Majorana states and magnetic orbital motion in planar hybrid nanowires

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The Majorana phase boundaries in planar 2D hybrid (semiconductor-superconductor) nanowires are modified by orbital effects due to off plane magnetic components. We show that Majorana zero modes survive sizable vertical field tiltings, uncovering a remarkable phase diagram. Analytical expressions of the phase boundaries are given for the strong orbital limit. It can be fulfilled with attainable setups, such as an InAs nanowire of 150 nm in transverse width.

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Introduction. The physics of 2D electron gases in magnetic fields has proved invaluable for the Condensed Matter field with, e.g., the celebrated quantum Hall effects ¹ as well as with many devices based on quantum wells, wires and dots ². On the other hand, Majorana zero modes in quasi 1D wires have recently attracted strong interest, both as exotic quasi-particles and as candidates for topological quantum computing ³ ⁴. In this Letter we relate 2D-gas properties and Majorana physics, showing a remarkable role of the orbital motions characteristic of 2D systems in magnetic fields. Orbital motions open a novel avenue to the manipulation of Majorana modes, with telltale phase transition boundaries.

Majorana modes in quasi 1D wires are effectively chargeless, zero-energy quasiparticles. They arise from the splitting, through a phase transition, of bulk electronic states into pairs of quasiparticles on the wire ends, each one being its own antiparticle ⁵ ⁶. Several experiments with hybrid superconductor-semiconductor nanowires using tunneling spectroscopy from a normal conductor to the nanowire have observed a zero bias peak consistent with a Majorana state ¹⁹ ²². An important feature of the Majorana peak is that it appears only when a critical value of the external magnetic field, a phase-transition threshold, has been surpassed.

In practice, the control of Majorana states as required, e.g., in braiding for quantum computation ²³ ²⁵, relies on the details of the phase diagram of these modes in each particular physical realization. Therefore, it is highly relevant knowing how Majorana physics is affected by the extra dimension in 2D, with respect to 1D. This question has already been addressed with quasi 1D multiband wires ²⁶ ²⁹, but the role of magnetic orbital motion has been usually disregarded (Ref. ³⁰ is an exception). Addressing Majorana physics in 2D systems with orbital motion is also relevant as a way to discard alternative scenarios that have been suggested, such as attributing the observations to Kondo-like interaction effects ²⁰.

In this Letter we show that in a planar nanowire Majoranas survive sizable vertical field tiltings (Fig. 1), even reaching the purely perpendicular orientation in some cases. In the strong orbital limit, the phase transitions occur for critical values of the polar angle, following a simple analytical law that does not depend on sample details. With parallel (x) field orientation the transition law is also analytical, while for intermediate regimes the phase transitions are obtained numerically. We assess the consistency between the phase diagram and direct calculations of the Majorana modes in semi-infinite and finite 2D wires, emphasizing the importance of covariant grid discretizations for the latter ³¹ ³².

Physical model. The combination of s-wave superconductivity, Rashba interaction and an external magnetic field is a well known source of Majorana Fermions ². We consider electronic motion restricted to the ẑ (longitudinal) and y (transverse) directions in presence of these three effects. The homogeneous magnetic field points in an arbitrary direction and the edges are modeled as infinite square well potentials in the longitudinal and transversal directions (Fig. 1).

The nanowire physics is described by a Hamiltonian of the Bogoliubov-deGennes kind, split in the following way

\[
\mathcal{H}_{\text{BdG}} = \mathcal{H}_0 + \mathcal{H}_Z + \mathcal{H}_\mu + \mathcal{H}_{\text{orb}} .
\]

(1)

The successive contributions to Eq. (1) are the zero-field and superconducting energies

\[
\mathcal{H}_0 = \left( \frac{p_x^2 + p_y^2}{2m} + V(x, y) - \mu \right) \tau_z + \Delta, \tau_x ;
\]

(2)
the Zeeman term
\[ \mathcal{H}_Z = \Delta_B \left( \sin \theta \cos \phi \sigma_x + \sin \theta \sin \phi \sigma_y + \cos \theta \sigma_z \right) ; \]  
the Rashba coupling term
\[ \mathcal{H}_R = \frac{\alpha}{\hbar} \left( p_x \sigma_y - p_y \sigma_x \right) \tau_z , \]  
and, finally, the magnetic orbital terms
\[ \mathcal{H}_{orb} = \frac{\hbar^2}{2mL_z^2} y^2 \tau_z - \frac{\hbar}{mL_z} yp_x - \frac{\alpha}{L_z^2} y \sigma_y . \]  

