Exact soliton-like probability measures for interacting jump processes

M.-O. Hongler, (max.hongler at epfl.ch)
EPFL/ IMT/ LPM
Station 17
CH-1015 Lausanne, (Switzerland)

Abstract
The cooperative dynamics of a 1-D collection of Markov jump, interacting stochastic processes is studied via a mean-field approach. In the time-asymptotic regime, the resulting nonlinear master equation is analytically solved. The nonlinearity compensates jumps induced diffusive behavior giving rise to a soliton-like stationary probability density. The soliton velocity and its sharpness both intimately depend on the interaction strength. Below a critical threshold of the strength of interactions, the cooperative behavior cannot be sustained leading to the destruction of the soliton-like solution. The bifurcation point for this behavioral phase transition is explicitly calculated.

Keywords: jump Markov processes, exact solution of a nonlinear master equation, mean-field description of interacting stochastic processes, soliton-like propagating probability measures.

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1 Introduction

Interacting stochastic agents are modeled by a collection of nonlinearly coupled Markovian stochastic processes. Inspired by the dynamics recently exposed in [Balázs 2014], we focus on pure, right-oriented jump processes. For large and homogeneous swarms, the mean-field description offers a powerful method to characterize the resulting nonlinear global dynamics. Adopting the MF approach, the swarm behavior is summarized into a field density variable obeying
a nonlinear master equation. Such partial differential integral equations are in general barely completely solvable. Nevertheless, several explicitly solvable models have been recently studied [Hongler 2014, Balázs 2014]. Our present goal is to enrich this yet available collection by proposing an intrinsically nonlinear extension of the recent models introduced by [Balázs 2014]. Models involving pure jumps complete the solvable models with dynamics driven either by Brownian Motion and or by alternating Markov renewal processes [Hongler 2014]. For strong enough mutual interactions, we explicitly observe the existence of a stationary probability measure propagating like a soliton. This soliton-like dynamics can be formed since the underlying nonlinear mechanism due to interactions exactly compensates the jump induced diffusion. This exhibits a close analogy with nonlinear wave dynamics where nonlinearity compensates the velocity dispersion. Since the model is uni-dimensional, long-range interactions between the agents are mandatory for the existence of cooperative behaviors here described by soliton-like probability measures. Decreasing the strength of the mutual interactions, via a barycentric modulation function similar to the one used in [Balázs 2014], we reach a critical threshold below which no stable cooperative behavior can be sustained. The critical threshold where the behavioral phase transition occurs can here be exactly calculated.

2 Linear pure jump stochastic processes

Let us first describe the dynamics of a single, isolated jump process which later in section 2, will enter into the composition of our interacting swarm. On \( \mathbb{R} \), we consider the right-oriented jump Markovian process \( X(t) \) characterized by the (linear) master equation:

\[
\partial_t P(x, t) = -P(x, t) + \int_{-\infty}^x P(y, t) \varphi(x - y) dy,
\]

where \( P(x, t) \) with \( P(x, 0) = f(x) \) stands for the transition probability density. The function \( \varphi(x) : \mathbb{R} \to \mathbb{R}^+ \) defines the probability density for the (right oriented) lengths of the process jumps. Taking the \( x \)-Laplace transform of Eq. (1) and taking into account of the convolution structure, we obtain directly:

\[
\partial_t \tilde{P}(s, t) = -[1 - \tilde{\varphi}(s)] \tilde{P}(s, t),
\]

which solution reads:

\[
\tilde{P}(s, t) = e^{-t + \tilde{\varphi}(s)t},
\]
where in writing Eq. (3), we already did assume the initial condition:

\[ P(x, t) \mid t=0 = \delta(x). \] (4)

**Example.** Consider the dynamics obtained when \( \varphi(x) = \lambda e^{-\lambda x} \) yielding \( \tilde{\varphi}(s) = \frac{\lambda}{\lambda + s} \) and when the initial probability density is \( f(x) = \delta(x) \). Accordingly Eq. (3) reads:

\[ \tilde{P}(s, t) = e^{-t} \left[ e^{t \left( \frac{\lambda}{\lambda + s} \right)} \right] = e^{-t} \left\{ \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \left[ \frac{1}{\lambda + s} \right]^n \right\}. \] (5)

