Synchronization in Networks with Nonlinearly Delayed Couplings on Example of Neural Mass Model

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Abstract: The problem of synchronization in heterogeneous networks of linear systems with nonlinear delayed diffusive coupling is considered. The network is presented in new coordinates mean-field dynamics and synchronization errors. Thus the problem of network synchronization is reduced to the studying of synchronization-error system stability. The circle criterion for time-delay systems is used to derive the stability conditions of synchronization-error system. Obtained results are applied to a network of neural mass model populations, and the synchronization conditions are established. Simulation results are provided to illustrate the obtained analytical results.

Keywords: Synchronization, time-delay systems, circle criterion, oscillation, neural mass model.

1. INTRODUCTION

Synchronization in networks of coupled oscillators is attractive phenomenon to study for specialists in various fields of science. Synchronization underlies many natural phenomena and is the cornerstone of many technical concepts and engineering approaches. Examples of synchronization, include, among others, numerous forms of collective behavior in complex biological and artificial systems, such as flocks of birds, swarming, and rendezvous Herbert-Read (2016); Sumpter (2010); ensembles of oscillators Hong and Strogatz (2011) and a group of mobile robots Ren and Beard (2008). Special attention is paid to synchronization in neural network dynamics. Synchronization depends on various network parameters, and a time delay in a signal propagation between the nodes plays a crucial role in this phenomenon.

Time delays are always present in real physical systems, therefore, in order to develop adequate realistic models of dynamic networks, one should to take into account delays in signal propagation in order to properly analyze the design of their dynamics. In neural networks, time delays can induce various rhythmic spatiotemporal patterns Coombes and Laing (2009); Song et al. (2009), change the stability of existing patterns Ermentrout and Ko (2009), and play a crucial role in synchronization behavior Dahlem et al. (2009); Schnitzler et al. (2009). Various works are devoted to study synchronization in delay coupled networks, just to mention a few, Steur et al. (2012); Proskurnikov (2013); Selivanov et al. (2015); Plotnikov and Fradkov (2018).

This paper continuous the work started in Plotnikov and Fradkov (2021). Here synchronization in heterogeneous networks of linear systems with nonlinear delayed diffusive coupling is considered. The network of neural mass model (NMM) populations Jansen and Rit (1995) is an example of the networks of this type. The problem of network synchronization is reduced to studying the stability of synchronization-error system which can be obtained using a coordinate transformation proposed in Panteley and Loria (2017). The analogue of the circle criterion for time-delay systems (TDS) proposed in Churilova (1995) and generalized in Bryntseva and Fradkov (2019) can be used to study the system stability.

The rest of the paper is organized as follows. Section 2 reminds some important concepts related to synchronization and the circle criterion for TDS. In Sec. 3 synchronization conditions of linear network with nonlinear delayed diffusive couplings are obtained, and NMM network is considered as an example Section 4 provides numerical results on synchronization. Finally, conclusions are given in Sec. 5.

Notation. Throughout the paper the superscript $\cdot^*$ stands for matrix transposition (complex conjugate); $\mathbb{R}^n$ denotes the $n$ dimensional real Euclidean space with vector norm $\| \cdot \|$; $j = \sqrt{-1}$ is the imaginary unit; the notation $z = \text{col}(x, y)$ means that $z$ is a vector of two components $x, y$; the notation $D = \text{diag}\{d_1, \ldots, d_n\}$ means that $D$ is a $n \times n$ diagonal matrix, where $d_i$ is its $i$th diagonal element; $A \otimes B$ means the Kronecker product of matrices $A$ and $B$; $I_n$ is an identity $n \times n$ matrix, while $0_n$ is a $n \times n$ matrix of zeros.

