Reproducing Kernels of Sobolev Spaces on $\mathbb{R}^d$
and Applications to
Embedding Constants and Tractability

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Abstract

The standard Sobolev space \( W^s_2(\mathbb{R}^d) \), with arbitrary positive integers \( s \) and \( d \) for which \( s > d/2 \), has the reproducing kernel

\[
K_{d,s}(x,t) = \int_{\mathbb{R}^d} \frac{\prod_{j=1}^d \cos \left( 2\pi (x_j - t_j)u_j \right)}{1 + \sum_{0<|\alpha|\leq s} \prod_{j=1}^d (2\pi u_j)^{2\alpha_j}} \, du
\]

for all \( x, t \in \mathbb{R}^d \), where \( x_j, t_j, u_j, \alpha_j \) are components of \( d \)-variate \( x, t, u, \alpha \), and \( |\alpha| = \sum_{j=1}^d \alpha_j \) with non-negative integers \( \alpha_j \). We obtain a more explicit form for the reproducing kernel \( K_{1,s} \) and find a closed form for the kernel \( K_{d,\infty} \).

Knowing the form of \( K_{d,s} \), we present applications on the best embedding constants between the Sobolev space \( W^s_2(\mathbb{R}^d) \) and \( L_\infty(\mathbb{R}^d) \), and on strong polynomial tractability of integration with an arbitrary probability density. We prove that the best embedding constants are exponentially small in \( d \), whereas worst case integration errors of algorithms using \( n \) function values are also exponentially small in \( d \) and decay at least like \( n^{-1/2} \). This yields strong polynomial tractability in the worst case setting for the absolute error criterion.

Key words: Reproducing kernels; Tractability; Sobolev space.

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1 Introduction and results

One of the most studied spaces in mathematical analysis are Sobolev spaces \( W^s_p(\Omega) \) for a positive integer \( s \), \( p \in [1, \infty] \) and \( \Omega \subseteq \mathbb{R}^d \). In this paper we consider \( p = 2 \) and \( \Omega = \mathbb{R}^d \) for arbitrary integers \( s \) and \( d \). Then \( W^s_2(\mathbb{R}^d) \) is a separable Hilbert space equipped with the inner product

\[
\langle f, g \rangle_{W^s_2(\mathbb{R}^d)} = \sum_{|\alpha| \leq s} \langle D^\alpha f, D^\alpha g \rangle_{L^2(\mathbb{R}^d)} \quad \text{for all } f, g \in W^s_2(\mathbb{R}^d).
\]

Here, \( D^\alpha \) is the differential operator

\[
D^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}} f(x) \quad \text{for all } x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d
\]

and \( L^2(\mathbb{R}^d) \) is the standard space of square integrable functions with the inner product

\[
\langle f, g \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x) \overline{g(x)} \, dx.
\]
The embedding condition \( s > d/2 \) implies that we can treat \( W^s_2(\mathbb{R}^d) \) as a space of continuous functions and function values are continuous linear functionals. This means that \( W^s_2(\mathbb{R}^d) \) is a reproducing kernel Hilbert space with a reproducing kernel \( K_{d,s} \), i.e. \( W^s_2(\mathbb{R}^d) = H(K_{d,s}) \). This is a function defined on \( \mathbb{R}^d \times \mathbb{R}^d \) such that \( (K_{d,s}(x, t))_{k,j=1,2,\ldots,n} \) is hermitian and semi-positive definite for all choices of \( n \) and \( x_j \in \mathbb{R}^d \), and most importantly

\[
f(t) = \langle f, K_{d,s}(\square, t) \rangle_{W^s_2(\mathbb{R}^d)} \quad \text{for all } f \in W^s_2(\mathbb{R}^d) \text{ and for all } t \in \mathbb{R}^d.
\]

Here, \( \square \) is used as the placeholder for the variable of a function we consider. Sometimes we use the shorter notation \( \delta_t(x) = K(x, t) \), hence

\[
f(t) = \langle f, \delta_t \rangle \quad \text{for all } f \in H(K).
\]

The knowledge of the reproducing kernels is very useful in the analysis of many computational problems. Examples include multivariate integration and approximation, scattered data approximation, statistical and machine learning, the numerical solution of partial differential equations, see for instance [1, 2, 4, 5, 8, 10, 12, 13, 16]. We only mention one application of the kernel \( K \) to the best linear estimation (or optimal recovery or Kriging). The problem is to find \( f \in H \) with minimal norm such that \( f(x_i) = y_i \) for \( i = 1, 2, \ldots, n \). The solution is an abstract spline of the form \( f^* = \sum_{j=1}^n \alpha_j \delta_{x_j} \), where \( \alpha_j \)'s are chosen such that \( f^*(x_i) = y_i \) for \( i = 1, 2, \ldots, n \).

It is usually enough to analyse reproducing kernels instead of the corresponding Hilbert spaces. It is therefore somehow surprising that it is difficult to find in the literature explicit formulas for the reproducing kernels of the Sobolev spaces \( W^s_2(\mathbb{R}^d) \) except for the univariate case \( d = 1 \) with \( s = 1 \) and \( s = 2 \). For \( s = 1 \), we have

\[
K_{1,1}(x, t) = \frac{1}{2} \exp \left( -|x - t| \right) \quad \text{for all } x, t \in \mathbb{R},
\]

see for example [12], and for \( s = 2 \) we have

\[
K_{1,2}(x, t) = \frac{\sqrt{3}}{3} e^{-|x-t|/\sqrt{3}} \sin \left( \frac{|x-t|}{2} + \frac{\pi}{6} \right) \quad \text{for all } x, t \in \mathbb{R},
\]

see [4].

We want to add that reproducing kernels of the Sobolev spaces with an equivalent norm to \( (1) \) or reproducing kernels of generalized Sobolev spaces can be found in the literature, see for instance [4, 5, 10, 16]. We will return to this point later.

The definition of the Sobolev space \( W^s_2(\mathbb{R}^d) \) makes sense even for infinite smoothness \( s = \infty \), hence we take, in the definition (1) of the norm, all partial derivatives of any order.
Observe that this is now a tensor product Sobolev space. This and more general Sobolev spaces of infinite order were studied by Dubinskij [3].

We comment what we mean by explicit formulas of the kernels $K_{d,s}$. It is well known that reproducing kernels are related to complete orthonormal basis’s of their corresponding Hilbert spaces. We illustrate this point for the space $W^s_2(\mathbb{R}^d)$. Let $\{e_k\}_{k=1}^\infty$ be its complete orthonormal basis. Since $K_{d,s}(\square, t) \in W^s_2(\mathbb{R}^d)$ for all $t \in \mathbb{R}^d$ then

$$K_{d,s}(\square, t) = \sum_{n=1}^{\infty} \langle K_{d,s}(\square, t), e_k \rangle_{W^s_2(\mathbb{R}^d)} e_k = \sum_{k=1}^{\infty} e_k(t) e_k.$$  

Hence,

$$K_{d,s}(x, t) = \sum_{n=1}^{\infty} e_k(t) e_k(x) \text{ for all } x, t \in \mathbb{R}^d.$$  

We hope that the reader would agree with us that the last formula is not very explicit and more explicit formulas of the reproducing kernels $K_{d,s}$ are indeed needed.

The following theorem is essentially from Hegland and Marti [7] in the case of finite smoothness $s$. In fact, it is only one sentence on page 608 in their paper that the kernel is the Fourier transform of the rational function given by (7) on page 614 without even giving the formula for the kernel. Therefore we give a complete presentation here.

**Theorem 1.** The reproducing kernel of $W^s_2(\mathbb{R}^d)$ with $s > d/2$ is

$$K_{d,s}(x, t) = \int_{\mathbb{R}^d} \frac{\exp(2\pi i (x - t) \cdot u)}{1 + \sum_{0 < |\alpha| \leq s} \prod_{j=1}^d (2\pi u_j)^{2\alpha_j}} \, du \text{ for all } x, t \in \mathbb{R}^d,$$

where $x_j, t_j, u_j$ are components of $x, t, u \in \mathbb{R}^d$, $i = \sqrt{-1}$, and $(x - t) \cdot u = \sum_{j=1}^d (x_j - t_j)u_j$ is the usual Euclidean inner product over $\mathbb{R}^d$.

