A COMPLETE DESCRIPTION OF THE COHOMOLOGICAL INVARIANTS OF EVEN GENUS HYPERELLIPTIC CURVES

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Abstract. When the genus $g$ is even, we extend the computation of the cohomological invariants of $\mathcal{H}_g$ to non algebraically closed fields, we give an explicit functorial description of the invariants and we completely describe their multiplicative structure.

Introduction

Notation: we fix a prime number $p$ and a field $k_0$ of characteristic not dividing $p$ and different from 2. Every scheme is assumed to be of finite type over $\text{Spec}(k_0)$. If $X$ is a variety, with the notation $H^•(X)$ we will always mean the graded-commutative ring $\bigoplus_i H^i_\mathbb{Z}(X, \mu_p^\otimes i)$. Sometimes, we will write $H^•(R)$, where $R$ is a finitely generated $k_0$-algebra, to indicate $H^•(\text{Spec}(R))$.

Cohomological invariants of algebraic groups are a well-known arithmetic analogue to the theory of characteristic classes for topological groups. The category of topological spaces is replaced with extensions of a base field $k_0$, and singular cohomology is replaced with Galois cohomology. More precisely, given an algebraic group $G$, write $P_{BG}$ for the functor that associates to a field $K/k_0$ the set of isomorphism classes of $G$-torsors over $K$. Then:

Definition. A cohomological invariant of $G$ is a natural transformation $P_{BG} \to H^•$ of functors from fields over $k_0$ to sets.

The set of cohomological invariants has a natural structure of graded-commutative ring induced by the structure of $H^•$.

The first appearance of cohomological invariants can be traced back to the seminal paper [Wit37] and since then they have been extensively studied. The book [GMS03], by Garibaldi, Merkurjev and Serre provides a detailed introduction to the modern approach to this theory.

One can think of the cohomological invariants of $G$ as invariants of the classifying stack $BG$ rather than the group $G$. Following this idea, in [Pir18] the second author extended the notion of cohomological invariants to arbitrary smooth algebraic stacks over $k_0$:

Definition. Let $X$ be a smooth algebraic stack, and let $P_X : (\text{Field}/k_0) \to (\text{Set})$ be its functor of points. A cohomological invariant of $X$ is a natural transformation $P_X \to H^•$ satisfying a certain continuity condition (see [Pir18, definition 1.1]).

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The graded-commutative ring of cohomological invariants of a smooth algebraic stack $\mathcal{X}$ is denoted $\text{Inv}^\bullet(\mathcal{X})$. Note that this definition recovers the classical invariants by taking $\mathcal{X} = BG$.

By [Pir18, 4.9] the cohomological invariants of a smooth scheme $X$ are equal to its zero-codimensional Chow group with coefficients $A^0(X)$, an extension of ordinary Chow groups introduced by Rost [Ros96]. Given a smooth quotient stack $\mathcal{X} = [X/G]$ we can construct the equivariant Chow ring with coefficients $A^*_\mathcal{X}(X)$ following Edidin and Graham’s construction [EG98] and we have the equality $A^*_\mathcal{X}(X) = \text{Inv}^\bullet(\mathcal{X})$ by [Pir17, 2.10].

In [Pir18] the second author also computed the cohomological invariants of $\mathcal{M}_{1,1}$, the moduli stack of smooth elliptic curves, and in the subsequent works [Pir17] and [Pir] he computed the cohomological invariants of $\mathcal{H}_g$, the moduli stack of smooth hyperelliptic curves, when $g$ is even or equal to 3 and the base field is algebraically closed. The first author then extended the result to arbitrary odd genus [DL], using a new presentation of the stack $\mathcal{H}_g$ he developed in [DL19]. When $p$ is odd, the invariants turn out to be (almost) trivial, but when $p = 2$ they get up to degree $g + 2$. Some relevant questions are still open:

- Does the result work for non algebraically closed fields?
- Can we get an explicit description of the invariants?
- What is the multiplicative structure of $\text{Inv}^\bullet(\mathcal{H}_g)$?

This paper answers the three questions above in the case when $g$ is even (see section 3). The main idea is rather simple: given an hyperelliptic curve $C$ over a field $K$, consider the curve’s Weierstrass divisor $W_C$, i.e. the ramification divisor of the quotient map $C \to C/\tau$ given by the hyperelliptic involution. Then $W_C$ is an étale algebra of degree $2g + 2$ over $K$, which is equivalent to a $S_{2g+2}$-torsor.

