Third Family Corrections to Quark and Lepton Mixing in SUSY Models with non-Abelian Family Symmetry

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ABSTRACT: We analyse the effect on quark and lepton mixing of third family wavefunction corrections in the framework of realistic SUSY models based on non-Abelian family symmetry. Such models are capable of providing a successful description of quark and lepton masses and mixing including neutrino masses and tri-bimaximal mixing. Such third family wavefunction effects can arise from either the canonical normalisation of the kinetic terms or renormalisation group running effects. At leading order we show that both sorts of corrections can be subsumed into a single universal parameter. With hierarchical neutrinos and tri-bimaximal mixing in the neutrino sector, and Cabibbo-like mixing in the charged lepton sector, we show that the solar mixing angle deviates from its tri-bimaximal value due to the effect of both charged lepton mixing and third family wavefunction corrections, leading to the lepton mixing sum rule $s = r \cos \delta + \frac{2}{3}a$ (where $s, r, a$ describe the deviations of solar, reactor and atmospheric mixing angles from their tri-bimaximal values, and $\delta$ is the observable Dirac CP phase). We discuss in some detail the question of the magnitude of the canonical corrections that are expected in realistic SUSY models based on non-Abelian family symmetry.

KEYWORDS: Beyond Standard Model, Quark Masses and SM Parameters.
1. Introduction

Since the discovery of neutrino masses and large lepton mixing angles, the flavour problem of the Standard Model (SM) has received much attention. As the precision of the neutrino data has improved, it has become apparent that lepton mixing is consistent with the so-called Tri-bimaximal (TB) mixing pattern [1], and many models attempt to reproduce this as a theoretical prediction [2, 3, 4, 5, 6, 7, 8, 9]. Since the forthcoming neutrino experiments may be sensitive to relatively small deviations from TB mixing, an interesting question is the amount of “theoretical error” inherent such TB mixing predictions.

In many classes of models TB mixing arises purely from the neutrino sector [10] subject to deviations due to charged lepton sector corrections [2, 3]. If these charged lepton corrections are “Cabibbo-like” in nature (i.e. dominated by a 1-2 mixing), then the deviation from the tri-maximal solar mixing angle is related to the Cabibbo-like charged lepton mixing angle, which also gives rise to a non-zero reactor angle. This allows the deviation of the solar mixing angle from its tri-maximal value to be expressed in terms of the reactor angle, leading to a predictive sum rule [2] which, in terms of the parametrisation in [11], is

\[ s = r \cos \delta, \]

where \( s \) and \( r \) describe the deviations of solar and reactor mixing angles from their tri-bimaximal values, and \( \delta \) is the observable Dirac CP phase in the standard parameterisation [12].

Another source of theoretical uncertainty in TB mixing schemes is the renormalisation group (RG) running [13] of the relevant quantities from the high energy (usually the unification scale \( M_G \)), where the theory is defined, to the electroweak scale \( M_Z \) appropriate for experimental measurements. The dominant source of RG corrections to lepton mixing arises typically from the large tau lepton and third family neutrino Yukawa couplings, leading to relatively large wave-function corrections in the framework of supersymmetric models. Such RG corrections can be readily estimated analytically [14, 15] for the TB mixing case with hierarchical light neutrinos considered here. Diagrammatically, such RG corrections correspond to loop diagrams involving third family matter and Higgs fields and their superpartners. Although suppressed by the loop factor of \( 1/16\pi^2 \), they can be relevant since the loop factor is multiplied by a large logarithm of the ratio of energy scales.

Apart from RG effects there is another type of third family wave-function correction which emerges at tree-level in certain classes of models, and thus can potentially be rather large. These corrections modify the kinetic terms in the Lagrangian causing them to deviate from the standard (or canonical) form. Before the theory can be reliably interpreted, field transformations must be performed in order to return the kinetic terms back to canonical form which, however, leads to appropriate modifications of the Yukawa couplings [16]. It is interesting that these effects are largest in many of the theories that predict TB mixing, especially those based on non-Abelian family symmetries spanning all three families of
SM matter (see e.g. [17, 18, 19, 20]). In such models the canonical normalisation (CN) corrections can in certain cases even exceed the effects due to RG running.

