Exact interferometers for the concurrence and residual 3-tangle

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(Dated: September 4, 2006)

In this paper we describe a set of circuits that can measure the concurrence of a two qubit density matrix without requiring the deliberate addition of noise. We then extend these methods to obtain a circuit to measure one type of three qubit entanglement for pure states, namely the 3-tangle.

PACS numbers: 03.67.Mn, 03.67.-a, 03.65.-w
Keywords:

1. INTRODUCTION

Techniques for measuring entanglement without first reconstructing the state have recently attracted a lot of attention. Some of these approaches have been based on the Structural Physical Approximation (SPA) [10], followed by measuring the spectrum of the resulting density matrix [6, 12, 14] (but see also [2, 8]). One method for measuring that spectrum was inspired by [18], in which it was shown that while the physical evolution of the density matrix must be \( \rho \rightarrow U \rho U^\dagger \), the interference pattern is proportional to \( \text{Re}(ve^{-i\varphi}) = \text{Re}(\text{Tr}(U\rho)) \), where \( v \) is the visibility and \( \varphi \) is the phase shift. Thus the interferometer circuit in Figure 1 can modify the expectation value of the measured qubit, which corresponds to the path in a Mach-Zehnder interferometer. Note that \( U \) can be any unitary operation.

In [4] we showed how to measure the spectrum of \( \rho^T \) without using the SPA to shift the spectrum to be non-negative. The “Wootters spin-flip” used to define the concurrence is also an unphysical operation. The techniques in [4] cannot be used directly to measure this invariant, but a more general type of circuit can.

We can measure the 3-tangle [5] in a similar way, but some of the algebra used in the design of these circuits is only valid for pure states; the function conflates subsystem mixing with entanglement, and so these circuits will have the same restrictions on their validity.

2. A GENERALIZED INTERFEROMETER

We will begin by generalizing the circuit construction in [18]. This studied the behaviour of a Mach-Zehnder interferometer. By assumption, our particle has an internal degree of freedom, or spin. The circuit couples the

\[ |0\rangle \xrightarrow{H} \rho \xrightarrow{U} \rho \xrightarrow{H} \rho \]

FIG. 1: General interferometer circuit
internal and external degrees of freedom, by the operation
\[ U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U^{(i)} + \begin{pmatrix} e^{i\chi} & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}^{(i)}. \] (1)

When the control qubit is “on”, the internal degrees of freedom \( \rho_0 \) undergo the evolution \( \rho_0 \rightarrow U^{(i)} \rho_0 U^{(i)\dagger} \) where the label \( (i) \) signifies that only the internal degrees of freedom are affected. Note that \( U^{(i)} \) can be any unitary matrix that only acts on the internal degrees of freedom. A little algebra then gives us that the output intensity,
\[ I \propto 1 + |\text{Tr}(U^{(i)} \rho_0)| \cos[\chi - \arg \text{Tr}(U^{(i)} \rho_0)]. \] (2)

We will now modify this circuit to generalise (1) to a completely positive (CP) map. (While this question has been considered before in [7], this had a different motivation and was not concerned with measuring entanglement.) We will limit ourselves to convex combinations of unitary operators, and introduce a family of unitary transformations labelled by the index \( k \),
\[ U_k = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U^{(i)}_k + \begin{pmatrix} e^{i\chi} & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}^{(i)}. \] (3)

Note that these operators differ only in the unitary operator \( U^{(i)}_k \) acting on the internal degrees of freedom. If we weight these with probabilities \( p_k \), then the controlled evolution of the internal degree of freedom is now
\[ \rho_0 \rightarrow \sum_k p_k U^{(i)}_k \rho_0 U^{(i)\dagger}_k. \] (4)

When we redo the calculation in [15] with our generalized internal dynamics, we obtain the measured intensity along the \( |0\rangle\langle 0| \) arm of the interferometer, which will be proportional to
\[ 1 + \left| \text{Tr} \left( \sum_k p_k U^{(i)}_k \rho_0 \right) \right| \cos \left[ \chi - \arg \text{Tr} \left( \sum_k p_k U^{(i)}_k \rho_0 \right) \right]. \] (5)

We will now use this to measure the concurrence.

