A Conceptual Shift to Rectify a Defect in the Lorentz–Dirac Equation

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Abstract

In his analysis of the Classical Theory of Radiating Electrons, Dirac (1938) draws attention to the characteristic instability of solutions to the third order equation of motion. He remarks that changing the sign of the self-force eliminates the runaway solutions and gives ‘reasonable behaviour’. Dirac rejects such a change and proceeds with an ad hoc modification to the solutions of the initial value problem that is not consistent with the principle of causality. We argue that his reasons for rejecting the change of sign are invalid on both physical and mathematical grounds.

The conceptual shift is to treat the physical particle as a composite of the source particle and the energy-momentum that is reversibly generated in its self-field by its motion. The reversibly generated energy in the self-field is interpreted as kinetic energy, and the changes that follow result in Dirac’s change of sign. Several exact solutions to the new equation of motion and its linearisation are given. For a particle in orbital motion the self-force enables the applied force to generate radiation and kinetic energy in the self-field that results in an outward spiral motion. The theory is consistent with all well-established principles of physics, including the principle of causality.

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1 INTRODUCTION

The system consists of a point particle with an applied force, the particle having the physical attributes of mass $m$ and charge $e$ with its field being defined by the Liénard retarded potential. The energy-momentum that is generated in its self-field by the acceleration of the source particle causes a very small perturbation to the motion, and we have to find the corresponding force.

1.1 THE LORENTZ–DIRAC EQUATION

Solving the Lorentz–Dirac equation of motion as an initial value problem results in unstable solutions in which the very small perturbation of the self-force gives rise to an enormous effect on the motion. Dirac (1938, p.157) remarked that this runaway behaviour indicates an error in the sign of the self-force; he points out that changing the sign results in solutions giving ‘reasonable behaviour’. However, he rejected the change because it makes a particle in a Coulomb field ‘spiral outwards, instead of spiralling inwards … as it should in the classical theory’.

Dirac does not explain why the particle should spiral inwards in the classical theory. Of course, if it is assumed that the radiation reaction can be modelled by a frictional force proportional to the velocity then the solution of the resulting second order equation of motion does indeed predict the particle spiralling inwards. But the radiation reaction is, without doubt, proportional to the hyper-acceleration, and the behaviour of a singular third order differential equation cannot be inferred from one of second order; thus, the objection is invalid.

In order to eliminate runaway behaviour, Dirac proposed an ad hoc method of solution that results in acausal behaviour, manifested as pre-acceleration. Rohrlich (1965) gives a detailed development. However, these are not solutions of the initial value problem, and they exhibit acausal behaviour; they are, therefore, unacceptable.

1.2 THE PHYSICS OF CLASSICAL ELECTRODYNAMICS

The energy-momentum generated in the self-field of a charged particle by its motion consists of two distinct parts. The part irreversibly generated is the radiated energy-momentum; it is propagated from the particle world-line as it is generated thereby gaining its independence from its source particle. Whereas, as Teitelboim (1970) has shown in detail, the reversibly generated part is bound to its source particle. Our conceptual shift is to treat the physical particle as a composite of the source particle together with that part of the energy-momentum that is reversibly generated in its self-field, which, therefore, contributes to the total kinetic energy of the physical particle. Clearly, this has some similarity to the self-energy in quantum electrodynamics, though different in detail.
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Energy-momentum generated in an applied field by the motion of a particle does not contribute to the self-field kinetic energy of the physical particle. Thus, there is a clear distinction between the process that generates energy-momentum in the self-field and the process that generates energy-momentum in an applied field: the two physical processes are different.

Two postulates—constitutive assumptions—are required in order to relate these energy-momentum flows to the corresponding forces acting on their generating particle. The force due to an applied field is postulated to be the familiar Lorentz force. The force due to the self-field is postulated to be proportional to and opposite to the Abraham four-vector. The choice of sign ensures that the reversible energy-momentum generated by and bound to the source particle in the form of kinetic energy is added to the inertial kinetic energy. Thus, within the conventional methodological framework, the theory is modified and we get Dirac’s change of sign.

### 1.3 The Self-Force

The self-force is the highest order (third) derivative in the equation of motion and is, in general, a very small perturbation on the other forces, therefore, the equation of motion is singular in the sense of singular perturbation theory. It is characteristic of such singular equations that the sign of the highest derivative determines whether the perturbation term has a small effect on the motion or a destabilising effect leading to ‘runaway’ behaviour.

Another characteristic is the existence of two time scales. During a brief initial period, the initial motion is exponentially damped—this is justly termed radiation damping. In the longer time scale the motion is dominated by the applied force with the self-force perturbing the motion by generating energy-momentum in the self-field. The self-force interacts with the applied force, extracting energy-momentum that is radiated and contributes to the total kinetic energy of the physical particle. Evidently, the self-force is not a simple ‘resistive’ force.

Solutions of the new equation of motion, solved as an initial value problem, have ‘reasonable behaviour’. Their physical interpretation is straightforward, in terms of the balance of energy-momentum flows. The theory is mathematically well-defined, self-consistent, and conforms to all well-established principles, including the principle of causality.

### 1.4 Structure of the Paper

In Section 2, we review the physical interpretation of how the electrodynamic field depends on the motion of its source particles. The field, the charge current density, and the energy-momentum density tensor are, ultimately, defined in terms of the

*The sign of this self-force is the opposite of that in the Lorentz–Dirac equation.*
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Liénard potential. Lorentz invariant identities for pairwise interactions and self-interactions are established for the energy-momentum density tensor. In Section 3, we are concerned with the action of the electrodynamic field on its source particle and on other particles. This requires two postulates: one for the relation between a particle and the applied field—the Lorentz force; and the other for the relation between a source particle and its self-field—the self-force. Hence, the equation of motion. Finally, in Section 4, the initial value problem for the equation of motion is solved for some important special cases and their physical interpretations given.

