Minimum-Time Earth-to-Mars Interplanetary Orbit Transfer Using Adaptive Gaussian Quadrature Collocation

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Abstract

The problem of minimum-time, low-thrust, Earth-to-Mars interplanetary orbital trajectory optimization is considered. The minimum-time orbital transfer problem is modeled as a four-phase optimal control problem where the four phases correspond to planetary alignment, Earth escape, heliocentric transfer, and Mars capture. The four-phase optimal control problem is then solved using an adaptive Gaussian quadrature collocation method. The following three models are used in the study: (1) circular planetary motion; (2) elliptic planetary motion; and (3) elliptic planetary motion with gravity perturbations. Results for all three cases are provided, and one particular case is studied in detail to show the key features of the optimal solutions. It was found that the minimum times for cases (1), (2), and (3) are, respectively, 215 d, 196 d, and 198 d with departure dates, respectively, of 1 July 2020, 30 June 2020, and 28 June 2020. Finally, the problem formulation developed in this study is compared against prior work on an Earth-to-Mars interplanetary orbit transfer where it is found that the results of this research show significant improvement in transfer time relative to the prior work.

Nomenclature

\( a \): thrust specific force  
\( a_{pr} \): radial component of gravity perturbations  
\( a_{p\theta} \): transverse component of gravity perturbations  
\( D \): length unit  
\( e_E \): eccentricity of Earth  
\( e_E \): eccentricity vector of Earth  
\( e_M \): eccentricity of Mars  
\( e_M \): eccentricity vector of Mars  
\( f_E \): second modified equinoctial orbital element of Earth  
\( f_M \): second modified equinoctial orbital element of Mars  
\( g_E \): third modified equinoctial orbital element of Earth  
\( g_M \): third modified equinoctial orbital element of Mars

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\[ L_E \] true longitude of Earth
\[ L_M \] true longitude of Mars
\[ L_{M_0} \] initial true longitude value of Mars
\[ p_E \] semi-latus rectum of Earth
\[ p_M \] semi-latus rectum of Mars
\[ P \] phase number
\[ r \] distance from central body to spacecraft
\[ r_E \] distance from Earth to Sun
\[ r_M \] distance from Mars to Sun
\[ r_{pe} \] distance from the spacecraft to Earth in the heliocentric transfer phase
\[ r_{pm} \] distance from the spacecraft to Mars in the heliocentric transfer phase
\[ r_{psE} \] distance from the spacecraft to Sun in the Earth escape phase
\[ r_{psM} \] distance from the spacecraft to Sun in the Mars capture phase
\[ R_E \] radius of Earth
\[ R_M \] radius of Mars
\[ R_{SOI}^E \] sphere of influence of Earth
\[ R_{SOI}^M \] sphere of influence of Mars
\[ R_{SE} \] distance from Sun to Earth
\[ R_{SM} \] distance from Sun to Mars
\[ t \] time
\[ t_0 \] initial time
\[ t_f \] terminal time
\[ T \] time unit
\[ v_r \] radial component of spacecraft velocity
\[ v_\theta \] transverse component of spacecraft velocity
\[ V \] speed unit
\[ w_r \] radial component of thrust direction
\[ w_\theta \] transverse component of thrust direction
\[ \theta \] longitude of spacecraft measured from the line of Aries
\[ \mu \] gravitational parameter of central body
\[ \mu_E \] gravitational parameter of Earth
\[ \mu_S \] gravitational parameter of Sun
\[ \mu_M \] gravitational parameter of Mars
\[ \nu_E \] true anomaly of Earth
\[ \nu_M \] true anomaly of Mars
\[ \omega_E \] longitude of perihelion of Earth orbit
\[ \omega_M \] longitude of perihelion of Mars orbit
\[ \varphi \] line of Aries
\[ \epsilon \] line of Cancer

1 Introduction

Interplanetary space travel has been a topic of interest to the space community for several decades. Due to the large distances between planets, most interplanetary orbital transfers require the expen-
diture of a large amount of propellant. Traditionally, large spacecraft that used high-thrust chemical propulsion were used for interplanetary travel. More recently, high-thrust chemical propulsion has been replaced with low-thrust propulsion using either electric propulsion or solar electric propulsion. The main benefit of using a low-thrust propulsion system is that the fuel consumption is significantly lower when compared with the fuel consumption using high-thrust chemical propulsion. One drawback to using low-thrust propulsion over high-thrust propulsion is that an orbital transfer using low-thrust propulsion is significantly longer than an orbital transfer using high-thrust propulsion due to the fact that a low-thrust engine produces significantly less thrust than a high-thrust engine. This research focuses on the use of low-thrust propulsion to accomplish an interplanetary orbital transfer from Earth to Mars.

A great deal of research has been done previously on the design of low-thrust orbital transfers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Refs. [1, 2] both study specific problems to further expand the knowledge of low-thrust spacecraft and interplanetary orbital transfers. Ref. [1] studied the Deep Space 1 mission because it was the first interplanetary mission to be propelled by a low-thrust solar electric propulsion system. A key result of Ref. [1] is that constraints on the spacecraft attitude and periods of coasting or thrusting of the propulsion system cause various trajectory design issues. Ref. [2] studied the Aldrin orbit, which is the simplest cy-cler orbit between Earth and Mars, and determines the minimum-impulse optimal solution from a multiconic optimization program for a 15 year cycle. Refs. [3, 4, 5, 6] solved minimum-fuel optimal control problems by solving the Hamiltonian boundary-value problem (HBVP) arising from the calculus of variations. In particular, Ref. [5] used an indirect multiple-shooting method to solve a three-dimensional minimum-fuel Earth-to-Mars orbital trajectory for a low-thrust spacecraft, while Ref. [6] transformed the HBVP into the Cauchy problem through the use of the continuation (homotopic) method. In a manner similar to that used in Ref. [6], Refs. [7, 8] utilized the homotopic approach to solve an optimal control problem. Ref. [7] implements the homotopic approach to decrease the computational time and increase the chance of determining the global optimal solution of the minimum-fuel low-thrust optimal control problem. Ref. [8] combined both the homotopic approach and the pseudospectral method to create a novel hybrid scheme by first discretizing the optimal control problem into a nonlinear programming problem (NLP) and solving the optimization problem using a homotopic algorithm once the costate is estimated through the use of Karush-Kuhn-Tucker (KKT) multipliers. Refs. [9, 10] also converted the optimal control problem into a NLP through the use of different methods. Ref. [9] employed an implicit integration scheme of the cubic polynomials used to represent the state variables, linearly interpolated the control variables, and used collocation to satisfy the differential equations of the system. Ref. [10] created a novel method to satisfy the differential equations of the system by using an explicit Runge-Kutta parallel-shooting approach, using piece-wise polynomials to approximate the control and state variables, and determine the accuracy of the NLP by using the Lagrange multipliers. Refs. [11, 12, 13, 14] employed a hybrid optimization method in order to solve their desired interplanetary orbital transfer optimal control problem by implementing genetic algorithms, basin hopping, calculus of variations, and/or an adaptive neighborhood global optimization algorithm. Specifically, Ref. [14] solved a three-phase minimum-fuel Earth-to-Mars orbital trajectory optimal control problem by creating a novel algorithm that forced the spacecraft to arrive at Mars and then optimized the final mass of the fuel. Refs. [15, 16, 17, 18] developed novel methods to solve orbital transfer optimal control problems involving low-thrust spacecraft. Ref. [15] shaped the velocity components as a function of time and polar angle during the transfer, Ref. [16] showed the robustness of a higher order collocation 7th degree system, Ref. [17] utilized the Legendre pseudospectral method and SNOPT, a sparse nonlinear optimization software, to solve a minimum-fuel satellite formation maneuver, and Ref. [18] solved constrained interplanetary trajectory optimization problems by developing a computer program, PLATO (Planetary Trajectory Optimization). Ref. [19] solved a four-phase interplanetary orbital transfer of a spacecraft from a low Earth orbit
to a low Martian orbit by utilizing a variable-stepsize integration technique implemented with the sequential gradient-restoration algorithm. Ref. [20] numerically solved a minimum-time, low-thrust interplanetary transfer trajectory optimal control problem by using collocation on a NLP and used a two-body gravitational model for each of the three phases: escape from the departure planet, heliocentric flight, and capture at the destination planet.

This research is inspired by the work of Ref. [20]. While both Ref. [20] and the work in this paper focus on the design of a minimum-time Earth-to-Mars interplanetary orbital transfer using low-thrust propulsion, the work presented in this paper is significantly different from the work of Ref. [20] in the following ways. First, Ref. [20] employs a three-phase structure consisting of Earth escape, heliocentric transfer, and Mars capture under the assumption that the planets move in circular orbits. On the other hand, the research described in this paper employs a four-phase structure consisting of Earth and Mars alignment, Earth escape, heliocentric transfer, and Mars capture and studies the problem for the cases where the planets move in either circular or elliptic orbits (where it is noted that the alignment of the planets in the first phase of the problem ensures that the planets are in the correct position to reduce the time required to complete the interplanetary transfer). In particular, the case of elliptic planetary motion introduces complexity in to the problem due to the fact that the timing of the transfer is critical in order to obtain the minimum-time orbital transfer. Additionally, Ref. [20] uses two-body dynamics to model the spacecraft’s motion in each phase, whereas in this study both the cases of two-body motion and motion with solar and planetary gravity perturbations are considered. Next, a close examination of Ref. [21] (which is the basis for the work of Ref. [20]) revealed two issues with the coordinate transformations used to define the Earth-to-heliocentric and heliocentric-to-Mars event constraints in Ref. [20]. First, the coordinate transformations in Ref. [20] contained singularities which are inconsistent with the behavior of two-dimensional coordinate transformations. Second, an error was found in the derivation of the inertial velocity in Eq. (2.21) on page 15 of Ref. [21]. Due to the issues which arose in studying Refs. [20] and [21], a key part of this research is the re-derivation of these coordinate transformations and the corresponding event constraints. These newly derived coordinate transformations and event constraints are used to connect the Earth escape and Mars capture phases to the heliocentric phase.

