ROUGH INTEGRATORS ON BANACH MANIFOLDS

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Abstract. We introduce a notion of p-rough integrator on any Banach manifolds, for any $p \geq 1$, which plays the role of weak geometric Hölder $p$-rough paths in the usual Banach space setting. The awaited results on rough differential equations driven by such objects are proved, and a canonical representation is given if the manifold is equipped with a connection.

1. Introduction

So far there has been only a few works dealing with rough paths in a geometrical setting, starting with the work [1] of Lyons and Qian. This seminal work investigated the well-posedness problem for the ordinary differential equation on the path space of a compact manifold generated by Itô vector fields, with an eye on probabilistic applications related to path space versions of the Cameron-Martin theorem and Driver’s flow equation. It was enriched by the work [2] of Li and Lyons, showing that the Itô-Lyons solution map to a Young differential equation is Fréchet regular under appropriate conditions, when the driving signal has finite $p$-variation, with $1 < p < 2$, and by the work [3] of the author providing a general regularity result for the Itô-Lyons solution map, for any $p \geq 2$, in the setting of controlled paths. Another work [4] of Lyons and Qian addressed again well-posedness issues for ordinary differential equations on path space associated with Itô vector fields obtained by varying the driving rough signal.

These works only use rough paths as an ingredient to construct some dynamics in a geometric configuration space. Cass, Litterer and Lyons made a step further in putting rough paths theory in a geometrical setting and proposed in [5] a notion of rough path on a manifold extending the classical notion defined in a linear setting. In the same way as a vector field on a manifold $M$ can be understood in an analytic/algebraic setting as a differentiation in the ring of smooth functions on the manifold $M$, a rough path is abstractly defined as a linear form on the space of sufficiently regular 1-forms on $M$, which is required to have some continuity property; call it an integrator. This functional analytic definition rests on a basic chain rule which eventually enables to understand their notion of rough path on a manifold as an equivalence class of classical rough paths, related by some chain rule under change of coordinates. This situation is exactly similar to representing a tangent vector on a $d$-dimensional manifold as an equivalence class of vectors in $\mathbb{R}^d$, indexed by local...
diffeomorphisms of a neighbourhood of 0 (that is local changes of coordinates), and related by a change of coordinate rule which exactly balances the changes in the numerical representation of a given 1-form $\alpha$ on $\mathbf{M}$ associated with local coordinates, so the quantity $\alpha(u)$ is independent of any choice of coordinates used to compute it. Their approach rests however on a notion of Lip-$\gamma$ manifold which prevents its easy use even with non-compact finite dimensional manifolds, not to speak about infinite dimensional manifolds.

The ideas of [5] have been reloaded in a different and more accessible form in the recent work [6] by Cass, Driver and Litterer, in which they define a weak geometric Hölder $p$-rough path on a finite dimensional compact embedded submanifold of $\mathbb{R}^d$ as an integrator, obtained by "projection" of a weak geometric Hölder $p$-rough path in the amiant Euclidean space, for $2 \leq p < 3$ only. Nothing is lost in working with submanifolds, and this notion is eventually shown to be independent of the embedding of the manifold in its environment, while these objects can be used to define and solve uniquely rough differential equations driven by weak geometric Hölder $p$-rough paths on compact manifolds. They can construct parallel transport along manifold-valued rough paths in their sense, and use it to show a one-to-one correspondence between rough paths on a finite $d$-dimensional manifold and rough paths on $d$-dimensional Euclidean space.

We propose in this note an elementary and flexible notion of weak geometric Hölder $p$-rough path on Banach manifolds that goes beyond the previous works. Roughly speaking, a weak geometric Hölder $p$-rough path on a manifold $\mathbf{M}$ is a triple made up of a vector field valued 1-form $F$ defined on some Banach space $U$, a weak geometric Hölder $p$-rough path $X$ over $U$, and the maximal solution to the rough differential on $\mathbf{M}$ constructed from $F$ and $X$. The data of these three objects is sufficient to define and solve uniquely rough differential equations driven by "manifold-valued" rough paths, or better, by $p$-rough integrators. One shows in addition that such rough integrators have a canonical representation if the tangent bundle of $\mathbf{M}$ is equipped with a connection, in which case one can always choose for Banach space $U$ in the preceeding description of a weak geometric Hölder $p$-rough path on $\mathbf{M}$ the Banach space on which the manifold is modelled.

