Perturbative relations between $e^+e^-$ annihilation and $\tau$ decay observables including resummation effects

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Abstract

By exploiting the analyticity properties of the two-point current-current correlator we obtain numerical predictions for the $e^+e^-$ moments in terms of the $\tau$ decay rate. We perform a partial resummation of the pertinent perturbative series expansion by solving the renormalization group equation for Adler’s function. Our predictions are renormalization scheme independent but depend on the order of the perturbative $\beta$-function expansion. The analysis involves the unknown five-loop coefficient $k_3$ for which we give some new estimates.
The interpretation of experimental strong interaction data in terms of QCD and the Standard Model requires higher and higher accuracy in the theoretical input [1, 2]. There is a need to improve on the precision of the predictions of perturbative QCD by developing a framework that allows one to go beyond finite orders of perturbation theory. This is indispensable if one wants to make progress in precision fits to data of existing and especially future experiments. New high order results in fixed order perturbation theory are becoming more difficult to come because of overwhelming technical difficulties. What is needed is to develop techniques that allow one to go beyond the existing finite order perturbative results. Attempts in this direction include finite order predictions based on the Padé approximation [5], different optimizations of perturbation theory [6], and infinite resummation procedures based on particular properties of perturbation series like renormalon methods [7, 8]. Another approach consists in the use of the renormalization scheme freedom to parameterize infrared contributions to physical observables in the renormalon approximation [9, 10, 11]. Latter approaches can serve as an alternative to the renormalon calculus.

The analysis of the moments of the $e^+e^-$ rate and, in conjunction with it, the analysis of the $\tau$ decay rate has a long-standing history both in theory and experiment. The accuracy in the determination of the $\tau$ decay rate and its decay characteristics has been continuously improving while there is now hope that there will be more precise data on the $e^+e^-$ rate in the low energy domain in the near future [12]. Because of the availability of a large number of terms in the perturbative QCD expansion and the simplicity of the analyticity structure of the underlying Green’s function, the analysis of the $e^+e^-$ annihilation process has advanced to a highly sophisticated stage. The analyticity structure is simple because the process is related to the two-point correlator of gauge invariant current correlators, the analyticity structure of which is determined by the Källén–Lehmann representation.

In a recent paper we advocated the idea to directly compare physical observables within fixed order perturbative QCD. Using this approach we determined moments of the $e^+e^-$ annihilation rate in terms of the $\tau$ decay rate [13]. This eliminates the problem of scheme dependence within finite order perturbative QCD (especially if the moments are taken at the same or comparable scale). In a second paper we exploited analyticity properties of the two-point current correlator to partially resum the perturbation series for a new analysis of $\alpha_s$ [14] (see also [15]). In the present paper we combine the approaches of [13] and [14] and present the results of an analysis which expresses the moments of $e^+e^-$ rate functions in terms of the $\tau$ decay rate including resummation effects. We also discuss some general features of the solution of the renormalization group equation in the complex plane and speculate on estimates of higher order coefficients of the perturbation series in a renormalization group invariant manner.

We closely follow the notation introduced in [14]. Let us begin by defining moments of the $e^+e^-$ annihilation rate in terms of the spectral density $R(s)$ according to

$$R_n(s_0) = (n + 1) \int_0^{s_0} ds \left( \frac{s}{s_0} \right)^n R(s). \quad (1)$$

For convenience we have normalized the moments such that $R_n(s_0) = 1$ for $R(s) = 1$. We define reduced moment functions by factoring out moments of the partonic Born term.

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1 One of the recent achievements was the calculation of the four-loop $\beta$-function [3] and the four-loop anomalous mass dimension [4].
contribution $R^0_n(s_0)$. The moments of the Born term contribution $R^0_n$ are given by the leading order in the strong interaction. Nonperturbative (power suppressed) corrections and corrections due to other interactions (i.e. electro-weak corrections in the case of the semileptonic $\tau$ decay) may be absorbed in $R^0_n$ [16]. After factorization of the Born term moments the reduced moment functions $r_n(s_0)$ are given by

$$ R_n(s_0) = R^0_n(s_0) \left( 1 + \frac{4}{9} r_n(s_0) \right). \quad (2) $$

Our aim is to establish relations between different sets of observables (and not relations between observables and powers of the strong coupling constant). Further the relations should be independent of the choice of a particular renormalization scheme. It is therefore convenient to deal with scheme independent quantities from the outset.

