Reprogramming static deformation patterns in mechanical metamaterials
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Abstract

This paper discusses an x-braced metamaterial lattice structure with the unusual property of exhibiting bandgaps in their deformation decay spectrum and hence, the capacity for reprogramming deformation patterns. The design of polarizing non-local lattice arising from the scenario of repeated zero eigenvalues of a system transfer matrix is also introduced. We develop a single mode fundamental solution for lattices with \textit{multiple degrees of freedom} per node in the form of static Raleigh waves. This wave can be blocked at the material boundary when the solution is constructed with the polarization vectors of the bandgap. This single mode solution is used as a basis to build analytical displacement solutions for any applied essential and natural boundary condition. Subsequently we address the bandgap design leading to a comprehensive approach for predicting deformation pattern behavior within the interior of an x-braced plane lattice. Overall, we show that the stiffness parameter and unit-cell aspect ratio of the x-braced lattice can be tuned to completely block or filter static boundary deformations, and to reverse dependence of deformation or strain energy attenuation parameter on the Raleigh wavenumber, a behavior known as the reverse Saint Venant’s edge effect (RSV). These findings could guide future research in engineering smart materials and structures with interesting functionalities such as load pattern recognition, strain energy redistribution and stress alleviation.

1. Introduction

The existence of bandgaps and their effect on static Raleigh wave mode propagation in periodic lattices could be compared to acoustic metamaterials and how their bandgap characteristics define wave propagation and translation. Such acoustic metamaterial behavior has spurred on the interest where recent studies have concentrated on how bandgap characteristics can be altered in materials to achieve desirable functionalities like vibration isolation, noise control and impact absorption [1,2,3]. Lubenski, Bertoldi and others have also studied topological metamaterials [4,5,6], lattice materials that are on the verge of mechanical instability due to their topological index and so exhibit zero-frequency modes or floppy modes. More interesting is how the release of self-stress in these lattices can lead to localization of those zero modes at a boundary surface without

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propagation. Such surface qualities allow for applications such as adhesion and grip enhancement in rubber tires and compliant and energy absorption of rigid load bearing elements [7,8]. Even though the x-braced lattice discussed in this paper does not have floppy modes, their nonlocality has the property of boundary static Raleigh wave blockage and the unique RSV behavior [9]. Our current interest is to be able to program the x-braced lattice to highlight relevant functionalities such as boundary deformation localization, deformation pattern recognition, strain energy redistribution and stress and strain alleviation inside the lattice. The use of periodic material systems has showed a great importance in structural and materials engineering where design and analysis are undertaken at the repetitive substructure level to provide elegant approaches to cost-effective solutions. Geometrical periodicity has helped in analyzing many interesting micro-structural properties exhibited by metamaterials. Some of the earlier quasistatic analysis of patterned and repetitive structures involved the use of the discrete field analysis as a solution method for the governing system of finite difference equations [10,11]. A compact matrix form of these equations with a discrete convolution operator was also suggested [12] and discrete Fourier transform (DFT) methods were used to write computationally efficient semianalytical solutions for arbitrary force loads in terms of lattice Green’s function operators [13]. The Green’s function approach also allowed for various domain reduction techniques in molecular mechanics of materials including single- and multilayered graphene and carbon nanotubes [14,15,16,17].

Studies on static deformation of periodic lattices have been done in recent works [18,19,20,21] using several methods such as the transfer matrix approach for 1D (beam like) structures. Using the transfer matrix approach, these authors [21] realized exponential decay of self-equilibrated end-load components, and rather simple polynomial behavior of tensile and bending displacement fields, even in complex architectural trusses and microstructured beams. Exponential decay of self-equilibrated sets of forces and couple moments at the beam ends was discussed as a manifest of the Saint-Venant principle in a discrete elastic media, and the corresponding decay rates were related to the transfer matrix eigenvalues.

Fully analytical solutions for discrete 2D materials and structures are interesting due to their immense technological significance realized in the recent decades. It was recently shown [9] that with a combination of the Fourier and transfer matrix methods we can produce a fundamental solution as a static surface mode or harmonic which propagates in the material volume without any shape transformation. Only the amplitude of these modes decays exponentially at a known rate \( \lambda < 1 \) depending on the Fourier parameter \( q \in (-\pi, \pi) \), termed the static Raleigh wave solution:

\[
d_{nm} = h(q) \lambda^n(q) e^{iqm} \tag{1}
\]

Here, index \( n \) increases toward material interior, and index \( m \) varies along the edge of a discrete two-dimensional material or structure, and \( d_{nm} \) is a vector of all displacement components in an arbitrary group of repetitive structural nodes numbered \((n, m)\). The decay rate \( \lambda(q) \) and polarization
vector $\mathbf{h}$ are the eigenvalue and normalized eigenvector of a transfer matrix written in terms of partial Fourier images of lattice force constants.

A monotonous increase of $\lambda(q)$ with $q$ in (1) is a manifest of the Saint-Venant principle in discrete 2D material systems, because $q$ is a basic measure of unevenness of the Raleigh mode (1), whose mean square deformation gradient over index $m$ is proportional to $q^2$. Thus, any acute mode with a higher $q$ should generally decay faster in the material volume than a smooth mode with a smaller $q$. However, anomalous behavior of the $\lambda(q)$ dependence is possible as was recently shown [9], where a group of vertical asymptotes (asymptotic bandgaps), and therefore negative slopes intervals can be introduced. This imply that certain small-scale unevenness may in fact prevail over coarse ones and propagate farer in the material volume. Ability of an engineering structural material to support anomalous propagation of fine-scale details of deformation in the material volume was named the reverse Saint-Venant’s effect (RSV) [9] and the material itself – the RSV metamaterial. Another practical significance of the asymptotic bandgap in static Fourier spectra of RSV metamaterials is the ability to completely detain other types of deformation on the material surface, including some rather smooth ones.

A systematic understanding of static load processing and modification in RSV metamaterials require simple analytic and numerical tools for reconstruction and testing of various deformation patterns in the material volume for any type of practical natural and essential boundary conditions. In this paper, we outline a simple semianalytical approach to reconstruct analytical solutions for any essential boundary conditions applied at one edge of any nonlocal lattice; the opposite edge is free and indefinitely remote, and periodical boundary conditions are applied at two other opposite edges. A bandgap design approach is also presented with which a designer has the flexibility to program the x-braced lattice load bearing and strain energy storage capabilities.

