A DECOMPOSITION FORMULA FOR J-STABILITY AND ITS APPLICATIONS

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ABSTRACT. For algebro-geometric study of J-stability, a variant of K-stability, we prove a decomposition formula of non-archimedean $J$-energy of $n$-dimensional varieties into $n$-dimensional intersection numbers rather than $(n + 1)$-dimensional ones, and show the equivalence of slope $J^H$-(semi)stability and $J^H$-(semi)stability for surfaces when $H$ is pseudoeffective. Among other applications, we also give a purely algebro-geometric proof of a uniform K-stability of minimal surfaces due to [23], and provides examples which are J-stable (resp., K-stable) but not uniformly J-stable (resp., uniformly K-stable).

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1. Introduction

Let $(X, L)$ be a polarized complex manifold. It is conjectured that the K-polystability of $(X, L)$ is equivalent to the existence of a constant scalar curvature Kähler (cscK for short) metric in $c_1(L)$ (the Yau-Tian-Donaldson conjecture). $(X, L)$ is K-stable if the non-Archimedean Mabuchi functional $M^{NA} > 0$ for any non-almost trivial ample test configuration over $(X, L)$. We can decompose $M^{NA}$ into the non-Archimedean entropy and the $(J^K)^{NA}$-energy as non-Archimedean functional (cf. [5]). Chi Li [30] proved that if $(J^K)^{NA}$-energy is non-negative, the Mabuchi functional in differential geometry is coercive and hence $(X, L)$ has a cscK metric. Thus $(J^H)^{NA}$-energy plays an important role in studying K-stability. On the other hand, Simon K. Donaldson [15] introduced J-flow and so-called J-equation (see Fact D). Donaldson proposed the question when there is a solution to J-equation, i.e. there is a stationary solution of J-flow. Xiuxiong Chen [7]
defined independently J-flow, which he called J-equation, and pointed out the solvability of this equation implies the coerciveness of Mabuchi functional of Kähler surfaces with ample canonical classes. This question is studied in [45], [29], [9], [22], [20], [6], [10], [44]. Moreover, Mehdi Lejmi and Gábor Székelyhidi [29] conjectured that the solvability of J-equation to $J^H$-stability (i.e. $(J^H)^{NA}(\mathcal{X}, \mathcal{L}) > 0$ for any non-almost trivial ample test configuration $(\mathcal{X}, \mathcal{L})$ over $(X, L)$, the notion of J-stability is introduced in [29], [20] and to another algebro-geometric condition. The conjecture is proved in [6], [10] and [44].

We prove that if $X$ is an integral surface and $H$ is pseudoeffective, $(X, L)$ is $J^H$-semistable iff 
\[
\frac{2L \cdot H}{L^2} L - H
\]
is nef. Then $(X, L)$ is $J^H$-semistable.

On the other hand, examples in [11] show that K-stability may not ensure the existence of a cscK metric and G. Székelyhidi [46] proposed that uniform notion of K-stability should be equivalent to the existence of a cscK metric. Ruadhaí Dervan [11] and Boucksom-Hisamoto-Jonsson [5] define the uniform K-stability independently. In [5], $(X, L)$ is uniformly K-stable iff there exists a positive real number $\delta$ such that $M^{NA}(\mathcal{X}, \mathcal{L}) \geq \delta J^{NA}(\mathcal{X}, \mathcal{L})$ for any semiample test configuration $(\mathcal{X}, \mathcal{L})$ over $(X, L)$. This definition is equivalent to the definition of uniform K-stability in [11]. S. K. Donaldson [16] extend the definition of K-stability to log pairs. There is a folklore conjecture that the uniform K-stability is equivalent to the K-stability for log pairs. There is a partial consequence: a log Fano pair $(X, B, -K(X, B))$ is K-stable if and only if it is uniformly K-stable due to [33] (for smooth Fano manifolds, cf. [8], [47] and [3]).

The notations are as in §2. In this paper, our main results on possibly singular surfaces are the followings with a purely algebro-geometric proof:

**Theorem A** (Theorem 6.3). Let $(X, L)$ be a polarized integral surface and $H$ be a pseudoeffective $\mathbb{Q}$-Cartier divisor such that 
\[
\frac{2L \cdot H}{L^2} L - H
\]
is nef. Then $(X, L)$ is $J^H$-semistable.

Theorem A tells us that the ampleness of $\frac{2L \cdot H}{L^2} L - H$ is equivalent to uniform $J^H$-stability if $H$ is big. To be precise, we have the following:

**Corollary B** (Corollary 6.4 Corollary 6.13 cf. Theorem 2 of [7], Theorem 1.1 of [6]). For any polarized deminormal surface $(X, L)$ with a big (resp. pseudoeffective) $\mathbb{Q}$-line bundle $H$ such that for any irreducible components $X_i$, \[
\frac{L \cdot H}{(L \cdot X_i)^2} = \frac{L \cdot H}{L^2},
\]
then the following are equivalent.

1. $(X, L)$ is uniformly $J^H$-stable (resp. $J^H$-semistable). In other words, there exists $\epsilon > 0$ such that for any semiample test configuration $(\mathcal{X}, \mathcal{L})$
\[
(J^H)^{NA}(\mathcal{X}, \mathcal{L}) \geq \epsilon J^{NA}(\mathcal{X}, \mathcal{L}) \quad (\text{resp.} \geq 0).
\]

2. $(X, L)$ is uniformly slope $J^H$-stable (resp. slope $J^H$ semistable). In other words, there exists $\epsilon > 0$ such that for any semiample deformation to the normal cone $(\mathcal{X}, \mathcal{L})$ along integral curve
\[
(J^H)^{NA}(\mathcal{X}, \mathcal{L}) \geq \epsilon J^{NA}(\mathcal{X}, \mathcal{L}) \quad (\text{resp.} \geq 0).
\]

3. There exists $\epsilon > 0$ such that for any integral curve $C$, 
\[
\left(2 \frac{H \cdot L}{L^2} L - H\right) \cdot C \geq \epsilon L \cdot C \quad (\text{resp.} \geq 0).
\]
X.X. Chen [7] proves this for Kähler surfaces $X$ when $H = K_X$ is ample. We emphasize that we show that Corollary [3] holds not only when $H$ is ample but big for singular surfaces.

The assumption that for any irreducible components $X_i$, $\frac{L_{|X_i} \cdot H|_{X_i}}{(L_{|X_i})^n} = \frac{L \cdot H}{L^n}$ is necessary. Indeed, if the assumption does not hold, $X$ is $J^H$-unstable as the following generalization of Theorem 7.1 of [32] shows.

**Theorem C** (Theorem 6.6). Let $(X, \Delta; L)$ be an $n$-dimensional polarized deminormal pair with a $\mathbb{Q}$-line bundle $H$ such that $X = \bigcup_{i=1}^l X_i$ be the irreducible decomposition. Let also $L_i = L|_{X_i}$ and $H_i = H|_{X_i}$.

Suppose that

$$\frac{H \cdot L_{i}^{n-1}}{L_{i}^{n}} \neq \frac{H_i \cdot L_{i}^{n-1}}{L_{i}^{n}}$$

for some $1 \leq i \leq l$. Then $(X, L)$ is $J^H$-unstable.

Furthermore, let $\nu : \bigcup_{i=1}^l X_i \to X$ be the normalization, $\overline{L_i} = L|_{X_i}$, $\overline{H_i} = H|_{X_i}$ and $K(\overline{X}, \Delta) = \nu^*(K(X, \Delta))|_{\overline{X}}$. Suppose that

$$\frac{K(\overline{X}, \Delta) \cdot L_{i}^{n-1}}{L_{i}^{n}} \neq \frac{K(\overline{X}, \Delta) \cdot \overline{L_i}^{n-1}}{\overline{L_i}^{n}}$$

for some $1 \leq i \leq l$. Then $(X, \Delta; L)$ is $K$-unstable.

Compare Corollary [3] with the following, which is originally conjectured by Lejmi and Székelyhidi. Gao Chen [6] proved the uniform version of this conjecture, Datar-Pingali [10] proved for projective manifolds and Jian Song [44] proved generally:

**Fact D** (Theorem 1.1 of [6], Theorem 1.2 of [10], Corollary 1.2 of [44]). Notations as in [6]. Given an $n$-dimensional Kähler manifold $M$. Let $\chi$ and $\omega_0$ be Kähler metrics on $M$ and $c_0 > 0$ be the constant such that

$$\int_M \chi \wedge \frac{\omega_0^{n-1}}{(n-1)!} = c_0 \int_M \frac{\omega_0^n}{n!}.$$

Then the followings are equivalent:

1. There exists a smooth function $\varphi$ such that
   $$\omega_\varphi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0$$
   satisfies the $J$-equation
   $$\text{tr}_{\omega_\varphi} \chi = c_0.$$
   Moreover, such $\varphi$ is unique up to a constant;
2. There exists a smooth function $\varphi$ such that $\varphi$ is the critical point of the $J_\chi$ functional. Moreover, such $\varphi$ is unique up to a constant;
3. The $J_\chi$ functional is coercive. In other words, there exist a positive constant $\epsilon$ and another constant $C$ such that $J_\chi(\varphi) \geq \epsilon J_{\omega_0}(\varphi) - C$;
4. $(M, [\omega_0], [\chi])$ is uniformly $J$-stable. In other words, there exists a positive constant $\epsilon$ such that for any Kähler test configuration $(X, \Omega)$ (cf. [13] Definition 2.10), [44] Definitions 3.2 and 3.4), the invariant $J_{[\chi]}(X, \Omega)$ (cf. [13] Definition 6.3) satisfies $J_{[\chi]}(X, \Omega) \geq \epsilon J_{[\omega_0]}(X, \Omega)$;
5. $(M, [\omega_0], [\chi])$ is uniformly slope $J$-stable. In other words, there exists a positive constant $\epsilon$ such that for any subvariety $V$ of $M$, the deformation to normal cone of $V (X, \Omega)$ (cf. [13] Example 2.11 (ii)) satisfies $J_{[\chi]}(X, \Omega) \geq \epsilon J_{[\omega_0]}(X, \Omega)$;
There exists a positive constant $\epsilon$ such that
\[
\int_V (c_0 - (n - p)\epsilon)\omega_0^p - p\chi \land \omega_0^{p-1} \geq 0
\]
for any $p$-dimensional subvariety $V$ with $p = 1, 2, \cdots, n-1$.

(7) \[
\int_V c_0\omega_0^p - p\chi \land \omega_0^{p-1} > 0
\]
for any $p$-dimensional subvariety $V$ with $p = 1, 2, \cdots, n-1$.

Recall that for K-stability case, slope stability was strictly weaker (cf. [39, Example 7.6]). On the other hand, the novelty of Fact D and Corollary B is to show the validity of slope type stability for J-stability case, in the spirit of Ross-Thom [40]. Our purely algebro-geometric proof of Theorem A is inspired from Fact D and from [40, Theorems 6.1 and 6.4]. We remark that the assumption of Theorem A is weaker than one of Fact D for smooth projective surfaces.

On the other hand, (3) $\Rightarrow$ (2) in Corollary B is not so trivial since \((2H \cdot L - H) \cdot C\) is not a slope \((J^H)^{NA}\)-energy of the deformation to the normal cone of $C$ as we see in Example 3.3. Indeed, we have the following:

**Theorem E** (Theorem 7.3, Proposition 7.4). There exists a smooth polarized surface \((X, L)\) with an ample divisor $H$ such that \((X, L)\) is $J^H$-stable but not uniformly slope $J^H$-stable. In particular, there does not necessarily exists a solution to $J$-equation even when \((X, L)\) is $J^H$-stable.

J. Song [44] calls the conditions (6) and (7) of Fact D uniform J-positivity and J-positivity respectively (cf. [44, Definition 1.1]), and actually proves that J-positivity and uniform J-positivity are equivalent to uniform J-stability. However, we prove J-positivity is not equivalent to J-stability.

Although K-stability and uniform K-stability are equivalent in the case of Fano manifolds, we show that the equivalence does not hold in general. In fact, we prove the following:

**Corollary F** (Corollary 7.5, Corollary 7.7). There exist a polarized normal pair \((X, \Delta; L)\) and a polarized deminormal surface \((Z, M)\) such that they are K-stable but not uniformly K-stable.

On the other hand, we extend (4) $\Leftrightarrow$ (6) in Fact D to higher dimensional deminormal schemes over $\mathbb{C}$ (see Theorem 8.12 and Remark 8.14). We apply this to obtain the following extension of Jian-Shi-Song [23, Theorem 1.1]:

**Theorem G** (Theorem 8.15). Let \((X, \Delta; L)\) be an $n$-dimensional klt log minimal model over $\mathbb{C}$, i.e., $K_{(X, \Delta)}$ is nef. Then \((X, \Delta; K_{(X, \Delta)} + \epsilon L)\) is uniformly K-stable for sufficiently small $\epsilon > 0$. Furthermore, if $X$ is smooth and $\Delta = 0$, \((X, K_X + \epsilon L)\) also has a cscK metric.

The existence of cscK metrics of smooth minimal models was already proved by Zakarias Sjöström Dyrefelt [43] and J. Song [44] but our proof is more algebro-geometric. In particular when $n = 2$, we have a purely algebro-geometric proof of Theorem G.

The point of the proof of Theorem A is the following:
Theorem H (Theorem 5.1). Given an $n$-dimensional polarized variety $(X, L)$, and an flag ideal $a = \sum_{i=0}^{r} a_i t^i \subset O_{X \times \mathbb{A}^1}$ such that $a_0 \neq 0$. Then there exists an alternation $\pi : X' \to X$ (i.e. $\pi$ is a generically finite and proper morphism) such that $X'$ is smooth and irreducible, $D_0$ is an snc divisor corresponding to $\pi^{-1}a_0$ and the integral closure $(\pi \times \text{id}_{\mathbb{A}^1})^{-1}a$ of the inverse image of $a$ to $X' \times \mathbb{A}^1$ satisfies the following condition (*)

\[
(\pi \times \text{id}_{\mathbb{A}^1})^{-1}a = \mathcal{I}_{D_0} + \mathcal{I}_{D_1} t + \cdots + \mathcal{I}_{D_{r-1}} t^{r-1} + t^r,
\]

where each $\mathcal{I}_{D_i}$ is a coherent ideal sheaf corresponding to a Cartier divisor $D_i$ of $X'$. Furthermore, for each $m \in \mathbb{Z}_{>0}$,

\[
\left( (\pi \times \text{id}_{\mathbb{A}^1})^{-1}a \right)^m = \sum_{k=0}^{mr} t^k \mathcal{J}_{m,k},
\]

where $\mathcal{J}_{m,k} = \mathcal{J}_{D_j}^{m-i} \cdot \mathcal{J}_{D_{j+1}}^i$ for $j = \lfloor \frac{k}{m} \rfloor$ and $i = k - mj$.

Moreover, if $(\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - E)$ is a semiample test configuration, where $\Pi : \text{Bl}_a(X \times \mathbb{A}^1) \to X \times \mathbb{A}^1$ is the blow up along $a$ with the exceptional divisor $E = \Pi^{-1}(a)$, $(\pi \times \text{id}_{\mathbb{A}^1})^{-1}a \cdot (\pi \times \text{id}_{\mathbb{A}^1})^{*}L_{\mathbb{A}^1}$ is semiample and $\pi^*L - D_0$ is nef.

This theorem is the technical heart of this note. In the above condition (*), we remark that

\[
\left( (\pi \times \text{id}_{\mathbb{A}^1})^{-1}a \right)^m \subset \sum_{k=0}^{\infty} t^k \mathcal{J}_{m,k}
\]

is not trivial. Due to Theorem H we have the following decomposition formula:

Theorem I (Theorem 3.8 Theorem 5.9). $(X, L)$, $a$ and $\pi$ are as in Theorem H. Suppose that $(\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - E)$ is a semiample test configuration, $n \geq 2$ and $\deg \pi = l$. Then we can calculate $(\mathcal{J}^H)^{\text{NA}}(\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - E)$ by using the mixed multiplicities of $\mathcal{I}_{D_k}$ (see Definition 3.7):

\[
L^n (\mathcal{J}^H)^{\text{NA}}(\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - E) = \frac{1}{l} \left( \frac{n}{n+1} H \cdot L^{n-1} \sum_{k=0}^{r-1} \sum_{j=0}^{n} e_{\pi^*L}(\mathcal{J}_{D_k}^j, \mathcal{J}_{D_{k+1}}^{n-j}) - \sum_{k=0}^{r-1} \sum_{j=0}^{n} e_{\pi^*L}(\mathcal{J}_{D_k}^j, \mathcal{J}_{D_{k+1}}^{n-j}) \right)
\]

\[
- \sum_{k=0}^{r-1} \sum_{j=0}^{n} (\pi^*L - D_k)^j \cdot (\pi^*L - D_{k+1})^{n-j}.
\]

Due to Theorem H we can calculate $(\mathcal{J}^H)^{\text{NA}}$-energy by using the intersection numbers of $X$ itself rather than of its test configuration.

Outline of this paper. In §2 we prepare many terminology and facts about K-stability and J-stability. From §3 we state our original result.

In §4 we state the definition of the mixed multiplicities and prove the modification of the consequences of Mumford [35]. Furthermore, we prove Theorem H when $\pi$ is an isomorphism.
In §4, we prepare many terminology about toroidal embeddings in [24] and introduce the notion of Newton polyhedron. Moreover, we explain about our proof of Theorems A and I briefly in Example 4.8.

In §5, we restate Theorems A and I and prove them by using the consequences of §3 and §4. We remark that Theorem I follows from Theorem H.

In §6, we study the J-stability of surfaces. In §6.1, we study irreducible surfaces and check that the mixed multiplicities are non-negative when $L$ and $H$ satisfy some conditions. Next, we apply Theorem I to obtain Theorem A and Corollary B for irreducible and normal surfaces.

In §6.2, we discuss about J-stability for reducible surfaces and construct a deminormal surface on which Hodge index theorem does not hold. Furthermore, we prove Theorem C and Corollary B completely.

In §7, we construct a smooth polarized surface $(X, L)$ and its ample divisor $H$ such that $(X, L)$ is slope $J^H$-stable but not uniformly $J^H$-stable. Then we apply Theorem I and prove that $(X, L)$ is also $J^H$-stable. Finally, Corollary C follows immediately from Theorem I.

In §8, we extend the consequences of Hashimoto-Keller [20, Theorem 3] and of Kento Fujita [17, Theorem 6.5]. On the other hand, we extend Lejmi-Székelyhidi conjecture (see Theorem 8.12) to non-smooth varieties over $\mathbb{C}$ and discuss on the application of J-stability for higher dimensional varieties. We apply the criteria to obtain the extension of the notion of stability threshold of Sjöström Dyrefelt [42] to deminormal schemes and to prove Theorem C.

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2. Notations

Unless otherwise stated, we work over an algebraically closed field $k$ of characteristic 0. We follow the notations used in [5]. If $V$ is a scheme, we assume $V$ to be an algebraic scheme (i.e. a scheme of finite type over $k$) that might not be proper. $V$ is a variety if $V$ is an irreducible and reduced algebraic scheme. On the other hand, $(X, L)$ be an $n$-dimensional polarized scheme if $X$ is an algebraic scheme and $L$ is an ample $\mathbb{Q}$-line bundle on $X$. Furthermore, we assume that $X$ is proper, reduced and equidimensional. Furthermore, $X$ is deminormal if $X$ satisfies that Serre’s condition $S_2$ and codimension 1 points of $X$ are either regular points or nodes (cf. Definition 5.1 of [26]). A flat and proper family $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ is a (semi)ample test configuration over $(X, L)$ if $\pi$ is $\mathbb{G}_m$-equivariant when $\mathbb{G}_m$ acts on $\mathbb{A}^1$ canonically, $\mathcal{L}$ is a $\mathbb{G}_m$-linearized (semi)ample $\mathbb{Q}$-line bundle, and there exists a $\mathbb{G}_m$-equivariant isomorphism

$$(\mathcal{X}, \mathcal{L}) \times_{\mathbb{A}^1} (\mathbb{A}^1 - \{0\}) \cong (X \times (\mathbb{A}^1 - \{0\}), L \otimes \mathcal{O}_{\mathbb{A}^1 - \{0\}})$$

(cf. Definitions 2.1, 2.2 of [5]). Denote the trivial test configuration by $(X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$. 
Definition 2.1. Let $B$ be a boundary of $X$ if $X$ is deminormal. In other words, there exists $l \in \mathbb{Z}_{\geq 0}$ and a finite number of $B = \sum_{i=1}^{l} a_i D_i$ for $0 \leq a_i \leq 1$ such that $K_{(X,B)} = K_X + B$ is $\mathbb{Q}$-Cartier, where $D_i$'s are integral closed subschemes of codimension 1. Suppose that $(\mathcal{X}, \mathcal{L})$ dominates $(X_{A1}, L_{A1})$. In other words, there exists a $\mathbb{G}_m$-equivariant morphism $\rho : \mathcal{X} \rightarrow X_{A1}$. Let $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ be the canonical compactification over $\mathbb{P}^l$ (cf. Definition 2.7 of loc.cit). Let also $p : \overline{\mathcal{X}} \rightarrow X$ be the canonical projection and $\overline{\mathcal{B}}$ be the closure of $B \times (\mathbb{P}^l \setminus \{0\})$ in $\overline{\mathcal{X}}$.

$$V(L) = L^n, \quad S(X, B; L) = -nV(L)^{-1}(K_{(X,B)} \cdot L^{n-1}).$$

If there is no confusion, we denote $L^n = L^n$. Then, we define as follows:

- $E^{NA}(\mathcal{X}, \mathcal{L}) = V(L)^{-1}(\overline{\mathcal{L}}^{n+1})_{(n+1)}$;
- $R_B^{NA}(\mathcal{X}, \mathcal{L}) = V(L)^{-1}(p^n K_{(X,B)} \cdot \overline{\mathcal{L}}^n)$;
- $I^{NA}(\mathcal{X}, \mathcal{L}) = V(L)^{-1}(\overline{\mathcal{L}} \cdot (p^n L)^n - (\overline{\mathcal{L}} - p^n L) \cdot \overline{\mathcal{L}}^{n+1})$;
- $J^{NA}(\mathcal{X}, \mathcal{L}) = V(L)^{-1}(\overline{\mathcal{L}} \cdot (p^n L)^n - E^{NA}(\mathcal{X}, \mathcal{L}))$;
- $\delta J^{NA}(\mathcal{X}, \mathcal{L}) = V(L)^{-1}(p^n H^\ast \cdot \overline{\mathcal{L}}^n - nH^\ast \cdot \overline{\mathcal{L}}^{n+1})$.

where $H$ is a $\mathbb{Q}$-line bundle on $X$. If there is no confusion, we denote $p^n L = \rho^\ast L_{\mathbb{P}^l}$ by $L_{\mathbb{P}^l}$.

On the other hand, we can see the functionals above are invariant under taking a pullback. In other words, if $\mu : (\mathcal{X}', \mathcal{L}') \rightarrow (\mathcal{X}, \mathcal{L})$ is a $\mathbb{G}_m$-equivariant morphism of test configurations and $\mathcal{L}' = \mu^\ast \mathcal{L}$, the functionals take the same value on them. For example, we have $\overline{\mathcal{L}}_{n+1} = \overline{\mathcal{L}}_{n+1}$. Therefore, we can introduce the equivalence relation generated by pulling back and let $\mathcal{H}^{NA}(L)$ be the set consists of all the equivalence class of semiample test configurations. We call $\mathcal{H}^{NA}(L)$ the set of all non-Archimedean positive metrics on $L$ (cf. Definition 6.2 of loc.cit). According to the fact we will explain after Definition 2.7 that for any metric $\phi \in \mathcal{H}^{NA}(L)$, we can necessarily take a semialgebraic test configuration $(\mathcal{X}', \mathcal{L}')$ that is a representative of $\phi$ and satisfies that $\mathcal{X}'$ dominates $X_{A1}$, we can consider $E^{NA}, R_B^{NA}, I^{NA}, J^{NA}$ and $(\delta J)^{NA}$ as well-defined functionals of $\mathcal{H}^{NA}(L)$.

Definition 2.2 (J-stability). The polarized pair $(X, B; L)$ is $J^H$-semistable (resp. $J^H$-stable, uniformly $J^H$-stable) if $(\delta J)^{NA} \geq 0$ (resp. $(\delta J)^{NA} > 0$, $(\delta J)^{NA} \geq \delta J^{NA}$ for some $\delta > 0$) on nontrivial metrics of $\mathcal{H}^{NA}(L)$. If there is no confusion, we say that J-semistable (resp. J-stable, uniformly J-stable).

Definition 2.3 (K-stability for deminormal pairs). If $X$ is deminormal and $K_{(X,B)}$ is $\mathbb{Q}$-Cartier, we can define

$$K_{(\overline{\mathcal{X}}, \overline{\mathcal{L}})/\mathbb{P}^l} = K_{(\overline{\mathcal{X}}, \overline{\mathcal{L}})} - (X_0 - X_{0,\text{red}})$$

for any deminormal semiample test configuration $(\mathcal{X}, \mathcal{L})$ that dominates $X_{A1}$ (since $\mathcal{X}$ is Gorenstein of codimension 1 and $S_2$. see also [17, §3]). Therefore we define the non-Archimedean entropy functional as

$$H_B^{NA}(\mathcal{X}, \mathcal{L}) = V(L)^{-1}(K_{(\overline{\mathcal{X}}, \overline{\mathcal{L}})/\mathbb{P}^l} \cdot \overline{\mathcal{L}}^{n+1}) - R_B^{NA}(\mathcal{X}, \mathcal{L})$$

For any non-Archimedean metric $\phi$ and its representative $(\mathcal{X}, \mathcal{L})$, take the partial normalization $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ of $(\mathcal{X}, \mathcal{L})$ (cf. [37, Definition 3.7, Proposition 3.8]). Note that $\overline{\mathcal{X}}$ is a deminormal test configuration by [17, Proposition 3.3] and define

$$H_B^{NA}(\phi) := H_B^{NA}(\overline{\mathcal{X}}, \overline{\mathcal{L}}).$$

Then, we also define the non-Archimedean Mabuchi functional as

$$M_B^{NA} = H_B^{NA} + R_B^{NA} + S(X, B; L)E^{NA}.$$
The polarized pair \((X, B; L)\) is K-semistable (resp. K-stable, uniformly K-stable) if \(M^\text{NA} \geq 0\) (resp. \(M^\text{NA} > 0\), \(M^\text{NA}_B \geq \delta J^\text{NA}\) for some \(\delta > 0\)) on \(\mathcal{H}^\text{NA}(L)\).

**Remark 2.4.** As usual, to define K-stability for general polarized schemes, we use the Donaldson-Futaki invariant (cf. [5, Definition 3.3, Proposition 3.12]). Denote the Donaldson-Futaki invariant for log polarized pairs \((X, B; L)\) by \(DF_B\). Due to [5, Remark 3.19, §7.3, Proposition 8.2], \(DF_B \geq 0\) (resp. \(> 0\), \(\geq \delta J^\text{NA}\) for some \(\delta > 0\)) iff \(M^\text{NA}_B \geq 0\) (resp. \(> 0\), \(\geq \delta J^\text{NA}\)), and hence we can define K-stability by using non-Archimedean Mabuchi functional.

**Definition 2.5** (Log discrepancy cf. §1.5 [5]). For any divisorial valuation \(v\) on a normal pair \((X, B; L)\), we can take a proper birational morphism \(\mu : Y \to X\) of normal varieties such that there exists a prime divisor \(F\) of \(Y\) such that \(v = c \text{ord}_F\). We define the log discrepancy of \(v\) as

\[ A_{(X,B)}(v) = c(1 + \text{ord}_F(K_Y - K_{(X,B)})). \]

This is independent of the choice of \(\mu : Y \to X\). Moreover, by Theorem 4.6 and Corollary 7.18 of [5],

\[ H^\text{NA}_B(X, \mathcal{L}) = V(L)^{-1} \sum_E b_E A_{(X,B)}(v_E)(E \cdot \mathcal{L}^n), \]

where \(E\) runs over the irreducible components of \(\mathcal{X}_0\), \(b_E = \text{ord}_E(\mathcal{X}_0) > 0\) and \(v_E\) is the restriction of \(b_E^{-1}\text{ord}_E\) to \(X\). If \(E_0\) is the trivial divisor i.e. \(E_0\) is the strict transformation of \(X \times \{0\}\), \(v_{E_0} = 0\) and we define \(A_{(X,B)}(v_{E_0}) = 0\).

**Definition 2.6** (Conductor subscheme). Let \((X, B; L)\) be a deminormal polarized pair and \(\nu : (\mathcal{X}, \mathcal{L}) \to (X, B; L)\) be the normalization. The **conductor ideal** of \(X\) is defined as

\[ \text{cond}_X := \text{Hom}_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\mathcal{X}}, \mathcal{O}_X). \]

This is a coherent ideal sheaf of both of \(\mathcal{O}_{\mathcal{X}}\) and \(\mathcal{O}_X\) and defines the closed subschemes

\[ D_X \subset X, \quad D_{\mathcal{X}} \subset \mathcal{X}. \]

They are reduced of codimension 1 and supported on nodes.

