Automorphisms groups for \( p \)-cyclic covers of the affine line

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Abstract

Let \( k \) be an algebraically closed field of positive characteristic \( p > 0 \) and \( C \to \mathbb{P}_k^1 \) a \( p \)-cyclic cover of the projective line ramified in exactly one point. We are interested in the \( p \)-part of the full automorphism group \( \text{Aut}_k C \). First we prove that these groups are exactly the extraspecial \( p \)-groups and groups \( G \) which are subgroups of an extraspecial group \( E \) such that \( Z(E) \subseteq G \). The paper also describes an efficient algorithm to compute the \( p \)-part of \( \text{Aut}_k C \) starting from an Artin-Schreier equation for the cover \( C \to \mathbb{P}_k^1 \).

The interest for these objects initially came from the study of the stable reduction of \( p \)-cyclic covers over the \( p \)-adics. There the covers \( C \to \mathbb{P}_k^1 \) naturally arise and their automorphism groups play a major role in understanding the arithmetic monodromy. Our methods rely on previous work by Stichtenoth [St1], [St2] whose approach we have adopted.

1 Introduction

When considering semi-stable models for \( p \)-cyclic covers of the projective line over a \( p \)-adic field \( K \), we get as irreducible components of the special fiber \( p \)-cyclic covers (we mean étale) of the affine line (see [Le1], [Ma]). An interesting arithmetico geometric object is the monodromy, i.e. the minimal Galois extension \( K'/K \) (say with group \( G \)) such that a semi-stable model is defined over \( K' \). A classical result (see [De], [Liu, theorem 4.44, p.551]) asserts that this group acts faithfully on the special fiber of the semi-stable model as an automorphism group; so the complexity of the monodromy group is intimately related with the automorphism group of \( p \)-cyclic covers of the affine line over the residue field \( k \). The aim of this paper is first to report and then to complete the literature on the subject ([St1], [St2], [vdG-vdV]) with the objective to use the results in order to study the monodromy group which occurs when considering \( p \)-cyclic covers of the projective line over a \( p \)-adic field (see [Le-Ma1], [Le-Ma2]).

Our choice to disjoin this paper from the one on monodromy came from the new interest in automorphism groups of curves (see [Gu], [Leo], [Po1], [Po2]).

Let us discuss the content of the paper. In the following theorem we have gathered the bounds proved in the paper.
**Theorem 1.1.** Let $k$ be an algebraically closed field of char $p > 0$ and $f(X) \in k[X]$ a polynomial of degree $m := \deg f$ prime to $p$. Let $k(X,W)/k(X)$ be an extension of degree $p$ defined by $W^p - W = f(X)$ and denote by $\infty$ the place in $k(X,W)$ above $X = \infty$. We write $\text{Aut}_k(X,W)$ for the full automorphism group, $\supset G_\infty(f)$ (the inertia group at $\infty$) $\supset G_{\infty,1}(f)$ (the wild inertia group at $\infty$). Let $g(f) = \frac{(m-1)(p-1)}{2}$ be the genus of $k(X,W)/k$ and assume $g(f) > 1$. Then

a) [St2 Satz 4], [Lemma 4.3 i)] $|G_{\infty,1}(f)| \leq p(m-1)^2$.

b) [St2 Satz 4], [Lemma 4.3 ii)] If $(m-1, p) = 1$, then $|G_{\infty,1}(f)| = p$.

c) [St2 Satz 5], [Proposition 4.12] If $f(X) = X^m$, then $|G_{\infty,1}(f)| = p$ for $m - 1 \neq 0$ and $|G_{\infty,1}(f)| = p^{2s+1}$ for $m - 1 = p^s$.

d) [Lemma 4.12] If $m - 1 = \ell p^s$, $s > 0$, $\ell > 1$, $(\ell, p) = 1$, then $|G_{\infty,1}(f)| \leq p^{s+1}$ for $p > 2$ and $|G_{\infty,1}(f)| \leq 2^s$ for $p = 2$; moreover these bounds are optimal.

e) [Proposition 4.16] If $\frac{|G_{\infty,1}(f)|}{g(f)} > \frac{p}{p-1}$ then $f(X) = XR(X)$, where $R(X)$ is an additive polynomial.

In order to answer the question “What is the structure of $G_{\infty,1}(f)$”, we face the following: For a given $f \in Xk[X]$ as above determine the set:

$$S(f) := \{y \in k \mid \exists P(X,y) \in Xk[X], \quad f(X+y) - f(X) - f(y) = (\text{Id} - F)P(X,y)\}$$

It is easy to see that giving $S(f)$ is equivalent to finding the additive polynomial $\text{Ad}_f(Y) = \prod_{y \in S(f)} (Y - y)$. Here we give an algorithm (cf. section 4.8) which for a given $f \in Xk[X]$ produces the polynomial $\text{Ad}_f(Y)$ and for $y \in S(f)$ the polynomial $P(X,y) \in Xk[X]$. From this we deduce the structure of the group $G_{\infty,1}(f)$. Moreover we use this algorithm in order to produce polynomials $f(X) \in k[X]$ or families $f_\ell \in k[X,\ell]$, with prescribed group $G_{\infty,1}(f)$. Next we describe the groups $G_{\infty,1}(f)$.

Let us first define 3 sets of isomorphism classes of $p$-groups.

$$C_1 := \{G \mid \exists N \text{ } p\text{-cyclic and normal subgroup } \subset G \mid G/N \text{ is } p\text{-elementary abelian}\}$$

$$C_2 := \{G \mid \exists E \text{ extraspecial group, } V \subset \frac{E}{Z(E)} \text{ } \mathbb{F}_p\text{-subspace } \mid G \simeq \pi^{-1}(V), \text{ where } \pi : E \to \frac{E}{Z(E)} \text{ is the canonical map}\}$$

$$C_3 := \{G \mid \exists f \in Xk[X], \ (\deg f, p) = 1 \text{ such that } G \simeq G_{\infty,1}(f)\}$$

**Theorem 1.2.** The 3 classes $C_i, i = 1, 2, 3$ are equal.
2 Notations

Throughout this paper we use the following notation:

- $k$ is an algebraically closed field of characteristic $p > 0$.
- $F$ design the Frobenius endomorphism for a $k$-algebra.
- $f(X) \in k[X]$ a polynomial, the equation $W^p - W = f(X)$ defines an étale cover of the affine line that we denote $C_f$; moreover each étale cover of the affine line can be presented like this. Let $f(X) \in k[X]$; there is a unique polynomial $\text{red}(f)(X) \in k[X]$ called the reduced representative of $f$ which is $p$-powers free, i.e. $\text{red}(f)(X) \in k \bigoplus_{i=1}^{s} kX^i$ and such that $\text{red}(f)(X) = f(X) \mod (F - \text{Id})k[X]$. Clearly the covers $C_f$ and $C_{\text{red}(f)}$ are the same $p$-cyclic cover of the affine line. The curve $C_f$ is irreducible iff $\text{red}(f) \neq 0$. In the sequel we assume that $\text{red}(f) \neq 0$, the degree of the reduced polynomial $\text{red}(f)$ is called the conductor of the cover it is prime to $p$ and equal to the degree of $f$ if $(\deg f, p) = 1$.
- By $C_f$ we also denote the non singular projective curve with function field $k(X, W)$. If $m = \deg f$ is prime to $p$ then the genus of $C_f$ is $g(C_f) = \frac{(m-1)(m-1)}{2}$.
- For $M \in C_f$ and $n \in \mathbb{N}$ let $L(nM) = \{ \varphi \in k(C_f) \mid (\varphi) + nM \geq 0 \}$, $\mathcal{P}(M) := \{-v_M(\varphi) \mid \varphi \in \bigcup_{n \in \mathbb{N}} L(nM) - \{0\}\}$ is the polar semi-group at $M$.
- We denote by $\infty \in C_f$ the point $X = \infty$, $W = \infty$ and by $M_{x,w}$ the point $X = x$, $W = w$ with $w^p - w = f(x)$.
- We denote $G(f) = \text{Aut}_kC_f$, $G_\infty(f)$ the inertia group at $\infty$ and $G_{\infty,1}(f)$ the wild inertia group at $\infty$. Let $\rho \in G(f)$ be such that $\rho(X) = X$ and $\rho(W) = W + 1$. Then $\rho$ generates a $p$-cyclic subgroup in $G_\infty(f)$.

3 Review and improvements of Stichtenoth’s results.

Stichtenoth proves the following:

Theorem 3.1 ([ST2], Satz 6). Assume the genus $g(C_f) \geq 2$, i.e. $\{m, p\} \neq \{2, 3\}$. Then $\infty$ is the only point $M \in C_f$ such that $\mathcal{P}(M) = \deg f\mathbb{N} + p\mathbb{N}$, with the following two exceptions:

a) $m < p$, $m|1 + p$, $f(X) = X^m$. In this case in addition to $\infty$, exactly the $p$ zeroes of $X$ have the same semi-group $\mathcal{P}(M)$. More precisely for $i \in \mathbb{F}_p$, let $\sigma_i$ be given by $\sigma_i(X) = \frac{X}{(W-1)^i}$ and $\sigma_i(W) = -\frac{1}{W-1}$, where $\delta = \frac{1+p}{m}$. Then $\sigma_i \in \text{Aut}_kC_f$ and $\sigma_i(M_{0,i}) = \infty$, in particular $\text{Aut}_kC_f$ acts transitively on the $p + 1$ points $M$ such that $\mathcal{P}(M) = \mathcal{P}(\infty)$. 

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b) \( f(X) = X^{1+p} \). Now exactly the \( p^2 \) points \( M_{a,b} \) where \( \alpha^p = \alpha \) and \( \beta^p - \beta = \alpha \) have the same semi-group at \( \infty \). Let \( \zeta \in k \) with \( \zeta^{p^2-1} = -1 \),

\[
\sigma_{a,b}(X) = \alpha + \frac{X}{\zeta^{1+p}W}, \quad \text{and} \quad \sigma_{a,b}(W) = \beta + \frac{1}{\zeta^{2(1+p)}W} - \frac{\alpha^p X}{\zeta^2 W}.
\]

Then \( \sigma_{a,b} \in \text{Aut}C_f \) and \( \sigma_{a,b}(\infty) = M_{a,b} \). In particular \( \text{Aut}C_f \) acts transitively on the \( p^2 + 1 \) points \( M \), such that \( P(M) = P(\infty) \).

**Remark 3.2.** Stichtenoth considers equations of type \( W^p + W = f(X) \), the expressions of the automorphisms \( \sigma_i \) and \( \sigma_{a,b} \) in case a) and b) are then simpler. For a description of the full automorphism group in case b) we refer to [Leo].

We deduce the following

**Proposition 3.3.** Let \( f \in Xk[X], \ g \in Tk[T] \) and \( C_g : V^p - V = g(T) \) such that \( \deg f, \ \deg g \) are prime to \( p \) and \( \deg(C_f) \geq 2 \). We assume \( \exists \sigma : C_f \to C_g \), a \( k \)-isomorphism. Then \( \exists \sigma \in \text{Aut}C_f \) such that \( \varphi \circ \sigma(\infty) = \infty \) and \( \varphi \circ \sigma \) descends to \( \mathbb{P}^1 \), i.e. \( \exists \tilde{\varphi} : \mathbb{P}^1 \to \mathbb{P}^1 \) and a commutative diagram

\[
\begin{array}{ccc}
C_f & \xrightarrow{\varphi \circ \sigma} & C_g \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow{\tilde{\varphi}} & \mathbb{P}^1.
\end{array}
\]

Moreover \( \exists (a,b) \in (k^\times, k) \) and \( c \in \mathbb{F}_p^\times \) such that \( \tilde{\varphi}(X) = aX + b \) and \( cf(X) - g(aX + b) \in (F - 1)k[X] \). If \( f \) is not in case a) or b) of Theorem we can take \( \sigma = \text{Id} \). Reciprocally any triple \( (a,b,c) \in k^\times \times k \times \mathbb{F}_p^\times \) such that \( cf(X) - g(aX + b) \in (F - 1)(k[X]) \) induces such a commutative diagram.