2. Displacement transfer matrix and polarization vectors

We start with the governing equation of equilibrium of an arbitrary nonlocal elastic medium with only pair-wise elastic interactions in the form of non-buckling bars, springs, or linearized interatomic bonds,

$$ (\mathbf{k} \ast \mathbf{d})_{nm} = \sum_{n' m'} k_{n-n' m-m'} \mathbf{d}_{n' m'} = \mathbf{f}_{nm} $$

(2)

Vectors $\mathbf{d}_{nm}$ and $\mathbf{f}_{nm}$ are comprised of all displacement and external force components in an arbitrary repetitive group of lattice nodes numbered $(n, m)$. Matrices $\mathbf{k}$ represent configuration and intensity of the elastic interactions between nodes of the current group and all its neighbor groups $(n', m')$. The repetitive group is selected large enough for the $n'$-summation to run from $n-1$ to $n+1$ only, while the $m'$-summation (along the lattice edge) may run for an arbitrary range. Structural periodicity implies dependence of $\mathbf{k}$ only on the differences $n-n'$ and $m-m'$, rather than separate dependences on the current and running indices, which would be the case for non-periodic lattices.
Assume that we know an essential boundary condition $d_{0m}$ or natural boundary $f_{0m}$ on the lattice edge and we want to determine the displacement solution $d_{nm}$, where $n > 0$. This solution will describe the state of free static deformation in the lattice interior, arising in response to the boundary conditions.

For all $n > 0$, we may write the homogenous governing equation

$$\Sigma_{n' m'} k_{n-n'm'-m} d_{n'm'} = 0$$

(3)

Taking summation of $n'$ in Eq. (3) from $n-1$ to $n+1$, we may write

$$\Sigma_m k_{1 m-m} d_{n-1} + k_{0 m-m} d_{nm} + k_{-1 m-m} d_{n+1} = 0$$

(4)

The Fourier domain form of equation (4) is obtained after performing the discrete Fourier transform (DFT) over its terms to get

$$K_1(q) d_{n-1} + K_0(q) d_n + K_{-1}(q) d_{n+1} = 0$$

(5)

Where $K_n(q) = \Sigma m k_{nm} e^{-iqm}$ and $d_n(q) = h(q) \lambda^n(q)$. Hence the displacement transfer matrix $H(q)$ is formulated from (5) as

$$H(q) = \begin{bmatrix} 0 & I \\ -K_{-1}(q)^{-1} K_1(q) & -K_{-1}(q)^{-1} K_0(q) \end{bmatrix} \begin{bmatrix} d_{n-1} \\ d_{n} \end{bmatrix} = \begin{bmatrix} d_{n} \\ d_{n+1} \end{bmatrix}$$

(6)

The static Raleigh wave mode solution in (1) as mentioned earlier, is dependent on the eigensystem of matrix $H(q)$ and since matrix $H(q)$ is of the order $2R \times 2R$ where $R$ is the number of degrees of freedom at a node, there will be $R$ distinct eigenvalues $\lambda(q)$. However, these eigenvalues would come in reciprocal pairs as consequence of the symplectic [22,23,24] nature of the displacement transfer matrix $H(q)$. That if $\lambda(q)$ is an eigenvalue then $1/\lambda(q)$ is also an eigenvalue and they could be either real or a complex conjugate pair. The 2R-component eigenvectors will have the form $\{h(q), \lambda(q) h(q)\}$ where the bottom half-vector is the top half-vector multiplied by $\lambda(q)$ [9].

Considering a 2DoF per node x-braced lattice, the 4 x 4 transfer matrix $H(q)$ is consistent with the structure

$$H(q) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_1 & \beta_3 i & \beta_4 & \beta_6 i \\ \beta_2 i & \beta_1 & \beta_5 i & \beta_7 \end{bmatrix}$$

(7)

$$\beta_1 = \frac{(\sqrt{2} \cos q + k \cos 2q)}{k + \sqrt{2} \cos q}, \quad \beta_2 = \frac{2(\sqrt{2} + k \cos q) \sin q}{k + \sqrt{2} \cos q}, \quad \beta_3 = \frac{k \sin q}{k + \sqrt{2} \cos q}, \quad \beta_4 = \frac{2(\sqrt{2} + k) \cos q}{k + \sqrt{2} \cos q}$$
\[
\beta_5 = -\frac{2(\sqrt{2} + k) \sin q}{k + \sqrt{2} \cos q}, \quad \beta_6 = -\frac{2\left(k - \sqrt{2}(k \cos q - 1)\right) \sin q}{k + \sqrt{2} \cos q}, \quad \beta_7 = \frac{(2 + k\sqrt{2} - 2 \cos q)(2 + \sqrt{2} k \cos q)}{k + \sqrt{2} \cos q},
\]

Where \(k\) represents the relative stiffness of the diagonal bar with a vertical or horizontal bar.

Solving the eigenvalue problem \((H(q) - I \lambda(q)) \begin{bmatrix} h(q) \\ \lambda(q) h(q) \end{bmatrix} = 0\), the eigenvector is derived (A1-A.4) in a compact form as

\[
\begin{bmatrix} h(q) \\ \lambda(q) h(q) \end{bmatrix} = C \begin{bmatrix} i(\beta_3 + \beta_6 \lambda) \\ \lambda^2 - \beta_4 \lambda - \beta_1 \\ i \lambda(\beta_3 + \beta_6 \lambda) \\ \lambda(\lambda^2 - \beta_4 \lambda - \beta_1) \end{bmatrix}
\]

(8)

Here \(C\) could be any real or complex number. It can be seen in equation (8) that the bottom half of the eigenvector is equal to the top half-vector multiplied by the eigenvalue \(\lambda(q)\). The polarization vector for a complex eigenvalue can be derived by substituting the complex conjugate eigenvalue pair \(\lambda(q) = \mu \pm \omega i\) into the top half of equation (8) and regroup real and imaginary parts to get

\[
h(q) = C_1 \begin{bmatrix} a + ib \\ -c + id \end{bmatrix}
\]

(9)

\[
a = -\beta_6 \omega, \quad b = (\beta_3 + \beta_6 \mu), \quad c = (-\beta_1 - \beta_4 \mu + \mu^2 - \omega^2), \quad d = (2\mu \omega - \beta_4 \omega)
\]

