Inhomogeneous Dust Collapse in 5D Einstein-Gauss-Bonnet Gravity

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We consider a Lemaître - Tolman - Bondi type space-time in Einstein gravity with the Gauss-Bonnet combination of quadratic curvature terms, and present exact solution in closed form. It turns out that the presence of the coupling constant of the Gauss-Bonnet terms $\alpha > 0$ completely changes the causal structure of the singularities from the analogous general relativistic case. The gravitational collapse of inhomogeneous dust in the five-dimensional Gauss-Bonnet extended Einstein equations leads to formation of a massive, but weak, timelike singularity which is forbidden in general relativity. Interestingly, this is a counterexample to three conjecture viz. cosmic censorship conjecture, hoop conjecture and Seifert’s conjecture.

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I. INTRODUCTION

The gravitational collapse of an incoherent spherical dust cloud is described by the metric satisfying Einstein equations $G_{ab} = \kappa T_{ab}$ with $T_{ab} = \epsilon(t, r) u_a u_b$, where $u_a$ is a velocity (i.e., unit-time-like) vector field and $\epsilon$ is the energy density of the system. This space-time is described by the Lemaître-Tolman-Bondi (LTB) solution [1], which, in a reference frame ($t$, $r$, $\theta$, $\phi$) comoving with the collapsing matter ($u_a = \delta_a^t$), for the marginally bound case, reads

$$ds^2 = -dt^2 + \left(\frac{\partial R}{\partial r}\right) dr^2 + R^2(d\theta^2 + \sin(\theta)^2 d\phi^2)$$

where $R(t, r) = r \left(1 - \frac{3}{2} \sqrt{\frac{2F(r)}{r^3}}\right)^{2/3}$, (1)

This two parameter family solution was a natural extension of the seminal work of Oppenheimer and Snyder, which led to the “establishment viewpoint” that the end state of continued gravitational collapse, for a remnant mass of a collapsing star beyond the threshold neutron star mass limit. The absence of analytical results has led to several conjectures as well, namely the (weak and strong) cosmic censorship conjecture (CCC) by Penrose [4], hoop conjecture (HC) by Thorne [3] and Seifert’s conjecture, which to date remain unproven.

The LTB solution has been extensively used not only to study the formation of naked singularities and black holes in spherical collapse, but in cosmology as well. It is well known that the LTB solutions admit both naked and covered singularities depending upon the choice of initial data and there is a smooth transition from one phase to the other [3, 5, 12]. These results have led to strong evidence against the weak CCC [4], which asserts that there can be no singularity visible from future null infinity. In other words, light rays emanate from singularity but are completely blocked by the event horizon and hence they could only lay bare singularity to observers who are co-falling with the collapsing star and not to external observers, while the strong CCC prohibits its visibility by any observer. That means no light rays emanate out of singularity, i.e., singularity is never naked. In precise mathematical terms it demands that space-time should be globally hyperbolic. Despite almost 40 years of effort we are still far from a general proof of CCC (for recent reviews and references, see [13]).

Many studies of gravitational collapse, particularly in cylindrical symmetry, were also motivated by Thorne’s HC of the necessary and sufficient conditions for the horizon formation: Horizons form when and only when a mass $M$ becomes compacted into a region whose circumference in every direction $C \leq 4\pi M^{2/3}$. Thus, planar or cylindrical matter will not form a black hole (black plane or black string) [3]. Unlike CCC, the HC does not suffer from counterexamples. The HC was originally given for four-dimensional (4D) space-times in general relativity. It was modified for higher dimensions by Ida and Nakao [14]: Black holes with horizons form when and only when a mass $M$ gets compacted into a region whose $(D-3)$-dimensional area $V_{D-3}$ in every direction is $V_{D-3} \leq G_D M$, where $G_D$ is the gravitational constant in the D-dimensional theory of gravity, and the $(D-3)$-dimensional area means the volume of the $(D-3)$-dimensional closed submanifold of a spacelike hypersurface. The HC is related to the trapped surface conjecture of Seifert [5] that massive singularities have to be trapped. It should be interesting to see if these conjectures still hold in Einstein gravity with the Gauss-Bonnet combination of quadratic curvature terms or in a higher dimensional space-time.