Similar to Eq. (2) we define the reduced Adler’s function $d(Q^2)$

$$ D(Q^2) = \frac{1}{4\pi^2} \left( 1 + \frac{4}{9} d(Q^2) \right) \quad (3) $$

which we take to be an effective coupling constant. In the Euclidean domain the reduced Adler’s function $d(Q^2)$ has the expansion

$$ d(Q^2) = a(Q^2) + k_1a^2(Q^2) + k_2a^3(Q^2) + k_3a^4(Q^2) + \ldots \quad (4) $$

where $a = 9\alpha_s/4\pi$.

The running of the effective coupling constant $d(Q^2)$ is determined by the renormalization group equation which we write as [17]

$$ Q^2 \frac{dd(Q^2)}{dQ^2} = \beta(d(Q^2)), \quad (5) $$

where the perturbative series expansion of the $\beta$-function reads

$$ \beta(d) = -d^2(1 + \rho_1d + \rho_2d^2 + \rho_3d^3 + \ldots). \quad (6) $$

Due to the fact that we have chosen the reduced Adler’s function as our expansion parameter, the coefficients $\rho_i$ are renormalization scheme independent quantities (see also [18]). Numerically they are given by ($N_c = 3$)

$$ \begin{align*}
\rho_1 &= \frac{64}{81} \approx 0.790 \\
\rho_2 &= \frac{16531}{2916} + \frac{728}{81} \zeta(3) - 16\zeta(3)^2 + \frac{200}{27} \zeta(5) \approx 1.035 \\
\rho_3 &= \frac{37096148}{59049} + \frac{4820288}{6561} \zeta(3) + \frac{12352}{81} \zeta(3)^2 - 256\zeta(3)^3 \\
&\quad - \frac{59800}{243} \zeta(5) + \frac{1600}{9} \zeta(3)\zeta(5) + 2k_3 \approx 2k_3 - 2.97953 \quad (7)
\end{align*} $$

where we have taken the numerical values for the known coefficients in the $\overline{MS}$ scheme from [19] together with the recently calculated four-loop $\beta$-function coefficient given in [3]. It should be kept in mind though that the final values of the coefficients $\rho_i$ are scheme independent even if they have been calculated in a specific scheme. The five-loop contribution $k_3$ in the coefficient $\rho_3$ is not yet known. Later on we shall present some estimates
for \(k_3\). We have set \(n_f = 3\) in the present application. Here we are interested in moments of \(e^+e^-\) annihilation spectral density at the scale \(Q^2 = m_\tau^2\). The extension of our approach to other scales is straightforward.

The reduced Adler’s function \(d(m_\tau^2)\) itself can be determined from experimental data in terms of the integral representation

\[
d(m_\tau^2) = m_\tau^2 \int_0^\infty \frac{r(s)ds}{(s + m_\tau^2)^2}
\]

where \(r(s)\) is the reduced spectral density. Note that after factorization of the Born term contribution, \(r(s)\) need not be positive definite anymore. One therefore may encounter dramatic cancellations in the evaluation of the integral in Eq. (8) at the cost of the precision with which \(d(m_\tau^2)\) can be determined. This is the reason why we do not use the integral representation Eq. (8) for our numerical estimates of \(d(m_\tau^2)\).

