On Sobolev instability of the interior problem of tomography

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Abstract

As is known, solving the interior problem with prior data specified on a finite collection of intervals $I_i$ is equivalent to analytic continuation of a function from $I_i$ to an open set $J$. In the paper we prove that this analytic continuation can be obtained with the help of a simple explicit formula, which involves summation of a series. Our second result is that the operator of analytic continuation is not stable for any pair of Sobolev spaces regardless of how close the set $J$ is to $I_i$. Our main tool is the singular value decomposition of the operator $H^{-1}_e$ that arises when the interior problem is reduced to a problem of inverting the Hilbert transform from incomplete data. The asymptotics of the singular values and singular functions of $H^{-1}_e$, the latter being valid uniformly on compact subsets of the interior of $I_i$, was obtained in [BKT14]. Using these asymptotics we can accurately measure the degree of ill-posedness of the analytic continuation as a function of the target interval $J$. Our last result is the convergence of the asymptotic approximation of the singular functions in the $L^2(I_i)$ sense. We also present a preliminary numerical experiment, which illustrates how to use our results for reducing the instability of the analytic continuation by optimizing the position of the intervals with prior knowledge.

1 Introduction

Suppose one is interested in imaging a small region of interest (ROI) inside an object using tomography. In order to acquire a complete data set that enables stable reconstruction, one needs to send multiple x-rays through the object from many different directions. In particular, the x-rays that do not pass through the ROI are required as well. The interior problem of tomography arises when only the x-rays through the ROI are measured. In this case the tomographic data are incomplete, and image reconstruction becomes a challenging problem. In what follows, image reconstruction from x-ray data tailored to an ROI will be called the interior problem, and the corresponding data will be called interior data. Practical importance of the interior problem is clear, since tailoring the x-ray exposure to an ROI results in a reduced x-ray dose to the patient in medical applications of tomography. See [WY13] for a nice review of the state of the art in interior tomography.

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One of the most powerful tools for investigating the interior problem from the theoretical point of view is the Gelfand-Graev formula, which relates the tomographic data of an object with its one-dimensional Hilbert transform along lines [GG91]. With the help of this formula, the interior problem of tomography can be reduced to the problem of inverting the Hilbert transform from incomplete data, see e.g. [NCP04, DNCK06, ZPS05, YYWW07, YYW07, YYW08, KCND08, CNDK08]. A more recent line of research based on a differential operator that commutes with the Finite Hilbert Transform (FHT) is represented by the papers [Kat10, Kat02, KT12, AAK14, AADK15].

Let $f$ be sufficiently smooth and compactly supported. The (restricted) cone beam transform of $f$ is defined as a collection of integrals of $f$ along lines intersecting a curve $\Gamma$:

$$D_f(y, \beta) = \int_0^\infty f(y + t\beta) dt, \ y \in \Gamma,$$

where $\beta$ is a unit vector. We assume that $\Gamma$ is piecewise smooth and does not intersect the support of $f$. In practice, $\Gamma$ is called the source trajectory. The data $D_f$ are collected by moving the x-ray source along $\Gamma$, irradiating the object with multiple x-ray beams, and measuring the intensities of the beams after they exit the object.

Let $y(s)$ be a parametrization of $\Gamma$. We assume that $\Gamma$ does not self-intersect and is traversed in one direction as $s$ varies over some interval $I$. Pick any two values $s_1, s_2 \in I, s_1 \neq s_2$. Let $\alpha$ be a unit vector along the chord $y(s_1), y(s_2)$. Then one has [GG91]:

$$\frac{1}{2} \int_{s_1}^{s_2} \frac{1}{|x - y(s)|} \frac{\partial}{\partial \lambda} D_f \left( y(\lambda), \frac{x - y(s)}{|x - y(s)|} \right) \bigg|_{\lambda = s} ds = \int f(x + t\alpha) dt,$$

where $x$ is located on the chord between $y(s_1)$ and $y(s_2)$. Equation (1.2) implies that knowing the cone beam transform of $f$ one can compute the Hilbert transform of $f$ on the chords of $\Gamma$.

Fix any chord $[y(s_1), y(s_2)]$ of $\Gamma$, and let $L$ be the line determined by the chord. Source trajectories that are commonly used in practice have the property that for any point in the object support there is a chord of $\Gamma$ containing that point. In what follows we regard $L$ as the $x$-axis. Fix some $2g + 2, g \in \mathbb{N}$, distinct points $a_i$ on $L$: $a_i < a_{i+1}, i = 1, 2, \ldots, 2g + 1$. Points $a_1$ and $a_{2g+2}$ mark the boundaries of the support of $f$ along $L$. Points $a_2$ and $a_{2g+1}$ mark the boundaries of the ROI along $L$. Consider the FHT

$$(Hf)(x) := \frac{1}{\pi} \int_{a_1}^{a_{2g+2}} \frac{f|_L(y)}{y - x} dy, \ \ f|_L \in L^2([a_1, a_{2g+2}]).$$

Here $f|_L$ is the restriction of $f$ to $L$, and $Hf$ is the one-dimensional Hilbert transform of $f|_L$.

Throughout the paper the line $L$ is always the same, so with some abuse of notation we write $f$ instead of $f|_L$. In the case of interior tomographic data, the Gelfand-Graev formula allows computation of $Hf$ only on $[a_2, a_{2g+1}]$, but not on all $[a_1, a_{2g+2}]$. Thus the interior problem of tomography is reduced to finding $f$ inside the ROI, i.e. on $[a_2, a_{2g+1}]$, by solving the equation

$$(Hf)(x) = \varphi(x), \ x \in [a_2, a_{2g+1}].$$

Consider the operator $H : L_2([a_1, a_{2g+2}]) \to L_2([a_2, a_{2g+1}])$. Unique recovery of $f$ on $[a_2, a_{2g+1}]$ is impossible since $H$ has a non-trivial kernel (see [KT12] for its complete description). Therefore, to achieve unique recovery the data $\varphi$ should be augmented by some additional information. One
type of information that guarantees uniqueness is the knowledge of $f$ on some interval or intervals inside $[a_2, a_{2g+1}]$. This is the so-called interior problem with prior knowledge ([YYW07, KCND08, CNDK08, WY13]) that will be considered below. Let us assume that $f$ is known on the intervals

$$I_i := [a_3, a_4] \cup [a_5, a_6] \cup \cdots \cup [a_{2g-1}, a_{2g}],$$

which we call “interior” (inside the ROI). Denote by $I_e := [a_1, a_2] \cup [a_{2g+1}, a_{2g+2}]$ the remaining “exterior” intervals (they are outside the ROI). Applying the FHT inversion formula (see e.g. [OE91]), we get

$$f(y) = -\frac{w(y)}{\pi} \left( \int_{a_1}^{a_2} + \int_{a_{2g+1}}^{a_{2g+2}} \right) \frac{\varphi(x)}{w(x)(x-y)} dx \frac{w(y)}{\pi} \int_{a_2}^{a_{2g+1}} \frac{\varphi(x)}{w(x)(x-y)} dx,$$

where $w(x) := \sqrt{(a_{2g+2} - x)(x - a_1)}$ and $\varphi(x) = (Hf)(x), \ x \in [a_1, a_{2g+2}]$.

The left side of (1.6) is known on $I_i$. The last integral on the right is known everywhere. Combining these known quantities we get an integral equation:

$$(H_{e}^{-1}\varphi)(y) := -\frac{w(y)}{\pi} \int_{I_e} \frac{\varphi(x)}{w(x)(x-y)} dx = \psi(y), \ y \in I_i,$$

where

$$\psi(y) = f(y) + \frac{w(y)}{\pi} \int_{a_2}^{a_{2g+1}} \frac{\varphi(x)}{w(x)(x-y)} dx, \ y \in I_i$$

is a known function.

The main problem we study in this paper is the stability of finding $f$ from the data. Several approaches to finding $f$ on $[a_2, a_{2g+1}]$ are possible. The first one consists of two steps. In step 1 we solve equation (1.7) for $\varphi(x)$ on $I_e$. In step 2 we substitute the computed $\varphi(x)$ into (1.6) and recover $f(y)$ on $[a_2, a_{2g+1}]$. It is clear that solving (1.7), i.e. inverting $H_{e}^{-1}$, is the most unstable step. Consider the operator $H_{e}^{-1}$ in (1.7) as a map between two weighted $L^2$-spaces:

$$H_{e}^{-1} : L^2(I_e, 1/w) \rightarrow L^2(I_i, 1/w).$$

(1.9)

Its adjoint is the Hilbert transform:

$$(H_{i}\psi)(x) := \frac{1}{\pi} \int_{I_e} \frac{\psi(y)}{y-x} dy, \ x \in I_e.$$  

(1.10)

In [BKT14] the authors studied the singular value decomposition (SVD) for the operator $H_{e}^{-1}$. Namely, we were interested in the singular values $2\lambda = 2\lambda_n > 0, \ n \in \mathbb{N}$, and the corresponding left and right singular functions $f = f_n, \ h = h_n$, satisfying

$$(H_{e}^{-1}h)(y) = -\frac{w(y)}{\pi} \int_{I_e} \frac{h(x)}{w(x)(x-y)} dx = 2\lambda f(y), \ y \in I_i,$$

$$(H_{i}f)(x) = \frac{1}{\pi} \int_{I_e} \frac{f(y)}{y-x} dy = 2\lambda h(x), \ x \in I_e.$$  

(1.11)

See (2.1)–(2.3) and Theorem 2.1, which show that the SVD is well-defined. It is well known that the rate at which $\lambda_n$’s approach zero is related with the ill-posedness of inverting $H_{e}^{-1}$. Because
of the symmetry \((\lambda, f, h) \leftrightarrow (-\lambda, -f, h)\) of (1.11), we are interested only in positive \(\lambda_n\). The main result of the paper [BKT14] is the large \(n\) asymptotics of \(\lambda_n\) (see (1.14)), \(f_n\) and \(h_n\) (Section 2). Let us introduce a \(g \times g\) matrix \(\mathcal{A}\) by

\[
(\mathcal{A})_{kj} = 2 \int_{a_{2k}}^{a_{2k+1}} \frac{z^{j-1}dz}{R(z)}, \quad k = 1, \ldots, g - 1, \quad \text{and} \quad (\mathcal{A})_{gj} = 2 \int_{a_1}^{a_{2g+2}} \frac{z^{j-1}dz}{R(z)}, \quad j = 1, \ldots, g,
\]

where \(R(z) = \prod_{j=1}^{2g+2} (z - a_j)^{\frac{1}{2}}\) is an analytic function on \(\mathbb{C} \setminus (I_c \cup I_i)\) behaving as \(z^{g+1}\) at infinity, and define

\[
\tau_{11} = -2 \sum_{j=1}^{g} (\mathcal{A}^{-1})_{j1} \int_{I_c} \frac{z^{j-1}dz}{R(z)}.
\]

Here and throughout the paper the subscripts \(\pm\) routinely denote limiting values of functions (vectors, matrices) from the left/right side of the corresponding oriented arcs. In particular, \(R_+\) means the limiting value of \(R\) on \(I = I_c \cup I_i\) from \(\Im z > 0\). We also want to note that, according to the well-known Riemann’s Theorem on periods of holomorphic differentials [FK92], \(\tau_{11}\) is a purely imaginary number with positive imaginary part. Then the asymptotics of \(\lambda_n\) is given by

\[
\lambda_n = e^{-\frac{\pi n}{\tau_{11}} + O(1)}, \quad n \to \infty.
\]

The rapid decay of singular values in (1.14) indicates that finding \(\varphi\) from \(\psi\) is very unstable. This, however, does not imply that finding \(f\) on \([a_2, a_{2g+1}]\) is unstable, since \(f\) is computed by applying a smoothing operator to \(\varphi\). The second approach to finding \(f\) is based on the observation that the function \(\psi\) defined by (1.8) is analytic in \(\mathbb{C} \setminus I_c\) (cf. (1.7)). Hence, analytically continuing \(\psi\) from \(I_i\) to \((a_2, a_{2g+1})\), we can find \(f\) using (1.8) with \(y \in (a_2, a_{2g+1})\). Note that any method that gives \(f\) on \((a_2, a_{2g+1})\) is equivalent to analytic continuation of \(\psi\) in view of (1.8). Thus, analytic continuation of \(\psi\) is at the heart of any method for solving the interior problem of tomography with prior knowledge.

In this paper we obtain two results regarding the analytic continuation of \(\psi\). We show that this analytic continuation can be obtained with the help of an explicit formula, which involves summation of a series, see Corollary 3.4. We prove that the series is absolutely convergent if \(\psi\) is in the range of \(H^{-1}_c\). Our second result is that the operator of analytic continuation is not stable for any pair of Sobolev spaces: \(H^{s_1}(I_i) \to H^{-s_2}(J)\), where \(J\) is any open set containing \(I_i\). In other words, the procedure is unstable no matter how close to \(I_i\) we perform the continuation. The importance of our results is two-fold. First, we prove that interior reconstruction with prior knowledge that includes only knowing \(f\) on a subinterval (or, even several subintervals) cannot be stable. In contrast, the results of [DNCK06, CNDK08] show that by using additional prior knowledge stability can be restored. The additional information in [DNCK06, CNDK08] includes a restriction on the global behavior of \(Hf\). Thus, our results and those of [DNCK06, CNDK08] complement each other. Global information can be used in an algorithm only if the latter reconstructs \(f\) on all of its support (otherwise \(Hf\) cannot be computed). In some practical applications one may be interested in reconstructing only a small ROI inside a relatively large object. In this case algorithms based on global recovery of \(f\) may be impractical. Since we obtain the behavior of singular functions at different distances from the intervals with prior knowledge and we know the
rate of decay of singular values, our results may be useful for developing appropriate purely local regularized reconstruction algorithms.

