The $N = 2$ supersymmetric matrix GNLS hierarchies

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Abstract

We construct the matrix generalization of the $N = 2$ supersymmetric GNLS hierarchies. This is done by exhibiting the corresponding matrix super Lax operators in terms of $N = 2$ superfields in two different superfield bases. We present the second Hamiltonian structure and discrete symmetries. We then extend our discussion by conjecturing the Lax operators of different reductions of the $N = 2$ supersymmetric matrix KP hierarchy and discuss the simplest examples.

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1. **Introduction.** The $N = 2$ supersymmetric $(n, m)$–Generalized Nonlinear Schrödinger (GNLS) hierarchies were introduced in [1]. We recall that they form a very large family of $N = 2$ supersymmetric hierarchies, one out three known families of $N = 2$ supersymmetric hierarchies with $N = 2$ W's second Hamiltonian structure. The $N = 2$ $(n, m)$–GNLS hierarchies were subsequently studied in a number of papers, [2–9]. In particular their discrete symmetries were derived in [6], and the Hamiltonian structures and recursion operator were constructed in [7] in two different superfield bases with local evolution equations. The $N = 2$ $(n, m)$–GNLS hierarchy involves $n + m$ pairs of chiral and antichiral $N = 2$ superfields, $n$ pairs of them being bosonic and $m$ pairs fermionic. In order to define the matrix generalization, we combine them into a single row and a single column of $(n + m)$ length. The aim of the present letter is to generalize the $N = 2$ $(n, m)$–GNLS hierarchy to the case when the row and column are replaced by a rectangular matrix of an arbitrary $(k \times (n + m))$–size and its transposed matrix, respectively. It appears that such integrable generalization actually exists, and many results of Refs. [3, 4] concerning the $N = 2$ super $(n, m)$–GNLS hierarchy can be straightforwardly extended to this case. This permits to present here the main facts concerning the new hierarchy, called $N = 2$ supersymmetric $(k|n, m)$ matrix GNLS (MGNLS) hierarchy, in a telegraphic style and refer the reader to Refs. [1, 6, 7] for more details. In section 2 we introduce the new matrix hierarchies by means of their Lax operators. In section 3 we present their Hamiltonian structures. In section 4 we write the same hierarchies in a new basis, the so–called KdV basis, while in section 5 we discuss the discrete symmetries of these hierarchies. Section 6 is devoted to a generalization: we start from the $N = 2$ matrix KP hierarchy (which contains an infinite number of fields) and conjecture the existence of an infinite family of reductions (with a finite number of fields), which includes in particular the $N = 2$ matrix GNLS hierarchies as well as almost all known $N = 2$ scalar KP reductions.

2. **The $N = 2$ super $(k|n, m)$–MGNLS hierarchy.** The Lax operator of the $N = 2$ supersymmetric MGNLS hierarchies has the following form

$$L = I\partial - \frac{1}{2}(FF + F\bar{D}\partial^{-1}[DF]), \quad [D, L] = 0. \quad (1)$$

Here, $F \equiv F_{Aa}(Z)$ and $\bar{F} \equiv \bar{F}_{aA}(Z)$ ($A, B = 1, \ldots, k$; $a, b = 1, \ldots, n + m$) are chiral and antichiral rectangular matrix-valued $N = 2$ superfields,

$$DF = 0, \quad \bar{D} \bar{F} = 0, \quad (2)$$

respectively. In [1] the matrix product is understood, for example $(F\bar{F})_{AB} \equiv \sum_{a} F_{Aa}\bar{F}_{aB}$. Moreover the square brackets mean that the relevant operators act only on the superfields inside the brackets, and $I$ is the unity matrix, $I \equiv \delta_{A,B}$. The matrix entries are bosonic superfields for $a = 1, \ldots, n$ and fermionic superfields for $a = n + 1, \ldots, n + m$, i.e., $F_{Aa}\bar{F}_{bB} = (-1)^{d_{a}\bar{d}_{b}}F_{aB}F_{bA}$, where $d_{a}$ and $\bar{d}_{b}$ are the Grassmann parities of the matrix elements $F_{Aa}$ and $\bar{F}_{bB}$, respectively, $d_{a} = 1 \ (d_{a} = 0)$ for fermionic (bosonic) entries; $Z = (z, \theta, \bar{\theta})$ is a coordinate of the $N = 2$ superspace, $dZ \equiv dzd\theta d\bar{\theta}$ and $D, \bar{D}$ are the $N = 2$ supersymmetric fermionic covariant derivatives

$$D = \frac{\partial}{\partial \theta} - \frac{1}{2}\bar{\theta}\frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2}\theta \frac{\partial}{\partial \bar{z}}, \quad D^{2} = \bar{D}^{2} = 0, \quad \{D, \bar{D}\} = -\frac{\partial}{\partial z} \equiv -\partial. \quad (3)$$

Let us stress that the chosen grading guarantees that the Lax operator $L$ is Grassman even, and it is consistent with the important property that $L$ commutes with fermionic derivative $D$ (see, eqs. (1)).
For \( p = 0, 1, 2, \ldots \), the Lax operator \( L \) provides the consistent flows

\[
\frac{\partial}{\partial t_p} L = [A_p, L], \quad A_p = (L^p)_{\geq 1}.
\]  

(4)

For the particular case \( k = 1 \), the Lax operator \( (4) \) coincides with the scalar Lax operator of the \( N = 2 \) super \((n,m)\)-GNLS hierarchy while for \( k \geq 2 \) it defines the new \( N = 2 \) super \((k|n,m)\)-MGNLS hierarchy. Like in the scalar case, the infinite number of Hamiltonians can be obtained as follows:

\[
H_p = \int dZ H_p, \quad H_p \equiv tr(L^p)_0,
\]  

(5)

where the subscripts \( \geq 1 \) and 0 denote the sum of the purely derivative terms and the constant part of the operator, respectively, and \( tr \) means the usual matrix trace. The evolution equations for the \( F \) and \( \overline{F} \) derived from (4) admit the involution

\[
F^* = i\overline{F}^T I, \quad \overline{F}^* = iF^T, \quad \theta^* = \overline{\theta}, \quad t^* = (-1)^{p+1} t_p, \quad z^* = z, \quad i^* = -i,
\]  

(6)

where \( i \) is the imaginary unit, the symbol \( T \) means the operation of matrix transposition, and the matrix \( I \) is

\[
I \equiv (-i)^{da} \delta_{ab},
\]  

(7)

with the properties

\[
II^* = I, \quad I^3 = I^*, \quad I^2 = (-1)^{da} \delta_{ab}.
\]  

(8)

The formulae (6)–(8) provide the following relation

\[
(F\overline{F})^* = -(F\overline{F})^T,
\]  

(9)

which is exploited in what follows. The origin of the involution \( (6) \) is explained in the next section.

