Towards Understanding Label Smoothing

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Abstract

Label smoothing regularization (LSR) has a great success in training deep neural networks by stochastic algorithms such as stochastic gradient descent and its variants. However, the theoretical understanding of its power from the view of optimization is still rare. This study opens the door to a deep understanding of LSR by initiating the analysis. In this paper, we analyze the convergence behaviors of stochastic gradient descent with LSR for solving non-convex problems and show that an appropriate LSR can help to speed up the convergence by reducing the variance of labels. More interestingly, we proposed a simple and efficient strategy, namely Two-Stage Label smoothing algorithm (TSLA), that uses LSR in the early training epochs and drops it off in the later training epochs. We observe from the improved convergence result of TSLA that it benefits from LSR in the first stage and essentially converges faster in the second stage. To the best of our knowledge, this is the first work for understanding the power of LSR via establishing convergence complexity of stochastic methods with LSR in non-convex optimization. We empirically demonstrate the effectiveness of the proposed method in comparison with baselines on training ResNet models over public data sets.

1 Introduction

In training deep neural networks, one common strategy is to minimize cross-entropy loss with one-hot label vectors, which may lead to overfitting during the training progress that would lower the generalization accuracy [Müller et al., 2019]. To overcome the overfitting issue, several regularization techniques such as ℓ1-norm or ℓ2-norm penalty over the model weights, Dropout which randomly sets the outputs of neurons to zero [Hinton et al., 2012b], batch normalization [Ioffe and Szegedy, 2015], and data augmentation [Simard et al., 1998], are employed to prevent the deep learning models from becoming over-confident. However, these regularization techniques conduct on the hidden activations or weights of a neural network. As an output regularizer, label smoothing regularization (LSR) [Szegedy et al., 2016] is proposed to improve the generalization and learning efficiency of a neural network by replacing the one-hot vector labels with the smoothed labels that average the hard targets and the uniform distribution of other labels. Specifically, for a K-class classification problem, the one-hot label is smoothed by \( \hat{y} = (1 - p)y + pu(K) \), where \( y \) is the one-hot label, \( p \in [0, 1] \) is the smoothing strength and \( u(K) = \frac{1}{K} \) is a uniform distribution for all labels. Extensive experimental results have shown that LSR has significant successes in many deep learning applications including image classification [Zoph et al., 2018, He et al., 2019], speech recognition [Chorowski and Jaitly, 2017, Zeyer et al., 2018], and language translation [Vaswani et al., 2017, Nguyen and Salazar, 2019].

Due to the importance of LSR, researchers try to explore its behavior in training deep neural networks. Müller et al. [2019] have empirically shown that the LSR can help improve model calibration, however, they also have found that LSR could impair knowledge distillation, that is, if one trains a teacher model with LSR, then a student model has worse performance. Yuan et al. [2019a] have proved that knowledge distillation is a special case of LSR and LSR provides a virtual teacher model for knowledge distillation. Although LSR is known as a powerful technique in real applications, its theoretical understanding is still unclear. As a widely...
used trick, people believe LSR works because it may reduce the noise in the assigned class labels. However, to the best of our knowledge, it is unclear, at least from a theoretical viewpoint, how the introduction of label smoothing will help improve the training of deep learning models, and to what stage, it can help. In this paper, we aim to provide an affirmative answer to this question and try to deeply understand why and how the LSR works from the view of optimization. We believe that an appropriate LSR can essentially reduce the variance in the assigned class labels. Moreover, we will propose a new efficient strategy of employing LSR that tells when to use LSR. We summarize the main contributions of this paper as follows.

- It is the first work that establishes improved iteration complexities of stochastic gradient descent (SGD) [Robbins and Monro, 1951] with LSR for finding an $\epsilon$-approximate stationary point in solving a smooth non-convex problem in the presence of an appropriate label smoothing. The results theoretically explain why an appropriate LSR can help speed up the convergence. (Section 4)

- We propose a simple and efficient strategy, namely Two-Stage Label smoothing (TSLA) algorithm, where in the first stage it trains models for certain epochs using a stochastic method with LSR while in the second stage it runs the same stochastic method without LSR. The proposed TSLA is a generic strategy that can incorporate many existing stochastic algorithms. We show that TSLA integrated with SGD has an improved iteration complexity, compared to the SGD with LSR and the SGD without LSR. (Section 5)

2 Related Work

In this section, we introduce some related work. A closely related idea to LSR is confidence penalty proposed by Pereyra et al. [2017], an output regularizer that penalizes confident output distributions by adding its negative entropy to the negative log-likelihood during the training process. The authors [Pereyra et al., 2017] presented extensive experimental results in training deep neural networks to demonstrate better generalization comparing to baselines with only focusing on the existing hyper-parameters. They have shown that LSR is equivalent to confidence penalty with a reversing direction of KL divergence between uniform distributions and the output distributions.

DisturbLabel introduced by Xie et al. [2016b] imposes the regularization within the loss layer, where it randomly replaces some of the ground truth labels as incorrect values at each training iteration. Its effect is quite similar to LSR that can help to prevent the neural network training from overfitting. The authors have verified the effectiveness of DisturbLabel via several experiments on training image classification tasks.

Recently, many works [Zhang et al., 2017, Bagherinezhad et al., 2018, Goibert and Dohmatob, 2019, Shen et al., 2019, Li et al., 2020b] explored the idea of LSR technique. Ding et al. [2019] extended an adaptive label regularization method, which enables the neural network to use both correctness and incorrectness during training. Pang et al. [2018] used the reverse cross-entropy loss to smooth the classifier’s gradients. Wang et al. [2020] proposed a graduated label smoothing method that uses the higher smoothing penalty for high-confidence predictions than that for low-confidence predictions. They found that the proposed method can improve both inference calibration and translation performance for neural machine translation models. By contrast, in this paper, we will try to understand the power of LSR from an optimization perspective and try to study how and when to use LSR.

3 Preliminaries and Notations

In this section, we first give a mathematical definition of the considered problem. We set $(x, y)$ as an instance-label pair. Let $P$ be the distribution of input instance $x \in \mathbb{R}^d$. For any $x \sim P$, its output label $y = h(x; \xi)$ follows a distribution $Q(x)$ conditional on $x$, and we denote by $g(x) = E_\xi[h(x; \xi)|x]$, where $E_\xi[\cdot]$ is the expectation that takes over a random variable $\xi$. When the randomness is obvious, we write $E[\cdot]$ for simplicity. Our goal is to learn a prediction function $f(w; x)$ that is as close as possible to $g(x)$. For the
simplicity of analysis, following by Allen-Zhu et al. [2019], we want to minimize the following optimization problem:

$$
\min_{w \in \mathbb{R}^d} \left\{ F(w) := \mathbb{E}_x \left[ \frac{1}{2} (f(w; x) - g(x))^2 \right] \right\}.
$$

(1)