The polarization vector in the instance of a real eigenvalue \(\lambda(q)\) is derived directly from the polarization vector \(h(q)\) of the complex eigenvalue by eliminating the imaginary part \(\omega\) in equation (9) to obtain the form

\[
h(q) = C_2 \begin{bmatrix} ib \\ c \end{bmatrix}
\]

(10)

3. 2DoF Static Raleigh Wave Solution

Since the Raleigh mode solution is being built for a periodic spatial domain along the index \(m\) of a lattice, the static Raleigh wave solution (1) must be constructed such that it is a real-valued cyclic solution by obeying the following symmetry condition

\[
d_{nm} = \begin{bmatrix} U_{nm} \\ V_{nm} \end{bmatrix} = \begin{bmatrix} U_{nm} \\ -V_{nm} \end{bmatrix}
\]

(11)

Constructing \(d_{nm}\), we first substitute \(\lambda(q)\) and \(h(q)\) into (1) and obtain a solution having both real and imaginary parts. Basically, that part of the solution satisfying (11) is considered the real cyclic static Raleigh wave solution. Should the eigenvalues be complex-valued, \(\lambda(q) = \mu \pm \omega i\), it would be convenient to utilize the polar coordinate form \(\lambda(q) = \rho e^{i\theta}\), where \(\rho = |\lambda(q)|\) and \(\theta = \)
Arg(\(\lambda(q)\)) and since eigenvalues are complex conjugate, after substituting into \(d_{nm}\) we obtain four (4) possible solutions (2 real parts, 2 imaginary parts) but none would satisfy (11). However, we can reduce this to the solution in equation (12) that satisfy the condition in equation (11) by summing the 2 real parts and subtracting the 2 imaginary parts. Below we state the real-cyclic solution forms for the complex and real eigenvalues as (A5-A12):

**Complex Eigenvalue:**

\[
\mathbf{h}(q) = C_1 \begin{bmatrix} a \pm ib \\ -c + id \end{bmatrix} \quad \mathbf{d}_{nm} = \begin{bmatrix} C_1 \rho^n(q) & \{a \cos qm \} \\
-C_1 \rho^n(q) & \{-d \sin qm \} \end{bmatrix} \begin{bmatrix} b \cos qm \\ c \sin qm \end{bmatrix}
\]  

(12)

**Real Eigenvalue:**

\[
\mathbf{h}(q) = C_2 \begin{bmatrix} ib \\ c \end{bmatrix} \quad \mathbf{d}_{nm} = C_2 \lambda^n(q) \begin{bmatrix} b \cos qm \\ c \sin qm \end{bmatrix}
\]  

(13)

In equation (12) and (13), the coefficients \((a, b, c, d)\) form vectors that are termed polarization vectors in Raleigh mode equation (1).

### 4. Essential boundary condition solution

Our discussion till now has been concerned with static Raleigh wave solution forms which are essentially harmonic (cosine, sine) but in practice an applied boundary conditions could be in several forms such as a gaussian, impact, triangular etc. and therefore this section is dedicated to finding a general solution form that would cater for any essential boundary condition. Considering eqn. (1), a general solution is obtained by the summation of all possible modes of the decomposed essential boundary condition as

\[
\mathbf{d}^\prime_{nm} = \frac{1}{M} \sum_{q=0}^{M-1} \mathbf{h}_1(q) \mathbf{h}_2(q)^* \begin{bmatrix} \lambda_1(q) & 0 \\
0 & \lambda_2(q) \end{bmatrix}^n \begin{bmatrix} C_1(q) \\ C_2(q) \end{bmatrix} e^{i q m} \]  

(14)

In equation (14), \(C_1(q)\) and \(C_2(q)\) are unknown Fourier dependent parameters. Considering an arbitrary boundary deformation \(\mathbf{d}_{0,m}\), a semi-analytical procedure is developed for finding \(C_1\) and \(C_2\) by multiplying both sides of equation (14) with the conjugate transpose of a normalized polarization vector \(\mathbf{h}_1^*(q')\) and performing the DFT on both sides of the same equation.

\[
\frac{1}{M} \sum_{m=0}^{M-1} \mathbf{h}_1^*(q') \mathbf{d}^\prime_{0,m} e^{-i q' m} = \frac{1}{M^2} \sum_{m=0}^{M-1} \sum_{q=0}^{M-1} \mathbf{h}_1^*(q') \mathbf{h}_1(q) \mathbf{h}_2(q)^* \begin{bmatrix} C_1(q) \\ C_2(q) \end{bmatrix} e^{i q m} e^{-i q' m} \]  

(15)

Rearranging,
\[ \sum_{m=0}^{M-1} \mathbf{h}_1(q') \mathbf{d}'_{0m} e^{-iq'm} = \frac{1}{M} \sum_{q'=0}^{M-1} \left\{ \mathbf{h}_1(q') \mathbf{h}_1(q) \right\} \cdot \left\{ \mathbf{C}_1(q) \right\} \left( \sum_{m=0}^{M-1} e^{-iq'm} e^{iqm} \right) \]  \hspace{1cm} (16)

Since,
\[ \sum_{m=0}^{M-1} e^{-iq'm} e^{iqm} = M \delta_{qq'} \]  \hspace{1cm} (17)

equation (16) can be simplified as
\[ \sum_{m=0}^{M-1} \mathbf{h}_1(q) \mathbf{d}'_{0m} e^{-iqm} = \left\{ \mathbf{h}_1(q') \mathbf{h}_2(q) \right\} \cdot \left\{ \mathbf{C}_1(q) \right\} \cdot \left\{ \mathbf{C}_2(q) \right\} \]  \hspace{1cm} (18)

Repeating the above procedure with \( \mathbf{h}_2(q') \) we obtain a similar expression as in (18) and solve for \( \mathbf{C}_1(q) \) and \( \mathbf{C}_2(q) \) as
\[ \left\{ \mathbf{C}_1(q) \right\} \cdot \left\{ \mathbf{C}_2(q) \right\} = \left[ \mathbf{h}_1(q) \mathbf{h}_2(q) \right]^{-1} \sum_{m=0}^{M-1} \left\{ \mathbf{h}_2(q') \mathbf{d}'_{0m} e^{-iqm} \right\} \]  \hspace{1cm} (19)

Substituting \( \mathbf{C}_1(q) \) and \( \mathbf{C}_2(q) \) into (14), a general solution is obtained that can fully represent static deformation in the lattice interior.

