ON BLOW-UP SOLUTIONS TO THE NONLINEAR SCHröDINGER EQUATION IN THE EXTERIOR OF A CONVEX OBSTACLE

OUSSAMA LANDOULSI

Abstract. In this paper, we consider the Schrödinger equation with a mass-supercritical focusing nonlinearity, in the exterior of a smooth, compact, convex obstacle of $\mathbb{R}^d$ with Dirichlet boundary conditions. We prove that solutions with negative energy blow up in finite time. Assuming furthermore that the nonlinearity is energy-subcritical, we also prove (under additional symmetry conditions) blow-up with the same optimal ground-state criterion than in the work of Holmer and Roudenko on $\mathbb{R}^d$. The classical proof of Glassey, based on the concavity of the variance, fails in the exterior of an obstacle because of the appearance of boundary terms with an unfavorable sign in the second derivative of the variance. The main idea of our proof is to introduce a new modified variance which is bounded from below and strictly concave for the solutions that we consider.

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1. Introduction

We consider the focusing nonlinear Schrödinger equation with Dirichlet boundary conditions.

\[
\begin{aligned}
\left\{ \begin{array}{ll}
i \partial_t u + \Delta \Omega u = -|u|^{p-1} u & \quad (t, x) \in \mathbb{R} \times \Omega, \\
u(t_0, x) = u_0(x) & \quad x \in \Omega, \\
u(t, x) = 0 & \quad (t, x) \in \mathbb{R} \times \partial \Omega,
\end{array} \right.
\end{aligned}
\]

where \( \Omega = \mathbb{R}^d \setminus \Theta \) is the exterior of a smooth, compact and convex obstacle on \( \mathbb{R}^d \), \( \Delta \Omega \) is the Dirichlet Laplace operator, \( \partial_t \) is the derivative with respect to the time variable and \( t_0 \in \mathbb{R} \) is the initial time. The function \( u \) is complex-valued, \( u : \mathbb{R} \times \Omega \to \mathbb{C} \), \( (t, x) \mapsto u(t, x) \).

The local well-posedness for the NLS\( _\Omega \) equation in the exterior of a smooth, compact and convex domain was studied in several articles and is now well understood in many cases. Local existence and uniqueness are usually proved by contraction mapping methods via Strichartz estimates. The Cauchy theory for the NLS\( _\Omega \) equation with initial data in \( H^1_0(\Omega) \), was initiated in [4]. The authors proved first a Strichartz estimates under some restriction, which led to a local existence result for \( p < 3, d = 3 \). This results was later extended for all \( 1 < p \leq 5 \), in dimension \( d = 3 \), in particular, see [1] for the cubic nonlinearity, [18] for \( 1 < p < 5 \) and [14] for \( p = 5 \), see also [2]. The full range of Strichartz estimates for NLS\( _\Omega \) except the end point was obtained in [12]. Therefore, the NLS\( _\Omega \) equation is locally-well posed in \( H^1_0(\Omega) \), for \( 1 < p < \frac{4+2}{d-2} \), \( d \geq 4 \) and \( 1 < p < \infty \), for \( d = 2 \) (Cf. Proposition 1.1 below for a precise local well posedness statement needed for our purpose).

The solutions of (NLS\( _\Omega \)) satisfy the mass and energy conservation laws:

\[
\begin{aligned}
M[u(t)] := \int_\Omega |u(t, x)|^2 dx = M[u_0], \\
E[u(t)] := \frac{1}{2} \int_\Omega |
abla u(t, x)|^2 dx - \frac{1}{p+1} \int_\Omega |u(t, x)|^{p+1} dx = E[u_0].
\end{aligned}
\]

In [15], the authors proved global existence and scattering of solutions for the focusing 3d cubic NLS\( _\Omega \) equation, whenever the initial data satisfies a smallness criterion given by the ground state threshold, see also [22] for \( \frac{7}{3} < p < 5, d = 3 \). The criterion is expressed in terms of the scale-invariant quantities \( ||u_0||_{L^2} \), \( ||\nabla u_0||_{L^2} \) and \( M[u]E[u] \). Moreover, in [5] the authors revisited the proof of scattering using Dodson and Murphy’s approach [6], [7] and the dispersive estimate established in [13]. In [17], we constructed a solitary wave solution for (NLS\( _\Omega \)) behaving asymptotically as a soliton on \( \mathbb{R}^3 \), as large time. This solution is global, does not scatter and prove the optimality of the threshold for scattering given above.

All results mentioned above for (NLS\( _\Omega \)) concern global solutions but the existence of blow-up solutions is still an open question which is the purpose of this paper.

Before stating our blow-up results for the NLS\( _\Omega \) equation, let us recall the proof of the classical blow-up criterion of Vlasov-Petrisshev-Talanov [19], Zakharov [20] and Glassey [8] which states that finite variance, negative energy solutions break down in finite time. This proof is a convexity argument on the variance \( V(t) \) defined as the following,

\[
V(t) := V(u(t)) = \int_{\mathbb{R}^d} |x|^2 |u(t, x)|^2 dx.
\]
Assuming $V(0) < \infty$, the following virial identity holds:

$$
\frac{1}{16} \frac{d^2}{dt^2} V(t) = E_{\mathbb{R}^d}(u) - \frac{1}{2} \left( \frac{d}{2} - \frac{d+2}{p+1} \right) \|u\|_{L^{p+1}(\mathbb{R}^d)}^{p+1},
$$

where $E_{\mathbb{R}^d}(u) = \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^d)}^{p+1}$. If $p > 1 + \frac{d}{4}$ and $E_{\mathbb{R}^d}(u) < 0$ then $u$ blows up in finite time. As proved in [10], in the energy subcritical case ($d \leq 2$ or $p < 1 + \frac{4}{d-2}$), the assumption $E_{\mathbb{R}^d}(u) < 0$ can be weakened to a condition on the initial data which can be formulated in term of the ground state (see Theorem 1.7 below).

This proof does not adapt directly to the case of an exterior domain because the boundary term in the virial identity does not have a favorable sign,

$$
\frac{1}{16} \frac{d^2}{dt^2} V(u(t)) = E[u] - \frac{1}{2} \left( \frac{d}{2} - \frac{d+2}{p+1} \right) \int_\Omega |u|^{p+1} dx - \frac{1}{4} \int_{\partial \Omega} |\nabla u|^2 (x,\vec{n}) d\sigma(x),
$$

where $\vec{n}$ is the unit outward normal vector. One can see that $x,\vec{n} \leq 0$ on $\partial \Omega$. In this work, we will define a new shifted variance $V(t)$ which will allow us to control the boundary term and to prove the existence of blow-up solution for the NLS$_\Omega$ equation. In the energy subcritical case, with an additional symmetry assumption on the initial data, we will prove blow-up with the sufficient condition obtained in [10] on the Euclidean space.

Next, we recall the needed local well-posedness property for the NLS$_\Omega$ equation posed outside a convex obstacle.

**Proposition 1.1.** Assume $p > 1$ if $d = 2$ and $1 < p < \frac{d+2}{d-2}$ if $d \geq 3$. Let $u_0 \in H^1_0(\Omega)$ then there exists $T > 0$ and a unique solution $u(t)$ of (NLS$_\Omega$) equation with $u \in C([-T,T], H^1_0(\Omega))$. Assume $d = 3$ and $p > 2$. Let $u_0 \in H^2 \cap H^1_0(\Omega)$ then there exists $T > 0$ and a unique solution $u(t)$ of (NLS$_\Omega$) equation with $u \in C([-T,T], H^2 \cap H^1_0(\Omega))$.

We omit the standard proof of Proposition 1.1. The local existence and uniqueness in $H^1_0(\Omega)$ can be carried out by classical methods, using fixed point argument via Strichartz estimates. The proof is very similar to the one for the NLS equation posed on the whole Euclidean space. Moreover, the local existence of solutions for the NLS$_\Omega$ equation in $H^2 \cap H^1_0(\Omega)$ can be established using the fact that $H^2$ is an algebra and the following continuous embedding for any smooth domain $\Omega \subset \mathbb{R}^3$, $H^2(\Omega) \subset L^\infty(\Omega)$, see [4, Proposition 2.1]. Thus we don’t have to control the nonlinearity growth but we just need regularity of the nonlinear term.

It is classical that the solution $u$ can be extended to a maximal time interval $I = (-T_-, T_+)$ of existence. If $T_+ = +\infty$ (respectively $T_- = -\infty$) then the solution is global for positive time (respectively for negative time) and if $T_+ < \infty$ (respectively $|T_-| < \infty$) then the solution blows up in finite time and

$$
\lim_{t \to T_+} \|u(t,\cdot)\|_{H^1_0(\Omega)} = +\infty, \quad \text{respectively} \quad \lim_{t \to T_-} \|u(t,\cdot)\|_{H^1_0(\Omega)} = +\infty.
$$

Let $\mathcal{H}(\Omega) \subset H^1_0(\Omega)$ be a space where a Cauchy theory for (NLS$_\Omega$) is available, in particular, from Proposition 1.1, for $d = 2$ one can consider $\mathcal{H}(\Omega) = H^1_0(\Omega)$ and for $d = 3$, and $p > 2$, it suffices to take $\mathcal{H}(\Omega) = H^2 \cap H^1_0(\Omega)$.
Now we state our main results.

