Local rigidity of homogeneous actions of parabolic subgroups of rank-one Lie groups

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Abstract

We show the local rigidity of the natural action of the Borel subgroup of $SO_+(n,1)$ on a cocompact quotient of $SO_+(n,1)$ for $n \geq 3$.

1 Introduction

Rigidity theory of actions of non-compact groups has been rapidly developed in the last two decades. It is found that many actions related to Lie groups of real-rank greater than one exhibit rigidity. See Fisher’s survey paper [2], for example. However, there are only few results on actions related to Lie groups of real-rank one. The aim of this paper is to show the local rigidity of some natural actions related to such groups.

Let $G$ be a Lie group and $M$ be a $C^\infty$ manifold. By $C^\infty(M \times G, M)$, we denote the space of $C^\infty$ maps from $M \times G$ to $M$ with the compact-open $C^\infty$-topology. Let $A(M, G)$ be the set of $C^\infty$ right actions of $G$ on $M$. It is a closed subset of $C^\infty(M \times G, M)$. We say two actions $\rho_1 : M_1 \times G \to M_1$ and $\rho_2 : M_2 \times G \to M_2$ are $C^\infty$-conjugate if there exists a $C^\infty$ diffeomorphism $h$ and an automorphism $\sigma$ of $G$ such that $h(\rho_1(x,g)) = \rho_2(h(x),\sigma(g))$ for any $x \in M_1$ and $g \in G$. An action $\rho \in A(M, G)$ is called $C^\infty$-locally rigid if the $C^\infty$-conjugacy class of $\rho$ is a neighborhood of $\rho$ in $A(M, G)$. We say an action $\rho \in A(M, G)$ is locally free if the isotropy subgroup $\{g \in G \mid \rho(x,g) = x\}$ is a discrete subgroup of $g$ for any $x \in M$.

Let $H$ be its closed subgroup of a Lie group $G$ and $\Gamma$ a cocompact lattice of $G$. We define the standard $H$-action $\rho_0$ on $\Gamma\backslash G$ by $\rho_0(\Gamma g, h) = \Gamma(gh)$. It is a locally free action. We say an action is homogeneous if it is $C^\infty$-conjugate to the standard action associated with some cocompact lattice.

Suppose that the Lie group $G$ is connected and semi-simple. Let $G = KAN$ be its Iwasawa decomposition. The dimension of the abelian subgroup $A$ is called the real-rank of $G$. Let $M$ be the centralizer of $A$ in $K$. The group $P =$

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Man is called the Borel subgroup associated with the Iwasawa decomposition $G = KAN$. It is known that the conjugacy class of the Borel subgroup does not depend on the choice of the Iwasawa decomposition. When $G = SL(2, \mathbb{R})$ for example, a Borel subgroup $P$ is conjugate to the group $GA$ of the upper triangular matrices in $SL(2, \mathbb{R})$. Fix a cocompact lattice $\Gamma$ of $SL(2, \mathbb{R})$ and put $M\Gamma = \Gamma \backslash SL(2, \mathbb{R})$. Let $\rho_0$ be the standard $P$-action on $M\Gamma$. It is not locally rigid since deformation of lattice $\Gamma$ gives a non-trivial deformation of actions. However, there are several rigidity results on $\rho_0$. Ghys [3] proved that if a locally free $GA$-action on $M\Gamma$ admits an invariant volume, then it is homogeneous. In [4], he remove the assumption on invariant volume when $H^1(M\Gamma)$ is trivial. As a consequence, there exists a non-homogeneous locally free $GA$-action on $M\Gamma$ when $H^1(M\Gamma)$ is non-trivial. In a forthcoming paper, he will also show that the standard $GA$-action $\rho_0$ admits a $C^\infty$ deformation into non-homogeneous actions in this case.

By Mostow’s rigidity theorem, any deformation of a cocompact lattice is trivial if $G$ is a higher-dimensional Lie group of real-rank one. So, it is natural to ask whether the standard $P$-action is $C^\infty$-locally rigid or not in this case. The main result of this paper answers this question when $G$ is $SO_+(n, 1)$.

**Theorem 1.1.** Let $P$ be a Borel subgroup of $SO_+(n, 1)$ and $\Gamma$ be a torsion-free cocompact lattice of $SO_+(n, 1)$. If $n \geq 3$, then the standard $P$-action on $\Gamma \backslash SO_+(n, 1)$ is $C^\infty$-locally rigid.

To ending the introduction, we remark on the local rigidity of the orbit foliation. Let $\mathcal{F}_\Gamma$ be the orbit foliation of the standard $P$-action on $\Gamma \backslash SO_+(n, 1)$. Ghys [5] showed a global rigidity result of $\mathcal{F}_\Gamma$ for $n = 2$. For $n \geq 3$, Yue [11] proved a partial result and Kanai [6] claimed the local rigidity of $\mathcal{F}_\Gamma$. However, Kanai’s proof contains a serious gap and it is not fixed so far. Hence, the local rigidity of $\mathcal{F}_\Gamma$ is still open. If any foliation sufficiently close to $\mathcal{F}_\Gamma$ carries an action of $P$, then the local rigidity of $\mathcal{F}_\Gamma$ follows from our theorem.

