THE 20-60-20 RULE

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ABSTRACT. In this paper we discuss an empirical phenomena known as the 20-60-20 rule. It says that if we split the population into three groups, according to some arbitrary benchmark criterion, this particular ratio often implies some sort of balance. From practical point of view, this feature leads to efficient management or control. We provide a mathematical illustration, justifying the occurrence of this rule in many real world situations. We show that for any population, which could be described using multivariate normal vector, this fixed ratio leads to a global equilibrium state, when dispersion and linear dependance measurement is considered.

Introduction. The 20-60-20 rule is an empirical statement. According to it, splitting the population into three separate groups proves to give an efficient partition, when the ratio of 20%, 60% and 20% is satisfied. The division is usually based on the performance of each element in the population and the groups are referred to as negative, neutral and positive, respectively. The first group relates to elements of population which positively contribute to the considered subject (e.g. effective workers, top sale managers, productive members), while the last one denotes the opposite. The middle set corresponds to the middle part of the population, having average performance. Putting it another way we cluster the population basing on a notion of effectiveness.

The importance of this rule comes from the fact that this particular partition seems to be the most effective one, for many empirical problems. To make this idea more transparent, let us present in details two common illustrations of this phenomena and then comment on the efficiency.

The first example considers sales departments. In almost any big company, the employees of the sales department could be split into three groups, maintaining the 20-60-20 ratio. The first group are top performers, who make big profits, even without supervision. The middle group are people who need to be managed to

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make average but stable profits. The last group are people who are heading towards termination or resignation. They produce no good income, even when supervised.

The second example relates to change capability. If you are willing to make substantial changes in any big institution, then on average 20% of the people are ready, willing and able to change, while 20% of people would not accept the change, whatever the cost. The middle 60% will wait to see how the situation turns out.

Corporations use the 20-60-20 rule widely in management and sales departments [15, 13]. One of the practical aspects of this phenomena relates to the fact that different procedures and methods are created to handle the efficiency in positive, negative and neutral group and the 20-60-20 ratio proves to be the most efficient partition. For example, in many problems related to human resource management, one should identify and focus his attention on the middle 60%, as this group could and should be managed efficiently.

Of course there are countless illustrations of this phenomena. One could consider financial market overall condition, fraud and theft capability among group of people, the structure of electorate, sport performance among athletes, potential of students, patient handling, medical treatments, etc. Please see e.g. [14, 3, 1, 5, 7, 8, 11, 2, 4], where the 20-60-20 ratio is used and the detailed procedures are proposed to handle many practical problems.

The natural question is why this specific 20-60-20 ratio is valid in so many situations? Why not 10-80-10 or 30-40-30? Is this a coincidence, or does it follow from some underlying and fundamental structure of the population?

While very popular among practitioners, no scientific evidence of the 20-60-20 principle has been presented yet, due to the authors knowledge. Consequently, this noteworthy rule become more of a slogan, than the scientific fact.

The possible mathematical illustration of this phenomena, based on the dispersion and linear dependance measurement is the main topic of this paper. We show that if a (multivariate) random vector is distributed normally and we do conditioning based on the quantile function of the first coordinate (or in fact any linear combination of the coordinates), then the ratio close to 20-60-20 imply a global equilibrium state, when dispersion and linear dependance measurement is considered. In particular, we prove that this particular partition implies the equality of covariance matrices, for all conditional vectors, implying some sort of global balance in the population. We also discuss the case of monotone dependance using conditional Kendall $\tau$ and Spearman $\rho$ matrices.

The material is organized as follows. The introduction is followed by a short Preliminaries, where we establish basic notations used throughout this paper. Next, in Section 2 we introduce a mathematical model for the 20-60-20 rule and define the equilibrium state, using conditional covariance matrices. The 20-60-20 rule for multivariate normal vectors is discussed in Section 3. Theorem 3.1 might be considered as the main result of this paper. Section 4 is devoted to the study of different equilibrium states, obtained using correlation matrices, Kendall $\tau$ matrices and Spearman $\rho$ matrices. In particular, we present here some theoretical results, when Spearman $\rho$ matrices are considered and a numerical example, illustrating the 20-60-20 rule for sample data. In Section 5 we discuss shortly what happens if we loose the normality assumption. The general elliptic case is considered here.

1. Preliminaries. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $n \in \mathbb{N}$. Let us fix an $n$-dimensional continuous random vector $X = (X_1, \ldots, X_n)$. We use
to denote the corresponding joint distribution function and
\[ H_1(x) = \mathbb{P}[X_1 \leq x], \quad i = 1, 2, \ldots, n, \]
to denote the marginal distribution functions. Given a Borel set \( B \) in \( \mathbb{R}^n \) such that \( \mathbb{P}[\{\omega \in \Omega : (X_1(\omega), \ldots, X_n(\omega)) \in B\}] > 0 \) we can define the conditional distribution \( H_B \) for all \((x_1, \ldots, x_n) \in B\) by
\[ H_B(x_1, \ldots, x_n) = \mathbb{P}[X_1 \leq x_1, \ldots, X_n \leq x_n \mid X \in B]. \]
Putting it in another words, we truncate the random vector \( X \) to the Borel set \( B \). If necessary, we assume the existence of regular conditional probabilities. In this paper we assume that \( B \) is a non-degenerate rectangle, i.e. \( B \in \mathcal{R} \), where
\[ \mathcal{R} := \{ A \in \mathbb{R}^n : A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n], \text{ where } a_n, b_n \in \mathbb{R} \text{ and } a_n < b_n \}. \]
As we are mainly interested in quantile-based conditioning on the first coordinate, for \( q_1, q_2 \in [0, 1] \) such that \( q_1 < q_2 \), we shall use notation
\[ H_{[q_1, q_2]}(x_1, \ldots, x_n) := H_{B(q_1, q_2)}(x_1, \ldots, x_n), \]
where the conditioning set is given by
\[ B(q_1, q_2) := [H_1^{-1}(q_1), H_1^{-1}(q_2)] \times \mathbb{R} \times \cdots \times \mathbb{R}. \]
We refer to \( H_{[q_1, q_2]} \) as the truncated distribution, while \( B(q_1, q_2) \) is called truncation interval (see [10]).