The flows (4) are local, and can be represented in the following form:

\[
\frac{\partial}{\partial t_p} F = ((L^p)_{\geq 1} F)_{0}, \quad \frac{\partial}{\partial t_p} \overline{F} = (-1)^{p+1} (\overline{F}(L^p)^{\leq 1})_{0},
\]  

(10)

where the arrows mean that the corresponding operators act on the left and \( \dagger \) denotes hermiten conjugation, i.e. involution plus transposition,

\[
\overleftarrow{L}^\dagger \equiv (\overleftarrow{L}^*)^T = I \overleftarrow{\partial} + \frac{1}{2}(F\overline{F} + [\overline{D}F] \overleftarrow{D} \overleftarrow{\partial}^{-1} F), \quad [\overleftarrow{D}, \overleftarrow{L}] = 0.
\]  

(11)

This Lax operator also provides consistent flows.

The first three flows from (10) and the first three nontrivial Hamiltonian densities from (5) are:

\[
\frac{\partial}{\partial t_0} F = F, \quad \frac{\partial}{\partial t_0} \overline{F} = -\overline{F}; \quad \frac{\partial}{\partial t_1} F = F'; \quad \frac{\partial}{\partial t_1} \overline{F} = \overline{F}';
\]

\[
\frac{\partial}{\partial t_2} F = F'' + D(F\overline{F} \overleftarrow{DF}), \quad \frac{\partial}{\partial t_2} \overline{F} = -\overline{F}'' + \overleftarrow{D}(\overleftarrow{D}F \overleftarrow{F}),
\]  

(12)
\[ H_1 = \frac{1}{2} \text{tr}(FF), \quad H_2 = \frac{1}{2} \text{tr}(FF') + \frac{1}{4} (FF')^2, \]
\[ H_3 = \frac{1}{2} \text{tr}(FF'') - \frac{1}{2} [DFF] [DFF] + FFF'F + \frac{1}{12} (FF')^3, \] (13)

respectively, where \( ' \) means the derivative with respect to \( z \).

Now we consider the bosonic limit of the second flow equations (12) both for the case of pure fermionic \((n = 0)\) and pure bosonic \((m = 0)\) matrices \(F, F'\), and establish their relations with the \( gl(k + m)/(gl(k) \times gl(m)) \) bosonic matrix NLS equations introduced in [10].

To derive the bosonic limit, let us define the matrix components of the fermionic superfield matrices
\[ f = \overline{DF}, \quad \overline{f} = D\overline{F}, \quad \psi = F, \quad \overline{\psi} = \overline{F}, \] (14)
and the matrix components of the bosonic superfield matrices
\[ \xi = \overline{DF}, \quad \overline{\xi} = D\overline{F}, \quad b = F, \quad \overline{b} = \overline{F}, \] (15)
where \(|\) means the \((\theta, \bar{\theta}) \to 0\) limit. So, \(\psi, \overline{\psi}, \xi, \overline{\xi}\) are fermionic matrix components while \(f, \overline{f}, b, \overline{b}\) are bosonic ones. To get the bosonic limit we have to put all the fermionic matrices \(\psi, \overline{\psi}, \xi, \overline{\xi}\) to zero. This leaves us with the following set of matrix equations
\[
\frac{\partial}{\partial t_2} f = f'' - f\overline{f}f, \quad \frac{\partial}{\partial t_2} \overline{f} = -\overline{f}'' + \overline{f}f\overline{f}, \] (16)
\[
\frac{\partial}{\partial t_2} b = b'' - b\overline{b}b', \quad \frac{\partial}{\partial t_2} \overline{b} = -\overline{b}'' - \overline{b}'b\overline{b} \] (17)
for the bosonic matrix components. The set of equations (16) form the bosonic matrix NLS equations which can be produced via \( gl(k + m)/(gl(k) \times gl(m)) \)-coset construction [10]. They can be viewed as the second flow of the bosonic matrix NLS hierarchies with the Lax operators \( L_1 \)
\[ L_1 = I\partial - \frac{1}{2} f\partial^{-1}\overline{f}, \quad \frac{\partial}{\partial t_2} L_1 = [(L_1^p)_{\geq 0}, L]. \] (18)
The latter can be easily derived from the Lax operator (11) in the bosonic limit[1]. In the same way one can derive the Lax operator for the equations (17),
\[ L_2 = (I - \frac{1}{2} b\partial^{-1}\overline{b})\partial, \] (19)
which generates the hierarchy which we call modified matrix NLS hierarchy.

