Abstract. We modify Grayson’s model of $K_1$ of an exact category to give a presentation whose generators are binary acyclic complexes of length at most $k$ for any given $k \geq 2$. As a corollary, we obtain another, very short proof of the identification of Nenashev’s and Grayson’s presentations.

1. Introduction

Let $\mathcal{N}$ be an exact category. Algebraic descriptions of Quillen’s $K_1$-group of $\mathcal{N}$ in terms of explicit generators and relations have been given by Nenashev [Nen98] and Grayson [Gra12]. The generators in both descriptions are so-called binary acyclic complexes in $\mathcal{N}$. While Nenashev uses complexes of length at most 2, Grayson’s generators are of arbitrary (finite) length. An algebraic proof of the fact that these two descriptions agree has been given in [KW]. In this paper, we will give another presentation of $K_1(\mathcal{N})$: this time the generators are binary acyclic complexes of length at most $k$ for any $k \geq 2$, see Definition 2.2 and Theorem 2.4. A motivating question behind this new description is to determine the precise relations that in addition to Grayson’s relations need to be divided out when restricting the generators in Grayson’s description to complexes of length at most $k$. If $k = 2$, our relations are special cases of Nenashev’s relations. In this sense, our presentation simplifies Nenashev’s presentation. All this leads to a new, natural and sleek algebraic proof of the fact that Nenashev’s and Grayson’s descriptions agree, see Section 4.

The proof of our main result, see Section 3, basically proceeds by induction on $k$. The crucial ingredient in the inductive step is a shortening procedure for binary acyclic complexes, see Definition 3.4, as discovered by Grayson and also used in [KW]. The main new idea in this paper is the comparatively short and simple way of showing how this shortening procedure yields an inverse to passing from complexes of length $k$ to complexes of length $k + 1$, see Proposition 3.7.

Acknowledgements. C. Winges acknowledges support by the Max Planck Society and Wolfgang Lück’s ERC Advanced Grant “KL2MG-interactions” (no. 662400). D. Kasprowski and C. Winges are members of the Hausdorff Center for Mathematics at the University of Bonn.

2. Background and Statement of Main Theorem

We recall a binary acyclic complex $\mathcal{P} = (P_*, d, d')$ in $\mathcal{N}$ is a graded object $P_*$ in $\mathcal{N}$ supported on a finite subset of $[0, \infty)$ together with two degree $-1$ maps $d, d' : P_* \to P_*$ such that both $(P, d)$ and $(P_*, d')$ are acyclic chain complexes in $\mathcal{N}$.
Here, *acyclic* means that each differential $d_n : P_n \to P_{n-1}$ admits a factorisation into an admissible epimorphism followed by an admissible monomorphism

$$P_n \twoheadrightarrow J_n \rightarrowtail P_{n-1}$$

such that $J_n \rightarrowtail P_n \rightarrowtail J_{n-1}$ is a short exact sequence in $N$ for every $n$. The differentials $d$ and $d'$ are called the *top* and *bottom* differential. We also write $P^\top$ and $P^\perp$ for the complexes $(P_\ast, d)$ and $(P_\ast, d')$. If $d = d'$, we call $P$ a *diagonal binary acyclic complex*.

A *morphism between binary acyclic complexes* $P$ and $Q$ is a degree 0 map between the underlying graded objects which is a chain map with respect to both differentials. According to [Gra12, Section 3], the obvious definition of short exact sequences turns the category of binary acyclic complexes into an exact category. We denote its Grothendieck group by $B_1(N)$.

**Definition 2.1.** The quotient of $B_1(N)$ obtained by declaring the classes of diagonal binary acyclic complexes to be zero is called *Grayson's $K_1$-group of $N$* and denoted by $K_1(N)$.

Grayson proves in [Gra12, Corollary 7.2] that $K_1(N)$ is naturally isomorphic to Quillen’s $K_1$-group of $N$. This justifies our notation. Note that Grayson uses complexes supported on $[-\infty, \infty]$. By [HKT17, Proposition 1.4], defining $K_1(N)$ with complexes supported on $[0, \infty]$ as above yields the same $K_1$-group.

If we restrict the generators to be complexes supported on $[0, k]$ (for $k \geq 0$), we write $B_{k1}(N)$ and $K_{k1}(N)$ for the resulting abelian groups.

