On Taylor series of zeros with general base function

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Abstract

We prove a formula for the Taylor series coefficients of a zero of the sum of a complex-exponent polynomial and a base function which is a general holomorphic function with a simple zero. Such a Taylor series is more general than a Puiseux series. We prove an integrality result about these coefficients which implies and generalizes the integrality result of Sturmfels (“Solving algebraic equations in terms of $A$-hypergeometric series”. Discrete Math. 210 (2000) pp. 171-181). We also prove a transformation rule for a special case of these Taylor series.

1 Introduction

Determining the zeros of a polynomial is a fundamental objective throughout pure and applied mathematics. By the work of Abel [1], Ruffini [9], and Galois [4], there is no general solution by radicals for polynomials of degree five or more. Nevertheless, formal series expansions may provide a unified way to consider zeros of polynomials of any degree.

Many authors have studied such formal series. Birkeland [2] obtained formulas for Taylor series coefficients using Lagrange inversion. Mayr [8] also studied these series using a system of differential equations. Sturmfels [10] obtained the same formulas using GKZ-systems (Gelfand-Kapranov-Zelevinsky) [5], [6]. Herrera [7] used reversion of Taylor series.

In this paper, using Taylor series directly, we prove a formula (Theorem 3.2) for the Taylor series coefficients of a zero of a function that may be a polynomial or, more generally, a sum of a complex-exponent polynomial and a holomorphic function (the base function). We have defined a complex-exponent polynomial and the term “base function” in [3] (also defined below in Definition 1.1), and our proofs use similar techniques to those of [3], including the use of Theorem 2.6 here. We next explain more about the set-up and relation to previous work.

First we recall complex-exponent polynomials. For two complex numbers $z \neq 0$ and $\gamma$, we define complex exponentiation $z^\gamma$ in the conventional way by taking $z$ to be an element of the Riemann surface $L$ for the logarithm. This surface $L$ is parametrized by

$$L = \{(r, \theta, n): r \in \mathbb{R}^+, \theta \in (-\pi, \pi], n \in \mathbb{Z}\}.$$
Then for $z \in L$ corresponding to $(r, \theta, n)$, define $z^{\gamma}$ by
\[ z^{\gamma} = e^{\gamma \ln(r) + i\gamma \theta + 2\pi i n \gamma} \in \mathbb{C}. \]

**Definition 1.1.** For an integer $d \geq 1$, let $\vec{a}$ and $\vec{\gamma}$ be two $d$-tuples of complex numbers
\[
\vec{a} = (a_1, \ldots, a_d)
\]
\[
\vec{\gamma} = (\gamma_1, \ldots, \gamma_d).
\]

Define a complex-exponent polynomial $p(z; \vec{a}, \vec{\gamma})$ to be a function of the form
\[ p: L \to \mathbb{C} \]
\[ z \mapsto \sum_{k=1}^{d} a_k z^{\gamma_k} \]
which we abbreviate as $p(z)$.

Let $g(z): \mathbb{C} \to \mathbb{C}$ be a holomorphic function. We assume that $g(z)$ has a simple zero $\alpha \neq 0$, and we write its expansion about $\alpha$ as
\[ g(z) = \sum_{k=1}^{\infty} c_k (z - \alpha)^k \]
for some $c_k \in \mathbb{C}$ with $c_1 \neq 0$. We call $g(z)$ the “base function”. We fix an element of $L$ that corresponds to $\alpha$ and also denote it by $\alpha$. Then $g(z)$ also determines a function from a neighborhood $V$ of $\alpha$ in $L$ to $\mathbb{C}$, via the mapping
\[ z \mapsto z^1 \in \mathbb{C} \text{ for } z \in V. \]

Now fix an integer $d \geq 1$ and $\vec{\gamma} \in \mathbb{C}^d$. For an $\vec{a} \in \mathbb{C}^d$, define the function
\[ f(z; \vec{a}, \vec{\gamma}, g): L \to \mathbb{C} \]
\[ z \mapsto g(z) + \sum_{i=1}^{d} a_i z^{\gamma_i} \]

Assume, for some neighborhood $U \subset \mathbb{C}^d$ of the origin $\vec{0}$, that there exists a smooth function
\[ \phi(\vec{a}; \vec{\gamma}, g, \alpha): U \to L \]
\[ \vec{a} \mapsto \phi(\vec{a}; \vec{\gamma}, g, \alpha) \]
that satisfies for all $\vec{a} \in U$
\[ f((\phi(\vec{a}; \vec{\gamma}, g, \alpha); \vec{a}, \vec{\gamma}, g)) = 0 \]  
\[ \phi(0; \vec{\gamma}, g, \alpha) = \alpha. \]
Equations (2) and (3) are sufficient to determine the Taylor series coefficients of \( \phi(\vec{a}; \gamma, g, \alpha) \) about \( \vec{0} \) in the variables \( \vec{a} \).

