Twisting of $N=1$ SUSY Gauge Theories and Heterotic Topological Theories

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Abstract

It is shown that $D = 4$ $N = 1$ SUSY Yang-Mills theory with an appropriate supermultiplet of matter can be twisted on compact Kähler manifold. The conditions of cancellation of anomalies of BRST charge are found. The twisted theory has an appropriate BRST charge. We find a non-trivial set of physical operators defined as classes of the cohomology of this BRST operator. We prove that the physical correlators are independent on external Kähler metric up to a power of a ratio of two Ray-Singer torsions for the Dolbeault cohomology complex on a Kähler manifold. The correlators of local physical operators turn out to be independent of anti-holomorphic coordinates defined with a complex structure on the Kähler manifold. However a dependence of the correlators on holomorphic coordinates can still remain. For a hyperkähler metric the physical correlators turn out to be independent of all coordinates of insertions of local physical operators.

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1 Introduction

It is well known that in supersymmetric (SUSY) theories certain correlators do not depend on coordinates of inserted operators due to the supersymmetric Ward identities in the flat space-time $\mathbb{R}^4$. These are correlators of operators which are the lowest components of superfields of the same chirality. In particular in the $D = 4 \ N = 1, 2$ QCD there are non-trivial correlators of this type which are proved to be non-zero due to contributions of the instanton-like configuration in the path integral $\mathbb{R}^4$. On the other hand it has been discovered $\mathbb{R}^4$ that in the $N = 2$ SUSY theories the sector of theory which contains only the operators of the same chirality with correlators independent of coordinates (from now on they will be referred as the topological correlators) can be mapped onto the space of local physical observables of an appropriate topological theory. Furthermore an independence of coordinates of the topological correlators still holds in the topological theories even in the presence of arbitrary non-trivial metric of space-time. Usually these correlators do not depend also on coupling constants and on external metric. In topological theories one can also find non-local physical operators which can be represented as integrals over non-trivial cycles of local operator-valued forms. The independence of physical correlators on external metric and gauge coupling constant turns out to be important for description of various moduli spaces and smooth structures of manifolds in two and four dimensions in terms of quantum field theory.

The procedure of translation of a SUSY model to a topological theory is a twisting. By twisting of supersymmetric models one can get a wide class of topological theories. In particular the most popular probably are topological Yang-Mills theories $\mathbb{R}^4$, topological $\sigma$ models $\mathbb{R}^4$ and topological conformal theories $\mathbb{R}^4$. In turn studying a ring of local observables in twisted theory one can get much information about physical correlators in untwisted SUSY theory $\mathbb{R}^4$. $\mathbb{R}^4$.

In the present paper we focus on $D=4$ twisted SUSY Yang-Mills theories. The topological Yang-Mills theory can be constructed through a twisting of $N=2$ SUSY Yang-Mills theory $\mathbb{R}^4$. $\mathbb{R}^4$. $\mathbb{R}^4$. Such a twisting procedure can be understood as a modification of the energy-momentum tensor of the theory by adding to it a derivative of an appropriate non-anomalous (axial) current. Such a modification of the energy-momentum tensor leads to a change of dimensions and spins of the quantum fields. As a consequence one of supergenerators becomes a scalar one and can be interpreted as the BRST operator of the twisted theory.

Technically for the case of $N=2$ SUSY the twisted version of the theory can be obtained in the framework of the $N=2$ SUGRA+YM theory $\mathbb{R}^4$. $\mathbb{R}^4$. $\mathbb{R}^4$. The supermultiplet of the supergravity fields contains vierbein field $e^a_{\mu}$, gravitino field $\psi_{\mu}^a$, $SU(2)$ valued vector field $V^{ij}_{\mu}$ (this $SU(2)$ symmetry generalizes the $U(1)$ $R$-symmetry of $N = 1$ SUSY for the case of $N = 2$ SUSY) and other (auxiliary) components ($\mu$, $\nu...$ are world indices, $a$, $b,..$ are indices in the tangent frame, while
i and j stand for $SU(2)$ indices corresponding to the $SU(2)$ symmetry of the model. The physical components of the Yang-Mills supermultiplet are the gauge vector field $A_\mu$, gluino $\lambda^\alpha_i$, $\bar{\lambda}_j^\dot{\alpha}$ and complex scalar field $\phi$ (here $\alpha$ and $\dot{\alpha}$ are the spinor indices). The gauge group is assumed to be compact, finite dimensional.

This theory is invariant under simultaneous supergravity transformations (which include localized SUSY and $SU(2)$ transformations) of both supergravity and Yang-Mills multiplets. To construct a topological theory we are to consider the supergravity multiplet as a set of external fields. In general all the symmetries generated by the $N = 2$ supercharges are broken in the presence of an external supergravity fields. However for a special choice of the supergravitational fields some of supersymmetries may still remain. The prescription for such a choice is that one should put all components of supergravity multiplet equal to zero except of the vierbein field $e^a_\mu$ and $SU(2)$ vector component $V^i_{\mu j}$. The auxiliary field $V^i_{\mu j}$ is coupled to a current that is used for twisting of the Lagrangian of the SUSY theory on the curved manifold. We are looking for a supercharge that survives as a symmetry of the theory. It is clear that such a supercharge should not generate any other components of the supergravity multiplet in addition to the vierbein and the vector fields. It is easy to see that this condition is equivalent to vanishing of the variation of the gravitino component under such a supertransformation:

\[
(\delta \bar{\psi}^i_\mu)_{L,R} = D^i_{\mu j}(\omega, V)\epsilon^j_{L,R} = [\delta^i_j(\partial_\mu - \frac{i}{4} \omega^a_{\mu} \sigma_{ab}) \pm \delta^i_{\mu j}]\epsilon^j_{L,R} = 0,
\]

where the indices $L$ and $R$ stand for the left- and right-handed components of spinor. Here $\omega^a_{\mu}$ is the usual spin-connection

\[
\omega^a_{\mu} = \frac{1}{2}[e^{av}_b e^{b\mu}_c (\partial_\nu e^v_\mu - \partial_\mu e^v_\nu) - e^{av}_b (\partial_\nu e^v_\mu - \partial_\mu e^v_\nu)].
\]
This theory is also intriguing from the physical point of view. In particular in the heterotic string theory the scattering amplitudes of spacetime axions at zero momenta are found to be proportional to the Donaldson invariants \[^{17}\] in the form as they are represented in Witten’s theory \[^{3}\].

In this paper we consider the case of $N = 1$ SUSY gauge theories and their twisted versions. The motivation of this work is the following. This theory also has a sector of chiral operators whose correlators do not depend on coordinates in the flat external metric \[^{1,2,3,4}\]. The problem that we study is to try to pick out the topological sector of the theory as space of physical states selected by a BRST operator in an appropriately twisted theory. We argue that the twisting procedure can be correctly done for certain theories with non-anomalous $U(1)$ current on compact, without boundaries, $D = 4$ Kähler manifolds. The condition of absence of anomaly of twisting $U(1)$ current is shown to be equivalent to a certain condition to a gauge group representation of the fields matter. It comes from a condition of cancellation of a mixed anomaly of gauge and gravitational fields \[^{2}\]. We find a non-trivial set of physical operators defined as classes of the cohomology of the BRST operator. In such a twisted theory physical correlators turn out to be invariant under smooth deformations of external Kähler metrics up to a power of a ratio of two Ray-Singer torsions for the Dolbeault cohomology on Kähler manifold. The correlators of the local observables turn out to be meromorphic sections of a tensor power of a holomorphic bundle over $D=4$ Kähler manifold $M$ with respect to the chosen complex structure and independent on gauge coupling constant. As a result the theory allows for a localization similar to Witten’s theory \[^{3}\]. As in the original SUSY theory the physical correlators can be non-vanishing due to contributions of instanton (or anti-instanton) classical configurations. For the case of hyperkähler manifolds the correlators of local physical operators turn out to be independent on coordinates of the operator.

We will henceforth generically name such (non-anomalous) twisted theories on Kähler manifold heterotic topological theories. A heterotic topological theory for a particular case when the fields of matter belong to an adjoint representation of the gauge group turns out to be equivalent to Witten’s theory. However it is different from Witten’s theory when the fields of matter transform as non-adjoint representation. In particular in a generic situation the correlators in heterotic topological theory depend on complex structure. Therefore we can hope that the correlators in the heterotic theory can give additional information about invariants of smooth and complex structures of $D=4$ Kähler manifolds.

It is also interesting that a heterotic theory seems to give a possibility for geometrical interpretation of zero modes of fields of matter in non-adjoint representation of the gauge group.

The paper is organized as follows. In sect.2 we discuss the conditions under which

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\[^{2}\]The idea of a possibility of such a cancellation of anomaly was suggested by A.Lossev.
there is an unbroken supersymmetry on a curved manifold and describe the twisting procedure in terms of the fields of SUSY theory at the classical level. In sect.3 the physical observables are defined and the Ward identities for physical correlators are found. The model is reformulated in terms of sections of holomorphic and anti-holomorphic vector bundles on the 4-manifold. In sect.4 and 5 we analyse the semiclassical representation of the physical correlators and deduce the conditions of cancellation of anomalies of BRST charge. The structure of the physical correlators is discussed. In Conclusions the results of this paper are summarized and some possibilities for further investigations are discussed.

2 N=1 SUSY Yang-Mills theory in an external gravitational field

We start with the N=1 Yang-Mills theory coupled to external supergravity multiplet in the framework of ”new minimal supergravity” (see also [13]). The supergravity multiplet contains vierbein field $e^a_\mu$, gravitino field $\psi^\alpha$, $\bar{\psi}\dot{\alpha}$, the $U(1)$ vector field $V_\mu$ and auxiliary fields. The physical components of Yang-Mills superfield are the gauge field $A_\mu$ and gaugino $\lambda^a$, $\bar{\lambda}\dot{\alpha}$. This theory is invariant under simultaneous local SUSY transformations for super Yang-Mills fields and external supergravity multiplet.

For definiteness the external metrics is assumed to be euclidean with the signature $(+,+,+,+)$ (that means that one should consider $\lambda^a$ and $\bar{\lambda}\dot{\alpha}$ as independent fields). We are just looking now for a generator of SUSY transformations that can still correspond to a symmetry of the model if we reduce the supergravity multiplet to the usual metric and an external vector field built out of the vierbein field (these fields of supergravity multiplet are considered as external ones). If such a generator does exist it should correspond to a global supertransformation which does not change the external supergravity fields. It can be easily checked that only gravitino field can be induced under infinitesimal supertransformations. Such a variation of the gravitino fields reads as follows

$$(\delta\psi_\mu)_{L,R} = \nabla_\mu(\omega, V)\epsilon_{L,R},$$  \hspace{1cm} (2.1)$$

where

$$\nabla_\mu(\omega, V) = (\partial_\mu - i \frac{1}{4} \omega^{ab}_\mu \sigma_{ab} \pm V_\mu)\epsilon_{L,R}.$$  \hspace{1cm} (2.2)$$

Here the latters $\mu, \nu...$ stand for world indices while $a$, $b$, $c...$ correspond to the Lorentz indices in the tangent frame.

From eq.(2.1) one can see that the residual global supersymmetry still holds if the matrix in the twisted covariant derivative for spinor representation has the same...
constant eigenvector $\epsilon$ for all $\mu$, i.e.

$$
\left( \frac{i}{4} \omega_{\mu}^{ab} \sigma_{ab} + V_{\mu} \gamma_5 \right) \epsilon = 0.
$$

(2.3)

It is clear that this equation can not be satisfied for a generic external metric (the difference with the twisting of $N = 2$ case is that here we have a smaller number of parameters in the twisting vector field). However for some special manifolds it can have solutions. To classify all such situations let us recall that the euclidean Lorentz group $SO(4)$ is isomorphic to $SU(2) \times SU(2)$ group that acts in the spinor space as $SU(2)$ for both chiralities. That means that in the presence of a non-flat external metric the holonomy group is generically $SU(2) \times SU(2)$. However for special manifolds it can be reduced. There are two cases (see, e.g. [20]) of reduction of holonomy group which are interesting for us: $SU(2) \times SU(2) \rightarrow U(2)$ and $SU(2) \times SU(2) \rightarrow SU(2)$. The first case corresponds to the Kähler metrics while the second defines the hyperkähler one. It is worth emphasizing that on Kähler manifolds, dotted and undotted spinor indices are essentially different. A Kähler structure distinguishes one of chiralities.