Current experimental results involving tests of the inverse square law do not rule out extra dimensions even...
as large as a tenth of a millimeter. It is important to consider the evolution of the extra dimensions since the observed strength of the gravitational force is directly dependent on the size of the extra dimensions. As a consequence, there is a renewed interest towards an understanding of the general relativity in more than four dimensions, as a growing volume of recent literature indicates. In particular, several solutions to the Einstein equations of localized sources in higher dimensions have been obtained in the recent years [15, 16], from viewpoint of gravitational collapse [17] and in particular LTB-like solutions [18, 22].

In recent years a renewed interest has grown in higher order gravity, which involves higher derivative curvature terms. Among the higher curvature gravities, the most extensively studied theory is the so-called Einstein-Gauss-Bonnet (EGB) gravity. The EGB gravity is a special case of Lovelocks theory of gravitation, whose Lagrangian contains just the first three terms. The Gauss-Bonnet term yields nontrivial dynamics in dimensions greater than or equal to 5. It appears naturally in the low-energy effective action of heterotic string theory [23]. Boulware and Deser [24] found exact black hole solutions in $N(\geq 5)$-dimensional gravitational theories with a four dimensional Gauss-Bonnet term modifying the usual Einstein-Hilbert action. These solutions are generalizations of the N-dimensional spherically symmetric black hole solution found by Tangherlini [15], and Myers and Perry [25]. Other spherically symmetric black hole solutions in the Gauss-Bonnet gravity have been found and discussed in [26–28]. Topologically nontrivial black holes have been studied in [29]. The effects of Gauss-Bonnet terms on the Vaidya solutions have been investigated in [30–33], and on the LTB solutions in [34]. These papers show that the appearance of a Gauss-Bonnet term in the field equations has no effect on the occurrence of locally naked singularity, while it has some effects on the strength of the curvature.

Recently, Maeda [34] considered the spherically symmetric gravitational collapse of an inhomogeneous dust with the $N(\geq 5)$-dimensional action including the Gauss-Bonnet term. He investigated its effects on the final fate of gravitational collapse without finding the explicit form of the solution. In this paper, we consider the 5D action with the Gauss-Bonnet terms for gravity and give a \textit{exact} model of the gravitational collapse of an inhomogeneous dust including the second order perturbative effects of quantum gravity. A 5D space-time is particularly relevant because both 10D and 11D supergravity theories yield solutions where a 5D space-time results after dimensional reduction [35].

This paper is organized as follows. In the next section, we derive the general solutions, in a closed form, for marginally bound case, which is a kind of generalized LTB space-time in the 5D EGB gravity with the energy-momentum tensor of a dust. For definiteness we shall call it 5D-LTB-EGB. The nature of singularities of such a space-time in terms of its being hidden within a black hole, or whether it would be visible to outside observers, and the consequence of EGB on 5D-LTB collapse are analyzed in Sec. III. The detailed analysis on apparent horizon is a subject of Sec. IV, Sec. V is devoted to strength of singularity, and is followed by a discussion.

We have used units which fix the speed of light and the gravitational constant via $8\pi G = c^4 = 1$.

II. 5D LEMAITRE-TOLMAN-BONDINI SOLUTIONS IN EINSTEIN GAUSS-BONNET GRAVITY

We begin with the following 5D action:

$$ S = \int d^5x \sqrt{-g} \left[ \frac{1}{2\kappa_5^2} (R + \alpha L_{\text{GB}}) \right] + S_{\text{matter}}, \quad (2) $$

where $R$ is a 5D Ricci scalar and $\kappa_5 \equiv \sqrt{8\pi G_5}$ is 5D gravitational constant. The Gauss-Bonnet Lagrangian is of the form

$$ L_{\text{GB}} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}, \quad (3) $$

where $\alpha$ is the coupling constant of the Gauss-Bonnet terms. This type of action is derived in the low-energy limit of heterotic superstring theory [23]. In that case, $\alpha$ is regarded as the inverse string tension and positive definite and we consider only the case with $\alpha \geq 0$ in this paper. In the 4D space-time, the Gauss-Bonnet terms do not contribute to the field equations. The action (2) leads to the following set of field equations:

$$ G_{ab} = G_{ab} + \alpha H_{ab} = T_{ab}, \quad (4) $$

where

$$ G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \quad (5) $$

is the Einstein tensor and

$$ H_{ab} = 2RR_{ab} - 2R^c_{ac} R^{c}_{ab} - 2R^{\alpha\beta} R_{a\alpha b\beta} + R^\alpha_{\alpha\beta\gamma} R_{\beta\alpha\gamma} - \frac{1}{2} g_{ab} L_{\text{GB}} \quad (6) $$

is the Lanczos tensor.

