Initial Value Problems and Signature Change

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Abstract

We make a rigorous study of classical field equations on a 2-dimensional signature changing spacetime using the techniques of operator theory. Boundary conditions at the surface of signature change are determined by forming self-adjoint extensions of the Schrödinger Hamiltonian. We show that the initial value problem for the Klein–Gordon equation on this spacetime is ill-posed in the sense that its solutions are unstable. Furthermore, if the initial data is smooth and compactly supported away from the surface of signature change, the solution has divergent $L^2$-norm after finite time.

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1 Introduction

A natural generalisation of general relativity is obtained by relaxing the restriction that the metric tensor is everywhere Lorentzian. This subject has produced a spirited debate on the nature of the junction conditions which must be satisfied by the metric tensor and matter fields at the surface of signature change [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Part of the difficulty in finding a consensus in this debate is that much depends on what one regards as pathological.

One approach [4, 5, 7, 8, 9] has been to assume certain differentiability conditions on the spacetime metric and matter fields and then choose junction conditions which provide a bounded energy momentum tensor. For example, consider a discontinuously signature changing metric that satisfies the Einstein equations, and assume that the metric is $C^2$ everywhere (except at the surface of signature change where only the induced metric is required to be $C^2$). Then the energy momentum tensor is bounded at the surface of signature change if and only if the second fundamental form of that surface vanishes [1, 4, 8]. As a second example, consider a scalar field propagating on a spacetime that continuously changes signature. Assuming that the scalar field is everywhere $C^3$ then the component of its momentum normal to the surface of signature change must vanish at that surface (see lemma 3 in [9]). This also implies that the energy momentum tensor of the scalar field is bounded at the surface of signature change.

A second approach [2, 6, 10] (and the one we will adopt in this paper) is to relax some of the assumptions under which the above junction conditions are derived. We will study signature change, with no coupling between gravity and matter, under the assumption that the matter fields are continuous, but not necessarily differentiable, at the surface of signature change [‡]. In this situation, there are no pre-ordained junction conditions on the matter fields.

[†] For a continuously signature changing metric, the conditions are slightly stronger, and we recommend that the reader notes the rigorous work in [8] (in particular see theorems 3 and 4).

[‡] This assumption is not unreasonable and there are many parallels in physics; for example, a sound wave that passes between two different media is only $C^0$ at the interface between the media.
at the surface of signature change. This philosophy was used in [10], where a variety of different junction conditions were used to study the propagation of classical fields on a fixed spacetime whose signature changed from Lorentzian to Kleinian. It was shown that a wide range of field behaviour could be generated in this way.

Ultimately, of course, one wants to choose the junction conditions which best model the physical scenario. Unfortunately, the scattering properties of a surface of signature change are unknown, and in this situation, it makes sense to choose boundary conditions which provide the most well-behaved system. If, with such a choice, the system is still poorly defined then one has grounds for concluding that the situation which is being modelled is unphysical. However, if the system is fairly well-defined, one should obtain some reasonable physical predictions, which can then be used to reassess the theory. One method of obtaining a well-behaved system is to impose smoothness conditions as mentioned above. In this paper, we investigate another possibility which utilises Hilbert space techniques.

For simplicity, we study a 2-dimensional spacetime with signature change from \((+-)\) to \((++),\) and examine classical fields that propagate, with no interaction, on the background spacetime\(\ddagger\). We identify + with time, and therefore this type of signature change has more similarity with a Lorentzian to Kleinian signature transition [10] than a Lorentzian to Riemannian signature transition. The existence of a global time parameter allows us to write the matter field equations in Hamiltonian form. We then define the matter field Hamiltonians as Hilbert space operators so that any junction conditions which are applied to a matter field are equivalent to boundary conditions satisfied by functions in the domain of its Hamiltonian.

In order to obtain a well behaved system, we require the Schrödinger Hamiltonian (ie. minus the Laplacian on the background spacetime) to be self-adjoint, and we also require the boundary conditions to take a general form which enforces continuity of the functions in its domain. This choice proves to be highly successful, as it singles out a unique set of boundary conditions for the

\(\ddagger\) The background spacetime that we study satisfies the vacuum Einstein equations exactly (ie. there are no distributional terms at the surface of signature change). This can be checked either by direct calculation or by using theorem 4 of [8].
matter fields. These boundary conditions were previously isolated in [6] using rather different methods. Moreover, self-adjointness allows us to construct a complete set of (generalised) eigenfunctions for the Laplacian, which facilitates the construction of $L^2$-solutions to the Schrödinger and Klein–Gordon equations.

In section 2 we briefly review the necessary operator theory, before proceeding in section 3 to examine the properties of the Dirac, Schrödinger and Klein–Gordon Hamiltonians on the signature changing spacetime. Using these properties, we explain why our methods cannot be applied to the Dirac Hamiltonian. In section 4 we determine the boundary conditions corresponding to self-adjoint extensions of the Schrödinger Hamiltonian. Solutions of the Schrödinger and Klein–Gordon equations are constructed and analysed in section 5. We show that the initial value problem for the Klein–Gordon equation is ill-posed in the sense that its solutions are unstable, and that the evolution of reasonable initial data leads to runaway solutions. In section 6 we generalise our results and show that they hold for a large class of signature changing metrics.

## 2 Operator Theory

Physical systems are often modelled using operators on a Hilbert space. If such a system exhibits singular points (where the formal expression of an operator does not apply in an obvious way), it is often initially convenient to define the action of the operators on a space of ‘nice’ functions supported away from the singular points. In our case of interest, this occurs at the surface of signature change; although it is clear how to define our operators away from this surface, it is not \textit{a priori} clear how to define them on the surface. The initial choice of domain is not generally sufficient to effectively model the physical situation, in particular, the operator may not be self-adjoint when restricted to this domain. If self-adjointness is required, one can try to extend the domain and action of our operator in order to determine a rigorously defined self-adjoint operator, which is called a \textit{self-adjoint extension} of the original operator. For differential operators, this is usually equivalent to the specification of appropriate boundary conditions at singular points (in
our case, at the surface of signature change).

## 2.1 Preliminary definitions

Although they are standard (see eg. [11]), the following definitions have been included so that the work presented here is self-contained.

Let $A$ be an operator defined on a dense subspace $D$ of a Hilbert space $H$ with an inner product $\langle \cdot | \cdot \rangle$ which is anti-linear in the first slot and linear in the second slot.

### Definition.
If for some $\phi \in H$ there exists $\eta$ such that $\langle \phi | A \psi \rangle = \langle \eta | \psi \rangle$ for all $\psi \in D$, then $\phi$ is said to be in the domain of the adjoint $A^*$ of $A$, and $A^*$ is defined on $\phi$ by $A^* \phi = \eta$.

### Definition.
Let $A'$ be any operator whose domain includes $D$, then $A'$ will be called an extension of $A$ if $A' \psi = A \psi$ for all $\psi \in D$. $A$ is said to be symmetric if $A^*$ is an extension of $A$, (ie. if $\langle \psi | A \phi \rangle = \langle A \psi | \phi \rangle$ for all $\psi, \phi \in D$).

### Definition.
$A$ is said to be self-adjoint if $A = A^*$, (ie. if $A$ is symmetric and has the same domain as its adjoint).

### Definition.
If the domain of $A^*$ is dense, then we define the closure $\bar{A}$ of $A$ by $\bar{A} = A^{**}$. $A$ is said to be essentially self-adjoint if $\bar{A}$ is self-adjoint. Moreover if $A$ is essentially self-adjoint then $\bar{A} = A^*$, and $A^*$ is the unique self-adjoint extension of $A$.

