On the dynamics of interacting populations in presence of state dependent fluctuations

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Abstract

We discuss several models of the dynamics of interacting populations. The models are constructed by nonlinear differential equations and have two sets of parameters: growth rates and coefficients of interaction between populations. We assume that the parameters depend on the densities of the populations. In addition the parameters can be influenced by different factors of the environment. This influence is modelled by noise terms in the equations for the growth rates and interaction coefficients. Thus the model differential equations become stochastic. In some particular cases these equations can be reduced to a Fokker-Planck equation for the probability density function of the densities of the interacting populations.

Keywords: interacting populations, density fluctuations, multiplicative white noise, probability density functions for populations densities

1 Introduction

In this paper we shall discuss several models of the dynamics of interacting biological populations. Usually such models consist of nonlinear ordinary differential equations for the population densities [1]-[6]. Two sets of parameters are presented in the models: growth rates and coefficients of interaction between the populations. The basic assumption in the discussed below models is that the model parameters depend on the densities of the populations.
The new point in this study is the assumption that the model parameters can depend also on the environment. This influence will be modelled by noise terms. Thus the model equations will become nonlinear stochastic differential equations. The kind of noise will be multiplicative noise (noise that depends on the populations densities) or more complicated kind of noise.

The result of the influence of the environment fluctuations is that instead of equations for the trajectories of the populations in the phase space of the population densities we will have to write and solve equations for the probability density functions of the densities of the interacting populations.

Below we shall discuss the models in order of their increasing mathematical complexity. We shall start with inclusion of additive noise only in the growth rates of populations. This will lead to arising of multiplicative noise in the model equations. Then we shall consider a model with additive noise in the coefficients of interaction between the populations. The third model will contain additive noise in the both sets of parameters: in growth rates and in the interaction coefficients. Next we shall consider a model with multiplicative state dependent noises in the growth rates and in the interaction coefficients of the model equations. Finally we shall show a part of methodology for reduction of the nonlinear stochastic differential equations to a Fokker-Planck equation for the probability density function of the spatial densities of the populations. Several concluding remarks are given at the end of the paper.

2 Model equations without influence of environmental fluctuations

The classical model of interacting populations is based on a system of nonlinear ordinary differential equations of the Lotka-Volterra kind:

\[ \dot{\rho}_i = r_i \rho_i(t) \left( 1 - \sum_{j=1}^{n} \alpha_{ij} \rho_j(t) \right), \]  

in Eqs. \( \text{(1)} \) \( \rho_i \) are the densities of the population members, \( r_i \) are the growth rates (that can be negative if the number of deaths in the corresponding population is larger than the number of births). \( \alpha_{ij} \) are coefficients of interaction between the populations \( i \) and \( j \). \( \dot{\rho}_i \) denotes the time derivative of the density \( \rho_i \).

Let us now suppose [4]-[8] that the birth rates and interaction coefficients
depend on the density of the populations:

\[ r_i = r_i^0 \left( 1 + \sum_{j=1}^{n} r_{ij} \rho_j \right); \quad \alpha_{ij} = \alpha_{ij}^0 \left( 1 + \sum_{j=1}^{n} \alpha_{ijk} \rho_k \right) \] (2)

In Eq. (2) \( r_{ij} \) and \( \alpha_{ij} \) are parameters. The substitution of Eq. (2) in Eq. (1) leads to a system of model equations of the kind

\[
\dot{\rho}_i = F_i(\rho_1, \ldots, \rho_n);
\]

\[
F_i(\rho_1, \ldots, \rho_n) = r_i^0 \rho_i \left\{ 1 - \sum_{j=1}^{n} (\alpha_{ij}^0 - r_{ij}) \rho_j - \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_{ij}^0 (\alpha_{ijl} + r_{il}) \rho_j \rho_l - \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_{ij}^0 r_{ik} \alpha_{ijl} \rho_j \rho_k \rho_l \right\}.
\] (3)

We note that the system (3) consists of nonlinear ordinary differential equations with polynomial nonlinearities up to the order 4.

