ON FILTERING OF MARKOV CHAINS IN STRONG NOISE

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ABSTRACT. The filtering problem for finite state Markov chains is revisited in the low signal-to-noise regime. We give a description of conditional measure concentration around the invariant distribution of the signal and derive asymptotic expressions for the performance indices of the MMSE and MAP filtering estimates.

1. INTRODUCTION

Consider the discrete time signal/observation pair \((X,Y) = (X_n,Y_n)_{n \in \mathbb{Z}_+}\), where the signal \(X\) is a finite state Markov chain with values in a real alphabet \(S = \{a_1, ..., a_d\}\), transition probabilities \(\lambda_{ij} = P(X_n = a_j | X_{n-1} = a_i)\) and initial distribution \(\nu\). The observation sequence \(Y\) is generated by

\[ Y_n = \sum_{i=1}^{d} 1_{\{X_n = a_i\}} \xi_n(i), \quad n \geq 1 \quad (1.1) \]

where \(\xi\) is a sequence of i.i.d. random vectors, independent of \(X\). Without loss of generality the probability laws of the entries of \(\xi_1\) can be assumed to have densities \(g_i(u), i = 1, ..., d, u \in \mathbb{R}\) with respect to a \(\sigma\)-finite measure \(\psi(du)\) on \(\mathbb{R}\) (typically the Lebesgue measure or purely atomic measure). Hereafter all the random variables are assumed to be supported on a complete probability space \((\Omega, \mathcal{F}, P)\).

This setting is often referred as Hidden Markov Model and is frequently encountered in information sciences (see e.g. the recent survey [5]). An important statistical problem related to HMM is filtering, i.e. estimation of the signal \(X_n\), given the observation trajectory \(Y\) up to time \(n\). The main building blocks of this estimation problem are the conditional probabilities \(\pi_n(i) = P(X_n = a_i | F_Y^n)\), where \(F_Y^n = \sigma\{Y_m, m \leq n\}\) is the \(\sigma\)-algebra of events generated by the observations. In particular the minimum mean square error (MMSE) and maximum a posterior probability (MAP) estimates of \(X_n\) are given by

\[ \hat{X}_{n,\text{mse}} = \sum_{i=1}^{d} a_i \pi_n(i) \quad \text{and} \quad \hat{X}_{n,\text{map}} = \arg\max_{a_i \in S} \pi_n(i). \quad (1.2) \]

The vector \(\pi_n\) satisfies the recursive Bayes formula, called the filtering equation,

\[ \pi_n = \frac{G(Y_n)\Lambda^* \pi_{n-1}}{|G(Y_n)\Lambda^* \pi_{n-1}|}, \quad \pi_0 = \nu, \quad (1.3) \]

where \(\Lambda^*\) is the transposed matrix of transition probabilities \(\lambda_{ij}, G(y), y \in \mathbb{R}\) is the scalar matrix with entries \(g_i(y)\) and \(|x|\) stands for the \(\ell_1\)-norm, i.e. \(|x| = \sum_{i=1}^{d} |x_i|\). As usual we identify the probability measures and functions on \(S\) with vectors from

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the simplex \( S^{d-1} = \{ x \in \mathbb{R}^d : x_i \geq 0, \sum_{i=1}^d x_i = 1 \} \) and \( \mathbb{R}^d \) respectively and use the notation \( \eta(f) = \sum_{i=1}^d f(a_i) \eta_i = f^* \eta \) for \( f : \mathbb{S} \to \mathbb{R} \) and \( \eta \in S^{d-1} \).

While the recursion (1.3) provides an efficient way to calculate the estimates in (1.2), no closed form formulae are known for the corresponding performance indices: the minimal mean square error

\[
\mathcal{E}_n = \min_{\theta \in L^2(\Omega, \mathcal{F}_n\mathcal{Y}^d, \mathcal{P})} \mathbb{E}(X_n - \theta)^2 = \mathbb{E}(X_n - \hat{X}_{n\text{mse}})^2 = \mathbb{E}X_n^2 - \mathbb{E}(\pi_n(a))^2
\]

and the minimum a posterior error probability

\[
\mathcal{P}_n = \min_{\theta \in L^\infty(\Omega, \mathcal{F}_n\mathcal{Y}^d, \mathcal{P})} \mathbb{P}(X_n \neq \theta) = 1 - \max_{a_i \in \mathcal{S}} \pi_n(i).
\]

and hence approximations of these quantities are of significant interest.

It is not hard to see that the random sequence \( \pi_n \) is a Markov process with values in \( S^{d-1} \). Under mild assumptions it is also a Feller process and hence it has at least one invariant measure \( \mathcal{M}_\pi(d\eta) \) (on the Borel field of \( S^{d-1} \)). The uniqueness of this measure is not at all obvious and in fact may fail if no restrictions are imposed on the noise densities, even when the signal \( X \) itself is ergodic (see a discussion in [2]). Recall that a Markov chain on \( \mathcal{S} \) is ergodic, if the limit probabilities \( \mu_i := P(X_n = a_i), i = 1, ..., d \) exist, are unique and positive. The sufficient and necessary condition for ergodicity is that the matrix \( \Lambda^q \) has positive entries for some integer \( q \geq 1 \) and then \( \mu \) is the unique solution of \( \Lambda^q \mu = \mu \) in \( S^{d-1} \). The invariant measure \( \mathcal{M}_\pi \) of \( \pi_n \) is unique, i.e. independent of \( \nu \), if \( X \) is ergodic and the noise densities are bounded and have the same support (see [1]). In this case the limits

\[
\mathcal{E} := \lim_{n \to \infty} \mathcal{E}_n \quad \text{and} \quad \mathcal{P} := \lim_{n \to \infty} \mathcal{P}_n
\]

exist and do not depend on \( \nu \).