2. PRELIMINARIES

2.1 Circle Criterion for Time-Delay Systems

In this section the details about studying the stability of linear system with multiple nonlinearities with time-varying delays are given. The analogue of the circle criterion proposed in Churilova (1995) and its generalization for multi input – multi output (MIMO) systems proposed
Consider the system, which is described by the following equations:

\[ \dot{x}(t) = Ax(t) + B \varphi(t, \sigma(t - \tau(t))), \]
\[ \sigma(t - \tau(t)) = C^T x(t - \tau(t)), \]  \( (1) \)

where \( A \) is a constant \( n \times n \) matrix, \( B, C \) are constant \( n \times m \) matrices and \( x \in \mathbb{R}^n \) is a state vector, \( \sigma = \text{col}\{\sigma_1, \ldots, \sigma_m\} \in \mathbb{R}^m \) is an input of the system, \( \tau_i(t) \in [0, T], i = 1, \ldots, m, \forall t \) is a bounded time-varying delay, \( \varphi = \text{col}\{\varphi_1, \ldots, \varphi_m\} \) is a vector function of sector-bounded nonlinearities, i.e.

the following inequalities are fulfilled:

\[ \mu_{11} \leq \varphi_1(t, \sigma_1)/\sigma_1 \leq \mu_{21}, \]
\[ \vdots \]
\[ \mu_{1m} \leq \varphi_m(t, \sigma_m)/\sigma_m \leq \mu_{2m}, \]

for \( \sigma_i \neq 0, i = 1, \ldots, m. \)

**Theorem 1.** For the system (1) denote the transfer function of its linear part \( W(p) = C^T(A - pI_m)^{-1}B \) and the characteristic polynomial \( \Delta(p) = \det(pI_m - A) \). Introduce the following diagonal matrices \( \mu_1 = \text{diag}\{\mu_{11}, \ldots, \mu_{1m}\}, \mu_2 = \text{diag}\{\mu_{21}, \ldots, \mu_{2m}\} \). Let the following assumptions be fulfilled:

1. Nonlinearity \( \varphi(\sigma) \) in system (1) satisfies the inequalities (2) for \( \sigma_i \neq 0, i = 1, \ldots, m; \)
2. There exists diagonal \( m \times m \) matrix \( \mu = \text{col}\{\mu_1, \mu_2\} \) such that each element \( \mu_{ij} \) lies between \( \mu_{11} \) and \( \mu_{21} \), \( i = 1, \ldots, m \), and matrix \( A + B \mu \) is Hurwitz;
3. For some diagonal \( m \times m \) matrix \( \nu \) with positive diagonal elements \( \nu_i \) such that \( 1 - 4\nu_i \mu_{11} \mu_{22} > 0, \ i = 1, \ldots, m \) the function

\[ \pi(\omega) = W(j\omega)^* (\mu_1 \mu_2 \omega - \omega^2 I_m/4(I_m - 4\nu_1 \mu_2)) \times W(j\omega) + \text{Re}[W(j\omega)(\mu_1 + \mu_2)(I_m - 4\nu_1 \mu_2)] \]
\[ + \nu_1 (I_m - \nu_1 (\mu_1 + \mu_2)^2), \]

satisfies the following conditions

\[ \lim_{\omega \to \infty} \pi(\omega) > 0, \]  \( (3a) \)
\[ |\Delta(j\omega)|^2 \pi(\omega) > 0, \ \forall \omega \geq 0, \]  \( (3b) \)

(for \( \omega \) such that \( |\Delta(j\omega)| = 0 \) holds the inequality \( (3b) \) is understood as a limiting one).

Then there exist positive constants \( C_1, C_2, \epsilon \) depending only on the coefficients of the linear part of the system (1) \( A, B, C \) and matrices \( \mu_1, \mu_2, T\mu_1 \), such that for all solutions \( x(t) \) of the system (1) with a continuous initial function \( x_0(t) \) defined for \( t \in [-\tau_{\text{max}}, 0] \), where \( \tau_{\text{max}} = \max_{i=1, \ldots, m} \tau_i(0) \), the following inequality holds

\[ \|x(t)\| \leq (C_1\|x_0(0)\| + C_2 \max_{-\tau_{\text{max}} \leq t < 0} \|C^Tx_0(t)\|) e^{-\epsilon t}, \]

for all \( t \geq 0. \)