3 Model equations when the birth rates are influenced by environmental fluctuations

Let us now suppose that the birth rates and interaction coefficients depend on the density of the populations and in addition the birth rates fluctuate. If the number of the populations in the studied system is \( n \) then in general the number of external influences that we have to account for will be \( n \) too.

The equations for the growth rates and interaction coefficients become

\[ r_i = r_i^0 \left( 1 + \sum_{j=1}^{n} r_{ij} \rho_j \right) + \eta_i; \]

\[ \alpha_{ij} = \alpha_{ij}^0 \left( 1 + \sum_{j=1}^{n} \alpha_{ijk} \rho_k \right). \] (4)

In Eq. (4) \( r_{ij} \) and \( \alpha_{ij} \) are parameters and \( \eta_i \) are noises (Below we shall assume that \( \eta_i \) are Gaussian white noises but in general there is no restriction on the probability density function and on the correlation properties of the noises).
The substitution of Eq. (4) in Eq. (1) leads to a system of model equations of the kind

\[ \dot{\rho}_i = F_i(\rho_1, \ldots, \rho_n) + \eta_i G_i(\rho_1, \ldots, \rho_n); \]

\[ F_i(\rho_1, \ldots, \rho_n) = r_i^0 \rho_i \left( 1 - \sum_{j=1}^{n} (\alpha_{ij}^0 - r_j) \rho_j - \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_{ij}^0 (\alpha_{ijl} + r_l) \rho_j \rho_l - \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_{ij}^0 r_{ik} \alpha_{ijl} \rho_j \rho_k \rho_l \right) \]

\[ G_i(\rho_1, \ldots, \rho_n) = \rho_i \left( 1 - \sum_{j=1}^{n} \alpha_{ij}^0 \rho_j - \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{ij}^0 \alpha_{ijk} \rho_j \rho_k \right) \]

(5)

Thus the presence of noise in the growth rates leads to change of the kind of the system of model equations. The system of nonlinear ordinary deterministic differential equations (2) is converted to a system of nonlinear stochastic differential equations (5). In addition the stochastic terms \( \eta_i G_i(\rho_1, \ldots, \rho_n) \) in the system (5) depend on the state of the system. The additive noise from Eqs. (3) leads to multiplicative noise in the system of equations (5). If all \( \eta \) are Gaussian white noises then the system (5) can be converted to a Fokker-Planck equation for the probability density function of the densities of the populations.

4 Model equations when the interaction coefficients are influenced by environmental fluctuations

This case is more complicated as the number of interaction coefficients in general is \( n^2 \) where \( n \) is the number of interacting populations. The additive noises \( \sigma_{ij} \) are included in the equations for \( \alpha_{ij} \)

\[ r_i = r_i^0 \left( 1 + \sum_{j=1}^{n} r_{ij} \rho_j \right); \]
\[ \alpha_{ij} = \alpha_{ij}^0 \left(1 + \sum_{j=1}^{n} \alpha_{ijk}\rho_k\right) + \sigma_{ij}. \]  
(6)

In Eq. (6) \( r_{ij} \) and \( \alpha_{ijk} \) are parameters and \( \eta_i \) are Gaussian white noises.

The substitution of Eq. (6) in Eq. (1) leads to a system of model equations of the kind

\[ \dot{\rho}_i = F_i(\rho_1, \ldots, \rho_n) - \sum_{j=1}^{n} \sigma_{ij} G_{ij}(\rho_1, \ldots, \rho_n); \]

\[ F_i(\rho_1, \ldots, \rho_n) = r_{i0}\rho_i \left\{1 - \sum_{j=1}^{n} (\alpha_{ij}^0 - r_{ij})\rho_j - \right. \]
\[ \left. \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_{ij}^0 (\alpha_{ijl} + r_{il})\rho_j\rho_l - \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_{ij}^0 r_{ik}\alpha_{ijl}\rho_k\rho_l \right\}; \]

\[ G_{ij}(\rho_1, \ldots, \rho_n) = \rho_i\rho_j r_{i0}^0 \left(1 + \sum_{k=1}^{n} r_{ik}^0\rho_k \right). \]
(7)

In the general case the system (7) can be solved only numerically. But in the particular cases (where each equations contains only a single multiplicative noise and this multiplicative noise is Gaussian white noise) the analytical treatment is possible on the basis of the theory of Markov processes and forward Kolmogorov (Fokker-Planck) equation.