This paper is organized as follows. Section 2 presents the four-phase Earth-to-Mars interplanetary orbital transfer optimal control problem by providing the modeling assumptions, phase descriptions, dynamics, path constraints, boundary conditions, variable bounds, the units used to solve the problem, the unit conversions needed to connect sequential phases, and event constraints. Section 3 presents the results and discussion of the numerical optimization study using the MATLAB optimal control software, GPOPS – II. Section 4 presents the conclusions on this research. Finally, Appendix A provides the derivations of the coordinate transformations used to connect the Earth escape and Mars capture phases to the heliocentric phase while Appendix B provides the derivation of the gravity perturbations.

2 Problem Description

This section develops the assumptions, constraints, boundary conditions, and equations of motion that define the four-phase Earth-to-Mars optimal control problem. Section 2.1 presents the simplifications and assumptions that were used to model the motion of the spacecraft and planets in each phase. Section 2.2 provides a description of the four phases of the optimal control problem. Section 2.3 covers the differential equations of motion for the spacecraft and planets as well as the path constraints that constrain the motion. Section 2.4 presents the initial and terminal boundary conditions as well as the lower and upper bounds on the time, control, and state in each phase. Section 2.5 presents the units used to solve the problem. Section 2.6 presents the unit conversions needed to connect sequential phases. Section 2.7 presents the event constraints that connect the
state and time of sequential phases to one another. Finally, a description of the complete optimal control problem is provided in Section 2.8.

2.1 Modeling Assumptions

First, it is assumed that the initial true longitudes of Earth and Mars are known from ephemeris data for 1 January 2019. Second, the spacecraft and the planets move in the same inertially fixed plane. Third, the spacecraft remains in its initial parking orbit during phase 1. As a result, phase 1 does not have any control. Fourth, in phases 2, 3, and 4 the only forces acting on the spacecraft are those due to thrust and the gravity acting on the spacecraft (either two-body or multi-body, depending on the case studied). Fifth, in phases 2, 3, and 4 the control is the thrust direction. Sixth, the thrust magnitude is constant in phases 2, 3, and 4. Lastly, the longitude of the spacecraft as it starts its departure from Earth is assumed to be free.

2.2 Phase Descriptions

Using the assumptions presented in Section 2.1, the minimum-time trajectory optimization problem is proposed as a four-phase optimal control problem. Phase 1 is a planetary alignment phase that starts with Earth and Mars at particular locations at a specified epoch and terminates when the planets are in positions that minimize the transfer time. Phase 2 is an Earth escape phase that begins with the spacecraft in a circular orbit relative to Earth and terminates either at the sphere of influence of Earth for the circular and elliptic cases, or is allowed to be a free parameter in the elliptic with gravity perturbations case. Phase 3 is a heliocentric transfer phase that begins with the spacecraft either at the sphere of influence of Earth for the circular and elliptic cases, or is allowed to be a free parameter in the elliptic with gravity perturbations case and terminates either at the sphere of influence of Mars for the circular and elliptic cases, or is allowed to be a free parameter in the elliptic with gravity perturbations case. Phase 4 is a Mars capture phase that begins either at the sphere of influence of Mars for the circular and elliptic cases, or is allowed to be a free parameter in the elliptic with gravity perturbations case and terminates with the spacecraft in a circular orbit relative to Mars.

2.3 Equations of Motion

The following sets of differential equations describe the motion of Earth, Mars, and the spacecraft. First, the positions of the planets are defined by their true longitudes relative to the line of Aries. The true longitudes are governed by the following differential equations in phases $P = (1, 2, 3, 4)$:

\[
\begin{align*}
\dot{L}_E^{[P]}(t) &= \frac{\mu_S^{[P]}}{\left(P_E^{[P]}\right)^{\frac{3}{2}}} \left(1 + f_E \cos \left(L_E^{[P]}(t)\right) + g_E \sin \left(L_E^{[P]}(t)\right)\right), \\
\dot{L}_M^{[P]}(t) &= \frac{\mu_S^{[P]}}{\left(P_M^{[P]}\right)^{\frac{3}{2}}} \left(1 + f_M \cos \left(L_M^{[P]}(t)\right) + g_M \sin \left(L_M^{[P]}(t)\right)\right).
\end{align*}
\]
Next, the differential equations that describe the motion of the spacecraft in phases \( P = (2, 3, 4) \) are given as:

\[
\begin{align*}
\dot{r}^{[P]}(t) &= v^{[P]}_r(t), \\
\dot{\theta}^{[P]}(t) &= \frac{v^{[P]}_\theta(t)}{r^{[P]}(t)}, \\
v^{[P]}_r(t) &= a^{[P]}u^{[P]}_r(t) - \frac{\mu^{[P]}(r^{[P]}(t))^2}{r^{[P]}(t)} + \frac{(v^{[P]}_\theta(t))^2}{r^{[P]}(t)} + a^{[P]}_{\rho r}, \\
v^{[P]}_\theta(t) &= a^{[P]}u^{[P]}_\theta(t) - \frac{v^{[P]}_r(t)v^{[P]}_\theta(t)}{r^{[P]}(t)} + a^{[P]}_{\rho \theta}.
\end{align*}
\]

Using Cowell’s method in Ref. [22], the gravity perturbation terms are given below:

\[
\begin{align*}
a^{[2]}_{\rho r} &= -\mu_S \left[ \frac{r_2 E}{r_2 E^{psE}} + \left( \frac{r_2 E}{r_2 E^{psE}} - \frac{1}{(r_2 E)^2} \right) \cos \left( \theta[2] - L[2] \right) \right], \\
a^{[2]}_{\rho \theta} &= -\mu_S \left( -\frac{r_2 E^{psE}}{r_3 E^{psE}} + \frac{1}{r_2 E} \right) \sin \left( \theta[2] - L[2] \right), \\
a^{[3]}_{\rho r} &= -\mu_E \left[ \frac{r_3 E}{r_3 E^{pe}} - \frac{r_3 E}{r_3 E^{pe}} \cos \left( \theta[3] - L[3] \right) \right] - \mu_M \left[ \frac{r_3 E^{psM}}{r_3 E^{psM}} - \frac{r_3 E^{psM}}{r_3 E^{psM}} \cos \left( \theta[3] - L[3] \right) \right], \\
a^{[3]}_{\rho \theta} &= -\mu_E \left( \frac{r_3 E^{psE}}{r_3 E^{pe}} \sin \left( \theta[3] - L[3] \right) \right) - \mu_M \left( \frac{r_3 M^{psM}}{r_3 M^{psM}} \sin \left( \theta[3] - L[3] \right) \right), \\
a^{[4]}_{\rho r} &= -\mu_S \left[ \frac{r_4 E}{r_4 E^{psM}} + \left( \frac{r_4 E}{r_4 E^{psM}} - \frac{1}{(r_4 E)^2} \right) \cos \left( \theta[4] - L[4] \right) \right], \\
a^{[4]}_{\rho \theta} &= -\mu_S \left( -\frac{r_4 E^{psM}}{r_4 E^{psM}} + \frac{1}{(r_4 E)^2} \right) \sin \left( \theta[4] - L[4] \right),
\end{align*}
\]

where

\[
\begin{align*}
& r_{psE} = \sqrt{\left( \frac{r_2 E}{r_2 E^{psE}} \right)^2 + \left( \frac{r_2 E^{psE}}{r_2 E} \right)^2 + 2r_2 E^{psE} \cos \left( \theta[2] - L[2] \right)}, \\
& r_{pe} = \sqrt{\left( \frac{r_3 E}{r_3 E^{pe}} \right)^2 + \left( \frac{r_3 E^{pe}}{r_3 E} \right)^2 - 2r_3 E^{pe} \cos \left( \theta[3] - L[3] \right)}, \\
& r_{pmn} = \sqrt{\left( \frac{r_3 E^{psM}}{r_3 E^{psM}} \right)^2 + \left( \frac{r_3 E^{psM}}{r_3 E^{psM}} \right)^2 - 2r_3 E^{psM} \cos \left( \theta[3] - L[3] \right)}, \\
& r_{psM} = \sqrt{\left( \frac{r_4 E^{psM}}{r_4 E^{psM}} \right)^2 + \left( \frac{r_4 E^{psM}}{r_4 E^{psM}} \right)^2 + 2r_4 E^{psM} \cos \left( \theta[4] - L[4] \right)},
\end{align*}
\]

It is noted that \( a_{\rho r} \) and \( a_{\rho \theta} \) are both zero when solving the circular and elliptic cases because they use two-body dynamics. Table 2 summarizes the state and control variables utilized in each phase. Next, in order to ensure that the thrust direction is a unit vector, the following equality path constraint is enforced in each of the transfer phases:

\[
\left( w^{[P]}_r(t) \right)^2 + \left( w^{[P]}_\theta(t) \right)^2 = 1
\]

(5)
Table 2: Variables used to represent the state and control in each phase.

