Covariant Formulation of the Dynamics in a Dissipative Dielectric Obtained from a Simplified Lagrangian

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Abstract

Equations of motions and energy-momentum density tensors are obtained for a dispersive and dissipative medium sustaining electric and magnetic polarizations, using Lagrangian formalisms. A previous work on the subject by the authors has been simplified by reduction in the Lagrangian of the number of independent vector fields, with which the sink modes (associated with the dissipation) interact. The paper also formulates the dynamics of the electromagnetic field in the medium in a covariant (relativistic) manner. We discuss (and compare) the results, especially from the point of view of the positivity of the energy density.

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1 Introduction

Lagrangian methods provide a systematic and unified approach to dynamic phenomena. Every mature subject in physics can and should be described
by a Lagrangian. To name a few: particle physics is described by the "standard Model" Lagrangian, General relativity by the "Curvature Scalar" Lagrangian and so on.

Nevertheless, there are some phenomenological treatments that are not obviously expressible by a Lagrangian formalism and have led to ambiguities and controversies. An instance of this is the dynamics of polarization in dispersive-dissipative materials. While the equations of motion (shown below in equation (10) and equation (11)) have been in frequent use, the derivation of these equations from a Lagrangian has been wanting for some time. As a consequence, basic quantities like energy and momentum densities, that are integral parts of the Lagrangian method, have become variously formulated and subject to dispute. A brief account of the history of energy densities is found in [1]. Following that account and adopting a Lagrangian formulation, that work has unambiguously derived an expression for the energy density.

Relativistic treatments of dissipative electromagnetic media, but not based on a Lagrangian formalism, were given in [2]. A comparison of three different approaches for relativistic treatments of dissipative electromagnetic media was undertaken in a book by Fano, Chu and Adler [3]. In this book the authors compared the approach they endorse with two other approaches, denoted the Minkowski and Amperian approaches. The Minkowskian approach was criticized by the authors because it assumes a linear scalar relation between the displacement field $\vec{D}$ and the electric field $\vec{E}$, and between the magnetic field $\vec{H}$ and the magnetic flux density field $\vec{B}$. The authors argue that this relation is not invariant under a general Lorentz transformation and thus is not objective but subjective and can only be regarded as true in a a particular frame of reference. Furthermore, even according to an observer in rest with respect the electromagnetic medium, the linear scalar relations are not generally true since some magnetic materials have a hysteresis curve and thus $\vec{B}$ and $\vec{H}$ are not linearly related, while other materials possess a tensorial dielectric constant instead of a scalar one. Fano, Chu & Adler also claim that the Lorentz force resulting from the Minkowskian formalism contradicts experimental evidence regarding polarized and magnetized materials. The Amperian approach is criticized in the book on the basis that the resulting Lorentz force contradicts experimental evidence regarding magnetized materials. The authors of this paper do not follow the approach of Fano, Chu & Adler, since it is not based on a variational approach and thus one cannot canonically derive an energy expression. Furthermore, Fano, Chu & Adler do not regard the magneti-
zation and polarization of a material as degrees of freedom subject to their own dynamics; rather, these are treated as given source terms for the the electric and magnetic fields. It should be noted however, that both the energy density obtained by them and that using the Amperian approach are positive.

Other approaches, notably for physical systems containing both viscous flows and electromagnetic fields are discussed by Felderhof [4] and Rosensweig [5]. The first work postulates an energy-momentum tensor (equation (3.12) in [4]); in our formalism we will aim at deriving the energy momentum tensor from a Lagrangian using the well known canonical formalism (see below). The second work [5] is based on an "engineering" type of approach. Rosensweig has postulated balance equations in a moving frame for linear momentum, angular momentum and energy and has derived "constitutive equations" using the first and second laws of thermodynamics. The energy momentum tensor was not considered in this work, nor a Lagrangian to derive the energy momentum tensor from.

It is evident that the energy density, written as $T_{00}$, is a component of the energy momentum tensor $T_{ij}$ ([9]-[12]). This is uniquely derivable from the Lagrangian density $\mathcal{L}(u_k, u_{k,j})$, which is composed of the field variables $u_k$ and of their derivatives $u_{k,j}$. (A symbol after the comma represents a derivative.) The formal definition of the energy-momentum density is given by

$$T_{ij} = \frac{\partial \mathcal{L}}{\partial u_{i,k}} u_{i,j} - \mathcal{L} \delta_{ij}$$

Summation is implied for repeated indexes. In our convention the indexes take the value 0 for the time component and the numbers (1, 2, 3) for the remaining, space components. The dot over a quantity will signify a time derivative.

The expression for the energy found in [1] was positive. The essential idea there was the introduction of added degrees of freedom (named sinks), which are depositories of the the energy residing in the electromagnetic field and in the polarization of the medium. Physically, the sinks are macroscopic modes of motion in the solid (and less obviously, in its surrounding) different from the polarization modes, but weakly interacting with them. These degrees of freedom have been given microscopic interpretation in the thermodynamic limit, in terms of a set of interacting oscillators with a special interaction matrix [6]. The need to include additional degrees of freedom to treat the irreversible behavior of polarizable systems was noted in a classic text [7].
The proof that any set of differential equations can be embedded in a larger set that can be derived from a variational principle was supplied by Bateman [8].

Our aim in the present work is twofold: first we simplify our previous formulation by reducing the number of vector degrees of freedom. Thus, the formalism presented here contains two vector degrees of freedom less than the previous one. The resulting Lagrangian is considerably simpler without compromising the power of our formalism. Secondly, in later sections, we give a relativistic covariant formalism, making the theory valid also in a moving system of reference. The motivation for this is twofold:

1. It is well known that any physical theory should obey the laws of special relativity and should be invariant under Lorentz transformation - in this paper we construct a theory along the same lines as our previous theory [1], which satisfied these requirements.

2. Our previous theory did not contain spatial derivatives of certain quantities; the principle of covariance has helped us put those derivatives in.

Further, we discuss the implications. Some of the equations in the earlier sections have already appeared in [1]; however, they are indispensable both for a self-contained reading of the simplified theory and as an introduction to and comparison with the covariant theory.