Let us emphasise that an operator (on a Hilbert space) does not exist independently of either its domain or the Hilbert space on which it is defined. For example, when one says that the operator $A$ is self-adjoint, it is implicit that one is working with the the domain $D$ and the Hilbert space $H$ on which $A$ is defined, as well as the inner product $\langle | \cdot \rangle$ of $H$. 

2.2 Deficiency indices

The von Neumann theory of deficiency indices [11] can be used to determine whether a symmetric operator $A$ with domain $\mathcal{D}$ has any self-adjoint extensions, and if so, to construct them.

We define the \textit{deficiency subspaces} $\mathcal{K}^+$ and $\mathcal{K}^-$ by

$$\mathcal{K}^\pm = \ker(A^* \mp i),$$

(2.1)

(i.e. $\mathcal{K}^\pm$ is the subspace of $L^2$-solutions of equation $A^*\psi = \pm i\psi$) and define the \textit{deficiency indices} by $n^\pm = \dim(\mathcal{K}^\pm)$. There are three possible cases:

(1) $n^+ \neq n^-$ $\iff$ $A$ has no self-adjoint extensions.
(2) $n^+ = n^- = 0$ $\iff$ $A$ has a unique self-adjoint extension.
(3) $n^+ = n^- = N > 0$ $\iff$ $\exists$ a unique family of self-adjoint extensions of $A$ labelled by $U(N)$.

In case (2), $A$ is essentially self-adjoint and its unique self-adjoint extension is equal to $A^*$. In case (3), which will be our case of interest, the self-adjoint extensions are parametrised as follows: representing $U(N)$ as the space of unitary maps from $\mathcal{K}^+$ to $\mathcal{K}^-$, we define for each $U \in U(N)$ the domain $\mathcal{D}_U$ by

$$\mathcal{D}_U = \{ \psi + \chi^+ + U\chi^+ | \psi \in \mathcal{D}, \chi^+ \in \mathcal{K}^+ \}.$$  

(2.2)

The operator $A^*|_{\mathcal{D}_U}$ is clearly an extension of $A$, moreover, $A^*|_{\mathcal{D}_U}$ is essentially self-adjoint with unique self-adjoint extension $A_U$ given by

$$A_U = (A^*|_{\mathcal{D}_U})^*.$$  

(2.3)

The full set of self-adjoint extensions of $A$ is then provided by the family $\{A_U | U \in U(N)\}$. 

5
3 Hamiltonian Form of the Classical Wave Equations

We will consider a simple 2-dimensional model in which the spacetime signature changes from \((+-)\) to \((++\)). We identify ‘+’ with time; thus this type of signature change, in which a spatial coordinate turns into a time coordinate, has more similarity with four-dimensional Lorentzian to Kleinian signature change \([10]\), than with four-dimensional Lorentzian to Riemannian signature change.

To begin our investigation we will examine discontinuous signature change with the flat metric

\[ ds^2 = g_{ab}dx^a dx^b = dt^2 + \text{sign}(z)dz^2. \] (3.1)

We show in section 6 that the results gained by studying this metric also apply to more general signature change metrics, including metrics which exhibit continuous signature change.

Let us briefly consider the measure density appropriate to surfaces of constant \(t\) in this and similar metrics. Since our intention is to use Hilbert space techniques, we seek a real, positive measure. If \(g_{11}\) were everywhere negative (ie. the usual Lorentzian case), one would use \((-g_{11})^{1/2}dz\). However, for signature changing metrics, this measure becomes complex, and hence we employ the measure \(|g_{11}|^{1/2}dz\), which is real and positive. For the metric (3.1), this reduces to \(dz\). It is suggested in \([6]\) that one could allow the orientation of the volume element to change across the surface of signature change, which would lead to a measure \(\text{sign}(z)|g_{11}|^{1/2}dz\). The inner product space defined using this measure would be a Krein space \([12]\) rather than a Hilbert space. Analysis in such indefinite inner product spaces is considerably more involved than in Hilbert spaces, and we will not discuss this possibility further in this paper.
3.1 The Dirac Hamiltonian

In ordinary Minkowski space one can easily define a self-adjoint Dirac Hamiltonian [13]. We now show that this is not possible in the flat signature changing spacetime (3.1). In this section, the Hilbert space of spinors is taken to be $L^2(\mathbb{R}, dz) \otimes \mathbb{C}^4$ with its usual inner product. Where no confusion can arise, we will often write $L^2(\mathbb{R})$ for $L^2(\mathbb{R}, dz)$. In addition, for any subset $S$ of $\mathbb{R}$, we denote the space of smooth functions compactly supported in $S$ by $C^\infty_0(S)$.

For $z < 0$ define the gamma matrices $\gamma^a_L$ by $\{\gamma^a_L, \gamma^b_L\} = 2g^{ab}$, and similarly, for $z > 0$ define the Kleinian gamma matrices $\gamma^a_K$ by $\{\gamma^a_K, \gamma^b_K\} = 2g^{ab}$. Choosing $\gamma^0_K = \gamma^0_L$, there are two possibilities for $\gamma^1_K$, given by $\gamma^1_K = \pm i\gamma^1_L$. We will use the standard representation, in which

$$
\gamma^0_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma^1_L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

(3.2)

The Dirac equation for the signature changing spacetime is

$$
(\gamma^a_L \partial_a + im)\Psi = 0 \quad z < 0,
$$

(3.3)

$$
(\gamma^a_K \partial_a + im)\Psi = 0 \quad z > 0.
$$

(3.4)

Defining

$$
\xi_\pm(z) = \begin{cases} 1 & z < 0 \\ \pm i & z > 0, \end{cases}
$$

(3.5)

the Dirac equation may be written in the Hamiltonian form

$$
i \frac{\partial}{\partial t} \Psi = D_\pm \Psi,$$

(3.6)

where the two candidate Dirac Hamiltonians

$$D_\pm = \gamma^0_L \left[ m - i\xi_\pm(z)\gamma^1_L \frac{\partial}{\partial z} \right]
$$

(3.7)

correspond to the two possible choices of Kleinian gamma matrices.
Initially, we define $D_\pm$ on the dense domain $\mathcal{D} = C_0^\infty(\mathbb{R}\setminus\{0\}) \otimes \mathbb{C}^4$ of smooth spinors compactly supported away from the origin in the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^4$. A short integration by parts argument shows that

$$D_\pm^*|_{\mathcal{D}} = \gamma_0^1 \left[ m - i \xi_\pm(z)^* \gamma_0 \frac{\partial}{\partial z} \right] = D_\pm^\dagger.$$  \hspace{1cm} (3.8)

Thus neither $D_+$ nor $D_-$ is symmetric; accordingly, neither admits any self-adjoint extensions. Since any reasonable local Dirac Hamiltonian should certainly contain $\mathcal{D}$ in its domain, we conclude that the spacetime (3.1) does not admit a self-adjoint Dirac Hamiltonian.

### 3.2 The Schrödinger Hamiltonian

The two candidate Dirac Hamiltonians both square to give the same operator, namely

$$D_\pm^2 = (H + m^2) \otimes 1$$ \hspace{1cm} (3.9)

on $C_0^\infty(\mathbb{R}\setminus\{0\}) \otimes \mathbb{C}^4$ in $L^2(\mathbb{R}) \otimes \mathbb{C}^4$, where the operator $H$ is given by

$$H = \text{sign}(z) \frac{\partial^2}{\partial z^2}$$ \hspace{1cm} (3.10)

with domain $C_0^\infty(\mathbb{R}\setminus\{0\}) \subset L^2(\mathbb{R})$. Note that since we are only concerned with functions that are compactly supported away from the origin, we need not concern ourselves with derivatives of $\xi(z)$ at $z = 0$. We will refer to $H$ as the Schrödinger Hamiltonian. Since $-\frac{\partial^2}{\partial z^2}$ is unbounded from above on $L^2(\mathbb{R}^-)$ and $\frac{\partial^2}{\partial z^2}$ is unbounded from below on $L^2(\mathbb{R}^+)$, we note that $H$ is unbounded from both above and below.