### 2.2 Synchronization

Here the mathematical notion of synchronization will be introduced. The networks of linear systems with heterogeneous delayed nonlinearities are considered throughout this paper. Each system of this type can be presented in normal form as the following:

\[ \dot{y}_i(t) = A^i \dot{x}_i(t) + \varphi_i(\sigma_i(t - \tau_i(t))), \]
\[ \dot{z}_i(t) = A^i z_i(t), \ i = 1, \ldots, N, \]  \( (4) \)

where \( y_i \in \mathbb{R}^m, z_i \in \mathbb{R}^{n-m}, x_i = \text{col}\{y_i, z_i\} \in \mathbb{R}^n \) and \( \sigma_i \in \mathbb{R}^m \) denote the input, the zero-dynamics, the state and the output of the \( i \text{th} \) system, respectively. \( A^i \in \mathbb{R}^{m \times n} \) and \( A^2 \in \mathbb{R}^{(m-n) \times n} \) are constant matrices.

\[ \varphi_i = \text{col}\{\varphi_1, \ldots, \varphi_m\} : \mathbb{R}^m \to \mathbb{R}^m \] is a vector function, while \( \tau_i = \text{col}\{\tau_1, \ldots, \tau_N\} \) is a time-varying delay. As input diffusive coupling is considered, which is described by

\[ \sigma_i(t - \tau_i(t)) = \sum_{k=1}^N \gamma_{ik} [y_i(t - \tau_k(t)) - y_k(t - \tau_k(t))]. \]  \( (5) \)

Suppose that the graph of the network under consideration is connected and undirected, therefore \( \gamma_{ik} = \gamma_{ki} \forall i \neq k, i, k = 1, \ldots, N \). For this type of coupling both signals are time-delayed. Such type of coupling may be observed, for instance, when the systems are interconnected by a centralized control law.

Synchronization phenomenon occurs often defined as the asymptotically identical evolution of the systems. One can easily introduce the notion of the asymptotic coordinate synchronization Fradkov (2007):

\[ \lim_{t \to \infty} \|x_i(t) - x_k(t)\| = 0, \ i, k = 1, \ldots, N. \]  \( (6) \)

The fulfilment of (6) means that the asymptotic behavior of all nodes of the network (4), (5) is identical and can be described by the function \( x_s \in \mathbb{R}^n \). Thus by defining the synchronization errors as \( e_i = x_i - x_s \) one can study the problem of network synchronization as the problem of stability of synchronization-error system (SES).

To obtain the equations of SES the approach proposed in Panteley and Loréa (2017) can be used. The idea behind this approach is the following: the system state space is decomposed in two orthogonal subspaces, one on which is projected the behavior of the mean-field state and one in which lay the synchronization errors. To obtain the network equations in new coordinates suppose that the graph of considered network is connected and undirected. Following the steps described in Plotnikov and Fradkov (2021) up to a time-delay one can present the equations of the network (4), (5) in the following form:

\[ \hat{x}_1(t) = A \hat{x}_1(t) + (1_N \otimes E_m) \]
\[ \times \Phi[(LU_1 \otimes E^T_n) \hat{x}_1(t - \tau(t))], \]
\[ \hat{x}_2(t) = (I_{n-1} \otimes A) \hat{x}_2(t) + (U_1 \otimes E_m) \]
\[ \times \Phi[(LU_1 \otimes E^T_m) \hat{x}_2(t - \tau(t))], \]  \( (7a) \)

\[ A = \begin{bmatrix} A^y & A^z \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad \tau(t) = \begin{bmatrix} \tau_1(t) \\ \vdots \\ \tau_N(t) \end{bmatrix} \in \mathbb{R}^{N^2}, \]  \( (8) \)