2 Time independent polarizations

Our first task is to formulate the Lagrangian density of the electromagnetic and polarization and magnetization fields for a material whose polarization are static. We use CGS Gaussian units. Connections between the electric displacement field \( \vec{D} \) and the electric field \( \vec{E} \) and polarization \( \vec{P} \), and, likewise, between the magnetic field \( \vec{H} \) and the magnetic induction field \( \vec{B} \) and magnetization \( \vec{M} \) are then provided by the relations:

\[
\vec{D} = \vec{E} + 4\pi \vec{P},
\]

\[
\vec{H} = \vec{B} - 4\pi \vec{M}.
\]

The above fields satisfy both the homogeneous Maxwell’s equations:

\[
curl \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0, \quad div \vec{B} = 0
\]
and the inhomogeneous equations:
\[
curl \vec{H} - \frac{1}{c} \dot{\vec{D}} = \frac{4\pi}{c} \vec{J}, \quad div \vec{D} = 4\pi \rho
\]  

(5)

We further make the assumption that both free charges $\rho$ and currents $\vec{J}$ are zero.

It is not possible to derive these equations from a Lagrangian density expressed in terms of the fields. However, this difficulty is traditionally overcome by representing the fields in terms of vector $\vec{A}$ and scalar $\Phi$ potentials, as follows:

\[
\vec{E} = -\vec{\nabla} \Phi - \frac{1}{c} \dot{\vec{A}}, \quad \vec{B} = curl \vec{A}
\]  

(6)

Using these definitions one can see that the homogeneous equations (4) are satisfied automatically. The inhomogeneous equations (5) are the outcome of the functional derivation of the Lagrangian below:

\[
\mathcal{L} = \mathcal{L}_{EB} + \mathcal{L}_{PE} + \mathcal{L}_{MH}
\]

\[
\mathcal{L}_{EB} = \frac{1}{8\pi} [\vec{E}^2 - \vec{B}^2]
\]

\[
\mathcal{L}_{PE} = \vec{P} \cdot \vec{E}
\]

\[
\mathcal{L}_{MH} = \vec{M} \cdot \vec{H}
\]  

(7)

In forming the derivatives it is understood that all field quantities as given in terms of the potentials of equation (6). The energy density is then obtained from equation (1):

\[
T^0_0 = \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} \cdot \dot{\vec{A}} - \mathcal{L}.
\]  

(8)

After some algebra this yields for the Lagrangian of equation (7) the result:

\[
T^0_0 = \frac{1}{8\pi} [\vec{E}^2 + \vec{H}^2 + (4\pi \vec{M})^2]
\]  

(9)

The last term $\vec{M}^2$ can be usually discarded for static magnetization, since it represents nothing but a time-independent constant. However, we include it for reference in future sections, where the electric polarization and the magnetization will become dynamic degrees of freedom.
3 Equations of motion with dissipative terms

The equations that treat the development of the polarization $\vec{P}$ by an applied electric field $\vec{E}$, and the magnetization $\vec{M}$ as induced by an external magnetic field $\vec{H}$ are, as appearing in e.g., [13], the following:

$$\ddot{\vec{P}} + \Gamma_e \dot{\vec{P}} + \omega_r^2 \vec{P} = \omega_p^2 \vec{E}$$  \hspace{1cm} (10)

$$\ddot{\vec{M}} + \Gamma_h \dot{\vec{M}} + \omega_0^2 \vec{M} = F \omega_0^2 \vec{H}$$  \hspace{1cm} (11)

with all symbols and notation as defined in [13]. These equations hold for $t \geq 0$. Fourier transformed to the frequency ($\omega$) domain, the equations for the frequency components of the fields are written, after a transposition of terms as

$$[-\omega^2 - i\omega \Gamma_e + \omega_r^2] \vec{P}(\omega) = \omega_p^2 \vec{E}(\omega) + \dot{\vec{P}}(t = 0) - (i\omega - \Gamma_e) \vec{P}(t = 0)$$ \hspace{1cm} (12)

and similarly for the magnetization. (Here time-derivatives of the polarization were integrated by parts in the manner: $\int_0^\infty \dot{P}(t)e^{i\omega t}dt = [P(t)e^{i\omega t}]_0^\infty - i\omega \int_0^\infty P(t)e^{i\omega t}dt$ and neglecting the field at infinity, etc. As discussed in [7], there can be a polarization at $t = 0$ due to permanent dipoles.) One can introduce the electric and magnetic susceptibilities, defined by

$$\chi_e(\omega) = \frac{\omega_p^2/\omega_r^2}{1 - (\omega/\omega_r)^2 - i\omega\Gamma_e/\omega_r^2}$$ \hspace{1cm} (13)

$$\chi_m(\omega) = \frac{F}{1 - (\omega/\omega_0)^2 - i\omega\Gamma_h/\omega_0^2}$$ \hspace{1cm} (14)

to write the equations of motion for the frequency components as:

$$\vec{P}(\omega) = \chi_e(\omega)[\vec{E}(\omega) + \frac{\dot{\vec{P}}(0) - (i\omega - \Gamma_e)\vec{P}(0)}{\omega_p^2}]$$

$$\vec{M}(\omega) = \chi_m(\omega)[\vec{H}(\omega) + \frac{\dot{\vec{M}}(0) - (i\omega - \Gamma_h)\vec{M}(0)}{F\omega_0^2}]$$ \hspace{1cm} (15)

The electric and magnetic susceptibilities are complex functions with related imaginary ($\Im$) and real ($\Re$) parts. When the electric losses are sufficiently low $\Gamma_e << 1$, the electric susceptibility can be approximated in the vicinity of the resonance by:

$$\Im[\chi_e] = \frac{\omega_p^2}{\omega_r \Gamma_e (\omega - \omega_r)^2 + 1}, \quad \Re[\chi_e] = -\frac{\omega - \omega_r}{\Gamma_e^2} \Im[\chi_e]$$ \hspace{1cm} (16)
Analogous relations can be derived for the real and imaginary parts of the magnetic susceptibility for $\Gamma_h << 1$:

$$\Im[\chi_m] = \frac{F\omega_0}{\Gamma_h} \frac{1}{(\frac{\omega - \omega_0}{\Gamma_h})^2 + 1}, \quad \Re[\chi_m] = -\frac{\omega - \omega_0}{\Gamma_h} \Im[\chi_e]$$  \hspace{1cm} (17)

Returning now to equation (10) and equation (11), for self-consistency these have to be supplemented by the Maxwell-equations given in the previous section and derived from the Lagrangian of equation (7) that contain the contravariant four-vectors $A^\alpha = (\vec{A}, \Phi)$.