The standard LTB solution [11] represents an interior of a collapsing inhomogeneous dust sphere. The solution we seek is collapse of a spherical dust in 5D-EGB. The energy-momentum tensor for dust is

$$ T_{ab} = \epsilon(t,r) \delta^t_a \delta^r_b, \quad (7) $$

where $\epsilon_a = \delta^t_a$ is the 5D velocity. The metric for the 5D case, in comoving coordinates, is [18, 21]:

$$ ds^2 = -dt^2 + A(t,r) dr^2 + R(t,r)^2 d\Omega_3^2, \quad (8) $$

where $d\Omega_3^2 = (d\theta^2 + \sin^2 \theta d\phi^2)$, is a metric on three-sphere. The coordinate $r$ is the comoving radial
coordinate, $t$ is the proper time of freely falling shells, $R$ is a function of $t$ and $r$ with $R \geq 0$, and $A$ is also a function of $t$ and $r$. For the metric (6), with energy-momentum tensor (7), the Einstein field equations take the form

\[ G^t_t = \frac{12(R^2 - A^2(1 + R^2))}{R^4 A^6} [R'^2 + 2R^2 \dot{A}^2 - AR'^2] = \frac{3}{A^4 R^2} \left[ A^2(1 + R^2) + 2 A^2 R \dot{A} R' + 2 \dot{A} R'^2 \right] - A R R'^2 + R'^2 = -\epsilon(t, r), \quad (9) \]

\[ G^\theta_\theta = G^\phi_\phi = \frac{4\alpha}{A^4 R^2} \left[ -2 A(AR' + A^2 \dot{A} R) \dot{R} + A(R^2 - A^2(1 + R^2)) \dot{A} + 2(AR' - A^2 \dot{A}) \right] \]

\[ -\frac{1}{A^3 R^2} \left[ A^3(1 + R^2 + 2 \dot{R} R) + A^2 R(2 \dot{R} A + R \dot{A}) + 2R \dot{R} A' - 2A(R R'^2 + R'^2) \right] = 0, \quad (11) \]

\[ G^t_r = \frac{12\alpha}{A^6 R^2} [AR' - A^2 \dot{R}] R^2 (1 + \dot{R}^2) = 0, \quad (12) \]

where an over-dot and prime denote the partial derivative with respect to $t$ and $r$, respectively. Since space-time is nonradiating Eq. (13), leads to two families of solutions

\[ A(t, r) = \frac{R^\prime}{W}, \quad (13) \]

and

\[ A(t, r) = \pm \frac{2\sqrt{\alpha} R^\prime}{\sqrt{R^2 + 4\alpha (R^2 + 1)^{1/2}}}, \quad (14) \]

where $W = W(r)$ is an arbitrary function of $r$. Note the striking similarity of function $A$ with analogous 5D-LTB solutions [18, 20, 21]. In what follows we shall consider the case $A(t, r) = R'/W$, since in the other case the $\alpha \to 0$ leads to a trivial solution.

It is straightforward to check that if the $G^r_r = 0$ condition is satisfied the other Einstein equations (namely, $G^\theta_\theta = G^\phi_\phi = G^t_t = 0$) are automatically satisfied. Finally, in order for $G^r_r = 0$ to hold it is necessary that $R$ satisfies

\[ \frac{\ddot{R}}{\dot{R}} = \frac{\dot{R}^2 - (W^2 - 1)}{4\alpha(W^2 - 1 - \dot{R}^2) - \dot{R}^2}, \quad (15) \]

which can be easily integrated to yield

\[ \frac{\dot{R}^2}{R^2} \left[ 1 - 4\alpha \frac{W^2 - 1}{R^2} \right] = (W^2 - 1) + \frac{F}{R^2} - 2\frac{\dot{R}}{R^2}, \quad (16) \]

This is the master equation of the system. Here $F = F(r)$ is an arbitrary function of $r$ and is referred to as mass function. Substituting Eqs. (13) and (16) into Eq. (9) we obtain

\[ F' = \frac{2}{3} \epsilon R^3 R', \quad (17) \]