Let us consider, for example, the case when the $SU(2)$ group acting on the right-handed sector is reduced. Notice that in the case of hyperkähler metric no vector field $V_{\mu}$ is necessary to make a twisting because the right-handed projection of the matrix $\omega_{\mu}^{ab} \sigma_{ab}$ vanishes (up to a pure gauge transformation of the Lorentz group). Moreover there are two global supersymmetries with right-handed generators which do not affect the external metric. In turn for the Kähler manifold the matrix $\omega_{\mu}^{ab} \sigma_{ab}$ in the right-handed sector can be taken locally proportional to a fixed constant $\sigma$ matrix, for example, to the $\sigma_3$ matrix in an open region on the 4-manifold. Then it is easy to see that eq.(2.3) can be satisfied by an appropriate choice of vector field $V_m$ and a spinor $\epsilon$. Therefore the supercharge corresponding to $\epsilon$ generates a supersymmetry of the system.

The lagrangian of the twisted SUSY Yang-Mills theory in the external gravitational field reads as

$$
L = \sqrt{g} Tr \left[ \frac{1}{4} F^2 + \bar{\lambda} i \gamma_5 \lambda + \frac{1}{2} D^2 \right]
$$

(2.4)

where $\gamma_5 = e^a_\mu D^\mu \gamma_a$, $D_\nu = \nabla_\nu (\omega, V) - i A_\nu$; $\lambda$ and $\bar{\lambda}$ stand for the right and left-handed Weyl spinor correspondingly, $F_{\mu\nu}$ is the strength tensor for the Yang-Mills field $A_\mu$ and $D$ is an auxiliary scalar field which is necessary for SUSY algebra to be closed. We assume that the spin-connection $\omega_{\mu}^{ab}$ in the sector of right-handed fermions is proportional to the generator of the $U(1)$ group or vanishes. The global transformation with a right-handed supergenerator which corresponds to the constant spinor $\bar{\epsilon}$ obeying eq.(2.3) is as follows (we use the Weyl notations)

$$
\delta A_\mu = (\bar{\epsilon} \sigma_5 \lambda) e^a_\mu , \; \delta \lambda = 0 ,
$$

(2.5)
One can show by a direct calculation that the Lagrangian (2.4) is invariant under the transformation (2.5). Actually the variation of the lagrangian with an arbitrary constant spinor \( \epsilon \) is proportional to the same expression that appears for the variation of gravitino field (eq.(2.1))

\[
\hat{F}^{\mu\nu} \bar{\epsilon} \left[ \frac{i}{4} \omega^{ab}_{\mu} \sigma_{ab} + V_\mu \right] e^c_\nu \sigma_c \lambda
\]

which is zero precisely for the \( \bar{\epsilon} \) obeying eq.(2.3). Here \( \hat{F}^{\mu\nu} \) stands for self-dual part of the Yang-Mills strength.

It can be shown that the operator \( Q \) which corresponds to the supergenerator associated with the spinor \( \bar{\epsilon} \) is nilpotent, i.e.

\[ Q^2 = 0. \quad (2.6) \]

Another important fact is that this lagrangian is \( Q \)-exact. By a direct calculation one can check that

\[
L = \frac{1}{2} \left\{ Q, \bar{\lambda} \left( D + \frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu} \right) \bar{\eta} \right\} . \quad (2.7)
\]

Here the constant spinor \( \bar{\eta} \) is linearly independent of \( \bar{\epsilon} \) and obeys eq.(2.3) with a change \( V_\mu \rightarrow -V_\mu \). Its normalization is defined by the condition \( \bar{e}_\alpha \bar{\eta}_\beta = 1 \) (here \( \bar{e}^{\alpha\beta} \) is the conventional metric in the usual spinor bundle). Eq.(2.7) for the Lagrangian is equivalent to eq.(2.4) up to the terms that vanish for the spinor \( \epsilon \) obeying eq.(2.3) and a term which is proportional to a topological charge of the gauge field.

A similar procedure can be also done for a chiral supermultiplet of matter in any representation of the gauge group and coupled to external supergravity. It is important however that the axial current of fermions which belong to the supermultiplet of matter is coupled to the twisting vector field \( V_\mu \) with an opposite charge as compared to the charge of the gaugino current. This fact will allow us to cancel an anomaly of a total axial current which is used for twisting. An absence of anomaly for this current is important when we want to interpret the twisted theory as a topological one.

A chiral supermultiplet of matter in a representation \( R \) of the gauge group contains a complex scalar field \( \phi \) and a Weyl fermion \( \psi_\alpha \) and also an auxiliary complex scalar field \( F \) (\( \bar{\phi}, \bar{\psi}_\dot{\alpha} \)) and \( \bar{F} \) are the complex conjugated fields, respectively). The lagrangian of matter coupled to external metric \( g_{\mu\nu} \) and the vector field \( V_\mu \) and to the Yang-Mills supermultiplet reads

\[
L_{\text{matter}} = \sqrt{g} \left( -D_\mu \bar{\phi} D^\mu \phi + \bar{\psi} i \bar{\lambda} \psi + FF - \bar{\phi} D\phi - 2\bar{\phi} \lambda \psi + \bar{\psi} \lambda \phi \right). \quad (2.8)
\]
It is invariant under the transformations (2.5) and the transformations of the fields of matter which read as follows

$$\delta \phi = 0, \quad \delta \psi = i\sigma_\mu \bar{\epsilon} D^\mu \phi, \quad \delta F = -\bar{\epsilon} i \bar{D} \psi - \bar{\lambda} \phi,$$

$$\delta \bar{\phi} = \bar{\epsilon} \psi, \quad \delta \bar{\psi} = \bar{F} \bar{\epsilon}, \quad \delta \bar{F} = 0.$$  \hspace{1cm} (2.9)

Here $D_\mu$ is a covariant derivative with respect to the gauge field when acting to a scalar field, while it contains the twisted spin-connection when acting to a spinor field. The lagrangian of matter (2.8) is $Q$-exact

$$L_{\text{matter}} = \left\{ Q, \bar{\eta} \left( -\bar{\psi} F + \bar{\psi} i D \psi + \bar{\phi} \bar{\lambda} \phi \right) \right\}.$$  \hspace{1cm} (2.10)

The total supercharge $Q$ is nilpotent.

It is clear now that any physical correlators with insertions of $Q$-exact operators vanish due to $Q$-invariance of the action. In turn it is easy to see that the left-handed gluino field $\lambda$ is $Q$-invariant and is not $Q$-exact. Therefore there is a natural candidate for the gauge-invariant non-trivial local observable given by the operator

$$Tr \lambda \lambda,$$  \hspace{1cm} (2.10)

where the scalar product is defined with the conventional constant antisymmetric metric $\epsilon_{\alpha\beta}$. However this operator is not quite appropriate for our purpose. The point is that in general it is not a scalar operator. Actually while in the untwisted theory the operator $Tr \lambda \lambda$ is scalar with respect to the usual spin-connection, in a twisted version of the model it acquires a non-trivial axial charge due to coupling of $\lambda$ to the external vector field $V_\mu$ which enters the definition of the paralell transport on the spinor bundle on $M$ and hence it is not a scalar. Actually one can check by a direct calculation that

$$(\partial_\nu + 2V_\nu) (Tr \lambda \lambda) \sigma^a \epsilon^a \bar{\epsilon} = -2i \left\{ Q, Tr \left[ \left( D + \frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu} \right) \lambda \right] \right\}. \hspace{1cm} (2.11)$$

This is not quite that equation which would allow us to prove an independence of the correlators of the operators $Tr \lambda \lambda$ on coordinates. We have to define an operator the usual derivative of which is given by a $Q$-exact operator. Thus one should redefine fields and metric on the spinor bundle in order to eliminate the vector field $V_\nu$ from the left hand side of this. We consider this problem in the next section.

In the sector of fields of matter the local observables are given by gauge invariant combinations of the scalar fields, e.g. $\langle \phi \phi \rangle$ (here $\langle ... \rangle$ stands for a scalar product for a representation $R$ of the gauge group). From eqs.(2.9) we can see that some linear combinations of usual derivatives in coordinates of $\langle \phi \phi \rangle$ are $Q$-exact. Therefore the correlators with insertions of these derivatives of local operators vanish. This is a generalization of supersymmetric Ward identities to the case of curved manifold. Actually the set of local scalar operators depends on the gauge group
and on a representation of the multiplet of matter and can include “mesons” and “baryons” [21]. These local operators correspond to flat directions in the classical moduli space of vacua in SUSY theory.

It is possible to define also non-local $Q$-invariant operators as integrals of gauge invariant operators integrated with some appropriate functions, but we postpone a description of such operators for the next sections.

Notice also that if the structure of the sector of the fields of matter allows for $N=2$ SUSY of the whole untwisted theory in flat space then this SUSY theory can be of course twisted in a different way due to $SU(2)$ global symmetry of $N=2$ SUSY. Such a twisting can be done on any 4D manifold. However we shall not consider this case in the present paper.

3 Observables and Topological Correlators

To understand better eq. (2.11) it is useful to define a complex structure $J^\mu_\nu$ on the manifold $M$ in terms of the (chosen locally to be constant) spinors $\bar{\epsilon}$ and $\bar{\eta}$:

$$J^\mu_\nu = (\bar{\eta}^a \delta^b_\nu) e^\mu_a e^\nu_b.$$  \hspace{1cm} (3.1)

Indeed it is easy to check that

$$D_\lambda J^\mu_\nu = 0$$ \hspace{1cm} (3.2)

where $D_\lambda$ is the nontwisted covariant derivative, and hence $J^{\mu \nu}$ is a closed two-form

$$dJ = 0.$$ \hspace{1cm} (3.3)

Moreover $J^{\mu \nu}$ is anti-selfdual so that the two-form $J$ is harmonic

$$(^* d^* d + d^* d^*) J = 0,$$ \hspace{1cm} (3.4)

and

$$(J^2)_\nu^\mu = -\delta^\mu_\nu.$$ \hspace{1cm} (3.5)

The complex structure allows to define holomorphic and anti-holomorphic coordinates

$$J^m_n \ z^n = i \bar{z}^m, \ J_{\bar{n}}^\bar{m} \ \bar{z}^{\bar{m}} = -i \bar{z}^\bar{m},$$ \hspace{1cm} (3.6)

where $m, n$ and $\bar{m}, \bar{n}$ takes values 1, 2 and $\bar{1}, \bar{2}$ respectively. In the well adapted frame (see, e.g., [20]) the complex structure has the simplest form

$$J^m_n = i \delta^m_n, \ J_{\bar{n}}^{\bar{m}} = -i \delta^{\bar{m}}_{\bar{n}},$$ \hspace{1cm} (3.7)

and hence

$$J^{m \bar{n}} = J^{\bar{m}n} = 0,$$ \hspace{1cm} (3.8)
while the metric has only $g^{mn}$ non-vanishing components. Now we can rewrite eq.(2.11) in an equivalent form in terms of $z_n$ and $\bar{z}_\bar{n}$ coordinates. To this aim let us consider the scalar product of both sides of this equation with a spinor $\bar{\eta}\sigma_k$. Then one can get the following equation

$$e^{-1}\partial_k(e\text{Tr}\lambda\lambda) = 2 \left\{ Q, \text{Tr} \left[ \bar{\eta}\sigma_k \left( D + \frac{1}{2}\sigma^{mn}F_{mn} \right) \lambda \right] \right\}. \quad (3.9)$$

Here $e = \text{det}e^a_m$ while $e^a_m$ stands for components of the vierbein field with both holomorphic world index $m$ and Lorentz index $a$ (we use the tangent frame where the real vierbein field $e^a_m$ is block-diagonal; from now on we shall use the letters $a, b...$ for holomorphic indices and $\bar{a}, ...$ for anti-holomorphic ones). Thus we find that the derivative of the operator

$$O_0 = \exp(-2 \int \bar{z}V_n dz^\bar{n})\text{Tr}\lambda\lambda = e\text{Tr}\lambda\lambda \quad (3.10)$$

in anti-holomorphic coordinates is $Q$-exact. Here we used that the twisting vector field $V_\mu$ locally is a total derivative: $V_\mu = (1/2)\partial_\mu \log \bar{e}$ and $V_{\bar{n}} = -(1/2)\partial_\bar{n} \log e$. However globally on $M$ the exponential in the definition of the operator $O_0$ can depend on the contour of integration, and therefore in general this operator is not well defined globally. As it will be discussed in the next section there could be still an interesting interpretation of the operator $O_0$. There is no such a problem for the local operators constructed of the field $\phi$ of matter since it has no axial charge and, hence, these operators are well defined globally.