1.5 Notation

The speed of propagation of the electrodynamic field is \( c \). We define constants 
\[ b = \frac{2}{3} \frac{e^2}{c^3}, \quad \tau_c = b/m, \quad \nu_c = \frac{1}{\tau_c}, \]
the particle position is given by \( \mathbf{r} = (x, y, z) \), the velocity \( \mathbf{v} = \dot{\mathbf{r}} \), acceleration \( \ddot{\mathbf{v}} \), and hyper-acceleration \( \dddot{\mathbf{v}} \), where an overdot denotes the derivative with respect to time \( t \); for a bracketed expression the dot is a superscript on the closing bracket, for example, \( (\dot{\mathbf{v}} \cdot \mathbf{v}) \).

The metric tensor, \( \eta \), has signature \((-,-,-,+)\). The scalar product is usually written as a dot-product. Greek indices range from 1 to 4, and the summation convention applies. A point \( x^\lambda \) has Euclidean coordinates \((x, y, z, ct)\), or \((\mathbf{r}, ct)\), and the invariant world-line parameter is defined by \( ds = \gamma^{-1} dt \), where \( \gamma = (1 - \mathbf{v} \cdot \mathbf{v}/c^2)^{-\frac{1}{2}} \). The particle position is given by the function \( X \); the velocity by \( \nu = dX/ds \), with components \( \nu^\lambda = (\nu_x, \nu_y, \nu_z) \), acceleration by \( a = d\nu/ds \), and hyper-acceleration by \( h = da/ds \). From these, \( \nu \cdot \nu = c^2 \) and \( a \cdot a \leq 0 \). The gradient operator has components \( \partial_{\alpha} = (\partial_x, \partial_y, \partial_z, \partial_{ct}) \).

2 Electrodynamic Field

This section is concerned with how the electrodynamic field—defined in terms of the Liénard retarded potential—depends on the motion of its source particles.

2.1 Preliminaries

Let the particle world-line be given by \( x = X(s) \), with \(-\infty < s < +\infty\). From an arbitrary field point \( x \) we draw the backward null cone to intersect the particle world-line at a point denoted by \( X(s_R) \). The null-vector from this point on the particle world-line to the field point is \( \mathbf{R} = x - X(s_R) \) so that

\[ \mathbf{R} \cdot \mathbf{R} = (x - X(s_R)) \cdot (x - X(s_R)) = 0 \]

Given the particle world-line, the value of the world-line parameter \( s_R \) depends on the choice of the field point \( x \), that is, \( s_R = s_R(x) \). The point \( X(s_R) \) and value \( s_R \) thus obtained are said to be retarded relative to \( x \).
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2.1.1 Liénard Potential and its Field

We start with the definition of the Liénard potential,

$$\Phi(x) = e \frac{v}{R \cdot v}$$

where $v = \frac{dX}{ds}(s_R)$. Following directly from its definition,

$$\partial \cdot \Phi = 0 \quad \text{and} \quad \Box \Phi = \frac{4\pi}{c} j$$ (1)

where the charge current density is defined by

$$j(x) = ec \int_{-\infty}^{\infty} ds \, \delta^{(4)}(x - X(s))v(s)$$ (2)

Hence, from (1), $\partial \cdot j = 0$.

The tensor field that is associated with a particle is defined by

$$F^{\mu\nu} = \partial^{\mu} \Phi^{\nu} - \partial^{\nu} \Phi^{\mu}$$

in terms of which the Maxwell–Lorentz identities are

$$\partial_{\lambda} F^{\mu\nu} + \partial_{\nu} F^{\lambda\mu} + \partial_{\mu} F^{\nu\lambda} = 0, \quad \partial_{\mu} F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$ (3)

2.1.2 Energy-Momentum Density Tensor

In a system of $N$ particles, energy-momentum is generated by the motion of the particles in two distinct ways. Firstly, the motion of one particle in the field of another particle generates energy-momentum in the field through their pairwise interaction. Secondly, the motion of an individual particle generates energy-momentum in its self-field.

We consider a system of $N$ particles, with the particles identified by upper case Latin subscripts. The total potential is the sum of the individual Liénard potentials, and similarly their corresponding currents and fields:

$$\Phi_{\text{total}} = \sum_{A=1}^{N} \Phi_A, \quad j_{\text{total}} = \sum_{A=1}^{N} j_A, \quad F_{\text{total}} = \sum_{A=1}^{N} F_A$$ (4)

Clearly, the identities (1), and (3) are satisfied by $\Phi_{\text{total}}$, $j_{\text{total}}$, and $F_{\text{total}}$.

The energy-momentum density tensor is defined as a function of two fields,

$$\Theta(F_A, F_B) = \frac{1}{4\pi} \left( F_A \cdot F_B - \frac{1}{4} \eta \, \text{tr}(F_A \cdot F_B) \right)$$
On noting the bilinear structure of $\Theta$, and writing $\Theta_{AB}$ for $\Theta(F_A, F_B)$, we get

$$\Theta_{\text{total}} = \Theta(F_{\text{total}}, F_{\text{total}}) = \sum_{A=1}^{N} \sum_{B=1}^{N} \Theta_{AB}$$

Since the $N$ fields $F_A$, and therefore $F_{\text{total}}$, satisfy the Maxwell–Lorentz identities, we have the identities

$$\partial \cdot \Theta_{\text{total}} = \frac{1}{c} \sum_{A=1}^{N} \sum_{B=1}^{N} j_A \cdot F_B$$

(5)

and

$$\partial \cdot \Theta_{AA} = \frac{1}{c} j_A \cdot F_A , \quad A = 1, \ldots, N$$

(6)

The pairwise interaction energy-momentum density tensor is defined by

$$\Theta_{\text{pairs}} = \Theta_{\text{total}} - \sum_{A=1}^{N} \Theta_{AA}$$

(7)

Hence, with the aid of (5) and (6), from (7) we get

$$\partial \cdot \Theta_{\text{pairs}} = \frac{1}{c} \sum_{A=1}^{N} \sum_{B=1}^{N} j_A \cdot F_B$$

(8)

### 2.2 Physical Significance of the Field Interactions

Both (6) and (8) are interpreted as conservation laws. Here, we examine physical significance of what is being conserved, which is not always clear in the literature although the mathematical development is well-known.

