Analysis of internal and external mathematical representation ability to senior high school students in Indonesia

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Abstract. This article discusses the students' ability to solve mathematical representation problems based on the student's initial ability level. The goal is to know the ability of internal representation that is how the students think in solving mathematical problems and external representation, how to construct new ideas in poured by students in the form of solving the problem description obtained from internal representation. This study used descriptive qualitative research, by giving three problems of mathematical representation to three students with female gender in Senior High School 1 Cibadak Sukabumi class X based on different Mathematical Preliminary Ability, with age about 15 years. The data in this research is test and interview to each student. The results of interviews with students are described in descriptive form or words to describe the facts, definition or answers questions that are then taken its essence alone. The results obtained that the ability of the external representation is influenced by internal representation, students whose initial ability is high, have a good external mathematical representation and the ability of internal mathematical representation is positive. Students who have the moderate ability, the result of external representations is sufficient and internal representations are positive. While students with low ability, external representation ability are less and their internal representation is negative. So the internal representation in this case of student's thinking habits influences the student's success in solving mathematical problems in his own way from the abilities he possesses in his mind. But not all of the students' thoughts can be expressed in their external representations since the limitation of the students in expressing the results of thought in the form of writing is still less than optimal. This is due to the thinking habits of students who still have to be developed in the learning process. So in need of intervention from teachers to solve it, one of them by applying an effective learning model to be able to improve the ability of representation.

1. Introduction

In Indonesia in the national curriculum high school level puts mathematics as subjects with the most hourly portion, that's mean to develop students' mathematical abilities not only to solve problems in mathematics, but students are trained how to develop their thinking ability to solve problems related to other subjects And problems in daily life. All the problems presented in the book students have linked the material as well as the problems associated with the real problem. The objects in mathematics are objects that can only be accessed through its representation and mathematical thinking requires the use of multiple representations \cite{1}. The statement indicates that in learning mathematics requires the ability to interpret and construct a representation. However, students have not been able to present the problems given by the teacher to different contexts, that mean students are only able to understand
knowledge but have not been able to apply the knowledge they have maximally into other contexts such as in daily life and can not choose alternative answers or solution is considered the most effective to use in accordance with their ability.

Teachers often emphasize only aspects of the mathematical process compared to their application in everyday life [1]. Students can not predict the phenomenon given by the teacher to be interpreted, to understand the information given so that the concept of students who previously owned as if it can not be used and maximized during the process. Based on constructivist theory, learning is seen as the compilation of knowledge from concrete experience, activity, collaborative, reflection and interpretation [2].

Mathematics as one of the disciplines has an important role in developing the ability of students including the ability to think. Thinking habits as a tendency to behave intellectually or intelligently when faced with problems, especially problems that are not immediately known to the solution [3]. When faced with problems, students tend to form certain intellectual behavior patterns that can encourage individual success in solving the problem.

The ability to read math problems to understand the problems that the real world context of the students and the story is still not as good as when they read mathematical problems in the context of algebra or direct count, in this case reading math done the students while thinking how to understand the problem, then translate the understanding into the form of symbols/ symbols of mathematics, as well as tell / write back the contents of the matter with his own words. This happens not only in low-ability students, even high-grade students when faced with this become the same problem.

According to Jones and Knuth suggests that a model or a substitute form of a problem situation or aspect of a problem situation used to find a solution is a representation [4]. In general, all mathematical material requires the ability of mathematical representation, that is the way someone used to communicate answers or mathematical ideas in question [5].

The representations raised by the students are expressions of the mathematical ideas that students display in their quest to find a solution to the problem they are facing [6]. The actual representation not only refers to the results or products embodied in new or different configurations or constructs but also the thought processes are undertaken to capture and understand the concepts, operations, and the mathematical relation of a configuration. That is, the process of mathematical representation takes place in two stages that is internal and external. In line with Hiebert and Wearne that the development of relations between external representations will foster a more integrated understanding of concepts and internal representations of mathematical ideas [7].

Internal representation can not be observed visually and can not be judged directly because it is a person's mental activity in his mind. In other words, a person who performs an internal representation process in learning mathematics will think about the ideas, notions, or mathematical concepts he is studying in order to clearly understand and understand the problem, linking and linking the problem with the knowledge it already possesses, and strategizing the solution. To understand the more important mathematical concept is not the storage of past experiences but how to recover the knowledge that has been stored in the memory and relevant to the needs and can be used when necessary. Furthermore, the process of gaining relevant knowledge and its use is closely related to the coding of past experiences.

Thinking of a mathematical idea which is then communicated requires an external representation that manifests what students do internally or representation internal. The results of the embodiment are expressed in the form of words, graphs, tables, and statements which are an approach which gives a thought in the free translation by the students to learn mathematical concepts [8].