There is a large body of results on tomographic image reconstruction from interior data, see e.g. the introduction in [CNDK08]. One line of research deals with the singular value decomposition for the interior Radon transform, see e.g. [LR89, Maa92]. Another direction deals with the reconstruction of singularities of a function, see e.g. [GU89, Qui93, RK96, KLM95, FBH+01]. A third direction uses wavelets to localize the Radon transform inversion [BW96]. None of these algorithms is designed to reconstruct $f$ exactly. Thus, analysis of stability of exact reconstruction from interior data is a practically important problem.

The paper is organized as follows. Since the derivation of our main results strongly depends on the results in [BKT14], the latter are briefly reviewed in Section 2. Analytic continuation of $\psi$ is studied in Section 3. In Section 3.1 we show that $\psi$ can be continued analytically with the help of a convergent series. Instability of the continuation in the scale of Sobolev spaces is established in Section 3.2. The asymptotics of the singular values and singular functions of (1.11) allows us to accurately estimate the degree of instability of the continuation. In Section 3.2 we also use an exponentially growing weight to introduce a Hilbert space $\mathcal{A}$ of functions defined on $I_i$. We show how fast the weight must grow to ensure that the analytic continuation from $I_i$ to an open set $J$ be a continuous map from $\mathcal{A} \to L^2(J)$. Thus, this rate of growth measures the degree of ill-posedness of the analytic continuation as a function of the target interval $J$. A comparison of our results with those of [CNDK08] is presented in Section 3.3.

In [BKT14] it is shown that the asymptotic approximation to the exact singular functions $f_n$ is valid uniformly on compact subsets of the interior of $I_i$ as $n \to \infty$. In Section 4 we show that the approximation is valid in the $L^2(I_i)$ sense as well. This is the third result obtained in this paper. We do not consider the set of right singular functions defined on $I_e$, since they are not needed for the analytic continuation of $\psi$. To simplify reading the paper, only an outline of the proof of the $L^2$ approximation property is given in Section 4. All the remaining technical proofs are collected in Appendix B. A preliminary numerical experiment, which illustrates the use of our results for reducing the instability of the analytic continuation by optimizing the position of the intervals with prior knowledge, is described in Section 5. The degree of flexibility in choosing the intervals with prior knowledge depends on a particular application. Our approach can be used not only for solving the interior problem of tomography, but whenever the incomplete Hilbert transform data needs to be inverted and prior knowledge is available to guarantee uniqueness.

To make the paper self-contained and for the convenience of the reader, the key results of [BKT14] that are needed for our proofs are summarized in Appendix A.

2 Necessary preliminaries

This section contains a brief review of the results from [BKT14] that are necessary for our analysis. For convenience of the reader, most of the statements below are provided with direct references (in square brackets) to the corresponding results in [BKT14]. Additional related information can be found in Appendix A.
The SVD system (1.11) can be represented in the form

\[
(H_{e}^{-1}\tilde{h})(y) := \frac{\sqrt{w(y)}}{2\pi i} \int_{I_{e}} \frac{\tilde{h}(x)}{\sqrt{w(x)}(x - y)} \, dx = \lambda y, \quad y \in I_{e},
\]

\[
(H_{i}\tilde{f})(x) := \frac{1}{2\pi i} \frac{1}{\sqrt{w(x)}} \int_{I_{e}} \frac{\tilde{f}(y)\sqrt{w(y)}}{(y - x)} \, dy = \lambda \tilde{h}(x), \quad x \in I_{e},
\]

where \(\tilde{h} = \frac{h}{\sqrt{w}} \in L^{2}(I_{e}), \tilde{f} = \frac{f}{\sqrt{w}} \in L^{2}(I_{i}),\) and the operators \(H_{e}^{-1}, H_{i}\) act on the corresponding unweighted \(L^{2}\) spaces. It can be checked directly that the triple \((\lambda, \tilde{f}, \tilde{h})\) satisfies the system (2.1) if and only if \(\lambda, \psi\) is the eigenvalue/eigenvector of the integral operator \((\hat{K}\phi)(z) = \int_{I} K(z, x)\phi(x)dx\) from \(L^{2}(I)\) to \(L^{2}(I)\), where

\[
K(z, x) = w^{\frac{i}{2}}(x)w^{-\frac{1}{2}}(z)\chi_{e}(z)\chi_{i}(x) + w^{\frac{i}{2}}(z)w^{-\frac{1}{2}}(x)\chi_{e}(z)\chi_{i}(x), \quad \psi = \tilde{f}(z)\chi_{i}(z) + \tilde{h}(z)\chi_{e}(z).
\]

Here and henceforth \(\chi_{i}(z), \chi_{e}(z)\) denote the characteristic (indicator) functions of the sets \(I_{i}, I_{e}\), respectively. Thus, the SVD problem for the system (2.1) is reduced to the spectral problem for the integral operator \(\hat{K} : L^{2}(I) \to L^{2}(I)\). It follows directly from (2.2) that

\[
\hat{K}|_{L^{2}(I_{i})} = H_{i}, \quad \hat{K}|_{L^{2}(I_{e})} = H_{e}^{-1}.
\]

**Theorem 2.1.** [Thm.3.1 and Cor.3.8] \(\hat{K}\) is a self-adjoint and a Hilbert–Schmidt operator. Moreover, all the eigenvalues of \(\hat{K}\) are simple.

According to Theorem 2.1, the eigenvalues of \(\hat{K}\) are real with the only possible point of accumulation \(\lambda = 0\). Because of the symmetry \((\lambda, \tilde{f}, \tilde{h}) \mapsto (-\lambda, -\tilde{f}, \tilde{h})\) in (2.1), we may assume that the singular values \(\lambda_{n}, n \in \mathbb{N},\) of \(\hat{K}\) are positive and ordered \(\lambda_{0} > \lambda_{1} > \cdots > 0\).

An important object of the spectral theory is the resolvent operator \(\hat{R}\) of \(\hat{K}\), defined by

\[
(\text{Id} + \hat{R})(\text{Id} - \frac{1}{\lambda} \hat{K}) = \text{Id}.
\]

In our case \(\hat{R}\) is an integral operator with the kernel of the form

\[
R(z, x; \lambda) = \frac{\bar{g}^{T}(x)\Gamma^{-1}(x; \lambda)\Gamma(z; \lambda)\tilde{f}(z)}{2i\pi \lambda(z - x)}, \quad \text{where} \quad \tilde{f}(z) := \begin{bmatrix} i\chi_{e}(z) \\ \sqrt{w(z)}\chi_{i}(z) \end{bmatrix}, \quad \bar{g}(x) := \begin{bmatrix} \frac{\sqrt{w(x)}\chi_{i}(x)}{\lambda_{0}(x)} \\ -i\sqrt{w(x)}\chi_{e}(x) \end{bmatrix},
\]

\(\bar{g}^{T}\) denotes the transposition of \(\bar{g}\), and the matrix \(\Gamma(z; \lambda)\) satisfies the following Riemann-Hilbert Problem (RHP) 2.2.

**RHP 2.2.** Find a \(2 \times 2\) matrix-function \(\Gamma = \Gamma(z; \lambda), \lambda \in \mathbb{C} \setminus \{0\},\) which is analytic in \(\mathbb{C} \setminus I,\) where \(I = I_{i} \cup I_{e}\), admits non-tangential boundary values from the upper/lower half-planes that belong to
Proposition 2.4. \( \Gamma_+(z; \lambda) = \Gamma_-(z; \lambda) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad z \in I_i; \quad \Gamma_+(z; \lambda) = \Gamma_-(z; \lambda) \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{\sqrt{w}} \end{bmatrix}, \quad z \in I_e, \) (2.6)

\[
\Gamma(z; \lambda) = 1 + O(z^{-1}) \quad \text{as} \quad z \to \infty, \quad (2.7)
\]

\[
\Gamma(z; \lambda) = \left[ O(1), O((z-a_j)^{-\frac{1}{2}}) \right], \quad z \to a_j, \quad j = 1, 2g + 2, \quad (2.8)
\]

\[
\Gamma(z; \lambda) = \left[ O(1), O(\ln(z-a_j)) \right], \quad z \to a_j, \quad j = 2, 2g + 1, \quad (2.9)
\]

\[
\Gamma(z; \lambda) = \left[ O(\ln(z-a_j)), O(1) \right], \quad z \to a_j, \quad j = 3, \ldots, 2g. \quad (2.10)
\]

Here the endpoint behavior of \( \Gamma \) is described column-wise. We will frequently omit the dependence on \( \lambda \) from notation and write simply \( \Gamma(z) \) for convenience.

The latest fact links the resolvent operator \( \hat{R} \) for \( \hat{K} \) with the RHP for the matrix \( \Gamma \) from (2.5).

Theorem 2.3. [Thm.3.17 and Prop.3.12] The RHP 2.2 has a solution \( \Gamma(z; \lambda) \), where \( \lambda \in \mathbb{C} \setminus \{0\} \), if and only if \( \lambda \) is not an eigenvalue of \( \hat{K} \). Moreover, for any fixed \( \lambda \in \mathbb{C} \setminus \{0\} \) the RHP 2.2 has at most one solution.

Connection of our spectral problem with the RHP 2.2 is remarkable, as the RHP 2.2 is a much more convenient object for rigorous asymptotic analysis (in small \( \lambda \)) than the spectral problem for \( \hat{K} \). The eigenfunctions of \( \hat{K} \) corresponding to a fixed eigenvalue \( \lambda_n \) are given by two proportional expressions

\[
\phi_{n,j}(z) = \frac{\lambda_{
u}^{c}(z)}{\sqrt{w(z)}} \text{res} \left. \Gamma_1(z; \lambda) \right|_{\lambda=\lambda_n} \frac{1}{\lambda} + i \frac{\sqrt{w(z)}}{\lambda} \text{res} \left. \Gamma_2(z; \lambda) \right|_{\lambda=\lambda_n} \frac{1}{\lambda}, \quad j = 1, 2, \quad (2.11)
\]

in terms of the entries of the matrix \( \Gamma(z, \lambda) \). In (2.11), for every \( n \in \mathbb{N} \) at least one of \( \phi_{n,j}, j = 1, 2 \), is not identical zero on \( I \).

Once the connection between the spectral problem for \( \hat{K} \) and the RHP 2.2 is established, we use the nonlinear steepest descent method of Deift and Zhou (see, for example, [DZ92, Dei99]) to construct an explicit leading order approximation of \( \Gamma(z, \lambda) \) as \( \lambda \to 0^+ \) in terms of the Riemann Theta functions. The crucial element of this construction is the so-called \( g \)-function given by

\[
g(z) = \frac{1}{2} - 2 \int_{a_1}^{z} \omega_1 dz, \quad \text{where} \quad \omega_1(z) = [\omega_1(z), \ldots, \omega_g(z)] = \left[ 1, \ldots, z^{g-1} \right] A^{-1} \quad (2.12)
\]

and the matrix \( A \) is defined by (1.12). Important properties of \( g \) are stated in the following proposition.

Proposition 2.4. [Prop.4.2] \( g(z) \) satisfies the jump conditions

\[
g_+(z) + g_-(z) = -1 \quad \text{on} \quad I_i, \quad g_+(z) + g_-(z) = 1 \quad \text{on} \quad I_e, \quad (2.13)
\]

and \( g_+(z) - g_-(z) = i\Omega_{\mu(j)} \quad \text{on} \quad [a_{2j}, a_{2j+1}], \quad j = 1, \ldots, g, \) (2.14)

where \( \Omega_0 = \frac{4}{7} \sum_{k=1}^{g} \int_{a_{2k-1}}^{a_{2k}} \omega_1 dz \in \mathbb{R} \) and \( \Omega_j = \frac{4}{7} \sum_{k=1}^{j} \int_{a_{2k-1}}^{a_{2k}} \omega_1 dz \in \mathbb{R} \).
3 Instability of the interior problem in Sobolev spaces

3.1 Continuation of \( \psi \) from \( I_i \)

The function \( \psi(y) \) in (1.7) is analytic in \( \mathbb{C} \setminus I_e \) and is known on \( I_i \). If we can find the analytic continuation of \( \psi(y) \) on \((a_2, a_{2g+1})\), then, according to (1.8), we can solve the problem of reconstructing \( f \) on \((a_2, a_{2g+1})\).

The idea of such reconstruction is straightforward. The eigenfunctions \( \phi_n = \frac{1}{\sqrt{2}}(\hat{f}_n \chi_i + \hat{h}_n \chi_e) \)
of the self-adjoint Hilbert-Schmidt integral operator \( \hat{K} : L^2(I) \rightarrow L^2(I) \) form an orthonormal basis in \( L^2(I) \). Thus, \( \hat{f}_n, \hat{h}_n \) form orthonormal bases in \( L^2(I_i), L^2(I_e) \) respectively, so that \( f_n, h_n \) form orthonormal bases in the corresponding spaces \( L^2(I_i, 1/w), L^2(I_e, 1/w) \). Note that the former coincides with \( L^2(I) \). Given \( \psi \in L^2(I_i, 1/w) \) and \( \varphi \in L^2(I_e, 1/w) \) we have

\[
\psi = \sum \phi_n f_n \text{ on } I_i \quad \text{and} \quad \varphi = \sum \phi_n h_n \text{ on } I_e,
\]

where \( \sum \phi_n^2 < \infty, \sum \phi_n^2 < \infty \). According to (2.1), \( \mathcal{H}_e^{-1} h_n = 2\lambda_n f_n \), so that \( \mathcal{H}_e^{-1} \varphi = \psi \) and (3.1) imply \( \psi_n = 2\lambda_n \varphi_n \). In view of the asymptotics (1.14) of \( \lambda_n \), the coefficients \( \psi_n \) decay exponentially fast. According to (1.11), the singular functions \( f_n \) are analytic in \( \mathbb{C} \setminus I_e \). Thus the question of analytic continuation of \( \psi \) from \( I_i \) to \((a_2, a_{2g+1})\) is reduced to the question of convergence of

\[
\psi = \sum \psi_n f_n \text{ in } (a_2, a_{2g+1}) \setminus I_i.
\]

Let \( I_\omega, \omega > 0 \), denote the set of all \( z \in (a_2, a_{2g+1}) \setminus I_i \) that are at least distance \( \omega \) away from the nearest branchpoint \( a_j, j = 2, 3, \ldots, 2g + 1 \). Below, we consider some fixed \( \omega > 0 \), such that \( a_j + \omega < a_{j+1} - \omega \) for all \( j = 2, \ldots, 2g \).

