Heterotic strings in a uniform magnetic field

J.G. Russo*

* Theory Division, CERN
CH-1211 Geneva 23, Switzerland

and

A.A. Tseytlin†

† Theoretical Physics Group, Blackett Laboratory
Imperial College, London SW7 2BZ, U.K.

Abstract

An exact conformal model representing a constant magnetic field background in heterotic string theory is explicitly solved in terms of free creation/annihilation operators. The spectrum of physical states is examined for different possible embeddings of the magnetic $U(1)$ subgroup. We find that an arbitrarily small magnetic field gives rise to an infinite number of tachyonic excitations corresponding to charged vector states of the massless level and to higher level states with large spins and charges.

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* e-mail address: jrusso@vxcern.cern.ch

† On leave from Lebedev Physics Institute, Moscow.
1. Introduction

Recently, a class of conformal models representing $D = 4$ axially symmetric magnetic field backgrounds in closed bosonic string theory was shown to be exactly solvable \cite{1,2}. Like string theory in flat space, or on orbifold \cite{3}, or open string theory in constant magnetic field \cite{4}, these quantum string models can be represented in terms of free creation/annihilation operators, so that the physical spectrum, partition function, vertex operators, etc., can be explicitly determined. In contrast to some other solvable models, here the underlying space-time geometry is non-trivial (e.g. the curvature is non-zero and may be singular in some cases). These backgrounds generalize exact string solutions found in \cite{3,4}.

The physical spectrum has exhibited the presence of tachyonic instabilities for infinitesimal values of the magnetic field. Instabilities at finite values of magnetic fields (observed previously in Yang-Mills gauge theory \cite{7} and in open string theory \cite{4,8}) indicate the presence of a phase transition \cite{7,1}. In the case of unbroken gauge theory the constant magnetic background is unstable already for an infinitesimal magnetic field \cite{7} (this infinitesimal instability goes away once the gauge symmetry is spontaneously broken; the magnetic field which is necessary to produce an instability is then of the order of the mass of the charged vector bosons).

Such infinitesimal instability is expected for charged massless string states (members of Yang-Mills multiplet)\footnote{Indeed, the thermodynamical partition function of a string gas cannot be defined beyond the Hagedorn temperature and beyond the critical magnetic field where tachyons appear in the spectrum \cite{9}. It was argued in \cite{9} that, like the Hagedorn transition at zero field \cite{10}, this is a first-order phase transition with a large latent heat.} and, similarly, should disappear after these states acquire some masses through spontaneous symmetry breaking. Although higher-spin string states may seem to be protected from this instability by large masses, it nevertheless turns out that an infinite number of them become tachyonic when an infinitesimal magnetic field is turned on \cite{1}. This does not happen in the case of the open string theory \cite{4,8}, where, as in the broken gauge theory, one needs a finite (Planck-order) magnetic field in order to make originally positive (mass)\footnote{It was absent in the open (super)string models considered in \cite{4,8} since there the Chan-Paton symmetry was assumed to be Abelian but should of course appear in the non-Abelian case.} string states tachyonic. The important feature of closed string theory is that, in contrast to the open string case, here the charges of states are not fixed but (like masses and spins) can take arbitrarily large values. As a result \cite{1}, there are states for which the gyromagnetic coupling term ($\sim fQJ$) overpowers the free string mass term for a magnetic field $f \sim 1/Q$, which can thus be arbitrarily small for large enough charge $Q$. 

\[1\]
The question that will be addressed in the present paper is whether these instabilities appear also in the heterotic string case \[11\]. We shall show that the magnetic field necessary to produce tachyonic states in the heterotic string models is indeed arbitrarily small. In addition to the tachyonic charged vector modes of the ‘massless’ level there is an infinite number of tachyons corresponding to higher spin and charge states of the free string theory.\[3\]

The heterotic string models discussed below which describe constant magnetic background were already introduced in \[1\]. Starting with 10-dimensional heterotic string theory one can embed the Abelian magnetic field either in the Kaluza-Klein sector (assuming that 6 dimensions are compactified on a torus, one of the periodic coordinates being used to couple the magnetic field) or in the internal \(E_8 \times E_8\) or \(SO(32)\) gauge sector. The two heterotic models realizing these two options will be discussed below. They appear to be closely related and have similar properties.

The Kaluza-Klein (KK) embedding option is the only one available in the bosonic string and closed superstring cases \[1\]. The type II superstring model based on direct \((1,1)\) supersymmetric generalization of the bosonic model of \[1\] turns out to have residual space-time supersymmetry and thus no tachyons in its spectrum.\[4\] The same applies to its ‘left’ \((1,0)\) truncation: the corresponding heterotic string model is stable. It should be noted that the ‘magnetic’ interpretation of these models is rather artificial, since the Abelian KK gauge field here cannot be identified with the usual Maxwell field.

The two ‘right’ \((0,1)\) heterotic models (with KK and with gauge sector embedding) have no residual space-time supersymmetry and exhibit tachyonic instabilities. Tachyonic instabilities in the presence of an \textit{infinitesimal} magnetic field are inevitable in any theory containing massless Yang-Mills vector bosons with non-zero \(U(1)_{\text{em}}\) charges. What is new in closed string theory is that these infinitesimal instabilities are associated also with higher level string states and should thus survive even after gauge vector bosons become massive.\[5\]

\[3\] This conclusion remains valid even if one introduces small mass corrections \(\sim M_s \ll M_{\text{Planck}}\) (e.g. originating from supersymmetry or gauge symmetry breaking) to the masses of all string states.

\[4\] Other magnetic (monopole-type) solutions in heterotic string theory were discussed, e.g., in \[12,13,14\].

\[5\] In this paper we shall use the fermionic string (NSR) formalism. The models we consider can be solved also using the Green-Schwarz approach: the exact conformal invariance of the bosonic background implies the existence of \(\kappa\)-supersymmetry and the existence of the covariantly constant null Killing vector makes it straightforward to fix the light-cone gauge both for bosons and space-time fermions.

\[6\] The instability could only be removed by Planck-order mass corrections to massive states (for a further discussion see Section 5).
We shall start in Section 2 by describing the actions of the heterotic models associated with a uniform magnetic background of \([1,5]\). We shall present the corresponding actions both in the manifestly Lorentz-invariant and in chiral boson forms, the latter being useful for establishing the relation between the two ‘right’ heterotic models (which can be interpreted as two special cases of the \(O(6,22)\) duality-invariant action \([R]\) of toroidally compactified heterotic string).

To find the spectra of states of these models we shall follow the same method as used in the bosonic case in \([1]\): solving explicitly the classical equations, quantizing the theory canonically and expressing the quantum Virasoro constraints in terms of free creation/annihilation operators. We shall first consider the ‘right’ heterotic model with KK embedding (Section 3) and demonstrate the presence of tachyonic instabilities in its spectrum. We shall also explain why these instabilities are absent in type II superstring, ‘left-right symmetric’ and ‘left’ heterotic string models with KK embedding (in agreement with space-time supersymmetry of these models).

Using the results of Section 3 we shall finally determine in Section 4 the spectrum of the ‘realistic’ heterotic string model with magnetic \(U(1)\) subgroup embedded in the \(E_8 \times E_8\) or \(SO(32)\) internal gauge symmetry group. As in the case of the ‘right’ heterotic model with KK embedding, there is an infinite number of tachyonic states for any given arbitrarily small value of the magnetic field strength.

Section 5 will contain a summary and concluding remarks.

2. Actions of the heterotic string models

As discussed above, our aim will be to solve the superstring and heterotic string versions of the bosonic constant magnetic field model studied in \([1]\). To embed an Abelian magnetic field in a closed superstring theory one is to consider a toroidal compactification (“Kaluza-Klein” embedding). In addition to KK embedding, in the heterotic string theory there is also an option to interpret the magnetic field as belonging to an Abelian subgroup of an internal gauge group (gauge sector embedding). The models we shall discuss below are thus type II closed superstring with KK embedding (and closely related ‘left-right symmetric’ heterotic model), its two inequivalent ‘left’ and ‘right’ heterotic truncations and the ‘right’ heterotic model with gauge sector embedding of the magnetic field. Many technical details of the solution of these models will be similar.