## 2 Preliminaries

In this section, we introduce some notations and review several known facts which we will use in the proof of Theorem 1.1.

### 2.1 The group $SO_+(n, 1)$

Fix $n \geq 3$ and let $I_{n, 1}$ be the diagonal matrix of size $(n + 1)$ whose diagonal elements are $1, \ldots, 1, -1$. Let $SO_+(n, 1)$ be the identity component of the subgroup of $GL(n + 1, \mathbb{R})$ consisting of matrices $A$ satisfying $^tAI_{n, 1}A = I_{n, 1}$. For

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1. It is isomorphic to the group of orientation preserving affine transformations of the real line.
2. The $C^1$-regularity of the strong unstable foliation claimed in the last sentence of p.677 does not hold in general.
any $1 \leq n' \leq n$, the standard embedding $GL(n', \mathbb{R}) \hookrightarrow GL(n+1, \mathbb{R})$ induces an embedding of $SO(n')$ into $SO_+(n, 1)$.

Let $\mathfrak{so}(n, 1)$ be the Lie algebra of $SO_+(n, 1)$. By $E_{i,j}$, we denote the square matrix of size $(n+1)$ such that the $(i, j)$-entry is one and other entries are zero. Put $X = E_{n(n+1)} - E_{(n+1)n}$, $Y_i = (E_{i(n+1)} + E_{(n+1)i}) - (E_{in} - E_{ni})$, and $Y'_i = (E_{i(n+1)} + E_{(n+1)i}) - (E_{in} - E_{ni})$ for $i, j = 1, \cdots, n$. Then, $\mathfrak{so}(n, 1)$ is generated by $X, Y_1, \cdots, Y_{n-1}, Y'_1, \cdots, Y'_{n-1}$ and the Lie subalgebra corresponding to the subgroup $SO(n-1)$ of $SO_+(n, 1)$. It is easy to check that

$$[Y_i, X] = -Y_i, \quad [Y'_i, X] = Y'_i, \quad [Y_i, Y_j] = [Y'_i, Y'_j] = 0 \tag{1}$$

for any $i, j = 1, \cdots, n-1$, and

$$\text{Ad}_m(X) = X \quad \text{Ad}_m(Y_1, \cdots, Y_{n-1}) = (Y_1, \cdots, Y_{n-1}) \cdot m \tag{2}$$

and for any $m \in SO(n-1)$.

Let $SO_+(n, 1) = KAN$ be the Iwasawa decomposition of $SO_+(n, 1)$ associated with the involution $\theta_0 : g \mapsto (^t g)^{-1}$. Then, $K = SO(n)$ and, $A$ and $N$ are the subgroups of $SO_+(n, 1)$ corresponding to the Lie subalgebras spanned by $X$ and $\{Y_1, \cdots, Y_{n-1}\}$, respectively. Since the centralizer of $A$ in $SO(n)$ is $SO(n-1)$, the Borel subgroup corresponding to $\theta_0$ is $SO(n-1)AN$.

\subsection{Anosov flows}

A $C^1$ flow $\Phi$ on a closed manifold $M$ is called Anosov, if it has no stationary points and there exists a continuous splitting $TM = T\Phi \oplus E^{ss} \oplus E^{uu}$, a constant $\lambda > 0$, and a continuous norm $\| \cdot \|$ on $TM$ which satisfy the following properties:

- $T\Phi$ is the one-dimensional subbundle tangent to the orbit of $\Phi$.
- $E^{ss}$ and $E^{uu}$ are $D\Phi$-invariant subbundles.
- $\|D\Phi^t(v^s)\| \leq e^{-\lambda t}\|v^s\|$ and $\|D\Phi^t(v^u)\| \geq e^{\lambda t}\|v^u\|$ for any $v^s \in E^{ss}, v^u \in E^{uu}$, and $t \geq 0$.

The subbundles $E^{ss}, E^{uu}, T\Phi \oplus E^{ss},$ and $T\Phi \oplus E^{uu}$ are called the strong stable, strong unstable, weak stable, and weak unstable subbundles, respectively. It is known that they generate continuous foliations with $C^\tau$ leaves, if $\Phi$ is a $C^\tau$ flow. The foliations are called the strong stable foliation, etc.

The following proposition may be well-known for experts, but we give a proof for convenience of the readers.

**Proposition 2.1.** Let $\Phi_1$ and $\Phi_2$ be Anosov flows on a closed manifold $M$. Suppose that $\Phi_1$ and $\Phi_2$ have the common strong unstable foliation $\mathcal{F}^{uu}$ and $\mathcal{F}^{uu}(\Phi_1^t(x)) = \mathcal{F}^{uu}(\Phi_2^t(x))$ for any $x \in M$ and $t \in \mathbb{R}$. Then, there exists a homeomorphism $h$ of $M$ such that $\Phi_2^t \circ h = h \circ \Phi_1^t$ for any $t \in \mathbb{R}$ and $h(\mathcal{F}^{uu}(x)) = \mathcal{F}^{uu}(x)$ for any $x \in M$. 


