Numerical Solutions of Jump Diffusions with Markovian Switching

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Abstract

In this paper we consider the numerical solutions for a class of jump diffusions with Markovian switching. After briefly reviewing necessary notions, a new jump-adapted efficient algorithm based on the Euler scheme is constructed for approximating the exact solution. Under some general conditions, it is proved that the numerical solution through such scheme converge to the exact solution. Moreover, the order of the error between the numerical solution and the exact solution is also derived. Numerical experiments are carried out to show the computational efficiency of the approximation.

Keywords Jump diffusion, Markovian switching, Numerical solutions, Poisson measure

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1 Introduction

Stochastic hybrid systems arise in numerous applications of systems with frequent unpredictable structural changes, e.g. flexible manufacturing systems, air traffic management, electric power networks, mathematical finance and risk management etc. In typical hybrid system, the state space consists of two parts: one component takes values in Euclidean space while the other takes discrete values. For various applications of stochastic hybrid systems, we refer to [8, 13, 17, 18].

One important classes of stochastic hybrid systems is the following jump diffusion with Markovian switching:

\[
y(t) = y_0 + \int_0^t b(y(s), r(s)) ds + \int_0^t \sigma(y(s), r(s)) dW(s) + \int_0^t \int_{\Gamma} g(y(s^-), r(s), v) N(ds, dv),
\]

where \( r(t) \) is a continuous-time Markov chain represents the environment or modes. Such processes have been increasingly used in modelling of the stochastic systems which are affected by randomly occurring impulses as well as frequent regime switching usually modelled by Markov chain. A
typical application of (1.1) stems from risk theory in insurance and finance where $y(t)$ is regarded as a surplus process in Markov-modulated market. Applications have also been reported in a wide range of fields such as option pricing [3,19] and flexible manufacturing systems [1].

Since in many problems such Lévy stochastic differential equations with Markovian switching cannot be solved explicitly, it is important, from theoretical point of view, and even more for the sake of various applications, to find their approximate solutions in an explicit form or in a form suitable for application of some numerical methods. Although the numerical solutions of stochastic differential equations have been the focus of enormous research (see e.g. [5,9,14,15,16]), there has been significantly less work on the numerical methods for diffusion processes or jump diffusion processes with Markovian switching. In the literatures addressing these problems, Yuan and Mao [21] firstly considered the numerical solutions of the following stochastic differential equations with Markovian switching

$$
\begin{align*}
    dy(t) &= f(y(t), r(t))dt + g(y(t), r(t))dW(t), \\
    P\{r(t + \delta) = j | r(t) = i\} &= q_{ij}\delta + o(\delta) \quad i \neq j.
\end{align*}
$$

It was proved that the numerical solutions of (1.2) under Euler method converge to the exact solutions, and the order of error was also estimated. Krystul and Bagchi [12] extended (1.2) to a more general switching diffusion processes with state-dependent switching rates, and an approximation scheme for first passage time of the processes was derived. Recently, Yin, Song and Zhang [20] proposed a numerical algorithm for (1.1), and proved that the algorithm converges to the desired limit by means of a martingale problem formulation.

In this paper, we try to develop a jump-adapted numerical algorithm for (1.1), which is based on the Euler scheme. Different from [20], the algorithm we proposed does not rely on a constant-step-size scheme, instead, a series of jump-times driven by a Poisson random measure are taken into account. Another difference is that convergence properties in [20] was proved in weak sense while we prove it in strong sense. Meanwhile this paper focus on the order of convergence, which is enlightened by the methods in [21].

The rest of the paper is organized as follows. Section 2 briefly reviews the existence and uniqueness of equation (1.1) and describe the modified Euler algorithm of numerical solutions. Section 3 shows that the numerical solution under such algorithm converge to the exact solution in $L^2$, and the order of the error between the numerical and exact solution is derived. Section 4 gives two numerical examples to examine the performance of the algorithm described in Section 3.
2 Preliminary and algorithm

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions. Suppose that there is a finite set \(S = \{1, 2, \ldots, N\}\), representing the possible regimes of the environment. Let \(\Gamma\) be a compact subset of \(\mathbb{R}\) not including the origin, and \(b(\cdot, \cdot) : \mathbb{R}^d \times S \to \mathbb{R}^d, \sigma(\cdot, \cdot) : \mathbb{R}^d \times S \to \mathbb{R}^{d \times d}, g(\cdot, \cdot, \cdot) : \mathbb{R}^d \times S \times \Gamma \to \mathbb{R}^d\). Consider the following jump diffusion with Markovian switching of the form

\[
y(t) = y_0 + \int_0^t b(y(s), r(s))ds + \int_0^t \sigma(y(s), r(s))dW(s) + \int_0^t \int_{\Gamma} g(y(s^-), r(s), v)\tilde{N}(ds, dv),
\]

with initial value \(y(0) = y_0 \in \mathbb{R}^d\) and \(r(0) = i_0 \in S\), where \(W(t) = (W^1(t), \cdots, W^d(t))^T\) is a \(d\)-dimensional \(\mathcal{F}_t\)-adapted standard Brownian motion, \(N(dt, dv)\) is a \(\mathcal{F}_t\)-Poisson point process on \(\mathbb{R}^+ \times \Gamma\) with determinis-}
2.1 Existence and uniqueness

Under certain conditions, we can establish the existence of a pathwise unique solution of (2.1). Here we make the following global Lipschitz and linear growth assumptions:

(H1) For all \((i, x, y, v) \in S \times \mathbb{R}^d \times \mathbb{R}^d \times \Gamma\), there exists a constant \(L_1 > 0\) such that

\[
|b(x, i) - b(y, i)|^2 + |\sigma(x, i) - \sigma(y, i)|^2 + \int_{\Gamma} |g(x, i, v) - g(y, i, v)|^2 \Pi(dv) \leq L_1|x - y|^2.
\]

(H2) For all \((i, x, v) \in S \times \mathbb{R}^d \times \Gamma\), there exists a constant \(L_2 > 0\) such that

\[
\int_{\Gamma} |g(x, i, v)|^2 \Pi(dv) \leq L_2(1 + |x|^2).
\]

(H3) Let \(K\) be the support of \(g(\cdot, \cdot, \cdot)\) and \(U\) be the projection of \(K\) on \(\Gamma \subset \mathbb{R}\), then assume that \(\Pi(U) < \infty\).

