ON SYMMETRIES OF THE GIBBONS–TSAREV EQUATION

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ABSTRACT. We study the Gibbons–Tsarev equation $z_{yy} + z_x z_{xy} - z_y z_{xx} + 1 = 0$ and, using the known Lax pair, we construct infinite series of conservation laws and the algebra of nonlocal symmetries in the covering associated with these conservation laws. We prove that the algebra is isomorphic to the Witt algebra. Finally, we show that the constructed symmetries are unique in the class of polynomial ones.

INTRODUCTION

The Gibbons–Tsarev equation considered in this paper was introduced in [6] to classify finite reductions of the infinite Benney system. The Gibbons–Tsarev equation is undoubtedly integrable [7, 21, 1]. It is known to have infinitely many conservation laws and infinitely many symmetries ([5, § 3.3] and [8]). However, unlike the majority of integrable equations with two independent variables, the Gibbons–Tsarev equation has only few local symmetries [19, § 1], thus escaping symmetry-based integrability tests [14, 15]. In this respect the equation resembles the integrable Ernst equation of general relativity [4].

Systematic computation of nonlocal symmetries soon reveals that infinitely many nonlocal symmetries can be obtained through commutation. This was first observed for the unreduced Benney system in [18], where five symmetries were written out explicitly and the structure of the symmetry algebra was revealed.

The aim of this paper is to provide an explicit description of symmetries of the Gibbons–Tsarev equation and an exact proof that the symmetries constitute the Witt algebra. As the reader will see, a rigorous proof is far from being simple. Moreover, although the symmetry algebra has two generators $Z^{(-1)}$ and $Z^{(1)}$ (constructed in Sections 2 and 4 respectively), they are not of much help, because obtaining them and their commutators requires essentially the same effort as obtaining all symmetries and commutators at once.

We present the results as follows. In Section 1 we introduce main notions and the notation. Section 2 deals with the local properties of the Gibbons–Tsarev equation. Coverings and nonlocal conservation laws are dealt with in Section 3. We introduce an appropriate infinite system of nonlocal conservation laws in two different but equivalent ways, which is convenient from the computational point of view. The corresponding infinite-dimensional covering is the common ‘ground’ for all the nonlocal symmetries and their commutators to be constructed in the sequel. We also consider a one-dimensional covering that allows to treat the equation as an evolutionary two-component system, which is also convenient for some proofs. In Section 4 we construct the nonlocal symmetries, starting with their
shadows. The shadows were found in a way that can be reused in other similar situations. The rest of the section is devoted to the explicit description of full nonlocal symmetries derived from these shadows and to a proof that the symmetries constitute the Witt algebra. Finally, Section 5 is devoted to the proof of uniqueness of the constructed symmetries in the class of polynomial ones.

1. Preliminaries and notation

We expose here briefly the fundamentals of local [3] and nonlocal [11] geometry of PDEs. Consider a PDE given by a system of relations \( \{ F = 0 \} \), where \( F = (F^1, \ldots, F^r) \) is a vector function in \( x = (x^1, \ldots, x^n) \), \( u = (u^1, \ldots, u^m) \) and finite number of partial derivatives of \( u \) with respect to \( x \). To any such a system we put into correspondence a locus \( \mathcal{E} \subset J^\infty(\pi) \) in the space of infinite jets, where \( \pi: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \) is the trivial bundle and \( \mathbb{R}^m, \mathbb{R}^n \) are Euclidean spaces with the coordinates \( u^1, \ldots, u^m, x^1, \ldots, x^n \), respectively. This locus is defined by all the differential consequenc es of the system and called the infinitely prolonged equation.

When the coordinates \( x^i, u^j \) are chosen, the adapted coordinates \( u^j_\sigma \) arise in \( J^\infty(\pi) \) and correspond to the partial derivatives \( \partial|_\sigma|u^j/\partial x^\sigma \), where \( \sigma \) is a symmetric multi-index whose entries are the integers \( 1, \ldots, n \). We always assume that the system is presented in the passive orthonomic form, see [13], which allows to choose internal coordinates on \( \mathcal{E} \).

When we say that an object is restricted from \( J^\infty(\pi) \) to \( \mathcal{E} \), we mean that it is rewritten in terms of the internal coordinates.

The key role in the geometry of PDEs is played by the total derivative operators

\[
D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{i,\sigma} u^j_\sigma \frac{\partial}{\partial u^j_\sigma}.
\]

These operators can be restricted to any infinitely prolonged equation. Consequently, any differential operator in total derivatives (a \( \mathcal{C} \)-differential operator) is restrictable to \( \mathcal{E} \) as well. We preserve the same notation for the restrictions if no contradiction arises. We say that \( \mathcal{E} \) is differentially connected if \( D_{x^i}(f) = 0, i = 1, \ldots, n \) implies \( f = \text{const} \). The distribution spanned by the total derivatives is called the Cartan distribution and denoted by \( \mathcal{C} \).

A vector field

\[
S = \sum_{\sigma} s^j_\sigma \frac{\partial}{\partial u^j_\sigma}
\]

on \( \mathcal{E} \) is a symmetry of \( \mathcal{E} \) if \( [S, D_{x^i}] = 0 \) for all \( i \). The notation \( \sum_{I} \) means that the sum is taken over the set \( I \) of all internal coordinates \( u^j_\sigma \). Symmetries form a Lie algebra denoted by \( \text{sym}(\mathcal{E}) \). To describe symmetries, consider the following construction. Let \( G = (G^1, \ldots, G^r) \) be a function on \( \mathcal{E} \). Define its linearisation as the matrix \( \mathcal{C} \)-differential operator

\[
\ell_G = \left( \sum_{\sigma} \frac{\partial G^\alpha_\sigma}{\partial u^\beta_\sigma} D_\sigma \right)_{\alpha=1,\ldots,r, \beta=1,\ldots,m},
\]

where \( D_\sigma \) denotes the composition of \( D_{x^i} \) corresponding to the multi-index \( \sigma \). We also use the notation \( \ell_\mathcal{E} \) for \( \ell_F|_\mathcal{E} \). Then the following result is valid: any symmetry is an
evolutionary vector field

\[ E_\varphi = \sum_1 D_\varphi(\varphi_i) \frac{\partial}{\partial u_i}, \]

where the generating section \( \varphi = (\varphi^1, \ldots, \varphi^m) \) satisfies the equation \( \ell_\varphi(\varphi) = 0 \). The commutator of symmetries induces the Jacobi bracket

\[ \{ \varphi, \varphi' \} = E_\varphi(\varphi') - E_{\varphi'}(\varphi). \]

We do not distinguish between symmetries and their generating sections below. A symmetry \( S \) is called classical if it is projectable to \( J^1(\pi) \). We say that \( S \) is a point symmetry if it is projectable to \( J^0(\pi) \).

A conservation law of \( \mathcal{E} \) is a horizontal \((n - 1)\)-form

\[ \omega = a_1 dx^2 \wedge dx^3 \wedge \cdots \wedge dx^n + a_2 dx^1 \wedge dx^3 \wedge \cdots \wedge dx^n + \cdots + a_n dx^2 \wedge dx^3 \wedge \cdots \wedge dx^{n-1} \]

closed with respect to the horizontal de Rham differential \( d_h = \sum_{i=1}^n dx^i \wedge D_{x^i} \), i.e., such that \( \sum_{i=1}^n (-1)^i D_{x^i}(a_i) = 0 \). A conservation law is trivial if \( \omega = d_h \rho \) for some \((n-2)\)-form \( \rho \).

The quotient group of all conservation laws modulo trivial ones is denoted by \( \text{Cl}(\mathcal{E}) \).

To compute conservation laws, their generating sections are used. Let \( \omega \) be a conservation law and \( \tilde{\omega} \) be its extension to the ambient space \( J^\infty(\pi) \supset \mathcal{E} \). Then \( d_h(\tilde{\omega}) = \Delta(F) \) for some \( \mathcal{E} \)-differential operator \( \Delta \) and the vector-function \( \psi = (\psi^1, \ldots, \psi^n) = \Delta^*(\tilde{\omega}) \), where \( \Delta^* \) denotes the adjoint operator, is the generating section of \( \omega \). It possesses two important properties: (a) \( \psi = 0 \) if and only if \( \omega \) is trivial, (b) \( \ell_{\tilde{\omega}}(\psi) = 0 \). Any solution of the last equation is called a cosymmetry. The space of cosymmetries is denoted by \( \text{cosym}(\mathcal{E}) \).

Let \( \tilde{\mathcal{E}} \), \( \mathcal{E} \) be equations. We say that a smooth map \( \tau: \tilde{\mathcal{E}} \to \mathcal{E} \) is a morphism if for any point \( \tilde{\theta} \in \tilde{\mathcal{E}} \) one has \( \tau_*|_{\tilde{\mathcal{E}}} \subset \mathcal{E}_{\tau(\tilde{\theta})} \). A morphism is a (differential) covering if \( \tau_*|_{\tilde{\mathcal{E}}} \) is a isomorphism for any \( \tilde{\theta} \in \tilde{\mathcal{E}} \). Two coverings \( \tau_1, \tau_2 \) over \( \mathcal{E} \) are equivalent if there exists an isomorphism \( f: \tilde{\mathcal{E}}_1 \to \tilde{\mathcal{E}}_2 \) such that \( \tau_2 \circ f = \tau_1 \). Assume that \( \mathcal{E} \) is differentially connected. Then we say that \( \tau \) is irreducible if \( \tilde{\mathcal{E}} \) is differentially connected as well.

Take coverings \( \tau_1 \) and \( \tau_2 \) and consider the Whitney product \( \tau_1 \times \tau_2 \) of the corresponding bundles. It carries a natural structure of a covering, which is called the Whitney product of these coverings.

Let \( \tau: \tilde{\mathcal{E}} = \mathcal{E} \times \mathbb{R}^s \) be the trivial bundle. Define the plane \( \tilde{\mathcal{E}}_{\tilde{\theta}} \) at a point \( \tilde{\theta} \in \tilde{\mathcal{E}} \) as the parallel lift of \( \mathcal{E}_{\theta} \) for \( \theta = \tau(\tilde{\theta}) \). This is a covering, and any covering is said to be trivial if it is equivalent to \( \tau \).

A one-dimensional covering \( \tau \) is called Abelian if it is either trivial or there exists a nontrivial conservation law \( \omega \) of \( \mathcal{E} \) such that its lift \( \tau^*(\omega) \) becomes trivial on \( \tilde{\mathcal{E}} \). In general, a covering is Abelian if it is equivalent to the Whitney product of the necessary number of one-dimensional Abelian coverings.

**Proposition 1** (see [9]). A finite-dimensional Abelian covering over \( \mathcal{E} \) is irreducible if and only if the corresponding system of conservation laws is linearly independent modulo trivial ones. Consequently, equivalence classes of irreducible \( s \)-dimensional, \( s < \infty \), Abelian coverings are in one-to-one correspondence with \( s \)-dimensional subspaces in \( \text{Cl}(\mathcal{E}) \).

We say that a symmetry of the covering equation \( \tilde{\mathcal{E}} \) is a nonlocal symmetry of \( \mathcal{E} \). Denote by \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \) the algebras of smooth functions on \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \). Due to \( \tau \), one has the embedding \( \mathcal{F} \subset \tilde{\mathcal{F}} \). A derivation \( \tilde{\mathcal{F}} \to \tilde{\mathcal{F}} \) that preserves the Cartan distributions is
called a nonlocal shadow. In particular, for any nonlocal symmetry \( \tilde{S} \) its restriction \( \tilde{S}|_\mathcal{F} \) is a shadow; \( \tilde{S} \) is said to be \textit{invisible} if its shadow vanishes. Local symmetries of \( \mathcal{E} \) can be treated as shadows in every covering. A shadow is called reconstructible if there exists a nonlocal symmetry such that its shadow is the given one.

We also say that a conservation law of \( \tilde{\mathcal{E}} \) is a nonlocal conservation law of \( \mathcal{E} \).

Let us pass to local coordinates. Since the Gibbons–Tsarev equation is two-dimensional, we shall confine ourselves to this case for simplicity. Consider the equation \( \mathcal{E} \) given by

\[
F(x, y, u, u_x, u_y, \ldots) = 0
\]

and let \( \tau: \tilde{\mathcal{E}} = \mathcal{E} \times \mathbb{R}^s \rightarrow \mathcal{E} \) be a covering (the case \( s = \infty \) is allowed). Let \( \{w^\alpha\} \) be coordinates in the fiber (they are called nonlocal variables). Then the total derivatives on \( \tilde{\mathcal{E}} \) are of the form

\[
\tilde{D}_x = D_x + \sum_\alpha X_\alpha \frac{\partial}{\partial w^\alpha}, \quad \tilde{D}_y = D_y + \sum_\alpha Y_\alpha \frac{\partial}{\partial w^\alpha},
\]

\( X_\alpha, Y_\alpha \) being smooth functions in all the internal variables and \( w^\alpha \). Then \( \tau \) is a covering if and only if

\[
[\tilde{D}_x, \tilde{D}_y] = 0,
\]

or, equivalently,

\[
D_x(Y_\alpha) - D_y(X_\alpha) + \sum_\beta \left( X_\beta \frac{\partial Y_\alpha}{\partial w^\beta} - Y_\beta \frac{\partial X_\alpha}{\partial w^\beta} \right) = 0, \quad \alpha = 1, \ldots, s.
\]

Equivalently, the system

\[
w^\alpha_x = X_\alpha, \quad w^\alpha_y = Y_\alpha,
\]

is compatible modulo \( \mathcal{E} \). If the functions \( X_\alpha, Y_\alpha \) do not depend on the nonlocal variables, then the covering is Abelian.

Any nonlocal symmetry in \( \tau \) is defined by its generating section \( \Phi = (\varphi, \ldots, \psi^\alpha, \ldots) \), where \( \varphi = (\varphi^1, \ldots, \varphi^m) \) and \( \psi^\alpha \) are functions on \( \tilde{\mathcal{E}} \) satisfying

\[
\tilde{D}_x(\psi^\alpha) = \tilde{\ell}_x(\varphi) + \sum_\beta \frac{\partial X_\alpha}{\partial w^\beta} \psi^\beta, \quad \tilde{D}_y(\psi^\alpha) = \tilde{\ell}_y(\varphi) + \sum_\beta \frac{\partial Y_\alpha}{\partial w^\beta} \psi^\beta,
\]

\[
\tilde{\ell}_E(\varphi) = 0,
\]

where the ‘tilde’ over a \( \mathcal{E} \)-differential operator denotes its natural lift from \( \mathcal{E} \) to \( \tilde{\mathcal{E}} \). Nonlocal shadows are given by functions \( \varphi \) that satisfy Equation (2), while invisible symmetries are sections \( \Phi \) with \( \varphi = 0 \) and \( \psi^\alpha \) satisfying

\[
\tilde{D}_x(\psi^\alpha) = \sum_\beta \frac{\partial X_\alpha}{\partial w^\beta} \psi^\beta, \quad \tilde{D}_y(\psi^\alpha) = \sum_\beta \frac{\partial Y_\alpha}{\partial w^\beta} \psi^\beta.
\]

Assume that the right-hand sides of (1) depend on a parameter \( \lambda \) (which is called the spectral parameter). A parameter is non-removable (essential) if the coverings \( \tau_\lambda \) are pairwise inequivalent (cf. [12]). Having a family of coverings with an essential parameter, one can expand the functions \( X_\alpha, Y_\alpha \) in formal series in \( \lambda \). If substitution of \( \psi^\alpha = \sum_{i \in \mathbb{Z}} \psi_i^\alpha \lambda^i \) to this expansion is well defined, one obtains an infinite-dimensional covering with the nonlocal variables \( \psi_i^\alpha \). In the case when this covering is Abelian we get an infinite family of conservation laws (perhaps, trivial or dependent). A classical example of this procedure is the construction of the infinite series of conservation laws for the Korteweg–de Vries equation, [16].
But even if the covering at hand does not depend on a parameter, there exists a standard way to insert such a parameter formally (the so-called reversion procedure, see [17]). Namely, assume for simplicity that $\tau$ is one-dimensional and is given by

$$w_x = X(x, y, w, u, u_x, u_y, \ldots), \quad w_y = Y(x, y, w, u, u_x, u_y, \ldots).$$

Then

$$v_x = -X(x, y, \lambda, u, u_x, u_y, \ldots)v_\lambda, \quad v_y = -Y(x, y, \lambda, u, u_x, u_y, \ldots)v_\lambda$$

is a covering as well, see [10] for the geometric interpretation. We use this construction below to construct infinite series of nonlocal conservation laws for the Gibbons–Tsarev equation.

