Sufficient spectral conditions for graphs being $k$-edge-Hamiltonian or $k$-Hamiltonian*

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Abstract

A graph $G$ is $k$-edge-Hamiltonian if any collection of vertex-disjoint paths with at most $k$ edges altogether belong to a Hamiltonian cycle in $G$. A graph $G$ is $k$-Hamiltonian if for all $S \subseteq V(G)$ with $|S| \leq k$, the subgraph induced by $V(G) \setminus S$ has a Hamiltonian cycle. These two concepts are classical extensions for the usual Hamiltonian graphs. In this paper, we present some spectral sufficient conditions for a graph to be $k$-edge-Hamiltonian and $k$-Hamiltonian in terms of the adjacency spectral radius as well as the signless Laplacian spectral radius. Our results could be viewed as slight extensions of the recent theorems proved by Li and Ning [Linear Multilinear Algebra 64 (2016)], Nikiforov [Czechoslovak Math. J. 66 (2016)] and Li, Liu and Peng [Linear Multilinear Algebra 66 (2018)]. Moreover, we shall prove a stability result for graphs being $k$-Hamiltonian, which could be regarded as a complement of two recent results of Füredi, Kostochka and Luo [Discrete Math. 340 (2017)] and [Discrete Math. 342 (2019)].

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1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V$ and edge set $E$. The order of $G$ is defined by $|V|$ and the size by $|E|$. We usually write $m$ and $n$ for the size and the order of $G$ respectively. For disjoint subsets $A, B \subseteq V$, we let $e(A, B)$ denote the number of edges of $G$ with one end-vertex in $A$ and the other in $B$. Let $d_G(v)$ (or $d(v)$ if there is no confusion) be the degree of a vertex $v$ in $G$, and let $\delta(G)$ be the minimum degree of $G$. We write $K_s$ for the complete graph on $s$ vertices, and $I_t$ for the independent set with $t$ vertices. Let $\omega(G)$ be the number of vertices of a largest complete subgraph in $G$. For two vertex-disjoint graphs $G$ and $H$, we use $G \cup H$ to denote the disjoint union of $G$ and $H$, and we write $G \vee H$ for the join graph of $G$ and $H$, which is a graph obtained from $G \cup H$ by adding all edges between $G$ and $H$.

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The adjacency matrix of $G$ is $A(G) = (a_{ij})_{n \times n}$, whose entries satisfy $a_{ij} = 1$ if two vertices $i$ and $j$ are adjacent in $G$, and $a_{ij} = 0$ otherwise. The characteristic polynomial of $G$ is $P_G(x) = \det(xI - A(G))$, and the eigenvalues of $G$ are the roots of $P_G(x)$ (with multiplicities). Clearly $A(G)$ is a real symmetric matrix, so the eigenvalues of $G$ are real. The largest eigenvalue of $G$ is called the spectral radius of $G$ and is denoted by $\lambda(G)$.

Let $d_i$ be the degree of vertex $v_i$ and $D(G)$ be the diagonal matrix of degrees, that is, $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$. The signless Laplacian matrix of $G$ is defined as $Q(G) = D(G) + A(G)$, which is also a real symmetric matrix, so the eigenvalues of $Q(G)$ are real numbers. The eigenvalues of $Q(G)$ are said to be the signless Laplacian eigenvalues of $G$. The largest eigenvalue of $Q(G)$ is called the signless Laplacian spectral radius of $G$, and denoted by $q(G)$.

In the study of spectral graph theory, there are various matrices that are associated with a graph, such as the adjacency matrix, the Laplacian matrix, signless Laplacian matrix and distance matrix. One of the main problems of algebraic graph theory is to determine the combinatorial properties of graphs that are reflected from the algebraic properties of such matrices; see [1, 5, 16] for more details. In this paper, we mainly focus on the adjacency spectral radius and signless Laplacian spectral radius.

1.1 Hamiltonicity of graphs

A cycle passing through all vertices of a graph is called a Hamilton cycle. A graph containing a Hamilton cycle is called a Hamiltonian graph. A path passing through all vertices of a graph is called a Hamiltonian path and a graph containing a Hamiltonian path is said to be traceable.

Every complete graph on at least three vertices is evidently Hamiltonian, as the vertices of a Hamilton cycle can be selected one by one in an arbitrary order. Conditions to guarantee the existence of a Hamilton cycle have been studied actively. In particular, we may ask how large the minimum degree can be in order to guarantee the existence of a Hamilton cycle. The celebrated Dirac theorem [9] answered this question. It states that every graph with $n \geq 3$ vertices and minimum degree at least $\frac{n}{2}$ has a Hamilton cycle. The condition is sharp when we consider the complete bipartite graph with the parts of sizes $\lfloor \frac{n-1}{2} \rfloor$ and $\lfloor \frac{n+1}{2} \rfloor$.

The following result is due to Ore [35] and Bondy [2] independently. It is a direct consequence of Chvátal’s theorem on degree sequences; see [4] p. 60 for more details.

**Theorem 1.1** (Ore [35], Bondy [2]). Let $G$ be a graph on $n \geq 3$ vertices. If

$$e(G) \geq \left( \frac{n-1}{2} \right) + 1,$$

then $G$ has a Hamilton cycle or $G = K_1 \lor (K_1 \cup K_{n-2})$ or $n = 5$ and $G = K_2 \lor I_3$.

In 1962, Erdős improved the above result for graphs with given minimum degree.

**Theorem 1.2** (Erdős [10]). Let $G$ be a graph of order $n$. If the minimum degree $\delta(G) \geq \delta$ where $1 \leq \delta \leq \frac{n-1}{2}$ and

$$e(G) > \max \left\{ \left( \frac{n-\delta}{2} \right)^2, \left( \frac{n-\lfloor \frac{n-1}{2} \rfloor}{2} \right)^2 + \left( \frac{n-\lfloor \frac{n-1}{2} \rfloor}{2} \right)^2 \right\},$$

(1)

then $G$ has a Hamilton cycle.
We remark here that the condition $\delta \leq \frac{n - 1}{2}$ is reasonable since if $\delta > \frac{n - 1}{2}$, then $\delta \geq \frac{n}{2}$, the well-known Dirac theorem guarantees that $G$ must be Hamiltonian. To see the sharpness of the bound in Theorem 1.2 we consider the graph $H_{n,\delta}$ obtained from a copy of $K_{n-\delta}$ by adding an independent set of $\delta$ vertices with degree $\delta$ each of which is adjacent to the same $\delta$ vertices in $K_{n-\delta}$. In the language of graph join and union, that is,

$$H_{n,\delta} := K_\delta \lor (K_{n-2\delta} \cup I_\delta).$$  \hspace{1cm} (2)

Clearly, $H_{n,\delta}$ does not contain a Hamilton cycle and $e(H_{n,\delta}) = \left(\frac{n-\delta}{2}\right) + \delta^2$. When $n \geq 6\delta$, we can see that $e(H_{n,\delta})$ attains the maximum in (1). Thus, we can get the following corollary.

**Corollary 1.3 (Erdős).** Let $\delta \geq 1$ and $n \geq 6\delta$. If $G$ is an $n$-vertex graph with $\delta(G) \geq \delta$ and $e(G) \geq e(H_{n,\delta})$, then either $G$ has a Hamilton cycle or $G = H_{n,\delta}$.

### 1.2 Spectral conditions for Hamiltonicity

In 2010, Fiedler and Nikiforov proved a spectral version of Theorem 1.1.

**Theorem 1.4 (Fiedler–Nikiforov [13]).** If $G$ is a graph on $n \geq 3$ vertices and

$$\lambda(G) > n - 2,$$

then either $G$ has a Hamilton cycle or $G = K_1 \lor (K_1 \cup K_{n-2})$.

This result motivated a large number of researches on the topic of finding a spectral condition to guarantee the existence of a Hamilton cycle and path; see, e.g., [37, 17, 18, 27, 12, 28, 24, 25, 34]. In 2013, Yu and Fan [37] gave the corresponding version for the signless Laplacian spectral radius. Recall that $q(G)$ stands for the signless Laplacian spectral radius, i.e., the largest eigenvalue of the signless Laplacian matrix $Q(G) = D(G) + A(G)$, where $D(G) = \text{diag}(d_1, \ldots, d_n)$ is the degree diagonal matrix and $A(G)$ is the adjacency matrix.