Thus we can conclude that the \( N = 2 \) supersymmetric \((k|n, m)\)-MGNLS hierarchies with Lax operators (11) are the \( N = 2 \) supersymmetric extensions of the bosonic matrix NLS and modified matrix NLS hierarchies. In a companion paper [13] we show that the \( N = 2 \) \((k|n, m)\)-MGNLS hierarchies can be obtained via a suitable coset construction applied to the \( sl(s|s-1) \) \( N = 2 \) affine superalgebras.

\[ ^1 \text{For more details concerning bosonic matrix NLS equations, see [11, 12] and references therein.} \]
3. Hamiltonian structure of the $N = 2$ super $(k|n,m)$-MGNLS hierarchy. The system of evolution equations ([12] is Hamiltonian and can be represented as

$$\frac{\partial}{\partial t_p} \left( \frac{F_{Aa}}{F_{aA}} \right) = (J_2)_{Aa,Bb} \left( \frac{\delta/\delta F_{Bb}}{\delta/\delta F_{bB}} \right) H_p,$$

where summation over repeated indices is understood, and $J_2$ is the second Hamiltonian structures, which has the following form:

$$(J_2)_{Aa,Bb} = \begin{pmatrix} (J_{11})_{Aa,Bb}, & (J_{12})_{Aa,Bb} \\ (J_{21})_{Aa,Bb}, & (J_{22})_{Aa,Bb} \end{pmatrix},$$

$$(J_{11})_{Aa,Bb} = (-1)^{d_a}d_b F_{Ab} D \partial^\dagger D \partial (F_{Ba} - F_{Ba} D \partial^\dagger D \partial (F_{Ab} - F_{Ab}),$$

$$(J_{12})_{Aa,Bb} = (-1)^{d_b}(2 D D \partial_\delta D_F - F_{Ac} D D \partial^\dagger D_F \delta_{ab} + F_{Ca} D D \partial^\dagger D_F \delta_{AB}),$$

$$(J_{21})_{Aa,Bb} = (2 D D \partial_\delta D_F + (-1)^{d_b} F_{Ac} D D \partial^\dagger D_F \delta_{ab} - F_{Ca} D D \partial^\dagger D_F \delta_{AB}),$$

$$(J_{22})_{Aa,Bb} = F_{aB} D D \partial^\dagger D_F b_A - (-1)^{d_a}d_b F_{bA} D D \partial^\dagger D_F a_B. \quad (21)$$

This formula should be expressible in a more compact way by using the $r$–matrix language, but we postpone this development to another occasion.

In terms of $J_2$, the $N = 2$ supersymmetric Poisson brackets algebra of the matrices $F$ and $\overline{F}$ are given by the formula:

$$\{ \left( \frac{F_{Aa}}{F_{aA}}(Z_1) \right), \left( \frac{F_{Bb}}{F_{bB}}(Z_2) \right) \} = (J_2)_{Aa,Bb}(Z_1) \delta^{N=2}(Z_1 - Z_2), \quad (22)$$

where $\delta^{N=2}(Z) \equiv \theta \overline{\theta} \delta(z)$ is the delta function in the $N = 2$ superspace and the notation ‘$\otimes$’ stands for the tensor product. $J_2$ satisfies the Jacobi identities and the symmetry properties related to the statistics of the matrix entries. In addition $J_2$ also satisfies the chiral consistency conditions

$$J_2 \Pi = \Pi J_2 = 0, \quad J_2 \overline{\Pi} = \overline{\Pi} J_2 = J_2, \quad (23)$$

where we introduced the matrices $\Pi$ and $\overline{\Pi}$

$$\Pi \equiv - \begin{pmatrix} D D \partial^\dagger D_F \delta_{ab} & 0 \\ 0 & D D \partial^\dagger D_F \delta_{ab} \end{pmatrix}, \quad \overline{\Pi} \equiv - \begin{pmatrix} D D \partial^\dagger D_F \delta_{AB} & 0 \\ 0 & D D \partial^\dagger D_F \delta_{AB} \end{pmatrix},$$

$$\Pi \overline{\Pi} = \Pi, \quad \Pi \overline{\Pi} = \Pi, \quad \Pi \Pi = \Pi = 0, \quad \Pi + \overline{\Pi} = I \quad (24)$$

which project on the chiral/antichiral and antichiral/chiral subspaces, respectively. Eqs.(23) show that the second Hamiltonian structure $J_2$ is represented by a degenerate matrix. One should stress that this is not a pathology of the Hamiltonian structure but a peculiarity of the $N = 2$ superfield description, which can be easily dealt with, see [7].

Let us turn now to the involution properties announced in the previous section. Under the action of the involution ([8] the Poisson brackets algebra ([21]) ([22]) change the overall sign while the Hamiltonians $H_p$ ([13]) transform as

$$H_p^* = (-1)^p H_p, \quad (25)$$

as one can check. This shows that the Hamiltonian system ([24]) of the $N = 2 (k|n,m)$-MGNLS flows is invariant under the involution ([14], as announced above.
Let us note that besides the Hamiltonians $H_p$ (13), which are in involution with respect to the Poisson structure (21)–(22), there are integrals of the flows (11), (20), which form a non-abelian algebra. Some of them are matrix-valued local integrals, for example,

$$H_{1,ab} = \int dZ (TF)_{ab}. \quad (26)$$

For the flows (12), this can be checked by direct calculations. Using the algebra (23) one can calculate the Poisson brackets between them and in this way generate new integrals. By repeatedly applying the same procedure, one can produce new series of both local and nonlocal matrix-valued integrals, [7].

4. The $N = 2$ super $(k|n,m)$-MGNLS hierarchy in the KdV-basis. One can rewrite the Lax operator (1) in a slightly different, but equivalent, form,

$$L = I\partial - \frac{1}{2} (B\overline{B} + B\overline{D} \partial^{-1} [D\overline{B}] + S\overline{S} + S\overline{D} \partial^{-1} [D\overline{S}]), \quad (27)$$

where $B \equiv B_{AC}$ and $\overline{B} \equiv \overline{B}_{AC}$ are purely bosonic or purely fermionic square-matrix extracted from the initial matrices $F$ and $\overline{F}$, respectively, and $S \equiv F - B$ and $\overline{S} \equiv \overline{F} - \overline{B}$. So $B$ and $\overline{B}$ are $k \times k$ matrices, while $S$ ($\overline{S}$) is a $k \times (n + m - k)$ ($(n + m - k) \times k$) matrix. This can always be done for the bosonic (fermionic) matrices $B$ and $\overline{B}$ provided $n \geq k$ ($m \geq k$). Actually there is an ambiguity, for instance, in the choice of the $k$ columns among $n$ columns (when $k < n$) that form the matrix $B$. However such ambiguity is irrelevant as it corresponds to an internal symmetry of the Lax operator. In the following let us consider the two cases $n \geq k$ and $m \geq k$ separately.