Proof. Let $\mathcal{H}$ be the set of continuous maps $h : M \to M$ which preserves each leaf of $\mathcal{F}^{uu}$. Fix a Riemannian metric $g$ of $M$. Let $d^u$ be the leafwise distance on leaves of $\mathcal{F}^{uu}$ which is determined by the restriction of the metric $g$ to each leaf. We define a distance $d$ on $\mathcal{H}$ by $d(h, h') = \sup_{x \in N_t} d^u_x(h(x), h'(x))$. It is a complete metric on $\mathcal{H}$.

For $i, j \in \{1, 2\}$, we define continuous flows $\Theta_{ij}$ on $\mathcal{H}$ by $\Theta_{ij}^t(h) = \Phi_i^{-t} \circ h \circ \Phi_j^t$. Since $\Phi_i$ and $\Phi_j$ expand $\mathcal{F}^{uu}$ uniformly, $\Theta_{ij}^t$ is a uniform contraction for any sufficiently large $t > 0$. By the contracting mapping theorem, there exists a unique fixed point $h_{ij} \in \mathcal{H}$ of the flow $\Theta_{ij}$. Since both $h_{ij} \circ h_{ji}$ and the identity map of $M$ are fixed point of $\Theta_{ii}$, $h_{ij} \circ h_{ji}$ is the identity map for $i, j \in \{1, 2\}$. In particular, $h_{ij}$ is the inverse of $h_{ji}$. Therefore, $h_{21}$ is a homeomorphism in $\mathcal{H}$ such that $h_{21} \circ \Phi_{1}^t = \Phi_{2}^t \circ h_{21}$ for any $t \in \mathbb{R}$.

Let $\Psi$ be a flow on a manifold $M$. A $C^\infty$ function $\alpha$ on $M \times \mathbb{R}$ is a cocycle over $\Psi$ if $\alpha(x, 0) = 0$ and $\alpha(x, t + t') = \alpha(x, t) + \alpha(\Psi^t(x), t')$ for any $x \in M$ and $t, t' \in \mathbb{R}$. We say $\Psi$ is topologically transitive if there exists $x_0 \in M$ whose orbit $\{\Psi^t(x_0) : t \in \mathbb{R}\}$ is a dense subset of $M$.

**Theorem 2.2** (The $C^\infty$ Livschitz Theorem [7]). Let $\Phi$ be a $C^\infty$ topologically transitive Anosov flow on a closed manifold $M$ and $\alpha$ be a $C^\infty$ cocycle over $\Phi$. If $\alpha(x, T) = 0$ for any $(x, t) \in M \times \mathbb{R}$ satisfying $\Psi^T(x) = x$, then there exists a $C^\infty$ function $\beta$ on $M$ such that $\alpha(x, t) = \beta(\Phi^t(x)) - \beta(x)$ for any $x \in M$ and $t \in \mathbb{R}$. Moreover, if $\alpha$ is sufficiently $C^\infty$-close to $0$, then we can choose $\beta$ so that it is $C^\infty$-close to $0$.

We say an Anosov flow $\Phi$ is $s$- (resp. $u$-)conformal if $D\Phi^t$ is conformal on $E^{ss}(x)$ (resp. $E^{uu}(x)$) for any $x \in M$ with respect to some continuous metric on $E^{ss}$. The following result plays fundamental role in the proof of Theorem 2.3.

**Theorem 2.3** (de la Llave [5]). Let $\Phi_1$ and $\Phi_2$ be $C^\infty$ $s$-conformal topologically transitive Anosov flows on a closed manifold $M$. For $i = 1, 2$, let $\mathcal{F}_i^{ss}$ be of $\Phi_i$. Suppose that the dimensions of the strong stable foliation of $\Phi_1$ and $\Phi_2$ are greater than one. If a homeomorphism $h$ of $M$ satisfies $\Phi_1^t \circ h = h \circ \Phi_2^t$ for any $t \in \mathbb{R}$, then the restriction of $h$ to a leaf of the strong stable foliation of $\Phi_1$ is a $C^\infty$ diffeomorphism to a leaf of the strong stable foliation of $\Phi_2$. Moreover, if both $\Phi_1$ and $\Phi_2$ are $u$-conformal in addition, then $h$ is a $C^\infty$ diffeomorphism of $M$.

We say that an Anosov flow is contact if it preserves a $C^1$-contact structure. It is easy to see that any contact structure invariant under an Anosov flow is the direct sum of the strong stable subbundle and the strong unstable subbundle.

**Proposition 2.4.** Let $\Phi$ be a contact Anosov flow on a closed manifold $M$. If $\Phi$ is $s$-conformal, then it is $u$-conformal.
Proposition 3.1. If \( \rho : M_1 \times P \to M_1 \) is sufficiently \( C^\infty \)-close to \( \rho_1 \) then \( \rho \) is \( C^\infty \)-conjugate to an action in \( \mathcal{A}_*(M_1, P) \) which is \( C^\infty \)-close to \( \rho_1 \).
Proof. It is an immediate corollary of Palais' stability theorem of compact group action (2).