**Theorem 1.5 (Yu–Fan [37]).** If $G$ is a graph on $n \geq 3$ vertices and

$$q(G) > 2(n - 2),$$

then $G$ has a Hamilton cycle or $G = K_1 \lor (K_1 \cup K_{n-2})$, or $n = 5$ and $G = K_2 \lor I_3$.

In [37], the counterexample of $n = 5, G = K_2 \lor I_3$ is missed. This tiny flaw has already been pointed out by Liu, Shiu and Xue [27] and by Li and Ning [24] as well.

By introducing the minimum degree of a graph as a new parameter, Li and Ning [24] extended Fiedler and Nikiforov’s results [13] in some sense and obtained a spectral analogue of Theorem 1.2 of Erdős on the existence of Hamilton cycles.

**Theorem 1.6 (Li–Ning [24]).** Suppose $\delta \geq 1$ and $n \geq \max\{6\delta + 5, (\delta^2 + 6\delta + 4)/2\}$. If $G$ is an $n$-vertex graph with $\delta(G) \geq \delta$ and

$$\lambda(G) \geq \lambda(H_{n,\delta}),$$

then either $G$ has a Hamilton cycle or $G = H_{n,\delta}$.
Theorem 1.7 (Li–Ning [24]). Suppose $\delta \geq 1$ and $n \geq \max\{6\delta + 5, (3\delta^2 + 5\delta + 4)/2\}$. If $G$ is an $n$-vertex graph with $\delta(G) \geq \delta$ and

$$q(G) \geq q(H_{n,\delta}),$$

then either $G$ has a Hamilton cycle or $G = H_{n,\delta}$.

Although these results of Li and Ning seem to be algebraic, their proof of Theorem 1.6 and Theorem 1.7 also need detailed graph structural analysis. The key ingredients of the proof of these theorems are mainly based on a stability result [24, Lemma 2]. We denote

$$L_{n,\delta} := K_1 \lor (K_\delta \cup K_{n-\delta-1}).$$

Clearly $L_{n,\delta}$ contains no Hamilton cycle and $e(L_{n,\delta}) = \binom{n-\delta}{2} + \binom{\delta+1}{2} < e(H_{n,\delta})$.

Soon after, Nikiforov [34, Theorem 1.4] proved the following theorem.

Theorem 1.8 (Nikiforov [34]). Suppose that $\delta \geq 1$ and $n \geq \delta^3 + \delta + 4$. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and

$$\lambda(G) \geq n - \delta - 1,$$

then $G$ has a Hamilton cycle, or $G = H_{n,\delta}$, or $G = L_{n,\delta}$.

We now introduce an important operation of graphs, which is known as the celebrated Kelmans operation; see, e.g., [22] or [5, p. 36]. Let $G$ be a graph and $u, v \in V(G)$ be distinct vertices. We define a new graph $G^*$ obtained from $G$ by replacing the edge $\{v, x\}$ by a new edge $\{u, x\}$ for all $x \in N(v) \setminus (N(u) \cup \{u\})$, and all vertices different from $u, v$ remain unchanged. Note that vertices $u, v$ are adjacent in $G^*$ if and only if they are adjacent in $G$. An isomorphic graph is obtained if the roles of $u$ and $v$ are interchanged. Furthermore, Csikvári [8] showed that the Kelmans operation does not decrease the spectral radius of a graph. Correspondingly, the same result also holds for the signless Laplacian spectral radius as well [24, Theorem 2.12]. Additionally, an analogous variant of the Kelmans operation can be seen in [36].

Theorem 1.8 is a slight improvement on Theorem 1.6. Indeed, we observe that $K_{n-\delta}$ is a proper subgraph of $H_{n,\delta}$ and $L_{n,\delta}$, which yields $\lambda(H_{n,\delta}) > \lambda(K_{n-\delta}) = n - \delta - 1$ and $\lambda(L_{n,\delta}) > \lambda(K_{n-\delta}) = n - \delta - 1$. Moreover, by applying the Kelmans operations on $L_{n,\delta}$, we can obtain a proper subgraph of $H_{n,\delta}$. Hence we can get $\lambda(H_{n,\delta}) > \lambda(L_{n,\delta})$. By calculation, we know that $\lambda(H_{n,\delta})$ is very close to $n - \delta - 1$ with $n$ sufficiently large.

Although Nikiforov’s Theorem 1.8 strengthened slightly Theorem 1.6. Unfortunately, one dissatisfaction in Theorem 1.8 is that the requirement of the order of graph is stricter than that in Theorem 1.6. One open problem is to relax this requirement. In addition, a natural question is that whether the value bound $q(G) \geq 2(n - \delta - 1)$ corresponding to Theorems 1.7 hold or not. The similar problems under the conditions of signless Laplacian spectral radius of graph seems much more complicated since we can delete some edges from the clique $K_{n-\delta}$ and still keep the signless Laplacian spectral radius no less than $2(n - \delta - 1)$.

In 2018, Li, Liu and Peng [26] solved this problem completely. They gave the corresponding improvement on the result of the signless Laplacian spectral radius in Theorem
Interestingly, the extremal graphs in their theorems are quite different from those in Theorems \ref{thm:1.7}. Recalling the definition in (2) and (3), we denote \(X = \{v \in V(H_{n,\delta}) : d(v) = \delta\}, \ Y = \{v \in V(H_{n,\delta}) : d(v) = n - 1\} \) and \(Z = \{v \in V(H_{n,\delta}) : d(v) = n - \delta - 1\} \). Let \(E_1(H_{n,\delta})\) be the set of those edges of \(H_{n,\delta}\) whose both endpoints are from \(Y \cup Z\). We define
\[
\mathcal{H}^{(1)}_{n,\delta} = \{H_{n,\delta} \setminus E' : E' \subseteq E_1(H_{n,\delta}) \text{ with } |E'| \leq \lfloor \delta^2 / 4 \rfloor \}.
\]
Similarly, for the graph \(L_{n,\delta} \), we denote \(X = \{v \in V(L_{n,\delta}) : d(v) = \delta\}, \ Y = \{v \in V(L_{n,\delta}) : d(v) = n - 1\} \) and \(Z = \{v \in V(L_{n,\delta}) : d(v) = n - \delta - 1\} \). The notation is clear although we used the same alphabets to denote the sets of vertices. It is easy to see that \(Y\) contains only one vertex. We use \(E_1(L_{n,\delta})\) to denote the set of edges of \(L_{n,\delta}\) whose both endpoints are from \(Y \cup Z\). We define
\[
\mathcal{L}^{(1)}_{n,\delta} = \{L_{n,\delta} \setminus E' : E' \subseteq E_1(L_{n,\delta}) \text{ with } |E'| \leq \lfloor \delta / 4 \rfloor \}.
\]

**Theorem 1.9** (Li–Liu–Peng [26]). Assume that \(\delta \geq 1\) and \(n \geq \delta^4 + \delta^3 + 4\delta^2 + \delta + 6\). Let \(G\) be a connected graph with \(n\) vertices and minimum degree \(\delta(G) \geq \delta\). If
\[
q(G) \geq 2(n - \delta - 1),
\]
then \(G\) has a Hamilton cycle unless \(G \in \mathcal{H}^{(1)}_{n,\delta}\) or \(G \in \mathcal{L}^{(1)}_{n,\delta}\).

The paper is organized as follows. In Section \ref{sec:2} we shall present our results on the problems involving the existence of Hamilton cycles. This paper is mainly motivated by the aforementioned works [24, 34, 26]. Our theorems extend Theorems \ref{thm:1.6}–\ref{thm:1.9} slightly, we shall provide the sufficient conditions on graphs being \(k\)-edge-Hamiltonian and \(k\)-Hamiltonian. In Section \ref{sec:3} we review some basic preliminaries for our use. Moreover, we shall prove a stability result on graphs being \(k\)-Hamiltonian (Theorem \ref{thm:3.6}). This theorem can be regarded as the complement of two recent results showed by Füredi, Kostochka and Luo [14, 15]. In Section \ref{sec:4} we shall give the complete proofs of our main results stated in Section \ref{sec:2}.

## 2 Main results

### 2.1 Spectral conditions for \(k\)-edge-Hamiltonicity

A graph \(G\) is called \(k\)-edge-Hamiltonian if any collection of at most \(k\) edges consisting of vertex-disjoint paths is contained in a Hamilton cycle in \(G\). In other words, each linear forest with at most \(k\) edges in \(G\) can be extended to a Hamilton cycle of \(G\). In particular, being \(0\)-edge-Hamiltonian is equivalent to being Hamiltonian.