**Proof.** \( \varphi : k(C_g) \to k(C_f) \) is a \( k \)-isomorphism which induces a graded isomorphism between the linear spaces \( L(\infty \varphi(M)) \) and \( L(\infty M) \) and so the polar semi-groups are preserved. It follows that \( \varphi^{-1}(\infty) \in C_f \) has a polar semi-group equal to that of \( \infty \in C_f \). We deduce the existence of \( \sigma \) and can assume from now on that \( \varphi(\infty) = \infty \).

Let \( m = \deg f \), it is classical that \( \deg(C_f) = \frac{(m-1)(p-1)}{2} \), so as \( \deg(C_f) = \deg(C_G) \) it follows that \( \deg f = \deg g \). Now we follow the discussion in Stichtenoth’s paper [St2, Satz 4].

Case 1. \( m > p \). Then 1, \( T, \ldots, T^{\left\lceil \frac{p}{r} \right\rceil} \), \( V \) is a \( k \)-basis for \( L(p\infty) \subset k(C_g) \) resp. \( L(m\infty) \subset k(C_g) \). Idem 1, \( X \) and \( 1, X, \ldots, X^{\left\lceil \frac{p}{r} \right\rceil} \), \( W \) are a basis for \( L(p\infty) \subset k(C_f) \) resp. \( L(m\infty) \subset k(C_f) \). As \( \varphi(L(p\infty)) = L(m\infty) \) resp. \( \varphi(L(m\infty)) = L(m\infty) \), we get \( \exists a \in k^\times, b \in k \), such that \( \varphi(T) = aX + b \) and \( \varphi(V) = cW + Q(X) \) where \( c \in k^\times \) and \( Q(X) \in k[X] \) with \( \deg Q(X) \leq \left\lfloor \frac{m}{p} \right\rfloor \). Then \( (c^p - c)W + c^p f(X) + (Q(X)^p - Q(X)) = g(aX + b) \), so \( c^p = c \) and \( cf(X) - g(aX + b) \in (F - 1)(k[X]) \). Reciprocally such relations define an isomorphism.

Case 2. \( m < p \). It works similarly, namely 1, \( V \), resp. 1, \( V, \ldots, V^r \), \( T \) with \( rm < p < (r+1)m \) is a basis for \( L(m\infty) \) resp. \( L(p\infty) \) in \( k(C_g) \). Then \( \varphi(T) = aX + P(W) \), where \( P(W) \in k[W] \) with \( m \deg P < p \) and \( \varphi(V) = cW + Q \) where \( c \in k^\times \) and \( Q \in k \). We get the equation \( (c^p - c)W + c^p f(X) + (Q(X)^p - Q(X)) = g(aX + P(W)) \) =
$a^m X^m + (ma^{m-1}P(W) + b_{m-1})X^{m-1} + \ldots + P(W)^m$, where $g(T) = T^m + b_{m-1}T^{m-1} + \ldots$.

Comparing the $W$ degrees it follows that $P(W) = b \in k$ and $c^p = c$. Now we get again the condition $e f(X) - g(aX + b) \in (F - Id)k[X]$.}

**Corollary 3.4.** Let $f \in Xk[X]$ such that $g(C_f) \geq 1$ and $S(f) := \{(a, b, c) \in k^x \times k \times \mathbb{F}_p^x \mid \exists P_{a, b, c}(X) \in Xk[X], cf(X) - f(aX + b) = P_{a, b, c}(X)^p - P_{a, b, c}(X)\}$ moreover such a polynomial $P_{a, b, c}(X)$ in $Xk[X]$ is uniquely determined by the relation. Then for $(a, b, c) \in S(f)$, the formulas $\sigma_{a, b, c}(X) = aX + b$, $\sigma_{a, b, c}(W) = cW + P_{a, b, c}(X)$ define an automorphism of $C_f$ which lies in $G\infty(f)$. Further $G\infty(f) = < \rho, \sigma_{a, b, c} >$ for $(a, b, c) \in S(f)$.

**Proof.** If $g(C_f) > 1$, this follows from the proposition. If $g(C_f) = 1$ then $\{p, m\} = \{2, 3\}$; if $p = 2$ then $f(X) = aX + X^3$ and if $p = 3$ then $f(X) = aX + X^2$; in each case we can give a basis for $L(m\infty)$ and $L(p\infty)$ of the same kind of those given in the two cases in the proof of the proposition; we conclude in the same way. Note that if $g(C_f) = 0$ then $G\infty(f) = \kappaX + k$.

**Notations.** In order to simplify the notations for $(1, b, 1) \in S(f)$ we denote by $\sigma_b$ the element $\sigma_{1,b,1}$ and by $P_f(X, b)$ the corresponding polynomial $P_{1,b,1}(X)$. For $(1, y, 1)$ and $(1, z, 1)$ in $S(f)$ we define the following function $\epsilon_f(y, z) = P_f(X, y) + P_f(X + y, z) - P_f(X, z) - P_f(X + z, y)$.

Now we can give the precise structure of the wild automorphism group:

**Corollary 3.5.** Assume $g(C_f) > 0$, then the element $\rho \in Z(G\infty,1(f))$ and $G\infty,1(f) = < \rho, \sigma_b >$. The commutation rule is given by $[\sigma_y, \sigma_z] = \rho^{\epsilon_f(y, z)}$ where $\epsilon_f(y, z) = P_f(X, y) + P_f(X + y, z) - P_f(X, z) - P_f(X + z, y) \in \mathbb{Z}/p\mathbb{Z}$. Moreover we have the following exact sequence $0 \rightarrow< \rho \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow G\infty,1(f) \xrightarrow{\pi} k$ where $\pi(\sigma_y) = y \in k$ and the image of $\pi$ is finite dimensional as $\mathbb{F}_p$-vector space.

**Proof.** Clear.

## 4 Universal family - Modifications of covers - Algorithm

### 4.1 Universal family

**Notations.** In order to be able to treat families of covers it is useful for a given conductor $m$ prime to $p$ to work over the ring $A := \mathbb{F}_p[t_i, 1 \leq i \leq m]$ and consider $f(X) = \sum_{1 \leq i \leq m} t_iX^i + X^m$.

A specialization homomorphism is an homomorphism $\varphi : A \rightarrow k$, where $k$ is an algebraically closed field of characteristic $p > 0$; then $\varphi(f)(X) = \sum_{1 \leq i \leq m} \varphi(t_i)X^i + X^m \in k[X]$.

Let $i < m$ and $(i, p) = 1$ and denote by $n(i) := \max\{n \in \mathbb{N} \mid ip^n < m\}$. The following lemma measures the defect for the polynomial $\varphi(f)$ to be additive. For this we introduce $\Delta(f)(X, Y) := f(X + Y) - f(X) - f(Y)$. 
Lemma 4.1.  With the notations above there is a unique polynomial $F(X,Y) \in \bigoplus_{1 \leq i < m, (i,p)=1} A[Y]X^{ip^{n(i)}}$ and a unique polynomial $P_f(X,Y) \in XA[Y][X]$ such that

$$\Delta(f)(X,Y) = F(X,Y) + (\text{Id} - F)P_f(X,Y)$$

(1)

The polynomial $P_f(X,Y)$ is characterized by the following:

$$P_f(X,Y) = (\text{Id} + F + \cdots + F^n)\Delta(f) \mod (X^{\frac{m-1}{p}})$$

(2)

for any $n$ such that $p^n > \frac{m-1}{p}$.

Proof. Existence. Note that $\deg X \Delta(f) = m - 1$ and $\Delta f \in (X,Y)A[X,Y]$. Let $(i,p) = 1$ with $1 \leq i < m$, and $0 \leq j < n(i)$. For a monomial $a_{ip^j}(Y)X^{ip^j}$ of $\Delta(f)$ where $a_{ip^j}(Y) \in A[Y]$ and total degree $m$ we write $a_{ip^j}(Y)X^{ip^j} = (a_{ip^j}(Y))^{p^n(i)-1}X^{ip^{n(i)}} + (\text{Id} - F)(P_{ip^j}(X))$ with $P_{ip^j}(X) = (\text{Id} + F + \cdots + F^{n(i)-1}) (a_{ip^j}(Y)X^{ip})$,

For the unicity we remark that if $P_f(X,Y)$ satisfies the formula (1) then $\deg X P_f(X,Y) \leq \frac{m-1}{p}$, so it is sufficient to prove the formula (2) in the lemma. We have the identity:

$$(\text{Id} + F + \cdots + F^{n-1})\Delta(f) = (\text{Id} + F + \cdots + F^{n-1})F(X,Y) + (\text{Id} - F^n)P_f(X,Y)$$

for any $n$. Now $F(X,Y) \in (X^{\frac{m-1}{p}})$ as $ip^{n(i)} < m < ip^{n(i)+1}$ and as $P_f(X,Y) \in XA[Y][X]$. Then for $p^n > \frac{m-1}{p}$ we obtain the formula (2). \qed

Definition 4.2. Let $\varphi : A \to k$ be a specialization homomorphism. We denote by the same letter the induced homomorphism on polynomials via the action on the coefficients. Let $\text{Ad}_{\varphi(f)}(Y) \in k[Y]$ be the monic generator of the ideal of coefficients of $\varphi(F)(X,Y)$ in $k[Y]$. As usual we denote by $Z(\text{Ad}_{\varphi(f)}(Y))$ the set of zeroes in the algebraically closed field $k$.

Lemma 4.3. Write $F(X,Y) = \sum_j a_j(Y)X^j = \sum_{1 \leq i < m, (i,p)=1} a_{ip^{n(i)}}(Y)X^{ip^{n(i)}}$.

i) The coefficient $a_{m-1}(Y) \in mY + (YA[Y])^p$, and $\deg a_{m-1}(Y) \leq (m-1)^2$.

ii) If $m = 1 + \ell p^s$ where $\ell > 1$, $\ell \equiv 1$, $s = v_p(m-1) > 0$, let $j_0 = 1 + (\ell - 1)p^s$. Then $j_0$ is prime to $p$ (also if $s = 0$) and $n(j_0) = 0$. Moreover the coefficient $a_{j_0}(Y) = \ell Y^{p^s} + \cdots + s t_{j_0+1}Y$ if $p > 2$ and if $p = 2$ one has $a_{j_0}(Y) = \ell Y^{p^s} + \cdots + t_{j_0+1}Y^2$. If $(p, m-1) = 1$ (so $s = 0$ and $p > 2$) one has $a_{j_0}(Y) = (\ell + 1)Y = mY$.

Proof. For i) we remark that $m - 1 = \ell p^s = \ell p^{n(\ell)}$ is the highest representative $< m$ of $\ell$ modulo multiplication by a power of $p$. As $\Delta(X^m) = mYX^{m-1}$ + lower degree terms and as lower degree monomials give contributions in $YA[Y]$ that one needs to raise to some $p$-power, the result follows. Concerning the degree, we remark that $\Delta(f)(X,Y) = \sum_{i \leq m-1} \delta_i(Y)X^i$ and $\deg \delta_i(Y) \leq m - 1$. Now write $i = jp^{n(i)}$ then the contribution of $\delta_i(Y)X^i$ in $F(X,Y)$ is $(\delta_i(Y)X^i)^{p^n(i)}$ whose $Y$-degree is $\leq (m-1)^2$.