Let \(\widetilde{B}\) be the divisorial part of \(\nu^{-1}B\) and then we have

\[ K_{(\mathcal{X}, \mathcal{B} + D_{\mathcal{X}})} \sim_{Q} \nu^*(K_{(X, B)}). \]

Here, we remark that if \((\mathcal{X}, \mathcal{L}) \to (X, \mathcal{L})\) is the normalization, then we have

\[ H^\text{NA}_B(\mathcal{X}, \mathcal{L}) = H^\text{NA}_B(X, \mathcal{L}). \]

Thanks to [38] and [36], if a polarized normal (resp. deminormal) pair \((X, B; L)\) is K-semistable, then \((X, B)\) has only lc (resp. slc) singularities (see [26]). Recall that if \((X, B; L)\) is an lc polarized pair, \(H^\text{NA}_B \geq 0\) on \(\mathcal{H}^\text{NA}(L)\) (cf. [5]). Furthermore, if \((X, B; L)\) is klt, \(H^\text{NA}_B \geq \alpha(X, B; L)\). Here, \(\alpha(X, B; L) = \inf_{D,v} \frac{A_{(X,B)}(v)}{v(D)} > 0\)

is the alpha invariant of \((X, B; L)\) where the infimum is taken over all the effective \(\mathbb{Q}\)-Cartier divisor \(D\) on \(X\) that is \(\mathbb{Q}\)-linearly equivalent to \(L\) and all the divisorial valuation \(v\) on \(X\) such that \(v(D) > 0\) (cf. Theorem 9.14 of loc.cit). On the other hand, \(J^H\)-stability is irrelevant to singularities of \(X\) and its boundary divisor.

We can also define \(M^\text{NA}_B\) and \(H^\text{NA}_B\) when \(X\) is deminormal similarly. Note also that we can define \((J^H)^\text{NA}\)-energy for any non-normal but integral polarized variety \((V, M)\).
Since the $J^H$-stability of $(V, M)$ coincides with one of the normalization of $(V, M)$, we have to discuss only about its normalization.

Here, we remark that $I_{NA}$ and $J_{NA}$ are nonnegative functionals on $H_{NA}(L)$ that vanish only on the metric of the trivial test configuration and satisfy
\[
\frac{1}{n}J_{NA} \leq I_{NA} - J_{NA} \leq nJ_{NA}
\]
(cf. Proposition 7.8 of [5]). In other words, $I_{NA} - J_{NA}, I_{NA}$ and $J_{NA}$ are equivalent norms on $H_{NA}(L)$. $E_{NA, R_B}, I_{NA}, J_{NA}, (J^H)_{NA}, H_{B_{NA}}$ and $M_{B_{NA}}$ are all homogeneous in $L$. In other words, $E_{NA}(X, tL) = tE_{NA}(X, L)$ for $t > 0$ for example. We can conclude that the stability of $(X, B; L)$ is equivalent to the stability of $(X, B; tL)$. $(J^H)_{NA}$ is also homogeneous in $H$.

We also remark that
\[
(J^{XL})_{NA}(X, L) = \lambda(I_{NA}(X, L) - J_{NA}(X, L))
\]
for $\lambda \in \mathbb{Q}$. It follows from an easy computation as in the proof of Lemma 7.25 of loc.cit. On the other hand,
\[
M_{B_{NA}}^{NA} - H_{B_{NA}}^{NA} = R_{B_{NA}}^{NA} + S(X, B; L)E_{NA} = (J^{K(x, m)})_{NA}.
\]

If $(X, B; L)$ is a klt polarized variety and there exists $\delta < \alpha(X, B; L)$ such that
\[
(J^{K(x, m)} (\phi))^{NA} \geq -\delta I^{NA}(\phi)
\]
for all $\phi \in H^{NA}(L)$, this polarized pair is uniformly K-stable by Proposition 9.16 of loc.cit. Furthermore, it follows from [30, Theorem 6.10] that if $X$ is smooth, $B = 0$ and there exists $\delta < \alpha(X, L)$ such that
\[
(J^{Kx}(\phi))^{NA} \geq -\delta I^{NA}(\phi)
\]
for all $\phi \in H^{NA}(L)$, then there exists a cscK metric on $(X, L)$.

Next, we give the definition of flag ideals in this section.

**Definition 2.7** (Flag Ideals (cf. Definition 3.1 of [37])). Let $(X, L)$ be an $n$-dimensional polarized variety. A coherent sheaf of ideals $a$ of $X \times \mathbb{A}^1$ is called a flag ideal if
\[
a = I_0 + I_1 t + \cdots + I_{r−1} t^{r−1} + (t'),
\]
where $t$ is the coordinate function of $\mathbb{A}^1$ and $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{r−1} \subseteq \mathcal{O}_X$ are coherent ideals. Note that it is equivalent to that $a$ is $\mathbb{G}_m$-invariant under the natural action of $\mathbb{G}_m$ on $X \times \mathbb{A}^1$.

According to Proposition 3.10 of [37] or the proof of Theorem 2.9 of [35] (cf. [40]), we can see that for any ample test configuration $(\mathcal{X}, \mathcal{L})$, there exists a flag ideal $a$ such that $(\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - sE)$ is a semiample test configuration, where $s \in \mathbb{Q}_{\geq 0}$ and $E$ is the exceptional divisor on $\text{Bl}_a(X \times \mathbb{A}^1)$, which is the blow up of $X \times \mathbb{A}^1$ along $a$, and a pullback of $(\mathcal{X}, \mathcal{L})$. If $(\mathcal{X}, \mathcal{L})$ is semiample, there exists a sufficiently divisible integer $k$ such that $k\mathcal{L}$ is globally generated. Then, we can see that the ample model $(\text{Proj}_{\mathbb{A}^1}(\bigoplus_{m=0}^{\infty} \pi_* \mathcal{O}_X(km\mathcal{L})), \mathcal{O}(1))$ is an ample test configuration equivalent to $(\mathcal{X}, k\mathcal{L})$ (cf. Proposition 2.17 of [33]). Therefore, there exists a flag ideal $a$ such that $(\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - sE)$ is a semiample test configuration equivalent to $(\mathcal{X}, \mathcal{L})$.

We can easily see that the deformations to the normal cone along $a$ and $\overline{a}$ define the same positive metric.

Note that multiplying $a$ by $t^m$ coincides with replacing $(\mathcal{X}, \mathcal{L})$ by $(\mathcal{X}, \mathcal{L} - m\mathcal{X}_0)$. On the other hand, $(J^H)_{NA}, J_{NA}, I_{NA}$ and $M_{B_{NA}}$ are translation invariant. In other words,
\((J^H)^{NA}(X, L) = (J^H)^{NA}(X, L + m\mathcal{X}_0)\) for \(m \in \mathbb{Q}\). To study \(J^H\)-stability and K-stability, we may assume that \(I_0 \neq 0\) for \(a\).

Finally, note that if \((X, L)\) is reducible, \((X_1, L_1, H_1)\) is an irreducible component of \((X, L, H)\), \((X_1, L_1)\) is the strict transformation of \(X_1 \times \mathbb{A}^1\) in \((X, L)\), then

\[(J^H)^{NA}(X, L) \neq (J^H)^{NA}(X, L + m\mathcal{X}_1, 0)\]

in general for \(m \in \mathbb{Q}\), where \(\mathcal{X}_1, 0\) is the central fibre of \(\mathcal{X}\) unless \(L_I^{-1}, H_1 = H_I L^{-1} I^n\).

**Definition 2.8.** If a polarized reducible scheme \((X, L)\) with a \(\mathbb{Q}\)-divisor \(H\) satisfies that

\[
\frac{L_I^{n-1} \cdot H}{L^n} = \frac{(L|_{X_1})^{n-1} \cdot H|_{X_1}}{(L|_{X_1})^n}
\]

for any irreducible component \(X_1\), then we say that all irreducible components of \((X, L)\) have the same average scalar curvature with respect to \(H\).

## 3. Mixed Multiplicity

In this section \(X\) is a variety. First, recall the definition of mixed multiplicities, which is used to express our decomposition formula, Theorem 5.9.

**Definition 3.1.** Let \(X\) be an \(n\)-dimensional variety, \(L\) be a \(\mathbb{Q}\)-line bundle on \(X\) and \(\mathcal{I}_1, \ldots, \mathcal{I}_n\) be coherent ideals of \(\mathcal{O}_X\) such that \(\text{supp}(\mathcal{O}_X/\mathcal{I}_j)\) is proper or \(\emptyset\). Choose a compactification \(\overline{X}\) of \(X\) on which \(L\) extends to a line bundle \(\overline{L}\) and let \(\pi : \overline{X} \to \overline{X}\) be the blowing up of \(\overline{X}\) along \(\prod \mathcal{I}_j\) so that \(\pi^{-1}(\mathcal{I}_j) = \mathcal{O}_{\overline{X}}(E_j)\). Let \(\pi^* L = \mathcal{O}_{\overline{X}}(D)\). We define

\[e_L(\mathcal{I}_1, \ldots, \mathcal{I}_n) = (D^n) - ((D-E_1) \cdots (D-E_n)).\]

\(e_L(\mathcal{I}_1, \ldots, \mathcal{I}_n)\) is independent of the choice of \(\overline{X}\) and \(\overline{L}\).

For the proof of the main theorem, we prepare the following modification of Proposition 4.10 of [35].

**Theorem 3.2.** Given an \(n\)-dimensional proper variety \(X\), a line bundle \(L\) on \(X\) and an flag ideal \(a \subset \mathcal{O}_{X \times \mathbb{A}^1}\). Suppose that \(a\) satisfies that the following condition (*):

\[(*)\]

\[a = \mathcal{I}_{D_0} + \mathcal{I}_{D_1} t + \cdots + \mathcal{I}_{D_r-1} t^{r-1} + t^r,\]

where each \(\mathcal{I}_{D_i}\) is a coherent ideal sheaf corresponding to a Cartier divisor \(D_i\) of \(X\). Furthermore, for each \(m \in \mathbb{Z}_{\geq 0}\),

\[
\left( (\pi \times \text{id}_{\mathbb{A}^1})^{-1} a \right)^m = \sum_{k=0}^{m} t^k \mathcal{I}_{m,k},
\]

where \(\mathcal{I}_{m,k} = \mathcal{I}_{D_j}^{m-i} \mathcal{I}_{D_{j+1}}^i\) for \(j = \lfloor \frac{k}{m} \rfloor \) and \(i = k - mj\).

Suppose also that \(L\) is semiample and \(aL_{\mathbb{A}^1}\) and all the \(\mathcal{I}_{D_k} L\) are nef. Then,

\[e_{L_{\mathbb{A}^1}}(a) = \sum_{k=0}^{r-1} \sum_{j=0}^{n} e_L(\mathcal{I}_{D_k}^{[j]} : \mathcal{I}_{D_{k+1}}^{[n-j]}),\]

where \(\mathcal{I}_{D_k} = \mathcal{O}_X, \mathcal{I}_{D_k}^{[j]}\) indicates that \(\mathcal{I}_{D_k}\) appears \(r_i\) times and \(e_{L_{\mathbb{A}^1}}(a) = e_{L_{\mathbb{A}^1}}(a^{[n+1]}).\)
In Theorem 3.2 let $\mathcal{X}$ be the blow up of $X \times \mathbb{A}^1$ along $a$ with the exceptional divisor and $\mathcal{L} = L_{\mathbb{A}^1} - E$. Then it is easy to see that

$$e_{L_{\mathbb{A}^1}}(a) = -\mathcal{L}^{n+1}.$$  

Here, $\mathcal{I}$ is nef (resp. ample) if $\pi : B \rightarrow X$ is the blow up along $\mathcal{I}$, $E$ is the exceptional divisor corresponding $\pi^{-1}\mathcal{I}$ and $\pi^*L - E$ is nef (resp. ample) on $B$. Note that the theorem holds even when $\mathcal{I}$ is not necessarily globally generated.

**Example 3.3.** Suppose that $(X, L)$ is an $n$-dimensional polarized proper variety, and $\mathcal{I}$ is a coherent ideal on $X$. Let $\mathcal{X}$ be the deformation to the normal cone along $\mathcal{I}$ and $E$ be the exceptional divisor. Suppose that $\mathcal{L} = p^*L - cE$ is a semiample line bundle where $p : \mathcal{X} \rightarrow X$ is the canonical projection for $c > 0$. Then, the fiber of $0 \in \mathbb{P}^1$,

$$\mathcal{X}_0 = \mathcal{X} + E,$$

where $\pi : \mathcal{X} \rightarrow X$ is the blow up along $\mathcal{I}$. Let $D$ be the exceptional divisor on $\mathcal{X}$ and then $D = \mathcal{X} \cap E$ scheme-theoretically. Therefore,

$$\mathcal{L}^{n+1} = (p^*L - cE)^{n+1}$$

$$= -c \sum_{i=0}^{n} E \cdot (p^*L^i \cdot (p^*L - cE)^{n-i})$$

$$= c \sum_{i=0}^{n} (\mathcal{X} - \mathcal{X}_0) \cdot (p^*L^i \cdot (p^*L - cE)^{n-i})$$

$$= c \sum_{i=0}^{n} \pi^*L^i \cdot (\pi^*L - cD)^{n-i} - c(n + 1)L^n.$$

Therefore, if $t$ is a parameter of $\mathbb{A}^1$,

$$e_{L_{\mathbb{A}^1}}(\mathcal{I} + (t)) = \sum_{j=0}^{n} e_L(\mathcal{I}^j, \mathcal{O}_{\mathcal{X}}^{[n-j]}).$$

On the other hand, if $H$ is a line bundle on $X$, it is easy to see that

$$V(L)(\mathcal{I}^H)^{NA}(\mathcal{X}, \mathcal{L}) = c \left( \pi^*H \cdot \left( \sum_{i=0}^{n-1} (\pi^*L - cD)^i \cdot \pi^*L^{n-i} \right) \right)$$

$$- \frac{nH \cdot L^{n-1}}{(n + 1)L^n} \sum_{i=0}^{n} (\pi^*L - cD)^i \cdot \pi^*L^{n-i}.$$  

We call $(\mathcal{I}^H)^{NA}$-energy of a semiample deformation to the normal cone a $(\mathcal{I}^H)^{NA}$-slope. We also remark that $e_{L_{\mathbb{A}^1}}(\mathcal{I} + (t)) = (p^*L)^{n+1} - \mathcal{L}^{n+1} = -\mathcal{L}^{n+1}$ if $c = 1$.

Finally, if $n = 2$ and $\mathcal{I}$ is invertible, then

$$V(L)(\mathcal{I}^H)^{NA}(\mathcal{X}, \mathcal{L}) = c^2 \left( \frac{H \cdot L}{2} - H \right) \cdot D - c^3 \frac{2H \cdot L}{3L^2}D^2.$$  

We show that this is nonnegative when $2\frac{H \cdot L}{L^2}L - H$ is nef and $H$ is pseudoeffective in Proposition 6.1.

In the condition (*), the fact that the right hand side is included in the left hand side is trivial. However, the opposite inclusion does not necessarily hold for an arbitrary flag ideal $a$ and for large $m$. Therefore, the inequality in Proposition 4.10 of [35] is not
necessarily an equality for an arbitrary flag ideal \( \mathfrak{a} \) either. However, if (*) holds, we show the inequality is an equality. Theorem 3.2 follows from Propositions 4.3, 4.8, 4.9 and 4.10 of loc.cit. However, we need the following modification of Proposition 4.9 of loc.cit.

**Proposition 3.4.** Let \( V^n \) be a proper variety over \( k \) and \( L \) be a line bundle on \( V \).

1. Let \( \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_r \) be coherent ideals in \( \mathcal{O}_V \). If \( L, \mathcal{I}_1L, \ldots, \mathcal{I}_rL \) are nef and all the \( \mathcal{I}_j \) is invertible, then

\[
\left| \chi(V, L\sum_{j=1}^r m_j / (\prod_{j=1}^r \mathcal{I}_j^{m_j}) \mathcal{O}_V) - \dim(\Gamma(V, L\sum_{j=1}^r m_j) / \Gamma(V, (\prod_{j=1}^r \mathcal{I}_j^{m_j}) \mathcal{O}_V)) \right| = O((\sum m_j)^{n-1}).
\]

2. Let \( \mathfrak{a} \) be a flag ideal in \( \mathcal{O}_{V \times \mathbb{A}^1} \) and hence \( \text{supp}(\mathcal{O}_{V \times \mathbb{A}^1}/\mathfrak{a}) \) is proper. If \( L \) is semiample and \( \mathfrak{a}L_{\mathbb{A}^1} \) is nef, then

\[
\left| \chi(V \times \mathbb{A}^1, L_{\mathbb{A}^1}^{\otimes m} / (\mathfrak{a}^m L_{\mathbb{A}^1}^{\otimes m})) - \dim(\Gamma(V \times \mathbb{A}^1, L_{\mathbb{A}^1}^{\otimes m}) / \Gamma(V \times \mathbb{A}^1, \mathfrak{a}^m L_{\mathbb{A}^1}^{\otimes m})) \right| = O(m^n).
\]

**Proof.** (1): It follows from the fact we will prove that

\[
h^i(V, \mathcal{O}_V) = O((\sum m_j)^{n-1})
\]

and

\[
h^i(V, (\prod \mathcal{I}_j^{m_j}) \mathcal{O}_V) = O((\sum m_j)^{n-1})
\]

for \( i > 0 \) as [35 Proposition 4.9]. The former is easier and we will only show the latter in this proof. More generally, we prove \( h^i(V, (\prod \mathcal{I}_j^{m_j})(\mathcal{O}_V)) = O((\sum m_j)^{n-i}) \) for any coherent sheaf \( \mathcal{F} \) by the induction on \( n \). Let \( \mathcal{O}_V(-E_i) = \mathcal{I}_i \). We can prove that there exists \( C' > 0 \) such that

\[
h^i(V, \mathcal{F} \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)}) < C'((\sum m_j)^n)
\]

by the fact that \( L - E_i \) are nef, using Fujita vanishing theorem (cf. Theorem 1.4.35 of [28]) and the induction on \( n \). In fact, if \( i = 0 \) we can take an ample divisor \( H \) that does not pass through any associated point of \( \mathcal{F} \) such that

\[
H^0(\mathcal{F} \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)}) = 0
\]

for \( i > 0 \) and any \( m_j \geq 0 \) by Fujita’s theorem and

\[
H^0(\mathcal{F} \otimes \mathcal{O}_V^{\otimes (\sum m_j)}) = H^0(\mathcal{F} \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)})
\]

\[
h^0(\mathcal{F} \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)}) = \chi(\mathcal{F} \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)}) = O((\sum m_j)^n)
\]

by Snapper’s theorem ([2] Theorem 1.1). Otherwise, take an ample and integral divisor \( H \) does not pass through any associated point of \( \mathcal{F} \) and consider the following exact sequence:

\[
0 \rightarrow \mathcal{F} \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)} \rightarrow \mathcal{F} \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)} \rightarrow \mathcal{F}(H) \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)} \rightarrow 0.
\]

Thanks to Fujita’s theorem, we can choose \( H \) so that \( H^i(\mathcal{F} \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)}) = 0 \) for \( i > 0 \) and any \( m_j \geq 0 \). By the long exact sequence and the induction hypothesis of \( n \), we have

\[
h^i(\mathcal{F} \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)}) \leq h^{i-1}(\mathcal{F}(H) \otimes \mathcal{O}_V^{\otimes (\sum m_j - \sum E_j)}) = O((\sum m_j)^{(n-1)-(i-1)}).
\]
We want to show that
\[ i > \text{supp}(\pi) \quad \text{and} \quad \pi \quad \text{for} \quad k \quad \text{and} \quad \langle \pi \rangle \quad \text{that} \quad k \quad \text{Hence, we can apply Fujita's theorem again to obtain} \]
\[ \dim(\text{Coker}(H^0(V \times \mathbb{A}_1, L_{\mathbb{A}_1}^{\otimes m}) \to H^0(V \times \mathbb{A}_1, L_{\mathbb{A}_1}^{\otimes m} / a^m \cdot L_{\mathbb{A}_1}^{\otimes m}))) = O(m^n) \]
and
\[ h^i(V \times \mathbb{A}_1, L_{\mathbb{A}_1}^{\otimes m} / a^m \cdot L_{\mathbb{A}_1}^{\otimes m}) = O(m^n) \]
for \( i > 0 \). There exists \( N' > 0 \) such that \( L_{\mathbb{A}_1}^{\otimes m} \) is globally generated for \( m \geq N' \). Then for \( k > 0 \) and \( m \geq N' \), since \( H^0(X \times \mathbb{A}_1, L_{\mathbb{A}_1}^{\otimes m}) \) is generated by
\[ H^0(V, L^m) \subset H^0(V \times \mathbb{P}_1, L_{\mathbb{P}_1}^{\otimes m} (km(V \times \{0\}))) \]
\[ \exists f \mapsto f \cdot t^{-km}, \]
we have only to prove that
\[ \dim(\text{Coker}(H^0(V \times \mathbb{P}_1, L_{\mathbb{P}_1}^{\otimes m} (mk(V \times \{0\}))) \to H^0(V \times \mathbb{P}_1, L_{\mathbb{P}_1}^{\otimes m} / a^m \cdot L_{\mathbb{P}_1}^{\otimes m}))) = O(m^n) \]
where \( L_{\mathbb{P}_1} = L \otimes O_{\mathbb{P}_1} \). Note that \( L_{\mathbb{P}_1}^{\otimes m} / a^m . L_{\mathbb{P}_1}^{\otimes m} = (L_{\mathbb{P}_1}^{\otimes m} \otimes O_{V \times \mathbb{P}_1} (mk(V \times \{0\})) \) since \( \text{supp}(O_{V \times \mathbb{A}_1} / a) \) is proper. Let \( \pi : B \to V \times \mathbb{P}_1 \) be the blow up of \( V \times \mathbb{P}_1 \) along \( a \) and \( E \) be the exceptional divisor. As in the proof of [3] Proposition 4.8, there is a large integer \( N \) such that
\[
\begin{align*}
& a) \ R^i \pi_* (O_B(-mE)) = 0, \ i > 0 \\
& b) \ \pi_* (\pi^* O_B(-mE)) = a^m
\end{align*}
\]
when \( m \geq N \). Then \( h^i(V, L^m) = h^i(B, \pi^* L_{\mathbb{P}_1}^{\otimes m} (mE)) \) by Leray spectral sequence. By the assumption, since \( \pi^* L_{\mathbb{P}_1} - E \) is \( \mathbb{P}_1 \)-nef, there exists sufficiently large \( k \) that \( \pi^* L_{\mathbb{P}_1} (k(V \times \{0\}) - E) \) is nef. In fact, we can prove that it is nef when we choose \( k \) so large that \( k \pi^* (V \times \{0\}) - E \) is effective. Let \( C \) be an integral curve on \( B \) and if \( C \cdot (\pi^* L - E) < 0, \ C \) is not contained in any fibre over \( \mathbb{P}_1 \). Since \( \pi^* L \) is nef and \( C \not\subseteq k \pi^* (V \times \{0\}) - E, \)
\[ C \cdot (\pi^* L_{\mathbb{P}_1} (k(V \times \{0\}) - E) \geq 0. \]
Hence, we can apply Fujita’s theorem again to obtain
\[ h^i(B, \pi^* L_{\mathbb{P}_1}^{\otimes m} (mk(V \times \{0\}) - mE)) = O(m^{n+1-i}) \]
and
\[ h^i(V \times \mathbb{P}_1, L_{\mathbb{P}_1}^{\otimes m} (mk(V \times \{0\}))) = O(m^{n+1-i}) \]
We can easily see that
\[ \dim(\text{Coker}(H^0(V \times \mathbb{P}_1, L_{\mathbb{P}_1}^{\otimes m} (mk(V \times \{0\}))) \to H^0(V \times \mathbb{P}_1, L_{\mathbb{P}_1}^{\otimes m} / a^m \cdot L_{\mathbb{P}_1}^{\otimes m}))) \leq h^1(B, \pi^* L_{\mathbb{P}_1}^{\otimes m} (mk(V \times \{0\}) - mE)) \]
and
\[ h^i(V \times \mathbb{A}_1, L_{\mathbb{A}_1}^{\otimes m} / a^m \cdot L_{\mathbb{A}_1}^{\otimes m}) = O(m^n) \]
for \( i > 0 \). We complete the proof. \( \square \)

**Proof of Theorem 2.** By the assumption, \( a \) satisfies the condition (*). For each \( m \), we denote \( a^m = \bigoplus_{k=0}^{\infty} t^k J_{m,k} \), where \( J_{m, mk+i} = J_{m+i}^{mk} \cdot J_{m,i} \). Since
\[ H^0(X \times \mathbb{A}_1, a^m L_{\mathbb{A}_1}^{\otimes m}) = \bigoplus_{k=0}^{\infty} H^0(X, J_{m,k} L_{\mathbb{A}_1}^{\otimes m}) \cdot t^k, \]
it follows that
\[
\dim(H^0(X \times \mathbb{A}^1, L_{k_1}^{\otimes m})/H^0(X \times \mathbb{A}^1, a^m L_{k_1}^{\otimes m})) = \sum_{k=0}^{\infty} \dim(H^0(X, L^{\otimes m})/H^0(X, \mathcal{M}_{k,L}^{\otimes m})).
\]
The rest of the proof follows from Proposition 4.10 of [35] immediately. For the reader’s convenience, we prove as follows. Apply Proposition 3.5 and we get the estimates
\[
\dim[H^0(X \times \mathbb{A}^1, L_{k_1}^{\otimes m})/H^0(X \times \mathbb{A}^1, a^m L_{k_1}^{\otimes m})]
\]
for \(\chi(L_{k_1}^{\otimes m}/a^m L_{k_1}^{\otimes m})\) and
\[
\dim(H^0(X, L^{\otimes m})/H^0(X, \mathcal{M}_{D_{k} \cdot D_{k+1} \cdot L^{\otimes m}}))
\]
for \(\chi(L^{\otimes m}/\mathcal{M}_{D_{k} \cdot D_{k+1} \cdot L^{\otimes m}}).\) By the weak form of Riemann-Roch Theorem ([27, Theorem 1.36]),
\[
\chi(L_{k_1}^{\otimes m}/a^m L_{k_1}^{\otimes m}) = \chi(L_{P^1}^{\otimes m}/a^m L_{P^1}^{\otimes m})
\]
\[
= \frac{1}{(n+1)!} e_{L_{k_1}}(a) m^{n+1} + O(m^n).
\]
and
\[
\chi(L^{\otimes m}/\mathcal{M}_{D_{k} \cdot D_{k+1} \cdot L^{\otimes m}}) = \chi(L^{\otimes m}) - \chi(L^{\otimes m}(-(m-i)D_k - iD_{k+1}))
\]
\[
= \sum_{j=0}^{n} j!(n-j)! e_L(\mathcal{M}_{D_{k}}^{[n-j]} \cdot \mathcal{M}_{D_{k+1}}^{[j]})(m-i)^{n-j} i^j + O(m^{n-1}).
\]
Therefore,
\[
\sum_{k=0}^{\infty} \dim(H^0(X, L^{\otimes m})/H^0(X, \mathcal{M}_{L}^{\otimes m}))
\]
\[
= \sum_{k=0}^{r-1} \sum_{i=0}^{m-1} \dim(H^0(X, L^{\otimes m})/H^0(X, \mathcal{M}_{D_{k} \cdot D_{k+1} \cdot L^{\otimes m}}))
\]
\[
= \sum_{k=0}^{r-1} \sum_{i=0}^{m-1} \left[ \sum_{j=0}^{n} j!(n-j)! e_L(\mathcal{M}_{D_{k}}^{[n-j]} \cdot \mathcal{M}_{D_{k+1}}^{[j]})(m-i)^{n-j} i^j + R_i \right],
\]
where \(R_i = O(m^{n-1}).\) As in the proof of [35, Proposition 4.8], \(\sum_{i=0}^{m-1} R_i = O(m^n).\) Then [35, Lemma 4.5] gives:
\[
\frac{1}{(n+1)!} m^{n+1} = \frac{1}{j!(n-j)!} \sum_{i=0}^{m-1} (m-i)^{n-j} i^j + O(m^n)
\]
and we see the theorem holds.

We have the following the extension of Theorem [35] to the case when \(L\) is a \(\mathbb{Q}\)-line bundle.

**Proposition 3.5.** Given an \(n\)-dimensional variety \(X,\) an ample \(\mathbb{Q}\)-line bundle \(L\) on \(X\) and an flag ideal \(a \subset \mathcal{O}_{X \times \mathbb{A}^1}.\) If \(a\) satisfies (*), then we have
\[
e_s L_{k_1}(a) = \sum_{k=0}^{r-1} \sum_{j=0}^{n} e_s L(\mathcal{M}_{D_{k}}^{[j]} \cdot \mathcal{M}_{D_{k+1}}^{[n-j]}),
\]
for any \(s \in \mathbb{Q}.)
Proof. In fact, if $\text{Bl}_a(X \times \mathbb{P}^1)$ is the blow up of $X \times \mathbb{P}^1$ along $a$ and $E$ is the exceptional divisor,

$$e_{sL a_1}(a) = (sL_{\mathbb{P}^1})^{n+1} - (sL_{\mathbb{P}^1} - E)^{n+1}$$

is a polynomial in $s$ whose degree is at most $n$. On the other hand,

$$\sum_{k=0}^{r-1} \sum_{j=0}^{n} e_{sL}(\mathcal{I}_{D_k}^{[j]}, \mathcal{I}_{D_{k+1}}^{[n-j]}) = \sum_{k=0}^{r-1} \sum_{j=0}^{n} ((sL)^{n} - (sL - D_k)^j \cdot (sL - D_{k+1})^{n-j})$$

is a polynomial in $s$ whose degree is at most $n - 1$. Since $L$ is an ample $\mathbb{Q}$-line bundle, there exists a sufficiently divisible integer $m$ such that $mL$ is a line bundle and $a(mL_{\mathbb{P}^1})$ is ample. Then we have all the $\mathcal{I}_{D_k}(mL)$ is nef by Corollary 5.8 of [40]. Then, we can see that $mL, 2mL, \cdots, (n+1)mL$ satisfy the assumption of Theorem 3.2. Therefore, we can apply Theorem 3.2 and we have

$$e_{sL a_1}(a) = \sum_{k=0}^{r-1} \sum_{j=0}^{n} e_{sL}(\mathcal{I}_{D_k}^{[j]}, \mathcal{I}_{D_{k+1}}^{[n-j]})$$

for $s = m, 2m, \cdots, (n+1)m$. There exists $n + 1$ zeroes of the polynomial in $s$

$$e_{sL a_1}(a) - \sum_{k=0}^{r-1} \sum_{j=0}^{n} e_{sL}(\mathcal{I}_{D_k}^{[j]}, \mathcal{I}_{D_{k+1}}^{[n-j]})$$

and hence we have

$$e_{sL a_1}(a) = \sum_{k=0}^{r-1} \sum_{j=0}^{n} e_{sL}(\mathcal{I}_{D_k}^{[j]}, \mathcal{I}_{D_{k+1}}^{[n-j]})$$

for any $s$. \qed

**Proposition 3.6.** Given an $n$-dimensional polarized normal variety $(X, L)$. Suppose that $n \geq 2$, $H$ is an ample line bundle on $X$ and a flag ideal $a \subset O_{X \times \mathbb{A}^1}$ satisfies (*) in Theorem 3.2. Then for general normal and connected divisor $D \in |lH|$ for $l \gg 0$, the restriction of $a$ to $D$ satisfies (*). Furthermore, let $\Pi : \text{Bl}_a(X \times \mathbb{A}^1) \to X \times \mathbb{A}^1$. Then, $\Pi^*(D \times \mathbb{A}^1)$ is a prime divisor for general $D$ and is isomorphic to the blow up of $D \times \mathbb{A}^1$ along $a|D \times \mathbb{A}^1$.