In a recent Letter [21] we provided a unified treatment of all the above sources of theoretical corrections to the TB mixing mixing, namely due to: i) RG corrections, ii) CN corrections and iii) charged lepton corrections. We also presented a novel testable neutrino mixing sum rule which, at leading order, is stable under all these effects. To be precise, we presented a unified formalism for dealing with both renormalisation group running effects and canonical normalisation corrections and then we used this formalism to investigate the third family wave-function corrections to the theoretical predictions of tri-bimaximal neutrino mixing. We found that at leading order both effects can be subsumed into a single universal parameter $\eta$. Including also the leading order Cabibbo-like charged lepton mixing corrections, which typically arise in unified flavour models, we found the theoretically stable sum rule $s = r \cos \delta + \frac{2}{3} a$ where $s$, $r$ and $a$ parameterise the deviations of the solar, reactor and atmospheric mixing angles from their tri-bimaximal values and $\delta$ is the leptonic Dirac CP phase. Such a sum rule is testable in future high precision neutrino experiments [22].

In this paper we shall generalise the above formalism to include quark as well as lepton mixing. As announced in [21], we shall also provide the anticipated rigorous derivation of the new theoretically stable sum rule, and a detailed discussion of it. Another important question which we shall address in this paper concerns the magnitude of the CN effects. In order to do this, it is necessary to be quite specific about the sorts of models under consideration. Here we shall work in the framework of realistic SUSY models based on non-Abelian family symmetry, which are of particular interest since they provide a well defined strategy for accounting for TB mixing. The essential starting point of such models is to invoke a non-Abelian family symmetry which spans all three families (like e.g. gauged SO(3) or $SU(3)$, or their discrete subgroups such as $A_4$ or $\Delta_{27}$ [6]) and which is subsequently broken by extra Higgs scalars (called flavons, generically denoted by $\phi$). The flavons typically couple to the SM matter fermions via heavy messenger fields giving rise (upon integrating out the messenger sector) to effective Yukawa operators proportional to powers of the flavon fields suppressed by powers of the messenger mass $M$. The effective Yukawa couplings are then expressed in terms of ratios of flavon vacuum expectation values (VEVs) $\langle \phi \rangle$ to these messenger mass scales $M$, which defines a set of expansion parameters often denoted as $\varepsilon = \langle \phi \rangle / M$.

If the neutrino masses are assumed to originate from the seesaw mechanism [23], the TB mixing pattern receives a natural explanation by means of the so called constrained sequential dominance (CSD) mechanism [24, 25]. The basic idea is that only one right-handed (RH) neutrino contributes dominantly to the atmospheric neutrino mass and thus the atmospheric mixing angle corresponds to a simple ratio of Yukawa couplings of just the
dominant RH neutrino. One of the subdominant RH neutrinos is then assumed to govern the solar neutrino mass, in which case the solar mixing angle corresponds to another simple ratio of Yukawa couplings associated to this RH state. The TB mixing pattern can then be implemented by means of simple constraints on the Yukawa couplings. Since these emerge from flavon VEVs, CSD is then achieved from a proper vacuum alignment of flavons in the family space, for example $|\langle \phi_3 \rangle| \approx (0, 0, 1)$, $|\langle \phi_{23} \rangle| \approx (0, 1, 1)$, $|\langle \phi_{123} \rangle| \approx (1, 1, 1)$, up to phases.