Moreover, we denote by \( \mu = (\mu_1, \ldots, \mu_n) \) and \( \Sigma = \{ \sigma_{ij}^2 \}_{i,j=1,\ldots,n} \), the mean vector and covariance matrix of \( X \). Similarly as in formula 1, given \( B \), we use \( \mu_B \) and \( \Sigma_B \) to denote the conditional mean vector and the conditional covariance matrix, i.e. mean vector and conditional covariance matrix of a random vector with distribution \( H_B \). Consequently, as in 2, we write
\[ H_{[q_1, q_2]} := \mu_B(q_1, q_2) \quad \text{and} \quad \Sigma_{[q_1, q_2]} := \Sigma_B(q_1, q_2). \]
We also use \( \Phi \) and \( \phi \) to denote the distribution and density function of a standard univariate normal distribution, respectively.

2. The global balance. To split the whole population into three separate groups basing on a notion of effectiveness, we need to make an assumption about the probability distribution of the whole population and the given benchmark, which measures the effectiveness of each element in the population. We assume that \( X \sim \mathcal{N} (\mu, \Sigma) \), i.e. the population is described using \( n \)-dimensional random vector \( X = (X_1, \ldots, X_n) \), that is normally distributed with mean vector \( \mu \) and covariance matrix \( \Sigma \). Furthermore, we assume that the benchmark level is determined by the first coordinate, i.e. \( X_1 \). Please note that for multivariate normal this may be a linear combination of all other coordinates, so \( X_1 \) is fixed as benchmark only for brevity. Moreover, one could look at other coordinates as various factors, which could influence the main benchmark. Note that, if we talk about people measures or abilities, then Gaussian functions, often described as bell curves, are a natural choice.

We seek for two real numbers \( q_1, q_2 \in (0, 1) \), such that \( q_1 + q_2 < 1 \) and the corresponding partition
\[ B(0, q_1), \quad B(q_1, 1 - q_2), \quad B(1 - q_2, 1), \]
admits some sort of equilibrium. In other words, we want to divide the whole population into three subgroups, corresponding to the lower 100$q_1$%, the middle 100$(1 - q_1 - q_2)$% and the upper 100$q_2$% of the population, where the effectiveness is measured by the benchmark. To do so, let us give a definition of global balance.

**Definition 2.1.** We say that a global balance is achieved in $X$ if

\[
\Sigma_{[0,q_1]} = \Sigma_{[q_1,1-q_2]} = \Sigma_{[1-q_2,1]},
\]

for some $q_1, q_2 \in (0, 1)$, such that $q_1 + q_2 < 1$.

Sometimes, instead of global balance we use term equilibrium state. In Definition 2.1, the equality of conditional covariance matrices say that:

1. The dispersion measured by variance is the same in each subgroup for any coordinate $X_i$, for $i = 1, 2, \ldots, n$. In particular the dispersion of the benchmark is the same everywhere.
2. The linear dependance structure, measured by the conditional correlation matrices, is the same in all three subgroups.

The first property creates a natural equilibrium state, as any perturbation leads to irregularity, when the square distance from the average member of each group is considered. The choice of this measure of dispersion seems to be natural, because people awareness of any differences should be high, as variance (or standard deviation) seems to be the simplest measure of variability.

The second property relate to the linear dependence structure. The equality of correlation matrices imply a natural equilibrium between groups, as people tend to notice the simplest (linear) dependancies first. Any shift between groups cause dependence instability between them.

In general (i.e. when we loose assumption about normality) the global balance might not exist or strongly depend on initial $\Sigma$, when we consider some family parametrised by covariance matrices.

3. **The 20/60/20 principle.** If $X$ is multivariate normal, it is reasonable to set $q_1 = q_2$, due to the symmetry of the Gaussian density. Thus, for simplicity, we set $q = q_1 = q_2$. Then, we seek for $q \in (0, 0.5)$ such that the conditional covariance matrix for the lower 100$q_1$% of the population coincide with the conditional covariance matrices of the middle 100$(1 - 2q)$% and upper 100$q_2$%.

We are now ready to present the main result of this paper. We show that if $X \sim N(\mu, \Sigma)$, then the equilibrium state is achieved for a unique $q \in (0, 0.5)$. This is a statement of Theorem 3.1.

**Theorem 3.1.** Let $X \sim N(\mu, \Sigma)$. Then there exists a unique $q \in (0, 0.5)$ such that the global balance in $X$ is achieved, i.e. the equality 3 is true for $q = q_1 = q_2$. Moreover, the value of $q$ is independent of $\mu$ and $\Sigma$ and the approximate value of $q$ is 0.19809.

The proof of Theorem 3.1 is surprisingly simple. It is a direct consequence of Lemma 3.2 and Lemma 3.3, which we now present and prove. Before we do this, let us give a comment on Theorem 3.1. It says that if we split the whole population into three separate groups, then the ratio close to 20-60-20 (and in fact only this ratio) implies the equality of conditional covariance matrices for all groups, creating a natural equilibrium. To prove Theorem 3.1 we need an analytic formula for conditional covariance structure, given any conditioning Borel set $B$ of positive measure. This is the purpose of Lemma 3.2.
Lemma 3.2. Let $X \sim \mathcal{N}(\mu, \Sigma)$. Then for any Borel subset $\mathcal{B}$ of $\mathbb{R}$ with positive measure,
\[
\Sigma_{\mathcal{B}} = \Sigma + (D^2[X_1 \mid X_1 \in \mathcal{B}] - D^2[X_1])\beta \beta^T,
\]
where
\[
\beta^T = (\beta_1, \ldots, \beta_n), \quad \beta_i = \frac{\text{Cov}[X_1, X_i]}{D^2[X_1]}.
\]

Proof of Lemma 3.2. Being in Gaussian world we can describe each random variable $X_i$ as a combination of the random variable $X_1$ and a random variable $Y_i$ independent of $X_1$. Indeed, we put for $i = 1, \ldots, n$
\[
Y_i = X_i - \beta_i X_1, \quad \text{where } \beta_i = \frac{\text{Cov}[X_1, X_i]}{D^2[X_1]},
\]
Obviously $\beta_1 = 1$ and $Y_1 = 0$. Since for $i = 2, \ldots, n$, the newly defined variable $Y_i$ is uncorrelated with $X_1$, they are independent.

