The Geometry of Self-Dual Gauge Fields

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Abstract
Self-dual 2-forms in $D = 2n$ dimensions are characterised by an
eigenvalue criterion. The equivalence of various definitions of self-
duality is proven. We show that the self-dual 2-forms determine a
$n^2 - n + 1$ dimensional manifold $S_{2n}$ and the dimension of the max-
imal linear subspaces of $S_{2n}$ is equal to the Radon-Hurwitz number
of linearly independent vector fields on the sphere $S^{2n-1}$. The rela-
tion between the maximal linear subspaces and the representations of
Clifford algebras is noted. A general procedure based on this relation
for the explicit construction of linearly self-dual 2-forms is given. The
construction of the octonionic instanton solution in $D = 8$ dimensions
is discussed.
1. Self-duality as an eigenvalue criterion

Let $M$ be a $D = 2n$ dimensional differentiable manifold, and $E$ be a vector bundle over $M$ with standard fiber $R^N$ and structure group $G$. The gauge potentials can be represented by a $G$-valued connection 1-form $A$ on $E$, where $G$ is a linear representation of the Lie algebra of the gauge group $G$. Then the gauge fields are represented by the curvature of the connection $A$ that is given locally by the $G$-valued 2-form

$$F = dA - A \wedge A.$$ 

The Yang-Mills action is the $L^2$ norm of the curvature 2-form $F$

$$\|F\|^2 = \int_M \text{tr}(F \wedge *F)$$

where $*$ denotes the Hodge dual defined relative to a positive definite metric on $M$. The Yang-Mills equations

$$d_E F = 0, \quad *d_E^* F = 0,$$

where $d_E$ is the bundle exterior covariant derivative and $-d_E^*$ is its formal adjoint, determine the critical points of the action.

In $D = 4$ dimensions $F$ is called self-dual or anti-self-dual provided

$$*F = \pm F.$$ 

In this case the self-dual or anti-self-dual 2-forms are the global extrema of the Yang-Mills action. This is due to the fact that the Yang-Mills action has a topological lower bound:

$$\|F\|^2 \geq \int_M \text{tr}(F \wedge F).$$

The term $\text{tr}(F \wedge F)$ is related to the Chern classes of the bundle. Actually if $E$ is a complex 2-plane bundle with $c_1(E) = 0$, then the topological bound is proportional to $c_2(E)$ and this lower bound is realised by a (anti-)self-dual connection. Furthermore, $SU(2)$ bundles over a four manifold are classified by $\int c_2(E)$, hence self-dual connections are minimal representatives of the connections in each equivalence class of $SU(2)$ bundles. This is a generalisation of the fact that an $SU(2)$ bundle admits a flat connection if and only if it is trivial.
In order to derive topological bounds in higher dimensions we briefly recall the computation of the characteristic classes of a vector bundle $E$ in terms of the local curvature 2-forms [1]. Let $F^\alpha$ be the matrix of the local curvature 2-form with respect to a local basis of sections of $E$ on a trivialising neighbourhood $U_\alpha$. The invariant polynomials $\sigma^\alpha_k$ of $F^\alpha$ are defined by

$$\det(I + tF^\alpha) = \sum_{k=0}^{n} \sigma^\alpha_k t^k.$$ 

Hence they are independent of the basis of local sections. Thus if $\sigma^\alpha_k$ and $\sigma^\beta_k$ are invariant polynomials of the local curvature 2-forms $F^\alpha$ and $F^\beta$, on $U_\alpha$ and $U_\beta$, respectively, then they agree on the intersection $U_\alpha \cap U_\beta$. Hence these locally defined 2$k$-forms patch up to give globally defined 2$k$-forms $\sigma_k$. Furthermore it can be shown that the $\sigma_k$'s are closed 2$k$-forms. They realise the de Rham cohomology classes in $H^{2k}$. These cohomology classes depend only on the bundle, i.e. they are independent of the connection. For a complex vector bundle, the cohomology class of $\sigma_k$ is proportional to the Chern class $c_k$, while for a real vector bundle, the $\sigma_{2k+1}$'s are exact forms, and $\sigma_{2k}$'s are proportional to the Pontrjagin classes $p_k$'s. Furthermore, for an $SO(N)$ bundle, the square root of the determinant of $F$ (which is a ring element) defines the Euler class $\chi$. In order to avoid proportionality constants, we will work with the quantities $\sigma_k$'s instead of the Chern or Pontrjagin classes. The $\sigma_k$'s can also be written as linear combinations of $\text{tr} F^k$ (see for example [2] Vol.1, p.87), where $F^k$ means the product of the matrix $F$ with itself $k$ times, with the wedge multiplication of the entries.