The fact that $H$ is symmetric can be demonstrated with an integration by parts argument, and hence we can construct the self-adjoint extensions of $H$ using the von Neumann theory of deficiency indices.
3.3 The Klein–Gordon Hamiltonian

The Klein–Gordon equation on the discontinuously signature changing space-time (3.1) is

\[
\left( g^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} + m^2 \right) \phi = 0 .
\]  

(3.11)

Let

\[
\Phi(t) = \begin{pmatrix} \dot{\phi} \\ \phi \end{pmatrix},
\]

(3.12)

where \( \dot{\phi} = \frac{\partial \phi}{\partial t} \), then we may write the Klein–Gordon equation in ‘first-order form’ as

\[
\frac{\partial \Phi}{\partial t} = K \Phi ,
\]

(3.13)

where the operator

\[
K = \begin{pmatrix} 0 & 1 \\ -(H + m^2) & 0 \end{pmatrix}
\]

(3.14)

on \( C^\infty_0(\mathbb{R}\backslash\{0\}) \oplus L^2(\mathbb{R}) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) will be called the Klein–Gordon Hamiltonian.

Clearly \( K \) is not symmetric (on this domain and Hilbert space) and therefore does not admit any self-adjoint extensions. Despite this, the special form of \( K \) and the fact that \( H \) has self-adjoint extensions will allow us to construct rigorously defined solutions to (3.13).

4 Boundary Conditions

In this section we construct self-adjoint extensions of the Schrödinger Hamiltonian \( H \), and study the boundary conditions which provide these extensions. There are many possible sets of boundary conditions for a matter field \( \phi \), however, we are interested in the subset which describe how \( \phi \) and \( \frac{\partial \phi}{\partial z} \) behave as they cross the surface of signature change. Since we have no data to guide us,
a reasonable physical assumption is that the field $\phi$ is everywhere continuous. This leaves the behaviour of $\frac{\partial \phi}{\partial z}$ to be determined. Thus we are looking for boundary conditions of the form

$$
\phi(0^-) = \phi(0^+), \quad (4.1)
$$

$$
\frac{\partial \phi}{\partial z}(0^-) = \omega \frac{\partial \phi}{\partial z}(0^+)
$$

for some $\omega \in \mathbb{C}, \omega \neq 0$.

### 4.1 Self-adjoint extensions

We will now construct the self-adjoint extensions of $H$. Applying the von Neumann theory of deficiency indices we try to find $L^2$-bases for $\mathcal{K}^+$ and $\mathcal{K}^-$. We find that $\mathcal{K}^+$ is spanned by

$$
\chi^+_1 = \begin{cases} 
2^{\frac{i}{4}} \exp(-e^{i\frac{3\pi}{4}z}) & z < 0 \\
0 & z > 0 
\end{cases} 
\chi^+_2 = \begin{cases} 
0 & z < 0 \\
2^{\frac{i}{4}} \exp(-e^{i\frac{\pi}{4}z}) & z > 0 
\end{cases}
$$

and $\mathcal{K}^-$ is spanned by

$$
\chi^-_1 = \begin{cases} 
2^{\frac{i}{4}} \exp(+e^{i\frac{\pi}{4}z}) & z < 0 \\
0 & z > 0 
\end{cases} 
\chi^-_2 = \begin{cases} 
0 & z < 0 \\
2^{\frac{i}{4}} \exp(+e^{i\frac{3\pi}{4}z}) & z > 0 
\end{cases}
$$

where the normalisation $||\chi^+_1|| = ||\chi^+_2|| = ||\chi^-_1|| = ||\chi^-_2|| = 1$ has been implemented. The deficiency indices are thus $n^+ = n^- = 2$, and hence there is a family of self-adjoint extensions of $H$ labelled by $U(2)$.

As described in section 2, for each $U \in U(2)$ we have a unique self-adjoint extension $H_U$ of $H$ given by

$$
H_U = (H^*|_{D_U})^* 
$$

where

$$
D_U = \{ \phi + \chi^+ + U\chi^+ | \phi \in C^\infty_0(\mathbb{R}\setminus\{0\}), \chi^+ \in \mathcal{K}^+ \}. 
$$
We now construct the boundary conditions which correspond to $H_u$. For any $\rho, \psi \in \mathcal{D}(H^*)$ it is possible to show that

$$\langle \rho | H^* \psi \rangle = \langle H^* \rho | \psi \rangle + \lim_{z \to 0^-} \left[ \frac{\partial \rho}{\partial z} \psi - \frac{\partial \psi}{\partial z} \rho \right] + \lim_{z \to 0^+} \left[ \frac{\partial \rho}{\partial z} \psi - \frac{\partial \psi}{\partial z} \rho \right]. \quad (4.6)$$

We note in passing that if we were studying $-\frac{\partial^2}{\partial z^2}$ instead of $H = \text{sign}(z) \frac{\partial^2}{\partial z^2}$ on $C_0^\infty(\mathbb{R}\{0\}) \subset L^2(\mathbb{R})$, then the boundary terms in the analogue to (4.6) would differ by a relative sign [14].

The domain of $H_u$ consists of those $\rho \in \mathcal{D}(H^*)$ for which the boundary terms in (4.6) vanish for all $\psi \in \mathcal{D}_u$. Clearly for $\psi \in C_0^\infty(\mathbb{R}\{0\})$ the boundary terms are zero, hence it suffices to consider $\psi \in \{ \chi^+ + U\chi^+ | \chi^+ \in \mathcal{K}^+ \}$. By choosing such $\psi$ we are able to determine the boundary conditions which $\rho^+$ and $\frac{\partial \rho}{\partial z}$ must satisfy at $z = 0$.

Choosing the basis (4.2) for $\mathcal{K}^+$ and the basis (4.3) for $\mathcal{K}^-$, the unitary map $U$ can be written as a matrix

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (4.7)$$

where

$$a\bar{a} + b\bar{b} = 1, \quad c\bar{c} + d\bar{d} = 1, \quad a\bar{c} + b\bar{d} = 0. \quad (4.8)$$

By substituting for $\psi$ in (4.6), it can now be shown that the boundary terms vanish if and only if

$$A \begin{pmatrix} \frac{\partial \rho^+}{\partial z}(0^-) \\ \frac{\partial \rho^+}{\partial z}(0^+) \end{pmatrix} = e^{i\pi/4} B \begin{pmatrix} \rho^+(0^-) \\ \rho^+(0^+) \end{pmatrix} \quad (4.9)$$

where the matrices $A$ and $B$ are given by

$$A = \begin{pmatrix} a + 1 & c \\ b & d + 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a - i & ic \\ b & id - 1 \end{pmatrix}. \quad (4.10)$$

Thus for each unitary matrix $U$ there is a self-adjoint extension $H_u$ of $H$ with a corresponding set of boundary conditions (4.9).
Finally, we note that the addition of a constant term to $H$ does not alter this class of boundary conditions, and therefore the self-adjoint extensions of $H + m^2$ follow immediately from those of $H$.

### 4.2 Determining the boundary conditions

We wish to find self-adjoint extensions of $H$ with corresponding boundary conditions of the form (4.1). In order to complete this task, it is clear from (4.9) that $A$ and $B$ must be singular, since if either $A$ or $B$ is invertible, we will obtain boundary conditions which relate the field to its derivative. In the appendix we prove the surprising result that there is only one self-adjoint extension satisfying the above restrictions. Its corresponding boundary conditions are

\[
\rho(0^-) = \rho(0^+) ,
\]

(4.11)

\[
\frac{\partial \rho}{\partial z}(0^-) = -\frac{\partial \rho}{\partial z}(0^+) .
\]

This is the set of boundary conditions which we will adopt in section 5 for the construction of wave equation solutions. These boundary conditions were previously isolated, using rather different methods, in [6]. In fact, the authors of [6] also consider the case where the orientation of the volume form is different on the two sides of the boundary, in which case the relative minus between the normal derivatives $\frac{\partial \rho}{\partial z}(0^+)$ is removed. As we mentioned in section 3, we exclude this possibility from our treatment as we require the integration measure to be both real and positive.

