On the exact solutions of the Bianchi IX cosmological model in the proper time

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Abstract
It has recently been argued that there might exist a four-parameter analytic solution to the Bianchi IX cosmological model, which would extend the three-parameter solution of Belinskii et al. to one more arbitrary constant. We perform the perturbative Painlevé test in the proper time variable, and confirm the possible existence of such an extension.

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1 Introduction

The Bianchi IX cosmological model results from the assumption of a universe with a homogeneous anisotropic space, and an important question concerns the singularities in the complex $t$-plane of the components $(a^2, b^2, c^2)$ of the metric tensor \[1, 2, 3\]. This system is made of three coupled second order nonlinear ordinary differential equations:

\[
2\sigma^2 abc \frac{d}{dt} \left( abc \frac{d(ln a)}{dt} \right) = a^4 - (b^2 - c^2)^2 \quad \text{and cyclically,} \tag{1}
\]

with the first integral

\[
I = \sigma^2 \left[ a^2 \frac{d(b^2)}{dt} \frac{d(c^2)}{dt} + b^2 \frac{d(c^2)}{dt} \frac{d(a^2)}{dt} + c^2 \frac{d(a^2)}{dt} \frac{d(b^2)}{dt} \right] + a^4 + b^4 + c^4 - 2(b^2 c^2 + c^2 a^2 + a^2 b^2). \tag{2}
\]

Here $\sigma^2$ is $+1$ or $-1$ according as the metric is asymptotically Euclidean or Minkowskian. The study of these singularities is easier in the so-called logarithmic time variable $\tau$, related to the proper time variable $t$ through the transformation:

\[
dt = abc \, d\tau. \tag{3}
\]

The main result of the Painlevé test \[4 \rightarrow 7\] is a probable chaotic behaviour of the system. Moreover, one can prove \[8\] the inexistence of rational first integrals other than \(2\). As a by-product, this test can also be used to detect all possible particular analytic solutions, in addition to the three particular solutions which were already known.

The first exact solution results from the existence of the Euler subsystem (1750):

\[
\sigma \frac{d\omega_1}{d\tau} = \omega_2 \omega_3 \quad \text{and cyclically} \tag{4}
\]

\[
\sigma \frac{d(bc)}{dt} = a \quad \text{and cyclically} \tag{5}
\]

with $\omega_1 = bc$ and cyclically, reducible to Weierstrass’ elliptic equation in the variable $\omega_1^2(\tau)$:

\[
\sigma^2 \left( \frac{d(\omega_1^2)}{d\tau} \right)^2 = 4\omega_1^2(\omega_1^2 + k_2)(\omega_1^2 + k_3). \tag{6}
\]

\[
\left( \frac{d(\omega_1^2)}{dt} \right)^4 = 16\omega_1^2(\omega_1^2 + k_2)(\omega_1^2 + k_3) \tag{7}
\]

The three–parameter solution of system \[9\], single-valued in $\tau$, was found by Abel and Jacobi, and later in the study of the Bianchi IX system by Belinskii \emph{et al.} \[9\].

A second subsystem exists, the so-called Darboux system \[10\] :

\[
\sigma \frac{d\omega_1}{d\tau} = \omega_2 \omega_3 - \omega_1 \omega_2 - \omega_1 \omega_3 \quad \text{and cyclically} \tag{8}
\]

\[
\sigma \frac{d(bc)}{dt} = a - b - c \quad \text{and cyclically} \tag{9}
\]
which has been integrated by Halphen \cite{11, 12}. This three-parameter solution, single-valued in $\tau$ was rediscovered in the context of Bianchi IX, by Gibbons et al. \cite{13}. This system is equivalent to the third order Chazy equation of class III \cite{14}

\begin{equation}
\frac{d^3y}{d\tau^3} - 2y \frac{d^2y}{d\tau^2} + 3 \left(\frac{dy}{d\tau}\right)^2 = 0 \tag{10}
\end{equation}

with $y = -2(\omega_1 + \omega_2 + \omega_3)/\sigma$.

Last, a third particular solution was found by Taub \cite{15}, when the metric tensor is axially symmetric (e.g. $b^2 = c^2$), leading to a four-parameter trigonometric solution.