**Definition 2.2.**

(a) A *binary ladder* is a quadruple $(P, Q, \sigma, \tau)$ consisting of two binary complexes $P$ and $Q$ together with isomorphisms $\sigma : P^\top \cong Q^\top$ and $\tau : P^\perp \cong Q^\perp$.

(b) For any object $P$ in $N$ let

$$\tau_P := \begin{pmatrix} 0 & \text{id} \\ \text{id} & 0 \end{pmatrix} : P \oplus P \cong P \oplus P$$

denote the automorphism of $P \oplus P$ which switches the two summands.

(c) Any two isomorphisms $\alpha, \beta : P \cong Q$ in $N$ define a binary acyclic complex

$$P \xrightarrow{\alpha} Q \xleftarrow{\beta} Q$$

in $N$ supported on $[0, 1]$. The corresponding element in any of the groups $B_1(N)$, $K_1(N)$ or $B_{k1}(N)$, $K_{k1}(N)$ for $k \geq 1$ is denoted by $\langle \alpha, \beta \rangle$.

(d) Define $L_{k1}(N)$ to be the quotient of $K_{k1}(N)$ obtained by additionally imposing the relation

$$P - Q = \sum_{i=0}^{k} (-1)^i \langle \sigma_i, \tau_i \rangle$$

for every binary ladder $(P, Q, \sigma, \tau)$ in $N$ such that $P$ and $Q$ are supported on $[0, k]$, $P^\top = Q^\top$, $\sigma = \text{id}$ and each $\tau_i$ is the direct sum of some automorphism $\tau_P$ with the identity on some object $\tilde{P}$.

**Remark 2.3.** The object $L_{k1}(N)$ has of course nothing to do with $L$-Theory; the $L$ here is rather meant to refer to ‘ladder’.
In our definition of $L^k_1(\mathcal{N})$, we use the smallest class of binary ladders such that the proofs below work. The proofs below show that choosing any other class of binary ladders of length $k$ containing this smallest class doesn’t change $L^k_1(\mathcal{N})$.

The following theorem is the precise formulation of our main result. Note that Lemma 3.2 below implies that we have a natural map $L^k_1(\mathcal{N}) \to K_1(\mathcal{N})$.

**Theorem 2.4.** The canonical map

$$L^k_1(\mathcal{N}) \to K_1(\mathcal{N})$$

is an isomorphism for every $k \geq 2$.

Section 3 contains the proof of Theorem 2.4.

### 3. Proof of the main theorem

Note that assigning $\langle \text{id}_P \oplus P, \tau_P \rangle$ with any object $P \in \mathcal{N}$ defines a homomorphism $K_0(\mathcal{N}) \to B^1_1(\mathcal{N})$. In particular, $\langle \text{id}_P \oplus P, \tau_P \rangle = \langle \text{id}_Q \oplus Q, \tau_Q \rangle$ if $P$ and $Q$ represent the same element in $K_0$. As an aside, we remark that $\langle \text{id}_P \oplus P, \tau_P \rangle$ is equal to $\langle \text{id}_P, -\text{id}_P \rangle$ in $L^1_1(\mathcal{N})$ and to its negative in $L^1_1(\mathcal{N})$; both of these two facts are easy to prove but won’t be used in this paper.

**Lemma 3.1.** Let $P$ be a binary acyclic complex in $\mathcal{N}$ supported on $[0, k]$ and let $\text{sw}(P)$ denote the binary complex obtained from $P$ by switching top and bottom differential. Then we have

$$\text{sw}(P) = -P \quad \text{in} \quad L^k_1(\mathcal{N}).$$

**Proof.** We have $P + \text{sw}(P) = P \oplus \text{sw}(P)$ in $B^1_1(\mathcal{N})$. The latter complex represents 0 in $L^k_1(\mathcal{N})$. To see this, consider the binary ladder $(P \oplus \text{sw}(P), D, \sigma, \tau)$ where $D$ is the diagonal complex with $D^\top = D^\perp = (P \oplus \text{sw}(P))^\top$, $\sigma = \text{id}$ and $\tau$ switches the two summands $P$ and $\text{sw}(P)$, and note that $\sum_{i=0}^k (-1)^i P_i = 0$ in $K_0(\mathcal{N})$. □

Regarding binary acyclic complexes supported on $[0, k]$ as complexes supported on $[0, k+1]$ defines a homomorphism

$$i_k: L^k_1(\mathcal{N}) \to L^{k+1}_1(\mathcal{N}).$$

The following lemma is basically a special case of the generalised Nenashev relation, see Definition 4.1 below and [Har15, Proposition 2.12]. We include a short proof to convince the reader that complexes supported on $[0, k+1]$ suffice to prove the desired relation. As usual, we write $\mathbb{P}[1]$ for the complex shifted by 1 (without changing the sign of the differentials $d$ and $d'$) so that $\mathbb{P}[1]_0 = 0$.