The contributions in Eq. 2 are, in left to right order, the kinetic, electrical potential \( V \), chemical potential \( \mu \) and superconducting \( \Delta \) energies. The Pauli operators for isospin (particle-hole) are represented by \( \tau_{x,y,z} \) while those for spin are \( \sigma_{x,y,z} \). The superconductor term represents an effective mean field approximation to more complicated interactions with a nearby s-wave superconductor. The Zeeman term, Eq. 3, depends on parameter \( \Delta_B \) and models the coupling of the spin with a magnetic field of arbitrary polar and azimuthal angles \( (\theta, \phi) \equiv \hat{n} \).

The Rashba coupling Eq. 4 is the result of the self-interaction between the quasiparticle spin with its own motion. This interaction is due to the presence of a transverse electric field representing an internal asymmetry in the confinement along \( z \) that may be either intrinsic or externally induced. The first Rashba contribution, depending on \( p_x \sigma_y \) is called the 1D Rashba term while the second one, \( p_y \sigma_x \), is the Rashba mixing term. The joint effects of superconductivity, Zeeman and 1D Rashba terms give rise to independent Majorana states, one from each transverse band. On the other hand, the Rashba mixing term couples different transverse bands and its role is to effectively allow only one Majorana zero mode in parameter regions where the 1D Rashba term would yield an odd number of them (even-odd effect) [28, 33].

In a planar nanowire the perpendicular component of the magnetic field induces orbital motions of the nanowire quasiparticles. The magnetic orbital terms, Eq. 5, describe this motion and their effect on the Majorana states is the central point of this Letter. These terms depend on the magnetic length \( l_z \), defined as \( l_z^2 = \hbar c/eB_z \), and they stem from the kinetic and Rashba energies with the magnetic substitution \( p_x \rightarrow p_x - \hbar y/l_z^2 \) and adding the required Pauli matrix \( \tau_z \) for proper particle-hole symmetry. In Eq. 5 we assumed the Landau gauge centered on \( y_c = 0 \), although our results are independent on this choice as discussed in the supplementary material. We note here that the above mentioned experiments [10, 22] used cylindrical nanowires, for which the 2D orbital effects discussed here are not present.

All parameters of the complete Hamiltonian are constant inside the nanowire, modeled as a perfectly confining box with \( L_x \gg L_y \). The numerical results of this work are presented in characteristic units of the problem obtained by taking \( \hbar, m \) and the width of the nanowire \( L_y \) as reference values. That is, our length and energy units are, respectively, \( L_U = L_y \) and \( E_U = \hbar^2/mL_y^2 \).

The matching method. In topological systems it is in general possible to relate the states of the bulk with the behavior at the boundaries. It has been demonstrated that a Majorana phase transition occurs in a semi-infinite nanowire when the propagating bands for the infinite nanowire (i.e., same Hamiltonian but disregarding edges) perform a gap closing and reopening at vanishing wavenumber when increasing the magnetic field [8, 34]. In our case, this implies searching numerically the solutions of
\[ \sum_{s',\sigma} \langle s'\sigma| h| s\sigma'\sigma \rangle \Psi_{s'\sigma'}(y) = 0 , \]  
where \( s = \pm \), \( s' = \pm \) are spin-isospin labels and \( h \) is obtained neglecting all \( p_x \)-dependent terms in Eq. 11.

\[ h = \left( \frac{p_y^2}{2m} + V(y) - \mu \right) \tau_z + \Delta s \tau_x - \frac{\alpha}{\hbar} p_y \sigma_x \tau_z \]  
\[ + \Delta_B \hat{n} \cdot \tau_+ \frac{\hbar^2}{2mL_z^2} y^2 \tau_z - \frac{\alpha}{L_z^2} y \sigma_y . \]  

We use the algorithm devised in Refs. [33, 35] selecting an arbitrary matching point \( y_m \), and spin-isospin components \( (s, t) \), say \( (+, +) \), to determine a physical measure
\[ F = \left| \frac{dL}{dy} - \frac{dR}{dy} \right| \Psi_{st}(y_m) \right|^2 , \]  
where \( dL/dy, dR/dy \) denote grid derivatives using only left (L) or right (R) grid neighbors. Only those parameter sets for which \( F = 0 \) represent physical solutions. Therefore, the \( F \) zeroes with varying parameters yield the desired gap closing boundaries.