In this section, we shall present our theorems on the sufficient spectral conditions for graphs being \(k\)-edge-Hamiltonian. For convenience, we denote
\[
H_{n,k,\delta} := K_{\delta} \cup (K_{n-2\delta+k} \cup I_{\delta-k})
\]
and
\[
L_{n,k,\delta} := K_{k+1} \cup (K_{n-\delta-1} \cup K_{\delta-k}).
\]
It is easy to see that both \(H_{n,k,\delta}\) and \(L_{n,k,\delta}\) have minimum degree \(\delta(H_{n,k,\delta}) = \delta(L_{n,k,\delta}) = \delta\). Moreover, \(H_{n,k,\delta}\) is not \(k\)-edge-Hamiltonian since no linear forest with \(k\) edges within the
Now, we are ready to present our results in this paper. To avoid unnecessary calculations, we do not attempt to get the best bound on the order of graphs in the proof.

**Theorem 2.1.** Let \( k \geq 0, \delta \geq k + 2 \) and \( n \) be sufficiently large. If \( G \) is an \( n \)-vertex graph with minimum degree \( \delta(G) \geq \delta \) and 

\[
\lambda(G) \geq n - \delta + k - 1,
\]

then \( G \) is \( k \)-edge-Hamiltonian unless \( G = H_{n,k,\delta} \) or \( G = L_{n,k,\delta} \).

Since \( H_{n,k,\delta} \) contains \( K_{\delta} \) as a proper subgraph, we have \( \lambda(H_{n,k,\delta}) > n - \delta + k - 1 \). Moreover, applying the Kelmans operations on \( L_{n,k,\delta} \), we get a proper subgraph of \( H_{n,k,\delta} \), this implies \( \lambda(H_{n,k,\delta}) > \lambda(L_{n,k,\delta}) \); see, e.g., \[24, Theorem 2.12\]. With this observation in mind, Theorem 2.1 implies the following corollary, which is an extension on Theorem 1.6.

**Corollary 2.2.** Let \( k \geq 0, \delta \geq k + 2 \) and \( n \) be sufficiently large. If \( G \) is an \( n \)-vertex graph with minimum degree \( \delta(G) \geq \delta \) and 

\[
\lambda(G) \geq \lambda(H_{n,k,\delta}),
\]

then \( G \) is \( k \)-edge-Hamiltonian unless \( G = H_{n,k,\delta} \).

Recall that \( H_{n,k,\delta} = K_{\delta} \vee (K_{n-2\delta+k} \cup I_{\delta-k}) \). Let \( X \) be the set of \( \delta - k \) vertices with degree \( \delta \) forming by the independent set \( I_{\delta-k} \), \( Y \) be the set of \( \delta \) vertices with degree \( n - 1 \) corresponding to the clique \( K_{\delta} \) and \( Z \) be the set of the remaining \( n - 2\delta + k \) vertices with degree \( n - \delta + k - 1 \) corresponding to the clique \( K_{n-2\delta+k} \). We write \( E_1(H_{n,k,\delta}) \) for the set of edges of \( H_{n,k,\delta} \) whose both endpoints are from \( Y \cup Z \). Furthermore, we define the family \( \mathcal{H}^{(1)}_{n,k,\delta} \) of graphs as below.

\[
\mathcal{H}^{(1)}_{n,k,\delta} = \{ H_{n,k,\delta} \setminus E' : E' \subseteq E_1(H_{n,k,\delta}) \text{ with } |E'| \leq \lfloor \delta(\delta - k)/4 \rfloor \}.
\]

Here, we write \( H_{n,k,\delta} \setminus E' \) for the graph obtained from \( H_{n,k,\delta} \) by deleting all edges of \( E' \). Similarly, for the graph \( L_{n,k,\delta} = K_{k+1} \vee (K_{n-\delta-1} \cup K_{\delta-k}) \), we denote \( X \) by the set of vertices with degree \( \delta \) corresponding to the clique \( K_{\delta-k} \), \( Y \) by the set of vertices with degree \( n - 1 \)
corresponding to the clique $K_{k+1}$, and $Z$ by the set of the remaining $n - \delta - 1$ vertices with degree $n - \delta + k - 1$. We write $E_1(L_{n,k,\delta})$ for the set of edges of $L_{n,k,\delta}$ whose both endpoints are from $Y \cup Z$. Moreover, we define the family $L^{(1)}_{n,k,\delta}$ of graphs as follows.

$$L^{(1)}_{n,k,\delta} = \left\{ L_{n,k,\delta} \setminus E' : E' \subseteq E_1(L_{n,k,\delta}) \text{ with } |E'| \leq \left\lfloor (k + 1)(\delta - k)/4 \right\rfloor \right\}.$$ 

In this paper, we also present the following sufficient conditions on the signless Laplacian spectral radius for $k$-edge-Hamiltonian graphs with large minimum degree.

**Theorem 2.3.** Let $k \geq 0$, $\delta \geq k + 2$ and $n$ be sufficiently large. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and

$$q(G) \geq 2(n - \delta + k - 1),$$

then $G$ is $k$-edge-Hamiltonian unless $G \in H^{(1)}_{n,k,\delta}$ or $G \in L^{(1)}_{n,k,\delta}$.

As a consequence, we get the following corollary.

**Corollary 2.4.** Let $k \geq 0$, $\delta \geq k + 2$ and $n$ be sufficiently large. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and

$$q(G) \geq q(H_{n,k,\delta}),$$

then $G$ is $k$-edge-Hamiltonian unless $G = H_{n,k,\delta}$.

### 2.2 Spectral conditions for $k$-Hamiltonicity

A graph $G = (V, E)$ is called $k$-Hamiltonian if for all $X \subseteq V$ with $|X| \leq k$, the subgraph induced by the set $V \setminus X$ is Hamiltonian. In particular, 0-Hamiltonian graph is the same as the general Hamiltonian graph. In [6, 7], it is obtained that for a graph $G$, if $\delta(G) \geq \frac{n+k}{2}$, then $G$ is $k$-Hamiltonian. Clearly, when $k = 0$, it reduces to the Dirac theorem.

Recently, by utilizing the degree sequences and the closure concept, Liu, Liu, Zhang and Feng [28] generalized Theorem 1.8 to $k$-Hamiltonian graphs. Moreover, Liu, Lai and Das [29] proved some further results on $k$-Hamiltonian graphs independently. The theorems in our paper could be viewed as slight improvements on partial results of [29], since the conditions in our theorems are more concise and the extremal graphs seems more accurate.

We mention here that there is a tiny typo at the end of the proof in [28, Theorem 4] since the extremal graph is not the only one. Clearly, the graph $H_{n,k,\delta} = K_{\delta} \vee (K_{n-2\delta+k} \cup I_{\delta-k})$ is not $k$-Hamiltonian and $\lambda(H_{n,k,\delta}) > n - \delta + k - 1$. By a careful modification in [28], the correct result should be the following theorem.

**Theorem 2.5.** Let $k \geq 0$, $\delta \geq k + 2$ and $n$ be sufficiently large. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and

$$\lambda(G) \geq n - \delta + k - 1,$$

then $G$ is $k$-Hamiltonian unless $G = H_{n,k,\delta}$ or $G = L_{n,k,\delta}$.
In this paper, we shall provide another different way of the proof of Theorem 2.5 by applying a stability result on the number of edges (see Theorem 3.6). Since \(H_{n,k,\delta}\) contains \(K_{n-\delta+k}\) as a proper subgraph, it follows that \(\lambda(H_{n,k,\delta}) > n-\delta + k - 1\). On the other hand, by applying the Kelmans operation many times on \(L_{n,k,\delta}\), we can get a proper subgraph of \(H_{n,k,\delta}\), which leads to \(\lambda(H_{n,k,\delta}) > \lambda(L_{n,k,\delta})\). So we can immediately get the following corollary, which extended Theorem 1.6 slightly.

**Corollary 2.6** (Liu et al. [28]). Let \(k \geq 0, \delta \geq k + 2\) and \(n\) be sufficiently large. If \(G\) is an \(n\)-vertex graph with minimum degree \(\delta(G) \geq \delta\) and \(\lambda(G) \geq \lambda(H_{n,k,\delta})\), then \(G\) is \(k\)-Hamiltonian unless \(G = H_{n,k,\delta}\).