**Proof.** Let

$$a = \sum_{k=0}^{r} \mathcal{I}_k t^k,$$

where each $\mathcal{I}_k$ is invertible as in Theorem 3.2. For $l \gg 0$, note that $|lH|$ is base point free on $X$ and $H^0(X \times \mathbb{P}^1, lH|_{\mathbb{P}^1}) = H^0(X, lH)$. Thus, we may assume that $D$ and $D \times \mathbb{A}^1$ are normal and connected, and do not pass through the associated points of all the $\mathcal{I}_k$ and $a$ due to Bertini’s theorem for base point free linear systems. We remark that the connectedness follows from the assumption of dim $X$. Therefore, $\mathcal{I}_k$ is an invertible ideal sheaf if $\iota : D \hookrightarrow X$ is the canonical inclusion of $D$. By the assumption, $a$ satisfies (*). In other words, for $m \in \mathbb{Z}_{\geq 0}$,

$$a^m = \sum_{k=0}^{\infty} t^k \mathcal{I}_{m,k}$$
on $X \times \mathbb{A}^1$, where $\mathcal{I}_{m,j+i} = \mathcal{I}_j^{m-i} \cdot \mathcal{I}_{j+1}^i$. Then for each $m \in \mathbb{Z}_{\geq 0}$,

$$[a|_{D \times \mathbb{A}^1}]^m = \sum_{k=0}^{\infty} t^k (\mathcal{I}_m^k)$$
on $D \times \mathbb{A}^1$. Note that $t^{-1}(\mathcal{I}_{m,j+i}) = t^{-1}(\mathcal{I}_j^{m-i}) \cdot t^{-1}(\mathcal{I}_{j+1}^i)$. Hence, $a|_{D \times \mathbb{A}^1}$ also satisfies (*).

To prove the last assertion, we take $D$ so general that $\Pi^*(D \times \mathbb{A}^1)$ is also integral by Bertini’s theorem. $\Pi^*(D \times \mathbb{A}^1)$ is birational to $D \times \mathbb{A}^1$ since it is not contained in the image of the exceptional locus of $\Pi$. On the other hand, let $D$ be the blow up of $D \times \mathbb{A}^1$ along $a|_{D \times \mathbb{A}^1}$ and a closed immersion $D \to \text{Bl}_a(X \times \mathbb{A}^1)$ \cite{19} II Corollary 7.15 factors through

$$\varphi : D \to \Pi^*(D \times \mathbb{A}^1).$$

Since $\Pi^*(D \times \mathbb{A}^1)$ is integral, $\varphi$ is a dominant closed immersion. Therefore, $\varphi$ is an isomorphism. \hfill \square

**Definition 3.7.** If $H$ and $D$ are as in Proposition 3.6,

$$e_{L | H}(\mathcal{I}_j^i, \mathcal{I}_{k+1}^{n-j}) = t^{-1} e_{L | D}(\mathcal{I}_k^i, \mathcal{I}_{k+1}^{n-j}).$$

If $H$ is a $\mathbb{Q}$-Cartier divisor, we define as follows:

$$e_{L | H}(\mathcal{I}_j^i, \mathcal{I}_{k+1}^{n-j}) = H \cdot (L^{n-1} - (L - D_k)^j \cdot (L - D_{k+1})^{n-j})$$

where $D_k$ corresponds to $\mathcal{I}_k$. We can check that the definition coincides with the one we gave before if $H$ is as in Proposition 3.6.

Thanks to Proposition 3.5 and Proposition 3.6, we can calculate $(\mathcal{I}^H)^{\text{NA}}$ of $(\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - E)$ as follows, where $E$ is the exceptional divisor and $L_{\mathbb{A}^1} - E$ is ample.

**Theorem 3.8.** Suppose that $(X, L)$ is an $n$-dimensional variety with a $\mathbb{Q}$-line bundle $H$ and $n \geq 2$. If

$$a = \sum_{k=0}^r \mathcal{I}_k t^k$$

is a flag ideal that satisfies the condition (*) and $(\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - E)$ is an ample test configuration over $(X, L)$, then

$$V(L)(\mathcal{I}^H)^{\text{NA}}(\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - E) = \frac{nH \cdot L^{n-1}}{(n+1)L^n} \sum_{k=0}^{r-1} \sum_{j=0}^n e_L(\mathcal{I}_k^i, \mathcal{I}_{k+1}^{n-j})$$

$$- \sum_{k=0}^{r-1} \sum_{j=0}^{n-1} e_{L | H}(\mathcal{I}_k^j, \mathcal{I}_{k+1}^{n-j}).$$

**Proof.** By the assumption, we can apply Proposition 3.5 and we have

$$-\mathcal{L}^{n+1} = e_{L | D}(a)$$

$$= \sum_{k=0}^r \sum_{j=0}^n e_L(\mathcal{I}_{D_k}^j, \mathcal{I}_{D_{k+1}}^{n-j}).$$

On the other hand, the left hand side and the right hand side of the equation we want to prove is linear in $H$. Therefore, we may assume that $H$ is ample. Since $L | D$ is ample and
Then the following are equivalent:

\[ a \in |lH| \text{ for sufficiently general } D \in |lH| \text{ for } l \gg 0 \text{ due to Proposition 3.6.} \]

We can also apply Proposition 3.5 to the computation of the mixed multiplicity of \( D \):

\[
-p^* D \cdot \mathcal{L}^n = (\mathcal{L}|_D)^n
\]

\[
= e_{L_{1}|D}^n(a|_{D \times \mathbb{A}^1})
\]

\[
= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} e_{L|D}^{[j]}(\mathcal{I}_{D_k|D} \cdot \mathcal{I}_{D_{k+1}|D}^{[n-j]}),
\]

where \( p : \mathcal{X} \to X \) is the canonical projection and \( D \) is the blow up of \( D \times \mathbb{A}^1 \) along \( a|_{D \times \mathbb{A}^1} \).

Remark 3.8 (Theorem 1.1 of [6]). Notations as in [6]. Given a Kähler manifold \( M^n \) with Kähler metrics \( \chi \) and \( \omega_0 \). Let \( c_0 \) be the positive constant such that

\[
\int_M \chi \wedge \frac{\omega_0^{n-1}}{(n-1)!} = c_0 \int_M \frac{\omega_0^n}{n!}.
\]

Then the following are equivalent:

1. There exists a smooth function \( \varphi \) such that
   \[ \omega_\varphi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \]
   satisfies the J-equation
   \[ \text{tr} \omega_\varphi \chi = c_0. \]
   Moreover, such \( \varphi \) is unique up to a constant;
2. There exists a smooth function \( \varphi \) such that \( \varphi \) is the critical point of the \( J_\chi \) functional. Moreover, such \( \varphi \) is unique up to a constant;
3. The \( J_\chi \) functional is coercive. In other words, there exist a positive constant \( \epsilon \) and another constant \( C \) such that \( J_\chi(\varphi) \geq \epsilon J_{\omega_0}(\varphi) - C; \)
4. \((M, [\omega_0], [\chi])\) is uniformly J-stable. In other words, there exists a positive constant \( \epsilon \) such that for any Kähler test configuration \((X, \Omega)\) (cf. [13], Definitions 3.2 and 3.4), the invariant \( J_{[\chi]}(X, \Omega) \) (cf. [13], Definition 6.3) satisfies
   \[ J_{[\chi]}(X, \Omega) \geq \epsilon J_{[\omega_0]}(X, \Omega); \]
5. \((M, [\omega_0], [\chi])\) is uniformly slope J-stable. In other words, there exists a constant \( \epsilon \) such that for any subvariety \( V \) of \( M \), deformation to normal cone of \( V \) \((X, \Omega)\) (cf. [13], Example 2.11 (ii)) satisfies \( J_{[\chi]}(X, \Omega) \geq \epsilon J_{[\omega_0]}(X, \Omega); \)
6. There exists a positive constant \( \epsilon \) such that
   \[
   \int_V (c_0 - (n-p)\epsilon)\omega_0^p - p\chi \wedge \omega_0^{p-1} \geq 0
   \]
   for any \( p \)-dimensional subvariety \( V \) with \( p = 1, 2, \ldots, n. \)

Remark 3.11. Note that uniform “J-stability” in (4) in Theorem 3.10 is “analytic” in the sense of [13], [11]. However, we remark that for any polarized smooth variety \((X, L)\) with an ample divisor \( H \), uniform \( J^H \)-stability of \((X, L)\) is equivalent to the following condition:
In fact, if \((X,L)\) is uniformly \(J^H\)-stable, it is easy to see that \((X,L)\) is uniformly slope \(J^H\)-stable. By \[29\] Proposition 13 (see \[6, Remark 1.4\]), we have
\[
\int_V (c_0 - (n-p)\epsilon)c_1(L)^p - p c_1(H) \cdot c_1(L)^{p-1} \geq 0
\]
for all \(p\)-dimensional (algebraic) subvarieties \(V\) with \(p = 1, 2, \ldots, n\). Therefore, \((6)'\) holds.

To show the converse, suppose that \((6)'\) holds. Take Kähler forms \(\omega_0 \in c_1(L)\) and \(\chi \in c_1(H)\). Then, \((X, [\omega_0],[\chi])\) is uniformly \(J\)-stable in the sense of \[13\] by Theorem 3.10. In particular, \((X,L)\) is uniformly \(J^H\)-stable.

We also remark that if the solution to the \(J\)-equation exists, it is unique up to a constant by Proposition 2 of \[7\] (see \[6\] Remark 1.4)).

Due to Theorem 3.10 uniform slope \(J^H\)-stability implies uniform \(J^H\)-stability in the case when \(X\) is smooth and \(H\) is ample. On the other hand, Ross and Thomas proved Theorems 6.1 and 6.4 of \[10\]. Thanks to the theorems, we can decompose \((J^H)^{NA}(X,L)\) into finitely many \((J^H)^{NA}\)-energy of semiample deformations to the normal cone (we say that \((J^H)^{NA}\)-slopes) under certain conditions. Unfortunately, the assumptions of Theorem 6.1 and 6.4 do not hold in general and we do not know that we can always decompose \((J^H)^{NA}\)-energy into a finite number of \((J^H)^{NA}\)-slopes. However, we can decompose \((J^H)^{NA}(X,L)\) into the mixed multiplicities instead of \((J^H)^{NA}\)-slopes in general by taking an alternation as we show in \[8\].

4. Newton Polyhedron and Toroidal Embeddings

In this section, we prepare the notion of Newton polyhedron to prove our decomposition formula. Here, recall the definitions and some facts of \[24\] Chapter II.

**Definition 4.1** (Toroidal embeddings). Suppose that \(Z\) is a \(p\)-dimensional normal variety, \(U\) is a smooth open subvariety of \(Z\) and \(U \rightarrow Z\) is a toroidal embedding without intersection in the sense of \[24\]. That is, for any closed point \(z\) of \(Z\), there exist an affine toric variety \(X_\sigma\), its closed point \(t\) and an isomorphism \(\widehat{\mathcal{O}}_{Z,z} \simeq \widehat{\mathcal{O}}_{X_\sigma,t}\) mapping the ideal corresponding to \(Z - U\) onto the ideal corresponding to \(X_\sigma - T\), where \(T\) is the \(p\)-dimensional torus acting on \(X_\sigma\), and if \(E_i\) is any irreducible component of \(Z - U\), then it is a normal Weil divisor on \(Z\). By replacing \(X_\sigma\) by its open toric subvariety, we may assume that \(t\) is closed in \(X_\sigma\) and then we call this a local model of \((Z,z)\). Then we define as following:

- \(Y\) is a stratum of \(Z\) if it is a locally closed subset that is an irreducible component of \(\bigcap_{i \in I} E_i - \bigcup_{j \notin I} E_j\). It is well-known that \(Z\) is the disjoint union of all of its strata. Now fix a stratum \(Y\).
- A star of \(Y\) if it is an open subset of \(Z\) that is the union of all the stratum \(Y'\) whose closure contains \(Y\). Let us denote it by \(\text{Star } Y\).
- \(M_{\text{Star } Y}\), which we call a lattice with respect to \(\text{Star } Y\), is a group of Cartier divisors supported on \(\text{Star } Y - U\). It is a subgroup of the free abelian group of Weil divisors supported on \(\text{Star } Y - U\). If there is no confusion, we will denote it by \(M_Y\) or \(M\). We denote \(M^+_{\text{Star } Y} = M_{\text{Star } Y} \otimes_\mathbb{Z} \mathbb{R}\). \(M^+_Y\) is a subsemigroup of non-negative Cartier
divisors and we define $M^Y_{\mathbb{R}^+}$ similarly. We call $M^Y_{\mathbb{R}^+}$ the positive cone of $M^Y$. We remark that $M^Y_{\mathbb{R}^+} + (-M^Y_{\mathbb{R}^+}) = M^Y$, where $M^Y_{\mathbb{R}^+} + (-M^Y_{\mathbb{R}^+})$ denotes the Minkowski sum.

- $N^Y = \text{Hom}(M^Y, \mathbb{Z})$ is a dual lattice and $\sigma^Y = \{ n \in N^Y_{\mathbb{R}} = N^Y \otimes \mathbb{R}; \forall m \in M^Y_{\mathbb{R}}, \langle n, m \rangle \geq 0 \}$ is a polyhedral cone corresponding to Star $Y$. If $D$ is a Cartier divisor supported on Star $Y - U$, we denote $\text{ord}_D \in M^Y$. We define that $m \leq m'$ if $m' - m \in M^Y_{\mathbb{R}^+}$ for $m, m' \in M^Y_{\mathbb{R}}$. It is easy to see that $m \leq m'$ iff $m' - m$ is a nonnegative function on $\sigma^Y$. Since $\sigma^Y$ is not contained in any hyperplane, $M^Y_{\mathbb{R}^+} \cap (-M^Y_{\mathbb{R}^+}) = 0$ (cf. Corollary 1 p.61 loc.cit).

- $\mathfrak{b}$ is a toroidal fractional ideal on $Z$ if it is a coherent sheaf of fractional ideals invariant under an isomorphism $\alpha : \widehat{\mathcal{O}}_{Z,z_1} \simeq \widehat{\mathcal{O}}_{Z,z_2}$ preserving strata (i.e. if $Y \subset Y^*$ for some stratum $Y^*$, then $\alpha$ maps the ideal sheaf corresponding to $Y^*$ isomorphically onto the one corresponding to $Y^*$ cf. p.73 of loc.cit.) where $z_1$ and $z_2$ are closed points of $Z$ in the same stratum $Y$. Here, $\widehat{\mathcal{O}}_{Z,z_1}$ means the completion of the local ring at $z_1$. It is well-known that the restriction of $\mathfrak{b}$ to Star $Y$ is a finite sum,

$$\sum_{i=1}^{l} \mathcal{O}_{\text{Star}Y}(-D^Y_i)$$

where $D^Y_i$ is a Cartier divisor supported on Star $Y - U$ (cf. [24, p.83] Lemma 3). Then we say that $\mathfrak{b}$ is generated by $D^Y$ or $\text{ord}_{D^Y} \in M^Y$. Here, $\text{ord}_{\mathfrak{b}}$ is the order function of $\mathfrak{b}$ if its restriction to each $\sigma^Y_i$ is a convex function $\min_{1 \leq i \leq l} \text{ord}_{D^Y_i}$. Then we remark that $\widehat{\mathfrak{b}}$, the integral closure of $\mathfrak{b}$, is also toroidal by Theorem 9 of [24].

In fact,

$$\widehat{\mathfrak{b}} = \sum_{\text{ord}_D \geq \text{ord}_{\mathfrak{b}} \text{ on } \sigma^Y} \mathcal{O}_{\text{Star}Y}(-D)$$

on a star of each stratum $Y$ (cf. Remark 4.2).

- Consider a normal variety $V$, an affine birational morphism $\varphi : V \rightarrow \text{Star}Y$ and the following commutative diagram:

$$
\begin{array}{ccc}
U & \xrightarrow{\varphi} & V \\
\downarrow & & \downarrow \\
\text{Star}Y & \xleftarrow{\varphi} & V
\end{array}
$$

where $U \hookrightarrow V$ is an open immersion. Then $\varphi$ is an affine toroidal morphism if for any isomorphism $\alpha : \widehat{\mathcal{O}}_{Z,z_1} \rightarrow \widehat{\mathcal{O}}_{Z,z_2}$ preserving strata for closed points $z_1, z_2$ of $Z$ in $Y$, $\alpha$ lifts to:

$$
\begin{align*}
V \times_Z \text{Spec} \widehat{\mathcal{O}}_{Z,z_2} & \xrightarrow{\alpha} V \times_Z \text{Spec} \widehat{\mathcal{O}}_{Z,z_1} \\
\text{Spec} \widehat{\mathcal{O}}_{Z,z_2} & \xrightarrow{\text{Spec } \alpha} \text{Spec} \widehat{\mathcal{O}}_{Z,z_1}.
\end{align*}
$$
By Theorem 1* of loc. cit., it is known that there is a 1-1 correspondence between the set of \((V, \varphi)\) and the set of rational polyhedral cones \(\tau \subseteq \sigma^V\) given by
\[
\tau \mapsto V_{\tau} = \operatorname{Spec}_{\operatorname{Star} Y} \mathcal{O}_\tau,
\]
where \(\mathcal{O}_\tau = \operatorname{subsheaf} \sum_{D \in \tau \cap M_Y} \mathcal{O}_{\operatorname{Star} Y}(-D)\) of the rational function field \(K(Z)\). Here, \(\tau^V = \{m \in M^Y_R; \forall n \in \tau, \langle n, m \rangle \geq 0\}\).

The definitions were given in [24] and the facts stated here were proved there.

**Remark 4.2.** We remark about [24]. Notations as in loc. cit.

- We point out a small error in the proof of Theorem 1* of [24] p.81]. To be precise, \(M_Y \cong M_Y^{\tau}\) is not always true but only the surjectivity \(M_Y^{\tau} \rightarrow M_Y^{\tau}\) holds in general. For example, consider the blow up \(\operatorname{Bl}_0(\mathbb{A}^2)\) of an affine plane \(\mathbb{A}^2 \cong \operatorname{Spec} k[x, y]\) at \((0, 0)\). Let \(F\) be the strict transformation of \((xy = 0)\), \(\tilde{Y} = \operatorname{Bl}_0(\mathbb{A}^2) \setminus F\) and \(Y = \mathbb{A}^2\). When we consider \(\tilde{Y}\) and \(Y\) as toric varieties, we have \(M_Y \cong M_Y^{\tau}\) (cf. [24] Chapter I). However, \(\tilde{Y}\) corresponds to a ray \(\mathbb{R}_{\geq 0}(1, 1)\) if we identify \(M_Y = \mathbb{Z} \cdot (x) + \mathbb{Z} \cdot (y)\), and hence \(M_Y^{\tau} = \mathbb{R} \cdot (1, 1)\) when we consider \(\mathbb{A}^2 \setminus \{(xy = 0)\} \leftrightarrow \tilde{Y}\) as a toroidal embedding.

The mistake does not affect their discussions in [24, Chapter II] so. In fact, we can easily see \(M_Y^{\tau}\) coincides with the image of \(M_Y^{\tau} \cap \tau^V\) and \(\tau^{\tau}\) holds via the canonical inclusion \(N^Y_R \hookrightarrow N^Y_R\).

- For the sake of the completeness, we explain the proof of [24] Chapter II, Theorem 9*]. The proof of all the assertion works as in toric varieties except the assertion of I that \(\mathcal{F}_f\) is integrally closed, where the notations are as in Theorem 9* of loc. cit. Since the conclusion is locally, we may assume that \(U \hookrightarrow Z = \operatorname{Star} Y\) is a toroidal embedding without self intersection, \(f : N^Y_R \rightarrow \mathbb{R}\) is a convex and piecewise-linear function such that \(f(N^Y) \subseteq \mathbb{Z}\) and \(f(\lambda x) = \lambda f(x)\) for \(\lambda \in \mathbb{R}\), and
\[
\mathcal{F}_f = \sum_{\operatorname{ord}_D \geq f} \mathcal{O}_{\operatorname{Star} Y}(-D)
\]
on \(Z\). We can easily see that \(\mathcal{F}_f\) is a coherent sheaf of fractional ideals. \(\mathcal{F}_f\) is integrally closed if and only if its Rees algebra \(\bigoplus_{n \geq 0} \mathcal{F}_f^n \otimes \mathcal{O}_{Z, z}\) is normal. To prove the latter, it is easy to see that \(\bigoplus_{n \geq 0} \mathcal{F}_f^n \otimes \mathcal{O}_{Z, z}\) is normal for any closed point \(z \in Z\). By replacing \(Y\), we may assume that \(z \in Y\). Since \(\mathcal{O}_{Z, z}\) is excellent, \(\beta : \bigoplus_{n \geq 0} \mathcal{F}_f^n \otimes \mathcal{O}_{Z, z} \rightarrow \bigoplus_{n \geq 0} \mathcal{F}_f^n \otimes \mathcal{O}_{\operatorname{pec} Y, z}\) is a regular homomorphism (cf. Lemma 4 of [24] p.253]). Then, \(\operatorname{Spec} \mathcal{F}_f\) has regular fibre and hence we have \(\bigoplus_{n \geq 0} \mathcal{F}_f^n \otimes \mathcal{O}_{Z, z}\) is normal if and only if so is \(\bigoplus_{n \geq 0} \mathcal{F}_f^n \otimes \mathcal{O}_{\operatorname{pec} Y, z}\). On the other hand, \(\mathcal{F}_f\) analytically corresponds to an integrally closed torus-invariant ideal on a normal toric variety. Therefore, \(\bigoplus_{n \geq 0} \mathcal{F}_f^n \otimes \mathcal{O}_{Z, z}\) is normal.

Here, we prepare the following definition motivated by the notion of Newton polygon used in [24] Theorem 6.4]:

**Definition 4.3** (Newton polyhedron). Suppose that \(U \hookrightarrow Z\) is a toroidal embedding without self intersection, \(V = \operatorname{Star} Y\) where \(Y\) is a stratum of \(Z\) and \(\mathfrak{b}\) is a toroidal fractional ideal sheaf on \(V\). Then we define in this paper as follows:

- We call a conical polyhedron \(P\) in \(M_{V, \mathbb{R}}\) the **Newton polyhedron associated with** \(\mathfrak{b}\) **on** \(V\) denoted by \(NP_P^V\) if \(\mathfrak{b} = \sum_{i=1}^n \mathcal{O}_V(-D_i)\) and
\[
P = \operatorname{Conv}(\operatorname{ord}_{D_1}, \ldots, \operatorname{ord}_{D_n}) + M^V_{\mathbb{R}, +}.
\]
where Conv\(\langle\text{ord}_{D_1}, \cdots, \text{ord}_{D_n}\rangle\) means the convex hull of \(\{\text{ord}_{D_1}, \cdots, \text{ord}_{D_n}\}\). Here, \(P\) is independent of the choice of \(D_1, \cdots, D_n\).

- If \(C\) is a compact polyhedron that has finite vertices and satisfies that
  \[ P = C + M_{\mathbb{R},+}, \]
  we call it a generating convex set of \(P\) and then say that the vertices of \(C\) generates \(P\).

- \(F\) is the bounded part of faces of \(P\) if it is the union of all of the bounded faces of \(P\). Since \(M_{\mathbb{R},+}^V \cap (-M_{\mathbb{R},+}^V) = 0\), we have \(F \neq \emptyset\).

**Proposition 4.4.** Notations as in Definition 4.3. Then the followings are equivalent:

1. \(b\) is integrally closed.
2. For each \(D \in M\),
   \[ D \in NP_b^V \iff O_V(-D) \subset b. \]

**Proof.** Let \(b = \sum O(-D_i)\). We only have to show the latter condition is equivalent to the following:

3. For each \(D \in M\),
   \[ \text{ord}_D \geq \text{ord}_b \quad \text{on} \quad \sigma^Y \iff O_V(-D) \subset b. \]

In fact, (3) is equivalent to (1) by Theorem 9* of [24, Chapter II]. Therefore, we only have to prove that

(1) \(\{h \in M_{\mathbb{R}}; h \geq \text{ord}_b\} = NP_b^V\).

\(\supset:\) If \(h \in NP_b^V\), we have by the definition \(\text{ord}_b = \min \text{ord}_{D_i}\) on \(\sigma^Y\) and
\[ h \geq \sum_{a_i \geq 0; \sum a_i = 1} a_i \text{ord}_{D_i} \geq \min \text{ord}_{D_i}. \]

\(\subset:\) Suppose that \(h \in M_{\mathbb{R}}\) satisfies \(h \geq \text{ord}_b\). Equivalently, \(h \geq \min \text{ord}_{D_i}\) on \(\sigma^Y\). Take finite elements \(\rho_j \in M_+ \setminus \{0\}\) defining \(\sigma^Y\). Then,
\[ \min\{\text{ord}_{D_i} - h, \rho_j\} \leq 0. \]
on \(N_{\mathbb{R}}\). This means that the polyhedral cone in \(N_{\mathbb{R}}\) defined by inequalities \(\text{ord}_{D_i} - h \geq 0, \rho_j \geq 0\) for all \(i, j\) is contained in a hyperplane. Hence, the cone generated by \(\text{ord}_{D_i} - h, \rho_j\) in \(M_{\mathbb{R}}\) contains a certain linear subspace and there exist \(m_i, n_j \geq 0\) but some \(m_i\) or \(n_j\) is not 0 such that
\[ \sum m_i(\text{ord}_{D_i} - h) + \sum n_j \rho_j = 0. \]
If all \(m_i = 0\), then \(\sum n_j \rho_j\) does not vanish on the relative interior of \(\sigma^Y\). Then, there exists \(m_i \neq 0\) and
\[ h \in \frac{\sum m_i \text{ord}_{D_i}}{\sum m_i} + M_{\mathbb{R},+}. \]
Therefore, \(h \in NP_b^V\). \(\square\)

**Remark 4.5.** Notations as in Definition 4.3. We can easily show that the bounded part of faces \(F\) equals to \(F'\), which is the union of faces of \(C\) that are also of \(P\). We can prove this by the following.

**Claim.** If \(x \in P\), then there exists \(y \in F'\) and \(x - y \in M_{\mathbb{R},+}\).
In fact, it is easy to see that $F' \subset F$. Assume that $H$ is a bounded face of $P$ but not contained in $C$. Let $h_1, \cdots, h_k$ be all the vertex of $H$. By the assumption, some $h_j$ is not contained in $C$. Then there exists $y \in F'$ such that $h_j - y \in M_{\mathbb{R}_+}$ by the claim. Since $h_j$ is also a vertex of $P$, $2h_j - y \in P$ and $h_j = \frac{1}{2}(2h_j - y) + y)$, we have $h_j = y \in C$. It is a contradiction. Therefore, $F \subset F'$.

**Proof of Claim.** Let $C = \text{Conv}(y_1, \cdots, y_n)$ be a generating convex set of $P$. We may assume that $P \subset M_{\mathbb{R}_+}$ by translation by a sufficiently large $w \in M_{\mathbb{R}_+}$. Assume that the claim does not hold for $x \in P$. As we saw in the last part of the proof of Proposition 4.4, there exists $z \in C$ such that $x \geq z$. Therefore, we may assume $x \in C$. Let $H$ be the minimal face of $P$ contains $x$. Then $H \cap C$ is a face of $C$. Assume that the face $H \cap C$ of $C$ was not a face of $P$. We see that there exist $y \in H \setminus C$ and $w_i \in M_{\mathbb{R}_+}$ such that

$$\eta = \sum_{a_i \geq 0, \sum a_i = -1} a_i(y_i + w_i).$$

If $H$ is a face defined by a linear function $f$ and $f(\sum a_iy_i) > 0$, then $f(\sum a_iw_i) < 0$ and it contradicts to the fact that $f(\sum a_iy_i + R(\sum a_iw_i)) \geq 0$ for any $R > 0$. Therefore, $\sum a_iy_i \in H \cap C$ and $\xi = \eta - \sum a_iy_i \in M_{\mathbb{R}_+} \setminus \{0\}$. Since $x$ is contained in the relative interior of $H$ and $M_{\mathbb{R}_+} \cap (-M_{\mathbb{R}_+}) = 0$, there exists $s > 0$ such that $x - s\xi \in H$ but $x - (s + \epsilon)\xi \notin P$ for $\epsilon > 0$. Then $x - s\xi$ is contained in a proper face $H'$ of $H$ and it follows from the similar argument as above that there exists $\gamma \in H' \cap C$ such that $x - s\xi \geq \gamma$. Since $\dim(H' \cap C) < \dim(H \cap C)$, we can iterate the above arguments by replacing $x$ and $H$ by $\gamma$ and $H'$ to obtain a face $H''$ of $P$ that is also a face of $C$ and $y \in H''$ such that $x \geq y$. It is a contradiction. 