The standard technique of contour integration in the complex plane allows one to obtain closed formulas for the reduced moments \(r_n\) in terms of \(d(m_\tau^2) \equiv d_\tau\) and the coefficients \(\rho_i\) appearing in the perturbative expansion of the \(\beta\)-function in Eq. (6). The moments are given by

\[
r_n(m_\tau^2) = \frac{n+1}{2\pi i} \oint_{|x|=1} x^n p(m_\tau^2 x)dx,
\]

where \(p(z)\) is the reduced vacuum polarization function. It is related to the reduced Adler’s function by

\[
d(Q^2) = -Q^2 \frac{dp(-Q^2)}{dQ^2}.
\]

After integrating Eq. (9) by parts using Eq. (10), the moments can be represented by

\[
r_n(m_\tau^2) = r_{\text{circ}}(m_\tau^2) + \Delta_n(m_\tau^2).
\]

The first term on the right hand side of Eq. (11) represents the surface term contribution. It does not depend on \(n\) and is, in fact, the renormalization group improved spectral density (the discontinuity across the cut at \(s = m_\tau^2 \pm i0\)) which can be calculated from \(d(Q^2) = d(m_\tau^2 e^{i\phi})\) using

\[
r_{\text{circ}}(m_\tau^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d(m_\tau^2 e^{i\phi})d\phi.
\]

The second term in the partial integration is given by

\[
\Delta_n(m_\tau^2) = \frac{(-1)^n}{2\pi} \int_{-\pi}^{\pi} e^{i(n+1)\phi} d(m_\tau^2 e^{i\phi})d\phi
\]

and depends on the order \(n\) of the moment under consideration. Latter contributions tend to be numerically suppressed because of the oscillatory factor \(e^{i(n+1)\phi}\). Considered as functions of the variable \(n\), the quantities \(\Delta_n\) are oscillating functions. At discrete values for \(n\) they can give quite irregular contributions to the moments. Numerically these contributions are suppressed but are not negligible.

We mention that the reduced moments cannot be approximated with any precision by an asymptotic expansion in \(d_\tau\). For small enough values of \(d_\tau\) the corresponding series converge [14]. However, if we take the numerical values for \(d_\tau\) derived below, they lie outside the circle of convergence and so the terms in Eq. (11) cannot be represented as
series in $d_\tau$. That is why we prefer to work directly with the integral representation given by Eqs. (12) and (13) and not with a series expansion.

To leading order in the $\beta$-function the reduced moments can be presented in explicit form. With $\beta(d) = -d^2$ one can integrate the renormalization group equation (5) in the complex $Q^2$-plane using the starting value $d_\tau$ and obtains $d(m_\tau^2 e^{i\phi}) = d_\tau (1 + i d_\tau \phi)^{-1}$. The moments can then be represented as

$$r_n(m_\tau^2) = \frac{1}{\pi} \arctan(\pi d_\tau) + \frac{(-1)^n}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i(n+1)\phi} d\phi}{1 + i d_\tau \phi}. \quad (14)$$

If more terms are included in the $\beta$-function expansion, we write the general result in symbolic form as

$$r_n(m_\tau^2) = r_n(d_\tau; \beta) = r_n(d_\tau; \rho_1, \rho_2, \rho_3, \ldots). \quad (15)$$

When expanded for small $d_\tau$ and a given order of the $\beta$-function expansion, these moments reproduce the known results of fixed order perturbation theory. However, here we consider a different approach which is not based on a series expansion. The functions $r_n(m_\tau^2)$ can be computed from Eq. (9) to any given order of the $\beta$-function expansion without ever having to invoke a series expansion. When one goes beyond the leading order approximation of the $\beta$-function, the renormalization group equation (5) can still be solved, but the integrations in Eqs. (12) and (13) become unwieldy. In these cases we proceed in numerical fashion.

In order to make contact with the semileptonic $\tau$ decay rate we take a particular combination of moments, namely

$$r_\tau = 2r_0(m_\tau^2) - 2r_2(m_\tau^2) + r_3(m_\tau^2). \quad (16)$$

Using the partially integrated representation Eq. (11), this results in

$$r_\tau(m_\tau^2) = r_{\text{circ}}(m_\tau^2) + 2\Delta_0(m_\tau^2) - 2\Delta_2(m_\tau^2) + \Delta_3(m_\tau^2) = r_{\text{circ}}(m_\tau^2) + \Delta_\tau(m_\tau^2). \quad (17)$$

This formula holds for perturbative QCD in the case of massless quarks where axial-vector and vector contributions are equal.

