Breakdown of a renormalized perturbation expansion around mode-coupling theory of the glass transition

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Abstract – We analyze a renormalized perturbation expansion around the mode-coupling theory of the glass transition. We focus on the long-time limit of the irreducible memory function. We discuss a renormalized diagrammatic expansion for this function and re-sum two infinite classes of diagrams. We show that the resulting contributions to the irreducible memory function diverge at the mode-coupling transition. A further re-summation of ladder diagrams constructed by iterating these divergent contributions gives a finite result which cancels the mode-coupling theory’s expression for the irreducible memory function.

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Introduction. – Since its introduction almost thirty years ago, the mode-coupling theory (MCT) [1–4] has significantly contributed to our understanding of the slowing-down of a fluid’s dynamics upon approaching the glass transition. In particular, the theory accounts for the cage effect: in a super-cooled fluid a given particle spends considerable time in its solvation shell before making any further motion. This physical picture results in intermediate time plateaus in the mean-square displacement and the intermediate scattering function. Both the values of the plateaus of these functions and their time dependence in the region of the plateaus are well described by the mode-coupling theory [5–8]. More generally, the theory accurately describes the initial stages of the slowing-down upon super-cooling. However, it also predicts a spurious dynamic transition, the so-called mode-coupling transition. It predicts that upon further super-cooling the time scale of the relaxation from the plateau region diverges as a power law. Extensive simulational studies showed that instead of this transition there is a smooth cross-over, with relaxation times and transport coefficients following mode-coupling–like power laws only for a few decades [9].

The microscopic derivation of the mode-coupling theory allows, in principle, for its extension, which could address the above-mentioned fundamental problem. Two such extensions were proposed shortly after the original theory was derived [10,11]. More recently, these extensions were critically assessed and found somewhat inadequate [12]. In addition to the arguments presented in ref. [12], the reason why neither of these extensions is satisfactory is that they rely upon couplings to current modes that are defined for systems with Newtonian dynamics. Thus, these extensions cannot be applied to systems with Brownian dynamics. In contrast, computer simulations showed that deviations from mode-coupling–like power laws are qualitatively the same in systems with both dynamics [13].

In fact, most of our current understanding of the significance of the processes neglected within the mode-coupling theory is indirect. Within the so-called Franz-Parisi potential approach (originally introduced in the context of mean-field spin glasses [14] but later generalized to super-cooled fluids [15]), the mode-coupling transition is identified with the appearance of a metastable state with a non-zero value of the overlap between the fluid and a template. This metastable state becomes the absolute minimum of the Franz-Parisi potential only at the thermodynamic glass transition. This approach suggests that the mode-coupling theory treats the metastable state as absolutely stable, thus neglecting two types of dynamic events. First, near the transition there should be critical fluctuations. At the very least, these fluctuations should move the mode-coupling transition towards lower temperatures and/or higher densities. Second, if the transition survives the inclusion of the fluctuations, it should be cut off by activated events, which are sometimes referred to as “hopping processes”. This general physical picture is supported
by results obtained for spin glass models with long-but-finite-range interactions [16,17]. However, explicit calculations on particle-based models have been lacking until the recent analysis of critical fluctuations [19]. This analysis is based on a static replica field theory approach.

The main problem with extending the dynamic approach beyond the mode-coupling theory comes from the original, projection operator derivation of the theory [4]. This derivation is rather opaque and, therefore, it is difficult to generalize. A more promising avenue is to start from a diagrammatic expansion and then resort to re-summations. However, most diagrammatic expansions derived to date [23–26] are quite complicated and, therefore, are unlikely to produce results going beyond the mode-coupling theory.

Recently, one of us developed an alternative diagrammatic approach [27]. To make some technical steps easier, it was assumed that the microscopic dynamics is Brownianian. The same assumption will be used in this letter. The starting point of the derivation presented in ref. [27] is the hierarchy of equations of motion for correlation functions of many-particle densities orthogonalized with respect to the equilibrium probability distribution. Using orthogonalized densities results in two advantages. First, bare inter-particle interactions get replaced by renormalized interactions, which can be expressed in terms of the derivatives of the equilibrium correlation functions. Second, the initial condition for the set of correlation functions of orthogonalized densities is very simple, which simplifies the structure of the diagrams. The second step of the derivation is a perturbative solution of the hierarchy of equations of motion. The terms in the expansion around the mode-coupling theory are represented by diagrams. After the expansion is derived, one can use all standard diagrammatic techniques including re-summations and Dyson equation-type analyses. Within this approach, the mode-coupling theory amounts to a self-consistent one-loop approximation for the so-called irreducible memory function [28]. Notably, the structure of the diagrammatic expansion of ref. [27] is relatively simple. This made it possible to use this approach to show that a four-point dynamic density correlation function contains a divergent contribution [29] and to evaluate two corrections to the mode-coupling expression for the long-time limit of the memory function [30].

