STABILITY OF STEIN STRUCTURES ON THE EUCLIDEAN SPACE

HERVÉ GAUSSIER AND ALEXANDRE SUKHOV

Abstract. We give a characterization of the space $\mathbb{C}^n$, proving a global version of Newlander-Nirenberg’s theorem on the integrability of complex structures close to the standard complex structure on $\mathbb{C}^n$.

1. Introduction

There exist essentially two different types of Stein structures. The first type consists of complex varieties with boundary: a model example is provided by strongly pseudoconvex domains. A typical example of a Stein manifold of second type is given by the complex Euclidean space $\mathbb{C}^n$. In many aspects (the hyperbolicity, the automorphism groups, etc.) the properties of these two types of Stein manifolds are totally different. Our work is motivated by Hamilton’s theorem [7] asserting that Stein structures of the first type are stable under small smooth deformations (up to the boundary) of the complex structure. The focus of this paper is to prove the same sort of stability result for the space $\mathbb{C}^n$. This question is closely related to the well-known problem of complex analysis and geometry on the integrability of almost complex structures. A first fundamental result here is due to Newlander-Nirenberg [15]. Later on many different proofs of this result and more general results have been obtained by various methods (see for instance [1, 4, 6, 8, 10, 13, 20]). The main result of the present work can be viewed as a global version of the Newlander-Nirenberg theorem for almost complex structures defined on the whole space $\mathbb{R}^{2n}$. Our approach is inspired by Hörmander’s proof [10] of the Newlander-Nirenberg theorem. We combine Hörmander’s $L_2$-techniques with Lempert’s and Kiremidjian’s results on extendibility of CR structures [12, 13] and with a global version of Nijenhuis-Woolf’s theorem on the existence of pseudoholomorphic discs for certain almost complex structures on $\mathbb{R}^{2n}$.

Let $M$ be a smooth real manifold of dimension $2n$. An almost complex structure $J$ on $M$ is a tensor field of type $(1,1)$ on $M$ (that is a section of $\text{End}(TM)$ satisfying $J^2 = -I$). It is called integrable if any point in $M$ admits an open neighborhood $U$ and a diffeomorphism $z : U \to \mathbb{B}$ between $U$ and the unit ball of $\mathbb{C}^n$ such that $(z_*)(J) := dz \circ J \circ dz^{-1} = J_{st}$ where $J_{st}$ denotes the standard complex structure of $\mathbb{C}^n$. In other words the coordinate $z$ is biholomorphic with respect to $J$ and $J_{st}$ and $M$ admits local complex holomorphic coordinates near every point. The Nijenhuis tensor of $J$ is defined by

$$N(X,Y) = [X,Y] - [JX,JY] + J[X,JY] + J[JX,Y].$$

A structure $J$ is called formally integrable if $N$ vanishes at every point of $M$. The theorem of Newlander-Nirenberg [15] states that formal integrability is equivalent to integrability.
Let $J$ be an integrable smooth ($C^\infty$) almost complex structure on $\mathbb{R}^{2n}$. We assume that for some real positive number $\theta > 1$ we have:

\begin{equation}
\| D^\alpha J(z) - D^\alpha J_{st}(z) \| \leq \frac{\lambda}{1 + \| z \|^{\theta+\theta}} \text{ for } 0 \leq |\alpha| \leq 1 \text{ and } z \in \mathbb{C}^n
\end{equation}

and for some positive integer $K = K(n)$:

\begin{equation}
\| D^\alpha J(z) - D^\alpha J_{st}(z) \| \leq \frac{\lambda}{1 + \| z \|^{\theta}} \text{ for } 2 \leq |\alpha| \leq K(n) \text{ and } z \in \mathbb{C}^n
\end{equation}

where $\lambda > 0$ is small enough. Here and everywhere below we use the notation $\| z \|^2 = \sum_{j=1}^{n} |z_j|^2$. The norm in the left hand-side is an arbitrary fixed norm on the space of real $(2n \times 2n)$ matrices.

In this paper we prove the following result.

**Theorem 1.1.** For every $n$ there exist a positive integer $K(n)$ and a positive real number $\lambda$ such that for every smooth ($C^\infty$) integrable almost complex structure $J$ on $\mathbb{R}^{2n}$ satisfying Conditions (1.1) and (1.2) the complex manifold $(\mathbb{R}^{2n}, J)$ is biholomorphic to $\mathbb{C}^n = (\mathbb{R}^{2n}, J_{st})$.

As we pointed out above, one can view Theorem 1.1 as a global version of the classical Newlander-Nirenberg theorem [15] or as an analog of Hamilton’s theorem [7] for the whole complex affine space. Similar results were obtained by I.V. Zhuravlev [20], Ch. L. Epstein and Y. Ouyang [11] and E. Chirka [1] under different hypothesis and by quite different techniques. We will discuss these results in the last section.

2. The $\overline{\partial}$-problem on $(\mathbb{C}^n, J)$

In this section we use the standard techniques of $L^2$ estimates for the $\overline{\partial}_J$-operator [9, 10]. We suppose that $J$ is a smooth ($C^\infty$) integrable almost complex structure on $\mathbb{R}^{2n}$ such that

\begin{equation}
\| J - J_{st} \|_{C^2(\mathbb{R}^{2n})} \leq \gamma,
\end{equation}

where $\gamma$ is a sufficiently small real positive number.

Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ we denote by $z = (z_1, \ldots, z_n)$ the standard complex coordinates. Fix a basis $\omega_j$, $j = 1, \ldots, n$, of differential forms of type $(1,0)$ with respect to $J$ on $\mathbb{R}^{2n}$. Since $J$ is close to $J_{st}$ we can choose $\omega_j$ in the form $\omega_j = dz_j + \sum_{k=1}^{n} b_{jk} dz_k$ where the smooth coefficients $b_{jk}$ are small enough. Then $\overline{\partial}_J u = \sum_j u \overline{\omega}_j$ for every $u \in C^\infty(\mathbb{R}^{2n})$. Consider the Hermitian metric $ds^2 = \sum \delta_{ij} \overline{\omega}_j \wedge \overline{\omega}_j$ on $\mathbb{R}^{2n}$ (here $\delta_{ij}$ denotes the Kroneker symbol). This metric is compatible with $J$. Since $\overline{\partial}_J \overline{\omega}_j$ is a $(0,2)$-form, we have $\overline{\partial}_J \overline{\omega}_j = \sum_{1 \leq k < \ell \leq n} b_{jk\ell} \overline{\omega}_k \wedge \overline{\omega}_\ell$ where the coefficients $b_{jk\ell}$ can be explicitly expressed in term of the almost complex structure tensor $J$ and its first order derivatives. Similarly $\overline{\partial}_J \omega_j$ is a $(1,1)$-form and

\begin{equation}
\overline{\partial}_J \omega_j = \sum_{1 \leq k < \ell \leq n} c_{jk\ell} \overline{\omega}_k \wedge \omega_\ell
\end{equation}

