ZERO-HOPF BIFURCATION IN A 3-D JERK SYSTEM

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Abstract. We consider the 3-D system defined by the jerk equation \( \dot{x} = -a \ddot{x} + x \dot{x}^2 - x^3 - bx + c \dot{x} \), with \( a, b, c \in \mathbb{R} \). When \( a = b = 0 \) and \( c < 0 \) the equilibrium point localized at the origin is a zero-Hopf equilibrium. We analyze the zero-Hopf Bifurcation that occur at this point when we persuade a quadratic perturbation of the coefficients, and prove that one, two or three periodic orbits can born when the parameter of the perturbation goes to 0.

1. Introduction

Motivated by the development of the Chua circuit [6], many researchers have been interested in finding other circuits that chaotically oscillate. Some simple third-order ordinary differential equations of the form

\[ \dot{x} = J(\ddot{x}, \dot{x}, x, t), \]

whose solutions are chaotic are example of such circuits [7], [18]. In classical mechanics, the function \( J \) is called jerk, and corresponds to the rate of change of acceleration, or equivalently to the third-time derivative of the position \( x \). A jerk flow so is an explicit third order differential equation as above describing the evolution of the position \( x(t) \) with the time \( t \).

The following non-linear third-order differential equation is the jerk flow studied by Vaidyanathan [19]:

\[ \dot{x} = -a \ddot{x} + x \dot{x}^2 - x^3 - bx + c \dot{x}, \]

where \( a, b \) and \( c \) are parameters. This equation generalizes the one studied by Sprott [18], where \( b = c = 0 \). In system form, the differential equation (1) corresponds to the 3-D jerk system

\[ \begin{align*}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= -ax - bx + cy + xy^2 - x^3.
\end{align*} \]

In [19], Vaidyanathan shows that system (2) is chaotic when \( a = 3.6, b = 1.3 \) and \( c = 0.1 \). The aim of the present paper is to study this system depending on the parameters \( (a, b, c) \in \mathbb{R}^3 \) from another point of view.

A zero-Hopf equilibrium of a 3-dimensional autonomous differential system is an isolated equilibrium point of the system, which has a zero eigenvalue and a pair of purely imaginary eigenvalues. In general, the zero-Hopf bifurcation is a parametric unfolding of a 3-dimensional autonomous differential system with a zero-Hopf equilibrium. The unfolding can exhibit different topological type of dynamics in the

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small neighborhood of this isolated equilibrium as the parameter varies in a small neighborhood of the origin. For more information on the zero-Hopf bifurcation, we address the reader to Guckenheimer, Han, Holmes, Kuznetsov, Marsden and Scheurle in \[10, 11, 12, 13, 17\], respectively.

As far as we know nobody has studied the existence or non-existence of zero-Hopf equilibria and zero-Hopf bifurcations in the 3-D jerk system \([2]\). The objective of this paper is to persuade this study.

Usually the main tool for studying a zero-Hopf bifurcation is to pass the system to the normal form of a zero-Hopf bifurcation. Our analysis, however, will use the averaging theory (see Section 2 for the results on this theory needed for our study). The averaging theory has already been used to study Hopf and zero-Hopf bifurcations in some others differential systems, see for instance \[3, 5, 8, 9, 14, 15, 16\].

Our main results are the following. We first characterize the zero-Hopf equilibrium point of system \([2]\) in Proposition 1.

**Proposition 1.** The differential system \([2]\) has a zero-Hopf equilibrium point if and only if \(a = b = 0\) and \(c < 0\). In this case, the zero-Hopf equilibrium is the only singular point of the system, and it is localized at the origin.

Then we study when the 3-D jerk system \([2]\) having a zero-Hopf equilibrium point at the origin of coordinates has a zero-Hopf bifurcation producing some periodic orbit in Theorem 2.