For ii) we remark that $j_0 > \frac{m-1}{p} + 1$ and $j_0$ is prime to $p$. So the coefficient of the monomial $X^{j_0}$ in $F(X,Y)$ is the same as that of $\Delta(f)$ and so equal $\sum_{j_0 < i \leq m, (i,p)=1} (\delta_j)^t Y^{i-j_0} = \ell Y^{p^s} + \cdots + (j_0 + 1)t_{j_0+1}Y$. The result follows. \qed
Proposition 4.4. Let \( \varphi : A \to k \) be a specialization homomorphism. Then \( \text{Ad}_{\varphi(f)}(Y) \) is a separable and additive polynomial and \( Z(\text{Ad}_{\varphi(f)}) = \{ y \in k \mid \Delta(\varphi \circ f)(X, y) \in (Id - F)k[X] \} \). Moreover if \( y \in Z(\text{Ad}_{\varphi(f)}) \) then there is a unique \( P(X) \in Xk[X] \) such that \( \Delta(\varphi(f))(X, y) = (Id - F)P(X) \) and \( P(X) = \varphi(P_f)(X, y) \). Then \( Z(\text{Ad}_{\varphi(f)}) = \{ y \in k \mid \sigma_y \in G_{\infty, 1}(f) \} \); in particular \( |G_{\infty, 1}(\varphi(f))| = \deg \text{Ad}_{\varphi(f)}(Y) \) and for \( y, z \in Z(\text{Ad}_{\varphi(f)}) \) the commutation rule for \( \sigma_y, \sigma_z \in G_{\infty, 1}(\varphi(f)) \) is \( \epsilon_{\varphi(f)}(y, z) = \varphi(P_f)(X, y) + \varphi(P_f)(X + y, z) - \varphi(P_f)(X, z) - \varphi(P_f)(X + z, y) \).

Proof. \( \text{Ad}_{\varphi(f)}(Y) \) is separable because we know from Lemma 4.3 i) it divides the polynomial \( a_{m-1}(Y) \in mY + (Y[A][Y])^p \) which is separable. Now to prove it is additive, it suffices to show its set of roots is stable under addition. We first remark that \( Z(\text{Ad}_{\varphi(f)}) \subset \{ y \in k \mid \sigma_y \in G_{\infty, 1}(f) \} \). For the reverse inclusion we remark that in the equality
\[
\Delta \varphi(f)(X, y) = (Id - F)P(X)
\]
(3) where \( P(X) \in k[X] \), we can assume that \( P(X) \in Xk[X] \) as \( \Delta(\varphi(f))(0, y) = 0 \). Note that \( \deg P \leq \left\lfloor \frac{m-1}{p} \right\rfloor \); then for \( n \gg 0 \) \( (Id + \ldots + F^n)\Delta(\varphi(f))(X, y) = P(X) \) mod \( X^{\frac{m-1}{p}+1} \), so \( P(X) = \varphi(P_f)(X, y) \). Then from 4.1 and (3) it follows that \( \varphi(F)(X, y) = 0 \), i.e. \( \text{Ad}_{\varphi(f)}(y) = 0 \).

Let \( y, z \in k \), such that \( \Delta(\varphi(f))(X, y), \Delta(\varphi(f))(X, z) \in (Id - F)k[X] \). We have the following general identity \( \Delta(\varphi(f))(X, y + z) = \Delta(\varphi(f))(X + y, z) + \Delta(\varphi(f))(X, z) - \Delta(\varphi(f))(y, z) \) and for our choice of \( y, z \) each term on the right hand side is in \( (Id - F)k[X] \), so \( \text{Ad}_{\varphi(f)}(y + z) = 0 \).

Corollary 4.5. We denote by \( \text{Id} : A \to A \) the inclusion homomorphism. Then

A. i) If \( p = 2 \) and \( m = 3 \) then \( \text{Ad}_{\text{Id}(f)}(Y) = Y^4 + Y \).

ii) If \( p = 2 \) and \( m = 5 \) then \( \text{Ad}_{\text{Id}(f)}(Y) = Y^{16} + t_1^2 Y^8 + t_3 Y^2 + Y \).

iii) If \( p = 3 \) and \( m = 4 \) then \( \text{Ad}_{\text{Id}(f)}(Y) = Y^9 + 2t_2^2 Y^3 + Y \).

B. Outside case A. One has \( \text{Ad}_{\text{Id}(f)}(Y) = Y \). Moreover \( \exists D(t) \in \mathbb{F}_p[t] \setminus \{0\} \) such that for any specialization \( \varphi : A \to k \) with \( \varphi(D(t)) \neq 0 \) one has \( \text{Ad}_{\varphi}(Y) = Y \) and so \( G_{\infty, 1} \simeq \mathbb{Z}/p\mathbb{Z} \).

Proof. A. A direct calculation gives the formulas.

B. If \( p \) doesn’t divide \( m - 1 \) the result follows from Lemma 4.3 ii). Therefore we can write \( m = 1 + \ell p^s \) with \( (\ell, p) = 1, s > 0 \). We distinguish two cases.

1. \( \ell > 1 \); let \( j_0 = 1 + (\ell - 1)p^s \), write \( F(X, Y) = \sum_j a_j(Y)X^j \) then \( a_{j_0}(Y) = \ell Y^p + \text{lower degree terms} \). The ring \( A \) is factorial and \( F(X, Y) \in A[Y][X] \), then the content of \( F(X, Y) \in A[Y][X] \) is unitary and so equal to \( \text{Ad}_{\text{Id}}(Y) \). This show that \( \text{Ad}_{\text{Id}}(Y) \in A[Y][X] \) is a unitary polynomial. Let \( \varphi : A \to k \) be a specialization morphism then \( \varphi(\text{Ad}_{\text{Id}}) \) is unitary and \( \deg \varphi(\text{Ad}_{\text{Id}}) = \deg \text{Ad}_{\text{Id}}(Y) \). Now we remark that \( \varphi(\text{Ad}_{\text{Id}}) \) divides \( \varphi(a_{m-1}(Y)) \in mY + (Y[A][Y])^p \) and so is still separable in \( k[X] \) (cf. Lemma 4.3 i). Let us now consider the specialization homomorphism \( \varphi_0 : A \to \mathbb{F}_p \) defined by \( \varphi_0(t_i) = 0 \) for \( i = 1, \ldots, m - 1 \). Then \( \varphi_0(f)(X) = X^m \) and \( \text{Ad}_{\varphi_0}(Y) \) divides \( \ell Y^p \) (see the proof of Lemma 4.3 ii). It follows that \( \text{Ad}_{\text{Id}}(Y) = Y \).

2. \( \ell = 1 \), then \( m = 1 + p^s \); with \( s > 0 \). We show that \( \text{Ad}_{\text{Id}}(Y) \) is unitary: A look at the monomials in \( \Delta(f)(X, Y) \) shows that the highest contribution in \( a_{m-1}(Y) \)
comes from \((X + Y)^p^s\) and more precisely from the linear term \(Y^p^sX\) which we need to raise to the power \(p^s\). So finally \(a_{m-1}(Y) = Y^p^sX\) lower degree monomials. Now like in case 1 we look for good specialization morphisms. For \(p > 1\), we consider \(\varphi_0(f)(X) = X^3 + X^m\), then \(\text{Ad}_{\varphi_0}(Y)|\varphi_0(a_{2p^s-1}(Y)) = Y^p^{s-1}\), we conclude as in case 1.

For \(p = 2\) and \(s > 2\), we consider \(\varphi_0(f)(X) = X^7 + X^m\), then \(\text{Ad}_{\varphi_0}(Y)|\varphi_0(a_{6p^s-2}(Y)) = Y^p^{s-2}\), we conclude as in case 1.

For \(p = 3\) and \(s > 1\), we consider \(\varphi_0(f)(X) = X^9 + X^m\), then \(\text{Ad}_{\varphi_0}(Y)|\varphi_0(a_{4p^s-1}(Y)) = (2Y)^p^{s-1}\), we conclude as in case 1.

In order to exhibit a polynomial \(D(t)\) we recall that \(Y\) is the content of the polynomial \(F(X,Y) = \sum_j a_j(Y)X^j\) in \(A[Y][X]\); so there are \(b_j(Y) \in \text{Fr}A[Y]\) such that \(\sum_j b_j(Y)a_j(Y) = Y\).

Any \(D(t) \in \mathbb{F}_p[t] - \{0\}\) such that \(D(t)b_j(Y) \in A[Y]\) for all \(j\) works.

\[\square\]

**Remark 4.6.** The method used here is a special case of \([De-Mn] \) Theorem 1.11 and lemma 1.12. As \(A\) is a UFD it is a natural question to ask for the best \(D(t)\).

### 4.2 Modifications of covers

In this paragraph \(k\) is an algebraically closed field (\(\text{char} \ p > 0\)). In the previous paragraph we fixed the conductor which is equivalent to fixing the genus of the cover. We study here how for \(f(X) \in k[X]\) the additive polynomial \(\text{Ad}_f(Y)\) changes through natural algebraic transformations which don’t necessarily preserve the genus.

**Definition 4.7.** Let \(S(X)\) be an additive polynomial \(\in k[X]\). We say that the cover \(C_{f \circ S}\) is a modification of type 1 of \(C_f\). Although \(f(X)\) is reduced, in general \(f \circ S\) is not reduced.

**Proposition 4.8.** Let \(S(X) \in k[X]\) be a separable and additive polynomial. Then \(\text{Ad}_f(S(Y))\) divides \(\text{Ad}_{\text{red}(f \circ S)}(Y)\). Further for \(y, z \in Z(\text{Ad}_f(S(Y)))\), we have \(y, z \in Z(\text{Ad}_{\text{red}(f \circ S)}(Y))\) and \(\epsilon_f(S(y), S(z)) = \epsilon_{\text{red}(f \circ S)}(y, z)\) (see 4.4).

**Proof.** One can write \(\Delta(f)(X, Y) = \text{Ad}_f(Y)G(X, Y) + P_f(X, Y) - P_f(X, Y)^p\), where \(G(X, Y) \in k[Y][X]\) has content equal to 1, and \(P(X, Y) \in k[Y][X]\). Then \(\Delta(f \circ S)(X, Y) = \Delta(f)(S(X), S(Y)) = \text{Ad}_f(S(Y))G(S(X), S(Y)) + P_f(S(X), S(Y)) - P_f(S(X), S(Y))^p\).

Let \(y \in Z(\text{Ad}_f(S(Y)))\), then \(\Delta(f \circ S)(X, y) = P(S(X), S(y)) - P(S(X), S(y))^p\) and as \(\text{red}(f \circ S)(X) = f \circ S(X) + Q(X) - Q(X)^p\) for some \(Q(X) \in k[X]\) we can write \(\Delta(\text{red}(f \circ S))(X, y) = (1-F)(P(S(X), S(y)) + \Delta(Q)(X, y))\) which by proposition 4.4 gives \(\text{Ad}_{f \circ S}(y) = 0\). The divisibility follows as \(S(Y)\) and so \(\text{Ad}_f(S(Y))\) is separable. Moreover by the same proposition for \(y \in Z(\text{Ad}_f(S(Y)))\), we have \(P_{\text{red}(f \circ S)}(X, y) = P(S(X), S(y)) + \Delta(Q)(X, y)\). We remark that \(\Delta(Q)(X, y) + \Delta(Q)(X + y, z) - \Delta(Q)(X, z) - \Delta(Q)(X + z, y) = 0\); it follows that for \(y, z \in Z(\text{Ad}_f(S(Y)))\)

\(\epsilon_f(S(y), S(z)) = \epsilon_{\text{red}(f \circ S)}(y, z)\). \(\square\)

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Remark 4.9. The divisibility can be strict. For example let \( f(X) = X^{1+p^n} \), then \( \text{Ad}_f(Y) = Y + Y^{p^n} \). Let \( S(X) = X + X^p \), then \( f \circ S(X) = (X + X^p)(X^{p^n} + X^{p^{n+1}}) = X^{1+p^n} + X^{1+p^{n+1}} + X^{p^n} + X^{p^n+p^{n+1}} \) and so \( \text{red}(f \circ S) = X^{1+p^{n-1}} + 2X^{1+p^n} + X^{1+p^{n+1}}. \) Further, by Proposition 4.14 below, \( \text{Ad}_{f \circ S}(Y) = Y + \ldots + Y^{p^{(n+1)}} \) and \( \text{Ad}_f(S(Y)) = Y + Y^p + Y^{p^n} + Y^{p^{n+1}}. \)

Definition 4.10. Let \( f(X), g(X) \in k[X] \), we assume that none of \( \text{red}(f) \), \( \text{red}(g) \), \( \text{red}(f+g) \) is zero. If \( \text{Ad}_{\text{red}(g)}(Y) \mid \text{Ad}_{\text{red}(f)}(Y) \) we say that the cover \( C_{f+g} \) is a modification of type 2 of \( C_f \).