As a next step we invert Eq. (17) numerically to obtain numerical values for $d_\tau$ by solving the differential equation (5) with $d_\tau$ as starting value. This is done by a systematic trial and error procedure using as many iterations as necessary. The whole procedure is repeated for successive orders in the $\beta$-function expansion. Using the experimental value $r_\tau = 0.487 \pm 0.011$ [20], we obtain

$$d_\tau^{(0)} = 0.3431 \pm 0.0069, \quad d_\tau^{(1)} = 0.3466 \pm 0.0081, \quad d_\tau^{(2)} = 0.3561 \pm 0.0097 \quad (18)$$

for increasing orders in the $\beta$-function expansion, where the superscript labels the order of the expansion. Once these values are known, we can predict the contributions of $r_{\text{circ}}$ and $\Delta_\tau$ to the different reduced moments $r_n$. Predictions for the different contributions using the leading, first and second order accuracy for the $\beta$-function are given in the first three columns of Table 1 whereas in Table 2 we present the results in terms of the full moments.

For the orders $n = 0, 1$ and $2$ of the moments the values in Table 2 agree with estimates given in [13] on the basis of fixed order expansion in perturbation theory. We cannot give a prediction for the moment $r_{-1/2}$ within the present approach because the weight function
used for constructing moments has to be analytically continued and must be single-valued in the vicinity of the cut along the positive semi-axis. This is the only singularity in the complex plane allowed for the vacuum polarization two-point function as given by the spectral representation. Though $\sqrt{z}$ is an analytical function in the complex plane with a cut, its values on different sides of the cut have opposite signs. Because of this, one cannot extract the discontinuity of the vacuum polarization function across the cut.

As a note we want to illustrate the advantage of the resummation technique in comparison with ordinary perturbation theory. This illustration is done by giving a very simple example concerning the really non-trivial question whether results can be obtained by directly keeping larger number of terms. Consider two observables given by perturbative series in the same given scheme, namely

\[ f(a) = a(1 - a + a^2 - \ldots) = \frac{a}{1 + a} \quad (19) \]

and

\[ g(a) = a(1 - 2a + 4a^2 - \ldots) = \frac{a}{1 + 2a}. \quad (20) \]

The functions $f(a)$ and $g(a)$ can be seen to be related by

\[ g(f) = \frac{f}{1 + f} = f(1 - f + f^2 - \ldots). \quad (21) \]

If we fit the right hand side of Eq. (19) to an experimental value of about $f = 0.6$, we get $a = 1.5$. But for this value of the coupling, the series in Eq. (19) diverges. So we cannot

| $i$ | $0$ | $1$ | $2$ | $3$ |
|-----|-----|-----|-----|-----|
| $d_\tau$ | 0.3431 | 0.3466 | 0.3561 | 0.3667 |
| $r_{\text{circ}}(m_T^2)$ | 0.2619 | 0.2396 | 0.2352 | 0.2354 |
| $\Delta_0(m_T^2)$ | 0.1255 | 0.1369 | 0.1411 | 0.1448 |
| $\Delta_1(m_T^2)$ | 0.0290 | 0.0174 | 0.0108 | 0.0073 |
| $\Delta_2(m_T^2)$ | 0.0204 | 0.0196 | 0.0210 | 0.0227 |
| $\Delta_3(m_T^2)$ | 0.0148 | 0.0129 | 0.0118 | 0.0106 |
| $\Delta_4(m_T^2)$ | 0.0116 | 0.0105 | 0.0103 | 0.0106 |
| $\Delta_\tau(m_T^2)$ | 0.2251 | 0.2474 | 0.2519 | 0.2552 |