3. MEASURING THE CONCURRENCE

The concurrence [20] is a bipartite entanglement measure defined on mixed states of two qubits to be
\[ C_{AB} = \max \{ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0 \} \] (6)
where the \( \lambda_i \)s are, in decreasing order of magnitude, the positive square roots of the eigenvalues of \( \rho_{AB} \tilde{\rho}_{AB} \), where
\[ \tilde{\rho}_{AB} = \sigma_y \otimes \sigma_y \rho_{AB}^\dagger \sigma_y \otimes \sigma_y \] (7)
and where \( ^\dagger \) denotes matrix transposition in the computational basis. The method in [4] will not work, because we will obtain acausal circuits in which some of the input rails double back on each other. If we had wanted the spectrum of \( \tilde{\rho}_{AB} \) on its own, we could measure that in the same way as in [4]. However, when we try to pre-multiply by \( \rho_{AB} \), we find that we have to sum two pairs of input indices with two other input indices and likewise with two pairs of output indices. The SPA [11] would also have serious problems because of that matrix multiplication, as it will instead measure the spectrum of
\[ \rho_{AB} \Lambda(\rho_{AB}) = \mu \rho_{AB} + \nu \rho_{AB} \sigma_y \otimes \sigma_y \rho_{AB}^\dagger \sigma_y \otimes \sigma_y, \] (8)
where \( \mu >> \nu \) are the parameters defining the SPA for this map. In effect, the SPA circuits will measure the eigenspectrum of \( \rho_{AB} \), with a small correction term proportional to \( \rho_{AB} \tilde{\rho}_{AB} \). Without prior knowledge of \( \rho_{AB} \) there is no way to correct this error.

We will now describe two alternative methods for measuring the concurrence. Recall that
\[ \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = i\varepsilon, \quad \text{where} \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] (9)
Rewriting equation (7) in index notation and using the Einstein summation convention, we obtain
\[ \rho_{ij}^{kl} = (\varepsilon \otimes \varepsilon)^{rs}_{ij}(\rho^T)_{pq}^{st}(\varepsilon \otimes \varepsilon)_{pq}^{kl}, \]
where we have omitted the \( AB \) subscripts to reduce clutter. Converting the \( \varepsilon \) matrices to vectors, we can write
\[ \rho_{ij}^{kl} = \varepsilon_{ir}^{st} \varepsilon_{js}^{kl} \rho_{pq}^{rs}. \]
Now use the identity \( \varepsilon_{ab} \varepsilon^{cd} = \delta^d_a \delta^b_c - \delta^b_a \delta^d_c \) to obtain \[3\]
\[ \rho_{ij}^{kl} = \rho_{ij}^{kl} - \rho_{rj}^{kl} \delta^k_i - \delta^l_i \rho_{is}^{rs} + \delta^l_i \rho_{is}^{rs}. \]
In the more familiar matrix notation, this is
\[ \rho_{\tilde{A} \tilde{B}} = \rho_{AB} - \mathbb{I}_A \otimes \rho_B - \rho_A \otimes \mathbb{I}_B + \mathbb{I}_{AB}. \] (13)
We can now safely perform the matrix multiplication:
\[ \rho_{\tilde{A} \tilde{B}} \rho_{\tilde{A} \tilde{B}} = \rho_{\tilde{A} \tilde{B}} - \rho_{\tilde{A} \tilde{B}}(\mathbb{I}_A \otimes \rho_B - \rho_A \otimes \mathbb{I}_B) + \rho_{AB}. \] (14)

We will need to find a way to measure the various moments of \( \rho_{\tilde{A} \tilde{B}} \rho_{\tilde{A} \tilde{B}} \) to determine the eigenspectrum and thus obtain the \( \lambda_i \)s for \[6\]. There are two ways in which we can convert equation (14) into an interferometer. The first method uses the modified interferometry circuit introduced in the previous section directly, as it enables us to measure the effects of sums of unitary matrices, despite the fact that no physical evolution can be written as the sum of unitary matrices in this way. It should be noted that we will be needing the moments of \( \rho_{\tilde{A} \tilde{B}} \) rather than \( \rho_{\tilde{A} \tilde{B}} \) itself. So, the fact that some of the terms in \( \tilde{\rho} \) do not have trace 1 will not require us to rescale those terms, as \( \rho \mathbb{I} = \rho \), so the second set of rails can just be ignored, without needing to find a way to change the relative weights of the terms.

The permutations required for each term can be decomposed into two operations which are variously “on” or “off” for each term in (14). The set of probabilities also factorizes conveniently, so we will need only two auxiliary control ancillas. Figure 2 shows the terms of (14) as a summation diagram, which is a pictorial version of index notation. The nodes represent indices and the links represent summations. Unlinked nodes are free indices. The nodes on each copy of \( \rho \) are ordered as for standard tensor index notation. The summation pattern for the last term is defined to have both ancillas in the “off” position. The corresponding circuit diagram for this single term could be obtained by rotating the Figure clockwise by \(-90^\circ\), omitting the \( \rho \) symbols and inserting the resulting wiring pattern into the box marked “\( U \)” in Figure 1. In this case, all the rails go straight through and this sub-circuit would measure the trace norm if implemented on its own. Contrast this with the first term which will have both ancillas “on” and is proportional to \( \text{Tr}(\rho_{\tilde{A} \tilde{B}}^2) \). Note that the only difference between these two terms is a pair of swaps.