However, with an extra symmetry breaking sector in the theory, one should take account of the normalisation of the matter sector kinetic terms. Indeed, after the flavour symmetry gets spontaneously broken, the higher order effective operators emerging from the matter-flavon-messenger couplings give rise to extra contributions to the matter sector kinetic terms. The effect is rather similar in form to wavefunction corrections coming from loop diagrams, but here it is due to operators arising in the low energy effective theory after the heavy fields are integrated out, rather than loop corrections. In the framework of supersymmetry (SUSY) and supergravity (SUGRA), the quark and lepton kinetic terms are encoded in the matter sector of the Kähler potential which (unlike the superpotential protected by holomorphicity) is sensitive to disturbances arising from the flavon VEVs that can lead to a wrong normalisation of the propagators in the effective theory (ET) below the flavour symmetry breakdown scale. Thus, in order to fix the asymptotic propagator behavior one has to redefine appropriately the ET matter superfields with subsequent effects in the other parts of the theory, and in particular the Yukawa sector emerging from the superpotential. This rather technical, but necessary procedure has been discussed in [16] and more recently in [17, 18, 20].

With the above discussion in hand, we can now be more precise about the class of models that we shall study in this paper, namely we shall consider the CN and RG effects in classes of flavour models in which all the Yukawa couplings vanish and the matter sector Kähler metric becomes proportional to the unit matrix (in the family space) in the exact flavour symmetry limit. Examples of such models include those based on (gauged) $SO(3)$ or $SU(3)$ non-Abelian family symmetries (as indicated by the correlations among all three family entries of the tri-bimaximal lepton mixing matrix), or their discrete subgroups like $A_4$, $\Delta_{27}$. Since the irreducible representations accommodating the matter sector in these models span all three families, the rather large third generation Yukawa couplings (in particular $y_t$) must originate from a vertex containing an insertion of at least one flavon field (say $\phi_3$) with only a very mildly suppressed expansion parameter $\varepsilon_3 = |\langle \phi_3 \rangle|/M_3$, typically $\varepsilon_3 \approx 0.5$. Such a flavon would also be expected to enter at leading order the corresponding matter sector Kähler potential (which is practically unprotected by flavour symmetries) leading to family-dependent effects in the corresponding Kähler metric, with consequences for in particular the third family normalisation of matter sector propagators.
It is namely this effect that will mainly concern us here.

Although this has been to some extent addressed previously in [17] and [18], the important effects due to the large expansion parameter associated with the third family in these classes of models have not been fully appreciated. As we shall show, the potentially large expansion parameter $\varepsilon_3$ present in the superpotential, and in particular its counterpart in the Kähler metric, can be responsible for large CN effects in the physical quantities. Especially, given the rescaling nature of the canonical transformation, the effects should be visible in measurable that are not suppressed by small Yukawa couplings, i.e. namely the quark and lepton sector mixing angles encoded in the relevant Cabibbo-Kobayashi-Maskawa (CKM) and Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrices. For example, we shall show that the $(2, 3)$ quark mixing angles may become corrected by much larger effects than the factor of $1 + O(\varepsilon^2)$ where $\varepsilon \approx 0.05$ claimed in [18].

Remarkably enough, though we expect the main changes due to CN to occur in the third family Yukawa couplings, even the 12 sector of the the bi-large TB mixing structure of the PMNS matrix becomes sensitive to such changes. One can then expect significant deviations from tri-bimaximal mixing not only in the atmospheric, but also in the solar mixing angle, as we shall discuss in great detail in this paper. In particular, due to a very simple origin of the leading order CN correction in the matter sector, one can calculate the leading correction to the aforementioned mixing sum-rule $s = r \cos \delta + 2 \varepsilon_3$ in terms of the atmospheric mixing TB mixing deviation parameter $a$ in the form $s = r \cos \delta + \frac{2}{3} a$.

As discussed [21], one of the virtues of this ‘theoretically improved sum-rule’ is its relative stability with respect to not only CN but also RG corrections that makes this relation potentially testable at the up-coming neutrino experiments. As we shall discuss in this paper, the individual corrections to the TB mixing parameters $a$, $s$ and $r$ can differ in specific flavour models, because the magnitude of the CN effects depend on the situation in the messenger sector. In particular, it is important to discuss carefully the relation between the expansion parameters in the Kähler potential and those in the superpotential. Though there is a generic tendency of ‘mirroring’ all the underlying theory vertices from the superpotential to the relevant parts of the Kähler metric, the large expansion parameter $\varepsilon_3$ in the superpotential need not be the same as that in the Kähler, and under certain conditions the latter might be very suppressed. This can lead to negligible Kähler corrections to the physical mixing matrices and, as we shall see, it is even possible to construct models where the expansion parameters driving the Yukawa couplings enter only the right-handed part of the matter sector Kähler metric, which is entirely harmless¹ for the CKM and PMNS mixings (governing the matter fields with left-handed chirality). This, in turn, can be exploited to identify classes of flavour models where the CN corrections to TB mixing can be made very small by a suitable choice of the messenger sector structure, thus restoring

