On \( \text{CH} + 2^{\aleph_1} \to (\alpha)_2^2 \) for \( \alpha < \omega_2 \)

by

Saharon Shelah\(^1\)

1. Introduction

We prove the consistency of

\[
\text{CH} + 2^{\aleph_1} \text{ is arbitrarily large } + 2^{\aleph_1} \not\to (\omega_1 \times \omega)_2^2
\]

(Theorem 1). If fact, we can get \( 2^{\aleph_1} \not\to [\omega_1 \times \omega]_{\aleph_0}^2 \), see 1A. In addition to this theorem, we give generalizations to other cardinals (Theorems 2 and 3). The \( \omega_1 \times \omega \) is best possible as CH implies

\[
\omega_3 \to (\omega \times n)_2^2.
\]

We were motivated by the question of Baumgartner [B1] on whether CH implies \( \omega_3 \to (\alpha)_2^2 \) for \( \alpha < \omega_2 \) (if \( 2^{\aleph_1} = \aleph_2 \), it follows from the Erdos–Rado theorem). He proved the consistency of positive answer with \( \text{CH} + 2^{\aleph_1} > \aleph_3 \), and proved in ZFC a related polarized partition relation (from CH)

\[
\left( \begin{array}{c} \aleph_3 \\ \aleph_2 \end{array} \right) \to \left( \begin{array}{c} \aleph_1 \\ \aleph_1 \end{array} \right)^{1,1}_{\aleph_0}.
\]

Note: The main proof here is that of Theorem 1. In that proof, in the way things are set up, the main point is proving the \( \aleph_2 \)-c.c. The main idea in the proof is using \( \mathbb{P} \) (defined in the proof). It turns out that we can use as elements of \( \mathcal{P} \) (see the proof) just pairs \((a, b)\). Not much would be changed if we used \( \{ (a_n, a_n) : n < \omega \} \), \( a_n \) a good approximation of the \( n \)th part of the suspected monochromatic set of order type \( \omega_1 \times \omega \). In 1A, 2 and 3 we deal with generalizations and in Theorem 4 with complementary positive results.

2. The main result

Theorem 1. Suppose

(a) CH.
(b) \( \lambda^{\aleph_1} = \lambda \).

Then there is an \( \aleph_2 \)-c.c. \( \aleph_1 \)-complete forcing notion \( \mathbb{P} \) such that

(i) \( |\mathbb{P}| = \lambda \).
(ii) \( \not\Vdash_{\mathbb{P}} "2^{\aleph_1} = \lambda, \lambda \not\to (\omega_1 \times \omega)_2^2" \).
(iii) \( \Vdash_{\mathbb{P}} \text{CH} \).
(iv) Forcing with \( \mathbb{P} \) preserves cofinalities and cardinalities.

\(^1\) Publication number 424. Partially supported by BSF.
Proof. By Erdos and Hajnal [EH] there is an algebra $\mathcal{B}$ with $2^{|\aleph_0|} = \aleph_1$ \omega-place functions, closed under composition (for simplicity only), such that

$\otimes$ If $\alpha_n < \lambda$ for $n < \omega$, then for some $k$

$\alpha_k \in \text{cl}_\mathcal{B}\{\alpha_l : k < l < \omega \}.$

[$\otimes$ implies that for every large enough $k$, for every $m$, $\alpha_k \in \text{cl}_\mathcal{B}\{\alpha_k : m < l < \omega \}$.] Let

$$\mathcal{R}_\delta = \{ b : b \subseteq \lambda, \otp(b) = \delta, \alpha \in b \Rightarrow b \subseteq \text{cl}_\mathcal{B}(b \setminus \alpha) \}.$$ 

So by $\otimes$ we have

$\oplus$ If $\alpha$ is a limit ordinal, $b \subseteq \lambda$, $\otp(b) = \alpha$, then for some $\alpha \in b$, $b \setminus \alpha \in \bigcup \delta \mathcal{R}_\delta$.