We must also take care to ensure that we reproduce the minus signs in equation (14), so the qubit we are going to measure will be able to “see” the minus signs from the ancillas. This can be done either by initializing the control ancillas with the appropriate relative phase, together with an initial controlled-\( \sigma_Z \) gate, or by including the desired phase in the gate itself \[1\]. We will use the first method and we will therefore need to initialize both auxiliary ancillas in the state \( |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \). The fact that the ancillas must be normalized means that each auxiliary ancilla will decrease the visibility of the interference fringes by a factor of 2. If we now insert these components into a modified version of Figure 1 so they are controlled by the two auxiliary ancillas, we will obtain the full circuit for \( \text{Tr}(\rho_{\tilde{A} \tilde{B}} \rho_{\tilde{A} \tilde{B}}) \) in Figure 3.

The same principles can be used to construct the circuit for \( \text{Tr}((\rho_{\tilde{A} \tilde{B}} \rho_{\tilde{A} \tilde{B}})^2) \). This has 16 terms and will require four auxiliary control ancillas, initialized in the same way. The structure of this circuit can be parsed as follows. The box in dotted lines with no auxiliary controls performs the permutation for the matrix multiplication of the two copies of \( \rho_{\tilde{A} \tilde{B}} \). The term with all the auxiliary control ancillas off will look like Figure 4. Figure 5 is the diagram for the term with all the ancilla-controlled operations “on”. Again, the difference between

\[ \rho \circ \bullet \rho \circ \bullet - 2 \rho \circ \bullet \rho \circ \bullet + 4 \rho \circ \bullet \rho \circ \bullet \]

FIG. 2: Summation diagram for \( \rho_{AB} \tilde{\rho}_{AB} \).
the two is just a set of simple swaps, which can be controlled by auxiliary ancillas as before. Thus we obtain the circuit for measuring $\frac{1}{16}\text{Tr}((\rho_1\bar{\rho})^2)$, in Figure 6. The circuits for $\frac{1}{64}\text{Tr}((\rho_2\bar{\rho})^3)$ and $\frac{1}{256}\text{Tr}((\rho_3\bar{\rho})^4)$ can be constructed in a similar way. We can now calculate the spectrum $\{\eta_i\}$ of $\rho_2\bar{\rho}_AB$ using the recipe given in [12, 13]. The positive square roots of the $\eta_i$s give the concurrence spectrum and equation (6) can then be used to compute the concurrence.

This first method shows that the concurrence (and hence the entanglement of formation) can be measured without any prior knowledge of the state. However, this method is not very sensitive, because the visibility drops by a factor of $\frac{1}{4}$ for the $m$th order concurrence moment. Fortunately, there is a second method for measuring the moments of the concurrence spectrum. This uses the principle of implicit measurement [16], whereby we can assume the auxiliary ancillas have been measured at the end of the circuit without affecting the statistics of the principal measurement. These circuits are examples of “non-erasing quantum erasers” [17], albeit very simple ones. In effect we will be measuring each term in the various summations separately. If we choose to measure the ancillas, we will be partitioning the output of the primary ancilla into subensembles with greatly enhanced visibility. We can also simplify the circuits by initializing the auxiliary ancillas using only a simple Hadamard gate, and omitting the controlled-$\sigma_Z$ gates altogether.

Alternatively, we can implement the circuits for each term separately, and insert the necessary weight factors by hand. This has the additional advantage that we can exploit the cyclicity of the trace and the fact that $\text{Tr}\rho_{AB} = 1$ and its first four moments completely define its spectrum, to reduce the number of circuits required from 340 to no more than 111. This does not compare well to full state tomography (which needs only 16 parameters) but this upper bound is unlikely to be tight, as this system can be characterised by 18 non-local parameters [15], of which only 9 are
fully algebraically independent; the other 9 are discrete valued.

4. MEASURING THE 3-TANGLE

The residual 3-tangle as defined in [5] is the modulus of a complex number. It is only defined in closed form for pure states of three qubits. If we define the 2-tangle $\tau_{AB}$ to be the square of the concurrence [5], then the residual 3-tangle is defined as entanglement between the qubits that cannot be accounted for in terms of bipartite entanglement:

$$\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{AC}. \tag{15}$$

The measure is symmetric under relabelling of the three parties [5]. We cannot use (15) directly, because we cannot define the equivalent of $\sigma_y$ on the merged party $(BC)$ without knowing the support of $\rho_{BC}$, which would require detailed knowledge of the state.

