THE BI-GRADED STRUCTURE OF SYMMETRIC ALGEBRAS WITH APPLICATIONS TO REES RINGS

ANDREW KUSTIN, CLAUDIA POLINI, AND BERND ULRICH

ABSTRACT. Consider a rational projective plane curve \( C \) parameterized by three homogeneous forms of the same degree in the polynomial ring \( R = k[x, y] \) over a field \( k \). The ideal \( I \) generated by these forms is presented by a homogeneous \( 3 \times 2 \) matrix \( \varphi \) with column degrees \( d_1 \leq d_2 \). The Rees algebra \( \mathcal{R} = R[I] \) of \( I \) is the bi-homogeneous coordinate ring of the graph of the parameterization of \( C \); and accordingly, there is a dictionary that translates between the singularities of \( C \) and algebraic properties of the ring \( \mathcal{R} \) and its defining ideal. Finding the defining equations of Rees rings is a classical problem in elimination theory that amounts to determining the kernel \( \mathcal{A} \) of the natural map from the symmetric algebra \( \text{Sym}(I) \) onto \( \mathcal{R} \). The ideal \( \mathcal{A}_{\geq d_2 - 1} \), which is an approximation of \( \mathcal{A} \), can be obtained using linkage. We exploit the bi-graded structure of \( \text{Sym}(I) \) in order to describe the structure of an improved approximation \( \mathcal{A}_{\geq d_1 - 1} \) when \( d_1 < d_2 \) and \( \varphi \) has a generalized zero in its first column. (The latter condition is equivalent to assuming that \( C \) has a singularity of multiplicity \( d_2 \).) In particular, we give the bi-degrees of a minimal bi-homogeneous generating set for this ideal. When \( 2 = d_1 < d_2 \) and \( \varphi \) has a generalized zero in its first column, then we record explicit generators for \( \mathcal{A} \). When \( d_1 = d_2 \), we provide a translation between the bi-degrees of a bi-homogeneous minimal generating set for \( \mathcal{A}_{d_1 - 2} \) and the number of singularities of multiplicity \( d_1 \) that are on or infinitely near \( C \). We conclude with a table that translates between the bi-degrees of a bi-homogeneous minimal generating set for \( \mathcal{A} \) and the configuration of singularities of \( C \) when the curve \( C \) has degree six.

1. INTRODUCTION.

TABLE OF CONTENTS

1. Introduction.
2. Duality, perfect pairing, and consequences.
   2.A. The abstract duality relating \( \mathcal{A} \) and \( \text{Sym}(I) \).
   2.B. The torsionfreeness and reflexivity of the \( S \)-module \( \text{Sym}(I)_i \) and how these properties are related to the geometry of the corresponding curve.
   2.C. The duality is given by multiplication.
   2.D. Explicit \( S \)-module generators for \( \mathcal{A}_i \), when \( i \) is large.
3. The case of a generalized zero in the first column of \( \varphi \).
4. Morley forms.
5. Explicit generators for \( \mathcal{A} \) when \( d_1 = 2 \).
6. The case of \( d_1 = d_2 \).

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7. An Application: Sextic curves.

Our basic setting is as follows: Let $k$ be an algebraically closed field, $R = k[x, y]$ a polynomial ring in two variables, and $I$ an ideal of $R$ minimally generated by homogeneous forms $h_1, h_2, h_3$ of the same degree $d > 0$. Extracting a common divisor we may harmlessly assume that $I$ has height two. We will keep these assumptions throughout the introduction, though many of our results are stated and proved in greater generality.

On the one hand, the homogeneous forms $h_1, h_2, h_3$ define a morphism

\[ \eta : \mathbb{P}^1_k \xrightarrow{[h_1 : h_2 : h_3]} \mathbb{P}^2_k \]

whose image is a curve $C$. After reparametrizing we may assume that the map $\eta$ is birational onto its image or, equivalently, that the curve $C$ has degree $d$.

On the other hand, associated to $h_1, h_2, h_3$ is a syzygy matrix $\varphi$ that gives rise to a homogeneous free resolution of the ideal $I$,

\[ 0 \rightarrow R(-d - d_1) \oplus R(-d - d_2) \xrightarrow{\varphi} R(-d)^3 \rightarrow I \rightarrow 0. \]

Here $\varphi$ is a 3 by 2 matrix with homogeneous entries in $R$, of degree $d_1$ in the first column and of degree $d_2$ in the second column. We may assume that $d_1 \leq d_2$. Notice that $d = d_1 + d_2$ by the Hilbert-Burch Theorem.

The two aspects, the curve $C$ parametrized by the forms $h_1, h_2, h_3$ and the syzygy matrix $\varphi$ of these forms, are mediated by the Rees algebra $R$ of $I$. The Rees algebra is defined as the subalgebra $R[I] = R[h_1t, h_2t, h_3t]$ of the polynomial ring $R[t]$. It becomes a standard bi-graded $k$-algebra if one sets $\deg x = \deg y = (1, 0)$ and $\deg t = (-d, 1)$, which gives $\deg h_t = (0, 1)$. The bi-homogeneous spectrum of $R$ is the graph $\Gamma \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ of the morphism $\eta = [h_1 : h_2 : h_3]$. Projecting to the second factor of $\mathbb{P}^1_k \times \mathbb{P}^2_k$ one obtains a surjection $\Gamma \twoheadrightarrow C$, which corresponds to an inclusion of coordinate rings

\[ R \hookrightarrow k[h_1t, h_2t, h_3t]. \]

Thus the coordinate ring $A(C)$ of the curve $C$ can be recovered as a direct summand of the Rees algebra $R$, namely

\[ A(C) = \bigoplus_i R(0, i). \]

The same holds for the ideal $I$,

\[ I \simeq It = \bigoplus_i R(i, 1). \]

Finally, the inclusion $\Gamma \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ corresponds to a homogeneous epimorphism $R \twoheadrightarrow B$, where $B = k[x, y, T_1, T_2, T_3]$ is a bi-graded polynomial ring with $\deg x = \deg y = (1, 0)$ and $\deg T_i = (0, 1)$, and the variables $T_i$ are mapped to $h_t$. The kernel of this epimorphism is a bi-homogeneous ideal $J$ of $B$, the ‘defining ideal’ of the Rees algebra $R$. Now the syzygy module of $I$ can be recovered as
well,

$$\text{syz}(I) \simeq \bigoplus_j J_{(i,1)}.$$  

Thus, the philosophy underlying this work can be summarized as follows: One wishes to study local properties of the rational plane curve $C$, such as the types of its singularities, by means of the syzygies of $I$, since linear relations among polynomials are easier to handle than polynomial relations. The mediator is the Rees algebra, which in turn carries more information than the coordinate ring $A(C)$ of the curve, just like the graph of a map reveals more than the image of the map. One may therefore hope that even relatively simple numerical data associated to this algebra, such as the (first) bi-graded Betti numbers, say a great deal about the curve. The syzygies of $I$ appear in the defining ideal $J$, which leads one to study defining ideals of Rees algebras. Finding such ideals or, equivalently, describing Rees rings explicitly in terms of generators and relations, is a fundamental problem in elimination theory, that has occupied commutative algebraists, algebraic geometers, and, more recently, applied mathematicians. The problem is wide open, even for ideals of polynomial rings in two variables.

Write $m = (x, y)$ for the homogeneous maximal ideal of $R = k[x, y]$ and $S$ for the polynomial ring $k[T_1, T_2, T_3]$. Recall that $B = k[x, y, T_1, T_2, T_3] = R \otimes_k S$ and that $R = B/J$. To study the Rees algebra of an ideal one customarily maps the symmetric algebra onto it,

$$0 \longrightarrow \mathcal{A} \longrightarrow \text{Sym}(I) \longrightarrow R \longrightarrow 0.$$  

One readily sees that $\mathcal{A} = H^0_m(\text{Sym}(I)) = 0 : \text{Sym}(I) \cdot m^\infty$. A presentation of the symmetric algebra is well understood, $\text{Sym}(I) \simeq B/(g_1, g_2)$, where

$$[g_1, g_2] = [T_1, T_2, T_3] \cdot \varphi.$$  

The polynomials $g_i$ are homogeneous of bi-degree $(d_i, 1)$ and together they form a $B$-regular sequence. The symmetric algebra $\text{Sym}(I)$, the Rees algebra $R$, and the ideal $\mathcal{A}$ of $\text{Sym}(I)$ that defines $R$ all are naturally equipped with two gradings: the $T$-grading and the $xy$-grading. Both gradings play crucial roles in our work. The $T$-grading is often used in the study of symmetric algebras and Rees algebras. For example, an ideal is said to be of “linear type” if $\mathcal{A} = 0$, which means that the defining ideal $J$ of the Rees algebra is generated by polynomials of $T$-degree 1. Ideals of linear type are much studied in the literature; see, for example [18, 29, 14, 19, 12, 26]. Much of our work is focused on the $xy$-grading. We view $\mathcal{A}$ as $\bigoplus \mathcal{A}_i$, where $\mathcal{A}_i$ is the $S$-submodule of $\mathcal{A}$ which consists of all elements homogeneous in $x$ and $y$ of degree $i$; in other words, $\mathcal{A}_i = \bigoplus_j \mathcal{A}_{i,j}$. One major advantage of this decomposition is the fact that $\mathcal{A}_i$ is non-zero for only finitely many values of $i$. Both gradings come into play in the proof of Theorem 3.3, which is one of the main results of the paper. In Theorem 3.3 we identify the degrees of the minimal generators of each $\mathcal{A}_i$ in the range $i \geq d_i - 1$ (see also Table 3.5). In particular, for each fixed $i$ we must determine the minimum value of $j$ for which $\mathcal{A}_{i,j}$ is not zero. Curiously enough, the key point in our proof is that we revert to the $T$-grading for this.
The mathematics that sets the present project in motion is due to Jouanolou \[22, 21\]; see also Busé \[5\]. Jouanolou proved that the multiplication map

\[(1.0.2) \quad \mathcal{A}_i \otimes \text{Sym}(I)_{g-i} \longrightarrow \mathcal{A}_b \cong S(-2)\]

gives a perfect pairing of \(S\)-modules for \(\delta = d - 2\). Jouanolou uses Morley forms to exhibit dual bases for the modules of \(1.0.2\). The perfect pairing \(1.0.2\) shows that the \(S\)-module structure of \(\mathcal{A}_i\) is completely determined by the \(S\)-module structure of \(\text{Sym}(I)_{g-i}\). The symmetric algebra \(\text{Sym}(I)\) is a complete intersection defined by the regular sequence \(g_1, g_2\); so, the \(S\)-module structure of \(\text{Sym}(I)_{g-i}\) depends on the relationship between \(\delta - i, d_1, \) and \(d_2\).

Ultimately we offer three proofs of Jouanolou’s perfect pairing \(1.0.2\). Two of our arguments are different from Jouanolou’s; furthermore, our arguments are self-contained, and we obtain results not obtained by Jouanolou. In particular, we relate the entries of \(\varphi\) to module-theoretic properties of \(\mathcal{A}_i\) and \(\text{Sym}(I)_{g-i}\), (see especially Theorem 2.11 and Corollary 2.12 in Subsection 2.B) and also to information about the singularities of the curve parameterized by a minimal generating set for \(I\); see especially Sections 6 and 7. A very quick proof of the abstract duality relating \(\mathcal{A}\) and \(\text{Sym}(I)\) is given in Subsection 2.A. This proof computes the local cohomology with support in \(m\) along the Koszul complex which resolves \(\text{Sym}(I)\) as a \(B\)-module. In Subsection 2.C we take advantage of the module theoretic properties of the \(\mathcal{A}_i\) (in particular the fact that they are reflexive \(S\)-modules as is shown in Subsection 2.B) to prove that the abstract isomorphism of Subsection 2.A is actually given by multiplication. Finally, in Theorem 4.2, as part of our review of the theory of Morley forms in Section 4, we give Jouanolou’s own proof of the perfect pairing \(1.0.2\).

In Theorem 3.3 we describe the \(S\)-module structure of \(\mathcal{A}_{\geq d_1-1}\) (according to the convention described above, \(\mathcal{A}_{\geq d_1-1}\) means \(\bigoplus_j \mathcal{A}_{(\geq d_1-1,j)}\)) under the hypothesis that \(d_1 < d_2\) and \(\varphi\) has a generalized zero in its first column. This module is free and we identify the bi-degrees of a bi-homogeneous basis for it; see also Table 3.5. In Corollary 3.10 we identify the bi-degrees of a minimal bi-homogeneous generating set of \(\mathcal{A}_{\geq d_1-1}\) as an ideal of \(\text{Sym}(I)\). When one views this result in the geometric context of \(1.0.1\), then the hypothesis concerning the existence of a generalized zero is equivalent to assuming that \(C\) has a singularity of multiplicity \(d_2\), and the ideal \(\mathcal{A}_{\geq d_1-1}\) of the conclusion is an approximation of the ideal that defines the graph \(\Gamma \subseteq \mathbb{P}_k^1 \times \mathbb{P}_k^2\) of the parameterization \(\eta : \mathbb{P}^1_k \rightarrow C\). The part of \(\text{Sym}(I)\) that corresponds to \(\mathcal{A}_{\geq d_1-1}\), under the duality of \(1.0.2\), is \(\text{Sym}(I)_{\leq d_2-1}\). There is no contribution from \(g_2\) to the \(S\)-module \(\text{Sym}(I)_{\leq d_2-1}\) in the bi-homogeneous \(B\)-resolution of \(\text{Sym}(I)\). So, basically, we may ignore \(g_2\). Furthermore, the hypothesis that the first column of \(\varphi\) has a generalized zero allows us to make the critical calculation over a subring \(U\) of \(S\), where \(U\) is a polynomial ring in two variables.

In Section 5 we focus on the situation \(2 = d_1 < d_2\). Busé \[5\] has given explicit formulas for the generators of \(\mathcal{A}\) if the first column of \(\varphi\) does not have a generalized zero. In Theorem 5.11 we carry out the analogous project in the case when the first column of \(\varphi\) does have a generalized zero. These hypotheses about generalized zeros in the first column have geometric implications for the corresponding curve. In Busé’s case all of the singularities of \(C\) have multiplicity at most \(d_1\);
whereas, in the situation of Theorem 5.11, \( C \) has at least one singularity of multiplicity \( d_2 \). The proof of Theorem 5.11 is based on the results of Section 3 (since \( \mathcal{A}_{d_1-1} \) is equal to \( \mathcal{A}_{2} \) when \( d_1 = 2 \) and \( \mathcal{A}_0 \) is always well understood), an analysis of the kernel of a Toeplitz matrix of linear forms in two variables (see Lemmas 5.7 and 5.10), and Jouanolou’s theory of Morley forms. We review the theory of Morley forms in Section 4.

In Section 6 we completely describe the \( S \)-module structure of \( \mathcal{A}_{d_1-2} \) when \( d_1 = d_2 \). A preliminary version of this section initiated the investigation that culminated in [7]. The geometric significance of these calculations are emphasized in [7] and are reprised in the present paper; however the main focus of Section 6 is on the Rees algebras.

The results in Sections 1 – 6 suffice to provide significant information about the defining equations for \( \mathcal{R} \) if \( d = d_1 + d_2 \leq 6 \), since then \( d_1 \leq 2 \) (see, especially, Section 5) or \( d_1 = d_2 \) (see, especially, Section 6). Section 7 is concerned with the case \( d = 6 \), the case of a sextic curve. We show that there is, essentially, a one-to-one correspondence between the bi-degrees of the defining equations of \( \mathcal{R} \) on the one hand and the types of the singularities on or infinitely near the curve \( C \) on the other hand.

If \( R \) is a ring, we write Quot(\( R \)) for the total ring of quotients of \( R \); that is, \( \text{Quot}(R) = U^{-1}R \), where \( U \) is the set of non zero-divisors on \( R \). If \( R \) is a domain, the total ring of quotients of \( R \) is usually called the quotient field of \( R \).

If \( M \) is a matrix, then \( M^T \) denotes the transpose of \( M \). If \( M \) has entries in a \( k \)-algebra, where \( k \) is a field, then a generalized zero of \( M \) is a product \( pMq^T = 0 \), where \( p \) and \( q \) are non-zero row vectors with entries from \( k \).

If \( M \) is a \( \ell - 1 \times \ell \) matrix with entries in a ring, then the ring elements \( m_1, \ldots, m_\ell \) are the signed maximal minors of the matrix \( M \); that is, \( m_i \) is \((-1)^{i+1}\) times the determinant of the submatrix of \( M \) obtained by removing column \( i \). We notice that the product \( M [m_1, \ldots, m_\ell]^T \) is zero.

If \( S \) is a ring and \( A, B, \) and \( C \) are \( S \)-modules, then the \( S \)-module homomorphism \( F : A \otimes S B \to C \) is a perfect pairing if the induced \( S \)-module homomorphisms \( A \to \text{Hom}_S(B, C) \) and \( B \to \text{Hom}_S(A, C) \), given by \( a \mapsto F(a \otimes \_ ) \) and \( b \mapsto F(\_ \otimes b) \), are isomorphisms.

2. Duality, perfect pairing, and consequences.

**Data 2.1.** Let \( k \) be a field, \( R = k[x,y] \) a polynomial ring in 2 variables over \( k \), \( m = (x,y)R \) the homogeneous maximal ideal of \( R \), and \( I \) a height 2 ideal of \( R \) minimally generated by 3 forms of the same positive degree \( d \). Let \( \delta = d - 2 \). Let \( \varphi \) be a homogeneous Hilbert-Burch matrix for \( I \); each entry in column \( i \) of \( \varphi \) has degree \( d_i \) with \( d_1 \leq d_2 \). Let \( \mathcal{A} \) be the kernel of the natural surjection

\[
\text{Sym}(I) \to \mathcal{R}
\]

from the symmetric algebra of \( I \) to the Rees algebra of \( I \), and let \( S \) and \( B \) be the polynomial rings \( S = k[T_1, T_2, T_3] \) and \( B = R \otimes_k S = k[x, y, T_1, T_2, T_3] \). View \( B \) as a bi-graded \( k \)-algebra, where \( x \) and \( y \) have bi-degree \((1,0)\) and each \( T_i \) has bi-degree \((0,1)\). A presentation of the symmetric algebra is
given by $\text{Sym}(I) \simeq B/(g_1, g_2)$, where

$$[g_1, g_2] = [T_1, T_2, T_3] \cdot \varphi.$$ 

**Remarks 2.2.** Adopt Data 2.1. The Hilbert-Burch Theorem guarantees that $d_1 + d_2 = d$. One readily sees that the $\text{Sym}(I)$-ideals

$$\mathcal{A} \quad \text{and} \quad H^0_m(\text{Sym}(I)) = 0 : \text{Sym}(I) \cdot m$$

are equal. The polynomial $g_m$ is homogeneous of bi-degree $(d_m, 1)$. The polynomials $g_1, g_2$ form a regular sequence on $B$ because the dimension of $\text{Sym}(I)$ is equal to 3 by [20]. Thus, the Koszul complex provides a bi-homogeneous $B$-resolution of the symmetric algebra:

$$(2.2.1) \quad \mathcal{K}^\bullet(\mathcal{A}_I; B) \longrightarrow \text{Sym}(I) \longrightarrow 0.$$ 

**Remark 2.3.** When the bi-graded $B$-modules $\mathcal{A} = \bigoplus_{i,j} \mathcal{A}_{(i,j)}$ and $\text{Sym}(I) = \bigoplus_{i,j} \text{Sym}(I)_{(i,j)}$ are viewed as $S$-modules, then we write $\mathcal{A} = \bigoplus_i \mathcal{A}_i$ and $\text{Sym}(I) = \bigoplus_i \text{Sym}(I)_i$, where $\mathcal{A}_i$ represents the $S$-module $\mathcal{A}_{(i,j)}$ and $\text{Sym}(I)_i$ represents the $S$-module $\text{Sym}(I)_{(i,j)}$.

The goal of this section is to prove that $\mathcal{A}_8$ is a free $S$-module generated by an explicit element $\text{syl} \in \text{Sym}(I)_{(8,2)}$ and that the multiplication map

$$(2.3.1) \quad \mathcal{A}_i \otimes \text{Sym}(I)_{8-i} \longrightarrow \mathcal{A}_8 = S \cdot \text{syl}$$

gives a perfect pairing of $S$-modules. Both of these results are due to Jouanolou [22, 21]; see also Busé [5]. Our arguments are different from Jouanolou’s; furthermore, our arguments are self-contained, and we obtain results not obtained by Jouanolou. In particular, we relate the entries of $\varphi$ to module-theoretic properties of $\mathcal{A}_i$ and $\text{Sym}(I)_{8-i}$, and also to information about the singularities of the curve parameterized by a minimal generating set for $I$. The section consists of four subsections:

2.A The abstract duality relating $\mathcal{A}$ and $\text{Sym}(I)$;

2.B The torsionfreeness and reflexivity of the $S$-module $\text{Sym}(I)_i$ and how these properties are related to the geometry of the corresponding curve;

2.C The duality is given by multiplication; and

2.D Explicit $S$-module generators for $\mathcal{A}_i$, when $i$ is large.

### 2.A  The abstract duality relating $\mathcal{A}$ and $\text{Sym}(I)$.

The goal of this subsection is to relate the $S$-modules $\mathcal{A}_i$ and $\text{Sym}(I)_{8-i}$ and to express $\mathcal{A}_i$ as the kernel of a homomorphism of free $S$-modules. This goal is attained in Corollary 2.5 and Theorem 2.7. The first step toward (2.3.1) is to establish an abstract isomorphism between $\mathcal{A}$ and a shift of $\text{Hom}_S(\text{Sym}(I), S)$, where $\text{Hom}$ denotes the graded dual. Toward that aim, one computes local cohomology with support in $m$ along the resolution (2.2.1), uses the symmetry of the Koszul complex, and the isomorphism $H^0_m(R) \cong \text{Hom}_k(R, k)(2)$. 
\textbf{Theorem 2.4.} If Data 2.1 is adopted, then there is an isomorphism of bi-graded $B$-modules

$$\mathcal{A} \simeq \text{Hom}_S(\text{Sym}(I), S)(-\delta, -2).$$

\textbf{Proof.} We first establish two isomorphisms that are essential for our proof. From the self-duality of the Koszul complex one obtains an isomorphism of complexes of bi-graded $B$-modules,

\begin{equation}
\text{Hom}_S(K_\bullet(g_1, g_2; B), S) \simeq K_\bullet(g_1, g_2; \text{Hom}_S(B, S))[2](d, 2).
\end{equation}

The symbol $[2]$ indicates homological degree shift. The internal bi-degree shift is written $(d, 2)$. We also use the following isomorphisms of bi-graded $B$-modules,

$$H^2_m(B) \simeq H^2_m(R \otimes_R B)$$

$$\simeq H^2_m(R) \otimes_R B \quad \text{since } B \text{ is } R\text{-flat}$$

$$\simeq H^2_m(R) \otimes_R S \quad \text{since } B \simeq R \otimes_R S$$

$$\simeq \text{Hom}_R(R, k)(2) \otimes_k S \quad \text{by Serre duality}$$

$$\simeq \text{Hom}_S(B, S)(2, 0).$$

We deduce that

\begin{equation}
H^2_m(B) \simeq \text{Hom}_S(B, S)(2, 0).
\end{equation}

To prove the assertion of the theorem we decompose $K_\bullet = K_\bullet(g_1, g_2; B)$ into short exact sequences

$$0 \to K_2 \to K_1 \to \mathcal{J} \to 0 \quad \text{and} \quad 0 \to \mathcal{J} \to K_0 \to \text{Sym}(I) \to 0.$$

The second sequence gives

\begin{equation}
0 = H^0_m(K_0) \to H^0_m(\text{Sym}(I)) \to H^1_m(\mathcal{J}) \to H^1_m(K_0) = 0,
\end{equation}

where the first and last modules vanish because grade $mB > 1$. The first sequence above yields

\begin{equation}
0 = H^0_m(K_1) \to H^1_m(\mathcal{J}) \to H^2_m(K_2) \to H^2_m(K_1).
\end{equation}

Notice that $\partial$ is the second differential of the Koszul complex $K_\bullet(g_1, g_2; H^2_m(B))$ because the formation of local cohomology commutes with taking direct sums. Thus from (2.4.3) and (2.4.4) we obtain a bi-graded isomorphism

$$H^0_m(\text{Sym}(I)) \simeq H_2(K_\bullet(g_1, g_2; H^2_m(B)).$$

On the other hand,

$$\begin{aligned}
H_2(K_\bullet(g_1, g_2; H^2_m(B)) &\simeq H_2(K_\bullet(g_1, g_2; \text{Hom}_S(B, S)(2, 0)) \quad \text{by (2.4.2)} \\
&\simeq H_0(\text{Hom}_S(K_\bullet(g_1, g_2; B), S))(-d, -2)(2, 0) \quad \text{by (2.4.1)} \\
&\simeq \text{Hom}_S(\text{Sym}(I), S)(2 - d, -2).
\end{aligned}$$

The last isomorphism holds because $K_\bullet(g_1, g_2; B)$ is a resolution of $\text{Sym}(I)$. $\Box$

\textbf{Corollary 2.5.} Adopt Data 2.1. The following statements hold.

