Black holes, first-order flow equations and geodesics on symmetric spaces

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For both extremal and non-extremal spherically symmetric black holes in theories with massless scalars and vectors coupled to gravity, we derive a general form of first-order gradient flow equations, equivalent to the equations of motion. For theories that have a symmetric moduli space after a dimensional reduction over the timelike direction, we discuss the condition for such a gradient flow to exist.

This note reviews the results of [1].

1 Introduction

Black hole solutions to extensions of general relativity, such as the various kinds of supergravity naturally occurring in the low-energy effective description of superstrings, often exhibit features unknown from pure Einstein’s theory. One such feature, distinctive for extremal black holes in gravity coupled to scalar and vector fields, is the attractor phenomenon [2–5]. It causes the end-points of the radial evolution of the scalars – their values on the event horizon – to be determined by the charges associated with the vectors and to be insensitive to the values of the scalars at spatial infinity. In particular, for extremal solutions that are supersymmetric, the evolution is governed by first-order (BPS) equations: the scalar fields follow a gradient flow in target space. Recently it was noticed, however, that non-supersymmetric, extremal black holes may also obey first-order gradient flows [6]. Moreover, examples of first-order equations have been found for some non-extremal (and hence neither supersymmetric nor attractive) black holes [7–10].

Two main questions arise: What is the general form of first-order flow equations for black holes? When does a gradient flow exist? For extremal black holes, these questions were first addressed in [6, 11–13], while [12] suggested in addition a possible extension to non-extremal black holes. Our work [1], which we shall briefly review here, offers a general answer, valid for static and spherically symmetric solutions – extremal and non-extremal alike.

In section 2 we derive the generalised form of first-order flow equations by demanding that the action be written as a sum of perfect squares. The conditions for the scalar fields to obey a first-order gradient flow are then found in section 3. In section 4 we analyse the case when the theory after dimensional reduction over time describes a non-linear sigma model on a symmetric space. We end with a discussion of some explicit examples in four and five dimensions in section 5.

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2 Effective action and flow equations

Consider gravity coupled to a number of neutral scalars \( \phi^a \) and vector fields \( A^I \) in \( D + 1 \) dimensions,

\[
S = \int d^{D+1}x \sqrt{|g|} \left( R_{D+1} - \frac{1}{2} G_{ab} \partial_{\mu} \phi^a \partial^\mu \phi^b - \frac{1}{2} \mu_{IJ} F^I_{\mu \nu} F^J{\mu \nu} \right),
\]

where \( G_{ab} \) and \( \mu_{IJ} \) are functions that depend on the scalars \( \phi^a \), and \( F^I_{\mu \nu} \) are Abelian field strengths.\(^1\) Greek indices are raised and lowered with the spacetime metric \( g_{\mu \nu} \) and \( g = \det g_{\mu \nu} \). The most general metric describing static, spherically symmetric black hole solutions of the theory described by the action (1) is

\[
ds_{D+1}^2 = -e^{2\beta \varphi} dt^2 + e^{2\alpha \varphi} \left( e^{2(D-1)\alpha} d\tau^2 + e^{2\alpha \varphi} d\Omega_D^2 \right), \quad e^{-(D-2)\alpha} = \gamma^{-1} \sinh[(D-2)\gamma \tau],
\]

where \( \alpha = -1/\sqrt{2(D-1)(D-2)} \), \( \beta = -(D-2) \alpha \), \( \gamma \) is a constant, and the scalars depend solely on the radial coordinate: \( \varphi = \varphi(\tau) \), \( \phi^a = \phi^a(\tau) \). The equations of motion for scalar fields of this system can be derived from a one-dimensional effective action

\[
S = \int d\tau \left( -\frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} G_{ab} \dot{\phi}^a \dot{\phi}^b - e^{2\beta \varphi} V(\phi^a) \right),
\]

where a dot denotes a derivative with respect to \( \tau \) and the effective potential \( V(\phi^a) \) results from solving for the vector fields in terms of the charges. This action is supplemented with a Hamiltonian constraint, which states that the radial evolution of the fields happens on a slice of constant total energy

\[
(D-1)(D-2)\gamma^2 = \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} G_{ab} \dot{\phi}^a \dot{\phi}^b - e^{2\beta \varphi} V(\phi) \equiv E.
\]

The constraint is the remnant of the original \( D + 1 \)-dimensional Einstein equations that is not reproduced by the effective action (3).

