An Atlas of Convection in Main-sequence Stars

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Abstract

Convection is ubiquitous in stars and occurs under many different conditions. Here we explore convection in main-sequence stars through two lenses: dimensionless parameters arising from stellar structure and parameters that emerge from the application of mixing length theory. We first define each quantity in terms familiar to both the 1D stellar evolution community and the hydrodynamics community. We then explore the variation of these quantities across different convection zones, different masses, and different stages of main-sequence evolution. We find immense diversity across stellar convection zones. Convection occurs in thin shells, deep envelopes, and nearly spherical cores; it can be efficient or inefficient, rotationally constrained or not, transsonic or deeply subsonic. This atlas serves as a guide for future theoretical and observational investigations by indicating which regimes of convection are active in a given star, and by describing appropriate model assumptions for numerical simulations.

Unified Astronomy Thesaurus concepts: Stellar physics (1621); Stellar evolutionary models (2046); Stellar convective zones (301)

1. Introduction

Convection is central to a wide range of mysteries in stars. How are pulsations excited (Goldreich & Kumar 1990)? How are magnetic fields generated in the Sun and stars in general (Parker 1955; Brandenburg & Subramanian 2005)? What drives stellar activity and stellar spots (Strassmeier 2009; Lehtinen et al. 2020)? How do stars spin down (Kraft 1967)? What sets the radii of M dwarfs (Kesseli et al. 2018; Morrell & Naylor 2019)? Where does stochastic low-frequency photometric variability come from (Bowman 2020; Cantiello et al. 2021)? What causes mass loss in luminous blue variables (Jiang et al. 2018)?

Convection occurs not only during core hydrogen burning but also in later evolutionary phases, including the red (super)giant phase (Chiavassa et al. 2009; Trampedach & Stein 2011; Goldberg et al. 2022), during the helium flash (e.g., Dearborn et al. 2006; Mocák et al. 2009), in the thermally pulsating asymptotic giant branch (e.g., Herwig 2005; Freytag & Höfner 2008; Siess 2010), in white dwarfs (e.g., Freytag et al. 1996; Cunningham et al. 2020), in classical novae (e.g., Denissenkov et al. 2013), during late phases of nuclear burning (e.g., Meakin & Arnett 2006, 2007; Müller et al. 2016), and during core collapse (e.g., Arnett & Meakin 2011; Burrows 2013; Müller 2016).

For some goals, such as modeling main-sequence stellar evolution, it is sufficient to use the steady-state 1D convective formulation of mixing length theory (MLT; Böhm-Vitense 1958; Henyey et al. 1965; Ludwig et al. 1999), but for questions of dynamics (Smolec 2016), rotation (Antia et al. 2008), waves (Zhou et al. 2019), and precision stellar evolution (Joyce & Chaboyer 2018), this approach is insufficient.

To that end, significant effort has gone into improved 1D theories (e.g., Gough 1977; Stellingwerf 1982; Kuhfuss 1986; Canuto & Mazzitelli 1991; Deng et al. 2006; Houdek & Dupret 2015; Jermyn et al. 2018) as well as into dynamical numerical simulations of convection zones (CZs; e.g., Chiavassa et al. 2011; Stein & Nordlund 2012; Gilet et al. 2013; Trampedach et al. 2013; Hotta et al. 2014; Rempel & Cheung 2014; Jiang et al. 2015; Augustsson et al. 2016; Yadav et al. 2016; Cristini et al. 2017; Strugarek et al. 2018; Cunningham et al. 2019; Edelmann et al. 2019; Andrassy et al. 2020; Brown et al. 2020; Horst et al. 2020; Pratt et al. 2020; Korr & Featherstone 2021; Anders et al. 2022; Schultz et al. 2022, among many others). A challenge in these efforts is that different CZs can lie in very different parts of parameter space. To name just a few axes of variation, convection can occur in thin shells or deep envelopes or nearly spherical cores, it can carry most of the energy flux or very little, it can be rotationally constrained or not, and it can be transsonic or deeply subsonic.

Here we explore the dimensionless parameters that set the stage for convection in main-sequence stars, as well as those that emerge with the application of MLT. We begin in Section 2 by describing the different CZs that appear in main-sequence stars. In Section 3 we then define each quantity of interest in terms familiar to both the 1D stellar evolution community and the hydrodynamics community. In Section 4 we explain how we construct our stellar models. We then explore how these quantities vary across different CZs, different masses, and different stages of main-sequence evolution in Section 5. Finally, we discuss the different regimes that arise in main-sequence stellar convection and the prospects for realizing these regimes in numerical simulations.

2. Convection Zones

Convection occurs in different regions of stars on the main sequence, which we depict schematically in Figure 1 for stars of different mass and spectral types. For a more quantitative picture obtained using 1D stellar evolution calculations, we refer to Figure 2 in Cantiello & Braithwaite (2019).
When we refer to a CZ below, we mean a Schwarzschild-unstable layer, assuming that convection rapidly homogenizes the composition of the zone.

Generally speaking, convection in stars can occur in the presence of a steep temperature gradient due to nuclear energy generation (e.g., core convection), a large value of the opacity, and/or a high heat capacity and consequently low adiabatic gradient. An increase of the opacity and a decrease of the adiabatic gradient tend to occur in stellar envelopes at temperatures corresponding to the ionization of different species (Cantiello & Braithwaite 2019).

We classify these CZs first into three broad classes:

1. Core CZs (e.g., the core of a 20 \(M_\odot\) star).
2. Deep Envelope CZs (e.g., the Sun’s envelope).
3. Thin ionization-driven CZs (e.g., the He II ionization zone in a 2 \(M_\odot\) star).

There is some ambiguity in assigning some CZs to these classes. This arises primarily in two scenarios. First, is the CZ in an M dwarf, which can span the entire star, a Deep Envelope CZ or a core CZ? Second, at what point does the Deep Envelope CZ seen in solar-like stars become a thin zone driven by H I/He I/He II ionization?

In the first instance we want to have as continuous a definition as possible across the Hertzsprung–Russell (H-R) diagram. So we say that an M dwarf has a Deep Envelope CZ, because the CZ gradually retreats upward with increasing mass.

In the case of the transition from deep envelope to thin ionization-driven convection, we make the decision based on the temperature at the base of the CZ, because ionization zones occur at nearly fixed temperatures.

In full then, our classification scheme is as follows:

1. Core CZ, if \(m_{\text{inner}}/M_* < 0.03\) and \(M_* > 2M_\odot\).
2. Deep Envelope CZ, if \(T_{\text{inner}} > 500,000\) K and \(M_* < 2M_\odot\).
3. Thin ionization-driven CZ otherwise.

Here \(m_{\text{inner}}\) and \(T_{\text{inner}}\) are, respectively, the mass coordinate and temperature at the inner boundary of the CZ. Note that this classification scheme does not distinguish whether a thin ionization-driven CZ is driven by a decrease in the adiabatic gradient or an opacity increase. Lower numbers in the above list receive priority if a CZ is validly described by multiple classes (e.g., a CZ that satisfies 1 and 2 is classified as 1). Next, if the zone is ionization driven, we select the subclass by

(a) H I CZ—if \(T_{\text{outer}} < 11,000\) K.
(b) He I—if the CZ contains a point with \(11,000 < T < 35,000\) K.
(c) He II—if the CZ contains a point with \(35,000 < T < 100,000\) K.
(d) FeCZ—if the CZ contains a point with \(100,000 < T < 500,000\) K.

This approach works because the ionization state is much more strongly controlled by temperature than by density, so temperature is actually a good proxy for ionization state. If a zone meets multiple of these criteria, we classify it by the hottest subclass it fits.
3. Dimensionless Parameters

We describe convection using two categories of dimensionless parameters: “inputs” (Table 1), which depend on the stellar models and microphysics, and “outputs” (Table 2), which include a measure of the convective velocity or mixed temperature gradient and therefore depend on a theory of convection. For the latter we employ MLT following the prescription of Cox & Giuli (1968); we emphasize that this theory reflects a drastic simplification of reality and is in many regards unsatisfactory, but it does allow us to provide order-of-magnitude estimates for these quantities. We note also that even the inputs in this paper depend on our choice to employ MLT, because the stratification achieved in convective regions can subtly alter the stratification and values throughout the model. Nonetheless, MLT reflects the current standard in stellar modeling and points the way to many open questions, so it is valuable to use as a baseline.

3.1. Input Parameters

3.1.1. Microphysics

We begin with quantities set solely by the microphysics. The Prandtl number is the ratio

\[ \text{Pr} \equiv \frac{\nu}{\chi}, \quad (1) \]

where \( \nu \) is the kinematic viscosity and \( \chi \) is the thermal diffusivity. The magnetic Prandtl number is similarly

\[ \text{Pm} \equiv \frac{\nu}{\eta}, \quad (2) \]

where \( \eta \) is the magnetic diffusivity. Our prescriptions for these diffusivities are given in Appendix A.

We further write the ratio of radiation pressure to total pressure as

\[ \beta_{\text{rad}} \equiv \frac{P_{\text{rad}}}{P}. \quad (3) \]

3.1.2. Stellar Structure

Next, we present quantities determined by stellar structure. The first of these is the Rayleigh number,

\[ \text{Ra} \equiv \delta r^4 \left( \frac{\langle s \rangle}{\langle \chi \rangle} \right) \left( \frac{1}{\langle h \rangle} \right) \left( \frac{4 - \langle \beta_{\text{gas}} \rangle}{\langle \beta_{\text{gas}} \rangle} \right). \quad (4) \]

which measures how unstable a region is to convection. When \( \text{Ra} < \text{Ra}_{\text{crit}} \approx 10^3 \), convective motions are diffusively stabilized, so the convective velocity is zero (Chandrasekhar 1961). Here, \( g \) is the acceleration of gravity, \( \delta r \) is the thickness of the CZ, \( h \equiv -dr/d\ln P \) is the pressure scale height, \( \beta_{\text{gas}} \equiv P_{\text{gas}}/P = 1 - \beta_{\text{rad}} \) is the ratio of gas pressure to total pressure,

\[ \nabla_{\text{rad}} \equiv \frac{\partial \ln T}{\partial \ln P} \bigg|_s \quad (5) \]

is the adiabatic temperature gradient, \( s \) is the entropy,

\[ \nabla_{\text{rad}} \equiv \frac{3 \kappa L}{64 \pi G M \sigma T^4} \quad (6) \]

is the radiative temperature gradient, \( \kappa \) is the opacity, \( L \) is the luminosity, \( P \) is the pressure, \( T \) is the temperature, \( G \) is the gravitational constant, \( M \) is the mass coordinate, and \( \sigma \) is the Stefan–Boltzmann constant. The factor in Equation (4) depending on \( \beta \) arises from the thermal expansion coefficient and is equal to the ratio of the density susceptibility to the temperature susceptibility (\( \chi_{\rho}/\chi_T \)). Note that the Rayleigh number is a global quantity defined over the whole CZ. \( \text{Ra} \) depends on the depth of the zone (\( \delta r \)), but the other quantities on the right-hand side of Equation (4) are generally functions of radius and so must be averaged, which we denote with angled brackets. The details of this averaging are described in Section 4.2.

Stellar structure provides the geometry of the CZ. The aspect ratio is

\[ A \equiv \frac{r_{\text{outer}}}{\delta r}, \quad (7) \]

where \( r_{\text{outer}} \) is the radial coordinate of the outer boundary of the CZ. Thin-shell CZs have large aspect ratios and can be adequately modeled with local Cartesian simulations, but CZs with small aspect ratios require global (spherical) geometry to capture realistic dynamics.

The density contrast is

\[ D \equiv \frac{\rho_{\text{inner}}}{\rho_{\text{outer}}}, \quad (8) \]

where \( \rho_{\text{inner}} \) and \( \rho_{\text{outer}} \) are, respectively, the density on the inner and outer boundaries of the CZ. When \( D \) is small, density stratification can be neglected, as in the Boussinesq approximation.
The optical depth across the CZ,
\[ \tau_{\text{CZ}} \equiv \int_{\text{CZ}} \kappa \rho \, dr, \]  
and the optical depth from the outer boundary of the CZ to infinity,
\[ \tau_{\text{outer}} \equiv \int_{r_{\text{outer}}}^{\infty} \kappa \rho \, dr, \]
together tell us whether or not radiation can be treated diffusively in the CZ. Note that our stellar models do not extend into the base of the atmosphere. In our models this minimum surface optical depth is \( \tau_{\text{min}} = 2/3 \).

Relatedly, the Eddington ratio is
\[ \Gamma_{\text{Edd}} \equiv \frac{L}{L_{\text{Edd}}}, \]  
where
\[ L_{\text{Edd}} = \frac{4\pi GMc}{\kappa}, \]  
and \( c \) is the speed of light. Here \( M \) is the mass coordinate, and both \( \kappa \) and \( L_{\text{Edd}} \) are functions of \( M \).

Finally, we measure the ratio of the rotational and viscous timescales via the Ekman number
\[ E_k \equiv \frac{\nu}{2\Omega \delta r^2}. \]  
Here, \( \Omega \) is the angular velocity of the CZ and we assume solid body rotation. We choose \( \Omega \) as a function of mass and evolutionary state to reproduce typical observed rotation periods. Details on this choice are described in Appendix D. Results for very different periods may be obtained by scaling our results accordingly.