Next, we calculate the conditional covariance matrix. Using (4), for $i, j = 1, \ldots, n$, we get
\[
\text{Cov}[X_i, X_j \mid X_1 \in \mathcal{B}] = \text{Cov}[\beta_i X_1 + Y_i, \beta_j X_1 + Y_j \mid X_1 \in \mathcal{B}].
\]
Since $Y_i$ and $Y_j$ do not dependent on $X_1$, we get
\[
\text{Cov}[Y_i, X_1 \mid X_1 \in \mathcal{B}] = 0 = \text{Cov}[Y_j, X_1 \mid X_1 \in \mathcal{B}],
\]
and
\[
\text{Cov}[Y_i, Y_j \mid X_1 \in \mathcal{B}] = \text{Cov}[Y_i, X_1] = \text{Cov}[X_i, X_j] - \beta_i \beta_j D^2[X_1].
\]
Therefore, we obtain
\[
\text{Cov}[X_i, X_j \mid X_1 \in \mathcal{B}] = \text{Cov}[X_i, X_j] + \beta_i \beta_j (D^2[X_1 \mid X_1 \in \mathcal{B}] - D^2[X_1]).
\]
Since $\beta_i \beta_j$ is the $i, j$-th entry of the $n \times n$ matrix $\beta \beta^T$, we finish the proof of the lemma.

Using Lemma 3.2 we can parametrise $\Sigma_{\mathcal{B}}$ in such a way, that it only depends on the conditional variance of $X_1$. Thus, we only need to show that there exists $q \in (0, 0.5)$ such that the (conditional) dispersion of $X_1$ in all three groups determined by sets $B(0, q)$, $B(q, 1 - q)$ and $B(1 - q, 1)$ coincide. This is the statement of Lemma 3.3.

Lemma 3.3. Let $X_1 \sim \mathcal{N}(\mu_1, \sigma^2_{11})$. Then there exist a unique $q \in (0, 0.5)$ such that
\[
D^2[X_1 \mid X_1 \in \mathcal{B}(0, q)] = D^2[X_1 \mid X_1 \in \mathcal{B}(q, 1 - q)] = D^2[X_1 \mid X_1 \in \mathcal{B}(1 - q, 1)].
\]
Moreover, $q = \Phi(x)$, where $x < 0$ is the unique negative solution of the following equation
\[
-x \Phi(x) = \phi(x)(1 - 2\Phi(x)),
\]
where $\phi$ and $\Phi$ denote the density and distribution function of standard normal, respectively. The approximate value of $q$ is 0.19809.

Proof of Lemma 3.3. Without any loss of generality we assume that $X_1$ has the standard normal distribution $\mathcal{N}(0, 1)$. Indeed, for $X_1' = \frac{X_1 - \mu_1}{\sigma_{11}}$, and $q_1, q_2 \in [0, 1]$, such that $q_1 < q_2$, we get
\[
D^2[X_1 \mid H_1(X_1) \in [q_1, q_2]] = D^2[\sigma_{11} X_1' + \mu_1 \mid \Phi(X_1') \in [q_1, q_2]]
= \sigma^2_{11} D^2[X_1' \mid \Phi(X_1') \in [q_1, q_2]].
\]
To proceed, we compute the first two moments of the truncated normal distribution of $X_1$. For transparency, we show full proofs (compare [10, Section 13.10.1]).
Let us calculate the tail and central conditional expectations for any $x \in (-\infty, 0)$, i.e. $E[X_1 \mid X_1 < x]$ and $E[X_1 \mid x < X_1 < -x]$. Since $\phi'(x) = -x\phi(x)$, we get

$$E[X_1 \mid X_1 < x] = \frac{1}{\Phi(x)} \int_{-\infty}^{x} \xi \phi(\xi) d\xi = \frac{1}{\Phi(x)} (-\phi(\xi)|_{-\infty}^{x}) = -\frac{\phi(x)}{\Phi(x)}.$$

$$E[X_1 \mid x < X_1 < -x] = 0.$$

To get the corresponding second moments we integrate by parts.

$$E[X_1^2 \mid X_1 < x] = \frac{1}{\Phi(x)} \int_{-\infty}^{x} \xi^2 \phi(\xi) d\xi$$

$$= \frac{1}{\Phi(x)} \left( -\xi \phi(\xi)|_{-\infty}^{x} + \int_{-\infty}^{x} \phi(\xi) d\xi \right)$$

$$= \frac{1}{\Phi(x)} (-x\phi(x) + \Phi(x))$$

$$= 1 - \frac{x\phi(x)}{\Phi(x)},$$

$$E[X_1^2 \mid x < X_1 < -x] = \frac{1}{1 - 2\Phi(x)} \int_{x}^{-\infty} \xi^2 \phi(\xi) d\xi$$

$$= \frac{1}{1 - 2\Phi(x)} \left( -\xi \phi(\xi)|_{x}^{-\infty} + \int_{x}^{-\infty} \phi(\xi) d\xi \right)$$

$$= \frac{1}{1 - 2\Phi(x)} (2x\phi(x) + 1 - 2\Phi(x))$$

$$= 1 + \frac{2x\phi(x)}{1 - 2\Phi(x)}.$$

Therefore,

$$D^2[X_1 \mid X_1 < x] = 1 - \frac{x\phi(x)}{\Phi(x)} - \phi(x)^2 \frac{1}{\Phi(x)^2},$$

$$D^2[X_1 \mid x < X_1 < -x] = 1 + \frac{2x\phi(x)}{1 - 2\Phi(x)}.$$

Since the conditional expected value behaves like a weighted arithmetic mean, we get that $E[X_1 \mid X_1 < x]$ is strictly increasing in $x$, while $E[X_1^2 \mid x < X_1 < -x]$ and $E[X_1^2 \mid X_1 < x]$ are strictly decreasing with respect to $x$. Consequently, the central conditional variance $D^2[X_1 \mid x < X_1 < -x]$ is strictly decreasing. Next, we show that the tail conditional variance $D^2[X_1 \mid X_1 < x]$ is strictly increasing. Indeed,

$$\frac{d}{dx} D^2[X_1 \mid X_1 < x] = -\frac{\phi(x)}{\Phi(x)} + x^2 \frac{\phi(x)}{\Phi(x)} - x \frac{\phi(x)^2}{\Phi(x)} + 2x \frac{\phi(x)^2}{\Phi(x)} + 2 \frac{\phi(x)^2}{\Phi(x)^2}$$

$$= \frac{\phi(x)}{\Phi(x)} \left( x^2 - 1 + x \frac{\phi(x)}{\Phi(x)} + 2 \frac{\phi(x)^2}{\Phi(x)^2} \right)$$

$$= \frac{\phi(x)}{\Phi(x)} \left( \left( x^2 - \frac{1}{2} \frac{\phi(x)}{\Phi(x)} \right)^2 + \frac{7}{4} \frac{\phi(x)^2}{\Phi(x)^2} - 1 \right) > 0.$$

