A SEMICLASSICAL APPROACH TO GEOMETRIC X-RAY TRANSFORMS IN THE PRESENCE OF CONVEXITY

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Abstract. In this short paper we introduce a variant of the approach to inverting the X-ray transform that originated in the author’s work with Uhlmann. The new method is based on semiclassical analysis and eliminates the need for using sufficiently small domains and layer stripping for obtaining the injectivity and stability results, assuming natural geometric conditions are satisfied.

1. Introduction

In this short paper we introduce a variant of the approach to inverting the X-ray transform that originated in the author’s work with Uhlmann [21]. Here recall that on a compact Riemannian manifold with boundary $M$ the X-ray transform is the map $I$ that assigns to each $f \in C^0(M)$ the function $If$ on $SM$, the unit sphere bundle, defined by

$$(If)(\beta) = \int_{\gamma_{\beta}} f(\gamma_{\beta}(t)) \, dt,$$

where $\gamma_{\beta}$ is the geodesic whose lift to $SM$ goes through $\beta$. In fact, other similar families of curves work equally well, as observed by H. Zhou in the appendix of [21]. There is also no need to consider $SM$ the unit sphere bundle; indeed it is convenient to consider $If$ defined on $TM \setminus o$ as a homogeneous function of degree $-1$; in addition, compactness can be relaxed. Here the geodesics are assumed to be sufficiently well-behaved so that the integrals are over finite intervals, i.e. the geodesics reach the boundary in finite affine parameter (i.e. are of finite length over $M$); the more strict requirements later on make this automatic. It is also useful to consider $M$ as a smooth domain in a manifold with boundary $\bar{M}$; in this case we regard $f$ as a function supported in $M$ (via extension by $0$); one can also work more generally with distributions on $\bar{M}$ supported in $M$. The inverse problem is to recover $f$ from $If$, i.e. to construct a left inverse, or at least show that $I$ is injective with suitable stability estimates. Under appropriate hypotheses, discussed below, this was achieved in [21], in the strong sense that the method was in fact constructive, at least on suitably (depending on both geometric and analytic information) small subdomains of $M$; given this, a layer stripping method allowed one to proceed to a global determination. In the present paper we provide a new semiclassical method that under slightly stronger, but geometrically natural, hypotheses eliminates the need for sufficiently small domains and layer stripping, and obtains these results in a single step globally. Moreover, with this new approach, even if the natural geometric
conditions are not satisfied for the one-step global determination, it can be used for the determination in geometrically natural subdomains (so typically larger subdomains than those of [21]), and combined with layer stripping. We state the result slightly informally first, but spelling out the geometric hypotheses; a more precise version is given in Theorem [12] and the role of the geometric hypotheses is also explained in the discussion following this statement.

**Theorem 1.1** (Simplified version of Theorem [12]). Suppose \( M \) is a compact Riemannian manifold with boundary of dimension \( \geq 3 \) equipped with a function \( x \) with strictly convex level sets and \( dx \) non-zero. Suppose also that geodesics do not have points conjugate to their points of tangency to the level sets of \( x \). Then the geodesic X-ray transform \( I \) is left invertible on \( \mathcal{H}^s, s \in \mathbb{R} \).

Here \( \mathcal{H}^s \) is the space of distributions in the Sobolev space \( H^s \) supported in the domain, using Hörmander’s notation [5]. The concave analogue of the convexity condition is defined in (1.1); see the next paragraph for an example.

An example for the hypotheses of the theorem being satisfied is domains in a non-positively curved simply connected manifold, and more generally in a simply connected manifold with no focal points: one can use the distance function from a point outside \( M \) as \( x \). The level sets are then strictly convex from the side of the sub-level sets, and strictly concave from the side of the super-level sets; it is the latter that is directly relevant below. This example, as well as examples for which the convexity conditions on \( x \) are satisfied, but not necessarily the lack of conjugate points, are discussed in [18, 21], see also a thorough study in [12, Section 2] and references therein. As mentioned already, our new approach has advantages even in the latter setting; see Theorem [13] for details.

In order to explain the context of this theorem, let us recall that typically one approaches such inverse problems by considering the normal operator \( I^* I \), or some modification of it. In the present context (using the above over-parameterization, in that many \( \beta \) correspond to the same geodesic) \( I^* \) is replaced by a closely related operator of the form

\[
(Lv)(z) = \int_{S_2 M} v(\beta) \, d\nu(\beta)
\]

of integration along the geodesics through \( z \) (here \( \nu \) is a smooth positive measure whose precise choice is actually irrelevant); ideally one would like \( LI \) invertible, at least up to ‘trivial’ errors.

The key idea of [21] was to introduce an artificial boundary, which is a hypersurface in \( M \), which meant that rather than working on all of \( M \), one initially attempts to recover \( f \) from \( If \) in the region on one side of this hypersurface. More precisely, one is working with a family of hypersurfaces which are level sets of a function \( \tilde{x} \). This function \( \tilde{x} \) is required to be strictly concave from the side of the super-level sets, i.e. for any \( t_0 \) we demand that

\[
(1.1) \quad \frac{d(\tilde{x} \circ \gamma_\beta)}{dt}(t_0) = 0 \implies \frac{d^2(\tilde{x} \circ \gamma_\beta)}{dt^2}(t_0) > 0.
\]

\(^1\)If the perspective is changed and \( If \) is regarded as a function on the inward or outward pointing unit sphere bundle over the boundary, then \( L \) would be the actual adjoint with respect to an appropriate measure, see e.g. [4, 13, 13].
If \( \hat{x} \) is normalized so that \( M \) is contained in \( \hat{x} \leq 0 \), then the main result of [21] was invertibility, in the above sense, in a region \( \hat{x} \geq -c \), where \( c > 0 \) was sufficiently small, depending on various analytic quantities. Here \( \hat{x} = -c \) is the artificial boundary, and the strict concavity is in fact only required in \( \hat{x} \geq -c \). This could be repeated in a layer stripping argument, allowing a global result after a multi-step process. The analytic heart of this artificial boundary argument involves Melrose’s algebra of scattering pseudodifferential operators [9], which are a geometric generalization of natural pseudodifferential operators on (asymptotically) Euclidean spaces; while the artificial boundary is at a geometrically finite place, the analytic way of obtaining a modified ‘normal operator’ effectively pushes it to infinity, where the scattering algebra can be used for the analysis. This is done by, most crucially (another modification is also needed), inserting a localizer \( \chi(\beta) \) into the formula for \( L \) that concentrates on geodesics almost tangential to the level sets of \( \hat{x} \), with the approximate tangency becoming more strict as \( \hat{x} \) approaches the artificial boundary, i.e. as \( x = \hat{x} + c \to 0^+ \); the precise way this happens determines the analytic structure:

\[
(Lv)(z) = \int_{S_{z,M}} \chi(\beta)v(\beta) \, d\nu(\beta).
\]

The new method introduced in this paper is based on semiclassical analysis and eliminates the need for using sufficiently small domains and layer stripping for obtaining the injectivity and stability results, assuming natural geometric conditions are satisfied. If these conditions are satisfied, it works in one step globally and it uses a variant of the standard semiclassical algebra. On the other hand, it can also be used for localized problems (in the sense of the artificial boundary), in which case it uses a semiclassical version of Melrose’s scattering algebra [8]. As final (after perhaps layer stripping) injectivity or stability statements, the results are the same as one would obtain with the original techniques of [21], but with a more transparent and streamlined proof. This is reflected by the stronger technical theorems on the modified normal operator for the X-ray transform. The analytic heart of the approach is again to introduce a localizer \( \chi_h \), which now also depends on \( h \); for \( h \) small this again localizes very close to geodesics tangential to level sets of \( \hat{x} \):

\[
(L_hv)(z) = \int_{S_{z,M}} \chi_h(\beta)v(\beta) \, d\nu(\beta);
\]

see the proof of Proposition 3.2 for the concrete \( \chi_h \).

Concretely, on manifolds of dimension \( \geq 3 \), in the case of no conjugate points but with a convex foliation still, we directly obtain a modified normal operator that is invertible; this involves the use of a semiclassical foliation pseudodifferential algebra (the aforementioned variant), but not the scattering algebra, and it also eliminates the need for making small steps (thin layers) in the layer stripping approach. One in fact needs a weaker requirement on the lack of conjugate points for curves from point of tangency to the foliation, which in dimension \( > 3 \) can be further weakened similarly to the work of Stefanov and Uhlmann [16]. Since in this case there is no need to renormalize \( \hat{x} \) as there is no artificial boundary, in order to simplify the notation we write \( x = \hat{x} \); this allows a notationally uniform treatment later. We then have the following global (in the sense of global hypotheses and conclusions) theorem:
**Theorem 1.2.** Suppose $M$ is a compact Riemannian manifold with boundary of dimension $\geq 3$ equipped with a function $x$ with strictly convex level sets and $dx$ non-zero. Suppose also that geodesics do not have points conjugate to their points of tangency to the level sets of $x$. Then the semiclassically modified normal operator $A$, see (3.6) with $\Phi(x) = -x$, of the geodesic X-ray transform is a left invertible elliptic order $-1$ pseudodifferential operator.

Here left invertible means that there is an order 1 pseudodifferential operator $G$ on $M$ such that $GA = 1d$ on $\dot{H}^s$ for all $s$. An immediate consequence, due to the fact that the operator $L$ (see (3.5)) used in the definition of our modified normal operator $A$ is a standard Fourier integral operator for fixed non-zero $h$, with appropriate order and canonical relation, is:

**Corollary 1.1.** Let $s \in \mathbb{R}$. Under the hypotheses of the theorem, the X-ray transform $I$ is injective on $\dot{H}^s$ and we have stability estimates: there exists $C > 0$ such that for all $f \in \dot{H}^s$, we have $\|f\|_{\dot{H}^s(M)} \leq C\|I f\|_{\dot{H}^{s+1/2}(SM)}$.