5 Model equations for the general case when all parameters are influenced by environmental fluctuations

In the general case the environment fluctuations can influence both the growth rates and the interaction coefficients. In this case the additive noises \( \sigma_{ij} \) are present in the equations for \( \alpha_{ij} \) and additive noises \( \eta_i \) are present in the equation for \( r_i \). Thus the equations for the growth rates and for the competition coefficients become

\[ r_i = r_{i0}^0 \left(1 + \sum_{j=1}^{n} r_{ij}\rho_j \right) + \eta_i; \]
\[
\alpha_{ij} = \alpha^0_{ij} \left(1 + \sum_{j=1}^{n} \alpha_{ijk} \rho_k\right) + \sigma_{ij}
\]  \quad (8)

The substitution of Eqs. (8) in Eq. (1) leads to a system of model equations of the kind

\[\dot{\rho}_i = F_i(\rho_1, \ldots, \rho_n) - \sum_{j=1}^{n} \sigma_{ij} G^{(1)}_{ij}(\rho_1, \ldots, \rho_n) + \eta_i G^{(2)}_i - G^{(3)}_i;\]

\[F_i(\rho_1, \ldots, \rho_n) = r_i^0 \rho_i \left(1 - \sum_{j=1}^{n} (\alpha^0_{ij} - r_j) \rho_j - \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha^0_{ijl} (\alpha_{ijkl} + r_{il}) \rho_j \rho_l - \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha^0_{ijl} r_{ij} \alpha_{ijkl} \rho_j \rho_l \rho_k \right)\]

\[G^{(1)}_{ij}(\rho_1, \ldots, \rho_n) = \rho_i \rho_j r_i^0 \left(1 + \sum_{k=1}^{n} r_{ik}^0 \rho_k\right)\]

\[G^{(2)}_i = \rho_i \left(1 - \sum_{j=1}^{n} \alpha^0_{ij} \rho_j - \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha^0_{ij} \alpha_{ijkl} \rho_j \rho_k \right)\]

\[G^{(3)}_i = \eta_i \rho_i \sum_{j=1}^{n} \sigma_{ij} \rho_j.\]  \quad (9)

We observe three kinds of noise terms in the system of equations (9). \(G^{(1)}_{ij}\) is a result of the action of the environment on the coefficients of interaction between the populations. \(G^{(2)}_i\) is a result of the action of the environment on the growth rates. And because of the nonlinearity of the model equations there exist third kind of terms \(G^{(3)}_i\) that is a result of the joint action of the two influences. If one kind of influence is not present \(G^{(3)}_i\) is 0. In general the system (9) can be studied only numerically. Analytical treatment is possible only when one of the two kinds of influences is missing and the noises that account for the environment influences are Gaussian white noises.
6 General case for presence of multiplicative white noise in the coefficients

Even more general case of influence by the environment is when this influence depends on the state of the system. In this case instead of additive noises we have to add multiplicative noises at the equations for the growth rates and interaction coefficients. The equations become

\[
    r_i = r_i^0 \left( 1 + \sum_{j=1}^{n} r_{ij} \rho_j \right) + \eta_i H_i(\rho_1, \ldots, \rho_n);
\]

\[
    \alpha_{ij} = \alpha_{ij}^0 \left( 1 + \sum_{j=1}^{n} \alpha_{ijk} \rho_k \right) + \sigma_{ij} I_{ij}(\rho_1, \ldots, \rho_n).
\]

(10)