**Lemma 3.2.** The homomorphism $i_k$ naturally factorises as

$$i_k: L^k_1(\mathcal{N}) \to K^{k+1}_1(\mathcal{N}) \to L^{k+1}_1(\mathcal{N})$$

where the second map is the canonical epimorphism.

**Proof.** Let $(P, Q, \sigma, \tau)$ be a binary ladder with $P, Q$ supported on $[0, k]$. Then all rows and columns of the diagram

\[
\begin{array}{cccc}
\ldots & \xrightarrow{\sigma_2} & P_2 & \xrightarrow{\sigma_1} & P_1 & \xrightarrow{\sigma_0} & P_0 \\
\downarrow{\tau_2} & & \downarrow{\tau_1} & & \downarrow{\tau_0} & & \\
\ldots & \xrightarrow{\sigma_2} & Q_2 & \xrightarrow{\sigma_1} & Q_1 & \xrightarrow{\sigma_0} & Q_0
\end{array}
\]



are binary acyclic complexes, top differentials commute with top differentials and bottom differentials commute with bottom differentials. Filtering the associated total complex $T$ (which is a binary acyclic complex supported on $[0, k+1]$) “horizontally and vertically” then yields the relation

$$Q + P[1] = T = \sum_{i=0}^{k} \langle \sigma_i, \tau_i \rangle [i]$$

in $B^{k+1}_1(\mathcal{N})$. If $P = Q$ and $\sigma = \tau = \text{id}$, this shows that

(3.3) $$P[1] = -P$$

in $K^{k+1}_1(\mathcal{N})$. The two equalities above finally show that

$$Q - P = \sum_{i=0}^{k} (-1)^i \langle \sigma_i, \tau_i \rangle$$

in $K^{k+1}_1(\mathcal{N})$, as desired. $\square$

**Definition 3.4.** Let $k \geq 2$ and let $P = (P_*, d, d')$ be a binary acyclic complex supported on $[0, k+1]$, and choose factorisations

$$d_2: P_2 \to J \to P_1 \quad \text{and} \quad d'_2: P_2 \to K \to P_1$$

witnessing that $(P_*, d)$ and $(P_*, d')$ are acyclic.

The **Grayson shortening of $P$** is the binary acyclic complex $\text{sh}(P)$ supported on $[0, k]$ whose top component is given by

$$\begin{array}{c}
\cdots \to P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} J \\
\oplus \quad K \xrightarrow{id} K \\
\oplus \quad J \xrightarrow{id} J \\
\oplus \qquad K \to P_1 \xrightarrow{d'_1} P_0
\end{array}$$

and whose bottom component is

$$\begin{array}{c}
\cdots \to P_3 \xrightarrow{d'_3} P_2 \xrightarrow{d'_2} K \\
\oplus \quad J \xrightarrow{id} J \\
\oplus \qquad K \xrightarrow{id} K \\
\oplus \quad J \to P_1 \xrightarrow{d_1} P_0
\end{array}$$

Note that we have permuted the summands in the bottom component for better legibility, but that we consider the summation order in the top component to be the definitive one.

**Remark 3.5.** The complex $\text{sh}(P)$ appears in handwritten notes by Grayson and has also been used in [KW, Section 5]. It may help to think of $\text{sh}(P)$ in the following way although Grayson seems not to have found it this way.

Suppose we want to (non-naively) truncate $P$ at $P_1$. As the image $J$ of $d_2$ is normally different from the image $K$ of $d'_2$, we cannot just replace $P_1$ with $J$ (or $K$) after omitting $P_0$. We therefore first replace $P_1$ with $J \oplus K$ which, in general, is the
smallest object containing both $J$ and $K$. In order to obtain an acyclic complex again, the smallest possible next step is then to add the isomorphism in the second line of each differential above. Then, in order to make the columns containing $P_2$ equal, we add $J$ or $K$ in those columns. Similarly to above, in order to obtain an acyclic complex again, we add the third lines; it is crucial here that we add an incoming rather than an outgoing identity. We are finally in the lucky position that adding the fourth lines makes all columns equal (up to permutation of summands). As $P_0$ appears again in the fourth line, we haven’t unfortunately really truncated $P$, but we have at least achieved that the object at place 0 has disappeared.