| Phase | State | Control |
|-------|-------|---------|
| Phase 1 | \( \left( L_{E}^{[1]}, L_{M}^{[1]} \right) \) | – |
| Phase 2 | \( \left( r_{2}^{[2]}, \theta_{2}^{[2]}, v_{r}^{[2]}, v_{\theta}^{[2]}, L_{E}^{[2]}, L_{M}^{[2]} \right) \) | \( \left( \Omega_{2}^{[2]}, w_{2}^{[2]} \right) \) |
| Phase 3 | \( \left( r_{3}^{[3]}, \theta_{3}^{[3]}, v_{r}^{[3]}, v_{\theta}^{[3]}, L_{E}^{[3]}, L_{M}^{[3]} \right) \) | \( \left( \Omega_{3}^{[3]}, w_{3}^{[3]} \right) \) |
| Phase 4 | \( \left( r_{4}^{[4]}, \theta_{4}^{[4]}, v_{r}^{[4]}, v_{\theta}^{[4]}, L_{E}^{[4]}, L_{M}^{[4]} \right) \) | \( \left( \Omega_{4}^{[4]}, w_{4}^{[4]} \right) \) |

Furthermore, the radii of the planetary orbits are given as:

\[
\begin{align*}
\dot{r}_{E}^{[P]}(t) &= \frac{p_{E}^{[P]} \nu_{E}^{[P]}(t)}{1 + e_{E} \cos(\nu_{E}^{[P]}(t))}, \\
\dot{r}_{M}^{[P]}(t) &= \frac{p_{M}^{[P]} \nu_{M}^{[P]}(t)}{1 + e_{M} \cos(\nu_{M}^{[P]}(t))},
\end{align*}
\]  

(6)

where the true anomaly of each planet is given as:

\[
\begin{align*}
\nu_{E}^{[P]}(t) &= L_{E}^{[P]}(t) - \varpi_{E}, \\
\nu_{M}^{[P]}(t) &= L_{M}^{[P]}(t) - \varpi_{M}.
\end{align*}
\]  

(7)

Finally, the rates of change of the radii of the planetary orbits are given as:

\[
\begin{align*}
\dot{r}_{E}^{[P]}(t) &= \frac{p_{E}^{[P]} e_{E} \nu_{E}^{[P]}(t) \sin(\nu_{E}^{[P]}(t))}{\left(1 + e_{E} \cos(\nu_{E}^{[P]}(t))\right)^{2}} = \frac{p_{E}^{[P]} \nu_{E}^{[P]}(t) \sin(\nu_{E}^{[P]}(t))}{\left(1 + e_{E} \cos(\nu_{E}^{[P]}(t))\right)^{2}}, \\
\dot{r}_{M}^{[P]}(t) &= \frac{p_{M}^{[P]} e_{M} \nu_{M}^{[P]}(t) \sin(\nu_{M}^{[P]}(t))}{\left(1 + e_{M} \cos(\nu_{M}^{[P]}(t))\right)^{2}} = \frac{p_{M}^{[P]} \nu_{M}^{[P]}(t) \sin(\nu_{M}^{[P]}(t))}{\left(1 + e_{M} \cos(\nu_{M}^{[P]}(t))\right)^{2}},
\end{align*}
\]  

(8)

where it is noted from Eq. (7) that

\[
\begin{align*}
\nu_{E}^{[P]}(t) &= \dot{\nu}_{E}^{[P]}(t), \\
\nu_{M}^{[P]}(t) &= \dot{\nu}_{M}^{[P]}(t).
\end{align*}
\]  

(9)

It is noted that Eq. (8) is used as part of the event constraints that connected the various phases in the problem.
2.4 Boundary Conditions and Bounds

Bounds are placed on the time, control, and state in all four phases and are given as:

\[
\begin{align*}
0_{\text{min}} & \leq t_0^{[P]} \leq 0_{\text{max}}^{[P]} \\
0_{\text{min}} & \leq t_f^{[P]} \leq 0_{\text{max}}^{[P]} \\
E_{\text{min}}^{[P]} & \leq E^{[P]} \leq E_{\text{max}}^{[P]} \\
M_{\text{min}}^{[P]} & \leq M^{[P]} \leq M_{\text{max}}^{[P]}, \quad P = (1, 2, 3, 4),
\end{align*}
\]

\[
\begin{align*}
\text{Variable} & \quad \text{Value} \\
t_0 & \quad 0 \text{ d} \\
E_0 & \quad 101.14 \text{ deg} \\
M_0 & \quad 41.23 \text{ deg}
\end{align*}
\] (10)

All initial and terminal boundary conditions for all phases are considered to be free parameters except for the variables specified in Tables 3a - 3c. Tables 4a - 4c show those variables that are constrained alongside the corresponding lower and upper limits. All other variables are free.

Finally, the physical constants and other numerical data used to model and solve the problem are given in Table 5.
Table 4: Lower and upper bounds during phases 1, 2, and 4.

(a) Lower and upper bounds for phase 1.

| Variable | [Lower Bound, Upper Bound] |
|----------|---------------------------|
| $t$      | [0, Free]                 |

(b) Lower and upper bounds for phase 2.

| Variable                        | [Lower Bound, Upper Bound] |
|---------------------------------|---------------------------|
| $r$ (without perturbations)     | $[6.6R_E, R_{E}^{SOI}]$   |
| $r$ (with perturbations)        | $[6.6R_E, \text{Free}]$   |
| $v_r$                           | [0, Free]                 |

(c) Lower and upper bounds for phase 4.

| Variable                        | [Lower Bound, Upper Bound] |
|---------------------------------|---------------------------|
| $r$ (without perturbations)     | $[6.0R_M, R_{M}^{SOI}]$   |
| $r$ (with perturbations)        | $[6.0R_M, \text{Free}]$   |
| $v_r$                           | [0, Free]                 |

Table 5: Physical constants.

| Quantity | Value          | Units     |
|----------|----------------|-----------|
| $R_E$    | $6.3781 \times 10^6$ | m         |
| $R_M$    | $3.3895 \times 10^6$ | m         |
| $R_{E}^{SOI}$ | $9.2455 \times 10^8$ | m         |
| $R_{M}^{SOI}$ | $5.7717 \times 10^8$ | m         |
| $R_{SE}$ | $1.4960 \times 10^{11}$ | m         |
| $R_{SM}$ | $2.2794 \times 10^{11}$ | m         |
| $\mu_E$ | $3.9860 \times 10^{14}$ | m$^3 \cdot \text{s}^{-2}$ |
| $\mu_S$ | $1.3271 \times 10^{20}$ | m$^3 \cdot \text{s}^{-2}$ |
| $\mu_M$ | $4.2828 \times 10^{13}$ | m$^3 \cdot \text{s}^{-2}$ |
| $\varpi_E$ | 102.9         | deg       |
| $\varpi_M$ | 336.0         | deg       |

2.5 Scale Factors

The units used in each phase $P \in (1, 2, 3, 4)$ of the four-phase optimal control problem were chosen such that the gravitational parameter of the central body in that phase is equal to unity. In order to attain a gravitational parameter of unity in each phase, the distance unit, $D^{[P]}$, $(P = 1, 2, 3, 4)$, speed unit, $V^{[P]}$, $(P = 1, 2, 3, 4)$, and time unit, $T^{[P]}$, $(P = 1, 2, 3, 4)$, were chosen as follows:

$$
\begin{bmatrix}
D^{[P]} \\
V^{[P]} \\
T^{[P]}
\end{bmatrix} = \begin{bmatrix}
R^{[P]} \\
\sqrt{\mu^{[P]}/D^{[P]}} \\
\sqrt{(D^{[P]})^3/\mu^{[P]}}
\end{bmatrix}, \quad P \in (1, 2, 3, 4),
$$

(11)

where

$$
\begin{bmatrix}
R^{[1]} \\
R^{[2]} \\
R^{[3]} \\
R^{[4]}
\end{bmatrix} = \begin{bmatrix}
R_{SE} \\
R_E \\
R_{SE} \\
R_M
\end{bmatrix}.
$$

(12)
2.6 Unit Conversions

A set of unit conversions are created to connect all variables across the sequential phases with identical units provided in Section 2.5. First, in the Alignment-to-Earth event constraint, the variables at the start of phase 2 are converted from Earth units \((D^2_r, V^2_r, T^2_r)\) to heliocentric units \((D^1_r, V^1_r, T^1_r)\) in order to connect the terminus of phase 1 to the start of phase 2. Next, in the Earth-to-Heliocentric event constraint, the variables at the terminus of phase 2 are converted from Earth units \((D^2_r, V^2_r, T^2_r)\) to heliocentric units \((D^3_r, V^3_r, T^3_r)\) in order to connect the terminus of phase 2 to the start of phase 3. Lastly, in the Heliocentric-to-Mars event constraint, the variables at the start of phase 4 are converted from Mars units \((D^4_r, V^4_r, T^4_r)\) to heliocentric units \((D^3_r, V^3_r, T^3_r)\) in order to connect the terminus of phase 3 to the start of phase 4. The necessary conversion factors to transform Earth units to heliocentric units are given as:

\[
\begin{bmatrix}
D^1_{SE} \\
V^1_{SE} \\
T^1_{SE}
\end{bmatrix} = \begin{bmatrix}
D^2_r/D^1_r \\
V^2_r/V^1_r \\
T^2_r/T^1_r
\end{bmatrix} = \begin{bmatrix}
D^2/D^3 \\
V^2/V^3 \\
T^2/T^3
\end{bmatrix}.
\] (13)