#### 2.2.1 $\Theta$ in Terms of $E$ and $B$

In terms of the spatial field vectors $E = (E_x, E_y, E_z)$ and $B = (B_x, B_y, B_z)$, the field components are

$$F^{\lambda\nu} = \begin{bmatrix}
0 & -B_z & B_y & E_x \\
B_z & 0 & -B_x & E_y \\
-B_y & B_x & 0 & E_z \\
-E_x & -E_y & -E_z & 0
\end{bmatrix}$$

In terms of these variables, the energy-momentum density tensor has the form

$$\Theta^{\mu\nu} = \begin{bmatrix}
-T & \mathbf{P} \\
\mathbf{P} & U
\end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix}
\mathbf{E} \cdot \mathbf{E} + \frac{1}{2} (\mathbf{E} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{B}) & \mathbf{E} \times \mathbf{B} \\
\mathbf{E} \times \mathbf{B} & \frac{1}{2} (\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B})
\end{bmatrix}$$
where \( U = U(E, B) \), \( P = P(E, B) \), and \( T = T(E, B) \) are the scalar energy density, the Poynting vector, and the Maxwell stress tensor, respectively. In terms of \( E \) and \( B \) the identities (6) and (8) yield differential identities with the form

\[
\begin{align*}
\frac{1}{c} \frac{\partial P}{\partial t} - \text{div} T &= -\left( \rho E + \frac{1}{c} j \times B \right) \\
\frac{1}{c} \frac{\partial U}{\partial t} + \text{div} P &= -\frac{1}{c} j \cdot E
\end{align*}
\] (9)

The terms on the left-hand sides have a well-known interpretation: they give the rates of change of field momentum and energy densities at a space-time point, plus the influx of field momentum and energy densities to the point. Evidently, from (2), the right-hand sides are zero except on particle world-lines.

Noting (2), the four-momentum is obtained from the four-momentum density by spatial integration, and similarly for other quantities. From (4),

\[
E_{\text{total}} = \sum_{A=1}^{N} E_A, \quad B_{\text{total}} = \sum_{A=1}^{N} B_A
\]

and comparing (9) with (5) we see that, for all \( A \) and \( B \),

\[
\frac{e}{c} v_A \cdot F_B \quad \text{and} \quad -e \left( E_B + \frac{1}{c} v_A \times B_B, \ v_A \cdot E_B \right)
\] (10)

are equivalent, the field being evaluated on the world-line of its source particle. These expressions give the flow of energy-momentum generated in the field labelled \( B \) by the motion of the particle labelled \( A \).

2.2.2 Energy-Momentum of the Self-Field

The self-field is propagated from its source particle on the forward light-cone, its value at any field point depending on the state of the motion of its source particle at the apex of the light-cone. From (10)—see also (6)—the rate at which the energy-momentum of the self-field of a source particle is generated by the motion of the source particle is \( (e/c) v_A \cdot F_A \). After a non-trivial calculation based on the Liénard potential, this quantity is given by the well-known expression:

\[
\frac{e}{c} v_A \cdot F_A = -bK
\] (11)

where \( b = \frac{2}{3} e^2 / c^3 \) and

\[
K = \left( \eta - \frac{1}{c^2} vv \right) \cdot h = h + \frac{1}{c^2} a \cdot a v
\] (12)

Rowe (1978) derived (11) using distribution theory; clear derivations are given by Teitelboim (1971) and Hogan (1973), albeit with renormalisation; Dirac (1938) involved both the retarded and advanced potentials together with renormalisation.
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The contravariant components of $K$ are given by $K^\lambda = \left( \gamma K_1, \frac{1}{c^2} \gamma K \cdot v \right)$, with

$$K = \gamma^2 \dddot{v} + \frac{\gamma^4}{c^2} \dot{v} \cdot v v + 3 \frac{\gamma^4}{c^2} \dot{v} \cdot v \dddot{v} + 3 \frac{\gamma^6}{c^4} (\dot{v} \cdot v)^2 v$$ \hspace{1cm} (13)

first obtained by Abraham (1903) within a classical kinematical framework. With the aid of the identity $\dot{\gamma} = \frac{\gamma^3}{c^2} \dot{v} \cdot v$, we find

$$K \cdot v = \gamma^4 \dddot{v} \cdot v + 3 \frac{\gamma^6}{c^2} (\dot{v} \cdot v)^2$$ \hspace{1cm} (14)

On rearranging (14), with the aid of (11), we can write the temporal component of $(e/c)v_\lambda \cdot F_\lambda$ as

$$-b K \cdot v = S^{\text{(rad)}} - S^{\text{(rev)}}$$ \hspace{1cm} (15)

where

$$S^{\text{(rad)}} = b\gamma^4 \left( \dot{v} \cdot v + \frac{\gamma^2}{c^2} (\dot{v} \cdot v)^2 \right) = -b a \cdot a \geq 0$$ \hspace{1cm} (16)

is the radiated power, first obtained by Lienard (1898), irreversibly generated in the self-field by the motion of its source particle. And

$$S^{\text{(rev)}} = b(\gamma^4 \dot{v} \cdot v)$$ \hspace{1cm} (17)

is the power reversibly generated in the self-field by the motion of its source particle.