Let \( \vec{n} \) denote a \( d \)-tuple of non-negative integers

\[
\vec{n} = (n_1, \ldots, n_d)
\]

and denote

\[
\Sigma \vec{n} = \sum_{i=1}^{d} n_i,
\]

Let \( \partial_{\vec{n}} \) denote the partial derivative operator

\[
\partial_{\vec{n}} = \prod_{i=1}^{d} \left( \frac{\partial}{\partial a_i} \right)^{n_i},
\]

and for any function

\[
\psi : U \to \mathbb{C}
\]

denote

\[
\partial(\psi, \vec{n}) = \partial_{\vec{n}}\psi(\vec{a})|_{\vec{0}}.
\]

The Taylor series coefficient formula of Theorem 1.3, [3] (stated as Theorem 2.5 here) is obtained using a base function \( g(z) \) of the form

\[
g(z) = 1 + bz^\beta \] (5)

for non-zero \( b, \beta \in \mathbb{C} \). As shown in [3], Birkeland [2] and Sturmfels [10] essentially find the Taylor series coefficients using the base function \( g(z) \) with positive integer \( \beta \) and with \( \gamma_i \) distinct integers not equal to \( \beta \), though they do not use that terminology or the methods of Taylor series. Their formulas may be expressed in terms of falling factorials. We generalize these formulas in [3] by allowing \( \gamma_i \) and \( \beta \) to be arbitrary complex numbers \( (\beta \neq 0) \) and show that the factorization is preserved using falling factorials (see Theorem 2.5 here).

In this paper, we consider the base function (1) and prove in Theorem 3.2 that \( \partial(\phi, \vec{n}) \) is a polynomial in \( c_{i_{1}}^{-1}, c_{i}, 2 \leq i \leq \Sigma \vec{n} \), and that the coefficients of this polynomial factorize as well, using similar falling factorials.

Now, the base function (5) is a special case of (1) obtained via

\[
c_k = b \binom{\beta}{k} \alpha^{\beta-k}.
\]

Using this fact combined with Theorem 3.4 gives another proof of Theorem 4.1 in Sturmfels [10]. His proof of Theorem 4.1 uses Hensel’s Lemma, and our proof of Theorem 3.4 counts terms in a derivative.

We also note with base function (6), Theorem 3.2 here and Theorem 1.3 of [3] give two different formulas for the quantity \( \partial(\phi, \vec{n}) \). We include in section 5 the objective of finding a direct proof of this identity.
2 Definitions and cited theorems

2.1 Definitions

Most of the following definitions occur in [3].

First we present notation for multisets and multiset partitions. For an integer \( M \geq 0 \), we let \([1, M]\) denote
\[
[1, M] = \{ i \in \mathbb{Z} : 1 \leq i \leq M \}.
\]

**Definition 2.1.** For a positive integer \( N \), define an ordered multiset \( I \) of \([1, d]\) to be an \( N \)-tuple of integers
\[
I = (I(1), \ldots, I(N))
\]
with \( 1 \leq I(i) \leq d \). We say that the order \(|I|\) is \( N \). We define the multiplicity \( \#(n, I) \) of \( n \) in \( I \) as the number of indices \( i \) such that \( I(i) = n \). For a multiset \( X \) we also denote the multiplicity of \( n \) in \( X \) as \( \#(n, X) \).

Let \text{Multiset}(d)\) denote the set of these ordered multisets.

For a positive integer \( k \), define a set partition \( s \) of \([1, N]\) with \( k \) parts to be a \( k \)-tuple
\[
s = (s_1, \ldots, s_k)
\]
where \( s_i \) are pairwise disjoint non-empty subsets of \([1, N]\),
\[
\bigcup_{i=1}^{k} s_i = [1, N],
\]
and
\[
\min(s_i) < \min(s_j) \text{ for } i < j.
\]

We also write a set \( s_i \) as an \( m \)-tuple
\[
s_i = (s_i(1), \ldots, s_i(m))
\]
where \( m = |s_i| \) and
\[
s_i(j) < s_i(l) \text{ for } j < l.
\]

Let \( S(N, k) \) denote the set of such \( s \). If \( H \) is any finite set of integers, we similarly denote \( S(H, k) \) to be the set of all set partitions of \( H \) into \( k \) non-empty parts.

For a multiset \( I \) and a set partition \( s \in S(|I|, k) \), define a multiset partition \( J \) of \( I \) with \( k \) parts to be a \( k \)-tuple
\[
J = (J_1, \ldots, J_k)
\]
where \( J \in \text{Multiset}(d) \) is given by
\[
J_i = (I(s_i(1)), \ldots, I(s_i(m)))
\]
where \( m = |s_i| \). Thus the multiset partitions of \( I \) with \( k \) parts are in bijection with the set partitions in \( S(|I|, k) \). Let \( \text{Parts}(I, k) \) denote the set of such multiset partitions \( J \).
Each part \( J_i \) of a \( J \in \text{Parts}(I, k) \) is thus a multiset. For \( J, J' \in \text{Parts}(I, k) \), we say that \( J \) and \( J' \) are equivalent if there exists a permutation \( \sigma \) of \([1, k]\) such that

\[
#(l, J_i) = #(l, J'_\sigma(i))
\]

for each \( 1 \leq l \leq d \).

We let \( I(\hat{h}) \) denote the ordered multiset obtained from \( I \) by removing the element at the \( h \)-th index:

\[
I(\hat{h}) = (I(1), \ldots, I(h-1), I(h+1), \ldots, I(N)).
\]

We use the notation

\[
\sum_{m \in I} \gamma_m = \sum_{i=1}^{N} \gamma_{I(i)}
\]

where \( N = |I| \).

For any function \( \psi(\vec{a}) \) of the form \( \psi \) and \( I \in \text{Multiset}(d) \), denote

\[
\partial(\psi, I) = \left( \prod_{i=1}^{|I|} \frac{\partial}{\partial a_{I(i)}} \psi(\vec{a}) \right)|_{\vec{a}=0}
\]

We also use the falling factorial applied to indeterminates, where “indeterminate” refers to an arbitrary element of some polynomial ring over \( \mathbb{Z} \).