The exact conformal invariance of these models as well as their space-time interpretation were already discussed in \([3,4]\). The corresponding 4-dimensional space-time background which solves the heterotic string (as well as compactified \(D = 5\) superstring and bosonic string) equations of motion, in particular \((\mu, \nu = 0, 1, 2, 3)\)

\[
R_{\mu\nu} - \frac{1}{4} H_{\mu\lambda\rho} H_{\nu}^{\lambda\rho} - \frac{1}{4} \alpha^\prime \mathcal{F}(V)_{\mu\lambda} \mathcal{F}(V)_{\nu}^{\lambda} + 2D_{\mu} D_{\nu} \Phi + ... = 0 , \quad (2.1)
\]
is given by \[5\] (we list only the non-vanishing components of the fields)

\[
    ds^2 = -[dt + A_i(x)dx^i]^2 + dx_i dx^i + dx^3 dx^3, \quad B_{it} = A_i(x), \quad \Phi = \Phi_0, \quad (2.2)
\]

\[
    V_i = e_0 A_i(x) = -\frac{1}{\sqrt{2\alpha'}} f\epsilon_{ij} x^j, \quad e_0 = \sqrt{2/\alpha'}, \quad i, j = 1, 2. \quad (2.3)
\]

The magnetic field is constant in this natural frame where the metric is stationary (it is covariantly constant in a general frame). The dilaton is trivial and the curvature and the antisymmetric tensor vanish in the absence of the magnetic field.

Let us first recall the form of the actions of these models \[1\] (we shall use 2d fermionic NSR formulation). The \((1,1)\) extension \[1\] of the bosonic model of \[5,1\] is

\[
    I_{(1,1)} = \frac{1}{\pi\alpha'} \int d^2\sigma \left[ \partial_+ u \partial_- v + \partial_+ x_m \partial_- x^m + 2A_i(x) \partial_+ u \partial_- x^i \right] (2.4)
\]

\[
    + \lambda^u_L \partial_- \lambda^u_L + \lambda_{Lm} \partial_- \lambda^m_L + F_{ij} \partial_- x^j \lambda^u_L \lambda^j_L + \lambda^v_R \partial_+ \lambda^u_R + \lambda_{Rm} \partial_+ \lambda^m_R - F_{ij} \partial_+ u \lambda^i_R \lambda^j_R].
\]

Here \(u \equiv y - t, \ v \equiv y + t\) whereas \(y \equiv y + 2\pi R\) is the internal KK coordinate. This model and its truncations discussed below are ‘self-dual’ with respect to duality in \(y\) direction (with \(R \to \alpha' / R\)). \(t, x^m \ (m = 1, 2, 3; \ i, j = 1, 2)\) are the 4-dimensional space-time coordinates. The isometry coordinate \(x^3\) is the direction of the constant magnetic field,

\[
    A_i = -\frac{1}{2} F_{ij} x^j, \quad F_{ij} = f\epsilon_{ij}. \quad (2.5)
\]

\(\lambda^u_L\) and \(\lambda^u_R\) are Majorana-Weyl fermions (we omit additional free 5 bosonic and 5 left and 5 right fermionic coordinates). This model corresponds to an exact solution of type II superstring theory. It preserves space-time supersymmetry \[10,1\] and the action \((2.4)\) has, in fact, extended \((4,1)\) world-sheet supersymmetry.\[8\]

There are four ‘magnetic’ heterotic models which are closely related to this superstring model \((2.4)\), and, in particular, correspond to the same space-time background \((2.2),(2.3)\):

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7 Performing the electromagnetic duality on this background one can find its \(S\)-dual, which will have non-constant dilaton (note that the axion is non-trivial here) and only electric component of the vector field. The resulting string model will not, however, be conformal to all orders in \(\alpha'\) (\(S\)-duality may be expected to be a symmetry of the \(D = 4\) heterotic string only non-perturbatively in string coupling).

8 We shall use the following notation: \(\sigma_0 = \sigma_0 \pm \sigma_1 \equiv \tau \pm \sigma, \ \partial_0 = \frac{1}{2} (\partial_0 \pm \partial_1), \ \sigma \in (0, \pi]\). The fermionic indices are coordinate ones with \(\lambda^0 \equiv \lambda^v + 2A_i \lambda^i\).

9 This is not surprising given that for the non-compact \(y\) the bosonic model admits a plane-wave interpretation and is equivalent to a non-semisimple WZW model \[17\].
three (‘left-right’ symmetric, ‘left’ and ‘right’) models with KK embedding of the magnetic field and the ‘right’ model with the magnetic field embedded in the internal gauge sector.

The model (2.4) can be also interpreted as a heterotic σ-model \[18,19\] corresponding to a ‘left-right symmetric’ heterotic solution obtained by the standard embedding of a closed superstring solution into the heterotic string theory.\[10\] This solution also preserves one half of maximal space-time supersymmetry \[16\] and the corresponding σ-model is (4, 1) supersymmetric.

In addition, there are two non-trivial, inequivalent heterotic string models which are obtained by (1, 0) and (0, 1) supersymmetric truncations of (2.4) \[5\]. Both models represent exact heterotic string solutions when combined with a free internal fermionic sector (there is no need to introduce non-vanishing internal gauge field background \[5\]). The (1, 0) (but not the (0, 1) one) truncation also preserves ‘one half’ of space-time supersymmetry \((N = 2, D = 4)\) and has extended \((4, 0)\) world-sheet supersymmetry \[5\]. Omitting the additional free space-time and internal fermionic contributions the corresponding actions can be written as follows:

\[
I_{(1,0)} = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ \partial_+ u \partial_+ v + \partial_+ x_m \partial_- x^m + 2A_i(x) \partial_+ u \partial_- x^i \right] \tag{2.6}
\]

\[
I_{(0,1)}^{(kk)} = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ \partial_+ u \partial_- v + \partial_+ x_m \partial_- x^m + 2A_i(x) \partial_+ u \partial_- x^i \right] \tag{2.7}
\]

\[
I_{(0,1)}^{(int)} = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ - \partial_+ t \partial_- t - 2A_i \partial_+ t \partial_- x^i + \left( \delta_{ij} - A_i A_j \right) \partial_+ x^i \partial_- x^j \right.
\]

\[
+ \partial x^j \partial x^3 - \lambda_R^{ij} \partial_+ \lambda_R^{ij} + \lambda_R^{ij} \partial_+ \lambda_R^{ij} + \lambda_R^{ij} \partial_+ \lambda_R^{ij} + F_{ij} \partial_+ t \lambda_R^{ij} \lambda_R^{ij} + \frac{1}{2} F_{ij} A_k \partial_+ x^k \lambda_R^{ij} \lambda_R^{ij} + \bar{\psi} (\partial_+ - ie_0 A_i \partial_- x^i) \psi + \frac{1}{2} ie_0 F_{ij} \bar{\psi} \psi \lambda_R^{ij} \lambda_R^{ij} \right]. \tag{2.8}
\]

Here \(e_0 \equiv \sqrt{2/\alpha'}\) and \(\lambda_R^{ij} = \frac{1}{2}(\lambda_R^{ij} - \lambda_R^{ji})\). The \(\lambda_R^{ij}\) are the four right Majorana-Weyl fermions of the supersymmetric sector and \(\psi\) is the left Weyl fermion of the internal sector which is coupled to the magnetic field. The complete anomaly-free heterotic string model

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\[\lambda_R^{ij}\] then play the role of the internal fermions and \(V^{ij} = \tilde{V}^{ij} = -F^{ij}\) the role of the internal gauge field.
is obtained by adding to (2.8) extra free fields: 6 scalars and 6 right and 30 left Majorana-Weyl fermions.

The model (2.8) admits also an alternative description with the coupling in the internal sector represented by a chiral boson. This representation will be useful for the solution of this heterotic string model, so let us discuss it in some detail. The reason why we can give a conformal and (on-shell) Lorentz-invariant chiral boson description of this model is that the coupling term \((A_i \partial_- x^i - \frac{1}{2} F_{ij} \lambda^i_R \lambda^j_R) \partial_+ y\) in the closely related model (2.7) is linear in the KK coordinate \(y\) and is chiral. As a result, \(y\) can be consistently truncated to its ‘chiral’ part.