The objective function $F(w)$ is not necessary convex since $f(w; x)$ is usually non-convex in terms of $w$ in many machine learning applications such as deep neural networks. To solve the problem (1), one can simply use some iterative methods such as stochastic gradient descent (SGD). Specifically, at each training iteration $t$, SGD updates solutions iteratively by

$$
w_{t+1} = w_t - \eta (f(w_t; x_t) - h(x_t; \xi_t)) \nabla f(w_t; x_t).
$$

If the objective function $F(w)$ is $L$-smooth (define shortly) and we will have the following descent result in expectation

$$
\mathbb{E}[F(w_{t+1}) - F(w_t)] \leq -\eta \mathbb{E} \left[ \| \nabla F(w_t) \|^2 \right] + \frac{\eta^2 L}{2} \mathbb{E} \left[ \| (f(w_t; x_t) - h(x_t; \xi_t)) \nabla f(w_t; x_t) \|^2 \right],
$$

where $\|z\|$ denote the Euclidean norm of a vector $z \in \mathbb{R}^d$. The variance term in the above expression can be further expanded as follows

$$
\mathbb{E} \left[ \| (f(w_t; x_t) - h(x_t; \xi_t)) \nabla f(w_t; x_t) \|^2 \right] = \mathbb{E} \left[ \| (f(w_t; x_t) - g(x_t)) \nabla f(w_t; x_t) \|^2 \| \right] + \mathbb{E} \left[ \| (h(x_t; \xi_t) - g(x_t; \xi_t)) \nabla f(w_t; x_t) \|^2 \right] := A_t + B_t.
$$

Note that when the prediction function is close to output, i.e., $f(w_t; x_t) \approx g(x_t)$, term $B_t$ will be significantly larger than $A_t$ since the variance in output label $h(x_t; \xi_t)$ is independent of the prediction model $f(w; x)$ and will become the dominant factor that slows down the convergence. As will be seen in the next section, by introduced appropriate smoothed labels, we can significantly reduce the impact of the variance in the output label.

Next, we present some notations and assumptions that will be used in the convergence analysis. We define the output variance $\sigma^2$ as

$$
\mathbb{E}_{x, \xi} \left[ (g(x) - h(x; \xi))^2 \right] = \sigma^2.
$$

(2)

Let $\hat{h}(x; \xi)$ be a noise prediction function introduced for smoothing label. The smoothed label for any instance $x_t$ is given by

$$
\tilde{h}(x_t; \xi_t) = (1 - p)h(x_t; \xi_t) + p\hat{h}(x_t; \xi_t),
$$

(3)

where $p \in (0, 1)$ is a smoothing parameter. Then the stochastic gradient $\nabla F(w_t)$ is given by

$$
\nabla F(w_t) := \left( f(w_t; x_t) - \tilde{h}(x_t; \xi_t) \right) \nabla f(w_t; x_t).
$$

(4)

Please note that $\hat{h}(x; \xi)$ is not necessary an unbiased estimator of $g(x)$, that is, $\mathbb{E}_\xi [\hat{h}(x; \xi)|x] \neq g(x)$ and $\mathbb{E}_\xi [\hat{h}(x; \xi)|x] \neq g(x)$. In the first paper of label smoothing [Szegedy et al., 2016] and the following related studies [Müller et al., 2019, Yuan et al., 2019a], researchers consider a uniform distribution over all $K$ classes of labels as the noise, i.e., set $\tilde{h}(x_t; \xi_t) = \frac{1}{K}$. In this paper, we make the following assumption on $\hat{h}(x; \xi)$, which is the key to our analysis.
Assumption 1. There exists a constant $\delta \in (0, 1)$ such that
\[
E_{x,\xi} \left[ (g(x) - \hat{h}(x; \xi))^2 \right] \leq \delta \sigma^2,
\]
where the constant $\sigma^2$ is the variance of output defined in (2).

Remark. The assumption shows that the noise label is closer to the ground truth label, comparing to the one-hot label. Although a simple selection of the noise label $\hat{h}(x; \xi)$ is the uniform distribution, our theoretical analysis shows that it can be extended to any noise label satisfying Assumption 1. Instead of a uniform distribution, for example, one can smooth labels with a teacher model [Hinton et al., 2015] or the model’s own distribution [Reed et al., 2014].

Throughout this paper, we also make the following assumptions for solving the problem (1).

Assumption 2. Assume the following conditions hold:

(i) The stochastic gradient of the objective function is unbiased, i.e.,
\[
E_x[(f(w; x) - g(x))\nabla f(w; x)] = \nabla F(w),
\]
and there exists a constant $G > 0$, such that $\|\nabla f(w, x)\| \leq G$.

(ii) $F(w)$ is smooth with an $L$-Lipchitz continuous gradient, i.e., it is differentiable and there exists a constant $L > 0$ such that
\[
\|\nabla F(w) - \nabla F(u)\| \leq L\|w - u\|, \forall w, u \in \mathbb{R}^d.
\]

Remark. Assumption 2 is standard and widely used in many existing non-convex optimization literatures [Ghadimi and Lan, 2013, Yan et al., 2018, Yuan et al., 2019b, Wang et al., 2019, Li et al., 2020a]. Assumption 2 (i) assures that the stochastic gradient of the objective function is unbiased and the gradient of $f(w; x)$ in terms of $w$ is upper bounded. Assumption 2 (ii) says the objective function is $L$-smooth, and it has an equivalent expression which is $F(w) - F(u) \leq \langle \nabla F(u), w - u \rangle + \frac{L}{2}\|w - u\|^2, \forall w, u \in \mathbb{R}^d$.

We now introduce an important assumption regarding $F(w)$, i.e. there is no very bad local optimum on the surface of objective function $F(w)$. More specifically, the following assumption holds.

Assumption 3. There exists a constant $\mu > 0$ such that
\[
2\mu F(w) \leq \|\nabla F(w)\|^2, \forall w \in \mathbb{R}^d.
\]

Remark. This property has been observed in training deep and shallow neural networks [Allen-Zhu et al., 2019, Xie et al., 2016a]. In many existing non-convex optimization studies, a similar condition is used to establish convergence, please see [Yuan et al., 2019b, Wang et al., 2019, Li et al., 2020a] and references therein.

To measure the convergence of non-convex and smooth optimization problems as in [Nesterov, 1998, Ghadimi and Lan, 2013, Yan et al., 2018], we need the following definition of the first-order stationary point.

Definition 1 (First-order stationary point). For the problem of $\min_{w \in \mathbb{R}^d} F(w)$, a point $w \in \mathbb{R}^d$ is called a first-order stationary point if $\|\nabla f(w)\| = 0$. Moreover, if $\|\nabla f(w)\| \leq \epsilon$, then the point $w$ is said to be an $\epsilon$-stationary point, where $\epsilon \in (0, 1)$ is a small positive value.