¹However, these corrections can still have big impact in the supersymmetric context, where they can potentially worsen the SUSY flavour issue.
The consistency of the results with [18].

The remainder of the paper is set out as follows: In the subsequent section we shall comment on the generalities of the canonical normalisation procedure focusing on ambiguities in the definition of the canonical normalisation transformation. We develop a perturbative technique to deal with effects of the canonical redefinition of fields in the Yukawa sector focusing in particular on its impact to the CKM and PMNS mixing parameters. Sections 3 and 4 are then devoted to a set of examples of the CN effect in the quark and lepton sectors respectively. Focusing on the large solar and atmospheric mixing, we derive the corrections to the TB mixing pattern and consider the CN effects in the setting with a small charged lepton sector contribution taken into account, leading to a new lepton mixing sum-rule. In section 5 we discuss RG corrections and show how, at leading order, they may be combined with CN corrections leading to no further modifications of the new sum rule. In section 6 we present a more in-depth discussion of the expected magnitude of the CN corrections expected in a particular class of realistic SUSY models based on non-Abelian family symmetry. We also provide an alternative simplified derivation of the effects of CN on the neutrino mixing angles by exploiting the CSD results which allow the corrections to the TB mixing angles to be directly read-off. We also present a detailed discussion of the messenger sector in the class of considered models. Section 7 concludes the paper. Some technical aspects of the discussion in the main body of the paper may be found in the Appendices.

2. The Kähler potential and effects of canonical normalisation

Whenever the Kähler potential of a given SUSY model is nontrivial there are extra effects coming from the canonical normalisation procedure bringing the generic kinetic terms

\[ L^{\text{kin}}_f = \partial_\mu \tilde{Q}_i \alpha \left( K_Q \right)_{ij} \partial^\mu \tilde{Q}_\alpha j + \partial_\mu \tilde{u}^c_i \alpha \left( K_u \right)_{ij} \partial^\mu \tilde{u}^c_j + \partial_\mu \tilde{d}^c_i \alpha \left( K_d \right)_{ij} \partial^\mu \tilde{d}^c_j + \ldots , \]

\[ L^{\text{kin}}_{f,c} = \partial_\mu \tilde{f}_i \alpha \left( K_f \right)_{ij} \partial^\mu \tilde{f}_{\alpha j} + \partial_\mu \tilde{f}^c_i \alpha \left( K_{f,c} \right)_{ij} \partial^\mu \tilde{f}^c_j + \ldots , \]

(2.1)

(where \((K_f)_{ij}\) denotes the Kähler metric for the given scalar \(\tilde{f}, \tilde{f}^c\) and fermionic \(f, f^c\) degrees of freedom) into the canonical form \(K_f = \delta_{ij}\langle \tilde{f}^{\text{can}}_i \rangle \langle \tilde{f}^{\text{can}}_j \rangle\) and \(\delta_{ij}\langle \tilde{f}^{\text{can}}_i \rangle \langle f^{\text{can}}_j \rangle\) respectively.