**Theorem 1.2.** Assume $\Theta = B(0, R)$ and $p \geq 5$.

- for $d = 2$, let $u_0 \in H^1_0(\Omega)$ such that $E[u_0] + \frac{1}{8R^2}M[u_0] < 0$ and $|x|u_0 \in L^2(\Omega)$,
- for $d \geq 3$, let $u_0 \in \mathcal{H}(\Omega)$ such that $E[u_0] < 0$ and $|x|u_0 \in L^2(\Omega)$,

and let $u$ be the corresponding solution of $(\text{NLS}_\Omega)$. Then the solution $u$ blows up in finite time.

Next, we extend the above result to any smooth, compact and convex obstacle $\Theta$, such that the following holds in dimension $d \geq 2$ : Let $M := \max_{x \in \partial \Theta}(|x|)$ and $m := \min_{x \in \partial \Theta}(|x|)$, then

$$
(1.1) \quad \frac{M}{m} < \frac{d}{d-1}.
$$

**Theorem 1.3.** Assume that $\Theta$ satisfies (1.1) and $p \geq 1 + \frac{4}{d-2}$.

- for $d = 2$, let $u_0 \in H^1_0(\Omega)$ such that $E[u_0] + \frac{M}{8m^2}M[u_0] < 0$ and $|x|u_0 \in L^2(\Omega)$,
- for $d \geq 3$, let $u_0 \in \mathcal{H}(\Omega)$ such that $E[u_0] < 0$ and $|x|u_0 \in L^2(\Omega)$,

and let $u$ be the corresponding solution of $(\text{NLS}_\Omega)$. Then the solution $u$ blows up in finite time.

**Remark 1.4.** Let us mention that, if $\Theta$ and $p$ satisfy the assumptions of the Theorems and $u_0 \in \mathcal{H}(\Omega) \setminus \{0\}$, then the solution with initial data $\lambda u_0$ blows up in finite time for large $\lambda$.

In the following result, we consider $\Theta$ to be a smooth, compact and convex obstacle which is invariant with respect to the transformations $x_j \mapsto -x_j$, for $j = 1, \ldots, d$, that is, if $x = (x_j)_{1 \leq j \leq d} \in \Theta$, then $(x_1, \ldots, -x_j, \ldots, x_d) \in \Theta$. For example, $\Theta$ might a ball or the volume delimited by an ellipsoid. We define $\mathcal{S}_d$ the set of initial data $u_0 \in \mathcal{H}$, which satisfy the additional symmetry conditions:

$$
\mathcal{S}_d := \{u_0 \in \mathcal{H}(\Omega) \setminus \{0\} \mid u_0(x_1, \ldots, -x_i, \ldots, x_d) = -u_0(x_1, \ldots, x_i, \ldots, x_d), \; i = 1, \ldots, d\}.
$$

By uniqueness for the Cauchy problem, the symmetry properties of $u_0 \in \mathcal{S}_d$ are conserved, that is,

$$
u(t, x_1, \ldots, -x_i, \ldots, x_d) = -u(t, x_1, \ldots, x_i, \ldots, x_d), \; i = 1, \ldots, d.
$$

**Theorem 1.5.** Assume $d \geq 2$, $p \geq 1 + \frac{4}{d}$ and $\Theta$ is invariant under the above symmetry. Let $u_0 \in \mathcal{S}_d$ and let $u$ be the corresponding solution of $(\text{NLS}_\Omega)$ with maximal time interval $I$ of existence. If $E[u] < 0$ and $|x|u_0 \in L^2(\Omega)$, then the length of $I$ is finite and thus the solution $u$ blows up in finite time.

**Remark 1.6.** Theorems 1.2 and 1.5 remain true for the NLS$_\Omega$ equation outside an obstacle centered at any point $x_0$. For Theorem 1.5, one would have to use a symmetry around $x_0$ instead of the origin. Moreover, we conjecture that the Theorems are still valid with weaker assumption $p > 1 + \frac{4}{d}$.

Now we introduce the concept of ground state. Let $Q$ be the solution of the following nonlinear elliptic equation

$$
(1.2) \quad -Q + \Delta Q + |Q|^{p-1}Q = 0, \quad Q = Q(x), \; x \in \mathbb{R}^d.
$$

For $1 < p < \frac{d+2}{d-2}$, this nonlinear equation has infinite number of solutions in $H^1(\mathbb{R}^d)$. Among these there is exactly one solution which is positive and radial, called the ground state solution. It is the unique minimal mass solution up to space translation and phase shift and exponentially decaying, see [16]. We henceforth denote by $Q$ this ground state solution.
Theorem 1.7. Assume that $\Theta$ is invariant by the symmetry defined above and $s_c = \frac{q}{2} - \frac{2}{p-1}$.

Let $u_0$ be such that
- for $d = 2$, $u_0 \in S_2$ and $s_c > 0$, i.e., $p > 3$.
- for $d \geq 3$, $u_0 \in S_d$ and $0 < s_c < 1$, i.e., $1 + \frac{4}{d} < p < \frac{4+2}{d}$.

and let $u$ be the corresponding solution of (NLS$_\Omega$) with maximal time interval of existence $I$. Suppose that

$$M[u_0]^{1-s_c} E[u_0] < M[Q]^{1-s_c} E[Q].$$

If (1.3) holds and

$$\|u_0\|_{L^2(\Omega)}^{1-s_c} \|\nabla u_0\|_{L^2(\Omega)}^{s_c} > \|Q\|_{L^2(\mathbb{R}^d)}^{1-s_c} \|\nabla Q\|_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

Then for $t \in I$,

$$\|u(t)\|_{L^2(\Omega)}^{1-s_c} \|\nabla u(t)\|_{L^2(\Omega)}^{s_c} > \|Q\|_{L^2(\mathbb{R}^d)}^{1-s_c} \|\nabla Q\|_{L^2(\mathbb{R}^d)}. \tag{1.5}$$

Furthermore, if $|x|u_0 \in L^2(\Omega)$ then the length of $I$ is finite and thus the solution blows up in finite time.

Let us mention that in the $L^2$-critical case we can find an almost explicit blow-up solution for the NLS$_\Omega$ equation using pseudo-conformal transformation. In this case, we can construct a blow-up solution for the NLS equation by adapting the argument of N. Burq, P. Gérard and N. Tzvetkov in [3], for the NLS equation inside a domain.

Indeed, assume $p = 1 + \frac{4}{d}$. Let $\Psi$ be a $C^{\infty}$-function such that $\Psi = 0$ near $\Theta$ and $\Psi = 1$ for $|x| >> 1$ and let $Q$ be any solution of the nonlinear elliptic equation (1.2), (it does not have to be the ground state) then there exists $T > 0$ and a smooth function $r(t,x)$ defined on $[0,T) \times \Omega$ and exponentially decaying as $t \to T$ such that

$$u(t,x) := \frac{1}{(T-t)} Q\left(\frac{x-x_0}{T-t}\right) \Psi(x) e^{i\left(\frac{4|x-x_0|^2}{4T-t}\right)} + r(t,x)$$

is solution for (NLS$_\Omega$) satisfying the Dirichlet boundary conditions, which blow-up in finite time $T$. The proof is similar to one given in [3] for (NLS) equation inside a domain in $\mathbb{R}^d$. We need to construct the smooth correction $r$ such that $u$ is a solution of (NLS$_\Omega$) satisfying Dirichlet boundary conditions. To achieve this, one define a contraction mapping using the Duhamel formula on a closed ball in the Banach space $(E, \|\cdot\|_E)$ defined by

$$E := \{f \in C([0,T), H^2(\Omega) \cap H^1_0(\Omega)); \|f\|_E < \infty\},$$

equipped with the norm

$$\|f\|_E := \sup_{t \in [0,T]} \left\{e^{\frac{1}{4(1-d)}} \|f\|_{L^2(\Omega)} + e^{\frac{3}{4(1-d)}} \|f\|_{H^2(\Omega)}\right\}.$$

The existence of the smooth correction $r$ follows from the fixed point theorem.

The paper is organized as follows. In section 2, we give a review of some properties related to the ground state $Q$. In section 3, we prove Pohozaev’s identities outside an obstacle. In section 4, we prove the existence of blow-up solution in the exterior of a ball for $p \geq 5$ and outside a convex obstacle that satisfies (1.1) for $p \geq 1 + \frac{4}{d-\frac{4}{m}(d-1)}$, using a convexity argument on the modified variance. In section 5, we study the existence of symmetric blow-up solution for $p \geq 1 + \frac{4}{d}$ using a different variance. Finally, in section 6, we study the behavior of the
solutions, in particular, the blow-up criteria for the solutions with initial data beyond the ground state threshold.

