Constraining the Cardoso–Pani–Rico metric with future observations of SgrA*  

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Abstract  
SgrA*, the supermassive black hole (BH) candidate at the center of our Galaxy, seems to be one of the most promising objects to test the Kerr BH hypothesis with near future observations. In a few years, it will hopefully be possible to measure a number of relativistic effects around this body, and the combination of different observations can be used to constrain possible deviations from the Kerr solution. In this paper, I discuss the combination of three promising techniques in the framework of the Cardoso–Pani–Rico parametrization: the observation of blobs of plasma orbiting near the innermost stable circular orbit, the detection of the BH shadow, and timing observations of a radio pulsar in a compact orbit. The observations of blobs of plasma and of the shadow can probe the strong gravitational field around SgrA*, while the radio pulsar would be sensitive to the weak field region at larger radii. In the case of a fast-rotating object, the combination of the three measurements could provide strong constraints on the actual nature of SgrA*. For a non-rotating or slow-rotating object, the bounds would be weak.  

Keywords: Kerr metric, alternative theories of gravity, SgrA*  

(Some figures may appear in colour only in the online journal)  

1. Introduction  
Astrophysical black hole (BH) candidates are thought to be the Kerr BHs of general relativity, but the actual nature of these objects has still to be confirmed. Robust dynamical measurements of their masses clearly indicate that these bodies cannot be explained in the framework of conventional physics as compact stars or clusters of compact stars [1], but the observational evidence that the geometry of the spacetime around them is described by the Kerr solution is
still lacking. In the last few years, there has been significant work to study the possibility of testing the Kerr BH hypothesis with present and future observations [2]. Current x-ray data can already rule out some alternative candidates, namely some exotic compact stars without an event horizon [3] and some kind of wormholes [4]. However, other candidates like non-Kerr BHs in alternative theories of gravity are very difficult to distinguish from the Kerr BHs of general relativity [5].

The Kerr BH hypothesis can be potentially tested by studying the properties of the electromagnetic radiation emitted by the gas of the accretion disk. Strictly speaking, such an approach can only test the Kerr geometry and measure possible deviations from it. It cannot test the Einstein equations, as the Kerr metric is a BH solution even in some alternative theories of gravity [6] and the electromagnetic spectrum of the accretion disk only depends on the background geometry. Bearing this in mind, the ideal strategy would be to use an approach similar to the PPN formalism of the weak field regime [7], where the most general background metric is characterized by a number of parameters to be measured from observations, and a posteriori one can verify if astrophysical data require that the values of these parameters is consistent with the predictions of general relativity. The observation of a spectrum that looks like the one of a Kerr BH with a certain spin is not enough to confirm the Kerr BH hypothesis, because there is typically a degeneracy between the measurement of the spin and possible deviations from the Kerr solution, with the result that a non-Kerr object may mimic a Kerr BH with a different spin [5].

Unfortunately, at present there is no general and satisfactory formalism as the PPN one to test the Kerr nature of BH candidates. In the strong field regime, an expansion in \( M/r \) does not work and deviations from the Kerr background can easily generate pathological features in the spacetime. A first attempt in this direction was proposed in [8], in which the background metric was characterized by a set of an infinite number of deformation parameters \( \{ \epsilon_k \} \) that measure possible deviations from the Kerr solution. Recently, Cardoso, Pani and Rico have proposed an extension of the Johannsen–Psaltis metric of [8], in which there are two sets of deformation parameters, namely \( \{ \epsilon^t_k \} \) and \( \{ \epsilon^r_k \} \) [9]. The Johannsen–Psaltis parametrization is recovered when \( \epsilon^t_k = \epsilon^r_k \), while the metric reduces to the Kerr solution when \( \epsilon^t_k = \epsilon^r_k = 0 \). As shown in [10], the Cardoso–Pani–Rico parametrization can describe non-Kerr objects with qualitatively different features with respect to the Johannsen–Psaltis one and it is therefore a more suitable framework to test the Kerr nature of BH candidates.