In addition, we shall prove the following signless Laplacian spectral version.

**Theorem 2.7.** Let \(k \geq 0, \delta \geq k + 2\) and \(n\) be sufficiently large. If \(G\) is an \(n\)-vertex graph with minimum degree \(\delta(G) \geq \delta\) and \(q(G) \geq 2(n - \delta + k - 1)\), then \(G\) is \(k\)-Hamiltonian unless \(G \in H^{(1)}_{n,k,\delta}\) or \(G \in L^{(1)}_{n,k,\delta}\).

Similarly, we have the following corollary.

**Corollary 2.8.** Let \(k \geq 0, \delta \geq k + 2\) and \(n\) be sufficiently large. If \(G\) is an \(n\)-vertex graph with minimum degree \(\delta(G) \geq \delta\) and \(q(G) \geq q(H_{n,k,\delta})\), then \(G\) is \(k\)-Hamiltonian unless \(G = H_{n,k,\delta}\).

It is worth noting that we spent a lot of efforts in characterizing the extremal families in terms of the signless Laplacian radius, which is one of the main parts in our paper (see Section 4). Furthermore, we have proved that our characterization of extremal graphs is sharp. In the proof of Theorems 2.3 and 2.7, we prove that the extremal graphs are contained in \(H_{n,k,\delta}\) or \(L_{n,k,\delta}\). Furthermore, the sharpness of our result can be seen from Lemmas 4.1 and 4.2.

### 3 Preliminaries and stability results

We need to use the following bounds on spectral radius. The first bound was found by Hong, Shu and Fang [19] for connected graphs. Independently, Nikiforov [33] Theorem 4.1 published a quite different method of this result for all graphs (not necessarily connected). Moreover, Zhou and Cho [38] determined the graphs which attain the upper bound.

**Theorem 3.1** ([19, 33]). Let \(G\) be a graph on \(n\) vertices with \(\delta(G) \geq \delta\). Then

\[
\lambda(G) \leq \frac{1}{2} \left( \delta - 1 + \sqrt{8e(G) - 4\delta n + (\delta + 1)^2} \right).
\]
The following theorem gives an upper bound on $q(G)$.

**Theorem 3.2** (Feng–Yu [11]). Let $G$ be a graph on $n$ vertices. Then

$$q(G) \leq \frac{2e(G)}{n-1} + n - 2.$$  

We also need the following operation for signless Laplacian spectral radius.

**Lemma 3.3** (Hong–Zhang [20]). Let $G$ be a connected graph and $q(G)$ be its signless Laplacian spectral radius corresponding to the Perron eigenvector $x$. Suppose that $u, v$ are two vertices of $G$ and $w_1, w_2, \ldots, w_s$ are distinct vertices in $N(v) \setminus (N(u) \cup \{v\})$ where $1 \leq s \leq d(v)$. If $x_u \geq x_v$ and $G^*$ is the graph obtained from $G$ by deleting the edges $vw_i$ and adding the edges $uw_i$ for all $1 \leq i \leq s$, then $q(G) < q(G^*)$.

The above operation is different from the Kelmans operation since we need to compare the coordinates of the Perron eigenvector. In addition, we remark that the same statement is also valid for the adjacency spectral radius; see, e.g., [30].

We next present some graph notations. The closure operation introduced by Bondy and Chvátal [3] is a powerful tool for the problems of Hamiltonicity of graphs. Let $G$ be a graph of order $n$. The $s$-closure of $G$, denoted by $\text{cl}_s(G)$, is the graph obtained from $G$ by recursively joining pairs of non-adjacent vertices whose degree sum is at least $s$ until no such pair remains. It is not hard to prove that the $s$-closure of $G$ is uniquely determined; see, e.g., [3]. Clearly, $G$ is a subgraph of $\text{cl}_s(G)$ for every $s$, and for any two non-adjacent vertices in $\text{cl}_s(G)$, the sum of their degrees is less than $s$.

**Theorem 3.4.** Let $G' = \text{cl}_{n+k}(G)$ be the $(n+k)$-closure graph of $G$.

1. [23, 3] A graph $G$ is $k$-edge-Hamiltonian if and only if $G'$ is $k$-edge-Hamiltonian.
2. [6, 3] A graph $G$ is $k$-Hamiltonian if and only if $G'$ is $k$-Hamiltonian.

**Remark.** From the above discussion, we know that if $d_G(u) + d_G(v) \geq n + k$ for all distinct vertices $u, v \in V(G)$, then the closure graph $\text{cl}_{n+k}(G)$ is a complete graph, hence it is $k$-Hamiltonian and $k$-edge-Hamiltonian, so is $G$ by Theorem 3.4.

To prove our theorems, we also need the following stability result, which is the main theorem proved by Füredi, Kostochka and Luo in [15, Theorem 5] and is also a generalization of the stability result on Hamilton cycle proved early in [24, Lemma 2] and [14, Theorem 3] independently.

**Theorem 3.5** (Füredi et al. [15]). Let $\delta > k \geq 0$ and $n \geq 6\delta - 5k + 5$. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and

$$e(G) > e(H_{n,k,\delta+1}),$$

then $G$ is $k$-edge-Hamiltonian unless $G \subseteq H_{n,k,\delta}$ or $G \subseteq L_{n,k,\delta}$.

In this section, we shall prove the next stability result for $k$-Hamiltonian graphs, which is a complement of Theorem 3.5 and a generalization of the result in [24] and [14]. Interestingly, the extremal graphs in Theorem 3.6 are the same as those in Theorem 3.5.
**Theorem 3.6.** Let \( \delta > k \geq 0 \) and \( n \) be sufficiently large. If \( G \) is an \( n \)-vertex graph with minimum degree \( \delta(G) \geq \delta \) and
\[
e(G) > e(H_{n,k,\delta+1}),
\]
then \( G \) is \( k \)-Hamiltonian unless \( G \subseteq H_{n,k,\delta} \) or \( G \subseteq L_{n,k,\delta} \).

We remark here that both Theorems 3.5 and 3.6 can be proved similarly by applying the techniques stated in [24] or [14, 15] or a variant of the proof in [34, Theorem 1.4]. We next include a proof using the method in [24] with slight differences.

**Proof.** Let \( G' = cl_{n+k}(G) \) be the \((n+k)\)-closure of \( G \). By Theorem 3.4, we know that if \( G' \) is \( k \)-Hamiltonian, then so is \( G \). Thus, we now assume that \( G' \) is not \( k \)-Hamiltonian. By the definition of closure, we know that any two distinct vertices in \( G' \) with degree sum no less than \( n+k \) are adjacent. Obviously, we have \( \delta(G') \geq \delta(G) \geq \delta \) and \( e(G') \geq e(G) \).

**Claim 3.1.** \( \omega(G') = n - \delta + k \).

**Proof of Claim.** A vertex of \( G' \) is called heavy if it has degree at least \( \frac{n+k}{2} \). Since every two vertices whose degree sum is at least \( n+k \) are adjacent, any two heavy vertices are adjacent in \( G' \). Namely, the set of all heavy vertices forms a clique in \( G' \). Let \( C \) be the set of vertices of a maximal clique of \( G' \) containing all heavy vertices. We denote \( t = |C| \) and \( H = G' \setminus C \) the subgraph of \( G' \) induced by \( V(G') \setminus C \).

We can observe the following two facts:

- For every \( v \in V(H) \), we have
  \[
d_{G'}(v) \leq \frac{n+k-1}{2}. \tag{6}
\]
  Indeed, otherwise, we may assume \( d_{G'}(v) > (n+k-1)/2 \), because \( d_{G'}(v) \) is a positive integer, then \( d_{G'}(v) \geq (n+k-1)/2 + 1/2 = (n+k)/2 \), so \( v \) is contained in \( C \), a contradiction.

- Moreover, for every \( v \in V(H) \), we have
  \[
d_{G'}(v) \leq n+k-t. \tag{7}
\]
  Otherwise, we assume that \( d_{G'}(v) \geq n+k-t+1 \). For each \( u \in C \), we have \( d_{G'}(u) \geq |C| - 1 = t - 1 \) since \( C \) is a clique. Note that \( d_{G'}(u) + d_{G'}(v) \geq n+k \). Thus \( v \) is adjacent to \( u \) for every \( u \in C \). The maximality of \( C \) implies that \( v \in C \), a contradiction.