Here, we significantly extend this last contribution. Again, we focus on the long-time limit of the irreducible memory function. Thus, like in static approaches we deal with time-independent quantities. However, in contrast to the latter approaches, our theory is derived from dynamics. We use the new, renormalized diagrammatic expansion for the memory function, which was suggested in ref. [30]. We discuss two infinite classes of renormalized diagrams, which have a clear physical interpretation. After re-summation, these two classes of diagrams result in corrections to the mode-coupling approximation that diverge upon approaching the mode-coupling transition in dimensions \( D < 4 \). We note that a subsequent re-summation of a whole series of divergent contributions produces a result, which is finite at the mode-coupling transition, but cancels the original mode-coupling contribution to the irreducible memory function. Our findings suggest a breakdown of a renormalized perturbation expansion around the mode-coupling theory.

**Diagrammatic expansion for the irreducible memory function.** – Here, we discuss two different expansions for the irreducible memory function. We start with diagrams introduced in ref. [27]. Briefly, in these diagrams, bonds represent bare propagation of density fluctuations, vertices represent renormalized interactions and diagrams with an odd number of four-leg vertices contribute with a negative sign indicated next to the diagram. The bonds are defined only for positive times. The time direction is indicated by an arrow attached to a bond.

The irreducible memory function \( M_{\text{irr}}^t(k;t) \) contains all non-trivial information about the dynamics of the system. Using a projection operator approach one can derive an exact but formal equation for \( M_{\text{irr}}^t \), which involves the so-called irreducible evolution operator that has single-particle dynamics projected out [28]. It was showed in ref. [27] that the irreducible memory function is represented by a sum of all diagrams that start with the right vertex and end with the left vertex, and are one-particle irreducible, *i.e.* they do not separate into disconnected components upon removal of a single bond or a single four-leg vertex. The first few diagrams contributing to the irreducible memory function are showed in fig. 1. Furthermore, it was showed in ref. [27] that the mode-coupling approximation amounts to including only those diagrams that separate into two disconnected components upon removing the left and right vertices. Thus, out of diagrams showed in fig. 1 one keeps only diagrams (a), (b). After all such diagrams are re-summed, one gets a diagram whose topology is identical to that of diagram (a) in fig. 1, but with bare propagators replaced with the full propagators.

![Fig. 1: The first few diagrams contributing to the irreducible memory function \( M_{\text{irr}}^t(k;t) \).](image)

From the point of view of the topology of the diagrams, the simplest diagrams neglected in the mode-coupling...
approximation$^3$ are the diagrams that separate into disconnected components upon removing the left and right vertices, and subsequent removing of a single propagator or a single four-leg vertex, and which satisfy the following condition: each of the components should be one of the diagrams included in the mode-coupling approximation. Diagrams (c)–(e) and (g), (h) in fig. 1 belong to this class. Diagram (f) separates into disconnected components upon removing the left and right vertices, and removing of two successive four-leg vertices. We will argue below that this diagram needs to be included together with the simplest non–mode-coupling diagrams. In contrast, diagrams (i), (j) in fig. 1 are not the simplest non–mode-coupling diagrams. Diagram (i) separates into two components upon removing the left and right vertices, and removing of a single propagator, but one of the resulting components is a diagram not included in the mode-coupling approximation. Diagram (j) separates into disconnected components upon removing the left and right vertices, and removing both a four-leg vertex and a single propagator.