To take account of the dissipative nature of the processes we introduce a set of new fields, designated canonical fields. These will be formally distinguished from the previous, physical fields by writing them in lower case symbols. The introduction of these fields follows theories by [14] -[16]. The canonical fields (taken to be real) are made to be part of a Lagrangian formulation. We note that for the preceding physical quantities (written in upper case symbols), this cannot be achieved [14].

To make our new notation explicit, we shall work with electric polarization $p^\alpha$ and magnetization $m^\alpha$ and the electromagnetic fields $e^\alpha, h^\alpha$ and the (contravariant) vector-potentials $a^\alpha$. It will be later demonstrated that these new, canonical fields contain a time dependent factor which (partly, but not fully) neutralizes the dissipative process. Otherwise, the choice of the fields (and of the various constant factors) is guided by the requirement that we recapture the form of the equations of motion and the energy density currently widely employed in solid state optics in the appropriate limiting case of no dissipation. (This requirement impairs at times the formal symmetry between the modes linked with the polarization and the magnetization, respectively.) The physical meaning of the canonical fields will be made clear later by comparing their equations of motion with those of the physical fields. (For a similar procedure see [14].)

In addition, we shall introduce two new scalar fields, the so called "sink" fields: the first being $\Psi$, that incorporates macroscopic degrees of freedom associated with the dissipation of the polarization (analogous to the degree of freedom denoted with the same symbol in [16]) and the second, the sink-field $\Phi$ which accounts for the dissipation in the magnetic mode (and originates physically in atomic spin or angular momenta in the solid).
4 The Lagrangian with sink terms

The addition of sink fields requires the extension of the Lagrangian as follows:

$$\mathcal{L} = \mathcal{L}_{eb} + \mathcal{L}_{pe} + \mathcal{L}_{mh} + \mathcal{L}_{p\Psi} + \mathcal{L}_{m\Phi} + \mathcal{L}_{\Psi} + \mathcal{L}_{\Phi}$$

(18)

In this Lagrangian we have first the Lagrangian of equation (7), there written in terms of the electromagnetic fields, and now rewritten in the new lower-case variables as $\mathcal{L}_{eb} + \mathcal{L}_{pe} + \mathcal{L}_{mh}$. The above expression also contains the Lagrangians in the polarization modes, the magnetization modes, the electromagnetic fields, the $\Psi$ and $\Phi$ sink fields and then the interaction-Lagrangians between polarization and electric field, magnetization and magnetic field and, finally, the polarization and magnetization with their respective sink fields. The various terms are written out as:

$$\mathcal{L}_{eb} = \frac{1}{8\pi} [e_1^2 - b_1^2]$$

(19)

$$\mathcal{L}_{pe} = pe_1$$

(20)

$$\mathcal{L}_{mh} = mh_1$$

(21)

$$\mathcal{L}_{p\Psi} = \frac{1}{2\omega_p^2} (\dot{p}^2 - (\omega_p^2 - \dot{\Psi}^2) p^2 - 2pp\dot{\Psi})$$

(22)

$$\mathcal{L}_{m\Phi} = \frac{1}{2F\omega_0^2} (\dot{m}^2 - (\omega_0^2 - \dot{\Phi}^2) m^2 - 2m\dot{m}\dot{\Phi})$$

(23)

$$\mathcal{L}_{\Psi} = \frac{1}{2} \kappa \dot{\Psi}^2 - \frac{1}{2} \rho \Psi^2$$

(24)

$$\mathcal{L}_{\Phi} = \frac{1}{2} \lambda \Phi^2 - \frac{1}{2} \nu \Phi^2$$

(25)

having chosen the $x$ (or 1) axis as the direction of the polarization and magnetization vectors. In the above, we have expressed $\mathcal{L}_{\Psi}$ and $\mathcal{L}_{\Phi}$ as harmonic modes of motion (when there is no coupling to other fields.) This choice seems to allow the simplest type of representation of unspecified (and, so far, arbitrary) degrees of freedom. The symbols $\kappa, \rho, \lambda, \nu$ are constant, non-negative coefficients, whose values depend on the nature of the relaxation mechanisms.

We next derive $\mathcal{L}$ with respect to each degree of freedom. In accordance with the Euler-Lagrange equations, this procedure leads to the equations of motion. In detail, for the variables $p$ and $m$ the equations of motion take
the following form:

\[
\ddot{p} - (\dot{\Psi} + \dot{\Psi}^2 - \omega_p^2)p = \omega_p^2 e_1 \tag{26}
\]

\[
\ddot{m} - (\dot{\Phi} + \dot{\Phi}^2 - \omega_0^2)m = F \omega_0^2 h_1 \tag{27}
\]

The electromagnetic equations of motion are identical to Maxwell’s equations in the new (small letter) variables. Lastly, for the sink coordinates one finds

\[
\ddot{\Psi} + \frac{\rho}{\kappa} \Psi = \frac{1}{\kappa \omega_p^2} \frac{\partial (p(\dot{p} - p\dot{\Psi}))}{\partial t} \tag{28}
\]

\[
\ddot{\Phi} + \frac{\nu}{\lambda} \Phi = \frac{1}{\lambda F \omega_0^2} \frac{\partial (m(\dot{m} - m\dot{\Phi}))}{\partial t} \tag{29}
\]

The expression for the energy density \( T_{0}^{0} \) is found from equation (1), namely,

\[
T_{0}^{0} = \sum_{k} \frac{\partial L}{\partial \dot{u}_k} u_k - L \tag{30}
\]

with the summation over all degrees of freedom. We separate the electromagnetic, the polarization and the magnetization parts

\[
T_{0}^{0} = (T_{0}^{0})_{e,h} + (T_{0}^{0})_{p,\Psi} + (T_{0}^{0})_{m,\Phi} \tag{31}
\]

where the first term has the well known form of the electromagnetic energy density, already given in equation (9):

\[
(T_{0}^{0})_{e,h} = \frac{1}{8\pi} [e^2 + h^2 + (4\pi m)^2] \tag{32}
\]