### 5. Natural boundary condition solution

Having formulated a general solution form for analyzing essential boundary conditions, it would be appropriate to deal with scenarios of natural or forced boundary condition which has a greater practical appeal. Considering the Fourier form of the equilibrium governing equation (4), at \( n = 0 \) we can write
\[ \frac{1}{2} \mathbf{K}_0(q) \mathbf{d}_0(q) + \mathbf{K}_{-1}(q) \mathbf{d}_1(q) = \mathbf{f}_0(q) \]  \hspace{1cm} (20)

Such a form represents the effect of neglecting all the nodal set to the left of the boundary where the natural boundary condition is applied and ignoring boundary stiffness interaction with same by having the \( \frac{1}{2} \mathbf{K}_0(q) \) in equation (20). Decomposing \( \mathbf{d}_0(q) \) and \( \mathbf{d}_1(q) \) from equation (20) into their Fourier components and solving for \( \mathbf{C}_1(q) \) and \( \mathbf{C}_2(q) \):
\[ \left\{ \mathbf{C}_1(q) \right\} \cdot \left\{ \mathbf{C}_2(q) \right\} = \left[ \frac{1}{2} \mathbf{K}_0(q) \mathbf{h}_1(q) \right] + \mathbf{K}_{-1}(q) \mathbf{h}_2(q) \cdot \left[ \mathbf{\lambda}_1(q) \mathbf{\lambda}_2(q) \right]^{-1} \mathbf{f}_0(q) \]  \hspace{1cm} (21)

Substituting \( \mathbf{C}_1(q) \) and \( \mathbf{C}_2(q) \) into (14) and adding \( \mathbf{G}(n) = n \mathbf{K}_{-1}(0)^{-1} \mathbf{f}_0(0) \), a linear polynomial term to account for uniform deformation [21] after considering the possible canonical modes of
static deformation of the Jordan block of the displacement transfer matrix \( \mathbf{H}(q) \). Hence the general solution for any natural boundary condition is stated as

\[
d'_{n,m} = \frac{1}{M} \sum_{q=0}^{M-1} \begin{bmatrix} h_1(q) \\ h_2(q) \end{bmatrix} \begin{bmatrix} \lambda_1(q) & 0 \\ 0 & \lambda_2(q) \end{bmatrix}^n \begin{bmatrix} c_1(q) \\ c_2(q) \end{bmatrix} e^{i\eta m} + G(n)
\] (22)

6. Raleigh wave mode bandgap design

Boundary deformation blockage or localization relates to the feature of asymptotic bandgaps in the deformation decay spectrum for a periodic lattice structure where \( q \) corresponds with a zero eigenvalue \( \lambda = 0 \) and subsequently the RSV effect since \( \eta(q) = -\log \lambda(q) \) starts to decrease in value as we increase \( q \): growth in fineness of static Raleigh wave mode corresponding to a slower deformation decay. The deformation decay spectrum is a map of the distribution of the decay parameter \( \eta(q) \) over \( q \) as shown in Fig. 3. The aim of this section is to develop a relationship between \( k \) and \( q \) for finding polarization vectors \( \mathbf{h}(q) \) that correspond with a bandgap in the lattice deformation decay spectrum. Knowing that \( \mathbf{H}(q) \) is a square matrix, it is valid that its determinant is equal to the product of its eigenvalues, \( \det \mathbf{H} = \prod_{i=1}^{n} \lambda_i(q) \) and so applying this property, a condition for attaining a zero eigenvalue \( \lambda(q) = 0 \) could be stated as when \( \mathbf{H}(q) = 0 \). For a 2DoF x-braced lattice, applying this condition, a zero-eigenvalue relationship (A13-14) is derived as

\[
k + \sqrt{2} \cos q = 0
\] (23)

Fig.1: Zero-eigenvalue plot (arrows represent orientation of polarization vectors at that \( q \) value)

Fig. 1 shows a plot of \( k \) against \( q \) from equation (23), this plot prescribes the stiffness parameter \( k \) of a 2DoF x-braced lattices that can be designed as a deformation blocker or filter and the dark
arrows shown describe the direction of polarization vectors $\mathbf{h}(q)$ that would be arrested at the boundary surface of chosen x-braced lattice.

It is also possible to create a bandgap phase diagram for design purposes by introducing aspect ratio $\alpha$ as a system parameter (A15). In such a scenario, equation (22) becomes

$$2k\alpha^3\sqrt{1 + \alpha^2 + \cos q} + 2\alpha^2 \cos q + \alpha^4H \cos q = 0 \quad \alpha = \frac{\text{breadth}}{\text{height}}$$

(24)

A plot of $\alpha$ against $k$ from equation (17) as shown in Fig. 2, represents a design map for generating bandgaps when modelling a 2DoF x-braced periodic lattice. It also shows the range of $k$ permissible in design for a specific $\alpha$. A typical x-brace lattice as in Fig. 1 with $\alpha = 1$, from Fig. 2 has $k$ ranging from 0-1.41 which is validated by Fig. 1. A look at Fig. 2 shows that as $\alpha \to 0$, the range of $k \to \infty$ for the region where bandgap exists and when $\alpha$ is in between 0 and 0.5, there is no restriction on $k$ at which bandgap would exist.

![Bandgap Phase Diagram](image)

**Fig. 2: Bandgap Phase Diagram**

7. **Polarizer structures: case of repeated zero eigenvalues ($\lambda_1 = \lambda_2 = 0$)**

In the above section, we dealt with boundary displacement blockage because of bandgap existence in the deformation decay spectrum of a lattice structure where in the analysis of 2DoF systems we
required only a single $\lambda = 0$ to reprogram a lattice to possess this displacement blocking feature. However, a case of repeated zero eigenvalues where $\lambda_1 = 0$ and $\lambda_2 = 0$ presents a unique class of structures that could be termed as polarizers with the mechanical property of reprogramming an arbitrary vector of a Raleigh of mode deformation at $n = 0$ into a desired polarization vector $\mathbf{h}(q)$ at $n = 1$ that would be completely blocked by the lattice structure at that point. Implementing a numerical searching procedure, it is possible to obtain repeated zero eigenvalues but for a 2DoF lattice structure the system transfer matrix $\mathbf{H}(q)$ generates only a single independent eigenvector $\{ \mathbf{h}(q), \lambda(q)\mathbf{h}(q) \}$ instead of two. Such a structure is deemed to have two (2) modes of static deformation according to the Jordan canonical form of $\mathbf{H}(q)$ [21] written as