Note that our definition of $\text{sh}(P)$ includes a shift by $-1$ so $\text{sh}(P)$ is supported on $[0, k]$ rather than on $[1, k+1]$. This avoids bulky notations later.

For $P$, $J$ and $K$ as in the definition above, we have $\langle \text{id}, \tau_J \rangle = \langle \text{id}, \tau_K \rangle$ because $J = K$ in $K_0(\mathcal{N})$. We denote the latter element by $\tau_P$. If in fact $J \cong K$, we replace the morphisms $P_1 \rightarrow K$ and $K \rightarrow P_1$ with $P_2 \rightarrow J$ and $J \rightarrow P_1$ by composing them with a fixed isomorphism between $J$ and $K$. Then the ordinary (non-naive) truncations

$$t_{\geq 1}(P) := ( \cdots \longrightarrow P_3 \overset{d_2}{\longrightarrow} P_2 \overset{\text{id}}{\longrightarrow} J )$$

and

$$t_{\leq 2}(P) := ( J \overset{\text{id}}{\longrightarrow} P_1 \overset{\text{id}}{\longrightarrow} P_0 )$$

are binary acyclic complexes again. In this case, the following crucial lemma computes $\text{sh}(P) \in L_k^1(\mathcal{N})$ in terms of these truncations and $\tau_P$.

**Lemma 3.6.** Let $P$ be a binary acyclic complex supported on $[0, k+1]$ and suppose that $J \cong K$. Then we have

$$\text{sh}(P) = t_{\geq 1}(P)[-1] - t_{\leq 2}(P) - \tau_P \quad \text{in} \quad L_k^1(\mathcal{N}).$$

In particular:

(a) If $P$ is a diagonal complex, then $\text{sh}(P) = -\tau_P \in L_k^1(\mathcal{N})$.

(b) If $P_0 = 0$, then $\text{sh}(P) = P[-1] - \tau_P \in L_k^1(\mathcal{N})$.

(c) If $P$ is supported on $[0, 2]$, let $\text{sh}(P)$ be defined as above, but $\text{sh}(P)$ is now supported on $[0, 2]$ again rather than on $[0, 1]$. Then $\text{sh}(P) = -P - \tau_P \in L_2^1(\mathcal{N})$.

**Proof.** Without permuting any summands in the bottom component, $\text{sh}(P)$ has top component

$$\cdots \longrightarrow P_3 \overset{d_2}{\longrightarrow} J \overset{\text{id}}{\longrightarrow} J \overset{\text{id}}{\longrightarrow} J \overset{\text{id}}{\longrightarrow} J$$

and bottom component

$$\cdots \longrightarrow P_3 \overset{d_3}{\longrightarrow} P_2 \overset{d_2}{\longrightarrow} J \overset{\text{id}}{\longrightarrow} J \overset{\text{id}}{\longrightarrow} J \overset{\text{id}}{\longrightarrow} J \overset{\text{id}}{\longrightarrow} J$$

$$\cdots \longrightarrow P_0 \overset{d_1}{\longrightarrow} P_1 \overset{\text{id}}{\longrightarrow} J \overset{\text{id}}{\longrightarrow} J \overset{\text{id}}{\longrightarrow} J \overset{\text{id}}{\longrightarrow} J$$
Now consider the binary ladder \((\text{sh}(\mathbb{P}), t_{\geq 1}(\mathbb{P})[-1] \oplus \mathbb{J} \oplus \mathbb{J}[1] \oplus \text{sw}(t_{\leq 2}(\mathbb{P})), \sigma, \tau)\), where \(\mathbb{J}\) is the diagonal binary complex supported on \([0, 1]\) given by \(\mathbb{J}^T = \mathbb{J}^+ = (J \xrightarrow{id} J)\), \(\sigma = \text{id}\) and \(\tau\) is the automorphism switching the two copies of \(J\) in degrees 0, 1 and 2 and the identity in all higher degrees. From this binary ladder we obtain the following equality in \(L^{1}_{2}(\mathcal{N})\):

\[
t_{\geq 1}(\mathbb{P})[-1] + \text{sw}(t_{\leq 2}(\mathbb{P})) = \text{sh}(\mathbb{P}) + \tau_{\mathbb{P}}.
\]

