CERTAIN SUBFAMILY OF HARMONIC FUNCTIONS RELATED TO SĂLĂGEAN $q$-DIFFERENTIAL OPERATOR

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Abstract. The theory of $q$-calculus operators are applied in many areas of sciences such as complex analysis. In this paper we apply Sălăgean $q$-differential operator to harmonic functions and introduce sharp coefficient bounds, extreme points, distortion inequalities and convexity results.

1. Introduction

We state some notations regarding to $q$-calculus used in this article, see [1, 4] and [6]. For $0 < q < 1$ and positive integer $n$, the $q$-integer number is denoted by $[n]_q$ and introduced by:

$$[n]_q = 1 - q^n = 1 + q + q^2 + \ldots + q^{n-1}. \quad (1.1)$$

We can easily conclude that:

$$\lim_{q \to 1^-} [n]_q = n.$$

If $f(z)$ be analytic in this open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and normalized by $f(0) = f'(0) - 1 = 0$, then the $q$-difference operator of $q$-calculus operated on $f$ given by:

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1 - q)}, \quad (1.2)$$

where $\lim_{q \to 1^-} \partial_q f(z) = f'(z)$, for example see [2, 5] and [8].

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For \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \), the Sălăgean \( q \)-differential operator is defined by:

\[
S_q^0 f(z) = f(z) \\
S_q^1 f(z) = z \partial_q f(z) = \frac{f(z) - f(qz)}{(1-q)} \\
\vdots \\
S_q^m f(z) = z \partial_q (S_q^{m-1} f(z)) = f(z) * (z + \sum_{k=2}^{\infty} [k]^m z^k) = z + \sum_{k=2}^{\infty} [k]^m a_k z^k,
\]

where \( m \) is a positive integer and “\(*\)” is the familiar Hadamard product or convolution of two analytic functions.

Since \( \lim_{q \to 1} S_q^m (z) = z + \sum_{k=2}^{\infty} k^m a_k z^k \), is the famous Sălăgean operator \([9]\), so the operator \( S_q^m \) is called Sălăgean \( q \)-differential operator.

Let \( S_h \) denote the class of functions:

\[
f = h + g
\]

which are harmonic, univalent and sense-preserving in \( U \) and normalized by \( f(0) = f'(0) - 1 = 0 \), where \( h \) and \( g \) are analytic in \( U \) take the form:

\[
h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad (0 \leq b_1 < 1).
\]

Also, we call \( h \) and \( g \) analytic part and co-analytic part of \( f \) respectively, see \([3]\).

Hence \( f \in S_h \) is of the type:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k
\]

Now, we consider the Sălăgean \( q \)-differential operator of harmonic functions \( f = h + g \), by:

\[
S_q^m f(z) = S_q^m h(z) + (-1)^m S_q^m g(z),
\]

where \( S_q^m \) is defined by (1.3) and \( h \) and \( g \) are of the type (1.5). For more details see \([7]\).

We denote by \( S^*_h \) the family of functions of the type (1.4) where:

\[
h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad , \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1.
\]

For \( A \geq 0, 0 \leq B, C \leq 1, 0 \leq D < 1 \) and \( \gamma \in \mathbb{R} \) let \( S^*_h(\gamma)(A,B,C,D) \) denote the class of functions in \( S^*_h \) of the type (1.5) such that:

\[
\text{Re} \left\{ (1 - A)(1 - B) \frac{S_q^0 f(z)}{z} + (A + B) \frac{(S_q^m f(z))^\prime}{z^\prime} - C e^{\gamma} \frac{(S_q^m f(z))^\prime}{z^\prime} + (Ce^{\gamma} - AB) \right\} \geq D,
\]

(1.9)
where

\[ z' = \frac{\partial}{\partial \theta}(z) = iz, \quad z'' = \frac{\partial^2}{\partial \theta^2}(z) = -z, \]

(1.10) \quad \text{and letting:}

\[ (S_q^m f(z))' = \frac{\partial}{\partial \theta}(S_q^m f(re^{i\theta})) = iz(S_q^m h)' - iz(S_q^m g)' \]

\[ (S_q^m f(z))'' = \frac{\partial^2}{\partial \theta^2}(S_q^m f(re^{i\theta})) = -z(S_q^m h)' - z^2(S_q^m h)'' - z(S_q^m g)' - z^2(S_q^m g)'' \]

(1.11)

We further denote by \( S_{h(\gamma)}^*(A, B, C, D) \) the subclass of \( S_{h(\gamma)}(A, B, C, D) \) consisting of harmonic functions \( f = h + \overline{g} \) so that \( h \) and \( g \) are of the form (1.8) and satisfying (1.9).

