THE PERRON-FROBENIUS THEOREM FOR
MULTI-HOMOGENEOUS MAPPINGS

ANTOINE GAUTIER∗, FRANCESCO TUDISCO†, AND MATTHIAS HEIN†

Abstract. The Perron-Frobenius theory for nonnegative matrices has been generalized to order-preserving homogeneous mappings on a cone and more recently to nonnegative multilinear forms. We unify both approaches by introducing the concept of order-preserving multi-homogeneous mappings, their associated nonlinear spectral problems and spectral radii. We show several Perron-Frobenius type results for these mappings addressing existence, uniqueness and maximality of nonnegative and positive eigenpairs. We prove a Collatz-Wielandt principle and other characterizations of the spectral radius and analyze the convergence of iterates of these mappings towards their unique positive eigenvectors. On top of providing a remarkable extension of the nonlinear Perron-Frobenius theory to the multi-dimensional case, our contribution poses the basis for several improvements and a deeper understanding of the current spectral theory for nonnegative tensors. In fact, in recent years, important results have been obtained by recasting certain spectral equations for multilinear forms in terms of homogeneous maps, however as our approach is more adapted to such problems, these results can be further refined and improved by employing our new multi-homogeneous setting.

Key words. Perron-Frobenius theorem, nonlinear power method, nonlinear eigenvalue, nonlinear singular value, Collatz-Wielandt principle, Hilbert projective metric

AMS subject classifications. 47H07, 47J10, 15B48, 47H09, 47H10

1. Introduction. The classical Perron-Frobenius theory addresses properties such as existence, uniqueness and maximality of eigenvectors and eigenvalues of matrices with nonnegative entries. Two important generalizations of this theory arise in the study of eigenvectors of order-preserving homogeneous mappings defined on cones and in multilinear algebra where spectral problems involving nonnegative tensors are considered. In this work we consider a framework allowing the unified study of both directions by introducing the concept of order-preserving multi-homogeneous mappings. While some multi-homogeneous spectral problems can be reformulated in terms of standard homogeneous maps (see e.g. [10]), the novel multi-homogeneous formulation allows us to go further and prove several results that either hold for a larger class of problems or that require weaker assumptions. In particular, we provide a notion of eigenvalue and spectral radius for multi-homogeneous mappings and prove several Perron-Frobenius type results. These results include the existence of a nonnegative eigenvector corresponding to the spectral radius, the existence and uniqueness of a positive maximal eigenvector, and a Collatz-Wielandt characterization of the spectral radius. Furthermore, we investigate the simplicity of the spectral radius and the convergence of the iterates of the mapping towards its unique positive eigenvector. The latter result is particularly relevant from a computational viewpoint as it naturally gives rise to an efficient and general algorithm for the computation of the positive eigenvector, with a linear convergence rate.

As on the one side linear algebra can be seen as a special case of multilinear algebra, on the other side eigenvectors and eigenvalues of nonnegative matrices are a special case of those of order-preserving homogeneous mappings on $\mathbb{R}_+^n = \{ \mathbf{u} \in \mathbb{R}^n : u_i \geq 0, \forall i \}$. Following a similar analogy, the nonlinear Perron-Frobenius theory for homogeneous mappings is a special case of that for multi-homogeneous mappings.

∗Department of Mathematics and Computer Science, Saarland University, 66041 Saarbrücken, Germany (ag@cs.uni-saarland.de, hein@math.uni-sb.de).
†Department of Mathematics and Statistics, University of Strathclyde, G11XH Glasgow, UK (f.tudisco@strath.ac.uk).

and the study of spectral problems induced by nonnegative multi-linear forms is a special case of the study of eigenvectors of order-preserving multi-homogeneous mappings on the product $\mathbb{R}_{+}^{n_1} \times \ldots \times \mathbb{R}_{+}^{n_d}$. Therefore, when $d = 1$, our results reduce to their counterparts in the existing linear and nonlinear finite dimensional Perron-Frobenius theories. However, when $d > 1$, the use of the proposed multi-homogeneous setting allows us to improve and unify many results and definitions in the study of spectral problems induced by nonnegative multi-linear forms, as for instance the $\ell^p$-eigenvector problem for (square) nonnegative tensors, the $\ell^{p,q}$-singular vector problem for nonnegative (rectangular) tensors and the $\ell^{p,q,r}$-singular vector problem for nonnegative tensors [13, 27, 28]. In [15] we discuss several of these implications in detail.

In recent years, the nonlinear Perron-Frobenius theory and spectral theory of nonnegative multi-linear forms have been successfully employed in a variety of applications ranging from signal processing [24] to multi-variate low rank approximation [8], mathematical economics [11] and dynamical systems [7]. The use of the multi-homogeneous framework opens the avenue to several challenging applications too. For instance, the techniques proposed in this paper have recently inspired the first practicable algorithm for the training of a class of generalized polynomial neural networks to global optimality [14], and have been employed in network science in order to extend eigenvector-based centrality measures to multi-dimensional graphs [25].

The nonlinear Perron-Frobenius theory has been developed for order-preserving mappings on general cones. However, for the sake of simplicity and in order to make our ideas more transparent, we restrict ourself to cones of the form $\mathbb{R}_{+}^n$ and their Cartesian product. Nevertheless, we took special care to use as few as possible the particular structure of $\mathbb{R}_{+}^n$ in order to facilitate the generalization of our results to general cones.

The paper is organized as follows: In Section 2, we introduce and motivate the class of order-preserving multi-homogeneous mappings. We propose a way to define eigenvectors and eigenvalues for multi-homogeneous mappings. Furthermore, we discuss characteristics of these mappings. In Section 3, we prove a contraction principle for our class of mappings in Theorem 3.1. In particular, this theorem implies the existence and uniqueness of a positive eigenvector under very mild conditions. In Section 4, we propose a generalized notion of spectral radius and prove, in Theorem 4.1, a weak form of the Perron-Frobenius theorem which implies the existence of a nonnegative eigenvector corresponding to the spectral radius. Then, we discuss a generalized notion of irreducibility allowing us to give in Theorem 4.3 a sufficient condition for the existence of a positive eigenvector of non-expansive mappings. In Section 5, we prove a Collatz-Wielandt formula for the spectral radius (Theorem 5.1) and discuss the simplicity and uniqueness of positive eigenvectors and their associated eigenvalues (Theorem 5.2). Finally, in Section 6, we discuss a method for computing the positive eigenvector of order-preserving multi-homogeneous mappings. The convergence of this method (with a linear rate) is discussed in Theorem 6.1. In Sections 3–6 we give for better readability first the main results and discuss them before we proceed with the proofs. For the sake of brevity, we shall prove only the results whose generalization from the homogeneous case is not straightforward.

2. Motivation, overview and notation. We start by motivating the class of mappings considered in this paper. Then, we introduce notation in order to facilitate its study. This notation will be used throughout the paper. We then recall and prove characterizations of order-preserving and multi-homogeneous mappings in Theorem 2.4 and Lemma 2.5 respectively. Furthermore, we discuss in Lemma 2.7 a number of
relevant operations under which the considered class of mappings is closed.

2.1. Multi-homogeneous mappings. Let us first define the class of mappings considered here. To this end, let \( n_1, \ldots, n_d \) be positive integers and consider \( \mathcal{K}_+ \) the product cones defined as \( \mathcal{K}_+ = \mathbb{R}_{++}^{n_1} \times \cdots \times \mathbb{R}_{++}^{n_d} \). Let \( [d] = \{1, \ldots, d\} \) and for \( i \in [d] \), let \( F_i: \mathcal{K}_+ \to \mathbb{R}_{++}^{n_i} \) be a continuous mapping and define \( F: \mathcal{K}_+ \to \mathcal{K}_+ \) as \( F(x) = (F_1(x), \ldots, F_d(x)) \). We say that \( F \) is (positively) multi-homogeneous if there exists a \( d \times d \) nonnegative matrix \( A \) such that for every \( x_j \in \mathbb{R}_{++}^{n_j} \) and \( \alpha_j \geq 0, j \in [d] \), it holds

\[
F_i(\alpha_1 x_1, \ldots, \alpha_d x_d) = \prod_{j=1}^{d} \alpha_j^{A_{i,j}} F_i(x_1, \ldots, x_d) \quad \forall i \in [d].
\]

We refer to \( A \) as the homogeneity matrix of \( F \). Clearly, every \( p \)-homogeneous mapping \( F: \mathbb{R}^n \to \mathbb{R}^n \) is multi-homogeneous with homogeneity matrix \( A = p \in \mathbb{R}^{1 \times 1} \). A multi-homogeneous mapping \( F \) is said to be order-preserving if it preserves the order induced by \( \mathcal{K}_+ \), that is

\[
F(x) - F(y) \in \mathcal{K}_+, \quad \forall x, y \in \mathcal{K}_+ \quad \text{with} \quad x - y \in \mathcal{K}_+.
\]

Finally, we consider non-degenerated mappings, that is we assume that

\[
F(\mathcal{K}_{++}) \subset \mathcal{K}_{++} \quad \text{and} \quad A \mathbb{R}_{++}^d \subset \mathbb{R}_{++}^d
\]

where \( \mathbb{R}_{++}^d \) and \( \mathcal{K}_{++} = \mathbb{R}_{++}^{n_1} \times \cdots \times \mathbb{R}_{++}^{n_d} \) are the interiors of \( \mathbb{R}_{++}^d \) and \( \mathcal{K}_+ \) respectively. In order to shorten the statements of our results, we introduce the following:

**Definition 2.1.** Let \( \mathcal{H}^d \) be the class of mappings defined as

\[
\mathcal{H}^d = \{ F: \mathcal{K}_+ \to \mathcal{K}_+ \mid F \text{ is continuous and satisfies } (1), (2), (3) \}.
\]

For \( F \in \mathcal{H}^d \), we write \( \mathcal{A}(F) \) to denote the homogeneity matrix \( A \) defined in (1).

As discussed in the preface of [19], the development of the nonlinear Perron-Frobenius theory has its origin in an observation made independently by Birkhoff and Samelson. They remarked that one can use Hilbert's projective metric and results of fixed point theory to prove some of the theorems of Perron and Frobenius concerning eigenvectors and eigenvalues of nonnegative matrices. More precisely, they made the following observation: For any matrix \( M \in \mathbb{R}^{n \times n} \) with positive entries, it holds

\[
\mu(Mx, My) \leq \mu(x, y) \quad \forall x, y \in \mathbb{R}_{++}^n,
\]

where \( \mu: \mathbb{R}_{++}^n \times \mathbb{R}_{++}^n \to \mathbb{R}_+ \) is the Hilbert metric defined as

\[
\mu(x, y) = \ln \left( \max_{i,j \in [n]} \frac{x_i}{y_i} \frac{y_j}{x_j} \right).
\]

In particular, it is known that for any norm \( \| \cdot \| \) on \( \mathbb{R}^n \), the pair \( \{ x \in \mathbb{R}_{++}^n : \| x \| = 1 \} , \mu \) forms a complete metric space (see for instance Proposition 4.4 in [23], p.82) and so one can use results of fixed point theory to analyze the eigenvectors of \( M \). This observation was then extended to a wider class of mappings, namely the class of mappings \( F: \mathbb{R}_{++}^n \to \mathbb{R}_{++}^n \) which are positively \( p \)-homogeneous, order-preserving and leave \( \mathbb{R}_{++}^n \) invariant. It can then be shown that \( p \) is a Lipschitz constant of \( F \) with respect to \( \mu \) (see for instance Theorem 3.1 in [5]). A generalization of this observation
can be proved for mappings in \( \mathcal{H}^d \). More precisely, if \( F \in \mathcal{H}^d \), \( A = \mathcal{A}(F) \) and \( \mathbf{b} \in \mathbb{R}^d_{++} \), then

\[
\mu_{\mathbf{b}}(F(\mathbf{x}), F(\mathbf{y})) \leq \min_{\mathbf{i} \in [d]} \frac{(A^\top \mathbf{b})_i}{b_i} \mu_{\mathbf{b}}(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K}_{++},
\]

where \( \mu_{\mathbf{b}} : \mathcal{K}_{++} \times \mathcal{K}_{++} \to \mathbb{R}_+ \) is the weighted product metric defined as

\[
\mu_{\mathbf{b}}((\mathbf{x}_1, \ldots, \mathbf{x}_d), (\mathbf{y}_1, \ldots, \mathbf{y}_d)) = \sum_{i=1}^{d} b_i \mu(\mathbf{x}_i, \mathbf{y}_i).
\]

Clearly, (4) is a special case of (5). The upper bound on the Lipschitz constant of \( F \) in (5) depends on the choice of the weights \( \mathbf{b} \in \mathbb{R}^d_{++} \). This upper bound can be minimized using the Collatz–Wielandt formula. For example, if \( A \) in (5) is irreducible, then the spectral radius \( \rho(A) \) of \( A \) satisfies

\[
\rho(A) = \min_{\mathbf{b} \in \mathbb{R}^d_{++}} \max_{i \in [d]} \left( \frac{(A^\top \mathbf{b})_i}{b_i} \right)
\]

and the minimum is attained at \( \mathbf{b} \in \mathbb{R}^d_{++} \) such that \( A^\top \mathbf{b} = \rho(A) \mathbf{b} \). This is a key property of the Perron-Frobenius theory for order-preserving multi-homogeneous mappings. Furthermore, we note that (5) is tight in \( \mathcal{H}^d \) in the following sense: For every nonnegative matrix \( A \in \mathbb{R}^{d \times d}_{++} \) with \( \mathbb{AR}^d_{++} \subset \mathbb{R}^d_{++} \), there exists \( F \in \mathcal{H}^d \) such that \( \mathcal{A}(F) = A \) and for every \( \mathbf{b} \in \mathbb{R}^d_{++} \) (5) holds with equality for some \( \mathbf{x}, \mathbf{y} \in \mathcal{K}_{++} \) with \( \mathbf{x} \neq \mathbf{y} \) (see Example 3.3).

One may wonder why the Perron–Frobenius theorem is not applied to identify \( \mathcal{K}_+ \) with \( \mathbb{R}^{n_1 + \cdots + n_d}_{++} \) and then consider the Hilbert metric on \( \mathbb{R}^{n_1 + \cdots + n_d}_{++} \) for the study of mappings in \( \mathcal{H}^d \). Such an approach is used in [10] and [13] where a spectral problem for nonnegative tensor is transformed into a spectral problem for order-preserving 1-homogeneous mapping. However, it turns out that, for equivalent results, the assumptions induced by these kind of transformations are much more restrictive than if one treats the problem in its original formulation, namely as a multi-homogeneous problem. Details on such improvement on the literature of nonnegative tensors can be found in [15].

As, the Perron-Frobenius theorem is concerned with eigenvectors and eigenvalues of mappings, we propose a generalization of these objects in the context of multi-homogeneous mappings:

Definition 2.2. Let \( F = (F_1, \ldots, F_d) \in \mathcal{H}^d \). We say that \( \mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_d) \in \mathcal{K}_+ \) is an eigenvector of \( F \) if \( \mathbf{x}_i \neq 0 \) for every \( i \) and there exists \( \lambda \in \mathbb{R}_+^d \) such that \( F(\mathbf{x}) = (\lambda_1 \mathbf{x}_1, \ldots, \lambda_d \mathbf{x}_d) \).

For consistency, in the above definition, we call \( \lambda \) an eigenvalue of \( F \) although it is a vector. We have several motivations for such a definition:

- If \( d = 1 \), then Definition 2.2 coincides with the usual definition of (nonnegative) eigenvector.
- If \( F \in \mathcal{H}^d \) and \( \mathbf{x} \in \mathcal{K}_+ \) satisfies \( \mathbf{x}_i \neq 0 \) for every \( i \) and \( F(\mathbf{x}) = \lambda \mathbf{x} \) for some \( \lambda \in \mathbb{R}_+ \), then \( \mathbf{x} \) is an eigenvector of \( F \) with eigenvalue \( \lambda = (\lambda, \ldots, \lambda) \).
- Likewise the eigenvectors of a \( p \)-homogeneous mapping \( F \in \mathcal{H}^1 \) are its fixed point in the projective space, the eigenvectors of a multi-homogeneous mapping \( F \in \mathcal{H}^d \) are its fixed points in the product of projective spaces.
- Unlike the case \( d = 1 \), if \( F \in \mathcal{H}^d \) with \( d > 1 \) and \( \mathbf{x} \in \mathcal{K}_+ \), \( \lambda \in \mathbb{R}_+^d \) satisfy \( F(\mathbf{x}) = (\lambda_1 \mathbf{x}_1, \ldots, \lambda_d \mathbf{x}_d) \), it not necessarily true that \( F(\alpha \mathbf{x}) = \beta(\lambda_1 \mathbf{x}_1, \ldots, \lambda_d \mathbf{x}_d) \).
for some $\beta \geq 0$. However, it is the case that for every $\alpha \in \mathbb{R}^{d+}_+$ there exists $\beta \in \mathbb{R}^d_+$ such that $F(\alpha_1 x_1, \ldots, \alpha_d x_d) = (\beta_1 \lambda_1 x_1, \ldots, \beta_d \lambda_d x_d)$.

