THE DIFFUSION PHENOMENON FOR DAMPED WAVE EQUATIONS WITH SPACE-TIME DEPENDENT COEFFICIENTS

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Abstract. We introduce a method to study the long-time behavior of solutions to damped wave equations, where the coefficients of the equations are space-time dependent. We show that solutions exhibit the diffusion phenomenon, connecting their asymptotic behaviors with the asymptotic behaviors of solutions to corresponding parabolic equations. Sharp decay estimates for solutions to damped wave equations are given, and decay estimates for derivatives of solutions are also discussed.

1. Introduction. We introduce a method to study solutions to damped wave equations, where the coefficients of the equations depend on space and time. To demonstrate this method, we consider the problem

\[
\begin{aligned}
& u_{tt} + u_t - \nabla \cdot (a(x,t) \nabla u) = 0, \quad x \in \mathbb{R}^N, \quad t > 0, \\
& (u, u_t)(x,0) = (u_0, u_1)(x), \quad x \in \mathbb{R}^N.
\end{aligned}
\]  

(1)

The global well-posedness and regularity of (1) are found in Ikawa [3].

We prove that the solution to (1) exhibits the diffusion phenomenon, meaning that \( u \) asymptotically behaves like a solution to

\[
\begin{aligned}
& v_t - \nabla \cdot (a(x,t) \nabla v) = 0, \quad x \in \mathbb{R}^N, \quad t > 0, \\
& v(x,0) = u_0(x) + u_1(x), \quad x \in \mathbb{R}^N.
\end{aligned}
\]  

(2)

As a corollary to the diffusion phenomenon, a sharp decay estimate for \( \|u(x,t)\|_{L^2_x} \) is obtained. Decay estimates for \( L^2_x \) norms of derivatives of \( u(x,t) \) are also discussed.

Matsumura [14] considered solutions to the equation \( u_{tt} + u_t - \Delta u = 0 \) and used Fourier methods to establish \( L^2_x - L^2_t \) sharp decay estimates for \( u(x,t) \) and its space and time derivatives. Following Matsumura's results, many authors have considered many variants of the equation \( u_{tt} + u_t - \Delta u = 0 \), including variants where \( \Delta \) is replaced with a more general, time-independent operator. Many authors have also considered variants of the form

\[
u_{tt} + a(x)b(t)u_t - \Delta u = 0.
\]  

(3)

Note that until recently, (3) was considered with either space- or time-dependent damping coefficients, and the methods used for these two types of problems are incompatible.

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Ikehata [5, 6] and Ono [21] studied (3) with constant damping on an exterior domain. Nakao [17], Ikehata [7, 8], Mochizuki and Nakao [15] showed energy decay for solutions to (3) with space-dependent damping coefficients, giving results comparable to Matsumura’s in low spatial dimensions. Mochizuki and Nakazawa [16] showed energy decay for solutions to (3) with a nonseparable damping coefficient.

Ikehata and Nishihara [9], followed by Chill and Haraux [1], considered the problem \( u_{tt} + u_t + Bu = 0 \) in a Hilbert space \( \mathcal{H} \), where \( B \) is a nonnegative self-adjoint operator in \( \mathcal{H} \). For \( B = \Delta \), their results are comparable to Matsumura’s in low spatial dimensions.

Reissig and Wirth [25], Wirth [29, 30] considered (3) with a slowly changing, time-dependent damping coefficient, and using Fourier methods, Wirth obtained \( L^p_x - L^q_x \) sharp decay estimates for solutions. Yamazaki [31] studied abstract wave equations with time-dependent damping.

For damped wave equations with slowly changing, space-dependent coefficients, Radu, Todorova and Yordanov [23] proved the exact gain in the decay rate for all higher order energies in terms of the first order energy.

Todorova and Yordanov [27] considered (3) with a constant damping coefficient and the nonlinear term \( |u|^p \). They developed a weighted energy method that uses a special weight related to the fundamental solution of the equation \( u_{tt} + u_t - \Delta u = 0 \).

Todorova and Yordanov [28] considered (3) with a slowly decaying, space-dependent damping coefficient. They used a weighted energy method with a special weight. Similar weighted energy methods were implemented in [11, 12, 13, 18, 19, 26]. Nishihara and Zhai [19] and Nishihara [18] considered (3) with a defocusing nonlinear term \(-|u|^{p-1}u\) and a slowly changing damping coefficient that was either space- or time-dependent. This was followed by Lin, Nishihara and Zhai [12, 13], and Khader [11] who studied (3) with a slowly changing, radially symmetric damping coefficient and a defocusing nonlinear term \(-|u|^{p-1}u\).

Radu, Todorova and Yordanov [22] proved the diffusion phenomenon for the abstract problem \( u_{tt} + u_t + Bu = 0 \) in a Hilbert space \( \mathcal{H} \). Ikehata, Todorova and Yordanov [10] showed a more complex diffusion phenomenon for abstract wave equations with strong damping. Then Radu, Todorova and Yordanov [24] proved the diffusion phenomenon for the problem \( Cu_{tt} + u_t + Bu = 0 \) in a Hilbert space \( \mathcal{H} \), where \( B \) and \( C \) are two noncommuting self-adjoint operator on \( \mathcal{H} \), which excludes the use of the spectral theorem. Instead, they used consecutive approximations with conveniently defined diffusion solutions. They also expanded their decay gains that originated in [23], giving the exact gain in the decay rate for \( \|\partial^p_t u\| \) in terms of \( \|u\| \).

By resolvent arguments, Nishiyama [20] showed the diffusion phenomenon for the problem \( u_{tt} + Au_t + Bu = 0 \) in a Hilbert space \( \mathcal{H} \). Here \( A \) and \( B \) are two noncommuting self-adjoint operator on \( \mathcal{H} \), satisfying some additional conditions.

Sobajima and Wakasugi [26] considered (3) with radially symmetric, slowly decaying space-dependent damping. They gave a sharp decay estimate for a modified \( L^2_x \) norm of the solution.

We present the \( L^2_x \) sharp decay rate for the solution to (1) and decay rates for its derivatives. The coefficient \( a(x,t) \) is neither assumed to be separable in space and time, nor radially symmetric.

There are three key tools used in this present work. The first tool is the improved decay, which specifically refers to the gains in the decay rates for space and time.
derivatives of \( u \) in terms of \( u \). This gain in decay is expressed in a weighted average sense. Note that the improved decay was discussed in [23] and [24].

The *weighted energy method* developed in [27] in the second key tool. Note that a special weight is used. One important consequence of this weighted energy method is that solutions to damped wave equations decay exponentially for \( x \) outside of the ball \( B_0((t + 1)^{(1+\delta)/2}) \), where \( \delta > 0 \) can be arbitrarily small.

The third key tool is the fundamental solution of (2). The fundamental solution of (2) encodes desirable parabolic decay properties. We prove that \( u - v \), the difference between the solutions of (1) and (2), can be expressed in terms of the fundamental solution of (2) acting on derivatives of \( u \). This representation of \( u - v \) permits the three key tools to work together.

1.1. **Assumptions for the hyperbolic problem (1).** We assume the data \((u_0, u_1)\) are in \( H^3(\mathbb{R}^N) \times H^2(\mathbb{R}^N) \), and \( \text{supp}(u_0, u_1) \subset \{ x \in \mathbb{R}^N : |x| < R_0 \} \) for some \( R_0 > 0 \). We also assume that \( a(x, t) \in C^2(\mathbb{R}^{N+1}) \) and its first and second order derivatives are bounded and continuous in \( \mathbb{R}^{N+1} \); this includes the mixed space-time derivatives. In addition, we assume

\[
\begin{align*}
  a_1 & \leq a(x, t) \leq a_2, \\
  |a_t(x, t)| & \leq a_3 \frac{a(x, t)}{t+1}, \\
  |a_{tt}(x, t)| & \leq a_4 \frac{a(x, t)}{(t+1)^2},
\end{align*}
\]

(A1) (A2) (A3)

where the constants \( a_1, a_2, a_3, \text{ and } a_4 > 0 \).

1.2. **Existence, uniqueness and regularity for the hyperbolic problem (1).**

These are given by

**Lemma 1.1.** (*Existence, uniqueness and regularity*) Under the assumptions of subsection 1.1, problem (1) admits a unique solution such that

\[
  u \in \bigcap_{i=0}^3 C^i([0, \infty); H^{3-i}(\mathbb{R}^N));
\]

(4)

see Ikawa [3, Theorem 2].