On the other hand, if conjugate points are present, one can still work in appropriately small layers as determined by the geometry (to eliminate the conjugate points), but without the need to further shrink the size of the steps to obtain invertibility as required in [21]; this approach still uses the scattering algebra at the artificial boundary. This gives rise to the following local (in the sense that the hypotheses and conclusions are local, on a subdomain of $M$) theorem:

**Theorem 1.3.** Suppose $M$ is a compact Riemannian manifold with boundary of dimension $\geq 3$ equipped with a function $\tilde{x}$ with strictly concave level sets in $\tilde{x} \geq -c$, from the super-level sets, and $d\tilde{x}$ non-zero. Suppose also that geodesics contained in the region $\tilde{x} \geq -c$ do not have points conjugate to their points of tangency to the level sets of $\tilde{x}$. Then, with $x = \tilde{x} + c$, the semiclassically modified normal operator $A$, see (3.6), with $\Phi(x) = x^{-1}$ and with cutoff $\tilde{\chi}$ given in (3.13), of the geodesic X-ray transform is a left invertible elliptic order $(-1, -2)$ scattering pseudodifferential operator.

One can then proceed inductively, using layer stripping, to turn this into a global result, as in [21]; as already mentioned, then the ‘headline result’ obtained is that of [21], but with a geometrically more natural proof.

The method is also applicable to other X-ray transform problems, such as the X-ray transform on asymptotically conic, e.g. asymptotically Euclidean, spaces, studied in work with Zachos [22], based in part on Zachos’ work [25], where it is harder to implement the original ‘thin layer’ approach of [21].

Notice that for non-linear problems there is an additional role in localizing near $\partial M$, not addressed by the semiclassicalization, namely if one uses a Stefanov-Uhlmann pseudolinearization formula [15], one needs to make sure that the coefficients of the transform in the formula are close to known values, typically at the boundary. This need for localization can be eliminated if the unknown quantity is a priori globally close to a given background, in which case one can obtain injectivity results provided one has injectivity results for the background, without having to introduce the additional small semiclassical parameter, but in order to obtain the prerequisite injectivity results for the background, the semiclassical approach is still very useful.

The plan of this short paper is the following. In Section 2 we introduce the analytic ingredients, namely the semiclassical foliation pseudodifferential algebras, and then
in Section 3 we use this for the analysis of the X-ray transform. The whole of Section 3 consists of the proofs of Theorems 1.2 and 1.3 with Corollary 1.1 deduced as a consequence of the former.

2. THE SEMICLASSICAL ALGEBRA

In this section we discuss an inhomogeneous pseudodifferential semiclassical algebra associated to a foliation $\mathcal{F}$ on a manifold $M$. As usual, it depends on a semiclassical parameter, traditionally denoted by $\hbar \in [0, 1]$; it is an $\hbar$-dependent family of operators on $M$. The standard semiclassical algebra is built from vector fields $hV, V \in \mathcal{V}(M)$ (see e.g. [26]), so semiclassical differential operators are in the algebra over $C^\infty(M)$ generated by these, i.e. locally finite sums of finite products of these with $C^\infty(M)$. Thus, in local coordinates $z \in O \subset \mathbb{R}^n$, $P \in \text{Diff}_h^m(M)$ means that

$$P = \sum_{|\alpha| \leq m} a_{\alpha}(z, h)hD_z^\alpha,$$

with $a_{\alpha} \in C^\infty(\mathbb{R}^n \times [0, 1])$, supported in $O$. Thus, as a traditional differential operator, $P$ degenerates at $h = 0$, but as a semiclassical operator it does not; its semiclassical principal symbol is

$$p(z, \zeta) = \sum_{|\alpha| \leq m} a_{\alpha}(z, 0)\zeta^\alpha,$$

obtained by replacing $hD_z$ by $\zeta$, and evaluating the coefficients at $h = 0$, and the operator is semiclassically elliptic if there is $c > 0$ such that

$$|p(z, \zeta)| \geq c\langle \zeta \rangle^m,$$

i.e. $p$ is non-vanishing and is elliptic in the standard sense.

Our new semiclassical foliation algebra is built from vector fields which are either semiclassical in the sense above, i.e. $hV, V \in \mathcal{V}(M)$, or $h^{1/2}$-semiclassical and tangent to the foliation:

$$\mathcal{V}_{h,\mathcal{F}}(M) = h\mathcal{V}(M) + h^{1/2}\mathcal{V}(M; \mathcal{F}),$$

where $\mathcal{V}(M; \mathcal{F})$ denotes the Lie algebra of vector fields tangent to the foliation. (Here $h$ is used as the actual parameter, and $\hbar$ is used to denote the semiclassical nature of the Lie algebra; in other references the subscript $\text{scl}$ may be used for ‘semiclassical’. In local coordinates, in which the foliation is locally given by $x = (x_1, \ldots, x_k)$ being constant, and remaining coordinates (which are thus coordinates along the leaves) are $y_1, \ldots, y_{n-k}$ (we also write $z = (x, y)$), this means that the semiclassical foliation vector fields are

$$\sum_{j=1}^k a_j(x, y, h)hD_{x_j} + \sum_{j=1}^{n-k} b_j(x, y, h)h^{1/2}D_{y_j}.$$

Correspondingly, elements of the algebra of semiclassical foliation differential operators of order $m$, $\text{Diff}_{h,\mathcal{F}}^m(M)$, are of the form

$$P = \sum_{|\alpha| + |\beta| \leq m} a_{\alpha\beta}(x, y, h)(hD_x)^\alpha(h^{1/2}D_y)^\beta,$$

and the semiclassical foliation principal symbol is

$$p(x, y, \xi, \eta) = \sum_{|\alpha| + |\beta| \leq m} a_{\alpha\beta}(x, y, 0)\xi^\alpha\eta^\beta,$$
with semiclassical ellipticity meaning that there is $c > 0$ such that
\[ |p(x, y, \xi, \eta)| \geq c \langle (\xi, \eta) \rangle^m. \]

A somewhat different perspective of $\mathcal{V}(M)$ is that it is a conformal version of the adiabatic Lie algebra. In the latter, with $\epsilon$ the adiabatic parameter, one considers a fibration (rather than just a foliation) $\mathcal{F}$, and the sum of $\epsilon \mathcal{V}(M)$ and $\mathcal{V}(M; \mathcal{F})$, so in local coordinates as above the vector fields are
\[
\sum_{j=1}^{k} a_j(x, y, \epsilon)\epsilon D_{x_j} + \sum_{j=1}^{n-k} b_j(x, y, \epsilon)D_{y_j}.
\]
Thus, with $\epsilon = h^{1/2}$, our Lie algebra is $\epsilon$ times the adiabatic Lie algebra, i.e. a conformal, more precisely, a $1$-conformal (in that the conformal factor is the first power of the adiabatic parameter $\epsilon$) version of the adiabatic Lie algebra. The conformal factor makes this algebra more localized, just like in the comparison of the more microlocalized scattering [9] and the more global $b$-algebras of Melrose [7,8]; this is what allows for relaxing the requirements on $\mathcal{F}$ to being a foliation.

Returning to our semiclassical perspective, we turn this into a pseudodifferential operator algebra $\Psi(\mathcal{F})$ as follows. First starting locally with $\mathbb{R}^n$ and the foliation $\mathcal{F}_{\mathbb{R}^n}$ given by the joint level sets of the $x_j$, we consider symbols $a$ with
\[
|(D_\xi^\alpha D_\eta^\beta a)(z, \xi, \eta, h)| \leq C_{\alpha\beta} \langle (\xi, \eta) \rangle^{m-|\beta|},
\]
i.e. the standard semiclassical class (one can also require differentiability in $h$; since this is a parameter, i.e. there is no differentiation in it, the choice is mostly irrelevant), but quantizing it according to the foliation as
\[
(A_h u)(x, y) = (Au)(x, y, h) = (2\pi)^{-n} h^{-n/2-k/2} \int e^{i(x-x') \cdot \xi/h + i(y-y') \cdot 1/2} a(x, y, \xi, \eta, h)u(x', y', h) \, d\xi \, d\eta \, dx' \, dy'.
\]
This gives a class $\Psi(h, \mathcal{F})(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n})$, where the uniformity statement of the symbolic estimates in $z$ on $\mathbb{R}^n$ is not shown in the notation.

Of course, one can change variables as $\tilde{\eta} = h^{1/2}\eta$, to obtain the usual semiclassical quantization
\[
(2\pi)^{-n} h^{-n} \int e^{i(x-x') \cdot \xi/h + i(y-y') \cdot \eta/h} a(x, y, \xi, \eta, h)u(x', y', h) \, d\xi \, d\eta \, dx' \, dy'.
\]
of the symbol
\[
a(x, y, \xi, \eta, h) = a(x, y, \xi, h^{-1/2}\eta, h); \]
regarded as a semiclassical symbol, $\tilde{a}$ is of a weaker type:
\[
|D_\xi^\alpha D_\eta^\beta D_{\tilde{\eta}}^\delta \tilde{a}(x, y, \xi, \eta, h)| \leq C_{\alpha\beta\gamma} h^{-|\delta|/2} \langle (\xi, h^{-1/2}\eta) \rangle^{m-|\gamma|-|\delta|},
\]
and while this is sufficient to push the semiclassical algebra through, it is more precise to consider the foliation setup above.

**Remark 2.1.** As an aside, one can also consider this as a ‘blown-down’ 2-microlocal coisotropic semiclassical algebra corresponding to the coisotropic $\tilde{\eta} = 0$, which corresponds exactly to the joint characteristic set of the semiclassical vector fields tangent to $\mathcal{F}$. The type of this algebra is a $1/2$-type, in that from the semiclassical perspective
one blows up of the coisotropic parabolically at \( h = 0 \) (the parabolic direction being tangent to \( h = 0 \)), so the scaling is \( h^{-1/2} \). More singular (with homogeneity 1) and thus delicate coisotropic algebras, corresponding to hypersurfaces, were introduced by Sjöstrand and Zworski [14]. (Though it was used for a different purpose and from a different perspective, the work [3] of Gannot and Wunsch introduced semiclassical paired Lagrangian distributions to extend the work of de Hoop, Uhlmann and Vasy [2] from the non-semiclassical setting, and this relates closely to 1-homogeneous 2-microlocalization at a coisotropic.) The ‘blown-down’ adjective refers to the fact that from this blow-up perspective, the standard semiclassical behavior (i.e. what happens away from \( \eta \neq 0 \)) is blown down, since \( \eta \to \infty \) as \( h \to 0 \), and we have placed joint symbolic demands on \( a \) in \((\xi, \eta)\).