The resulting map $\mathcal{H}_g \to BS_{2g+2}$ produces an inclusion $\text{Inv}^\bullet(S_{2g+2}) \subset \text{Inv}^\bullet(\mathcal{H}_g)$ which yields $H^\bullet(k_0)$-linearly independent invariants $\alpha_0 = 1, \alpha_1, \ldots, \alpha_{g+1}$, respectively of degree $0, \ldots, g + 1$ (see section 1).

These invariants turn out to almost generate $\text{Inv}^\bullet(\mathcal{H}_g)$: there is only one missing generator, of which we give an explicit description.

Specifically, we can do the following. Assume that $g$ is even. An hyperelliptic curve over $K$ comes equipped with a rational conic $C' = C/\tau$ over $K$, an invertible sheaf of degree $-g - 1$ on $C'$, and a section $s$ of $H^0(L^{g-2})$. We can (smooth-Nisnevich) locally on $\mathcal{H}_g$ choose a section $s_0$ of $L^{g-2}$. Then the element $t(C) := s/s_0$ can be seen as belonging to $H^1(K) = K^*/(K^*)^2$. The product $t \cdot \alpha_{g+1}$ does not depend on the choices we made and provides a new invariant $\beta_{g+2}$.

Another way of seeing the same invariant is that locally we can assume that our section does not pass through a given point $\infty$ of $C'$. Then $s(\infty)$ is well defined up to squares and the product $s(\infty) \cdot \alpha_{g+1}$ can be extended to our last invariant $\beta_{g+2}$.

This approach works over any field, solving the first two questions. For the last one, the multiplicative structure of $\text{Inv}^\bullet(S_{2g+2})$ is known, and their products with $\beta_{g+2}$ can be easily obtained from the explicit description, completely describing the multiplicative structure of $\text{Inv}^\bullet(\mathcal{H}_g)$ when $g$ is even (see theorem 3.1).

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1. Cohomological invariants from Weierstrass divisors

We start by recalling some basic notions on families of hyperelliptic curves. A more detailed discussion can be found in [KL79].
A family of hyperelliptic curves $C \to S$ of genus $g$ is defined as a proper and smooth morphism whose fibres are curves of genus $g$, and moreover there exists an hyperelliptic involution $\iota: C \to C$ of $S$-schemes such that the quotient $C' := C/\iota$ is a family of conics over $S$.

The ramification divisor of the projection $C \to C'$, equipped with the scheme structure given by the zeroth Fitting ideal of $\Omega_{C/C'}$, is called the Weierstrass subscheme of $C/S$, and it is denoted $W_{C/S}$. The morphism $W_{C/S} \to C'$ is a closed immersion, so we will use the same notation for the divisor on $C$ and on $C'$ when no confusion is possible.

The scheme $W_{C/S}$ is finite and étale over $S$ of degree $2g + 2$. The functor sending a family $C/S$ to its Weierstrass subscheme $W_{C/S}$ defines a morphism from $\mathcal{H}_g$ to $\text{Ét}_{2g+2}$, the stack of étale algebras of degree $2g + 2$, which is in turn isomorphic to the classifying stack $BS_{2g+2}$ of $S_{2g+2}$-torsors.

More generally, consider $\mathbb{A}^{n+1}$ as the space of binary forms of degree $n$, and let $\mathbb{A}_{\text{sm}}^{n+1}$ be the open subset of non degenerate forms. Then there is a morphism $\mathbb{A}_{\text{sm}}^{n+1} \to \text{Ét}_n$ obtained by sending a form $f$ to the zero locus $V_f \subset P^1$. This map factors through the projectivization $P_{\text{sm}}^n$. Arsie and Vistoli [AV04] constructed a presentation of $\mathcal{H}_g$ as $[\mathbb{A}_{\text{sm}}^{2g+1}/G]$, where $G$ is either $GL_2$ or $PGL_2 \times \mathbb{G}_m$ depending on parity of $g$.