The necessary conversion factors to transform Mars units to heliocentric units are given as:

\[
\begin{bmatrix}
D^1_{SM} \\
V^1_{SM} \\
T^1_{SM}
\end{bmatrix} = \begin{bmatrix}
D^4_r/D^1_r \\
V^4_r/V^1_r \\
T^4_r/T^1_r
\end{bmatrix} = \begin{bmatrix}
D^4/D^3 \\
V^4/V^3 \\
T^4/T^3
\end{bmatrix}.
\] (14)

2.7 Event Constraints

Event constraints are utilized to connect subsequent phases to one another by enforcing the constraints at the endpoints, so that the variables at the terminus of one phase can be transformed to the variables at the start of the subsequent phase. The event constraints in this research connect phase 1 to phase 2, phase 2 to phase 3, and phase 3 to phase 4. The following variables are utilized in the event constraints to ensure continuity: time, true longitude of Earth and Mars, and the spacecraft’s position and velocity components. The unit conversions stated in Section 2.6 are implemented in the event constraints to provide continuous units. It is noted in this research that the event constraints are formulated different from those found in Ref. [20] and is constructed in such a way to allow for non-circular planetary motion unlike in Ref. [20]. First, the Alignment-to-Earth event constraints that connect the terminus of phase 1 to the start of phase 2 are given as:

\[
t^{[1]}_f = t^{[2]}_0,
\]

\[
L^1_E(t^{[1]}_f) = L^2_E(t^{[2]}_0),
\]

\[
L^1_M(t^{[1]}_f) = L^2_M(t^{[2]}_0).
\] (15)

The geometry of phase 1 and phase 2 is shown in Fig. I. Second, the event constraints that connect the terminus of phase 2 to the start of phase 3 are given as (see A for derivation of the coordinate
The geometry of phase 2 and phase 3 is shown in Fig. 2. Finally, the event constraints that connect the terminus of phase 3 to the start of phase 4 are given as (see [A] for the derivation of the coordinate transformations):

\begin{align}
    t_{f}^{[2]}T_{SE} &= t_{0}^{[3]},
    
    L_{E}^{[2]}(t_{f}^{[2]}) &= L_{E}^{[3]}(t_{0}^{[3]}),
    
    L_{M}^{[2]}(t_{f}^{[2]}) &= L_{M}^{[3]}(t_{0}^{[3]}),
    
    r^{[2]}(t_{f}^{[2]})D_{SE} &= -r_{E}^{[3]}(t_{0}^{[3]}) \cdot \cos(\alpha) + r_{E}^{[3]}(t_{0}^{[3]}) \cdot \cos(\beta),
    
    0 &= -r_{E}^{[3]}(t_{0}^{[3]}) \cdot \sin(\alpha) + r_{E}^{[3]}(t_{0}^{[3]}) \cdot \sin(\beta),
    
    v_{r}^{[2]}(t_{f}^{[2]})V_{SE} &= -r_{E}^{[3]}(t_{0}^{[3]}) \cdot \sin(\alpha) - r_{E}^{[3]}(t_{0}^{[3]}) \cdot \sin(\beta),
    
    v_{\theta}^{[2]}(t_{f}^{[2]})V_{SE} &= -v_{r}^{[2]}(t_{f}^{[2]}) - v_{\theta}^{[2]}(t_{f}^{[2]}),
\end{align}

where

\begin{align}
    \alpha &= \theta^{[2]}(t_{f}^{[2]}) - L_{E}^{[3]}(t_{0}^{[3]}),
    
    \beta &= \theta^{[2]}(t_{f}^{[2]}) - \theta^{[3]}(t_{0}^{[3]}).
\end{align}

The geometry of the planetary alignment and Earth escape phases.
2.8 Optimal Control Problem

For the Earth-to-Mars transfer, the optimal control problem is stated as follows. Determine the state \( \left( L_E^P, L_M^P \right) \), \( P = 1 \) and \( \left( r^P, \theta^P, v_r^P, v_\theta^P, r_E^P, L_M^P \right) \), \( P = (2, 3, 4) \), the control \( \left( w_r^P, w_\theta^P \right) \), \( P = (2, 3, 4) \), as well as the initial and terminal times \( \left( t_0^P, t_f^P \right) \), \( P = (1, 2, 3, 4) \) which minimize

\[
J = t_f^4 - t_0^2 \tag{20}
\]
Figure 3: Geometry of the heliocentric transfer and Mars capture phases.

while satisfying the dynamics and path constraints in Section 2.3, the variable bounds and boundary conditions in Section 2.4, and the event constraints in Section 2.7.

3 Results and Discussion

This section presents the results acquired by solving the four-phase Earth-to-Mars orbital transfer optimal control problem described in Section 2. For all results presented in this research, it is assumed that the planetary alignment phase starts with Earth and Mars at the locations defined by the NASA HORIZONS J2000 basis [23] on 1 January 2019 at 00:00 Coordinated Universal Time (UTC) and that the longitudes of perihelion of Earth and Mars, $\varpi_E$ and $\varpi_M$, are 102.9 deg and 336.0 deg, respectively (see Fig. 4). Next, the cases of circular planetary motion, elliptic planetary motion, and elliptic planetary motion with gravity perturbations are considered. For the case of elliptic planetary motion, it is assumed that eccentricities of Earth and Mars orbit are $e_E = 0.0167$ and $e_M = 0.0935$. Solutions are obtained for the three different cases of planetary motion for $a = (9.8, 9.9, 10.0, 10.1, 10.2) \times 10^{-4}$ m·s$^{-2}$. Finally, the particular case of elliptic planetary motion with gravity perturbations and $a = 9.8 \times 10^{-4}$ m·s$^{-2}$ is used to show the key features of the optimized solutions.

The Earth-to-Mars transfer problem is solved using the general-purpose MATLAB optimal control software GPOPS – II [24] with the following settings. First, the mesh refinement method described in Ref. [25] was used with a mesh refinement accuracy tolerance $10^{-7}$. Second the nonlinear programming problem (NLP) solver IPOPT [26] was employed in full Newton (second derivative) mode with an NLP solver tolerance $\epsilon_{NLP}$, of $10^{-10}$, and all first and second derivatives required by IPOPT were obtained using the open-source algorithmic differentiations software ADiGator [27]. All computations were performed using a 2.9 GHz Intel Core i9 MacBook Pro running Mac OS version 10.13.6 (High Sierra) with 32GB 2400MHz DDR4 RAM and MATLAB Version R2018b (build 9.5.0.944444).

Next, an initial guess must be supplied in order to solve the four-phase optimal control problem using GPOPS – II. In this research the following initial guesses were generated using GPOPS – II for each of the four phases of the problem. First, for the planetary alignment phase GPOPS – II
was used to determine the planetary trajectories starting at \((L_E(0), L_M(0)) = (101.14, 41.23)\) deg and terminating 150 d later. Second, for the Earth escape phase GPOPS II was used to determine the minimum-time trajectory and control starting with the spacecraft in an Earth circular orbit of radius \(6.6R_E\) and terminating in a circular orbit of radius \(R_E^{SOI}\) relative to the Earth. Third, for the heliocentric phase GPOPS II was used to determine the minimum-time trajectory and control starting in a heliocentric circular orbit of radius 1 AU and terminating with the spacecraft in a heliocentric circular orbit with a radius 1.5 AU. Finally, for the Mars capture phase GPOPS II was used to determine the trajectory and control starting with the spacecraft in Mars circular orbit of radius \(R_M^{SOI}\) and terminating in Mars circular orbit of radius 6.0\(R_M\). Each of the aforementioned four one-phase solutions were then used as an initial guess for the four-phase circular planetary motion problem with \(a = 9.8 \times 10^{-4} \, m \cdot s^{-2}\). Finally, the solution obtained for each case of planetary motion and each value of \(a\) was obtained using the previously obtained solution as an initial guess.

### 3.1 Overall Performance

Table 6a shows the minimum-time solutions obtained from Ref. [20]. The minimum-time results obtained in this study for the circular, elliptic, and elliptic with gravity perturbations cases are shown in Tables 6b–6d respectively for \(a = (9.8, 9.9, 10.0, 10.1, 10.2) \times 10^{-4} \, m \cdot s^{-2}\). For each value of \(a\), the optimal duration for each phase of the problem along with the total minimum interplanetary transfer time and the corresponding start date for the transfer are presented. It is seen from Tables 6b and 6c that, as \(a\) increases, the time for the planetary alignment phase also increases while the time for each of the transfer phases and the total transfer time decreases. Furthermore, Table 6d shows that, as \(a\) increases, the time for the planetary alignment phase also increases, the transfer phases do not follow a distinct pattern for either increasing or decreasing, and the total transfer time decreases. It is noted that, when gravity perturbations are included, the phases do not follow a pattern similar to those of the circular and elliptic cases because when gravity perturbations are included the radius at the terminus of phase 2 and start of phase 4 are free. Next, for the case of circular planetary motion, Tables 6a and 6b show the differences between
the results of Ref. [20] and the results of this study. First, it is not possible to compare the durations of the individual phases or the start dates of the transfer because Ref. [20] does not contain this data. Next, the results of this study show that the total transfer time for the circular case is shorter by an average of 8.89 d for all values of $a$ when compared with the results of Ref. [20].