Thus if the BRST symmetry does not have any anomalies at the quantum level the correlators of local operators are independent of anti-holomorphic coordinates by the standard arguments.

Notice that for the case of hyperkähler manifold there is also another equation due to the second BRST charge (the twisting field $V_\mu$ can be gauged out in this case). Using the second BRST charge one can show that the correlator of these physical operators does not depend on all coordinates.

Now we want to understand if there is any dependence of physical correlators on external Kähler metric. We observe that the metric enters both the definition of the $Q$ transformation and the action. Furthermore the variation of the action in the metric is not $Q$-exact since the definition of the operator $Q$ depends on the metric. To overcome this difficulty it is necessary to redefine the quantum fields in the theory and to reformulate the definition of the Lagrangian, observables and the BRST charge in terms of sections of holomorphic and anti-holomorphic vector bundles over the manifold $M$.

Let us consider first the sector of the gauge multiplet. Let us introduce the following fields:

$$\chi_n = \bar{\epsilon}\sigma_n\lambda, \quad \bar{\lambda}_{\bar{m}n} = e_a^m e_b^n (\bar{\eta}\sigma_a \bar{\eta}) (\bar{\epsilon}\lambda), \quad \bar{\lambda} = \bar{\eta}\bar{\lambda}, \quad (3.11)$$
\[ D' = D + i J^{\bar{m}n} F_{\bar{m}n}, \]

where \( F_{\bar{m}n} \) is the strength tensor components of the Yang-Mills field. Now the right-handed spinor \( \lambda_\alpha \) is splitted into zero-form \( \lambda \) and the anti-holomorphic two-form \( \bar{\lambda}_{\bar{m}n} \). The left-handed spinor \( \lambda_\alpha \) becomes a section \( \chi_n \) of the holomorphic vector bundle over \( M \). The field \( D \) is still an auxiliary zero-form but is shifted by the strength tensor of the gauge field.

The BRST transformation rules in terms of these forms do not now depend on external metric:

\[ \delta A_n = \chi_n, \quad \delta A_{\bar{n}} = 0, \quad \delta \chi_n = 0, \quad \delta \bar{\lambda} = -D', \quad \delta \bar{\lambda}_{\bar{m}n} = 2i F_{\bar{m}n}, \quad \delta D' = 0. \]

One can see that all the fields are combined into three different supermultiplets: \((A_n, \chi_n), (\bar{\lambda}, D')\) and \((\bar{\lambda}_{\bar{m}n}, F_{\bar{m}n})\). Fixing the ghost number of the BRST charge to be 1 we have the following dimensions \((d)\) and ghost numbers \((G)\) of the fields:

\[ \begin{align*}
(d, G)(A_n) &= (d, G)(A_{\bar{n}}) = (1, 0), \quad (d, G)(\chi_n) = (1, 1) \\
(d, G)(\bar{\lambda}) &= (d, G)(\bar{\lambda}_{\bar{m}n}) = (2, -1), \quad (d, G)(D') = (2, 0).
\end{align*} \]

The conservation of the ghost number is broken by an anomaly.

The Lagrangian for the gauge multiplet reads now as follows

\[ L = \sqrt{g} Tr[F^{\bar{m}n} F_{\bar{m}n} + i \bar{\lambda}_{\bar{m}n} \nabla_m \chi_n + \frac{1}{2} D'^2 + J^{mn}(i D' F_{\bar{m}n} + \bar{\lambda} \nabla_m \chi_n)] = \sqrt{g} \{Q, Tr[-\bar{\lambda} J^{\bar{m}n} F_{\bar{m}n} - \frac{1}{2} D' \bar{\lambda} - i \bar{\lambda}_{\bar{m}n} F^{\bar{m}n}]}, \]

where \( g \) is the determinant of the metric. It is easy to see that a variation of this Lagrangian in external metric is \( Q \)-exact.

Let us consider now the fields of matter. After an appropriate redefinition of the fields of matter we get two scalar fields \( \phi \) and \( \tilde{\phi} \), a scalar fermion \( \tilde{\psi} \), the fermionic fields \( \psi_{\bar{m}} \) and \( \bar{\psi}_{\bar{m}n} \) which are \((0, 1)\) and \((2, 0)\) forms respectively, and the auxiliary \((0, 2)\) and \((2, 0)\) forms \( N_{\bar{m}n} \) and \( \bar{N}^{\bar{m}n} \). The BRST transformations (for matter coupled to the gauge supermultiplet) read now as follows

\[ \begin{align*}
\delta \phi &= 0, \quad \delta \psi_{\bar{m}} = D_{\bar{m}} \phi, \quad \delta N_{\bar{m}n} = D_{\bar{m}} \psi_{\bar{n}} - D_{\bar{n}} \psi_{\bar{m}} + \frac{1}{2} \bar{\lambda}_{\bar{m}n} \phi, \\
\delta \tilde{\phi} &= \tilde{\psi}, \quad \delta \tilde{\psi} = 0, \quad \delta \bar{\psi} = -2 \bar{N}^{\bar{m}n}, \quad \delta \bar{N}^{\bar{m}n} = 0.
\end{align*} \]

The derivative \( D_{\bar{m}} \) and \( D_{\bar{m}} \) are covariant with respect to both external metric and the gauge field. The lagrangian of matter has the following form

\[ L_{\text{matter}} = \sqrt{g}(\bar{\psi} \psi_{\bar{m}} + \bar{\psi} \psi_{\bar{m}} D_{\bar{m}} \phi + \bar{\psi} \psi_{\bar{m}} + N_{\bar{m}n} \bar{N}^{\bar{m}n}) \]
This lagrangian is $Q$-exact

$$L_{\text{matter}} = \{Q, -\frac{1}{2} \bar{\psi}^{\dot{m} \dot{n}} N_{\dot{m}\dot{n}} + \bar{\phi} D_{\dot{m}} \psi_{\dot{m}} - \bar{\phi} \bar{\lambda} \phi \}$$

(3.17)

while the total BRST charge is nilpotent.

Thus the total lagrangian of the gauge multiplet + matter is BRST exact and formally allows for a localization because its variations in the external metric and in the gauge coupling constant are BRST exact. The fields of the multiplet of matter have the following dimensions $(d)$ and ghost numbers $(G)$

$$(d, G)(\phi) = (0, 2), \quad (d, G)(\bar{\phi}) = (2, -2), \quad (d, G)(\bar{\psi}_{\dot{m}}) = (1, 1),$$

$$(d, G)(\bar{\psi}) = (d, G)(\bar{\psi}_{\dot{m}\dot{n}}) = (2, -1), \quad (d, G)(N_{\dot{m} \dot{n}}) = (2, 0), \quad (d, G)(\bar{N}_{\dot{m} \dot{n}}) = (2, 0).$$

(3.18)

The local observable for the sector of the gauge multiplet becomes a $(2,0)$ form (of dimension 2)

$$O_{mn}^{(0)} = Tr_{\chi_m \chi_n},$$

(3.19)

instead of $O_0$ (see eq.(3.10)), which is related to $O_{mn}^{(0)}$ by the following equation

$$O_0 = e Tr_{\chi_m \chi_n} (\bar{\epsilon} \sigma_{ab} \epsilon) e^{am} e^{bn}.$$

(3.20)

It is to be noticed that the situation is different from the ordinary topological theories where the local observables are zero-forms; non-zero forms should usually be integrated over closed cycles to get non-local observables (in the case of highest forms one gets moduli of the topological theory). The difference here is due to the splitting of four coordinates into holomorphic and anti-holomorphic ones, so that the $(2,0)$ form is effectively a scalar with respect to anti-holomorphic derivatives. We have

$$\partial_{\bar{k}} Tr_{\chi_m \chi_n} = \{Q, \ldots \}. $$

(3.21)

It follows from this equation that the physical correlators with insertions of this operator are holomorphic with respect to its coordinate. However we will see that in a semiclassical representation such a correlator is given by an integral over the moduli space of instanton. In turn such an integration could induce singularities in the correlator. Notice also that as we saw in the SUSY version of the theory the analog of the operator $O_{mn}^{(0)}$ in SUSY theory is not in general globally defined. A manifestation of that here is that the operator $O_{mn}^{(0)}$ is a section of a holomorphic bundle over $M$. The correlators with insertions of such an operator can have a nontrivial monodromy on $M$. It is tempting to try to interprete these correlators as a generalization of conformal blocks of 2D conformal theories (which usually have a non-trivial monodromy on 2D manifold; see, e.g. [22]) to the 4D case. We can also
integrate such an operator with an appropriate closed \((0,2)\) form: then we get a well defined non-local operator.

One can construct also the non-local observables using the method of descent equations \([5]\). We have

\[
\partial_{\bar{k}} O^{(0)}_{mn} = \{Q, H^{(1)}_{mn,\bar{k}}\}, \\
\partial_{\bar{p}} O^{(1)}_{mn,\bar{k}} = \{Q, H^{(2)}_{mn,\bar{k}\bar{p}}\}
\]

where

\[
H^{(1)}_{mn,\bar{k}} = Tr(F_{km}\chi_n - F_{kn}\chi_m), \\
H^{(2)}_{mn,\bar{k}\bar{p}} = \frac{1}{2} Tr(F_{km}F_{\bar{p}n} - F_{kn}F_{\bar{p}m} + F_{\bar{p}k}F_{mn}) + i\frac{1}{4}(\partial_{m}Tr\bar{\lambda}_{\bar{p}k}\chi_{n} - \partial_{n}Tr\bar{\lambda}_{\bar{p}k}\chi_{m}).
\]

The operator \(H^{(2)}_{mn,\bar{k}\bar{p}}\) is obviously the density of the topological charge of the gauge field up to an exact form. These relations allows us to construct the following non-local observables

\[
O^{(1)} = \int \bar{\omega}H^{(1)}, \quad O^{(2)} = \int H^{(2)},
\]

where \(\omega\) is a closed \((0,1)\) form. Here we used that the forms \(H^{(1)}\) and \(H^{(2)}\) are closed up to BRST-exact operators according to eqs.\((3.22)\) and \((3.23)\).

The local operators in the sector of matter are given by the same gauge invariant functions of the (dimensionless) scalar field \(\phi\) as in the supersymmetric version of the theory (because this scalar field has zero axial charge) which correspond to flat directions of the classical moduli space of vacua \([21]\). It is also possible to construct non-local operators. In this paper we focus on the BRST-invariant operator

\[
O_{\text{matter}} = \sum_{IJ} a_{IJ} \int E \wedge \langle \psi^I \wedge \psi^J + N^I \phi^J \rangle,
\]

where \(\langle ... \rangle\) stands for a gauge invariant pairing of the fields (its definition depends on a representation of the gauge group for the fields of matter), the indices \(I\) and \(J\) stand for different irreducible multiplets of matter, and \(E\) is a holomorphic \((2,0)\) form on \(M\). \(a_{IJ}\) is a constant matrix (a choice of it depends on a representation of the fields of matter). For example for the case when the multiplet of matter is in an adjoint representation of the gauge group the matrix \(a_{IJ}\) is just 1. For the case of the gauge group \(SU(2)\) and the fields of matter in two copies of a spinor representation \(a_{IJ}\) is antisymmetric.