The interest of working with Banach manifolds comes from the fact that they naturally pop up in a number of geometric situations, as path or loop spaces over some finite dimensional manifold, as in the works [7, 8] of Brzezniak, Carroll and Elworthy, or the works [9, 10] of Inahama and Kawabi, or as manifolds of maps of a given finite dimensional manifold, as in the works [11, 12] of Elworthy and Brzezniak, to mention but a few works from the probability community. Note however that many interesting infinite dimensional manifolds are Fréchet manifolds, for which no theory of rough paths is presently available.

As a matter of fact, working with Banach manifolds will not bring any additional difficulty along the way, as compared to working with finite dimensional manifolds. We refer the reader to the books [13] of Lang, and [14] of Abraham, Marsden and Ratiu, for the basics of differential geometry in an infinite dimensional setting. Let just recall that it makes perfect sense in a manifold setting to say that a (Banach
space or real-valued) function $f$ of class $C^k$, defined on the domain of some local chart $\varphi$ of $M$, has bounded derivatives if the function $f \circ \varphi^{-1}$, defined on a neighbourhood of 0 in some Banach space, has bounded derivatives, as this boundedness character will not depend on the precise chart used to define it, while the bounds themselves depend on $\varphi$.

**Notations.** We gather here a few notations that will be used throughout the note.

- We shall denote by $U$ a generic Banach space; the notation $T^{[p]}(U)$ will be used for the truncated tensor product of order $[p]$, completed for some choice of tensor norms. The letter $X$ will stand for a weak geometric Hölder $p$-rough path over $U$, for some $p \geq 2$, and for the above choice of tensor norm on $T^{[p]}(U)$.
- We shall denote by $M$ a Banach manifold modelled on some Banach space $E$. The set of continuous linear homomorphisms of $E$ will be denoted by $L(E)$, and the set of continuous linear isomorphisms of $E$ will be denoted by $GL(E)$.

### 2. Rough integrators on Banach manifolds

We define in this section what will play the role in a manifold setting of weak geometric Hölder $p$-rough paths. Rough integrators are basically defined as a triple, consisting of a traditional weak geometric Hölder $p$-rough path, a vector field valued 1-form and a solution to an associated rough differential equation on the manifold. This definition, given in section 2.2, requires that we recall some facts on rough differential equations with values in manifolds; this is done in section 2.1. We explain in section 2.3 how rough integrators can be used to define and solve rough differential equations.

#### 2.1. Rough differential equations.** We adopt in this work the definition of a solution path to an $M$-valued rough differential equation given in [16] in a Banach space setting, as it is perfectly suited for our needs. It essentially amounts to requiring from a solution path that it satisfies some uniform Taylor-Euler expansion formulas, in the line of Davie’ seminal work [15]. Let $U$ be a Banach space, and $F$ be a 1-form on $U$ with values in the space of vector fields on a Banach manifold $M$, of class $C^{[p]+1}$. Given $u \in U$, we identify the vector field $F(\cdot; u)$ on $M$ with its associated first order differential operator $F^\otimes(u)$, and extend the definition of $F^\otimes$ to $T^{[p]}(U)$ setting $F^\otimes(1) = \text{Id}$, and

$$F^\otimes(u_1 \otimes \cdots \otimes u_k) = F^\otimes(u_1)\cdots F^\otimes(u_k),$$

for $1 \leq k \leq [p]$, and $u_i \in U$, and by linearity; so $F^\otimes(u_1 \otimes \cdots \otimes u_k)$ is a differential operator of order $k$. Given a weak geometric Hölder $p$-rough path $X$ over $U$, recall that an $M$-valued continuous path $(x_t)_{0 \leq t < \zeta}$ is said to solve the rough differential equation

$$dx_t = F^\otimes(x_t; X(dt))$$

(2.1)
if there exists a constant \( a > 1 \) such that, for any \( 0 \leq s < \zeta \), there exists an open neighbourhood \( V_s \) of \( x_s \) such that we have the Taylor-Euler expansion

\[
    f(x_t) = (F^\otimes(X_{ts}) f)(x_s) + O(|t - s|^a)
\]