Next, when comparing the results of the circular and elliptic cases obtained in this study, it is seen in Tables 6b and 6c that, for all values of $a$, the elliptic case has a shorter phase 1 duration by an average of 1.63 d, a longer phase 2 duration by approximately 0.01 d, a shorter phase 3 duration by an average of 18.98 d, a shorter phase 4 duration by an average of 0.01 d, and a shorter total transfer time by an average of 18.98 d. When comparing the results of the elliptic and elliptic with gravity perturbations cases obtained in this study, Tables 6c and 6d show that, for all values of $a$, the elliptic with gravity perturbations case has a shorter phase 1 duration by an average of 1.13 d. Furthermore, because the transfer phases of the elliptic with gravity perturbations case do not follow an increasing or decreasing pattern, some of the phases have a longer or shorter duration when compared with the elliptic case depending upon the value of $a$. Consequently, the phase 2 duration was found to differ by an average of 0.83 d, the phase 3 duration differed by an average of 1.33 d, and the phase 4 duration differed by an average of 0.32 d. For all values of $a$, the elliptic with gravity perturbations case has a longer total transfer time by an average of 1.83 d. Finally, as $a$ increases, the start date of the transfer for the circular case lies between 1 July 2020 to 3 July 2020, the elliptic case lies between 30 June 2020 to 2 July 2020, and the elliptic with gravity perturbations case lies between 28 June 2020 to 1 July 2020.

### 3.2 Key Features of Optimized Solutions

This section shows the key features of all optimized solutions using the particular case $a = 9.8 \times 10^{-4} \text{ m} \cdot \text{s}^{-2}$ for the case of elliptic planetary motion with gravity perturbations. These key features of the solutions are shown for each phase of the four-phase problem. Particular attention is given to the behavior of the spacecraft trajectory and the control that produces that behavior.

Figure 5 shows the planetary movement of both Earth and Mars during phase 1 (planetary alignment). It is seen that Earth and Mars start at true longitudes 101.14 deg and 41.23 deg. It can be seen that, during the planetary alignment phase, Earth makes approximately 1.50 orbits around the Sun while Mars makes approximately 0.79 orbits around the Sun. During this phase the spacecraft remains in its initial orbit (that is, no propulsive force is exerted on the spacecraft). The optimized duration of the planetary alignment phase is 544.88 d and terminates on 28 June 2020.

![Figure 5: Optimal two-dimensional planetary movement for the planetary alignment phase.](image-url)
Table 6: Transfer times obtained from Ref. [20] and transfer times obtained in this research for the cases of circular planetary motion, elliptic planetary motion, and elliptic planetary motion with gravity perturbations.

(a) Total transfer times from Ref. [20].

| $a$ (m s$^{-2}$) $\times 10^{-4}$ | Transfer Time (d) |
|----------------------------------|-------------------|
| 9.8                              | 229.15            |
| 9.9                              | 224.37            |
| 10.0                             | 222.14            |
| 10.1                             | 218.18            |
| 10.2                             | 214.51            |

(b) Transfer times for circular planetary motion.

| $a$ (m s$^{-2}$) $\times 10^{-4}$ | Phase 1 (d) | Phase 2 (d) | Phase 3 (d) | Phase 4 (d) | Transfer Time (d) | Start Date   |
|----------------------------------|-------------|-------------|-------------|-------------|-------------------|--------------|
| 9.8                              | 547.63      | 33.27       | 162.48      | 19.31       | 215.05            | 1 July 2020  |
| 9.9                              | 548.20      | 32.98       | 161.76      | 19.16       | 213.90            | 2 July 2020  |
| 10.0                             | 548.75      | 32.69       | 161.06      | 19.01       | 212.76            | 2 July 2020  |
| 10.1                             | 549.30      | 32.40       | 160.37      | 18.87       | 211.64            | 3 July 2020  |
| 10.2                             | 549.84      | 32.11       | 159.70      | 18.73       | 210.53            | 3 July 2020  |

(c) Transfer times for elliptic planetary motion.

| $a$ (m s$^{-2}$) $\times 10^{-4}$ | Phase 1 (d) | Phase 2 (d) | Phase 3 (d) | Phase 4 (d) | Transfer Time (d) | Start Date   |
|----------------------------------|-------------|-------------|-------------|-------------|-------------------|--------------|
| 9.8                              | 546.04      | 33.28       | 143.36      | 19.30       | 195.94            | 30 June 2020 |
| 9.9                              | 546.58      | 32.99       | 142.72      | 19.15       | 194.86            | 30 June 2020 |
| 10.0                             | 547.12      | 32.70       | 142.08      | 19.00       | 193.78            | 1 July 2020  |
| 10.1                             | 547.66      | 32.41       | 141.46      | 18.86       | 192.72            | 1 July 2020  |
| 10.2                             | 548.19      | 32.12       | 140.84      | 18.72       | 191.68            | 2 July 2020  |

(d) Transfer times for elliptic planetary motion with gravity perturbations.

| $a$ (m s$^{-2}$) $\times 10^{-4}$ | Phase 1 (d) | Phase 2 (d) | Phase 3 (d) | Phase 4 (d) | Transfer Time (d) | Start Date   |
|----------------------------------|-------------|-------------|-------------|-------------|-------------------|--------------|
| 9.8                              | 544.88      | 33.31       | 146.25      | 18.27       | 197.83            | 28 June 2020 |
| 9.9                              | 545.44      | 33.05       | 146.27      | 17.39       | 196.71            | 29 June 2020 |
| 10.0                             | 546.00      | 36.76       | 138.97      | 19.89       | 195.61            | 30 June 2020 |
| 10.1                             | 546.55      | 32.43       | 144.31      | 17.79       | 194.53            | 30 June 2020 |
| 10.2                             | 547.09      | 32.08       | 141.29      | 20.09       | 193.46            | 1 July 2020  |
Next, Fig. 8a shows the optimized two-dimensional trajectory in Cartesian coordinates \((x^2(t), y^2(t)) = (r^2(t) \cos(\theta^2(t)), r^2(t) \sin(\theta^2(t)))\) during phase 2 (Earth escape). It is seen for phase 2 that the spacecraft starts in a circular orbit of radius \(6.6R_E\) relative to Earth and terminates a distance of \(1.01R_E^{SOI}\) from the Earth. Next, Fig. 6b shows the eccentricity of the spacecraft as a function of time. It is seen that the spacecraft makes multiple revolutions around the Earth in phase 2 such that the eccentricity remains between zero and 0.2 for approximately 2/3 of the phase. Then, during the last \(1/3\) of the phase, the eccentricity quickly grows to larger than unity at which point the spacecraft orbit transitions from elliptic to hyperbolic relative to the Earth. Escape from Earth (that is, an eccentricity that exceeds unity) occurs at approximately 75 percent of the way into the phase. Finally, the eccentricity at the end of the phase is approximately two. Next, Fig. 6c shows the radial and transverse components of the thrust direction, \(w_r^2(t)\) and \(w_p^2(t)\), respectively. During the portion of the phase where the eccentricity is small, the radial component of thrust remains close to zero while the transverse component of the thrust remains near unity. It is noted that during this early part of the phase the radial component of the thrust oscillates with decreasing amplitude about zero, thereby indicating that the spacecraft is being propelled further from the Earth. Then, during the hyperbolic segment of the phase, the radial component of thrust increases to approximately 0.9 while the transverse component of thrust decreases steadily to approximately 0.55. The optimized duration of the Earth escape phase is 33.31 d.

Next, Fig. 7a shows the optimal two-dimensional trajectory in Cartesian coordinates \((x^3(t), y^3(t)) = (r^3(t) \cos(\theta^3(t)), r^3(t) \sin(\theta^3(t)))\) during phase 3 (heliocentric transfer from Earth to Mars). It is seen that the spacecraft only makes a partial orbit around the Sun. Figure 7b shows the eccentricity of the spacecraft over the duration of phase 3 where it stays in a range of 0.07 to 0.37. Figure 7c shows the radial and transverse components of the thrust direction, \(w_r^3(t)\) and \(w_p^3(t)\), respectively. The radial thrust component begins at approximately 0.5, increases to 1, rapidly drops to -1, then increases to approximately -0.85, while the transverse thrust component rapidly decreases from approximately 0.85 to -1 and then rapidly increases to approximately 0.5, in order to satisfy the path constraints on the system. The thrust components behave in this way because of the requirement for a minimum-time solution, where the spacecraft needs the thrust direction to point towards Earth so it can accelerate away from the planet to escape from the gravitational force and then needs the thrust direction to rapidly switch so that the thrust direction is pointed towards Mars so that the spacecraft will decelerate in order to be captured by the gravitational force of the planet. The duration of this phase is 146.25 d.

Next, Fig. 8a shows the optimized two-dimensional trajectory of the spacecraft in Cartesian coordinates \((x^4(t), y^4(t)) = (r^4(t) \cos(\theta^4(t)), r^4(t) \sin(\theta^4(t)))\) during phase 4 (Mars capture). It is seen for phase 4 that the spacecraft starts at a distance of \(0.86R_M^{SOI}\) from Mars and terminates in a circular orbit of radius \(6.0R_M\) relative to Mars. Next, Fig. 8b shows the eccentricity of the spacecraft as a function of time. It is seen that the spacecraft starts hyperbolic relative to Mars for the first 1/4 of the phase such that the eccentricity decreases from 3.5 to unity, where the spacecraft orbit transitions from hyperbolic to elliptic relative to Mars. Then, during the last 3/4 of the phase, the spacecraft makes multiple revolutions around Mars such that the eccentricity decreases to 0, which corresponds to the terminal circular orbit. Capture to Mars (that is, an eccentricity that does not exceed unity) occurs at approximately 32 percent of the way into the phase. Next, Fig. 8c shows the radial and transverse components of the thrust direction, \(w_r^4(t)\) and \(w_p^4(t)\), respectively. During the hyperbolic segment of the phase, the radial component of thrust decreases from approximately 0.95 to 0.6 while the transverse component of thrust decreases from approximately -0.35 to -0.8. Then, during the portion of the phase where the eccentricity is small, the radial component of thrust remains close to zero while the transverse component of the thrust remains near -1. It is noted that during this later part of the phase the radial component of the thrust oscillates with decreasing amplitude about zero, thereby indicating that the spacecraft is being propelled towards Mars. The optimized duration of the Mars capture phase is 18.27 d.
Lastly, a higher-level view of the orbital transfer between Earth and Mars is shown in Fig. 9. In particular, Fig. 9 shows the motion of the planets and the spacecraft during the Earth escape, heliocentric, and Mars capture phases. It is noted that, using a date of 1 January 2019 for the start of the planetary alignment phase, the interplanetary orbital transfer starts on 28 June 2020. The interplanetary transfer has a minimum-time solution of 197.83 d.