We now return to a discussion of the basic properties of the semiclassical foliation algebra. One can more generally allow \( a \) in \((2.1)\) to depend on \( z' = (x', y') \) as well. The standard left- and right-reduction arguments, removing the \( z' \), resp. \( z \), dependence apply, see e.g. [6, 24], and give asymptotic expansions, so for instance the right-reduced version of \( a(z, z', \eta) \) is

\[
b(z', \zeta, h) \sim \sum \frac{1}{\alpha! \beta!} (-hD_\xi)^\alpha (-h^{1/2}D_\eta)^\beta \partial_x^\alpha \partial_y^\beta a|_{z = z' = (x', y')},
\]

while the left-reduced version is

\[
c(z, \zeta, h) \sim \sum \frac{1}{\alpha! \beta!} (hD_\xi)^\alpha (h^{1/2}D_\eta)^\beta \partial_x^\alpha \partial_y^\beta a|_{z' = z = (x, y')}.
\]

This gives (again, see [3, 24]) that the semiclassical foliation pseudodifferential operators form a filtered \(*\)-algebra (with respect to the \( L^2\)-inner product):

\[
A \in \Psi^m_{r, h,F}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}), \quad B \in \Psi^{m'}_{r, h,F}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}) \Rightarrow AB \in \Psi^{m+m'}_{r, h,F}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}),
\]

\[
A \in \Psi^m_{r, h,F}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}) \Rightarrow A^* \in \Psi^m_{r, h,F}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}),
\]

and moreover that with \( \sigma_m(A) = [a] \in S^m/h^{1/2}S^{m-1} \), so for smooth (in \( h^{1/2} \)) \( a \),

\[
\sigma_m(A)(z, \zeta)|_{h = 0} = a(z, \zeta, 0),
\]

we have

\[
\sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B), \quad \sigma_m(A^*) = \overline{\sigma_m(A)}.
\]

One also has the standard elliptic parametrix construction. One says that \( A \), and its principal symbol \( a \), are elliptic if \( a \) has an inverse \( b \in S^{-m}/h^{1/2}S^{-m-1} \) in the sense that \( ab - 1 \in h^{1/2}S^{-1} \); this is equivalent to the lower bound

\[
|a(z, \zeta, h)| \geq c(\zeta)^m, \quad c > 0,
\]

for \( h \) small (i.e. there exists \( h_0 > 0 \) such that the estimate holds for \( h < h_0 \)). Then there is a parametrix \( B \in \Psi^{-m}_{r, h,F}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}) \) such that \( AB - I, BA - I \in h^{\infty}\Psi^{-\infty}_{r, h,F}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}) \).

Furthermore, \( \Psi^m_{r, h,F}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}) \) is invariant under local diffeomorphisms preserving the foliation as is easily seen by the standard Kuranishi trick; this allows the introduction of the class \( \Psi^m_{r, h,F}(M, \mathcal{F}) \) on manifolds, which has all the analogous properties to \( \Psi^m_{r, h,F}(\mathbb{R}^n; \mathcal{F}_{\mathbb{R}^n}) \) discussed above. In addition, with \( H^s_{r, h,F}(M) \) the foliation semiclassical
Sobolev space, i.e. the standard Sobolev space $H^s(M)$ but with the natural $h$-dependent family of norms, so for $s \geq 0$ integer, locally,

$$
\|u\|^2_{H^s_{h, \mathcal{F}}} = \sum_{|\alpha| + |\beta| \leq s} \|(hD_x)^{\alpha}(h^{1/2}D_y)^{\beta} u\|^2_{L^2},
$$

for negative integer $s$ by duality, in general by interpolation (or via the foliation Fourier transform, or via elliptic ps.d.o’s),

$$
\Psi^m_{h, \mathcal{F}}(M; \mathcal{F}) \subset \mathcal{L}(H^s_{h, \mathcal{F}}(M), H^{s-m}_{h, \mathcal{F}}(M))
$$

uniformly in $h$.

A great advantage of the semiclassical algebra, which is maintained by the foliation semiclassical algebra, is that the error of the elliptic parametrix construction is $O(h^\infty)$, thus small for $h$ small, as an element of $\mathcal{L}(H^s_{h, \mathcal{F}}(M), H^s_{h, \mathcal{F}}(M))$, so e.g. $BA = I + E$ as the output of the elliptic parametrix construction means that there exists $h_0 > 0$ such that $(I + E)^{-1}$ exists for $h < h_0$ (and differs from $I$ by an element of $h^\infty \Psi^\infty_{h, \mathcal{F}}(M, \mathcal{F})$), and thus $A$ actually has a left inverse, and similarly it also has a right inverse.

As in [21], we actually work on an ambient manifold $\tilde{M}$ with $M$ a domain with smooth boundary in it, and $A \in \Psi^m_{h, \mathcal{F}}(\tilde{M}; \mathcal{F})$ is elliptic on a neighborhood of $M$. Then there exists $B \in \Psi^-_{h, \mathcal{F}}(\tilde{M}; \mathcal{F})$ such that $AB - I, BA - I \in \Psi^0_{h, \mathcal{F}}(\tilde{M}; \mathcal{F})$ are in fact in $h^\infty \Psi^\infty_{h, \mathcal{F}}(\tilde{M}; \mathcal{F}_{\mathbb{R}^n})$ when localized to a sufficiently small neighborhood of $M$, i.e. for suitable cutoffs $\psi$, identically 1 in $M$,

$$
\psi(AB-I)\psi, \psi(BA-I)\psi \in h^\infty \Psi^\infty_{h, \mathcal{F}}(\tilde{M}, \mathcal{F}).
$$

So in particular, with $E = BA - I$, for $v$ supported in $M$, $\psi v = v$, so $\psi BA \psi = \psi^2 + \psi E \psi$ shows that

$$
(\text{Id} + \psi E \psi)v = \psi BA v.
$$

Now $\text{Id} + \psi E \psi$ is invertible for sufficiently small $h$, so $(\text{Id} + \psi E \psi)^{-1}\psi B$ is a left inverse for $A$ on distributions supported in $M$.

There is an immediate extension of this algebra to the scattering setting of Melrose [9]; this algebra actually can be locally reduced to a standard Hörmander algebra, which in turn was studied earlier by Parenti [10] and Shubin [20]. For simplicity, since this is the only relevant case for us, we consider only the case of a codimension one foliation given by a boundary defining function $x$. Recall that scattering vector fields $V \in \mathcal{V}_\text{sc}(M)$ on a manifold with boundary $M$ are of the form $xV'$, $V'$ is a $b$-vector field, i.e. a vector field on $M$ tangent to $\partial M$, so in local coordinates, they are of the form

$$
a_0(x, y)x^2D_x + \sum a_j(x, y)xD_{y_j}.
$$

As mentioned, we take our foliation to be given by the level sets of $x$, so the foliation tangent sc-vector fields are locally

$$
\sum a_j(x, y)xD_{y_j}.
$$

The semiclassical version of $\mathcal{V}_\text{sc}(M)$ is simply $\mathcal{V}_{\text{sc}, h}(M) = h \mathcal{V}_\text{sc}(M)$ (for which pseudodifferential operators were introduced by Vasy and Zworski [23], but in the local, Euclidean, setting this has a much longer history); the semiclassical foliation version is

$$
\mathcal{V}_{\text{sc}, h, \mathcal{F}}(M; \mathcal{F}) = h \mathcal{V}_\text{sc}(M) + h^{1/2} \mathcal{V}_\text{sc}(M; \mathcal{F}).
$$
Thus, the semiclassical foliation scattering differential operators take the form
\[ \sum_{\alpha + |\beta| \leq m} a_{\alpha \beta}(x, y, h)(hx^2D_x)^\alpha(h^{1/2}xD_y)^\beta. \]

The corresponding pseudodifferential operators \( A \in \Psi_{sc,h,\mathcal{F}}^{m,m}(M, \mathcal{F}) \) again arise by a modified semiclassical quantization of standard semiclassical symbols \( a \), i.e. ones satisfying (conormal in \( x \)) symbol estimates
\[ |(xD_x)^\alpha D_y^\beta a(x, y, \tau, \mu, h)| \leq C_{\alpha \beta \delta}((\tau, \mu))^m - \gamma - |\delta| x^{-l}, \]
namely
\[
A_h u(x, y) = A u(x, y, h) = (2\pi)^{-n}h^{-n/2 - 1/2} \int e^{i \left( \frac{x-x'}{\pi} + \frac{y-y'}{x} \frac{\mu}{h^{1/2}} \right)} a(x, y, \tau, \mu, h) u(x', y') \frac{dx'}{x'} \frac{dy'}{x'} d\tau d\mu.
\]
Thus, in \( x > 0 \), these are just the standard semiclassical foliation operators, in \( h > 0 \) the standard scattering pseudodifferential operators, with the combined behavior near \( x = h = 0 \). In particular we have an elliptic theory as in the semiclassical foliation setting: if \( A \) is elliptic, meaning
\[ |a(x, y, \tau, \mu, h)| \geq c x^{-l}((\tau, \mu))^m, \quad c > 0, \]
for \( h \) sufficiently small, then there is a parametrix \( B \in \Psi_{sc,h,\mathcal{F}}^{-m,m-1}(M, \mathcal{F}) \) with
\[ AB - \text{Id}, BA - \text{Id} \in \mathcal{H}_{-\infty,-\infty}^\infty(M, \mathcal{F}), \]
and there exists \( h_0 > 0 \) such that for \( h < h_0 \), \( A \in \mathcal{L}(H_{sc,h,\mathcal{F}}^{s,r}, H_{sc,h,\mathcal{F}}^{s-m,r-l}) \) is invertible with uniform bounds. One can again proceed with localizing the elliptic parametrix construction as above in case one has a smooth domain \( M \) in an ambient space \( \tilde{M} \).