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{25}$$

The two possible solution modes $j_1(n)$ and $j_2(n)$ resulting from the above Jordan block would have the forms

$$j_1(n) = 0^n \mathbf{h} \tag{26}$$

$$j_2(n) = 0^n \tilde{\mathbf{h}} + n0^{n-1} \mathbf{h} \tag{27}$$

Where $\mathbf{h}$ is the only independent eigenvector and $\tilde{\mathbf{h}}$ is the subsequently obtained generalized eigenvector. In the case of a 2DoF system with repeated zero eigenvalues, when the polarization vector $\mathbf{h}(q)$ of the lattice structure is applied, we observe that mode $j_1(n)$ control Raleigh mode static deformation and displacements are blocked at $n = 0$. Applying an arbitrary vector in equation (1), we note that $j_2(n)$, a combination of an exponential and polynomial modes rather control static deformation in the lattice and displacements are blocked at $n = 1$ instead as shown below.

At $n = 1$,

$$j_2(1) = 0 \tilde{\mathbf{h}} + \mathbf{h} = \mathbf{h} \tag{28}$$

At $n = 2$,

$$j_2(2) = 0 \tilde{\mathbf{h}} + 2 \times 0 \mathbf{h} = 0 \tag{29}$$

From the above analysis, we can state that a lattice structure yielding repeated zero eigenvalue polarizes an arbitrary Raleigh mode at position $n = 1$ and this wave completely disappears at $n = 2$. 
8. Illustrative examples

In section, we study several examples of a 2DoF x-braced lattice with relative stiffness parameter $k = 0.93$ and having a deformation decay spectrum as shown in Fig. 3. First we consider a Raleigh mode solution (1) for a Fourier parameter $q = \frac{4}{5}\pi$, where the generated eigenvalues are $\lambda_1 = 0.10$ and $\lambda_2 = -0.06$ and their corresponding eigenvectors are $h_1 = \{0.9335, -0.3442\}$ and $h_2 = \{0.4870, 0.8710\}$ respectively. Since the eigenvalues are real, the real-cyclic static Raleigh wave solutions are calculated from equation (13) as

$$d_{nm}^{(1)}(q) = C_2 \lambda_1^n(q) \begin{pmatrix} b \cos qm \\ -c \sin qm \end{pmatrix} = 0.10^n \begin{pmatrix} 0.9335 \cos \frac{4}{5}\pi m \\ 0.3442 \sin \frac{4}{5}\pi m \end{pmatrix}$$

$$d_{nm}^{(2)}(q) = C_2 \lambda_2^n(q) \begin{pmatrix} b \cos qm \\ c \sin qm \end{pmatrix} = (-0.06)^n \begin{pmatrix} 0.4870 \cos \frac{4}{5}\pi m \\ 0.8710 \sin \frac{4}{5}\pi m \end{pmatrix}$$

Fig. 3: Deformation decay spectrum for an x-braced lattice ($k = 0.93$)

Fig. 4 shows the deformation configuration of the Raleigh mode solutions in equations (30-31) for a vertical lattice dimension $M = 10$. Now taking $q$ as $\frac{1}{5}\pi$ for the same stiffness parameter $k = 0.93$, a pair of complex conjugate eigenvalue $\lambda = 0.5439 \pm 0.1425i$ is obtained and the possible real cyclic solutions are constructed from equation (12) as
\[ h = \left\{ \begin{array}{l} 0.3389 \pm 0.0888i \\ -0.1736 - 0.2955i \end{array} \right\} : \]

\[ d_{nm}^{(1)} = C \rho^n(q) \begin{pmatrix} a \cos qm \\ -d \sin qm \end{pmatrix} = 0.32^n \begin{pmatrix} 0.3389 \cos \frac{1}{5} \pi m \\ 0.2955 \sin \frac{1}{5} \pi m \end{pmatrix} \]  
\hspace{1cm} (32)

\[ d_{nm}^{(2)} = C \rho^n(q) \begin{pmatrix} b \cos qm \\ c \sin qm \end{pmatrix} = 0.32^n \begin{pmatrix} 0.0888 \cos \frac{1}{5} \pi m \\ -0.1736 \sin \frac{1}{5} \pi m \end{pmatrix} \]  
\hspace{1cm} (33)

Fig. 5 also shows the deformation configuration of the Raleigh mode solutions in equations (32-33) for a lattice vertical dimension \( M = 10 \).

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**Fig. 4:** Deformation configuration of 2DoF x-braced lattice \( (k = 0.93, q = \frac{4}{5} \pi, m = 0 \to 10, n = 0 \to 4) \): a picture showing deformation configuration of solutions (30) and (31) in (a) and (b) respectively (scaled)
Fig. 5: Deformation configuration of 2DoF x-braced lattice \((k = 0.93, \frac{q}{\pi} = \frac{1}{5}, m = 0 \rightarrow 10, n = 0 \rightarrow 4)\): a picture showing deformation configuration of solutions (32) and (33) in (a) and (b) respectively (scaled)

Now we would like to program an x-braced lattice having a deformation decay spectrum shown in Fig. 3 to observe a block in Raleigh mode propagation and the RSV effect. Since the bandgap \((\lambda = 0)\) in Fig. 3 exists at \(q = \frac{8\pi}{11} \approx 0.73\), the RSV effect must be seen in the subsequent Fourier parameters and so we consider Fourier parameters \(q = \frac{9\pi}{11}\) and \(q = \frac{10\pi}{11}\). The Raleigh mode solutions for all three (3) cases are constructed as

\[
q = \frac{8\pi}{11}: \quad \mathbf{d}_{nm} = 0.0015^n \begin{pmatrix} 0.6609 \cos \frac{8}{11} \pi m \\ 0.7504 \sin \frac{8}{11} \pi m \end{pmatrix} \quad (33)
\]

\[
q = \frac{9\pi}{11}: \quad \mathbf{d}_{nm} = -0.0756^n \begin{pmatrix} 0.4495 \cos \frac{9}{11} \pi m \\ 0.8901 \sin \frac{9}{11} \pi m \end{pmatrix} \quad (34)
\]

\[
q = \frac{10\pi}{11}: \quad \mathbf{d}_{nm} = 0.1039^n \begin{pmatrix} 0.9823 \cos \frac{10}{11} \pi m \\ 0.1524 \sin \frac{10}{11} \pi m \end{pmatrix} \quad (35)
\]
Fig. 6: Deformation configuration of solutions (35), (34) and (33) respectively for half the cyclic domain of an x-braced lattice ($k = 0.93$, $m = 0 \rightarrow 11$, $n = 0 \rightarrow 4$ (scaled)).

