Factorization approach for barotropic FRW model with a cosmological constant

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We apply the technique of standard supersymmetric factorization for any q factor ordering in the Wheeler-DeWitt (WDW) equation for barotropic Friedmann-Robertson-Walker (FRW) minisuperspace model, including the cosmological term. The resulting wave functions of the universe are exhibited as one-parameter families, which for particular values of the γ parameter of the barotropic model are exactly found.

PACS numbers: 02.30.Hq, 04.20.Jb, 04.40.Nr, 98.80.Hw.

I. INTRODUCTION

The study of eigenvalue problems associated with second-order differential operators found a renewed impulse with the application of the factorization technique and its generalizations [1–4], which was fundamented on the inverse scattering method of Gelfand and Levitan [5].

The use of the strictly isospectral scheme based on the general Riccati solution [1–4], has been applied from classical and quantum physics [1] to relativistic models [6]. This technique has been known since about a decade in one-dimensional supersymmetric quantum mechanics (SUSY-QM) and usually requires nodeless, normalizable states of Schrödinger equation. However, Pappademos, Sukhatme, and Pagnamenta [8] showed that the strictly isospectral construction can also be performed on non-normalizable states.

SUSY-QM may be considered an equivalent formulation of the Darboux transformation method, which is well-known in mathematics from the original paper of Darboux [7], book by Ince [9], and book by Matveev and Salle [10], where the method is widely used in the context of the soliton theory. An essential ingredient of the method is a particular choice of a transformation operator [5] in the form of a differential operator [11] which intertwines two hamiltonians and relates their eigenfunctions. When this approach is applied in quantum theory it allows one to generate a huge family of exactly solvable local potential starting with a given exactly solvable local potential [4].

In nonrelativistic one-dimensional supersymmetric quantum mechanics, the factorization technique was applied to the q = 0 factor ordered WDW equation corresponding to the FRW cosmological models without matter field [12], where a one-parameter class of strictly isospectral cosmological FRW solutions was exactly found, representing the wave functions of the universe for that case. Also, in Ref. [13] a one-parameter family of closed, radiation-filled FRW quantum universe for any q factor order was found.

In this work, we extend the application of the factorization technique to the case q ≠ 0 factor ordering in the WDW equation for the barotropic FRW minisuperspace model, including a cosmological term.
The work is organized as follows. In section II we describe the supersymmetric factorization using the hamiltonian of the barotropic FRW cosmological model, including any factor ordering in the hamiltonian operator. In section III, we present the general solution to the Riccati equation obtained when supersymmeterizing this hamiltonian, as a strictly isospectral one-parameter family of both FRW cosmological potential and wave function of the model. Section IV is devoted to explicitly obtain the exact cosmological solutions of the WDW equation for some typical values of the $\gamma$ and $\kappa$ parameters in the literature. We give final remarks in section V.

## II. SUPERSYMMETRIC FACTORIZATION SCHEME

We start with the hamiltonian that appears in the study of the barotropic FRW cosmological model, with cosmological constant (Wheeler-DeWitt equation),

$$\hat{H}|\Psi\rangle = \frac{1}{24A} \left[ -\frac{d^2}{dA^2} + 144\kappa A^2 + 48\Lambda A^4 - 384\pi GM\gamma A^{-3\gamma+1} \right]|\Psi\rangle = 0,$$

(1)

where $A$ is the scale factor, $\kappa$ is the curvature index of the universe ($\kappa = 0$, $+1$, $-1$ plane, close and open, respectively), $\Lambda$ the cosmological term and $\gamma$ the parameter that describes the state equation of the fluid.

In principle, the order ambiguity in equation (1) should be taken into account. This is quite a difficult problem to be treated in all its generality, since the hamiltonian operator in (1) must be written in a very general form in order to take into account all possible orders. For the semi-general Hartle-Hawking factor ordering \[14\], we have \(^1\)

$$A^{-1}\frac{d^2\Psi}{dA^2} \rightarrow A^{-1+q} \frac{d}{dA} A^{-q} \frac{d\Psi}{dA} = A^{-1} \left( \frac{d^2\Psi}{dA^2} - qA^{-1} \frac{d\Psi}{dA} \right),$$

(2)

where the real parameter $q$ measures the ambiguity in the factor ordering. In this approach, the Wheeler-DeWitt equation can be written as follows

$$\mathcal{H}_0 \Psi = -A \frac{d^2\Psi}{dA^2} + q \frac{d\Psi}{dA} + V(A)\Psi = 0,$$

(3)

where

$$V(A) = 144\kappa A^3 + 48\Lambda A^5 - 384\pi GM\gamma A^{-3\gamma+2}.$$

(4)

Our interest here is to study the quantum solutions of equation (3) when using the supersymmetric factorization scheme. As we shall show, the solutions can be exactly found for particular values of the $\gamma$ parameter and the factor ordering parameter $q$.