Let $\mathcal{R}_{<\omega_1} = \bigcup_{\alpha < \omega_1} \mathcal{R}_\alpha$. Let $\mathcal{P}$ be the set of forcing conditions

$$(w, c, \mathcal{P})$$

where $w$ is a countable subset of $\lambda$, $c : [w]^2 \rightarrow \{\text{red}, \text{green}\} = \{0, 1\}$ (but we write $c(\alpha, \beta)$ instead of $c(\{\alpha, \beta\})$), and $\mathcal{P}$ is a countable family of pairs $(a, b)$ such that

(i) $a, b$ are subsets of $w$
(ii) $b \in \mathcal{R}_{<\omega_1}$ and $a$ is a finite union of members of $\mathcal{R}_{<\omega_1}$
(iii) $\sup(a) < \min(b)$
(iv) If $\sup(a) \leq \gamma < \min(b)$, $\gamma \in w$, then $c(\gamma, \cdot)$ divides $a$ or $b$ into two infinite sets.

We use the notation

$$p = (w^p, c^p, \mathcal{P}^p)$$

for $p \in \mathcal{P}$. The ordering of the conditions is defined as follows:

$$p \leq q \iff w^p \subseteq w^q \& c^p \subseteq c^q \& \mathcal{P}^p \subseteq \mathcal{P}^q.$$ 

Let

$$\mathcal{G} = \bigcup\{c^p : p \in G_{\mathcal{P}} \}.$$ 

Fact A. $\mathcal{P}$ is $\aleph_2$-complete.

Proof. Trivial—take the union. 

Fact B. For $\gamma < \lambda$, $\{ q \in \mathcal{P} : \gamma \in w^q \}$ is open dense.

Proof. Let $p \in \mathcal{P}$. If $\gamma \in w^p$, we are done. Otherwise we define $q$ as follows: $w^q = w^p \cup \{\gamma\}$, $\mathcal{P}^q = \mathcal{P}^p$, $c^q | w^p = c^p$ and $c^q(\gamma, \cdot)$ is defined so that if $(a, b) \in \mathcal{P}^q$, then $c^q(\gamma, \cdot)$ divides $a$ and $b$ into two infinite sets.
Fact C. $\|\mathbb{P}\| \geq \lambda$ and $c : [\lambda]^{2} \to \{\text{red, green}\}$

Proof. The second phrase follows from Fact B. For the first phase, define $\rho_{\alpha} \in \omega_{1}2$, for $\alpha < \lambda$, by: $\rho_{\alpha}(i) = c(0, \alpha + i)$. Easily

\[ \|\mathbb{P}\| \rho_{\alpha} \in \omega_{1}2 \text{ and for } \alpha < \beta < \lambda, \rho_{\alpha} \neq \rho_{\beta}; \text{ so } \|\mathbb{P}\| \geq 2^{\aleph_{1}}. \]

\[\square\]

Fact D. $\mathbb{P}$ satisfies the $\aleph_{2}$-c.c.

Proof. Suppose $p_{i} \in \mathbb{P}$ for $i < \aleph_{2}$. For each $i$ choose a countable family $\mathcal{A}^{i}$ of subsets of $w^{p_{i}}$ such that $\mathcal{A}^{i} \subseteq \mathcal{R}_{<\omega_{1}}$ and $(a, b) \in \mathcal{P}^{p_{i}}$ implies $b \in \mathcal{A}^{i}$ and $a$ is a finite union of members of $\mathcal{A}^{i}$. For each $\gamma \in c \in \mathcal{A}^{i}$ choose a function $F^{i}_{\gamma,c}$ s.t. $F^{i}_{\gamma,c}(c \setminus (\gamma + 1)) = \gamma$. Let $v_{i}$ be the closure of $w_{i}$ (in the order topology).

We may assume that $(v_{i} : i < \omega_{2})$ is a $\Delta$-system (we have CH) and that otp($v_{i}$) is the same for all $i < \omega_{2}$. W.l.o.g. for $i < j$ the unique order-preserving function $h_{i,j}$ from $v_{i}$ onto $v_{j}$ maps $p_{i}$ onto $p_{j}$, $w^{p_{i}} \cap w^{p_{j}} = w^{p_{0}} \cap w^{p_{1}}$ onto itself, and

\[ F^{i}_{\gamma,c} = F^{j}_{h_{i,j}(\gamma),h_{i,j} \cdot c} \]

for $\gamma \in c \in \mathcal{A}^{i}$ (remember: $\mathbb{IB}$ has $2^{\omega_{0}} = \aleph_{1}$ functions only). Hence

$\otimes_{1}$

$h_{i,j}$ is the identity on $v_{i} \cap v_{j}$ for $i < j$.