2. LOCAL SYMMETRIES AND CONSERVATION LAWS

Consider the Gibbons–Tsarev equation [6] in the form

$$z_{yy} + z_x z_{xy} - z_y z_{xx} + 1 = 0$$

(obtained from [6, eq. (15)] by the exchange $x \leftrightarrow y$ and $z \leftrightarrow -z$). For a monomial $X = x^i y^j$, let us use the notation

$$z_X = \frac{\partial^i z}{\partial x^i \partial y^j}.$$

In particular, $z_X = z$ when $i = j = 0$. For internal coordinates on $E$ we choose $x, y, z_X$ such that $X = x^k$ or $X = x^k y, k \geq 0$, while

$$z_{yy}X = D_X(z_y z_{xx} - z_x z_{xy} - 1)$$

are functions of the internal coordinates for every monomial $X$.

If not stated otherwise, sums are taken over all internal coordinates. The total derivatives on $E$ are

$$D_x = \frac{\partial}{\partial x} + \sum z_{xX} \frac{\partial}{\partial z_X}, \quad D_y = \frac{\partial}{\partial y} + \sum z_{yX} \frac{\partial}{\partial z_X},$$

(summation over all internal coordinates $z_X$). It is straightforward to check that that [3] is a differentially connected equation.

2.1. Weights. The Gibbons–Tsarev equation becomes homogeneous if we assign the weights $|x| = 3, |y| = 2, |z| = 4$ (due to the scaling symmetry, see Subsection 2.2 below) and

$$|z_{x^k}| = |z| - k |x| = 4 - 3k,\quad |z_{x^ky}| = |z| - k |x| - |y| = 2 - 3k.$$  

To any monomial in $x, y, z_{x^k},$ and $z_{x^ky}$ we assign the weight that equals the sum of weights of its factors. The total derivatives preserve the space of polynomials and, as operators, have the weights $|D_x| = -3, |D_y| = -2.
2.2. Local symmetries. Let $\mathcal{L} = \mathbb{E}_Z$ be a symmetry of $\mathcal{E}$. Then the defining equation for the generating sections of symmetries is

$$\ell_\mathcal{E}(Z) \equiv D^2_y(Z) + z_x D_x D_y(Z) - z_y D^2_x(Z) + z_{xy} D_x(Z) - z_{xx} D_y(Z) = 0. \quad (4)$$

Solving (4) for functions $Z$ of small jet order, we found that Equation (3) possesses five local symmetries

\[
\begin{align*}
Z^{(-4)} &= 1, & \text{z-translation}, \\
Z^{(-3)} &= z_x, & \text{x-translation}, \\
Z^{(-2)} &= z_y, & \text{y-translation}, \\
Z^{(-1)} &= yz_x - 2x, & \text{generalized Galilean boost}, \\
Z^{(0)} &= 3xz_x + 2yzy - 4z, & \text{scaling}.
\end{align*}
\]

All these symmetries are point ones. In Section 5 it will be shown that this is the complete set of local symmetries. The vector field $\mathcal{L}^{(i)} = \mathbb{E}_{Z^{(i)}}$, as an operator, has the weight $|\mathcal{L}^{(i)}| = i$.

All commutators of the symmetries $\mathcal{L}^{(-4)}, \ldots, \mathcal{L}^{(0)}$ vanish except for

\[
\begin{align*}
[\mathcal{L}^{(0)}, \mathcal{L}^{(-4)}] &= 2\mathcal{L}^{(-4)}, & [\mathcal{L}^{(0)}, \mathcal{L}^{(-3)}] &= \frac{3}{2}\mathcal{L}^{(-3)}, \\
[\mathcal{L}^{(0)}, \mathcal{L}^{(-2)}] &= \mathcal{L}^{(-2)}, & [\mathcal{L}^{(0)}, \mathcal{L}^{(-1)}] &= \frac{1}{2}\mathcal{L}^{(-1)}, \\
[\mathcal{L}^{(-1)}, \mathcal{L}^{(-3)}] &= -2\mathcal{L}^{(-4)}, & [\mathcal{L}^{(-1)}, \mathcal{L}^{(-2)}] &= -\mathcal{L}^{(-3)}.
\end{align*}
\]

Remark 1. Note that changing the basis by $\mathcal{L}^{(0)} \mapsto -\frac{1}{2}\mathcal{L}^{(0)}$ we arrive to the commutator relations $[\mathcal{L}^{(i)}, \mathcal{L}^{(j)}] = (j - i)\mathcal{L}^{(i+j)}$, where formally $\mathcal{L}^{(\alpha)} = 0$ for $\alpha < -4$. In what follows, we use the latter choice of the basic symmetries.

It will be shown in Section 5 that this set of five symmetries can be extended to a hierarchy of nonlocal symmetries infinite in both positive and negative directions.

2.3. Cosymmetries. The defining equation for cosymmetries of (3) is

$$\ell_\mathcal{E}(\mathcal{R}) \equiv D^2_y(\mathcal{R}) + z_x D_x D_y(\mathcal{R}) - z_y D^2_x(\mathcal{R}) - 2z_{xy} D_x(\mathcal{R}) + 2z_{xx} D_y(\mathcal{R}) = 0.$$

Solutions of lower order include six local symmetries of the first order

\[
\begin{align*}
\mathcal{R}^{(0)} &= 1, \\
\mathcal{R}^{(1)} &= 2z_x, \\
\mathcal{R}^{(2)} &= 3z_x^2 + 2z_y + 3y, \\
\mathcal{R}^{(3)} &= 4z_x^3 + 6z_xz_y + 8yz_x + 2x, \\
\mathcal{R}^{(4)} &= 5z_x^4 + 12z_x^2z_y + 15yz_x^2 + 3z_y^2 + 6xz_x + 10yz_xz_y + z + \frac{15}{2}y^2, \\
\mathcal{R}^{(5)} &= 6z_x^5 + 20z_x^3z_y + 24yz_x^3 + 12z_x^2z_y^2 + 12x^2z_x^2 + 36yz_xz_y^2 + 4(z + 6y^2)z_x + 8xz_y + 12xy
\end{align*}
\]

(compare with [20], p. 156]) and a single one of the third order

$$\mathcal{R}^{(-5)} = z_{xxx}.$$
We have verified by direct computation that Equation (3) has no local generating section of order 2 and 4.

2.4. Conservation laws. All the above listed cosymmetries are the generating sections of conservation laws $\rho^{(i)} = P^{(i)} \, dx + Q^{(i)} \, dy$, where

\[
P^{(0)} = z_x^2 + z_y + y,
\]
\[
Q^{(0)} = z_x z_y,
\]
\[
P^{(1)} = z_x^3 + 2z_x z_y - x,
\]
\[
Q^{(1)} = z_x^2 z_y + z_y^2 - 2z,
\]
\[
P^{(2)} = z_x^3 + 3z_x^2 z_y + 3y z_x^2 + z_y^2 + 3y z_y - z,
\]
\[
Q^{(2)} = z_x^3 + 2z_x z_y^2 + 3y z_x z_y - 3xy,
\]
\[
P^{(3)} = z_x^4 + 4z_x^3 z_y + 4y z_x^3 + 3z_x z_y^2 + 2x z_x^2 + 8yz_x z_y - 2z z_y + 2x z_y - 4xy,
\]
\[
Q^{(3)} = z_x^4 + 3z_x^3 z_y + 4y z_x^3 z_y + z_y^3 + 2x z_x z_y + 4y z_y^2 - 2z z_y - 8yz - 3x^2,
\]
\[
P^{(4)} = z_x^5 + 5z_x^4 z_y + 5y z_x^4 + 6z_x^2 z_y^2 + 3z_x z_y^3 + 15y z_x z_y^2 + z_y^3 + (z + \frac{15}{2} y^2) z_x^2 + 6x z_x z_y^2 + 5y z_y^2 + (z + \frac{15}{2} y^2) z_y - 5yz - 4x^2,
\]
\[
Q^{(4)} = z_x^5 + 4z_x^4 z_y + 5y z_x^4 z_y + 3z_x z_y^3 + 3z_x z_y^2 + 10y z_x z_y^2 + (z + \frac{15}{2} y^2) z_x z_y + 3z_x^2 z_y + 3z_x z_y^2 - \frac{3}{2} x(4z + 5y^2),
\]
\[
P^{(5)} = z_x^6 + 6z_x^5 z_y + 6y z_x^5 + 10z_x^3 z_y^2 + 4x z_x^4 + 24y z_x^3 z_y + 4z_x z_y^3 + 4(\frac{1}{2} z + 3y^2) z_x^3 z_y + 12x z_x^2 z_y + 18yz_x z_y^2 + 4(z + \frac{15}{2} y^2) z_x z_y + 4x z_y^2 + 12xyz_y - 4xz,
\]
\[
Q^{(5)} = z_x^6 z_y + 5z_x^5 z_y + 6y z_x^5 z_y + 6z_x^3 z_y^3 + 4x z_x^4 z_y + 18yz_x z_y^2 + z_y^4 + 4(z + 3y^2) z_x^2 z_y + 8x z_x z_y^2 + 6yz_x z_y + 4(\frac{1}{2} z + 3y^2) z_y^2 - 2z^2 - 24y^2 z - 6x^2 y
\]

and

\[
P^{(-5)} = -\frac{1}{2} z_x z_y^2 - z_y z_x z_y,
\]
\[
Q^{(-5)} = -\frac{1}{2} z_y z_x^2 - \frac{1}{2} z_x z_y.
\]

Note that $|\rho^{(i)}| = i + 5$, $i = -5, 0, \ldots, 5$.

In the next section we construct an infinite series of nonlocal conservation laws for the Gibbons–Tsarev equation (3).

3. Coverings and the infinite series of nonlocal conservation laws

Using two known coverings [6, 7] of the Gibbons–Tsarev equation, we construct here an infinite series of (nonlocal) conservation laws that later (Section 4) will be used to construct the corresponding infinite-dimensional Abelian covering and describe the algebra of nonlocal symmetries in this covering. It will also be shown that the obtained infinite dimensional coverings are equivalent.
3.1. Coverings. Consider the nonlinear non-Abelian covering \( \tau_z : \tilde{E} \to E \) over Equation (3) given by
\[
\varphi_x = \frac{1}{z_y + z_x \varphi - \varphi^2} \quad \varphi_y = - \frac{z_x - \varphi}{z_y + z_x \varphi - \varphi^2}.
\] (6)
The covering introduced by Gibbons and Tsarev in \([7]\) can be rewritten in this way.

To simplify the subsequent computations, let us introduce new variables \( u \) and \( v \) such that
\[
z_x = u + v, \quad z_y = -uv.
\] (7)
Due to the compatibility condition
\[
(u + v)_y + (uv)_x = 0
\] (8)
and by Equation (3) we deduce that the new variables enjoy the system of evolution equations
\[
u_y + vu_x = \frac{1}{v - u}, \quad v_y + uv_x = \frac{1}{u - v}
\] (9)
Denote this equation by \( E_1 \). The equation is homogeneous with respect to the weights \(|x| = 3, |y| = 2, |u| = |v| = 1\). Due to \([5]\), the form \((u + v) \ dx - uv \ dy\) is a conservation law of the equation \( E_1 \) while \([7]\) defines the covering \( E \to E_1 \) associated with this conservation law.

The covering \( \tau_z \) defined by (6) generates the covering \( \tau_{uv} : \tilde{E}_1 \to E_1 \) given by the relations
\[
\varphi_x = - \frac{1}{(\varphi - u)(\varphi - v)}, \quad \varphi_y = \frac{u + v - \varphi}{(\varphi - u)(\varphi - v)}
\] (10)
and the diagram of coverings
\[
\begin{array}{ccc}
\tilde{E} & \xrightarrow{\tau_z} & \tilde{E}_1 \\
\tau_{uv} & & \tau_{uv}
\end{array}
\] is commutative.

3.2. Nonlocal conservation laws. We construct an infinite hierarchy of nonlocal conservation laws for the Gibbons-Tsarev equation using two different but related ways.

3.2.1. The first way. Consider an arbitrary gauge symmetry \( \varphi \mapsto \psi(\varphi) \) of the covering \( \tau_{uv} \). For the sake of convenience, relabel the variable \( \varphi \) to \( \lambda \). Then, applying the reversion procedure described in Section 1 to the covering \( \tau_{uv} \), one obtains
\[
\psi_x = \frac{1}{(\lambda - u)(\lambda - v)} \cdot \psi_\lambda, \quad \psi_y = \frac{\lambda - (u + v)}{(\lambda - u)(\lambda - v)} \cdot \psi_\lambda.
\] (11)
Now, we consider \( \lambda \) as a formal parameter and expand \( \psi \) in the Laurent series
\[
\psi = \psi(-1) \lambda + \psi(0) \frac{\psi}{\lambda} + \cdots + \psi(k) \frac{\psi}{\lambda^k} + \ldots
\] (12)
One also has the obvious expansions
\[
\frac{1}{\lambda - u} = \frac{1}{\lambda} \sum_{i \geq 0} \frac{u^i}{\lambda^i}, \quad \frac{1}{\lambda - v} = \frac{1}{\lambda} \sum_{i \geq 0} \frac{v^i}{\lambda^i}
\]
which imply

\[
\frac{1}{(\lambda - u)(\lambda - v)} = \frac{1}{\lambda^2} \left( 1 + \frac{\sigma_1}{\lambda} + \cdots + \frac{\sigma_k}{\lambda^k} + \cdots \right),
\]

where

\[
\sigma_k = \sum_{i+j=k} u^i v^j.
\]

Remark 2. Note that since the quantities \(\sigma_k\) are symmetric in the variables \(u\) and \(v\), they can be rewritten as polynomials in \(z_x = u + v\) and \(z_y = -uv\). See formula (24) below.