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2. Properties of the Ground State

Weinstein [21] proved that the sharp constant \( C_{GN} \) in the Gagliardo-Nirenberg estimate

\[
\|f\|_{L^{p+1}}^{p+1} \leq C_{GN} \left( \|\nabla f\|_{L^2}^{\frac{d(p-1)}{2}} \|f\|_{L^2}^{\frac{(d-2)(p-1)}{2}} \right)
\]

is attained at the function \( Q \) (the ground state described in the introduction), i.e.

\[
C_{GN} = \frac{\|Q\|_{L^{p+1}(\mathbb{R}^d)}}{\|\nabla Q\|_{L^2(\mathbb{R}^d)}^{\frac{d(p-1)}{2}} \|Q\|_{L^2(\mathbb{R}^d)}^{\frac{(d-2)(p-1)}{2}}}
\]

Multiplying (1.2) by \( Q \) and integrating by parts, we obtain

\[
\|Q\|_{L^2(\mathbb{R}^d)}^2 - \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 + \|Q\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} = 0.
\]

Multiplying (1.2) by \( x.\nabla Q \) and integrating by parts, we obtain the following identity

\[
\frac{d}{2} \|Q\|_{L^2(\mathbb{R}^d)}^2 + \frac{d-2}{2} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 - \frac{d}{p+1} \|Q\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} = 0.
\]

These two identities (2.2) and (2.3) enable us to obtain these relations

\[
\|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 = \frac{d(p-1)}{d+2-p(d-2)} \|Q\|_{L^2(\mathbb{R}^d)}^2
\]

\[
\|Q\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} = \frac{2(p+1)}{d(p-1)} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2
\]

and thus, reexpress

\[
C_{GN} = \left( \frac{2(p+1)}{d(p-1)} \|\nabla Q\|_{L^2(\mathbb{R}^d)} \|Q\|_{L^2(\mathbb{R}^d)}^{\frac{4-(d-2)(p-1)}{d(p-1)-4}} \right) - \frac{d(p-1)-4}{2}
\]

We also compute

\[
E[Q] := \frac{1}{2} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 - \frac{1}{p+1} \|Q\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} = \frac{d(p-1)-4}{2d(p-1)} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2.
\]

3. Pohozaev’s identities outside obstacle

This section is devoted to the proof of the Pohozaev’s Identity in exterior domain. In the following Proposition \( \Omega \) can be the exterior of any regular obstacle.
Proposition 3.1 (Pohozaev’s identity). Let \( u \in H^2 \cap H^1_0(\Omega) \), \( |x| u \in L^2(\Omega) \) then we have,

\[
(3.1) \quad \text{Re} \int_\Omega \Delta \overline{u} \left( \frac{d}{2} u + x \cdot \nabla u \right) dx = - \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 \left( x \cdot \overline{n} \right) d\sigma(x).
\]

\[
(3.2) \quad \text{Re} \int_\Omega \Delta \overline{u} \left( \nabla u \cdot \frac{x}{|x|} + \left( \frac{d-1}{2} \right) \frac{u}{|x|} \right) dx = - \frac{(d-1)(d-3)}{4} \int_\Omega \frac{|u|^2}{|x|^3} dx - \int_\Omega \frac{\nabla u^2}{|x|} dx + \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 \left( x \cdot \overline{n} \right) d\sigma(x),
\]

where \( |\nabla u|^2 := |\nabla u|^2 - \left| \frac{x}{|x|} \cdot \nabla u \right|^2 \) and \( \overline{n} \) is the outward unit normal vector.

Proof. Using integration by parts and the fact that \( u \) satisfies Dirichlet boundary condition (i.e \( u = 0 \) on \( \partial \Omega \)), we obtain

\[
\text{Re} \int_\Omega \Delta \overline{u} (x \cdot \nabla u) dx = - \sum_{j=1}^d \text{Re} \int_\Omega \partial_{x_j} \overline{u} \partial_{x_j} u dx - \sum_{j,k=1}^d \text{Re} \int_\Omega x_k \partial_{x_j} \partial_{x_k} u \partial_{x_j} \overline{u} dx
\]

\[
+ \int_{\partial \Omega} |\nabla u|^2 \left( x \cdot \overline{n} \right) d\sigma(x)
\]

\[
= - \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \sum_{k=1}^d \int_\Omega |\nabla u|^2 dx - \frac{1}{2} \sum_{k=1}^d \int_{\partial \Omega} |\nabla u|^2 x_k n_k d\sigma(x)
\]

\[
+ \int_{\partial \Omega} |\nabla u|^2 \left( x \cdot \overline{n} \right) d\sigma(x)
\]

\[
= - \int_\Omega |\nabla u|^2 dx - \frac{d}{2} \text{Re} \int_\Omega \Delta \overline{u} u dx + \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 \left( x \cdot \overline{n} \right) d\sigma(x).
\]

This concludes the proof of (3.1). Now let us prove (3.2), using the same argument as above and the fact that,

\[
\partial_{x_k} \left( \frac{x_j}{|x|} \right) = \begin{cases} \frac{1}{|x|^3} - \frac{x_j^2}{|x|^5} & \text{if } j = k, \\ -\frac{x_j x_k}{|x|^5} & \text{if } j \neq k, \end{cases}
\]

we obtain

\[
\text{Re} \int_\Omega \nabla \overline{u} \cdot \frac{x}{|x|} dx = - \frac{1}{2} \sum_{j,k=1}^d \int_\Omega \partial_{x_k} \overline{u} \partial_{x_k} u \frac{x_j}{|x|^3} dx + \partial_{x_k} u \partial_{x_j} \overline{u} \frac{x_j}{|x|^3} dx
\]

\[
- \text{Re} \sum_{j=1}^d \int_\Omega \partial_{x_j} \overline{u} \partial_{x_j} u \left( \frac{1}{|x|^3} - \frac{x_j^2}{|x|^5} \right) dx + \text{Re} \sum_{j,k=1}^d \int_{\partial \Omega} \partial_{x_k} \overline{u} \partial_{x_j} u \frac{x_j x_k}{|x|^5} dx
\]

\[
+ \int_{\partial \Omega} |\nabla u|^2 \left( x \cdot \overline{n} \right) d\sigma(x)
\]

This concludes the proof of (3.2).
\[
\text{Re} \int_\Omega \Delta \pi \nabla u \cdot \frac{x}{|x|} \, dx = -\frac{1}{2} \sum_{j=1}^d \int_\Omega \frac{x_j}{|x|} \partial_{x_j} \left( |\nabla u|^2 \right) \, dx - \int_\Omega \frac{|\nabla u|^2}{|x|} \, dx + \int_\Omega \frac{x}{|x|} \nabla u \cdot \nabla u \left( \frac{1}{|x|} \right) \, dx
\]

\[
+ \int_{\partial \Omega} \frac{|\nabla u|^2}{|x|} \, d\sigma(x)
\]

\[
= \left( d - \frac{1}{2} \right) \int_\Omega \frac{|\nabla u|^2}{|x|} \, dx - \int_\Omega \frac{\nabla u^2}{|x|} \, dx + \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 \frac{(x, \vec{n})}{|x|} \, d\sigma(x)
\]

\[
= \left( d - \frac{1}{2} \right) \text{Re} \int_\Omega \Delta \pi \, u \frac{1}{|x|} \, dx + \left( d - \frac{1}{2} \right) \text{Re} \sum_{k=1}^d \partial_{x_k} \pi u \frac{x_k}{|x|^3} \, dx
\]

\[- \int_\Omega \frac{\nabla u^2}{|x|} \, dx + \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 \frac{(x, \vec{n})}{|x|} \, d\sigma(x).
\]

Using the fact that,

\[
\left( d - \frac{1}{2} \right) \text{Re} \sum_{k=1}^d \partial_{x_k} \pi u \frac{x_k}{|x|^3} \, dx = -\frac{(d - 1)(d - 3)}{4} \int_\Omega \frac{|u|^2}{|x|^5} \, dx,
\]

we obtain (3.2), which concludes the proof of Proposition 3.1. \(\square\)

4. Existence of blow-up solution

This section is devoted to the proofs of Theorem 1.2 and 1.3. We assume \(d \in \{2, 3\}\). Nevertheless, the computations below still valid for \(d \geq 4\) if an appropriate Cauchy theory is available.

Denote:

\[
\Upsilon_1(u(t)) := \int_\Omega |x| |u(t, x)|^2 \, dx, \quad \Upsilon_2(u(t)) := \int_\Omega |x|^2 |u(t, x)|^2 \, dx.
\]

We will start by proving the following virial identities in the exterior of a convex obstacle, in particular in the exterior of a ball which is needed in the proof of Theorem 1.2.

4.1. Virial identities in exterior domain.

**Proposition 4.1.** Assume that \(\Theta\) is a smooth, compact and convex obstacle. Let \(u_0 \in H^2 \cap H^3_0(\Omega), |x| u_0 \in L^2(\Omega)\) and let \(u\) be the corresponding solution of the NLS\(_\Omega\) equation. Then

\[
\frac{d}{dt} \Upsilon_2(u(t)) = 4 \text{ Im} \int_\Omega \pi(t, x) \cdot \nabla u(t, x) dx.
\]

\[
\frac{1}{16} \frac{d^2}{dt^2} \Upsilon_2(u(t)) = E[u] - \frac{1}{2} \left( \frac{d}{2} - \frac{d + 2}{p + 1} \right) \int_\Omega |u|^{p+1} \, dx - \frac{1}{4} \int_{\partial \Omega} |\nabla u|^2 \cdot (x, \vec{n}) \, d\sigma(x).
\]

And

\[
\frac{d}{dt} \Upsilon_1(u(t)) = 2 \text{ Im} \sum_{j=1}^d \int_\Omega \pi(t, x) \frac{x_j}{|x|} \partial_{x_j} u(t, x) \, dx.
\]
\[
(4.4) \quad \frac{1}{16} \frac{d^2}{dt^2} \Upsilon_1(u(t)) = \frac{(d-1)(d-3)}{16} \int_\Omega \frac{|u|^2}{|x|^3} \, dx - \frac{(d-1)(p-1)}{8(p+1)} \int_\Omega \frac{|u|^{p+1}}{|x|} \, dx \\
+ \frac{1}{4} \int_\Omega \frac{\nabla u^2}{|x|} \, dx - \frac{1}{8} \int_{\partial \Omega} |\nabla u|^2 \frac{x \bar{n}}{|x|} \, d\sigma(x).
\]

where \(|\nabla u|^2 := |\nabla u|^2 - \frac{x \cdot \nabla u}{|x|}^2| \) and \(\bar{n}\) is the outward unit normal vector.