for all \( t \) close enough to \( s \) for \( x_t \) to belong to \( V_s \), and any function \( f \) of class \( \mathcal{C}^{[p]+1} \) defined on \( V_s \), where it has bounded derivatives. The results of \([16]\) show that such a rough differential equation has a unique maximal solution started from any given point, as awaited. It also follows from the results of \([16]\), or other classical works, that the solution path \( x_\bullet \) depends continuously on the driving signal \( X \) in the following sense. Fix \( T < \zeta \) and cover the compact support of the path \((x_t)_{0 \leq t \leq T}\) by finitely many local chart domains \((O_i')_{1 \leq i \leq N}\) and \((O_i)_{1 \leq i \leq N}\), with \( O_i' \subset O_i \) for all \( 1 \leq i \leq N \). Then, there exists a positive constant \( \delta \) such that for \( Y \) \( \delta \)-close to \( X \) in the Hölder rough path distance, the solution path \( y_\bullet \) to the rough differential equation

\[
    dy_t = F^\otimes(y_t ; Y(dt))
\]

started from \( x_0 \), will be well-defined on the time interval \([0, T]\) and will remain in the open neighbourhood \( \bigcup_{i=1}^N O_i \) of the support of \((x_t)_{0 \leq t \leq T}\), with \( y_t \) in \( O_i \) whenever \( x_t \) is in \( O_i' \).

Note here that the rough path setting allows to work with any Banach manifold, without any restriction on the Banach space on which it is modelled, unlike what needs to be done in a stochastic setting where a robust theory of stochastic integration is only available for some special types of Banach spaces, the so-called M-type 2 and UMD spaces, see \([17, 18]\).

**An example.** Rough differential equations with values in Banach Lie groups. Let \( G \) be a Banach Lie group, with Lie algebra \( \mathfrak{lie}(G) \). Think for instance to the loop groups, made up of \( C^k \) maps from some finite dimensional manifold \( M_0 \) to some finite dimensional Lie group \( G_0 \), with pointwise multiplication and inversion operations. Their Lie algebra is the set \( C^k(M_0, \mathfrak{g}_0) \) of \( C^k \) maps from \( M_0 \) to the Lie algebra \( \mathfrak{g}_0 \) of \( G_0 \), with bracket defined pointwise by the relation \([u, v](x) := [u(x), v(x)]\), for any \( u, v \) in \( C^k(M_0, \mathfrak{g}_0) \) and \( x \in M_0 \). These groups are extensively used in gauge theory or quantum field theory.

Write \( L_g \) for the left translation by \( g \) in \( G \). One defines a continuous linear map from \( \mathfrak{lie}(G) \) to the space of smooth vector fields on \( G \) setting

\[
    F(g; u) := (D_{id} L_g)(u),
\]

for any \( g \in G \) and \( u \in \mathfrak{lie}(G) \); so \( F(\cdot; u) \) is for any \( u \in \mathfrak{lie}(G) \) a smooth left invariant vector field on \( G \). It is elementary to proceed as in the proof of theorem 4.20 in \([9]\) and see that for any weak geometric Hölder p-rough path \( X \) over \( \mathfrak{lie}(G) \), defined on some interval \([0, T]\), for some \( 0 \leq T \leq \infty \), the solution path to equation \((2.2)\) cannot explode, so it is also defined on the interval \([0, T]\).
2.2. Rough integrators on Banach manifolds.

Definition 1. Let \( p \geq 2 \) be given. A \textbf{weak geometric Hölder \( p \)-rough path on} \( M \) is a triple \( \Theta = ((x_t)_{0 \leq t < \zeta}, F, X) \), where

- \( X \) is a weak geometric Hölder \( p \)-rough path over some Banach space \( U \), defined on the time interval \([0, \zeta)\),
- \( F \) is a continuous linear map from \( U \) to the space of vector fields on \( M \) of class \( C^{[p]+1} \),
- the path \( (x_t)_{0 \leq t < \zeta} \) solves the rough differential equation

\[
\frac{dx_t}{dt} = F(x_t; X dt).
\]

We also call \( \Theta \) a \textbf{basic \( p \)-rough integrator}.

Given a \( p \)-rough integrator \( \Theta \) as above, the triple \( ((x_t)_{0 \leq t \leq T}, F, X) \), with \( T < \zeta \), is also called a basic \( p \)-rough integrator. Let us insist on the fact that the Banach space \( U \) in the above definition is not fixed a priori, and depends on the rough path \( \Theta \). So any weak geometric Hölder \( p \)-rough path \( Y \) over some Banach space \( V \) can be canonically seen as a weak geometric Hölder \( p \)-rough path in the above sense by choosing \( U = V \) and \( X = Y \), the identity map for \( F \), and the first level of the rough path \( X_t \) for \( x_t \).