Phase diagrams. Figure 2 shows the phase transition boundaries obtained with the matching method for an \( L_y = 150 \) nm nanowire with material parameters typical of InAs and a magnetic field strength between 0 and 6 T (\( \Delta_B = 0 - 25 E_U \)). As mentioned, the energy unit scales as \( L_y^2 \), such that in a 300 nm wire \( 25 E_U \) would correspond to 1.5 T. The other panels of Fig. 2 correspond to lower (b) and higher (c.d) values of the Rashba coupling strength. We will explicitly calculate the zero modes for particular sets of parameters below. Here, let us anticipate that the phases in Fig. 2 contain either no Majoranas or at most one Majorana mode in regions labeled with an M. The main result of this Letter is that orbital effects do not always destroy Majoranas, but lead to a characteristic phase map where Majoranas survive sizable vertical tiltings (even up to \( \theta = 0 \) in Fig. 21).

It is possible to give analytical expressions of the phase boundaries in particular limits (Supplementary Material). For \( \theta = 90^\circ \) and \( \phi = 0 \) the critical magnetic fields read
\[ \Delta^c_B \equiv \sqrt{\left( \mu - \epsilon_n + \frac{m\alpha^2}{2\hbar^2} \right)^2 + \Delta^2} , \]  

where \( \mu \) is the chemical potential, \( \epsilon_n \) the chemical potential of the third band, \( \Delta \) the superconducting gap and \( \Delta_B \) the Rashba term.
FIG. 2. \( \mathcal{F} \) measure in a color (gray) scale as a function of \( \Delta_B \) and polar angle \( \theta \). The azimuthal angle remains \( \phi = 0 \). Zero values of \( \mathcal{F} \) (red) signal the phase transition boundaries. Phases with a Majorana mode are labeled with an M. From top to bottom the panels correspond to \( \mathcal{F}(b) \) and \( \mathcal{F}(c) \), where the allowed complex wave numbers \( \{\Psi_{s}\} \) are obtained with the matching method discussed previously. Equation (12) can be reworked into

\[
\sum_k \mathcal{M}_{k'k} C_k = 0 , \tag{13}
\]

with the matrix

\[
\mathcal{M}_{k'k} = \sum_{s,s'} \int dy \Psi_{s,\sigma}^{(k')*}(y) \Psi_{s,\sigma}^{(k)}(y) . \tag{14}
\]
Equation (13) shows that, when enough wave numbers are included, each Majorana state is represented by a null-space eigenvector of matrix $\mathcal{M}$. In Fig. 4a we can see the convergence of the $\mathcal{M}$ eigenvalues with the cut off in wave number for a particular point of Fig. 2. Clearly, the lower eigenvalue vanishes asymptotically indicating that for this point of the phase diagram a Majorana mode is present as expected. In Fig. 4b we can see the corresponding density function, confirming the edge character of the mode, as also expected for a Majorana. We have also found that the decay length does not change in a relevant way until $\theta$ is near a critical value (Supplementary Material).

The phase diagram can also be checked with full diagonalizations of nanowires with large, but finite, $L_x$. Though more realistic, this approach is conceptually more qualitative, since finite-nanowire Majoranas are not exact zero modes but small energy modes (the smaller the energy the larger $L_x$). Equivalently, the phase boundaries become blurred due to the finite size effect. Figure 4c shows the density of the finite nanowire Majorana corresponding to the semi infinite one of Fig. 3b. Differences are small, just a slight distortion and a somewhat longer decay tail of the finite-nanowire density.

In the finite nanowire diagonalization with orbital terms we have found it crucial using a covariant grid discretization [31, 32]. Otherwise, numerical artificial biases wrongly suggest that Majoranas are always destroyed by orbital terms [30], in clear contradiction with the phase diagram (Fig. 2) and the semi-infinite wire analysis. In essence, the covariant discretization amounts to expressing the canonical momentum components as symmetry-like transformations. For instance,

$$\Pi_x \equiv -i\hbar \frac{\partial}{\partial x} - \hbar \frac{y}{L_z} e^{i y x / L_z^2} \left(-i\hbar \frac{\partial}{\partial x}\right) e^{-i y x / L_z^2}. \quad (15)$$

Although these two representations of the canonical operator are equivalent in the continuous limit, they are not on a discrete grid. As demonstrated in Ref. [32] the covariant derivative preserves by construction the gauge invariance of the solutions while a non-covariant treatment only does that for extremely fine discretizations, unfeasible in our case.