Following [20,21] one can describe this coupling by a chiral scalar action which is not manifestly Lorentz invariant but defines a Lorentz-invariant theory on-shell. Starting with the bosonic \(y\)-dependent part of (2.4) or (2.7)

\[ I(y, A_-) = \frac{1}{\pi \alpha'} \int d^2 \sigma (\partial_+ y \partial_- y + 2 A_- \partial_+ y) \]  

and introducing the dual field \(\tilde{y}\) one finds the following ‘doubled’ action (see [21] and section 2.4 in [1])

\[ I(y, \tilde{y}, A_-) = \frac{1}{4 \pi \alpha'} \int d^2 \sigma [\partial_0 y \partial_1 \tilde{y} + \partial_0 \tilde{y} \partial_1 y - \partial_1 y \partial_1 y - \partial_1 \tilde{y} \partial_1 \tilde{y}] + 4 A_- (\partial_1 y + \partial_1 \tilde{y}) - 4 A_- A_- \]  

Equation (2.10) is the same as the phase-space action with momentum replaced by \(\partial_1 \tilde{y}\). Integration over \(\tilde{y}\) leads back to (2.10). Written in terms of \(y^\pm = \frac{1}{2} (y \pm \tilde{y})\) (2.10) becomes

\[ I(y, \tilde{y}, A_-) = I(y^-) + I(y^+, A_-) , \quad I(y^-) = -\frac{1}{\pi \alpha'} \int d^2 \sigma \partial_0 y^- \partial_1 y^- \]  

\[ I(y^+, A_-) = \frac{1}{\pi \alpha'} \int d^2 \sigma (\partial_1 y^+ \partial_- y^+ + 2 A_- \partial_1 y^+ - A_- A_-) \]  

The equations that follow (under proper boundary conditions) from (2.11) and (2.12) are \(\partial_+ y^- = 0, \quad \partial_- y^+ + A_- = 0\). Since \(y^-\) is decoupled from the rest of the fields it can be consistently set equal to zero. Like the original theory (2.9), (2.11), and the theory of the free chiral scalar \(y^-\), the theory defined by \(I(y^+, A_-)\) is also Lorentz-invariant on shell (this can be easily checked by computing its stress energy tensor on the equations of

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11 In general, the fermionic description of the internal sector of a heterotic string model is more fundamental: it can be replaced by a chiral bosonic one only in special cases.

12 As discussed in [1], (2.8) can be obtained from (2.7) (with decoupled fermionic components \(\lambda^i_L, \lambda^i_R\)) by ‘fermionising’ the compact internal coordinate \(y\) and dropping extra free field terms in the action.
motion. Since the equation of motion that follows from (2.9) is \( \partial_+ (\partial_- y + A_-) = 0 \) the chiral truncation corresponds to choosing only the solutions which satisfy \( \partial_- y + A_- = 0 \) (note that for generic \( A_- \) such \( y = y^+ \) will depend on both \( \tau - \sigma \) and \( \tau + \sigma \)). The action (2.12) can be rewritten also as \( (D_p y \equiv \partial_p y + A_p) \)

\[
I(y^+, A_-) = \frac{1}{\pi \alpha'} \int d^2 \sigma (D_1 y^+ D_- y^+ + \frac{1}{2} \epsilon^{pq} A_p D_q y^+ - A_- A_+) .
\] (2.13)

The equivalent form of the heterotic action (2.8) with the bosonic representation of the internal sector is obtained by replacing the fermionic \( \psi \)-terms, together with the \( A_+ \hat{A}_- \)-term in (2.8), by \( I(y^+, \hat{A}_-) \),

\[
I_{(0,1)}^{(\text{int})} = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ - \partial_+ t \partial_- t - 2 \hat{A}_- \partial_+ t + \partial_+ x_m \partial_- x^m - \lambda^i_R \partial_+ \lambda^i_R + \lambda_{Rm} \partial_+ \lambda^m_R 
+ \partial_1 y^+ \partial_- y^+ - 2 \hat{A}_- \partial_1 y^+ - \hat{A}_- \hat{A}_- \right] , \quad \hat{A}_- \equiv A_i \partial_- x^i - \frac{1}{2} F_{ij} \lambda^i_R \lambda^j_R .
\] (2.14)

\( y^+ \) should be identified with one of the coordinates \( y^I_L \) of the ‘left’ 16-torus of the internal sector of the free heterotic string. In general, the coupling to the 16 Abelian vector fields \( A^I_\mu \) of Cartan subalgebra is described by the action

\[
I = \frac{1}{\pi \alpha'} \int d^2 \sigma \sum_{l=1}^{16} [\partial_1 y^I_L \partial_- y^I_L + 2 \hat{A}^I_-(x) \partial_1 y^I_L - \hat{A}^I_- \hat{A}^I_] \]
\] (2.15)

\[
= \frac{1}{\pi \alpha'} \int d^2 \sigma \sum_{a,b=1}^{16} g_{ab} [\partial_1 y^a_L \partial_- y^b_L + 2 \hat{A}^a_- (x) \partial_1 y^a_L - \hat{A}^a_- \hat{A}^a_] ,
\]

where

\[
y^I_L \equiv y^I_L + 2 \pi R \sum_{a=1}^{16} n_a e^I_a , \quad y^I_L = \sum_{a=1}^{16} e^I_a y^a_L , \quad \hat{A}^I_- = \sum_{a=1}^{16} e^I_a \hat{A}^a ,
\] (2.16)

\[
R = \sqrt{\alpha' / 2} , \quad g_{ab} = \sum_{I=1}^{16} e^I_a e^I_b , \quad g_{aa} = 2 ,
\]

and \( e^I_a \) are the generators of the even self-dual 16-lattice (\( \Gamma_8 \times \Gamma_8 \) or \( \Gamma_{16} \)).

The two ‘right’ heterotic models (2.7) and (2.14) are closely related. Indeed, (2.7) can be put into the form similar to (2.14) by first using that \( u = y - t , \quad v = y + t \) and then replacing the \( y \)-dependent part of the action (i.e. (2.9) with \( A_- \rightarrow \hat{A}_- \)) by the equivalent form (2.10):

\[
I_{(0,1)}^{(\text{kk})} = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ - \partial_+ t \partial_- t - 2 \hat{A}_- \partial_+ t + \partial_+ x_m \partial_- x^m + \lambda^u_R \partial_+ \lambda^u_R + \lambda_{Rm} \partial_+ \lambda^m_R 
+ \partial_1 y^+ \partial_- y^+ - 2 \hat{A}_- \partial_1 y^+ - \hat{A}_- \hat{A}_- \right] , \quad \hat{A}_- \equiv A_i \partial_- x^i - \frac{1}{2} F_{ij} \lambda^i_R \lambda^j_R .
\] (2.14)
\[ + \frac{1}{4} ( \partial_0 y \partial_1 \tilde{y} + \partial_0 \tilde{y} \partial_1 y - \partial_1 y \partial_0 \tilde{y} - \partial_1 \tilde{y} \partial_0 y) + \hat{A}_- (\partial_1 y + \partial_1 \tilde{y}) - \hat{A}_- \hat{A}_- ] . \]  

(2.17)

In view of (2.11) one can also trade the \((y, \tilde{y})\)-terms for the \((y^-, y^+)\) ones. Then it becomes explicit that (2.14) is just the \(y^- = 0, \lambda^\tilde{y} = 0\) truncation of (2.17).

The actions (2.14) and (2.17) are the special cases of the action [15] of the \(D = 4\) heterotic string compactified on a torus [22]. Let \(y^\alpha(\tau, \sigma)\) be 28 fields that parametrize 28-torus conjugate to an even self-dual lattice of signature \((6, 22)\). The invariant metric of \(O(6, 22)\) can be chosen as

\[
\mathcal{L}_{\alpha\beta} = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & I_{16} \end{pmatrix}.
\]

Introducing 28 Abelian vector fields \(A_\mu^\alpha\) and the matrix \(M^{\alpha\beta}\) of moduli fields \((M^T \mathcal{L} M = \mathcal{L}, \ M^T = M)\) one finds that the (bosonic part of) \(y^\alpha\)-dependent terms in the manifestly \(O(6, 22)\) T-duality invariant heterotic string action are given by [13, 21]

\[
(D_p y^\alpha = \partial_p y^\alpha + A_\mu^\alpha \partial_\mu x^\mu)_{13}
\]

\[
I = \frac{1}{4\pi \alpha'} \int d^2 \sigma [ \mathcal{L}_{\alpha\beta} D_0 y^\alpha D_1 y^\beta - (\mathcal{L} M \mathcal{L})_{\alpha\beta} D_1 y^\alpha D_1 y^\beta + \epsilon^{pq} A_\rho^\alpha \mathcal{L}_{\alpha\beta} D_q y^\beta ] . \]  