The last inequality follows from the fact that since $\frac{\phi(x)}{\Phi(x)} = -E[X_1 \mid X_1 < x]$
is decreasing and positive, we get

\[
\frac{\phi(x)^2}{\Phi(x)^2} \geq \frac{\phi(0)^2}{\Phi(0)^2} = \frac{2}{\pi} > \frac{4}{7}.
\]

Next, note that (compare [9, Lemma 8.1])

\[
\lim_{x \to -\infty} D^2[X_1 \mid X_1 < x] = 0 \quad \text{and} \quad D^2[X_1 \mid X_1 < 0] = 1 - \frac{2}{\pi},
\]

while

\[
\lim_{x \to -\infty} D^2[X_1 \mid x < X_1 < -x] = 1 \quad \text{and} \quad \lim_{x \to 0} D^2[X_1 \mid x < X_1 < -x] = 0.
\]

Hence there exists a unique \( x < 0 \) such that

\[
D^2[X_1 \mid X_1 < x] = D^2[X_1 \mid x < X_1 < -x].
\]

Compare Figure 1 for visualisation.

Figure 1. The graph of conditional tail variance \( D^2[X_1 \mid X_1 \in B(0,q)] \) and conditional central variance \( D^2[X_1 \mid X_1 \in B(q,1-q)] \) as functions of \( q \in (0,0.5) \), under the assumption \( X_1 \sim \mathcal{N}(0,1) \).

Moreover,

\[
D^2[X_1 \mid X_1 < x] - D^2[X_1 \mid x < X_1 < -x] = 1 - \frac{x\phi(x)}{\Phi(x)} - \frac{\phi(x)^2}{\Phi(x)^2} - 1 - \frac{2x\phi(x)}{1 - 2\Phi(x)}
\]

\[
= \frac{\Phi(x)}{\Phi(x)^2(1 - 2\Phi(x))} (-x\Phi(x) - \phi(x)(1 - 2\Phi(x))),
\]

which shows that \( x \) is a (negative) solution of equation 5. Using basic numerical tools we checked that 5 is satisfied for \( x \approx -0.848464848, \) for which \( \Phi(x) \approx 0, 198089615. \)

\( \square \)
Theorem 3.1 provides an illustration to the empirical 20-60-20 rule. In particular we have shown that for any multivariate normal vector, this fixed ratio leads to a global equilibrium state, when dispersion and linear dependence measurement is considered. Nevertheless, please note, that the equality of conditional variances does not imply the equality of conditional distributions, as could be seen in Figure 2.

Also, while linear dependence structure is the same, the overall dependence in each subgroup, measured e.g. by the copula function \cite{12} is different. See Figure 3 for an illustrative example.

Remark 1. The equilibrium level $q$ calculated in Lemma 3.3 depends neither on $\mu$ nor $\Sigma$. Therefore, if we consider correlation matrices instead of covariance matrices in \eqref{3}, then the optimal value of $q$ from Theorem 3.1 also implies the corresponding equilibrium state, for correlation matrices.\footnote{Please note we need additional assumption that $X_1$ is not independent of $(X_2, \ldots, X_n)$ as otherwise any $q \in (0, 0.5)$ will satisfy \eqref{3} for correlation matrices instead of covariance matrices.}

Remark 2. The value $\|\Sigma_{(0,q]} - \Sigma_{(q,1-\eta]}\|$, for $q = 0.198$ and some arbitrary matrix norm (e.g. Frobenius norm) might be used to test how far $X$ is from a multivariate normal distribution. This test is particularly interesting, as it shows the impact of tails on the central part of the distribution, as usually (for empirical data) the dependence (correlation) structure in the tails significantly increases, revealing non-normality.

Remark 3. We can also consider more than three states, when clustering the population (e.g. having 5 states we might relate to them as critical, bad, normal, good and outstanding performance, based on selected benchmark). The ratios implying equilibrium state (similar to the one from Definition 2.1) for 5 and 7 different states are close to

\[
0.027/0.243/0.460/0.243/0.027 \quad \text{and} \quad 0.004/0.058/0.246/0.384/0.246/0.058/0.004,
\]

respectively. Those values could be easily computed using results from Lemma 3.2 and Lemma 3.3.
Figure 3. The conditional samples (upper row) and their conditional copula functions (lower row) from the bivariate normal with \( \mu = (0, 0) \) and \( \Sigma = \sigma_{ij} \), where \( \sigma_{11} = \sigma_{22} = 1 \) and \( \sigma_{12} = \sigma_{21} = 0.8 \).

The conditioning is based on the first coordinate and relates to the lower 20%, middle 60% and upper 20% of the whole population.

4. **Equilibrium for monotonic dependence.** In the definition of the equilibrium state (Definition 2.1) we have in fact measured the distance between conditional covariance matrices to compare the variability and linear dependence structure between the groups. As explained in Remark 1, one could use conditional correlation matrices instead of covariance matrices and focus on the comparison of the linear dependence structure. Of course there are also other measures of dependence that could be used to reformulate Definition 2.1.

Among most popular ones are so called measures of concordance, where Kendall \( \tau \) and Spearman \( \rho \) are usually picked representatives for two dimensional case (see [12, Section 5] for more details). Instead of measuring the linear dependence, they focus on the monotone dependence, being invariant to any strictly monotone transform of a random variable (note that correlation is only invariant wrt. positive linear transformation).

Thus, instead of covariance matrices \( \Sigma_{[0,q], \Sigma_{[q,1-q], \Sigma_{[1-q,1]} in Definition 2.1 \) we can consider the corresponding matrices of conditional Kendall \( \tau \) and conditional Spearman \( \rho \) denoted by \( \Sigma_{\tau}^{[0,q], \Sigma_{\tau}^{[q,1-q], \Sigma_{\tau}^{[1-q,1] \) and \( \Sigma_{\rho}^{[0,q], \Sigma_{\rho}^{[q,1-q], \Sigma_{\rho}^{[1-q,1] \) respectively. For comparison, we also consider conditional correlation matrices, for which we use notation \( \Sigma_{r}^{[0,q], \Sigma_{r}^{[q,1-q], \Sigma_{r}^{[1-q,1] \).