2. Main results

In our first theorem, we introduce a sufficient coefficient condition for functions in \( S_{h(\gamma)}(A, B, C, D) \) and then we show that this condition is also necessary for \( f(z) \in S_{h(\gamma)}^*(A, B, C, D) \).

**Theorem 2.1.** Suppose \( f = h + \overline{g} \), where \( h \) and \( g \) be given by (1.5) and:

\[ \sum_{k=1}^{\infty} |(A + B)k - (1 - A - B + AB) - Ck^2| |k|^m_q |a_k| + \]

\[ \sum_{k=1}^{\infty} |(A + B)k - (1 - A - B + AB) - Ck^2| |k|^m_q |b_k| \leq 1 - D. \]  

(2.1)

Then \( f(z) \in S_{h(\gamma)}(A, B, C, D) \).

**Proof.** In view of the fact that:

\[ \text{“Re}\{W\} \geq 0 \iff |W + 1 - D| \geq |w - 1 - D|, \]

and letting:

\[ W = (1 - A)(1 - B)\frac{S_q^m f(z)}{z} + (A + B)\frac{(S_q^m f(z))'}{z'} - Ce^{i\gamma} \frac{(S_q^m f(z))''}{z''} + (Ce^{i\theta} - AB), \]

it is enough to show that:

\[ |W + 1 - D| - |W - 1 - D| \geq 0. \]

But by using (1.10) and (1.11) we have:

\[ |W + 1 - D| = \left| (1 - A)(1 - B)\left(1 + \sum_{k=2}^{\infty} a_k[k]^m_q z^{k-1} + \sum_{k=1}^{\infty} b_k[k]^m_q (\overline{\tau})^{k-1}\right) \right| \]

\[ + (A + B)\left(1 + \sum_{k=2}^{\infty} k\overline{a}_k[k]^m_q z^{k-1} - \sum_{k=1}^{\infty} k\overline{b}_k[k]^m_q (\overline{\tau})^{k-1}\right) \]

\[ - Ce^{i\gamma}\left(1 + \sum_{k=2}^{\infty} k\overline{a}_k[k]^m_q z^{k-1} + \sum_{k=2}^{\infty} k(k - 1)a_k[k]^m_q z^{k-1} \right) \]

\[ + \sum_{k=1}^{\infty} kb_k[k]^m_q (\overline{\tau})^{k-1} + \sum_{k=1}^{\infty} k(k - 1)b_k[k]^m_q (\overline{\tau})^{k-1}\right) \]
\[ + Ce^{i\gamma} - AB + 1 - D \]
\[ \leq 2 - D - \sum_{k=1}^{\infty} \left| (A + B)(k - 1) + AB - CK^2 |k|_q^m |a_k| \right| \frac{s_k}{z} \]
\[ - \sum_{k=1}^{\infty} \left| 1 - (A + B)(k - 1) + AB - CK^2 |k|_q^m |b_k| \right| \frac{s_k}{z} , \]
and
\[ |W - 1 - D| \leq D + \sum_{k=2}^{\infty} \left| (A + B)(k - 1) + AB - CK^2 |k|_q^m |a_k| \right| \frac{s_k}{z} \]
\[ + \sum_{k=1}^{\infty} \left| 1 - (A + B)(k - 1) + AB - CK^2 |k|_q^m |b_k| \right| \frac{s_k}{z} . \]

So by using (2.1), we get:
\[ |W + 1 - D| - |W - 1 - D| \geq 2 \left[ 1 - D - \sum_{k=2}^{\infty} \left| (A + B)k - (1 - A - B + AB) - CK^2 |k|_q^m |a_k| \right| - \sum_{k=1}^{\infty} \left| (A + B)k - (1 - A - B + AB) - CK^2 |k|_q^m |b_k| \right| \right] \geq 0. \]

Remark 2.1. The coefficient bound (2.1) is sharp for the function:
\[ F(z) = z + \sum_{k=2}^{\infty} \frac{x_k}{|k|_q^m |a_k|} z^k + \sum_{k=1}^{\infty} \frac{y_k}{|k|_q^m |b_k|} z^k, \]
where
\[ \frac{1}{1 - D} \left( \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| \right) = 1. \]

Theorem 2.2. Let \( f = h + \overline{g} \in \mathcal{S}_h^* \). Then \( f(z) \in \mathcal{S}_h^*(A, B, C, D) \) if and only if:
\[ \sum_{k=2}^{\infty} \left| (A + B)k - (1 - A - B + AB) - CK^2 |k|_q^m |a_k| \right| \]
\[ + \sum_{k=1}^{\infty} \left| (A + B)k - (1 - A - B + AB) - CK^2 |k|_q^m |b_k| \right| \leq 1 - D. \]

Proof. From Theorem 2.1, and since \( \mathcal{S}_h^*(A, B, C, D) \subset \mathcal{S}_h(A, B, C, D) \), we conclude the “if” part.