**Definition 2.2.** For an integer \( k \geq 0 \) and an indeterminate \( x \), define the falling factorial

\[
(x)_k = \prod_{i=1}^{k} (x - i + 1).
\]

Let \( \mu \) denote an infinite sequence \( (\mu_i)_{i=1}^{\infty} \) of non-negative integers \( \mu_i \) such that \( \mu_i = 0 \) for sufficiently large \( i \). For integer \( r \geq 1 \), let \( C(r) \) denote the set of all such \( \mu \) such that

\[
\sum_{i \geq 1} \mu_i = r.
\]

Thus \( C(r) \) is the set of compositions of \( r \) with non-negative parts.

### 2.2 Cited theorems

These results are used here in Theorems 3.2, 3.3, and 4.1.

**Lemma 2.3.** For indeterminates \( a \) and \( b \) and an integer \( n \geq 0 \),

\[
(a + b)_n = \sum_{i=0}^{n} \binom{n}{i} (a)_i (b)_{n-i}
\]

and equivalently

\[
\binom{a + b}{n} = \sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i}.
\]
Proof. This is Lemma 6.1 of [3].

**Lemma 2.4.** Suppose $F(x)$ is a polynomial of degree $m$. Then

$$F(x) = \sum_{k=1}^{m+1} \frac{(x - 1)_{k-1}}{(k - 1)!} \sum_{r=0}^{k-1} (-1)^{k-1-r} \binom{k-1}{r} F(r+1).$$

Proof. This is Lemma 2.4 of [3].

**Theorem 2.5.** With the above notation and base function (5), for $\Sigma n \geq 1$,

$$\partial_{\alpha} \phi(\alpha) \big|_0 = -\frac{\alpha^{1+\sum_{i=1}^{d} n_i (\gamma_i - 1)} \Sigma_n^{-1}}{g'(\alpha) \Sigma_n} \prod_{i=1}^{d} (-1 + i \beta - \sum_{i=1}^{d} n_i \gamma_i)$$

where

$$g'(\alpha) = b \beta \alpha^{\beta - 1}.$$

Proof. This is Theorem 1.3 of [3].

**Theorem 2.6.** Let $R$ be a commutative ring and let $\delta: R \rightarrow R$ be a derivation. For an integer $M \geq 0$ and elements $f_A, f_B, f_i \in R, 1 \leq i \leq M$,

$$\sum_{w \subseteq [1,M]} \delta^{|w| - 1} (f_A^{(1)} \prod_{i \in w} f_i) \delta^{|w|} (f_B \prod_{i \in w} f_i) = \delta^M (f_A f_B \prod_{i=1}^{M} f_i) \quad (6)$$

where $w^c$ denotes the complement of $w$ in $[1,M]$; $f_A^{(1)}$ denotes $\delta f_A$; and in the case $w^c$ is empty, $\delta^{-1} f_A^{(1)}$ denotes $f_A$.

Proof. This is Theorem 4.3 of [3].

**Theorem 2.7.** For integers $1 \leq k \leq N$, an indeterminate $\nu$, and $N$ indeterminates $x_i, 1 \leq i \leq N$,

$$\frac{1}{(k-1)!} \sum_{r=0}^{k-1} (-1)^{k-1-r} \binom{k-1}{r} ((r+1)\nu - 1 + \sum_{i=1}^{N} x_i)_{N-1} \quad (7)$$

$$= \nu^{k-1} \sum_{s \in S(N,k)} \prod_{i=1}^{k} (\nu - 1 + \sum_{m \in s_i} x_m)_{|s_i|-1}. \quad (8)$$

Proof. This is Theorem 4.1 of [3].
3 Taylor coefficient theorem

Definition 3.1. Let \( \alpha \) be the zero of base function (1). For an \( x \in \mathbb{C} \) and integers \( r \geq 0 \) and \( a \geq 1 \), define

\[
F(x, r, a) = -\frac{\alpha^x}{(a - 1)!} \sum_{\mu \in C(r)} (-1)^{\mu_1} \left( r - \sum_{i \geq 2} (i - 1)\mu_i \right) \left( x^r - \sum_{i \geq 2} (i - 1)\mu_i \right) \left( r + \sum_{i \geq 2} \mu_i \right) !
\]

\[
\times \frac{\alpha^{-(r - \sum_{i \geq 2} (i - 1)\mu_i - (a - 1))}}{r! + \sum_{i \geq 2} \mu_i !} \prod_{i \geq 2} \frac{\mu_i}{\mu_i !}
\]

and also

\[
F(x, -1, a) = 0.
\]

Theorem 3.2. With the above notation and base function (1), and for an \( I \in \text{Multiset}(d) \) with \( |I| \geq 1 \),

\[
\partial(\phi, I) = F(\sum_{m \in I} \gamma_m, |I| - 1, 1).
\]

Proof. We use induction on \( |I| \). When \( I = (i) \), we differentiate equation (2) with respect to \( a \) and evaluate at \( a = 0 \) to obtain

\[
0 = \alpha^{\gamma_i} + c_1 \partial(\phi, I).
\]

Solving for \( \partial(\phi, I) \) proves the base case of \( |I| = 1 \).