We remember that in Eq. (10) \( r_{ij} \) and \( \alpha_{ijk} \) are parameters; \( \eta_i \) and \( \sigma_{ij} \) are Gaussian white noises; and \( H_i \) and \( I_{ij} \) are functions depending on the densities of the populations. The substitution of Eq. (10) in Eq. (1) leads to a system of model equations of the kind

\[
    \dot{\rho}_i = F_i(\rho_1, \ldots, \rho_n) - \sum_{j=1}^{n} \sigma_{ij} I_{ij}(\rho_1, \ldots, \rho_n) \times \]

\[
    G_{ij}^{(1)}(\rho_1, \ldots, \rho_n) \]

\[
    + \eta_i H_i(\rho_1, \ldots, \rho_n) G_{i}^{(2)} - G_{i}^{(3)};
\]

\[
    F_i(\rho_1, \ldots, \rho_n) = r_i^0 \rho_i \left( 1 - \sum_{j=1}^{n} (\alpha_{ij}^0 - r_j) \rho_j - \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_{ij}^0 (\alpha_{ijl} + r_{il}) \rho_j \rho_l - \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_{ij}^0 r_{ik} \alpha_{ijl} \rho_j \rho_k \rho_l \right)
\]

\[
    G_{ij}^{(1)}(\rho_1, \ldots, \rho_n) = \rho_i \rho_j r_i^0 \left( 1 + \sum_{k=1}^{n} r_{ik}^0 \rho_k \right)
\]

\[
    G_{i}^{(2)} = \rho_i \left( 1 - \sum_{j=1}^{n} \alpha_{ij}^0 \rho_j - \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{ij}^0 \alpha_{ijk} \rho_j \rho_k \right)
\]
\[ G^{(3)}_i = \eta_i H_i(\rho_1, \ldots, \rho_n) \rho_i \sum_{j=1}^{n} \sigma_{ij} \times I_{ij}(\rho_1, \ldots, \rho_n) \rho_j. \] (11)

7 An example for reduction of model stochastic differential equations to a Fokker-Planck equation

Let us discuss the model system (5) for the case of one population (we set \( r^0 = r; r_{11} = 0; \alpha^0_{11} = \alpha; \alpha_{111} = 0 \)). The model equation is

\[ \dot{\rho} = F(\rho) + \eta G(\rho) \]

\[ F(\rho) = r\rho - \alpha \rho^2; \quad G(\rho) = \rho - \alpha \rho^2. \] (12)

Eq. (12) is a particular case of a more general equation. We shall discuss the case where \( F(\rho) \) and \( G(\rho) \) are polynomials of arbitrary orders \( p_1 \) and \( p_2 \), i.e.,

\[ F(\rho) = \sum_{i=1}^{p_1} \mu_i \rho^i; \quad G(\rho) = \sum_{i=1}^{p_2} \theta_i \rho^i. \] (13)

where \( \mu_i \) and \( \theta_i \) are parameters. In this case Eq. (12) becomes

\[ \dot{\rho} = \sum_{i=1}^{p_1} \mu_i \rho^i + \eta \sum_{i=1}^{p_2} \theta_i \rho^i. \] (14)

The formal integration of Eq. (14) leads to the equation

\[ \rho(t) = \rho(t = 0) + \int_{0}^{t} d\tau \ F[\rho(\tau)] + \int_{0}^{t} dW_\tau \ G[\rho(\tau)], \] (15)

where \( W_\tau \) is a Wiener process. The integral \( \int_{0}^{t} dW_\tau \ G(\rho(\tau)) \) can be integral of Ito kind or integral of Stratonovich kind. Let us assume that the integral is an integral of Ito kind. For this case Eq. (15) can be written as

\[ d\rho_t = F(\rho_t) dt + G(\rho_t) dW_t, \] (16)

where we wrote the time dependence as subscript and in general \( F \) and \( G \) are given by Eqs. (13). The Fokker-Planck equation that corresponds to Eq. (16)
is
\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} \left\{ p(x, t) \left[ \sum_{i=1}^{p_1} \mu_i x^i \right] \right\} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left\{ p(x, t) \left[ \sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j x^{i+j} \right] \right\}.
\]

We can formulate the following statement (the proof will be presented elsewhere): Let \( b_1 \) and \( b_2 \) be natural boundary points \((-\infty \leq b_1 < b_2 \leq \infty)\). Let in addition \( \sigma(x) = \sum_{i=1}^{p_2} \theta_i x^i > 0 \) in \((b_1, b_2)\). Then the diffusion process \( X_t \) that is solution of the stochastic differential equation Eq. (16) has unique invariant distribution with p.d.f.