Clearly by the definition of $\mathcal{R}_{<\omega_{1}}$ and the condition on $F^{i}_{\gamma,c}$:

$\otimes_{2}$

If $a \in \mathcal{A}^{i}$, $i \neq j$ and $a \notin w^{p_{i}} \cap w^{p_{j}}$,

then $a \setminus (w^{p_{i}} \cap w^{p_{j}})$ is infinite.

We define $q$ as follows.

$w^{q} = w^{p_{0}} \cup w^{p_{1}}$.

$\mathcal{P}^{q} = \mathcal{P}^{p_{0}} \cup \mathcal{P}^{p_{1}}$.

$c^{p}$ extends $c^{p_{0}}$ and $c^{p_{1}}$ in such a way that, for $e \in \{0, 1\}$,

(*) for every $\gamma \in w^{p_{e}} \setminus w^{p_{1-e}}$ and every $a \in \mathcal{A}^{1-e}$, $w^{p}(\gamma, \cdot)$ divides $a$ into two infinite parts, provided that

(**) $a \setminus w^{p_{e}}$ is infinite.

This is easily done and $p_{0} \leq q$, $p_{1} \leq q$, provided that $q \in \mathbb{P}$. For this the problematic part is $c^{q}$ and, in particular, part (iv) of the definition of $\mathbb{P}$. So suppose $(a, b) \in \mathcal{P}^{q}$, e.g., $(a, b) \in \mathcal{P}^{p_{0}}$. Suppose also $\gamma^{*} \in w^{q}$ so that sup($a$) $\leq \gamma^{*} < \sup(b)$. If $\gamma^{*} \in w^{p_{0}}$, there is no problem, as $p_{0} \in \mathbb{P}$. So let us assume $\gamma^{*} \in w^{q} \setminus w^{p_{0}} = w^{p_{1}} \setminus w^{p_{0}}$. If $a \setminus w^{p_{1}}$ or $b \setminus w^{p_{1}}$ is infinite, we are through in view of condition (*) in the definition of $c^{q}$. Let us finally assume $a \setminus w^{p_{1}}$ is finite. But $a \subseteq w^{p_{0}}$. Hence $a \setminus (w^{p_{0}} \cap w^{p_{1}})$ is finite and $\otimes_{2}$ implies it is empty, i.e. $a \subseteq w^{p_{0}} \cap w^{p_{1}}$. Similarly, $b \subseteq w^{p_{0}} \cap w^{p_{1}}$. So $h_{0,1}$ is the identity. But $(a, b) \in \mathcal{P}^{p_{0}}$. But $h_{i,j}$ maps $p_{i}$ onto $p_{j}$. Hence $(a, b) \in \mathcal{P}^{p_{1}}$. As $p_{1} \in \mathbb{P}$, we get the desired conclusion. $\square$
Fact E. \[ \vdash \text{“There is no c-monochromatic subset of } \lambda \text{ of order-type } \omega_1 \times \omega.” \]

Proof. Let \( p \) force the existence of a counterexample. Let \( G \) be \( \mathbb{I} \mathbb{P} \)-generic over \( V \) with \( p \in G \). In \( V[G] \) we can find \( A \subseteq \lambda \) of order-type \( \omega_1 \times \omega \) such that \( c \mid [A]^2 \) is constant. Let \( A = \bigcup_{n<\omega} A_n \) where \( \text{otp}(A_n) = \omega_1 \) and \( \sup(A_n) \leq \min(A_{n+1}) \). We can replace \( A_n \) by any \( A'_n \subseteq A_n \) of the same cardinality. Hence we may assume w.l.o.g.

\[ (\ast)_1 \quad A_n \in \mathcal{R}_{\omega_1} \quad \text{for } n < \omega. \]

Let \( \delta_n = \sup(A_n) \) and

\[ \beta_n = \min \{ \beta : \delta \leq \beta < \lambda, \ d(\beta, \cdot) \text{ does not divide } \bigcup_{l \leq n} A_l \text{ into two infinite sets} \}, \]

where \( d = c^G \). Clearly \( \beta_n \leq \min(A_{n+1}) \). Hence \( \beta_n < \beta_{n+1} \). Let \( d_n \in \{0, 1\} \) be such that \( d(\beta_n, \gamma) = d_n \) for all but finitely many \( \gamma \in \bigcup_{l \leq n} A_l \). Let \( u \) be an infinite subset of \( \omega \) such that \( \{ \beta_n : n \in u \} \in \mathcal{R}_\omega \). Let \( A_l = \{ \alpha_i^l : i < \omega_1 \} \) in increasing order. So \( p \) forces all this on suitable names

\[ \langle \beta_n : n < \omega \rangle, \langle \alpha_i^l : i < \omega_1 \rangle, \langle \delta_n : n < \omega \rangle. \]