4 Convergence Analysis of Stochastic Gradient Descent with LSR

To understand LSR from the optimization perspective, we consider SGD with LSR in Algorithm 1 for the sake of simplicity. The only difference between Algorithm 1 and standard SGD is the use of the output label for constructing a stochastic gradient. The following theorem shows that Algorithm 1 converges to an approximate stationary point in expectation under some conditions. We include its proof in the Appendix.
Algorithm 1 SGD with Label Smoothing Regularization

1: Initialize: $w_0 \in \mathbb{R}^d$, $p \in (0, 1)$, set $\eta$ as the value in Theorem 4.
2: for $t = 0, 1, \ldots, T - 1$ do
3:    set $\tilde{h}(x_t) = (1 - p)h(x_t; \xi_t) + p\hat{h}(x_t; \xi_t)$
4:    update $w_{t+1} = w_t - \eta \nabla F(w_t)$, where the stochastic gradient $\nabla F(w_t)$ is defined as (4)
5: end for

Theorem 4. Under Assumptions 1, 2, 3, run Algorithm 1 with $\eta = \min (\frac{\mu}{2LG^2\sigma^2}, \frac{1}{T})$ and $p = \frac{1}{1+\delta}$, then

$$E_R[\|\nabla F(w_R)\|^2] \leq \frac{2F(w_0)}{\eta T} + 6\delta G^2\sigma^2,$$

where $R$ is uniformly sampled from $\{0, 1, \ldots, T - 1\}$. Furthermore, we have the following two results.

(1) when $\delta \leq \frac{\epsilon^2}{12G^2\sigma^2}$, if we set $T = \frac{4F(w_0)}{\eta \epsilon^2}$, then Algorithm 1 converges to an $\epsilon$-stationary point in expectation, i.e., $E_R[\|\nabla F(w_R)\|^2] \leq \epsilon^2$. The total sample complexity is $T = O\left(\frac{1}{\epsilon^4}\right)$.

(2) when $\delta > \frac{\epsilon^2}{12G^2\sigma^2}$, if we set $T = \frac{4F(w_0)}{\eta \delta G^2\sigma^2}$, then Algorithm 1 does not converge to an $\epsilon$-stationary point, but we have $E_R[\|\nabla F(w_R)\|^2] \leq 12\delta G^2\sigma^2 \leq O(\delta)$.

Remark. We observe that the variance term is $6\delta G^2\sigma^2$, instead of $\eta LC^2\sigma^2$ for standard analysis of SGD without LSR (i.e., $p = 0$, please see the detailed analysis in the Appendix). For the convergence analysis, the different between SGD with LSR and SGD without LSR is that $\tilde{h}(x; \xi)$ is not an unbiased estimator of $g(x)$ when using LSR. The convergence behavior of Algorithm 1 heavily depends on the parameter $\delta$. When $\delta$ is small enough, say $\delta \leq O(\epsilon^2)$ with a small positive value $\epsilon \in (0, 1)$, then Algorithm 1 converges to an $\epsilon$-stationary point with the total sample complexity of $O\left(\frac{1}{\epsilon^4}\right)$. Recall that the total sample complexity of standard SGD without LSR for finding an $\epsilon$-stationary point is $O\left(\frac{1}{\epsilon^4}\right)$. The convergence result shows that if we could learn a prediction function $\tilde{h}(x; \xi)$ that has a reasonably small amount of bias $\delta$, through the label smoothing trick, we will be able to reduce sample complexity for training a learning model from $O\left(\frac{1}{\epsilon^4}\right)$ to $O\left(\frac{1}{\epsilon^2}\right)$. Thus, the reduction in variance will happen when an appropriate label smoothing with $\delta \in (0, 1)$ is introduced. We may consider a simple linear model from data first, and then by using the label smoothing trick to help train a large-scale deep model. On the other hand, when the parameter $\delta$ is large such that $\delta > \Omega(\epsilon^2)$, that is to say, if an inappropriate label smoothing is used, then Algorithm 1 does not converge to an $\epsilon$-stationary point, but it converges to a worse level of $O(\delta)$.

5 TSLA: A Generic Two-Stage Label Smoothing Algorithm

Despite superior outcomes in training deep neural networks, some real applications have shown the adverse effect of LSR. Müller et al. [2019] have empirically observed that LSR impairs distillation, that is, after training teacher models with LSR, student models perform worse. The authors believed that LSR reduces mutual information between input example and output logit. Kornblith et al. [2019] have found that LSR impairs the accuracy of transfer learning when training deep neural network models on ImageNet data set. Seo et al. [2020] trained deep neural network models for few-shot learning on miniImageNet and found a significant performance drop with LSR. This motivates us to investigate a strategy that combines the algorithm with and without LSR during the training progress. Recall that the original purpose of using LSR is to avoid overfitting in training deep neural networks. In Figure 1, we plot the training loss and testing loss versus the number of epochs both for SGD with and without LSR on training ResNet-18 over CIFAR-100 dataset. Although the gap of training loss and testing loss with LSR is smaller than the gap without LSR, the testing loss with LSR is essentially worse than the testing loss without LSR. This may cause the issue of “underfitting”. Let think in another way, one possible scenario is that training one-hot label is “easier” than training smoothed label. Nevertheless, training deep neural networks is usually getting harder and harder with the increase of training epochs. It seems that training smoothed label in the late
epochs makes the learning progress more difficult. In addition, the figure shows that the testing loss with LSR is smaller than the testing loss without LSR at the beginning of training progress. One question is whether LSR helps at the early training epochs but it has less (even negative) effect during the later training epochs? This question encourages us to propose and analyze a simple strategy with LSR dropping that switches a stochastic algorithm with LSR to the algorithm without LSR.

5.1 The TSLA Algorithm

In this subsection, we propose a generic framework that consists of two stages, wherein the first stage it runs a stochastic algorithm $\mathcal{A}$ (e.g., SGD) with LSR in $T_1$ iterations and the second stage it runs the same algorithm without LSR up to $T_2$ iterations. This framework is referred to as Two-Stage LAbel smoothing (TSLA) algorithm, whose updating details are presented in Algorithm 2. The notation $\mathcal{A}$-step($\cdot$, $\eta$) is one update step of a stochastic algorithm $\mathcal{A}$ with learning rate $\eta$. For example, if we select SGD as algorithm $\mathcal{A}$, then

\[
\text{SGD-step}(w_t; \tilde{h}(x_t; \xi_t), \eta_1) = w_t - \eta_1 \left( f(w_t; x_t) - \tilde{h}(x_t; \xi_t) \right) \nabla f(w_t; x_t), \tag{5}
\]

\[
\text{SGD-step}(w_t; h(x_t; \xi_t), \eta_2) = w_t - \eta_2 \left( f(w_t; x_t) - h(x_t; \xi_t) \right) \nabla f(w_t; x_t). \tag{6}
\]

The proposed TSLA is a generic strategy where the subroutine algorithm $\mathcal{A}$ can be replaced by any stochastic algorithms such as momentum SGD [Polyak, 1964], Stochastic Nesterov’s Accelerated Gradient [Nesterov, 2013].
of 120 breeds of dogs, where 100 images from each breed is used for training. CUB-2011 data set with 11,788 images of 200 birds species. The ResNet-18 model [He et al., 2016] is applied as the backbone in the experiments. We compare the proposed TSLA incorporating with SGD (TSLA) with two

to LSR method, i.e., a standard stochastic algorithm $A$, reduces to the baseline, i.e., a standard stochastic algorithm $A$ without LSR; while if $T_2 = 0$, TSLA becomes to LSR method, i.e., a standard stochastic algorithm $A$ with LSR.