4 Conclusions

A numerical optimization study of minimum-time Earth-to-Mars orbital trajectory optimization was performed. Three different cases of planetary motion were considered: circular, elliptic, and elliptic with gravity perturbations. The trajectory optimization problem was formulated as a four-phase optimal control problem where the phases consisted of planetary alignment, Earth escape, heliocentric transfer, and Mars capture. The time, state components, and control components of each phase were optimized. Because the problem was divided into four phases, interior-point event constraints were imposed to transform the variables at the terminus of one phase to the variables used at the start of the subsequent phase. It was found that the duration of the planetary alignment phases increased as a function of thrust specific force while the overall transfer time decreased. For
all values of thrust specific force considered, it was found that on average the duration of phases 1, 2, 3, and 4 for the elliptic with gravity perturbations case were, respectively, 545.99 d, 33.53 d, 143.42 d, and 18.69 d, while the average transfer time was 195.63 d. Moreover, the departure dates from Earth for these values of thrust specific force were found to lie between 28 June 2020 and 1 July 2020. In addition, the results for circular planetary motion were found to be qualitatively similar to those for elliptic planetary motion. Finally, it was found that the circular results of this study showed an average reduction in transfer time of 8.89 d when compared with a previous study on minimum-time Earth-to-Mars interplanetary orbital transfer.

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Figure 8: Optimal trajectory and control for the Mars capture phase.

Figure 9: Optimized motion of planets and spacecraft during earth escape, heliocentric, and Mars capture phases of the four-phase minimum-time interplanetary orbit transfer.
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Appendix

A Derivation of Coordinate Transformations

The derivations are now provided for the coordinate transformations from planet-centered coordinates to heliocentric coordinates. Note that the results of this section can be applied to transform from either the Earth or the Mars coordinates to the heliocentric coordinates. As a result, the derivation is provided only once for both coordinate transformations, leading to either Eq. (16) or (18).

A.1 Notation, Conventions, and Coordinate Systems

Let $S$, $Q$, and $P$ denote, respectively, the locations of the Sun, the planet (central body), and the spacecraft. Furthermore, assume that the Sun ($S$) is an inertially fixed point. Next, assume that $S$, $Q$, and $P$ all lie in the ecliptic plane where the ecliptic plane is assumed to be an inertial reference frame. Also, assume that the $N = U_{\varpi} \times U_{\varpi}$ is the normal to the ecliptic plane, where $U_{\varpi}$ and $U_{\varpi}$ are the unit vectors along the First Point of Aries and the First Point of Cancer, respectively. Finally, let $U_{SQ}$, $U_{SP}$, and $U_{QP}$, denote, respectively, the unit vectors in the directions from $S$ to $Q$, $S$ to $P$, and $Q$ to $P$. The aforementioned unit vectors are then used to define the following reference frames:

1. $I = \{U_{\varpi}, N\}$-plane = inertial reference frame.
2. $Q = \{U_{SQ}, N\}$-plane = (Sun, central body) reference frame.
3. $H = \{U_{SP}, N\}$-plane = (Sun, spacecraft) reference frame.
4. $U = \{U_{QP}, N\}$-plane = (central body, spacecraft) reference frame.

The following coordinate systems are then fixed, respectively, in reference frames $I$, $Q$, $H$, and $U$:

1. $I$-fixed coordinate system: $(S, i_1, i_2, i_3)$
   
   Origin: $S$,
   
   $i_1 = \text{Along } \Upsilon,$
   
   $i_3 = \text{Normal to Ecliptic Plane},$
   
   $i_2 = i_3 \times i_1.$

2. $Q$-fixed coordinate system: $(S, q_1, q_2, q_3)$
   
   Origin: $S$,
   
   $q_1 = \text{Along } U_{SQ},$
   
   $q_3 = N,$
   
   $q_2 = q_3 \times q_1.$

3. $H$-fixed coordinate system: $(S, h_1, h_2, h_3)$
   
   Origin: $S$,
   
   $h_1 = \text{Along } U_{SP},$
   
   $h_3 = N,$
   
   $h_2 = h_3 \times h_1.$
4. \(U\)-fixed coordinate system: \((Q, u_1, u_2, u_3)\)

\[
\begin{align*}
\text{Origin: } & Q, \\
u_1 & = \text{Along } U_{QP}, \\
u_3 & = N, \\
u_2 & = u_3 \times u_1.
\end{align*}
\]

(24)

Then, using Fig. 10, the bases \(\{q_1, q_2, q_3\}\) and \(\{h_1, h_2, h_3\}\) are related to the basis \(\{u_1, u_2, u_3\}\), respectively, as

\[
\begin{align*}
q_1 &= \cos(\theta - L) u_1 - \sin(\theta - L) u_2, \\
q_2 &= \sin(\theta - L) u_1 + \cos(\theta - L) u_2, \\
q_3 &= u_3,
\end{align*}
\]

(25)

and

\[
\begin{align*}
h_1 &= \cos(\theta - \phi) u_1 - \sin(\theta - \phi) u_2, \\
h_2 &= \sin(\theta - \phi) u_1 + \cos(\theta - \phi) u_2, \\
h_3 &= u_3.
\end{align*}
\]

(26)

Using the aforementioned coordinate systems, the following variables are used to define the position and inertial velocity of the spacecraft relative to the Sun in terms of the coordinate systems \((S, h_1, h_2, h_3)\):

\[
\begin{align*}
\rho &= \text{Distance from } S \text{ to } P, \\
\phi &= \text{Angle from } i_1 \text{ to } h_1, \\
v_\rho &= \dot{\rho}, \\
v_\phi &= \rho \dot{\phi}.
\end{align*}
\]

(27)

Also, the following variables are used to define the position and inertial velocity of the spacecraft relative to the Sun in terms of the coordinate systems \((S, q_1, q_2, q_3)\) and \((Q, u_1, u_2, u_3)\):

\[
\begin{align*}
r &= \text{Distance from } Q \text{ to } P, \\
\theta &= \text{Angle from } i_1 \text{ to } u_1, \\
v_r &= \dot{r}, \\
v_\theta &= r \dot{\theta}.
\end{align*}
\]

(28)

Finally, because the position of the planet is known, the quantities \(R\) and \(L\), and their rate of changes, \(\dot{R}\) and \(\dot{L}\), are considered to be known. The variables \((\rho, \phi)\) and \((r, \theta)\) are then used to derive independent expressions for the position of the spacecraft relative to the Sun and the inertial velocity of the spacecraft. Section A.2 derives these independent expressions and arrives at relationships between \((\rho, \phi, v_\rho, v_\phi)\) and \((r, \theta, v_r, v_\theta)\). A schematic of the various reference frames, coordinate systems, and variables is shown in Fig. 10.

A.2 Relationship Between Planet-Centered and Sun-Centered Coordinates

The transformation from planet-centered to Sun-centered coordinates is now derived using Fig. 10. First, one expression for the position of the spacecraft relative to the Sun is expressed in the basis \(\{h_1, h_2, h_3\}\) as

\[
r_{P/S} = \rho h_1.
\]

(29)

Now, because the angular velocity of reference frame \(H\) as viewed by an observer in reference frame \(I\) is \(\mathbf{I} \omega^H = \dot{\mathbf{h}}_3\), the inertial velocity of the spacecraft is obtained from the transport theorem \[28\] as

\[
\dot{r}_P = \mathbf{I} \frac{d}{dt} (r_{P/S}) = \frac{d}{dt} (r_{P/S}) + \mathbf{I} \omega^H \times r_{P/S} = \dot{\rho} h_1 + \dot{\phi} h_3 \times \rho h_1 = \dot{\rho} h_1 + \rho \dot{\phi} h_2 = v_\rho h_1 + v_\phi h_2.
\]

(30)
Figure 10: Schematic of Sun (S), planet (Q), spacecraft (P) along with the variables \( \rho, \phi, r, \) and \( \theta \) that define the position of the spacecraft relative to the Sun.