We conclude the section with simple examples which illustrate the notions discussed above: Let $M \in \mathbb{R}^{n \times n}$ be a positive matrix.

i) Define $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ as $F(x) = Mx$. Then, we have $F \in \mathcal{H}^1$ with $A(F) = 1$ and the eigenvectors of $F$ are the nonnegative eigenvectors of $M$.

ii) Define $G : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+ \times \mathbb{R}^n_+$ as $G(x, y) = (M^T y, Mx)$. Then, we have $G \in \mathcal{H}^2$ with

$$A(G) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the eigenvectors of $G$ are the nonnegative singular vectors of $M$.

iii) Define $H : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+ \times \mathbb{R}^n_+$ as $H(x, y) = ((M^T y)^{(p-1)/p}, (Mx)^{(q-1)/q})$ where the powers are taken component-wise and $p, q \in (1, \infty)$. Then, we have $H \in \mathcal{H}^2$ with $A(H) = \text{diag}(1/(p-1), 1/(q-1)) A(G)$ and the eigenvectors of $H$ are the so-called nonnegative $p^q$-singular vectors of $M$ [3]. In particular, it can be shown that the eigenvectors of $H$ are exactly the nonnegative critical points of the function

$$(x, y) \mapsto \frac{(x, My)}{\|x\|_p \|y\|_q},$$

where $\|\cdot\|_p$ is the $p$-norm on $\mathbb{R}^n$. Furthermore, if $p \neq q$, then the eigenvalues of $H$ are of the form $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \neq \lambda_2$.

Based on the above observations, we introduce notation, on top of the standard one, which allows to better capture the structure of mappings in $\mathcal{H}^d$.

2.2. Notation. For $n \in \mathbb{N}$, define $[n] = \{1, \ldots, n\}$, $\mathbb{R}^n_+ = \{z \in \mathbb{R}^n \mid z_i \geq 0, \forall i \in [n]\}$, $\mathbb{R}^n_+ - 0 = \mathbb{R}^n_+ \setminus \{0\}$, $\mathbb{R}^n_+^\ast = \{z \in \mathbb{R}^n \mid z_i > 0, \forall i \in [n]\}$ and $\Delta^n_+ = \{z \in \mathbb{R}^n_+ \mid \sum_{i=1}^n z_i = 1\}$. For $p \in [1, \infty]$, we write $\|z\|_p$ to denote the usual $p$-norm of $z \in \mathbb{C}^n$. We write $|z|$ to denote the component-wise absolute value of $z$, i.e. $|z| = (|z_1|, \ldots, |z_n|)$.

On $\mathbb{R}^n_+$ we consider the partial ordering induced by $\leq$, i.e. for every $y, z \in \mathbb{R}^n$ we write $z \leq y$, $z \not< y$ if $y - z \in \mathbb{R}^n_+ \setminus \{0\}$ and $y - z \in \mathbb{R}^n_+^\ast$, respectively. We write $I \in \mathbb{R}^{n \times n}$ and $1 \in \mathbb{R}^n$ to denote the identity matrix and the vector of all ones respectively. We write $p(A)$ for the spectral radius of a matrix $A \in \mathbb{R}^{n \times n}$. Let us recall that a matrix $A \in \mathbb{R}^{n \times n}_+$ is irreducible if $(I + A)^{n-1} \in \mathbb{R}^{n \times n}_+$ and primitive if there exists $\nu \in \mathbb{N}$ such that $A^\nu \in \mathbb{R}^{n \times n}_+$ where $\mathbb{R}^{n \times n}_+$ and $\mathbb{R}^{n \times n}_+^\ast$ denote the sets of matrices with nonnegative, respectively positive, entries.

Now, for $d \in \mathbb{N}$ and $n_1, \ldots, n_d \in \mathbb{N}$, define $V = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d}$, $K_+ = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$, $K_{++} = \mathbb{R}^{n_1+}_+ \times \ldots \times \mathbb{R}^{n_d+}_+$ and $K_++ = \mathbb{R}^{n_1+}_+ \times \ldots \times \mathbb{R}^{n_d+}_+$. We use bold letters without index to denote elements of $V$, bold letters with index $i \in [d]$ denote vectors in $\mathbb{R}^{n_i}$, whereas components of $x_i$ are written in normal font. Namely

$$x = (x_1, \ldots, x_d) \in V, \quad x_i = (x_{i,1}, \ldots, x_{i,n_i}) \in \mathbb{R}^{n_i} \quad \text{and} \quad x_{i,j} \in \mathbb{R}.$$ 

To express entries of $x \in V$, it is convenient to consider the following sets of indices

$$\mathcal{I} = \bigcup_{i=1}^d [i] \times [n_i], \quad \mathcal{J} = [n_1] \times [n_2] \times \ldots \times [n_d].$$

The cone $K_+$ induces a partial ordering on $V$. We write $x \leq_K u$, $x \leq_K u$, $x <_K u$ if $u - x \in K_+$, $u - x \in K_+ \setminus \{0\}$ and $u - x \in K_{++}$ respectively. In particular, we note that (2) can be rewritten as $F(x) \leq_K F(y)$ for every $x, y \in K_+$ with $x <_K y$. 

For $F: V \to V$ we use the same notation as for vectors, i.e. $F = (F_1, \ldots, F_d)$ and $F_i = (F_i, \ldots, F_{i,n})$ with $F_i: V \to \mathbb{R}^n$ and $F_{i,j}: V \to \mathbb{R}$. For $k \in \mathbb{N}$, we denote the iterates of $F$ as $F^k$, where $F^0(x) = x$ and $F^k(x) = F(F^{k-1}(x))$. If $F$ is differentiable at $v \in V$, we write $D_kF(v) \in \mathbb{R}^{n \times n_k}$ to denote the Jacobian matrix of the mapping $x_k \mapsto F_i(x)$ at $x = v$. Similarly, if $f: V \to \mathbb{R}$, $\nabla_i f(x)$ denotes the gradient of $x_i \mapsto f(x)$.

For technical reasons we will need to consider different types of product of unit spheres. For $i \in [d]$, let $\| \cdot \|_{\gamma_i}$ be a norm on $\mathbb{R}^n$ and let $\phi \in \mathcal{C}_++$. We consider

$$S_+ = \{ x \in \mathcal{C}_+ \mid \| x_i \|_{\gamma_i} = 1, \forall i \in [d] \}, \quad S_{++} = S_+ \cap \mathcal{C}_{++},$$

$$S^\phi_+ = \{ x \in \mathcal{C}_+ \mid \langle x_i, \phi_i \rangle = 1, \forall i \in [d] \}, \quad S^\phi_{++} = S^\phi_+ \cap \mathcal{C}_{++}.$$  

Note also that $S^\phi_+ = S_{++}$ if for every $x \in \mathcal{C}_+$, it holds $\| x_i \|_{\gamma_i} = \langle \phi_i, x_i \rangle$ for all $i \in [d]$. In some proofs, it useful to assume that the norms $\| \cdot \|_{\gamma_1}, \ldots, \| \cdot \|_{\gamma_d}$ are monotonic, thus we make a general assumption in this paper.

**Assumption 2.3.** We assume that the norms $\| \cdot \|_{\gamma_1}, \ldots, \| \cdot \|_{\gamma_d}$ are monotonic, i.e. for all $i \in [d]$ and every $x_i, y_i \in \mathbb{R}^n$, $\| x_i \|_{\gamma_i} \leq \| y_i \|_{\gamma_i}$ whenever $|x_i| \leq |y_i|$. In particular, for simplicity, the reader may assume that $\| \cdot \|_{\gamma_i}$ is the Euclidean norm on $\mathbb{R}^n$ for every $i \in [d]$.

For $\alpha \in \mathbb{R}_{++}^n$ and $B \in \mathbb{R}^{n \times n}$ (or $\alpha \in \mathbb{R}^n_+$ and $B \in \mathbb{R}^{n \times n}_+$), define $\alpha^B \in \mathbb{R}^n_+$ as

$$\alpha^B = \left( \prod_{k=1}^n \alpha_{B_{1,k}}^{1,k}, \ldots, \prod_{k=1}^n \alpha_{B_{n,k}}^{B_{n,k}} \right).$$

A direct computation shows that for every $\alpha, \beta \in \mathbb{R}_{++}^n$ and every $B, C \in \mathbb{R}^{n \times n}$, the following identities hold

$$\alpha^B \circ \alpha^C = \alpha^{B+C}, \quad (\alpha^C)^B = \alpha^{BC} \quad \text{and} \quad (\alpha \circ \beta)^B = \alpha^B \circ \beta^B,$$

where $\circ$ denotes the entrywise product, i.e. $\alpha \circ \beta = (\alpha_1 \beta_1, \ldots, \alpha_n \beta_n)$. Moreover, if $\alpha \in \mathbb{R}_{++}^n$ and $\lambda > 0$, then

$$\prod_{i=1}^n (\alpha_i)^{a_i} = \prod_{i=1}^n \alpha_i^{(B^T a)_i} \quad \text{and} \quad (\lambda^{a_1}, \ldots, \lambda^{a_n})^B = (\lambda(B a)_1, \ldots, \lambda(B a)_n).$$

We use the symbol $\otimes$ to denote the following operation

$$\alpha \otimes x = (\alpha_1 x_1, \ldots, \alpha_d x_d) \quad \forall \alpha \in \mathbb{R}^d, x \in V.$$

As discussed in the previous section, $\otimes$ arises naturally when considering spectral problems for multi-homogeneous mappings $F \in \mathcal{H}^d$. For instance, (1) can be rewritten as $F(x) = \alpha^A \otimes F(x)$ for every $x \in \mathcal{C}_{++}$ and $\alpha \in \mathbb{R}_{++}^d$. Furthermore, we have that $x \in \mathcal{C}_+$ is an eigenvector of $F$ if $x \in \mathcal{C}_{++}$ and there exists $\lambda \in \mathbb{R}_+^d$ such that $F(x) = \lambda \otimes x$.

We consider the mappings $\mathcal{M}(\cdot / \cdot), m(\cdot / \cdot) : \mathcal{C}_{++} \times \mathcal{C}_{++} \to \mathbb{R}^d_{++}$ defined as

$$\mathcal{M}(x/y) = (\mathcal{M}_1(x/y), \ldots, \mathcal{M}_d(x/y)) = \left( \max_{j_1 \in [n_1]} \frac{x_{1,j_1}}{y_{1,j_1}}, \ldots, \max_{j_d \in [n_d]} \frac{x_{d,j_d}}{y_{d,j_d}} \right),$$

$$m(x/y) = (m_1(x/y), \ldots, m_d(x/y)) = \left( \min_{j_1 \in [n_1]} \frac{x_{1,j_1}}{y_{1,j_1}}, \ldots, \min_{j_d \in [n_d]} \frac{x_{d,j_d}}{y_{d,j_d}} \right).$$
for every \( x, y \in \mathcal{K}_{++} \). These mappings are useful for two main reasons: First we have

\[
m(x/y) \otimes y \leq_{\mathcal{K}} x \leq_{\mathcal{K}} m(x/y) \quad \forall x, y \in \mathcal{K}_{++}.
\]

Second, they appear in the definition of the weighted Hilbert metric and the weighted Thompson metric on \( \mathcal{K}_{++} \). That is, for \( b \in \mathbb{R}^d_{+} \), we consider the weighted Hilbert and Thompson metric \( \mu_b, \mu_b : \mathcal{K}_{++} \times \mathcal{K}_{++} \to \mathbb{R}_{+} \), defined respectively as

\[
\mu_b(x, y) = \sum_{i=1}^{d} b_i \ln \left( \frac{m_i(x/y)}{m_i(x/y)} \right) = \ln \left( \prod_{i=1}^{d} \frac{m_i(x/y)}{m_i(x/y)} \right),
\]

\[
\overline{\mu}_b(x, y) = \sum_{i=1}^{d} b_i \ln \left( \max \{ m_i(x/y), m_i(y/x) \} \right).
\]

In particular, it follows from Corollary 2.5.6 in [19] that for any \( b \in \mathbb{R}^d_{+} \) and any \( \phi \in \mathcal{K}_{++} \), the metric spaces \( (\mathbb{S}^{\phi}_{++}, \mu_b), (\mathbb{S}^{++}_{++}, \mu_b) \) and \( (\mathcal{K}_{++}, \overline{\mu}_b) \) are complete and their topology coincide with the norm topology.

### 2.3. Preliminaries

With the notations above, we can describe the set \( \mathcal{H}^d \) of Definition 2.1 as follows: Let \( F : \mathcal{K}_{+} \to \mathcal{K}_{+} \) be continuous, then \( F \in \mathcal{H}^d \) if and only if the following conditions are all satisfied:

(a) \( F(x) \leq_{\mathcal{K}} F(y) \) for all \( 0 \leq_{\mathcal{K}} x \leq y \).

(b) There exists a nonnegative matrix \( A \in \mathbb{R}^{d \times d}_{+} \) such that \( A \mathbb{R}^{d \times d}_{+} \subset \mathbb{R}^{d \times d}_{+} \) and \( F(\alpha \otimes x) = \alpha A \otimes F(x) \) for every \( x \in \mathcal{K}_{+} \) and \( \alpha \in \mathbb{R}^d_{+} \).

(c) \( F(\mathcal{K}_{++}) \subset \mathcal{K}_{++} \).

We briefly discuss each of the three points above: First, we recall a known theorem that characterizes property (a). In particular, it implies that the differential of a mapping \( F \in \mathcal{H}^d \) is order-preserving.

**Theorem 2.4 (Theorem 1.3.1, [19]).** Let \( U \subset \mathcal{K}_{+} \) be an open convex set. If \( F : U \to \mathcal{K}_{+} \) is locally Lipschitz, then \( \text{DF}(x) \) exists for Lebesgue almost all \( x \in U \), and \( F \) is order-preserving if and only if \( \text{DF}(x) \mathcal{K}_{+} \subset \mathcal{K}_{++} \) for all \( x \in U \) for which \( \text{DF}(x) \) exists.

Next, we prove a lemma that generalizes Euler’s theorem for homogeneous mappings to multi-homogeneous mappings. More precisely, this lemma gives a characterization of multi-homogeneous mappings and provides information on the multi-homogeneity of their derivatives.

**Lemma 2.5.** Let \( U \subset V \) be open and such that \( \alpha \otimes x = (\alpha_1 x_1, \ldots, \alpha_d x_d) \in U \) for all \( \alpha \in \mathbb{R}^{d \times +} \) and \( x \in U \). Let \( a \in \mathbb{R}^d \) and \( f : U \to \mathbb{R} \), a differentiable mapping. The following are equivalent:

1. It holds \( f(\alpha \otimes x) = f(x) \prod_{k=1}^{d} \alpha_k^{a_k} \) for every \( \alpha \in \mathbb{R}^{d \times +}, x \in U \).

2. It holds \( \nabla_i f(x, x_i) = a_i f(x) \) for every \( i \in [d], x \in U \).

Moreover, if \( f \) satisfies (1) or (2), then:

3. It holds \( \nabla_i f(\alpha \otimes x) = \nabla_i f(x) \alpha_i^{-1} \prod_{k=1}^{d} \alpha_k^{a_k} \) for all \( i \in [d], \alpha \in \mathbb{R}^{d \times +}, x \in U \).