We also remark that vertices of $F$ coincides with vertices of $P$ and hence generate $P$.

**Remark 4.6.** In Proposition 4.4, if $b$ is integrally closed, then the following are equivalent.

1. $b$ is invertible;
2. $NP_b^V$ has only one vertex.

In fact, (2) $\Rightarrow$ (1) is easy and if $b$ is invertible, there exists a Cartier divisor $D$ supported on $V - U$ such that $b = \mathcal{O}_V(-D)$ due to Lemma 3 of [24, p. 83].

**Lemma 4.7.** Notations as in Proposition 4.4. Suppose that $NP_b^V$ has integral vertices $x_1, x_2, \cdots, x_k$ and $g : V' \to V$ is an affine toroidal morphism corresponding to a polyhedral cone $\tau \subset \sigma^V$ (cf. Definition 4.1). Then $NP_{g^{-1}b}^{V'}$ is generated by $x_1, x_2, \cdots, x_k$. If $g^{-1}b$ is invertible, then one of $y_1, y_2, \cdots, y_k$, which are the images of $x_1, x_2, \cdots, x_k$ under the canonical map $M_V \to M_{V'}$, is the unique vertex of $NP_{g^{-1}b}^{V'}$.

**Proof.** By the equation (11) in the proof of Proposition 4.4 and Theorem 9$^*$ of [24], we have $NP_b^V = NP_{\mathcal{B}}^\tau$ if $b$ is the integral closure of $b$. Let $c$ be an ideal on $V$ generated by $x_1, x_2, \cdots, x_k$. Then $\mathcal{B} = \mathcal{C}$ and hence $g^{-1}\mathcal{B} = g^{-1}\mathcal{C}$. Hence, $NP_{g^{-1}b}^{V'} = NP_{g^{-1}c}^{V'}$. We can see $NP_{g^{-1}c}^{V'}$ is generated by $y_1, y_2, \cdots, y_k$. For the second assertion, $NP_{g^{-1}b}^{V'}$ has the unique vertex by the previous remark. Since $g^{-1}b$ is invertible, $g$ factors through the normalized blow up $W$ along $b$, $W$ is the normalized blow up along $c$ due to Lemma 1.8 of [5]. Hence, $g^{-1}b = g^{-1}c$ and is generated by $y_1, y_2, \cdots, y_k$. Then $g^{-1}b$ is generated by one of $y_1, y_2, \cdots, y_k$ since $g^{-1}b$ is invertible and whose bounded part of faces is one point. Thus, one of $y_1, y_2, \cdots, y_k$ is the unique vertex. \(\square\)
Example 4.8. We illustrate the proof of Theorems 5.1 and 5.9 briefly by a few examples. Let $U \hookrightarrow X$ be a toroidal embedding and take a flag ideal $a = \sum_{k=0}^{r} \mathcal{O}_X(-D_k)t^k$, where each $D_k$ is a Cartier divisor supported on $X - U$. It is necessary for (*) in Theorem 3.2 to seek out what is a generator of $a^n$. Then we can consider the Newton polyhedron of $a$. Note that $NP_{\mathfrak{a}^n} = kNP_{\mathfrak{a}}$. As we saw, the Newton polyhedron tells us what is a generator of $a^n$ for $n \geq 0$ if $a^n$ is integrally closed. First, as in [40, Theorem 6.4], let us consider when each $D_k$ is some multiple of one Cartier divisor $D$. When $a = t^4 + \mathcal{O}(3D)$, $\bar{a}$ contains $\mathcal{O}(-2D)t^2$, which is not on the bounded part of faces $F$. Therefore, we can conclude that the integral points of $F$ do not necessarily generate $a$. However, if there exists $D'$ such that $D = 2D'$, the integral points of $nF$ generate $a^n$. In general, we will prove in Proposition 5.3 that there exists $\pi : X' \rightarrow X$, where $\pi$ is a composition of its 2-cyclic coverings and its Bloch-Gieseker covering [4] such that there exists $\pi^*D = 2D'$. Here, we also remark that $F$ is one-dimensional then.

Next, let us consider $a = t^2 + \mathcal{O}(-E_1)t + \mathcal{O}(-E_1 - E_2)$. Then $a^2 = t^4 + \mathcal{O}(-E_1)t^3 + (\mathcal{O}(-E_1) + \mathcal{O}(-E_2))\mathcal{O}(-E_1)t^2 + \mathcal{O}(-2E_1 - E_2)t + \mathcal{O}(-2E_1 - 2E_2)$ and hence $F$ is not one-dimensional. We can easily see that if $a$ does not satisfy (*). If $\frac{1}{2}E_1$ and $\frac{1}{2}E_2$ are also Cartier divisors, $\pi = t^2 + \left(\mathcal{O}(-\frac{1}{2}E_1) + \mathcal{O}(-\frac{1}{2}E_2)\right)\mathcal{O}(-\frac{1}{2}E_1)t + \mathcal{O}(-E_1 - E_2)$. Then the intersection of $F$ and the hyperplane defined by $t^1$ is generated by $E_1$ and $\frac{1}{2}(E_1 + E_2)$. Assume that $f_1, f_2$ are local generators of $\frac{1}{2}E_1$ and $\frac{1}{2}E_2$ respectively. Take the blow up $\pi : X' \rightarrow X$ along $\mathcal{O}(-\frac{1}{2}E_1) + \mathcal{O}(-\frac{1}{2}E_2)$ with the exceptional divisor $\hat{E}$. Assume that $F_1$ and $F_2$ are the strict transformations of $\frac{1}{2}E_1$ and $\frac{1}{2}E_2$ respectively. Then $(\pi \times \text{id}_{\mathfrak{a}^1})^{-1}\pi = t^2 + \mathcal{O}(-2\hat{E} - F_1)t + \mathcal{O}(-4\hat{E} - 2F_1 - 2F_2)$. We can easily see that the bounded part of faces $F$ is one-dimensional and $(\pi \times \text{id}_{\mathfrak{a}^1})^{-1}\bar{a}$ satisfies that (*). However, if $L$ is a polarization of $X$ and $L' = \pi^*L$, $L'$ is big and semiample but not ample and we can not apply Theorem 3.3 immediately. On the other hand, $L'$ is an accumulation point of ample $\mathbb{Q}$-line bundles to which we can apply Theorem 3.3 and hence we can calculate $(\mathcal{J}^H)^{NA}$-energy by using the mixed multiplicity of Cartier divisors on $X'$.

5. Construction of the Alternation

In this section we prove Theorems 11 and 11 that we stated in 11. We want to show the following theorem motivated by Theorems 6.1 and 6.4 of 11.
Theorem 5.1. Given an $n$-dimensional polarized variety $(X, L)$, and an flag ideal $a = \sum a_i t_i \subset \mathcal{O}_{X \times \mathbb{A}^1}$ such that $a_0 \neq 0$. Then there exists an alternation $\pi : X' \to X$ (i.e. $\pi$ is a generically finite and proper morphism) such that $X'$ is smooth and irreducible, $D_0$ is an snc divisor corresponding to $\pi^{-1}a_0$ and the integral closure $(\pi \times \text{id}_{\mathbb{A}^1})^{-1}a$ of the inverse image of $a$ to $X' \times \mathbb{A}^1$ satisfies (*) in Theorem 3.2. Moreover, if $(\mathcal{B}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - E)$ is a semample test configuration, where $E = \Pi^{-1}(a)$ and $\Pi : \mathcal{B}_a(X \times \mathbb{A}^1) \to X \times \mathbb{A}^1$ is the blow up, $(\pi \times \text{id}_{\mathbb{A}^1})^{-1}a \cdot (\pi \times \text{id}_{\mathbb{A}^1})^*L_{\mathbb{A}^1}$ is semample and $\pi^*L - D_0$ is nef.

The proof consists of three steps. First, we replace $X$ by a resolution of singularities $X_1$ of $X$. Second, we construct $X'$ as a toroidal blow up of $X_2$. Abusing the notation, we denote $\pi \times \text{id}_{\mathbb{A}^1}$ by $\pi$.

First, we can take a log resolution of $X$ by the theorem of Hironaka [22] and by the assumption that char $k = 0$, and may assume that each $Z_i$ the closed subscheme corresponding to $a_i$ is a simple normal crossing divisor (snc for short). We can easily see if $(Y, D)$ is a log smooth variety, which means $Y$ is smooth and $D$ is an snc divisor in this paper, and $U = Y - D$, $U \leftarrow Y$ is a toroidal embeddings without self intersection (cf. 41). In fact, we can take $(\mathbb{A}^k \times (\mathbb{A}^1 - \{0\})^{-k}, (0, 1))$ as a local model. Hence, if $U = X - Z_0$, $U \leftarrow X$ and $U \times (\mathbb{A}^1 - 0) \leftarrow X \times \mathbb{A}^1$ are toroidal embeddings without self intersection. Furthermore, we can easily check $a_i$ and $a$ are toroidal ideal sheaves in Definition 4.1. Fix $r > 0$ such that $a_i = \mathcal{O}_X$. Moreover, the integral closure $\bar{a}$ of $a$ is also a toroidal and flag ideal. However, $\mathcal{J}_m$ is not necessarily invertible on $X$ but toroidal, where

$$\bar{a} = \sum_{m=0}^r \mathcal{J}_m t^m.$$ 

Suppose that $Y_i$ is any stratum of $X$ and $X = \coprod Y_i$. We can see the open sets $\text{Star } Y_i \times \mathbb{A}^1$ are stars of strata $Y_i \times \{0\}$ in $X \times \mathbb{A}^1$ and

$$X \times \mathbb{A}^1 = \bigcup \text{Star } Y_i \times \mathbb{A}^1.$$ 

Moreover, $M^\text{Star } Y_i \times \mathbb{A}^1 = M^\text{Star } Y_i \times \mathbb{Z}_{\geq 0}(\text{ord}_{X \times \{0\}})$. Let $P_i = \text{NP}^\text{Star } Y_i \times \mathbb{A}^1$ and $(\text{ord}_{X \times \{0\}})^*$ be the linear function that takes $1$ at $1$ at $\text{ord}_{X \times \{0\}}$ and vanishes on $M^\text{Star } Y_i$. Let also $F_i$ be the bounded part of faces of $P_i$ and $C_i = \text{Conv}(F_i)$. Then the following holds:

**Lemma 5.2.** If the notations are as above, then $P_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\}$ is a rational conical polyhedron of $M^\text{Star } Y_i$ such that

$$P_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\} = (C_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\}) + M^\text{Star } Y_i$$ 

and hence has finite rational vertices for any $0 \leq m \leq r$.

**Proof.** Note that $C_i$ must contain $(0, r)(= (0, \text{ord}_{X \times \{0\}}))$. If $(x_i, t_i)$ are vertices of $C_i$, there exists $(y_i, s_i) \in M^\text{Star } Y_i \times \mathbb{A}^1$ for $(z, m) \in P_i$ such that

$$(z, m) = \sum_{a_i \geq 0} a_i (x_i + y_i, t_i + s_i).$$

Note that $s_i, t_i \geq 0$ and let $u = \sum a_i t_i \leq m$. Then

$$\left(\frac{r - m}{r - u} \sum a_i x_i, m\right) = \frac{r - m}{r - u} \left(\sum a_i x_i, u\right) + \frac{m - u}{r - u} (0, r) \in C_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\}$$

and

$$z \geq \sum a_i x_i \geq \frac{r - m}{r - u} \sum a_i x_i.$$
since \( a \subset O_{X \times \mathbb{A}^1} \) and \( C_i \) contains \((0, r)\). The rest of the assertion follows from the fact that \( C_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\} \) is a compact rational polyhedron. \( \square \)

Next, we want to multiply \( M^Y_{+} \) by a sufficiently divisible integer \( l \) that \( C_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\} \) has the integral vertices for \( m \in \mathbb{Z}_{\geq 0} \) on any \( \text{Star}_Y \). For this, we need the following proposition.

**Proposition 5.3.** Suppose that \((V, E)\) is a log smooth variety and \( E = \bigcup_{i=1}^r E_i \) is the irreducible decomposition. We consider \( V - E \hookrightarrow V \) to be a toroidal embedding without self-intersection. Let \( l \) be an integer. Then there exists a finite covering \( \tilde{\pi} : \tilde{V} \to V \) such that \((\tilde{V}, \text{red}(\tilde{\pi}^* E))\) is also a log smooth variety and

\[
\tilde{\pi}^* E_i = l \text{ red}(\pi^* E_i).
\]

In other words, if \( Y \) is a stratum of \( V \) containing a closed point \( x \) of \( V \) and a closed point \( x' \in V \) satisfies \( \tilde{\pi}(x') = x \), the irreducible component \( Y' \) of \( \tilde{\pi}^{-1}(Y) \) contains \( x' \) is a stratum of \( \tilde{V} \) and the canonical map gives an isomorphism

\[
M^Y_{+} \cong \frac{1}{l} M^Y_{+}.
\]

Moreover, if \( b \) is a toroidal fractional ideal sheaf on \( V \), \( NP^{\text{Star}}_{\tilde{\pi}^{-1} b} = NP^{\text{Star}}_b \subset M^Y_{+} \subset \frac{1}{l} M^Y_{+} \) via the isomorphism above.

We prove the following two lemmas for the proposition.

**Lemma 5.4.** Suppose that \((V, E)\) is a log smooth variety and \( D \) is a smooth divisor supported on \( E \) such that there exists a line bundle \( L \) on \( V \) that satisfies \( D \in |lL| \). We remark that \( D \) might be reducible. Let \( \pi : V' \to V \) be an \( l \)-cyclic covering branched along \( D \). Then \((V', \text{red}(\pi^* E))\) is also a log smooth variety,

\[
\pi^* D = l \text{ red}(\pi^* D)
\]

and \( \pi^* D' \) is also reduced for any irreducible component of \( E \) not contained in \( \text{supp} \, D \). In other words, for any closed point \( x \in V \) and any stratum \( Y \) contains \( x \), if \( \pi(x') = x \), then \( Y' \) the irreducible component of \( \pi^{-1}(Y) \) contains \( x' \) is the stratum of \( V' \) and the canonical map gives

\[
M^Y_{+} \cong \begin{cases} 
M^Y_{+} + \mathbb{Z}_{\geq 0} \frac{1}{l} D & (Y \subset D) \\
M^Y_{+} & (Y \cap D = \emptyset).
\end{cases}
\]

Moreover, if \( b \) is a toroidal fractional ideal sheaf on \( V \), \( NP^{\text{Star}}_{\pi^{-1} b} = NP^{\text{Star}}_b \) via the isomorphism above.

**Proof of Lemma 5.4.** First, we can easily show that \((V', (\bigcup_{i=1}^k \pi^* D_i))\) is log smooth where \( D_i \)'s are integral divisors on \( V \) supported on \( E \) and different from each other and irreducible components of \( D \). In fact, this holds by the following:

**Claim.** For any subset \( J \) of \( \{1, \cdots, k\} \), \( \bigcap_{j \in J} \pi^* D_j \) is isomorphic to the \( l \)-cyclic covering of \( \bigcap_{j \in J} \text{supp} \, D_j \) branched along a smooth divisor \( D \cap \bigcap_{j \in J} \text{supp} \, D_j \) if the intersection is not void. Let \( t \) be an indeterminate. If \( D \cap \bigcap_{j \in J} D_j = \emptyset \), \( \bigcap_{j \in J} \pi^* D_j \) is isomorphic to the \( l \)-cyclic unramified covering

\[
\text{Spec}_{\bigcap_{j \in J} D_j} \left( \bigoplus_{i=0}^{\infty} (L^{-i}|_{\bigcap_{j \in J} D_j}) t^i / (t^1 - s_1) \right)
\]
where \( s_1 \) is a nowhere-vanishing section of \( L^i|_{\bigcap_{j \in J} D_j} \) (cf. Definition 2.49 of [27]). In particular, \( \bigcap_{j \in J} \pi^* D_j \) is smooth.

To prove the claim, \( V' \cong \text{Spec}_V(\bigoplus_{i=0}^{\infty} L^{-i} t^i/(t^i - s)) \) where \( t \) is an indeterminate and \( s \) is the global section of \( L^i \) corresponding to \( D \) by the assumption (cf. Definition 2.50 of [27]). If \( D \cap \bigcap_{j \in J} D_j \neq \emptyset \), \( s|_{\bigcap_{j \in J} D_j} \in \Gamma(L^i|_{\bigcap_{j \in J} D_j}) \) corresponds to this since the intersection is transversal. Therefore, by

\[
\bigcap_{j \in J} \pi^* D_j \cong \text{Spec}_{\bigcap_{j \in J} D_j} \left( \bigoplus_{i=0}^{\infty} (L^{-i}|_{\bigcap_{j \in J} D_j}) t^i/(t^i - s|_{\bigcap_{j \in J} D_j}) \right),
\]

it is isomorphic to the \( l \)-cyclic covering of \( \bigcap_{j \in J} D_j \) branched along a smooth divisor \( D \cap \bigcap_{j \in J} D_j \). Hence, it is smooth by Lemma 2.51 of [27]. On the other hand, if \( D \cap \bigcap_{j \in J} D_j = \emptyset \), \( s|_{\bigcap_{j \in J} D_j} \) is a nowhere-vanishing section. Therefore, \( L^i|_{\bigcap_{j \in J} D_j} \) is trivial and \( \bigcap_{j \in J} \pi^* D_j \) is isomorphic to the \( l \)-cyclic unramified covering defined by \( s|_{\bigcap_{j \in J} D_j} \). Since \( \bigcap_{j \in J} \pi^* D_j \) is étale over a smooth variety, it is also smooth. We complete the proof of the claim. In particular, \( V' \) is a smooth variety. In fact, let a closed point \( x \in D \). Then \( \mathcal{O}_{V,x} \) is a regular local ring and hence is a unique factorization domain ([34 Theorem 48]). The restriction of \( s \) is an irreducible element of \( \mathcal{O}_{V,x} \). Note that \( \text{Spec} \mathcal{O}_{V,x} \times_V V' \cong \text{Spec} (\mathcal{O}_{V,x}[t]/(t^i - s)) \) and \((t^i - s)\) is a prime ideal. Therefore, the generic fibre is irreducible and so is \( V' \).

Next, we prove the first assertion. We can easily see that

\[
\text{red}(\pi^* D) \cap \bigcap_{j \in J} \pi^* D_j
\]

is smooth where \( D_j \) and \( J \) as above if \( \text{red}(\pi^* D) \cap \bigcap_{j \in J} \pi^* D_j \neq \emptyset \) by Lemma 2.51 of loc.cit. In fact, \( \pi^{-1}\bigcap_{j \in J} D_j \to \bigcap_{j \in J} D_j \) is an \( l \)-cyclic covering branched along \( D|_{\bigcap_{j \in J} D_j} \) by the claim. Let \( Z = \bigcap_{j \in J} D_j \) and \( F = D|_{\bigcap_{j \in J} D_j} \). If \( s \in \Gamma(Z, L^i|_Z) \) is a global generator of \( F \), \( \pi^{-1}Z \) is isomorphic to \( \text{Spec} \mathcal{O}_Z[t]/(t^i - s) \) affine locally. Hence, \( \text{red}(\pi^* F) \) is isomorphic to \( F \) and it is smooth. On the other hand, we can see \( \pi^* F = \text{red}(\pi^* F) \) by \( \pi^{-1}Z \cong \text{Spec} \mathcal{O}_Z[t]/(t^i - s) \). In particular, we have \( \pi^* D = \text{red}(\pi^* D) \) when \( J = \emptyset \). Since

\[
(\text{red}(\pi^* D)) \cap \pi^{-1} Z = \pi^* D \cap \pi^{-1} Z
= \pi^* F
= \text{red}(\pi^* F),
\]

we have \( \text{red}(\pi^* D) \cap \pi^{-1} Z = \text{red}(\pi^* F) \). Hence, \( \text{red}(\pi^* D) \cap \bigcap_{j \in J} \pi^* D_j \) is smooth and we finish the proof of the first assertion. For the second assertion, if \( Y' \) is an irreducible component of \( \pi^{-1}Y \) contains \( x' \), it is the unique stratum contains \( x' \) defined \( \text{red}(\pi^* D) \) and the unique irreducible component of each \( \pi^* D_j \) contains \( x' \). For any toroidal fractional ideal sheaf \( b \), there exists a finite number of Cartier divisors \( F_j \) supported on \( E \) such that

\[
b = \sum \mathcal{O}_{\text{Stat} Y}(-F_j)
\]

locally. Then \( \pi^{-1}b = \sum \mathcal{O}_{\text{Stat} Y'}(-\pi^* F_j) \). Since \( \pi^* D = \text{red}(\pi^* D) \) and \( \pi^* D' \) is reduced for any divisor different from irreducible components of \( D \), the rest of the assertion follows immediately.

\[\square\]

Lemma 5.5. Suppose that \((V,E)\) is log smooth and there exists a finite covering \( \pi : V_1 \to V \) such that \((V_1, \pi^* E)\) is also log smooth. For any closed point \( x \in V \), take the stratum
\(Y\) contains \(x\). Then \(Y'\) that is the irreducible component of \(\pi^{-1}(Y)\) contains \(x\) such that \(\pi(x_1) = x\) is the stratum of \(V_1\) and
\[M_+^Y \cong M_+^Y.\]

We can prove Lemma 5.5 similarly to Lemma 5.4.

Proof of Proposition 5.3. By Theorem 4.1.10 of [28] (cf. [1], [27]), we can take a smooth variety \(V_1\) and a finite covering \(p_1 : V_1 \to V\) such that there exists a line bundle \(L_i\) such that \(p_1^*E_i \in |L_i|\) for any \(E_i\), and \(p_1^*E\) is an snc divisor. We emphasize that \(V_1\) is irreducible due to the theorem of Fulton-Hansen [28, Example 3.3.10].

Therefore, we can take \(p_2 : V_2 \to V_1\) the \(l\)-cyclic covering of \(V_1\) branched along \(p_1^*E_1\). Due to Lemma 5.4, \(p_2^*p_1^*E_2 \in |(p_2^*L_2)|\) is also smooth and hence we can construct \(p_3 : V_3 \to V_2\) the \(l\)-cyclic covering of \(V_2\) branched along \(p_2^*p_1^*E_2\).

Therefore, we can construct \(p_{i+1}\) for \(E_i\) inductively and let \(\tilde{\pi} = p_1 \circ \cdots \circ p_{i+1} : \tilde{V} = V_{i+1} \to V\). We can see \((\tilde{V}, \text{red}(\tilde{\pi}^*E))\) is also a log smooth variety by Lemma 5.4. Take any \(x \in \tilde{V}\) and let \(x_i = p_1 \circ \cdots \circ p_{i+1}(\tilde{x})\) and \(x = \tilde{\pi}(\tilde{x})\). Let \(I \subset \{1, \ldots, s\}\) and \(J\) be its complement and suppose that the stratum \(Y\) that contains \(x\) is the irreducible component of \(\bigcap_{i \in I} E_i - \bigcup_{j \in J} E_j\). If \(Y_k\) is the irreducible component of \((p_1 \circ \cdots \circ p_k)^{-1}(Y)\), which contains \(x_k\), then \(Y_k\) is a stratum of \(V_k\) and
\[M_{+Y_k} \cong M_{+Y_k} + \sum_{t \in I \cap \{1, \ldots, k-1\}} \mathbb{Z}_{\geq 0} E_t\]
by Lemmas 5.4 and 5.5. Therefore, if \(\tilde{Y}\) is the irreducible component of \(\tilde{\pi}^{-1}(Y)\) contains \(\tilde{x}\), then \(\tilde{Y}\) is the stratum of \(\tilde{V}\) and \(M_{+Y} \cong lM_{+\tilde{Y}}\).

The rest is also easy. \(\square\)

According to Proposition 5.3, by taking a finite covering \(\tilde{\pi} : \tilde{X} \to X\) and replacing \(X\) by \(\tilde{X}\), we may assume that \(C_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\}\) has the integral vertices for \(n \in \mathbb{Z}_{\geq 0}\) on \(\text{Star} Y_i \times \mathbb{A}^1\). For the third step of the proof of Theorem 5.1 we want the following.

Lemma 5.6. Notations as above. If \(P_i\) has a one-dimensional bounded part of faces \(F_i\) and \(F_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\}\) consists of an integral point for each \(m\) and \(i\), the normalization of \(a\) satisfies the condition (*).

Proof. We may replace \(a\) by its integral closure and may assume that \(a = \sum \mathcal{I}_m^{tm}\) contains all the integral point of \(F_i\) on \(\text{Star} Y_i \times \mathbb{A}^1\). It follows from the assumption \(F_i\) is one-dimensional and the fact \(\text{NP}^\text{Star} Y_i = k\text{NP}^\text{Star} Y_i\) that each \(\mathcal{I}_m\) is invertible and \(a^k\) also contains all the integral point of \(kF_i\), which is the bounded part of faces of \(\text{NP}^\text{Star} Y_i\) and is also one-dimensional. On the other hand, if \(a^k\) is generated by the integral points of \(kF_i\) for \(k \geq 1\), \(a\) satisfies (*). Therefore, we only have to prove that if \(a\) contains all integral points of \(F_i\) and \(F_i\) is one-dimensional, then \(a\) is generated by the integral points of \(F_i\). It follows from the next claim immediately:

Claim. Suppose that \(r' \leq r\) and satisfies that \((0, r') \in F_i\), and let \(\{(y_m, m)\} = F_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\}\) for \(0 \leq m \leq r'\). Then, \(x \geq y_m\) for \((x, m) \in P_i \cap M_{\text{Star} Y_i}\) and there exists \(0 < \epsilon < 1\) such that \((1 - \epsilon)y_m - y_{m+1} \in M_{+Y_i, \mathbb{R}}\) for \(0 \leq m \leq r' - 1\).

We prove the claim by the induction on \(m\). If \(m = 0\) and \((x, m) \in P_i \cap M_{\text{Star} Y_i}\), there exists a point \((y, s)\) of \(F_i\) such that \((x, 0) \geq (y, s)\) by the claim of Remark 4.5. Since \(s \geq 0, s = 0\) and \(y = y_0\). Moreover, \(r'-1/r'y_0 \in P_i\) and we can conclude that there exists
a point \((y, s)\) of \(F_i\) such that \(0 < s \leq 1\) and \(y \leq \frac{r' - 1}{r'}y_0\). Since \(F_i\) is one-dimensional, \(y = (1 - s)y_0 + sy_1\). Therefore,
\[
0 \leq y_1 \leq \frac{r's - 1}{r's}y_0.
\]

We see that if \(y_1 \neq 0\), we have \(r's > 1\) and hence the second assertion follows when \(m = 0\). Assume that \(0 \leq m' \leq r' - 1\) satisfies that the claim holds for \(m < m'\). For \((x, m') \in P_i \cap M_{\text{Star}\, Y' \times \mathbb{A}^1}\), there exists a point \((y, s)\) of \(F_i\) such that \((x, m') - (y, s) \in M_{\text{Star}\, Y' \times \mathbb{A}^1, \mathbb{R}}^+\) by Remark 4.5. By the induction hypothesis and the assumption that \(F_i\) is one-dimensional, we can see \(y \geq y_m\). The second assertion when \(m = m'\) follows from the similar argument as when \(m = 0\). Therefore the claim holds also for \(m'\).

\[\square\]

**Remark 5.7.** From the first of the proof of this lemma, \(P_i \cap \{(\text{ord}_{X \times \{0\}})^* = 0\}\) has the unique vertex \(D_0\).

**Proof of Theorem 5.7.** We may assume that all the \(C_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\}\) has finitely many integral vertices by Proposition 5.3. Let \(\mathcal{J}_m\) be an ideal sheaf globally defined on \(X\) such that
\[
\overline{a} = \sum_{m=0}^{r} \mathcal{J}_m t^m.
\]

Let also the integral vertices of \(C_i \cap \{(\text{ord}_{X \times \{0\}})^* = m\}\) be \(x_1^{(m)}, \ldots, x_k^{(m)}\). Then each \(\mathcal{J}_m\) is a toroidal ideal sheaf whose Newton polyhedron is generated by \(\text{Conv}(x_1^{(m)}, \ldots, x_k^{(m)})\) on \(\text{Star} Y_i\) by Remark 4.5 and Lemma 5.2. Since \(X\) is toroidal, we can take its toroidal log resolution \(\pi : X' \to X\) such that \((X', \pi^* D_0)\) is log smooth and the inverse images of \(\mathcal{J}_m\) are line bundles by Theorems 10* and 11* of [21, Chapter II]. Take Star \(Y'\) of \(X'\) corresponds to a subcone \(\sigma^{Y'} \subset \sigma^{Y_i}\). Then there exists a canonical surjection \(M_{Y_i} \to M_{Y'}\) and \(\text{Image} M_{Y_i}^+ \subset M_{Y'}^\times\). \(C_i\) is a generating convex set of \(P_i\) and the image of \(C_i\) also generates \(\text{NP}_{\text{Star} Y' \times \mathbb{A}^1}\) by Lemma 4.7. Therefore, the bounded part of faces \(F'\) on \(\text{Star} Y'\) is contained in the image of \(F_i\) and the image of \(C_i\), which we call \(C'\). Furthermore, one of the images of \(x_1^{(m)}, \ldots, x_k^{(m)}\) is smaller than the others on \(\sigma^{Y'}\) also by Lemma 4.7. Therefore, \(F'\) meets \(\{(\text{ord}_{X \times \{0\}})^* = m\}\) at one point \(x_j^{(m)}\). On the other hand, \(C' \cap \{m \leq (\text{ord}_{X \times \{0\}})^* \leq m + 1\}\) is the convex hull of all \(x_j^{(m)}, x_j^{(m+1)}\) since \(C'\) is generated by integral vertices. Therefore, \(F'\) meets \(\{(\text{ord}_{X \times \{0\}})^* = m + s\}\) only at one point \((1-s)x_j^{(m)} + sx_j^{(m+1)}\) for \(0 < s < 1\) and hence \(F'\) is one-dimensional. Then the first assertion follows from Lemma 5.6 and the rest follows from Corollary 5.8 of [11] since \(\mathcal{J}_0\) is a line bundle from the first. \[\square\]

We have the following consequence by Theorem 5.1.