The map $\mathbb{A}_{\text{sm}}^{n+1} \to \text{Ét}_n$ factors through $\mathbb{A}_{\text{sm}}^{n+1} \to \mathcal{H}_g$: in fact, if we pull back the universal family $C_g \to \mathcal{H}_g$ to $\mathbb{A}_{\text{sm}}^{n+1}$ we see that given a morphism $S \to \mathbb{A}_{\text{sm}}^{n+1}$ we obtain a family $C_S/S$ such that $C_S/S = P^1$ and $V_f = W_{C/S}$.

Recall from [Pir18, Def 3.2] that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is smooth-Nisnevich if it is smooth, representable and every map from the spectrum of a field to $\mathcal{Y}$ lifts to $\mathcal{X}$. Cohomological invariants form a sheaf with respect to the topology induced by smooth-Nisnevich morphisms [Pir18, Thm 3.8].

**Proposition 1.1.** The morphism $\mathbb{A}_{\text{sm}}^{n+1} \to BS_n$ is smooth-Nisnevich.

**Proof.** Write down a form of degree $n$ as $f(\lambda_1, \lambda_2) = x_0\lambda_1^n + x_1\lambda_1^{n-1}\lambda_2 + \ldots + x_n\lambda_2^n$. Then we can factor $f = (\lambda_1 + \lambda_2)\ldots(\lambda_1 + \alpha_n\lambda_2)$.

Consider the subscheme $V \subset \mathbb{A}_{\text{sm}}^{n+1}$ given by $x_0 = 1$. Denote by $\Delta$ the subset of $\mathbb{A}^n$ where the coordinates are not distinct. We have a map $(\mathbb{A}^n \setminus \Delta) \to V$ given by $(\alpha_1, \ldots, \alpha_n) \mapsto (\lambda_1 + \alpha_1\lambda_2)\ldots(\lambda_1 + \alpha_n\lambda_2)$.

This map is clearly the $S_n$-torsor inducing the map $V \to BS_n$. As the action of $S_n$ on $\mathbb{A}^n \setminus \Delta$ is free the torsor is versal [GMS03, 5.1-5.3], which implies our claim.

In particular, given a splitting $\mathbb{A}_{\text{sm}}^{n+1} \to \mathcal{X} \to BS_n$ where the stack $\mathcal{X}$ is smooth over the base field and the second morphism is representable, the morphism $\pi$ is smooth-Nisnevich. Then the pullback $\pi^*$ on cohomological invariants is injective.

**Corollary 1.2.** The pullback $\text{Inv}^*(BS_{2g+2}) \to \text{Inv}^*(\mathcal{H}_g)$ is injective.

A complete description of the cohomological invariants of $BS_n$ can be found in [GMS03, CH. VII]. We briefly recall here some of their properties, in particular the ones that will be relevant for our work.

Let $E$ be an étale algebra over a field $K$ of degree $n$. We denote $m_x : E \to E$ the multiplication morphism by an element $x$ of $E$. We can then define a morphism of classifying stacks $\varphi : BS_n \to BO_n$ by sending an étale algebra $E$ to the quadratic form on $E$ defined by the formula $x \mapsto \text{Tr}(m_x)$. Let $a_i$ be the degree $i$ cohomological invariant obtained by pulling
back the $i$th Stiefel-Whitney class along $\varphi$. Then $\text{Inv}^\bullet(BS_n)$ is a free $H^\bullet(k_0)$-module generated by
\[ \alpha_0 = 1, \alpha_1, \ldots, \alpha_{[n/2]}, \]
where the degree of $\alpha_i$ is $i$.

Before proceeding further, let us explain how to explicitly compute the value of the cohomological invariants $\alpha_i$.

As already said, we can associate to $E$ a quadratic form as follows: given two elements $x$ and $y$ of $E$, we define
\[ q_E(x,y) := \text{Tr}(m_{xy}) \]
Regarding $E$ as a vector space of dimension $n$, choose a basis $e_1, \ldots, e_n$ of $E$ such that $q_E(x,x) = \sum_{i=1}^n \lambda_i x_i^2$, where $x = x_1 e_1 + \cdots + x_n e_n$. If $\sigma_i$ denotes the elementary symmetric polynomial of degree $i$ in $n$ variables, we have:
\[ \alpha_i(E) = \sigma_i(\{\lambda_1, \ldots, \lambda_n\}) \in H^i(K) \]
where $\{\lambda_i\}$ are the corresponding classes in $H^i(K) \simeq K^*/(K^*)^2$ and the product is the one defined in cohomology.