The definition of a solution to the rough differential equation (2.2) makes it clear that if \( g \) is any sufficiently regular diffeomorphism between \( M \) and another manifold \( M' \), one defines a weak geometric Hölder \( p \)-rough path on \( M' \) setting

\[
g^*\Theta := \left( (g(x_t))_{0 \leq t < \zeta}, g^*F, X \right),
\]

where \( g^*F(y; u) := (Dg^{-1}(y))g(F(g^{-1}(y); u)) \), is the push forward on \( M' \) by \( g \) of the vector field \( F(\cdot; u) \) on \( M \).

Definition 1 encompasses the characterization of a weak geometric Hölder \( p \)-rough path on a submanifold of \( \mathbb{R}^d \) given in [6], only for \( 2 \leq p < 3 \), in terms of a priori extrinsic considerations. Roughly speaking, they define their class of weak geometric \( p \)-rough paths on a compact submanifold \( M \) of \( \mathbb{R}^d \), for \( 2 \leq p < 3 \), as the class of all weak geometric rough paths \( Y \) over \( \mathbb{R}^d \) with the property that \( Y_0 = (1, y_0, 0, \ldots) \) with \( y_0 \in M \), and

\[
dY_t = Q(y_t) dY_t
\]

for any projector-valued sufficiently regular map \( Q \) such that \( Q(y) \) has range included in \( T_y M \) if \( y \in M \); this equation ensures that \( y_t \in M \). If \( M' \subset \mathbb{R}^d \) is any other embedded manifold diffeomorphic to \( M \) through \( \varphi : M \rightarrow M' \), equation (2.3) and the chain rule for rough integrals shows that one defines an image rough path \( Z \) over \( \mathbb{R}^{d'} \) setting

\[
dZ_t = d\varphi(dY_t)
\]

and that it satisfies the identity

\[
\int_0^1 (\varphi^* \alpha)(dZ_t) = \int_0^1 \alpha(dY_t),
\]
for any sufficiently regular 1-form \( \alpha \) on \( TM \). (The expression \( \varphi^*\alpha \) stands for the push-forward of the 1-form \( \alpha \) by \( \varphi \), and both integrals make sense because of condition (2.3).) So the class of weak geometric \( p \)-rough paths is intrinsically defined as a class of rough integrators, independently of any particular embedding of \( M \), seen as an abstract manifold. Note that they use general controls \( \omega(s, t) \) to measure the size of their rough paths, while we simply use \( \omega(s, t) = t - s \) and Hölder scales.

A weak geometric \( p \)-rough path in the sense of [6] is simply described in our terms as a special kind of weak geometric \( p \)-rough path in the ambient linear space via equation (2.3). The interest of working with the intrinsic definition [1] is clear in an infinite dimensional setting where embeddings of Banach manifolds in larger Banach spaces are less natural and rarely happen, unless \( M \) is parallelizable [19].

One of the nice features of the notion of rough integrator on a manifold proposed in [5] and [6] is the possibility to push forward a rough integrator by a sufficiently regular map \( g \) from \( M \) to another manifold \( N \), giving as a result a rough integrator on \( N \). The rough integral of a 1-form \( \alpha \) on \( N \) against the image rough integrator is simply defined as the rough integral of \( \alpha \circ dq \) against the rough integrator on \( M \). A similar transport operation can be done in our framework, by seeing the push forward of a \( p \)-rough integrator \( \Theta \) on \( M \) by \( g : M \to N \) as the \( p \)-rough integrator \( g\Theta \) on \( M \times N \) associated with \((F, dq \circ F)\) and \( X \).

It may happen that the vector fields \( F(\cdot; u) \) and \( F(\cdot; v) \) commute for any \( u, v \in U \), and \( X \) has null first level, so the path \( x_\cdot \) is constant. We say in that case that \( \Theta \) is a pure rough path. It can still give rise to some dynamics, when seen as a rough integrator in a rough differential equation, as shown in the next section.

2.3. Weak geometric Hölder \( p \)-rough paths as integrators. The next proposition shows that a weak geometric Hölder \( p \)-rough path on \( M \) can be used as an integrator in a rough differential equation, which justifies calling it a \( p \)-rough integrator. Recall the tangent space of the cartesian product of two manifolds \( M \) and \( N \) is canonically identified with \( TM \times TN \).