Given an \( I \) with \( |I| \geq 1 \), suppose equation (9) is true for all \( I' \) with \( |I'| < |I| \). Given any function \( \psi(\vec{a}) \) of the form (4), it follows from the definitions that

\[
\partial(f \circ \psi, I) = \sum_{\mu \in \text{Parts}(I(h), k)} \prod_{i = 1}^k \partial(\psi, J_i)
\]

\[
\sum_{k=1}^{|I|} g^{(k)}(\psi(\vec{0})) \sum_{J \in \text{Parts}(I, k)} \prod_{i = 1}^k \partial(\psi, J_i).
\]

Setting \( \psi \) to be \( \phi \), we obtain

\[
0 = \sum_{h=1}^{|I|} \sum_{k=1}^{|I| - 1} (\gamma_I(h))_k \alpha^{\gamma_I(h) - k} \sum_{J \in \text{Parts}(I(h), k)} \prod_{i = 1}^k \partial(\phi, J_i)
\]

\[
+ \sum_{k=2}^{|I|} k! c_k \sum_{J \in \text{Parts}(I, k)} \prod_{i = 1}^k \partial(\phi, J_i)
\]

\[
+ c_1 \partial(\phi, I).
\]

Using the induction hypothesis we may express the sum on the right side of the equation at line (12) as

\[
\sum_{h=1}^{|I|} \sum_{k=1}^{|I| - 1} (\gamma_I(h))_k \alpha^{\gamma_I(h) - k} \prod_{J \in \text{Parts}(I(h), k)} \prod_{m \in J_i} F(\sum_{m \in J_i} x_m, |J_i| - 1, 1).
\]
To the sum over \( J \) we apply Theorem 3.3 and obtain

\[
\sum_{h=1}^{[I]} \sum_{k=1}^{[I]-1} (\gamma_{I(h)})_k \alpha^{\gamma_{I(h)}-k} F(\sum_{m \in I(h)} x_m, |I| - 2, k).
\]

Substituting in the definition of \( F(x, r, a) \) and then using

\[
(\gamma_{I(h)})_k = (\gamma_{I(h)})_k (\gamma_{I(h)} - 1)_{k-1}
\]

yields

\[
\sum_{h=1}^{[I]} -\alpha \sum_{i \in I} \gamma_i \sum_{\mu \in C(|I|-2)} \frac{(\gamma_{I(h)} - 1)_{k-1}}{(k-1)!} \sum_{i \geq 2} (-1)^{\mu_i} \left( |I| - 2 - \sum_{i \geq 2} (i-1)\mu_i - (k-1) \right)
\]

\[
\times (|I| - 2 + \sum_{i \geq 2} \mu_i) \alpha^{\sum_{i \geq 2} (i-1)\mu_i} c_{\mu_1}^{\mu_1} \prod_{i \geq 2} \mu_i.
\]

Interchanging the sum over \( k \) and \( \mu \) and applying Lemma 2.3 gives

\[
\sum_{h=1}^{[I]} -\alpha \sum_{i \in I} \gamma_i \sum_{\mu \in C(|I|-2)} \frac{(-1)^{\mu_i}}{(|I| - 2 - \sum_{i \geq 2} (i-1)\mu_i)!} \frac{\alpha^{\sum_{i \geq 2} (i-1)\mu_i}}{c_{\mu_1}^{\mu_1} \prod_{i \geq 2} \mu_i}
\]

and summing over \( h \) gives

\[
-\alpha \sum_{i \in I} \gamma_i \sum_{\mu \in C(|I|-2)} (-1)^{\mu_i} \frac{\sum_{i \in I} \gamma_i |I| - 1 - \sum_{i \geq 2} (i-1)\mu_i}{(|I| - 2 - \sum_{i \geq 2} (i-1)\mu_i)!}.
\]

(15)

Now to each term at line \((13)\), we again apply the induction hypothesis and Theorem 3.3 and at line \((14)\) we assume the result

\[
\partial(\phi, I) = F(\sum_{m \in I} x_m, |I| - 1, 1).
\]

Now combining the result \((15)\) it is sufficient to prove

\[
0 = -\sum_{\mu \in C(|I|-2)} \frac{(x)|I| - 1 - \sum_{i \geq 2} (i-1)\mu_i}{(|I| - 2 - \sum_{i \geq 2} (i-1)\mu_i)!} (-1)^{\mu_i} \frac{(|I| - 2 + \sum_{i \geq 2} \mu_i)!}{c_{\mu_1}^{\mu_1} \prod_{i \geq 2} \mu_i} \prod_{i \geq 2} \mu_i
\]

(16)

\[
+ \sum_{i \geq 2} i c_i F(\sum_{m \in I} x_m, |I| - 1, i)
\]

(17)

\[
+ c_1 F(\sum_{m \in I} x_m, |I| - 1, 1).
\]

(18)

Take the coefficient of \( \prod_{i \geq 2} \mu_i \) from each line. Consider the expression

\[
\frac{(x)|I| - 1 - \sum_{i \geq 2} (i-1)\mu_i}{(|I| - 1 - \sum_{i \geq 2} (i-1)\mu_i)!} (-1)^{|I| - 1 + \sum_{i \geq 2} \mu_i} \frac{(|I| - 2 + \sum_{i \geq 2} \mu_i)!}{c_{\mu_1}^{\mu_1} \prod_{i \geq 2} \mu_i}.
\]

(19)
The coefficient from line (16) is equal to the expression (19) times
\[ |I| - 1 - \sum_{i \geq 2} (i - 1)\nu_i. \tag{20} \]

The coefficient from the \( i \)-th term at line (17) is equal to the expression (19) times
\[ \frac{i!}{(i-1)!} \nu_i. \tag{21} \]

The coefficient from line (18) is equal to the expression (19) times
\[ - (|I| - 1 + \sum_{i \geq 2} \nu_i). \tag{22} \]

Adding the three expressions yields 0:
\[ |I| - 1 - \sum_{i \geq 2} (i - 1)\nu_i + \sum_{i \geq 2} i\nu_i - (|I| - 1 + \sum_{i \geq 2} \nu_i) = 0. \]

This completes the proof. \( \square \)