3 Particle-Field Interactions

Following the introduction of our conceptual shift, the forces acting on a particle are identified and, thereby, we obtain the equation of motion for a charged particle.

3.1 The ‘Dressed’ Particle

The reversibly generated energy-momentum in the self-field, $S^{\text{(rev)}}$, is bound to the source particle (Teitelboim: 1970, Section III). Our conceptual shift is to treat the physical particle as a composite of the source particle together with that part of the energy-momentum that is reversibly generated in its self-field. On integrating (17) we get

$$T_{\text{charge}} = b\gamma^4 \dot{v} \cdot v$$

where $T_{\text{charge}}$ is interpreted as the kinetic energy generated in the self-field of the charged particle. Accordingly, the total kinetic energy of the physical particle is
defined as the sum of the inertial kinetic energy of the source particle and the kinetic energy generated in its self-field:

\[ T = T_{\text{mass}} + T_{\text{charge}} = mc^2(\gamma - 1) + b\gamma^4 \dot{v} \cdot v \]  

(18)

We remark that if \( \dot{v} \cdot v \geq 0 \) and \( (\gamma^4 \dot{v} \cdot v)^4 \geq 0 \) then the self-field kinetic energy is increasing, and that the sign of \( S^{(\text{rev})} \) in (15) reflects the fact that it is bound to the particle, whereas \( S^{(\text{rad})} \) is propagated away from the particle.

It is tempting to say that the ‘bare’ source particle is ‘dressed’ with the reversibly generated energy-momentum in its self-field. However, any analogy to the self-energy in quantum electrodynamics cannot be taken too literally.

3.2 Balance of Forces on a Particle

The fundamental law of particle dynamics states that for each particle the forces that act on it must balance. In the classical electrodynamics of \( N \) particles there are \( N \) equations of the form

\[ f_{\text{charge}} + f_{\text{mass}} + f_{\text{Lorentz}} + f_{\text{applied}} = 0 \]

(19)

where \( f_{\text{charge}} \) is the four-force on the particle due to the changes that its motion induces on its self-field, \( f_{\text{mass}} \) is the inertial four-force due to the particle mass, \( f_{\text{Lorentz}} \) is the four-force on the particle due to pairwise interactions with the other \( N - 1 \) particles, and \( f_{\text{applied}} \) is an externally applied four-force.

3.3 Inertial Force

The constitutive law for the inertial force:

**Postulate I (Inertial Force).** The inertial force acting on a particle of mass \( m \) is

\[ f_{\text{mass}} = -ma \]

(20)

where \( a \) is the four-acceleration of the particle.

Concerning the concept of inertial force see [Noll] (2007).

3.4 Electrodynamical Forces

In (10) we have the expression \( (e/c)v_A \cdot F_B \) for the flow of energy-momentum generated in the field labelled \( B \) by the motion of the particle labelled \( A \)—the expression being valid for all \( A \) and \( B \). There are two physical processes to consider:

- For \( B \neq A \) we have a pairwise field interaction in which the energy-momentum \( (e/c)v_A \cdot F_B \) is generated in the field of another particle.
• For $B = A$ we have the self-field interaction in which the energy-momentum $(e/c)v_A \cdot F_A$ is generated in the field of the particle. As it is generated, part of this self-field energy-momentum separates from its source as radiation, with the remaining part being bound to its source particle thereby contributing to the total kinetic energy of the physical particle—see (11) and (15).

Manifestly, the physical interaction between a particle and its self-field differs radically from the physical interaction between a particle and the field of another particle.

We now need to relate the rate at which energy-momentum is generated by the motion of the particle to the resulting forces acting on the particle. Since we have two distinct physical processes it is evident that each has to be dealt with separately: one for the pairwise field interaction and the other for the self-field interaction.

### 3.4.1 Lorentz Force

For the pairwise field interaction: we make the assumption that the flow of energy-momentum that is generated in the field $F_B$ by the motion of particle $A$ is balanced by a four-force $f_A$ acting on the particle $A$.

$$f_A + \frac{e}{c}v_A \cdot F_B = 0$$

Hence,

**Postulate II** (Lorentz Force). The four-force exerted on a particle at the point $x$ by the field of another particle is given by

$$f_{Lorentz}(F) = -\frac{e}{c}v \cdot F(x) \quad (21)$$

where $e$ is the particle charge, $v$ its four-velocity, and $F$ the field of another particle.

### 3.4.2 Self-Force

For the self-field interaction, we make the physical assumption that the energy-momentum generated in the self-field by the motion of its source particle is, at the point at which it is generated, part of the particle–self-field composite—the radiation then being propagated away from its point of generation. Accordingly, the flow of inertial energy-momentum to the particle is $-ma$, giving a four-force of $f_{mass}$ on the particle. And from (10), the energy-momentum generated in—that is, flowing to—the self-field is $(e/c)v_A \cdot F_A$, giving a four-force of $f_{charge}$ on the particle. Thus, noting (11), the mass and the charge give rise to the four-force

$$f_{mass} + f_{charge} = -ma - bK \quad (22)$$

acting on the particle. Thus,
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**Postulate III (Self-Force).** The four-force that the self-field, \( F \), exerts on its source particle is given by

\[
\mathbf{f}_{\text{charge}} = \frac{e}{c} \mathbf{v} \cdot \mathbf{F}
\]  

where \( e \) is the particle charge and \( \mathbf{v} \) its four-velocity, the product \( \mathbf{v} \cdot \mathbf{F} \) being evaluated on the world-line. Without approximation, we have

\[
\mathbf{f}_{\text{charge}} = -bK
\]

where \( K \) is the Abraham four-vector and \( b = \frac{2}{3} \frac{e^2}{c^3} \).

The contrast between (21) and (23) is evident: in the former, the flow of energy-momentum is to the applied field, whereas, in the latter, the energy-momentum is generated as a part of the particle–self-field composite.