The others take the forms:

\[
(T_{0}^{0})_{p,\Psi} = \frac{1}{2\omega_p^2} [(\dot{p} - p\dot{\Psi})^2 + \omega_p^2 p^2] + \frac{\kappa}{2} \dot{\Psi}^2 + \frac{\rho}{2} \Psi^2 \tag{33}
\]

\[
= \frac{1}{2\omega_p^2} (\dot{p}^2 + \omega_p^2 p^2) + \frac{\kappa}{2} \dot{\Psi}^2 + \frac{\rho}{2} \Psi^2 + \frac{1}{2\omega_p^2} \dot{p} \dot{\Psi} \tag{34}
\]

and

\[
(T_{0}^{0})_{m,\Phi} = \frac{1}{2F \omega_0^2} [(\dot{m} - m\dot{\Phi})^2 + \omega_0^2 m^2] + \frac{\lambda}{2} \dot{\Phi}^2 + \frac{\nu}{2} \Phi^2 \tag{35}
\]
\[
\begin{align*}
\frac{1}{2F_0^2} (\dot{m}^2 + \omega_0^2 m^2) + \frac{\lambda}{2} \Phi^2 + \frac{\nu}{2} \Phi^2 + \frac{1}{2F_0^2} m^2 \dot{\Phi}^2 - \frac{1}{2F_0^2} \frac{\partial m^2}{\partial t} \Phi
\end{align*}
\] (36)

The first-written forms are sums of squares with non-negative coefficients; this ensures that each part of the energy density is positive (non-negative). The usual expressions for the energy density, e.g. in [12] or [13], differ from the above by the presence of the terms in \( \Phi \) and \( \Psi \) and their time derivatives. In the following section we shall eliminate these variables by making use of the equations of motion, equation (28) and equation (29).

5 A simple solution

To work out a fully solvable case, we postulate that the coefficients \( \rho \) and \( \nu \) in equation (24) and equation (25) vanish. Then the equations of motion for the sink-modes, equation (28) and equation (29), can be integrated. We shall carry through the electric polarization case, but an analogous development holds for the magnetization sink-variable. We assume the following initial conditions for \( \Psi(t) \)

\[
\Psi(0) = 0, \quad \dot{\Psi}(0) = \Gamma_e/2
\] (37)

With these choices the differential equation (10) is regained for short times, \( t << \frac{2}{\Gamma_e} \), as will be shown presently. (This is similar to the procedure in [14]). Then, from equation (28),

\[
\ddot{\Psi} = \frac{1}{2\kappa_\omega^2} \frac{\partial^2}{\partial t^2} p^2(t) - \frac{1}{\kappa_\omega^2} \frac{\partial (p^2(t) \dot{\Psi})}{\partial t}
\] (38)

A first integration gives

\[
\dot{\Psi} = \frac{1}{2\kappa_\omega^2} \frac{\partial p^2(t)}{\partial t} - \frac{p^2(t) \dot{\Psi}}{\kappa_\omega^2} + C
\] (39)

which leads to

\[
(1 + \frac{p^2(t)}{\kappa_\omega^2}) \dot{\Psi} = \frac{1}{\kappa_\omega^2} \frac{\partial p^2(t)}{\partial t} + C
\] (40)

where the constant is

\[
C = \frac{\Gamma_e}{2} (1 + \frac{p^2(0)}{\kappa_\omega^2}) - \frac{p(0) \dot{p}(0)}{\kappa_\omega^2}
\] (41)
Integrating once more and fitting the constant so as to satisfy the first initial condition in equation (37), we finally obtain:

\[ \Psi(t) = \frac{1}{2} \ln \left[ \frac{1 + \frac{p^2(t)}{\kappa \omega_p^2}}{1 + \frac{p^2(0)}{\kappa \omega_p^2}} \right] + \frac{\Gamma_e}{2} \left( 1 + \frac{p^2(0)}{\kappa \omega_p^2} \right) - \frac{p(0) \dot{p}(0)}{\kappa \omega_p^2} \int_0^t \frac{dt'}{1 + \frac{p^2(t')}{\kappa \omega_p^2}} \]  

(42)

One can now substitute this expression and equation (39) into equation (34) to obtain, after considerable simplification, an expression for the energy density arising from the time varying "canonical" polarization \( p(t) \) in the form:

\[ (T_0^0)_{p,\psi} = \frac{1}{2 \omega_p^2} \left( \frac{\dot{p}^2(t) + \kappa \omega_p^2 C^2}{1 + \frac{p^2(t)}{\kappa \omega_p^2}} + \omega_v^2 p^2(t) \right) \]  

(43)

In the case of slow relaxation, when \( \Gamma_e \) is small (quantitatively, when \( \Gamma_e \sqrt{\kappa} << 1 \), then so is \( \dot{p}(0) \). (This will be confirmed in the next section.) This results in the quantity \( C^2 \) being a small quantity of the second order, which can be neglected:

\[ (T_0^0)_{p,\psi} = \frac{1}{2 \omega_p^2} \left( \frac{\dot{p}^2(t)}{1 + \frac{p^2(t)}{\kappa \omega_p^2}} + \omega_v^2 p^2(t) \right) \]  

(44)

For the part of the energy density involving the "canonical" magnetization \( m(t) \), the variables \( n(t) \) and \( \Phi(t) \) having been eliminated through their equations of motion, a similar procedure gives

\[ (T_0^0)_{m,\Phi} = \frac{1}{2 F \omega_0^2} \left( \frac{\dot{m}^2(t)}{1 + \frac{m^2(t)}{\Lambda F \omega_0^2}} + \omega_v^2 m^2(t) \right) \]  

(45)

The above two expressions for the energy densities, equation (44) and equation (45), resemble those in equation (11) of Ruppin [13], except that they are written in the canonical (small letter) variables, rather than in the physical variables (the relations between these will be presently obtained), and that in the denominators they contain the polarization fields squared. The role of these is to damp out "kinetic" energy in the polarization motion, associated with the motion of charged (or spinning) matter.

The main results of this section, equation (43) and its analogue for the magnetization energy density, are exact and contain non-perturbative corrections to the energy density, due to the presence of the sink degrees of
freedom. While exact, they are model dependent in the sense that sinks represented by different Lagrangians would lead to different energy densities. This is clear, due to the presence in the energy densities of the parameters $\kappa$ and $\lambda$ that were introduced in the Lagrangian in equation (24) and equation (25). This outcome was anticipated some time ago in [17]. (We also note the opposing view in [18].) It is of interest to note that the non-dissipative limit is not regained when $\Gamma_e, \Gamma_h \to 0$, but only when also $\kappa, \lambda \to \infty$. It can be shown that in these limits the canonical fields are identical to the physical fields.