The smooth coefficients $c_{jk\ell}$ are determined by the derivatives of $J$ up to the first order. For a real function $\varphi$ of class $C^\infty$ the coefficients $\varphi_{kj}$ of the Levi form of the function $\varphi$ with respect to the structure $J$ are given by:

\begin{equation}
\varphi_{kj} = \frac{\partial^2 \varphi}{\partial \omega_j \partial \omega_k} + \sum_i c_{ij} \frac{\partial \varphi}{\partial \omega_i} = \frac{\partial^2 \varphi}{\partial \omega_k \partial \omega_j} + \sum_i c_{kj} \frac{\partial \varphi}{\partial \omega_i}.
\end{equation}
Denote by $dV_0$ the standard euclidean volume form on $\mathbb{R}^{2n}$ and by $dS_0$ its restriction to the boundary of $\Omega$. The volume form $dV = \sum_{j=1}^{n} \omega_j \wedge \overline{\omega}_j$ defined by $ds^2$ is given by the expression

$$(2.4) \quad dV = (1 + \Phi)dV_0$$

where the density $\Phi$ can be written explicitly in terms of $b_{jk}$.

From now on we set

$$\varphi(z) = \|z\|^2$$

and we suppose that $J$ is a smooth integrable almost complex structure satisfying Conditions (1.1), (1.2). In particular we assume everywhere that $\lambda$ in Conditions (1.1), (1.2) is small enough.

We point out that under these conditions there exists a positive constant $\tau_0$ such that:

$$(2.5) \quad \tau_0 \sum_{j=1}^{n} dz_j d\overline{z}_j \leq \sum_{j,k=1}^{n} \varphi_{jk} \omega_k \overline{\omega}_j.$$

Indeed, it follows from (2.3) that the functions $\varphi_{jk}$ are linear combinations of the partial derivatives of $\varphi$ up to the second order, the coefficients of these linear combinations consisting of the entries of the matrix $J - J_{st}$ and of their first order partial derivatives (moreover, the coefficients of the first order partial derivatives of $\varphi$ always contain first order partial derivatives of the entries of $J - J_{st}$). So (2.5) is a consequence of (1.1), (1.2).

The expression in the left (resp. right) hand-side of (2.5) is just the Levi form of $\varphi$ with respect to the structure $J_{st}$ (resp. $J$). So Condition (2.5) means that the function $\varphi$ remains strictly plurisubharmonic on $\mathbb{R}^{2n}$ with respect to $J$ and its Levi form is uniformly strictly positive.

The following proposition is implicitly contained in [9, 10].

**Proposition 2.1.** Fix a real number $\alpha$ such that $0 < \alpha < \tau_0$. Then for every sufficiently small positive real number $\lambda$, for every smooth integrable structure $J$ on $\mathbb{R}^{2n}$ satisfying condition (1.2) and for every $g \in L^2_{0,1}(\mathbb{R}^{2n}, \varphi)$ satisfying

$$ \overline{\partial}_J g = 0$$

there exists $u \in L^2(\mathbb{R}^{2n}, \varphi)$ such that

$$ \overline{\partial}_J u = g$$

and

$$ \int_{\mathbb{R}^{2n}} |u|^2 e^{-\varphi} dV \leq 4 \int_{\mathbb{R}^{2n}} |g|^2 e^{-\varphi} / (\tau_0 - \alpha) dV.$$

**Proof.** We keep the same notations as in [10]. Consider the following two maps of the $\overline{\partial}$-complex on $(\Omega, J)$:

$$ T = \overline{\partial}_J : L^2(\mathbb{R}^{2n}, \varphi) \rightarrow L^2_{0,1}(\mathbb{R}^{2n}, \varphi) $$

and

$$ S = \overline{\partial}_J : L^2_{0,1}(\mathbb{R}^{2n}, \varphi) \rightarrow L^2_{0,2}(\mathbb{R}^{2n}, \varphi), $$
where the space $L^2_{0,k}(\mathbb{R}^{2n}, \varphi)$, equipped with the weighted scalar product and norm, is defined as in [9] or [10] pp.108-109. More precisely, let a form $f$ of type $(p, q)$ be written in the form

$$f = \sum' \sum' f_{I,J} \omega^I \wedge \overline{\omega}^J$$

where $f_{I,J}$ are skew-symmetric both in $I$ and in $J$ and $\sum'$ means that the summation is extended only over increasing multi-indices. Then

$$|f|^2 = \frac{1}{p!q!} \sum |f_{I,J}|^2$$

and

$$\|f\|_\varphi^2 = \int_{\mathbb{R}^{2n}} |f|^2 e^{-\varphi} dV.$$

Fix a smooth function $\eta$ with compact support in $\mathbb{R}^{2n}$ and equal to 1 in a neighborhood of the origin. Then the functions $\eta_{\nu}(z) = \eta(z/\nu)$ satisfy Condition (2.6)

$$\sum_{k=1}^n |\partial \eta_{\nu}/\partial \omega_k| \leq \text{const.}$$

Using the estimates (2.6), it follows from Lemma 5.2.1 of [10] that the space $D_{0,1}(\mathbb{C}^n)$ of smooth $(0,1)$-forms with compact support in $\mathbb{C}^n$ is dense in $D_T \cap D_S$ for the graph norm

$$f \mapsto \|f\|_\varphi + \|T^*f\|_\varphi + \|Sf\|_\varphi.$$

Following the classical method of [9] or [10] pp.77-85,107-114, we obtain for any $f \in D_T \cap D_S$:

$$\int_{\mathbb{R}^{2n}} (\tau_0 - \alpha)|f|^2 e^{-\varphi} dV \leq 4(\|T^*f\|_\varphi^2 + \|Sf\|_\varphi^2)$$

if $\lambda$ is small enough. Inequality (2.7) enables to conclude similarly to the proof of Theorem 4.4.1 of [10].

### 3. Deformation of the complex structure on $\mathbb{C}^n$

Our proof of Theorem 1.1 is based on isotropic dilations of coordinates “at infinity”. We assume everywhere that we are in the hypothesis of this Theorem.

#### 3.1. Dilations

Our goal is to find a solution $u_k$ of the equation

$$(3.1) \quad \overline{\partial}_J u_k = \overline{\partial}_J z_k$$

for $k = 1, \ldots, n$ with suitable properties. Consider the linear transformations (isotropic dilations at the ”point at infinity”) of the form :

$$d_\varepsilon : \mathbb{C}^n \rightarrow \mathbb{C}^n \quad \varepsilon z =: z'.$$

Set $J_\varepsilon := (d_\varepsilon)_*(J)$ and $u_k(\varepsilon z') := u_k(\varepsilon^{-1}z')$. If we denote by $(z'_1, \ldots, z'_n)$ the coordinates of $z'$ then for every $k = 1, \ldots, n$ :

$$(3.2) \quad \overline{\partial}_{J_k} u_k = \overline{\partial}_{J_k} z_k.$$

Since $J_\varepsilon(z') = J(\varepsilon^{-1}z')$ and $J_{st}(z') = J_{st}(\varepsilon^{-1}z')$, Conditions (1.1) and (1.2) imply :
L. Lempert \[13\] (in the case

Condition (2.5) holds in the coordinates

Lemma 3.2.