**Theorem 3.3.** For an integers \( 1 \leq a \leq M \) and indeterminates \( x_i, 1 \leq i \leq M \),
\[
\sum_{s \in S(M,a)} \prod_{i=1}^{a} F(\sum_{m \in s_i} x_m, |s_i| - 1, 1) = F(\sum_{i=1}^{M} x_i, M - 1, a). \tag{23}
\]

**Proof.** We use induction on \( M \). Equation (23) is true when \( M = 1 \). Assume it is true for all values less than or equal to some \( M \geq 1 \). We note the equation is true when \( a = 1 \) for any \( M \). For \( 1 \leq a \leq M \), we have by the induction hypothesis
\[
\sum_{s \in S(M+1,a+1)} \prod_{i=1}^{a+1} F(\sum_{m \in s_i} x_m, |s_i| - 1, 1)
= \sum_{w \subseteq [1, M+1], w \cup M+1 \in w} F(\sum_{m \in w} x_m, |w| - 1, a) F(\sum_{m \in w} x_m, |w| - 1, 1) \tag{24}
\]
where \( w \) is the set of the set partition \( s \) that contains \( M + 1 \). For elements \( \nu \in C(M) \) and \( \mu \in C(j) \) for some \( j \), we say \( \mu \leq_1 \nu \) if
- \( \mu_i \leq \nu_i \) for \( i \geq 2 \)
- \( \mu_1 \leq \nu_1 - 1 \).

If \( \mu \leq_1 \nu \), denote \( \mu' \in C(M - 1 - j) \) by
- \( \mu'_i = \nu_i - \mu_i \) for \( i \geq 2 \)
- \( \mu'_1 = \nu_1 - 1 - \mu_1 \).
Fix an element $\nu \in C(M)$. Then the coefficient of

$$\prod_{i \geq 2} c_i^{\nu_i}$$

in the right side of equation (21) is

$$-\frac{\alpha \sum_{i=1}^{M+1} x_i}{(a-1)!} \sum_{w \subset [1,M+1], M+1 \leq w \mu \leq \nu, \mu \in C(|w|-1)} \left(\prod_{i=2}^{M+1} \left(\begin{array}{c}
\alpha \left(|w^c| - 1 - \sum_{i \geq 2} (i-1) \mu_i - (a-1)\right) \\
\prod_{i \geq 2} \mu_i!
\end{array}\right)^i\right)$$

$$\times \left(\prod_{i=2}^{M+1} \left(\sum_{m \in w^c} x_m \right) \right)^i$$

$$\times \left(\prod_{i=2}^{M+1} \left(\sum_{m \in w^c} x_m \right) \right)^i$$

$$\times \left(\prod_{i=2}^{M+1} \left(\sum_{m \in w^c} x_m \right) \right)^i$$

Therefore it is sufficient to prove that sum at lines (30) and (31) is equal to expression (32). We prove this next by comparing the coefficients of the variables $x_i$.

Fix integers $l$ and $n_i \geq 0$ such that $l \leq M$. Consider the monomial

$$\prod_{i=1}^{l} x_i^{n_i}.$$
The coefficient of monomial \((33)\) in the sum at lines \((30)\) and \((31)\) is

\[
\frac{1}{(a-1)!b!} \sum_{v \subseteq [1,l]} \sum_{j=0}^{M} \sum_{\mu \leq \nu, \mu \in C(j)} \left[ \frac{M-j-1-(a-1)-\sum_{i \geq 2}(i-1)\mu'_i}{\sum_{i \in v} n_i} \right] \left( \frac{\prod_{i \in v} n_i!}{\prod_{i \geq 2} \nu_i!} \right) \left( \frac{\sum_{i \geq 2}(i-1)\mu'_i!}{\prod_{i \geq 2} \mu'_i!} \right)
\]

\[
(M - j - 1 - (a - 1) - \sum_{i \geq 2}(i-1)\mu'_i) \left( \frac{\prod_{i \in v} n_i!}{\prod_{i \geq 2} \nu_i!} \right) \left( \frac{(M - j - 1 + \sum_{i \geq 2} \mu'_i)!}{\prod_{i \geq 2} \mu'_i!} \right)
\]

\[
\times \left( \frac{\sum_{i \geq 2}(i-1)\mu'_i!}{\prod_{i \geq 2} \mu'_i!} \right) \left( \frac{\sum_{i \geq 2}(i-1)\mu'_i!}{\prod_{i \geq 2} \mu'_i!} \right)
\]

where we have set \(j = |u|\) and used the fact that there are \(\binom{M - l}{j - |v|}\) ways to choose a set \(u \in [1,M]\) such that \(v \subseteq u\) and \(v^c \subseteq u^c\).

Now each term in the above sum is independent of \(\nu_1, \mu_1, \) and \(\mu'_1\). We thus re-arrange the summation at line \((34)\) to be

\[
\sum_{v \subseteq [1,l]} \sum_{\tilde{\nu}} \sum_{\tilde{\mu}=0}^{M}
\]

where \(\tilde{\nu}\) is the fixed sequence \((\nu_i)_{i \geq 2}\); \(\tilde{\mu}\) is any sequence of non-negative integers \((\tilde{\mu}_i)_{i \geq 2}\) with

\[
\tilde{\mu}_i \leq \nu_i \quad \text{for} \quad i \geq 2
\]

and

\[
\tilde{\mu}'_i = \nu_i - \tilde{\mu}_i \quad \text{for} \quad i \geq 2.
\]

