symbol $\leq$, we hope this causes no confusion.

**Definition 5.** Let $C$ be a symmetric distributive category, $F,G \in C^{L^d}$, $G_1,\ldots,G_d \in C^{L^d}_+$, and $(x,\leq,f) \in L^d$. The following formulae define, respectively, sum, product, composition and derivative of non-commutative species:
1. \((F + G)(x, \leq, f) = F(x, \leq, f) \oplus G(x, \leq, f)\).

2. \((FG)(x, \leq, f) = \bigoplus_{p, g} F(p, \leq, g) \otimes G(p(b, \leq, f)|b)\), where the sum runs over all pairs \((x_1, \leq)\) and \((x_2, \leq)\) such that \((x_1, \leq) \sqcup (x_2, \leq) = (x, \leq)\).

3. \(F(G_1, \ldots, G_d)(x, \leq, f) = \bigoplus_{p \in opar(x, \leq)} F(x_1, \leq, f|_{x_1}) \otimes G(x_2, \leq, f|_{x_2})\), where the sum runs over all \(p \in opar(x, \leq)\) and all maps \(g: p \to [d]\).

4. \(\partial_i : \mathcal{L}^d \to \mathcal{L}^d\) where \(\partial_i F\) is given by

\[
\partial_i F(x, \leq, f) = \bigoplus_{\leq^*} F(x \sqcup \{\ast\}, \leq^*_x, f \sqcup \{(\ast, i)\})
\]

where the sum runs over all extensions \(\leq^*_x\) of the order on \(x\) to a linear order on \(x \sqcup \{\ast\}\).

Figure 1 explains the graphical meaning of the derivative. The sum on the right hand side runs over all possible ways to insert the edge \((\ast, i)\) in the order set \((x, \leq, f)\). Let \(x_1, \ldots, x_d\) be a family of non-commutative variables. Then for \((x, \leq, f) \in \mathbb{L}_d\) we write \(x_f = x_f(1) \cdots x_f(d)\).

Let \(R\) be a ring and let \(C\) be a symmetric distributive category provided with a valuation map \(|\ | : C \to R\).

**Theorem 6.** Under the conditions above \(\mathcal{L}^d\) is a distributive category and the map

\[
|\ | : \mathcal{L}^d \to R\langle\langle x_1, \ldots, x_d \rangle\rangle
\]

given for \(F \in \mathcal{L}^d\) by \(|F| = \sum_{f : [m] \to [d]} |F([m], f)| x_f\)

is a valuation on \(\mathcal{L}^d\). Moreover we have that

\(|\partial_i F| = \partial_i |F|\) and \(|F(G_1, \ldots, G_n)| = |F|(|G_1|, \ldots, |G_n|)|.

**Proof.** For \(F, G \in \mathcal{L}^d\) we have that:

\[
|F + G| = \sum_{f : [m] \to [d]} |F \oplus G([m], f)| x_f = \sum_{f : [m] \to [d]} |F([m], f) \oplus G([m], f)| x_f = \sum_{f : [m] \to [d]} |F([m], f)| x_f + \sum_{f : [m] \to [d]} |G([m], f)| x_f = |F| + |G|
\]
\[ |FG| = \sum_{f: [m] \to [d]} |FG([m], f)| x_f \]
\[ = \sum_{f: [m] \to [d]} \sum_{m_1 \sqcup m_2 = m} |F(m_1, f|m_1)| x_{f|m_1} |G(m_2, f|m_2)| x_{f|m_2} \]
\[ = |F||G|. \]

For \( G_1, \ldots, G_d \in \mathcal{C}^m_+ \) we have that:
\[ |F(G_1, \ldots, G_d)| = \sum_{f: [m] \to [d]} |F(G_1, \ldots, G_d)([m], f)| x_f \]
\[ = \sum_{f: [m] \to [d]} \left( \bigoplus F(p, g) \otimes \bigotimes_{b \in p} G_{p(b)}(b, f|b) \right) x_f \]
\[ = \sum_{f: [m] \to [d]} \prod_{p, g} (|F(p, g)| |G_{p(b)}(b, f|b)| x_f) \]
\[ = \sum_{f: [m] \to [d]} |F([m], f)| \prod_{i=1}^n \left( \sum_{g: [m_i] \to [d]} |G_i([m_i], g_i)| x_{g_i} \right) \]
\[ = |F|([G_1, \ldots, G_d]). \]

\[ |\partial_i F| = \sum_{f: [m] \to [d]} |\partial_i F([m], f)| x_f \]
\[ = \sum_{f: [m] \to [d]} \sum_{|m| \leq * \& \sum f \sqcup \{(*, i)\}} |F([m] \sqcup \{\}, \leq, f \sqcup \{(\cdot, i)\})| x_f = \partial_i |F|, \]

since \( \partial_i x_g = x_f \) if and only if the domain of \( g \) is isomorphic to \([m] \sqcup \{\} \) and \( g = f \sqcup \{\cdot, i\}\). \( \square \)

**Example 7.** Let \((x, \leq, f) \in \mathbb{L}_d\).

1. For \( i \in [d] \) the non-commutative singleton species \( X_i \in \mathcal{C}_+ \) is given by \( X_i(x, \leq, f) = 1 \) if \(|x| = 1\) and \( f(x) = i \), otherwise \( X_i(x, \leq, f) = 0 \). We have that

\[ |X_i| = \sum_{f: [m] \to [d]} |X_i([m], \leq, f)| x_f = x_i. \]

where \( \leq \) denotes the standard linear order on \([m]\).

2. The species \( 1 \in \mathcal{C}_+ \) is given by \( 1(x, \leq, f) = 1 \) if \( x = \emptyset \) and \( 1(x, \leq, f) = 0 \) otherwise. We have that

\[ |1| = \sum_{f: [m] \to [d]} |1([m], \leq, f)| x_f = 1. \]
3. The non-commutative species $NE_d \in C^{d,d}$ is such that $NE_d(x, \leq, f) = 1$ for $(x, \leq, f) \in \mathbb{L}_d$.

We have that:

$$|NE_d| = \sum_{f : [m] \rightarrow [d]} |NE_d([m], \leq, f)| x_f = \sum_{f : [m] \rightarrow [d]} x_f.$$ 

As motivation for the study of non-commutative species we consider the problem of finding the analogue of the notion of operads in the non-commutative context. The reader may consult the next section for a brief summary on operads. Notice that our definition of non-commutative operads is actually an analogue of the notion of non-symmetric operads. For the next proposition, and in other similar situations, we regard $[d]$ as the category with identity morphisms only.

**Proposition 8.** Let $C$ be a symmetric distributive category, then $(C^{[d] \times \mathbb{L}_d}, \circ, (X_1, \ldots, X_d))$ is a monoidal category where for $(F_1, \ldots, F_d), (G_1, \ldots, G_d) \in C^{[d] \times \mathbb{L}_d}$ and $i \in [d]$ we set:

$$(F_1, \ldots, F_d) \circ (G_1, \ldots, G_d) = (F_1(G_1, \ldots, G_d), \ldots, F_i(G_1, \ldots, G_d), \ldots, F_n(G_1, \ldots, G_d)).$$

Let $(C, \odot, 1)$ be a monoidal category. A monoid $M$ in $C^{[d] \times \mathbb{L}_d}$ is an object $M \in C$ together with morphisms $m \in C(M \odot M, M)$ and $u \in C(1, M)$ such that the following diagrams commute

$$\begin{align*}
M \odot M \odot M &\xrightarrow{1 \odot m} M \odot M \\
m \odot 1 &\xrightarrow{m} M
\end{align*}$$

$$\begin{align*}
1 \odot M &\xrightarrow{u \odot 1} M \odot M \\
m &\xrightarrow{m} M
\end{align*}$$

where the diagonal arrows are the canonical isomorphisms coming from the properties of the unit element in a symmetric monoidal category.

**Definition 9.** A non-commutative $d$-operad in $C$ is a monoid in $(C^{[d] \times \mathbb{L}_d}, \circ, (X_1, \ldots, X_d))$.

Our next result gives an explicit description of non-commutative operads.

**Theorem 10.** A non-commutative $d$-operad in $C$ is given by a collection of objects $O = \{O^j_i\}$ where $O^j_i \in C$, $j \in [d]$, and $f : [m] \rightarrow [d]$ is a map with domain $[m]$ for some $m \geq 1$. In addition there should be unit maps $u_j : 1 \rightarrow O^j_{1,j}$ and composition maps

$$\gamma : O^j_i \otimes \bigotimes_{i=1}^m O^f_{g_{i,j}} \rightarrow O^j_{\sqcup g_1},$$

where $g_i : [k] \rightarrow [d]$, and $\sqcup g_i : \sqcup [k] \rightarrow [d]$ is given by

$$\sqcup g_i|_[k] = g_j.$$

These data should satisfy the associativity axiom:
Let $1_j : [1] \to [d]$ be given by $1_j(1) = j$ for $j \in [d]$. The following unity axioms must hold:

$$
O_f^j = O_f^j \otimes 1 \otimes \cdots \otimes 1 \xrightarrow{1 \otimes u_f(1) \otimes \cdots \otimes u_f(1)} O_f^j \otimes O_f^j \\
1 \otimes O_f^j = O_f^j \xrightarrow{u_j} O_f^j \\
O_f^j \otimes O_f^j \xrightarrow{\gamma} O_{1j} \otimes O_f^j
$$

Figure 2 illustrates the meaning of the composition maps $\gamma$.

Figure 2: Example of an application of the map $\gamma$.

Figure 3 explains graphically the associativity axiom. Next example provides a simple construction, a non-commutative analogue of the endomorphisms operad, that shows that there are plenty of non-commutative operads.

Example 11. Let $C$ be a monoidal category. There is a non-commutative $d$-operad $E$ associated with any $d$-sequence $(a_1, \ldots, a_d)$ of objects in $C$ given for $i \in [d]$ and $f : [m] \to [d]$
Figure 3: Example of the associativity for $\gamma$.

by

$$E^i_f = C(\bigotimes_{j \in [m]} a_{f(j)}, a_j).$$

**Example 12.** If $F$ is a non-commutative species, then we let $F_+$ be the species such that $F_+((x, \leq, f)) = F((x, \leq, f))$ if $x$ is non-empty and $F_+(\emptyset) = 0$. The $n$-tuple $(NE_{d,+}, ..., NE_{d,+})$ is a non-commutative $n$-operad.

Next we shall define a non-commutative analogue for the binomial coefficients [45]. Let $R$ be a commutative ring.