**Lemma 3.1.** There exists a constant \( C_\omega > 0 \), such that for all \( n \in \mathbb{N} \) and all \( z \in I_\omega \)

\[
|f_n(z)| \leq C_\omega e^{\kappa_n (\Re g(z) + \frac{1}{2})},
\]

where \( \kappa_n = -\ln \lambda_n \).

Lemma 3.1 follows from Lemma B.4, Theorem A.13 and (2.1).

**Lemma 3.2.** \( |\Re g(z)| < \frac{1}{2} \) for any \( z \in \mathbb{C} \setminus I \), with \( \Re g(z) \equiv \frac{1}{2} \) on \( I_e \) and \( \Re g(z) \equiv -\frac{1}{2} \) on \( I_i \).

**Proof.** Note that \( g(z) \) is Schwarz symmetrical and satisfies the jump conditions \( g_+ + g_- \equiv 1 \) on \( I_e \) and \( g_+ + g_- \equiv -1 \) on \( I_i \), see Proposition 2.4. Thus, \( \Re g(z) \equiv \frac{1}{2} \) on \( I_e \) and \( \Re g(z) \equiv -\frac{1}{2} \) on \( I_i \). The remaining statement follows from the maximum principle for harmonic functions.

**Theorem 3.3.** For a given \( \omega > 0 \), the series \( \psi(z) = \sum \psi_n f_n(z) \) converges absolutely and uniformly on \( I_\omega \).

**Proof.** As a consequence of Lemma 3.1, we have

\[
\sum |\psi_n f_n(z)| \leq 2C_\omega \varphi_* \sum e^{\kappa_n (\Re g(z) - \frac{1}{2})},
\]

where \( \varphi_* = \max_{n} \{|\varphi_n|\} < \infty \). In light of (1.14) and Lemma 3.2, the series in the right hand side of (3.3) converges absolutely and uniformly on \( I_\omega \).

**Corollary 3.4.** The series \( \psi(z) = \sum \psi_n f_n(z) \) provides analytic continuation of \( \psi \) onto \((a_2, a_{2g+1})\).

Indeed, by choosing a sufficiently small \( \omega \), one can analytically continue \( \psi(z) \) to any point in \((a_2, a_{2g+1}) \setminus I_i \) through this series.
3.2 Instability of analytic continuation in Sobolev norms

In the previous section we obtained a formula for analytic continuation of $\psi(y)$ from $I_i$ to all of $(a_2, a_{2g+1})$. Next we prove that analytic continuation of $\psi$ from $I_i$ is unstable for any pair of Sobolev spaces: $H^{s_1}(I_i) \rightarrow H^{-s_2}(\mathbf{J})$, where $\mathbf{J}$ is any open set containing at least one of the intervals that makes up $I_i$ (cf. (1.5)). Clearly, it makes sense to consider $s_1, s_2 > 0$. For simplicity we will assume that $s_1$ and $s_2$ are integers, so (see Chapter 1 in [ES97]):

$$\|f\|_{H^{s_1}(I_i)}^2 := \sum_{j=0}^{s_1} \int_{I_i} |f^{(j)}(y)|^2 dy,$$

and

$$\|f\|_{H^{-s_2}(\mathbf{J})} := \inf_{\tilde{f} \in H^{-s_2}(\mathbf{R}), f = \tilde{f}|_{\gamma \in C_c(\mathbf{R})}} \|\tilde{f}\|_{H^{-s_2}(\mathbf{R})}.$$

Let $\gamma$ be a collection of simple loops in the complex plane so that $I_i$ is contained in the union of the interiors of the loops. We take $\gamma$ to be sufficiently close to $I_i$. By the Cauchy integral theorem using the analyticity of $f_n$ one can show that

$$\|f_n\|_{H^{s_1}(I_i)} \leq c(s_1, \gamma) \max_{z \in \gamma} |f_n(z)|$$

for some $c(s_1, \gamma) > 0$. Analogously to Lemma 3.1, it follows from Lemma B.4 that

$$\max_{z \in \gamma} |f_n(z)| \leq c_{\gamma} \exp(\kappa_{\gamma}(\max_{z \in \gamma} \Re g(z) + \frac{1}{2}))$$

for some $c_{\gamma} > 0$. By taking $\gamma$ sufficiently close to $I_i$, we can make $\max_{z \in \gamma} \Re g(z) + \frac{1}{2}$ as close to zero as we want.

**Lemma 3.5.** One can find a sequence of intervals $J_n \subset \mathbf{J}$ with the following properties:

1. The length of each $J_n$ is greater than a fixed positive constant independent of $n$;
2. The distance of each $J_n$ to $I_i$ is greater than a fixed positive constant independent of $n$; and
3. There exists $N > 0$ large enough such that

$$|f_n(y) - f_0(y)| \geq c \exp(\kappa N \Re(y) + \frac{1}{2}), \quad n \geq N, \quad y \in J_n,$$

for some $c > 0$ independent of $n$.

Lemma 3.5 is proven in Appendix B. By property 1 in Lemma 3.5 we can find $L > 0$ such that the length of each interval $J_n$ is greater than or equal to $L$. Then we select a real-valued function $\phi \in C_c^\infty([-L/2, L/2])$, $\phi \geq 0$, $\phi \neq 0$. By shifting $\phi$ appropriately, we get a collection of functions $\phi_n \in C_c^\infty(J_n)$ and they all have the same $H^{s_2}(\mathbf{R})$-norm. Using the facts that: (i) $f$ and $\tilde{f}$ coincide on $\mathbf{J}$ (cf. (3.5)); (ii) $f_n$’s are real-valued on $\mathbf{J}$, and; (iii) $f_n$’s do not change sign on $J_n$ for $n$ large (cf. (3.8)), equation (3.5) immediately yields

$$\|f_n\|_{H^{-s_2}(\mathbf{J})} \geq c_\phi \min_{y \in J_n} |f_n(y)|, \quad n \geq N,$$

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for some \( c_o > 0 \).

From the second property in Lemma 3.5, by choosing \( \gamma \) sufficiently close to \( I_i \) so that all \( J_n \) are in the exterior of \( \gamma \) and \( \text{dist}(\gamma, \cup_n J_n) > 0 \), we get \( \inf_{y \in \cup_n J_n} \Re g(y) > \max_{z \in \gamma} \Re g(z) \). Hence,

\[
\frac{\exp(\kappa_n(\inf_{y \in J_n} \Re g(y) + \frac{1}{2}))}{\exp(\kappa_n(\max_{z \in \gamma} \Re g(z) + \frac{1}{2}))} \to \infty, \quad n \to \infty. \tag{3.10}
\]

Hence it follows from (3.7) and (3.8) that the quantity \( \|f_n\|_{H^{-2}(J)} \) cannot be bounded in terms of \( \|f_n\|_{H^{\gamma_1}(I_i)} \). Since the Sobolev norm \( \|f\|_{H^s} \) is a monotonically increasing function of \( s \) (provided that \( f \) belongs to the appropriate spaces), our argument proves the following result.

**Theorem 3.6.** Fix an open set \( J \) that contains at least one of the intervals that makes up \( I_i \). The operation of analytic continuation from \( I_i \) to \( J \) described in Corollary 3.4 cannot be extended to a continuous operator \( H^{s_1}(I_i) \to H^{-s_2}(J) \) for any \( s_1, s_2 \).

Theorem 3.6 shows that analytic continuation is more unstable than calculation of any number of derivatives. An interesting question is to estimate the degree of ill-posedness of analytic continuation. This can be done, for example, by finding a Hilbert space \( \mathcal{A} \) on which the operator of analytic continuation is bounded. It is clear that the space \( \mathcal{A} \) should contain at least all functions in the range of \( \mathcal{H}_{e}^{-1} : L^2(I_e, 1/w) \to L^2(I_i, 1/w) \). If \( \psi \in \mathcal{A} \), but \( \psi \) is not in the range of \( \mathcal{H}_e^{-1} \), then the analytic continuation of \( \psi \) is understood via the summation of the series in Corollary 3.4.

Let \( w_n \) be a sequence of positive numbers. Introduce the following space:

\[
\mathcal{A} := \{ \psi \in L^2(I_i) : \sum_{n \geq 0} w_n^2 |\psi_n|^2 < \infty \}, \tag{3.11}
\]

where

\[
\psi_n := \langle \psi, f_n \rangle := \int_{I_i} \psi(y) f_n(y) \frac{1}{w(y)} \, dy. \tag{3.12}
\]

It is obvious that \( \mathcal{A} \) is a Hilbert space with the inner product defined by the formula

\[
\langle \psi^{(1)}, \psi^{(2)} \rangle := \sum_{n \geq 0} w_n^2 \psi_n^{(1)}(\psi_n^{(2)}). \tag{3.13}
\]

**Theorem 3.7.** Fix an open set \( J \), whose closure is a subset of \((a_2, a_{2g+1})\). Suppose that each connected component of \( J \) contains at least one of the intervals that make up \( I_i \). Suppose the sequence of \( w_n \)'s is such that the limit below exists and satisfies

\[
0 < \lim_{n \to \infty} \left\{ \frac{w_n}{n} \exp(-\kappa_n(\sup_{z \in J} \Re g(z) + \frac{1}{2})) \right\} < \infty. \tag{3.14}
\]

Then one has: (1) \( \mathcal{H}_e^{-1}(L^2(I_e, 1/w)) \subset \mathcal{A} \), and; (2) the operator of analytic continuation acting between the spaces \( \mathcal{A} \to L^2(J) \) is continuous.

**Proof.** Similarly to the proof of Theorem 3.3, it is easily seen that assertion (1) holds. Now we prove assertion (2). First we show that

\[
\max_{z \in J} |f_n(z)| \leq c_3 \exp(\kappa_n(\sup_{z \in J} \Re g(z) + \frac{1}{2})) \tag{3.15}
\]
for some \( c_J > 0 \). Denote \( G := \sup_{z \in J} \Re g(z) \). Let \( \gamma \) be a collection of simple contours in \( \mathbb{C} \) containing the components of \( J := \sup_{x \in J} \Re g(z) \) in their interior. By making \( \gamma \) as close to these components as we need and using Lemma 3.2, we can find \( \gamma \) such that \( \sup_{z \in \gamma} \Re g(z) < G \). Now (3.15) follows immediately by using inequalities (3.2) and (3.7) combined with the maximum modulus principle. Finally, to prove (2) we fix any \( N > 0 \). Then

\[
\int \left| \sum_{n=0}^{N} \psi_n f_n(z) \right|^2 \, dz \leq |J| \left( \sum_{n=0}^{N} \left| \psi_n \sup_{z \in J} |f_n(z)| \right| \right)^2 \leq c \left( \sum_{n=0}^{N} |\psi_n| w_n \right)^2 \leq c \sum_{n=0}^{N} (|\psi_n| w_n)^2 \sum_{n=0}^{N} \frac{1}{n^2},
\]

where \( c > 0 \) is some constant. By taking the limit \( N \to \infty \) the desired assertion follows immediately.

\[\Box\]

**Remark 3.8.** Suppose the geometry of the problem, i.e. points \( a_1, \ldots, a_{2g+2} \), are fixed, and we would like to estimate how unstable it is to perform analytic continuation from \( I_i \) to \( J \) for different \( J \). From (3.10) and Theorem 3.7 we see that the quantity \( Q(J) := \sup_{x \in J} \Re g(z) + \frac{1}{2} \) can be viewed as a measure of instability. Observe that if \( J \) is located between \( a_3 \) and \( a_{2g} \), then \( Q(J) \) is strictly less than and separated from 1. However, if \( J \) extends to the left of \( a_3 \) or to the right of \( a_{2g} \), then the closer \( J \) is to \( I_e \), the closer \( Q(J) \) gets to 1.

**Remark 3.9.** Using the fact that the singular functions \( f_n \) are analytic on \( J \) and the coefficients \( \psi_n \) go to zero sufficiently fast, similarly to the proof of Theorem 3.3 and (3.16) it is easy to see that each \( \psi \in A \) defined on \( J \) via the series in Corollary 3.4 is a uniform limit of analytic functions. Hence the continuation of \( \psi \) from \( I_i \) to \( J \) via the series and via the conventional analytic continuation coincide.