Now, from the expansion (12) one obtains

\[
\begin{align*}
\psi_x &= \psi_x^{(-1)} \lambda + \psi_x^{(0)} + \frac{\psi_x^{(1)}}{\lambda} + \cdots + \frac{\psi_x^{(k)}}{\lambda^k} + \cdots, \\
\psi_y &= \psi_y^{(-1)} \lambda + \psi_y^{(0)} + \frac{\psi_y^{(1)}}{\lambda} + \cdots + \frac{\psi_y^{(k)}}{\lambda^k} + \cdots,
\end{align*}
\]

and

\[
\psi_\lambda = \psi^{(-1)} - \frac{\psi^{(1)}}{\lambda} - 2 \frac{\psi^{(2)}}{\lambda^2} - \cdots - k \frac{\psi^{(k)}}{\lambda^{k+1}} + \cdots
\]

Substituting all the above expansions to Equations (11), one obtains

\[
\begin{align*}
\psi_x^{(-1)} \lambda + \psi_x^{(0)} + \frac{\psi_x^{(1)}}{\lambda} + \cdots + \frac{\psi_x^{(k)}}{\lambda^k} + \cdots &= \frac{1}{\lambda^2} \left( 1 + \frac{\sigma_1}{\lambda} + \cdots + \frac{\sigma_k}{\lambda^k} + \cdots \right) \left( \psi^{(-1)} - \frac{\psi^{(1)}}{\lambda^2} - 2 \frac{\psi^{(2)}}{\lambda^3} - \cdots - k \frac{\psi^{(k)}}{\lambda^{k+1}} + \cdots \right), \\
\psi_y^{(-1)} \lambda + \psi_y^{(0)} + \frac{\psi_y^{(1)}}{\lambda} + \cdots + \frac{\psi_y^{(k)}}{\lambda^k} + \cdots &= \left( \frac{1}{\lambda} - \frac{\sigma_1}{\lambda^2} \right) \left( 1 + \frac{\sigma_1}{\lambda} + \cdots + \frac{\sigma_k}{\lambda^k} + \cdots \right) \left( \psi^{(-1)} - \frac{\psi^{(1)}}{\lambda^2} - 2 \frac{\psi^{(2)}}{\lambda^3} - \cdots - k \frac{\psi^{(k)}}{\lambda^{k+1}} + \cdots \right),
\end{align*}
\]

Denote by

\[
A_0 + \frac{A_1}{\lambda} + \cdots + \frac{A_k}{\lambda^k} + \cdots
\]

the result of multiplication of the last two factors in the previous expressions, i.e., \(A_0 = \psi^{(-1)}\), \(A_1 = \sigma_1 \psi^{(-1)}\), \(A_2 = \sigma_2 \psi^{(-1)} - \psi^{(1)}\), and

\[
A_k = \sigma_k \psi^{(-1)} - \sigma_{k-2} \psi^{(1)} - 2 \sigma_{k-3} \psi^{(2)} - \cdots - (k-2) \sigma_{k-2} \psi^{(k-2)} - (k-1) \psi^{(k-1)}, \quad k \geq 3.
\]

Consequently,

\[
\begin{align*}
\psi_x^{(-1)} \lambda + \psi_x^{(0)} + \frac{\psi_x^{(1)}}{\lambda} + \cdots + \frac{\psi_x^{(k)}}{\lambda^k} + \cdots &= \frac{1}{\lambda^2} \left( A_0 + \frac{A_1}{\lambda} + \cdots + \frac{A_k}{\lambda^k} + \cdots \right), \\
\psi_y^{(-1)} \lambda + \psi_y^{(0)} + \frac{\psi_y^{(1)}}{\lambda} + \cdots + \frac{\psi_y^{(k)}}{\lambda^k} + \cdots &= \left( \frac{1}{\lambda} - \frac{\sigma_1}{\lambda^2} \right) \left( A_0 + \frac{A_1}{\lambda} + \cdots + \frac{A_k}{\lambda^k} + \cdots \right)
\end{align*}
\]

and thus

\[
\psi_x^{(-1)} = 0, \quad \psi_x^{(0)} = 0, \quad \psi_x^{(1)} = 0, \quad \psi_x^{(-1)} = 0, \quad \psi_y^{(0)} = 0, \quad \psi_y^{(1)} = A_1
\]
A_{k-2}, \quad \psi_y^{(k)} = A_{k-1} - \sigma_1 A_{k-2} \tag{14}
for k \geq 2. Without loss of generality we can set \(\psi^{(-1)} = 1\) and skip the variable \(\psi^{(0)}\), since the coefficients \(A_k\) are independent of it. Then, using the obtained expressions for \(A_0\) and \(A_1\), we obtain \(\psi_x^{(1)} = 0\), \(\psi_y^{(1)} = 1\), \(\psi_x^{(2)} = 1\), \(\psi_y^{(2)} = 0\) and set
\[
\psi^{(1)} = y, \quad \psi^{(2)} = x, \tag{15}
\]
without loss of generality as well. Thus, we have
\[
A_0 = 1, \quad A_1 = \sigma_1, \quad A_2 = \sigma_2 - y, \quad A_3 = \sigma_3 - \sigma_1 y - 2x
\]
and
\[
A_k = \sigma_k - \sigma_{k-2} y - 2\sigma_{k-3} x - 3\sigma_{k-4} \psi_3^{(3)} - \cdots - (k-2)\sigma_1 \psi_1^{(k-2)} - (k-1)\psi_1^{(k-1)}
\]
for \(k \geq 3\).

Then, using the obvious identities \(\sigma_1 \sigma_k - \sigma_{k+1} = u v \sigma_{k-1}\), we obtain from \((14)\)
\[
\begin{align*}
\psi_x^{(3)} &= \sigma_1, & \psi_y^{(3)} &= -u v - y; \\
\psi_x^{(4)} &= \sigma_2 - y, & \psi_y^{(4)} &= -u v \sigma_1 - 2 x; \\
\psi_x^{(5)} &= \sigma_3 - \sigma_1 y - 2 x, & \psi_y^{(5)} &= -u v (\sigma_2 - y) - 3 \psi_3^{(3)}; \\
\psi_x^{(6)} &= \sigma_4 - \sigma_2 y - 2 \sigma_1 x - 3 \psi_3^{(3)}, & \psi_y^{(6)} &= -u v (\sigma_3 - \sigma_1 y - 2 x) - 4 \psi_4^{(4)}; \\
\psi_x^{(7)} &= \sigma_5 - \sigma_3 y - 2 \sigma_2 x - 3 \sigma_1 \psi_3^{(3)} - 4 \psi_4^{(4)}, & \psi_y^{(7)} &= -u v (\sigma_4 - \sigma_2 y - 2 \sigma_1 x - 3 \psi_3^{(3)}) - 5 \psi_5^{(5)}
\end{align*}
\]
and
\[
\begin{align*}
\psi_x^{(k)} &= \sigma_{k-2} - \sigma_{k-4} y - 2 \sigma_{k-3} x - \sum_{i=3}^{k-3} i \sigma_{k-i-3} \psi^{(i)}; \\
\psi_y^{(k)} &= -u v (\sigma_{k-3} - \sigma_{k-5} y - 2 \sigma_{k-6} x - \sum_{i=3}^{k-4} i \sigma_{k-i-4} \psi^{(i)}) - (k-2) \psi_1^{(k-2)}.
\end{align*}
\]
for \(k \geq 7\). Denote by \(X^{(k)}\) and \(Y^{(k)}\) the right-hand sides of the obtained equations, i.e.,
\[
\psi_x^{(k)} = X^{(k)}, \quad \psi_y^{(k)} = Y^{(k)}, \quad k \geq 3. \tag{18}
\]
Obviously, we have \(|X^{(k)}| = k - 2\), \(|Y^{(k)}| = k - 1\), \(|\psi^{(k)}| = k + 1\).

Let us now return back to the equation \(E_1\) given by \((9)\) and consider the spaces
\[
E_2 = E_1 \times R^{(3)}, \ldots, E_k = E_{k-1} \times R^{(k+1)}, \ldots,
\]
where \(R^{(k)}\) is \(\mathbb{R}^1\) with the distinguished coordinate \(\psi^{(k)}\), \(k \geq 3\). Consider also the natural projections
\[
\tau_{k,k-1}: E_k \rightarrow E_{k-1}, \quad \tau_k: E_k \rightarrow E_1.
\]
Let \(E_*\) be the inverse limit of the infinite sequence
\[
E_1 \leftarrow \ldots \leftarrow E_{k-1} \leftarrow E_k \leftarrow \ldots
\]
Proposition 3. Let \( D^{(k)}_x = D_x + \sum_{i=3}^{k+1} X^{(i)} \frac{\partial}{\partial \psi_i}, \quad D^{(k)}_y = D_y + \sum_{i=3}^{k+1} Y^{(i)} \frac{\partial}{\partial \psi_i}, \)
where \( D_x \) and \( D_y \) are the total derivatives on \( \mathcal{E}_1 \). Similarly, we define the fields \( D^{(s)}_x \) and \( D^{(s)}_y \) on \( \mathcal{E}_s \).

Proposition 2. For all \( k \), including the case \( k = * \), one has \([D^{(k)}_x, D^{(k)}_y] = 0\).

Proof. This is an immediate consequence of the fact that (10) is a covering over \( \mathcal{E}_1 \). \( \square \)

Hence, all the maps \( \tau_k \) carry covering structures; these coverings are irreducible:

Proposition 3. Let \( f \in \mathcal{F}(\mathcal{E}_k) \) be a function such that \( D^{(k)}_x(f) = D^{(k)}_y(f) = 0 \). Then \( f = \text{const.} \)

Proof. Let \( x, y, \ldots, u_i = \frac{\partial^i u}{\partial x^i}, \quad v_i = \frac{\partial^i v}{\partial x^i}, \ldots \)
be coordinates on \( \mathcal{E}_1 \) and
\[
D_x = \frac{\partial}{\partial x} + \sum_{i \geq 0} \left( u_{i+1} \frac{\partial}{\partial u_i} + v_{i+1} \frac{\partial}{\partial v_i} \right),
\]
\[
D_y = \frac{\partial}{\partial y} + \sum_{i \geq 0} (D^i_x \left( \frac{1}{v - u} + v u_1 \right) \frac{\partial}{\partial u_i} + D^i_x \left( \frac{1}{u - v} + u v_1 \right) \frac{\partial}{\partial v_i})
\]
be the total derivatives in these coordinates. Consider a function
\[
f = f(x, y, u, v, \ldots, u_i, v_j, \psi^{(3)}, \ldots, \psi^{(k)})
\]
on \( \mathcal{E}_k \) and assume that
\[
D_x(f) + X^{(3)} \frac{\partial f}{\partial \psi^{(3)}} + \cdots + X^{(k)} \frac{\partial f}{\partial \psi^{(k)}} = D_y(f) + Y^{(3)} \frac{\partial f}{\partial \psi^{(3)}} + \cdots + Y^{(k)} \frac{\partial f}{\partial \psi^{(k)}} = 0. \tag{19}
\]
Since the coefficients \( X^{(3)}, Y^{(3)}, \ldots, X^{(k)}, Y^{(k)} \) are independent of the variables \( u_\alpha, v_\beta \) for all \( \alpha \) and \( \beta > 0 \), from the above formulas for \( D_x \) and \( D_y \) it follows that \( f \) cannot depend on these variables either as well as on \( u \) and \( v \) and thus Equation (19) reads now
\[
\frac{\partial f}{\partial x} + X^{(3)} \frac{\partial f}{\partial \psi^{(3)}} + \cdots + X^{(k)} \frac{\partial f}{\partial \psi^{(k)}} = \frac{\partial f}{\partial y} + Y^{(3)} \frac{\partial f}{\partial \psi^{(3)}} + \cdots + Y^{(k)} \frac{\partial f}{\partial \psi^{(k)}} = 0.
\]
But \( X^{(\alpha)} \) and \( Y^{(\beta)} \) are polynomials in \( u \) and \( v \) of degrees \( \alpha - 2 \) and \( \beta - 1 \), respectively, and this finishes the proof. \( \square \)

Obviously, every map \( \tau_{k,k-1} : \mathcal{E}_k \to \mathcal{E}_{k-1} \) is also a covering; moreover, it is an Abelian covering associated to the conservation law
\[
\omega^{(k)} = X^{(k)} \, dx + Y^{(k)} \, dy \in \text{Cl}(\mathcal{E}_{k-1})
\]
and \( |\omega^{(k)}| = k + 1 \).

Proposition 4. The conservation law \( \omega^{(k)} \) is nontrivial on \( \mathcal{E}_{k-1} \).
Proof. This readily follows from general properties of coverings (see Section 1) and Propositions.

Remark 3. By the very construction, the equation $\mathcal{E}_2$ is equivalent to the Gibbons–Tsarev equation (3). Moreover, it can be checked that the conservation laws $\omega^{(4)}, \ldots, \omega^{(9)}$ are equivalent to the conservation laws $\rho^{(0)}, \ldots, \rho^{(5)}$, respectively, described in Subsection 2.4.

Remark 4. Of course, the initial choice (15) for the values of $\psi^{(-1)}$, $\psi^{(1)}$, and $\psi^{(2)}$ is not unique. Nevertheless, one can easily show that other admissible values lead to equivalent results.

3.2.2. The second method. Consider now the covering (10) and assume that

$$\varphi = \frac{\varphi^{(-1)}}{\lambda} + \varphi^{(0)} + \varphi^{(1)} \lambda + \cdots + \varphi^{(k)} \lambda^k + \cdots$$

Then, rewriting (11) in the form

$$(\varphi - u)(\varphi - v)\varphi_x = -1, \quad (\varphi - u)(\varphi - v)\varphi_y = u + v - \varphi$$

and substituting expansion (20), one obtains the following defining system for the coefficients $\varphi^{(i)}$:

$$B_{-2} \varphi^{(-1)} = 0,$$
$$B_{-2} \varphi^{(0)} + B_{-1} \varphi^{(-1)} = 0,$$
$$B_{-2} \varphi^{(1)} + B_{-1} \varphi^{(0)} + B_0 \varphi^{(-1)} = 0,$$
$$B_{-2} \varphi^{(2)} + B_{-1} \varphi^{(1)} + B_0 \varphi^{(0)} + B_1 \varphi^{(-1)} = -1,$$
$$B_{-2} \varphi^{(3)} + B_{-1} \varphi^{(2)} + B_0 \varphi^{(1)} + B_1 \varphi^{(0)} + B_2 \varphi^{(-1)} = 0,$$
$$\vdots$$
$$B_{-2} \varphi^{(k+2)} + B_{-1} \varphi^{(k)} + \cdots + B_{k+1} \varphi^{(-1)} = 0,$$

where

$$(\varphi - u)(\varphi - v) = \frac{B_{-2}}{\lambda^2} + \frac{B_{-1}}{\lambda} + B_0 + B_1 \lambda + \cdots + B_k \lambda^k + \cdots$$

is the expansion of the product $(\varphi - u)(\varphi - v)$, i.e.,

$$B_{-2} = (\varphi^{(-1)})^2,$$
$$B_{-1} = \varphi^{(-1)} (2\varphi^{(0)} - u - v),$$
$$B_0 = 2\varphi^{(-1)} \varphi^{(1)} + (\varphi^{(0)} - u) (\varphi^{(0)} - v),$$
$$B_1 = 2\varphi^{(-1)} \varphi^{(2)} + (2\varphi^{(0)} - u - v) \varphi^{(1)},$$
$$B_2 = 2\varphi^{(-1)} \varphi^{(3)} + (2\varphi^{(0)} - u - v) \varphi^{(2)} + (\varphi^{(1)})^2.$$
Then choice of coefficients is possible:

\[ B_3 = 2\varphi(-1)\varphi^{(4)} + (2\varphi(0) - u - v)\varphi^{(3)} + 2\varphi(1)\varphi^{(2)}, \]

\[ \ldots \]

\[ B_{2k} = 2\varphi(-1)\varphi^{(2k+1)} + (2\varphi(0) - u - v)\varphi^{(2k)} + 2\varphi(1)\varphi^{(2k-1)} + \ldots + 2\varphi(k-1)\varphi^{(k+1)} + (\varphi^{(k)})^2, \]

\[ B_{2k+1} = 2\varphi(-1)\varphi^{(2k+2)} + (2\varphi(0) - u - v)\varphi^{(2k+1)} + 2\varphi(1)\varphi^{(2k)} + \ldots + 2\varphi(k)\varphi^{(k+1)}, \]

\[ \ldots \]

Hence, the initial defining system transforms to

\[ \varphi^{(-1)} = 1, \quad \varphi^{(0)} = 0, \quad \varphi^{(1)} = -y, \quad \varphi^{(2)} = -x. \]  \hspace{1cm} (21)