**Definition 13.**

1. Consider a family $\{s_f\}$, where $s_f : \mathbb{N} \rightarrow R$ is a map, and $f$ is a map from some $[m]$ into $[d]$. We called such a family a non-commutative multiplicative sequence if it is such that for $a, b \in \mathbb{N}$ the following identity holds:

$$s_f(a + b) = \sum_{i=1}^{m} s_{f_{<i}}(a)s_{f_{\geq i}}(b),$$

where $f_{<i} : [1, i - 1] \rightarrow [d]$ and $f_{\geq i} : [i, m] \rightarrow [d]$ are the restrictions of $f$ to the appropriated domains.

2. Consider a family $\{s_{f,i}\}$, where $s_{f,i} : \mathbb{N} \rightarrow R$ is a map, $f : [m] \rightarrow [d]$ is a map with domain $[m]$, for some $m \in \mathbb{N}$, and $i \in [d]$. We call such a family a non-commutative compositional sequence if for $a, b \in \mathbb{N}$ the following identity holds:

$$s_{f,i}(a + b) = \sum_{p} s_{p,i}(a) \prod_{j=1}^{k} s_{f,j,p(j)}(b),$$
where the sum runs the ordered partitions

\[(x_1, \leq, x_k) \sqcup \cdots \sqcup (x_k, \leq, f_k) = ([m], \leq, f)\]

and the maps \(p : [k] \to [d]\).

The following result gives a simple construction that generates non-commutative multiplicative sequences, and provides a categorical interpretation for it. Let \(s = \sum_{f : [m] \to [d]} s_f x_f\) be a non-commutative formal power series. For \(a \in \mathbb{N}\) we set \(s^0 = 1, s^{a+1} = s^a s\), and

\[s^a = \sum_{f : [m] \to [d]} s_f(a) x_f.\]

For \(S \in \mathcal{C}_{\mathbb{N}}d\) we define recursively \(S^0 = 1, S^{a+1} = S^a S\).

**Theorem 14.**

1. For \(s \in R\langle\langle x_1, \ldots, x_d\rangle\rangle\) the family \(\{s_f\}\) defined above is a noncommutative multiplicative sequence.

2. Let \(S \in \mathcal{C}_{\mathbb{N}}d\) be such that \(|S| = s\), for \(a \in \mathbb{N}^+\) let \(opar([m], a)\) be the set of ordered partitions of \([m]\) into \(a\) blocks. Then we have that

\[s_f(a) = \bigoplus_{\pi \in opar([m], a)} \bigotimes_{b \in \pi} S(b, \leq, f|_b).\]

3. Let \(S \in \mathcal{C}_{\mathbb{N}}d\) be given by \(S = 1 - F\) where \(F \in \mathcal{C}_{\mathbb{N}}^d\). The non-commutative species \(S^{-1}\) sending the empty set into 1 and a non-empty linearly order colored set \((x, \leq, f)\) into

\[S^{-1}(x, \leq, f) = \bigoplus_{\pi \in opar(x, \leq)} \bigotimes_{b \in \pi} F(b, \leq, f|_b)\]

is such that

\[|S^{-1}||S| = 1 = |S||S^{-1}|.\]

**Proof.** We proof the second part.

\[
\sum_{f : [m] \to [d]} s_f(a) x_f = |S|^a = |S^a| = \sum_{f : [m] \to [d]} |S^a([m], \leq, f)| x_f = \\
= \sum_{f : [m] \to [d]} \left| \bigoplus_{\pi \in opar([m], a)} \bigotimes_{b \in \pi} S(b, \leq, f|_b) \right| x_f.
\]

Notice that we are not claiming that \(SS^{-1} = 1\). It would be nice to have such an identity, but in most cases we have to deal with the weaker identity \(|S||S^{-1}| = 1\).
We let Digraph be the category whose objects are directed graphs. A directed graph is a triple \((V, E, (s, t))\) where \(V\) is the set of vertices, \(E\) is the set of edges and \((s, t) : E \rightarrow V \times V\) is a map. A rooted tree is a directed graph such that there is a distinguished vertex \(r\) called the root; and for each vertex \(v\) there is a unique directed path from \(v\) to \(r\). Notice that necessarily \(|s^{-1}(r)| = 0\). A vertex \(v\) is a leaf if \(|t^{-1}(v)| = 0\). A planar rooted tree is a tree together with a linear order on \(t^{-1}(v)\) for each vertex \(v\). We shall consider colored or labeled planar trees, i.e. planar trees provided with a map \(l : V \rightarrow [n]\). Figure 4 shows an example of a colored planar rooted tree. Let \(T\) be the category whose objects are colored planar rooted trees, and whose morphisms are morphisms between the underlying directed graphs that preserve labels and the linear orders associated with each vertex. A colored set is a set \(x\) together with a map \(f : x \rightarrow [n]\). Given a linear ordered colored set \((x, \leq, f)\) and \(a \in \mathbb{N}\), we let \(T_i^a(x)\) be the full subcategory of \(T\) whose objects are colored planar rooted trees \(\gamma\) such that:

- \(l(r) = i\).
- The set of leaves is \(x\) and \(l(i) = f(i)\) for \(i \in x\).
- The linear order on \(x\) agrees with the order induced by the planar structure on \(\gamma\).

Also we define \(T_i^a(x)\) as the full subcategory of \(T_i(x)\) whose objects are such that any path from a leave to the root has length \(a\).

Next result provides a source of non-commutative compositional sequences, and also a categorical interpretation for it. Moreover we show that, in a sense made clear in the statement of the next theorem, most \(C\)-valued non-commutative species have a compositional inverse. Let \(s = (s_1, \ldots, s_d) \in R\langle\langle x_1, \ldots, x_d\rangle\rangle^d\) be such that \(s(0) = 0\) and \(\partial_j s_i = \delta_{ij}\). For \(a \in \mathbb{N}\) we set \(s^{<0>} = x = (x_1, \ldots, x_d)\), and \(s^{<a+1>} = s^a \circ s\). We also set

\[s_i^{<a>} = \sum_{f : [m] \rightarrow [d]} s_f, i(a)x_f.\]

For \(S \in C_{+}^{[d] \times 1_d}\) we define recursively \(S^{(0)} = (X_1, \ldots, X_d)\) and \(S^{(a+1)} = S^{(a)} \circ S\).
Theorem 15. 1. The sequence \( \{ s_{f,i} \} \) defined above is a non-commutative compositional sequence.

2. Suppose that \( S \in C_{+}^{d} \) satisfies \( |S| = s \), then

\[
s_{f,i}(a) = \bigoplus_{\gamma \in T_{a}(x)} \bigotimes_{v \in V_{\gamma} \setminus x} C_{l(v)}(t^{-1}(v), \leq, l),
\]

where \( l \) denotes the coloring of the graph \( \gamma \).

3. If \( S \in C_{+}^{d} \) is such that \( S_{i} = X_{i} - F_{i} \), where \( F_{i} \in C_{+}^{d} \) and \( F_{i}(x, \leq, f) = 0 \) for \( |x| \leq 1 \), then the non-commutative species \( S_{<}^{<} \in C_{+}^{d} \) given by

\[
S_{i}^{<}(x, \leq, f) = \bigoplus_{\gamma \in T_{i}(x)} \bigotimes_{v \in V_{\gamma} \setminus x} F_{l(v)}(t^{-1}(v), \leq, l),
\]

is such that

\[
|S| \circ |S^{<}| = (X_{1}, ..., X_{d}) = |S^{<}| \circ |S|.
\]

4 Categorification of affine space

We begin this section recalling the construction of the category of \( C \)-valued commutative species, following the approach introduced by Joyal [30, 31], and a fully developed by Bergeron, Labelle and Leroux [6]. The notion of species, under the name of collections, has also appeared in algebraic topology, for example, in the works of Boardman and Vogt [10]. We show that commutative and non-commutative \( C \)-species are intertwined by a pair of adjoint functors. Let \( B^{d} \) be the category whose objects are pairs \( (x, f) \) where \( x \) is a finite set and \( f : x \rightarrow [d] \) is a map. Morphisms in \( B^{d} \) are given by

\[
B^{d}((x, f), (y, g)) = \{ \alpha : x \rightarrow y \mid \alpha \text{ is bijective and } g \circ \alpha = f \}.
\]

Definition 16. The category \( C_{B}^{d} \) of functors from \( B^{d} \) to \( C \) is called the category of \( C \)-species of type \( d \). We let \( C_{B}^{d} \) be the full subcategory of \( C_{B}^{d} \) whose objects are functors \( F \) such that \( F(\emptyset) = 0 \).

Consider the species \( \text{par} : B \rightarrow \text{set} \) that sends a finite set \( x \) into the set of all its partitions, i.e. families of non-empty disjoints subsets of \( x \) with union equal to \( x \).

Definition 17. Let \( C \) be a symmetric distributive category. Let \( F, G \in C_{B}^{d} \) and \( G_{1}, \ldots, G_{d} \in C_{B}^{d} \) and \( (x, f) \in B^{d} \), the following formulae defines sum, product, composition and derivative for commutative species:

1. \( (F + G)(x, f) = F(x, f) \oplus G(x, f) \).
2. \( FG(x, f) = \bigoplus_{y \in x} F(y, f|_y) \otimes G(x - y, f|_{x-y}) \).

3. \( F(G_1, \ldots, G_d)(x, f) = \bigoplus_{p, g} F(p, g) \otimes \bigotimes_{b \in p} G_{p(b)}(b, f|_b) \), where \( p \in \text{par}(x) \) and \( g : p \to [d] \) is a map.

4. \( \partial_i : C^d \to C^d \) where \( \partial_i F \) is given by the formula \( \partial_i F(x, f) = F(x \sqcup \{i\}, f \sqcup \{(i, i)\}) \).

For \( (a_1, \ldots, a_d) \in \mathbb{N}^d \) we write \( a! = a_1! \cdots a_d! \), \([a] = ([a_1], \ldots, [a_d])\) and \( x^a = x_1^{a_1} \cdots x_d^{a_d} \). Let \( R \) be a ring of characteristic 0 and \( C \) be a symmetric monoidal category provided with a valuation map \( | \cdot : C \to R \).