For the time being, the spacetime geometry around BH candidates can be probed with available x-ray data, by studying the thermal spectrum of thin accretion disks [11] or the shape of the iron \( K\alpha \) line [12]. However, there is a strong correlation between the measurements of the spin and of possible deviations from the Kerr solution, with the result that a single measurement can only constrain a combination of the spin and the deformation parameter [10, 13]. In order to really test the Kerr BH hypothesis it would be necessary to combine several measurements of the same object. Unfortunately, the measurements of the disk’s thermal spectrum and of the iron line are very sensitive to the same feature, namely the position of the innermost stable circular orbit (ISCO), and only with very high quality data, not available today, it may be possible to break the degeneracy between the estimates of the spin and of the deformation parameter [5]. Other approaches, like the study of QPOs [14] or the estimate of the jet power [15], are not yet mature techniques, and therefore they cannot yet be used to test fundamental physics.

In [10], it was studied how to constrain the Cardoso–Pani–Rico deformation parameters from x-ray observations of stellar-mass BH candidates with the analysis of the thermal spectrum of thin accretion disks. These data are already available and constraints were obtained for two specific sources, namely A0620 and Cygnus X-1. The conclusion was that
there is a fundamental degeneracy between the measurement of the spin and the deformation parameters and therefore we cannot put any bound on possible deviations from the Kerr solution. The aim of the present work is to study the possibility of constraining the values of these deformation parameters with future observations of SgrA*, the supermassive BH candidate at the center of our Galaxy. For the time being, there are no observational data to probe the spacetime geometry around this object and no reliable approaches to estimate its spin. However, the situation could soon change, and SgrA* may become one of the best candidates to test the Kerr BH hypothesis. There is indeed the realistic possibility that it will be soon possible to test the geometry around this object with a number of different observations. The key point is that such different observations could measure the relativistic effects at different radii, which would break the degeneracy between the estimate of the spin and of the deformation parameters. In particular, the instrument GRAVITY may allow us to observe blobs of plasma around SgrA* within a few years [16, 17]. Experiments like the Event Horizon Telescope may observe the shadow of SgrA* and thus measure its apparent photon capture sphere [18, 19]. The possible discovery of a radio pulsar in a compact orbit around SgrA* would permit precise measurements of its mass and spin, independently of the actual nature of this object [20]. The combination of all these measurements can potentially test the nature of SgrA*, which is not the case for stellar-mass BH candidates.

The content of the paper is as follows. In section 2, I briefly review the Cardoso–Pani–Rico parametrization, namely the theoretical framework that will be used here to study how to test the Kerr nature of SgrA*, and the possible future observations of blobs of plasma, a BH shadow, and pulsar in a compact orbit. Section 3 discusses the constraints from these measurements, and the summary and conclusions are reported in section 4. Throughout the paper, I use units in which $G_N = c = 1$.

2. Testing the nature of SgrA*

2.1. Theoretical framework

In Boyer–Lindquist coordinates, the Cardoso–Pani–Rico parametrization reads [9]

$$
\begin{align*}
\begin{vmatrix}
\frac{dr}{\Delta + h' a^2 \sin^2 \theta} \\
\frac{d\phi}{2 \sqrt{(1 + h')(1 + h')}} \\
\frac{d\theta}{\sqrt{(1 + h')(1 + h')}} \\
\end{vmatrix}
\end{align*}
$$

where $M$ is the BH mass, $a = J/M$ is the BH spin parameter, $J$ is the BH spin angular momentum, $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$, and

$$
\begin{align*}
\Sigma_0 &= \sum_{k=0}^{+\infty} \left( e^{2k} + e^{2k+1} \frac{M}{\Sigma} \right) \left( \frac{M^2}{\Sigma} \right)^k, \\
\Sigma_1 &= \sum_{k=0}^{+\infty} \left( e^{2k} + e^{2k+1} \frac{M}{\Sigma} \right) \left( \frac{M^2}{\Sigma} \right)^k.
\end{align*}
$$

There are two infinite sets of deformation parameters, $\{e^2\}$ and $\{e^4\}$. The line element in (1) reduces to the Johannsen–Psaltis one for $h' = h''$, and to the Kerr line element when $h' = h'' = 0$. 

In the Johannsen–Psaltis background, $\epsilon_0 = 0$ to have an asymptotically flat spacetime, while $\epsilon_1$ and $\epsilon_2$ must be small to meet the solar system constraints [9]. $\epsilon_3$ is the first unbounded deformation parameter and there are no qualitative differences between $\epsilon_3$ and higher order terms [21]. In the Cardoso–Pani–Rico background, the first unconstrained deformation parameters are $\epsilon_1'$ and $\epsilon_2'$ [9].