1. The graded $S$-modules $A_i$ and $\text{Hom}_S(\text{Sym}(I), S(-2))$ are isomorphic for all $i$.
2. The $S$-module $A_i$ is zero for all $i > \delta$.
3. The graded $S$-module $A_{\delta}$ is isomorphic to $S(-2)$.
4. The $S$-module $A_{\delta}$ is reflexive for all $i$. 

Proof. Assertion (1) follows directly from Theorem 2.4; (2) follows from (1) since $\text{Sym}(I)_{\ell}$ is zero when $\ell$ is negative; (3) holds because $\text{Sym}(I)_0 = S$; and (4) holds because the $S$-dual of every finitely generated $S$-module is reflexive.

Theorem 2.4 shows that the $S$-module structure of $A_i$ is completely determined by the $S$-module structure of $\text{Sym}(I)_{\delta-i}$. The symmetric algebra $\text{Sym}(I)$ is a complete intersection defined by the regular sequence $g_1, g_2$; so, the $S$-module structure of $\text{Sym}(I)_{\delta-i}$ depends on the relationship between $\delta-i, d_1$, and $d_2$. Theorem 2.7 describes the $S$-module structure of $\text{Sym}(I)_{\delta-i}$ and $A_i$ as a function of where $\delta-i$ sits with respect to $d_1 \leq d_2$. We set up the relevant notation in the next definition.

Definition 2.6. The polynomials $g_1$ and $g_2$ in $S[x,y]$ are defined in Data 2.1. At this point we name their coefficients by writing
\begin{equation}
(2.6.1) \quad g_m = \sum_{\ell=0}^{d_m} c_{\ell,m} x^{d_m-\ell},
\end{equation}
with $c_{\ell,m} \in S_1$, for $m$ equal to 1 or 2. For positive integers $n$ and $m$, with $m$ equal to 1 or 2, let $\Upsilon_{n,m}$ be the $(d_m+n) \times n$ matrix
\[
\Upsilon_{n,m} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
c_{d_m,m} & \cdots & \cdots & \cdots & \cdots \\
0 & c_{d_m,m} & \cdots & \cdots & \cdots \\
0 & 0 & c_{d_m,m} & \cdots & \cdots \\
0 & 0 & 0 & c_{d_m,m} & \cdots \\
0 & 0 & 0 & 0 & c_{d_m,m}
\end{bmatrix},
\]
with entries from $S_1$. The matrix $\Upsilon_{n,m}$ represents the map of free $S$-modules $S[x,y]_{n-1} \to S[x,y]_{n-1+d_m}$ which is given by multiplication by $g_m$ when the bases $y^{n-1}, \ldots, x^{n-1}$ and $y^{n-1+d_m}, \ldots, x^{n-1+d_m}$ are used for $S[x,y]_{n-1}$ and $S[x,y]_{n-1+d_m}$, respectively.

Theorem 2.7. Adopt Data 2.1. The following statements hold.

1. If $0 \leq i \leq d_1 - 2$, then the $S$-modules $A_i$ and $\text{Sym}(I)_{\delta-i}$ both have rank $i+1$; furthermore, the following sequences of $S$-modules are exact:
\[
0 \to S(-1)^{d_2-i-1} \oplus S(-1)^{d_1-i-1} \xrightarrow{(\Upsilon_{d_2-i-1,1} \quad \Upsilon_{d_1-i-1,2})} S^{\delta-i+1} \to \text{Sym}(I)_{\delta-i} \to 0
\]
The statements about the modules $\mathcal{A}_i$ and $\text{Sym}(I)_{\delta-i}$ both have rank $d_1$; furthermore, the following sequences of $S$-modules are exact:

$$0 \to S(-2)^{\delta-i+1} \overset{\begin{pmatrix} Y_{d_2-i+1,1}^T \\ Y_{d_1-i+1,2}^T \end{pmatrix}}{\longrightarrow} S(-1)^{d_2-i-1} \oplus S(-1)^{d_1-i-1} \to 0.$$  

(2) If $d_1 - 1 \leq i \leq d_2 - 2$, then the $S$-modules $\mathcal{A}_i$ and $\text{Sym}(I)_{\delta-i}$ both have rank $d_1$; furthermore, the following sequences of $S$-modules are exact:

$$0 \to S(-1)^{d_2-i-1} \overset{Y_{d_2-i+1,1}}{\longrightarrow} S^{\delta-i+1} \longrightarrow \text{Sym}(I)_{\delta-i} \to 0.$$  

and

$$0 \to \mathcal{A}_i \to S(-2)^{\delta-i+1} \overset{Y_{d_2-i+1,1}^T}{\longrightarrow} S(-1)^{d_2-i-1} \to 0.$$  

(3) If $d_2 - 1 \leq i \leq \delta$, then the $S$-modules $\mathcal{A}_i$ and $\text{Sym}(I)_{\delta-i}$ both have rank $\delta-i+1$; furthermore, $\text{Sym}(I)_{\delta-i} \simeq S^{\delta-i+1}$ and $\mathcal{A}_i \simeq S(-2)^{\delta-i+1}$.

**Proof.** The homogeneous $B$-resolution of $\text{Sym}(I)$

$$0 \to B(-d_1-d_2,-2) \to B(-d_1,-1) \oplus B(-d_2,-1) \to B \to \text{Sym}(I) \to 0,$$

which is given in (2.2.1), may be decomposed into the graded strands recorded in the statement of the Theorem. The rank of each $S$-module $\text{Sym}(I)_{\delta-i}$ can be read immediately from its resolution. The statements about the modules $\mathcal{A}_i$ follow from part (1) of Corollary 2.5.

**2.B** THE TORSIONFREENESS AND REFLEXIVITY OF THE $S$-MODULE $\text{Sym}(I)_i$ AND HOW THESE PROPERTIES ARE RELATED TO THE GEOMETRY OF THE CORRESPONDING CURVE.

We are now going to investigate the torsionfreeness and reflexivity of the graded components of $\text{Sym}(I)$. To do so we need to estimate the height of ideals of minors of the matrices that appear in parts (1) and (2) of Theorem 2.7.

**Lemma 2.8.** Adopt Data 2.1. Let $n$ be a positive integer and $Y_{n,1}$ be the $(d_1 + n) \times n$ matrix introduced in Definition 2.6. The following statements hold:

1. $\text{ht}_n(Y_{n,1}) \geq 2$;
2. $\text{ht}_n(Y_{n,1}) = 3$ if and only if the first column of $\varphi$ does not have a generalized zero.

**Proof.** The ideal $I_n(Y_{n,1})$ is equal to the $n^{th}$ power of the ideal $I_1(Y_{1,1})$; and, for any given ideal $J$ in the polynomial ring $S$, the ideals $J$ and $J^n$ have the same height. Therefore, it suffices to prove the result when $n = 1$. On the other hand, the ideal $I_1(Y_{1,1})$ is generated by linear forms in $S_1$; so the height of $I_1(Y_{1,1})$ is equal to the minimal number of generators of $I_1(Y_{1,1})$. Recall that

$$[y^{d_1}, xy^{d_1-1}, \ldots, x^{d_1}] Y_{1,1} = g_1 = [T_1, T_2, T_3] \varphi_1,$$
where \( \varphi_1 \) is the first column of \( \varphi \). The entries of \( \varphi_1 \) generate an ideal of height at least 2 because \( \text{ht} I_2(\varphi) = 2 \). To complete the proof it suffices to show that

\[
(2.8.1) \quad \mu(I_1(\varphi_1)) = \mu(I_1(\varphi_1)).
\]

Indeed, suppose, for the time being, that (2.8.1) has been established. Then

\[
2 \leq \text{ht}(I_1(\varphi_1)) \implies 2 \leq \mu(I_1(\varphi_1)) = \mu(I_1(\varphi_1)) = \text{ht}(I_1(\varphi_1)) = \text{ht}(I_\mu(\varphi_1))
\]

and (1) holds. Also,

\[
\text{ht} I_\mu(\varphi_1) \leq 2 \iff \mu(I_1(\varphi_1)) \leq 2 \iff \mu(I_1(\varphi_1)) \leq 2 \iff \varphi_1 \text{ has a generalized zero.}
\]

Now we prove (2.8.1). Suppose that \( I_1(\varphi_1) \) is minimally generated by \( \lambda_1, \ldots, \lambda_s \) in \( S_1 \). It follows that

\[
\varphi_1, \varphi_1 = \lambda_1 \rho_1 + \cdots + \lambda_s \rho_s
\]

for column vectors \( \rho_\ell \) in \( \text{Mat}(d_1+1) \times (k) \). For \( 1 \leq \ell \leq s \), let \( \xi_\ell \) be the homogeneous form

\[
\xi_\ell = [y^{d_\ell}, x y^{d_\ell-1}, \ldots, x^{d_\ell}] \rho_\ell
\]

in \( R_{d_\ell} \) and let \( Z_\ell \) be the column vector of three constants with \( \lambda_\ell = [T_1, T_2, T_3] Z_\ell \). We have

\[
[T_1, T_2, T_3] \varphi_1 = g_1 = [y^{d_1}, x y^{d_1-1}, \ldots, x^{d_1}] \varphi_1 = [y^{d_1}, x y^{d_1-1}, \ldots, x^{d_1}] \lambda_1 \rho_1 + \cdots + \lambda_s \rho_s
\]

\[
= \xi_1 \lambda_1 + \cdots + \xi_s \lambda_s = \xi_1 [T_1, T_2, T_3] Z_1 + \cdots + \xi_s [T_1, T_2, T_3] Z_s = [T_1, T_2, T_3] \left( \sum_{\ell=1}^s \xi_\ell Z_\ell \right).
\]

The entries of the \( 3 \times 1 \) vector \( \varphi_1 - \sum \xi_\ell Z_\ell \) are homogeneous forms of degree \( d_1 \) in \( R \); hence, this vector cannot be in the kernel of \( [T_1, T_2, T_3] \) unless it is already zero. Thus, \( \varphi_1 = \sum \xi_\ell Z_\ell \). Let \( V \) be the subspace of \( R_{d_1} \) which is spanned by the entries of \( \varphi_1 \). We have shown that \( V \) is a subspace of the vector space spanned by \( \xi_1, \ldots, \xi_s \). It follows that

\[
\mu(I_1(\varphi_1)) = \dim V \leq s = \mu(I_1(\varphi_1)).
\]

One may read the calculation in the other direction to see that \( \mu(I_1(\varphi_1)) \leq \mu(I_1(\varphi_1)) \).

\[ \square \]

**Remark 2.9.** Adopt Data 2.1 and assume \( k \) is algebraically closed. The signed maximal minors \( h_1, h_2, h_3 \) of \( \varphi \) define a morphism

\[
\mathbb{P}^1_k \xrightarrow{[h_1, h_2, h_3]} \mathbb{P}^2_k
\]

whose image is a rational plane curve \( C \). The degree of the curve \( C \) satisfies the equality \( \text{deg} C = d/r \), where \( r \) is the degree of the field extension \( [\text{Quot}(k[R_d]) : \text{Quot}(k[I_d])] \). In particular, \( r = 1 \) if and only if the parametrization is birational onto its image.

As it turns out, the heights of various ideals of minors of interest can be expressed in terms of the singularities of the curve \( C \).
Lemma 2.10. Adopt Data 2.1 with \( d_1 = d_2 \). Let \( C \) be the \((d_1 + 1) \times 2\) matrix \( C = (\mathcal{Y}_{1,1} \quad \mathcal{Y}_{1,2}) \) for \( \mathcal{Y}_{1,1} \) and \( \mathcal{Y}_{1,2} \) as introduced in Definition 2.6, and let \( C \) be the curve of Remark 2.9. The following statements hold:

1. \( \text{ht} I_2(C) \geq 2 \) if and only if \( I_1(\phi) \) is not a complete intersection; furthermore, if \( k \) is algebraically closed, then the previous conditions hold if and only if the curve \( C \) is singular;
2. \( \text{ht} I_2(C) = 3 \) if and only if \( \phi \) does not have a generalized zero; furthermore, if \( k \) is algebraically closed, then the previous conditions hold if and only if the curve \( C \) is singular and its singularities have multiplicity at most \((\deg C)/2 - 1\).

Proof. We may harmlessly assume that \( k \) is algebraically closed. Let \( r \) be the degree of the field extension \([\text{Quot}(k[R_d]) : \text{Quot}(k[I_d])]\), as described in Remark 2.9. Then there exists a regular sequence \( u, v \) in \( R_r \) so that \( I = I_2(\phi')R \) for some \( 3 \times 2 \) matrix \( \phi' \) whose entries are homogeneous polynomials of degree \((\deg C)/2 = d/2r \) in the variables \( u, v \) (see [25]). The signed maximal minors of \( \phi' \) provide a birational parametrization of the same curve \( C \). Let \( R' = k[u, v] \) and \( I' \) be the ideal \( I_2(\phi') \) of \( R' \). Define elements \( g'_1, g'_2 \in R'[T_1, T_2, T_3] \) via the equation \([g'_1, g'_2] = [T_1, T_2, T_3] \cdot \phi' \). Use these data to obtain matrices \( Y_{n,m}' \) as in Definition 2.6. Finally, let \( C' \) be the matrix \((\mathcal{Y}'_{1,1} \quad \mathcal{Y}'_{1,2})\).

From [7, 3.14(2)] we know that \( \text{ht} I_2(C') \geq 2 \) if and only if the curve \( C \) is singular, and \( \text{ht} I_2(C') = 3 \) if and only if the curve \( C \) is singular and its singularities have multiplicity at most \((\deg C)/2 - 1\). (The result from [7] is stated assuming the birationality of the parametrization. A complete proof of the geometric interpretation of \( \text{ht} I_2(C') \geq 2 \) uses the fact that a rational plane curve of degree at least three is singular.)

The curve \( C \) is nonsingular if and only if its homogeneous coordinate ring \( k[I_{d/r}] \) is normal. Since the parametrization is birational, the latter obtains if and only if \( k[I'_{d/r}] = k[R'_{d/r}] \) or, equivalently, \( 3 = \dim_k I'_{d/r} = \dim_k R'_{d/r} \). This holds if and only if \( d/r = 2 \). The last equality means that \( I_1(\phi') \) is generated by linear forms, equivalently \( I_1(\phi') = (u, v)R' \). The latter holds if and only if \( I_1(\phi') \) is a complete intersection, again because the parametrization is birational. Finally, the \( R' \)-ideal \( I_1(\phi') \) is a complete intersection if and only if the \( R \)-ideal \( I_1(\phi) \) is.

On the other hand, the curve \( C \) is singular and its singularities have multiplicity at most \((\deg C)/2 - 1\) if and only if \( \phi' \) does not have a generalized zero, as was shown in part (4) of [7, 1.9]. Notice that \( \phi' \) has a generalized zero if and only if \( \phi \) does.

It remains to show that \( I_2(C') = I_2(C) \). Extend the ordered set \( v^{d/r}, \ldots, u^{d/r} \) of monomials in \( u, v \) of degree \( d/r \) to an ordered basis of \( R_d \), which we call \( b_0, \ldots, b_d \). Define a \((d + 1) \times 2\) matrix \( D \) with entries in \( S_1 \) via the equality \([g'_1, g'_2] = [b_0, \ldots, b_d] \cdot D \). Notice that \( D = \begin{bmatrix} C' \n 0 \end{bmatrix} \), and hence \( I_2(C') = I_2(D) \). Finally, the matrix \( D \) is obtained from \( C \) by elementary row operations that correspond to the transition from \( b_0, \ldots, b_d \) to the monomial basis \( y^d, \ldots, x^d \) of \( R_d \). Therefore \( I_2(D) = I_2(C) \).

Theorem 2.11. Adopt Data 2.1 and let \( C \) be the curve of Remark 2.9. The following statements hold.

1. If \( d_1 \leq i \leq d_2 - 1 \), then
   a. the \( S \)-module \( \text{Sym}(I)_i \) is torsion free; and
(b) the $S$-module $\text{Sym}(I)_i$ is reflexive if and only if the first column of $\varphi$ does not have a generalized zero; furthermore, if $k$ is algebraically closed, then the previous conditions hold if and only if the singularities of the curve $C$ have multiplicity at most $d_1(\deg C)/d$.

(2) If $i = d_1 = d_2$, then
(a) the $S$-module $\text{Sym}(I)_i$ is torsionfree if and only if $I_1(\varphi)$ is not a complete intersection; furthermore, if $k$ is algebraically closed, then the previous conditions hold if and only if the curve $C$ is singular; and
(b) the $S$-module $\text{Sym}(I)_i$ is reflexive if and only if $\varphi$ does not have a generalized zero; furthermore, if $k$ is algebraically closed, then the previous conditions hold if and only if the curve $C$ is singular and its singularities have multiplicity at most $(\deg C)/2 - 1$.

**Proof.** We first argue the third equivalence in item (1.b). Again, as in the proof of Lemma 2.10, one obtains a regular sequence $u, v$ of forms of degree $d/(\deg C)$ so that $I = I_2(\varphi')R$ for some $3 \times 2$ matrix $\varphi'$ whose entries in position $i, j$ are homogeneous polynomials of degree $d_j(\deg C)/d$ in the variables $u, v$ (see [25]). Thus one reduces to the case of a birational parametrization. Now part (4) of [7, 1.9] shows that $C$ has a singularity of multiplicity at least $d_1(\deg C)/d + 1$ if and only if the first column of $\varphi'$ has a generalized zero, or, equivalently, the first column of $\varphi$ has a generalized zero.

Write $n = i - d_1 + 1$. From Theorem 2.7 we know that the $S$-module $\text{Sym}(I)_i$ has projective dimension at most one and that it is presented by $\Upsilon_{n,1}$ in the setting of (1) and by $C = (\Upsilon_{1,1} \ \ Upsilon_{1,2})$ in the setting of (2). Thus $\text{Sym}(I)_i$ is torsionfree if and only if $I_n(\Upsilon_{n,1})$ or $I_2(C)$, respectively, has height at least two. Likewise, $\text{Sym}(I)_i$ is reflexive if and only if this height is at least 3. Now it remains to appeal to Lemmas 2.8 and 2.10.

**Corollary 2.12.** Adopt Data 2.1 and let $C$ be the curve of Remark 2.9. The following statements hold.

(1) If $d_1 - 1 \leq i \leq d_2 - 2$, then the $S$-module $\mathcal{A}_i$ is free if and only if the first column of the matrix $\varphi$ has a generalized zero; furthermore, if $k$ is algebraically closed, then the previous conditions hold if and only if the curve $C$ has a singularity of multiplicity equal to $d_2^2(\deg C)/d$.

(2) If $i = d_1 = d_2$, then the $S$-module $\mathcal{A}_i$ is free if and only if $\mu(I_2(C)) \leq 4$; furthermore, if $k$ is algebraically closed, then the previous conditions hold if and only if there are at least two singularities of multiplicity $(\deg C)/2$ on or infinitely near $C$.

**Proof.** We prove part (1). If the first column of the matrix $\varphi$ does not have a generalized zero, then $\text{Sym}(I)_{d-i}$ is reflexive according to Theorem 2.11 part (1.b). Thus, part (1) of Corollary 2.5 shows that $\text{Sym}(I)_{d-i} \simeq \text{Hom}_S(\mathcal{A}_i, S(-2))$. Since $\text{Sym}(I)_{d-i}$ is not free it follows that $\mathcal{A}_i$ cannot be free either. Conversely, if the first column of the matrix $\varphi$ has a generalized zero then $\mathcal{A}_i$ is free as will be shown in Theorem 3.3. The third equivalence in item (1) is parts (1), (2), and (4) of [7, 1.9], after reducing to the case of a birational parameterization.
The first equivalence of part (2) will be proved in Theorem 6.2. The second equivalence follows from [7, 3.22] after reducing to the case of a birational parametrization.

\[ \square \]

2.C \hspace{1em} THE DUALITY IS GIVEN BY MULTIPLICATION.

In (2.3.1), we promised an explicit perfect pairing \( \mathcal{A}_i \otimes S \text{Sym}(I)_{\delta-i} \rightarrow A_\delta = s \cdot \text{syl} \). So far, in part (1) of Corollary 2.5, we showed that \( \mathcal{A}_i \) is isomorphic to \( \text{Hom}_S(\text{Sym}(I)_{\delta-i}, S(-2)) \). In Theorem 2.13 we prove that the abstract isomorphism of Corollary 2.5 is given by multiplication. The other highlight of the present subsection is Corollary 2.14, where we prove that the \( \text{Sym}(I) \)-ideals \( \mathcal{A}_{\geq i} \) and \( 0 : \text{Sym}(I) \cdot m^{d-1-i} \) are equal. We use this equality in subsection 2.D to record explicit generators for the \( S \)-modules \( \mathcal{A}_i \) when the equality represents linkage. The Sylvester element \( \text{syl} \) is one of these explicit generators. It is introduced in Remark 2.17 of subsection 2.D.

**Theorem 2.13.** Adopt Data 2.1. For each \( i \), the multiplication map \( \mathcal{A}_i \otimes \text{Sym}(I)_{\delta-i} \rightarrow \mathcal{A}_\delta \) induces a homogeneous isomorphism of \( S \)-modules

\[ \mathcal{A}_i \rightarrow \text{Hom}_S(\text{Sym}(I)_{\delta-i}, \mathcal{A}_\delta). \]

**Proof.** If \( i < 0 \) or \( \delta < i \), then the assertion is trivial because \( \text{Hom}_S(\text{Sym}(I)_{\delta-i}, \mathcal{A}_\delta) \) and \( \mathcal{A}_i \) both vanish due to (1) and (2) from Corollary 2.5. Hence, it suffices to prove the assertion for \( i \) in the range \( 0 \leq i \leq \delta \). We fix such an \( i \) and we denote the map induced by multiplication by

\[ \Phi : \mathcal{A}_i \rightarrow \text{Hom}_S(\text{Sym}(I)_{\delta-i}, \mathcal{A}_\delta). \]

Write \( \Sigma \) for \( \text{Sym}(I) \). If \( p \in \text{Spec}(S) \), then the ring \( \Sigma_p = S_p \otimes_S \Sigma \) is a standard graded \( S_p \)-algebra with irrelevant ideal \( (\Sigma_p)_+ = m\Sigma_p \). Furthermore, \( \Sigma_p \) is a complete intersection with \( \dim \Sigma_p = \dim S_p \). The source and the target of the homomorphism \( \Phi \) are reflexive \( S \)-modules, see part (4) of Corollary 2.5. Thus, to prove that \( \Phi \) is injective it suffices to show that \( \Phi_p = S_p \otimes \Phi \) is injective when \( p \) is the zero ideal of \( S \), and then to prove that \( \Phi \) is surjective one only needs to check that \( \Phi_p \) is surjective for every \( p \in \text{Spec}(S) \) with \( \dim S_p = 1 \).

First, let \( p \) be the zero ideal. In this case, \( \Sigma_p \) is an Artinian standard graded Gorenstein algebra over a field with homogeneous maximal ideal \( m\Sigma_p \). Therefore \( \mathcal{A}_p = 0 : \Sigma_p \cdot m^{\infty} = \Sigma_p \). In particular, \( [\Sigma_p]_\delta \neq 0 \) and \( [\Sigma_p]_i = 0 \) for \( i > \delta \) by Corollary 2.5, parts (1) and (2). In other words, \( [\Sigma_p]_\delta \) is the socle of the Gorenstein algebra \( \Sigma_p \). Thus, multiplication induces an isomorphism

\[ [\Sigma_p]_i \rightarrow \text{Hom}_{S_p}([\Sigma_p]_{\delta-i}, [\Sigma_p]_\delta). \]

As \( \Sigma_p = \mathcal{A}_\delta \), we conclude that \( \Phi_p \) is an isomorphism.

Next, let \( p \in \text{Spec}(S) \) with \( \dim S_p = 1 \). We need to show that \( \Phi_p \) is surjective. Let \( \theta \) be any element of \( \text{Hom}_{S_p}([\Sigma_p]_{\delta-i}, [\mathcal{A}_p]_\delta) \). We prove that the map \( \theta \) is multiplication by some element of \( [\mathcal{A}_p]_i \). Notice that

\[ \text{Hom}_{S_p}([\Sigma_p]_{\delta-i}, [\mathcal{A}_p]_\delta) = \text{Hom}_{S_p}((m^{\delta-i} \Sigma_p)/(m^{\delta-i+1} \Sigma_p), [\mathcal{A}_p]_\delta) \subset \text{Hom}_{S_p}((m^{\delta-i} \Sigma_p)/(m^{\delta-i+1} \Sigma_p), \mathcal{A}_p) \subset \text{Hom}_{S_p}((m^{\delta-i} \Sigma_p)/(m^{\delta-i+1} \Sigma_p), \Sigma_p), \]

where the next-to-last inclusion holds because \( m[\mathcal{A}_p]_\delta = 0 \) by Corollary 2.5, part (2).
We will prove that \( \theta \in [\text{Hom}_{\Sigma_p}(\langle m^{\delta-i}\Sigma_p/(m^{\delta-i+1}\Sigma_p), \Sigma_p \rangle)]_i \) can be lifted to a map
\[ \tilde{\theta} \in [\text{Hom}_{\Sigma_p}(\Sigma_p/(m^{\delta-i+1}\Sigma_p), \Sigma_p)]_i. \]

Any such \( \tilde{\theta} \) is induced by multiplication by an element \( \lambda \in [\Sigma_p]_i \). The element \( \lambda \) is necessarily annihilated by \( m^{\delta-i+1} \). Recall that \( A_p = 0 : \Sigma_p m^\infty \). Thus \( \lambda \) lies in \( [A_p]_i \), and therefore \( \tilde{\theta} \) and \( \theta \) are both induced by multiplication by an element \( \lambda \in [A_p]_i \).