Integrating Eq. (17) leads to

\[ F(r) = \frac{2}{3} \int \epsilon R^3 dR, \quad (18) \]

where the constant of integration is taken as zero since we want a finite distribution of matter at the origin $r = 0$. We note that $F'$ (as well as $F$) must be positive. Indeed the energy density, $\epsilon$, must be non-negative. It is easy to see that as $\alpha \to 0$ the master solution (16) of the system reduces to the corresponding 5D-LTB solution in [18, 20]

\[ \dot{R}^2 = W^2 - 1 + \frac{F}{R^2}. \quad (19) \]

It may be noted that in a general relativistic case ($\alpha \to 0$), Eq. (10) or (19) has three types of solutions, namely, hyperbolic, parabolic and elliptic solutions depending on whether $W(r) > 1$, $W(r) = 1$ or $W(r) < 1$ respectively [20, 21]. Analogously, here the condition $W(r) = 1$, is the marginally bound condition, meaning collapsing shell is at rest at spatial infinity ($R = \infty$). From Eq. (16), we obtain

\[ \frac{\dot{R}^2}{R^2} = (W^2 - 1) - \frac{R'^2}{4\alpha} \left[ 1 + \frac{16\alpha^2}{R^4} (W^2 - 1)^2 + \frac{8\alpha F(r)}{R^4} \right]^{-1/2}. \quad (20) \]

There are two families of solutions which correspond to the sign in front of the square root in Eq. (20). We call the family which has the minus (plus) sign the minus (plus) branch solution. In the general relativistic limit $\alpha \to 0$, we recover the 5D-LTB solution in Einstein gravity [18, 20]. There is no such limit for the plus-branch solution. We consider the minus-branch solution in order to compare with general relativistic case. Eq. (20) is a modified Friedmann-like equation in 5D-EBG and is a bit complicated compared to the corresponding general relativistic Eq. (19). Maeda [34] has analyzed LTB models near the center ($r \sim 0$) in EGB without finding an explicit solution. Here we present the 5D-LTB-EBG exact solution in closed form, which facilitates us to explicitly analyze the final fate of gravitational collapse.

Henceforth, we shall confine ourselves to the marginally bound case ($W(r) = 1$). In the present discussion, we are concerned with gravitational collapse, which requires $\dot{R}(t, r) < 0$. Eq. (20) can be integrated to

\[ t_c(r) - t = \frac{\sqrt{\alpha}}{2\sqrt{2}} \tan^{-1} \left[ \frac{3R^2 - \sqrt{R^4 + 8\alpha F}}{2\sqrt{2} R \sqrt[4]{R^4 + 8\alpha F - R^2}} \right] \]

\[ + \sqrt{\alpha} R^2 \sqrt[4]{R^4 + 8\alpha F - R^2}, \quad (21) \]
where \( t_\zeta(r)(r) \) is an arbitrary function of integration. As it is possible to make an arbitrary relabeling of spherical dust shells by \( r \to g(r) \), without loss of generality, we fix the labeling by requiring that, on the hypersurface \( t = 0 \), \( r \) coincide with the area radius \( R(0, r) = r \).

This corresponds to the following choice of \( t_\zeta(r) \):

\[
t_\zeta(r) = \frac{\sqrt{\alpha}}{2\sqrt{2}} \tan^{-1} \left[ \frac{3 - \sqrt{1 + 8\alpha F}}{2\sqrt{2} \sqrt{1 + 8\alpha F} - 1} \right]
\]

\[+ \sqrt{1 + 8\alpha F} - 1, \]  

where \( F = F/r^4 \).

In order to study the collapse of a finite spherical body in EGB, we have to match the solution along the time-like surface at some \( r = r_c > 0 \) to the 5D-EGB Schwarzschild exterior discovered by Boulware and Deser \[24\] and Wheeler \[26\]. On a spherical hypersurface \( \Sigma \), the junction conditions yield \( F = M_s \) \[34\]. Here \( r_s = R(t, r_c) \), and \( M_s \) is the total mass enclosed within the coordinate radius \( r_c \) \[34\].

Also, in the general relativistic case the energy-momentum tensor given by Eq. \[7\] satisfies the weak energy condition. It means that the energy density as measured by any local observer is non-negative. However, this may true in EGB because the Gauss-Bonnet term itself violates the energy condition (like a negative cosmological constant).