**Theorem 18.** The map \( | \cdot : C^d \to R[[x_1, x_2, \ldots, x_d]] \) given by

\[
|F| = \sum_{a \in \mathbb{N}^d} |F[a]| \frac{x^a}{a!}
\]

is a valuation on \( C^d \). Moreover \( |\partial_i F| = |\partial_i| |F| \) and \( |F(G_1, \ldots, G_d)| = |F|(|G_1|, \ldots, |G_d|) \), for \( F \in C^d \) and \( G_1, \ldots, G_d \in C^d \).

**Example 19.** Let \( (x, f) \in \mathbb{B}^d \).

- The singleton specie \( X_i \in C^d \), for \( i \in [d] \), is such that \( X_i(x, f) = 1 \) if \( |x| = 1 \) and \( f(x) = i \), otherwise \( X_i(x, f) = 0 \).
- The species \( 1 \in C^d \) is given by \( 1(x, f) = 1 \) if \( x = \emptyset \), and \( 1(x, f) = 0 \) otherwise.
- The exponential species \( E \in C^d \) is given by \( E(x, f) = 1 \).

It should be clear that \( |X_i| = x_i, |1| = 1 \) and \( |E| = e^{x_1 + \cdots + x_n} \).

**Example 20.** Let us give an example of a combinatorial species with a biological flavor. Let \( ADN \in \text{set}^d \) be the species such that for \( A, T, C, G \in \mathbb{B} \) we have:

1. \( ADN(A, T, C, G) \) is the set of ordered restricted matchings on \( A \sqcup T \sqcup C \sqcup G \).

2. An ordered restricted matching \( \alpha \) is a map \( \alpha : \{0, 1\} \times [n] \to A \sqcup T \sqcup C \sqcup G \) such that:

   - \( \alpha \) is a bijection.
   - \( \alpha(0, i) \in A \) (respectively \( C \)) if and only if \( \alpha(1, i) \in T \) (respectively \( G \)).
   - \( \alpha(1, i) \in A \) (respectively \( C \)) if and only if \( \alpha(0, i) \in T \) (respectively \( G \)).

3. It is not hard to check that the valuation \( |ADN| \in \mathbb{N}[[a, t, c, g]] \) of \( ADN \) is given by

\[
|ADN|(a, t, c, g) = \frac{1}{1 - 2at - 2cg}.
\]
Recall [37, 42] that an operad $O$ in a symmetric monoidal category $(C, \odot, 1)$ consists of a family $O = \{O_d\}$, with $d \in \mathbb{N}$ and $O_d \in C$, together with the following structural maps:

1. A composition law $\gamma : O_k \odot O_{j_1} \odot \cdots \odot O_{j_k} \rightarrow O_j$, for $k \geq 1$ and $j_s \geq 0$ such that $\sum_{s=1}^{k} j_s = j$.
2. A right action $O_d \times S_d \rightarrow O_d$ of the symmetric group $S_d$ on $O_d$.
3. An unit map $\eta : 1 \rightarrow O_1$.

The structural maps are required to be associative, unital and equivariant in the appropriated sense [28, 37]. Non-symmetric operads are defined omitting the actions of the symmetric groups. Let Cat be the category of essentially small categories.

**Proposition 21.** The collection $\{C_{B_d}^+\}_{d \geq 0}$ is an operad in Cat.

**Proof.** For $k \geq 1$ the composition map

$$\gamma_k : C_{B^k} \times C_{B^{n_1}} \times \cdots \times C_{B^{n_k}} \rightarrow C_{B^{n_1+\cdots+n_k}}$$

is given by

$$\gamma_k(F, G_1, \ldots, G_k) = F(G_1, \ldots, G_k).$$

Given $\sigma \in S_d$ and $F \in C_{B^d}$, let $F\sigma \in C_{B^d}$ be such that $F\sigma(x, f) = F(x, \sigma f)$. The map

$$C_{B^d} \times S_d \rightarrow C_{B^d}$$

given by $(F, \sigma) \mapsto F\sigma$ provides $C_{B^d}$ with a $S_d$-action. \hfill \Box

For the next result we assume the sum functor $\oplus$ on $C$ behaves as a coproduct, i.e. any morphisms $\varphi : \bigoplus_i c_i \rightarrow d$ in $C$ is uniquely determined by a family of morphisms $\varphi : c_i \rightarrow d$.

**Theorem 22.** Let $C$ be a symmetric distributive category.

1. The following maps are functorial:
   
   - $S : C_{B^d} \rightarrow C_{L^d}$ given by $SG(x, \leq, f) = G(x, f)$.
   
   - $\Pi : C_{L^d} \rightarrow C_{B^d}$ given by $\Pi F(x, f) = \bigoplus_{\leq} F(x, \leq, f)$ where the sum runs over the linear orders $\leq$ on $x$.

2. $\Pi$ is a left adjoint of $S$. 

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Theorem 23. Let $S \in C^{\mathbb{B}^d}$. We must show that $C^{\mathbb{B}^d}(\Pi F, G) = C^{\mathbb{B}^d}(F, SG)$. An element $S \in C^{\mathbb{B}^d} \Pi F \rightarrow G$. $S$ is given by morphisms $S(x, f) : \Pi F(x, f) \rightarrow G(x, f)$, one for each pair $(x, f) \in \mathbb{B}^d$, such that for any morphisms $\varphi : (x, f) \rightarrow (y, g)$ in $\mathbb{B}^d$ the diagram

$$\Pi F(x, f) \xrightarrow{S(x, f)} G(x, f)$$

$$\Pi F(\varphi) \downarrow \quad \downarrow G(\varphi)$$

$$\Pi F(y, g) \longrightarrow G(y, g)$$

is commutative. A morphism $S(x, f) : \bigoplus F(x, \leq, f) \rightarrow G(x, f)$ is the same as a family of morphisms $S(x, \leq, f) : F(x, \leq, f) \rightarrow SG(x, \leq, f)$, which in turns defines a natural transformation $S : F \rightarrow SG$.

Our next result gives and interpretation in terms of formal power series for the couple of adjoint functors given above. Consider the $R$-linear map $\pi : R\langle x_1, \ldots, x_d \rangle \rightarrow R[[x_1, \ldots, x_d]]$ given on monomials by

$$\pi(x_f) = \prod_{i=1}^n x_i^{f^{-1}(i)}$$

for $x_f \in R\langle x_1, \ldots, x_d \rangle$. Consider also the linear map $s : R[[x_1, \ldots, x_d]] \rightarrow R\langle x_1, \ldots, x_d \rangle$ given by

$$s\left(\frac{x^a}{a!}\right) = \sum_{f \in \{a_1 + \cdots + a_d\}} \prod_{i \in \{a_1 + \cdots + a_d\}} x_i^{f(i)},$$

where the sum runs over the maps $f : [a_1 + \cdots + a_d] \rightarrow [d]$ such that $f^{-1}(i) = a_i$.

**Theorem 23.** Let $C$ be a symmetric distributive category provided with a $R$-valuation. The following diagrams commute:

$$\xymatrix{ C^{\mathbb{B}^n} \ar[r]^{\|} \ar[d]^{S} & R[[x_1, \ldots, x_d]] \ar[d]^{\pi} \\
C^{\mathbb{I}^n} \ar[r]^{\|} & R\langle x_1, \ldots, x_d \rangle }$$

$$\xymatrix{ C^{\mathbb{B}^d} \ar[r]^{\|} \ar[d]^{\Pi} & R[[x_1, \ldots, x_d]] \ar[d]^{\pi} \\
C^{\mathbb{I}^n} \ar[r]^{\|} & R\langle x_1, \ldots, x_d \rangle }$$

**Proof.** For $F \in C^{\mathbb{B}^d}$ we have that

$$|SF| = \sum_{f : [m] \rightarrow [d]} |SF([m], \leq, f)|x_f = \sum_{f : [m] \rightarrow [d]} |F([m], f)|x_f$$

$$= \sum_{a \in \mathbb{N}^d} \sum_{f : [m] \rightarrow [d], f^{-1}(i) = a_i} |S(F([m], f)|x_f$$

$$= \sum_{a \in \mathbb{N}^d} F[a] \sum_{f : [m] \rightarrow [d], f^{-1}(i) = a_i} \prod_{i \in \{a_1 + \cdots + a_d\}} x_i^{f(i)} = \sum_{a \in \mathbb{N}^d} |F[a]| s\left(\frac{x^a}{a!}\right) = s|F|.$$
Let \( F \in C^{Ld} \) then we have that:

\[
\pi |F| = \sum_{f : [m] \to [d]} F([m], \leq, f) \pi(x_f) = \sum_{a \in \mathbb{N}^d} \left( \sum_{f^{-1}(i) = a_i} |F([m], \leq, f)| \right) \prod_{i=1}^d a_i^{f^{-1}(i)}
\]

\[
= \sum_{a \in \mathbb{N}^d} \sum_{|f^{-1}(i)| = a_i} |F([m], \leq, f)| \cdot a! x^a \frac{a!}{a!} = \sum_{a \in \mathbb{N}^d} \sum_{\sigma \in S_{[a_1 + \cdots + a_d]}} |F([a_1 + \cdots + a_d], \leq, f \circ \sigma)| \cdot a! \frac{x^a}{a!} = \Pi F.
\]

Next we define the shuffle bifunctor which intertwines the product on commutative and non-commutative species.

**Definition 24.** The shuffle bifunctor \( \text{Sh} : C^{Ld} \times C^{Ld} \rightarrow C^{Ld} \) is given on objects by

\[
\text{Sh}(F, G)(x, \leq, f) = \bigoplus_{y \subseteq x} F(y, \leq, f|_y) \otimes G(x - y, \leq, f|_{x-y})
\]

and is defined on morphisms in the natural way.

**Theorem 25.** 1. For \( F, G \in C^{Ld} \) the functor \( \Pi \) satisfies:

(a) \( \Pi(F + G) = \Pi(F) + \Pi(G) \).

(b) \( \Pi(FG) = \Pi(F) \Pi(G) \).

(c) \( \partial_i F = \partial_i \Pi F \).

2. For \( F, G \in C^{Bd} \) the functor \( S \) satisfies:

(a) \( S(F + G) = S(F) + S(G) \) for \( F, G \in C^{Bd} \).

(b) \( S(FG) = \text{Sh}(SF, SG) \).

(c) \( \partial_i SF = S \partial_i F \).