The multiplicative structure of the invariants $\alpha_i$ is described the following way. Given $s, r \leq [n/2]$, write $s = \sum_{i \in S} 2^i, r = \sum_{i \in R} 2^i$ and let $m = \sum_{i \in S \cap R} 2^i$. Then
\[ \alpha_s \cdot \alpha_r = (-1)^m \cdot \alpha_{s+r-m}. \]

Let $E$ denote an étale algebra over a scheme $S$ of degree $n$ and write $\alpha_{\text{tot}} = \sum_i \alpha_i$.

Then the following properties hold:
1. $\alpha_i(E) = 0$ if $i > [n/2] + 1$.
2. $\alpha_{[n/2]+1}(E) = \{2\} \cdot \alpha_{[n/2]}$ if $[n/2] + 1$ is even, and $0$ otherwise.
3. $\alpha_{\text{tot}}(K^*) = 1$.
4. $\alpha_{\text{tot}}(K[x]/(x^2-a)) = 1 + \{a\}$.
5. $\alpha_{\text{tot}}(E \times E') = \alpha_{\text{tot}}(E) \alpha_{\text{tot}}(E')$.

The fact that we know the existence of a large subalgebra of $\text{Inv}^\bullet(H_g)$ allows for a vast simplification of the original computation. Let $P^n_{\text{sm}}$ be the quotient of $A^n_{\text{sm}}$ by the multiplicative group, and set $G := GL_2$ or $PGL_2 \times \mathbb{G}_m$, depending on the parity of $g$. One of the most challenging steps in the inductive proofs of $[\text{Pir17, Pir}]$ lay in showing that the last map in the exact sequence of equivariant Chow groups with coefficients
\[ 0 \rightarrow A_G^0(P^n) \rightarrow A_G^n(P^n_{\text{sm}}) \overset{\partial}{\rightarrow} A_G^0(\Delta_n) \rightarrow A_G^1(P^n_{\text{sm}}) \]
was zero for every even $n$. In fact, for $g$ even this step forced the second author to assume that the base field was algebraically closed, and it required a completely different construction by the first author for odd genus $g > 3$.

Knowing that $\text{Inv}^\bullet(S_{n/2}) \subset \text{Inv}^\bullet([P^n_{\text{sm}}/G]) = A_G^0(P^n_{\text{sm}})$ lets us prove it easily, just by comparing the elements that we know must be in $A_G^0(P^n_{\text{sm}})$ and those that are allowed by the exact sequence.

**Corollary 1.3.** Let $n \geq 0$ be even. Then the last morphism of the exact sequence
\[ 0 \rightarrow A_G^0(P^n) \rightarrow A_G^n(P^n_{\text{sm}}) \overset{\partial}{\rightarrow} A_G^0(\Delta_n) \]
is surjective.

The inclusion $\text{Inv}^\bullet(S_{n/2}) \subset \text{Inv}^\bullet([P^n_{\text{sm}}/G])$ is an isomorphism when $n/2$ is odd; when $n/2$ is even the cokernel of the inclusion is a free $H^\bullet(k_0)$-module generated by the 2-torsion Brauer class coming from the cohomological invariants of $PGL_2$. 
Proof. We proceed by induction on the even integer $n$, the case $n = 0$ being trivial. By [Pir17, 3.3, 3.4] we know that $A^n_G(\Delta_n) \cong A^n_G(P^n_{\text{sm}} \times P^1)$, which by the inductive hypothesis and the projective bundle formula is freely generated as a $H^\bullet(k_0)$-module by $1, \alpha_1, \ldots, \alpha_{n-2/2}$.

Using the fact that $A^n_G(P^n_{\text{sm}})$ has to contain the cohomological invariants of $\mathcal{B}S_n$, we see that the cokernel of $A^n_G(P^n) \to A^n_G(P^n_{\text{sm}})$ is freely generated by elements $y_1, \ldots, y_{[n/2]}$, of degree $\deg(y_i) = i$.

Comparing the two graded modules we immediately obtain that the map

$$\partial : A^0_G(P^n_{\text{sm}}) \to A^0_G(\Delta_n)$$

which lowers degree by one, must be surjective. \qed

In the next section we will explicitly construct another invariant of $\mathcal{H}_g$ when $g$ is even. This will allow us to conclude the generalization of the proof in [Pir17] in section 3.