In the following we consider real $SO(N)$ bundles.

In $D = 4$ dimensions the topological bound we wrote above is the only one that is available. On the other hand in $D = 8$ dimensions it is possible to introduce two independent topological bounds. The topological lower bound on the action

$$\int_M \text{tr}(F^2 \wedge^* F^2) \geq \int_M p_2(E)$$

is well-known. The self-duality of $F^2$ in the Hodge sense gives global minima of this action involving the second Pontryagin number $\int p_2(E)$. We introduced [3] another topological lower bound on the action

$$\int \text{tr}(F \wedge^* F)^2 \geq \frac{2}{3} \int_M p_1(E)^2.$$
This involves the square of the first Pontryagin number and has to be taken into account as the topology of the Yang-Mills bundle on an eight manifold has to be characterised by both the first and the second Pontryagin numbers. In general, similar topological bounds could be given in higher dimensions. We now turn to the definition of self-dual gauge fields in higher dimensions [3],[4] that would saturate these topological bounds.

**Definition 1.** Suppose $\omega$ is a real 2-form in $D = 2n$ dimensions, and let $\Omega$ be the corresponding $2n \times 2n$ skew-symmetric matrix with respect to some local orthonormal basis. Let $\pm i\lambda_1, \ldots, \pm i\lambda_n$ be the eigenvalues of $\Omega$. $\omega$ is said to be (anti-)self-dual if

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_n|.$$  

There are $2^n$ possible ways to satisfy the above set of equalities. The half of these correspond to self-dual 2-forms, while the remaining half correspond to anti-self-dual 2-forms.

It is not difficult to check that in $D = 4$ dimensions, the above definition coincides with the usual definition of self-duality in the Hodge sense. Let $\Omega$ be the skew symmetric matrix representing a 2-form $F$ in four dimensions. Then it can be seen that the eigenvalues of the matrix $\Omega$ satisfy

$$\lambda_1 \mp \lambda_2 = \sqrt{(\Omega_{12} \mp \Omega_{34})^2 + (\Omega_{13} \pm \Omega_{24})^2 + (\Omega_{14} \mp \Omega_{23})^2}.$$  

Thus for self-duality $\lambda_1 = \lambda_2$, while for anti-self-duality $\lambda_1 = -\lambda_2$. In both cases the absolute values of the eigenvalues are equal. Two cases are distinguished by the sign of the Pfaffian of $\Omega$:

$$\Omega_{12}\Omega_{34} - \Omega_{13}\Omega_{24} + \Omega_{14}\Omega_{23}.$$  

There were several attempts to generalize the notion of self-duality to higher dimensions:

i) A 2-form $\omega$ in $D = 2n$ dimensions is called self-dual if the Hodge dual of $\omega$ is proportional to $\omega^{n-1}$. Here wedge product of $\omega$’s should be understood. This notion is introduced by Trautman [5], and Thcrakian [6] and used widely by others.

ii) A self-dual 2-form $\omega$ in $D = 4k$ dimensions is defined to be the one such that $\omega^k$ is self-dual in the Hodge sense, that is $^*\omega^k = \pm \omega^k$. This notion is also introduced by Thcrakian [6] and adopted by Grossman, Kephardt and Stasheff (GKS) in their study of octonionic instantons in eight dimensions [7].
iii) Both the criteria above are non-linear. Alternatively, (anti-)self-dual 2-forms in \( D = 2n \) dimensions can be defined as eigen-bivectors of a completely antisymmetric fourth rank tensor that is invariant under a subgroup of \( SO(2n) \). The set of such self-dual 2-forms would span a linear space. This notion of self-duality is introduced by Corrigan, Devchand, Fairlie and Nuyts (CDFN) who studied the first-order equations satisfied by Yang-Mills fields in spaces of dimension greater than four and derived \( SO(7) \) self-duality equations in \( R^8 \) [8].