6 Interpretation of the fields

We now find the relation of the canonical fields to the physical fields in the presence of dissipation and sink modes. We regain the original equations of motion, equation (10) and equation (11), for the physical polarization variable, as follows: We postulate

$$p(t) = e^{\Psi(t)}P(t)$$ \hspace{1cm} (46)
$$m(t) = e^{\Phi(t)}M(t)$$ \hspace{1cm} (47)

This turns (the vector form of) equation (26) into the following:

$$\ddot{\vec{P}} + 2\dot{\Psi}(t)\dot{\vec{P}}(t) + \omega_r^2\vec{P} = \omega_p^2 e^{-\Psi(t)}\vec{e}$$ \hspace{1cm} (48)

and likewise for the magnetization variables. Then, from equation (37), for short times $0 < t << \frac{2}{\Gamma_e}$,

$$\dot{\Psi}(t) \approx \dot{\Psi}(0) = \Gamma_e/2$$ \hspace{1cm} (49)

We then obtain

$$\ddot{\vec{P}} + \Gamma_e\dot{\vec{P}}(t) + \omega_r^2\vec{P} = \omega_p^2 e^{-\Psi(t)}\vec{e}$$ \hspace{1cm} (50)

Recalling equation (10) and equation (11), we can thus extrapolate to later times so as to identify

$$\vec{e}(t) = e^{\Psi(t)}\vec{E}(t)$$ \hspace{1cm} (51)
$$\vec{h}(t) = e^{\Phi(t)}\vec{H}(t)$$ \hspace{1cm} (52)

and regain equation (10) and equation (11). This provides a physical meaning for all the "canonical" variables, as those fields in which the decay has
been reinstated. On the other hand, the decay is itself dependent on the fields. (Cf. [19].) Furthermore, the Maxwell equations for the physical fields are also modified in the dissipative-polarizable medium. This feature (of a modified Maxwell equation) also appears in [15] and [16] (eq. (14) and eq. (12), respectively).

7 A covariant dissipative Lagrangian density

In this section we introduce a covariant Lagrangian density which is a Lorentz invariant generalization of the Lagrangian density described in equation (18). Again we introduce a set of new fields, which are formally distinguished from the previous, physical fields by writing them in lower case symbols. Explicitly, we shall work with the scaled polarization and electromagnetic tensors $p^{\alpha\beta}$ and $f^{\alpha\beta}$ and other lower case quantities that were introduced earlier. Notice that the Lagrangian introduced here will contain less terms since both the magnetization and the polarization are now part of the polarization tensor $p^{\alpha\beta}$. This Lagrangian has the following parts:

$$L = L_{fp} + L_{p\Psi} + L_\Psi$$

In which $L_{fp}$ is defined in equation (89) in the appendix (although an upper case symbols to lower case symbols transformation is needed). $L_{p\Psi}$ is obtained by generalizing equation (22) as follows:

$$L_{p\Psi} = \frac{1}{2\omega_p^2} \left[ \partial_\mu p^{\alpha\beta} \partial^{\mu} p_{\alpha\beta} - (\dot{\omega}_p^2 - \partial_\mu \Psi \partial^\mu \Psi) p^{\alpha\beta} p_{\alpha\beta} - 2p_{\alpha\beta} \partial_\mu p^{\alpha\beta} \partial^\mu \Psi \right]$$

$L_{p\Psi}$ can be written as a sum of two terms: one depending on the magnetization, while the other depends on the polarization.

$$L_p = L_{p\Psi} + L_{m\Psi}$$

$$L_{p\Psi} = \frac{1}{c^2} \left[ -\frac{1}{c^2} (\partial_t \vec{p})^2 + (\partial_i \vec{p})^2 + (\dot{\omega}_p^2 - \frac{1}{c^2} (\partial_t \Psi)^2 + (\vec{\nabla} \Psi)^2) \vec{p}^2 ight]$$

$$L_{p\Psi} = \frac{2}{c^2} \vec{p} \cdot \dot{\vec{p}} \Psi - 2 \vec{p} \cdot \partial_i \vec{p} \partial_i \Psi$$

$$L_{m\Psi} = -\frac{1}{\omega_p^2} \left[ -\frac{1}{c^2} (\partial_t \vec{m})^2 + (\partial_i \vec{m})^2 + (\dot{\omega}_p^2 - \frac{1}{c^2} (\partial_t \Psi)^2 + (\vec{\nabla} \Psi)^2) \vec{m}^2 \right]$$

$$L_{m\Psi} = \frac{2}{c^2} \vec{m} \cdot \dot{\vec{m}} \Psi - 2 \vec{m} \cdot \partial_i \vec{m} \partial_i \Psi$$

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Finally $\mathcal{L}_\Psi$ is obtained by generalizing equation (24) as follows:

$$\mathcal{L}_\Psi = \frac{1}{2} [\hat{k} \partial_\mu \Psi \partial^\mu \Psi - \rho \Psi^2] = \frac{1}{2} [\hat{k} \left(\frac{1}{c^2} (\partial_t \Psi)^2 - (\vec{\nabla} \Psi)^2\right) - \rho \Psi^2]$$  \hspace{1cm} (56)

The field equations with respect to $p^{\alpha \beta}$, $\Psi$ and the electromagnetic fields can be obtained by taking the variational derivative of $\mathcal{L}$ given in equation (53) with respect to $p^{\alpha \beta}$, $\Psi$ and $a^\alpha$. The electromagnetic field equations were already derived by taking the variational derivative of equation (89) with respect to $a^\alpha$ and further change is unnecessary (except for that from upper to lower case symbols). Taking the variational derivative of $\mathcal{L}$ with respect to $\Psi$ leads to the equation:

$$\hat{k} \partial_\mu \partial^\mu \Psi + \rho \Psi = \frac{1}{\omega_p^2} \partial^\mu [p^{\alpha \beta} (\partial_\mu p_{\alpha \beta} - \partial_\mu \partial^\mu \Psi)]$$  \hspace{1cm} (57)

Taking the variational derivative of $\mathcal{L}$ with respect to $p_{\alpha \beta}$ leads to the equation:

$$\partial_\mu \partial^\mu p_{\alpha \beta} + (\hat{\omega}_r^2 - \partial_\mu \Psi \partial^\mu \Psi - \partial_\mu \partial^\mu \Psi) p_{\alpha \beta} = -\frac{1}{2} \omega_p^2 f_{\alpha \beta}$$  \hspace{1cm} (58)