In the case of infinite smoothness $s = \infty$, we obtain the kernel

$$K_{d,\infty}(x, t) = \prod_{j=1}^d \frac{2}{\pi(x_j - t_j)^3} (\sin(x_j - t_j) - (x_j - t_j) \cos(x_j - t_j)) \text{ for all } x, t \in \mathbb{R}^d.$$  

We obtain these formulas by using the Fourier transform and a few of its standard properties. In particular, we find a formula which relates the inner products of $W^s_2(\mathbb{R}^d)$ and $L_2(\mathbb{R}^d)$. This relation allows us to find a complete orthonormal basis of $W^s_2(\mathbb{R}^d)$ in terms of a complete orthonormal basis of $L_2(\mathbb{R}^d)$. 

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We now comment on the form of \( K_{d,s} \). Obviously, \( K_{d,s} \) takes real values since we can replace
\[
\exp(2\pi i (x - t) \cdot u) = \cos(2\pi (x - t) \cdot u) + i \sin(2\pi (x - t) \cdot u)
\]
and the integral of the imaginary part \( \sin(2\pi (x - t) \cdot u) \) is zero since the corresponding integrand is odd with respect to \( u \). We can do even more. Namely,
\[
e^{2\pi i (x - t) \cdot u} = \prod_{j=1}^{d} e^{2\pi i (x_j - t_j) u_j} = \prod_{j=1}^{d} \left( \cos(2\pi (x_j - t_j) u_j) + i \sin(2\pi (x_j - y_j) u_j) \right)
\]
and all terms with \( \beta_j = 0 \) for some \( j \) will disappear after integration as an odd function of \( u_j \). Therefore, we can rewrite \( K_{d,s} \) for all \( x, t \in \mathbb{R}^d \) as
\[
K_{d,s}(x, t) = \int_{\mathbb{R}^d} \frac{\cos(2\pi (x - t) \cdot u)}{1 + \sum_{0<|\alpha|_1 \leq s} \prod_{j=1}^{d} (2\pi u_j)^{2\alpha_j}} \, du = \int_{\mathbb{R}^d} \frac{\prod_{j=1}^{d} \cos(2\pi (x_j - t_j) u_j)}{1 + \sum_{0<|\alpha|_1 \leq s} \prod_{j=1}^{d} (2\pi u_j)^{2\alpha_j}} \, du.
\]
(2)

Clearly, \( K_{d,s}(x, x) \) is independent of \( x \) and
\[
K_{d,s}(x, x) = \int_{\mathbb{R}^d} \frac{1}{1 + \sum_{0<|\alpha|_1 \leq s} \prod_{j=1}^{d} (2\pi u_j)^{2\alpha_j}} \, du.
\]
Note that \( K_{d,s}(x, x) < \infty \) iff \( s > d/2 \). This shows the importance of the embedding condition for the existence of the reproducing kernel.

Obviously, it would be useful to find an even more explicit form of \( K_{d,s} \) than that presented in Theorem. Ideally, we would like to find a closed form for the integral defining \( K_{d,s} \). We succeeded with this problem only for \( d = 1 \). In this case, the integral over \( \mathbb{R} \) can be explicitly computed by the residual method and for all \( x, t \in \mathbb{R} \) we obtain
\[
K_{1,s}(x, t) = -\sum_{j=1}^{s} \frac{e^{-|x-t| \sin(j\pi/(s+1))}}{s+1} \sin \left( \frac{j\pi}{s+1} \right) \cos \left( \frac{|x-t|}{s+1} \right) \cos \left( \frac{j\pi}{s+1} + \frac{2j\pi}{s+1} \right).
\]
(3)

It is interesting that, for fixed \( x \) and \( t \to \infty \), the function \( K_{1,s}(x, t) \) decays exponentially for all \( s < \infty \) but only polynomially for \( s = \infty \), see Theorem. The proofs of all these formulas are provided in Section 2.
In Section 3 we present two applications based on the form of the reproducing kernel. The first application is on the best embedding constants between the Sobolev spaces $W^s_2(\mathbb{R}^d)$ with $s > d/2$ and $L_\infty(\mathbb{R}^d)$. It is easy to show that the best embedding constant is $K_{d,s}(0,0)^{1/2}$ and it is exponentially small in $d$.

The second application is on integration problems

$$S_{\varrho_d}(f) = \int_{\mathbb{R}^d} f(x) \varrho_d(x) \, dx$$

for $f \in W^s_2(\mathbb{R}^d)$ and a probability density $\varrho_d : \mathbb{R}^d \to \mathbb{R}^+_0$, where $s > d/2$. We prove that worst case integration errors of some algorithms that use $n$ function values is exponentially small in $d$ and decay at least as $n^{-1/2}$. This implies strong polynomial tractability of integration for the absolute error criterion. In addition, we also consider strong polynomial tractability of integration for tensor product Sobolev spaces.

The final Section 4 of this paper contains concluding remarks on how the results can be generalized to weighted Sobolev spaces, Sobolev spaces with equivalent norms, as well as more general reproducing kernels Hilbert spaces.

## 2 Proofs

We will be using standard properties of the Fourier transform which can be found, for example, in [11]. For integrable functions $f$ over $\mathbb{R}^d$, the Fourier transform is defined as

$$[\mathcal{F}f](z) = \int_{\mathbb{R}^d} f(u) e^{-2\pi i z \cdot u} \, du \quad \text{for all} \quad z \in \mathbb{R}^d,$$

where, as before, $z \cdot u = \sum_{j=1}^d z_j u_j$ for components $z_j, u_j$ of $z$ and $u$.

It is well known that for $f, g \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ we have

$$\langle f, g \rangle_{L_2(\mathbb{R}^d)} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L_2(\mathbb{R}^d)}.$$

Since $L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ is a dense subset of $L_2(\mathbb{R}^d)$ there is a unique extension of $\mathcal{F}$ to $L_2(\mathbb{R}^d)$. For simplicity we denote this extension also by $\mathcal{F}$. The mapping $\mathcal{F}$ is an isometry and

$$[\mathcal{F}^{-1}f](z) = [\mathcal{F}f](-z) \quad \text{for all} \quad f \in L_2(\mathbb{R}^d) \quad \text{and} \quad z \in \mathbb{R}^d.$$

For $f \in W^s_2(\mathbb{R}^d)$, we have $D^\alpha f \in L_2(\mathbb{R}^d)$ for all $|\alpha| \leq s$. It is known that

$$[\mathcal{F}(D^\alpha f)](z) = \left( \prod_{j=1}^d (2\pi i z_j)^{\alpha_j} \right) [\mathcal{F}f](z) \quad \text{for all} \quad z \in \mathbb{R}^d.$$
Let
\[ v_{d,s}(z) = \left( 1 + \sum_{0<|\alpha|_1 \leq s} \prod_{j=1}^{d} (2\pi z_j)^{2|\alpha_j|} \right)^{1/2} \text{ for all } z \in \mathbb{R}^d. \] (4)

Clearly, \( v_{d,s} \geq 1 \) and it is easy to verify that \( s > d/2 \) implies \( v_{d,s}^{-1} \in L_2(\mathbb{R}^d) \). We are ready to prove the following lemma which relates the inner products of \( W_2^s(\mathbb{R}^d) \) and \( L_2(\mathbb{R}^d) \).

**Lemma 2.**
\[ f \in W_2^s(\mathbb{R}^d) \iff v_{d,s} \mathcal{F}f \in L_2(\mathbb{R}^d), \]
\[ \langle f, g \rangle_{W_2^s(\mathbb{R}^d)} = \langle v_{d,s} \mathcal{F}f, v_{d,s} \mathcal{F}g \rangle_{L_2(\mathbb{R}^d)} \text{ for all } f, g \in W_2^s(\mathbb{R}^d). \]

**Proof.** For \( f \in W_2^s(\mathbb{R}^d) \) we have
\[ \| f \|^2_{W_2^s(\mathbb{R}^d)} = \sum_{|\alpha|_1 \leq s} \| D^\alpha f \|^2_{L_2(\mathbb{R}^d)} = \sum_{|\alpha|_1 \leq s} \| [\mathcal{F}(D^\alpha)] f \|^2_{L_2(\mathbb{R}^d)} \]
\[ = \int_{\mathbb{R}^d} \sum_{|\alpha|_1 \leq s} \prod_{j=1}^{d} (2\pi z_j)^{2|\alpha_j|} \| [\mathcal{F} f](z) \|^2 \, dz = \int_{\mathbb{R}^d} v_{d,s}(z)^2 \| [\mathcal{F} f](z) \|^2 \, dz \]
\[ = \| v_{d,s} \mathcal{F}f \|^2_{L_2(\mathbb{R}^d)}. \]

This means that \( f \in W_2^s(\mathbb{R}^d) \) implies that \( v_{d,s} \mathcal{F}f \in L_2(\mathbb{R}^d) \). Of course, if \( v_{d,s} \mathcal{F}f \in L_2(\mathbb{R}^d) \) then we can reverse our reasoning and claim that \( f \in W_2^s(\mathbb{R}^d) \). This proves the first part of Lemma 2.