Fig. 7: Deformation configurations of different stiffness parameters under a Raleigh mode solution for $q = \frac{8\pi}{11}$. Showing half the cyclic domain of the x-braced lattices, $m = 0 \rightarrow 11$, $n = 0 \rightarrow 4$ (scaled).
The Raleigh mode deformation configurations of equations (33-35) for a lattice vertical dimension, \(M = 22\) are as shown in Fig. 6. This figure shows a complete blockage of the Raleigh wave mode at \(q = \frac{8\pi}{11}\) which is as expected and comparing denser or finer Raleigh modes \(q = \frac{10\pi}{11}\) with coarser Raleigh mode \(q = \frac{9\pi}{11}\), we observe the RSV edge effect a coarse mode decays faster than a finer mode. In Fig. 7, the deformation configurations for the Raleigh mode of \(q = \frac{8\pi}{11}\) applied at \(n = 0\) are analyzed by varying the stiffness parameter \(k\). The ability to reprogram the x-braced lattice for blockage is realized as shown in Fig. 7 by tuning \(k\) to the value corresponding to the band gap which is \(k_3 = 0.93\). We also observe how the rate of Raleigh mode decay in the lattice interior is programmed by tuning the stiffness parameter where at \(k_1 = 0.15\), we see a slow Raleigh mode decay and at \(k_2 = 0.6\), a very fast decay.

![Strain Energy](image)

**Fig. 8:** Strain Energy along index \(n\) in an x-braced lattice with \(k = 0.93\): the fastest total strain energy decay is when \(q = \frac{8\pi}{11}\), followed by \(q = \frac{9\pi}{11}\) and then \(q = \frac{10\pi}{11}\) due to the RSV

The distribution of strain energy inside a lattice, when mapped could show interesting features and a spectrum of such a distribution could help when programming a lattice to harness the functionalities such as strain energy redistribution and resilience in design. The strain energy, \(W\) calculated by summing the strain energy, \(w\) stored at each associate substructure level (A16-17) along lattice index \(m\) for the x-braced lattice is plotted against the lattice index \(n\) for the example
in Fig. 6 in Fig. 8. In Fig. 8, the strain energy $W$ stored along the lattice index $n$ follows the same order as the decay of deformation along index $n$ where plot of strain energy $W$ for a Raleigh mode $q = \frac{6\pi}{11}$ exhibits the fastest decay followed by $q = \frac{9\pi}{11}$ and then $q = \frac{10\pi}{11}$ due to the RSV effect. To present a comprehensive picture to show that the pattern of strain energy decay in a lattice is analogous to its deformation decay, we plot in Fig. 9, the strain energy along index $n$ ($0 \to 4$) relative to $n = 1$ for all possible static Raleigh modes. Fig. 9 shows a monotonous decay of strain energy, $W$, as $q$ of the Raleigh mode is increased from 0 to $\frac{7\pi}{11}$. When $q$ is $\frac{8\pi}{11}$, there is much faster decay due to its corresponding bandgap ($\lambda = 0$) and after this point as we increase $q$ the normalized strain energy at each lattice index $n$ starts to increase depicting an RSV behavior.

![Fig. 9: Relative Strain Energy against Fourier parameter q. A plot of strain energy at $n$ relative to strain energy at $n=1$ against Fourier parameter $q$.](image)

At this instance, we illustrate the behavior of a polarizing lattice structure by analyzing two x-braced lattices of equal spatial dimensions, the first lattice has $k = 0.4714$ and the second lattice with $k = 1.0834$. The deformation decay spectrums of these two lattices are shown in Fig. 10(a) and Fig. 10(b) respectively.
Fig. 10: Deformation decay spectrum for $k = 0.4714$ and $k = 1.0834$ respectively.

From Fig. 10, it can be referenced that at $q = \frac{7}{9}\pi$, the first x-braced lattice ($k = 0.4714$) has eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$ and the second x-braced lattice ($k = 1.0834$) produce eigenvalues $\lambda_1 \approx 0$ and $\lambda_2 \approx 0$. We verify the polarization behavior of the second x-braced lattice, which has repeating eigenvalues that are approximately zero by applying an arbitrary Raleigh mode deformation in equation (1). So instead of using the required polarization vector $\mathbf{h} = \begin{bmatrix} 1.198 \\ 1 \end{bmatrix}$ for the zero eigenvalues in equation (1), we apply an arbitrary vector $\hat{\mathbf{h}} = \begin{bmatrix} 1.198 \\ 2 \end{bmatrix}$ to both lattices and calculate nodal displacements. The deformed shape of the two x-braced lattices can be seen in Figure 11, showing that displacements are propagate in the first x-brace since its eigenvalues are not zero but for the second x-braced we see that although displacements are not blocked at $n = 0$, they propagate, and we see blockage rather at $n = 1$. Therefore, the second x-braced lattice has been shown to possess the ability to polarize an arbitrary Raleigh mode into a polarized Raleigh mode. The polarization behavior can be checked by simply deriving the required polarization vector $\mathbf{h}$ from the calculated displacements at $n = 1$ using equation (1).

Lastly, we consider a natural boundary condition where $\mathbf{f}_{0m} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta_{m0}$ which describes a point load or an indentation force acting mid-point of the x-braced lattice having the same stiffness parameter as the examples above and a lattice vertical dimension $M = 22$. The solution was constructed from equations (14,20,21) and In Fig. 12, we show the lattice deformation configuration and the normalized strain energy distribution contour map. However, the x-braced lattice in this case experiences a very slow decay in deformation and strain energy compared to its boundary as deformation at a node would be composed of all possible static Raleigh modes. As we move along the lattice $n$, it is realized that the strain energy at the boundary is much focused and moving away from the boundary begins to take a Gaussian form till the distribution becomes uniform approaching zero as $n \to m$. 
Fig. 11: A picture showing the deformation configurations for half the cyclic domain of the x-braced lattice \((m = 0 \rightarrow 9, n = 0 \rightarrow 4\) (scaled)): (a) \(k = 0.4714\) (b) \(k = 1.0834\)

The examples shown in this paper can be modelled in ANSYS as a seamless cyclic model, the top and bottom edges of our model were constrained to have only horizontal translation. Comparing deformations from above examples to their model solutions from ANSYS, the difference was of the order \(10^{-6}\) which shows high accuracy of our analytical results.