3. Global X-ray Transform

We now consider the inverse problem for the X-ray transform
\[ If(\beta) = \int_{\gamma_\beta} f(\gamma_\beta(t)) dt, \]
where for \( \beta \in SM \), \( \gamma_\beta \) is the geodesic through \( \beta \) (or in fact other similar families of curves work equally well), i.e. \( \beta = (\gamma_\beta(0), \dot{\gamma}_\beta(0)) \in S_{\gamma_\beta(0)}M \) (with the dot denoting \( t \)-derivatives) utilizing the notation of [21]. We overall follow the approach of [19, Section 4-5] via oscillatory integrals, rather than the blow-up analysis of [21]. Concretely, the approach of the non-semiclassical proof of Proposition 4.2 in [19] underlies most of the local arguments near \( t = 0 \) and as we follow these quite closely, we will be relatively brief.

With \( x \) the function giving the foliation, writing \( x(\gamma_\beta(t)) = \gamma_\beta^{(1)}(t) \), the concavity hypothesis is that
\[ (3.1) \quad \dot{\gamma}_\beta^{(1)}(t) = 0 \implies \ddot{\gamma}_\beta^{(1)}(t) > 0. \]
By compactness considerations this implies that there exist \( \varepsilon > 0 \) and \( C_0 > 0 \) such that
\[ |\gamma_\beta^{(1)}(t)| \leq \varepsilon \implies \gamma_\beta^{(1)}(t) \geq C_0. \]
It is convenient to take advantage of this also holding in a neighborhood \( M' \) of \( M \) in \( \tilde{M} \).
Remark 3.1. We actually do not need to make any convexity assumptions on \( \partial M \). However, if it is not strictly convex, we need to consider the geodesic segments as those in \( M' \), and some of these may intersect \( M \) in a number of segments. This is not an issue below since knowing \( I f \) in the sense of integrals along geodesic segments in \( M \), one also obtains \( I' f \), the integrals along geodesic segments in \( M' \) when \( f \) is supported in \( M \). We do not make this distinction explicit below; thus I actually refers to \( I' \) from this point on. (There is no such issue if \( \partial M \) is strictly convex and one chooses \( \partial M' \) appropriately.)

We write, relative to our convex foliation and some coordinates, denoted by \( y \), along the level sets, \( \beta = (x, y, \lambda, \omega) \), so we write tangent vectors as

\[
\lambda \partial_x + \omega \partial_y,
\]

and use \( \gamma^{(1)} \) to denote the \( x \) component of \( \gamma \), and similarly \( \gamma^{(2)} \) to denote the \( y \) component of \( \gamma \) to avoid confusion. Then we have

\[
\gamma^{(1)}_{x,y,\lambda,\omega}(t) = (\gamma^{(1)}_{x,y,\lambda,\omega}(t), \gamma^{(2)}_{x,y,\lambda,\omega}(t))
\]

\[
= (x + \lambda t + \alpha(x, y, \lambda, \omega)t^2 + t^3 \Gamma^{(1)}(x, y, \lambda, \omega, t),
\]

\[
y + \omega t + t^2 \Gamma^{(2)}(x, y, \lambda, \omega, t))
\]

with \( \Gamma^{(1)}, \Gamma^{(2)} \) smooth functions of \( x, y, \lambda, \omega, t, \alpha \) a smooth function of \( x, y, \lambda, \omega \), and \( \alpha(x, y, 0, \omega) \geq C > 0 \) by the concavity from the super-level sets hypothesis; see [21, Section 3] and [19, Proof of Proposition 4.2]. For us the relevant regime will be \( \lambda \) small; we shall restrict to an arbitrarily small neighborhood of \( \lambda = 0 \) via the semiclassical localization.

This implies the following bound:

Lemma 3.1. There exists \( T > 0 \) such that every geodesic reaches \( \partial M' \) (thus \( \partial M \)) in affine parameter (parameter after identification as a Hamiltonian integral curve of the dual metric function in \( S^* M \)) \( \leq T \).

Moreover, there exist \( \lambda_0 > 0 \) and \( C > 0 \) such that for all \( \beta = (x, y, \lambda, \omega) \) with \( |\lambda| < \lambda_0 \) and for \( t \) in the closed interval on which \( \gamma_\beta \) is defined,

\[
\gamma^{(1)}_{x,y,\lambda,\omega}(t) \geq x + \lambda t + Ct^2/2.
\]

Proof. The concavity hypothesis implies that any critical point of \( \gamma^{(1)} \) in \( M' \) is a strict local minimum, and \( \dot{\gamma}^{(1)} \) can only change sign once and do so non-degenerately since immediately to the left of any zero of \( \dot{\gamma}^{(1)} \) it is negative, and immediately to the right it is positive by the concavity. Thus, for any geodesic either the sign of \( \dot{\gamma}^{(1)} \) is constant (non-zero) or there is a unique point on it with minimal \( \gamma^{(1)} \) in \( M' \) (so either in \( M' \) or on \( \partial M' \)).

Moreover, in case the minimum of \( \gamma^{(1)} \) is reached at some \( t_0 \), \( \dot{\gamma}^{(1)}(t) \) has the same sign as \( t - t_0 \). Indeed, this is so for sufficiently small \( |t - t_0| \) by either the concavity or the minimum being on \( \partial M' \), and if it vanishes for some \( t > t_0 \) (with \( t < t_0 \) similar), taking the infimum \( t_1 > t_0 \) of the values of \( t \) at which this happens one concludes that \( \dot{\gamma}^{(1)}(t_1) = 0 \), and \( \dot{\gamma}^{(1)}(t) > 0 \) for \( 0 < t < t_0 \), which is a contradiction in view of the concavity hypothesis.

In addition, by the uniform concavity estimate, if \( \dot{\gamma}^{(1)}(t_1) \geq \epsilon \), then \( \dot{\gamma}^{(1)}(t) \geq \epsilon \) for \( t \geq t_1 \), and similarly if \( \dot{\gamma}^{(1)}(t_1) \leq -\epsilon \) then \( \dot{\gamma}^{(1)}(t) \leq -\epsilon \) for \( t \leq t_1 \). Note that by the uniform concavity estimate, \( |\dot{\gamma}^{(1)}(t)| \leq \epsilon \) can only hold for an affine parameter interval
2\varepsilon/C_0; and if the minimum of \(\gamma^{(1)}\) is reached at \(t_0\), then for \(|t-t_0| \geq \varepsilon/C_0\) one necessarily has \(|\dot{\gamma}^{(1)}(t)| \geq \varepsilon\).

Taking into account that \(M'\) is compact so \(x\) is bounded, we conclude that there exists \(T > 0\) such that every geodesic \(\gamma_{\rho}\) reaches \(\partial M'\) in both directions in affine parameter \(\leq T\): if \(|x| \leq C_1\) on \(M'\), say, then this holds with \(T = 2\varepsilon/C_0 + 4C_1/\varepsilon\).

Turning to (3.3), it suffices to prove this for \(\lambda = 0\), and then it follows for sufficiently small \(\lambda\) by compactness taking into account that it holds near \(t = 0\) by (3.2). For \(\lambda = 0\), we have seen that \(\dot{\gamma}^{(1)}(t)\) has the same sign as \(t\). Then by compactness one obtains a positive lower bound for \(\dot{\gamma}^{(1)}\) on any compact subset of \((0, \infty)\). Since we have the estimate (3.3) for sufficiently small \(t\), say \(0 < t < \delta\), then this holds with \(T = 2\varepsilon/C_0 + 4C_1/\varepsilon\).

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**Remark 3.2.** We recall from [21] that we needed to work in a sufficiently small region so that there are no geometric complications, thus there the interval \([-T, T]\) of integration in \(t\), i.e. \(T\), is such that \(\ddot{\gamma}^{(1)}(t)\) is uniformly bounded below by a positive constant in the region over which we integrate, see the discussion in [21] above Equation (3.1), and then further reduced in Equations (3.3)–(3.4) so that the map sending \((x, y, \lambda, \omega, t)\) to the lift of \((x, y, \gamma_{x, y, \lambda, \omega}(t))\) in the resolved space \(\tilde{M}^2\) with the diagonal being blown up is a diffeomorphism in \(t \geq 0\), as well as \(t \leq 0\). In the present paper the appearance of no conjugate points assumptions occurs in a closely related manner, when dealing with the stationary phase expansion, though we use the weaker concavity condition (3.1), so even geometrically we reduce the conditions imposed. In addition, the extra restrictions in [21] that arise from making the smoothing (thus ‘trivial’) error remov-a-ble disappear here.

We now introduce the weight \(\Phi\) for the exponential conjugation of our normal operator. Below we consider weights \(\Phi = \Phi(x)\) which are decreasing functions of \(x\); in the global context, \(\Phi(x) = -x\) will be used, in the scattering context (in which \(x > 0\)) \(\Phi(x) = x^{-1}\). The effect of \(\Phi\) is to put greater weight on the unknown function for greater values of \(x\) since below the operator analyzed is \(A_h\) of (3.6) whose input is \(e^{-\Phi(x)/h}\) times the unknown function. Thus, as one proceeds towards lower level sets of the \(x\), the estimates become weaker in the semiclassical limit, but as we eventually fix a small positive value of \(h\), this weakening has no effect up to a fixed overall constant.

Now, for \(\Phi(x) = -x\),
\[
\Phi(\gamma_{x, y, \lambda, \omega}^{(1)}(t)) - \Phi(x) \leq -\lambda t - Ct^2/2 \leq -\frac{C}{2}(t + \frac{\lambda}{C})^2 + \frac{\lambda^2}{2C} \leq \frac{\lambda^2}{2C}.
\]
Hence, with \(\hat{\lambda} = \lambda/\sqrt{h}, \hat{t} = t/\sqrt{h}\), the rescaling which plays a key role below,
\[
h^{-1}(\Phi(\gamma_{x, y, \lambda, \omega}^{(1)}(t)) - \Phi(x)) \leq -\frac{C}{2}(\hat{t} + \frac{\hat{\lambda}}{C})^2 + \frac{\hat{\lambda}^2}{2C},
\]
so for \(\hat{\lambda}\) in a fixed compact set (and \(h\) sufficiently small, as is always assumed),
\[
(3.4) \quad \exp(h^{-1}(\Phi(\gamma_{x, y, \lambda, \omega}^{(1)}(t)) - \Phi(x)))
\]
is uniformly bounded above by a Gaussian in \(\hat{t}\).