For \( \rho \in A_\ast (M_\Gamma, P) \), we define a flow \( \Phi_\rho \) on \( N_\Gamma \) by \( \Phi_\rho^t(x) = \pi(\rho^{\exp(tX)}(x)) \). It is well-defined since \( \exp(tX) \) commutes with any element of \( SO(n - 1) \). We call the flow \( \Phi_\rho \) the flow induced by \( \rho \).

For \( \rho \in A_\ast (M_\Gamma) \), we define vector fields \( Y_1^\rho, \ldots, Y_{n-1}^\rho \) on \( M_\Gamma \) by \( Y_i^\rho(x) = (d/dt)\rho^{\exp(tY_i)}(x)|_{t=0} \).

**Lemma 3.2.** For any \( x \in M_\Gamma \) and \( m \in SO(n - 1) \),

\[
(D\pi(Y_1^\rho(x \cdot m)), \ldots, D\pi(Y_{n-1}^\rho(x \cdot m))) = (D\pi(Y_1^\rho(x)), \ldots, D\pi(Y_{n-1}^\rho(x))) \cdot m.
\]

**Proof.** For any \( x \in M_\Gamma \), \( t \in \mathbb{R} \), and \( m \in SO(n - 1) \),

\[
\pi \circ \rho(x \cdot m, \exp(tY_i)) = \pi(\rho(x, [m \exp(tY_i)m^{-1}]) \cdot m) = \pi(\rho(x, \exp(t \cdot Ad_m(Y_i)))).
\]

Hence, the equation (2) implies the lemma.

By the above lemma, we can define a \( C^\infty \) subbundle \( E^-_\rho \) of \( TN_\Gamma \) by

\[
E^-_\rho(\pi(x)) = D\pi((Y_1^\rho(x), \ldots, Y_{n-1}^\rho(x))).
\]

There exists a \( C^\infty \) metric \( g_\rho \) on \( E^-_\rho \) such that \( (D\pi(Y_1^\rho(x)), \ldots, D\pi(Y_{n-1}^\rho(x))) \) is an orthonormal basis of \( E^-_\rho(x) \) with respect to \( g_\rho \). The subbundle \( E^-_\rho \) is \( D\Phi_\rho \)-invariant and

\[
\|D\Phi_\rho^t(v)\|_{g_\rho} = e^{-t}\|v\|_{g_\rho}
\]

for any \( t \in \mathbb{R} \) and \( v \in E^-_\rho \).

For \( i = 1, \ldots, n-1 \), let \( Y_i^- \) be a vector field on \( M_\Gamma \) given by \( Y_i^-(x) = (d/dt)x \exp(tY_i)|_{t=0} \). Similar to the above, we can define a \( C^\infty \) subbundle \( E^+_\rho_0 \) of \( TN_\Gamma \) and its \( C^\infty \) metric \( g^+ \) such that

\[
E^+_\rho_0(\pi(x)) = D\pi((Y_1^+(x), \ldots, Y_{n-1}^+(x)))
\]

and \( (D\pi(Y_1^+(x)), \ldots, D\pi(Y_{n-1}^+(x))) \) is an orthonormal basis of \( E^+_\rho_0(x) \) with respect to \( g^+ \). The subbundle \( E^+_\rho_0 \) is \( D\Phi_\rho_0 \)-invariant and

\[
\|D\Phi_\rho_0^t(v')\|_{g^+} = e^t\|v'\|_{g^+}
\]

for any \( t \in \mathbb{R} \) and \( v' \in E^+_\rho_0 \). The flow \( \Phi_\rho_0 \) is an Anosov flow with the Anosov splitting \( TN_\Gamma = T\Phi \oplus E^-_\rho_0 \oplus E^+_\rho_0 \) and it is \( s \)- and \( u \)-conformal with respect to \( g_\rho_0 \) and \( g^+ \), respectively. It is known that \( E^-_\rho_0 \oplus E^+_\rho_0 \) is a \( \Phi_\rho_0 \)-invariant contact structure.

Since the set of Anosov flows is open in the space of \( C^1 \) flows, the induced flow \( \Phi_\rho \) is Anosov if \( \rho \in A_\ast (M_\Gamma, P) \) is sufficiently \( C^1 \)-close to \( \rho_0 \). In this case, \( E^-_\rho \) is the strong stable subbundle of \( \Phi_\rho \) and the Anosov flow \( \Phi_\rho \) is \( s \)-conformal with respect to \( g_\rho \).
3.2 Reduction to the conjugacy of induced flows

We reduce Theorem 1.1 to the smooth conjugacy problem of the induced flows.

**Theorem 3.3.** Let \( \rho \) be a locally free action in \( A_n(M_\Gamma, P) \). Suppose that a \( C^\infty \) diffeomorphism \( h \) of \( N_\Gamma \) satisfies \( \Phi_t^\rho \circ h = h \circ \Phi_t^\rho_0 \) for any \( t \in \mathbb{R} \). Then, \( \rho \) is \( C^\infty \)-conjugate to the standard \( P \)-action \( \rho_0 \).