In [10] particular attention was devoted to the boundary conditions $\rho^i$ continuous at $z = 0$, and either $\frac{\partial \rho}{\partial z}(0^-) = \frac{\partial \rho}{\partial z}(0^+)$ or $\frac{\partial \rho}{\partial z}(0^-) = \pm i \frac{\partial \rho}{\partial z}(0^+)$; it is worth emphasising that no self-adjoint extension corresponds to such boundary conditions.
5 Wave Equation Solutions

In this section we construct solutions to the Schrödinger and Klein–Gordon equations on the signature changing spacetime (3.1), satisfying the boundary conditions (4.11). Throughout this section \( H_U \) will represent the self-adjoint extension of \( H \) corresponding to these boundary conditions, and \( \mathcal{H} \) will denote the Hilbert space \( L^2(\mathbb{R}, dz) \).

5.1 Solutions of the Schrödinger equation

For our choice of self-adjoint extension, the Schrödinger equation on the signature changing spacetime (3.1) is

\[
i \frac{\partial \Psi}{\partial t} = H_U \Psi,
\]

and has general solution \( \Psi(t) = e^{-iH_\Omega t} \Psi(0) \). The evolution operator \( e^{-iH_\Omega t} \) is most conveniently constructed using the spectral representation of \( H_U \), which we now study.

Solving the time-independent Schrödinger equation

\[
H_U \tilde{\Psi} = E \tilde{\Psi}
\]

as an ODE subject to the boundary conditions (4.11), we obtain the generalised eigenfunctions (mode solutions)

\[
\tilde{\Psi}(z, E) = a(E) \left\{ (1-i) \theta(-z)\theta(-E) \exp(\sqrt{-E}z) + (1+i) \theta(+z)\theta(-E) \left[ \cos(\sqrt{-E}z) - \sin(\sqrt{-E}z) \right] \\
+ (1+i) \theta(-z)\theta(+E) \left[ \cos(\sqrt{+E}z) + \sin(\sqrt{+E}z) \right] + (1+i) \theta(+z)\theta(+E) \exp(-\sqrt{+E}z) \right\},
\]

where \( \theta \) is the usual step function and the function \( a(E) \) is

\[
a(E) = \frac{1}{\sqrt{2\sqrt{|E|}}}.
\]
Next, we define an integral transform $\mathcal{M} : L^2(\mathbb{R},dE) \to L^2(\mathbb{R},dz)$ by
\[
(\mathcal{M} f)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(E) \tilde{\Psi}(z,E) dE .
\] (5.5)

In appendix B, we show that $\mathcal{M}$ is unitary. Heuristically, this is equivalent to the continuum normalisation relations
\[
\int_{-\infty}^{\infty} dE \tilde{\Psi}(z,E)^* \tilde{\Psi}(z,E') = 2\pi\delta(E - E') ,
\] (5.6)
and
\[
\int_{-\infty}^{\infty} dz \tilde{\Psi}(z,E)^* \tilde{\Psi}(z,E') = 2\pi\delta(z - z') .
\] (5.7)

A simple calculation shows that $\mathcal{M}EM^{-1}$ agrees with $H$ on $C^0_\infty(0,\infty)$. It is therefore one of the self-adjoint extensions of $H$, and furthermore it is clear that it must be equal to $H_\nu$. Thus we have
\[
H_\nu = \mathcal{M}EM^{-1} ,
\] (5.8)

and using the spectral theorem [16], we see that the general solution of (5.1) satisfying the boundary conditions (4.11) may be written in initial value form as
\[
\Psi(z,t) = M e^{-iEt} M^{-1} \Psi(z,0) ,
\] (5.9)
for any $\Psi(z,0) \in \mathcal{H}$. For smooth compactly supported initial data, one may show that the solutions $\Psi(z,t)$ are $C^\infty$ everywhere except at $z = 0$ where they are $C^0$.

It would also be possible to give an explicit form for the integral kernel of the evolution operator $e^{-iH_\nu t}$. However, we will not do this here because our main focus is the Klein–Gordon equation.

## 5.2 Solutions of the Klein–Gordon equation

Given our choice of Schrödinger Hamiltonian $H_\nu$, the Klein–Gordon Hamiltonian becomes
\[
K_\nu = \begin{pmatrix} 0 & 1 \\ -(H_\nu + m^2) & 0 \end{pmatrix}
\] (5.10)
on $\mathcal{D}(H_u) \oplus L^2(\mathbb{R}) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Initially, we consider the massless case ($m = 0$) and observe, following [15], that the first order form of the Klein–Gordon equation
\[
\frac{\partial \Phi}{\partial t} = K_u \Phi \quad \text{ (5.11)}
\]
has a formal solution of the form
\[
\Phi(t) = \mathcal{T}(t)\Phi(0) \quad \text{ (5.12)}
\]
where
\[
\mathcal{T}(t) = \begin{pmatrix}
\cos(H_u^{1/2}t) & H_u^{-1/2} \sin(H_u^{1/2}t) \\
-H_u^{1/2} \sin(H_u^{1/2}t) & \cos(H_u^{1/2}t)
\end{pmatrix} \quad \text{ (5.13)}
\]
At first sight, $\mathcal{T}(t)$ seems to depend on some choice of square root. However, as noted in [15], the formal power series for $\cos(H_u^{1/2}t)$, $H_u^{-1/2} \sin(H_u^{1/2}t)$ and $H_u^{1/2} \sin(H_u^{1/2}t)$ are independent of any definition of the square root. Indeed, we have that
\[
\cos(H_u^{1/2}t) = 1 - \frac{H_u t^2}{2!} + \frac{H_u^2 t^4}{4!} - \cdots \quad \text{ (5.14)}
\]
\[
H_u^{-1/2} \sin(H_u^{1/2}t) = t - \frac{H_u t^3}{3!} + \frac{H_u^2 t^5}{5!} - \cdots \quad \text{ (5.15)}
\]
\[
H_u^{1/2} \sin(H_u^{1/2}t) = H_u t - \frac{H_u^2 t^3}{3!} + \frac{H_u^3 t^5}{5!} - \cdots \quad \text{ (5.16)}
\]
Thus $\mathcal{T}(t)$ can be unambiguously defined using the spectral theorem, with domain
\[
\mathcal{D}(\mathcal{T}(t)) = \mathcal{D}(H_u^{1/2} \sin(H_u^{1/2}t)) \oplus \mathcal{D}(\cos(H_u^{1/2}t)) \quad \text{ (5.17)}
\]
Note also that if $\Phi \in \mathcal{D}(\mathcal{T}(t_1))$ and $\mathcal{T}(t_1)\Phi \in \mathcal{D}(\mathcal{T}(t_2))$ then the group property
\[
\mathcal{T}(t_1 + t_2)\Phi = \mathcal{T}(t_2)\mathcal{T}(t_1)\Phi \quad \text{ (5.18)}
\]
is satisfied.