### 3.5 Equation of Motion

The exact equation of motion for a single particle is obtained by setting \( f_{\text{Lorentz}} = 0 \) in (19) and substituting from (20) and (24):

\[
bK + ma = f_{\text{applied}}
\]

The spatial component is

\[
bK + m(\gamma \dot{\mathbf{v}}) = f_{\text{applied}}
\]

and, on substituting from (15), using (17), and rearranging, the temporal component is

\[
mc^2 \dot{\gamma} + b(\gamma^4 \dot{\mathbf{v}} \cdot \mathbf{v}) - S^{(\text{rad})} = P_{\text{applied}}
\]

where \( P_{\text{applied}} = \mathbf{f}_{\text{applied}} \cdot \mathbf{v} \) is the power generated on the particle by the applied force. With the definition of the total kinetic energy given in (18), the balance of power, (27), is

\[
\dot{T} = S^{(\text{rad})} + P_{\text{applied}}
\]

If we interpret radiation as a flow of heat then (28) is, for one source particle and its self-field, an expression of the first law of thermodynamics.

### 4 Applications

The solutions to the applications reveal that the underlying physical explanation for the behaviour of the particle is due to the self-force extracting energy-momentum from the applied force and converting it to radiation and energy-momentum for the self-field of the particle. The applied forces considered here are constant and, therefore, provide an unlimited source of energy-momentum.
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For a free particle, the only energy-momentum available for radiation is from the initial acceleration. In all cases the self-force generates only a very small effect, which is consistent with its very small magnitude.

If no exact solution to the equation of motion can be found then we are reduced to singular perturbation methods: an accessible introductory account of the methods of Matched Asymptotic Expansions and of Multiple Scales is to be found in [Hinch (1991)]. However, for motion that is not too extreme, the equation of motion can be approximated by a linearised equation of motion that can be solved without further approximation in several important special cases.

We remark that the corresponding solutions to the Lorentz–Dirac equation, solved as an initial value problem, are obtained by setting $b \rightarrow -b$, which implies that we set $\tau_c \rightarrow -\tau_c$ and $\nu_c \rightarrow -\nu_c$. On recalling that, for an electron, the interval $\tau_c = \frac{2}{3} \frac{e^2}{mc^3} \approx 6 \times 10^{-24}$ seconds, the unstable behaviour of such solutions is manifest.

### 4.1 The Exact Equation of Motion

The integration of the exact equation of motion for a free particle, a particle given an applied impulse, and for a particle with a frictional four-force is straightforward.

#### 4.1.1 Free Particle

From (25), recalling (12), the equation of motion for a free particle is

$$h + \frac{1}{c^2} a \cdot av + \nu_c a = 0 \quad (29)$$

to be integrated with initial values $v(0) = v_0$ and $a(0) = a_0$.

Taking the scalar product of (29) with $a$, we get

$$\frac{d}{ds}(a \cdot a) + 2 \nu_c a \cdot a = 0$$

Integrating yields

$$a \cdot a = a_0 \cdot a_0 \ e^{-2\nu_c s} \quad (30)$$

Using this to eliminate the factor $a \cdot a$ in (29) yields the linear equation

$$\frac{d^2v}{dx^2} + \nu_c \frac{dv}{ds} - \sigma^2 v e^{-2\nu_c s} v = 0, \quad \sigma = \tau_c \sqrt{-a_0 \cdot a_0/c^2}$$

With a change of variable, setting $x = e^{-\nu_c s}$ and $v(s) = w(x)$, we find

$$\frac{d^2w}{dx^2} - \sigma^2 w = 0$$
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Hence,
\[ v = v_0 \cosh[\sigma(1 - e^{-\nu_c s})] + a_0 \frac{\tau_c}{\sigma} \sinh[\sigma(1 - e^{-\nu_c s})] + a_0 \tau_c a_0 \frac{\sinh \sigma}{\sigma}. \]  

For \( s \gg \tau_c \) the velocity has the constant value \( v_0 \cosh \sigma + \tau_c a_0 \frac{\sinh \sigma}{\sigma}. \)

Substituting (30) into (16), we get
\[ S^{(\text{rad})} = -ba_0 \cdot a_0 e^{-2 \nu_c s}, \]  and with (28),
\[ \dot{T} = S^{(\text{rad})}. \]

Thus, the self-field kinetic energy from the initial acceleration is converted into radiated energy. This phenomenon can rightly be described as radiation damping.

From \( s = 0 \) to \( s = \infty \), the total energy radiated is \( T(0) - T(\infty) = -\frac{1}{2} \tau_c b a_0 \cdot a_0 \geq 0. \)

4.1.2 Applied Impulse

The equation of motion for a free particle with an impulse \( f_{\text{applied}} = k \delta(s - s_i) \) is
\[ h + \frac{1}{c^2} a \cdot a v + \nu_c a = k \delta(s - s_i) \]

to be integrated with initial values \( v(0) = v_0 \) and \( a(0) = a_0 \). Noting the solution for a free particle, (31), we find
\[ v = v_0 \cosh[\sigma(1 - e^{-\nu_c s})] + a_0 \frac{\tau_c}{\sigma} \sinh[\sigma(1 - e^{-\nu_c s})] + k \frac{\tau_c}{\sigma} \sinh[\sigma(1 - e^{-\nu_c(s-s_i)})] \theta(s - s_i) \]

where \( \theta \) is the Heaviside step function and \( \sigma = \tau_c \sqrt{-a_0 \cdot a_0/c^2}. \)