The mode-coupling expression for the memory function has the following important property: if the full propagator has a non-zero long-time limit, the memory function also has a finite, non-zero long-time limit. This is consistent with the exact equation expressing the time derivative of intermediate scattering function $F(k;t)$ in terms of $M^{\text{irr}}(k;t)$ [27,28,31]. Namely, it follows from this equation that if the scattering function has a non-zero limit, $\lim_{\tau \to \infty} F(k;t) = F(k) > 0$, then the irreducible memory function also has a non-zero limit, $\lim_{\tau \to \infty} M^{\text{irr}}(k;t) = (D_0 k^2/S(k))n_m(k) > 0$, where $S(k)$ is the static structure factor and $D_0$ is the diffusion coefficient of an isolated Brownian particle, and $F(k)$ and $m(k)$ are related by the following equation, $F(k)/(S(k) - F(k)) = m(k)$. We shall emphasize that the only assumption used in writing this equation is that both the intermediate scattering function and the irreducible memory function have finite non-zero limits. To simplify the nomenclature we will hereafter refer to $m(k)$ as the long-time limit of the irreducible memory function.

As discussed in ref. [30], when calculating non–mode-coupling contributions to $m(k)$ one has to make sure that the above-discussed consistency is maintained. Thus, e.g., one cannot naively replace bare propagators by full propagators in diagram (d) in fig. 1, but one has to also include diagram (f) and a whole class of similar diagrams. In this way one gets a new diagrammatic expansion in which all diagrams remain finite even if $F(k;t)$ has a non-zero long-time limit. The first few diagrams of the resulting expansion for $m(k)$ are shown in fig. 2. In these diagrams bonds $\blacktriangleright$ represent the long-time limit of the full intermediate scattering function, $F(k)$, the right outside vertex $\triangleright$ represents $n_1 \tilde{v}_{k_1+k_2}(k_1,k_2)$ where $n$ is the number density and $\tilde{v}_{k_1+k_2}(k_1,k_2) = (k_1 + k_2) \cdot (c(k_1)k_1 + c(k_2)k_2)/|k_1 + k_2|^2$ where $c(k)$ is the direct correlation function, the left outside vertex $\triangleright$ represents $S(|k_1 + k_2|)\tilde{v}_{k_1+k_2}(k_1,k_2)$, the right vertex inside the diagram $\blacktriangleright$ represents $n_1 \tilde{v}_{k_1+k_2}(k_1,k_2)/m(|k_1 + k_2|)$ and the left vertex inside the diagram $\blacktriangleright$ represents $\tilde{v}_{k_1+k_2}(k_1,k_2)/m(|k_1 + k_2|)$. Finally, the four-leg vertex $\blacktriangleright$ is a sum of two parts. The more compact part reads $n_1 S(|k_1 + k_2|)\tilde{v}_{k_1+k_2}(k_1,k_2)/|k_1 + k_2|$ and the second part will be presented elsewhere [32].

Diagrams contributing to $m(k)$ in the expansion showed in fig. 2 have the following property: one cannot cut a part out of these diagrams by removing two bonds, one bond and one four-leg vertex, or two four-leg vertices.

Divergent corrections to the mode-coupling contribution. – The first diagram in fig. 2 represents the mode-coupling contribution to $m(k)$. The contributions represented by the second and third diagrams were calculated in ref. [30]. In that calculation, in the spirit of a perturbative expansion around mode-coupling theory, the exact long-time limits of the full scattering and memory functions, $F(k)$ and $m(k)$, were replaced by mode-coupling approximations for these functions. It was showed that these two diagrams make a non-negligible contribution, which is negative at the mode-coupling transition.

Here we re-sum two infinite classes of diagrams contributing to $m(k)$. Again, in the spirit of a perturbative expansion around mode-coupling theory, the exact long-time limits of the full scattering and memory functions will be replaced by mode-coupling approximations for these functions. The two classes of diagrams are showed in fig. 3. The original motivation for considering these diagrams comes from the fact that they originate from the simplest non–mode-coupling diagrams discussed in the preceding section (diagrams (d)–(h) in fig. 1). Each of these diagrams separates into disconnected components upon removing the left and right vertices and subsequently

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These diagrams originate from couplings between different dynamic modes that are neglected in the mode-coupling theory. For brevity we will refer to them as non–mode-coupling diagrams.
removing one bond or one four-leg vertex. In turn, each of the resulting components has a characteristic rainbow-like insertion.