Let $E_{[A,B]}$ denote the spectral projector of $H_u$ on $[A, B]$ (ie. the space of position space functions with $'A \leq$ energy $\leq B'$). By considering the positive
and negative energy subspaces of $H$, then we see that $\cos(H^{1/2} t)$ may be written

$$\cos((H^{1/2} t) = \cos((H|_{\mathcal{E}_{[0,\infty)}})^{1/2} t) + \cosh((H|_{\mathcal{E}_{(-\infty,0)}})^{1/2} t). \quad (5.19)$$

Similar decompositions clearly exist for $H^{1/2} \sin(H^{1/2} t)$ and $H^{1/2} \sin(H^{1/2} t)$. Note that it is the exponentially growing parts of these decompositions which force us to restrict the domain of $T(t)$ to the subset of $\mathcal{H} \oplus \mathcal{H}$ given in (5.17). Thus we see that, in contrast to the bounded Schrödinger evolution $e^{-iHt}$, $T(t)$ is unbounded.

Using (5.19) and its analogues, and dominated convergence arguments similar to those employed in the proof of Theorem VIII.7 in [16], it is possible to show that, for any $\tau > 0$ and $\Phi(0) \in \mathcal{D}(KU)$,

(i) $T(t)\Phi(0) \in \mathcal{D}(KU)$ for all $t \in [0, \tau)$,

(ii) The vector-valued function $\Phi(t) = T(t)\Phi(0)$ is differentiable with respect to $t$ for all $t \in [0, \tau)$, with derivative $KU\Phi(t)$.

Thus $\Phi(t)$ is an $L^2$-solution of the Klein–Gordon equation (3.13) satisfying the boundary conditions (4.11) and with initial data $\Phi(0)$. Note that a different self-adjoint extension would yield solutions satisfying different boundary conditions at $z = 0$.

### 5.3 Initial Data for the Klein–Gordon equation

**Instability**

It is clear from (5.19) and the fact that $H$ is unbounded from below that the operator $T(t)$ is unbounded. As a consequence, there exist sequences of initial data $\Phi_n(0)$ with $\Phi_n(0) \to 0$ but $T(t)\Phi_n(0)$ divergent for any $t > 0$. For example, simply choose any $\{\Phi_n(0)\}$ such that $||\Phi_n(0)|| \leq \frac{1}{n}$ and

$$\Phi_n(0) \in E_{[-n,1-n]}\mathcal{H} \oplus E_{[-n,1-n]}\mathcal{H}. \quad (5.20)$$

Clearly $\Phi_n(0) \in \mathcal{D}(T(t))$ for all $t \in \mathbb{R}$ and $\Phi_n(0) \to 0$ as $n \to \infty$. However,

$$||T(t)\Phi_n(0)|| \geq \cosh((n-1)^{1/2} t), \quad (5.21)$$
which diverges as $n \to \infty$. This problem is related to the fact that initial value problems for elliptic equations are ill-posed (see eg. Hadamard’s example in Chapter III §6.2 of [17]). However, as is noted in [17], many problems which are of physical interest are actually ill-posed, and so we will proceed, but with caution.

**Runaway solutions**

A reasonable experiment would be to observe the behaviour of matter which propagated through a Lorentzian region and scattered off a Kleinian region. Such an experiment is modelled by evolving data which is initially compactly supported in the Lorentzian region $z < 0$.

Consider any smooth initial data $\Phi(0)$ with compact support $[-z_1, -z_0]$ for some real constants $z_1 > z_0 > 0$. Thus $\Phi(0)$ can be written as

$$
\Phi(0) = \Phi(z, 0) = \begin{cases} 
    \begin{pmatrix} 
        \phi_1(z) \\
        \phi_2(z) 
    \end{pmatrix} & z \in [-z_1, -z_0] \\
    \begin{pmatrix} 
        0 \\
        0 
    \end{pmatrix} & \text{otherwise}
\end{cases}
$$

(5.22)

for smooth compactly supported functions $\phi_1(z)$ and $\phi_2(z)$, where $z_1 = -\inf \{z | \Phi(z, 0) \neq 0\}$ and $z_0 = -\sup \{z | \Phi(z, 0) \neq 0\}$.

Let us define $\Phi^-(E, t)$ to be the negative energy part of the evolved data at time $t$ in the energy representation, ie.

$$
\Phi^-(E, t) = \theta(-E)((\mathcal{M}^{-1} \oplus \mathcal{M}^{-1})\Phi(t))(E),
$$

(5.23)

then we find

$$
\Phi^-(E, t) = k(E) \begin{pmatrix} 
    \cosh(\sqrt{|E|}t)\hat{\phi}_1(E) + \frac{1}{\sqrt{|E|}} \sinh(\sqrt{|E|}t)\hat{\phi}_2(E) \\
    \sqrt{|E|} \sinh(\sqrt{|E|}t)\hat{\phi}_1(E) + \cosh(\sqrt{|E|}t)\hat{\phi}_2(E) 
\end{pmatrix},
$$

(5.24)

where

$$
k(E) = \frac{(1 + i)\theta(-E)a(E)}{\sqrt{2\pi}} ,
$$

(5.25)
and the \( \hat{\phi}_i(E) \) are defined by

\[
\hat{\phi}_i(E) = \int_{-z_1}^{-z_0} \phi_i(z) e^{\sqrt{|E|}z} dz \quad i = 1, 2.
\]  

(5.26)

One may easily establish two decay estimates for these functions. Firstly, for each \( N \in \mathbb{Z} \), there exists a constant \( C_N \) such that

\[
|\hat{\phi}_i(E)| \leq \frac{C_N}{1 + |E|^{N/2}} e^{-\sqrt{|E|}z_0}
\]

(5.27)

for all \( E < 0 \). For \( N = 0 \), this is obvious; for general \( N \), one replaces \( \phi_i(z) \) by \( \phi_i(z) + (-1)^N d^N \phi_i/dz^N \) and applies the same argument. Secondly, for all sufficiently small \( \epsilon > 0 \) there exists a constant \( C > 0 \) such that

\[
|\hat{\phi}_i(E)| \geq C e^{-\sqrt{|E|}(z_0+\epsilon)}
\]

(5.28)

for all sufficiently negative \( E \). This estimate is proved by taking \( \epsilon \) small enough so that the real and imaginary parts of \( \phi \) are single-signed on \((-z_0 - \epsilon, -z_0)\).

Comparing with (5.24), it is clear that \( \Phi^{-}(E,t) \) is in \( L^2(\mathbb{R}^+, dE) \oplus L^2(\mathbb{R}^-, dE) \) for \( t \leq z_0 \), but that it fails to be \( L^2 \) for \( t > z_0 \). Thus the initial data cannot be evolved beyond the instant at which it begins to arrive at the surface of signature change. Such behaviour suggests that there is a severe back-reaction just after \( t = z_0 \). Further analysis shows that the solution is \( C^\infty \) for \( t < z_0 \).

It is easy to see that the re-introduction of mass makes little difference to our analysis. Essentially, the point is that the addition of a finite mass term still leaves the Hamiltonian unbounded from below, which is the root of all the above problems. If we re-define \( \Phi^{-}(E,t) = \theta(-E + m^2)((\mathcal{M}^{-1} \oplus \mathcal{M}^{-1})\Phi(t))(E) \), one can see that this portion of the solution is evolved by factors such as \( \cosh \sqrt{|E + m^2|}t \). The same decay bounds as before show that \( \Phi^{-}(E,t) \) fails to be \( L^2 \) as soon as \( t > z_0 \).

6 Variations and Generalisations
6.1 General Kleinian signature change

We now consider the class of 2-dimensional metrics (which includes the metric (3.1)) of the form

\[ ds^2 = dt^2 + h(z)dz^2, \quad (6.1) \]

where \( h(z) \) is \( C^\infty \) and non-zero on all compact sets which exclude the origin, and satisfies \( h(z) = \text{sign}(z) \) outside a compact neighbourhood of the origin. Thus \( h(z) \) is bounded away from the origin. Initially, we will also assume that \( z|h(z)|^{1/2} \to 0 \) as \( z \to 0 \) (as a two-sided limit). Metrics in this class exhibit precisely one change of signature from Lorentzian to Kleinian, due to \( h(z) \) either vanishing or becoming (mildly) singular or discontinuous at the origin.