Let \( \text{Fr} E^-_\rho \) be the frame bundle of \( E^-_\rho \). It admits a natural right action of \( GL(n-1, \mathbb{R}) \). The flow \( \Phi_\rho \) induce a flow \( \text{Fr} \Phi_\rho \) on \( \text{Fr} E^-_\rho \). Let \( OE^-_\rho \) be the orthonormal frame bundle of \( (E^-_\rho, g_\rho) \). We define a map \( \psi_\rho : M_\Gamma \rightarrow OE^-_\rho \) by

\[
\psi_\rho(x) = (D\pi(Y_1^\rho(x)), \ldots, D\pi(Y_{n-1}(x))).
\]

By Lemma 3.2, \( OE^-_\rho(y) = \{ \psi_\rho(x) \mid x \in \pi^{-1}(y) \} \) for any \( y \in N_\Gamma \) and \( \psi_\rho \) is a diffeomorphism from \( M_\Gamma \) to \( OE^-_\rho \). By Equation (1), we have \( \text{Fr} \Phi_t^\rho(\psi_\rho(x)) = e^{-t}\psi_\rho(\rho^\exp(tX)(x)) \). Hence, we can define a flow \( O\Phi_\rho \) on \( OE^-_\rho \) by

\[
O\Phi_t^\rho(\psi_\rho(x)) = e^t \cdot \text{Fr} \Phi_t^\rho(\psi_\rho(x)) = \psi_\rho(\rho^\exp(tX)(x)).
\]

In particular, the map \( \psi_\rho \) is a \( C^\infty \) conjugacy between \( \rho^\exp(tX) \) and \( O\Phi_t^\rho \). By Moore’s ergodicity theorem, the flow \( \rho^\exp(tX) \) is topologically transitive. Hence, so the flow \( O\Phi_\rho \) is.

Fix \( \rho \in A_n(M_\Gamma, P) \) and suppose that there exists a \( C^\infty \) diffeomorphism \( h \) of \( N_\Gamma \) such that

\[
\Phi_t^\rho \circ h = h \circ \Phi_t^\rho_0
\]

for any \( t \in \mathbb{R} \).

**Lemma 3.4.** \( Dh(E^-_\rho_0) = E^-_\rho \) and there exists a constant \( c_h > 0 \) such that \( \| Dh(v) \|_{g_\rho} = c_h \cdot \| v \|_{g_\rho} \) for any \( v \in E^-_\rho_0 \).

**Proof.** Recall that the flow \( \Phi_\rho_0 \) is Anosov and \( E^-_\rho_0 \) is its strong stable subbundle. Since \( h \) is a \( C^\infty \) conjugacy between \( \Phi_\rho_0 \) and \( \Phi_\rho \), the flow \( \Phi_\rho \) is also Anosov and its strong stable subbundle is \( Dh(E^-_\rho_0) \). By Equation (3), the subbundle \( E^-_\rho \) is contained in the strong stable subbundle \( Dh(E^-_\rho_0) \). Since their dimensions are equal, we have \( Dh(E^-_\rho_0) = E^-_\rho \).

Let \( SE^-_\rho_0 \) be the unit sphere bundle \( \{ v \in E^-_\rho_0 \mid \| v \|_{g_\rho_0} = 1 \} \) of \( E^-_\rho_0 \) and \( \pi_O : OE^-_\rho_0 \rightarrow SE^-_\rho_0 \) be the projection defined by \( (v_1, \ldots, v_{n-1}) \mapsto v_1 \). By Equation (3) for \( \rho_0 \), we can define a flow \( S\Phi_\rho_0 \) on \( SE^-_\rho_0 \) by \( S\Phi_t^\rho_0 = e^t \Phi_t^\rho_0 \). Then, \( \pi_O \circ O\Phi_t^\rho_0 = S\Phi_t^\rho_0 \circ \pi_O \). Since \( O\Phi_\rho_0 \) is topologically transitive, \( S\Phi_\rho_0 \) also is. Take \( v_0 \in SE^-_\rho_0 \) such that the orbit \( \{ S\Phi_t^\rho_0(v_0) \mid t \in \mathbb{R} \} \) is dense in \( SE^-_\rho_0 \). Put \( c_h = \| Dh(v_0) \|_{g_\rho} \). By Equation (3),

\[
\| Dh \circ S\Phi_t^\rho_0(v_0) \|_{g_\rho} = c_h \cdot \| D\Phi_t^\rho_0(v_0) \|_{g_\rho}
\]

for any \( t \in \mathbb{R} \). It implies that \( \| Dh(v) \|_{g_\rho} = c_h \) for any \( v \in SE^-_\rho_0 \). \( \square \)
Proposition 3.5. There exists a $C^\infty$ diffeomorphism $H$ of $M_\Gamma$ such that $H \circ \rho_0^t = \rho^t \circ H$ for any $g \in \{\exp(tX)\}_t \in \mathbb{R}, m \in SO(n-1)$.