Proof. Denote by \(L^J(\psi, p, v)\) the value of the Levi form of a function \(\psi\) at \((p, v) \in T(\mathbb{R}^{2n})\). Then Condition (2.5) can be written

\[
L^{J_{st}}(\varphi, p, v) \leq L^J(\varphi, p, v).
\]
Set \( \varphi_{\varepsilon}(z') = \varphi \circ (d_\varepsilon)^{-1}(z') \) that is \( \varphi_{\varepsilon}(z') = \varepsilon^{-2}\varphi(z') \). The invariance of the Levi form and the inequality (3.8) then imply
\[
L^\ast_{st}(\varphi, p', v') \leq L^\ast(\varphi, p', v')
\]
for any \((p', v') \in T(\mathbb{R}^{2n})\). Since the structure \( \tilde{J}_{\varepsilon} \) is close to \( J_{st} \) in the \( C^2 \)-metric on the ball \((1/2)\mathbb{B} \), the last inequality also holds for \( \tilde{J}_{\varepsilon} \). This proves Lemma 3.2.

The form \( \tilde{\partial}_{\tilde{J}_{\varepsilon}}z'_k \) satisfies, for every real vector field \( Y \):
\[
\tilde{\partial}_{\tilde{J}_{\varepsilon}}z'_k(Y) := \frac{1}{2}(dz'_k(Y) + \tilde{J}_{\varepsilon}(z'_k)dz'_k(iY))
\]
\[
= \tilde{\partial}_{J_{st}}z'_k(Y) + \frac{1}{2}[\tilde{J}_{\varepsilon}(z'_k)dz'_k(iY) - J_{st}(z'_k)dz'_k(iY)]
\]
\[
= \frac{1}{2}[\tilde{J}_{\varepsilon}(z'_k)dz'_k(iY) - J_{st}(z'_k)dz'_k(iY)].
\]

It follows from the construction of \( \tilde{J}_{\varepsilon} \) (see Condition (3.6)) that there exists a positive constant \( M \) independent of \( \varepsilon \) such that for every \( k = 1, \ldots, n \) and for every \( 0 \leq |\alpha| \leq K(n) \):
\[
\| D^\alpha(\tilde{J}_{\varepsilon}z'_k) \|_{L^\infty(\mathbb{C}^n)} \leq M.
\]
Using (3.9) with \( |\alpha| = 0 \) there exists a positive constant \( M' \) such that:
\[
\| \tilde{\partial}_{\tilde{J}_{\varepsilon}}z'_k \|_{\varphi} \leq M'.
\]
Denote by \( dV_\varepsilon \) the volume form corresponding to the structure \( \tilde{J}_{\varepsilon} \), given by (2.4) (with \( \Phi \) being defined by means of \( J_{st} \)).

Fix \( 1 \leq k \leq n \) and apply Proposition 2.11 to the structure \( \tilde{J}_{\varepsilon} \), with \( g = \tilde{\partial}_{\tilde{J}_{\varepsilon}}z'_k \), by virtue of the inequalities (3.6) and (3.7). There exists a solution \( u^\varepsilon_k \) of the equation (3.2) satisfying:
\[
\int_{\mathbb{C}^n} |u^\varepsilon_k|^2 e^{-\varphi} dV_\varepsilon \leq \int_{\mathbb{C}^n} |\tilde{\partial}_{\tilde{J}_{\varepsilon}}z'_k|^2 e^{-\varphi} / \tau_1 dV_\varepsilon = (1/\tau_1) \| \tilde{\partial}_{\tilde{J}_{\varepsilon}}z'_k \|_{\varphi}^2,
\]
with \( \tau_1 := \tau_0 - \alpha \) for \( \tau_0 \) and \( \alpha \) from Proposition 2.1 independent of \( \varepsilon \).

Furthermore it follows from Conditions (3.5), (3.6) and (3.7) and from the expression of \( \tilde{\partial}_{\tilde{J}_{\varepsilon}}z'_k \) given above that for every compact subset \( X \) of \( \mathbb{C}^n \) we have:
\[
\lim_{\lambda \to 0} \sup_{0 \leq |\alpha| \leq K(n)} |D^\alpha(\tilde{\partial}_{\tilde{J}_{\varepsilon}}z'_k)| = 0,
\]
uniformly with respect to \( \varepsilon \) sufficiently small.

Then using (3.11) with \( |\alpha| = 0 \) we have by the Lebesgue Theorem:
\[
\lim_{\lambda \to 0} \| \tilde{\partial}_{\tilde{J}_{\varepsilon}}z'_k \|_{\varphi} = 0,
\]
uniformly with respect to \( \varepsilon \) sufficiently small.

It follows now from (3.10) and (3.12) :
\[
\lim_{\lambda \to 0} \int_{\mathbb{C}^n} |u^\varepsilon_k|^2 e^{-\varphi} dV_\varepsilon = 0
\]
which implies:
\[
\lim_{\lambda \to 0} \int_{\mathbb{C}^n} |u^\varepsilon_k|^2 dV_\varepsilon = 0,
\]
uniformly with respect to \( \varepsilon \) sufficiently small.
Finally according to \((3.11)\) we have for every \(0 \leq |\alpha| \leq K(n)\):

\[
\lim_{\lambda \to 0} \int_{2B} |D^\alpha (\bar{\partial}_j u^\varepsilon_k)|^2 dV_\varepsilon = \lim_{\lambda \to 0} \int_{2B} |D^\alpha (\bar{\partial}_j u^\varepsilon)|^2 dV_\varepsilon = 0,
\]

uniformly with respect to \(\varepsilon\) sufficiently small.

### 3.2. Estimate of the \(C^1\)-norm.
Since the right hand-side of the equality \((3.1)\) is smooth, according to the well-known results (for instance, Theorem 5.2.5 of [10]) the solution \(u^\varepsilon_k\) is also smooth. In order to control its \(C^1\) norm we follow classical arguments of [10]. The next statement is just a consequence of the Sobolev embedding Theorem:

**Lemma 3.3.** There exists a positive constant \(C_1\) independent of \(\varepsilon\) such that

\[
\| u^\varepsilon_k \|_{C^1(\mathbb{B})}^2 \leq C_1 \left( \int_{2B} |u^\varepsilon_k|^2 dV_\varepsilon + \int_{2B} |\bar{\partial}_j u^\varepsilon_k|^2 dV_\varepsilon + \sum_{1 \leq |\alpha| \leq n+1} \int_{2B} |D^\alpha (\bar{\partial}_j u^\varepsilon_k)|^2 dV_\varepsilon \right),
\]

where \(\mathbb{B}\) denotes the unit ball in \(\mathbb{C}^n\).