We start with the case when the matrices $B$ and $\overline{B}$ are bosonic ones. We call them $B$ and $\overline{B}$, respectively. We apply the gauge transformation

$$L^{KdV} = B^{-1}LB, \quad A^{KdV}_p = B^{-1}A_pB - B^{-1}\frac{\partial}{\partial t_p}B, \quad \frac{\partial}{\partial t_p}L^{KdV} = [A^{KdV}_p, L^{KdV}] \quad (28)$$

and substitute the $t_p$-derivative of $B$ obtained from (11) into (28). Then, introducing the new basis $\{J = J_{AC}, \Phi = \Phi_{aA}, \overline{\Phi} = \overline{\Phi}_{aA}; A = 1, \ldots, k; a = n - k, \ldots, n, \ldots, n + m\}$

$$J^T = \frac{1}{2}B^{-1}(\frac{1}{2}B\overline{B} + \frac{1}{2}S\overline{S} - \partial B), \quad \Phi^T = \frac{1}{\sqrt{2}}D(\overline{S}B), \quad \overline{\Phi}^T = \frac{1}{\sqrt{2}}\overline{D}(B^{-1}S), \quad (29)$$

and making obvious algebraic manipulations in the result, we obtain the following explicit expressions for the operators $L^{KdV}$ and $A^{KdV}_p$:

$$L^{KdV} = I\partial - 2J^T - 2\overline{D} \partial^{-1} \left[ D \left( J - \frac{1}{2} (\Phi \partial^{-1} \overline{\Phi})^T \right) \right] + \left[ D \partial^{-1} \Phi^T \right] \overline{D} \partial^{-1} \Phi^T,$$$$

A^{KdV}_p = (L^{KdV}_p)_{\geq 1}. \quad (30)$$

The flows (12) and Hamiltonian densities (13) now become

$$\frac{\partial}{\partial t_0} J = \frac{\partial}{\partial s_0} \overline{\Phi} = \frac{\partial}{\partial s_0} \Phi = 0; \quad \frac{\partial}{\partial t_1} J = J', \quad \frac{\partial}{\partial t_1} \overline{\Phi} = \overline{\Phi}', \quad \frac{\partial}{\partial t_1} \Phi = \Phi',$$

$$\frac{\partial}{\partial t_2} J = (-[D, \overline{D}] J - 2J^2 + \Phi \overline{\Phi})' - 2[J, [D, \overline{D}] J - \Phi \overline{\Phi}],$$

$$\frac{\partial}{\partial t_2} \Phi = -\Phi'' + 4D\overline{D}(J \Phi), \quad \frac{\partial}{\partial t_2} \overline{\Phi} = \overline{\Phi}'' + 4\overline{D}D(\overline{\Phi} J), \quad (31)$$
\[ \mathcal{H}_1 = -2 \text{tr}(J), \quad \mathcal{H}_2 = \text{tr}(2J^2 - \Phi \overline{\Phi}), \quad \mathcal{H}_3 = \text{tr}(\Phi' \overline{\Phi} + 4J \Phi \overline{\Phi} - 4\overline{\Phi}JDJ - \frac{8}{3}J^3), \]  

(32)

respectively, where the brackets \([,]\) represent the commutator. In addition to the first involution (4), hidden in this basis, they admit an extra, second involution

\[ \Phi^* = \overline{\Phi}^T \mathcal{T}, \quad \overline{\Phi}^* = \mathcal{T} \Phi^T, \quad J^* = -J^T, \quad \theta^* = \overline{\theta}, \quad \overline{\theta}^* = \theta, \quad t^*_p = (-1)^{p+1}t_p, \quad z^* = z, \]  

(33)

which is manifest in this basis, but is hidden in the former one. We call the basis (29) a KdV-basis, because in the scalar case it coincides with the KdV-basis introduced in (4). In the KdV-basis, the \( N = 2 (k|n,m) \)-MGNLS hierarchy of integrable equations, together with its Hamiltonians, can be calculated using formulas (4), (5), where the Lax operator \( L \) is replaced by the gauge related Lax operator \( L^{\text{KdV}} \) (30).

The second Hamiltonian structures \( J_{2}^{\text{KdV}} \) in the KdV-basis are related to \( J_2 \) (21) by the general rule

\[ J_{2}^{\text{KdV}} = \mathcal{G} J_2 \mathcal{G}^T, \]  

(34)

where \( \mathcal{G} \) is the matrix of Fréchet derivatives corresponding to the transformation \{ \( \mathcal{B}, \overline{\mathcal{B}}, \mathcal{S}, \overline{\mathcal{S}} \} \Rightarrow \{ J, \overline{\Phi}, \Phi \} \) (29) to the KdV-basis. The calculation of \( J_{2}^{\text{KdV}} \) via formula (34) is a simple exercise and we do not reproduce it here.

Now, let us consider the second case, i.e., when the matrices \( B \) and \( \overline{B} \) are fermionic ones. We relabel them \( \mathcal{F} \) and \( \overline{\mathcal{F}} \), respectively. Then introducing the new basis \{ \( J \equiv J_{AC}, \Phi \equiv \Phi_{Aa}, \overline{\Phi} \equiv \overline{\Phi}_{aA}; A = 1, \ldots, k; a = 1, \ldots, n, n+1, \ldots, n+m-k \) \}

\[ J = -\frac{1}{2} [D \mathcal{F}] (\frac{1}{2} \mathcal{F} \mathcal{F} + \frac{1}{2} \mathcal{S} \overline{\mathcal{S}} - \partial) [D \mathcal{F}]^{-1}, \quad \Phi = \frac{1}{\sqrt{2}} D(\mathcal{F} \mathcal{S}), \quad \overline{\Phi} = \frac{1}{\sqrt{2}} D(\overline{\mathcal{S}} [D \mathcal{F}]^{-1}), \]  