Using expressions for \( h_1 \) and \( h_2 \) in terms of \( u_1 \) and \( u_2 \) given in Eq. (26), the inertial velocity of the spacecraft is expressed in the basis \( \{u_1, u_2, u_3\} \) as

\[
\mathcal{T} v_P = v_\rho \cos(\theta - \phi) u_1 - \sin(\theta - \phi) u_2 + v_\phi \sin(\theta - \phi) u_1 + \cos(\theta - \phi) u_2. 
\]  

(31)

Next, a second expression for the position of the spacecraft relative to the Sun is given in terms of the bases \( \{q_1, q_2, q_3\} \) and \( \{u_1, u_2, u_3\} \) as

\[
r_{P/Q} = r_{Q/S} + r_{P/Q} = Rq_1 + ru_1. 
\]  

(32)

The inertial velocity of the spacecraft can then be written as

\[
\mathcal{T} v_P = \mathcal{T} \frac{d}{dt} (r_{Q/S}) + \mathcal{T} \frac{d}{dt} (r_{P/Q}) = \mathcal{T} v_Q + \mathcal{T} v_{P/Q}. 
\]  

(33)

Now, using the fact that the angular velocity of reference frame \( Q \) as viewed by an observer in reference frame \( I \) is \( \mathcal{T} \omega^Q = \dot{L}q_3 \), the inertial velocity of point \( Q \) is obtained from the transport theorem [28] as

\[
\mathcal{T} v_Q = \mathcal{T} \frac{d}{dt} (r_{Q/S}) = \frac{Q_d}{dt} (r_{Q/S}) + \mathcal{T} \omega^Q \times r_{Q/S} = \dot{R}q_1 + \dot{L}q_3 \times Rq_1 = \dot{R}q_1 + \dot{R}Lq_2. 
\]  

(34)

Furthermore, because the angular velocity of reference frame \( U \) as viewed by an observer in reference frame \( I \) is \( \mathcal{T} \omega^U = \dot{\theta}u_3 \), the inertial velocity of the spacecraft relative to the central body is obtained from the transport theorem [28] as

\[
\mathcal{T} v_{P/Q} = \mathcal{T} \frac{d}{dt} (r_{P/Q}) = \frac{U_d}{dt} (r_{P/Q}) + \mathcal{T} \omega^U \times r_{P/Q} = \dot{r}u_1 + \dot{\theta}u_3 \times r_{P/Q} = \dot{r}u_1 + r\dot{\theta}u_2 \equiv v_r u_1 + v_\theta u_2. 
\]  

(35)

Adding the results of Eqs. (34) and (35), the second expression for the inertial velocity of the spacecraft is given as

\[
\mathcal{T} v_P = \mathcal{T} v_Q + \mathcal{T} v_{P/Q} = \dot{R}q_1 + \dot{R}Lq_2 + \dot{r}u_1 + r\dot{\theta}u_2 = \dot{R}q_1 + \dot{R}Lq_2 + v_r u_1 + v_\theta u_2. 
\]  

(36)
It is seen that Eqs. (29) and (32) are two expressions for the position of the spacecraft relative to the Sun, while Eqs. (30) and (36) are two expressions for the inertial velocity of the spacecraft. Suppose now that the expressions given in Eqs. (29) and (32) are projected into the basis \{u_1, u_2, u_3\}. First, projecting the expression given in Eq. (29) into the basis \{u_1, u_2, u_3\} gives
\[
\begin{align*}
  r_{P/S} \cdot u_1 &= \rho \cos(\theta - \phi), \\
  r_{P/S} \cdot u_2 &= -\rho \sin(\theta - \phi).
\end{align*}
\] (37)

Next, projecting the expression given in Eq. (32) into the basis \{u_1, u_2, u_3\} gives
\[
\begin{align*}
  r_{P/S} \cdot u_1 &= R \cos(\theta - L) + r, \\
  r_{P/S} \cdot u_2 &= -R \sin(\theta - L).
\end{align*}
\] (38)

Setting the first and second expressions in Eq. (37) equal to the first and second expressions in Eq. (38), respectively, and rearranging gives
\[
\begin{align*}
  r &= -R \cos(\theta - L) + \rho \cos(\theta - \phi), \\
  0 &= R \sin(\theta - L) - \rho \sin(\theta - \phi).
\end{align*}
\] (39)

Next, projecting the expression given in Eq. (30) into the basis \{u_1, u_2, u_3\} gives
\[
\begin{align*}
  \mathcal{I}_V r_P \cdot u_1 &= v_\rho \cos(\theta - \phi) + v_\phi \sin(\theta - \phi), \\
  \mathcal{I}_V r_P \cdot u_2 &= -v_\rho \sin(\theta - \phi) + v_\phi \cos(\theta - \phi).
\end{align*}
\] (40)

Then, projecting the expression given in Eq. (36) into the basis \{u_1, u_2, u_3\} gives
\[
\begin{align*}
  \mathcal{I}_V r_P \cdot u_1 &= \ddot{R} \cos(\theta - L) + R \dot{L} \sin(\theta - L) + v_r, \\
  \mathcal{I}_V r_P \cdot u_2 &= -\ddot{R} \sin(\theta - L) + R \dot{L} \cos(\theta - L) + v_\phi.
\end{align*}
\] (41)

Setting the first and second expressions in Eq. (40) equal to the first and second expressions in Eq. (41), respectively, and rearranging gives
\[
\begin{align*}
  v_r &= -\ddot{R} \cos(\theta - L) - R \dot{L} \sin(\theta - L) + v_\rho \cos(\theta - \phi) + v_\phi \sin(\theta - \phi), \\
  v_\phi &= \ddot{R} \sin(\theta - L) - R \dot{L} \cos(\theta - L) + v_\rho \sin(\theta - \phi) + v_\phi \cos(\theta - \phi).
\end{align*}
\] (42)

Equation (39) provides the relationship between the variables \((\rho, \phi)\) and \((r, \theta)\), while Eq. (42) provides the relationship between the variables \((r, \theta, v_r, v_\phi)\) and the variables \((\rho, \phi, v_\rho, v_\phi)\). Equations (39) and (42) are used to develop the event constraints of Eqs. (16) and (18). It is noted that Eq. (16) connects the terminus of the Earth escape phase to the start of the heliocentric phase, while Eq. (18) connects the terminus of the heliocentric phase to the start of the Mars capture phase. When using Eqs. (39) and (42) in the context of the event constraints given in Section 2.7, the variables \((\rho, \phi, v_\rho, v_\phi)\) and \((r, \theta, v_r, v_\phi)\) have been replaced with the appropriate variables given in Table 2.

B Derivation of Gravity Perturbations

The derivations are now provided for the gravity perturbations given in Eq. (3). Note that for the Earth escape, Mars capture, and heliocentric phases (that is, phases 2, 3, and 4) Cowell’s method [22] is used. On the other hand, no gravity perturbations are included in phase 1 because the motion of the spacecraft is not included until phase 2.

26
B.1 Gravity Perturbations During Phases 2 and 4

During phases 2 and 4, where the central bodies are Earth and Mars, respectively, the Sun is the only perturbing body. Because the perturbing acceleration due to the Sun lies along $\mathbf{h}_1$ (see Fig. 10) while the dynamics are expressed in the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, the perturbing acceleration due to the Sun in the Earth escape and Mars captures phases is projected into the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Using Fig. 10 the position of the spacecraft relative to the Sun is given as

$$\mathbf{r}_{P/S} = \mathbf{r}_{Q/S} + \mathbf{r}_{P/Q} = R \mathbf{q}_1 + r \mathbf{u}_1 = \rho \mathbf{h}_1. \quad (43)$$

Expressing $\mathbf{q}_1$ in the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ gives

$$\rho \mathbf{h}_1 = R \left[ \cos (\theta - L_Q) \mathbf{u}_1 - \sin (\theta - L_Q) \mathbf{u}_2 \right] + r \mathbf{u}_1$$

$$= \left[ r + R \cos (\theta - L_Q) \right] \mathbf{u}_1 - R \sin (\theta - L_Q) \mathbf{u}_2, \quad (44)$$

where

$$\mathbf{h}_1 = \left[ \frac{r + R \cos (\theta - L_Q)}{\rho} \right] \mathbf{u}_1 - \frac{R}{\rho} \sin (\theta - L_Q) \mathbf{u}_2. \quad (45)$$

Next, using the first expression in Eq. (43), $\mathbf{h}_1$ can be written as

$$\mathbf{h}_1 = \frac{\mathbf{r}_{P/S}}{||\mathbf{r}_{P/S}||} = \frac{\left[ r + R \cos (\theta - L_Q) \right] \mathbf{u}_1 - R \sin (\theta - L_Q) \mathbf{u}_2}{\sqrt{(r + R \cos (\theta - L_Q))^2 + (-R \sin (\theta - L_Q))^2}}. \quad (46)$$

Setting the results of Eqs. (43) and (46) equal to one another gives

$$\rho = \sqrt{(r + R \cos (\theta - L_Q))^2 + (-R \sin (\theta - L_Q))^2} = \sqrt{r^2 + R^2 + 2rR \cos (\theta - L_Q)}. \quad (47)$$

Then, using Cowell’s method [22], the gravity perturbation of the Sun for use in phases 2 and 4 is given as

$$\tau \mathbf{a}_p = -\mu_S \left[ \frac{\mathbf{r}_{P/S}}{||\mathbf{r}_{P/S}||^3} - \frac{\mathbf{r}_{Q/S}}{||\mathbf{r}_{Q/S}||^3} \right] = -\mu_S \left[ \frac{\rho}{\rho^3} \mathbf{h}_1 - \frac{R}{R^3} \mathbf{q}_1 \right] = -\mu_S \left[ \frac{1}{\rho^3} \mathbf{h}_1 - \frac{1}{R^3} \mathbf{q}_1 \right]. \quad (48)$$

Now, using Eqs. (45) and (25), $\mathbf{a}_p$ can be expressed in the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ as

$$\tau \mathbf{a}_p = -\mu_S \left[ \frac{\left[ (r + R \cos (\theta - L_Q)) \mathbf{u}_1 - R \sin (\theta - L_Q) \mathbf{u}_2 \right]}{\rho^3} - \frac{\left[ \cos (\theta - L_Q) \mathbf{u}_1 - \sin (\theta - L_Q) \mathbf{u}_2 \right]}{R^3} \right]. \quad (49)$$

