Static Fundamental Solutions of Einstein Equations and Superposition Principle in Relativistic Gravity

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Abstract

We show that Einstein equations are compatible with the presence of massive point particle idealization and find the corresponding two parameter family of solutions. They are complete defined by the bare mechanical mass $M > 0$ and the Keplerian mass $m > 0$ ($m < M$) of the point source of gravity. The global analytical properties of these solutions in the complex plane define a unique preferable radial variable of the one particle problem.

These new solutions are fundamental solutions of the quasi-linear Einstein equations. We introduce and discuss a novel nonlinear superposition principle for solutions of Einstein equations and discover the basic role of the relativistic analog of the Newton gravitational potential. For the relativistic potential we introduce a simple quasi-linear superposition principle as a new physical requirement for the initial conditions for Einstein equations, thus justifying the instant gravistatic case for N particle system.

This superposition principle allows us to sketch a new theory of the gravitational mass defect. In it a specific Mach-like principle for the Keplerian mass $m$ is valid, i.e. it depends on the mass distribution in the universe, in contrast to the bare mass $M$, which remains a true constant. Several basic examples both of discrete and of continuous mass distributions are considered.

1 Introduction

1.1 Static Fundamental Solution and Superposition Principle in Newton Theory of Gravity

The notion of a static fundamental solution of a classical field equation appeared at first in Newton theory of gravity [I]. Such a solution solves the Poisson equation with source term, proportional to Dirac 3D $\delta$-function:

$$\Delta \varphi^{\text{Newton}}(\mathbf{r}) = 4\pi m \delta^{(3)}(\mathbf{r}).$$

Here and further on for simplification of the formulas we are using units in which the Newton gravitational constant $G^{\text{Newton}} = 1$ and velocity of light $c = 1$.
The static fundamental solution does not depend on arbitrary functions, or additional constants. It is unambiguously fixed, among all solutions of Eq. (1.1), if we require this solution to tend to zero at infinite distances. This unique solution describes the potential of the newtonian gravitational field

$$\phi_{\text{Newton}}(r) = -\frac{m}{r},$$

which is created by a classical point particle of gravitational (Keplerian) mass $m$, placed in Euclidean 3D space at the origin of coordinate system $r_0 = 0$. Here $r = |r| \geq 0$.

Analogous solutions are well known in the problem of static point source of electric field in Maxwell electrodynamics, as well. A proper generalization of the notion of fundamental solution for hyperbolic partial differential equations can be find, for example, in [2] and in the references therein.

According to well known mathematical results, the solutions (1.2) describe, too, the static field in vacuum, outside sources of finite dimension, assuming spherical symmetry of the corresponding distribution $\mu(r)$ of gravitational mass, or electric charge.

The fundamental role of the solutions (1.2) in Newton gravistatics is substantiate by the superposition principle, according to which in these linear theories the field of any aggregate of matter can be obtained as a sum of the fields of its constituent matter points at positions $r_A$ (or $r'$). For example, in Newton gravistatics

$$\phi_{\text{Newton}}(r; r_1, \ldots, r_N) = -\sum_{A=1}^{N} \frac{m_A}{|r - r_A|},$$

in the case of a set of $N$ discrete massive points, and

$$\phi_{\text{Newton}}(r) = -\int \frac{\mu_{\text{Kepler}}(r')}{|r - r'|} d^3r'$$

in the case of a continuous distribution of Keplerian mass $\mu_{\text{Kepler}}(r)$.

In a more general form the superposition principle in the newtonian gravity can be expressed as

**Proposition 1:** If $\mu_I(r)$ and $\mu_{II}(r)$ are two mass distributions, which create gravitational fields with corresponding potentials $\phi_{\text{Newton}}^I(r)$ and $\phi_{\text{Newton}}^{II}(r)$, then the potential of the field, created by mass distribution $\mu(r) = \mu_I(r) + \mu_{II}(r)$ is

$$\phi_{\text{Newton}}(r) = \phi_{\text{Newton}}^I(r) + \phi_{\text{Newton}}^{II}(r).$$

### 1.2 Massive Point Particle in General Relativity

#### 1.2.1 The Schwarzschild Solution

An attempt to solve the point-mass problem in general relativity (GR) was made at first ninety years ago, as early as in the pioneering article by Schwarzschild [3] and its subsequent modifications [4, 5].

Today it is well known that inconsistencies arise when we look at Schwarzschild solution as the space-time arising from localized point mass singularity [6]. Actually, the well known Schwarzschild metric in Hilbert gauge:
\[ ds^2 = \left( 1 - \frac{2m}{\rho} \right) dt^2 + \frac{dr^2}{1 - \frac{2m}{\rho}} - \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] (1.6)

solves the *vacuum* Einstein equations \( G^\nu_\mu = 0 \) in the spherically symmetric static case. It possesses an *event horizon* at \( \rho = \rho_G = 2m \) and a strong hidden *singularity* at \( \rho = 0 \). This solution describes a completely empty space-time with removed point \( \rho = 0 \) and nontrivial non-Euclidean topology. Indeed, in the Weyl’s isotropic coordinates with radial variable \( r_W = \frac{1}{2} \left( \rho - m + \sqrt{\rho(\rho - 2m)} \right) \) one can easily see that this solution describes a two sheeted space-time with a flat asymptotic at \( r_W = 0 \) and \( r_W = \infty \) connected by a specific bridge. Then the nonzero Keplerian mass \( m \) appears in the solution due to the nontrivial topology of the space-time, i.e. in the spirit of Einstein-Rosen-Misner-Wheeler geometrodynamics \[7\], as described by the sentence ”mass, without mass”.

The singularity at \( \rho = 0 \) is *not related* with a massive point particle with proper bare mass \( M \) and mechanical action \( A_M = -M \int ds \). Indeed, as we shall see in more detail below, the solution (1.6) does not solve the Einstein equations (EE)

\[ G^\nu_\mu = 8\pi T^\nu_\mu \] (1.7)

in presence of matter with stress-energy tensor \( T^\nu_\mu \sim M \delta^{(3)}(r) \). Here \( M \delta^{(3)}(r) \) describes the mass distribution of the point particle with proper bare mass \( M \).

In his pioneering article Schwarzschild has used another radial variable \( r \), which defers essentially from variable \( \rho \), i.e., his choice a radial gauge for the spherically symmetric static metric

\[ ds^2 = g_{tt}(r)^2 dt^2 + g_{rr}(r)^2 dr^2 - \rho(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] (1.8)

is different from the Hilbert’s one (1.6).

Borrowing from the Minkowskian flat space-time the gauge condition

\[ |^4g| := \det ||g_{\mu\nu}(t, r)|| = 1 \]

(which takes place there in Cartesian coordinates), Schwarzschild was able to fix the three unknown functions in the form:

\[ \rho(r) = \sqrt{r^3 + \rho_G^3} > 0, \quad g_{tt}(r) = 1 - \frac{2m}{\rho(r)} > 0, \quad \text{and} \quad g_{rr}(r) = -\rho'(r)^2/g_{tt}(r) < 0. \]

This solution of EE has *no event horizon*. Its peculiar feature is that it describes a *point like object* of Keplerian mass \( m > 0 \), zero radius, zero volume, but *nonzero area* \( A_{\rho_G} = 4\pi\rho_G^2 > 0 \). These unusual properties of the *original* Schwarzschild solution have been discussed by Brillouin \[8\] as early as in 1923.

At present the original Schwarzschild geometry and other similar geometries of space-time are widely ignored in GR. A main stream of articles in the last 40 years is strongly limited to consideration of the black hole interpretation of the Hilbert form (1.6) of vacuum Schwarzschild solution and its generalizations. In addition, essential features of the original Schwarzschild solution are not reproduced in the most of modern literature on this subject and remain hardly known.
1.2.2 On the Choice of Radial Variable in the GR Massive Point Particle Problem

One of the basic problems in the description of single massive point particle as a source of gravitational field in GR is the choice of proper radial variable \( r \).

The quantity \( \rho \geq 0 \) has a clear geometrical and physical meaning:

i) It is well known that \( \rho \) defines the area \( A_\rho = 4\pi \rho^2 \) of a centered at \( r = 0 \) sphere with ”area radius” \( \rho \) and the length of the big circle on it \( l_\rho = 2\pi \rho \). Thus we see that the quantity \( \rho \) has a well defined geometrical meaning and is a gauge invariant notion.

ii) The coordinate \( \rho \) measures the curvature \( \sim 1/\rho^2 \) of the 2D-manifolds (2D-spheres) in 3D Riemannian space, which are invariant under rotations around the center of spherical symmetry. It measures, too, the curvature of the 4D pseudo-Riemannian space-time: \( ^4R = ^4R(\rho) \) and of the corresponding 3D-space: \( ^3R = ^3R(\rho) \) in the spherically symmetric case, inside the matter source of finite dimension. Hence the name ”curvature radius”.

iii) From physical point of view one can refer to \( \rho \) as an optical ”luminosity distance”, because the luminosity \( L \) of distant physical objects is reciprocal to \( A_\rho \):

\[
L \sim \frac{1}{\rho^2}\.
\]

In contrast, the physical and geometrical meaning of the coordinate \( r \) is not defined by the spherical symmetry of the problem and is unknown a priori [9]. Its choice has been discussed from physical point of view by Eddington as early as in [10]. His conclusion was that all admissible variables \( r \) are practically equivalent at distances \( r >> \rho_G \), since under suitable coherent choice of their scales we have \( \rho/r \to 1 \) when \( r \to \infty \).

The following assumptions about the mathematical properties of the radial variable \( r \) of the single point particle problem seem to be natural from physical point of view:

i) Its value \( r = 0 \) is to correspond to the center of the symmetry, where one must place the physical source of the gravitational field – the massive point particle.

ii) The radial variable is to vary in the semi-bounded physical interval \( r \in [0, \infty) \).

iii) The luminosity variable \( \rho(r) \) is to increase monotonically to infinity in this interval, together with radial ones, i.e., \( d\rho/dr > 0 \), and, in addition, one has to impose the Eddington condition:

\[
\lim_{r \to \infty} \frac{\rho(r)}{r} = 1.
\] (1.9)

iv) The infinite value \( r = \infty \) of the radial variable is to be prescribed to the boundary of the asymptotically flat domain of space-time.

v) There must not exist non-physical singularities of the solution of EE in the whole compactified complex domain \( \tilde{C} \) of the radial variable \( r \in \tilde{C} \).

In the present article we show that these physical requirements define in a unique way the radial variable \( r \) of the problem at hand, thus solving the corresponding uniformization problem in the case of point particle source of gravity in GR.

1.2.3 Some Remarks on the Massive Point Particle Idealization in GR

A clear physical motivation for consideration of massive point particle sources of gravitational field in GR, both electrically neutral and charged ones, can be found in 1962-63 Feynman lectures on gravity [11]. The energy momentum tensor of a point particle has been used in the excellent textbooks by Landau & Lifshiz and by Weynberg [1] as a tool for treatment of many particle systems in GR. In spite of this fact the single particle case is still an open problem in GR.
Moreover, at present the vast majority of relativists do not accept the consideration of point particles in GR, assuming that it is an idealization, which is incompatible with EE \cite{12}. There are different reasons:

i) Some doubts about consistence of the theory of mathematical distributions (like 3D Dirac $\delta$-function $\delta^{(3)}(\mathbf{r})$) \cite{13} with the obviously nonlinear character of EE \cite{14}.

The formal mathematical problems, which emerge when one attempts to work with distributions in EE were successfully advanced in the last decade using Colombeau’s theory of generalized functions \cite{15}. Unfortunately, the published results on the point particle problem in GR, based on this approach, are physically incorrect (see Section 2.3.5).

ii) The clear understanding that an infinite concentration of energy in a single space-point will change drastically the geometry of the GR-Riemannian space-time $M^{(1,3)}\{g_{\mu\nu}\}$ in a small vicinity of the world-line of this point;

iii) Some attempts to neglect the role of classical description of matter in GR, replacing it by classical field description, or by quantum field description, according to the so called ”third approach” by Einstein, Wheeler and many others, see \cite{7} and references therein;

iv) The absence of understanding of necessity to use fundamental solutions of EE. These were unknown up to recently in GR, but may turn to be useful mathematical tool.

v) The absence of a general non-linear superposition principle for EE, which is to correspond to the linear superposition principle (1.3) – (1.5) in Newton theory of gravity. Note that a specific kind of superposition principle for initial conditions of black hole solutions is well known \cite{7}, but the general problem and other specific cases are still not studied, to the best of our knowledge.

On the other hand, it is obvious that in Nature the very distant stars look like ”points” of finite mass and finite luminosity. This fact has a proper mathematical description in the language of mathematical distributions in the Newton theory of gravity, but still not in GR.

In spite of the absence of proper description of the massive point particles, in the practical relativistic celestial mechanics, for example, in the calculations of the solar-system trajectories of the space crafts, even the Sun is considered often as a massive point source of gravity.

A formal mathematical problem is to find the corresponding correct treatment of such objects in GR, but up to recently no reasonable approach was known. Unfortunately, the most of the existing formal attempts to solve the point particle problem in GR do not take into account one essential physical difference between the GR and the Newtonian description of the massive objects. It is well known \cite{11}, that any body in GR has two different masses: the Kepelrian one $m$, as seen from distant observer, and the proper (bare) mass $M > m$, which is the sum of the masses of its constituents, when placed at infinite distances between them, i.e., with gravitational interaction – turned off. The difference $M - m$, or the ratio $\rho = m/M$ describe the gravitational defect of mass. One must include properly this specific feature of the relativistic theory of gravity in the GR-point-particle model. To the best of our knowledge, such attempts ware not made up to recently, with the only exception – \cite{16}.