Let \( \mathcal{H}' \) denote the Hilbert space \( L^2(\mathbb{R}, |h(z)|^{1/2}dz) \), and define

\[ H' = h(z)^{-1} \left( \frac{\partial^2}{\partial z^2} - \frac{h'(z)}{2h(z)} \frac{\partial}{\partial z} \right) \quad (6.2) \]

with domain \( \mathcal{D}' = C_0^\infty(\mathbb{R}\setminus\{0\}) \subset \mathcal{H}' \). This operator is equal to minus the Laplacian on the spacetime (3.1), and it is therefore the analogue of the operator \( H \) on \( \mathcal{D} = C_0^\infty(\mathbb{R}\setminus\{0\}) \subset L^2(\mathbb{R}, dz) \) on the spacetime (3.1). We now investigate the self-adjoint extensions of \( H' \).

Due to our assumption that the \( z|h(z)|^{1/2} \to 0 \) as a two-sided limit as \( z \to 0 \), we can define a new coordinate \( \omega \) by

\[ \omega(z) = \int_0^z dz'|h(z')|^{1/2} \quad (6.3) \]

which has the same sign as \( z \) and satisfies \( h(z)dz^2 = \text{sign}(\omega)d\omega^2 \). Thus we have a coordinate transformation from our new metric to the discontinuous case studied above. We note that, in general, this transformation is not smooth at \( z = 0 \).

Now consider the unitary operator \( W : L^2(\mathbb{R}, d\omega) \to \mathcal{H}' \) implementing the coordinate change by

\[ (W\phi)(z) = \phi(\omega(z)) \quad , \quad (6.4) \]
and note that $W$ preserves the space of smooth functions compactly supported away from the origin. It is then easy to check that

$$ (H'W\phi)(z) = (WH\phi)(z) $$

for all $\phi \in C_0^\infty(\mathbb{R}\setminus\{0\})$. The choice of domain ensures that the smoothness or otherwise of $\omega(z)$ at the origin is irrelevant. We therefore have $H' = WHW^*$. By the unitarity of $W$, $H'$ and $H$ have the same deficiency indices and their self-adjoint extensions are related by

$$ H'_U = WH_UW^* , $$

for $U \in U(2)$.

Accordingly, the results of sections 4 and 5 can be translated directly into results for metrics of the form (6.1) satisfying $z|h(z)|^{1/2} \to 0$ as $z \to 0$ as a two-sided limit. In particular, because $\mathcal{D}(H'_U) = W\mathcal{D}(H_U)$, the boundary conditions corresponding to unitary matrix $U$ are

$$ A \begin{pmatrix} |h(z)|^{-1/2} \partial \rho^\dagger(0^-) \\ |h(z)|^{-1/2} \partial \rho^\dagger(0^+) \end{pmatrix} = e^{i\pi/4} B \begin{pmatrix} \rho^\dagger(0^-) \\ \rho^\dagger(0^+) \end{pmatrix} , $$

where the matrices $A$ and $B$ are once again given by (4.10).

The distinguished boundary condition picked out in section 4 therefore becomes

$$ \rho(0^-) = \rho(0^+) , $$

$$ \lim_{z \to 0^-} |h(z)|^{-1/2} \partial \rho(z) = - \lim_{z \to 0^+} |h(z)|^{-1/2} \partial \rho(z) . $$

This boundary condition corresponds to a slight generalisation of one of the possible choices isolated in [6], as it includes the case of mildly singular $h(z)$.

Turning to section 5, the entire analysis goes through under the unitary equivalence $W$. The initial value problem is ill-posed for the distinguished boundary conditions, and the evolved data fails to be $L^2$ immediately after it begins to strike the surface of signature change.
For completeness, let us now relax the condition that \( z|h(z)|^{1/2} \to 0 \) as a two-sided limit. The limit may fail from above, below, or both above and below. The physical interpretation of this is easily understood as follows. Suppose the limit fails from above. Then any coordinate transformation \( z \mapsto \omega(z) \) with \( d\omega = |h(z)|^{1/2}dz \) necessarily maps \((0, \infty)\) onto the whole real line. This corresponds to the fact that no causal curve can cross the surface of signature change in finite coordinate time. Thus this metric is not really a model of signature change, because one cannot cross the boundary between the two regions. Nonetheless, it is instructive to see how this alters our analysis.

Writing \( H' \) as the direct sum of operators on \( L^2(\mathbb{R}^\pm, |h(z)|^{1/2}dz) \), we may compute its deficiency indices by considering the contributions from each half-line separately. Let us consider \( \mathbb{R}^+ \). If \( z|h(z)|^{1/2} \to 0 \) as \( z \to 0^+ \), then \( H' \) on \( C_0^\infty(0, \infty) \) is unitarily equivalent to \( \frac{\partial^2}{\partial \omega^2} \) on \( C_0^\infty(0, \infty) \subset L^2(\mathbb{R}^+, d\omega) \), which has deficiency indices \( \langle 1, 1 \rangle \). Alternatively, if \( z|h(z)|^{1/2} \not\to 0 \) as \( z \to 0^+ \) then \( H' \) on \( C_0^\infty(0, \infty) \) is unitarily equivalent to \( \frac{\partial^2}{\partial \omega^2} \) on \( C_0^\infty(\mathbb{R}) \subset L^2(\mathbb{R}, d\omega) \), which is essentially self-adjoint and therefore has deficiency indices \( \langle 0, 0 \rangle \).

Identical arguments for the negative half-line allow us to build up the full picture as follows:

Case (i): \( z|h(z)|^{1/2} \to 0 \) from neither above nor below. The total deficiency indices of \( H' \) are \( \langle 0, 0 \rangle \), so there is a unique self-adjoint extension, which may be shown to correspond to Dirichlet boundary conditions at \( z = 0 \).

Case (ii): \( z|h(z)|^{1/2} \to 0 \) from below but not from above. The total deficiency indices of \( H' \) are \( \langle 1, 1 \rangle \), so there is a 1-parameter family of self-adjoint extensions labelled by \( L \in \mathbb{R} \cup \{\infty\} \), corresponding to boundary conditions

\[
\lim_{z \to 0^-} \left( |h(z)|^{-1/2} \frac{\partial \rho}{\partial z}(z) + L^{-1} \rho(z) \right) = 0 \quad \text{for} \quad \rho(0^+) = 0. \tag{6.10}
\]

Case (iii): \( z|h(z)|^{1/2} \to 0 \) from above but not from below. Again, we have deficiency indices \( \langle 1, 1 \rangle \) and a 1-parameter family of boundary conditions obtained from (6.10) by reflection about \( z = 0 \).

*Obviously, we could also consider the case in which \( z|h(z)|^{1/2} \to 0 \) from both above and below. This replicates the results presented at the beginning of this section, i.e. \( H' \) on \( C_0^\infty(\mathbb{R}\setminus\{0\}) \) has deficiency indices \( \langle 2, 2 \rangle \).
Cases (i)–(iii) are distinguished by the fact that the two regions $z < 0$ and $z > 0$ are decoupled from one another. The surface of signature change is impenetrable to classical particles and also to classical fields. The initial value problem is well-posed for smooth initial data compactly supported in the Lorentzian region (although it is ill-posed for such data supported in the Kleinian region). However, as we have indicated, metrics which do not satisfy $z|h(z)|^{1/2} \to 0$ as $z \to 0$ are not really examples of signature change, so the well-posedness of this system is of little interest.