\[ \Phi(\sigma) = \begin{bmatrix} \varphi_1(\sigma_1) \\ \vdots \\ \varphi_N(\sigma_N) \end{bmatrix} : \mathbb{R}^{mN} \to \mathbb{R}^{mN} \]
are the matrix of the linear part of the individual system, the vector of delays and the
nonlinear vector function, respectively;
\[
L = \begin{bmatrix}
\sum_{k=2}^{N} \gamma_{1k} & -\gamma_{12} & \cdots & -\gamma_{1N} \\
-\gamma_{21} & \sum_{k=1,k\neq 2}^{N} \gamma_{2k} & \cdots & -\gamma_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
-\gamma_{N1} & -\gamma_{N2} & \cdots & \sum_{k=1}^{N-1} \gamma_{Nk}
\end{bmatrix} \in \mathbb{R}^{N \times N},
\] (9)
is the Laplace matrix defining the coupling links in the network, where \( \gamma_{i\ell} = \gamma_{\ell i} \) \( \forall i \neq \ell, i, \ell = 1, \ldots, N \) by the assumption;
\[
1_N = \begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix} \in \mathbb{R}^N, \quad E_m = \begin{bmatrix}
I_m \\
0_{(m-n) \times n}
\end{bmatrix} \in \mathbb{R}^{n \times m},
\]
(10)
is the vector of delays and the nonlinear vector function, \( \beta \) and \( k \) are the gain and the reciprocal of the synaptic/membrane time constant;
\[
\Phi(y) = \begin{bmatrix}
\sigma y_1(y_1) \\
\vdots \\
\sigma y_n(y_n)
\end{bmatrix}
\]
is a centered sigmoidal function relating the neuronal states; \( \gamma_1 = \gamma_{i\ell} \) \( \forall i \neq \ell, i, \ell = 1, \ldots, N \) by the assumption.

3. MAIN RESULT

3.1 Synchronization Conditions. General Case

This paper considers a network of \( N \) linear dynamical systems with heterogeneous delayed nonlinear couplings in normal form (4). The graph of considered network is supposed to be connected and undirected, and the connections between the nodes of the network are diffusive ones (5). As described in a subsection, such a network can be represented in the form of MFD (7a) and SES (7b).

The transfer function of the linear part of the system (7b) can be calculated just like in Plotnikov and Fradkov (2021);
\[
W(p) = L \otimes \left[ E_m^T (A - \mu L)^{-1} E_m \right],
\] (12)
where \( L \) is the Laplace matrix (9), \( A \) is the matrix of linear part (8) and \( E_m \) is the supplementary matrix (10).

The nonlinear part of the system (7b) are described by the functions \( \varphi_i = \text{col}\{\varphi_{i1}, \ldots, \varphi_{im}\}, i = 1, \ldots, N \). Suppose that they belong to the two-cavity sector between two straight lines, i.e. the following inequalities are fulfilled:
\[
\mu_{i1} \leq \varphi_{i1}(\sigma_{i1})/\sigma_{i1} \leq \mu_{2i1},
\]
\[
\vdots
\]
\[
\mu_{im} \leq \varphi_{im}(\sigma_{im})/\sigma_{im} \leq \mu_{2im},
\]
\( i = 1, \ldots, N \).

Introduce diagonal matrices matrices by the following way
\[
\mu_1 = \text{diag}\{\mu_{111}, \ldots, \mu_{1m1}, \mu_{11N}, \ldots, \mu_{1mN}\},
\]
\[
\mu_2 = \text{diag}\{\mu_{211}, \ldots, \mu_{2m1}, \mu_{21N}, \ldots, \mu_{2mN}\}.
\]

Suppose that all delays \( \tau_i = \text{col}\{\tau_{i1}, \ldots, \tau_{iN}\}, i = 1, \ldots, N \) in system (7b) are bounded functions, i.e. \( \tau_i(t) \in [0, T], i = 1, m, \forall t \).

Thus the theorem about synchronization of linear networks with nonlinear delayed couplings can be formulated.

