Quantum orders in an exact soluble model

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We find all the exact eigenstates and eigenvalues of a spin-1/2 model on square lattice: \( H = 16g \sum_i S_i^x S_{i+\bar{y}} S_i^y \). We show that the ground states for \( g < 0 \) and \( g > 0 \) have different quantum orders described by Z2A and Z2B projective symmetry groups. The phase transition at \( g = 0 \) represents a new kind of phase transitions that changes quantum orders but not symmetry. Both the Z2A and Z2B states are described by Z2 lattice gauge theories at low energies. They have robust topologically degenerate ground states and gapless edge excitations.

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Introduction

We used to believe that all phases of matter are described by Landau’s symmetry breaking theory.\(^1\)\(^2\)\(^3\)\(^4\) The symmetry and the related order parameters dominate our understanding of phases and phases transitions for over 50 years. In this respect, the fractional quantum Hall (FQH) states discovered in 1982\(^5\)\(^6\)\(^7\) opened a new chapter in condensed matter physics. The theory of phases and phase transitions entered into a new era. This is because all different FQH states have the same symmetry and hence cannot be described by the Landau’s theory. In 1989, it was realized that FQH states, having a robust topological degeneracy, contain a completely new kind of order - topological order.\(^4\)\(^7\)\(^8\) A whole new theory was developed to describe the topological orders in FQH liquids. (For a review, see Ref.\(^9\).

The Landau’s theory was developed for classical statistical systems which are described by positive probability distribution functions of infinite variables. FQH states are described by their ground state wave functions which are complex functions of infinite variables. Thus it is not surprising that FQH states contain addition structures (or a new kind of orders) that cannot be described by broken symmetries and the Landau’s theory. From this point of view, we see that any quantum states may contain new kind of orders that are beyond symmetry characterization. Such kind of orders was studied in Ref.\(^10\) and was called quantum order. Since we cannot use order parameter to describe quantum orders, a new mathematical object - projective symmetry group (PSG) - was introduced\(^11\) to characterize them. The topological order is a special case of quantum order - a quantum order with a finite energy gap.

One may ask why do we need to introduce a new concept quantum order? What use can it have? To answer such a question, we would like to ask why do we need the concept of symmetry breaking? Is the symmetry breaking description of classical order useful? Symmetry breaking is useful because (A) it leads to a classification of crystal orders, and (B) it determines the structure of low energy excitations without the needs to know the details of a system.\(^10\)\(^11\)\(^12\) The quantum order and its PSG description are useful in same sense: (A) PSG can classify different quantum states that have the same symmetry,\(^11\)\(^13\)\(^14\) and (B) quantum orders determine the structure of low energy excitations without the needs to know the details of a system.\(^11\)\(^13\)\(^14\)\(^15\)\(^16\)\(^17\)\(^18\) The main difference between classical orders and quantum orders is that classical orders produce and protect gapless Nambu-Goldstone modes\(^16\)\(^17\)\(^18\) which is a bosonic excitation, while quantum orders can produce and protect gapless gauge bosons and gapless fermions. Fermion excitations (with gauge charge) can even appear in pure bosonic models as long as the boson ground state has a proper quantum order.\(^16\)\(^17\)\(^18\)

Those amazing properties of quantum orders could fundamentally change our views on the universe and its elementary building blocks. The believed “elementary” particles, such as photons, electrons, etc, may not be elementary after all. Our vacuum may be a bosonic state with a non-trivial quantum order where the “elementary” gauge bosons and the “elementary” fermions actually appear as the collective excitations above the quantum ordered ground state. It may be the quantum order that protects the lightness of those “elementary” particles whose masses are \(10^{20}\) below the natural mass - the Plank mass. Those conjectures are not just wild guesses. A quantum ordered state for a lattice spin model has been constructed\(^19\) which reproduces a complete QED with light, electrons, protons, atoms, ...