Finally, let us consider the Dirac equation in the spacetime (6.1). In the notation of section 3, the two candidate Dirac Hamiltonians are

$$D'_\pm = \gamma_0^0 \left[ m - i \xi_\pm(z) \gamma_1^1 |h(z)|^{-1/2} \frac{\partial}{\partial z} \right]$$

(6.11)

on $D'_\pm = C^\infty_0(\mathbb{R}\{0\}) \otimes \mathbb{C}^4 \subset L^2(\mathbb{R}, |h(z)|^{1/2} dz) \otimes \mathbb{C}^4$, where the $\gamma$-matrices $\gamma_a$ obey the Minkowski space algebra. As in section 3, we find that $(D'_\pm)^*|_{D'} = D'_\mp$, and therefore neither operator admits any self-adjoint extensions.

### 6.2 Second Order Analysis

In case it is thought that the runaway solution of section 5 is simply due to the first order formalism and our use of self-adjoint extensions, we provide a second order analogue of the result using only Fourier analysis and elementary arguments. In the following, it will only be necessary to assume boundary conditions of the general form (4.1). This allows us to consider boundary conditions for which both the field and its derivative are continuous, as well as the boundary conditions employed in section 5.

Consider the Klein–Gordon equation (3.11) in the Kleinian region ($z > 0$) with $m = 0$. Fourier transforming in $t$ we obtain

$$\frac{\partial^2 \tilde{\phi}}{\partial z^2} = E^2 \tilde{\phi}.$$  

(6.12)

Taking the solution which decays as $z \to \infty$ we find

$$\tilde{\phi}(E, z) = \tilde{\phi}(E, 0)e^{-|E|z},$$

(6.13)
and hence
\[ \frac{\partial \tilde{\phi}}{\partial z}(E, 0^+) = -|E|\tilde{\phi}(E, 0) . \] (6.14)

Thus, on the Lorentzian side of the signature change surface,
\[ \frac{\partial \tilde{\phi}}{\partial z}(E, 0^-) = -\omega|E|\tilde{\phi}(E, 0) , \] (6.15)

where \( \omega \) parametrises the boundary condition (4.1).

If \( \tilde{\phi}(E, 0) \) is analytic in \( E \) (but not identically zero) then we see that \( \frac{\partial \tilde{\phi}}{\partial z}(E, 0^-) \) is not analytic, and vice versa. Furthermore, by the Paley–Wiener theorem (see eg. [11]), we then have that it is not possible for both \( \phi(t, 0) \) and \( \frac{\partial \phi}{\partial z}(t, 0^-) \) to be elements of \( C_0^\infty(\mathbb{R}) \).

Let us again model the experiment described in the section 5. Choose the initial data \( \phi(0, z) \) and \( \frac{\partial \phi}{\partial t}(0, z) \) to be in \( C_0^\infty(-\infty, 0) \), and assume that at least some of the matter propagates towards the Kleinian region at \( z = 0 \). For massless particles this would give data \( \phi(t, 0) \) and \( \frac{\partial \phi}{\partial z}(t, 0^-) \) on the surface of signature change which are both in \( C_0^\infty(\mathbb{R}) \), but such data will not provide a solution of the Klein–Gordon equation in the Kleinian region which decays as \( z \to \infty \). Hence the second order formalism also leads to divergent solutions from smooth compactly supported initial data.

7 Conclusion

We have seen that the Schrödinger Hamiltonian on the flat signature changing spacetime (3.1) admits self-adjoint extensions. The additional restrictions that (i) the boundary conditions corresponding to the self-adjoint extension have the form of junction conditions for the matter field and its first derivative, and (ii) that the matter field is continuous, were shown to pick out one particular set of boundary conditions (4.11). Using these results, we were able to construct solutions of the Schrödinger and Klein–Gordon matter field equations. The Dirac Hamiltonian is non-symmetric and it is highly likely that the Dirac equation is only satisfied by solutions of the type found in [10].
The boundary conditions (4.11) provide a well-behaved system to study, and furthermore, since a single set of boundary conditions are selected, the requirement of self-adjointness provides a possible way of solving the dilemma of which boundary conditions to choose for signature change. The boundary conditions (4.11) are physically very similar to what might be regarded as the 'natural boundary conditions' where both the field and its derivative are continuous, since on applying them to mode solutions (as in [10]) we find that a scalar field is completely reflected, whilst the Dirac field is completely absorbed.

A reasonable physical experiment would be to observe the result of throwing matter at a Kleinian region. In our model, this type of experiment corresponds to taking smooth compactly supported initial data in the Lorentzian region \( z < 0 \) at \( t = 0 \). The results for the Schrödinger equation were very promising. Smooth compactly supported initial data in the Lorentzian region could be evolved indefinitely. However, the more physically relevant Klein–Gordon solutions gave very different results. Although we could define such solutions, in section 5 we saw that smooth compactly supported initial data could only be evolved for a finite time. Indeed, for a massless scalar field this time was equal to the earliest possible time that classical matter could reach the surface of signature change. As soon as matter reaches the signature change surface the spatial integral of the energy momentum tensor of that matter field becomes unbounded and our assumption that the matter fields do not interact with the background spacetime breaks down. Moreover, we have seen that these results apply to general Kleinian signature change systems.

One possible resolution would be to impose Dirichlet boundary conditions at the surface of signature change [18]. These boundary conditions may also be studied using the self-adjoint extension approach (in fact they correspond to setting the unitary map (4.7) to \( U = -1 \)), and they provide a well-posed system for initial data compactly supported in the Lorentzian region. The cost of obtaining this well-posed system is that the regions of differing signature are essentially decoupled. When there is only one change of signature, this is tantamount to ignoring the Kleinian region, and it is therefore questionable whether this is a good model for signature change. However, in systems with more than one change of signature, it is possible that such boundary
conditions might lead to interesting effects [18].

Returning therefore to boundary conditions (4.11); the consequent severe behaviour of the Klein–Gordon field suggests that, if interaction were allowed, the back-reaction of matter fields on the metric might annihilate the region of signature change, and thus regions of signature change would be unable to form. These results are consistent with what could be called the “Signature Protection Conjecture”, or in other words, the hypothesis that fluctuations in the signature of the spacetime metric are suppressed. In light of the fact that even this ‘well-behaved’ signature change system predicts its own downfall, it may be prudent to reassess the inclusion of signature changing metrics in quantum gravity theories.

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Appendix A

The general form of the boundary conditions for the Schrödinger Hamiltonian are given by equation (4.9). Let $A$ and $B$ be the matrices defined by (4.10), then we are interested in the subset of boundary conditions given by the restriction that both $A$ and $B$ are singular. For such $A$ and $B$, the of the boundary conditions are of the form

$$\omega_1 \rho^\dagger(0^-) = \omega_2 \rho^\dagger(0^+) ,$$

(A.1)

$$\omega_3 \frac{\partial \rho^\dagger}{\partial z}(0^-) = \omega_4 \frac{\partial \rho^\dagger}{\partial z}(0^+) ,$$

for some $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{C}$. We also wish to exclude any cases for which any of the $\omega_i$ are zero.

We can calculate the general form of the unitary matrix $U$ which satisfies $\det(A) = 0$, $\det(B) = 0$, and the constraints (4.8) on $a$, $b$, $c$ and $d$. We find the two parameter family

$$U = \left(\begin{array}{cc}
-\frac{1}{2}(1 - i)(1 + ie^{i\theta}) & \sqrt{\sin(\theta)} e^{i\kappa} \\
-i\sqrt{\sin(\theta)} e^{i(\theta - \kappa)} & -\frac{1}{2}(1 + i)(1 + ie^{i\theta})
\end{array}\right) \quad (A.2)
$$

for any $\theta \in [0, \pi]$ and $\kappa \in [0, 2\pi]$.