Unfortunately, the analog of Theorem 3.1 is not true, if we substitute covariance matrices with the Spearman \( \rho \) or Kendall \( \tau \) matrices in Definition 2.1. Because of that we need different kind of notation for the equilibrium state, as stated in Definition 4.1.
Definition 4.1. Let us assume that $X$ is symmetric and let $\kappa \in \{r, \rho, \tau\}$. We say that a quasi-global balance (or quasi-equilibrium state) is achieved in $X$ for $\kappa$ and $\hat{q} \in (0, 0.5)$ if
\[
\|\Sigma_{[0,q]} - \Sigma_{[q,1-q]}\|_F = \inf_{q \in (0,0.5)} \|\Sigma_{[0,q]} - \Sigma_{[q,1-q]}\|_F,
\]
where $\| \cdot \|_F$ is a standard Frobenius matrix norm given by
\[
\|A\|_F := tr AA^T = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2},
\]
for any $n$-dimensional matrix $A = \{a_{ij}\}_{i,j=1,...,n}$.
Similarly as in Definition 2.1, we say that a global balance (or equilibrium state) is achieved in $X$ if the value in 6 is equal to 0.

With slight abuse of notation, we write
\[
\hat{q}^r = \arg \min_{q \in (0,0.5)} \|\Sigma_{[0,q]} - \Sigma_{[q,1-q]}\|_F, \tag{7}
\]
\[
\hat{q}^\tau = \arg \min_{q \in (0,0.5)} \|\Sigma_{[0,q]} - \Sigma_{[q,1-q]}\|_F, \tag{8}
\]
\[
\hat{q}^\rho = \arg \min_{q \in (0,0.5)} \|\Sigma_{[0,q]} - \Sigma_{[q,1-q]}\|_F, \tag{9}
\]
to denote ratios, which imply quasi-equilibrium states given in 6.

As expected, for $X \sim \mathcal{N}(\mu, \Sigma)$, the values $\hat{q}^r$ and $\hat{q}^\rho$ also seem to be very close to 0.2, for almost any value of $\mu$ and $\Sigma$. To illustrate this property, we have picked 1000 random covariance matrices $\{\Sigma_i\}_{i=1}^n$ for $n = 4^5$ and computed the values of functions
\[
f^r_i(q) = \|\Sigma_{[0,q]} - \Sigma_{[q,1-q]}\|_F, \tag{10}
\]
\[
f^\tau_i(q) = \|\Sigma_{[0,q]} - \Sigma_{[q,1-q]}\|_F, \tag{11}
\]
\[
f^\rho_i(q) = \|\Sigma_{[0,q]} - \Sigma_{[q,1-q]}\|_F. \tag{12}
\]
To do so, for each $i \in \{1, 2, \ldots, 1000\}$ we have taken 1,000,000 Monte Carlo sample from $X \sim \mathcal{N}(0, \Sigma^i)$ and computed values of 10, 11 and 12 using MC estimates of the corresponding conditional matrices. The graphs of $f^r_i$, $f^\tau_i$ and $f^\rho_i$ for $i = 1, 2, \ldots, 50$ are presented in Figure 4. In Figure 5, we also present the smoothed histogram function of points $\{\hat{q}^r_i\}_{i=1}^{1000}$, $\{\hat{q}^\tau_i\}_{i=1}^{1000}$ and $\{\hat{q}^\rho_i\}_{i=1}^{1000}$, for which the minimum is attained in 10, 11 and 12, for $i = 1, 2, \ldots, 1000$.

Unfortunately, in general the values $\hat{q}^r$ and $\hat{q}^\rho$ defined in 8 and 9 are not constant and independent of $\Sigma$. In particular, if the dependence inside $X$ is very strong, e.g.

\footnote{That is $X$ is symmetric wrt. $E[X] = (E[X_1], \ldots, E[X_n])$; note that it implies that $\Sigma_{[0,q]} = \Sigma_{[1-q,1]}$, for any $q \in (0,0.5)$}

\footnote{This relate to the conditional correlation matrices, Spearman $\rho$ matrices or Kendall $\tau$ matrices, respectively.}

\footnote{For simplicity, we use arg min and assume that the (quasi) equilibrium state exists and is unique.}

\footnote{With additional assumption that correlation coefficients are bigger than 0.2 and smaller than 0.8, to avoid computation problems resulting from independence or comonotonicity, respectively (see also Remark 1). Note also that the sign of correlation coefficient is irrelevant, due to symmetry of $X$, so without loss of generality, we can assume that the correlation matrix is positive. Moreover, the values of $\hat{q}^r$ and $\hat{q}^\rho$ are invariant wrt. $\mu$, so we can set $\mu = 0$ without loss of generality.}
The graphs of functions $f^r_i$, $f^\tau_i$ and $f^\rho_i$ for $i = 1, 2, \ldots, 50$, computed using 1.000.000 sample from $\mathcal{N}(0, \Sigma^j)$ and the corresponding estimates of conditional matrices.

Monte Carlo density functions constructed using points $\{\hat{q}^j_i\}_{i=1}^{1000}$, $\{\hat{q}^\tau_i\}_{i=1}^{1000}$ and $\{\hat{q}^\rho_i\}_{i=1}^{1000}$. For each $i = 1, 2, \ldots, 1000$ a 1.000.000 sample from $\mathcal{N}(0, \Sigma^j)$ was simulated and the corresponding estimates of conditional matrices were used for computations.

To illustrate this property, let us present some theoretical results, involving conditional Spearman $\rho$ and Kendall $\tau$. For simplicity, till the end of this subsection, we assume that $n = 2$.

Given $X \sim \mathcal{N}(\mu, \Sigma)$, we know that $\sigma_{12}^2 = \sigma_{21}^2 = r \sigma_{11} \sigma_{22}$, where $r \in [-1, 1]$ is the correlation between $X_1$ and $X_2$. It is easy to show (see [9]), that both unconditional and conditional values of Spearman $\rho$ as well as Kendall $\tau$s depend only on the copula of $X^7$, which is parametrised by the correlation coefficient. Thus, without loss of generality, instead of considering all $\mu$ and $\Sigma$, we might assume that

$$X = (X_1, X_2) \sim \mathcal{N}(\mu, \Sigma) \quad \text{where} \quad \mu = (0, 0) \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix},$$

for a fixed $r \in [-1, 1]$.

Note that in our numerical example we have assumed that the correlation for any pair is between 0.2 and 0.8, excluding extremal cases.