Though these “steady state” optimal errors cannot be calculated exactly, they are amenable to asymptotic approximations, as the one obtained by R.Khasminskii and O.Zeitouni in [7] and G.Golubev in [6]. Suppose that the transition probabilities satisfy

\[
\lambda_{ij}^\varepsilon = \begin{cases} 
1 - \varepsilon \sum_{\ell \neq j} \lambda_{\ell j}, & i = j \\
\varepsilon \lambda_{ij}, & i \neq j
\end{cases}
\]

with a small parameter \( \varepsilon \in (0,1) \), which controls the transitions rate of the corresponding slow chain \( X_n^\varepsilon \) (note that the invariant measure of \( X^\varepsilon \) does not depend on \( \varepsilon \) and equals \( \mu \)). The observation process \( Y^\varepsilon \) is given by (1.1) with \( X \) replaced with \( X^\varepsilon \) and \( \pi_n^\varepsilon \) is the solution of (1.3) with \( Y \) and \( \Lambda \) replaced with \( Y^\varepsilon \) and \( \Lambda^\varepsilon \) respectively. It is proved in [7], that if all the Kullback-Leibler divergences

\[
\mathcal{D}(g_i \| g_j) = \int_{\mathbb{R}} g_i(u) \log \frac{g_i(u)}{g_j(u)} \psi(du)
\]

are finite and positive, the error probability\(^1\) \( \mathcal{P}^\varepsilon \) converges to zero as \( \varepsilon \to 0 \) and

\[
\mathcal{P}^\varepsilon = \left( \sum_{i=1}^d \mu_i \sum_{j \neq i} \frac{\lambda_{ij}}{\mathcal{D}(g_j \| g_i)} \right) \varepsilon \log \varepsilon^{-1}(1 + o(1)), \quad \varepsilon \to 0. \quad (1.4)
\]

\(^1\)throughout superscripts are added to various quantities to emphasize their dependence on the corresponding parameter
Similar asymptotic holds for the minimal mean square error as shown in [3]:

\[ \mathcal{E} = \left( \sum_{i=1}^{d} \sum_{j \neq i}^{d} \mu_i \frac{\lambda_{ij}}{D(g_j \| g_i)} (a_i - a_j)^2 \right) \varepsilon \log \varepsilon^{-1} (1 + o(1)), \quad \varepsilon \to 0. \] (1.5)

These results give an idea of how fast the invariant measure \( M_\varepsilon \pi(d\eta) \) concentrates around \( M_0 \pi(d\eta) = \sum_{i=1}^{d} \mu_i \delta_{p_i}(d\eta) \), where \( p_i \) are probability vectors with 1 at the \( i \)-th entry.

In a sense the slow chain limit is the counterpart of the weak noise asymptotic \( \sigma \to 0 \) for the additive observation model (cf. (1.1))

\[ Y_\sigma^n = h(X_n) + \sigma \xi_n, \quad n \geq 1, \] (1.6)

where \( \xi \) is a sequence of i.i.d. random variables, independent of \( X \), \( h \) is an \( \mathbb{S} \to \mathbb{R} \) function and \( \sigma \) is the constant, controlling the noise intensity. Though less apparent in the discrete time setting, the analogy is complete for continuous time model, as explained in Section 2 below. In this paper the strong noise asymptotic is addressed, when the filtering probabilities \( \pi_\sigma \nu \) converge to the a priori distribution of the signal \( \nu_n = (\Lambda^*)^n \nu \) as \( \sigma \to \infty \). Thus in the stationary case we deal with the concentration of \( M_\sigma \pi(d\eta) \) around \( M_\infty \pi(d\eta) = \delta_\mu(d\eta) \) as \( \sigma \to \infty \). The precise formulation of the results is given in Section 2 which are proved in Sections 3 and 4.

2. Main results

2.1. Discrete time. Let \( (X, Y^\sigma) \) be the filtering model, with \( X \) being a finite state Markov chain on \( \mathbb{S} \) with transition probabilities matrix \( \Lambda \) and initial distribution \( \nu \) and suppose that \( Y^\sigma \) is generated by (1.6).

**Theorem 2.1.** Assume that the probability law of \( \xi_1 \) has a bounded twice continuously differentiable density \( g(u) \) with respect to the Lebesgue measure on \( \mathbb{R} \) with bounded continuous derivatives. Then the solution of \( (1.3) \) converges to \( \nu_n = (\Lambda^*)^n \nu \) as \( \sigma \to \infty \) and

\[ \sigma (\pi_\sigma^n - \nu_n) \xrightarrow{\text{P-}a.s.} Z_n, \quad n \geq 0 \]

where \( Z_n \) satisfies

\[ Z_n = \Lambda^* Z_{n-1} - (\text{diag}(\nu_n - \nu_n \nu_n^*) h \frac{g'(\xi_n)}{g(\xi_n)}, \quad Z_0 = 0. \] (2.1)

The following two theorems give asymptotic expressions for \( \mathcal{E}^\sigma \) and \( \mathcal{P}^\sigma \).