Using Lemma 3.1, we finally obtain the desired equality in \(L^{1}_{1}(\mathcal{N})\):

\[
\text{sh}(\mathbb{P}) = t_{\geq 1}(\mathbb{P})[-1] - t_{\leq 2}(\mathbb{P}) - \tau_{\mathbb{P}}.
\]

If \(\mathbb{P}\) is a diagonal complex, both truncations are diagonal again and part (a) follows. If \(P_0 = 0\), then \(t_{\geq 1}(\mathbb{P}) = \mathbb{P}\), \(J = P_1\) and \(t_{\leq 2}(\mathbb{P}) = (\text{id}_{P_1}, \text{id}_{P_1})[1] = 0\) in \(L^{1}_{2}(\mathcal{N})\); this shows part (b). If \(k = 1\), the main equality of Lemma 3.6 holds in \(L^{1}_{1}(\mathcal{N})\) rather than in \(L^{1}_{1}(\mathcal{N})\) and we have \(J = P_2\), \(t_{\leq 2}(\mathbb{P}) = \mathbb{P}\) and \(t_{\geq 1}(\mathbb{P}) = (\text{id}_{P_2}, \text{id}_{P_2})[1] = 0\); this proves part (c).

**Proposition 3.7.** The homomorphism

\[
i_k : L^{k}_{1}(\mathcal{N}) \rightarrow L^{k+1}_{1}(\mathcal{N})
\]

is an isomorphism for all \(k \geq 2\). Its inverse is induced by the assignment

\[
\mathbb{P} \mapsto - \text{sh}(\mathbb{P}) - \tau_{\mathbb{P}}.
\]

**Proof.** We begin by showing that the assignment \(\mathbb{P} \mapsto - \text{sh}(\mathbb{P}) - \tau_{\mathbb{P}}\) induces a well-defined homomorphism \(p_k : L^{k+1}_{1}(\mathcal{N}) \rightarrow L^{k}_{1}(\mathcal{N})\).

If \(\mathbb{P}' \rightarrow \mathbb{P} \rightarrow \mathbb{P}''\) is a short exact sequence of binary acyclic complexes supported on \([0, k + 1]\), then we have induced short exact sequences \(\mathbb{P}' \rightarrow J \rightarrow \mathbb{P}''\) and \(K' \rightarrow K \rightarrow K''\) by [Büh10, Corollary 3.6]. In particular, we obtain a short exact sequence \(\text{sh}(\mathbb{P}') \rightarrow \text{sh}(\mathbb{P}) \rightarrow \text{sh}(\mathbb{P}'')\). Since we also have \(\tau_{\mathbb{P}} = \tau_{\mathbb{P}'} + \tau_{\mathbb{P}''}\), we obtain an induced homomorphism

\[
p''_k : \mathcal{P}^{k+1}_{1}(\mathcal{N}) \rightarrow \mathcal{P}^{k}_{1}(\mathcal{N}).
\]

Let \(\mathbb{P}\) be a diagonal binary acyclic complex supported on \([0, k + 1]\), then we have \(p''_k(\mathbb{P}) = 0\) in \(L^{1}_{1}(\mathcal{N})\) by Lemma 3.6(a). In particular, \(p''_k\) induces a homomorphism

\[
p'_k : \mathcal{K}^{k+1}_{1}(\mathcal{N}) \rightarrow \mathcal{K}^{k}_{1}(\mathcal{N}).
\]

If \((\mathbb{P}, \mathbb{Q}, \sigma, \tau)\) is a binary ladder in \(\mathcal{N}\) with \(\mathbb{P}\) and \(\mathbb{Q}\) supported on \([0, k + 1]\), then there are induced isomorphisms \(\sigma_{\mathbb{J}} : J_{\mathbb{P}} \xrightarrow{\sim} J_{\mathbb{Q}}\) and \(\tau_{\mathbb{K}} : K_{\mathbb{P}} \xrightarrow{\sim} K_{\mathbb{Q}}\). These define a binary ladder \((\text{sh}(\mathbb{P}), \text{sh}(\mathbb{Q}), \text{sh}(\sigma), \text{sh}(\tau))\) with