The Laplace inversion of Eq. (5) yields:

\[ P(x, t) = e^{-t} \left\{ \delta(x) + e^{-\lambda x} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \frac{x^{n-1}}{(n-1)!} \right\} \] (6)

For \( J(x, t) \), we can write:

\[ J(x, t) = \frac{d}{dx} \left\{ \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \frac{x^n}{n!} \right\} = \frac{\sqrt{\lambda t}}{\sqrt{x}} I_1 \left( 2\sqrt{\lambda x t} \right) \] (7)

where \( I_m(z) \) stands for the \( m \)-modified Bessel’s functions. Hence the final probability density \( P(x, t) \) reads:

\[ P(x, t) = e^{-t} \left\{ \delta(x) + e^{-\lambda x} \sqrt{\lambda t} \sqrt{x} I_1 \left( 2\sqrt{\lambda x t} \right) \right\}, \quad x \in \mathbb{R}^+ \] (8)

and one may explicitly verify that one indeed has: \( \int_{\mathbb{R}^+} P(x, t)dx = 1 \), (use the entry 6.643(2) in [Gradshteyn 80]).

For time asymptotic regimes, Eq. (8) behaves as:

\[ \lim_{t \to \infty} P(x, t) \simeq \frac{1}{2\sqrt{\pi x}} \lambda \left( \frac{\lambda t}{4} \right)^{1/2} e^{-\left[ \sqrt{\lambda x} - \sqrt{\lambda t} \right]^2/4}, \] (9)

exhibiting therefore a diffusive propagating wave with vanishing amplitude and velocity \( V := \frac{1}{\lambda} \). Due to translation invariance of the dynamics, we note that \( P(x - y, t) \) fulfills a \( \delta(x - y) \) initial condition.
Hence, when \( P(x, 0) = f(x) \), the linearity of the dynamics Eq.\( \text{(1)} \) enables to write:

\[
\begin{align*}
\begin{cases}
P_f(x, 0) &= f(x), \\
P_f(x, t) &= \int_{\mathbb{R}^+} P((x - y), t)f(y)dy.
\end{cases}
\end{align*}
\]

(10)

3 Non-linear Markovian jump processes

Keeping the jumps probability density as \( \varphi(x) = \lambda e^{-\lambda x} \), let us now consider a large homogeneous collection of identical processes evolving like Eq.\( \text{(1)} \) now subject of mutual long-range interactions. The class of interactions we consider yields, in the mean-field limit, the nonlinear master equation:

\[
\begin{align*}
\Omega(x, t) &= \int_{\mathbb{R}^+} g(z - \langle X(t) \rangle) \partial_z G(z, t)dz \\
\partial_{x_t} G(x, t) &= -\Omega(x, t)\partial_x G(x, t) + \int_{-\infty}^x \Omega(y, t)\partial_y G(y, t)\lambda e^{-\lambda(x-y)}dy, \\
\langle X(t) \rangle &= \int_{\mathbb{R}^+} y \partial_y G(y, t)dy,
\end{align*}
\]

(11)

where \( G(x, t) \) stands for the cumulative distribution of the a nonlinear jump process, (i.e. \( G(x, t) \) is monotonously increasing with boundary conditions \( G(-\infty, t) = 0 \) and \( G(\infty, t) = 1 \)). Note that while in Eq.\( \text{(1)} \) the jumping rate is unity, in Eq.\( \text{(11)} \) it is replaced by \( \Omega(x, t) > 0 \) which is explicitly state-dependent. This is precisely where the mutual interaction introduce a strong nonlinearity into the dynamics. In the sequel, we focus on cases where \( g(x) = g(-x) > 0 \).