To show that two classes of diagrams presented in fig. 3 can be re-summed we start from the analysis of rainbow diagrams showed in fig. 4. If we take \( \mathbf{k} + \mathbf{q}/2 \) as the incoming wave vector, \( \mathbf{k} - \mathbf{q}/2 \) as the outgoing one and \( \mathbf{q} \) as a side wave vector, we see that the sum of the rainbow diagrams showed in fig. 4, which will be denoted by \( \chi^0(\mathbf{q}) \), satisfies the following linear integral equation:

\[
\chi^0(\mathbf{q}) = n F(\|\mathbf{k} - \mathbf{q}/2\|) \frac{\mathbb{I}_\mathbf{k} + \mathbb{I}_\mathbf{q}/2}{\mathbb{M}(\mathbf{k}, \mathbf{k}_1)\chi(\mathbf{k}_1)} + \int \frac{dk_1}{(2\pi)^3} \mathbb{M}(\mathbf{k}, \mathbf{k}_1)\chi^0(\mathbf{k}_1),
\]

(1)

where the source term, \( \chi^0(\mathbf{k}) \), is given by

\[
\chi^0(\mathbf{k}) = n F(\|\mathbf{k} - \mathbf{q}/2\|) \frac{\mathbb{I}_\mathbf{k} + \mathbb{I}_\mathbf{q}/2}{m(\|\mathbf{k} + \mathbf{q}/2\|)} F(\|\mathbf{k} + \mathbf{q}/2\|),
\]

(2)

and

\[
\mathbb{M}(\mathbf{k}, \mathbf{k}_1) = n F(\|\mathbf{k} - \mathbf{q}/2\|) \frac{\mathbb{I}_\mathbf{k} + \mathbb{I}_\mathbf{q}/2}{m(\|\mathbf{k} - \mathbf{q}/2\|)} F(\|\mathbf{k} - \mathbf{q}/2\|)\frac{\mathbb{I}_\mathbf{k} + \mathbb{I}_\mathbf{q}/2}{m(\|\mathbf{k} + \mathbf{q}/2\|)}.
\]

(3)

The linear integral operator at the right-hand side of eq. (1) coincides with the operator introduced by Biroli and Bouchaud [33] and subsequently re-derived by Biroli et al. [34] and by one of us [29]. This is consistent with the fact that the rainbow diagrams contribute to the divergence of both a three-point susceptibility introduced by Biroli et al. [34] and a four-point correlation function of ref. [29]. Thus, we can conclude that re-summing rainbow diagrams (and, more generally, re-summing two classes of diagrams showed in fig. 3) amounts to adding corrections to mode-coupling theory that originate from critical fluctuations.

At \( \mathbf{q} = 0 \), \( \mathbb{M}(\mathbf{k}, \mathbf{k}_1) \) becomes the stability matrix of the mode-coupling theory [4] (remember that mode-coupling expressions for \( F(\mathbf{k}) \) and \( m(\mathbf{k}) \) are used in eq. (3)). Upon approaching the mode-coupling transition its largest eigenvalue approaches 1,

\[
\int \frac{dk_1}{(2\pi)^3} \mathbb{M}(\mathbf{k}, \mathbf{k}_1)\hat{\mathbf{c}}(\mathbf{k}_1) = \left(1 - 2g(1 - \lambda)\epsilon^{1/2}\right)\hat{\mathbf{c}}(\mathbf{k}),
\]

where \( \hat{\mathbf{c}} \) is the right eigenvector of the stability matrix corresponding to the largest eigenvalue, \( \epsilon \) is the fractional distance from the transition, and \( g \) and \( \lambda \) are standard constants introduced in the mode-coupling analysis [4]. This translates into a divergence of \( \chi_0 \) at the transition. More generally, close to the transition and for small \( \mathbf{q} \), we have

\[
\chi_0(\mathbf{k}) = \epsilon^{-1/2} \frac{n a h^c(\mathbf{k}) S(\mathbf{k})}{1 + \epsilon^{-1/2} \Gamma^2 q^2},
\]

(5)

where \( a \) is proportional to the projection of the \( \mathbf{q} = 0 \) part of the source term on the left eigenvector of the stability matrix corresponding to the largest eigenvalue [4], \( \hat{\mathbf{c}} \),

\[
a = (2n g(1 - \lambda))^{-1} \int_0^\infty dk \hat{h}^c(k) \chi^0(k)/S(k).
\]

(6)

In eq. (5) \( \Gamma \) originates from the small-\( \mathbf{q} \) correction to the largest eigenvalue of \( \mathbb{M}(\mathbf{k}, \mathbf{k}_1) \). \( \Gamma \) can be expressed in terms of the right and left eigenvectors of the stability matrix and equilibrium correlation functions [32].