This type of effective action was first introduced in the context of supersymmetric black holes in \( \mathcal{N} = 2 \) supergravity in four dimensions [5, 14], where it was also observed that the black hole potential can be derived from a superpotential, proportional to the modulus of the central charge \( |Z| \)

\[
V = \frac{1}{2} \beta^2 W^2 + \frac{1}{2} G^{ab} \partial_a W \partial_b W .
\]

The terms in the effective action can then be rearranged as a sum of squares of the BPS equations, in this setting known as the attractor flow equations:

\[
\dot{\varphi} = \pm \beta e^{\beta \varphi} W ,
\]

\[
\dot{\phi}^a = \pm e^{\beta \varphi} \partial^a W .
\]

Only relatively recently has it been noticed [6] that the above rewriting is not unique, and when \( W \not\propto |Z| \) can describe extremal black holes that are not supersymmetric. It has also been demonstrated by [11] (and corroborated by [13]) that when these four-dimensional equations are viewed as dimensionally reduced five-dimensional flows, the ambiguity in defining \( W \) is not merely a residue of supersymmetry in one dimension higher. An ansatz for \( W \) reproducing all the known black hole attractors in \( \mathcal{N} > 2 \) supergravities in \( D + 1 = 4 \) has been constructed in [12].

Andrianopoli et al. [12] explored also the possibility of formulating a superpotential capable of describing both extremal (supersymmetric and non-supersymmetric) as well as non-extremal black holes. The required generalization of the superpotential \( W \) would consist in adding an explicit dependence on the radial parameter \( \tau \). Here we outline a different approach, presented and exemplified in [1].

\(^1\) When \( D + 1 = 4 \), there can be another term of the form \( -\frac{1}{2} \nu_{IJ}(\phi) F^I_{\mu \nu}(\ast F^J{\mu \nu}) \) in the action.
Assuming that there exists a ‘generalised superpotential’ \( Y(\varphi, \phi^a) \), such that
\[
e^{2\beta\varphi} V(\phi^a) = \frac{1}{2} \partial_2 Y \partial_{\varphi} Y + \frac{1}{2} \partial_4 Y \partial^2 Y + \Delta ,
\]
where \( \Delta \) is a constant, the effective action \( S \) can be written in the form
\[
S = -\frac{1}{2} \int d\tau \left[ (\dot{\varphi} + \partial_\varphi Y)^2 + (\dot{\phi}^a + \partial^a Y)^2 \right] ,
\]
plus a total derivative. We will address the question of existence in the next section.

Demanding a stationary point of the action produces generalised flow equations
\[
\dot{\varphi} + \partial_\varphi Y = 0 ,
\]
\[
\dot{\phi}^a + G^{ab} \partial_b Y = 0 .
\]
The Hamiltonian constraint \( \Delta \) yields \( \Delta = -E \).

Our general formulae reduce as desired to the familiar extremal expressions \( \ref{eqn:Delta} \), \( \ref{eqn:extreme} \) when \( \Delta = 0 \), in which case equation \( \ref{eqn:Y} \) implies that \( Y(\varphi, \phi^a) \) must factor as
\[
Y(\varphi, \phi^a) = e^{\beta \varphi} W(\phi^a) .
\]
This factorisation property is the main difference between extremal and non-extremal flow equations.

### 3 Existence of a generalised superpotential

For extremal black hole solutions involving one scalar field a superpotential always exists \[15\]: assume that the extremal solution exists, then equation \( \ref{eqn:W} \) defines the function \( W(\tau) \). Since the black hole is supported by a single scalar \( \phi \), and locally we can always invert \( \phi(\tau) \) to \( \tau(\phi) \), this defines \( W(\phi) \).