The Cardoso–Pani–Rico metric is not a solution of any known alternative theory of gravity, but it is an attempt to quantify generic deviations from the Kerr solutions through its deformation parameters $\{\epsilon_k\}$ and $\{\epsilon_k'\}$. This is the same approach as the PPN metric for solar system experiments, in which the deformation parameters are free and to be determined by observations. Only measurements of the Schwarzschild and Kerr metrics can constrain these deformation parameters, as, in the absence of a theory, it is not possible to relate them to bounds inferred in other contexts, like the expansion of the Universe or the emission of gravitational waves. The Cardoso–Pani–Rico metric is obtained by considering a generic deformation in the non-rotating Schwarzschild solution, and then it is performed by a Newman–Janis transformation to get a rotating BH. With such a procedure, $\sigma_{\theta\theta}$ is not altered and all the deviations are encoded into two functions, namely $h'$ and $h''$, but this does not mean that the result is the most general rotating BH solution—actually, we know it is not. As already mentioned, at present there is no satisfactory formalism like the PPN one for the strong field regime. The physical implications of this limitation are that we cannot really take into account generic deviations from the Kerr metric, and the line element in (1) may miss some important theoretical motivated BH solutions.

2.2. Orbiting hot-spots

General relativistic magneto-hydrodynamic simulations of accretion flows onto BHs indicate that temporary clumps of matter may be common in the region near the ISCO [22]. SgrA* exhibits powerful flares in the x-ray, NIR, and submillimeter bands, see e.g. [23]. A flare typically lasts 1–3 h and shows a quasi-periodic substructure with a time scale of about 20 min. While the exact mechanism responsible for these flares is not known, current data seem to favor the hot spot model, namely a blob of plasma orbiting near the ISCO radius [24]. Within a couple of years, the GRAVITY instrument for the ESO Very Large Telescope Interferometer is expected to test the actual nature of these flares and hopefully confirm the hot spot model [16, 17].

Light curves, centroid tracks, and direct images of hot spots orbiting around SgrA* can potentially carry a lot of information about the spacetime geometry around this object, but the situation with real data is much more complicated and, except in the case of substantial differences from the Kerr spacetime, eventually we can at most get an estimate of the Keplerian frequency at the ISCO radius [25] (but see [26]). The latter does depend on the spin parameter and on possible deviations from the Kerr solution and its measurement can thus constrain the nature of SgrA* on the spin parameter–deformation parameter plane.

2.3. BH shadow

The direct image of a BH surrounded by an optically thin accretion flow is characterized by a dark area over a bright background. Such a dark area is usually referred to as the shadow of the BH [27]. While the intensity map of the image depends on the exact accretion model and emission mechanisms, the shape of the shadow is only determined by the metric of the spacetime and corresponds to the boundary of the photon capture sphere as seen by a distant
observer. The observation of the shadow of a BH can thus be used to test the nature of the compact object [28, 29], and shadows of many non-Kerr BHs have been calculated [30–33].

SgrA* is the best candidate to observe the shadow of a BH for the first time, because it is the BH candidate with the largest angular size on the sky. At submillimeter wavelengths, its accretion flow is supposed to become optically thin and the interstellar scattering should be significantly lower [27]. At first approximation, the shape of the shadow is a circle, whose size is essentially set by the mass of the compact object and its distance from us. In the case of SgrA*, the radius of the shadow should be about 25 μas and not very sensitive to the exact background geometry, except in very special cases [30]. The first order correction to the circle is due to the spin, as the photon capture radius is different for co-rotating and counter-rotating particles. The boundary of the shadow has thus a dent on one side: the deformation is more pronounced for an observer on the equatorial plane (viewing angle \( i = 90° \)) and decreases as the observer moves towards the spin axis, to completely disappear when \( i = 0° \) or 180°. The shape of the shadow can be characterized by the Hioki–Maeda distortion parameter [34]. If the inclination angle is known, the Hioki–Maeda parameter only depends on the spin in the case of the Kerr background and its measurement can thus be used to estimate \( a/M \). However, if we relax the Kerr BH hypothesis, the Hioki–Maeda parameter depends on both the spin and possible deviations from the Kerr solution. Eventually there is a degeneracy between these two quantities and therefore its measurement can only constrain an allowed region on the spin parameter–deformation parameter plane [35].