The concept of topological/quantum order is also useful in the field of quantum computation. People has been designing different kinds of quantum entangled states to perform different computing tasks. When number of qubits becomes larger and larger, it is more and more difficult to understand the pattern of quantum entanglements. One needs a theory to characterize different quantum entanglements in many-qubit systems. The theory of topological/quantum order\(^14\)\(^15\) is just such a theory. In fact topological/quantum orders can be viewed as patterns of quantum entanglements and gauge bosons the fluctuations of quantum entanglements. Also the robust topological degeneracy in topological ordered states discovered in Ref.\(^10\)\(^11\)\(^12\)\(^13\) can be used in fault-tolerant quantum computation.\(^13\)

It is hard to convince people about the usefulness of quantum orders when their very existence is in doubt. However, a growing list of soluble or quasi soluble models\(^13\)\(^14\)\(^15\)\(^16\)\(^17\)\(^18\) indicates that topolog-
ical/quantum order do exist beyond FQH states. In particular, Kitaev has constructed exactly soluble spin models that realize both topological orders (ie quantum orders with finite energy gap) and gapless quantum orders.\cite{14}

In this paper, we study an exact soluble spin-1/2 model on square lattice: \( H = 16\sum S_i^x S_j^x S_{i+x}^y S_{j+y}^y \). We find that the ground states for \( g < 0 \) and \( g > 0 \) have the same symmetry but different quantum orders. The PSG’s for those quantum ordered states are identified. The phase transition at \( g = 0 \) represents a new kind of phase transitions that changes quantum orders but not symmetry. We show that the projective construction that is used to construct quantum ordered ground states\cite{12, 19, 20, 21, 22} not only gives us exact ground states for our model, but also all the exact excited states. Through this solvable model, we hope to put quantum order and its PSG description on a firm ground.

We would like to mention that the above spin-1/2 model, having one spin per unit cell, is different from Kitaev’s exact soluble spin-1/2 models on the links of square lattice and on the sites of honeycomb lattice (which have two spins per unit cell).\cite{3, 4} However, the \( g < 0 \) version of the above model corresponds to the low energy sector of Kitaev’s honeycomb lattice model in the \( J_z >> J_x, J_y \) limit.\cite{4, 22}

### Quantum orders in spin-1/2 and hard-core-boson models

In this section, we are going to give a brief and general review of the PSG description of quantum order. Readers who are interested in the exact soluble model can go directly to the next section. Let us consider a spin-1/2 system on a square lattice. Such a system can be viewed as a hard-core-boson model if we identify \( | ↓ \rangle \) state as zero-boson state \( | 0 \rangle \) and \( | ↑ \rangle \) state as one-boson state \( | 1 \rangle \). In the follow we will use the boson picture to describe our model.

To construct quantum ordered (or entangled) many-boson wave functions, we will use projective construction. We first introduce a “mean-field” fermion Hamiltonian:\cite{7}

\[
H_{\text{mean}} = \sum_{\langle ij \rangle} \left( \psi_{i,i}^\dagger \chi_{ij}^T \psi_{j,j} + \psi_{i,i}^\dagger \eta_{ij}^T \psi_{j,j} + \text{h.c.} \right) \tag{1}
\]

where \( I, J = 1, 2 \). We will use \( \chi_{ij} \) and \( \eta_{ij} \) to denote the \( 2 \times 2 \) complex matrices whose elements are \( \chi_{ij}^T \) and \( \eta_{ij}^T \). Let \( \Phi^{(\chi_{ij}, \eta_{ij})} \) be the ground state of the above free fermion Hamiltonian, then a many-body boson wave function can be obtained

\[
\Phi^{(\chi_{ij}, \eta_{ij})}(i_1, i_2, \ldots) = \langle 0| \prod_n b(n) \Phi^{(\chi_{ij}, \eta_{ij})} \rangle \tag{2}
\]

where

\[
b(i) = \psi_{i,1} \psi_{i,2} \psi_{i,3} \ldots \tag{3}
\]