### 3.3 Comparison with stability estimates from [CNDK08]

Important first results on stability of inverting the FHT from incomplete data were obtained in [DNCK06] and [CNDK08]. For convenience of the reader we present here a brief summary of the stability estimates of [CNDK08], where the geometry of the problem is very similar to ours. We start with an estimate that applies in the case of one interior interval, i.e. when \( g = 2 \) (see the paragraph preceding (1.3)). Consider four points \(-1 < a < b < c < e < 1\). Suppose the support of \( f \) is inside the interval \((-1, 1)\), the data \( g = \mathcal{H}f \) are known on the interval \([a, e]\), and the values of \( f \) (i.e., prior knowledge) are given on the interval \([b, c]\). To quantify errors in the data we fix any non-negative function \( E(x) \), which is continuous on \((a, e)\) and satisfies \( E(x) \leq 1 \) on \((b, c)\). Define the function

\[
h_1(x) = \frac{1}{\pi} \int_{-1}^{1} f(t) \, dt + \frac{1}{\pi} \int_{a}^{e} \left( \mathcal{H}f \right)(y) \sqrt{1 - y^2} \, dy.
\]

Recall that \( \mathcal{H}f \) can be computed from the tomographic data via the Gelfand-Graev formula. It is well known that the first integral on the right in (3.17) can also be computed from the tomographic data. Let \( h_1(x) \) denote an approximation to \( h_1(x) \), which is computed from the measurements. We assume that the errors in the data are such that

\[
|h_1(x) - h_1(x)| < \epsilon E(x), \ x \in (a, e).
\]
Suppose also that we know constants $M_1$ and $M_2$ such that

$$\frac{1}{\pi} |(\mathcal{H}f)(x)| \sqrt{1 - x^2} \leq \begin{cases} M_1/2, & x \in (-1, a), \\ M_2/2, & x \in (e, 1). \end{cases} \quad (3.19)$$

Let $f_r$ denote a function reconstructed from the data. In view of the above discussion, we assume that $f_r$ satisfies the following conditions

(i) $\text{supp}(f_r) \subset (-1, 1)$;

(ii) $f_r \equiv f$ on $(b, c)$, i.e. $f_r$ matches our prior knowledge about $f$;

(iii) $|h_{1,r}(x) - h_1(x)| < \epsilon E(x)$ for $x \in (a, e)$, where $h_{1,r}(x)$ is the function computed using (3.17) with $f$ replaced by $f_r$;

(iv) $\frac{1}{\pi} |(\mathcal{H}f_r)(x)| \sqrt{1 - x^2} \leq \begin{cases} M_1/2, & x \in (-1, a), \\ M_2/2, & x \in (e, 1). \end{cases}$

Theorem 4.2 of [CNDK08] asserts that

$$\sqrt{1 - x^2} |f(x) - f_r(x)| \leq \left[ \epsilon E(x) + \sqrt{2} M_1 \left( \kappa + \ln \left( \frac{x + 1}{x - a} \right) \right) \right]^{1-\omega(x)}, \quad x \in (a, b),$$

where

$$\kappa = \frac{M_2 \ln \left( \frac{1-e}{e-a} \right)}{M_1}, \quad \omega(x) = \frac{4}{\pi} \arctan \sqrt{\frac{2(b-x)(c-a)}{(c-a)^2 - (2b-a-c)(2x-a-c)}}. \quad (3.21)$$

A similar estimate holds for the interval $(c, e)$. Theorem 4.3 in [CNDK08] is an estimate that also looks similar to (3.20), but applies in the case when $f$ is known on two intervals $(b, c), (d, e) \subset (-1, 1)$, and holds for points between these intervals, $c < x < d$. The assumptions of the theorem are similar to (i)–(iv) above, with an obvious change that $f_r$ coincides with $f$ on both intervals.

We see that the assumptions on which stability estimates of [CNDK08] are based are stronger than those in Theorem 3.6. In our opinion, (3.19) is a key additional assumption that allows one to prove stability. This assumption controls the behavior of $\mathcal{H}f$ outside the field of view, where the Hilbert transform of $f$ cannot be computed from the data.

The difference between the assumptions made in [CNDK08] and in the present paper reflects the difference between the respective approaches to solving the interior problem. The main principle on which the results of [CNDK08] are based is the recovery of $f$ on all of its support. Of course, on the exterior intervals $(-1, a)$ and $(e, 1)$ the accuracy of finding $f$ is quite low, but nevertheless $f$ cannot be too wild there, since its Hilbert transform is controlled on $(-1, a)$ and $(e, 1)$ via the assumption (iv). On the contrary, our approach to finding $f$ is purely local: we analytically continue $\psi(y)$ (which then gives $f$, cf. (1.8)) only as far away from the known subintervals as needed; $f$ is never found everywhere.
Our results prove that interior reconstruction with prior knowledge that includes only knowing $f$ on a subinterval cannot be stable. The results of [DNCK06] and [CNDK08] show that with additional prior information stability can be restored. Thus, to obtain a stable solution to the interior problem in practice one has to use as much prior knowledge as possible. The latter may come in different forms. For instance, when one uses a reconstruction algorithm based on compressive sensing, one uses prior information that $f$ is sparse in some basis. There is a growing body of work showing that the interior problem can indeed be solved stably when the appropriate prior knowledge is used, see e.g. [KCND08, RBFK11, WY13].

Notice also that the stability estimate in (3.20) is relatively weak, since the exponent $1 - \omega(x)$ can be quite close to zero. It is interesting to determine whether it is possible to increase the exponent and thereby obtain a stronger estimate.

4 Approximation in $L^2(I_i)$

Let us now consider the asymptotics of normalized singular functions $\hat{f}_n$. According to (2.11) and (2.2), we have

$$\hat{f}_n = \frac{i(w(z))^{-\frac{1}{2}} \text{res}_{\lambda = \lambda_n} (\Gamma_{j\lambda}(z; \lambda))^{\frac{1}{2}}}{\|(w(z))^{-\frac{1}{2}} \text{res}_{\lambda = \lambda_n} (\Gamma_{j\lambda}(z; \lambda))\|}. \quad (4.1)$$

Here and in the rest of this section $\| \cdot \|$ denotes the $L^2(I_i)$ norm. We can choose any $j = 1, 2$ in (4.1) as long as the denominator there is not zero.

The normalized singular functions $\hat{f}_n, \hat{h}_n$ can be approximated by replacing rows of the matrix $\Gamma_{j\lambda}(z; \lambda), j, k \in \{1, 2\}$, in (2.11) by the corresponding rows of the approximate solution to the RHP 2.2. This approximate solution $\Psi$ satisfies the so-called model RHP A.6. An explicit expression for $\Psi$ in terms of the Riemann Theta functions is given in (A.27). Direct calculations show (Theorems A.27, A.13) that the normalized singular functions $\hat{f}_n$ are approximated by

$$\tilde{f}_n(z) := i\Im\left[2\Upsilon(z; f_n)e^{-i\pi n \Im(g_+(z)) - i\Im(d_+(z))}\right] \quad (4.2)$$

with accuracy $O(n^{-1})$ in the sup-norm (uniformly) on any compact subset of the interior of $I_i$. Notations of (4.2) are explained in Appendix A (see Remark A.26, (A.2)). In this brief section we prove that $\tilde{f}_n(z)$ approximate $\hat{f}_n(z)$ also in $L^2(I_i)$. For the benefit of the reader, the technical lemmas are moved to Appendix B.

**Lemma 4.1.** Let $\omega_0$ be so small that each interval $\langle a_k - \omega_0, a_k + \omega_0 \rangle$ contains no endpoints except $a_k$. Then there exists some $\eta > 0$, such that

$$\forall k \in \{3, \ldots, 2g\}, \forall n \in \mathbb{N}, \forall \omega \in (0, \omega_0) : \quad |\tilde{f}_n(z)| \leq \frac{\eta}{|z - a_k|^\frac{1}{2}} \quad \text{on} \quad \langle a_k - \omega, a_k + \omega \rangle. \quad (4.3)$$

Lemma 4.1 follows from Lemma B.2 and Corollary A.25.

**Lemma 4.2.** The norms of $\tilde{f}_n(z)$ given by (4.2) satisfy

$$\|\tilde{f}_n\| = 1 + O(n^{-1}) , \quad n \to \infty. \quad (4.4)$$
Thus, \( \tilde{L} \) where exterior intervals are fixed: \[ a \]

To illustrate one possible application of our results we conduct a numerical experiment. The 5 Numerical experiments It is clear that for a small \( \epsilon > 0 \) condition 2 implies \( \|f\|_t \leq \frac{1}{\sqrt{1 - \epsilon}} \) implies \( \|f\|_t \leq \epsilon \). Choose \( \omega = \frac{1}{\sqrt{1 - \epsilon}} \). Then the former inequality holds for all \( n > \frac{2}{1 - \epsilon^2} \). The proof is completed.

**Theorem 4.3.** \( \hat{f}_n \) approximate \( \hat{f}_n \) in \( L^2(I_i) \), that is, \( \forall \epsilon > 0 \exists n_0 \in \mathbb{N} \) such that \( \forall n > n_0 : \|\hat{f}_n - \hat{f}_n\| < \epsilon \).

**Proof.** According to (4.3), \( \|\hat{f}_n\|_b \leq 2/\sqrt{1 - 1/\sqrt{\epsilon}} \) for all \( n \in \mathbb{N} \). As implied by Lemma 4.2, there exists \( Q_\epsilon > 0 \) such that \( \|\hat{f}_n\| \geq 1 - \frac{Q_\epsilon}{n} \). Since \( 1 - \frac{Q_\epsilon}{n} \leq \|\hat{f}_n\| \leq \|\hat{f}_n\|_b + \|\hat{f}_n\|_t \), we obtain

\[
\|\hat{f}_n\|_b \geq 1 - 2/\sqrt{1 - 1/\sqrt{\epsilon}} = \frac{Q_\epsilon}{n},
\]

so that

\[
\|\hat{f}_n\|_b^2 = 1 - \|\hat{f}_n\|_b^2 \leq 2 \left( 2/\sqrt{1 - 1/\sqrt{\epsilon}} + \frac{Q_\epsilon}{n} \right).
\]

Thus,

\[
\|\hat{f}_n - \hat{f}_n\| \leq \|\hat{f}_n - \hat{f}_n\|_b + \|\hat{f}_n\|_t + \|\hat{f}_n\|_t \leq 2\sqrt{1 - 1/\sqrt{\epsilon}} + \frac{Q_\epsilon}{n} + \sqrt{2 \left( 2\sqrt{1 - 1/\sqrt{\epsilon}} + \frac{Q_\epsilon}{n} \right)}.
\]

It is clear that for a small \( \epsilon > 0 \) condition \( 2\sqrt{1 - 1/\sqrt{\epsilon}} + \frac{Q_\epsilon}{n} + \sqrt{2 \left( 2\sqrt{1 - 1/\sqrt{\epsilon}} + \frac{Q_\epsilon}{n} \right)} < \epsilon \) implies \( \|\hat{f}_n - \hat{f}_n\| \leq \epsilon \). Choose \( \omega = \frac{1}{\sqrt{1 - \epsilon}} \). Then the former inequality holds for all \( n > \frac{2}{1 - \epsilon^2} \). The proof is completed.

5 Numerical experiments

To illustrate one possible application of our results we conduct a numerical experiment. The exterior intervals are fixed: \([a_1, a_2] = [-10, -9]\) and \([a_7, a_8] = [9, 10]\). Prior knowledge is given on two intervals of a fixed length, whose positions can be arbitrary. We are interested in finding the positions of the intervals that provide minimal instability of analytic continuation to all of \([a_2, a_7]\).

Each singular function \( h_n \) and \( f_n \) has precisely the total of \( n \) zeroes on the intervals \( I_e \) and \( I_i \), respectively. The proof follows from the general theory of positive operators (a form of Sturm oscillation theorem) as referred to in \([BKT14]\); indeed the \( n \)-th eigenfunction of a positive operator has \( n \) sign changes.

The total length of the intervals \( I_i \) is the same in all three cases, so for a fixed \( n \) the characteristic frequency of oscillations of \( f_n \) also stays the same in these cases. Suppose the analytic continuation of \( \psi \) is performed via the series in Corollary 3.4. To regularize the series we approximate it by the sum of the first \( N \) terms. Our discussion implies that the frequency content of the data, which
Figure 1: Measure of instability for different locations of interior intervals. Blue - case 1, red - case 2, and green - case 3.

is used by such a regularized algorithm (cf. (3.1)), is also the same for all cases. Thus the ill-
posedness of analytic continuation is reflected by the coefficient in front of $n$ in the exponent in
the formula (3.2). Ignoring the common factor $\pi$, it follows from (1.14) and (3.2) that the quantity
$(\Re g(z) + \frac{1}{2})/\tau_{11}$ can be viewed as an indicator of instability of analytic continuation to the point $z$.

Figure 2: Exact and approximate singular values for different locations of interior intervals. From left to right: cases 1, 2, and 3.

In Figure 1 we see the plots of $(\Re g(z) + \frac{1}{2})/\tau_{11}$ on the interval $[a_1, a_8]$ for the following three cases. In case 1 the interior intervals are $[a_2, a_3] = [-1.4, -0.4]$ and $[a_4, a_5] = [0.4, 1.4]$. In case 2 the interior intervals are $[a_2, a_3] = [-3, -2]$ and $[a_4, a_5] = [2, 3]$. In case 3 the interior intervals are $[a_2, a_3] = [-7, -6]$ and $[a_4, a_5] = [6, 7]$. In Figure 2 we see the exact singular values $\ln \lambda_n$ and the corresponding line $n \to -\frac{\pi n}{\tau_{11}}$ (cf. (1.14)) for all three cases. The slope matches perfectly, and the shift of the line is due to the $O(1)$ term in (1.14). In Figure 3 we see the exact singular functions $h_8$ and $f_8$ for all three cases. The exact singular values and singular functions are computed
Figure 3: Exact singular functions $h_8$ and $f_8$ for different locations of interior intervals. From left to right: cases 1, 2, and 3.

numerically by discretizing the corresponding integral operator. The discretization is achieved by using Tchebycheff nodes and weights in each of the intervals, and thus reducing the operator to a matrix which we then diagonalize numerically. We used 50 nodes per interval (a total of 400 nodes) in variable precision calculations. Empirically we verified that the results were numerically stable up to about the thirtieth eigenfunction, and hence we plotted only the first 24 eigenvalues.\footnote{The singular values are the squares of the $\lambda_n$ appearing in (1.11). They have double degeneracy, corresponding to the symmetry $(\lambda_n, f_n, h_n) \mapsto (-\lambda_n, -f_n, h_n)$ as remarked immediately after (1.11). Hence the plots in Figure 3 are for the eigenfunction corresponding to the eighth positive $\lambda$.}

The graphs in Figure 1 indicate that placing the intervals $[a_2, a_3]$ and $[a_4, a_5]$ farther apart increases the instability of analytic continuation for points between them. However, for the interval $[a_2, a_7]$ the overall instability is reduced.

We would like to emphasize that the above experiment is of a preliminary nature. More detailed research is needed to study how much the ill-posedness of analytic continuation can be reduced by finding the optimal position of the intervals with prior knowledge. The degree of flexibility in choosing the intervals with prior knowledge depends on a particular application. Our approach can be used not only for solving the interior problem of tomography, but in any application where the problem of inverting the Hilbert transform with incomplete data arises.