For \( f, g \in W_2^s(\mathbb{R}^d) \) we have
\[ \langle f, g \rangle_{W_2^s(\mathbb{R}^d)} = \sum_{|\alpha|_1 \leq s} \langle D^\alpha f, D^\alpha g \rangle_{L_2(\mathbb{R}^d)} = \sum_{|\alpha|_1 \leq s} \langle [\mathcal{F}(D^\alpha)] f, [\mathcal{F}(D^\alpha)] g \rangle_{L_2(\mathbb{R}^d)} \]
\[ = \int_{\mathbb{R}^d} \sum_{|\alpha|_1 \leq s} \prod_{j=1}^{d} (2\pi z_j)^{2|\alpha_j|} \| [\mathcal{F} f](z) \| \left( \prod_{j=1}^{d} (2\pi z_j)^{2|\alpha_j|} \right) \| [\mathcal{F} g](z) \| \, dz \]
\[ = \int_{\mathbb{R}^d} v_{d,s}(z)^2 \langle [\mathcal{F} f](z), [\mathcal{F} g](z) \rangle \, dz \]
\[ = \langle v_{d,s} \mathcal{F}f, v_{d,s} \mathcal{F}g \rangle_{L_2(\mathbb{R}^d)}, \]

as claimed in the second part of Lemma 2. \( \square \)
From Lemma 2 it is easy to find a complete orthonormal basis of $W^s_2(\mathbb{R}^d)$ in terms of a complete orthonormal basis $\{e_k\}_{k=1}^\infty$ of the space $L^2(\mathbb{R}^d)$. Indeed, let

$$f_k = \mathcal{F}^{-1}(v_{d,s}^{-1}e_k) \quad \text{for all } k \in \mathbb{N}.$$  

Then $f_k \in L^2(\mathbb{R}^d)$ and $e_k = v_{d,s} \mathcal{F}f_k \in L^2(\mathbb{R}^d)$. Due to the first point of Lemma 2 we also have that $f_k \in W^s_2(\mathbb{R}^d)$. Clearly, due to the second point of Lemma 2 we have

$$\langle f_k, f_j \rangle_{W^s_2(\mathbb{R}^d)} = \langle v_{d,s} \mathcal{F}f_k, v_{d,s} \mathcal{F}f_j \rangle_{L^2(\mathbb{R}^d)} = \langle e_k, e_j \rangle_{L^2(\mathbb{R}^d)} = \delta_{k,j}. $$

Hence $\{f_k\}_{k=1}^\infty$ is orthonormal in $W^s_2(\mathbb{R}^d)$.

To show that the $\{f_k\}_{k=1}^\infty$ is complete, take an arbitrary $f \in W^s_2(\mathbb{R}^d)$. Then $v_{d,s} \mathcal{F}f \in L^2(\mathbb{R}^d)$ and

$$v_{d,s} \mathcal{F}f = \sum_{k=1}^\infty \langle v_{d,s} \mathcal{F}f, e_k \rangle_{L^2(\mathbb{R}^d)} e_k = \sum_{k=1}^\infty \langle f, \mathcal{F}^{-1}(v_{d,s}^{-1}e_k) \rangle_{W^s_2(\mathbb{R}^d)} e_k$$

$$= \sum_{k=1}^\infty \langle f, f_k \rangle_{W^s_2(\mathbb{R}^d)} e_k.$$  

Hence

$$f = \sum_{k=1}^\infty \langle f, f_k \rangle_{W^s_2(\mathbb{R}^d)} \mathcal{F}^{-1}(v_{d,s}^{-1}e_k) = \sum_{k=1}^\infty \langle f, f_k \rangle_{W^s_2(\mathbb{R}^d)} f_k,$$  

as claimed.

Due to $s > d/2$ we know that $W^s_2(\mathbb{R}^d)$ is a reproducing kernel Hilbert space and its reproducing kernel is denoted by $K_{d,s}$. We need to show that $K_{d,s}$ satisfies the formula of Theorem 1. Since $K_{d,s}(\square, t) \in W^s_2(\mathbb{R}^d)$, Lemma 2 yields for all $f \in W^s_2(\mathbb{R}^d)$

$$f(t) = \langle f, K_{d,s}(\square, t) \rangle_{W^s_2(\mathbb{R}^d)} = \langle v_{d,s} \mathcal{F}f, v_{d,s} \mathcal{F}[K_{d,s}(\square, t)] \rangle_{L^2(\mathbb{R}^d)}.$$  

On the other hand,

$$f(t) = [\mathcal{F}^{-1} \mathcal{F}f](t) = \int_{\mathbb{R}^d} e^{2\pi i t \cdot u} [\mathcal{F}f](u) \, du$$

$$= \int_{\mathbb{R}^d} v_{d,s}(u) [\mathcal{F}f](u) v_{d,s}(u) \frac{\exp(2\pi i t \cdot u)}{v_{d,s}^2(u)} \, du$$

$$= \left\langle v_{d,s} \mathcal{F}f, v_{d,s} \frac{\exp(-2\pi i t \cdot \square)}{v_{d,s}^2} \right\rangle_{L^2(\mathbb{R}^d)}.$$  

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From (5) and (6) we conclude
\[
\mathcal{F}[K_{d,s}(\Box,t)](u) = \frac{\exp(-2\pi i t \cdot u)}{v_{d,s}^2}
\]
or equivalently
\[
K_{d,s}(x,t) = \mathcal{F}^{-1}\left[\frac{\exp(-2\pi i t \cdot \Box)}{v_{d,s}^2}\right](x)
\]
almost everywhere.

Since we are dealing with continuous functions, the last relation must hold for all arguments, i.e.,
\[
K_{d,s}(x,t) = \int_{\mathbb{R}^d} \frac{\exp(2\pi i (x-t) \cdot u)}{v_{d,s}^2(u)} \, du,
\]
as claimed. This completes the proof of Theorem 1 for finite \(s\).

We turn to the case \(s = \infty\). Again we obtain
\[
K_{d,\infty}(x,t) = \int_{\mathbb{R}^d} \frac{\prod_{j=1}^d \cos(2\pi (x_j-t_j)u_j)}{\sum_{|\alpha|_1 < \infty} \prod_{j=1}^d (2\pi u_j)^{2\alpha_j}} \, du.
\]
Now
\[
\sum_{|\alpha|_1 < \infty} \prod_{j=1}^d (2\pi u_j)^{2\alpha_j} = \prod_{j=1}^d \left( \sum_{n=0}^{\infty} (2\pi u_j)^{2n} \right).
\]
For \(2\pi |u_j| \geq 1\) the last product is not finite and therefore we need to integrate only over \([-1/(2\pi), 1/(2\pi)]\), and we obtain
\[
K_{d,\infty}(x,t) = \prod_{j=1}^d \int_{-1/(2\pi)}^{1/(2\pi)} (1 - 4\pi^2 u^2) \cos(2\pi (x_j - t_j) u) \, du.
\]
Integration by parts yields
\[
K_{d,\infty}(x,t) = \prod_{j=1}^d 2 \pi (x_j - t_j)^3 \left( \sin(x_j - t_j) - (x_j - t_j) \cos(x_j - t_j) \right).
\]
(7)

Let\(^{1}\)
\[
\tilde{K}_{\infty}(x) := K_{1,\infty}(x,0) = \prod_{j=1}^d \frac{2}{\pi x^3} (\sin x - x \cos x) \quad \text{for all } x \in \mathbb{R},
\]

\(^{1}\)We propose to call the function \(\tilde{K}_{\infty}\) the \textit{Varenna} function since it was found during the discrepancy workshop in Varenna, Italy, in June 2016.
which is possibly the "simplest" function in the space $W^\infty_2(\mathbb{R})$, in particular, this is a $C^\infty$ function with small derivatives, see Figure 1. Using the series representation of sin and cos we obtain

$$\tilde{K}_\infty(x) = \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!(2j+3)} x^{2j}$$

$$= \frac{2}{3\pi} \left( 1 - \frac{x^2}{10} + \frac{x^4}{280} - \ldots \right).$$

The kernel $K_{d,\infty}$ is generated by the function $\tilde{K}_\infty$ since

$$K_{d,\infty}(x, t) = \prod_{j=1}^{d} \tilde{K}_\infty(x_j - t_j) \quad \text{for all} \quad x, t \in \mathbb{R}^d.$$

In particular, we obtain

$$K_{d,\infty}(x, x) = \left( \frac{2}{3\pi} \right)^d = (0.2122\ldots)^d. \quad (8)$$

We still need to prove (3). Assume now that $d = 1$. Then

$$K_{1,s}(x, t) = \int_{\mathbb{R}} \frac{e^{2\pi i (x-t)u}}{1 + \sum_{\ell=1}^{s}(2\pi u)^{2\ell}} \, du = \int_{\mathbb{R}} \frac{e^{2\pi i |x-t|u}}{1 + \sum_{\ell=1}^{s}(2\pi u)^{2\ell}} \, du.$$

We did not find an explicit formula for the kernel $K_{1,s}$ in the literature except for $s = 1$ and $s = 2$. In any case the derivation of the kernel is similar as in [12] for $s = 1$. 