We denote by \(|\cdot|\) the Euclidean norm for vectors or the trace norm for matrices.

Remarks 2.2. It’s easy to show that if \(b(\cdot, \cdot), \sigma(\cdot, \cdot)\) satisfy global Lipschitz condition, then they also satisfy linear growth condition, i.e. for all \((x, i) \in \mathbb{R}^d \times S\), there exists a constant \(L_3\) such that

\[
|b(x, i)|^2 + |\sigma(x, i)|^2 \leq L_3(1 + |x|^2).
\]

To show this, just let \(L_3 = 2L_1 \vee \max(2|b(0, i)|^2 + 2|\sigma(0, i)|^2 : i \in S)\). For convenience, we will set \(L = L_1 \vee L_2 \vee L_3\) in the following of this paper.

Theorem 2.1. If \(b(x, i), \sigma(x, i)\) and \(g(x, i, v)\) satisfy the conditions (H1), (H2), (H3), and suppose \(W(t), r(t), N(dt, dv)\) be independent. Then there exists a unique \(d\)-dimensional \(\mathcal{F}_t\)-adapted right-continuous process \(y(t)\) with left-hand limits which satisfies equation (2.1) and such that \(y(0) = y_0\) and \(r(0) = i_0\) a.s.

Proof. See [2, 7].

2.2 Algorithm

Now we turn our attention to numerical algorithm. Given \(\Delta > 0\) as a step size, denote \(\{t'_i\}_{i \geq 1}\) the usual equidistant time discretization of a bounded interval \([0, T]\), i.e. \(t'_0 = 0, t'_i - t'_{i-1} = \Delta,\) if \(t'_{n-1} < T \leq t'_n\) then set \(t'_n = T\). Suppose \(\tau'_j = \inf\{t \geq 0, N_t \geq j\}, j = 1, 2, \ldots\) are the jump times of \(N_t \triangleq N(t, \Gamma)\). Apparently, \(\tau_j = \tau'_j \wedge T, j = 1, 2, \ldots\) are also stopping times. Now we take a new time discretization \(\{t_k\}_{k \geq 1} = \{t'_i\}_{i \geq 1} \cup \{\tau_j\}_{j \geq 1}\)
satisfying $t_{k+1} - t_k \leq \Delta$ for all $k \geq 1$. This type of grid using the jump-times as additional grid points is usually referred to as a jump-adapted method, see for example [5].

Since $N(dt, dv)$ and $r(t)$ are independent, for given partition $\{t_k\}_{k \geq 1}$, $\{r(t_k), k = 1, 2, \cdots \}$ is a discrete Markov chain with transition probability matrix $(P(i, j))_{N \times N}$, here $P(i, j) = P(r(t_{k+1}) = j | r(t_k) = i)$ is the $ij$th entry of the matrix $e^{(t_{k+1} - t_k)Q}$, thus we could use following recursion procedure to simulate the discrete Markov chain $\{r(t_k)\}_{k \geq 1}$, suppose $r(t_k) = i_1$ and generate a random number $\xi$ which is uniformly distributed in $[0, 1]$, then we define

$$r(t_{k+1}) = \begin{cases} i_2, & \text{if } i_2 \in S - \{N\} \text{ and } \sum_{j=1}^{i_2-1} P(i_1, j) \leq \xi < \sum_{j=1}^{i_2} P(i_1, j), \\ N, & \text{if } \sum_{j=1}^{N-1} P(i_1, j) \leq \xi. \end{cases}$$

Repeating this procedure a trajectory of $\{r(t_k)\}_{k \geq 1}$ can be simulated.

Now we could define the Euler approximation solution of (2.1). Let

$$X_{t_0} = y_0, \quad r_{t_0} = r_0,$$

$$X_{t_{k+1}} = X_{t_k} + b(X_{t_k}, r_{t_k})(t_{k+1} - t_k) + \sigma(X_{t_k}, r_{t_k})(W(t_{k+1}) - W(t_k)) + \sum_{j=1}^{N_T} g(X_{t_k}, r_{t_k}, v_j)I_{\{t_{k+1} = \tau_j\}}. \tag{2.2}$$

Let

$$\bar{X}(t) = X_{t_k}, \quad \bar{r}(t) = r_{t_k} \quad (t_k \leq t < t_{k+1}).$$

Hence we can define the approximation solution $X(t)$ on the entire interval $[0, T]$ by

$$X(t) = y_0 + \int_0^t b(\bar{X}(s), \bar{r}(s))ds + \int_0^t \sigma(\bar{X}(s), \bar{r}(s))dW(s) + \int_0^t \int_{\Gamma} g(\bar{X}(s^-), \bar{r}(s), v)N(ds, dv). \tag{2.3}$$

Remarks 2.3. $\sum_{j=1}^{N_T} g(X_{t_k}, r_{t_k}, v_j)I_{\{t_{k+1} = \tau_j\}}$ only take nonzero values on stopping times $\{\tau_j\}_{j \geq 1}$, thus $X(t)$ is continuous on $[0, T] \setminus \{\tau_j\}_{j \geq 1}$. Moreover, we have $X(t_k) = \bar{X}(t_k) = X_{t_k}$.

3 Convergence with the Lipschitz and linear growth conditions

In this section, we will prove that the numerical solution $X(t)$ converges to the exact solution $y(t)$ in $L^2$ as step size $\Delta \downarrow 0$, and the order of convergence is one-half, i.e.

$$E(\sup_{0 \leq t \leq T} |X(t) - y(t)|^2) \leq C\Delta + o(\Delta). \tag{3.1}$$
To begin with, we need the following lemma.