According to Ref.\cite{3}, the quantum order in the boson wave function \( \Phi^{(\chi_{ij}, \eta_{ij})}(\{i_n\}) \) can be (partially) characterized by projective symmetry group (PSG). To define PSG, we first discuss two types of transformations. The first type is \( SU(2) \) gauge transformation

\[
(\psi_i, \chi_{ij}, \eta_{ij}) \rightarrow (G(i)\psi_i, G(i)\chi_{ij} G^T(j), G(i)\eta_{ij} G^T(j)) \tag{4}
\]

where \( G(i) \in SU(2) \). We note that the physical boson wave function \( \Phi^{(\chi_{ij}, \eta_{ij})}(\{i_n\}) \) is invariant under the above \( SU(2) \) gauge transformations. The second type is the usual symmetry transformation, such as the translations \( T_x: i \rightarrow i - \hat{x}, T_y: i \rightarrow i - \hat{y} \). A generic transformation is a combination of the above two types, say \( GT_x(\chi_{ij}) = G(i)\chi_{i-j} G^T(j) \). The PSG for an ansatz \( (\chi_{ij}, \eta_{ij}) \) is formed by all the transformations that leave the ansatz invariant.

Every PSG contains a special subgroup, which is called the invariant gauge group (IGG). An IGG is formed by pure gauge transformations that leave the ansatz unchanged \( IGG = \{ G | \chi_{ij} = G(i)\chi_{ij} G^T(j), \eta_{ij} = G(i)\eta_{ij} G^T(j) \} \). One can show that PSG, IGG, and the symmetry group (SG) of the many-boson wave function are related: \( PSG/IGG = SG \).\cite{5}

Different quantum orders in the ground states of our boson system are characterized by different PSG’s. In the following we will concentrate on the simplest kind of quantum orders whose PSG has a IGG=\( Z_2 \). We will call those quantum states \( Z_2 \) quantum states. We would like to ask how many different \( Z_2 \) quantum states are there that have translation symmetry. According to our PSG characterization of quantum orders, the above physical question becomes the following mathematical question: how many different PSG’s are there that satisfy \( PSG/Z_2 \) = translation symmetry group. This problem has been solved in Ref.\cite{6}. The answer is 2 for 2D square lattice. Both PSG’s are generated by three elements \( \{ G_x T_x, G_y T_y, G_y \} \), where \( G_y \) is a pure gauge transformation that generates the \( Z_2 \) IGG: \( IGG = \{ 1, G_y \} \). The gauge transformations in the three generators for the first \( Z_2 \) PSG are given by

\[
G_y(i) = -1, \quad G_x(i) = 1, \quad G_y(i) = 1. \tag{5}
\]

Such a PSG will be called a Z2A PSG. The quantum states characterized by Z2A PSG will be called Z2A quantum states. For the second \( Z_2 \) PSG, we have

\[
G_y(i) = -1, \quad G_x(i) = 1, \quad G_y(i) = (-1)^i. \tag{6}
\]

Such a PSG will be called a Z2B PSG.

If we increase the symmetry of the boson wave function, there can be more different quantum orders. A classification of quantum orders for spin-1/2 system on a square lattice is given in Ref.\cite{6} where hundreds of different quantum orders with the translation, parity, and time-reversal symmetries were found.

In the following we will study an exact soluble model whose ground states realize some of the constructed quantum orders. Our model has the following nice property: the projective construction Eq.\cite{6} gives us all the energy eigenstates for some proper choices of \( \chi_{ij} \) and \( \eta_{ij} \). All the energy eigenvalues can be calculated exactly.

### Exact soluble models on 2D square lattice
Our construction is motivated by Kitaev’s construction of soluble spin-1/2 models on honeycomb lattice. The key step in both constructions is to find a system of commuting operators. Let \( \hat{U}^a_{ij} = \lambda^a_{ij} U^a_{ij} \), where \( i, j \) label lattice sites, \( a \) is an integer index, \( U^a_{ij} \) is an \( n \times n \) matrix satisfying \( (U^a_{ij})^T = -U^a_{ji} \), and \( \lambda^a_i = (\lambda_{1,i}, \lambda_{2,i}, ..., \lambda_{n,i}) \) is a \( n \)-component Majorana fermion operator satisfying \( \{\lambda_{n,i}, \lambda_{n,j}\} = 2\delta_{n0}\delta_{ij} \). We require that all \( \hat{U}^a_{ij} \) to commute with each other: \( [\hat{U}^a_{i_1,i_2}, \hat{U}^b_{j_1,j_2}] = 0 \), which can be satisfied iff

\[
\begin{align*}
U^a_{i_1,i_2} U^b_{i_2,i_3} &= 0, \quad U^a_{i_1,i_2} U^b_{i_2,i_3} = (U^a_{i_1,i_2} U^b_{i_2,i_3})^T \\
U^a_{i_1,i_2} U^b_{i_1,i_2} &= 0 \\
\end{align*}
\tag{7}
\]

where \( i_1, i_2, i_3 \) are all different.