The coefficient of term \((38)\) in expression \((32)\) is

\[
\frac{(M + \sum_{i \geq 2} \nu_i)!}{a!b! \prod_{i \geq 2} \nu_i!} \left[ \frac{M-a-b-\sum_{i \geq 2}(i-1)\nu_i}{\sum_{i \geq 2} n_i} \right] \left( \frac{\prod_{i \geq 2} \nu_i!}{\prod_{i \geq 2} \nu_i!} \right) \left( \frac{\sum_{i \geq 2}(i-1)\nu_i!}{\prod_{i \geq 2} \nu_i!} \right).
\]

We equate expression \((39)\) with the sum beginning at line \((34)\) (using the or-
dering (38)) and simplify to obtain

\[
\sum_{v \in [1, l]} \sum_{j=0}^{M} \frac{M - j - 1 + \sum_{i \geq 2} \tilde{\mu}'_j}{(M - j - 1 - (|v|^e - 1))! \sum_{i \geq 2} (i - 1)\tilde{\nu}_i} \prod_{i \geq 1} n_i ! \sum_{i \geq 2} \tilde{\mu}_i ! (\sum_{i \in v^c} n_i) ! \tag{40}
\]

\[
a \frac{M - j - 1 + \sum_{i \geq 2} \tilde{\mu}'_j}{(M - j - 1 - (|v|^e - 1))! \sum_{i \geq 2} (i - 1)\tilde{\nu}_i} \prod_{i \geq 1} n_i ! \sum_{i \geq 2} \tilde{\mu}_i ! (\sum_{i \in v^c} n_i) ! \sum_{i \geq 2} \tilde{\mu}_i ! (\tilde{\mu}_i) ! (\tilde{\nu}_i) ! \tag{41}
\]

\[
\times \frac{(j + \sum_{i \geq 2} \tilde{\mu}_i)!}{(j - |v|)! (j - b - \sum_{i \geq 2} (i - 1)\tilde{\mu}_i)! (\sum_{i \in v} n_i) !} \prod_{i \geq 2} (\tilde{\mu}_i) ! (\tilde{\nu}_i) ! \tag{42}
\]

\[
\times \prod_{i \geq 2} \frac{(\tilde{\mu}_i + \tilde{\mu}'_i)!}{(\tilde{\mu}_i) ! (\tilde{\nu}_i) !} \prod_{i \geq 2} \frac{(M - a - b - \sum_{i \geq 2} (i - 1)\mu_i)!}{(M - a - b - \sum_{i \geq 2} (i - 1)\mu_i)! (\sum_{i \geq 1} n_i) !} \prod_{i \geq 2} (\tilde{\mu}_i) ! (\tilde{\nu}_i) ! \tag{43}
\]

\[
= \frac{(M + \sum_{i \geq 2} \mu_i)!}{(M - l)!} \frac{(M - a - b - \sum_{i \geq 2} (i - 1)\mu_i)!}{(M - a - b - \sum_{i \geq 2} (i - 1)\mu_i)! (\sum_{i \geq 1} n_i) !} \prod_{i \geq 2} (\tilde{\mu}_i) ! (\tilde{\nu}_i) ! \tag{44}
\]

We claim the above equation is true for any integer \( M \geq l \). Thus multiply both sides above the above equation by \( t^{M-l} \) and sum over \( M \geq l \). The claim is thus equivalent to the equality of power series

\[
\sum_{v \in [1, l]} \sum_{j=0}^{M} \frac{d}{dt} \sum_{i \geq 2} \tilde{\mu}_i \ln(1 + t^{i}) \prod_{i \geq 2} n_i ! \sum_{i \geq 2} \tilde{\mu}_i ! (\tilde{\nu}_i) ! \tag{45}
\]

\[
\times \frac{(j + \sum_{i \geq 2} \tilde{\mu}_i)!}{(j - |v|)! (j - b - \sum_{i \geq 2} (i - 1)\tilde{\mu}_i)! (\sum_{i \in v} n_i) !} \prod_{i \geq 2} (\tilde{\mu}_i) ! (\tilde{\nu}_i) ! \tag{46}
\]

\[
\times \prod_{i \geq 2} \frac{(\tilde{\mu}_i + \tilde{\mu}'_i)!}{(\tilde{\mu}_i) ! (\tilde{\nu}_i) !} \prod_{i \geq 2} \frac{(M - a - b - \sum_{i \geq 2} (i - 1)\mu_i)!}{(M - a - b - \sum_{i \geq 2} (i - 1)\mu_i)! (\sum_{i \geq 1} n_i) !} \prod_{i \geq 2} (\tilde{\mu}_i) ! (\tilde{\nu}_i) ! \tag{47}
\]

\[
= \frac{d}{dt} \sum_{i \geq 2} \tilde{\mu}_i \ln(1 + t^{i}) \prod_{i \geq 2} n_i ! \sum_{i \geq 2} \tilde{\mu}_i ! (\tilde{\nu}_i) ! \tag{48}
\]

This equality follows from Theorem 2.6 using the ring \( R \) of power series in \( t \); \( \delta \) differentiation with respect to \( t \); and

\[
f_A = t^a
\]

\[
f_B = t^b
\]

\[
f_i = (- \ln(1 - t))^{n_i} \text{ for } 1 \leq i \leq l,
\]

such that for each integer \( h \geq 2 \) there are exactly \( \nu_h \) indices \( i \) with \( l + 1 \leq i \leq l + \sum_{h \geq 2} \nu_h \) and

\[
f_i = t^h.
\]

Here we have used the fact that the number \( 48 \) counts the number of subsets of \([1, l + \sum_{h \geq 2} \nu_h]\) that contain \( v \), do not contain \( v^c \), and contain exactly \( \tilde{\mu}_h \) numbers.
such that \( f_i = t^h \), for each \( h \geq 2 \). This proves the claim and completes the proof. \( \square \)