(2.18)

The case of (2.10), (2.17) corresponds to \(y^\alpha = (y, \tilde{y}), \ M = I, \ A^\alpha_\mu = A_\mu\), while that of (2.12), (2.13), (2.14) corresponds to \(y^\alpha = \sqrt{2} y^+, \ A^\alpha_\mu = \sqrt{2} A_\mu\)

3. Solution of the heterotic string models with magnetic field in the Kaluza-Klein sector

To determine the spectrum of the conformal models defined by (2.4), (2.6), (2.7) one may follow the same strategy as used in the bosonic case in [1]. The simplest model to solve is (2.6). In the light-cone gauge \(\lambda^\mu = 0\) the bosonic and fermionic variables essentially decouple and the solution reduces to that of the bosonic model with trivial modifications due to the presence of the free fermionic oscillators. As a consequence, one finds that GSO projection [23] eliminates tachyons which were present in the bosonic case, the spectrum is symmetric between bosons and fermions and the partition function vanishes. This

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13 \(y^\alpha\) can be assumed to be compactified on circles of radii \(R = \sqrt{\alpha'}\) with information about the specific torus being encoded in moduli. Note that the upper block of \(\mathcal{L}_{\alpha\beta}\) is diagonalized by setting \(y = \frac{1}{\sqrt{2}} (y^1 + y^2), \ \tilde{y} = \frac{1}{\sqrt{2}} (y^1 - y^2)\) with \(y^{1,2}\) having the same normalization as \(y^\alpha\).

14 Note that the action (2.18) in [15] differs from (2.10), (2.12), (2.13) by the Lorentz-invariant ‘counterterm’ \(A_- A_+\), which can be absorbed in the \(x\)-dependent part of the action and must be present in (2.14), (2.17) (for a discussion of this term in connection with scheme dependence see [1]).
conclusion is consistent with space-time supersymmetry of the corresponding background [4,13]. The resulting stability of this heterotic string model may be surprising in view of the conclusion [4,8] that the constant magnetic field background in the open superstring theory is unstable for certain values of the magnetic field. Indeed, \( F_{ij} = \text{const} \) background breaks space-time supersymmetry of this theory and (as in the bosonic open string case) there are tachyonic states in its spectrum. The heterotic model (2.6) has world-sheet supersymmetry in the left (‘charge’) sector and, therefore, here the presence of the magnetic background does not spoil the space-time supersymmetry.\(^{15}\) The type II superstring model (2.4) inherits the space-time supersymmetry of its ‘left part’ and is also stable (has no tachyons and zero partition function after the GSO projection).

It is the model (2.7) that is the analogue of the open superstring model: here the world-sheet supersymmetry is present only in the right sector and as a result the space-time supersymmetry is broken (see also [4,13]). As we shall show below, this model has indeed tachyonic instabilities (in particular, the usual Yang-Mills ones). It will turn out that the heterotic (0, 1) model (2.8), (2.14) with the gauge sector embedding of the magnetic field will have similar properties. Since its solution can be obtained from the solution of (2.7) by a ‘chiral truncation’ (cf. (2.14), (2.17)) we shall first consider the latter model in detail.

3.1. Quantization and Virasoro conditions

Introducing \( x = x^1 + ix^2, \ x^* = x^1 - ix^2, \ \lambda_R = \lambda_R^1 + i\lambda_R^2, \ \lambda_R^* = \lambda_R^1 - i\lambda_R^2 \) one can represent the action (2.7), (2.5) in the form (we omit additional free field terms and the subscript \( R \) on fermions)

\[
I^{(kk)}_{(0,1)} = \frac{1}{\pi\alpha'} \int d^2 \sigma \left[ \partial_+ u \partial_- v + \partial_+ x \partial_- x^* + \frac{1}{2}i f \partial_+ u (x \partial_- x^* - x^* \partial_- x) + \lambda^5 \partial_+ \lambda^u + \lambda^* \partial_+ \lambda + i f \partial_+ u \lambda^* \lambda \right].
\]

The equations of motion are given by

\[
\partial_- \partial_+ u = 0, \quad \partial_+ \left[ \partial_- v + \frac{1}{2}i f (x \partial_- x^* - x^* \partial_- x + 2\lambda^* \lambda) \right] = 0, \quad \partial_+ \partial_- x + i f \partial_+ u \partial_- x = 0, \quad \partial_+ \partial_- x^* - i f \partial_+ u \partial_- x^* = 0,
\]

\[
\partial_+ \lambda^u = \partial_+ \lambda^v = 0, \quad \partial_+ \lambda + i f \partial_+ u \lambda = 0, \quad \partial_+ \lambda^* - i f \partial_+ u \lambda^* = 0.
\]

\(^{15}\) In particular, \( SO(32) \) or \( E_8 \times E_8 \) gauge vector bosons do not become tachyonic in the heterotic model (2.4) because they are singlets under this Kaluza-Klein \( U(1) \) group.
components corresponding to the model (3.1) are given by
\[ x = e^{-ip\sigma^+} X, \quad x^* = e^{ip\sigma^+} X^*, \quad X = X_+ + X_-, \quad (3.5) \]
\[ v = v_+ + v_- + \frac{1}{2}if(X_+^*X_+ - X_-^*X_-), \quad (3.6) \]
\[ \lambda = e^{-ip\sigma^+}\eta_-, \quad \lambda^* = e^{ip\sigma^+}\eta_+, \quad (3.7) \]

where the subscripts ± indicate dependence on \( \sigma_\pm = \tau \pm \sigma \), i.e. \( X_\pm = X_\pm(\sigma_\pm) \), etc. The closed string periodicity conditions \( x(\sigma + \pi, \tau) = x(\sigma, \tau) \), \( \lambda(\sigma + \pi, \tau) = \pm \lambda(\sigma, \tau) \) are easily implemented by setting
\[ X_+ = e^{ip\sigma^+}X_+, \quad X_- = e^{-ip\sigma^-}X_-, \quad \eta_- = e^{-ip\sigma^-}\eta_-, \quad (3.8) \]

where the new fields satisfy the standard “free-theory” boundary conditions, \( \mathcal{X}_\pm(\sigma, \tau) = \mathcal{X}_\pm(\sigma + \pi, \tau) \) and \( \chi_-(\sigma + \pi, \tau) = \pm \chi_-(\sigma, \tau) \), with the signs “±” corresponding to the Ramond (R) and Neveu-Schwarz (NS) sectors. Thus
\[ \mathcal{X}_+ = i\sqrt{\frac{1}{2}\alpha'}\sum_n \tilde{a}_n e^{-2in\sigma_+}, \quad \mathcal{X}_- = i\sqrt{\frac{1}{2}\alpha'}\sum_n a_n e^{-2in\sigma_-}, \quad (3.9) \]
\[ R: \quad \chi_- = \sqrt{2\alpha'}\sum_{n \in \mathbb{Z}} d_n e^{-2in\sigma_-}, \quad NS: \quad \chi_- = \sqrt{2\alpha'}\sum_{r \in \mathbb{Z}^+1/2} c_r e^{-2ir\sigma_-}. \quad (3.10) \]

The zero mode parts of the fields are \( (u \equiv y - t, \quad v \equiv y + t) \)
\[ y_{\text{zero}} = y_0 + 2L\sigma + k\tau, \quad t_{\text{zero}} = t_0 + p\tau, \quad (3.11) \]
\[ u_{\text{zero}} = u = u_0 + p_+\sigma_+ + p_-\sigma_-, \quad p_\pm = \pm L + \frac{1}{2}(k - p), \quad (3.12) \]
\[ v_{\text{zero}} = v_0 + q_+\sigma_+ + q_-\sigma_-, \quad q_\pm = \pm L + \frac{1}{2}(k + p). \quad (3.13) \]

If \( y \) is compactified on a circle of radius \( R \) then \( L = Rw, \quad w = 0, \pm 1, \ldots \). The stress tensor components corresponding to the model (3.1) are given by
\[ T_- = \partial_- u \partial_- x + \partial_- x \partial_- x^* + \frac{1}{2}if \partial_- u(x \partial_- x^* - x^* \partial_- x) \]
\[ + i\lambda^* \partial_- \lambda - f \partial_- u \lambda^* \lambda, \quad (3.14) \]
\[ T_+ = \partial_+ u \partial_+ x + \partial_+ x \partial_+ x^* + \frac{1}{2}if \partial_+ u(x \partial_+ x^* - x^* \partial_+ x). \quad (3.15) \]