1.3. **Main results.**

**Theorem 1.2.** (*Diffusion phenomenon*) Let \( u(x, t) \) be the solution to (1), where the assumptions in subsection 1.1 hold. Then for \( t \geq 0 \)

\[
  \|u(x, t) - v(x, t)\|^2_{L^2_x} \leq C (t + 1)^{-N+2} \| (u_0, u_1) \|^2_{H^2_x \times H^1(\mathbb{R}^N)} \, ,
\]

where \( v(x, t) \) is a prescribed solution to (2), and the constant \( C \) depends on \( a_1, \ldots, a_4, \) \( N, \text{ and } R_0. \)

The prescribed solution to (2) will be shown to have the property \( \|v(x, t)\|^2_{L^2_x} \leq C (t + 1)^{-N} \| (u_0, u_1) \|^2_{H^2_x \times H^1(\mathbb{R}^N)} \). Combining this with Theorem 1.2 gives the following corollary.
Corollary 1. Let $u(x, t)$ be the solution to (1), where the assumptions in subsection 1.1 hold. Then for $t \geq 0$, the following hold:

\[
\|u(x, t)\|_{L^2}^2 \leq C (t + 1)^{-\frac{N}{2}} \|(u_0, u_1)\|_{H^2 \times H^1(\mathbb{R}^N)}^2, \quad (i)
\]

\[
\|\nabla u(x, t)\|_{L^2}^2 \leq C \ln(t + 2) (t + 1)^{-\frac{N}{2} - 1} \|(u_0, u_1)\|_{H^2 \times H^1(\mathbb{R}^N)}^2, \quad (ii)
\]

\[
\|u_t(x, t)\|_{L^2}^2 \leq C \ln(t + 2) (t + 1)^{-\frac{N}{2} - 2} \|(u_0, u_1)\|_{H^2 \times H^1(\mathbb{R}^N)}^2, \quad (iii)
\]

where the constant $C$ depends on $a_1, \ldots, a_4, N,$ and $R_0$.

Remark 1. Using the same method, it is possible to consider a more general problem

\[
\begin{cases}
    c(x, t) u_{tt} + b(x, t) u_t - \nabla \cdot (a(x, t) \nabla u) = 0, & x \in \mathbb{R}^N, \ t > 0, \\
    (u, u_t)(x, 0) = (u_0, u_1)(x), & x \in \mathbb{R}^N,
\end{cases}
\]

where $a, b, c \in C^2(\mathbb{R}^{N+1})$ have bounded first and second order derivatives and each satisfies assumptions similar to A1, A2, and A3. Conclusions analogous to Theorem 1.2 and Corollary 1 can be achieved via analogous proofs.

This paper is structured as follows: section two is devoted to the proofs of the improved decay. Estimates coming from the weighted energy method are proved in section three. A representation formula for the difference between solutions of (1) and (2), in terms of the fundamental solution of (2), is derived in section four. In section five, the main result and its corollary are proved via the three key tools.

2. Improved decay for dissipative wave equations.

2.1. Preliminary lemmas. The lemmas in this subsection are standard and presented for completeness.

Lemma 2.1. (Energy inequality) Let $u(x, t)$ be the solution to (1), where the assumptions in subsection 1.1 hold, and let $T > 0$. Then for $0 \leq t \leq T$

\[
\|u(x, t)\|_{H^2(\mathbb{R}^N)}^2 + \|u_t(x, t)\|_{H^1(\mathbb{R}^N)}^2 + \|u_{tt}(x, t)\|_{L^2(\mathbb{R}^N)}^2 \leq C(T) \|(u_0, u_1)\|_{H^2 \times H^1(\mathbb{R}^N)}^2;
\]

see Ikawa [3, Proposition 2.6] or cf. [4, Theorem 2.15].

Lemma 2.2. (Finite speed of propagation) Let $u(x, t)$ be the solution to (1), where the assumptions in subsection 1.1 hold. Then $u$ has a finite speed of propagation; see Ikawa [4, Theorem 2.7].

2.2. Improved decay. The purpose of this subsection is to obtain the gains in the decay rates for derivatives of $u$ in terms of $u$. These gains in decay are expressed in a weighted average sense.

Definition 2.3. For $i = 1, 2$, respectively, we use the first and second energies:

\[
E_i(t; u) := \frac{1}{2} \int_{\mathbb{R}^N} \left( \partial_t^i u(x, t) \right)^2 + a(x, t) \left| \nabla \partial_t^{i-1} u(x, t) \right|^2 dx.
\]

The improved decay for the first energy $E_1(t; u)$ is proved in the following proposition.
**Proposition 1.** Let \( u(x, t) \) be the solution to (1), and let the assumptions in subsection 1.1 be satisfied. For \( r \geq 0 \) and \( \theta \geq 0 \),
\[
\int_0^r (t + 1)^\theta E_1(t; u) dt \leq C \|(u_0, u_1)\|_{H^2 \times H^1(\mathbb{R}^N)}^2 + C \int_0^r (t + 1)^{\theta - 1} \|u\|^2_{L^2} dt,
\]
where \( C \) depends on \( a_2, a_3, \) and \( \theta \).

**Proof.** We begin by taking the \( L^2(\mathbb{R}^N) \) inner product of equation (1) and \( 2u_t \). Then apply assumption (A2) and get
\[
\partial_t \left( \|u_t\|^2_{L^2} + \langle a\nabla u, \nabla u \rangle_{L^2} \right) \leq -2 \|u_t\|^2_{L^2} + \frac{a_3}{t + 1} \langle a\nabla u, \nabla u \rangle_{L^2}. \tag{6}
\]
Similarly, we take the \( L^2(\mathbb{R}^N) \) inner product of equation (1) and \( u \) to obtain
\[
\partial_t \left( \langle u_t, u \rangle_{L^2} + \frac{1}{2} \|u\|^2_{L^2} \right) = \|u_t\|^2_{L^2} - \langle a\nabla u, \nabla u \rangle_{L^2}. \tag{7}
\]
Next, define the continuously differentiable function
\[
Y(t) := \|u_t\|^2_{L^2} + \langle u_t, u \rangle_{L^2} + \frac{1}{2} \|u\|^2_{L^2} + \langle a\nabla u, \nabla u \rangle_{L^2};
\]
for the regularity, see Lemma 1.1. Then combine (6) with (7) and add \( \frac{\theta}{t + 1} Y(t) \) to both sides. This gives
\[
\frac{\theta}{t + 1} Y(t) + Y'(t) + E_1(t; u) \leq \frac{\theta}{t + 1} Y(t) + \frac{a_3}{t + 1} \langle a\nabla u, \nabla u \rangle_{L^2} - E_1(t; u). \tag{8}
\]
Notice that \( u_t u \geq -\frac{1}{2} (u^2 + u_t^2) \), giving
\[
0 \leq Y(t). \tag{9}
\]
Similarly,
\[
Y(t) \leq \|u\|^2_{L^2} + 3E_1(t; u), \tag{10}
\]
since \( u_t u \leq \frac{1}{2} (u^2 + u_t^2) \) and \( 2E_1(t; u) = \|u_t\|^2_{L^2} + \langle a\nabla u, \nabla u \rangle_{L^2} \). Apply (10) and \( \langle a\nabla u, \nabla u \rangle_{L^2} \leq 2E_1(t; u) \) to the RHS of (8) and obtain
\[
\frac{\theta}{t + 1} Y(t) + Y'(t) + E_1(t; u) \leq \frac{\theta}{t + 1} \|u\|^2_{L^2} + \left( \frac{3\theta + 2a_3}{t + 1} - 1 \right) E_1(t; u). \tag{11}
\]
Multiply both sides of (11) by the integrating factor \( (t + 1)^\theta \) to see that
\[
\partial_t \left( (t + 1)^\theta Y(t) \right) + (t + 1)^\theta E_1(t; u) \leq \theta(t + 1)^{\theta - 1} \|u\|^2_{L^2} + (t + 1)^\theta \left( \frac{3\theta + 2a_3}{t + 1} - 1 \right) E_1(t; u). \tag{12}
\]
Next integrate both sides of (12) with respect to \( t \), from 0 to \( r \). To complete the proof, we estimate the integrals of the first and last terms of (12) by the initial
data. Note that (9) and (10), followed by assumption (A1) give
\[
(t + 1)^{\theta}Y(t)|_{t=0} = (r + 1)^{\theta}Y(r) - Y(0) \\
\geq 0 - \left(\|u_0\|_{L^2}^2 + 3E_1(0; u)\right) \\
\geq -C(a_2)\|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}.
\]
Define \(T_0 := \max\{0, 3\theta + 2a_3 - 1\}\). Then for all \(r \geq 0\),
\[
\int_0^r (t + 1)^{\theta} \left(\frac{3\theta + 2a_3}{t + 1} - 1\right) E_1(t; u) dt \\
\leq \int_0^{T_0} (t + 1)^{\theta} \left(\frac{3\theta + 2a_3}{t + 1} - 1\right) E_1(t; u) dt,
\]
since \(\frac{3\theta + 2a_3}{t + 1} - 1 \leq 0\) for \(t \geq T_0\). Apply assumption (A1) and then the energy inequality Lemma 2.1 to the RHS of (13), obtaining
\[
\int_0^r (t + 1)^{\theta} \left(\frac{3\theta + 2a_3}{t + 1} - 1\right) E_1(t; u) dt \leq C(a_2, a_3, \theta) \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}
\]
for all \(r \geq 0\). Therefore, the proof of (5) is complete.

The improved decay for \(\|u_t\|_{L^2_t}^2\) is shown in the following proposition.