III. INITIAL DATA AND SINGULARITY

The final fate of gravitational collapse and the nature of the singularity continues to be among one of the most outstanding problems in general relativity. As pointed out earlier, the conjecture that such a collapse, leading to a singularity, under physically realistic conditions must end in the formation of a black hole, and that the eventual singularity must be covered by an event horizon is the CCC. Despite numerous attempts, this conjecture as such remains a major unsolved problem lying at the foundation of black hole physics today. From such a perspective, it is worthwhile to examine the nature of the singularity, in terms of its visibility for an observer, when it develops in the context of the 5D-LTB-EBG. In LTB space-time, shell crossing singularities are defined by \( R' = 0 \) and they can be naked. It has been shown in LTB case \[8\] that shell crossing singularities are gravitationally weak and hence such singularities cannot be considered seriously in the context of the CCC. On the other hand, in general relativity central shell focusing singularities (characterized by \( R = 0 \)) can also be naked and gravitationally strong as well. Thus, unlike shell crossing singularities, shell focusing singularities do not admit any metric extension through them and are considered to be the genuine singularities of space-time. This led us to investigate a similar situation in 5D-LTB-EBG space-time, because shell crossing singularities are assumed to be extendible in general relativity. Christodoulou \[3\] pointed out in the LTB case that the noncentral singularities cannot be naked. This is also true for all spherically symmetric models (including models in higher dimensions) for physically reasonable matter fields. It will be interesting to discuss if this feature gets modified by introduction of the Gauss-Bonnet term.

The easiest way to detect a singularity in a space-time is to observe the divergence of some invariants of the Riemann tensor. The Kretschmann scalar \( K = R_{abcd}R^{abcd} \), \( R_{abcd} \) is the Riemann tensor) for the metric \[8\] with the help of \[13\] reduces to

\[
K = 12 \frac{\ddot{R}^2}{R^2} + 12 \frac{\dot{R}^4}{R^4} + 4 \frac{\ddot{R}^2}{R^2} + 12 \frac{\ddot{R}^2}{R^2} \frac{\dot{R}^2}{R^2} \]  

(24)

It can be verified that the Kretschmann scalar is finite on the initial data surface. For our case the general expression for energy density is

\[
\epsilon(t, r) = \frac{3F'}{2R^3R} \]  

(25)

Hence, it is clear that if \( F' \) is regular and bounded away from zero, then energy density diverges when \( R' = 0 \) and \( R = 0 \). Hence, we have both shell crossing as well as shell focusing singularities for, respectively, \( R' = 0 \) and \( R = 0 \). For \( t = t_c(r) \), we have \( R(t, r) = 0 \), which is the time when the matter shell \( r = \) constant hits the physical singularity. Further, the Kretschmann scalar diverges at \( t = t_c(r) \) indicating the presence of a scalar polynomial curvature singularity \[30\].

The (shell focusing) singularity curve can be obtained using Eq. \[21\] as

\[
t_c(r) = t_\zeta(r) + \frac{\pi \sqrt{\alpha}}{4\sqrt{2}}, \]  

(26)

which represents the proper time for the complete collapse of a shell with coordinate \( r \). Interestingly, positive \( \alpha \) delays the formation of singularity. In the limit of vanishing \( \alpha \) we recover the crunch time for relativistic 5D-LTB. The two arbitrary functions \( F(r) \) and \( t_c(r) \) completely specify the dynamics of collapsing shells.

Analogous to LTB models, in the case of positive \( \alpha \), the evolution always leads to formation of a shell focusing curvature singularity. The mass function \( F \) can be related with initial data (density) at the scaling surface, \( t = 0 \) \( (R = r) \), where \[25\] reduces to form

\[
F(r) = \frac{2}{3} \int_0^r \epsilon(0, r) r^3 dr, \]  

(27)

which completely specifies the mass function in terms of the initial density profile. The function \( F \) must be positive, because \( F < 0 \) implies the existence of negative
mass. This can be seen from the mass function \( m(t, r) \) \[^{[20, 37]}\] which in the 5D-LTB-EGB case is given by

\[
m(t, r) = R^2 \left(1 - g^{\alpha \beta} R_{\alpha \beta} R_{,\alpha \beta}\right) = R^2 \left(1 - \frac{R^2}{A^2} + \dot{R}^2\right). \tag{28}
\]

Using Eqs. \(^{[13]}\) and \(^{[16]}\) into Eq. \(^{[28]}\) we get

\[
m(t, r) = F(r) - 2\alpha \dot{R}^4. \tag{29}
\]

It may noted that one can also calculate mass using the formula proposed by Maeda \[^{[34, 38]}\] for the generalized mass function in the EGB. The mass function \( F(r) = m(t, r) + 2\alpha \dot{R}^4 \), is equivalent, up to a constant factor, to the generalized mass function in EGB \[^{[34, 38]}\].