Note that the (conditional) Spearman $\rho$ and Kendall $\tau$ is invariant to any monotone transform of $X_1$ or $X_2$, and so is the copula function.
Let \( \rho_{[p,q]}(r) \) and \( \tau_{[p,q]}(r) \) denote the corresponding conditional Spearman \( \rho \) and Kendall \( \tau \), given truncation interval \( B(p,q) \). Note that \( \rho_{[p,q]}(r) \) and \( \tau_{[p,q]}(r) \) are odd functions of \( r \).

**Lemma 4.2.** For all \( 0 \leq p < q \leq 1 \) and \( r \in (-1,1) \),

\[
\rho_{[p,q]}(-r) = -\rho_{[p,q]}(r) \quad \text{and} \quad \tau_{[p,q]}(-r) = -\tau_{[p,q]}(r).
\]

**Proof.** Before we begin the proof, let us recall some basic facts from the copula theory (cf. [12] and references therein). We use \( C^r \) to denote the Gaussian copula, with parameter \( r \in (-1,1) \), which coincides with the correlation coefficient. Noting, that the copula could be seen as a distribution function (with uniform margins) let us assume that \((U, V)\) is a random vector with distribution \( C^r \). We denote by \( C^r_{[p,q]} \) the copula of the conditional distribution \((U, V)\) under the condition \( U \in [p, q] \), where \( 0 \leq p < q \leq 1 \). Due to Sklar’s Theorem we get the following description of \( C^r_{[p,q]} \):

\[
C^r_{[p,q]}(u, v) = C^r((p - q)u + p, v) - C^r(p, v), \quad u, v \in [0, 1].
\]

(13)

Next, it is easy to notice, that the distribution function of \((U, 1 - V)\) is equal to \( C^{-r} \). Hence the Gaussian copulas commute with flipping, i.e.

\[
C^{-r}(u, v) = u - C^r(u, 1 - v) \quad \text{for} \quad u, v \in [0, 1].
\]

On the other hand the flipping transforms the conditional distribution \((U, V)|_{U \in [p, q]}\) to \((U, 1 - V)|_{U \in [p, q]}\). Hence, we get

\[
C^{-r}_{[p,q]}(u, v) = u - C^r_{[p,q]}(u, 1 - v).
\]

Finally, basing on [12, Theorem 5.1.9], we conclude

\[
\rho_{[p,q]}(-r) = -\rho_{[p,q]}(r),
\]

\[
\tau_{[p,q]}(-r) = -\tau_{[p,q]}(r),
\]

\[
\square
\]

We recall that the Spearman \( \rho \) and Kendall \( \tau \) of the conditional copula \( C^r_{[p,q]} \) are given by formulas:

\[
\rho_{[p,q]}(r) = \rho(C^r_{[p,q]}) = -3 + 12 \int_0^1 \int_0^1 C^r_{[p,q]}(u, v) \, du \, dv,
\]

\[
\tau_{[p,q]}(r) = \tau(C^r_{[p,q]}) = -1 + 4 \int_{[0,1]^2} C^r_{[p,q]}(u, v) \, dC^r_{[p,q]}(u, v).
\]

To describe their behaviour for small \( r \) we need their Taylor expansions with respect to \( r \).

**Proposition 1.** For a fixed \( p, q \in (0,1) \) \((p < q)\) and \( r \in (-1,1) \), such that \( r \) is close to 0, we get

\[
\rho_{[p,q]}(r) = r \frac{3}{(q - p)^2 \pi} \left( \Phi(2x_2) - \Phi(2x_1) - (q - p)\sqrt{\pi}(\varphi(x_1) + \varphi(x_2)) \right) + O(r^3),
\]

\[
\tau_{[p,q]}(r) = \frac{2}{3} \rho_{[p,q]}(r) + O(r^3),
\]

where \( x_1 = \Phi^{-1}(p) \) and \( x_2 = \Phi^{-1}(q) \).
Proof. We use notation similar to the one introduced in Lemma 4.2. The proof is based on two facts. First, for \( r = 0 \) both \( C \) and \( C_{[p,q]} \) are equal to product copula \( \Pi(u, v) := uv \), i.e.

\[
C^0(u, v) = uv = C^0_{[p,q]}(u, v).
\]

Second, the derivative of the distribution function of a bivariate Gaussian distribution having standardised margins with respect to the parameter \( r \) is equal to its density, which implies

\[
\frac{\partial C''(u, v)}{\partial r} = \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(- \frac{\Phi^{-1}(u)^2 + \Phi^{-1}(v)^2 - 2r\Phi^{-1}(u)\Phi^{-1}(v)}{2(1-r^2)} \right).
\]

We calculate the Taylor expansion of \( \rho_{[p,q]}(r) \) at \( r = 0 \).

\[
\rho_{[p,q]}(0) = \rho(\Pi) = 0.
\]

\[
\frac{\partial \rho_{[p,q]}(0)}{\partial r} = 12 \int_0^1 \int_0^1 \frac{\partial C^0_{[p,q]}(u, v)}{\partial r} \, du \, dv.
\]

The derivative of \( C''_{[p,q]} \) is calculated in two steps. First we differentiate formula 13. We get

\[
\frac{\partial}{\partial r} C''_{[p,q]}(u, v) = C''(q, v) - C''(p, v)
\]

\[
+ \frac{\partial}{\partial r} C''_{[p,q]}(u, v) \left( \frac{C''(q, v) - C''(p, v)}{q - p} \right) - \frac{1}{q - p} \left( \frac{\partial C''(q, v)}{\partial r} - \frac{\partial C''(p, v)}{\partial r} \right).
\]

Next, setting \( r = 0 \), we obtain

\[
\frac{\partial}{\partial r} C''_{[p,q]}(u, v) = \frac{1}{q - p} \left( \varphi(\Phi^{-1}((q - p)u + p)) - u\varphi(\Phi^{-1}(q)) - (1 - u)\varphi(\Phi^{-1}(p)) \right) \varphi(\Phi^{-1}(v)).
\]

Finally, we get

\[
\frac{\partial \rho_{[p,q]}(0)}{\partial r} = 12 \int_0^1 \int_0^1 \left( \varphi(\Phi^{-1}((q - p)u + p)) - u\varphi(\Phi^{-1}(q)) - (1 - u)\varphi(\Phi^{-1}(p)) \right) \varphi(\Phi^{-1}(v)) \, du \, dv
\]

\[
= 12 \int_0^1 \left( \varphi(\Phi^{-1}((q - p)u + p)) - u\varphi(\Phi^{-1}(q)) - (1 - u)\varphi(\Phi^{-1}(p)) \right) \varphi(\Phi^{-1}(v)) \, dv
\]

\[
= \frac{12}{q - p} \frac{1}{2\sqrt{\pi}} \left( \frac{1}{q - p} \frac{1}{2\sqrt{\pi}} \left( \Phi(\sqrt{2}\Phi^{-1}(q)) - \Phi(\sqrt{2}\Phi^{-1}(p)) \right) - \frac{1}{2} \left( \varphi(\Phi^{-1}(q)) + \varphi(\Phi^{-1}(p)) \right) \right).
\]