The success of the effort to construct the problems of a representation is influenced by the student's reflection on the knowledge he possesses and the role of that knowledge in the constructed representation [9]. The statement indicates that the student's initial knowledge affects the student's representational abilities. In addition to the cognitive aspects of students, this psychological aspect contributes to and contributes to the success of a person in completing tasks/ problems well. Developing the ability of mathematical representation by cultivating the habit of creative thinking in
line with the opinion of Sternberg which views creativity as a habit [10]. This is in line with the objectives of the National Curriculum, which is preparing the nation's generation to have the ability to live as productive, creative, innovative and effective citizens and individuals.

The use of appropriate and adequate representations will have a very large contribution to the formation of understanding (understanding or meaning) concept. Appropriate in the sense of being suitable to represent the concept, and adequate in the sense of sufficient quantity to enable students to find a linkage, either between representation or in one type of representation.

2. Research Purposes
Based on the problems that have been explained earlier, it seems important to see the ability of internal mathematical representation of how students think in solving mathematical problems and external representation, how the new construction that pours students in the form of problem-solving obtained from its internal representation.

3. The Ability of Mathematical Representation
Representation is a configuration of forms or arrangements that can illustrate, represent or symbolize something in a way [11]. The actual representation not only refers to the results or products that are presented in new or different configurations or constructs but also the thought processes undertaken to acquire and understand the concepts, operations and mathematical relations of a configuration.

Thomas and Hong argue that a representation can be seen as a multi-faceted construction that assumes different roles depending on how the students interact with the representation [12]. Hiebert and Carpenter argue that basically representations can be distinguished in two forms, that is 1) internal representation, thinking about mathematical ideas that enable one's mind to work on the basis of the idea [7]. To understand the more important mathematical concept is not the storage of past experiences but how to recover the knowledge that has been stored in the memory and relevant to the needs and can be used when necessary. Furthermore, it is also explained that the process of gaining relevant knowledge and its use is closely related to the coding of the past experience. This process is called internal representation because it is one of mental activity; 2) external representation, thinking about mathematical ideas that are then communicated requires an external representation of verbal form, images, and concrete objects.

The form of outcomes in education consists of four levels: the first level mastered certain content, the second level mastered thinking skills, the third level mastered the cognitive tasks that demanded full thinking skills, and the fourth level is thinking habits [13]. Habit is not a behavior that we use or place it casually or at will [4]. Habits are behaviors that we show well at the right times and work just like that without us even bothering to try.

4. Methodology
This research is descriptive-qualitative research, with the objective to see the mathematical representation of the students ability in the form of external representation can be seen based on the result of the students answer in resolving the matter in the form of its internal representation that is based on the description of how their way of thinking in pouring the idea based on the habits of thinking to solve the problem that can’t be visible by naked eye and cannot be assessed directly because it is a person’s mental activity in their mind. In other words, a person who performs the process of internal representation in learning mathematics will think about ideas or mathematical concepts that can be revealed.

The subject consists of three students with female gender in Senior High School 1 Cibadak - Sukabumi, West Java, Indonesia, class X based on different Mathematical Preliminary Ability, with age about 15 years representing the Mathematical Preliminary Ability that are high, medium, and low. The score used to determine the category of Mathematical Preliminary Ability student’s come from some of the previous daily test scores. The reason for choosing High School students in this research
because at this level, they already have a broader way of thinking, already have a habit of thinking in solving a problem based on experience in the previous level of primary and secondary schools.

The given problem is designed based on the mathematical representation of capability indicators, namely: visual representation, mathematical expressions, and words or written text. The three students were asked to answer three representational questions. Then interviewed using semi-structural approach.

Data collection of interviews between with researchers and students is taken after the students finished answering the question. The interview process in which the researcher observe and discover information through some of the questions to view the internal representation of students namely how to think and find ideas in solving problems that are given based on the habits of thinking according to the indicators in the Clune consisting of 7 aspects: 1) striving for accuracy (Always do the best, set high standards, examine and find ways to improve continuously); 2) manage conscience (think before acting, stay calm); 3) habits of diligent (Diligent in the task through completion, given focused, finding ways to achieve goals when stuck, not giving up); 4) metacognitive thinking habits; 5) listening with great empathy (To devote mental energy to the thoughts and ideas of others, to make an effort to understand other points of view and emotion); 6) taking responsible risks (try new things constantly); And 7) applying past knowledge to new situations (Accessing prior knowledge, transferring knowledge outside of the situations where he studied) [3]. The results of interviews with students are described in descriptive form or words to describe the facts, definition or answers questions that are then taken its essence alone.