**Proof.** Let \( x \in U \) and define the differentiable functions \( g_x, h_x : \mathbb{R}^{d \times +} \to \mathbb{R} \) as

\[
g_x(\alpha) = f(\alpha \otimes x) - f(x) \prod_{k=1}^{d} \alpha_k^{a_k} \quad \text{and} \quad h_x(\alpha) = f(\alpha \otimes x) \prod_{k=1}^{d} \alpha_k^{-a_k} - f(x).
\]

(1) holds, then \( g_x \) is constant and (2) follows from \( \nabla g_x(1) = 0 \). If (2) holds, then \( \nabla h_x(\alpha) = 0 \) for every \( \alpha \) and (1) follows from \( h_x(\alpha) = h_x(1) = 0 \). To show the last part, let \( (i, j) \in \mathcal{J} \) and consider \( \alpha^{(i,j)}(x) \in \mathcal{K}_{+} \), the vector such that \( \alpha^{(i,j)}(x) = 1 \) if
(k, l_k) = (i, j_i) and (e^{i,j_i})_{k,l_k} = 0 else. Then, for every small enough h, it holds
\[
f(\alpha \otimes x + h e^{(i,j_i)}) - f(\alpha \otimes x) = \left( \alpha_i^{-1} \prod_{k=1}^d \alpha_k^{a_k} \right) \frac{f(x + \alpha_i^{-1} h e^{(i,j_i)}) - f(x)}{\alpha_i^{-1} h}.
\]
Letting \( h \to 0 \) concludes the proof. \qed

There exist order-preserving multi-homogeneous mappings which are naturally defined on \( \mathcal{K}_{++} \) rather than on \( \mathcal{K}_+ \). This frequently happens in the case \( d = 1 \) when considering the log-exp transform of a topological mapping (see e.g. \cite{[2]}) We also face such a situation when deriving a dual condition for the existence of a positive eigenvector in Corollary 4.8. It is then useful to know whether the considered mapping can be continuously extended to a mapping in \( \mathcal{H}^d \). In the case \( d = 1 \), such an extension has been proved to exist in Theorem 3.10 \cite{[4]} and Theorem 5.1.2 \cite{[19]}. As the proof of this result can be easily generalized for \( d > 1 \) (with the help of Lemma 3.2), we omit it here.

**Theorem 2.6.** Let \( F : \mathcal{K}_{++} \to \mathcal{K}_{++} \) be order-preserving and multi-homogeneous. If \( A(F) \) has at least one positive entry per row and there exists \( b \in \mathbb{R}_{++}^d \) such that \( A(F)^T b \leq b \), then there exists \( \overline{F} \in \mathcal{H}^d \) such that \( F = \overline{F}|_{\mathcal{K}_{++}} \) and \( A(\overline{F}) = A(F) \).

The following straightforward lemma describes operations under which \( \mathcal{H}^d \) is closed.

**Lemma 2.7.** Let \( F, G \in \mathcal{H}^d \), \( A = A(F) \) and \( B = A(G) \). Moreover, let \( D \in \mathbb{R}_{++}^{d \times d} \) with \( D \geq A, B \) and \( \xi_1, \ldots, \xi_d : \mathcal{K}_+ \to \mathbb{R}_+ \) be continuous, order-preserving, homogeneous mappings such that \( \xi_i(K_{++}, 0) \subset \mathbb{R}_{++} \) for every \( i \in [d] \). Define \( N : \mathcal{K}_+ \to \mathbb{R}_+^d \) as \( N(x) = (\xi_1(x), \ldots, \xi_d(x)) \). Finally, let \( H^{(1)}, H^{(2)}, H^{(3)} : \mathcal{K}_+ \to \mathcal{K}_+ \) with
\[
H^{(1)}(x) = F(G(x)), \quad H^{(2)}(x) = F(x) \circ G(x), \quad H^{(3)}(x) = N(x)^{D-A} \otimes F(x) + N(x)^{D-B} \otimes G(x).
\]
Then \( H^{(1)}, H^{(2)}, H^{(3)} \in \mathcal{H}^d \) with homogeneity matrices \( AB, A + B, D \) respectively. In particular, it follows that for every \( F \in \mathcal{H}^d \), we have \( A(F^k) = A(F)^k \). This observation is particularly useful when discussing the behavior of the iterates of multi-homogeneous mappings in Section 6.

3. Contraction principle for Multi-homogeneous mappings. Our first result is a combination of (5) with the Banach fixed point theorem. This result is particularly interesting as it shows that when we can build a metric so that \( F \in \mathcal{H}^d \) is a strict contraction then the existence and uniqueness of a positive eigenvector are always guaranteed without further assumptions. As discussed below (5), such a metric can be explicitly constructed using the left eigenvector of the homogeneity matrix of \( F \) in order to obtain the following:

**Theorem 3.1.** Let \( F \in \mathcal{H}^d \) and \( A = A(F) \). If \( \rho(A) < 1 \), then \( F \) has a unique positive eigenvector \( x \in \mathcal{K}_{++} \) up to rescaling of \( x_i \) for \( i \in [d] \).

The simplicity of the assumptions in the above theorem is remarkable. While this result was known in the case \( d = 1 \) (see for instance \cite{[4]}), it has strong novel implications in the Perron-Frobenius theory for spectral problems induced by nonnegative tensors, which we discuss in \cite{[19]}. A simple consequence of Theorem 3.1 is the following: Let \( M \in \mathbb{R}^{m \times n} \) be a nonnegative matrix, then this theorem implies that the nonlinear power method of [3] for the estimation of \( \|M\|_{p,q} = \max \{ \|Mx\|_p : \|x\|_q = 1 \} \) always
converges to the global maximum, whenever \( p < q \) and \( M^T M \) has at least one nonzero entry per row. The existing convergence result for this method requires \( M^T M \) to be irreducible which is much more restrictive.

Unfortunately, the eigenvalue problem \( Mx = \lambda x \) where \( M \in \mathbb{R}^{n \times n} \) is a positive matrix and \( x \in \mathbb{R}^n_+ \), does not fulfill the assumptions of Theorem 3.1 because in this particular case, \( F(x) = Mx \) is one homogeneous and so \( \mathcal{A}(F) = 1 \). That is, \( F \) is non-expansive but may not be a strict contraction. This explains to some extent why the linear Perron-Frobenius theorem requires \( M \) to be irreducible and not simply \( F(\mathbb{R}^n_+) \subset \mathbb{R}^n_+ \). To distinguish these cases and facilitate our discussion, for a mapping \( F \in \mathcal{H}^d \), we say that \( F \) is a (strict) contraction if \( \rho(F) < 1 \) and say that \( F \) is non-expansive if \( \rho(F) = 1 \). As for the case \( d = 1 \), when \( d > 1 \) the study of non-expansive mappings is more involved than that of strict contractions.

### 3.1. Lipschitz continuity and the contraction principle.

We prove the analogue of the observation of Birkhoff and Samelson stated in (5) and discuss why this generalization is tight by an example. Then we prove Theorem 3.1 and give an example to illustrate its relevance.

**Lemma 3.2.** Let \( F \in \mathcal{H}^d \), \( A = \mathcal{A}(F) \), \( b \in \mathbb{R}^d_+^d \). For every \( x, y \in \mathcal{K}_{++} \), it holds

\[
\mu_b(F(x), F(y)) \leq C \mu_b(x, y) \quad \text{and} \quad \overline{\mu}_b(F(x), F(y)) \leq C \overline{\mu}_b(x, y).
\]

where \( C = \max \{ (A^T b)_i / b_i \mid i \in [d] \} \).

**Proof.** For any \( x, y \in \mathcal{K}_{++} \), we have

\[
m(x/y)^A \otimes F(y) \leq F(x) \leq M(x/y)^A \otimes F(y).
\]

It follows that for every \( (j_1, \ldots, j_d) \in J \) it holds

\[
\prod_{i=1}^d m_i(x/y)^{(A^T b)_i} \leq \prod_{i=1}^d \left( \frac{F_{i,j_i}(x)}{F_{i,j_i}(y)} \right)^{b_i} \leq \prod_{i=1}^d M_i(x/y)^{(A^T b)_i}.
\]

Hence, we have

\[
\mu_b(F(x), F(y)) = \sum_{i=1}^d b_i \ln \left( \frac{m_i(F(x)/F(y))}{m_i(F(x)/F(y))} \right) \leq \sum_{i=1}^d (A^T b)_i \ln \left( \frac{M_i(x/y)}{m_i(x/y)} \right)
\]

\[
= \sum_{i=1}^d \frac{(A^T b)_i}{b_i} b_i \ln \left( \frac{M_i(x/y)}{m_i(x/y)} \right) \leq C \mu_b(x, y).
\]

Furthermore, Equation (9) implies that

\[
\overline{\mu}_b(F(x), F(y)) \leq \ln \left( \prod_{i=1}^d \max \left\{ \prod_{k=1}^d m_k(x/y)^{A_{i,k}} \prod_{k=1}^d m_k(y/x)^{A_{i,k}} \right\}^{b_i} \right)
\]

\[
\leq \ln \left( \prod_{k=1}^d \max \left\{ m_k(x/y), M_k(y/x) \right\}^{(A^T b)_k} \right) \leq C \overline{\mu}_b(x, y),
\]

which concludes the proof. \( \square \)

The constant \( C \) in the above lemma can not be improved further without additional assumptions on \( F \in \mathcal{H}^d \). This fact is illustrated by the following example where we show that for any matrix \( A \in \mathbb{R}^d_+ \) with \( A \mathbb{R}^d_+ \subset \mathbb{R}^d_+ \), there exists a mapping \( F \in \mathcal{H}^d \) such that \( \mathcal{A}(F) = A \) and we have equality in (8) for some \( x, y \in \mathcal{K}_{++} \) with \( x \neq y \).
Example 3.3. Let \( n_1, \ldots, n_d \geq 2 \) and \( A \in \mathbb{R}^{d \times d}_+ \) be any matrix such that \( A \mathbb{R}^d_+ \subset \mathbb{R}^d_+. \) Define \( F: \mathcal{K}_+ \to \mathcal{K}_+ \) as \( F_{i,j}(x) = \prod_{i=1}^{d} x_{i,j}^{A_{i,j}} \) for every \((i,j) \in \mathcal{I}\). Then, we have \( F \in \mathcal{H}^d \) with \( A(F) = A \). Let \( b \in \mathbb{R}^d_+ \). For every \( x, y \in \mathcal{K}_+ \) and \((j_1, \ldots, j_d) \in \mathcal{J}\) it holds

\[
\prod_{i=1}^{d} \left( \frac{F_{i,j_i}(x)}{F_{i,j_i}(y)} \right)^{b_i} = \prod_{i=1}^{d} \left( \frac{x_{i,j_i}^{A_{i,j_i}}}{y_{i,j_i}} \right)^{b_i} = \prod_{i=1}^{d} \left( \frac{x_{i,j_i}}{y_{i,j_i}} \right)^{(A^\top) b_i}.
\]

Now, let \( k \) be such that \( C = (A^\top b)_k/b_k = \max_{i \in [d]} (A^\top b)_i/b_i \). Furthermore, let \( x, y \in \mathcal{K}_+ \) with \( x_{k,1} \neq y_{k,1} \) and \( x_{i,j} = y_{i,j} \) otherwise. Then \( x \neq y \) and the above equality implies that \( \mu_b(F(x), F(y)) = C \mu_b(x, y) \) and \( \mathcal{P}_b(F(x), F(y)) = C \mathcal{P}_b(x, y) \).

Before proving Theorem 3.1, we note that when \( \rho(A) < 1 \), there is always a choice for the vector \( b \) so that \( C < 1 \) in Lemma 3.2.

Remark 3.4. The Collatz-Wielandt principle is useful to get bounds on the possible Lipschitz constant \( C \) in Lemma 3.2. Moreover, for any \( A \in \mathbb{R}^{d \times d}_+ \), if \( \rho(A) < 1 \), there exists \( r \in \rho(A), 1 \) and \( b \in \mathbb{R}^d_+ \), such that \( A^\top b \leq r b \). Indeed, if \( A^\top \) has a positive eigenvector \( c \in \mathbb{R}^d_+ \), then we can choose \( b = c \) so that \( r = \rho(A) \). Otherwise, if \( A \) has no positive eigenvector, define \( A(t) = A + t(I_1^\top) \) for \( t \in \mathbb{R}_+ \). As \( A < A(t) \) for any \( t > 0 \), by continuity, there exists \( t_0 > 0 \) such that \( 0 \leq \rho(A) \leq \rho(A(t_0)) < 1 \). The linear Perron-Frobenius theorem implies the existence of \( b \in \mathbb{R}^d_+ \) such that \( A(t_0)^\top b = rb \) with \( r = \rho(A(t_0)) \). It follows that \( A^\top b < A(t_0)^\top b = rb \).

We are now ready to prove Theorem 3.1 which is a combination of Lemma 3.2 and Remark 3.4.

Proof of Theorem 3.1. As \( \rho(A) < 1 \) by assumption, Remark 3.4 implies the existence of \( r \in \rho(A), 1 \) and \( b \in \mathbb{R}^d_+ \) such that \( A^\top b \leq r b \). By Lemma 3.2, we have \( \mu_b(F(x), F(y)) \leq r \mu_b(x, y) \) for all \( x, y \in \mathcal{K}_+ \). Now, consider the mapping \( G: \mathcal{S}_+ \to \mathcal{S}_+ \) defined as \( G(x) = (||F_1(x)||_{\gamma_1}, \ldots, ||F_d(x)||_{\gamma_d})^{-1} \otimes F(x) \), then we have \( \mu_b(G(x), G(y)) = \mu_b(F(x), F(y)) \) for every \( x, y \in \mathcal{S}_+ \). Thus, \( G \) is a strict contraction on the complete metric space \( (\mathcal{S}_+, \mu_b) \). The result is now a consequence of the Banach fixed point theorem (see e.g. Theorem 3.1 in [18]).

To conclude this section, we give an example of a mapping which is expansive with respect to the Hilbert and Thompson metrics on \( \mathbb{R}^{n_1 + \ldots + n_d}_+ \) but satisfies all the assumptions of Theorem 3.1. This example motivates the study of multi-homogeneous mappings and illustrates that several arguments involving homogeneous mappings do not hold anymore in the multi-homogeneous framework.

Example 3.5. Let \( \mathcal{K}_+ = \mathbb{R}^2_+ \times \mathbb{R}^2_+ \) and \( F: \mathcal{K}_+ \to \mathcal{K}_+ \) be defined as

\[
F((s, t), (u, v)) = ((u^2, v^2), (s^{1/8}, t^{1/8})).
\]

Clearly, \( F \) is not homogeneous. It is nevertheless subhomogeneous of degree 2, i.e., \( \lambda^2 F(z) \leq F(\lambda z) \) for \( \lambda \in (0, 1) \) and \( z \in \mathbb{R}^d_+ \). Now, let \( \mu \) and \( \overline{\mu} \) be respectively the Hilbert metric and the Thompson metric on \( \mathbb{R}^d_+ \). Then with \( x = (1, 1, 1, 1) \) and \( y = (1, 1, 1, 2) \) we have

\[
\overline{\mu}(F(x), F(y)) = \mu(F(x), F(y)) = \ln(4) = 2 \ln(2) = 2 \mu(x, y) = 2 \overline{\mu}(x, y),
\]
and thus $F$ is expansive with respect to $\mu$ and $\mu$. However, we have $F \in \mathcal{H}^d$ with

$$A(F) = A = \begin{pmatrix} 0 & 2 \\ 1/8 & 0 \end{pmatrix} \quad \text{and} \quad A^\top \begin{pmatrix} 1/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix}.$$ 

Hence, $\rho(A) = 1/2$ is a Lipschitz constant of $F$ with respect to the metric

$$\mu_b(x, y) = \frac{1}{4} \mu(x_1, y_1) + \mu(x_2, y_2) \quad \forall x, y \in K_+,$$

where $\mu$ is the Hilbert metric on $\mathbb{R}^d_+$. The same holds if we replace $\mu$ by the Thompson metric $\overline{\mu}$ on $\mathbb{R}^d_+$ in the above equation.