We consider now the operator \(L\) defined by
\[
(3.5) \quad L v(z) = \int \tilde{\chi}(x, y, \lambda/h^{1/2}, \omega) v(\gamma_{x, y, \lambda, \omega}) |d\nu|,
\]
with \( \tilde{\gamma} \) having compact support in the third variable, thus localizing to \( \lambda \sim h^{1/2} \), and hence to geodesics almost tangent to the level sets of \( x \) at the base point \((x, y)\), for \( h \) small. Here \(|dv|\) is a smooth positive density in \((\lambda, \omega)\), such as \(|d\lambda \, d\omega|\). (We usually do not distinguish between densities and differential forms of the relevant degree here.) Further we consider the operator

\[
A_h = e^{-\Phi(x)/h} L_h e^{\Phi(x)/h},
\]

which is thus given by

\[
A_h f(z) = \int e^{-\Phi(x(z))/h} e^{\Phi(x(y_{z,\lambda,\omega}(t)))/h} \tilde{\gamma}(z, \lambda/h^{1/2}, \omega) f(y_{z,\lambda,\omega}(t)) \, dt \, |dv|,
\]

where \( A_h \) is understood to apply only to \( f \) with support in \( M \), thus for which the \( t \)-integral is in a fixed finite interval, say \([-T, T]\). The first step is to prove:

**Proposition 3.1.** There exists \( h_0 > 0 \) such that for \( h < h_0 \), \( A_h \in h\Psi^{-1}(\hat{\mathcal{M}}; \mathcal{F}) \).

**Proof.** The operator \( A_h \) is the left quantization of the (a priori tempered distributional) symbol \( a_h \) where \( a_h \) is the inverse Fourier transform in the second variable \( z' \) of the integral kernel: if \( K_{A_h} \) is the Schwartz kernel of \( A_h \), then in the sense of oscillatory integrals (or directly if the order of \( a \) is sufficiently low)

\[
K_{A_h}(z, z') = (2\pi)^{-n} h^{-n/2 - 1/2} \int e^{i(x-x')\xi/h + i(y-y')\eta/h^{1/2}} a_h(x, y, \xi, \eta) \, d\xi \, d\eta,
\]

i.e. \((2\pi)^{-n}\) times the semiclassical foliation Fourier transform in \((\xi, \eta)\) of

\[
(x, y, \xi, \eta) \mapsto e^{i(x-x')\xi/h + i(y-y')\eta/h^{1/2}} a_h(z, \xi),
\]

so taking the semiclassical foliation inverse Fourier transform in \((x', y')\) yields

\[
(2\pi)^{-n} a_h(x, y, \xi, \eta) e^{i(x-x')\xi/h + i(y-y')\eta/h^{1/2}},
\]

i.e.

\[
a_h(z, \xi') = (2\pi)^{n} e^{-ix\cdot\xi/h - iy\cdot\eta/h^{1/2}} (\mathcal{F}_h^{-1})(x', y') \to (\xi, \eta) K_{A_h}(x, y, x', y').
\]

Here we are using local coordinates \( z = (x, y) \) and \( z' = (x', y') \); we comment below on the considerations when \( z \) and \( z' \) are far apart and cannot be analyzed in the same coordinate chart.

Now,

\[
K_{A_h}(x, y, x', y') = \int e^{-\Phi(x)/h} e^{\Phi(x(y_{z,\lambda,\omega}(t)))}/h \tilde{\gamma}(z, \lambda/h^{1/2}, \omega) \delta(z' - y_{z,\lambda,\omega}(t)) \, dt \, |dv|\]

\[
= (2\pi)^{-n} h^{-n/2 - 1/2} \int e^{-\Phi(x)/h} e^{\Phi(x(y_{z,\lambda,\omega}(t)))}/h \tilde{\gamma}(z, \lambda/h^{1/2}, \omega) e^{-i\xi' \cdot (x' - y_{z,\lambda,\omega}(t))}/h e^{-i\xi' \cdot (x' - y_{z,\lambda,\omega}(t))}/h^{1/2} \, dt \, |dv| \, |d\xi'| \, |d\eta'|;
\]

as remarked above, the \( t \) integral is actually over a fixed finite interval, say \(|t| < T\), or one may explicitly insert a compactly supported cutoff in \( t \) instead. (So the only non-compact domain of integration is in \((\xi', \eta')\), corresponding to the Fourier transform.) Thus, taking the semiclassical foliation inverse Fourier transform in \( x', y' \) and
evaluating at $\xi, \eta$ gives

$$a_h(x, y, \xi, \eta) = \int e^{-\Phi(x)/h}e^{\Phi(x(y_{x,\lambda,\omega}(t)))}/h \chi(z, \lambda/h^{1/2}, \omega)$$

$$e^{i\xi \cdot (y_{x,\lambda,\omega}^{(1)}(t) - x)/h}e^{i\eta \cdot (y_{x,\lambda,\omega}^{(2)}(t) - y)/h^{1/2}} dt |d\nu|.$$  

(3.8)

The proof of the proposition is completed by showing that the right hand side is actually $h$ times a symbol of order $-1$.

Notice that technically we are using local coordinates in (3.8). For the semiclassical foliation pseudodifferential operators for symbolic statements we should be considering $z, z'$ in the same chart as well as when $z$ and $z'$ are apart and cannot be analyzed in the same chart. In the latter case $|t|$ is bounded below by a positive constant, and we show that $K_A$ is smooth and $O(h^\infty)$. This is implied by the semiclassically Fourier transformed, in $z'$, expression being Schwartz, i.e. $a_h$ (and its derivatives) being $O(h^\infty((\xi, \eta)^{-\infty})$; for this we do not need to explicitly consider a coordinate chart in $z = (x, y)$. We prove this decay below by stationary phase considerations in $(\lambda, \omega, t)$, meaning that the phase is not actually stationary; in this case the oscillatory factor $e^{-i(\xi x/h + \eta \cdot y/h^{1/2})}$ is actually irrelevant.

We first consider the $|t|$ small behavior, say $|t| < T_0$, so a single chart can be used in the analysis ($z, z'$ are in the same chart). We change the variables of integration to $\hat{t} = t/\sqrt{h}$, and $\hat{\lambda} = \lambda/\sqrt{h}$, so the $\hat{\lambda}$ integral is in fact over a fixed compact interval, but the $\hat{t}$ one is over $|\hat{t}| < T_0/\sqrt{h}$ which grows as $h \to 0$; in this process we obtain an additional factor of $h$ from the change of density. We deduce that the phase is

$$\xi \cdot (y_{x,\lambda,\omega}^{(1)}(t) - x)/h + \eta \cdot (y_{x,\lambda,\omega}^{(2)}(t) - y)/h^{1/2}$$

$$= \xi \left(\hat{\lambda} \hat{t} + \alpha \hat{t}^2 + h^{1/2} \hat{\Gamma}^{(1)}(x, y, h^{1/2}\hat{\lambda}, \omega, h^{1/2}\hat{t})\right)$$

$$+ \eta \cdot (\omega \hat{t} + h^{1/2} \hat{\Gamma}^{(2)}(x, y, h^{1/2}\hat{\lambda}, \omega, h^{1/2}\hat{t})),$$

while the exponent of the exponential damping factor (which we regard as a Schwartz function, part of the amplitude, when one regards $\hat{t}$ as a variable on $\mathbb{R}$) is

$$-\Phi(x)/h + \Phi(x(y_{x,\lambda,\omega}(t)))/h$$

$$= x/h - y_{x,\lambda,\omega}^{(1)}(t)/h$$

$$= -h^{-1}(\hat{\lambda} \hat{t} + \alpha \hat{t}^2 + t^3 \hat{\Gamma}^{(1)}(x, y, \lambda, \omega, t))$$

$$= -(\hat{\lambda} \hat{t} + \alpha \hat{t}^2 + \hat{t}^3 h^{1/2} \hat{\Gamma}^{(1)}(x, y, h^{1/2}\hat{\lambda}, \omega, h^{1/2}\hat{t})),$$

with $\hat{\Gamma}^{(1)}$ a smooth function. Thus, after the rescaling of $t$ and $\lambda$ to $\hat{t}$ and $\hat{\lambda}$, the integrand of (3.8) is a smooth function of all variables. In view of the Gaussian decay of the exponential damping factor around (3.4), the lack of compactness in the $\hat{t}$ integration domain is not an issue, and we conclude that for $\xi, \eta$ in a bounded region we conclude that $a_h$ is $h$ times a $C^\infty$ function.