Proposition 4.11. Let \( f, g \) as above and assume that \( C_{f+g} \) be a modification of type 2 of \( C_f \). Then \( \text{Ad}_{\text{red}(g)}(Y) \mid \text{Ad}_{\text{red}(f)}(Y) \). Further for \( y, z \in Z(\text{Ad}_{\text{red}(g)}(Y)) \) we have \( y, z \in Z(\text{Ad}_{\text{red}(f+g)}(Y)) \) and \( \epsilon_{\text{red}(f+g)}(y, z) = \epsilon_{\text{red}(f)}(y, z) + \epsilon_{\text{red}(g)}(y, z) \) (see 4.4).

Proof. Clear from Proposition 4.4.

Proposition 4.12. Let \( k \) be an algebraically closed field of char \( p > 0 \) and \( f(X) \in k[X] \) whose degree deg \( f := m \) is prime to \( p \). Let \( g(f) = \frac{(m-1)(p-1)}{2} \), the genus of \( C_f \); we assume \( g(f) > 1 \). We write \( m = 1 + \ell p^s \) with \( \ell \geq 1, (\ell, p) = 1 \) and \( s > 0 \).

i) Let \( \ell > 1 \) and \( p = 2 \), then \( |G_{\infty,1}(f)| \leq 2^s \) and the ratio \( \frac{|G_{\infty,1}(f)|}{g(f)} \leq \frac{2}{3} \). The two inequalities are equalities for \( f(X) = \text{red}((X + X^{2^{s-1}})^7) \) and \( s > 1 \).

ii) Let \( \ell > 1 \) and \( p > 2 \), then \( |G_{\infty,1}(f)| \leq p^{s+1} \) and the ratio \( \frac{|G_{\infty,1}(f)|}{g(f)} \leq \frac{2}{\ell} \frac{p^s}{p^{s-1}} \leq \frac{p}{p^{s-1}}. \) We have equality for \( f(X) = X^{1+2p^s} - X^{2+p^s}. \)

iii) Let \( \ell = 1 \), i.e. \( m = 1 + p^s \), then \( |G_{\infty,1}(f)| \leq p^{2s+1} \), i.e. ratio \( \frac{|G_{\infty,1}(f)|}{g(f)} \leq 2p^s \frac{p^s}{p^{s-1}} \) and we have equality for \( f(X) = X^{1+p^s}. \)

Proof. i) and ii). We work with the universal family defined by \( f \in A[X] \) of degree \( m \) (cf. 4.3). Let \( j_0 = 1 + (\ell - 1)p^s \). We have seen in Lemma 4.3 that \( a_{j_0}(Y) = \psi(Y)^p + \ldots + 2t_{j_0}Y \). As \( \text{Ad}_f(Y) \) is a separable and additive polynomial by Proposition 4.4 it follows that \( \deg \text{Ad}_f(Y) \leq \deg a_{j_0}(Y) = p^s \) if \( p > 2 \) and \( \deg \text{Ad}_f(Y) \leq (1/2) \deg a_{j_0}(Y) = 2^{s+1} \) if \( p = 2 \).

For the equality of the bound we give examples.

Let \( p = 2, (\ell, 2) = 1 \) and \( f(X) = X^{1+\ell^2} \). In this case it is an easy consequence of Lemma 4.3 that \( \text{Ad}_f(Y) = Y \). Let \( S(X) = X + X^{2^{s-1}} \) where \( s > 1 \). Then \( f \circ S(X) = (X + X^{2^{s-1}})(X^2 + X^{2^s})^\ell = X(X^2 + X^{2^s})^\ell + X^{2^{s-1}}(X + X^{2^{s-1}})^\ell \mod (\text{Id} - F)k[X] \) has conductor \( 1 + \ell 2^s = m. \) From Proposition 4.8 we know that \( \text{Ad}_f(S(Y)) = Y + Y^{2s-1} \mid \text{Ad}_{f \circ S}(Y) \) and from the first part of the Proposition \( \deg \text{Ad}_{f \circ S}(Y) \leq 2^{s-1} \). So we have equality \( \text{Ad}_{f \circ S}(Y) = Y + Y^{2s} = \text{Ad}_f(S(Y)) \). Then the ratio \( \frac{|G_{\infty,1}(f \circ S)|}{g(f)} = \frac{2}{\ell} \leq \frac{2}{3} \).

So we have equality for \( \ell = 3 \), i.e. \( f(S(X)) = (X + X^{2^{s-1}})^7 \).

Let \( p > 2 \) we give an example for \( \ell = 2 \). As \( j_0 + 1 = 2 + (\ell - 1)p^s = 2 + p^s \) we need \( t_{j_0+1} \neq 0 \). So we consider \( f(X) = X^{1+2p^s} - X^{2+p^s} \) and show that \( \text{Ad}_f(Y) = Y^{p^s} - Y \).

As from the first part of the Proposition \( \deg \text{Ad}_f(Y) \leq p^s \) it suffices to prove a divisibility. Let \( y \in k \mid y^{p^s} = y, \) then \( \Delta(f)(X, y) = (X + y)(X^{p^s} + y)^2 - (X + y)^2(X^{p^s} + \ldots) \leq y^{p^s} \).
\( y - f(X) - f(y) = (X + y)(X^{2p^s} + 2yX^{p^s} + y^2) - (X^2 + 2yX + y^2)(X^{p^s} + y) - f(X) - f(y) = 0 \mod (\text{Id} - F)k[X] \)

iii) see Lemma 4.3.1).

\[ \square \]

**Remark 4.13.** In general for a given curve \( C \) in order to bound the automorphism group one considers \( \ell > 2 \) prime \( \neq p \) and uses the into homomorphism

\[ \text{Aut}C \hookrightarrow \text{Gl}(2g(C), \mathbb{F}_\ell) \]

where \( g(C) \) is the genus. Let us for example consider the case \( p = 2 \) and \( m = 1 + 2^k \). Then \( 2g(C_f) = 2^k \), and so \( |G_{\infty,1}(f)| \) divides the cardinality of a 2-Sylow of \( \text{Gl}(2^k, \mathbb{F}_\ell) \) which is equal to \( 2^{g(C)F(1) - 1} \), where \( F(1) = 1 + v_2(\ell^2 - 1) + v_2(\ell - 1) \) (see [C-F] for the structure of 2-Sylow subgroups of \( \text{Gl}(2^k, \mathbb{F}_\ell) \). From this one also obtains bounds for \( \text{Aut}C \) but they are far from being as good as the ones in Proposition 4.12.

### 4.3 Extraspecial groups

We recall some basic facts concerning extraspecial groups. We refer to [Hu] and [Su] for the structure of finite \( p \)-groups.

We saw in Corollary 3.5 that the groups \( G_{\infty,1}(f) \) belong to the class \( C_1 \) of \( p \)-groups:

\[ C_1 := \{ G \ | \ \exists C \ p - \text{cyclic } \subset Z(G) \text{ such that } G/C \text{ is } p\text{-elementary abelian} \} \]

Those non abelian groups in \( C_1 \) for which \( Z(G) \) is itself \( p \)-cyclic are called extraspecial. In particular a non abelian group \( G \) of order \( p^3 \) is extraspecial:

If \( p = 2 \), \( G \) is isomorphic to the dihedral group \( D_8 \) or the quaternion group \( Q_8 \).

These two groups have exponent \( 2^2 \). If \( p > 2 \), \( G \) is isomorphic to one of the two groups:

- \( E(p^3) = \langle x, y \mid x^p = y^p = [x, y]^p = 1, [x, y] \in Z(E(p^3)) \rangle \) with exponent \( p \), or
- \( M(p^3) = \langle x, y \mid x^{p^2} = y^p = 1, y^{-1}xy = x^{1+p} \rangle \) with exponent \( p^2 \).

More generally let \( G \) be an extraspecial \( p \)-group then (see [Su] Th. 4.18) \( |G| = p^{2n+1} \) for some \( n > 0 \) and the following 4 types occur.

I. If \( \exp G = p \), then \( p > 2 \) and \( G \) is a central product of \( n \) groups \( E(p^3) \).

II. If \( \exp G = p^2 \) and \( p > 2 \) then \( G \) is a central product of \( M(p^3) \) and \( n - 1 \) groups \( E(p^3) \).

III. If \( p = 2 \), then either

a. \( G \) is a central product of \( n \) groups \( D_8 \). Let \( q_a(n) \) (resp. \( d_a(n) \)) the number of elements with order 4 (resp. \( \leq 2 \)).

Or b. \( G \) is a central product of a group \( Q_8 \) and \( n - 1 \) groups \( D_8 \). Let \( q_b(n) \) (resp. \( d_b(n) \)) the number of elements with order 4 (resp. \( \leq 2 \)).

We have the following:

\[ q_a(n) = -2^n + 2^{2n} \quad \text{and so } \quad d_a(n) = 2^n + 2^{2n} \]
\[ q_b(n) = 2^n + 2^{2n} \quad \text{and so } \quad d_b(n) = -2^n + 2^{2n} \]

In each case, the structure of the central product is uniquely determined by the structure of the factors. In the sequel when speaking of the isomorphism type of an extraspecial group we will refer to the 4 types above, namely I, II, III.a and III.b.
Proposition 4.14. Let $n > 1$, $A_n := \mathbb{F}_{p^n}^q[t_i, 0 \leq i \leq n-1]$, and $f := t_0X^{1+p} + t_1X^{1+p} + ... + t_{n-1}X^{1+p^{n-1}} + X^{1+p^n} \in A_n[X]$. Let $k$ be an algebraically closed field and $\varphi : A_n \to k$, a specialization homomorphism. Then $\text{Ad}_f(Y) = \sum_{0 \leq i \leq n} (t_i^p Y^{p^{n-1}} + t_i^{p^2} Y^{p^{n-2}} + ... + t_i^{p^n} Y^{p^0}) = t_n Y + ... + t_i^{p^n} Y^{p^0}$ and $(G_{\infty, 1}(\varphi(f))) = p^{2n+1}$. Further

i) If $p > 2$, then $G_{\infty, 1}(\varphi(f))$ is an extraspecial group of type $1$.

ii) If $p = 2$, and $t_0 = 0$, then $G_{\infty, 1}(\varphi(f))$ is an extraspecial group of type $III.b$.