Next, we study the structure of singularities in 5D-LTB-EGB space-time and compare it with the general relativistic case by using solution obtained in the previous section. Consider a spherically symmetric dust cloud with density profile:

\[
\epsilon(0, r) = \epsilon_0 \left[1 - \left(\frac{r}{r_b}\right)^n\right]. \tag{30}
\]

Here, \( \epsilon_0 \) is the central density and \( r_b \) is the boundary of the collapsing cloud, which is a profile where energy density decreases as we move away from the center, as is expected inside a star. Initial data for a model are completely specified by \( \epsilon_0, r_b, n \) and \( \alpha \). Since matter is pressureless fluid matching to a suitable exterior requires matching of the mass function of the interior and exterior space-times. We would like to mention here that, like LTB models \[^{[11, 12]}\], we can choose an arbitrary profile of the form \( \epsilon(0, r) = \epsilon_0 + \epsilon_1 r + \cdots \). However, as it is well known \[^{[11, 12]}\], only the first nonvanishing term in the density gradient is important in deciding the causal structure of a singularity near the center. Hence our choice of density profile in no way restricts the generality of our analysis.

The mass function corresponding to the density profile given above is of the form:

\[
F(r) = \frac{\epsilon_0 r_b^4}{6} \left[1 - \frac{4r^n}{(n + 4)r_b^n}\right]. \tag{31}
\]

One of the important ingredients in singularity theorems is the assumption of trapped surfaces. The important issue in collapse is to show whether such trapped surfaces form during collapse or not. More importantly there should not be a \textit{a priori} trapped surfaces present in the initial data surface. The condition for the existence of an apparent horizon (the inner boundary of the region containing trapped surfaces), two-spheres with outward normals as null, is

\[
g^{\mu \nu} R_{\mu \nu} + \frac{\dot{R}^2}{A^2} = 0 \tag{32}
\]

Demanding the absence of trapped surfaces \(^{[32]}\) in initial data implies,

\[
g^{\mu \nu} R_{\mu \nu}|_{r=0} > 0.
\]

For the mass function of the form \(^{[25]}\) this condition reduces to

\[
r^2 \left[-1 + \frac{4\alpha \epsilon_0}{3} \left(1 - \frac{4r^n}{(n + 4)r_b^n}\right)\right] < 4\alpha.
\]

Therefore, for a given density profile and central density the size of the cloud is limited by the magnitude of parameter \( \alpha \). Note, however, that collapsing matter being dust the density need not vanish at the boundary for a smooth matching to the vacuum exterior.

### IV. DYNAMICS OF APPARENT HORIZON

One of the important constructions in general relativity is that of a trapped surface. In a 4D space-time, it is a compact 2D, smooth spacelike submanifold with the property that the expansion of future directed null geodesics (outgoing as well as ingoing), orthogonal to this submanifold, is negative everywhere \[^{[39]}\]. They are crucial in proving null-geodesic incompleteness in context of gravitational collapse. The apparent horizon (AH) is the outermost marginally trapped surface for the outgoing photons. The AH can be either null or spacelike, that is, it can `move’ causally or acausally \[^{[40]}\]. The main advantage of working with the apparent horizon is that it is local in time and can be located at a given spacelike hypersurface. Moreover, even if energy conditions hold the whole scenario of the event horizon still remains unclear in EGB \[^{[41]}\].