In the "homogeneous" case in which the spatial derivatives of both the polarization tensor $p^{\alpha \beta}$ and the scalar field $\Psi$ vanish, the tensor equation (58) can be written as two vector equations:

$$\ddot{\vec{p}} - (\ddot{\Psi} + \dot{\Psi}^2 - {\omega}_r^2) \vec{p} = \omega_p^2 \vec{e}$$  \hspace{1cm} (59)

$$\ddot{\vec{m}} - (\ddot{\Psi} + \dot{\Psi}^2 - {\omega}_p^2) \vec{m} = -\omega_p^2 \vec{b}$$  \hspace{1cm} (60)

In which $\omega_r^2 = c^2 \hat{\omega}_r^2$ and $\omega_p^2 = \frac{1}{2} c^2 \hat{\omega}_p^2$. We can clearly see that equation (59) is the same as equation (26). As for equation (60) by using the equality (3) this can be rewritten as:

$$\ddot{\vec{m}} - (\ddot{\Psi} + \dot{\Psi}^2 - {\omega}_0^2) \vec{m} = -\omega_p^2 \vec{b}$$  \hspace{1cm} (61)

In which $\omega_0^2 = \omega_r^2 + 4\pi \omega_p^2$. This equation can be identified with equation (27) by choosing $F = -\omega_p^2 \omega_0^2$ and equating the magnetic and electric dissipative modes ($\Phi \equiv \Psi$). Notice that in magnetized materials the magnetic field is in a direction opposite to the magnetization as can be seen from figure 1, which explains the negative sign in the above equation.
Figure 1: Magnetic field lines in a box of permanent uniform magnetization - a cross section view. The direction of magnetization can be inferred from the way the magnetic field lines look outside the box magnet. The normal component of the field changes sign across the material boundary, unlike in the dielectric case.

7.1 The energy momentum tensor of the dissipative Lagrangian density

Using the dissipative Lagrangian density defined in equation (53) we can calculate the energy momentum tensor by using the formula given in (1). Doing so we obtain the following expression:

\[ T^\beta_\alpha = T^\beta_\alpha[f\rho] + T^\beta_\alpha[p\Psi] + T^\beta_\alpha[\Psi] \] (62)

\( T^\beta_\alpha[f\rho] \) is calculated in Appendix (A.3) in equation (93). It remains to obtain expressions for \( T^\beta_\alpha[p\Psi] \) and \( T^\beta_\alpha[\Psi] \); these are given below:

\[ T^\beta_\alpha[p\Psi] = \frac{1}{\omega^2_p} [\partial^\beta p^\gamma\partial_\alpha p_\gamma + p^\gamma(\partial^\beta p\partial_\alpha p_\gamma + \partial_\alpha p \partial^\beta p_\gamma) - \delta^\beta_\alpha L_{p\Psi}] \] (63)

and

\[ T^\beta_\alpha[\Psi] = \hat{\kappa} \partial^\beta \Psi \partial_\alpha \Psi - \delta^\beta_\alpha L_{\Psi} \] (64)

Let us look at the energy density \( T^{0}_0 \). For \( T^{0}_0[\Psi] \) we obtain the positive quantity

\[ T^{0}_0[\Psi] = \frac{1}{2} \hat{\kappa} \dot{\Psi}^2 + \hat{\kappa}(\nabla \Psi)^2 + \rho \Psi^2 \] (65)

For \( T^{0}_0[p\Psi] \) we obtain

\[ T^{0}_0[p\Psi] = \frac{1}{\omega^2_p} [(\partial_\mu \bar{m} - \partial_\mu \Psi \bar{m})^2 - (\partial_\mu \bar{p} - \partial_\mu \Psi \bar{p})^2 + \omega^2_p (\bar{m}^2 - \bar{p}^2)] \] (66)
This expression contains positive magnetization contributions and negative polarization contributions. The positive magnetization contributions can overcome the negative magnetization contribution of \( T_0^p \) given in equation (94) of Appendix (A.3), provided that \( \frac{\hat{\omega}^2}{\omega^2_p} > 2\pi \). However, the negative polarization contribution to \( T_0^p \) results in the uncertainty whether the total polarization contribution is negative or positive, since this depends on the solution of equations (88,57,58). Those equations are difficult to solve since they are complicated non-linear partial differential equations. In the following subsection we will present a solution for a particular, simple situation. Notice that this situation is completely different from the situation in the non-covariant model for which the positiveness of the energy density was assured. To conclude this subsection we calculate the contributions to \( T_0^i \):

\[
T_0^i = T_0^i[p] + T_0^i[p\psi] + T_0^i[\psi]
\]

(67)

since \( T_0^i[p] \) given in equation (95) is the Poynting vector, we can consider \( T_0^i[p\psi] \) and \( T_0^i[\psi] \) as corrections to the Poynting vector. Those are given by:

\[
T_0^i[p\psi] = \frac{2}{c\hat{\omega}^2_p} \left[ \nabla p^k \dot{p}^k - \nabla m^k \dot{m}^k - p^k (\nabla \psi p^k + \dot{\psi} \nabla p^k) \right. \\
+ \left. m^k (\nabla \dot{\psi} m^k + \dot{\psi} \nabla m^k) - \nabla \psi \dot{\psi} (\dot{m}^2 - \dot{p}^2) \right]
\]

(68)

and

\[
T_0^i[\psi] = -\frac{\kappa}{c} \nabla \psi \dot{\psi}
\]

(69)

One should notice that both these corrections vanish for the homogeneous case in which the spatial derivatives vanish.

### 7.2 A covariant solvable case

In this final subsection we would like to elucidate the nature of the energy density component \( T_0^0 \) by introducing a simple solution of equation (57). We assume a homogeneous situation in which all spatial derivative vanish. Furthermore, we assume the \( \rho = 0 \) and introduce the constant \( \kappa = \frac{\dot{\psi}}{\psi} \). In this case we obtain the equation

\[
\kappa \dot{\psi} = \frac{1}{\omega^2_p} \partial_t [\dot{m} \cdot \ddot{m} - \dot{\psi} m^2 - \ddot{p} \cdot \ddot{p} + \dddot{p}^2]
\]

(70)

From this equation we can obtain an expression for \( \dot{\psi} \), as follows:

\[
\dot{\psi} = \left( \frac{1}{2} \right) \frac{\partial_t (\dot{m}^2 - \dot{p}^2)}{\kappa \omega^2_p + \dot{m}^2 - \dot{p}^2}
\]