5.2 Convergence Result of TSLA

In this subsection, we will give the convergence result of the proposed TSLA algorithm. For simplicity, we use SGD as the subroutine algorithm $A$ in the analysis. The convergence result in the following theorem shows the power of LSR from the optimization perspective. Its proof is presented in Appendix. It is easy to see from the proof that by using the last output of the first stage as the initial point of the second stage, TSLA can enjoy the advantage of LSR in the second stage with an improved convergence.

Theorem 5. Under Assumptions 1, 2, 3, suppose $6\sigma^2G^2\delta/\mu \leq F(w_0)$, run Algorithm 2 with $A = \text{SGD}$, $p = \frac{1}{1+\delta}$, $\eta_1 = \min \left(\frac{1}{2}, \frac{\mu}{4\beta L\mu}\right)$, $T_1 = 2\log \left(\frac{\mu F(w_0)}{(1+2\eta_1L\mu)\sigma^2}\right)/(\eta_1\mu)$, $\eta_2 = \min \left(\frac{\mu}{Lao\sigma^2}, \frac{\epsilon^2}{4Lao\sigma^2}\right)$ and $T_2 = \frac{48SG^2\sigma^2}{\Psi_2\epsilon^2}$, then $E_R[\|\nabla F(w_R)\|^2] \leq \epsilon^2$, where $R$ is uniformly sampled from $\{0, \ldots, T_2 - 1\}$.

Remark. It is obvious that the learning rate $\eta_2$ in the second stage is roughly smaller than the learning rate $\eta_1$ in the first stage, which matches the widely used learning rate decay scheme in training neural networks. To explore the total sample complexity of TSLA, we consider different conditions on $\delta$. We summarize the total sample complexities of finding $\epsilon$-stationary points for SGD with TSLA (TSLA), SGD with LSR (LSR), and SGD without LSR (baseline) in Table 1, where $\epsilon \in (0, 1)$ is the target convergence level, and we only present the orders of the complexities but ignore all constants. When $\Omega(\epsilon^2) < \delta < 1$, LSR does not converge to an $\epsilon$-stationary point (denoted by $\infty$), while TSLA reduces sample complexity from $O \left(\frac{1}{\epsilon^2}\right)$ to $O \left(\frac{1}{\epsilon}\right)$, compared to the baseline. When $\delta < \Omega(\epsilon^2)$, the total complexity of TSLA is between $\log(1/\epsilon)$ and $1/\epsilon^2$, which is always better than LSR and the baseline. In summary, TSLA achieves the best total sample complexity by enjoying the good property of an appropriate label smoothing.

6 Experiments

To further evaluate the performance of the proposed TSLA method, we trained deep neural networks on three benchmark data sets, CIFAR-100 [Krizhevsky and Hinton, 2009], Stanford Dogs [Khosla et al., 2011] and CUB-2011 [Wah et al., 2011], for image classification tasks. CIFAR-100 \(^1\) has 50,000 training images and 10,000 testing images of 32x32 resolution with 100 classes. Stanford Dogs data set \(^2\) contains 20,580 images of 120 breeds of dogs, where 100 images from each breed is used for training. CUB-2011 \(^3\) is a birds image data set with 11,788 images of 200 birds species. The ResNet-18 model [He et al., 2016] is applied as the backbone in the experiments. We compare the proposed TSLA incorporating with SGD (TSLA) with two

\(^1\)https://www.cs.toronto.edu/~kriz/cifar.html
\(^2\)http://vision.stanford.edu/aditya86/ImageNetDogs/
\(^3\)http://www.vision.caltech.edu/visipedia/CUB-200.html
Table 2: Comparisons of Testing Accuracy for Different Methods (mean ± standard deviation, in %).

| Algorithm* | Stanford Dogs | CUB-2011 |
|------------|---------------|----------|
|            | Top-1 accuracy | Top-5 accuracy | Top-1 accuracy | Top-5 accuracy |
| baseline   | 82.31 ± 0.18  | 97.76 ± 0.06 | 75.31 ± 0.25  | 93.14 ± 0.31  |
| LSR        | 82.80 ± 0.07  | 97.41 ± 0.09 | 76.97 ± 0.19  | 92.73 ± 0.12  |
| TSLA(20)   | 83.15 ± 0.02  | 97.91 ± 0.08 | 76.62 ± 0.15  | 93.60 ± 0.18  |
| TSLA(30)   | 83.89 ± 0.16  | 98.05 ± 0.08 | 77.44 ± 0.19  | 93.92 ± 0.16  |
| TSLA(40)   | **83.93 ± 0.13** | 98.03 ± 0.05 | 77.50 ± 0.20  | 93.99 ± 0.11  |
| TSLA(50)   | 83.91 ± 0.15  | **98.07 ± 0.06** | **77.57 ± 0.21** | **93.86 ± 0.14** |
| TSLA(60)   | 83.51 ± 0.11  | 97.99 ± 0.06  | 77.25 ± 0.29  | **94.43 ± 0.18** |
| TSLA(70)   | 83.38 ± 0.09  | 97.90 ± 0.09  | 77.21 ± 0.15  | 93.31 ± 0.12  |
| TSLA(80)   | 83.14 ± 0.09  | 97.73 ± 0.07  | 77.05 ± 0.14  | 93.05 ± 0.08  |

*TSLA(s): TSLA drops off LSR after epoch s.

baselines, SGD with LSR (LSR) and SGD without LSR (baseline). The mini-batch size of training instances for all methods is 256 as suggested by He et al. [2019] and He et al. [2016]. The momentum parameter is fixed as 0.9.