It is worth noticing that we could use the vector field \(-V_\mu\) for a twisting of the theory. Such a modification of the model corresponds to a change \(\epsilon, \eta \rightarrow \eta, \epsilon\). The local operators in this mirror model are antiholomorphic up to BRST exact operators (for example, \(\partial_n \phi = \{Q, ...\}\)) while their correlators operators are anti-meromorphic.
4 Instanton measure and Ray-Singer torsion

In this section we consider the condition of cancellation of BRST anomaly and discuss the structure physical correlators in semiclassical representation.

Let us consider a calculation of a physical correlator. First we observe that the term \((1/2)Tr \bar{D}^2\) in the Lagrangian for the sector of gauge multiplet is BRST exact and can be (at least formally) taken out from the Lagrangian without any change of the correlator. After this modification one can integrate out over the field \(D'\) that leads to the following constraint

\[
J^{n\bar{m}} F_{mn} = 0. \tag{4.1}
\]

It is easy to see that this condition is necessary for the anti-self-duality of the Yang-Mills field. To show that it is convenient to use the identity \(\varepsilon_{m\bar{n}k\bar{l}} = J_{m\bar{n}} J_{k\bar{l}} - J_{m\bar{l}} J_{k\bar{n}}\). Then one can check that

\[
\varepsilon_{nkm} F^{n\bar{m}} = -F_{\bar{n}k} + J_{\bar{n}k}(J^{\bar{p}q} F_{\bar{p}q}),
\]

\[
\varepsilon_{nkm} F^{km} = 2F_{\bar{n}l}, \quad \varepsilon_{nkm} F^{\bar{n}\bar{m}} = 2F_{\bar{k}m}. \tag{4.2}
\]

In turn the anti-self-duality condition means that

\[
\varepsilon_{nkm} F^{n\bar{m}} = -F_{\bar{n}k}, \quad \varepsilon_{nkm} F^{km} = -2F_{\bar{n}l}, \quad \varepsilon_{nkm} F^{\bar{n}\bar{m}} = -2F_{\bar{k}m}. \tag{4.3}
\]

From these equations it is easy to see that the anti-self-duality condition is equivalent to the following equations:

\[
J^{\bar{p}q} F_{\bar{p}q} = 0, \quad F^{km} = F^{\bar{n}\bar{m}} = 0. \tag{4.4}
\]

Eq.(4.1) coincides with the first of the anti-self-duality conditions. The other conditions in eq.(4.4) appear in the limit of weak gauge coupling constant because in this case the functional integral is dominated by the fields with \(F^{km} = F^{\bar{n}\bar{m}} = 0\) which correspond to the minimum of the action. In turn we can consider this limit since the action of the theory is BRST exact and, hence, the correlators are independent of the value of the coupling constant. Therefore the theory is semiclassical similar to Witten’s theory and the physical correlators can be computed semiclassically in the presence of anti-instanton field. In the presence of anti-instanton field some of fields of the model have zero modes which should be substituted into the preexponential factor in the path integral for correlator (directly or using Yukawa couplings). Moreover one should integrate over quadratic fluctuations near the anti-instanton field. Notice that the classical action for anti-instanton equals to zero.

Let us analyse the zero modes. Actually it is easy to see from eqs.(4.4) that the variation of the self-duality equations in the vector field \(A_n\) (fixing gauge of the variation of the gauge field to be \(D^u \delta A_u = 0\)) gives the following equations

\[
D_{[m} \delta A_{n]} = 0, \quad J^{mn} D_m \delta A_n = 0, \tag{4.5}
\]
\[ D_{[m}A_{n]} = 0, \quad J^{\bar{m}m}D_m \delta A_{\bar{m}}. \]  

(4.6)

We shall see that up to gauge transformations these equations determine the zero modes of the gauge field in the presence of anti-instanton field.

The equations of motion for the field \( \chi_n \) read as follows

\[ D_{[m}\chi_{n]} = 0, \quad J^{\bar{m}n}D_{\bar{m}}\chi_n = 0. \]  

(4.7)

Due to similarity of these equations to eqs. (4.5) we see that the zero modes of the fermionic field \( \chi_n \) and \( A_n \) coincide. Therefore the zero modes of \( \chi_n \) correspond to a half of tangent vectors to the moduli space \( \mathcal{M} \) of anti-instanton because there is no superpartners to \( A_{\bar{n}} \). As we shall see there is a natural complex structure on \( \mathcal{M} \) and the zero modes of \( \chi_n \) correspond to holomorphic tangent vectors on \( \mathcal{M} \).

For a compact manifold \( M \) the moduli space \( \mathcal{M} \) of anti-instanton is a manifold of dimension

\[ d = p_1(G) - \frac{1}{2} \dim G(\chi - \tau), \]  

(4.8)

where \( p_1(G) \) is the first Pontrjagin class of the adjoint bundle over \( M \), \( G \) stands for the gauge group, \( \chi \) is the Euler characteristic of \( M \), and \( \tau \) is the signature of \( M \). In this paper we focus on the case of generic irreducible anti-instanton field and assume that the dimension of the moduli space \( \mathcal{M} \) is given by \( p_1(G) \). For simplicity we consider the case of the gauge group \( SU(2) \) and an anti-instanton field with the Pontrjagin class \( p_1(SU(2)) = 8 \) (the topological charge equals to -1). In this case for a generic Kähler manifold there are 8 zero modes for the gauge field and 4 zero modes for the fermionic field \( \chi_n \) and no zero modes for other fields (see, e.g. citeabc).

Notice that the fermionic field \( \chi_m \) has 4 zero modes but not 8 despite the similarity of eqs. (4.5) and (4.7). The point is that for the gluonic field we should consider the pairs \( (\delta A_m, \delta \bar{A}_m) \) while the fermionic field has only components with holomorphic index. It is easy to show that if \( (\delta A_m, \delta \bar{A}_m) \) is a wave function for a gluonic zero mode then \( (i\delta A_m, -i\delta \bar{A}_m) \) also corresponds to a zero mode because the latter can be rewritten as \( J^\mu_\nu \delta A_\mu \) which satisfies the same equation as \( \delta A_\mu \) since the tensor \( J^\mu_\nu \) is covariantly constant.

Let us consider now the sector of the fields of matter. In the semiclassical limit the path integral is dominated by the contributions from solutions to the equations of motion. In the typical situation for \( SU(2) \) gauge group only the field \( \psi_m \) has a zero mode. The equations of motion for \( \psi_m \) read

\[ D_m \psi_\bar{m} - D_{\bar{m}} \psi_m = 0, \quad D_{\bar{m}}^m \psi_{\bar{m}} = 0. \]  

(4.9)

These equations are similar to those for the zero modes of the gauge field but the field of matter \( \psi \) can belong to an arbitrary representation \( \bar{R} \) of the gauge group (not necessarily to an adjoint one). The number of zero modes of the field \( \psi \) depends on the representation \( \bar{R} \) and is determined by the index theorem for a corresponding
Dirac operator. For example for the gauge group $SU(2)$ and spinor representation $R$ there is generically a single zero mode \[24\].

Let us consider the integration over quadratic fluctuations around the anti-instanton field. The result of such an integration provides us with a combination of determinants of the Laplace-type operators.

Let $M$ be a compact Kähler manifold without boundaries and let $G$ be a compact, finite dimensional Lie group. Let $\mathcal{D}^{p,q}$ denote the space of $C^\infty$ complex $(p,q)$ forms on the Kähler manifold $M$ with values in the direct product of a $G$ vector bundle $L_G$ and a flat holomorphic vector bundle $L(\zeta)$ associated with a finite dimensional representation $\zeta$ of the fundamental group $\pi_1(M)$ of the manifold $M$ \[24\].

Let us consider the anti-instanton gauge field $B_\mu$ obeying eqs.(4.4). Then we can introduce the exterior derivatives

$$ D : \mathcal{D}^{p,q} \to \mathcal{D}^{p+1,q}, \quad (4.10) $$

$$ \bar{D} : \mathcal{D}^{p,q} \to \mathcal{D}^{p,q+1}, $$

where $D$ and $\bar{D}$ are the covariant derivatives in the presence of gauge field. The operators $D$ and $\bar{D}$ depend on a flat connection on $L(\zeta)$. Obviously,

$$ D^2 = 0, \quad \bar{D}^2 = 0. \quad (4.11) $$

However $\{D, \bar{D}\} \neq 0$ for a non-trivial gauge field. Therefore $D + \bar{D}$ is not nilpotent and hence does not generate any real complex \[3\].

Let us now fix the background field gauge in the theory with the Lagrangian (3.14) by introducing the following terms

$$ L_{gf} = \frac{1}{2} \text{Tr}(D^m A_m + D_m A^m)^2 + \text{Tr} c^+ D^n D_n c, \quad (4.12) $$

where $A_m$ and $A^m$ are components of quantum part of the gluonic field, while $c^+$ and $c$ stand for the ghost fields (here we used that $F^m_m = 0$).

With this gauge fixing it is easy to check that the quadratic form for the gluonic field is given by

$$ L_{qu} = -2 \text{Tr} A^n (D^m D_m A_n - [D^m, D_n] A_m). \quad (4.13) $$

The differential operator in this expression can be represented using the Hodge star operator and exterior derivatives $D$ and $\bar{D}$ introduced above as the Laplace type operator $\Delta_{1,0}$ acting on $\mathcal{D}^{1,0}$, where

$$ \Delta_{p,q} = \star \bar{D}^* D + D^* \bar{D}. \quad (4.14) $$

\[3\] Notice however that a modified real complex can be defined \[20\].
The operator acting to the fermionic field $\chi_n$ reads as $D^* \bar{D}^*$ so that the relevant Laplace operator for the fermionic sector is $\Delta_{1,0} = (D^* \bar{D}^*)^2$.

Integrating over the non-zero modes of gluon field and fermions and over ghosts in the one-loop approximation we get the following ratio of the determinants of Laplace operators

$$Z = \det \Delta_{0,0} \det' \Delta_{1,0}^{-1/2}, \quad (4.15)$$

where $\det'$ stands for the determinant without zero eigenvalues. Here $\det \Delta_{0,0}$ corresponds to integration over ghost fields, and $\det' \Delta_{1,0}^{-1/2}$ comes from integration over fermions ($\det' \Delta_{1,0}^{1/2}$) and gluons ($\det' \Delta_{1,0}^{-1}$). It is worth emphasizing that the boundary conditions for fermionic fields are changed as compared to those in the untwisted SUSY theory and are the same as in the bosonic sector.

In the presence of non-flat external metric the non-zero modes of the fields of bosonic and fermionic fields of SUSY multiplet are not paired up. Therefore the dependence on metric of the ratio of determinants in eq.(4.15) can be non-trivial. This dependence can come also from integration over regulator fields. The corresponding contribution is an anomaly which can spoil the BRST invariance at the quantum level.

In the case of vanishing gauge field this combination (4.15) is given by the Ray-Singer torsion $T_2(M, \zeta)$ for the Dolbeault cohomology complex defined as

$$\log T_p(M, \zeta) = \frac{1}{2} \sum_{q=0}^{2} (-1)^q T_{p,q}(M, \zeta), \quad (4.16)$$

where

$$T_{p,q}(M, \zeta) = \int_{\epsilon}^{\infty} \frac{dt}{t} \text{Tr} \left( e^{t\tilde{\Delta}_{p,q}} - P_{p,q} \right), \quad (4.17)$$

where $\tilde{\Delta}_{p,q} = D^* \bar{D}^* + \bar{D}^* D^*$ and $\epsilon \to 0$ stands for a parameter of the ultraviolet cutoff. One of the important properties of the Ray-Singer torsion is that the ratio $T_p(\zeta_1)/T_p(\zeta_2)$ does not change under purely Kählerian deformations of the external Kähler metric, i.e. under deformations which do not change its Kähler class.