In the present article we show that a correct mathematical solutions of EE with $\delta^{(3)}(\mathbf{r})$ term in the rhs do exist. Such solutions describe a two parameter family of analytical space-times $M^{(1,3)}\{g_{\mu\nu}\}$ with a specific strong singularity at the place of the massive point source with bare mechanical mass $M > 0$ and Keplerian mass $m < M$. 
Indeed, to reproduce the δ topology in the spirit of the Einstein-Rosen-Wheeler geometrodynamics [7], the original Schwarzschild solution, which describes an empty space-time with nontrivial are essentially different from the ones of the most popular, at present, Hilbert form (1.6) of the original Schwarzschild solution, which describes an empty space-time with nontrivial topology in the spirit of the Einstein-Rosen-Wheeler geometrodynamics [7].

As we have stressed already, this geometry was introduced in GR for the first time actually in the original Schwarzschild article [3] and has been discussed by Brillouin [8]. The unusual geometry is essentially different from the geometry around the space-time points with finite energy density in them.

The global properties of the space-time manifolds, generated by massive point source, are essentially different from the ones of the most popular, at present, Hilbert form (1.6) of the original Schwarzschild solution, which describes an empty space-time with nontrivial topology in the spirit of the Einstein-Rosen-Wheeler geometrodynamics [7].

i) To accept the unusual geometry of the space-time around the matter point with infinite concentration of energy in it.

As we have stressed already, this geometry was introduced in GR for the first time actually in the original Schwarzschild article [3] and has been discussed by Brillouin [8].

ii) To allow consideration of metrics, whose coefficients are not a C³-smooth functions. Indeed, to reproduce the δ(r) term in the rhs of EE, the metric tensor, and/or its derivatives, related to the geometry of the Riemannian space-time, must have a definite singularities (discontinuities) at the place of the point source of gravity [16].

iii) To replace the mathematical theory of the real smooth manifolds, which is in current use in GR, with the theory of the analytical manifolds with proper singular points, considering the whole complex domain of the space-time variables.

1.2.4 The Nonlinear Superposition Principle in GR

In addition, here we formulate a GR-nonlinear superposition principle, analogous to the Newton one, described in Proposition 1.

Let’s consider for simplicity only the case of asymptotically flat space times, which correspond to energy-momentum stress tensors $T_\mu^\nu(x)$ with compact support in $M^{(1,3)} \{ g_{\mu\nu} \}$, i.e. let’s focus our attention on so called ”island universes”. It is well known that after a proper fixing of the gauge, the boundary conditions at infinity define the solutions of EE (1.7) in a unique way, see for example [14, 1] and the references therein.

Proposition 2: Let $g_{\mu\nu}(x)_I$ and $g_{\mu\nu}(x)_{II}$ are two metrics, which correspond via EE (1.7) to two energy-momentum tensor distributions $T_\mu^\nu(x)_I$ and $T_\mu^\nu(x)_{II}$ of compact supports. Then the metric $g_{\mu\nu}(x)$ of the GR gravitational field, created by energy-momentum tensor distribution $T_\mu^\nu(x) = T_\mu^\nu(x)_I + T_\mu^\nu(x)_{II}$ via EE (1.7) is uniquely defined by the two metrics $g_{\mu\nu}(x)_I$ and $g_{\mu\nu}(x)_{II}$.

Thus we obtain the unambiguous correspondence

\[ \{ g_{\mu\nu}(x)_I, g_{\mu\nu}(x)_{II} \} \mapsto g_{\mu\nu}(x) \quad (1.10) \]

and the metric $g_{\mu\nu}(x)$ deserves to be called a nonlinear superposition of the metrics $g_{\mu\nu}(x)_I$ and $g_{\mu\nu}(x)_{II}$.

The essence of the proof of the existence of such nonlinear superposition principle in GR is in the simple note that the support of the distribution $T_\mu^\nu(x) = T_\mu^\nu(x)_I + T_\mu^\nu(x)_{II}$ will be certainly compact, if both $T_\mu^\nu(x)_I$ and $T_\mu^\nu(x)_{II}$ have compact supports in $M^{(1,3)} \{ g_{\mu\nu} \}$. Then the EE (1.7) with $T_\mu^\nu(x)$ in rhs, supplied with asymptotically flat space-time boundary conditions, will have an unique solution $g_{\mu\nu}(x)$, which corresponds to the metrics $g_{\mu\nu}(x)_I$ and $g_{\mu\nu}(x)_{II}$ and ought to be named their (nonlinear) superposition due to obvious physical reasons. It is clear that $g_{\mu\nu}(x)$ is a very complicated functional of the metrics $g_{\mu\nu}(x)_I$ and $g_{\mu\nu}(x)_{II}$. The problem of reconstruction of $g_{\mu\nu}(x)$ in (1.10), using
two given metrics $g_{\mu\nu}(x)_I$ and $g_{\mu\nu}(x)_II$ is highly nontrivial. We shall use the symbol $\circledast$ to denote the composition (1.10) of the two metrics in the form

$$g_{\mu\nu}(x) = g_{\mu\nu}(x)_I \circledast g_{\mu\nu}(x)_II. \quad (1.11)$$

It is clear that by construction this new operation on the metrics is symmetric and associative:

$$g_{\mu\nu}(x)_I \circledast g_{\mu\nu}(x)_II = g_{\mu\nu}(x)_II \circledast g_{\mu\nu}(x)_I,$$

$$\left( g_{\mu\nu}(x)_I \circledast g_{\mu\nu}(x)_II \right) \circledast g_{\mu\nu}(x)_III = g_{\mu\nu}(x)_II \circledast \left( g_{\mu\nu}(x)_I \circledast g_{\mu\nu}(x)_III \right). \quad (1.12)$$

These properties are an immediate consequences of the corresponding properties of the summation of energy-momentum tensors of compact supports in $\mathbb{M}^{1,3}$, assuming that we are considering space-times with a fixed flat-geometry-boundary-conditions at space infinity.

Finally, we can define correctly the superposition of an arbitrary number $N$ of metrics, obeying the same boundary conditions at infinity, i.e. we can introduce a multiple superposition operation:

$$g_{\mu\nu}(x) = \circledast_N A=I g_{\mu\nu}(x)_A := g_{\mu\nu}(x)_I \circledast \ldots \circledast g_{\mu\nu}(x)_N. \quad (1.13)$$

To some extent the novel principle (1.11), (1.13) is unexpected, and certainly much more complicated than the simple linear superposition principle (1.5) in Newton gravity.

Indeed, in GR we have a very specific physical situation. It is clear that even if the metrics $g_{\mu\nu}(x)_I$ and $g_{\mu\nu}(x)_II$ are static, in general case their superposition $g_{\mu\nu}(x)$ is not a static metric. It contains the whole GR dynamics, including the possible radiation of gravitational waves, due to the gravitational interaction between the physical sources of the metrics $g_{\mu\nu}(x)_I$ and $g_{\mu\nu}(x)_II$. In contrast to the situation in electrodynamics, where we can introduce non-electrodynamical forces and stresses to keep the composite aggregate of charges in a static state without introducing new terms in the Maxwell equations, in GR any additional interactions, introduced for the same purpose, will have a nonzero energy-momentum tensor $T^\nu_\mu(x)_{\text{additional}}$, which enters the rhs of EE and changes the space-time geometry and the very problem. As a result we see that without introducing of non-gravitational interactions between particles in GR we have only a unique (whole-time) static case – the single point particle problem (see, for example, Fock in [11]).

Nevertheless, as we shall show in the present article, considering just the instant static case, one can introduce a simple quasi-linear superposition principle for static fundamental solutions in GR. It reveals the role of static fundamental solutions of EE in GR, which is much like the role of corresponding fundamental solutions in linear field theories like Newton gravity and Maxwell electrostatics, if one considers only a single 3D space-time surface $t = 0$ [17].

The resolution of the relativistic gravistatic problem requires the solution only of the well known suitable form of $tt$-EE. It does not contain second derivatives of metric with respect to the time variable $t$. Let us consider a space-time, which is a solution of EE. The 3D curvature $(3)R$ of arbitrary 3D space-like surface in it obeys the well known basic equation

$$(3)R + K_2 = 16\pi\mu. \quad (1.14)$$
Here \( K_2 = K^2 - K_{ij} K^{ij} \) (where \( K = g^{ij} K_{ij} \)) is the exterior curvature of the 3D surface and \( \mu \) is the relativistic density of mass distribution.

If the last equation is fulfilled at some time instant \( t \), as a consequence of the EE it will be fulfilled at all time instants \( t \in (-\infty, \infty) \), for which the problem is well defined. There exists an inverse theorem, too: The whole system of EE, which governs dynamics in GR, may be derived by the requirement to have the above relation \( (1.14) \) co-variantly valid at all time instants \([18]\).

According to articles \([17]\), one can define the relativistic gravistatics as a description of time-symmetric initial value problem for EE in proper coordinates, as well. To see this, it is enough to know that choosing appropriate coordinates outside the 3D surface \( t = 0 \) one obtains for the coefficients of the second fundamental form of this surface \( K_{ij} = -\frac{1}{2} \partial_t g_{ij} \). Then one defines the instantaneous-static solutions of EE, possessing a 3D space-like surface \( t = 0 \) on which \( K_{ij} = 0 \). The coordinate independent way to this definition implies existence of isometry of space-time: \( t \to -t, r \to r \) \([17]\). Then the basic equation reduces to the following simple form, valid at time instant \( t = 0 \):

\[
(3) R = 16\pi\mu. \tag{1.15}
\]

This equation is not a dynamical equation, but just a constraint on the initial conditions – a specific relativistic constraint equation (RCE). As a result of relativistic dynamics, governed by the other EE, the RCE will be automatically fulfilled for any time \( t \), if it will be valid at time instant \( t = 0 \) \([14][1]\). Hence, the time \( t \) is a simple auxiliary parameter in the RCE and in its solutions. Thus we see that it is enough to solve the RCE only at the initial time instant \( t = 0 \), i.e., it is enough to solve the Eq. \((1.15)\).

From pure mathematical point of view any of the solutions of Eq. \((1.15)\) may be considered as an initial condition of a proper initial value problem for EE. One of the basic purposes of present article is to find the physically meaningful solutions of RCE among the whole variety of its possible mathematical solutions. These physical initial conditions were not known until now. Their discovery calls for reconsideration of many well studied problems in GR, including the gravitational collapse problem.

We present here an instant static solutions of the RCE with singularities, which correspond to presence of arbitrary number of massive matter points, both of discrete or of continuous distribution. These solutions define the physically meaningful initial conditions for EE, which describe the real matter, made of massive point particles. They present a very special class in the variety of all initial condition, which are admissible from pure mathematical point of view.

To obtain the non-stationary gravitational field of moving matter point sources, the instant solutions of RCE can be modified in a manner, which is well known from relativistic electrodynamics. Of course, the whole problem is a highly complicated and still not solved. One can hope that in GR a procedure, which is analogous to the introduction of Liéanard-Wiechert potentials may take place. Here we shall stress that in electrodynamics for this purpose the static Colomb potential is in use. We believe that in a similar way our fundamental solutions may turn to be the key tool for the correct treatment of the nonlinear GR dynamics of many-particle systems, which is still an open problem.

As already mentioned, in GR, due to gravitational mass defect, we have to distinguish two different masses of every body – the Keplerian gravitational mass \( m \) and the proper bare mass \( M > m \) of the body \([1]\). Under some additional natural assumptions our new quasi-linear superposition principle for the fundamental solution of EE yields a novel
theory of the relativistic gravitational mass defect of systems of discrete matter points and composite bodies of continuous mass distribution. It is based on a specific integral equation for the relativistic gravitational potential, derived for the first time here. It turns out that the GR mass defect is governed completely by the RCE, with time $t$, playing the role of an auxiliary parameter in its solutions. Thus, our quasi-linear superposition principle for the fundamental solution of EE has a basic impact on the relativistic theory of gravitational defect of mass. We give here for the first time a number of solutions of the integral equation for the relativistic mass defect. These describe some basic physical problems: two-point-particle problem, some special cases of three and four point particle problems, the general properties of the $N$-point-particle problem, as well as some basic examples of continuous mass distribution: homogeneous massive circles, spheres and balls.

Further important physical consequences, which can be derived using the new fundamental solutions of EE and the corresponding nonlinear superposition principle for them, will be considered elsewhere.

2 The Mathematical Problem of Single Point Particle in GR

2.1 The Total Action and Introduction of Coordinates

Let us suppose that in the whole universe there exist only a single massive point particle of bare mechanical mass $M$, and that it creates its own gravitational field according to the laws of GR. This problem is described by the total action $A_{tot} = A_{GR} + A_{M}$. The first term describes the action of the gravitational field, created by the single particle. The second term adds to the total action the pure matter (mechanical) action of the massive particle. Thus in GR the total action acquires the well known explicit form:

$$A_{tot} = -\frac{1}{16\pi} \int d^4x \sqrt{|g|} R - M \int ds.$$  

(2.1)

In the rest frame of the point particle both the total action (2.1) and the formed by this particle GR space-time manifold $\mathbb{M}^{(1,3)}\{g_{\mu\nu}\}$ have an obvious group of symmetry $SO(3) \times T_t(1)$. As a result, the problem can be reduced not only on the orbits of the group $SO(3)$, i.e. on the 2D quotient space $\mathbb{M}^{(1,1)} = \mathbb{M}^{(1,3)}/SO(3)$, with natural global coordinates $t$ and $r$, but even on the orbits of the whole group $SO(3) \times T_t(1)$, i.e. on the 1D quotient space $\mathbb{M}^{(1)} = \mathbb{M}^{(1,3)}/(SO(3) \times T_t(1))$, with some natural radial coordinate $r$.