In case of multiple scalars, the above argument for the existence of extremal flow equations does not apply \[16\].

For non-extremal solutions carried by an arbitrary number of scalar fields, the superpotential can be proven to exist under certain conditions \[1\], generalising the results of \[6\]. Since both the ‘warp factor’ \( \varphi \) of the \((D + 1)\)-dimensional metric and the \((D + 1)\)-dimensional scalars \( \phi^a \) appear on the same footing in equations \( \ref{eqn:phi} \) and \( \ref{eqn:phi^a} \), we combine them in a vector \( \phi^A \):
\[
\phi^A = \{ \varphi, \phi^a \} .
\]

In the following section we study a class of theories with a symmetric moduli space when reduced over one dimension; their equations of motion are known to be integrable. The integrability of the effective action allows to explicitly write down the velocity vector field \( f \) on the enlarged scalar manifold in \( D \) dimensions
\[
\dot{\phi}^A = f^A(\phi, \chi) ,
\]
\[
\dot{\chi}^\alpha = f^\alpha(\phi, \chi) ,
\]
where the \( \chi^\alpha \) are the scalars descending from the vector potentials upon dimensional reduction. One can demonstrate that there are enough ‘integrals of motion’ to fully eliminate the \( \chi^\alpha \) in terms of the \( \phi^A \), such that one can write down a velocity field on the original target space in \( D + 1 \) dimensions:
\[
\dot{\phi}^A = f^A(\phi, \chi(\phi)) .
\]

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2 Without loss of generality (redefinition of variables) we choose the plus sign within the squares.

3 Having constructed the fake superpotential \( W(\phi) \) for the extremal solution, we could then attempt the deformation technique of \[8\] to obtain the function \( Y(\varphi, \phi) \) in the non-extremal case. This approach, however, requires the Lagrangian to have certain properties (see \[8\] for details), which do not hold in general.

4 Unless some complicated conditions are satisfied, as explained in the case of domain walls in \[16, 17\].
Having obtained the velocity field \ref{16} on the moduli space in \( D + 1 \) dimensions, it suffices to show that the velocity one-form \( f_A \) is locally exact
\[
 f_A(\phi, \chi(\phi)) \equiv \tilde{G}_{AB}(\phi) f^B(\phi, \chi(\phi)) = \partial_A Y(\phi),
\] (17)
where \( \tilde{G} \) is the metric on the scalar manifold in the \( D \)-dimensional theory. A necessary and sufficient condition for this to hold locally is, by Poincaré’s lemma, that the one-form is closed
\[
 \partial[A f_B] = 0. \tag{18}
\]

For specific non-supersymmetric solutions it might be very difficult in practice to find the superpotential \( Y \). In spite of this, by verifying the vanishing curl condition \ref{18} one can demonstrate the existence of a gradient flow. For this reason we restrict ourselves to those theories that have a symmetric moduli space after timelike reduction, where we know that \( f \) exists.

### 4 Black holes and geodesics

To arrive at an explicit expression for the velocity field \( \dot{\phi}^A = \{ \dot{\phi}, \dot{\phi}^a \} \) for theories with symmetric moduli spaces after dimensional reduction \cite{1}, we first consider a timelike reduction of the \( D + 1 \)-dimensional theory and then give the necessary background on geodesics on symmetric spaces.

#### 4.1 Timelike dimensional reduction

There is another way to interpret the one-dimensional effective action given above, first described in the \( D + 1 = 4 \) case \cite{20}. It is based on the observation that a static solution in \( D + 1 \) dimensions can be dimensionally reduced over time (a Killing direction) to a Euclidean \( D \)-dimensional instanton solution. Because of the assumed spherical symmetry, the resulting instanton solutions are carried only by the metric and the scalars in \( D \) dimensions. We interpret the metric \ref{2} as the ansatz for a dimensional reduction over time. The scalar field equations of motion are found from the following effective one-dimensional action
\[
 S = -\frac{1}{2} \int d\tau \tilde{G}_{ij} \dot{\phi}^i \dot{\phi}^j , \tag{19}
\]
which describes the free geodesic motion of a particle in an enlarged target space of scalar fields \( \tilde{\phi}^i = \{ \phi^A, \chi^a \} \), where the \( \phi^A \) contain both the scalars \( \phi^a \) of the \( (D, 1) \)-dimensional theory and the ‘warp factor’ \( \varphi \), and the \( \chi^a \) are axions, consisting of electric potentials (and magnetic potentials when \( D + 1 = 4 \)).