\[
p^0(x) = \frac{N}{\sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j x^{i+j}} \exp \left( \int_c^x dy \left[ \frac{2}{\sum_{i=1}^{p_1} \mu_i y^i} \right] \right),
\]

\( \forall x \in (b_1, b_2) \) \hspace{1cm} (18)

if the quantity

\[
\int_{b_1}^{b_2} dx \frac{1}{\sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j x^{i+j}} \exp \left( \int_c^x dy \left[ \frac{2}{\sum_{i=1}^{p_1} \mu_i y^i} \right] \right),
\]

\( b_1 < c < b_2 \) \hspace{1cm} (19)

has finite value. In addition each time-dependent solution \( p(x, t) \) of the Fokker-Planck equation \((17)\) in \((b_1, b_2)\) satisfies

\[
\lim_{t \to \infty} p(x, t) = p^0(x)
\]

(20)

For the case of more than one population we have to solve the system of stochastic differential equations

\[
dX_i(t) = F_i[X_1(t), \ldots, X_n(t)] + G_i[X_1(t), \ldots, X_n(t)] dW_i(t), \ i = 1, \ldots, n,
\]

(21)
where \( W_j(t) \) are independent Wiener processes and 

\[
F_i(\rho_1, \ldots, \rho_n) = r^0_i \rho_i \left( 1 - \sum_{j=1}^{n} (\alpha^0_{ij} - r_j) \rho_j - \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha^0_{ij} (\alpha_{ijl} + r_l) \rho_j \rho_l - \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha^0_{ij} r_{ik} \alpha_{ijl} \rho_j \rho_k \rho_l \right) 
\]

\[
G_i(\rho_1, \ldots, \rho_n) = \rho_i \left( 1 - \sum_{j=1}^{n} \alpha^0_{ij} \rho_j - \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha^0_{ij} \alpha_{ijk} \rho_j \rho_k \right). 
\]

(22)

The corresponding Fokker-Planck equation is: 

\[
\frac{\partial}{\partial t} p = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [p F_i(x_1, \ldots, x_n, t)] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} [p \times G_{ij}(x_1, \ldots, x_n, t) G_{ji}(x_1, \ldots, x_n, t)].
\]

(23)

8 Concluding remarks

In this paper we discuss the influence of environment fluctuations on the dynamics of interacting populations modelled by system of nonlinear differential equations. The problem is interesting as the fluctuations are often present in the complex systems [9] - [11] and in particular in the systems of populations [12] - [14].

In the discussed above models the growth rates and the interaction coefficients depend on the density of the populations. The influence of environment led to terms containing multiplicative noise or more complicated kind of noise. This research is continuation of our research on presence of additive noise in the model equations of the population dynamics [15], [16]. There are two main approaches to deal with fluctuations. The first approach is based on appropriate averaging and in this way one investigates mean quantities connected to the problem. In addition this approach can lead to reduction of the spatial dimensions of the problem if such dimensions are present as well.
as it can lead to relatively simple mathematical description of somlex media such as porous media (for examples see [17][18]).

Finally let us note that one possible extension of the above research is to include spatial dimensions in the model equations [19]-[21]. For the case without environment influence we can obtain even analytical solutions of the model nonlinear PDEs [22]-[24]. In general the case when environment influence is present can be treated only numerically. We shall discuss these problems in more detail elsewhere.

References

[1] J.D. Murray. Lectures on Nonlinear Differential Equation Models in Biology. Oxford, England: Oxford University Press, 1977.

[2] R. M. May, W. J. Leonard. Nonlinear Aspects of Competition between Three Species. SIAM J. Appl. Math., 29, 243-253, 1975.

[3] H. W. Hethcote. The Mathematics of Infectious Diseases. SIAM Review, 42 599-653, 2000.

[4] Z. I. Dimitrova, N. K. Vitanov. Influence of Adaptation on the Nonlinear Dynamics of a System of Competing Populations. Phys. Lett. A 272, 368 - 380, 2000.

[5] Z. I. Dimitrova, N. K. Vitanov. Adaptation and its Impact on the Dynamics of a System of Three Competing Populations. Physica A 300, 91 - 115, 2001.