2.4. Pulsar timing

It is thought that a large population of pulsars can orbit near SgrA*, and some of them may be in compact orbits to allow for precise measurements of the spacetime geometry around this object. Some deep pulsar searches have already been conducted [36]. Current results are consistent with the expectation of such a large population, but a pulsar sufficiently close to SgrA* to test general relativity has still to be found.

As discussed in [20], the possible observation of radio pulsars with an orbital period shorter than about half year could provide a precise and clean measurement of the spin parameter of the central object from the Lense–Thirring effect. As the pulsar would still be far from the BH, such a technique would really measure the spin parameter independently of the exact nature of the BH candidate, just because possible deviations from the Kerr background correspond to higher-order terms in the multipole moment expansion of the metric. In the case of a pulsar with an orbital period of one month or less, it would also be possible to measure the mass quadrupole moment of the compact object and thus have a first test of the no-hair theorem [20], as in the case of a Kerr BH, the mass quadrupole moment \( Q \) is related to the mass \( M \) and the spin angular momentum \( J \) by the relation \( Q = -J^2/M \). In what follows, such a possibility will be neglected and only the measurement of the spin will be considered. The measurement of \( Q \) is indeed much more challenging, a pulsar with such a short orbital period may be difficult to find, and the approach can only test possible deviations from the Kerr quadrupole moment, while the measurements of the hot-spot period and the BH shadow are sensitive even to higher order corrections, as they depend on the spacetime properties in the strong field regime.

It is worth noting that a similar measurement could be obtained from accurate astrometric data of ordinary stars orbiting at sub-mpc radii from SgrA*, see e.g. [37]. Even in this case, we would probe the weak gravitational field of the BH candidate and therefore we would measure the actual value of the spin from the frame-dragging precession, independently of the nature of the compact object. Like in the pulsar case, for very compact orbits it may be
possible to infer the mass-quadrupole moment. In the next section, I will talk about pulsar timing, but the same kind of constraints can potentially be obtained from astrometric observations, which have their own advantages and disadvantages.

3. Simulations

The aim of this section is to consider some examples of possible future measurements with the three techniques discussed above and to study the constraints that can be obtained on the Cardoso–Pani–Rico deformation parameters. Of course, it is now impossible to predict the actual accuracy of these observations, so the final results have to be seen as a general guide for future work, qualitatively correct but quantitatively to be taken with caution. While the first unconstrained deformation parameters are $\epsilon^t_3$ and $\epsilon^r_2$ [9], here the attention will be focused on $\epsilon^t_3$ and $\epsilon^r_3$, in order to consider the same order of deformation and quickly recover the Johannsen–Psaltis case when $\epsilon^t_3 = \epsilon^r_3$. The results and the conclusions found for $\epsilon^t_3$ are qualitatively the same as the ones that would be obtained for a higher order $\epsilon^t_i$-type deformation parameter. The same is true for $\epsilon^r_3$ and higher order $\epsilon^r_i$-type deformation parameters. However, the more general possibility in which several deformation parameters can be non-vanishing at the same time is more complicated and requires a larger number of observations. The last example in this section considers the possibility of constraining $\epsilon^t_3$ and $\epsilon^r_3$ at the same time. In the following examples, it will be assumed that SgrA* has a mass of $4 \times 10^6 \odot$ and our viewing angle is $i = 60^\circ$.

It is well known that the Kerr metric can describe either BH (when $a/M \leq 1$) or naked singularity ($a/M > 1$) solutions. The two scenarios can be experimentally distinguished by specific observational signatures [29, 38]. In the case of the Cardoso–Pani–Rico metric, the situation is similar, but there are now three qualitatively different possibilities: BH solutions with a regular exterior, BH solutions with naked singularities, and naked singularity solutions. As in the Kerr case, the three classes of spacetimes have their own observational signatures. An example is given in [26]. In what follows, I will not look for similar observational signatures, which are beyond the aim of this work, and I will pay attention to the objects that may be interpreted as Kerr BHs.