**Proof.** Fix a smooth \(C^\infty\) function \(\xi\) in \(\mathbb{C}^n\) with compact support in \(2\mathbb{B}\), and such that the restriction of \(\xi\) to \(\mathbb{B}\) is identically equal to 1. The classical arguments of [10], Lemma 5.7.2 page 140 shows that there exists a positive constant \(C_3\) independent of \(\varepsilon\) such that:

\[
\| \partial_{J^*} (\xi u^\varepsilon_k) \|_{L^2(2\mathbb{B})} \leq C_3 \left( \| \xi u^\varepsilon_k \|_{L^2(2\mathbb{B})} + \| \bar{\partial}_{J^*} (\xi u^\varepsilon_k) \|_{L^2(2\mathbb{B})} \right)
\]

where the \(L^2\)-norms are considered with respect to the standard volume form \(dV_0\) on \(\mathbb{C}^n\).

It follows from \((3.15)\) and from Proposition \((2.1)\) that there exist positive constants \(C_3\), \(C_4\) and \(C_5\) such that:

\[
\| \partial_{J^*} (\xi u^\varepsilon_k) \|_{L^2(2\mathbb{B})} \leq C_3 \left( \| \xi u^\varepsilon_k \|_{L^2(2\mathbb{B})} + \| u^\varepsilon_k \|_{L^2(2\mathbb{B})} \times \| \bar{\partial}_{J^*} \xi \|_{L^2(2\mathbb{B})} + \| \bar{\partial}_{J^*} u^\varepsilon_k \|_{L^2(2\mathbb{B})} \right)
\]

\[
\leq C_4 \left( \| u^\varepsilon_k \|_{L^2(2\mathbb{B})} + \| \bar{\partial}_{J^*} u^\varepsilon_k \|_{L^2(2\mathbb{B})} \right)
\]

\[
\leq C_5 \| \bar{\partial}_{J^*} u^\varepsilon_k \|_{L^2(2\mathbb{B})}.
\]

Thus we have:

\[
\| \partial_{J^*} (u^\varepsilon_k) \|_{L^2(2\mathbb{B})} \leq C_5 \| \bar{\partial}_{J^*} u^\varepsilon_k \|_{L^2(2\mathbb{B})}.
\]

In particular, we may impose that the \(L^2\)-Sobolev norm \(W^{1,2}(2\mathbb{B})\) of \(u^\varepsilon_k\) is arbitrarily small if \(\lambda\), provided by Condition \((1.2)\), is small enough. In order to obtain an estimate of the \(C^1\) norm, we iterate this argument. For instance, set \(g^k_i = \partial u^\varepsilon_k / \partial \omega_i\). Then the above argument gives the existence of positive constants \(C_6\) and \(C_7\) such that:

\[
\| \partial_{J^*} g^k_i \|_{L^2(2\mathbb{B})} \leq C_6 \left( \| g^k_i \|_{L^2(2\mathbb{B})} + \| \bar{\partial}_{J^*} g^k_i \|_{L^2(2\mathbb{B})} \right) \leq C_7 \sum_{|\alpha| \leq 1} \| D^\alpha \bar{\partial}_{J^*} u^\varepsilon_k \|_{L^2(2\mathbb{B})}.
\]

Iterating this argument we obtain estimates on the \(L^2\)-norms of the derivatives of \(u^k\) up to order \(n + 1\). Then the Sobolev embedding theorem implies the desired statement.

\[\square\]

### 4. \(J\)-complex curves and Montel’s property

In this section we conclude the proof of Theorem \((1.1)\).
4.1. Montel’s property. Let $r$ be a sufficiently small real positive number. According to Conditions (3.13), (3.14) and to Lemma 3.3 if $K(n) ≥ n + 1$ then we can choose $λ$ small enough such that:

\[(4.1) \quad \| u_κ^ε \|_{C^1(B)} ≤ r,\]

uniformly with respect to $ε$ sufficiently small.

Consider the smooth $C^∞$ almost complex structure $\tilde{J}$ defined on $\mathbb{C}^n$ by $\tilde{J} = (d_ε^{-1})_*(\tilde{J}_ε)$. In our situation we have $J(z) = \tilde{J}_ε(εz)$. Then the structure $\tilde{J}$ coincides with $J$ on $\mathbb{C}^n \setminus \frac{1}{2ε}\mathbb{B}$. In particular the function $\tilde{v}_κ^ε$ defined by $\tilde{v}_κ^ε(z) := u_κ^ε(εz)$ satisfies the equation:

$$\overline{\partial}_J \tilde{v}_κ^ε = \overline{\partial}_J z_k$$

on $\mathbb{C}^n \setminus \frac{1}{2ε}\mathbb{B}$ This means that the function

$$f_κ^ε := z_k - \tilde{v}_κ^ε$$

is $J$-holomorphic on $\mathbb{C}^n \setminus \frac{1}{2ε}\mathbb{B}$. Since the unit ball $\mathbb{B}$ equipped with the structure $J$ is a Stein manifold, by the removal of compact singularities the function $f_κ^ε$ extends on $\mathbb{C}^n$ to a function still denoted by $f_κ^ε$ and $J$-holomorphic on $\mathbb{C}^n$. Moreover according to the estimate (4.1) we have:

\[(4.2) \quad \| v_κ^ε \|_{C^1((1/2ε)\mathbb{B})} ≤ r.\]

Consider now a sequence $(ε_j)_j$ decreasing to 0 and set $f^j := f_κ^ε$. It follows from (4.2):

\[(4.3) \quad \| f^j - Id \|_{C^1((1/2ε_j)\mathbb{B})} ≤ r.\]

In order to prove that the sequence $(f^j)$ is a normal family it is sufficient to show that the sequence $(f^j)$ is uniformly bounded on $r_0\mathbb{B}$ for every $r_0 > 0$. A natural idea is to use the estimate (4.3) and the maximum principle. The obstacle here is that the identity map $Id : (\mathbb{C}^n, J) → (\mathbb{C}^n, J_{st})$ is not holomorphic. For this reason we use the techniques of $J$-holomorphic curves similarly to M.Gromov’s work [5] (basic local results were obtained in [16]).

In what follows we denote by $\Delta$ the unit disc of $\mathbb{C}$. We remind that for $R > 0$ a smooth map $L_J : R\Delta → \mathbb{C}^n$ is called $J$-holomorphic if $dL_J \circ J_{st} = J \circ dL_J$. In other words, $L_J$ is holomorphic with respect to the structure $J_{st}$ on $\mathbb{C}$ and the structure $J$ on $\mathbb{C}^n$. We also call such maps $J$-holomorphic discs. We always suppose that a $J$-holomorphic disc is continuous on $R\overline{\Delta}$. In the case where a $J$-holomorphic map $L_J$ is defined on the whole complex plane $\mathbb{C}$ we call it $J$-complex line.