The proof of the Kendall \( \tau \) case follows from the symmetry

\[
\iint C_1 \, dC_2 = \iint C_2 \, dC_1.
\]
Theorem 4.3. For \( A \) decreasing in \( q \) where \( q \) for each \( r \) are strictly increasing in \( q \) \( q \) be a function, which assigns appropriate \( r > \) fixed \( \kappa \) for \( \tau \). We have

\[
\frac{\partial r_{q,p}(r)}{\partial r} = 4 \frac{\partial}{\partial r} \int_{[0,1]^2} C_{[p,q]}^r(u, v) \, dC_{[p,q]}^r(u, v)
= 8 \int_{[0,1]^2} \frac{\partial}{\partial r} C_{[p,q]}^r(u, v) \, dC_{[p,q]}^r(u, v).
\]

Setting \( r = 0 \) we get

\[
\frac{\partial r_{q,p}(0)}{\partial r} = 8 \int_{[0,1]^2} \frac{\partial}{\partial r} C_{[p,q]}^0(u, v) \, dC_{[p,q]}^0(u, v)
= 8 \int_{[0,1]^2} \frac{\partial}{\partial r} C_{[p,q]}^0(u, v) \, du \, dv
= \frac{8}{12} \frac{\partial \rho_{q,p}(0)}{\partial r}.
\]

For \( \kappa \) denoting either \( \rho \) or \( \tau \), using Proposition 1, we can compare values of \( \kappa_{[0,q]}(r) \) and \( \kappa_{[q,1-q]}(r) \), changing both \( q \in (0,0.5) \) and \( r \in (-1,1) \). Note that for \( n = 2 \) the equilibrium state corresponding to 9 is achieved, if and only if \( \kappa_{[0,q]}(r) - \kappa_{[q,1-q]}(r) = 0 \). In [9, Theorems 4.1 and 4.4], it was shown that for any fixed \( r > 0 \), the conditional copulas \( C_{[0,q]}^r \) are increasing in \( q \) while \( C_{[q,1-q]}^r \) are decreasing in \( q \). Hence, the differences

\[
\Delta_\rho(q, r) = \rho_{[0,q]}(r) - \rho_{[q,1-q]}(r) \quad \text{and} \quad \Delta_\tau(q, r) = \tau_{[0,q]}(r) - \tau_{[q,1-q]}(r)
\]

are strictly increasing in \( q \) and changing the sign. Using Lemma 4.2 we know that for each \( r \in (-1,1) \), such that \( r \neq 0 \), there exists exactly one \( q \in (0,0.5) \) for which \( \Delta_\rho(q, r) = 0 \), and one \( q \in (0,0.5) \) for which \( \Delta_\tau(q, r) = 0 \). Let

\[
A_\kappa : (-1,1) \to (0,0.5), \quad \kappa = \rho, \tau,
\]

be a function, which assigns appropriate \( q \) for any \( r \neq 0 \), and let

\[
A_\kappa(0) = \liminf_{t \to 0} A_\kappa(t).^8
\]

We now show that the graphs of \( A_\rho \) and \( A_\tau \) are orthogonal to the line \( r = 0 \).

**Theorem 4.3.** For \( r \) close to 0, we get

\[
A_\rho(r) = A_\tau(r) + O(r^2) = q^* + O(r^2),
\]

where \( q^* \approx 0.2132413 \) is a solution of the following equation

\[
(1 - 4q + 6q^2) \Phi(\sqrt{2} \Phi^{-1}(q)) - q(1 - 6q + 8q^2) \sqrt{\pi} \varphi(\Phi^{-1}(q)) - q^2 = 0.
\]

**Proof.** If \( r = 0 \), then for any \( q \in (0,0.5) \), we get that 14 is equal to 0, so for clarity we might set \( A_\kappa(0) = q^* \). Using Lemma 4.2, without loss of generality, we might assume that \( r > 0 \). Due to Proposition 1, for small \( r \), we get

\[
\rho_{[0,q]}(r) - \rho_{[q,1-q]}(r) = \frac{3}{\pi} r^2 q^{-2} \left( \Phi(\sqrt{2} \Phi^{-1}(q)) - q \sqrt{\pi} \varphi(\Phi^{-1}(q)) \right) + O(r^3)
- r^2 (1 - 2q)^2 \left( 1 - 2 \Phi(\sqrt{2} \Phi^{-1}(q)) \right)
\]

Note, that for \( r = 0 \), any \( q \in (0,0.5) \) implies equilibrium state, the reason we define \( A(0) \) in that way.
\[ -2(1 - 2q)\sqrt{\pi \varphi(\Phi^{-1}(q))} + O(r^3) \]
\[ = \frac{3}{\pi q^2(1 - 2q)^2} \left( (1 - 4q + 6q^2)\Phi(\sqrt{2}\Phi^{-1}(q)) - q(1 - 6q + 8q^2)\sqrt{\pi \varphi(\Phi^{-1}(q)) - q^2} \right) + O(r^3) \]

and a similar formula for \( \tau \).

In particular, Theorem 4.3 implies that \( A_\rho(0) = A_\tau(0) = q^* \). Using basic numerical calculations, for \( \kappa \) denoting \( \rho \) or \( \tau \), we get
\[ 0.213 < A_\kappa(r) < 0.271, \]
for any \( r \in (-1, 1) \). Nevertheless, usually, this bond is much tighter, which could have been already observed in our previous numerical example (see e.g. Figure 4). With some easy calculations, we get
\[ 0.213 < A_\kappa(r) < 0.230, \]
for \( r \in (-0.9, 0.9) \). The graph of function \( \Delta_\rho(q, r) = \rho_{[0, q]}(r) - \rho_{[q, 1-q]}(r) \) for various fixed values of \( q \in (0, 0.5) \) is presented in Figure 6. See also Figure 7 for the corresponding graph of \( \Delta_\tau \).