To be able to use some coordinates $x = \{x^\mu\}$ in the Riemannian space-time $\mathbb{M}^{(1,3)}\{g_{\mu\nu}\}$ of the point particle problem, one actually presupposes to have a flat Minkowskian space-time $\mathbb{E}^{(1,3)}\{\eta_{\mu\nu}\}$, endowed with the same coordinates. For example, one assumes to borrow the Cartesian coordinates: $\{t, r\}$, or the spherical ones: $\{t, r, \theta, \phi\}$ from the flat Minkowskian space-time for the use in the Riemannian space-time. Thus we have at our disposal simultaneously a flat metric $\eta_{\mu\nu}(x)$, and a Riemannian metric $g_{\mu\nu}(x)$, expressed in the same coordinates.

One of the basic results of present article is that the auxiliary flat space-time $\mathbb{E}^{(1,3)}\{\eta_{\mu\nu}\}$ plays much more profound role in the problem at hand, than the usually expected formal one. In particular, it turns out that the real geometrical points of the two space-times: $\mathbb{E}^{(1,3)}\{\eta_{\mu\nu}\}/W_0$, with the world line $W_0$ of one point (the origin $r = 0$) removed, and
the space-time of the GR massive point particle problem, are in one-to-one correspondence. In particular, the last has the same topology as the first one. Note that the Galilean space-time of a single point particle problem in Newtonian gravity has precisely the same topology as the one of the manifold $E^{(1,3)}\{\eta_{\mu\nu}\}/W_0$. Thus, in the real domain of variables, the space-time in the Newton gravity and in the GR, formed by a single massive point particle, have the same topology. This observation makes it clear that even because of pure topological reasons the black hole solutions of EE are not compatible with the matter point sources of gravity, since they have a different topology.

In its proper frame the single massive point particle, placed at the origin of the standard spherical coordinate system in the 3D Riemannian space $M^{(3)}\{g_{ij}\} \subset M^{(1,3)}\{g_{\mu\nu}\}$ yields the familiar static metric \[ (1.8) \] with three unknown functions $g_{tt}(r) \geq 0$, $g_{rr}(r) \leq 0$, and $\rho(r) \geq 0$ of the radial variable $r \geq 0$. The variable $r$ is not defined by the $SO(3)$ symmetry of the problem, nor by its global-time translation invariance with respect to the group $T_t(1)$. From geometrical point of view the choice of the function $\rho(r)$ fixes the imbedding of the quotient space $M^{(1)} = M^{(3)}/SO(3)$ into the 3D space $M^{(3)}$. We assume that by definition the value $r = 0$ of the radial variable $r$ corresponds to the center of spherical symmetry, $C$. There the massive matter point is placed. We also accept other assumptions about the mathematical properties of the radial variable $r$, listed in the Subsection 1.2.2 of the Introduction.

2.2 On the Role of the Gauge Fixing in the Massive Point Problem

General relativity is a gauge theory. The fixing of the gauge in GR is described by a proper choice of the quantities

\[ \bar{\Gamma}_\mu = -\frac{1}{\sqrt{|g|}}g_{\mu\nu}\partial_\lambda \left( \sqrt{|g|}g^{\lambda\nu} \right) \]

in the 4D d’Alembert operator $g^{\mu\nu}\nabla_\mu \nabla_\nu = g^{\mu\nu} \left( \partial_\mu \partial_\nu - \bar{\Gamma}_\mu \partial_\nu \right)$, and actually is a fixing of the coordinates. In our problem the choice of spherical coordinates and static metric dictates the form of three of the quantities $\bar{\Gamma}_\mu$: $\bar{\Gamma}_t = 0$, $\bar{\Gamma}_\theta = -\cot \theta$, $\bar{\Gamma}_\phi = 0$, but the function $\rho(r)$ and, equivalently, the form of the quantity

\[ \bar{\Gamma}_r = \left( \ln \left( \frac{\sqrt{-g_{rr}}}{\sqrt{g_{tt}}} \rho^2 \right) \right)' \]

are still not fixed. Here and further on, the prime denotes differentiation with respect to the variable $r$. We refer to the freedom of choice of the function $\rho(r)$ as a rho-gauge freedom in a broad sense, and to the choice of the $\rho(r)$ function as a rho-gauge fixing.

At first glance the function $\rho(r)$ may be chosen in quite arbitrary way, thus fixing the remaining (radial) gauge freedom of the problem – the only one, which is not fixed by symmetry reasons. We show that choosing a definite class of functions $\rho(r)$ one can solve correctly the EE (1.7) with stress-energy tensor

\[ T^\nu_\mu = M \delta^{(3)}_\delta(r) \delta^0_\nu \delta^0_\mu = M \frac{\delta^{(3)}(r)}{\sqrt{\left| g(r) \right|}} \delta^\nu_\nu \delta^0_\mu, \quad (2.2) \]
which describes a massive point source with bare mass $M$ at rest in stationary and static coordinates. It may seem strange that for solving this problem, one needs to fix the class of coordinates by a proper choice of the radial gauge. As we shall see, the choice of the admissible class of radial coordinates $r$ is a consequence of the boundary conditions. In the problem at hand these conditions are masked in 3D Dirac $\delta$-function in (2.2). It describes in a formal mathematical way the properties of source of gravity and its boundary.

The following comments throw an additional light on this delicate issue:

1. The strong believe in the independence of the GR results on the choice of coordinates $x$ in the space-time $M^{(1,3)}\{g_{\mu\nu}(x)\}$ predisposes us to a somewhat light-head attitude towards the choice of the coordinates for a given specific problem. Indeed, it is obvious that physical results of any theory must not depend on the choice of the variables and, in particular, these results must be invariant under any admissible changes of the coordinates. This requirement is a basic principle in GR. It is fulfilled in any already fixed mathematical problem.

2. Nevertheless, the change of the interpretation of the variables may change the formulation of the very mathematical problem and thus, the physical results. This can happen, because we are using the variables according to their meaning. For example, if we are considering the luminosity distance $\rho$ as a radial variable of the problem, it seems natural to put the point source at the point $\rho = 0$. In general, we may obtain a physically different problem, if we are considering another variable $r$ as a radial one. In this case we shall place the source at a different geometrical point $r = 0$, which now seems to be the natural position for the center $C$, but does not coincide with the previous one – $\rho = 0$. Imposing the same physical requirements, i.e. the same boundary conditions at different places in the space, we obviously will obtain different physical problems and results. Of course, as in any gauge theory, in GR there exist a classes of physically equivalent gauges. All gauges (coordinates) in such class yield the same physical results. The real problem is how to find the correct class of the gauges, proper for the given physical configuration.

3. The relation between the two geometrical ”points”: $\rho = 0$ and $r = 0$, and between the corresponding physical models of a point particle, strongly depends on the choice of the class of functions $\rho(r)$, i.e. on the class of the radial gauges. Thus, applying the same physical requirements in essentially different ”natural” variables, we arrive at different physical models, because we are solving EE under different boundary conditions, coded in corresponding 3D Dirac $\delta$-functions in (2.2). One has to find a theoretical or an experimental reasons to resolve this essential ambiguity.

4. The choice of the radial coordinate in the single point particle problem in GR needs a careful analysis. It is essential for the description of the very source of gravitational field, not for the description of the field in surrounding this source vacuum domain. A well known mathematical fact is that in the vicinity of a definite singular point of a mathematical functions one must use a definite special type of coordinates for adequate description of the character of the singularity, i.e., one is to solve the corresponding uniformization problem.

5. The solutions of EE in essentially different coordinates have different singularities somewhere in the whole complex domain of the corresponding variables. The essentially different coordinates may be equivalent only locally – in the spirit of the widely used theory of smooth manifolds. One ought to make a reservation, speaking about ”essentially different coordinates”, because there exist a coordinate changes, which alter only the place of the singularities of the solutions of EE in the complex domain of the variables, without
varying the character and the number of these singularities. Such changes are precisely the linear ones and the fractional-linear ones. All other, more general coordinate changes, do not possess such property and yield essentially different coordinates in the whole complex domain.

6. In our particular problem, according to Birkhoff theorem, the spherically symmetric solution with given Keplerian mass \( m \) is unique in the vacuum domain. The coordinates, which are essentially different somewhere else, may be locally equivalent in the vacuum domain. As a result, all local GR effects, like gravitational redshift, perihelion shift, deflection of light rays, time-delay of signals, etc., will have their standard exact values in static spherically symmetric gravitational field with given Keplerian mass \( m \). These physical values do not depend on the admissible coordinate form of the solution.

We will use this local gauge freedom in description of the gravitational field outside the source to reach an adequate mathematical modelling of the very point source.

2.3 The Gravitational Field Equations and Their Solution

2.3.1 The Vacuum Solution in an Arbitrary Radial Gauge

The EE (1.7) for our problem with metric (1.8) can be easily derived from the following form of the nonzero components of Einstein tensor:

\[
G^t_t = \frac{1}{-g_{rr}} \left( -2 \left( \frac{\rho'}{\rho} \right)' - 3 \left( \frac{\rho'}{\rho} \right)^2 + 2 \frac{\rho' \sqrt{-g_{rr}}}{\rho \sqrt{-g_{rr}}} \right) + \frac{1}{\rho^2},
\]

\[
G^r_r = \frac{1}{-g_{rr}} \left( - \left( \frac{\rho'}{\rho} \right)^2 + 2 \frac{\rho' \sqrt{g_{tt}}}{\rho \sqrt{g_{tt}}} \right) + \frac{1}{\rho^2},
\]

\[
G^\theta_\theta = G^\phi_\phi = \frac{1}{-g_{rr}} \left( \left( \frac{\rho'}{\rho} \right)' - \left( \frac{\rho'}{\rho} \right)^2 - \frac{\rho' \sqrt{g_{tt}}}{\rho \sqrt{g_{tt}}} + \frac{\rho' \sqrt{-g_{rr}}}{\rho \sqrt{-g_{rr}}} \right),
\]

using, in addition, the corresponding components of the energy-momentum tensor (2.2) of a single matter point.

In particular, solving the EE for the static, spherically symmetric case in vacuum, one easily obtains the following most general solution:

\[
g_{tt}(r) = 1 - \frac{2m}{\rho(r)} > 0, \quad g_{rr}(r) = - (\rho'(r))^2 / g_{tt}(r) < 0,
\]

\[
\rho(r) \text{ an arbitrary } C^1 \text{ function.}
\]

Using, in addition, the corresponding components of the energy-momentum tensor (2.2) of a single matter point.

It was derived for the first time already in the articles [19].

If one uses the Hilbert gauge \( \rho_H(r) = r \) in the EE with \( \delta(r) \) term in the rhs, one easily reaches a contradiction [6, 16]. Hence, now the question is as to how to choose the radial-gauge-fixing-function \( \rho(r) \), to be able to comply with the specific boundary conditions at \( r = 0 \), coded in the \( \delta(r) \) term in the rhs of EE. In other words, we have to find a radial gauge, which makes the boundary problem for EE consistent with the presence of matter point source of gravity.
2.3.2 Normal Coordinates for Static Spherically Symmetric Gravitational Field

The expressions (2.3) demonstrate a very important feature of EE: In spite of their non-linearity, which may yield doubts in the applicability of the theory of mathematical distribution, EE are quasi-linear differential equations. After all, the higher (second) order derivatives of the unknown functions enter these equations linearly. This makes possible the usage of mathematical distributions \[13\] in the GR massive point particle problem in some specific coordinates and the usage of the Colombeau’s theory of the generalized functions \[15\], hopefully in all admissible coordinates.

The fundamental solutions of EE were found for the first time in the articles \[16\], introducing (in a slightly different notations) proper normal field variables \(\varphi(r), \varphi_2(r)\) and \(\bar{\varphi}(r)\) according to the formulas

\[
\begin{align*}
g_{tt} &= \exp(2\varphi), \quad \rho = \bar{\rho} \exp(-\varphi + \varphi_2), \quad g_{rr} = -\exp(-2\varphi + 4\varphi_2 - 2\bar{\varphi}).
\end{align*}
\]

(2.5)

Here \(\bar{\rho} = \text{const} > 0\) defines the scale of the luminosity variable.

In the present article we develop a more general approach to the fundamental solutions of EE, which allows consideration of arbitrary number of massive point sources of gravity. We shall see that an essential ingredient of this approach remains the Fock conformal transformation of the space \(M^{(3)}\), see Fock in \[1\]. It arises naturally when one puts the gravitational action of the static spherically symmetric problem to a canonical form \[16\]. Indeed, after reduction of the Hilbert-Einstein action \(A_{GR}\) on the orbits of the group \(SO(3) \times T_t(1)\), we arrive at one dimensional variational problem with "Lagrangian"

\[
\mathcal{L} = \frac{1}{2} \left( \frac{2\rho \rho'}{\sqrt{-g_{rr}}} + (\rho')^2 \sqrt{g_{tt}} \right) + \sqrt{g_{tt}} \sqrt{-g_{rr}}.
\]

(2.6)

The corresponding Euler-Lagrange equations for pure gravitational field in vacuum read:

\[
\begin{align*}
\left( \frac{2\rho \rho'}{\sqrt{-g_{rr}}} \right)' - \frac{\rho'^2}{\sqrt{-g_{rr}}} - \sqrt{-g_{rr}} &= 0, \quad (2.7a) \\
\left( \frac{(\rho \sqrt{g_{tt}})'}{\sqrt{-g_{rr}}} \right)' - \frac{\rho' (\sqrt{g_{tt}})'}{\sqrt{-g_{rr}}} &= 0, \quad (2.7b) \\
\frac{2\rho \rho'}{\sqrt{-g_{rr}}} + (\rho')^2 \sqrt{g_{tt}} - \sqrt{g_{tt}} \sqrt{-g_{rr}} = 0, \quad (2.7c)
\end{align*}
\]

where the symbol \(\equiv\) denotes a weak equality in the sense of theory of constrained dynamical systems. As a result of the rho-gauge freedom the field variable \(\sqrt{-g_{rr}}\) is not a true dynamical variable but rather plays the role of a (specific nonlinear) Lagrange multiplier, which is needed in a description of constrained dynamics. Its derivative with respect to the radial variable \(r\) does not enter the Lagrangian \(2.6\). An advantage of such derivation of field equation is that it makes transparent this fact. Of course, the equations \(2.7\) are completely equivalent to the vacuum EE, considered in the previous subsection.