**Lemma 3.1.** Under conditions \((H_1), (H_2)\) and \((H_3)\), there exists a constant \(M\) which is dependent on \(T, L, y_0\), but independent of \(\Delta\), such that

\[
E\left(\sup_{0 \leq t \leq T} |y(t)|^2\right) \vee E\left(\sup_{0 \leq t \leq T} |X(t)|^2\right) \leq M. \tag{3.2}
\]

**Proof.** From Hölder inequality, we have

\[
|y(t)|^2 = |y_0 + \int_0^t b(y(s), r(s))ds + \int_0^t \sigma(y(s), r(s))dW(s)
+ \int_0^t \int_{\Gamma} g(y(s^-), r(s), v)N(ds, dv)|^2
\leq 4|y_0|^2 + 4|\int_0^t b(y(s), r(s))ds|^2 + 4|\int_0^t \sigma(y(s), r(s))dW(s)|^2
+ 4|\int_0^t \int_{\Gamma} g(y(s^-), r(s), v)N(ds, dv)|^2
\leq 4|y_0|^2 + 4t|\int_0^t |b(y(s), r(s))|^2ds + 4|\int_0^t \sigma(y(s), r(s))dW(s)|^2
+ 4|\int_0^t \int_{\Gamma} g(y(s^-), r(s), v)N(ds, dv)|^2.
\]

Thus for any \(0 \leq T' \leq T\), we have

\[
\sup_{0 \leq t \leq T'} |y(t)|^2 \leq 4|y_0|^2 + 4T|\int_0^{T'} |b(y(s), r(s))|^2ds
+ 4\sup_{0 \leq t \leq T'} |\int_0^t \sigma(y(s), r(s))dW(s)|^2
+ 4\sup_{0 \leq t \leq T'} |\int_0^t \int_{\Gamma} g(y(s^-), r(s), v)N(ds, dv)|^2. \tag{3.3}
\]

From \((H_1)\) and Remark 2.2, we have

\[
E\int_0^{T'} |b(y(s), r(s))|^2ds \leq E\int_0^{T'} L(1 + |y(s)|)^2ds
\leq C_1 + C_1\int_0^{T'} E|y(s)|^2ds. \tag{3.4}
\]

Here \(C_1\) is a positive constant, in fact, \(C_1 = T'L\vee L\) in \((3.4)\). For convenience, in the following of this paper, we may frequently denote the related constants by \(C_k\). In that case, we only mean there exist such a positive constant rather than some specific one.
Since $\int_0^t \sigma(y(s), r(s))dW(s)$ is martingale (see [6]), therefore, from Doob martingale inequality and (H1), we have

$$E(\sup_{0 \leq t \leq T'} |\int_0^t \sigma(y(s), r(s))dW(s)|^2) \leq 4E|\int_0^{T'} \sigma(y(s), r(s))dW(s)|^2$$

$$= 4E\int_0^{T'} |\sigma(y(s), r(s))|^2 ds \leq 4E\int_0^{T'} L(1 + |y(s)|^2)ds$$

$$\leq C_2 + C_2\int_0^{T'} E|y(s)|^2 ds.$$ \hspace{1cm} (3.5)

For the last term of (3.3), notice that $\tilde{N}(dt, dv) = N(dt, dv) - \Pi(dv)dt$ is the compensated Poisson random measure, and $\int_0^t \int_\Gamma g(y(s^-), r(s), v)\tilde{N}(ds, dv)$ is martingale (see [10]), we have

$$E(\sup_{0 \leq t \leq T'} |\int_0^t \int_\Gamma g(y(s^-), r(s), v)\tilde{N}(ds, dv)|^2) \leq 4E|\int_0^{T'} \int_\Gamma g(y(s^-), r(s), v)\tilde{N}(ds, dv)|^2$$

$$= 4E\int_0^{T'} \int_\Gamma |g(y(s^-), r(s), v)|^2 \Pi(dv)ds.$$ \hspace{1cm} (3.6)

Hence, by Doob martingale inequality, we have

$$E(\sup_{0 \leq t \leq T'} |\int_0^t \int_\Gamma g(y(s^-), r(s), v)\tilde{N}(ds, dv)|^2) \leq 4E|\int_0^{T'} \int_\Gamma g(y(s^-), r(s), v)\tilde{N}(ds, dv)|^2$$

$$\leq 4E|\int_0^{T'} \int_\Gamma |g(y(s^-), r(s), v)|^2 \Pi(dv)ds.$$ \hspace{1cm} (3.7)
On the other hand, we can get
\[
E \left( \sup_{0 \leq t \leq T'} \left| \int_0^t \int_{\Gamma} g(y(s^-), r(s), v) \Pi(dv) ds \right|^2 \right) 
\leq E \left( \sup_{0 \leq t \leq T'} t \int_0^t \int_{\Gamma} |g(y(s^-), r(s), v)|^2 \Pi(dv) ds \right) \tag{3.8}
\leq T E \int_0^{T'} \int_{\Gamma} |g(y(s^-), r(s), v)|^2 \Pi(dv) ds.
\]

Combining (3.6), (3.7), (3.8) and (H2), we obtain that
\[
E \left( \sup_{0 \leq t \leq T'} \left| \int_0^t \int_{\Gamma} g(y(s^-), r(s), v) N(ds, dv) \right|^2 \right) 
\leq (8 + 2T)E \int_0^{T'} \int_{\Gamma} |g(y(s^-), r(s), v)|^2 \Pi(dv) ds \tag{3.9}
\leq (8 + 2T)E \int_0^{T'} L(1 + |y(s^-)|)^2 ds 
\leq C_3 + C_3 \int_0^{T'} E|y(s^-)|^2 ds.
\]