Conclusions. In this work we have shown that the orbital motions caused by perpendicular components of the magnetic field in planar 2D nanowires give rise to a rich phase diagram, with regions containing Majoranas for sizable vertical tilting of the magnetic field. In fact, with proper parameters, it is possible to find Majoranas in a fully perpendicular field. We have developed a general numerical method to obtain the Majorana phases in nanowires in a computer efficient way and we have checked this method against alternative calculations for semi-infinite and finite nanowires. Analytical expressions of the transition boundaries in asymptotic regions have been found. For realistic parameter values (weak $\alpha$) these analytical expressions are a good approximation in general and not only asymptotically. In the strong orbital limit the critical angles are independent of sample details. Finally, the relevance of the covariant grid discretization for the finite nanowire diagonalization has been pointed out.

FIG. 3. a) Evolution of the lower eigenvalues of matrix $\mathcal{M}$ when increasing the number of evanescent modes, as given by a cutoff $|k|$. b) Majorana density function in a semi-infinite nanowire for the null eigenvector of $\mathcal{M}$. c) Majorana density function in a finite nanowire with $L_x = 20L_U$ calculated by direct diagonalization of the Hamiltonian using covariant derivative discretization. The three panels correspond to $\Delta_B = 10E_U$, $\theta = 75^\circ$ and the rest of parameters as in Fig. 2.

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SUPPLEMENTARY MATERIAL

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I. ANALYTICAL LIMITS

A. Longitudinal magnetic field

When the magnetic field is along x (see axis orientations in Fig. 1 of the paper) the phase transition law is fully analytical. As discussed in the paper, finding the phase transition implies searching for the zero energy eigenstates of the simplified Hamiltonian

\[ h = \left( \frac{\hat{p}_y^2}{2m} - \mu \right) \tau_z + \Delta_s \tau_x + \left( \Delta_B - \frac{\alpha}{\hbar} \hat{p}_y \tau_z \right) \sigma_x , \quad (S1) \]

where we have taken into account that orbital terms in Eq. (7) of the paper vanish for this particular field direction.

The eigenstates of Eq. (S1) can be obtained analytically noticing that the linear \( \hat{p}_y \) term from the Rashba interaction can be absorbed in the kinetic term

\[ h = \left( \frac{\hat{p}_y^2}{2m} - \frac{m\alpha^2}{2\hbar^2} - \mu \right) \tau_z + \Delta_s \tau_x + \Delta_B \sigma_x , \quad (S2) \]

where \( \hat{p}_y = p_y - m\alpha \sigma_x / \hbar \). Using a basis of square-well eigenstates of energies \( \epsilon_n = \hbar^2 \pi^2 n^2 / (2mL_y^2) \), the matrix to diagonalize is

\[ h \equiv \begin{pmatrix} \epsilon_n - \mu - \frac{m\alpha^2}{2\hbar^2} & \Delta_s \\ \Delta_s & -(\epsilon_n - \mu - \frac{m\alpha^2}{2\hbar^2}) \end{pmatrix} . \quad (S3) \]

The diagonalization of this simplified Hamiltonian yields the eigenenergies

\[ E_{ns_1s_2} = s_1 \Delta_B + s_2 \sqrt{(\mu - \epsilon_n + \frac{m\alpha^2}{2\hbar^2})^2 + \Delta_s^2} , \quad (S4) \]

with \( n = 1, 2, \ldots \), \( s_1 = \pm 1 \) and \( s_2 = \pm 1 \). Of the four eigenenergies only the two with opposite \( s_1 \) and \( s_2 \) can lead to a zero energy solution at the critical values

\[ \Delta_{B,n}^{(c)} = \sqrt{(\mu - \epsilon_n + \frac{m\alpha^2}{2\hbar^2})^2 + \Delta_s^2} . \quad (S5) \]

Equation (S5) with \( n = 1, 2, \ldots \) gives the critical Zeeman parameter of phase transitions for a two dimensional nanowire in parallel magnetic field. This analytical result extends recent findings from other authors [26,29] who assumed that the contribution in parenthesis in Eq. (S5) is an effective chemical potential from subband \( n \), without specifying its detailed \( \alpha \) dependence. Notice that in Eq. (S5) \( n \) has to be interpreted simply as an ordering index of the successive transitions, and not as a label of independent transverse modes. These latter interpretation would be wrong, since different transverse modes are coupled through the Rashba mixing term and one cannot associate a particular transverse mode with a particular transition point. Shaded regions in Fig. S1 contain one Majorana mode, while white regions have none. There are no regions with multiple Majoranas due to the energy splittings induced by the Rashba mixing in planar nanowires [28,33].