**Definition 2.** Let \( \Theta \) be a weak geometric Hölder \( p \)-rough path on \( M \) in the sense of definition [1]. Let \( N \) be another Banach manifold, and \( G : N \times TM \to TN \) be a function of class \( C^{[p] + 1} \) which defines for each \( x \in M \) a vector field-valued 1-form on \( T_xM \), with values in \( TN \). An \( N \)-valued path \((y_t)_{0 \leq t < \zeta'}\) is said to solve the rough differential equation

\[
dy_t = G(y_t, x_t; \Theta(dt))
\]

(2.4)

driven by the \( p \)-rough integrator \( \Theta \) on \( M \) if \( \zeta' \leq \zeta \), and the pair \((x_t, y_t)_{0 \leq t < \zeta'}\) of \( M \times N \)-valued paths solves the rough differential equations

\[
d(x_t, y_t) = \mathcal{G}(x_t, y_t; X(dt)),
\]

(2.5)

on the time interval \([0, \zeta')\), where \( \mathcal{G}(\cdot; \cdot; \cdot) \) is the 1-form on \( U \) with values in the space of \( C^{[p] + 1} \) vector fields on \( M \times N \), given for any \( u \in U \) by the formula

\[
\mathcal{G}(x, y; u) = \left(F(x; u), G(y, x; F(x; u))\right) \in T(M \times N).
\]
This definition is reminiscent of the approach chosen by Cass, Litterer and Lyons, in section 5 of [5], to define in their setting rough differential equations driven by manifold-valued rough paths. Note however that their theory requires the use of the rigid and non-trivial notion of Lip $\gamma$ manifold; no such constraint holds in our setting.

This definition can in particular be used with a 1-form $G$ independent of its $y$-component, and with $M$ and $N$ some Banach spaces, $M = U$ and the identity for $F$; it gives back in that case the classical definition of the rough integral setting.

The fact that the two vector fields $\Theta(\cdot, \cdot; u)$ and $\Theta(\cdot, \cdot; v)$ may not commute while $F(\cdot; u)$ and $F(\cdot; v)$ may commute, for some $u, v \in U$, explains why pure rough paths can generate dynamics. The results on rough differential equations recalled in the introduction of this section apply and show that

**Proposition 3.** The rough differential equation (2.4) has a unique maximal solution started from any given point.

Solving successively some rough differential equations of the form (2.4) with some basic $p$-rough integrators $\Theta^{(1)} = (\langle x_t^{(1)} \rangle_{0 \leq t \leq t_0}, F^{(1)}(t, \mathbf{X}(t)), \ldots, \Theta^{(k)}$, with $x_0^{(j)} = x_{t_{j-1}}^{(j-1)}$, $y_0^{(j)} = y_{t_{j-1}}^{(j-1)}$ and $t_{j-1} < \zeta_{j-1}^{(j)} \leq \zeta_{j-1}^{(j)}$, for $2 \leq j \leq k$, defines, whenever this makes sense, the concatenation $\Theta^{(k)} \ldots \Theta^{(1)}$ of the basic $p$-rough integrators $\Theta^{(i)}$. We call such an object a $p$-rough integrator.

### 3. Canonical representation of rough integrators

We show in this section that $p$-rough integrators have a canonical representation when the tangent bundle of $M$ is equipped with a connection. This representation is the analogue of the representation of a regular path $\gamma$ on $M$ by a regular path in $T_{\gamma_0}M$ using Cartan’s development map.

#### 3.1. Cartan’s moving frame method.

Let $M$ be a manifold of finite dimension $d$. One owes to E. Cartan the introduction in differential geometry of the moving frame method, which provides a chart for the set of $M$-valued paths of class $C^1$ based at some fixed starting point, in terms of $C^1$ paths in $\mathbb{R}^d$ started from 0. Its construction requires the use of a connection on $TM$ and of the frame bundle of $M$; it can basically be described as follows. Given a $C^1$ path $\gamma_\bullet = (\gamma_t)_{0 \leq t \leq 1}$ on $M$ and a frame $e_0$ at $\gamma_0$, it parallel transport along the path $\gamma_\bullet$ defines a section $(e_t)_{0 \leq t \leq 1}$ of the frame bundle above $\gamma_\bullet$. One can use these frames to describe $\dot{\gamma}_t$ at any time in terms of its coordinates $(\dot{u}_t^i)_{1 \leq i \leq d}$ in $e_t$. The $\mathbb{R}^d$-valued path $u_\bullet$ defined by the formula