In what follows, we will show that \( t \geq n - \delta + k \).

**Case 1.** Suppose first that \( 1 \leq t \leq n/3 - k/3 + \delta + 4/3 \). We mention here that this threshold is determined in the forthcoming Case 2. Clearly, for every \( v \in V(G') \), we have \( d_{C}(v) \leq t - 1 \), which together with (3) yields
\[
e(H) + e(V(H),C) = \frac{1}{2} \sum_{v \in V(H)} (d_{G'}(v) + d_{C}(v)) \leq \frac{1}{2} (n-t) \left( \frac{n+k-1}{2} + t - 1 \right).
\]
Then we have
\[
e(G') = e(G'[C]) + e(H) + e(V(H), C)
\leq \left(\frac{t}{2}\right) + \frac{1}{2}(n-t) \left(\frac{n+k-1}{2} + t - 1\right) = \frac{n-k+1}{4}t + \frac{n(n+k-3)}{4}
\leq \frac{n-k+1}{4} \left(\frac{n}{3} - \frac{k}{4} + \frac{1}{3}\right) + \frac{n(n+k-3)}{4}
= \frac{1}{3}n^2 + \frac{3k+k-4}{12} n + \frac{4}{3} + k^2 - \frac{5k}{4} - \frac{5k}{12} + \frac{1}{3}
< e(H_{n,k,\delta+1}),
\]
the last inequality follows since \( n \) is large enough, which leads to a contradiction.

**Case 2.** Secondly, suppose that \( n/3 - k/3 + \delta + 4/3 \leq t \leq n - \delta + k - 1 \). Note that
\[
e(H) + e(V(H), C) \leq \sum_{v \in V(H)} d_G'(v) \leq (n-t)(n+k-t),
\]
where the last inequality follows by using (7). Therefore
\[
e(G') = e(G'[C]) + e(H) + e(V(H), C)
\leq \left(\frac{t}{2}\right) + (n-t)(n+k-t) = \frac{3t^2}{4} - (2n+k-\frac{1}{2})t + n(n+k)
\leq \frac{3}{4} (n-\delta+k-1)^2 - (2n+k+\frac{1}{2})(n-\delta+k-1) + n(n+k)
= e(H_{n,k,\delta+1}),
\]
where the last inequality holds since the quadratic function on variable \( t \) attains the maximum at \( t = n - \delta + k - 1 \). This is also a contradiction.

From the above discussion, we know that \( \omega(G') \geq |C| \geq n - \delta + k \). Suppose that \( C' \) is a largest clique in \( G' \) with \( |C'| \geq n - \delta + k + 1 \). We denote by \( H' = G' \setminus C' \) the subgraph of \( G' \) induced by \( V(G') \setminus C' \). Since \( G' \) is not a clique (otherwise, \( G' \) is \( k \)-Hamiltonian), we get that \( V(H') \) is not empty. Note that \( d_G'(v) \geq \delta(G) \geq \delta \) for every \( v \in V(H') \) and \( d_G'(u) \geq |C'| - 1 \geq n + k - \delta \) for every \( u \in C' \), hence \( d_G'(v) + d_G'(u) \geq n + k \), this means that every vertex in \( H' \) is adjacent to every vertex of \( C' \), this contradicts the fact that \( C' \) is a maximum clique.

From the above case analysis, we now obtain that \( \omega(G') = n - \delta + k \). In addition, the argument also showed that the set \( C \) of vertices with degree at least \( \frac{n+k}{2} \) induces a maximum clique of \( G' \) and \( |C| = n - \delta + k \). \( \square \)

Let \( C \) be the set of vertices of a largest clique in \( G' \) and \( H = G' \setminus C \). By Claim 3.1, we have \( |C| = n - \delta + k \) and \( |V(H)| = \delta - k \). By the definition of \( G' \), we can see that every vertex of \( H \) has degree exactly \( \delta \) in \( G' \). We say that a vertex in \( C \) is a frontier vertex if it has degree at least \( n - \delta + k \) in \( G' \), that is, it has at least one neighbor in \( H \). We denote by \( F = \{u_1, u_2, \ldots, u_s\} \) the set of all frontier vertices in \( C \). Since \( d_G'(u_i) \geq n - \delta + k \) and \( d_G'(v) \geq \delta \) for every \( v \in F \), we know that every vertex in \( H \) is adjacent to every vertex in \( H \), and then \( N(v) \cap C = F \) for every \( v \in V(H) \). In fact, we have \( d_G'(u_i) = n - 1 \) for every \( u_i \in F \). Since \( d(v) = \delta \) for every \( v \in H \), we then get \( k + 1 \leq s \leq \delta \). Since \( C \) forms a clique on \( n - \delta + k \) vertices, we can choose a path \( P \) in \( C - F \) with two end-vertices \( u_1 \) and \( u_s \).

**Claim 3.2.** We claim that \( s = k + 1 \) or \( s = \delta \).
Proof of Claim. If \( k + 2 \leq s \leq \delta - 1 \), we will show that \( G' \) is \( k \)-Hamiltonian in this case. Let \( S \subseteq V(G') \) be any set of vertices with size at most \( k \). Set \( S_1 = S \cap (F \cup V(H)) \) and \( S_2 = S \cap (C \setminus F) \). Since \( |F| = s \geq k + 2 \), we have \( |F \setminus S_1| \geq 2 \), so we can fix two vertices \( u, v \in F \setminus S_1 \). We consider the induced subgraph \( G^* = G'[F \cup V(H)] \) and will show that there exists a Hamilton path in \( G^* \) that connects vertices \( u, v \) and lies outside \( S_1 \). Note that for any \( x, y \in F \cup V(H) \), we have

\[
d_{G^*}(x) + d_{G^*}(y) \geq 2\delta \geq \delta + s + 1 = |F \cup V(H)| + k + 1.
\]

By noting the remark of Theorem 3.4, we obtain that \( G^* \) is \( (k + 1) \)-Hamiltonian. Since \( |S_1 \cup \{u\}| \leq k + 1 \), there exists a Hamilton cycle in the induced subgraph \( G^* \setminus (S_1 \cup \{u\}) \), say \( v_1 v_2 \cdots v_r v \). Note that \( u \in F \) is adjacent to every vertex in \( G^* \). In particular, \( \{u, v_1\} \in E(G^*) \). Therefore the path \( P_1 = uv_1 v_2 \cdots v_r v \) passes through all vertices in \( (F \cup V(H)) \setminus S_1 \). Note that the subgraph of \( G' \) induced by the vertex set \( (C \setminus F) \cup \{u, v\} \) is a complete graph. Thus there exists a Hamilton path \( P_2 \) that connects vertices \( u, v \) and passes through all vertices in \( (C \setminus F) \setminus S_2 \). We conclude that \( P_1 \cup P_2 \) is a Hamiltonian cycle in \( G' \setminus S \), so \( G' \) is \( k \)-Hamiltonian, this is a contradiction.

If \( s = k + 1 \), then \( |F \cup V(H)| = |F| + |V(H)| = k + 1 + \delta - k = \delta + 1 \). Note that \( G'[F, V(H)] \) forms a complete bipartite graph and \( d_{G'}(v) = \delta \) for each \( v \in V(H) \). This implies that \( H \) is a complete subgraph on \( \delta - k \) vertices and \( F \cup V(H) \) is a clique on \( \delta + 1 \) vertices. In this case, we have \( G' = L_{n,k,\delta} \) and then \( G \subseteq L_{n,k,\delta} \).

If \( s = \delta \), then by noticing \( G'[F, V(H)] \) forms a complete bipartite graph and \( d_{G'}(v) = \delta \) for each \( v \in V(H) \), we know that \( V(H) \) is an independent set of order \( \delta - k \). Thus we get \( G' = H_{n,k,\delta} \) and then \( G \subseteq H_{n,k,\delta} \). The proof is now complete.

Remark. We remark that the stability Theorem 3.6 was partially proved in [29], although the line of the proofs seems similar with that in [24, Lemma 2]. However, the extremal graphs are characterized in [29, Theorem 1.10] by using the terminology of graph-closure, which states that the closure graph \( \text{cl}_{n+k}(G) \subseteq G_n(p, k + 1, \delta) \), where \( G_n(p, k + 1, \delta) \) is a family of many graphs; see [29] for the exact definition. While in Theorem 3.6 of the present paper, we have shown that there are only two possible extremal graphs: \( \text{cl}_{n+k}(G) = H_{n,k,\delta} \) or \( \text{cl}_{n+k}(G) = L_{n,k,\delta} \), which implies \( G \subseteq H_{n,k,\delta} \) or \( G \subseteq L_{n,k,\delta} \). Thus, Theorem 3.6 could be viewed as a slight improvement of [29, Theorem 1.10] in some sense.