**Proposition 2.** Let \(u(x, t)\) be the solution to (1), and let the assumptions in subsection 1.1 be satisfied. For \(r \geq 0\) and \(\theta \geq 0\),
\[
\int_0^r (t + 1)^{\theta + 1} \|u_t\|_{L^2_t}^2 dt \leq C \|(u_0, u_1)\|^2_{H^1 \times L^2(\mathbb{R}^N)} \\
+ C \int_0^r (t + 1)^{\theta - 1} \|u\|_{L^2_t}^2 dt,
\]
where \(C\) depends on \(a_2, a_3, \) and \(\theta\).

**Proof.** Add \(\frac{\theta + 1}{t + 1}E_1(t; u)\) to both sides of (6) to obtain
\[
\frac{\theta + 1}{t + 1}E_1(t; u) + \partial_t E_1(t; u) \leq -2\|u_t\|_{L^2_t}^2 + \frac{\theta + 1}{t + 1}E_1(t; u) + \frac{a_3}{t + 1} \langle a\nabla u, \nabla u \rangle_{L^2_x}.
\]
Next, bound \(\langle a\nabla u, \nabla u \rangle_{L^2_x}\) from above by \(2E_1(t; u)\), and then multiply both sides of the resulting inequality by the integrating factor \((t + 1)^{\theta + 1}\). This gives
\[
\partial_t \left((t + 1)^{\theta + 1}E_1(t; u)\right) \leq -2(t + 1)^{\theta + 1}\|u_t\|_{L^2_t}^2 + (\theta + 1 + 2a_3)(t + 1)^\theta E_1(t; u).
\]
Next, integrate both sides of this inequality with respect to \(t\), from 0 to \(r\), and note that
\[
(t + 1)^{\theta + 1}E_1(t; u)|_{t=0} = -E_1(0; u).
\]
To complete the proof of (14), apply the improved decay Proposition 1 to the term \(\int_0^r (\theta + 1 + 2a_3)(t + 1)^\theta E_1(t; u) dt\), obtaining the last term on the RHS of (14).

The next two propositions show the improved decay for \(E_2(t; u)\) and \(\|u_{tt}\|_{L^2_t}^2\), respectively. These propositions are analogous to Propositions 1 and 2, except with larger weights on their left-hand sides.
Proposition 3. Let \( u(x,t) \) be the solution to (1), and let the assumptions in subsection 1.1 be satisfied. For \( r \geq 0 \) and \( \theta \geq 0 \),

\[
\int_0^r (t + 1)^{\theta+2} E_2(t; u) dt \leq C \|(u_0, u_1)\|_{H^2 \times H^1(\mathbb{R}^N)}^2 + C \int_0^r (t + 1)^{\theta-1} \|u\|_{L_2^2}^2 dt,
\]

(15)

where \( C \) depends on \( a_2, a_3, a_4, \) and \( \theta \).

Proof. Begin by taking the \( L_2^2(\mathbb{R}^N) \) inner product of \( \partial_t (u_{tt} + u_t - \nabla \cdot (a \nabla u)) = 0 \) and \( 2u_{tt} \) to obtain

\[
\partial_t \left( \|u_{tt}\|_{L_2^2}^2 + \langle a \nabla u_t, \nabla u_t \rangle_{L_2^2} + 2 \langle a_t \nabla u, \nabla u_t \rangle_{L_2^2} \right)
= -2 \|u_{tt}\|_{L_2^2}^2 + \langle 3a_t \nabla u_t + 2a_{tt} \nabla u, \nabla u_t \rangle_{L_2^2}.
\]

(16)

Similarly we take the \( L_2^2(\mathbb{R}^N) \) inner product of \( \partial_t (u_{tt} + u_t - \nabla \cdot (a \nabla u)) = 0 \) and \( u_t \) to obtain

\[
\partial_t \left( \|u_{tt}\|_{L_2^2}^2 + \frac{1}{2} \|u_t\|_{L_2^2}^2 \right)
= \|u_{tt}\|_{L_2^2}^2 - \langle a \nabla u_t, \nabla u_t \rangle_{L_2^2} - \langle a_t \nabla u, \nabla u_t \rangle_{L_2^2}.
\]

(17)

Now, define the functions

\[
Z_1(t) := \frac{(a_3)^2}{(t + 1)^2} \langle a \nabla u, \nabla u \rangle_{L_2^2}, \quad Z_2(t) := \langle a_t \nabla u_t, \nabla u_t \rangle_{L_2^2}, \quad \text{and}
\]

\[
Z_3(t) := \langle (2a_{tt} - a_t) \nabla u, \nabla u_t \rangle_{L_2^2}.
\]

Next, define the continuously differentiable function

\[
Y(t) := \|u_{tt}\|_{L_2^2}^2 + \langle u_{tt}, u_t \rangle_{L_2^2} + \frac{1}{2} \|u_t\|_{L_2^2}^2
+ \langle a \nabla u_t, \nabla u_t \rangle_{L_2^2} + 2 \langle a_t \nabla u, \nabla u_t \rangle_{L_2^2} + Z_1(t),
\]

noting that \( Z_1(t) \) is also continuously differentiable. Importantly, the presence of \( Z_1(t) \) in \( Y(t) \) ensures that \( Y(t) \geq 0 \), which will be shown later.

Observe that the left-hand sides of (16) and (17) sum to \( Y'(t) - Z_1'(t) \). Thus (16) combined with (17) gives \( Y'(t) + 2E_2(t; u) = Z_1'(t) + 3Z_2(t) + Z_3(t) \), and adding \( \frac{\theta+2}{t+1} Y(t) \) to both sides gives

\[
\frac{\theta+2}{t+1} Y(t) + Y'(t) + 2E_2(t; u)
= \frac{\theta+2}{t+1} Y(t) + Z_1'(t) + 3Z_2(t) + Z_3(t).
\]

(18)

To estimate the RHS of (18) from above, we estimate \( Z_1'(t) \), \( Z_2(t) \), and \( Z_3(t) \). First, notice that

\[
Z_1(t) \leq \frac{a_3 - 1}{t + 1} Z_1(t) + \frac{(a_3)^2}{t + 1} \langle a \nabla u_t, \nabla u_t \rangle_{L_2^2},
\]

(19)
by assumption (A2) and $2a |\nabla u| |\nabla u_t| \leq \frac{a}{t+1} |\nabla u|^2 + (t+1)a |\nabla u_t|^2$. Next, assumption (A2) gives

$$Z_2(t) \leq \frac{a^2}{t+1} \langle a \nabla u_t, \nabla u_t \rangle_{L^2_x}.$$  

(20)

Observe

$$Z_3(t) \leq \left( \frac{4(a_4)^2}{(a_3)^2(t+1)^2} + 1 \right) Z_1(t) + \frac{1}{2} \langle a \nabla u_t, \nabla u_t \rangle_{L^2_x},$$  

(21)

via $|2a_{tt} - a_t \nabla u| |\nabla u_t| \leq \frac{2(a_{tt} - a_t)^2 |\nabla u|^2 + a |\nabla u_t|^2}{2a}$, with $(2a_{tt} - a_t)^2 \leq \frac{8(a_4)^2}{(t+1)^2} a^2 + \frac{2(a_3)^2}{(t+1)^2} a^2$ by assumptions (A2) and (A3). Now use (19) - (21) to estimate the RHS of (18) from above by

$$\frac{\theta + 2}{t+1} Y(t) + C(a_3, a_4) Z_1(t) + \left( \frac{C(a_3)}{t+1} + \frac{1}{2} \right) \langle a \nabla u_t, \nabla u_t \rangle_{L^2_x}. $$  

(22)

The following estimates hold:

$$Z_1(t) \leq \frac{2(a_3)^2}{(t+1)^2} E_1(t; u),$$  

(23)

$$0 \leq Y(t),$$  

(24)

$$Y(t) \leq \|u_t\|^2_{L^2_x} + \frac{3}{2} \|u_{tt}\|^2_{L^2_x} + 2 \langle a \nabla u_t, \nabla u_t \rangle_{L^2_x} + 2 Z_1(t).$$  

(25)

The proof of (23) follows from the definitions of $Z_1(t)$ and $E_1(t; u)$. To show (24), note $Y(t) \geq \langle a \nabla u_t, \nabla u_t \rangle_{L^2_x} + 2 \langle a \nabla u, \nabla u_t \rangle_{L^2_x} + Z_1(t)$ since $u_t u_t \geq -\frac{1}{2} (u_t^2 + u_{tt}^2)$. Also, $-2a_t \nabla u \cdot \nabla u_t \leq \frac{(a_3)^2}{(t+1)^2} a |\nabla u|^2 + a |\nabla u_t|^2$, and $(a_t)^2 \leq \frac{(a_3)^2 a^2}{(t+1)^2}$ by assumption (A2). Hence $-2a \nabla u \cdot \nabla u_t \leq \frac{(a_3)^2}{(t+1)^2} a |\nabla u|^2 + a |\nabla u_t|^2$, meaning that $0 \leq Y(t)$. The proof of (25) is similar to the proof of (24).