We want now to analyze the dependence of $Z$ on the external Kähler metric in the presence of non-trivial (anti-instanton) gauge field. To this end let us consider a more general gauge field $B_\mu$ obeying $F_{m\bar{n}} = 0$ and $\bar{F}_{\bar{m}\bar{n}} = 0$ while the value $g^{m\bar{n}} F_{m\bar{n}}$ can be non-zero. Then we can still define two cohomology complexes with the exterior derivatives $D$ and $\bar{D}$, respectively, since these derivatives are nilpotent. It is crucial that in this case the external gauge field can be assumed to be independent of external metric.

We define a generalized torsion $\log \tilde{T}_p(M, \zeta, B, Ad)$ in the presence of the gauge field by the same formula as for the case of vanishing gauge field ($Ad$ stands for

4The idea of a possibility of a generalization of the formalizm of Ray and Singer in the presence of instanton field was suggested by A.Lossev
the adjoint representation of the gauge group). For our problem we are interested in $\hat{T}_2$. Indeed $\hat{T}_2 = \det \Delta_{2,2}/(\det \Delta_{2,1})^{1/2}$. By duality we have $\Delta_{2,2}^* = \Delta_{0,0}$ and $\Delta_{2,1}^* = \Delta_{1,0}$. Hence $\hat{T}_2 = Z$.

Let us consider a variation of this torsion under a smooth deformations of the metric $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$. Using a nilpotence of the operators $D$ and $\bar{D}$ it is easy to check [25] the following identity

$$\delta \sum_{q=0}^{2} (-1)^q T_{p,q} = \sum_{q=0}^{2} (-1)^q \int_\epsilon^\infty \frac{d}{dt}(\alpha e^{t\Delta_{p,q}})dt =$$

$$= \sum_{q=0}^{2} (-1)^q \text{Tr}(\alpha e^{t\Delta_{p,q}})|_\epsilon^\infty,$$

where

$$\alpha = -1 \delta \star$$

and $\star$ is Hodge star. From the above equation we get

$$\delta \hat{T}_p = -\frac{1}{2} \sum_{q=0}^{2} (-1)^q \text{Tr}(\alpha e^{t\Delta_{p,q}}) + \frac{1}{2} \sum_{q=0}^{2} (-1)^q \text{Tr}(\alpha P_{p,q}).$$

(4.20)

Here $P_{p,q}$ is a projector to $(p,q)$ zero modes. For $(1,0)$ zero modes (which are related by duality to zero modes of $\Delta_{2,1}$) we have

$$P_{m\bar{m}}(x, y) = A^i_m(x)P^{-1}_{ij}\bar{A}^j_{\bar{m}}(y)\sqrt{g(y)}$$

(4.21)

where

$$P^{ij}_{p,q} = \int d^4x\sqrt{g}g^{m\bar{m}}\text{Tr}A^i_m\bar{A}^j_{\bar{m}}$$

(4.22)

and $A^i_m$ and $\bar{A}^j_{\bar{m}}$ stand for zero modes of the Laplace operator in external gauge field. Notice that $P_{m\bar{m}}$ is a matrix in the adjoint representation of the gauge group. As it was mentioned above we assume that there are only zero modes for $(1,0)$ and $(0,1)$ sectors (and by duality in the $(2,1)$ and $(1,2)$ sectors) while there are no zero modes in sectors $(0,0)$, $(0,2)$ and $(2,0)$.

The first term in eq.(4.20) is an ultraviolet contribution while the second one corresponds to the infrared domain. This formula is analogous to the anomalous equation for the divergence of the axial current where the ultraviolet contribution corresponds to the axial anomaly while the infrared contribution comes from the ‘soft’ divergence induced by the mass terms for fermions.

The anomalous ultraviolet contribution in eq.(4.20) can be represented as follows

$$\delta S_{gr} + V_{mix}[A_m, A_{\bar{m}}, g_{m\bar{m}}].$$

Here $\delta S_{gr}(g)$ does not depend on the gauge field. It is easy to see that $\zeta$ does not enter this ultra-violet contribution because the corresponding connection on $L(\zeta)$ is
flat. (A non-trivial dependence on $\zeta$ can appear only from infrared contributions). It is obvious that $S_{\text{gr}}$ is proportional to the dimension of representation of fields over which we integrate in the path integral. In the present case this is adjoint representation. We denote $S_{\text{gr}}$ as $\dim G \log T_p(0, M)$. If there is no zero modes in the absence of the gauge field and for a trivial representation of the fundamental group (the corresponding connection equals to zero) then $T_p(0, M)$ is a particular Ray-Singer torsion.

If we normalize the path integral dividing it by the same path integral without external gauge field then we get the following factor which does not depend on the gauge field

\[
\left( \frac{T_2(0, M)}{T_2(\zeta, M)} \right)^{\dim G}.
\]

It can be shown that this factor depends only on the Kähler class of the metric [25].

The second part of the variation $V_{\text{mixed}}[A_m, A_{\bar{m}}, g_{m\bar{m}}]$ is a local functional of the external gauge field because this contribution is determined by ultra-violet contributions. Therefore its form can be determined by a calculation for a gauge field for which the Laplace operator has no zero modes, in particular for small values of the gauge field (however we still have to assume that $(2, 0)$ and $(0, 2)$ components of the strength vanish). It is useful to compare this situation with the supersymmetric version of the theory (with twisted covariant derivative) where two anomalies are present. The first one is a conformal anomaly which is proportional to $\int \text{Tr}^a F \wedge F \log g$ while the second anomaly appears due to coupling of the anomalous axial current to the external vector field $V_\mu$. The axial anomaly gives the following term in the effective action $\log Z$

\[
\frac{1}{16\pi^2} \int \text{Tr}_{Ad} F \wedge F \frac{1}{\Delta} D_\mu V_\mu = \frac{1}{32\pi^2} \int \text{Tr}_{Ad} F \wedge F \log e/\bar{e}.
\]

(4.23)

where we used that locally the vector field $V$ is a total derivative. Here the trace $\text{Tr}_{Ad}$ is taken in an adjoint representation while the generators $t^a$ of the gauge group are normalized as $\text{Tr} t^a t^b = C_G \delta^{ab}$.

For the twisted theory formulated in terms of sections of holomorphic and antiholomorphic vector bundles over $M$ the anomaly can be directly calculated by integration of the ultraviolet contribution in eq.(4.20). For $p = 2$ we get the following expression for $V_{\text{mixed}}$ (notice that $\alpha$ vanishes in the sector of $(2, 0)$ forms)

\[-\frac{1}{16\pi^2} \text{Tr}_{Ad} [g^{\bar{m}m} \delta_{\bar{n}l} (-F^{k\bar{l}} F_{kn} + F_{p}^{\bar{p}} F_{\bar{p}n}^k) - g^{\bar{m}m} \delta_{\bar{l}n} (-F^{k\bar{l}} F_{kn} + (F_{p}^{\bar{p}})^2)].\]

In general it is not clear whether this anomaly integrable for an arbitrary variation of metric. Let the variation of metric be purely Kählerian, i.e. $\delta g_{m\bar{m}} = \partial_m \omega_{\bar{m}} + \partial_{\bar{m}} \omega_m$ where $\omega_m$ and $\omega_{\bar{m}}$ are $(1, 0)$ and $(0, 1)$ forms so that the Kähler forms $J$ and $J + d\omega$ belong to the same cohomology class in $H^2(M, R)$. Then this variation can
be integrated and we get the following expression for an anomalous contribution to the effective action \( \log Z \)

\[-\frac{1}{32\pi^2} \int \text{Tr}_{\text{Ad}} F \wedge F \log g. \]  \hspace{1cm} (4.24)

The anomaly in this form takes into account the change of the path integral measure when we translate the supersymmetric theory into the twisted one.

For external anti-instanton field this term in the effective action obviously mixes the dependence of metric and of moduli of instanton. Hence to get a topological theory we have to cancel this anomaly by contributions of matter in an appropriate representation of the gauge group.

Let us consider now the infrared contribution. It is given by an expression

\[
\text{Tr}_{\text{Ad}} \int \sqrt{g} (g^{nm} \delta g_{km} - \delta^m_k g^{mn} \delta g_{mn}) P_n(x,x).
\]

This part of variation is also integrable. To show that let us demonstrate that the zero modes obey the following conditions

\[
DA^i = 0, \quad \bar{D} \bar{A}^i = 0, \quad * \bar{D}^* A^i = 0, \quad * \bar{D}^* \bar{A}^j = 0. \hspace{1cm} (4.25)
\]

where the indices \( i, j \) label zero modes of the gauge field.

The equation for zero mode can be read off from the quadratic form for fluctuations near the background field \( B_\mu \). We have

\[
(* \bar{D}^* D + D^* \bar{D}^*) A^i = 0. \hspace{1cm} (4.26)
\]

Let us split the wave function for zero mode as follows

\[
A^i = \bar{A}^i + D\phi^i, \hspace{1cm} (4.27)
\]

where \( \phi^i \) is a scalar field and

\[
* \bar{D}^* \bar{A}^i = 0. \hspace{1cm} (4.28)
\]

Then we easily get from eq.(4.26)

\[
* \bar{D}^* D \phi^i = 0. \hspace{1cm} (4.29)
\]

Since we assume that there are no non-trivial scalar zero modes (we consider the gauge fields which are relatively close to the anti-instanton field with respect to a natural metric on the space of gauge fields while the anti-instanton field is assumed to be irreducible) we immediately get \( \phi = 0 \). In turn the field \( \bar{A}^i_n \) obeys the following equation

\[
* \bar{D}^* (D \bar{A}^i) = 0. \hspace{1cm} (4.30)
\]
Acting to the left hand side of this equation by the operator $D$ we get
\[ \Delta_{2,0}(D\bar{A}^i) = 0, \] (4.31)
which is equivalent to the condition that $\bar{A}^i$ is closed
\[ D\bar{A}^i = 0 \] (4.32)
because we assumed that there is no zero modes in the $\cal D^{2,0}$ space. Thus $A^i = \bar{A}^i$. Similarly one can show that $D\bar{A}^i = * D^*\bar{A}^i = 0$. Thus we can say that $A^i$ and $\bar{A}^i$ are “harmonic”.

Let us now consider the variation in metric of the conditions $D^n A^i_n = 0$ and $D_m A^i_n - D_n A^i_m = 0$. They read
\[ D^n \delta A^i_n + \delta g^{nm} D_m A^i_m = 0, \] (4.33)
\[ D_m \delta A^i_n - D_n \delta A^i_m = 0. \]
Notice that in the case of Kähler metric we should not vary the covariant derivatives because the corresponding gravitational connection vanishes and the external gauge field is assumed to be independent of metric. The solution to these equations with respect to $\delta A^i_n$ is given by
\[ \delta A^i_n = D_n \Phi^i + c^i_j A^j_n, \quad \delta A^i_n = D_n \bar{\Phi}^i + \bar{c}^i_j A^j_n. \] (4.34)
Here $\Phi^i$ and $\bar{\Phi}^i$ are complex (conjugated to each other) scalar fields,
\[ \Phi^i = -\frac{1}{D^n D_m} \delta g^{nm} D_m A^i_n, \] (4.35)
and $c^i_j$ and $\bar{c}^i_j$ are the components of a (real) connection on the bundle $\cal K \times \cal G$ where $\cal K$ and $\cal G$ are the spaces of zero modes of gauge field and of Kähler metrics consistent with the complex structure on $M$, respectively. An integrability condition for eqs.(4.34) is equivalent to
\[ D\phi = 0, \quad [\cal D, \cal D] = 0, \] (4.36)
where $\cal D^i_j = \delta^i_j \delta - c^i_j$. Hence $c^i_j$ is a flat connection and can be locally included into normalization of zero modes. Thus the infrared variation of the torsion can be locally represented as follows
\[ \text{Tr}_{Ad}(\alpha P_{1,0}) = \delta \log \det P_{1,0}^{ij}. \] (4.37)
Therefore this infrared variation is locally integrable and we get for the torsion the following expression
\[ Z = \frac{\hat{T}_2(M, \zeta, B, Ad)}{\hat{T}_2(M, \zeta, 0, Ad)} = \] (4.38)
Here $Y_G$ is a factor which does not depend on external metric $g$. The factor $(T_2(0, M)/T_2(\zeta, M))^{\dim G}$ does not change under smooth deformations of the Kähler metric which do not change its Kähler class. If the infrared contribution is not globally integrable this torsion is a section of a line bundle over $\mathcal{G}$.