4.1.3 Frictional Force

From (12) and (25), the equation of motion for a particle with the frictional four-force \( f_{\text{applied}} = -k v \), where \( k \) is a constant, is
\[ h + \frac{1}{c^2} a \cdot a v + \nu_c a + \frac{k}{b} v = 0 \]  \hspace{1cm} (32)

with initial values of \( v_0 \) and \( a_0 \) for the velocity and acceleration, respectively. Following the same procedure as for the free particle, using (30) to eliminate the factor \( a \cdot a \) in (32), yields the linear equation
\[ \frac{d^2 v}{ds^2} + \nu_c \frac{dv}{ds} + \left( \frac{k}{b} - \sigma^2 \nu_c^2 e^{-2 \nu_c s} \right) v = 0, \quad \sigma = \tau_c \sqrt{-a_0 \cdot a_0/c^2}. \]

With the substitution \( v(s) = w(x(s)) \exp(-\frac{1}{2} \nu_c s) \) and \( x = \sigma e^{-\nu_c s} \), we find
\[ x^2 \frac{d^2 w}{dx^2} + x \frac{dw}{dx} - (x^2 + a^2) w = 0 \]  \hspace{1cm} (33)
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where $\alpha^2 = \frac{1}{4}(1 - 4\tau,k/m)$. The modified Bessel equation, (33), has the solution

$$w(x) = C_1 I_{\alpha}(x) + C_2 K_{\alpha}(x)$$

where $I_{\alpha}$ and $K_{\alpha}$ are modified Bessel functions of order $\alpha$. Hence,

$$v(s) = \{C_1 I_{\alpha}(\sigma e^{-\nu c s}) + C_2 K_{\alpha}(\sigma e^{-\nu c s})\} \exp(-\frac{1}{2} \nu c s)$$

where, applying the initial conditions, the constants $C_1$ and $C_2$ are given by

$$C_1 = \frac{1}{\Delta} \left( \frac{1}{\sigma \nu c} K_{\alpha}(\sigma) a_0 + \frac{1}{2} \nu c v_0 + K'_{\alpha}(\sigma) v_0 \right)$$

$$C_2 = -\frac{1}{\Delta} \left( \frac{1}{\sigma \nu c} I_{\alpha}(\sigma) a_0 + \frac{1}{2} \nu c v_0 + I'_{\alpha}(\sigma) v_0 \right)$$

with

$$\Delta = I_{\alpha}(\sigma) K'_{\alpha}(\sigma) - K_{\alpha}(\sigma) I'_{\alpha}(\sigma)$$

4.2 The Linearised Equation of Motion

For the linearised approximation only those terms that are linear in $v$ and its derivatives are retained—for approximately circular motion, this is equivalent to dropping the terms of order $v \cdot v/c^2$. From (13) we find $K = \ddot{v}$, and the equation of motion (26) reduces to

$$b \ddot{v} + m \dot{v} = f_{\text{applied}}$$

(34)

to be integrated with initial conditions $r(0) = r_0$, $v(0) = v_0$, and $a(0) = a_0$. The radiated power, (16), and the total kinetic energy, (18), reduce to

$$S^{(\text{rad})} = b \dot{v} \cdot \dot{v} \quad \text{and} \quad T = \frac{1}{2} m v \cdot v + b \dot{v} \cdot v$$

(35)

For the first three applications, the source of the particle energy is finite and is expended by the change in particle motion and radiation. By contrast, in the remaining four applications the particle is in a field that is held constant, from which there is no limit to the available power. With the exception of the Coulomb field, the problems are solved without further approximation.

4.2.1 Free Particle

From (34), the equation of motion for a free particle is

$$b \ddot{v} + m \dot{v} = 0$$

with the solution

$$v = v_0 + \tau_c(1 - e^{-\nu c t})a_0$$

(36)

Hence, from (28) and (35) we find that

$$\dot{F} = S^{(\text{rad})} = b a_0 \cdot a_0 e^{-2\nu c t}$$

In the initial interval of $O(\tau_c)$ the power contained in the initial acceleration is dissipated as radiation, and for $t \gg \tau_c$ the velocity is constant: $v \rightarrow v_0 + \tau_c a_0$. 

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4.2.2 Applied Impulse

With the applied force an impulse at \( t = t_i > 0 \), equation (34) becomes

\[
b \ddot{\mathbf{v}} + m \dot{\mathbf{v}} = \delta(t - t_i) \tau_c \mathbf{k}
\]

where \( \mathbf{k} \) is constant. The solution is

\[
\mathbf{v}(t) = \mathbf{v}_0 + \tau_c (1 - e^{-\nu_c t}) \mathbf{a}_0 + \tau_c (1 - e^{-\nu_c (t - t_i)}) \left( (1/m) \mathbf{k} \right) \theta(t - t_i)
\]

where \( \theta \) is the Heaviside step function. The impulse gives the particle an additional acceleration of \( (1/m) \mathbf{k} \). The final steady-state velocity is \( \mathbf{v}_0 + \tau_c \mathbf{a}_0 + (\tau_c/m) \mathbf{k} \), following the expenditure of the finite energy supplied by the initial acceleration and the impulse. The self-force both moderates the effects of the initial acceleration and the impulse while generating radiation.

4.2.3 Frictional Force

Substituting the frictional force \( \mathbf{f}_{\text{applied}} = -k \mathbf{v} \) into (34), we get the equation of motion

\[
b \ddot{\mathbf{v}} + m \dot{\mathbf{v}} + k \mathbf{v} = 0
\]

Integrating,

\[
\mathbf{v}(t) = \{ \mathbf{v}_0 \cosh \alpha \nu_c t + (1/\alpha \nu_c) (\mathbf{a}_0 + \frac{1}{2} \nu_c \mathbf{v}_0) \sinh \alpha \nu_c t \} \exp(-\frac{1}{2} \nu_c t)
\]

where \( \alpha = \frac{1}{2} \left( 1 - \sqrt{1 - 4 \tau_c k/m} \right) \). For any system of interest we have \( \tau_c k/m < 1 \). The rate of decay of the initial velocity is increased slightly by the self-force generating radiation.