For the “only if” part, suppose that \( f(z) \in \mathcal{S}_h^*(A, B, C, D) \). Thus for \( z = re^{i\theta} \in U \), we have:
\[ \operatorname{Re} \left\{ (1 - A)(1 - B) \frac{S_m f(z)}{z} + (A + B) \frac{\left(S_m^\gamma f(z)\right)^\prime}{z^\prime} - Ce^{i\gamma} \frac{\left(S_m^\gamma f(z)\right)^\prime}{z^\prime} + Ce^{i\gamma} + Ce^{i\gamma} - AB \right\} \]
\[ = \operatorname{Re} \left\{ (1 - A)(1 - B) \left( 1 + \sum_{k=2}^{\infty} a_k [k]_q^m z^{-k-1} + \sum_{k=1}^{\infty} b_k [k]_q^m (z)^{-k-1} \right) \right\} \]
Theorem 3.1. \( \square \)

Now the proof is complete.

In this section, we first introduce extreme points of \( S_{\ast}(A,B,C,D) \) and then we obtain the distortion bounds for \( f \in S_{\ast}(A,B,C,D) \). Finally we show that the class \( S_{\ast}(A,B,C,D) \) is a convex set.

**Theorem 3.1.** \( f = h + \overline{g} \in S_{\ast}(A,B,C,D) \) if and only if it can be expressed:

\[
    f(z) = X_1 z + \sum_{k=2}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_k(z), \quad (z \in \mathbb{U}),
\]

where

\[
    h_k(z) = z - \frac{1 - D}{|(A + B)k - (1 - A - B + AB) - Ck^2| |k|^m_q} z^k, \quad (k = 2, 3, \ldots),
\]

\[
    g_k(z) = \frac{1 - D}{|(A + B)k - (1 - A - B + AB) - Ck^2| |k|^m_q} (\varphi(z))^k, \quad (k = 1, 2, \ldots),
\]

\( X_1 \geq 0, Y_1 \geq 0, X_1 + \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1, X_k \geq 0 \text{ and } Y_k \geq 0 \text{ for } k = 2, 3, \ldots. \)

**Proof.** If \( f \) is given by (3.1), then:

\[
    f(z) = z - \sum_{k=2}^{\infty} \frac{1 - D}{|(A + B)k - (1 - A - B + AB) - Ck^2| |k|^m_q} X_k z^k + \sum_{k=1}^{\infty} \frac{1 - D}{|(A + B)k - (1 - A - B + AB) - Ck^2| |k|^m_q} Y_k (\varphi(z))^k.
\]

Since by (2.2), we have:

\[
    \sum_{k=2}^{\infty} \frac{1 - D}{|(A + B)k - (1 - A - B + AB) - Ck^2| |k|^m_q} X_k \leq \sum_{k=1}^{\infty} \frac{1 - D}{|(A + B)k - (1 - A - B + AB) - Ck^2| |k|^m_q} Y_k
\]

3. Geometric properties of \( S_{\ast}(A,B,C,D) \)

In this section, we first introduce extreme points of \( S_{\ast}(A,B,C,D) \) and then we obtain the distortion bounds for \( f \in S_{\ast}(A,B,C,D) \). Finally we show that the class \( S_{\ast}(A,B,C,D) \) is a convex set.
\[ (1 - D) \left( \sum_{k=2}^{\infty} |X_k| + \sum_{k=1}^{\infty} |Y_k| \right) = (1 - D)(1 - X_1) \leq 1 - D. \]

So \( f(z) \in S_{r(c)}^*(A, B, C, D) \).