Apply (23) and (25) to bound (22), and hence the RHS of (18), from above by

$$\frac{\theta + 2}{t+1} \|u_t\|^2_{L^2_x} + \frac{C(a_3, a_4, \theta)}{(t+1)^2} E_1(t; u) + \frac{1}{2} \left( \frac{C(a_3, \theta)}{t+1} + 1 \right) \left( \langle a \nabla u_t, \nabla u_t \rangle_{L^2_x} + \|u_{tt}\|^2_{L^2_x} \right).$$  

(26)

Replace the RHS of (18) with (26) to obtain

$$\frac{\theta + 2}{t+1} Y(t) + Y'(t) + \frac{1}{2} E_2(t; u) \leq \frac{\theta + 2}{t+1} \|u_t\|^2_{L^2_x} + \frac{C(a_3, a_4, \theta)}{(t+1)^2} E_1(t; u) + \left( \frac{C(a_3, \theta)}{t+1} - \frac{1}{2} \right) E_2(t; u),$$  

(27)

recalling that $\langle a \nabla u_t, \nabla u_t \rangle_{L^2_x} + \|u_{tt}\|^2_{L^2_x} = 2 E_2(t; u)$. Multiply both sides of (27) by the integrating factor $(t+1)^{\theta+2}$ and integrate in $t$, from 0 to $r$. Then apply the improved decay Propositions 1 and 2 to the integrals involving $E_1(t; u)$ and $\|u_t\|^2_{L^2_x}$, respectively. This way, we get the last term on the RHS of (15). Thus we only need to bound the integrals involving the first two terms and the last term of (27) by the initial data.
The integral involving the first two terms of (27) is bounded via inequalities (24) and (25), i.e., observe that
\[ \int_{0}^{r} (t + 1)^{\theta + 2} \left( \frac{\theta + 2}{t + 1} Y(t) + Y'(t) \right) dt = (t + 1)^{\theta + 2} Y(t) \bigg|_{t=0}^{r} \geq -C(a_2, a_3) \| (u_0, u_1) \|_{H^2 \times H^1(\mathbb{R}^N)}^2. \]
To bound the integral involving the last term of (27), define \( T_0 := \max \{ 0, 2C(a_3, \theta) - 1 \} \). Then for all \( r \geq 0 \),
\[ \int_{0}^{r} (t + 1)^{\theta + 2} \left( \frac{C(a_3, \theta)}{t + 1} - \frac{1}{2} \right) E_2(t; u) dt \leq \int_{0}^{T_0} (t + 1)^{\theta + 2} \left( \frac{C(a_3, \theta)}{t + 1} - \frac{1}{2} \right) E_2(t; u) dt, \]
from which it follows that \( C(a_3, \theta) - \frac{1}{2} \leq 0 \) for \( t \geq T_0 \). To complete the proof, apply assumption (A1) and then the energy inequality Lemma 2.1 to the RHS of (28), obtaining
\[ \int_{0}^{r} (t + 1)^{\theta + 2} \left( \frac{C(a_3, \theta)}{t + 1} - \frac{1}{2} \right) E_2(t; u) dt \leq C(a_2, a_3, \theta) \| (u_0, u_1) \|_{H^2 \times H^1(\mathbb{R}^N)}^2 \]
for all \( r \geq 0 \). \( \square \)

**Proposition 4.** Let \( u(x, t) \) be the solution to (1), and let the assumptions in subsection 1.1 be satisfied. For \( r \geq 0 \) and \( \theta \geq 0 \),
\[ \int_{0}^{r} (t + 1)^{\theta + 3} \| u_{tt} \|_{L^2}^2 dt \leq C \| (u_0, u_1) \|_{H^2 \times H^1(\mathbb{R}^N)}^2 + C \int_{0}^{r} (t + 1)^{\theta - 1} \| u \|_{L^2}^2 dt, \]
where \( C \) depends on \( a_2, a_3, a_4 \), and \( \theta \).

**Proof.** We use the functions \( Z_3(t) \) and \( Z_4(t) \) defined in the improved decay Proposition 3. Define the functions
\[ Z_4(t) := 2 \langle a_t \nabla u, \nabla u_t \rangle_{L^2} \] and
\[ Y(t) := \| u_{tt} \|_{L^2}^2 + \langle a \nabla u_t, \nabla u_t \rangle_{L^2} + 2 \langle a_t \nabla u, \nabla u_t \rangle_{L^2} + Z_1(t). \]
Notice that (16) gives \( Y'(t) + 2 \| u_{tt} \|_{L^2}^2 = Z_1'(t) + 3Z_2(t) + Z_4(t) \), and adding \( \frac{\theta + 3}{t + 1} Y(t) \) to both sides gives
\[ \frac{\theta + 3}{t + 1} Y(t) + Y'(t) + 2 \| u_{tt} \|_{L^2}^2 = \frac{\theta + 3}{t + 1} Y(t) + Z_1'(t) + 3Z_2(t) + Z_4(t). \]

Similar to (21), (24) and (25), respectively, we have
\[ Z_4(t) \leq \frac{(a_4)^2}{(a_3)^2(t + 1)} Z_1(t) + \frac{\langle a \nabla u_t, \nabla u_t \rangle_{L^2}}{t + 1}, \]
\[ 0 \leq Y(t), \]
\[ Y(t) \leq \| u_{tt} \|_{L^2}^2 + 2 \langle a \nabla u_t, \nabla u_t \rangle_{L^2} + 2Z_1(t). \]
To show (31), use $2|a_t\nabla u||\nabla u_t| \leq \frac{(t+1)a_t^2}{a}||\nabla u_t||^2 + a||\nabla u_t||^2$, with $a_t^2 \leq \frac{(a)^2}{(t+1)^2}a^2$ by assumption (A3). Next, apply (19), (20), (23), (31) and (33) to the RHS of (30) and get
\[
\frac{\theta + 3}{t + 1} Y(t) + Y'(t) + 2||u_t||^2_{L^2} \leq \frac{C(a_3, a_4, \theta)}{(t + 1)^{\theta}} E_1(t; u) + \frac{C(a_3, \theta)}{t + 1} E_2(t; u).
\]
Then multiply both sides of this inequality by the integrating factor $(t + 1)^{\theta + 3}$ and integrate with respect to $t$, from 0 to $r$. To obtain the last term on the RHS of (29), apply the improved decay Propositions 1 and 3 to the integrals involving $E_1(t; u)$ and $E_2(t; u)$, respectively. Note that inequalities (32) and (33) give
\[
\int_0^r (t + 1)^{\theta + 3} \left( \frac{\theta + 2}{t + 1} Y(t) + Y'(t) \right) dt = (t + 1)^{\theta + 3} Y(t) \bigg|_{t=0}^r
\]
\[
\geq -C(a_2, a_3) \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}.
\]
Therefore, the proof is complete.

3. Weighted energy method. Let $\delta > 0$ and $A(t) = \mathbb{R}^N \setminus B_0 ((t + 1)^{(1+\delta)/2})$. One goal of this section is to prove that the derivatives of the solution to (1) decay exponentially for $x \in A(t)$. More precisely,
\[
\|
\frac{\partial^n}{\partial t^n} \nabla u \|_{L^2(A(t))}^2 \leq C e^{-k(t+1)^{\delta}} \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}
\]
for some $k > 0$, where $n = 0, 1$ and $m = 1 - n, 2 - n$; see Proposition 7.

Definition 3.1. The weight we use in this section is
\[
W(x, t) := e^{\gamma \frac{|x|^2}{W^2}}, \text{ where } \gamma > 0 \text{ will be chosen conveniently.}
\]

The following observations will be employed later:
\[
0 \leq -W_t \leq \frac{W^2}{t + 1},
\]
\[
|\nabla W|^2 = -4\gamma W W_t.
\]
Note that $-W_t \leq \frac{W^2}{t + 1}$ because $m \leq e^m$ for all $m \in \mathbb{R}$.

Definition 3.2. For $i = 1, 2$, respectively, we define the first and second weighted energies:
\[
E_{i,W}(t; u) := \int_{\mathbb{R}^N} W(x, t) \left( (\partial_t^i u(x, t))^2 + a(x, t)|\nabla \partial_t^{i-1} u(x, t)|^2 \right) dx.
\]

The first weighted energy estimate is given by the following proposition.

Proposition 5. Let $u(x, t)$ be the solution to (1), and let the assumptions in subsection 1.1 be satisfied. Assume $\gamma$ in (34) is such that $0 < \gamma \leq \frac{1}{2a_2}$. Then for $t \geq 0$,
\[
E_{1,W}(t; u) \leq (t + 1)^{a_3} E_{1,W}(0; u).
\]

Proof. For $W(x, t)$ as in (34), define the functions
\[
Z_1(t) := 2\nabla \cdot (a u_t W \nabla u) \text{ and } Z_2(t) := a W_t |\nabla u|^2 - 2W u_t^2 - 2u_t a \nabla W \cdot \nabla u.
\]
Also define the function
\[
Y(t) := W u_t^2 + a W |\nabla u|^2,
\]
and note that $Y(t)$ is continuously differentiable as an $L^1_\gamma (\mathbb{R}^N)$ function by Lemma 1.1. Next, multiply equation (1) by $2Wu_t$ to get
\[
Y'(t) = Z_1(t) + Z_2(t) + a_1 W |\nabla u|^2 + W_t u_t^2. \tag{38}
\]
Note that $Z_2(t) \leq 0$; to see this, observe that
\[-2 (u_t) (a \nabla W \cdot \nabla u) \leq 2 W u_t^2 + \frac{a_1^2 |\nabla W|^2 |\nabla u|^2}{2W} = 2 W u_t^2 - (2a_1) a W |\nabla u|^2
\]
via Young’s inequality. Then recall that $-W_t \geq 0$ by (35), and notice that $2a_1 \leq 1$ since $a \leq a_2$ and $0 < \gamma \leq \frac{1}{2a_2}$.