(35)

and applying the gauge transformation

\[ L^{\text{KdV}} = [D \mathcal{F}] L [D \mathcal{F}]^{-1} \equiv \partial + 2J - 2 \left[ D(J - \frac{1}{2}\Phi \partial^{-1} \overline{\Phi}) \right] \overline{\partial} \partial^{-1} + \Phi \overline{\partial} \partial^{-1} \left[ D \partial^{-1} \overline{\Phi} \right] \]  

(36)

one sees that (13) coincides with eqs. (31), up to change of sign of \( t_2 \), and admit the involution (33). Thus, in addition to the transformation (29), relating eqs. (31) to (12), there exists the transformation (33), which relate them up to the \( t_2 \)-sign. Actually, there are two more sets of such transformations which can be derived by applying in consecutive order the involutions (4) and (33) to the transformations (29) and (33),

\[ J^T = \frac{1}{2} \mathcal{B}(\frac{1}{2} \mathcal{B} \mathcal{B} + \frac{1}{2} \overline{\mathcal{S}} \mathcal{S} \overline{\partial} \mathcal{B})^{-1}, \quad \Phi^T = -i \sqrt{2} D(\mathcal{S} \mathcal{B}^{-1}), \quad \overline{\Phi}^T = i \sqrt{2} \overline{D}(\overline{\mathcal{B}} \mathcal{S}), \]  

(37)

\[ J = -\frac{1}{2} [D \mathcal{F}]^{-1} (\frac{1}{2} \mathcal{F} \mathcal{F} + \frac{1}{2} \overline{\mathcal{S}} \mathcal{S} \overline{\partial} \mathcal{F}) [D \mathcal{F}], \quad \Phi = i \sqrt{2} D([D \mathcal{F}]^{-1} \mathcal{S}), \quad \overline{\Phi} = i \sqrt{2} \overline{D}(\overline{\mathcal{F}}^2 \mathcal{S} \mathcal{F}), \]  

(38)

\(^2\)Let us recall that Hamiltonian densities are defined up to terms which are fermionic or bosonic total derivatives of arbitrary nonsingular, local functions of the superfield matrices.

\(^3\)Let us recall the rules for the adjoint conjugation operation \( T \): \( D^T = -D, \overline{D}^T = -\overline{D}, (QP)^T = (-1)^{dp^T} P^T Q^T \), where \( Q \) and \( P \) are arbitrary operators. For matrices, this operation means the matrix transposition. All other rules can be derived using these.
respective.

5. Discrete symmetries of the $N = 2$ super $(k|n,m)$-MGNLS hierarchy. Following the method developed in [2, 3] and using the results of section 4 one can easily derive the discrete symmetries of the $N = 2$ $(k|n,m)$-MGNLS hierarchy. We recall that the discrete symmetries of any integrable system represent as a rule integrable lattice dynamical systems [4, 5, 6]. Without going into details, let us present only the results. We refer to the Lax operator (27) and quote a basic proposition. If the matrices $\{B_j, \overline{B}_j, S_j, \overline{S}_j\}$ labeled by index $j$ ($j \in \mathbb{Z}$) at some given $j$ form a solution of the $N = 2$ super $(k|n,m)$-MGNLS hierarchy, then the matrices $\{B_{j+1}, \overline{B}_{j+1}, S_{j+1}, \overline{S}_{j+1}\}$ also form a solution provided they are connected with the former ones by the following relations:

$$B_{j+1}\overline{B}_{j+1} + S_{j+1}\overline{S}_{j+1} - (B_{j+1}\overline{B}_j)(B_j\overline{B}_j + S_j\overline{S}_j)(B_{j+1}\overline{B}_j)^{-1} = 2(B_{j+1}\overline{B}_j)'(B_{j+1}\overline{B}_j)^{-1},$$

$$\overline{D}(\overline{B}_jS_j + i\overline{B}_{j+1}^{-1}S_{j+1}) = 0, \quad D(\overline{S}_jB_{j+1} + i\overline{S}_{j+1}^{-1}\overline{B}_j) = 0,$$

for the case of bosonic matrices $B_j \equiv B_j$ and $\overline{B}_j \equiv \overline{B}_j$, or

$$D(\overline{F}_jS_j - i[\overline{D}F_{j+1}]^{-1}\overline{D}S_{j+1}) = 0, \quad \overline{D}(\overline{D}^2S_{j+1}F_{j+1} + iD\overline{S}_j[D\overline{F}_j])^{-1} = 0,$$

$$F_{j+1}\overline{F}_{j+1} + S_{j+1}\overline{S}_j - ([\overline{D}F_{j+1}][D\overline{F}_j])(\overline{F}_j\overline{F}_j + S_j\overline{S}_j)([\overline{D}F_{j+1}][D\overline{F}_j])^{-1}$$

$$= ([\overline{D}F_{j+1}][D\overline{F}_j])'([\overline{D}F_{j+1}][D\overline{F}_j])^{-1},$$

for the case when they are fermionic, $B_j \equiv F_j$ and $\overline{B}_j \equiv \overline{F}_j$. The relations (33) and (34) represent the matrix generalization of a wide class of $N = 2$ supersymmetric generalized Toda-lattice equations, constructed in [3]. The detailed analysis of them is out the scope of the present letter and will be discussed elsewhere. Let us only mention that among these systems one has the minimal $N = 2$ supersymmetric generalization of the bosonic non-abelian Toda lattice equations [17] as well as one important property of theirs, i.e. the involution $\sigma_l$ ($l \in \mathbb{Z}$) defined by

$$\sigma_lB_j\sigma_l^{-1} = i\overline{B}_j^T\overline{I}_{l-j}, \quad \sigma_l\overline{B}_j\sigma_l^{-1} = i\overline{I}_jB_j^T,$$

$$\sigma_lS_j\sigma_l^{-1} = i\overline{S}_j^T\overline{I}_{l-j}, \quad \sigma_l\overline{S}_j\sigma_l^{-1} = i\overline{I}_jS_j^T,$$

$$\sigma_l\theta\sigma_l^{-1} = \overline{\theta}, \quad \sigma_l\overline{\theta}\sigma_l^{-1} = \theta, \quad \sigma_lz\sigma_l^{-1} = z, \quad \sigma_li^*\sigma_l^{-1} = -i$$

which is very useful for analyzing them.