5. Results and Discussion

Based on the test score ability of mathematical representation to know the ability of external representation with the interpretation based on Purwanto from each student obtained data [14]:

| No | Code | Mathematical Preliminary Ability | Score (Max = 12) | Percentage | Interpretation |
|----|------|----------------------------------|------------------|------------|----------------|
| 1  | S1   | High                             | 10               | 83%        | Good          |
| 2  | S2   | Medium                           | 8                | 67%        | Enough        |
| 3  | S3   | Low                              | 4                | 33%        | Less          |

To analyze the ability of the external representation of students based on test result. Students with a high Mathematical Preliminary Ability (S1) are presented in Figure 1.

In Figure 1, the results of the test on the external representation of the mathematical ability on writing S1 on the number 1 with a form of visual representation, S1 is able to make the triangle images appropriately based on the information provided in the story and create procedures and conclude the issue contextually related to trigonometric ratio. Question the number 2 with mathematical expression, S1 is able to make mathematical equations using the formula of cosine rules and by correctly calculate the length of the tunnel in question. Whereas, in question the number 3 with the form of words, S1 is capable of writing the contextual interpretation of the question in the form of the image with his steps with the words but the solution and the calculations are less precise.

To analyze the ability of the internal representation of the student’s way that is how students solve a test by using their habit of thinking. An analysis of the student's external representation with medium Mathematical Preliminary Ability (S2) is presented in Figure 2.
Figure 1. Results of answers about the test of mathematical representation ability S1

The external representation of the mathematical ability of S2 as seen from the result of the answer, question no. 1 with a form of visual representation, S2 is able to make the image of triangle appropriately based on the provided on the story and create procedures and conclude the issue contextually related to trigonometric ratio. Problem the number 2 with the form of mathematical expression, S2 is able to create mathematical equations using the formula of cosine rules and less precise in calculating the length of the tunnel in question. Whereas, the number 3 with a form of words, S2 is capable of writing the contextual interpretation of the question in the form of the image with the steps are either the words but the solution and calculations are not precise. The written test result of students with a low Mathematical Preliminary Ability (S3) is presented in Figure 3.

In Figure 3, for the matter of no 1 with the visual representation, S3 just draw a triangle but is not able to make the proper image of the triangle based on the information given on the story and could not make the procedure and conclude the issue contextually related to trigonometric ratio. For question the number 2, S3 does not fill it at all. As for the question the number 3 with the form of words, the S3 is not capable of writing the contextual interpretation of the question of the form of the image with the steps with words could not finish the calculation, S3 just write down what is known of the problem only.

Based on the results of interviews can be inferred that the managing impulsivity S1 and S2 which is done the first time after the problem get reserved, they read the first question that there is a new estimate used to be the answer, and then solving it. But on S3 which is done first is to directly answer the question according to what is known without thinking, so in terms of managing the conscience, S1 and S2 tend to positive and negative for S3. Of the three students, all do the same thing in applying the knowledge of the past, they work on the problem by changing the problem into the form of images to make it easier in the calculation based on the concept that he had known before.

When the students faced a situation of mathematical problems in classroom learning, they will try to understand the problem and solve it in the ways they know. These ways are closely linked to previously existing knowledge relating to the problem presented. One part of the effort that students can do is to create a model or representation of the problem. Models or representations that are made can vary depending on the ability of each individual in interpreting existing problems. Habituation in thinking to represent a problem is based on the design of learning that commonly used in the
classroom. Learning design is useful for student empowerment, involvement, motivation and improves communication and thinking skills [15].

At striving for accuracy, S1 and S2 always do the reexamination of answers they have done, even S2 finds errors in calculating and repeating again, but S3 did not do the re-examination of answers from the steps in doing. In the habit of surviving, the three students are always looking for other alternatives in completing all the given questions, especially in S1 and S2 when they forget what kind of concept that used to challenge the problem, they will use other alternatives that they remember.

When working on the questions, S1 and S2 are always thought and ask themselves whether the way of completion is done right and suitable to work on any number of questions. However, based on S3’s admission, he never asks whether he thinks fit or not the solution he is working on, the most important thing about that he can fill even though the answer is true or false. The need for positive attitudes and perceptions of learned knowledge and ability [16]. Building the habit of thinking is a tool for self-management, by turning knowledge into taking initiative on its own behalf, resulting in active learning exercises [17].

![Figure 2. Results of answers about the mathematical representation test of S2](image-url)
On listening with understanding and empathy, based on the recognition of S1 after completing the question, he did not discuss the results of his answer and forget the given problem because he felt the answer and his results are sure to be true. In contrast to S2 and S3 after the completion of the finished work, they discuss the results of his answer and exchange opinions on their different answers and discuss which answers are most appropriate. S1 working on the three questions given and confident with the results obtained, S2 working on these three questions but he did not confident of number 3 due to insufficient time and long enough to understand the purpose given problem, while the S3 he tried to do 2 questions, but he was not sure of what he had done and he collected the results of the answer 15 minutes before the given time was over because the S3 felt can not do it anymore.