\[ \begin{align*}
\text{Proof.} & \quad \text{Write } f_i = t_i X^{1+p^i}, \text{ with } t_n = 1 \text{ and } t_0 = 0 \text{ for } p = 2. \text{ Then } \Delta(f)(X, Y) = \sum_{0 \leq i \leq n} \Delta(f_i), \text{ where } \Delta(f_i) = A_i + B_i, \text{ where } A_i = t_i Y X^{p^i}, \text{ and } B_i = t_i Y X^{p^i}. \text{ We can write } \\
\Delta(f_i) = (t_i^{p^{n-i}} Y^{p^{n-i}} + t_i^{p^i} Y^{p^{n-i+1}}) X^{p^i} + P_i - P_i^p. \text{ Let } A_i = t_i Y + ... + t_i^{p^{n-1}} + B_i + ... + B_i^{p^{n-1}}. \text{ One obtains } \\
\text{Ad}_f(Y) = \sum_{0 \leq i \leq n} (t_i^{p^{n-i}} Y^{p^{n-i}} + t_i^{p^i} Y^{p^{n-i+1}}) = t_n Y + ... + t_i^{p^n} Y^{p^0}. \text{ The polynomial } P(X, Y) = \sum_{0 \leq i \leq n} P_i(X, Y) \text{ is here an additive polynomial, so if } \\
\sigma_{c,Y}(W) = W + P(X, Y) + c, \text{ then } \sigma_{c,Y}(W) = W + P(X, Y) + P(X + Y, Y) + ... + P(X + (p-1)Y, Y) = W + \frac{p(p^n-1)}{2} Y. \text{ Let } p > 2, \text{ and } \\
\sigma_{c,Y}(W) = 1 \text{ and the exponent of } G_{\infty, 1}(\varphi(f)) \text{ is } p. \text{ Now if } p = 2, \text{ and } \\
\sigma_{c,Y}(W) = W + P(Y, Y) \text{ and the exponent of } G_{\infty, 1}(\varphi(f)) \text{ is } 2^3. \text{ Let } y, z \in Z(\text{Ad}_{\varphi(f)}(Y)), \text{ then } \\
\epsilon_f(y, z) = P(X, z) + P(X, y, z) - P(X, y) - P(X + y, z) = P(z, y) - P(y, z) = \sum_{0 \leq i \leq n} \sum_{n-i \leq j \leq n-1} (B_i(z, y) - A_i(z, y))^{p^i} = ... - t_i^{p^{n-1}} z^{p^{n-1}} - t_i^{p^n} z^{p^0}. \text{ If } y \neq 0, \text{ is such that } \sigma_{y,z} \text{ is in the center this polynomial in } z \text{ should have } p^{2n} \text{ roots; so the center is } \text{< } \rho \text{ > where } \rho(X) = X + Y \text{ and } \rho(W) = W + 1. \text{ We conclude that } \\
G_{\infty, 1}(\varphi(f)) \text{ is an extraspecial group of order } p^{2n+1} \text{ and exponent } p \text{ for } \text{p > 2 and exponent } p^2 \text{ for } p = 2. \text{ In the case } p = 2 \text{ this is not yet sufficient to determine the isomorphism class of this group; we need some evaluation of the number of elements of order } \leq 2. \text{ Let us } \text{first consider the case } f_0(X) = X^{1+2^n}. \text{ We have the following parametrization for the elements in } G_{\infty, 1}(f_0). \text{ For } y \in Z(\text{Ad}_{f_0}(Y)) = Z(Y + Y^{2^n}) \text{ and } c \in Z/pZ, \text{ where } P(X, Y) = Y^{2^n} X + Y^{2^n+1} X^2 + ... + Y^{2^n-1} X^{2^n-1}. \text{ Let } d(n) \text{ be the number of elements of order } \leq 2; \text{ they correspond to those } y \text{ such that } P(y, y) = 0 \text{ (this condition doesn't depend on } c). \text{ One has } P(Y, Y) = S + S^2 + ... + S^{2^n-1}, \text{ where } S = Y^{2^n+1} \text{ and so } Q(Y) := P(Y, Y) Y^{-2^n} \text{ is a separable polynomial of degree } 2^{2n-1} + 2^n - 2^n. \text{ Moreover } P(Y, Y)^2 - P(Y, Y) = S^{2^n} - S = Y^{2^n}(Y^{2^{2n}} + Y). \text{ The roots of } Q(Y) \text{ are simple and among those of } Y^{2^n} + Y. \text{ It follows that } \\
d(n) = 2(2^{2n-1} + 2^n - 2^n) = 2^{2n} - 2^n = d_b(n). \text{ Now let us consider the general case. The result follows from the previous special case if we remark that the number of elements of order } 4 \text{ can only decrease via a specialization } \varphi : A_n \to k. \text{ And as } q_b(n) > q_a(n), \text{ only extraspecial groups of type } b) \text{ can degenerate in type } b). \]

\[ \Box \]

Remark 4.15. It was surprising for us when we saw that a special case appears in literature in connection with coding theory; namely \text{[vdG-vdV]} consider the specialization homomorphisms $\varphi$ taking values in $\mathbb{F}_q$ where $q = p^n$. The additive polynomial $\text{Ad}_{\varphi(f)}$ is then their polynomial $E_n(Y) = R(Y)^{p^n} + \sum_{0 \leq i \leq n} (t_i Y)^{p^{n-1}}$ where
\[ R(X) = \sum_{0 \leq i \leq n} t_i Y^{p^i} \]. The zeroes are interpreted as the \( \mathbb{F}_p \) vector space which is the kernel of the \( \mathbb{F}_p \)-bilinear form \( \text{Tr}_{E_i/\mathbb{F}_p}(xR(y) + yR(x)) \) (which is symmetric if \( p > 2 \) and alternating if \( p = 2 \)). In case \( p = 2 \) they prove a factorization of \( E_n(Y) = YE_n^{-1}(Y)E_n^n(Y) \) which corresponds to a partition of the roots depending on the order of the corresponding automorphism of \( C_f \). Such a decomposition works in general.

### 4.4 Application to the moduli space of curves

We keep the notations of Proposition 4.14. For a fixed \( n > 1 \) let \( g_n = \frac{p^n(p-1)}{2} \), \( A_n := \mathbb{F}_p^{alg}[t_i, 0 \leq i \leq n-1] \) and \( f := t_0 X^{1+p^0} + t_1 X^{1+p} + \ldots + t_{n-1} X^{1+p^{n-1}} + X^{1+p^n} \in A_n[X] \) with \( t_0 = 0 \) for \( p = 2 \). Let \( \theta \in \mathbb{F}_p^{alg} \) be a primitive \((p-1)(p^n+1)\)-th root of 1. Then \( \Theta : (t_0, \ldots, t_{n-1}) \rightarrow (\theta^p t_0, \theta^{p^2} t_1, \ldots, \theta^{p^{n-1}} t_{n-1}) \) induces an \( \mathbb{F}_p \)-automorphism of \( \text{Spec} A_n \) of order \( p^n + 1 \). Let \( B_n := A_n^{<\Theta>} \) be the quotient of the affine space by the cyclic group of automorphisms \( < \Theta > \). The structural morphism \( \pi : C_f \rightarrow \text{Spec} A_n \) is a family of curves of genus \( g_n \). Moreover by Proposition 4.16 two specialization morphisms \( \varphi_i : A_n \rightarrow k \), for \( i = 1, 2 \) will give isomorphic \( k \)-curves iff \( \exists c \in \mathbb{F}_p^*, (a, b) \in (k^*, k) \) such that \( \sum_{0 \leq i \leq n-1} \varphi_2(t_i)(aX + b)^{1+p^i} + (aX + b)^{1+p^n} = c(\sum_{0 \leq i \leq n-1} \varphi_1(t_i)X^{1+p^i} + X^{1+p^n}) \mod (\text{Id} - F) k[X] \), i.e. \( a^{1+p^n} = c \in \mathbb{F}_p^* \) and for \( i \geq 0 \) one has \( \varphi_2(t_i) = ca^{-(1+p^i)} \varphi_1(t_i) = a^{p^i} \varphi_1(t_i) \) (note that for \( p = 2 \) we assumed \( t_0 = 0 \) and so \( f \) is reduced). Finally this shows that the two specialization morphisms are in the same orbit under the action of the group \( \Theta \). By the definition of the coarse moduli space \( M_{g_n} \) we deduce from the existence of the family \( C_f \rightarrow \text{Spec} A_n \) a map from \( \text{Spec} A_n \) to \( M_{g_n} \), which factorizes through \( \text{Spec} B_n \) in an injective morphism. The image is an algebraic subset of \( M_{g_n} \), of dimension that of \( \text{Spec} B_n \). A measure of the size of families of curves which are étale covers of the affine line and with given extraspecial group of type I and order \( p^{2n+1} \) as an automorphism group is given by the dimension of this image which is \( O(\log(g_n)) \).

It is remarkable that these families for varying \( n \) can be characterized by the following Hurwitz-type bound.

**Proposition 4.16.** Let \( k \) be an algebraically closed field of char \( p > 0 \) and \( f(X) \in Xk[X] \) a polynomial of degree \( m := \deg f \) prime to \( p \). We assume \( f \) is reduced. If
\[ \frac{|G_{\infty,1}(f)|}{g(f)} > \frac{p}{p-1} \left( \frac{2}{3} \text{ for } p = 2 \right) \] then \( f(X) = XR(X) \), where \( R(X) \) is an additive polynomial. Moreover if \( \deg f = 1 + p^n \), then \( \frac{|G_{\infty,1}(f)|}{g(f)} = 2p^n \frac{p}{p-1} \).

**Proof.** We will show only the case \( p > 2 \) and point out that for \( p = 2 \) a similar argument works. The proof works by elimination of bad monomials. We saw in Proposition 4.12 that only for \( m = 1 + p^s \) with \( s > 0 \) the ratio \( \frac{|G_{\infty,1}(f)|}{g(f)} \) can be \( > \frac{p}{p-1} \). Now we show that any other monomial in \( f(X) \) has exponent \( 1 + p^t \) with \( t < s \).

Let us assume that this is not the case and denote by \( X^a \) the monomial of highest degree which is not of the above form. We first assume that \( p \) doesn’t divide \( a - 1 \) and consider the integer \( k \) such that \( p^k < a - 1 < p^k \). Then \( k \leq s \) and \( p^{s-1} < (a-1)p^{s-k} < p^s \). Then \((*)X^a \) has a contribution in \( F(X,Y) \) which is equal to
Let us now assume that $a-1 = \ell p^t$, where $\ell > 1$ and $(\ell, p) = 1$. Let $j_0 = 1+(\ell-1)p^t$ and say $p^{s-1} < j_0 < p^k$. Then $p^{s-1} < j_0 p^{s-k} < p^s$ and the monomial $X^a$ contributes to $F(X,Y)$ in the monomial $X^{j_0 p^{s-k}}$ the term $(Y^p + ... + t_{j_0+1}Y)^{p^{s-k}} X^{j_0 p^{s-k}}$. Note that $p j_0 < a$ iff $\ell \leq \frac{p^t}{p^t - 1} \frac{1}{p}$ which is not the case; so any other contribution in $F$ in the monomial $X^{j_0 p^{s-k}}$ can only occur from a monomial $X^b$ with $j_0 < b < a$. Now such a contribution will be $((b)^{b-j_0})^{p^{s-k}}$ whose degree is $(b-j_0)p^{s-k} < (a-j_0)p^{s-k} = p^{t+s-k}$, so finally we get that $\text{deg} \ Adf(Y) \leq p^{t+s-k}$. From this we get the ratio $\frac{|G_{f,1}(f)|}{g(f)} \leq \frac{2 p^{t+s-k+1}}{p^s (p-1)}$ and so $1 < 2p^{t-k}$, i.e. $t \geq k$. On the other hand $(\ell - 1)p^t < p^k$, so $(\ell - 1) < 1\frac{1}{p}$ - a contradiction. \[\square\]

4.5 Realization of the other extraspecial groups

Case $p > 2$.

Let $G$ be an extraspecial group of type II. We will use Witt vectors of length 2 and modifications of covers.

Let $c(X,Y) = \frac{(X+Y)^p - X^p - Y^p}{p}= \sum_{1 \leq i \leq p-1} \frac{(-1)^{i-1}}{i} X^i Y^{p-i}$ and $f_1(X) := c(X^p,-X) = \sum_{1 \leq i \leq p-1} \frac{1}{i} X^i Y^{p-1}$. A straightforward calculation in $W_2(F_p)$ shows that:

$\Delta(f_1)(X,Y) = c((F-\text{Id})X,(F-\text{Id})Y) + (F-\text{Id})c(X,Y)$. As $c((F-\text{Id})X,(F-\text{Id})Y) = (Y^p - Y)(X^p - X)^{-1}$ + lower degree terms $\in (Y^p - Y)F_2[X,Y]$ it follows that $\text{Ad}_f(Y) = Y^p - Y$ and for $y \in Z(\text{Ad}_f(Y))$ (i.e. $y \in F_p$) $P(X,y) = -c(X,y) = \sum_{1 \leq i \leq p-1} \frac{(-1)^{i-1} y^{p-1} Y^{i}}{i}$.