**Proof.** Multiplying the equation by \(|x|^2 \bar{u}\) and taking the imaginary part we get,

\[
\text{Im} \int_\Omega i \partial_t u |x|^2 \bar{u} \, dx + \text{Im} \int_\Omega \Delta u |x|^2 \bar{u} \, dx = -\text{Im} \int_\Omega |u|^{p-1} u |x|^2 \bar{u} \, dx = 0.
\]

Which yields,

\[
\frac{1}{2} \frac{d}{dt} \Upsilon_2(u(t)) = \frac{1}{2} \frac{d}{dt} \int_\Omega |x|^2 |u(t, x)|^2 \, dx = -\text{Im} \int_\Omega |x|^2 \Delta u \bar{u} \, dx.
\]

Integration by parts ensures

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |x|^2 |u(t, x)|^2 \, dx = 2 \sum_{k=1}^d \text{Im} \int_\Omega x_k \partial x_k u \bar{u} \, dx = 2 \text{Im} \int_\Omega \bar{u} \cdot x \nabla u \, dx.
\]

This implies (4.1). Now let us compute the second derivative of \(\Upsilon_2\).

\[
\frac{d^2}{dt^2} \Upsilon_2(u(t)) = 4 \frac{d}{dt} \text{Im} \int_\Omega \bar{u} \cdot x \nabla u \, dx
\]

\[
= 4 \left( \text{Im} \int_\Omega \partial_t \bar{u} \cdot x \nabla u \, dx + \text{Im} \int_\Omega \bar{u} \cdot x \nabla (\partial_t u) \, dx \right)
\]

\[
= 4 \left( \text{Im} \int_\Omega (-i \Delta \bar{u} - i |u|^{p-1} \bar{u}) \cdot x \nabla u \, dx + \text{Im} \int_\Omega \bar{u} \cdot x \nabla (i \Delta u + i |u|^{p-1} u) \, dx \right)
\]

\[
= 4 \left[ \text{Re} \int_\Omega -\Delta \bar{u} \cdot x \nabla u \, dx + \text{Re} \int_\Omega \bar{u} \cdot x \nabla (\Delta u) \, dx + \text{Re} \int_\Omega -|u|^{p-1} \bar{u} \cdot x \nabla u \, dx \right]
\]

\[
+ \text{Re} \int_\Omega \bar{u} \cdot x \nabla (|u|^{p-1} u) \, dx \right].
\]

Next, we compute each integral apart, using integration by parts and the Dirichlet boundary condition, i.e., \(u = 0\) on \(\partial \Omega\).

\[
I_2 := \text{Re} \sum_{k=1}^d \int_\Omega \bar{u} x_k \partial x_k \Delta u \, dx = \sum_{k=1}^d \text{Re} \int_\Omega -\partial x_k (\bar{u} x_k) \Delta u \, dx + \text{Re} \sum_{k=1}^d \int_{\partial \Omega} \bar{u} \Delta u (x_k n_k) \, d\sigma(x)
\]

\[
= \text{Re} \int_\Omega -\nabla \bar{u} \cdot x \Delta u \, dx - d \text{Re} \int_\Omega \bar{u} \Delta u \, dx.
\]

Hence,

\[
I_1 + I_2 := -2 \text{Re} \int_\Omega \Delta \bar{u} \left( x \nabla u + \frac{d}{2} u \right) \, dx.
\]
Using Pohozaev’s Identity (3.1), we get

\[
I_1 + I_2 := 2 \int_{\Omega} |\nabla u|^2 \, dx - \int_{\partial \Omega} |\nabla u|^2 (x, \vec{n}) \, d\sigma(x).
\]

\[
I_4 := \text{Re} \int_{\Omega} \nabla x. \nabla (|u|^{p-1} u) \, dx = - \text{Re} \int_{\Omega} \nabla \nabla x \, |u|^{p-1} u \, dx - d \int_{\Omega} |u|^{p+1} \, dx.
\]

Using the fact that

\[
\nabla (|u|^{p+1}) = (p+1) |u|^{p-1} \text{ Re} (\nabla u),
\]

we obtain

\[
I_3 + I_4 = -2 \text{Re} \int_{\Omega} |u|^{p-1} \nabla x. \nabla u \, dx - d \int_{\Omega} |u|^{p+1} \, dx
\]

\[
= - \frac{2}{p+1} \text{Re} \int_{\Omega} x. \nabla (|u|^{p+1}) \, dx - d \int_{\Omega} |u|^{p+1} \, dx
\]

\[
= \left( \frac{2d}{p+1} - d \right) \int_{\Omega} |u|^{p+1} \, dx.
\]

Which yields

\[
\frac{d^2}{dt^2} \Upsilon_2(u(t)) = 8 \int_{\Omega} |\nabla u|^2 \, dx + \left( \frac{8d}{p+1} - 4d \right) \int_{\Omega} |u|^{p+1} \, dx - 4 \int_{\partial \Omega} |\nabla u|^2 (x, \vec{n}) \, d\sigma(x).
\]

Thus

\[
0 = E[u] - \frac{1}{2} \left( \frac{d}{2} - \frac{d+2}{p+1} \right) \int_{\Omega} |u|^{p+1} \, dx - \frac{1}{4} \int_{\partial \Omega} |\nabla u|^2 (x, \vec{n}) \, d\sigma(x).
\]

This concludes the proof of (4.2). Now let us compute the first derivative of \( \Upsilon_1 \). Similarly, multiplying the equation by \(|x| \nabla u\) and taking the imaginary part we get,

\[
\frac{1}{2} \frac{d}{dt} \Upsilon_1(u(t)) = \frac{d}{dt} \int_{\Omega} |x||u(t,x)|^2 \, dx = \text{Im} \int_{\Omega} -\Delta u |x| \nabla u \, dx = \text{Im} \sum_{j=1}^{d} \int_{\Omega} \frac{x_j}{|x|} \partial_{x_j} u \nabla u \, dx.
\]

Thus, we obtain (4.3)

\[
\frac{d}{dt} \Upsilon_1(u(t)) = 2 \text{ Im} \sum_{j=1}^{d} \int_{\Omega} \nabla u(x) \frac{x_j}{|x|} \partial_{x_j} u \, dx = 2 \mathcal{W}(u(t)).
\]

For the second derivative of \( \Upsilon_1 \), using the (NLS\(_{\Omega}\)) we get

\[
\frac{d^2}{dt^2} \Upsilon_1(u(t)) = 2 \text{ Im} \sum_{j=1}^{d} \int_{\Omega} \partial_{x_j} \frac{x_j}{|x|} \partial_{x_j} u \, dx + 2 \text{ Im} \sum_{j=1}^{d} \int_{\Omega} \frac{x_j}{|x|} \partial_{x_j} (\partial_{x_j} u) \, dx
\]

\[
= 2 \text{ Re} \sum_{j=1}^{d} \int_{\Omega} -\Delta \frac{x_j}{|x|} \partial_{x_j} u \, dx + 2 \text{ Re} \sum_{j=1}^{d} \int_{\Omega} \frac{x_j}{|x|} \partial_{x_j} (\Delta u) \, dx
\]

\[
+ 2 \text{ Re} \sum_{j=1}^{d} \int_{\Omega} |u|^{p-1} \frac{x_j}{|x|} \partial_{x_j} u \, dx + 2 \text{ Re} \sum_{j=1}^{d} \int_{\Omega} \frac{x_j}{|x|} \partial_{x_j} (|u|^{p-1} u) \, dx.
\]
We will compute each integral apart, using again integration by parts and the Dirichlet boundary condition, i.e., $u = 0$ on $\partial \Omega$.

\[
J_2 := 2 \text{Re} \sum_{j=1}^{d} \int_{\Omega} \frac{x_j}{|x|} \partial_{x_j} (\Delta u) \, dx = 2 \text{Re} \sum_{j=1}^{d} \int_{\Omega} -\partial_{x_j} \left( \frac{x_j}{|x|} \right) \Delta u \, dx
\]
\[
= -2 \text{Re} \int_{\Omega} \nabla u \cdot \frac{x}{|x|} \Delta u \, dx - 2 \text{Re} \int_{\Omega} \frac{(d-1)}{|x|} \Delta u \, dx.
\]
Hence,

\[
J_1 + J_2 := -4 \text{Re} \int_{\Omega} \Delta \nabla u \cdot \frac{x}{|x|} \, dx - 2 \text{Re} \int_{\Omega} \Delta \left( \frac{d-1}{|x|} \right) u \, dx
\]
\[
= -4 \left[ \text{Re} \int_{\Omega} \Delta \left( \nabla u \cdot \frac{x}{|x|} + \frac{(d-1)}{2} \frac{1}{|x|} \right) \, dx \right].
\]

Using (3.2), we get

\[
J_1 + J_2 = -(d-1)(d-3) \int_{\Omega} \frac{|u|^2}{|x|^3} \, dx + 4 \int_{\Omega} \frac{\nabla u \cdot \nabla u}{|x|} \, dx - 2 \int_{\partial \Omega} |\nabla u|^2 \frac{x}{|x|} \cdot \vec{n} \, d\sigma(x)
\]
\[
= (d-1)(d-3) \int_{\Omega} |u|^2 \, dx - 2(d-1) \int_{\Omega} |u|^{p+1} \, dx.
\]

Due to (4.5), we have

\[
J_3 + J_4 := -4 \text{Re} \int_{\Omega} \nabla \left( \frac{x}{|x|} \right) u \, dx - 2(d-1) \int_{\Omega} |u|^{p+1} \, dx
\]
\[
= -4 \int_{\Omega} \frac{x}{|x|} \cdot \nabla (|u|^{p+1}) \, dx - 2(d-1) \int_{\Omega} |u|^{p+1} \, dx
\]
\[
= -2(d-1)(p-1) \int_{\Omega} |u|^{p+1} \, dx.
\]

Summing all terms, we get

\[
\frac{d^2}{dt^2} \mathcal{T}_1(u(t)) = \frac{d^2}{dt^2} \int_{\Omega} |x||u|^2 \, dx
\]
\[
= (d-1)(d-3) \int_{\Omega} |u|^2 \, dx + 4 \int_{\Omega} \frac{\nabla u \cdot \nabla u}{|x|} \, dx - 2(d-1)(p-1) \int_{\Omega} |u|^{p+1} \, dx
\]
\[
- 2 \int_{\partial \Omega} |\nabla u|^2 \frac{x}{|x|} \cdot \vec{n} \, d\sigma(x).
\]

This concludes the proof of Proposition 4.1. \[\square\]
4.2. Existence of blow-up solution in the exterior of a ball.