Projecting $\tau \mathbf{a}_p$ into the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ gives

$$a_{pr} = \tau \mathbf{a}_p \cdot \mathbf{u}_1, \quad a_{p\theta} = \tau \mathbf{a}_p \cdot \mathbf{u}_2. \quad (50)$$

The quantities $a_{pr}$ and $a_{p\theta}$ are then given as

$$a_{pr} = -\mu_S \left[ \frac{r}{\rho^3} + \left( \frac{R}{\rho^3} - \frac{1}{R^3} \right) \cos (\theta - L_Q) \right], \quad (51)$$

$$a_{p\theta} = -\mu_S \left( -\frac{R}{\rho^3} + \frac{1}{R^3} \right) \sin (\theta - L_Q).$$

It is noted again that the result given in Eq. (51) can be used in both the Earth escape phase (that is, phase 2) and the Mars capture phase (that is, phase 4).
B.2 Gravity Perturbations During Phase 3

During phase 3, where the central body is the Sun, both Earth and Mars are perturbing bodies. It is noted that the derivation shown below considers a generic perturbing body which could be either Earth or Mars. The total perturbing acceleration for the heliocentric phase, which includes perturbations from both Earth and Mars, is given in [B.3]. Because the perturbing acceleration due to a perturbing body \( Q \) lies along \( \mathbf{u}_1 \) (see Fig. [10]) while the dynamics are expressed in the basis \( \{ \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \} \), the perturbing acceleration due to \( Q \) in the heliocentric transfer phase is projected into the basis \( \{ \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \} \).

Using Fig. [10] the basis \( \{ \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \} \) is related to the basis \( \{ \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \} \) as

\[
\begin{align*}
\mathbf{q}_1 & = \cos (\phi - L_Q) \mathbf{h}_1 - \sin (\phi - L_Q) \mathbf{h}_2, \\
\mathbf{q}_2 & = \sin (\phi - L_Q) \mathbf{h}_1 + \cos (\phi - L_Q) \mathbf{h}_2, \\
\mathbf{q}_3 & = \mathbf{h}_3.
\end{align*}
\]

Next, from Eq. ([43]), the position of the spacecraft relative to the planet is given as

\[
\mathbf{r}_{P/Q} = \mathbf{r}_{P/S} - \mathbf{r}_{Q/S} = \rho \mathbf{h}_1 - R \mathbf{q}_1 = r \mathbf{u}_1.
\]

Expressing \( \mathbf{q}_1 \) in the basis \( \{ \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \} \) using Eq. (52) gives

\[
\begin{align*}
r \mathbf{u}_1 & = \rho \mathbf{h}_1 - R [\cos (\phi - L_Q) \mathbf{h}_1 - \sin (\phi - L_Q) \mathbf{h}_2] \\
& = [\rho - R \cos (\phi - L_Q)] \mathbf{h}_1 + R \sin (\phi - L_Q) \mathbf{h}_2,
\end{align*}
\]

where

\[
\mathbf{u}_1 = \left[ \frac{\rho - R \cos (\phi - L_Q)}{r} \right] \mathbf{h}_1 + \frac{R \sin (\phi - L_Q)}{r} \mathbf{h}_2.
\]

Next, using the first expression in Eq. (53), \( \mathbf{u}_1 \) can be written as

\[
\mathbf{u}_1 = \frac{\mathbf{r}_{P/Q}}{\| \mathbf{r}_{P/Q} \|} = \left[ \frac{\rho - R \cos (\phi - L_Q)}{\rho - R \cos (\phi - L_Q)^2 + R \sin (\phi - L_Q)^2} \right] \mathbf{h}_1 + \frac{R \sin (\phi - L_Q)}{\rho - R \cos (\phi - L_Q)} \mathbf{h}_2.
\]

Setting the results of Eqs. (55) and (56) equal gives

\[
r = \sqrt{\rho - R \cos (\phi - L_Q)^2 + R \sin (\phi - L_Q)^2} = \sqrt{\rho^2 + R^2 - 2 R \rho \cos (\phi - L_Q)}.
\]

Then, using Cowell’s method [22], the gravity perturbation due to \( Q \) in phase 3 is given as

\[
\mathbf{\tau} \mathbf{a}_p = -\mu_Q \left[ \frac{\mathbf{r}_{P/Q}}{\| \mathbf{r}_{P/Q} \|^3} \right] = -\mu_Q \left[ \frac{r}{r^3} \mathbf{u}_1 \right] = -\mu_Q \left[ \frac{1}{r^2} \mathbf{u}_1 \right].
\]

Now, using Eqs. (52) and (55), \( \mathbf{\tau} \mathbf{a}_p \) can be expressed in the basis \( \{ \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \} \) as

\[
\mathbf{\tau} \mathbf{a}_p = -\mu_Q \left[ \frac{[(\rho - R \cos (\phi - L_Q)) \mathbf{h}_1 + R \sin (\phi - L_Q) \mathbf{h}_2]}{r^3} \right].
\]

Projecting \( \mathbf{\tau} \mathbf{a}_p \) into the basis \( \{ \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \} \) gives

\[
\begin{align*}
a_{pr} & = \mathbf{\tau} \mathbf{a}_p \cdot \mathbf{h}_1, \\
a_{\rho \theta} & = \mathbf{\tau} \mathbf{a}_p \cdot \mathbf{h}_2.
\end{align*}
\]

The quantities \( a_{pr} \) and \( a_{\rho \theta} \) are then given as

\[
\begin{align*}
a_{pr} & = -\mu_Q \left[ \frac{\rho}{r^3} - \frac{R}{r^3} \cos (\phi - L_Q) \right], \\
a_{\rho \theta} & = -\mu_Q \frac{R}{r^3} \sin (\phi - L_Q).
\end{align*}
\]

It is noted again that the result given in Eq. (61) can be used to obtain the perturbing acceleration of either Earth or Mars.
B.3 Simplified Gravity Perturbations Using Cowell’s Method

Using Cowell’s method in Ref. 22, the radial and transverse components of the gravity perturbations for phases 2, 3, and 4 are given as follows:

\[
\begin{align*}
\alpha_{pr}^{[2]} &= -\mu_2 \left[ \frac{r^{[2]}_E}{r_{psE}^3} + \left( \frac{r^{[2]}_E}{r_{psE}^3} - \frac{1}{\left( r^{[2]}_E \right)^2} \right) \cos \left( \theta^{[2]} - L^{[2]}_E \right) \right], \\
\alpha_{p\theta}^{[2]} &= -\mu_2 \left( -\frac{r^{[2]}_E}{r_{psE}^3} + \frac{1}{\left( r^{[2]}_E \right)^2} \right) \sin \left( \theta^{[2]} - L^{[2]}_E \right), \\
\alpha_{pr}^{[3]} &= -\mu_3 \left[ \frac{r^{[3]}_E}{r_{pe}^3} - \frac{r^{[3]}_E}{r_{pe}^3} \cos \left( \theta^{[3]} - L^{[3]}_E \right) \right] - \mu_3 \left[ \frac{r^{[3]}_M}{r_{pm}^3} - \frac{r^{[3]}_M}{r_{pm}^3} \cos \left( \theta^{[3]} - L^{[3]}_M \right) \right], \\
\alpha_{p\theta}^{[3]} &= -\mu_3 \frac{r^{[3]}_E}{r_{pe}^3} \sin \left( \theta^{[3]} - L^{[3]}_E \right) - \mu_3 \frac{r^{[3]}_M}{r_{pm}^3} \sin \left( \theta^{[3]} - L^{[3]}_M \right), \\
\alpha_{pr}^{[4]} &= -\mu_4 \left[ \frac{r^{[4]}_E}{r_{psM}^3} + \frac{r^{[4]}_M}{r_{psM}^3} - \frac{1}{\left( r^{[4]}_M \right)^2} \right] \cos \left( \theta^{[4]} - L^{[4]}_M \right), \\
\alpha_{p\theta}^{[4]} &= -\mu_4 \left( -\frac{r^{[4]}_M}{r_{psM}^3} + \frac{1}{\left( r^{[4]}_M \right)^2} \right) \sin \left( \theta^{[4]} - L^{[4]}_M \right),
\end{align*}
\]

where

\[
\begin{align*}
\text{r}_{psE} &= \sqrt{\left( r^{[2]}_E \right)^2 + \left( r^{[2]}_E \right)^2 + 2r^{[2]} r^{[2]}_E \cos \left( \theta^{[2]} - L^{[2]}_E \right)}, \\
\text{r}_{pe} &= \sqrt{\left( r^{[3]}_E \right)^2 + \left( r^{[3]}_E \right)^2 - 2r^{[3]} r^{[3]}_E \cos \left( \theta^{[3]} - L^{[3]}_E \right)}, \\
\text{r}_{pm} &= \sqrt{\left( r^{[3]}_M \right)^2 + \left( r^{[3]}_M \right)^2 - 2r^{[3]} r^{[3]}_M \cos \left( \theta^{[3]} - L^{[3]}_M \right)}, \\
\text{r}_{psM} &= \sqrt{\left( r^{[4]}_M \right)^2 + \left( r^{[4]}_M \right)^2 + 2r^{[4]} r^{[4]}_M \cos \left( \theta^{[4]} - L^{[4]}_M \right)},
\end{align*}
\]