4. Spectral radii and existence of eigenvectors. Maximality plays an important role in the Perron-Frobenius theory. For example, if the eigenvectors of a mapping $F \in \mathcal{H}^d$ are the critical points of some potential $f : K_+ \to \mathbb{R}$, then we want to assert that the positive eigenvector coincides with the global maximizer of $f$ constrained on some product of unit balls. The function $f$ can be regarded as the numerator of a nonlinear Rayleigh quotient. In order to keep such connections, we propose the following way to compare the “spectral magnitude” of eigenvectors. The main idea is to fix the scaling of eigenvectors by imposing unit norm constraints on $x_i$ and then take the weighted geometric mean of the eigenvalues $\lambda_1, \ldots, \lambda_d$ associated to $x \in K_+$. Note in particular that the eigenvectors of $F \in \mathcal{H}^d$ can always be rescaled so that they belong to $S_+$. So, for $b \in \mathbb{R}^d_+$, we introduce the following notion of spectral radius of $F \in \mathcal{H}^d$

$$r_b(F) = \sup \left\{ \prod_{i=1}^d \lambda_i^{b_i} \mid F(x) = \lambda \otimes x \text{ for some } x \in S_+ \right\}.$$ 

Note that $r_b(F)$ is always nonnegative, as $F(K_+) \subset K_+$ and so $F(x) = \lambda \otimes x$ implies $\lambda_i \geq 0$ for all $i \in [d]$. By Theorem 3.1, it is clear that $r_b(F)$ is well defined for strict contractions in $\mathcal{H}^d$. It is however less clear that, in the case when $F$ is non-expansive, the supremum above is not taken over an empty set. This issue is addressed by the next theorem which can be seen as a generalization of the weak Perron-Frobenius theorem. In particular, it is shown that every non-expansive mapping $F \in \mathcal{H}^d$ for which there exists $b \in \mathbb{R}^d_+$ with $A(F)^\top b = b$, has a nonnegative eigenvector with eigenvalue corresponding to $r_b(F)$. A proof for the case $d = 1$ can be found in Theorem 5.4.1 [19] and essentially relies on the fact that the spectral radius of an order-preserving homogeneous mapping can be characterized in terms of its Bonsall spectral radius [2] and in terms of its cone spectral radius [20]. By generalizing these characterizations, we obtain the following:

**Theorem 4.1.** Let $F \in \mathcal{H}^d$ and $A = A(F)$. If there exists $b \in \Delta_{++}^d$ such that $A^T b = b$, then there exists $u \in S_+$ and $\lambda \in \mathbb{R}^d_+$ such that $F(u) = \lambda \otimes u$ and $r_b(F) = \prod_{i=1}^d \lambda_i^{b_i}$. Furthermore, it holds

$$r_b(F) = \sup_{x \in K_+} \lim_{m \to \infty} \prod_{i=1}^d \| F_i^m(x) \|_{\gamma_i}^{b_i/m} = \lim_{m \to \infty} \sup_{x \in S_+} \prod_{i=1}^d \| F_i^m(x) \|_{\gamma_i}^{b_i/m}.$$ 

As in the linear case, we need to introduce a concept of irreducibility in order to ensure that a non-expansive mapping has a positive eigenvector $F$. To this end, we extend the definition of directed graph associated to order-preserving homogeneous
mappings, proposed in [11], to multi-homogeneous mappings. For the definition of this graph, we consider, for all \( i \in [d], j_i \in [n_i] \), the mapping \( u^{(i,j_i)}: \mathbb{R}^+ \to \mathcal{K}_+ \) defined as

\[
(u^{(i,j_i)}(t))_{k,l_k} = \begin{cases} t & \text{if } (k,l_k) = (i,j_i) \\ 1 & \text{otherwise}, \end{cases} \quad \forall (k,l_k) \in \mathcal{I} = \bigcup_{\nu=1}^{d} \left( \{\nu\} \times [n_\nu] \right).
\]

Then, the graph associated with a mapping in \( \mathcal{H}^d \) is given by the following:

**Definition 4.2.** For \( F \in \mathcal{H}^d \), the directed graph \( G(F) = (\mathcal{I}, \mathcal{E}) \) is defined as follows: There is an edge from \((k,l_k)\) to \((i,j_i),\) i.e. \((k,l_k), (i,j_i)\) \( \in \mathcal{E} \), if

\[
\lim_{t \to \infty} F_{k,l_k}(u^{(i,j_i)}(t)) = \infty.
\]

In particular, we note that if \( F(x) = Mx \) for some nonnegative matrix \( M \in \mathbb{R}^{n \times n} \), then \( G(F) = ([n], \mathcal{E}) \) is the graph with \( M \) as adjacency matrix. Furthermore, if \( G(x,y) = (My, M^T x) \) for a nonnegative matrix \( M \in \mathbb{R}^{m \times n} \), then \( G(G) = ([m] \times [n], \mathcal{E}) \) is the bipartite graph with \( M \) as biadjacency matrix. With this definition, we prove the following generalization of Theorem 2 in [11]:

**Theorem 4.3.** Let \( F \in \mathcal{H}^d \) and \( b \in \mathbb{R}^d_+ \) be such that \( A(F)^T b = b \). If for all \( \nu \in [d], i_\nu \in [n_\nu] \) and \((j_1, \ldots, j_d) \in [n_1] \times \cdots \times [n_d] \), there exists \( i_\nu \in [d] \) so that there is a path from \((i_\nu, j_i)\) to \((i, j_i)\) in \( G(F) \), then \( F \) has an eigenvector in \( \mathbb{S}^d_+ \).

While the assumption in the above theorem is equivalent to requiring that \( G(F) \) is strongly connected when \( d = 1 \), this is not the case anymore when \( d > 1 \). This is shown for instance by Example 4.7. Indeed, if \( G(F) \) is strongly connected, then \( G(F) \) satisfies the assumption of Theorem 4.3 but the converse is not true in general for \( d > 1 \).

### 4.1. Spectral radius of non-expansive mappings.

We consider the notions of Bonsall spectral radius and cone spectral radius for mappings \( F \in \mathcal{H}^d \) such that there exists \( b > 0 \) with \( A(F)^T b = b \). This allows us to show that the supremum in the definition of \( r_b(F) \) is attained.

Let \( x \in \mathcal{K}_+ \), \( F \in \mathcal{H}^d \), \( A = A(F), b \in \Delta^d_+ = \{ c \in \mathbb{R}^d_+ : \sum_{i=1}^{d} c_i = 1 \} \) and assume that \( A^T b = b \). Define

\[
\|x\|_b = \prod_{i=1}^{d} \|x_i\|_{\gamma_i}^b \quad \text{and} \quad \|F\|_b = \sup_{x \in \mathbb{S}_+} \|F(x)\|_b.
\]

Then, for every \( \alpha \in \mathbb{R}^d_+ \) and \( x \in \mathcal{K}_{+0} \), it holds

\[
\|F(\alpha \otimes x)\|_b = \|\alpha^A \otimes F(x)\|_b = \|F(x)\|_b \prod_{i=1}^{d} \alpha_i^{(A^T b)_i} = \|F(x)\|_b \prod_{i=1}^{d} \alpha_i^{b_i}
\]

Hence, with \( \beta = (\|x_1\|_{\gamma_1}^{-1}, \ldots, \|x_d\|_{\gamma_d}^{-1}) \), we have

\[
\|F(x)\|_b = \|F(\beta \otimes x)\|_b \|x\|_b \leq \|F\|_b \|x\|_b \quad \forall x \in \mathcal{K}_{+0}.
\]

Now, consider

\[
\tau_b(F) = \sup_{x \in \mathcal{K}_{+0}} \limsup_{m \to \infty} \|F^m(x)\|_b^{1/m} \quad \text{and} \quad \hat{r}_b(F) = \lim_{m \to \infty} \|F^m\|_b^{1/m}.
\]
In the case $d = 1$, $\hat{r}_b$ is known as Bonsall spectral radius [2] and $\tau_b$ is known as cone spectral radius [20]. Note that for every $\lambda > 0$, we have

$$\tau_b(\lambda F) = \lambda \tau_b(F) \quad \text{and} \quad \hat{r}_b(\lambda F) = \lambda \hat{r}_b(F).$$

Moreover, if $M \in \mathbb{R}^{n \times n}$ and $F(x) = Mx$, then the Gelfand formula [16] implies that $\rho(M) = \tau_1(F) = \hat{r}_1(F)$. The proof of Theorem 5.31 [19], a special case of Theorem 2.2 [20], can be easily adapted to obtain the following:

**Theorem 4.4.** Let $F \in \mathcal{H}^d$, $A = A(F)$ and $b \in \Delta^d_+$ with $A^\top b = b$, then it holds $0 \leq \tau_b(F) = \hat{r}_b(F) < \infty$.

In the following proposition we extend the second part of Theorem 2.2 [20] to the multi-homogeneous case. In particular, it implies that if $F \in \mathcal{H}^d$ is non-expansive and has a positive eigenvector $u \in S_+$ with $F(u) = \lambda \otimes u$, then $\tau_b(F) = \prod_{i=1}^d \lambda_i^b_i$.

Moreover, we use this proposition for the proof of the Collatz-Wielandt formula in Section 5.1.

**Proposition 4.5.** Let $F \in \mathcal{H}^d$, $A = A(F)$ and $b \in \Delta^d_+$ with $A^\top b = b$. Then, $\tau_b(F) = \lim_{m \to \infty} \|F^m(x)\|_b^{1/m}$ for all $x \in K_+$. Moreover, for every $y \in S_+$ and $\theta \in \mathbb{R}^d_+$ with $\theta \otimes y \leq_F F(y)$, we have $\prod_{i=1}^d \theta_i^b_i \leq \tau_b(F)$.

**Proof.** Let $x \in K_+$, there exists $s \in \mathbb{R}^d_+$ such that for every $y \in S_+$, it holds $y \leq_F s \otimes x$. Let $\sigma = \prod_{i=1}^d s_i^b_i$. For $k \in \mathbb{N}$ and $y \in S_+$ we have

$$\|F^k(y)\|_b \leq \|F^k(s \otimes x)\|_b = \|s^A \otimes F^k(x)\|_b = \|F^k(x)\|_b \sigma.$$  

Taking the supremum over $y \in S_+$, we get

$$\|F^k\|_b \sigma^{-1} \leq \|F^k(x)\|_b \leq \|F^k\|_b \|x\|_b \quad \forall k \in \mathbb{N}.$$  

Theorem 4.4 implies $\lim_{k \to \infty} \|F^k\|_b^{1/k} = \tau_b(F)$, hence $\tau_b(F) = \lim_{k \to \infty} \|F^k(x)\|_b^{1/k}$.

Now, if $\theta \otimes y \leq_F F(y)$, then for all $k \in \mathbb{N}$

$$\|F^k(y)\|_b \geq \|y\|_b \prod_{i=1}^d \theta_i^b_i \left(\sum_{j=0}^{k-1} (A^\top)^j b_i\right) = \|y\|_b \prod_{i=1}^d \theta_i^{kb_i}.$$  

Hence,

$$\prod_{i=1}^d \theta_i^{kb_i} = \lim_{k \to \infty} \left(\|y\|_b \prod_{i=1}^d \theta_i^{kb_i}\right)^{1/k} \leq \limsup_{k \to \infty} \|F^k(y)\|_b^{1/k} \leq \tau_b(F),$$

which concludes the proof.

The last tool we need to prove our weak Perron-Frobenius Theorem 4.1. is the next result which is a generalization of Theorem 5.4.1 [19]. We prove the existence of an eigenvector corresponding to the spectral radius for a class of mappings in $\mathcal{H}^d$. Although being of interest in its own, this theorem will also be helpful in Section 4.2 for the proof of the existence of a positive eigenvector. Furthermore, we will use it in Section 5 to show that the Collatz-Wielandt characterization of the spectral radius holds without the assumption that there exists a positive eigenvector.
Theorem 4.6. Let \( F \in \mathcal{H}^d \), \( A = A(F) \) and \( b \in \Delta^d_+ \) with \( A^\top b = b \). For each \( \delta > 0 \), define \( F^{(\delta)} : K_+ \rightarrow K_+ \) as

\[
F^{(\delta)}(x) = F(x) + \delta (\|x_1\|_{\gamma_1}, \ldots, \|x_d\|_{\gamma_d})^A \otimes 1,
\]

where \( 1 \) is the vector of all ones. Then, the following statements hold:

1. For every \( \delta > 0 \), we have \( F^{(\delta)} \in \mathcal{H}^d \), \( A(F^{(\delta)}) = A \) and there exists \( (\lambda^{(\delta)}, x^{(\delta)}) \) in \( \mathbb{R}^d_+ \times \mathbb{S}_+ \) such that \( F^{(\delta)}(x^{(\delta)}) = \lambda^{(\delta)} \otimes x^{(\delta)} \) and \( \prod_{i=1}^d (\lambda_i^{(\delta)}) b_i = \tau_b(F^{(\delta)}) \).

2. If \( 0 < \eta < \varepsilon \), then \( \tau_b(F^{(\eta)}) < \tau_b(F^{(\varepsilon)}) \) and \( \lim_{\eta \rightarrow 0} \tau_b(F^{(\delta)}) = r \) exists.

3. There exists \( (F^{(\delta)})_{k=1}^\infty \subset \{ F^{(\delta)} \}_{\delta > 0} \) such that \( \lim_{k \rightarrow \infty} F^{(\delta_k)} = F \) and the corresponding sequence \( (\lambda^{(\delta_k)}, x^{(\delta_k)})_{k=1}^\infty \) obtained from (1), converges to a maximal eigenpair of \( F \). That is, there exists a pair \( (\lambda, x) \in \mathbb{R}^d_+ \times \mathbb{S}_+ \) such that it holds \( \lim_{k \rightarrow \infty} (\lambda^{(\delta_k)}, x^{(\delta_k)}) = (\lambda, x) \), \( F(x) = \lambda \otimes x \) and \( \tau_b(F) = \prod_{i=1}^d \lambda_i^{b_i} = r \).

Proof. We prove (1): Let \( \delta > 0 \), then \( F^{(\delta)} \in \mathcal{H}^d \) and \( A = A(F^{(\delta)}) \) follow from Lemma 2.7. Let \( \phi \in K_+ \), as \( F^{(\delta)}(K_+, 0) \subset K_+ \), the mapping \( G^{(\delta)} : \mathbb{S}^d_+ \rightarrow \mathbb{S}^d_+ \):

\[
G^{(\delta)}(z) = ((\phi_1, F_1^{(\delta)}(z)), \ldots, (\phi_d, F_d^{(\delta)}(z)))^{-1} \otimes F^{(\delta)}(z).
\]

is well defined and continuous. It follows from the Brouwer fixed point theorem (see for instance [17]) that \( G^{(\delta)} \) has a fixed point \( \bar{x}^{(\delta)} \in \mathbb{S}^d_+ \). We have \( \bar{x}^{(\delta)} \in K_+ \) as \( F^{(\delta)}(K_+, 0) \subset K_+ \). By rescaling \( \bar{x}^{(\delta)} \) and using \( G^{(\delta)}(\bar{x}^{(\delta)}) = \bar{x}^{(\delta)} \) we obtain the existence of \( (\lambda^{(\delta)}, x^{(\delta)}) \in \mathbb{R}^d_+ \times \mathbb{S}_+ \) such that \( F^{(\delta)}(x^{(\delta)}) = \lambda^{(\delta)} \otimes x^{(\delta)} \). By Proposition 4.5, we know \( \prod_{i=1}^d (\lambda_i^{(\delta)}) b_i = \tau_b(F^{(\delta)}) \).

We prove (2): Let \( 0 < \eta < \varepsilon \). As \( F^{(\eta)}(x^{(\eta)}) = (\lambda^{(\eta)} \otimes x^{(\eta)}) \), we have

\[
F^{(\varepsilon)}(x^{(\eta)}) = F(x^{(\eta)}) + \varepsilon 1 = F^{(\eta)}(x^{(\eta)}) + (\varepsilon - \eta) \mathbf{1} = \lambda^{(\eta)} \otimes x^{(\eta)} + (\varepsilon - \eta) \mathbf{1}.
\]

There exist \( \zeta > 0 \) such that \( \zeta x^{(\eta)} \leq (\varepsilon - \eta) \mathbf{1} \) and \( \xi > 0 \) such that \( \tau_b(F^{(\eta)}) + \xi < \prod_{i=1}^d (\lambda_i^{(\eta)} + \zeta) b_i \). We have \( (\lambda^{(\eta)} + \zeta) \otimes x^{(\eta)} \leq \tau_b(F^{(\varepsilon)}(x^{(\eta)})) \). So, Proposition 4.5 implies

\[
\tau_b(F^{(\eta)}) + \xi < \prod_{i=1}^d (\lambda_i^{(\eta)} + \zeta) b_i \leq \tau_b(F^{(\varepsilon)}).
\]

Hence, \( \tau_b(F^{(\eta)}) < \tau_b(F^{(\varepsilon)}) \) for every \( 0 < \eta < \varepsilon \) and \( \lim_{\eta \rightarrow 0} \tau_b(F^{(\delta)}) = r \) exists.

Finally, we prove (3). There exists \( C > 0 \) such that \( y \leq C \mathbf{1} \) for every \( y \in \mathbb{S}_+ \). It follows that for every \( 0 < \varepsilon \leq 1 \), it holds

\[
0 \leq \lambda^{(\varepsilon)} \otimes x^{(\varepsilon)} = F^{(\varepsilon)}(x^{(\varepsilon)}) \leq \tau_b(F^{(1)}(x^{(\varepsilon)}) \leq \tau_b(F^{(1)}(C \mathbf{1})),
\]

and thus \( \{ \lambda^{(\varepsilon)} \mid 0 < \varepsilon \leq 1 \} \) is bounded in \( \mathbb{R}^d_+ \) as well as \( \{ x^{(\varepsilon)} \mid 0 < \varepsilon \leq 1 \} \subset \mathbb{S}_+ \).