In a wide class of non-Abelian flavour models, the dominant contributions to \(K_{f,f^c}\)'s come from insertions of the flavon field associated to the third family Yukawas\(^2\) (usually

\(^2\)In the flavour models based on \(SU(3)\) family symmetry, the third family Yukawa couplings are governed by operators of the type \(\frac{1}{M}(f,\phi_3)(f^{\ast},\phi_3)H\) exploiting the triplet nature of matter of both chiralities \(f, f^{\ast}\), while in the \(SO(3)\) theories the structure of the leading order operators typically looks like \(\frac{1}{M}(f,\phi)\tilde{f}^2_i H\) due to the singlet nature of the right-handed spinors. The dot product corresponds to the simplest bilinear invariants of the flavour symmetry under consideration.
denoted by $\phi_3$) yielding

$$K_{f,f^c} \approx \overline{f}_i \left[ k_{f}^c \delta_{ij} + k_{c}^f \left( \frac{1}{M_\psi^2} \langle \phi_3 \rangle_i \langle \phi_3 \rangle_j \right) \right] f_j + \overline{f}^c_i \left[ k_{f}^c \delta_{ij} + k_{c}^f \left( \frac{1}{M_\chi^2} \langle \phi_3 \rangle_i \langle \phi_3 \rangle_j \right) \right] f^c_j$$

(2.2)

where $k_{0,3}^{f,f^c}$ are real constants and $M_\psi$ and $M_\chi$ denote masses of the messenger fields relevant for the left- and right-chirality matter sectors respectively. If the flavon VEV $\langle \phi_3 \rangle$ is comparable to either $M_\psi$ and/or $M_\chi$, one can expect a potentially large deviations from the leading order universality (governed by the $\delta_{ij}$ factors above) in the relevant part of the Kähler metric.

It is important that one can hardly trim all these contributions to zero simultaneously by fiddling around with the messenger masses because there is no symmetry that could prevent every Yukawa sector relevant messenger from entering either $K_f$ or $K_{f^c}$. This$^3$, however, depends strongly on the character of the Yukawa operators, and can, in turn, single out a class of particular “Kähler-corrections-safe” flavour models; for more detailed discussion see section 6.2.

2.1 Definitions and ambiguities

Canonical normalisation consists in redefining the defining basis fields $f$ and $f^c$ so that the original (for instance scalar sector) kinetic terms $\mathcal{L}_{\text{kin}} = \partial \hat{f}^\dagger K_f \partial \hat{f} + \partial \hat{f}^c \partial K_{f^c} \partial \hat{f}^c$ receive the canonical form $\mathcal{L}_{\text{kin}}^{\text{can}} = \partial f^\dagger \partial f + \partial f^c \partial f^c$. This is achieved$^4$ by transforming the defining superfields by $\hat{f} \rightarrow \hat{f} \equiv f_{\text{can}}$ where $P_f$ is a matrix bringing the relevant Kähler metric $K_f$ into the diagonal form:

$$P_f^\dagger K_f P_f = 1, \quad \text{i.e.} \quad K_f = P_f^{-1} P_f^{-1}.$$  

(2.3)

This can be easily done in two steps:

- First, one can always diagonalise the Hermitean Kähler metric by means of a unitary transformation $U_f K_f U_f^\dagger = K_f^D$ where $K_f^D$ is a real diagonal matrix.

- Second, a diagonal rescaling by $\sqrt{K_f^D}$ from both sides of $U_f K_f U_f^\dagger = K_f^D$ drives its RHS to unity: $\left( \sqrt{K_f^D} \right)^{-1} U_f K_f U_f^\dagger \left( \sqrt{K_f^D} \right)^{-1} = 1$. Moreover, one can multiply this formula by any unitary matrix from the left (and its inverse from the right) with no effect on this unity matrix.

$^3$As we shall see, it is namely the left-handed sector (i.e. nonuniversalities in $K_f$) that could have large effects on quark and lepton mixing in the charged currents.

$^4$In what follows, we shall focus on the left-chiral matter sector, i.e. we shall often give the results only for $K_f$; the relevant formulae for $K_{f^c}$ can be obtained upon replacing all the super-(sub-)scripts $f \rightarrow f^c$.
Thus, the most generic form of $P_f$ reads

$$ P_f = U_f^\dagger \left( \sqrt{K_D^f} \right)^{-1} \bar{U}_f, \quad (2.4) $$

with the freedom to choose the unitary $\bar{U}_f$ matrix arbitrarily. Thus, one can for instance have $P_f$ Hermitian by choosing $\bar{U}_f = U_f$ or exploit this freedom to bring the $P_f$ into a triangular form as e.g. in [18].