The classical expressions for the Virasoro operators \( L_0, \tilde{L}_0 \) are, in the R-sector,
\[ L_0^{(R)} = \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma \ T_- = \frac{p-q}{4\alpha'} + \frac{1}{2} \sum_n (n + \frac{1}{2}fp_+)(n + fL)a^*_n a_n \quad (3.16) \]
The expressions in the NS sector are similar. Using (3.9) we obtain from (3.1) the canonical momentum of \( y \)
\[
p_y = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \left[ \partial_0 y + \frac{1}{2} i f (x \partial_- x^* - x^* \partial_- x + 2\lambda^* \lambda) \right] = \frac{1}{2} \alpha'^{-1} k + f J_R .
\] (3.18)

\( J_R \) is the ‘right’ part of the angular momentum
\[
J_R = -\frac{1}{2} \sum_n (n + \frac{1}{2} f p_+) a_n^* a_n + K ,
\] (3.19)

\[
K^{(\text{NS})} = -\sum_r c_r^* c_r , \quad K^{(\text{R})} = -\sum_n d_n^* d_n .
\] (3.20)

Since the background is stationary, the string also has conserved energy
\[
E = \int_0^\pi d\sigma P_t = -\frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \left[ \partial_0 t + \frac{1}{2} i f (x \partial_- x^* - x^* \partial_- x + 2\lambda^* \lambda) \right]
\]
\[
= -\frac{1}{2} \alpha'^{-1} p - f J_R ,
\] (3.21)

\[
p_+ = L + \alpha' (p_y + E) = \alpha' (Q_L + E) , \quad Q_{L,R} \equiv p_y \pm \alpha'^{-1} L .
\] (3.22)

Expressing (3.16) and (3.17) in terms of \( E, p_y, L \) (or \( E, Q_{L,R} \)) and oscillators, we obtain
\[
L_0^{(\text{R})} = -\frac{1}{4} \alpha' E^2 + \frac{1}{4} \alpha' Q_R^2 + \sum_n \left[ \frac{1}{2} (n + \frac{1}{2} f p_+) a_n^* a_n + d_n^* d_n \right] - \frac{1}{2} f p_+ J_R
\] (3.23)

\[
= -\frac{1}{4} \alpha' E^2 + \frac{1}{4} \alpha' Q_R^2 + \sum_n \left[ (n + \frac{1}{2} f p_+) \left[ \frac{1}{2} (n + \frac{1}{2} f p_+) a_n^* a_n + d_n^* d_n \right] \right] ,
\]

\[
\tilde{L}_0^{(\text{R})} = -\frac{1}{4} \alpha' E^2 + \frac{1}{4} \alpha' Q_R^2 + \sum_n \left[ \frac{1}{2} (n - \frac{1}{2} f p_+) \tilde{a}_n^* \tilde{a}_n - \frac{1}{2} f p_+ J_R \right] .
\] (3.24)

We can now quantize the theory in a standard way by promoting the Fourier modes to operators acting in Fock space and imposing the canonical commutation relations. They imply the commutation relations for the zero modes (\( [y_0, p_y] = i \) so that the momentum eigenvalues are \( p_y = m R^{-1} , \ m = 0, \pm 1, ... \)) and
\[
[a_n, a_m^*] = 2(n + \frac{1}{2} f p_+)^{-1} \delta_{nm} , \quad [\tilde{a}_n, \tilde{a}_m^*] = 2(n - \frac{1}{2} f p_+)^{-1} \delta_{nm} .
\] (3.25)

\[
\{d_n, d_m^*\} = \delta_{nm} , \quad \{c_r, c_s^*\} = \delta_{rs} .
\] (3.26)
Symmetrizing the classical expressions for $L_0, \bar{L}_0$ and $J_R$ we then normal-order them and use the generalized $\zeta$-function prescription\[4\]. In the R-sector the bosonic and fermionic normal ordering constants in $L_0$ cancel out completely, i.e. one finds only that $\bar{L}_0 \to \bar{L}_0 - 1$. In the NS sector one obtains: $L_0 \to L_0 - \frac{1}{2} + \frac{1}{4}fp_+$ and $\bar{L}_0 \to \bar{L}_0 - 1 + \frac{1}{4}fp_+$. \[\]

To write down the resulting expressions for the Virasoro operators it is convenient to introduce the creation and annihilation operators as follows

\[
[b_{n\pm}, b^\dagger_{m\pm}] = \delta_{nm}, \quad [b_{n\pm}, b^\dagger_{m\pm}] = \delta_{nm}, \quad [b_0, b^\dagger_0] = 1, \quad [\bar{b}_0, \bar{b}^\dagger_0] = 1,
\]

\[
b^\dagger_{n+} = a_{-n}\omega_-, \quad b_{n+} = a^*_{-n}\omega_-, \quad b^\dagger_{n-} = a^*_{n}\omega_+, \quad b_{n-} = a_n\omega_+,
\]

\[
\tilde{b}^\dagger_{n+} = \tilde{a}_{-n}\omega_+, \quad \tilde{b}_{n+} = \tilde{a}^*_{-n}\omega_+, \quad \tilde{b}^\dagger_{n-} = \tilde{a}^*_{n}\omega_-, \quad \tilde{b}_{n-} = \tilde{a}_n\omega_-,
\]

\[
b^\dagger_0 = \frac{1}{2}\sqrt{fp_+a_0^*}, \quad b_0 = \frac{1}{2}\sqrt{fp_+a_0}, \quad \tilde{b}^\dagger_0 = \frac{1}{2}\sqrt{fp_+\tilde{a}_0^*}, \quad \tilde{b}_0 = \frac{1}{2}\sqrt{fp_+\tilde{a}_0},
\]

where $\omega_\pm \equiv \sqrt{\frac{1}{2}(n \pm \frac{1}{2}fp_+)}$, $n = 1, 2, \ldots$. The subscripts $\pm$ correspond to components with spin ‘up’ and ‘down’ respectively. We have assumed that $0 < fp_+ < 2$. For $fp_+ > 2$ or $fp_+ < 0$ the creation/annihilation roles of some operators change but the analysis remains essentially the same (see \[3\] for a detailed discussion of this point). The Fock vacuum obeys also $d^*_n|0\rangle = d_n|0\rangle = 0$, $n > 0$ and $c^*_r|0\rangle = c_r|0\rangle = 0$, $r > 0$. Symmetrizing and normal-ordering the classical expression for $J_R$ (3.19) we get

\[
\hat{J}_R = -b^\dagger_0 b_0 - \frac{1}{2} + \sum_{n=1}^\infty (b^\dagger_{n+}b_{n+} - b^\dagger_{n-}b_{n-}) + \hat{K} = J_R - \frac{1}{2},
\]

\[
\hat{K}^{(NS)} = -\sum_{r=1/2}^\infty (c^*_r c_r + c_- r c^*_{-r}), \quad \hat{K}^{(R)} = -\frac{1}{2}[d^*_0, d_0] - \sum_{n=1}^\infty (d^*_n d_n + d_- n d^*_{-n}).
\]

The Virasoro operators (3.23), (3.24) should include also the contributions of additional free degrees of freedom. In the standard bosonic description of the heterotic string theory \[11,24\] there are 16 left (internal sector) chiral bosons $y^I_L$ ($I = 1, \ldots, 16$) (see (2.15), (2.16)) compactified on a torus corresponding to the even self-dual 16-lattice

\[
y^I_L = y^I_0 + \sqrt{2\alpha'} p^I_L \sigma_+ + i \sqrt{\frac{1}{2}\alpha'} \sum_{n\neq 0} \frac{1}{n} \tilde{a}^I_n e^{-2in\sigma_+}, \quad p^I_L = \sum_{a=1}^{16} n_a c^I_a,
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\[ y_L^I \equiv y_L^I + 2\pi L^I, \quad L^I = \sqrt{\frac{1}{2} \alpha'} \sum_{a=1}^{16} n_a e_a^I = \sqrt{\frac{1}{2} \alpha'} p_L^I. \]  

(3.33)

Including also the contribution of the remaining 5 free non-chiral bosonic fields of the supersymmetric sector (\(\alpha = 5, ..., 9\)) we get from (3.23), (3.24)

\[ \hat{L}_0 = \frac{1}{4} \alpha' (E^2 + p_L^2 + Q_R^2) + \hat{N}_R - \frac{1}{2} \alpha' f(Q_L + E) \hat{J}_R, \]  