We now consider the behavior of $|((\xi, \eta)| \to \infty$ to complete showing that $a_h$ is a symbol. The only subtlety in applying the stationary phase lemma is that the domain of integration in $\hat{t}$ is not compact, so we need to explicitly deal with the region $|\hat{t}| \geq 1$, say, assuming that the amplitude is Schwartz in $\hat{t}$, uniformly in the other variables (as it is in our case thanks to the exponential weight factor). Notice that as long as the first derivatives of the phase in the integration variables have a lower bound $c|((\xi, \eta)| |\hat{t}|^{-k}$
for some $k$, and for some $c > 0$, the standard integration by parts argument gives the rapid decay of the integral in the large parameter $|\xi|$. At $h = 0$ the phase is $\xi(\hat{\lambda}\hat{t} + \alpha\hat{t}^2) + i\eta \cdot \omega$, if $|\hat{t}| \geq 1$, say, the $\hat{\lambda}$ derivative is $\xi\hat{t}$, which is thus bounded below by $|\xi|$ in magnitude, so the only place where one may not have rapid decay is at $\xi = 0$ (meaning, in the spherical variables, $\xi = 0$). In this region one may use $|\eta|$ as the large variable to simplify the notation slightly. The phase is then with $\hat{\lambda} = \xi|\eta|$, $\hat{\eta} = \eta|\eta|$

\[\eta|\xi(\hat{\lambda} + \alpha\hat{t}^2) + i\hat{t} \cdot \omega),\]

with parameter differentials (ignoring the overall $|\eta|$ factor)

\[\xi\hat{t} d\hat{\lambda}, (i\hat{\eta} + i^2\xi\partial_{\omega}\alpha) \cdot d\omega, (\xi(\hat{\lambda} + 2\alpha\hat{t}) + \hat{\eta} \cdot \omega) d\hat{t}.\]

With $\hat{\Xi} = \xi\hat{t}$ and $\rho = \hat{t}^{-1}$ these are

\[\hat{\Xi} d\hat{\lambda}, i(\hat{\eta} + \hat{\Xi}\partial_{\omega}\alpha) \cdot d\omega, (\hat{\Xi}(\hat{\rho} + 2\alpha) + \hat{\eta} \cdot \omega) d\hat{t}.\]

and now for critical points $\hat{\Xi}$ must vanish (as we already knew from above), then the last of these gives that $\hat{\eta} \cdot \omega$ vanishes, but then the second gives that there cannot be a critical point (in $|\hat{t}| \geq 1$). While this argument was at $h = 0$, the full phase derivatives are

\[\xi(\hat{\lambda} + 2\alpha\hat{t} + 3h^{1/2}\hat{t}\Gamma(1) + h^{1/2}\hat{t}\partial_{\omega}\Gamma(1)) \cdot d\omega,
\]

\[\xi\hat{t} \cdot d\hat{\lambda}, (i\hat{\eta} + i^2\hat{\xi}\partial_{\omega}\alpha) \cdot d\omega, (\hat{\Xi}(\hat{\rho} + 2\alpha) + \hat{\eta} \cdot \omega) d\hat{t},\]

i.e.

\[\hat{\Xi}(1 + t\partial_{\alpha} + t^2\partial_{\lambda}\Gamma(1)) + \hat{\eta} \cdot t^2\partial_{\lambda}\Gamma(2)) d\hat{\lambda},\]

\[i(\hat{\eta} + \hat{\xi} \cdot t\partial_{\omega}\Gamma(2) + \hat{\Xi}\partial_{\omega}\alpha + t\hat{\Xi}\partial_{\omega}\Gamma(1)) \cdot d\omega,
\]

\[\hat{\Xi}(\hat{\rho} + 2\alpha + 3t\Gamma(1) + t^2\partial_{\omega}\Gamma(1)) + \hat{\eta} \cdot \omega + 2t\Gamma(2) + t^2\partial_{\omega}\Gamma(2)) d\hat{t},\]

and now all the additional terms are small if $T_0$ is small, where we assume $|\hat{t}| < T_0$, so the lack of critical points in the $h = 0$ computation implies the analogous statement (in $|\hat{t}| > 1$) for the general computation assuming $T_0$ is sufficiently small.

Now, if $t \in [T_0, T]$ is not so small, the same result can be achieved under a no-conjugate points assumption, i.e. the Jacobian $\frac{\partial \gamma}{\partial (t, \lambda, \omega)}$ is full rank for $t$ away from 0. Notice that we might use different coordinate charts for $z, z'$ as discussed after the paragraph of (3.8), with the factor $e^{-i((\xi x/h + \eta y)/h^{1/2})}$ of the integrand irrelevant if one is to prove rapid decay (since it is oscillatory and can be pulled out of the integral). In this case one can run the non-stationary phase argument directly for $t$ (as opposed to $\hat{t}$) away from 0 and $(\lambda, \omega)$, showing that there are no stationary points, and thus, as $|(\xi, \eta)| \to \infty$ or $h \to 0$, one can reduce to the case $\hat{t} = 0$ discussed below (as there are no other non-trivial contributions). Concretely, we need to keep in mind that the exponential weight is bounded by $e^{-\epsilon/(h^2)}$ for some $\epsilon > 0$ when $t$ is bounded away from 0 (and $h$ is sufficiently small), thus is rapidly decaying as $h \to 0$; in particular this assures the smoothness of $a_h$, and its rapid decay in $h$, for bounded $(\xi, \eta)$. Otherwise $z$ only enters via the parameterization of $\gamma$, and potentially smooth dependence in $\hat{\chi}$, apart from the factor $e^{-i((\xi x/h + \eta y)/h^{1/2})}$ which can be pulled out of the integral; thus effectively we can consider only the $z'$ (and not the $z$) coordinates explicitly. Now, there is $C_0 > 0$
such that if $|\xi|/h^{1/2} > C_0|\eta|$ then $\partial_t \gamma(1)$ is positive, since $\partial_t \gamma(1) \neq 0$ by the convexity properties of the foliation as shown in the last paragraph of the proof of Lemma 3.1. Thus, in this case we deduce rapid decay of the integral in $|\xi|/h$, hence also in $|\eta|/h^{1/2}$. So let $\xi = \xi/h^{1/2}$, and assume that $|\xi| < 2C_0|\eta|$. Then the phase becomes

$$h^{-1/2}(\xi(\gamma(1)_{x,y,\lambda,\omega}(t) - x) + \eta \cdot (\gamma(2)_{x,y,\lambda,\omega}(t) - y)),$$

which is a standard $h^{1/2}$-semiclassical phase, whose non-stationarity is assured by the no-conjugate points hypothesis, i.e. that $\partial_t \gamma(1)_{x,y,\lambda,\omega}$ is full rank for $t$ away from 0. Due to the exponential weight, the amplitude is rapidly decreasing in $h$ regardless of the singular $\lambda$-dependence of $\chi$, which makes stationary phase applicable, giving the desired result of rapid decay in $|\eta|/h^{1/2}$, hence also in $|\xi|/h^{1/2}$ in view of the constraint on $\xi$.

This discussion implies that we may work in $|\xi| < 2$, say, and one can use the standard parameter-dependent stationary phase lemma, see e.g. [5, Theorem 7.7.6] for our stationary phase computation in $(\xi, \eta)$. At $h = 0$, the stationary points of the phase are $\xi = 0, \eta = 0, \lambda = 0$, which remain critical points for $h$ non-zero due to the $h^{1/2}\xi^2$ vanishing of the other terms, and when $|\xi| < 2$ and $h$ is sufficiently small, so $h^{1/2}\xi$ is small, there are no other critical points. (One can see this in a different way: above we worked with $|\xi| \geq 1$, but for any $\varepsilon > 0, |\xi| \geq \varepsilon$ would have worked equally.) These critical points lie on a smooth codimension 2 submanifold of the parameter space. For the following argument it is useful to consider $(\xi, \eta)$ jointly, and write $\xi = \frac{\xi}{|\xi, \eta|}, \eta = \frac{\eta}{|\xi, \eta|}$.

Moreover, we write $\theta = (\lambda, \omega)$, and decompose it into a parallel and orthogonal component relative to $(\xi, \eta): \theta^\parallel = (\xi, \eta) \cdot (\lambda, \omega), \theta^\perp; \theta^\parallel$ as $(\xi, \eta)$ is a unit vector, this is a valid change of coordinates. At $h = 0$, the $(\xi, \theta^\parallel)$-Hessian matrix of the phase

$$|(\xi, \eta)|(\xi(\lambda \xi + \lambda \xi^2) + \lambda \xi \cdot \omega) = |(\xi, \eta)|((\xi, \theta^\parallel) + \xi \alpha t^2)$$

is always invertible. This also implies, by continuity (and homogeneity in $(\xi, \eta)$) of the Hessian the invertibility for small $h$. We thus use the stationary phase lemma in the $(\xi, \theta^\parallel)$ variables, which shows that $a_h$ is $(h$ times, due to the integration variable change, already mentioned for the finite $\xi, \eta$ discussion!) a symbol of order $-1$, since the stationary phase is with respect to a 2-dimensional space.

We now proceed to compute the semiclassical principal symbol:

**Proposition 3.2.** There exist $h_0 > 0$ and $\tilde{\chi}$ of compact support such that the operator $A_h \in h\Psi^{-1}_{h^{-1}}(\tilde{M}; F)$ is elliptic on $M$ for $h < h_0$.

**Proof.** For the semiclassical principal symbol computation we may simply set $h = 0$ in the above rescaled expression used for the stationary phase argument; the latter also implies that we can reduce our considerations to $\tilde{\chi}$ close to 0. Thus, with $\tilde{\chi}$ =
\( \chi(\lambda/h^{1/2}) = \chi(\hat{\lambda}) \), we have that
\[
(3.9) \quad a_h(x, y, \xi, \eta) = h \int e^{i(\xi h^{-1}x^{(1)}_{x,y,h^{1/2}\hat{\lambda},\omega}(h^{1/2}I) - x) + \eta h^{-1/2}x^{(2)}_{x,y,h^{1/2}\hat{\lambda},\omega}(h^{1/2}I) - y)}
\]
\[
e^{-\left(\hat{\lambda} t + at^2\right)} \chi(\hat{\lambda}) \, dt \, d\hat{\lambda} \, d\omega
\]
\[
= h \int e^{i(\xi (\hat{\lambda} t + at^2 + h^{1/2}I^{(1)}(x, y, h^{1/2}\hat{\lambda}, \omega, h^{1/2}I)) + \eta \omega t + h^{1/2}I^{(2)}(x, y, h^{1/2}\hat{\lambda}, \omega, h^{1/2}I))}
\]
\[
e^{-\left(\hat{\lambda} t + at^2\right)} \chi(\hat{\lambda}) \, dt \, d\hat{\lambda} \, d\omega,
\]
up to errors that are \( O(h^{1/2}(\xi, \eta)^{-1}) \) relative to the a priori order, \(-1\), arising from the 0th order symbol in the oscillatory integral and the 2-dimensional space in which the stationary phase lemma is applied, and semiclassical order \(-1\), corresponding to the factor of \( h \) in front. Factoring out the overall \( h \), this in turn becomes modulo \( O(h^{1/2}) \) errors, i.e. at the semiclassical foliation principal symbol level,

\[
(3.10) \quad a_0(x, y, \xi, \eta) = \int e^{i(\xi (\hat{\lambda} t + at^2 + \eta \omega t))} e^{-\left(\hat{\lambda} t + at^2\right)} \chi(\hat{\lambda}) \, dt \, d\hat{\lambda} \, d\omega.
\]

Now computing the principal symbol of this in the standard differential order sense, i.e. the behavior as \( |(\xi, \eta)| \to \infty \), we consider the critical points of the phase, \( t = 0, \theta^\parallel \equiv \hat{\xi} \hat{\lambda} + \hat{\eta} \cdot \omega = 0 \), where \( \theta^\perp \) is the variable along the critical set (where \( t \) and \( \theta^\parallel \) vanish), which gives, up to an overall elliptic factor, keeping in mind that \( \hat{\lambda} \) depends on \( \theta^\perp \) along this equator \( S^{n-2}_{\theta^\parallel = 0} \) (namely along \( \theta^\parallel = 0 \)),

\[
\int_{S^{n-2}_{\theta^\parallel = 0}} \chi(\hat{\lambda}(\theta^\perp)) \, d\theta^\perp,
\]

which is elliptic for \( \chi \geq 0 \) with \( \chi(0) > 0 \) since the codimension one planes (intersected with a sphere) \( \theta^\parallel = 0 \) and \( \hat{\lambda} = 0 \) necessarily intersect in at least a line (intersected with the sphere) as the dimension is \( n \geq 2 + 1 = 3 \). This proves ellipticity in the standard differential order sense at \( h = 0 \) hence for sufficiently small \( h > 0 \).