We always use the notation \( \tilde{G} \) for the moduli space metric in the reduced (Euclidean) gravity theory. Note that in this procedure the vectors (or equivalently, the axions) are not eliminated by their equations of motion. This action has to be complemented by the Hamiltonian constraint \cite{9} (compare with \ref{3})
\[
 \frac{1}{2} \tilde{G}_{ij} \dot{\phi}^i \dot{\phi}^j \equiv E . \tag{20}
\]

Those \( D \)-dimensional solutions that lift to extremal black holes in \( (D, 1) \) dimensions have flat \( D \)-dimensional geometries, or equivalently \( E = 0 \), which implies that the geodesic is null: \( \tilde{G}_{ij} \dot{\phi}^i \dot{\phi}^j = 0 \).

For static, spherically symmetric solutions one can eliminate the axions \( \chi^a \) from the action, since the moduli space metric has the following crucial properties: \( \tilde{G}_{\alpha A} = 0, \partial_\alpha \tilde{G}_{ij} = 0 \). These identities stem from the fact that the shift symmetries of the scalars \( \phi^a \) in \( D + 1 \) dimensions commute.

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5 In some cases a direct integration turns out to be possible for an extremal ansatz, as in \cite{18, 19}. One can readily check that the velocity field is irrotational in these examples.

6 We refer to \cite{21} for a recent discussion and application of this formalism for black holes in symmetric supergravities.

7 When going from four to three dimensions, there is also a scalar dual to the Taub-NUT vector. Since we only consider static black hole solutions, we restrict to geodesics for which the NUT charge vanishes.

8 In fact, for \( D > 3 \) these properties also hold for stationary solutions. In \( D = 3 \), the shift symmetries associated with electric and magnetic charges \( q_i, p^j \) no longer commute for solutions with a non-vanishing NUT-charge. However, the mentioned properties
4.2 Geodesics on symmetric spaces

Let us now assume that the target space in $D$ dimensions is a symmetric coset space $G/H$, where $G$ is a Lie group and $H$ some subgroup subject to certain conditions that we state below. This assumption is always valid for supergravity theories with more than eight supercharges and for some theories with less supersymmetry. Nevertheless, our analysis here is independent of any supersymmetry considerations.

The Lie algebras associated to $G$ and $H$ are denoted by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The defining property of a symmetric space $G/H$ is that there exists a Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{f},$$

with respect to the Cartan automorphic involution $\theta$, such that $\theta(\mathfrak{f}) = -\mathfrak{f}$ and $\theta(\mathfrak{h}) = +\mathfrak{h}$. Take a coset representative $L(\tilde{\phi}) \in G$. We first define the group multiplication from the left, $L \rightarrow gL$, $\forall g \in G$, and we let the local symmetry act from the right $L \rightarrow hL$, $\forall h \in H$. From the Cartan involution we can construct the symmetric coset matrix $M = LL^\dagger$, where $\dagger$ is the generalised transpose

$$L^\dagger = \exp[-\theta(\log L)].$$

The matrix $M$ is invariant under $H$-transformations that act from the right on $L$. Under $G$-transformations from the left, $M$ transforms as $M \rightarrow gMg^\dagger$.