As \( \mathbb{I} \mathbb{P} \) is \( \aleph_1 \)-complete, we can find \( p_0 \in \mathbb{I} \mathbb{P} \) with \( p \leq p_0 \) so that \( p_0 \) forces \( \beta_l = \beta_l \) and \( \delta_n = \delta_n \) for some \( \beta_l \) and \( \delta_n \). We can choose inductively conditions \( p_k \in \mathbb{I} \mathbb{P} \) such that \( p_k \leq p_{k+1} \) and there are \( i_k < j_k \) and \( \alpha_i^l \) (for \( i < j_k \)) with

\[ p_{k+1} \models “\alpha_i^l > \sup \{ i : \alpha_i^l \in w^p \}, \]

\[ \alpha_i^l \in w^{p_{k+1}} \text{ for } i < j_k, \]

\[ \{ \alpha_i^l : i < i_k \} \subseteq c_\mathbb{I} \mathbb{B} \{ \alpha_i^l : i < j_k \}, \]

\[ \alpha_i^l = \alpha_i^l \text{ for } i < j_k \] and

\[ \gamma \in [\delta_m, \beta_m] \cap w^p \text{ implies } c(\gamma, \cdot) \text{ divides } \{ \alpha_i^l : i < j_k, l \leq m \} \text{ into two infinite sets.”} \]

(remember our choice of \( \beta_m \)). Let

\[ l(\ast) = \min(u) \]

\[ a = \{ \alpha_i^l : l \leq l(\ast), i < \bigcup_k j_k \} \]

\[ b = \{ \beta_l : l \in u \} \]

\[ q = (\bigcup_k w^p, \bigcup_k c^p, \bigcup_k p^p \cup \{(a, b)\}). \]
Now \( q \in \mathcal{I} \). To see that \( q \) satisfies condition (iv) of the definition of \( \mathcal{I} \), let 
\[ \sup(a) \leq \gamma < \min(b). \]
Then 
\[ \sup\{ \alpha_{i_k}^{l(*)} : k < \omega \} \leq \gamma < \beta_{l(*)}. \]
But \( \gamma \in w^p = \bigcup_k w^{p_k} \), so for some \( k, \gamma \in w^{p_k} \). This implies
\[ \gamma \notin \left( \alpha_{i_{k+1}}^{l(*)}, \delta_{l(*)} \right), \]
whence \( \gamma \geq \delta_{l(*)} \) and
\[ \{ \alpha_i^l : l \leq l(*), i < j_k \} \subseteq a, \]
which implies the needed conclusion.

Also \( q \geq p_k \geq p \). But now, if \( r \geq q \) forces a value to \( \alpha_{l(*)}^{l(*)} \), we get a contradiction. \( \square \)

**Remark 1A.** Note that the proof of Theorem 1 also gives the consistency of \( \lambda \not\rightarrow [\omega_1 \times \omega_1]^2 \) \( \kappa \): replace “\( c(\gamma, \cdot) \) divides a set \( x \) into two infinite parts” by “\( c(\gamma, \cdot) \) gets all values on a set \( x \).”

### 3. Generalizations to other cardinals

How much does the proof of Theorem 1 depend on \( \aleph_1 \)? Suppose we replace \( \aleph_0 \) by \( \mu \).