Take expectation of (3.3) and substitute (3.4), (3.5), (3.9) into it, we have
\[
E \left( \sup_{0 \leq t \leq T'} |y(t)|^2 \right) 
\leq 4|y_0|^2 + (C_1 + C_2) \int_0^{T'} E|y(t)|^2 dt + C_3 \int_0^{T'} E|y(t^-)|^2 dt \tag{3.10}
\leq C_4 + C_4 \int_0^{T'} E( \sup_{0 \leq s \leq t} |y(s)|^2) dt.
\]

From Gronwall inequality, we can show that
\[
E \left( \sup_{0 \leq t \leq T'} |y(t)|^2 \right) \leq C_4(1 + C_4 Te^{C_4 T}) = M_1.
\]

Similar to above procedure, we know that for \(X(t)\)
\[
E \left( \sup_{0 \leq t \leq T'} |X(t)|^2 \right) \leq 4|y_0|^2 + (C_1 + C_2) \int_0^{T'} E|\bar{X}(t)|^2 dt + C_3 \int_0^{T'} E|\bar{X}(t^-)|^2 dt.
\]

According to Remark 2.3
\[
\bar{X}(t) = \bar{X}(t^-) = X_{t_k} = X(t_k), \quad t_k < t < t_{k+1},
\]
\[
\bar{X}(t) = X_{t_k} = X(t_k), \quad \bar{X}(t^-) = X_{t_{k-1}} = X(t_{k-1}), \quad t = t_k.
\]

8
In both cases we have
\[ |\dot{X}(t)|^2 \vee |\dot{X}(t^-)|^2 \leq \sup_{0 \leq s \leq t} |X(s)|^2. \]

Thus by Gronwall inequality
\[ E(\sup_{0 \leq t \leq T} |X(t)|^2) \leq M_2. \]

Finally there exists a positive constant \( M \) such that
\[ E(\sup_{0 \leq t \leq T} |y(t)|^2) \vee E(\sup_{0 \leq t \leq T} |X(t)|^2) \leq M. \]

\[ \square \]

Now we come back to our main theorem, the proof below mainly refers to Yuan and Mao [21].

**Theorem 3.1.** Assume the jump diffusion with Markovian switching (2.1) defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) satisfying

\[ a) \ W(t), r(t), N(dt, dv) \text{ are independent,} \]
\[ b) \ b(\cdot, \cdot), \sigma(\cdot, \cdot), \ g(\cdot, \cdot, \cdot) \text{ satisfy conditions } (H1), (H2) \text{ and } (H3), \]

then the unique strong solution \( y(t) \) and numerical solution \( X(t) \) obtained in section 2.2 satisfying:

\[ E(\sup_{0 \leq t \leq T} |X(t) - y(t)|^2) \leq C\Delta + o(\Delta). \]

where \( C \) is a positive constant independent of \( \Delta \).

**Proof.** For every \( 0 \leq T' \leq T \), similar to Lemma 3.1 we can easily verify that
\[ E(\sup_{0 \leq t \leq T'} |X(t) - y(t)|^2) \]
\[ \leq 3TE \int_0^{T'} |b(\dot{X}(s), \bar{r}(s)) - b(y(s), r(s))|^2 ds \]
\[ + 12E \int_0^{T'} |\sigma(\dot{X}(s), \bar{r}(s)) - \sigma(y(s), r(s))|^2 ds \]
\[ + CE \int_0^{T'} |g(\dot{X}(s^-), \bar{r}(s), v) - g(y(s^-), r(s), v)|^2 \Pi(dv) ds \] \( (3.11) \)

9
We focus on the third part of the right side

\[
E \int_0^T \int_{\Gamma} |g(\bar{X}(s^-), \bar{r}(s), v) - g(y(s^-), r(s), v)|^2 \Pi(dv) ds
\]

\[
\leq 2E \int_0^T \int_{\Gamma} |g(\bar{X}(s^-), r(s), v) - g(y(s^-), r(s), v)|^2 \Pi(dv) ds \tag{3.12}
\]

\[
+ 2E \int_0^T \int_{\Gamma} |g(\bar{X}(s^-), \bar{r}(s), v) - g(\bar{X}(s^-), r(s), v)|^2 \Pi(dv) ds.
\]

By condition (H1), we have

\[
E \int_0^T \int_{\Gamma} |g(\bar{X}(s^-), r(s), v) - g(y(s^-), r(s), v)|^2 \Pi(dv) ds
\]

\[
\leq 2L^2 \int_0^T E|\bar{X}(s^-) - y(s^-)|^2 ds. \tag{3.13}
\]

By (H2), we can compute that

\[
E \int_0^T \int_{\Gamma} |g(\bar{X}(s^-), \bar{r}(s), v) - g(\bar{X}(s^-), r(s), v)|^2 \Pi(dv) ds
\]

\[
= E[E[\int_0^T \int_{\Gamma} |g(\bar{X}(s^-), \bar{r}(s), v) - g(\bar{X}(s^-), r(s), v)|^2 \Pi(dv) ds \mid N_T, \tau_1, \ldots, \tau_{N_T}]]
\]

\[
= E[\sum_{k \geq 0} E[\int_{t_k}^{t_{k+1}} \int_{\Gamma} |g(\bar{X}(s^-), \bar{r}(s), v) - g(\bar{X}(s^-), r(s), v)|^2
\]

\[
\times I_{\{r(s)\neq \bar{r}(t_k)\}} \Pi(dv) ds \mid N_T, \tau_1, \ldots, \tau_{N_T}]
\]

\[
\leq 2E[\sum_{k \geq 0} E[\int_{t_k}^{t_{k+1}} \int_{\Gamma} (|g(\bar{X}(t_k), r(t_k), v)|^2 + |g(\bar{X}(t_k), r(s), v)|^2)
\]

\[
\times I_{\{r(s)\neq \bar{r}(t_k)\}} \Pi(dv) ds \mid N_T, \tau_1, \ldots, \tau_{N_T}]
\]

\[
\leq CE[\sum_{k \geq 0} E[\int_{t_k}^{t_{k+1}} [1 + |\bar{X}(t_k)|^2] I_{\{r(s)\neq \bar{r}(t_k)\}} ds \mid N_T, \tau_1, \ldots, \tau_{N_T}]]
\]