Proof. Let $Fr \h$ be the lift of $h$ to $Fr E_{\rho_0}^{-}$. By the above lemma, we can define a diffeomorphism $Oh : O\h \rightarrow OE_{\rho_0}^{-}$ by $Oh = c_h^{-1} Fr \h$. Then, $O\Phi^t \circ Oh = Oh \circ O\Phi^t_{\rho_0}$ for any $t \in \mathbb{R}$. Since $Fr \h$ commutes with the action of $SO(n-1)$, we have $Oh(z \cdot m) = Oh(z) \cdot m$ for any $z \in OE_{\rho_0}^{-}$ and $m \in SO(n-1)$. Put $H = \psi_p^{-1} \circ Oh \circ \psi_{\rho_0}$. Then, we have

$$H \circ \rho_0^t \exp(tX) = \rho^t \exp(tX) \circ H,$$

$$H(x \cdot m) = H(x) \cdot m$$

for any $t \in \mathbb{R}$ and any $m \in SO(n-1)$. \qed

The proof of Theorem \ref{thm:main} will finish once we show the following

Proposition 3.6. There exists an automorphism $\theta$ of $P$ such that $\rho(H(x), \theta(g)) = \rho_0(x, g)$ for any $x \in M_\Gamma$ and $g \in P$.

Proof. Since $\rho^t \exp(tX) \circ H = H \circ \rho_0^t \exp(tX)$ for any $t \in \mathbb{R}$, we have

$$D\rho^t \exp(tX) (DH(Y_{\rho_0}^t(x))) = DH(D\rho_0^t \exp(tX) (Y_{\rho_0}^t(x))) = e^{-t} DH(Y_{\rho_0}^t(h^t X(x))).$$

We also have

$$DH(\langle Y_{\rho_0}^1(x), \ldots, Y_{\rho_0}^{n}(x) \rangle) = DH(\{v \in T_x M_\Gamma | \lim_{t \rightarrow +\infty} \|D\rho_0^t \exp(tX)(v)\| = 0\})$$

$$= \{v' \in T_{H(x)} M_\Gamma | \lim_{t \rightarrow +\infty} \|D\rho^t \exp(tX)(v')\| = 0\}$$

$$\supset \langle Y_{\rho}^1(H(x)), \ldots, Y_{\rho}^{n-1}(H(x)) \rangle$$

for any $x \in M_\Gamma$. In particular,

$$DH(\langle Y_{\rho_0}^1(x), \ldots, Y_{\rho_0}^{n}(x) \rangle) = \langle Y_{\rho}^1(H(x)), \ldots, Y_{\rho}^{n-1}(H(x)) \rangle.$$

Since the flow $(\rho_0^t \exp(tX))_{t \in \mathbb{R}}$ is topologically transitive, there exists $x_0 \in M_\Gamma$ such that $\{\rho_0^t X(x_0) | t \in \mathbb{R}\}$ is a dense subset of $M_\Gamma$. Let $b = (b_{ij})_{i,j=1,\ldots,n-1}$ be the square matrix given by $Y_j^\rho(H(x_0)) = \sum_{i=1}^{n-1} b_{ij} DH(Y_{\rho_0}^i(x_0))$. Remark that it is an invertible matrix. For any $t \in \mathbb{R}$, we have

$$Y_j^\rho(H(\rho_0^t \exp(tX)(x_0))) = Y_j^\rho(\rho^t \exp(tX)(H(x_0)))$$

$$= e^t D\rho^t \exp(tX) (Y_j^\rho(H(x_0)))$$

$$= \sum_{i=1}^{n-1} b_{ij} DH(Y_{\rho_0}^i(\rho_0^t \exp(tX)(x_0))).$$

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Since the orbit of $x_0$ is dense, $Y^p_i(H(x)) = \sum_{i=1}^{n-1} b_{ij} DH(Y^p_{i+1})(x)$ for any $x \in M$. In particular,

$$DH(Y^p_1, \ldots, Y^p_{n-1}) = (Y^p_1, \ldots, Y^p_{n-1}) \cdot b^{-1} \tag{7}$$

Recall that

$$D\rho^m(Y^p_1, \ldots, Y^p_{n-1}) = (Y^p_1, \ldots, Y^p_{n-1}) \cdot m,$$
$$D\rho^m(Y_1, \ldots, Y_{n-1}) = (Y_1, \ldots, Y_{n-1}) \cdot m$$

for any $m \in SO(n-1)$. Since $\rho^m \circ H = H \circ \rho^m_0$ for any $m$, Equation (7) implies

$$(Y^p_1, \ldots, Y^p_{n-1}) \cdot mb^{-1} = (Y^p_1, \ldots, Y^p_{n-1}) \cdot b^{-1} m.$$ 

Hence, $b$ commutes with any $m \in SO(n-1)$. It is easy to check that

- if $n \geq 4$, then there exists $\alpha \in \mathbb{R} \setminus \{0\}$ such that $b = \alpha I_{n-1}$, where $I_{n-1}$ is the unit matrix of size $(n-1)$,
- if $n = 3$, then there exists $\alpha \in \mathbb{R} \setminus \{0\}$ and $m_0 \in SO(2)$ such that such that $b = \alpha m_0$.