Proof of Theorem 1.2. Assume \( \Theta = B(0, R) \) and \( p \geq 5 \). Let \( u_0 \in \mathcal{H}(\Omega) \) (for \( d = 2, 3 \), it suffices to take \( u_0 \in H^2 \cap H^1_0(\Omega) \)), we will later relax the assumption to \( u_0 \in H^1_0(\Omega) \) if \( d = 2 \), \( |x|u_0 \in L^2(\Omega) \), \( E[u_0] + \frac{1}{8R^2} M[u_0] < 0 \) if \( d=2 \) and \( E[u] < 0 \) if \( d \geq 3 \). Let \( u \) be the corresponding solution of \((\text{NLS}_\Theta)\) outside the ball \( B(0, R) \), with maximal time interval \( I \) of existence. Define the variance used in this proof:

\[
\mathcal{V}(u(t)) := \int_{\Omega} (|x|^2 - 2R|x| + 10) |u(t, x)|^2 \, dx.
\]

From Proposition 4.1 we have

\[
(4.6) \quad \frac{1}{16} \frac{d^2}{dt^2} \mathcal{V}(u(t)) = E[u] - \frac{R}{2} \int_{\Omega} \frac{\nabla u}{|x|}^2 \, dx - \frac{1}{2} \left( \frac{d}{2} - \frac{d + 2}{p + 1} \right) \int_{\Omega} |u|^{p+1} \, dx
\]

\[
+ \frac{R(d - 1)(p - 1)}{4(p + 1)} \int_{\Omega} |u|^{p+1} \, dx - \frac{R(d - 1)(d - 3)}{8} \int_{\Omega} |u|^2 \, dx
\]

\[
- \frac{1}{4} \int_{\partial \Omega} |\nabla u|^2 (x, \vec{n}) \, d\sigma(x) + \frac{R}{4} \int_{\partial \Omega} |\nabla u|^2 \frac{x \cdot \vec{n}}{|x|} \, d\sigma(x) = \frac{1}{4} \int_{\partial \Omega} |\nabla u|^2 (x, \vec{n}) \left( \frac{R}{|x|} - 1 \right) \, d\sigma(x) = 0.
\]

Now, we will estimate the nonlinear terms. Using the fact that \( \frac{1}{|x|} \leq \frac{1}{R} \), for all \( x \in \Omega \), and \( p \geq 5 \), we have

\[
- \frac{1}{2} \left( \frac{d}{2} - \frac{d + 2}{p + 1} \right) \int_{\Omega} |u|^{p+1} \, dx + \frac{R(d - 1)(p - 1)}{4(p + 1)} \int_{\Omega} \frac{|u|^{p+1}}{|x|} \, dx
\]

\[
\leq \left[ - \frac{1}{2} \left( \frac{d}{2} - \frac{d + 2}{p + 1} \right) + \frac{(d - 1)(p - 1)}{4(p + 1)} \right] \int_{\Omega} |u|^{p+1} \, dx
\]

\[
= - \left( \frac{p - 5}{4(p + 1)} \right) \int_{\Omega} |u|^{p+1} \, dx \leq 0.
\]

Finally, for all \( d \neq 2 \) one can see that,

\[
(4.7) \quad \frac{-R(d - 1)(d - 3)}{8} \int_{\Omega} \frac{|u|^2}{|x|^3} \, dx \leq 0.
\]

In particular, for \( d = 3 \) we have \( \frac{- (d-1)(d-3)}{8} \int_{\Omega} \frac{|u|^2}{|x|^3} \, dx = 0 \).

For \( d = 2 \), we use the fact that, \( E[u] + \frac{1}{8R^2} M[u] < 0 \) and \( \frac{1}{|x|} \leq \frac{1}{R} \) for all \( x \in \Omega \). Indeed,

\[
E[u] + \frac{R}{8} \int_{\Omega} \frac{|u|^2}{|x|^3} \, dx \leq E[u] + \frac{1}{8R^2} M[u] < 0.
\]
This implies that the second derivative of the variance is bounded by a negative constant, for all \( t \in I \).

\[
\frac{d^2}{dt^2} \mathcal{V}(u(t)) \leq -A, \quad \text{where} \quad -A = \begin{cases} 
E[u] + \frac{1}{8R^2} M[u] & \text{if } d = 2, \\
E[u] & \text{if } d = 3.
\end{cases}
\]

Moreover, integrating twice over \( t \), we have that

\[
\mathcal{V}(u(t)) \leq -At^2 + Bt + C, \quad \text{where } B = \frac{d}{dt} \mathcal{V}(u_0) \text{ and } C = \mathcal{V}(u_0).
\]

By density (4.8) remains true, if \( d = 2 \), assuming that \( u_0 \in H^1_0(\Omega) \) and \( |x|u_0 \in L^2(\Omega) \). Due to (4.8), there exists \( T^* \) such that \( \mathcal{V}(u(T^*)) < 0 \), which is a contradiction. Then the length of the maximal time interval of existence \( I \) is finite and one can prove that the solution \( u \) blows up in finite time. This concludes the proof of Theorem 1.2.

\Square

4.3. Existence of blow-up solution in the exterior of a convex obstacle. In this section, we extend the previous results in the exterior of a ball to a smooth, compact, convex obstacle \( \Theta \). We prove Theorem 1.3, assuming without loss of generality that \( 0 \in \Theta \), and we suppose that \( \Theta \) satisfies the following property for \( d \geq 2 \):

\[
\frac{M}{m} < \frac{d}{d-1}, \quad \text{where } M = \max_{x \in \partial \Theta} (|x|) \text{ and } m = \min_{x \in \partial \Theta} (|x|).
\]

We use the following variance identity:

\[
\mathcal{V}(u(t)) := \int_{\Omega} (|x|^2 - 2M |x| + 10) |u(t, x)|^2 dx.
\]

**Proof of Theorem 1.3.** Assume that \( \Theta \) satisfies (4.9) and \( p \geq 1 + \frac{4}{d-4} \). Let \( u_0 \in \mathcal{H} \), \( |x|u_0 \in L^2(\Omega) \), and suppose that \( E[u_0] + \frac{M}{m} M[u_0] < 0 \), if \( d = 2 \), and \( E[u_0] < 0 \), if \( d = 3 \). Let \( u \) be the corresponding solution of the NLS\(_{\Omega} \) equation in the exterior of a convex obstacle \( \Theta \), such that the assumption (4.9) holds, with maximal time interval \( I \) of existence.

From Proposition 4.1, we have

\[
\frac{1}{16} \frac{d^2}{dt^2} \mathcal{V}(u(t)) = E[u] - \frac{M}{2} \int_{\Omega} \frac{\nabla u^2}{|x|} dx - \frac{1}{2} \left( \frac{d}{2} - \frac{d+2}{p+1} \right) \int_{\Omega} |u|^{p+1} dx
\]

\[
+ \frac{M(d-1)(p-1)}{4(p+1)} \int_{\Omega} \frac{|u|^{p+1}}{|x|} dx - \frac{M(d-1)(d-3)}{8} \int_{\Omega} |u|^2 dx - \frac{1}{4} \int_{\partial \Omega} |\nabla u|^2 (x, \bar{n}) d\sigma(x) + \frac{M}{4} \int_{\partial \Omega} |\nabla u|^2 \frac{x, \bar{n}}{|x|} d\sigma(x).
\]

We first control the boundary terms. Recall that \( \Omega \) is the exterior of a convex obstacle \( \Theta \) and \( 0 \in \Theta \), so that \( x \cdot \bar{n} \leq 0 \) for all \( x \in \partial \Omega \). As \( M = \max_{x \in \partial \Omega} (|x|) = \max_{|x|} (|x|) \), then \( \left( \frac{M}{|x|} - 1 \right) \geq 0 \) for all \( x \in \partial \Omega \). Thus,

\[
- \frac{1}{4} \int_{\partial \Omega} |\nabla u|^2 (x, \bar{n}) d\sigma(x) + \frac{M}{4} \int_{\partial \Omega} |\nabla u|^2 \frac{x, \bar{n}}{|x|} d\sigma(x) = \frac{1}{4} \int_{\partial \Omega} |\nabla u|^2 (x, \bar{n}) \left( \frac{M}{|x|} - 1 \right) d\sigma(x) \leq 0.
\]
Next, we control the nonlinear terms using the fact that $\frac{M}{m} < \frac{d}{d-1}$, $p \geq 1 + \frac{4}{m(d-1)}$ and $\frac{1}{|x|} \leq \frac{1}{m}$, for all $x \in \Omega$.