Note that given \{N_T, \tau_1, \ldots, \tau_{N_T}\}, \(1 + |\bar{X}(t_k)|^2\) and \(I_{\{r(s)\neq \bar{r}(t_k)\}}\) are condition-
ally independent with respect to \( r(t) \), thus
\[
E \int_{t_k}^{t_{k+1}} [1 + |\bar{X}(t_k)|^2] I_{\{|(r(s)) \neq r(t_k)| \}} ds \\
= \int_{t_k}^{t_{k+1}} E[E[(1 + |\bar{X}(t_k)|^2) I_{\{|(r(s)) \neq r(t_k)| \}} | r(t_k)] ds \\
= \int_{t_k}^{t_{k+1}} E[E[(1 + |\bar{X}(t_k)|^2) | r(t_k)] E[I_{\{|(r(s)) \neq r(t_k)| \}} | r(t_k)] ds.
\]

On the other hand, by the Markov property, we get

\[ E[I_{\{|(r(s)) \neq r(t_k)| \}} | r(t_k)] = \sum_{i \in S} I_{\{|(r(t_k)) = i\}} P(r(s) \neq i | r(t_k) = i) \]
\[
= \sum_{i \in S} I_{\{|(r(t_k)) = i\}} (-q_{ii}(s - t_k) + o(s - t_k)) \\
\leq \sum_{i \in S} I_{\{|(r(t_k)) = i\}} (\max(-q_{ii}) \Delta + o(\Delta)) \]
\[
\leq C \Delta + o(\Delta).
\]

Substituting (3.16) into (3.15) gives

\[
E \int_{t_k}^{t_{k+1}} [1 + |\bar{X}(t_k)|^2] I_{\{|(r(s)) \neq r(t_k)| \}} ds \\
\leq \int_{t_k}^{t_{k+1}} E[E[(1 + |\bar{X}(t_k)|^2) | r(t_k)] (C \Delta + o(\Delta))] ds \\
\leq (C \Delta + o(\Delta)) \int_{t_k}^{t_{k+1}} E[E[(1 + |\bar{X}(t_k)|^2) | r(t_k)] ds \\
= (C \Delta + o(\Delta)) \int_{t_k}^{t_{k+1}} E[1 + |\bar{X}(t_k)|^2] ds.
\]

By Lemma 3.1 there exists a positive constant \( M \) such that
\[
E[1 + |\bar{X}(t_k)|^2] \leq 1 + E( \sup_{0 \leq t \leq T} |X(t)|^2) \leq M,
\]
then
\[
E \int_{t_k}^{t_{k+1}} [1 + |\bar{X}(t_k)|^2] I_{\{|(r(s)) \neq r(t_k)| \}} ds \leq (t_{k+1} - t_k)(C \Delta + o(\Delta)). \tag{3.17}
\]
Substitute (3.17) into (3.14)

\[
E \int_0^T \int_\Gamma |g(\bar{X}(s^-), \bar{r}(s), v) - g(\bar{X}(s^-), r(s), v)|^2 \Pi(dv)ds \\
\leq CE \sum_{k \geq 0} E[(t_{k+1} - t_k)(C\Delta + o(\Delta)) \mid N_T, \tau_1, ..., \tau_{N_T}] \\
= CE \sum_{k \geq 0} (t_{k+1} - t_k)(C\Delta + o(\Delta)) \\
\leq C\Delta + o(\Delta).
\]

(3.18)

Here we use the fact that \((t_{k+1} - t_k)(C\Delta + o(\Delta))\) are measurable with respect to the \(\sigma\)-algebra generated by \(\{N_T, \tau_1, ..., \tau_{N_T}\}\).

Now substitute (3.13), (3.18) into (3.12), we obtain that

\[
E \int_0^{T'} \int_\Gamma |g(\bar{X}(s^-), \bar{r}(s), v) - g(y(s^-), r(s), v)|^2 \Pi(dv)ds \\
\leq 2L^2 \int_0^{T'} E|\bar{X}(s^-) - y(s^-)|^2 ds + C\Delta + o(\Delta).
\]

(3.19)

Similar to (3.19), we have

\[
E \int_0^{T'} |b(\bar{X}(s), \bar{r}(s)) - b(y(s), r(s))|^2 ds \\
\leq 2L^2 \int_0^{T'} E|\bar{X}(s) - y(s)|^2 ds + C\Delta + o(\Delta),
\]

(3.20)

\[
E \int_0^{T'} |\sigma(\bar{X}(s), \bar{r}(s)) - \sigma(y(s), r(s))|^2 ds \\
\leq 2L^2 \int_0^{T'} E|\bar{X}(s) - y(s)|^2 ds + C\Delta + o(\Delta).
\]

(3.21)

Substituting (3.19), (3.20), (3.21) into (3.11) shows that

\[
E(\sup_{0 \leq t \leq T'} |X(t) - y(t)|^2) \\
\leq C \int_0^{T'} E|\bar{X}(s) - y(s)|^2 ds \\
+ C \int_0^{T'} E|\bar{X}(s^-) - y(s^-)|^2 ds + C\Delta + o(\Delta).
\]

(3.22)

Note that

\[
E|\bar{X}(s) - y(s)|^2 \leq 2E|X(s) - y(s)|^2 + 2E|\bar{X}(s) - X(s)|^2,
\]

(3.23)

\[
E|\bar{X}(s^-) - y(s^-)|^2 \leq 2E|X(s^-) - y(s^-)|^2 + 2E|\bar{X}(s^-) - X(s^-)|^2.
\]