**Note on notation:**
In what follows, whenever appropriate we use hats to denote quantities in the defining basis (i.e. before canonical normalisation) while the un-hatted symbols correspond to their physical counterparts, i.e. to quantities after the CN effects were already taken into account.

### 2.2 Effect of canonical normalisation in the Yukawa couplings

**Lepton sector:**
Suppose the original charged lepton and the light Majorana neutrino mass matrices are diagonalised by means of the biunitary transformations $\hat{V}_L^l \hat{M}_l \hat{V}_R^l = \hat{M}_D^l$ and $\hat{V}_L^\nu \hat{m}_\nu \hat{V}_L^\nu = \hat{m}_\nu^D$ so that the lepton mixing matrix (before canonical normalisation) obeys $\hat{U}_{PMNS} = \hat{V}_L^\nu \hat{V}_L^l$. The effect of canonical normalisation on $\hat{M}_l$ and $\hat{M}_\nu$:

$$ \hat{M}_l \rightarrow P_L^l \hat{M}_l P_e^c \equiv M_l \quad \text{and} \quad \hat{m}_\nu \rightarrow P_L^\nu \hat{m}_\nu P_L \equiv m_\nu \quad (2.5) $$

induces a relevant change on $\hat{V}_L^l \rightarrow V_{L,R}^l$ and $\hat{V}_L^\nu \rightarrow V_L^\nu$ so that $V_{L,R}^l M_l V_{L,R}^l = M_D^l$ and $V_L^\nu m_\nu V_L^\nu = m_D^\nu$ is fulfilled, i.e.:

$$ V_L^l P_L^T \hat{M}_l P_e^c V_R^l = M_D^l \quad \text{and} \quad V_L^\nu P_L^T \hat{m}_\nu P_L V_L^\nu = m_D^\nu \quad (2.6) $$

should be satisfied. Then the physical lepton mixing matrix obeys (up to the rephasing bringing it into the standard PDG form [12]) $U_{PMNS} = V_L^l V_L^\nu$.

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5The situation in the quark and lepton sectors is different: since the quark sector diagonalisation transformation is **biunitary**, one can always absorb the would-be phases of the diagonal entries of $M_D^f$ by a suitable redefinition of $V_L^l$ and $V_R^l$ and get rid of all but one CP phase in the CKM matrix. This is not possible for Majorana neutrinos as there is only one unitary matrix in the relevant formula. This, in turn, gives rise to extra phase factors associated to PMNS mixing - the Majorana phases.

6Note that the $P_{e^c}$ actually does not enter the effective light neutrino matrix because it cancels among the right-handed components of the neutrino Yukawas and the inverse of the Majorana mass matrix in the seesaw formula $\hat{m}_\nu = M_D^\nu M_M^{-1} (M_D^\nu)^T$. 

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Quark sector:
The reasoning for the quark sector goes along the same lines as above - the original basis up and down-type Yukawa matrices \( \hat{M}_{u,d} \) diagonalizable by biunitary transformations \( \hat{V}_L^u \hat{M}_u \hat{V}_R^{u\dagger} = \hat{M}_u^D \) and \( \hat{V}_L^d \hat{M}_d \hat{V}_R^{d\dagger} = \hat{M}_d^D \) (leading to \( \hat{V}_{CKM} = \hat{V}_L^u \hat{V}_L^{d\dagger} \) before canonical normalisation) change upon canonical normalisation into

\[
\hat{M}_u \rightarrow P_Q^T \hat{M}_u P_{u^c} \equiv M_u \quad \text{and} \quad \hat{M}_d \rightarrow P_Q^T \hat{M}_d P_{d^c} \equiv M_d
\]

(2.7)

inducing a change on \( \hat{V}_L^u \rightarrow V_L^{u,R} \) and \( \hat{V}_L^d \rightarrow V_L^{d,R} \) so that

\[
V_L^u P_Q^T \hat{M}_u P_{u^c} V_R^{u\dagger} = M_u^D \quad \text{and} \quad V_L^d P_Q^T \hat{M}_d P_{d^c} V_R^{d\dagger} = M_d^D.
\]

(2.8)

The physical CKM matrix then obeys \( V_{CKM} = V_L^u V_L^{d\dagger} \).