Table 1: Moment contributions for increasing $\beta$-function accuracy $i$

| $i$ | $0$ | $1$ | $2$ | $3$ |
|-----|-----|-----|-----|-----|
| $d_\tau$ | 0.3431 | 0.3466 | 0.3561 | 0.3667 |
| $R_0(m_T^2)$ | 2.3444 | 2.3347 | 2.3344 | 2.3380 |
| $R_1(m_T^2)$ | 2.2586 | 2.2284 | 2.2187 | 2.2157 |
| $R_2(m_T^2)$ | 2.2509 | 2.2304 | 2.2277 | 2.2295 |
| $R_3(m_T^2)$ | 2.2459 | 2.2245 | 2.2195 | 2.2190 |
| $R_4(m_T^2)$ | 2.2432 | 2.2223 | 2.2182 | 2.2187 |

Table 2: Full moments for increasing $\beta$-function accuracy $i$
get $a$ from it without a proper resummation procedure that in this case is trivially given by the appended exact formula. Consequently we cannot get a prediction for $g$ using the series in Eq. (20) in terms of $a$. On the other hand, the direct relation in terms of the series in Eq. (21) converges perfectly and gives an unambiguous result for $g$ in terms of measured $f$. Of course, for such an improvement to occur one has to analyze in detail the underlying theory and the origin of the series. Within our resummation procedure we are able to do so though we still could not answer whether one can reexpand moments through $r_\tau$ directly to get a convergent series. We could not prove the opposite either.

Comparing Eq. (19) with Eq. (16) and Eq. (20) with Eq. (15), the example demonstrates that we can obtain a dependence between the reduced $e^+e^-$ moments $r_n$ and the $\tau$ decay rate $r_\tau$ without using an expansion in $d_\tau$. This implicit dependence is shown in Fig. 1 for different moments.

In contrast to our previous study [13] the $e^+e^-$ moments can be computed without having to perform a sophisticated analysis of divergent series and estimating the truncation errors. The only question that remains is the question of errors in the present approach. The statistical error resulting from the uncertainty in the experimental number for $r_\tau$ can easily be taken into account, while for the perturbation series itself we suggest to take the difference between results for different $\beta$-function accuracy as the resulting error. When analyzed along the contour in the complex $Q^2$-plane, the worst pattern of convergence for the $\beta$-function expansion is given in the vicinity of the Euclidean point $Q^2 = m_\tau^2$ and reads

$$\beta(0.36) = -(0.36)^2(1 + 0.79(0.36) + 1.035(0.36)^2 + \rho_3(0.36)^3)$$

$$= -(0.36)^2(1 + 0.284 + 0.134 + 0.0467\rho_3). \quad (22)$$

To estimate the error of this expansion we have to estimate the value of $\rho_3$. This is also necessary to obtain a feeling for the accuracy of the perturbative approximation for the $\beta$-function in Eq. (22). In the $\overline{\text{MS}}$ scheme, this coefficient is given by the coefficients of the $\beta$-function up to the known coefficient $\beta_3$ [3] and the coefficients of the Adler’s function up to the yet unknown five-loop coefficient $k_3$. In the $\overline{\text{MS}}$ scheme there exist estimates for this quantity. They are essentially Padé estimates valid within this particular scheme and result in a value of $k_3 = 2.17$ or $\rho_3 = 1.36$. Other estimates are based on various optimization procedures for the perturbation series. They give values close to the Padé estimate [3].

Let us add to the above estimates of $\rho_3$ and present our own analysis. The first estimate is a Padé approximation for the $\beta$-function itself, which gives $\rho_3 = \rho_3^2/\rho_1 = 1.3548$. We obtain a second estimate by considering a one parameter subgroup of the renormalization group which leaves the $\beta$-function invariant. It is given by

$$d'(m_\tau^2) = d(e^\gamma m_\tau^2) = d_\tau - \gamma d_\tau^2 + (\gamma^2 - \rho_1 \gamma) - (\gamma^3 - \frac{5}{2}\rho_1 \gamma^2 + \rho_2 \gamma) d_\tau^4 + \ldots \quad (23)$$