Figure 1: The function $\tilde{K}_\infty$
To compute the integral that appears in the formula for $K_{1,s}$, we use the residual method. Let $\xi = |x - t|$. Then the integrand is

$$f(u) := \frac{\exp(2\pi i \xi u)}{1 + \sum_{\ell=1}^{s} (2\pi u)^{2\ell}} = \exp(2\pi i \xi u) \frac{(2\pi u)^{2} - 1}{(2\pi u)^{2s+2} - 1} \quad \text{for} \quad (2\pi u)^{2} \neq 1.$$  

The poles of $f$ in the upper half plane are

$$u_{j} = \frac{1}{2\pi} \exp \left( i \frac{j\pi}{s+1} \right) \quad \text{for} \quad j = 1, 2, \ldots, s.$$  

Note that $u_{0} = \frac{1}{2\pi}$ and $u_{s+1} = -\frac{1}{2\pi}$ on the real line are not poles. The integral is then equal to the product of $2\pi i$ by the sum of the residues of the integrand at the poles. We have

$$\text{Res}_{f(u_{j})} = \lim_{u \to u_{j}} (u - u_{j}) f(u) = \lim_{u \to u_{j}} (u - u_{j}) \frac{\exp(2\pi i \xi u)((2\pi u)^{2} - 1)}{(2\pi u)^{2s+2} - 1}$$

$$= \lim_{u \to u_{j}} (u - u_{j}) \frac{\exp(2\pi i \xi u)((2\pi u)^{2s+2} - 1) - ((2\pi u_{j})^{2s+2} - 1)}{(2\pi u)^{2s+2} - (2\pi u_{j})^{2s+2}}$$

$$= \lim_{u \to u_{j}} \frac{(2\pi u)^{2} - 1 \exp(2\pi i \xi u)}{2(2s + 2)(2\pi u_{j})^{2s+1}} \exp(2\pi i \xi u_{j}).$$

This yields

$$K_{1,s}(x, t) = \frac{i}{2s + 2} \sum_{j=1}^{s} \exp \left( |x - t| \frac{\exp(i \frac{j\pi}{s+1} + i \frac{\pi}{2})}{\exp(i \frac{j\pi}{s+1})} \left( \exp \left( i \frac{2j\pi}{s+1} \right) - 1 \right) \right). \quad (9)$$

The kernel is real valued and we may write $\tilde{K}_{s}(x - t) = K_{1,s}(x, t)$ as

$$\tilde{K}_{s}(t) = -\frac{1}{2s + 2} \sum_{j=1}^{s} e^{-|t| \sin \left( \frac{\pi j}{s+1} \right)} \left( \sin \left( |t| \cos \left( \frac{j\pi}{s+1} \right) + \frac{3j\pi}{s+1} \right) \right.$$

$$\left. - \sin \left( |t| \cos \left( \frac{j\pi}{s+1} \right) + \frac{j\pi}{s+1} \right) \right) \quad \text{for} \quad |t| \cos \left( \frac{j\pi}{s+1} \right) + \frac{j\pi}{s+1} \right). \quad (10)$$

This proves (9), and completes the proofs of all results mentioned in the previous section.
We illustrate \( K_{1,s} \) for \( s = 1, 2, 3, 4 \). We have
\[
\tilde{K}_1(t) = \frac{1}{2} e^{-|t|},
\]
\[
\tilde{K}_2(t) = \frac{\sqrt{3}}{3} e^{-|t|\sqrt{3}/2} \sin \left( \frac{|t|}{2} + \frac{\pi}{6} \right),
\]
\[
\tilde{K}_3(t) = \frac{1}{4} \left( e^{-|t|} + \sqrt{2} e^{-|t|/\sqrt{2}} \sin \frac{|t|}{\sqrt{2}} \right),
\]
\[
\tilde{K}_4(t) = -\frac{2}{3} \left( e^{-|t|\sin \frac{\pi}{3}} \cos \left( |t| \cos \frac{\pi}{5} + \frac{2\pi}{3} \right) \sin \frac{\pi}{5} + e^{-|t|\sin \frac{\pi}{3}} \cos \left( |t| \cos \frac{2\pi}{5} + \frac{4\pi}{5} \right) \sin \frac{2\pi}{5} \right).
\]
The function \( \tilde{K}_1 \) is positive on \( \mathbb{R} \), while the functions \( \tilde{K}_2, \tilde{K}_3 \) and \( \tilde{K}_4 \) also take negative values.

**Remark 3.** The above formulas, see (10), show that the functions \( \tilde{K}_s \) decay exponentially fast for finite \( s \), while for \( s = \infty \) we only have quadratic decay.

Clearly, and as one can see from the formula in Theorem 1 the value of \( K_{1,\infty}(0,0) = \tilde{K}_s(0) \) is monotonically decreasing with \( s \). Using the explicit formula for \( \tilde{K}_1 \) from above together with (8), we obtain the following lemma.

**Lemma 4.** Let \( K_{d,s} \) be the reproducing kernel from Theorem 1. Then, for \( d = 1 \) and \( s \in \mathbb{N} \), we have
\[
\frac{2}{3\pi} = K_{1,\infty}(0,0) \leq K_{1,s}(0,0) = \frac{1}{s + 1} \cos \frac{\pi}{s+2} \leq K_{1,1}(0,0) = \frac{1}{2}.
\]
For finite smoothness \( s \), the explicit equality above was shown by Hegland and Marti [7, Corollary 1]. These authors also computed the limit for \( s \to \infty \). See also [14, 15] for more representations.

### 3 Applications

We briefly discuss two applications for which the knowledge of the form of the reproducing kernel is very helpful.

#### 3.1 Embedding constants

It is well-known that all information of a reproducing kernel Hilbert space \( H(K) \) is given by the reproducing kernel \( K : D \times D \to \mathbb{R} \) of the space. In particular, one can give an
explicit formula for the embedding constant in $L_\infty(D)$, i.e., the maximal absolute function value that can be attained by a function in the unit ball of $H(K)$. This constant is the norm of the identity operator $I_K : H(K) \to L_\infty(D)$, hence

$$\|I_K\| = \sup_{f \neq 0} \frac{\|f\|_{L_\infty(D)}}{\|f\|_{H(K)}}.$$  \hfill (11)

The following result is known, for convenience we give a short proof.

**Lemma 5.** Let $H(K)$ be a reproducing kernel Hilbert space with reproducing kernel $K : D \times D \to \mathbb{R}$ for a nonempty $D \subseteq \mathbb{R}^d$. For the embedding $I_K : H(K) \to L_\infty$ we have

$$\|I_K\| = \sup_{x \in D} K(x, x)^{1/2}$$

and, in particular, $\|I_K\| = K(0,0)^{1/2}$ if $H(K)$ is translation invariant.

**Proof.** We denote by $\delta_x(t) = K(t, x)$ the (representer of the) Dirac functional in $H(K)$, i.e., $f(x) = \langle f, \delta_x \rangle_{H(K)}$. Clearly, $K(x,t) = \langle \delta_x, \delta_t \rangle_{H(K)}$ and hence $\|\delta_x\|_{H(K)} = K(x,x)^{1/2}$. From this we obtain the formula for the norm of $I_K$. Finally, if $H(K)$ is translation invariant, we clearly have $K(x,t) = K(x+s,t+s)$ and hence $K(x,x) = K(0,0)$. This completes the proof. \hfill $\square$

The embedding constant is important for many applications and so we have another reason to know the kernel $K$ of a Hilbert space. We will discuss one of these applications in the next section.