(3.34)

\[ \hat{L}_0 = \frac{1}{4} \alpha' (E^2 + p_L^2 + Q_L^2) + \frac{1}{2} (p_L^I)^2 + \hat{N}_L - \frac{1}{2} \alpha' f(Q_L + E) \hat{J}_R, \]  

(3.35)

\[ \hat{H} = \hat{L}_0 + \hat{L}_0 = \frac{1}{2} \alpha' [E^2 + p_L^2 + \frac{1}{2} (Q_L^2 + Q_R^2)] + \frac{1}{2} (p_L^I)^2 + \hat{N}_R + \hat{N}_L - \alpha' f(Q_L + E) \hat{J}_R, \]  

(3.36)

where

\[ \hat{N}_R = N_R - a, \quad \hat{N}_L = N_L - 1, \quad a^{(R)} = 0, \quad a^{(NS)} = \frac{1}{2}, \]  

(3.37)

\[ Q_{L,R} = m R^{-1} \pm \alpha'^{-1} w R, \quad \frac{1}{4} \alpha' (Q_L^2 - Q_R^2) = mw, \]  

(3.38)

and the free-theory operators \(N_L\) and \(N_R\) are (e.g. in the Ramond sector):

\[ N_R^{(R)} = \sum_{n=1}^{\infty} \left[ n (b_{n+}^1 b_{n+} + b_{n-}^1 b_{n-} + b_{n\alpha}^1 b_{n\alpha}) + d_n^* d_n + d_{-n}^* d_{-n} + d_{-n\alpha} d_{n\alpha} \right], \]  

(3.39)

\[ N_L^{(R)} = \sum_{n=1}^{\infty} \left[ n (b_{n+}^* b_{n+} + b_{n-}^* b_{n-} + b_{n\alpha}^* b_{n\alpha}) + \bar{a}_{n+} \bar{a}_{n+} \right]. \]  

(3.40)

The Virasoro conditions are thus \(\hat{L}_0 = \hat{L}_0 = 0\), i.e.,

\[ \hat{H} = 0, \quad \hat{N}_R + \frac{1}{4} \alpha' Q_R^2 = N_L - 1 + \frac{1}{4} \alpha' Q_L^2 + \frac{1}{2} (p_L^I)^2. \]  

(3.41)

Separating the spin part of the angular momentum, \(\hat{J}_R = -b_0^1 b_0^1 - \frac{1}{2} + S_R \rightarrow -(l + \frac{1}{2}) + S_R\), we obtain from (3.41):

\[ E^2 = p_L^2 + Q_R^2 + 4 \alpha'^{-1} \hat{N}_R + f(Q_L + E)(2l + 1) - 2 f(Q_L + E) S_R, \]  

(3.42)

where \(l = 0, 1, 2, ..., \) is the Landau level.\(^{13}\)

\(^{17}\) We are assuming that the \(D = 10\) heterotic string is compactified on a 6-torus \(T^6 = S^1 \times T^5\) where \(S^1\) is the \(y = x^4\)-circle used to embed the Abelian magnetic field and \(T^5\) corresponds to the additional free coordinates. For simplicity, we shall consider only the states which have zero winding number in these 5 additional free dimensions. In particular, various generalizations along the lines of \(^{22}\) are straightforward.

\(^{18}\) In the non-relativistic limit one finds the following expression for the gyromagnetic factor of an arbitrary physical state, \(g = 2(1 + \frac{M}{Q_L}) \frac{(s_R)}{(5)}\), which was discussed in \(^{13}\) in the context of the bosonic model. As was pointed out in \(^{2}\), the presence of the term \(\hat{O}(M/Q_L)\) in this model is accidental and is due to the non-vanishing antisymmetric tensor with strength proportional to the magnetic field. The universal expression for the \(g\)-factor associated to the intrinsic magnetic moment of the particle in heterotic string theory is \(^{25}\): \(g = 2 \frac{(s_R)}{(5)}\). This expression was confirmed in \(^{2}\) for a general class of exactly solvable models describing magnetic backgrounds.
3.2. Energy spectrum and tachyonic instabilities

The analysis similar to the one carried out in the bosonic case shows that this model has tachyonic states in its spectrum. Indeed, the resulting form of the Hamiltonian and level matching constraint is very similar to that in the bosonic case: the only differences are the presence of the fermionic terms in the operators $N_R, J_R$, different normal ordering constants and the standard free heterotic string term $(p_L^I)^2$ in the left Virasoro operator $(3.35)$. Since the constraints $(3.41)$ are expressed in terms of free creation/annihilation operators and are diagonal in Fock space the spectrum is found in the same way as in the free heterotic string theory.

As follows from $(3.41)$, the equation for the energy spectrum can be represented as

$$E^2 = 4\alpha' - 1 \hat{N}_R + Q_L^2 - 2f(Q_L + E)\hat{J}_R,$$

(3.43)

and

$$(E + f\hat{J}_R)^2 = 4\alpha' - 1 \hat{N}_R + (Q_L - f\hat{J}_R)^2 + Q_R^2 - Q_L^2,$$

(3.44)

or, equivalently,

$$(E + f\hat{J}_R)^2 = 4\alpha' - 1 (N_L - 1) + (Q_L - f\hat{J}_R)^2.$$

(3.45)

The GSO projection in the supersymmetric right sector implies that $\hat{N}_R$ can take only non-negative integer values ($\hat{N}_R$ corresponds to the number of states operator of the light-cone Green-Schwarz formulation). As a result, there are no tachyons in the free ($f = 0$) heterotic string theory. For a non-zero field $f$, the energy levels of the free heterotic string split according to the value of the ‘right’ contribution to the angular momentum $J_R$ and the ‘left’ charge $Q_L$. As follows from $(3.43),(3.44),(3.45)$, for $f \neq 0$ there are states for which $E$ is complex. This indicates the presence of a tachyonic instability. Equation $(3.44)$ implies that the tachyonic states must have $\alpha'(Q_L^2 - Q_R^2) = 4mw > 0$, i.e. belong to the winding sector. From $(3.45)$ one learns that such states necessarily must have $N_L = 0$ ($N_L$ can take only values $0, 1, 2, ...$).

One particular choice of parameters and quantum numbers that leads to tachyonic states is

$$R = \sqrt{\alpha'}, \quad m = w = 1, 2, ..., \quad Q_R = 0, \quad Q_L = 2m/\sqrt{\alpha'}, \quad p_L^I = 0,$$

(3.46)

$\hat{N}_R = -1 + m^2, \quad N_L = 0.$

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19 For the purpose of identifying some tachyonic states in the spectrum it is enough to consider only the states with vanishing momenta in extra free directions, $p_\alpha = 0$.

20 This instability is also reflected in the partition function which has infrared divergences at those values of $f$ for which the energy gets an imaginary component. 

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Consider components with $\hat{J}_R > 0$. It follows from (3.45) that a given state becomes tachyonic in the range $f_{cr(1)} > f > f_{cr(2)}$,

$$\sqrt{\alpha'} f_{cr(1,2)} = \frac{2(m \pm 1)}{\hat{J}_R}.$$  (3.47)

In particular, $m = 1$, $\hat{N}_R = 0$ gives the standard charged vector (Yang-Mills) instability which appears already for an infinitesimal $f$. For the states with large $m$ and lying on the leading Regge trajectory (with maximal $\hat{J}_R$, i.e. with zero orbital quantum number $l = 0$ and maximal spin at a given level), $\hat{N}_R \sim \hat{J}_R \sim m^2$, we find that $\sqrt{\alpha'} f_{cr} \simeq 2/m$. Thus the higher the charge and spin of a given state, the smaller the magnetic field needed to make it tachyonic.

Also, for any given arbitrarily small $f$ there exists an infinite number of tachyonic states with large enough charges and spins. For large $m$, these are states which in the Regge diagram lie between the parabolas, $\hat{J}_R = c\sqrt{\hat{N}_R} + 1 \pm c$, $c = 2/(\sqrt{\alpha'} f)$. Indeed, for fixed $f$ all states with

$$\frac{1}{2} \sqrt{\alpha'} f \hat{J}_R - 1 < m < \frac{1}{2} \sqrt{\alpha'} f \hat{J}_R + 1, \quad m \geq m_0,$$  (3.48)

where $m_0 = \frac{1}{2} \left( c + \sqrt{c^2 - 4(c - \frac{3}{2} + a)  \right) \equiv c$, are tachyonic. The condition $m \geq m_0$ comes from the requirement $S_R \leq N_R$. Since $m$ (with $\hat{J}_R$ satisfying (3.48)) can take infinitely many possible values, there are an infinite number of tachyons for any given magnetic field. Similar results were found in the bosonic string case [1] and will apply also to the heterotic string model (2.14) considered in next section. This pattern of tachyonic instabilities is different from the one found in the open superstring theory [8]. In particular, it reflects the fact that in closed string theory there are states with arbitrarily large values of charges.