To show that \( \theta \) can be lifted, we first apply \( \text{Hom}_{\Sigma_p}(-, \Sigma_p) \) to the short exact sequence of graded \( \Sigma_p \)-modules,
\[ 0 \to (m^{\delta-i}\Sigma_p)/(m^{\delta-i+1}\Sigma_p) \to \Sigma_p/(m^{\delta-i+1}\Sigma_p) \to \Sigma_p/(m^{\delta-i}\Sigma_p) \to 0. \]

The corresponding long exact sequence of cohomology induces the following exact sequence of \( S_p \)-modules
\[ [\text{Hom}_{\Sigma_p}(\Sigma_p/(m^{\delta-i+1}\Sigma_p), \Sigma_p)]_i \to [\text{Hom}_{\Sigma_p}(\langle m^{\delta-i}\Sigma_p/(m^{\delta-i+1}\Sigma_p), \Sigma_p \rangle)]_i \to [\text{Ext}^1_{\Sigma_p}(\Sigma_p/(m^{\delta-i}\Sigma_p), \Sigma_p)]_i. \]

It suffices to prove that \([\text{Ext}^1_{\Sigma_p}(\Sigma_p/(m^{\delta-i}\Sigma_p), \Sigma_p)]_i = 0\). To do so, recall that \( \Sigma_p = \text{Sym}((x, y)/(g_1, g_2)) \), where \( g_1, g_2 \) is a regular sequence of forms of degrees \( d_1, d_2 \). Therefore,
\[ \omega_{\Sigma_p} \simeq \Sigma_p(-2 + d_1 + d_2) = \Sigma_p(-d). \]

It follows that
\[ [\text{Ext}^1_{\Sigma_p}(\Sigma_p/(m^{\delta-i}\Sigma_p), \Sigma_p)]_i \simeq [\text{Ext}^1_{\Sigma_p}(\Sigma_p/(m^{\delta-i}\Sigma_p), \omega_{\Sigma_p})]_{i-d}. \]

As \( \dim \Sigma_p = 1 \), local duality implies that the latter module vanishes if and only if
\[ [H^0_{\mathfrak{M}}(\Sigma_p/(m^{\delta-i}\Sigma_p))]_{\delta - i} = 0, \]
where \( \mathfrak{M} \) denotes the homogeneous maximal ideal of \( \Sigma_p \). To finish, we note that
\[ [H^0_{\mathfrak{M}}(\Sigma_p/(m^{\delta-i}\Sigma_p))]_{\delta - i} \subset [\Sigma_p/(m^{\delta-i}\Sigma_p)]_{\delta - i} = 0. \]

\( \square \)

**Corollary 2.14.** Adopt Data 2.1. For each integer \( i \), the \( \text{Sym}(I) \)-ideals \( A_{\geq i} \) and \( 0: \text{Sym}(I) m^{d-1-i} \) are equal.

**Proof.** Assume first that \( d - 1 \leq i \). In this case, \( A_{\geq i} = 0 \) by Corollary 2.5, part (2). On the other hand, in this case, \( d - 1 - i \leq 0 \); so, \( m^{d-1-i} = R \) and \( 0: \text{Sym}(I) m^{d-1-i} = 0: \text{Sym}(I) R = 0 \).

Now assume that \( i \leq 0 \). In this case, \( A_{\geq i} = A \). On the other hand, in this case, \( d - 1 \leq d - 1 - i \); thus, \( A m^{d-1-i} \subset A_{d-1-i} \subset A_{d-1} = 0 \) and
\[ A \subset 0: \text{Sym}(I) m^{d-1-i} \subset 0: \text{Sym}(I) m^\infty = A. \]

Finally, assume that \( 1 \leq i \leq d - 2 \). In this case, \( m^{d-1-i} A_{\geq i} \subset A_{d-1} = A_{d-1} = 0 \), where the last equality holds by part (2) of Corollary 2.5. It follows that \( A_{\geq i} \subset 0: \text{Sym}(I) m^{d-1-i} \). To see the other
inclusion, let $\lambda \neq 0$ be a homogeneous element in $0 : \text{Sym}(I) \, m^{d-1-i}$. Clearly $\lambda \in 0 : \text{Sym}(I) \, m^w = A$. We prove that $\lambda$ has degree at least $i$. Suppose otherwise. In this case, we observe that $\lambda [\text{Sym}(I)]_{d-2-\deg \lambda} \subset \lambda m^{d-2-\deg \lambda} \text{Sym}(I) \subset \lambda m^{d-1-i} \text{Sym}(I) = 0,$

where the last inclusion holds because $d - 2 - \deg \lambda \geq d - 1 - i$. This shows that $\lambda$ is a non-zero homogeneous element with the property that multiplication by $\lambda$ induces the zero homomorphism from $[\text{Sym}(I)]_{d-2-\deg \lambda}$ to $A_{d-2}$. This contradicts the injectivity of the isomorphism established in Theorem 2.13.

2.D \hspace{1em} \text{Explicit $S$-module generators for $A_i$, when $i$ is large.}

When $i$ is chosen so that $g_1$ and $g_2$ lie in $m^{d-1-i}B$, then the equality of Corollary 2.14 has an interpretation in terms of linkage. In the present subsection we exploit that interpretation in order to exhibit explicit generators. Hong, Simis, and Vasconcelos [17, Sect. 3] have used linkage in a similar manner.

**Definition 2.15.** For each positive index $\ell$, define $\Lambda_\ell$ to be the $(\ell + 1) \times \ell$ matrix

$$
\Lambda_\ell = \begin{bmatrix}
-x & 0 & 0 & \cdots & 0 & 0 \\
y & -x & 0 & \cdots & 0 & 0 \\
0 & y & \cdots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & -x \\
0 & 0 & 0 & \cdots & 0 & y
\end{bmatrix}.
$$

Notice that $\Lambda_\ell$ is a Hilbert-Burch matrix for the row vector $[y^\ell \ xy^{\ell-1} \ \cdots \ x^\ell]$. 

If $\ell$ is an index with $1 \leq \ell \leq d_1$, then the polynomials $g_1, g_2$ of Data 2.1 are both in $(x, y)^\ell B$. Let $\Xi_\ell$ be an $(\ell + 1) \times 2$ matrix of bi-homogeneous elements of $B$, of bi-degree $(d_1 - \ell, 1)$ in column 1 and bi-degree $(d_2 - \ell, 1)$ in column 2, with

$$
[\begin{array}{c}
g_1 \\
g_2
\end{array}] = [y^\ell \ xy^{\ell-1} \ \cdots \ x^\ell] \Xi_\ell
$$

and $\Psi_\ell$ be the $(\ell + 1) \times (\ell + 2)$ matrix

$$
\Psi_\ell = [\Lambda_\ell \ \Xi_\ell]
$$

of bi-homogeneous elements of $B$.

**Corollary 2.16.** Adopt Data 2.1. Let $i$ be an index with $d_2 - 1 \leq i \leq \delta$ and $\Psi_{d-1-i}$ be a matrix as described in (2.15.2).

1. The ideals

$$
A_{2i}, \quad A_{(i,2)} \text{Sym}(I), \quad \text{and} \quad I_{d-i}(\Psi_{d-1-i}) \text{Sym}(I)
$$

of $\text{Sym}(I)$ are equal. In particular, the minimal bi-homogeneous generators of the $\text{Sym}(I)$-ideal $A$ all have $\{x, y\}$-degree at most $d_2 - 1$. 

(2) The $S$-module $\mathcal{A}_i$ is free of rank $d - i - 1$; a basis consists of the maximal minors of the matrix $\Psi_{d-1-i}$ that involve the last two columns.

Note. The bound on the generator degrees that is given in item (1) is also given in [3, Thm. 4.6].

Proof. Recall that $\mathcal{A}_{\ge i} = 0 :_{\text{Sym}(I)} m^{d-1-i}$ by Corollary 2.14. The latter ideal equals

$$\frac{(g_1, g_2)_B :_B m^{d-1-i}}{(g_1, g_2)_B}.$$ 

The homogeneous $B$-ideal $m^{d-1-i}B$ is perfect of grade 2 and it contains the homogeneous $B$-regular sequence $g_1, g_2$ because the hypothesis $d_2 - 1 \le i \le \delta$ guarantees that

$$(2.16.1) \quad 1 \le d - 1 - i \le d_1.$$ 

The matrix $\Lambda_{d-1-i}$ of Definition 2.15 is a homogeneous matrix of relations among the generators $y^{d-1-i}, xy^{d-2-i}, \ldots, x^{d-1-i}$ of $m^{d-1-i}B$, and $\Xi_{d-1-i}$ is a homogeneous matrix of coefficients when writing $g_1, g_2$ in terms of these generators. The inequalities of (2.16.1) guarantee that the matrix $\Xi_{d-1-i}$ of (2.15.1) is defined. Now, according to [10], the linked ideal $\frac{(g_1, g_2)_B :_B m^{d-1-i}}{(g_1, g_2)_B}$ is generated by the maximal minors of the $d - i$ by $d - i + 1$ matrix $\Psi_{d-1-i} = [\Lambda_{d-1-i} \Xi_{d-1-i}]$. Thus indeed,

$$\frac{(g_1, g_2)_B :_B m^{d-1-i}}{(g_1, g_2)_B} = I_{d-i}(\Psi_{d-1-i})B = I_{d-i}(\Psi_{d-1-i})\text{Sym}(I).$$ 

The maximal minors of $\Psi_{d-1-i}$ are $g_1, g_2$ (up to sign), together with the minors $\Delta_1, \ldots, \Delta_{d-1-i}$ that involve the last two columns. We conclude that

$$\mathcal{A}_{\ge i} = (\Delta_1, \ldots, \Delta_{d-1-i})\text{Sym}(I)$$

Observe that each $\Delta_i$ has bi-degree $(i, 2)$. This completes the proof of (1).

Assertion (2) follows from Theorem 2.7, part (3), and the fact that the elements $\Delta_1, \ldots, \Delta_{d-1-i}$ generate $\mathcal{A}_i$ as an $S$-module.

Remark 2.17. Corollary 2.16 describes $\mathcal{A}_{i_0}$ for each $i_0$ with $d_2 - 1 \le i_0 \le \delta$. We highlight the content of Corollary 2.16 at the boundaries $i_0 = d_2 - 1$ and $i_0 = \delta$.

Take $i_0 = \delta$. In this situation, the $S$-module $\mathcal{A}_\delta$ is free of rank 1 with basis element any Sylvester form “syl”, where syl is the image in $\text{Sym}(I)$ of the determinant of any fixed $2 \times 2$ matrix $\Xi_1$ described in (2.15.1). (The $k$-vector space $\mathcal{A}_{(\delta, 2)}$ is one-dimensional and we use the name “syl” for any basis element of this vector space.) Notice that the entries of column $m$ of $\Xi_1$ are homogeneous of bi-degree $(d_m - 1, 1)$ and syl is homogeneous of bi-degree $(\delta, 2)$. Let $i$ be arbitrary. One explicit realization of the isomorphism

$$\mathcal{A}_i \simeq \text{Hom}_S(\text{Sym}(I)_{\delta-i}, S(-2))$$

of Corollary 2.5 is that this isomorphism is induced by the composition

$$(2.17.1) \quad \mathcal{A}_i \otimes \text{Sym}(I)_{\delta-i} \xrightarrow{\text{mult}} \mathcal{A}^\delta \xrightarrow{\sigma} S(-2),$$

where $\sigma$ is the isomorphism induced by the isomorphism

$$(2.18.1) \quad \frac{(g_1, g_2)_B :_B m^{d-1-i}}{(g_1, g_2)_B} = I_{d-i}(\Psi_{d-1-i})B = I_{d-i}(\Psi_{d-1-i})\text{Sym}(I).$$
where mult is the multiplication in Sym(I), as described in Theorem 2.13, and σ is the inverse of the isomorphism $S(-2) \to \mathcal{A}_0$ which sends 1 to syl.

Take $i_0 = d_2 - 1$. Corollary 2.16 guarantees that $A_{\geq d_2 - 1}$ is generated as a $B$-module by the maximal minors of $\Psi_{d_1} = [\Lambda_{d_1}, \Xi_{d_1}]$ which involve both columns of $\Xi_{d_1}$. Each of these minors is homogeneous of bi-degree $(d_2 - 1, 2)$, and one choice for $\Xi_{d_1}$ is

$$
\begin{bmatrix}
  c_{0,1} & c_{0,2}x^{d_2-d_1} + \cdots + c_{d_2-d_1,2}x^{d_2-d_1} \\
  c_{1,1} & c_{d_2-d_1+1,2}x^{d_2-d_1} \\
  \vdots & \vdots \\
  c_{d_1,1} & c_{d_1,2}x^{d_2-d_1}
\end{bmatrix},
$$

for $c_{\ell,m}$ as described in (2.6.1).

3. The case of a generalized zero in the first column of $\varphi$.

Data 3.1. Adopt Data 2.1 with $d_1 < d_2$. Assume that there is a generalized zero in the first column of $\varphi$.

In this section we are in the situation of Data 3.1 and we describe $A_{\geq d_1 - 1}$ as an $S$-module and as a $B$-module. The hypothesis that there is a generalized zero in the first column of $\varphi$ has geometric significance. Indeed, as described in Remark 2.9, the homogeneous minimal generators of $I$ describe a morphism $\eta : \mathbb{P}^1_k \to \mathbb{P}^2_k$ whose image is a rational curve $C$. If the morphism $\eta$ is birational onto $C$ (or equivalently, if the curve $C$ has degree $d$), then the assumption that $\varphi$ has a generalized zero in the first column is equivalent to the assumption that $C$ has a singularity of multiplicity $d_2$; see the General Lemma in [7, 1.9]; or [28, Thm. 3] or [6, Thm. 1].

The $S$-module structure of $A_{\geq d_1 - 1}$ is completely described in Theorem 3.3: $A_{\geq d_1 - 1}$ is a free $S$-module of finite rank and a complete list of the bi-degrees of an $S$-module basis for $A_{\geq d_1 - 1}$ is given; see also Table 3.5.1. The $B$-module structure of $A_{\geq d_1 - 1}$ is described in Corollary 3.10.

The part of Sym(I) that corresponds to $A_{\geq d_1 - 1}$, under the duality of Theorem 2.4, is Sym(I)$_{\leq d_2 - 1}$. There is no contribution from $g_2$ to the $S$-module Sym(I)$_{\leq d_2 - 1}$ in the bi-homogeneous $B$-resolution of Sym(I). So, basically, we may ignore $g_2$ in the present section. Furthermore, the hypothesis that the first column of $\varphi$ has a generalized zero allows us to make the critical calculation over a subring $U$ of $S$, where $U$ is a polynomial ring in two variables. In the proof of Theorem 3.3, we decompose various bi-graded complexes over $R \otimes_k U$ into their $R$-graded components and their $U$-graded components. Ultimately, the critical calculation is to produce a lower bound for the degrees of the syzygies of a $U$-module homomorphism.

Lemma 3.2 is a statement about $U$-module homomorphisms. This lemma explains how, sometimes, lower bounds for the degrees of syzygies suffice to determine these degrees. The statement of Lemma 3.2 may be deduced from a classification of matrices whose entries are linear forms from $U$. This classification was known by Weierstrass (in the singular case) and Kronecker (in the general case); see [11, Chapt XII]. The proof we give for Lemma 3.2 uses Hilbert series to relate the twists...
in a homogeneous resolution to the betti numbers. This technique, which is now standard, was introduced by Peskine and Szpiro [27] and was used with great success by Herzog and Kühl [13]; it was also a motivation for Boij-Söderberg theory.

**Lemma 3.2.** Let $M$ be a graded module of finite length with a linear presentation over the polynomial ring $U = k[T_1, T_2]$. If a homogeneous resolution of $M$ has the form

$$0 \longrightarrow \bigoplus_{\ell=1}^{m} U(-b_\ell) \longrightarrow U(-1)^n \longrightarrow U^{n-m} \longrightarrow M \longrightarrow 0,$$

then $\sum_{\ell=1}^{m} b_\ell = n$.

**Proof.** The Hilbert series of $M$ is $h_M(t) / (1 - t)^2$, where $h_M(t) = n - m - nt + \sum_{\ell=1}^{m} t^{b_\ell}$. Since $M$ has finite length, this series is a polynomial; hence, $(1 - t)^2$ divides $h_M(t)$. Therefore, $h'_M(1) = 0$, which gives the assertion.

**Theorem 3.3.** Adopt Data 3.1. If $d_1 - 1 \leq i \leq d_2 - 1$, then

$$A_i \simeq \bigoplus_{\ell=1}^{d_1} S(-a_\ell),$$

where

$$\left\lfloor \frac{d + d_1 - 1 - i}{d_1} \right\rfloor = a_1 \leq \cdots \leq a_{d_1} = \left\lfloor \frac{d + d_1 - 1 - i}{d_1} \right\rfloor$$

and

$$\sum_{\ell=1}^{d_1} a_\ell = d + d_1 - 1 - i.$$

**Remark 3.4.** We offer an alternate phrasing for Theorem 3.3. If $d_1 - 1 \leq i \leq d_2 - 1$ and $\alpha_i$ and $\beta_i$ are integers with

$$d + d_1 - 1 - i = \alpha_i d_1 + \beta_i \quad \text{and} \quad 0 \leq \beta_i \leq d_1 - 1,$$

then

$$A_i \simeq S(-\alpha_i)^{d_1 - \beta_i} \oplus S(-\alpha_i - 1)^{\beta_i}.$$

Of course, in this language, $\alpha_i$ is equal to $\left\lfloor \frac{d + d_1 - 1 - i}{d_1} \right\rfloor$ and $\beta_i$ is the “remainder that is obtained when $d + d_1 - 1 - i$ is divided by $d_1$”. Observe that the parameter $\alpha_i$ is always at least 2, the exponent $d_1 - \beta_i$ is positive, and the other exponent, $\beta_i$, is non-negative.

**Proof of Theorem 3.3.** The case $i = d_2 - 1$ is covered in part (3) of Corollary 2.7. Fix an integer $i$ with

$$d_1 - 1 \leq i \leq d_2 - 2.$$

We prove the following four ingredients.

(3.4.1) The $S$-module $A_i$ is free of rank $d_1$. 

Once (3.4.1) is established, then we define the shifts $a_\ell$ by $A_i \simeq \bigoplus_{\ell=1}^{d_i} S(a_\ell)$ and we prove

(3.4.2) \[ \sum_{\ell=1}^{d_i} a_\ell = d + d_1 - 1 - i, \]

(3.4.3) \[ A_{i(j)} = 0 \quad \text{for} \quad j \leq \alpha_i - 1, \quad \text{and} \]

(3.4.4) \[ \dim_k A_{i(\alpha_i)} = d_1 - \beta_i, \]

where we have used the language of Remark 3.4. Once the four ingredients have been established, then it is not difficult to complete the proof. If $\beta_i = 0$, then the result follows immediately from (3.4.1) and (3.4.4); and if $0 < \beta_i$, then one may apply the pigeon hole principle. Indeed, if (3.4.1) – (3.4.4) hold and $0 < \beta_i$, then

\[ \alpha_i + 1 \leq a_{d_1 - \beta_i + 1} \leq \cdots \leq a_{d_i} \]

and

\[ (\alpha_i + 1)\beta_i \leq a_{d_1 - \beta_i + 1} + \cdots + a_{d_1} = \sum_{\ell=1}^{d_1} a_\ell - (d_1 - \beta_i)\alpha_i = (d + d_1 - 1 - i) - (d_1 - \beta_i)\alpha_i \]

\[ = (\alpha_i d_1 + \beta_i) - (d_1 - \beta_i)\alpha_i = (\alpha_i + 1)\beta_i; \]

hence, $a_\ell = \alpha_i + 1$ for $d_1 - \beta_i + 1 \leq \ell \leq d_1$.

The assertion about the rank of $A_i$ is established in part (2) of Theorem 2.7.

The hypothesis that the first column of $\varphi$ has a generalized zero ensures that after performing row operations on $\varphi$ and renaming the generators of $S_1$, we have that $\varphi$ has the form

$\varphi = \begin{pmatrix} f_1 & \ast \\ f_2 & \ast \\ 0 & \ast \end{pmatrix}$

and $g_1$, which is equal to $[T_1, T_2, T_3]$ times the first column of $\varphi$, only involves $T_1$ and $T_2$. Indeed, $g_1 = f_1 T_1 + f_2 T_2$. The hypothesis from Data 2.1 that $I$ has height two guarantees that $f_1$ and $f_2$ are a regular sequence of forms of degree $d_1$ in $R$. At this point, we introduce the subrings $U = k[T_1, T_2]$ of $S$ and $C = R \otimes_k U$ of $R \otimes_k S = B$. Notice that $C[T_3] = B$ and $g_1$ is in $C$.

Let $K_\ast$ be the following bi-graded complex of $C[T_3]$-modules:

(3.4.5) \[ K_\ast : \quad C[T_3](-d_1, -1) \overset{g_1}{\longrightarrow} C[T_3] \longrightarrow \text{Sym}(I) \rightarrow 0. \]

The graded strand $[K_\ast]_{(\delta - i, \ast)}$ of $K_\ast$ is exact because $\delta - i \leq d_2 - 1$ by our assumption $d_1 - 1 \leq i$. Taking graded $S$-duals we obtain the complex

$\text{Hom}_S(K_\ast, S) : \quad 0 \rightarrow \text{Hom}_S(\text{Sym}(I), S) \rightarrow \text{Hom}_S(C[T_3], S) \overset{g_1}{\longrightarrow} \text{Hom}_S(C[T_3], S)(d_1, 1).$

Furthermore, the graded strand $\text{Hom}_S(K_\ast, S)_{(i, \delta)}$ is exact. Theorem 2.4 guarantees that

(3.4.6) \[ \text{Hom}_S(\text{Sym}(I), S) \simeq A(\delta, 2). \]

Observe that

(3.4.7) \[ \text{Hom}_S(C[T_3], S) \simeq \text{Hom}_S(R \otimes_k S, S) = \text{Hom}_k(R, k) \otimes_k S \simeq \text{Hom}_k(R, k) \otimes_k U \otimes_k k[T_3]. \]
Combine (3.4.6) and (3.4.7) to see that the complex $\text{Hom}_S(K_*, S)$ may be identified with

$$0 \to A(\delta, 2) \to \text{Hom}_K(R, k) \otimes_k U \otimes_k k[T_3] \xrightarrow{\text{Gr}} \text{Hom}_K(R, k)(d_1) \otimes_k U(1) \otimes_k k[T_3].$$

Since $g_1 \in R \otimes_k U$, multiplication by $g_1$ gives a bi-homogeneous $(R \otimes_k U)$-module homomorphism

$$\psi : \text{Hom}_K(R, k) \otimes_k U \xrightarrow{g_1} \text{Hom}_K(R, k)(d_1) \otimes_k U(1).$$

We focus on the kernel and cokernel of various $R$-graded and $U$-graded components of $\psi$. We have

$$\ker \psi_{i, \delta}(-2)[T_3] \simeq A_i, \quad \text{for } d_1 - 1 \leq i \leq d_2 - 2.$$  

It follows immediately that $A_i$ is free over $S$ because $\ker \psi_{i, \delta}$ is a second syzygy over $U$, a polynomial ring in 2 variables. Thus, (3.4.1) is established. Notice that $(\ker \psi)_{i, \delta} \simeq \bigoplus_{\ell=1}^{d_1} U(-a_{\ell} + 2)$. The left-most map in the complex $K_*$ of (3.4.5) is injective; consequently, a calculation similar to the one that produced (3.4.8) yields

$$(\ker \psi)_{i, \delta}(-2)[T_3] \simeq \text{Ext}_S^1(\text{Sym}(I)_{i, \delta - i}, S) \quad \text{for } d_1 - 1 \leq i \leq d_2 - 2.$$  

According to part (1.a) of Theorem 2.11, $\text{Ext}_S^1(\text{Sym}(I)_{i, \delta - i}, S)$ vanishes locally in codimension one whenever $d_1 - 1 \leq i \leq d_2 - 2$; hence, $(\ker \psi)_{i, \delta}$ has finite length as a $U$-module. Now we can apply Lemma 3.2 to $(\ker \psi)_{i, \delta}(-1)$, which has a homogeneous free $U$-resolution of the form:

$$0 \to (\ker \psi)_{i, \delta}(-1) \to \text{Hom}_K(R, k)_{i-\delta \otimes} U(1) \to \text{Hom}_K(R, k)_{i-\delta + d_1 \otimes} U \to (\ker \psi)_{i, \delta}(-1) \to 0.$$  

As $(\ker \psi)_{i, \delta}(-1) \simeq \bigoplus_{\ell=1}^{d_1} U(-a_{\ell} + 1)$ and

$$\dim_k \text{Hom}_K(R, k)_{i-\delta} = \dim_k R_{\delta-i} = \delta + 1 - i,$$

Lemma 3.2 gives that $\sum_{\ell=1}^{d_1} (a_{\ell} - 1) = \delta + 1 - i$ or equivalently, $\sum_{\ell=1}^{d_1} a_{\ell} = d + d_1 - 1 - i$. The second ingredient, (3.4.2), has been established.