Hence, there exists \( (\varepsilon_k)_{k=1}^\infty \subset \mathbb{S}_+ \) with \( \varepsilon_k \rightarrow 0 \), \( x^{(\varepsilon_k)} \rightarrow x \) and \( \lambda^{(\varepsilon_k)} \rightarrow \lambda \) as \( k \rightarrow \infty \). Note that \( r = \lim_{k \rightarrow \infty} \prod_{i=1}^d (\lambda_i^{(\varepsilon_k)}) b_i = \prod_{i=1}^d \lambda_i^{b_i} \).

Now,

\[
F(x^{(\varepsilon_k)}) = F(x^{(\varepsilon_k)}) - \varepsilon_k \mathbf{1} = \lambda^{(\varepsilon_k)} \otimes x^{(\varepsilon_k)} - \varepsilon_k \mathbf{1}.
\]

follows from \( F^{(\varepsilon_k)}(x^{(\varepsilon_k)}) = \lambda^{(\varepsilon_k)} \otimes x^{(\varepsilon_k)} \). So, by continuity of \( F \), we get

\[
F(x) = \lim_{k \rightarrow \infty} F(x^{(\varepsilon_k)}) = \lim_{k \rightarrow \infty} \lambda^{(\varepsilon_k)} \otimes x^{(\varepsilon_k)} - \varepsilon_k \mathbf{1} = \lambda \otimes x.
\]
On the one hand, by definition, we have
\[
\tau_b(F) \geq \limsup_{m \to \infty} \|F^m(x)\|_b^{1/m} = \limsup_{m \to \infty} \left(\|\lambda \sum_{j=0}^{m-1} \lambda^j \otimes x\|_b\right)^{1/m} = \prod_{i=1}^d \lambda_i^{b_i}.
\]
On the other hand, Proposition 4.5 implies \( \lambda \otimes x = F(x) \leq_K F(x_k)(x) \) so that
\[
\tau_b(F) \leq \tau_b(F(x_k)) = \prod_{i=1}^d (\lambda_i^{(x_k)})^{b_i} \quad \forall k \in \mathbb{N}.
\]
Letting \( k \to \infty \), we finally get \( \tau_b(F) \leq \prod_{i=1}^d \lambda_i^{b_i} \).

The proof of Theorem 4.1 is now a collection of the results above.

**Proof of Theorem 4.1.** Theorem 4.6 implies the existence of \( (\lambda, u) \in \mathbb{R}_+^d \times S_+ \) such that \( F(u) = \lambda \otimes u \) and \( \prod_{i=1}^d \lambda_i^{b_i} = \tau_b(F) \). Hence, we have \( \tau_b(F) \leq r_b(F) \). For the reverse inequality, note that if \( v \in S_+ \) is an eigenvector of \( F \) with \( F(v) = \theta \otimes v \), then by Proposition 4.5 we have \( \prod_{i=1}^d \theta_i^{b_i} \leq \tau_b(F) \). It follows that \( \tau_b(F) = r_b(F) \) and the characterizations of \( r_b(F) \) follow from Theorem 4.4.

**4.2. Existence of positive eigenvectors for non-expansive mappings.** In order to obtain a sufficient condition for the existence of a positive eigenvector of a non-expansive mapping in \( \mathcal{H}^d \), we propose a notion of irreducibility adapted from [11]. When \( d = 1 \), the graph of Definition 4.2 coincides with the one proposed in [11] and our existence Theorem 4.3 coincides with Theorem 2 [11] where the graph is required to be strongly connected. However, this is not true anymore when \( d > 1 \). The following is an example of a mapping \( F \in \mathcal{H}^2 \) such that the graph \( G(F) \) of Definition 4.2 is not strongly connected but satisfies the connectivity assumption of Theorem 4.3.

**Example 4.7.** Let \( n_1 = n_2 = 2 \) and \( F \in \mathcal{H}^2 \) with
\[
F((s,t), (u,v)) = \left((s,t), \left(\max\{s,v\}^{1/2}, \max\{t,u\}^{1/2}\right)\right).
\]
Then, \( F(1,1) = (1,1) \) and \( G(F) \) is given by

```
   α  b
  / \  /   \  \    /
 s → 1 → t
```

The proof of Theorem 4.3 relies on the following construction which is a generalization of the technique proposed in Section 3.2 of [11]: Let \( F \in \mathcal{H}^d, b \in \mathbb{R}_+^d, G(F) = (I, E) \) and, for \( r > 0 \), define
\[
\Psi(r) = \sup \left\{ t \geq 0 \mid \min_{(i,j),(k,l) \in E} F_{i,j} \left(u^{(k,l)}(t)\right)^{b_i} \prod_{a_1 \neq i}^d F_{s,a} \left(u^{(k,l)}(t)\right)^{b_s} \leq r \right\}.
\]
Note that, by definition of \( G(F) \), \( \Psi(r) < \infty \) for any \( r > 0 \) and \( \Psi \) is an increasing function. Moreover, note that \( \Psi \) has the following property: Let \( (j_1, \ldots, j_d) \in J \),...
\[ i \in [d], (k, l_k) \in I \text{ and } t > 0, \text{ if } ((i, j_i), (k, l_k)) \in E \text{ then} \]

\[ \prod_{s=1}^{d} F_{s, j_s} \left( u^{(k, l_k)}(t) \right)^{b_s} \leq r \quad \text{implies} \quad t \leq \Psi(r). \]

In the case \( d = 1 \), the proof of Theorem 6.2.3 [19] relies on the following idea: if \( F \in \mathcal{H}^1 \) is homogeneous, \( \mathcal{G}(F) \) is strongly connected and its maximal nonnegative eigenvector \( x \in \mathbb{R}^{n+} \setminus \{0\} \) (given by Theorem 4.5) has a zero entry, then one gets the contradiction \( x = 0 \). Following the same idea, for a non-expansive mapping \( F \in \mathcal{H}^d \) with \( d \geq 1 \), the condition on \( \mathcal{G}(F) \) in Theorem 4.3 is so that, if its maximal eigenvector \( x \in \mathcal{K}_{+, 0} \) has a zero entry, then \( x_i = 0 \) for some \( i \in [d] \) contradicting \( x \in \mathcal{K}_{+, 0} \).

**Proof of Theorem 4.3.** By Theorem 4.6, there exists \( (\lambda^{(x_k)}, x^{(x_k)})_{k=1}^{\infty} \subset [\mathbb{R}^{d}] \times \mathbb{S}_{++} \) such that \( \lim_{k \to \infty} (\lambda^{(x_k)}, x^{(x_k)}) = (\lambda, x^*) \) and \( F(x^*) = \lambda \otimes x^* \in \mathbb{S}_{+} \). Since \( x^* \to \lambda \), there exists a constant \( M_0 > 0 \) such that

\[ \prod_{s=1}^{d} (\lambda_s^{(x_k)})^{b_s} \leq M_0 \quad \forall k \in \mathbb{N}. \]

Suppose by contradiction that \( x^* \in \mathbb{S}_{+} \setminus \mathbb{S}_{++} \). By taking a subsequence if necessary, we may assume that there exists \((j_1, \ldots, j_d) \in J \) and \( \omega \in [d] \) such that

\[ \min_{k \in [s]} x_{s, j_s}^{(x_k)} = x_{j_s}^{(x_k)} \quad \forall s \in [d], k \in \mathbb{N} \quad \text{and} \quad \lim_{k \to \infty} x_{\omega, j_{\omega}}^{(x_k)} = x_{\omega, j_{\omega}}^* = 0. \]

In particular, as \( x^{(x_k)} \in \mathbb{S}_{+} \), there exists \( \tilde{C} > 0 \) such that \( x_{s, j_s}^{(x_k)} \leq \tilde{C} \) for every \((s, t_s) \in I \) and \( k \in \mathbb{N} \). Thus,

\[ 0 \leq \lim_{k \to \infty} \prod_{s=1}^{d} \left( x_{s, j_s}^{(x_k)} \right)^{b_s} \leq \tilde{C}^{1-b_{\omega}} \lim_{k \to \infty} \left( x_{\omega, j_{\omega}}^{(x_k)} \right)^{b_{\omega}} = 0. \]

Since \( x^* \in \mathbb{S}_{+} \), there exists \((i_1, \ldots, i_d) \in J \) with \( x_{s, l_s}^* > 0 \) for all \( s \in [d] \). Thus,

\[ \lim_{k \to \infty} \prod_{s=1}^{d} \left( x_{s, l_s}^{(x_k)} \right)^{b_s} = \prod_{s=1}^{d} \left( x_{s, l_s}^* \right)^{b_s} > 0. \]

Let \( \nu \in [d] \), by assumption on \( \mathcal{G}(F) \), there exists \( i_{\nu} \in [d] \) and a path \((i_{\nu, j_{i_{\nu}}} = (m_1, \xi_{m_1}) \to (m_2, \xi_{m_2}) \to \ldots \to (m_{N_{\nu}}, \xi_{m_{N_{\nu}}}) = (\nu, l_{\nu}) \) in \( \mathcal{G}(F) \) with \( N_{\nu} \leq n_1 + \ldots + n_d \). Define \( i(1), i(2), \ldots, i(N_{\nu}) \in J \) as

\[ i_{\nu}(a) = \begin{cases} \xi_{m_a} & \text{if } s = m_a, \\ j_s & \text{otherwise.} \end{cases} \quad \forall s \in [d], a \in [N_{\nu}]. \]

Fix \( k \in \mathbb{N} \) and let \( t = x_{m_2, \xi_{m_2}}^{(x_k)} / x_{m_2, j_2}^{(x_k)} \) and \( \alpha = ((x_{1, j_1}^{(x_k)})^{-1}, \ldots, (x_{d, j_d}^{(x_k)})^{-1}) \). We have \( u^{(m_2, \xi_{m_2})}(t) \leq x_{1, j_1} \otimes x_{m_2, j_2}^{(x_k)} \) and

\[ \prod_{s=1}^{d} F_{s, j_s}(u^{(m_2, \xi_{m_2})}(t))^{b_s} \leq \prod_{s=1}^{d} F_{s, j_s}(\alpha \otimes x_{m_2, j_2}^{(x_k)})^{b_s} \]

\[ = \left( \prod_{s=1}^{d} (x_{s, j_s}^{(x_k)})^{b_s} \right)^{-1} \prod_{s=1}^{d} F_{s, i_s}(x_{s, j_s}^{(x_k)})^{b_s} \leq M_0. \]
where \( M_0 > 0 \) satisfies (13). Hence, by (12), \( t = x_{m_2,\xi_{m_2}}^{(e_k)}/x_{m_2,j_2}^{(e_k)} \leq \Psi(M_0) \) and
\[
\prod_{s=1}^{d} (x_{s,i_s(2)}^{(e_k)})^{b_s} \leq M_1 \prod_{s=1}^{d} (x_{s,j_s}^{(e_k)})^{b_s} \quad \text{with} \quad M_1 = \Psi(M_0)^{b_{m_2}}.
\]
Applying this procedure again to \((m_3,\xi_{m_3})\), we get the existence of a constant \( M_2 > 0 \) independent of \( k \), such that
\[
\prod_{s=1}^{d} (x_{s,i_s(3)}^{(e_k)})^{b_s} \leq M_2 \prod_{s=1}^{d} (x_{s,j_s}^{(e_k)})^{b_s}.
\]
Indeed, let \( t = x_{m_3,\xi_{m_3}}^{(e_k)}/x_{m_3,j_3}^{(e_k)} \), then \( u^{(m_3,\xi_{m_3})}(t) \leq \alpha \otimes x^{(e_k)} \) and
\[
\prod_{s=1}^{d} F_{s,i_s(2)}(u^{(m_3,\xi_{m_3})}(t))^{b_s} \leq \prod_{s=1}^{d} F_{s,i_s(2)}(\alpha \otimes x^{(e_k)})^{b_s} = \left( \prod_{s=1}^{d} (x_{s,j_s}^{(e_k)})^{b_s} \right)^{-1} \prod_{s=1}^{d} F_{s,i_s(2)}(x^{(e_k)})^{b_s} \leq M_0 M_1.
\]
Hence, with \( M_2 = \Psi(M_0 M_1)^{b_{m_2}} \), we get the desired inequality. Repeating this process at most \( N_\nu \) times, we obtain \( C_\nu > 0 \) independent of \( k \), such that
\[
(x_{s,i_s(3)}^{(e_k)})^{b_s} \prod_{s=1}^{d} (x_{s,j_s}^{(e_k)})^{b_s} = \prod_{s=1}^{d} (x_{s,i_s(\nu_s)}^{(e_k)})^{b_s} \leq C_\nu \prod_{s=1}^{d} (x_{s,j_s}^{(e_k)})^{b_s} \quad \forall k \in \mathbb{N}.
\]
Taking the product over \( \nu \in [d] \) in (16) and dividing by \( \prod_{s=1}^{d} (x_{s,j_s}^{(e_k)})^{(d-1)b_s} \) shows
\[
\prod_{\nu=1}^{d} (x_{s,j_s}^{(e_k)})^{b_s} \leq C \prod_{s=1}^{d} (x_{s,j_s}^{(e_k)})^{b_s} \quad \forall k \in \mathbb{N},
\]
where \( C = \prod_{\nu=1}^{d} C_\nu \). Finally, using (14) and (15) we get a contradiction. \( \Box \)

As noted in Corollary 6.2.4 [19] for the case \( d = 1 \), there exists a dual version of Theorem 4.3 which follows by considering the mapping \( \tau: \mathbb{R}^N_+ \to \mathbb{R}^N_+ \) defined as \( \tau(z) = (z_1^{-1}, \ldots, z_N^{-1}) \) with \( N = n_1 + \ldots + n_d \). More precisely, let \( F \in \mathcal{H}^d \) and define \( \tilde{F}: K_{++} \to K_{++} \) as \( \tilde{F}(x) = \tau(F(\tau(x))) \) for all \( x \in K_{++} \). Then, \( \tau \) is a bijection between the positive eigenvectors of \( F \) and \( \tilde{F} \). Moreover, by Theorem 2.6, \( \tilde{F} \) can be continuously extended on \( K_+ \) so that \( \tilde{F} \in \mathcal{H}^d \). The following corollary follows directly from Theorem 4.3 applied to \( \tilde{F} \).

**Corollary 4.8.** Let \( F \in \mathcal{H}^d \), \( A = A(F) \) and \( b \in \Delta_{++}^d \) with \( A^T b = b \). Let \( \tilde{F} \) be defined as above. Suppose that, for every \( (\nu, i_\nu) \in \mathcal{I} \) and \( (j_1, \ldots, j_d) \in \mathcal{J} \) there exists \( i_\nu \in [d] \) such that there is a path from \((i_\nu, j_\nu)\) to \((i_\nu, l_\nu)\) in \( \mathcal{G}(\tilde{F}) \). Then \( F \) has an eigenvector in \( S_{+++} \).

We make some observations on \( \mathcal{G}(\tilde{F}) = (\mathcal{I}, \tilde{E}) \). First, note that if \( u^{(k,j_k)}(t) \) is defined as in (10), then we have \(((k,l_k),(i,j_i))\in \tilde{E}, if and only if \( \lim_{t \to 0} F_{k,l_k}(u^{(i,j_i)}(t)) = 0 \). Furthermore, \( \mathcal{G}(F) \) and \( \mathcal{G}(\tilde{F}) \) can be very different. In fact, consider again the mapping \( F \in \mathcal{H}^1 \) of Example 4.7. Then, \( \mathcal{G}(\tilde{F}) \) contains only two self-loops. On the
other hand, if we substitute the max’s with min’s in the definition of \( F \), we obtain a mapping \( H \in \mathcal{H}^1 \) with \( G(H) = G(F) \) and \( G(H) = G(\hat{F}) \).

We conclude this section by noting that unlike the linear case, the assumption that \( G(F) \) is strongly connected does not imply the uniqueness of positive eigenvectors.

**Example 4.9.** Let \( \varepsilon \in (0, 1) \), \( d = 1 \), \( n_1 = 3 \) and \( F \in \mathcal{H}^d \) with

\[
F(a, b, c) = (\max(a, b, c), \max(\varepsilon a, b), \max(\varepsilon b, c)).
\]

Then, \( G(F) \) is strongly connected and \((1, b, c)\) is an eigenvector of \( F \) for all \( b, c \in [\varepsilon, 1] \).