In the presence of rotation, \( E_k, Ra, \) and \( Pr \) combined determine the diffusive instability of a CZ. Rotation dominates viscous effects when \( E_k \ll 1 \), but the output Rossby number (below) more reliably describes how effectively rotation deflects convective flows.

### 3.2. Outputs

Using MLT we obtain the convection speed \( v_c \) and temperature gradient \( \nabla \) (Böhm-Vitense 1958; Cox & Giuli 1968). We define the Reynolds number
\[ \text{Re} \equiv \frac{v_c \delta r}{\nu}, \]  
which is the ratio of the viscous timescale to the convective turnover time, so convection at large Re is turbulent.

We further define the Péclet number
\[ \text{Pe} \equiv \frac{v_c \delta r}{\chi}, \]  
which is the ratio of the thermal diffusion timescale across the zone to the turnover time and so measures the relative importance of convective and radiative heat transfer.

Next we define the Mach number
\[ Ma \equiv \frac{v_c}{c_s}, \]  
where \( c_s \) is the adiabatic sound speed. \( Ma \) measures the magnitude of thermodynamic fluctuations, and soundproof (e.g., anelastic) models are valid at low \( Ma \). In radiation-dominated \( (\beta_{\text{rad}} \approx 1) \), low-Pe gases, the relevant fluctuations become isothermal and we should replace \( c_s \) with the isothermal sound speed \( c_{\text{iso}}^2 = \partial P/\partial \rho |_{\text{iso}} \) (Grassitelli et al. 2015; Jiang et al. 2015). For CZs where this is relevant we will show both the adiabatic and isothermal Mach numbers \( (Ma_{\text{iso}} \equiv v_c/c_{\text{iso}}) \).

With \( \nabla \) we obtain the ratio of convective flux to total flux \( F_{\text{conv}}/F \), which is closely related to both the Nusselt number \( Nu \) and the efficiency parameter \( \Gamma \) in MLT (see the discussion in Jermyn et al. 2022b). We also construct the radiative Eddington ratio
\[ \Gamma_{\text{Edd}}^r \equiv \frac{L_{\text{rad}}}{L_{\text{Edd}}}, \]  
where \( L_{\text{rad}} \) is the radiative luminosity, which depends on the steady-state temperature gradient \( \nabla \) achieved by the convection and is therefore distinct from the Eddington ratio, \( \Gamma_{\text{Edd}} \).

We further write the stiffness
\[ S \equiv \frac{N_{\text{KZ}}^2}{f_{\text{conv}}}, \]  
of the convective-radiative boundary, which measures how difficult it is for convective motions to proceed past the boundary. Here, \( N_{\text{KZ}} \) is the Brünt-Väisälä frequency in the radiative zone and the convective frequency \( f_{\text{conv}} = v_c/h \). Inner and outer boundaries in shell CZs are distinguished as \( S_{\text{inner/outer}} \). We only report values for CZs that have the relevant boundaries.\(^5\)

We measure rotation using the Rossby number
\[ \text{Ro} \equiv \frac{v_c}{2\Omega \delta r}. \]  
We use the same uniform \( \Omega \) to calculate Ro as we did for \( E_k \) (Equation (13)).

Finally, we compute the dimensional turnover time
\[ t_{\text{conv}} \equiv \int_{\text{CZ}} \frac{dr}{v_c}, \]  
which is useful for reasoning about the effects of rotation as a function of \( \Omega \).

### 4. Methods

#### 4.1. Stellar Evolution

We calculated stellar evolutionary tracks for stars ranging from 0.3 to 60 \( M_\odot \), using release r21.12.1 of the Modules for Experiments in Stellar Astrophysics software instrument (MESA; Paxton et al. 2011, 2013, 2015, 2018, 2019). Details on the MESA microphysics inputs are provided in

\(^5\) Core CZs do not have inner boundaries. Likewise, when CZs reach the surface of our models, we cannot calculate an outer stiffness because there is no outer boundary inside of the model. Physically there ought to be a location where the energy transport becomes radiative, but this occurs in the atmosphere, which we treat with a simple boundary condition.
Appendix B. Our models were run at the Milky Way metallicity of \(Z = 0.014\), use convective premixing (Paxton et al. 2019, Section 5.2) and the Cox MLT option (Cox & Giuli 1968) with \(\alpha_{\text{MLT}} = 1.6\), and determine the convective boundary using the Ledoux criterion. All data and scripts used in compiling this atlas are available publicly. See Appendix C for details.

4.2. Averaging

We are interested in obtaining a global measure of each parameter (e.g., Re) across a CZ, yet these parameters depend on quantities that themselves vary across the CZ (e.g., \(v_c, \nu\)), so we require an averaging procedure. We denote the radial average of a quantity \(q\) over a CZ by

\[
\langle q \rangle \equiv \frac{1}{\delta r} \int_{CZ} q \, dr.
\]

With this, we calculate the averages

\[
\Pr = \left\langle \frac{\nu}{\chi} \right\rangle, \\
\Pm = \left\langle \frac{\nu}{\eta} \right\rangle, \\
\beta_{\text{rad}} = \left\langle \frac{P_{\text{rad}}}{\Pr} \right\rangle,
\]

\[
Ra = \delta r^4 \frac{g}{(\nu/\chi)} \left( \frac{1}{h} \right) \left( 1 - \left\langle \frac{\beta_{\text{gas}}}{\beta_{\text{gas}}} \right\rangle \right),
\]

\[
\Gamma_{\text{Edd}} = \left\langle \frac{L}{L_{\text{Edd}}} \right\rangle, \\
\Gamma_{\text{rad Edd}} = \left\langle \frac{L_{\text{rad}}}{L_{\text{Edd}}} \right\rangle, \\
\frac{F_{\text{conv}}}{F} = \left\langle \frac{F_{\text{conv}}(\nu)}{F(\nu)} \right\rangle, \\
Ek = \left\langle \frac{\nu}{2\Omega r^2} \right\rangle, \\
Re = \left\langle \frac{\nu}{\delta r} \right\rangle, \\
Pe = \left\langle \frac{\nu}{\chi} \right\rangle, \\
Ma = \left\langle \frac{\nu}{c_s} \right\rangle, \\
Ro = \left\langle \frac{\nu}{2\delta r} \right\rangle.
\]

To compute the stiffness we radially average quantities over as much as we can without crossing another convective-radiative boundary, where the scale height is measured at the boundary.

Once more denoting radial averages by \(\langle \ldots \rangle\), we write

\[
S = \frac{\langle N^2/\nu \rangle_{\text{RZ}}}{\langle (v_c/h)^2 \rangle_{\text{CZ}}},
\]

where the subscript RZ denotes an average on the radiative side of the boundary and CZ denotes an average on the convective side.

We chose to compute averages radially because convection involves radial heat transport, making averaging in one dimension (radially) seem more appropriate than a volume average, which would pick up much more contribution from low-density outer layers, or a mass average, which is dominated by motions very deep down. Arguments could well be made for other choices, including mass, volume, or pressure weighting, and in specific applications it may be clearer which kind of average is most appropriate.

In Appendix E we examine the effects of averaging profiles of, for example, \(\Pr\) directly (i.e., \(\Pr = \langle \nu/\chi \rangle\)), rather than averaging the diffusivities individually. For most CZs we find small \((\lesssim 2\times)\) differences between the two, but for Deep Envelope CZs quantities can differ by orders of magnitude depending on the averaging procedure because those zones are strongly stratified. To better capture this variation we additionally report averages over the outer and inner pressure scale height of the Deep Envelope CZs.

5. Discussion

5.1. Summary of Results

In Appendix F we present plots of the quantities in Tables 1 and 2 for each CZ across the H-R diagram. We group quantities by the class and subclass of CZ, as our aim is to provide as complete a picture as possible about the properties of each kind of CZ. Our aim is for each subsection of Appendix F to stand on its own, and so the same points are often repeated between these.

For each zone we begin with a figure showing the aspect ratio to introduce the geometry of the zone. We then study each of our other parameters in Tables 1 and 2. Along the way we discuss implications for studying these CZs in simulations, as well as the prospects for answering specific science questions.

Table 3 shows the ranges of key input parameters we obtain for each CZ, and Table 4 shows the same for output parameters. For some CZs we show both the full mass range and subsets meant to highlight properties that vary strongly with mass. Moreover, the He I CZ has subcritical Rayleigh numbers over certain mass ranges. In those ranges the He I zone has no convective motions because it is not unstable (Jermyn et al. 2022a). As such, we have restricted the mass range for the He I CZ in our table to just that which supports supercritical Rayleigh numbers.\(^7\) The same applies to the He II CZ, but for it, subcritical Rayleigh numbers occur at high masses with large Eddington ratios, which may reflect instability (Jiang et al. 2015, 2018), so we retain the full mass range for the He II CZ.

Tables 3 and 4 tell a story of enormous diversity. For a single class of CZ, many properties, like Ra, Re, and Pe, span five or

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\(^6\) If the CZ or RZ does not extend for a full pressure scale height, we average over as much as we can without crossing another convective-radiative boundary.

\(^7\) Here we have used the simplest possible stability analysis: comparing the average Ra from Equation (25) to the canonical critical value. A more robust stability analysis could be performed by solving for the eigenvalues associated with the stellar stratification, but such an analysis is beyond the scope of this atlas and we encourage future authors to explore this.
more orders of magnitude as a function of mass. Across different classes of CZs, the diversity is even more striking. We see aspect ratios ranging from unity to $10^3$, density ratios from unity to $10^7$, optical depths from unity up to $10^{12}$, and Rossby numbers from $10^{-3}$ up to $10$. In Sections 5.2 and 5.3 we examine the implications of these parameter ranges for the choice of simulation techniques and for the scientific questions that each CZ poses.

5.2. Physical Regimes

We have classified CZs in solar metallicity main-sequence stars into six categories:

1. Deep Envelope ($M \lesssim 1.2 \, M_\odot$)
2. H I ($1.3 \, M_\odot \lesssim M \lesssim 3 \, M_\odot$)
3. He I ($2 \, M_\odot \lesssim M \lesssim 2.5 \, M_\odot$)
4. He II ($M \gtrsim 1.5 \, M_\odot$)
5. FeCZ ($M \gtrsim 7 \, M_\odot$)
6. Core ($M \gtrsim 1.1 \, M_\odot$)

Different physical processes dominate in each of these (Table 5), which suggests using different kinds of numerical experiments for studying them.

5.3. Scientific Questions

Different regimes of convection pose distinct scientific questions. Table 6 summarizes some of the topics, which are interesting to study for each class of CZ.

5.3.1. Rotation

Rapid global rotation can stabilize the convective instability (Chandrasekhar 1961). This effect is most prominent in the regime of $E_k \ll 1$ and $Pr \gtrsim 1$; since stellar CZs have $Pr \ll 1$, we do not expect rotation to substantially modify the value of the critical Rayleigh number.

Assuming convective instability, rotation influences convection by deflecting its flows via the Coriolis force. The Rossby number $Ro$ measures how large advection is compared to the Coriolis force, and when $Ro \ll 1$, the Coriolis force dominates over the nonlinear inertial force. In the regime of $Ro \ll 1$ attained in Deep Envelope and core CZs, rotation two-dimensionalizes the convection, deflecting flows into tall, skinny columns that align with the rotation axis (discussed in the solar context in, e.g., Featherstone & Hindman 2016; Vasil et al. 2021). In the regime of $Ro \gg 1$, rotation does not
appreciably modify convection compared with a nonrotating system.

It is unclear how rotation influences convective dynamics in the transitional Ro \sim 1 regime. A brief and modern review describing the modern theory of rotating convection is provided by Aurnou et al. (2020); we note that the regimes of Ro \gg 1 and Ro \ll 1 are described in detail, but Ro \sim 1 does not have a clean theoretical understanding. On the scale of whole CZs, rotation can couple with convection to drive differential rotation, and the associated latitudinal shear can be crucial in driving dynamos (Brun et al. 2017).

### 5.3.2. Magnetism

Main-sequence CZs consist of strongly ionized fluid, so magnetism should be universally important. Dynamos are some of the most exciting applications of magnetocoonvection (Brandenburg & Subramanian 2005). Deep Envelope CZs are known to support complex dynamo cycles (as in the Sun), and a full understanding of these cycles remains elusive (e.g., Brown et al. 2010; Hotta et al. 2016; Brun et al. 2022). Dynamos are likewise of interest in the He II CZ because it has the highest kinetic energy density of all subsurface CZs for most A/B stars (Cantiello & Braithwaite 2019; Jermyn & Cantiello 2020), making dynamo action there a good candidate to explain the observed magnetic fields of, for example, Vega and Sirius A (Petit et al. 2010, 2011). The FeCZ is similarly of interest because FeCZ dynamos probably set the magnetic field strengths of most O stars (Cantiello & Braithwaite 2011; MacDonald & Petit 2019). In core CZs, dynamo-generated fields could be the progenitors of those observed in red giant cores and compact stellar remnants (Fuller et al. 2015; Cantiello et al. 2016), so understanding their configuration and magnitude is important for connecting to observations.