We re-phrase (3.4.8) as

$$\mathcal{A}_{i, j} \simeq (\ker \psi_{i, \delta, j - 2})[T_3], \quad \text{for } d_1 - 1 \leq i \leq d_2 - 2 \text{ and all } j,$$

in order to focus on the individual components $\mathcal{A}_{i, j}$ of the free $S$-module $\mathcal{A}_i$. Each such component is a finite dimensional $k$-vector space. We see that

Claim (3.4.3) holds $\iff$ $\mathcal{A}_{i, j} = 0$ for $j \leq \left\lfloor \frac{d-1-i}{d_1} \right\rfloor$

$\iff$ $(\ker \psi)_{i, \delta, j - 2} = 0$ for $j - 2 \leq \left\lfloor \frac{d-1-i}{d_1} \right\rfloor - 2$

$\iff$ $(\ker \psi)_{(m, n)} = 0$ for $n \leq \left\lfloor \frac{1-m}{d_1} \right\rfloor - 2$.

To prove the most recent version of (3.4.3) we fix an integer $n$ and consider the homogeneous $R$-module homomorphism

$$\psi_{(n, n)} : \text{Hom}_K(R, k) \otimes_k U_n \to \text{Hom}_K(R, k)(d_1) \otimes_k U_{n+1}.$$  

Recall that $\psi$ is multiplication by $f_1 T_1 + f_2 T_2$. We choose the $k$-bases $T_1^n, \ldots, T_2^n$ and $T_1^{n+1}, \ldots, T_2^{n+1}$ for $U_n$ and $U_{n+1}$, respectively and we see that the $R$-module homomorphism $\psi_{(n, n)}$ is represented by
the \((n + 2) \times (n + 1)\) matrix
\[
\begin{bmatrix}
f_1 & 0 & 0 & \cdots & 0 \\
f_2 & f_1 & 0 & \cdots & 0 \\
0 & f_2 & f_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & f_2 & f_1 \\
0 & \cdots & 0 & 0 & f_2
\end{bmatrix}.
\]

Take the graded \(k\)-dual of \(\psi_{(-,n)}\) to obtain a homogeneous \(R\)-module homomorphism
\[
n\chi : R(-d_1) \otimes_k U_{n+1}^* \longrightarrow R \otimes_k U_n^*;
\]
given by the \((n + 1) \times (n + 2)\) matrix
\[
\begin{bmatrix}
f_1 & f_2 & 0 & \cdots & 0 \\
f_1 & f_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & f_2 & f_1 \\
0 & \cdots & 0 & 0 & f_2
\end{bmatrix},
\]
with entries in \(R\), where \((-)^*\) means \(\text{Hom}_k((-),k)\). (The name \(n\chi\) emphasizes that this map depends on \(n\); however, this map is not created as a graded strand of some other, previously named, object.) Notice that \(\psi_{(m,n)}\) is injective if and only if the graded component \(n\chi_{-m}\) of \(n\chi\) is surjective; moreover, in general,
\[
(3.4.10) \quad \dim_k \ker \psi_{(m,n)} = \dim_k \operatorname{coker} n\chi_{-m}.
\]
The Buchsbaum-Rim complex yields a homogeneous free resolution of \(\operatorname{coker} n\chi\):
\[
(3.4.11) \quad 0 \rightarrow R(-(n + 2)d_1) \rightarrow R(-d_1) \otimes_k U_{n+1}^* \xrightarrow{n\chi} R \otimes_k U_n^*;
\]
This resolution shows that the socle of \(\operatorname{coker} n\chi\) is surjective if \(-m \geq (n + 2)d_1 - 1\). However \(-m \geq (n + 2)d_1 - 1 \iff n \leq \frac{1 - m}{d_1} - 2\), and therefore Claim (3.4.3) holds.

We now compute \(\dim_k \mathcal{A}_{(i,\alpha_i)}\), as required by Claim (3.4.4). The generators of the free \(S\)-module \(\mathcal{A}_i\) all have degree at least \(\alpha_i\). Therefore, the number of minimal generators of degree \(\alpha_i\) for the \(S\)-module \(\mathcal{A}_i\) is the dimension of the \(k\)-vector space \(\mathcal{A}_{(i,\alpha_i)}\), and, by (3.4.9), the vector spaces \(\mathcal{A}_{(i,\alpha_i)}\) and \(\ker \psi_{(i-\delta,\alpha_i-2)}\) have the same dimension. In this calculation, we dig more deeply into the sequence of ideas that was used in the proof of (3.4.3). In particular, (3.4.9) and (3.4.10) give
\[
(3.4.12) \quad \dim_k \mathcal{A}_{(i,\alpha_i)} = \dim_k \ker \psi_{(i-\delta,\alpha_i-2)} = \dim_k (\operatorname{coker}(\alpha_i Z_{-\delta - i})).
\]

We again use (3.4.11) to make this computation:
\[
(3.4.13) \quad \dim_k (\operatorname{coker}(\alpha_i Z_{-\delta - i})) = \dim_k (R \otimes_k U_{\alpha_i-2}^*)_{\delta - i} - \dim_k (R(-d_1) \otimes_k U_{\alpha_i-1}^*)_{\delta - i} + \dim_k R(-\alpha_id_1)_{\delta - i}.
\]
We notice that
\[
(3.4.14) \quad (R(-d_1) \otimes_k U_{\alpha_i-1}^*)_{\delta - i} \neq 0 \quad \text{and}
\]
\[
(3.4.15) \quad R(-\alpha_id_1)_{\delta - i} = 0,
\]
for \(d_1 - 1 \leq i \leq d_2 - 2\). Indeed, the hypothesis \(i \leq d_2 - 2\) guarantees that \(0 \leq d_2 - 2 - i = \delta - d_1 - i\); hence, \(d_1 \leq \delta - i\) and (3.4.14) holds. Also, \(a - (b - 1) = \lfloor \frac{a}{b} \rfloor b\) for all positive integers \(a\) and \(b\); hence,
\[
\delta - i < d - i = (d + d_1 - 1 - i) - (d_1 - 1) \leq \left\lfloor \frac{d + d_1 - 1 - i}{d_1} \right\rfloor d_1 = a_i d_1
\]
and (3.4.15) holds. Combine (3.4.12), (3.4.13), (3.4.14), and (3.4.15) to see that for \(\delta \geq 1\) and if \(\delta > 0\), then the generators of \(B_i\) remain constant at \(\delta - 1\) and the generators of \(B_i\) live in two degrees, or, occasionally, only one degree. As one looks from right to left, one free rank one summand of \(\text{lowest}\) shift in \(A_i\) is replaced by a free rank one summand with shift \(\text{one}\) higher in \(A_{i-1}\).

One continues this pattern all the way until the left boundary of \(A_{\geq d_1 - 1}\); namely \(i = d_1 - 1\). The shifts in \(A_{d_1 - 1}\) are given by \(\left\lfloor \frac{d}{d_1} \right\rfloor : (d_1 - r)\) and \((\left\lfloor \frac{d}{d_1} \right\rfloor + 1)\) : \(r\) where \(r\) is the remainder of \(d\) upon division by \(d_1\); that is, \(d = d_1 \left\lfloor \frac{d}{d_1} \right\rfloor + r\), with \(0 \leq r \leq d_1 - 1\), and
\[
A_{d_1 - 1} = S(\left\lfloor \frac{d}{d_1} \right\rfloor) d^{d_1 - r} \oplus S(\left\lfloor \frac{d}{d_1} \right\rfloor - 1)^r.
\]

Remark 3.6. The “exterior corner points” \((i, j)\) in Table 3.5.1 are very important when one considers the \(B\)-module structure of \(A_{\geq d_1 - 1}\) because degree considerations show that the corresponding basis element of the \(k\)-vector space \(A_{(i,j)}\) is part of a minimal bi-homogeneous generating set for the \(B\)-module \(A_{\geq d_1 - 1}\). For this reason we carefully record where these corner points occur. Continue to write \(d = d_1 \left\lfloor \frac{d}{d_1} \right\rfloor + r\), with \(0 \leq r \leq d_1 - 1\). Let \((i, j)\) be integers. We claim that
\[
d_1 \leq i \leq d_2 - 1 \quad \text{and} \quad A_i = S(-j)^i \oplus S(-j - 1)^{d_i - 1}
\]
The left hand inequalities $d_1 \leq \lambda d_1 + r$ and $1 \leq \lambda$ are equivalent because $0 \leq r \leq d_1 - 1$. To see that

$$\lambda d_1 + r \leq d_2 - 1 \iff \lambda \leq \left\lfloor \frac{d_1}{r} \right\rfloor - 2,$$

add $d_1 - r$ to both sides of the left hand inequality and use $d_1 + d_2 - r = d_1 \left\lfloor \frac{d_1}{r} \right\rfloor$ to see that

$$\lambda d_1 + r \leq d_2 - 1 \iff \lambda d_1 + d_1 \leq d_1 \left\lfloor \frac{d_1}{r} \right\rfloor - 1 \iff \lambda \leq \left\lfloor \frac{d_1}{r} \right\rfloor - \frac{1}{d_1} - 1 \iff \lambda \leq \left\lfloor \frac{d_1}{r} \right\rfloor - 2.$$

\[\square\]

**Remark 3.7.** In the notation of Table 3.5, we see that
(1) \( \mathcal{A}_{(d_1-1, \lceil \frac{d}{d_1} \rceil)} \) is a \( k \)-vector space of dimension \( d_1 - r \),
(2) \( \mathcal{A}_{(d_1-1, \lfloor \frac{d}{d_1} \rfloor + 1)} \) is a \( k \)-vector space of dimension \( r \), and
(3) if \( \lambda \) is an index with \( 1 \leq \lambda \leq \lfloor \frac{d}{d_1} \rfloor - 2 \), then \( \mathcal{A}_{(\lambda + 1 - r, \lceil \frac{d}{d_1} \rceil - \lambda)} \) is a \( k \)-vector space of dimension \( 1 \).

**Definition 3.8.** Let \( v_1, \ldots, v_r \) be a \( k \)-basis for \( \mathcal{A}_{(d_1-1, \lceil \frac{d}{d_1} \rceil + 1)} \); \( u_1, \ldots, u_{d_1-r} \) be a \( k \)-basis for \( \mathcal{A}_{(d_1-1, \lfloor \frac{d}{d_1} \rfloor)} \); for each \( \lambda \), with \( 1 \leq \lambda \leq \lfloor \frac{d}{d_1} \rfloor - 2 \), let \( w_\lambda \) be a \( k \)-basis for \( \mathcal{A}_{(\lambda + 1 - r, \lceil \frac{d}{d_1} \rceil - \lambda)} \).

In Corollary 3.10 we prove that the elements of Definition 3.8 form a minimal bi-homogeneous generating set for the \( B \)-module \( \mathcal{A}_{d_1-1} \). In the proof of Corollary 3.10 we appeal to the following fact about Hilbert functions that may well be known to experts.

**Lemma 3.9.** Let \( k \) be an algebraically closed field, \( R \) a positively graded \( k \)-algebra with \( \dim_k R_1 > 1 \), \( M \) a graded \( R \)-module and \( i \geq 0 \) a fixed integer. Assume that for every non-zero element \( \ell \in R_1 \) the map \( M_i \to M_{i+1} \) induced by multiplication with \( \ell \) is injective. Then for any non-zero finite dimensional \( k \)-subspace \( V \) of \( M_1 \) one has \( \dim_k R_1 V > \dim_k V \).

**Proof.** Suppose that \( \dim_k R_1 V \leq \dim_k V \). Choose two \( k \)-linearly independent elements \( x \) and \( y \) in \( R_1 \). Since \( \dim_k xV = \dim_k yV = \dim_k V \geq \dim_k R_1 V \), it follows that \( xV = R_1 V = yV \). Let \( e_1, \ldots, e_n \) be a \( k \)-basis of \( V \). One has

\[
(x e_1, \ldots, x e_n) = (y e_1, \ldots, y e_n) \cdot \Phi
\]

for some \( n \times n \) matrix \( \Phi \) with entries in \( k \). Let \( v \in k^n \) be a non-zero eigenvector of \( \Phi \) belonging to an eigenvalue \( \lambda \in k \) and write \( z = [e_1, \ldots, e_n] \cdot v \). Notice that \( z \) is a non-zero element of \( M_1 \). Multiplying equation (3.9.1) by \( v \) from the right we obtain \( xz = \lambda yz \). Thus \( (x - \lambda y)z = 0 \in M_{i+1} \). Since \( x - \lambda y \) is a non-zero element in \( R_1 \), we have a contradiction to our assumption on \( M \). \( \square \)

**Corollary 3.10.** Adopt Data 3.1. The set \( \{v_1, \ldots, v_r\} \cup \{u_1, \ldots, u_{d_1-r}\} \cup \{w_\lambda \mid 1 \leq \lambda \leq \lfloor \frac{d}{d_1} \rfloor - 2 \} \) of elements from Definition 3.8 is a minimal bi-homogeneous generating set for the \( B \)-module \( \mathcal{A}_{d_1-1} \).

**Proof.** We may harmlessly assume that \( k \) is algebraically closed. Write \( X = \{v_\delta \} \cup \{u_\mu \} \cup \{w_\lambda \} \) for the set described in the statement. First notice that no element of \( X \) is \( B \)-linear combination of the others. This is a consequence of following facts: \( \mathcal{A}_{d_1-1} \) is concentrated in \( R \)-degrees \( \geq d_1 - 1 \), \( \{v_\delta \} \cup \{u_\mu \} \) forms a homogeneous \( S \)-basis of \( \mathcal{A}_{d_1-1} \) concentrated in \( S \)-degrees \( \geq \lceil \frac{d}{d_1} \rceil \) (see Theorem 3.3), the \( w_\lambda \) have \( R \)-degrees \( \geq d_1 - 1 \) and \( S \)-degrees \( < \lfloor \frac{d}{d_1} \rfloor \), and as \( \lambda \) increases the \( R \)-degrees of the \( w_\lambda \) increase strictly whereas their \( S \)-degrees decrease strictly.

Thus it suffices to show that \( \mathcal{A}_{d_1-1} = XB \). To do so we prove by induction on \( i \geq d_1 - 1 \) that

\[
\mathcal{A}_i \subset \mathcal{A}_{i-1} + R_i + XB.
\]

Notice that from part (1) of Corollary 2.16 we have \( \mathcal{A}_{d_2} \subset \mathcal{A}_{d_2} R \). Hence we may assume that \( i \) is in the range \( d_1 - 1 \leq i \leq d_2 - 2 \). For \( i = d_1 - 1 \) equation (3.10.1) follows from Theorem 3.3. As for the induction step let \( i \geq d_1 \). From Table 3.5.1 we know that \( \mathcal{A}_{d_1} \) is a graded free \( S \)-module with \( t \)
homogeneous basis elements in degree $j$ and $d_1 - t$ homogeneous basis elements in the next higher degree $j + 1$, where $t$ is in the range $0 \leq t \leq d_1 - 1$. Write $U$ for the $k$-subspace of $\mathcal{A}_{(i-1,j)}$ spanned by the basis elements of degree $j$ and $V$ for the $k$-subspace of $\mathcal{A}_{(i-1,j+1)}$ spanned by the basis elements of degree $j + 1$. Likewise, the graded $S$-module $\mathcal{A}_i$ has $t + 1$ homogeneous basis elements of degree $j$ and $d_1 - t - 1$ homogeneous basis elements of degree $j + 1$. These basis elements span $k$-subspaces $W \subset \mathcal{A}_{(i,j)}$ and $Z \subset \mathcal{A}_{(i,j+1)}$, respectively. Notice that $W = \mathcal{A}_{(i,j)}$. Finally, write $Y$ for the $k$-subspace of $\mathcal{A}_{(i,j)}$ spanned by the elements $w_\lambda$ of bi-degree $(i, j)$. Accordingly it suffices to prove that

$$R_1U + Y = W$$

and

$$S_1W + xV = S_1W + Z.$$  

If $U = 0$ then $t = 0$ and hence $Y = W$ by definition, which shows (3.10.2) in this case. Now we turn to the case $U \neq 0$. We first argue that we may apply Lemma 3.9. Recall that the element $g_1 \in B = R[T_1, T_2, T_3]$ is of the form $T_1 f_1 + T_2 f_2 + T_3 f_3$, where $f_1, f_2, f_3$ are the entries of the first column of $\varphi$. These entries generate an $R$-ideal of height 2 because $I = I_2(\varphi)$ has height two, and therefore $g_1 = T_1 f_1 + T_2 f_2 + T_3 f_3$ is a prime element of $B$ according to [15, Theorem]. Thus $B/(g_1)$ is a domain. Since $\text{Sym}(I)_{(m, \_)} = [B/(g_1)]_{(m, \_)}$ for $m \leq d_2 - 1$ and since $i \leq d_2 - 1$, we deduce that multiplication by any linear form in $R$ induces an injective $S$-linear map from $\text{Sym}(I)_{(i-1, \_)}$ to $\text{Sym}(I)_{(i, \_)}$. Now Lemma 3.9 shows that $\dim_k R_1U \geq \dim_k U + 1$ since $U \neq 0$. On the other hand, we have $R_1U \subset \mathcal{A}_{(i,j)} = W$, $\dim_k U = t$, and $\dim_k W = t + 1$. Thus Claim (3.10.2) has been proved.

To show (3.10.3) we suppose that the containment $S_1W + xV \subset S_1W + Z$ is strict. In this case there exists a proper $k$-subspace $Z'$ of $Z$, so that

$$S_1W + xV \subset S_1W + Z'.$$

In particular, $x(S_1U + V) \subset S_1W + Z'$ and then

$$x(SU + SV) \subset SW + SZ'.$$

However, the first $S$-module is isomorphic to $\mathcal{A}_{(i-1, \_)}$ and thus has rank $d_1$, whereas the number of generators of the second $S$-module is at most

$$\dim_k W + \dim_k Z' \leq \dim_k W + \dim_k Z - 1 = d_1 - 1$$

and hence this module has rank at most $d_1 - 1$. This is a contradiction to the inclusion (3.10.4). \hfill $\square$

4. Morley forms.

Adopt Data (2.1). The multiplication map

$$\mathcal{A}_i \otimes_S \text{Sym}(I)_{8-i} \longrightarrow \mathcal{A}_s \cong S(-2)$$

is a perfect pairing of $S$-modules. This perfect pairing is the starting point for the entire theory. It was first established by Jouanolou [22, 21]. We found Busé’s description [5] of Jouanolou’s work to
be very helpful. Our proof of this perfect pairing is given in Theorem 2.13. We have already highlighted this perfect pairing in (1.0.2) and (2.3.1). The above perfect pairing induces a homogeneous isomorphism of graded $S$-modules

\[(4.0.1)\quad \mathcal{A}_i \to \text{Hom}_S(\text{Sym}(I)_{\delta-i}, S(-2)).\]

There are situations where we are able to identify an explicit basis for $\text{Hom}_S(\text{Sym}(I)_{\delta-i}, S(-2))$; see, for example, Lemmas 5.7 and 5.10. Our proof of Theorem 2.13 is highly non-constructive. On the other hand, Jouanolou’s proof is constructive. He uses Morley forms to exhibit an explicit inverse for the isomorphism (4.0.1). We summarize Jouanolou’s theory of Morley forms in the present section, and then we apply these ideas in Theorem 5.11 to exhibit an explicit generating set for $\mathcal{A}$ when, in the language of Data 2.1, $2 = d_1 < d_2$ and $\varphi$ has a generalized zero in the first column.

Most of the present section is purely expository. We include this material for the reader’s convenience and in order to put the ideas of Morley forms into the ambient notation. As far as we know, the calculation of “$q_{\beta,\delta-i-\beta}$” in part (5) of Observation 4.6 does not appear elsewhere in the literature; on the other hand, this calculation is straightforward. These “$q_{\beta,\delta-i-\beta}$” are the ingredient from the present section that is used in the proof of Theorem 5.11; see Corollary 4.5. We have calculated the “$q_{\beta,\delta-i-\beta}$” in greater generality than we use in the present paper because the calculation of these “$q_{\beta,\delta-i-\beta}$” is not the obstruction to generalizing Theorem 5.11; the obstruction is finding the appropriate generalization of Lemmas 5.7 and 5.10.

Begin with Data 2.1 and consider the ring $B \otimes_S B$. The ideal of the diagonal,

\[(x \otimes 1 - 1 \otimes x, y \otimes 1 - 1 \otimes y),\]

is the kernel of the multiplication map $B \otimes_S B \to B$. It is clear that the elements $g_j \otimes 1 - 1 \otimes g_j$ of $B \otimes_S B$ belongs to this ideal for $1 \leq j \leq 2$. Let $H$ be a $2 \times 2$ matrix with entries in $B \otimes_S B$ so that

\[(4.0.2)\quad [g_1 \otimes 1 - 1 \otimes g_1, g_2 \otimes 1 - 1 \otimes g_2] = [x \otimes 1 - 1 \otimes x, y \otimes 1 - 1 \otimes y] \cdot H.

Define $\Delta$ to be the image of $\det H$ in $\text{Sym}(I) \otimes_S \text{Sym}(I)$ under the natural map

\[B \otimes_S B \xrightarrow{\rho \otimes \rho} \text{Sym}(I) \otimes_S \text{Sym}(I),\]

where $\rho$ is the natural quotient map

\[B \longrightarrow B/(g_1, g_2) = \text{Sym}(I).\]

Notice that $\Delta$ is bi-homogeneous of bi-degree $(\delta, 2)$ in $\text{Sym}(I) \otimes_S \text{Sym}(I)$, where $x \otimes 1, 1 \otimes x, y \otimes 1, 1 \otimes y$ have bi-degree $(1, 0)$ and $T_1, T_2, T_3$ have bi-degree $(0, 1)$. The element $\Delta$ is uniquely determined up to multiplication by a unit in $k$ because $\Delta$ generates the image in $\text{Sym}(I) \otimes_S \text{Sym}(I)$ of the ideal

\[(g_1 \otimes 1 - 1 \otimes g_1, g_2 \otimes 1 - 1 \otimes g_2) : (x \otimes 1 - 1 \otimes x, y \otimes 1 - 1 \otimes y) \subset B \otimes_S B.

One may also view $\text{Sym}(I) \otimes_S \text{Sym}(I)$ as having three degrees: $x \otimes 1, y \otimes 1$ have tri-degree $(1, 0, 0)$, $1 \otimes x, 1 \otimes y$ have tri-degree $(0, 1, 0)$, and $T_1, T_2, T_3$ have tri-degree $(0, 0, 1)$. Write

\[\Delta = \sum_{i=0}^{\delta} \text{morl}_{i,\delta-i},\]
where each
\[
\text{morl}_{(i,\delta-i)} \in (\text{Sym}(I) \otimes \text{Sym}(I))_{(i,\delta-i,2)}.
\]
The tri-homogeneous elements
\[
\{\text{morl}_{(i,\delta-i)} \mid 0 \leq i \leq \delta\}
\]
are the Morley forms associated to the regular sequence \(g_1, g_2\) in \(B\).

**Observation 4.1.** Adopt 2.1 and let \(syl\) be a fixed generator for the one-dimensional vector space \(A_{(\delta,2)}\), as described in Remark 2.17. Then the following statements hold:

1. \(\text{morl}_{(0,0)} = \alpha_1 \cdot \text{syl} \otimes 1 \in \text{Sym}(I)_{\delta} \otimes \text{Sym}(I)_0\), for some unit \(\alpha_1\) in \(k\),
2. \(\text{morl}_{(0,\delta)} = 1 \otimes \alpha_2 \cdot \text{syl} \in \text{Sym}(I)_0 \otimes \text{Sym}(I)_\delta\), for some unit \(\alpha_2\) in \(k\),
3. if \(L\) is an element of \(B\), then
   \[
   (L \otimes 1 - 1 \otimes L)\Delta = 0
   \]
in \(\text{Sym}(I) \otimes S \text{Sym}(I)\), and
4. if \(b\) is an element of the \(S\)-module \(\text{Sym}(I)_\ell\), for some non-negative integer \(\ell\), then
   \[
   (b \otimes 1) \text{morl}_{(i,\delta-i)} = (1 \otimes b) \text{morl}_{(i+\ell,\delta-i-\ell)}
   \]
in \(\text{Sym}(I)_{i+\ell} \otimes S \text{Sym}(I)_{\delta-i}\).

**Proof.** To prove (1), observe that \(\text{morl}_{(0,0)}\) is equal to the image of \(\Delta\) under the natural ring surjection
\[
\text{Sym}(I) \otimes S \text{Sym}(I) \twoheadrightarrow \text{Sym}(I) \otimes \frac{S \text{Sym}(I)}{\text{mSym}(I)} = \text{Sym}(I) \otimes S = \text{Sym}(I).
\]
On the other hand, the image of (4.0.2) in \(B \otimes S B/ (B \otimes S \text{mB}) = B\) is \([g_1, g_2] = [x, y] \cdot \hat{H}\), where \(\hat{H}\) denotes the image of \(H\), and after permuting the rows of \(\hat{H}\) this becomes the equation that is used, in Remark 2.17, to define \(syl\).