2. The last invariant

Consider the open subset $U_0 = \{x_0 \neq 0\}$ inside of $P_{\text{sm}}^{2g+2}$, and let $U_0$ be its preimage in $\mathbb{H}^{2g+3}$. The $G_m$-torsor $U_0 \to \overline{U}_0$ is clearly trivial. Consequently we have

$$A^0(U_0) = A^0(U_0) \oplus A^0(U_0) \cdot t$$

where $t$ is the cohomological invariant that sends a $K$-point $(x_0 : x_1 : \cdots : x_{2g+2})$ to $\{x_0\}$ in $H^1(K) \cong K^*/(K^*)^2$. The multiplicative structure is defined by the single additional relation $t^2 = \{-1\} \cdot t$.

The invariant $t$ clearly does not extend to a cohomological invariant of $\mathbb{H}^{2g+3}_{\text{sm}}$, but we claim that the element $\beta_{g+2} := t \cdot \alpha_{g+1}$ does.

Proposition 2.1. The element $\beta_{g+2}$ defined above extends to a cohomological invariant of $\mathbb{H}^{2g+3}_{\text{sm}}$. Moreover, $\beta_{g+2}$ is $H^\bullet(k_0)$-linearly independent from the invariants coming from $\mathcal{B}S_{2g+2}$.

Proof. We have an exact sequence

$$0 \to A^0(\mathbb{H}_{\text{sm}}^{2g+3}) \to A^0(U_0) \oplus A^0(U_0) \cdot t \xrightarrow{0} A^0(V_0)$$

where $V_0$ is the complement to $U_0$. We claim that the element $t \cdot \alpha_{g+1}$ maps to zero. As $\partial(t \cdot \alpha) = \alpha$ for any $\alpha$ coming from $A^0(\mathbb{H}_{\text{sm}}^{2g+3})$, this is equivalent to saying that $\alpha_{g+1}$ becomes zero when restricted to $V_0$.

Consider the universal conic $C'/\mathbb{H}_{\text{sm}}^{2g+3} \cong \mathbb{H}_{\text{sm}}^{2g+3} \times P^1$. Restricting to the open subset $U_0$ is equivalent to requiring the Weierstrass divisor of a curve $C/S$ to not contain the divisor at infinity $S \times \infty$. Conversely, given a curve mapping to the complement $V_0$, the Weierstrass divisor will always have a section $S \to W_C$ given by $S \to S \times \infty$. In other words, given a field $K$ and a curve $C/K$ lying over $V_0$, the étale algebra $R_C/K$ will split as $R'_C \times K$.

Now we apply property (4) of the Stiefel-Whitney classes. Looking at the part of degree $g + 1$ we get

$$\alpha_{g+1}(R'_C \times K) = \sum_{i+j=g+1} \alpha_i(R'_C) \cdot \alpha_j(K).$$

By property (1) the right hand side is zero, concluding our proof. \qed

Now we want to prove that this element glues to a cohomological invariant of $\mathcal{H}_g$. We will show two different approaches to the problem.
The first is a straight up computation that reduces the problem to a maximal torus inside $GL_2$. The second is more subtle: we produce an invariant on a projective bundle over $\mathbb{A}^{2g+3}_{\text{sm}}$ which is trivially equivariant, but which we cannot a priori show to be nonzero.

Then we show that after restricting to a locally closed subset it is equal to $\beta_{g+2}$, proving that it is independent from the invariants coming from $\mathcal{B}S_{2g+2}$ (and in particular nonzero).

### 2.1. First proof: reduction to the torus action.

**Lemma 2.2.** Let $X \xrightarrow{f} Y$ be a map of algebraic spaces such that Zariski locally on $Y$ we have $X = Y \times Z$, where $Z$ is a smooth proper scheme admitting a cell decomposition $Z = \sqcup_{i \in I} (\sqcup_{j \in J_i} \mathbb{A}^n_i)$. Then we have

$$A^\bullet(X) \simeq A^\bullet(Y) \otimes CH^\bullet(Z).$$

**Proof.** We begin with the case where $X = Y \times Z$, proceeding by induction on the dimension of $Z$. Note that at this point we do not need the proper and smooth assumption on $Z$.