Let us consider now the sector of fields of matter in the representation $R$ of the gauge group. For the fields of matter in the semiclassical approximation get the following expression

$$Z_R = \det \hat{\Delta}_{0,0}^{-1} \det ' \hat{\Delta}_{1,0}^{1/2} = \det \Delta_{0,0}^{-1} \det ' \Delta_{1,0}^{1/2}. \tag{4.39}$$

Here we used the fact that we can represent the determinant $\det'\Delta_{1,0}$ as a path integral over $(1,0)$ and $(0,1)$ forms with a bilinear action (with operator $\Delta_{1,0}$ in the quadratic form) and integrate by parts. Then we get an operator $\hat{\Delta}_{0,1}$ while no boundary terms appear under integration by parts on a compact manifold without boundaries. Hence their determinants coincide. The same argument can be used for $(0,0)$ sector.

Thus we can represent this ratio of determinants as a generalized torsion $\hat{T}_2(M, \zeta, B, R)$ for the representation $R$. It is defined by eq.(4.16) with the covariant derivatives in the representation $R$. By the same arguments as for the adjoint representation one can show that the ultraviolet and infrared contributions to the variation in metric of $Z_R$ are integrable separately. Thus we get for $Z_R$ the following expression

$$Z_R = \frac{\hat{T}_2(M, \zeta, 0, R)}{\hat{T}_2(M, \zeta, B, R)} = f_R \left( \frac{T_2(\zeta, M)}{T_2(0, M)} \right)^{\dim R} \left( \det \hat{P} \right)^{1/2} \exp \left( \frac{1}{32\pi^2} \int \text{Tr}_R F \wedge F \log g \right), \tag{4.40}$$

where the trace $\text{Tr}_R$ is taken in the representation $R$ while the generators $t^a$ of the gauge group are normalized as $\text{Tr} t^a t^b = C_R \delta^{ab}$; $f_R$ is a factor independent on external metric, $\dim R$ is a dimension of a representation of the multiplet of matter (it can be reducible) and $\hat{P}$ is a matrix of bilinear integrals of fermionic zero modes (we assume that there is no scalar field zero modes, i.e. the instanton field is irreducible). In the simplest case when only $(0,1)$ zero modes exist we have

$$\hat{P}^{IJ} = \int d^4 x \sqrt{|g|} g^{mn} < \bar{\psi}^I_m \psi^J_m >, \tag{4.41}$$

where $\psi^I_m$ is a zero mode wave function (labelled by an index $J$) while $\bar{\psi}^I_m$ is a complex conjugated zero mode.

Notice that though $Z$ and $Z_R$ are the ratios of the generalized torsions they can depend on a representative of the Kähler class of external metric in contrast to the case of absence of external gauge field [25]. The point is that the argument of Ray and Singer in ref.[25] is essentially based on a nilpotence of the operator $D + \bar{D}$ while it is not nilpotent in the presence of external gauge field.
The total contribution of non-zero modes reads

$$ZZ_R = \frac{\hat{T}_2(M, \zeta, 0, R) \hat{T}_2(M, \zeta, B, Ad)}{\hat{T}_2(M, \zeta, B, R) \hat{T}_2(M, \zeta, 0, Ad)}.$$ 

Now we can see that the mixed anomaly is cancelled if

$$C_G - C_R = C_G - \sum_i C_{R_i} = 0,$$  \quad (4.42)

where $R = \sum_i R_i$, $R_i$ are irreducible representations of the gauge group. The condition of cancellation of the Ray-Singer torsion is the following

$$\dim G - \sum_i \dim R_i = 0.$$  \quad (4.43)

However since the Ray-Singer torsion does not depend on the gauge field and can depend only on the Kähler class of the metric we do not impose the condition (4.43). Thus the whole physical correlator does not change under smooth variations of external Kahler metric which do not change its Kähler class if the mixed anomaly is cancelled. The dependence on the Kähler class is factorized out into a power of a ratio of two Ray-Singer torsions.

The ultraviolet contribution to the generalized torsions has been shown to be integrable under an assumption that the variation of metric is purely Kählerian, i.e. it does not change the Kähler class of metric. Actually it is easy to check that the cancellation of a mixed anomaly does not depend on this assumption if the condition (4.42) is satisfied.

There is also another restriction to the sector of matter: the theory should not have (both local and global) gauge anomalies.

Notice that if the multiplet of matter is in the adjoint representation of the gauge group ($C_G = C_R$) both the anomaly and the infrared contributions are cancelled in the product $ZZ_R$ and we get Witten’s theory.

Now we can substitute the anti-instanton field into the expressions for $Z$ and $Z_R$ and study the physical correlators. If $t^i$ are coordinates on the moduli space the wave functions for zero modes of gauge field can be represented as follows [17]

$$A^i_\mu = \partial^i B_\mu + D_\mu \epsilon^i,$$  \quad (4.44)

where $\epsilon(t^i, x)$ are gauge parameters ($x \in M$), $\partial^i = \partial/\partial t^i$ and $B_\mu$ stands for anti-instanton field. This parameter $\epsilon^i$ plays a role of a natural connection on the moduli space $M$ [17].

It is useful to introduce a natural metric on $M$ which is induced by a metric on $M$ [27]:

$$\hat{G}^{ij} = \int_M \sqrt{g} \text{Tr} A^i_\mu A^{\mu j}.$$  \quad (4.45)
It is convenient to define also the Kähler form
\[ \hat{J}^{ij} = \int d^4x \sqrt{g} J^{\mu\nu} \text{Tr} A_i^\mu A_j^\nu. \] (4.46)

Moreover we can define a natural complex structure on \( \mathcal{M} \) induced by the complex structure on \( M \) :
\[ \hat{J}^i_j = \hat{j}^{il} \hat{G}^{lj}. \] (4.47)

It is easy to check that there is the following relation between the complex structures on \( M \) and \( \mathcal{M} \)
\[ \hat{J}^i_j A^j_\mu = J^\nu_\mu A^i_\nu. \] (4.48)

This means that the only non-vanishing components of the wave functions of zero modes for the gauge field are \( A^i_\mu \) and \( \bar{A}^\bar{k}_\bar{\mu} \), where the indices \( i, \bar{k} \) are holomorphic and anti-holomorphic, respectively. From now on we shall use the letters \( i, j, ... \) for holomorphic and \( \bar{k}, \bar{l}, ... \) for anti-holomorphic indices on \( \mathcal{M} \).

The metric on the moduli space turns out to be Kähler one so that in the well adapted frame the metric has only components with one holomorphic (\( i \)) and one anti-holomorphic indices (\( \bar{k} \))
\[ \hat{G}^{ik} = P^{ik} = \int d^4x \sqrt{g} g^{m\bar{m}} \text{Tr} A_i^m \bar{A}^{\bar{k}}_{\bar{m}}. \] (4.49)

With respect to this complex structure the coordinates on the moduli space can be splitted into holomorphic and anti-holomorphic ones, \( t^i \) and \( \bar{t}^{\bar{k}} \) respectively. The wave functions for zero modes can be represented as follows \( A^i_\mu = \partial^i B_n + D_n \epsilon^i \) and \( \bar{A}^{\bar{k}}_{\bar{\mu}} = \partial^{\bar{k}} \bar{B}_n + D_n \epsilon^{\bar{k}} \), where \( B_\mu = (B_n, \bar{B}_n) \) is the anti-instanton field. It follows from eq.(4.48) that \( \partial^k B_n = -D_n \epsilon^k \) and \( \partial^{\bar{k}} \bar{B}_n = -D_n \epsilon^{\bar{k}} \).

The Kähler form \( \hat{J}^{ik} \) is closed. This can be easily seen if we locally (on \( \mathcal{M} \)) fix the gauge of the anti-instanton field by a condition \( \epsilon^i = \epsilon^{\bar{k}} = 0 \). Then \( \hat{G}^{ik} = \partial_i \partial^{\bar{k}} \int_M \sqrt{g} \text{Tr} B_n \bar{B}^n \). However in general the Kähler potential \( \hat{K} (\hat{G}^{ik} = \partial^i \partial^{\bar{k}} \hat{K}) \) is a non-trivial functional of the instanton field [17]
\[ \hat{K} = \frac{1}{2} \int_M \mathcal{F} \wedge J - \log \int_M J \wedge J, \] (4.50)
where \( \mathcal{F} \) is a solution to the following equation
\[ \text{Tr} F \wedge F = i \partial \bar{\partial} \mathcal{F}. \] (4.51)
Here \( \partial \) and \( \bar{\partial} \) are the Dolbeault operators on \( M \).

It is easy to see that the Kähler class of \( \hat{J}^{ik} \) depends only on the Kähler class of the Kähler form \( J \) on \( M \). Indeed in the case of Kählerian variation of metric \( \delta J = \partial \bar{\omega} + \bar{\partial} \omega = d\Omega \) (\( \omega \) and \( \bar{\omega} \) are (1,0) and (0,1) forms on \( M \), respectively) we have
\[ \delta \hat{J}^{ik} = \partial^i \hat{\omega}^k - \partial^k \hat{\omega}_i, \] (4.52)
where
\[ \hat{\omega}^k = \int_M \left( J \wedge \text{Tr} B \wedge \partial^k B + \Omega \wedge \text{Tr} F \wedge \partial^k B \right), \quad (4.53) \]
\[ \hat{\omega}^i = \int_M \left( J \wedge \text{Tr} B \wedge \partial^i B + \Omega \wedge \text{Tr} F \wedge \partial^i B \right). \]
In eqs.(4.53) we used the total field \( B^\mu = (B_n, \bar{B}_n) \). Here \( \delta B^\mu \) is a variation of \( B^\mu \) in metric and we used that the Kähler form \( \hat{J} \) (eq.(4.46)) can be rewritten as follows
\[ \hat{J}^i \bar{k} = \frac{1}{2} \partial^i \int \sqrt{g} J^\mu \nu \text{Tr} B^\mu \partial^{\bar{k}} B^\nu - \frac{1}{2} \partial^{\bar{k}} \int \sqrt{g} J^\mu \nu \text{Tr} B^\mu \partial^i B^\nu, \]
and the condition \( J \wedge F = 0 \).

It is worth noticing that the connection \( \epsilon^i (\epsilon^k) \) is holomorphically (anti-holomorphically) flat. Indeed with the complex structure on \( M \) defined above we can see that the integrability condition for the equation \( \partial^i \bar{B}_n = -D_n \epsilon^i \) reads
\[ \bar{D}(\partial^i \epsilon^j - \partial^j \epsilon^i + i[\epsilon^i, \epsilon^j]) = 0. \quad (4.54) \]
This equation is equivalent to
\[ \partial^i \epsilon^j - \partial^j \epsilon^i + i[\epsilon^i, \epsilon^j] = 0 \quad (4.55) \]
because we assume that the anti-instanton field is irreducible. Moreover
\[ (\partial^i + i \epsilon^i) \bar{A}^k = \bar{D} \Phi^{ik}, \quad (\partial^k + i \epsilon^k) A^i = -D \Phi^{ik}, \]
where \( \Phi^{ik} = \partial^i \epsilon^k - \partial^k \epsilon^i + i[\epsilon^i, \epsilon^k] \). Notice that the strength \( \Phi^{ik} \) can be non-zero.