6. Reductions of the $N = 2$ supersymmetric matrix KP hierarchy. The generic form of the $N = 2$ matrix KP Lax operator is

$$L_{KP} = I \partial + \sum_{j=-\infty}^{0} (a_j + \omega_jD + \overline{\omega}_j\overline{D} + b_j[D, \overline{D}]) \partial^j,$$

where $a_j, b_j$ ($\omega_j, \overline{\omega}_j$) are generic bosonic (fermionic) square matrix $N = 2$ superfields. The Lax operators (31) and (34), introduced above, represent reductions of the $N = 2$ matrix KP hierarchy, characterized by a finite number of superfields. In the following we want to show that these examples are particular cases of an infinite class of reductions (with a finite number of superfields).

\footnote{For details concerning bosonic matrix KP and matrix (extended) Gelfand-Dickey hierarchies, see the recent papers [18, 19, 20] and references therein.}
The idea of the construction of this class is based on previous work.

In [21] a new type of \( N = 2 \) supersymmetric pseudo-differential Lax operators \( L \) was introduced. It was characterized by a non-standard \( N = 2 \) super-residue equal to the \( N = 2 \) superfield integral of the constant part of the operator. In [3] it was observed that all these Lax operators possess the following important property: they commute with one of the two fermionic covariant derivatives, \([D, L] = 0 \) or \([\overline{D}, L] = 0 \).

In other words, in [3] it was established that actually they are not \( N = 2 \), but \( N = 1 \) supersymmetric pseudo-differential operators. Consequently, the residue of these operators coincides with the residue of \( N = 1 \) supersymmetric pseudo-differential operators, i.e. with the \( N = 1 \) superfield integral of the coefficient of the operator \( \overline{D}\partial^{-1} \) or \( D\partial^{-1} \). This would seem to lead to \( N = 1 \) supersymmetric hierarchies, rather then to \( N = 2 \) ones. Nevertheless the coefficients of such operators are expressed in a special way in terms of \( N = 2 \) superfields and their fermionic derivatives, and, as a result, this fact leads to \( N = 2 \) supersymmetric systems. Due to commutativity of the Lax operator and the fermionic derivative, the \( N = 1 \) super-residue coincides with the \( N = 2 \) superfield integral of the constant part of the operator (see eq.\((13)\)), and, it reproduces and justifies the definition of the super-residue given in [21] for the case of degenerated Lax operators.

So, the first property of the reduced Lax operator we want to maintain is commutativity with the fermionic derivative. The constraint \([D, L_{KP}] = 0\) for the generic \( N = 2 \) matrix KP Lax operator ([12]), can be be solved in general and takes the form

\[
L^{\text{red}}_{KP} = I\partial + a_0 + \omega_0 D + \sum_{j=-\infty}^{1} (a_j \partial - [D a_j] \overline{D} + \omega_j D \partial - \frac{1}{2} [D \omega_j] [D, \overline{D}]) \partial^{-j},
\]

where \( a_0 \) and \( \omega_0 \) are chiral superfields. This Lax operator defines a reduction of KP with an infinite number of matrix superfields.

In addition in [1] the bosonic limits of the Lax operators corresponding to the \( N = 2 \) supersymmetric integrable hierarchies with \( N = 2 \) \( W_{s+1} \) second Hamiltonian structure were conjectured. The third input are the results we obtained in the previous sections for the \( N = 2 \) supersymmetric \((k|n, m)\)-MGNLS hierarchy.

Based on these three inputs, we are lead to the following conjecture for the expression of the matrix-valued pseudo-differential operator with a finite number of superfields, representing a reduction of \( N = 2 \) matrix KP hierarchy,

\[
(I_{KP}^{\text{red}})^s \equiv L_s = I\partial^s + \sum_{j=1}^{s-1} (J_{s-j} \partial - [DJ_{s-j} \overline{D}] ) \partial^{-j-1} - J_s - \overline{D} \partial^{-1} [DJ_s] - F\overline{F} - F\overline{D} \partial^{-1} [DF],
\]

\[
[D, L_s] = 0, \quad \frac{\partial}{\partial p} L_s = [(L^{p/s}_s)_{>1}, L_s], \quad H_p = \int dZ \mathcal{H}_p \equiv \int dZ \text{tr}(L^{p/s}_s)_{0}.
\]

Here, \( s \in \mathbb{N} \), the \( J_j \) are \( k \times k \) matrix-valued functions with the scaling dimension in length \([J_j] = -j\), and \([F] = [\overline{F}] = -s/2\). One can easily verify that its bosonic limit in the scalar case, i.e., at \( k = 1 \), in fact reproduces the Lax operator \( L^{(2)}_{[s|\alpha]} \) \( (L^{(3)}_{[s|\alpha]}) \) conjectured in [1] at \( F = \overline{F} = 0 \) \((F = F = J_s = 0)\).

In fact, before going into a more detailed discussion of the Lax operator ([14]) in the matrix case, it is instructive to examine it in the simpler and more studied scalar case (i.e., \( k = 1 \)). To start with let us mention that in the scalar case the Lax operators ([14]) at \( F = \overline{F} = J_s = 0 \)

---

\(^5\)Let us recall that standard \( N = 2 \) super-residue is defined as the \( N = 2 \) superfield integral of the coefficient of the operator \([D, \overline{D}] \partial^{-1}\).

\(^6\)In the case when the \( N = 1 \) super-residue vanishes, one can use the bosonic residue to construct the Hamiltonians, see [1].
reproduce the scalar hierarchies first studied in [22] in terms of $N = 1$ superfields. One can say that in general in the scalar case the Lax operator (44) describes almost all known $N = 2$ supersymmetric hierarchies. Let us present the first few cases in explicit form.