Now using $Z_2(t) \leq 0$, assumption (A2) and $W_t \leq 0$, we refine (38) and get
\[
Y'(t) \leq Z_1(t) + \frac{a_3}{t+1} a W |\nabla u|^2.
\]
Thus $Y'(t) \leq Z_1(t) + \frac{a_3}{t+1} Y(t)$, since $a W |\nabla u|^2 \leq Y(t)$, and integrating the former inequality with respect to $x$, in $\mathbb{R}^N$ gives
\[
\partial_t E_{1,W}(t; u) \leq \int_{\mathbb{R}^N} Z_1(t) dx + \frac{a_3}{t+1} E_{1,W}(t; u).
\]
Now, $\int_{\mathbb{R}^N} Z_1(t) dx = 0$ since $u$ has compact support in $x$, and we thus have
\[
\partial_t E_{1,W}(t; u) \leq \frac{a_3}{t+1} E_{1,W}(t; u).
\]
To complete the proof, multiply both side of this inequality by the integrating factor $(t+1)^{-a_3}$, and then integrate with respect to $t$, on $[0, r]$. $\square$

The following proposition gives the second weighted energy estimate.

**Proposition 6.** Let $u(x,t)$ be the solution to (1), and let the assumptions in subsection 1.1 be satisfied. Assume $\gamma$ in (34) is such that $0 < \gamma \leq \frac{1}{6a_2}$. Then for $t \geq 0$,
\[
E_{2,W}(t; u) \leq C(a_3, a_4) \ (t+1)^{4a_3} \ (E_{1,W}(0; u) + E_{2,W}(0; u)). \tag{39}
\]

**Proof.** For $W(x,t)$ as in (34), define the functions
\[
Z_1(t) := 2 \nabla \cdot \partial_t (a \nabla u) W u_t, \quad Z_2(t) := a W_t |\nabla u_t|^2 - 2 W u_t^2 - 2u_t a \nabla W \cdot \nabla u_t,
\]
\[
Z_3(t) := a_1 \nabla W \cdot \nabla u, \quad \text{and} \quad Z_4(t) := \left( a_t W + a_t W_t - \frac{4a_3}{t+1} a_t W \right) \nabla u.
\]
Also define the function
\[
Y(t) := W u_t^2 + a W |\nabla u_t|^2 + 2a_t W \nabla u \cdot \nabla u_t,
\]
and note that $Y(t)$ is continuously differentiable as an $L^1_\gamma (\mathbb{R}^N)$ function by Lemma 1.1. Next, multiply $\partial_t (u_t + u_t - \nabla \cdot (a \nabla u) = 0$ by $2W u_t$ to get
\[
Y'(t) = Z_1(t) + Z_2(t) - 2Z_3(t) u_t + 2Z_4(t) \cdot \nabla u_t
\]
\[
+ 3a_t W |\nabla u_t|^2 + W_t u_t^2 + \frac{8a_3}{t+1} a_t W \nabla u \cdot \nabla u_t.
\]
As in the proof of the weighted energy estimate Proposition 5, we get $Z_2(t) \leq 0$. Consequently, $Z_2(t) \leq 0$, assumption (A2) and $W_t \leq 0$ give
\[
Y'(t) \leq Z_1(t) - 2Z_3(t) u_t + 2Z_4(t) \cdot \nabla u_t
\]
\[
+ 3a_3 \frac{a W |\nabla u_t|^2 + \frac{8a_3}{t+1} a_t W \nabla u \cdot \nabla u_t. \tag{40}
\]

To refine (40), apply the following inequalities, which are proved via Young’s inequality:
\[
2|Z_3(t)| |u_{tt}| \leq \frac{t + 1}{4a_3W} Z_3(t)^2 + \frac{4a_3W}{t + 1} u_{tt},
\]
\[
2|Z_4(t) \cdot \nabla u_t| \leq \frac{t + 1}{a_3W} Z_4(t)^2 + \frac{a_3W}{t + 1} |\nabla u_t|^2.
\]
The refinement is
\[
Y'(t) \leq Z_1(t) + \frac{t + 1}{4a_3W} Z_3(t)^2 + \frac{t + 1}{a_3W} Z_4(t)^2 + \frac{4a_3}{t + 1} Y(t). \tag{41}
\]
Recall that \(m \leq e^m\) for all \(m \in \mathbb{R}\). Thus \(|\nabla W|^2 \leq \frac{4t^4W^4}{(t + 1)^2}\) and \(W_t^2 \leq \frac{W^4}{(t + 1)^2}\). Therefore, \(Z_3(t)^2 \leq C(a_3) \frac{aW^3|\nabla u|^2}{(t + 1)^2}\) and \(Z_4(t)^2 \leq C(a_3, a_4) \frac{a^2W^4|\nabla u|^2}{(t + 1)^2}\), by assumptions (A1) - (A3) and \(0 < \gamma \leq \frac{1}{a^2}\). Apply these estimates for \(Z_3(t)^2\) and \(Z_4(t)^2\) to (41) and obtain
\[
Y'(t) \leq Z_1(t) + C_1(a_3, a_4) \frac{aW^3|\nabla u|^2}{(t + 1)^2} + \frac{4a_3}{t + 1} Y(t). \tag{42}
\]
For \(\gamma' = 3\gamma\), define the weight \(W' := \exp\left(\gamma' \frac{|x|^2}{t + 1}\right)\), and notice that \(W^3 = W'\). Thus (42) is the same as
\[
(t + 1)^{4a_3} \partial_t \left((t + 1)^{-4a_3} Y(t)\right) \leq Z_1(t) + C_1(a_3, a_4) \frac{aW'|\nabla u|^2}{(t + 1)^2}. \tag{43}
\]
Integrate both sides of (43) with respect to \(x\), in \(\mathbb{R}^N\). Now \(\int_{\mathbb{R}^N} Z_1(t) dx = 0\) since \(u\) has compact support in \(x\). Also, \(\int_{\mathbb{R}^N} aW' |\nabla u|^2 dx \leq (t + 1)^{a_3} E_{1,W}(0; u)\) by weighted energy estimate Proposition 5, since \(0 < \gamma' \leq \frac{1}{a^2}\). Therefore,
\[
\partial_t \left((t + 1)^{-4a_3} \int_{\mathbb{R}^N} Y(t) dx\right) \leq C_1(a_3, a_4) \frac{a^{1/2} E_{1,W}(0; u)}{(t + 1)^{2 + 3a_3}}.
\]
Now integrate this inequality with respect to \(t\), on \([0, r]\) and get
\[
\int_{\mathbb{R}^N} Y(r) dx \leq (r + 1)^{4a_3} C_2(a_3, a_4) \left(E_{1,W}(0; u) + \int_{\mathbb{R}^N} Y(0) dx\right). \tag{44}
\]
Observe that
\[
\int_{\mathbb{R}^N} Y(r) dx = E_{2,W}(r; u) + \int_{\mathbb{R}^N} 2a_3 W \nabla u \cdot \nabla u_r, dx \tag{45}
\]
for \(r \geq 0\). Hence, we estimate the second term on the RHS of (45). Notice that Young’s inequality and assumption (A2) give
\[
|2a_3 W \nabla u \cdot \nabla u_r| \leq \frac{aW}{2} |\nabla u_r|^2 + \frac{2a^2}{a} W |\nabla u|^2 \leq \frac{aW}{2} |\nabla u_r|^2 + \frac{2(a_3)^2}{(r + 1)^2} aW |\nabla u|^2.
\]
Thus, by the weighted energy estimate Proposition 5,
\[
\int_{\mathbb{R}^N} 2a_3 W \nabla u \cdot \nabla u_r, dx \leq \frac{1}{2} E_{2,W}(r; u) + \frac{2(a_3)^2}{(r + 1)^2} E_{1,W}(0; u). \tag{46}
\]
To complete the proof, apply (45) and (46) to estimate the LHS and RHS of (44) from below and above, respectively. We obtain
\[ \frac{1}{2} E_{2,W}(r; u) \leq (r + 1)^{4a_3} C_3(a_3, a_4) \left( E_{1,W}(0; u) + E_{2,W}(0; u) \right). \]

The next proposition shows that derivatives of the solution to (1) decay exponentially outside of a ball.

**Proposition 7.** (Exponential decay) Let \( u(x,t) \) be the solution to (1), and let the assumptions in subsection 1.1 be satisfied. For \( \delta > 0 \) and \( A(t) = R^N \setminus B_0 \left( (t + 1)^{(1+\delta)/2} \right) \),
\[ \| \partial_t^m \nabla^n u \|_{L^2(A(t))}^2 \leq C e^{-k(t+1)^{\delta}} \| (u_0, u_1) \|_{H^2 \times H^1(R^N)}^2 \]  
(47)
for some \( k > 0 \), where \( n = 0, 1 \) and \( m = 1 - n, 2 - n \). The constant \( C \) depends on \( a_1, a_2, a_3, a_4, R_0, \) and \( \delta \).