The proof of (2) is completely analogous to the proof of (1). One sets \(x \otimes 1\) and \(y \otimes 1\) to zero instead of setting \(1 \otimes x\) and \(1 \otimes y\) to zero.

To show (3), notice that \(L \otimes 1 - 1 \otimes L\) belongs to the ideal of the diagonal \((x \otimes 1 - 1 \otimes x, y \otimes 1 - 1 \otimes y)\). Multiply both sides of (4.0.2) on the right by the classical adjoint of \(H\) to see that the ideal
\[
(x \otimes 1 - 1 \otimes x, y \otimes 1 - 1 \otimes y) \det H
\]
is contained in the ideal \((g_1 \otimes 1 - 1 \otimes g_1, g_2 \otimes 1 - 1 \otimes g_2)\). The second ideal is sent to zero under the homomorphism \(\rho \otimes \rho : B \otimes S B \twoheadrightarrow \text{Sym}(I) \otimes S \text{Sym}(I)\).

We now prove part (4). If \(\ell = 0\), then \(b\) is in \(S\) and there is nothing to show. If \(\ell\) is positive, then assertion (3) guarantees that \((b \otimes 1 - 1 \otimes b)\Delta = 0\). One completes the proof by examining the component of \((b \otimes 1 - 1 \otimes b)\Delta = 0\) in \(\text{Sym}(I)_{i+\ell} \otimes S \text{Sym}(I)_{\delta-i}\). \(\square\)

Now that the Morley forms have been defined, we set up the rest of the notation that is used in the statement of Theorem 4.2, where we establish that the Morley forms provide an inverse to the
isomorphism of (4.0.1). If \( u \in \text{Hom}_S(\text{Sym}(I)_\delta-i, S) \), then \( u \) induces a map
\[
\text{Sym}(I) \otimes_S \text{Sym}(I)_\delta-i \xrightarrow{1 \otimes u} \text{Sym}(I) \otimes_S S = \text{Sym}(I).
\]
When this map is applied to \( \text{morl}_{(i,\delta-i)} \in \text{Sym}(I)_i \otimes_S \text{Sym}(I)_\delta-i \), the result is
\[
(1 \otimes u)(\text{morl}_{(i,\delta-i)}) \in \text{Sym}(I)_i.
\]
It is shown in the proof of Theorem 4.2 that
\[
(1 \otimes u)(\text{morl}_{(i,\delta-i)}) \text{ actually is in } \mathcal{A}_i.
\]
Once (4.1.2) has been established, then it makes sense to define the \( S \)-module homomorphism
\[
\nu_1 : \text{Hom}_S(\text{Sym}_{\delta-i}(I), S) \longrightarrow \mathcal{A}_i
\]
by
\[
\nu_1(u) = (1 \otimes u)(\text{morl}_{(i,\delta-i)}).
\]
We also define the \( S \)-module homomorphism
\[
\nu_2 : \mathcal{A}_i \longrightarrow \text{Hom}_S(\text{Sym}(I)_{\delta-i}, S).
\]
If \( a \) is in \( \mathcal{A}_i \), then multiplication by \( a \) is an \( S \)-module homomorphism
\[
\mu_a : \text{Sym}(I)_{\delta-i} \longrightarrow \mathcal{A}_\delta.
\]
It is well-known that \( \mathcal{A}_\delta \) is the free \( S \)-module generated by any fixed Sylvester element \( \text{syl} \). (Our proof of this statement may be found in Remark 2.17.) Let \( \mu_{\text{syl}}^{-1} : \mathcal{A}_\delta \rightarrow S \) be the inverse of the isomorphism \( \mu_{\text{syl}} : S \rightarrow \mathcal{A}_\delta \). (The notation is consistent because \( \text{Sym}(I)_0 = S \).) In Remark 2.17, the homomorphism \( \mu_{\text{syl}}^{-1} \) is called \( \sigma \). If \( a \) is in \( \mathcal{A}_i \), then we define \( \nu_2(a) \) to be the homomorphism in \( \text{Hom}_S(\text{Sym}(I)_{\delta-i}, S) \) which is given by
\[
\text{Sym}(I)_{\delta-i} \xrightarrow{\mu_a} \mathcal{A}_\delta \xrightarrow{\mu_{\text{syl}}^{-1}} S.
\]
We point out that the homomorphisms \( \nu_1 \) and \( \nu_2 \) are not homogeneous; see, however, Remark 4.3.

**Theorem 4.2.** (Jouanolou [21, §3.6] and [22, §3.11]) Adopt Data 2.1 and let \( i \) be an integer with \( 0 \leq i \leq \delta \).

1. If \( u \in \text{Hom}(\text{Sym}(I)_{\delta-i}, S) \), then \( (1 \otimes u)(\text{morl}_{(i,\delta-i)}) \in \mathcal{A}_i \).
2. The homomorphisms
\[
\nu_2 : \mathcal{A}_i \longrightarrow \text{Hom}_S(\text{Sym}(I)_{\delta-i}, S) \quad \text{and} \quad \nu_1 : \text{Hom}_S(\text{Sym}(I)_{\delta-i}, S) \longrightarrow \mathcal{A}_i,
\]

as described in (4.1.4) and (4.1.3), respectively, are inverses of one another (up to multiplication by a unit of \( k \)).
Remark 4.4. Adopt Data 2.1 and let \( q \) be minimally generated by the monomials \( \alpha_1 \) and \( \alpha_2 \) with
\[
\text{morl}_{(\delta,0)} = \alpha_1 \cdot \text{syl} \otimes 1 \quad \text{and} \quad \text{morl}_{(0,\delta)} = 1 \otimes \alpha_2 \cdot \text{syl},
\]
as described in parts (1) and (2) of Observation 4.1. Apply parts (4) and (1) of Observation 4.1 to see that
\[
(b \otimes 1) \text{morl}_{(i,\delta-i)} = (1 \otimes b) \text{morl}_{(i,\delta-i)} = \alpha_1 \cdot \text{syl} \otimes b \in \text{Sym}(I)_{\delta} \otimes S \text{Sym}(I)_{\delta-i},
\]
for all \( b \) in the \( S \)-module \( \text{Sym}(I)_{\delta-i} \).

Let \( u \) be an arbitrary element of \( \text{Hom}_S(\text{Sym}(I)_{\delta-i}, S) \). Apply
\[
1 \otimes u : \text{Sym}(I) \otimes S \text{Sym}(I)_{\delta-i} \to \text{Sym}(I),
\]
as described in (4.1.1), to each side of (4.2.1) to obtain
\[
b \cdot (1 \otimes u) (\text{morl}_{(i,\delta-i)}) = \alpha_1 \cdot \text{syl} \cdot u(b) \in \mathcal{A}_\delta.
\]
Notice that this holds in particular for all \( b \) in \( R_{\delta-i} \subset \text{Sym}(I)_{\delta-i} \). Therefore,
\[
(1 \otimes u) (\text{morl}_{(i,\delta-i)}) \in \mathcal{A}_\delta : \text{Sym}(I) \text{ m}^{\delta-i} = \mathcal{A}.
\]
The last equality holds because \( \mathcal{A} = 0 : \text{Sym}(I) \text{ m}^{\infty} \); see Remark 2.2. We have established assertion (1). We have also established half of assertion (2) because we have shown that \( (v_2 \circ v_1)(u) \) sends the element \( b \) of \( \text{Sym}(I)_{\delta-i} \) to
\[
\mu_{\text{syl}}^{-1}(b \cdot v_1(u)) = \mu_{\text{syl}}^{-1}(b \cdot (1 \otimes u)(\text{morl}_{(i,\delta-i)})) = \alpha_1 \cdot \mu_{\text{syl}}^{-1}(\text{syl} \cdot u(b)) = \alpha_1 \cdot u(b);
\]
and therefore, \( v_2 \circ v_1 \) is equal to multiplication by the unit \( \alpha_1 \) on \( \text{Hom}_S(\text{Sym}(I)_{\delta-i}, S) \).

Now we prove the rest of (2). Let \( a \in \mathcal{A}_\delta \). We compute
\[
(v_1 \circ v_2)(a) = (1 \otimes v_2(a))(\text{morl}_{(i,\delta-i)})
\]
\[
= (1 \otimes \mu_{\text{syl}}^{-1} \circ \mu_a)(\text{morl}_{(i,\delta-i)})
\]
\[
= ((1 \otimes \mu_{\text{syl}}^{-1}) \circ (1 \otimes \mu_a))(\text{morl}_{(i,\delta-i)})
\]
\[
= (1 \otimes \mu_{\text{syl}}^{-1})(1 \otimes a) \text{morl}_{(i,\delta-i)}
\]
\[
= (1 \otimes \mu_{\text{syl}}^{-1})(a \otimes \alpha_2 \otimes \text{syl} = \alpha_2 \cdot \text{syl}) \quad \text{by Observation 4.1, part (4)}
\]
\[
= (1 \otimes \mu_{\text{syl}}^{-1})(a \otimes \alpha_2 \otimes \text{syl}) = \alpha_2 \cdot \text{syl} \quad \text{by Observation 4.1, part (2)}.
\]

\[\square\]

Remark 4.3. If one replaces \( S \) by \( S(-2) \) in part (2) of Theorem 4.2, then the isomorphism of \( S \)-modules \( v_2 \) becomes homogeneous and hence does \( v_1 \).

Remark 4.4. Adopt Data 2.1 and let \( i \) be an integer with \( 0 \leq i \leq \delta \). As an \( S \)-module \( \text{Sym}(I)_{\delta-i} \) is minimally generated by the monomials \( x^\beta y^{\delta-i-\beta} \), with \( 0 \leq \beta \leq \delta-i \). Thus, there are elements \( q_{\beta,\delta-i-\beta} \) in \( \text{Sym}(I)_i \), with \( 0 \leq \beta \leq \delta-i \), so that
\[
\text{morl}_{(i,\delta-i)} = \sum_{\beta=0}^{\delta-i} q_{\beta,\delta-i-\beta} \otimes x^\beta y^{\delta-i-\beta} \quad \text{in} \quad \text{Sym}(I)_i \otimes S \text{Sym}(I)_{\delta-i}.
\]
Furthermore, if \( u \in \text{Hom}_S(\text{Sym}(I)^{\delta-i}, S) \), then
\[
v_1(u) = (1 \otimes u)(\text{morl}_{(i,\delta-i)}) = \sum_{\beta=0}^{\delta-i} q_{\beta,\delta-i-\beta} \cdot u(x^\beta y^{\delta-i-\beta}).
\]

**Corollary 4.5.** Retain the notation and hypotheses of Remark 4.4 with \( d_1 - 1 \leq i \leq d_2 - 2 \). Recall the exact sequence
\[
(4.5.1) \quad 0 \to \text{Hom}_S(\text{Sym}(I)^{\delta-i}, S) \to S^{\delta-i+1} \xrightarrow{\Upsilon_{d_2-i,1}^T} S(1)^{d_2-i-1} \to 0,
\]
from part (2) of Theorem 2.7. If \( \chi \) is an element of \( S^{\delta-i+1} \) with \( \Upsilon_{d_2-i,1}^T \cdot \chi = 0 \), then \( \chi \) represents an element \( u_{\chi} \) of \( \text{Hom}_S(\text{Sym}(I)^{\delta-i}, S) \) and
\[
v_1(u_{\chi}) = [q_{0,\delta-i} \cdots q_{\delta-i,0}] \cdot \chi.
\]

**Proof.** From part (2) of Theorem 2.7 we have an exact sequence
\[
0 \to S^{d_2-i-1}(-1) \xrightarrow{\Upsilon_{d_2-i,1}^T} S^{\delta-i+1} \to \text{Sym}(I)^{\delta-i} \to 0,
\]
where \( S^{\delta-i+1} \simeq B_{\delta-i} := B(\delta-i,1) \) is considered with the \( S \)-basis \( y_1^{\delta-i}, \ldots, y^{\delta-i} \). Apply \( \text{Hom}_S(\_, S) \) to this sequence and use Remark 4.4. \( \square \)

An explicit formula for each \( q_{\beta,\delta-i-\beta} \) is given in part (5) of the following Observation.

**Observation 4.6.** Adopt Data 2.1. Statements (1) – (5) below hold; (1) – (4) take place in \( B \otimes_S B \) and (5) takes place in \( \text{Sym}(I) \otimes_S \text{Sym}(I) \).

1. If \( a \) and \( b \) are non-negative integers, then
\[
x^a y^b \otimes 1 - 1 \otimes x^a y^b = (x \otimes 1 - 1 \otimes x) \sum_{\beta=0}^{a-1} x^{a-1-\beta} \otimes x^\beta y^b + (y \otimes 1 - 1 \otimes y) \sum_{\gamma=0}^{b-1} x^a y^{b-1-\gamma} \otimes y^\gamma.
\]

2. If \( g = \sum_{\ell=0}^d c_{\ell} x^\ell y^{d-\ell} \) is an element of \( B \), with each \( c_{\ell} \) in \( S \), then \( g \otimes 1 - 1 \otimes g \) is equal to
\[
(x \otimes 1 - 1 \otimes x) \left( \sum_{\ell=0}^d \sum_{\beta=0}^{\ell-1} c_{\ell} x^{\ell-1-\beta} \otimes x^\beta y^{d-\ell} \right) + (y \otimes 1 - 1 \otimes y) \left( \sum_{\lambda=0}^d \sum_{\gamma=0}^{d-\lambda-1} c_{\lambda} x^\lambda y^{d-\lambda-1-\gamma} \otimes y^\gamma \right).
\]

3. If \( g_1 = \sum_{\ell=0}^{d_1} c_{\ell,1} x^\ell y^{d_1-\ell} \) and \( g_2 = \sum_{\ell=0}^{d_2} c_{\ell,2} x^\ell y^{d_2-\ell} \), then
\[
\begin{bmatrix} g_1 \otimes 1 - 1 \otimes g_1 \quad g_2 \otimes 1 - 1 \otimes g_2 \end{bmatrix} = \begin{bmatrix} x \otimes 1 - 1 \otimes x \quad y \otimes 1 - 1 \otimes y \end{bmatrix} H,
\]
for
\[
H = \begin{bmatrix}
\sum_{\ell=0}^{d_1} \sum_{\beta=0}^{\ell-1} c_{\ell,1} x^{\ell-1-\beta} \otimes x^\beta y^{d_1-\ell} & \sum_{\ell=0}^{d_1} \sum_{\beta=0}^{\ell-1} c_{\ell,2} x^{\ell-1-\beta} \otimes x^\beta y^{d_2-\ell} \\
\sum_{\lambda=0}^{d_1-\lambda-1} \sum_{\gamma=0}^{\lambda} c_{\lambda,1} x^\lambda y^{d_1-\lambda-1-\gamma} \otimes y^\gamma & \sum_{\lambda=0}^{d_2-\lambda-1} \sum_{\gamma=0}^{\lambda} c_{\lambda,2} x^\lambda y^{d_2-\lambda-1-\gamma} \otimes y^\gamma
\end{bmatrix}.
\]
(4) If $H$ is the matrix of (3), then the determinant of $H$ is equal to

$$\delta \sum_{i=0}^{\delta-i} \sum_{m=0}^{\delta-i} \sum_{w=0}^{\delta-i} \left[ \sum_{(\ell,m) \in \mathcal{S}_1} c_{\ell,1}c_{m,2} - \sum_{(\ell,m) \in \mathcal{S}_2} c_{m,1}c_{\ell,2} \right] x^w y^{i-w} \otimes x^y y^\delta-i-\beta,$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are the following sets of pairs of non-negative integers:

$$\mathcal{S}_1 = \left\{ (\ell, m) \mid \ell + m = w + 1 + \beta, \beta + 1 \leq \ell \leq \lambda, 0 \leq m \leq \delta - i - 1 + w \right\} \quad \text{and} \quad \mathcal{S}_2 = \left\{ (\ell, m) \mid \ell + m = w + 1 + \beta, \beta + 1 \leq \ell \leq \lambda, 0 \leq m \leq \delta - i - 1 + w \right\}.$$

(5) In the language of Remark 4.4, once $i$ is fixed with $0 \leq i \leq \delta$, then the Morley form $\text{morl}(i, \delta - i)$ is equal to $\sum_{\beta=0}^{\delta-i} q_{\beta, \delta-i-\beta} \otimes x^y y^\delta-i-\beta$ in $\text{Sym}(I)_i \otimes_S \text{Sym}(I)_{\delta-i}$ with

$$q_{\beta, \delta-i-\beta} = \sum_{w=0}^{\delta-i} \left[ \sum_{(\ell,m) \in \mathcal{S}_1} c_{\ell,1}c_{m,2} - \sum_{(\ell,m) \in \mathcal{S}_2} c_{m,1}c_{\ell,2} \right] x^w y^{i-w} \in \text{Sym}(I)_i,$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are the sets of (4.61).

**Proof.** Assertions (1) – (3) are straightforward calculations. We prove (4). Once (4) is established, then (5) follows immediately because the image of $\text{det} H$ in $\text{Sym}(I) \otimes_S \text{Sym}(I)$ is equal to

$$\sum_{i=0}^{\delta} \text{morl}(i, \delta - i) = \sum_{i=0}^{\delta-i} q_{\beta, \delta-i-\beta} \otimes x^y y^\delta-i-\beta.$$

We calculate $\text{det} H = A - B$ with

$$A = \left( \sum_{\beta=0}^{d_1} \sum_{\ell=0}^{d_1} c_{\ell,1}x^{\ell-1-\beta} \otimes x^y d_1 - \ell \right) \left( \sum_{\lambda=0}^{d_2-1} \sum_{\gamma=0}^{d_2-1} c_{\lambda,2}x^\lambda y^{d_2 - \lambda - 1 - \gamma} \otimes y^\gamma \right),$$

$$B = \left( \sum_{\beta=0}^{d_1} \sum_{\ell=0}^{d_1} c_{\ell,1}x^{\ell-1-\beta} \otimes x^y d_1 - \ell \right) \left( \sum_{\lambda=0}^{d_2-1} \sum_{\gamma=0}^{d_2-1} c_{\lambda,2}x^\lambda y^{d_2 - \lambda - 1 - \gamma} \otimes y^\gamma \right).$$

We put the summation signs that involve $\beta$ on the left in order to see that

$$A = \sum_{\beta=0}^{d_1-1} \sum_{\ell=0}^{d_1} \sum_{\lambda=0}^{d_1} \sum_{\gamma=0}^{d_1} c_{\ell,1}c_{\lambda,2}x^{\ell-1-\beta}y^{d_1 - \ell - \lambda - 1 - \gamma} \otimes x^y d_1 - \ell + \gamma,$$

$$B = \sum_{\beta=0}^{d_2-1} \sum_{\ell=0}^{d_2} \sum_{\lambda=0}^{d_2} \sum_{\gamma=0}^{d_2} c_{\ell,1}c_{\lambda,2}x^{\ell-1-\beta}y^{d_1 - \ell - \lambda - 1 - \gamma} \otimes x^y d_1 - \ell + \gamma.$$
in $A$. The analogous inequalities hold in $B$; in particular, $i$ also satisfies $0 \leq i \leq \delta$. The old constraints on $i$ now become constraints on $\lambda$ and $\ell$. It follows that

$$A = \sum_{i=0}^{\delta} \sum_{\beta=0}^{d_i-1} \sum_{\ell=0}^{d_i} \min \{d_i, \beta+i+1-\ell\} \sum_{\lambda=0}^{\delta-i} c_{\ell,1} c_{\lambda,2} x_{\ell-1-\beta+i+1-\ell-\lambda} \otimes \beta y_{\delta-i-\beta}$$

$$B = \sum_{i=0}^{\delta} \sum_{\beta=0}^{d_i-1} \sum_{\ell=0}^{d_i} \min \{d_i, \beta+i+1-\ell\} \sum_{\lambda=0}^{\delta-i} c_{\lambda,1} c_{\ell,2} x_{\ell-1-\beta+i+1-\ell-\lambda} \otimes \beta y_{\delta-i-\beta}.$$ 

In $A$, the third summation sign represents the empty sum unless $\beta \leq d_1 - 1$ and $\beta \leq \delta - i$. Thus, the following four choices for the second summation sign all yield the same value for $A$:

$$\sum_{\beta=0}^{d_1-1} \sum_{\beta=0}^{d_i-1} \sum_{\beta=0}^{\delta-i} \min \{d_1, \delta-i\}.$$ 

An analogous statement holds for $B$. We conclude that

$$A = \sum_{i=0}^{\delta} \sum_{\beta=0}^{\delta-i} \sum_{\ell=0}^{d_i-1} \min \{d_i, \beta+i+1-\ell\} \sum_{\lambda=0}^{\delta-i} c_{\ell,1} c_{\lambda,2} x_{\ell-1-\beta+i+1-\ell-\lambda} \otimes \beta y_{\delta-i-\beta}$$

$$B = \sum_{i=0}^{\delta} \sum_{\beta=0}^{\delta-i} \sum_{\ell=0}^{d_i-1} \min \{d_i, \beta+i+1-\ell\} \sum_{\lambda=0}^{\delta-i} c_{\lambda,1} c_{\ell,2} x_{\ell-1-\beta+i+1-\ell-\lambda} \otimes \beta y_{\delta-i-\beta}.$$ 

Replace $\lambda$ with $w - \ell + 1 + \beta$ to obtain

$$A = \sum_{i=0}^{\delta} \sum_{\beta=0}^{\delta-i} \sum_{\ell=0}^{d_i} \min \{d_i, \ell+1-\beta\} \sum_{\omega=\ell-1-\beta} c_{\ell,1} c_{\omega,2} x_{\ell-1-\beta+i+1-\ell-\omega} \otimes \beta y_{\delta-i-\beta}$$

$$B = \sum_{i=0}^{\delta} \sum_{\beta=0}^{\delta-i} \sum_{\ell=0}^{d_i} \min \{d_i, \ell+1-\beta\} \sum_{\omega=\ell-1-\beta} c_{\omega,1} c_{\ell,2} x_{\ell-1-\beta+i+1-\ell-\omega} \otimes \beta y_{\delta-i-\beta}.$$ 

Exchange the third and fourth summation signs. Keep in mind that $0 \leq \ell - 1 - \beta \leq w \leq i$. We see that

$$A = \sum_{i=0}^{\delta} \sum_{\beta=0}^{\delta-i} \sum_{\ell=0}^{d_i} \sum_{\omega=\ell-1-\beta} c_{\ell,1} c_{\omega,2} x_{\ell-1-\beta+i+1-\ell-\omega} \otimes \beta y_{\delta-i-\beta}$$

$$B = \sum_{i=0}^{\delta} \sum_{\beta=0}^{\delta-i} \sum_{\ell=0}^{d_i} \sum_{\omega=\ell-1-\beta} c_{\omega,1} c_{\ell,2} x_{\ell-1-\beta+i+1-\ell-\omega} \otimes \beta y_{\delta-i-\beta}.$$ 

The parameter $w$ satisfies $0 \leq w \leq i$; hence, $w + 1 + \beta - d_s \leq i - d_s + 2 + \beta$, for $s$ equal to 1 or 2, and

$$A = \sum_{i=0}^{\delta} \sum_{\beta=0}^{\delta-i} \sum_{\ell=0}^{d_i} \sum_{\omega=\ell-1-\beta} c_{\ell,1} c_{\omega,2} x_{\ell-1-\beta+i+1-\ell-\omega} \otimes \beta y_{\delta-i-\beta}$$

$$B = \sum_{i=0}^{\delta} \sum_{\beta=0}^{\delta-i} \sum_{\ell=0}^{d_i} \sum_{\omega=\ell-1-\beta} c_{\omega,1} c_{\ell,2} x_{\ell-1-\beta+i+1-\ell-\omega} \otimes \beta y_{\delta-i-\beta}.$$ 

Let $m = w - \ell + 1 + \beta$. The four constraints

$$\beta + 1 \leq \ell, \quad i - d_2 + 2 + \beta \leq \ell \leq d_1, \quad \ell \leq w + 1 + \beta$$

on $\ell$ in $A$ are equivalent to

$$\beta + 1 \leq \ell, \quad m \leq d_s - i + 1 + w, \quad \ell \leq d_1, \quad 0 \leq m.$$
respectively; and the four constraints
\[ \beta + 1 \leq \ell, \quad i - d_1 + 2 + \beta \leq \ell, \quad \ell \leq d_2, \quad \ell \leq w + 1 + \beta \]
on \ell in B are equivalent to
\[ \beta + 1 \leq \ell, \quad m \leq d_1 - i + w, \quad \ell \leq d_2, \quad 0 \leq m, \]
respectively. It follows that
\[ A = \sum_{i=0}^{\delta} \sum_{\beta=0}^{\delta-i} \sum_{w=0}^{i} c_{\ell,1} c_{m,2} x^w y^{j-w} \otimes x^\beta y^{\delta-i-\beta} \]
\[ B = \sum_{i=0}^{\delta} \sum_{\beta=0}^{\delta-i} \sum_{w=0}^{i} \sum_{(\ell,m) \in \mathfrak{S}_2} c_{m,1} c_{\ell,2} x^w y^{j-w} \otimes x^\beta y^{\delta-i-\beta}, \]
as desired. \qed