This of course implies that for finite, but sufficiently large, \( (\xi, \eta) \), the semiclassical symbol \( a_0 \) is elliptic; this region is uniform if \( \chi \) is bounded in Schwartz functions with uniform positive lower bound on \( \chi(0) \). For general finite \( (\xi, \eta) \) it is harder to compute \( a_0 \) explicitly for general \( \chi \). However, when \( \chi \) is a Gaussian, the computation is straightforward. We write

\[
a_0(x, y, \xi, \eta) = \int e^{-\left(\alpha(1-i\xi) \hat{\lambda}^2 + i(\hat{\lambda}(1-i\xi) - i\eta \cdot \omega)\right)} \chi(\hat{\lambda}) \, d\hat{\lambda} \, d\omega
\]

\[
= \int e^{-\alpha(1-i\xi)(\hat{\lambda} + \frac{(\hat{\lambda}(1-i\xi) - i\eta \cdot \omega)^2}{2\alpha(1-i\xi)})} e^{-\frac{(\hat{\lambda}(1-i\xi) - i\eta \cdot \omega)^2}{4\alpha(1-i\xi)}} \chi(\hat{\lambda}) \, d\hat{\lambda} \, d\omega
\]

\[
= c \int e^{-\alpha(1-i\xi)^{-1/2}} \frac{(\hat{\lambda}(1-i\xi) - i\eta \cdot \omega)^2}{4\alpha(1-i\xi)} \chi(\hat{\lambda}) \, d\hat{\lambda} \, d\omega
\]

\[
= c \int \alpha^{-1/2}(1-i\xi)^{-1/2} e^{-\frac{(\hat{\lambda}(1-i\xi) - i\eta \cdot \omega)^2}{4\alpha(1-i\xi)}} \chi(\hat{\lambda}) \, d\hat{\lambda} \, d\omega
\]

\[
= c \int \alpha^{-1/2}(1-i\xi)^{-1/2} e^{-\frac{(\hat{\lambda}(1-i\xi) - i\eta \cdot \omega)^2}{4\alpha(1-i\xi)}} \chi(\hat{\lambda}) \, d\hat{\lambda} \, d\omega.
\]
with \(c\) a non-zero constant. Now a particularly helpful choice is \(\chi(\hat{\lambda}) = e^{-\hat{\lambda}^2/(2\alpha)}\), for then we have
\[
a_0(x, y, \xi, \eta) = c \int_{\alpha^{-1/2}} (1 - i \xi)^{-1/2} e^{\frac{\hat{\lambda}^2(1+\xi)}{4\alpha} - \frac{i \hat{\lambda} \eta \omega}{2\alpha(1-\xi^2)}} d\hat{\lambda} d\omega
\]
and the integral is now positive since the integrand is such, while \(c'\) is a new non-zero constant. It follows immediately that the same positivity property is maintained if \(\chi\) is close to the Gaussian in the space of Schwartz functions, which can be achieved by taking a compactly supported \(\chi\).

In view of the errors of the elliptic parametrix construction being small in the semiclassical Sobolev spaces, for sufficiently small \(h\) (but \(h\) can be fixed to such a small value, so \(A_h\) is a standard pseudodifferential operator then, and the Sobolev spaces are standard Sobolev spaces with an equivalent norm!), as discussed in Section 2, see in particular (2.2) with
\[(3.11) \quad G = (\text{Id} + \psi E\psi)^{-1} \psi B\]
being the left inverse on distributions supported in \(M\) in its notation, this proves Theorem 1.2.

As a consequence we deduce:

\textit{Proof of Corollary 1.1.} Injectivity follows from Theorem 1.2 using the local left inverse \(G\) for fixed sufficiently small \(h\) (dropped from the notation on occasion). Indeed, this left inverse gives
\[(3.12) \quad f = e^{\hat{\Phi}/h} G A e^{-\hat{\Phi}/h} f = e^{\hat{\Phi}/h} G e^{-\hat{\Phi}/h} L_h I f;\]

note here that as \(h\) is fixed, \(e^{\pm \hat{\Phi}/h}\) is a bounded smooth function. This proves injectivity.

Under the stronger assumption of the metric being simple (rather than our weaker no points conjugate to their point of tangency to level sets of \(x\)), the stability estimate for \(s = 0\) follows from the work of Assylbekov and Stefanov [1] by considering (as we do in general) \(I f\) on the extended manifold \(\tilde{M}\); the hypothesis they need is exactly the injectivity of \(I\). Note that the space of invariant (under the lifted geodesic flow) \(H^s\) functions on \(\tilde{M}\) with support along lifted geodesics that intersect \(SM\) can be identified with \(H^s\) functions on the boundary of an extension \(M_1\) of \(M\) with an analogous support condition (the latter is the parameterization used in [1]) since solving the homogeneous transport equation has this property and since we stay away from geodesics tangential to \(\partial M_1\) by the support condition. Note also that the key result this reference uses is due to Stefanov and Uhlmann [17] where the required mapping properties of \(I\) and \(I^*\) are established for \(s \geq 0\) (indeed \(s \geq -1/2\) as the proof shows).

In the general case of our setting, the mapping properties of \(L = L_h\) (for fixed \(h\)) follow from the work of Holman and Uhlmann [4, Lemma 7], where the analogous operator is denoted by \(\mathcal{X}_\phi\) (keeping in mind the identification discussed in the previous paragraph), with \(\phi\) being a phase space weight corresponding to our \(\hat{\chi}\) (and arbitrary measure \(|d\nu|\)). While the authors make general assumptions on conjugate points
(Assumption 2), which are satisfied if there are no conjugate points, these are only needed on \( \text{supp } \phi \) (in their notation), i.e. \( \text{supp } \tilde{\chi} \) in ours. For sufficiently small \( h \), this is then in turn implied (in the strong form of no conjugate points in the relevant set) by our assumption. Thus, \( L_h : H^s \to H^{s+1/2} \), as stated in [4, Lemma 7] for \( s = 0 \), so \( e^{\Phi/h} G e^{-\Phi/h} L_h : H^s \to H^{s-1/2} \), proving Corollary 1.1 for \( s = -1/2 \) in view of (3.12). For general \( s \), the proof of [4, Lemma 7] applies ‘mutatis mutandis’: one composes \( \mathcal{X}_\phi \) with elliptic pseudodifferential operators of order \( s \) from the left and \( -s \) from the right (and dually with \( \mathcal{X}_\phi \)), which does not affect the composition properties thanks to the FIO argument used there (unlike in [17]), to obtain the desired result (when combined with the \( s = 0 \) case if one uses merely elliptic, rather than invertible, operators).

The scattering version is quite similar; recall that \( x = \tilde{x} + c \) in terms of the original foliation function \( \tilde{x} \). The cutoff scaling we use in this case is

\[
\tilde{\chi}(z, \lambda/(x h^{1/2}), \omega).
\]

Thus, writing scattering covectors as \( \xi_{sc} \frac{dx}{\tilde{x}} + \eta_{sc} \frac{dy}{\tilde{x}} \), i.e. substituting

\[
\xi_{sc} = x^2 \xi, \, \eta_{sc} = x \eta,
\]

into (3.8)

\[
(3.13)
\]

\[
(3.14)
\]

\[
a_h(x, y, \xi_{sc}, \eta_{sc}) = \int e^{-\Phi(x)/h} e^{\Phi(x(y, \lambda, \omega(t)))}/h \tilde{\chi}(z, \lambda/(x h^{1/2}), \omega) e^{ix - \xi_{sc}(\gamma^{(1)}(x, y, \lambda, \omega, \xi_{sc}(I))) - y} e^{ix - \eta_{sc}(\gamma^{(2)}(x, y, \lambda, \omega, \eta_{sc}(I)))}/h dt |dv|,
\]

with \( \Phi(x) = x^{-1} \) in this case.

**Proposition 3.3.** Let \( \tilde{M}_c = \tilde{M} \cap \{ \tilde{x} \geq -c \} = \tilde{M} \cap \{ x \geq 0 \} \). Then \( A_h \in h\Psi_{sc, h, \mathcal{F}}^{1-2}(\tilde{M}_c; \mathcal{F}) \).