A Approximation of singular values and singular functions

The main idea of the approach in [BKT14] is to reduce the SVD problem (1.11) to a matrix RHP (2.2), which, in turn, is asymptotically reduced to a simpler RHP, see (A.6) below. That simpler (model) RHP has an explicit solution, which can be expressed in terms of the Riemann Theta function. In this section we outline the approach of [BKT14] by concentrating on the results that will be needed for the proofs in Appendix B. For convenience, direct references to the appropriate results from [BKT14] are provided in square brackets.

Let $\mathcal{R}$ denote the hyperelliptic Riemann surface with branchcuts along the intervals $I = I_e \cup I_k$. 
Then
\[ u(z) = \int_{a_1}^{z} \omega(\zeta) d\zeta, \quad z \in \mathbb{C} \setminus [a_1, \infty), \]  
(A.1)
is known as the Abel map on \(\mathcal{R}\). We now introduce the function \(d(z)\) that is used in the leading order solution of the RHP 2.2:
\[ d(z) = \frac{R(z)}{2\pi i} \left( -\sum_{j=1}^{g+1} \int_{a_{2j-1}}^{a_{2j}} \frac{\ln w(\zeta) d\zeta}{(\zeta - z) R_+ (\zeta)} + \sum_{j=1}^{g} \int_{a_{2j}}^{a_{2j+1}} \frac{i \delta_{\mu(j)} d\zeta}{(\zeta - z) R_+ (\zeta)} \right), \]  
(A.2)
where: \(\mu(g) = 0\) and \(\mu(j) = j\) for all \(j \neq g\); the vector \(\vec{\delta} = [\delta_1, \ldots, \delta_{g-1}, \delta_0]^t\) is given by \(\vec{\delta} = 2\pi L^{-1} (2u(\infty) - u(a_{2g+2}))\) and
\[ L = \begin{bmatrix} 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & 0 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}. \]  
(A.3)

**Proposition A.1.** The function \(d(z)\) given by (A.2) is analytic on \(\overline{\mathbb{C}} \setminus [a_1, a_{2g+2}]\) (in particular, analytic at infinity) and satisfies the jump conditions
\[ d_+ + d_- = -\ln w \quad \text{on} \quad I, \quad d_+ - d_- = i \delta_{\mu(j)} \quad \text{on} \quad c_j, \quad [a_{2j}, a_{2j+1}], \quad j = 1, \ldots, g. \]  
(A.4)

### A.1 Reduction to the of the RHP 2.2 to the leading order (model) RHP

Construction of the leading order approximation of the solution \(\Gamma(z; \lambda)\) of the RHP 2.2 in the limit \(\lambda \to 0^+\) is at the heart of our method. We also have to control the accuracy of such approximation. We employ the nonlinear steepest descent method of Deift and Zhou, that allows to asymptotically reduce the original RHP (RHP 2.2) to a certain RHP with constant jumps (RHP A.6) that one can solve explicitly. The asymptotic reduction consists of a sequence of transformations of the RHP 2.2, some of them equivalent and some asymptotic. The key idea is a factorization of the jump matrix with a subsequent contour deformation, where each factor “acquires” its own jump-contour in the process. In this appendix we only briefly outline some main points of the reduction of the RHP 2.2 and provide a solution to the corresponding “reduced” RHP with constant jumps. The details can be found in [BKT14]. There exists a large and rapidly growing literature about the method of Deift and Zhou and its various applications, see, for example, [Dei99], [BKL+08]. We also include some facts about theta divisors as well as some further results from [BKT14] that are used in the proof of technical lemmas in Appendix B.

Let \(\Sigma\) be an oriented collection of contours that partition \(\mathbb{C}\) into a finite number of open regions and let \(V(z)\) be an \(n \times n\) matrix-valued function defined on \(\Sigma\), satisfying certain conditions at the nodes of \(\Sigma\). A (somewhat) general formulation of a matrix RHP can be stated as follows. We do not get here into the details of the smoothness of \(\Sigma\) and \(V(z)\).
RHP A.2. Find an \( n \times n \) matrix-function \( M(z) \) that:

- is analytic in each element of partition, induced by the contour \( \Sigma \);
- for any \( z \in \Sigma \) that is not a node \( M(z) \) admits non-tangential boundary values \( M_{\pm}(z) \) from the corresponding sides of \( \Sigma \) and

\[
M_{+}(z) = M_{-}(z)V(z);
\]  
\[ (A.5) \]

\[
\lim_{z \to \infty} M(z) = 1. 
\]  
\[ (A.6) \]

In general, the existence of a solution to the RHP (A.2) is not guaranteed. The nonlinear steepest descent method is based upon the following “small norm theorem”.

Theorem A.3. Let \( N_{p} \) denotes the norms of \( V(z) - 1 \) in \( L^{p}(\Sigma,dz) \). Then

- There is a constant \( C_{\Sigma} \) such that if \( N_{\infty} < C_{\Sigma}^{-1} \) the solution of the RHP A.2 exists;
- In this case

\[
||M(z) - 1|| \leq \frac{1}{2\pi \text{dist}(z,\Sigma)} \left( N_{1} + \frac{C_{\Sigma}N_{2}^{2}}{1 - C_{\Sigma}N_{\infty}} \right)
\]  
\[ (A.7) \]

for every \( z \in \mathbb{C} \setminus \Sigma \).

The name of this theorem reflects the fact that the solution \( M(z) \) of the RHP A.2 is close (pointwise) to the identity matrix \( \mathbf{1} \) if the norms \( N_{1,2} \) are small.

Let \( \kappa = -\ln \lambda \). Then \( \kappa > 0 \) when \( \lambda \in (0,1) \) and \( \kappa \to \infty \) as \( \lambda \to 0 \). The first transformation is replacing \( \Gamma(z;\lambda) \) with \( Y(z;\kappa) \) by

\[
Y(z;\kappa) = e^{-(\kappa g(z) + d(z))}\sigma_{1}\Gamma(z;e^{-\kappa})e^{(\kappa g(z) + d(z))}\sigma_{3},
\]  
\[ (A.8) \]

where \( g(z),d(z) \) are defined by (A.2), and the Pauli matrices are defined as

\[
\sigma_{1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_{2} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_{3} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Then direct calculations show that the RHP 2.2 for \( \Gamma(z;\lambda) \) is reduced to the following equivalent RHP for \( Y \).

RHP A.4. Find a \( 2 \times 2 \) matrix-function \( Y(z;\kappa) \) with the following properties:

(a) \( Y(z;\kappa) \) is analytic in \( \mathbb{C} \setminus [a_{1},a_{2g+2}] \);

(b) \( Y(z;\kappa) \) satisfies the jump conditions

\[
Y_{+} = \begin{cases}
Y_{-} & z \in I_{i}, \\
Y_{-} & z \in I_{e},
\end{cases}
\]

\[ \begin{array}{c}
Y_{+} = Y_{-} e^{(\kappa g(z) + d(z))..} & \kappa (z) \in I_{i}, \\
Y_{+} = Y_{-} \kappa (z) \in I_{e}, \\
\end{array} \]

\[ Y_{+} = Y_{-} \kappa (z) \in [a_{2j},a_{2j+1}], j = 1, \ldots, g; \]  
\[ (A.9) \]
Figure 4: The regions of the lenses $\mathcal{L}_{i,e}^{(\pm)}$ (above) and the jumps of the error matrix $\mathcal{E}$ (below).

(c) $Y = 1 + O(z^{-1})$ as $z \to \infty$, and;
(d) Near the branchpoints (we indicate the behavior for the columns if these have different behaviors)

$$Y(z; \varkappa) = [O(1), O(z - a_j)^{-\frac{1}{2}}], \quad j = 1, 2g + 2; \quad Y(z; \varkappa) = O(\ln(z - a_j)), \quad j = 2, \ldots, 2g + 1. \quad (A.10)$$

In the next transformation (now of the RHP A.4) we first factorize the triangular jump matrices in (A.9) as

$$Y_+ = Y_- \begin{bmatrix} 1 & e^{-\kappa(2g+1)+2d} & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}$$

on $I_i$, and

$$Y_+ = Y_- \begin{bmatrix} 1 & e^{-\kappa(2g+1)+2d} & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}$$

on $I_e$, \quad (A.11)

and then put each of the three factors (for $I_e$ and for $I_i$) on its own jump contour as described below.

The validity of the factorization can be checked directly, taking into the account the identities

$$-\kappa(g_+ + g_- + 1) - \ln w - d_+ - d_- \equiv 0 \quad \text{that hold on } I_i \text{ and } I_e \text{ respectively, see (2.13), (A.4).}$$

The left and right (triangular) matrices in both factorizations (on $I_i$ and on $I_e$) admit analytic extension in the left/right vicinities of the corresponding segments because they are the boundary values of analytic matrices in those vicinities. This suggests opening of the lenses $\partial L_e^{(\pm)}$, $\partial L_i^{(\pm)}$ around the corresponding intervals of $I_e \cup I_i$, see Figure 4 upper panel, and introduction of the new unknown matrix

$$Z(z; \varkappa) = \begin{cases} Y(z; \varkappa) & \text{outside the lenses,} \\ Y(z; \varkappa) \begin{bmatrix} 1 & 0 \\ -iw e^{-\kappa(2g-1)+2d} & 1 \end{bmatrix} & z \in L_e^{(\pm)}, \\ Y(z; \varkappa) \begin{bmatrix} 1 & 0 \\ -iw e^{-\kappa(2g+1)+2d} & 1 \end{bmatrix} & z \in L_i^{(\pm)}. \end{cases} \quad (A.12)$$

Consequently, after the second (equivalent) transformation we obtain the following RHP for the matrix $Z$. 

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RHP A.5. Find the matrix \( Z \), analytic on the complement of the arcs of Figure 4, satisfying the jump conditions (note also the orientations marked in Figure 4)

\[
Z_+(z; \varkappa) = Z_-(z; \varkappa) \begin{cases} 
\begin{aligned}
e^{(a \Omega_{\mu(j)} + \delta_{\mu(j)}) \sigma_3} & z \in [a_{2j}, a_{2j+1}], \ j = 1, \ldots, g, \\
1 & z \in \partial L^{(+) \setminus \mathbb{R}}, \\
1 & z \in \partial L^{(+) \setminus \mathbb{R}}, \\
\end{aligned}
\end{cases}
\]

normalized by

\[
Z(z; \varkappa) \to 1, \ z \to \infty, \quad (A.14)
\]

and with the same endpoint behavior as \( Y \) near the endpoints \( a_j \)'s, see (A.10). Here

\[
\tilde{\delta} = [\delta_1, \ldots, \delta_{g-1}, \delta_0]^t = 2\pi L^{-1}(2u(\infty) - u(a_{2g+2})), \quad \tilde{\Omega} = [\Omega_1, \ldots, \Omega_{g-1}, \Omega_0]^t = -2iL^{-1}\tau_1
\]

and \( \mu(g) = 0, \mu(j) = j \) for all \( j \neq g \), see Proposition A.1.

In the third and final transformation we would like to (asymptotically) reduce the RHP A.5 for \( Z(z; \varkappa) \) to the following RHP for \( \Psi = \Psi(z; \varkappa) \).

RHP A.6 (Model problem). Find a matrix \( \Psi = \Psi(z; \varkappa) \), analytic on \( \mathbb{C} \setminus [a_1, a_{2g+2}] \) and satisfying the following conditions:

\[
\begin{aligned}
\Psi_+ = \Psi_- (-1)^s(j) (i \sigma_1), & \quad z \in [a_{2j-1}, a_{2j}], \ j = 1, \ldots, g + 1, \\
\Psi_+ = \Psi_- e^{(a \Omega_{\mu(j)} + \delta_{\mu(j)}) \sigma_3} & \quad z \in [a_{2j}, a_{2j+1}], \ j = 1, \ldots, g, \\
\Psi(z) = O(|z - a_j|^{-\frac{1}{2}}) & \quad z \to a_j, \ j = 1, \ldots, 2g + 2, \\
\Psi(z) = 1 + O(z^{-1}) & \quad z \to \infty, \ \text{and} \ \Psi_\pm(z) \in L^2([a_1, a_{2g+2}]). \\
\end{aligned}
\]

Here \( s(j) = \delta_{j,1} + \delta_{j, g+1} \), where \( \delta_{j,k} \) denotes the Kronecker delta.

The RHP A.6 is not equivalent to the RHP A.5 since the former does not have jumps on the lenses \( \partial L_e^{(+) \setminus \mathbb{R}}, \partial L_i^{(+) \setminus \mathbb{R}} \). Also, a different behavior is required near the branchpoints \( a_j \), \( j = 1, 2, \ldots, 2g + 2 \). However, as a consequence of Theorem A.3, the solution \( \Psi(z; \varkappa) \) of the RHP A.6 will approximate the solution \( Z(z; \varkappa) \) of the RHP A.5, if the jump matrices on the lenses \( \partial L_e^{(+) \setminus \mathbb{R}}, \partial L_i^{(+) \setminus \mathbb{R}} \) in (A.13) will be small in the norms \( N_1, N_2 \) and \( N_\infty \). The choice of the so-called \( g \)-function \( g(z) \) in the transformation (A.8), as well as of the lenses \( L_e^{(+) \setminus \mathbb{R}} \), is defined by the requirements that

\[
\Re(2g - 1) < 0 \ \text{on} \ \partial L_e^{(+) \setminus \mathbb{R}} \ \text{and} \ \Re(2g + 1) > 0 \ \text{on} \ \partial L_i^{(+) \setminus \mathbb{R}}. \quad (A.16)
\]

If these requirements hold, the jump matrices (A.13) on the contours \( \partial L_e^{(+) \setminus \mathbb{R}}, \partial L_i^{(+) \setminus \mathbb{R}} \) approach \( 1 \) exponentially fast as \( \varkappa \to \infty \) in any \( L^p, \ p < \infty \) but not in \( L^\infty \), because this convergence is uniform away from small vicinities of the branchpoints. These vicinities which require special consideration. Namely, to match RHP A.13 with RHP A.6, we need to construct special local parametrices in these vicinities of the branchpoints.
The discrepancy between \( Z(z; \varkappa) \) and \( \Psi(z; \varkappa) \), the latter modified by the parametrices near the branchpoints, is represented by the so-called error matrix \( \mathcal{E}(z; \varkappa) \). The error matrix also satisfies a certain RHP; the jump contours of this RHP are shown on Figure 4. We already know that the jump matrices on the arcs \( \partial \mathcal{L}^{(\pm)}_e \setminus \mathbb{R}, \partial \mathcal{L}^{(\pm)}_i \setminus \mathbb{R} \) away from the branchpoints should approach \( 1 \) exponentially fast in \( \varkappa \to \infty \). The parametrices, constructed in [BKT14], ensure that the jump matrices on the circles, shown in Figure 4, behave like \( 1 + O(\varkappa^{-1}) \) as \( \varkappa \to \infty \) in any \( L^p, 1 \leq p \leq \infty \). Thus, according to Theorem A.3, \( \Psi(z; \varkappa) \) is \( O(\varkappa^{-1}) \) close to \( Z(z; \varkappa) \) uniformly in \( z \) outside small circles around the branchpoints.