Let \( \{1, \ldots, 4\} \) be a basis of a 4 dimensional real linear space. Then the following \( \hat{U}_{ij} \) on a square lattice \( \hat{U}_{i+\hat{x},i} = |1\rangle \langle 3| \), \( \hat{U}_{i+\hat{y},i} = -|3\rangle \langle 1| \), \( \hat{U}_{i+\hat{x}+\hat{y},i} = |2\rangle \langle 4| \), and \( \hat{U}_{i-\hat{y},i} = -|4\rangle \langle 2| \) form a solution of Eq. (7). We find that

\[
\hat{U}_{i,i+\hat{x}} = \lambda_{1,i} \lambda_{3,i+\hat{x}}, \quad \hat{U}_{i,i+\hat{y}} = \lambda_{2,i} \lambda_{4,i+\hat{y}}
\tag{8}
\]

form a commuting set of operators.

After obtaining a commuting set of operators, we can easily see that the following Hamiltonian

\[
H = g \sum_i \hat{F}_i, \quad \hat{F}_i = \hat{U}_{i,i} \hat{U}_{i,i+\hat{x}} \hat{U}_{i+\hat{x},i} \hat{U}_{i+\hat{y},i} \hat{U}_{i,i+\hat{y}} = \sum_i \hat{F}_i \hat{Z}
\tag{9}
\]

commutes with all the \( \hat{U}_{ij} \)'s, where \( i_1 = i + \hat{x}, i_2 = i + \hat{x} + \hat{y}, \) and \( i_3 = i + \hat{y} \). We will call \( \hat{F}_i \) a \( Z_2 \) flux operator. Let \( |s_{ij}\rangle \) be the common eigenstate of \( \hat{U}_{ij} \) with eigenvalue \( s_{ij} \). Since \( (\hat{U}_{ij})^2 = -1 \), \( s_{ij} \) satisfies \( s_{ij} = \pm i \) and \( s_{ij} = -s_{ji} \). \(|s_{ij}\rangle \) is also an energy eigenstate with energy

\[
E = g \sum_i \hat{F}_i, \quad \hat{F}_i = s_{i,i} s_{i,i+\hat{x}} s_{i+\hat{x},i} s_{i+\hat{x},i+\hat{y}} s_{i+\hat{y},i+\hat{x}}
\tag{10}
\]

Let us discuss the Hilbert space within which the above \( H \) acts. On each site, we group \( \lambda_{1,2,3,4} \) into two fermion operators

\[
\psi_{1,i} = \lambda_{1,i} + i\lambda_{3,i}, \quad \psi_{2,i} = \lambda_{2,i} + i\lambda_{4,i}
\tag{11}
\]

\( \psi_{1,2} \) generate a four dimensional Hilbert space on each site. Let us assume the 2D square lattice to have \( N \) lattice sites and a periodic boundary condition in both directions. Since there are total of \( 2^{2N_f} \), different choices of \( s_{ij} \) (two choices for each link), the states \( |s_{ij}\rangle \) exhaust all the \( 4^{2N_f} \) states in the Hilbert space. Thus the common eigenstates of \( \hat{U}_{ij} \) is not degenerate and the above approach allows us to obtain all the eigenstates and eigenvalues of the \( H \).

We note that the Hamiltonian \( H \) can only change the fermion number on each site by an even number. Thus the \( H \) acts within a subspace which has an even number of fermions on each site. The subspace has only two states per site. When defined on the subspace, \( H \) actually describes a spin-1/2 or a hard-core boson system under the operator mapping Eq. (8). The subspace is formed by states that are invariant under local \( Z_2 \) gauge transformations: \( \psi_{ij} \rightarrow G(i)\psi_{ij}, \ G(i) = \pm 1 \). We will call those states physical states and call the subspace the physical Hilbert space.