In each case, $\alpha^{-1}b$ is contained in the center of $SO(n-1)$.

We define a map $\theta : P \to SO_{+}(n,1)$ by $\theta(g) = bg^{-1}$. Since $\alpha^{-1}b$ is contained in the center of $SO(n-1)$, we have $\theta(P) = P$. In particular, $\theta$ is an automorphism of $P$ such that

$$\theta(Y_1, \ldots, Y_{n-1}) = (Y_1, \ldots, Y_{n-1}) \cdot b^{-1},$$

where $\theta_*$ is the induced automorphism of the Lie algebra of $P$. By Equation (7), $\rho(H(x), \theta(\exp(Y_i))) = H(\rho_0(x, \exp(Y_i)))$ for any $x \in M$ and $i = 1, \ldots, n-1$.

On the other hand, $\theta(g') = g'$ and $\rho^t \circ H = H \circ \rho^t_0$ for any $g' \in \{\exp(tX)m \mid t \in \mathbb{R}, m \in SO(n-1)\}$. Therefore, $\rho(H(x), \theta(g)) = H(\rho_0(x, g))$ for any $x \in M$ and $g \in P$, \hfill \square

### 3.3 Smooth conjugacy between induced flows

In this subsection, we show the following theorem. With Proposition 3.3 and Theorem 3.3 it completes the proof of the main theorem.

**Theorem 3.7.** If $\rho \in \mathfrak{A}_s(M, P)$ is sufficiently $C^\infty$-close to $\rho_0$, then there exists a $C^\infty$ diffeomorphism $h$ of $N$ such that $\Phi_h^t \circ h = h \circ \Phi_0^t$ for any $t \in \mathbb{R}$.

Choose $\rho \in \mathfrak{A}_s(M, P)$ such that $\Phi_\rho$ is an $s$-conformal Anosov flow with respect to $g\rho$ and $E_\rho^\perp$ is transverse to $T\Phi_\rho \oplus E_\rho^\perp$. By $\mathcal{F}_\rho^\text{ss}$, $\mathcal{F}_\rho^\text{su}$, $\mathcal{F}_\rho^\text{su}$, $\mathcal{F}_\rho^\text{uu}$, we denote the strong stable, strong unstable, weak stable, weak unstable foliations of $\Phi_\rho$, respectively. Similarly, by $\mathcal{F}_\rho^\text{ss}$, $\mathcal{F}_\rho^\text{su}$, we denote the strong stable and weak stable foliations of $\Phi_\rho$, respectively. Remark that all of them are $C^\infty$ foliations, but the strong unstable and weak unstable foliations of $\Phi_\rho$ may not be $C^\infty$. 

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By $X_{p_0}$ and $X_p$, we denote the vector fields generating the flows $\Phi_{p_0}$ and $\Phi_p$. Let $\sigma_1: T N_1 \to E^+_{p_0}$ and $\sigma_2: T N_1 \to E^-_{p_0}$ be the projection with respect to the splittings $E^+_{p_0} \oplus (T \Phi_{p_0} \oplus E^-_{p_0})$ and $E^-_{p_0} \oplus (T \Phi_{p_0} \oplus E^+_{p_0})$ of $TN_1$, respectively. Put $X_1 = X_{p_0} - \sigma_1(X_{p_0})$ and $X_2 = X_p - \sigma_2(X_p)$. They generates flows $\Psi_1$ and $\Psi_2$ on $N_1$. If $\rho$ is sufficiently $C^\infty$-close to $p_0$, then $\Psi_1$ and $\Psi_2$ are $C^\infty$-close to $\Phi_{p_0}$. Hence, we may assume that they are Anosov flows. Since $[X_1, E^+_{p_0}] \subset E^+_{p_0}$ and $X_1 \in T \Phi_{p_0} \oplus E^+_{p_0}$, we have

$$\Psi_1^t(x) \in F^u_{p_0}(\Phi_{p_0}(x)) \cap F^s_ho(x).$$

for any $x \in M_1$ and $t \in \mathbb{R}$. If $\rho$ is sufficiently close to $p_0$, then $D\Psi_1$ expands $E^u_{p_0}$ uniformly. So, we may assume that $E^u_{p_0}$ is the strong unstable subbundle of $\Psi_1$. Similarly, we may assume

$$\Psi_2^t(x) \in F^s_{\rho}(\Phi_{\rho}(x)) \cap F^u_{p_0}(x)$$

for any $x \in N_1$ and $t \in \mathbb{R}$, and $E^u_{\rho}$ is the strong stable subbundle of $\Psi_2$. Since both $\Psi_1^t(x)$ and $\Psi_2^t(x)$ are contained in $F^u_{p_0}(x) \cap F^s_{\rho}(x)$, the orbits of $\Psi_1$ and $\Psi_2$ coincide. Hence, there exists a $C^\infty$ cocycle over $\Psi_2$ such that

$$\Psi_1^t(x) = \Psi_2^{\rho(x,t)}(x)$$

for any $x \in N_1$ and $t \in \mathbb{R}$. Since each leaf of $F^u_{p_0}$ is $\Phi_{p_0}$- and $\Phi_\rho$-invariant and it is transverse to both $E^u_{p_0}$ and $E^u_\rho$, we have

$$\det D\Psi_1^t|_{E^u_{p_0}(x)} = \det D\Psi_2^{\rho(x,t)}|_{E^u_{\rho}(x)}$$

for any $(x, t) \in N_1 \times \mathbb{R}$ satisfying $\Psi_2^{\rho(x,t)}(x) = x$.