**Remark 4.** When we consider the equilibrium state for conditional Spearman \( \rho \) matrices (or Kendall \( \tau \)), we only need to know the dependance structure of \( X \), given by the copula function. Thus, we can set any marginal distributions of \( X_1, \ldots, X_n \), without changing the equilibrium. This allow us to consider much more general class of multivariate distributions, for which the 20-60-20 rule holds.

5. **Abandoning Gaussian world.** When the assumption \( X \sim \mathcal{N}(\mu, \Sigma) \) is loosed, the existence of equilibrium is no longer guaranteed. A natural question is if for any elliptical distribution the equivalent of 20-60-20 rule holds. In this section we discuss this matter shortly.

We say that \( X \) has the elliptic distribution if it can be defined in terms of a characteristic function
\[ \phi_X(t) = e^{it'\mu}\Psi(t'\Sigma t), \]
where $\mu$ is a vector (which coincides with mean vector, if it exists), $\Sigma$ is a scale matrix (which is proportional to covariance matrix, if it exists) and $\Psi$ is so called characteristic generator of the elliptical distribution (cf. [6] and references therein for a general survey about elliptic distributions). For simplicity, we use so called stochastic representation of an elliptic distribution. It is well known (see [6]) that if $X$ has the density, then it is elliptic if and only if it can be represented as

$$X = \mu + \sqrt{\Sigma}RU,$$

where $\sqrt{\Sigma}$ is any square matrix such that $\sqrt{\Sigma}^t \sqrt{\Sigma} = \Sigma$ (e.g. obtained using Cholesky decomposition), $U$ is an $n$-dimensional random vector, uniformly distributed on the unit $n$-sphere, and $R$ is a nonnegative random vector, corresponding to the radial density, independent of $U$. Moreover, we assume that the first two moments of $R$ exists, which ensures the existence of mean vector and covariance matrix of $X$. Now we can ask, if for given $U$ and $R$ the equilibrium state of $X$ always exists and if it is invariant wrt. $\mu$ and $\Sigma$.

Unfortunately, it is easy to show, that the equilibrium state (with covariance matrices) is not always achieved and the quasi-equilibrium state might strongly depend on $\Sigma$, even when we consider only the class of multivariate t-student distributions (i.e. we can consider appropriate radial distributions and covariance matrices in Algorithm 1).

On the other hand, if we substitute covariance matrices with correlation matrices in 3, then the result similar to Theorem 3.1 hold for a much more general class of elliptic distributions.

To illustrate this property, we have conducted simple computational experiment, using multivariate t-student distribution, as it is commonly used by practitioners. Assuming $n = 4$, for any $\nu \in \{2, 3, \ldots, 20\}$ we have picked 100 random matrices $\Sigma^i_\nu$ and for each $i = 1, 2, \ldots, 100$ we simulated 1,000,000 Monte Carlo sample, assuming $X \sim t_\nu(0, \Sigma^i_\nu)$. Next, we have calculated the values of $q^i_\nu \in (0, 0.5)$, for which (quasi-)equilibrium state is attained (i.e. for estimates of conditional correlation matrices; see Algorithm 1). In Figure 8 we present the graph of 0.1, 0.5 and 0.9 quantiles of the sample $\{q^i_\nu\}_{i=1}^{100}$, for $\nu = 2, 3, \ldots, 20$. The value of $q$ for which (quasi-)equilibrium state is achieved clearly depends on the degrees of freedom. As
supposed, it increase wrt. \( \nu \) up to 0.198 (note that t-student distribution converge to normal distribution, when \( \nu \to \infty \)).

![Figure 8](image_url)

**Figure 8.** The graph of 0.1, 0.5 and 0.9 quantiles of \( \{q^i_\nu\}_{i=1}^{100} \) for \( \nu = 2, 3, \ldots, 20 \).

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Algorithm 1 Compute quasi-equilibrium state for elliptic distribution

Require:
\( n \in \mathbb{N}_+ \) – dimension
\( N \in \mathbb{N}_+ \) – size of Monte Carlo sample
\( \text{radial.Dist} \) – radial distribution (e.g. \( \sqrt{\chi(n)} \) for multivariate normal)

1: procedure Equilibrium(\( n, N, \text{radial.Dist} \))
2: Generate \( U \): \( N \) independent samples from \( n \)-dimensional unit sphere (uniform density)
3: Generate \( R \): \( N \) independent samples from (univariate) \( \text{radial.dist} \)
4: Generate \( \Sigma = \{ \sigma_{ij} \} \): \( n \times n \) scale matrix (proportional to covariance matrix)
5: while \( \min_{i \neq j} \left( \frac{\sigma_{ij}^2}{|\sigma_{ii}\sigma_{jj}|} \right) < 0.2 \lor \max_{i \neq j} \left( \frac{\sigma_{ij}^2}{|\sigma_{ii}\sigma_{jj}|} \right) > 0.8 \) do
6: Generate (new) \( \Sigma = \{ \sigma_{ij} \} \): \( n \times n \) scale matrix
7: end while
8: Compute \( \sqrt{\Sigma} \), e.g. using Cholesky decomposition
9: Compute \( X = \{ X_{ik} \} = (\sqrt{\Sigma})RU \) (i.e. matrix \( n \times N \); random sample from elliptic distribution)
10: Define function \( \text{Dist}(q) \), for \( q \in (0, 0.5) \)
11: function Dist(\( q \))
12: Compute \( q^1 \), sample lower \( q \)-quantile of \( \{ X_{1k} \} \)
13: Compute \( q^2 \) sample lower \( (1-q) \)-quantile of \( \{ X_{1k} \} \)
14: Compute conditional tail sample \( X^1 \), by selecting all \( 1 \leq k \leq N \), for which \( X_{1k} \leq q^1 \)
15: Compute conditional central sample \( X^2 \), by selecting all \( 1 \leq k \leq N \), for which \( q^1 \leq X_{1k} \leq q^2 \)
16: Compute \( \Sigma_{(0,q]} \), a (conditional) covariance matrix of \( X^1 \)
17: Compute \( \Sigma_{(q,1−q]} \), a (conditional) covariance matrix of \( X^2 \)
18: Compute \( d = \|\Sigma_{(0,q]} - \Sigma_{(q,1−q]}\|_F \)
19: return \( d \)
20: end function
21: Compute \( \hat{q} = \arg \min_{0 \leq q \leq 0.5} \text{Dist}(q) \)
22: return \( \hat{q} \)
23: end procedure

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