**Theorem 2.2.** Assume that \( X \) is an ergodic chain and \( g \) satisfies the following conditions

- \( (a_1) \) \( g(u) \) does not vanish on \( \mathbb{R} \), is bounded and has two bounded derivatives
- \( (a_2) \) there is a \( \delta > 0 \), so that

\[ \int_{-\infty}^{\infty} \left( \frac{g'(x)}{\min_{|u| \leq \delta} g(x + u)} \right)^2 g(x)dx < \infty, \]

and

\[ \int_{-\infty}^{\infty} \left( \frac{\max_{|u| \leq \delta} |g''(x + v)|}{\min_{|u| \leq \delta} g(x + u)} \right)^2 g(x)dx < \infty. \]
Let $I$ denote the Fisher information of $g$:

$$I = \int_{-\infty}^{\infty} \frac{(g'(x))^2}{g(x)} dx < \infty.$$ 

Then the algebraic Lyapunov equation

$$P = \Lambda^* P \Lambda + \left( \text{diag}(\mu) - \mu^* \right) \Lambda \Lambda^* \left( \text{diag}(\mu) - \mu^* \right)$$

(2.2)

has a unique solution $P$ in the class of nonnegative definite matrices with $\sum_{i,j} P_{ij} = 0$ and

$$\lim_{\sigma \to \infty} \sigma^2 (E^\infty - E^\sigma) = a^* P a,$$

(2.3)

where $a$ is a vector with entries $a_1, \ldots, a_d$ and $E^\infty = \mu(\sigma^2(a)) - \mu^2(a)$ is the a priori mean square error.

**Remark 2.3.** The assumption (A1) and ergodicity of $X$ guarantee uniqueness of the invariant measure $\mathcal{M}_\pi^\sigma(d\eta)$ (see [4]). The assumption (A2) is satisfied for many frequently encountered densities. For Gaussian density $g(x) = (2\pi)^{-1/2} \exp\{-x^2/2\}$

$$\frac{|g'(x)|}{\min_{|u| \leq \delta} g(x + u)} = \frac{|x| e^{-x^2/2}}{\min_{|u| \leq \delta} e^{-(x+u)^2/2}} = \frac{|x|}{\min_{|u| \leq \delta} e^{-x u - u^2/2}} \leq \frac{|x|}{e^{-|x| \delta - \delta^2/2}}$$

and hence

$$\int_{-\infty}^{\infty} \left( \frac{|g'(x)|}{\min_{|u| \leq \delta} g(x + u)} \right)^p \, g(x) \, dx \leq \int_{-\infty}^{\infty} \left( \frac{|x|}{e^{-|x| \delta - \delta^2/2}} \right)^p e^{-x^2/2} \, dx < \infty$$

for any $p \geq 0$ and not just $p = 2$ as required by the first part of (A2). Similarly

$$\frac{|g''(x + v)|}{g'(x + u)} = \frac{\max_{|u| \leq \delta} |(x + v)^2 - 1| e^{-(x+v)^2/2}}{\min_{|u| \leq \delta} e^{-(x-u)^2/2}} \leq (2x^2 + 2\delta^2 + 1)e^{2|x|\delta + \delta^2}$$

and the second condition of (A2) holds with any power $p \geq 0$ as well. It is not hard to verify that (A2) also holds for e.g. Cauchy density $g(x) = \pi^{-1}(1 + x^2)^{-1}$, which fails to have the first moment.

**Theorem 2.4.** Assume that $X$ is ergodic and $\xi_1$ is a standard Gaussian random variable, then for any continuous function $F : \mathbb{R}^d \to \mathbb{R}$, growing not faster than polynomially,

$$\int_{\mathbb{R}^d} F(\sigma(\eta - \mu)) \mathcal{M}_\pi^\sigma(d\eta) \xrightarrow{\sigma \to \infty} EF(Z),$$

(2.4)

where $Z$ is a zero mean Gaussian vector with covariance matrix $P$, defined by (A4) with $I \equiv 1$. In particular

$$\lim_{\sigma \to \infty} \sigma \left( \mathcal{P}^\infty - \mathcal{P}^\sigma \right) = \mathbb{E} \max_{j \in \mathcal{J}} Z_j,$$

(2.5)

where $\mathcal{P}^\infty := 1 - \max_{i \in \mathcal{I}} \mu_i$ is the a priori error probability and $\mathcal{J} = \{i : \mu_i = \max_j \mu_j\}$. If $\mu$ has a unique maximal atom, then for any integer $p \geq 1$

$$\lim_{\sigma \to \infty} \sigma^p \left( \mathcal{P}^\infty - \mathcal{P}^\sigma \right) = 0.$$
Example 2.5. Let \( X \) be a binary chain with the transition matrix
\[
\Lambda = \begin{pmatrix}
\lambda & 1 - \lambda \\
1 - \gamma & \gamma
\end{pmatrix}, \quad \lambda, \gamma \in (0, 1).
\]
The equation (2.2) is one dimensional and \( P := P_{11} = P_{22} = -P_{12} = -P_{21} \) satisfies
\[
P = P(1 - \lambda - \gamma)^2 + \mu_1^2 \mu_2^2 (h_1 - h_2)^2
\]
and hence
\[
P = \frac{\mu_1^2 \mu_2^2 (h_1 - h_2)^2}{(\lambda + \gamma)(1 - \lambda + 1 - \gamma)} = \frac{(1 - \lambda)^2 (1 - \gamma)^2 (h_1 - h_2)^2}{(\lambda + \gamma)(1 - \lambda + 1 - \gamma)^5}.
\]
Now by Theorem 2.2
\[
\lim_{\sigma \to \infty} \sigma \left( \mathcal{E}^\infty - \mathcal{E}^\sigma \right) = (a_1 - a_2)^2 P.
\]
By Theorem 2.4 if \( \gamma \neq \lambda \)
\[
\lim_{\sigma \to \infty} \sigma^p (\mathcal{P}^\infty - \mathcal{P}^\sigma) = 0, \quad p \geq 1
\]
and if \( \gamma = \lambda \),
\[
\lim_{\sigma \to \infty} \sigma (\mathcal{P}^\infty - \mathcal{P}^\sigma) = E \max(Z, -Z) = E |Z| = 2 \sqrt{P} \int_0^\infty \frac{x}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{|h_1 - h_2|}{4 \sqrt{\lambda(1 - \lambda)}} \cdot 0.3839...
\]
2.2. Continuous time. The continuous time analogue of the aforementioned setting consists of a time homogeneous Markov chain with values in \( S \), transition intensities \( \lambda_{ij} \) and initial distribution \( \nu \) and the observation process \( Y^\sigma = (Y^\sigma)_t \in \mathbb{R}_+ \) satisfying
\[
Y^\sigma_t = \int_0^t h(X_s) ds + \sigma B_t, \quad t \geq 0,
\]
where \( h \) is an \( S \mapsto \mathbb{R} \) function, \( \sigma > 0 \) is a real constant and \( B = (B_t)_{t \geq 0} \) is a Brownian motion, independent of \( X \). We treat the continuous time case separately and hence use the same notations for transition intensities and transition probabilities, etc.