The ability of students' internal representation of mathematics is seen from the habit of thinking, students with high initial ability and moderate have positive thinking habits, whereas students whose initially low ability of thinking habit tend to negative. In S3 negative thinking habits that arise among him, the first thing that is did after getting the question, the student immediately do without thinking long according to what he knows without predicting the answer, do not think and ask yourself whether the answer is done already True or suitable in solving the given problem and not check back the results of answers that he did, so that when there is mistake write or calculate is not corrected, the only answer the 2 questions and did not complete and collect the answers when time is not over, even though the result is not maximum.

So the solution of student representation which is poured in the form of description problem is very dependent on the internal representation of the students, in this case, the habit of thinking. It appears that the actual representation not only refers to the results or products that are presented in new or different configurations or constructs but also the thought processes undertaken to acquire and understand the concepts, operations and mathematical relationships of a configuration. Because the students who ability initially medium, the habit of thinking is positive but the result of its external representation is sufficient because the time in doing the problem is not well organized and careless in the calculation of representation. Because, good and positive thinking habits must comprise seven aspects according to Clune in the seven aspects referred to in this case are indicated in the interview guidelines [3].

Ultimately the habitual setting of the thinking process is a way to open the mind space as the place of the process takes place. In line with the statement of Costa and Kallick that the habit of thinking
students really become the foundation of students in the course of learning [3]. A person who performs an internal representation process in learning mathematics will think about the ideas, ideas, or mathematical concepts he is studying in order to clearly understand the problem, to link the problem with the knowledge it has, and to develop its solution strategy. Through the disclosure of student-made completion methods, encouraging their conceptual understanding, and by developing the child's meta-rational thinking ability can be effectively increased [18].

Students need to have good thinking habits to be able to respond to any problems that arise in learning. Student thinking habits at the time of learning become fundamental when they get a little problem, represent the problem and they have to find a solution to what the solution is. Habits are also very supportive of student performance in everyday life. The habit of thinking is the root of students' strength in exercising their ability in determining the solution of a problem. The classroom is a condition or environment that they occupy as they study.

According to Aristotle, individual success is determined by the habits he does [19]. Meanwhile, according to Slavin revealed that a person is considered to have learned something if he can show changes in his behavior [20]. This suggests that the orientation of learning in addition to developing the knowledge and skills of students, namely to develop attitudes. Attitude is a factor that greatly influences a person's behavior or actions in dealing with tasks, including academic tasks. Learning approach with peer tutors can improve students' mathematics learning regardless of gender, can add students' abilities, interest in mathematics learning and their own mathematical efficiency [21]. This means that in the process of learning mathematics, the realm of attitude is necessary and continues to be developed optimally in the students themselves. This is because in learning mathematics and living everyday life, students are always dealing with multiple problems.

Mathematics learning in the classroom should provide sufficient opportunities for students to be able to train and develop the ability of mathematical representation as an important part of solving the problem. The problems presented are tailored to the content and depth of the material at each level by paying attention to the initial knowledge or the prerequisites and habits of thinking that the students have.

In the process of learning and learning in the classroom, the student must be accustomed to solving the problem of learning independently to find something useful in himself and with ideas because the essence of constructivist theory is the idea that students, individually, must find and transform information Complex if they make it themselves [2][22][23][24]. When students have been able to find information and be able to convey complex information to other situations, then learning and learning must be packaged into a process of "constructing," not "receiving" knowledge. Therefore, according to Marzano shows that the habit of thinking is one dimension of learning that needs to be developed and measured further than just mastering the concept and application [16][25].

6. Conclusion
The results of this study indicate that students who have different mathematical initial abilities in solving problems to find out the ability of external mathematical representation produce answers with various representations. This diversity is influenced by the ability of early mathematics and is a manifestation of the strategy of completion of students in solving the problem of representation test that visual ability, mathematical expression, and words given by the habits of thinking of each student. In addition, external representation is influenced by internal representation, students whose initial ability is high in both mathematical external representation and the ability of internal mathematical representation is positive. Students whose initial abilities are adequately externalized and internal representations are positive. Whereas students with low external representation ability are poor and their internal representation is negative. Each student has a distinctive way of thinking. Habits of thinking affect the process of internal representation of students. This student's thinking habits influence the student's success in solving mathematical problems in his own way from the abilities he possesses in his mind. But not all of the students' thoughts can be expressed in their external representations since the limitation of the students in expressing the results of thought in the form of
writing is still less than optimal. This is due to the thinking habits of students who still have to be developed in the learning process. So in need of intervention from teachers to solve it, one of them by applying an effective learning model to be able to improve the ability of representation. Such as cooperative learning or peer instruction.

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