6.1 Stanford Dogs and CUB-2011

We separately train ResNet-18 [He et al., 2016] up to 90 epochs over two data sets Stanford Dogs and CUB-2011. We use weight decay with the parameter value of $10^{-4}$. For all algorithms, the initial learning rates for FC are set to be 0.1, while that for the pre-trained backbones are 0.001 and 0.01 for Standford Dogs and CUB-2011, respectively. The learning rates are divided by 10 every 30 epochs. For LSR, we fix the value of smoothing strength $p = 0.4$ for the best performance, and the noise prediction function used for label smoothing is set to be a uniform distribution over all $K$ classes, i.e., $\hat{h}(x; \xi) = \frac{1}{K}$. The same values of the smoothing strength and the same noise prediction function are used during the first stage of TSLA. For TSLA, we drop off the LSR (i.e., let $p = 0$) after $s$ epochs during the training process, where $s \in \{20, 30, 40, 50, 60, 70, 80\}$. We first report the highest top-1 and top-5 accuracy on the testing data sets for different methods. All top-1 and top-5 accuracy are averaged over 5 independent random trails with their standard deviations. The results of the comparison are summarized in Table 2, where the notation “TSLA(s)” means that the TSLA algorithm drops off LSR after epoch $s$. It can be seen from Table 2 that under an appropriate hyperparameter setting the models trained using TSLA outperform that trained using LSR and baseline, which supports the convergence result in Section 5. We notice that the best top-1 accuracy of TSLA are TSLA(40) and TSLA(50) for Stanford Dogs and CUB-2011, respectively, meaning that the performance of TSLA(s) is not monotonic over the dropping epoch $s$. For CUB-2011, the top-1 accuracy of TSLA(20) is smaller than that of LSR. This result matches the convergence analysis of TSLA showing that it can not drop off LSR too early. For top-5 accuracy, we found that TSLA(80) is slightly worse than baseline. This is because of dropping LSR too late so that the update iterations (i.e., $T_2$) in the second stage of TSLA is too small to converge to a good solution. We also observe that LSR is better than baseline regarding top-1 accuracy but the result is opposite as to top-5 accuracy. We then plot the averaged top-1 accuracy, averaged top-5 accuracy, and averaged loss among 5 trails of different methods in Figure 2. We remove the results for TSLA(20) since it dropped off LSR too early as mentioned before. The figure shows TSLA improves the top-1 and top-5 testing accuracy immediately once it drops off LSR. Although TSLA may not converges if it drops off LSR too late, see TSLA(60), TSLA(70), and TSLA(80) from the third column of Figure 2, it still has the best performance compared to LSR and baseline. TSLA(30), TSLA(40), and TSLA(50) can converge to lower objective levels, comparing to LSR and baseline.
TSLA (30)

6.2 CIFAR-100

The total epochs of training ResNet-18 [He et al., 2016] on CIFAR-100 is set to be 200. The weight decay with the parameter value of $5 \times 10^{-4}$ is used. We use 0.1 as the initial learning rates for all algorithms and divide them by 10 every 60 epochs suggested in [He et al., 2016, Zagoruyko and Komodakis, 2016]. For LSR and the first stage of TSLA, the value of smoothing strength is fixed as $p = 0.1$, which shows the best performance for LSR. We use two different noise prediction functions to smooth the one-hot label, the uniform distribution over all labels and the distribution predicted by an ImageNet pre-trained model which downloaded directly from PyTorch [Paszke et al., 2019]. For TSLA, we try to drop off the LSR after s epochs during the training process, where $s \in \{120, 140, 160, 180\}$. All top-1 and top-5 accuracy on the testing data set are averaged over 5 independent random trails with their standard deviations. We summarize the results in Table 3, where LSR-pre and TSLA-pre indicate that LSR and TSLA use the noise prediction function

Figure 3: Testing Top-1, Top-5 Accuracy and Loss on ResNet-18 over CIFAR-100. TSLA(s)/TSLA-pre(s) means TSLA/TSLA-pre drops off LSR/LSR-pre after epoch s.
Table 3: Comparison of Testing Accuracy for Different Methods (mean ± standard deviation, in %).

| Algorithm | CIFAR-100 |
|-----------|-----------|
|           | Top-1 accuracy | Top-5 accuracy |
| baseline  | 76.87 ± 0.04   | 93.47 ± 0.15   |
| LSR       | 77.77 ± 0.18   | 93.55 ± 0.11   |
| TSLA(120) | 77.92 ± 0.21   | 94.13 ± 0.23   |
| TSLA(140) | 77.93 ± 0.19   | 94.11 ± 0.22   |
| TSLA(160) | 77.96 ± 0.20   | 94.19 ± 0.21   |
| TSLA(180) | 78.04 ± 0.27   | 94.23 ± 0.15   |
| LSR-pre   | 78.07 ± 0.31   | 94.70 ± 0.14   |
| TSLA-pre(120) | 78.34 ± 0.31 | 94.68 ± 0.14   |
| TSLA-pre(140) | 78.39 ± 0.25 | 94.73 ± 0.11   |
| TSLA-pre(160) | 78.55 ± 0.28 | 94.83 ± 0.08   |
| TSLA-pre(180) | 78.53 ± 0.23 | **94.96 ± 0.23** |

*TSLA(s)/TSLA-pre(s): TSLA/TSLA-pre drops off LSR/LSR-pre after epoch s.*

by the ImageNet pre-trained model. The results show that LSR-pre/TSLA-pre has a better performance than LSR/TSLA. The reason might be that the pre-trained model-based prediction is closer to the ground truth than the uniform prediction and it has lower variance (smaller $\delta$). Then, TSLA (LSR) with such pre-trained model-based prediction converges faster than TSLA (LSR) with uniform prediction, which verifies our theoretical findings in Sections 5 (Section 4). This observation also empirically tells us the selection of the prediction function $\hat{h}(x; \xi)$ used for smoothing label is the key to the success of TSLA as well as LSR. Among all methods, the performance of TSLA-pre is the best. For top-1 accuracy, TSLA-pre(160) outperforms all other algorithms, while for top-5 accuracy, TSLA-pre(180) has the best performance. Finally, we observe from Figure 3 that both TSLA and TSLA-pre converge, while TSLA-pre converges to the lowest objective value. Similarly, the results of top-1 and op-5 accuracy show the improvements of TSLA and TSLA-pre at the point of dropping off LSR.

7 Conclusions

In this paper, we have studied the power of LSR in training deep neural networks by analyzing SGD with LSR in different non-convex optimization settings. The convergence results show that an appropriate LSR with reduced label variance can help speed up the convergence. We have proposed a simple and efficient strategy so-called TSLA that can incorporate many existing stochastic algorithms. The basic idea of TSLA is to switch the training from smoothed label to one-hot label. Integrating TSLA with SGD, we observe from its improved convergence result that TSLA benefits by LSR in the first stage and essentially converges faster in the second stage. Throughout extensive experiments, we have shown that TSLA improves the generalization accuracy of ResNet-18 models on benchmark data sets.

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References

Zeyuan Allen-Zhu, Yuanzhi Li, and Zhao Song. A convergence theory for deep learning via over-parameterization. In *International Conference on Machine Learning*, pages 242–252, 2019.
Hessam Bagherinezhad, Maxwell Horton, Mohammad Rastegari, and Ali Farhadi. Label refinery: Improving imagenet classification through label progression. *arXiv preprint arXiv:1805.02641*, 2018.

Jan Chorowski and Navdeep Jaitly. Towards better decoding and language model integration in sequence to sequence models. *Proc. Interspeech 2017*, pages 523–527, 2017.

Qianggang Ding, Sifan Wu, Hao Sun, Jiadong Guo, and Shu-Tao Xia. Adaptive regularization of labels. *arXiv preprint arXiv:1908.05474*, 2019.

Timothy Dozat. Incorporating nesterov momentum into adam. 2016.

John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. *Journal of Machine Learning Research*, 12:2121–2159, 2011.

Saeed Ghadimi and Guanghui Lan. Stochastic first-and zeroth-order methods for nonconvex stochastic programming. *SIAM Journal on Optimization*, 23(4):2341–2368, 2013.

Morgane Goibert and Elvis Dohmatob. Adversarial robustness via adversarial label-smoothing. *arXiv preprint arXiv:1906.11567*, 2019.

Kaiming He, Xiangyu Zhang, Shaoqing Ren, and Jian Sun. Deep residual learning for image recognition. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pages 770–778, 2016.