As a direct consequence of Theorems 3.5 and 3.6, we get the following corollary since we can verify that \( e(H_{n,k,\delta+1}) < e(H_{n,k,\delta}) \) and \( e(L_{n,k,\delta}) < e(H_{n,k,\delta}) \) for \( n \) sufficiently large.

Corollary 3.7. Let \( \delta > k \geq 0 \) and \( n \) be sufficiently large. If \( G \) is an \( n \)-vertex graph with minimum degree \( \delta(G) \geq \delta \) and

\[
e(G) \geq e(H_{n,k,\delta}),
\]

then \( G \) is \( k \)-edge-Hamiltonian and \( k \)-Hamiltonian unless \( G = H_{n,k,\delta} \).

We know that the proof strategy of Theorem 1.8 presented in [34] does not apply the stability theorem directly, it seems more algebraic than that in [29]. We remark here that there is another way to prove Theorem 1.8. In fact, by a tiny modification of the proof of Theorem 1.6 in [24], we can prove that if \( G \) is non-Hamiltonian and \( \lambda(G) \geq n - \delta - 1 \), then \( G \) is a subgraph of \( H_{n,\delta} \) or \( L_{n,\delta} \). As pointed out by Nikiforov [34], the crucial point of the
argument of Theorem 1.6 is based on proving that for large $n \geq \delta^3 + \delta + 4$, if $G$ is a subgraph of $H_{n,\delta}$ with $\delta(G) \geq \delta$, then $\lambda(G) < n - \delta - 1$, unless $G = H_{n,\delta}$. The same argument holds for the subgraph of $L_{n,\delta}$. We observe that both $H_{n,\delta}$ and $L_{n,\delta}$ consist of a large clique $K_{n-\delta}$ together with a few number of outgrowth edges, this implies that $\lambda(H_{n,\delta})$ and $\lambda(L_{n,\delta})$ are slightly greater than $\lambda(K_{n-\delta}) = n - \delta - 1$. Roughly speaking, for sufficiently large $n$ with respect to $\delta$, the key idea of Nikiforov exploits the fact that when $G$ is a subgraph of $H_{n,\delta}$ or $L_{n,\delta}$ with $\delta(G) \geq \delta$, then $G$ is obtained by deleting edges from the subgraph $K_{n-\delta}$. One can further show that all the outgrowth edges contribute to $\lambda(G)$ much less than a single edge within $K_{n-\delta}$. Thus the deleting of any one edge from the clique $K_{n-\delta}$ can lead to $\lambda(G) < \lambda(K_{n-\delta}) = n - \delta - 1$; see [34, Theorem 1.6] for more details.

With the above observation, one can prove the following analogues.

**Theorem 3.8** (see [28]). Let $k \geq 1, \delta \geq k + 2$ and $n$ be sufficiently large. If $G$ is a subgraph of $H_{n,k,\delta}$ or $L_{n,k,\delta}$ and the minimum degree $\delta(G) \geq \delta$, then

$$\lambda(G) < n - \delta + k - 1,$$

unless $G = H_{n,k,\delta}$ or $G = L_{n,k,\delta}$.

**Remark.** When $G$ is a subgraph of $L_{n,k,\delta}$, this theorem was partially proved in [28, Theorem 1.6]. The remaining case that $G$ is a subgraph of $H_{n,k,\delta}$ can be proved similarly, so we leave the details for interested readers.

### 4 Proofs of main results

Recall that the signless Laplacian matrix $Q(G)$ associated with graph $G$ is given as $D + A$, where $D$ is the diagonal matrix of degrees and $A$ is the adjacency matrix of $G$. Let $q(G)$ denote the largest eigenvalue of $Q(G)$. The Rayleigh principle yields

$$q(G) = \max_{x \neq 0} \frac{x^T Q(G) x}{x^T x},$$

where

$$x^T Q(G) x = \sum_{v \in V(G)} d(v) x_v^2 + 2 \sum_{\{u,v\} \in E(G)} x_u x_v.$$

Let $f$ be an eigenvector corresponding to $q(G)$, i.e., $Q(G)f = q(G)f$. By the celebrated Perron–Frobenius theorem (see [5, p. 22] or [16, p. 178]), we may assume that $f_v > 0$ for each $v \in V(G)$ when $G$ is connected. It is easy to see from the eigen-equation that for any $u, v \in V(G)$,

$$(q(G) - d(u))f_u = \sum_{w \in N(u)} f_w, \quad \text{and} \quad (q(G) - d(v))f_v = \sum_{z \in N(v)} f_z.$$

Therefore, we obtain

$$(q(G) - d(u))(f_u - f_v) = (q(G) - d(u))f_u - (q(G) - d(v))f_v + (d(u) - d(v))f_v.$$
Proof. In the following proof, we shall assume that

Similarly, we can show that

**Lemma 4.2.** If \( G \in \mathcal{H}^{(1)}_{n,k,\delta} \) or \( G \in \mathcal{L}^{(1)}_{n,k,\delta} \), then \( G \) is neither \( k \)-edge-Hamiltonian nor \( k \)-Hamiltonian. Moreover, we have

\[
q(G) \geq 2(n - \delta + k - 1).
\]

Proof. For each graph \( G \in \mathcal{H}^{(1)}_{n,k,\delta} \) or \( G \in \mathcal{L}^{(1)}_{n,k,\delta} \), we can clearly see that \( G \) is not \( k \)-edge-Hamiltonian. Next we shall prove that \( q(G) \geq 2(n - \delta + k - 1) \). Recall the subsets \( X, Y \) and \( Z \) defined as above. For each case, we define a vector \( h \) such that \( h_v = 1 \) for every \( v \in Y \cup Z \) and \( h_v = 0 \) for every \( v \in X \). Note that \( q(K_{n-\delta+k} \cup I_{\delta-k}) = q(K_{n-\delta+k}) = 2(n - \delta + k - 1) \) and \( h \) is the corresponding eigenvector. If \( G \in \mathcal{H}^{(1)}_{n,k,\delta} \), then we get

\[
h^T Q(G) h - h^T Q(K_{n-\delta+k} \cup I_{\delta-k}) h = \delta(\delta - k) - 4|E'| \geq 0.
\]

By the Rayleigh Formula, we have

\[
q(G) \geq \frac{h^T Q(G) h}{h^T h} \geq \frac{h^T Q(K_{n-\delta+k} \cup I_{\delta-k}) h}{h^T h} = 2(n - \delta + k - 1).
\]

Similarly, we can show that \( q(G) \geq 2(n - \delta + k - 1) \) for every \( G \in \mathcal{L}^{(1)}_{n,k,\delta} \).

We give the definitions of two families of graphs.

\[
\mathcal{H}^{(2)}_{n,k,\delta} = \{ H_{n,k,\delta} \setminus E' : E' \subseteq E_1(H_{n,k,\delta}) \text{ with } |E'| \geq \lfloor \delta(\delta - k)/4 \rfloor + 1 \},
\]

and

\[
\mathcal{L}^{(2)}_{n,k,\delta} = \{ L_{n,k,\delta} \setminus E' : E' \subseteq E_1(L_{n,k,\delta}) \text{ with } |E'| \geq \lceil (k + 1)(\delta - k)/4 \rceil + 1 \}.
\]

**Lemma 4.2.** If \( n \) is sufficiently large and \( G \in \mathcal{H}^{(2)}_{n,k,\delta} \) or \( G \in \mathcal{L}^{(2)}_{n,k,\delta} \), then

\[
q(G) < 2(n - \delta + k - 1).
\]

Proof. In the following proof, we shall assume that \( G \in \mathcal{H}^{(2)}_{n,k,\delta} \). Since the proof for the case of \( G \in \mathcal{L}^{(2)}_{n,k,\delta} \) is similar, we only give the sketch in this case. Let \( G \) be a graph from \( \mathcal{H}^{(2)}_{n,k,\delta} \) with maximum signless Laplacian spectral radius. This means that \( G \) is obtained from \( H_{n,k,\delta} \) by deleting exactly \( \lfloor \delta(\delta - k)/4 \rfloor + 1 \) edges from \( E_1(H_{n,k,\delta}) \) as the monotonicity of the signless Laplacian spectral radius. Let \( f \) be the eigenvector corresponding to \( q(G) \). Furthermore, we assume that \( \max_{v \in V(G)} |f_v| = 1 \).