\[
\text{sh}(\sigma)_{0} = \sigma_{\mathbb{J}} \oplus \sigma_{\mathbb{K}} \oplus \tau_{\mathbb{Q}}, \quad \text{sh}(\sigma)_{1} = \sigma_{\mathbb{J}} \oplus \sigma_{\mathbb{K}} \oplus \tau_{\mathbb{Q}}, \quad \text{sh}(\sigma)_{2} = \sigma_{\mathbb{J}} \oplus \sigma_{\mathbb{K}} \oplus \tau_{\mathbb{Q}}
\]

\[
\text{sh}(\tau)_{0} = \sigma_{\mathbb{J}} \oplus \sigma_{\mathbb{K}} \oplus \tau_{\mathbb{Q}}, \quad \text{sh}(\tau)_{1} = \sigma_{\mathbb{J}} \oplus \sigma_{\mathbb{K}} \oplus \tau_{\mathbb{Q}}, \quad \text{sh}(\tau)_{2} = \sigma_{\mathbb{J}} \oplus \sigma_{\mathbb{K}} \oplus \tau_{\mathbb{Q}}
\]

and \(\text{sh}(\sigma)_{i} = \sigma_{i+1}\) and \(\text{sh}(\tau)_{i} = \tau_{i+1}\) for \(i \geq 3\). Hence we have

\[
\sum_{i=0}^{k} (-1)^i (\text{sh}(\sigma)_{i}, \text{sh}(\tau)_{i})
\]

\[
= \langle \tau_{0}, \sigma_{0} \rangle - \langle \sigma_{2}, \tau_{2} \rangle - \langle \tau_{1}, \sigma_{1} \rangle + \langle \sigma_{3}, \tau_{3} \rangle + \sum_{i=3}^{\infty} (-1)^i (\sigma_{i+1}, \tau_{i+1})
\]

\[
= - \sum_{i=0}^{k+1} (-1)^i (\sigma_{i}, \tau_{i})
\]
in $K_1^2(N)$ by Lemma 3.1. If the given ladder is as in Definition 2.2(d), it follows that

\[ p_k'(Q - P) = -sh(Q) - \tau_Q + sh(P) + \tau_P \]

\[ = -\sum_{i \geq 0} (-1)^i (sh(\sigma)_i, sh(\tau)_i) = p_k' \left( \sum_{i \geq 0} (-1)^i (\sigma_i, \tau_i) \right) \]

in $L_1^k(N)$ since $J_P \cong J_Q$, hence $\tau_P = \tau_Q$, and since, by Lemma 3.6(c), $p_k'$ maps $\langle \sigma_i, \tau_i \rangle \in K_1^{k+1}(N)$ to $\langle \sigma_i, \tau_i \rangle \in L_1^k(N)$ for each $i$. Consequently, $p_k'$ induces the desired homomorphism

\[ p_k : L_1^{k+1}(N) \rightarrow L_1^k(N). \]

We now show that $p_k \circ i_k = \text{id}$ for all $k \geq 2$. Let $P$ be a binary acyclic complex in $N$ supported on $[0, k]$. By Equation (3.3) and Lemma 3.6(b), we have

\[ p_k(i_k(P)) = -p_k(P[1]) = sh(P[1]) + \tau_{P[1]} = (P - \tau_{P[1]}) + \tau_{P[1]} = P. \]

in $L_1^k(N)$, as was to be shown.

We are left with showing that $p_k$ is also a right-inverse to $i_k$. Let $P$ be a binary acyclic complex in $N$ supported on $[0, k + 1]$. Since the definition of $sh(P)$ is independent of whether we regard $P$ as a complex supported on $[0, k + 1]$ or $[0, k + 2]$, we have

\[ i_k(p_k(P)) = p_{k+1}(i_{k+1}(P)) = P \]

in $L_1^{k+1}(N)$, as desired. \qed

**Proof of Theorem 2.4.** From Lemma 3.2 we obtain the directed system

\[ K_1^2(N) \rightarrow L_1^2(N) \rightarrow K_1^3(N) \rightarrow L_1^3(N) \rightarrow \ldots \]

Since the colimit of the cofinal sub-system $K_1^2(N) \rightarrow K_1^3(N) \rightarrow \ldots$ is $K_1(N)$, the colimit of the displayed system is $K_1(N)$ as well. Hence, the colimit of the cofinal sub-system $L_1^2(N) \rightarrow L_1^3(N) \rightarrow \ldots$ is also $K_1(N)$. Furthermore, all the connecting maps in this sub-system are isomorphisms by Proposition 3.7. The claim follows. \qed

**Remark 3.8.** Grayson shows in the handwritten notes mentioned in Remark 3.5 that $sh(P) \in K_1(N)$ differs from $P$ by classes of binary acyclic complexes of length at most 2. By induction, this proves that the canonical map $K_1^2(N) \rightarrow K_1(N)$ is surjective. While Grayson uses slightly involved double complex arguments, we use simpler and at the same time more potent arguments and also prove the simple relation $P + sh(P) = -\tau_P$ in $K_1(N)$.