The fact that the two-party density matrices are all rank 2 means that $\lambda_3 = \lambda_4 = 0$. So we can write

$$\tau_{AC} = (\lambda_1 - \lambda_2)^2 = \lambda_1^2 - \lambda_2^2 - 2\lambda_1\lambda_2. \tag{16}$$
Wootters et al. then obtain that $\tau_{A(BC)} = 4 \det \rho_A$. This, together with the definition in (15) gives us that

$$\tau_{AB} + \tau_{AC} = 4 \det \rho_A - \tau_{ABC}. \tag{17}$$

Using the relabelling symmetry and the fact that $\det \rho_i = \frac{1}{2} (1 - \Tr(\rho_i^2))$ for $2 \times 2$ matrices, we can write [19]

$$\tau_{ABC} = 2(1 - \Tr(\rho_A^2) - \Tr(\rho_B^2) + \Tr(\rho_C^2) - \tau_{AB}). \tag{18}$$

We can calculate this from the rescaled expectation values measured by the circuits for $\tau_i$. These one-party circuits need two copies of their single particle $\rho$s each and the controlled-$U$ operation for these is the SWAP gate [12]. We can combine these three one-party circuits into a single three party circuit with three control qubits, thus saving four copies of $\rho_{123}$ per measurement. We can therefore measure the 3-tangle using this family of five circuits, provided the state is pure.

We can improve both the cost and sensitivity for the 3-tangle as follows. Recall that [5]

$$\Tr(\rho_{AB}\bar{\rho}_{AB}) + \Tr(\rho_{AC}\bar{\rho}_{AC}) = 4 \det \rho_A, \tag{19}$$

where the superscripts refer to which two-party density matrix was used to obtain the concurrence spectrum. We know that $\lambda_1^{(AB)}\lambda_2^{(AB)} = \lambda_1^{(BC)}\lambda_2^{(BC)} = \lambda_1^{(AC)}\lambda_2^{(AC)}$ from the relabelling symmetry, so we can omit the superscripts and write $\tau_{ABC} = 4\lambda_1\lambda_2$. The square root in the definition of the $\lambda$s prevents us from measuring this but we can construct a circuit for $|\tau_{ABC}|^2 = 16\lambda_1^2\lambda_2^2$. Now we just need to notice that

$$2\lambda_1^2\lambda_2^2 = (\lambda_1^2 + \lambda_2^2)^2 - \lambda_1^4 - \lambda_2^4, \tag{20}$$

where $\tau_{ABC}$ is given by

$$\tau_{ABC} = 2(1 - \Tr(\rho_A^2) - \Tr(\rho_B^2) + \Tr(\rho_C^2) - \tau_{AB}). \tag{18}$$

A circuit with visibility $\propto |\tau_{ABC}|^2$ can be obtained from Figure 6. We need an equal superposition of the sub-circuits (the faint dotted box) switched on and off so we will insert a new ancilla in the state $|{-}\rangle$. The intensity along the $|0\rangle\langle 0|$ arm for this naïve circuit will be [18]

$$I_{|\tau_{ABC}|^2} \propto 1 + |\tau_{ABC}|^2/32. \tag{21}$$

This circuit needs four copies of the state per run, compared with the first method which needs 22 copies. The sensitivity of this naïve circuit can also be improved by measuring the terms separately. This can in fact be done with just 14 circuits, compared with the 16 parameters that must be determined to learn the state of a three qubit state that is known to be pure [9].

5. DISCUSSION

We have presented circuits for both the concurrence and the residual 3-tangle. The circuits for the concurrence should work for any two qubit state. The visibility of these circuits is not much better than that for the SPA circuits, however, the former do not introduce state-dependent errors. The visibility can be improved by decomposing the original moment circuits into unitary circuits, but there will then be many more of them.

The two methods for finding the residual 3-tangle are, strictly speaking, only valid for pure states. In any laboratory setting the states will have a certain amount of mixing. The effect of this on the residual 3-tangle is incompletely understood so it is unclear how useful these circuits would be in practice, but at least these circuits do not aggravate the mixing beyond that caused by unavoidable experimental noise. They also require fewer circuits than the corresponding state tomography problem.

Acknowledgments

I would like to thank Todd Brun, Martin Rötteler and Michele Mosca for useful feedback on this manuscript, and Stephen Bullock and Dianne O’Leary for spotting an error in the circuit diagrams in an earlier version of this paper. This research was supported by MITACS, The Fields Institute, and the NSERC CRO project “Quantum Information and Algorithms.”

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