(3.24)
Suppose $t_k \leq s < t_{k+1}$, then

$$|X(s) - X(s)|^2 = |X(s) - X(t_k)|^2$$

$$= |\int_{t_k}^s b(\bar{X}(u), \bar{r}(u))du + \int_{t_k}^s \sigma(\bar{X}(u), \bar{r}(u))dW(u)$$

$$+ \int_{t_k}^s \int_{\Gamma} g(\bar{X}(u^-), \bar{r}(u), v)N(du, dv)|^2$$

$$= |b(X_{t_k}, r_{t_k})(s - t_k) + \sigma(X_{t_k}, r_{t_k})(W(s) - W(t_k))$$

$$+ \int_{t_k}^s \int_{\Gamma} g(X_{t_k}, r_{t_k}, v)N(du, dv)|^2$$

$$\leq 3|b(X_{t_k}, r_{t_k})|^2(s - t_k)^2 + 3|\sigma(X_{t_k}, r_{t_k})|^2(W(s) - W(t_k))^2$$

$$+ 3\int_{t_k}^s \int_{\Gamma} g(X_{t_k}, r_{t_k}, v)N(du, dv)|^2.$$

Since for given $\{t_k\}_{k \geq 1}$, $W(s) - W(t_k)$ and $(X(t_k), r_{t_k})$ are independent, we have

$$E|X(s) - X(s)|^2$$

$$\leq 3E[|b(X_{t_k}, r_{t_k})|^2(s - t_k)^2 + |\sigma(X_{t_k}, r_{t_k})|^2(W(s) - W(t_k))^2$$

$$+ |\int_{t_k}^s \int_{\Gamma} g(X_{t_k}, r_{t_k}, v)N(du, dv)|^2]$$

$$= 3E[|b(X_{t_k}, r_{t_k})|^2(s - t_k)^2 + |\sigma(X_{t_k}, r_{t_k})|^2(W(s) - W(t_k))^2$$

$$+ |\int_{t_k}^s \int_{\Gamma} g(X_{t_k}, r_{t_k}, v)N(du, dv)|^2 | t_k]]$$

$$\leq 3E[|b(X_{t_k}, r_{t_k})|^2\Delta^2 | t_k] + E[|\sigma(X_{t_k}, r_{t_k})|^2(W(s) - W(t_k))^2 | t_k]$$

$$+ E[|\int_{t_k}^s \int_{\Gamma} g(X_{t_k}, r_{t_k}, v)N(du, dv)|^2 | t_k]]$$

$$= 3\Delta^2 E[|b(X_{t_k}, r_{t_k})|^2] + 3E[|\sigma(X_{t_k}, r_{t_k})|^2 | t_k]E[(W(s) - W(t_k))^2 | t_k]$$

$$+ 3E[|\int_{t_k}^s \int_{\Gamma} g(X_{t_k}, r_{t_k}, v)|^2\Pi(du)dv | t_k]]$$

$$= 3\Delta^2 E[|b(X_{t_k}, r_{t_k})|^2] + 3E[|\sigma(X_{t_k}, r_{t_k})|^2 | t_k](s - t_k)].$$
\[ + 3E[E\int_{\Gamma} |g(X_{t_k}, r_{t_k}, v)|^2 \Pi(dv) | t_k](s - t_k)] \]
\[ \leq 3E[|b(X_{t_k}, r_{t_k})|^2] + 3E[E[|\sigma(X_{t_k}, r_{t_k})|^2 | t_k]\Delta] \]
\[ + 3E[E\int_{\Gamma} |g(X_{t_k}, r_{t_k}, v)|^2 \Pi(dv) | t_k]\Delta \]
\[ = 3\Delta^2 E[|b(X_{t_k}, r_{t_k})|^2] + 3\Delta E[|\sigma(X_{t_k}, r_{t_k})|^2] \]
\[ + 3\Delta E\int_{\Gamma} |g(X_{t_k}, r_{t_k}, v)|^2 \Pi(dv)]. \]

Hence by (H1) and Lemma 3.1, we get
\[ E|\bar{X}(s) - X(s)|^2 \leq C\Delta + o(\Delta). \] (3.25)

Similarly, we can show that
\[ E|\bar{X}(s^-) - X(s^-)|^2 \leq C\Delta + o(\Delta). \] (3.26)

Substituting (3.23), (3.24), (3.25), (3.26) into (3.22) immediately shows that
\[ E(\sup_{0 \leq t \leq T^\prime} |X(t) - y(t)|^2) \]
\[ \leq C \int_0^{T^\prime} (E|\bar{X}(s) - y(s)|^2 + E|\bar{X}(s^-) - y(s^-)|^2)ds + C\Delta + o(\Delta) \]
\[ \leq C \int_0^{T^\prime} E(\sup_{0 \leq s \leq t} |X(s) - y(s)|^2)dt + C\Delta + o(\Delta). \]

Therefore, from Gronwall inequality we obtain that
\[ E(\sup_{0 \leq t \leq T} |X(t) - y(t)|^2) \leq C\Delta + o(\Delta). \]

The proof is complete. \(\square\)

4 Numerical examples

In this section, we discuss two numerical examples to illustrate our theory established in the previous sections. It is shown that the computer simulation based on our numerical method is feasible and efficient.
Example 4.1. Consider the following one-dimensional geometric Lévy processes

\[ y(t) = y_0 + \int_0^t y(s)\mu(r(s))\,ds + \int_0^t y(s)\sigma(r(s))\,dW(s) + \int_0^t y(s^-)g(r(s))\,dN(s), \tag{4.1} \]

with initial value \( y(0) = y_0, \) \( r(0) = \bar{r}_0, \) where \( W(t) \) is a scalar Brownian motion, \( N(t) \) is a Poisson process with intensity \( \lambda > 0 \) and \( r(t) \) is a right-continuous Markov chain taking values in \( S \), suppose that \( r(t), W(t), N(t) \) are independent. It is obvious that the assumptions \( (H1), (H2), (H3) \) are satisfied for equation \((4.1)\). It is well known that equation \((4.1)\) has an explicit solution

\[ y(t) = y_0 \exp\left[ \int_0^t \mu(r(s))\,ds + \int_0^t \sigma(r(s))\,dW(s) - \frac{1}{2} \int_0^t \sigma^2(r(s))\,ds \right] \times \prod_{n=1}^{N(t)} (1 + g(r(\tau_n))), \tag{4.2} \]

where \( \tau_n \) are jump times determined by Poisson process \( N(t) \).