The above discussion implies that the sum of the class of diagrams showed in the first line of fig. 3 reads

\[
S(k) \int \frac{dk_1 dk}{(2\pi)^6} \hat{\mathbb{I}}_\mathbf{k}(\mathbf{k} - \mathbf{q}/2, \mathbf{k} - \mathbf{k}_1 + \mathbf{q}/2) \times \chi_0(\mathbf{k} - \mathbf{k}_1) F(q) \hat{\mathbb{I}}_\mathbf{q}(\mathbf{k} + \mathbf{q}/2, \mathbf{k} - \mathbf{k}_1 - \mathbf{q}/2).
\]

(7)

Close to the transition the integral is dominated by the diverging small-\( \mathbf{q} \) contribution. In three spatial dimensions, \( D = 3 \), we can use asymptotic formula (5) for all wave vectors. In this way we can show that the singular part of (7) is equal to \( a^2 F(0)d(k) \) where \( a \) is defined in eq. (6) and the function \( d(k) \) is given by the following formula

\[
d(k) = \frac{n^2 S(k)}{\epsilon^{1/2} 8 \pi^{3/2}} \int \frac{dk_1}{(2\pi)^3} \hat{h}^c(k_1, \mathbf{k} - \mathbf{k}_1) \\
\times S(k_1) S((|\mathbf{k} - \mathbf{k}_1|)h^c(|\mathbf{k} - \mathbf{k}_1|).
\]

(8)

Explicit calculation (see fig. 5) shows that \( h^c(\mathbf{k}) \) does not change the sign and this makes \( d(k) \) a positive function. We note that this function diverges as \( \epsilon^{-1/4} \) upon approaching the mode-coupling transition.

The analysis of the class of diagrams showed in the second line of fig. 3 follows the same line of reasoning. The final result is that close to the transition the singular part of the sum of the class of diagrams showed in the second line of fig. 3 is equal to \( bd(k) \) where the coefficient \( b \) is a sum of two terms, \( b = b^{(1)} + b^{(2)} \), which originate from the two parts of the four-leg vertex. The first term reads

\[
b^{(1)} = -\frac{1}{(2g(1 - \lambda))^2} \int_0^\infty dk_1 \frac{\hat{h}^c(k_1)^2}{S(k_1)} \\
\times \left( \int_0^\infty dk_2 \frac{\hat{h}^c(k_2)^2 F^2(k_2)}{S(k_2)} \int \frac{dk_1}{4\pi} \frac{\hat{\mathbb{I}}_\mathbf{k}(k_1, k_2)}{m(|k_1 + k_2|)} \right).
\]

(9)

The second term is given by a lengthier expression and it will be presented elsewhere [32].
Breakdown of a renormalized perturbation expansion around MCT

We numerically calculated all quantities needed to evaluate the singular part of the total contribution, $\delta m(k) = (a^2 F(0) + b)d(k)$. The only input in this calculation is the static structure factor $S(k)$, which we calculated for the hard-sphere interaction potential using the Percus-Yevick approximation. We used 300 equally spaced wave vectors with spacing $\delta = 0.2$, between $k_0 = 0.1$ and $k_{\text{max}} = 59.9$. For the hard-sphere system the only control parameter is volume fraction $\varphi = n\sigma^3/6$, where $\sigma$ is the hard-sphere diameter. For our discretization the mode-coupling transition is located at $\varphi_c = 0.515866763$. In fig. 5 we show the non-ergodicity parameter, $f(k) = F(k)/S(k)$, and the left and right eigenvectors, $h^c(k)$ and $h^c(k)$, calculated at the mode-coupling transition. We used these functions to calculate mode-coupling constants $g = 2.41$, $\lambda = 0.735$, and $\Gamma = 0.0708$, and coefficients $a^2 F(0) = 1.02 \times 10^{-6}$, $b^{(1)} = -0.1065$, and $b^{(2)} = -0.07074$. We see that the contribution of the class of diagrams showed in the second line of fig. 3 dominates and makes $\delta m(k)$ negative.

In fig. 6 we show the mode-coupling result for the long-time limit of the irreducible memory function, $m_{\text{MCT}}(k)$, and the singular part of the contribution from two classes of diagrams showed in fig. 3, $\delta m(k)$. The latter quantity is calculated for $\epsilon = (\varphi - \varphi_c)/\varphi_c = 0.1$. We see that even at such a large $\epsilon$ $m_{\text{MCT}}(k)$ and $\delta m(k)$ are comparable.