**Theorem 3.4.** Suppose \( \gamma_i \in \mathbb{Z} \) for \( 1 \leq i \leq d \). Then

\[
\frac{\partial(\phi, \mathbf{n})}{\prod_{i=1}^{d} n_i!} \in \alpha^{\mathbf{n}} \gamma \mathbb{Z}[\alpha^{-1}, c_1^{-1}, c_2, c_3, \ldots, c_{\Sigma_{\mathbf{n}}^{-1}}].
\]  

(50)

**Proof.** We use induction on \( \Sigma_{\mathbf{n}} \). From Theorem 3.2, statement (50) is true when \( \Sigma_{\mathbf{n}} = 1 \). Assume it is true for all \( \mathbf{n} \) with \( \Sigma_{\mathbf{n}} \leq m \) for some \( m \geq 1 \). Given a \( \mathbf{n} \) with \( \Sigma_{\mathbf{n}} = m + 1 \), fix an \( r \) such that \( n_r \geq 1 \). Consider

\[
\frac{\partial(\phi, \mathbf{r})}{\mathbf{r}^{\gamma}} (51)
\]

After setting \( \mathbf{r} \) to \( \mathbf{0} \), expression (51) is equal to

\[
n_r \sum_{k=1}^{\Sigma_{\mathbf{n}} - 1} (\gamma_r)_k \alpha^{\gamma_r - k} \sum_{J \in \text{Parts}(I, k)} \prod_{i=1}^{k} \partial(\phi, J_i)
\]

(52)

where \( I \) is the ordered multiset satisfying \( I(h) \leq I(h+1) \), and with \( r \) occurring with multiplicity \( n_r - 1 \) and \( i \) occurring with multiplicity \( n_i \) for all other \( i \).

Now fix a \( k \) and \( J \). Suppose that there are \( v \) distinct sets out of the \( k \) sets of \( J \). Call these sets \( J_1, \ldots, J_v \), with \( J_i \) appearing \( b_i \) times in \( J \). The number of \( J' \in \text{Parts}(I, k) \) equivalent to \( J \) is then

\[
\frac{1}{\prod_{i=1}^{d} b_i!} \prod_{i=1}^{d} \frac{\#(i, J)!}{\prod_{i=1}^{k} \#(i, J_i)!}
\]

Collect these terms for \( J' \) equivalent to \( J \) and divide by \( \prod_{i=1}^{d} n_i! \). The total coefficient is then

\[
\frac{n_r (\gamma_r)_k \alpha^{\gamma_r - k}}{n_r \prod_{i=1}^{d} \#(i, J)!} \frac{1}{\prod_{i=1}^{k} \#(i, J_i)!} \prod_{i=1}^{k} \partial(\phi, J_i)
\]

\[
= \binom{\gamma_r}{k} \text{multinomial}((b_i)_{i=1}^v) \prod_{i=1}^{k} \frac{\partial(\phi, J_i)}{\prod_{i=1}^{k} \#(i, J_i)!}
\]

where have used \( \sum_{i=1}^{v} b_i = k \).

By the induction hypothesis

\[
\frac{\partial(\phi, J_i)}{\prod_{i=1}^{d} \#(i, J)!} \in \alpha^{\sum_{i \in I} \gamma_i} \mathbb{Z}[\alpha^{-1}, c_1^{-1}, c_2, c_3, \ldots, c_{\#(I, J)}],
\]

and \( \binom{\gamma_r}{k} \) is an integer by the assumption that \( \gamma_r \) is an integer.

Next, consider a fixed \( k \)-th term at line (53) where \( I \) is the ordered multiset satisfying \( I(h) \leq I(h+1) \), and with \( n_i = \#(i, I) \). Collecting terms for all equivalent \( J \) and applying similar reasoning above yields an element in the set at line (50), with \( k! \) taking the role of \( (\gamma_r)_k \). This completes the proof. \( \square \)
4 A transformation rule

Now we present definitions for Theorem 4.1.

Using base function $f(z)$, we rename its zero $\alpha$ by $\alpha_1$, and we denote the corresponding zero $\phi(a, g, \alpha_1)$ of $f(z)$ by

$$\phi(a, g, \alpha_1).$$

For a non-zero $\beta_2 \in \mathbb{C}$, we denote $\alpha_2$ by

$$\alpha_2 = \alpha_1 \beta_2.$$

(53)

Let $M(d)$ denote the set of $d$-tuples $\vec{n}$

$$\vec{n} = (n_i)_{i=1}^d$$

of non-negative integers $n_i$ such that $\sum n_i \geq 1$.

For $\vec{n} \in M(d)$, let $V(k, \vec{n})$ denote the set of $k \times d$ arrays $\vec{v}$

$$\vec{v} = (v_{i,j})$$

for $1 \leq i \leq d$ and $1 \leq j \leq k$

where

$$\sum_{j=1}^k v_{i,j} = n_i$$

and $v_{j} \in M(d)$ where

$$v_{j} = (v_{i,j})_{i=1}^d.$$

Let $\text{perm}(\vec{v})$ denote the number of permutations of the multiset $(v_{j})_{j=1}^k$. Let $w(\vec{n}, \vec{\gamma}, \beta)$ denote

$$w(\vec{n}, \vec{\gamma}, \beta) = (\beta_1^{-1} - 1 + \beta_1^{-1} \sum_{i=1}^d n_i \gamma_i) \prod_{i=1}^d a_{n_i}^\gamma.$$ 

Theorem 4.1. With the above notation, as Taylor series about $\vec{a} = \vec{0}$

$$\phi(\vec{a}, \vec{\gamma}, b, \beta_1) = \phi(\vec{a}, \vec{\gamma}, b, \beta_2 \beta_1).$$