**Theorem 2.** Assume \( 2^\mu = \mu^+ < \lambda = \lambda^\mu \) and \( 2 < \kappa \leq \mu \). Then for some \( \mu^+-\)complete \( \mu^{++}-\)c.c. forcing notion \( \mathcal{I} \) of cardinality \( 2^\mu \):
\[ \models_{\mathcal{I}} 2^\mu = \lambda, \quad \lambda \not\rightarrow [\mu^+ \times \mu]^2_\kappa. \]

**Proof.** Let \( \mathcal{I} \) and \( \mathcal{R}_\delta \) be defined as above (for \( \delta \leq \mu^+ \)). Clearly \( \oplus \)

If \( a \subseteq \lambda \) has no last element, then for some \( \alpha \in a, \ a \setminus \alpha \in \bigcup_\delta \mathcal{R}_\delta. \)
Hence, if \( \delta = \text{otp}(a) \) is additively indecomposable, then \( a \setminus \alpha \in \mathcal{R}_\delta \) for some \( \alpha \in a. \)

Let \( \mathcal{I}_\mu \) be the set of forcing conditions
\[ (w, c, \mathcal{P}) \]
where \( w \subseteq \lambda, \ |w| \leq \mu, \ c : [w]^2 \rightarrow \{ \text{red}, \text{green} \}, \) and \( \mathcal{P} \) is a set of \( \leq \mu \) pairs \( (a, b) \) such that
(i) \( a, b \) are subsets of \( w \).
(ii) \( b \in \mathcal{R}_\mu \), and \( a \) is a finite union of members of \( \bigcup_{\mu \leq \delta < \mu^+} \mathcal{R}_\delta. \)
(iii) \( \sup(a) < \min(b) \).
(iv) If \( \sup(a) \leq \gamma < \min(b), \ \gamma \in w, \) then the function \( c(\gamma, \cdot) \) gets all values \( (< \kappa) \)
on \( a \) or \( b \).

With the same proof as above we get
\[ \mathcal{I}_\mu \text{ satisfies the } \mu^{++}-\text{c.c.}, \]
\[ \mathcal{I}_\mu \text{ is } \mu^+-\text{complete}, \]
(so cardinal arithmetic is clear) and
\[ \models_{\mathcal{I}_\mu} \lambda \not\rightarrow [\mu \times \mu]^2_\kappa. \]
\[ \square \]
What about replacing $\mu^+$ by an inaccessible $\theta$? We can manage by demanding
\[
\{ a \cap (\alpha, \beta) : (a, b) \in \mathcal{P}, \bigcup_n \text{otp}(a \cap (\alpha, \beta)) \times n = \text{otp}(a) \}
\]
$(\alpha, \beta)$ maximal under these conditions $\}$
is free (meaning there are pairwise disjoint end segments) and by taking care in defining the order. Hence the completeness drops to $\theta$-strategical completeness. This is carried out in Theorem 3 below.

**Theorem 3.** Assume $\theta = \theta^{<\theta} > \aleph_0$ and $\lambda = \lambda^{<\theta}$. Then for some $\theta^+-$c.c. $\theta$-strategically complete forcing $\mathbb{P}$, $|\mathbb{P}| = \lambda$ and
\[
\models \mathbb{P} 2^\theta = \lambda, \lambda \not\rightarrow (\theta \times \theta)^2.
\]

**Proof.** For $W$ a family of subsets of $\lambda$, each with no last element, let
\[
\text{Fr}(W) = \{ f : f \text{ is a choice function on } W \text{ s.t.} \}
\]
\[
\{ a \setminus f(a) : a \in W \} \text{ are pairwise disjoint } \}.
\]

If $\text{Fr}(W) \neq \emptyset$, $W$ is called free.

Let $\mathbb{P}_{<\theta}$ be the set of forcing conditions
\[
(w, c, \mathcal{P}, W)
\]
where $w \subseteq \lambda$, $|w| < \theta$, $c : [w]^2 \to \{\text{red}, \text{green}\}$, $W$ is a free family of $< \theta$ subsets of $w$, each of which is in $\bigcup_{\delta < \theta} \mathcal{R}_\delta$, and $\mathcal{P}$ is a set of $< \theta$ pairs $(a, b)$ such that
(i) $a, b$ are subsets of $w$.
(ii) $b \in \mathcal{R}_\omega$.
(iii) $\sup(a) < \min(b)$ and for some $\delta_0 < \delta_1 < \cdots < \delta_n$, $\delta_0 < \min(a)$, $\sup(a) \leq \delta_n$, $a \cap [\delta_i, \delta_{i+1}) \in W$.
(iv) If $\sup(a) \leq \gamma < \min(b)$, $\gamma \in w$, then $c(\gamma, \cdot)$ divides $a$ or $b$ into two infinite sets.