Then \( B_{-2} = 1 \), while

\[ B_{-1} = -(u + v), \]
\[ B_0 = -2y + uv, \]
\[ B_1 = -2x + y(u + v), \]
\[ B_2 = 2\varphi^{(3)} + x(u + v) + y^2, \]
\[ B_3 = 2\varphi^{(4)} - (u + v)\varphi^{(3)} + 2xy, \]
\[ B_4 = 2\varphi^{(5)} - (u + v)\varphi^{(4)} - 2y\varphi^{(3)} + 2x^2, \]
\[ B_5 = 2\varphi^{(6)} - (u + v)\varphi^{(5)} - 2y\varphi^{(4)} - 2x\varphi^{(3)}, \]
\[ B_6 = 2\varphi^{(7)} - (u + v)\varphi^{(6)} - 2y\varphi^{(5)} - 2x\varphi^{(4)} + (\varphi^{(3)})^2, \]
\[ B_7 = 2\varphi^{(8)} - (u + v)\varphi^{(7)} - 2y\varphi^{(6)} - 2x\varphi^{(5)} + 2\varphi^{(3)}\varphi^{(4)}, \]

\[ \ldots \]

\[ B_{2k} = 2\varphi^{(2k+1)} - (u + v)\varphi^{(2k)} - 2y\varphi^{(2k-1)} - 2x\varphi^{(2k-2)} + 2\varphi^{(3)}\varphi^{(2k-3)} + \ldots + 2\varphi^{(k-1)}\varphi^{(k+1)} + (\varphi^{(k)})^2, \]

\[ B_{2k+1} = 2\varphi^{(2k+2)} - (u + v)\varphi^{(2k+1)} - 2y\varphi^{(2k)} - 2x\varphi^{(2k-1)} + 2\varphi^{(3)}\varphi^{(2k-2)} + \ldots + 2\varphi^{(k)}\varphi^{(k+1)}, \]

\[ \ldots \]

Hence, the initial defining system transforms to

\[ \varphi^{(3)}_x = -(u + v), \quad \varphi^{(3)}_y = u - y, \]
\[ \varphi^{(4)}_x = -2y - u^2 - uv - v^2, \quad \varphi^{(4)}_y = -x + uv(u + v), \]

while for \( k > 4 \) we have the recurrent relations

\[ \varphi^{(k)}_x = B_{k-1} - B_{k-5}\varphi^{(3)}_x - \ldots - B_{-1}\varphi^{(k-1)}_x, \]
\[ \varphi^{(k)}_y = B_{k-3} - B_{k-5}\varphi^{(3)}_y - \ldots - B_{-1}\varphi^{(k-1)}_y - \varphi^{(k-2)}. \]  \hspace{1cm} (22)

Denote by \( \bar{X}^{(k)} \) and \( \bar{Y}^{(k)} \) the right-hand sides of equations \( [22] \), i.e.,

\[ \varphi^{(k)}_x = \bar{X}^{(k)}, \quad \varphi^{(k)}_y = \bar{Y}^{(k)}, \quad k \geq 3. \]  \hspace{1cm} (23)

We have \( |\varphi^{(k)}| = k + 1. \)
Now, exactly as in Subsection 3.2.1 we introduce the spaces $\tilde{E}_k = \tilde{E}_{k-1} \times \tilde{R}^{(k+1)}$, $k = 2, \ldots$, where $\tilde{R}^{(k)} = \mathbb{R}^1$ with the coordinate $\varphi^{(k)}$, the projections

$$\tilde{r}_{k,k-1} : \tilde{E}_k \to \tilde{E}_{k-1}, \quad \tilde{r}_k : \tilde{E}_{k} \to \tilde{E}_{1}$$

and $\tilde{r}_z : \tilde{E}_z \to \tilde{E}_1$ as the inverse limit. We endow these spaces with the vector fields

$$\tilde{D}_{x}^{(k)} = D_x + \sum_{i=3}^{k+1} \tilde{X}^{(i)} \frac{\partial}{\partial \varphi^{(i)}}, \quad \tilde{D}_{y}^{(k)} = D_y + \sum_{i=3}^{k+1} \tilde{Y}^{(i)} \frac{\partial}{\partial \varphi^{(i)}}.$$  

Similarly, we define $\tilde{D}_{x}^{(*)}$ and $\tilde{D}_{y}^{(*)}$.

**Proposition 5.** For all $k \geq 2$ and $k = *$ one has $[\tilde{D}_{x}^{(k)}, \tilde{D}_{y}^{(k)}] = 0$, i.e., all the maps $\tilde{r}_k$ and $\tilde{r}_{k,k-1}$ are coverings. All these coverings are irreducible.

Consider the forms

$$\tilde{\omega}^{(k)} = \tilde{X}^{(k)} \, dx + \tilde{Y}^{(k)} \, dy.$$  

One has $|\tilde{\omega}^{(k)}| = k + 1$ and

**Proposition 6.** For every $k \geq 3$, the form $\tilde{\omega}^{(k)}$ is a nontrivial conservation law of the equation $\tilde{E}_{k-1}$.

**Remark 5.** As before, the choice (21) of initial values for $\varphi^{(-1)}, \ldots, \varphi^{(2)}$ is not unique, but all admissible choices lead to equivalent results.

Finally, the following statement is valid:

**Proposition 7.** The pairs of coverings $\tau_{k,k-1}$ and $\tilde{r}_{k,k-1}$, $\tau_k$ and $\tilde{r}_k$, $\tau_*$ and $\tilde{r}_*$ are equivalent.

We provide the proof in the next subsection.

3.3. **Proof of Proposition 7** Let us turn back to the Gibbons–Tsarev equation (3). For reader’s convenience, we summarise the results of the previous section in terms of the variables $x, y, z$. We recall that

$$\psi^{(0)} = 0, \quad \psi^{(1)} = y, \quad \psi^{(2)} = x, \quad \psi^{(3)} = z - \frac{1}{2}y^2,$$

while $\psi^{(k)}$, $k > 3$, are genuine nonlocal variables of the Gibbons–Tsarev equation, satisfying

$$\psi_x^{(k)} = \sigma_{k-2} - \sum_{i=1}^{k-3} i \sigma_{k-i-3} \psi^{(i)}, \quad \psi_y^{(k)} = z_y \psi_x^{(k-1)} - (k - 2) \psi^{(k-2)}.$$  

In terms of $z$, we have

$$\sigma_k = \sum_{0 \leq j \leq k-j} \binom{k-j}{j} z_x^{k-2j} z_y^j, \quad k > 0. \tag{24}$$

To prove formula (24), we consider the formal power series in an auxiliary variable $\lambda$ with coefficients taken from the two sides of formula (24) and show that they coincide. Using the left-hand side, we have, according to formula (13),

$$\sum_{k \geq 0} \sigma_k \lambda^k = \sum_{i,j \geq 0} u^i v^j \lambda^{i+j} = \sum_{i,j \geq 0} (u \lambda)^i (v \lambda)^j = \frac{1}{1 - u \lambda} \cdot \frac{1}{1 - v \lambda}.$$
Using the right-hand side, where we substitute for $z_x, z_y$ from formulas (7), we obtain the same series:

$$
\sum_{k \geq 0} \sum_{0 \leq j \leq k-j} \binom{k-j}{j} z_x^{k-2j} z_y^j \lambda^k = \sum_{i \geq 0} \sum_{0 \leq j \leq i} \binom{i-j}{j} z_x^{i-j} z_y^j \lambda^{i+j} \\
= \sum_{i \geq 0} \sum_{0 \leq j \leq i} \binom{i-j}{j} (z_x \lambda)^{i-j} (z_y \lambda^2)^j = \sum_{i \geq 0} (z_x \lambda + z_y \lambda^2)^i = \sum_{i \geq 0} ((u + v) \lambda - uv \lambda^2)^i \\
= \sum_{i \geq 0} (1 - (1 - u \lambda)(1 - v \lambda))^i = \frac{1}{(1 - u \lambda)(1 - v \lambda)}.
$$

Thus, formula (24) is proved.

The first method of the previous section uses the expansion (12), i.e.,

$$
\psi(\lambda) = \lambda + \frac{\psi^{(1)}}{\lambda} + \cdots + \frac{\psi^{(k)}}{\lambda^k} + \cdots, \quad (25)
$$

where $\psi$ satisfies the linear system (11), which we rewrite in terms of $z_x, z_y$:

$$
\psi_x = \frac{1}{\lambda^2 - z_x \lambda - z_y} \cdot \psi', \quad \psi_y = \frac{\lambda - z_x}{\lambda^2 - z_x \lambda - z_y} \cdot \psi', \quad (26)
$$

where the ‘prime’ denotes the $\lambda$-derivative. The second method uses the expansion (20), i.e.,

$$
\varphi(\lambda) = \frac{1}{\lambda} + \varphi^{(1)} \lambda + \cdots + \varphi^{(k)} \lambda^k + \cdots, \quad (27)
$$

where $\varphi$ satisfies the nonlinear system (6), i.e.,

$$
\varphi_x = -\frac{1}{\varphi^2 - z_x \varphi - z_y}, \quad \varphi_y = -\frac{\varphi - z_x}{\varphi^2 - z_x \varphi - z_y}. \quad (28)
$$

Recall that composition $b \circ a$ of formal series $b(\mu) = \sum_{j \geq s} b_j \mu^j$ and $a(\lambda) = \sum_{i \geq r} a_i \lambda^i$, i.e.,

$$
b(a(\lambda)) = \sum_{j \geq s} b_j \left( \sum_{i \geq r} a_i \lambda^i \right)^j \\
= \sum_{j \geq s} b_j \left( \sum_{i \geq r} a_i \lambda^i \right) \cdots \left( \sum_{i \geq r} a_i \lambda^i \right) \\
= \sum_{i_{j} \geq r} \sum_{i \geq r} a_{i_{1}} \cdots a_{i_{j}} \lambda^{i_{1}+\cdots+i_{j}}
$$

is a formal series if and only if the coefficients at powers of $\lambda$ are finite sums. This is certainly the case when $\sum_{j \geq s} b_j \mu^j$ is a polynomial or when $r \geq 1$, i.e., when $\sum_{i \geq r} a_i \lambda^i$ is a power series without the constant term.

Computing

$$
\frac{1}{\varphi(\lambda)} = \lambda - \varphi^{(1)} \lambda^{k+2} + \cdots
$$
we see that \(1/\phi(\lambda)\) is a power series without the constant term and therefore, the composition series \(\psi \circ \phi\), i.e.,
\[
\psi(\phi(\lambda)) = \frac{\psi^{(1)}}{\phi(\lambda)} + \frac{\psi^{(2)}}{\phi(\lambda)^2} + \cdots,
\]
is well defined.

**Proposition 8.** Let \(\psi(\lambda)\) and \(\phi(\lambda)\) be the formal expansions (25) and (27), respectively. Then each pair of the conditions

1. equation (26);
2. equation (28);
3. \(\psi(\phi) = c(\lambda)\), where \(c(\lambda)\) is a constant (possibly depending on \(\lambda\)),

implies the remaining condition.

**Proof.** Assume that (26) and (28) hold. Substituting \(\phi\) for \(\lambda\) in (26), an easy computation yields
\[
(\psi(\phi))_x = \psi_x(\phi) + \psi'(\phi)\phi_x = 0,
\]
\[
(\psi(\phi))_y = \psi_y(\phi) + \psi'(\phi)\phi_y = 0
\]
by virtue of (28). Then \(\psi(\phi)\) is a constant with respect to \(x\) and \(y\), since the covering (28) is differentially connected.

Conversely, assume that \(\psi(\phi) = c(\lambda)\), where \(c(\lambda)\) does not depend on \(x\) and \(y\). Then \(\psi(\phi)_x = \psi(\phi)_y = 0\) and
\[
\psi'(\phi)\phi_x, \quad \psi'(\phi)\phi_y,
\]
which yields the equivalence of Equations (26) and (28).

Under the substitution \(\lambda \to 1/\lambda\), the expansion (25) acquires the form
\[
\psi\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda} + \psi^{(1)}(1)\lambda + \cdots + \psi^{(k)}(k)\lambda^k + \cdots,
\]
i.e., \(\phi(\lambda)\) and \(1/\psi(1/\lambda)\) are Laurent series of the lowest degree \(-1\). Consequently, \(1/\phi(\lambda)\) and \(1/\psi(1/\lambda)\) are power series without a constant term. So, they are composable with each other.

There is a preferable choice of the constant \(c(\lambda)\) in Proposition 8.

**Proposition 9.** The expansions \(\phi, \psi\) can be chosen so that
\[
\psi(\phi(\lambda)) = 1/\lambda,
\]
i.e., the power series \(1/\psi(1/\lambda)\) and \(1/\phi(\lambda)\) are compositionally inverse one to another.

**Proof.** According to Proposition 8, we are free to choose \(c(\lambda) = 1/\lambda\), i.e., \(\psi(\phi(\lambda)) = 1/\lambda\). Substituting \(1/\phi(\lambda)\) for \(\lambda\) in \(1/\psi(1/\lambda)\), we obtain \(1/\psi(\phi(\lambda)) = 1/c(\lambda) = 1/\lambda\). Hence the statement.

With this choice of \(c(\lambda)\), the \(k\)-tuples of coefficients \(\psi^{(1)}, \ldots, \psi^{(k)}\) and \(\phi^{(1)}, \ldots, \phi^{(k)}\) determine each other uniquely, thereby providing the induction step in the proof of the
equalities $e_k = \bar{e}_k$. It is, however, necessary to check that the condition $c(\lambda) = 1/\lambda$ is compatible with the choices

$$\begin{align*}
\psi^{(1)}(y) &= y, & \phi^{(1)} &= -y, \\
\psi^{(2)}(x) &= x, & \phi^{(2)} &= -x, \\
\psi^{(3)}(z - \frac{1}{2}y^2) &= z - \frac{1}{2}y^2, & \phi^{(3)} &= z - \frac{1}{2}y^2.
\end{align*}$$

(29)

made in Section 3. To this end, we compute

$$\begin{align*}
\frac{1}{\lambda} &= \psi(\phi) = \varphi(\lambda) + \frac{\psi^{(1)}}{\varphi(\lambda)} + \cdots + \frac{\psi^{(k)}}{\varphi(\lambda)^k} + \cdots \\
&= \frac{1}{\lambda} + (\psi^{(1)} + \varphi^{(1)})\lambda \\
&\quad + (\psi^{(2)} + \varphi^{(2)})\lambda^2 \\
&\quad + (\psi^{(3)} + \varphi^{(3)} - \psi^{(1)}\varphi^{(1)})\lambda^3 \\
&\quad + (\psi^{(4)} + \varphi^{(4)} - \psi^{(1)}\varphi^{(2)} - 2\psi^{(2)}\varphi^{(1)})\lambda^4 \\
&\quad + \cdots
\end{align*}$$

One easily sees that the coefficients at $\lambda^i$, $i = 1, 2, 3$, vanish under the above mentioned choices and we obtain the recurrent formulas

$$\begin{align*}
\psi^{(k)} &= -\sum_{m \geq 1} (-1)^m \sum_{i_1 + \cdots + i_m = k+1} \frac{1}{k} \binom{k}{m} \varphi^{(i_1 - 1)} \cdots \varphi^{(i_m - 1)}, \\
\phi^{(k)} &= -\sum_{m \geq 1} \sum_{i_1 + \cdots + i_m = k+1} \frac{1}{k} \binom{k}{m} \psi^{(i_1 - 1)} \cdots \psi^{(i_m - 1)}
\end{align*}$$

that provide the needed equivalence of coverings.

4. Nonlocal symmetries

It is straightforward to compute the first-degree nonlocal shadows depending on any number of nonlocal variables. It may seem to be insignificant whether we use $\psi^{(i)}$ or $\phi^{(i)}$, but the formulas to follow turn out to be simpler if the latter choice is made. Thus, we give here an explicit description of nonlocal symmetries in the covering $\bar{\tau}^*$ and prove that they form the Witt algebra. As the first step, we obtain the shadows.

4.1. The hierarchy of symmetry shadows. Consider the covering $\bar{\tau}$ with the nonlocal variables $\varphi^{(i)}$ and present the total derivatives in the form

$$\begin{align*}
\bar{D}_x &= D_x + \sum_i \bar{X}^{(i)} \frac{\partial}{\partial \varphi^{(i)}}, \\
\bar{D}_y &= D_y + \sum_i \bar{Y}^{(i)} \frac{\partial}{\partial \varphi^{(i)}},
\end{align*}$$

where $\bar{X}^{(i)}$ and $\bar{Y}^{(i)}$ are the right-hand sides in (22).