We also note that Deep Envelope CZs uniformly have Pm \ll 1, while each other CZ class has Pm \gg 1 in some mass range, so these could exhibit very different dynamo behaviors and likely need to be studied separately. Many fundamental studies into scaling laws and force balances in magnetocoonvection employ the quasistatic approximation for MHD (e.g., Yan et al. 2019). The quasistatic approximation assumes that Rm = PmRe \gg 1; in doing so, this approximation assumes a global background magnetic field is dominant and neglects the nonlinear portion of the Lorentz force. This approximation breaks down in CZs with Pm > 1, and future numerical experiments should seek to understand how magnetocoonvection operates in this regime.

Understanding which lessons from these reduced experiments apply in the Pm \ll 1 regime may help unravel some mysteries of core CZs. A brief and modern review of dynamo processes in the HI regime is given by Brun et al. (2017), so we do not discuss it further here. We also note that Deep Envelope CZs uniformly have Pm \ll 1, while each other CZ class has Pm \gg 1 in some mass range, so these could exhibit very different dynamo behaviors and likely need to be studied separately. Many fundamental studies into scaling laws and force balances in magnetocoonvection employ the quasistatic approximation for MHD (e.g., Yan et al. 2019). The quasistatic approximation assumes that Rm = PmRe \gg 1; in doing so, this approximation assumes a global background magnetic field is dominant and neglects the nonlinear portion of the Lorentz force. This approximation breaks down in CZs with Pm > 1, and future numerical experiments should seek to understand how magnetocoonvection operates in this regime. Understanding which lessons from these reduced experiments apply in the Pm > 1 regime may help unravel some mysteries of dynamo processes in the HI/HeI/HeII/Fe/core CZs.

Beyond dynamos, convection can interact with fossil magnetic fields (Zeldovich 1957), and it has been suggested that this can serve to erase or hide the near-surface fields of early-type stars. For more on fossil magnetic fields and convection see Jermyn & Cantiello (2021) and references therein. “Rad.” means radiation.

### Table 5

| Mass Range | Geometry | Radiation Pressure? | Radiative Transfer? | Rotation? | Fluid Dynamics |
|------------|----------|---------------------|---------------------|-----------|----------------|
| M ≤ 1.2 M_☉ | Global | Neglect | Diff. Bulk, Full Bound. | Include | Mixed^a |
| 1.3 M_☉ ≤ M ≤ 1.5 M_☉ | Local | Neglect | Diff. Bulk, Full Bound. | Neglect | Compressible |
| 1.5 M_☉ ≤ M ≤ 2.5 M_☉ | Local | Neglect | Mixed^b | Include | Boussinesq |
| 2 M_☉ ≤ M ≤ 2.5 M_☉ | Local | Neglect | Diffuse | Include | Boussinesq |
| 1.5 M_☉ ≤ M ≤ 9 M_☉ | Local | Neglect | Diffuse | Include | Boussinesq |
| 30 M_☉ ≤ M ≤ 30 M_☉ | Global | Include | Diffuse | Include | Boussinesq |

### Table 6

| Mass Range | Topics |
|------------|--------|
| M ≤ 1.2 M_☉ | Rotational constraints. Dynamo action. Impact of stratification. |
| 1.3 M_☉ ≤ M ≤ 1.5 M_☉ | IGW. Wave mixing. Impact of sonic motion. |
| 1.5 M_☉ ≤ M ≤ 2.5 M_☉ | Marginal Ra. Weak turbulence. Interaction with fossil fields. |
| 2 M_☉ ≤ M ≤ 2.5 M_☉ | Marginal Ra. Weak turbulence. Rotational constraints. Optically thin convection. |
| 1.5 M_☉ ≤ M ≤ 9 M_☉ | Marginal Ra. Weak turbulence. Interaction with fossil fields. Dynamo action. |
| 7 M_☉ ≤ M ≤ 30 M_☉ | Interaction with fossil magnetic fields. Low-stiffness boundaries. Dynamo action. |
| 30 M_☉ ≤ M ≤ 30 M_☉ | Impact of rad. pressure. Sonic motion. Super-Eddington limit. Low Pe. Dynamo action. |
| 1.1 M_☉ ≤ M ≤ 30 M_☉ | IGWs. Dynamo action. Rotational constraints. Stiff boundaries. |
| 30 M_☉ ≤ M ≤ 30 M_☉ | IGWs. Dynamo action. Impact of radiation pressure. |

Note. “Bound.” refers to the outer boundary; “Diff.” refers to the diffusive approximation. All mass ranges are approximate and metallicity dependent. We do not include mass ranges in which CZs that have subcritical Rayleigh numbers as these zones do not undergo convective motions (Jermyn et al. 2022a).

^a The anelastic approximation is appropriate in the deeper portions of these zones, where the Mach numbers are low. The compressible equations are needed near the surface, where the Mach numbers can approach unity.

^b Diffusion is appropriate in the bulk, but full radiative transfer is important near the surface.
stars (Jermyn & Cantiello 2020, 2021). In the limit of $\nabla_{\text{rad}} \gg \nabla_{\text{ad}}$ and weak magnetic fields the result is likely that the fossil field is erased and replaced with the result of a convective dynamo (Tobias et al. 2001; Korre et al. 2021), although considerable uncertainties remain in this story (Featherstone et al. 2009). In the limit of weak convection ($\nabla_{\text{rad}} \sim \nabla_{\text{ad}}$) and strong magnetic fields, convection is actually stabilized and so prevented from occurring in the first place (Gough & Tayler 1966; MacDonald & Petit 2019; Jermyn & Cantiello 2020). These interactions are least understood and so most interesting in the intermediate limit where the kinetic and magnetic energy densities are comparable, which occurs only in the He I, He II, and FeCZs, which are very diffusive. These will organize into “roll” structures with predominantly horizontal vorticity, which exhibit dynamics such as spiral defect chaos (e.g., Vitral et al. 2020). Although the thermal structures may be diffusive, the flows are still highly turbulent (e.g., Pandey et al. 2018, 2021) due to the values of $Pr \ll 1$ and $Re \gg 1$. This may be important for studying the convective zone boundary, mixing, and/or interaction with waves.

### 5.3.3. Internal Gravity Waves

Internal gravity waves (IGWs) are generated at the boundaries of CZs (Goldreich & Kumar 1990). The power in these waves peaks in frequency near the convective turnover frequency, and the overall flux of an IGW scales as $F_{\text{core}} \alpha c$ for some order-unity $\alpha$ (Goldreich & Kumar 1990; Lecoanet & Quataert 2013; Rogers et al. 2013; Cousten et al. 2018). This makes it particularly important to study wave generation at the low-mass end of the H I CZ, because the H I CZ has $F_{\text{core}} \sim F$ and $\alpha \sim 0.3$, so the wave flux is expected to be a substantial fraction of the total stellar luminosity. This could produce substantial wave mixing (Garcia Lopez & Spruit 1991) or even alter the thermal structure of the star. Wave mixing may also be induced by core CZs (Rogers & McElwaine 2017), where the smaller Mach number is offset by the larger stellar luminosity, resulting in even larger IGW fluxes.

Since $g$ modes are even-odd in convective regions, they have small predicted amplitudes at the surface of stars with convective envelopes, making them difficult to detect (Appourchaux et al. 2010). There are claims of detection of solar $g$ modes in the literature (Garca et al. 2007; Fossat et al. 2017), but so far none has been fully verified (Schunker et al. 2018; Böning et al. 2019; Scherrer & Gough 2019).

Even in early-type stars, it is usually difficult to observe IGW with very long periods, because these require long observing campaigns. Because the IGW excitation spectrum peaks near the convective turnover time, IGW from the core, Fe, and low-mass H I CZs should have the most readily observable timescales, with typical periods less than 10 days. However, the transmission of waves to the surface may make IGWs difficult to observe from core CZs (Lecoanet et al. 2019; Cantiello et al. 2021).

### 5.3.4. Marginal Stability

Convective instability requires a Rayleigh number greater than the critical value of $\sim 10^5$ (Chandrasekhar 1961), and the character of marginally unstable convection can be quite different from that of high-Ra convection. This is most relevant for the He I, He II, and high-mass H I CZs, as these have Rayleigh numbers that cross the threshold from subcritical (stable) to supercritical (unstable) (Jermyn et al. 2022a).

The actual stability of a CZ is determined by the radial profile of the Rayleigh number. For cases where the average Rayleigh number is close to the critical value, it is important to solve the global stability problem to determine whether the CZ is stable or not. If the Rayleigh number is everywhere subcritical, the putative CZ should be stable.

Stellar convection at moderate Rayleigh numbers is likely dominated by thermal structures with aspect ratios near unity, which are very diffusive. These will organize into “roll” structures with predominantly horizontal vorticity, which exhibit dynamics such as spiral defect chaos (e.g., Vitral et al. 2020). Although the thermal structures may be diffusive, the flows are still highly turbulent (e.g., Pandey et al. 2018, 2021) due to the values of $Pr \ll 1$ and $Re \gg 1$. This may be important for studying the convective zone boundary, mixing, and/or interaction with waves.

### 5.3.5. Stratification

Stratification effects are strongest in Deep Envelope CZs, where the density can vary by four to seven orders of magnitude across the CZ. Fundamentally, density stratification breaks the symmetry of convective upflows and downflows, leading to intense, fast, narrow downflows and broad, slower, diffuse upflows. Global simulations suggest that these broad upflows, named “giant cells,” should imprint on surface convection flows, but there is debate and disagreement regarding whether or not these flows are observed in the Sun (Hanasoge et al. 2016). It is possible that rotation influences giant cells in a way that changes their observational signature (Featherstone & Hindman 2016; Vasil et al. 2021). It is also possible that high-density stratification turns cold downflows into small, fast features called “entropy rain” (Brandenburg 2016; Anders et al. 2019). The very low diffusivities present in stars could allow small downflows to traverse the full depth of these stratified CZs without diffusing, but modern convection simulations generally do not have enough spatial resolution (or have too large diffusivity) to resolve these motions. In summary, stratification breaks symmetry, but the precise consequences of this symmetry breaking on convective dynamics and observed phenomena is not clear.

### 5.3.6. Super-Eddington Convection

When the radiative energy flux through the system is locally super-Eddington, the hydrostatic solution develops a density inversion and becomes unstable to convection (Joss et al. 1973). In deep stellar interiors the thermal energy density is large enough that convection can carry most of the energy flux. This lowers the radiative flux, allowing it to remain sub-Eddington and eliminating the density inversion (Jiang et al. 2015). Convection is then sustained by the usual superadiabatic temperature gradient. In this way the convective cores of very massive stars can remain nearly hydrostatic despite super-Eddington total energy fluxes.

On the other hand, in near-surface layers where convection becomes inefficient, convection may not able to carry enough heat and the radiative flux can stay super-Eddington. Radiative acceleration can then drive a density inversion in late-type stars and in luminous stars can even drive outflows (Owocki & Shaviv 2012; Jiang et al. 2018). This is what happens in the FeCZ at high masses.

### 5.3.7. Radiation Pressure

Even in sub-Eddington systems radiation pressure can play a role by changing the effective polytropic index of the adiabat (i.e., $\gamma \rightarrow 4/3$ as $\beta_{\text{rad}} \rightarrow 1$). We are not aware of studies addressing the effects of high-$\beta$ convection, which suggests it could be interesting to study this limit and in particular to compare otherwise identical systems both including and excluding radiation pressure to understand what differences arise.
5.3.8. Radiative Transfer

Radiation pressure and a high Eddington ratio can substantially affect the nature of convection (Joss et al. 1973; Shaviv 1998; Paxton et al. 2013). The resulting dynamics depends sensitively on the ratio of the photon diffusion time to the dynamical time (Jiang et al. 2015). This determines whether calculations can use a simplified diffusive treatment of radiation or if they need to use full radiative transfer.

In core CZs both $\beta_{\text{rad}}$ and $\Gamma_{\text{Edd}}$ approach unity. The photon diffusion time is long so it is suitable to treat radiation in the diffusion approximation (e.g., as done by Augustson et al. 2016).

In massive stars the FeCZ has also $\beta_{\text{rad}}$, $\Gamma_{\text{Edd}} \sim 1$. The photon diffusion time in the FeCZ is comparable to or shorter than the dynamical timescale. Such zones must thus be studied using full radiative transfer (Jiang et al. 2015, 2017, 2018; Schultz et al. 2020, 2022).

5.3.9. Stiffness and the Péclet Number

The stiffness of a convective boundary determines the extent of overshooting past the edge of the boundary. All of the upper boundaries we examined are extremely stiff except for the H i CZ, which has $S_{\text{outer}} \sim 0.1$–1, and Deep Envelope CZs, which have $S_{\text{outer}} \sim 0.1$–10 when they do not extend all the way to the surface of the model. This suggests that motions could carry a decent fraction of a scale height past the upper boundary of the H i CZ, which might cause observable motions at the photosphere and so could conceivably explain some of the observed macro-/microturbulence (Landstreet et al. 2009). The connection between (near-)surface convection and surface velocity fields is well established in the regime of late-type stars (e.g., Asplund et al. 2000; Collet et al. 2007; Mathur et al. 2011; Bergemann et al. 2012; Steffen et al. 2013; Trampedach et al. 2013) but is still under investigation for early-type stars (Cantiello et al. 2009; Jiang et al. 2015; Schultz et al. 2020, 2022). Most of the lower boundaries are similarly stiff, although the low-mass H i CZs and the FeCZ both have relatively low-stiffness lower boundaries with $S_{\text{inner}} \sim 10$–100. This could be responsible for some chemical mixing beneath these CZs.