The answer of part (5) of Observation 4.6 can be simplified when \( d_1 = 2 \) and \( 1 \leq i \leq d_2 - 1 \). This hypothesis is in effect when we apply Observation 4.6 in the proof of Theorem 5.11. Once \( d_1 = 2 \), then \( \delta \) is equal to \( d_2 \) and we use \( d_2 \) in place of \( \delta \) in our simplification. If \( P \) is a statement, then define
\[ \chi(P) = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false.} \end{cases} \]

**Corollary 4.7.** If \( d_1 = 2, \ 1 \leq i \leq d_2 - 1, \) and \( 0 \leq \beta \leq d_2 - i \), then the element \( q_{\beta,d_2-i-\beta} \) of (4.6.2) is equal to
\[ q_{\beta,d_2-i-\beta} = \chi(\beta = 0) \left( \sum_{w=0}^{i} c_{1,1} c_{w,2} x^w y^{j-w} + \sum_{w=1}^{i} c_{2,1} c_{w-1,2} x^w y^{j-w} \right) + \chi(\beta = 1) \sum_{w=0}^{i} c_{2,1} c_{w,2} x^w y^{j-w} \]
\[ - \chi(\beta \leq d_2 - i - 1) \sum_{w=0}^{i} c_{0,1} c_{w+\beta,2} x^w y^{j-w} - c_{1,1} c_{i+\beta,2} x^i y^{j-1} y. \]

**Proof.** The parameter \( d_1 \) is equal to 2; so \( \delta = d_2 \). Write \( q_{\beta,d_2-i-\beta} \) from (4.6.2), as \( A + B \), where
\[ A = \sum_{w=0}^{i} \sum_{(\ell,m) \in \mathfrak{S}_1} c_{\ell,1} c_{m,2} x^w y^{j-w}, \quad \text{and} \quad B = \sum_{w=0}^{i} \sum_{(\ell,m) \in \mathfrak{S}_2} c_{m,1} c_{\ell,2} x^w y^{j-w}. \]
Since \( d_1 = 2 \), the constraint \( \beta + 1 \leq \ell \leq d_1 \) in the definition of \( \mathfrak{S}_1 \) allows only three possible values for the pair \((\beta, \ell)\); namely, \((\beta, \ell)\) is equal to \((0, 1)\), or \((0, 2)\), or \((1, 2)\). For each of these pairs, one sets \( m = w + 1 + \beta - \ell \) and then one verifies that \( 0 \leq m \leq d_2 - i + w \) becomes \( 1 \leq w \) when \((\beta, \ell) = (0, 2)\) and automatically holds otherwise. It follows that
\[ A = \chi(\beta = 0) \left( \sum_{w=0}^{i} c_{1,1} c_{w,2} x^w y^{j-w} + \sum_{w=1}^{i} c_{2,1} c_{w-1,2} x^w y^{j-w} \right) + \chi(\beta = 1) \sum_{w=0}^{i} c_{2,1} c_{w,2} x^w y^{j-w}. \]
Now we simplify \( B \). The constraint \( 0 \leq m \leq d_1 - i + w \) in the definition of \( \mathfrak{S}_2 \) becomes
\[ 0 \leq m \leq w + 1 + i, \]
when \( d_1 = 2 \). On the other hand, the parameter \( w \) in \( B \) is always at most \( i \). Thus, the pair \((w, m)\) must satisfy \( 0 \leq m \leq w + 1 - i \) and \( w \leq i \leq w + 1 \). It follows that there are only three possible values for the pair \((w, m)\), namely, \((w, m)\) is equal to \((i, 0)\) or \((i, 1)\), or \((i - 1, 0)\). Use \( \ell + m = w + 1 + \beta \) to define
ℓ. Verify that \( \beta + 1 \leq \ell \leq d_2 \) becomes \( \beta \leq d_2 - i - 1 \) when \((w, m) = (i, 0)\) and holds automatically otherwise. It follows that
\[
B = -\chi(\beta \leq d_2 - i - 1)c_{0,1}c_{i+1} + \beta c_{i+2}x^i - c_{1,1}c_{i+1} + \beta c_{i+2}x^i - c_{0,1}c_{i+1} + \beta c_{i+2}x^i - 1y.
\]

5. Explicit Generators for \( A \) when \( d_1 = 2 \).

Adopt Data (2.1) with \( d_1 = 2 \). If \( d_2 = 2 \), then the generators of \( A_{\geq 1} \) are explicitly described in Corollary 2.16. If \( \varphi \) does not have a generalized zero in the first column, then Busé [5, Prop. 3.2] gave explicit formulas for the generators of \( A_{\geq 1} \). The present section is concerned with the following situation.

**Data 5.1.** Adopt Data (2.1) with \( 2 = d_1 < d_2 \). Assume also that \( \varphi \) has a generalized zero in the first column.

Let \( C \) be the curve of Remark 2.9. We recall that if the parameterization \( \mathbb{P}^1_k \to C \) is birational, then the hypothesis that \( \varphi \) has a generalized zero in the first column is equivalent to the statement that there is a singularity of multiplicity \( d_2 \) on \( C \).

In this section we assume that Data 5.1 is in effect and we describe explicitly all of the defining equations of the Rees algebra \( R \). Of course, the results of Section 3 apply in the present section; so we know the degrees of the generators of \( A_{d_1-1} = A_{\geq 1} \), a priori, from Table 3.5.1. Indeed, in the context of the present section, Table 3.5.1 is given in Tables 5.1.1 and 5.1.2. There are two ways in which the present tables are simpler than the general table. First of all, the description of the generator degrees depends on the remainder of a division by \( d_1 \). When \( d_1 \) is 2, there are only two possible remainders: 0 or 1. Secondly, Table 3.5.1, together with Corollary 3.10, describes the degrees of the generators of \( A_{d_1-1} \) as an \( S \)-module and a \( B \)-module. Furthermore, the \( S \)-module \( A_{d_1-1} \) is free according to part (1) of Corollary 2.12. When \( d_1 = 2 \), then \( A_{d_1-1} \) is, in fact, equal to all of \( A_{\geq 1} \). At any rate, in the present section we give much more than the degrees of the generators. We give explicit formulas for the minimal generators of \( A_{\geq 1} \).

**Remark 5.2.** We notice, with significant interest, how similar Tables 5.1.1 and 5.1.2 are to the degree tables of [3, 5.6]. It appears that the tables of [3] are the transpose of the tables given here. This observation is particularly striking because there is virtually no overlap between the data of [3] and the data used here. In the present section, \( d_1 = 2 \) and \( d_2 = d_2 \); so the two parameters \( d_1 \) and \( d_2 \) are almost as far apart as possible. On the other hand, in [3], the parameters are as close as possible: \( d_1 = \lfloor \frac{d_2}{2} \rfloor \) and \( d_2 = \lceil \frac{d_2}{2} \rceil \).

We recall that the \( S \)-module \( A_0 \) is free of rank 1 and is generated in degree \((0, \deg C)\) by the implicit equation \( F(T_1, T_2, T_3) \) of the curve \( C \) of Remarks 2.9. Furthermore, \( F \) is an \( r \)-th root of the resultant of \( g_1 \) and \( g_2 \), where \( r = 1 \) if the parameterization \( \mathbb{P}^1_k \to C \) of Remark 2.9 is birational, and \( r = 2 \) otherwise. Degree considerations show that, when the parameterization \( \mathbb{P}^1_k \to C \) is birational, then \( F \)
Table 5.1.1. The generator degrees for the free $S$-module $A$, in the presence of Data 5.1, when $d$ is odd. (The elements that correspond to the generator degrees marked by $\ast$ are minimal generators for the Sym$(I)$-ideal $A$.)

| $T$-deg $d$ | $1^\ast$ | $\vdots$ | $\frac{d}{2}$ | $1^\ast$ | $2$ | $1$ | $\vdots$ | $\frac{d}{2} - 1$ | $1^\ast$ | $\vdots$ | $4$ | $\vdots$ | $3$ | $1^\ast$ | $2$ | $1$ | $2$ | $0$ | $1$ | $2$ | $3$ | $\cdots$ | $d_2 - 4$ | $d_2 - 3$ | $d_2 - 2$ | $d_2 - 1$ | $d_2$ | $xy$-deg |
|-------------|-----------|----------|-------------|-----------|-----|-----|----------|-------------|-----------|----------|-----|----------|-----|-----------|-----|-----|-----|-----|-----|----------|------------|------------|-----------|-----------|---|---------|

Table 5.1.2. The generator degrees for the free $S$-module $A$, in the presence of Data 5.1, when $d$ is even and and the morphism $P^1_k \to C$ of Remark 2.9 is birational. (The elements that correspond to the generator degrees marked by $\ast$ are minimal generators for the Sym$(I)$-ideal $A$.)

| $T$-deg $d$ | $1^\ast$ | $\vdots$ | $\frac{d}{2}$ | $2^\ast$ | $1$ | $\vdots$ | $\frac{d}{2} - 1$ | $1^\ast$ | $2$ | $\vdots$ | $4$ | $\vdots$ | $3$ | $1^\ast$ | $2$ | $1$ | $2$ | $0$ | $1$ | $2$ | $3$ | $\cdots$ | $d_2 - 4$ | $d_2 - 3$ | $d_2 - 2$ | $d_2 - 1$ | $d_2$ | $xy$-deg |

Together with the minimal generators of the Sym$(I)$-ideal $A_{\geq 1}$, form a minimal generating set for the Sym$(I)$-ideal $A$. The parameterization $P^1_k \to C$ is guaranteed to be birational when $d_2$ is odd; see [25, 4.6(1)] or [7, 0.10]. If the parameterization $P^1_k \to C$ is not birational, then one can reparameterize in order to obtain a birational parameterization. The column degrees of the Hilbert-Burch matrix $\varphi'$, which corresponds to the new parameterization, are $d'_1 = 1$ and $d'_2 = \frac{d_2}{2}$. The matrix $\varphi'$ is “almost linear”. The defining equations of the Rees algebra associated to $\varphi'$ are recorded explicitly in [24, Sect. 3]; see also [8, 2.3]. Thus we may assume in any case that the parametrization is birational or, equivalently, that the curve $C$ has degree $d$.

We first show how to modify the arbitrary Data 5.1 into data in a canonical form.
Observation 5.3. If Data 5.1 is adopted with \( k \) a field which is closed under taking square roots, then one may assume that the first column of \( \varphi \) is either \([x^2 + y^2, xy, 0]^T\) or \([y^2, x^2, 0]^T\).

Proof. Let \( \varphi_1 \) represent the first column of \( \varphi \). We may apply invertible row operations to \( \varphi \) and linear changes of variables to \( R = k[x, y] \) without changing the ideal \( I \), the symmetric algebra \( \text{Sym}(I) \), the Rees algebra \( R_\alpha \), or any other essential feature of Data 5.1. The hypothesis about the generalized zero allows us to use invertible row operations to put a zero into the bottom position of \( \varphi_1 \). The hypothesis about the height of \( I_2(\varphi) \) implies that the two remaining entries of \( \varphi_1 \) are non-zero. They each factor into a product of linear forms. If both of these entries are perfect squares, then, after a linear change of variables \((5.4.2)\) is about the height of \( I \). In the language of Definition 5.5, the matrix of \((5.4.1)\) is \( A_\ell \) with \( \ell = d_2 - i + 1 \) and the matrix of \((5.4.2)\) is \( A_\ell \) with \( \ell = d_2 - i + 1 \).

Corollary 5.4. If Data 5.1 is adopted and \( i \) is an integer with \( 1 \leq i \leq d_2 - 2 \), then the matrix \( \Upsilon_{d_2-i-1,1}^T \) of Corollary 4.5 is the \( d_2 - i - 1 \times d_2 - i + 1 \) matrix

\[
\begin{bmatrix}
T_1 & T_2 & T_1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & T_1 & T_2 & T_1
\end{bmatrix}
\]

if the first column of \( \varphi \) is \([x^2 + y^2, xy, 0]^T\).

\[
\begin{bmatrix}
T_1 & 0 & T_2 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & T_1 & 0 & T_2
\end{bmatrix}
\]

if the first column of \( \varphi \) is \([y^2, x^2, 0]^T\).

Proof. According to Definition 2.6, \( \Upsilon_{d_2-i-1,1}^T \) is the \( d_2 - i - 1 \times d_2 - i + 1 \) matrix

\[
\begin{bmatrix}
c_{0,1} & c_{1,1} & c_{2,1} & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & c_{0,1} & c_{1,1} & c_{2,1}
\end{bmatrix},
\]

where \( g_1 = c_{0,1}y^2 + c_{1,1}xy + c_{2,1}x^2 \). On the other hand,

\[
g_1 = [T_1, T_2, T_3] \begin{bmatrix}
x^2 + y^2 \\
x y \\
x 0
\end{bmatrix} = T_1(x^2 + y^2) + T_2xy \quad \text{or} \quad g_1 = [T_1, T_2, T_3] \begin{bmatrix}
y^2 \\
x^2 \\
x 0
\end{bmatrix} = T_1y^2 + T_2x^2.
\]

Our intention is to apply the technique of Corollary 4.5 when the hypotheses of Observation 5.3 are in effect. For that reason, we next find the relations on matrices like those of \((5.4.1)\) and \((5.4.2)\). In the language of Definition 5.5, the matrix of \((5.4.1)\) is \( A_\ell \) with \( \ell = d_2 - i + 1 \) and the matrix of \((5.4.2)\) is \( A_\ell \) with \( \ell = d_2 - i + 1 \).
**Definition 5.5.** For each integer \( \ell \), with \( 3 \leq \ell \), let \( A_\ell \) and \( \mathcal{A}_\ell \) be the following \((\ell - 2) \times \ell\) matrices with entries in the polynomial ring \( U = k[T_1, T_2] \):

\[
A_\ell = \begin{bmatrix}
T_1 & T_2 & T_1 & 0 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & T_1 & T_2 & T_1
\end{bmatrix}
\quad \text{and} \quad
\mathcal{A}_\ell = \begin{bmatrix}
T_1 & 0 & T_2 & 0 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & T_1 & 0 & T_2
\end{bmatrix}.
\]

In Lemmas 5.7 and 5.10 we resolve \( \text{coker} A_\ell \) and \( \text{coker} \mathcal{A}_\ell \), respectively. In each case, the answer depends on the parity of \( \ell \). We decompose each \( A_\ell \) into four pieces of approximately equal size. The relations on \( A_\ell \) are constructed from maximal minors of these smaller matrices. In fact, up to re-arrangement of the rows and columns, there are only two constituent pieces for the \( A_\ell \). We call the two primary constituent pieces \( B_k \) and \( L_k \times b \).

\[
(5.5.1) \quad \text{For each integer } k, \text{ with } 2 \leq k, \text{ let } B_k \text{ be the } (k - 1) \times k \text{ matrix } A_{k+1} \text{ with the last column removed.}
\]

For example,

\[
B_2 = \begin{bmatrix} T_1 & T_2 \end{bmatrix}, \quad \text{and} \quad B_3 = \begin{bmatrix} T_1 & T_2 & T_1 \\
0 & T_1 & T_2 \end{bmatrix}.
\]

For each pair of positive integers \( a, b \), let \( L_{a \times b} \) be the \( a \times b \) matrix with \( T_1 \) in the lower left hand corner and zero everywhere else.

\[
(5.5.2) \quad \text{If } C \text{ is a matrix, then let } C^\dagger \text{ represent the matrix that is obtained from } C \text{ by rearranging both the rows and the columns of } C \text{ in the exact opposite order.}
\]

In other words, if \( J_k \) is the \( k \times k \) matrix with

\[
(J_k)_{i,j} = \begin{cases} 
1 & \text{if } i + j = k + 1 \\
0 & \text{otherwise,}
\end{cases}
\]

and \( C \) is an \( a \times b \) matrix, then

\[
C^\dagger = J_a C J_b.
\]

For example,

\[
B_3^\dagger = \begin{bmatrix} T_2 & T_1 & 0 \\
T_1 & T_2 & T_1 \end{bmatrix},
\]

and the matrix \( L_{a \times b}^\dagger \) is the \( a \times b \) matrix with \( T_1 \) in the upper right hand corner and zero everywhere else. We find it mnemonically helpful to write \( u_{a \times b} = L_{a \times b}^\dagger \).

Observe that

\[
A_\ell = \begin{cases} 
\begin{bmatrix} B_k & L_{k-1 \times k} \\
u_{k-1 \times k} & B_k^\dagger \end{bmatrix} \quad & \text{if } \ell = 2k \\
\begin{bmatrix} B_k & L_{k-1 \times k+1} \\
u_{k \times k} & B_k^\dagger \end{bmatrix} \quad & \text{if } \ell = 2k + 1.
\end{cases}
\]

We collect a few properties of the objects we have defined.
Observation 5.6. The following statements hold.

1. The operation $\dagger$ respects matrix multiplication in the sense that $(AB)\dagger = A\dagger B\dagger$.
2. The last column of $A_{k+1}$ is $[0, \ldots, 0, T_1]^T$.
3. If the first column of $A_{k+1}$ is deleted, then the resulting matrix is $B_k^\dagger$.
4. If the first row of $B_k^\dagger$ is deleted, then the resulting matrix is $A_k$.
5. If the last row of $B_k$ is deleted, then the resulting matrix is $A_k$.
6. If $v = [v_1, \ldots, v_k]^T$ is a vector and $B_k^\dagger v = 0$, then
   a. the matrix that is obtained from $B_k^\dagger$ by deleting the first row sends $v' = [0, v_1, \ldots, v_k-1]^T$ to zero, and
   b. the matrix that is obtained from $B_k^\dagger$ by deleting the first row sends $v'' = [0, v_1, \ldots, v_k]^T$ to zero.
7. If $v = [v_1, \ldots, v_{k+1}]^T$ is a vector and $B_{k+1}^\dagger v = 0$, then the matrix that is obtained from $B_k$ by deleting the last row sends $v' = [v_{k-1}, \ldots, v_1, 0]^T$ to zero.

Proof. Assertions (1) – (5) are obvious. We prove (6). Start with $B_k^\dagger v = 0$. Apply (3) to see that

\[(5.6.1) \quad A_{k+1} \begin{bmatrix} 0 \\ v \end{bmatrix} = 0.\]

This is assertion (6.b) according to (4). The only non-zero entry in the last column of $A_{k+1}$ lives in the last row, see (2). It follows from (5.6.1) that $A_{k+1}$, with the last row and column deleted, sends $v'$ to zero. This means that $A_k v' = 0$. Now (4) implies (6.a).

We now prove (7). The only non-zero entry in the last column of $B_{k+1}^\dagger$ lives in the last row according to (3) and (2), and the matrix obtained from $B_{k+1}^\dagger$ by removing the last row and column is $B_k^\dagger$. Thus $B_k^\dagger v = 0$ implies that $B_k^\dagger v'' = 0$ with $v'' = [v_1, \ldots, v_k]^T$. Now (6.a) and (4) give $A_k v''' = 0$ with $v''' = [0, v_1, \ldots, v_{k-1}]^T$. Apply the operation $\dagger$ to $A_k v''' = 0$ and use (1). The desired conclusion follows because $A_k^\dagger = A_k$, $v''' \dagger = v'$, and $A_k$ is obtained from $B_k$ by deleting the last row according to (5).

\[\square\]

Lemma 5.7. Adopt the notation of Definition 5.5 and (5.5.1).

1. If $\ell$ is the even integer $2k$, $m_1, \ldots, m_k$ are the signed maximal order minors of $B_k$, and

\[C = \begin{bmatrix} m_1 & \ldots & m_{k-1} & m_k & 0 & -m_k & \ldots & -m_2 \\ -m_2 & \ldots & -m_k & 0 & m_k & m_{k-1} & \ldots & m_1 \end{bmatrix}^T,\]

then

\[(5.7.1) \quad 0 \to U(-k)^2 \xrightarrow{\cdot C} U(-1)^\ell \xrightarrow{A_k^{\ell-2}} U^{\ell-2} \]

is exact.
(2) If \( \ell \) is the odd integer \( 2k+1 \), \( m_1, \ldots, m_k \) are the signed maximal order minors of \( B_k, M_1, \ldots, M_{k+1} \) are the signed maximal order minors of \( B^*_{k,1} \), and

\[
C = \begin{cases}
  \left[ \begin{array}{cccc}
  m_1 & \ldots & m_{k-1} & m_k \\
  -M_{k-1} & \ldots & -M_1 & 0 \\
  1 & 0 & -1 & T_1 \\
  0 & T_1 & -T_2 \\
  \end{array} \right]^T & \text{for } 5 \leq \ell \\
  \left[ \begin{array}{ccc}
  1 & 0 & -1 \\
  0 & T_1 & -T_2 \\
  \end{array} \right]^T & \text{for } 3 = \ell,
\end{cases}
\]

then

\[
0 \to U(-k) \oplus U(-k-1) \xrightarrow{C} U(-1)^\ell \xrightarrow{A_\ell} U^{\ell-2}
\]

is exact.

**Proof.** Denote the \( i \)-th column of each matrix \( C \) by \( C_i \).

We prove (1). We first show that (5.7.1) is a complex. Separate the first column of \( C \) into two pieces:

\[
C_1 = \begin{bmatrix} t \\ b \end{bmatrix} \quad \text{with} \quad t = [m_1, \ldots, m_k]^T \quad \text{and} \quad b = [0, -m_k, \ldots, -m_2]^T.
\]

The definition of the \( m \)'s gives \( B_k t = 0 \). The product \( L_{k-1 \times k} b \) is also zero because the only non-zero entry in \( L_{k-1 \times k} \) is multiplied by zero. The product of row \( k \) from \( A_\ell \) times \( C_1 \) is \( T_1 m_k - T_1 m_k = 0 \). The last \( k-2 \) rows of \( u_{k-1 \times k} \) are zero. Apply (1) from Observation 5.6 to the equation \( B_k t = 0 \) to see that \( B^*_{k,1} \) sends \( t^i = [m_k, \ldots, m_1]^T \) to zero. It follows from (6.a) in Observation 5.6 that the last \( k-2 \) rows of \( B^*_{k,1} \) kill \( b \). (The minus signs do not cause any difficulty.) We have shown that \( A_\ell C_1 = 0 \). It follows from (1) of Observation 5.6 that \( A_\ell C_1^i = 0 \); but \( A_\ell C_1^i = C_2 \). It follows that \( A_\ell C_2 = 0 \) and therefore \( A_\ell C = 0 \) and (5.7.1) is a complex. We apply the Buchsbaum-Eisenbud criterion to show that (5.7.1) is exact. The ideal of 2 \( \times \) 2 minors of \( C \) has grade two since

\[
\begin{bmatrix} m_k & 0 \\
 0 & m_k \end{bmatrix} = T_1^{\ell-2} \quad \text{and} \quad \begin{bmatrix} m_1 & -m_2 \\
 -m_2 & m_1 \end{bmatrix} = T_2^{\ell-2} \mod (T_1).
\]

Assertion (1) has been established.

Assertion (2) is obvious when \( \ell = 3 \). Henceforth, we assume \( 5 \leq \ell \). We next show that (5.7.2) is a complex. Write

\[
C_1 = \begin{bmatrix} t \\ b \end{bmatrix} \quad \text{with} \quad t = [m_1, \ldots, m_k]^T \quad \text{and} \quad b = [0, -m_k, \ldots, -m_1]^T.
\]

The definition of the \( m \)'s ensures that \( B_k t = 0 \). The product \( L_{k-1 \times k+1} b \) is zero because the only non-zero entry in \( L_{k-1 \times k+1} \) is multiplied by zero. The product of row \( k \) of \( A_\ell \) times \( C_1 \) is \( T_1 m_k - T_1 m_k = 0 \). The last \( k-1 \) rows of \( u_{k \times k} \) are identically zero. Apply (6.b) from Observation 5.6 to \( B^*_{k,1} t^i = 0 \) in order to see that the last \( k-1 \) rows of \( B^*_{k,1} \) times \( b \) are equal to zero. (Again, the signs play no role in this part of the calculation.) We have shown that \( A_\ell C_1 = 0 \). Write

\[
C_2 = \begin{bmatrix} t \\ b \end{bmatrix} \quad \text{with} \quad t = [-M_{k-1}, \ldots, -M_1, 0]^T \quad \text{and} \quad b = [M_1, \ldots, M_{k+1}]^T.
\]
The definition of the $M$'s guarantees that $B^T_{k+1}b = 0$. The product $u_k \times k$ is zero because the only non-zero entry of $u_k \times k$ is multiplied by zero. The product of row $k - 1$ of $A_\ell$ times $C_2$ is $-T_1M_1 + T_1M_1 = 0$. The top $k - 2$ rows of $L_{k-1 \times k+1}$ are identically zero. Assertion (7) of Observation 5.6 shows that the top $k - 2$ rows of $B_k$ times $t$ are equal to zero. We have shown that $A_\ell C = 0$. Again the ideal $I_2(C)$ has grade two since

$$
\begin{vmatrix}
m_k & 0 \\
0 & M_1
\end{vmatrix} = \pm T_1^{\ell - 2} \quad \text{and} \quad 
\begin{vmatrix}
m_1 & -M_{k-1} \\
-m_1 & M_{k+1}
\end{vmatrix} \equiv \pm T_2^{\ell - 2} \mod (T_1).
$$

The complex (5.7.2) is also exact by the Buchsbaum-Eisenbud criterion.

We next resolve coker $\mathfrak{A}_\ell$ for the matrices $\mathfrak{A}_\ell$ of Definition 5.5.