As a result equations \(2.7\) are solved by the functions \(2.4\).

Let us consider the formal 2D space \(M^{(2)}\) of the field variables \(\sqrt{g_{tt}}\) and \(\rho\), endowed with the quadratic metric form \(\frac{2\rho}{\sqrt{-g_{rr}}(\rho)} d\rho d(\sqrt{g_{tt}}) + \frac{\sqrt{g_{tt}}}{\sqrt{-g_{rr}}(\rho)} d\rho^2\). It is easy to check that
its Riemannian curvature tensor is zero. Hence, in this space one can introduce a normal field variables, transforming its 2D metric into canonical form. The above change of variables yields the corresponding diagonal form of the Lagrangian:

$$\mathcal{L} = \frac{1}{2} e^{\varphi} \left[ -(\bar{\varphi} \varphi')^2 + (\bar{\varphi} \varphi')^2 \right] + e^{-\varphi} e^{2\varphi_2}.$$ (2.8)

Hence, the new field variables play the role of a normal fields’ variables for the problem at hand. In these variables the metric acquires the form

$$ds^2 = e^{2\varphi} dt^2 - e^{-2\varphi + 4\varphi_2 - 2\varphi} dr^2 - r^2 e^{-2\varphi + 2\varphi_2} (d\theta^2 + \sin^2 \theta d\phi^2) = e^{2\varphi} dt^2 - e^{-2\varphi} dl^2$$ (2.9)

where $\varphi(r)$, $\varphi_2(r)$ and $\bar{\varphi}(r)$ are still unknown functions of the variable $r$.

Obviously, the variable $\varphi$ describes the Fock conformal transformation to the 3D space with infinitesimal distance $dl$. The variable $\bar{\varphi}$ is not a dynamical one and fixes the radial gauge. We define a basic radial gauge (BRG) via the relation $\bar{\varphi}_{BRG}(r) \equiv 0$. In BRG the coefficients of the diagonal kinetic term in (2.8) are constant.

### 2.3.3 Solution of Einstein Equations for Single Massive Point Source

**a) Distributional form of EE for Point Particle.**

Let us consider EE (1.7), rewritten in the form

$$R^\mu_\nu - 8\pi \left( T^\mu_\nu - \frac{1}{2} T \delta^\mu_\nu \right) = 0.$$ (2.10)

Since the energy-momentum tensor (2.2) of the problem is a distribution: $T^\mu_\nu(x) \in \mathcal{D}'(\mathbb{M}^{1,3}) \{g_{\mu\nu}(x)\}$, the correct mathematical treatment requires to consider tensor-valued test functions $\Psi^\mu_\nu(x) \in \mathcal{D}'(\mathbb{M}^{1,3}) \{g_{\mu\nu}(x)\}$ and to rewrite equations (2.10) in the form

$$\int_{\mathbb{M}^{1,3}} d^4x \sqrt{|g(x)|} \left( R^\mu_\nu - 8\pi \left( T^\mu_\nu - \frac{1}{2} T \delta^\mu_\nu \right) \right) \Psi^\mu_\nu(x) = 0.$$ (2.11)

For a static problem the expression in the lhs in equation (2.11) does not depend on the variable $x^0$ and we can use test functions of the form $\Psi^\mu_\nu(x) = \psi^\mu_\nu(r) \chi^\mu_\nu(x^0)$, where $\int_{-\infty}^{\infty} dx^0 \chi^\mu_\nu(x^0) = 1$ and $\psi^\mu_\nu(r) \in \mathcal{D}'(\mathbb{M}) \{g_{ij}(r)\}$. Then equation (2.11) reduces to

$$\int_{\mathbb{M}} d^3r \sqrt{|g_{tt}(r)|} \left( R^\mu_\nu - 8\pi \left( T^\mu_\nu - \frac{1}{2} T \delta^\mu_\nu \right) \right) \psi^\mu_\nu(r) = 0.$$ (2.12)

These equations must be fulfilled for any test functions $\psi^\mu_\nu(r)$.

**b) Solution of the relativistic constraint equation (RCE).**

Taking into account:

i) the only nonzero component of the energy momentum tensor (2.2) $T^0_0 = M \delta^0_0(r)$;

ii) the expression $R^0_0 = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j \right)$, which is valid in the static case, where $\Delta_g = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j \right)$ is the Laplacian in $\mathbb{M}^{1,3} \{g_{ij}(r)\}$; and

iii) using the specific test functions of the type $\psi^\mu_\nu(r) = \psi(r) \delta^\mu_0 \delta^\nu_0$,
we obtain the RCE on the initial conditions in the form

$$
\int_{M^{(3)}} d^3 r \sqrt{|\Delta g(r)|} \Delta_g \left( \sqrt{g_{tt}} \right) \psi(r) = 4\pi m \psi(0). \tag{2.13}
$$

Here emerges a new constant

$$
m := M \sqrt{g_{tt}(0)}. \tag{2.14}
$$

In spherical coordinates one easily finds \( \sqrt{|\Delta g(r)|} = \sqrt{-g_{rr}(r)r^2/r^2} \). Then using the normal field’s variables \((2.5)\) and 3D Euclidean space notations, we reach the final form of the RCE:

$$
\bar{\rho}^2 \int_{E^{(3)}} d^3 r \nabla \cdot \left( \phi' e^{\phi} \nabla \left( -\frac{1}{r} \right) \right) \psi(r) = 4\pi m \psi(0). \tag{2.15}
$$

Its solution determines the dependence of the function \( \phi(r) \) on the radial gauge function \( \bar{\phi}(r) \) in the form:

$$
\phi(r) = \frac{m}{\bar{\rho}^2} \int_{r_{\infty}}^r dr e^{-\phi(r)}. \tag{2.16}
$$

The value \( r_{\infty} > 0 \) of the radial variable, used in formula \((2.16)\), defines the place, where \( \phi(r_{\infty}) = 0 \), i.e., where \( g_{tt}(r_{\infty}) = 1 \). This value \( r_{\infty} \) obviously may depend on the choice of the gauge function \( \bar{\phi}(r) \). The value of the \( \phi(r) \) at the place of the point source is

$$
\phi(0) = -\frac{m}{\bar{\rho}^2} \int_0^{r_{\infty}} dr e^{-\phi(r)} < 0. \tag{2.17}
$$

Now we see that the solution of RCE translates the differential 3-form \( \omega_{g}^3 := d^3 r \sqrt{|\Delta g(r)|} \Delta_g \left( \sqrt{g_{tt}} \right) \) on \( M^{(3)} \{ g_{ij}(r) \} \) to the distribution-valued differential 3-form \( \omega_{\delta}^3 := d^3 r \Delta \left( -\frac{m}{\bar{\rho}^2} \right) = d^3 r \ m \delta(r) \), defined on the Euclidean space \( E^{(3)} \{ \delta_{ij} \} \), i.e., on the solution of RCE we have the relation

$$
\omega_{g}^3 = \omega_{\delta}^3. \tag{2.18}
$$

One can consider this relation \((2.18)\) as a new form of the RCE.

The correspondence between the spaces \( M^{(3)} \{ g_{ij}(r) \} \) and \( E^{(3)} \{ \delta_{ij} \} \) was stressed in Section 2.1. The extension of this correspondence, obtained here, is the geometrical basis for application of the mathematical theory of distributions in the massive point particle problems in GR.

c) Solution of the Rest of EE.

Since the other components of the energy-momentum tensor of point particle are zero in its proper frame, one can use for them any of the forms \((1.7), (2.10), \) or \((2.11)\) of EE, and we arrive at the following ordinary differential equations for normal field variables:

$$
\varphi'' + \varphi' \frac{\varphi'}{r_2} = \frac{1}{\bar{\rho}^2} e^{2(\varphi_2 - \bar{\varphi})}, \tag{2.19a}
$$

$$
(\varphi')^2 - (\varphi_2')^2 + \frac{1}{\bar{\rho}^2} e^{2(\varphi_2 - \bar{\varphi})} \frac{m}{\bar{\rho}^2} = 0 \tag{2.19b}
$$
Using the relation (2.16), we can exclude the gauge function \( \varphi \) from this system, obtaining its radial-gauge-invariant form:

\[
\frac{d^2 \varphi}{d\varphi^2} = \frac{\bar{\rho}^2}{m^2} e^{2\varphi}, \\
1 - \left( \frac{d\varphi}{d\varphi} \right)^2 + \frac{\bar{\rho}^2}{m^2} e^{2\varphi} w = 0
\]  

(2.20a)

(2.20b)

Note that meanwhile we have excluded the radial-gauge-dependent variable \( r \) replacing it with the radial-gauge-independent one – \( \varphi \), which now plays the role of the independent "radial" variable.

The first equation (2.20a) can be integrate immediately in quadratures. The second equation (2.20b) imposes a constraint on the two integration constants in the general solution of the first one. Thus we remain with only one integration constant \( \varphi_\infty \) in the solution of the system (2.20):

\[
\varphi_2(\varphi) = \ln \left( \frac{m/\bar{\rho}}{|\sinh (\varphi - \varphi_\infty)|} \right), \quad \rho(\varphi) = \frac{m \exp(-\varphi)}{|\sinh (\varphi - \varphi_\infty)|}
\]  

(2.21)

d) Fixing the emerging constants in the general solution of the problem.

The second expression in equations (2.21) is derived using formulas (2.5). It shows that \( \rho(\varphi_\infty) = \infty \). Hence, the value \( \varphi_\infty \) corresponds to the physical infinity, where the space-time is asymptotically flat and we must have \( g_{tt} = 1 \). Thus we see that the value \( \varphi_\infty \) must be reached for the value of the radial variable \( r_\infty \), i.e. the relation \( \varphi_\infty = \varphi(r_\infty) = 0 \) must take place. For the value of the radial variable \( r_\infty \) the space-time \( M^{(1,3)} \{ g_{\mu\nu}(x) \} \) is asymptotically flat. This value corresponds to the physical infinity. Hence, under our conventions, described in the Introduction, the physically admissible interval of the values of the radial variable is \( r \in [0, r_\infty] \).

As a result we remain with only two arbitrary constants \( m \) and \( r_\infty \) in the solution of the whole system of EE for the massive point particle. The 4-D metric in normal field’s variables acquires the final gauge invariant form in which its coefficients are functions only of \( \varphi \) in the role of a radial variable:

\[
ds^2 = e^{2\varphi} dt^2 - e^{-2\varphi} m^2 \left( \frac{d\varphi^2}{\sinh(\varphi)^4} + \frac{d\theta^2 + \sin(\theta)^2 d\phi^2}{\sinh(\varphi)^2} \right).
\]  

(2.22)

It’s remarkable that in the metric (2.22) appears only the integration constant \( m \). As a result only the value of this constant will influence the local dynamics of any test particles and fields, which probe the metric in the space-time of single massive point source. This important conclusion is independent of the choice of radial variable \( r \), i.e. is gauge invariant, as well as the whole equation (2.22).

According to the definition (2.14) of the constant \( m \) and the formulas (2.5) we have \( \varphi(0) = \ln(m/M) \). The relation (2.17) shows that \( \varphi(0) < 0 \). Thus we obtain, that in the GR massive point particle problem the variable \( \varphi \) varies in the interval \( \varphi \in [\ln(m/M), 0] \). After all, the bare mechanical mass \( M \) shows up in GR massive point particle solution, defying the interval of the physical values of the variable \( \varphi \).

It is convenient to introduce the mass ratio

\[
\varrho = m/M \in (0, 1).
\]  

(2.23)
Then in the problem at hand the basic quantity $\sqrt{g_{tt}} = \exp(\varphi)$ varies in the physical interval $\sqrt{g_{tt}} \in [\varrho, 1]$.

The final formula

$$\rho(\varphi) = \frac{2m}{1 - \exp(2\varphi)} \geq \frac{2m}{1 - \varrho^2} > 2m$$

(2.24)

shows that the luminosity variable $\rho$ in the gravitational field of massive point particle cannot take values, less than $\frac{\rho_G}{1 - \varrho^2} > \rho_G$, since $2m = \rho_G$, as we shall see in the next subsection. This is in strong contrast to the situation with Schwarzschild solution in Hilbert gauge (1.6) and in complete accord with Dirac’s suggestion [21].

d) Fixing of the physical radial variable.

The previous consideration gives a correct mathematical ground for our conclusion about the topology of the space-time of single massive point particle problem in GR, as described in Section 2.1. Indeed, it is easy to obtain from EE (2.10) the 4D scalar curvature of space-time with single massive point source:

$$4R = -8\pi M \delta^{(3)}_g(r),$$

(2.25)

and its 3D scalar curvature

$$3R = 16\pi M \delta^{(3)}_g(r).$$

(2.26)

Clearly, the last equation is the concrete form of the RCE (1.15) in the point particle case. Hence, the 3D space $M(3)\{g_{ij}(r)\}$ has a strong singularity at the geometrical point $r = 0$, where the massive matter point is placed, and the solution of EE cannot be extended behind this point, both from physical and from mathematical reasons. The space-time of the problem $M(1,3)\{g_{\mu\nu}(x)\}$ has a singular line – the world line $W_0$ of the massive matter point. This singular line must be removed from the manifold $M(1,3)\{g_{\mu\nu}(x)\}$ and we remain with the topology, described in Section 2.1.