If the dimension of $Z$ is zero, the statement is trivially true. Now let the dimension of $Z$ be equal to $n$, and let $Z' \subset Z$ be the union of all lower dimensional components, which is a closed subset of $Z$. For any $V \subseteq Z$ there is a map $A^\bullet(Y) \otimes CH^\bullet(V) \to A^\bullet(X)$ given by $(a, b) \to a \times b$. We have a long exact sequence

$$\ldots \to A^\bullet(Y \times Z) \to A^\bullet(Y \times (\sqcup_{j \in J_i} \mathbb{A}^n_i)) \xrightarrow{\partial} A^\bullet(Y \times Z') \to A^{n-1}(Y \times Z) \to \ldots$$

As the Chow groups with coefficients of an affine bundle are isomorphic to those of $X$, proceeding by induction on the dimension of $Z'$ we can conclude by comparing the long exact sequence above and the exact sequence

$$\ldots \to \bigoplus_{i+j=s} A^i(Y) \otimes CH^j(Z) \to \bigoplus_{i+j=s} A^i(Y) \otimes \mathbb{Z}^#J_n \xrightarrow{\partial} \bigoplus_{i+j=s} A^i(Y) \otimes CH^j(Z') \ldots$$

For the general case, note that we know the result to hold true for ordinary Chow groups [EG97, Prop. 1]. Thus we have a subring of $A^\bullet(X)$ isomorphic to $\text{CH}^\bullet(Z) \otimes \mathbb{F}_p$, and by taking multiplication this induces a map $\text{CH}^\bullet(Z) \otimes A^\bullet(Y) \to A^\bullet(X)$.

Now let $U \subseteq Y$ be a Zariski open subset over which the fibration is trivial, and assume by induction that the formula holds on the complement $V$. The map $\text{CH}^\bullet(Z) \otimes A^\bullet(Y) \to A^\bullet(X)$ is compatible with the isomorphisms $A^\bullet(f^{-1}(V)) \simeq A^\bullet(V) \otimes \text{CH}^\bullet(Z)$, $A^\bullet(f^{-1}(U)) \simeq A^\bullet(U) \otimes \text{CH}^\bullet(Z)$, so we can compare the two corresponding long exact sequences and conclude by the five lemma as above. □

**Proposition 2.3.** Let $G$ be an affine, smooth, special algebraic group, and let $T \subseteq G$ be a maximal torus. Then for any $G$-scheme $X$ we have

$$A^0_G(X) \simeq A^0_T(X).$$

**Proof.** After picking an equivariant approximation for $[X/G]$, we may assume that $[X/G]$ is an algebraic space. Let $T \subseteq B \subseteq G$ be a Borel subgroup. The map $[X/T] \to [X/G]$ splits as

$$[X/T] \xrightarrow{g} [X/B] \xrightarrow{f} [X/G]$$

where the map $g$ is an affine bundle, and $f$ admits a cell decomposition Zariski locally. Then we can use lemma 2.2 to conclude. □

**Proposition 2.4.** The cohomological invariant $\beta_{g+2}$ glued to an invariant of $H_g$. 

Proposition 2.6. The element $\lambda E_{x}$ from $B_{2}$ is invariant under the action of $GL_{2}$ over $H_{2}$. Moreover, it suffices to check it on the generic point of $\mathbb{A}^{2g+3}_{\mathbb{A}^{2g+3}} \times \mathbb{A}^{2g}_{\mathbb{A}^{2g+3}}$.

Note that we already know that $\gamma_{g+1}$ is invariant, so the question boils down to whether $\gamma_{g+1} \cdot \{\{x_{0}\} - \{x(0)\}\} = 0$ for every element $\lambda = (\lambda_{1}, \lambda_{2})$ of $\mathbb{A}^{2g+3}_{\mathbb{A}^{2g+3}}$.

Recall that $GL_{2}$ acts by $A(f) = det(A)^{g}f(A^{-1})$, so in particular the element $\lambda$ sends $x_{0}$ to $(\lambda_{1}\lambda_{2})^{g}(\lambda_{1}-x_{0}) = \lambda_{2}^{g}x_{0}$. As $g$ is even, we have

$$\{\lambda_{2}^{g}x_{0}\} = \{\lambda_{2}^{g}\} + \{x_{0}\} = g_{1}\{\lambda_{2}\} + \{x_{0}\} = \{x_{0}\},$$