$$\frac{1}{2} \left( \frac{d}{2} - \frac{d + 2}{p + 1} \right) \int_\Omega |u|^{p+1} \, dx + \frac{M(d-1)(p-1)}{4(p+1)} \int_\Omega \frac{|u|^{p+1}}{|x|} \, dx \leq \left[ \frac{1}{2} \left( \frac{d}{2} - \frac{d + 2}{p + 1} \right) + \frac{M(d-1)(p-1)}{4m(p+1)} \right] \int_\Omega |u|^{p+1} \, dx$$

$$= - \left( \frac{(p-1)(d - \frac{M}{m}(d-1))}{4(p+1)} \right) \int_\Omega |u|^{p+1} \, dx \leq 0.$$  

For all $d \neq 2$, one can see that all other terms are negative. For $d = 2$, we use the fact that, $\frac{1}{|x|} \leq \frac{1}{m}$ for all $x \in \Omega$ and $E[u] + \frac{M}{8m^3}M[u] < 0$, to obtain

$$E[u] + \frac{M}{8} \int_\Omega \frac{|u|^2}{|x|^3} \, dx \leq E[u] + \frac{M}{8m^3}M[u] < 0.$$  

This implies that the second derivative of the variance is bounded by a negative constant, for all $t \in I$.

$$\frac{d^2}{dt^2} \mathcal{V}(u(t)) \leq -A, \quad \text{where} \quad A = \begin{cases} E[u] + \frac{M}{8m^3}M[u] < 0 & \text{if } d = 2, \\ E[u] < 0 & \text{if } d \geq 3. \end{cases}$$

Using the same argument as in the proof of Theorem 1.2, one can prove that the solution $u$ blows up in finite time and this concludes the proof of Theorem 1.3.

5. Existence of blow-up symmetric solution

In this section we prove Theorem 1.5. For the sake of simplicity, we will give the proof for $d = 2$, then we will generalize the result to any dimension $d \geq 3$.

Assume that $d = 2$, $p \geq 3$ and $\Theta$ is invariant by the symmetry, $x_j \mapsto -x_j$, Recall that define $S_2$ is defined as the following

$$S_2 := \{ u_0 \in \mathcal{H}(\Omega) \setminus u_0(-x_1, x_2) = u_0(x_1, -x_2) = -u_0(x_1, x_2) \}.$$  

Here, we can consider $\mathcal{H}(\Omega) = H^1_0(\Omega)$. By uniqueness for the Cauchy problem, we have

$$u(t, -x_1, x_2) = u(t, x_1, -x_2) = -u(t, x_1, x_2).$$

Due to the two symmetry assumptions, we will reduce the problem to a quarter of the space. We define

$$\Omega := \bigcup \Omega^{\pm \pm} := \Omega^+ \cup \Omega^- \quad \text{and} \quad \Omega^{\pm \pm} := \{(x_1, x_2) \in \Omega \setminus x_1 \in \mathbb{R}^\pm \text{ and } x_2 \in \mathbb{R}^\pm \},$$

$$\Omega^+ : = \{(x_1, x_2) \in \Omega \setminus x_1 \in \mathbb{R}^+ \} \quad \text{and} \quad \Omega^- : = \{(x_1, x_2) \in \Omega \setminus x_2 \in \mathbb{R}^+ \}.$$  

Furthermore, since $u|_{x_1 = 0} = 0$ and $u|_{x_2 = 0} = 0$, one can see that $u$ satisfies the Dirichlet boundary conditions on each set defined above, $\Omega^{\pm \pm}$ and $\Omega^{\pm}$.

The variance identity here is the following: Let $C > 0$ be a positive constant to be specified later,

$$V(u(t)) := \int_\Omega (|x|^2 - C|x_1| - C|x_2| + C^2) |u(t, x)|^2 \, dx.$$
Denote
\[ \Gamma_1(u(t)) := \int_{\Omega} |x_1||u(t,x)|^2 dx, \quad \Gamma_2(u(t)) := \int_{\Omega} |x_2||u(t,x)|^2 dx. \]

**Proposition 5.1.** Let \( u_0 \in H^2 \cap H_0^1(\Omega) \) and \( |x| u_0 \in L^2(\Omega) \) such that \( u_0 \in S_2 \) and let \( u \) be the corresponding solution of the NLS\( _\Omega \) equation. Then

\[
\frac{d}{dt} \Gamma_1(u(t)) = 8 \text{Im} \int_{\Omega} \partial_{x_1} u(t,x) \overline{u}(t,x) dx, \\
\frac{d^2}{dt^2} \Gamma_1(u(t)) = 8 \int_{\partial \Omega} \|
abla u(t,x)\|^2 |n_1| dx,
\]

and

\[
\frac{d}{dt} \Gamma_2(u(t)) = 8 \text{Im} \int_{\Omega} \partial_{x_2} u(t,x) \overline{u}(t,x) dx, \\
\frac{d^2}{dt^2} \Gamma_2(u(t)) = 8 \int_{\partial \Omega} \|
abla u(t,x)\|^2 |n_2| dx.
\]

**Proof.** Multiply the equation by \( |x_1| \bar{u} \) and take the imaginary part to get:

\[
\text{Im} \int_{\Omega} i \partial_t |x_1| \bar{u} dx + \text{Im} \int_{\Omega} \Delta |x_1| \bar{u} dx = - \text{Im} \int_{\Omega} |u|^{p-1} |x_1| \bar{u} dx,
\]

which yields

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |x_1||u(t,x)|^2 dx = - \text{Im} \int_{\Omega} |x_1| \Delta u \bar{u} dx.
\]

Integration by parts ensures,

\[
\text{Im} \int_{\Omega} |x_1| \Delta u \bar{u} dx = - \text{Im} \int_{\Omega} \partial_{x_1}(|x_1|) \partial_{x_1} u \bar{u} dx
\]

\[
= - \left( \text{Im} \int_{\Omega^+} \partial_{x_1} u \bar{u} dx - \text{Im} \int_{\Omega^-} \partial_{x_1} u \bar{u} dx \right)
\]

\[
= -4 \text{Im} \int_{\Omega^+} \partial_{x_1} u \bar{u} dx,
\]

which yields (5.1). Now, let us compute the second derivative of \( \Gamma_1(u(t)) \).

Denote:

\[
\alpha(t) := \text{Im} \int_{\Omega^+} \bar{u}(t,x) x. \nabla u(t,x) dx, \quad \beta(t) := \text{Im} \int_{\Omega^+} \bar{u}(t,x) (x - e_1). \nabla u(t,x) dx.
\]

Thus, we have

\[
\frac{d}{dt} \Gamma_1(u(t)) = 8(\alpha(t) - \beta(t)).
\]
Due to the symmetry properties of \( u \), one can see that \( \alpha(t) \) is equal to \( \frac{1}{16} \frac{d}{dt} \mathcal{Y}_2(t) \) and \( \beta(t) \) is equal to (4.1) applied to \( (x - e_1) \), where \( e_1 = (1, 0) \). By (4.2), we obtain,

\[
\frac{d}{dt} \alpha(t) = 4E_+[u] - 2 \left( \frac{d}{2} - \frac{d+2}{p+1} \right) \int_{\Omega^+} |u|^{p+1} dx - \int_{\partial \Omega^+} |\nabla u|^2 (x, \bar{n}) d\sigma(x),
\]

\[
\frac{d}{dt} \beta(t) = 4E_+[u] - 2 \left( \frac{d}{2} - \frac{d+2}{p+1} \right) \int_{\Omega^+} |u|^{p+1} dx - \int_{\partial \Omega^+} |\nabla u|^2 ((x - e_1), \bar{n}) d\sigma(x).
\]

where, \( E_+[u] = \frac{1}{2} \| \nabla u \|_{L^2(\Omega^+)} - \frac{1}{p+1} \int_{\Omega^+} |u|^{p+1} dx \). Taking the difference between these two equalities, we get

\[
\frac{d}{dt} (\alpha(t) - \beta(t)) = - \int_{\partial \Omega^+} |\nabla u|^2 (e_1, \bar{n}) dx = - \int_{\partial \Omega^+} |\nabla u|^2 n_1 dx,
\]

which yields

\[
\frac{d^2}{dt^2} \Gamma_1(u(t)) := -8 \int_{\partial \Omega^+} |\nabla u|^2 n_1 dx.
\]

Using the symmetry properties, the convexity of the obstacle \( \Theta \) and the fact that \( \bar{n} \) is the unit outward normal vector, we have \( n_1 \leq 0 \) on \( \partial \Omega^+ \). Then, we obtain

\[
\frac{d^2}{dt^2} \Gamma_1(u(t)) = 8 \int_{\partial \Omega^+} |\nabla u|^2 |n_1| dx.
\]

Recall that

\[
\Gamma_2 := \int_{\Omega} |x_2| |u(t, x)|^2 dx.
\]

We multiply the equation by \( |x_2| \bar{u} \) and take the imaginary part to get:

\[
\text{Im} \int_{\Omega} i \partial_t u |x_2| \bar{n} dx + \text{Im} \int_{\Omega} \Delta u |x_2| \bar{n} dx = - \text{Im} \int_{\Omega} |u|^{p-1} u |x_2| \bar{n} dx = 0.
\]