As described in Section 2, we get the numerical solution of \((4.1)\)

\[ X(t) = y_0 + \int_0^t \bar{X}(s)\mu(\bar{r}(s))\,ds + \int_0^t \bar{X}(s)\sigma(\bar{r}(s))\,dW(s) + \int_0^t \bar{X}(s)g(\bar{r}(s))\,dN(s). \tag{4.3} \]

By virtue of Theorem 3.1, it follows that the solutions \( y(t) \) and \( X(t) \) are close in the \( L^2 \)-norm, with the rate less than \( C\Delta \) when \( \Delta \downarrow 0 \). To examine this convergence result, we firstly simulate the trajectory of \( y(t) \) and \( X(t) \) for sufficient small stepsize \( \Delta > 0 \), then use simulated value \( \hat{E}\{ \sup_{0 \leq t \leq T} |X(t_k) - y(t_k)|^2 \} \) at new division points \( t_k \) to estimate \( E\{ \sup_{0 \leq t \leq T} |X(t) - y(t)|^2 \} \), where \( \hat{E} \) denotes sample mean.

For simulation reason, it is convenient to transform \((4.2)\) into following recursion form with \( y(t_0) = y_0 \),

\[ y(t_{k+1}) = y(t_k)\exp[(t_{k+1} - t_k)\mu(r_k) + (W(t_{k+1}) - W(t_k))\sigma(r_k)] - \frac{1}{2}(t_{k+1} - t_k)\sigma^2(r_k) \times \sum_{j=1}^{N(T)} (1 + g(r_{t_{k+1}}))I(t_{k+1} = \tau_j). \tag{4.4} \]

Notice that \( y(t_{k+1}) \) in \((4.4)\) is not the exact value of \( y(t) \) at the division points \( t_{k+1} \), because \( r(s) \) is not necessarily constant on \([t_k, t_{k+1}]\). However,
since

\[ P\{r(t_{k+1}) = i | r(t_k) = i\} = 1 + q_{ii}(t_{k+1} - t_k) + o(t_{k+1} - t_k) \to 1, \]

as \( \Delta \downarrow 0 \), for sufficiently small \( \Delta \), it is reasonable to use (4.4) as an approximation of the exact solution of \( y(t) \).

For the discrete approximate solution (4.3) at the division points \( t_{k+1} \), we have

\[
X_{t_{k+1}} = X_{t_k} + X_{t_k} \mu(r_{t_k})(t_{k+1} - t_k) + X_{t_k} \sigma(r_{t_k})(W(t_{k+1}) - W(t_k)) \\
+ \sum_{j=1}^{N(T)} X_{t_k} g(r_{t_k}) I_{\{t_{k+1} = \tau_j\}}. \tag{4.5}
\]

Now we take specific data to computer the above example. Let \( r(t) \) be a right-continuous Markov chain taking values in \( S = \{1, 2\} \) with generator

\[
Q = \begin{bmatrix}
-0.5 & 0.5 \\
0.5 & -0.5
\end{bmatrix},
\]

therefore, the one step transition probability from \( r(t_k) \) to \( r(t_{k+1}) \) is \( e^{(t_{k+1} - t_k)Q} \).

Let \( \mu(r(t)), \sigma(r(t)), g(r(t)) \) take values as follows

\[
\mu(1) = 0.15, \quad \sigma(1) = 0.1, \quad g(1) = -0.2;
\]

\[
\mu(2) = 0.05, \quad \sigma(2) = 0.1, \quad g(2) = -0.1.
\]

Let the Poisson intensity \( \lambda = 1 \), and choose initial values \( y_0 = 10, r_0 = 1 \), \( T \) is fixed at 10. By applying the previously described procedure, in Figure 1 the trajectory of the approximate solution \( X(t) \) with \( \Delta = 0.01 \) is constructed.

![Figure 1. The trajectory of the approximate solution when \( \Delta = 0.01 \)](image)

16
To carry out the numerical simulation we successively choose the stepsize \( \Delta \) as the following Table 1, and for each \( \Delta \), we repeatedly simulate and compute \( \sup_{t_k \in [0, 10]} |X(t_k) - y(t_k)|^2 \) for 1000 times, then calculate the sample mean \( \hat{E}(\sup_{t_k \in [0, 10]} |X(t_k) - y(t_k)|^2) \). The results are listed in the following Table 1.

| \( \Delta \) | \( \hat{E}(\sup_{t_k \in [0, 10]} |X(t_k) - y(t_k)|^2) \) |
|----------|------------------|
| 0.001    | 0.000179654      |
| 0.005    | 0.000908019      |
| 0.01     | 0.001793076      |
| 0.02     | 0.003626408      |
| 0.03     | 0.005427134      |
| 0.05     | 0.009155912      |
| 0.08     | 0.014817250      |
| 0.1      | 0.018204170      |

**Table 1.** Estimation of the error between the numerical and exact solutions

Clearly, the numerical method reveals that the numerical solution \( X(t) \) converges to the exact solution \( y(t) \) in \( L^2 \) as step size \( \Delta \downarrow 0 \), and the order of convergence is one-half, i.e.,

\[
E(\sup_{0 \leq t \leq T} |X(t) - y(t)|^2) \leq C \Delta + o(\Delta),
\]

which strongly demonstrate our result.