[6] N. K. Vitanov, Z. I. Dimitrova, M. Ausloos. Verhulst-Lotka-Volterra (VLV) model of ideological struggle. Physica A 389, 4970 - 4980, 2010.

[7] Z. I. Dimitrova, N. K. Vitanov. Dynamical consequences of adaptation of the growth rates in a system of three competing populations. J. Phys. A: Math. Gen. 34, 7459 - 7473, 2001.

[8] Z. I. Dimitrova, N. K. Vitanov. Chaotic Pairwise Competition Theoretical Population Biology 66, 1 - 12, 2004.

[9] N. S. Goel, N. Richter-Dyn. Stochastic Models in Biology. Academic Press, New York, 1974.

[10] J. M. Pedraza, A. van Oudenaarden. Noise Propagation in Gene Networks. Science 307, No. 5711, 1965-1969, 2005.
[11] H. Kantz, D. Holstein, M. Ragwitz, N. K. Vitanov. Markov Chain Model for Turbulent Wind Speed Data. Physica A 342, 315-321, 2004.

[12] P. Turchin. Complex Population Dynamics: A Theoretical/Empirical Synthesis. Princeton University Press, Princeton, NJ, 2003.

[13] B. Spagnolo, M. Cirone, A. La Barbera, F. de Pasquale. Noise-induced Effects in Population Dynamics. J. Phys.: Condens. Matter, 14, 2247-2255, 2002.

[14] R. Lande, S. Engen, B.-E. Saether. Stochastic Population Dynamics in Ecology and Conservation. Oxford University Press, Oxford, 2003.

[15] N. K. Vitanov, Z. I. Dimitrova. On Waves and Distributions in Population Dynamics. BIOMATH 1, No. 1, Article ID: 1209253, 2012.

[16] N. K. Vitanov, Z. I. Dimitrova, K. N. Vitanov. Traveling waves and statistical distributions connected to systems of interacting populations Computers & Mathematics with Applications, (in press), doi: 10.1016/j.camwa.2013.04.002, 2013.

[17] N. K. Vitanov. Upper Bounds on Convective Heat Transport in a Rotating Fluid Layer of Infinite Prandtl Number: Case of Intermediate Taylor Numbers. Phys. Rev. E 62, 3581-3591, 2000.

[18] N. K. Vitanov. Upper Bounds on the Heat Transport in a Porous Layer. Physica D 136, 322-339, 2000.

[19] N. K. Vitanov, I. P. Jordanov, Z. I. Dimitrova. On Nonlinear Dynamics of Interacting Populations: Coupled Kink Waves in a System of Two Populations. Commun. Nonlinear Sci. Numer. Simulat. 14 2379 - 2388, 2009.

[20] N. K. Vitanov, I. P. Jordanov, Z. I. Dimitrova. On Nonlinear Population Waves. Applied Mathematics and Computation. 215 2950 - 2964, 2009.

[21] N. K. Vitanov, Z. I. Dimitrova. Application of the Method of Simplest Equation for Obtaining Exact Traveling-Wave Solutions for Two Classes of Model PDEs from Ecology and Population Dynamics. Commun. Nonlinear Sci. Numer. Simulat. 15, 2836 - 2845, 2010

[22] N. K. Vitanov, Z. I. Dimitrova, H. Kantz. Modified Method of Simplest Equation and its Application to Nonlinear PDEs. Applied Mathematics and Computation, 216, 2587 - 2595, 2010.
[23] N. K. Vitanov. *Modified Method of Simplest Equation: Powerful Tool for Obtaining Exact and Approximate Traveling-Wave Solutions of Nonlinear PDEs*. Commun. Nonlinear Sci. Numer. Simulat. **16**, 1176 - 1185, 2011.

[24] N. K. Vitanov. *Application of Simplest Equations of Bernoulli and Riccati Kind for Obtaining Exact Traveling Wave Solutions for a Class of PDEs with Polynomial Nonlinearity*. Commun. Nonlinear Sci. Numer. Simulat. **15**, 2050 - 2060, 2010.