Since \( \hat{U}_{ij} \) do not act within the physical Hilbert space, they do not have definite values for physical states. However, we note that the \( Z_2 \) flux operator \( \hat{F}_i \) act within the physical Hilbert space. The \( Z_2 \) flux operators commute with each other and the \( H \) is a function of \( \hat{F}_i \). To obtain the common eigenstates of the \( Z_2 \) flux operators in the physical Hilbert space, we note that the \( Z_2 \) flux operators are invariant under the \( Z_2 \) gauge transformation:

\[
\psi_{ij} \rightarrow G(i)\psi_{ij}.
\]

A \( Z_2 \) gauge transformation changes one eigenstate \(|s_{ij}\rangle\) to another eigenstate \(|s_{ij}\rangle \propto |G(i)s_{ij}\rangle \), where \( \hat{N}_i = \sum s_{ij} \) is the fermion number operator at site \( i \). We will call those two eigenstates gauge equivalent. The common eigenstates of the \( Z_2 \) flux operators within the physical Hilbert space can be obtained by summing over all the gauge equivalent eigenstates with equal amplitude:

\[
|s_{ij}\rangle_{phy} \equiv \sum_i (\prod G_i^{N_i}) |s_{ij}\rangle
\tag{12}
\]

(Note \(|s_{ij}\rangle_{phy} \) can also be viewed as the projection of \(|s_{ij}\rangle \) onto the physical Hilbert space.)

Let us count the physical states that can be constructed this way, again assuming a periodic boundary condition in both directions. We note that the terms in the above sum can be grouped into pairs, where the \( Z_2 \) gauge transformations in a pair differs by a uniform \( Z_2 \) gauge transformation \( G(i) = -1 \). We see that the sum of the two terms in a pair is zero if the total number of the fermions \( N_f = \sum N_i \) is odd. Thus only states \(|s_{ij}\rangle \) with even number of fermions leads to physical states in the above construction. For states with even number of fermions, there are only \( 2^{N_f}/2 \) distinct terms in the above sum. Thus each physical eigenstate comes from \( 2^{N_f}/2 \) gauge equivalent eigenstates. Since there are \( 4^{2N_f}/2 \) states with even number of fermions, we find the number of physical states is \( 2^{N_f} \), which is the dimension of the physical Hilbert space. Thus we can obtain all the eigenstates and eigenvalues of the \( H \) in the physical Hilbert space from our construction.

Let us introduce a notion of \( Z_2 \) flux configuration. Among all the possible \( s_{ij} \)'s, we can use the \( Z_2 \) gauge transformation \( s_{ij} \rightarrow G(i)s_{ij}G(j) \) to define an equivalence relation. A \( Z_2 \) flux configuration is then an equivalence class under the \( Z_2 \) gauge transformation. From the above picture, we see that for every choice of \( Z_2 \) flux configuration \( F_i = \pm 1 \), we can choose a \( s_{ij} \) that reproduces the \( Z_2 \) flux on each plaquette. If the state \(|s_{ij}\rangle \) has even numbers of fermions, it will leads to a physical eigenstate (see Eq. (12)). The energy of the physical eigenstate is given by Eq. (10). The explicit many-boson wave function of the eigenstate is given by

\[
\Phi(i_n) = \langle 0| \prod_n \hat{b}(i_n)|\Psi_{mean}\rangle
\]

where \(|\Psi_{mean}\rangle \) is the ground state of

\[
H_{mean} = \sum_{(ij)} (s_{ij} \hat{U}_{ij} + h.c.)
\tag{13}
\]
Here we see that all the eigenstates of our model can be obtained from the projective construction Eq. (3).