We can now write down the integration measure on the moduli space \( \mathcal{M} \) for the correlator. Taking the measure \( \prod_{x, \mu, a} dA^\mu_a(x) \) on the space of gauge connections modulo gauge transformations we get an induced measure on the moduli space
\[ \prod_i dt_i \prod_k dt_{\bar{k}} \det P, \quad (4.56) \]
where the matrix \( P^{ik} \) is evaluated for the anti-instantonic background field and corresponds simply to the Jacobian of the change of variables in the path integral. However there is a contribution from integration over non-zero modes. We actually calculated it through its variation in metric. If the condition of cancellation of anomaly fulfilled then the correct measure reads
\[ \prod_i dt_i \prod_k dt_{\bar{k}} (\det P \det P)^{1/2} \left( \frac{T_2(0, M)}{T_2(\zeta, M)} \right)^{\dim G - \dim R} Y_G f_R \quad (4.57) \]
We take into account here that the classical action equals to zero for instanton configuration. The integrand eq.(4.57) stands for a generalization of square root of a determinant of real metric on the moduli space.
The physical correlator is given now by an integral over $M$ of a product of the operators where the quantum fields are replaced by the normalized zero modes (a normalization factor comes from the Jacobian of a change of variables for fermionic zero modes in the fermionic path integral) and the instanton measure. Let us consider for example the gauge group $SU(2)$ and the matter in the representation given by four copies of a spinor representation. Then the mixed anomaly is cancelled because $C_G = 2$ and $C_R = 4 \times 1/2 = 2$, while the whole correlator is proportional to $T_2(\zeta, M)/T_2(0, M)$. The fermions of matter have altogether 4 zero modes which are (at least naively) $(0,0)$ forms on the moduli space (generically each fermionic doublet $\psi_n$ has a single zero mode). We take a physical operator for the sector of matter defined by eq.(3.25) with a non-degenerate antisymmetric matrix $a^i_{ij}$ and for the sector of the gauge multiplet we take $O^{(0)}_{mn}(x_1)O^{(0)}_{kl}(x_2)$ ($x_{1,2}$ are coordinates on $M$). Then the physical correlator reads as follows (we substitute the zero modes $A^n_i$ for the fields $\chi_n$)

$$
\frac{T_2(\zeta, M)}{T_2(0, M)} \int_M \text{Tr}(A_m \wedge A_n)(z_1) \wedge \text{Tr}(A_k \wedge A_l)(z_2)(\int E \wedge \psi^0 \wedge \psi^0)^2 Y_G f_R, \tag{4.58}
$$

where $\psi^0$ is a zero mode wave function of fermionic field $\psi^I$ (the same for all 4 copies of a spinor representation of the gauge group). Notice that the wave functions for zero modes in eq.(4.58) are not normalized because they absorbed the factor $(\det P \det \tilde{P})^{1/2}$.

With the complex structure on $M$ the operator constructed of the fields $\chi_n$ becomes a $(4,0)$ form on the moduli space (notice that $A^i$ are $(1,0)$ forms on $M$). To keep the covariance on the moduli space with respect to the Kähler metric we need to get for a correlator a volume $(4,4)$ form integrated over all the moduli space. That means that the product of zero modes of the fields of matter times $Y_G f_R$ should transform as $(0,4)$ form on the moduli space. However it is not clear in general case of an arbitrary irreducible representation of the gauge group how do the zero modes of matter transform with respect to the moduli space, i.e. whether they are forms or section of a spinor bundle over $M$. The number of zero modes in general case is given by the index theorem while an unravelling of the geometrical properties of these zero modes is an interesting problem.

5 Heterotic topological model as a deformation of Witten’s theory

It is possible however to show that the integrand in eq.(4.58) is indeed a $(4,4)$ form on the moduli space. To this end we shall deduce the same result using a different ultraviolet regularization. This approach will be also more transparent for the question of dependence of the physical correlator on external metric. Let us start
with Witten’s theory which of course does not have any anomalies. It is equivalent to our theory with matter in the adjoint representation of the gauge group. We can consider a deformation of such a theory by the mass term operator

$$\Lambda \{ Q, \int S \wedge \text{Tr} \bar{\psi} \phi \} + \Lambda \int E \wedge \text{Tr}(\psi \wedge \psi + N\phi),$$  

(5.1)

where $S$ is an arbitrary (non-singular) (0,2) form, $E$ is a holomorphic (2,0) form and $\Lambda$ is a parameter (a mass). The fields of matter are in an adjoint representation of the gauge group. This deformation is $Q$-closed and does not spoil $Q$ invariance of the theory.

In Witten’s theory the physical correlator with two insertions of this operator can be interpreted up to a constant factor as a correlator without such insertions in a deformed theory with the mass operator for the fields of matter (eq.(5.1)) with small coefficient (we change $\Lambda \to m$ in eq.(5.1)) $m \to 0$. In semiclassical approximation the deformed theory with $\Lambda \to \infty$ differs from Witten’s one by a factor which is a ratio of two path integrals for the fields of matter with masses $\Lambda$ and $m$, $Z(\Lambda)/Z(m)$, where $Z(m)$ and $Z(\Lambda)$ stand for these two path integrals. It is clear that such a ratio is well defined since the fields with mass $\Lambda$ can be understood as the regulator fields for the fields with mass $m$. In turn one can see that when evaluated in the external anti-instanton field this ratio is (0,0) form on the moduli space because we integrate over zero modes both in the nominator and in the denominator. Furthermore the ratio $Z(\Lambda)/Z(m)$ does not depend on external metric because the variation in metric of both non-deformed and deformed lagrangians is $Q$-exact. However this ratio depends on a choice of (2,0) form $E$.

The variation of $\log Z(\Lambda)/Z(m)$ in the (2,0) form $E$ can be easily calculated. It is worth noticing that the form $E$ can have zeros. Therefore for this calculation it is useful to regularize $E \to E_r$ where $E_r$ does not have zeros, and to put $S = E^{-1}$. At the end of calculations we have to switch off the regularization. In the case $F_{mn} = F_{\bar{m}n} = 0$ we get for the variation

$$\frac{1}{32\pi^2} \text{Tr}_A(dF_n^{\bar{m}} F_{\bar{m}m} - (F_n^m)^2) \delta \log \det E - \frac{1}{2} \delta \log \det \hat{E}. $$

Here

$$\hat{E}^{\bar{k}l} = \int E \wedge \text{Tr}_A \psi^k \wedge \psi^l, $$

(5.2)

and $\psi^k$ stands for a normalized wave function for $\bar{k}$ zero mode of fermionic field of matter (the zero modes are normalized by a factor $(\det \text{Tr}^* \bar{\psi}^{\bar{k}} \psi^k)^{1/2}$ where $\bar{\psi}^{\bar{k}}$ is complex conjugated to $\psi^k$). Notice that in the case of the anti-instanton background $\psi^k$ is antiholomorphic vector on $\mathcal{M}$. The first term in eq.(5.2) is the ultraviolet contribution from the regulator fields (with mass $\Lambda$) while the second one corresponds to the contribution of fields with mass $m$. Notice that in the limit $m \to 0$ only zero modes contribute to the infrared part of the variation.
This variation can be integrated up to a factor $\tilde{f}_G$ which does not depend on $E$. Thus we get
\[ Z(\Lambda)/Z(m) = \tilde{f}_G(\tilde{T}_2(0, E, M))^{\dim G} (\det \tilde{E})^{-1/2} \exp \left( -\frac{1}{32\pi^2} \int \text{Tr} \text{Ad} F \wedge F \log \det E \right). \quad (5.3) \]
The factor $\tilde{T}_2(0, E, M)$ is a “torsion” which does not depend on external gauge field but can be a non-trivial functional of $E$. If we normalize the path integral to the same path integral without gauge field we get the same factor $(T_2(0, M)/T_2(\zeta, M))^{\dim G}$ as in eq.(4.38) because the dependence on $E$ of the path integral in absence of the gauge field is given by the same ultraviolet contribution as $\tilde{T}_2(0, E, M)$.

One can see that the expression (5.3) is explicitly invariant under rescaling of the $(2,0)$ form $E$. This is a manifestation of the fact that this ratio $Z(\Lambda)/Z(m)$ is $(0,0)$ form.

Let us now analyse a dependence of the ratio (5.3) on a choice of the form $E$. For the case of hyperkähler there is a canonical $(2,0)$ form which is not degenerate everywhere and can be used to construct the mass terms. However for an arbitrary Kähler manifold the form $E$ can have zeros. Therefore we shall consider a regularized form $E$ which is non-zero everywhere but it can be non-holomorphic. Then the theory is not $Q$-invariant due to mass terms. Let us consider an arbitrary non-singular (not necessarily holomorphic) variation $\delta E$ of $E$. The corresponding variation of the mass term can be removed by an appropriate simultaneous change of the variables of integration in the path integrals (both for physical and regulator fields): the fields $\psi_\bar{m}$, $\phi$ and $N_{\bar{m}\bar{n}}$ are rescaled by $1 + h$, where $h = -(E^{-1})^{mn} \delta E_{mn}$, while other fields are rescaled by $1 - h$ (of course we have to consider the dependence of $h$ in linear approximation). Then instead of the lagrangian with a variation of the mass term we get the lagrangian with an additional deformation
\[ \{ Q, \int \sqrt{g} \partial_m h \text{Tr} \bar{\phi} \psi^m \} + \int \sqrt{g} \partial_m h \text{Tr}(\bar{\psi}^{\bar{m}\bar{n}} \psi_n - \phi D^{\bar{m}} \bar{\phi}). \quad (5.4) \]
The first term in the above expression is $Q$-exact while the second one is not even $Q$-invariant. Now let us continuously switch off the regularization so that the forms $E$ and $\delta E$ may have zeros. The first term above gives vanishing contributions in the physical correlators while the second one vanishes only if $h$ is holomorphic. That means that the ratio $Z(\Lambda)/Z(m)$ does not depend on deformations of $E$ which do not change the positions of zeros of $E$. For the case of hyperkähler manifold with a canonical holomorphic form $E$ without zeros or poles $Z(\Lambda)/Z(m)$ does not depend on a choice of $E$.

Let now the deformation $h$ be meromorphic with poles. Generically these poles determine 2D surfaces (the equation $h^{-1} = 0$ is equivalent to two real equations for 4 real variables). Then taking into account an identity $\bar{\partial}(1/z) = \delta^{(2)}(z)$, where $z$ is a complex variable, we get for a variation of $Z(\Lambda)/Z(m)$ a sum of correlators
with insertions of the current $< \bar{\psi}^m \psi_n - \phi D^m \bar{\phi} >$ integrated over 2-dimensional hypersurfaces on $M$.

This analysis can be easily generalized for an arbitrary representation $R$ for the multiplet of matter. Again we can introduce mass terms into the lagrangian of matter with the same (2,0) form

$$\sum_{IJ} a_{IJ}(\Lambda \{Q, \int S \wedge < \bar{\psi}^I \phi^J > \} + \Lambda \int E \wedge < \psi^I \wedge \psi^J + N^I \phi^J >),$$

(5.5)

where $a_{IJ}$ is an appropriate nondegenerate constant matrix (the indices $I, J$ correspond to different irreducible multiplets of matter). A choice of this matrix depends on the representation $R$. For the case when the gauge group is $SU(2)$ and there is an even number of copies of a spinor representation for the fields of matter this matrix should be antisymmetric.

One can show that the ratio of two path integrals $Z_R(m)/Z_R(\Lambda)$ with different masses $m$ and $\Lambda$ (they are regularizing each other) depends only on the equivalence class of $E$ (two forms $E$ are equivalent if they differ by a nonsingular holomorphic factor). This ratio is again a (0,0) form on the moduli space when considered in the external anti-instanton field. The dependence on an arbitrary variation of $E$ can be calculated as well. We have

$$Z_R(\Lambda)/Z_R(m) = \tilde{f}_R(\bar{T}_2(0, E, M))^{\dim R}(\det \hat{E}_R)^{-1/2} \exp \left( -\frac{1}{32\pi^2} \int \text{Tr}_R F \wedge F \log \det E \right),$$

(5.6)

where $\hat{E}_R$ is a matrix of bilinear combinations of the zero mode wave functions of fermions of matter integrated with $E$. The factor $\tilde{f}_R$ does not depend on $E$ while $\bar{T}_2(0, E, M)$ is a “torsion” that does not depend on the external gauge field a representation $\zeta$ of the fundamental group of $M$ but it can be a functional of $E$. If we normalize the path integral dividing it by the same integral without gauge field we get $(\bar{T}_2(0, E, M)/\bar{T}_2(\zeta, E, M))^{\dim R} = (T_2(0, M)/T_2(\zeta, M))^{\dim R}$ which does not depend on $E$.