It can be shown that any self-dual 2-form defined by the above criteria satisfies the Yang-Mills equations. However, the corresponding Yang-Mills action need not be extremal. For details we refer to review articles [9],[10]. Here we will prove that our eigenvalue criterion encompasses all these three notions of self-duality.

We start by noting that the invariant polynomials \( s_{2k} \) of \( \omega \) can be expressed in terms of the elementary symmetric functions of the \( (\lambda_k)^2 \)'s. Then the inner products \( (\omega^k, \omega^k) \) and \( s_{2k} \)'s are related as follows:

\[
(\omega, \omega) = s_2 = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2,
\]
\[
\frac{1}{(2!)}(\omega^2, \omega^2) = s_4 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \ldots + \lambda_{n-1}^2\lambda_n^2,
\]
\[
\frac{1}{(3!)}(\omega^3, \omega^3) = s_6 = \lambda_1^2\lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_2^2\lambda_4^2 + \ldots + \lambda_{n-2}^2\lambda_{n-1}\lambda_n^2,
\]
\[
\vdots
\]
\[
\frac{1}{(n!)}(\omega^n, \omega^n) = \frac{1}{(n!)}|\star \omega^n|^2 = s_{2n} = \lambda_1^2\lambda_2^2\ldots\lambda_n^2.
\]

If we define the weighted elementary symmetric polynomials by

\[
\binom{n}{k} q_k = s_{2k},
\]
we have the inequalities (see for example [11], Ch.2, Sec. 3).

\[
q_1 \geq q_2^{1/2} \geq q_3^{1/3} \geq \ldots \geq q_n^{1/n}, \quad q_{r-1}q_{r+1} \leq q_r^2, \quad 1 \leq r < n,
\]

and the equalities hold iff all the \( \lambda_k \)'s are equal, hence in the case of self-duality. We have the following lemma.

**Lemma 2.** Let \( \omega \) be a 2-form in \( 2n \) dimensions. Then

\[
(n-1)(\omega, \omega)^2 - \frac{n}{2}(\omega^2, \omega^2) \geq 0
\]
and
\[(\omega^{n/2}, \omega^{n/2}) \geq *\omega^n,\]
provided \(n\) is even. Equality holds if and only if all eigenvalues of \(\omega\) are equal.

**Proof.** To obtain the first inequality we use,

\[
\frac{q_1^2}{n^2 s_2^2} \geq \frac{2}{n(n-1)} s_4
\]

from which gives the desired result. Similarly using

\[
\left(\frac{(n/2)! (n/2)!}{n! s_{n/2}}\right)^2 \geq s_{2n}
\]

we obtain the second inequality. e.o.p.

From Lemma 2, we immediately have

**Corollary 3.** A 2-form \(\omega\) is self-dual iff \(\omega^{n/2}\) is self-dual in the Hodge sense.

**Proposition 4.** Let \(\omega\) be a 2-form in \(2n\) dimensions.

\[
\omega^{n-1} = \kappa * \omega
\]

where \(\kappa\) is a constant, iff \(\omega\) is self-dual and \(\kappa = \frac{n!}{n^{n/2}} (\omega, \omega)^{\frac{n}{2} - 1} \).

**Proof.**

If \(\omega\) is self-dual, we can choose an orthonormal basis such that \(\omega = e_1 e_2 + e_3 e_4 + e_5 e_6 + e_7 e_8\) with respect to this basis, and it can easily be seen that the identity holds. Conversely, if the identity holds, then multiplying it with \(\omega\) and taking Hodge duals, we obtain, \(\omega^n = \kappa (\omega, \omega)\). Since \((\omega, \omega) = s_2 = nq_4\) and \(|*\omega^n| = n! q_4^{1/2} = n! q_{n/2}\), we obtain \(\kappa = (n-1)!q_4^{1/2}/q_1\). Then taking inner products of both sides of the identity with themselves, we obtain \((\omega^{n-1}, \omega^{n-1}) = k^2(*\omega, *\omega) = k^2(\omega, \omega)\). Substituting the value of \(\kappa\) obtained above, and using \((\omega^{n-1}, \omega^{n-1}) = ((n-1)!)^2 n q_{n-1}\), we obtain
\[ q_n = q_{n-1}q_1. \] But since \( q_1 \geq q_1^{1/n} \), we have \( q_n \geq q_{n-1}q_1^{1/n} \), which leads to \( q_n^{n-1} \geq (q_1^n)^{-1} \). This is just the reverse of the inequality proved in the previous lemma, hence equality must hold and furthermore all eigenvalues of \( \omega \) are equal in absolute value. Thus \( \omega \) is self-dual and it can also be seen that \( \kappa = \frac{n!}{n^n} (\omega, \omega)^{\frac{n}{2}} \). e.o.p.