**Theorem 2.** Let \(a_2, b_2, c_1, c_2, \delta \in \mathbb{R}\) such that \(3 - \delta^2 \neq 0\) and \(2a_2\delta^2 \neq b_2\) and set \((a, b, c) = (\varepsilon^2a_2, \varepsilon^2b_2, -\delta^2 + \varepsilon c_1 + \varepsilon^2c_2)\). Then the 3-D jerk system \([2]\) has a zero-Hopf bifurcation at the equilibrium point localized at the origin of coordinates in the following situations:

1. \(a_2\delta^2 + 2b_2 < 0\) and \(a_2\delta^2 - b_2 > 0\). In this case three periodic orbits born at the equilibrium point when \(\varepsilon \to 0\).
2. \(a_2\delta^2 + 2b_2 < 0\) and \(a_2\delta^2 - b_2 < 0\). In this case two periodic solutions born at the equilibrium point when \(\varepsilon \to 0\).
3. \(a_2\delta^2 + 2b_2 > 0\) and \(a_2\delta^2 - b_2 > 0\). In this case one periodic orbit borns at the equilibrium point when \(\varepsilon \to 0\).

We illustrate a case of Theorem 2 in Fig. 1. We prove our results in Section 3.

**Remark 1.** We prove Theorem 2 by applying averaging theory of order 2. Since we prove that the averaged function cannot be identically zero, it follows that with averaging of higher order we will not find more periodic solutions.

2. THE AVERAGING THEORY OF FIRST AND SECOND ORDER

In this section we summarize the main results on the theory of averaging which will be used in the proof of Theorem 2. For a proof of the following theorem and more information on averaging theory, we address the reader to \[11, 14\].

**Theorem 3.** Let \(D\) be an open subset of \(\mathbb{R}^n\), \(\varepsilon_f > 0\) and consider the differential system

\[
\dot{x}(t) = \varepsilon F_1(x, t) + \varepsilon^2 F_2(x, t) + \varepsilon^3 R(x, t, \varepsilon),
\]
Figure 1. Three periodic solutions emanating from the origin of coordinates. Here $\delta = 2, \ a_2 = 1, \ b_2 = 5, \ c_1 = c_2 = 0$ and $\varepsilon = 1/10$.

where $F_1, F_2 : D \times \mathbb{R} \to \mathbb{R}^n$, $R : D \times \mathbb{R} \times (\varepsilon f, \varepsilon f) \to \mathbb{R}^n$ are continuous functions, $T$-periodic in the second variable, $F_1(\cdot, t) \in C^1(D)$ for all $t \in \mathbb{R}$, $F_1, F_2, R$ and $D_x F_1$ are locally Lipschitz with respect to $x$, and $R$ is differentiable with respect to $\varepsilon$. Define $f, g : D \to \mathbb{R}^n$ as

\[(4) \quad f(z) = \frac{1}{T} \int_0^T F_1(z, s) ds, \]

\[(5) \quad g(z) = \frac{1}{T} \int_0^T \left[ D_x F_1(z, s) \cdot \int_0^s F_1(z, t) dt + F_2(z, s) \right] ds, \]

and assume that for an open and bounded subset $V \subset D$ and for each $\varepsilon \in (-\varepsilon f, \varepsilon f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that $f(a_\varepsilon) + \varepsilon g(a_\varepsilon) = 0$ and $d_B(f + \varepsilon g, V, a_\varepsilon) \neq 0$. Then for $|\varepsilon| > 0$ sufficiently small, there exists a $T$-periodic solution $\varphi(\cdot, \varepsilon)$ of system (3) such that $\varphi(0, \varepsilon) = a_\varepsilon$.

The expression $d_B(f + \varepsilon g, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $f + \varepsilon g : V \to \mathbb{R}^n$ at the fixed point $a_\varepsilon$ is not zero. We recall that a sufficient condition for this is that the Jacobian determinant of the function $f + \varepsilon g$ at $a_\varepsilon$ is not zero. For the definition, the mentioned and other properties of Brouwer degree we address the reader to Browder’s paper [2].