For a particular case of the gauge group $SU(2)$ with $2N$ copies of spinor representation for the fields of matter we have

$$\det \hat{E}_R = (\int E \wedge \psi(0) \wedge \bar{\psi}(0))^N.$$  

(5.7)

Notice that here the wave function $\psi(0)$ is normalized ($\int * \bar{\psi}(0) \wedge \psi(0) = 1$, where $\bar{\psi}(0)$ stands for complex conjugated zero mode).

The exponential of $\int \text{Tr}_R F \wedge F \log \det E$ in the above expression is a manifestation of the anomaly that was already discussed in this paper. Here we see once again that if $C_G = C_R$ (i.e. $\text{Tr}_{Ad} F \wedge F = \text{Tr}_R F \wedge F$) then the anomaly is cancelled in the product

$$\left( \frac{Z(\Lambda)}{Z(m)} \right) \left( \frac{Z_R(m)}{Z_R(\Lambda)} \right) = \left( \frac{\det \hat{E}_R}{\det E} \right)^{1/2} \left( \frac{T_2(0, M)}{\bar{T}_2(\zeta, M)} \right)^{\dim G - \dim R} \frac{\tilde{f}_G}{\tilde{f}_R}.$$  

(5.8)
If we now consider in Witten’s non-deformed theory a physical correlator with two insertions of the mass term operator for the matter multiplet (in adjoint representation of the gauge group) we can translate it into the theory where the matter in adjoint representation is replaced by the matter in an appropriate representation \( R \) of the gauge group. Indeed the theory has non-anomalous BRST symmetry and therefore allows for a localization of the path integral to the anti-instanton configurations. Therefore for such a translation we should multiply the integrand in the integral over \( \mathcal{M} \) for the correlator in Witten’s theory by an above factor where we substitute the anti-instanton field. Then we get the expression for the correlator in this heterotic theory given by the same formula of eq.(4.58) (with some change of notations).

Notice that the ratio \( Z_R(m)/Z_R(\Lambda) \) in the external anti-instanton field does not depend on the holomorphic modulii (for any representation \( R \) of the gauge group) because the BRST charge \( Q \) does not depend on \( A_n \) gauge field (the parameter \( \epsilon^p \) which appears for the derivative in \( t_p \) enters as a gauge transformation and does not affect the result of calculations). Therefore we get an identity

\[
\partial^p \log \det \hat{E}_R = \frac{1}{16\pi^2} \partial^p \int \text{Tr}_R F \wedge F \log E + 2\partial^p \log \tilde{f}_R = 2\partial^p \log \tilde{f}_R. \tag{5.9}
\]

It is interesting that this derivative does not depend on the holomorphic (2,0) form \( E \).

Let us consider for simplicity the case of \( SU(2) \) gauge group and spinor representation for the matter fields. Let us consider \( \partial^p \psi \). It obeys the following equations

\[
\bar{D}((\partial^p + i\epsilon^p)\psi) = 0, \quad *D^*((\partial^p + i\epsilon^p)\psi) - i* A^p * \psi = 0. \tag{5.10}
\]

A general solution to this equation reads (we use a normalization of zero modes in the adjoint representation determined by eq.(4.44))

\[
(\partial^p + i\epsilon^p)\psi = \tilde{D} u^p + c^p \psi, \tag{5.11}
\]

where \( u^p \) is (0,1) form on \( M \) while \( c^p \) are some coefficients depending on modulii and external metric. Substituting this expression into \( \partial^p \hat{E}_R \) we get

\[
\partial^p \hat{E}_R = 2c^p \hat{E}_R. \tag{5.12}
\]

Hence \( \partial^p \tilde{f}_R = c^p \) and \( \tilde{f}_R \) is a section of a line bundle over \( \mathcal{M} \times \mathcal{G} \) while \( c^p \) is flat connection on a bundle over \( \mathcal{M} \times \mathcal{G} \) with a fibre given by the space of zero modes for a representation \( R \).

We can now (locally) redefine the wave functions for zero modes and include \( c^p \) into their normalization. Then \( \hat{E}_R \) is locally anti-holomorphic

\[
\partial^p \hat{E}_R = 0. \tag{5.13}
\]
For the zero modes of adjoint representation using the fact that the connection $\epsilon^p$ is flat (with a normalization of zero modes determined by eq. (4.44)) and $(\partial^p + i \epsilon^p) \bar{A}^k = \bar{D} \Phi^b$ we get

$$\partial^p \det \hat{E} = \partial^p \hat{j}_G = 0. \quad (5.14)$$

Let us now consider the variation of $Z_R(m)/Z_R(\Lambda)$ in metric. Since the dependence of it on metric is due to a dependence of metric of the anti-instanton field we have to consider first the variation of the anti-instanton gauge field $B_\mu$ under the variation of metric. The variation of equation $F_{mn} = 0$ gives us

$$D_m \delta B_n - D_n \delta B_m = 0, \quad (5.15)$$

while from equation $g^{mn} F_{m\bar{n}} = 0$ we get

$$\delta g^{mn} F_{m\bar{n}} + g^{mn} (D_m \delta B_{\bar{n}} - D_{\bar{n}} \delta B_m) = 0. \quad (5.16)$$

The solution to these equations reads

$$\delta B_n = D_n \Phi + c_i A^i_n, \quad \delta B_{\bar{n}} = D_{\bar{n}} \Phi + \bar{c}_i \bar{A}^i_{\bar{n}}, \quad (5.17)$$

where $\Phi$ and $\bar{\Phi}$ are complex conjugated to each other scalar fields obeying the following condition

$$\Phi - \bar{\Phi} = -\frac{1}{\Delta} \delta g^{mn} F_{m\bar{n}} \quad (5.18)$$

while $\Delta = D_n D^n$. The constant coefficients $c_i$ and $\bar{c}_i$ are complex conjugated to each other and can depend on the moduli of instanton and on metric.

The variation of the zero mode wave function $\delta \psi^{(0)}$ under a variation of metric obeys the following equations

$$D \delta \psi^{(0)} - i \delta B \wedge \psi^{(0)} = 0, \quad (5.19)$$

$$^*D^* \delta \psi^{(0)} - i^* \delta B^* \psi^{(0)} + (\delta^*) D^* \psi^{(0)} + ^*D(\delta^*) \psi^{(0)} = 0.$$

A general solution to these equations reads

$$\delta \psi^{(0)} = i (\bar{\Phi} + c_p \epsilon^p) \psi^{(0)} + c_p \partial^p \psi^{(0)} + \bar{D} \epsilon + c \psi^{(0)}, \quad (5.20)$$

where $c$ is a section of a line bundle over $\mathcal{M} \times \mathcal{G}$. Here we used the expression for variation in metric of the gauge field. Then for the variation of $K$ we get

$$\delta \hat{E}_R = 2c \hat{E}_R + c_p \partial^p \hat{E}_R. \quad (5.21)$$

For the adjoint representation of the gauge group we can deduce in a similar way

$$\delta A^k_{\bar{m}} = i (\bar{\Phi} + c_p \epsilon^p) A^k_{\bar{m}} + \bar{D}_m \epsilon^k + c_p \partial^p A^k + \bar{c}_l A^l, \quad (5.22)$$
where $\tilde{\epsilon}^k$ is a (0,0) form on $M$ and $\tilde{c}_l^k$ are some coefficients which can depend on the coordinates on the moduli space and external metric. Then we get for the variation of $\det \hat{E}$ in external metric

$$
\delta \det \hat{E} = 2\tilde{c}_l^k \hat{E} + c_p \partial_p \det \hat{E}.
$$

(5.23)

The integrability condition for eq.(5.23) gives a linear (variational) equation which relates $c$, $c_p$ and $\det \hat{E}$

$$(2\delta c + \delta c_p \partial^p c + c_q \partial^q c_p \partial^p \det \hat{E}) \hat{E} = 0,$$

(5.24)

and a similar equation for the adjoint representation. We see here that the vector field $c_p \partial^p$ on $M$ plays a role of connection for the variation in metric. Now we notice that the whole correlator in the heterotic theory does not depend on external metric. Therefore if we include all the factors independent on $E$ in $Z_R(m)/Z_R(\Lambda)$ into a normalization of $\psi^{(0)}$ then we can deduce one more relation between connections $c$ and $Z(\Lambda)/Z(m)$ and the correlator in Witten’s theory. Indeed the correlator in Witten’s theory is given by an integral of a (4,4) form $\Omega$ over $\mathcal{M}$. Under a variation in metric of $M$ this form changes by an exact (4,4) form $\partial \omega + \partial \bar{\omega}$ ($\partial$ and $\bar{\partial}$ are the Dolbeault operators on $\mathcal{M}$ and $\omega$ and $\bar{\omega}$ are (3,4) and (4,3) forms on $\mathcal{M}$, respectively) which in Witten’s theory does not contribute to the correlator. In the heterotic theory the physical correlator is given essentially by an integral of $\Omega \times (\det \hat{E}_R/ \det \hat{E})^{1/2}$ over $\mathcal{M}$. Because the correlator in the heterotic theory does not depend on Kählerian deformations of the metric we can assume that the variation of the integrand should be exact too. This gives us a certain equation which includes $c$ and some parameters related to the geometry of zero modes in adjoint representation.

We discussed above a special class of physical correlators in the heterotic model. Of course we can choose a different physical operator for the sector of matter, for example, the local operator constructed of the scalar field $\phi$ (however for the regulator path integral it is convenient to leave the mass operator considered above). In this case we get physical correlators which can not be obtained from Witten’s theory by the described procedure. However since the anomaly in the BRST charge is cancelled we can generalize the above discussion to this case.

It is to be emphasized that the physical correlator in the heterotic model is given by an integral of a product of factors each of which is not in general globally defined on $\mathcal{M}$. However the whole correlator should be well defined because the original SUSY theory seems to be well defined on Kähler manifold.

6 Conclusions

To conclude we have shown that N=1 SUSY gauge theory with an appropriate representation of supermultiplet of matter can be twisted on Kähler manifold $M$. 

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The twisted theory is shown to have a BRST charge $Q$ with a nontrivial cohomology ring which determines the ring of physical operators. The lagrangian of such a heterotic model is $Q$-exact. The physical correlators turn out to be independent on Kählerian deformations of metric (the dependence on the Kähler class of the metric is factorized out into a power of a ratio of two Ray-Singer torsions for the Dolbeault complex) and on the gauge coupling constant. Therefore they can be calculated in a semiclassical approximation near anti-instanton configurations.

The correlators of the physical local operators do not depend on anti-holomorphic coordinates but they can depend on holomorphic coordinates on $M$. However due to integration over the moduli space of instanton these correlators can have singularities. This is an interesting possibility because these correlators could be interpreted as correlators in some two-dimensional theory. Such an effective two-dimensional theory (if it exists) is not pure topological in a sense that the physical correlators can depend on holomorphic coordinates and therefore can correspond to two-dimensional propagating degrees of freedom.

Thus the physical correlators are sections of a holomorphic bundle on $M$ with coefficients which are topological invariants. Actually the physical correlators depend on the complex structure on $M$. It is an open question if this dependence can be factorized out from the dependence on smooth structure.

There is also an interesting possibility that such a twisting induces a set of Ward identities for topological correlators in an untwisted theory which could allow to calculate some of the correlators in an untwisted theory in terms of holomorphic sections of a vector bundle on the Kähler manifold. In particular the interesting problem is also whether the heterotic topological theory allows to formulate an analog of the $tt^*$ fusion equations [10] for non-topological amplitudes in SUSY theory in 4D.

As we have seen in the heterotic model some connection between zero modes for adjoint and non-adjoint representations appears. Therefore there is a tempting possibility that such a heterotic topological theory can be a tool for a study of zero modes of fields of matter in non-adjoint representation of the gauge group from the point of view of geometry of the moduli space of instanton.

An open question is also if the correlators in the heterotic topological model can be interpreted in terms of $S$-matrix elements in heterotic string theory.

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