When $P_e$ is small, radiative diffusion dominates over advection and the motion becomes effectively isothermal rather than adiabatic. Overshooting convective flows feel a reduced stiffness and so can extend further than the stiffness alone would suggest. This may be why 3D calculations show that motions from the FeCZ extend far beyond 1D predictions and up to the stellar surface (Jiang et al. 2015, 2018) and could account for some of the observed stellar variability and surface turbulence in stars with near-surface thin ionization-driven CZs (Cantiello et al. 2021; Elliott et al. 2022; Schultz et al. 2022). That said, processes other than near-surface convection could be important in driving the observed surface velocity fields (see, e.g., Aerts et al. 2009; Rogers et al. 2013).

5.3.10. Transonic Convection

Thermodynamic perturbations in a CZ scale like the square Mach number (Anders & Brown 2017). Flows with Mach numbers approaching one at the photospheres of Deep Envelope CZs can therefore add to the stellar photometric variability, participating in the “bolometric flicker” observed in light curves (Trampedach et al. 1998; Ludwig 2006; Mathur et al. 2011; Chiavassa et al. 2014; Van Kooten et al. 2021). Improved models of high-Mach-number surface convection can therefore assist in cleaning photometric lightcurves and assist in exoplanet detection.

We also observe high-Mach-number flows in the opacity-driven H i and FeCZs. Local models of these regions have been studied by, for example, Jiang et al. (2015) and Schultz et al. (2022), who observed that the large thermal perturbations associated with this convection created low-density, optically thin “chimneys” through which radiation could escape the star directly. Turbulent motions driven by these high-Mach-number regions are a candidate for the origin of observed low-frequency variability (Cantiello et al. 2021).

In summary, transonic (high-Mach-number) convection occurs in various convective regions near the surfaces of both early- and late-type stars, and the large thermodynamic perturbations associated with this convection can generate short-timescale variability of the stellar luminosity.

5.4. The Road to 3D Stellar Evolution

The study of stars has relied heavily on 1D stellar evolution calculations. Three-dimensional stellar evolution calculations are not yet feasible due to the formidable range of spatial and temporal scales characterizing the problem. For example, the average Reynolds number in the solar CZ is about $10^{12}$. Assuming a Kolmogorov spectrum of turbulence, the range of scales that need to be resolved to achieve an accurate and resolved direct numerical simulation is then $\ell_{\text{max}} / \ell_{\text{min}} \approx \text{Re}^{3/4} \sim 10^9$. Such a simulation would need about $10^{27}$ resolution elements, while the largest hydrodynamic simulations currently use $\sim 10^{12}$.

Similarly, simulations that aim to capture the full dynamics of convection on the longer timescales of stellar evolution (thermal and nuclear) would require an enormous number of time steps. For example, the dynamical timescale in the solar CZ is of order an hour, while the thermal and nuclear timescales are $10^7$ yr and $10^{10}$ yr, respectively. Assuming one time step per dynamical timescale (a vast underestimate), this implies that $\sim 10^{11}$–14 time steps would be needed to resolve the solar CZ for a thermal (nuclear) timescale, exceeding the number of steps achieved in state-of-the-art multidimensional numerical simulations (e.g., $\sim 4 \times 10^8$, Anders et al. 2022) by many orders of magnitude.

Even assuming that Moore’s law will continue to hold, a fully resolved stellar turbulence calculation of solar convection will not be achievable for $1.5 \log_2 10^{15} \sim 50$ yr (e.g., Meakin 2008). A similar calculation protracted for a full thermal (nuclear) timescale is $1.5 \log_2 10^{20} \sim 70$ yr ($1.5 \log_2 10^{23} \sim 80$ yr) away.

It is clear, then, that 1D stellar evolution calculations will remain an important tool for decades to come. At the same time, the last decade has marked a transition to a new landscape in the theoretical study of stars.

Driven by progress in both hardware and numerical schemes, multidimensional calculations are becoming common tools for studying stellar interiors. Even though these calculations do not resolve the full range of relevant scales, it is likely that many of the flow features they reveal are robust to varying resolution. While this is not guaranteed to be true in all cases (e.g., in MHD systems or other setups with inverse cascades), and so must be checked carefully, numerical simulations can provide valuable insight into real astrophysical situations.
As computers and numerical methods become more and more powerful, new problems in stellar convection become accessible to numerical study. Our aim with this atlas is to provide a guide useful for the next generation of stellar physicists performing numerical simulations of stellar convection. Our “maps” illuminate the range of parameters involved in such numerical efforts, and our hope is that they will be used by modelers to better navigate this landscape.

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Software: MESA (Paxton et al. 2011, 2013, 2015, 2018, 2019, http://mesa.sourceforge.net), MESASDK (Townsend 2021a, 2021b), matplotlib (Hunter 2007), NumPy (van der Walt et al. 2011).

Appendix A

A.1. Viscosity

Computing the viscosity of a plasma is complicated. For simplicity we use the viscosity of pure hydrogen plus radiation, so that

\[ \nu \approx \nu_{\text{ie}} + \nu_{\text{rad}}. \]  

(A1)

The radiation component is (Spitzer 1962)

\[ \nu_{\text{rad}} = \frac{4aT^4}{15\kappa r^2}, \]  

(A2)

where \( a \) is the radiation gas constant. We obtain the hydrogen viscosity using the Braginskii–Spitzer formula (Braginskii 1957; Spitzer 1962)

\[ \nu_{\text{ie}} \approx 2.21 \times 10^{-15} \frac{(T/K)^{5/2}}{\left(\rho/g/cm^3\right) \ln \Lambda} \text{cm}^2 \text{s}^{-1}, \]  

(A3)

where \( \ln \Lambda \) is the Coulomb logarithm, given for hydrogen by (Wendell et al. 1987)

\[ \ln \Lambda = -17.9 + 1.5 \ln \frac{T}{K} - 0.5 \ln \frac{\rho}{g/cm^3} \]  

(A4)

for temperatures \( T < 4.2 \times 10^5 \text{K} \) and

\[ \ln \Lambda = -11.5 + \ln \frac{T}{K} - 0.5 \ln \frac{\rho}{g/cm^3} \]  

(A5)

for \( T > 4.2 \times 10^5 \text{K} \). Corrections owing to different compositions are generally small relative to the many orders of magnitude we are interested in here. For instance, the difference between pure hydrogen and a cosmic mixture of hydrogen and helium is under 30% (Balbus & Henri 2008).

A.2. Thermal Diffusion

Thermal diffusion in main-sequence stars is dominated by photons, resulting in the radiative diffusivity

\[ \chi \equiv \frac{16aT^3}{3\kappa r^2}. \]  

(A6)

A.3. Electrical Conductivity

We use routines built into MESA to compute the conductivity \( \sigma \). These routines were provided by S. C. Yoon (2003, personal communication) and follow the formula of Spitzer (1962), with a Coulomb logarithm computed by Wendell et al. (1987). Given \( \sigma \), the magnetic diffusivity is computed via

\[ \eta = \frac{\sigma^2}{4\pi \nu}. \]  

(A7)

Appendix B

MESA

The MESA EOS is a blend of the OPAL (Rogers & Nayfonov 2002), SCVH (Saumon et al. 1995), FreeEOS (Irwin 2004), HELM (Timmes & Swesty 2000), PC (Potekhin & Chabrier 2010), and Skye (Jermyn et al. 2021) EOSes.

Radiative opacities are primarily from OPAL (Iglesias & Rogers 1993, 1996), with low-temperature data from Ferguson et al. (2005) and the high-temperature, Compton scattering–dominated regime by Poutanen (2017). Electron conduction opacities are from Cassisi et al. (2007).

Nuclear reaction rates are from JINA REACLIB (Cubyrt et al. 2010), NACRE (Angulo et al. 1999), and additional tabulated weak reaction rates (Fuller et al. 1985; Oda et al. 1994; Langanke & Martínez-Pinedo 2000). Screening is included via the prescription of Chugunov et al. (2007). Thermal neutrino loss rates are from Itoh et al. (1996).

Models were constructed on the pre–main sequence with \( Z = 0.014, Y = 0.24 + 2Z \), and \( X = 1 \)–\( Y \)–\( Z \) and evolved from there. We neglect rotation and associated chemical mixing.

Appendix C

Data Availability

The inlists and run scripts used in producing the H–R diagrams in this work are available in this other GitHub repository\(^8\) on the atlas branch in the commit with short-sha 701d74d9. The data those scripts produced are available at doi: 10.5281/zenodo.6533097, along with the plotting scripts used in this work.

Appendix D

Rotation Law

Throughout this work we choose a typical \( \Omega \) corresponding to a surface velocity of 150 km s\(^{-1}\) for O–F stars (down to 1.2 \( M_\odot \)). From 1.2 to 0.8 \( M_\odot \), we vary the surface velocity linearly down to 3 km s\(^{-1}\) and treat it as constant below this point. The resulting angular velocities are shown

\( ^8 \)https://github.com/adamjermyn/conv_trends
in Figure 2. This simple form approximately reproduces the inferred equatorial velocities from $v \sin i$ measurements by Glebocki & Gnacinski (2005), the typical equatorial velocities reported by Ramírez-Agudelo et al. (2013), and the rotation period inferred from Kepler observations by (Nielsen et al. 2013) for B–M stars. Binary interactions can, of course, change the relevant velocities (de Mink et al. 2013), although in general we expect the dominant variation in rotation-related quantities to be due to variation in the properties of convection and not variation in the rotation periods.

**Figure 2.** The angular velocity $\Omega$ is shown in terms of log $T_{\text{eff}}$/spectral type and log $L$ for stellar models with Milky Way metallicity $Z = 0.014$. Note that for Sun-like stars we predict angular velocities of approximately $10^{-4}$ rad s$^{-1}$, which is ~50 times faster than the current solar rotation rate.

### Appendix E

**Averaging Sensitivity**

We had many choices in how to average different quantities, so it is worth performing a sensitivity analysis. Figure 3 shows the Prandtl number computed both by averaging $\nu$ and $\chi$ independently and by averaging $\nu/\chi$ directly. This was chosen as a particularly simple quantity, and we show it for each CZ. For all but the Deep Envelope CZs, the differences are small ($\lesssim 2\times$), but for the Deep Envelope zones we see differences of up to $10^5$, motivating our choice to additionally show quantities near the inner and outer boundaries of Deep Envelope CZs.
Appendix F
Convection Zone H-R Diagrams

F.1. Deep Envelope CZ

Here we examine Deep Envelope CZs, which occur in low-mass stars ($M_\odot \lesssim 1.2 M_\odot$). Because these zones are strongly density stratified, we first examine averages over the entire CZ and then compare those with averages over just the innermost and outermost pressure scale heights.

F.1.1. Whole-zone Averages

We begin with the bulk structure of Deep Envelope CZs. Figure 4 shows the aspect ratio $A$, which ranges from 3 to 10.
The aspect ratio $A$ is shown in terms of Figure 4:

$\log_{10} L/\lambda_{\odot}$ vs $\log_{10} T_{\text{eff}}/K$

Figure 4. The aspect ratio $A$ is shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that the aspect ratio is an input parameter and does not depend on a specific theory of convection.

These small-to-moderate aspect ratios suggest that the global (spherical shell) geometry could be important.

Next, the density ratio $D$ (Figure 5, left) and Mach number $Ma$ (Figure 5, right) inform which physics the fluid equations must include to model these zones. Deep Envelope CZs are strongly density stratified, so the Boussinesq approximation is likely insufficient to study them. For the most part convection in these zones is also highly subsonic. This, along with the density ratio, suggests it is appropriate to use the anelastic approximation. However, near the surface $v_{\text{r}}$ becomes large and the fully compressible equations may be necessary (Section F.1.2).

The Rayleigh number $Ra$ (Figure 6, left) determines whether or not a putative CZ is actually unstable to convection, and the Reynolds number $Re$ determines how turbulent the zone is if instability sets in (Figure 6, right). In these zones both numbers are enormous, so we should expect convective instability to result in highly turbulent flows.

The optical depth across a CZ $\tau_{\text{CZ}}$ (Figure 7, left) indicates whether or not radiation can be handled in the diffusive approximation, while the optical depth from the outer boundary of the CZ to infinity $\tau_{\text{outer}}$ (Figure 7, right) indicates the nature of radiative transfer and cooling in the outer regions of the CZ. We see that the optical depth across these zones is enormous ($\tau_{\text{CZ}} \sim 10^3$) but their outer boundaries lie at very small optical depths ($\tau_{\text{outer}} \sim 1$). This means that the bulk of the CZ can be modeled in the limit of radiative diffusion, but the dynamics of the outer regions likely require radiation hydrodynamics.