We remark that if \((\omega, \omega)\) is nonzero everywhere on \( M \), then it can be normalised to have constant norm and it defines an almost complex structure. In this case \( \ast \omega = \kappa \omega^{n-1} \), where \( \kappa \) is constant. Consequently, if \( \omega \) is closed and has constant norm, then \( \ast \omega \) is harmonic.

2. The explicit construction of self-dual 2-forms[4]

Let \( S_{2n} \) be the set of self-dual 2-forms in \( 2n \) dimensions. If \( A_{2n} \) denotes the set of antisymmetric matrices in \( 2n \) dimensions, then \( S_{2n} = \{ A \in A_{2n} | A^2 + \lambda^2 I = 0, \lambda \in \mathbb{R}, \lambda \neq 0 \} \). Here and in the following \( I \) denotes an identity matrix of appropriate dimension.

**Proposition 5.** \( S_{2n} \) is diffeomorphic to the homogeneous manifold \((O(2n) \times \mathbb{R}^+) / U(n) \times \{1\})\), and \( \dim S_{2n} = n^2 - n + 1 \).

As \( O(2n) \) has two connected components ( \( SO(2n) \) and \( O(2n) \setminus SO(2n) \)), \( U(n) \) is connected and \( U(n) \subset SO(2n) \), \( S_{2n} \) has two connected components. One of which consists of the self-dual forms and the other of the anti-self-dual forms.

Let \( L_\alpha^{2n} \) be a maximal linear subspace of \( S_{2n} \), where \( \alpha \) is a real parameter. The elements of \( L_\alpha^{2n} \) are skew-symmetric and non-degenerate. Therefore,

**Proposition 6.** The dimension of the maximal linear subspaces of \( S_{2n} \) is equal to the number of linearly independent vector fields on \( S^{2n-1} \).

The maximal number of pointwise linearly independent vector fields on the sphere \( S^N \) is given by the Radon-Hurwitz number \( k \). If \( N + 1 = 2n = (2a + 1)2^{d+1} + c \) with \( c = 0, 1, 2, 3 \), then \( k = 8d + 2^c - 1 \). Using this formula it can be seen that there are three vector fields on \( S^3 \), seven on \( S^7 \), three on \( S^{11} \), eight on \( S^{15} \) and so on. In particular there is only one vector field on the spheres \( S^{2n-1} \) for odd \( n \). This property shows that there is an intimate relationship between generalised self-duality and Clifford algebras [12].

We shall now discuss a general procedure for constructing linear subspaces of self-dual forms. Note that \( S_{2n} \) is the set of skew-symmetric matrices in \( O(2n) \times \mathbb{R} \). We define \( P_{2n} \) to be the set of symmetric matrices in \( O(2n) \times \mathbb{R} \). Recall that an orthonormal basis for a \( k \)-dimensional linear subspaces of \( S_{2n} \) corresponds to the representation of \( Cl_k \) in the skew-symmetric
matrices. Similarly an orthonormal basis for a \( k \)-dimensional linear subspace of \( \mathcal{P}_{2n} \) corresponds to a representation of the dual Clifford algebra \( \mathcal{C}l'_k \) in the symmetric matrices. These bases will be the building blocks for self-dual forms in the double dimension.

We have already shown that in dimensions \( 2n = 2(2a + 1) \) the maximal linear subspaces of \( \mathcal{S}_{2n} \) are one dimensional. Similarly, in dimensions \( 2n = 4(2a + 1) \), the dimension of maximal linear subspaces of \( \mathcal{S}_{2n} \) are three dimensional. It can be seen that the matrices

\[
J_0 = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix},
\]

where \( I \) is the identity matrix, form an orthonormal basis for three dimensional linear subspaces of \( \mathcal{S}_{4(2a+1)} \).