**Proposition 4.1.** There exists a positive constant $C$ such that for every $v ∈ \mathbb{S}$ there is a $J$-complex line $L_J^v : \mathbb{C} → \mathbb{C}^n$ satisfying:

$$\sup_{ζ ∈ \mathbb{C}} \| L_J^v(ζ) - ζv \| ≤ Cλ,$$

$λ$ being a constant from (1.1), (1.2). Furthermore

$$\mathbb{C}^n \setminus \mathbb{B} ⊂ \{ L_J^v(ζ), \ v ∈ \mathbb{S}, \ ζ ∈ \mathbb{C} \}.$$

A weaker version (but still sufficient for the proof of Lemma 4.3) of Proposition 4.1 giving the existence of arbitrary large $J$-complex discs instead of lines follows from the results of [2]. On the other hand Proposition 4.1 is a consequence of a more general result, concerning smooth almost complex deformations of the standard complex structure. Since this result is of independent interest we state and prove it in the next section for the convenience of
the reader (see Theorem 5.1). We will apply Proposition 4.1 together with the following Lemma. This frequently used statement is just a variation of the classical Nijenhuis-Woolf’s Theorem (see [17]).

Lemma 4.2. For a \( a \in \mathbb{C}^{n-1} \), \( |a| \leq 1 \), let the map \( N_a : 2\Delta \to \mathbb{C}^n \) be defined by \( N^a(\zeta) = (a, 0) + \zeta(0, 1) \) where \((a, 0), (0, 1) \in \mathbb{C}^{n-1} \times \mathbb{C} \). For every \( \lambda \) sufficiently small there exists a \( J \)-holomorphic curve \( N^\lambda_j : 2\Delta \to \mathbb{C}^n \) such that \( N^\lambda_j(0) = (a, 0) \) and \( \sup_{\zeta \in 2\Delta} \| N^\lambda_j(\zeta) - N^a(\zeta) \| \leq C\lambda \). These curves form a foliation of a neighborhood of \( \mathbb{B} \).

We continue now the proof of Theorem 1.1.

The crucial consequence of Proposition 4.1 is the following

Lemma 4.3. We have

\[
\| f^j(z) - z \| \leq r + C\lambda
\]

for every \( j \) and every \( z \in (1/2\varepsilon_j)\mathbb{B} \). In particular, for every \( r_0 > 0 \) the sequence \( (f^j) \) is uniformly bounded on the ball \( r_0 \mathbb{B} \).

Proof. Let \( p \in (1/2\varepsilon_j)\mathbb{B} \setminus \mathbb{B} \). Consider the \( J \)-complex line \( L^j_p \) given by Proposition 4.1 passing through \( p \), i.e. satisfying \( L^j(\zeta_p) = p \) for some \( \zeta_p \in \mathbb{C} \). Denote by \( L^v \) the usual complex line \( \zeta \in \mathbb{C} \mapsto \zeta v \). Since the maps \( L^v : (\mathbb{C}, J_{st}) \to (\mathbb{C}^n, J) \) and \( f^j : (\mathbb{C}^n, J) \to (\mathbb{C}^n, J_{st}) \) are holomorphic, the map \( \zeta \mapsto f^j(L^v_j(\zeta)) - L^v(\zeta) \) is holomorphic with respect to \( J_{st} \). It follows by Proposition 4.1 that the domain \( D_j := (L^v_j)^{-1}((1/2\varepsilon_j)\mathbb{B}) \) is a bounded domain in \( \mathbb{C} \) containing the point \( \zeta_p \). Furthermore, the function \( \varphi(z) = \| z \|^2 \) is \( J \)-plurisubharmonic and so the composition \( \varphi \circ L^j \) is a subharmonic function on \( \mathbb{C} \). Applying the maximum principle to this function we obtain that the domain \( D_j \) is simply connected and its boundary coincides with the level set \( \{ \zeta \in \mathbb{C} : \| L^v_j(\zeta) \| = 1/(2\varepsilon_j) \} \). For \( \zeta \in \mathbb{C} \) satisfying \( \| L^v_j(\zeta) \| = 1/(2\varepsilon_j) \) we have:

\[
\| f^j(L^v_j(\zeta)) - L^v(\zeta) \| \leq \| f^j(L^v_j(\zeta)) - L^j(\zeta) \| + \| L^v_j(\zeta) - L^v(\zeta) \| .
\]

The first quantity in the right hand-side is bounded from above by \( r \) in view of (4.1) and the second one is bounded by \( C\lambda \). Applying on the domain \( D_j \) the maximum principle to the map \( \zeta \mapsto f^j(L^v_j(\zeta)) - L^v(\zeta) \) we conclude that the estimate (4.4) holds on \( (1/2\varepsilon_j)\mathbb{B} \setminus \mathbb{B} \).

If \( p \in \mathbb{B} \) we repeat the same argument, replacing respectively \( L^v \) by \( N^a \) and \( L^j_p \) by \( N^j_p \), from Lemma 4.2. This ends the proof of Lemma 4.3. \( \square \)

4.2. End of the proof of Theorem 1.1. Denote by \( \text{Jac}(f^j)(z) \) the Jacobian determinant of the map \( f^j \) at \( z \).

Lemma 4.4. There exists a positive constant \( \alpha \) such that

\[
|\text{Jac}(f^j)(z)| \geq \alpha
\]

for every \( j \) and every \( z \in (1/2\varepsilon_j)\mathbb{B} \).

Proof. It follows from (4.3) that the Jacobian determinant \( \text{Jac}(f^j) \) of \( f^j \) is a holomorphic (with respect to \( J \) and \( J_{st} \)) function not vanishing on \((1/2\varepsilon_j)\mathbb{S}\). Hence it follows from the removal compact singularities theorem that \( \text{Jac}(f^j) \) does not vanish on \( (1/2\varepsilon_j)\mathbb{B} \). Applying the maximum principle to the function \( 1/\text{Jac}(f^j) \) we conclude. \( \square \)

Lemma 4.3 implies that the sequence \( (f^j) \) contains a subsequence (still denoted by \( (f^j) \)) uniformly converging (with all partial derivatives) on any compact subset of \( \mathbb{C}^n \). Denote by \( f \)
the limit map. Then the map $f$ satisfies the equation $\overline{\partial}_J f = 0$ that is $f : (\mathbb{C}^n, J) \to (\mathbb{C}^n, J_{st})$ is holomorphic. Furthermore according to Lemma 4.4 the map $f$ is locally biholomorphic. On the other hand, it follows by Lemma 4.3 that for every compact subset $K$ of $\mathbb{C}^n$ we have $\| f - Id \|_{\mathcal{C}^\infty(K)} \leq r + C\lambda$ and therefore $f : \mathbb{C}^n \to \mathbb{C}^n$ is a proper map. Now by the classical theorem of J.Hadamard [6] $f$ is a global diffeomorphism of $\mathbb{R}^{2n}$ and so is globally biholomorphic.

This completes the proof of Theorem 1.1. □

5. Stability of curves under a deformation of $J_{st}$

In this Section we establish a general version of Proposition 4.1. This does not use the integrability of $J$ and is valid for any almost complex structure on $\mathbb{C}^n$ satisfying Conditions (1.1), (1.2). We point out that one of the results of Gromov [5] gives the existence of a $J$-complex map from the Riemann sphere to the complex projective space $\mathbb{CP}^n$ equipped with an almost complex structure tamed by the standard symplectic form of $\mathbb{CP}^n$. One can also view the next statement as a global analog of the Nijenhuis-Wolf theorem [16, 17].