Now, by reversing the chain of transformations, one obtains the following summary of the steepest descent analysis of [BKT14]: let constants \( \epsilon, \rho_z, \rho_0 > 0 \) be fixed and sufficiently small. Then

\[
\Gamma(z; e^{-\varkappa}) = e^{(\varkappa g(\infty) + d(\infty)) \sigma_3} \mathcal{E}(z; \varkappa) \Psi(z; \varkappa) \left[ \begin{array}{cc} 1 & \pm \frac{i}{16} e^{-\varkappa(2g+1) - 2d} \hat{\chi}_{e,\pm} \\ \pm iw e^{\varkappa(2g-1) + 2d} \hat{\chi}_{e,\pm} & 1 \end{array} \right] (A.17)
\]

where \( \hat{\chi}_{e,\pm}^{\pm}, \hat{\chi}_{i,\pm}^{\pm} \) are the characteristic (indicator) functions of the sets \( \mathcal{L}^{(\pm)}_e, \mathcal{L}^{(\pm)}_i \) respectively, see Figure 4, and

\[
\mathcal{E}(z; \varkappa) = 1 + O(\varkappa^{-1}) \left( 1 + |z| \right)^{-1} (A.18)
\]

uniformly in the domain

\[
|\Im \varkappa| < \epsilon, \quad |\Theta(W(\varkappa) - W_0)| > \rho_0, \quad |z - a_j| > \rho_z, \quad j = 1, \ldots, 2g + 2. \quad (A.19)
\]

The matrix \( \mathcal{E}(z; \varkappa) \) solves an auxiliary RHP, where all the jumps satisfy the assumption of Theorem A.3: in particular it is important for us that \( \mathcal{E} \) does not have a jump on the main arcs \( I_e \cup I_i \). This implies that the following matrix

\[
\Gamma^{(\infty)}(z; e^{-\varkappa}) = e^{(\varkappa g(\infty) + d(\infty)) \sigma_3} \Psi(z; \varkappa) \left[ \begin{array}{cc} 1 & \pm \frac{i}{16} e^{-\varkappa(2g+1) - 2d} \hat{\chi}_{e,\pm} \\ \pm iw e^{\varkappa(2g-1) + 2d} \hat{\chi}_{e,\pm} & 1 \end{array} \right] e^{-(\varkappa g(z) + d(z)) \sigma_3} (A.20)
\]

has the exact same jumps as \( \Gamma(z; \lambda) \) on \( I_e \cup I_i \).

To solve the model RHP A.6 we need to introduce Riemann Theta functions.

A.2 Solution of the model RHP A.6 and asymptotics of \( \lambda_n \)

The Riemann Theta function associated with a symmetric matrix \( \tau \) with strictly positive definite imaginary part (that guarantees convergence) is the function of the vector argument \( \hat{z} \in \mathbb{C}^g \) given by

\[
\Theta(\hat{z}, \tau) := \sum_{\hat{n} \in \mathbb{Z}^g} \exp \left( i \pi \hat{n}^t \cdot \tau \cdot \hat{n} + 2i \pi \hat{n}^t \hat{z} \right). \quad (A.21)
\]

Often the dependence on \( \tau \) is omitted from the notation. We will consider the matrix \( \tau \) given by

\[
\tau = [\tau_{ij}] = \left[ \int_{B_i} \omega_j d\zeta \right]_{i,j = 1, \ldots, g}, \quad (A.22)
\]

where \( \omega^i(z) \) is defined in (2.12) and the loops (cycles) \( B_i, i = 1, \ldots, g \) are shown in Figure 5.
Figure 5: Hyperelliptic Riemann surface $R$ built on intervals $I = I_e \cup I_i$ with the choice of $A$ and $B$ cycles.

**Theorem A.7** (Riemann [FK92]). The matrix $\tau$ is symmetric and its imaginary part is strictly positive definite.

Matrix $\tau$ is an important object in the theory of compact Riemann surfaces. Indeed, consider the hyperelliptic Riemann surface $R$, defined by the segments $[a_{2k-1}, a_{2k}]$, $k = 1, 2, \ldots, g+1$, that form $I$, with canonical $A$ and $B$ cycles shown in Figure 5. Then $\omega(z)dz$ (see (2.12)) is known as the vector of normalized holomorphic differentials on $R$ and $\tau$ is called the normalized matrix of $B$-periods of $R$. Note that $[A]_{ji} = \oint_{A_j} \zeta^{i-1} d\zeta_R(\zeta)$, and $\tau_{11}$ in (1.13) is the $(1, 1)$ entry of the matrix $\tau$.

**Remark A.8.** It follows from (A.22), (2.12) and (1.12) that the entries of the matrix $\tau$ are purely imaginary.

**Proposition A.9.** For any $\lambda, \mu \in \mathbb{Z}^g$, the Theta function has the following properties:

\begin{align*}
\Theta(\bar{z}, \tau) &= \Theta(-\bar{z}, \tau); \quad \text{(A.23)} \\
\Theta(\bar{z} + \mu + \tau \lambda, \tau) &= \exp \left( -2i\pi \lambda^t \bar{z} - i\pi \lambda^t \tau \lambda \right) \Theta(\bar{z}, \tau). \quad \text{(A.24)}
\end{align*}

According to (A.21) and Proposition A.9, the Theta function is an even function of $g$ complex variables, periodic on the lattice $\mathbb{Z}^g$ and quasi-periodic on the lattice $\tau \mathbb{Z}^g$. A hypersurface ($\Theta \subset \mathbb{C}^g$, defined by $\Theta(\bar{z}, \tau) = 0$, is called a theta divisor. This is a hypersurface of complex codimension one or real codimension two. According to Proposition A.9, the theta divisor ($\Theta$) is periodic in $\mathbb{Z}^g$ and $\tau \mathbb{Z}^g$.

Let

\[ r(z) := \sqrt[4]{\prod_{j \in J}(z - a_j) / \prod_{\ell \in J'}(z - a_\ell)}, \quad z \in \mathbb{C} \setminus [a_1, a_{2g+2}], \quad \text{(A.25)} \]

where $J = \{1, 5, 7, 9, \ldots, 2g - 1\}$ and $J' = \{1, 2, 3, \ldots, 2g + 2\} \setminus J$ (so that $|J| = g - 1$ and $|J'| = g + 3$). The function $r(z)$ is defined so that it is analytic in $\mathbb{C} \setminus [a_1, a_{2g+2}]$ and behaves like $\frac{1}{z}$ at infinity. Let

\[ W = W(\zeta) = \frac{\zeta}{i\pi} \tau_1 + 2u(\infty) + \frac{e_1}{2}, \quad W_0 = \frac{\tau_1}{2} - \frac{e_1 + e_g}{2}, \quad \text{(A.26)} \]
where $\tau_1$ is the first column of matrix $\tau$ and $e_k$ denotes the $k$th vector of the standard basis in $\mathbb{C}^g$. Then [Theorem 5.7] the explicit solution to the RHP A.6 is given by

$$
\Psi(z; \kappa) = C_0 \begin{bmatrix}
\Theta(u(z) - u(\infty) - W_0 + W)r(z) & \Theta(-u(z) - u(\infty) - W_0 + W)r(z) \\
\Theta(W - W_0)\Theta(u(z) - u(\infty) - W_0) & \Theta(W - W_0)\Theta(-u(z) - u(\infty) - W_0)
\end{bmatrix}
$$

(A.27)

where the vectors $W = W(\kappa)$ and $W_0$ are defined in (A.26), and $C_0 = [A^{-1}\nabla\Theta(W_0)]_g \neq 0$ is the $g$-th (last) entry of the vector $A^{-1}\nabla\Theta(W_0)$.

**Proposition A.10** (Symmetry). *If $\Psi(z; \kappa)$ satisfies the RHP A.6, then $\det \Psi = 1$ and $\tilde{\Psi}(z) = \Psi(z)$, where $\tilde{\Psi}(z; \kappa) = \overline{\Psi(z; \overline{\kappa})}$. In particular, for $\kappa \in \mathbb{R}$, $\Psi_{j+1}(z; \kappa) = \overline{\Psi_{j-1}(z; \kappa)}$ for any $z \in I = I_1 \cup I_2$.*

**Lemma A.11.** *The endpoints $a_n$, $n \in J$, and infinity (on one of the sheets) are the only zeroes in $z$ of the functions $\Theta \left(-(-1)^{j}u(z) + (-1)^{k}u(\infty) + W_{0}\right)$, $j,k = 1,2$. All these zeroes are simple.*

According to Lemma A.11, $\Psi(z; \kappa)$ is well defined if $\Theta(W - W_0) \neq 0$ in the denominators in (A.27).

**Theorem A.12.** [Theorem 5.3] *The RHP A.6 has a solution if and only if $\Theta(W - W_0) \neq 0$.*

As a consequence, the approximate eigenvalues $\tilde{\lambda}_n$ are defined via the zeroes $\tilde{\lambda}_n$ of

$$
\Theta(W(z) - W_0) = 0
$$

(A.28)

by $\tilde{\lambda}_n = -\ln \tilde{\lambda}_n$. Geometrically, condition (A.28) determines the points of intersection of the line $W(z) - W_0 \subset \mathbb{C}^g$ with the theta divisor. Let us consider this question in a little more details. Direct calculations show that all the terms of $W(z)$ in (A.26) are real, provided that $\kappa \in \mathbb{R}$. Thus, the line $\{W(z): \kappa \in \mathbb{R}\} \subset \mathbb{R}^g \subset \mathbb{R}^{2g}$, if we identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$. So, the line $W(z) - W_0$, $\kappa \in \mathbb{R}$, is a subset of the shifted hyperplane $\Pi = W_0 + \mathbb{R}^g$. Let $(\Theta)_R := (\Theta) \cap \Pi$.

**Theorem A.13.** [Corollary 6.3] *If $n \in \mathbb{N}$ is sufficiently large, then there is exactly one eigenvalue $\lambda_n = e^{-\tilde{\lambda}_n}$ within a distance

$$
|\tilde{\lambda}_n - \xi| = O(\tilde{\lambda}_n^{-\frac{1}{2}})
$$

(A.29)

of each approximate eigenvalue $e^{-\tilde{\lambda}_n}$.*

**Lemma A.14.** [Lemma 7.5] *Each connected component of $(\Theta)_R$ is a smooth $g - 1$ (real) dimensional hypersurface in $\Pi$.*

Moreover, since $(\Theta)_R$ is $Z^g$ periodic on $\Pi$, it is sufficient to study $(\Theta)_R$ in a $g$ (real) dimensional torus $\mathbb{T}_g$. Numerically simulated surfaces $(\Theta)_R \cap \mathbb{T}_g$ for $g = 2, 3$, and their intersections with the line $W(z) - W_0$ are shown on Figure 6. In the case $g = 2$ we proved that the line $W(z) - W_0$ has one and only one intersection with $(\Theta)_R$ in $\mathbb{T}_2$. It is likely (but not proven yet) that this statement holds for a general $g \in \mathbb{N}$. However, the following lemma is sufficient to obtain the asymptotics (1.14) for $\lambda_n$ with any $g \in \{2,3,\ldots\}$.
Figure 6: Intersection of the line $W(\kappa) - W_0$ (blue or lighter line) with the theta divisor $(\Theta)_R$ in $T_g$, where $g = 2$ (left panel) and $g = 3$ (right panel). On the left panel ($g = 2$) $(\Theta)_R$ is represented by a curve, on the right panel ($g = 3$) $(\Theta)_R$ is represented by a surface. In both cases the point of intersection of $W(\kappa) - W_0$ with $(\Theta)_R$ determines some $\kappa = \tilde{\kappa}_n$.

**Lemma A.15.** [Proposition 7.11] Let $\kappa_0 \in \mathbb{R}^+$ and $g \in \{2, 3, \ldots \}$. For any $N \in \mathbb{N}$ the number $m(N)$ of intersections of the segment of the line $W(\kappa) - W_0$, where $\kappa \in [\kappa_0, \kappa_0 + \frac{N(g-1)\pi}{\tau_1}]$, with $(\Theta)_R$ is bounded by

$$ (N - 1)(g - 1) \leq m(N) \leq (N + 1)(g - 1). \quad (A.30) $$

Now the asymptotics (1.14) of singular values $\lambda_n$ follows from Lemma A.15 and Theorem A.13.

**A.3 Approximation of the normalized singular functions**

According to (2.11), in order to approximate the singular functions, we need to calculate the residues of (A.27) at $\tilde{\lambda}_n$. We also need some information about zeroes of the Theta function.