We now turn to specific estimates of the embedding constant $\|I_{d,s}\|$ for the Sobolev spaces $W^{s}_2(\mathbb{R}^d)$ with reproducing kernel $K_{d,s}$, where $I_{d,s} := I_{K_{d,s}}$ is the embedding from $W^{s}_2(\mathbb{R}^d)$ to $L_\infty(\mathbb{R}^d)$. Recall that the diagonal values of $K_{d,s}$ are given by

$$K_{d,s}(x,x) = K_{d,s}(0,0) = \int_{\mathbb{R}^d} \frac{1}{1 + \sum_{0<|\alpha|_1 \leq s} \prod_{j=1}^d (2\pi u_j)^{2\alpha_j}} \, du,$$

see Theorem \[. We change variables by $t_j = 2\pi u_j$, and obtain from Lemma \[ that

$$\|I_{d,s}\|^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{dt}{1 + \sum_{0<|\alpha|_1 \leq s} \prod_{j=1}^d t_j^{2\alpha_j}}.$$ 

We use the multinomial identity for $\ell \in \{1, 2, \ldots, s\}$ and obtain

$$1 + \sum_{0<|\alpha|_1 \leq s} \prod_{j=1}^d t_j^{2\alpha_j} \geq 1 + \sum_{0<|\alpha|_1 \leq \ell} \prod_{j=1}^d t_j^{2\alpha_j} \geq \frac{1}{\ell!} \left( 1 + \sum_{j=1}^\ell \frac{t_j^2}{j} \right)^\ell.$$ 

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Therefore

\[ \|I_{d,s}\|^2 \leq \frac{\ell!}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{dt}{(1 + \sum_{j=1}^{d} t_j^2)\ell} = \frac{\ell!}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \frac{y^{d-1}}{(1+y^2)\ell} dy. \]

The last integral is finite iff \(2\ell - d \geq 1\).

Let \(d = 1\). Then \(s \geq 1\) and we can take \(\ell = 1\). Then

\[ \int_0^\infty \frac{y^{d-1}}{(1+y^2)\ell} \, dy = \int_0^\infty \frac{1}{1+y^2} \, dy = \frac{\pi}{2}, \]

and since \(\Gamma(1/2) = \sqrt{\pi}\) we have

\[ \|I_{1,s}\|^2 \leq \frac{1}{2} \leq \frac{(d+1)^2}{2} \frac{1}{2d\pi^{d/2}} \bigg|_{d=1}. \]

For \(d \geq 1\), we have

\[ \int_0^\infty \frac{y^{d-1}}{(1+y^2)\ell} \, dy \leq \int_0^1 y^{d-1} \, dy + \int_1^\infty y^{-2\ell+d-1} \, dy = \frac{1}{d} + \frac{1}{2\ell - d}. \]  \hspace{1cm} (12)

Let \(d = 3\). Then \(s \geq 2\) and we can take \(\ell = 2\). Since \(\Gamma(3/2) = \sqrt{\pi}/2\) then

\[ \|I_{3,s}\|^2 \leq \frac{4}{3\pi^2} \leq \frac{(d+1)^2}{2} \frac{1}{2d\pi^{d/2}} \bigg|_{d=3}. \]

Assume now that \(d\) is even, \(d = 2k\) with \(k \geq 1\). Then \(s > d/2\) means that \(s \geq k + 1\) and we may take \(\ell = k + 1\). Hence \(2\ell - d = 2\). Using this value of \(\ell\), and remembering that \(\Gamma(d/2) = \Gamma(k) = (k-1)!\) we obtain

\[ \|I_{d,s}\|^2 \leq \frac{2k(k+1)}{2d\pi^{d/2}} \left( \frac{1}{d} + \frac{1}{2} \right) = \frac{(d+2)^2}{4} \frac{1}{2d\pi^{d/2}} \leq \frac{(d+1)^2}{2} \frac{1}{2d\pi^{d/2}}. \]

Assume now that \(d\) is odd, \(d = 2k + 1\) with \(d \geq 5\). Then \(s > d/2\) means that, again, we may take \(\ell = k + 1\) and \(k \geq 2\). The Gamma function \(\Gamma\) is monotone increasing for \(x \geq 2\). Therefore \(\Gamma(k+1/2) \geq \Gamma(k) = (k-1)!\), and we obtain

\[ \|I_{d,s}\|^2 \leq \frac{2k(k+1)}{2d\pi^{d/2}} \left( \frac{1}{d} + 1 \right) \leq \frac{(d+1)^2}{2} \frac{1}{2d\pi^{d/2}}. \]

Hence, for all \(d\) we have

\[ \|I_{d,s}\|^2 \leq \frac{(d+1)^2}{2d\pi^{d/2}}. \]
Clearly, this means that $\|I_{d,s}\|$ goes exponentially fast to zero when $d$ approaches infinity. Asymptotically, the speed of convergence with respect to $d$ is $(2^{1/2} \pi^{1/4})^{-d} = (0.531 \ldots)^d < (6/11)^d = (0.545 \ldots)^d$. It can be verified numerically that for all values of $d$ we have $\|I_{d,s}\| \leq 10.03 (6/11)^d$.

We can also obtain a lower bound on $\|I_{d,s}\|$. It is clear that $\|I_{d,s}\|$ is a decreasing function of $s$ and therefore it is lower bounded for $s = \infty$, which is $(2/(3\pi))^{d/2} = (0.460 \ldots)^d \geq (5/11)^d = (0.454 \ldots)^d$. We summarize these estimates in the following theorem.

**Theorem 6.** Let $I_{d,s}$ be the embedding from $W_2^s(\mathbb{R}^d)$ to $L_{\infty}(\mathbb{R}^d)$. Then for $d, s \in \mathbb{N}$ with $s > d/2$, we have

$$\left(\frac{5}{11}\right)^d \leq \left(\frac{2}{3\pi}\right)^{d/2} = \|I_{d,\infty}\| \leq \|I_{d,s}\| \leq \frac{d+1}{2(d+1)^2 \pi^{d/4}} \leq 10.03 \left(\frac{6}{11}\right)^d.$$ 

### 3.2 Strong polynomial tractability of integration

We now study the integration problem

$$S_\rho(f) = \int_D f(x) \rho(x) \, dx \quad \text{for} \quad f \in H(K),$$

where $H(K)$ is a reproducing kernel Hilbert space of integrable functions defined on $D \subseteq \mathbb{R}^d$ with kernel $K$, and a probability density $\rho : \mathbb{R}^d \to [0, \infty)$, i.e., $\int_D \rho(x) \, dx = 1$.

Consider a QMC algorithm

$$A_n(f) = \frac{1}{n} \sum_{j=1}^n f(x_j) \quad \text{for} \quad f \in H(K)$$

for some points $x_1, x_2, \ldots, x_n \in D$. It is well known that the worst case error of $A_n$ is

$$e_K(x_1, x_2, \ldots, x_n) := \sup_{\|f\|_{H(K)} \leq 1} \left| S_\rho(f) - \frac{1}{n} \sum_{j=1}^n f(x_j) \right|$$

$$= \left\| \int_{\mathbb{R}^d} \delta_x \rho(x) \, dx - \frac{1}{n} \sum_{j=1}^n \delta_{x_j} \right\|_{H(K)}$$

with $\delta_x(t) = K(t, x)$. The function $h = \int_{\mathbb{R}^d} \delta_x \rho(x) \, dx$, i.e.,

$$h(t) = \int_{\mathbb{R}^d} K(t, x) \rho(x) \, dx,$$
is the representer of $S_\varphi$, hence $S_\varphi(f) = \langle f, h \rangle$. It is also known that if we average the square of the worst case error with respect to $x_1, x_2, \ldots, x_n$ distributed according to the densities $\varphi$ then

$$\int_{D^n} e_K^2(x_1, x_2, \ldots, x_n) \varphi(x_1) \cdots \varphi(x_n) \, dx_1 \cdots dx_n \leq \frac{1}{n} \int_{\mathbb{R}^d} K(t, t) \varphi(t) \, dt.$$  

Hence, there exist points $x_1^*, x_2^*, \ldots, x_n^* \in D$ such that

$$e_K(x_1^*, x_2^*, \ldots, x_n^*) \leq \frac{1}{\sqrt{n}} \left( \int_{\mathbb{R}^d} K(t, t) \varphi(t) \, dt \right)^{1/2}.  \tag{13}$$

From Lemma 5 we obtain

$$e_K(x_1^*, x_2^*, \ldots, x_n^*) \leq \frac{\|I_K\|}{\sqrt{n}}.  \tag{13}$$

Let $n(\varepsilon, H(K))$ be the information complexity of integration, i.e., the minimal number of function values needed to find an algorithm with the (absolute) worst case error at most $\varepsilon$. Then (13) yields

$$n(\varepsilon, H(K)) \leq \left\lceil \left( \frac{\|I_K\|}{\varepsilon} \right)^2 \right\rceil.  \tag{14}$$

We now apply the last estimates to integration in the space $W_2^s(\mathbb{R}^d)$. The following theorem follows from (13), (14) and Theorem 6.