As to the Painlevé analysis, it identifies three possible local behaviours of the general solution in the $\tau$ variable

1. a simple pole for $a^2$ and a simple zero for $b^2$ and $c^2$;
2. a simple pole for $a^2$, $b^2$ and $c^2$;
3. a regular behaviour for $a^2$ and a double pole for $b^2$ and $c^2$.

Finally, the only possible new solution isolated by the Painlevé analysis is \cite{1} an extension of Belinskii’s elliptic solution to four parameters, whose closed form is yet unknown and which we denote for convenience BGPP4. The finding of this closed form has a strong interest in cosmology, since it would ipso facto yield a six-parameter single-valued solution to the Brans-Dicke Bianchi IX cosmological model \cite{16, 17}.

The aim of this paper is to further examine this four-parameter solution. The connection between the local behaviours in $t$ and those in $\tau$ is established in section 2. We perform in section 3 the perturbative Painlevé test \cite{18, 19} on system (1) in the proper time variable $t$, in order to see whether or not the existence of the four-parameter solution can be denied.

## 2 The local study

The successive steps are the following. First we determine all possible leading behaviours of the Laurent series for the dependent variables $a(t), b(t)$ and $c(t)$. This is done by substituting respectively $a_0(t-t_0)^{p_1}, b_0(t-t_0)^{p_2}, c_0(t-t_0)^{p_3}$ into the system (1), and by checking all the different cases for the constants $p_1, p_2, p_3$, under the condition that the constants $a_0, b_0, c_0$ are all different from zero.

Then, for each leading behaviour, we have to look for the so-called indices or resonances. These indices can be found by inserting:

\[ a(t) = a_0 \chi^{p_1}(1 + \varepsilon a_r \chi^r); \quad b(t) = b_0 \chi^{p_2}(1 + \varepsilon b_r \chi^r); \quad c(t) = c_0 \chi^{p_3}(1 + \varepsilon c_r \chi^r); \]

with $\chi = t - t_0$, into the system (1), and by taking only those terms which are linear in $\varepsilon$. This leads to a coupled system of three linear equations in $a_r, b_r, c_r$, whose solution is unique iff the determinant of the coefficients differs from zero. The indices are the zeroes $r$ of this determinant.
If all the leading exponents are positive, it is recommended to invert at least one of the dependent variables, for example $a(t) \rightarrow a(t)^{-1}$ in order to calculate the indices. For a more detailed explanation we refer to [20].

If we now apply the scheme explained above, we obtain only three possibilities, listed in Table 1.

The arbitrary values of the two opposite indices of the third family (III) immediately imply an infinite amount of logarithmic branching, hence probably chaos.

The last row represents the local transformations, after integration of relation (3) in which we introduced the leading behaviours for $a(t)$, $b(t)$ and $c(t)$.

Table 1: Families of singularities of $a$, $b$ and $c$ in the variable $t$. Notation is $\varepsilon^2 = \varepsilon_i^2 = 1$.

|        | (I)                        | (II)                        | (III)                      |
|--------|----------------------------|-----------------------------|----------------------------|
| $p_1$, $p_2$, $p_3$ | $(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ | $(1, 1, 1)$                 | $(1, 0, 0)$                 |
| $a_0$, $b_0$, $c_0$ | $(2\varepsilon b_0 c_0 / 3, b_0, c_0)$ | $(\varepsilon_1, \varepsilon_2, \varepsilon_3) / (2\sigma)$ | $(a_0, b_0, \varepsilon b_0)$ |
| $r$    | $-1, 0, 0, \frac{2}{3}, \frac{4}{3}$ | $-4, -4, -4, -1, 2, 2$     | $-1, 0, 0, 0, -2 / (\sigma a_0), 2 / (\sigma a_0)$ |
| $\tau - \tau_0$ | $(t - t_0)^{2/3}$            | $(t - t_0)^{-2}$            | $\ln(t - t_0)$             |

Analogously, Table 2 gathers the results of [4]–[6], who performed the same kind of analysis in the $\tau$-variable and for the dependent variables $A = a^2$, $B = b^2$, $C = c^2$.

The comparison of the two tables immediately shows the specificity of family (III). Indeed, the local transformation between the two times is different for family (III).