Now, using the expansion (27), let us introduce a new set of nonlocal variables $\varphi^{\lambda^i} = d^i \varphi/d\lambda^i$ and consider the product $\bar{e}_\lambda = \bar{e} \times J(\lambda; \varphi)$, where $J(\lambda; \varphi)$ is the space with the
coordinates $\lambda$ and $\varphi^\lambda$, and the covering $\tilde{\tau}^\lambda: \tilde{\mathcal{E}}_\lambda \to \mathcal{E}$. In what follows we abbreviate the ‘index’ $\lambda^n$ as $\Lambda$. We equip $\tilde{\mathcal{E}}_\lambda$ with the total derivatives

$$
\tilde{D}_x = D_x + \sum_{\lambda} \varphi_{x\lambda} \frac{\partial}{\partial \varphi^\lambda}, \quad \tilde{D}_y = D_y + \sum_{\lambda} \varphi_{y\lambda} \frac{\partial}{\partial \varphi^\lambda}, \quad \tilde{D}_\lambda = \frac{d}{d\lambda} + \sum_{\lambda} \varphi_{\lambda\lambda} \frac{\partial}{\partial \varphi^\lambda},
$$

(30)

where the coefficients $\varphi_{x\lambda}$ and $\varphi_{y\lambda}$ can be computed by means of Equations (28). Then $\tilde{\mathcal{E}}$ endowed with the vector fields (30) is equivalent to the system consisting of the Gibbons–Tsarev equation (3), the condition

$$
z_\lambda = 0,
$$

(31)

and the pair (28) over the extended set of independent variables $x, y, \lambda$.

**Proposition 10.** Denote

$$
Z = (\varphi^2 - z_x \varphi - z_y)^2,
$$

(32)

Under the expansion (27), $Z$ is a formal Laurent series of the form

$$
Z = \sum_{n=-4}^{\infty} Z^{(n)} \lambda^{-2n}.
$$

(33)

Then $Z^{(n)}$ are shadows of symmetries of the Gibbons–Tsarev equation in the covering $\tilde{\tau}_\lambda$.

**Proof.** It is a routine computation to insert (32) into the linearisation

$$
\tilde{\ell}_\mathcal{E}(Z) \equiv \tilde{D}_y^2(Z) + z_x \tilde{D}_x \tilde{D}_y(Z) - z_y \tilde{D}_x^2(Z) + z_{xy} \tilde{D}_x(Z) - z_{xx} \tilde{D}_y(Z)
$$

(34)

and check that $\tilde{\ell}_\mathcal{E}(Z) = 0$ modulo equations (3), (28) and (31). If $Z$ is replaced with its expansion (33), we obtain

$$
0 = \tilde{\ell}_\mathcal{E}(Z) = \sum_{n=-4}^{\infty} \tilde{\ell}_\mathcal{E}(Z^{(n)}) \lambda^{-2n}.
$$

Since $\tilde{\ell}_\mathcal{E}(Z^{(n)})$ do not depend on $\lambda$, they have to vanish modulo equation (3) and expanded system (28), i.e., equations (23). Hence the statement. □

**Remark 6.** It is easy to compute functions $Z$ such that $\tilde{\ell}_\mathcal{E}(Z) = 0$ modulo equations (3), (28) and (31) (cf. the proof of Proposition 10). Besides the expression (32), another such function is $Z = \varphi^\lambda$, which, however, generates just the invisible symmetries (see Sect. 1).

Moreover, if some $Z$ satisfies $\tilde{\ell}_\mathcal{E}(Z) = 0$, then so does $f(\lambda)Z$ for any function $f(\lambda)$. This does not extend the linear space of generated shadows $Z^{(i)}$, however.

**Remark 7.** Although the condition $z_\lambda = 0$ is necessary for $Z$ given by (32) to be a shadow of the Gibbons–Tsarev equation, the same $Z$ does not satisfy $\tilde{D}_\lambda Z = 0$ and, therefore, is not a shadow of the system consisting of the Gibbons–Tsarev equation and the equation $z_\lambda = 0$.

Proposition 10 says that $Z$ is the generating section for an infinite hierarchy of shadows of the Gibbons–Tsarev equation. These shadows are easy to obtain explicitly. Let $\sum^{(*)}$ denote summation where indices run through all integers from $-1$ to infinity, possibly subject to additional requirements written under the symbol.
Proposition 11. Let
\[ A_2^{(k,n)} = \sum_{i_1+\ldots+i_{k+2}=n}^{(*)} i_1 i_2 \varphi^{(i_1)} \cdots \varphi^{(i_{k+2})}, \quad k \geq 0. \] (35)

Then
\[ Z^{(n)} = A_2^{(1,n)} z_x + A_2^{(0,n)} z_y - A_2^{(2,n)}. \]

Proof. Considering the expansion (27), we have
\[ \varphi^k \varphi^2 = \left( \sum_{i_1}^{(*)} \varphi^{(i_1)} \lambda^{i_1} \right) \cdots \left( \sum_{i_k}^{(*)} \varphi^{(i_k)} \lambda^{i_k} \right) \]
\[ \times \frac{1}{\lambda^2} \left( \sum_{i_{k+1}}^{(*)} \varphi^{(i_{k+1})} \lambda^{i_{k+1}} \right) \left( \sum_{i_{k+2}}^{(*)} \varphi^{(i_{k+2})} \lambda^{i_{k+2}} \right) \]
\[ = \sum_n \sum_{i_1+\ldots+i_{k+2}=n}^{(*)} i_{k+1} i_{k+2} \varphi^{(i_{k+1})} \ldots \varphi^{(i_{k+2})} \lambda^{n-2} = \sum_n A_2^{(k,n)} \lambda^{n-2}. \]

Inserting into Z given by formula (32), we obtain the result immediately. \( \square \)

4.2. The hierarchy of full symmetries. Here the shadows \( Z^{(n)} \) obtained in the previous section will be extended to full symmetries of the covering \( \bar{\tau}_* \). To this end, consider a nonlocal symmetry in the form
\[ \mathcal{S} = \frac{a}{\partial x} + \frac{b}{\partial y} + \frac{c}{\partial z} + \sum_{i>3} f^{(i)} \frac{\partial}{\partial \varphi^{(i)}} + \cdots, \]
where \( a, b, c, f^{(i)} \) are functions on \( \mathcal{E}_* \). Then the corresponding vertical field, obtained by subtracting \( aD_x + bD_y \), is
\[ \mathcal{S} = \sum_{z} \hat{D}_z (c - az_x - bz_y) \frac{\partial}{\partial z} + \sum_{i>3} f^{(i)} \varphi^{(i)}(a \varphi^{(i)} - b \varphi^{(i)}) \frac{\partial}{\partial \varphi^{(i)}}, \] (36)
where \( \varphi_x^{(i)} = \bar{X}^{(i)}, \varphi_y^{(i)} = \bar{Y}^{(i)} \) are given by recurrent relations (22) and \( z_{\Xi} \) are internal coordinates in \( \mathcal{E} \) (see Section 2). Then \( \mathcal{S} \) is a symmetry of \( \mathcal{E}_* \) if and only if
\[ \mathcal{S}(z_{yy} + z_x z_{xy} - z_y z_{xx} + 1) = 0, \quad \mathcal{S}(\varphi_x^{(i)} - \bar{X}^{(i)}) = 0, \quad \mathcal{S}(\varphi_y^{(i)} - \bar{Y}^{(i)}) = 0 \] (37)
modulo equations (3) and (24). Using formulas (29), variables \( x, y, z \) can be expressed in terms of \( \varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)} \). Consequently, we can rewrite (36) and (37) in terms of \( \varphi^{(i)} \) and \( z_{\Xi}, |\Xi| > 0, \) alone.

Proposition 12. In terms of coordinates \( \varphi^{(i)}, i > 0, \) and \( z_{\Xi}, \) a vertical evolutionary field in the covering \( \bar{\tau}_* \) can be written as
\[ \mathcal{S} = \sum_{i>0} \Phi^{(i)} \frac{\partial}{\partial \varphi^{(i)}} + \sum_{|\Xi|>0} \hat{D}_\Xi Z \frac{\partial}{\partial z_{\Xi}}, \]
\[ Z = (z_y - \varphi^{(1)}) f^{(1)} + z_x f^{(2)} - f^{(3)}, \]
\[ \Phi^{(i)} = f^{(i)} + f^{(2)} X^{(i)} + f^{(1)} Y^{(i)}. \] (38)
The field $\mathcal{S}$ is a symmetry if and only if $\tilde{\ell}_E Z = 0$ and
\[ \tilde{D}_x \Phi^{(i)} - \mathcal{S} \tilde{X}^{(i)} = 0, \quad \tilde{D}_y \Phi^{(i)} - \mathcal{S} \tilde{Y}^{(i)} = 0. \quad (39) \]

**Proof.** Formulas (38) are obtained by direct computation, while (39) follows from (37) immediately. □

Let us now pass from the covering $\bar{\tau}_*\varphi^{(i)}$ with the nonlocal variables $\varphi^{(i)}$ to the covering $\bar{E}_\lambda \to E$ obtained from the covering (28) by means of the expansion (27), i.e.,
\[ \varphi = \frac{1}{\lambda^2} + \sum_i \varphi^{(i)} \lambda^i. \]
Then $\varphi^{(i)} = -1/\lambda^2 + \sum_i i \varphi^{(i)} \lambda^{i-1}$, $\lambda^{li} = 2/\lambda^3 + \sum_i i(i-1) \varphi^{(i)} \lambda^{i-2}$, etc. Hence,
\[ \frac{\partial \varphi}{\partial \varphi^{(i)}} = \lambda^i, \quad \frac{\partial \varphi_{\lambda}}{\partial \varphi^{(i)}} = i \lambda^{i-1}, \quad \frac{\partial \varphi_{\lambda \lambda}}{\partial \varphi^{(i)}} = i(i-1) \lambda^{i-2} = \frac{d^2 \lambda^i}{d\lambda^2}, \ldots \]
and, therefore,
\[ \frac{\partial}{\partial \varphi^{(i)}} = \sum_\Lambda \partial_{\lambda^i} \frac{\partial}{\partial \varphi}, \]
where, as above, $\Lambda$ stands for $\lambda^n, n \geq 0$, and $\partial_{\lambda^n} = d^n/d\lambda^n$. Alternatively speaking, the vector field $\partial/\partial \varphi^{(i)}$, when rewritten in the coordinates $\lambda, \varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}, \ldots$ becomes
\[ \frac{\partial}{\partial \lambda} + \varphi^{(1)} \frac{\partial}{\partial \varphi} + \varphi^{(2)} \frac{\partial}{\partial \varphi_{\lambda}} + \ldots = \frac{\partial}{\partial \lambda} + \sum_\Lambda \varphi_{\lambda} \frac{\partial}{\partial \varphi_{\lambda}} = D_{\lambda} \]
in the coordinates $\lambda, \varphi_{\lambda}$. 

**Proposition 13.** In terms of the coordinates $\varphi_{\lambda}$ and $z_{\Xi}, |\Xi| > 0$, a vertical infinitely prolonged field in the covering $\bar{\tau}_\lambda$ can be written as
\[ \mathcal{S} = \sum_\Lambda \partial_{\lambda^i} \Phi \frac{\partial}{\partial \varphi_{\lambda}} + \sum_{|\Xi| > 0} \tilde{D}_z Z \frac{\partial}{\partial z_{\Xi}}, \quad (40) \]
where
\[ Z = (z_y - \varphi^{(1)}) f^{(1)} + z_x f^{(2)} - f^{(3)}; \]
\[ \Phi = f - \frac{f^{(2)} + f^{(1)} (\varphi - z_x)}{\varphi^2 - z_x \varphi - z_y}; \quad (41) \]
\[ f = \sum_{i>0} f^{(i)} \lambda^i. \]

The field $\mathcal{S}$ is a symmetry if and only if $\tilde{\ell}_E Z = 0$, see (34), and
\[ \tilde{D}_x \Phi + \frac{\varphi \tilde{D}_x Z + \tilde{D}_y Z - (2 \varphi - z_x) \Phi}{(\varphi^2 - z_x \varphi - z_y)^2} = 0, \]
\[ \tilde{D}_y \Phi + \frac{z_y \tilde{D}_x Z + (\varphi - z_x) \tilde{D}_y Z - ((\varphi - z_x)^2 + z_y) \Phi}{(\varphi^2 - z_x \varphi - z_y)^2} = 0. \quad (42) \]
Proof. Formula (38) can be rewritten as

\[ S = \sum_{i>0} \Phi^{(i)} \frac{\partial}{\partial \varphi^{(i)}} \cdot \sum_{|\Xi|>0} \tilde{D}_z Z \frac{\partial}{\partial z}, \]

where

\[ Z = (z_y - \varphi^{(1)}) f^{(1)} + z_x f^{(2)} - f^{(3)}, \quad \text{and} \]

\[ \sum_{i>0} \Phi^{(i)} \lambda^i = \sum_{i>0} f^{(i)} \lambda^i + f^{(2)} \sum_{i>0} \tilde{X}^{(i)} \lambda^i + f^{(1)} \sum_{i>0} \tilde{Y}^{(i)} \lambda^i \]

\[ = f + f^{(2)} \sum_{i>0} \varphi_x^{(i)} \lambda^i + f^{(1)} \sum_{i>0} \varphi_y^{(i)} \lambda^i \]

\[ = f + f^{(2)} \varphi_x + f^{(1)} \varphi_y = f - \frac{f^{(2)} + f^{(1)} (\varphi - z_x)}{\varphi^2 - z_x \varphi - z_y}, \]

where \( f = \sum_{i>0} f^{(i)} \lambda^i \). Hence formulas (41).

Now, \( S \) is a symmetry if and only if

\[ \tilde{D}_x \Phi + S \left( \frac{1}{\varphi^2 - z_x \varphi - z_y} \right) = 0, \quad \tilde{D}_y \Phi + S \left( \frac{\varphi - z_x}{\varphi^2 - z_x \varphi - z_y} \right) = 0. \]

These are formulas (42). \qed

The last proposition suggests the following construction. Let

\[ f = \sum_{i>0} f^{(i)} \lambda^i, \]

where the coefficients \( f^{(i)} \) are independent of \( \lambda \). Then we set

\[ \mathcal{S}_f = \sum_{i>0} f^{(i)} \frac{\partial}{\partial \varphi^{(i)}}, \]

Transforming to the coordinates \( \lambda, \varphi, \ldots, \varphi_{\Lambda}, \ldots \), we obtain

\[ \mathcal{S}_f = \sum_{\Lambda} \partial_{\lambda} f \frac{\partial}{\partial \varphi_{\lambda}}, \]

which is the usual prolongation of a vertical generator \( f \frac{\partial}{\partial \varphi} \). Obviously,

\[ [\mathcal{S}_f, \mathcal{S}_g] = \mathcal{S}_{\{f,g\}}, \quad \{f, g\} = \mathcal{S}_f g - \mathcal{S}_g f. \quad (43) \]

Using this notation, the symmetries we are looking for can be written as

\[ \mathcal{J} = \mathcal{S}_\Phi + \sum_{|\Xi|>0} \tilde{D}_z Z \frac{\partial}{\partial z}, \]
where
\[
Z = (z_y - \varphi(1))f(1) + z_x f(2) - f(3),
\]
\[
\Phi = f - \frac{f(2) + f(1)(\varphi - z_x)}{\varphi^2 - z_x \varphi - z_y},
\]
\[
f = \sum_{i>0} f^{(i)} \lambda^i,
\]
and should satisfy \(\tilde{\ell}_\varphi(Z) = 0\), see (34), as well as Equations (42).

Now, using the formal series
\[
f = \sum_{i \geq 1} i \varphi^{(i)} \lambda^{n+i-1}
\]
and the field \(\mathcal{S}_f\) we shall show that all the shadows described in Subsection 4.1 are lifted to a nonlocal symmetry in \(\bar{\tau}_s\). The proof depends on the integer \(n\).

The case \(n \geq 3\). The series \(\lambda^n \varphi\lambda\) is of the form required by the definition of \(\mathcal{S}_f\). In all these cases, conditions (34) and (42) are easily checked by straightforward computation, which is omitted.