(d) \( \Pi S = d! \).
Proof. Let \( F, G \in C^{R_d} \) then we have that:

\[
\Pi(F + G)(x, f) = \bigoplus_{\leq} F \oplus G(x, \leq, f) = \bigoplus_{l} F(x, \leq, f) \oplus G(x, \leq, f)
\]

\[
= (\Pi F \oplus \Pi G)(x, f).
\]

\[
\Pi(FG)(x, f) = \bigoplus_{\leq} FG(x, \leq, f) = \bigoplus_{x} \bigoplus_{x_1 \sqcup x_2 = x} F(x_1, \leq, f|_{x_1}) \otimes G(x_2, \leq, f|_{x_2})
\]

\[
= \Pi F \Pi G(x, f).
\]

\[
\Pi \partial_i F(x, f) = \Pi F(x \sqcup \{\ast\}, f \sqcup \{(*, i)\}) = \bigoplus_{\leq} F(x \sqcup \{\ast\}, \leq, f \sqcup \{(*, i)\})
\]

\[
= \partial_i \Pi F(x, f).
\]

Let \( F, G \in C^{R_d} \) then we have that:

\[
S(F + G)(x, \leq, f) = F \oplus G(x, \leq, f) = F(x, f) \oplus G(x, f) = (SF \oplus SG)(x, \leq, f).
\]

\[
S(FG)(x, \leq, f) = FG(x, f) = \bigoplus_{y \subseteq x} F(y, f|_{y}) \otimes G(x \setminus y, f|_{x \setminus y})
\]

\[
= \bigoplus_{y \subseteq x} SF(y, \leq, f|_{y}) \otimes SG(x \setminus y, \leq, f|_{x \setminus y})
\]

\[
= \Sh(SF, SG)(x, \leq, f).
\]

\[
S \partial_i F(x, \leq, f) = \bigoplus_{\leq} SF(x \sqcup \{\ast\}, \leq, f \sqcup \{(*, i)\}) = F(x \sqcup \{\ast\}, f \sqcup \{(*, i)\})
\]

\[
\Pi SF(x, f) = \bigoplus_{\leq} SF(x, \leq, f) = \bigoplus_{\leq} F(x, f) = d!F(x, f).
\]

\[\Box\]

**Definition 26.**

1. Consider a sequence \( \{s_n\}_{n \in \mathbb{N}^d} \) where \( s_n : \mathbb{N} \rightarrow R \) is a map. We call such a sequence a polybinomial sequence if for \( a, b \in \mathbb{N} \) we have:

\[
s_n(a + b) = \sum_{j \in \mathbb{N}^d} \binom{n_1}{j_1} \ldots \binom{n_d}{j_d} s_j(a)s_{n-j}(b).
\]

2. Consider a sequence of maps \( \{s_{n,i}\}_{n \in \mathbb{N}^d} \), where \( s_{n,i} : \mathbb{N} \rightarrow R \) is a map and \( i \in [d] \). We call such a sequence a polymultinomial sequence if it is such that for \( a, b \in \mathbb{N} \) the following identity holds:

\[
s_{n,i}(a + b) = \sum_{m, p} \frac{s_{m,i}(a)}{m!} \prod_{l=1}^{d} \binom{n_l}{p_{l,t}} \prod_{j=1}^{m_j} \prod_{k=1}^{s_{j,k}} (p_{l,j,k})
\]

where \( m = (m_1, \ldots, m_d) \in \mathbb{N}^d \), and \( p \) is a map that sends triples \( l, j, k \) such that \( l, j \in [d] \) and \( k \in [m_j] \) into \( \mathbb{N} \). The map \( p \) must be such that \( \sum_{j,k} p_{l,j,k} = n_t \). Also we set

\[
m! = m_1! \ldots m_d!, \quad \overline{p}_{l,t} = (p_{l,1,1}, \ldots, p_{l,d,m_d}) \quad \text{and} \quad \overline{p}_{j,k} = (p_{l,j,k}, \ldots, p_{d,j,k}).
\]
For one variable, \( d = 1 \), the conditions on the coefficients are, respectively, as follows:

\[
s_n(a + b) = \sum_{k=0}^{n} \binom{n}{k} s_k(a)s_{n-k}(b),
\]

\[
s_n(a + b) = \sum_{i_1 + \ldots + i_k = n} \frac{1}{k!} \left( \binom{n}{i_1, \ldots, i_k} s_k(a)s_{i_1}(b) \ldots s_{i_k}(b) \right).
\]

In this case these coefficient are called binomial coefficients and multinomial coefficients, respectively \([45]\). Given \( s = \sum_{n \in \mathbb{N}} s_n \frac{x^n}{n!} \in R[[x_1, \ldots, x_d]] \) we set \( s^0 = 1 \), and for \( a \in \mathbb{N} \) we set \( s^{a+1} = s^a s \), and \( s^a = \sum_{n \in \mathbb{N}^d} s_n(a) \frac{x^n}{n!} \). Also for \( S \in C^d \) we set \( S^0 = 1 \) and \( S^{a+1} = S^a S \).

**Theorem 27.** 1. The sequence \( \{s_n\}_{n \in \mathbb{N}} \) defined as above is polybinomial sequence.

2. Assume that there exists \( S \in C^d \) such that \( |S| = s \), then

\[
s_n(a) = \bigg| \biggoplus_{x_1 \cup \ldots \cup x_a = x} \bigotimes_{i=1}^a S(x_i, f|x_i) \bigg|.
\]

3. Suppose that \( S \in C^d \) is such that \( S = 1 - F \) where \( F \in C^d_+ \), then the species \( S^{-1} \) sending the empty set into 1 and a non-empty set into

\[
S^{-1}(x, f) = \biggoplus_{n \in \mathbb{N}} \biggoplus_{x_1 \cup \ldots \cup x_n = x} \bigotimes_{i=1}^n F(x_i, f|x_i)
\]

is such that \( |S^{-1}||S| = 1 = |S||S^{-1}| \).

We define categories of directed trees \( T, T_i(x, f) \), and \( T^a_i(x, f) \) pretty much as we did in the planar case, but now we omit the planar condition. Let \( s = (s_1, \ldots, s_d) \in R[[x_1, \ldots, x_d]]^d \) be such that \( s_i(0) = 0 \) and \( \partial_j x_i = \delta_{i,j} \). Set \( s^{<0>} = (x_1, \ldots, x_d) \), \( s^{a+1} = s^a \circ s \), and

\[
s_i^{(a)} = \sum_{n \in \mathbb{N}^d} s_{n,i}(a) \frac{x^n}{n!},
\]

If \( S \in C^d_+ \) then we set \( S^0 = (X_1, \ldots, X_d) \) and \( S^{(a)} = S \circ S \circ \cdots \circ S \).

**Theorem 28.** 1. The sequence \( \{s_n, i\} \) defined above is a polymultinomial sequence.

2. Suppose that \( S \in C^d_+ \) satisfies \( |S|_i = s_i \), and let \( (x, f) \in \mathbb{B}^d \) be such that \( f^{-1}(i) = n_i \). Then

\[
s_{n,i}(a) = \bigg| \biggoplus_{\gamma \in T^a_i(x)} \left( \bigotimes_{v \in V \setminus x} S_{I(v)}(t^{-1}(v), l) \right) \right|_{\text{Aut}(\gamma)}.
\]
3. If $S \in C_{d}^{[d] \times B_{d}}$ is such that $S_{i} = X_{i} - F_{i}$ with $F_{i}(x, f) = 0$ for $|x| \leq 1$, then the $C$-species $S^{-1}$ given by

$$S^{-1}(x, f) = \bigoplus_{\gamma \in T_{0}(x, f)} \left( \bigotimes_{v \in V_{\gamma} \setminus x} F_{l(v)}(t^{-1}(v), l) \right)_{\text{Aut} (\gamma)}$$

is such that

$$|S| \circ |S^{-1}| = (X_{1}, ..., X_{d}) = |S^{-1}| \circ |S|.$$

In the statement above the notation

$$\left( \bigotimes_{v \in V_{\gamma} \setminus x} S_{l(v)}(t^{-1}(v), l) \right)_{\text{Aut} (\gamma)}$$

represents the colimit, which we assume that exists, of the functor that sends $\gamma \in T_{0}(x)$ into

$$\bigotimes_{v \in V_{\gamma} \setminus x} S_{l(v)}(t^{-1}(v), l).$$

5 Categorification of Feynman integrals

In order to find an appropriated categorification of finite dimensional Feynman integrals we need to generalize the notion of species to the context of $G$-$C$ species; here $G$ is a semisimple groupoid such that for $x \in G$ the cardinality of $G(x, x)$ is finite, and $C$ is a symmetric distributive category provided with a $R$-valuation. We call a groupoid semisimple if:

1. It is provided with a bifunctor $\oplus : G \times G \rightarrow G$, turning $G$ into a symmetric monoidal category. We choose an unit object and denote it by 0.

2. Every object in $G$ is isomorphic to a finite sum, unique up to reordering, of simple objects.

   An object $x \in G$ is simple if $x = x_{1} \oplus x_{2}$ implies that either $x_{1}$ is isomorphic to $x$ and $x_{2}$ is isomorphic to 0, or $x_{1}$ is isomorphic to 0 and $x_{2}$ is isomorphic to $x$.

Let $G$ be a semisimple groupoid and $g_{1}, ..., g_{n}, ...$ a countable family of formal variables. A $R[[g_{1}, ..., g_{n}, ...]]$-weight on $G$ is a map $\omega : G \rightarrow R[[g_{1}, ..., g_{n}, ...]]$ satisfying for $x, y \in G$ the following conditions:

$$\omega(x) = \omega(y) \text{ if } x \text{ is isomorphic to } y, \quad \omega(x \oplus y) = \omega(x)\omega(y), \text{ and } \omega(0) = 1.$$

We define the category of $G$-$C$ species to be the category $C^{G}$ of functors from $G$ to $C$. With these notation one defines a map $| \cdot | : C^{G} \rightarrow R[[g_{1}, ..., g_{n}, ...]]$ by

$$|F| = \sum_{x \in G} |F(x)| \frac{\omega(x)}{|G(x, x)|},$$

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Recall that a matching on a set \( x \) is a partition of \( x \) with blocks of cardinality two. Let \( M : B \to B \) be the species that sends \( x \) to the set \( M(x) \) of all matchings of \( x \). For applications to Feynman integrals we consider the groupoid of graphs \( \text{Gr} \). An object \( \gamma \in \text{Gr} \) is a triple \((F,V,E)\) such that:

- \( F \) is a finite set whose elements are called flags.
- \( V \) is a partition of \( F \); blocks of \( V \) are the vertices of \( \gamma \).
- \( E \) is a matching on \( F \); blocks of \( E \) are the edges of \( \gamma \).