Theorem 5.1. Let $J$ be a smooth almost complex structure in $\mathbb{C}^n$ satisfying Conditions (1.1) and (1.2), where $\lambda$ is sufficiently small. Then there exists a positive constant $C$ such that for every $v \in \mathbb{S}$ there is a $J$-complex line $L^v_J : \mathbb{C} \to \mathbb{C}^n$ satisfying:

\[
\sup_{\zeta \in \mathbb{C}} \| L^v_J(\zeta) - \zeta v \| \leq C\lambda.
\]

Furthermore

$$\mathbb{C}^n \setminus B = \{ L^v_J(\zeta), \ v \in \mathbb{S}, \ \zeta \in \mathbb{C} \}.$$
Proof of Theorem 5.1 This is based on a bijective correspondence between certain classes of $J$-complex lines and standard complex lines. The crucial observation is the following. Let $L : \mathbb{C} \to \mathbb{C}^n$, $L : \zeta \mapsto z(\zeta)$ be a $J$-holomorphic map. Then the map $L$ is a solution of the following quasi-linear elliptic system of partial differential equations:

\[(5.3) \quad z_{\overline{\zeta}} - A(z)\overline{z_{\zeta}} = 0\]

where $A(z)$ is a complex $(n \times n)$-matrix function. It corresponds to the matrix representation of the real endomorphism $(J(z) + J_{st})^{-1}(J(z) - J_{st})$ of $\mathbb{R}^{2n}$; it is easy to check that this endomorphism is anti-linear with respect to $J_{st}$ (cf. for instance [16]). This equation is equivalent to

\[(5.4) \quad (z - TA(z)\overline{z_{\zeta}})_{\overline{\zeta}} = 0\]

that is to the $J_{st}$-holomorphicity of the map $w : \zeta \mapsto z(\zeta) - TA(z(\zeta))\overline{z(\zeta)}$ on $\mathbb{C}$.

Consider the $J_{st}$-holomorphic map $L^v : \zeta \mapsto \zeta v$, $\zeta \in \mathbb{C}$. Our goal is to consider $J$-holomorphic maps defined on $\mathbb{C}$ close enough to $L^v$. Since the map $L^v$ is not bounded and so does not belong to the space $C^{1,\gamma}(\mathbb{C}, \mathbb{C}^n)$ equipped with the standard norm introduced above, we need to modify slightly this norm in order to enlarge this space. Fix $\gamma \in ]0, 1[$ and consider a continuous map $z : \mathbb{C} \to \mathbb{C}^n$. We define its weighted $C^0$ norm by

\[(5.5) \quad \|z\|_{w} := \|z(\zeta)(1 + |\zeta|^2)^{-1/2}\|_{c^0(\mathbb{C}, \mathbb{C}^n)}.
\]

For a function $z : \mathbb{C} \to \mathbb{C}^n$ of class $C^1$ with the $\gamma$-Hölderian first order partial derivatives we define the weighted $C^{1,\gamma}$-norm by

\[(5.6) \quad \|z\|_{C^{1,\gamma}(\mathbb{C}, \mathbb{C}^n)} := \|z\|_{w} + \|z\|_{C^0,\gamma}(\mathbb{C}, \mathbb{C}^n) + \|z_{\zeta}\|_{C^0,\gamma}(\mathbb{C}, \mathbb{C}^n)\]

Thus we just add the weight to the $C^0$-term in the standard norm (5.2). We denote by $C^{1,\gamma}_w(\mathbb{C}, \mathbb{C}^n)$ the space of the above maps $z : \mathbb{C} \to \mathbb{C}^n$ equipped with this norm. Then obviously

(i) The space $C^{1,\gamma}_w(\mathbb{C}, \mathbb{C}^n)$ is Banach

(ii) For every $z \in C^{1,\gamma}_w(\mathbb{C}, \mathbb{C}^n)$ one has

\[(5.7) \quad \|z\|_{C^{1,\gamma}_w(\mathbb{C}, \mathbb{C}^n)} \leq \|z\|_{C^1,\gamma}(\mathbb{C}, \mathbb{C}^n)\]

In particular the identity map

\[id : C^{1,\gamma}(\mathbb{C}, \mathbb{C}^n) \to C^{1,\gamma}_w(\mathbb{C}, \mathbb{C}^n)\]

is continuous.

For $\varepsilon_0 > 0$ small enough consider the open subset $\mathcal{U}_{\varepsilon_0}$ of the space $C^{1,\gamma}_w(\mathbb{C}, \mathbb{C}^n)$ defined by:

\[\mathcal{U}_{\varepsilon_0} := \{z \in C^{1,\gamma}_w(\mathbb{C}, \mathbb{C}^n) : \|z - L^v\|_{C^{1,\gamma}_w(\mathbb{C}, \mathbb{C}^n)} < \varepsilon_0, \|v\| = 1\}.
\]

We point out that $\mathcal{U}_{\varepsilon_0}$ is independent of $v$ which runs over the unit sphere in the above definition.

Given $\theta > 1$ from Condition (1.1) fix $p \in ]1, 2[$ such that $\theta p > 2$. Then $A(z)\overline{z_{\zeta}} \in L^p(\mathbb{C}, \mathbb{C}^n) \cap C^{0,\gamma}(\mathbb{C}, \mathbb{C}^n)$ for every $z \in \mathcal{U}_{\varepsilon_0}$. Thus the operator:

\[\Phi_J : \mathcal{U}_{\varepsilon_0} \subset C^{1,\gamma}_w(\mathbb{C}, \mathbb{C}^n) \to C^{1,\gamma}_w(\mathbb{C}, \mathbb{C}^n)_{z} \mapsto w = z - TA(z)\overline{z_{\zeta}}.\]
is correctly defined by Proposition 5.2 if $\varepsilon_0 > 0$ is small enough. In what follows we always assume that this assumption is satisfied.

**Proposition 5.3.** If $\lambda$ given by Conditions (1.1), (1.2) is sufficiently small then the operator $\Phi_J$ is a smooth local $C^1$ diffeomorphism from $U_{\varepsilon_0}$ to $\Phi_J(U_{\varepsilon_0})$. Moreover $L^v \in \Phi_J(U_{\varepsilon_0})$ for every $v \in \mathbb{S}$.

**Proof of Proposition 5.3.** First we show that $\Phi_J$ is a small $C^1$ deformation of the identity on $U_{\varepsilon_0}$.

**Lemma 5.4.** There exists a positive constant $D_1(\varepsilon_0)$ such that for every $z \in U_{\varepsilon_0}$:

$$\| \Phi_J(z) - z \|_{C^1(\varepsilon, \mathbb{C}^n)} \leq D_1 \lambda. \tag{5.8}$$

We point out that we employ the standard non-weighted norm in (5.8).