Consider the equation (modified WDW)

\(^1\)There are other possibilities depending on different considerations on the operators, see refs. \[15,16\]
\[ H^+ \Psi = -A^{-q} \frac{d^2 \Psi}{dA^2} + qA^{-1-2q} \frac{d\Psi}{dA} + A^{-1-2q}V(A)\Psi = 0. \] (5)

It is easy to show that the first order differential operators

\[ A^+ = -A^{-q} \frac{d}{dA} + W(A), \] (6)

\[ A^- = A^{-q} \frac{d}{dA} + W(A), \] (7)

where \( W \) plays the role of a superpotential function, factorize the Hamiltonian (5) as

\[ H^+ = A^+ A^- . \] (8)

The potential term \( V(A) \) is related to the superpotential function \( W(A) \) via the Ricatti equation

\[ V_+(A, \gamma) = A^{1+2q}W_\gamma^2 - A^{1+q} \frac{dW_\gamma}{dA} . \] (9)

Making the transformation

\[ W_\gamma = -A^{-q} \frac{u_\gamma'}{u_\gamma} \] (10)

where the \( \gamma \) means \( \frac{d}{dA} \), (9) is transformed into the original Hamiltonian applied to the function \( u_\gamma \), which means that the superpotential function is know once we have a solution to the original WDW equation. If we choose the ordinary factor ordering \( q = 0 \), we recover the relation found in [12].

In the supersymmetric factorization scheme, \( V_- \), the partner superpotential of \( V_+ \), is obtained by performing the product

\[ H^- = A^- A^+, \quad H^- f(A) = 0, \] (11)

where \( f \) is the wave function related at the Hamiltonian \( H^- \). Then, the isospectral potential to \( V_+(A, \gamma) \) is

\[ V_-(A, \gamma) = A^{1+2q}W^2 + A^{1+q}W_\gamma' = V_+(A, \gamma) + 2A^{1+q}W_\gamma'. \] (12)

In addition, the solution \( f(A) \) is found using the fact that \( A^+ A^- u_\gamma = 0 \). Multiplying both sides of Eq. (12) by \( A^- \), we have that \( A^- A^+(A^- u) = H^-(A^- u) = 0 \); then \( f(A) = A^- u \). In this way, the structure for \( f(A) \) becomes

\[ f(A) = W_\gamma u_\gamma + A^{-q} u_\gamma'. \] (13)

Hence, knowing the superpotential function we can find the wave functions of the partner Hamiltonian \( H^- \). However, this is not the most general solution, as will be shown in the following section.

\[ ^2 \text{Hereafter we shall use } W_\gamma \text{ to explicitly denote the superpotential functions depend on the actual choice of the } \gamma \text{ parameter, and we shall call } V_+ \text{ to } V \text{ in (5).} \]
III. GENERAL SOLUTION

The general solution to the Ricatti equation (12) is found when we propose the following scheme [1,4]

\[ V_-(A, \gamma) = A^{1+2q}W^2 + A^{1+q}W' \equiv A^{1+2q}W^2 + A^{1+q}W', \tag{14} \]

which by choosing

\[ \hat{W} \equiv W + \frac{1}{y}, \tag{15} \]

leads to a Bernoulli equation for \( y \gamma \),

\[ y' - 2W y^a = A^q, \tag{16} \]

whose solution is

\[ y_\gamma(A) = u_\gamma^{-2} [I_\gamma + \lambda], \tag{17} \]

where \( I_\gamma(A) = \int_0^A x^a u_\gamma^2 dx \).

In this way, (15) can be written as

\[ \hat{W}(A) = W + \frac{u_\gamma^2}{I_\gamma + \lambda}, \tag{18} \]

and the entire family of bosonic potentials can be built as

\[ \hat{V}_+(A, \gamma, \lambda) = A^{1+2q}W^2(A, \gamma, \lambda) - A^{1+q}W'(A, \gamma, \lambda), \tag{19} \]

\[ \hat{V}_+(A, \gamma, \lambda) = V_+ - 2A^{-q}W' = V_+(A, \gamma) - 4 \frac{A^{1+q}u_\gamma u_\gamma'}{I_\gamma + \lambda} + 2 \frac{A^{1+2q}u_\gamma^4}{(I_\gamma + \lambda)^2}. \tag{20} \]

Finally,

\[ \hat{u}_\gamma \equiv g(\lambda) \frac{u_\gamma}{I_\gamma + \lambda}, \tag{22} \]

is the isospectral solution of the Schrödinger equation (3) for the new family potential (21), with the condition on the function \( g(\lambda) = \sqrt{\lambda(\lambda + 1)} \), though in the limit

\[ \lambda \to \pm \infty \quad g(\lambda) \to \lambda \quad \text{and} \quad \hat{u}_\gamma \to u_\gamma. \tag{23} \]

This \( \lambda \) parameter is included not for factorization reasons, because the wave functions in quantum cosmology are still nonnormalizable, but as decoherence parameter embodying a sort of quantum cosmological dissipation (or damping) distance.