The simple correspondence between the Riemannian space-time of point particle and Minkowskian space-time gives a good reason to adopt, as much as possible, the basic properties of the Minkowskian radial variable $r$ for the Riemannian case, as described in Section 1.2.2. Then, according to results in the previous Section, we have to fix the gauge function $\bar{\varphi}(r)$ in such a manner, that as a result of relation (2.16) we will have the mapping:

$$[\ln(m/M), 0]_\varphi \xrightarrow{\varphi(r)} [0, \infty],$$

(2.27)

A very important additional requisite of the mapping (2.27) is the requirement to preserve the number and the character of the original singularities of the solution (2.22) in the whole compactified complex domain of variable $\varphi$. The mapping (2.27) is allowed only to change the positions of these singularities in the compactified complex domain of variable $r$.

The only way to fulfill this requirement is to use a fractional-linear function $\varphi(r) = \frac{ar+b}{r+c}$ with some constant coefficients $a, b, c$, which are unambiguously fixed by the mapping (2.27) in the form: $\varphi(r) = \frac{b}{r+b/\ln \varrho}$. Taking into account that at $r \to \infty$ the asymptotic of the function $\varphi(r) \sim b/r$ yields an asymptotic $g_{tt} \sim 1 + 2b/r$, one sees that the standard
comparison with the real observations imply $b = -m_{\text{Kepler}}$, where $m_{\text{Kepler}}$ is the Keplerian mass of the particle, as observed by a distant observer. Then the formula (2.24) gives
\[
\lim_{r \to \infty} \frac{\rho(r)}{r} = m/m_{\text{Kepler}} \quad \text{and the Edington’s coherent scale condition: } \lim_{r \to \infty} \frac{\rho(r)}{r} = 1 \quad \text{(see Section 1.2.2)}
\]
fixes the value of our integration constant $m = m_{\text{Kepler}}$. Hence, in this physical gauge the function
\[
\varphi(r) = -\frac{m}{|r - r_0| + m/\ln \frac{1}{\varrho}} = -\frac{M\varrho}{|r - r_0| + R} = \varphi(r; M, R, \varrho) \quad (2.28)
\]
presents a proper GR generalization of the Newton potential (1.2)
\[
\varphi_{\text{Newton}}(r) = -\frac{m}{|r - r_0|}
\]
of matter point with Keplerian mass $m$, placed at the position $r_0$, which describes the fundamental solution in Newton theory of gravity.\footnote{The form of the Newton potential can be derived, following the same consideration with only one difference: the physical values of the Newton potential cover the whole semi-constrained interval $\varphi_{\text{Newton}} \in (-\infty, 0]$. This interval has to be mapped onto the interval $[0, \infty) \ni r$ by fractional-linear function $\varphi_{\text{Newton}}(r)$ and this gives $\varphi_{\text{Newton}}(r) = -\frac{m}{r - r_0}$.}

Now we see that the ratio $\varrho = m/M \in (0, 1)$ describes the relativistic gravitational defect of mass for massive point particle. It was introduced for the first time in [16], where the solution of the problem was derived using a different mathematical technic. As a final result we obtain a two parameter family of solutions (2.28) to the massive point particle problem in GR. This family can be parameterized by any two of the three constants $m$, $M$, and $\varrho$.

In the formula (2.28) we use a short notation $R = m/\ln \frac{1}{\varrho} = M/\left(\frac{1}{\varrho} \ln \frac{1}{\varrho}\right)$ for the GR correction to the Newton potential. Further on we shall refer to the correction $R$ as "a relativistic shift" in the Newtonian potential. The potential $\varphi_{\text{Newton}}(r)$ can be derived as a limit $R \to 0$ of the relativistic one (2.28).\footnote{One easily obtains the following instructive estimates: a) If $\varrho \lesssim 1/\sqrt{\varv} \approx 0.60653$, then $R \lesssim \rho_G$. b) If, according to Birkhoff’s theorem, one applies the formula (2.28) outside the spherically symmetric body of finite radius $r_B$, the quantity $R$ will not exceed the radius of the body $r_B$, when $\varrho_B \lesssim \exp(-\rho_G/r_B)$. This restriction is very weak, since $\rho_G/r_B \ll 1$ for real bodies, and for any of them $1 - \exp(-\rho_G/r_B) \ll 1$.}

Therefore we shall refer to the very function $\varphi(r)$ (2.28) as "a relativistic gravitational potential". From mathematical point of view the relativistic gravitational potential defines the Fock conformal mapping. It is clear that this potential plays a basic physical role in the relativistic theory of gravity.

2.3.4 Some Remarks on the Non-Relativistic Limit $c \to \infty$.

One can expect that in the non-relativistic limit $c \to \infty$ our GR solution for single particle will reproduce the results of the Newton theory. For study this limit it is necessary to restore the physical units in corresponding formulae. Then we obtain for the mass ratio:
\[
\varrho = e^{\varphi(0)/c^2}, \quad (2.29)
\]
and
\[
\varphi(r) = -\frac{G_{\text{Newton}} M e^{\varphi(0)/c^2}}{|r - r_0| - G_{\text{Newton}} M e^{\varphi(0)/c^2}/\varphi(0)} \quad (2.30)
\]
- for the relativistic potential.
Unfortunately, at present we do not have a theory of the relativistic collapse, which has to describe in detail the origin of the relativistic gravitational defect of mass of a single point particle and the value of the mass ratio $\rho$. Here we are considering $\rho$ just as an additional free parameter of the class of point particle solutions, studied in the present article. If one considers, instead, as a free parameter $\varphi(0)$ and assumes that it does not depend on the velocity of light, in the limit $c \to \infty$ one obviously obtains from the relation (2.29) the limit:

$$\lim_{c \to \infty} \rho = 1.$$  

(2.31)

This result sounds physically right. Indeed, one expects that in the non-relativistic limit $c \to \infty$ the gravitational mass defect will disappear and we will return back to the Newton theory of gravity with $m = M$.

Despite of this physically reasonable result, the assumption that $\varphi(0)$ does not depend on the velocity of light $c$ gives a wrong limit $\lim_{c \to \infty} \varphi(r) = -G^{\text{Newton}} M / (|r - r_0| - G^{\text{Newton}} M / \varphi(0))$ in the formula (2.31).

To obtain the physically right results in the both cases, one has to assume that:

i) actually the quantity $\varphi(0)$ depends on the velocity of light $c$ in some specific way, and

ii) the unknown at present function $\varphi(0, c)$ fulfills simultaneously two additional conditions:

$$\lim_{c \to \infty} \varphi(0, c) = \varphi^{\text{Newton}}(0) = -\infty,$$

$$\lim_{c \to \infty} \left( \varphi(0, c) / c^2 \right) = 0.$$  

(2.32)

As a result of these conditions, which are obviously compatible, one obtains both the relation (2.31) and the right non-relativistic limit

$$\lim_{c \to \infty} \varphi(r) = -\frac{G^{\text{Newton}} M}{|r - r_0|} = \varphi^{\text{Newton}}(r).$$  

(2.33)

One can hope that the future theory of the relativistic gravitational collapse, accompanied by a proper treatment of the gravitational mass defect, or some other additional considerations, will be able to derive the precise form of the function $\varphi(0, c)$ and to confirm the physically natural relations (2.32). In the present article we will assume these relations to be fulfilled.

### 2.3.5 Some Remarks on the Properties of Static Fundamental Solutions

#### 1. On the three dimensional form of the fundamental solutions of EE.

An unexpected and remarkable feature of the relativistic gravitational potential (2.28) is that it has a finite negative value $\varphi(r_0) = \ln \rho$ at the place of the very point source $r_0$. This unique property is in a sharp contrast to the case of the Newton potential $\varphi^{\text{Newton}}(r)$ (1.2), which diverges as $-m / |r - r_0|$, when $r \to r_0$. Thus we see that in GR we have a self-regularizing mechanism for gravitational interaction, based on the influence of matter on the space-time curvature. In the article [22] we have shown that the same phenomenon comes into being in GR electrostatic problem of single massive point charge.
As a result of the relativistic self-regularization all components of the metric tensor \( \sqrt{g_{tt}} = e^\varphi \), \( \sqrt{-g_{rr}} = \frac{e^{\varphi}}{\sinh(r)^2} \), \( \rho = \frac{mc^{-2}}{\sinh(r)} \) in spherical coordinates are regular at the place of the point source \( r_0 \):

\[
\sqrt{g_{tt}(r_0)} = \varrho, \quad \sqrt{-g_{rr}(r_0)} = \frac{4}{\varrho} \left( \frac{\varrho \ln \varrho}{1 - \varrho^2} \right)^2, \quad \rho(r_0) = \frac{2m}{1 - \varrho^2}. \tag{2.34}
\]

Now it becomes clear that the singular term \( \delta^{(3)}(r - r_0) \) in the lhs of Eq. \( \text{(2.15)} \) originates from the singularity of the 3D Cartesian determinant:

\[
\sqrt{|\Delta g(r)|} = \sqrt{-g_{rr}(r)\rho(r)^2/|r - r_0|^2} \sim 1/|r - r_0|^2, \quad \text{when} \quad r \to r_0. \tag{2.35}
\]

The singularity of metric coefficients at the place of the point source of gravity becomes transparent in Cartesian coordinates. Indeed, one can write down the 3D distance in the tensorial form \( dl^2 = -dr \frac{3}{2} y(r) dr \), using the 3D Cartesian metric tensor

\[
-\frac{3}{2} y(r) = \frac{\rho^2(r)}{|r - r_0|^2} (I - e_r \otimes e_r) - g_{rr}(r) e_r \otimes e_r, \quad \text{where} \quad e_r := \frac{r - r_0}{|r - r_0|}. \tag{2.36}
\]

The components of this tensor are obviously singular at the point \( r = r_0 \). This is precisely because in the specific geometry, defined by fundamental solutions of EE, we have \( \rho(r_0) > 0 \). The expression \( \text{(2.35)} \) defines the square root of the determinant of tensor \( \text{(2.36)} \).

2. One-dimensional-like representation of the fundamental solutions of EE.

The Dirac \( \delta \)-function is a linear functional. Its representation depends on the class of the test functions in use. We can take advantage of spherical coordinates in description of the test functions of the GR point particle problem. Starting with Cartesian coordinate test functions \( \psi(r) \in \mathcal{D}'(\mathbb{R}^3) \), in spherical coordinates we obtain a specific class of test functions \( \psi_{sph}(r, \theta, \phi) := \psi(r e_r, (\theta, \phi)) \in \mathcal{D}'_{sph}(\mathbb{R}^{(1)} \times SO(3)) \). These must be distinguish form the arbitrary test functions \( \psi(r, \theta, \phi) = \psi(r, e_r, (\theta, \phi)) \in \mathcal{D}'(\mathbb{R}^{(1)} \times SO(3)) \) on the manifold \( \mathcal{M}^{(3)} = \mathbb{R}^{(1)} \times SO(3) \). Now \( r \in \mathbb{R}^{(1)} \) is considered as an independent variable, not just as a short notation for \( |r| := \sqrt{x^2 + y^2 + z^2} \). The difference between the functions \( \psi_{sph}(r, \theta, \phi) \) and \( \psi(r, \theta, \phi) \) is of critical importance for our problem, since \( \psi_{sph}(r = 0, \theta, \phi) := \psi(0) = \text{const} \) for any values of the angle variables \( \theta \) and \( \phi \). The functions \( \psi(r, \theta, \phi) = \psi(r, e_r, (\theta, \phi)) \) do not have such property. Instead, the functions \( \psi(r = 0, \theta, \phi) = \psi(0, e_r, (\theta, \phi)) \neq \text{const} \) keep the dependence on the variables \( \theta \) and \( \phi \).

Let us use the class of test function \( \mathcal{D}_{sph}'(\mathbb{R}^{(1)} \times SO(3)) \). The standard restriction of the Euclidean Laplacean \( \Delta \varphi(r) = \frac{1}{r^2} \partial_r^2 (r \varphi(r)) \) on the functions, which depend only on variable \( r \), brings us to 1D formulation of the problem. One can write down the solution of the Eq. \( \text{(2.15)} \) with added point source of gravitational field, described by function \( \delta(r) \), in the following one-dimensional form (see for details \[16\]):

\[
\varphi(r) = \ln \varrho \left( 1 - \frac{r}{r + R} \Theta \left( \frac{r}{r + R} \right) \right). \tag{2.37}
\]

Here we are using the Heaviside steep function \( \Theta(r) \) with regularization \( \Theta(0) = 1 \).

The fundamental solutions of EE were found for first time in \[16\] in this form. Its advantage is that it makes transparent the jump in the derivatives of the metric coefficients.
in spherical coordinates. This jump reproduces via the Einstein tensor the \( \delta \)-function in the rhs of EE with point source.

In the case of 1D representation \((2.37)\) the form of the metric \((2.22)\) must be considered as valid only in the vacuum domain, outside the point source. In this domain the form \((2.22)\) does not make difference between 3D and 1D representation of the fundamental solutions.

### 2.3.6 On the Choice of Radial Gauge in the Single Particle Problem

The above consideration solves on a clear theoretical basis the longstanding problem of the choice of radial variable \( r \) for point source of gravity in GR. The unambiguously obtained physical radial variable \( r \) is obviously a preferable one, both from mathematical and and physical point of view.