The Eddington ratio $\Gamma_{\text{Edd}}$ (Figure 8, left) indicates whether or not radiation hydrodynamic instabilities are important in the nonconvecting state, and the radiative Eddington ratio $\Gamma_{\text{Edd}}^r$ (Figure 8, right) indicates the same in the developed convective state. Here we see that in the absence of convection Deep Envelope CZs would reach moderate $\Gamma_{\text{Edd}} \sim 0.3$, but because convection transports some of the flux this is reduced to $\lesssim 0.1$ and radiation hydrodynamic instabilities are unlikely to matter.

The Prandtl number $Pr$ (Figure 9, left) measures the relative importance of thermal diffusion and viscosity, and the magnetic Prandtl number $Pm$ (Figure 9, right) measures the same for magnetic diffusion and viscosity. We see that both are very small, so the thermal diffusion and magnetic diffusion length scales are much larger than the viscous scale.

The radiation pressure ratio $\beta_{\text{rad}}$ (Figure 10) measures the importance of radiation in setting the thermodynamic properties of the fluid. We see that this is uniformly small and so radiation pressure likely plays a subdominant role in these zones.

The Ekman number $Ek$ (Figure 11) indicates the relative importance of viscosity and rotation. This is tiny across the H-R diagram, so we expect rotation to dominate over viscosity, except at very small length scales.

The Rossby number $Ro$ (Figure 12, left) measures the relative importance of rotation and inertia. This is small, so these zones are strongly rotationally constrained for typical rotation rates (Nielsen et al. 2013), although becoming less so toward higher masses. This is also strongly depth dependent (Section F.1.2), and near the surface $v_{\text{r}}$ becomes large and flows are typically not rotationally constrained.

We have assumed a fiducial rotation law to calculate $Ro$. Stars exhibit a variety of different rotation rates, so we also show the convective turnover time $t_{\text{conv}}$ (Figure 12, right), which may be used to estimate the Rossby number for different rotation periods.

The Péclet number $Pe$ (Figure 13, left) measures the relative importance of advection and diffusion in transporting heat, and the flux ratio $F_{\text{conv}}/F$ (Figure 13, right) reports the fraction of the energy flux that is advected. The former is large, and the latter is near unity, so we conclude that convection in these zones is highly efficient and heat transport is dominated by advection.

Figure 14 shows that the base of the envelope CZ is always extremely stiff, with $S \sim 10^{4-5}$. We expect very little mechanical overshooting as a result, although there could still be convective penetration (Anders et al. 2022).

F.1.2. Inner and Outer Scale Heights

Because Deep Envelope CZs are strongly stratified their properties vary tremendously with depth. We now examine this variation by considering some of the same quantities averaged just over the innermost or outermost pressure scale heights. For this section only we evaluate $Ek$, $Re$, $Pe$, and $Ro$ using the pressure scale height at the relevant boundary rather than the full size of the CZ $\delta r$. We do this to focus on the dynamics of motions near the boundary.

Figure 15 shows the Mach number $Ma$. The Mach number shows important differences, being much larger in the outermost pressure scale (left) than the innermost one (right), and reaches 0.3 at the outer boundaries of more massive stars ($M \approx 1.1 M_\odot$). So, while the innermost regions of these zones are well modeled by the anelastic approximation, the near-surface regions likely require using the fully compressible equations.

Figure 16 shows the Reynolds number $Re$. The Reynolds number shows a larger range of values near the outer boundary than near the inner one but centered on similar typical values of $10^{12}$. This indicates well-developed turbulence near both boundaries.

Figure 17 shows the Prandtl number $Pr$ (upper) and magnetic Prandtl number $Pm$ (lower). Both are much smaller near the outer boundary of the CZ (left) than the inner boundary (right), although the difference is much more stark ($\sim 10^2-10^3$) for the Prandtl number than for the magnetic Prandtl number ($\sim 10^3-10^4$). Both remain small near both boundaries, however, so the qualitative features of convection that they reflect are unchanged through the zone.

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Figure 5. The density ratio $D$ (left) and Mach number $Ma$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that while the density ratio is an input parameter and does not depend on a specific theory of convection, the Mach number is an output of such a theory.

Figure 6. The Rayleigh number $Ra$ (left) and Reynolds number $Re$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that while the Rayleigh number is an input parameter and does not depend on a specific theory of convection, the Reynolds number is an output of such a theory.

Figure 7. The convection optical depth $\tau_{\text{CZ}}$ (left) and the optical depth to the surface $\tau_{\text{outer}}$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that both of these are input parameters and do not depend on a specific theory of convection.
Figure 8. The Eddington ratios with the full luminosity $\Gamma_{\text{Edd}}$ (left) and the radiative luminosity (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that while $\Gamma_{\text{Edd}}$ is an input parameter and does not depend on a specific theory of convection, $\Gamma_{\text{Edd}}^{\text{rad}}$ is an output of such a theory.

Figure 9. The Prandtl number Pr (left) and magnetic Prandtl number Pm (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that both Pr and Pm are input parameters and so do not depend on a specific theory of convection.

Figure 10. The radiation pressure ratio $\beta_{\text{rad}}$ is shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that $\beta_{\text{rad}}$ does not depend on the steady-state convective velocity or temperature gradient, so it is not sensitive to the choice of convection theory.

Figure 11. The Ekman number Ek is shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that the Ekman number is an input parameter and does not depend on a specific theory of convection.
Figure 12. The Rossby number Ro (left) and turnover time $t_{\text{conv}}$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that both Ro and $t_{\text{conv}}$ are outputs of a theory of convection.

Figure 13. The Péclet number $Pe$ (left) and $F_{\text{conv}}/F$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that both $Pe$ and $F_{\text{conv}}/F$ are outputs of a theory of convection.

Figure 14. The stiffness of the inner convective boundary is shown in terms of $\log T_{\text{eff}}$ and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that the stiffness is an output of a theory of convection.

Figure 18 shows the Eddington ratios $\Gamma_{\text{Edd}}$ (upper) and $\Gamma_{\text{Edd}}^{\text{rad}}$ (lower). These are both small at both boundaries, so there is no significant difference in the regime of convection between the two boundaries.

Figure 19 shows the radiation pressure ratio $\beta_{\text{rad}}$. The radiation pressure ratio is smaller by a factor of 10–100 near the outer boundary of the CZ (left) than the inner boundary (right), although it remains less than 1% in both cases, so radiation pressure may be safely ignored in these CZs.

Figure 20 shows the Rossby number $Ro$ (upper) and Ekman number $Ek$ (lower). The Rossby number is somewhat larger near the outer boundary than near the inner one. Importantly, the Rossby number near the outer boundary is larger than unity, meaning that the flows are not rotationally constrained. This is in contrast to both the average and inner boundary values, which have $Ro \ll 1$ and indicate strong rotational constraints. This suggests that rotation is quite important for the bulk of these zones but can be safely neglected in their outer regions.

The Ekman number, by contrast, is similar between the two boundaries. Although the Ekman number is much smaller at the
inner boundary than the outer one, it is tiny in both regions. Hence throughout the zone we expect rotation to dominate over viscosity, except at very small length scales.

Figure 21 shows the Péclet number $\text{Pe}$ (upper) and $F_{\text{conv}}/F$ (lower). The Péclet number is quite a bit smaller near the outer boundary than near the inner one, by factors of $10^2$–$10^4$. Both quantities lie in the same qualitative regimes near both boundaries: the Péclet number indicates that advection dominates diffusion in the heat equation, and the flux ratio indicates that convection carries a substantial fraction of the flux. The flux ratio is similar near the inner and outer boundaries of the CZ, although it takes on a wider range of values near the outer boundary than the inner one. Near both boundaries, however, it is smaller than in the bulk of the zone, which matches the intuition that convection ought to be more efficient in the bulk than near the boundaries. The difference between the boundaries and bulk is as large as it is because the ratio $\nabla_{\text{rad}}/\nabla_{\text{ad}}$ varies gradually near both boundaries.

$F.2. \ \text{H} \, \text{I CZ}$

We now examine the bulk structure of H I CZs, which occur in the subsurface layers of stars with masses $1.2 \ M_\odot \lesssim M_\ast \lesssim 3 \ M_\odot$. Unlike in the case of Deep Envelope CZs, the H I CZs show large enough variation in all studied parameters to encompass many different regimes. In this section the boundary between “low” and “high” masses is $\sim 1.5 \ M_\odot$.

Note that in some regions of the H-R diagram this CZ has a Rayleigh number below the $\sim 10^5$ critical value (Chandrasekhar 1961). As a result, while the region is superadiabatic, it is not unstable to convection. We therefore neglect these stable regions in our analysis and shade them in gray in our figures.

Figure 22 shows the aspect ratio $A$, which ranges from 10 to 3000. These large aspect ratios suggest that local simulations are likely sufficient to capture their dynamics.

Next, the density ratio $D$ (Figure 23, left) and Mach number $\text{Ma}$ (Figure 23, right) inform which physics the fluid equations must include to model these zones. At low masses the density
ratio is large, while at higher masses it is near unity. Likewise, at low masses the Mach number is moderate (\(\sim 0.3\)), while at high masses it is small (\(10^{-5}\)). This, along with the density ratio, suggests it is appropriate to use the Boussinesq approximation at high masses, while the fully compressible equations are necessary at low masses.

The Rayleigh number \(R_a\) (Figure 24, left) determines whether or not a putative CZ is actually unstable to convection, and the Reynolds number \(Re\) determines how turbulent the zone is if instability sets in (Figure 24, right). At low masses the Rayleigh number is large (\(10^{25}\)), and at high masses it plummets and eventually becomes subcritical, which we show in gray. Likewise at low masses the Reynolds number is large (\(10^{13}\)), while at high masses it is quite small (\(\sim 1\)). These putative CZZs then span a wide range of properties, from being subcritical and stable (Chandrasekhar 1961) at high masses, to being marginally unstable and weakly turbulent at intermediate masses (\(\sim 2 M_\odot\)), to eventually being strongly unstable and having well-developed turbulence at low masses.

The optical depth across a CZ \(\tau_{\text{CZ}}\) (Figure 25, left) indicates whether or not radiation can be handled in the diffusive approximation, while the optical depth from the outer boundary to infinity \(\tau_{\text{outer}}\) (Figure 25, right) indicates the nature of radiative transfer and cooling in the outer regions of the CZ. The surface of the H I CZ is always at low optical depth (\(\tau_{\text{outer}} \sim 1\)), meaning that radiation hydrodynamics is likely necessary near the outer boundary of this zone. By contrast, the optical depth across the H I CZ is low at high masses (\(\sim 1\)–\(10\)) and large at low masses (\(10^{3+}\)). This implies that radiation hydrodynamics is necessary to model the bulk of the H I CZ at high masses but not at low masses.

The Eddington ratio \(\Gamma_{\text{Edd}}\) (Figure 26, left) indicates whether or not radiation hydrodynamic instabilities are important in the nonconvecting state, and the radiative Eddington ratio \(\Gamma_{\text{Edd}}^{\text{rad}}\) (Figure 26, right) indicates the same in the developed convective state. Both ratios are small in the H I CZ, so radiation hydrodynamic instabilities are unlikely to matter.

The Prandtl number \(Pr\) (Figure 27, left) measures the relative importance of thermal diffusion and viscosity, and the magnetic Prandtl number \(Pm\) (Figure 27, right) measures the same for magnetic diffusion and viscosity. The Prandtl number is always small in these models, so the thermal diffusion length scale is much larger than the viscous scale. By contrast, the magnetic
Figure 18. The Eddington ratio with the full luminosity $\Gamma_{\text{Edd}}$ (upper) and the radiative luminosity (lower) are shown averaged over the outermost pressure scale height (left) and innermost one (right) in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that while $\Gamma_{\text{Edd}}$ is an input parameter and does not depend on a specific theory of convection, $\Gamma_{\text{Edd}}^{\text{rad}}$ is an output of such a theory.

Figure 19. The radiation pressure ratio $\beta_{\text{rad}}$ is shown averaged over the outermost pressure scale height (left) and innermost one (right) in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$.
Prandtl number varies from small at low masses to large at high masses. The fact that $Pm$ is large at high masses is notable because the quasistatic approximation for MHD has frequently been used to study magnetoconvection in minimal 3D MHD simulations of planetary and stellar interiors (e.g., Yan et al. 2019) and assumes that $\mathbf{R} = \mathbf{P} m R \mathbf{m} e \rightarrow \mathbf{0}$; in doing so, this approximation assumes a global background magnetic field is dominant and neglects the nonlinear portion of the Lorentz force. This approximation breaks down in CZs with $Pm > 1$, and future numerical experiments should seek to understand how magnetoconvection operates in this regime.

The radiation pressure ratio $\beta_{\text{rad}}$ (Figure 28) measures the importance of radiation in setting the thermodynamic properties of the fluid. We see that this is uniformly small ($\sim 0.1$), so radiation pressure likely plays a subdominant role in these zones.

The Ekman number $\mathbf{E}k$ (Figure 29) indicates the relative importance of viscosity and rotation. This is tiny across the H-R diagram, so we expect rotation to dominate over viscosity, except at very small length scales.

The Rossby number $\mathbf{R}o$ (Figure 30, left) measures the relative importance of rotation and inertia. This is small at high masses but greater than unity at low masses, meaning that the H1 CZ is rotationally constrained at high masses but not at low masses for typical rotation rates (Nielsen et al. 2013).