For \(n = 3\), we have \(f = \lambda^3 \varphi\lambda = -\lambda^3 + \sum_{i \geq 1} i \varphi^{(i)} \lambda^{2+i}\), i.e., \(f^{(1)} = -1, f^{(2)} = 0, f^{(3)} = \varphi^{(1)}\).

In this case, \(Z = -z_y = -Z^{(-2)}\), i.e., we obtain the lift
\[
\mathcal{S}_{\lambda^3 \varphi\lambda} = - \frac{\partial}{\partial \varphi^{(1)}} + \sum_{i \geq 1} i \varphi^{(i)} \frac{\partial}{\partial \varphi^{(2+i)}}
\]
of the \(y\)-translation.

For \(n = 4\), we have \(f = \lambda^4 \varphi\lambda = -\lambda^4 + \sum_{i \geq 1} i \varphi^{(i)} \lambda^{3+i}\), i.e., \(f^{(1)} = f^{(3)} = 0, f^{(2)} = -1\).

In this case, \(Z = -z_x = -Z^{(-3)}\), i.e., we obtain the lift
\[
\mathcal{S}_{\lambda^4 \varphi\lambda} = - \frac{\partial}{\partial \varphi^{(2)}} + \sum_{i \geq 1} i \varphi^{(i)} \frac{\partial}{\partial \varphi^{(3+i)}}
\]
of the \(x\)-translation.

If \(n = 5\), then \(f = \lambda^5 \varphi\lambda = -\lambda^5 + \sum_{i \geq 1} i \varphi^{(i)} \lambda^{4+i}\), i.e., \(f^{(1)} = f^{(2)} = 0, f^{(3)} = -1\).

Obviously, \(Z = 1 = Z^{(-4)}\), i.e., we recover the first classical symmetry and obtained its lift
\[
\mathcal{S}_{\lambda^5 \varphi\lambda} = - \frac{\partial}{\partial \varphi^{(3)}} + \sum_{i \geq 1} i \varphi^{(i)} \frac{\partial}{\partial \varphi^{(4+i)}}.
\]

If \(n \geq 6\), then the coefficients \(f^{(1)}, f^{(2)}, f^{(3)}\) are zero. Obviously, \(Z = 0\) and we obtain the invisible symmetries
\[
\mathcal{S}_{\lambda^n \varphi\lambda} = - \frac{\partial}{\partial \varphi^{(n-2)}} + \sum_{i \geq 1} i \varphi^{(i)} \frac{\partial}{\partial \varphi^{(n+i-1)}}, \quad n \geq 6.
\]

The case \(n < 3\). In this case, the series (44) contains non-positive terms and so we cannot construct the corresponding field \(\mathcal{S}_f\) directly. To overcome this problem, we do the following.
Lemma 1. The generating function for the first three coefficients $f, g, h$ satisfies equations (42), then, by linearity, (42) will be satisfied for all $f_n$. To turn this observation into a proof, we need an analytic expression for $f$ and also for the first three coefficients $f^{(i)}$ of the expansion $f = \sum_i f^{(i)}(\xi)\lambda^i$.

**Proposition 14.** For any $n \leq 2$, all the vector fields $\mathcal{S}$ of the form (40) with

$$f = f_n = \Psi_\varphi (\varphi_\lambda \lambda^n)$$

are nonlocal symmetries in $\bar{\tau}_\sigma$.

**Example.** To illustrate how the construction works, let us discuss the case of $n = 1$ in more detail. We have

$$\lambda \varphi_\lambda = -\frac{1}{\lambda} + \varphi^{(1)}(1) + 2\varphi^{(2)}(2) \lambda^2 + 3\varphi^{(3)}(3) \lambda^3 + \ldots$$

and

$$f_1 = \Psi_{\lambda \varphi_\lambda} = \lambda \varphi_\lambda + \varphi = 2\varphi^{(1)}(1) + 3\varphi^{(2)}(2) \lambda^2 + 4\varphi^{(3)}(3) \lambda^3 + \ldots$$

Consequently, $f_1^{(i)} = (i + 1)\varphi^{(i)}(i)$. In particular, $f^{(1)} = 2\varphi^{(1)}(1) = -2y$, $f^{(2)} = 3\varphi^{(2)}(2) = -3x$, $f^{(3)} = 4\varphi^{(3)}(3) = -4z - 2y^2$. Substituting into formulas (41), we get

$$Z = (zy - \varphi^{(1)}(1)f^{(1)} + zx f^{(2)} - f^{(3)} = -2yz - 3xz + 4z = -Z^{(0)}),$$

$$\Phi = \lambda \varphi_\lambda + \varphi + \frac{3x + 2y (\varphi - zx)}{\varphi^2 - zx \varphi - zy}.$$

Thus, we have the lift of the scaling symmetry.

To prove Proposition 14, we introduce the generating function

$$f = f(\lambda, \xi) = \sum_{n=-4}^{\infty} \xi^{n-2} f_{1-n}.$$ (47)

If we show that $f$ satisfies equations (42), then, by linearity, (42) will be satisfied for all $f_n$. To turn this observation into a proof, we need an analytic expression for $f$ and also for the first three coefficients $f^{(i)}$ of the expansion $f = \sum_i f^{(i)}(\xi)\lambda^i$.

**Lemma 1.** The generating function (47) admits the representation

$$f = \left(\frac{\lambda}{\xi}\right)^6 \frac{\varphi_\lambda(\lambda)}{\lambda - \xi} + \frac{\varphi_\xi(\xi)^2}{\varphi(\xi) - \varphi(\lambda)}.$$

In addition,

$$f^{(1)} = -\varphi_\xi(\xi)^2, \quad f^{(2)} = -\varphi(\xi)^2 \varphi(\xi), \quad f^{(3)} = \varphi_\xi(\xi)^2 (\varphi^{(1)} - \varphi(\xi)^2).$$

**Proof.** We break the proof into several steps.

**Step 1.** We show that

$$[\varphi(\xi)^k] \frac{\varphi_\xi(\xi)^2}{\xi^{n-1}} = -[\xi^{n-2}] \frac{\varphi_\xi(\xi)^2}{\varphi(\xi)^{k+1}}.$$
Recall that the ‘formal residue’ of the Laurent series $g(\xi) = \sum_{k=-\infty}^{\infty} g_k \xi^k$ is defined by

$$\text{Res } g = [\xi^{-1}] g(\xi) = g_{-1}.$$  

It is straightforward to check that it has the following properties:

$$\text{Res } \alpha g + \beta h = \alpha \text{Res } g + \beta \text{Res } h, \quad \alpha, \beta = \text{const},$$

$$\text{Res } g' = 0,$$

$$[\xi^n] g = \text{Res } \frac{g}{\xi^{n+1}},$$

$$\text{Res } (g(h))h' = \frac{h'}{h} \cdot \text{Res } g,$$

where the ‘prime’ denotes the $\xi$-derivative and $g(h)$ is the composition of formal series. The last property is valid for all Laurent series $g$ of the form $g = g_{-n} \xi^{-n} + g_{1-n} \xi^{1-n} + \cdots + g_0 + g_1 \xi + g_2 \xi^2 + \cdots$, i.e., whose principal part is finite, and for $h$ of the form $h = \xi^m (h_0 + h_1 \xi + h_2 \xi^2 + \cdots)$ or $h = \xi^{-m} (h_0 + h_1 \xi^{-1} + h_2 \xi^{-2} + \cdots)$, where $m > 0$.

We are going to use the last property in the form

$$\text{Res } g = \frac{\text{Res } (g \circ h)h'}{\text{Res } (h'/h)},$$

with $g(\omega) = \psi(\omega)$ and $h(\xi) = \varphi(\xi)$, where $\psi$ is the compositional inverse of $\varphi$, i.e. $\psi(\varphi(\lambda)) = \varphi(\psi(\lambda)) = \lambda$. In other words $\omega = \varphi(\xi)$, $\xi = \psi(\omega)$. Since $\varphi$ and $\psi$ are compositionally mutually inverse, one has

$$\varphi'(\psi(\omega)) = \frac{1}{\psi'(\omega)}, \quad \psi'(\varphi(\xi)) = \frac{1}{\varphi'(\xi)},$$

and using the obvious identity $\text{Res } (\varphi'(\xi)/\varphi(\xi)) = -1$, we obtain

$$[\varphi(\xi)^k] \frac{\varphi'(\xi)}{\xi^{n-1}} = [\omega^k] \frac{1}{\psi'(\omega)\psi(\omega)n^{-1}} = \text{Res } \frac{1}{\psi'(\omega)\psi(\omega)n^{-1}\omega^{k+1}}$$

$$= - \text{Res } \frac{\varphi'(\xi)}{\psi'(\varphi(\xi))}\xi^{n-1}\varphi(\xi)^{k+1} = - \text{Res } \frac{\varphi'(\xi)^2}{\xi^{n-1}\varphi(\xi)^{k+1}} = - [\xi^{n-2}] \frac{\varphi'(\xi)^2}{\varphi(\xi)^{k+1}}.$$  

**Step 2.** Substituting this result into the definition of $f_n$ we obtain

$$f_{1-n} = \frac{\varphi'(\lambda)}{\lambda^{n-1}} + \sum_{k=0}^{n+1} \varphi(\lambda)^k [\xi^{n-2}] \frac{\varphi'(\xi)^2}{\varphi(\xi)^{k+1}},$$

where, as a matter of fact, we can extend the upper summation bound to infinity since

$$[\xi^{n-2}] \frac{\varphi'(\xi)^2}{\varphi(\xi)^{k+1}} = 0, \quad k > n + 1.$$  

Thus, we obtain

$$f_{1-n} = \frac{\varphi'(\lambda)}{\lambda^{n-1}} + [\xi^{n-2}] \sum_{k=0}^{\infty} \varphi(\lambda)^k \frac{\varphi'(\xi)^2}{\varphi(\xi)^{k+1}} = \frac{\varphi'(\lambda)}{\lambda^{n-1}} + [\xi^{n-2}] \frac{\varphi'(\xi)^2}{\varphi(\xi) - \varphi(\lambda)}.$$  

(49)
Step 3. To get the closed formula sought for the generating function
\[ f(\lambda, \xi) = \sum_{n=-4}^{\infty} \xi^{n-2} f_{1-n}, \]
it remains to use the identities
\[ \sum_{n=-4}^{\infty} \xi^{n-2} \frac{\varphi'(\lambda)}{\lambda^{n-1}} = \left(\frac{\lambda}{\xi}\right)^6 \frac{\varphi'(\lambda)}{\lambda - \xi}, \quad \sum_{n=-4}^{\infty} \xi^{n-2} \varphi' \left[ \frac{\varphi'(\xi)^2}{\varphi(\xi)} \right] = \frac{\varphi'(\xi)^2}{\varphi(\xi) - \varphi(\lambda)}. \]

Step 4. Computation of the first three coefficients
\[ f^{(1)} = [\lambda] f(\lambda, \xi), \quad f^{(2)} = [\lambda^2] f(\lambda, \xi), \quad f^{(3)} = [\lambda^3] f(\lambda, \xi) \]
is straightforward. □

Proof of Proposition 14. Let us verify equations (42) for the generating function \[ f = f(\lambda, \xi). \]
In this case, we have
\[ Z = -(z_y + z_x \varphi(\xi) - \varphi(\xi)^2) \varphi(\xi)^2. \]
This is a shadow of symmetries of the Gibbons–Tsarev equation as proved in Proposition 10 so Equation (34) is satisfied. We have
\[ \Phi = \left(\frac{\lambda}{\xi}\right)^6 \frac{\varphi(\lambda)}{\lambda - \xi} + \frac{\varphi(\xi)^2}{\varphi(\xi) - \varphi(\lambda)} \varphi(\xi)^2 - z_x \varphi(\xi) - z_y. \]
These expressions can be put directly into equations (42). The proof that equations (42) indeed hold is a matter of direct computation with the help of the identities
\[ \frac{\partial \varphi(t)}{\partial x} = \frac{1}{\varphi(t)^2 - z_x \varphi(t) - z_y}, \]
\[ \frac{\partial \varphi(t)}{\partial y} = \frac{\varphi(t) - z_x}{\varphi(t)^2 - z_x \varphi(t) - z_y}, \]
\[ \frac{\partial \varphi_1(t)}{\partial x} = \frac{(2 \varphi(t) - z_x) \varphi_1(t)}{(\varphi(t)^2 - z_x \varphi(t) - z_y)^2}, \]
\[ \frac{\partial \varphi_1(t)}{\partial y} = \frac{(\varphi(t)^2 - z_x \varphi(t) - z_y)^2}{(\varphi(t) - z_x)^2 + z_y \varphi_1(t)} - 1, \]
where \( t \) is either \( \lambda \) or \( \xi \).

Collecting together the above facts, we obtain the main result of this section:

**Theorem 1.** In the notation of Proposition 13 the vector fields \( \mathcal{S} \) with \( f = f_n \) are symmetries for all \( n \in \mathbb{Z} \).

Remarkably, we are able to obtain explicit formulas for the symmetries. Denote by
\[ A_r^{(k,n)} = [\lambda^{n-r}] \varphi^k \varphi^r, \]
the coefficient at $\lambda^{n-r}$ in the product $\varphi^k \varphi^r$, where $k, r$ are arbitrary integers, cf. \((33)\). For $r = 0$ we have

$$A_0^{(k,n)} = \sum_{j_1 + \cdots + j_k = n} \varphi^{(j_1)} \varphi^{(j_2)} \cdots \varphi^{(j_k)}, \quad k > 0$$

(recall that the notation $\sum_{\bullet}$ means summation where indices run through all integers from $-1$ to infinity).

**Proposition 15.** Vector fields $\mathcal{S}^{(n)} = \mathfrak{S}_{f_1 - n}$ admit the explicit formula

$$\mathcal{S}^{(n)} = \sum_{m \geq 1} \left( (n + m) \varphi^{(n+m)} + \sum_{k=0}^{n+1} A_0^{(k,m)} A_2^{(-k-1,n)} \right) \frac{\partial}{\partial \varphi^{(m)}}. \tag{50}$$

**Proof.** This is a direct consequence of the representation of $f_1 - n$, see \((48)\), and the definition of $\mathfrak{S}_f$. \(\square\)

**Remark 8.** Alternatively, we can also write

$$\mathcal{S}^{(n)} = -\sum_{m \geq 1} \sum_{k=0}^m A_0^{(-k-1,m)} A_2^{(k,n)} \frac{\partial}{\partial \varphi^{(m)}}.$$  

4.3. **The Lie algebra structure.** The main result of this part is

**Theorem 2.** The vector fields $\mathcal{S}^{(n)} = \mathfrak{S}_{f_1 - n}$ satisfy

$$[\mathcal{S}^{(n)}, \mathcal{S}^{(m)}] = (m - n) \mathcal{S}^{(n+m)},$$

i.e., constitute a basis of the Witt algebra.

For the proof we are going to make use of the following lemma:

**Lemma 2.** Let

$$g = g(\varphi, \varphi', \lambda), \quad h = h(\varphi, \varphi', \lambda),$$

be two formal series in $\lambda$ of the lowest order $1$. Then

$$[\mathfrak{S}_g, \mathfrak{S}_h] = \mathfrak{S}_{\{g,h\}}, \tag{51}$$

where

$$\{g, h\} = h_\varphi g - g_\varphi h + h_{\varphi'} D_\lambda g - g_{\varphi'} D_\lambda h,$$

where, as before, $D_\lambda = \partial / \partial \lambda + \sum_\lambda \varphi_\lambda \partial / \partial \lambda$.