Considering Eq. \(^{[16]}\), the apparent horizon condition \(^{[32]}\) becomes

\[
R(t_{AH}(r), r) = \sqrt{F(r) - 2\alpha}. \tag{33}
\]

It is clear that the presence of the coupling constant of the Gauss-Bonnet terms \( \alpha \) produces a change in the location of these horizons. Such a change could have a significant effect in the dynamical evolution of these horizons. In the relativistic limit, \( \alpha \to 0 \), \( R_{AH} \to \sqrt{F(r)} \) \[^{[19]}\]. For nonzero \( \alpha \) the structure of the apparent horizon is non-trivial. Interestingly the theory demands \( \alpha \) to be a positive number which forbids apparent horizon from reaching the center thereby making the singularity massive and eternally visible, which is forbidden in the corresponding general relativistic scenario. In general relativity noncentral singularity is always covered \[^{[3]}\] (see also \[^{[42]}\]). However, in the presence of the Gauss-Bonnet term we find that even the noncentral singularity is naked, in spite of being massive \( |F(r > 0) > 0\). Further, Eq. \(^{[33]}\) has a mathematical similarity for the analogous situation in null fluid collapse where the expression for the apparent horizon is \( r_{AH} = \sqrt{m(r) - 2\alpha} \). \[^{[33]}\].
Equation (33) implicitly defines a curve \( t_{ah}(r) \) and represents the apparent horizon, i.e. the time at which the shell gets trapped. Since the collapse is spherical the whole framework can be expressed by a 2D picture, where the singularity curve Eq. (26) represents the time of complete collapse of the shell labeled \( r \). To further analyze the horizon curve, we combine Eqs. (21) and (26) giving
\[
t_c(r) - t = \frac{\pi \sqrt{\alpha}}{4\sqrt{2}} + \frac{\alpha R^2}{\sqrt{R^2 + 8\alpha F - R^2}} + \frac{\sqrt{\alpha}}{2\sqrt{2}} \tan^{-1} \left[ \frac{3R^2 - \sqrt{R^2 + 8\alpha F - R^2}}{2\sqrt{2}R(\sqrt{R^2 + 8\alpha F - R^2})^{1/2}} \right].
\]

Then, the apparent horizon condition (33) reduces Eq. (34) to form
\[
t_c(r) - t_{AH}(r) = \frac{\pi \sqrt{\alpha}}{4\sqrt{2}} + \frac{\sqrt{\alpha}}{2\sqrt{2}} \tan^{-1} \left[ \frac{F - 4\alpha}{2\sqrt{2}(F - 2\alpha)} \right] + \frac{1}{2} \sqrt{F - 2\alpha}, \quad (35)
\]

Clearly, for a positive \( \alpha \), the central shell doesn’t get trapped, and the untrapped region around the center increases with increasing \( \alpha \), for both homogeneous and inhomogeneous models, and are respectively illustrated in Fig. (1).

V. STRENGTH OF SINGULARITY

Finally, we need to determine the curvature strength of the naked singularity, which is an important aspect of a singularity \[15, 44\]. A singularity is gravitationally strong or simply strong if volume elements defined through Jacobi fields get crushed to zero volume at the singularity, and weak otherwise \[30, 44, 45\]. It is widely believed that a space-time does not admit an extension through a singularity if it is a strong curvature singularity in the sense of Tipler \[15, 44\]. Clarke and Królak \[46\] have shown that in four dimensions a sufficient condition for a strong curvature singularity as defined by Tipler \[44\] is that for at least one nonspacelike geodesic with affine parameter \( \tau \), in the limiting approach to the singularity, we must have
\[
\lim_{\tau \to \tau_0} (\tau - \tau_0)^2 \psi = \lim_{\tau \to \tau_0} (\tau - \tau_0)^2 R_{ab} K^a K^b > 0 \quad (36)
\]
where \( R_{ab} \) is the Ricci tensor. This provides a sufficient condition for all the two-forms, defined along the singular null geodesic, to vanish as the singularity is approached, and implies a very powerful curvature growth establishing a strong curvature singularity.

Following \[10\], we consider a timelike causal curve \( K^a = dx^a/d\tau \) where \( \tau \) is the proper time along particle trajectory and \( K^a \) satisfies condition \( K^a K_a = -1 \). The radial timelike geodesics must satisfy \[10\]:
\[
\frac{dK^t}{d\tau} + \frac{R}{R'} [ (K^t)^2 - 1 ] = 0. \quad (37)
\]

![FIG. 1: A 2D picture of the EGB collapse showing formation of singularity and apparent horizon for the parameter values \( \epsilon_0 = 1, \alpha = 0.2, r_b = 5.5 \). The dotted curves represent singularity whereas the apparent horizons are the continuous one](image)