B. Strong orbital limit

When the kinetic orbital effect overcomes both the confinement by the transverse square well and the Rashba term, the magnetic length \( l_z = \sqrt{\hbar c/eB_z} \) is smaller than \( l_y \) (here we define \( l_y = L_y \) of the paper) and also smaller than the Rashba length \( l_\alpha = \hbar^2 / ma \). In the limit \( l_z \ll (l_y, l_\alpha) \) it is possible to derive an analytical expression of the transition boundaries. Neglecting the square well \( V(y) \) and the Rashba terms in Eq. (7) of the paper we find

\[ h = \left( \frac{\hat{p}_y^2}{2m} + \frac{\hbar^2}{2ml_z^2} y^2 - \mu \right) \tau_z + \Delta_B \bar{\sigma} \cdot \hat{n} + \Delta_s \tau_x . \quad (S6) \]

The eigenvalues of Eq. (S6) are straightforward in a basis \( \{|n s_\sigma s_\tau\} \), where \( n = 1, 2, \ldots \) represent now harmonic
oscillator eigenstates, \( s_\sigma = \pm \) indicates spin eigenstates in direction \( n \), while \( s_r = \pm \) indicates isospin in direction \( z \). Since the \( h \) matrix is diagonal in spin, we can diagonalize each subspace independently. For instance, the matrix for \( s_\sigma = + \) reads

\[
\begin{pmatrix}
\epsilon_n^{(ho)} - \mu + \Delta_B & \Delta_s \\
\Delta_s & - (\epsilon_n^{(ho)} - \mu) + \Delta_B
\end{pmatrix},
\tag{S7}
\]

with \( \epsilon_n^{(ho)} = (n-1/2)\hbar^2/ml_z^2 \). The eigenvalues of Eq. (S7) are easily found, as well as those of the analogous matrix for spin \( s_\sigma = - \).

The null-eigenvalue condition for \( h \) is now

\[
\Delta_B = \sqrt{\left[ (n-1/2)\hbar^2/ml_z^2 - \mu \right]^2 + \Delta_s^2} ,
\tag{S8}
\]

that looks similar to Eq. (S5). An essential difference, however, is that the r.h.s. in Eq. (S8) depends itself on the Zeeman parameter \( \Delta_B \) through \( l_z \). It is

\[
\frac{\hbar^2}{ml_z^2} = \frac{4}{gm^*}\Delta_B \cos \theta ,
\tag{S9}
\]

where \( m^* \) is the ratio of effective to bare mass, \( m = m^*m_e \), while \( g \) is the gyromagnetic factor defined from the Zeeman parameter by \( \Delta_B \equiv g \mu_B B/2 \). From Eq. (S8) we finally arrive at the following relation

\[
\cos \theta = \frac{gm^*}{4(n-\frac{1}{2})}\sqrt{\Delta_s^2 - \Delta_B^2 + \mu} .
\tag{S10}
\]

For large enough \( \Delta_B \), as compared to \( \Delta_s \) and \( \mu \), this leads to the prediction of field-independent critical angles

\[
\cos \theta_n^{(c)} = \frac{gm^*}{4(n-\frac{1}{2})} ,
\tag{S11}
\]

as given in Eq. (10) of the paper.