**Remark 5.8.** We can use Theorem 4.8 and Theorem 5.1 for computations of \((\mathcal{J}^H)^{\text{NA}}\). In fact, for any \(n(\geq 2)\)-dimensional polarized variety \((X, L)\) with a divisor \(H\) and any class \(\phi \in H^{\text{NA}}(L)\), there exists a flag ideal \(a = \sum a_k t^k\) and \(s \in \mathbb{Q}_{\geq 0}\) such that a semialgebraic test configuration \((\mathcal{X}, \mathcal{L}) = (\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - sE)\), where \((\cdot)\) means the normalization, \(s \geq 0\) and \(E\) is the exceptional divisor. If \(s = 0\), \(\phi\) is trivial. Hence, we may assume that \(s > 0\). Since \((\mathcal{J}^H)^{\text{NA}}\) is homogeneous in \(\mathcal{L}\),
\[
(\mathcal{J}^H)^{\text{NA}}(\mathcal{X}, \mathcal{L}) = s(\mathcal{J}^H)^{\text{NA}}(\mathcal{X}, L_{\mathbb{A}^1} - E).
\]

Then replace \(L\) by \(\frac{1}{s}L\) and we may assume that \(s = 1\). If necessary, replace \((\mathcal{X}, L_{\mathbb{A}^1} - E)\) by \((\mathcal{X}, L_{\mathbb{A}^1} - E + mX_0)\) and we may assume that \(a_0 \neq 0\). Now, \(a\) satisfies the assumption
of Theorem 5.1. Then there exists an alternation \( \pi : X' \to X \) in Theorem 5.1 whose degree is \( l \) over \( X \). We can see

\[
(J^\pi H)^{\text{NA}}(\Bl_{\pi^{-1}a}(X' \times A^1), \pi^*L_{A^1} - E') = (J^H)^{\text{NA}}(X, \mathcal{L}).
\]

where \( E' \) is the exceptional divisor of \( \Bl_{\pi^{-1}a}(X' \times A^1) \). In fact, \( \Bl_{\pi^{-1}a}(X' \times A^1) \to X \times A^1 \) factors through \( \Bl_a(X \times A^1) \) by the universal property of blowing up (cf. Proposition 7.14 of [19]). Let \( \pi' : \Bl_{\pi^{-1}a}(X' \times A^1) \to \mathcal{X} \) and \( E \) be the exceptional divisor of \( \mathcal{X} \). Then \( E' = \pi^*E \) and the above equality follows from

\[
\pi^*H_{A^1}(\pi^*L_{A^1} - E')^n - \frac{n \pi^*H \cdot (\pi^*L)^{n-1}}{(n + 1)(\pi^*L)^n}(\pi^*L_{A^1} - E')^{n+1} = l(H_{A^1}(L_{A^1} - E))^n - \frac{nH \cdot L^{n-1}}{(n + 1)L^n}(L_{A^1} - E)^{n+1}
\]

(cf. [2 Lemma 1.18]) since \( \pi' \) is also proper and a generically finite morphism whose degree is \( l \). Let

\[
\frac{\pi^{-1}a}{\sum_{k=0}^{r} \mathcal{I}_k t^k},
\]

where each \( \mathcal{I}_k \) is an invertible ideal of \( X' \). Now, we want to show the following.

**Theorem 5.9.** Notations as in Remark 5.8 Then we have

\[
V(\pi^*L)((J^\pi H)^{\text{NA}}(\Bl_{\pi^{-1}a}(X' \times A^1), \pi^*L_{A^1} - E')) = \sum_{k=0}^{r} \sum_{j=0}^{n-1} e_{\pi^*L}(\mathcal{I}_k^{[j]}, \mathcal{I}_{k+1}^{[n-j]}) - \sum_{k=0}^{r-1} \sum_{j=0}^{n-1} e_{\pi^*L}(\mathcal{I}_k^{[j]}, \mathcal{I}_{k+1}^{[n-j]})
\]

In other words, we can calculate \((J^H)^{\text{NA}}\)-energy of \((X, \mathcal{L}) = (\Bl_a(X \times A^1), L_{A^1} - E)\) as follows:

\[
(J^H)^{\text{NA}}(X, \mathcal{L}) = \frac{1}{lV(L)} \left( \sum_{k=0}^{r} \sum_{j=0}^{n-1} e_{\pi^*L}(\mathcal{I}_k^{[j]}, \mathcal{I}_{k+1}^{[n-j]}) - \sum_{k=0}^{r-1} \sum_{j=0}^{n-1} e_{\pi^*L}(\mathcal{I}_k^{[j]}, \mathcal{I}_{k+1}^{[n-j]}) \right).
\]

Furthermore, there exists a sequence of \((J^{\pi^*H})\)-energy of semistable test configurations that converges to \((J^{\pi^*H})^{\text{NA}}(\Bl_{\pi^{-1}a}(X' \times A^1), \pi^*L_{A^1} - E')\).

**Proof of Theorem 5.9** Let \((X', \mathcal{L}') = (\Bl_{\pi^{-1}a}(X' \times A^1), \pi^*L_{A^1} - E')\) be the normalized blow up. First, note \( X \) is the normalized blow up along \( a \) and \( X' \) is the normalized blow up along \( \pi^{-1}a \) in Remark 5.8. They are normal test configurations of \( X \) and \( X' \) respectively. Now, we have the following commutative diagram.

\[
\begin{array}{ccc}
X' & \xrightarrow{\pi'} & X \\
\rho' \downarrow & & \rho \downarrow \\
X' \times A^1 & \xrightarrow{\pi} & X \times A^1
\end{array}
\]

\( X' \) is isomorphic to the strict transformation of the normalization of \( \mathcal{X} \times X \). In fact, let the latter be \( \overline{X} \) and we can find a birational morphism from \( X' \) to \( \mathcal{X} \times X \) induced by the diagram. The morphism induces a birational morphism \( \alpha : X' \to \overline{X} \). On the other
hand, the inverse image of $\pi^{-1}a$ under $\rho'$ coincides the inverse image of $\rho^{-1}a$ under $\pi'$ on $X' \times_X X'$ and is hence an invertible sheaf on $\mathcal{X}'$ since it is an inverse image of an invertible ideal sheaf and $\mathcal{X}'$ is integral. Therefore, there exists a birational morphism $\beta: \mathcal{X}' \to X'$ by the universal property of blowing up and hence $\alpha$ and $\beta$ are the inverse of each other, i.e. there is a canonical isomorphism

$$\mathcal{X}' \cong \mathcal{X}'.$$

Let $F$ be a $\pi$-ample divisor on $X'$. Then $\rho'^*F$ is $\pi'$-ample. We can show that $\pi^*L + \epsilon F$ are ample for sufficiently small $\epsilon > 0$. On the other hand, we can show that $\rho'^{-1}\pi^{-1}a = \pi'^* (\rho^{-1}a)$ and hence

$$\rho'^* (\pi^*L_{\mathcal{A}^1} + s\rho_{\mathcal{A}^1}) + s\pi'^* (\rho^{-1}a)$$

is $\mathbb{A}^1$-ample on $X' \times_X X'$ for sufficiently small positive rational number $s$. We can easily show that it holds also on $\mathcal{X}'$ by the isomorphism above. For sufficiently small positive rational number $\delta$,

$$\rho'^* \left( \pi^*L_{\mathcal{A}^1} + \frac{s\delta}{1+\delta}F_{\mathcal{A}^1} \right) + \frac{1 + s\delta}{1+\delta} \pi'^* (\rho^{-1}a)$$

is also $\mathbb{A}^1$-ample since

$$(1 + \delta) \left( \rho'^* (\pi^*L_{\mathcal{A}^1} + \frac{s\delta}{1+\delta}F_{\mathcal{A}^1}) + \frac{1 + s\delta}{1+\delta} \pi'^* (\rho^{-1}a) \right) = \rho'^* (\pi^*L_{\mathcal{A}^1}) + \pi'^* (\rho^{-1}a) + \delta(\rho'^* (\pi^*L_{\mathcal{A}^1} + sF_{\mathcal{A}^1}) + s\pi'^* (\rho^{-1}a))$$

is the summation of a semiample divisor and ample one. Let $\epsilon = \frac{s\delta}{1+\delta}$ and note that $\lim_{\delta \to 0} s = 0$ and $s$ is independent of $\delta$. If we could prove the theorem for $(\mathcal{X}', \rho'^* (\pi^*L_{\mathcal{A}^1} + \frac{s\delta}{1+\delta}F_{\mathcal{A}^1}) + \frac{1 + s\delta}{1+\delta} \pi'^* (\rho^{-1}a))$, then we have

$$V(\pi^*L)(\mathcal{J}^{\pi^*H})^{\mathcal{NA}}(\mathcal{X}', \rho'^* (\pi^*L_{\mathcal{A}^1} + \pi'^* (\rho^{-1}a))$$

(2)

$$= \lim_{\delta \to 0} \left( \frac{1 + s\delta}{1+\delta} \right)^{n+1} V \left( \frac{1 + \delta}{1 + s\delta} \pi^*L + \epsilon F \right) (\mathcal{J}^{\pi^*H})^{\mathcal{NA}} \left( \mathcal{X}', \frac{1 + \delta}{1 + s\delta} \rho'^* (\pi^*L_{\mathcal{A}^1} + \epsilon F_{\mathcal{A}^1}) + \pi'^* (\rho^{-1}a) \right)$$

$$= \lim_{\delta \to 0} \left( \frac{1 + s\delta}{1+\delta} \right)^{n+1} \left( \frac{n(\pi^*H) \cdot (\pi^*L + \epsilon F)^{n-1}}{(n+1)(\pi^*L + \epsilon F)^{n}} \sum_{k=0}^{r-1} \sum_{j=0}^{n-1} \epsilon_1^{1+\delta} \epsilon_2^{(\pi^*L + \epsilon F)(\mathcal{J}_{k}^{|j|}, \mathcal{J}_{k+1}^{[n-1-j]})} \right)$$

$$- \sum_{k=0}^{r-1} \sum_{j=0}^{n-1} \epsilon_1^{1+\delta} \epsilon_2^{(\pi^*L + \epsilon F)(\mathcal{J}_{k}^{|j|}, \mathcal{J}_{k+1}^{[n-1-j]})}$$

$$= \frac{n\pi^*H \cdot \pi^*L^{n-1}}{(n+1)\pi^*L^n} \sum_{k=0}^{r-1} \sum_{j=0}^{n-1} \epsilon_1 \epsilon_2 (\mathcal{J}_{k}^{|j|}, \mathcal{J}_{k+1}^{[n-1-j]}) - \sum_{k=0}^{r-1} \sum_{j=0}^{n-1} \epsilon_1 \epsilon_2 (\mathcal{J}_{k}^{|j|}, \mathcal{J}_{k+1}^{[n-1-j]})$$

Therefore, we may assume that $X' \cong X$, $L$ is an ample $\mathbb{Q}$-line bundle and $\mathcal{X}$ is an ample test configuration. Take $\mathcal{A}$ the integral closure of $a$. This is an integrally closed flag ideal and $\mathcal{X}$ is the normalized blow up along this. Furthermore, $\mathcal{A} = \sum \mathcal{J}_{D_i}$ satisfies (*), and $L$ is an ample $\mathbb{Q}$-line bundle. Therefore, we apply Theorem 3.8 and complete the proof.

### 6. J-Stability for Surfaces

In this section, we discuss about the applications of Theorems 5.1 and 5.9 to J-stability for surfaces.
6.1. J-stability for irreducible surfaces. If \((X, L)\) is a polarized normal and irreducible surface with a \(\mathbb{Q}\)-Cartier divisor \(H\), \((J^H)^{NA}\) is decomposed into non-negative values due to Theorems 5.1 and 5.9 as we will see in this section. First, we prepare the following:

**Proposition 6.1.** Let \(X\) be an integral surface, \(L\) be a big and nef \(\mathbb{Q}\)-line bundle and \(H\) be a \(\mathbb{Q}\)-Cartier divisor such that \(L \cdot H \geq 0\) and

\[
2 \frac{L \cdot H}{L^2} L - H
\]

is nef. Let also \(C\) be a pseudoeffective \(\mathbb{Q}\)-Cartier divisor such that

\[
(L - C) \cdot H \geq 0.
\]

Then,

\[
2(C \cdot L)(L \cdot H) - (C \cdot H)(L^2) - (L \cdot H)(C^2) \geq 0.
\]

**Proof.** If \(L \cdot H = 0\), \(-H\) is nef and

\[
2(C \cdot L)(L \cdot H) - (C \cdot H)(L^2) - (L \cdot H)(C^2) = -(C \cdot H)(L^2) \geq 0.
\]

Thus, we may assume that \(L \cdot H > 0\). Let \(B = \frac{L \cdot H}{L^2} L - H\). If \(B \equiv 0\), where \(\equiv\) means the numerical equivalence, let \(F = C - \frac{C \cdot L}{L^2} L\). Then \(F \cdot L = 0\) and \(F^2 \leq 0\) by the Hodge Index Theorem. Hence, we see the proposition is true since

\[
2(C \cdot L)(L \cdot H) - (C \cdot H)(L^2) - (L \cdot H)(C^2) = (C \cdot L)(L \cdot H) - (L \cdot H)(C^2)
\]

\[
\geq (C \cdot L)(L \cdot H) - (C \cdot L) \frac{H \cdot L}{L^2} \frac{L}{L^2} = \frac{(C \cdot L)(L \cdot H)}{L^2} (L \cdot (L - C)) \geq 0.
\]

The last inequality holds by the assumption since \(H \equiv \frac{L \cdot H}{L^2} L\).

Hence, we may assume that \(B \not\equiv 0\). Note that \(L \cdot B = 0\) and let

\[E = C - \left( \frac{C \cdot L}{L^2} L + \frac{C \cdot B}{B^2} B \right).\]

Then \(E \cdot L = E \cdot B = 0\). Now, \(B^2 < 0\) and \(E^2 \leq 0\) by Hodge Index Theorem. Therefore,

\[
2(C \cdot L)(L \cdot H) - (C \cdot H)(L^2) - (L \cdot H)(C^2)
\]

\[
\geq 2(C \cdot L)(L \cdot H) - (L^2) \left( \frac{(H \cdot L)(C \cdot L)}{L^2} - (C \cdot B) \right) - (L \cdot H) \left( \frac{(C \cdot L)^2}{L^2} + \frac{(C \cdot B)^2}{B^2} \right)
\]

\[
= (C \cdot L)(L \cdot H) + (L^2)(C \cdot B) - (L \cdot H) \left( \frac{(C \cdot L)^2}{L^2} + \frac{(C \cdot B)^2}{B^2} \right).
\]

Here, we used \(-(L \cdot H)(E^2) \geq 0\). By assumption, we have that

\[
\frac{H \cdot L}{L^2} L + B = \frac{L \cdot H}{L^2} L - H
\]

is nef and

\[
\left( 1 - \frac{C \cdot L}{L^2} \right) L - \frac{C \cdot B}{B^2} B - E = L - C.
\]

Then, we have

\[
B^2 \geq - \frac{(H \cdot L)^2}{L^2}
\]

and

\[
(H \cdot L) \left( 1 - \frac{C \cdot L}{L^2} \right) + (C \cdot B) \geq 0.
\]
by \((\frac{H \cdot L}{L^2} + B)^2 \geq 0\) and \(H \cdot (L - C) \geq 0\) respectively, and
\[
(C \cdot L)(L \cdot H) + (L^2)(C \cdot B) - (L \cdot H) \left(\frac{(C \cdot L)^2}{L^2} + \frac{(C \cdot B)^2}{B^2}\right)
\geq (C \cdot L)(L \cdot H) + (L^2)(C \cdot B) - (L \cdot H) \left(\frac{(C \cdot L)^2}{L^2} - \frac{(C \cdot B)^2 L^2}{(H \cdot L)^2}\right)
= (L^2)(L \cdot H) \left(\frac{C \cdot L}{L^2} + \frac{C \cdot B}{H \cdot L}\right) \cdot \left(1 - \frac{C \cdot L}{L^2} + \frac{C \cdot B}{H \cdot L}\right).
\]
Here, we used \(-\frac{(C \cdot B)^2}{B^2} \geq (C \cdot B)^2 \frac{L^2}{(H \cdot L)^2}\). Then, we have
\[
\frac{C \cdot L}{L^2} + \frac{C \cdot B}{H \cdot L} = (H \cdot L)^{-1} \left(2 \frac{L \cdot H}{L^2} - L - H\right) \cdot C \geq 0.
\]
On the other hand,
\[
1 - \frac{C \cdot L}{L^2} + \frac{C \cdot B}{H \cdot L} = (H \cdot L)^{-1} \left((H \cdot L) \left(1 - \frac{C \cdot L}{L^2}\right) + (C \cdot B)\right) \geq 0.
\]
We accomplish the proof. \(\square\)

**Remark 6.2.** The assumption that \(L \cdot H \geq 0\) and \((L - C) \cdot H \geq 0\) is satisfied when one of \(\{L - C, H\}\) is nef and the other is pseudoeffective. Moreover, if \(L \cdot H > 0\) and \(B \neq 0\), we have proved that
\[
2(C \cdot L)(L \cdot H) - (C \cdot H)(L^2) - (L \cdot H)(C^2) \geq L^2(L \cdot H) \left(\frac{C \cdot L}{L^2} + \frac{C \cdot B}{H \cdot L}\right) \cdot \left(1 - \frac{C \cdot L}{L^2} + \frac{C \cdot B}{H \cdot L}\right) \geq 0.
\]
in the above proof. We see that if the equality holds, we have
\[
E \equiv 0
\]
and
\[
\left(B^2 + \frac{(H \cdot L)^2}{L^2}\right) (C \cdot B)^2 = 0.
\]
We make use of these facts in the proof of Proposition 7.3 below.

**Theorem 6.3.** Let \((X, L)\) be a polarized normal surface and \(H\) be a pseudoeffective \(\mathbb{Q}\)-Cartier divisor on \(X\) such that
\[
2 \frac{L \cdot H}{L^2} L - H
\]
is nef. Then \((X, L)\) is \(J^H\)-semistable.

**Proof.** First, note that \((J^H)^{NA}\) is linear in \(H\) and \(H + aL\) is a big divisor such that
\[
2 \frac{L \cdot (H + aL)}{L^2} L - (H + aL) = 2 \frac{L \cdot H}{L^2} L - H + aL
\]
is ample for \(a > 0\). Therefore, we may assume that \(H\) is big and \(2 \frac{L \cdot H}{L^2} L - H\) is ample. Furthermore, replace \(H\) by \(bH\) for some \(b > 0\) and we may also assume that \(L \cdot H = L^2\). We must show that \((J^H)^{NA}(\phi) \geq 0\) for any \(\phi \in \mathcal{H}^{NA}(L)\). By replacing \(L\) by some multiple, we may assume as in Remark 5.5 that a flag ideal \(a = \sum a_k t^k\) that satisfies \(a_0 \neq 0\) and induces a semiample test configuration \((X', \mathcal{L}) = (\text{Bl}_a(X \times \mathbb{A}^1), L_{\mathbb{A}^1} - E)\), where \(E\) is the exceptional divisor, that is a representative of \(\phi\). By Theorem 5.9 there exists an alternation \(\pi: X' \to X\) and
\[
\pi^{-1}a = \mathcal{I}_{D_0} + \mathcal{I}_{D_1} t + \cdots + \mathcal{I}_{D_{r-1}} t^{r-1} + t^r,
\]
where each \( D_i \) is an snc divisor, satisfies that the condition (*) and \((\pi^{-1}\cdot a)\pi^*L_1\) is semiample. Next, we take \( F, \epsilon, s \) and \( \delta \) as in the proof of Theorem 5.9. We can check that \( \pi^*H + \epsilon F \) is big and \( \pi^*L + \epsilon F \) and

\[
2\left(\frac{\pi^*L + \epsilon F}{\pi^*L + \epsilon F^2}\right)(\pi^*L + \epsilon F) - (\pi^*H + \epsilon F) = \pi^*(2L - H) + O(\epsilon)\pi^*L + (\epsilon + O(\epsilon^2))F
\]

are ample for sufficiently small \( \epsilon \). Then if we could prove the theorem in the case where \( \pi \) is isomorphic and \( a \) induces an ample deformation to the normal cone, we have by the equation (2) in the proof of Theorem 5.9

\[
(J^H)\!^\text{NA}(X, \rho^*L_{ha} + \rho^{-1}(a)) = \lim_{\delta \to 0} (J^{\pi^*H + \epsilon F})\!^\text{NA} \left(X', \rho^* (\pi^*L_{ha} + \frac{s\delta}{1 + \delta} F_{ha}) + \frac{1 + s\delta}{1 + \delta} \pi^* (\rho^{-1}(a)) \right)
\]

\[
\geq 0,
\]

where \( \epsilon = \frac{s\delta}{1 + \delta} \). Thus, we may also assume that \( \pi \) is an isomorphism and \( a \) induces an ample deformation to the normal cone. Then all the \( L - D_i \) is nef by Corollary 5.8 of [40]. Let us denote \( D_r = 0 \). Now, it is easy to see by Theorem 3.8 that \( V(L)(J^H)\!^\text{NA}(X, \mathcal{L}) \) is as follows:

\[
V(L)(J^H)\!^\text{NA}(X, \mathcal{L}) = p^*H \cdot \mathcal{L}^2 - \frac{2H \cdot L \mathcal{L}^3}{3L^2}
\]

\[
= (3L^2)^{-1} \left( \sum_{i=0}^{r-1} \left( 6((D_i + D_{i+1}) \cdot L)(L \cdot H) - 3((D_i + D_{i+1}) \cdot H)(L^2) - 2(L \cdot H)(D_i + D_{i+1}) \right) \right).
\]

We can prove that

\[
6((D_i + D_{i+1}) \cdot L)(L \cdot H) - 3((D_i + D_{i+1}) \cdot H)(L^2) - 2(L \cdot H)(D_i + D_{i+1}) \geq 0
\]

as follows. Note that \( L - \frac{D_i + D_{i+1}}{2} = \frac{1}{2}((L - D_i) + (L - D_{i+1})) \) is nef. Then by Proposition 6.1

\[
4((D_i + D_{i+1}) \cdot L)(L \cdot H) - 2((D_i + D_{i+1}) \cdot H)(L^2) - (L \cdot H)(D_i + D_{i+1})^2
\]

\[
= 4 \left( 2 \left( \frac{D_i + D_{i+1}}{2} \cdot L \right)(L \cdot H) - \left( \frac{D_i + D_{i+1}}{2} \cdot H \right)(L^2) - (L \cdot H) \left( \frac{D_i + D_{i+1}}{2} \right)^2 \right)
\]

\[
\geq 0.
\]

Therefore, we have that

\[
6((D_i + D_{i+1}) \cdot L)(L \cdot H) - 3((D_i + D_{i+1}) \cdot H)(L^2) - 2(L \cdot H)(D_i + D_{i+1})^2
\]

\[
= 4((D_i + D_{i+1}) \cdot L)(L \cdot H) - 2((D_i + D_{i+1}) \cdot H)(L^2) - (L \cdot H)(D_i + D_{i+1})^2
\]

\[
+ 2(D_i \cdot L)(L \cdot H) - (D_i \cdot H)(L^2) - (L \cdot H)(D_i^2)
\]

\[
+ 2(D_{i+1} \cdot L)(L \cdot H) - (D_{i+1} \cdot H)(L^2) - (L \cdot H)(D_{i+1}^2)
\]

\[
\geq 0.
\]
by Proposition 6.1. Hence,
\[
\left( J^H \right)^{\text{NA}}(X, L) = (3V(L)^2)^{-1} \left( \sum_{i=0}^{r-1} \left( 6((D_i + D_{i+1}) \cdot L)(L \cdot H) - 3((D_i + D_{i+1}) \cdot H)(L^2) - 2(L \cdot H)(D_i^2 + D_i \cdot D_{i+1} + D_{i+1}^2) \right) \right).
\]
is nonnegative. □

**Corollary 6.4.** For any polarized integral surface \((X, L)\) with a big (resp. pseudoeffective) \(\mathbb{Q}\)-Cartier divisor \(H\), the following are equivalent.

1. \((X, L)\) is uniformly \(J^H\)-stable (resp. \(J^H\)-semistable). In other words, there exists \(\epsilon > 0\) such that for any semiample test configuration \((X, L')\)
\[
\left( J^H \right)^{\text{NA}}(X, L') \geq \epsilon \left( J^{\text{NA}} \right)(X, L) \quad \text{(resp.} \geq 0).\n\]

2. \((X, L)\) is uniformly slope \(J^H\)-stable (resp. slope \(J^H\)-semistable). In other words, there exists \(\epsilon > 0\) such that for any semiample deformation to the normal cone \((X, L)\) along any integral curve
\[
\left( J^H \right)^{\text{NA}}(X, L') \geq \epsilon \left( J^{\text{NA}} \right)(X, L) \quad \text{(resp.} \geq 0).\n\]

3. There exists \(\epsilon > 0\) such that for any integral curve \(C\)
\[
\left( 2 \frac{H \cdot L}{L^2} - L - H \right) \cdot C \geq \epsilon L \cdot C \quad \text{(resp.} \geq 0).\n\]

**Proof.** (1) \(\Rightarrow\) (2) is trivial. We have proved that (3) \(\Rightarrow\) (1) when \(2 \frac{H \cdot L}{L^2} - L - H\) is nef. If \(2 \frac{H \cdot L}{L^2} - L - H \geq \epsilon L \cdot C\), then
\[
\left( 2 \frac{H - \epsilon L}{L^2} L - (H - \epsilon L) \right) \cdot C \geq 0.
\]
Therefore, take \(\epsilon\) so small that \(H - \epsilon L\) still be big and
\[
\left( J^H \right)^{\text{NA}} = \left( J^{H-\epsilon L} \right)^{\text{NA}} + \epsilon \left( I^{\text{NA}} - J^{\text{NA}} \right) \geq \epsilon \left( I^{\text{NA}} - J^{\text{NA}} \right) .
\]

(2) \(\Rightarrow\) (3) follows from the next generalized lemma of Lejmi-Székelyhidi [29]. We reprove it for possibly singular varieties. □

**Lemma 6.5 (cf. Proposition 13 [29]).** For any polarized \(n\)-dimensional variety \((X, L)\) with a \(\mathbb{Q}\)-Cartier divisor \(H\), if there exists a \(p\)-dimensional subvariety \(V\) such that
\[
\left( n \frac{H \cdot L^{n-1}}{L^n} L - pH \right) \cdot L^{p-1} \cdot V < 0,
\]
then \((X, L)\) is slope \(J^H\)-unstable. Furthermore, if
\[
\left( n \frac{H \cdot L^{n-1}}{L^n} - (n-p)\epsilon L - pH \right) \cdot L^{p-1} \cdot V < 0
\]
for any \(\epsilon > 0\), then \((X, L)\) is not uniformly slope \(J^H\)-stable.
Proof. To prove the first assertion of this lemma, let $(\mathcal{X}, L_{\Delta^1} - E)$ be a deformation to the normal cone of $V$ and $\hat{X}$ be the strict transformation of $X \times \{0\}$. $\hat{X}$ is isomorphic to the blow up of $X$ along $V$ and let $\pi : \hat{X} \to X$ be the canonical projection and $D$ be the exceptional divisor. Now, we can compute $(\mathcal{J}^H)^{NA}(\mathcal{X}, L_{\Delta^1} - \delta E)$. Note that $\pi_*(D^k)$ is a zero cycle when $k < n - p$. Then for $\delta > 0$,

$$V(L)(\mathcal{J}^H)^{NA}(\mathcal{X}, L_{\Delta^1} - \delta E)$$

$$= \delta \pi^*H \cdot \left( \sum_{i=0}^{n-1} (\pi^*L - \delta D)^i \cdot \pi^*L^{n-i-1} \right) - \delta \frac{nH \cdot L^{n-1}}{(n+1)L^n} \sum_{i=0}^{n} (\pi^*L - \delta D)^i \cdot \pi^*L^{n-i}$$

$$= \delta^{-p+1}(-1)^{n-p-1} \frac{n!}{(n+p+1)!p!} D^{n-p} \cdot \left( \frac{nH \cdot L^{n-1}}{L^n} \pi^*L^p - p\pi^*H \cdot \pi^*L^{p-1} \right) + O(\delta^{-p+2}).$$

It is well-known that $\pi_*((-1)^{n-p-1}D^{n-p}) = e_VX[V]$ where $e_VX$ is the multiplicity of $X$ along $V$ (cf. [13 §4.3]) and $e_VX > 0$. Therefore, we apply the projection formula to obtain

$$(-1)^{n-p-1} D^{n-p} \cdot \left( \frac{nH \cdot L^{n-1}}{L^n} \pi^*L^p - p\pi^*H \cdot \pi^*L^{p-1} \right) = e_VX \left( \frac{nH \cdot L^{n-1}}{L^n} L^p - pH \cdot L^{p-1} \right) \cdot V$$

by the assumption. Hence, for sufficiently small $\delta$, $(\mathcal{J}^H)^{NA}(\mathcal{X}, L - \delta E) < 0$. The last assertion follows from the above argument and replacing $H$ by $H - \epsilon L$ for sufficiently small $\epsilon$. 

\[\square\]

6.2. Remarks on reducible J-stable surfaces.

6.2.1. Stability of reducible schemes. J-stability for reducible schemes behave in a more complicated way as the following generalization of [32 Theorem 7.1] shows:

**Theorem 6.6.** Let $(X, \Delta; L)$ be an $n$-dimensional polarized reducible deminormal pair with a $\mathbb{Q}$-line bundle $H$ such that $X = \bigcup_{i=1}^l X_i$ be the irreducible decomposition. Let also $L_i = L_{|X_i}$ and $H_i = H_{|X_i}$. Suppose that

$$\frac{H \cdot L^{n-1}}{L^n} \neq \frac{H_i \cdot L_i^{n-1}}{L_i^n}$$

for some $1 \leq i \leq l$. Then $(X, L)$ is $J^H$-unstable.