Which yields

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |x_2| |u(t, x)|^2 dx = - \text{Im} \int_{\Omega} |x_2| \Delta u \bar{n} dx
\]

\[
\frac{d}{dt} \Gamma_2(u(t)) := \frac{d}{dt} \int_{\Omega} |x_2| |u(t, x)|^2 dx = -2 \text{Im} \int_{\Omega} |x_2| \Delta u \bar{n} dx.
\]

Integration by parts ensures,

\[
\text{Im} \int_{\Omega} |x_2| \Delta u \bar{n} dx = - \text{Im} \int_{\Omega} \partial_{x_2} (|x_2|) \partial_{x_2} u \bar{n} dx
\]

\[
= - \left( \text{Im} \int_{\Omega^+} \partial_{x_2} u \bar{n} dx - \text{Im} \int_{\Omega^-} \partial_{x_2} u \bar{n} dx \right)
\]

\[
= -4 \text{Im} \int_{\partial \Omega^+} \partial_{x_2} u \bar{n} dx,
\]

which yields (5.3). The computation of \( \frac{d^2}{dt^2} \Gamma_2(t) \) is similar to the one of \( \frac{d^2}{dt^2} \Gamma_1(t) \), the only difference is to apply (4.1) and (4.2) to \( (x - e_2) \), where \( e_2 = (0, 1) \) instead of \( (x - e_1) \). Thus, we get

\[
\frac{d^2}{dt^2} \Gamma_2(u(t)) := -8 \int_{\partial \Omega^+} |\nabla u|^2 n_2 dx,
\]
As $\vec{n}$ is the unit outward normal vector, we have $n_2 \leq 0$ on $\partial \Omega^{++}$ using the symmetry properties and the convexity of the obstacle $\Theta$. Then we obtain,

$$\frac{d^2}{dt^2} \Gamma_2(u(t)) := 8 \int_{\partial \Omega^{++}} |\nabla u|^2 |n_2| \, dx.$$ 

This concludes the proof of Proposition 5.1.

**Proof of Theorem 1.5.** Assume $d = 2$ and $p \geq 3$. Let $u_0 \in S_2$ (i.e., $u_0 \in H_0^1(\Omega)$), $u_0(-x_1, x_2) = u_0(x_1, -x_2) = -u_0(x_1, x_2)$, $|x|u_0 \in L^2(\Omega)$ and $E[u_0] < 0$ and let $u$ be the corresponding solution of (NLS$_{\Omega}$) with maximal time interval $I$ of existence. From Proposition 5.1 and 4.1, we deduce the second derivative of the variance for $d = 2$,

\begin{equation}
\frac{d^2}{dt^2} V(u(t)) = 16E[u] - 8 \left( \frac{p-3}{p+1} \right) \int_\Omega |u|^{p+1} \, dx
+ \int_{\partial \Omega^{++}} |\nabla u|^2 \left[ 16|x.\vec{n}| - 8C \left( |n_1| + |n_2| \right) \right] \, d\sigma(x).
\end{equation}

Using the fact that $p \geq 3$ and $E[u] < 0$, one can see that the first two terms are negative, i.e., $16E[u] - 8 \left( \frac{p-3}{p+1} \right) \int_\Omega |u|^{p+1} \, dx \leq 0$. Choosing $C \geq 2 \max_{x \in \partial \Omega} \left( \frac{|x.\vec{n}|}{|n_1| + |n_2|} \right)$ this implies that,

$$\frac{d^2}{dt^2} V(u(t)) \leq -A, \quad \text{where} \quad A > 0.$$ 

Using the same argument as in the first proof, one can prove that the length of the maximal time interval of existence $I$ is finite. Therefore, the solution $u$ blows up in finite time and this concludes the proof of Theorem 1.5 in dimension 2. For any dimension $d \geq 3$, we should suppose that, $u_0 \in S_d$, i.e., $u_0(x_1, \ldots, x_i, \ldots, x_d) = -u_0(x_1, \ldots, -x_i, \ldots, x_d)$, for $i = 1, 2, \ldots, d$ and we use the following variance:

$$V(u(t)) = \int_\Omega \left( |x|^2 - C \sum_{i=1}^d |x_i| + C^2 \right) |u(t, x)|^2 \, dx.$$ 

One can check that

\begin{equation}
\frac{d^2}{dt^2} V(u(t)) = 16E[u] - 8 \left( \frac{d}{2} - \frac{d+2}{p+1} \right) \int_\Omega |u|^{p+1} \, dx
+ \int_{\partial \{x_i \geq 0, 1 \leq i \leq d \}} |\nabla u|^2 \left[ 2^{d+2}|x.\vec{n}| - 2^{d+1}C \sum_{i=1}^d |n_i| \right] \, d\sigma(x).
\end{equation}

Using the fact that $p \geq 1 + \frac{4}{d}$, $E[u] < 0$ and choosing $C$ such that

$$C \geq 2 \max_{x \in \partial \Omega} \left( |x.\vec{n}| \left( \sum_{i=1}^d |n_i| \right)^{-1} \right),$$

we get

$$\frac{d^2}{dt^2} V(u(t)) \leq -A, \quad \text{where} \quad A > 0.$$ 

Then $u$ blows up in finite time for any dimension $d \geq 3$ and this concludes the proof of Theorem 1.5. □
6. Ground state threshold for blow-up

This section is devoted to the proof of Theorem 1.7. First, assume that \( d = 2 \) and \( s_c > 0 \), i.e., \( p > 3 \) and \( \Theta \) is invariant by the symmetry. Let \( u_0 \in S_2 \), \( u_0 \in H^1_0(\Omega) \), \( u_0(-x_1,x_2) = u_0(x_1,-x_2) = -u_0(x_1,x_2) \), and \( |x| u_0 \in L^2(\Omega) \). Let \( u \) be the corresponding solution of the NLS equation with maximal time interval \( I \) of existence. Moreover, we consider the same modified variance as in the proof of Theorem 1.5. Let \( C > 0 \) be a positive constant to be specified later,

\[
V(u(t)) := \int_{\Omega} \left( |x|^2 - C|x_1| - C|x_2| + C^2 \right) |u(t,x)|^2 \, dx
\]

Lemma 6.1. Let \( u_0 \in H^1_0(\Omega) \) satisfy

\[
M[u_0]^{1-s_c} E[u_0]^{s_c} < M[Q]^{1-s_c} E[Q]^{s_c}.
\]

(6.1)

\[
\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} > \|Q\|_{L^2(\mathbb{R}^2)} \|\nabla Q\|_{L^2(\mathbb{R}^2)}^{s_c}.
\]

(6.2)

Then the corresponding solution \( u \) to (NLS) satisfies,

\[
\forall t \in I, \quad \|u_0\|_{L^2(\Omega)} \|\nabla u(t)\|_{L^2(\Omega)} > \|Q\|_{L^2(\mathbb{R}^2)} \|\nabla Q\|_{L^2(\mathbb{R}^2)}^{s_c}.
\]

(6.3)

Proof. The proof of the lemma is the same as in [10], [11] for the proof of blow-up solutions of (NLS) equation on \( \mathbb{R}^d \). We give it for the sake of completeness and for the convenience of the reader. The key point is that a function \( f \in H^1_0(\Omega) \) extended by 0 outside \( \Omega \) can be identified to an element of \( H^1(\mathbb{R}^2) \). Thus, it satisfies the same Gagliardo-Nirenberg inequality as an element of \( H^1(\mathbb{R}^2) \).

Multiplying the energy by \( M[u]^{1-s_c} \) and applying Gagliardo-Nirenberg’s inequality for \( d = 2 \), we have,

\[
M[u]^{1-s_c} E[u] = \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 \|u\|_{L^{2(1+s_c)}(\Omega)}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\Omega)}^{p+1} \|u\|_{L^{2(1+s_c)}(\Omega)}^{2(1+s_c)}
\]

\[
\geq \frac{1}{2} \left( \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^{2(1+s_c)}(\Omega)} \right)^2 - \frac{C_{GN}}{p+1} \left( \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^{2(1+s_c)}(\Omega)} \right)^{p+1}
\]

\[
\geq f \left( \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^{2(1+s_c)}(\Omega)} \right),
\]

where \( f(x) = \frac{1}{2} x^2 - \frac{C_{GN}}{p+1} x^{p+1} \). Then, \( f'(x) = x - \frac{C_{GN}(p-1)}{p+1} x^{p-2} \), and thus, \( f'(x) = 0 \) for \( x_0 = 0 \) and \( x_1 = \left( \frac{C_{GN}(p-1)}{p+1} \right)^{-\frac{1}{p-2}} = \|\nabla Q\|_{L^2(\mathbb{R}^2)} \|Q\|_{L^{2(1+s_c)}(\mathbb{R}^2)} \) by (2.4). Since (2.1) is attained at ground state \( Q \) then we have, \( f(\|\nabla Q\|_{L^2(\mathbb{R}^2)} \|Q\|_{L^{2(1+s_c)}(\mathbb{R}^2)}) = M[Q]^{1-s_c} E[Q] \), we also have \( f(0) = 0 \). Thus, the function \( f \) is increasing on \((0,x_1)\) and decreasing on \((x_1,\infty)\). Using the energy conservation, we get

\[
f \left( \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^{2(1+s_c)}(\Omega)} \right) \leq M[u]^{1-s_c} E[u(t)] < f(x_1).
\]

(6.4)