We order $\mathbb{P}_{<\theta}$ as follows:
\[
p \leq q \text{ iff } w^p \subseteq w^q, c^p \subseteq c^q, \mathcal{P}^p \subseteq \mathcal{P}^q, W^p \subseteq W^q
\]
and every $f \in \text{Fr}(W^p)$ can be extended to a member of $\text{Fr}(W^q)$.

\[\square\]

4. A provable partition relation

**Claim 4.** Suppose $\theta > \aleph_0$, $n, r < \omega$ and $\lambda = \lambda^{<\theta}$. Then
\[
(\lambda^+)^r \times n \rightarrow (\theta \times n, \theta \times r)^2.
\]
Proof. We prove this by induction on \( r \). Clearly the claim holds for\( r = 0,1 \). So w.l.o.g. we assume \( r \geq 2 \). Let \( c \) be a 2-place function from \((\lambda^+)^r \times n\) to \{red,green\}. Let \( \chi = \mathit{beth}_2(\lambda)^+ \). Choose by induction on \( l \) a model \( N_l \) such that 

\[
N_l \prec (H(\chi), \in, <^*) ,
\]

\(|N_l| = \lambda, \lambda + 1 \subseteq N_l, N_l^{<\theta} \subseteq N_l, c \in N_l \text{ and } N_l \in N_{l+1}. \] Here \( <^* \) is a well-ordering of \( H(\chi) \). Let 

\[
A_l = [(\lambda^+)^r \times l, (\lambda^+)^r \times (l+1)] ,
\]

and let \( \delta_l \in A_l \setminus N_l \) be such that \( \delta_l \notin x \) whenever \( x \in N_l \) is a subset of \( A_l \) and \( \text{otp}(x) < (\lambda^+)^{r-1} \). W.l.o.g. we have \( \delta_l \in N_{l+1} \). Now we shall show

\[
(*) \quad \text{If } Y \in N_0, Y \subseteq A_m, |Y| = \lambda^+ \text{ and } \delta_m \in Y, \text{ then we can find } \beta \in Y \text{ such that } c(\beta, \delta_m) = \text{red.}
\]

Why \((*)\) suffices? Assume \((*)\) holds. We can construct by induction on \( i < \theta \) and for each \( i \) by induction on \( l < n \) an ordinal \( \alpha_{i,l} \) s.t.

(a) \( \alpha_{i,l} \in A_l \) and \( j < i \Rightarrow \alpha_{j,l} < \alpha_{i,l} \).

(b) \( \alpha_{i,l} \in N_0 \).

(c) \( c(\alpha_{i,l}, \delta_m) = \text{red} \) for \( m < n \).

(d) \( c(\alpha_{i,l}, \alpha_{i_1,l_1}) = \text{red} \) when \( i_1 < i \) or \( i_1 = i \) & \( l_1 < l \).

Accomplishing this suffices as \( \alpha_{i,l} \in A_l \) and 

\[
l < m \Rightarrow \text{sup } A_l \leq \text{min } A_m .
\]

Arriving in the inductive process at \((i,l)\), let 

\[
Y = \{ \beta \in A_l : c(\beta, \alpha_{j,m}) = \text{red} \text{ if } j < i, m < n, \text{ or } j = i, m < l \} .
\]

Now clearly \( Y \subseteq A_l \). Also \( Y \in N_0 \) as all parameters are from \( N_0 \), their number is \( < \theta \) and \( N_0^{<\theta} \subseteq N_0 \). Also \( \delta_l \in Y \) by the induction hypothesis (and \( \delta_l \in A_l \)). So by \((*)\) we can find \( \alpha_{i,l} \) as required.

Proof of \((*)\): \( Y \nsubseteq N_0 \), because \( \delta_m \in Y \) and \( Y \in N_0 \). As \( |Y| = \lambda^+ \), we have \( \text{otp}(Y) \geq \lambda^+ \). But \( \lambda^+ \rightarrow (\lambda^+ , \theta)^2 \), so there is \( B \subseteq A_m \) s.t. \( |B| = \lambda^+ \) and \( c \mid B \times B \) is constantly red or there is \( B \subseteq A_m \) s.t. \( |B| = \theta \) and \( c \mid B \times B \) is constantly green. In the former case we get the conclusion of the claim. In the latter case we may assume \( B \in N_0 \), hence \( B \nsubseteq N_0 \), and let \( k \leq n \) be maximal s.t.