Since this discussion applies to both Neveu-Schwarz and Ramond sectors, there are also an infinite number of space-time fermions with an imaginary part in the energy. This conclusion is also different from what happens in the open superstring theory where there are no tachyons in the R-sector [8]. As expected, the massless spin $\frac{1}{2}$ fermions do not become tachyonic for $f \neq 0$ (for them the contribution of the gyromagnetic coupling cancels against the energy of the zeroth Landau level) [7].

Now let us consider the model (2.4). It has supersymmetric left and non-supersymmetric right sector, so that the free-theory parts of the Virasoro operators (3.34) and (3.35) are interchanged (with $p^I_R$ replacing $p^I_L$). The interaction ($f$-dependent) term

\footnote{The presence of higher spin fermionic states with complex energy does not seem to be in conflict with the standard field-theory expectation that ‘tachyonic’ fermions contradict unitarity of the theory since here the background metric is non-static.}
is now purely bosonic (there is no fermionic contribution in $\hat{J}_R$). As a result, the analogs of the conditions (3.43), (3.41) are

$$E^2 = 4\alpha'^{-1}\tilde{N}_L + Q_L^2 - 2f(Q_L + E)\hat{J}_R,$$

(3.49)

$$N_R - 1 + \frac{1}{2}(p_R^I)^2 = \tilde{N}_L + mw, \quad \tilde{N}_L \equiv N_L - a.$$  \hfill (3.50)

The GSO projection here applies to the left sector implying that $\tilde{N}_L = 0, 1, 2, ...$. Now $Q_L$ appears both in the interacting and the free part of the energy relation (cf. (3.43) and (3.49)) so that (3.49) can be put in manifestly non-negative form (cf. (3.44))

$$(E + f\hat{J}_R)^2 = 4\alpha'^{-1}\tilde{N}_L + (Q_L - f\hat{J}_R)^2.$$  \hfill (3.51)

The expression for the energy spectrum in type II superstring model (2.4) is found by combining the above expressions and again is manifestly non-negative. The obvious difference with respect to the heterotic model (2.7) is in the form of the level matching constraint (now $\tilde{N}_R = \tilde{N}_L + mw$). Apart from the fact that $\hat{J}_R$ again contains the fermionic part, the expression for $E^2$ is identically the same as (3.49) or (3.51).

Since the magnetic field couples to the spin, a priori one expects that in any magnetic field background there will be a mass splitting between fermions and bosons, and hence supersymmetry will be necessarily broken. One may wonder how the ‘left-right symmetric’ and ‘left’ heterotic (and type II superstring) models managed to preserve supersymmetry and hence avoid tachyonic instabilities. The reason is that here the magnetic field does not couple to the total spin, but only to the right part of it, and the latter may happen to be the same for fermions and bosons. Both the ‘left-right symmetric’ (or type II superstring) and ‘left’ heterotic models still have an equal number of bosons and fermions with the same $\hat{J}_R$ and, as a result, an equal number of bosons and fermions at each level. The formal mechanism responsible for avoiding tachyons in these models is GSO projection. It eliminated not only ground state tachyon but also certain higher level states of the free bosonic string spectrum which otherwise would become tachyonic in the presence of the magnetic field. For example, the electrically charged massless vector states which appear in the bosonic string compactified on a circle of radius $R = \sqrt{\alpha'}$, and which become tachyonic in the presence of the magnetic field, are actually projected out by GSO in the above two theories.

\footnote{Note that in heterotic string theory it is not possible to couple the magnetic field to the total spin.}
4. Heterotic string model with a magnetic field in the internal gauge sector

Let us now describe the solution of the ‘right’ heterotic model (2.8) or (2.14) where the magnetic field appears in the internal gauge symmetry sector. The two ‘right’ heterotic models (2.7),(2.17) and (2.14) are closely related: as discussed in Section 2, (2.14) is just a chiral truncation of (2.17). Given that the two actions (2.14) and (2.17) are special cases of the action (2.18) of the heterotic string compactified on 28-torus, which is manifestly invariant under the $T$-duality group $O(6,22)$, it is natural to expect that the two models have similar properties, in particular, the heterotic model (2.14) with the gauge sector (Cartan subalgebra) embedding of the Abelian magnetic field also contains tachyonic states in its spectrum.

The solution of the model (2.14) is found by repeating the discussion of the previous section while dropping the $y^-$ part of $y = y^+(\tau,\sigma)+y^-(\tau-\sigma)$, i.e. by choosing the special solution $\partial_-y^+ + \hat{A}_- = 0$ of the equation $\partial_+(\partial_+y + \hat{A}_-) = 0$ in (3.2),

$$\partial_-y^+ + \frac{1}{2}if(x\partial_-x^* - x^*\partial_-x + 2\lambda^*\lambda) = 0 .$$  \hspace{1cm} (4.1)

Then eqs. (3.5)-(3.13) still apply, in particular,

$$y^+_{zero} = y^+_0 + 2L^+\sigma + k^+\tau ,$$  \hspace{1cm} (4.2)

Integrating eq.(4.1) over $\sigma$ we now get (cf. (3.18),(3.19))

$$k^+ - 2L^+ + 2\alpha'fJ_R = 0 .$$  \hspace{1cm} (4.3)

The definition of the momentum (3.18) is also modified ($\partial_0y^+$ does not appear in the interaction term in the action, see (2.12)), but the final expression is still formally the same as in (3.18) after we use (4.3)

$$p^+_y = \frac{1}{2\pi\alpha} \int_0^\pi d\sigma \partial_1y^+ = \alpha'^{-1}L^+ = \frac{1}{2}\alpha'^{-1}k^+ + fJ_R .$$  \hspace{1cm} (4.4)

As a result, the ‘right’ charge in (3.22) is now equal to zero, i.e.

$$Q_R = 0 , \quad Q_L = 2\alpha'^{-1}L^+ .$$  \hspace{1cm} (4.5)

The expressions for the Virasoro operators and the Hamiltonian are still given by (3.34)-(3.36) with $Q_R = 0$ and $Q_L^2$ being now part of the lattice momenta term $(p_L^1)^2$. Indeed, in this section we are assuming that the Abelian magnetic field is embedded in the internal gauge symmetry group by identifying $y^+$ with one of the coordinates of the 16-torus, e.g. the first one, $y^+ = y^+_L$ (cf. (2.14),(2.15)). Then (see (3.32),(3.33))

$$L^+ = L^1 = \sqrt{\frac{1}{2}\alpha'^{-1}p_L^1} , \quad Q_L = \sqrt{2\alpha'^{-1}}p_L^1 ,$$  \hspace{1cm} (4.6)
\[(p_L^I)^2 = \frac{1}{2}\alpha' Q_L^2 + (p_L^{I'})^2, \quad I' = 2, ..., 16.\]

The values of \(L^+\) and \(Q_L\) are determined by the allowed values of \(p_L^I\) which depend on the choice of one of the two possible integral even self-dual 16-lattices \([11]\). The final expressions for the Virasoro operators are \([23]\)

\[\hat{L}_0 = \frac{1}{4}\alpha'(-E^2 + p_\alpha^2) + \hat{N}_R - \frac{1}{2}\alpha' f(Q_L + E)\hat{J}_R, \quad (4.7)\]

\[\hat{L}_0 = \frac{1}{4}\alpha'(-E^2 + p_\alpha^2) + \frac{1}{2}(p_L^I)^2 + \hat{N}_L - \frac{1}{2}\alpha' f(Q_L + E)\hat{J}_R, \quad (4.8)\]

\[\hat{H} = \frac{1}{2}\alpha'(-E^2 + p_\alpha^2) + \frac{1}{2}(p_L^I)^2 + \hat{N}_R + \hat{N}_L - \alpha' f(Q_L + E)\hat{J}_R, \quad (4.9)\]

so that the analogues of the constraint \((3.41)\) and the energy spectrum relation \((3.43),(3.44)\) are

\[\hat{N}_R = \hat{N}_L + \frac{1}{2}(p_L^I)^2, \quad \hat{N}_L = N_L - 1, \quad (4.10)\]

\[E^2 = 4\alpha'^{-1}\hat{N}_R + p_\alpha^2 - f(Q_L + E)\hat{J}_R. \quad (4.11)\]

As in the case of bosonic string and heterotic string with Kaluza-Klein embedding \((2.7)\) discussed above, the expression for \(E^2\) is not manifestly positive so that tachyonic instabilities are expected to appear.