If $f$ is not identically zero, then the zeros of $f + \varepsilon g$ are mainly the zeros of $f$ for $\varepsilon$ sufficiently small. In this case the previous result provides the averaging theory of first order.

If $f$ is identically zero and $g$ is not identically zero, then clearly the zeros of $f + \varepsilon g$ are the zeros of $g$. In this case the previous result provides the averaging theory of second order.

3. Proofs

**Proof of Proposition** if $(x, 0, 0)$ with $x = 0$ or $x^2 = -b$, if $b \leq 0$, are the singular points of system [2].

The characteristic polynomial of the linear part of the system at $(x, 0, 0)$ is

\[ p(\lambda) = -\lambda^3 - a\lambda^2 + c\lambda - b - 3x^2. \]
In order to have a zero-Hopf equilibrium, we need one null and two purely imaginary, say $\pm i\delta$, with $\delta > 0$, eigenvalues. Imposing that 

$$p(\lambda) = -\lambda(\lambda^2 + \delta^2),$$

we obtain $a = b = 0$ and $c = -\delta^2$. In particular, the only zero-Hopf equilibrium is $(0, 0, 0)$ and we are done.

Proof of Theorem 2. With the parameters $(a, b, c) = (\varepsilon a_1 + \varepsilon^2 a_2, \varepsilon b_1 + \varepsilon^2 b_2, -\delta^2 + \varepsilon c_1 + \varepsilon^2 c_2)$, the 3-D jerk system (2) takes the form

$$\begin{align*}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= -\varepsilon (a_1 + \varepsilon a_2) z - \varepsilon (b_1 + \varepsilon b_2) x + (-\delta^2 + \varepsilon c_1 + \varepsilon^2 c_2)y + xy^2 - x^3;
\end{align*}$$

As in the proof of Proposition 1 when $\varepsilon = 0$, the eigenvalues at the origin of system (6) are 0 and $\pm i\delta$.

With the change of variables $(x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$ and rescaling of time, system (6) writes

$$\begin{align*}
\dot{X} &= Y \\
\dot{Y} &= Z \\
\dot{Z} &= -\delta^2 Y + \varepsilon (-a_1 Z - b_1 X + c_1 Y) + \varepsilon^2 (-a_2 Z - b_2 X + c_2 Y + X Y^2 - X^3).
\end{align*}$$

Now we make a linear change of variables in order to have the matrix of the linear part of system (7) at the origin when $\varepsilon = 0$ written in its real Jordan normal form

$$J = \begin{pmatrix} 0 & -\delta & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

It is simple to see that the change

$$\begin{align*}
X &= w + \frac{v}{\delta}, \\
Y &= u, \\
Z &= -\delta v
\end{align*}$$

makes what we want. In these new variables, system (9) writes

$$\begin{align*}
\dot{u} &= -\delta v \\
\dot{v} &= \delta u + \varepsilon \delta (h_1 + \varepsilon h_2) \\
\dot{w} &= -\varepsilon (h_1 + \varepsilon h_2),
\end{align*}$$

with

$$\begin{align*}
h_1 &= h_1(u, v, w) = \frac{b_1 v}{\delta^3} - \frac{c_1 u - b_1 w}{\delta^2} - \frac{a_1 v}{\delta}, \\
h_2 &= h_2(u, v, w) = \frac{v^3}{\delta^3} + \frac{3u v^2 w}{\delta^4} + \frac{(b_2 - u^2 + 3w^2)v}{\delta^3} - \frac{c_2 u + (u^2 - b_2)w - w^3}{\delta^2} - \frac{a_2 v}{\delta}.
\end{align*}$$
Now we pass the differential system (11) to cylindrical coordinates \((r, \theta, w)\) defined by \(u = r \cos \theta, v = r \sin \theta, w = w\), obtaining