Tong He, Zhi Zhang, Hang Zhang, Zhongyue Zhang, Junyuan Xie, and Mu Li. Bag of tricks for image classification with convolutional neural networks. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 558–567, 2019.

Geoffrey Hinton, Nitish Srivastava, and Kevin Swersky. Neural networks for machine learning lecture 6a overview of mini-batch gradient descent. 2012a.

Geoffrey Hinton, Oriol Vinyals, and Jeff Dean. Distilling the knowledge in a neural network. *arXiv preprint arXiv:1503.02531*, 2015.

Geoffrey E Hinton, Nitish Srivastava, Alex Krizhevsky, Ilya Sutskever, and Ruslan R Salakhutdinov. Improving neural networks by preventing co-adaptation of feature detectors. *arXiv preprint arXiv:1207.0580*, 2012b.

Sergey Ioffe and Christian Szegedy. Batch normalization: Accelerating deep network training by reducing internal covariate shift. *arXiv preprint arXiv:1502.03167*, 2015.

Aditya Khosla, Nityananda Jayadevaprakash, Bangpeng Yao, and Fei-Fei Li. Novel dataset for fine-grained image categorization: Stanford dogs. In *Proc. CVPR Workshop on Fine-Grained Visual Categorization (FGVC)*, volume 2, 2011.

Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2014.

Simon Kornblith, Jonathon Shlens, and Quoc V Le. Do better imagenet models transfer better? In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pages 2661–2671, 2019.

Alex Krizhevsky and Geoffrey Hinton. Learning multiple layers of features from tiny images. *Technical report, University of Toronto*, 2009.

Xiaoyu Li, Zhenxun Zhuang, and Francesco Orabona. Exponential step sizes for non-convex optimization. *arXiv preprint arXiv:2002.05273*, 2020a.

Xingjian Li, Haoyi Xiong, Haozhe An, Dejing Dou, and Chengzhong Xu. Colam: Co-learning of deep neural networks and soft labels via alternating minimization. *arXiv preprint arXiv:2004.12443*, 2020b.
Rafael Müller, Simon Kornblith, and Geoffrey E Hinton. When does label smoothing help? In Advances in Neural Information Processing Systems, pages 4696–4705, 2019.

Yurii Nesterov. A method of solving a convex programming problem with convergence rate $O(1/k^2)$. Soviet Mathematics Doklady, 27:372–376, 1983.

Yurii Nesterov. Introductory lectures on convex programming volume i: Basic course. 1998.

Toan Q Nguyen and Julian Salazar. Transformers without tears: Improving the normalization of self-attention. arXiv preprint arXiv:1910.05895, 2019.

Tianyu Pang, Chao Du, Yinpeng Dong, and Jun Zhu. Towards robust detection of adversarial examples. In Advances in Neural Information Processing Systems, pages 4579–4589, 2018.

Adam Paszke, Sam Gross, Francisco Massa, Adam Lerer, James Bradbury, Gregory Chanan, Trevor Killeen, Zeming Lin, Natalia Gimelshein, Luca Antiga, et al. Pytorch: An imperative style, high-performance deep learning library. In Advances in Neural Information Processing Systems, pages 8024–8035, 2019. URL https://pytorch.org/docs/stable/torchvision/models.html.

Gabriel Pereyra, George Tucker, Jan Chorowski, Lukasz Kaiser, and Geoffrey Hinton. Regularizing neural networks by penalizing confident output distributions. arXiv preprint arXiv:1701.06548, 2017.

Boris T Polyak. Some methods of speeding up the convergence of iteration methods. USSR Computational Mathematics and Mathematical Physics, 4(5):1–17, 1964.

Sashank J Reddi, Satyen Kale, and Sanjiv Kumar. On the convergence of adam and beyond. arXiv preprint arXiv:1904.09237, 2019.

Paul Seo, Hong-Gyu Jung, and Seong-Whan Lee. Self-augmentation: Generalizing deep networks to unseen classes for few-shot learning. arXiv preprint arXiv:2004.00251, 2020.

Chaomin Shen, Yaxin Peng, Guixu Zhang, and Jinsong Fan. Defending against adversarial attacks by suppressing the largest eigenvalue of fisher information matrix. arXiv preprint arXiv:1909.06137, 2019.

Patrice Y Simard, Yann A LeCun, John S Denker, and Bernard Victorri. Transformation invariance in pattern recognitiontangent distance and tangent propagation. In Neural networks: tricks of the trade, pages 239–274. Springer, 1998.

Christian Szegedy, Vincent Vanhoucke, Sergey Ioffe, Jon Shlens, and Zbigniew Wojna. Rethinking the inception architecture for computer vision. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pages 2818–2826, 2016.

Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones, Aidan N Gomez, Lukasz Kaiser, and Illia Polosukhin. Attention is all you need. In Advances in Neural Information Processing Systems, pages 5998–6008, 2017.

C. Wah, S. Branson, P. Welinder, P. Perona, and S. Belongie. The Caltech-UCSD Birds-200-2011 Dataset. Technical report, 2011.

Shuo Wang, Zhaopeng Tu, Shuming Shi, and Yang Liu. On the inference calibration of neural machine translation. arXiv preprint arXiv:2005.00963, 2020.
Zhe Wang, Kaiyi Ji, Yi Zhou, Yingbin Liang, and Vahid Tarokh. Spiderboost and momentum: Faster variance reduction algorithms. In *Advances in Neural Information Processing Systems*, pages 2403–2413, 2019.

Bo Xie, Yingyu Liang, and Le Song. Diverse neural network learns true target functions. *arXiv preprint arXiv:1611.03131*, 2016a.

Lingxi Xie, Jingdong Wang, Zhen Wei, Meng Wang, and Qi Tian. Disturblabel: Regularizing cnn on the loss layer. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 4753–4762, 2016b.

Yan Yan, Tianbao Yang, Zhe Li, Qihang Lin, and Yi Yang. A unified analysis of stochastic momentum methods for deep learning. In *International Joint Conference on Artificial Intelligence (IJCAI)*, pages 2955–2961, 2018.

Li Yuan, Francis EH Tay, Guilin Li, Tao Wang, and Jiashi Feng. Revisit knowledge distillation: a teacher-free framework. *arXiv preprint arXiv:1909.11723*, 2019a.

Zhuoning Yuan, Yan Yan, Rong Jin, and Tianbao Yang. Stagewise training accelerates convergence of testing error over sgd. In *Advances in Neural Information Processing Systems*, pages 2604–2614, 2019b.

Sergey Zagoruyko and Nikos Komodakis. Wide residual networks. *arXiv preprint arXiv:1605.07146*, 2016.

Matthew D Zeiler. Adadelta: an adaptive learning rate method. *arXiv preprint arXiv:1212.5701*, 2012.

Albert Zeyer, Kazuki Irie, Ralf Schluter, and Hermann Ney. Improved training of end-to-end attention models for speech recognition. *Proc. Interspeech 2018*, pages 7–11, 2018.