To determine the presence of states with complex energy let us consider the simplest configuration with zero momenta in 6 extra dimensions \(p_\alpha = 0\). Then \((4.10),(4.11)\) imply (cf. \((3.44),(3.45)\))

\[(E + f\hat{J}_R)^2 = 4\alpha'^{-1}\hat{N}_R + (Q_L - f\hat{J}_R)^2 - Q_L^2, \quad (4.12)\]

\[(E + f\hat{J}_R)^2 = 4\alpha'^{-1}[N_L - 1 + \frac{1}{2}(p_L^I)^2] + (Q_L - f\hat{J}_R)^2 - Q_L^2. \quad (4.13)\]

Note that \((4.12)\) is the same as the condition \((3.44)\) on the spectrum of another ‘right’ heterotic model \((2.7)\) in the special case of \(Q_R = 0\) \((3.46)\) discussed in the previous section. As in that model, here the tachyonic states may also appear only in the sector with \(N_L = 0\). From the Virasoro constraint \((4.10)\) we learn that the condition \(N_L = 0\) (and the fact that after GSO projection \(\hat{N}_R \geq 0\)) implies that \((p_L^I)^2 \geq 2\).

In the simplest case of \((p_L^I)^2 = 2\), \(\hat{N}_R = 0\), \(N_L = 0\), which is analogous to the \(m = 1\) case in \((3.46)\) and corresponds to the charged vector bosons of the massless heterotic string level, \((4.12)\) reproduces the standard infinitesimal magnetic instability of non-Abelian theory (considering \(Q_L, f\hat{J}_R > 0\), one has \((E + f\hat{J}_R)^2 = -f\hat{J}_R(2Q_L - f\hat{J}_R) < 0\) for \(f\) infinitesimal). Another special choice that demonstrates the presence of tachyons at higher string

\[23\] As in \((3.36)\) \(p_\alpha\) are momenta of extra (here 6) dimensions which may be assumed, e.g., to be compactified on a torus.
levels is $p^I_L = 0$ (cf. (1.6)). Then $(p^I_L)^2 = \frac{1}{2} \alpha' Q^2_L$ and (4.10),(4.13) become identically the same as the conditions (3.41),(3.45) with $Q_R = 0$, $p^I_L = 0$ (3.46), i.e. we get

$$\hat{N}_R = -1 + \frac{1}{4} \alpha' Q^2_L \ , \quad N_L = 0 \ , \quad \sqrt{\alpha'} Q_L = \sqrt{2} p^1_L \ ,$$

(4.14)

$$(E + f \hat{J}_R)^2 = -4 \alpha'^{-1} + (Q_L - f \hat{J}_R)^2 \ .$$

(4.15)

To find which values $p^1_L$ are actually possible let us express $p^I_L$ in terms of the dual generators, $p^I_L = \sum_{a=1}^{16} m_a e^I_a$, where $m_a$ are integers. Then $m_a = \sum_I p^I_L e^I_a = p^1_L e^1_a$, i.e. $p^1_L = m_1 / e^1_1$. In the basis of generators used in [11] the components $e^1_a$ are either $\pm 1/2$ or $\pm 1$ (this applies to both $\Gamma_8 \times \Gamma_8$ and $\Gamma_{16}$ lattices). Typical charge configurations thus give $p^1_L = 2m$, as can be explicitly checked (note that $(p^I_L)^2$ must be even). In this case the analogs of the conditions in (3.46) are

$$p^1_L = 2m \ , \quad Q_L = 2 \sqrt{\frac{2}{\alpha'}} m \ , \quad \hat{N}_R = -1 + 2m^2 \ , \quad m = 1, 2, \ldots \ .$$

(4.16)

These states become tachyonic for $f_{\text{cr}(1)} > f > f_{\text{cr}(2)}$ with (cf. (3.47))

$$\sqrt{\alpha'} f_{\text{cr}(1,2)} = \frac{2(\sqrt{2}m \pm 1)}{\hat{J}_R} \ .$$

(4.17)

The inequality analogous to (3.48) shows that as the bosonic model or the heterotic model with KK embedding, the heterotic model with gauge sector embedding also has an infinite number of tachyons for any (e.g. arbitrarily small) value of the magnetic field strength $f$.

5. Concluding remarks

Generalizing the previous work [1] we have shown here that, as the model of open superstrings, the models of closed superstrings and heterotic strings in constant magnetic field are also exactly solvable. The resulting structure of the string Hamiltonian is very simple: it is given by the free-theory part plus the gyromagnetic-type interaction term, which is linear in the magnetic field strength (see (3.36),(4.9)).

We have studied in turn the two non-supersymmetric heterotic string models (2.7) and (2.14) (with world-sheet supersymmetry in the ‘right’ sector), which correspond to the two possible ways to embed the Abelian magnetic field into heterotic string theory: (i) Kaluza-Klein embedding in the case of toroidal compactification from 10 to 4 dimensions, and (ii) embedding in the internal gauge symmetry sector of the 10-dimensional theory (which can be further compactified on some manifold $M^6$). While the second case is closer to realistic magnetic field backgrounds, the two models are related (with the latter being essentially a ‘truncation’ of the former). This is not surprising given that the internal gauge symmetry
group of the heterotic string also originates from a (chiral) compactification on a special 16-torus \([11]\). The two types of \(U(1)\) gauge fields are indeed particular members of the set of 28 Abelian vector fields which are present in the case of toroidal compactification of the heterotic string \([22]\).

The two heterotic models have similar properties. In particular, both exhibit a tachyonic instability, i.e. contain states with complex energy in their spectra. The novel feature of the closed string theory compared to the open string one is the presence of states with arbitrarily large values not only of masses and spins but also of charges. This leads to a remarkable closed string generalization of the well-known magnetic instability of non-Abelian gauge theory: there exist an infinite number of closed string tachyonic states for any value of the magnetic field strength \(f\). Since the gyromagnetic coupling term in \((\text{mass})^2 \sim M_0^2 - 2fQ_L \hat{J}_R + \ldots\) is given by the product of the magnetic field strength \(f\) with charge \(Q_L\) and angular momentum (= spin \(S_R\) minus the Landau orbital momentum number), the states with the free string mass term \(M_0^2 \sim m^2/\alpha'\), spin \(S_R \sim m^2\) and charge \(Q_L \sim m/\sqrt{\alpha'}\) will become tachyonic for \(\sqrt{\alpha'} f \sim 1/m\).

This instability should apply to 10-dimensional heterotic string as well as to any of its compactifications to 4 dimensions. It can be eliminated only if massive states receive Planck-mass corrections to their free-theory masses. Thus in heterotic string theory there are directions in the space of possible backgrounds along which an infinitesimal (supersymmetry breaking) deformation produces infrared instabilities (which, being associated with both massless and massive level states of the free theory, remain even after states of the massless level get small masses).

It should be noted that since these infinitesimal instabilities are due to states with large charges \(Q\), whose tree-level masses may receive important loop corrections, it might disappear at the string loop level. For example, if we restrict consideration to states with \(gQ << 1\), where \(g\) is the string coupling, then the minimal critical magnetic field will be of order \(\sqrt{\alpha'} f \sim g\), i.e. will no longer be infinitesimal (once massless level particles also become massive as a result of symmetry breaking).

In the context of the bosonic string theory it was shown \([2]\) that the same pattern of instabilities appears also in a more general class of models describing magnetic field configurations (in particular, with vanishing antisymmetric tensor, like \(a = 1\) or \(a = \sqrt{3}\) dilatonic Melvin backgrounds). In these cases the mass \(M^2 = E^2 - p_a^2\) is invariant with respect to the residual Lorentz group acting in directions orthogonal to the \((x_1, x_2)\)-plane. We expect similar tachyonic instabilities to be present also in the heterotic string versions of these models (which do not preserve space-time supersymmetry either).

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