\[
\begin{align*}
\dot{r} &= \varepsilon(1 + \varepsilon h_2) + \frac{\varepsilon h_1}{\varepsilon + \varepsilon h_2} r 
\end{align*}
\]

or

\[
\begin{align*}
\dot{r} &= \varepsilon(1 + \varepsilon h_2) + \frac{\varepsilon h_1}{\varepsilon + \varepsilon h_2} r 
\end{align*}
\]

with \(h_1 = h_1(r \cos \theta, r \sin \theta, w)\) and \(h_2 = h_2(r \cos \theta, r \sin \theta, w)\). By introducing \(\theta\) as the new independent variable, we obtain

\[
\begin{align*}
\frac{dr}{d\theta} &= \frac{\varepsilon(1 + \varepsilon h_2)}{\varepsilon + \varepsilon h_2} r \sin \theta 
\end{align*}
\]

and

\[
\begin{align*}
\frac{dw}{d\theta} &= -\frac{\varepsilon(1 + \varepsilon h_2)}{\varepsilon + \varepsilon h_2} r 
\end{align*}
\]

In the notation of Theorem 3 by taking \(t = \theta, T = 2\pi\) and \(z = (r, w)^t\), we have

\[
F_1(r, w, \theta) = h_1 \left( \sin \theta, -1/\delta \right), \quad F_2(r, w, \theta) = h_2 \left( \sin \theta, -1/\delta \right).
\]

We calculate \(f(r, w)\) obtaining

\[
\begin{align*}
f(r, w) = \frac{1}{2\pi \delta^3} \int_0^{2\pi} F_1(r, w, \theta) d\theta = \left( \frac{r}{2\delta^3} (b_1 - a_1 \delta^2), \ -b_1 w \delta \right).
\end{align*}
\]

The solutions of \(f(r, w) = 0\) with \(b_1 - a_1 \delta^2 \neq 0\) are contained in \(r = 0\), and so they are not allowed, because \(r\) must be positive. On the other hand, if \(b_1 - a_1 \delta^2 = 0\), the zeros of \(f(r, w)\) are not isolated. In particular we can not apply the averaging of first order. We pass then to the averaging of second order, assuming \(f \equiv 0\). This makes \(a_1 = b_1 = 0\). We now calculate \(g(r, w)\):

\[
\begin{align*}
g(r, w) &= \frac{1}{2\pi} \int_0^{2\pi} \left( D_{(r,w)} F_1(r, w, \theta) \int_0^\theta F_1(r, w, s) ds + F_2(r, w, \theta) \right) d\theta 
\end{align*}
\]

\[
\begin{align*}
&= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{r^2}{2\delta^3} \left( \delta \cos \theta \sin^3 \theta - \cos \theta \sin^2 \theta \right) + F_2(r, w, \theta) \right) d\theta 
\end{align*}
\]

\[
\begin{align*}
&= \frac{1}{2\delta^3} \left( r \left( (3 - \delta^2) r^2 + 4b_2 \delta^2 - 4a_2 \delta^4 + 12b_2 \delta^2 w^2 \right) / 4 + w \left( (3 - \delta^2) r^2 + 2b_2 \delta^2 + 2b_2 \delta^2 w^2 \right) \right).
\end{align*}
\]

We analyze the solutions of \(g(r, w) = 0\), with \(r > 0\). We first observe that in order to obtain isolated solutions according to Theorem 3, we ought to have \(\delta^2 \neq 3\). With this assumption in force, we readily obtain the following two group of solutions

\[
r^2 = \frac{4(a_2 \delta^2 - b_2) \delta^2}{3 - \delta^2}, \quad w = 0,
\]

or

\[
r^2 = \frac{-4(a_2 \delta^2 + 2b_2) \delta^2}{5(3 - \delta^2)}, \quad w^2 = \frac{2a_2 \delta^2 - b_2}{5}.
\]

"
The Jacobian determinant of $g$ at the solutions above take the values

$$-rac{(a_2\delta^2 - b_2)(2a_2\delta^2 - b_2)}{\delta^6}$$

and

$$-\frac{2(a_2\delta^2 + 2b_2)(2a_2\delta^2 - b_2)}{5\delta^6},$$

respectively.