Hongyi Zhang, Moustapha Cisse, Yann N Dauphin, and David Lopez-Paz. mixup: Beyond empirical risk minimization. *arXiv preprint arXiv:1710.09412*, 2017.

Barret Zoph, Vijay Vasudevan, Jonathon Shlens, and Quoc V Le. Learning transferable architectures for scalable image recognition. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 8697–8710, 2018.

### A Technical Lemma

**Lemma 1.** Under Assumption 1 and Assumption 2 (i), we have

$$E \left[ \left\| \left( \tilde{h}(x_t; \xi_t) - g(x_t) \right) \nabla f(w_t; x_t) \right\|^2 \right] \leq (1 - p) G^2 \sigma^2 + p G^2 \delta \sigma^2.$$

**Proof.** By the fact of $\tilde{h}(x_t; \xi_t) = (1 - p) h(x_t; \xi_t) + p \hat{h}(x_t; \xi_t)$, we have

$$E \left[ \left\| (1 - p) (h(x_t; \xi_t) - g(x_t)) + p (\hat{h}(x_t; \xi_t) - g(x_t)) \right\| \nabla f(w_t; x_t) \right\|^2 \right]$$

$$(a) \leq (1 - p) E \left[ \left\| (h(x_t; \xi_t) - g(x_t)) \nabla f(w_t; x_t) \right\|^2 \right] + p E \left[ \left\| (\hat{h}(x_t; \xi_t) - g(x_t)) \nabla f(w_t; x_t) \right\|^2 \right]$$

$$(b) \leq (1 - p) G^2 \sigma^2 + p G^2 \delta \sigma^2,$$

where (a) uses the convexity of norm, i.e., $\|(1 - p) X + p Y\|^2 \leq (1 - p) \|X\|^2 + p \|Y\|^2$; (b) uses the fact that of $E_{x,\xi} \left[ (g(x) - h(x; \xi))^2 \right] = \sigma^2$, Assumption 1, and Assumption 2 (i).
Proof of Theorem 4

Proof. By the smoothness of objective function $F(w)$ we have

$$
E[F(w_{t+1}) - F(w_t)] 
\leq E[(\nabla F(w_t), w_{t+1} - w_t)] + \frac{L}{2} E[||w_{t+1} - w_t||^2] 
$$

(a) $= -\eta E[(\nabla F(w_t), (f(w_t; x_t) - g(x_t)) \nabla f(w_t; x_t))]$

(b) $= -\eta E[(\nabla F(w_t), (g(x_t) - \bar{h}(x_t; \xi_t)) \nabla f(w_t; x_t))]$

$$
+ \frac{\eta^2 L}{2} E[||f(w_t; x_t) - \bar{h}(x_t; \xi_t)||^2] 
$$

(c) $\leq -\eta E[||\nabla F(w_t)||^2] + \frac{\eta^2 L}{2} E[||\bar{h}(x_t; \xi_t) - g(x_t)||^2]$

$$
+ \frac{\eta^2 L}{2} E[||f(w_t; x_t) - \bar{h}(x_t; \xi_t)||^2] 
$$

(d) $\leq -\frac{\eta^2}{2} E[||\nabla F(w_t)||^2] + \frac{\eta + 2\eta^2 L}{2} E[||\bar{h}(x_t; \xi_t) - g(x_t)||^2]$

$$
+ \frac{\eta^2 L}{2} E[||(f(w_t; x_t) - g(x_t)) \nabla f(w_t; x_t)||^2] 
$$

(e) $\leq -\frac{\eta^2}{2} E[||\nabla F(w_t)||^2] + \frac{\eta + 2\eta^2 L}{2} E[||\bar{h}(x_t; \xi_t) - g(x_t)||^2]$

$$
+ \frac{\eta^2 L}{2} E[||(f(w_t; x_t) - g(x_t)) \nabla f(w_t; x_t)||^2] + 2\eta^2 LG^2 E[F(w_t)] 
$$

(f) $\leq -\frac{\eta^2}{2} E[||\nabla F(w_t)||^2] + \frac{\eta + 2\eta^2 L}{2} E[||\bar{h}(x_t; \xi_t) - g(x_t)||^2]$

$$
+ \frac{\eta^2 L}{2} E[||(f(w_t; x_t) - g(x_t)) \nabla f(w_t; x_t)||^2] + 2\eta^2 LG^2 E[F(w_t)]. 
$$

where (a) is due to the update of $w_{t+1} = w_t - \eta \left(f(w_t; x_t) - \bar{h}(x_t; \xi_t)\right) \nabla f(w_t; x_t)$; (b) is due to Assumption 2 (i); (c) and (d) are due to the Young’s inequality; (e) uses Assumption 2 (i); (f) is due to Lemma 1.

Since $\eta \leq \frac{\mu}{2L \sigma^2}$, using the condition in Assumption 3 we can simplify the inequality from (7) as

$$
E[F(w_{t+1}) - F(w_t)] 
\leq -\frac{\eta^2}{2} E[||\nabla F(w_t)||^2] + \frac{\eta + 2\eta^2 L}{2} E[||\bar{h}(x_t; \xi_t) - g(x_t)||^2]$

$$
+ \frac{\eta^2 L}{2} E[||(f(w_t; x_t) - g(x_t)) \nabla f(w_t; x_t)||^2] + 2\eta^2 LG^2 E[F(w_t)] 
$$

which implies

$$
\frac{1}{T} \sum_{t=0}^{T-1} E[||\nabla F(w_t)||^2] \leq \frac{2F(w_0)}{\eta T} + (1 + 2\eta L) \left((1 - p)G^2 \sigma^2 + pG^2 \delta \sigma^2\right)$

$$
= \frac{2F(w_0)}{\eta T} + (1 + 2\eta L) \frac{2\delta}{1 + \delta} G^2 \sigma^2$

$$
\leq \frac{2F(w_0)}{\eta T} + 6\delta G^2 \sigma^2. 
$$
C Convergence Analysis of SGD without LSR ($p = 0$)

**Theorem 6.** Under Assumptions 1, 2, 3, the solutions $w_t$ from Algorithm 1 with $p = 0$ satisfy

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \nabla F(w_t) \|^2 \right] \leq \frac{2F(w_0)}{\eta T} + \eta LG^2 \sigma^2.
$$

In order to have $\mathbb{E}_R[\| \nabla F(w_R) \|^2] \leq c^2$, it suffices to set $\eta = \min \left( \frac{\mu}{LG^2}, \frac{c^2}{2LG^2\sigma^2} \right)$ and $T = \frac{4F(w_0)}{\eta^2}$. The total complexity is $O \left( \frac{1}{\eta^2} \right)$.