Now we consider the only remaining case of self-dual 2-forms in \( 8n \) dimensions. The matrix of a self-dual form can be written in the form

\[
\Omega = \begin{pmatrix} A_a & B_a + B_s \\ B_a - B_s & D_a \end{pmatrix},
\]

where the matrices \( A_a, B_a, D_a \)'s are anti-symmetric and \( B_s \) is symmetric. The requirement that \( \Omega^2 \) be proportional to the identity matrix gives the following equations:

\[
A_a^2 = D_a^2, \quad A_a^2 + B_a^2 - B_s^2 = kI, \quad [B_a, B_s] = 0,
\]

\[
A_a B_a + B_a D_a = 0, \quad B_a A_a + D_a B_a = 0,
\]

\[
A_a B_s + B_s D_a = 0, \quad B_s A_a + D_a B_s = 0.
\]

If we furthermore require that \( \Omega \) be build up from the linear subspaces of \( \mathcal{S}_{4n} \) and \( \mathcal{P}_{4n} \), then we see that \( A_a, D_a, B_a, B_s \) have to be nondegenerate.

We shall give now an explicit construction of various linear subspaces of \( \mathcal{S}_8 \). Let \( \mathcal{A}^- \) and \( \mathcal{A}^+ \) be orthonormal bases for linear subspaces of \( \mathcal{S}_{2n} \) and \( \mathcal{P}_{2n} \), respectively.

In two dimensions we have the following structure.

\[
\mathcal{A}^- = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad \mathcal{A}^+_{(1)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \mathcal{A}^+_{(2)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.
\]

From the commutation relations it can be seen that the orthonormal bases for linear subspaces of self-dual 2-forms in four dimensions are determined
by the choice of $B_s$. The choice $B_s \in A_{(1)}^+$ leads to the usual anti self-dual 2-forms, while the choice $B_s \in A_{(2)}^+$ leads to the self-dual 2-forms. Hence in four dimensions we obtain two different sets of orthonormal bases for linear subspaces of $S_4$. By similar considerations, we obtain seven different bases for linear subspaces of $P_4$. The elements of these bases are listed below:

\[
\begin{align*}
a_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
b_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
c_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
c_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
p_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
p_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
q_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
q_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
e_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{align*}
\]

Using the commutation relations it can be shown that in four dimensions we have the following orthonormal bases for the linear subspaces of $S_4$.

\[
\begin{align*}
A_{(1)}^- &= \{a_1, a_2, a_3\}, \\
A_{(2)}^- &= \{b_1, b_2, b_3\}, \\
A_{(1)}^+ &= \{I\}, \\
A_{(2)}^+ &= \{c_1, c_2, e_1\}, \\
A_{(3)}^+ &= \{p_1, q_2, d_2\}, \\
A_{(4)}^+ &= \{p_2, q_1, d_1\}, \\
A_{(5)}^+ &= \{c_1, p_1, p_2\}, \\
A_{(6)}^+ &= \{c_2, q_2, q_1\}, \\
A_{(7)}^+ &= \{e_1, d_2, d_1\}.
\end{align*}
\]
Orthonormal bases for linear subspaces of \( S_8 \) can be constructed using the sets given above. We now show that the basis obtained by choosing \( B_s = I \) corresponds to the representation of \( Cl_7 \) using octonionic multiplication. Let us describe an octonion by a pair of quaternions \((a, b)\). Then the octonionic multiplication rule is \( (a, b) \circ (c, d) = (ac -\overline{db}, da + bc) \). If we represent an octonion \((c, d)\) by a vector in \( \mathbb{R}^8 \), its multiplication by imaginary octonions correspond to linear transformations on \( \mathbb{R}^8 \). Using the multiplication rule above, it is easy to see that we have the following correspondences:

\[
(i,0) \rightarrow \begin{pmatrix} b_1 & 0 \\ 0 & -b_1 \end{pmatrix}, \quad (j,0) \rightarrow \begin{pmatrix} b_2 & 0 \\ 0 & -b_2 \end{pmatrix}, \\
(k,0) \rightarrow \begin{pmatrix} b_3 & 0 \\ 0 & -b_3 \end{pmatrix}, \quad (0,1) \rightarrow \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \equiv J, \\
(0,i) \rightarrow \begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix}, \quad (0,j) \rightarrow \begin{pmatrix} 0 & a_2 \\ a_2 & 0 \end{pmatrix}, \quad (0,k) \rightarrow \begin{pmatrix} 0 & a_3 \\ a_3 & 0 \end{pmatrix}.
\]