The triple inequality \( l_z \ll (y, l_\alpha, l_\delta) \), where we define \( l_\delta \equiv \sqrt{\hbar^2/ml_s^2} \), leads, when written in effective units, to Eq. (11) of the paper. In this situation the phase diagram does not deviate much from the straight lines of the analytical limits, Eqs. (S5) and (S11). Table I contains the numerical values of the inequality conditions for the four panels in Fig. 2 of the paper. While panel a) fulfills all conditions, for the rest of panels the second inequality degrades as \( \alpha \) increases from panel b) to d). This explains the deviations in those panels from the analytical limits.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Panel & \( \Delta_{B,1}^{(c)} \cos \theta_1^{(c)} \) & \( \frac{gm^*}{4E_U} \) & \( \frac{gm^* \omega_s^2}{4E_U^2} \) & \( \frac{gm^* \Delta_s}{4E_U} \) \\
\hline
a) & 1.43 & 0.12 & 0.001 & 0.36 \\
b) & 1.03 & 0.12 & 0.48 & 0.36 \\
c) & 0.74 & 0.12 & 1.22 & 0.36 \\
d) & 1.06 & 0.12 & 1.98 & 0.36 \\
\hline
\end{tabular}
\caption{Numerical values in effective units of the inequalities, Eq. (11) of the paper, corresponding to the four panels in Fig. 2 of the paper.}
\end{table}

FIG. S2. \( \mathcal{F} \) measure in a color (gray) scale showing the position of the allowed wave numbers as zeroes (red islands). The parameters used are \( g = 15 \), \( m^* = 0.033 \), \( \alpha = \pi E_U L_U^2 \), \( \Delta_s = 3 \), \( \Delta_B = 10 E_U \). Panels from top to bottom are for different polar angles: \( \theta = 68^\circ \) (a), \( \theta = 67^\circ \) (b), \( \theta = 66^\circ \) (c).

II. GAUGE INVARINACE

In presence of magnetic orbital effects a relevant check for approximate descriptions (both analytical and numerical) is the preservation of gauge invariance. Changing the gauge origin usually constitutes a severe difficulty for numerical discretizations not using covariant derivative formulations \([31, 32]\). This limitation led to a flawed estimation of the orbital effect in Ref. [31]. In our case, we can introduce an arbitrary gauge center \( y_c \) for the canonical momentum, generalizing Eq. (15) of the paper to

\[
\Pi_z = e^{i(y-s_\sigma y_c)x/\ell_z^2} \left( -i\hbar \frac{\partial}{\partial x} \right) e^{-i(y-s_\sigma y_c)x/\ell_z^2} ,
\tag{S12}
\]

where the isospin sign \( s_\sigma = \pm \) is introduced in order to preserve the particle-hole symmetry of the Bogoliubov-deGennes equation.

We have checked that our numerical results for the finite nanowire diagonalization as, e.g., in the lower pannel of Fig. 3 of the paper, do not depend on the choice of \( y_c \), thus proving the gauge invariance of the finite system results. We have also obtained good agreement of the
FIG. S3. Mode lengths \( L_m \) (defined in Sec. III) for the two wave numbers shown in Fig. S2. Note that the required nanowire length at each \( \theta \) is the higher of both curves.

finite nanowire diagonalizations and the results of the semi-infinite system regarding the existence or absence of a zero mode in the different regions of the phase diagrams (Fig. 2 of the paper), again proving the reliability of the method. Notice that the semi-infinite solution, being purely 1D, can be obtained in very dense \( y \) grids, while the finite system 2D diagonalization requires much coarser \( xy \) grids.

III. DECAY LENGTHS

Within our complex-band-structure approach to the semi-infinite nanowire we can estimate the length of the Majorana decay tail from the imaginary part of the allowed wave numbers. The lower the imaginary part, the longer the Majorana decay tail (and thus the required length of the nanowire to contain it without distortion). Figure S2 shows a typical evolution of the wave numbers (red islands) in the complex plane as the polar angle is approaching the critical value. In the sequence from upper to lower panels, one of the wave numbers moves along the imaginary axis towards the origin; the phase transition being signaled by one mode touching the origin (lower panel).

We calculate the required nanowire length with the smallest imaginary wave number of the set of all allowed wave numbers \( \{k^{(m)}\} \). However, as shown in Fig. S2, the smallest imaginary part \( \Im(k^{(m)}) \equiv k_i^{(m)} \) changes from a fixed mode to the one touching the origin when approaching the phase transition. We define the mode length \( L_m \) as two times the length needed for the wave function to drop to one percent of its maximum, that is \( e^{-k_i^{(m)} L_m / 2} = 0.01 \). An estimate of the nanowire length for undistorted Majoranas is simply the maximum of all mode lengths.

In Fig. S3 we show the mode lengths of the two allowed wave numbers of Fig. S2. As we decrease the polar angle from 90 degrees, the needed nanowire length (the higher of the two curves) remains more or less stable until \( \theta \) approaches the critical value. A few degrees before the transition the Majorana contracts before diverging to infinity at the phase transition angle.