**Physical properties**

In terms of the hard-core-boson operator Eq. (3), the Hamiltonian Eq. (3) of our soluble model has a form $H = -g \sum_{i} (b_{i} - b_{j})_{i} (b_{j} + b_{i})_{i} + \eta_{i} (b_{i} + b_{i})_{i}$. In terms of spin-1/2 operator $\tau_{x} = \sigma_{x} + 1$ and $\tau_{y} = i (\sigma_{x} - 1)$, the Hamiltonian has a form

$$H = g \sum_{i} F_{i} = g \sum_{i} \tau_{i}^{y} \tau_{i}^{x} \tau_{i}^{y}$$

(14)

In the bulk, the above model corresponds to the following simple Ising model $H_{\text{Ising}} = g \sum_{i} \tau_{i}^{z}$. However, the two models are different for finite systems and for systems with edges (see discussions below).

When $g < 0$, the ground state of our model is given by $Z_{2}$ flux configuration $F_{i} = 1$. To produce such a flux, we can choose $s_{i, i + x} = s_{i, i + y} = i$. In this case, Eq. (3) becomes Eq. (14) with $-\eta_{i, i + x} = \chi_{i, i + x} = 1 + \tau_{x}$ and $-\eta_{i, i + y} = \chi_{i, i + y} = 1 - \tau_{y}$. The PSG for the above ansatz turns out to be the $Z_{2}$ PSG in Eq. (3). Thus the ground state for $g < 0$ is a $Z_{2}$ state.

When $g > 0$, the ground state is given by configuration $F_{i} = -1$ which can be produced by $(-)^{x} s_{i, i + x} = s_{i, i + y} = i$. The ansatz now has a form $-\eta_{i, i + x} = \chi_{i, i + x} = 1 + \tau_{x}$ and $-\eta_{i, i + y} = \chi_{i, i + y} = 1 - \tau_{y}$. Its PSG is the $Z_{2}$ PSG in Eq. (3). Thus the ground state for $g > 0$ is a $Z_{2}$ state.

Both the $Z_{2}$A and the $Z_{2}$B states have translation $T_{x,y}$, parity $P_{xy}$ : $(i_{x}, i_{y}) \rightarrow (i_{y}, i_{x})$, and time reversal (since $\chi_{ij}$ and $\eta_{ij}$ are real) symmetries. The low energy excitations in both states are $Z_{2}$ vortices generated by flipping the signs of an even numbers of $F_{i}$'s. Those $Z_{2}$ vortex excitations have a finite energy gap $\Delta = 2|g|$. The low energy sector of our model is identical to the low energy sector of a $Z_{2}$ lattice gauge theory. However, our model is not equivalent to a $Z_{2}$ lattice gauge theory. This is because a $Z_{2}$ lattice gauge theory has $4 \times 2^{N_{e}}/2$ states on a torus while our model has $2^{N_{e}}$ states.

Due to the low energy $Z_{2}$ gauge structure, both the $Z_{2}$A and the $Z_{2}$B states have four degenerate ground states on an even by even lattice with periodic boundary condition. The degeneracy is topological and is protected by the topological order in the two states. The degeneracy is robust against arbitrary perturbations. This is because perturbations are $Z_{2}$ gauge invariant physical operators which cannot break the low energy $Z_{2}$ gauge structure that produces the degeneracy.

On an even by even lattice, the ground state energy per site is given by $-|g|$. The singularity at $g = 0$ implies a phase transition between two states with the same symmetry and the same ground state degeneracy! Thus the transition is a new type of continuous transitions that only changes the quantum orders (from $Z_{2}$A to $Z_{2}$B). The transition is continuous since gap vanishes at $g = 0$. Although the $Z_{2}$A and the $Z_{2}$B states share many common properties on an even by even lattice, the two states are quite different on an odd by odd lattice. On an odd by odd lattice, the $Z_{2}$A state has an energy $-|g|N_{s}$ and a 2 fold degeneracy, while the $Z_{2}$B state, containing a single $Z_{2}$ vortex to satisfy the constraint $\prod_{i} F_{i} = 1$, has an energy $-|g|(N_{s} - 2)$ and a 2$N_{s}$ fold degeneracy. On lattice with edges in $(1,0)$ and/or $(0,1)$ directions, our model has $\sim 2^{N_{e} / 2}$ gapless edge states, where $N_{e}$ is the number of edge sites.

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