For families (I) and (II), the local transformation yields the correspondence between the exponents $p_j$ and the indices $r$. On the contrary, for family (III), the two local laws $\tau - \tau_0 = (t - t_0)^{-1}$ and $\tau - \tau_0 = \ln(t - t_0)$ are incompatible, and the deep reason for that is the non-Fuchsian nature of family (III) in the logarithmic time $\tau$. 

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Table 2: Families of singularities of $A = a^2$, $B = b^2$ and $C = c^2$ in the variable $\tau$.

|       | (I)     | (II)      | (III)     |
|-------|---------|-----------|-----------|
| $2(p_1, p_2, p_3)$ | $(-1, 1, 1)$ | $(-1, -1, -1)$ | $(0, -2, -2)$ |
| $(A_0, B_0, C_0)$ | $(\sigma, \sigma, \sigma)$ | $(A_0, \sigma^2/A_0, \sigma^2/A_0)$ |
| $r$   | $-1, 0, 0, 1, 1, 2$ | $-1, -1 - 1, 2, 2, 2$ | $-1, 0, 0, 2$ |
| $\tau - \tau_0$ | $(t - t_0)^{2/3}$ | $(t - t_0)^{-2}$ | $(t - t_0)^{-1}$ |

3 Recovering the exact solutions

Let us now perform the perturbative Painlevé analysis [18, 19] in order, mainly, to test the existence of the assumed four-parameter extension to the BGPP solution.

Each of the three families is separately a local representation of the general solution, so each family must describe every closed form particular solution.

For the family (I), we have checked the absence of any movable logarithm, so none of the known closed form particular solutions in $\tau$ is excluded.

For the family (III), the matrix $P(r)$ of the usual recurrence relation

$$\forall r\quad P(r) \begin{pmatrix} a_r \\ b_r \\ c_r \end{pmatrix} + Q_r = 0$$  \hspace{1cm} (11)

is

$$P(r) = 2b_0^3 \begin{pmatrix} a_0 b_0 \sigma^2 r (r + 1) & a_0^2 \sigma^2 r & a_0^2 \sigma^2 r \\ 0 & a_0^2 \sigma^2 r^2 - 2 & 2 \varepsilon \\ 0 & 2 \varepsilon & (a_0^2 \sigma^2 r^2 - 2) \end{pmatrix}.$$  \hspace{1cm} (12)

So the number of movable logarithms is: 2 (from the two irrational indices), plus 1 (from the difference between the multiplicity 3 and dim(Ker($P(r)$)) = 2 at index $r = 0$). When one returns to the $\tau$ variable, the local transformation $\tau - \tau_0 = \ln(t - t_0)$ suppresses one of these three logarithms, leaving open the possibility of a four-parameter solution, single valued in the $\tau$ variable.

Let us now process the family (II), by expanding $a, b, c$ in the double series

$$a = \sum_{n=0}^{+\infty} \varepsilon^n \sum_{j=n\rho}^{+\infty} a_j^{(n)} \rho^j + p = \sum_{n=0}^{+\infty} \varepsilon^n a^{(n)}, \quad \text{and cyclically},$$  \hspace{1cm} (13)

where $\rho$ stands for the lowest index $\rho = -4$.

Due to the invariance under permutation, it is convenient [3] to introduce the seven new variables $(X, Y, Z, U, V, W, T)$

$$\begin{align*}
a_2^{(0)} &= -\sqrt{2}/3 Y + \sqrt{3}/6 Z \\
b_2^{(0)} &= \sqrt{2}/2 X + \sqrt{3}/6 Y + \sqrt{3}/6 Z \\
c_2^{(0)} &= -\sqrt{2}/2 X + \sqrt{3}/6 Y + \sqrt{3}/6 Z
\end{align*}$$  \hspace{1cm} (14)
\[
\begin{align*}
\left\{ \begin{aligned}
a^{(1)}_{-4} &= -\sqrt{\frac{2}{3}}V + \frac{1}{\sqrt{3}}W \\
b^{(1)}_{-4} &= \frac{1}{\sqrt{2}}U + \frac{1}{\sqrt{6}}V + \frac{1}{\sqrt{3}}W \\
c^{(1)}_{-4} &= -\frac{1}{\sqrt{2}}U + \frac{1}{\sqrt{6}}V + \frac{1}{\sqrt{3}}W \\
\end{aligned} \right. \\
\right. \\
a^{(1)}_{-1} + b^{(1)}_{-1} + c^{(1)}_{-1} = T
\end{align*}
\]

where the six arbitrary variables are: \((a^{(1)}_{-4}, b^{(1)}_{-4}, c^{(1)}_{-4})\), two out of \((a^{(0)}_{2}, b^{(0)}_{2}, c^{(0)}_{2})\), and one out of \((a^{(1)}_{-1}, b^{(1)}_{-1}, c^{(1)}_{-1})\). The necessary conditions for the absence of movable branching are the following.