It has a simple solution \( K^a = dx^a/d\tau = \delta_0^a, r = 0 \), which is the worldline of the center of the collapsing cloud. In terms of proper time we can describe it as
\[
t_c(0) - t = \tau_0 - \tau. \quad (38)
\]

We consider the expansion of \( R \) near the center
\[
R(t, r) = R_0(t)r + R_1(t)r^2 + \cdots \quad (39)
\]
where \( R_0(t) \) is unity at \( t = 0 \) and vanishes at \( t = t_c(0) \). The expression for \( \psi \), with the help of eqs. (37), (38) and (39), becomes:
\[
\psi = -\frac{1}{\alpha} + \frac{20F_0}{[R_0(t)^4 + 8\alpha F_0]^{3/2}} R_0^2 + \mathcal{O}(R_0^2) \quad (40)
\]
where \( \mathcal{O}(R_0^2) \) signifies terms which vanish faster than \( R_0(t)^2 \) in the limit \( R_0(t) \to 0 \). Thus one finds that \( \lim_{\tau \to \tau_0} (\tau - \tau_0)^2 \psi = 0 \) and therefore the strong curvature condition is not satisfied. Thus the Gauss-Bonnet term weakens the strength of singularity.
VI. DISCUSSION

The low-energy expansion of supersymmetric string theory suggests that the leading correction to Einstein action is given by Gauss-Bonnet invariant. In this paper, in 5D, we have found exact spherically symmetric LTB solutions, for the marginally bound case, to Gauss-Bonnet extended Einstein equations (5D-LTB-EBG). This describes gravitational collapse of spherically symmetric inhomogeneous dust in a 5D space-time in EGB gravity.

The solution in turn is utilized to bring down the effect of the Gauss-Bonnet term on the final fate of the 5D relativistic gravitational collapse of a dust cloud. It may be noted that the analogous 5D-LTB case exhibits critical behavior governing the formation of black holes or naked singularities. The natural questions would be, for instance, whether such solutions remain naked with the correction terms of second order in the curvature? Do they get covered? Does the nature of the singularity change in a more fundamental theory preserving censorship?

We found that, as in the case of general relativity, a naked singularity is inevitably formed. In the general relativistic case, a naked singularity will form only when $M_0$ takes a sufficiently small value, and therefore turning on the Gauss-Bonnet term worsens the situation from the viewpoint of CCC. Our analysis shows that the Gauss-Bonnet contribution has a profound influence on the nature of the singularity and the whole picture of gravitational collapse changes drastically. While there may be an apparent horizon about this singularity, for $\alpha > 0$, the singularity always remains visible to any observer as the apparent horizon lies beyond singularity which is actually not in the space-time. It is interesting that the coupling constant of the Gauss-Bonnet terms produces a change in the location of the apparent horizon by the factor $2\alpha$ is exactly the same as in the case of 5D null fluid collapse in EGB.

The most interesting consequence of the second order curvature corrections is that the final fate of gravitational collapse is quite different in the sense that a massive naked singularity is formed, which is disallowed in 5D-LTB. Thus we have shown here that there exist regular initial data which lead to a massive naked singularity violating CCC. However, since the strength singularity is weaker as compared to the corresponding 5D-LTB, this may not be a serious threat to CCC. According to Seifert conjecture [6] any singularity that occurs, if a finite nonzero amount of matter tends to collapse, into one point is always hidden. Hence, this is a counterexample to Seifert conjecture as well. The singularities are always naked as they are formed prior to the formation of apparent horizon and there is no black hole formation at least for the marginally bound case, and hence they must violate HC. Thus we have a unique counterexample to all three conjectures. It would be interesting to investigate further gravitational collapse in EGB theory to see if these features are generic [17]. However, it would be difficult to say that this is serious threat to CCC, since the strength of singularity is weaker in 5D-LTB-EBG than in 5D-LTB.

It is seen here that the Gauss-Bonnet term modifies the time of formation of singularities, and the time lag between singularity formation and apparent horizon formation, in contrast to the 5D dust models. Indeed, the time for the occurrence of the central shell focusing singularity for the collapse is increased as compared to the autologous 5D-LTB case. The reason may be, there is relatively less mass-energy [see Eq. (24)] collapsing in the 5D-LTB-EBG space-time as compared to the 5D-LTB case. In particular, our results in the limit $\alpha \to 0$ reduce vis-à-vis to 5D relativistic case.

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