4.2.4 Constant Uniform Electric Field

We have \( \mathbf{f}_{\text{applied}} = e \mathbf{E} \) with the electric field \( \mathbf{E} \) constant. The equation of motion (34) is

\[
b \ddot{\mathbf{v}} + m \dot{\mathbf{v}} = e \mathbf{E}
\]

Integrating,

\[
\mathbf{v} = \mathbf{v}_0 + \tau_c \mathbf{a}_0 + (e/m)(t - \tau_c) \mathbf{E} - \tau_c e^{-\nu_c t} (\mathbf{a}_0 - (e/m) \mathbf{E})
\]

Following the initial interval of \( O(\tau_c) \), the acceleration is constant and the radiated power, \( S^{(\text{rad})} > 0 \). Also, \( S^{(\text{rev})} = S^{(\text{rad})} \), since \( \dot{\mathbf{v}} = 0 \). Thus, the particle is accelerated by the applied force and, although there is no overall working by the self-force, the self-force creates the radiation and the increasing kinetic energy of the self-field.
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4.2.5 Harmonic Oscillator Field

The force $f_{\text{applied}} = -kr$, with $k > 0$, is the harmonic force centred at the origin. With this applied force, the equation of motion (34) is

$$b \ddot{\mathbf{r}} + m\dot{\mathbf{r}} + kr = 0$$ (37)

On taking the scalar product with $\mathbf{r} \times \dot{\mathbf{r}}$, and noting that $\mathbf{r} \times \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \ddot{\mathbf{r}}$, we find

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) + \nu_c \mathbf{r} \times \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = 0$$ (38)

with the solution

$$\mathbf{r} \times \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = \mathbf{r}_0 \times \dot{\mathbf{r}}_0 \cdot a_0 e^{-\nu_c t}$$ (39)

Therefore, any initial acceleration parallel to the angular velocity decays, and the motion settles down to lie in the plane of the angular velocity.

The equation of motion, (37), on setting $k = m\Omega^2$ and multiplying by $\tau_c^3/b$ is

$$\tau_c^3 \ddot{\mathbf{r}} + \tau_c^2 \dot{\mathbf{r}} + \lambda^2 \mathbf{r} = 0$$

where the parameter $\lambda = \tau_c \Omega$—it is evident that for any system of interest $\lambda \ll 1$.

With the trial solution $\mathbf{r}(t) = e^{\alpha \nu_c t}$, we find $\alpha^3 + \alpha^2 + \lambda^2 = 0$. This cubic has one real root and two complex roots that we write in the form

$$\alpha_0 = -1 - 2\kappa \tau_c, \quad \alpha_1 = \kappa \tau_c - i \omega \tau_c, \quad \alpha_2 = \kappa \tau_c + i \omega \tau_c$$

where the constants $\omega$ and $\kappa$ are given by

$$\omega \tau_c = \frac{\sqrt{3}}{2}(\Lambda^+ - \Lambda^-), \quad \kappa \tau_c = \frac{1}{2}(\Lambda^+ + \Lambda^-) - \frac{1}{3}$$ (40)

with

$$\Lambda^\pm = \left[\frac{1}{27} + \frac{1}{2} \lambda^2 \pm \sqrt{\frac{1}{27} \lambda^2 + \frac{1}{4} \lambda^4}\right]^{1/3}$$

and the constants satisfy the identity $\omega^2 = 3\kappa^2 + 2\nu_c \kappa$.

The general solution of (37) can be written in the form

$$\mathbf{r}(t) = a e^{(-\nu_c+2\kappa) t} + e^{\kappa t}[b \cos \omega t + c \sin \omega t]$$

where $a$, $b$, and $c$ are constants of integration. On expanding $\Lambda^\pm$ in a power series, from (40), we find

$$\omega \tau_c = \lambda + O(\lambda^3), \quad \kappa \tau_c = \frac{1}{2} \lambda^2 + O(\lambda^3)$$

from which, to terms $O(\lambda^2)$, we find $\omega = \Omega$ and $\kappa = \frac{1}{2} \lambda \Omega$.

Following the initial interval of $O(\tau_c)$, the motion takes place in the plane defined by $b$ and $c$ in a slowly expanding elliptical logarithmic spiral with constant angular velocity. The source of energy for this spiral motion, with the slowly increasing speed of the particle and the radiation generated, is the interaction of the self-force and the harmonic field. Both the kinetic energy and the harmonic potential energy increase with time.
4.2.6 Constant Uniform Magnetic Field

With an applied magnetic field the equation of motion \((34)\) is

\[
b \ddot{v} + m \dot{v} = \frac{e}{c} v \times B
\]

Let the magnetic field be \(B = (0, 0, B)\), with \(B\) constant, and write \(v = (u, v, w)\) so that \(v \times B = (Bv, -Bu, 0)\). Accordingly, the motion in the direction of the \(z\)-axis is that of a free particle; from \((36)\),

\[
v_z = v_{z0} + \tau c (1 - e^{-\nu c t}) a_z0
\]

For the motion in the \(xy\)-plane we use matrix notation. Let

\[
v = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

then, with \(\Omega = eB/mc\), the equation of motion, projected onto the \(xy\)-plane, is

\[
\tau c \ddot{v} + \dot{v} = \Omega Av \tag{41}
\]

The solution to the equation of motion \((41)\) is

\[
v(t) = e^{\lambda t} e^{A\omega t} h + e^{-(\nu c + \kappa) t} e^{-A\omega t} k \tag{42}
\]

where the column matrices \(h\) and \(k\) are to be determined by the initial conditions,

\[
v(0) = v_0 , \quad \dot{v}(0) = a_0
\]

We remark that

\[
e^{A\omega t} = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}
\]

The two constants, both positive, are given by

\[
\omega = \frac{1}{2} \nu c \sqrt{\frac{1}{2} \sqrt{1 + 16\lambda^2} - \frac{1}{2}} , \quad \kappa = \frac{1}{2} \nu c \left[ \sqrt{1 + 4\tau \omega^2} \right]
\]

where \(\lambda = \tau c \Omega\); they satisfy the identity \(\omega^2 = \kappa^2 + \nu c \kappa\).