Furthermore, let $\nu : \bigcup_{i=1}^l X_i^\nu \to X$ be the normalization, $\overline{L}_i = \nu^*L_{|X_i^\nu}$, $\overline{H}_i = \nu^*H_{|X_i^\nu}$, and $K(X_i^\nu, \Delta_i^\nu) = \nu^*(K(X, \Delta)|_{X_i^\nu})$. Suppose that

$$\frac{K(X, \Delta) \cdot L^{n-1}}{L^n} \neq \frac{K(X_i^\nu, \Delta_i^\nu) \cdot L_i^{n-1}}{L_i^n}$$

for some $1 \leq i \leq l$. Then $(X, \Delta; L)$ is $K$-unstable.

**Remark 6.7.** We point out a small error in [17]. Notations as in loc.cit. In the proof of [17 Proposition 6.1], it is asserted that $\psi_{L^{-1}}^n \cdot (\psi_N - \mu_N(L)\psi_L) \equiv 0$. To be precise, $\psi_{L^{-1}}^n \cdot (\psi_N - \mu_N(L)\psi_L) \neq 0$ if there exists an irreducible component $(X_i, L_i)$ of $(X, L)$ such that $\mu_{N_{|X_i}}(L_i) = \mu_N(L)$. Theorem 1.1 of loc.cit does not hold for reducible deminormal schemes.

See the definition of deminormal schemes at [32]. To prove Theorem 6.6 recall [51 §9.3] briefly.
**Definition 6.8** (Definition 9.10 of [5]). Let \((X, L)\) be a polarized normal and irreducible variety, \(\phi \in \mathcal{H}^{NA}(L)\) be a positive metric and \((\mathcal{X}, \mathcal{L})\) be a normal, semiample test configuration representing \(\phi\). Suppose that \(\mathcal{X}\) dominates \(X_{A^1}\). For each irreducible component \(E\) of \(X_0\), let \(Z_E \subset X\) be the closure of the center of \(v_E\) on \(X\), and let \(r_E := \text{codim}_X Z_E\). Then the canonical morphism \(\mathcal{X} \to X \times A^1\) maps \(E\) onto \(Z_E \times \{0\}\). Let \(F_E\) be the general fibre of the induced morphism \(E \to Z_E\). Then, set the local degree \(\deg_E(\phi)\) as

\[
\deg_E(\phi) := (F_E \cdot L^{r_E}).
\]

It is independent of the choice of representatives of \(\phi\).

Since \(\mathcal{L}\) is semiample on \(E \subset X_0\), \(\deg_E(\phi) \geq 0\). Furthermore, \(\deg_E(\phi) > 0\) iff \(E\) is not contracted on the ample model of \((\mathcal{X}, \mathcal{L})\). Here, we have the following:

**Proposition 6.9** (Lemma 9.11, Proposition 9.12 of [5]). Notations as above. Given \(0 \leq j \leq n\), \(\mathbb{Q}\)-line bundles \(M_1, \ldots, M_{n-j}\) on \(X\) and a Cartier divisor \(D\) on \(X\) such that \(D = \mathcal{L} - L_{A^1}\). Then, we have, for \(0 < \epsilon \ll 1\) rational:

\[
\epsilon^{r_E} \left[ \deg_E(\phi) \left( \frac{j}{r_E} \right) (Z_E \cdot L^{j-r_E} \cdot M_1 \cdots M_{n-j}) + O(\epsilon^{r_E+1}) \right] \quad \text{for } j \geq r_E
\]

\[
0 \quad \text{for } j < r_E.
\]

The proof of this proposition is straightforward.

Recall the following facts of [5, §7]. Let \(H\) be a \(\mathbb{Q}\)-line bundle on \(X\) and \((\mathcal{X}, L_{A^1} - \epsilon D)\) be a semiample test configuration, where \(D = \sum D_i\). Then

\[
(L_{p_1} - \epsilon D)^{n+1} = -\sum_{j=0}^{n} \left( \epsilon \sum_i D_i \cdot (L_{p_1} - \epsilon D)^j \cdot L_{p_1}^{n-j} \right)
\]

\[
H_{p_1} \cdot (L_{p_1} - \epsilon D)^n = -\sum_{j=0}^{n-1} \left( \epsilon \sum_i D_i \cdot (L_{p_1} - \epsilon D)^j \cdot L_{p_1}^{n-1-j} \right) \cdot H_{p_1}
\]

\[
H_B^{NA}(\mathcal{X}, L_{p_1} - \epsilon D) = V(L)^{-1} \sum_E A_{(X, B)}(v_E) b_E (E \cdot (L_{p_1} - \epsilon D)^n),
\]

where \(E\) runs over the irreducible components of \(X_0\) and \(b_E = \text{ord}_E(\mathcal{X}_0)\), where \(X_0\) is the central fibre. As in [5, Proposition 9.12], if \(r = \min \{r_{D_i}\}\), then

\[
(L_{p_1} - \epsilon D)^{n+1} = O(\epsilon^{r+1});
\]

\[
H_{p_1} \cdot (L_{p_1} - \epsilon D)^n = O(\epsilon^{r+1}).
\]

On the other hand, we remark that

\[
V(L)^{-1} A_{(X, B)}(v_E) b_E (E \cdot (L_{p_1} - \epsilon D)^n) = O(\epsilon^{r_E}).
\]

Finally, we recall the following notion to prove Theorem 6.6:

**Definition 6.10** (Rees Valuation cf. Definition 1.9 of [5]). Let \(X\) be a normal variety and \(Z\) be its closed subscheme. Let also \(\widetilde{X} \to X\) be the normalized blow up along \(Z\) with the exceptional divisor \(E\). A divisorial valuation \(v\) on \(X\) is a Rees valuation with respect to \(Z\) if there exists a prime divisor \(F\) of \(\widetilde{X}\) contained in \(E\) such that

\[
v = \frac{\text{ord}_F}{\text{ord}_F(E)}.
\]
We denote the set of all Rees valuations by \( \text{Rees}(Z) \). Here, we remark that thanks to [5, Theorem 4.8], the following holds. Let \( \mathcal{X} \) be the normalization of the deformation to the normal cone of \( X \) along \( Z \). Then, \( \text{Rees}(Z) \) coincides with the set of the valuations \( v_E \), where \( E \) runs over the non-trivial irreducible components of \( X_0 \). For the definition of \( v_E \), see Definition 2.5.

**Proof of Theorem 6.6.** Note that
\[
\sum_{i=1}^{l} \left( H_i \cdot L_i^{n-1} - \frac{H \cdot L^{n-1}}{L} L_i^n \right) = 0.
\]
Thus, we may assume that
\[
\left( H_1 \cdot L_1^{n-1} - \frac{H \cdot L^{n-1}}{L} L_1^n \right) < 0.
\]
Let \( Z = X_1 \cap \bigcup_{i \geq 2} X_i \) and \( \mathcal{X} \) be the blow up of \( X_{A_1} \) along \( Z \times \{0\} \) with the exceptional divisor \( E \). Note that \( Z \times \{0\} = X_1 \times \{0\} \cap \bigcup_{i \geq 2} X_i \times A_1 \) and hence the strict transformation \( F \) of \( X_1 \times \{0\} \) is disjoint from the strict transformations of \( X_i \times A_1 \) for \( i \geq 2 \). On the other hand, \( F \) is a Cartier divisor on the strict transformation of \( X_1 \times A_1 \). Hence, \( F \) is a Cartier divisor on \( \mathcal{X} \). Choose \( \eta > 0 \) such that \(-E + \eta F\) is \( X_{A_1}\)-ample. For sufficiently small \( \epsilon > 0 \), we consider an ample test configuration
\[
(\mathcal{X}, L_{A_1} - \epsilon(E - \eta F))
\]
over \((X, L)\).

Take the normalization \((\overline{\mathcal{X}} = \coprod_{i} \mathcal{X}_i^\nu, \overline{L}_{A_1} - \epsilon(E - \eta F))\) and \((\overline{\mathcal{X}} = \coprod_{i} \mathcal{X}_i^\nu, \overline{L})\) of \((\mathcal{X}, L_{A_1} - \epsilon(E - \eta F))\) and \((X, L)\) respectively. Then
\[
(J^H)^{\text{NA}}(\mathcal{X}, L_{A_1} - \epsilon(E - \eta F)) = (J^{(\coprod_{i} \mathcal{X}_i^\nu)\cdot \overline{L}_{A_1} - \epsilon(E - \eta F)}) = V(L)^{-1} \sum_{i=1}^{l} \left( \mathcal{P}_{i, \mathcal{X}_i^\nu} \cdot (\mathcal{L}_i^\nu)^n - \frac{nH \cdot L^{n-1}}{(n+1)L^n} (\mathcal{L}_i^\nu)^{n+1} \right),
\]
where \( \mathcal{L}_i^\nu = \overline{L}_{A_1} - \epsilon(E - \eta F) \big|_{\mathcal{X}_i^\nu} \). Note that the image of each irreducible component of \( E \) on \( X \times \{0\} \) has codimension \( \geq 1 \). Then we apply [5, Lemma 9.11] to each \( \mathcal{X}_i^\nu \) and obtain
\[
(J^H)^{\text{NA}}(\mathcal{X}, L_{A_1} - \epsilon(E - \eta F)) = (J^{(\coprod_{i} \mathcal{X}_i^\nu)}\cdot \overline{L}_{A_1} + \epsilon\eta\mathcal{X}_i^{1,0} + O(\epsilon^2)) = \epsilon\eta \frac{n}{V(L)} \left( H_1 \cdot L_1^{n-1} - \frac{H \cdot L^{n-1}}{L} L_1^n \right) + O(\epsilon^2),
\]
where \( \mathcal{X}_i^{1,0} \) is the central fibre of \( \mathcal{X}_i^\nu \). Since \((H_1 \cdot L_1^{n-1} - \frac{H \cdot L^{n-1}}{L} L_1^n) < 0 \), we have \((J^H)^{\text{NA}}(\mathcal{X}, L_{A_1} - \epsilon(E - \eta F)) < 0 \) for \( 0 < \epsilon \ll 1 \).

For K-unstability, we may assume that \((X, L)\) is slc and then the second assertion immediately follows from [5, Lemma 9.11]. Indeed, by the above argument and replacing \( H \) by \( K_{(X, \Delta)} \),
\[
(J^{K_{(X, \Delta)}})^{\text{NA}}(\mathcal{X}, L_{A_1} - \epsilon(E - \eta F)) = \epsilon\eta \frac{n}{V(L)} \left( K_{(X_1^\nu, \Delta_1^\nu)} \cdot L_1^{n-1} - \frac{K_{(X, \Delta)} \cdot L^{n-1}}{L} L_1^n \right) + O(\epsilon^2).
\]
Assume that
\[
\left( K_{(X_1^\nu, \Delta_1^\nu)} \cdot L_1^{n-1} - \frac{K_{(X, \Delta)} \cdot L^{n-1}}{L} L_1^n \right) < 0
\]
and then we have only to show
\[
V(L)H^{\Delta \text{NA}}_\nu(\tilde{X}, \tilde{\mathcal{L}}) = \sum_{i=1}^l V(L_i)H^{\Delta \text{NA}}_{\nu_i}(X_i^\nu, \mathcal{L}_i) = O(\epsilon^2),
\]
where \((\tilde{X}, \tilde{\mathcal{L}})\) is the partial normalization (see Definition 2.3). The proof of the first equality is straightforward. To prove the second equality, note that \(X_i^\nu\) is the normalization of the deformation of \(X_i^\nu\) to the normal cone of \(Z_i\), where \(Z_i\) is the inverse image of \(Z\) under \(X_i^\nu \to X\). Hence, we have only to prove that \(\text{Rees}(Z_i)\) has valuations whose log discrepancies are 0 or whose center on \(X \times \{0\}\) has codimension \(r \geq 2\) thanks to [5, Theorem 4.8]. Indeed, since the points of codimension 1 in \(X_i \cap X_j\) for \(i \neq j\) are nodes, \(Z_i\) is the union of the restrictions of irreducible components of the conductor subscheme to \(X_i^\nu\) and closed subschemes of \(X_i^\nu\) of codimension more than 1 for each \(i\). Note that the valuations of prime divisors of the conductor subscheme have the log discrepancies 0. Therefore, by [5, Proposition 9.12] we have
\[
H^{\Delta \text{NA}}_{\nu_i}(X_i^\nu, \mathcal{L}_i) = O(\epsilon^2).
\]
We complete the proof. \(\square\)

**Definition 6.11.** Let \(V\) be a reducible scheme and \(V = \bigcup V_i\) is the irreducible decomposition. A \(\mathbb{Q}\)-line bundle \(L\) on \(V\) is ample (resp. nef, big, pseudoeffective) if so is each \(L|_{V_i}\).

Reflecting Theorem 6.6, we have only to consider about polarized deminormal schemes \((X, L)\) with \(\mathbb{Q}\)-line bundle on \(X\) such that all irreducible components of \((X, L)\) have the same average scalar curvature with respect to \(H\) (see Definition 2.8). For such \((X, L)\), \(J^H\)-stability behave similarly as for normal varieties. Indeed, we have the following decomposition formula:

**Lemma 6.12.** Let \((X, L)\) be a polarized deminormal scheme with a \(\mathbb{Q}\)-line bundle \(H\) such that all irreducible components \(\{X_i\}_{i=1}^l\) of \(X\) have the same average scalar curvature with respect to \(H\). If the restrictions of \(L\) and \(H\) to each \(X_i\) are \(L_i\) and \(H_i\), respectively, then for any semiample test configuration \((X, \mathcal{L})\),

\[
V(L)(J^H)^{\text{NA}}(X, \mathcal{L}) = \sum_{i=1}^l V(L_i)(J^{H_i})^{\text{NA}}(X_i, \mathcal{L}_i);
\]

\[
V(L)J^{\text{NA}}(X, \mathcal{L}) = \sum_{i=1}^l V(L_i)J^{\text{NA}}(X_i, \mathcal{L}_i);
\]

\[
V(L)I^{\text{NA}}(X, \mathcal{L}) = \sum_{i=1}^l V(L_i)I^{\text{NA}}(X_i, \mathcal{L}_i),
\]

where \(X_i\) is the strict transformation of \(X_i \times \mathbb{A}^1\) and \(\mathcal{L}_i\) is the restriction of \(\mathcal{L}\) to \(X_i\). Moreover, let \((X, B)\) be a deminormal pair, \(\nu : (X^\nu = \coprod_{i=1}^l X_i^\nu, L^\nu) \to (X, L)\) be the normalization and \(\mathcal{B}\) be the conductor subscheme of \(X^\nu\). Suppose that \(\mathcal{B}\) is the divisorial part of \(\nu^{-1}B\) and \(X_i^\nu\) has the same average scalar curvature with respect to \(K_{(X^\nu, \mathcal{B} + \mathcal{B})}\). If
\((\mathcal{X}^{\nu}, \mathcal{L}^{\nu})\) is the normalization of \((\mathcal{X}, \mathcal{L})\), then
\[
V(L)H_B^{\text{NA}}(\mathcal{X}, \mathcal{L}) = \sum_{i=1}^{l} V(L_i^{\nu})H_B^{\text{NA}}|_{\mathcal{X}^{\nu}_i + D_{\mathcal{X}^{\nu}_i}}(\mathcal{X}^{\nu}_i, \mathcal{L}^{\nu}_i);
\]
\[
V(L)M_B^{\text{NA}}(\mathcal{X}, \mathcal{L}) = \sum_{i=1}^{l} V(L_i^{\nu})M_B^{\text{NA}}|_{\mathcal{X}^{\nu}_i + D_{\mathcal{X}^{\nu}_i}}(\mathcal{X}^{\nu}_i, \mathcal{L}^{\nu}_i),
\]
where \(\mathcal{X}^{\nu}_i\) is the strict transformation of \(X^{\nu}_i \times \mathbb{A}^1\), \(L^{\nu}_i\) is the restriction of \(L^{\nu}\) to each \(X^{\nu}_i\) and \(\mathcal{L}^{\nu}_i\) is the restriction of \(\mathcal{L}^{\nu}\) to \(\mathcal{X}_i^{\nu}\).

The proof is straightforward. Therefore we have the following generalization of Corollary 6.4.

**Corollary 6.13.** For any polarized deminormal surface \((X, L)\) with a big (resp. pseudo-effective) \(\mathbb{Q}\)-line bundle \(H\) such that all irreducible components of \((X, L)\) have the same average scalar curvature with respect to \(H\) (see Remark 2.3), the following are equivalent.

1. \((X, L)\) is uniformly \(J^H\)-stable (resp. \(J^H\)-semistable). In other words, there exists \(\epsilon > 0\) such that for any semistable test configuration \((\mathcal{X}, \mathcal{L})\)
\[
(J^H)^{\text{NA}}(\mathcal{X}, \mathcal{L}) \geq \epsilon J^{\text{NA}}(\mathcal{X}, \mathcal{L}) \quad (\text{resp. } \geq 0).
\]

2. \((X, L)\) is uniformly slope \(J^H\)-stable (resp. slope \(J^H\)-semistable). In other words, there exists \(\epsilon > 0\) such that for any semistable deformation to the normal cone \((\mathcal{X}, \mathcal{L})\) along any integral curve
\[
(J^H)^{\text{NA}}(\mathcal{X}, \mathcal{L}) \geq \epsilon J^{\text{NA}}(\mathcal{X}, \mathcal{L}) \quad (\text{resp. } \geq 0).
\]

3. There exists \(\epsilon > 0\) such that for any integral curve \(C\),
\[
\left(2 \frac{H \cdot L}{L^2} L - H\right) \cdot C \geq \epsilon L \cdot C \quad (\text{resp. } \geq 0).
\]

**Proof.** (1) \(\Rightarrow\) (2) is trivial. On the other hand, (3) \(\Rightarrow\) (1) holds by the following argument.
For any semistable test configuration \((\mathcal{X}, \mathcal{L})\), take the normalization \(\left(\coprod \mathcal{X}_i, \coprod \mathcal{L}_i\right)\), where \(\mathcal{X}_i\) is the strict transform of each irreducible component \(X_i \times \mathbb{A}^1\) of \(X \times \mathbb{A}^1\) and \(\mathcal{L}_i = L|_{\mathcal{X}_i}\).
Then we have
\[
V(L)(J^H)^{\text{NA}}(\mathcal{X}, \mathcal{L}) = \sum_i V(L_i)(J^{H_i})^{\text{NA}}(\mathcal{X}_i, \mathcal{L}_i)
\]
by Lemma 6.12 where \(L_i = L|_{\mathcal{X}_i}\) and \(H_i = H|_{\mathcal{X}_i}\). Similarly, we have
\[
V(L)J^{\text{NA}}(\mathcal{X}, \mathcal{L}) = \sum_i V(L_i)J^{\text{NA}}(\mathcal{X}_i, \mathcal{L}_i).
\]
Therefore, the assertion follows from Corollary 6.4.

Finally, to prove (2) \(\Rightarrow\) (3), we have to show the following generalization of Lemma 6.5. Indeed, the corollary follows immediately from Lemma 6.14 below.

**Lemma 6.14.** For any polarized \(n\)-dimensional deminormal scheme \((X, L)\) with a \(\mathbb{Q}\)-line bundle \(H\) such that all irreducible components of \((X, L)\) have the same average scalar curvature, if there exists a \(p\)-dimensional subvariety \(V\) such that
\[
\left(\frac{n H \cdot L^{n-1}}{L^n} L - pH\right) \cdot L^{p-1} \cdot V < 0,
\]
then \((X, L)\) is slope \(J^H\)-unstable. Furthermore, if
\[
\left( \frac{nH \cdot L^{n-1}}{L^n} - (n-p)(1-p)\right) \cdot L^{p-1} \cdot V < 0
\]
for any \(\epsilon > 0\), then \((X, L)\) is not uniformly slope \(J^H\)-stable.

**Proof.** Recall that Lemma 6.6 holds for nonnormal varieties and note that we have only to prove the lemma when \(\epsilon = 0\). Take \(V\) as in the assumption. Let \(X_i\)'s be irreducible components of \(X\). If \(\mathcal{X}\) is the blow up of \(X \times \mathbb{A}^1\) along \(V \times \{0\}\), the strict transformation \(X_i\) of each \(X_i \times \mathbb{A}^1\) is the blow up along \(V_i \times \{0\}\), where \(V_i\) is the restriction of \(V\) to \(X_i\). Note that if \(V \subset X_i\),
\[
\left( \frac{nH_i \cdot L_i^{n-1}}{L_i^n} - pH_i\right) \cdot L_i^{p-1} \cdot V_i = \left( \frac{nH \cdot L^{n-1}}{L^n} - pH\right) \cdot L^{p-1} \cdot V < 0,
\]
where \(L_i\) and \(H_i\) are the restrictions of \(L\) and \(H\) to \(X_i\) respectively. Moreover, if \(E\) is the exceptional divisor of \(\mathcal{X}\), then the restriction \(E|_{X_i}\) to \(X_i\) is the one of \(X_i\). Hence, it follows from the computation of the proof of Lemma 6.5 that for sufficiently small \(\delta > 0\),
\[
V(L_i)((\mathcal{J}^H_i)^{\text{NA}}(X_i, L_i, \mathbb{A}^1-\delta E|_{X_i})) = \frac{n!}{(n-p+1)!} \left( \frac{nH \cdot L^{n-1}}{L^n} - pH \cdot L^{p-1} \right) \cdot V_i + O(\delta^{n-p+2})
\]
for \(X_i\) such that \(V \subset X_i\) and
\[
V(L_j)((\mathcal{J}^H_j)^{\text{NA}}(X_j, L_j, \mathbb{A}^1-\delta E|_{X_j})) = O(\delta^{n-p+2})
\]
for \(X_j\) such that \(V \not\subset X_j\). We complete the proof. \(\square\)

### 6.2.2 Counter examples of Proposition 6.4 for reducible schemes.
In fact, it is easy to see that Proposition 6.4 also hold when \((X, L)\) is deminormal surface with a pseudoeffective \(\mathbb{Q}\)-line bundle \(H\) such that all irreducible components of \((X, L)\) have the same average scalar curvature. Indeed, we have only to take the normalization and apply Proposition 6.4 to each irreducible component. However, Hodge index theorem does not hold even for connected deminormal surfaces in general as the following example shows. We prove that Proposition 6.4 does not hold in general either.

**Example 6.15.** Let \(X\) be a projective, smooth and irreducible surface and \(P\) be a closed point of \(X\). Let \(\pi: Y \to X\) be the blow up a closed point \(P\) of \(X\). Then glue two copies of \(Y\) by [25, Theorem 33], say \(Y_1\) and \(Y_2\), along the exceptional curve \(E\), as a deminormal surface
\[
Z = Y_1 \cup_E Y_2.
\]
The construction as follows:

**Lemma 6.16.** Suppose that \(V\) is a smooth irreducible variety and \(F\) is an snc divisor on \(V\). Let \(V_1\) and \(V_2\) be two copies of \(V\). Then, there exists an slc scheme \(W \cong V_1 \cup_F V_2\).

**Proof.** Take an ample \(\mathbb{Q}\)-divisor \(\Delta\) on \(V\) such that \((V, F + \Delta)\) is an lc pair and \(K_V + \Delta + F\) is ample. For \(i = 1, 2\), let \((V_i, F_i + \Delta_i)\) be copies of \((V, F + \Delta)\). We can easily take the canonical involution \(\tau\) of \((F_i + F_2, \text{Diff}_{F_1} \Delta_1 + \text{Diff}_{F_2} \Delta_2)\) by changing indices. By Kollár’s gluing theorem (Theorem 23 of [25]), there exists an slc pair \((W, \overline{\Delta})\). In fact, \(W = V_1 \cup_F V_2\) set-theoretically. Finally, \(W\) is slc since \(\overline{\Delta}\) is effective. \(\square\)
Remark 6.17. We can also glue $Y_1$ and $Y_2$ directly as follows. Take an affine open neighborhood $U \cong \text{Spec } A$ of $X$ contains $P$. Let $x, y \in A$ form a system of parameter around $P$. Then

$$U' = \text{Proj } ((A \otimes A/(x_1x_2, y_1y_2, x_1y_2, y_1x_2))[u, v]/(x_1v - y_1u, x_2v - y_2u))$$

is étale equivalent to

$$\text{Proj } ((k[T_1, W_1] \otimes k[T_2, W_2]/(T_1T_2, W_1W_2, T_1W_2, W_1T_2))[u, v]/(T_1v - W_1u, T_2v - W_2u)),$$

where $x_1 = x \otimes 1, x_2 = 1 \otimes x$. Hence, $U'$ is deminormal. We can construct $Z$ by gluing together $U'$ and $(Y_1 - E_1) \cup (Y_2 - E_2)$. Finally, we can easily check that $Z$, which we have constructed just now, coincides with the deminormal surface we obtain by Lemma 6.16 thanks to [26, Proposition 5.3].

We can see that the following lemma.

Lemma 6.18. Suppose that $M_1 = \pi^*F_1 - \epsilon E$ and $M_2 = \pi^*F_2 - \delta E$ are line bundles on $Y_1$ and $Y_2$ respectively, where $F_1$ and $F_2$ are line bundles on $X$. If $\epsilon = \delta$, there exists $M$ a line bundle on $Z$ whose restriction to $Y_1$ and $Y_2$ are $M_1$ and $M_2$ respectively.

Proof. First, we prove the lemma in the case where $\epsilon = \delta = 0$. We may assume that $F_1$ and $F_2$ are ample and there are general global sections $s_1$ and $s_2$ whose zero loci do not pass through $P$. If $D_i = (s_i)_0$, $\pi^*D_i$ is disjoint from $E_i$ and we can get $D$ by gluing $\pi^*D_1$ and $\pi^*D_2$. Hence, $D$ is a Cartier divisor on $Z$ and we obtain $F = O_Z(D)$ such that $F|_{Z_i} = F_i$ for $i = 1, 2$. Next, we consider an affine neighborhood of $P$ of $X, \text{Spec}(A)$. Let $x, y \in A$ form a system of parameter around $P$. As we saw in Remark 6.17 $Z$ is

$$\text{Proj } ((A \otimes A/(x_1x_2, y_1y_2, x_1y_2, y_1x_2))[u, v]/(x_1v - y_1u, x_2v - y_2u))$$

locally around $E$ where $x_1 = x \otimes 1, x_2 = 1 \otimes x$. On $(\text{Spec } A \cup \text{Spec } A) \setminus P$, $Z$ is isomorphic to this. $O_{\text{Proj}(1)}$ can be extended globally such that it is trivial on $Z \setminus E$ and the restriction to $Y_i$ is $O(-E_i)$. \hfill \Box

Then, Hodge index theorem does not hold on $Z$, which we have constructed. Take an ample line bundle $L$ on $X$, let two line bundles $L_1, L_2$ are two copies of $\pi^*L$ on $Y_1, Y_2$. We construct line bundles $M_1$ and $M_2$ on $Z$ by gluing $L_1$ and $O_{Y_2}$ together, and $O_{Y_1}$ and $L_2$ together respectively. Then $M_1^2, M_2^2 > 0$ but $M_1 \cdot M_2 = 0$.