The singularities of the metric coefficients in the whole compactified complex plain \( \tilde{\mathbb{C}}_\varphi \) are placed at the positions \( \varphi_n = i\pi n, \ n \in \mathbb{Z} \). The points of finite \( n \) are poles and the infinite point \(|n| = \infty\) is an essentially singular one.

The singular points of the solution in the whole compactified complex plain \( \tilde{\mathbb{C}}_r \) of the physical variable \( r \) are of two essentially different types:

1. The place of the point source of gravity at \( r = 0 \) where the curvature of space-time has a strong singularity, proportional to \( \delta^{(3)}(r) \). This singularity is seen in the differential 3-forms \((2.18)\), too. Surprisingly, the relativistic potential \((2.28)\) and the metric coefficients, when written in 3D form, are regular at this point.

2. In Einstein theory of gravity an unavoidable singular points of the metric coefficient \( g_{tt}(r) \) are the (complex) points \( r_n = -R + i\frac{\pi}{n}, \ n \in \mathbb{Z} \). These singular points are placed in the nonphysical domain of the physical variable \( r \). For finite \( n \in \mathbb{Z} \) the singular points are poles of the 3D metric coefficients in the expression \((2.22)\). The sequence of these singular points has a limiting point \( r = -R \) for \(|n| = \infty\). This is a real essentially singular point, which is not an isolated one. In contrast, the real singular point \( r = -R \) is a simple pole of the relativistic potential \((2.28)\), which has no other singular points. This pole is placed in non-physical domain \( r < 0 \) and corresponds to the real pole at \( r = 0 \) of the gravitational potential in Newton theory of gravity.

The multiplication of this single simple pole of the relativistic potential \( \varphi(r; M, R, \varrho) \) to an infinite series of singularities of GR metric coefficients is produced by the specific exponential mapping \((2.22)\) of \( \varphi(r; M, R, \varrho) \) onto these coefficients. This is a specific feature of the relativistic description of gravitational field of massive point particle and may be considered as a price, one has to pay, for the self-regularization mechanism, discussed in the previous subsection.

Under gage transformation to some other radial variable \( r_{\text{other}} \), related with the physical one \( r \) by a coordinate transformation \( r_{\text{other}} = r_{\text{other}}(r) \) of general type, which is not fractional-linear one, in the solution of the problem will appear additional nonphysical singularities in the corresponding compactified complex plane \( \tilde{\mathbb{C}}_{r_{\text{other}}} \).

From the relation \((2.16)\) one easily obtains the physical gauge function \( \tilde{\varphi}_{\text{phys}}(r) = 2\ln((r + R)/\varrho) \), which is compatible with boundary conditions of the problem and with all additional requirements on the physical radial variable \( r \), as formulated in the Introduction.

We shall call a regular gauges of the problem all gauges, for which the integral \((2.16)\) makes sense in the physical interval \( r \in [0, r_{\infty}] \). This is just the necessary and sufficient
condition for the radial gauge to be compatible with the boundary conditions, coded in the Dirac $\delta$-function in the RCE of the massive point particle problem.

This condition fixes a large class of admissible gauges for this problem. One of them is BRG $\bar{\varphi}(r) \equiv 0$ in which the relativistic potential has the form $\varphi_{BRG}(r) = m(r - r_\infty)/\bar{\rho}^2$, $r \in [0, r_\infty]$.

Between the regular gauges for the one-particle problem in GR is the gauge by Droste [4]. It has a clear geometrical meaning, since in this gauge the radial variable $r$ measures the radial geometrical distance in the 3D Schwarzschild metric. This gauge reproduces only a very special value of the mass defect ratio $\varrho = (\sqrt{5} - 1)/2 \approx .6180$ [16]. Quite curiously, under such geometrical choice of the radial gauge $\varrho$ equals precisely the famous mathematical golden ratio.

All other known radial gauges, probed for spherically symmetrical static solutions of EE and described in [16], are not regular. Therefore they cannot be used for solution of the point mass problem. As we have seen in the previous subsections, only a certain combination of gauge function $\bar{\varphi}(r)$ and corresponding form of the relativistic potential $\varphi(r)$ can obey the specific boundary conditions of this problem.

For example, in the most popular at present Hilbert gauge: $\rho_H(r) \equiv r$ the static spherically symmetric problem has a relativistic potential $\varphi_H(\rho) = \ln \sqrt{1 - \rho_G/\rho}$ and a radial gauge function $\bar{\varphi}_H(\rho) = \ln \left(\rho_0(\rho - \rho_G)/(\bar{\rho})^2\right)$. Hence, for Hilbert gauge the integral (2.16) diverges logarithmically: 1) when $\rho$ approaches the center $\rho = 0$, where the point source of gravity must be placed in this gauge, if one insist on the point particle interpretation of the the form of the Schwarzschild solution; 2) when $\rho$ approaches the event horizon $\rho = \rho_G$. In addition, the value of this integral becomes an imaginary number for $\rho < \rho_G$.

This means that Hilbert gauge is incompatible with the specific boundary conditions for EE in presence of massive point particle. Therefore one cannot use the Hilbert gauge to solve the point particle problem in GR. This gauge yields the well known nonphysical singularity at the point $\rho = 0$, i.e. on the boundary of the to-be-physical domain of the radial variable $\rho \in [0, \infty)$. More over, the meaning of the variable $\rho$ radically changes in the interval $[0, \rho_G]$. Here it plays the role of a specific time variable and the point $\rho = 0$ describes the future infinity of the internal time $t_{in} = x - 1/x \in (-\infty, \infty)$, where $x = \rho + \rho_G \ln \left(|\rho/\rho_G - 1|\right)$ is the Regge-Wheeler “tortoise” coordinate in the interior of Hilbert solution [23]. It becomes clear that even if we will be able to reproduce mathematically a term $\sim \delta(\rho)$ in the rhs of EE (see the articles by P. Parker, by H. Belasin & H. Nachbagauer, and by J. M. Heinzle & R. Steinbauer in [15]), its interpretation as a source of the gravitational field and the curvature of the Schwarzschild solution is physically unacceptable. Such term may describe only a $\delta$-shaped – with respect to the time, “impulse” at the time instant $\rho = 0 (\Rightarrow t_{in} = \infty)$ and has a complete unclear physical meaning. In any case it is not able to describe the usual physical 3D-space-point source of static gravitational field.

### 2.4 The Total Energy of the Aggregate of Massive Point Source and its Gravitational Field

In the problem at hand we have an extreme example of an ”island universe“. In it a privileged reference system and a well defined global time exist. It is well known that under these conditions the energy of the gravitational field can be defined unambiguously [1]. Moreover, we can calculate the total energy of the aggregate of a mechanical particle.
and its gravitational field in a canonical way, considering the corresponding 1D variational problem for total action (2.1) in the spherically symmetric static case [16]. The canonical procedure produces a total Hamilton density

\[ H_{\text{tot}} = \sum_{a=1,2; \mu=t,r} \pi_{a,\mu} \varphi_{a,\mu} - L_{\text{tot}} = \frac{1}{2} \left( -\bar{\rho}^2 \varphi'^2 + \rho^2 \varphi'^2 - e^{2\varphi_2} \right) + M e^{\varphi} \delta(r). \]

Using the equations (2.19), one immediately obtains for the total energy of the GR universe with one point particle in it:

\[ E_{\text{tot}} = \int_0^\infty H_{\text{tot}} dr = m = \varrho M < M. \] (2.38)

This result completely agrees with the strong equivalence principle of GR. The energy of the static longitudinal gravitational field, created by a point particle at rest is a negative quantity:

\[ E_{\text{GR}} = E_{\text{tot}} - E_{\text{M}} = m - M = -M(1 - \varrho) < 0. \]

Since both matter point and its gravitational field have nonzero proper energies, this result proves that the ratio \( \varrho \) must belong to the open interval \((0, 1)\), see [16] for more details.

The above consideration gives a clear physical explanation of the gravitational mass defect of a point particle.

### 3 Quasi-Linear Superposition Principle for Static Fundamental Solutions of Einstein Equations

#### 3.1 Justification of the Quasi-Linear Superposition Principle for the Relativistic Gravitational Potential \( \varphi \)

##### 3.1.1 Some General Arguments

As a result of the GR dynamics, the relativistic constraint equation (RCE) is a restriction, which will be fulfilled at any time instant \( t \), if it is valid at the initial time instant \( t = 0 \). Therefore it is enough to solve RCE only at the initial time instant \( t = 0 \).

One can use this specific feature of the RCE to simplify it, imposing special additional conditions at the initial time instant, which can not be fulfilled during the further evolution of the physical system. Thus one can impose different physical conditions on the initial state of the system under consideration.

For example, as stressed already by Misner in [17] and a bit later by Fock in [1], it is impossible to have a permanent static solution for N-particle system in GR, if \( N > 1 \). Nevertheless, it is possible to find an initial-instant-static solutions of the problem with any number of particles \( N \). As a rule, the initial conditions will contain some initial amount of gravitational waves. One of the basic open problems is how to exclude the presence of initial gravitational waves. One can expect that for such solutions RCE will take its simplest form.

It turns out that under the conditions \( \partial_t g_{\mu\nu}|_{t=0} = 0 \) on the time-derivatives of the metric and some weak additional constraint on \( \partial^2 g_{\mu\nu}|_{t=0} \) the RCE, together with other EE, yields a quasi-linear equation for the relativistic potential of any mass distribution. This equation can be considered as a relativistic analog to the linear equation (1.1) for Newton potential in the classical theory of gravity [17].

Let us consider a system of N point particles with bare masses \( M_A, A = 1, \ldots, N \) at positions \( r_A(t) \). Here we suppose to work with space-time manifold, which allows an
existence of global time $t$. The energy-momentum tensor of such system at time instant $t$ is:

$$T^\nu_\mu(t, r) = \sum_{A=1}^{N} M_A \delta^{(3)}(r - r_A(t)) u_A^\nu(t) u_A^\mu(t). \quad (3.1)$$

According to the relativistic nonlinear superposition principle, described in Section 1.2.4, the metric $g_{\mu\nu}(t, r)$ of this N-particle problem is given by formula (1.13). This formula does not yield any simple practical results. It describes in a formal way the solution of the very complicated GR problem under consideration and demonstrates its existence and uniqueness under proper boundary conditions.

As we shall show in this Section, in contrast, one can introduce a simple quasi-linear superposition principle for the relativistic gravitational potential $\varphi(r)$. We shall consider the special case in which at the initial time instant $t = 0$ all particles are at rest and have initial positions $r_A(0) = r_A$. Then

$$T^\nu_\mu(0, r) = \left( \sum_{A=1}^{N} M_A \delta^{(3)}(r - r_A) \right) \delta^\nu_0 \delta^\mu_0. \quad (3.2)$$

According to articles [17], the generalization of the covariant form of RCE (2.26) for N point particles at rest and under additional conditions $\partial_t g_{\mu\nu}|_{t=0} = 0$, i.e. in the case of the N-particle instant-gravistatics, is:

$$3R = 16\pi \sum_{A=1}^{N} M_A \delta^{(3)}(r - r_A). \quad (3.3)$$

The proper generalization of the relation (2.13), which follows in the instant-gravistatic case from the EE for N-particles reads:

$$\int_{\mathcal{M}^{(3)}} d^3r \sqrt{3g(r)} |\Delta_g(\sqrt{g_{tt}}) \psi(r)| = 4\pi \sum_{A=1}^{N} m_A \psi(r_A). \quad (3.4)$$

Here $m_A = M_A \sqrt{g_{tt}(r_A)}$ are the corresponding Keplerian masses of the point particles.

A remarkable feature of the equation (3.4) is that under conformal Fock transformation:

$$g_{ij} = e^{-2\varphi} h_{ij}, \quad g^{ij} = e^{2\varphi} h^{ij}, \quad \sqrt{|3g|} = e^{-3\varphi} \sqrt{|3h|}, \quad (3.5)$$

one obtains the quasi-linear equation for the relativistic potential of the N-particle problem:

$$\partial_i \left( \sqrt{|3h|} h^{ij} \partial_j \varphi \right) = 4\pi \sum_{A=1}^{N} M_A e^{\varphi_A} \delta^{(3)}(r - r_A), \quad \text{where} \quad \varphi_A := \varphi(r_A). \quad (3.6)$$

Here $h_{ij}$ are functions which define the unknown metric in the Fock conformal space of the N-particle case. In Eq. (3.6) we have used the substitution $\sqrt{g_{tt}} = e^{\varphi}$.

After the pioneering work by Lichnerowicz, one usually supposes the metric $h_{ij}$ to be conformally flat (see in [17]). This leads to a well known superposition principle in the case of instant gravistatics of N Schwarzschild black holes [17].
Today it is well known that the conjecture of conformal flatness of $h_{ij}$ is too restrictive and does not allow one to obtain the solutions of real physical problems (see the review article by Cook in [17] and the references therein). Unfortunately, at present we do not know an alternative assumption, which fixes the metric $h_{ij}$ in a physically acceptable way for the case of N matter bodies.

### 3.1.2 The Superposition Principle for the Potential $\varphi$

Here we outline a complete different approach to the problem at hand. It is based on a specific superposition principle in GR. This new approach may turn to be a more physical alternative to the conformally flat one, mention in the previous Section. The novel superposition principle may play the role of the additional physical requirement, needed to select the proper initial conditions for the GR N-body-problem between all mathematically admissible and formal initial conditions.