The WDW (3) equation has particular solutions for the \( \gamma \) parameter and different universes, which we shall explore in the following section.
IV. SOLUTION OF THE WDW EQUATION FOR PARTICULAR $\gamma$

Quantum solutions can be readily found for particular choices of the $\gamma$ parameter. Here, we list some of them.

1. $\gamma = -1$ corresponds to inflationary sceneary. For this stadium (3) can be written

$$Au''_{-1} - qu'_{-1} + 144A^3 \left( m^2 A^2 - \kappa \right) u_{-1} = 0,$$  \hspace{1cm} (24)

where $m^2 = -\frac{A}{3} + \frac{8}{3} \pi GM_{-1}$.

Now, we have three possible cases in our analysis:

a) $m^2 > 0$,

The differential equation for this subcase is

$$Au''_{-1} - qu'_{-1} + 144 \kappa A^3 \left( m^2 A^2 - \kappa \right) u_{-1} = 0,$$  \hspace{1cm} (25)

which, after the consecutive substitutions $v = m^2 A^2 - \kappa$, $u_{-1} = v^{1/2}y(v)$, and $z = \frac{4}{m^2}v^{3/2}$, yields an ordinary Bessel equation for $q = 1$, with solution

$$u_{-1} = \left( m^2 A^2 - \kappa \right)^{1/2} \left[ a_0 J_{\frac{1}{2}}(z) + b_0 J_{-\frac{1}{2}}(z) \right],$$ \hspace{1cm} (26)

where $z = \frac{4}{m^2} \left[ m^2 A^2 - \kappa \right]^{3/2}$, and $a_0$ and $b_0$ are superposition constants.

b) $m^2 < 0$,

The differential equation for this subcase is

$$-Au''_{-1} + qu'_{-1} + 144 \kappa A^3 \left( |m^2| A^2 + \kappa \right) u_{-1} = 0,$$ \hspace{1cm} (27)

being the solution the modified Bessel equation for $q = 1$.
\[ u_{-1} = \left(|m^2|A^2 + \kappa\right)^{\frac{1}{2}} \left[a_1 I_{\frac{\nu}{2}}(z) + b_1 K_{\frac{\nu}{2}}(z)\right], \] (28)

where \( z = \frac{4}{|m^2|} \left[|m^2|A^2 + \kappa\right]^{3/2} \), and \( a_1 \) and \( b_1 \) are superposition constants.

c) For \( m^2 = 0 \), and the differential equation for this situation is

\[-Au_{-1}'' + qu_{-1}' + 144\kappa A^3 u_{-1} = 0,\] (29)

which, for \( \kappa = 1 \) has as solutions the modified Bessel functions of order \( \nu = \frac{1+q}{4} \)

\[ u_{-1} = A^{2\nu} \left[A_0 I_{\nu}(6A^2) + B_0 K_{\nu}(6A^2)\right], \] (30)

where \( A_0 \) and \( B_0 \) are superposition constants, whereas for \( \kappa = -1 \), the solutions become the ordinary Bessel functions

\[ u_{-1} = A^{2\nu} \left[A_1 J_{\nu}(6A^2) + B_1 Y_{\nu}(6A^2)\right], \] (31)

where \( A_1 \) and \( B_1 \) are superposition constants.

2. Dust era, \( \gamma = 0 \), and plane universe, \( \kappa = 0 \):

By use of the transformations \( z = \frac{8}{3}\sqrt{3A} A^3 \) and \( u_0 = e^{-z/2} w(z) \), we find the hypergeometric differential equation for \( w(z) \)

\[ z \frac{d^2w}{dz^2} + (\alpha - z) \frac{dw}{dz} - nw = 0, \] (32)

where \( n = \frac{2-q}{6} - \frac{16\pi G M_0}{\sqrt{-3A}} \) and \( \alpha = \frac{2-q}{3} \). Therefore, the independent solutions are [17]

\[ w_1 = {}_1F_1(n, \alpha; z), \] (33)
\[ w_2 = z^{1-\alpha} {}_1F_1(n - \alpha + 1, 2 - \alpha; z), \] (34)

where \( {}_1F_1 \) is the degenerate hypergeometric function. In this way, the solution for this case become

\[ u = e^{-z/2} \left[A_2 w_1(z) + B_2 w_2(z)\right], \] (35)

where \( A_2 \) and \( B_2 \) are superposition constants.