(71)
To this we need to add the homogeneous energy density

\[ T_0^0[p\Psi] = \frac{1}{2\omega_p^2}[(\dot{\vec{m}} - \vec{m}\dot{\vec{n}})^2 - (\dot{\vec{p}} - \vec{n}\dot{\vec{p}})^2 + \omega_p^2(m^2 - p^2)] \quad (73) \]

We will consider to separate cases:

1. \( \vec{p} = 0, \vec{m} = m\hat{x} \). In this case we obtain

\[ T_0^0[\Psi] + T_0^0[p\Psi] = \frac{1}{2\omega_p^2}[\omega_p^2 m^2 + \frac{\dot{m}^2}{1 + \frac{m^2}{\kappa\omega_p^2}}] \quad (74) \]

which is very similar to equation (45). This expression is positive and with the correct choice of parameters can overcome the negative magnetization contribution of \( T_0^0[p\Psi] \)

2. \( \vec{p} = p\hat{x}, \vec{m} = 0 \). In this case we obtain

\[ T_0^0[\Psi] + T_0^0[p\Psi] = \frac{1}{2}\left[\frac{\kappa\dot{p}^2}{p^2 - \kappa\omega_p^2} - \frac{\omega_p^2}{\omega_p^2}p^2\right] \quad (75) \]

which is very different from equation (44). This expression is positive only if \( p^2 > \kappa\omega_p^2 \) and if \( p \) changes fast enough to overcome the negative part \(-\frac{\omega_p^2}{\omega_p^2}p^2\); otherwise, the energy density is negative.

8 Conclusion

In this paper the Lagrangian method for a dissipative medium (capable also of sustaining electric and/or magnetic polarizations) was carried out to obtain equations of motion, energy-momentum densities, etc. To apply the method it has been found necessary to introduce additional degrees of freedom ("sink-modes"), associated with decay mechanisms in the electrical and magnetic modes.

Two versions have been formulated: The first one, which is not invariant under Lorentz transformation, has led to equations for both the electromagnetic fields (a slightly corrected set of Maxwell equations), polarization and
magnetization and, using a model for the sink variables, has unambiguously yielded energy densities, shown in equation (43) - equation (45). These are positive and resemble the corresponding results in [13]. The results obtained are similar to the ones obtained in [1], however, they have been obtained here with a much simpler Lagrangian, containing fewer degrees of freedom. To be precise, two vector degrees of freedom (or six independent quantities) have been omitted from the beginning, thus making the formalism more useful for numerical calculations.

Secondly, a covariant Lagrangian was formulated, one that is invariant under the Lorentz transformation and contains spatial as well as temporal derivatives. From this, we have derived equations for both the electromagnetic fields, polarization and magnetization. Calculating the energy momentum tensor we have derived expression for both the energy density and the generalized Poynting vector. However, we find that with a specific model for the sink modes, the energy density derived from this covariant Lagrangian is not positive definite. This points to the need to introduce a better model for the sink degrees of freedom, perhaps with better physical underpinning.

APPENDIX

A The covariant formalism

In a covariant formalism of the electromagnetic theory one uses a four dimensional space-time formalism instead of the usual distinction that is practised between spatial and temporal coordinates. In this formalism the invariance of the Lagrangian of the electromagnetic theory under Lorentz transformation is clear.

A.1 Four dimensional notations

The four dimensional coordinate are defined as:

\[ x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z \]  \hspace{1cm} (76)

In this formalism there is a difference between coordinates with upper and lower indexes. For example:

\[ x_0 = ct, \quad x_1 = -x, \quad x_2 = -y, \quad x_3 = -z \]  \hspace{1cm} (77)
The connection between upper and lower coordinates is established through the metric matrix $g_{\alpha\beta}$, which is defined as

$$g_{\alpha\beta} = g^{\alpha\beta} = \text{diag}(1, -1, -1, -1)$$  \hspace{1cm} (78)

(we shall use Greek letters for 4 coordinates according to the well established custom) such that

$$g_{\alpha\gamma}g^{\gamma\beta} = \delta_{\alpha}^{\beta} = \text{diag}(1, 1, 1, 1)$$  \hspace{1cm} (79)

Using $g_{\alpha\beta}$ we can write $x_{\alpha} = g_{\alpha\beta}x^{\beta}$. This notation will be adapted for any four vector

$$A^{\alpha} = (A^{0}, \vec{A}), \quad A_{\alpha} = (A^{0}, -\vec{A}), \quad A_{\alpha} = g_{\alpha\beta}A^{\beta}$$  \hspace{1cm} (80)

For the four dimensional derivatives we will use the notation

$$\partial^{\alpha} = \frac{\partial}{\partial x^{\alpha}} = (\frac{\partial}{\partial x^{0}}, -\nabla), \quad \partial_{\alpha} = \frac{\partial}{\partial x_{\alpha}} = (\frac{\partial}{\partial x^{0}}, \nabla)$$  \hspace{1cm} (81)

A.2 Electromagnetic theory in four dimensions

The vector and scalar electromagnetic potentials are replaced by a single four-vector

$$A^{\alpha} = (\Phi, \vec{A})$$  \hspace{1cm} (82)

Using equations (6) and (82) we can define the contravariant and covariant antisymmetric tensors:

$$F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha} = \begin{pmatrix} 0 & -E^{x} & -E^{y} & -E^{z} \\ E^{x} & 0 & -B^{z} & B^{y} \\ E^{y} & B^{z} & 0 & -B^{x} \\ E^{z} & -B^{y} & B^{x} & 0 \end{pmatrix}$$  \hspace{1cm} (83)

$$F_{\alpha\beta} = g_{\alpha\gamma}F^{\gamma\eta}g_{\eta\beta} = \begin{pmatrix} 0 & E^{x} & E^{y} & E^{z} \\ -E^{x} & 0 & -B^{z} & B^{y} \\ -E^{y} & B^{z} & 0 & -B^{x} \\ -E^{z} & -B^{y} & B^{x} & 0 \end{pmatrix}$$  \hspace{1cm} (84)