**Theorem 7.** Consider the integration problem $S_\varphi$ given by

$$S_\varphi(f) = \int_{\mathbb{R}^d} f(x) \varphi_d(x) \, dx \quad \text{for} \quad f \in W_2^s(\mathbb{R}^d)$$

with $s > d/2$ and a probability density $\varphi_d$. There exist $x_1^*, x_2^*, \ldots, x_n^* \in \mathbb{R}^d$ such that

$$e_{K_d,s}(x_1^*, x_2^*, \ldots, x_n^*) \leq \frac{10.03}{\sqrt{n}} \left( \frac{6}{11} \right)^d.  \tag{13}$$

Furthermore,

$$n(\varepsilon, d) := n(\varepsilon, W_2^s(\mathbb{R}^d)) \leq \left\lceil 100.6009 \left( \frac{6}{11} \right)^{2d} \frac{1}{\varepsilon^2} \right\rceil.  \tag{14}$$

Since $n(\varepsilon, d) = O(\varepsilon^{-2})$ with the factor in the big $O$ notation independent of $d$, this means that the integration problem is strongly polynomially tractable independently of the probability densities $\varphi_d$’s.
Remark 8. We stress that this positive tractability result holds for the *absolute error criterion*. When we use the *normalized error criterion* then we compare the error with the initial error \(|S_{\varrho_d}|\) for the given density \(\varrho_d\), and consider

\[ n \left( \varepsilon |S_{\varrho_d}| \right) = n \left( \varepsilon |S_{\varrho_d}|, W^s_2(\mathbb{R}^d) \right). \]

It is well known that \(S_{\varrho_d}\) is a well defined continuous linear functional on \(W^s_2(\mathbb{R}^d)\) with

\[ |S_{\varrho_d}|^2 = \int_{\mathbb{R}^{2d}} K_{d,s}(x,t) \varrho_d(x) \varrho_d(t) \, dx \, dt \leq \sup_{x,t \in \mathbb{R}^d} K_{d,s}(x,t) = K_{d,s}(0,0). \]

We now show that this bound is optimal for some \(\varrho_d\). Indeed, consider an arbitrary continuous probability density \(\varrho_d\) on \(\mathbb{R}^d\) with compact support that contains the origin. Now, with \(\varrho_{d,\delta}(x) := \varrho_d(x/\delta)/\delta\), we obtain \(\lim_{\delta \to 0} |S_{\varrho_{d,\delta}}|^2 = K_{d,s}(0,0)\).

In general, if there exists a number \(c \in (0,1]\) such that

\[ |S_{\varrho_d}| \geq c K_{d,s}(0,0)^{1/2} \quad \text{for all } d \in \mathbb{N}, \]

then strong polynomial tractability also holds for the normalized error criterion.

However, if \(|S_{\varrho_d}|/K_{d,s}(0,0)^{1/2}\) goes to zero with \(d\) approaching infinity then we cannot conclude whether the integration problem is tractable or not for the normalized error criterion.

Remark 9. Observe that (13) holds for arbitrary kernels and arbitrary probability density functions \(\varrho\). In particular, it holds for the tensor product Sobolev spaces with the kernels

\[ \tilde{K}_{d,s}(x,t) = \prod_{j=1}^d K_{1,s}(x_j,t_j) \quad \text{for all } x,t \in \mathbb{R}^d. \tag{15} \]

In the last formula we can take arbitrary natural numbers \(s\) and \(d\), the space is always a space of bounded continuous functions, and again the embedding constant is given by \(\tilde{K}_{d,s}(0,0)^{1/2}\).

For such spaces and arbitrary densities \(\varrho_d\) we obtain the existence of \(x_1^*, x_2^*, \ldots, x_n^* \in \mathbb{R}^d\) such that

\[ e(x_1^*, x_2^*, \ldots, x_n^*) \leq \frac{\|\tilde{I}_{d,s}\|}{\sqrt{n}} = \frac{\tilde{K}_{d,s}(0,0)^{1/2}}{\sqrt{n}} = \frac{K_{1,s}(0,0)^{d/2}}{\sqrt{n}} \leq \frac{K_{1,1}(0,0)^{d/2}}{\sqrt{n}} \leq \frac{2^{-d/2}}{\sqrt{n}}. \]

Again, all such integration problems are strongly polynomially tractable for the absolute error criterion.
Remark 10. Some readers might be puzzled since it is well known that, for example, the integration problem
\[ S_d(f) = \int_{[0,1]^d} f(x) \, dx \]
is not polynomially tractable and suffers from the curse of dimensionality for many classical spaces, see [8] for a survey of such results.

In particular, this holds for the spaces \( H^1_{\text{mix}}([0,1]^d) = W_2^1([0,1]) \otimes \cdots \otimes W_2^1([0,1]) \), i.e., the \( d \)-fold tensor product of the space \( W_2^1([0,1]) \), where the domain of functions is restricted to the unit cube \([0,1]^d\). For a detailed discussion of this, see Chapter 20 of [8] and papers cited there. The corresponding space on \( \mathbb{R}^d \) is \( H^1_{\text{mix}}(\mathbb{R}^d) = W_2^1(\mathbb{R}) \otimes \cdots \otimes W_2^1(\mathbb{R}) \), which is the (reproducing kernel) Hilbert space discussed in Remark 9. For this space we consider integration with a probability measure \( \rho_d \) and achieve even strong polynomial tractability. Observe that a function \( f \in H^1_{\text{mix}}([0,1]^d) \) can always be extended to a function \( \tilde{f} \in H^1_{\text{mix}}(\mathbb{R}^d) \) with \( \tilde{f} \mid_{[0,1]^d} = f \). However, the norms of \( f \) and \( \tilde{f} \) can be quite different and hence from \( f \) being in the unit ball of \( H^1_{\text{mix}}([0,1]^d) \) we cannot conclude that \( \tilde{f} \) is in the unit ball of \( H^1_{\text{mix}}(\mathbb{R}^d) \). In some sense, the unit ball of \( H^1_{\text{mix}}(\mathbb{R}^d) \) is “quite small” and admits strong polynomial tractability whereas the unit ball of \( H^1_{\text{mix}}([0,1]^d) \) is “quite large” and causes the curse of dimensionality.

4 Concluding remarks

In the final section we discuss a few issues related to the previous considerations.

4.1 Sobolev spaces with a different norm

Due to Lemma 2, we can write the Sobolev space as
\[ W^s_2(\mathbb{R}^d) = \{ f \in L_2(\mathbb{R}^d) : v_{d,s} \mathcal{F} f \in L_2(\mathbb{R}^d) \}, \]
and the norm can be expressed as
\[ \| f \|_{W^s_2(\mathbb{R}^d)} = \| v_{d,s} \mathcal{F} f \|_{L_2(\mathbb{R}^d)}. \]

In [16], p.133, the following Sobolev space was used for a real \( s \) with \( s > d/2 \)
\[ H^s(\mathbb{R}^d) = \{ f \in L_2(\mathbb{R}^d) : (1 + \| \cdot \|_2^2)^{s/2} \hat{f} \in L_2(\mathbb{R}^d) \} \]
with the inner product
\[ \langle f, g \rangle_{H^s(\mathbb{R}^d)} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} (1 + \|u\|^2)^s \hat{f}(u) \overline{\hat{g}(u)} \, du, \]
and the Fourier transform of \( f \) given by
\[ \hat{f}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(u) e^{-ix \cdot u} \, du, \]
which differs from \( F \) by a factor depending on \( d \).

Neglecting a slightly different role of the factors, the basic difference between \( W^s_2(\mathbb{R}^d) \) and \( H^s(\mathbb{R}^d) \) is that the function \( \nu_{d,s} \) for the space \( W^s_2(\mathbb{R}^d) \) is now replaced by \( \nu_s = (1 + \| \cdot \|^2)^{s/2} \) for the space \( H^s(\mathbb{R}^d) \). Obviously \( \nu_{d,s} \neq \nu_s \). Therefore, although the norms of \( W^s_2(\mathbb{R}^d) \) and \( H^s(\mathbb{R}^d) \) are equivalent, they have different reproducing kernels. Namely, it is proved in [16] that the reproducing kernel of \( H^s(\mathbb{R}^d) \) is
\[ K^*_{d,s}(x, t) = \frac{2^{1-s}}{(s-1)!} \|x - t\|^{s-d/2} B_{d/2-s}(||x - t||_2) \quad \text{for all } x, t \in \mathbb{R}^d, \]
where \( B_{d/2-s} \) is the modified Bessel function of the third kind.