The vector of conditional probabilities \( \pi_t \) satisfies the Wonham filtering Itô stochastic differential equation (3.1, see also 3.3)
\[
d\pi_t = \Lambda^* \pi_t dt + \sigma^2 \left( \text{diag}(\pi_t) - \pi_t \pi_t^* \right) h(dY_t - \pi_t(h) dt), \quad (2.6)
\]
subject to \( \pi_0 = \nu \), where \( \Lambda \) is the transition intensities matrix, \( \text{diag}(x) \), \( x \in \mathbb{R}^d \) stands for the scalar matrix with \( x \) on the diagonal, \( h \) is a column vector with entries \( h(a_i) \) and \( x^* \) is the transposed of \( x \).

Recall that \( X = (X_t)_{t \in \mathbb{R}_+} \) is ergodic, if \( \exp(\Lambda) \) has positive entries or equivalently if all of its states communicate. For ergodic chains the Markov process \( \pi_t \) has a unique invariant measure \( \mathcal{M}_\sigma^\pi(d\eta) \) for any \( \sigma > 0 \) (see 3.4). In the case \( d = 2 \) the exact expressions are known for both \( \mathcal{P} \) and \( \mathcal{E} \) in terms of integrals with respect to the density of \( \mathcal{M}_\sigma^\pi(d\eta) \), which can be explicitly found by solving the corresponding Kolmogorov-Fokker-Plank equation (see 3.1, 3.3). In higher dimension the closed form solution for KFP equation is unavailable, which makes the direct analysis of 2.6 intractable.

The slow chain \( X^\varepsilon \) is obtained by the time scaling \( X^\varepsilon_t = X_{\varepsilon t}, \ t \geq 0 \) and its transition intensities matrix equals \( \varepsilon \Lambda \). As in the discrete time the invariant measure of \( X^\varepsilon \) is independent of \( \varepsilon \) and solves \( \Lambda^* \mu = 0 \) in \( S^{d-1} \). The asymptotic expressions
and 1.5 remain valid with \( \mathcal{D}(g_i \Vert g_j) \) replaced by \( (h_i - h_j)^2 / 2 \) (see [7], [8]). It is not hard to see, either by appropriate time change or directly from the KFP equation, that the weak noise asymptotic \( \sigma \to 0 \) is obtained by replacing \( \varepsilon = \sigma^2 \) in these expressions. The strong noise asymptotic also turns to be similar to the discrete time case:

**Theorem 2.6.** The solution of (2.6) converges to \( \nu_t = e^{A^* t} \nu \) as \( \sigma \to \infty \) and for any \( p \geq 1 \)

\[
\sigma \left( \pi_i^2 - \nu_t \right) \xrightarrow{\sigma \to \infty} Z_t, \quad t \geq 0,
\]

where \( Z_t \) is the Gaussian diffusion process:

\[
dZ_t = \Lambda^* Z_t dt + (\text{diag}(\nu_t) - \nu_t \nu_t^*) hd\tilde{B}_t, \quad Z_0 = 0,
\]

with \( \tilde{B} = \sigma^{-1} \left( Y_t^\sigma - \int_0^t \pi_s^\sigma(h) ds \right) \) being the innovation Brownian motion. If \( X \) is ergodic, the algebraic Lyapunov equation

\[
0 = \Lambda^* P + PA + (\text{diag}(\mu) - \mu \mu^*) hh^*(\text{diag}(\mu) - \mu \mu^*)
\]

has a unique solution \( P \) in the class of nonnegative definite matrices satisfying \( \sum_{ij} P_{ij} = 0 \) and for any \( F : \mathbb{R}^d \to \mathbb{R} \), growing not faster than polynomially,

\[
\int_{\mathbb{S}^{d-1}} F(\sigma(\eta - \mu)) \mathcal{M}_n^\sigma(d\eta) \xrightarrow{\sigma \to \infty} EF(Z),
\]

where \( Z \) is a zero mean Gaussian random vector with covariance matrix \( P \).