Irrelevance of \( \hat{U}_f \) matrices

It is easy to see that the arbitrary \( \hat{U}_f \) matrices in the definition of \( P_f \) do not play any role in either the mixing matrices \( U_{PMNS}, V_{CKM} \) or the physical spectra. Indeed, under any unitary change \( \hat{U}' \) of the relevant \( \hat{U}_f \) matrices in the definition \( \hat{U}_f = (2.4), \) i.e. \( P_f \rightarrow P_f U_f', \) the effects in \( (2.6) \) can be absorbed into redefinitions \( V_L^f \rightarrow V_L^f \hat{U}_L^f \) and \( V_R^f \rightarrow V_R^f \hat{U}_R^f \) so that \( (2.4) \) remains unaffected. However, \( \hat{U}_L^f \) cancels in \( U_{PMNS} \) and the physical spectra remain intact, because the would-be effects of \( \hat{U}_L \) and \( \hat{U}_{u^c} \) matrices in \( (2.4) \) can be absorbed into the biunitary transformation revealing the spectrum of the charged lepton Yukawa matrix. Similarly, one can justify the irrelevance of the particular choice of \( P_Q \) and \( P_{u^c,d^c} \) for \( V_{CKM} \) and the quark sector spectra.

Exploiting the freedom in definition of \( P_{f,f^c} \)

Thus, one can exploit the freedom in choosing \( \hat{U}_{f,f^c} \) matrices in the definition of \( P_{f,f^c} \) to simplify the structure of \( (2.4) \) so that the \( P_{f,f^c} \)-factors are particularly easy to handle. The convenient choice is indicated by the fact that even if the original Kähler metric is just a slight perturbation of the unity matrix \( \hat{1} \) (up to an irrelevant overall normalisation \( k_0^f \)), the diagonalisation matrix \( U_f \) in \( P_f = U_f^\dagger (\sqrt{K_f^D})^{-1} \hat{U}_f \) could still be large. The intention to write \( (\sqrt{K_f^D})^{-1} \) as \( (\hat{1} + \Delta K_f^D)/\sqrt{k_0^f} \), exploiting the limited departure of the Kähler metric spectrum from unity, then gives \( P_f = (U_f^\dagger \hat{U}_f + U_f^\dagger \Delta K_f^D \hat{U}_f)/\sqrt{k_0^f} \) that could be brought to a particularly convenient form for \( \hat{U}_f = U_f \), and we can benefit from \( P_f = (\hat{1} + U_f^\dagger \Delta K_f^D U_f)/\sqrt{k_0^f} \), i.e. hermiticity of \( \Delta P_f \equiv U_f^\dagger \Delta K_f^D U_f \) and simplicity of \( P_f = (\hat{1} + \Delta P_f)/\sqrt{k_0^f} \).
2.3 Perturbative prescription for the physical rotation matrices

For Hermitian $P_{L,e}$ and $P_{Q,u,d}$ and $P_{f,f'} = (1 + \Delta P_{f,f'})/\sqrt{k_0^{f,f'}}$ (assuming a small departure of $K_f$ and $K_{f'}$ from unity), one obtains from (2.6):

$$V_L^f (1 + \Delta P_L^e)^* \hat{M}_i (1 + \Delta P_{e'}) V_R^i = \sqrt{k_0^L k_0^{e'}} M_i^D,$$

$$V_L^e (1 + \Delta P_L^e)^* \hat{m}_{\nu} (1 + \Delta P_{\nu}) V_R^{\nu} = k_0^L m_{\nu}^D$$

for the lepton sector and from (2.8):

$$V_L^u (1 + \Delta P_Q^u)^* \hat{M}_u (1 + \Delta P_{u'}) V_R^{u} = \sqrt{k_0^Q k_0^{u'}} M_u^D,$$

$$V_L^d (1 + \Delta P_Q^d)^* \hat{M}_d (1 + \Delta P_{d'}) V_R^{d} = \sqrt{k_0^Q k_0^{d'}} M_d^D$$

for the quarks. If all the (high-scale) physical spectra are sufficiently hierarchical\textsuperscript{7}, the smallness of $\Delta P_{f,f'}$ factors ensures only small differences between the hatted and un-hatted diagonalisation matrices, i.e.