Theorem 2. If the following conditions are fulfilled

(1) The network (4), (5) nonlinearities lie in the sector, i.e. inequalities (13) hold;
(2) The graph of the network (4), (5) is connected and undirected.
(3) There exists matrix \( \mu_0 \in \mathbb{R}^{mN \times mN} \) such that matrices \( \mu_0 - \mu_1 \) and \( \mu_2 - \mu_0 \) have only nonnegative elements, and matrix
\[
\Psi = \mu_{N-1} \otimes A + [U_1^T \otimes E_m] \mu_0 [U_1 \otimes E_m^T]
\]
is Hurwitz;
(4) For some diagonal \( mN \times mN \) matrix \( \nu \) with positive diagonal elements such that matrix \( I_{mN} - 4\nu \mu_1 \mu_2 \) has positive diagonal elements, the function
\[
\pi(\omega) = W(j\omega)^T (\mu_1 \mu_2 - T^2 \omega^2 I_{mN})/4
\]
\[
\times (I_{mN} - 4\nu \mu_1 \mu_2) W(j\omega)
\]
\[
+ \text{Re}[W(j\omega)(\mu_1 + \mu_2)(I_{mN} - 4\nu \mu_1 \mu_2)\nu]
\]
\[
+ \nu(I_{mN} - \nu(\mu_1 + \mu_2)^2)
\]
satisfies the following conditions.
\[
\lim_{\omega \to \infty} \pi(\omega) > 0, \quad (14a)
\]
\[
|\Delta(j\omega)|^2 \pi(\omega) > 0, \quad \forall \omega \geq 0, \quad (14b)
\]
(for \(\omega\) such that \(|\Delta(j\omega)| = 0\) holds the inequality (14b) is understood as a limiting one), where \(W(p)\) is the transfer function (12) and
\[
\Delta(p) = \det(pI_n(N-1) - I_{N-1} \otimes A) = \det((I_{N-1} \otimes (pI_n - A))^{N-1})
\]
is the characteristic polynomial of the matrix of the linear part (7b).

Then the systems in the network (4), (5) are asymptotically synchronized.

### 3.2 Synchronization Conditions of Neural Mass Model Populations

This section considers the heterogeneous network of non-linearly delayed coupled NMM populations
\[
\begin{align*}
\dot{y}_{i1}(t) &= -2\alpha g_{i1}(t) - \alpha^2 y_{i2}(t) \\
&\quad + \alpha \beta \varphi_i \left\{ \sum_{k=1}^N \gamma_{ik}[y_{i1}(t - \tau_{ik}(t)) - y_{ik}(t - \tau_{ik}(t))] \right\}, \\
\dot{y}_{i2}(t) &= y_{i1}(t), \quad i = 1, \ldots, N,
\end{align*}
\]
(15)

where \(x = \{y_{i1}, y_{i2}, \ldots, y_{iN}, \gamma_{iN}\}\) is a state vector; \(\alpha, \beta\) are system parameters; \(\gamma_{ik}\) are coupling coefficients; \(\tau_{ik}(t) \in [0, T], \forall t, \forall k, i = 1, \ldots, N\) are bounded time-varying delays (all delay functions have the same upper bound \(T\)). Functions \(\varphi_i\) are sigmoidal ones, which are described by (11) with parameters \(g_i, k_0_i, i = 1, \ldots, N\).

The network of NMMs can be presented in new coordinates (7a), (7b) with matrices
\[
A = \begin{bmatrix}
-2\alpha - \alpha^2 & 0 \\
1 & 0
\end{bmatrix}, \quad E_m = \begin{bmatrix} 1 \\
0
\end{bmatrix}.
\]

To study the network (15) synchronization one should check the conditions of Theorem 2:

Sigmoidal functions (11) are sector ones, which lie between two straight lines 0 and 0.5\(g_i, \sigma\) (see explanation in Gorskho et al. (2017), \(i = 1, \ldots, N\). One can find \(g_{\max} = \max_{i=1, \ldots, N} g_i\). Suppose that \(\alpha \beta > 0\), then the matrices from the condition (1) of the Theorem 2 can be expressed as \(\mu_1 = 0\) and \(\mu_2 = 0.25g_{\max} \alpha \beta I_N\).

Supposing that the graph of considered network (15) is connected and undirected guarantees the fulfillment of the condition (2) of the Theorem 2.

The condition (3) of Theorem 2 is the same as for the case without delays: this fact follows from the conditions of Circle criterion. This condition was previously checked in Plotnikov and Fradkov (2021): If \(\alpha > 0\), then the matrix \(\Psi\) in the condition (3) of the Theorem 2 is Hurwitz.