Theorems 2.2 and 2.4 remain valid in continuous time case with obvious adjustments, namely

\[
\lim_{\sigma \to \infty} \sigma^2 (\mathcal{E}_\infty - \mathcal{E}_\sigma) = a^* Pa \quad \text{and} \quad \lim_{\sigma \to \infty} \sigma (\mathcal{P}_\infty - \mathcal{P}_\sigma) = E \max_j Z_j,
\]

where \( P \) is the solution of (2.8) and \( Z \) is the Gaussian vector defined in Theorem 2.6. If the maximal atom of \( \mu \) is unique,

\[
\lim_{\sigma \to \infty} \sigma^p (\mathcal{P}_\infty - \mathcal{P}_\sigma) = 0, \quad p \geq 1.
\]

3. **Proofs in discrete time**

3.1. **Proof of Theorem 2.1** The entries of the diagonal matrix \( G(y) \) in (1.6) in case of the observations (1.6) have the form

\[
g \left( \frac{y - h_i}{\sigma} \right), \quad i = 1, ..., d.
\]

To emphasize the dependence on \( \sigma \) write \( G^\sigma(y) \) and let \( T^\sigma(y)x = G^\sigma(y)\Lambda^* x / |G^\sigma(y)\Lambda^* x| \), \( y \in \mathbb{R} \), \( x \in \mathcal{S}^{d-1} \).

Since the density \( g(u) \) is continuous, for any \( i = 1, ..., d \)

\[
g \left( \frac{Y_n^\sigma - h_i}{\sigma} \right) = g \left( \xi_n + \frac{h(X_n) - h_i}{\sigma} \right) \xrightarrow{\sigma \to \infty} g(\xi_n),
\]

and hence \( \lim_{\sigma \to \infty} T^\sigma(Y_n)x = \Lambda^* x \), P-a.s. for any \( x \in \mathcal{S}^{d-1} \). Then for any fixed \( n \geq 1 \)

\[
\pi_n^\sigma = T^\sigma(Y_n^\sigma) \circ \cdots \circ T^\sigma(Y_1^\sigma) \circ \nu \xrightarrow{\sigma \to \infty} (\Lambda^*)^n \nu = \nu_n.
\]

Since both \( \pi_n^\sigma \) and \( \nu_n \) are bounded, the convergence also holds in \( L^p \) for any \( p \geq 1 \).
Let \( q_n^\sigma \) be the solution of

\[
q_n^\sigma = \Lambda^* q_{n-1}^\sigma - \sigma^{-1} (\text{diag}(\nu_n) - \nu_n^* \nu_n^*) h \frac{g'(\xi_n)}{g(\xi_n)}, \quad q_0^\sigma = \nu. \quad (3.2)
\]

The process \( \Delta_n^\sigma = \sigma (\pi_n^\sigma - q_n^\sigma) \) satisfies

\[
\Delta_n^\sigma = \Lambda^* \Delta_{n-1}^\sigma + \sigma \left( \frac{G^\sigma(Y_n^\sigma) \Lambda^* \pi_{n-1}^\sigma}{G^\sigma(Y_n^\sigma) \Lambda^* \pi_{n-1}^\sigma} - \Lambda^* \pi_{n-1}^\sigma \right) + (\text{diag}(\nu_n) - \nu_n^* \nu_n^*) h \frac{g'(\xi_n)}{g(\xi_n)} \quad (3.3)
\]

subject to \( \Delta_0^\sigma = 0 \). Denote \( \pi_{n|n-1}^\sigma = \Lambda^* \pi_{n-1}^\sigma \), then

\[
g\left( \sigma^{-1}(Y_n^\sigma - h_i) \right) \pi_{n|n-1}^\sigma(i) - \pi_{n|n-1}^\sigma(i) =
\frac{g(\xi_n + \sigma^{-1}(h(X_n) - h_i)) - \sum_{j=1}^d g(\xi_n + \sigma^{-1}(h(X_n) - h_j)) \pi_{n|n-1}^\sigma(j)}{\sum_{j=1}^d g(\xi_n + \sigma^{-1}(h(X_n) - h_j)) \pi_{n|n-1}^\sigma(j)} \pi_{n|n-1}^\sigma(i) =
\frac{-\sigma^{-1} g'(\xi_n) \left( h_i - \sum_{j=1}^d h_j \pi_{n|n-1}^\sigma(j) \right) + K_n \sigma^{-2}}{\sum_{j=1}^d g(\xi_n + \sigma^{-1}(h(X_n) - h_j)) \pi_{n|n-1}^\sigma(j)} \pi_{n|n-1}^\sigma(i),
\]

where \( K_n \) are bounded random variables (recall that \( g''(u) \) is assumed bounded). Hence by \( 3.1 \) and continuity of \( g \)

\[
\sigma \left( \frac{G^\sigma(Y_n^\sigma) \pi_{n|n-1}^\sigma}{G^\sigma(Y_n^\sigma) \pi_{n|n-1}^\sigma} - \pi_{n|n-1}^\sigma \right) \xrightarrow{\sigma \to \infty} (\text{diag}(\nu_n) - \nu_n^* \nu_n^*) h \frac{g'(\xi_n)}{g(\xi_n)}.
\]

Iterating \( 3.3 \) one gets \( \lim_{\sigma \to \infty} \Delta_n^\sigma = 0, \ P - a.s. \) for any fixed \( n \geq 0 \) and the statement of the theorem follows:

\[
\sigma (\pi_n^\sigma - \nu_n) = \sigma (\pi_n^\sigma - q_n^\sigma) + \sigma (q_n^\sigma - \nu_n) \xrightarrow{\sigma \to \infty} Z_n,
\]

where \( Z_n := \sigma (q_n^\sigma - \nu_n) \) clearly satisfies \( 2.1 \), which doesn’t depend on \( \sigma \). \( \square \)

3.2. Proof of Theorem 2.2. Note that to verify \( 2.3 \) one should first take the limit \( n \to \infty \) and then \( \sigma \to \infty \) and thus cannot use the statement of Theorem 2.1 per se. The proof relies on stability property of the matrix \( \Lambda \), provided by ergodicity of \( X \).