Let \( h \) be the vector defined as in the proof of Lemma 4.1. First of all, we can show the following claim, which is a lower bound on \( q(G) \).

**Claim 1.** \( q(G) > 2(n - \delta + k - 1) - 1 \).
Proof of Claim 1. If $G \in \mathcal{H}_{n,k,\delta}^{(2)}$, then we obtain
\[ h^T Q(G) h - h^T Q(K_{n-\delta+k} \cup I_{\delta-k}) h = \delta(\delta - k) - 4|E'| \geq -4. \]

By the Rayleigh Formula, we have
\[ q(G) \geq \frac{h^T Q(G) h}{h^T h} \geq \frac{h^T Q(K_{n-\delta+k} \cup I_{\delta-k}) h}{h^T h} - \frac{4}{h^T h} = 2(n - \delta + k - 1) - \frac{4}{h^T h}, \]
which implies that $q(G) \geq 2(n - \delta + k - 1) - 1$. \qed

Recall that
\[
X = \{ v \in V(H_{n,k,\delta}) : d(v) = \delta \}, \\
Y = \{ v \in V(H_{n,k,\delta}) : d(v) = n - 1 \}, \\
Z = \{ v \in V(H_{n,k,\delta}) : d(v) = n - \delta + k - 1 \}.
\]

We next show that all entries of $f$ corresponding to the vertices of $X$ are tiny (or small) since $q(G) > 2n - 2\delta + 2k - 3$ by Claim 1, and the condition that $n$ is large enough.

Claim 2. For each $x \in X$, we have
\[ f_x \leq \frac{\delta}{q(G) - \delta} = o(n). \]

Proof of Claim 2. The following equality
\[ (q(G) - d(x)) f_x = \sum_{y \in Y} f_y, \]
together with $d(x) = \delta$, yields the required inequality. \qed

Note that $E'$ is the edge set in which both endpoints are in $Y \cup Z$, we define two subsets of $Y$ and $Z$, respectively.
\[
Y_1 = \{ y \in Y : d(y) = n - 1 \} \text{ and } Y_2 = \{ y \in Y : d(y) \leq n - 2 \}.
\]

Similarly, we define two subsets of $Z$ as follows.
\[
Z_1 = \{ z \in Z : d(v) = n - \delta + k - 1 \} \text{ and } Z_2 = \{ z \in Z : d(v) \leq n - \delta + k - 2 \}.
\]

Note that the number of edges removed from $Y \cup Z$ is exactly $|\delta(\delta - k)/4| + 1$. Since $|Z| = n - 2\delta + k$ and $n$ is sufficiently large, we get $Z_1 \neq \emptyset$. We next compare the entries of eigenvector $f$ corresponding to these subsets. Roughly speaking, the vertex with large degree always has large value at the corresponding entry.

Claim 3. We have the following statements.
(a) If $Y_2 \neq \emptyset$, then $f_{y_2} < f_{z_1}$ for all $y_2 \in Y_2$ and $z_1 \in Z_1$.
(b) If $Z_2 \neq \emptyset$, then $f_{z_2} < f_{z_1}$ for all $z_2 \in Z_2$ and $z_1 \in Z_1$.
(c) If $Y_1, Y_2 \neq \emptyset$, then $f_{y_2} < f_{y_1}$ for all $y_2 \in Y_2$ and $y_1 \in Y_1$.
(d) If $Y_1 \neq \emptyset$, then $f_{z_1} < f_{y_1}$ for all $z_1 \in Z_1$ and $y_1 \in Y_1$. 

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Proof of Claim 3. (a) Suppose on the contrary that there exist some \( y_2 \in Y_2 \) and \( z_1 \in Z_1 \) such that \( f_{y_2} \geq f_{z_1} \). Let \( v \in Z_2 \cup Y_2 \) be a vertex not adjacent to \( y_2 \). Since \( z_1 \in Z_1 \), we have \( \{z_1, y_2\} \in E(G) \) and \( \{z_1, w\} \in E(G) \). We define a new graph \( G^* \in \mathcal{H}^{(2)}_{n,k,\delta} \) by deleting \( \{z_1, w\} \) and adding \( \{y_2, w\} \). By Lemma 3.3, we get \( q(G^*) > q(G) \), which contradicts with the choice of \( G \). Thus we have \( f_{y_2} < f_{z_1} \) for all \( y_2 \in Y_2 \) and \( z_1 \in Z_1 \).

(b) For \( z_2 \in Z_2 \) and \( z_1 \in Z_1 \), we have \( \{z_1, z_2\} \in E(G) \). So \( z_2 \in N(z_1) \setminus N(z_2) \) and \( z_1 \in N(z_2) \setminus N(z_1) \). We denote \( N[z_2] = N(z_2) \cup \{z_2\} \) and \( N[z_1] = N(z_1) \cup \{z_1\} \). We can get from (8) that

\[
(q(G) - d(z_1) + 1)(f_{z_1} - f_{z_2}) = (d(z_1) - d(z_2))f_{z_2} + \sum_{s \in N(z_1) \setminus N[z_2]} f_s - \sum_{t \in N(z_2) \setminus N[z_1]} f_t.
\]

Since \( N(z_2) \setminus \{z_1\} \subseteq N(z_1) \setminus \{z_2\} \), we then get

\[
(q(G) - d(z_1) + 1)(f_{z_1} - f_{z_2}) = (d(z_1) - d(z_2))f_{z_2} + \sum_{s \in N(z_1) \setminus N[z_2]} f_s > 0, \tag{9}
\]

which implies \( f_{z_1} > f_{z_2} \) since \( q(G) \geq 2(n - \delta + k - 1) - 1 > d(z_1) - 1 \) and \( d(z_1) > d(z_2) \). The proofs of (c) and (d) are similar with that of (a) and (b), so we omit the details. \( \square \)

Claim 4. For each \( w \in Y \cup Z \), we have

\[
f_w \geq 1 - \frac{(\delta + 2)(\delta - k) + 6}{2(q(G) - n + 2)} = 1 - o(n).
\]

Proof of Claim 4. By Claim 3, it is sufficient to prove this claim in the case \( w \in Y_2 \cup Z_2 \) whenever \( Y_2 \neq \emptyset \) or \( Z_2 \neq \emptyset \). Since \( Z_1 \) is non-empty, by Claim 3 again, we know that \( \max_{v \in V(G)} f_v \) is attained by vertices from \( Y_1 \) or \( Z_1 \). We next proceed in two cases.

Case 1. \( Y_1 = \emptyset \). By Claim 3, we get that \( \max_{v \in V(G)} f_v \) is attained by vertices from \( Z_1 \). Choose a vertex \( z_1 \) from \( Z_1 \) with \( f_{z_1} = 1 \). We notice that \( z_1 \) is adjacent to all other vertices in \( Y \cup Z \).