With the aid of the matrix $M$ the line element on the space $G/H$ with coordinates $\tilde{\phi}^i$ can be written as

$$\text{d}s^2 = \tilde{G}_{ij} \text{d}\tilde{\phi}^i \text{d}\tilde{\phi}^j = -\frac{1}{2} \text{Tr} (\text{d}M \text{d}M^{-1}).$$

A local action of $H$ on $L$ from the right and a global action of $G$ on $L$ from the left leave the metric invariant. The latter implies that $G$ is the isometry group of $G/H$. The action (19) of the dimensionally reduced theory then describes the geodesic curves on $G/H$ and the resulting equations of motion are

$$\frac{d}{d\tau} (M^{-1} \frac{d}{d\tau} M) = 0 \quad \Rightarrow \quad M^{-1} \frac{d}{d\tau} M = Q,$$

with the matrix of Noether charges $Q$ being a constant matrix in some representation of $g$. We now see that the geodesic equations are indeed integrable and their general solution is

$$M(\tau) = M(0)e^{Q\tau}.$$  

The affine velocity squared of the geodesic curve is (the dot stands for ordinary matrix multiplication)

$$\tilde{G}_{ij} \frac{\dot{\tilde{\phi}}^i}{\tau} \frac{\dot{\tilde{\phi}}^j}{\tau} = \frac{1}{2} \text{Tr} (Q \cdot Q),$$

and coincides with the Hamiltonian constraint (20).

An integrable geodesic motion on an $n$-dimensional space is specified by $2n$ constants: the initial position and velocity of the geodesic curve. So the geodesic motion on $G/H$ is specified by $2(\dim G - \dim H)$ integration constants. In eq. (25) $M(0)$ contains $\dim (G - \dim H)$ constants corresponding the initial position. The number of arbitrary constants in $Q$ (the initial velocity) is reduced from $\dim G$ to $\dim (G - \dim H)$ through the constraint $M^2(\tau) = M(\tau)$, which gives $\theta(Q) = -M(0)^{-1}QM(0)$.

The first-order equation (24) can be written compactly as $M^{-1} \frac{d}{d\tau} M \frac{\dot{\phi}}{\dot{\phi}} = Q$ or equivalently,

$$\frac{\dot{\phi}}{\dot{\phi}} = \frac{1}{2} \tilde{G}^{ij} \text{Tr} (M^{-1} \partial_i M \cdot Q).$$

These are only $\dim (G - \dim H)$ equations. After substituting (27) into eq. (24), the remaining $\dim H$ components become non-differential equations. This shows the power of (24): we split the $\dim G$ differential equations in $M^{-1} \frac{d}{d\tau} M = Q$ into $\dim (G - \dim H)$ first-order equations and $\dim H$ equations without any derivatives. In the context of section 3 these non-differential equations are precisely what is needed to eliminate the additional scalars resulting from dimensional reduction, so that we obtain first-order equations in terms of the scalars in $D + 1$ dimensions, as in eq. (16).
5 Applications and discussion

The gradient flow equations described here descend from a generalised superpotential and are equally applicable to extremal (whether supersymmetric or not) as well as non-extremal black holes (necessarily non-supersymmetric). They naturally encompass previously known partial results, albeit differ from the form conjectured in [12].

For theories with scalar manifolds being symmetric spaces after a timelike dimensional reduction, we give a method of verifying whether a generalised superpotential exists. It relies on the fact that the black hole solutions trace out integrable geodesics on the moduli space of the theory when reduced over time.

We applied these general results to two examples (for details we refer the reader to [1]). For a dilatonic Einstein–Maxwell black hole in four dimensions, where we generalised the earlier work of [9], we were able to show the existence of a superpotential even when the procedure of deforming the extremal solution [8] cannot be employed. For the Kaluza–Klein black hole in five dimensions, which was our second test application, we demonstrated that the condition for the existence of a generalised superpotential, which can be viewed as a restriction on the charges, is nontrivial and independent of extremality.

These findings show that, although it is possible to introduce a (generalised) superpotential also for non-extremal solutions, not all black holes of the class discussed, not even all extremal ones, can be described by a gradient flow, but only those that carry a specific combination of charges. It would be interesting to investigate if such charge configurations distinguish themselves through other physically or mathematically significant properties.

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