Figure 5 shows how objects in \( \text{Gr} \) are represented in pictures. A morphism \( \varphi : \gamma_1 \to \gamma_2 \) in \( \text{Gr} \) is a bijection \( \varphi : F_1 \to F_2 \) such that \( \varphi(V_1) = V_2 \) and \( \varphi(E_1) = E_2 \).

On \( C^{\text{Gr}} \), the category of functors from \( \text{Gr} \) into \( C \), we define the sum of functors as usual

\[
(F + G)(\gamma) = F(\gamma) \oplus G(\gamma).
\]

The product functor on \( C^{\text{Gr}} \) is given by

\[
(FG)(\gamma) = \bigoplus_{\gamma_1 \sqcup \gamma_2 = \gamma} F(\gamma_1) \otimes G(\gamma_2)
\]

where \( \gamma_1 \sqcup \gamma_2 \) denotes the disjoint union of graphs. These definitions turn \((C^{\text{Gr}},+,\cdot)\) into a distributive category. Moreover each \( R[[g_1,\ldots,g_n,\ldots]]\)-weight on \( \text{Gr} \) induces a valuation map on \((C^{\text{Gr}},+,\cdot)\).

**Proposition 29.** The map \(|\cdot| : C^{\text{Gr}} \to R[[g_1,\ldots,g_n,\ldots]]\) given by

\[
|F| = \sum_{\gamma \in \text{Gr}} |F(\gamma)| \frac{\omega(\gamma)}{|\text{Gr}(\gamma, \gamma)|}
\]

defines a valuation on \( C^{\text{Gr}} \).

Notice that a vertex \( v \) of a graph is a subset of the set of flags, thus it makes sense to compute its cardinality \( |v| \). We shall use the following type of \( R[[g_1,\ldots,g_n,\ldots]]\)-weight on \( \text{Gr} \):

\[
\omega(\gamma) = \prod_{v \in V} g_{|v|} \text{ for } \gamma \in \text{Gr}.
\]

Below we shall also need the following map:

\[
\text{Pert} : \mathbb{C}[[x_1,\ldots,x_d]] \to \mathbb{C}[[x_1,\ldots,x_d, g_0, g_1,\ldots, g_k,\ldots]]
\]

\[
\sum_{a \in \mathbb{N}^d} f_a x^a \mapsto \sum_{a \in \mathbb{N}^d} f_a x^a \frac{g_{|a|}}{|a|!}
\]

where for \( a \in \mathbb{N}^d \) we set \( |a| = \sum_{i \in [d]} a_i \).
A fundamental property of the Gaussian measure is that it has a clear combinatorial meaning; in contrast, a similar understanding for the Lebesgue measure is lacking. The combinatorial meaning of Gaussian integrals may be summarized in the remarkable identity:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} dx = |M[n]|.$$

For example, see Figure 5, we have that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2}} dx = |M[4]| = 3.$$

Using polarization and diagonalization, the formula above implies the following identity for any positive definite symmetric $n^2$ real matrix $a$ and any $(x, f) \in \mathbb{B}^d$:

$$\frac{(2\pi)^{-\frac{d}{2}}}{\sqrt{\det a}} \int_{\mathbb{R}^d} \prod_{i \in x} x_{f(i)} e^{-\sum a_{ij}^{-1} x_i x_j} dx_1 ... dx_d = \sum_{\sigma \in M(x)} \prod_{m \in \sigma} a_{f(m)},$$

where for $m = \{i, j\} \in \sigma$ we set $f(m) = \{f(i), f(j)\}$.

Assume we are given $A_{ij} \in C$, for $i, j \in [d]$, such that $A_{ij} \simeq A_{ji}$ and $|A_{ij}| = a_{ij}$. Given $(x, f) \in \mathbb{B}^n$ we define $\text{Gr}(x, f)$ to be the full subcategory of $\text{Gr}$ whose objects are graphs such that: $x \subset F$ and $\{i\} \in V$ for all $i \in x$. Thus $\text{Gr}(x, f)$ denotes the category of graphs that include $x$ as a subset of the vertices of cardinality 1.
**Definition 30.** Fix $(x, f) \in \mathbb{B}^n$. The Feynman functor $\mathbf{F} : C_{\text{Gr}}^d \rightarrow C_{\text{Gr}}(x, f)$ sends $S \in C_{\text{Gr}}^d$ to $\mathbf{F}S \in C_{\text{Gr}}(x, f)$ given by

$$\mathbf{F}S(\gamma) = \bigoplus \left( \bigotimes_{v \in V} S(v, \hat{f}_v) \bigotimes_{e \in E} A_f(e) \right) \quad \text{for} \quad \gamma \in \text{Gr}(x, f),$$

where the sum runs over the extensions $\hat{f} : F \rightarrow [d]$ of $f : x \rightarrow [d]$.

**Theorem 31.** Fix $(x, f) \in \mathbb{B}^n$ and let $S \in C_{\text{Gr}}^d$ be such that $S(y, g) = 0$ if $|y| \leq 2$. The following identity holds

$$\frac{(2\pi)^{-\frac{d}{2}}}{\sqrt{\det a}} \int_{\mathbb{R}^d} e^{-a_{ij} x_i x_j + \sum_{|t| \geq 3} s_t \sum_{k_t \in k_t !} g_{|t|}} \prod_{i \in x} x_f(i) dx_1 ... dx_d = |\mathbf{F}S|.$$ 

**Proof.** Let the valuation of $S$ be given by

$$|S| = \sum_{t \in \mathbb{N}^d} s_t \frac{x^t}{t!}.$$ 

Then we have that

$$\frac{(2\pi)^{-\frac{d}{2}}}{\sqrt{\det a}} \int_{\mathbb{R}^d} e^{-a_{ij} x_i x_j + \sum_{|t| \geq 3} s_t \sum_{k_t \in k_t !} g_{|t|}} \prod_{i \in x} x_f(i) dx_1 ... dx_d =$$

$$\frac{(2\pi)^{-\frac{d}{2}}}{\sqrt{\det a}} \int_{\mathbb{R}^d} e^{-a_{ij} x_i x_j} \prod_{|t| \geq 3} \left( \sum_{k_t = 0}^{\infty} \frac{(s_t \sum_{k_t \in k_t !} g_{|t|})^{k_t}}{k_t !} \right) \prod_{i \in x} x_f(i) dx_1 ... dx_d =$$

$$\frac{(2\pi)^{-\frac{d}{2}}}{\sqrt{\det a}} \int_{\mathbb{R}^d} e^{-a_{ij} x_i x_j} \prod_{|t| \geq 3} \left( \sum_{k_t = 0}^{\infty} \frac{s_t \sum_{k_t \in k_t !} g_{|t|}}{k_t ! |t| |t|} \right) \prod_{i \in x} x_f(i) dx_1 ... dx_d.$$ 

Let $k$ be a map $k : \mathbb{N}^d \rightarrow \mathbb{N}$, such $k_t = 0$ for almost all $t \in \mathbb{N}^d$. The formula above is equal to

$$\frac{(2\pi)^{-\frac{d}{2}}}{\sqrt{\det a}} \sum_{k} \left( \prod_{|t| \geq 3} \frac{s_t \sum_{k_t \in k_t !} g_{|t|}}{k_t ! |t| |t|} \right) \int_{\mathbb{R}^d} e^{-a_{ij} x_i x_j} \prod_{i \in x} x_f(i) dx_1 ... dx_d.$$ 

Let $z = y \cup x$ and $(y, g) \cup (x, f)$ be such that $(y, g)$ is any colored set with $g^{-1}(i) = \sum_{t=3}^{\infty} k_t t_i$. Then the previous formula becomes

$$\sum_{k} \sum_{\sigma \in M(z)} \left( \prod_{|t| \geq 3} \frac{s_t \sum_{k_t \in k_t !} g_{|t|}}{k_t ! |t| |t|} \right) \prod_{m \in \sigma} a_{f(m)} = \sum_{\gamma \in \Gamma(z)} \left| \mathbf{F}S(\gamma) \right| \frac{\omega(\gamma)}{\text{Gr}(\gamma, \gamma)} = |\mathbf{F}S|.$$ 

\[\square\]

In the computations above we assumed that we could interchange infinite sums and integrals; that is the formal step in the definition of Feynman integrals. The formalism introduce in this section will be further developed in [11, 12, 19].

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6 Categorification of deformation quantization

In this section we assume that the reader is familiar with the notations and results from [36]. A Poisson manifold is a pair \((M, \{, \})\) where \(M\) is a \(d\)-dimensional smooth manifold provided with a bracket \(\{, \} : C^\infty(M) \otimes C^\infty(M) \rightarrow C^\infty(M)\) satisfying for all \(f, g, h \in C^\infty(M)\) the following identities:

1. \(\{f, g\} = -\{g, f\}\).
2. \(\{f, gh\} = \{f, g\}h + g\{f, h\}\).
3. \(\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}\).
4. \(\{, \}\) is a local bidifferential operator.

The axioms above imply that the bracket can be written in local coordinates as

\[
\{f, g\}(x) = \sum_{i,j} \alpha_{ij} \partial_i f \partial_j g
\]

where \(\alpha_{ij}\) is an antisymmetric \(m^2\) matrix with entries in \(C^\infty(M)\). The bivector

\[
\alpha = \sum_{i,j} \alpha_{ij} \partial_i \otimes \partial_j \in \Gamma(M, \bigwedge^2 TM)
\]

is called the Poisson bivector associated with the Poisson manifold \((M, \{, \})\).

**Definition 32.** Let \(M\) be a Poisson manifold. A formal deformation of \(C^\infty(M)\) is a star product \(\star : C^\infty(M)[[\hbar]] \otimes \mathbb{R}[[\hbar]] C^\infty(M)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]]\) such that:

1. \(\star\) is associative.
2. \(f \star g = \sum_{n=0}^\infty B_n(f, g)\hbar^n\), where \(B_n(, )\) are bidifferential operators.
3. \(f \star g = fg + \frac{1}{2}\{f, g\}\hbar + O(\hbar^2)\), where \(O(\hbar^2)\) are terms of order \(\hbar^2\).