**Definition A.16.** Let $a_1$ be a base-point of the Abel map $u(z)$ (see (A.1)) on the hyperelliptic Riemann surface $R$ of $\sqrt{\prod_{j=1}^{2g+2}(z - a_j)}$. Then the vector of Riemann constants $\mathcal{K}$ is

$$ \mathcal{K} = \sum_{j=1}^{g} u(a_{2j+1}). \quad (A.31) $$

**Theorem A.17** (p. 308, [FK92]). Let $f \in \mathbb{C}^g$ be arbitrary, and denote by $u(p)$ the Abel map extended to the whole Riemann surface. The (multi-valued) function $\Theta(u(z) - f)$ on the Riemann
surface either vanishes identically or vanishes at \( g \) points \( p_1, \ldots, p_g \) (counted with multiplicity). In the latter case we have
\[
f = \sum_{j=1}^{g} u(p_j) + \mathcal{K}.
\] (A.32)

Let us denote by \( \Lambda = \mathbb{Z}^g + \tau \mathbb{Z}^g \subset \mathbb{C}^g \) the lattice of periods. The Jacobian is the quotient \( J_\tau = \mathbb{C}^g / \Lambda \) and it is a compact torus of real dimension 2\( g \) on account of Theorem A.7.

**Definition A.18.** The theta divisor is the locus \( e \in J_\tau \) such that \( \Theta(e) = 0 \). It will be denoted by the symbol \( \Theta \).

**Proposition A.19.** [Proposition 7.1] If \( W \in \mathbb{R}^g \) and \( W_0 \) is given as in (A.26), then
\[
\Theta(W - W_0) = 0 \iff W = \sum_{\ell=1}^{g-1} (u(p_{\ell+1}) - u(a_{\ell})) \mod \mathbb{Z}^g,
\] (A.33)
where \( p_{\ell+1} = (z_{\ell+1}, R_{\ell+1}), \ell = 1, \ldots, g-1 \), are arbitrary points with \( z_{\ell+1} \in [a_{2\ell}, a_{2\ell+1}] \), \( \ell = 1, \ldots, g-2 \), and \( z_0 \in \mathbb{R} \setminus [a_1, a_{2g+2}] \) (i.e. belonging to the cycles \( A_{1+\ell}, \ell = 1, \ldots, g-1 \)), and \( f_\ell \in J \).

**Remark A.20.** Proposition A.19 explicitly parametrizes the hypersurface \( \Theta(W - W_0) = 0 \), \( W \in \mathbb{R}^g \) in terms of \( g-1 \) points \( p_2, \ldots, p_g \) belonging to the cycles \( A_2, \ldots, A_g \). For the special values \( x = \tilde{z}_n \), when the line \( W(x) \) (given by (A.26)) intersects with this hypersurface, we shall denote the corresponding points on the cycles \( A_2, \ldots, A_g \) by \( p_2^{(n)}, \ldots, p_g^{(n)} \) with \( \tilde{p}_n = (p_2^{(n)}, \ldots, p_g^{(n)}) \). According to (A.2) and Theorem A.17,
\[
f_n := W(\tilde{z}_n) - W_0 = \sum_{j=2}^{g} u(p_j^{(n)}) + \mathcal{K}.
\]
For this reason it makes sense to consider \( f(p) := \sum_{j=2}^{g} u(p_j) + \mathcal{K} \), where \( p = (p_2, \ldots, p_g) \), as a function on the (universal cover) of the torus \( A_2 \times \cdots \times A_g \). Then we have \( f_n = f(\tilde{p}_n) \).

**Lemma A.21.** [Lemma 7.14] (1) For \( \Psi(z; x) \) from (A.27) we have
\[
\text{res}_{x = \tilde{z}_n} \Psi(z; x) = C_0 \left[ \begin{array}{c}
\frac{i\pi \Theta(u(z) - u(\infty) + f(\tilde{p}_n)) r(z)}{\tau_1 \Theta(f(\tilde{p}_n)) \Theta(u(z) - u(\infty) + W_0)} \\
\frac{i\pi \Theta(-u(z) - u(\infty) + f(\tilde{p}_n)) r(z)}{\tau_1 \Theta(f(\tilde{p}_n)) \Theta(-u(z) - u(\infty) + W_0)} \\
\frac{-i\pi \Theta(u(z) + u(\infty) + f(\tilde{p}_n)) r(z)}{\tau_1 \Theta(f(\tilde{p}_n)) \Theta(u(z) + u(\infty) + W_0)} \\
\frac{-i\pi \Theta(-u(z) + u(\infty) + f(\tilde{p}_n)) r(z)}{\tau_1 \Theta(f(\tilde{p}_n)) \Theta(-u(z) + u(\infty) + W_0)} \\
\end{array} \right].
\] (A.34)
(2) For any \( \tilde{p}_n \in A_2 \times \cdots \times A_g \) the matrix in (A.34) is not identically zero. (3) The two rows of the matrix in (A.34) are proportional to each other for any \( \tilde{p}_n \in A_2 \times \cdots \times A_g \).

**Lemma A.22.** (1) The following identities hold for \( j = 1, 2 \):
\[
N_j(\tilde{p}_n) := -\frac{i}{\pi^2} \int_{J_\tau} \text{res}_{z = \tilde{z}_n} \text{res}_{x = \tilde{z}_n} \Psi_j(z; x) \Psi_j(z; x) \, dz = \frac{\Theta(f(\tilde{p}_n) + (-1)^j 2u(\infty)) [A^{-1} \nabla \Theta(W_0)]_g}{\Theta(W_0 + (-1)^j 2u(\infty))} \frac{\tau_1 \Theta(f(\tilde{p}_n))}{i\tau_1 \Theta(f(\tilde{p}_n))}.
\] (A.35)
(2) The function \( N_j(\tilde{p}) \) is a (real) analytic function of \( \tilde{p} \in A_2 \times \cdots \times A_g \). It vanishes to second order at \( p_{g-1} = \infty_l \), where \( \infty_l \) is the point at \( z = \infty \) on the sheet \( l = 1, 2 \), and has no other zeroes.
According to (2.11), (2.2), the normalized singular function \( \hat{f}_n(z) \) is proportional to
\[
\varphi_{n,j}(z) = i\sqrt{w(z)} \lim_{\lambda=\lambda_n} \Gamma_{j2}(z;\lambda) \frac{1}{\lambda}, \quad j = 1, 2,
\] (A.36)
where at least one of the latter expressions is not zero. Note that \( \varphi_{n,j} \) corresponds to the second term of \( \varphi_{n,j} \) from (2.11).

**Proposition A.23.** The norms in \( L^2(I) \) of the singular functions \( \phi_{n,j} \) are given by
\[
\|\phi_{n,j}\|^2 = 2e^{(-1)^{j+1}2(d_{\infty}+\bar{r}_n\Theta_{\infty})-\bar{r}_n} \left( \pi^2 N_j(\bar{p}_n) + O(\bar{r}_n^{-1}) \right), \quad j = 1, 2.
\] (A.37)
Moreover, \( \|\varphi_{n,j}\|^2 = \frac{1}{2} \|\phi_{n,j}\|^2 \), where \( \|\varphi_{n,j}\| \) is the \( L^2(I_1) \) norm of \( \varphi_{n,j} \).

**Corollary A.24.** (1) The functions \( N_j(\bar{p}) \) have constant sign on the torus \( A_2 \times \ldots A_g \). The function \( \sqrt{N_j(\bar{p})} \) can be defined analytically on the double cover of \( A_2 \times \ldots A_g \). (2) There exists \( \nu > 0 \) such that for all \( \bar{p} \in A_2 \times \ldots A_g \)
\[
\max_{j=1,2} |N_j(\bar{p})| > \nu.
\] (A.38)

**Corollary A.25.** [Corollary 7.20] The functions
\[
\Upsilon_{j,k}(z;\bar{p}) = \frac{\Theta(W_0-(-1)^j2u(\infty) \times [\bar{A}^{-1}\nabla\Theta(W_0)]_\rho \Theta(-1)^{k+1}u(z)+(-1)^j2u(\infty)+f(\bar{p})} {\Theta(f(\bar{p}))+(-1)^j2u(\infty)} \frac{\Theta((-1)^{k+1}u(z)+(-1)^j2u(\infty)+W_0)} {\Theta((-1)^{k+1}u(z)+(-1)^j2u(\infty)+W_0)},
\] (A.39)
j, k = 1, 2, are analytic in \( z \) on \( Z_0 \) and in \( \bar{p} \) on the double covering of the torus \( A_2 \times \ldots A_g \), where \( Z_0 = \bar{C} \setminus [a_1,a_{2g+2}] \) together with the boundary points on both sides of the interval \( (a_1,a_{2g+2}) \). Moreover, \( \Upsilon_{1,k}(z;\bar{p}) \) coincides with \( \Upsilon_{2,k}(z;\bar{p}) \), \( k = 1, 2, \) on \( Z_0 \times A_2 \times \ldots A_g \) modulo factor \(-1)\).

**Remark A.26.** Denote \( \Upsilon^{(1)}(z;f_n) := (\Upsilon_{1,j}+\Upsilon_{1,j+x})(z;\bar{p}_n), \) \( j = 1, 2 \). The subscript “+” indicates that the limiting value on the upper side of \( z \in I \) in \( Z_0 \) is taken. In view of Corollary A.25, for every \( n \in \mathbb{N} \) we have \( \Upsilon^{(1)}(z;f_n) \equiv \pm \Upsilon^{(2)}(z;f_n) \), where the choice of sign depends on \( n \). It turns out that this sign is not essential, since the normalized singular functions \( \hat{f}_n(z) \) and \( \hat{h}_n(z) \), approximated through \( \Upsilon^{(1)}(z;f_n) \) (see below), are determined only up to a sign. Thus, we introduce \( \Upsilon(z;f_n) \) that, for a given \( n \in \mathbb{N} \), coincides with both \( \Upsilon^{(1)}(z;f_n) \), \( j = 1, 2 \), modulo factor \(-1)\).

Now the asymptotics of the singular functions is described by the following theorem.

**Theorem A.27.** [Theorem 7.22] The singular functions \( \hat{f}_n(z) \) and \( \hat{h}_n(z) \) of the system in (2.1) normalized in \( L^2(I_i) \) and \( L^2(I_e) \), respectively, are asymptotically given by
\[
\hat{f}_n(z) = i3 \left[ 2\Upsilon(z;f_n)e^{-i\bar{r}_n\Theta(\bar{g}_+(z))} - i3(\bar{g}_+(z)) \right] + O(\bar{r}_n^{-1}), \quad z \in I_i,
\] (A.40)
\[
\hat{h}_n(z) = \Re \left[ 2\Upsilon(z;f_n)e^{-i\bar{r}_n\Theta(\bar{g}_+(z))} - i3(\bar{g}_+(z)) \right] + O(\bar{r}_n^{-1}), \quad z \in I_e,
\]
where the approximation is uniform in any compact subset of the interior of \( I_i, I_e \), respectively.
B Proofs of the technical lemmas

In this section, we use \( \hat{f}_n \) to denote the \( n \)-th normalized singular function for the system (2.1), as well as its analytic continuation on \( \mathbb{C} \setminus I_c \). It follows from (2.1) that each \( \hat{f}_n \) is purely imaginary on \( I \), and defined uniquely modulo the factor \(-1\). According to (A.17),

\[
\varphi_{n,j}(z) = i\sqrt{w(z)} \hat{\varphi}_n(g(z))^{(-1)^j\theta_\infty} + (d(z))^{(-1)^j\delta_\infty} \left( \res_{\lambda=\lambda_n} \Psi_{j,2}(z; \kappa) + O(\pi_n^{-1}) \right) \quad \text{(B.1)}
\]

uniformly on any compact set not intersecting the lenses \( \mathcal{L}_c^{(\pm)} \), see Figure 4, and

\[
\varphi_{n,j}(z) = i\sqrt{w(z)}m_{n,j} \left[ \res_{\lambda=\lambda_n} \left( \pm \frac{1}{i\sqrt{w(z)}} \Psi_{j,1}(z)e^{-\kappa(g(z)+1)-d(z)} + \Psi_{j,2}(z)e^{\kappa g(z)+d(z)} \right) + O(\pi_n^{-1}) \right],
\]

where \( m_{n,j} := e^{-(1)^j\kappa_0 g(\infty) - (1)^j\delta_\infty} \), \( j = 1, 2 \), uniformly on any compact subset of \( \mathcal{L}_c^{(\pm)} \) not containing the endpoints.

We now define the approximations \( \varphi_{n,j}^{(\infty)}(z) \) of \( \varphi_{n,j}(z) \) as \( \varphi_{n,j}^{(\infty)} = i\sqrt{w(z)} \res_{\lambda=\lambda_n} \Gamma_{j,2}^{(\infty)}(z; \lambda) \frac{1}{2}, \)

for \( z \in \mathbb{C} \setminus \bigcup_{\pm} \mathcal{L}_c^{(\pm)} \) and

\[
\varphi_{n,j}^{(\infty)}(z) = im_{n,j} \sqrt{w(z)} \res_{\lambda=\lambda_n} \left( \pm \frac{1}{i\sqrt{w(z)}} \Psi_{j,1}(z)e^{-\kappa(g(z)+1)-d(z)} + \Psi_{j,2}(z)e^{\kappa g(z)+d(z)} \right) \quad \text{(B.3)}
\]

for \( z \in \mathcal{L}_c^{(\pm)} \) (we will not need an expression in \( \mathcal{L}_c^{(\pm)} \)).