**Proof of Lemma 5.4.** Indeed, in view of Condition (1.1) and the choice of $p$ we have the estimate

$$\| A(z)\zeta^\ast \|_{L_p(\varepsilon, \mathbb{C}^n)} + \| A(z)\zeta^\ast \|_{C^0(\varepsilon, \mathbb{C}^n)} \leq D_2 \lambda \tag{5.9}$$

for any $z \in U_{\varepsilon_0}$, for some $D_2 > 0$. So Proposition 5.2 implies the statement of Lemma. □

Next we prove that the Fréchet derivative $\Phi_J$ of $\Phi_J$ is close to identity as an operator. The derivative of $\Phi_J$ at $z \in U_{\varepsilon_0}$ is defined by:

$$\Phi_J(z) : C_w^{1,\gamma}(\varepsilon, \mathbb{C}^n) \rightarrow C_w^{1,\gamma}(\varepsilon, \mathbb{C}^n) \quad \hat{z} \mapsto \hat{w} = \hat{z} - T A(z)\zeta^\ast - T(B(\hat{z})\zeta^\ast)$$

where $B$ is given by

$$B(\hat{z}) = [J(z) + J_{st}]^{-1}DJ(z)(\hat{z}) - [J(z) + J_{st}]^{-1}DJ(z)(\hat{z})[J(z) + J_{st}]^{-1}J(z) - J_{st}.$$  

**Lemma 5.5.** There exists a positive constant $D_3(\varepsilon_0)$ such that for every $z \in U_{\varepsilon_0}$:

$$\|\|\Phi_J(z) - Id\|\| \leq D_3 \lambda. \tag{5.10}$$

**Proof of Lemma 5.5.** It follows from Condition (1.1) that there exist positive constants $D_4, D_5$ such that for every $z \in U_{\varepsilon_0}$ and for every $\hat{z} \in C_w^{1,\gamma}(\varepsilon, \mathbb{C}^n)$ we have:

$$\| B(\hat{z})\zeta^\ast \|_{L_p(\varepsilon, \mathbb{C}^n)} + \| B(\hat{z})\zeta^\ast \|_{C^0(\varepsilon, \mathbb{C}^n)} \leq D_4 \lambda \|\hat{z}\|_{C_w^{1,\gamma}(\varepsilon, \mathbb{C}^n)}, \tag{5.11}$$

and

$$\| A(z)\zeta^\ast \|_{L_p(\varepsilon, \mathbb{C}^n)} + \| A(z)\zeta^\ast \|_{C^0(\varepsilon, \mathbb{C}^n)} \leq D_5 \lambda \|\hat{z}\|_{C_w^{1,\gamma}(\varepsilon, \mathbb{C}^n)}. \tag{5.12}$$

Now Proposition 5.2 implies the desired statement. □

**Remark 5.6.** We point out that the estimates (5.9), (5.11), (5.12) and the existence of the Fréchet derivative of $\Phi_J$ rely deeply on the definition of $U_{\varepsilon_0}$. Indeed every map $z \in U_{\varepsilon_0}$ has the same asymptotic behaviour at infinity as $|\zeta|$. This allows to use Conditions (1.1) and (1.2).

Proposition 5.3 is now a consequence of Lemmas 5.4 and 5.5 provided $\lambda$ is sufficiently small. We just need to justify the last statement, i.e. the condition $L^v \in \Phi_J(U_{\varepsilon_0})$ for every $v \in \mathbb{S}$. We must show that the equation:

$$z = TA(z)\zeta^\ast + L^v \tag{5.13}$$
admits a solution \( z \in U_{\varepsilon_0} \). Consider the operator \( Q : z \mapsto TA(z)\overline{T} + L^v \) defined on the closure \( U_{\varepsilon_0} \). It follows by Lemma 5.4 and the estimate (5.1) that for \( \lambda \) small enough the image \( Q(U_{\varepsilon_0}) \) is contained in \( U_{\varepsilon_0} \). Furthermore Lemma 5.5 implies that \( Q \) is a contracting map. So by the Fixed Point Theorem the equation (5.13) has a unique solution in \( U_{\varepsilon_0} \).

We can conclude now the proof of Theorem 5.1. According to Proposition 5.3 if \( \lambda \) is sufficiently small then for every \( v \in S \) the map \( L^v \) belongs to \( \Phi_J(U_{\varepsilon_0}) \). Then the map \( L^v_J := \Phi^{-1}_J(L^v) \) is \( J \)-holomorphic on \( C \), satisfying the condition (5.11). Finally consider the evaluation map

\[
ev_J : \mathbb{R}^+ \times S \rightarrow \mathbb{C}^n
\]

\[
(t, v) \mapsto L^v_J(t)
\]

smoothly depending on \( J \) as a parameter. The map \( \ev_J \) is a smooth small deformation of the map \( \ev_{J_{st}} : (t, v) \mapsto tv \). Let the projection \( \pi : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{R}^+ \times S \) be defined by \( \pi(z) = (\| z \|^2, \| z \|^{-2}) \) so that \( (\ev_{J_{st}} \circ \pi) : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{C}^n \setminus \{0\} \) is the identity map. Now we extend the restriction \( (\ev_J \circ \pi)|_S \) smoothly on the unit ball \( B \) and obtain a smooth map \( \ev_J : \mathbb{C}^n \rightarrow \mathbb{C}^n \) coinciding with \( (\ev_{J_{st}} \circ \pi) \) on \( \mathbb{C}^n \setminus B \). It follows by the condition (5.1) that we have \( \| \ev_J(z) \| \rightarrow \infty \) as \( \| z \| \rightarrow \infty \) i.e. this map is proper and so is surjective.

6. Examples and remarks

5.1. It is not difficult to produce examples of structures satisfying the assumptions of Theorem 5.1. One may consider a smooth diffeomorphism \( F : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \) such that \( F \) converges “fast enough” to the identity map as \( \sum_j |z_j|^2 \) tends to infinity, and set \( J = F_*(J_{st}) \).

The integer \( K(n) \) in Theorem 5.1 depends on results of G.Kirchmijian and L.Lempert. In the case \( n \geq 3 \) it follows by [12] that one can take \( K(n) = n + 3 \). In the case \( n = 2 \) the assumptions on the initial regularity are more involved (see Proposition 13.1 of [13]).

We point out that Charles L. Epstein and Yong Ouyang [4] obtained another version of Theorem 5.1. Roughly speaking they require the convergence of \( J \) and its partial derivatives up to the third order to \( J_{st} \) at infinity, with “the third degree plus \( \varepsilon \)” polynomial decrease. Their approach is based on the study of the asymptotic behavior of sectional curvatures near a pole in a complete Kähler manifold. Thus our assumption on the asymptotic behavior of the partial derivatives of \( J \) up to the third order are weaker, but we need assumptions on higher order derivatives.