Kontsevich [36] constructed a canonical \(\star\)-product for any finite dimensional Poisson manifold. For the manifold \((\mathbb{R}^d, \alpha)\) with Poisson bivector \(\alpha\) the \(\star\)-product is given by the formula

\[
\sum_{n=0}^\infty \frac{\hbar^n}{n!} \sum_{\gamma \in G_{n,2}} \omega_\gamma B_{\gamma, \alpha}(f, g),
\]

where \(G_{n,2}\) is the category of admissible graphs and \(\omega_\gamma\) are some constants which are independent of \(M\) and \(\alpha\). Let us proceed to define in details the category of admissible graphs.
Definition 33. For $k, n \in \mathbb{N} \times \mathbb{N}_{\geq 2}$ we let $G_{k,n}$ be the full subcategory of Digraph whose objects, called admissible graphs of type $(k, n)$, are directed graphs $\gamma$ such that:

1. $V_\gamma = V_\gamma^1 \sqcup V_\gamma^2$ where $V_\gamma^1$ and $V_\gamma^2$ are totally ordered sets with $|V_\gamma^1| = k$, $|V_\gamma^2| = n$.

2. $E_\gamma = V_\gamma^1 \times [2]$.

3. $t(e) \neq s(e)$ for $e \in E_\gamma$.

4. $s(v, i) = v$ for $(v, i) \in E_\gamma$.

Next we define a couple of partial orders $\leq_L$ and $\leq_R$ on $\bigsqcup_{k \geq 0} G_{k,n}$. First we need some graph theoretical notions. Let $\gamma_1$ and $\gamma_2$ be directed graphs. We say that $\gamma_1$ is included in $\gamma_2$ and write $\gamma_1 \subset \gamma_2$, if $V_{\gamma_1} \subset V_{\gamma_2}$, $E_{\gamma_1} \subset E_{\gamma_2}$ and $(s_1, t_1) = (s_2|_{E_{\gamma_1}}, t_2|_{E_{\gamma_1}})$. If $\gamma_1 \subset \gamma_2$ then we define the graph $\gamma_2/\gamma_1$ as follows:

1. $V_{\gamma_2/\gamma_1} = (V_{\gamma_2} \setminus V_{\gamma_1}) \sqcup \{\ast\}$.

2. $E_{\gamma_2/\gamma_1} = E_{\gamma_2} \setminus (s_2^{-1}(V_{\gamma_1}) \cap t_2^{-1}(V_{\gamma_1}))$.

3. $s_{\gamma_2/\gamma_1} = s_2$.

4. $t_{\gamma_2/\gamma_1}$ is equal to $t_2$ on $E_{\gamma_2/\gamma_1} \setminus t_2^{-1}(V_{\gamma_1})$ and to $\{\ast\}$ on $E_{\gamma_2/\gamma_1} \cap t_2^{-1}(V_{\gamma_1})$.

Let $\gamma \in \bigsqcup_{k \geq 0} G_{k,n}$ and assume that $V_\gamma^2 = \{i_1 < i_2 \ldots < i_n\}$. For $\gamma_1 \in \bigsqcup_{k=0}^{m} G_{k,n}$ we let $\gamma_1 \leq_L \gamma$ if and only if:

$\gamma_1 \subset \gamma$, $V_{\gamma_1}^0 = \{i_1 < i_2 \ldots < i_s\}$ for some $s \leq n$, and $\gamma/\gamma_1$ is an admissible graph.

Similarly we let $\gamma_1 \leq_R \gamma$ if and only if:

$\gamma_1 \subset \gamma$, $V_{\gamma_1}^0 = \{i_s < i_{s+1} < \ldots < i_n\}$ for some $s \leq n$, and $\gamma/\gamma_1$ is an admissible graph.

In order to categorify the Poisson manifold $(\mathbb{R}^d, \alpha)$ we need to find a distributive category with a natural valuation on the ring $(R[[x_1, \ldots, x_d, \hbar]], \ast)$. We make the following assumptions:

1. A categorification $|\ | : C \longrightarrow R$ of $R$ is given.

2. For $1 \leq i \neq j \leq d$ we are given $A^{ij} \in C^{\mathbb{R}^d}$ such that $A^{ij} \simeq -A^{ji}$ and $|A^{ij}| = \alpha^{ij}$.

3. We are given a functor $\Omega : \bigsqcup_{k \geq 0} G_{k,2} \longrightarrow C$ which sends $\gamma$ into $\Omega_\gamma$. We assume that there are natural isomorphisms:

$$\sum_{\gamma_1 \leq \gamma} \Omega_{\gamma_1} \otimes \Omega_{\gamma/\gamma_1} \simeq \sum_{\gamma_2 \geq \gamma} \Omega_{\gamma_2} \otimes \Omega_{\gamma_2}$$

for which Mac Lanes’s pentagon axiom holds.
Recall that an object in $\mathbb{B}^{d+1}$ may be identified with a triple $(x, f, y)$ where $x, y \in \mathbb{B}$ and $f : x \to [d]$.

**Theorem 34.** $(\mathbb{C}\mathbb{B}^{d+1}, +, \star)$ is a distributive category with the $\star$-product on functors given by:

$$F \star G(x, f, y) = \bigoplus_{y_1 \sqcup y_2 \sqcup y_3 = y} \bigoplus_{\gamma \in G_{[y_2, y_3]} \cup E_{y \to [d]}} \Omega_{\gamma} \otimes A_{\gamma, I, b} \otimes B_{\gamma, I, b}$$

where

1. $I : E_{\gamma} \to [d]$ and $b : x \to V_{\gamma}$ are maps.
2. $A_{\gamma, I, b} = \bigotimes_{v \in V_{\gamma}} A_{I_{b^{-1}(v)}}(v) \cup E_{v} \cup f_{|b^{-1}(v)} \cup I_{E_{v}}$.
3. $B_{\gamma, I, b} = F(b^{-1}(1) \cup E_{E_{1}} \cup f_{|b^{-1}(1)} \cup I_{E_{1}}) \otimes G(b^{-1}(2) \cup E_{E_{2}} \cup f_{|b^{-1}(2)} \cup I_{E_{2}})$.

**Proof.** The key issue is that on the one hand we have that:

$$F \star (G \star H)(x, y, f) = \bigoplus_{\gamma_1 \leq \gamma \leq \gamma_3} \Omega_{\gamma_1} \otimes \Omega_{\gamma_3}$$

where the sum runs over all $y_1 \sqcup y_2 \sqcup y_3 = y$, $\gamma \in G_{[y_2, y_3]}$ and $I : E_{\gamma} \to [m]$. On the other hand we have that:

$$(F \star G) \star H(x, f, y) = \bigoplus_{\gamma_1 \leq \gamma \leq \gamma_3} \Omega_{\gamma_1} \otimes \Omega_{\gamma_3}$$

where the sum runs over all $y_1 \sqcup y_2 \sqcup y_3 = y$, $\gamma \in G_{[y_2, y_3]}$ and $I : E_{\gamma} \to [d]$. Above $A, B$ are defined as in the statement of the Theorem 37.
**Definition 35.** Let \( (R[[x_1, \ldots, x_d, h]], \star) \) be the ring for formal power series in the variables \( x_1, \ldots, x_d, h \) with the star product \( \star \) given by the Kontsevich’s formula above with the constants \( \omega_\gamma \) given by \( \omega_\gamma = |\Omega_\gamma| \in R \).

**Theorem 36.** The map \( |\cdot| : (C_B^{d+1}, +, \star) \rightarrow (R[[x_1, \ldots, x_d, h]], \star) \) given by

\[
|F| = \sum_{(a,b) \in \mathbb{N}^d \times \mathbb{N}} |F([a], [b])| \frac{x^a h^b}{a! b!}
\]

defines a valuation on \( (C_B^{d+1}, +, \star) \).

**Proof.** One checks that

\[
F \star G = \sum_{n=0}^{\infty} \left( \sum_{\gamma \in G_{n,2}} \Omega_\gamma B_\gamma(F, G) \right) \frac{H^n}{n!}
\]

where \( B_\gamma(F, G) \) is given by

\[
\sum_{I:E_\gamma \rightarrow [d]} \prod_{v \in V_\gamma^0} \left( \prod_{e \in E_\gamma, t(e)=v} \partial I(e) \right) A^{I(v,1)} A^{I(v,2)} \left( \prod_{e \in E_\gamma, t(e)=1} \partial I(e) \right) F \left( \prod_{e \in E_\gamma, t(e)=v} \partial I(e) \right) G.
\]

Taking valuations and looking at page 5 of [36] ones obtains the desired result. \( \square \)

![Figure 8: Example of a graph \( \gamma \in G_{3,2} \) and \( B_\gamma(F, G) \).](image)

Finding a category \( C \) with an appropriated family of objects \( \Omega_\gamma \) is by no means an easy matter, fortunately Kontsevich’s have shown that there are indeed examples [36]. We hope that the methods developed in this section may be of some use in order to find further examples.

Using induction and the formula from Theorem [34] one can show that:
Theorem 37. Let $F_1 \ast F_2 \ast F_3 \cdots \ast F_n \in C^{B^{d+1}}$, then we have that:

$$F_1 \ast F_2 \ast F_3 \cdots \ast F_n(x, f, h) \simeq \bigoplus_{\gamma \in \mathcal{G}(h_{n+1}/n)} \Omega_\gamma \otimes A_\gamma \otimes B_\gamma$$

where the sum runs over the decompositions $h_1 \sqcup \cdots \sqcup h_{n+1} = h$ of $h$ into $n + 1$ disjoint blocks and

1. $\bigoplus_{\gamma \leq \delta \cdots \leq \delta \gamma_2 \leq \delta \gamma} \Omega_{\gamma_1} \otimes \bigotimes_{i=2}^{n-2} \Omega_{\gamma_i/\gamma_{i-1}} \otimes \Omega_{\gamma/\gamma_{n-2}}.$

2. $A_\gamma = \bigotimes_{v \in V_\gamma} A^{I(v_1)}(e^2_v)(b^{-1}(v) \sqcup E_v, f|_{b^{-1}(v)} \sqcup I|_{E_v})$.

3. $B_\gamma = \bigotimes_{i=1}^{d} F_i(b^{-1}(\overline{v}) \sqcup E_v, f|_{b^{-1}(v)} \sqcup I|_{E_v})$.

7 Categorification of quantum phase space

In this section we consider the categorification of the quantum phase space of a free particle with $n$-degrees of freedom. Before developing the details of our approach we like to mention that there are other attempts to try to understand quantum mechanics using category theory, for example the reader may consult [43, 53]. Quantum phase space in this case is the deformation quantization of the classical phase space, which may be identified with the symplectic manifold $(\mathbb{R}^{2n}, \{ , \})$ with bracket:

$$\{x_i, x_j\} = \delta_{ij}, \quad \{x_i, x_j\} = \{y_i, y_j\} = 0$$

for $x_1, \ldots, x_d, y_1, \ldots, y_d$ coordinates on $\mathbb{R}^{2n}$. The ring of functions on the associated quantum phase space is isomorphic to the formal Weyl algebra $W_d$ which we proceed to introduce.