**Corollary 3.9 ([KW, Theorem 1.4]).** The canonical map $K_1^3(N) \rightarrow K_1(N)$ is onto and admits a canonical section.

**Proof.** The right inverse is given by the inverse of the bijection $L_1^2(N) \rightarrow K_1(N)$ from Theorem 2.4 composed with the map $L_1^3(N) \rightarrow K_1^3(N)$ from Lemma 3.2. \qed

**Remark 3.10.** The inverse of the isomorphism $L_1^2(N) \rightarrow K_1(N)$ from Theorem 2.4 admits an explicit description. This agrees with the map $\Psi$ appearing in the proof of [KW, Theorem 1.1].
4. The relation to Nenashev’s description

In this section, we compare Nenashev’s and Grayson’s descriptions of $K_1$.

**Definition 4.1.** Nenashev’s $K_1$-group $K_N^N(N)$ of $N$ is defined as the abelian group generated by binary acyclic complexes $P$ of length 2 subject to the following relations:

1. If $P$ is a diagonal complex, then $P = 0$.
2. If

$$P_2' \rightarrow P_1' \rightarrow P_0'$$

$$P_2 \rightarrow P_1 \rightarrow P_0$$

$$P_2'' \rightarrow P_1'' \rightarrow P_0''$$

is a diagram in $N$ such that all rows and columns are binary acyclic complexes, top differentials commute with top differentials and bottom differentials commute with bottom differentials, then

$$P_0 - P_1 + P_2 = P' - P + P''$$.

Nenashev proves in [Nen98] that $K_N^N(N)$ is canonically isomorphic to Quillen’s $K_1$-group of $N$. The following corollary purely algebraically proves that $K_N^N(N)$ is isomorphic to $K_1(N)$, i.e., to Grayson’s $K_1$-group of $N$. By [Gra12, Remark 8.1], regarding a binary acyclic complex of length 2 as a class in $K_1(N)$ defines a map $K_N^N(N) \rightarrow K_1(N)$.

**Corollary 4.2 ([KW, Theorem 1.1]).** The canonical map

$$K_N^N(N) \rightarrow K_1(N)$$

is an isomorphism.

**Proof.** Since the relations used to define $L_2^2(N)$ are special cases of Nenashev’s relation, the canonical surjection $K_1^N(N) \rightarrow K_N^N(N)$ factors via $L_2^2(N)$, yielding a surjection $L_2^2(N) \rightarrow K_N^N(N)$. Since we have a commutative diagram

$$\begin{array}{ccc}
L_2^2(N) & \rightarrow & K_1(N) \\
\downarrow & & \downarrow \\
K_N^N(N) & \rightarrows & \\
\end{array}$$

it follows from Theorem 2.4 that $L_2^2(N) \rightarrow K_N^N(N)$ and $K_N^N(N) \rightarrow K_1(N)$ are isomorphisms.

**References**

[Büh10] Theo Bühler, *Exact categories*, Expositiones Math. 28 (2010), 1–69.

[Gra12] Daniel R. Grayson, *Algebraic K-theory via binary complexes*, J. Amer. Math. Soc. 25 (2012), no. 4, 1149–1167. MR 2947948

[Har15] Thomas K. Harris, *Binary complexes and algebraic K-theory*, http://eprints.soton.ac.uk/383999/, 2015, PhD Thesis, Southampton.

[HKT17] Tom Harris, Bernhard Kock, and Lenny Taelman, *Exterior power operations on higher K-groups via binary complexes*, Ann. K-Theory 2 (2017), no. 3, 409–449. MR 3658990

[KW] Daniel Kasprowski and Christoph Winges, *Shortening binary complexes and the commutativity of K-theory with infinite products*, arXiv:1705.09116.
[Nen98] A. Nenashev, $K_1$ by generators and relations, J. Pure Appl. Algebra 131 (1998), no. 2, 195–212. MR 1637539

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