concluding the proof. $\square$

2.2. Second proof: invariants of the universal conic.

Let $C_{g} \to H_{g}$ be the universal conic bundle over $H_{g}$. It is the projectivization of a rank two vector bundle over $H_{g}$, so it has the same cohomological invariants. Pulling it back to $\mathbb{A}^{2g+3}_{\mathbb{A}^{2g+3}}$, we obtain the $GL_{2}$-equivariant projective bundle $C_{g}'' = P^{1} \times \mathbb{A}^{2g+3}_{\mathbb{A}^{2g+3}} \to \mathbb{A}^{2g}_{\mathbb{A}^{2g+3}}$. Consider a $K$ point $(p, f)$ on $C_{g}''$ such that $f$ is not zero at $p$, that is $p$ does not belong to the image of the Weierstrass divisor of the corresponding curve. Then $f(p)$ is well defined up to squares, so it defines an element in $K^{*}/(K^{*})^{2} = H^{1}(K)$.

Let $U''$ be the $GL_{2}$-equivariant open subset $\{(p, f) \mid f(p) \neq 0\}$ of $C_{g}''$. The natural transformation $(p, f) \to f(p)$ defines a cohomological invariant on $U''$. This element clearly cannot extend to $C_{g}''$, but we claim that it does after multiplying it by $\gamma_{g+1}$:

Proposition 2.5. The element $\gamma_{g+1} \cdot f$ is unramified on the universal conic over $\mathbb{A}^{2g+3}_{\mathbb{A}^{2g+3}}$, and it glue to a cohomological invariant of $H_{g}$.

Proof. To show that the element extends, we need to check the boundary map $A^{0}(U'') \to A^{0}(C_{g}'' \setminus U'')$. It is immediate that $\partial(\gamma_{g+1} \cdot f) = \gamma_{g+1}$. Now we note that on the complement of $U''$ the Weierstrass divisor contains a rational point, so $\gamma_{g+1}$ restricts to zero due to the same argument as proposition 2.1.

To check $GL_{2}$-invariance, let $A \in GL_{2}$. Then $A$ acts trivially on $\gamma_{g+1}$ and sends $f(p)$ to $det(A)^{g}f(A^{-1}(A(p))$. The determinant is raised to an even power, and $f(A^{-1}(A(p))$ is just a rescaling of $f(p)$ by an even power, so the class in cohomology does not change, concluding our proof.

We still have to prove a rather relevant point: that the invariant we have created is not zero. For this, consider the open subset $U_{0} \subset \mathbb{A}^{2g}_{\mathbb{A}^{2g+3}}$ defined earlier. The coefficient $x_{0}$ of a form is equal, up to squares, to its value at infinity, so taking the copy of $U_{0}$ inside $U''$ given by $U_{0} \times \infty$, the invariant $\gamma_{g+1} \cdot f$ we defined in proposition 2.1 is just the restriction of $\gamma_{g+1} \cdot f$. We have proven:

Proposition 2.6. The element $\gamma_{g+1} \cdot f$ restricts to $\gamma_{g+1} \cdot f$ on $U_{0} \times \infty$. In particular, it is nonzero and $H^{*}(k_{0})$-linearly independent from the invariants coming from $B_{2}S_{2g+2}$.

Remark 2.7. It is easy to see that every non-zero element $\hat{\xi} \cdot t$ of $A^{1}_{\mathbb{A}^{2g+2}}(P^{2g+2}_{\mathbb{A}^{2g+3}}) \cdot t$ which is not a multiple of $\beta_{u-2}$, regarded as an invariant of $U_{0}$, cannot be extended to a global invariant.

Indeed, the generic point of $V_{0}$ defines the étale algebra $E_{gen} \times k$, where $E_{gen}$ is the generic étale algebra of degree $2g + 1$. The boundary of $\xi \cdot t$ is equal to an invariant of $B_{2}S_{2g+1}$, whose value on $E_{gen} \times k$ is zero if and only if $\xi = 0$. 
3. Multiplicative structure of $\text{Inv}^\bullet(H_g)$

In this last section we put together the results of the previous sections so to give a complete description of the multiplicative structure of $\text{Inv}^\bullet(H_g)$.

Recall that $\alpha_i$ denotes the degree $i$ cohomological invariant obtained by pulling back the $i^{th}$ Stiefel-Whitney invariant along the morphism of stacks

$$H_g \longrightarrow BS_{2g+2} \longrightarrow BO_{2g+2}$$

Recall also that in proposition 2.1 we introduced a cohomological invariant $\beta_{g+2}$ of $\mathbb{A}^{2g+3}_{sm}$ which descend to a cohomological invariant of $H_g$.