3. Stiff fluid, \( \gamma = 1 \) and plane universe

The WDW equation for this stadium is written as

\[ Au''_1 - qu'_1 + 48 \left(-\Lambda A^5 + 8\pi G M_1 A^{-1}\right) u_1 = 0, \] (36)

whose solutions become
\[ u = A^{\frac{1+q}{3}} Z_{\mu} \left( \frac{4\sqrt{-3\Lambda}}{3} A^3 \right); \quad \text{with} \quad \mu = \frac{1}{3} \sqrt{\left( \frac{1+q}{2} \right)^2 - 384\pi GM_1}, \tag{37} \]

where \( Z_{\mu} \) is a generic Bessel function with real or imaginary order \( \mu \) [18].

In this case, the exact expression of the solutions will depend on the signs of the cosmological constant and the parameter \( \mu \).

i) \( \mu \) real and \( \Lambda > 0 \), the function \( Z_{\mu} \) become the modified Bessel functions, either \( I_{\mu} \) or \( K_{\mu} \), depending on the boundary conditions.

ii) \( \mu \) real and \( \Lambda < 0 \), the functions \( Z_{\mu} \) turn into the ordinary Bessel function, either \( J_{\mu} \) or \( Y_{\mu} \); depending on the boundary conditions.

iii) \( \Lambda > 0 \) and \( \mu \) pure imaginary, the functions \( Z_{\mu} \) become the modified Bessel functions of pure imaginary order [18], either \( I_{\mu} \) or \( K_{\mu} \), depending on the boundary conditions.

iv) \( \Lambda < 0 \) and \( \mu \) pure imaginary, the functions \( Z_{\mu} \) turn into the ordinary Bessel functions of pure imaginary order [18], either \( J_{\mu} \) or \( Y_{\mu} \), depending on the boundary conditions.

V. REMARKS

SUSY-QM has allowed us to show that new classes of exact solutions are found for the WDW equation, extending the class of exactly solvable spectral problem for the Schrödinger like hamiltonian in one space dimension (in quantum cosmology the WDW equation plays this role). In this work, we found isospectral cosmological potential and a one-parameter family of wavefunctions of the universe for barotropic FRW models with a cosmological constant, including the factor ordering problem. We were able to find exact particular solutions for different choices of the \( \gamma \) parameter in the WDW equation (3) and different universes. The parameter \( \Lambda \) looks like a decoherence parameter embodying a sort of quantum cosmological dissipation (or damping) distance (see Fig. (1)).

[1] B. Mielnik, J. Math. Phys. 25, 3387 (1984).
[2] M.M. Nieto, Phys. Lett. B 145, 208 (1984).
[3] D.J. Fernández, Lett. Math. Phys. 8, 337 (1984).
[4] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995).
[5] I.M. Gelfand and B.M. Levitan, Am. Math. Soc. Transl. bf 1, 253 (1955).
[6] B.F. Samsonov and A.A. Suzko, Discrete supersymmetries of the Schrödinger equation and non-local exactly solvable potential, quant-ph/0301109; G. Junker, Supersymmetric Methods in Quantum and Statistical Physics, Springer, Berlin (1996); B.K. Bagchi, Supersymmetry in Quantum and Classical Mechanics, Chapman & Hall, New York, (2001).
[7] G. Darboux, C.R. Acad. Sci. (Paris) 94, 1456 (1882).
[8] J. Pappademos, U. Sukhatme and A. Pagnamenta, Phys. Rev. A 48, 3525 (1993).
[9] E. L. Ince, *Ordinary Differential Equations*, Dover, New York (1926).
[10] V.B. Matveev and M.A. Salle, *Darboux transformations and Solitons*, Springer, Berlin (1991).
[11] V.G. Bagrov and B.F. Samsonov, Theor. Math. Phys. 104, 356 (1995).
[12] H. Rosu and J. Socorro, Il Nuovo Cimento B 113, 683 (1998).
[13] H. Rosu and J. Socorro, Phys. Lett. A 223, 28 (1996).
[14] J. Hartle and S. W. Hawking, *Phys. Rev. D* 28, 2960 (1983).
[15] T. Christodoulakis and J. Zanelli, *Phys. Lett. A* 102, 227 (1984); *Phys. Rev. D* 29, 2738 (1984)
[16] J.E. Lidsey and P.V. Moniz, *Class. Quantum Grav.* 17 4823 (2000).
[17] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, (Academic Press, 1980), page 1059.
[18] T.M. Dunster, *Siam J. Math. Anal* 21(4), 995 (1990).