We have assumed a fiducial rotation law to calculate $\mathbf{R}o$. Stars exhibit a variety of different rotation rates, so we also show the convective turnover time $t_{\text{conv}}$ (Figure 30, right), which may be used to estimate the Rossby number for different rotation periods.

The Péclet number $\mathbf{P}e$ (Figure 31, left) measures the relative importance of advection and diffusion in transporting heat, and the flux ratio $F_{\text{conv}} / F$ (Figure 31, right) reports the fraction of the energy flux that is advected. Both exhibit substantial

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Figure 20. The Rossby number $\mathbf{R}o$ (upper) is shown averaged over the outermost pressure scale height (left) and innermost one (right) in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with Deep Envelope CZs and Milky Way metallicity $Z = 0.014$. Note that both numbers were calculated here using the pressure scale height at the relevant boundary rather than the full width of the CZ δr.

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Note that, because the Prandtl number is also very small, this does not significantly alter the critical Rayleigh number (see chapter 3 of Chandrasekhar 1961 and Appendix D of Jermyn et al. 2022a).
variation with mass. The Péclet number varies from large ($10^4$) at low masses to very small at high masses ($10^{-9}$), and the flux ratio similarly varies from near unity at low masses to tiny ($10^{-16}$) at high masses. That is, there is a large gradient in convective efficiency with mass, with efficient convection at low masses and very inefficient convection at high masses.

Of particular interest at intermediate masses ($1.5 \, M_\odot \lesssim M \lesssim 1.7 \, M_\odot$) are stars for which the Reynolds number is still large ($R_e > 10^4$) but the Péclet number is small ($Pe < 1$). In these stars the H I CZ should exhibit turbulent velocity fields but very laminar thermodynamic fields, which could be quite interesting to study numerically.

We further note that at the low-mass end the Mach number is moderate ($\sim 0.3$) high and the $F_{\text{conv}}/F$ is near unity, so convection likely produces a significant luminosity in IGWs (Goldreich & Kumar 1990; Lecoanet & Quataert 2013).

Finally, Figure 32 shows the stiffness of both the inner and outer boundaries of the H I CZ. Both range from very stiff ($S \sim 10^{4-8}$) to very weak ($S \sim 1$), with decreasing stiffness toward decreasing mass. For instance, for masses $M \gtrsim 1.5 \, M_\odot$ we do not expect much mechanical overshooting, whereas for...
Figure 23. The density ratio $D$ (left) and Mach number $Ma$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with H I CZs and Milky Way metallicity $Z = 0.014$. Note that while the density ratio is an input parameter and does not depend on a specific theory of convection, the Mach number is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 24. The Rayleigh number $Ra$ (left) and Reynolds number $Re$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with H I CZs and Milky Way metallicity $Z = 0.014$. Note that while the Rayleigh number is an input parameter and does not depend on a specific theory of convection, the Reynolds number is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 25. The convection optical depth $\tau_{\text{CZ}}$ (left) and the optical depth to the surface $\tau_{\text{outer}}$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with H I CZs and Milky Way metallicity $Z = 0.014$. Note that both of these are input parameters and do not depend on a specific theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.
Figure 26. The Eddington ratios with the full luminosity $\Gamma_{\text{Edd}}$ (left) and the radiative luminosity (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with H I CZs and Milky Way metallicity $Z = 0.014$. Note that while $\Gamma_{\text{Edd}}$ is an input parameter and does not depend on a specific theory of convection, $\Gamma_{\text{Edd}}^{\text{rad}}$ is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 27. The Prandtl number $Pr$ (left) and magnetic Prandtl number $Pm$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with H I CZs and Milky Way metallicity $Z = 0.014$. Note that both $Pr$ and $Pm$ are input parameters and so do not depend on a specific theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 28. The radiation pressure ratio $\beta_{\text{rad}}$ is shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with H I CZs and Milky Way metallicity $Z = 0.014$. Note that this ratio is an input parameter and does not depend on a specific theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 29. The Ekman number $Ek$ is shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with H I CZs and Milky Way metallicity $Z = 0.014$. Note that the Ekman number is an input parameter and does not depend on a specific theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.
$M \lesssim 1.5\ M_\odot$ both boundaries should show substantial overshooting, because their low stiffness causes convective flows to decelerate over large length scales.

Note that at the inner boundary the stiffness shows sharp changes along evolutionary tracks. This is because the emergence of the He I CZ just below the H I CZ makes the typical radiative $N^2$ near the lower boundary of the H I CZ much smaller, thereby reducing the stiffness.

### F.3. He I CZ

We now examine the bulk structure of He I CZs, which occur in the subsurface layers of stars with masses $2\ M_\odot \lesssim M_\ast \lesssim 5\ M_\odot$. Note that in some regions of the H-R diagram this CZ has a Rayleigh number below the $\sim 10^3$ critical value (Chandrasekhar 1961). As a result, while the region is superadiabatic, it is not unstable to convection. We therefore neglect these stable regions in our analysis and shade them in gray in our figures.

Figure 33 shows the aspect ratio $A$, which is of order $10^3$. These large aspect ratios suggest that local simulations spanning the full depth of the CZ and only a fraction of $4\pi$ angularly can capture the convective dynamics.

Next, the density ratio $D$ (Figure 34, left) and Mach number $Ma$ (Figure 34, right) inform which physics the fluid equations must include to model these zones. The density ratio is always of order unity, and the Mach number is always small ($\approx 10^{-4}$). This suggests it is always appropriate to use the Boussinesq approximation in modeling these CZs.

The Rayleigh number $Ra$ (Figure 35, left) determines whether or not a putative CZ is actually unstable to convection, and the Reynolds number $Re$ determines how turbulent the zone is if convection sets in (Figure 35, right). At low masses the Rayleigh number is slightly supercritical ($10^4$–$10^5$), and at high masses it plummets and eventually becomes subcritical, which we show in gray. Likewise at low masses the Reynolds number is around the threshold for turbulence to develop ($\sim 10^5$), while at high masses it is quite small ($\sim 1$). These putative CZs then span a wide range of properties, from being subcritical and stable at high masses, to being weakly unstable and weakly turbulent at low masses ($\sim 2\ M_\odot$).
The optical depth across a CZ $\tau_{\text{CZ}}$ (Figure 36, left) indicates whether or not radiation can be handled in the diffusive approximation, while the optical depth from the outer boundary to infinity $\tau_{\text{outer}}$ (Figure 36, right) indicates the nature of radiative transfer and cooling in the outer regions of the CZ. The surface of the HeI CZ is always at moderate low optical depth in the unstable region ($\tau_{\text{outer}} \sim 10$), and the optical depth across the HeI CZ is of the same order. This means that both the bulk and outer boundary of the HeI CZ can be treated within the diffusive approximation for radiation.

The Eddington ratio $\Gamma_{\text{Edd}}$ (Figure 37, left) indicates whether or not radiation hydrodynamic instabilities are important in the nonconvecting state, and the radiative Eddington ratio $\Gamma^{\text{rad}}_{\text{Edd}}$ (Figure 37, right) indicates the same in the developed convective state. Both ratios are small in the HeI CZ, so radiation hydrodynamic instabilities are unlikely to matter.

The Prandtl number $\text{Pr}$ (Figure 38, left) measures the relative importance of thermal diffusion and viscosity, and the magnetic Prandtl number $\text{Pm}$ (Figure 38, right) measures the same for magnetic diffusion and viscosity. The Prandtl number is always small in these models, so the thermal diffusion length scale is much larger than the viscous scale. By contrast, the magnetic Prandtl number is always greater than unity and reaches nearly $10^4$ in the unstable regions.

The fact that $\text{Pm}$ is large is notable because the quasistatic approximation for MHD has frequently been used to study magnetoconvection in minimal 3D MHD simulations of planetary and stellar interiors (e.g., Yan et al. 2019) and assumes that $\text{Rm} = \text{PmRe} \rightarrow 0$; in doing so, this approximation assumes a global background magnetic field is dominant and neglects the nonlinear portion of the Lorentz force. This approximation breaks down in CZs with $\text{Pm} > 1$, and future numerical experiments should seek to understand how magnetoconvection operates in this regime.

The radiation pressure ratio $\beta_{\text{rad}}$ (Figure 39) measures the importance of radiation in setting the thermodynamic properties of the fluid. We see that this is uniformly small ($\lesssim 0.1$), so radiation pressure likely plays a subdominant role in these zones.

The Ekman number $\text{Ek}$ (Figure 40) indicates the relative importance of viscosity and rotation. This is tiny across the H-R diagram, so we expect rotation to dominate over viscosity, except at very small length scales.

The Rossby number $\text{Ro}$ (Figure 41, left) measures the relative importance of rotation and inertia. This is uniformly small, meaning that the HeI CZ is rotationally constrained for typical rotation rates (Nielsen et al. 2013).

We have assumed a fiducial rotation law to calculate $\text{Ro}$. Stars exhibit a variety of different rotation rates, so we also show the convective turnover time $t_{\text{conv}}$ (Figure 41, right), which may be used to estimate the Rossby number for different rotation periods.

The Péclet number $\text{Pe}$ (Figure 42, left) measures the relative importance of advection and diffusion in transporting heat, and the flux ratio $F_{\text{conv}}/F$ (Figure 42, right) reports the fraction of...
Figure 34. The density ratio \( D \) (left) and Mach number \( M_a \) (right) are shown in terms of \( \log T_{\text{eff}}/\text{spectral type} \) and \( \log L \) for stellar models with He I CZs and Milky Way metallicity \( Z = 0.014 \). Note that while the density ratio is an input parameter and does not depend on a specific theory of convection, the Mach number is an output of such a theory. Regions with \( \text{Ra} < \text{Ra}_{\text{crit}} \) are stable to convection and shaded in gray.

Figure 35. The Rayleigh number \( \text{Ra} \) (left) and Reynolds number \( \text{Re} \) (right) are shown in terms of \( \log T_{\text{eff}}/\text{spectral type} \) and \( \log L \) for stellar models with He I CZs and Milky Way metallicity \( Z = 0.014 \). Note that while the Rayleigh number is an input parameter and does not depend on a specific theory of convection, the Reynolds number is an output of such a theory. Regions with \( \text{Ra} < \text{Ra}_{\text{crit}} \) are stable to convection and shaded in gray.

Figure 36. The convection optical depth \( \tau_{\text{CZ}} \) (left) and the optical depth to the surface \( \tau_{\text{outer}} \) (right) are shown in terms of \( \log T_{\text{eff}}/\text{spectral type} \) and \( \log L \) for stellar models with He I CZs and Milky Way metallicity \( Z = 0.014 \). Note that both of these are input parameters and do not depend on a specific theory of convection. Regions with \( \text{Ra} < \text{Ra}_{\text{crit}} \) are stable to convection and shaded in gray.
the energy flux that is advected. Both are extremely small, meaning that these CZs are very inefficient.

Finally, Figure 43 shows the stiffness of both the inner and outer boundaries of the He I CZ. Both are very stiff at all masses ($S \sim 10^5 - 10^6$), so we do not expect much mechanical overshooting, although there could still be convective penetration (Anders et al. 2022).

**F.4. He II CZ**

We now examine the bulk structure of He II CZs, which occur in the subsurface layers of stars with masses $1.3 M_\odot \lesssim M_* \lesssim 60 M_\odot$. Note that in some regions of the H-R diagram this CZ has a Rayleigh number below the $\sim 10^3$ critical value (Chandrasekhar 1961). As a result, while the region is superadiabatic, it is not unstable to convection. We therefore neglect these stable regions in our analysis and shade them in gray.

Figure 44 shows the aspect ratio $A$, which ranges from $10^2$ to $10^3$. These large aspect ratios suggest that local simulations are likely sufficient to capture their dynamics.
The Ekman number $E_k$ is shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with He I CZs and Milky Way metallicity $Z = 0.014$. Note that the Ekman number is an input parameter and does not depend on a specific theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

The Rossby number $Ro$ (left) and turnover time $t_{\text{conv}}$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with He I CZs and Milky Way metallicity $Z = 0.014$. Note that both $Ro$ and $t_{\text{conv}}$ are outputs of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray. Note that the turnover time exhibits numerical noise related to the model mesh resolution because the integrand $1/\nu_c$ diverges toward the convective boundaries.