**Proof.** Consider the notation \( aW := a(x,t)W(x,t) \), where \( W \) is the weight from (34), with \( \gamma = \frac{1}{6a_2} \). Then observe that
\[ \| \partial_t^m \nabla^n u \|_{L^2(A(t))}^2 \leq \| (\sqrt{aW})^{-1} \|_{L^2(A(t))}^2 \| (\sqrt{aW}) \partial_t^m \nabla^n u \|_{L^2(A(t))}^2 \]  
(48)
by Hölder’s inequality. Consequently, for \( \gamma = \frac{1}{6a_2} \) in the weight function \( W \),
\[ \| (\sqrt{aW}) \partial_t^m \nabla^n u \|_{L^2(A(t))}^2 \leq (t + 1)^{4a_3} C(a_3, a_4) \left( E_{1,W}(0; u) + E_{2,W}(0; u) \right) \]  
(49)
by the weighted energy estimates Propositions 5 and 6. Notice that
\[ \| (\sqrt{aW})^{-1} \|_{L^2(A(t))}^2 \leq (t + 1)^{\frac{\gamma}{(1+\delta)}} \frac{C(N)}{a_1} \int_1^\infty e^{-\gamma(t+1)^{\delta} r^2} r^{N-1} dr \]  
by assumption (A1) and polar coordinates. Additionally, since \( r^{N-2} e^{-\frac{\gamma}{2} r^2} \leq C(\gamma, N) \) for a sufficiently large \( C(\gamma, N) \) and \( r \geq 1 \),
\[ \int_1^\infty e^{-\gamma(t+1)^{\delta} r^2} r^{N-1} dr \leq C(\gamma, N) \int_1^\infty e^{-\frac{\gamma}{2} (t+1)^{\delta} r^2} r^2 dr = e^{-\frac{\gamma}{2} (t+1)^{\delta} \frac{C_1(\gamma, N)}{(t+1)^{\delta}}} \]  
Therefore,
\[ \| (\sqrt{aW})^{-1} \|_{L^2(A(t))}^2 \leq e^{-\frac{\gamma}{2} (t+1)^{\delta}} (t + 1)^{\frac{\gamma}{(1+\delta)} - \delta} C(a_1, \gamma, N). \]  
(50)
Combine (48) - (50) to get
\[ \| \partial_t^m \nabla^n u \|_{L^2(A(t))}^2 \leq e^{-\frac{\gamma}{2} (t+1)^{\delta}} (t + 1)^{\frac{\gamma}{(1+\delta)} - \delta + 4a_3} C(a_1, a_3, a_4, \gamma, N) \left( E_{1,W}(0; u) + E_{2,W}(0; u) \right). \]

Note that \( e^{-\frac{\gamma}{2} (t+1)^{\delta}} (t + 1)^{\frac{\gamma}{(1+\delta)} - \delta + 4a_3} \leq C(a_3, \gamma, N, \delta) \) for a sufficiently large \( C(a_3, \gamma, N, \delta) \). Also, \( E_{1,W}(0; u) + E_{2,W}(0; u) \leq C(a_2, \gamma, R_0) \| (u_0, u_1) \|_{H^2 \times H^1(R^N)} \), where \( R_0 \) is the size of the support for the data \( (u_0, u_1) \). Therefore, to complete
the proof, choose $k = \frac{\gamma}{4}$, and replace the dependence on $\gamma$ with a dependence on $a_2$ since $\gamma = \frac{1}{6a_2}$.

4. The representation of the difference between solutions of (1) and (2) in terms of the fundamental solution of the parabolic problem (2). The differential equation in (2) has a pointwise, classical fundamental solution $\Gamma(x, t; \xi, s)$ for $x, \xi \in \mathbb{R}^N$ and $0 \leq s < t$. The fundamental solution allows the transfer of decay from the solution of (2) to the solution of (1). The properties of $\Gamma(x, t; \xi, s)$ are in Friedman [2, Chapter 1]. Importantly, Friedman [2, Chapter 1, (6.12)] and [2, Chapter 1, (8.14) and Theorem 15] give the following lemma.

Lemma 4.1. (Fundamental solution decay properties) Let $\Gamma(x, t; \xi, s)$ be the fundamental solution of (2). Then

$$|\Gamma(x, t; \xi, s)| \leq C (t-s)^{-\frac{N}{2}} \exp \left(-C \frac{|x-\xi|^2}{t-s} \right),$$

(i)

$$|\partial_\xi \Gamma(x, t; \xi, s)| \leq C (t-s)^{-\frac{N+1}{2}} \exp \left(-C \frac{|x-\xi|^2}{t-s} \right).$$

(ii)

Definition 4.2. We use the notations:

$$\Gamma^{t,s}_x := \int_{\mathbb{R}^N} \Gamma(x, t; \xi, s)f(\xi, s)d\xi,$$

$$(a\nabla_\xi \Gamma)^{t,s}_x g := \int_{\mathbb{R}^N} a(\xi, s)\nabla_\xi \Gamma(x, t; \xi, s) \cdot g(\xi, s)d\xi,$$

for scalar $f(\xi, s)$ and vector $g(\xi, s)$, with $f(\xi, s), |g(\xi, s)| \in L^p_1$ for any $1 \leq p \leq \infty$.

The following lemma makes use of Lemma 4.1 to get bounds for the operators $\Gamma^{t,s}_x$ and $(a\nabla_\xi \Gamma)^{t,s}_x$.

Lemma 4.3. (Diffusion operator estimates) Let $\Gamma(x, t; \xi, s)$ be the fundamental solution of (2). Then for $f(x, \cdot), |g(x, \cdot)| \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and $0 \leq s < t$, the following properties hold:

$$\left\| \Gamma^{t,s}_x f \right\|_{L^2_1} \leq C \left\| f(x, s) \right\|_{L^2_1},$$

(i)

$$\left\| \Gamma^{t,s}_x f \right\|_{L^2_1} \leq C (t-s)^{-\frac{N}{4}} \left\| f(x, s) \right\|_{L^2_1},$$

(ii)

$$\left\| (a\nabla_\xi \Gamma)^{t,s}_x g \right\|_{L^2_1} \leq C(a_2) (t-s)^{-\frac{1}{2}} \left\| g(x, s) \right\|_{L^2_1},$$

(iii)

$$\left\| (a\nabla_\xi \Gamma)^{t,s}_x g \right\|_{L^2_1} \leq C(a_2) (t-s)^{-\frac{N+2}{4}} \left\| g(x, s) \right\|_{L^2_1}.$$ 

(iv)

Proof. We prove (i) and (ii). By Lemma 4.1(i),

$$\left| \Gamma^{t,s}_x f \right| \leq C (t-s)^{-\frac{N}{2}} \exp \left(-C \frac{|x|^2}{t-s} \right) \left| f(x, s) \right|. $$

Take the $L^2_1$ norm of both sides of this inequality. To get (i), apply Young’s convolution inequality $\left\| h \ast x k \right\|_{L^2_1} \leq \left\| h \right\|_{L^1_1} \left\| k \right\|_{L^1_1}$, where $h = \exp \left(-C \frac{|x|^2}{t-s} \right)$ and $k = |f(x, s)|$. To get (ii), let $h = |f(x, s)|$ and $k = \exp \left(-C \frac{|x|^2}{t-s} \right)$. To prove (iii) and (iv), repeat the proof of (i) and (ii), except use Lemma 4.1(ii) instead of (i).
Next, we rewrite the solution \( v(x,t) \) of (2) as
\[
v(x,t) = \Gamma^{t,0}_x (u + u_t).
\]
(51)
This solution satisfies
\[
v(x,t) \in C \left( [0, \infty); L^2_x(\mathbb{R}^N) \right),
\]
(52)
by Lemma 4.3(i) and the continuity of \( \Gamma(x,t;\xi,s) \).

The next proposition precisely determines the difference between the solutions of (1) and (2) in terms of the fundamental solution of (2).

**Proposition 8.** (Integral identity) Let \( u(x,t) \) be the solution to (1), where the assumptions in subsection 1.1 hold. Then for \( t > 0 \)
\[
u^t(x,t) = v^t(x,t) - \Gamma^{t,t/2}_x u_t - \int_{t/2}^t \Gamma^{t,s}_x u_{ss} ds
\]
\[
+ \int_0^{t/2} (a \nabla_\xi \Gamma)^{t,s}_x (\nabla_\xi u_s) d\xi ds.
\]
(53)
holds in the \( L^2_x(\mathbb{R}^N) \) sense, where \( v(x,t) \) is the solution to (2) rewritten as in (51).