Now we show that $G_{\infty,1}(f_1)$ is $p^2$-cyclic. Let $\sigma_g(W) = W + P(X,y)$. Then $\sigma_g(W) = W - \frac{(-1)^{p-1} Y^{p-1}}{p^2} Y = W + y$, i.e. $\sigma_g = \rho^1$, so $G_{\infty,1}(f_1) = < \sigma_g >$ is $p^2$-cyclic.

Note that in this case the conductor is $m = 1 + \ell p$ with $\ell = p - 1$; then the ratio is $\frac{|G_{\infty,1}(f_1)|}{g(f_1)} = \frac{2p^2}{p-1}$ as $p > 2$. In order to get an extraspecial group of exponent $p^2$ we use modifications of $f_1$. Let $q := p^n$, $f_2(X) = X^{1+q}$, $\theta$ a primitive $q^2-1$-th root of unity and $A(X) := \theta X^q - \theta^2 X$. Then $S(X) := A(X) + A(X)^q + ... + A(X)^{q/p}$ is an additive polynomial and $S(X)^p - S(X) = A(X)^q - A(X) = \theta q X + Y X^q$.

Set $f(X) := f_1(S(X)) + f_2(X)$.

Proposition 4.17. $G_{\infty,1}(f)$ is an extraspecial group of type II and the ratio $\frac{|G_{\infty,1}(f)|}{g(f)} = \frac{2p^2}{(p-1)^2} < \frac{p}{p-1}$.

Proof. We have $\Delta(f_1)(S(X),S(Y)) = c(S(X)^p - S(X),S(Y)^p - S(Y)) + (F-\text{Id})c(S(X),S(Y)) = c(A(X)^q - A(X), A(Y)^q - A(Y)) + (F-\text{Id})c(S(X),S(Y))$ and $\Delta(f_2)(X,Y) = Y^q X + Y X^q$.

We claim that $\text{Ad}_{f_1 OS}(Y) = \theta^{-q} \text{Ad}_{f_1}(S(Y)) = Y + Y q^2$ and for $y + q^2 = 0$ one has $P_{f_1 OS}(X,y) = c(S(X),S(y))$. To this end notice that $f_1(S(X)) = c(S(X)^p, - S(X)) = \sum_{1 \leq i \leq p-1} \frac{1}{i} S(X)^{p+(p-1)i}$ has conductor $1 + (p-1)q^2$, hence $\text{deg} \text{Ad}_{f_1 OS} \leq q^2$. As by
Proposition 4.8. \( \text{Ad}_{f_1}(S(Y)) = S(Y)^p - S(Y) = A(Y)^q - A(Y) = \theta^q(X + X^{q^2})|\text{Ad}_{f_1}S(Y) \) we get the equality.

Note that \( f_1 \circ S \) and \( f_2 \) have the same additive polynomial, so due to the property of second type modifications, we obtain \( Y + Y^{q^2}|\text{Ad}_f(Y) \). As \( f \) and \( f_1 \circ S \) have the same conductor we get \( \deg \text{Ad}_f(Y) \leq q^2 \). Finally we conclude \( \text{Ad}_f(Y) = Y + Y^{q^2} \) showing the claim.

Next we claim that \( G_{\infty,1}(f) \) has exponent \( p^2 \). Let \( y \) be such that \( \text{Ad}_f(y) = \text{Ad}_{f_1}(y) = \text{Ad}_{f_2}(y) = 0 \). Then \( P_f(X, y) = P_1(X, y) + P_2(X, y) \), where \( P_1(X, Y) = c(S(X, S(Y)) \) and \( P_2(X, Y) = -(y^qX + (y^qX)^p + ... + (y^qX)^q/p) \).

Note that \( c(S(X), S(Y)) = \sum_{1 \leq i \leq p-1} (-1)^{i-1}S(y)^iS(X)^{p-i} \) and as \( S \) is an additive polynomial we set \( Z := S(y)S(X)^p - S(y)^pS(X) \). Then \( \sum_{1 \leq i \leq p-1} P_1(X + iy, y) = \text{Tr}_{k(S(X))/k(z)}c(S(X), S(y)) = S(y) \) where \( k = \mathbb{F}_p^{alg} \). Note that \( f_2 \) induces an extraspecial group of exponent \( p \) so \( \sum_{1 \leq i \leq p-1} P_2(X + iy, y) = 0 \).

Finally if \( y^{q^2} + y = 0 \) and \( \sigma_y(W) = W + P_f(X, y) \) then \( \sigma_y(W) = W + S(y) \).

Now we show that the center is \( \langle \rho \rangle \). Let \( y, z \in Z(\text{Ad}_f(Y)) \); then \( \epsilon_j(y, z) = P_j(X, z) + P_j(X + z, y) - P_1(X, y) - P_j(X + y, z) \) for \( j = 1, 2 \).

We have seen that \( S(X)^p - S(X) = \theta^q(X^{q^2} + X) \), so if \( y^{q^2} + y = 0 \) then \( S(y)^p - S(y) = 0 \). We also saw that \( \text{Ad}_{f_1}(Y) = Y^{p^2} - Y \) and \( G_{\infty,1}(f_1) \) is cyclic, so \( \epsilon_{f_1}(S(Y), S(z)) = \epsilon_f_y(S(Y), S(z)) = 0 \). Finally \( \epsilon(y, z) = \epsilon_2(y, z) = -(z^qy + z^{q^2}y^p + ... + z^{q^2/p}y^{q^2/p}) + (z^qy + z^qy^{q^2} + ... + z^{q^2/p}y^{q^2/p}) \). For \( z \neq 0 \) this is a polynomial in \( y \) of degree \( q^2/p \), so it has at most \( q^2/p \) roots and hence \( z \in Z(\text{Ad}_{\infty,1}(f)) \) iff \( z = 0 \).

\[ \square \]

**Remark 4.18.** One can follow the same method in order to produce a family of curves with automorphism group an extraspecial group of exponent \( p^2 \).

Say \( q = p^n, f_2 = (t_0)^qX^1 + p^0 + (t_1)^qX^1 + p^0 + ... + (t_{n-1})^qX^{1+p^{n-1}} + X^{1+p^n} \). Then \( \text{Ad}_{f_2}(Y) = Y^{p^2n} + Y + (t_{n-1}^qY + t_{n-1}^{p^{n-1}}Y^{2(n-1)})^p + ... + (t_1^qY + t_1^{p^{n-1}}Y^{p^2})^{p^{n-1}} \).

Now we look for a \( \theta \) such that \( \theta \text{Ad}_{f_2}(Y) \in (F - \text{Id})k[X] \). Write \( \theta = \alpha^{p^2} \); then the condition can be simplified by using the equivalence \( A^p = A \mod (F - \text{Id})k[X] \).

We get

\[ (\alpha^{p^2} + \alpha) + (\alpha^{p^2-1}t_1^{p^n} + \alpha t_1^{p^n}) + ... + (\alpha^{p^{n+1}}t_1^{p^n} + \alpha^{p^{n-1}}t_1^{p^n}) = 0. \]

Choose for \( \alpha \) any root. We then can write \( \theta \text{Ad}_{f_2}(Y) = S(Y)^p - S(Y) \) where \( S(Y) \) is an additive polynomial, and this fact is essential for further calculations. Now \( f = f_1 \circ S + f_2 \) gives a family of curves with automorphism group an extraspecial group of exponent \( p^2 \).

If \( p = 2 \), it is not so easy to distinguish between the two classes of extraspecial groups. In fact we have realized those extraspecial groups which are a central product of a group \( Q_8 \) and \( n-1 \) groups \( D_8 \), i.e. the so called type III.b. A realization of type III.a. will be a consequence of the following general method.
4.6 Saturated subgroups of extraspecial groups and their realization

Now we give the main result of this paper which describes the set of groups $G_{\infty,1}(f)$. Let us first define 2 sets of isomorphism classes of $p$-groups.

$C_1 := \{ G \mid \exists N \text{ p-cyclic and normal subgroup } \subset G \mid G/N \text{ is } p\text{-elementary abelian} \}$

$C_2 := \{ G \mid \exists E \text{ extraspecial group, } V \subset E \text{ } \mathbb{Z}(E) \text{ F}_p\text{-subspace } \mid G \simeq \pi^{-1}(V),$

$\text{where } \pi : E \to \frac{E}{\mathbb{Z}(E)} \text{ is the canonical map} \}$.

The class $C_1$ has been described in the following result (cf. [Su, 4.16]):

**Proposition 4.19.** Any $G \in C_1$ is isomorphic to one of the groups in the following list.

a) An elementary abelian $p$-group.

b) An abelian group of type $(p, p, \ldots, p, p^2)$.

c) A central product of an extraspecial $p$-group $E$ and an abelian group $A$. If $A$ is not elementary abelian, then

$E \cap A = \mathbb{Z}(E) = A^p$

We will need the following result from group theory. It seems to us that it should be somewhere in the literature but we have no reference. Although it is possible to give a proof using the classification of extraspecial groups in section 4.3 and Proposition 4.19, we give a direct proof which deals with factor systems and so is related with the algorithm proved for class $C_3$.

**Proposition 4.20.** The 2 classes $C_i, i = 1, 2$ are equal.

**Proof.** Set $G_0 = G$, $G_i = [G, G_{i-1}]$, in particular $G_1 = [G, G] = G'$ and $G/N$ being elementary abelian implies $G_1 \subset N$, so either $|G_1| = p$ or $|G_1| = 1$. As $G$ is solvable, the sequence of the $G_i$ is strictly decreasing with $G_n = \{1\}$ for $n \gg 0$. So $G_2 = [G_1, G] = \{1\}$ and $G_1 = G' \subset N \subset \mathbb{Z}(G)$. This last condition allows to define a skew-symmetric bilinear form on the $\mathbb{F}_p$-vector space $G/N$: if $\vec{x}, \vec{y} \in G/N$, then $[x, y] \in G' = < \rho > \subset N$, i.e. $[x, y] = \rho^e$ where $e \in \mathbb{Z}/p\mathbb{Z}$. Note that $[x, y]$ is independent of the lifts of $\vec{x}$ and $\vec{y}$ to $G$ as $N \subset \mathbb{Z}(G)$. We define $< \vec{x}, \vec{y} > := e$. Note that $x \in \mathbb{Z}(G)$ iff $< \vec{x}, \vec{y} > = 0$ for all $y \in G$. Therefore $< . , >$ is non degenerate iff $G' = N = \mathbb{Z}(G)$, i.e. $G$ is extraspecial.

Consider the extension of groups

$1 \to N \to G \to V := G/N \to 1$
Let $s$ be a set theoretical section. To any two $v_1, v_2 \in V$, we have a $c(v_1, v_2) \in N$ such that $s(v_1)s(v_2) = s(v_1v_2)c(v_1, v_2)$ and $c(\ldots)$ is the 2-cocycle corresponding to the equivalence class of the above extension in $H^2(V, N) = H^2(V, \mathbb{F}_p)$. The extension is central, so $N$ has trivial action by $V$. From $c(\ldots)$, we recover $G$ in the following way: on the set $V \times \mathbb{F}_p$, one defines a group structure via:

$$(v_1, \alpha)(v_2, \beta) = (v_1 + v_2, \alpha + \beta + c(v_1, v_2)).$$

The form $<\ldots>$ on $V$ can be expressed in terms of $c$: $<v_1, v_2> = [s(v_1), s(v_2)] = s(v_1)^{-1}s(v_2)^{-1}s(v_1)s(v_2) = (s(v_2)s(v_1))^{-1}s(v_1)s(v_2) = c(v_2, v_1)^{-1}s(v_2v_1)^{-1}s(v_1v_2)c(v_1, v_2) = c(v_2, v_1)^{-1}c(v_1, v_2)$. Identifying $N$ with $\mathbb{F}_p$ we write $<v_1, v_2> = c(v_1, v_2) - c(v_2, v_1)$.