Thus we obtain in \( D = 8 \) dimensions the following matrix of a self-dual 2-form:

\[
\Omega = F_{12} J + \begin{pmatrix} \Omega' & \Omega'' \\ \Omega'' & -\Omega' \end{pmatrix}
\]

where \( \Omega' \) is a D=4 self-dual 2-form, \( \Omega'' \) is a D=4 anti-self-dual 2-form and \( F_{12} \) is a real function.

On the other hand a CDFN self-dual 2-form \( F \) is obtained by imposing the following conditions among its components [8]:

\[
F_{12} - F_{34} = 0 \quad F_{12} - F_{56} = 0 \quad F_{12} - F_{78} = 0 \\
F_{13} + F_{24} = 0 \quad F_{13} - F_{57} = 0 \quad F_{13} + F_{68} = 0 \\
F_{14} - F_{23} = 0 \quad F_{14} + F_{67} = 0 \quad F_{14} + F_{58} = 0 \\
F_{15} + F_{26} = 0 \quad F_{15} + F_{37} = 0 \quad F_{15} - F_{48} = 0 \\
F_{16} - F_{25} = 0 \quad F_{16} - F_{38} = 0 \quad F_{16} - F_{47} = 0 \\
F_{17} + F_{28} = 0 \quad F_{17} - F_{35} = 0 \quad F_{17} + F_{46} = 0 \\
F_{18} - F_{27} = 0 \quad F_{18} + F_{36} = 0 \quad F_{18} + F_{45} = 0
\]

We will refer to the plane consisting of these forms as the CDFN-plane. The
skew-symmetric matrix $\Omega$ of such a self-dual 2-form is

$$
\begin{pmatrix}
0 & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} & \Omega_{17} & \Omega_{18} \\
0 & \Omega_{14} & -\Omega_{13} & \Omega_{16} & -\Omega_{15} & \Omega_{18} & -\Omega_{17} & \\
0 & \Omega_{12} & \Omega_{17} & -\Omega_{18} & -\Omega_{15} & \Omega_{16} & \Omega_{15} & \\
0 & -\Omega_{18} & -\Omega_{17} & \Omega_{16} & \Omega_{15} & \\
0 & \Omega_{12} & \Omega_{13} & -\Omega_{14} & \\
0 & -\Omega_{14} & -\Omega_{13} & \\
0 & \Omega_{12} & \\
0 & 0 & 
\end{pmatrix}
$$

It is easy to show that the above matrix is related to our self-dual 2-form by conjugation $R^t \Omega R$ with

$$
R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

In fact, the choice $B_s \in \{d_2, p_1, q_2\}$ determines the possible choices for $B_a$’s, $A_a$’s and $D_a$’s and leads to the CDFN plane [4].

Finally, we would like to point out that these constructions can be generalised to dimensions which are multiples of eight, by replacing the unit element with identity matrices of appropriate size. In dimensions which are multiples of 16, one can make use of the property $Cl_{k+8} = Cl_k \otimes Cl_8$ to obtain a $Cl_{k+8}$ representation on $R^{16n}$, using an already known representation of $Cl_k$ on $R^n$. Hence linear subspaces of $S_{16n}$ can be obtained from the knowledge of the linear subspaces of $S_n$.

3. The octonionic instanton solution in $D = 8$ dimensions

In this section we shall discuss the construction of the octonionic instanton solution [7],[13-17]. Before that let us give identities concerning self-dual 2-forms. We have already shown that a self-dual 2-form $\omega$ satisfies the basic equalities:

$$(\omega, \omega)^2 = \frac{2}{3}(\omega^2, \omega^2) = \frac{2}{3} * \omega^4,$$
and

\[ \omega^3 = \frac{3}{2}(\omega, \omega) \star \omega. \]

We can obtain from these a series of identities concerning the product of two self-dual 2-forms. We give in what follows some essential results without proof [18].