As the reproducing kernel of \( W^s_2(\mathbb{R}^d) \), the reproducing kernel \( K^*_{d,s} \) of \( H^s(\mathbb{R}^d) \) can also be expressed in the form of an integral, see [16, Theorem 10.12],
\[ K^*_{d,s}(x, t) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp \left( i \cdot \frac{x - t}{2} \cdot u \right) \frac{1}{\left( 1 + \sum_{j=1}^d u_j^2 \right)^s} \, du, \quad (16) \]
for all \( x, t \in \mathbb{R}^d \), where \( x_j, t_j, u_j \) are components of \( x, t, u \in \mathbb{R}^d \).

Note that \( K^*_{d,s} \) depends on \( \|x - t\|_2 \), whereas \( K_{d,s} \) depends on \( x - t \). Indeed, the norm (and/or scalar product) of \( H^s(\mathbb{R}^d) \) is isotropic. As the norm for the space \( W^s_2(\mathbb{R}^d) \), the norm of \( H^s(\mathbb{R}^d) \) can also be given by \( L_2 \) norms of the derivatives. One can show this as follows, see also [9, Section 1.3.5] for similar calculations.

Using the formulas of the Fourier transform for derivatives, we have
\[ \widehat{D^\beta f}(x) = \hat{f}(x) \cdot \prod_{j=1}^d (i \cdot x_j)^{\beta_j}, \quad \beta \in \mathbb{N}^d_0, \]
and

$$(1 + \|u\|_2^2)^s = \sum_{\ell=0}^s \binom{s}{\ell} \cdot \|u\|_2^{2\ell} = \sum_{\ell=0}^s \binom{s}{\ell} \cdot \frac{\ell!}{\beta!} \cdot \prod_{j=1}^d u_j^{2\beta_j}$$

$$= \sum_{\ell=0}^s \frac{s!}{(s-\ell)!} \sum_{\beta \in N_0^{d\ell}} \prod_{|\beta|_1=\ell} (iu_j)^{\beta_j} \beta_j!,$$

where $|\beta|_1 = \beta_1 + \ldots + \beta_d$ and $\beta! = \prod_{j=1}^d (\beta_j!)$.

By Parseval’s relation, see Grafakos [6, Theorem 2.2.14], we obtain

$$\int_{\mathbb{R}^d} \hat{f}(u) \hat{g}(u) \, du = \int_{\mathbb{R}^d} f(x) g(x) \, dx.$$ 

Hence,

$$\langle f, g \rangle_{H^s} = (2\pi)^{-d/2} \sum_{\ell=0}^s \frac{s!}{(s-\ell)!} \sum_{\beta \in N_0^{d\ell}} \prod_{|\beta|_1=\ell} \langle D^\beta f, D^\beta g \rangle_{L^2(\mathbb{R}^d)}$$

$$= (2\pi)^{-d/2} \sum_{|\beta|_1 \leq s} \frac{|\beta|_1!}{\beta!} \cdot \binom{s}{|\beta|_1} \langle D^\beta f, D^\beta g \rangle_{L^2(\mathbb{R}^d)}. \quad (17)$$

We now estimate the embedding constant $\|I_{d,s}^*\|$ for the Sobolev spaces $H^s(\mathbb{R}^d)$ with reproducing kernel $K_{d,s}^*$, where $I_{d,s}^* := I_{K_{d,s}}$ is the embedding from $H^s(\mathbb{R}^d)$ to $L_\infty(\mathbb{R}^d)$. From Lemma 5 and (16) we have

$$\|I_{d,s}^*\|^2 = K_{d,s}^*(0,0) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} (1 + \|u\|_2^2)^{-s} \, du = \frac{2}{2^{d/2} \Gamma(d/2)} \int_0^\infty \frac{t^{d-1}}{(1 + t^2)^s} \, dt.$$

Note that the embedding constant tends to infinity for $s \to d/2$. Obviously, if we vary $d$ we must also vary $s = s(d)$ so that $(d/2) > d/2$. Assume that

$$\beta := \inf_{d \in \mathbb{N}} (2s(d) - d) > 0. \quad (18)$$

This assumption allows us to find a bound on $\|I_{d,s}^*\|^2$ only in terms of $d$. Indeed, we estimate the last integral by (12) with $\ell = s(d)$. Then

$$\int_0^\infty \frac{t^{d-1}}{(1 + t^2)^s} \, dt \leq \frac{1}{d} + \frac{1}{2s(d) - d} \leq 1 + \frac{1}{\beta}.$$ 

This yields the following bound on $\|I_{d,s(d)}^*\|^2$. 

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Theorem 11. Let $I_{d,s}^*$ be the embedding from $H^s(\mathbb{R}^d)$ to $L_\infty(\mathbb{R}^d)$ for a real $s = s(d)$ with $s > d/2$ and satisfying (18). Then

$$\|I_{d,s}^*\|^2 \leq \frac{2(1 + 1/\beta)}{2d/2 \Gamma(d/2)}.$$ 

It is well-known that $\Gamma(d/2)$ is super-exponentially large in $d$. Therefore, the embedding constant for the norm of $H^s(\mathbb{R}^d)$ is super-exponentially small in $d$, and it is much smaller than the one for $W^s_2(\mathbb{R}^d)$.

Obviously, we can also consider the integration problem for the space $H^s(\mathbb{R}^d)$. Then (18) implies strong polynomial tractability of integration in the worst case setting and the absolute error criterion.

4.2 Another Sobolev space for $s = \infty$

For the space $W^s_2(\mathbb{R}^d)$ we can take $s = \infty$, whereas for the space $H^s(\mathbb{R}^d)$ the choice $s = \infty$ does not make much sense since $v_s(u) = \infty$ for all $u \neq 0$ and the space $H^\infty(\mathbb{R}^d)$ consists only of the zero function.

There are, however, other Sobolev spaces of functions of infinite smoothness which formally corresponds to $s = \infty$. Furthermore, the reproducing kernel of such a space can be given by a properly normalized Gaussian kernel. In this section we present two definitions of the norm of such a Sobolev space that lead to the Gaussian kernel. We do not know whether these results are known but we could not find a suitable reference in the literature.

For $s = \infty$, we can define a Sobolev space by

$$H^\infty_2(\mathbb{R}^d) = \{ f \in L_2(\mathbb{R}^d) : e^{\|u\|^2/4} \hat{f}(u) \in L_2(\mathbb{R}^d) \}$$

with the radial symmetric inner product

$$\langle f, g \rangle_{H^\infty_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} e^{\|u\|^2/4} \hat{f}(u) \overline{\hat{g}(u)} \, du,$$

where $\| \cdot \|$ denotes the Euclidean norm and the Fourier transform $\hat{f}$ is defined as in the previous section. Expanding $e^{\|u\|^2/2}$, we obtain

$$\langle f, g \rangle_{H^\infty_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \hat{f}(u) \overline{\hat{g}(u)} \left( \sum_{\ell=0}^{\infty} \frac{\|u\|^{2\ell}}{2^\ell \cdot \ell!} \right) \, du$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{2^\ell \cdot \ell!} \int_{\mathbb{R}^d} \hat{f}(u) \overline{\hat{g}(u)} \|u\|^{2\ell} \, du.$$
We now use
\[ \|u\|^2_\ell = \sum_{\beta \in \mathbb{N}_0^d, |eta|_1 = \ell} \ell! \prod_{j=1}^d u_j^{\beta_j} = \sum_{\beta \in \mathbb{N}_0^d, |eta|_1 = \ell} \ell! \prod_{j=1}^d (i u_j)^{\beta_j} (i u_j)^{\beta_j}, \]
as well as \( \left( \prod_{j=1}^d (i \cdot u_j)^{\beta_j} \right) \hat{f}(u) = \widehat{D^\beta f}(u), \) and
\[
\int_{\mathbb{R}^d} \hat{f}(u) \overline{g(u)} \, du = \int_{\mathbb{R}^d} f(x) \overline{g(x)} \, dx.
\]

We observe that
\[
\langle f, g \rangle_{H^\infty_2(\mathbb{R}^d)} = \sum_{\ell=0}^{\infty} \frac{1}{2\ell} \sum_{\beta \in \mathbb{N}_0^d, |eta|_1 = \ell} \frac{1}{\ell!} \int_{\mathbb{R}^d} \hat{D^\beta f}(u) \overline{\hat{D^\beta g}(u)} \, du
\]
\[= \sum_{\beta \in \mathbb{N}_0^d} \frac{1}{2^{|eta|_1} \prod_{j=1}^d (\beta_j)!} \langle D^\beta f, D^\beta g \rangle_{L^2(\mathbb{R}^d)}. \tag{21} \]

These inner products are invariant under orthogonal transformations in the sense that
\[
\left| \langle f, g \rangle_{H^\infty_2(\mathbb{R}^d)} \right| = \left| \langle f \circ O, g \circ O \rangle_{H^\infty_2(\mathbb{R}^d)} \right|
\]
for any orthogonal transformation \( O : \mathbb{R}^d \rightarrow \mathbb{R}^d, \) and every \( f, g \in H^\infty_2(\mathbb{R}^d). \) This is easily seen by the formulas (19) and (20).