**Example 4.2.** In this example, as an application, we turn our attention to the expected ruin time for a surplus process with regime switching. Denote by \( r(t) \) the external environment process, which influences the frequency of claims and the distribution of claims. Suppose that the process \( r(t) \) is a homogeneous, irreducible and recurrent Markov process with finite state space \( S = \{1, 2, \cdots, N\} \) with generator \( Q = (q_{ij})_{N \times N} \) and stationary distribution \( \pi = (\pi_1, \pi_2, \cdots, \pi_N) \). \( N(t) \) is a Markov-modulated Poisson process which accounts the arrival of claims, i.e. at time \( t \), claim arrival occur accord to a Poisson process with constant intensity \( \lambda_i \) when \( r(t) = i \), and the corresponding claim size \( U_j \) have distribution \( F_i(x) \) with means \( \mu_i \). The surplus process is given by

\[
Y(t) = u + t - \sum_{n=1}^{N(t)} U_n, \tag{4.6}
\]

where the aggregate premium received with rate 1 during interval \((0, t]\) and \( u \) is the initial reserve, the i.i.d. random variables \( U_n \) and \( N(t) \) are conditionally independent given \( r(t) \).
Define \( T^* = \inf \{ t > 0 \mid Y(t) < 0 \} \) to be the time of ruin of (4.0), and define the expected ruin time, given that the initial environment state is \( i \) and the initial reserve is \( u \), by \( \xi_i(u) = E[T^* \mid r(0) = i, Y(0) = u]. \) Let \( \eta = \sum_{i=1}^{N} \pi_i \lambda_i \mu_i \) and \( \rho = \eta^{-1} - 1. \) It’s known that \( \rho < 0 \) ensures \( \xi_i(u) < \infty. \)

Now we consider the special case of two environmental states. Let

\[
Q = \begin{pmatrix}
-q_1 & q_1 \\
q_2 & -q_2
\end{pmatrix},
\]

then the stationary distribution of \( r(t) \) is \( \pi_1 = \frac{q_2}{q_1 + q_2}, \pi_2 = \frac{q_1}{q_1 + q_2}. \) Suppose that claim sizes \( U_1, U_2, \ldots \) are independent and exponentially distributed with mean \( \mu, \) which are independent of the environment process for convenience. Thus according to [4], we have

**Proposition 4.1.** If \( \rho < 0, \) then it holds

\[
\xi_1(u) = A_1 + \frac{u}{\eta - 1} + B e^{ku},
\]

\[
\xi_2(u) = A_2 + \frac{u}{\eta - 1} + B D(k) e^{ku},
\]

where \( k \) is the unique negative root of the following equation

\[
P(k) = k^3 + k^2 (\rho_1 + \rho_2 - q_1 - q_2) + k (\rho_1 \rho_2 - \rho_1 q_2 - \rho_2 q_1 - \frac{q_1}{\mu} - \frac{q_2}{\mu}) - \frac{\rho_1 q_2}{\mu} - \frac{\rho_2 q_1}{\mu} = 0,
\]

with \( \rho_i = \frac{1}{\mu} - \lambda_i, \ i = 1, 2, \) and \( (A_1, B, A_2) \) is the unique solution of the following linear equation

\[
\begin{pmatrix}
q_1 & 0 & -q_1 \\
1 & 1/(k \mu + 1) & 0 \\
0 & D(k)/(k \mu + 1) & 1
\end{pmatrix}
\begin{pmatrix}
A_1 \\
B \\
A_2
\end{pmatrix} = \frac{1}{\eta - 1}
\begin{pmatrix}
\eta - \lambda_1 \mu \\
\mu \\
\mu
\end{pmatrix},
\]

where

\[
D(k) = \frac{q_1 + k (\mu q_1 + \mu \lambda_1 - 1) - k^2 \mu}{q_1 + k \mu q_1}.
\]
Let
\[ Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (4.11) \]
and \( \lambda_1 = 1, \lambda_2 = 2, \mu = 1 \), then it is easy to obtain that \( \pi = \left( \frac{1}{2}, \frac{1}{2} \right), \eta = \frac{3}{2} \).
\( \rho = -\frac{1}{3} < 0, \rho_1 = 0, \rho_2 = -1 \). Solving equations (4.8) and (4.9) respectively, we get \( k = -0.6751309, A_1 = 2.712742, B = 1.481194 \). Substituting into (4.7) gives the exact value of \( \xi_1(u) \)
\[ \xi_1(u) = 2.712742 + 2u + 1.481194e^{-0.6751309u}. \quad (4.12) \]

For comparing the exact value (4.12) with the simulated value of \( \xi_1(u) \), we adopt the numerical algorithm in Section 2. After simulating the trajectory of \( Y(t) \), which is also denoted by \( X(t) \), it is easy to explore the trajectory behavior on \([0, T]\). To avoid unnecessary error, we choose a sufficient large interval \([0, T]\) and record the ruin time \( T^* \) of \( X(t) \) for every trajectory. Set \( T^* = T \) if \( T^* > T \). In the following, let the stepsize \( \Delta = 0.01 \) and \( T = 100 \), for each initial reserve \( u = 5, 8, 10, 15, 20, \) we repeatedly simulate and compute \( T^* \) for 1000 times respectively, then calculate the sample mean \( \bar{\xi}_1(u) \). The results are listed in the following Table 2.

| \( u \) | \( \xi_1(u) \) | \( \bar{\xi}_1(u) \) |
|-----|-------|-------|
| 5   | 12.76339 | 12.4102 |
| 8   | 18.71942 | 18.7526 |
| 10  | 22.71447 | 23.1208 |
| 15  | 32.71280 | 32.5327 |
| 20  | 42.71274 | 41.9659 |

Table 2. Exact and simulated value of the expected ruin time for initial reserve \( u \)

Clearly, the simulated value \( \bar{\xi}_1(u) \) is considerably close to \( \xi_1(u) \), which again demonstrate the efficiency of our numerical method.

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