**5. Maximality and uniqueness of positive eigenvectors.** Theorems 3.1 and 4.3 provide sufficient conditions for the existence of a positive eigenvector. In the linear case, it is known that the eigenvalue associated to a positive eigenvector of a nonnegative matrix always coincides with its spectral radius. This can be deduced by the Collatz-Wielandt formula. A generalization of this characterization to the spectral radius of non-expansive mappings in \( \mathcal{H}^1 \) can be found in Theorem 5.6.1 [19] and Theorem 1 [12]. In the context of nonnegative multi-linear forms, Collatz-Wielandt formulas were established for different types of spectral problems [10, 13, 27]. By combining techniques from the proofs of Theorem 5.6.1 in [19] and of Theorem 1 in [13], we obtain the following Collatz-Wielandt characterization of the spectral radius for mappings in \( \mathcal{H}^d \):

**Theorem 5.1.** Let \( F \in \mathcal{H}^d \), \( A = A(F) \) and \( b \in \Delta_{++}^d \). If either \( A^\top b = b \) or \( \rho(A) < 1 \) and \( (A^\top b)_i \leq b_i \) for every \( i \in [d] \), then

\[
\inf_{\nu \in \mathcal{S}_+^d} \prod_{i=1}^d \mu_i(F(\nu)/\nu)^{b_i} = r_b(F) = \max_{\nu \in \mathcal{S}_+^d} \prod_{i=1}^d m_i(F(\nu)/\nu)^{b_i}.
\]

In particular, we note that if \( d = 1 \) and \( F \) is linear, then the left hand side of (17) reduces to the classical Collatz-Wielandt formula (6).

Our next result is concerned with the simplicity of the positive eigenvector of a multi-homogeneous mappings and its eigenvalue. In the linear case, it is known that every nonnegative irreducible matrix has a unique real eigenvector corresponding to its spectral radius and this vector must have positive entries. We have seen in Theorem 4.3 a possible way to generalize the notion of irreducibility to mappings in \( \mathcal{H}^d \) which ensures existence of a positive eigenvector. However, as shown in Example 4.9, already in the case \( d = 1 \), this assumption does not guarantee that this positive eigenvector is unique in \( \mathcal{S}_+ \). This suggests that the notion of irreducibility needs to be generalized in a different way in order to obtain uniqueness results. A possible approach is proposed in Theorem 2.5 [22] and Theorem 6.1.7 [19], which have assumptions on the derivative of the mapping. More precisely, let \( F \in \mathcal{H}^1 \) be such that \( A(F) = 1 \), \( F \) has a positive eigenvector \( u \in \mathcal{S}_+ \) and \( F \) is differentiable at \( u \in \mathcal{S}_+ \).

Let \( DF(u) \) be the Jacobian of \( F \) at \( u \). If \( DF(u) \) is irreducible, then Theorem 2.5 [22] implies that \( u \) is the unique eigenvector of \( F \) in \( \mathcal{S}_+ \) and Theorem 6.1.7 [19] implies that for any eigenvector \( v \in \mathcal{S}_+ \setminus \mathcal{S}_+ \) with \( F(v) = \theta v \) we have \( \theta < r_1(F) \). The combination of these results can therefore be interpreted as a result on the simplicity of the spectral radius. Indeed the first one implies that the positive eigenvector is unique and the second one implies that the spectral radius of \( F \) can only be attained by a positive eigenvector. The following theorem generalizes the results above to the multi-homogeneous setting.
Theorem 5.2. Let \( F \in \mathcal{H}^d \), \( A = \mathcal{A}(F) \) and \( b \in \Delta^d_+ \). Suppose that \( F \) has a positive eigenvector \( u \in \mathcal{S}_{++} \). Then, \( u \) is the unique eigenvector of \( F \) in \( \mathcal{S}_{++} \) if either \( \rho(A) < 1 \) or \( A^T b = b \). \( F \) is differentiable at \( u \) and \( DF(u) \) is irreducible. Furthermore, suppose that \( F \) has an eigenvector \( v \in \mathcal{S}_+ \setminus \mathcal{S}_{++} \) and let \( \theta \in \mathbb{R}_+^d \) be such that \( F(v) = \theta \otimes v \). If \( A^T b = \rho(A)b \), \( F \) is differentiable at \( u \) and \( DF(u) \) is irreducible, then \( \prod_{i=1}^d \theta_i^{b_i} < r_b(F) \).

It turns out that the assumptions in the theorem above can be refined. On the one hand, as in Theorem 2.5 \cite{22}, in order to guarantee the uniqueness of a positive eigenvector the requirement that \( DF(u) \) is irreducible can be relaxed to a condition on the eigenspace of \( DF(u) \) corresponding to its spectral radius. On the other hand, for \( d > 1 \), it can be shown that the spectral radius cannot be attained in \( \mathcal{S}_+ \setminus \mathcal{S}_{++} \) under a weaker assumption than irreducibility. These relaxed assumptions are given in Theorems 5.5 and 5.3 at the end of the section.

5.1. Collatz-Wielandt formulas. For convenience in the proof of Theorem 5.1, for a given \( b \in \Delta^d_+ \), we introduce the functions \( \tilde{c_w}_b : \mathcal{H}^d \times \mathcal{K}_{++,0} \to \mathbb{R}_+ \) and \( \tilde{c_w}_b : \mathcal{H}^d \times \mathcal{K}_{++,} \to \mathbb{R}_+ \), defined as

\[
\text{(18)} \quad \tilde{c_w}_b(F, u) = \prod_{i=1}^d \left( \max_{j_i \in [n_i]} \frac{F_{i,j_i}(u)}{u_{i,j_i}} \right)^{b_i}, \quad \tilde{c_w}_b(F, x) = \prod_{i=1}^d \left( \min_{j_i \in [n_i]} \frac{F_{i,j_i}(x)}{x_{i,j_i}} \right)^{b_i}.
\]

With this notation, the characterization of Theorem 5.1 can be reformulated as

\[
\text{(19)} \quad \inf \left\{ \tilde{c_w}_b(F, u) \mid u \in \mathcal{S}_{++} \right\} = r_b(F) = \max \left\{ \tilde{c_w}_b(F, v) \mid v \in \mathcal{S}_+ \right\}.
\]

Note also that for \( F \in \mathcal{H}^d \), \( b \in \mathbb{R}_+^d \) and \( x \in \mathcal{K}_{++,} \), it holds \( \tilde{c_w}_b(F, x) = \tilde{c_w}_b(F, x) \) if and only if \( x \) is an eigenvector of \( F \).

The proof of Theorem 5.1 contains two cases, namely the case where \( F \in \mathcal{H}^d \) is non-expansive and the one where \( F \) is a strict contraction. For the first case we generalize Theorem 5.6.1 in [19] which holds for the case \( d = 1 \). For the second case, we generalize the Collatz-Wielandt formula of Theorem 21 in [13].

Proof of Theorem 5.1. First assume that \( A^T b = b \). Let \( x \in \mathcal{S}_{++} \) and \( k \in \mathbb{N} \), then we have \( F^k(x) \leq_k \mathcal{M}(F(x)/x) \sum_{i=0}^{k-1} A^i \otimes x \). Proposition 4.5 implies

\[
r_b(F) = \lim_{k \to \infty} \| F^k(x) \|_b^{1/k} \leq \lim_{k \to \infty} \prod_{i=1}^d \mathcal{M}_i(F(x)/x)^{\left( \frac{1}{k} \sum_{j=0}^{k-1} A^j b_i \right)} = \tilde{c_w}_b(F, x).
\]

Hence, \( r_b(F) \leq \inf \{ \tilde{c_w}_b(F, u) \mid u \in \mathcal{S}_{++} \} \). To show equality, assume first that \( F \) has an eigenvector \( u \in \mathcal{S}_{++} \). Then \( \tilde{c_w}_b(F, u) = r_b(F) \) and we are done. Now, suppose that \( F \) does not have an eigenvector in \( \mathcal{S}_{++} \), let \( F(\delta_k) \) and \( (\lambda(\delta_k), x(\delta_k)) \in \mathbb{R}_+^d \times \mathcal{S}_{++} \) be as in Theorem 4.6. Note that \( \tilde{c_w}_b(F, x) \leq \tilde{c_w}_b(F(\delta_k), x) \) as \( F(x) \leq_k F(\delta_k)(x) \) for every \( k \in \mathbb{N} \) and \( x \in \mathcal{K}_{++,} \). It follows that

\[
r_b(F) = \lim_{k \to \infty} r_b(F(\delta_k)) = \lim_{k \to \infty} \inf_{x \in \mathcal{S}_{++}} \tilde{c_w}_b(F(\delta_k), x) \geq \inf_{x \in \mathcal{S}_{++}} \tilde{c_w}_b(F, x).
\]

Now, we prove \( r_b(F) = \max \{ \tilde{c_w}_b(F, v) \mid v \in \mathcal{S}_+ \} \). To this end, let \( y \in \mathcal{S}_+ \), if there exists \( (i, j_i) \in I \) such that \( y_{i,j_i} > 0 \) and \( F_{i,j_i}(y) = 0 \), then \( \tilde{c_w}_b(F, y) = 0 \leq r_b(F) \).

If this is not the case, then we have \( \theta \otimes y \leq F(y) \) with \( \theta \in \mathbb{R}_+^d \) defined as \( \theta_i = \)
Now, note that \( F(\lambda, u) = \mathbb{R}_+ \times \mathbb{S}_+ \) is irreducible assumption (\( \rho(A) < 1 \), Theorem 3.1 implies the existence of \( (\lambda, u) \in \mathbb{R}_+^d \times \mathbb{S}_+ \) such that \( F(u) = \lambda \otimes u \). Clearly, we have \( \tilde{w}_b(F, u) = \lambda \). To prove the right-hand side of (19), it suffices to prove that for every \( y \in \mathbb{S}_+ \), we have \( \tilde{w}_b(F, y) \leq \lambda_b(\mathbb{F}) \). So, let \( y \in \mathbb{S}_+ \), if there exists \( (i, j) \in \mathbb{I} \) such that \( y_{i,j} > 0 \) and \( F_{i,j}(y) = 0 \), then the inequality is clear. Thus, we may assume without loss of generality that \( F_{i,j}(y) > 0 \) for every \( (i, j) \in \mathbb{I} \) such that \( y_{i,j} > 0 \). Let \( \theta \in \mathbb{R}_+^d \) be defined as \( \theta_i = \min\{u_{i,j}, y_{i,j} : y_{i,j} > 0, j \in [n_i] \} \) for all \( i \in [d] \). Then \( \theta \leq 1 \) because \( \theta_i = \|\theta_i y_i\|_{\gamma_i} \leq \|\theta_i u_i\|_{\gamma_i} = 1 \) for all \( i \in [d] \). Let \( \Theta = \theta^{-1} \), then \( \Theta \geq 1 \) and \( y \leq \theta \otimes u \). Thus, for \( s = b - A^t b \in \mathbb{R}_+^d \), we have \( \prod_{i=1}^d \Theta_i^{-s_i} \leq 1 \). Now, note that \( F(\Theta \otimes u) = (\lambda \Theta) \otimes u \) and thus

\[
\tilde{w}_b(F, y) \leq \prod_{i=1}^d \left( \min_{y_{i,j} > 0} \frac{F_{i,j}(\Theta \otimes u)}{y_{i,j}} \right)^{b_i} = \prod_{i=1}^d \Theta_i^{-s_i} \lambda_i \leq \lambda_b(\mathbb{F}).
\]

The left-hand side of (19) can be proved in a similar way. Indeed, if \( y \in \mathbb{S}_+ \), then

\[
\tilde{w}_b(F, u) \geq \prod_{i=1}^d \mathbb{M}_i \left( F(\mathbb{m}(y/u) \otimes u) / y \right)^{b_i} = \prod_{i=1}^d \mathbb{m}_i \left( (y/u)^{-s_i} \lambda_i \right) \geq \lambda_b(\mathbb{F}),
\]

as \( \prod_{i=1}^d \mathbb{m}_i \left( (y/u)^{-s_i} \right) \geq 1 \).

5.2. Uniqueness and simplicity of positive eigenvectors. We prove the following theorem which gives a condition ensuring that the eigenvalue corresponding to an eigenvector which has some zero entry can not be maximal.

**Theorem 5.3.** Let \( F \in \mathcal{H}_d \) and \( A = \mathcal{A}(F) \). Suppose that there exists \( b \in \Delta_+^d \), \( \lambda \in \mathbb{R}_+^d \) and \( u \in \mathbb{S}_+^d \) such that \( A^t b = \rho(A) b \) and \( F(u) = \lambda \otimes u \). Assume \( \rho(A) \leq 1 \), \( F \) is differentiable at \( u \) and there exist \( i \in [d] \) and \( \tau \in \mathbb{N} \) such that

\[
(20) \quad \forall w \in \mathbb{K}_{\tau} \setminus \{0\}, \quad \text{if} \quad x = \sum_{k=1}^{\tau} DF(u)^k w, \quad \text{then} \quad x_i \in \mathbb{R}^n_{++}.
\]

Then, for every eigenpair \( (\theta, v) \in \mathbb{R}_+^d \times (\mathbb{S}_+ \setminus \mathbb{K}_{++}) \) with \( F(v) = \theta \otimes v \), it holds \( \prod_{i=1}^d \theta_i^{b_i} < \prod_{j=1}^d \lambda_j^{b_j} \).

Before giving a proof of this theorem, we note that while in the case \( d = 1 \) the irreducibility assumption (20) is equivalent to requiring \( DF(u) \) to be irreducible, this is not the case anymore when \( d > 1 \). Indeed, if \( DF(u) \) is irreducible, then (20) is satisfied, however the converse might not be true as shown by the following example. In fact, for any \( d \geq 1 \), \( DF(u) \) is irreducible if and only if (20) holds and \( A(F) \) is irreducible.

**Example 5.4.** Let \( d = 2 \), \( n_1 = n_2 = 2 \) and \( F \in \mathcal{H}_d \) with

\[
F((s, t), (u, v)) = \left( ((st)^{1/4}u^{1/2}, (st)^{1/4}v^{1/2}), ((uv)^{1/2}, (uv)^{1/2}) \right).
\]

Then, \( F(1) = 1 \), Theorem 5.3 applies to \( F \), but \( DF(1) \) is not irreducible.
We now prove Theorem 5.3. The techniques used are inspired by the proof of Theorem 6.1.7 in [19] which implies the same result for the case \( d = 1 \).

**Proof of Theorem 5.3.** Let \( \| \cdot \| \) be any norm on \( \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d} \) and \( \lambda \in \mathbb{R}^d_+ \) be such that \( F(u) = \lambda \otimes u \). We first prove the statement for \( \lambda = 1 \), then we show how to transfer the proof to the case \( \lambda \neq 1 \). By the chain rule, we have
\[
DF(u)^k = DF(u)^k \quad \text{for every} \ k \in \mathbb{N}.
\]
Suppose by contradiction that there exists \((\theta, \nu) \in \mathbb{R}^d_+ \times (S_+ \setminus K_+)\) with \( F(\nu) = \theta \otimes \nu \) and \( \prod_{l=1}^d \theta_{l}^{b_l} = 1 \). Let \( \alpha \in \mathbb{R}^d_+ \) be defined as \( \alpha_k = \min \{ u_{k,l_k}/v_{k,l_k} \mid l_k \in [n_k], \ v_{k,l_k} > 0 \} \) for every \( k \in [d] \), then \( 0 \leq \nu - \alpha \otimes \nu \leq \nu \). Hence \( -\left( \sum_{k=1}^d DF(u)^k(\alpha \otimes \nu - u) \right) \in \mathbb{R}^n_+ \). For \( t \in (0, 1] \), define \( y(t) = (1 - t)u + t\alpha \otimes \nu \leq \nu \) and note that
\[
F^k(y(t)) = F^k(u) + t DF(u)^k(\alpha \otimes \nu - u) + t \| \alpha \otimes \nu - u \| \varepsilon_k(t(\alpha \otimes \nu - u))
\]
where \( \lim_{\| w \| \to 0} \varepsilon_k(w) = 0 \). If follows that, with \( z = \alpha \otimes \nu - u \), we have
\[
\sum_{k=1}^\tau \left( F^k(u) - F^k(y(t)) \right) = t \left( - \sum_{k=1}^{\tau} DF(u)^k z - \| z \| \sum_{k=1}^{\tau} \varepsilon_k(tz) \right).
\]
Since \( \lim_{t \to 0} \sum_{k=1}^\tau \varepsilon_k(tz) = 0 \) and \( - \sum_{k=1}^\tau (DF(u)^k z) \in \mathbb{R}^n_+ \), there exists \( s \in (0, 1] \) such that for every \( t \in (0, s] \), it holds \( \sum_{k=1}^\tau (F^k(u) - F^k(y(t))) \in \mathbb{R}^n_+ \). For all \( t \in (0, 1] \), we have \( \alpha \otimes \nu \leq \nu \) and thus \( \sum_{k=1}^\tau (F^k(y(t)) - F^k(\alpha \otimes \nu)) \in K_+ \). It follows with \( \lambda = 1 \) and \( F(u) = u \) that \( \sum_{k=1}^\tau F^k(\alpha \otimes \nu) \leq \tau u \) and \( \sum_{k=1}^\tau F^k(\alpha \otimes \nu) < \tau u \).