**Example 5.8.** When $\ell$ is 5 or 6, the syzygy module for the matrices $\mathfrak{A}_5$ and $\mathfrak{A}_6$ are generated by the columns of $\mathcal{C}_5$ and $\mathcal{C}_6$, respectively, for

$$
\mathcal{C}_5 = \begin{bmatrix}
0 & T_2^2 \\
T_2 & 0 \\
0 & -T_1 T_2 \\
-T_1 & 0 \\
0 & T_1^2
\end{bmatrix} \quad \text{and} \quad \mathcal{C}_6 = \begin{bmatrix}
0 & T_2^2 \\
T_2 & 0 \\
0 & -T_1 T_2 \\
-T_1 T_2 & 0 \\
0 & T_1^2
\end{bmatrix},
$$

Indeed, the rows and columns of $\mathfrak{A}_5$ and $\mathfrak{A}_6$ may be rearranged to convert these matrices into the block matrices

$$
\begin{bmatrix}
T_1 & T_2 & 0 & 0 & 0 \\
0 & 0 & T_1 & T_2 & 0 \\
0 & 0 & 0 & T_1 & T_2
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
T_1 & T_2 & 0 & 0 & 0 \\
0 & T_1 & T_2 & 0 & 0 \\
0 & 0 & 0 & T_1 & T_2
\end{bmatrix},
$$

respectively. The syzygies of the constituent pieces are well understood.

In Lemma 5.10 we prove that the pattern established in Example 5.8 holds for all $\ell$ with $3 \leq \ell$. We find it convenient to let $V$ be a free module of rank two with basis $s,t$ over the polynomial ring $U = k[T_1, T_2]$. The $(\ell - 2) \times \ell$ matrix $\mathfrak{A}_\ell$ represents the composition

$$
\text{Sym}_{\ell - 1} V \longrightarrow \text{Sym}_{\ell + 1} V \longrightarrow \text{Sym}_{\ell + 1} V_{(T^{\ell+1}, st^{\ell}, s't, s'^{\ell+1})},
$$

where the first map is multiplication by $\xi = T_2 s^2 + T_1 s^2$ and the second map is the natural quotient map. The basis for $\text{Sym}_{\ell - 1} V$ is $t^{\ell - 1}, st^{\ell - 2}, \ldots, s'^{\ell - 2}t, s'^{\ell - 1}$ and the basis for $\text{Sym}_{\ell + 1} V_{(T^{\ell+1}, st^{\ell}, s't, s'^{\ell+1})}$ is $s^2t^{\ell-1}, s^3t^{\ell-2}, \ldots, s'^{\ell-1}t^2$. For each positive integer $\alpha$, let

$$
\kappa_\alpha = \sum_{a+b=\alpha} (T_2^2)^a (-T_1 s^2)^b \in \text{Sym}_{2\alpha} V.
$$

Notice that the columns of $\mathcal{C}_5$ are $sT_1 \kappa_1$ and $\kappa_2$ in $\text{Sym}_3 V$ and the columns of $\mathcal{C}_6$ are $sT_2$ and $tT_2$ in $\text{Sym}_3 V$. 

Lemma 5.9. If \( \ell \geq 3 \) is an integer, then the kernel of the composition of (5.8.1) is generated by

\[
\begin{cases}
&sf_{s+1} \text{ and } \kappa_{s+1} \text{ in } \Sym_{s-1} V \quad \text{if } \ell \text{ is odd} \\
&sk_{s+1} \text{ and } tk_{s+1} \text{ in } \Sym_{s-1} V \quad \text{if } \ell \text{ is even}.
\end{cases}
\]

Proof. Observe that

\[
\xi \cdot \kappa_{s} = (T_{2}t^{2} + T_{1}s^{2}) \sum_{a+b=s} (T_{2}t^{2})^{a}(-T_{1}s^{2})^{b} = (T_{2}t^{2})^{a+1} + (-T_{1}s^{2})^{a+1} \in \Sym_{2a+2} V.
\]

It is now clear that

\[
0 \to U(-\left(\frac{t}{2}\right)) \oplus U(-\left(\frac{s}{2}\right)) \to \Sym_{s-1} V(-1) \to \Sym_{s+1} V(\ell) \quad (5.8.1)
\]

and

\[
0 \to U(-\left(\frac{s}{2}\right))^2 \to \Sym_{s-1} V(-1) \to \Sym_{s+1} V(\ell) \quad (5.8.1)
\]

are complexes, when \( \ell \) is odd or even, respectively. Apply the Buchsbaum-Eisenbud criterion to see that these complexes are exact: the determinant of the first two rows of the matrices \( \begin{bmatrix} sf_{s+1} & \kappa_{s+1} \end{bmatrix} \) and \( \begin{bmatrix} sk_{s+1} & t\kappa_{s+1} \end{bmatrix} \) is plus or minus a power of \( T_{2} \), and the determinant of the last two rows of these matrices is plus or minus a power of \( T_{1} \).

Lemma 5.10. Adopt the notation of Definition (5.5).

1. If \( \ell \) is the even integer \( 2k \) and

\[
\mathcal{C}_{\ell} = \begin{bmatrix}
0 & T_{2}^{k-1} \\
T_{2}^{k-1} & 0 \\
0 & (-T_{1})T_{2}^{k-2} \\
(-T_{1})T_{2}^{k-2} & 0 \\
0 & (-T_{1})^{2}T_{2}^{k-3} \\
\vdots & \vdots \\
0 & (-T_{1})^{k-1} \\
(-T_{1})^{k-1} & 0
\end{bmatrix}
\]

then

\[
0 \to U(-k)^{2} \xrightarrow{\mathcal{C}_{\ell}} U(1)^{\ell} \xrightarrow{\mathcal{A}_{\ell}} U^{\ell-2}
\]

is exact.

2. If \( \ell \) is the odd integer \( 2k + 1 \) and

\[
\mathcal{C}_{\ell} = \begin{bmatrix}
0 & T_{2}^{k} \\
T_{2}^{k-1} & 0 \\
0 & (-T_{1})T_{2}^{k-1} \\
(-T_{1})T_{2}^{k-2} & 0 \\
0 & (-T_{1})^{2}T_{2}^{k-2} \\
\vdots & \vdots \\
0 & (-T_{1})^{k-1} \\
(-T_{1})^{k-1} & 0
\end{bmatrix}
\]
then

\[ 0 \rightarrow U(-k) \oplus U(-k-1) \xrightarrow{\varphi} U(-1)^\ell \xrightarrow{\pi} U^{\ell-2} \]

is exact.

**Proof.** The present Lemma is a restatement of Lemma 5.9.

We are now ready to prove the main result of this section. Adopt Data 5.1 with \( k \) a field closed under taking square roots. According to Remark 5.2, we may assume that the parametrization \( \mathbb{P}^1_k \rightarrow C \) is birational. We know from Corollary 3.10 -- see also Tables 5.1.1 and 5.1.2 -- that there exist bi-homogeneous elements \( g_{(i,j)} \in \mathcal{A}_{(i,j)} \) such that the minimal generating set of the \( B \)-module \( \mathcal{A} \) is given by (5.10.1)

\[
\{g_{(0,d_1+1)}\} \cup \left\{ g_{(i,d_2+2-i)} \right\}_{1 \leq i \leq d_2-2} \text{ and } d_2 - i \text{ is even} \cup \left\{ g_{(1,d_2+2)} \right\} \right\} \quad \text{if } d_2 \text{ is odd}
\]

\[
\{g_{(1,d_2+2)} \cup g'_{(1,d_2+2)}\} \quad \text{if } d_2 \text{ is even}.
\]

The element \( g_{(0,d_1+1)} \) is the resultant of \( g_1 \) and \( g_2 \), when these polynomials are viewed as homogeneous forms in \( S[x,y] \) of degree \( d_1 \) and \( d_2 \), respectively. In Theorem 5.11 we record explicit formulas for the rest of the \( g_{(i,j)} \) from (5.10.1). According to Observation 5.3, we may assume that the first column of \( \varphi \) has one of two forms. The linear form \( c_{w,2} \) of \( S \) is defined in (2.6.1) by the equation

\[
g_2 = \sum_{w=0}^{d_2} c_{w,2} x^w y^{d_2-w}.
\]

The results of [4] and Theorem 5.11 were obtained more or less simultaneously; indeed, [4] and this article were posted on the arXiv three days apart.

**Theorem 5.11.** Adopt Data 5.1 with \( k \) a field closed under taking square roots. Without loss of generality, the parametrization \( \mathbb{P}^1_k \rightarrow C \) is birational. We give explicit formulas for the elements of (5.10.1). The element \( g_{(i,j)} \) or \( g'_{(i,j)} \) is in \( \mathcal{A}_{(i,j)} \) and the set \( \{g_{(i,j)}, g'_{(i,j)}\} \) of (5.10.1) is a bi-homogeneous minimal generating set for the \( B \)-module \( \mathcal{A} \).

1. Assume that the first column of \( \varphi \) is \([x^2 + y^2, xy, 0]^T\).

   (a) If \( 1 \leq i \leq d_2 - 4 \) and \( d_2 - i \) is even, then

   \[
   g'_{(i,d_2+2-i)} = \left\{ \begin{array}{l}
   \sum_{w=0}^{i} T_2 c_{w,2} x^w y^{d_2-w} m_1 + \sum_{w=1}^{i} T_1 c_{w-1,2} x^w y^{d_2-w} m_1 + \sum_{w=0}^{i} T_1 c_{w,2} x^w y^{d_2-w} m_2 \\
   + T_1 x^j \left( \sum_{\beta=2}^{d_2-i} c_{d_2+2-\beta,2} m_1 - \sum_{\beta=1}^{d_2-i} c_{\beta+2,2} m_1 \right) \\
   + (T_2 x^j + T_1 x^{j-1} y) \sum_{\beta=1}^{d_2-i} (c_{d_2+1-\beta,2} - c_{\beta+2-1,2}) m_1 \\
   \end{array} \right.
   \]

   where \( m_1, \ldots, m_{d_2-i} \) are the signed maximal minors of the matrix \( B_{\frac{d_2-i}{2}} \) of (5.5.1).
(a') If $i = d_2 - 2$, then
\[
g_{(d_2-2,2)} = \begin{cases} 
\sum_{w=0}^{d_2-3} T_2 c_{w,2} x^w y^{d_2-2-w} + \sum_{w=1}^{d_2-2} T_1 c_{w-1,2} x^w y^{d_2-2-w} - T_1 c_{d_2-1,2} x^{d_2-2} \\
-T_1 c_{d_2-2,2} x^{d_2-3} y + T_2 c_{d_2,2} x^{d_2-2} + T_1 c_{d_2,2} x^{d_2-3} y.
\end{cases}
\]

(b) If $i = 1$ and $d_2$ is odd, then
\[
g_{(1,d_2+1/2)} = \begin{cases} 
+\chi(d_2 = 3)(T_1 c_{0,2} y + T_1 c_{1,2} x) M_1 \\
-\chi(5 \leq d_2) (T_2 c_{0,2} y + T_2 c_{1,2} x + T_1 c_{0,2} x) M_{d_2-1} \\
-\chi(7 \leq d_2)(T_1 c_{0,2} y + T_1 c_{1,2} x) M_{d_2-5} - \sum_{\beta=1}^{d_2-1} T_1 c_{d_2+1-\beta,2} x M_{\beta} \\
+ \sum_{\beta=1}^{\frac{d_2-1}{2}} (T_1 c_{\frac{d_2+1}{2}+\beta,2} x^2 + T_2 c_{\frac{d_2+1}{2}-\beta,2} x^2 + T_1 c_{\frac{d_2+1}{2}+\beta,2} y) M_{\beta} \\
- \sum_{\beta=1}^{\frac{d_2-1}{2}} (T_2 c_{\frac{d_2+1}{2}+\beta,2} x + T_1 c_{\frac{d_2+1}{2}+\beta,2} y) M_{\beta},
\end{cases}
\]

where $M_1, \ldots, M_{\frac{d_2+1}{2}}$ are the signed maximal order minors of the matrix $B_{\frac{d_2+1}{2}}$, and the matrix $B_{\frac{d_2+1}{2}}$ and the operation “+” are defined at (5.5.1) and (5.5.2), respectively.

(c) If $i = 1$ and $d_2$ is even, then
\[
g_{(1,d_2+3)} = \begin{cases} 
(T_2 c_{0,2} y + T_2 c_{1,2} x + T_1 c_{0,2} x) m_1 + (T_1 c_{0,2} y + T_1 c_{1,2} x) m_2 \\
- \sum_{\beta=1}^{\frac{d_2}{2}} (T_1 c_{\beta+1,2} x + T_2 c_{\beta,2} x + T_1 c_{\beta,2} y) m_{\beta} \\
+ \sum_{\beta=3}^{\frac{d_2}{2}} T_1 c_{d_2+3-\beta,2} x m_{\beta} + \sum_{\beta=2}^{\frac{d_2}{2}} (T_2 c_{d_2+2-\beta,2} x + T_1 c_{d_2+2-\beta,2} y) m_{\beta}
\end{cases}
\]

and
\[
g'_{(1,d_2+3/2)} = \begin{cases} 
-(T_2 c_{0,2} y + T_2 c_{1,2} x + T_1 c_{0,2} x) m_2 - \chi(6 \leq d_2)(T_1 c_{0,2} y + T_1 c_{1,2} x) m_3 \\
+ \sum_{\beta=2}^{\frac{d_2}{2}} (T_1 c_{\beta,2} x + T_2 c_{\beta-1,2} x + T_1 c_{\beta-1,2} y - T_1 c_{d_2-\beta+2,2} x) m_{\beta} \\
- \sum_{\beta=1}^{\frac{d_2}{2}} (T_2 c_{d_2+1-\beta,2} x + T_1 c_{d_2+1-\beta,2} y) m_{\beta},
\end{cases}
\]

where $m_1, \ldots, m_{\frac{d_2}{2}}$ are the signed maximal order minors of the matrix $B_{\frac{d_2}{2}}$ of (5.5.1).

(2) Assume that the first column of $\varphi$ is $[y^2, x^2, 0]^T$.

(a) If $1 \leq i \leq d_2 - 2$ and $d_2 - i$ is even, then
\[
g_{(i,d_2+1/2)} = \sum_{w=0}^{i} c_{w,2} x^w y^{i-w} T_2^{d_2-1} + \sum_{\lambda=1}^{\frac{d_2-i}{2}} (c_{i+2\lambda-1,2} y + c_{i+2\lambda,2} x) x^{-i} (-T_1)^{\lambda} T_2^{d_2-1-\lambda}.
\]

(b) If $i = 1$ and $d_2$ is odd, then
\[
g_{(1,d_2+3/2)} = \sum_{\lambda=1}^{\frac{d_2+1}{2}} c_{2\lambda-1,2} y (-T_1)^{\lambda} T_2^{d_2+1-\lambda} + \sum_{\lambda=0}^{\frac{d_2+1}{2}} c_{2\lambda,2} x (-T_1)^{\lambda} T_2^{d_2+1-\lambda}.
\]
(c) If \( i = 1 \) and \( d_2 \) is even, then

\[
g'(1,\frac{d_2}{2}) = \sum_{\lambda=\frac{d_2}{2}}^{d_2} c_{2\lambda-1,2} g(-T_1)^{\lambda} T_2^{d_2-\lambda} + \sum_{\lambda=0}^{\frac{d_2}{2}-1} c_{2\lambda+1,2} g(-T_1)^{\lambda} T_2^{d_2-\lambda}
\]

and

\[
g'(1,\frac{d_2}{2}) = \sum_{\lambda=0}^{\frac{d_2}{2}} c_{2\lambda+1,2} g(-T_1)^{\lambda} T_2^{d_2-\lambda} + \sum_{\lambda=0}^{\frac{d_2}{2}-1} c_{2\lambda+1,2} g(-T_1)^{\lambda} T_2^{d_2-\lambda}
\]

**Proof.** We first prove (1.a) and (1.a'). Apply the technique of Corollary 4.5. The matrix \( \Upsilon_{d_2-i-1,1}^T \) of (4.5.1) is given in (5.4.1) and this matrix is called \( A_\ell \), with \( \ell = d_2 - i + 1 \), in Definition 5.5. The kernel of \( A_\ell \) is calculated in part (2) of Lemma 5.7 because the index \( \ell = d_2 - i + 1 \) is odd. This kernel is free of rank two. The homogeneous minimal generating set for the kernel of \( A_\ell \) has one generator in each of two consecutive degrees. Let \( \chi = [\chi_0, \ldots, \chi_{\ell-1}]^T \) be the generator with the smaller degree. The technique of Corollary 4.5 and the formula (5.10.1) then give

\[
g(i,\frac{d_2-i-1}{2}) = \sum_{\beta=0}^{\ell-1} q_{\beta,d_2-i-\beta} \chi_\beta.
\]

An explicit formula for \( q_{\beta,d_2-i-\beta} \) is given in Corollary 4.7 and

\[
\chi = \begin{cases} [m_1, \ldots, m_k, 0, -m_k, \ldots, -m_1]^T & \text{if } 4 \leq d_2 - i \\ [1, 0, -1]^T & \text{if } 2 = d_2 - i \end{cases}
\]

is given in part (2) of Lemma 5.7, where \( k = \frac{\ell - 1}{2} = \frac{d_2 - i}{2} \), \( B_k \) is the matrix described in (5.5.1), and \( m_i \) is the \( i \)th signed maximal order minor of \( B_k \).

If \( i = d_2 - 2 \), then (5.11.1) yields \( g(d_2-2,2) = q_{0,2} - q_{2,0} \), with

\[
q_{0,2} = \sum_{w=0}^{d_2-3} c_{1,1} c_{w,2} x^w y^{d_2-2-w} + \sum_{w=1}^{d_2-2} c_{2,1} c_{w,1,2} x^w y^{d_2-2-w}
\]

and \( q_{2,0} = -c_{1,1} c_{d_2,2} x^{d_2-2} - c_{0,1} c_{d_2,2} x^{d_2-3} y \). The polynomial \( g_1 = c_{0,1} y^2 + c_{1,1} xy + c_{2,1} x^2 \) is equal to

\[
T_1(x^2 + y^2) + T_2 xy; \text{ so}
\]

\[
c_{0,1} = T_1, \quad c_{1,1} = T_2, \quad \text{and} \quad c_{2,1} = T_1,
\]

and the computation of (1.a') is complete.

If \( 4 \leq d_2 - i \), then (5.11.2) gives

\[
\chi_\beta = \begin{cases} m_{\beta+1} & \text{if } 0 \leq \beta \leq \frac{d_2-i}{2} - 1 \\ 0 & \text{if } \beta = \frac{d_2-i}{2} \\ -m_{d_2-i+1-\beta} & \text{if } \frac{d_2-i}{2} + 1 \leq \beta \leq d_2 - i; \end{cases}
\]

and therefore, \( g(i,\frac{d_2-i}{2}) = \theta_1 + \theta_2 \), with

\[
\theta_1 = \sum_{\beta=0}^{\frac{d_2-i}{2}-1} q_{\beta,d_2-i-\beta} m_{\beta+1} \quad \text{and} \quad \theta_2 = -\sum_{\beta=\frac{d_2-i}{2}+1}^{d_2-i} q_{\beta,d_2-i-\beta} m_{d_2-i+1-\beta}.
\]
Use Corollary 4.7 to calculate
\[
\theta_1 = \begin{cases} 
\sum_{w=0}^{i} c_{1,1} c_{w,2} x^{i-w} m_1 + i \sum_{w=1}^{2} c_{1,1} c_{w-1,2} x^{i-w} m_1 + \sum_{w=0}^{i} c_{2,1} c_{w,2} x^{i-w} m_2 \\
- \sum_{\beta=1}^{d_{2}^{i-1}} c_{0,1} c_{i+\beta,2} x^i m_\beta - \sum_{\beta=1}^{d_{2}^{i-1}} c_{1,1} c_{i+\beta-1,2} x^i m_\beta - \sum_{\beta=1}^{d_{2}^{i-1}} c_{0,1} c_{i+\beta-1,2} x^{i-1} y m_\beta
\end{cases}
\]
and
\[
\theta_2 = \sum_{\beta=1}^{d_{2}^{i-1}} c_{0,1} c_{d_2+1-\beta,2} x^i m_\beta + \sum_{\beta=1}^{d_{2}^{i-1}} c_{1,1} c_{d_2+1-\beta,2} x^i m_\beta + \sum_{\beta=1}^{d_{2}^{i-1}} c_{0,1} c_{d_2+1-\beta,2} x^{i-1} y m_\beta.
\]
Use (5.11.3) to complete the proof of (1.a).

To prove (1.b), we consider the minimal syzygy of degree \(k\)
\[
\chi = \begin{bmatrix} -M_{k-1} & \cdots & -M_1 & 0 & M_1 & \cdots & M_{k+1} \end{bmatrix}^T
\]
on the matrix \(A_{d_2}\), where \(k = \frac{d_{2}^{i-1} - 5}{2}\), as given in part (2) of Lemma 5.7. This formulation makes sense and gives
\[
\chi = \begin{bmatrix} 0 & T_1 & -T_2 \end{bmatrix}^T
\]
when \(d_2 = 3\). In other words, \(\chi = [\chi_0, \ldots, \chi_{d_2-1}]^T\), with
\[
\chi_\beta = \begin{cases} 
-M_{\beta - \frac{d_{2}^{i-1} - 3}{2}} & \text{if } 0 \leq \beta \leq \frac{d_{2}^{i-1} - 3}{2} \\
0 & \text{if } \beta = \frac{d_{2}^{i-1} - 3}{2} \\
M_{\beta + 1 - \frac{d_{2}^{i-1} - 1}{2}} & \text{if } \frac{d_{2}^{i-1} - 1}{2} \leq \beta \leq d_2 - 1.
\end{cases}
\]
The techniques of Corollary 4.5 give
\[
g_{\left(\frac{d_{2}^{i-1} + 1}{2}\right)} = \sum_{\beta=0}^{d_{2}^{i-1} - 2} q_{\beta, d_2 - 1 - \beta} \chi_\beta = -\sum_{\beta=0}^{d_{2}^{i-1} - 2} q_{\beta, d_2 - 1 - \beta} M_{\beta - \frac{d_{2}^{i-1} - 3}{2}} + \sum_{\beta=\frac{d_{2}^{i-1} - 1}{2}}^{d_{2}^{i-1} - 1} q_{\beta, d_2 - 1 - \beta} M_{\beta - \frac{d_{2}^{i-1} - 1}{2}} + 1.
\]
We apply Corollary 4.7 and (5.11.3) as we simplify this expression.

The computation of (1.c) proceeds in the same manner. One begins with the relations
\[
\chi = \begin{bmatrix} m_1 & \cdots & m_{\frac{d_{2}^{i-1} + 1}{2}} & -m_{\frac{d_{2}^{i-1} + 1}{2}} & \cdots & -m_2 \end{bmatrix}^T \text{ and } \chi' = \begin{bmatrix} -m_2 & \cdots & -m_{\frac{d_{2}^{i-1} + 1}{2}} & 0 & m_{\frac{d_{2}^{i-1} + 1}{2}} & \cdots & m_1 \end{bmatrix}^T
\]
on the matrix \(A_{d_2}\), as given in part (1) of Lemma 5.7, where the matrix \(B_{\frac{d_{2}^{i-1} + 1}{2}}\) is defined in (5.5.1) and \(m_1, \ldots, m_{d_2}\) are the maximal order minors of \(B_{\frac{d_{2}^{i-1} + 1}{2}}\). The relations \(\chi\) and \(\chi'\) are used to produce \(g_{\left(\frac{d_{2}^{i-1} + 1}{2}\right)}\) and \(g'_{\left(\frac{d_{2}^{i-1} + 1}{2}\right)}\), respectively.

Now we prove (2.a). Again we use the method of Corollary 4.5. Now the matrix \(Y_{d_2 - i - 1, 1}^T\) of (4.5.1) is given in (5.4.2) and this matrix is called \(\mathfrak{X}_{\ell}\), with \(\ell = d_2 - i + 1\), in Definition 5.5. The
relation

\[
\chi = \begin{bmatrix}
0 \\
T_2^\alpha \\
0 \\
(-T_1)T_2^{\alpha-1} \\
\vdots \\
(-T_1)T_2^{\alpha-1} \\
0
\end{bmatrix},
\]

with \( \alpha = \frac{d_2 - i - 2}{2} \), is read from part (2) of Lemma 5.10. We see that

\[
\chi_{\beta} = \begin{cases}
0 & \text{if } \beta \text{ is even} \\
(-T_1)^\lambda T_2^{\frac{d_2 - i - 2}{2} - \lambda} & \text{if } \beta = 2\lambda + 1 \text{ and } 0 \leq \lambda \leq \frac{d_2 - i - 2}{2}.
\end{cases}
\]

Use Corollary 4.5 and Corollary 4.7 to see that \( g\left(i, \frac{d_2 - i - 2}{2}\right) \) is equal to

\[
\frac{d_2 - i - 1}{2} \sum_{\lambda=0}^{d_2 - i - 1} q_{2\lambda + 1, d_2 - i - (2\lambda + 1)} \left( -T_1 \right)^\lambda T_2^{\frac{d_2 - i - 2}{2} - \lambda - 1} \\
+ \sum_{\lambda=0}^{d_2 - i - 1} \left( -c_{0,1}c_{i+2\lambda+1,2}x^i - c_{1,1}c_{i+2\lambda+1,2}x^i - c_{0,1}c_{i+2\lambda,2}x^{i-1}y \right) \left( -T_1 \right)^\lambda T_2^{\frac{d_2 - i - 2}{2} - \lambda - 1}.
\]

The polynomial \( g_1 = c_{0,1}y^2 + c_{1,1}xy + c_{2,1}x^2 \) is equal to \( T_1y^2 + T_2x^2 \); so

\( c_{0,1} = T_1, \quad c_{1,1} = 0, \quad \text{and} \quad c_{2,1} = T_2, \)

and the computation of (2.a) is complete.