3.1. Example 1: slow-rotating Kerr BH

As a first example, let us consider the case in which SgrA* is a Kerr BH with dimensionless spin parameter $a_*=a/M = 0.25$. The observation of a blob of plasma orbiting close to SgrA* could provide an estimate of the Keplerian orbital period at the ISCO radius. For a four million solar mass Kerr BH with $a_*=0.25$, $T_{\text{ISCO}} = 25 \text{ min}$. Let us suppose that the measurement is between 21.5 and 28.5 min, which would correspond to a spin estimate $0.10 < a_ < 0.40$ in the Kerr background. If we do not assume the Kerr background, such a measurement of $T_{\text{ISCO}}$ provides an allowed region on the spin parameter–deformation parameter plane. Figure 1 shows this region (the area between the two red solid lines) in the case where the only non-vanishing deformation parameter is $\epsilon^t_3$ (left panel) and in the case in which the only non-vanishing deformation parameter is $\epsilon^r_3$ (right panel). In the former case, there is a clear correlation between $\epsilon^t_3$ and $a_*$. In the latter case, the correlation is weak and actually the measurement of the hot spot orbital period is not really sensitive to the value of $\epsilon^r_3$.

To quantify the shape of the shadow, it is convenient to use the Hioki–Maeda distortion parameter $\delta$ [34]. While future observations will use a more accurate approach, the Hioki–Maeda distortion parameter is a simple proxy to estimate the shadow departure from the shape...
of a circle. Let us also assume that our viewing angle is $i = 60^\circ$. For $i$ close to $0^\circ$, the boundary of the shadow would be necessarily a circle and its detection would provide no information about the nature of SgrA*. $i$ close to $90^\circ$ would be the most favorable position for a measurement, because the Hioki–Maeda distortion parameter would assume the highest value and any measurement would be more accurate. The measurement of $i$ can be independently obtained from the other two techniques. For a Kerr BH with $a* = 0.25$ and a viewing angle $i = 60^\circ$, $\delta = 0.006$, where $\delta = D/R$, $D$ is the ‘dent’ and $R$ is the ‘radius’ of the shadow (see [34] for the definitions). Since our distance from the galactic center is about 8 kpc, the expected value of $R$ is around $25 \mu$as. In this example, $\sim D = 0.15 \mu$as, but such a measurement is out of reach. An optimistic accuracy on the measurement of the shape of the shadow is $\sim 0.5 \mu$as. It can thus make sense to consider a measurement $|\delta| < 0.023$. The latter corresponds to an estimate $|a*| < 0.50$ in the Kerr background and to the areas between the blue dotted lines in figure 1. The constraints from the shadow will be surely weak in the case of a slow-rotating object. As one can see from the plots in the second paper in [30], in non-Kerr metrics the Hioki–Maeda distortion parameter may be negative, so the sign of $\delta$ does not necessary fix the spin orientation.

Lastly, there is the pulsar measurement. Let us assume we get the measurement $0.22 < a* < 0.28$. This would be the actual measurement, independently of the spacetime geometry, because the pulsar orbit is at relatively large radii, where possible deviations from the Kerr solution are more suppressed than the spin term in the expansion of the background metric. In figure 1, the pulsar measurement is the region between the two green dashed lines. The lines are vertical because the measurement is independent of the values of the deformation parameters.

### 3.2. Example 2: fast-rotating Kerr BH

Let us now assume that SgrA* is a Kerr BH with spin parameter $a* = 0.85$. The result of the three measurements are shown in figure 2, respectively for $\epsilon^j_3$ (left panel) and $\epsilon^j_1$ (right panel), with all the other deformation parameters set to zero. The hot spot measurement is still represented by the area between the two red solid lines. For a Kerr BH with $a* = 0.85$,
\(T_{\text{ISCO}} = 10.5 \text{ min},\) and here it has been assumed an orbital period measurement between 9 and 12 min, which corresponds to a spin estimate \(0.80 < a_\bullet < 0.90\) in Kerr. The constraint from the shadow is reported by the blue dotted lines. With the viewing angle \(i = 60^\circ,\) the Hioki–Maeda parameter would be \(\delta = 0.093.\) Here, the measurement \(0.071 < \delta < 0.120\) has been assumed. In the Kerr metric, it would correspond to a spin estimate \(0.78 < a_\bullet < 0.91.\) Lastly, the observation from a radio pulsar could provide the bound \(0.84 < a_\bullet < 0.86,\) and it is independent of the value of the deformation parameters.