**Proof.** Let \( \epsilon > 0 \). Mollify \( u(x,t) \) in space, i.e., use a standard mollifier \( \eta_\epsilon(x) = \epsilon^{-N} \eta(\epsilon^{-1}x) \) and define 
\[
u^\epsilon(x,t) := \int_{\mathbb{R}^N} \eta_\epsilon(y) u(x-y,t) dy.
\]
Note that \( \nu^\epsilon \in C^3 (\mathbb{R}^N \times [0, \infty)) \) because of the regularity (4) and the mollification. Moreover, \( \text{supp} (u^\epsilon) \cap (\mathbb{R}^N \times [0,t]) \) is compact because of the finite speed of propagation Lemma 2.2. Now, define
\[
f^\epsilon(x,t) := u_{tt}^\epsilon(x,t) + u_t^\epsilon(x,t) - \nabla \cdot (a(x,t) \nabla u^\epsilon(x,t)).
\]
Write the above identity as
\[
u^\epsilon_t = \Gamma^{t,0}_x u^\epsilon - \int_0^t \Gamma^{t,s}_x (u_{ss}^\epsilon - f^\epsilon) ds.
\]
(54)
Apply integration by parts in \( s \) to \( \int_0^{t/2} \Gamma^{t,s}_x u_{ss}^\epsilon ds \), moving the derivative from \( u_{ss}^\epsilon \) to \( \Gamma^{t,s}_x \), and get
\[
\int_0^{t/2} \Gamma^{t,s}_x u_{ss}^\epsilon ds = \Gamma^{t,t/2}_x u_t^\epsilon - \Gamma^{t,0}_x u_t^\epsilon - \int_0^{t/2} \int_{\mathbb{R}^N} (\partial_s \Gamma(x,t;\xi,s) (\partial_x \Gamma(x,t;\xi,s))) u_s^\epsilon(\xi,s) d\xi ds.
\]
(55)
For the last term on the RHS of (55), use the fact that \( \Gamma(x,t;\xi,s) \) is a classical solution to the backwards problem, i.e., use
\[
\partial_s \Gamma(x,t;\xi,s) = -\nabla_\xi \cdot (a(\xi,s) \nabla_\xi \Gamma(x,t;\xi,s))
\]
where \( x, \xi \in \mathbb{R}^N \) and \( 0 \leq s < t \). Then apply the divergence theorem to the last term on the RHS of (55) and get
\[
\int_0^{t/2} \Gamma^{t,s}_x u_{ss}^\epsilon ds = \Gamma^{t,t/2}_x u_t^\epsilon - \Gamma^{t,0}_x u_t^\epsilon - \int_0^{t/2} (a \nabla_\xi \Gamma)^{t,s}_x (\nabla_\xi u_s^\epsilon) d\xi ds.
\]
Using this identity, rewrite (54) as
\[
\begin{align*}
u^\epsilon(x,t) &= \nu^\epsilon(x,t) - \Gamma^{\epsilon,t/2}_x u^\epsilon_t - \int_{t/2}^t \Gamma^{\epsilon,s}_x u^\epsilon_s ds \\
& \quad + \int_0^{t/2} (a \nabla \Gamma^{\epsilon,t/2}_x (\nabla \xi u^\epsilon_s)) ds + \int_0^t \Gamma^{\epsilon,s} f^\epsilon ds,
\end{align*}
\] (56)
where \( v^\epsilon(x,t) = \Gamma^{\epsilon,0}_x (u^\epsilon + u^\epsilon_t) \). Take \( \epsilon \to 0 \). Using the regularity (4), the first three terms on the RHS of (56) converge, respectively, in \( L^2_x \) to the first three terms on the RHS of (53) because of the diffusion operator estimate Lemma 4.3(i). Similarly, the fourth term on the RHS of (56) converges in \( L^2_x \) to the fourth term on the RHS of (53) because of the diffusion operator estimate Lemma 4.3(iv) of Proposition 7, followed by another application of the improved decay.

Note that the diffusion operator estimate Lemma 4.3(i) gives
\[
\left\| \int_0^t \Gamma^{\epsilon,s}_x f^\epsilon ds \right\|_{L^2_x} \leq C \int_0^t \| f^\epsilon(x,s) \|_{L^2_x} ds.
\] (57)
By using the regularity (4) and the boundedness of \( a \) and \( \nabla \), we get:
\[
\| f^\epsilon(x,s) \|_{L^2_x} \leq \| u_{ss}(x,s) \|_{L^2_x} + \| u_s(x,s) \|_{L^2_x} + C(a) \| u(x,s) \|_{H^1_x} \leq M(t),
\]
for \( 0 \leq s \leq t \), where \( M(t) \) is a real-valued function of \( t \), and
\[
\| f^\epsilon - (u_{ss} + u_s - \nabla \cdot (a \nabla u)) \|_{L^2_x} \to 0 \quad \text{as} \quad \epsilon \to 0.
\]
By the properties of \( u \), see Ikawa [3, Equation (1.5)], we have \( \| u_{ss} + u_s - \nabla \cdot (a \nabla u) \|_{L^2_x} = 0 \), hence \( \| f^\epsilon \|_{L^2_x} \to 0 \) as \( \epsilon \to 0 \). Application of the dominated convergence theorem to the RHS of (57) completes the proof.

5. The diffusion phenomenon and decay. In the proof of Theorem 1.2, the improved decay is used to extract decay from the second and third terms on the RHS of the integral identity (53), after using the diffusion operator estimate Lemma 4.3(i). Then the diffusion operator estimate Lemma 4.3(iv) is used to extract decay from the fourth term on the RHS of the integral identity. This comes at the price of having to estimate \( \| \nabla u_{ss}(x,s) \|_{L^1_x} \), which is paid by the exponential decay Proposition 7, followed by another application of the improved decay.

Proof of Theorem 1.2. We consider the cases when \( t < 1 \) and \( t \geq 1 \). First, assume that \( t < 1 \). Then by the energy inequality Lemma 2.1,
\[
\| u(x,t) \|_{L^2_x}^2 \leq C(1) \| (u_0, u_1) \|_{H^2 \times H^1(R^N)}^2,
\] (58)
and the diffusion operator estimate Lemma 4.3(i) gives
\[
\| \nu(x,t) \|_{L^2_x}^2 = \| \Gamma^{\epsilon,0}_x (u + u_1) \|_{L^2_x}^2 \leq C \| u_0 + u_1 \|_{L^2_x}^2.
\]
Therefore, \( \| u(x,t) - \nu(x,t) \|_{L^2_x}^2 \leq C \| (u_0, u_1) \|_{H^2 \times H^1(R^N)}^2 \), and Theorem 1.2 is verified for \( t < 1 \). Now, assume that \( t \geq 1 \). Define the functions
\[
\begin{align*}
Z_1(t) := \int_0^t (s + 1) \frac{\epsilon}{s + \epsilon} \| u(x,s) \|_{L^2_x}^2 ds, \quad Z_2(t) := \int_0^t (s + 1) \frac{\epsilon}{s + \epsilon} \| u_s(x,s) \|_{L^2_x}^2 ds, \\
Z_3(t) := \int_0^t (s + 1) \frac{\epsilon}{s + \epsilon} \| \nabla u_s(x,s) \|_{L^2_x}^2 ds, \quad Z_4(t) := \int_0^t (s + 1) \frac{\epsilon}{s + \epsilon} \| u_{ss}(x,s) \|_{L^2_x}^2 ds.
\end{align*}
\]
Also define the function
\[ Y(t) := \int_1^t (s + 1)^{\frac{N-3}{2}} \| u(x, s) - v(x, s) \|_{L^2}^2 \, ds, \]
which has continuous derivative \( Y'(t) = (t + 1)^{\frac{N-3}{2}} \| u(x, t) - v(x, t) \|_{L^2}^2 \) ds via the regularity (4) and (52).

In the integral identity (53), subtract \( v(x, t) \) from both sides, and then apply \( \| \cdot \|_{L^2}^2 \) to obtain
\[ (t + 1)^{\frac{N}{2}} Y'(t) \leq C (t + 1)^{\frac{N+2}{2}} (I_1 + I_2 + I_3), \]
(59)
where
\[ I_1 = \left\| \Gamma_{L^2}^{t, t/2} u_t \right\|_{L^2}^2, \quad I_2 = \left\| \int_{t/2}^t \Gamma_{L^2}^{s} u_{ss} \, ds \right\|_{L^2}^2, \quad \text{and} \]
\[ I_3 = \left\| \int_0^{t/2} \left( a \nabla_x \Gamma_{L^2}^{t, s} (\nabla_x u_s) \right) \, ds \right\|_{L^2}^2. \]

We estimate each of \( I_1, I_2, \) and \( I_3 \). For \( I_1 \), the diffusion operator estimate Lemma 4.3(i) gives \( C \, I_1 \leq \| u_t(x, t/2) \|_{L^2}^2 \). Thus
\[ C \, (t + 1)^{\frac{N+2}{2}} I_1 \leq (t/2 + 1)^{\frac{N+2}{2}} \| u_t(x, t/2) \|_{L^2}^2 \]
\[ = \int_{t/2}^t \partial_s \left( (s + 1)^{\frac{N+2}{2}} \| u_s(x, s) \|_{L^2}^2 \right) \, ds + \| u_1(x) \|_{L^2}^2, \]
and using \( 2u_s u_{ss} \leq \frac{\nu^2}{s+1} + (s + 1)u_{ss}^2 \) gives
\[ C \, (t + 1)^{\frac{N+2}{2}} I_1 \leq Z_2(t) + Z_4(t) + \| u_1(x) \|_{L^2}^2. \]
Let \( \theta = \frac{N-1}{2} \). Then by the improved decay Propositions 2 and 4, respectively,
\[ Z_2(t) \leq C \, Z_1(t) + C \, \| (u_0, u_1) \|_{H^2 \times H^1([\theta, \infty[N)}^2, \]
\[ Z_4(t) \leq C \, Z_1(t) + C \, \| (u_0, u_1) \|_{H^2 \times H^1([\theta, \infty[N)}^2. \]
Hence
\[ (t + 1)^{\frac{N+2}{2}} I_1 \leq C \, Z_1(t) + C \, \| (u_0, u_1) \|_{H^2 \times H^1([\theta, \infty[N)}^2, \]
(60)
where the constant \( C \) depends on \( a_2, a_3, a_4, \) and \( N \).