Analog tensors can also be defined for the magnetic field $\vec{H}$ and electric displacement field $\vec{D}$ such that

$$G^{\alpha\beta} = \begin{pmatrix} 0 & -D^{x} & -D^{y} & -D^{z} \\ D^{x} & 0 & -H^{z} & H^{y} \\ D^{y} & H^{z} & 0 & -H^{x} \\ D^{z} & -H^{y} & H^{x} & 0 \end{pmatrix}, \quad G_{\alpha\beta} = \begin{pmatrix} 0 & D^{x} & D^{y} & D^{z} \\ -D^{x} & 0 & -H^{z} & H^{y} \\ -D^{y} & H^{z} & 0 & -H^{x} \\ -D^{z} & -H^{y} & H^{x} & 0 \end{pmatrix}$$  \hspace{1cm} (85)
Finally we introduce the polarization tensors which contain both the magnetization $\vec{M}$ and polarization $\vec{P}$

$$P^{\alpha\beta} = \begin{pmatrix}
0 & P^x & P^y & P^z \\
-P^x & 0 & -M^z & M^y \\
-P^y & M^z & 0 & -M^x \\
-P^z & -M^y & M^x & 0
\end{pmatrix}, \quad P_{\alpha\beta} = \begin{pmatrix}
0 & -P^x & -P^y & -P^z \\
P^x & 0 & -M^z & M^y \\
P^y & M^z & 0 & -M^x \\
P^z & -M^y & M^x & 0
\end{pmatrix}$$

(86)

In terms of equations (83), (85) and (86) we can rewrite equation (2) and equation (3) as a single tensor equation

$$G^{\alpha\beta} = F^{\alpha\beta} - 4\pi P^{\alpha\beta}$$

(87)

Furthermore, the equations appearing in (5) can be rewritten as a single tensor equation

$$\partial_\alpha G^{\alpha\beta} = \frac{4}{c} J^\beta_f$$

(88)

In this the four current is defined as $J^\alpha = (c\rho, \vec{J})$ and the subscript $f$ refers to free currents and charges. In what follows we assume that the material under study does not contain any free currents or charges. The covariant Lagrangian density for the electromagnetic field can be written as

$$L_{FP} = L_F + L_{FJ}$$

$$L_F = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta}$$

$$L_{FJ} = -\frac{1}{c} J_\alpha A^\alpha$$

(89)

In matter $J_\alpha$ can be written as

$$J_\alpha = c\partial^\beta P_{\beta\alpha}$$

(90)

This expression automatically satisfies the conservation law $\partial^\alpha J_{\alpha} = 0$, due to the antisymmetry of $P_{\beta\alpha}$. The field equations (88) can be derived by taking the variational derivative of equation (89) with respect to the four potential $A^\alpha$. Writing $L$ given in equation (89) in terms of the standard notation leads to the expression

$$L_{FP} = \frac{1}{8\pi} [\vec{E}^2 - \vec{B}^2] + \vec{P} \cdot \vec{E} + \vec{M} \cdot \vec{B} + \vec{\nabla} \cdot (\Phi \vec{P} + \vec{M} \times \vec{A}) + \frac{1}{c} \partial_t (\vec{A} \cdot \vec{P})$$

(91)

Notice the differences between equation (7) and the covariant result given in equation (91):
1. The covariant formalism contains the divergence term $\vec{\nabla} \cdot (\Phi \vec{P} + \vec{M} \times \vec{A})$ which is absent in the previous formalism. This term will contribute only on the boundary of the domain of integration and can be ignored if we assume that the domain of integration is over the entire space and the fields vanish at infinity.

2. The covariant formalism contains the total time derivative $\frac{1}{c} \partial_t (\vec{A} \cdot \vec{P})$ which is absent in the previous formalism. This term will not contribute to the field equations and can be ignored.

3. The coupling between the magnetization and magnetic field is different. In the covariant formalism the induction density $\vec{B}$ is coupled to the magnetization instead of the magnetic field $\vec{H}$.

A.3 The energy momentum tensor of the electromagnetic field

The energy momentum tensor in terms of arbitrary fields $\eta_\rho$ is given by

$$T^\beta_\alpha = \frac{\partial \mathcal{L}}{\partial (\partial_\beta \eta_\rho)} \partial_\alpha \eta_\rho - \delta^\beta_\alpha \mathcal{L}$$ (92)

For the electromagnetic field, when we calculate the energy momentum tensor $T^\beta_\alpha [FP]$ using $\mathcal{L}_{FP}$, this results in

$$T^\beta_\alpha [FP] = \frac{1}{4\pi} \partial_\alpha A_\mu F^{\mu\beta} + A_\mu \partial_\alpha P^{\mu\beta} - \delta^\beta_\alpha \mathcal{L}_{FP}$$ (93)

The energy density is given by $T^0_0 [FP]$ which can be written as follows:

$$T^0_0 [FP] = \frac{1}{8\pi} [\vec{E}^2 + \vec{H}^2 - (4\pi \vec{M})^2] + \vec{\nabla} \cdot (\frac{\Phi \vec{E}}{4\pi} + \vec{A} \times \vec{M})$$ (94)

The divergence term $\vec{\nabla} \cdot (\frac{\Phi \vec{E}}{4\pi} + \vec{A} \times \vec{M})$ will only contribute as a boundary term to the electromagnetic energy and can be ignored if we assume that the domain of integration is over the entire space and the fields vanish at infinity. Notice that for the covariant Lagrangian density the energy density is different from the result of equation (9): it contains a negative quadratic term in the magnetization instead of a positive one. If the magnetization is a given static quantity, this is just a constant that can be ignored. On the other hand, if the magnetization is a dynamic degree of freedom, this term
can render the energy negative. Of course, if the magnetization is dynamic, it will have its own Lagrangian which will have its own contribution to the energy momentum tensor; this will be discussed in the following subsections.

(One would expect that the relativistic energy density in equation (93) would smoothly approach the non-relativistic expression in equation (9) as \( c \to \infty \). Apparently this is not the case, because of the different signs of the quadratic magnetization term. In truth, when one goes deeper, into the atomic theory of magnetization, one recognizes that both the orbital and spin contributions to the magnetization \( M \) are proportional to the Bohr magneton \( \frac{\hbar}{2mc} \). This vanishes in a theory where \( c \to \infty \).)

Finally we derive the Poynting vector by calculating \( T^i[F,P] \). This results in

\[
T^i[F,P] = \left[ \frac{1}{4\pi} \vec{E} \times \vec{H} + \frac{1}{c} \partial_t (\vec{M} \times \vec{A} - \frac{\Phi \vec{E}}{4\pi}) + \nabla \times \left( \frac{\Phi \vec{H}}{4\pi} \right) \right]^i
\]

which is the "correct" form if we ignore boundary terms and total time derivatives.

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