1. The $s = 1$ scalar case.

In this case the Lax operator has the following form:

$$L_1 = \partial - J_1 - \overline{D} \partial^{-1} [DJ_1] - FF - F \overline{D} \partial^{-1} [DF].$$

(45)

In the generic case this Lax operator describes the GNLS hierarchy in the KdV basis [4, 7]. There are two possible reductions, one with $F = \overline{F} = 0$ corresponds to the $N = 2 \ a = 4$ KdV hierarchy [21], the second, with $J_1 = 0$, corresponds to the $N = 2$ GNLS hierarchy [1]. In the special case when we deal only with one pair of bosonic chiral-anti-chiral superfields $F, \overline{F}$, the Lax operator (45) describes the $N = 4$ KdV hierarchy [4].

2. The $s = 2$ scalar case.

In this case the Lax operator

$$L_2 = \partial^2 + J_1 \partial - [DJ_1] \overline{D} - J_2 - \overline{D} \partial^{-1} [DJ_2] - F \overline{F} - F \overline{D} \partial^{-1} [DF].$$

(46)

includes spin 1 superfields – a general superfield $J_1$ and chiral/anti–chiral $F, \overline{F}$ ones, as well as the spin 2 general superfield $J_2$. This operator has been considered in [3]. It describes the extension of the $N = 2, a = -2$ super Boussinesq hierarchy, because in the case of $F = \overline{F} = 0$ the operator (46) corresponds just to the $a = -2$ Boussinesq hierarchy. Another possible reduction, $J_2 = 0$, gives us the Lax operator for the extended quasi-$N = 4$ KdV hierarchy [23], while the maximal possible reduction, $J_2 = F = \overline{F} = 0$, reproduces the Lax operator for the $N = 2 \ a = -2$ KdV hierarchy [24].

3. The $s = 3$ scalar case.

The last case we are going to consider explicitly, corresponds to the $N = 2$ supersymmetric extension of the bosonic system involving also a spin 4 field – i.e. the bosonic 4–KdV hierarchy. The corresponding Lax operator

$$L_3 = \partial^3 + J_1 \partial^2 - [DJ_1] \overline{D} \partial + J_2 \partial - [DJ_2] \overline{D} - J_3 - \overline{D} \partial^{-1} [DJ_3] - F \overline{F} - F \overline{D} \partial^{-1} [DF].$$

(47)

includes general superfields $(J_1, J_2, J_3)$ with spin $(1, 2, 3)$, respectively, and chiral/anti–chiral superfields $F, \overline{F}$ with spin $3/2$. The generic Lax operator (47) describes one of the three integrable systems possessing the $N = 2 \ W_4$ algebra as the second Hamiltonian structure (in the limit $F = \overline{F} = 0$) [25] and it is presented here for the first time. The reduction $J_3 = 0$ gives the Lax operator for the extension of the $N = 2 \ a = -1/2$ Boussinesq hierarchy [4], while the maximal reduction $J_3 = F = \overline{F} = 0$ reproduces the $N = 2 \ a = -1/2$ Boussinesq hierarchy itself [26].

Let us finish this excursion in the scalar $N = 2$ KP reductions with two comments which hold also in the matrix case.

First of all, it comes as a surprise that two out of three families of integrable hierarchies with $N = 2 \ W_4$ algebra as the second Hamiltonian structure are naturally combined in the Lax operators (44) – one is given by the $L_s$ operator itself while the second is the $J_{s+1} = 0$ reduction of the $L_{s+1}$ operator (in the limit $F = \overline{F} = 0$). The remaining family of hierarchies,
which starts from $N = 2, a = 5/2$ Boussinesq equation, has a completely different Lax operator whose closed form is unknown yet, see for details [1].

As a second remark, we would like to stress that the gauge transformations (28), (36) give the possibility to recover higher spin superfield $J_s$ in the Lax operator $L_s$ (14) starting from the reduced case $J_s = 0$. As a consequence the superalgebras which are the second Hamiltonian structures for these hierarchies (the reduced and unreduced ones) are closely related. In other words the superalgebra with the currents $(J_1, \ldots, J_s)$ and $n + m$ pairs of superfields $(F, \overline{F})$ can be realized in terms of the superalgebra formed by the supercurrents $(J_1, \ldots, J_{s-1})$ and $n + m + 1$ pairs of $(F, \overline{F})$. The first example of such relations between $N = 2$ NLS and $N = 2$ $a = 4$ KdV hierarchies was elaborated in [27].

Let us now return to the matrix hierarchies. In what follows we verify that the operator (14) is actually a reduction of the $N = 2$ matrix KP hierarchy in the simplest cases corresponding to $s = 1, s = 2$ and $s = 3$, by showing that the second flows are consistently produced via the Lax pair representation (14).

1. The $s = 1$ matrix case.

It is obvious that in this case the operator (14) reproduces the Lax operators (1) and (33) [1].

2. The $s = 2, p = 2$ matrix case.

\[
\begin{align*}
\frac{\partial}{\partial t_2} J_1 &= 2(J_2 + F\overline{F})' + [J_1, J_2 + F\overline{F}], \\
\frac{\partial}{\partial t_2} J_2 &= [D, \overline{D}] J_2' - J_1 J_2' + [DJ_1]\overline{D} J_2 - [DJ_2]\overline{D} J_1 - [J_1, \overline{D}D J_2], \\
\frac{\partial}{\partial t_2} F &= -F'' + D(J_1\overline{D} F), \quad \frac{\partial}{\partial t_2} \overline{F} = \overline{F}'' + \overline{D}([D\overline{F}] J_1), \\
\mathcal{H}_2 &= tr(J_2 + F\overline{F}).
\end{align*}
\]

The second flow equations, in the limit $F = \overline{F} = 0$, provides the matrix extension of the $N = 2$ $a = -2$ Boussinesq equation.