$$u_t := \int_0^t \dot{u}_s^i \, ds$$

is called the **anti-development of the path** $\gamma$. One finds back $\gamma_\bullet$ from $u_\bullet$ and $(x_0, e_0)$ as the projection on $M$ of the unique solution of the ordinary differential equation in the frame bundle

$$\nabla_{\dot{\gamma}_t} e_t = 0, \quad \dot{\gamma}_t := e_t(\dot{u}_t).$$
The interest of Cartan’s moving frame method is that it somehow provides a dimensionally-optimal coding of a path in \( M \) in terms of vector space-valued paths. Here is a trivial illustration of this fact. Let \( u_\bullet \) be an \( \mathbb{R}^{10} \)-valued path and \( \gamma_t = \sum_{i=1}^{10} u_i^t \) be an \( \mathbb{R} \)-valued path. The coding of \( \gamma_\bullet \) by the 10 components of \( u_\bullet \) is optimized by coding it with its real value at each time. Replacing \( \mathbb{R}^{10} \) and \( \mathbb{R} \) by an infinite dimensional spaces emphasizes the importance of such parcimonious representations.

Recall \( M \) is modelled on some Banach space \( E \). One can actually give a parcimonious description of weak geometric Hölder \( p \)-rough paths on Banach manifolds similar to the above one, providing a description of these objects in terms of weak geometric Hölder \( p \)-rough paths in the tensor space \( T^{[p]}(E) \), as opposed to the a priori unrelated Banach space \( U \) involved in the primary definition of a weak geometric Hölder \( p \)-rough path, see definition \([1]\). As in Cartan’s moving frame method, this requires the tangent bundle \( TM \to M \) to be equipped with a connection; this is a non-trivial assumption in an infinite dimensional setting, linked to the fact that there exists some Banach spaces that do not even admit a smooth partition of unity. No such pathology happens in finite dimension, and finite dimensional manifolds can always be endowed with an arbitrary connection; so the results of the forthcoming section \([3.2]\) always hold for rough integrators on finite dimensional manifolds.

The frame bundle \( GL(M) \) of \( M \) will play a crucial role in that play. This is the collection of all isomorphisms from \( E \) to \( T_m M \), for \( m \in M \); it has a natural manifold structure modelled on \( GL(E) \times E \). We shall denote by \( e \) a generic element of \( GL(M) \), and by \( VGL(M) \) the vertical sub-bundle of \( TGL(M) \), canonically identified with the Lie algebra \( gl(E) = L(E) \) of \( GL(E) \). The connection \( \nabla \) on the bundle \( TM \to M \) is naturally lifted into a connection on the bundle \( \pi : GL(M) \to M \), still denoted by the same symbol. Remark that \( \pi_* \) is an isomorphism between the horizontal sub-bundle \( H^\pi GL(M) \) in \( TGL(M) \) and \( TM \). The horizontal distribution in \( TGL(M) \) can be used to define a continuous linear map \( F^\pi \) from \( E \) to the space of horizontal vector fields on \( TGL(M) \), defined by the requirement that \( F^\pi(e_\bullet ; a) \in T_e GL(M) \) is horizontal and corresponds to \( e(a) \in TM \), for any \( a \in E \) and \( e \in GL(M) \). This vector field valued 1-form \( F^\pi \) on \( E \) is called the canonical horizontal 1-form.

Once again, we refer the reader to the nice books \([13]\) and \([14]\) for the basics of differential geometry on Banach manifolds.

3.2. Canonical representation of rough integrators. Theorem \([4]\) below gives a canonical and parcimonious representation of a given weak geometric Hölder \( p \)-rough path, seen as a rough integrator. We assume for that purpose that the manifold \( M \) is endowed with a connection \( \nabla \). Given a 1-form \( F \) on \( U \) with values in the space of vector fields on \( M \) as in the above definition of a weak geometric Hölder \( p \)-rough path on \( M \), we denote by \( F_\bullet \) its lift to a 1-form with values in the space of \( \nabla \)-horizontal vector fields on \( GL(M) \). Parallel translation along a weak geometric Hölder \( p \)-rough path \( \Theta_\bullet = (x_t)_{0 \leq t < \zeta}, F, X \) is defined as the solution path to the rough differential equation in \( GL(M) \)

\[
de_t = F(e_t, X(dt))
\]
started from any given frame \( e_0 \in \text{GL}(M) \) above \( x_0 \). It is elementary to see that the paths \( e_* \) and \( x_* \) are defined on the same maximal interval.