**Order zero :** \(n = 0\)

We obtain at \((n, j) = (0, 2)\) the single condition \(Z = 0\), i.e. \(a^{(0)}_{2} + b^{(0)}_{2} + c^{(0)}_{2} = 0\). Parity implies \(a^{(0)}_{2j+1} = b^{(0)}_{2j+1} = c^{(0)}_{2j+1} = 0\) with \(j = 0, 1, \ldots\).

**Higher orders :** \(n \geq 1\)

At first order one must distinguish three cases, and examine them separately at higher orders.

**Case 1 :** \(X = Y = 0\)

Because \(a^{(1)}_{-4}, b^{(1)}_{-4}, c^{(1)}_{-4}\) and for instance \(a^{(1)}_{-1}\) are arbitrary, one can set for \(n > 1\) \(a^{(n)}_{-4}, b^{(n)}_{-4}, c^{(n)}_{-4}\) and \(a^{(n)}_{-1}\) equal to zero; the same can been done for \(b^{(n)}_{2}\) and \(c^{(n)}_{2}\) with \(n > 0\). Each series \(a^{(n)}\) then terminates

\[\forall n : \ a^{(n)} = t \sum_{j=0}^{n} a^{(n)}_{3j-4n} t^{3j-4n} \quad \text{with} \quad a^{(1)}_{-1} = b^{(1)}_{-1} = c^{(1)}_{-1} = \frac{T}{3},\]

and we have checked the absence of any logarithm up to perturbation order \(n = 8\).

This results in a four–parameter Laurent series depending on \(U, V, W, T\). The three–parameter solution of Belinskii 

et al. corresponds to \(W = 0\), as can be checked by substituting the series (13) into the Euler system (5).

**Case 2 :** \((3V^2 - U^2)U = 0, (3Y^2 - X^2)X = 0\).

No additional no–log conditions are encountered at higher orders \(n\), and this case isolates the three Taub reductions : \(b = c \neq a\) and cyclically.

**Case 3 :** \(U = V = 0\).

At second order we find the only additional condition \(W = 0\), which implies : \(a^{(n)}_{j} = b^{(n)}_{j} = c^{(n)}_{j} = 0\) with \(j < 0, n = 1, 2, \ldots\) except for \(a^{(1)}_{-1} = b^{(1)}_{-1} = c^{(1)}_{-1} = T/3\).

This represents a three–parameter solution depending on \(X, Y, T\), which is precisely the Halphen solution to the Darboux system (9), as can be checked by direct substitution.

The above results are summarized in Table 3, together with their interpretation.
Table 3: Results of the perturbative Painlevé test.

| order (n) | \( Z = 0 \) |
|-----------|--------------|
| \( n = 0 \) | \( Z = 0 \) |
| \( n = 1 \) | \( X = Y = 0 \) |
| \( (3V^2 - U^2)U = 0 \) |
| \( (3Y^2 - X^2)X = 0 \) |
| \( U = V = 0 \) |
| \( n = 2 \) | no conditions |
| \( (b - c)(c - a)(a - b) = 0 \) |
| \( W = 0 \) |
| arbitrary parameters | \( U, V, W, T \) |
| \( X, U, W, T \) |
| \( X, Y, T \) |
| identification | BGPP4? |
| | Taub |
| | Halphen |

4 Conclusion

We have applied the pertubative Painlevé analysis to the Bianchi IX cosmological model in the proper time variable and connected the results to the already existing ones in the logarithmic time variable. The main result is the absence, checked to a high perturbation order, of any obstacle to the existence of a four–parameter extension to the solution of Belinskii et al., which would be single valued in the logarithmic time.

However the challenge still remains to obtain the closed–form expression of this four–parameter extension.

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