Except for two special cases (when $\theta = 0$ and $\theta = \pi$), the boundary conditions corresponding to this $U$ are

$$\rho^\dagger(0^-) = \left(\frac{1 + \frac{1}{2}(1 + i)(1 + ie^{i\theta})}{\sqrt{\sin(\theta)} e^{i\kappa}}\right) \rho^\dagger(0^+) , \quad (A.3)$$

$$\frac{\partial \rho^\dagger}{\partial z}(0^-) = \left(\frac{\frac{1}{2}(1 + i)(1 + ie^{i\theta}) - 1}{\sqrt{\sin(\theta)} e^{i\kappa}}\right) \frac{\partial \rho^\dagger}{\partial z}(0^+) \quad (A.4)$$

In the special case $\theta = 0$, the boundary conditions are

$$\rho^\dagger(0^-) = 0 ,$$

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and the special case $\theta = \pi$ gives the boundary conditions

$$
\rho^+(0^+) = 0 ,
$$

and

$$
\rho^-(0^-) = 0 .
$$

These special cases correspond to setting $\omega_2 = \omega_3 = 0$ and $\omega_1 = \omega_4 = 0$ respectively, and hence we will ignore them.

It seems reasonable to assume that the matter field remains continuous across the surface of signature change. In this case, using (A.3) to eliminate $\kappa$ from (A.4) we find that the derivative boundary condition becomes

$$
\frac{\partial \rho^\dagger}{\partial z}(0^-) = \left( \frac{\sin(\theta)}{\cos(\theta) - 1} \right) \frac{\partial \rho^\dagger}{\partial z}(0^+) .
$$

Finally, we can employ something particular to this signature change system. We want every mode solution to satisfy the boundary conditions. Furthermore, for $E > 0$ and $z > 0$ the only bounded solution of $H \rho^\dagger = E \rho^\dagger$ is

$$
\rho^\dagger(z) = \rho^\dagger(0^+) e^{-\sqrt{E} z} .
$$

and hence we have the additional information that

$$
\frac{\partial \rho^\dagger}{\partial z}(0^+) = -\sqrt{E} \rho^\dagger(0^+) .
$$

Using equations (A.7) and (A.9), equation (4.9) may then be rewritten as

$$
-\sqrt{E} A \begin{pmatrix} \frac{\sin(\theta)}{\cos(\theta) - 1} \\ 1 \end{pmatrix} = e^{i\pi/4} B \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,
$$
but this must hold for all $E > 0$ and so we must have

$$A \left( \begin{array}{c} \sin(\theta) \\ \cos(\theta) - 1 \\ 1 \end{array} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$  \hspace{1cm} (A.11)

These equations are consistent only if $\theta = \frac{\pi}{2}$.

Thus, setting $\theta = \frac{\pi}{2}$ in (A.7), we have shown that the only set of boundary conditions of the form (4.1), which provide a self-adjoint extension of the Schrödinger operator on the signature changing spacetime (3.1), are

$$\rho^\dagger(0^-) = \rho^\dagger(0^+),$$  \hspace{1cm} (A.12)

$$\frac{\partial \rho^\dagger}{\partial z}(0^-) = -\frac{\partial \rho^\dagger}{\partial z}(0^+).$$
Appendix B

Here, we prove that the integral transform $\mathcal{M}$ is unitary. It is convenient to work with the operator $\tilde{\mathcal{M}}$ defined by

$$\tilde{\mathcal{M}} = VMU^*,$$  \hspace{1cm} (B.1)

where $U : L^2(\mathbb{R}, dE) \rightarrow L^2(\mathbb{R}^+, dk) \otimes \mathbb{C}^2$ and $V : L^2(\mathbb{R}, dz) \rightarrow L^2(\mathbb{R}^+, dz) \otimes \mathbb{C}^2$ are unitary operators given by

$$ (Uf)(k) = \left( \frac{(2k)^{1/2}f(k^2)}{(2k)^{1/2}f(-k^2)} \right) \quad \text{and} \quad (Vf)(z) = \left( \begin{array}{c} f(z) \\ f(-z) \end{array} \right).$$  \hspace{1cm} (B.2)

Explicitly, $\tilde{\mathcal{M}} : L^2(\mathbb{R}^+, dk) \otimes \mathbb{C}^2 \rightarrow L^2(\mathbb{R}^+, dz) \otimes \mathbb{C}^2$ takes the matrix form

$$ \tilde{\mathcal{M}} = \left( \begin{array}{cc} (1 + i)A & (1 - i)B \\ (1 - i)B & (1 - i)A \end{array} \right),$$  \hspace{1cm} (B.3)

where $A, B : L^2(\mathbb{R}^+, dk) \rightarrow L^2(\mathbb{R}^+, dz)$ are defined by

$$ (Af)(z) = \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} e^{-kz}f(k)dk$$  \hspace{1cm} (B.4)

and

$$ (Bf)(z) = \frac{1}{2}((C - S)f)(z),$$  \hspace{1cm} (B.5)

and $C$ and $S$ are the cosine and sine transforms defined by

$$ (Cf)(z) = \left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} dk f(k) \cos kz$$  \hspace{1cm} (B.6)

and

$$ (Sf)(z) = \left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} dk f(k) \sin kz.$$  \hspace{1cm} (B.7)

In order to prove that $\tilde{\mathcal{M}}$ (and hence $\mathcal{M}$) is unitary, it now suffices to establish the following identities:

$$ 2(A^*A + B^*B) = \mathbb{1} \quad 2(AA^* + BB^*) = \mathbb{1} \quad A^*B + B^*A = 0 \quad AB^* + BA^* = 0.$$  \hspace{1cm} (B.8)
Let \( f \in C_0^\infty(0, \infty) \). Then

\[
(A^*Bf)(k) = \frac{1}{2\pi} \int_0^\infty dz e^{-kz} \int_0^\infty dk'(\cos k'z - \sin k'z)f(k')
= \frac{1}{2\pi} \int_0^\infty dk' f(k')G(k, k'),
\]

where

\[
G(k, k') = \int_0^\infty dz e^{-kz}(\cos k'z - \sin k'z) = \frac{k - k'}{k^2 + k'^2},
\]

and Fubini’s theorem has been employed. Moreover, an analogous argument shows that

\[
(B^*Af)(k) = \frac{1}{2\pi} \int_0^\infty dk' f(k')G(k', k).
\]

Thus, because \( G(k, k') = -G(k', k) \), we conclude that \( A^*B + B^*A \) vanishes on \( C_0^\infty(0, \infty) \) and hence on the whole of \( L^2(\mathbb{R}^+, dk) \). An identical argument (with \( z \) and \( k \) interchanged) shows that \( AB^* + BA^* = 0 \).

Next, note that \( 2B^*B = 1 - \frac{1}{2}(S^*C + C^*S) \). Thus we need to show that \( \frac{1}{2}(S^*C + C^*S) = 2A^*A \). Let \( f \in C_0^\infty(0, \infty) \), then Fubini’s theorem may be used to show that

\[
2(A^*Af)(k) = \frac{1}{\pi} \int_0^\infty dk' \frac{f(k')}{k + k'},
\]

and also that

\[
\frac{1}{2}((S^*e^{-\epsilon z}C + C^*e^{-\epsilon z}S))f(k) = \frac{1}{\pi} \int_0^\infty dk' f(k') \frac{k + k'}{\epsilon^2 + (k + k')^2}
\longrightarrow \frac{1}{\pi} \int_0^\infty dk' \frac{f(k')}{k + k'}
\]

as \( \epsilon \to 0^+ \). Using the fact that \( S^*e^{-\epsilon z}C + C^*e^{-\epsilon z}S \to S^*C + C^*S \) strongly as \( \epsilon \to 0^+ \), we have \( 2(A^*A + B^*B) = 1 \) on \( C_0^\infty(0, \infty) \) and hence on \( L^2(\mathbb{R}^+, dk) \). An analogous argument shows that \( 2(AA^* + BB^*) = 1 \).
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