Subcase 1.1. If \( w \in Z_2 \), then \( \{z_1, w\} \in E(G) \) and \( N(w) \subseteq N(z_1) \), which yields \( d(z_1) - d(w) \leq \lfloor \delta(\delta - k)/4 \rfloor + 1 \) and \( |N(z_1) \setminus N[w]| \leq \lfloor \delta(\delta - k)/4 \rfloor + 1 \). By applying (9), we have

\[
(q(G) - d(z_1) + 1)(f_{z_1} - f_w) = (d(z_1) - d(w))f_w + \sum_{s \in N(z_1) \setminus N[w]} f_s \leq \frac{\delta(\delta - k)}{2} + 2.
\]

Note that \( d(z_1) = n - \delta + k - 1 \) and \( f_{z_1} = 1 \), we obtain

\[
1 - f_w \leq \frac{\delta(\delta - k) + 4}{2(q(G) - n + \delta - k + 2)}.
\]

Subcase 1.2. If \( w \in Y_2 \), then \( |d(z_1) - d(w)| \leq \frac{\delta(\delta - k)}{4} + 1 + (\delta - k) \). Moreover, we have

\[
|N(z_1) \setminus N[w]| \leq \frac{\delta(\delta - k)}{4} + 1, \quad |N(w) \setminus N[z_1]| = |X| = \delta - k.
\]
Applying (9), we get
\[ (q(G) - d(z_1) + 1)(f_{z_1} - f_w) = (d(z_1) - d(w))f_w + \sum_{s \in N(z_1) \setminus N[w]} f_s - \sum_{t \in N(w) \setminus N[z_1]} f_t \leq \frac{\delta(\delta - k)}{4} + 1 + (\delta - k) + \frac{\delta(\delta - k)}{4} + 1. \]

Note that \( d(z_1) = n - \delta + k - 1 \) and \( f_{z_1} = 1 \), we obtain
\[ 1 - f_w \leq \frac{(\delta + 2)(\delta - k) + 4}{2(q(G) - n + \delta - k + 2)}. \]

**Case 2.** \( Y_1 \neq \emptyset \). By Claim 3, we get that \( \max_{v \in V(G)} f_v \) is attained by vertices from \( Y_1 \). Let \( y_1 \) be a vertex from \( Y_1 \) such that \( f_{y_1} = 1 \). By repeating the argument in Case 1, we can prove that
\[ 1 - f_w \leq \frac{(\delta + 2)(\delta - k) + 6}{2(q(G) - n + 2)}. \]

The proof of Claim 4 is completed.

Claims 2 and 4 showed that the Perron eigenvector \( f \) has small values on the entries of \( X \) and large values on the entries of \( Y \). This reveals that vector \( f \) is entrywise close to \( h \) defined as in the proof of Lemma 4.1.

We are now ready to prove Lemma 4.2 completely. By Claims 2 and 4, we obtain
\[
\begin{align*}
&f^T Q(G)f - f^T Q(K_{n-\delta+k} \cup I_{\delta-k})f \\
= &\sum_{x \in X, y \in Y} (f_x + f_y)^2 - \sum_{\{u,v\} \in E'} (f_u + f_v)^2 \\
\leq &\delta(\delta - k) \left( \frac{\delta}{q(G) - \delta} + 1 \right)^2 - 4|E'| \left( 1 - \frac{(\delta - 2)(\delta - k) + 6}{2(q(G) - n + 2)} \right)^2 \\
< &0,
\end{align*}
\]

where the last inequality holds by using \( q(G) > 2(n - \delta + k - 1) - 1 \) in Claim 1 and \(|E'| = \lfloor \frac{\delta(\delta-k)}{4} \rfloor + 1 > \frac{\delta(\delta-k)}{4} \), then \( \frac{\delta(\delta-k)}{4|E'|} < \frac{1-\alpha(1)}{1+\alpha(1)} \) holds for sufficiently large \( n \). Hence, we have
\[ q(G) = \frac{f^T Q(G)f}{f^T f} < \frac{f^T Q(K_{n-\delta+k} \cup I_{\delta-k})f}{f^T f} \leq 2(n - \delta + k - 1). \]

This completes the proof.

With the help of these Lemmas, we can prove our theorems immediately.

**Proof of Theorems 2.1 and 2.5.** By Theorem 3.1, we can get
\[ n - \delta + k - 1 \leq \lambda(G) \leq \frac{1}{2} \left( \delta - 1 + \sqrt{8e(G) - 4n + (\delta + 1)^2} \right). \]

By calculation, and noticing that \( n \) is large enough, we have
\[
e(G) \geq \frac{n^2 - (2\delta - 2k + 1)n + 2\delta^2 - 3\delta k + \delta + k^2 - k}{2}
\]
\[ > \left( n - \delta - \frac{1+k}{2} \right) + (\delta + 1)(\delta + 1 - k) \]
\[ = e(H_{n,k,\delta+1}). \]

By Theorems 3.5 and 3.6, we know that \( G \) is \( k \)-edge-Hamiltonian and \( k \)-Hamiltonian unless \( G \subseteq H_{n,k,\delta} \) or \( G \subseteq L_{n,k,\delta} \). Note that \( \delta(G) \geq \delta \) and \( \lambda(G) \geq n - \delta + k - 1 \). By Theorem 3.8, we get \( G = H_{n,k,\delta} \) or \( G = L_{n,k,\delta} \). \( \square \)

**Proof of Theorems 2.3 and 2.7** By applying Theorem 3.2, we obtain

\[ 2(n - \delta + k - 1) \leq q(G) \leq \frac{2e(G)}{n-1} + n - 2. \]

Since \( n \) is sufficiently large, we have

\[ e(G) \geq \frac{n^2 - (2\delta - 2k + 1)n + 2\delta - 2k}{2} > e(H_{n,k,\delta+1}). \]

By Theorems 3.5 and 3.6, we get that \( G \) is \( k \)-edge-Hamiltonian and \( k \)-Hamiltonian unless \( G \subseteq H_{n,k,\delta} \) or \( G \subseteq L_{n,k,\delta} \). Note that \( \delta(G) \geq \delta \) and \( q(G) \geq 2(n - \delta + k - 1) \). By Lemma 4.2, we know that \( G \in \mathcal{H}_{n,k,\delta}^{(1)} \) or \( G \in \mathcal{L}_{n,k,\delta}^{(1)} \). \( \square \)

**5 Concluding remarks**

Let \( G \) be a bipartite graph with vertex sets \( X \) and \( Y \). The bipartite graph \( G \) is called balanced if \( |X| = |Y| \). If a bipartite graph has Hamilton cycle, then it must be balanced. So we consider the existence of Hamilton cycle only in balanced bipartite graphs.

Motivated by the work of Erdős [10] in Theorem 1.2, Moon and Moser [32] provided a corresponding result for the balanced bipartite graphs.

**Theorem 5.1** (Moon–Moser [32]). Let \( G \) be a balanced bipartite graph on \( 2n \) vertices. If the minimum degree \( \delta(G) \geq \delta \) for some \( 1 \leq \delta \leq n/2 \) and

\[ e(G) > n(n - \delta) + \delta^2, \]

then \( G \) has a Hamilton cycle.

The condition \( \delta \leq n/2 \) is well natural since Moon and Moser [32] also pointed out that if \( G \) is a balanced bipartite on \( 2n \) vertices with \( \delta(G) > n/2 \), then \( G \) must be Hamiltonian. This is a bipartite version of the Dirac theorem.

Let \( B_{n,\delta} \) be the bipartite graph obtained from the complete bipartite graph \( K_{n,n} \) by deleting all edges in its one subgraph \( K_{\delta,n-\delta} \). More precisely, the two vertex parts of \( B_{n,\delta} \) are \( V = V_1 \cup V_2 \) and \( U = U_1 \cup U_2 \) where \( |V_1| = |U_1| = \delta \) and \( |V_2| = |U_2| = n - \delta \), we join all edges between \( V_1 \) and \( U_1 \), and all edges between \( V_2 \) and \( U \). It is easy to see that \( e(B_{n,\delta}) = n(n - \delta) + \delta^2 \) and \( B_{n,\delta} \) contains no Hamilton cycle. This implies that the condition in Moon–Moser’s theorem is best possible.

For the Hamiltonicity of balanced bipartite graphs, Li and Ning [24, Theorem 1.10] also proved the spectral version of the Moon–Moser Theorem 5.1, that is, the bipartite version of Theorems 1.6 and 1.7. On the other hand, a bipartite graph is called nearly balanced if
\[|X| - |Y| \leq 1. \] Additionally, Li and Ning \cite{25} also considered the existence of Hamilton path in nearly balanced bipartite graphs. Later, Ge and Ning \cite[Theorem 1.4]{18}, and Jiang, Yu and Fang \cite[Theorem 1.2]{21} independently proved a further improvement on the adjacency spectral result of Li and Ning for balanced bipartite graphs. Correspondingly, Li, Liu and Peng \cite[Theorem 4]{26} also gave a further improvement on the signless Laplacian spectral result of Li and Ning. It is noteworthy that Liu, Wu and Lai \cite{30} unified several former spectral Hamiltonian results on balanced bipartite graphs. In addition, Lu \cite{31} extended some spectral conditions for the Hamiltonicity of balanced bipartite graphs.

**Question 5.2.** It is possible and interesting to extend the notations of \( k \)-edge-Hamiltonian and \( k \)-Hamiltonian, and establish the variants of our results in Section 2 for the balanced bipartite graphs or nearly balanced bipartite graphs.

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