We now discuss the reproducing kernel of \( H^\infty_2(\mathbb{R}^d) \) with respect to the inner product (19) or (21). It is well-known that the function
\[
\delta_0(x) = (2\pi)^{-d/2} e^{-\|x\|^2/2} \quad \text{for all } x \in \mathbb{R}^d
\]
is invariant under the Fourier transform, i.e.,
\[
\hat{\delta}_0(x) = \delta_0(x) \quad \text{for all } x \in \mathbb{R}^d.
\]

Hence, we obtain from the definition that the Dirac delta \( \delta_x, x \in \mathbb{R}^d \) in \( H^\infty_2(\mathbb{R}^d) \) is given by
\[
\delta_x(y) = (2\pi)^{-d/2} e^{-\|y-x\|^2/2} \quad \text{for all } x, y \in \mathbb{R}^d.
\]

For this, note that
\[
\hat{\delta}_x(u) = e^{-i u \cdot x} \delta_0(u) = e^{-i u \cdot x} (2\pi)^{-d/2} e^{-\|x\|^2/2}.
\]
The reproducing kernel of $H^\infty_2(\mathbb{R}^d)$ is therefore the famous Gaussian kernel,

$$K^{\text{rad}}_\infty(x, y) = (2\pi)^{-d/4} e^{-\|x-y\|^2/2}$$

for all $x, y \in \mathbb{R}^d$.

Let $I^{\text{rad}}_{d,\infty}$ be the embedding from $H^\infty_2(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. Then for $d \in \mathbb{N}$, we have

$$\|I^{\text{rad}}_{d,\infty}\| = K^{\text{rad}}_\infty(0, 0)^{1/2} = (2\pi)^{-d/4} = (0.6316 \ldots)^d,$$

which is larger than $\|I_{d,\infty}\| = (2/(3\pi))^d = (0.4606 \ldots)^d$.

### 4.3 Weighted multivariate Sobolev spaces

Each variable of $f \in W^s_2(\mathbb{R}^d)$ plays the same role. If we permute variables in an arbitrary way then we obtain another function from $W^s_2(\mathbb{R}^d)$ with the same norm as $f$. This property often leads to the curse of dimensionality for many computational problems, see [8], in the worst case setting for the normalized error criterion. That is why it seems reasonable to treat various variables and groups of variables differently. This can be achieved by weighted spaces. We illustrate this concept for weighted Sobolev multivariate spaces. Let

$$\lambda = \{\lambda_{d, s, \alpha}\}_{d \in \mathbb{N}, |\alpha| \leq s}$$

be a family of positive numbers. We then define the space $W^{s,\lambda}_2(\mathbb{R}^d)$ as the space $W^s_2(\mathbb{R}^d)$ with the redefined norm by

$$\|f\|_{W^{s,\lambda}_2(\mathbb{R}^d)}^2 = \sum_{|\alpha| \leq s} \lambda_{d, s, \alpha} \|D^\alpha f\|_{L^2(\mathbb{R}^d)}^2.$$

Note that for

$$\lambda_{d, s, \alpha} = \frac{|\alpha|!}{(2\pi)^{d/2} \alpha!} \left( \frac{s}{|\alpha|} \right)$$

we have the space $W^s_2(\mathbb{R}^d)$,

$$\lambda_{d, s, \alpha} = \frac{1}{2^{|\alpha|} \alpha!}$$

we have the space $H^s(\mathbb{R}^d)$,

$$\lambda_{d, \infty, \alpha} = \frac{1}{2^{|\alpha|} \alpha!}$$

we have the space $H^\infty_2(\mathbb{R}^d)$.

The assumption that all $\lambda_{d, s, \alpha} < \infty$ is essential. It is done for a good reason since if one of them $\lambda_{d, s, \alpha} = \infty$ and we adopt the convention that $\infty \cdot 0 = 0$ then we must assume that
$D^\alpha f = 0$ and $f$ must be in the kernel of $D^\alpha$, i.e., it must be a polynomial of degree at most of degree $\max(0, \alpha_j - 1)$ for each variables $x_j$. But the only polynomial that belongs to the space $W^s_2(\mathbb{R}^d)$ is the zero polynomial, and therefore in this case the whole space degenerates to the zero element.

For positive and finite $\lambda$, it is easy to check that the reproducing kernel of the weighted Sobolev space $W^{s,\lambda}_2(\mathbb{R}^d)$ is

$$K_{d,s,\lambda}(x,t) = \int_{\mathbb{R}^d} \frac{\exp(2\pi i (x-t) \cdot u)}{\lambda_{d,s,0} + \sum_{0<|\alpha| \leq s} \lambda_{d,s,\alpha} \prod_{j=1}^d (2\pi u_j)^{2\alpha_j}} \, du \quad \text{for all } x, t \in \mathbb{R}^d.$$  

In particular, if $\lambda_{d,s,0} = 1$ and $\lambda_{d,s,\alpha} = \beta$ for all $\alpha$ with $|\alpha| \in (0, s]$ then

$$K_{d,s,\lambda}(x,t) = \int_{\mathbb{R}^d} \frac{\exp(2\pi i (x-t) \cdot u)}{1 + \beta \sum_{0<|\alpha| \leq s} \prod_{j=1}^d (2\pi u_j)^{2\alpha_j}} \, du \quad \text{for all } x, t \in \mathbb{R}^d.$$  

Note that for large $\beta$, the unit ball $\|f\|_{d,s,\lambda} \leq 1$ must have all $\|D^\alpha f\|_{L_2(\mathbb{R}^d)}$ small for nonzero $\alpha$. Clearly, the larger $\beta$ the smaller the unit ball. In general, for appropriately chosen $\lambda_{d,s,\alpha}$ we have a chance to break the curse of dimensionality of many computational problems.

### 4.4 More general Hilbert spaces

For the Sobolev space $W^s_2(\mathbb{R}^d)$ the function $v_{d,s}$ from (4) was instrumental in obtaining the reproducing kernel. We now show that it is not a coincidence and a similar analysis can be done for many functions $\nu$ which generate corresponding reproducing kernel Hilbert spaces.

We now outline this approach. We opt for simplicity and consider the class of functions $\nu$ defined on $\mathbb{R}^d$ from the class $\mathcal{M}$ which is given by

$$\mathcal{M} := \{ \nu \in C(\mathbb{R}^d) : \nu \geq 1 \text{ and } \nu^{-1} \in L_2(\mathbb{R}^d) \}.$$  

For $\nu \in \mathcal{M}$, consider the space $\tilde{H}^\nu$ of all $f \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ such that the $L_2$-norm of $\nu [\mathcal{F}f] \in L_2(\mathbb{R}^d)$ is finite, and with the inner product

$$\langle f, g \rangle_\nu = \langle \nu [\mathcal{F}f], \nu [\mathcal{F}g] \rangle_{L_2(\mathbb{R}^d)},$$  

see also Lemma 2. We denote by $H^\nu$ the completion of $\tilde{H}^\nu$.

Using a similar analysis as before it can be checked that $H^\nu$ is a reproducing kernel Hilbert space and its reproducing kernel is

$$K_\nu(x,t) = \int_{\mathbb{R}^d} \frac{e^{2\pi i (t-x) \cdot u}}{\nu(u)^2} \, du \quad \text{for all } x, t \in \mathbb{R}^d.$$  

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Note that if we take $\nu = v_{d, s}$ then $H^\nu = W^2_s(\mathbb{R}^d)$ and the formula for $K_\nu$ is the same as in Theorem 1. Obtaining bounds or even explicit formulas for the embedding constants and upper bounds for the error of integration can be found in the same way as it was done in Section 3.

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