Fig. 12: A contour map of normalized strain energy of an x-braced lattice with lattice dimensions \(m = 22\) and \(n = 22\) for point load \(f_{0m} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta_{m0}\). The inset shows the strain energy \(W\) at \(n = 1\) to \(n = 4\).
9. Conclusions

In this paper, we have discussed in detail the 2DoF general Raleigh wave mode solutions for analyzing essential boundary and natural boundary conditions. The concept of bandgap design for deformation blockage and achieving RSV effect was also introduced. Such bandgap analysis has shown to be readily applicable to any fundamental Raleigh wave solution due to the solution dependence on a single zero eigenvalue \( \lambda = 0 \) corresponding with the Fourier parameter \( q \). The bandgap relationships presented can serve as tools for programming the unit-cell aspect ratio and stiffness parameter of an x-braced lattice to block a specific static Raleigh mode or filter out irrelevant modes when an applied boundary condition is a combination of several modes. The case of repeated zero eigenvalues have also been shown to present a unique class of nonlocal lattice that can serve as polarizers to induce blockage at \( n = 1 \) of a pseudo polarization vector of a Raleigh mode. Solutions to non-Raleigh mode boundary conditions were shown to depend on all possible Fourier modes with controllable decay parameters, providing an opportunity to program the overall strain energy distribution in the material sample. An equivalent continuum theory [25], analysis of strain energy spectral density and information entropy of deformation [26] for nonlocal mechanical metamaterials will be separate efforts to be undertaken in the near future.

An understanding of the methodologies presented in this study would be key in driving future research on RSV metamaterials, where uniqueness in deformation and strain energy distribution patterns are harnessed to engineer smart materials and structures with interesting functionalities and properties such as load pattern recognition, high resilience and stress alleviation.

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Appendix A

Finding eigenvectors of transfer matrix $\mathbf{H}(q)$ of 2DoF x-braced lattice:

$$
\mathbf{H}(q) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\beta_1 & \beta_3 & \beta_4 & \beta_6 \\
\beta_2 & \beta_1 & \beta_5 & \beta_7
\end{bmatrix}
$$

(A1)

$$
\beta_1 = -\frac{(\sqrt{2} \cos q + k \cos 2q)}{k + \sqrt{2} \cos q}, \quad \beta_2 = \frac{2(\sqrt{2} + k \cos q)}{k + \sqrt{2} \cos q}, \quad \beta_3 = \frac{k \sin q}{k + \sqrt{2} \cos q}, \quad \beta_4 = \frac{2(\sqrt{2} + k \cos q)}{k + \sqrt{2} \cos q}
$$

$$
\beta_5 = -\frac{2(\sqrt{2} + k) \sin q}{k + \sqrt{2} \cos q}, \quad \beta_6 = -\frac{2(k - \sqrt{2}(k \cos q - 1)) \sin q}{k + \sqrt{2} \cos q}, \quad \beta_7 = \frac{(2 + k \sqrt{2} - 2 \cos q)(2 + \sqrt{2} k \cos q)}{k + \sqrt{2} \cos q}
$$

To find eigenvectors $\{\mathbf{h}(q)\}$ of $\mathbf{H}(q)$ we use the row reduction echelon method to solve

$$
(\mathbf{H}(q) - \lambda \mathbf{I})\mathbf{h}(q) = \mathbf{0}
$$

and get the equation

$$
\begin{bmatrix}
1 & 0 & \frac{1}{-\lambda} \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \end{bmatrix}
\begin{bmatrix}
\beta_7 + \beta_1 \frac{1}{-\lambda} \\
\beta_6 i + \beta_3 i \frac{1}{-\lambda} \frac{1}{(\beta_6 - \beta_3 \lambda - \beta_1)} \\
\beta_4 i + \beta_3 i \frac{1}{-\lambda} \frac{1}{(\beta_4 - \beta_3 \lambda - \beta_1)} \\
\beta_5 i + \beta_2 i \frac{1}{-\lambda} \frac{1}{(\beta_5 - \beta_2 \lambda - \beta_1)}
\end{bmatrix}
= \begin{bmatrix}
x \\
y \\
w \\
z
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
$$

(A2)

Writing down equations, $z$ is 0 which makes the eigenvector zero but since an eigenvector cannot be zero we take $z$ as any real or complex number. However, taking $z = 1$ and solving for $(x, y, w, z)$ we get

$$
\left\{ \begin{array}{c}
\mathbf{h}(q) \\
\lambda(q) \mathbf{h}(q)
\end{array} \right\} = C \left\{ \begin{array}{c}
\frac{i(\beta_3 + \beta_6 \lambda)}{\lambda(\lambda^2 - \beta_4 \lambda - \beta_1)} \\
\frac{1}{\lambda} \\
\frac{i(\beta_3 + \beta_6 \lambda)}{\lambda^2 - \beta_4 \lambda - \beta_1}
\end{array} \right\}
$$

(A3)

The above expression can be rewritten as

$$
\left\{ \begin{array}{c}
\mathbf{h}(q) \\
\lambda(q) \mathbf{h}(q)
\end{array} \right\} = C \left\{ \begin{array}{c}
\frac{i(\beta_3 + \beta_6 \lambda)}{\lambda^2 - \beta_4 \lambda - \beta_1} \\
\frac{i(\beta_3 + \beta_6 \lambda)}{\lambda(\lambda^2 - \beta_4 \lambda - \beta_1)}
\end{array} \right\}
$$

(A4)
Constructing real-valued cyclic Raleigh wave solutions:

**Complex Eigenvalues:**

\[ h(q) = C_1 \begin{bmatrix} a + ib \\ -c + id \end{bmatrix}; \]

\[ d_{nm}^{(1)} = C_2 \rho_n(q) \left\{ \begin{bmatrix} a \cos (\theta_n q_m) - b \sin (\theta_n q_m) \\ c \cos (\theta_n q_m) - d \sin (\theta_n q_m) \end{bmatrix} + i \begin{bmatrix} a \sin (\theta_n q_m) + b \cos (\theta_n q_m) \\ c \sin (\theta_n q_m) + d \cos (\theta_n q_m) \end{bmatrix} \right\} \] (A5)

\[ d_{nm}^{(2)} = C_3 \rho_n(q) \left\{ \begin{bmatrix} a \cos (-\theta_n q_m) + b \sin (-\theta_n q_m) \\ -c \cos (-\theta_n q_m) - d \sin (-\theta_n q_m) \end{bmatrix} + i \begin{bmatrix} a \sin (-\theta_n q_m) - b \cos (-\theta_n q_m) \\ -c \sin (-\theta_n q_m) + d \cos (-\theta_n q_m) \end{bmatrix} \right\} \] (A6)

Possible cyclic harmonic solutions are obtained by summing and subtracting the corresponding real and imaginary parts of the above equations as shown below:

\[ d_{nm} = \text{Re} \; d_{nm}^{(1)} + \text{Re} \; d_{nm}^{(2)} = C_2 \rho_n(q) \begin{bmatrix} a \cos q_m \\ c \sin q_m \end{bmatrix} \] (A7)

\[ d_{nm} = \text{Im} \; d_{nm}^{(1)} - \text{Im} \; d_{nm}^{(2)} = C_2 \rho_n(q) \begin{bmatrix} b \cos q_m \\ c \sin q_m \end{bmatrix} \] (A8)

**Real Eigenvalues:**

Case 1: \( h(q) = C_2 \begin{bmatrix} ib \\ c \end{bmatrix} \)

\[ d_{nm} = C_2 \bar{R}^n(q) \left\{ \begin{bmatrix} -b \sin q_m \\ c \cos q_m \end{bmatrix} + i \begin{bmatrix} b \cos q_m \\ c \sin q_m \end{bmatrix} \right\} \] (A9)

The real-cyclic solution is the imaginary part of the solution above:

\[ d_{nm} = C_2 \bar{R}^n(q) \begin{bmatrix} b \cos q_m \\ c \sin q_m \end{bmatrix} \] (A10)

Case 2: \( h(q) = C_2 \begin{bmatrix} b \\ ic \end{bmatrix} \)

\[ d_{nm} = C_2 \bar{R}^n(q) \left\{ \begin{bmatrix} b \cos q_m \\ -c \sin q_m \end{bmatrix} + i \begin{bmatrix} b \sin q_m \\ c \cos q_m \end{bmatrix} \right\} \] (A11)

The real-cyclic solution is the real part of the solution above

\[ d_{nm} = C_2 \bar{R}^n(q) \begin{bmatrix} b \cos q_m \\ -c \sin q_m \end{bmatrix} \] (A12)

Finding the zero-eigenvalue relationship for a 2DoF x-braced lattice:
\[
\det \mathbf{H}(q) = \prod_{i=1}^{n} \lambda_i(q) = \frac{k(k+\sqrt{2} \cos q)^2}{k(k+\sqrt{2} \cos q)^2}
\]

(A13)

Since \( \lambda(q) = 0 \) when \( \det \mathbf{H}(q) = 0 \), we write the zero-eigenvalue relationship as
\[
k + \sqrt{2} \cos q = 0
\]

(A14)

The transfer matrix \( \mathbf{H}(q) \) when we introduce aspect ratio \( \alpha \):
\[
\mathbf{H}(q) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\gamma_1 & \gamma_3 & \gamma_4 & \gamma_6 \\
\gamma_2 & \gamma_4 & \gamma_5 & \gamma_7
\end{bmatrix}
\]

(A15)

\[
\gamma_1 = \frac{-(1 + \alpha^2)^2 \cos q + 2k\alpha^3\sqrt{1 + \alpha^2} \cos 2q}{2k\alpha^3\sqrt{1 + \alpha^2} + (1 + \alpha^2)^2 \cos q},
\]

\[
\gamma_2 = \frac{2\alpha((1 + \alpha^2)^2 + 2k\alpha^3\sqrt{1 + \alpha^2} \cos q) \sin q}{2k\alpha^3\sqrt{1 + \alpha^2} + (1 + \alpha^2)^2 \cos q},
\]

\[
\gamma_3 = \frac{4k\alpha^2\sqrt{1 + \alpha^2} \cos q \sin q}{2k\alpha^3\sqrt{1 + \alpha^2} + (1 + \alpha^2)^2 \cos q},
\]

\[
\gamma_4 = \frac{2(1 + \alpha^2(2 + \alpha^2 + 2k\alpha\sqrt{1 + \alpha^2})) \cos q}{2k\alpha^3\sqrt{1 + \alpha^2} + (1 + \alpha^2)^2 \cos q},
\]

\[
\gamma_5 = \frac{2\alpha((1 + \alpha^2)(2 + \alpha^2 + 2k\alpha\sqrt{1 + \alpha^2})) \sin q}{2k\alpha^3\sqrt{1 + \alpha^2} + (1 + \alpha^2)^2 \cos q},
\]

\[
\gamma_6 = \frac{2\alpha(-2k\alpha\sqrt{1 + \alpha^2} - (1 + \alpha^2)^2 + (1 + \alpha^2)^2 \cos q) \sin q}{2k\alpha^3\sqrt{1 + \alpha^2} + (1 + \alpha^2)^2 \cos q},
\]

\[
\gamma_7 = \frac{-\sqrt{1 + \alpha^2}((1 + \alpha^2)^{3/2} + 2k\alpha^3 \cos q)(-2k - (1 + \alpha^2)^{3/2} + (1 + \alpha^2)^{3/2} \cos q)}{k(2k\alpha^3\sqrt{1 + \alpha^2} + (1 + \alpha^2)^2 \cos q)}
\]

Finding total strain energy along index \( n \):
\[
\omega = \frac{1}{2} \sum_{n, m, m'} \mathbf{d}_{n, m} \cdot \mathbf{k}_{n-n, m-m} \cdot \mathbf{d}_{n, m'}
\]

(A16)

\[
W = \frac{1}{2} \sum_{M} \sum_{n, m, m'} \mathbf{d}_{n, m} \cdot \mathbf{k}_{n-n, m-m} \cdot \mathbf{d}_{n, m'}
\]

(A17)