**Proof.** By the smoothness of objective function $F(w)$ we have

$$
\mathbb{E} [F(w_{t+1}) - F(w_t)]
\leq \mathbb{E} [\langle \nabla F(w_t), (w_{t+1} - w_t) \rangle] + \frac{L}{2} \mathbb{E} \left[ \| w_{t+1} - w_t \|^2 \right]
\leq (a) - \eta \mathbb{E} [\langle \nabla F(w_t), (f(w_t; x_t; \xi) - h(x_t; \xi)) \nabla f(w_t; x_t) \rangle]
\quad + \frac{\eta^2 L}{2} \mathbb{E} \left[ \| (f(w_t; x_t; \xi) - h(x_t; \xi)) \nabla f(w_t; x_t) \|^2 \right]
\leq (b) - \eta \mathbb{E} \left[ \| \nabla F(w_t) \|^2 \right] + \frac{\eta^2 L}{2} \mathbb{E} \left[ \| (f(w_t; x_t; \xi) - h(x_t; \xi)) \nabla f(w_t; x_t) \|^2 \right]
\quad + \frac{\eta^2 L}{2} \mathbb{E} \left[ \| (f(w_t; x_t) - g(x_t)) \nabla f(w_t; x_t) \|^2 \right]
\leq (d) - \eta \mathbb{E} \left[ \| \nabla F(w_t) \|^2 \right] + \frac{\eta^2 L}{2} G^2 \sigma^2 + \eta^2 LG^2 \mathbb{E}[F(w_t)].
$$

where (a) is due to the update of $w_{t+1} = w_t - \eta (f(w_t; x_t) - h(x_t; \xi)) \nabla f(w_t; x_t)$; (b) uses the fact that $\nabla F(w_t) = \mathbb{E} [(f(w_t; x_t) - g(x_t)) \nabla f(w_t; x_t)]$ from Assumption 2 (i); (c) is due to $\mathbb{E}_x [h(x; \xi)|x] = g(x)$; (d) uses the fact that of $\mathbb{E}_{x, \xi} \left[ (g(x) - h(x; \xi))^2 \right] = \sigma^2$ and Assumption 2 (i).

Since $\eta \leq \frac{\mu}{2LG^2}$, using the condition in Assumption 3 we can simplify the inequality from (8) as

$$
\mathbb{E} [F(w_{t+1}) - F(w_t)]
\leq - \eta \left( 1 - \frac{\eta LG^2}{2\mu} \right) \mathbb{E} \left[ \| \nabla F(w_t) \|^2 \right] + \frac{\eta^2 L}{2} G^2 \sigma^2
\leq - \frac{\eta}{2} \mathbb{E} \left[ \| \nabla F(w_t) \|^2 \right] + \frac{\eta^2 L}{2} G^2 \sigma^2.
$$

The inequality (9) implies

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \nabla F(w_t) \|^2 \right] \leq \frac{2F(w_0)}{\eta T} + \eta LG^2 \sigma^2.
$$

Since $\eta \leq \frac{c^2}{2LG^2 \sigma^2}$ and $T = \frac{4F(w_0)}{\eta^2}$, we have $\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \nabla F(w_t) \|^2 \right] \leq c^2$.

\[ \square \]
D Proof of Theorem 5

Proof. Following the inequality (7) from the proof of Theorem 4, we have

\[
\mathbb{E}[F(w_{t+1}) - F(w_t)] \leq -\frac{\eta_t}{2} \mathbb{E}[\|
abla F(w_t)\|^2] + \frac{\eta_t + 2\eta_t^2 L}{2} ((1 - p)G^2 \sigma^2 + pG^2 \delta \sigma^2) + 2\eta_t^2 L G^2 \mathbb{E}[F(w_t)].
\] (10)

Since \( \eta_t \leq \frac{\mu}{4L_G^2} \), using the condition in Assumption 3 we can simplify the inequality from (10) as

\[
\mathbb{E}[F(w_{t+1})] \leq (1 + 2\eta_t^2 L G^2 - \eta_t \mu) \mathbb{E}[F(w_t)] + \frac{\eta_t + 2\eta_t^2 L}{2} ((1 - p)G^2 \sigma^2 + pG^2 \delta \sigma^2) \leq (1 - \eta_t \mu/2) \mathbb{E}[F(w_t)] + \frac{\eta_t + 2\eta_t^2 L}{2} ((1 - p)G^2 \sigma^2 + pG^2 \delta \sigma^2) \sum_{i=0}^{t} (1 - \eta_t \mu/2)^i.
\]

Since \( \eta_t \leq \frac{1}{1 + 2\eta_t} < \frac{1}{\mu} \), then \( (1 - \eta_t \mu/2)^{t+1} < \exp(-\eta_t \mu(t + 1)/2) \) and \( \sum_{i=0}^{t} (1 - \eta_t \mu/2)^i \leq \frac{2}{\eta_t \mu} \). As a result, for any \( T_1 \), we have

\[
\mathbb{E}[F(w_{T_1})] \leq \exp(-\eta_t \mu T_1/2) F(w_0) + \frac{1 + 2\eta_t L}{\mu} ((1 - p)G^2 \sigma^2 + pG^2 \delta \sigma^2). \] (11)

Let \( p = \frac{1}{1 + \sigma} \) and \( \delta^2 := (1 - p)\sigma^2 + p\delta \sigma^2 = \frac{2\sigma}{1 + \sigma} \delta^2 \) then \( \frac{1+2\eta_t L}{\mu} ((1 - p)G^2 \sigma^2 + pG^2 \delta \sigma^2) \leq F(w_0) \) since \( \delta \) is small enough and \( \eta_t L \leq 1 \). By setting

\[
T_1 = 2 \log \left( \frac{\mu F(w_0)}{(1 + 2\eta_t L)G^2 \sigma^2} \right) / (\eta_t \mu)
\]

we have

\[
\mathbb{E}[F(w_{T_1})] \leq \frac{2(1 + 2\eta_t L)\delta^2}{\mu} \leq \frac{12G^2 \sigma^2}{\mu}. \] (12)

After \( T_1 \) iterations, we drop off the label smoothing, i.e. \( p = 0 \), then we know for any \( t \geq T_1 \), following the inequality (9) from the proof of Theorem 6, we have

\[
\mathbb{E}[F(w_{t+1}) - F(w_t)] \leq -\frac{\eta_t}{2} \mathbb{E}[\|
abla F(w_t)\|^2] + \frac{\eta_t^2 L G^2 \sigma^2}{2},
\]

where uses the condition in Assumption 3 and the condition of \( \eta_2 \leq \frac{\mu}{12L_G^2} \). Therefore, we get

\[
\frac{1}{T_2} \sum_{t=T_1}^{T_1 + T_2 - 1} \mathbb{E}[\|
abla F(w_t)\|^2] \leq \frac{2}{\eta_2 T_2} \mathbb{E}[F(w_{T_1})] + \eta_2 L G^2 \sigma^2 \leq \frac{24\delta G^2 \sigma^2}{\mu \eta_2 T_2} + \eta_2 L G^2 \sigma^2. \] (13)

By setting \( \eta_2 \leq \frac{\epsilon^2}{2L_G^2 \sigma^2} \) and \( T_2 = \frac{48\delta G^2 \sigma^2}{\mu \eta_2 \epsilon^2} \), we have \( \frac{1}{T_2} \sum_{t=T_1}^{T_1 + T_2 - 1} \mathbb{E}[\|
abla F(w_t)\|^2] \leq \epsilon^2. \) \( \square \)