3. The $s = 3, p = 2$ matrix case.

\[
\begin{align*}
\frac{\partial}{\partial t_3} J_1 &= (J_1' + \frac{1}{3} J_1^2 - 2J_2)' + \frac{1}{3}[J_1, J_1' - 2J_2], \\
\frac{\partial}{\partial t_3} J_2 &= (2J_3 - J_2' + \frac{2}{3} J_1'' + 2F\overline{F})' + \frac{2}{3}(J_2 J_1' - J_1 J_2' + J_1 J_1'') \\
&- [D J_1]\overline{D} J_1' + [DJ_1]\overline{D} J_2 - [DJ_2]\overline{D} J_1 + [J_1, J_3 + F\overline{F}], \\
\frac{\partial}{\partial t_3} J_3 &= [D, \overline{D}] J_3' + \frac{2}{3}([DJ_1]\overline{D} J_3 - [DJ_3]\overline{D} J_1 - [J_1, \overline{D}D J_3] - J_1 J_3'), \\
\frac{\partial}{\partial t_3} F &= -F'' + \frac{2}{3} D(J_1\overline{D} F), \quad \frac{\partial}{\partial t_3} \overline{F} = \overline{F}'' + \frac{2}{3}\overline{D}([D\overline{F}] J_1), \\
\mathcal{H}_2 &= tr(J_2 - \frac{1}{6} J_1^2), \quad \mathcal{H}_3 = tr(J_3 + F\overline{F}).
\end{align*}
\]

The second flow equations, in the limit $J_3 = F = \overline{F} = 0$, define the matrix extension of the $N = 2, a = -\frac{1}{2}$ Boussinesq equation.

\footnote{This correspondence can be easily established after obvious transformation to the new basis in the space of the superfields.}
Let us make a few remarks about these hierarchies. The equations (48) admit the involution
\[ F^* = i^s F^T, \quad \overline{F}^* = i^s \overline{F}^T, \quad J_j^s = (-1)^j J_j^T, \]
\[ \theta^* = \theta, \quad \overline{\theta}^* = \theta, \quad t_p^* = (-1)^{p+1} t_p, \quad z^* = z, \quad i^* = -i, \]
for \( s = 2 \). The same involution property is not satisfied for \( s = 3 \). Nevertheless one can still think that there exists a basis in the space of the superfield matrices where the involution (50) is admitted for any given value of the parameter \( s \). Indeed, taking as an example the \( s = 3 \) equations (49), let us introduce a new basis with the superfield \( J_2 \) being replaced by
\[ J_2 \Rightarrow J_2 - \frac{1}{6} J_1^2 - \frac{1}{2} J_1', \]
while all the other superfields are unchanged. It is a simple exercise to verify that in this new basis the involution property (50) is satisfied.

From eqs. (48) and (49) we can explicitly see that it is consistent to set either the superfield \( J_s = 0 \) or \( F = \overline{F} = 0 \) or both simultaneously. These additional reduction properties are general, and, together with the involution properties of the superfield matrices they can be used to straightforwardly derive discrete symmetries of the Lax operators \( L_s \) at \( J_s = 0 \) and generic \( s \), following the method developed in [5, 6] and used in sections 4 and 5 for the case of \( s = 1 \).

To close this Letter let us remind that there is an alternative description of the reductions of the scalar \( N = 2 \) supersymmetric KP hierarchies based on Lax operators obeying the chirality preserving condition \( DL = L\overline{D} = 0 \), as opposed to the condition \( [D, L] = 0 \) we used here. This type of Lax operators has been introduced in [28] and then it was studied in details in [3], where a wide class of \( N = 2 \) scalar KP reductions was derived. One may wonder whether these Lax operators also admit a matrix generalization. The answer is positive and the construction is straightforward: one simply replaces in the Lax operators of [3] superfield functions by matrix valued superfields:
\[ L_s = D \left( I \partial^{s-1} + \sum_{j=0}^{s-2} J_j \partial^j + F \partial^{-1} \overline{F} \right) \overline{D} \]
and
\[ \tilde{L}_s = D \left( I \partial^{s-1} + \sum_{j=0}^{s-2} J_j \partial^j + \overline{D} \partial^{-1} (J_s + F \partial^{-1} F) \partial^{-1} D \right) \overline{D}, \]
where the \( J_j \) are \( k \times k \) matrix-valued general \( N = 2 \) superfields, and \( F, \overline{F} \) chiral/anti–chiral rectangular matrix superfields. The formulae for flows and Hamiltonians are the standard ones. The two operators above are not independent but gauge related.

We have checked that all particular cases explicitly presented in [3] admit a matrix generalization with Lax operators (52) and the corresponding flow equations and Hamiltonians coincide with those which come from the Lax operator (44). But a rigorous proof for generic \( s \) is still lacking.

Summary. In this paper we have defined the \( N = 2 \) generalization of the matrix GNLS hierarchies, referred to as \( N = 2 (k|n,m) \)-MGNLS hierarchies. Many results are simply stated without proof, since they are a straightforward generalization of the analogous results for the corresponding scalar hierarchies – but this is not a general rule. We have first introduced the
Lax operator realization of the $N = 2 (k|n,m)$–MGNLS hierarchies. Then we have explicitly calculated their second Hamiltonian structure as well as their conserved charges. Then we have produced a new basis for the matrix superfields, the so–called KdV basis, and defined the new hierarchies in this new basis, which also admit local flows. Furthermore we have spelled out the discrete symmetries of the $N = 2 (k|n,m)$–MGNLS hierarchies, which lead to the matrix analogs of the $N = 2$ Toda lattice hierarchies. Finally we have analyzed the possibility of viewing the previously introduced hierarchies as reductions of the matrix $N = 2$ KP hierarchy. This has led us to conjecture the existence of an infinite family of $N = 2$ matrix hierarchies, which are reductions of the KP hierarchy and are characterized by a finite number of fields. The family is parametrized by a natural number $s$. For $s = 1$ the hierarchy corresponds to the $N = 2 (k|n,m)$–MGNLS one. We have verified this conjecture for the first few cases. We have also derived the involution and reduction properties of these few cases and conjectured that they hold in general.

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