It is evident that for any system of interest \(\lambda \ll 1\). Expanding \(\kappa\) and \(\omega\) we find

\[
\omega = \Omega + O(\lambda^3) , \quad \kappa = \lambda \Omega + O(\lambda^3)
\]

With these approximations the solution \((42)\) simplifies to

\[
v(t) = e^{\lambda t} e^{A\omega t} h + e^{-\nu c t} e^{-A\omega t} k
\]

The motion of the particle, given in \((42)\), projected onto the \(xy\)-plane, is in a slowly expanding logarithmic spiral with constant angular velocity, modified in the initial interval of \(O(\tau_c)\). Since \(V^T Av = 0\), there is no work done by the magnetic field directly on the particle, and we have \(\tilde{T} = S^{(\text{rad})}\). From \((42)\), we find

\[
\tilde{T} = S^{(\text{rad})} = b(k^2 + \omega^2) e^{2\nu c t} h^T h + O(e^{-\nu c t})
\]

Manifestly, the source of energy for this spiral motion, with the slowly increasing speed of the particle and the radiation generated, is the magnetic field.
4.2.7  **Coulomb Field**

With the Coulomb force, the equation of motion (34) becomes

\[ b\ddot{r} + m\ddot{r} + \frac{e^2}{r^3} r = 0 \]

On taking the scalar product with \( \mathbf{r} \times \dot{\mathbf{r}} \) we again get the equation (38) with its solution (39). Therefore, any initial acceleration parallel to the angular velocity decays, and the motion settles down to lie in the plane through the origin and perpendicular to the angular velocity.

Taking the scalar product of the equation of motion with \( \dot{\mathbf{r}} \), noting (35) and rearranging, we get

\[ \frac{d}{dt} \left( T - \frac{e^2}{r} \right) = S^{(\text{rad})} > 0 \]

The total energy of the particle is slowly increasing: the source of this energy being the power derived from the self-force.

In polar coordinates, the radial and transverse components of the equation of motion are

\[ \ddot{r} - r\dot{\theta}^2 + \frac{e^2}{m} \frac{1}{r^2} = -\tau_c \left( \ddot{r} - 3\dot{r}\dot{\theta}^2 - 3r\dot{\theta}\ddot{\theta} \right) \]

\[ 2\dot{r}\dot{\theta} + r\ddot{\theta} = -\tau_c \left( 3\ddot{\theta} + 3\dot{r}\ddot{\theta} + r\dddot{\theta} - r\ddot{\theta}^3 \right) \]

We seek the motion for \( t \gg \tau_c \). The self-force, on the right-hand side, is to be treated as a perturbation term. In the lowest order of approximation, with the right-hand side set to zero, for circular motion the solution is

\[ r = a, \quad \dot{\theta} = \Omega \]

where \( a \) is the radius and \( \Omega = \sqrt{e^2/ma^3} \) is the circular frequency, and \( \tau_c \Omega \ll 1 \).

Iterating this solution into the right-hand side gives

\[ \ddot{r} - r\dot{\theta}^2 + a^3\Omega^2 \frac{1}{r^2} = 0 \]

\[ 2\dot{r}\dot{\theta} + r\ddot{\theta} = a\tau_c \Omega^3 \]

The trial solution \( r = a(1 + \alpha t), \dot{\theta} = \Omega(1 + \beta t) \), with \( \alpha \) and \( \beta \) constants, yields

\[ r = a(1 + 2\tau_c \Omega^2 t) \quad \text{and} \quad \dot{\theta} = \Omega(1 - 3\tau_c \Omega^2 t) \]

accurate to terms \( O(\tau_c, \Omega) \) in the interval \( \tau_c \ll t \ll 1/\tau_c \Omega^2 \).

Following an initial interval of \( O(\tau_c) \), the motion takes place in the plane defined by the angular momentum. Initial circular motion becomes an expanding spiral with decreasing particle speed and angular velocity. However, the total energy and the angular momentum both increase, driven by the Coulomb field.
5 Conclusion

Our conceptual shift is to treat the physical particle as a composite of the (bare) source particle together with the energy-momentum that is reversibly generated in its self-field by its motion. This energy-momentum reversibly generated in the self-field is bound to its source particle and is manifested as kinetic energy. Accordingly, the total kinetic energy of the physical particle is the sum of the inertial kinetic energy and the bound kinetic energy of the self-field. The energy-momentum irreversibly generated in the self-field is propagated from the point at which it is generated as radiation.

In order to be consistent with the total kinetic energy of the physical particle being defined as the sum of the kinetic energy generated in the self-field and the inertial kinetic energy of the source particle, the corresponding four-forces are summed—see (18) and (22)—and this results in the self-force having its sign opposite to that in the Lorentz–Dirac equation. Thus, as a direct consequence of our conceptual shift, we find Dirac’s change of sign for the self-force.

The self-force is the highest order (third) derivative in the equation of motion and is, in general, a very small perturbation on the other forces. The solutions to the applications reveal that the underlying physical explanation for the behaviour of the particle is that the self-force extracts energy-momentum from the applied force and converts it into radiation and kinetic energy for the particle. It is the latter characteristic of the self-force that causes a particle in orbital motion to spiral outwards, albeit very slowly.

The new equation of motion, solved as an initial value problem, has solutions giving ‘reasonable behaviour’ with the physical interpretation of the self-force and its effects being straightforwardly comprehensible. The theory is mathematically well-defined, self-consistent, and conforms to all well-established principles, including the principle of causality.

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