If condition (6.2) holds, i.e \( \|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} > x_1 = \|Q\|_{L^2(\mathbb{R}^2)} \|\nabla Q\|_{L^2(\mathbb{R}^2)}^{s_c} \), then by (6.4) and the continuity of \( \|\nabla u(t)\|_{L^2(\Omega)} \) in time we obtain (6.3) for all time \( t \in I \). \( \square \)
Moreover, if the conditions (6.1) and (6.2) holds, then there exists $\delta_1 > 0$ such that

\begin{equation}
M[u_0]^{1-s_c} E[u_0]^{s_c} < (1 - \delta_1) M[Q]^{1-s_c} E[Q]^{s_c}.
\end{equation}

Thus, there exists $\delta_2 := \delta_2(\delta_1) > 0$ such that

\begin{equation}
\forall t \in I, \quad \|u_0\|^{1-s_c}_{L^2(\Omega)} \|\nabla u(t)\|^{s_c}_{L^2(\Omega)} > (1 + \delta_2) \|Q\|^{1-s_c}_{L^2(\mathbb{R}^2)} \|\nabla Q\|^{s_c}_{L^2(\mathbb{R}^2)}.
\end{equation}

Now let us prove that

\begin{equation}
\frac{d^2}{dt^2} V(u(t)) \leq 8(p-1) E[u] - 4(p-3) \|\nabla u\|_{L^2(\Omega)}^2 + \int_{\partial\Omega^+} |\nabla u|^2 [16|x.\bar{n}| - 8C (|n_1| + |n_2|)] d\sigma(x).
\end{equation}

From (5.5), we have

\[
\frac{d^2}{dt^2} V(u(t)) = 16 E[u] - 8 \left( \frac{p-3}{p+1} \right) \int_{\Omega} |u|^{p+1} dx + \int_{\partial\Omega^+} |\nabla u|^2 [16|x.\bar{n}| - 8C (|n_1| + |n_2|)] d\sigma(x)
\leq 8 \|\nabla u\|_{L^2}^2 - \frac{8(p-1)}{p+1} \int_{\Omega} |u|^{p+1} dx + \int_{\partial\Omega^+} |\nabla u|^2 [16|x.\bar{n}| - 8C (|n_1| + |n_2|)] d\sigma(x)
\leq 8(p-1) E[u] - 4(p-3) \|\nabla u\|_{L^2(\Omega)}^2 + \int_{\partial\Omega^+} |\nabla u|^2 [16|x.\bar{n}| - 8C (|n_1| + |n_2|)] d\sigma(x).
\]

Multiplying (6.7) by $M[u]^{1-s_c}$ and using (2.5) for $d = 2$ with the two refined inequalities (6.5) and (6.6), we have

\[
M[u]^{1-s_c} \frac{d^2}{dt^2} V(u(t)) \leq \left( 8(p-1) E[u] - 4(p-3) \|\nabla u\|_{L^2(\Omega)}^2 \right) M[u]^{1-s_c}
+ \int_{\partial\Omega^+} |\nabla u|^2 [16|x.\bar{n}| - 8C (|n_1| + |n_2|)] d\sigma(x) M[u]^{1-s_c}
\leq 8(p-1)(1-\delta_1) E[Q] M[Q]^{1-s_c} - 4(p-3)(1+\delta_2) \|\nabla Q\|_{L^2(\mathbb{R}^2)}^2 M[Q]^{1-s_c}
+ \int_{\partial\Omega^+} |\nabla u|^2 [16|x.\bar{n}| - 8C (|n_1| + |n_2|)] d\sigma(x) M[u]^{1-s_c}
\leq 8(p-1)(1-\delta_1) \frac{(p-3)}{2(p+1)} \|\nabla Q\|_{L^2(\mathbb{R}^2)}^2 M[Q]^{1-s_c}
- 4(p-3)(1+\delta_2) \|\nabla Q\|_{L^2(\mathbb{R}^2)}^2 M[Q]^{1-s_c}
+ \int_{\partial\Omega^+} |\nabla u|^2 [16|x.\bar{n}| - 8C (|n_1| + |n_2|)] d\sigma(x) M[u]^{1-s_c},
\]

which yields

\[
M[u]^{1-s_c} \frac{d^2}{dt^2} V(u(t)) \leq \left[ 4(p-3) - 4(p-3) \right] \|\nabla Q\|_{L^2(\mathbb{R}^2)}^2 M[Q]^{1-s_c}
- \left[ 4(p-3)\delta_1 + 4(p-3)\delta_2 \right] \|\nabla Q\|_{L^2(\mathbb{R}^2)}^2 M[Q]^{1-s_c}
+ \int_{\partial\Omega^+} |\nabla u|^2 [16|x.\bar{n}| - 8C (|n_1| + |n_2|)] d\sigma(x) M[u]^{1-s_c}.
\]
Using the fact that \( p > 3 \) and choosing \( C \geq 2 \max_{x \in \partial \Omega} \left( \frac{|x, n|}{|n_1| + |n_2|} \right) \) imply that the second derivative of the variance is bounded by a negative constant, for all \( t \in I \),

\[
\frac{d^2}{dt^2} V(u(t)) \leq -A, \quad \text{where } A > 0.
\]

Thus the maximal time interval of existence \( I \) is finite and the solution \( u \) blows up in finite time. This concludes the proof of Theorem 1.7 in dimension 2.

Next, we will give the proof for dimension \( d \geq 3 \). For that, we suppose that \( u_0 \in \mathcal{S}_d \), i.e.,

\[
u_0(x_1, \ldots, x_i, \ldots x_d) = -u_0(x_1, \ldots, x_i, \ldots, x_d), \quad \text{for } i = 1, 2, \ldots, d.
\]

Using the following variance

\[
V(u(t)) = \int_{\Omega} \left( |x|^2 - C \sum_{i=1}^{d} |x_i| + C^2 \right) |u(t, x)|^2 \, dx,
\]

we have,

\[
(6.8) \quad \frac{d^2}{dt^2} V(u(t)) = 4d(p - 1) E[u] - (2d(p - 1) - 8) \| \nabla u \|^2_{L^2(\Omega)}
\]

\[
+ \int_{\partial \{x_i \geq 0, 1 \leq i \leq d \}} |\nabla u|^2 \left[ 2^{d+2}|x, n| - 2^{d+1}C \sum_{i=1}^{d} |n_i| \right] d\sigma(x).
\]

Using the same argument as above, one can check that Lemma 6.1 remains true for \( d \geq 3 \), see [10], [9]. If the conditions (6.1) and (6.2) hold then there exists \( \delta_1 > 0, \delta_2(\delta_1) > 0 \) such that (6.5) and (6.6) are valid for \( d \geq 3 \). Multiplying (6.8) by \( M[u]^{1-\frac{4c}{sc}} \) and using (2.5) with the two refined inequalities (6.5) and (6.6), we have

\[
M[u]^{\frac{1-4c}{sc}} \frac{d^2}{dt^2} V(u(t)) \leq 4d(p - 1)(1 - \delta_1) E[Q] M[Q]^{\frac{1-4c}{sc}}
\]

\[
- (2d(p - 1) - 8)(1 + \delta_2) \| \nabla Q \|^2_{L^2(\mathbb{R}^d)} M[Q]^{\frac{1-4c}{sc}}
\]

\[
+ \int_{\partial \{x_i \geq 0, 1 \leq i \leq d \}} |\nabla u|^2 \left[ 2^{d+2}|x, n| - 2^{d+1}C \sum_{i=1}^{d} |n_i| \right] d\sigma(x) M[u]^{\frac{1-4c}{sc}}
\]

\[
\leq (2d(p - 1) - 8)(1 - \delta_1) \| \nabla Q \|^2_{L^2(\mathbb{R}^d)} M[Q]^{\frac{1-4c}{sc}}
\]

\[
- (2d(p - 1) - 8)(1 + \delta_2) \| \nabla Q \|^2_{L^2(\mathbb{R}^d)} M[Q]^{\frac{1-4c}{sc}}
\]

\[
+ \int_{\partial \{x_i \geq 0, 1 \leq i \leq d \}} |\nabla u|^2 \left[ 2^{d+2}|x, n| - 2^{d+1}C \sum_{i=1}^{d} |n_i| \right] d\sigma(x) M[u]^{\frac{1-4c}{sc}},
\]

which yields

\[
M[u]^{\frac{1-4c}{sc}} \frac{d^2}{dt^2} V(u(t)) \leq -\left( (2d(p - 1) - 8)\delta_1 + (2d(p - 1) - 8)\delta_2 \right) \| \nabla Q \|^2_{L^2(\mathbb{R}^d)} M[Q]^{\frac{1-4c}{sc}}
\]

\[
+ \int_{\partial \{x_i \geq 0, 1 \leq i \leq d \}} |\nabla u|^2 \left[ 2^{d+2}|x, n| - 2^{d+1}C \sum_{i=1}^{d} |n_i| \right] d\sigma(x) M[u]^{\frac{1-4c}{sc}}.
\]
Thus
\[
\frac{d^2}{dt^2} V(u(t)) \leq -A, \quad \text{where} \quad A > 0.
\]
Provided \(p > 1 + \frac{4}{d}\) and \(C \geq 2 \max_{x \in \partial \Omega} \left( \frac{1}{|x.\vec{n}|} \left( \sum_{i=1}^{d} |n_i| \right)^{-1} \right)\). Then the solution \(u\) blows up in finite time and this concludes the proof of Theorem 1.7.

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*Email address*, Oussama Landoulsi: landoulsi@math.univ-paris13.fr