As shown in \( 3.1 \) the Markov process \( (X, \pi^\sigma) \) has the unique invariant measure \( M^\sigma(dx, d\eta) \) if \( X \) is ergodic and assumption \( 3.1 \) is satisfied. In particular \( M^\sigma_\pi(d\eta) = \sum_{i=1}^d M^\sigma(\{a_i\}, d\eta) \). If the equation \( 3.3 \) and \( X \) is started from a random variable with distribution \( M^\sigma(dx, d\eta) \), the process \( \pi^\sigma = (\pi_n^\sigma)_{n \geq 0} \) is stationary, which is assumed hereafter.

As in \( 3.1 \) \( \lim_{\sigma \to \infty} \pi_n^\sigma = (\Lambda^*)^n \pi_0^\sigma \), \( P - a.s. \) and so for any \( \varepsilon > 0 \) and any \( m \geq 0 \)

\[
\lim_{\sigma \to \infty} P(|\pi_0^\sigma - \mu| \geq \varepsilon) = \lim_{\sigma \to \infty} P(|\pi_m^\sigma - \mu| \geq \varepsilon) \leq \lim_{\sigma \to \infty} P(|(\Lambda^*)^m \pi_0^\sigma - \mu| \geq \varepsilon/2) + \lim_{\sigma \to \infty} P(|(\Lambda^*)^m \pi_0^\sigma - \mu| \geq \varepsilon/2) \xrightarrow{m \to \infty} 0 \quad (3.4)
\]

where the latter convergence holds since \( (\Lambda^*)^n x \to \mu \) for all \( x \in S^{d-1} \) by ergodicity of \( X \).
Let $q_n^\sigma$ denote the solution of (cf. (3.2))
\[ q_n^\sigma = \Lambda^* q_{n-1}^\sigma - \sigma^{-1} (\text{diag}(\mu) - \mu \mu^*) h \frac{g'(\xi_n)}{g(\xi_n)}, \quad q_0^\sigma = \mu. \]

We use the notations, introduced in the previous section, to denote random processes, playing the same role as in the proof of Theorem 2.1 but defined differently to fit the stationary setup under consideration.

The process $\Delta_n^\sigma = \sigma(\pi_n^\sigma - q_n^\sigma)$ satisfies (cf. (3.3))
\[ \Delta_n^\sigma = \Lambda^* \Delta_{n-1}^\sigma + \sigma \left( \frac{G^\sigma(Y_n^\sigma) \Lambda^* \pi_{n-1}^\sigma}{|G^\sigma(Y_n^\sigma) \Lambda^* \pi_{n-1}^\sigma|} - \Lambda^* \pi_{n-1}^\sigma \right) + \]
\[ (\text{diag}(\mu) - \mu \mu^*) h \frac{g'(\xi_n)}{g(\xi_n)} := \Lambda^* \Delta_{n-1}^\sigma + \theta_n^\sigma \]
subject to $\Delta_0^\sigma = \sigma(\pi_0^\sigma - \mu)$. Note that since the Fisher information is finite and $\pi_n^\sigma$ is stationary, for any fixed $\sigma > 0$, $E\theta_n^\sigma \theta_n^\sigma = E\theta_0^\sigma \theta_0^\sigma : = \Gamma^\sigma$ and hence $Q_n^\sigma = E\Delta_n^\sigma \Delta_n^\sigma$ satisfies
\[ Q_n^\sigma = \Lambda^* Q_{n-1}^\sigma \Lambda + \Gamma^\sigma, \quad n \geq 1, \]
subject to $Q_0^\sigma = \sigma^2 E(\pi_0^\sigma - \mu)(\pi_0^\sigma - \mu)^*$. If $X$ is ergodic, $\Lambda^*$ is a stability matrix, when restricted to the subspace $\{ x \in \mathbb{R}^d : \sum_{i=1}^d x_i = 0 \}$. Since $\Delta_n^\sigma$ belongs to this subspace for all $n \geq 0$, the Lyapunov equation (3.5) has a bounded solution, which converges to the unique limit $Q^\sigma = \sum_{i=0}^{\infty} \Lambda^* \Gamma^\sigma \Lambda^m$.

For brevity set $\pi_{1|0}^\sigma = \Lambda^* \pi_0^\sigma$ and define
\[ a^\sigma := \sigma \left( \frac{G^\sigma(Y_1^\sigma) \pi_{1|0}^\sigma - \pi_{1|0}^\sigma}{|G^\sigma(Y_1^\sigma) \pi_{1|0}^\sigma|} \right). \]