The Péclet number $Pe$ (left) and $F_{\text{conv}}/F$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with He I CZs and Milky Way metallicity $Z = 0.014$. Note that both $Pe$ and $F_{\text{conv}}/F$ are outputs of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.
Figure 43. The stiffnesses of the inner (left) and outer (right) convective boundaries are shown in terms of $\log T_{\text{eff}}$ and $\log L$ for stellar models with He I CZs and Milky Way metallicity $Z = 0.014$. Note that the stiffness is an output of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 44. The aspect ratio $A$ is shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with He II CZs and Milky Way metallicity $Z = 0.014$. Note that the aspect ratio is an input parameter and does not depend on a specific theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 45. The density ratio $D$ (left) and Mach number $Ma$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with He II CZs and Milky Way metallicity $Z = 0.014$. Note that while the density ratio is an input parameter and does not depend on a specific theory of convection, the Mach number is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.
Figure 46. The Rayleigh number $Ra$ (left) and Reynolds number $Re$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with He II CZs and Milky Way metallicity $Z = 0.014$. Note that while the Rayleigh number is an input parameter and does not depend on a specific theory of convection, the Reynolds number is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 47. The convection optical depth $\tau_{\text{CZ}}$ (left) and the optical depth to the surface $\tau_{\text{outer}}$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with He II CZs and Milky Way metallicity $Z = 0.014$. Note that both of these are input parameters and do not depend on a specific theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 48. The Eddington ratios with the full luminosity $\Gamma_{\text{Edd}}$ (left) and the radiative luminosity (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with He II CZs and Milky Way metallicity $Z = 0.014$. Note that while $\Gamma_{\text{Edd}}$ is an input parameter and does not depend on a specific theory of convection, $\Gamma_{\text{Edd}}^{\text{rad}}$ is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.
Next, the density ratio $D$ (Figure 45, left) and Mach number $Ma$ (Figure 45, right) inform which physics the fluid equations must include to model these zones. The density ratio is typically small, of order $2 – 3$, and the Mach number ranges from $\sim 0.1$ at $M \approx 2 M_\odot$ down to $10^{-4}$ at $M \approx 9 M_\odot$. This suggests that above $\approx 2 M_\odot$ the Boussinesq approximation is valid, whereas below this the fully compressible equations may be needed to capture the dynamics at moderate Mach numbers.

The Rayleigh number $Ra$ (Figure 46, left) determines whether or not a putative CZ is actually unstable to convection, and the Reynolds number $Re$ determines how turbulent the zone is if instability sets in (Figure 46, right). At low masses the Rayleigh number is large ($10^{10}$), and at high masses it plummets and eventually becomes subcritical, which we show in gray. Likewise, at low masses the Reynolds number is large ($10^9$), while at high masses it is quite small ($\sim 10^2$). These putative CZs then span a wide range of properties, from being subcritical and stable (Chandrasekhar 1961) at high masses, to being marginally unstable and weakly turbulent at intermediate masses ($\sim 9 M_\odot$), to eventually being strongly unstable and having well-developed turbulence at low masses ($\sim 2 M_\odot$).

The optical depth across a CZ $\tau_{CZ}$ (Figure 47, left) indicates whether or not radiation can be handled in the diffusive approximation, while the optical depth from the outer boundary to infinity $\tau_{outer}$ (Figure 47, right) indicates the nature of radiative transfer and cooling in the outer regions of the CZ. At high masses ($M \gtrsim 9 M_\odot$) the surface of the He II CZ is at low optical depth ($\tau_{outer} \sim 1 – 3$), while at lower masses the optical depth quickly becomes large. Similarly, the optical depth across the He II CZ is moderate at high masses ($\sim 10$) and becomes large toward lower masses. Overall, then, the bulk of the He II CZ can likely be treated in the diffusive approximation, as can the outer boundary for $M \lesssim 9 M_\odot$, while the outer boundary at higher masses likely requires a treatment with radiation hydrodynamics.
Figure 52. The Rossby number $\text{Ro}$ (left) and turnover time $t_{\text{conv}}$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with He II CZs and Milky Way metallicity $Z = 0.014$. Note that both $\text{Ro}$ and $t_{\text{conv}}$ are outputs of a theory of convection. Regions with $\text{Ra} < \text{Ra}_{\text{crit}}$ are stable to convection and shaded in gray. Note that the turnover time exhibits numerical noise related to the model mesh resolution because the integrand $1/v_c$ diverges toward the convective boundaries.

Figure 53. The Péclet number $\text{Pe}$ (left) and $F_{\text{conv}}/F$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with He II CZs and Milky Way metallicity $Z = 0.014$. Note that both $\text{Pe}$ and $F_{\text{conv}}/F$ are outputs of a theory of convection. Regions with $\text{Ra} < \text{Ra}_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 54. The stiffnesses of the inner (left) and outer (right) convective boundaries are shown in terms of $\log T_{\text{eff}}$ and $\log L$ for stellar models with He II CZs and Milky Way metallicity $Z = 0.014$. Note that the stiffness is an output of a theory of convection. Regions with $\text{Ra} < \text{Ra}_{\text{crit}}$ are stable to convection and shaded in gray.
Figure 55. The aspect ratio $A$ is shown in terms of $\log T_{\text{eff}}/\text{spectral type}$ and $\log L$ for stellar models with FeCZs and Milky Way metallicity $Z = 0.014$. Note that the aspect ratio is an input parameter and does not depend on a specific theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 56. The density ratio $D$ (left) and Mach number $Ma$ (right) are shown in terms of $\log T_{\text{eff}}/\text{spectral type}$ and $\log L$ for stellar models with FeCZs and Milky Way metallicity $Z = 0.014$. Note that while the density ratio is an input parameter and does not depend on a specific theory of convection, the Mach number is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 57. The Mach number computed with the isothermal sound speed is shown in terms of $\log T_{\text{eff}}$ and $\log L$ for stellar models with FeCZs and Milky Way metallicity $Z = 0.014$. 

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Figure 58. The Rayleigh number $Ra$ (left) and Reynolds number $Re$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with FeCZs and Milky Way metallicity $Z = 0.014$. Note that while the Rayleigh number is an input parameter and does not depend on a specific theory of convection, the Reynolds number is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 59. The convection optical depth $\tau_{\text{CZ}}$ (left) and the optical depth to the surface $\tau_{\text{outer}}$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with FeCZs and Milky Way metallicity $Z = 0.014$. Note that both of these are input parameters and do not depend on a specific theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 60. The Eddington ratios with the full luminosity $\Gamma_{\text{Edd}}$ (left) and the radiative luminosity (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with FeCZs and Milky Way metallicity $Z = 0.014$. Note that while $\Gamma_{\text{Edd}}$ is an input parameter and does not depend on a specific theory of convection, $\Gamma_{\text{Edd}}^{\text{rad}}$ is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.
The Eddington ratio $\Gamma_{\text{Edd}}$ (Figure 48, left) indicates whether or not radiation hydrodynamic instabilities are important in the nonconvecting state, and the radiative Eddington ratio $\Gamma_{\text{Edd rad}}$ (Figure 48, right) indicates the same in the developed convective state. Both ratios are moderate at high masses ($\Gamma \sim 0.3$ at $M \sim 10 M_\odot$), and radiation hydrodynamic instabilities could be important in this regime. By contrast, at lower masses ($M \lesssim 6 M_\odot$) these ratios are both small, and radiation hydrodynamic instabilities are unlikely to matter.

The Prandtl number $Pr$ (Figure 49, left) measures the relative importance of thermal diffusion and viscosity, and the magnetic Prandtl number $Pm$ (Figure 49, right) measures the same for magnetic diffusion and viscosity. The Prandtl number is always small in these models, so the thermal diffusion length scale is much larger than the viscous scale. By contrast, the magnetic Prandtl number varies from order unity at low masses to large ($10^4$) at high masses.

The fact that $Pm$ is large at high masses is notable because the quasistatic approximation for MHD has frequently been used to study magnetoconvection in minimal 3D MHD simulations of planetary and stellar interiors (e.g., Yan et al. 2019) and assumes that $Rm = PmRe \rightarrow 0$; in doing so, this approximation assumes a global background magnetic field is dominant and neglects the nonlinear portion of the Lorentz force. This approximation breaks down in CZs with $Pm > 1$, and future numerical experiments should seek to understand how magnetoconvection operates in this regime.

The radiation pressure ratio $\beta_{\text{rad}}$ (Figure 50) measures the importance of radiation in setting the thermodynamic properties of the fluid. We see that this is uniformly small ($\lesssim 0.1$), so radiation pressure likely plays a subdominant role in these zones.

The Ekman number $Ek$ (Figure 51) indicates the relative importance of viscosity and rotation. This is tiny across the H-R diagram, so we expect rotation to dominate over viscosity, except at very small length scales.

Note that, because the Prandtl number is also very small, this does not significantly alter the critical Rayleigh number (see chapter 3 of Chandrasekhar 1961 and Appendix D of Jermyn et al. 2022a).
Figure 64. The Rossby number $\text{Ro}$ (left) and turnover time $t_{\text{conv}}$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with FeCZs and Milky Way metallicity $Z = 0.014$. Note that both $\text{Ro}$ and $t_{\text{conv}}$ are outputs of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 65. The Péclet number $Pe$ (left) and $F_{\text{conv}}/F$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with FeCZs and Milky Way metallicity $Z = 0.014$. Note that both $Pe$ and $F_{\text{conv}}/F$ are outputs of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 66. The stiffnesses of the inner (left) and outer (right) convective boundaries are shown in terms of $\log T_{\text{eff}}$ and $\log L$ for stellar models with FeCZs and Milky Way metallicity $Z = 0.014$. Note that the stiffness is an output of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.
Figure 67. The density ratio $D$ (left) and Mach number $Ma$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with core CZs and Milky Way metallicity $Z = 0.014$. Note that while the density ratio is an input parameter and does not depend on a specific theory of convection, the Mach number is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 68. The Rayleigh number $Ra$ (left) and Reynolds number $Re$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with core CZs and Milky Way metallicity $Z = 0.014$. Note that while the Rayleigh number is an input parameter and does not depend on a specific theory of convection, the Reynolds number is an output of such a theory. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 69. The convection optical depth $\tau_{\text{CZ}}$ (left) and the optical depth to the surface $\tau_{\text{outer}}$ (right) are shown in terms of $\log T_{\text{eff}}$/spectral type and $\log L$ for stellar models with core CZs and Milky Way metallicity $Z = 0.014$. Note that both of these are input parameters and do not depend on a specific theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.
The Rossby number $Ro$ (Figure 52, left) measures the relative importance of rotation and inertia. This is small ($10^{-3}$) at high masses but greater than unity at low masses, meaning that the He II CZ is rotationally constrained at high masses but not at low masses for typical rotation rates (Nielsen et al. 2013). We have assumed a fiducial rotation law to calculate $Ro$.

Stars exhibit a variety of different rotation rates, so we also show the convective turnover time $t_{\text{conv}}$ (Figure 52, right), which may be used to estimate the Rossby number for different rotation periods.

The Péclet number $Pe$ (Figure 53, left) measures the relative importance of advection and diffusion in transporting heat, and the flux ratio $F_{\text{conv}}/F$ (Figure 53, right) reports the fraction of the energy flux that is advedted. Both exhibit substantial variation with mass. The Péclet number varies from order unity at low masses to very small at high masses ($10^{-5}$), and the flux ratio similarly varies from near unity at low masses to tiny ($10^{-14}$) at high masses. That is, there is a large gradient in convective efficiency with mass, with efficient convection at low masses and very inefficient convection at high masses.

Finally, Figure 54 shows the stiffness of both the inner and outer boundaries of the He II CZ. Both range from very stiff ($S \sim 10^4$) to very weak ($S \sim 1$), with decreasing stiffness toward decreasing mass. So, for instance, for masses $M \gtrsim 3 M_\odot$, we do not expect much mechanical overshooting, whereas for $M \lesssim 3 M_\odot$ both boundaries should show substantial overshooting, because their low stiffness causes convective flows to decelerate over large length scales.

**F.5. FeCZ**

We round out the ionization-driven CZs by turning to the FeCZs, which occur in the subsurface layers of solar metallicity stars with masses $M_\star \gtrsim 7 M_\odot$, and we first turn to our input parameters.

Figure 55 shows the aspect ratio $A$. The aspect ratios are typically large (10–1000) except for very massive ($M_\star \sim 60 M_\odot$) stars on the terminal age main sequence (TAMS$^{12}$), so local simulations are likely sufficient to capture their dynamics.

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12 The TAMS is defined by hydrogen exhaustion in the core.
At high masses on the TAMS the aspect ratio is high enough that global (spherical shell) geometry could be important.

Next, the density ratio D (Figure 56, left) and Mach number Ma (Figure 56, right) inform which physics the fluid equations must include to model these zones. The density ratio is typically small, of order 2–3, and the Mach number ranges from ≲0.1 at $M \lesssim 20 M_\odot$ up to 0.3 at $M \approx 50–60 M_\odot$. This suggests that below $\approx 20 M_\odot$ the Boussinesq approximation is valid, whereas above this the fully compressible equations may be needed to capture the dynamics at moderate Mach numbers.

In fact the Mach number in Figure 56 is an underestimate of the importance of density fluctuations because at high masses the zone is radiation pressure dominated ($\beta_{rad} \sim 1$) and has a moderate Péclet number ($Pe \sim 10$), so fluctuations occur isothermally and we should really be comparing the convection speed with the isothermal sound speed rather than the adiabatic one. Figure 57 shows this comparison, which reveals even larger Mach numbers ($Ma_{iso} \sim 1$) at high masses. Taking this into account, we suggest using the fully compressible equations down to $\approx 12 M_\odot$ to ensure that density fluctuations are correctly accounted for.

The Rayleigh number Ra (Figure 58, left) determines whether or not a putative CZ is actually unstable to convection, and the Reynolds number Re determines how turbulent the zone is if instability sets in (Figure 58, right). The Rayleigh number is generally large ($10^{13–10^{14}}$), as is the Reynolds number ($10^{3–10^{7}}$), suggesting that the FeCZ is strongly unstable to convection and exhibits well-developed turbulence.