\[
B' = \{ \xi \in B : \bigwedge_{l<k} c(\delta_l, \xi) = \text{red} \}
\]

has cardinality \( \theta \). If \( k = n \), any member of \( B' \) is as required in \((*)\). So assume \( k < n \). Now \( B' \in N_k \), since \( B \in N_0 \prec N_k \) and \( \{N_l, A_l\} \in N_k \) and \( \delta_l \in N_k \) for \( l < k \). Also

\[
\{ \xi \in B' : c(\delta_k, \xi) = \text{red} \}
\]
is a subset of $B'$ of cardinality $< \theta$ by the choice of $k$. So for some $B''_0 \subseteq N_0$, $c \mid \{\delta_k\} \times (B' \setminus B'')$ is constantly green (e.g., as $B' \subseteq N_0$, and $N_0^<\theta \subseteq N_0$). Let

$$Z = \{ \delta \in A_k : c \mid \{\delta\} \times (B' \setminus B'') \text{ is constantly green} \}$$

and

$$Z' = \{ \delta \in Z : (\forall \alpha \in B' \setminus B'') (\delta < \alpha \iff \delta_k < \alpha) \}.$$ 

So $Z \subseteq A_k$, $Z \in N_k$, $\delta_k \notin N_k$ and therefore $\text{otp}(Z) = \text{otp}(A_k) = (\lambda^+)^r$. Note that $k \neq l \Rightarrow Z' = Z$ and $k = l \Rightarrow Z' = Z \setminus \sup(B' \setminus B'')$, so $Z'$ has the same properties. Now we apply the induction hypothesis: one of the following holds (note that we can interchange the colours): (a) there is $Z'' \subseteq Z'$, $\text{otp}(Z'') = \theta \times n$, $c \mid Z'' \times Z''$ is constantly red, wlog $Z'' \in N_k$, or (b) there is $Z'' \subseteq Z'$, $\text{otp}(Z'') = \theta \times (r - 1)$, $c \mid Z'' \times Z''$ green and wlog $Z'' \in N_k$. If (a), we are done; if (b), $Z'' \cup (B' \setminus B'')$ is as required. □

**Remark 4A.** So $(\lambda^+)^{n+1} \to (\theta \times n)^2$ for $\lambda = \lambda^{<\theta}$, $\theta = \text{cf}(\theta) > \aleph_0$ (e.g., $\lambda = 2^{<\theta}$).

**Remark 4B.** Suppose $\lambda = \lambda^{<\theta}$, $\theta > \aleph_0$. If $c$ is a 2-colouring of $(\lambda^+)^s \times n$ by $k$ colours and every subset of it of order type $(\lambda^{+(r-1)})^s \times n$ has a monochromatic subset of order type $\theta$ for each of the colours, one of the colours being red, then by the last proof we get

(a) There is a monochromatic subset of order type $\theta \times n$ and of colour red or

(b) There is a colour $d$ and a set $Z$ of order type $(\lambda^+)^s$ and a set $B$ of order type $\theta$ s.t. $B < Z$ or $Z < B$ and

$$\{ (\alpha, \beta) : \alpha \in B, \beta \in Z \text{ or } \alpha \neq \beta \in B \}$$

are all coloured with $d$.

So we can prove that for 2-colourings by $k$ colours $c$

$$(\lambda^+)^s \times n \to (\theta \times n_1, \ldots, \theta \times n_k)^2$$

when $r, s, n$ are sufficiently large (e.g., $n \geq \min\{n_1 : l = 1, \ldots, k, s \geq \sum_{l=1}^k n_l \}$ by induction on $\sum_{l=1}^k n_l$.

Note that if $c$ is a 2-colouring of $\lambda^{+2k}$, then for some $l < k$ and $A \subseteq \lambda^{+2k}$ of order type $\lambda^{+(2l+2)}$ we have

(*) If $A' \subseteq A$, $\text{otp}(A') = \lambda^{+2l}$, and $d$ is a colour which appears in $A$, then there is $B \subseteq A'$ of order type $\theta$ s.t. $B$ is monochromatic of colour $d$.

We can conclude $\lambda^{+2k} \to (\theta \times n)^2_k$.

**References**

[B1] J. Baumgartner, ??

[EH] P. Erdos and A. Hajnal, ??
Institute of Mathematics
The Hebrew University
Jerusalem
Israel

Department of Mathematics
Rutgers University
New Brunswick, NJ
USA