By Proposition 2.1, there exist a homeomorphism $h_1$ of $N_1$ such that $\Psi_1^t \circ h_1 = h_1 \circ \Phi_{p_0}$ for any $t \in \mathbb{R}$ and $h_1(F^u_{p_0}(x)) = F^u_{p_0}(x)$ for any $x \in N_1$. Since $h_1$ preserves each leaf of $F^u_{p_0}$, we have

$$\det D\Psi_1^t|_{E^u_{p_0}(x)} = \det D\Phi_{p_0}^t|_{E^u_{p_0}(h_1^{-1}(x))} = e^{-(n-1)t}$$

for any $(x, T') \in N_1 \times \mathbb{R}$ satisfying $\Psi_1^{T'}(x) = x$.

By Proposition 2.1 again, there exist a homeomorphism $h_2$ of $N_1$ such that $\Psi_2^t \circ h_2 = h_2 \circ \Phi_\rho$, for any $t \in \mathbb{R}$ and $h_2(F^s_{\rho}(x)) = F^s_{\rho}(x)$ for any $x \in N_1$. Since $F^u_{p_0}$ is a transversely conformal foliation, $\Psi_2$ is $s$-conformal. By Theorem 2.3, the restriction of $h_2$ to each leaf of $F^s_{\rho}$ is smooth. Hence,

$$\det D\Psi_2^t|_{E^s_{\rho}(x)} = \det D\Phi_{\rho}^t|_{E^s_{\rho}(h_2^{-1}(x))} = e^{-(n-1)t}$$

for any $(x, T) \in N_1 \times \mathbb{R}$ satisfying $\Psi_2^T(x) = x$.

By Equations (11), (12), and (13), we have $\alpha(x, T) = T = 0$ for any $(x, T) \in N_1 \times \mathbb{R}$ satisfying $\Psi_2^T(x) = x$. By The $C^\infty$ Livshitz Theorem, there exists a $C^\infty$ function $\beta$ on $N_1$ such that

$$\alpha(x, t) - t = \beta(\Psi_2^t(x)) - \beta(x)$$

for any $(x, t) \in N_1$.
for any \( x \in N_{\Gamma} \) and \( t \in \mathbb{R} \). We define a map \( h_3 : N_{\Gamma} \to N_{\Gamma} \) by \( h_3(x) = \Psi_2^{-\beta}(x) \).

Remark that if \( \rho \) is sufficiently \( C^\infty \)-close to \( \rho_0 \), then \( \alpha \) is \( C^\infty \)-close to zero, and hence, we can choose \( \beta \) so that it is \( C^\infty \)-close to 0. So, we may assume that \( h_3 \) is a \( C^\infty \) diffeomorphism sufficiently \( C^\infty \)-close to the identity map. Since \( \Phi_{\rho_0} \) is a contact Anosov flow and the set of contact structures is open in the space of \( C^1 \) hyperplane fields, we also may assume that \( Dh_3(E^-_\rho) \oplus E^+_{\rho_0} \) is a contact structure.

By Equation (10), we have \( \Psi_1^t \circ h_3 = h_3 \circ \Psi_2^t \). In particular, \( Dh_3(E^-_\rho) \) is the strong stable subbundle of \( \Psi_1 \). Since \( E^s_{\rho_0} \) is the strong unstable subbundle of \( \Psi_1 \), the flow \( \Psi_1 \) is a contact Anosov flow. Since \( \Psi_1 \) is s-conformal, it is also \( u \)-conformal by Proposition 2.3. By Theorem 2.3 \( h_1 \) is a \( C^\infty \) diffeomorphism.

Since \( F^s_{\rho_0} \) is a transversely conformal foliation, \( F^s_{\rho_0} = h_1(F^s_{\rho_0}) \) also is. It implies that \( \Phi_{\rho} \) and \( \Psi_2 \) are \( u \)-conformal. Since \( \Phi_{\rho} \) and \( \Psi_2 \) are s-conformal, Theorem 2.3 implies that \( h_2 \) is a \( C^\infty \) diffeomorphism. Now, we put \( h = h_2^{-1} \circ h_3^{-1} \circ h_1 \). Then, \( h \) is a \( C^\infty \) diffeomorphism and \( \Phi_{\rho}^t \circ h = h \circ \Phi_{\rho_0}^t \) for any \( t \in \mathbb{R} \).

The proof of Theorem 3.7 is completed.

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