The optical depth across a CZ $\tau_{\text{CZ}}$ (Figure 59, left) indicates whether or not radiation can be handled in the diffusive approximation, while the optical depth from the outer boundary to infinity $\tau_{\text{outer}}$ (Figure 59, right) indicates the nature of radiative transfer and cooling in the outer regions of the CZ. We see that the optical depth across these zones is $\tau_{\text{CZ}} \sim 10^8$, as is that from the outer boundary to infinity ($\tau_{\text{outer}} \gtrsim 10$). This suggests that radiation can be treated in the diffusive approximation.

Our only reservation with this conclusion is that both the Mach number and Eddington ratios can be large in the FeCZ, so density fluctuations can be large and can open up low-density optically thin “tunnels” through the FeCZ. This is what is seen in 3D radiation hydrodynamics simulations of the FeCZ (Schultz et al. 2020), which show strong correlations between the radiative flux and the attenuation length $(\kappa \rho)^{-1}$. Thus radiation hydrodynamics seems to be essential for modeling the FeCZ, at least at the higher masses ($M \gtrsim 15 M_\odot$), which host near-unity Eddington ratios and moderate Mach numbers.

The Eddington ratio $\Gamma_{\text{Edd}}$ (Figure 60, left) indicates whether or not radiation hydrodynamic instabilities are important in the nonconvecting state, and the radiative Eddington ratio $\Gamma_{\text{rad Edd}}$ (Figure 60, right) indicates the same in the developed convective state. Both ratios are moderate at low masses ($\Gamma \sim 0.3$ at $M \sim 10 M_\odot$) and reach unity at high masses ($M \gtrsim 25 M_\odot$), so radiation hydrodynamic instabilities are almost certainly important in the FeCZ.

The Prandtl number Pr (Figure 61, left) measures the relative importance of thermal diffusion and viscosity, and the magnetic Prandtl number Pm (Figure 61, right) measures the same for magnetic diffusion and viscosity. The Prandtl number is always small in these models, so the thermal diffusion length scale is much larger than the viscous scale. By contrast, the magnetic Prandtl number is very large ($10^2–10^3$), so the viscous scale is much larger than the magnetic diffusion scale.

The fact that Pm is large at high masses is notable because the quasistatic approximation for MHD has frequently been used to study magnetoconvection in minimal 3D MHD simulations of planetary and stellar interiors (e.g., Yan et al. 2019) and assumes that $Rm = PrM_\text{Re} \rightarrow 0$; in doing so, this approximation assumes a global background magnetic field is dominant and neglects the nonlinear portion of the Lorentz force. This approximation breaks down in CZs with Pm > 1, and future numerical experiments should seek to understand how magnetoconvection operates in this regime.

The radiation pressure ratio $\beta_{rad}$ (Figure 62) measures the importance of radiation in setting the thermodynamic properties of the fluid. This is a 30%–100% correction and so is very important to capture in modeling the FeCZ.

The Ekman number Ek (Figure 63) indicates the relative importance of viscosity and rotation. This is tiny across the...
H-R diagram, so we expect rotation to dominate over viscosity, except at very small length scales. The Rossby number $Ro$ (Figure 64, left) measures the relative importance of rotation and inertia. This is of order unity, with a gradient from moderately smaller ($\sim 0.1$) to moderately larger ($\sim 3$) running from low to high mass. We conclude then that for typical rotation rates (Nielsen et al. 2013) the FeCZ is rotationally constrained at low masses ($M \approx 12 M_\odot$), weakly so at intermediate masses ($12 M_\odot \lesssim M \lesssim 30 M_\odot$), and not constrained at high masses ($M \gtrsim 30 M_\odot$).

We have assumed a fiducial rotation law to calculate $Ro$. Stars exhibit a variety of different rotation rates, so we also show the convective turnover time $t_{\text{conv}}$ (Figure 64, right), which may be used to estimate the Rossby number for different rotation periods.

The Péclet number $Pe$ (Figure 65, left) and $F_{\text{conv}}/F$ (right) are shown in terms of $T_{\text{eff}}$/spectral type and $L$ for stellar models with core CZs and Milky Way metallicity $Z = 0.014$. Note that both $Pe$ and $F_{\text{conv}}/F$ are outputs of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

The stiffness of the outer convective boundary is shown in terms of $T_{\text{eff}}$ and $L$ for stellar models with core CZs and Milky Way metallicity $Z = 0.014$. Note that the stiffness is an output of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray. H-R diagram, so we expect rotation to dominate over viscosity, except at very small length scales.

The Rossby number $Ro$ (Figure 64, left) measures the relative importance of rotation and inertia. This is of order unity, with a gradient from moderately smaller ($\sim 0.1$) to moderately larger ($\sim 3$) running from low to high mass. We conclude then that for typical rotation rates (Nielsen et al. 2013) the FeCZ is rotationally constrained at low masses ($M \approx 12 M_\odot$), weakly so at intermediate masses ($12 M_\odot \lesssim M \lesssim 30 M_\odot$), and not constrained at high masses ($M \gtrsim 30 M_\odot$).

We have assumed a fiducial rotation law to calculate $Ro$. Stars exhibit a variety of different rotation rates, so we also show the convective turnover time $t_{\text{conv}}$ (Figure 64, right), which may be used to estimate the Rossby number for different rotation periods.

The Péclet number $Pe$ (Figure 65, left) and $F_{\text{conv}}/F$ (right) are shown in terms of $T_{\text{eff}}$/spectral type and $L$ for stellar models with core CZs and Milky Way metallicity $Z = 0.014$. Note that both $Pe$ and $F_{\text{conv}}/F$ are outputs of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Note that, because the Prandtl number is also very small, this does not significantly alter the critical Rayleigh numbers (see chapter 3 of Chandrasekhar 1961 and Appendix D of Jernyn et al. 2022a).

Figure 74. The Rossby number $Ro$ (left) and turnover time $t_{\text{conv}}$ (right) are shown in terms of $T_{\text{eff}}$/spectral type and $L$ for stellar models with core CZs and Milky Way metallicity $Z = 0.014$. Note that both $Ro$ and $t_{\text{conv}}$ are outputs of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 75. The Péclet number $Pe$ (left) and $F_{\text{conv}}/F$ (right) are shown in terms of $T_{\text{eff}}$/spectral type and $L$ for stellar models with core CZs and Milky Way metallicity $Z = 0.014$. Note that both $Pe$ and $F_{\text{conv}}/F$ are outputs of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.

Figure 76. The stiffness of the outer convective boundary is shown in terms of $T_{\text{eff}}$ and $L$ for stellar models with core CZs and Milky Way metallicity $Z = 0.014$. Note that the stiffness is an output of a theory of convection. Regions with $Ra < Ra_{\text{crit}}$ are stable to convection and shaded in gray.
the flux ratio $F_{\text{conv}}/F$ (Figure 65, right) reports the fraction of the energy flux that is advected. Both exhibit substantial variation with mass. The Pécellet number varies from order 30 at high masses on the TAMS to very small at low masses ($10^{-2}$), and the flux ratio similarly varies from $\sim 0.3$ at high masses on the TAMS to tiny ($10^{-6}$) at low masses. That is, there is a large gradient in convective efficiency with mass, with moderately efficient convection at high masses and very inefficient convection at low masses.

Finally, Figure 66 shows the stiffness of both the inner and outer boundaries of the FeCZ. The outer boundary is considerably more stiff ($\sim 10^{2-6}$) than the inner boundary ($\sim 10^{0-3}$). Over most of the mass range and much of the main sequence we expect significant overshooting past the relatively weak inner boundary. Naively, we would expect the opposite for the outer boundary, but there is numerical evidence of significant motion past the convective boundary (Schultz et al. 2020). We suspect this is due to the moderate-to-small Pécellet numbers in and near the FeCZ, which mean that motions approach isothermal and so see a reduced entropy gradient. This lowers the effective stiffness of the outer boundary.

### F.6. Core CZ

We now examine the bulk structure of core CZs, which occur in stars with masses $M_e \gtrsim 1.1 M_\odot$. The aspect ratios are unity, by definition, so the global (spherical) geometry is important.

The density ratio $D$ (Figure 67, left) and Mach number $M_a$ (Figure 67, right) inform which physics the fluid equations must include to model these zones. The density ratio is typically small, of order 2–3, and the Mach number ranges from $\sim 0.1$ at $M \lesssim 2 M_\odot$ down to $10^{-4}$ at $M \approx 9 M_\odot$. This suggests that above $\approx 2 M_\odot$ the Boussinesq approximation is valid, whereas below this the fully compressible equations may be needed to capture the dynamics at moderate Mach numbers.

The Rayleigh number $Ra$ (Figure 68, left) determines whether or not a putative CZ is actually unstable to convection, and the Reynolds number $Re$ determines how turbulent the zone is if instability sets in (Figure 68, right). In these zones both numbers are enormous, so we should expect convective instability to result in highly turbulent flows.

The optical depth across a CZ $\tau_{\text{CZ}}$ (Figure 69, left) indicates whether or not radiation can be handled in the diffusive approximation, while the optical depth from the outer boundary to infinity $\tau_{\text{outer}}$ (Figure 69, right) indicates the nature of radiative transfer and cooling in the outer regions of the CZ. We see that the optical depth across these zones is enormous ($\tau_{\text{CZ}} \sim 10^{11}$) and that their outer boundaries lie at very large optical depths ($\tau_{\text{outer}} \gtrsim 10^{10}$). This means that both the bulk and the outer boundary of the core CZ can likely be modeled in the limit of radiative diffusion.

The Eddington ratio $\Gamma_\text{Edd}$ (Figure 70, left) indicates whether or not radiation hydrodynamic instabilities are important in the nonconvecting state, and the radiative Eddington ratio $\Gamma_\text{rad,Edd}$ (Figure 70, right) indicates the same in the developed convective state. The Eddington ratio with the full luminosity ($\Gamma_\text{Edd}$) approaches unity around masses of $20 M_\odot$, and continues to rise with increasing mass from there, resulting in a nominally super-Eddington system. Convection is able to carry much of this luminosity, however, so the radiative Eddington ratio $\Gamma_{\text{rad,Edd}}$ approaches but does not exceed unity.

These near-unity Eddington ratios suggest that radiation hydrodynamic instabilities are important. We have two reservations with this conclusion. First, the Mach numbers in the core CZ are tiny, so the velocity field does not drive large density fluctuations and hence we do not expect large opacity fluctuations like those reported in the FeCZ (Schultz et al. 2020). Second, the optical depth across the core CZ is enormous, so even if radiation hydrodynamics instabilities arise, they should remain limited to scales that are small compared with the size of the CZ.

The Prandtl number $Pr$ (Figure 71, left) measures the relative importance of thermal diffusion and viscosity, and the magnetic Prandtl number $Pm$ (Figure 71, right) measures the same for magnetic diffusivity and viscosity. The Prandtl number is always small in these models, so the thermal diffusion length scale is much larger than the viscous scale. By contrast, the magnetic Prandtl number varies from order unity at low masses to large ($10^4$) at high masses.

The fact that $Pm$ is large at high masses is notable because the quasistatic approximation for MHD has frequently been used to study magnetoconvection in minimal 3D MHD simulations of planetary and stellar interiors (e.g., Yan et al. 2019) and assumes that $Rm = PmRe \to 0$; in doing so, this approximation assumes a global background magnetic field is dominant and neglects the nonlinear portion of the Lorentz force. This approximation breaks down in CZs with $Pm > 1$, and future numerical experiments should seek to understand how magnetoconvection operates in this regime.

The radiation pressure ratio $\beta_{\text{rad}}$ (Figure 72) measures the importance of radiation in setting the thermodynamic properties of the fluid. This is small at low masses ($M_e \lesssim 9 M_\odot$) but reaches a 30%–100% correction at high masses ($M_e > 20 M_\odot$), so radiation pressure is very important to capture at the high-mass end.

The Ekman number $Ek$ (Figure 73) indicates the relative importance of viscosity and rotation. This is tiny across the H-R diagram, so we expect rotation to dominate over viscosity, except at very small length scales.

The Rossby number $Ro$ (Figure 74, left) measures the relative importance of rotation and inertia. This is small ($10^{-7} - 10^{-10}$), meaning that the core CZ is rotationally constrained for typical rotation rates (Nielsen et al. 2013).

We have assumed a fiducial rotation law to calculate $Ro$. Stars exhibit a variety of different rotation rates, so we also show the convective turnover time $t_{\text{conv}}$ (Figure 74, right), which may be used to estimate the Rossby number for different rotation periods.

The Pécellet number $Pe$ (Figure 75, left) measures the relative importance of advection and diffusion in transporting heat, and the flux ratio $F_{\text{conv}}/F$ (Figure 75, right) reports the fraction of the energy flux that is advected. The Pécellet number is always large ($10^6 - 10^7$), and the flux ratio varies from $\sim 0.2$ at the low-mass end ($M \lesssim 2 M_\odot$) to $\sim 0.6$ for $M \gtrsim 3 M_\odot$. In general then core convection is very efficient and carries a substantial fraction of the flux.

Finally, Figure 76 shows the stiffness of the outer boundary of the core CZ. Over the whole mass range and main sequence this boundary is very stiff ($S \sim 10^{5-6}$), so we do not expect much mechanical overshooting, although there could still be convective penetration (Anders et al. 2022).
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