Proof. We have by Theorem 2.5

$$\phi(\vec{a}, \vec{\gamma}, b, \beta_1) = \alpha_1 \left(1 + \sum_{\vec{n} \in M(d)} \frac{(-1)^{\sum n_i \gamma_i - 1}}{g'(\alpha)^{\sum n_i \gamma_i}} \beta_1^{\sum n_i - 1} \prod_{i=1}^d a_{n_i}^{\gamma_i} \right).$$

$$= \alpha_1 \left(1 + \beta_1^{-1} \sum_{\vec{n} \in M(d)} \alpha_1^{\sum n_i \gamma_i} w(\vec{n}, \vec{\gamma}, \beta) \prod_{i=1}^d a_{n_i}^{\gamma_i} \right).$$

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Therefore

\[
\phi(\tilde{a}; \tilde{\gamma}, b, \beta_1)^{\frac{1}{2}} = \alpha_1^{\frac{1}{2}} \left( 1 + \beta_1^{-1} \sum_{\vec{n} \in M(d)} \alpha_1^{\sum_{i=1}^d n_i \gamma_i} w(\vec{n}, \vec{\gamma}, \beta) \prod_{i=1}^d \frac{a_i^{n_i}}{n_i!} \right)^{\frac{1}{2}}
\]

\[
= \alpha_2 \left( 1 + \sum_{k=1}^{\infty} \beta_1^{-k} \binom{\beta_2^{-1}}{k} \left( \sum_{\vec{n} \in M(d)} \alpha_1^{\sum_{i=1}^d n_i \gamma_i} w(\vec{n}, \vec{\gamma}, \beta) \prod_{i=1}^d \frac{a_i^{n_i}}{n_i!} \right)^k \right)
\]

by the binomial theorem, using the fact that \( \tilde{a} \) is sufficiently close to \( \tilde{\nu} \).

For an \( \tilde{m} \in M(d) \), the coefficient of \( \prod_{i=1}^d \frac{a_i^{m_i}}{m_i!} \) in the above sum is

\[
\alpha_2 \frac{\sum_{i=1}^d m_i \gamma_i \Sigma_{\tilde{m}}}{\beta_1 \beta_2 \prod_{i=1}^d m_i!} \sum_{k=1}^{\Sigma_{\tilde{m}}} \beta_1^{-k} \binom{\beta_2^{-1}}{k} \sum_{v \in V(k, \tilde{m})} (\text{perm}(v) \prod_{i=1}^d \text{mult}((v_{i,j})_{j=1}^k)) \prod_{j=1}^k w(v^*_j, \vec{\gamma}, \beta)
\]

Let \( I \in \text{Multiset}(d) \) be the ordered multiset in which \#(i, I) = m_i and ordered so that \( I(j) \leq I(j+1) \). For \( J = \text{Parts}(I, k) \), let \( w(J, \vec{\gamma}, \beta) \) denote

\[
w(J, \vec{\gamma}, \beta) = \prod_{h=1}^k (\beta_1^{-1} - 1 + \beta_1^{-1} \sum_{i \in J_h} \gamma_i)_{|J_h|-1}.
\]

Then we may write expression \((54)\) as

\[
\alpha_2 \alpha_1 \frac{\sum_{i=1}^d m_i \gamma_i \Sigma_{\tilde{m}}}{\beta_1 \beta_2 \prod_{i=1}^d m_i!} \sum_{k=1}^{\Sigma_{\tilde{m}}} \beta_1^{-k} \binom{\beta_2^{-1}}{k} \sum_{J \in \text{Parts}(I, k)} w(J, \vec{\gamma}, \beta). \tag{55}
\]

By Theorem 2.7 we have

\[
(\beta_1^{-1})^{k-1} \sum_{J \in \text{Parts}(I, k)} w(J, \vec{\gamma}, \beta) = \frac{1}{(k-1)!} \sum_{r=0}^{k-1} (-1)^{k-1-r} \binom{k-1}{r} ((r+1)\beta_1^{-1} - 1 + \sum_{i \in I} \gamma_i)_{|J|-1}
\]

Let \( c_k \) denote the right side of the above equation. By Lemma 2.3 we have expression \((55)\) is equal to

\[
\alpha_2 \alpha_1 \frac{\sum_{i=1}^d m_i \gamma_i}{\beta_1 \beta_2 \prod_{i=1}^d m_i!} ((\beta_1 \beta_2)^{-1} - 1 + \beta_1^{-1} \sum_{i=1}^d m_i \gamma_i)_{\Sigma_{\tilde{m}}-1}.
\]

Using equation \((53)\), we see that this is the coefficient of \( \prod_{i=1}^d \frac{a_i^{m_i}}{m_i!} \) in the series for \( \phi(\tilde{a}; \tilde{\beta}_2 \vec{\gamma}, b, \beta_2 \beta_1) \). This completes the proof. \( \square \)
5 Further work

- Find a combinatorial meaning of the integer coefficients of the \( c_i \) terms using trees.
- Find a combinatorial meaning of the factored coefficients in \( g(z) \) when
  \[
g(z) = 1 + \frac{a_{i+d}}{a_i} z^d.
\]
- Generalize NRS using these combinatorial meanings.
- Directly prove the factorization of coefficients in \( g(z) \) from the formulas here.
- Generalize Theorem 4.1 for complex-exponent polynomials \( g(z) \) with multiple terms.
- Use the method of Lagrange or a modification to construct iterated radical formulas for polynomial zeros and relate them to Taylor series.

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