**Proof.** Obviously from \((27)\), one has

$$\frac{\partial \varphi}{\partial \varphi^{(m)}} = \lambda^m, \quad \frac{\partial \varphi'}{\partial \varphi^{(m)}} = m \lambda^{m-1}$$

for all integers $m > 0$. Therefore, $\partial_{\varphi^m} g(\lambda) = g_\varphi \lambda^m + g_{\varphi'} D_\lambda \lambda^m$. Hence

$$\mathfrak{S}_h g(\lambda) = \sum_{m=1}^\infty [\xi^m] h(\xi) \partial_{\varphi^m} g(\lambda) = \sum_{m=1}^\infty [\xi^m] h(\xi) (g_\varphi \lambda^m + g_{\varphi'} D_\lambda \lambda^m) = g_\varphi h(\lambda) + g_{\varphi'} D_\lambda h(\lambda).$$
The last equality stems from the fact that $h$ is of the lowest order one. Finally,

$$\left[ \mathcal{G}_g, \mathcal{G}_h \right] = \sum_{m=1}^{\infty} \left[ \lambda^m \right] (\mathcal{G}_g h(\lambda) - \mathcal{G}_h g(\lambda)) \partial_{\varphi^m}. $$

\[ \square \]

**Proof of Theorem** We have to prove the identity

$$\left[ \mathcal{G}_{f_{1+n}}, \mathcal{G}_{f_{1+m}} \right] = (n-m)\mathcal{G}_{f_{1+n+m}}. \quad (52)$$

Recall that

$$\mathcal{G}_g = \sum_{m=1}^{\infty} \left[ \lambda^m \right] g(\lambda) \partial_{\varphi^m},$$

where $g$ is a series in $\lambda$ of the lowest order one.

For $n > 1$, the quantity $f_{1+n} = \lambda^{n+1} \varphi'$ is of the lowest order one and depends on $\varphi'$ only, hence is admissible for the commutation rule above. It is easy to show that for $n, m > 1$ we have

$$\{ \lambda^{n+1} \varphi', \lambda^{m+1} \varphi' \} = \lambda^{m+1} D_{\lambda} \lambda^{n+1} \varphi' - \lambda^{n+1} D_{\lambda} \lambda^{m+1} \varphi' = (n-m)\lambda^{m+n+1} \varphi'.$$

In other words

$$\left[ \mathcal{G}_{f_{1+n}}, \mathcal{G}_{f_{1+m}} \right] = \mathcal{G}_{(n-m)\lambda^{m+n+1} \varphi} = (n-m)\mathcal{G}_{f_{1+n+m}}.$$

For $n \leq 1$, dependence of $f_{1-n}$ on $\varphi^{(i)}$ is different, thus another approach is needed. We are going to use the representation [19] for $f_{1-n}(\lambda)$, that is,

$$f_{1-n}(\lambda) = \frac{\varphi'(\lambda)}{\lambda^{n-1}} + [\xi^{n-2}] \frac{\varphi'(\xi)^2}{(\varphi(\xi) - \varphi(\lambda))^2}.$$

Note that $f_{1-n}(\lambda)$ does not depend on $\xi$.

Let us compute

$$\begin{align*}
\mathcal{G}_{f_{1-n}(\lambda)} f_{1-m}(\lambda) &= \mathcal{G}_{f_{1-n}(\lambda)} \frac{\varphi'(\lambda)}{\lambda^{m-1}} + [\xi^{m-2}] \mathcal{G}_{f_{1-n}(\lambda)} \frac{\varphi'(\xi)^2}{(\varphi(\xi) - \varphi(\lambda))^2} \\
&= \frac{1}{\lambda^{m-1}} f'_{1-n}(\lambda) + [\xi^{m-2}] \frac{f_{1-n}(\lambda) \varphi'(\xi)^2}{(\varphi(\xi) - \varphi(\lambda))^2} - [\xi^{m-2}] f_{1-n}(\xi) \frac{\varphi'(\xi)^2}{(\varphi(\xi) - \varphi(\lambda))^2} \\
&+ [\xi^{m-2}] f'_{2-n}(\xi) \frac{2\varphi'(\xi)^2}{(\varphi(\xi) - \varphi(\lambda))^2}.
\end{align*}$$

where the ' in $f'_{1-n}$ denotes the total derivative. Collecting terms that contain $\varphi'(\lambda)$, $\varphi''(\lambda)$ and the rest that depends only on $\varphi(\lambda)$, we can rewrite the last result as follows:

$$\mathcal{G}_{f_{1-n}(\lambda)} f_{1-m}(\lambda) = -(n-1) \frac{\varphi'(\lambda)}{\lambda^{m+n-1}} + \frac{\varphi''(\lambda)}{\lambda^{m+n-2}}$$

$$+ [\xi^{m-2}] \frac{\varphi'(\xi)^2}{(\varphi(\xi) - \varphi(\lambda))^2} \frac{\varphi'(\lambda)}{\lambda^{m-1}} + [\xi^{m-2}] \frac{\varphi'(\xi)^2}{(\varphi(\xi) - \varphi(\lambda))^2} \frac{\varphi'(\lambda)}{\lambda^{m-1}} + \sum_{k=0}^{\infty} c_k \varphi(\lambda)^k,$$

where $c_k$ are some coefficients depending on $\varphi^{(i)}$. 
Notice that all terms except the first one and the last one are symmetric with respect to swapping $n \leftrightarrow m$. Hence,
\[
\{f_{1-n}(\lambda), f_{1-m}(\lambda)\} = \mathfrak{S}_{f_{1-n}(\lambda)f_{1-m}(\lambda)} - \mathfrak{S}_{f_{1-m}(\lambda)f_{1-n}(\lambda)} \\
= (m - n) \frac{\phi'(\lambda)}{\lambda^{m+n-1}} + \sum_{k=0}^{\infty} c_k \phi(\lambda)^k,
\]
where, again, $c_k$ are some coefficients depending on $\phi^{(i)}$, $i \geq 1$.

Note now that for any formal series $g, h$ that are of lowest order 1, we have
\[
[\phi(\lambda)^j] \{g(\lambda), h(\lambda)\} = 0, \quad j \geq 0,
\]
since the operator $\mathfrak{S}_g$ does not act on $\lambda$ and thus cannot produce non-positive powers of $\lambda$. In our case, both $f_{1-n}(\lambda)$ and $f_{1-m}(\lambda)$ are of lowest order 1 and thus
\[
0 = [\phi(\lambda)^j] \{f_{1-n}(\lambda), f_{1-m}(\lambda)\} = [\phi(\lambda)^j] \left( (m - n) \frac{\phi'(\lambda)}{\lambda^{m+n-1}} + \sum_{k=0}^{\infty} c_k \phi(\lambda)^k \right) \\
= (m - n)[\phi(\lambda)^j] \frac{\phi'(\lambda)}{\lambda^{m+n-1}} + c_j.
\]
Hence,
\[
c_j = -(m - n)[\phi(\lambda)^j] \frac{\phi'(\lambda)}{\lambda^{m+n-1}}
\]
and
\[
\{f_{1-n}(\lambda), f_{1-m}(\lambda)\} \\
= (m - n) \frac{\phi'(\lambda)}{\lambda^{m+n-1}} - (m - n) \sum_{k=0}^{\infty} \phi(\lambda)^k \frac{\phi'(\xi)^k}{\xi^{m+n-1}} = (m - n)f_{1-m-n}(\lambda).
\]
Finally,
\[
[\mathfrak{S}_{g_{1-n}}, \mathfrak{S}_{g_{1-m}}] = \mathfrak{S}_{\{g_{1-n}, g_{1-m}\}} = (-n - (-m)) \mathfrak{S}_{g_{1-m-n}},
\]
as claimed. \(\square\)

5. Uniqueness of symmetries

We prove here some uniqueness results for the nonlocal symmetries of the Gibbons–Tsarev equation \([4]\) in the covering $\tau_*$ defined in Section 3.2.1. For technical reasons, it is more convenient for us to deal with System \([5]\), i.e.,
\[
u_y + uv_x = \frac{1}{u - v}, \quad v_y + uv_x = \frac{1}{u - v}.
\]
Due to \([7]\), the relation between shadows of \([3]\) and those of \([2]\) is established by
\[
U = u D_x(Z) + D_y(Z), \quad V = v D_x(Z) + D_y(Z),
\]
where $Z$ is a shadow for the Gibbons–Tsarev equation, while $(U, V)$ is a shadow for the system in $u$ and $v$. In particular, for the symmetries $\mathcal{F}^{-3}, \ldots, \mathcal{F}^{(9)}$ with the generating sections given by \([5]\) one has
\[
Z^{-3} \quad \mapsto \quad W^{-3} = (u_x, v_x),
\]
with the nonlocal variables and the defining equations for symmetries of (54) are

\[ Z^{(-2)} \mapsto W^{(-2)} = (u_y, v_y), \]
\[ Z^{(-1)} \mapsto W^{(-1)} = (1 - yu_x, 1 - yv_x), \]
\[ Z^{(0)} \mapsto W^{(0)} = (3xu_x + 3yu_y - u, 3xv_x + 3yv_y - v), \]

while the symmetry \( \mathcal{S}^{(-4)} \) becomes invisible.

In what follows, we, for convenience, use the notation

\[ u_y = f(u, v) - vu_x, \quad v_y = g(u, v) - uv_x, \quad (54) \]

where \( f \) and \( g \) may be considered as functions in \( u \) and \( v \) such that the partial derivatives \( f_u, f_v, g_u, g_v \) do not vanish. We choose the functions

\[ x, t, u_i = \frac{\partial^x u}{\partial x^i}, \quad v_i = \frac{\partial^v v}{\partial x^i}, \quad i \geq 0, \]

for the internal coordinates on \( \mathcal{E} \), and then the total derivatives are

\[ D_x = \frac{\partial}{\partial x} + \sum_{i \geq 0} \left( u_{i+1} \frac{\partial}{\partial u_i} + v_{i+1} \frac{\partial}{\partial v_i} \right), \]
\[ D_y = \frac{\partial}{\partial y} + \sum_{i \geq 0} \left( D_x^i (f - vu_1) \frac{\partial}{\partial u_i} + D_x^i (g - uv_1) \frac{\partial}{\partial v_i} \right). \]

Then

\[ \ell_{\mathcal{E}} = \begin{pmatrix} vD_x + D_y - f_u & u_1 - f_v \\ v_1 - g_u & uD_x + D_y - g_v \end{pmatrix} \]

and the defining equations for symmetries of (54) are

\[ D_y(U) = f_u U - vD_x(U) + (f_v - u_1)V, \]
\[ D_y(V) = g_v V - uD_x(V) + (g_u - v_1)U. \quad (55) \]

5.1. **Uniqueness of polynomial shadows.** Consider now the covering \( \tau_\mathcal{E} : \mathcal{E} \to \mathcal{E} \) with the nonlocal variables \( \psi^{(3)}, \ldots, \psi^{(k)}, \ldots \) defined in Subsection 3.2.1. We say that a function \( F \) on \( \mathcal{E} \) is of order \( k \) if at least one of the partial derivatives \( F_{u_k} \) or \( F_{v_k} \) does not vanish, while \( F_{u_i} = F_{v_i} = 0 \) for all \( i > k \).

Let us estimate the higher order terms of \( \tau_\mathcal{E} \)-shadows. The defining equations for \( \tau_\mathcal{E} \)-shadows is obtained from (55) by changing the total derivatives \( D_x \) and \( D_y \) to

\[ D_x^{(s)} = D_x + \sum_{i \geq 3} X^{(i)} \frac{\partial}{\partial \psi^{(i)}}, \quad D_y^{(s)} = D_y + \sum_{i \geq 3} Y^{(i)} \frac{\partial}{\partial \psi^{(i)}}, \]

i.e., they are of the form

\[ D_y^{(s)}(U) = f_u U - vD_x^{(s)}(U) + (f_v - u_1)V, \quad (56) \]
\[ D_y^{(s)}(V) = g_v V - uD_x^{(s)}(V) + (g_u - v_1)U. \quad (57) \]

Note that the coefficients \( X^{(i)} \) and \( Y^{(i)} \) are of order zero.

We shall need the following ‘asymptotics’ below:

\[ (D_x^{(s)})^p (f - vu_1) = -vu_{p+1} + (f_u - pu_1)u_p + (f_v - u_1)v_p + O(p - 1), \]
\[ (D_y^{(s)})^p (g - uv_1) = -uv_{p+1} + (g_v - pu_1)v_p + (g_u - v_1)u_p + O(p - 1) \quad (58) \]

for an arbitrary \( p > 1 \). Here and in what follows \( O(\alpha) \) denotes terms of order \( \leq \alpha \).
Proposition 16. Equation (54) admits no \( \tau \)-shadow of order > 1.

Proof. Let us assume that the components \( U \) and \( V \) of the shadow under consideration are of order \( k \) and, using (58), differentiate Equation (56) with respect to \( v_{k+1} \). The result is \(-uU_{vk} = -vU_v\). In a similar way, applying \( \partial/\partial u_{k+1} \) to (57), we get \(-vV_{uk} = -uV_u\). Consequently,

\[
U = U(\ldots, u_{k-1}, v_{k-1}, u_k), \quad V = V(\ldots, u_{k-1}, v_{k-1}, v_k),
\]

where ‘dots’ stand for the variables of order \( \leq k-2 \).

Apply the partial derivatives \( \partial/\partial u_k \) and \( \partial/\partial v_k \) to Equations (56) and (57):

\[
\frac{\partial}{\partial u_k} D_y(U) + D_y(U_{uk}) = f_u U_{uk} - v \left( \frac{\partial D_x(U)}{\partial u_k} + D_x(U_{uk}) \right) + (f_v - u_1) U_{uk},
\]

\[
\frac{\partial}{\partial v_k} D_y(U) + D_y(U_{vk}) = f_u U_{vk} - v \left( \frac{\partial D_x(U)}{\partial v_k} + D_x(U_{vk}) \right) + (f_v - u_1) U_{vk},
\]

\[
\frac{\partial}{\partial u_k} D_y(V) + D_y(V_{uk}) = g_v V_{uk} - u \left( \frac{\partial D_x(V)}{\partial u_k} + D_x(V_{uk}) \right) + (g_u - v_1) V_{uk},
\]

\[
\frac{\partial}{\partial v_k} D_y(V) + D_y(V_{vk}) = g_v V_{vk} - u \left( \frac{\partial D_x(V)}{\partial v_k} + D_x(V_{vk}) \right) + (g_u - v_1) V_{vk},
\]

(the partial derivatives above are applied to the coefficients of the corresponding operators). Using now (59), we see that the above equalities amount to

\[
\frac{\partial D_y(U)}{\partial u_k} = f_u U_{uk} - v \left( \frac{\partial D_x(U)}{\partial u_k} + D_x(U_{uk}) \right),
\]

\[
\frac{\partial D_y(V)}{\partial v_k} = g_v V_{vk} - u \left( \frac{\partial D_x(V)}{\partial v_k} + D_x(V_{vk}) \right)
\]

and

\[
\frac{\partial D_y(U)}{\partial v_k} = -v \frac{\partial D_x(U)}{\partial v_k} + (f_v - u_1) V_{vk},
\]

\[
\frac{\partial D_y(V)}{\partial u_k} = -u \frac{\partial D_x(V)}{\partial u_k} + (g_u - v_1) U_{uk}.
\]

Now, by (58), we have

\[
\frac{\partial D_y}{\partial u_k} = \frac{\partial}{\partial u_{k-1}}, \quad \frac{\partial D_y}{\partial v_k} = \frac{\partial}{\partial v_{k-1}},
\]

\[
\frac{\partial D_x}{\partial u_k} = (f_u - k v_1) \frac{\partial}{\partial u_k} + (g_u - v_1) \frac{\partial}{\partial v_k} - v \frac{\partial}{\partial u_{k-1}},
\]

\[
\frac{\partial D_x}{\partial v_k} = (f_u - u_1) \frac{\partial}{\partial u_k} + (g_v - k u_1) \frac{\partial}{\partial v_k} - u \frac{\partial}{\partial v_{k-1}}.
\]

and, using (59) again, we arrive to

\[
-k v_1 U_{uk} + D_y(U_{uk}) = -v D_x(U_{uk}),
\]

\[
-k u_1 V_{vk} + D_y(V_{vk}) = -u D_x(V_{vk})
\]
and

\[(f_v - u_1)U_{u_k} + (v - u)U_{v_{k-1}} = (f_v - u_1)V_{v_k}, \quad (60)
\]

\[(g_u - v_1)V_{v_k} + (u - v)V_{u_{k-1}} = (g_u - v_1)U_{u_k}. \quad (61)
\]

Then, differentiating Equation (60) with respect to \(v_k\) and Equation (61) with respect to \(u_k\), we obtain

\[(f_v - u_1)V_{v_k v_k} = (g_u - v_1)U_{u_k u_k} = 0,
\]

i.e.,

\[U = au_k + b, \quad V = bv_k + d, \quad (62)
\]

where the functions \(a, b, c, d\) are of order \(k - 1\). Let us substitute the obtained expressions (62) to the defining system (56)–(57):

\[
D_y^{(s)}(a)u_k + aD_y^{(s)}(u_k) + D_y^{(s)}(b) = f_u(au_k + b) - v (D_x^{(s)}(a)u_k + au_{k+1} + D^{(s)}(b))
+ (f_v - u_1)(cv_k + d)
\]

\[
D_y^{(s)}(c)v_k + cD_y^{(s)}(v_k) + D_y^{(s)}(d) = g_v(cv_k + d) - u (D_x^{(s)}(c)v_k + cv_{k+1} + D^{(s)}(d))
+ (g_u - v_1)(au_k + b).
\]

Using the estimates (58) and comparing the terms containing \(u_{k+1}, v_{k+1}\) and \(u_k, v_k\), we see that the terms with \(u_{k+1}, v_{k+1}\) and \(u_k, v_k^2\) are cancelling, while

- in Eq. (63) at \(u_k v_k\): \(-ua_{v_{k-1}} = -va_{v_{k-1}}\),
- in Eq. (64) at \(u_k v_k\): \(-vc_{u_{k-1}} = -uc_{u_{k-1}}\),
- in Eq. (65) at \(u_k\): \(-kv_1 a = 0\),
- in Eq. (66) at \(v_k\): \(-ku_1 c = 0\),
- in Eq. (63) at \(v_k\): \((f_v - u_1)(a - c) = (u - v)b_{v_{k-1}}\),
- in Eq. (61) at \(u_k\): \((g_u - v_1)(c - a) = (v - u)d_{u_{k-1}}\).