Having in mind the inhomogeneous quasi-linear equation (3.6), it seems natural to define its solution, we are looking for, by the formula

$$\varphi(r; r_1, \ldots, r_N) = -\sum_{A=1}^{N} \frac{m_A}{|r - r_A| + R_A}. \quad (3.7)$$

It follows a simple quasi-linear superposition principle for the static relativistic potential $\varphi$. Here the Keplerian masses are $m_A = \rho_A M_A$ and the mass-defect ratios $\rho_A(r_1, \ldots, r_N) = \exp \varphi_A(r_A; r_1, \ldots, r_N)$ of the A-th particle in presence of the other massive points can be obtained from the self-consistency condition for the relativistic potential

$$\varphi(r_A; r_1, \ldots, r_N) = \sum_{B=1}^{N} C_B \varphi(|r_A - r_B|, M_B, R_B), \quad (3.8)$$

$$\varphi(|r_A - r_B|, M_B, R_B) = \frac{m_B}{|r_A - r_B| + R_B}. \quad (3.9)$$

Note that the procedure, based on the relations (3.7) and (3.8), represents the only way to construct a relativistic gravitational potential $\varphi(r; r_1, \ldots, r_N)$, (and corresponding metric) of a system of N point particles with the following

**Fundamental property:** For a system of N particles at finite and nonzero mutual distances the total variety of singularities of the relativistic potential in the whole complex domain of the variables is just a superposition of the singularities of the relativistic potentials of the separate matter constituents of the system.

In other words, joining several point particles in a gravitationally interacting system, we remain just with the singularities of all independent particles. As a result of the integration of the independent particles in a joint interacting system no additional new singularities emerge, as well as no old singularities disappear in the whole complex domain of space variables. In addition, the singularities of the separate particles do not change their character.

From analytical point of view one can reach such simple preservation of the singularities only by constructing a linear combination of the corresponding analytical functions with some constant coefficients. The uniqueness of this construction, up to the choice of the constant coefficients, is guaranteed by the corresponding theorem of the complex analysis, which stays that every analytical function is unambiguously defined by its singularities.
In the quasi-linear superposition principle (3.7) we are using the single point particle solutions (2.28), denoting by \( \varrho_A^\infty = \exp(\varphi_A^\infty) \) the value of the mass defect ratio of the A-th particle in the case \( r_{AB} := |r_A - r_B| \to \infty \) for all \( B \neq A \), i.e., in the previously considered case of a single massive point particle in the whole universe. The constants \( C_A = \varrho_A / \varrho_A^\infty > 0 \) are unknown and have to be justified.

The above consistency condition (3.8) yields the following basic nonlinear algebraic system of \( N \) equations:

\[
\varphi_A = - \sum_{B=1}^{N} \frac{M_B e^{\varphi_B}}{r_{AB} + R_B}, \quad A = 1, \ldots, N; \tag{3.10}
\]

for the mass defect ratios \( \varrho_A = \exp(\varphi_A) \) of the A-th particle as a member of the N-particle system in finite space range, i.e., when all \( r_{AB} < \infty \). The system (3.10) is regular one when \( R_A \neq 0 \) for all values of \( A = 0, \ldots, N \).

In the \( N \) relations (3.10) we have too many free parameters, which have to be fixed using some proper physical assumptions. Taking into account that the bare mass \( M_A \) and the relativistic shift \( R_A = M_A / \left( \frac{1}{\varrho_A} \ln \frac{1}{\varrho_A} \right) \) are inner characteristics of the very A-th point particle, we can suppose both of them to be constants, whose values are independent of the N-particle configuration.

Then we see that our quasi-linear superposition principle (3.7) leads, after all, to a definite mathematical formulation of a novel relativistic Mach-like principle. It states that the Keplerian masses \( m_A = m_A(r_{12}, \ldots, r_{N-1,N}) \) of the bodies depend on the mass distribution in the universe, in contrast to their bare masses \( M_A \), which remain independent of matter distribution. These essentially different properties of the masses \( m_A \) and \( M_A \) seem to be natural in the relativistic theory of gravity and may have important physical consequences.

The independence of the bare particle masses \( M_A \) of the system configuration seems to be quite natural assumption. In addition we will assume the bare mass to be an additive quantity, i.e., the total bare mass \( M \) of the composite system of particles is just the sum of the bare masses of the constituent particles: \( M = \sum_{A=1}^{N} M_A \).

More speculative, from physical point of view, is the requirement of the independence of the relativistic shifts \( R_A \) from the particle configuration. In the present article we test this assumption as a way to restrict the number of the free parameters in the problem at hand and study some of its consequences, leaving for future developments its justification on a more profound physical basis.

### 3.1.3 The Non-Relativistic Limit of the N-Particle Potential \( \varphi \)

The restoration of the correct physical units draws an additional light on the proposed quasi-linear superposition principle (3.7). In physical dimension-full quantities Eq. (3.7) acquires the form

\[
\varphi(r; r_1, \ldots, r_N) = \sum_{A=1}^{N} \frac{-G_{\text{Newton}} M_A e^{\varphi_A} / c^2}{|r - r_A| - G_{\text{Newton}} M_A e^{\varphi_A^\infty} / c^2 / \varphi_A^\infty}, \tag{3.11}
\]
and the consistency conditions (3.10) read:

\[ \varphi_A = e^{\left(\varphi_A - \varphi_A^{\infty}\right)/c^2} \varphi_A^{\infty} - \sum_{B \neq A}^N \frac{G^{\text{Newton}} M_B e^{\varphi_B^{\infty}}}{r_{AB} - G^{\text{Newton}} M_B e^{\varphi_B^{\infty}}/c^2} \varphi_B^{\infty}, \quad A = 1, \ldots, N. \tag{3.12} \]

The last formulas show directly that under the two assumptions (2.32) in the limit \( c \to \infty \) one obtains precisely the Newtonian superposition principle (1.3), as it should be for the correct relativistic generalization of the non-relativistic theory. This observation increases our confidence in the approach, based on the superposition principle (3.7).

To prove this statement one should take into account that:

a) \( \lim_{c \to \infty} \varphi_A^{\infty} = -\infty \) and \( \lim_{c \to \infty} \left(\frac{\varphi_A^{\infty}}{c^2}\right) = 0 \) – according to the relations (2.32).

b) Using the last relations in the consistency condition (3.12) one obtains easily first

\[ \lim_{c \to \infty} \left(\frac{\varphi_A}{c^2}\right) = 0, \tag{3.13} \]

and then

\[ \lim_{c \to \infty} \varphi_A = -\infty. \tag{3.14} \]

c) As a result \( g_A = e^{\varphi_A/c^2} \to 1 \) and \( m_A = g_A M_A \to M_A \) when \( c \to \infty \).

### 3.1.4 The Case of Continuous Mass Distribution

The generalization of the equations (3.7) for continuous distribution of mass is straightforward and reads:

\[ \varphi(r) = -\int_{\mathcal{M}^{(3)}} d^3r' \frac{\mu(r') e^{\varphi(r')}}{|r - r'| + R(r')} \tag{3.15} \]

Here \( \mu(r) \) is the density of bare mass \( M \). In the case of continuous mass distribution the relativistic shift \( R \) is \( R(r) = \mu(r)/\chi^{\infty}(r) \). The local density \( \chi^{\infty}(r) \) of the quantity \( x^{\infty} \) is the second independent function, needed for description of the density of the relativistic point potential in the case of continuous mass distributions.

The relation (3.15) generalizes and replaces the non-relativistic superposition principle (1.4) for continuously distributed masses. The last can be derived from equation (3.15), taking the limit \( c \to \infty \) precisely in the same way, as in the discrete case.

From mathematical point of view the relation (3.15) is a nonlinear and nonsingular integral equation for the relativistic potential \( \varphi(r) \):

\[ \varphi(r) = \int_{\mathcal{M}^{(3)}} d^3r' K(r, r') e^{\varphi(r')} \tag{3.16} \]

with a nonsingular kernel

\[ K(r, r') = -\frac{\mu(r')}{|r - r'| + R(r')} \tag{3.17} \]

For continuous distribution of identical particles with fixed \( \chi^{\infty} = \text{const} \) one has to put \( R(r) = \mu(r)/\chi^{\infty} \) in (3.17), thus remaining with only one given function \( \mu(r) \geq 0 \) in the kernel \( K(r, r') \). The function \( \mu(r) \) reduces to a given constant \( \mu = \text{const} \) for homogeneous mass distributions. Hence, in the last case \( R = \text{const} \), too.
3.2 Some Basic Solutions of the Mass Defect Equation

In this Subsection we are testing our basic assumptions, described in the previous Sections, considering both the cases of a few point particles, and of continuous mass distributions.

For this purpose is convenient to introduce the quantity

$$x := \frac{M}{R} = \frac{1}{\varrho} \ln \frac{1}{\varrho} \geq 0,$$  \hspace{1cm} (3.18)

which turns to play a basic role in our considerations and demonstrates some simple properties. Making use of the Lambert function $W(z)$ [23], i.e., the solution of the equation

$$W(e^W) = z \Rightarrow W = W(z),$$

one obtains, solving the equation (3.18) with respect to the mass ratio $\varrho$, the basic relation (see Fig. 1)

$$\varrho = \varrho(x) = e^{-W(x)}.$$  \hspace{1cm} (3.19)

![Figure 1: The dependence (3.19) of the mass ratio $\varrho$ on the variable $x$.](image)

Now one can write down the basic equations (3.10) in the form:

$$\ln \left( \frac{1}{\varrho_A} \right) = \sum_{B=1}^{N} \frac{x_B^\infty \varrho_B}{1 + r_{AB}/R_B}.$$  \hspace{1cm} (3.20)

We shall call this equations mass defect equations (MDE). One immediately obtains from MDE two important consequences:

3.2.1 A Few Particle Solutions

1. Let us consider first the two-particle solutions ($N=2$) of the MDE:

$$\ln \left( \frac{1}{\varrho_1} \right) = x_1^\infty \varrho_1 + \frac{x_2^\infty \varrho_2}{1 + r_{12}/R_2},$$

$$\ln \left( \frac{1}{\varrho_2} \right) = \frac{x_1^\infty \varrho_1}{1 + r_{12}/R_1} + x_2^\infty \varrho_2.$$  \hspace{1cm} (3.21)
For them we have two interesting limiting cases:

i) Total decay of the two-particle system, when \( r_{12} \to \infty \). Then obviously

\[
\varrho_1(r_{12}) \to \varrho_1^\infty, \quad \varrho_2(r_{12}) \to \varrho_2^\infty,
\]
as one expects.

ii) Merger of two particles \( r_{12} \to 0 \):

\[
\varrho_{1,2}(\varrho_1^\infty; \varrho_2^\infty) = \exp \left( - W (x_1^\infty + x_2^\infty) \right).
\]

Note that for the quantities \( x \) we obtain a simple linear superposition:

\[
x_{1,2} = x_1^\infty + x_2^\infty. \tag{3.22}
\]

As a result

\[
\frac{1}{R_{1,2}} = \frac{\mu_1}{R_1} + \frac{\mu_2}{R_2} =: \left\langle \frac{1}{R} \right\rangle_\mu \quad \mu_{1,2} = \frac{M_{1,2}}{M_1 + M_2} \in [0,1]. \tag{3.23}
\]

Introducing the quantities

\[
\delta M := \frac{M_1 - M_2}{M_1 + M_2} \quad \text{and} \quad \delta R := \frac{R_1 - R_2}{R_1 + R_2},
\]

we obtain finally

\[
R_{1,2} = \langle R \rangle_{ar} \frac{1 - \delta R^2}{1 - \delta M \delta R}. \tag{3.25}
\]

Here \( \langle R \rangle_{ar} := \frac{1}{N} \sum_{A=1}^{N} R_A \) denotes the arithmetic average. In the present case \( N = 2 \).

Now we can derive easily the following basic properties of the function \( R_{1,2} = R_{1,2}(R_1, R_2; M_1, M_2) \):

1) \( R_{1,2}(R_1, R_2; M_1, M_2) = R_{1,2}(R_2, R_1; M_1, M_2) \).
2) \( R_{1,2}(R_1, R_2; M_1, M_2) = R_{1,2}(R_1, R_2; M_2, M_1) \).
3) \( R_{1,2}(kR_1, kR_2; M_1, M_2) = kR_{1,4}(R_1, R_2; M_1, M_2), \ \forall k > 0 \).
4) \( R_{1,2}(R_1, R_2; kM_1, kM_2) = R_{1,6}(R_1, R_2; M_1, M_2), \ \forall k > 0 \).
5) \( R_{1,2}(R_1, R_2; M_1, M_2) = R, \ \forall M_1, M_2 \).
6) \( R_{1,2}(R_1, R_2; M, M) = \frac{2R}{R_1 + R_2}, \text{ or} \)

\[
R_{1,2} \langle R \rangle_{ar} = \langle R \rangle_{gm}^2.
\]

Here \( \langle R \rangle_{gm} := \sqrt{R_1 R_2} \) denotes the geometric average.

9) \( R_{1,2}(R_1, 0; M_1, M_2) = 0 \).
10) \( R_{1,2}(R_1, \infty; M_1, M_2) = \frac{R_1}{\mu_1} = \frac{M_1 + M_2}{x_{1,2}}, \text{ where} \ x_{1,2} = x_1 \).

iii) In the case of two identical particles: \( M_1 = M_2 = M, \ R_1 = R_2 = R \), at finite distance \( r_{12} \) one easily obtains for \( \varrho_1(r_{12}) = \varrho_2(r_{12}) = \varrho(r_{12}) \)

\[
\varrho(r_{12}) = \exp \left( - W \left( \frac{2 + r_{12}/R}{1 + r_{12}/R} x_{1,2} \right) \right) \tag{3.26}
\]
Figure 2: The dependence (3.26) of the mass ratio $\rho(r_{12}) = m(r_{12})/M$ of system of the two identical particles with different individual mass ratios $\rho^\infty = 0.1, \ldots, 0.9$; on the distance $r_{12}$ between them. The distance $r_{12}$ is shown in units of $R$.

and

$$\rho_{1\cup2}(\rho^\infty) = \exp\left(-W\left(\frac{2}{\rho^\infty} \ln \frac{1}{\rho^\infty}\right)\right).$$

The small variation of the mass ratio of each of the two identical particles in the system with the change of the distance between the particles is shown in Fig. 2. As seen, at distances $r_{12} \gg R$ the measurable Keplerian mass of each particle is practically constant, since the value of $\rho$ is almost constant.