On the one hand, the result showed in fig. 6 confirms our intuitive expectations based on the Franz-Parisi potential picture. The sum of the contributions to the long-time limit of the irreducible memory function originating from critical fluctuations is negative and, thus, in a perturbative calculation, the transition would be shifted towards lower temperatures and/or higher densities. On the other hand, we see that these contributions diverge at the transition suggesting that a perturbative expansion around mode-coupling theory breaks down.

Generalizing our results to higher dimensions we see that for $D > 4$ the integral resulting from using asymptotic formula (5) in eq. (8) is IR convergent and UV divergent. The IR contribution is non-analytic in $\epsilon$ for $D < 8$ and thus $D = 8$ is the upper critical dimension, which agrees with earlier results [19,35].

One could argue on general grounds that the full memory function $m(k)$ should be positive, thus in addition to the negative contribution showed in fig. 6 there have to be additional, positive contributions. In three spatial dimensions it is possible to perform an additional re-summation of a class of renormalized ladder diagrams in which the “rungs” are the same as in the diagrams showed in fig. 3. The first few ladder diagrams are showed in fig. 7. One can show that the re-summation of diagrams showed in fig. 3 and of their ladder counterparts showed in fig. 7 results in the following contribution to $m(k)$:

$$
\epsilon^{-1/4} \frac{\eta^2 S(k)(a^2 F(0) + b) d(k)}{8\pi^{3/2}} \int \frac{dk_1 k_1^2}{(2\pi)^3} \left( \frac{1}{S(k_1)} S(|k - k_1|) h^c(k_1) h^c(|k - k_1|) \right) \times
\frac{S(k_1) S(|k - k_1|) h^c(k_1) h^c(|k - k_1|)}{1 - 2\epsilon^{-1/4} \frac{a^2 F(0) + b}{8\pi^{3/2}} \frac{2(k_1) S(|k - k_1|) h^c(k_1) h^c(|k - k_1|)}{S(k_1) S(|k - k_1|) h^c(k_1) h^c(|k - k_1|)}}
$$

(one should remember that $(a^2 F(0) + b) < 0$). We note that as $\epsilon \to 0$ the above contribution is finite and tends to $-m_{\text{MCT}}(k)$. Thus, it cancels the mode-coupling contribution to $m(k)$. This fact suggests that in $D = 3$ the spurious transition predicted by the mode-coupling theory is cut off by the inclusion of critical fluctuations.

**Discussion.** — We showed that re-summations of two infinite classes of diagrams contributing to the long-time limit of the irreducible memory function results in contributions that diverge at the mode-coupling transition. Since we are considering re-summation of terms that diverge as $\epsilon \to 0$, we perform Borel re-summation.
The origin of the divergences are rainbow-like diagrammatic insertions. The same insertions are responsible for the divergence of the three-point susceptibility [34] and the divergent part of a four-point correlation function [29]. This allows us to associate the divergences which we identified with critical fluctuations appearing at the transition. Our results suggest that in three spatial dimensions these fluctuations cut off the transition.

We emphasize that we have only considered the long-time limit of the irreducible memory function. In particular, we do not expect the cancellation that we found to also happen at finite times. We shall mention in this context a very recent preprint [36], which suggests, on the basis of an ingenious mapping of a diagrammatic expansion onto a stochastic field theory, that including all leading-order corrections cuts off the mode-coupling transition but does not significantly change the mode-coupling predictions in the region of the plateaus. Most interestingly, according to ref. [36] the re-summation of the leading-order corrections accounts for activated dynamics.

Our fully microscopic analysis could be compared with a recent static replica field theory investigation of critical correlations at the dynamic transition [19]. At present, both approaches deal with different quantities. We focused on the long-time limit of the irreducible memory function whereas refs. [19] analyzed critical correlations directly. Our approach could be used to investigate the three-point susceptibility of Biroli et al. [34], which also reflects critical correlations directly. This is left for future research.

Finally, from the point of view of the equation for the ergodicity breaking parameter the long-time limit of the irreducible memory function \( m(k) \) obtained from a dynamic theory plays the same role as \( nS(k)\tilde{c}(k) \), where \( \tilde{c}(k) \) is the off-diagonal direct correlation function of a static approach. Thus, our renormalized diagrammatic expansion can be compared with a recent systematic expansion in the non-ergodicity parameter obtained from a static replica approach [37]. We note that these two expansions are different and, thus, the relationship of dynamic and static approaches to the glass transition appears unclear.

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