If $\frac{a_2\delta^2 - b_2}{3 - \delta^2} > 0$, $\frac{a_2\delta^2 + 2b_2}{3 - \delta^2} < 0$ and $2a_2\delta^2 - b_2 \neq 0$, we have the isolated solutions $(r_1, w_1) = \left(2\delta \sqrt{\frac{a_2\delta^2 - b_2}{3 - \delta^2}}, 0\right)$, $(r_2, w_2) = \left(2\delta \sqrt{\frac{a_2\delta^2 + 2b_2}{5(3 - \delta^2)}}, \sqrt{\frac{2a_2\delta^2 - b_2}{5}}\right)$ and $(r_3, w_3) = (r_2, -w_2)$.

If $\frac{a_2\delta^2 - b_2}{3 - \delta^2} > 0$, $\frac{a_2\delta^2 + 2b_2}{3 - \delta^2} \geq 0$ and $2a_2\delta^2 - b_2 \neq 0$ then we have just the isolated solution $(r_1, w_1)$.

If $\frac{a_2\delta^2 - b_2}{3 - \delta^2} \leq 0$, $\frac{a_2\delta^2 + 2b_2}{3 - \delta^2} < 0$ and $2a_2\delta^2 - b_2 \neq 0$ we will have just the solutions $(r_2, w_2)$ and $(r_3, w_3)$.

Now Theorem 3 guarantees that, for $\varepsilon$ sufficiently small, to each root $(r_i, w_i)$ of $g$ it corresponds a periodic solution with period $2\pi$ of system (11) of the form $(r(\theta, \varepsilon), w(\theta, \varepsilon))$, with $(r(0, \varepsilon), w(0, \varepsilon)) = (r_1, w_1)$. Corresponding to this one, system (10) has the periodic solution of certain period $T_\varepsilon (\theta(t, \varepsilon), r(t, \varepsilon), w(t, \varepsilon))$ satisfying $(\theta(0, \varepsilon), r(0, \varepsilon), w(0, \varepsilon)) = (0, r_1, w_1)$. Then system (9) has the periodic solution of period $T_\varepsilon$

$$(u(t, \varepsilon), v(t, \varepsilon), w(t, \varepsilon)) = (r(t, \varepsilon) \cos \theta(t, \varepsilon), r(t, \varepsilon) \sin \theta(t, \varepsilon), w(t, \varepsilon)),$$

for $\varepsilon$ sufficiently small, with $(u(0, \varepsilon), v(0, \varepsilon), w(0, \varepsilon)) = (r_1, 0, w_1)$. Now we apply the change of variables (8) to this one and obtain for small $\varepsilon$ the periodic solution $(X(t, \varepsilon), Y(t, \varepsilon), Z(t, \varepsilon))$ of system (7) with the same period, such that $(X(0, \varepsilon), Y(0, \varepsilon), Z(0, \varepsilon)) = (w_1 + r_1/\delta, r_1, 0)$. Finally, for $\varepsilon \neq 0$ sufficiently small, system (6) has the periodic solution $(x(t, \varepsilon), y(t, \varepsilon), z(t, \varepsilon)) = (\varepsilon X(\theta), \varepsilon Y(\theta), \varepsilon Z(\theta))$, with $(x(0, \varepsilon), y(0, \varepsilon), z(0, \varepsilon)) = \varepsilon(w_1 + r_1/\delta, r_1, 0)$, and that clearly tends to the origin of coordinates when $\varepsilon \to 0$. Thus, this is a periodic solution emanating from the zero-Hopf bifurcation point located at the origin of coordinates when $\varepsilon = 0$. This concludes the proof of Theorem 2.

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