To calculate the function \(\pi(\omega)\) from the condition (4) of the Theorem 2 the frequency transfer function \(W(\omega)\) should be found using formula (12):
\[
W(j\omega) = L \otimes \begin{bmatrix} 1 & 0 \\
-2\alpha - j\omega - \alpha^2 & 1 \\
1 & -j\omega
\end{bmatrix} = \frac{-j\omega}{\alpha^2 - \omega^2 + 2j\alpha \omega} L = \frac{-2\alpha \omega^2 - j\omega(\alpha^2 - \omega^2)}{(\alpha^2 + \omega^2)^2} L. \quad (16)
\]

Meaning that \(\mu_1 = 0\) and \(\mu_2 = 0.25g_{\max} \alpha \beta I_N\) and choosing matrix \(v\) as \(v_0 I_N\) one obtains:
\[
\pi(\omega) = \frac{-T^2 \omega^4}{4}(W(j\omega)*) W(j\omega) + \frac{g_{\max} \alpha \beta v_0}{4} \Re \{W(j\omega)\}
\]
\[
+ v_0 \left(1 - \nu_0 g_{\max} \alpha^2 \beta^2 \right) I_N.
\]

Since the graph of considered network is undirected the corresponding Laplace matrix \(L\) is symmetric. Using this fact and (16) the function \(\pi(\omega)\) equals:
\[
\pi(\omega) = \frac{-T^2 \omega^4}{4(\alpha^2 + \omega^2)^2} - \frac{g_{\max} \alpha \beta v_0 \omega^2}{2(\alpha^2 + \omega^2)^2} L + v_0 \left(1 - \nu_0 g_{\max} \alpha^2 \beta^2 \right) I_N.
\]

Now check the matrix inequality (14a):
\[
\lim_{\omega \to \infty} \pi(\omega) = \frac{-T^2 \alpha^2}{4} L^2 + v_0 \left(1 - \nu_0 g_{\max} \alpha^2 \beta^2 \right) I_N > 0
\]

Consider some symmetric matrix \(P\) it is positive definite if the corresponding quadratic form \(x^T P x\) is positive \(\forall x \neq 0\). 3 matrix \(U: \ P = UD U^T\), where \(D\) is a diagonal matrix. Then for \(z = U^T x\) one obtains \(z^T D z > 0\). Therefore the obtained inequality is fulfilled if and only if the following inequalities are fulfilled:
\[
\frac{-T^2 \lambda_i^2}{4} + v_0 \left(1 - \nu_0 g_{\max} \alpha^2 \beta^2 \right) > 0, \quad i = 1, \ldots, N, \quad (17)
\]
where \(\lambda_i\) are the eigenvalues of Laplace matrix \(L\). Then the maximal value of the delay \(T\) can be estimated:
\[
T^2 < 4 \nu_0 g_{\max} \alpha \beta I^N, \quad i = 1, \ldots, N, \quad (18)
\]

There is a quadratic equation depending on \(v_0\) in the numerator of resulting fraction, which has the maximal value for \(v_0 = 8/(g_{\max}^2 \alpha^2 \beta^2)\). All equations (18) hold if
\[
T < \frac{4}{g_{\max} \lambda_{\max} \alpha \beta}, \quad (19)
\]
where \(\lambda_{\max}\) is the maximal eigenvalue of Laplace matrix \(L\).

From (14b) one obtains that inequality
\[
\frac{-T^2 \omega^4}{4} L^2 \quad \frac{g_{\max} \alpha \beta v_0 \omega^2}{2} L + v_0 \left(1 - \nu_0 g_{\max} \alpha^2 \beta^2 \right) (\alpha^2 + \omega^2)^2 I_N > 0
\]
should be fulfilled \(\forall \omega \geq 0\). As before the set of the following inequalities can be considered instead of the obtained matrix inequality:
The coefficient before $\omega^4$ is the same as (17), and it is positive if $\nu_0 = \frac{8}{(g_{\max}^2 \alpha^2 \beta^2)}$ and the inequality (19) is fulfilled. Zero order term is also positive for chosen value of $\nu_0$. The coefficient before $\omega^2$ is positive if $g_{\max} \beta \lambda < 2$. Thus the inequality (20) is fulfilled for chosen parameters.

All conditions of the Theorem 2 are fulfilled, therefore the network (15) is synchronized. The following theorem holds.