**Lemma 7.** Let \( \omega \) and \( \eta \) be self-dual 2-forms. If \((\omega, \eta) = 0\) then

\[ \omega^3 \eta = 0. \]

If furthermore \( \omega \pm \eta \) are also self-dual then

\[ (\omega, \omega)(\eta, \eta) = 2(\omega^2, \eta^2) = 2\omega^2 \eta^2, \]

\[ \omega^2 \eta = \frac{1}{2}(\omega, \omega) \star \eta, \]

and

\[ \omega \eta = \star(\omega \eta). \]

Further applying the above equalities to three mutually orthogonal (linear) self-dual 2-forms we obtain

**Lemma 8.** Let \( \omega, \eta \) and \( \alpha \) be mutually orthogonal self-dual 2-forms such that \( \omega + \eta + \alpha \) is also self-dual. Then

\[ \omega \eta \alpha = 0. \]

Collecting these results we state the following

**Proposition 9.** Let \( F = \sum \omega_a E_a \) where \( \{\omega_a\} \) is a set of mutually orthogonal (linear) self-dual 2-forms and \( \{E_a\} \) is a basis of the Lie algebra of a gauge group \( G \). Then

i) \( F^2 = \star F^2 \) for any \( G \),

ii) \( \star F \) is proportional to \( F^3 \) provided the Lie algebra is such that \( E^2_a E_b \) is proportional to \( E_b \).

We proceed now with the construction of the \( D = 8 \) octonionic instanton solution. We note however that the conditions of Proposition 9 are too strong and we can obtain an interesting result by considering a curvature 2-form where the entries in each row come from different linear subspaces.

Suppose \( F \) is an \( \text{so}(8) \)-valued gauge field 2-form. We may write \( F = \sum_{i,j} \omega_{ij} E_{ij} \) where \( \{E_{ij}\} \) is the standard basis of skew-symmetric orthogonal \( 8 \times 8 \) matrices. The \( \{\omega_{ij}\} \) is a set of mutually orthogonal self-dual 2-forms.
The requirement that $F^2$ is self-dual in the Hodge sense severely restricts the choice of $\omega_{ij}$'s. A possible choice can be found as follows. Let $\omega_{88}$ be a basis for any linear subspace of self-dual 2-forms. Then $\omega_{jk}$ for $j, k \neq 8$ are determined uniquely by the conditions $(\omega_{jk}, \omega_{i8}) = 0$ for $i = 1, \ldots, 7$ and $\omega_{j8} \wedge \omega_{jk} = *(\omega_{j8} \wedge \omega_{jk})$ and $\omega_{k8} \wedge \omega_{jk} = *(\omega_{k8} \wedge \omega_{jk})$. Then by construction, making use of the identities given above, we see that the entries of the squared matrix are 4-forms that are self-dual in the Hodge sense. Thus $F$ would saturate the topological lower bound $||F^2||^2 = \int_M tr(F \wedge F \wedge F \wedge F)$.

It also turns out that $F^3 = -90 \ast F$.

All that remains to be done is to check whether this $F$ could come from a potential $A$. To this end we use the Bianchi identity $AF - FA = -dF$. It turns out that $A$ can be determined in terms of the derivative of a function $\phi$. Substituting into the definition $F = dA - A^2$, we obtain a system of second order differential equations satisfied by $\phi$. In these equations all second derivatives are determined and the coefficients of the first order derivatives must be equal to each other. Thus the solution is unique, and it can be shown that in the Cartesian coordinate chart $\{x^i\}$ of $R^8$ we have

$$\phi = \left[\frac{1}{2} + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2\right]^{-2}.$$  

This expression yields the octonionic instanton solution [7]. It will be instructive to compare the above derivation of the octonionic instanton solution with the previous approaches based on spin(8) matrices [16]. Let $F = \frac{1}{2} F_{ij} dx^i \wedge dx^j$. We may set $F_{ij} = f \Sigma_{ij}$ where $f$ is a function on $R^8$ to be determined and $8 \times 8$ matrices $\Sigma_{ij}$'s form a basis for spin(8). It can be shown that the non-linear self-duality equation $\ast F^2 = F^2$ is satisfied as a consequence of the algebraic properties of the spin matrices $\Sigma_{ij}$. It must be clear from our exposition thus far that the two approaches are equivalent.

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