**Proof.** We change the variables of integration to \( \tilde{t} = t/(h^{1/2} x) \), and \( \tilde{\lambda} = \lambda/(h^{1/2} x) \), so again the \( \tilde{\lambda} \) integral is in fact over a fixed compact interval, but the \( \tilde{t} \) integral is over \( |\tilde{t}| < T/(x h^{1/2}) \) which grows as \( h \to 0 \) or \( x \to 0 \). We get that the phase is

\[
\xi_{sc}(\tilde{\lambda} \tilde{t} + \alpha \tilde{t}^2 + x h^{1/2} \tilde{t}^3 \Gamma^{(1)}(x, y, h^{1/2} \lambda, \omega, x h^{1/2} \tilde{t}))
+ \eta_{sc}(\omega \tilde{t} + x h^{1/2} \tilde{t}^2 \Gamma^{(2)}(x, y, h^{1/2} \lambda, \omega, x h^{1/2} \tilde{t}))
\]

while the exponential damping factor (which we regard as a Schwartz function, part of the amplitude, when one regards \( \tilde{t} \) as a variable on \( \mathbb{R} \)) is

\[
-1/(hx) + 1/(h \gamma^{(1)}(x, y, \lambda, \omega, \tilde{t}))
\]

\[
= -h^{-1}(\lambda t + \alpha t^2 + t^3 \Gamma^{(1)}(x, y, \lambda, \omega, t)) x^{-1}(x + \lambda t + \alpha t^2 + t^3 \Gamma^{(1)}(x, y, \lambda, \omega, t))^{-1}
\]

\[
= -(\tilde{\lambda} \tilde{t} + \alpha \tilde{t}^2 + \tilde{t}^3 x h^{1/2} \Gamma^{(1)}(x, y, x h^{1/2} \lambda, \omega, x h^{1/2} \tilde{t}))
\]

with \( \Gamma^{(1)} \) a smooth function. Thus, for \( \xi, \eta \) in a bounded region we conclude that \( a_h \) is a \( C^\infty \) function. Furthermore, we observe that with \( (\xi_{sc}, \eta_{sc}) \) in place of \( (\xi, \eta) \), and in the new integration variables \( \tilde{t} \) and \( \tilde{\lambda} \), (3.14) has the same form as (3.8), so identical stationary phase arguments are applicable. In particular, the \( t \) bounded away from 0 case proceeds analogously with \( x h^{1/2} \) playing the role of \( h^{1/2} \), keeping in mind that the exponential weight is bounded by \( e^{-c/(x h)} \) for \( t \) bounded away from 0, so is rapidly decaying in \( x h \); in this case \( |\xi_{sc}|/(x h^{1/2}) > C_0 \eta_{sc} \) assures the possibility of \( t \)-integration.
by parts to obtain rapid decay in \(xh\), while if \(|\tilde{\xi}_{sc}| = |\tilde{\xi}_{sc}|/(xh^{1/2}) < 2C_0|\eta_{sc}|\), then one can apply a standard no-stationary point argument as above under the no-conjugate point assumption since the phase is \(x^{-1}h^{-1/2}\) times a usual homogeneous degree 1 phase in \((\tilde{\xi}_{sc}, \eta_{sc})\), giving rapid decay in \(x^{-1}h^{-1/2}|\eta_{sc}|\), thus in \(x^{-2}h^{-1}|\xi_{sc}|\) as well. \(\square\)

Finally, it remains to compute the semiclassical foliation scattering principal symbol, which is, taking into account the density factor \(hx^2\) from the change of variables,

\[(3.15) \quad a_0(x, y, \xi_{sc}, \eta_{sc}) = hx^2 \int e^{i(\xi_{sc}(\hat{\lambda} \hat{t} + \alpha \hat{t}^2) + \eta_{sc} \cdot \omega \hat{t})} e^{-(\hat{\lambda} \hat{t} + \alpha \hat{t}^2)} \chi(\hat{\lambda}) d\hat{t} d\hat{\lambda} d\omega.\]

Then completely analogous arguments to the above computation yield that as \(|(\xi_{sc}, \eta_{sc})| \to \infty\), up to an overall elliptic factor, we have

\[\int_{S^{n-2}} \chi(\hat{\lambda}(\theta^\perp)) d\theta^\perp,\]

which is elliptic for \(\chi \geq 0\) with \(\chi(0) > 0\). Further, the ellipticity at finite points follows the same computation as above, for the same choice of \(\chi\), \(\chi(\hat{\lambda}) = e^{-\hat{\lambda}^2/(2\alpha)}\), with \((\xi, \eta)\) replaced by \((\xi_{sc}, \eta_{sc})\). Again, in view of the errors of the elliptic parametrix construction being small in the semiclassical Sobolev spaces as discussed in Section 2, this proves Theorem 1.3.

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REFERENCES

[1] Yernat M. Assylbekov and Plamen Stefanov, Sharp stability estimate for the geodesic ray transform, Inverse Problems 36 (2020), no. 2, 025013, 14, DOI 10.1088/1361-6420/ab3d12. MR3938205
[2] Maarten de Hoop, Gunther Uhlmann, and András Vasy, Diffraction from conormal singularities (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 2, 351–408, DOI 10.24033/asens.2247. MR3346174
[3] Oran Gannot and Jared Wunsch, Semiclassical diffraction by conormal potential singularities (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) 56 (2023), no. 3, 713–800, DOI 10.24033/asens.2543. MR4650162
[4] Sean Holman and Gunther Uhlmann, On the microlocal analysis of the geodesic X-ray transform with conjugate points, J. Differential Geom. 108 (2018), no. 3, 459–494, DOI 10.4310/jdg/1519959623. MR3770848
[5] Lars Hörmander, The analysis of linear partial differential operators. II, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 257, Springer-Verlag, Berlin, 1983. Differential operators with constant coefficients, DOI 10.1007/978-3-642-96750-4. MR705278
[6] R. B. Melrose, Lecture notes for 18.157: introduction to microlocal analysis\(\color{red}{\text{http://math.mit.edu/~rbm/18.157-F09/18.157-F09.html}}\), 2009.
[7] Richard B. Melrose, Transformation of boundary problems, Acta Math. 147 (1981), no. 3-4, 149–236, DOI 10.1007/BF02392873. MR639039
[8] Richard B. Melrose, The Atiyah-Patodi-Singer index theorem, Research Notes in Mathematics, vol. 4, A K Peters, Ltd., Wellesley, MA, 1993, DOI 10.1016/0377-0257(93)80040-i. MR1348401
[9] Richard B. Melrose, Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces, Spectral and scattering theory (Sanda, 1992), Lecture Notes in Pure and Appl. Math., vol. 161, Dekker, New York, 1994, pp. 85–130. MR1291640
[10] Cesare Parenti, *Operatori pseudo-differenziali in $\mathbb{R}^n$ e applicazioni*, Ann. Mat. Pura Appl. (4) **93** (1972), 359–389, DOI 10.1007/BF02412028. MR437917

[11] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann, *Geometric inverse problems—with emphasis on two dimensions*, Cambridge Studies in Advanced Mathematics, vol. 204, Cambridge University Press, Cambridge, 2023. With a foreword by András Vasy. MR3520155

[12] Gabriel P. Paternain, Mikko Salo, Gunther Uhlmann, and Hanning Zhou, *The geodesic X-ray transform with matrix weights*, Amer. J. Math. **141** (2019), no. 6, 1707–1750, DOI 10.1353/ajm.2019.0045. MR4030525

[13] V. A. Sharafutdinov, *Integral geometry of tensor fields*, Inverse and Ill-posed Problems Series, VSP, Utrecht, 1994, DOI 10.1515/9783110900095. MR1374572

[14] Johannes Sjöstrand and Maciej Zworski, *Fractal upper bounds on the density of semiclassical resonances*, Duke Math. J. **137** (2007), no. 3, 381–459, DOI 10.1215/S0012-7094-07-13731-1. MR2309150

[15] Plamen Stefanov and Gunther Uhlmann, *Rigidity for metrics with the same lengths of geodesics*, Math. Res. Lett. **5** (1998), no. 1-2, 83–96, DOI 10.4310/MRL.1998.v5.n1.a7. MR1618347

[16] Plamen Stefanov and Gunther Uhlmann, *Integral geometry on tensor fields on a class of non-simple Riemannian manifolds*, Amer. J. Math. **130** (2008), no. 1, 239–268, DOI 10.1353/ajm.2008.0003. MR2335214

[17] Plamen Stefanov and Gunther Uhlmann, *The geodesic X-ray transform with fold caustics*, Anal. PDE **5** (2012), no. 2, 219–260, DOI 10.2140/apde.2012.5.219. MR2970708

[18] Plamen Stefanov, Gunther Uhlmann, and Andhrás Vasy, *Inverting the local geodesic X-ray transform on tensors*, J. Anal. Math. **136** (2018), no. 1, 151–208, DOI 10.1007/s11854-018-0058-3. MR3892472

[19] Plamen Stefanov, Gunther Uhlmann, and Andhrás Vasy, *Local and global boundary rigidity and the geodesic X-ray transform in the normal gauge*, Ann. of Math. (2) **194** (2021), no. 1, 1–95, DOI 10.4007/annals.2021.194.1.1. MR4276284

[20] M. A. Šubin, *Pseudodifferential operators in $\mathbb{R}^n$* (Russian), Dokl. Akad. Nauk SSSR **196** (1971), 316–319. MR273463

[21] Gunther Uhlmann and Andhrás Vasy, *The inverse problem for the local geodesic ray transform*, Invent. Math. **205** (2016), no. 1, 83–120, DOI 10.1007/s00222-015-0631-7. MR3514959

[22] A. Vasy and E. Zachos, *The X-ray transform on asymptotically conic spaces*, Preprint, arXiv:2204.11706, 2022.

[23] Andhrás Vasy and Maciej Zworski, *Semiclassical estimates in asymptotically Euclidean scattering*, Comm. Math. Phys. **212** (2000), no. 1, 205–217, DOI 10.1007/s002200000207. MR1764368

[24] Andhrás Vasy, *A minicourse on microlocal analysis for wave propagation*, Asymptotic analysis in general relativity, London Math. Soc. Lecture Note Ser., vol. 443, Cambridge Univ. Press, Cambridge, 2018, pp. 219–374. MR3792086

[25] Evangeline Zachos, *The X-ray transform on asymptotically Euclidean spaces*, ProQuest LLC, Ann Arbor, MI, 2020. Thesis (Ph.D.)–Stanford University. MR4144687

[26] Maciej Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, vol. 138, American Mathematical Society, Providence, RI, 2012, DOI 10.1090/gsm/138. MR2952218

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