So, for every \((j_1, \ldots, j_d) \in J\), we have
\[
\tau \prod_{l=1}^d \prod_{i=1}^{b_l} u_{l,j_l}^{b_l} > \prod_{l=1}^d \left( \sum_{k=1}^\tau F^k_{l,j_l}(\alpha \otimes \nu) \right)^{b_l} = \prod_{l=1}^d \left( \sum_{k=1}^\tau \left( \alpha^A_k \right)_l \left( \theta^{A^*_l} \sum_{s=0}^{k-1} A^*_l \right)^{b_l} \right).
\]
Using the inequality relating arithmetic and geometric mean, we get
\[
\sum_{k=1}^\tau \left( \alpha^A_k \right)_l \left( \theta^{A^*_l} \sum_{s=0}^{k-1} A^*_l \right)^{b_l} \geq \tau \prod_{k=1}^\tau \left( \left( \alpha^A_k \right)_l \left( \theta^{A^*_l} \sum_{s=0}^{k-1} A^*_l \right)^{b_l} \right)^{1/\tau}.
\]
It follows that
\[
\prod_{l=1}^d \left( \sum_{k=1}^\tau \left( \alpha^A_k \right)_l \left( \theta^{A^*_l} \sum_{s=0}^{k-1} A^*_l \right)^{b_l} \right)^{b_l} \geq \tau \prod_{l=1}^d \prod_{k=1}^\tau \left( \left( \alpha^A_k \right)_l \left( \theta^{A^*_l} \sum_{s=0}^{k-1} A^*_l \right)^{b_l} \right)^{b_l/\tau} \geq \tau \prod_{l=1}^d \left( \prod_{i=1}^{b_l} \left( \frac{d(\lambda)^k}{\theta^A_k} \right)^{b_l} \sum_{s=0}^{k-1} \rho(A^*_l) \right)^{b_l/\tau} \geq \tau \prod_{l=1}^d \alpha_{l,i}^{b_l},
\]
where we have used that \( \alpha \leq 1 \) because \( u, \nu \in S_+ \). Thus, for all \((j_1, \ldots, j_d) \in J\) we have \( \prod_{l=1}^d u_{l,j_l}^{b_l} > \prod_{l=1}^d (\theta_{l,j_l}/\alpha_{l,i})^{b_l} \), a contradiction to the definition of \( \alpha \). Now, if \( F(u) = \lambda \otimes u \) with \( \lambda \neq 1 \), then \( F' \in H^d \) defined as \( F'(x) = \lambda^{-1} \otimes F(x) \) satisfies our assumptions and \( F'(u) = u \). So, if \((\theta, \nu) \in \mathbb{R}^d_+ \times (S_+ \setminus K_+)\) satisfies \( F(\nu) = \theta \otimes \nu \), then \( F'(\nu) = (\lambda^{-1} \circ \theta) \otimes \nu \) and thus \( \prod_{l=1}^d (\theta_{l,j_l}/\lambda)^{b_l} < 1 \) implies \( \prod_{l=1}^d \theta_{l,j_l}^{b_l} = r(F) \). \( \square \)

The following theorem is concerned with the uniqueness of positive eigenvectors in \( S_+^\Phi = \{ x \in K_+ \mid \langle x, \phi_i \rangle = 1, \ i \in [d] \} \).
Theorem 5.5. Let $\phi \in \mathcal{K}_{++}$, $F \in \mathcal{H}^d$ and $A = \mathcal{A}(F)$. Suppose that $A$ is irreducible, $\rho(A) = 1$, there exist $\lambda \in \mathbb{R}^d_{++}$ and $u \in \mathbb{S}^d_+$ with $F(u) = \lambda \otimes u$ and $F$ is differentiable at $u$. Consider the linear mapping $L: \mathcal{K}_+ \to \mathcal{K}_+$ defined as $L(x) = \lambda^{-1} \otimes DF(x)x$ for every $x$, then $\rho(L) = 1$ and if $\dim(\ker(I - L)) = 1$, then $u$ is the unique eigenvector of $F$ in $\mathbb{S}_+^d$.

If in the above theorem, $DF(u)$ is irreducible, then $L$ is irreducible and so $\dim(\ker(I - L)) = 1$ follows by the linear Perron-Frobenius theorem. However, it is known that there are cases where $\dim(\ker(I - L)) = 1$ is satisfied but $DF(u)$ is not irreducible (see e.g. [19] p. 143).

For the proof of Theorem 5.5 we first need to derive intermediary results. The first one is a theorem with a flavor of fixed point theory in the sense that it only requires $G: \mathbb{S}_+^d \to \mathbb{S}_+^d$ to be non-expansive under the metric $\rho_b$. The theorem states that if the mapping has two distinct positive eigenvectors $u, w \in \mathbb{S}_+^d$, then $DG(u)$ has a fixed point $v$ which is orthogonal to $\phi$. The second one is a lemma describing properties of $DF(u)$ and $DG(u)$ where $G: \mathbb{S}_+^d \to \mathbb{S}_+^d$ is defined as

\begin{equation}
G(x) = \left( \langle \phi_1, F_1(x) \rangle, \ldots, \langle \phi_d, F_d(x) \rangle \right)^{-1} \otimes F(x).
\end{equation}

The proof of Theorem 6.4.1 [19] can be easily adapted to obtain the following:

Theorem 5.6. Let $\phi \in \mathcal{K}_{++}$, $b \in \mathbb{R}^d_{++}$ and $G: \mathbb{S}_+^d \to \mathbb{S}_+^d$ be such that $\mu_b(G(x), G(y)) \leq \mu_b(x, y)$ for all $x, y \in \mathbb{S}_+^d$. If there exist $u, w \in \mathbb{S}_+^d$, $u \neq w$ such that $G(u) = u$, $G(w) = w$ and $G$ is differentiable at $u$, then there exists $v \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d}, v \neq 0$ such that $\langle v, \phi \rangle = 0$ and $DG(u)v = v$.

The next lemma shows that when $u$ is a fixed point of $F \in \mathcal{H}^d$ and $F$ is differentiable at $u$, then one can find $b \in \mathbb{R}^d_{++}$ such that $b \otimes u$ is an eigenvector of $DF(u)$.

Lemma 5.7. Let $\phi \in \mathcal{K}_{++}$, $F \in \mathcal{H}^d$, $A = \mathcal{A}(F)$ and $G$ as in (21). If there exists $u \in \mathbb{S}_+^d$ with $F(u) = u$, $F$ is differentiable at $u$ and $b \in \mathbb{R}_{++}^d$ satisfies $Ab = b$, then $DF(u)\tilde{u} = \tilde{u}$ with $\tilde{u} = b \otimes u$. Moreover, $G$ is differentiable at $u$ and for every $z \in V$,

\begin{equation}
DG(u)z = DF(u)z - \left( \langle DF_1(u)z, \phi_1 \rangle, \ldots, \langle DF_d(u)z, \phi_d \rangle \right) \otimes u.
\end{equation}

Proof. By Lemma 2.5, for all $k, i \in [d]$, we have $D_kF_i(u)u_i = A_{ki}u_k$. Hence,

\begin{equation}
DF_i(u)(\alpha \otimes u) = (A\alpha)_iu_i \quad \forall \alpha \in \mathbb{R}_{++}^d,
\end{equation}

implying $DF(u)\tilde{u} = (A\tilde{b}) \otimes u = \tilde{u}$. Now, if $F$ is differentiable at $x \in \mathcal{K}_{++}$, then

$$D_kG_i(x) = \frac{\langle F_i(x), \phi_i \rangle D_kF_i(x) - F_i(x)\phi_i^\top D_kF_i(x)}{\langle F_i(x), \phi_i \rangle^2}$$

$\forall k, i \in [d]$.

In particular, if $x = u \in \mathbb{S}_+^d$ and $F(u) = u$, the above equation simplifies to $D_kG_i(u) = D_kF_i(u) - u_i\phi_i^\top D_kF_i(u)$.

We now prove Theorem 5.5 which extends Theorem 6.4.6 in [19] to the case $d \geq 1$.

Proof of Theorem 5.5. Let $\tilde{b}, b \in \Delta_{++}^d$ be such that $A\tilde{b} = b$ and $A^\top \tilde{b} = b$. These vectors always exist because $A$ is assumed to be irreducible. Suppose by contradiction that there exists $w \in \mathbb{S}_+^d \setminus \{u\}$ and $\lambda \in \mathbb{R}_{++}^d$ such that $F(w) = \lambda \otimes w$. Let $\tilde{F} \in \mathcal{H}^d$ be defined as $\tilde{F}(x) = \lambda^{-1} \otimes F(x)$ for every $x \in \mathcal{K}_{++}$. Then, we have $\tilde{F}(u) = u$. ...
\[ L = D\tilde{F}(u) \text{ and } \tilde{F}(w) = (\lambda^{-I} \circ \lambda) \otimes w. \] Lemma 5.7 implies that \( \tilde{u} = \tilde{b} \otimes u \in K_{++} \) satisfies \( Lu = \tilde{u} \). Theorem 2.4 implies that \( L \) is a nonnegative matrix. Hence, Proposition 4.5 and \( L\tilde{u} = \tilde{u} \in K_{++} \) imply that \( \rho(L) = 1 \). Let \( G \) be defined as (21), then \( G \) is non-expansive by Lemma 3.2. By Theorem 5.6, there is a \( v \neq 0 \) with

\[ (24) \quad (v, \phi) = 0, \quad Lv - \alpha \otimes u = v \quad \text{where} \quad \alpha = \left( \langle Lv, \phi_1 \rangle, \ldots, \langle Lv, \phi_d \rangle \right). \]

First, suppose that \( \langle b, \alpha \rangle = 0 \). Then for \( \varphi \in K_{++,0} \) with \( \langle u_i, \varphi_i \rangle = 1 \) for each \( i \in [d] \), we have

\[ (25) \quad \sum_{i=1}^{d} \langle (Lv)_i, b_i \varphi_i \rangle = \sum_{i=1}^{d} \langle v_i, b_i \varphi_i \rangle + \sum_{i=1}^{d} \alpha_i b_i \langle u_i, \varphi_i \rangle = \sum_{i=1}^{d} \langle v_i, b_i \varphi_i \rangle. \]

Let \( (i, j_i) \in \mathcal{I} \) and define \( \tilde{e}^{(i, j_i)} \in \mathbb{R}^{n_i}_{++} \) as \( \tilde{e}^{(i,j_i)} = 1 \) if \( j_i = l_i \) and \( \tilde{e}^{(i,j_i)} = 0 \) otherwise. Furthermore, consider \( \varphi^{(i,j_i)} \in K_{++,0} \) defined as

\[ \varphi^{(i,j_i)} = \left( \frac{1}{(1, u_{i_1}), \ldots, \frac{1}{(1, u_{i - 1})}, \frac{1 - e^{(i,j_i)}}{(1, u_{i_{l}})}, \frac{1}{(1, u_{i_{l+1}})}, \ldots, \frac{1}{(1, u_d)}} \right). \]

Plugging \( \varphi^{(i,j_i)} \) into Equation (25) for each \( (i, j_i) \in \mathcal{I} \) implies the existence of \( M \in \mathbb{R}^{N \times N} \), with \( N = n_1 + \ldots + n_d \), such that \( MLv = Lv \) with \( M_{i,j} = 0 \) for every \( (i, j_i), (k, l_k) \in \mathcal{I} \) with \( (i, j_i) \neq (k, l_k) \) and \( M_{i,j}, (i, j_i) = 0 \) for every \( (i, j_i) \in \mathcal{I} \). In particular, \( M \) is invertible and thus \( Lv = v \). Hence, by assumption, there exists \( \beta \in \mathbb{R} \setminus \{0\} \) such that \( v = \beta \tilde{u} \). We obtain the contradiction \( 0 = \beta^{-1} (v, \phi) = \sum_{i=1}^{d} \tilde{b}_i = 1 \). Now, suppose that \( \langle b, \alpha \rangle \neq 0 \) and let \( \| \cdot \| \) be any monotonic norm on \( \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d} \). Note that \( A\alpha \neq 0 \) because it would imply the contradiction \( 0 = \langle A\alpha, b \rangle = \langle \alpha, A^\top b \rangle = \langle \alpha, b \rangle \). Let \( \nu \in \mathbb{N} \), with (23) and (24) we get

\[ (26) \quad L^\nu v - v = \sum_{k=0}^{\nu} L^k (L\nu v - v) = \sum_{k=0}^{\nu} L^k (\alpha \otimes u) = \sum_{k=0}^{\nu} (A^k \alpha) \otimes u. \]

On the one hand, as \( \tilde{u} > 0 \), there exists \( t > 0 \) with \( -t\tilde{u} \leq \nu \leq t\tilde{u} \). It follows that \( 0 \leq L^\nu v + t\tilde{u} \leq L^\nu v \leq 2t\tilde{u} \). Thus,

\[ (27) \quad \|L^\nu v\| \leq \|L^\nu v + t\tilde{u}\| + \|t\tilde{u}\| \leq 3t\|u\| \quad \forall \nu \in \mathbb{N}. \]

On the other hand, as \( A \) is irreducible, we know from Theorem 1.1 [26] that the sequence \( \frac{A^k}{k} \sum_{s=0}^{k} A^s \) converges towards \( \langle b, b \rangle^{-1} b b^\top \) as \( k \to \infty \). This implies that we have \( \lim_{\nu \to \infty} \| \sum_{k=0}^{\nu} (A^k \alpha) \otimes u \| = \infty \). A contradiction to (26) and (27).

We collect these results for the proof of Theorem 5.2.

**Proof of Theorem 5.2.** We have discussed that if \( DF(u) \) is irreducible, then the assumptions on \( DF(u) \) in Theorems 5.3 and 5.5 are satisfied. Hence, uniqueness of \( u \) follows from Theorem 3.1 if \( \rho(A) < 1 \) and Theorem 5.5 if \( \rho(A) = 1 \). Finally, Theorem 5.3 implies the second part of the claim.

6. **Convergence to the unique positive eigenvector.** We conclude the paper with a study of the convergence of the iterates of a mapping \( F \in H^d \) towards its unique positive eigenvector \( u \). Such analysis is particularly interesting in applications as it naturally induces an algorithm for the computation of \( u \) and \( r_u(F) \). For example, this allows to solve certain nonconvex optimization problems to global optimality [14], a
hard task in general, or can be used to efficiently identify important components in networks with multiple layers [25].

When $F$ is a strict contraction, convergence is a direct consequence of the Banach fixed point theorem, however when $F$ is non-expansive we need stronger assumptions on $F$. For example, if $F : \mathbb{R}_+^d \to \mathbb{R}_+^d$ is the linear mapping $F(x) = Mx$ where $M$ is the matrix representing the nontrivial permutation of two elements then, although $M$ is irreducible, the iterates of $F$ will never converge towards its eigenvector. For the case $d = 1$, it is proved in Theorem 2.3 [22] that the normalized iterates of a non-expansive mapping $F \in \mathcal{H}^1$ converge towards its positive eigenvector $u$ if $DF(u)$ is primitive. We prove in the following theorem that such a result can be extended for the case $d > 1$. Furthermore, taking inspiration from the study of nonnegative multilinear forms (see e.g. [3, 9, 13, 21]), we show that each of the iterates induce two monotonic sequences which are particularly useful for the estimation of the spectral radius. These results are summarized in the following:

**Theorem 6.1.** Let $F \in \mathcal{H}^d$, $A = A(F)$ and $b \in \Delta^d_{++}$. Suppose that $F$ has a positive eigenvector $u \in \mathbb{S}^{++}$ and define the sequence of normalized iterates given by $x^0 \in \mathbb{S}^{++}$ and

$$x^k = \left( \frac{F_1(x^{k-1})}{\|F_1(x^{k-1})\|_{\gamma_1}}, \ldots, \frac{F_d(x^{k-1})}{\|F_d(x^{k-1})\|_{\gamma_d}} \right) \quad \forall k = 1, 2, \ldots$$

Then, $\lim_{k \to \infty} x^k = u$ if either $\rho(A) < 1$ or $A^T b = b$, $F$ is differentiable at $u$ and $DF(u)$ is primitive. Furthermore, if $A^T b \leq b$, then

$$\bar{\alpha}_k \leq \bar{\alpha}_{k+1} \leq r_0(F) \leq \bar{\alpha}_{k+1} \leq \bar{\alpha}_k \quad \forall k = 0, 1, 2, \ldots$$

where $\bar{\alpha}_k = \prod_{i=1}^d m_i(F(x^k)/x^k)^{b_i}$, $\bar{\alpha}_k = \prod_{i=1}^d \mathfrak{m}_i(F(x^k)/x^k)^{b_i}$. Finally, if $A b < b$, then $\rho(A) < 1$ and the following bound on the convergence rate holds

$$\mu_b(x^k, u) \leq \rho(A)^k \frac{\mu_b(x^0, u)}{1 - \rho(A)} \quad \forall k \in \mathbb{N}.$$

**6.1. Convergence analysis.** First, we need the subsequent lemma which can be proved in the same way as Lemma 6.5.7 [19] dealing with the case $d = 1$.