Now, proceed to estimate \( I_2 \). Observe that the diffusion operator estimate Lemma 4.3(i) gives
\[ I_2 \leq \left( \int_{t/2}^t \| \Gamma_{L^2}^{s} u_{ss} \|_{L^2}^2 \, ds \right)^2 \leq C \, \left( \int_{t/2}^t \| u_{ss}(x, s) \|_{L^2}^2 \, ds \right)^2, \]
and the RHS is bounded from above by \( C \, (t + 1) \int_{t/2}^t \| u_{ss}(x, s) \|_{L^2}^2 \, ds \) via Hölder’s inequality. Thus
\[ (t + 1)^{\frac{N+2}{2}} I_2 \leq C \, (t/2 + 1)^{\frac{N+4}{2}} \int_{t/2}^t \| u_{ss}(x, s) \|_{L^2}^2 \, ds \leq C \, Z_4(t). \]
Therefore, as above, use the improved decay Proposition 4 with $\theta = \frac{N-1}{2}$ to get
\[
(t + 1)^{\frac{N+2}{2}} I_2 \leq C \ Z_4(t) + C \ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)},
\] (61)
where the constant $C$ depends on $a_2, a_3, a_4, N$, and $R_0$. Now we address $I_3$. Observe that the diffusion operator estimate Lemma 4.3(iv) gives
\[
I_3 \leq \left( \int_0^{t/2} C(a_2)(t - s)^{-\frac{N+2}{2}} \|\nabla u_s(x, s)\|_{L^2_x} \, ds \right)^2 \\
\leq C \ (t + 1)^{\frac{N+2}{2}} \left( \int_0^{t/2} C(a_2) \|\nabla u_s(x, s)\|_{L^2_x} \, ds \right)^2,
\] since $t \geq 1$. Thus by Hölder’s inequality,
\[
(t + 1)^{\frac{N+2}{2}} I_3 \\
\leq C \int_0^{t/2} (s + 1)^{-\frac{N}{2}} C(a_2)^2 \, ds \int_0^{t/2} (s + 1)^{\frac{N}{2}} \|\nabla u_s(x, s)\|^2_{L^2_x} \, ds.
\] (62)
Then the exponential decay Proposition 7 with $\delta = \frac{1}{2N}$ and $A(s) = \mathbb{R}^N \setminus B_0 \ ((s + 1)^{(1+\delta)/2})$ gives
\[
\|\nabla u_s(x, s)\|^2_{L^2_x(A(s))} \leq e^{-k(s+1)^{\delta}} C \ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}.
\]
Combine this with the estimate
\[
\|\nabla u_s(x, s)\|^2_{L^2_x(A(s)^c)} \leq |A(s)^c| \|\nabla u_s(x, s)\|^2_{L^2_x(A(s)^c)} \\
\leq C \ (s + 1)^{\frac{N}{2} + \frac{1}{4}} \|\nabla u_s(x, s)\|^2_{L^2_x(A(s)^c)},
\]
where $A(s)^c$ is the ball $B_0 \ ((s + 1)^{(1+\delta)/2})$ and $|A(s)^c|$ is the volume of the ball, and obtain the estimate
\[
\|\nabla u_s(x, s)\|^2_{L^2_x(\mathbb{R}^N)} \\
\leq (s + 1)^{\frac{N}{2} + \frac{1}{4}} C \ \|\nabla u_s(x, s)\|^2_{L^2_x(A(s)^c)} + e^{-k(s+1)^{\delta}} C \ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}.
\]
Apply this estimate to the RHS of (62) and get
\[
(t + 1)^{\frac{N+2}{2}} I_3 \leq C \ Z_3(t) + C \ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}.
\]
Now use the improved decay Proposition 3 with $\theta = \frac{N-1}{2}$ to obtain
\[
(t + 1)^{\frac{N+2}{2}} I_3 \leq C \ Z_1(t) + C \ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)},
\] (63)
where the constant $C$ depends on $a_1, a_2, a_3, a_4, N$, and $R_0$. Next, apply (60), (61) and (63) to the RHS of (59) to get
\[
(t + 1)^{\frac{N}{2}} Y''(t) \leq C \ Z_1(t) + C \ \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)},
\] (64)
where the constant \(C\) depends on \(a_1, a_2, a_3, a_4, N,\) and \(R_0.\) Then by (58) and \(\|u\|_L^2 \leq C \|u - v\|_L^2 + C \|v\|_L^2,\)

\[
Z_1(t) = \int_0^1 (s + 1) \frac{\Delta}{L} \|u(x, s)\|^2_L ds + \int_1^t (s + 1) \frac{\Delta}{L} \|u(x, s)\|^2_L ds
\]

\[
\leq C \|(u_0, u_1)\|^2_{H^2 \times H^1} + C Y(t) + C \int_1^t (s + 1) \frac{\Delta}{L} \|u(x, s)\|^2_L ds.
\]

Note that

\[
\|v(x, s)\|^2_L \leq C s^{-\frac{N}{2}} \|u_0 + u_1\|^2_L \leq C(R_0) s^{-\frac{N}{2}} \|u_0 + u_1\|^2_L
\]

by the diffusion operator estimate Lemma 4.3(ii). Thus

\[
Z_1(t) \leq C \|(u_0, u_1)\|^2_{H^2 \times H^1} + C Y(t).
\]

Therefore, estimates (64) and (66) give

\[
(t + 1)^{\frac{\Delta}{L}} Y(t) \leq C \|(u_0, u_1)\|^2_{H^2 \times H^1} + C Y(t).
\]

Multiply both sides of (67) by \((t + 1)^{-\frac{\Delta}{2}}.\) Then use the integrating factor \(e^{\frac{2C}{\Delta}(t + 1)^{-3/2}}\) to get \(Y(t) \leq C \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}.\) Therefore, the RHS of (67) is bounded by the initial data, giving

\[
(t + 1)^{\frac{\Delta}{L}} \|u(x, t) - v(x, t)\|^2_L = (t + 1)^{\frac{\Delta}{2}} Y(t) \leq C \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)},
\]

completing the proof.

Proof of Corollary 1. To prove (i) for \(t < 1,\) use (58). For \(t \geq 1,\) use the diffusion phenomenon Theorem 1.2 and (65).

To prove (ii), let \(\theta = \frac{N}{2},\) and notice that

\[
(t + 1)^{\theta + 1} \|\nabla u(x, t)\|^2_L - \|\nabla u_0\|^2_L = \int_0^t \frac{\partial}{\partial s} \left( (s + 1)^{\theta + 1} \|\nabla u\|^2_L \right) ds.
\]

Use 2 \(|\nabla u \cdot \nabla u_s| \leq \frac{|\nabla u|^2}{s+1} + (s + 1) |\nabla u_s|^2\) to obtain

\[
(t + 1)^{\theta + 1} \|\nabla u(x, t)\|^2_L - \|\nabla u_0\|^2_L \leq (\theta + 2) \int_0^t (s + 1)^{\theta} \|\nabla u\|^2_L ds + \int_0^t (s + 1)^{\theta + 2} \|\nabla u_s\|^2_L ds.
\]

Apply the improved decay Propositions 1 and 3, respectively, to the first and second terms on the RHS and get

\[
(t + 1)^{\theta + 1} \|\nabla u(x, t)\|^2_L \leq C \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)} + C \int_0^t (s + 1)^{\theta - 1} \|u\|_L^2 ds.
\]

Then use part (i) of this corollary, i.e., use \(\|u(x, s)\|^2_L \leq (s + 1)^{-\frac{N}{2}} C \|(u_0, u_1)\|^2_{H^2 \times H^1(\mathbb{R}^N)}\) to complete part (ii) of this proof.

To prove (iii), repeat the proof of (ii) with \(\theta = \frac{N}{2},\) except estimate \((t + 1)^{\theta + 2} \|u_1(x, t)\|^2_L\) instead of \((t + 1)^{\theta + 1} \|\nabla u(x, t)\|^2_L.\) Similarly to above, \((t + 1)^{\theta + 2} \|u_1(x, t)\|^2_L - \|u_1\|^2_L\)

\[
\leq (\theta + 3) \int_0^t (s + 1)^{\theta + 1} \|u_1\|^2_L ds + \int_0^t (s + 1)^{\theta + 3} \|u_1\|^2_L ds.
\]
Apply the improved decay Propositions 2 and 4, respectively, to the first and second terms on the RHS, and then use part (i) of this corollary.

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