**Theorem 3.1.** Let $g \geq 2$ be an even number. Then:

1. The $H^\bullet(k_0)$-module $\text{Inv}^\bullet(H_g)$ is freely generated by the invariants

$$1, \alpha_1, \alpha_2, \ldots, \alpha_{g+1}, \beta_{g+2}.$$ 

Moreover, the invariants $\alpha_i$ are zero for $i > g + 2$ and $\alpha_{g+2} = \{2\} \cdot \alpha_{g+1}$.

2. The ring structure of $\text{Inv}^\bullet(H_g)$ is determined by the following formulas:

$$\alpha_r \cdot \alpha_s = \{-1\}^{m(r,s)} \cdot \alpha_{r+s-m(r,s)}$$

$$\alpha_1 \cdot \beta_{g+2} = 0 \quad \text{for } i \neq g + 1$$

$$\alpha_{g+1} \cdot \beta_{g+2} = \{-1\}^{g+1} \cdot \beta_{g+2}$$

$$\beta_{g+2} \cdot \beta_{g+2} = \{-1\}^{g+2} \cdot \beta_{g+2}$$

where $m(r,s)$ is computed as follows: if we write $s = \sum_{i \in I} 2^i$ and $r = \sum_{j \in J} 2^j$, then $m(r,s) = \sum_{k \in I \cap J} 2^k$.

**Proof.** We will rely on the isomorphism $A^T_{2g} (\mathbb{A}^{2g+3}_{sm}) \simeq \text{Inv}^\bullet(H_g)$ given by proposition 2.3.

Recall that $U_0$ is the $T$-invariant open subscheme of $P^g_{sm}$ where the coordinate $x_0 \neq 0$, and $\overline{V}_0$ is its complement in $P^g_{sm}$. Let $U_0$ and $V_0$ be their preimages along the $G_m$-torsor $p: \mathbb{A}^{2g+3}_{sm} \to P^g_{sm}$.

This torsor induces an exact sequence of $T$-equivariant Chow groups with coefficients:

$$0 \to A^T_0 (P^g_{sm}) \to A^T_0 (\mathbb{A}^{2g+3}_{sm}) \to A^T_0 (P^g_{sm})$$

Therefore, the elements in $A^T_0 (\mathbb{A}^{2g+3}_{sm})$ are either of the form $p^* \eta$ for some $\eta$ in $A^T_0 (P^g_{sm})$ or their boundary is a non-zero element of $A^T_1 (P^g_{sm})$.

We also have the inclusion:

$$A^T_0 (\mathbb{A}^{2g+3}_{sm}) \hookrightarrow A^T_0 (U_0) \simeq A_T (\overline{U}_0) \oplus A_T (\overline{U}_0) \cdot t$$

where $t$ is the degree 1 cohomological invariant introduced at the beginning of section 2, i.e. the invariant that sends a form to its value at infinity.

Pick an element $\eta$ in $A^T_0 (\mathbb{A}^{2g+3}_{sm})$ such that $\partial(\eta) \neq 0$. Then its restriction to $U_0$ must be of the form $p^* \xi \cdot t$ for some $\xi$ in $A_T (\overline{U}_0)$. Moreover we know from the observations above that this $\xi$ must come from $A^T_0 (P^g_{sm})$.

By remark 2.7 the only possibility is that $\xi$ is a multiple of $\alpha_{g+1}$. Combining this with corollary 1.3 and proposition 2.1 we deduce that the elements

$$1, \alpha_1, \ldots, \alpha_{g+1}, \beta_{g+2}$$

form a basis for $\text{Inv}^\bullet(H_G)$ as $H^\bullet(k_0)$-module.

To prove point (2), we exploit the inclusion:

$$\text{Inv}^\bullet(H_g) \hookrightarrow A^T_0 (P^g_{sm})[t]/(t^2 - \{-1\} t)$$

and the fact that the restriction of $\beta_{g+2}$ is equal to $\alpha_{g+1} \cdot t$ by construction. Then the formulas above easily follow from the multiplicative structure of $\text{Inv}^\bullet (BS_{2g+2})$. □
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