and thus expresses $\rho_3$ as a function of $\gamma$. Although the overall value of $\rho_3$ is scheme independent by definition, one introduces a scheme dependence for $\rho_3$ through the estimation procedure because $k_3$ is scheme dependent. The dependence of $\rho_3$ in terms $\gamma$ is shown in Fig. 2. We see that the value of $\rho_3 = 1.36$ in the $\overline{\text{MS}}$ scheme (i.e. for $\gamma = 0$) is not stable against small variations of $\gamma$. There is, however, a region where $\rho_3$ is almost independent of $\gamma$ and yet close to the value given in the $\overline{\text{MS}}$ scheme. Choosing a scheme in the stability
region is known as the principle of minimal sensitivity (PMS) \[21\] and works well in a number of applications (see for instance \[22\]). Using this principle, we obtain the value \(\rho_3 = 2.4530\) in a wide range around the value \(\gamma = -1.3288\). It happens that this choice is close to the so-called G-scheme \[23\] with \(\gamma = -2\), where \(\rho_3 = 2.0518\). Although the above dependence is not the most general variation within the renormalization group, we believe that it gives an additional support for the obtained value of \(\rho_3\).

The last estimate may look a bit extravagant, nevertheless the result is consistent with that of the previous ones. We fix a scheme such that the first few terms of the reduced Adler’s function \(d(Q^2)\) are represented by a pure geometric series

\[
d(Q^2) = a_{\text{GS}}(Q^2) \left( 1 + k a_{\text{GS}}(Q^2) + k^2 a_{\text{GS}}^2(Q^2) + k^3 a_{\text{GS}}^3(Q^2) + \ldots \right)
\]

with coefficients of the \(\beta\)-function given in the \(\overline{\text{MS}}\) scheme. This is always possible up to order \(k^2\) because of the freedom of the one-dimensional reparametrization invariance given by Eq. (23). Numerically one finds

\[
d(Q^2) = a_{\text{GS}}(Q^2) \left( 1 - 0.1917 a_{\text{GS}}(Q^2) + 0.0367 a_{\text{GS}}^2(Q^2) + (k_3 - 2.602) a_{\text{GS}}^3(Q^2) + \ldots \right)
\]

where \(k_3\) is the unknown five-loop coefficient given in the \(\overline{\text{MS}}\) scheme. By demanding a geometric series behaviour for Eq. (25) one calculates \(k_3 = 2.595\). As an average of our three estimates we quote \(\rho_3 = 2.0 \pm 0.5\). With this final estimate for \(\rho_3\) we obtain \(d_\tau = 0.3667 \pm 0.0120\) and the results given in the last columns of Table 1 and Table 2.

The statistical error due to the input uncertainties in \(r_\tau\) is about 3% for all moments. This error is larger than the change resulting from adding the next (estimated) term of the \(\beta\)-function expansion. We therefore conclude that up to this order in the \(\beta\)-function the perturbative expansion for the \(\beta\)-function does not seem to limit the accuracy of the resummed predictions for the \(e^+e^-\) moments. The main uncertainty comes from the experimental error in the semileptonic \(\tau\) decay rate.

To conclude, we have obtained numerical predictions for the moments of the \(e^+e^-\) annihilation rate in terms of the known \(\tau\) decay rate. We exploited the analyticity properties of the two-point current-current correlator to perform a partial resummation of the perturbative series relating the two sets of observables. Our predictions for the \(e^+e^-\) moments are renormalization scheme independent but depend on the order of the perturbative \(\beta\)-function expansion. We have attempted to estimate the error in our prediction for the \(e^+e^-\) moments by estimating an unknown five-loop piece in the \(\beta\)-function expansion. Using this estimate, we found that the error in our predictions is dominated by the experimental error of the \(\tau\) decay rate.

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**Figure Captions**

Fig. 1: Implicit dependence of the reduced $e^+e^-$ moments $r_0$, $r_1$, $r_2$ and $r_3$ on the  
semileptonic $\tau$ decay rate

Fig. 2: Dependence of the $\beta$-function coefficient $\rho_3$ on the subgroup parameter $\gamma$  
which specifies the choice of the renormalization scheme
Figure 1
Figure 2