Here by using $Z$ above, we construct a counter-example for Theorem 6.3 when $X$ is reducible and all irreducible components of $X$ do not have the same average scalar curvature. Due to Lemma 6.18 we can construct two line bundles $\tilde{L}$ and $\tilde{H}$ by gluing

$$\delta L_1 - \epsilon E_1, L_2 - \epsilon E_2$$

and

$$\delta L_1 - \frac{1}{3}\epsilon E_1, \left(\frac{1}{2} + \eta\right) L_2 - \frac{1}{3}\epsilon E_2$$

respectively for $0 < \epsilon \ll \delta \ll \eta \ll 1$. Then

$$\frac{\tilde{L} \cdot \tilde{H}}{L^2} = \frac{1}{2} + \eta + O(\epsilon, \delta).$$
Therefore,
\[
\nu^* \left( \frac{2}{L^2} \hat{L} \cdot \hat{H} \hat{L} - \hat{H} \right) = (1 + 2\eta + O(\epsilon, \delta))(\delta L_1 - \epsilon E_1 \times L_2 - \epsilon E_2)
\]
\[
- \left( \delta L_1 - \frac{1}{3} \epsilon E_1 \times \left( \frac{1}{2} + \eta \right) L_2 - \frac{1}{3} \epsilon E_2 \right)
\]
\[
= \left( \delta(2\eta + O(\epsilon, \delta))L_1 - \epsilon \left( \frac{2}{3} + O(\epsilon, \delta, \eta) \right) E_1 \right.
\]
\[
\times \left( \frac{1}{2} + \eta + O(\epsilon, \delta) \right) L_2 - \epsilon \left( \frac{2}{3} + O(\epsilon, \delta, \eta) \right) E_2 \right)
\]
is ample for \(0 < \epsilon \ll \eta \delta \ll \delta \ll \eta \ll 1\), where \(\nu\) is the normalization. Hence, \(2\frac{\hat{L} \cdot \hat{H}}{L^2} \hat{L} - \hat{H}\), \(\hat{L}\) and \(\hat{H}\) are also ample. We may assume that \(\eta \delta L - \epsilon E\) is ample and \(\hat{L} - C\) is ample, where
\[
\nu^* C = (\delta(1 - \eta)L_1 \times \mathcal{O}_{Y_2}).
\]

Then
\[
\left( \frac{2}{L^2} \hat{L} \cdot \hat{H} \hat{L} - \hat{H} \right) \cdot C - \frac{\hat{L} \cdot \hat{H}}{L^2} C^2 = \frac{1}{2} \delta^2(-1 + O(\epsilon, \delta, \eta))L^2 < 0.
\]

7. **Stability versus Uniform Stability**

In this section, we construct polarized surfaces \((X, L)\) with ample divisors \(H\) such that \((X, L)\) are \(J^H\)-stable but not uniformly \(J^H\)-stable.

G. Chen proved the uniform version of Lejmi-Székelyhidi conjecture in [6]. The original conjecture (cf. [29], [9]) was also solved recently as follows:

**Theorem 7.1** (Theorem 1.2 of [10], Corollary 1.2 of [44]). Notations as in Theorem 3.10.\(\)

(1) is equivalent to the following:

(7)
\[
\int_V c_0\omega_0^p - p\chi \land \omega_0^{p-1} > 0
\]
for all \(p\)-dimensional subvarieties \(V\) with \(p = 1, 2, \cdots, n - 1\).

It is proved in [10, Corollary 1.3] when \(M\) is projective and in [44, Corollary 1.2] for general Kähler manifolds. We can easily prove the original conjecture in projective smooth surfaces holds as follows.

**Proposition 7.2.** In Corollary 6.4 (3), there exists \(\epsilon > 0\) such that for any integral curve \(C\),
\[
\left( 2 \frac{H \cdot L}{L^2} L - H \right) \cdot C \geq \epsilon L \cdot C
\]
iff
\[
\left( 2 \frac{H \cdot L}{L^2} L - H \right) \cdot C > 0
\]
for any integral curve \(C\).
Proof. The second assertion follows from the first assertion and Theorem 1.1 of [6]. For curvature with respect to $H$ such that ($2H - H^2 > 0$ and if $C \cdot (2H - H^2) > 0$ for any integral curve, $2H - H$ is ample by the Nakai-Moishezon criterion. Therefore, there exists $\epsilon > 0$ such that $(2H - H^2 - \epsilon)H - H$ is ample. Hence we have Proposition 7.2 by Corollary 6.4. For Kähler manifolds, the solvability of J-equation is equivalent to the class $2H - H^2 c_1(L) - c_1(H)$ is Kähler by [15].

However, J-stability and uniform J-stability is not equivalent as following.

**Theorem 7.3.** There are smooth polarized surfaces $(X,L)$ with ample divisors $H$ such that $(X,L)$ are $JH$-stable but not uniformly stable. In particular, if Kähler metrics $\omega_0$ and $\chi$ satisfy that $\omega_0 \in c_1(L)$ and $\chi \in c_1(H)$, there is no smooth function $\varphi$ that satisfies the J-equation:

$$\operatorname{tr}_{\omega_0} \chi = \frac{H \cdot L}{L^2},$$

where

$$\omega_\varphi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$$

is a positive $(1,1)$-form.

**Proof.** The second assertion follows from the first assertion and Theorem 1.1 of [19]. For the first assertion, we can construct an example such that $L, H$ is ample but there is an integral curve $C_0$ on $X = \mathbb{P}^1(\mathcal{O} \oplus \mathcal{O}(e))$ for $e > 0$, such that $(2H - H^2 - \epsilon)H - C_0 = 0$ as follows:

$C_0$ be the section of $\mathbb{P}^1$ satisfies that $C_0^2 = -e$ and $f$ denote the numerical class of fibre. By Corollary 2.18 in [19], $aC_0 + bf$ is ample $\iff b > ae > 0$. Therefore, $aC_0 + ae f$ is nef for $a > 0$. Let $L = mC_0 + nf$ and

$$H = \frac{2na}{2nm - m^2e}(mC_0 + nf) - (aC_0 + af)$$

$$= \frac{m^2ea}{2nm - m^2e}C_0 + \frac{(2n(n - me) + m^2e^2)a f}{2nm - m^2e}$$

for $0 < me < n$ and $a > 0$. Here, we see that $H$ is ample. Then we can show that $2H - H^2 = aC_0 + af$ by an easy computation. Note that $(aC_0 + af) \cdot C_0 = 0$ and $(X,L)$ is not uniformly $JH$-stable by Lemma 6.5. Next, we prove the following:

**Proposition 7.4.** For any polarized deminormal surface $(X,L)$ with an ample $\mathbb{Q}$-line bundle $H$ such that all irreducible components of $(X,L)$ have the same average scalar curvature with respect to $H$ and

$$2 \frac{L \cdot H}{L^2} H - H$$

is nef, $(X,L)$ is $JH$-stable.

**Proof of Proposition 7.4.** We can easily reduce to the case when $X$ is normal and irreducible by taking the normalization. Thus we may assume that $X$ is normal and irreducible. Suppose that $a$ is a flag ideal induces a nontrivial semiample test configuration. As in the proof of Theorem 6.3 there exists an alternation $\pi : X' \to X$ due to Theorem 5.1 and

$$\overline{\pi^{-1}a} = \mathcal{I}_{D_0} + \mathcal{I}_{D_1} t + \cdots + \mathcal{I}_{D_{r-1}} t^{r-1} + t^r,$$

where each $D_i$ is a Cariter divisor, satisfies that the condition (*) and $(\pi^{-1}a) \pi^* L_1$ is semiample and $\pi^* L - D_0$ is nef. Let $L' = \pi^* L$ and $H' = \pi^* H$. Now, we can see the
following by Theorem 5.9,
\[
p^*H' \cdot L'^2 - \frac{2H' \cdot L'}{3L'^2} L'^3 = (3L'^2)^{-1} \left( \sum_{i=0}^{r-2} \left( 6((D_i + D_{i+1}) \cdot L')(L' \cdot H') - 3((D_i + D_{i+1}) \cdot H')(L'^2) - 2(L' \cdot H')(D^2_i + D_i \cdot D_{i+1} + D^2_{i+1}) \right) + 6(D_{r-1} \cdot L')(L' \cdot H') - 3(D_{r-1} \cdot H')(L'^2) - 2(L' \cdot H')(D^2_{r-1}) \right).
\]

Note that \(D_0 \geq D_1 \geq \cdots \geq D_{r-1} \geq 0\), where \(D \geq D'\) if \(D - D'\) is effective, and we may assume that \(D_{r-1} \neq 0\). Therefore, we can prove that
\[
6((D_i + D_{i+1}) \cdot L')(L' \cdot H') - 3((D_i + D_{i+1}) \cdot H')(L'^2) - 2(L' \cdot H')(D^2_i + D_i \cdot D_{i+1} + D^2_{i+1}) \geq 0
\]
byproposition 6.1 as in the proof of Theorem 5.3, since \(L'\), \(H'\) and \(2L'H' L' - H'\) are nef and big and all the \(L' - D_i\) and \(L' - \frac{D_i + D_{i+1}}{2}\) are pseudoeffective (cf. Remark 6.2). Hence, we have only to prove
\[
6(D_{r-1} \cdot L')(L' \cdot H') - 3(D_{r-1} \cdot H')(L'^2) - 2(L' \cdot H')(D^2_{r-1}) > 0.
\]
As in the proof of Proposition 6.1, let \(B = \frac{L'H' L'}{L'^2} L' - H' \neq 0\) and \(E = D_{r-1} - \frac{D_{r-1} L'}{L'^2} L' + \frac{D_{r-1} B}{B^2} B.\) Since \(H'\) is nef and big, we have \(H'^2 > 0\) and \(B^2 > -\frac{(H' \cdot L')^2}{L'^2}\). Hence, it is necessary for the value (3) to be zero that \(E^2 = 0\) and \(D_{r-1} \cdot B = 0\). Equivalently, there exists \(t \in \mathbb{Q}\) such that
\[
D_{r-1} \equiv tL'.
\]
However, if \(D_{r-1} \equiv tL'\), then \(0 < t \leq 1\) since \(L' - D_{r-1}\) is pseudoeffective (if \(t = 0\), \(a\) induces an almost trivial test configuration) and we see
\[
6(D_{r-1} \cdot L')(L' \cdot H') - 3(D_{r-1} \cdot H')(L'^2) - 2(L' \cdot H')(D^2_{r-1}) \geq t(L'^2)(H' \cdot L') > 0.
\]
We complete the proof of the proposition. \(\square\)

Therefore, we have constructed \(J^H\)-stable smooth surfaces but they are not uniformly \(J^H\)-stable. We finish the proof of Theorem 5.3. \(\square\)

We apply Theorem 7.3 to construct the following log pairs. We remark that they are not log Fano (cf. [33, Theorem 1.5]).

**Corollary 7.5.** There exist polarized normal pairs \((X, \Delta; L)\) such that they are K-stable but not uniformly K-stable.

**Proof.** In the proof of Theorem 7.3, fix a polarization \(L = mC_0 + nf\). By Bertini’s theorem, we can take a general ample \(\mathbb{Q}\)-divisor \(\Delta\) and \(K_{(X, \Delta + C_0)} \equiv H\) for sufficiently large \(a\) so that \((X, \Delta + C_0)\) is lc and its lc center contains \(C_0\). Since \((X, \Delta + C_0; L)\) is \((J^{K(X, \Delta + C_0)})^{NA}\)-stable as we saw and has only lc singularities, we have \(H^{NA}_{\Delta + C_0} \geq 0\). Therefore, \((X, \Delta + C_0; L)\) is K-stable. However, if \(X\) is the deformation to the normal cone along \(C_0\), then \(H^{NA}_{\Delta + C_0}(X, L_{k^1} - \delta E) = 0\) for \(\delta > 0\), where \(E\) is the exceptional divisor (cf. [5, Theorem 4.8, Corollary 7.18]). For any \(\epsilon > 0\), there exists \(\delta > 0\) such that
\[
(J^{K(X, \Delta + C_0)})^{NA}(X, L_{k^1} - \delta E) < \epsilon J^{NA}(X, L_{k^1} - \delta E)
\]
due to Lemma 5.3 and hence \((X, \Delta + C_0; L)\) is not uniformly K-stable. Therefore we have the corollary. \(\square\)
Hence, we propose the following conjecture:

**Conjecture 7.6.** There exists a polarized normal variety \((X, L)\) such that it is K-stable but not uniformly K-stable.

On the other hand, if \(X\) is deminormal, we have the following result:

**Corollary 7.7.** There exist polarized connected and deminormal surfaces \((Z, M)\) such that \(K_Z\) are ample and \((Z, M)\) are \(J^{K_Z}\)-stable and K-stable but not uniformly either.

**Proof.** Take \((X, \Delta + C_0, L)\) as in the proof of Corollary 7.3 such that \(\Delta\) is an integral and ample \(Z\)-divisor. By Lemma 6.16 we obtain an sfc scheme \(Z \cong X \cup_{\Delta + C_0} X\) and let \(M\) be the line bundle on \(Z\) obtained by gluing \(L_1\) and \(L_2\) together i.e. \(\nu^*M = L_1 \times L_2\), where \(\nu : X_1 \sqcup X_2 \to Z\) is the normalization and \((X_1, L_i)\) is a copy of \((X, L)\) for \(i = 1, 2\).

We have \(\nu^*K_Z = K_{(X_1, \Delta + C_0)} \times K_{(X_2, \Delta + C_0)}\) and hence \(K_Z\) is ample. Note also that all irreducible components of \((Z, M)\) have the same average scalar curvature with respect to \(K_Z\). Recall from [5] Remark 3.19 that \((Z, M)\) is K-stable (resp. uniformly K-stable) iff for any semiample test configuration \((Z, \mathcal{M})\) over \((Z, M)\), \(M^{NA}_{(\Delta + C_0), (\Delta + C_0)}(Z, \mathcal{M}) > 0\) (resp. \(M^{NA}_{(\Delta + C_0), (\Delta + C_0)}(Z, \mathcal{M}) \geq \epsilon J^{NA}(Z, \mathcal{M})\) for some \(\epsilon > 0\)), where \((Z, \mathcal{M})\) is the normalization of \((Z, \mathcal{M})\). Since \((X, \Delta + C_0, L)\) is \(J^{K(X, \Delta + C_0)}\)-stable and K-stable, so is \((Z, M)\).

On the other hand, let \(\mathcal{Z}\) be the deformation to the normal cone of \(Z\) along a closed subscheme \(D_0\) contained in the node and \(E\) be the exceptional divisor. Here, \(D_0\) is the irreducible component of the conductor subscheme in \(Z\), which is the image of \((C_0)_1 \sqcup (C_0)_2\). It is easy to see that the inverse image \(\nu^{-1}D_0 \supset (C_0)_1 \sqcup (C_0)_2\). To prove the converse, let \(\mathcal{J}\) be the coherent ideal of \(\mathcal{O}_Z\) corresponding to \(D_0\) and \(\mathcal{J}\) be the one of \(\mathcal{O}_{X_1 \sqcup X_2}\) corresponding to \((C_0)_1 \sqcup (C_0)_2\). Note that \((C_0)_1 \sqcup (C_0)_2\) is a part of the conductor and it is easy to see that \(\nu^{-1}\mathcal{J} = \mathcal{J}\) holds for \(\prod_{k=1}^2 (X_k \setminus (\Delta \cap C_0)_k)\). Since \(\mathcal{J}\) is \(S_2\), we have \(\mathcal{J} \subset \nu^{-1} \mathcal{J}\).

Therefore, we have that \(\nu^{-1}D_0 = (C_0)_1 \sqcup (C_0)_2\) scheme-theoretically. Then \(\overline{\mathcal{Z}}\) is the normalization of the deformation to the normal cone of \(X_1 \sqcup X_2\) along \((C_0)_1 \sqcup (C_0)_2\) and the pullback \(\overline{\mathcal{E}}\) of \(E\) is the exceptional divisor. Since for any \(\epsilon > 0\), there exists sufficiently small \(\delta > 0\) such that

\[ M^{NA}_{(\Delta + C_0), (\Delta + C_0)}(\overline{\mathcal{Z}}, L_1 \times L_2 - \delta \overline{\mathcal{E}}) < \epsilon J^{NA}(\overline{\mathcal{Z}}, L_1 \times L_2 - \delta \overline{\mathcal{E}}) \]

and

\[ (\mathcal{J}^{K(X_1, \Delta + C_0) \times K(X_2, \Delta + C_0)})^{NA}(\overline{\mathcal{Z}}, L_1 \times L_2 - \delta \overline{\mathcal{E}}) < \epsilon J^{NA}(\overline{\mathcal{Z}}, L_1 \times L_2 - \delta \overline{\mathcal{E}}) \]

due to Proposition 7.4 and Corollary 7.5, \((Z, M)\) is neither uniformly \(J^{K_Z}\)-stable nor uniformly K-stable.

**Remark 7.8.** There is a more general way to construct such examples as in Theorem 7.3 by the application of the following consequences of Sjöström Dyrefelt [12]. We extend them to singular projective surfaces:

**Theorem 7.9** (cf. Corollary 7, Corollary 34 of [12]). Let \(X\) be a normal and irreducible surface, \(H\) be a fixed ample \(\mathbb{R}\)-divisor on \(X\), \(\text{Amp}_X\) be its ample cone and \(\text{Big}_X\) be the cone of \(NS(X) \otimes \mathbb{R}\) that consists of big \(\mathbb{R}\)-divisors, where \(NS(X)\) is Néron-Severi group. Let the subsets \(uJs_X^H\) and \(Js_X^H\) of \(\text{Amp}_X\) be

\[ uJs_X^H = \{ L \in \text{Amp}_X; 2 \frac{L \cdot H}{L^2} - H \text{ is ample} \} \]
and

\[ J_{s}^{H} = \{ L \in \text{Amp}_{X}; 2 \frac{L \cdot H}{L^2} L - H \text{ is nef} \} \]

respectively. Then \( uJ_{s}^{H} \) is open connected, and star convex, i.e. the line segment from \( L \) to \( H \) is contained in \( uJ_{s}^{H} \) for any \( L \in uJ_{s}^{H} \). Moreover, \( J_{s}^{H} \) is the closure of \( uJ_{s}^{H} \) in \( \text{Amp}_{X} \).

**Remark 7.10.** Abusing the notations, we denote also a numerical class of a \( \mathbb{Q} \)-line bundle \( L \) by \( L \). By Corollary [6.4], Proposition [7.4] and the fact \( H^2 > 0 \), we have if \( H \) is a \( \mathbb{Q} \)-line bundle, \( uJ_{s}^{H} \cap (NS(X) \otimes \mathbb{Q}) \) is the set of polarizations of \( X \) such that \( (X, L) \) is uniformly \( J^{H} \)-stable and \( J_{s}^{H} \cap (NS(X) \otimes \mathbb{Q}) \) is the set of \( L \) such that \( (X, L) \) is \( J^{H} \)-stable.

**Proof of Theorem 7.9.** By the definition, it follows that \( uJ_{s}^{H} \) is open and \( J_{s}^{H} \) is closed. Due to Proposition [7.4] we only have to show the following:

**Claim.** If \( L \in J_{s}^{H} \), then \( L + tH \in uJ_{s}^{H} \) for \( t > 0 \).

Indeed, \( 2(L \cdot H)L - (L^2)H \) is nef by the assumption and

\[ 2 \frac{(L + tH) \cdot H}{(L + tH)^2} (L + tH) - H = 2 \frac{L \cdot H + tH^2}{L^2 + 2tL \cdot H + t^2H^2} (L + tH) - H \]

\[ = \frac{1}{L^2 + 2tL \cdot H + t^2H^2} (2(L \cdot H)L - (L^2)H + 2t(H^2)L + t^2(H^2)H) \]

is ample.

If \( H \) is \( \mathbb{Q} \)-ample, it follows from Theorem 7.9 that all rational points \( L \) contained in the boundary \( \partial(uJ_{s}^{H}) \) in \( \text{Amp}_{X} \) satisfy that \( (X, L) \) is \( J^{H} \)-stable but not uniformly \( J^{H} \)-stable. Next, we extend the following to the case when \( X \) has singularities:

**Theorem 7.11** (cf. Theorem 6 [42]). Notations as in Theorem 7.9. If \( \text{Big}_{X} = \text{Amp}_{X} \),

\[ uJ_{s}^{H} = \text{Amp}_{X} \]

Otherwise, let \( M \in \partial \text{Amp}_{X} \) be \( M^2 = H^2 \) and \( L_t \) be any ample divisors on \( X \) such that \( L_t := (1 - t)M + tH \) for \( t \in (0, 1) \). Then

\[ uJ_{s}^{H} = \{ \lambda L_t : \lambda > 0, t \in (\frac{1}{2}, 1] \} \subset \text{Amp}_{X} \]

**Proof.** For the first assertion, it is easy to see that for any ample \( \mathbb{R} \)-divisor \( L \), there are \( t > 0 \) and a nef but not ample \( \mathbb{R} \)-divisor \( N \) such that

\[ L = tH + N. \]

Note that \( N \) is not big by the assumption. Then \( N^2 = 0 \). Therefore, as in the equation (4) in the proof of Theorem 7.11,

\[ 2 \frac{(N + tH) \cdot H}{(N + tH)^2} (N + tH) - H = \frac{1}{2tN \cdot H + t^2H^2} (2(N \cdot H)N + 2t(H^2)N + t^2(H^2)H) \]

is ample.

For the second assertion, we only have to show that \( L_t \in J_{s}^{H} \setminus uJ_{s}^{H} \) when \( t = \frac{1}{2} \) by Theorem 7.9. If \( t = \frac{1}{2} \), \( L = \frac{1}{2}(H + M) \) satisfies that

\[ 2 \frac{L \cdot H}{L^2} L - H = M \]

is nef but not ample. We complete the proof. \( \square \)
Thus, if there exists \( \mathbb{Q} \)-divisors \( H, M \) as in Theorem 7.11, we can construct examples as Theorem 7.3.

Finally, we remark that we can not extend Theorems 7.9 and 7.11 to the case when \( X \) is deminormal in general. In fact, let \( X \) be a deminormal surface and \( X_i \)'s are irreducible components of \( X \). In fact, in Theorem 7.11, if \( M |_{\mathcal{X}_i} \cdot H |_{\mathcal{X}_i} = M |_{\mathcal{X}_j} \cdot H |_{\mathcal{X}_j} \) for \( i, j \), then the theorem holds for \( X \). To be precise, we have Proposition 7.12 below.

Proposition 7.12. Let \( X \) be a projective, deminormal and reducible surface and \( L, H \) be \( \mathbb{Q} \)-ample line bundles. Suppose that \( L, H \) satisfy that

\[
\frac{M |_{\mathcal{X}_i}^2}{H |_{\mathcal{X}_i}^2} = \frac{M |_{\mathcal{X}_j}^2}{H |_{\mathcal{X}_j}^2}
\]

and

\[
\frac{M |_{\mathcal{X}_i} \cdot H |_{\mathcal{X}_i}}{H |_{\mathcal{X}_i}^2} = \frac{M |_{\mathcal{X}_j} \cdot H |_{\mathcal{X}_j}}{H |_{\mathcal{X}_j}^2}
\]

for \( i, j \), then the theorem holds for \( X \). To be precise, we have Proposition 7.12 below. Otherwise, the theorem does not hold since there exists \( t > 0 \) such that all irreducible components of \((X, H + tM)\) do not have the same average scalar curvature with respect to \( H \).

8. J-stability for Higher Dimensional Varieties

In this section, we prove two criteria of J-stability for higher dimensional polarized deminormal schemes, Theorems 8.3 and 8.12. We also explain applications of them and prove K-stability of log minimal models. Let \((X, L)\) be a deminormal polarized scheme and \( H \) be a \( \mathbb{Q} \)-line bundle on \( X \). Define that \((X, L)\) and \( H \) satisfy the condition (**), if all irreducible components of \((X, L)\) have the same average scalar curvature with respect to \( H \).
8.1. Song-Weinkove criterion and its applications to stability threshold. First, recall the following criterion, which is proved by Song-Weinkove [45].

**Theorem 8.1** (Theorem 1.1 [45]). Let \( X \) be a Kähler manifold and \( \chi, \omega \) be Kähler \((1,1)\)-forms on \( X \). Then the \((n-1, n-1)\)-form

\[
\left( n \int_X \chi \wedge \omega^{n-1} / \int_X \omega^n \right) \omega - (n-1) \chi \wedge \chi^{n-2}
\]

is positive iff \( J^X \)-energy is proper, where an \((n-1, n-1)\)-form \( \beta \) is positive if \( \beta \wedge \alpha \wedge \bar{\alpha} > 0 \) for any point of \( X \) and for any nonzero \((1,0)\)-form \( \alpha \).

From an algebro-geometric perspective, the next theorem follows from Theorems 8.1 and 8.10.

**Theorem 8.2.** Let \((X, L)\) be a smooth polarized variety over \(\mathbb{C}\) and \(H\) be an ample \(\mathbb{Q}\)-line bundle on \(X\). Suppose that

\[
\frac{n}{n^2} \frac{H \cdot L^{n-1}}{L^n} L - (n-1) H
\]

is ample. Then there exists \(\delta > 0\) such that

\[
(J^H)^{\operatorname{NA}} \geq \delta J^{\operatorname{NA}}
\]

on \(H^{\operatorname{NA}}(L)\).

We call this theorem Song-Weinkove criterion for J-stability. This is a weak form not only of Theorem 8.1 but also of Theorem 8.10.

We generalize the following variant of Song-Weinkove criterion, which is partially known by [20] and [17], with a purely algebro-geometric proof:

**Theorem 8.3** (cf. Theorem 3 [20], Theorem 6.5 [17]). Let \((X, L)\) be an \(n\)-dimensional polarized deminormal scheme with a \(\mathbb{Q}\)-line bundle \(H\) on \(X\) such that \((X, L)\) and \(H\) satisfy (**). Assume that \(L^{n-1} \cdot H > 0\) (resp. \(\geq 0\)) and

\[
n^2 \frac{H \cdot L^{n-1}}{L^n} L - (n^2 - 1) H
\]

is ample (resp. nef). Then there exists \(\epsilon > 0\) (resp. \(\geq 0\)) such that

\[
(J^H)^{\operatorname{NA}} \geq \epsilon (I^{\operatorname{NA}} - J^{\operatorname{NA}}).
\]

**Proof.** We may assume that \(X\) is normal and irreducible by taking the normalization due to Lemma 6.12. If \(n = 1\), \(L\) is numerically equivalent to a positive (resp. nonnegative) multiple of \(H\) by the assumption and hence the theorem immediately follows. Thus we may assume that \(n \geq 2\). Let

\[
A = \left( \frac{n^2}{n^2 - 1} \frac{H \cdot L^{n-1}}{L^n} - \delta \right) L - H
\]

for \(\delta \geq 0\). The assertion when \(L^{n-1} \cdot H > 0\) and \(\frac{n^2 H \cdot L^{n-1}}{L^n} L - (n^2 - 1) H\) is ample follows from when \(H \cdot L^{n-1} \geq 0\) and \(\frac{n^2 H \cdot L^{n-1}}{L^n} L - (n^2 - 1) H\) is nef. In fact, since \(\frac{n^2 H \cdot L^{n-1}}{L^n} L - (n^2 - 1) H\) is ample and \(H \cdot L^{n-1} > 0\), \(A\) is nef and \((H - (n^2 - 1) \delta L) \cdot L^{n-1} \geq 0\) for sufficiently small \(\delta > 0\). Replace \(H\) by \(H - (n^2 - 1) \delta L\) for such \(\delta\). Then, if the theorem when \(H \cdot L^{n-1} \geq 0\) and \(\frac{n^2 H \cdot L^{n-1}}{L^n} L - (n^2 - 1) H\) is nef would hold,

\[
(J^H)^{\operatorname{NA}} - (n^2 - 1) \delta (I^{\operatorname{NA}} - J^{\operatorname{NA}}) = (J^{H - (n^2 - 1) \delta L})^{\operatorname{NA}} \geq 0
\]

Therefore, we can take \(\epsilon = (n^2 - 1) \delta\).
Let $\delta = 0$. Then we have

$$\frac{A \cdot L^{n-1}}{L^n} = \frac{1}{n^2 - 1} \frac{H \cdot L^{n-1}}{L^n} \geq 0$$

and

$$\frac{n^2}{n^2 - 1} \frac{H \cdot L^{n-1}}{L^n} (I^\NA - J^\NA) = (J^A)^\NA + (J^H)^\NA.$$

For any non-Archimedean metric $\phi$ and any representative $(X, \mathcal{L})$ of $\phi$ that dominates $X_{\text{A}^1}$, we may assume that the support of $\mathcal{L} - L_{\text{A}^1}$ does not contain the strict transformation of $X \times \{0\}$ by translation. Then we have $A_{p_1} \cdot (\mathcal{L} - L_{p_1}) \cdot L_{p_1}^{n-1} = 0$. On the other hand,

$$V(L)(J^A)^\NA = A_{p_1} \cdot L^n + nV(L)\frac{A \cdot L^{n-1}}{L^n} J^\NA.$$

Since

$$A_{p_1} \cdot L^n = A_{p_1} \cdot (L^n - L_{p_1}^{n-1})
= \sum_{j=0}^{n-1} A_{p_1} \cdot (\mathcal{L} - L_{p_1}) \cdot (\mathcal{L}^j \cdot L_{p_1}^{n-1-j})
= \sum_{j=0}^{n-1} A_{p_1} \cdot (\mathcal{L} - L_{p_1}) \cdot (\mathcal{L}^j - L_{p_1}^j) \cdot L_{p_1}^{n-1-j}
= \sum_{j=0}^{n-1} A_{p_1} \cdot (\mathcal{L} - L_{p_1})^2 \cdot \left( \sum_{i=0}^{j-1} \mathcal{L}^i \cdot L_{p_1}^{n-2-i} \right)
\leq 0$$

by [31] Lemma 1] and $I^\NA \geq \frac{n+1}{n} J^\NA$, we have

$$(J^H)^\NA \geq \frac{n^2}{n^2 - 1} \frac{H \cdot L^{n-1}}{L^n} (I^\NA - J^\NA) - nA \frac{L^{n-1}}{L^n} J^\NA
\geq \frac{n}{n^2 - 1} \frac{H \cdot L^{n-1}}{L^n} J^\NA - \frac{n}{n^2 - 1} \frac{H \cdot L^{n-1}}{L^n} J^\NA
= 0.$$

\[\square\]

**Remark 8.4.** We do not assume that $H$ is nef or pseudoeffective in Theorem 8.3. Therefore, Theorem 8.3 is not weaker than Theorem 8.12 below over $\mathbb{C}$.