We distinguish the cases $p > 2$ and $p = 2$.

Case $p > 2$. Let $V \hookrightarrow W := V \bigoplus V$, and $\pi : W \rightarrow V$ the projection on the first factor. We denote by $c$ a cocycle corresponding to the given group extension and we extend the corresponding 2-form $<v_1, v_2> = c(v_1, v_2) - c(v_2, v_1)$ from $V$ to $W$ to a non degenerate skew form given by the matrix

$$
\begin{pmatrix}
-1 & & & \\
& \begin{pmatrix} & & \\
& A & \ddots \\
& \ddots & & -1 \\
& & 1 & \\
& & & 0 \\
& & & 1
\end{pmatrix} & \\
2 & & & \\
& & & -2 \\
\end{pmatrix}
$$

where $A$ is the matrix of $<\ldots>$ on $V$.

Now we obtain a 2-cocycle $d : W \times W \rightarrow \mathbb{F}_p$, via $(w_1, w_2) \rightarrow <w_1, w_2> + c(\pi(w_2), \pi(w_1))$. We remark that $d_W$ maps $(v_1, v_2)$ to $<v_1, v_2> + c(v_2, v_1) = c(v_1, v_2) - c(v_2, v_1) + c(v_2, v_1) = c(v_1, v_2)$. So $d_W = c$ and the group extension $E$ corresponding to $d$ therefore contains $G$ as a subgroup.

It remains to show that $|Z(E)| = p$. This amounts to the skew-form $\ll<\ldots>\gg$ associated to $d$ on $W$ to be non degenerate. We compute on $W$:

$$
\ll<w_1, w_2>\gg = d(w_1, w_2) - d(w_2, w_1)
= <w_1, w_2> + c(\pi(w_2), \pi(w_1)) - <w_2, w_1> - c(\pi(w_1), \pi(w_2))
= 2 <w_1, w_2> + <\pi(w_2), \pi(w_1)> - <w_2, w_1> - <\pi(w_1), \pi(w_2)>.
$$

Therefore $\ll<\ldots>\gg$ has the matrix

$$
\begin{pmatrix}
-2 & & & \\
& \begin{pmatrix} & & \\
& 2A & \ddots \\
& \ddots & & -2 \\
& & 2 & \\
& & & 0 \\
& & & 0
\end{pmatrix} & \\
2 & & & \\
& & & -2 \\
\end{pmatrix}
$$

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which has maximal rank as \( p > 2 \). We conclude \( E \) is extraspecial. We have
gained \( G \) as subgroup of the extraspecial group \( E \) and \( N = Z(E) \subset G \) follows from the fact that in the construction above, the second factor of \( V \times \mathbb{F}_p \) and \( W \times \mathbb{F}_p \) correspond to \( N \) and \( Z(E) \) respectively.

The case \( p = 2 \). Using the above notation, let \( n := \dim V \) and \( M_n(\mathbb{F}_2) \) the \( \mathbb{F}_2 \)-vector space of \( n \times n \) matrices. Any such matrix defines a bilinear form (hence a 2-cocycle) on \( V \). Therefore we have a map of \( \mathbb{F}_2 \)-vector spaces

\[
M_n(\mathbb{F}_2) \xrightarrow{\varphi} H^2(V, \mathbb{F}_2)
\]

Moreover a matrix \( A \) is in the kernel \( K \) of \( \varphi \) iff its associated 2-cocycle \( c \) defines the split extension, which is the elementary abelian 2-group of rank \( n + 1 \).

This is equivalent to \( c(v_1, v_2) = c(v_2, v_2) \) for all \( v_1, v_2 \in V \) and \( c(v, v) = 0 \) for all \( v \in V \) (here we use \( p = 2 \)). In other words \( A = A^t \) and \( A \) has only zeroes on its diagonal. We conclude \( \dim \ker \varphi = n - 1 + n - 2 + ... + 1 = \frac{n(n-1)}{2} \).

Therefore \( \dim H^2(V, \mathbb{F}_2) = \frac{n(n-1)}{2} \), \[10 \] p. 169.

It is known that \( \dim H^2(V, \mathbb{F}_2) = \frac{n(n-1)}{2} \).

Therefore \( 1 \rightarrow K \rightarrow M_n(\mathbb{F}_2) \xrightarrow{\varphi} H^2(V, \mathbb{F}_2) \rightarrow 1 \)
is exact. In particular \( \varphi \) is onto, so every element of \( H^2(V, \mathbb{F}_2) \) can be represented by a 2-cocycle that is a bilinear form.

Again we let the given extension correspond to the cocycle \( c \in Z^2(V, \mathbb{F}_2) \) and by the above we may assume \( c \) is bilinear corresponding to a matrix \( A \).

Let \( V \hookrightarrow W := V \oplus V \) be the first factor and consider \( d \in Z^2(V, \mathbb{F}_2) \) corresponding to the matrix

\[
B = \begin{pmatrix}
1 & & & \\
A & 1 & & \\
& 0 & 0 & \\
& & & \\
\end{pmatrix} \in M_{2n}(\mathbb{F}_2)
\]

and \( d|_V = c \), so the group defined by \( d \) contains \( G \).

Claim: \( E \) is an extraspecial group, i.e. the 2-form \( \langle \langle \ldots \rangle \rangle \) defined by \( d \) on \( W \) is non degenerate.

We calculate \( \langle \langle w_1, w_2 \rangle \rangle = d(w_1, w_2) - d(w_2, w_1) = w_1^t B w_2 + w_2^t B w_1 = w_1^t B w_2 + (w_2^t B w_1)^t = w_1^t B w_2 + w_1^t B^t w_2 = w_1^t (B + B^t) w_2 \) and

\[
B + B^t = \begin{pmatrix}
1 & & & \\
A + A^t & 1 & & \\
& 1 & & \\
& & & 0 \\
\end{pmatrix}
\]

has rank \( 2n \) (independently of what \( A \) is). \( \square \)
We define a third class
\[ C_3 := \{ G \mid \exists f \in Xk[X], \; \deg f = 1 \text{ such that } G \cong G_{\infty,1}(f) \} \]

**Theorem 4.21.** The 3 classes \( C_i, i = 1, 2, 3 \) are equal.

**Proof.** It is sufficient to realize any subgroup of the extraspecial groups of type I, II and III b. (see remark 4.18). So we distinguish these three cases. The method is the following: let \( E \) be an extraspecial group and \( G = \pi^{-1}(F \subset E/Z(E)) \) a saturated subgroup. For each type of extraspecial group we consider a realization \( C_f \) and to the sub-space \( F \) we associate an additive polynomial \( S(F) \) which we use in order to produce a convenient modification of the cover \( C_f \). The key point is that our modification will not change the commutation rule.

**Type I.** So \( p > 2 \). Let \( E \) the extraspecial group of exponent \( p \) and order \( p^{2n+1} \).

Consider the realization: \( W^p - W = f_1(X) := X^{1+q} \) where \( q = p^n \); then \( \text{Ad}_{f_1}(Y) = Yq^2 + Y \).

Note that \( E/Z(E) \) is the automorphism group of \( k[X] \) whose elements are \( \sigma_y(X) = X + y \) where \( y \) goes through the roots of \( \text{Ad}_{f_1}(Y) \). Then the subgroups correspond to those \( \sigma_y \) going through \( y \in F \) where \( F \) is a subgroup generated by a subset of such roots, i.e. there is an additive polynomial (monic) \( S_F \) which divides \( Y^2 + Y \) and \( y \) goes through these roots. Note that necessarily \( S_F \) has distinct roots so \( S_F = s_0X + s_1X^p + \ldots + X^r \) and \( s_0 \neq 0 \). As \( \pi^{-1}(W) = E \), we can assume that \( 0 < r < 2n \).

Let us assume that \( p > 3 \) and \( \ell > 1 \) such that \( (\ell(\ell + 1), p) = 1 \).

Let \( f(X) := S_F(X)^{\ell+1} + f_1(X) \). We remark that the conductor of \( \text{red}(S_F(X)^{\ell+1}) \) is \( 1 + \ell p^\ell \). We can use the same trick as in lemma 4.3 for this let \( j_0 = 1 + (\ell - 1)p^\ell \). The coefficient in \( \Delta(f)(X, Y) \) of \( X^{3\ell} \) is that of \( \Delta(S_F^{\ell+1})(X, Y) = (S_F(Y) + s_0X + \ldots + X^r)^{\ell+1} - S_F(Y)^{\ell+1} - S_F(X)^{\ell+1} \). For this we solve the system

\[ i + i_0 + \ldots + i_\ell = \ell + 1 \text{ and } i_0 + i_1p + i_2p^2 + \ldots + i_r p^\ell = j_0 \text{ where } i \in \{ 1, 2, \ldots, \ell \}. \]

We get \( p^\ell - 1 = (i - 1)p^\ell + i_0(p^\ell - 1) + i_1(p^\ell - p) + \ldots + i_{r-1}(p^\ell - p^{\ell-1}) \). It follows that \( p|i_0 - 1 \) and \( i_0 \leq 1 \) so \( i_0 = 1 \) and \( i = 1, i_1 = i_2 = \ldots = i_{r-1} = 0 \) and so \( i_r = \ell - 1 \). Finally we have shown that the desired coefficient is \( \frac{\ell+1}{1!(\ell-1)!} S_F(Y)s_0 \).

It follows that \( \text{Ad}_{f_1}(Y) \) divides \( S_F(Y) \) which itself divides \( \text{Ad}_{f_1}(Y) = Y^{q^2} + Y \) and so \( \text{Ad}_{f_1}(Y) = S_F(Y) \).

We remark that \( \text{Ad}_{X^{\ell+1}}(Y) = Y \) by lemma 4.3, so \( \text{Ad}_{S_F^{\ell+1}(Y)} = S_F(Y) \) by 4.8, then by 4.11 for \( y, z \in Z(S_F(Y)) \), one has \( \epsilon_f(y, z) = \epsilon_{S_F^{\ell+1}(y, z)} + \epsilon_{f_1}(y, z) \). As \( \epsilon_{S_F^{\ell+1}}(y, z) = 0 \) the commutation rule is that of \( E \).

The simplest choice for \( p > 3 \) is \( \ell = 2 \) and for \( p = 3, \ell = 4 \).

**Type II.** So \( p > 2 \). Let \( E \) be the extraspecial group of exponent \( p^2 \) and order \( p^{2n+1} \). We first recall the realization of \( E \) we gave in Proposition 4.17. Let \( q = p^n, \theta a q^2 - 1 \)-th root of \( -1 \) and \( A(X) = \theta X^q - \theta^2 X \). Then \( S(X) := A(X) + A(X)^q + \ldots + A(X)^{q/p} \) is an additive polynomial such that \( S(X)^p - S(X) = A(X)^q - A(X) = \theta^q(X^q + X) \) and \( f_0(X) := f_1(S(X)) + f_2(X) \) gives a realization of \( E \) where \( f_1(X) := c(X^p, -X) \) and \( f_2(X) = X^{1+q} \). Moreover \( \text{Ad}_{f_0}(Y) = Y^{q^2} + Y \) and so we can apply the same strategy as for type I.
As above we consider $\ell > 1$ such that $(\ell(\ell + 1), p) = 1$ and $f(X) := S_F(X)^{\ell+1} + f_1(S(X)) + f_2(X)$. If we compare to the type $I$ case, we need to show that $\Delta(f_1 \circ S)(X,Y)$ has no contribution which cancels \[ \frac{(\ell+1)!}{(\ell-1)!} S_F(Y)^{s_0} X^{3\partial q}. \]

We have (see \[ \Delta(f_1 \circ S)(X,Y) = c(A(X)^q - A(X), A(Y)^q - A(Y)) + (F - Id)c(S(X), S(Y)) \]
and \( c(A(X)^q - A(X), A(Y)^q - A(Y)) = c(\theta(q^2 + X), \theta(q^2 + Y)) \). We remark that the equation \( i_0 + i_1 q^2 = j_0 \) with \( i_0 + i_1 = i \) and \( 1 \leq i \leq p - 1 \) is equivalent to \( i_0 = 1 \) and \( \ell - 1 = i_1 p^{2n-r} \); so if \(((\ell - 1)\ell(\ell + 1), p) = 1\) there is no cancellation and we can conclude as in the previous case.