For asymptotic time, we now postulate that Eq.\( \text{(11)} \) admits \( \xi \)-functional dependent solutions with \( \xi = (x - Vt) \) and with the even symmetry:

\[
\int_{\mathbb{R}} \xi \partial_\xi G(\xi)d\xi = 0,
\]

(12)

where \( V \) is a propagating velocity parameter. In terms of \( \xi \), Eq.\( \text{(11)} \) can be rewritten as:

\[
V \left[ \partial^2_{\xi \xi} G(\xi) + \lambda \partial^2_{\xi} G(\xi) \right] = \partial_\xi \{ \Omega(\xi)\partial_\xi G(\xi) \}.
\]

(13)

Defining \( L(\xi) := \log [\partial_\xi G(\xi)] \), after one integration step where the integration constant is taken to be zero, Eq.\( \text{(13)} \) can be rewritten as:

\[
V \partial^2_{\xi} L(\xi) = -\lambda V + \int_{\xi}^\infty [g(\eta)\partial_\eta G(\eta)] d\eta.
\]

(14)
Assuming now a functional dependence \( g(\xi) = \cosh^{-n}(\xi) \) with \( n \in \mathbb{R} \), by direct substitution, it is immediate to see that Eq. (14) is solved by the (normalized) probability density \( \partial_\xi G(\xi) \):

\[
\begin{align*}
\partial_\xi G(\xi) &= \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{m}{2}\right)} \cosh^{-m}(\xi), \quad m > 0, \\
V &= \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{m}{2}\right)}.
\end{align*}
\]

Due to the \( \xi \)-symmetry of the probability density \( \partial_\xi G(\xi) \), Eq. (12) is trivially fulfilled.

For \( n \in ]2, -\infty[ \), Eq. (15) implies that a stationary propagating density \( \partial_x G(x) \) is sustained by the nonlinear dynamics Eq. (11). However, for short decaying \( g(x) \)-modulation, occurring when \( n > 2 \), no stationary propagating probability density exists, (i.e. for this parameter range, \( m < 0 \) in Eq. (14) and the solution cannot be normalized to unity as required for a probability measure). For this exactly solvable dynamics, we also observe that the average jump length \( \lambda^{-1} \) and the barycentric modulation strength controlled by the factor \( n \) are intimately dependent control parameters. In addition, we note that for large \( m \), the asymptotic expansion of the \( \Gamma \)-function implies that \( \lim_{m \to \infty} V \simeq \sqrt{m} \).

Illustration. Along the same lines as exposed in [Hongler 2014], the nonlinear dynamics given by Eq. (11) can be viewed as representing the mean-field evolution associated with a large population of stochastic jumping agents subject to a mutual imitation process. The swarm dynamics is described via the probability density function \( \partial_x G(x, t) \) obeying a nonlinear partial differential equations (PDE). Agents mutual interactions are responsible for the state-dependent jumping rate \( \Omega(x, t) \) in Eq. (11). The functional form of \( \Omega(x, t) \) simultaneously en-globes two distinct nonlinear features, namely:

a) imitation process. To isolate this process, we may consider the case \( g(x) \equiv 1 \), (i.e. \( n = 0 \)) implying that

\[
\Omega(x, t) = 1 - G(x, t).
\]

The resulting state-dependent jumping rate Eq. (16) induces a traveling and compacting tendency. As the agents are subject to pure right-oriented jump, Eq. (16) effectively describes situations where the laggard agents jump more frequently than the leaders, (i.e. laggards try to effectively imitate the leaders behavior).
b) barycentric range modulation of the mutual interactions. The modulation obtained when \( g(x) \neq 1 \) describes the relative importance attributed to interactions with agents remote from the barycenter \( \langle X(t) \rangle \) of the swarm. Here, we may separate two distinct tendencies:

i) when \( n \in [0, 2] \), far remote agents tend to not influence the dynamics. In this case, the resulting behavior can be referred as a **weak cooperate identity** and the propagating probability density given by Eq. (15) exhibits the shape of a **table-top soliton** with a plateau increasing when the limiting value 2 is approached. AOn observes a comparatively low propagating velocity \( V \) of these table-top like aggregates. Again, we emphasize that for \( n > 2 \), the cooperative interactions are not strong enough to sustain the propagation of a cooperative behavior in asymptotic time. This is well known in general for 1-D stochastic interacting system, (the Ising model being the paradigmatic example) where no cooperative phase can be formed when the interactions operate on too limited ranges.

ii) for \( n < 0 \), the \( g(x) \) modulation effectively gives rise to a **strong cooperate identity**. Far remote agents increasingly influence the swarm. This gives rise to sharply peaked solitons-like probability densities propagating with high propagating velocities.

References

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