We apply Theorem 8.3 to extend the following for deminormal schemes:

**Corollary 8.5** (cf. Theorem 3.15 [14], Lemma 6.9 [30]). For any polarized deminormal scheme $(X, \mathcal{L})$ and any $\mathbb{Q}$-line bundle $H$ on $X$ such that $(X, \mathcal{L})$ and $H$ satisfy (**), there exists a constant $\delta > 0$ such that

$$-\delta J^\NA \leq (J^H)^\NA \leq \delta J^\NA.$$

**Remark 8.6.** C. Li [30] proved Corollary 8.5 for smooth projective manifolds and $M$ is their canonical divisor $K_X$ over $\mathbb{C}$ by applying Theorem 8.2. On the other hand, Dervan and Ross show Corollary 8.5 for normal varieties in the proof of [14, Theorem 3.15].
Proof of Corollary 8.5. We may assume that $X$ is normal and irreducible. Because $J^{NA}$ and $I^{NA} - J^{NA}$ is comparable, in the above inequality we can use the latter to compare. Let $M = H + \delta L$. Then $\frac{M \cdot L^{n-1}}{L^n} = \frac{H \cdot L^{n-1}}{L^n} + \delta$. For sufficiently large $\delta$, $M$ is ample and
\[
n M \cdot L^{n-1} - \frac{n^2 - 1}{n} M = n\left(\frac{H \cdot L^{n-1}}{L^n} + \delta\right) L - \frac{n^2 - 1}{n} (H + \delta L) = \left(n^{-1} \delta + \frac{n H \cdot L^{n-1}}{L^n}\right) L - \frac{n^2 - 1}{n} H
\]
is ample. In this case, we get $(J^H)^{NA} + \delta(I^{NA} - J^{NA}) = (J^M)^{NA} \geq 0$ by Proposition 8.3.

On the other hand, let $M = -H + \delta L$. For sufficiently large $\delta$, we can prove the latter part of the corollary similarly. \hfill \Box

Thanks to Corollary 8.5, we can introduce the notion of the stability threshold of Sjöström Dyrefelt [42].

Definition 8.7. Let $(X, B; L)$ be a polarized deminormal pair. Then the non-Archimedean log $K$-stability threshold is
\[
\Delta_B(X, L) = \sup\{\delta \in \mathbb{R}; M_B^{NA} \geq \delta(I^{NA} - J^{NA})\}.
\]
On the other hand, let $H$ be a $\mathbb{Q}$-line bundle such that irreducible components of $(X, L)$ have the same average scalar curvature with respect to $H$. Then the non-Archimedean $J^H$-stability threshold is
\[
\Delta_{pp}^H(X, L) = \sup\{\delta \in \mathbb{R}; (J^H)^{NA} \geq \delta(I^{NA} - J^{NA})\}.
\]

By Corollary 8.5, we can easily show that both of $\Delta_B(X, L)$ and $\Delta_{pp}^H(X, L)$ are well-defined for slc pairs whose irreducible components have the same average scalar curvature. We can extend the basic properties of $\Delta_{pp}^H$ of [42] for possibly singular varieties as follows:

Theorem 8.8 (cf. Theorem 17 of [42]). Suppose that $X$ is a deminormal projective surface. For any two $\mathbb{Q}$-line bundles $L, H$, where $L$ is ample and $H$ is pseudoeffective such that $(X, L)$ and $H$ satisfy (**), then the stability threshold satisfies
\[
\Delta_{pp}^H(X, L) = 2\frac{H \cdot L}{L^2} - \inf\{\delta > 0 : \delta L - H \text{ is ample}\}.
\]

We mimic the proof of [42]. To prove the theorem, we prepare the followings.

Lemma 8.9 (cf. Lemma 14 [42]). Suppose that $H$ is a $\mathbb{Q}$-line bundle on $X$ and $L$ is an ample $\mathbb{Q}$-line bundle. Suppose that $(X, L)$ and $H$ satisfy (**). Let $a, b \in \mathbb{Q}$ with $a \geq 0$. Then
\[
\Delta_{pp}^{aH + bL}(X, L) = a \Delta_{pp}^H(X, L) + b.
\]

The proof of Lemma 8.9 is straightforward. Note that all irreducible components of $(X, L)$ have the same average scalar curvature with respect to $aH + bL$ for $a, b \in \mathbb{R}$ by the assumption.

Let $H$ be a $\mathbb{Q}$-line bundle on $X$ and $L$ is an ample $\mathbb{Q}$-line bundle. Suppose that $(X, L)$ and $H$ satisfy (**). Decompose the two dimensional subspace $\text{Span}(H, L)$ of $\text{NS}(X) \otimes \mathbb{R}$ spanned by $H$ and $L$ into two components
\[
\text{Span}(H, L) = \text{Span}(H, L)^+ \cup \text{Span}(H, L)^-,
\]
where
\[
\text{Span}(H, L)^+ := \{aH + bL : a \geq 0, b \in \mathbb{R}\}
\]
and 
\[ \text{Span}(H, L)^- := \{ aH + bL : a \leq 0, b \in \mathbb{R} \}. \]

Then the following holds.

**Lemma 8.10** (cf. Lemma 15 [12]). Notations as above. Suppose that $M_0$ and $M_1$ are $\mathbb{Q}$-line bundles on $X$ such that either $(M_0, M_1) \in \text{Span}(H, L)^+ \times \text{Span}(H, L)^+$ or $(M_0, M_1) \in \text{Span}(H, L)^- \times \text{Span}(H, L)^-$. Then

\[ \Delta_{M_0 + tM_1}^{pp}(X, L) = (1 - t)\Delta_{M_0}^{pp}(X, L) + t\Delta_{M_1}^{pp}(X, L) \]

for any $t \in [0, 1] \cap \mathbb{Q}$.

**Proof.** First, we prove the lemma when $(M_0, M_1) \in \text{Span}(H, L)^+ \times \text{Span}(H, L)^+$. Suppose that $M_0 = a_0H + b_0L$ and $M_1 = a_1H + b_1L$, where $a_0, a_1 \in \mathbb{Q}_{\geq 0}$ and $b_0, b_1 \in \mathbb{Q}$. For each $t \in [0, 1] \cap \mathbb{Q}$,

\[ (\mathcal{J}^{1-t}M_0 + tM_1)^{\mathrm{NA}} = (1 - t)(\mathcal{J}^{M_0})^{\mathrm{NA}} + t(\mathcal{J}^{M_1})^{\mathrm{NA}} = (a_0(1 - t) + a_1 t)(\mathcal{J}^{H})^{\mathrm{NA}} + (b_0(1 - t) + b_1 t)(I^{\mathrm{NA}} - J^{\mathrm{NA}}). \]

Therefore, we have

\[ \Delta_{M_0 + tM_1}^{pp}(X, L) = (a_0(1 - t) + a_1 t)\Delta_{M_0}^{pp}(X, L) + (b_0(1 - t) + b_1 t) \]

by Lemma 8.9.

On the other hand, if $(M_0, M_1) \in \text{Span}(H, L)^- \times \text{Span}(H, L)^-$, then $(M_0, M_1) \in \text{Span}(-H, L)^+ \times \text{Span}(-H, L)^+$. Hence the lemma holds. \hfill \Box

**Proof of Theorem 8.8**. Let $H_t = (1 - t)H + tL$ for $t \in [0, 1] \cap \mathbb{Q}$, $R(t) = \Delta_{H_t}^{pp}(X, L)$ and $L(t) = 2\frac{HtL}{L^2} - \inf\{ \delta > 0 : \delta L - H_t \text{ is ample} \}$. By Lemma 8.10, $R(t)$ is affine linear and extended to all of $[0, 1]$. On the other hand, $L(t)$ is also affine linear and extended to all of $[0, 1]$. It is easy to see that $R(1) = L(1) = 1$. We have only to show that there exists $t_0 < 1$ such that $R(t_0) = L(t_0)$. This follows immediately from the following claim:

**Claim.** For $t \in [0, 1] \cap \mathbb{Q}$, $L(t) < 0$ if and only if $R(t) < 0$.

Indeed, by Corollary 6.13, $R(t) \geq 0$ if and only if $2\frac{HtL}{L^2} - H_t$ is nef. The latter condition is equivalent to $L(t) \geq 0$. \hfill \Box

Finally, we remark that the following theorem:

**Theorem 8.11** (cf. Theorem 18 [12]). Suppose that $(X, L)$ is a polarized deminormal surface. If $H$ is a $\mathbb{Q}$-line bundle on $X$ such that $(X, L)$ and $H$ satisfy (***) and $\Delta_{H_t}^{pp}(X, L) < \mathcal{T}(H, L)$, where

\[ \mathcal{T}(H, L) := \sup\{ \delta \in \mathbb{R} : H - \delta L \text{ is pseudoeffective} \}, \]

then

\[ \Delta_{H_t}^{pp}(X, L) = \frac{2}{L^2} - \inf\{ \delta > 0 : \delta L - H \text{ is ample} \}. \]

**Proof.** Note that $\mathcal{T}(H, L) \geq 0$ and the left and the right hand sides are positive homogeneous in $H$. Let $\delta_0 \geq 0$ be a rational number such that $\Delta_{H_t}^{pp}(X, L) < \delta_0 \leq \mathcal{T}(H, L)$ and then we have $(X, L)$ is $\mathcal{J}$-$\delta_0$-unstable and $H - \delta_0 L$ is a pseudoeffective $\mathbb{Q}$-line bundle by the assumption. As in the proof of Theorem 8.8, let $H_t = (1 - t)(H - \delta_0 L) + tL$, $R(t) = \Delta_{H_t}^{pp}(X, L)$ and $L(t) = 2\frac{HtL}{L^2} - \inf\{ \delta > 0 : \delta L - H_t \text{ is ample} \}$. Note that $L(t)$ and
$R(t)$ is defined only on $t \in [0, 1]$ such that $H_t$ is a $\mathbb{Q}$-line bundle. However, we can extend these to affine linear functions defined on $[0, 1]$ similarly. Note that $L(1) = R(1) = 1$. On the other hand, we can find $t_0 < 1$ such that $L(t_0) = R(t_0)$. Therefore, $L(t) = R(t)$ for any $t \in [0, 1]$ and we complete the proof by $L(\frac{\delta}{1 + \delta_0}) = R(\frac{\delta}{1 + \delta_0})$.

8.2. Generalized Lejmi-Székelyhidi conjecture and K-stability of minimal models. In this subsection, we work over $\mathbb{C}$. We extend Lejmi-Székelyhidi conjecture to the case when $X$ is deminormal and explain its applications to K-stability of klt minimal models.

**Theorem 8.12** (Lejmi-Székelyhidi conjecture for deminormal polarized schemes). Let $(X, L)$ be a deminormal polarized scheme over $\mathbb{C}$ and $H$ be an ample (resp. nef) $\mathbb{Q}$-line bundle on $X$ such that $(X, L)$ and $H$ satisfy (**). Then the followings are equivalent.

1. $(X, L)$ is uniformly $J^H$-stable (resp. $J^H$-semistable). In other words, there exists $\epsilon > 0$ such that for any semiample test configuration $(\mathcal{X}, \mathcal{L})$

   $$(J^H)^{NA}(\mathcal{X}, \mathcal{L}) \geq \epsilon J^{NA}(\mathcal{X}, \mathcal{L}) \quad \text{(resp.} \geq 0).$$

2. $(X, L)$ is uniformly slope $J^H$-stable (resp. slope $J^H$-semistable). In other words, there exists $\epsilon > 0$ such that for any semiample deformation to the normal cone $(\mathcal{X}, \mathcal{L})$ along any integral curve

   $$(J^H)^{NA}(\mathcal{X}, \mathcal{L}) \geq \epsilon J^{NA}(\mathcal{X}, \mathcal{L}) \quad \text{(resp.} \geq 0).$$

3. There exists $\epsilon > 0$ such that for any $p$-dimensional subvariety $V$,

   $$\left( n \frac{H \cdot L^n - pH}{L^n} L - pH \right) \cdot L^{p-1} \cdot V \geq (n - p)\epsilon p^p \cdot V \quad \text{(resp.} \geq 0).$$

**Proof.** We have only to prove $(3) \Rightarrow (1)$ due to Lemma [6.14] First, we prove the assertion if $X$ is normal and irreducible and $\epsilon = 0$. For sufficiently small $a > 0$, $H + aL$ is ample and

$$\left( n \frac{H + aL \cdot L^{n-1}}{L^n} L - p(H + aL) \right) \cdot L^{p-1} \cdot V > 0.$$

Therefore, we may assume that $H$ and

$$\left( n \frac{H \cdot L^{n-1}}{L^n} L - pH \right) \cdot L^{p-1} \cdot V > 0.$$

Take any semiample test configuration $(\mathcal{X}, \mathcal{L})$ dominates $X_{\mathbb{A}^1}$ and a resolution of singularities $\tilde{X}$ of $X$. Since we want to show $(J^H)^{NA}$ is non-negative, we may assume that $\mathcal{L}$ is $\mathbb{A}^1$-ample by a small perturbation of $\mathcal{L}$. Let $E$ be an ample divisor on $\tilde{X}$ and $\tilde{X}$ be the strict transformation of $\mathcal{X}$ in the normalization of $\mathcal{X} \times \mathcal{X}$. Now, we have the following commutative diagram.

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\pi}} & \mathcal{X} \\
\tilde{\rho} \downarrow & & \downarrow \rho \\
\tilde{X} \times \mathbb{A}^1 & \xrightarrow{\tilde{\pi} \times id_{\mathbb{A}^1}} & X \times \mathbb{A}^1.
\end{array}$$

We can easily see that $(\tilde{X}, \tilde{\pi}^* \mathcal{L} + \epsilon \tilde{\rho}^* E_{\mathbb{A}^1})$ is an ample test configuration of the polarized variety $(\tilde{X}, \tilde{\pi}^* L + \epsilon E)$ for any sufficiently small $\epsilon \in \mathbb{Q}_+$ since $\mathcal{L}$ is $\mathbb{A}^1$-ample and $\tilde{\rho}^* E_{\mathbb{A}^1}$ is $\mathcal{X}$-ample. On the other hand,

$$\lim_{\delta, \epsilon \to 0} (J^{(\pi^* L + \delta (\pi^* L + \epsilon E)))^{NA}(\tilde{X}, \tilde{\pi}^* \mathcal{L} + \epsilon \tilde{\rho}^* E_{\mathbb{A}^1}) = (J^H)^{NA}(\mathcal{X}, \mathcal{L}).$$
Thus we only have to show that
\[(\mathcal{J}^{(\pi^* H + \delta(\pi^* L + \epsilon E))})^{NA}(\widetilde{X}, \pi^* \mathcal{L} + \epsilon \rho^* E_{H^1}) \geq 0\]
for rational $0 < \epsilon \ll \delta \ll 1$ since $(\mathcal{J}^H)^{NA}$-energy is linear in $H$. For any $p$-dimensional subvariety $V$ of $\widetilde{X}$,
\[
\left(\frac{\pi^* H \cdot (\pi^* L + \epsilon E)^{n-1}}{(\pi^* L + \epsilon E)^n}(\pi^* L + \epsilon E) - p\pi^* H, (\pi^* L + \epsilon E)^{p-1} \cdot V\right) = \left(\frac{H \cdot L^{n-1}}{L^n} + O(\epsilon)(\pi^* L + \epsilon E) - p\pi^* H, (\pi^* L + \epsilon E)^{p-1} \cdot V,\right)
\]
where $O(\epsilon)$ is independent of $V$. On the other hand, we prove that
\[
\left(\frac{H \cdot L^{n-1}}{L^n}(\pi^* L + \epsilon E) - p\pi^* H, (\pi^* L + \epsilon E)^{p-1} \cdot V \geq 0,\right)
\]
Indeed, if $p - r > q$, the $\epsilon^r$-term of the left hand side of the above inequality is $0$, where $q$ is the dimension of $\pi(V)$. Otherwise, we consider $\pi_*(E^r \cdot V)$ as a positive $(p-r)$-cycle of $X$ and the easy computation shows that the $\epsilon^r$-term is
\[
\epsilon^r \left(\frac{H \cdot L^{n-1}}{L^n} \cdot \pi^* L^r - p\left(\frac{p-1}{r}\right)\pi^* H \cdot \pi^* L^{p-r-1}\right) \cdot (E^r \cdot V) = \epsilon^r \left(\frac{H \cdot L^{n-1}}{L^n} \cdot L^{p-r} - (p-r)H\right) \cdot L^{p-r-1} \cdot \pi_*(E^r \cdot V) \geq 0
\]
by the assumption. Therefore, for any $\delta > 0$, there exists $\epsilon_0 > 0$ such that
\[
\left(\frac{(\pi^* H + \delta(\pi^* L + \epsilon E)) \cdot (\pi^* L + \epsilon E)^{n-1}}{(\pi^* L + \epsilon E)^n}(\pi^* L + \epsilon E) - p\pi^* H, (\pi^* L + \epsilon E)^{p-1} \cdot V \geq 0,\right)
\]
for any $0 < \epsilon \leq \epsilon_0$. Thanks to Theorem 3.10
\[\text{(}\mathcal{J}^{(\pi^* H + \delta(\pi^* L + \epsilon E))})^{NA}(\widetilde{X}, \pi^* \mathcal{L} + \epsilon \rho^* E_{H^1}) \geq 0.\]
Thus we prove $(3) \Rightarrow (1)$ when $\epsilon = 0$ and $X$ is normal and irreducible.

If $X$ is normal and irreducible, $H$ is ample and $\epsilon > 0$, uniform $J^H$-stability follows immediately from the same argument of the proof of Theorem 8.3. Finally, if $X$ is deminormal, the assertion follows immediately by taking its normalization. \hfill \Box

Remark 8.13. If $n = 2$, Theorem 8.12 follows from Corollary 6.4.

Remark 8.14. The proof of Theorem 8.12 also shows that the following. If $(3) \Rightarrow (1)$ holds for all smooth polarized varieties $(X, L)$ over any algebraically closed field $k$ of characteristic $0$ with nef $\mathbb{Q}$-Cartier divisors, then it also holds for all polarized deminormal schemes with nef $\mathbb{Q}$-line bundles that satisfy (**).

Now, we have the following without assuming the log canonical divisor is semiample due to Theorem 8.12 with a more algebro-geometric approach than Sjöström Dyrefelt [43] and J. Song [44].

Theorem 8.15 (cf. Theorem 1.1 of Jian-Shi-Song [23], Theorem 1.1 of Sjöström Dyrefelt [43], Corollary 1.3 of J. Song [44]). Let $(X, \Delta; L)$ be an $n$-dimensional polarized slc minimal model over $\mathbb{C}$ i.e. $K_{(X, \Delta)}$ is nef. Suppose that $(X, L)$ and $K_{(X, \Delta)}$ satisfy (**). If $K_{(X, \Delta)}$ is also big, $(X, \Delta; K_{(X, \Delta)} + \epsilon L)$ is uniformly $J^{K_{(X, \Delta)}}$-stable for sufficiently small $\epsilon > 0$. Otherwise, suppose that $(X, \Delta)$ is klt. Then $(X, \Delta; K_{(X, \Delta)} + \epsilon L)$ is uniformly $K$-stable for sufficiently small $\epsilon > 0$. If $\Delta = 0$ and $X$ is smooth, $(X, K_X + \epsilon L)$ also has a cscK metric.
Proof. We may assume that $X$ is normal and irreducible. If $K_{(X,\Delta)}^n \neq 0$, 

$$n\frac{K_{(X,\Delta)} \cdot (K_{(X,\Delta)} + \epsilon L)^n}{(K_{(X,\Delta)} + \epsilon L)^n} (K_{(X,\Delta)} + \epsilon L) - (n-1)K_{(X,\Delta)} = (1+O(\epsilon))K_{(X,\Delta)} + \epsilon(n+O(\epsilon))L$$

is ample for sufficiently small $\epsilon > 0$. Therefore, the first assertion follows from Theorem 5.12.

Next, suppose that $(X, \Delta)$ is klt and irreducible. Let $m = \nu(K_X) \leq n$. Suppose that $V \subset X$ is a $p$-dimensional subvariety and $\nu(K_X|_V) = j \leq \min\{m, p\}$. Then for sufficiently small $\epsilon > 0$, since 

$$n\frac{K_{(X,\Delta)} \cdot (K_{(X,\Delta)} + \epsilon L)^{n-1}}{(K_{(X,\Delta)} + \epsilon L)^n} = m + O(\epsilon),$$

we have 

$$\left(n\frac{K_{(X,\Delta)} \cdot (K_{(X,\Delta)} + \epsilon L)^{n-1}}{(K_{(X,\Delta)} + \epsilon L)^n} (K_{(X,\Delta)} + \epsilon L) - pK_{(X,\Delta)}\right) \cdot (K_{(X,\Delta)} + \epsilon L)^{p-1} \cdot V$$

$$= ((m-p)K_{(X,\Delta)} + m\epsilon L + O(\epsilon)(K_{(X,\Delta)} + \epsilon L)) \cdot (K_{(X,\Delta)} + \epsilon L)^{p-1} \cdot V,$$

where $O(\epsilon)$ is independent of $V$. Hence, there exists a constant $C > 0$ independent of $V$ such that 

$$\left(n\frac{K_{(X,\Delta)} \cdot (K_{(X,\Delta)} + \epsilon L)^{n-1}}{(K_{(X,\Delta)} + \epsilon L)^n} (K_{(X,\Delta)} + \epsilon L) - pK_{(X,\Delta)}\right) \cdot (K_{(X,\Delta)} + \epsilon L)^{p-1} \cdot V$$

$$> ((m-p)K_{(X,\Delta)} + m\epsilon L - C\epsilon(n-p)(K_{(X,\Delta)} + \epsilon L)) \cdot (K_{(X,\Delta)} + \epsilon L)^{p-1} \cdot V.$$

Now, we want to prove the following:

Claim. $V, m, p$ and $j$ as above. Then

$$((m-p)K_{(X,\Delta)} + m\epsilon L) \cdot (K_{(X,\Delta)} + \epsilon L)^{p-1} \cdot V \geq 0$$

for sufficiently small $\epsilon > 0$.

The theorem follows from this claim. In fact, it is easy to see that $(X, K_{(X,\Delta)} + \epsilon L)$ is $J_{(K_{(X,\Delta)} + \epsilon C)}^{K_{(X,\Delta)} + \epsilon L}$-semistable by Theorem 5.12 since $K_{(X,\Delta)} + \epsilon C(K_{(X,\Delta)} + \epsilon L)$ is ample. Then

$$(J_{(K_{(X,\Delta)})}^{K_{(X,\Delta)}})^{NA} \geq -C\epsilon(J^{NA} - J^{NA})$$

on $\mathcal{H}^{NA}(L)$. Therefore, the first assertion follows from the fact that

$$\alpha(X, \Delta; K_{(X,\Delta)} + \epsilon L) \geq \alpha(X, \Delta; K_{(X,\Delta)} + L) > 0$$

for $0 < \epsilon < 1$. Indeed, it is easy to see the fact holds by the definition of the alpha invariant. Then we have only to choose $\epsilon$ so small that $C\epsilon < \alpha(X, \Delta; K_{(X,\Delta)} + L)$. The second assertion follows from [30, Theorem 6.10].

Now, we prove the claim. If $m \geq p$, the claim holds. Otherwise, $j \leq m < p$ and hence the coefficient of $\epsilon^{p-i}$-term of $((m-p)K_{(X,\Delta)} + m\epsilon L) \cdot (K_{(X,\Delta)} + \epsilon L)^{p-1} \cdot V$ is

$$\left((m-p)\binom{p-1}{p-i} + m\binom{p-1}{p-i-1}\right) K_{(X,\Delta)} \cdot L^{p-i} \cdot V = \binom{p}{i} (m-i) K_{(X,\Delta)} \cdot L^{p-i} \cdot V \geq 0$$

for $1 \leq i \leq j$ or zero for $i > j$. On the other hand, we can easily see the term is positive when $i = 0$. \qed
Remark 8.16. If $n = 2$ in Theorem 8.15, we can replace Theorem 8.12 by Theorem 6.3 in the above proof. On the other hand, there is a purely algebro-geometric proof of uniform K-stability of $(X, \Delta; L)$ such that $K_{(X, \Delta)} \equiv 0$ for any polarization (cf. [37, 11, 5]). Thus we have a purely algebro-geometric proof of uniform K-stability of klt log minimal surfaces with certain polarizations.

Remark 8.17. Jian-Shi-Song [23] prove Theorem 8.15 when $\Delta = 0$ and $K_X$ is semiample. Theorem 8.15 is also proved without the assumption that $K_X$ is semiample by Z. Sjöström Dyrefelt [43] and by J. Song [44].

Remark 8.18. There is another application. We extend [23, Theorem 1.2] to the case for deminormal varieties in [21]. On the other hand, we prove in [21] the following another extension of [23]:

Theorem 8.19. Let $(X, \Delta; H)$ be an $n$-dimensional polarized klt variety over $\mathbb{C}$. Suppose that $X$ admits a flat fibration $f : X \to B$ over a polarized smooth curve $(B, L)$ and $H \equiv K_{(X, \Delta)} + f^*L_0$, where $L_0$ is a line bundle on $B$. If

$$n K_{(X, \Delta)} \cdot H^{n-1} - (n-1) \geq 0$$

for normal and irreducible fibre $(X_b, \Delta_b; H_b)$ over general point $b$ of $B$, then $(X, \Delta; f^*L + \delta H)$ is uniformly K-stable for sufficiently small $\delta > 0$. Furthermore, if $X$ is smooth and $\Delta = 0$, $(X, f^*L + \delta H)$ has a cscK metric.

References

[1] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, and C. W. Tønnesen-Friedman. Hamiltonian 2-forms in Kähler geometry, III Extremal metrics and stability. Inventiones mathematicae 173.3 (2008), pp. 547-601.

[2] L. Bădescu. Algebraic Surfaces. Springer-Verlag, New York-Heidelberg, 2001, Universitext.

[3] R. J. Berman, S. Boucksom, and M. Jonsson: A variational approach to the Yau-Tian-Donaldson conjecture. arXiv:1509.04561v3, 2020.

[4] S. Bloch and D. Gieseker. The positivity of the Chern classes of an ample vector bundle. Inv. Math., 12:112-117, 1971.

[5] S. Boucksom, T. Hisamoto, M. Jonsson. Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs, Ann. Inst. Fourier (Grenoble) 67(2), 743-841, 2017.

[6] G. Chen, The J-equation and the supercritical deformed Hermitian-Yang-Mills equation, Invent. math. https://doi.org/10.1007/s00222-021-01035-3, 2021.

[7] X. X. Chen, On the Lower Bound of the Mabuchi Energy and Its Application, Internat. Math. Res. Notices, 2000, no. 12, 607-623.

[8] X. X. Chen, S. K. Donaldson and S. Sun. Kähler-Einstein metrics on Fano manifolds, I-III. J. Amer. Math. Soc. 28 (2015), 183-197, 199-234, 235-278.

[9] T. Collins, G. Székelyhidi: Convergence of the J-Flow on Toric manifolds. J.Differential Geom. 107 (2017), no. 1, 47-81.

[10] V. Datar, V. Pingali. A numerical criterion for generalised Monge-Ampère equations on projective manifolds. Preprint [arXiv:2006.01530] 2020.

[11] R. Dervan. Uniform stability of twisted constant scalar curvature Kähler metrics, Int. Math. Res. Not. IMRN 15 (2016), p. 4728-4783.

[12] R. Dervan, J. Keller, A finite dimensional approach to Donaldson’s J-flow. Communications in Analysis and Geometry, 27 (2019), no. 5, p. 1025-1085.

[13] R. Dervan, J. Ross, K-stability for Kähler manifolds, Math. Res. Lett., 24 (2017), no. 3, 689-739.

[14] R. Dervan, J. Ross, Stable maps in higher dimensions. Mathematische Annalen volume 374, 1033-1073(2019)

[15] S. K. Donaldson, Moment maps and diffeomorphisms, Sir Michael Atiyah: a great mathematician of the twentieth century, Asian J. Math., 3 (1999), no. 1, 1-15.
[16] S. K. Donaldson, *Kähler metrics with cone singularities along a divisor*, Essays in mathematics and its applications, Springer, 2012, pp. 49-79.

[17] K. Fujita. *Openness results for uniform K-stability*, Math. Annalen, **373**, (2019), 1529-1548.

[18] W. Fulton. *Intersection Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer, 1984.

[19] R. Hartshorne. *Algebraic Geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[20] Y. Hashimoto, J. Keller, *About J-flow, J-balanced metrics, uniform J-stability and K-stability*, Asian J. Math., 22 (2018), n. 3, 391-412.

[21] M. Hattori. in preparation.

[22] H. Hironaka. *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. **79** (1964), I: 109-203, II: 205-326.

[23] W. Jian, Y. Shi and J. Song. *A remark on constant scalar curvature Kähler metrics on minimal models*, Proc. Amer. Math. Soc. 147 (2019), 3507-3513. MSC (2010).

[24] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. **339**, Springer-Verlag, Berlin-New York, 1973. viii+209pp.

[25] J. Kollár. *Sources of log canonical centers*, arXiv:1107.1863v3, 2012.

[26] J. Kollár. *Singularities of the Minimal Model Program*. Cambridge Tracts in Mathematics **200**, Cambridge University Press, Cambridge, 2015.

[27] J. Kollár and S. Mori. *Birational geometry of algebraic varieties*. Cambridge Tracts in Mathematics **134**, Cambridge University Press, Cambridge, 1998.

[28] R. Lazarsfeld. *Positivity in algebraic geometry. I*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer. (2004).

[29] C. Li. *Geodesic rays and stability in the cscK problem*, arXiv:2012.09405.