The approach of I.V.Zhuravlev [20] is based on an explicit solution of the \( \bar{\partial} \)-equation in \( \mathbb{C}^n \) by means of a suitable integral representation. He obtains an analogue of Theorem 5.1 assuming that the norm \( \| J - J_{st} \|_{L^\infty(\mathbb{C}^n)} \) is small enough and the matrix function \( J - J_{st} \) admits certain second order Sobolev derivatives in \( L^p(\mathbb{C}^n) \) for suitable \( p > 1 \). This result requires a quite low regularity of \( J \). However Sobolev’s type condition of \( L^p \) integrability on \( \mathbb{C}^n \) is somewhat restrictive: there are obvious examples of functions polynomially decreasing on \( \mathbb{C}^n \), which are not in \( L^p(\mathbb{C}^n) \). So our result is independent from the results of [4, 20]. It would be interesting to find a general statement which would contain the results of [4, 20] and the result of the present paper as special cases.

A strong result was obtained recently by E.Chirka [1] who proved an analogue of Theorem 5.1 in the case of \( \mathbb{C}^2 \), under the assumption that the norm \( \| J - J_{st} \|_{L^\infty(\mathbb{C}^2)} \) is small enough and that \( J - J_{st} \) is of class \( L^2 \) on \( \mathbb{C}^2 \). His method is based on the study of global foliations of \( \mathbb{C}^2 \) by pseudoholomorphic curves and strongly uses the dimension two. He also obtained a statement on the existence of \( J \)-complex lines in the spirit of Theorem 5.1 under assumptions
different from ours; his result can not be applied in our situation. We believe that the approaches of the present work and the works \cite{1, 20} will be useful for a further work concerning the natural questions on deformations of Stein structures.

5.2. We point out that there exist integrable almost complex structures on $\mathbb{R}^{2n}$ without non-constant plurisubharmonic functions (see \cite{3}). The simplest examples come from the complex projective space $\mathbb{C}P^n$.

**Proposition 6.1.** There exists an integrable almost complex structure $J$ on $\mathbb{R}^{2n}$ such that every $J$-plurisubharmonic function $u : \mathbb{R}^{2n} \to [-\infty, +\infty[$ is constant.

**Proof.** Consider the complex projective space $\mathbb{C}P^n$ and the affine space $\mathbb{C}^n$ equipped with the standard complex structures $J_{PS}^P$ and $J_{st}$ respectively. Let also $\pi : \mathbb{C}P^n \to \mathbb{C}^n$ be the canonical projection. Then $\mathbb{C}P^n = \pi^{-1}(\mathbb{C}^n) \cup S$ where $S$ ("the pullback of infinity") is a smooth compact complex hypersurface in $\mathbb{C}P^n$. After an arbitrary small perturbation of $S$, given by a global diffeomorphism $\Phi$ of $\mathbb{C}P^n$ we obtain a smooth compact real submanifold $\tilde{S} := \Phi(S)$ of codimension 2 in $\mathbb{C}P^n$.

For such a general perturbation, the manifold $\tilde{S}$ will be generic almost everywhere, meaning that the complex linear span of its tangent space in almost every point is equal to the tangent space of $\mathbb{C}P^n$ at this point. According to well known results (see for instance \cite{11}) such a manifold is a removable singularity for any plurisubharmonic function. More precisely, if $u$ is a plurisubharmonic function on $\mathbb{C}P^n \setminus \tilde{S}$ then there exists a plurisubharmonic function $\tilde{u}$ on $\mathbb{C}P^n$ such that $\tilde{u}|_{\mathbb{C}P^n \setminus \tilde{S}} = u$. Therefore any function which is plurisubharmonic (with respect to $J_{st}^P$) on $\mathbb{C}P^n \setminus \tilde{S}$ is constant. Then the complex structure $J := (\pi \circ \Phi^{-1})_* (J_{st}^P)$ satisfies the statement of Proposition 6.1.

**Acknowledgements.** We thank E.M.Chirka for bringing our attention to the subject of the present paper and to the works \cite{1, 20}. We are grateful to Ch.Epstein for bringing our attention to the article \cite{4}.

**References**

[1] E.M.Chirka, *Quasibiholomorphic mappings*, in ”Geometricheskij analiz i ego prilozhenia”, Volgograd. GU, 2005, 203-241.
[2] B.Coupet, H.Gaussier, A.Sukhov, *Riemann maps in almost complex manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), 761-785.
[3] K.Diederich, N.Sibony, *Strange complex structures on Euclidean space*, J. Reine Angew. Math. 311/312 (1979), 255-272.
[4] C.L.Epstein, Y.Ouyang, *Deformations of open Stein manifolds*, Comm. in PDE 25 (2000), 2333-2351.
[5] M.Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. math. 82 (1985), 307-347.
[6] J.Hadamard, *Sur les transformations pontuelles*, Bull. Soc. Math. France 34(1906), 71 - 84.
[7] R.Hamilton, *Deformation of complex structures on manifolds with boundary*, J. Diff. Geom.12(1977), 1-45, 14, 1979 (1979), 409-473.
[8] C.D.Hill, M.Taylor, *Integrability of rough almost complex structures* J. Geom. Anal. 13 (2003), 163-172.
[9] L.Hörmander, *$L^2$-estimates and existence theorems for $\overline{\partial}$-operator*, Acta Math. 113 (1965), 89-152.
[10] L.Hörmander, *An introduction to Complex Analysis in Several Variables*, D. Van Nostrand Company, Inc. 1966.
[11] N.Karpova, *On the removal of singularities of plurisubharmonic functions*, Math. Notes 49 (1991), 252-256.
[12] G.K.Kiremidjian, *A direct extension method for CR structures*, Math. Ann. 242 (1979), 1-19.
[13] L.Lempert, *On three-dimensional Cauchy-Riemann manifolds*, Journ. AMS 5(1992), 923-969.
[14] D.McDuff, D.Salamon, *J-holomorphic curves and symplectic topology*, AMS Coll. Publ. vol. 52, 2004.
[15] A.Newlander, L.Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. Math. **65** (1957), 391-404.
[16] A.Nijenhuis, W.B.Woolf, *Some integration problems in almost-complex and complex manifolds*, Ann. Math. **77**(1964), 424-489.
[17] J.-C.Sikorav, *Some properties of holomorphic curves in almost complex manifolds*, pp.165-189, in “Holomorphic curves in symplectic geometry”, M.Audin, J.Lafontaine eds. Birkhauser 1994.
[18] I.N.Vekua, *Generalized analytic functions*, Pergamon Press, 1962.
[19] S.Webster, *A new proof of Newlander - Nirenberg theorem*, Math. Z. **201** (1989), 303-316.
[20] I.V.Zhuravlev, *On the existence of a homeomorphic solution for a multidimensional analogue of the Beltrami equation* Russian Acad. Sci. Dokl. Math. **45**(1992), 649-652.

Hervé Gaussier
Université Joseph Fourier U.S.T.L.
100 rue des Maths Cité Scientifique
38402 Saint Martin d’Hères 59655 Villeneuve d’Ascq Cedex
FRANCE FRANCE
herve.gaussier@ujf-grenoble.fr sukhov@agat.univ-lille1.fr