**Lemma 6.2.** Let $F \in \mathcal{H}^d$ and $u \in \mathbb{K}^{++}$ with $F(u) = u$. If $F$ is differentiable at $u$ and $\nu$ is a positive integer such that $DF(u)^\nu$ has strictly positive entries, then $F^\nu(u) \leq F^\nu(x)$ for all $x \in \mathbb{K}^{++}$ with $u \leq x$.

We recall known results of fixed point theory: For $x \in \mathbb{K}^{++}$ and $F \in \mathcal{H}^d$, the orbit $O(F,x)$ of $x$ under $F$ is defined as $O(F,x) = \{ F^k(x) \mid k \in \mathbb{N} \}$. Furthermore, the $\omega$-limit set $\omega(F,x)$ of $x$ under $F$ is the set of accumulation points of $O(F,x)$. For $F \in \mathcal{H}^d$, Theorem 3.1.7 and Lemmas 3.1.2, 3.1.3 and 3.1.6 in [19] imply the following:

(I) If $F$ is non-expansive with respect to the weighted Thompson metric $\bar{p}_b$ on $\mathbb{K}^{++}$ and there exists $u \in \mathbb{K}^{++}$ such that $(F^k(u))_{k=1}^{\infty} \subset \mathbb{K}^{++}$ has a bounded subsequence, then $O(F,x)$ is bounded for each $x \in \mathbb{K}^{++}$.

(II) If $x \in \mathbb{K}^{++}$ is such that $O(F,x)$ has a compact closure, then $\omega(F,x)$ is a non-empty compact set and $F(\omega(F,x)) \subset \omega(F,x)$.

(III) If $x \in \mathbb{K}^{++}$ is such that $O(F,x)$ has a compact closure and $|\omega(F,x)| = p$, then there exists $z \in \mathbb{K}^{++}$ such that $\lim_{k \to \infty} F^{pk}(x) = z$ and $\omega(F,x) = O(F,z)$.

(IV) If $F$ is non-expansive with respect to $\bar{p}_b$, then for all $x \in \mathbb{K}^{++}$ and $y \in \omega(F,x)$, we have that $\omega(F,y) = \omega(F,x)$.
Property (I) is also known as Calka’s Theorem [6]. We are now ready to prove the following theorem which is a special case of Corollary 6.5.8 in [19] when $d = 1$.

**Theorem 6.3.** Let $F \in \mathcal{H}^d$, $x^0 \in S_++$ and $A = A(F)$. Suppose that $\rho(A) = 1$ and there exist $(\lambda, u) \in \mathbb{R}_{++}^d \times S_+$ such that $F(u) = \lambda u$. If $F$ is differentiable at $u$ and $DF(u)$ is primitive, then $u$ is the unique eigenvector of $F$ in $S_+$ and the sequence $(x^k)_{k=0}^\infty \subset S_+$ defined in Theorem 6.1 satisfies $\lim_{k \to \infty} x^k = u$.

**Proof.** First, note that the primitivity of $DF(u)$ implies that of $A$ by Lemma 2.5. Hence, by Theorem 5.5, $u$ is the unique positive eigenvector of $F$. Furthermore, there exist $b, b \in \Delta_{++}^d$ and $\nu \in \mathbb{N}$ such that $A^T b = b$, $Ab = b$ and $DF(u)^\nu > 0$. Now, let $\lambda \in \mathbb{R}_{++}^d$ with $F(u) = \lambda u$ and $\tilde{F} \in \mathcal{H}^d$ defined as $\tilde{F}(x) = \lambda^{-1} \otimes F(x)$. Then $A(\tilde{F}) = A$, $u$ is the unique eigenvector of $\tilde{F}$, $\tilde{F}$ is differentiable at $u$ and $D\tilde{F}(u)^\nu > 0$. We show that for every $x \in K_+$, there exists $\alpha \in \mathbb{R}_{++}^d$ such that $\omega(\tilde{F}, x) = \{\alpha \otimes u\}$. Let $x \in S_+^d$ and consider the sequence $\xi_k = \prod_{i=1}^d m_i(F^k(x)/u)^{b_i}$. Then, we have

$$
\prod_{i=1}^d m_i(x/u)^{b_i} \geq \xi_{k+1} \geq \prod_{i=1}^d m_i(F(m(F^k(x)/u) \otimes u)/u)^{b_i} = \xi_k
$$

which implies that the sequence $(\xi_k)_{k=1}^\infty$ converges towards some $\xi > 0$ as it is monotonic and bounded. In particular, it holds $\xi = \prod_{i=1}^d m_i(z/u)^{b_i}$ for every $z \in \omega(\tilde{F}, x)$. Now, by Lemma 3.2, we know that $\tilde{F}$ is non-expansive with respect to the weighted Thompson metric $\tilde{p}_\nu$ on $K_+$. Since $\tilde{F}(x) = u$, we have $\tilde{F}^k(x) = u$ for every $k \in \mathbb{N}$ and thus (I) implies that $O(\tilde{F}, x)$ is bounded. Now, let $\nu \in \mathbb{N}$ be such that $DF(u)^\nu > 0$. It follows from (II), that $\tilde{F}^\nu(\omega(\tilde{F}, x)) \subset \omega(\tilde{F}, x)$ and thus $\tilde{F}^\nu(z) \in \omega(\tilde{F}, x)$ for every $z \in \omega(\tilde{F}, x)$. Now, let $z \in \omega(\tilde{F}, x)$ and suppose by contradiction that $z \neq \beta \otimes u$ for every $\beta \in \Delta_+^d$. Then $m(z/u) \otimes u \leq K z$ and, with Lemma 6.2, we get $m(z/u)^{A^\nu} \otimes \tilde{F}^\nu(u) = \tilde{F}^\nu(m(z/u) \otimes u) < K \tilde{F}^\nu(z)$. Thus, with $\xi = \prod_{i=1}^d m_i(z/u)^{b_i}$ and $\tilde{F}^\nu(u) = u$, we obtain the contradiction

$$
\xi = \prod_{i=1}^d m_i(z/u)^{b_i} m_i(\tilde{F}^\nu(u)/u)^{b_i} < \prod_{i=1}^d m_i(\tilde{F}^\nu(z)/u)^{b_i} = \xi.
$$

Hence, there exists $\alpha \in \mathbb{R}_{++}^d$ such that $z = \alpha \otimes u$ and (IV) implies that $\omega(\tilde{F}, x) = \omega(\tilde{F}, \alpha \otimes u)$. As $A$ is primitive, we know from Theorem 1.1 [26] that it holds $\lim_{k \to \infty} A^k = B$ with $B = (\tilde{b}, \tilde{b}, \tilde{b}, \ldots)^T$. In particular, we have

$$
\lim_{k \to \infty} \tilde{F}^k(\alpha \otimes u) = \lim_{k \to \infty} \alpha A^k \otimes \tilde{F}^k(u) = \lim_{k \to \infty} \alpha A^k \otimes u = \xi B \otimes u.
$$

Hence, we have $\omega(\tilde{F}, x) = \omega(\tilde{F}, \alpha \otimes u) = \{\alpha B \otimes u\}$. So, $\lim_{k \to \infty} \tilde{F}^k(x) = \alpha B \otimes u$ follows from (III). To conclude the proof, note that for every $y \in K_+$ and $i \in [d]$ it holds $\|F_i(y)\|_{\gamma_i}^{-1} F_i(y) = \|F_i(y)\|_{\gamma_i}^{-1} F_i(y)$ and thus $\lim_{k \to \infty} x^k = u$. $
$

The following lemma generalizes Proposition 28 in [13]. It implies the monotonicity of the sequence $(\alpha_k)_{k=1}^\infty$ and $(\tilde{\alpha}_k)_{k=1}^\infty$.

**Lemma 6.4.** Let $F \in \mathcal{H}^d$ and $(\lambda, u) \in \mathbb{R}_{++}^d \times S_+$ be such that $F(u) = \lambda u$. Let $b \in \Delta_{++}^d$ with $A^T b \leq b$, consider the mapping $\tilde{G} : S_+^d \to S_+$ defined as $\tilde{G}(x) = (\|F_1(x)\|_{\gamma_1}, \ldots, \|F_d(x)\|_{\gamma_d})^{-1} \otimes F$ and let $\tilde{c}_b$, $\tilde{c}_b$ be as in (18). Then, for every $x \in S_+$, it holds $\tilde{c}_b(F, x) \leq \tilde{c}_b(F, \tilde{G}(x)) \leq r_b(F) \leq \tilde{c}_b(F, \tilde{G}(x)) \leq \tilde{c}_b(F, x)$. 

Proof. Let $x \in S_{++}$, then $m(\tilde{G}(x)/x) \leq 1$ because $\tilde{G}(x) \in S_{++}$. Thus, with $s = b - A^T b \in \mathbb{R}^d_+$, we have $1 \leq \prod_{i=1}^d m_i(\tilde{G}(x)/x)^{-s_i}$. It follows that

$$
\tilde{c}_{w_b}(F, \tilde{G}(x)) \geq \prod_{i=1}^d \|F(x)\|_{s_i} m_i(F(x)/x) \geq \tilde{c}_{w_b}(F, x).
$$

The inequality $\tilde{c}_{w_b}(F, \tilde{G}(x)) \leq \tilde{c}_{w_b}(F, x)$ can be proved in the same way by swapping the inequalities and exchanging the roles of $m$ and $2 \mathbf{1}$. The end of the proof follows from Theorem 5.1.

Proof of Theorem 6.1. Let $\tilde{G}$ be defined as in Lemma 6.4. If $\rho(A) < 1$, then by the proof of Theorem 3.1, $\tilde{G}$ is a strict contraction with respect to $\mu_b$. In particular, $\lim_{k \to \infty} x^k = u$ and the linear convergence rate follows from the Banach fixed point theorem (see Theorem 3.1 [18]). If $\rho(A) = 1$, then $\lim_{k \to \infty} x^k = u$ follows from Theorem 6.3. Finally, if $\tilde{c}_{w_b}, \tilde{c}_{w_b}$ are defined as in Section 5, then $\tilde{\alpha}_k = \tilde{c}_{w_b}(F(x^k), x^k)$ and $\tilde{\alpha}_k = \tilde{c}_{w_b}(F(x^k), x^k)$. Hence, the monotonicity of these sequences follows form Lemma 6.4 and $\lim_{k \to \infty} \tilde{\alpha}_k = \lim_{k \to \infty} \tilde{\alpha}_k = r_b(F)$ follows from the continuity of $\tilde{c}_{w_b}, \tilde{c}_{w_b}$.

Acknowledgments. The authors are grateful to Shmuel Friedland and Lek-Heng Lim for a number of insightful discussions and for pointing out relevant references. This work has been funded by the ERC starting grant NOLEPRO 307793. The work of F.T. has been also partially funded by the MSC individual fellowship MAGNET 744014.

REFERENCES

[1] M. Akian, S. Gaubert, and A. Guterman, Tropical polyhedra are equivalent to mean payoff games, Internat. J. Algebra Comput., 22 (2012), pp. 125001, 43.
[2] F. F. Bonsall, Linear operators in complete positive cones, Proc. Lond. Math. Soc., 8 (1958), pp. 53–75.
[3] D. W. Boyd, The power method for $\ell^p$ norms, Linear Algebra Appl., 9 (1974), pp. 95–101.
[4] D. D. Burbanks, R. D. Nussbaum, and T. Sparrow, Extension of order-preserving maps on a cone, Proc. Roy. Soc. Edinburgh Sect. A, 133 (2003), pp. 35–59.
[5] P. J. Bushell, Hilbert’s metric and positive contraction mappings in a Banach space, Arch. Rat. Mech. Anal., 52 (1973), pp. 330–338.
[6] A. Calka, On conditions under which isometries have bounded orbits, Colloq. Math., 48 (1984), pp. 219–227.
[7] V. Calvez, P. Gabriel, and S. Gaubert, Non-linear eigenvalue problems arising from growth maximization of positive linear dynamical systems, in 53rd IEEE CDC, 2014.
[8] K. C. Chang, K. J. Pearson, and T. Zhang, Some variational principles for $Z$-eigenvalues of nonnegative tensors, Linear Algebra Appl., 438 (2013), pp. 4166–4182.
[9] K. C. Chang, L. Qi, and G. Zhou, Singular values of a real rectangular tensor, J. Math. Anal. Appl., 370 (2010), pp. 284–294.
[10] S. Friedland, S. Gaubert, and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, Linear Algebra Appl., 438 (2013), pp. 738–749.
[11] S. Gaubert and J. Gunawardena, The Perron-Frobenius theorem for homogeneous, monotone functions, Trans. Amer. Math. Soc, 356 (2004), pp. 4931–4950.
[12] S. Gaubert and G. Vigeral, A maximin characterisation of the escape rate of non-expansive mappings in metrically convex spaces, Math. Proc. Cambridge Philos. Soc., 152 (2012), pp. 341–363.

[13] A. Gautier and M. Hein, Tensor norm and maximal singular vectors of nonnegative tensors – A Perron-Frobenius theorem, a Collatz-Wielandt characterization and a generalized power method, Linear Algebra Appl., 505 (2016), pp. 313–343.

[14] A. Gautier, Q. Nguyen, and M. Hein, Globally Optimal Training of Generalized Polynomial Neural Networks with Nonlinear Spectral Methods, in NIPS, 2016.

[15] A. Gautier, F. Tudisco, and M. Hein, A unifying Perron-Frobenius theorem for nonnegative tensors via multi-homogeneous maps, Submitted.

[16] I. Gelfand, Zur Theorie der Charaktere der abelschen topologischen Gruppen, Sb. Math., 9 (1941), pp. 49–50.

[17] R. B. Kellogg, T. Y. Li, and J. Yorke, A Constructive Proof of the Brouwer Fixed-Point Theorem and Computational Results, SIAM J. Numer. Anal., 13 (1976), pp. 473–483.

[18] M. A. Khamsi and W. A. Kirk, An Introduction to Metric Spaces and Fixed Point Theory, John Wiley & Sons, 2001.

[19] B. Lemmens and R. D. Nussbaum, Nonlinear Perron-Frobenius theory, Cambridge University Press, general ed., 2012.

[20] J. Mallet-Paret and R. D. Nussbaum, Eigenvalues for a class of homogeneous cone maps arising from max-plus operators, Discrete Contin. Dyn. Syst., 8 (2002), pp. 519–562.

[21] M. Ng, L. Qi, and G. Zhou, Finding the largest eigenvalue of a nonnegative tensor, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 1090–1099.

[22] R. D. Nussbaum, Hilbert’s projective metric and iterated nonlinear maps, Mem. Amer. Math. Soc., 75 (1988).

[23] A. Papadopoulos and M. Troyanov, Handbook of Hilbert Geometry, vol. 22, IRMA Lectures in Mathematics and Theoretical Physics, 2014.

[24] S. U. Pillai, T. Suel, and C. Seunghun, The Perron-Frobenius theorem: some of its applications, IEEE Signal Process. Mag., 22 (2005), pp. 62–75.

[25] F. Tudisco, F. Arrigo, and A. Gautier, Node and layer eigenvector centralities for multiplex networks, To appear in SIAM J. Appl. Math., (2018).

[26] F. Tudisco, V. Cardinali, and C. Fiore, On complex power nonnegative matrices, Linear Algebra Appl., 471 (2015), pp. 449–468.

[27] Q. Yang and Y. Yang, Further results for the Perron-Frobenius theorem for nonnegative tensors II, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 1236–1250.

[28] Y. Yang and Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2517–2530.