Given another Banach manifold \( N \), and a 1-form \( G : N \times TM \to TN \), as used above to write down a rough differential equation in definition 2, define \( \mathcal{S} \) on \( \text{GL}(M) \times N \times U \) by the formula

\[
\mathcal{S}(e, y; u) = \left( F(e; u), G(y, \pi(e); \pi_* F(e; u)) \right),
\]

and \( \mathcal{S} \) on \( \text{GL}(M) \times N \times E \) by the formula

\[
\mathcal{S}(e, y; a) = \left( F(\nabla e; a), G(y, \pi(e); \pi_* F(\nabla e; a)) \right).
\]

**Theorem 4** (Canonical representation of rough integrators). Let \( M \) be a Banach manifold endowed with a connection \( \nabla \), and \( \Theta = (x_t)_{0 \leq t < \zeta}, F, X \) be a basic \( p \)-rough integrator on \( M \).

1. One defines a weak geometric Hölder \( p \)-rough path \( Z \) over \( E \), on the time interval \([0, \zeta)\), by solving the rough differential equation

\[
\frac{d}{dt} e_t = F(e_t; X(dt)),
\]

\[
\frac{d}{dt} Z_t = Z_t \otimes e_t^{-1} F(\pi(e_t); X(dt)).
\]

in \( \text{GL}(M) \times T^{[p]}(E) \), started from \((e_0, \text{Id})\).

2. The solution paths \((e_t, y_t)\) and \((\overline{e}_t, \overline{y}_t)\) in \( \text{GL}(M) \times N \) to the rough differential equations

\[
\frac{d}{dt} (e_t, y_t) = \mathcal{S}(e_t, y_t; X(dt)),
\]

\[
\frac{d}{dt} (\overline{e}_t, \overline{y}_t) = \mathcal{S}(\overline{e}_t, \overline{y}_t; Z(dt)),
\]

coincide if they start from the same initial condition.

Note there is no difficulty in stating a similar result for any \( p \)-rough integrator. The interesting point is that although the rough paths in each \( \Theta^{(i)} \) may be defined on different Banach spaces, the canonical representation of the concatenation of the \( \Theta^{(i)} \) only involves a rough path over \( E \).

**Proof** – The first claim follows from general principles. For the second point, pick \( p < p' < [p]+1 \). Using the continuity result for the Itô map recalled in section 2.1 in the topology of Hölder \( p' \)-rough path, together with the continuous embedding of the space \( W G_p(U) \) of weak geometric Hölder \( p \)-rough paths in \( U \) into the space \( G_{p'}(U) \) of geometric Hölder \( p' \)-rough paths in \( U \), and the continuous embedding of \( G_{p'}(U) \) into \( W G_{p'}(U) \), we see that it suffices to prove the claim when \( X \) is the weak geometric Hölder \( (p') \)-rough path associated with a smooth \( U \)-valued control. The result is clear in that case as it appears as a rephrasing of Cartan’s moving frame method, as described above in section 3.1.

\( \triangleright \)

So one can always understand the solution of a rough differential equation driven by a \( p \)-rough integrator \( \Theta \) as the solution to another rough differential equation driven by a \( p \)-rough integrator involvng a weak geometric Hölder \( p \)-rough path over
the model space $E$, and the canonical horizontal 1-form $F^\nabla$. The dependence on $F$ in the original $p$-rough integrator is hidden in the definition of $Z$ in this reformulation. Given $e_0$ above $x_0$, the "GL($M$)-valued" $p$-rough integrator $(e_t)_{0 \leq t < \zeta}, (F^\nabla, Z)$ is said to be the canonical representation of $\Theta$.

This result echoes one of the main results of [6], corollary 6.19, where a one-to-one correspondence between the classes of weak geometric rough path on a given finite $d$-dimensional manifold, weak geometric horizontal rough paths on its frame bundle, and classical weak geometric rough paths on $\mathbb{R}^d$, is proved in their setting, using Cartan’s development map as an essential ingredient. Applications of our setting will be developed in a forthcoming work.

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