Definition 38. Let $R$ be a commutative ring. The Weyl algebra $W_d$ over $R$ is given by

$$W_d = R\langle\langle x_1, \ldots, x_d, y_1, \ldots, y_d\rangle\rangle[[h]]/I_d$$

where $I_d$ is the ideal generated for $i, j \in [d]$ by the following relations

$$[y_i, x_j] = \delta_{ij}h, \quad [x_i, x_j] = [y_i, y_j] = 0.$$
Definition 39. The distributive category \((\mathbb{C}^{\mathbb{B}^{2d+1}}, +, \star)\) is such that the sum and product functors are given by:

\[
F \oplus G(x, f, h) = F(x, f, h) \oplus G(x, f, h),
\]

\[
F \star G(x, f, h) = \bigoplus F(x_1 \sqcup h_3, f|_{x_1} \sqcup \{2\} \times g, h_1) \otimes G(x_2 \sqcup h_3, f|_{x_2} \sqcup \{1\} \times g, h_2)
\]

where the sum runs over all pairs \(x_1, x_2\) and all triples \(h_1, h_2, h_3\) such that

\[x_1 \sqcup x_2 = x, h_1 \sqcup h_2 \sqcup h_3 = h\] and \(g : h_3 \rightarrow [d]\).

Figure 9 illustrates with an example the graphical interpretation of the star product \(F \star G\), where \(F\) and \(G\) are functors from \(\mathbb{B}^3\) to \(C\).

Our next result is a direct consequence of the Proposition 47 shown below.

Theorem 40. The map \(|\ | : \mathbb{C}^{\mathbb{B}^{2d+1}} \rightarrow \mathbb{W}_d\) given by

\[
|F| = \sum_{(a,b,c) \in \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{N}} |F([a], [b], [c])| \frac{x^a}{a!} \frac{y^b}{b!} \frac{h^c}{c!}
\]

defines a valuation map on \((\mathbb{C}^{\mathbb{B}^{2d+1}}, +, \star)\).

Using Definition 42 inductively ones obtains the following result:

Proposition 41. Given \(F_1, \ldots, F_m \in \mathbb{C}^{\mathbb{B}^{2d+1}}\) we have that:

\[
F_1 \star \cdots \star F_m(x, f, h) = \bigoplus \bigotimes_{j=1}^m F_j(x_j, \bigcup_{i<j} h_{ij}, \bigcup_{j<k} h_{jk}, f|_{x_j} \bigcup_{i<j} \{1\} \times g, h_{jj})
\]

where the sum runs over the sets \(x_i, h_{ij}\), and the maps \(g\) such that:

\[
\bigcup_{i=1}^d x_i = x, \bigcup_{i \leq j} h_{ij} = h\] and \(g : \bigcup_{i < j} h_{ij} \rightarrow [d]\).
The result from the previous proposition may be rewritten as follows:

\[ F_1 \star \cdots \star F_m(x, f, h) = \bigoplus_{\gamma \in G_m} O_\gamma, \]

where \( G_m \) is the set of equivalence classes of graphs \( \gamma \) such that:

1. \( F_\gamma = x \uplus h_1 \uplus [2] \times h_2 \) where \( h_1 \uplus h_2 = h \).
2. \( g : h_2 \rightarrow [d] \) is any map. Below we use the natural extension map \( \hat{g} : [2] \times h_2 \rightarrow [2] \times [d] \).
3. \( V_\gamma = \{b_1, \ldots, b_d\} \) where \( \bigsqcup_{i=1}^d b_i = F_\gamma, b_i \cap b_j = \emptyset \) and if it happens that \( s \in h_2, (2, s) \in b_i \), and \( (1, s) \in b_j \), then it must also happen that \( i < j \).

4. \( E_\gamma = \{(s, 1), (s, 2) | s \in h_2\} \).

We associate to each \( \gamma \in G_m \) an object \( O_\gamma \in C \) as follows:

\[ O_\gamma = \bigotimes_{i=1}^m F_i(b_i \cap (x \uplus [2] \times h_2), f|_{b_i \cap x} \uplus \hat{g}|_{b_i \cap ([2] \times h_2)}, b_i \cap h_1). \]

With this notation it should be clear that the formula above for the star product \( F_1 \star \cdots \star F_m \) is just a reformulation of Proposition 11.

Next we consider the quantum analogue of the binomial sequences.

**Definition 42.** Let \( m, n, l \in \mathbb{N} \). A sequence \( \{s_{m,n,l}\} \) where \( s_{m,n,l} : \mathbb{N} \rightarrow R \) such that

\[ s_{m,n,l}(r + t) = \sum_{l_1,l_2,l_3,m_1,m_2,n_1,n_2} \binom{l}{l_1,l_2,l_3} \binom{m}{m_1} \binom{n}{n_1} s_{m_1,n_1+s_3,l_1}(r) s_{m_2+n_2,l_2}(t), \]

where \( l_1 + l_2 + l_3 = l, m_1 + m_2 = m \) and \( n_1 + n_2 = n \) is called a quantum multinomial sequence.

Our next results describe a natural source of quantum multinomial sequences and provide a categorical interpretation for such sequences. Let \( s = \sum_{a,b,c} s_{m,n,l} \frac{x^m y^n h^l}{m! n! l!} \in W_d \), then we set \( s^0 = 1 \), and for \( r \in \mathbb{N} \) set \( s^{r+1} = s^r \star s \), and

\[ s^r = \sum_{m,n,l} s_{m,n,l}(r) \frac{x^m y^n h^l}{m! n! l!}. \]

**Proposition 43.** 1. The sequence \( \{s_{m,n,l}\} \) defined above is a quantum multinomial sequence.

2. Assume that \( S \in C^{B^3} \) is such that \( |S| = s \), then

\[ s_{m,n,l}(r) = \left| \bigoplus_{\Gamma \in G_n} \bigotimes_{i=1}^m S(b_i \cap (x \uplus [2] \times h_2), f|_{b_i \cap x} \uplus \hat{g}|_{b_i \cap ([2] \times h_2)}, b_i \cap h_1) \right|. \]
Let us consider a particular example of quantum multiplicative sequence and provide a categorical interpretation for it.

**Definition 44.** Let \( n, a, b \) be integers such that \( n - a - b \) is zero or even. The quantum binomial is given by

\[
\binom{n}{a, b} = \frac{n!(n - a - b - 1)!!}{a!b!(n - a - b)!}.
\]

The reader should not confuse the integers \( \binom{n}{a, b} \) with the \( q \)-analogues of the binomial coefficients that are so often studied in the literature. Below we need the singleton species \( X \) and \( Y \), they are define just as in the case of commutative species.

**Proposition 45.** \(|(X + Y)^n([a], [b], [c])| = \binom{n}{a, b} a!b!c!\).

**Proof.** According to Proposition 41 in order to construct \((X + Y)^n([a], [b], [c])\), see Figure 11, one should:

1. Choose a partition of \([n]\) in 3 blocks with cardinalities \( a, b, \) and \( 2c \), respectively. There are \( \binom{n}{a, b, 2c} \) ways of doing this. Choose a linear order on the first and second blocks. There are \( a!b! \) such possible orderings.

2. Choose a pairing on \([2c]\); there are \( (2c - 1)!! \) possible choices.

3. Choose a bijection between \([c]\) and the pairing selected in the previous step. There are \( c! \) choices.

All together we see that

\[
|(X + Y)^n([a], [b], [c])| = \binom{n}{a, b, 2c} a!b!(2c - 1)!!c! = \binom{n}{a, b} a!b!c!.
\]

\(\square\)

Using the formula above one obtains that:

\[
(x + y)^n = |(X + Y)^n| = \sum_{a+b+2c=n} \frac{n!(2c - 1)!!c! x^a y^b h^c}{(2c)! a! b! c!} = \sum_{a+b+2c=n} \binom{n}{a, b} x^a y^b h^c.
\]

The quantum binomial coefficients satisfy a recursion relation which we describe next. It follows from identity \((X + Y)^{n+1} = (X + Y)(X + Y)^n\) and identity \(2.(b)\) below.

**Proposition 46.** For \( n, a, b, c \in \mathbb{N} \) such that \( a + b + 2c = n + 1 \) the following identity holds

\[
\binom{n+1}{a, b} = \binom{n}{a-1, b} + \binom{n}{a, b-1} + (a+1)\binom{n}{a+1, b}.
\]
**Proposition 47.** The following formulae hold in \((C^3, +, \ast)\):

1. \(X \ast X \ast \cdots \ast X \ast Y \ast Y \ast \cdots \ast Y = X^p Y^q\).

2. \(YX = XY + H\).

3. \(\frac{Y^n}{n!} \ast \frac{X^m}{m!} = \sum_{i=0}^{\min} \frac{X^{m-i}}{m-i!} \frac{Y^{n-i}}{(n-i)!} \frac{H^i}{i!}\), where \(\min = \min\{n, m\}\).

**Proof.** 1. is obvious. The second formula of Definition [12] implies that for all \(a, b, c \in \mathbb{B}\) the product \(Y \ast X(a, b, c) = 1\) if \(|a| = |b| = 1\) and \(|c| = 0\), or \(|a| = |b| = 0\) and \(|c| = 1\). Otherwise \(YX(a, b, c) = \emptyset\). This result exactly agree with \((XY + H)(a, b, c)\) for \(a, b, c \in \mathbb{B}^3\).

Recall that \(\frac{X^n}{n!}(a, b, c) = 1\) if \(|a| = n\) and \(|b| = |c| = 0\), and 0 otherwise. Since

\[
\frac{Y^n}{n!} \ast \frac{X^m}{m!} = \frac{Y^n}{n!}(b \cup c) \otimes \frac{X^m}{m!}(a \cup c),
\]

then \(\frac{Y^n}{n!} \ast \frac{X^m}{m!}(a, b, c) = 1\) if and only if \(|b| + |c| = n\) and \(|a| + |c| = m\) and zero otherwise. Therefore \(\frac{Y^n}{n!} \ast \frac{X^m}{m!}(a, b, c) = 1\) if and only if \(|b| = n - |c|, |a| = m - |c|\) and \(|c| \leq \min\{n, m\}\), that is

\[
\frac{Y^n}{n!} \ast \frac{X^m}{m!} = \sum_{i=0}^{\min} \frac{X^{m-i}}{m-i!} \frac{Y^{n-i}}{(n-i)!} \frac{H^i}{i!}.\]

\[\qed\]

Taking valuations in the Proposition above, setting \(a = 1, m = n\) for part 2, and \(b = n, m = 1\) for part 3 ones gets the following corollary.