Theorem 3. If the network (15) systems parameters $\alpha > 0$ and $\beta > 0$, $g_i > 0$, $i = 1, \ldots, N$, the graph of the network is connected and undirected, the maximum eigenvalue of the Laplace matrix $L$ is less than $2/(\beta g_{\max})$, and the delays in the signal propagation are bounded (19), then the network of NMMs is asymptotically synchronized.

Note that while $T \to 0$ and choosing $\nu_0 \to 0$ such that $T^2/\nu_0 \to 0$, we get the similar conditions of network synchronization as in Plotnikov and Fradkov (2021).

4. SIMULATION

This section presents the results of simulation. The network of NMM populations with $N = 10$ is considered. The system parameters $\alpha$ and $\beta$ are equal to 1 and 0.8, respectively. The parameters defining the shape of sigmoidal function have uniform distribution: $g_i$, $i = 1, \ldots, N$ are distributed on the interval $[0; 1]$, while $k_{hi}$, $i = 1, \ldots, N$, are distributed on the interval $[-1; 1]$. The graph of considered network is weighted, connected and undirected, meaning that its adjacency matrix is a symmetric sparse matrix with density 0.7, which means that it has approximately $0.7N^2$ nonzero entries. Let the delays $\tau_{ik}(t) \in [0; T]$, $i, k = 1, \ldots, N$ be time-varying functions

$$\tau_{ik}(t) = h_{1ik} + h_{2ik} \sin(h_{3ik} + h_{4ik}),$$

which are uniformly distributed on the interval $[0; T/2]$ such that $h_{1ik} > h_{2ik}$, $i, k = 1, \ldots, N$. The maximum value of the delay $T$ will be defined later. The initial functions $y_{1i}(t), y_{2i}(t), t \in [-T; 0]$, $i = 1, \ldots, N$ are constants, which are uniformly distributed on the interval $[-1; 1]$.

First of all consider the case, when NMM network has parameters satisfying the conditions of the Theorem 3. The maximum eigenvalue $\lambda_{\max}$ of the Laplace matrix $L$ is equal to 2.5080 is this case, while $g_{\max} = 0.9724$. For these parameters of the network the inequality $\lambda_{\max} < 2/(\beta g_{\max}) \approx 2.509$ is fulfilled. Choosing $T = 2$ one can ensure the fulfillment of the inequality (19), thereby guarantee the fulfillment of all condition of the Theorem 3. This means that for these parameters the network of NMMs will synchronize. Figure 1 presents the results of simulation. As one can see, for the chosen parameter values, there is synchronization among the state variables of the network, and all system trajectories tend to equilibrium point, which confirms Theorem 3.

Now consider the case, when the delays in signal propagation between the nodes are too large, that prevents the network synchronization. In this case $\lambda_{\max} = 1.7983$ and $g_{\max} = 0.9630$, which means that the inequality $\lambda_{\max} < 2/(\beta g_{\max}) \approx 2.596$ is also fulfilled. Choosing $T = 10$ one violates the Theorem 3 condition (19). Therefore for these parameters of the network, the Theorem 3 doesn’t guarantee the network synchronization. One can see the results of simulation in Fig 2: there is no synchronization among the state variables of the network.

5. CONCLUSION

In this paper, the problem of heterogeneous network synchronization of linear systems with delayed nonlinear diffusive couplings. The delays are supposed to be bounded time-varying functions. The nonlinear coupling functions can be different but their graphs should lie in two-cavity sector between two straight lines. As in Plotnikov and Fradkov (2021) the synchronization problem is reduced to
study the stability of the SES. The coordinate transformation approach proposed in Panteley and Loria (2017) is used to present the network in coordinates "MFD - SES". To find the conditions for SES stability the circle criterion for TDS was applied. The theorem about network synchronization of this type was formulated and proven.

As an example, the dynamics of NMM populations connected via delayed nonlinear diffusive coupling was considered. Using the obtained theorem, the simple condition for network synchronization was derived. In the case of the delay absence this condition coincide with the result obtained in Plotnikov and Fradkov (2021). Also, the simulation of NMM network dynamics was performed. In the case, when the Theorem 3 conditions are fulfilled, one can observe the synchronization between the network states. In the other case, when the delays are large enough, there is no synchronization between the network nodes.

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