The computations of (2.b) and (2.c) proceed in the same manner. One uses

\[
\chi = \begin{bmatrix}
T_2^{\frac{d_2 - i - 1}{2}} \\
0 \\
\vdots \\
0 \\
(-T_1)^{\frac{d_2 - i - 1}{2}}
\end{bmatrix}, \quad \chi = \begin{bmatrix}
T_2^{\frac{d_2 - 1}{2}} \\
0 \\
\vdots \\
0 \\
(-T_1)^{\frac{d_2 - 1}{2}}
\end{bmatrix}, \quad \text{and} \quad \chi = \begin{bmatrix}
0 \\
T_2^{\frac{d_2 - 1}{2}} \\
0 \\
\vdots \\
0 \\
(-T_1)^{\frac{d_2 - 1}{2}}
\end{bmatrix}
\]

to compute \( g\left(1, \frac{d_2 - 1}{2}\right) \) (when \( d_2 \) is odd), and \( g\left(1, \frac{d_2 - 2}{2}\right) \) and \( g\left(1, \frac{d_2 + 2}{2}\right) \) (when \( d_2 \) is even), respectively. \( \square \)

6. THE CASE OF \( d_1 = d_2 \).

The \( S \)-module structure of \( \mathcal{A}_{d_2, d_2 - 1} \) is completely described in Corollary 2.16 for all choices of \( d_1 \leq d_2 \) in Data 2.1. If \( d_1 < d_2 \) and the first column of \( \phi \) contains a generalized zero, then the \( S \)-module structure of \( \mathcal{A}_{d_2, d_2 - 1} \) is completely described in Theorem 3.3; see also Table 3.5. In Theorem 6.2 we assume that \( d_1 = d_2 \) and we describe \( \mathcal{A}_{d_1, d_2 - 2} \); hence, in this case, the \( S \)-module structure of \( \mathcal{A}_{d_2, d_2 - 2} \) is completely described by combining Theorem 6.2 and Corollary 2.16. The geometric significance of Theorem 6.2 is explained in Remark 6.3. A preliminary version of Theorem 6.2...
Theorem 6.2. If Data 6.1. are emphasized in [7]. Most of the calculations that are used in the proof of Theorem 6.2 have already been incorporated in [7]. The geometric applications of these calculations are emphasized in [7]. In the present work we focus on the application of these calculations to Rees algebras.

Data 6.1. Adopt Data 2.1 with \( d_1 = d_2 \). Let

\[
C = \begin{bmatrix}
c_{0,1} & c_{0,2} \\
\vdots & \vdots \\
c_{d_1,1} & c_{d_1,2}
\end{bmatrix}
\]

be the matrix \([\Upsilon_1,1|\Upsilon_1,2]\) of Definition 2.6. Notice that

\[
(6.1.1)
\begin{bmatrix}
T_1 & T_2 & T_3
\end{bmatrix}\phi = \begin{bmatrix} y^{d_1} & xy^{d_1-1} & \cdots & x^{d_1}
\end{bmatrix}C.
\]

Theorem 6.2. If Data 6.1 is adopted, then the following statements hold.

1. There are eight possible values for the pair of integers \((\mu(I_1(\phi)), \mu(I_2(C)))\). Indeed, the following chart gives the possible values for \((\mu(I_1(\phi)), \mu(I_2(C)))\) as a function of \(d_1\):

\[
\begin{array}{c|c}
d_1 & \text{possible values for } (\mu(I_1(\phi)), \mu(I_2(C))) \\
\hline
\text{5 or more} & (6,6), (5,6), (5,5), (4,6), (4,5), (4,4), (3,3), or (2,1) \\
4 & (5,6), (5,5), (4,6), (4,5), (4,4), (3,3), or (2,1) \\
3 & (4,6), (4,5), (4,4), (3,3), or (2,1) \\
2 & (3,3), or (2,1) \\
1 & (2,1)
\end{array}
\]

2. The \(S\)-module \(\mathcal{A}_{d_1-2}\) is resolved by

\[
\begin{align*}
0 & \to S(-4)^2 \to S(-2)^{d_1-5} \oplus S(-3)^6 & \text{if } (\mu(I_1(\phi)), \mu(I_2(C))) = (6,6) \\
0 & \to S(-5) \to S(-2)^{d_1-4} \oplus S(-3)^3 \oplus S(-4) & \text{if } (\mu(I_1(\phi)), \mu(I_2(C))) = (5,6) \\
0 & \to S(-4) \to S(-2)^{d_1-4} \oplus S(-3)^4 & \text{if } (\mu(I_1(\phi)), \mu(I_2(C))) = (5,5) \\
0 & \to S(-5)^2 \to S(-2)^{d_1-3} \oplus S(-3)^3 \oplus S(-4)^4 & \text{if } (\mu(I_1(\phi)), \mu(I_2(C))) = (4,6) \\
0 & \to S(-5) \to S(-2)^{d_1-3} \oplus S(-3)^2 \oplus S(-4)^2 & \text{if } (\mu(I_1(\phi)), \mu(I_2(C))) = (4,5) \\
0 & \to S(-2)^{d_1-2} \oplus S(-4) & \text{if } (\mu(I_1(\phi)), \mu(I_2(C))) = (3,3) \\
0 & \to S(-2)^{d_1-1} & \text{if } (\mu(I_1(\phi)), \mu(I_2(C))) = (2,1).
\end{align*}
\]

3. The \(S\)-module \(\mathcal{A}_{d_1-2}\) is free if and only if \(\mu(I_2(C)) \leq 4\).

Proof. We know from Theorem 2.7, part (1), with \(i = d_1 - 2\), that

\[
0 \to \mathcal{A}_{d_1-2} \to S(-2)^{d_1+1} \mathcal{C}^T \to S(-1)^2
\]

is an exact sequence of homogeneous \(S\)-module homomorphisms; thus, \(\mathcal{A}_{d_1-2} \simeq \ker \mathcal{C}^T\). We apply the results of Section 4 in [7].

The matrix \(\phi\) is an element of the space of matrices “BalH\(_d\)” from Definition 4.3 in [7]. The group \(G = \text{GL}_3(k) \times \text{GL}_2(k)\) acts on BalH\(_d\); and, in Theorem 4.9 of [7], BalH\(_d\) is decomposed into 11 disjoint orbits under the action of \(G\): BalH\(_d\) = \(\bigcup_{\# \in \text{ECP}} \text{DO}_{\#}^{\text{Bal}}\). These orbits are parameterized
by a poset called the Extended Configuration Poset (ECP). The value of \((\mu(I_1(\phi)),\mu(I_2(C)))\), as a function of \# with \(\phi \in \text{DO}_{\#}^{\text{Bal}}\), is given, in part (2) of Lemma 4.10 in [7], to be

\[
\begin{align*}
(\mu(I_1(\phi)),\mu(I_2(C))) = (6, 6) & \iff \# = (\emptyset, \mu_6) \\
(\mu(I_1(\phi)),\mu(I_2(C))) = (5, 6) & \iff \# = (\emptyset, \mu_5) \\
(\mu(I_1(\phi)),\mu(I_2(C))) = (5, 5) & \iff \# = (c, \mu_5) \\
(\mu(I_1(\phi)),\mu(I_2(C))) = (4, 6) & \iff \# = (0, \mu_4) \\
(\mu(I_1(\phi)),\mu(I_2(C))) = (4, 5) & \iff \# = (c, c, \mu_4) \\
(\mu(I_1(\phi)),\mu(I_2(C))) = (4, 4) & \iff \# = (c, c), \text{ or } (c : c) \\
(\mu(I_1(\phi)),\mu(I_2(C))) = (3, 3) & \iff \# = (c, c, c), \text{ or } (c : c, c), \text{ or } (c : c : c) \\
(\mu(I_1(\phi)),\mu(I_2(C))) = (2, 1) & \iff \# = \mu_2.
\end{align*}
\]

We notice that when \(d_1\) is small, then it is not possible for \((\mu(I_1(\phi)),\mu(I_2(C)))\) to take on all of the values listed so far. Indeed, the entries of \(\phi\) are elements of the vector space of homogeneous forms of degree \(d_1\) in 2 variables; consequently, \(\mu(I_1(\phi)) \leq d_1 + 1\). On the other hand, this is the only constraint that a small value for \(d_1\) imposes on the pair \((\mu(I_1(\phi)),\mu(I_2(C)))\), as is shown in Proposition 4.21 of [7]. This completes the proof of (1).

Now we prove (2). For each \# in ECP, there is a canonical matrix \(C_\#\) with the property that if \(\phi' \in \text{DO}_{\#'}^{\text{Bal}}\) and \(C'\) is the partner of \(\phi'\) in the sense of (6.1.1), then \(C_\#\) may be obtained from \(C'\) by a linear change of variables in \(S\), elementary row and column operations, and the suppression of zero rows; furthermore, \(I_i(C') = I_i(C_\#)\), and if \(X_\# = \begin{bmatrix} C_\# \\ \emptyset \end{bmatrix}\) is the matrix with the same number of rows as \(C'\), then \(\ker C'^T \simeq \ker X_\#^T\). We now record the matrices \(C_\#\) as given in [7, Lemma 4.10, part (1)]:

\[
\begin{align*}
C_{(\emptyset, \emptyset)} &= \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \\ 0 & 0 \\ 0 & T_2 \end{bmatrix}, & C_{(\emptyset, \emptyset)} &= \begin{bmatrix} T_1 & T_7 \\ T_2 & 0 \\ 0 & T_1 \\ 0 & T_2 \end{bmatrix}, & C_{(c, \mu_5)} &= \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \\ 0 & T_1 \\ 0 & T_2 \end{bmatrix}, & C_{(\emptyset, \mu_1)} &= \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \\ 0 & T_1 \\ 0 & T_2 \end{bmatrix}, \\
C_{(c, \mu_4)} &= \begin{bmatrix} T_1 & T_3 \\ T_2 & 0 \\ 0 & T_3 \\ 0 & T_2 \end{bmatrix}, & C_{c, c} &= \begin{bmatrix} T_1 & 0 \\ T_3 & T_2 \\ T_2 & T_1 \\ 0 & T_2 \end{bmatrix}, & C_{c, c} &= \begin{bmatrix} T_1 & 0 \\ T_3 & T_2 \\ T_2 & T_1 \\ 0 & T_2 \end{bmatrix}, & C_{c, c, c} &= \begin{bmatrix} T_1 & T_1 \\ T_2 & T_1 \\ T_1 & T_2 \\ 0 & T_2 \end{bmatrix}, \\
C_{c, c, c} &= \begin{bmatrix} T_1 & T_3 \\ T_2 & 0 \\ 0 & T_3 \\ 0 & T_2 \end{bmatrix}, & C_{c, c, c} &= \begin{bmatrix} T_1 & T_3 \\ T_2 & 0 \\ 0 & T_3 \\ 0 & T_2 \end{bmatrix}, & C_{\mu_2} &= \begin{bmatrix} T_1 & T_3 \\ T_2 & T_1 \\ T_1 & T_2 \\ 0 & T_2 \end{bmatrix}.
\end{align*}
\]

One readily checks that the kernel of \(X_\#^T\) is as claimed for each choice of \#. This completes the proof of (2). Assertion (3) follows immediately from (2). \(\Box\)

**Remark 6.3.** We explain the names of the elements of ECP. Adopt Data 6.1 with \(k\) algebraically closed. Let \(C_\#\) be the curve and \(\eta_\# : \mathbb{P}^1_k \to C_\#\) be the parameterization described in Remark 2.9.

1. If \(\phi \in \text{DO}_{\#_1}^{\text{Bal}}\), then \(\eta_\#\) is a birational parameterization of \(C_\#\) if and only if \(d_1 = 1\); in this case, \(C_\#\) is nonsingular.
2. If \(d_1\) is a prime integer, then \(\eta_\#\) is a birational parameterization of \(C_\#\) if and only if \(\mu(I_1(\phi))\) is at least 3; see [7, 0.11].

For the rest of our remarks, we assume that \(\eta_\#\) is a birational parameterization of \(C_\#\). (If one starts with a non-birational parameterization, then one can always re-parameterize in order to obtain a birational parameterization.)
(3) If $\varphi \in \text{DO}_\text{Bal}_\#$, for some $\# \in \text{ECP}$, and $\eta_\varphi$ is a birational parameterization of $C_\varphi$, then the number of $c$’s that appear in the name of $\#$ is equal to the number of singularities of multiplicity $d_1$ on, or infinitely near, $C$. (In [7], the degree of $C$ is $d = 2c$; in the present language, the degree of $C$ is $d = 2d_1$.) In particular,

(a) The $S$-module $\mathcal{A}_{d_1-2}$ is free and isomorphic to $S(-2)^{d_1-2} \oplus S(-4)$ if and only if there are exactly 3 singularities of multiplicity $d_1$ on, or infinitely near, $C$.

(b) The $S$-module $\mathcal{A}_{d_1-2}$ is free and isomorphic to $S(-2)^{d_1-3} \oplus S(-3)^2$ if and only if there are exactly 2 singularities of multiplicity $d_1$ on, or infinitely near, $C$.

(c) The $S$-module $\mathcal{A}_{d_1-2}$ is free if and only if there are at least 2 singularities of multiplicity $d_1$ on, or infinitely near, $C$.

(4) If $\# \in \text{ECP}$, then the punctuation describes the configuration of multiplicity $d_1$ singularities on $C$. A colon indicates an infinitely near singularity, a comma indicates a different singularity on the curve, and $0$ indicates that there are no multiplicity $d_1$ singularities on $C_\varphi$.

**Remark 6.4.** Again, we compare our results with the results of [3]. In [3] the ambient hypothesis forces $d_2$ to equal either $d_1$ or $d_1 + 1$ (see [3, Cor. 4.4]); and therefore, the hypothesis $d_1 = d_2$ of the present section agrees with one of the hypotheses of [3]. The style of answer; however, is completely different. We describe **all** of the possible bi-degrees for all possible bi-homogeneous minimal generating sets for exactly one $\mathcal{A}_\#$; namely, $\mathcal{A}_{d_1-2}$, and we explain how these bi-degrees reflect the configuration of singularities on the curve. On the other hand, [3] considers exactly one configuration of singularities and for this configuration [3] gives a complete set of minimal bi-homogeneous generators (notice generators rather than just degrees) for the entire ideal $\mathcal{A}$. The singularities of the curve of [3] can be read from [3, Thm. 3.11], together with [7, Cor. 1.9(1)]. The curve of [3] has four singularities: all of these singularities are on the curve, three of these singularities have multiplicity $d_1$, and the fourth singularity has multiplicity $d_1 - 1$. In the language of the present section, the singularities of the curve of [3] correspond to the element $\# = (c, c, c)$ of the Extended Configuration Poset and the Data 6.1 for the curves of [3] satisfies $(\mu(I_1(\varphi)), \mu(I_2(C)))$ is equal to $(3, 3)$.

7. **An Application: Sextic Curves.**

The results outlined in the previous sections suffice to provide significant information about the defining equations for $\mathcal{R}$ if $d = d_1 + d_2 \leq 6$, since then $d_1 \leq 2$ (see Section 5) or $d_1 = d_2$ (see Section 6). We focus on the case $d = 6$, the case of a sextic curve. The following data is in effect throughout this section.

**Data 7.1.** Adopt Data 2.1 with $d = 6$ and $k$ an algebraically closed field. Let $\eta : \mathbb{P}^1_k \rightarrow \mathbb{P}^2_k$ be the morphism determined by $\varphi$ and $C$ be the rational plane curve parameterized by $\eta$, as described in Remark 2.9. Assume that $C$ has degree 6, or equivalently, that the morphism $\eta$ is birational onto its
image \( C \). Let \( J \) be the ideal in \( B \) which is the kernel of the composition

\[
B \rightarrow \text{Sym}(I) \rightarrow \mathcal{R},
\]
as described in Data 2.1.

As it turns out, there is, essentially, a one-to-one correspondence between the bi-degrees of the defining equations of \( \mathcal{R} \) on the one hand and the types of the singularities on or infinitely near the curve \( C \) on the other hand. Here one says that a singularity is infinitely near \( C \) if it is obtained from a singularity on \( C \) by a sequence of quadratic transformations. This correspondence can be justified using the results of [7]. The information in Theorem 7.2 about the bi-degrees of the defining equations of \( \mathcal{R} \) is a compilation of results from many places as described below.

**Theorem 7.2.** Adopt Data 7.1. The correspondence between the bi-degrees of the defining equations of \( \mathcal{R} \) and the types of the singularities on or infinitely near the curve \( C \) is summarized in Table 7.2.1. The first column gives the possible values of \( d_1, d_2 \), namely 1, 5 or 2, 4 or 3, 3; the second column lists the corresponding bi-degrees of minimal homogeneous generators of \( J \) together with the multiplicities by which they appear, suppressing the obvious bi-degrees \( (d_1,1),(d_2,1) \) (of the equations defining \( \text{Sym}(I) \)) and \( (0,6) \) (of the implicit equation of \( C \)); and the third column gives the multiplicities of the singularities on or infinitely near \( C \).

**Remark 7.3.** Notice that in Table 7.2.1, the constellation of 10 double points on or infinitely near the curve corresponds to two distinct numerical types of Rees algebras. Thus, the Rees algebra provides a finer distinction. We would like to know a geometric interpretation of this algebraic distinction.

Before proving Theorem 7.2, we recall a few of the ingredients that are used. We make repeated use of Max Noether’s formula for the geometric genus of an irreducible plane curve. In the special case of a rational curve \( C \) of degree 6 it says that

| \( d_1 \) | \( d_2 \) | equations of \( \mathcal{R} \) | singularities of \( C \) |
|-----|-----|-----------------|-----------------|
| 1   | 5   | (1,5) (2,4) (3,3) (4,2) | 1 of multiplicity 5 on \( C \) |
| 2   | 4   | (1,3):2 (2,2)         | 1 of multiplicity 4 on \( C \) |
|     |     | (1,4):4 (2,3):3 (3,2)| 4 double points on or near \( C \) |
| 3   | 3   | (1,3) (1,4):2 (2,2):3 | 10 double points on or near \( C \) |
|     |     | (1,3):2 (2,2):3       | 1 of multiplicity 3 on \( C \) |
|     |     |                       | 7 double points on or near \( C \) |
|     |     | (1,2) (1,4) (2,2)     | 2 of multiplicity 3 and |
|     |     |                       | 4 double points on or near \( C \) |
|     |     |                       | 3 of multiplicity 3 and |
|     |     |                       | 1 double point on or near \( C \) |

Table 7.2.1. The correspondence between the Rees algebra and the singularities of a parameterized plane sextic.
where \( q \) ranges over all singularities on or infinitely near \( C \), and \( m_q \) is the multiplicity at \( q \). We also make repeated use of the General Lemma of [7] (see [7, 1.7, 1.8, and 1.9], or [1, 1.1] and [2, Lems. 1.3 and 1.5 and Prop. 1.5], or [28, Thm. 3]), which we again state under the special hypotheses of this section.

**Theorem 7.4.** Adopt Data 7.1.

1. If \( p \) is a point on \( C \), then the multiplicity \( m_p \) of \( C \) at \( p \) satisfies either \( m_p = d_2 \) or \( m_p \leq d_1 \).
2. If \( d_1 < d_2 \), then the first column of \( \varphi \) has a generalized zero if and only if \( C \) has a singularity of multiplicity \( d_2 \).
3. If \( d_1 = d_2 = 3 \) and \( C \) is the matrix of Data 6.1, then the number of singular points of multiplicity 3 that are either on \( C \) or infinitely near \( C \) is \( 6 - \mu(I_2(C)) \).
4. The infinitely near singularities of \( C \) have multiplicity at most \( d_1 \).

**Proof.** Item (1) follows from parts (1) and (2) of Corollary 1.9 of [7], and item (2) is a consequence of part (4) of the same corollary. Item (3) is established by combining parts (1) and (4) of [7, 3.22]. Notice that the equality \( \deg \gcd I_3(A) = 6 - \mu(I_2(C)) \) from (1) of [7, 3.22] holds even when \( \gcd I_3(A) \) is a unit; see, for example, item (2) of [7, 4.10]. Item (4) is [7, 2.3].

**Proof of Theorem 7.2.** We now justify the Table 7.2.1 in detail. When \((d_1, d_2) = (1,5)\), the numerical information about the Rees ring follows from [16, 4.3], [8, 2.3], or [24, 3.6]. Moreover, there is a generalized zero in the first column of \( \varphi \), hence the curve \( C \) has a singularity of multiplicity 5, as can be seen from item (2) of Theorem 7.4. There are no further singularities or infinitely near singularities of \( C \) by Max Noether’s formula (7.3.1).

Next assume that \((d_1, d_2) = (2,4)\) and that the first column of \( \varphi \) has a generalized zero. The numerical information about the Rees algebra is contained in Table 5.1.2. Item (1) of Theorem 7.4 shows that all of the singularities of \( C \) have multiplicity 2 or 4, and, according to (2), \( C \) has a singularity of multiplicity 4. All of the infinitely near singularities of \( C \) have multiplicity 2 by (4). The rest of the description of the singularities of \( C \) follows from (7.3.1).

If \((d_1, d_2) = (2,4)\) and the first column of \( \varphi \) does not have a generalized zero, then the numerical information about the Rees ring is given by [5, 3.2]. Since the first column of \( \varphi \) does not have a generalized zero, there is no point of multiplicity 4 on the curve. Hence all singularities and infinitely near singularities have multiplicity 2 and there are 10 of them by (7.3.1).

Finally we consider the case \((d_1, d_2) = (3,3)\). Notice that a quadratic transformation cannot increase the multiplicity of a singularity. By item (1) of Theorem 7.4, all singularities on or infinitely near \( C \) have multiplicity either 2 or 3. Therefore, we employ (7.3.1), once again, to see that the last column of Table 7.2.1 lists all possible configurations of singularities on or infinitely near \( C \). We use Theorem 6.2 to connect the configuration of singularities on or infinitely near \( C \) to the degrees of the
defining equations of \( \mathcal{R} \). Part (3) of Theorem 7.4 shows that the number of points of multiplicity 3 on or infinitely near \( C \) is \( 6 - \mu(I_2(C)) \). The other invariant that is used in Theorem 6.2 is \( \mu(I_1(\phi)) \). Observe that \( 3 \leq \mu(I_1(\phi)) \leq 4 \). Indeed, the first inequality holds by part (2) of Remark 6.3 because the ambient hypothesis of Data 7.1 guarantees that the morphism \( \eta \), which parameterizes the curve \( C \), is birational onto its image; and the second inequality holds because the entries of \( \phi \) are cubics in two variables. We read the \( T \)-degrees of a minimal \( S \)-module generating set for \( \mathcal{A}_1 \) from part (2) of Theorem 6.2. As seen above, we need only look at the rows with \( 3 \leq \mu(I_1(\phi)) \leq 4 \). We see that if there are no multiplicity 3 singularities on or infinitely near \( C \), then \( \mu(I_2(C)) = 6 \) and there are 4 minimal generators of the \( S \)-module \( \mathcal{A}_1 \) and each of these has \( T \)-degree 4. We record these generator degrees as \((1, 4) : 4\) since every element of \( \mathcal{A}_1 \) has \( xy \)-degree 1. Similarly, if there is exactly one multiplicity 3 singularity on or infinitely near \( C \), then the minimal generators of the \( S \)-module \( \mathcal{A}_1 \) have bi-degree \((1, 3) \) and \((1, 4) : 2\). If there are 2 such singularities, then the generators have bi-degree \((1, 3) : 2\), and if there are 3 such singularities, then the generators have bi-degree \((1, 2) \) and \((1, 4) \). Corollary 2.16 shows that the \( S \)-module \( \mathcal{A}_{\leq 2} \) is minimally generated by 3 elements of bi-degree \((2, 2) \). Degree considerations show that in three of the cases the minimal generators of the \( B \)-module \( J \) have been identified. When there are 3 singularities of multiplicity 3 on or infinitely near \( C \), then a further calculation is necessary. As \( \text{Sym}(I)_{\leq 2} \simeq B_{\leq 2} \), multiplying the generator of \( \mathcal{A} \) of bi-degree \((1, 2) \) yields two linearly independent elements of \( \mathcal{A} \) of bi-degree \((2, 2) \); and therefore \( \dim_k[\mathcal{A}/B\mathcal{A}(1, 2)]_{(2, 2)} = 1 \).

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**Department of Mathematics, University of South Carolina, Columbia, SC 29208**
*E-mail address:* kustin@math.sc.edu

**Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556**
*E-mail address:* cpolini@nd.edu

**Department of Mathematics, Purdue University, West Lafayette, IN 47907**
*E-mail address:* ulrich@math.purdue.edu