In particular, from Equations (65) and (66) we see that the coefficients \(a\) and \(c\) vanish and thus, by virtue of (62), the functions \(U\) and \(V\) are of order \(k - 1\). We repeat the procedure until the order of the shadows at hand becomes equal to 1. \(\square\)

Using Proposition 16, we shall now prove that the symmetries \(\mathcal{S}^{(i)} = E_{Z^{(i)}}, i = -4, -3, \ldots\) exhaust all the polynomial symmetries in the covering \(\tau_x\).

**Theorem 3.** Any \(\tau_x\)-nonlocal symmetry of the Gibbons–Tsarev equation of weight \(k\), polynomial in all variables, coincides with \(\mathcal{S}^{(k)}\) up to a constant factor, \(k \geq -4\).

**Proof.** Using Equations (53), we pass from symmetries of the Gibbons–Tsarev equation (3) to those of System (51). The proof is accomplished by induction on the weight.

For small weights (\(|\mathcal{S}| = -4, \ldots, 0\)) this fact can be checked by direct computations due to Proposition 16.

Let us fix a \(k > 0\) and assume that for all weights less than \(k\) the statement is true. To proceed with the proof, we need a number of auxiliary facts. The first two of them can be observed from the results of Section 4.
Fact 1. For a symmetry $\mathcal{S}^{(i)} = E_{Z^{(i)}}$, one has the following ‘asymptotics’ in $\psi$s:

$$Z^{(1)} = -3\psi^{(4)} + \text{local terms}$$

and, for $i > 1$,

$$Z^{(i)} = -(i + 2)\psi^{(i+3)} + \frac{2i + 3}{2} x \psi^{(i+2)} + \Upsilon(i + 1), \quad i \geq 0,$$

where $\Upsilon(\alpha)$ denotes the terms independent of $\psi^{(\beta)}$ for $\beta > \alpha$. This means, by (53), that the corresponding generating section for the system is of the form

$$W^{(1)} = \frac{5}{2}\psi^{(3)}W^{(-3)} + \text{local terms}$$

and, for $i > 1$,

$$U^{(i)} = \frac{2i + 3}{2} \psi^{(i+2)}U^{(-3)} + (i + 1)\psi^{(i+1)}U^{(-2)} + \Upsilon(i),$$

$$V^{(i)} = \frac{2i + 3}{2} \psi^{(i+2)}V^{(-3)} + (i + 1)\psi^{(i+1)}V^{(-2)} + \Upsilon(i),$$

or

$$W^{(i)} = \frac{2i + 3}{2} \psi^{(i+2)}W^{(-3)} + (i + 1)\psi^{(i+1)}W^{(-2)} + \Upsilon(i), \quad (67)$$

where

$$W^{(-3)} = (u_1, v_1), \quad W^{(-2)} = \left( \frac{1}{v - u}, \frac{1}{u - v} - uv_1 \right),$$

are the generating sections of the infinitesimal $x$- and $y$-translations, respectively.

Fact 2. The shadow $W^{(-1)} = (1 - yu_1, 1 - yv_1)$ of the generalised Galilean boost extends to $\mathcal{E}_*$ as follows

$$\mathcal{S}^{(-1)} = \sum_{l \geq 0} \left( D_x^l (1 - yu_1) \frac{\partial}{\partial u_1} + D_x^l (1 - yv_1) \frac{\partial}{\partial v_1} \right) + (2x - yX^{(3)}) \frac{\partial}{\partial \psi^{(3)}} + \sum_{j > 3} ((j - 1)\psi^{(j-1)} - yX^{(j)}) \frac{\partial}{\partial \psi^{(j)}}.$$

Since the last expression can be rewritten in the form

$$\mathcal{S}^{(-1)} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} + y \frac{\partial}{\partial x} - yD_x^{(s)} + 2x \frac{\partial}{\partial \psi^{(3)}} + \sum_{j > 3} (j - 1)\psi^{(j-1)} \frac{\partial}{\partial \psi^{(j)}},$$

while

$$\ell_{W^{(-1)}} = \begin{pmatrix} -yD_x^{(s)} & 0 \\ 0 & -yD_x^{(s)} \end{pmatrix},$$

we obtain

$$\{W^{(-1)}, W\} = \begin{pmatrix} \frac{\partial U}{\partial u} + \frac{\partial U}{\partial v} + y \frac{\partial U}{\partial x} + 2x \frac{\partial U}{\partial \psi^{(3)}} + \sum_{j > 3} (j - 1)\psi^{(j-1)} \frac{\partial U}{\partial \psi^{(j)}} \\ \frac{\partial V}{\partial u} + \frac{\partial V}{\partial v} + y \frac{\partial V}{\partial x} + 2x \frac{\partial V}{\partial \psi^{(3)}} + \sum_{j > 3} (j - 1)\psi^{(j-1)} \frac{\partial V}{\partial \psi^{(j)}} \end{pmatrix} \quad (68)$$
for any $W = (U, V)$, and, in particular,
\[
\{W^{(-1)}, W^{(i)}\} = (i + 1)W^{(i-1)} + \Upsilon(i - 2), \quad i \geq 2.
\]

**Fact 3.** A straightforward, but important consequence of (63) is that the adjoint action $W \mapsto \{W^{(-1)}, W\}$ is a derivation, i.e.,
\[
\{W^{(-1)}, hW\} = h\{W^{(-1)}, W\} + X_{W^{(-1)}}(h)W, \quad h \in \mathcal{F}(\mathcal{E}),
\]
where
\[
X_{W^{(-1)}} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} + y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial \psi(j)} + \sum_{j \geq 3} (j - 1)\psi^{(j-1)} \frac{\partial}{\partial \psi^{(j)}}
\]
is a vector field on $\mathcal{E}$.

**Fact 4.** Let $W = (U, V)$ be a solution of System (36)-(37) of weight $k$. Let also $l$ be the minimal integer such that $\partial U/\partial \psi^{(j)}$ and $\partial V/\partial \psi^{(j)}$ vanish for all $j > l$. Recall (see Subsection 3.2.1) that the ‘nonlocal tails’ of the total derivatives $D^*_{x}$ and $D^*_{y}$ on $\mathcal{E}$ are of the form
\[
X = \sum_{j \geq 3} X^{(j)} \frac{\partial}{\partial \psi^{(j)}}, \quad Y = \sum_{j \geq 3} Y^{(j)} \frac{\partial}{\partial \psi^{(j)}},
\]
where
\[
X^{(j)} = -(j - 3)\psi^{(j-3)} + \Upsilon(j - 4), \quad Y^{(j)} = -(j - 2)\psi^{(j-2)} + \Upsilon(j - 4).
\]
This implies that if $W = (U, V)$ and $l$ is chosen as above, then
\[
\frac{\partial W}{\partial \psi^{(l)}} = \left( \frac{\partial U}{\partial \psi^{(l)}}, \frac{\partial V}{\partial \psi^{(l)}} \right), \quad \frac{\partial W}{\partial \psi^{(l-1)}} = \left( \frac{\partial U}{\partial \psi^{(l-1)}}, \frac{\partial V}{\partial \psi^{(l-1)}} \right)
\]
are shadows as well (of weights $k - l - 1$ and $k - l$, respectively).

Let us now return to the main course of the proof. Since $k - l < k$, then due to the induction hypothesis we have
\[
\frac{\partial W}{\partial \psi^{(l)}} = \alpha W^{(k-l-1)} = \alpha \left( \frac{2k - 2l + 1}{2} \psi^{(k-l+1)}W^{(-3)} + (k - l)\psi^{(k-l)}W^{(-2)} \right) + \Upsilon(k + l - 1),
\]
where $\alpha \in \mathbb{R}$ is a nonvanishing constant. Note also that due to the definition of $l$ one has $l \geq k - l + 1$, or
\[
2l \geq k + 1.
\]
We now consider two cases: Inequality (70) is either strict or an equality.

**The case** $2l > k + 1$. In this case, Equation (69) implies
\[
W = \alpha W^{(k-l-1)}\psi^{(l)} + \Upsilon(l - 1).
\]
Let us apply the operator $\{W^{(-1)}, \cdot\}$ to both sides of (71):
\[
\{W^{(-1)}, W\} = \alpha \{W^{(-1)}, W^{(k-l-1)}\}\psi^{(l)} + \alpha W^{(k-l-1)}X_{W^{(-1)}}(\psi^{(l)}) + \Upsilon(l - 2)
\]
\[
= \alpha \{W^{(-1)}, W^{(k-l-1)}\}\psi^{(l)} + \alpha(l - 1)W^{(k-l-1)}\psi^{(l-1)} + \Upsilon(l - 2).
\]
But \( \{W^{(-1)}, W\} = k - 1 \) and, by the induction hypothesis, we must have \( \{W^{(-1)}, W\} = \beta W^{(k-1)} + \Upsilon(k - 2) \). Consequently, using (67) and (68), we obtain

\[
\beta \left( \frac{2k + 1}{2} W^{(k-1)} \psi^{(k+1)} + \Upsilon(k) \right) = \alpha(k - l) W^{(k-l-2)} \psi^{(l)} + \alpha(l - 2) W^{(k-l-1)} \psi^{(l-1)} + \Upsilon(l - 2).
\]

The last equality can hold only when

\[
l = k + 1, \quad l = -\frac{2k + 1}{2} \beta.
\]

Consider the shadow \( \tilde{W} = W - \beta W^{(k)} \). There are two possibilities: (a) \( \tilde{W} = 0 \) and then the proof is finished; (b) \( \tilde{W} \neq 0 \) and then there should exist the minimal integer \( \tilde{l} < l \) such that \( \partial \tilde{W}/\partial \psi^{(l)} \neq 0 \). The only possibility is \( \tilde{l} = (k + 1)/2 \) and thus we pass to the second case.

**The case** \( 2l = k + 1 \). Now Equation (69) reads

\[
\frac{\partial W}{\partial \psi^{(l)}} = \alpha \frac{2l - 1}{2} \psi^{(l)} W^{(l-3)} + \Upsilon(l - 1),
\]

or

\[
W = \alpha \frac{2l - 1}{4} (\psi^{(l)})^2 W^{(l-3)} + \text{terms linear in } \psi^{(l)} + \Upsilon(l - 1).
\]

Let us apply the operator \( \{W^{(-1)}, \cdot\} \) to the last equation. Then in the left-hand side we obtain a shadow of weight \( k - 1 = 2l - 2 \) which, by the induction hypothesis and Equation (67), must be proportional to \( \psi^{(2l)} W^{(-3)} + \Upsilon(2l - 1) \). But in the right-hand side such a term cannot appear. This contradiction finishes the proof. \( \square \)

5.2. **Uniqueness of invisible symmetries.** Consider a symmetry

\[
\mathcal{S} = \sum_{i \geq 0} \left( (D_x^{(s)})^i (U) \frac{\partial}{\partial u_i} + (D_x^{(s)})^i (V) \frac{\partial}{\partial v_i} \right) + \sum_{\alpha \geq 3} \Psi^{(\alpha)} \frac{\partial}{\partial \psi^{(\alpha)}}
\]

of \( \mathcal{E}_s \). Let us say that \( S \) is invisible of depth \( k \) if \( U = V = \Psi^{(3)} = \cdots = \Psi^{(k-1)} = 0 \), i.e.,

\[
\mathcal{S} = \sum_{\alpha \geq k} \Psi^{(\alpha)} \frac{\partial}{\partial \psi^{(\alpha)}}.
\]

The defining equations for such symmetries are\( \{\mathcal{S}, D_x^{(s)}\} = \{\mathcal{S}, D_y^{(s)}\} = 0 \), or

\[
D_x^{(s)}(\Psi^{(\alpha)}) = \sum_{\beta = k}^{\alpha - 3} \Psi^{(\beta)} \frac{\partial X^{(\alpha)}}{\partial \psi^{(\beta)}}, \quad D_y^{(s)}(\Psi^{(\alpha)}) = \sum_{\beta = k}^{\alpha - 2} \Psi^{(\beta)} \frac{\partial Y^{(\alpha)}}{\partial \psi^{(\beta)}}
\]

for all \( \alpha \geq k \), where, as before,

\[
D_x^{(s)} = D_x + \sum_{\alpha \geq 3} X^{(\alpha)} \frac{\partial}{\partial \psi^{(\alpha)}}, \quad D_y^{(s)} = D_y + \sum_{\alpha \geq 3} Y^{(\alpha)} \frac{\partial}{\partial \psi^{(\alpha)}},
\]

and \( X^{(\alpha)}, Y^{(\alpha)} \) are the right-hand sides of (16) and (17), respectively.
Theorem 4. Any nontrivial invisible symmetry of depth $k$ is of the form
\[
\frac{\partial}{\partial \psi^{(k)}} + \gamma \frac{\partial}{\partial \psi^{(k+1)}} + \sum_{\alpha \geq k+2} \Psi^{(\alpha)} \frac{\partial}{\partial \psi^{(\alpha)}},
\]
where $\gamma = \text{const}$.

Proof. Indeed, the right-hand sides of Equations (72) vanish for $\alpha = k$ and $\alpha = k+1$, i.e.,
\[
D_x^{(\alpha)}(\Psi^{(k)}) = D_x^{(\alpha)}(\Psi^{(k+1)}) = 0, \quad D_y^{(\alpha)}(\Psi^{(k)}) = D_y^{(\alpha)}(\Psi^{(k+1)}) = 0.
\]
But, by Proposition 3 the equation $\mathcal{E}_\ast$ is differentially connected and thus $\Psi^{(k)}$ and $\Psi^{(k+1)}$ are constants. \hfill \square

Remark 9. Actually, one can say more about the structure of the coefficients $\Psi^{(\alpha)}$ (see Equation (45)), but for our cause the above said is sufficient.

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