**Corollary 48.** The following identities hold in \(W_1\):

1. \(\frac{y^n}{n!} \ast \frac{x^m}{m!} = \sum_{i=0}^{\min} \frac{X^{m-i}}{m-i!} \frac{Y^{n-i}}{(n-i)!} \frac{H^i}{i!}\), where \(\min = \min\{n, m\}\).
2. \(yx^n = x^n y + nx^{n-1}h\).

3. \(y^n x = xy^n + ny^{n-1}h\).

Let us provide another application of our categorical approach to Weyl algebras.

**Proposition 49.**

1. \([XY, X^n] = nX^n H\).

2. \(e^Y e^X = e^X e^Y e^H\).

**Proof.**

1. We have

\[ Y \star X^n(a, b, c) = \bigoplus Y(a_1, b_1 \sqcup c_0, c_1) \otimes X^n(c_0 \sqcup a_2, b_2, c_2) \]

where the sum runs over all set \(a, b, c\) such that \(a_1 \sqcup a_2 = a, b_1 \sqcup b_2 = b\) and \(c_0 \sqcup c_1 \sqcup c_2 = c\).

- \(|b_1| = 1, c_0 = \emptyset\) implies that \(c = \emptyset\). We have the specie \(X^n Y\).
- \(|b_1| = \emptyset, |c_0| = 1, |a_2| = n - 1\). In this case we have the specie \(nYX^{n-1}H\).

finally, we have \([XY, X^n] = nX^n H\).

2. \(e^Y e^X(a, b, c) = \bigoplus e^Y(a, b \sqcup c) \otimes e^X(a \sqcup c, b) = e^Y(b \sqcup c) e^X(a \sqcup c) = 1_C = e^X e^Y e^H(a, b, c)\)

for \(a, b, c \in \mathbb{B}\).

\(\square\)

**Example 50.** Consider the star product \(Y^4 \star X^3\) of the species \(Y^4, X^3\). The possible graphs arising in this case are shown in Figure 11. Thus we see that in the Weyl algebra the following identity holds:

\[ y^4 \star x^3 = |Y^4 \star X^3| = x^3 y^4 + 12xy^3 h + 36xy^2 h^2 + 24yh^3. \]

\begin{figure}[h]
\centering
\begin{tabular}{ccc}
0 edges & \begin{tikzpicture}[baseline=-.5ex]
    \node (a) at (0,0) {$\leftarrow$};
    \node (b) at (1,0) {$\rightarrow$};
\end{tikzpicture} & x^3 y^4 \\
1 edge & \begin{tikzpicture}[baseline=-.5ex]
    \node (a) at (0,0) {$\leftarrow$};
    \node (b) at (1,0) {$\rightarrow$};
\end{tikzpicture} & 12xy^3 h \\
2 edges & \begin{tikzpicture}[baseline=-.5ex]
    \node (a) at (0,0) {$\leftarrow$};
    \node (b) at (1,0) {$\rightarrow$};
\end{tikzpicture} & 36xy^2 h^2 \\
3 edges & \begin{tikzpicture}[baseline=-.5ex]
    \node (a) at (0,0) {$\leftarrow$};
    \node (b) at (1,0) {$\rightarrow$};
\end{tikzpicture} & 24yh^3 \\
\end{tabular}
\caption{Graphs contributing to the computation of \(y^4 \star x^3\).}
\end{figure}

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8 Categorification of formal superspace

In this closing section we study the categorification of the formal supermanifold $\mathbb{R}^{d|m}$. Let $R$ be a commutative ring, then by definition the ring of $R$-valued functions on $\mathbb{R}^{d|m}$ is given by

$$R[[\mathbb{R}^{d|m}]] = R[[x_1, \ldots, x_d]] \otimes \bigwedge[\theta_1, \ldots, \theta_m].$$

An element $f \in R[[\mathbb{R}^{d|m}]]$ can be uniquely written as $f = \sum_{I \subset [m]} f_I \theta_I$ where $f_I \in R[[x_1, \ldots, x_d]]$, and for $I = \{i_1 < \cdots < i_k\}$ we set $\theta_I = \theta_{i_1} \cdots \theta_{i_k}$. The product on $C[[\mathbb{R}^{d|m}]]$ is given by

$$fg = \left( \sum_{I \subset [m]} f_I \theta_I \right) \left( \sum_{J \subset [m]} g_J \theta_J \right) = \sum_{K \subset [m]} \left( \sum_{I \cup J = K} \text{sgn}(I, J) f_I g_J \right) \theta_K,$$

where

$$\theta_I \theta_J = \text{sgn}(I, J) \theta_{I \cup J} \quad \text{and} \quad \text{sgn}(I, J) = (-1)^{\{i,j \in I \times J \mid j < i\}}.$$

More generally if $I_1 \cup I_2 \cup \cdots \cup I_p = I$ then we define

$$\text{sgn}(I_1, I_2, \ldots, I_p) = (-1)^{\{i,j \in I_k \mid j \in I_1, 1 \leq k < p \text{ and } j < i\}}.$$

**Definition 51.** Given $d, m \in \mathbb{N}$ we let $\mathbb{B}^d \times F_m$ be the category such that:

- $\text{Ob}(\mathbb{B}^d \times F_m) = \{(x, f, I) : x \in \mathbb{B}^d, f : x \to [d], I \subset [m]\}$.

- $\mathbb{B}^d \times F_m((x, f, I), (y, g, J)) = \{a : x \to y \mid \alpha \text{ is a bijection and } ga = f\}$ if $I = J$; otherwise it is the empty set.

**Definition 52.** Let $C$ be a symmetric distributive category. The category $C^{\mathbb{B}^d \times F_m}$ of functors from $\mathbb{B}^d \times F_m$ to $C$ is called the category of superspecies of type $d|m$.

Figure 12 shows the graphical representation of the action of a superspecies in $C^{\mathbb{B}^5 \times F_8}$, where the standard lines are bosons and the bold lines are fermions, i.e. represent commutative or anti-commutative variables.

**Definition 53.** Let $C$ be a symmetric distributive category and let $F, G \in C^{\mathbb{B}^d \times F_m}$. The following formulae define the sum and product for superspecies:

$$(F + G)(x, f, I) = F(x, f, I) \oplus G(x, f, I),$$

$$FG(x, f, I) = \bigoplus \text{sgn}(I_1, I_2) F(x_1, f|_{x_1}, I_1) \otimes G(x_2, f|_{x_2}, I_2)$$

where the sum runs over all $x_1 \cup x_2 = x$ and $I_1 \cup I_2 = I$.

**Theorem 54.** $(C^{\mathbb{B}^d \times F_m}, +, \cdot)$ is a symmetric distributive category, and the map

$$| | : C^{\mathbb{B}^d \times F_m} \longrightarrow R[[x_1, \ldots, x_d]] \otimes \bigwedge[\theta_1, \ldots, \theta_m]$$

given by $|F| = \sum_{(a, I) \in \mathbb{B}^d \times F_m} |F[a, I]| \frac{d^a}{dt} \theta_I$ is a valuation map on $C^{\mathbb{B}^d \times F_m}$.
Next we define the super analogue of the binomial coefficients \([15]\), and provided a combinatorial interpretation for them. We also include a combinatorial interpretation for the multiplicative inverse of a superspecies.

**Definition 55.** Let \( n \in \mathbb{N}^d \), \( I \subset [m] \) and \( a \in \mathbb{N} \). A sequence \( \{s_{n,I}\} \) where \( s_{n,I} : \mathbb{N} \rightarrow R \) is called a super multiplicative sequence if

\[
s_{n,I}(a + b) = \sum_{j,A} \text{sgn}(A, A') \binom{n}{j} s_{j,A}(a)s_{n-j,I-A}(b)
\]

where \( 0 \leq j \leq n \) and \( A \subset I \).

For \( s = \sum_{n,I} s_{n,I} \frac{x^n}{n!} \theta_I \in R[[\mathbb{R}^d|m]] \) we set \( s^0 = 1 \), \( s^{a+1} = s^a s \), and

\[
s^a = \sum_{n,I} s_{n,I}(a) \frac{x^n}{n!} \theta_I.
\]

**Proposition 56.** 1. The sequence \( \{s_{n,I}\} \) defined above is super multiplicative.

2. If \( S \in C_{\mathbb{R}^d \times F_m} \) is such that \(|S| = s\), then

\[
s_{n,I}(a) = \bigoplus_{i=1}^a \text{sgn}(I_1, \ldots, I_a) \bigotimes_{i=1}^a S(x_i, f_{x_i}, I_i)
\]

where the sum runs over all partitions \( x_1 \sqcup \cdots \sqcup x_a = x \) and \( I_1 \sqcup \cdots \sqcup I_a = I \).

3. If \( S = 1 - F \) where \( F \in C_{\mathbb{R}^d \times F_m} \) is such that \( F(\emptyset) = 0 \), then the superspecies

\[
S^{-1}(x, f, I) = \bigoplus_{a=1}^{|x|+|I|} \bigoplus_{\sqcup_i I_i = I} \bigotimes_{i=1}^a \text{sgn}(I_1, \ldots, I_a) \bigotimes_{I_i = I_i} S(x_i, f_{x_i}, I_i).
\]

is such that \(|S||S^{-1}| = 1 = |S^{-1}||S|\).
The methods and techniques introduced in this work will gradually find applications in a variety of settings. Applications of superspecies to the study of formal simple supersymmetries will be developed in [20]. For an introduction to Lie algebras with a view towards categorification the reader may consult [21]. One expects to find, along the lines developed in this work, categorifications of several variants of the Weyl algebra and their symmetric powers [22, 23, 24, 25]. It should be possible to find a categorical analogue of the perturbative methods developed by Díaz and Leal [18] in order to obtain topological and geometrical invariants from equivariant classical field theories.

Acknowledgments
This work owes much to conversations of the first author with a true teacher and friend Professor Gian-Carlo Rota. Part of this work was done while the first author was visiting ICTP, Italy, and Universidad de Sonora, México.

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