### 3.3. Example 3: constraining two deformation parameters

Lastly, I consider the case in which one wants to constrain \(\epsilon^t_3\) and \(\epsilon^r_3\) at the same time. As the pulsar measurement can provide the correct value of the spin parameter independently of the exact nature of SgrA*, it is possible to fix the value of \(a_\bullet\) with the pulsar and study the constraints on the \(\epsilon^t_3-\epsilon^r_3\) plane. Let us assume that the pulsar observations provide the estimate...
\(a_* = 0.6\). We then consider the following two scenarios: (i) SgrA* is a Kerr BH (left panel in figure 3), and (ii) SgrA* is a non-Kerr BH with \(\epsilon_1^f = 2\) and \(\epsilon_1^r = 6\) (right panel in figure 3).

In the former scenario of a Kerr BH, for a four million solar mass object the orbital period at the ISCO is \(T_{\text{ISCO}} = 17\) min. Let us assume that observations provide the measurement \(14.5 < T < 19.5\) min, corresponding to a spin estimate \(0.5 < a_0 < 0.7\) in the Kerr background. The shadow of a Kerr BH with spin \(a_0 = 0.60\) and seen with an inclination angle \(i = 60^\circ\) has Hioki–Maeda deformation parameter \(\delta = 0.036\). Here an uncertainty of 50\% is assumed, so the measurement would be \(0.018 < \delta < 0.054\) and the spin estimate in the Kerr metric would be \(0.45 < a_0 < 0.70\). The hot spot and shadow measurements are represented, respectively, by the red solid lines and blue dotted lines in figure 3.

In the scenario of a non-Kerr BH, the ISCO orbital period would be \(T_{\text{ISCO}} = 8.5\) min. Let us assume here a measurement \(7.5 < T_{\text{ISCO}} < 9.5\) min. The Hioki–Maeda distortion parameter would be \(\delta = 0.219\). The measurement could be \(0.200 < \delta < 0.240\). The final constraints are shown in the right panel in figure 3.

### 4. Concluding remarks

Astrophysical BH candidates are supposed to be the Kerr BHs of general relativity, but current observations cannot yet confirm this hypothesis. The key point is that there is a strong correlation between the estimate of the spin and possible deviations from the Kerr solution. Current techniques to test the nature of these objects are the analysis of the thermal spectrum of thin accretion disks and of the shape of the iron \(K\alpha\) line. However, they can only constrain a combination between the spin and the deformation parameters, and a non-Kerr object could be interpreted as a Kerr BH with a different spin parameter. The combination of the measurements from the disk’s spectrum and the iron line is not very fruitful, as both techniques are sensitive to the position of the ISCO radius.

If we want to test the Kerr BH hypothesis, it is necessary to combine several measurements of the same object and check whether the observational data require no deviations from the Kerr solution. For this reason, SgrA* may soon become the best object to test the actual nature of BH candidates. While there are currently no robust approaches to probe the spacetime geometry around SgrA*, there is the realistic possibility that, hopefully within a few years, a number of different techniques will be able to observe different relativistic effects from this source. The combination of these measurements can potentially allow us to verify the nature of the supermassive BH candidate at the center of our Galaxy.

In this paper, I have focused on three observations: the measurement of the ISCO frequency from the detection of a hot spot, the estimate of the distortion parameter of the shadow of SgrA*, and the determination of the BH spin from accurate pulsar timing. All three measurements could potentially be available within a few years. The hot spot and the shadow can test the geometry of the strong field regime, very close to the BH candidate. Pulsar measurements probe the weak field region at larger radii, where the perturbative regime holds. Possible constraints from future observations have been discussed within the Cardoso–Pani–Rico parametrization. In summary:

1. The discovery of a pulsar in a compact orbit can provide an accurate measurement of the spin parameter, independently of the exact nature of the compact object.
2. In the case of a slow-rotating object, the measurement of the shadow can only provide very weak bounds. The combination of the hot spot and pulsar measurements can constrain the \(\epsilon_1^r\)-type deformation parameters, but it is much more difficult to do the same with the \(\epsilon_1^f\)-type deformation parameters.
In the case of a fast-rotating object, the hot spot and the shadow can probably provide similar constraints, but their combination is helpful anyway to confirm the measurements, since it is challenging to have all the systematic effects perfectly under control. The combination of the three measurements can constrain the deformation parameter, and quite strong bounds can be obtained for the $\epsilon^r_k$-type deformation parameters.

The possibility of constraining both the $\epsilon^t_k$ and $\epsilon^r_k$ deformation parameters at the same time is more challenging, and the three techniques discussed in this paper may not be able to do this.

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