Then
\[ a^\sigma(i) = \sigma \left( \frac{g(\sigma^{-1}(Y_1^\sigma - h_i)) \pi_{1|0}^\sigma(i)}{|G^\sigma(Y_1^\sigma) \pi_{1|0}^\sigma|} - \pi_{1|0}^\sigma(i) \right) = \]
\[ \frac{g(\xi_1 + \sigma^{-1}(h(X_1) - h_i)) - \sum_{j=1}^d g(\xi_1 + \sigma^{-1}(h(X_1) - h_j)) \pi_{1|0}^\sigma(j)}{\sum_{j=1}^d g(\xi_1 + \sigma^{-1}(h(X_1) - h_j)) \pi_{1|0}^\sigma(j)} \pi_{1|0}^\sigma(i) = \]
\[ \frac{-g'(\xi_1)(h_i - \sum_{j=1}^d h_j \pi_{1|0}^\sigma(j))}{\sum_{j=1}^d g(\xi_1 + \sigma^{-1}(h(X_1) - h_j)) \pi_{1|0}^\sigma(j)} + \]
\[ \frac{\pi_{1|0}^\sigma(i) g''(\xi_1 + \alpha_i/\sigma)(h(X_1) - h_i)^2 \beta_i - \sum_{j=1}^d g''(\xi_1 + \alpha_j/\sigma)(h(X_1) - h_j)^2 \beta_j \pi_{1|0}^\sigma(j)}{2 \sigma \sum_{j=1}^d g(\xi_1 + \sigma^{-1}(h(X_1) - h_j)) \pi_{1|0}^\sigma(j)} \]
where the latter holds by the mean value theorem with $|\alpha_j| \leq |h(X_1) - h_j|$ and $\beta_i \in [0, 1]$. Since $g''$ is bounded and by (3.4) $\pi_{1|0}^\sigma \to \mu$ in probability as $\sigma \to \infty$
\[ a^\sigma(i) \xrightarrow{P_{\sigma \to \infty}} -\frac{g'(\xi_1)}{g(\xi_1)}(h_i - \sum_{j=1}^d h_j \mu_j) \mu_i. \]

Note that for $\sigma > \max_i |h_i - h_j|/\delta$,
\[ \sum_{j=1}^d g(\xi_1 + \sigma^{-1}(h(X_1) - h_j)) \pi_{1|0}^\sigma(j) \leq \frac{|g'(\xi_1)|}{\min_{|u| \leq \delta} g(\xi_1 + u)}, \]
where by the assumption \[ \| x \| \leq C(\sigma) \]
the right hand side is square integrable. Analogously for sufficiently small \( \sigma \),
\[
\frac{|g''(\xi_1 + \alpha_i/\sigma)|}{\sum_{j=1}^d g(\xi_1 + \sigma^{-1}(h(X_1) - h_j)) \pi^*_1(j)} \leq \frac{\max_{|v| \leq \delta} |g''(\xi_1 + v)|}{\min_{|u| \leq \delta} g(\xi_1 + u)},
\]
with a square integrable right hand side. Hence by the Lebesgue dominated convergence (3.6) implies
\[
a^\sigma \xrightarrow{L^2} -(\text{diag}(\mu) - \mu \mu^*) h \frac{g'(\xi_1)}{g(\xi_1)}
\]
and in turn
\[
\lim_{\sigma \to \infty} \text{tr}(\Gamma^\sigma) = 0 \implies \lim_{\sigma \to \infty} Q^\sigma = 0. \tag{3.7}
\]
On the other hand, the sequence \( Z_n = \sigma(q_n^* - \mu) \) does not depend on \( \sigma \) and satisfies
\[
Z_n = \Lambda^* Z_{n-1} - (\text{diag}(\mu) - \mu \mu^*) h \frac{g'(\xi_n)}{g(\xi_n)}; \quad Z_0 = 0. \tag{3.8}
\]
Again by the stability property of \( \Lambda^* \) on the subspace \( \{ x \in \mathbb{R}^d : \sum_{j=1}^d x_j = 0 \} \)
\[
\lim_{n \to \infty} E Z_n Z_n^* = P,
\]
where \( P \) uniquely solves (2.8) in the class of nonnegative matrices with \( \sum_{ij} P_{ij} = 0 \),
which is a well known property of the Lyapunov equation for stable matrices (see e.g. [1]). Hence,
\[
\sigma^2 E(\pi_0^\sigma - \mu)(\pi_0^\sigma - \mu)^* - P = \sigma^2 E(\pi_n^\sigma - \mu)(\pi_n^\sigma - \mu)^* - \lim_{n \to \infty} Z_n Z_n^* = \lim_{n \to \infty} E \Delta_n\sigma^\sigma \Delta_n^* \sigma^\sigma \to 0 \tag{3.9}
\]
where the latter convergence holds by (3.7). This in turn implies (2.8):
\[
\sigma^2 (\mathcal{E}^\sigma - \mathcal{E}^\sigma) = \sigma^2 E a^*(\pi_0^\sigma - \mu)(\pi_0^\sigma - \mu)^* a \xrightarrow{\sigma \to \infty} a^* P a. \tag{\Box}
\]

3.3. **Proof of Theorem 2.3.** For standard Gaussian \( \xi_1 \), \( p'(x)/p(x) = -x \) and \( I = 1 \), hence the process \( Z_n \), defined in (2.8) is Gaussian. By the Remark 2.3 Gaussian density satisfies the assumption 2.2 of Theorem 2.2 with square integrability replaced by integrability to any power \( p \geq 1 \) and hence, similarly to (3.3), for any continuous \( F : \mathbb{R}^d \to \mathbb{R} \) with the norm bounded by a polynomial function of any finite order
\[
E F(\sigma(\pi_0^\sigma - \mu)) \xrightarrow{\sigma \to \infty} E F(Z),
\]
where \( Z \) is a zero mean Gaussian vector with covariance matrix \( P \), defined by (2.2).