Figure 3: The dependence (3.27) of the mass ratio difference $\Delta\rho(x^\infty)$ on the individual value of $x^\infty$ for the two identical particles.

The formula

$$\Delta\rho(x^\infty) = \rho(0) - \rho(\infty) = e^{-W(x^\infty)} - e^{-W(2x^\infty)}$$

shows that the variation of the mass ratio of each particle $\rho(r_{12})$ is not bigger than
\[ \approx 0.1408, \text{ when the distance between them varies from zero to infinity: } r_{12} \in (0, \infty), \text{ see Fig. 3}. \]

![Figure 4: The dependence (3.28) of the energy release (in units $2Mc^2$) on the individual mass ratio $\varrho^\infty$ of the two identical particles.](image)

The energetic efficiency of the process of gravitational merger of two identical particles is described by the quantity:

\[
\frac{\Delta E_G}{2Mc^2} = 1 - \varrho_{1\cup2}(\varrho^\infty)/\varrho^\infty.
\] (3.28)

Here $\Delta E_G$ is the energy release in the gravitational collapse of the pair of the two identical point particles from infinite distance to their merger.

2. For three identical particles at the vortices of equilateral triangle $r_{AB} = r$ for all $A, B = 1, 2, 3; A \neq B$ one obtains:

\[
\varrho(r) = \exp \left( -W \left( \frac{3 + r/R}{1 + r/R} x^\infty_A \right) \right).
\] (3.29)

3. For four identical particles at the vortices of equilateral tetrahedron $r_{AB} = r$ for all $A, B = 1, 2, 4; A \neq B$ the result is:

\[
\varrho(r) = \exp \left( -W \left( \frac{4 + r/R}{1 + r/R} x^\infty_A \right) \right).
\] (3.30)

### 3.2.2 Some Basic Properties of the N-Particles Solutions

Consider now a system of $N$ particles:

i) If the system of $N$ point particles decays, i.e., when $r_{AB} \to \infty$ for all $B \neq A$ and we remain with only one such particle in the whole universe, then $x_A \to x^\infty_A$, as it should be.

ii) In the opposite case, when all $r_{AB} \to 0$, i.e. when the system $N$ particle collapses and they merge into one composite particle, from MDE one obtains $\varrho_A = \varrho_B = \text{const} = \varrho_{1\cup2\ldots\cup N}$.
for all A, B. For the resulting mass defect ratio one has
\[ \varrho_{1\cup\ldots\cup N} = \exp \left( -W \left( x_{1\cup\ldots\cup N} \right) \right) = \exp \left( -W \left( \sum_{A=1}^{N} x_A \right) \right) \] (3.31)
and the simple linear superposition:
\[ x_{1\cup\ldots\cup N} = \sum_{A=1}^{N} x_A. \] (3.32)

iii) For an aggregate of N fused particles of total bare mass M we obtain:
\[ 1/R_{1\cup\ldots\cup N} = \sum_{A=1}^{N} \mu_A/R_A =: \langle 1/R \rangle_\mu, \] (3.33)
\[ \mu_A = M_A/M \in (0, 1), \quad \sum_{A=1}^{N} \mu_A = 1. \]

iv) Using the asymptotic of function W(z) one easily obtains for the merger of N identical point particles, each of bare mass M and relativistic shift R
\[ \varrho_{1\cup\ldots\cup N} = \exp \left( -W \left( N \frac{M}{R} \right) \right) \sim \frac{\ln \left( N \frac{M}{R} \right)}{N \frac{M}{R}} - \text{when } N \to \infty \] (3.34)
and
\[ R_{1\cup\ldots\cup N} = R, \quad \forall N. \] (3.35)

Here R is the relativistic shift of the separate particle.

These formulas show that the accumulation of particles during the merger of N particles leads to increase of the mass defect, since \( \varrho_{1\cup\ldots\cup N} \to 0 \) when \( N \to \infty \) – much like the situation in nuclear physics, as it should be from physical point of view.

### 3.2.3 Some Solutions of the Mass Defect Equation for Continuous Distributions

To acquire some experience working with solutions of the MDE in the case of continuous mass distributions, we will consider in this subsection several simple examples of such distributions of identical particles with a constant density \( \mu \) and a simple geometry:

1. First we consider one-dimensional continuous mass distribution on a homogeneous circle of diameter \( d \), and linear mass density \( \mu_1 \), made of identical particles.

From a symmetry reasons at the very circle one has \( \varphi = \text{const.} \). Then one has to calculate a simple integral in the rhs of Eq. (3.15). Thus one obtains for the mass ratio the following expression:
\[ \varrho(d) = \exp \left( -W \left( \frac{\mu_1}{\sqrt{1-R^2/d^2}} \ln \frac{1+\sqrt{1-R^2/d^2}}{1-\sqrt{1-R^2/d^2}} \right) \right). \] (3.36)
Figure 5: The dependence of the mass ratio $\varrho_{\text{circle}} = m/M$ on the diameter $d$ of circles with different fixed total bare masses $M$. The diameter $d$ is shown in units of $R$.

Suppose that the total bare mass of the circle is $M$. Then $\mu_1 = M/\pi d$. Replacing $\mu_1$ in the relation (3.36) with this value, we obtain the result, shown in the Fig. 5. As seen, $\varrho(0) = 0$ and when the masses of the circle are dispersed at bigger distances, increasing its diameter, the mass ratio increases and goes to 1 for $d \to \infty$, as it should be from physical point of view.

2. Our second example is a two-dimensional continuous mass distribution on a homogeneous sphere of diameter $d$ and surface mass density $\mu_2 = 4M/\pi d^2$, made of identical particles. In this case one obtains in a similar way for the mass ratio:

$$\varrho(d) = \exp \left( -W \left( \frac{8M}{d} \left( 1 - \frac{R}{d} \ln \left( 1 + \frac{d}{R} \right) \right) \right) \right).$$  \hspace{1cm} (3.37)

As seen in Fig. 6 when the masses of the sphere are dispersed at bigger distances, increasing the diameter, the mass ratio increases and goes to 1, when $d \to \infty$, as it should be from physical point of view.

In addition we see that in this example the mass ratio goes to zero for the non-physical value of sphere diameter $d = -R$. In contrast to the previous example, now one has $\varrho(0) = e^{-W(4M/R)} > 0$. This new property reflects the peculiar geometry of the spheres around the point sources: the limit of the sphere surface area remains finite when the sphere diameter goes to zero.

3. For a homogeneous three-dimensional ball of radius $r_*$ one easily obtains the one dimensional integral equation:

$$\varphi(r) = \int_0^{r_*} dr' k(r, r') e^{\varphi(r')}$$  \hspace{1cm} (3.38)

with kernel

$$k(r, r') = -\frac{2\pi \mu}{r - r'} \left( r + r' - |r - r'| - R \ln \left( \frac{r + r' + R}{|r - r'| + R} \right) \right).$$  \hspace{1cm} (3.39)
Let us consider the limiting case of Eq. (3.38) with $R = 0$, and let us denote $\varphi[0] = \varphi|_{R=0}$. Then we obtain the following Debye-Hückel like boundary problem:

$$
\frac{d^2\varphi[0]}{d\xi^2} + \frac{2}{\xi} \frac{d\varphi[0]}{d\xi} = e^{\varphi[0]}, \quad \xi = \sqrt{4\pi \mu r},
$$

$$
\frac{d\varphi[0]}{d\xi}(0) = 0, \quad \xi_* \frac{d\varphi[0]}{d\xi}(\xi_*) + \varphi(\xi_*) = 0.
$$

(3.40)

This problem has no auxiliary parameters and defines an *universal* function $\varphi[0](\xi, \xi_*)$. The solution gives the zero order term in the exact solution of the integral equation (3.38): $\varphi(r) = \varphi[0](\sqrt{4\pi \mu r}) + O(R)$ and obviously describes the gravitational screening of the bare mass in the present simple case of three-dimensional continuous mass distribution.

It is not hard to find the numerical solutions of the problem (3.40).

4 Some Concluding Remarks

There exist at least three different approaches to the relation between physics and geometry:

A) The classical physics considers the space-time continuum only as arena for the struggles of fields and particles. "These entities are foreign to geometry. They must be added to geometry to permit any physics" (see, for example, Misner and Wheeler in [17]). In this approach the geometry is completely independent of physics.

B) According to original Einstein’s idea, the geometry of space-time is determined by the matter sources of gravity. Masses and non-gravitational fields are of non-geometrical origin. To some extend these sources of gravity are independent of geometry entities. In their presence the space-time continuum acquires the geometrical properties of curved 4D pseudo-Riemannian manifold. Its geometry determines the motion and evolution of the very matter. Hence, in this approach we have a complicated interplay between geometry and physics, based on nonlinear differential equations.
C) There is a third way, pioneered by Einstein and Rosen and developed in pure "geometrodynamics" by Wheeler and others. In this approach one may think that "There is nothing in the world except empty curved space. Matter, charge, electromagnetism and other fields are only manifestation of the bending of space. Physics is geometry.", see Misner and Wheeler in [17]. In this approach we have "mass without mass, charge without charge", etc. At present the most well known hypothetical notions, created in the framework of this approach are the "black holes" and the "wormholes" of different type in space-time. They are consequences of the assumption to have "everywhere empty curved space", see Wheeler in [17].

In the present article we accept the original Einstein’s idea, described in point B, about the relation between mater and geometry of space-time. Here we develop the mathematical realization of this idea considering matter point particles as sources of gravity in GR. We have reached the following basic results:

1. We have studied a new, two parameter class of solutions of Einstein equations. These static spherically symmetric solutions describe the gravitational field of massive point particle with bare mass $M > 0$ and Keplerian mass $m$ ($0 < m < M$). The difference between these masses, or their ratio $\rho = m/M \in (0, 1)$, defines the gravitational mass defect of the point particle. Such mass defect was not considered and studied until now, because for the standard Hilbert form (1.6) of the Schwarzschild solution "the bare rest-mass density is never even introduced" [27].

2. The new solutions form a two parameter family of metrics on singular manifolds $\mathbb{M}^{(1,3)}\{g_{\mu\nu}\}$, described in details in the present article, as well as in [16, 22].

3. We have shown the principal role of the massive point source of gravity. Its presence offers a natural cutting for the physical values of the luminosity variable $\rho \in [\rho_0, \infty)$, where $\rho_0 > \rho_G$. This happens because the infinite mass density of the matter point changes drastically the geometry of the space-time around it. This phenomenon is in sharp contrast to the situation in geometrodynamics, where luminosity variable may have an arbitrary close to zero value around the center of the black holes.

A geometry of space-time with $\rho_0 \equiv \rho_G > 0$ was discovered at first in the original article by Schwarzschild [3]. According to Eq. (2.24), such limiting value of the luminosity variable corresponds to zero value $\rho = 0$ of mass defect ratio. For a finite value of $m$ this is possible only if $M = \infty$. In this sense our work is a proper extension of the Schwarzschild one to the physically and mathematically admissible values of the mass defect ratio $\rho \in (0, 1)$.

4. The existing attempts to describe the point matter source using Schwarzschild solution in Hilbert gauge (1.6) do not take into account the mass defect and thus fail to present a point idealization of the real relativistic objects.

5. In full accord with Dirac’s suggestion [21] our cutting of the domain of luminosity variable places the event horizon in the nonphysical domain of the variables. This effect is well known from the solutions of Einstein equations with massive matter sources of finite dimension [1].

6. The mathematical and the physical properties of the new solutions are essentially different in comparison with the well known other spherically symmetric static solutions to the Einstein equations. All of the new solutions have a strong singularity at the center of the symmetry, which is surrounded by empty space. They describe the single point particle sources of gravity in GR. The previously known static spherically symmetric solutions were often erroneously considered as a solutions, which describe a point mass,
but this is not the case.

7. It is clear that our solutions in generalized functions define in mathematical sense the fundamental solutions of the quasi-linear Einstein equations. These solutions are complete analogous to the fundamental solutions of Poisson equation in Newton theory of gravity. Thus the problem, formulated by Feynman in [11] is solved.

8. A proper quasi-linear superposition principle for initial conditions of Einstein equations exist. It leads to a new theory of the relativistic gravitational defect of mass, illustrated in short in the present article. Our study shows that the hypotheses, used in this approach to the superposition principle in GR lead to physically reasonable consequences. One has to put on a more profound basis these hypotheses and to compare their consequences with the physical reality.

9. Our results show that it may turn to be possible to transform the original Einstein relativistic theory of gravity, without change of its dynamical equations, into a normal physical theory with basic properties, which are intrinsic to the other branches of physics. This may permit us to get out of the way some of the specific scientific fictions, which are widespread at present and are thought to be an unavoidable consequences of GR.

Especially, the correct theory of N matter particles systems seems to lead to the space-times with Euclidean topology, global time and standard physical causality. Hopefully, these important properties may transmute the now-days form of GR into a new version of the relativistic theory of gravity, compatible with the standard relativistic quantum mechanics of particles, much like the theories in the other branches of physics, and despite of the curvature of the space-time.

10. Our basic conclusion is that we urgently need critical experiments and observations, which can help us to choose the version of Einstein relativistic theory of gravity, compatible with the physical reality. Some idea of such type of experiments was recently proposed and discussed in [28].

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