If \( p > 3 \ell \) 2 works.

If \( p = 3 \) we need to look more carefully. In this case take \( \ell = 4 \). Then the equation above gives \( 3^{2n-r} i_1 = 3 \) which has a solution iff \( r = 2n - 1 \), and then \( i_1 = 1 \) and \( i = i_0 + i_1 = 2 \). Let us assume that \( r = 2n - 1 \). The contribution in \( X^{3\partial q} \) is \( c(Y) : = 2 S_F(Y)^{s_0} + 2\theta(q^2 + Y). \) We can write \( Y q^2 + Y = (Y - \alpha) S_F(Y) \). Then \( s_0 \alpha = -1 \) and \( c(Y) = 2 S(Y)^{\theta(q^2(Y - \alpha + \theta^{-3} q^2))}. \) We remark that \( \alpha q^2 + \alpha = 0 \) so \((-\theta^{-3} q^2 + \alpha)^2 + (-\theta^{-3} q^2 + \alpha) = 0; \) in particular \( c(Y) \) divides \( S(Y)^2 \) and we conclude as in the previous case.

Type III.b. So \( p = 2 \). Let \( E \) be the extraspecial group of type III.b. and cardinal \( p^{2n+1} \), i.e. it is the central product of \( Q_8 \) and \( n - 1 \) copies of \( D_8 \).

We have seen that the cover \( W^2 + W = X^{1+2^n} \) induces the extraspecial group \( E \). The corresponding additive polynomial is \( Y q^2 + Y \) where \( q = 2^m \). So if \( S_F(Y) \) is the additive polynomial corresponding to a saturated subgroup we take \( f(X) = S_F(X)^{q} + X^{1+2^n} \). We look at the contribution of \( \Delta(S_F^q)(X,Y) = (S_F(Y) + s_0 X + \ldots + X^{2^n})^q - S_F(Y) - S_F(X)^q \). The contribution in \( S(Y)^{s_0^2} X^{2+2^{n+1}} \) is the only one in degree \( 1 + 2^{n+1} \) modulo multiplication by \( 2^n \) and again we conclude as in the previous case.

\( \square \)

### 4.7 Application

In paragraph \[ \text{we haven’t given a realization of extraspecial groups of type III.a.} \]
Such a group with cardinal \( 2^{2(n-1)+1} \) is the central product \( D_8 \ast \ldots \ast D_8 \) \((n-1)\) times. Let us explain how we get a realization using the method above. Such a group is a saturated subgroup of the extraspecial group of type III.b. \( Q_8 \ast D_8 \ast \ldots \ast D_8 \) of cardinal \( 2^{2n+1} \). The construction above gives the existence of \( f(X) = S_F(X)^{q} + X^{1+2^n} \) where \( S_F \) is an additive polynomial of degree \( 2^{n-1} \) such that the automorphism group \( G_{\infty,1}(f) \) is the saturated subgroup \( D_8 \ast \ldots \ast D_8 \) \((n-1)\) times. Note that the conductor is \( \leq 1 + 6 \ast 2^{(n-1)} \). More concretely we now give an explicit realization of \( D_8 \) \((n = 2) \) with conductor 25 which is the minimal one as \( \frac{|G_{\infty,1}(f)|}{g(f)} = \frac{2^3}{(m-1)/2} \leq \frac{2}{3} \).

We view \( D_8 \) as a saturated subgroup \( \pi^{-1}(F) \) of the extraspecial group \( E := Q_8 \ast D_8 \) where \( \pi : E \to E/Z(E) = W \) for which we know that \( f_1(X) = X^{1+2^n} \) gives a realization. The corresponding additive polynomial is \( Ad_{f_1}(Y) = Y^{2^n} + Y \) and \( W = \mathbb{Z}(Ad_{f_1}) \subset \mathbb{F}_2^{ad_q}. \) Then \( F \) is a subgroup of order 4 and \( \mathbb{F} = Z(S(F)) \), where \( S(F) \) is an additive polynomial dividing \( Ad_{f_1} \). Therefore we can write \( S(F)(Y) = Y^4 + aY^2 + bY \).
where \(b \neq 0\).

The remainder of the division of \(Y^{16} + Y\) by \(Y^4 + aY^2 + bY\) is \((1 + b^5 + ba^6)Y + (b^2a^4 + ab^4 + a^7)Y^2\). Consequently we get the two equations \(1 + b^5 + ba^6 = 0\) and \(b^2a^4 + ab^4 + a^7 = 0\). For each couple \((a,b)\) satisfying these equations we consider \(f_{a,b} := (X^4 + aX^2 + bX)^7 + X^5\); then \(\text{Ad}_{f_{a,b}}(Y) = Y^4 + aY^2 + bY\). Let \(y \in W\) then \(P_{f_{1}}(X, y) + P_{f_{1}}(X + y, y) = y^4(y^6 + y)\). We remark that \(Y^6 + Y\) divides \(\text{Ad}_{f_{1}}(Y) = Y^{16} + Y\) and the quotient is \(Y^{10} + Y^5 + 1\). This gives a partition of \(W\) in two sets \(W_2\): The roots of \(Y^6 + Y\) corresponding to the 12 elements of \(G_{\infty,1}(f_{1})\) of order \(\leq 2\) and \(W_4\): The roots of \(Y^{10} + Y^5 + 1\) corresponding to the 20 elements of \(G_{\infty,1}(f_{1})\) of order 4.

Now \(F := Z(\text{Ad}_{f_{a,b}})\) is a subgroup of \(W\) and for \(y \in F\) one has \(P_{f_{1}}(X, y) = P_{f_{a,b}}(X, y)\) and so \(P_{f_{a,b}}(X, y) + P_{f_{a,b}}(X + y, y) = y^4(y^6 + y)\). Concerning the commutation rule for \(y, z \in F \subset W\) we have \(\epsilon_{f_{a,b}}(y, z) = \epsilon_{f_{1}}(y, z) = z^2y^8 + zy^4 + z^6y^2 + z^4y = yz(y + z)(y^2 + yz + z^2)(yz^4 + z^4y + 1)\).

1. \(a = 0\) and \(1 + b^5 = 0\). Note that in this case \(\text{Ad}_{f}(Y) = Y^4 + bY\) and only the roots \(y = 0\) and \(y = b^2\) are in \(W_2\). Moreover for \(y, z \in F\) one has \(yz(y + z)(y^2 + yz + z^2) = yz(y^3 + z^3) = 0\); it follows that the group \(G_{\infty,1}(f_{a,b})\) is abelian, isomorphic to \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}\).

2. Let \(A := b^5 + 6b^6 + 1\) and \(B := b^4 + a^3b^2 + a^6\). Then the resultant of \(A, B\) in \(b\) is \((b^5 + 1)(b^{10} + b^4 + 1)^2\). The case \(b^5 + 1 = 0\) is case 1. above. Now we can assume that \(b^{10} + b^5 + 1 = 0\), i.e. \(b\) is a primitive 15-th root of 1. The equations \(A = B = 0\) give three sets of covers.

i) \(ab = 1\), i.e. \(a = b^{14}\). In this case \(Y^4 + aY^2 + bY\) divides \(Y^6 + Y\), the group has exponent 2 and it is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^3\).

ii) \(ab = b^5\) i.e. \(a = b^4\). In this case \(Y^4 + aY^2 + bY\) has only one root \((b^5)\) in common with \(Y^{10} + Y^5 + 1\). It follows that the group \(G_{\infty,1}(f_{a,b})\) has 2 elements of order 4 so it is \(D_8\). We can write \(\text{red}(f_{a,b}) = (b^{14} + b^5)X + (b + b^8)X^3 + (1 + b^{14} + b^{13})X^5 + (b^7 + 1)X^7 + (b^{13} + b^{10})X^9 + (b^4 + b + b^6)X^11 + (b^2 + 1)X^{13} + b^3X^{17} + b^3X^{19} + b^9X^{21} + bX^{25}\), which is defined over \(\mathbb{F}_{16}\).

iii) \(ab = b^{10}\) i.e. \(a = b^{9}\). In this case \((Y^4 + aY^2 + bY)/Y\) divides \(Y^{10} + Y^5 + 1\) and it follows that the group \(G_{\infty,1}(f_{a,b})\) has 6 elements of order 4, so this is \(Q_8\).

**Remark 4.22.** We could as well obtain families. For this it suffices to deal with \(f_1\) giving a family, for example \(f_1 = tX^3 + X^5\). The corresponding discussion is thought more delicate as the above.

### 4.8 An Algorithm

Here we illustrate the algorithm which for a given \(f\) gives the structure of the group \(G_{\infty,1}(f)\). This example is a realization \(D_8\) over \(\mathbb{F}_2\). We have used the following Maple code:

```maple
> restart;
> f:=X^2*(1+2)+X^2*(1+2+2^2)+X^2*(1+2+2^4)+X^2*(1+2+2^5)+X^2*(1+2+2^3+2^5):
> F:=collect(subs(X=X+Y,f),X) mod 2:
> f1:=collect(F-subs(X=0,F)-subs(Y=0,F),[X,Y]) mod 2:
> f2:=rem(collect(f1+f2^2-subs(X=0,f1+f2^2),[X,Y]),2,X^21,X) mod 2:
```

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Note that $21 = 40/2 + 1$. Here one reiterates the command until it is stationary.

```latex
> p:=f2:
> G:=collect(f1+p^2-p,[X,Y,t]) mod 2;
G :=
(Y^24+Y^80+Y^132+Y^528+Y^192+Y^64+Y^576+Y^1280+Y^1088+Y^6+Y^3+Y^16+Y^9
+Y^272)*X^32+(Y^256+Y^128+Y^4+Y^32)*X^24+(Y^128+Y^2)*X^36+(Y^4+Y)*X^34
+(Y+Y^16)*X^40+(Y^8+Y^2)*X^33
```

Here we remark that $A_d(Y)$ divides the coefficient of $X^{34}$.

```latex
> G:=collect(rem(G,Y^4+Y,Y) mod 2,X);
G := 0
```

Conclusion: $A_d(Y) = Y^4 + Y$.

```latex
> p:=collect(rem(p,Y^4+Y,Y) mod 2,X);
p :=
Y^2*X^20+Y^2*X^17+X^10*Y+Y^2*X^9+Y^3*X^8+Y^2*X^5+X^3*Y^2+Y^3*X^2+Y^2*X
> rem(collect(p+subs(X=X+Y,p),X),Y^4+Y,Y) mod 2;
Y^3+Y^2+Y
> Gcd(Y^4+Y,Y^3+Y^2+Y) mod 2;
Y^3+Y^2+Y
```

It follows that the 3 roots of $Y^3 + Y^2 + Y$ induce 6 order 2 elements and the last root $Y = 1$ induces 2 order 4 elements.

```latex
> CY:=collect(subs(X=X+Z,p)+p,[Y]) mod 2:
> CZ:=subs([Y=Z,Z=Y],CY):
> C:=collect(CY-CZ,[X,Y,Z]) mod 2:
> CC:=collect(rem(C,Y^4+Y,Y) mod 2,Z):
> CCC:=collect(rem(CC,Z^4+Z,Z) mod 2,X);
CCC := Z^2*Y+Z*Y^2
```

The group is non abelian of order 8 with 2 elements of order 4; this is $D_8$. Note that $f := X^3 + X^7 + X^{19} + X^{35} + X^{41}$ is reduced with conductor > 25. More generally it is a good question to ask for realizations over $\mathbb{F}_2$ (i.e. $f \in \mathbb{F}_2[X]$) for groups in the class $C_1$.

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