Remark B.1. It follows from (2.13), (A.4) and that Schwarz symmetry of \( g(z), d(z) \) that \( g_+ + 1 = \frac{1}{2} \) and \( \frac{1}{2}(g_+ - g_-) = \frac{1}{2} + i\frac{\pi}{2} \) on \( I \), and \( d_+ = \frac{d_- - d_0 - \ln w}{2} = i\pi d_+ - \frac{1}{2} \ln w \) on \( I \). Then, taking the + boundary value of (B.4) and using Proposition A.10, we obtain the following chain of equalities valid for \( z \in I \) (we omit the dependence on \( z \) for brevity)

\[
\varphi_{n,j}^{(\infty)}(z) = im_{n,j} \sqrt{w(z)} \res_{\lambda=\lambda_n} \left( \frac{1}{i\sqrt{w(z)}} \Psi_{j,1}e^{-(\kappa g_+ + 1) + d_+} + \Psi_{j,2}e^{\kappa g_+ + d_+} \right) = \]

\[
= m_{n,j} \res_{\lambda=\lambda_n} \left( \Psi_{j,1}e^{-i\kappa g_+ - \frac{\pi}{2} - i\pi d_+} + \Psi_{j,2}e^{i\kappa g_+ - \frac{\pi}{2} + i\pi d_+} \right) = \]

\[
= m_{n,j} \sqrt{w(z)} \res_{\lambda=\lambda_n} \left( \Psi_{j,1}e^{-i\kappa g_+ - \frac{\pi}{2} - i\pi d_+} + \Psi_{j,2}e^{i\kappa g_+ + i\pi d_+} \right) = \]

\[
= 2im_{n,j} e^{-\frac{\pi}{2}} \Im \left( \res_{\lambda=\lambda_n} \Psi_{j,1}e^{-i\kappa g_+ - \frac{\pi}{2} - i\pi d_+} \right). \quad \text{(B.5)}
\]

Lemma B.2. The functions \( \prod_{j=1}^{2g+2}(z - a_j)^{\frac{1}{2}} \Upsilon_{jk}(z; p) \) are uniformly bounded on the compact set \( Z_0 \times A_2 \times \ldots A_g \).
Proof. The closure $\bar{Z}_0$ of $Z_0$ includes the endpoints $a_j$, $j = 1, \ldots, 2g+2$. Lemma A.22 and Corollary A.25 imply
\[
\Upsilon_{jk}(z; \vec{p}) = \frac{(-1)^j}{\sqrt{N_j(\vec{p})}} \cdot \frac{\Theta((-1)^{k+1}u(z) + (-1)^{j}u(\infty) + f(\vec{p})) \, r(z)}{\Theta((-1)^{k+1}u(z) + (-1)^{j}u(\infty) + W_0)}.
\] (B.6)
Moreover, according to Corollary A.25 and (A.38), we can always assume that for a given $\vec{p} \in A_2 \times \ldots A_g$ we have $\left| \frac{1}{\sqrt{N_j(\vec{p})}} \right| < \nu^{-\frac{1}{2}}$. Thus, it remains to estimate the second factor in (B.6).

The numerator $\Theta((-1)^{k+1}u(z) + (-1)^{j}u(\infty) + f(\vec{p}))$ is analytic (in all variables) on the compact set $(z; \vec{p}) \in \bar{Z}_0 \times A_2 \times \ldots A_g$ and, thus, is bounded there. The Theta function in the denominator depends only on $z$. According to Lemma A.11, it vanishes only at infinity (like $z^{-1}$) and at the $g-1$ points $z = a_j$, $j \in J$, where it vanishes like $\sqrt{z} - a_j$. Taking into account (A.25), we see that the ratio $\frac{r(z)}{\Theta((-1)^{k+1}u(z) + (-1)^{j}u(\infty) + W_0)}$ is bounded on $\bar{Z}_0$ away from the endpoints $a_j$, and behaves like $O(z - a_j)^{-\frac{1}{2}}$ near each endpoint $a_j$, $j = 1, \ldots, 2g+2$. Thus, the statement of the lemma is proven.

\[\square\]

Remark B.3. According to Lemmas A.22, A.21,
\[
\Upsilon_{j,k}(z; \vec{p}_n) = \frac{\text{res}_{\nu = \xi_n} \Psi_{jk}(z; \nu)}{\pi \sqrt{N_j(\vec{p}_n)}} \quad \text{for} \quad j, k = 1, 2.
\] (B.7)
Thus, (B.5) implies that $\varphi_{n,j}^{(\infty)}$ belongs to $L^2(I_i)$.

Let $\mathcal{J}^\omega$ denote the $\omega$-neighborhood of the endpoints of $I$.

Lemma B.4. For any $\omega > 0$ there exists $c_\omega > 0$ such that
\[
|\hat{f}_n(z)| \leq \frac{c_\omega}{1 + |\hat{z}|^2} e^{\xi_n(\Re(\hat{z}) + \frac{1}{2})} \quad \text{on} \quad \bar{C} \setminus \mathcal{J}^\omega.
\] (B.8)
Proof. As it was mentioned in Section A.3, $\hat{f}_n = \varphi_{n,j} / \| \varphi_{n,j} \|$, $j = 1, 2$, provided $\| \varphi_{n,j} \| > 0$. Note that for every $n \in \mathbb{N}$ at least one of $\| \varphi_{n,j} \| > 0$. Then, using the estimate (A.18) for $E(z; \nu)$ and taking the residue of (A.17), we have
\[
\hat{f}_n(z) = i e^{\xi_n(\hat{z} + \frac{1}{2}) + d(\hat{z})} \left( \sqrt{w(z)} \, \Upsilon_{2}(z; \vec{p}_n) + \frac{\sqrt{|w(z)| \mathcal{O}(\bar{\xi}_n^{-1})}}{1 + |\hat{z}|} \right)
\] (B.9)
uniformly on closed subsets of $C \setminus \mathcal{J}^\omega$. Here we also used Remarks B.3 and A.26. Near $z = \infty$ the function $\Upsilon_{2}(z; \vec{p}_n)$ has the behavior
\[
\Upsilon_{2}(z; \vec{p}_n) = \frac{K(\vec{p}_n)}{z} + \mathcal{O}(z^{-2})
\] (B.10) (see RHP A.6 and (B.7)) with some constant $K(\vec{p}_n) > 0$. Since $K(\vec{p}_n)$ is continuous on the compact set $\vec{p}_n \in A_2 \times \cdots \times A_g$, we conclude that there is $\bar{K} > 0$ and a neighborhood of $z = \infty$ such that $|\Upsilon_{2}(z; \vec{p}_n)| \leq \frac{\bar{K}}{|z|}$ in this neighborhood for all $n \in \mathbb{N}$. Then, according to Lemma B.2, there exists some $K > 0$, such that
\[
|\Upsilon_{2}(z; \vec{p}_n)| \leq \frac{K}{1 + |z|}
\] (B.11)
on $C \setminus \mathcal{J}^\omega$ uniformly in $n \in \mathbb{N}$. The statement thus follows from (B.9).
Remark B.5. The subtlety in proving the accuracy in (B.9) is that $\hat{J}_n$ is obtained by dividing (B.3) by (A.37) and, although $\max_{j=1,2} |N_j(\tilde{p}_n)|$ is separated from zero, each of the sequences $|N_1(\tilde{p}_n)|$, $|N_2(\tilde{p}_n)|$ is, in general, not. However, for each $n \in \mathbb{N}$ we can always use a particular choice of $j$ that provides the said maximum, which guarantees the uniformity of the estimate.

The following corollary is a direct consequence of Corollary A.25.

**Corollary B.6.** For any $\omega > 0$ and any closed interval $I \subset \mathbb{R} \setminus J^\omega$, the functions

$$\mu_I(\tilde{p}) = \max_{z \in I} |\tilde{Y}_2(z; \tilde{p})|, \quad \nu_I(\tilde{p}) = \max_{z \in I} \left|\frac{\partial}{\partial z} \tilde{Y}_2(z; \tilde{p})\right|,$$

(B.12)

are continuous on $A_2 \times \cdots \times A_g$.

**Proof of Lemma 3.5.** Let us choose $\omega > 0$ so that $J \setminus J^\omega$ contains some closed interval $I$. We construct intervals $J_n \subset I$, $n \in \mathbb{N}$. In view of Theorem A.13 and (B.9), it is sufficient to construct $J_n$ so that $|\tilde{Y}_2(z; F_n)|$ instead of $|f_n e^{-\kappa_n(\partial(\Gamma_z) + \frac{1}{\lambda})}|$ is separated from zero on $J_n$ uniformly in $n$. Let

$$\mu_* = \min_{\tilde{p} \in A_2 \times \cdots \times A_g} \mu_I(\tilde{p}), \quad \nu_* = \max_{\tilde{p} \in A_2 \times \cdots \times A_g} \nu_I(\tilde{p}),$$

(B.13)

and let the maximum $\mu_I(\tilde{p})$ of $|\tilde{Y}_2(z; \tilde{p})|$ in $z \in I$ be attained at some $\tilde{z} \in I$. Obviously, $\mu_*>0$ and $\nu_* \leq \frac{\mu_*}{2}$. Let us now define the intervals $J_n$ by

$$J_n = \left( z^{(n)} - \frac{\mu_*}{2\nu_*}, z^{(n)} + \frac{\mu_*}{2\nu_*} \right) \cap I.$$

(B.14)

Then the length of each $J_n$ is at least $\min(\frac{\mu_*}{2\nu_*}, |I|)$, where $|I|$ is the length of $I$. Then, according to (B.12), (B.13),

$$|\tilde{Y}_2(z; \tilde{p}^{(n)})| \geq \frac{\mu_*}{2}$$

(B.15)

for all $z \in J_n$. Thus, we completed the proof Lemma 3.5.

**Proof of Lemma 4.2.** The norm of $\varphi_{n,j}^{(\infty)}$ in $L^2(I_i)$ (here $\Lambda_n = e^{-\kappa_n}$ is the approximate singular-value) is given by

\[
\int_{I_i} \left( \frac{\operatorname{res}_{\lambda = \lambda_n} \Gamma_{j_2}^{(\infty)}(z; \lambda)}{\lambda} \right)^2 \frac{2}{1} w(z) dz = -i \int_{I_i} \left( \frac{\operatorname{res}_{\lambda = \lambda_n} \lambda J_{j_1}^{(\infty)}(z; \lambda)}{\lambda} \right) \left( \frac{\operatorname{res}_{\lambda = \lambda_n} \Gamma_{j_2}^{(\infty)}(z; \lambda)}{\lambda} \right) \frac{1}{\lambda} dz,
\]

(B.16)

where $J(F) = F_+ - F_-$. We can thus deform the two contours from the $\pm$ boundaries of $I_i$ to $\partial \mathcal{L}_1^{(\pm)}$. The latter consist of arcs joining the consecutive endpoints of $I_i$ (in the formula below, we omit the reference to the dependence on $\kappa, z$ for brevity):

\[
-i \int_{I_i} \left( \frac{\operatorname{res}_{\lambda = \lambda_n} \lambda J_{j_1}^{(\infty)}(z; \lambda)}{\lambda} \right) \frac{1}{\lambda} \left( \frac{\operatorname{res}_{\lambda = \lambda_n} \Gamma_{j_2}^{(\infty)}(z; \lambda)}{\lambda} \right) \frac{1}{\lambda} dz
\]

\[
= -i m_{n,j}^{(2)} e^{-\kappa_n} \sum_{\pm} \int_{\partial \mathcal{L}_1^{(\pm)}} \left( \frac{\operatorname{res}_{\kappa = \kappa_n} \Psi_{j_1}(z; \kappa)}{\kappa} \right) \left( \frac{\operatorname{res}_{\kappa = \kappa_n} \Psi_{j_2}(z; \kappa)}{\kappa} \right) \frac{1}{iw} \frac{\partial}{\partial w} W_{j_1,j_2}(w) \frac{1}{w} dz
\]

\[
= -i m_{n,j}^{(2)} e^{-\kappa_n} \left( \int_{B_1} \left( \frac{\operatorname{res}_{\kappa = \kappa_n} \Psi_{j_1}(z; \kappa)}{\kappa} \right)^2 \frac{e^{-\kappa_n}}{iw(z)} \frac{1}{\partial w} W_{j_1,j_2}(w) \frac{1}{w} dz + \sum_{\pm} \int_{\partial \mathcal{L}_1^{(\pm)}} \left( \frac{\operatorname{res}_{\kappa = \kappa_n} \Psi_{j_1}(z; \kappa)}{\kappa} \right)^2 \frac{e^{-\kappa_n}}{iw(z)} \frac{1}{\partial w} W_{j_1,j_2}(w) \frac{1}{w} dz \right).
\]

(B.17)

\[\text{(B.18)}\]
The expression (B.17) is precisely $m_{n,j}^2N_{n,j}$ from (A.35). The remaining terms on line (B.18) contribute to order $\mathcal{O}(\bar{\kappa}_n^{-1})$ as we now explain. Indeed, according to (B.7) and Lemma B.2, there exists some $C > 0$, such that the integrals in (B.18) are bounded by

$$\int_{\partial L(z)} \frac{C}{\prod_{j=1}^{2g+2} (z - a_j)} e^{-\bar{\kappa}_n \Re(2g(z)+1)-2\Re(d(z)} |w(z)| \left| \omega(z) \right|^2 \, dz \tag{B.19}$$

uniformly in $n \in \mathbb{N}$. Using that $\Re(2g(z)+1) = C_j |z - a_j|^2 (1 + \mathcal{O}(|z - a_j|))$, see (A.2), we can estimate (B.18) to be of order $\mathcal{O}(\bar{\kappa}^{-1})$ as $\bar{\kappa} \to +\infty$. Thus, according to Lemma A.22, we have proved that

$$\left\| \varphi_{n,j}^{(\infty)} \right\|^2_{I_i} = m_{n,j}^2 e^{-\bar{\kappa}_n \pi^2/2} N_j(p_n) \left( 1 + \mathcal{O}(\bar{\kappa}_n^{-1}) \right). \tag{B.20}$$

Using (4.2), (B.5) and (B.7), we obtain

$$\bar{f}_n(z) = \frac{\varphi_{n,j}^{(\infty)}(z)}{m_{n,j} e^{-\bar{\kappa}_n \pi/2} \sqrt{N_j(p_n)}}, \tag{B.21}$$

where the right hand side does not depend on $j = 1, 2$. Now (B.20), (B.21) and (1.14) imply that $\| \bar{f}_n(z) \|_{I_i} = 1 + \mathcal{O}(n^{-1})$.

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