Conversely, suppose \( f(z) \in S_{r(c)}^*(A, B, C, D) \). By putting:

\[ X_1 = 1 - \left( \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k \right), \]

where

\[ X_k = \frac{|(A + B)k - (1 - A - B + AB) - CK^2| |k|^m}{1 - D} |a_k|, \]
\[ Y_k = \frac{|(A + B)k - (1 - A - B + AB) - CK^2| |k|^m}{1 - D} |b_k|, \]

we conclude the required representation (3.1), so the proof is complete. \( \square \)

**Theorem 3.2.** If \( f(z) \in S_{r(c)}^*(A, B, C, D) \), \( |z| = r < 1 \), then:

\[ |f(z)| \geq (1 - |b_1|)r \]

\[ \geq r - |b_1|r - \frac{1 - D}{(A + B) + (1 + AB) - 4C} \left( \sum_{k=2}^{\infty} \left( \frac{(A + B) + (1 + AB) - 4C}{1 - D} |a_k| \right) \right. \]
\[ + \frac{(A + B) + (1 + AB) - 4C}{1 - D} \left| b_k \right|^r \left. \right) \geq (1 - |b_1|)r \]

\[ \geq \frac{1 - D}{(A + B) + (1 + AB) - 4C} \left( \sum_{k=2}^{\infty} \left( \frac{(A + B)(k - 1) + (1 + AB) - 4C}{1 - D} |a_k| \right) \right. \]
\[ + \frac{(A + B)(k - 1) - (1 + AB) - 4C}{1 - D} \left| b_k \right|^r \left. \right) \geq (1 - |b_1|)r \]

\[ \geq \frac{1 - D}{(A + B) + (1 + AB) - 4C} \left( \sum_{k=2}^{\infty} \left( \frac{2(A + B) - (1 + AB) - 4C}{1 - D} |b_1| \right) \right. \]
\[ \left. \right) \geq (1 - |b_1|)r \]

\[ = (1 - |b_1|) \left( \frac{1 - D}{(A + B) + (1 + AB) - 4C} \right) \left( 2(A + B) - (1 + AB) - 4C \right) |b_1| r^2. \]

Relation (3.5) can be proved by using the similar statements. So the proof is complete. \( \square \)
Theorem 3.3. If \( f_j(z), j = 1, 2, \ldots, \) belongs to \( S^*_h(\gamma)(A, B, C, D) \), then the function \( F(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z) \) is also in \( S^*_h(\gamma)(A, B, C, D) \), where \( f_j(z) \) defined by:

\[
f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k + \sum_{k=1}^{\infty} b_{k,j} (\overline{z})^k, \quad (j = 1, 2, \ldots, \sum_{j=1}^{\infty} \lambda_j = 1).
\]

In the other worlds, \( S^*_h(\gamma)(A, B, C, D) \), is a convex set.

Proof. Since \( f_j(z) \in S^*_h(\gamma)(A, B, C, D) \), so by (2.2), we get:

\[
\sum_{k=2}^{\infty} \left| (A + B)k - (1 - A - B + AB) - Ck^2 \right| |k|^{|m|} |a_k|
\]

\[
\sum_{k=1}^{\infty} \left| (A + B)k - (1 - A - B + AB) - Ck^2 \right| |k|^{|m|} |b_{k,j}| \leq 1 - D, \quad (j = 1, 2, \ldots).
\]

Also

\[
F(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z) = z - \sum_{k=2}^{\infty} \left( \sum_{j=1}^{\infty} \lambda_j a_{k,j} \right) z^k + \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} \lambda_j b_{k,j} \right) (\overline{z})^k.
\]

Now, according to Theorem 2.2, we have:

\[
\sum_{k=2}^{\infty} \left| (A + B)k - (1 - A - B + AB) - Ck^2 \right| |k|^{|m|} \left| \sum_{j=1}^{\infty} \lambda_j a_{k,j} \right|
\]

\[
+ \sum_{j=1}^{\infty} \left( \sum_{k=2}^{\infty} \left| (A + B)k - (1 - A - B + AB) - Ck^2 \right| |k|^{|m|} |a_{k,j}| \right)
\]

\[
= \sum_{j=1}^{\infty} \left( \sum_{k=2}^{\infty} \left| (A + B)k - (1 - A - B + AB) - Ck^2 \right| |k|^{|m|} |a_{k,j}| \right)
\]

\[
+ \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} \left| (A + B)k - (1 - A - B + AB) - Ck^2 \right| |k|^{|m|} |b_{k,j}| \right) \lambda_j
\]

\[\leq (1 - D) \sum_{j=1}^{\infty} \lambda_j = 1 - D.\]

Thus, \( F(z) \in S^*_h(\gamma)(A, B, C, D) \). \( \Box \)

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