Let \( J = \{ i : \mu_i = \max_j \mu_j \} \) and assume \( \mu_1 \in J \) for definiteness. Then
\[
\sigma(\mathcal{P}^\infty - \mathcal{P}^\sigma) = \sigma\left( E \max_{a_i \in S} (\pi_0^\sigma(i) - \max_i \mu_i) = E \sigma \max_{a_i \in S} (\pi_0^\sigma(i) - \mu_1) = E \max_{a_i \in S} \left( \sigma(\pi_0^\sigma(i) - \mu_i) + \sigma(\mu_i - \mu_1) \right) \xrightarrow{\sigma \to \infty} E \max_{j \in J} Z_j,
\]
where the convergence holds by (2.4), since \( \max_i(x_i), x \in \mathbb{R}^d \) is a continuous function and \( \mu_i - \mu_1 < 0 \) for \( i \notin J \).
Suppose now that \( \mu_1 \) is the unique maximal atom of \( \mu \) and let \( r = \min_{j \neq 1} |\mu_1 - \mu_j| > 0 \). Let \( A_\sigma := \{ |\pi_0^\sigma - \mu| \leq r/2 \} \), \( 1_{A_\sigma} \) be the indicator function of \( A_\sigma \) and \( A_\sigma^c = \Omega \backslash A_\sigma \). Then
\[
\max_{a_i \in S} \pi_0^\sigma(i) = 1_{A_\sigma} \pi_0^\sigma(1) + 1_{A_\sigma^c} = \pi_0^\sigma(i) = \pi_0^\sigma(1) + 1_{A_\sigma^c} \left( \max_{a_i \in S} \pi_0^\sigma(i) - \pi_0^\sigma(1) \right),
\]
Hence for any two integers \( q > p \geq 1 \),
\[
\sigma^p \left| E \max_{a_i \in S} \pi_0^\sigma(i) - \max_{a_i \in S} \mu_i \right| = \sigma^p \left| E \max_{a_i \in S} \pi_0^\sigma(1) - \mu(1) \right| + E \left| 1_{A_\sigma^c} \left( \max_{a_i \in S} \pi_0^\sigma(i) - \pi_0^\sigma(1) \right) \right| \leq 2e^\sigma \left| \pi_0^\sigma \right| q^{(p/q)} \frac{(r/2)^q}{q!} \to 0,
\]
since by (2.9), the limit \( \lim_{\sigma \to \infty} \sigma^q E |\pi_0^\sigma - \mu|^q \) exists and is finite.

4. PROOFS IN CONTINUOUS TIME

The continuous time filter \( \nu_t = \exp(\Lambda^* t) \nu \) solves \( \dot{\nu}_t = \Lambda^* \nu_t, \nu_0 = \nu \), the process \( \pi_t^\nu := \pi_t^\nu - \nu_t \) satisfies
\[
d\pi_t^\nu = \Lambda^* \pi_t^\nu dt + \sigma^{-1} \left( \text{diag}(\pi_t^\nu) - \pi_t^\nu \pi_t^{\sigma^*} \right)hd\tilde{B}_t, \quad \pi_0^\nu = 0,
\]
and hence
\[
\pi_t^\nu = \sigma^{-1} \int_0^t e^{\Lambda^*(t-s)} \left( \text{diag}(\pi_s^\nu) - \pi_s^\nu \pi_s^{\sigma^*} \right)hd\tilde{B}_s.
\]
Since the integrand is continuous and bounded for any \( t \geq 0 \),
\[
\lim_{\sigma \to \infty} \pi_t^\sigma = 0, \quad P - a.s.
\]
The convergence holds in \( L^p, p \geq 1 \) as well, since the integrand of the stochastic integral is uniformly bounded in \( \sigma \) and hence \( |\pi_t^\sigma| \) is uniformly integrable to any power as \( \sigma \to \infty \). Let \( q_t^\sigma \) be solution of the linear SDE
\[
dq_t^\sigma = \Lambda^* q_t^\sigma dt + \sigma^{-1} \left( \text{diag}(q_t^\nu) - q_t^\nu \nu_t^{\sigma^*} \right)hd\tilde{B}_t, \quad q_0^\nu = \nu.
\]
The process \( \Delta_t^\sigma = \sigma(\pi_t^\nu - q_t^\sigma) \) satisfies
\[
d\Delta_t^\sigma = \Lambda^* \Delta_t^\sigma dt + (\Gamma(\pi_t^\sigma) - \Gamma(\nu_t))hd\tilde{B}_t, \quad \Delta_0^\sigma = 0, \quad (4.1)
\]
where \( \Gamma(x) = \text{diag}(x) - xx^{\sigma^*} \) is defined for brevity. Then
\[
\Delta_t^\sigma = \int_0^t e^{\Lambda^*(t-s)} \left( \Gamma(\pi_s^\sigma) - \Gamma(\nu_s) \right) hd\tilde{B}_s \sigma \to \infty \to 0, \quad P - a.s \text{ and in } L^p,
\]
since \( \Gamma(\cdot) \) is continuous, \( \pi_t^\sigma \) and \( \nu_t \) are bounded and \( \pi_t^\sigma \to \nu_t \) P-a.s. as \( \sigma \to \infty \). The process \( Z_t = \sigma(q_t^\sigma - \nu_t) \) satisfies (2.17) and thus \( \sigma(\pi_t^\sigma - \nu_t) = \sigma(q_t^\sigma - \nu_t) + \sigma(\nu_t - \nu_t) \frac{L}{\sigma \to \infty} \to Z_t \).

If \( X \) is ergodic, \( \Lambda^* \) is a stability matrix on \( \{ x \in \mathbb{R} : \sum_i x_i = 0 \} \), which is an invariant subspace of (1.1) and (2.7). Then by the very same arguments, used in the proof of Theorem 2.2 and taking into account the integrability properties of the stochastic integral with respect to \( B \), one verifies (2.9).
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