$$V_{L,R}^f = W_{L,R}^f \hat{V}_{L,R}^f,$$ (2.11)

where $W_{L,R}^f$ are small unitary rotations in the unity neighborhood (up to a phase ambiguity to be discussed later):

$$W_{L,R}^f = 1 + i \Delta W_{L,R}^f,$$ (2.12)

with $\Delta W_{L,R}^f$ denoting their Hermitian generators. One can disentangle the left-handed and right-handed rotations in formulae (2.9) and (2.10) by considering $M_f \hat{M}_f^\dagger$:

$$W_{L,R}^f \hat{V}_{L,R}^f (1 + \Delta P_L^e)^* \hat{M}_i (1 + 2 \Delta P_{e'}) \hat{M}_i^\dagger (1 + \Delta P_L^e)^* \hat{V}_{L,R}^f W_{L,R}^f = k_0^L k_0^{e'} M_i^D M_i^D,$$

$$W_{L,R}^u \hat{V}_{L,R}^u (1 + \Delta P_Q^u)^* \hat{M}_u (1 + 2 \Delta P_{u'}) \hat{M}_u^\dagger (1 + \Delta P_Q^u)^* \hat{V}_{L,R}^u W_{L,R}^u = k_0^Q k_0^{u'} M_u^D M_u^D,$$

$$W_{L,R}^d \hat{V}_{L,R}^d (1 + \Delta P_Q^d)^* \hat{M}_d (1 + 2 \Delta P_{d'}) \hat{M}_d^\dagger (1 + \Delta P_Q^d)^* \hat{V}_{L,R}^d W_{L,R}^d = k_0^Q k_0^{d'} M_d^D M_d^D,$$ (2.13)

which yields (from the three complex off-diagonal zero conditions) at the leading order:

$$(\Delta W_L^f)^{ij, i\neq j} = \frac{i}{\hat{m}_{\bar{f}}^{ij} - \hat{m}_{\bar{f}}^{i\bar{j}}} \left[ (\hat{m}_{\bar{f}}^{ij})^2 + \hat{m}_{\bar{f}}^{i\bar{j}} \right] \left( \hat{V}_L^f (1 + \Delta P_L^e) \hat{V}_L^f \right)^{ij},$$

$$(\Delta W_L^u)^{ij, i\neq j} = \frac{i}{\hat{m}_{\bar{f}}^{ij} - \hat{m}_{\bar{f}}^{i\bar{j}}} \left[ (\hat{m}_{\bar{f}}^{ij})^2 + \hat{m}_{\bar{f}}^{i\bar{j}} \right] \left( \hat{V}_L^u (1 + \Delta P_Q^u) \hat{V}_L^u \right)^{ij},$$

$$(\Delta W_L^d)^{ij, i\neq j} = \frac{i}{\hat{m}_{\bar{f}}^{ij} - \hat{m}_{\bar{f}}^{i\bar{j}}} \left[ (\hat{m}_{\bar{f}}^{ij})^2 + \hat{m}_{\bar{f}}^{i\bar{j}} \right] \left( \hat{V}_L^d (1 + \Delta P_Q^d) \hat{V}_L^d \right)^{ij},$$ (2.14)

\textsuperscript{7}This is certainly true for all charged matter fermion spectra; for neutrinos we shall stick to the hierarchical spectrum case from now on.
where the eigenvalues \( \hat{m}_i^2 \) of the original \( \hat{M}_f \) matrices can be at the leading order identified with the physical charged fermion masses and the overall normalisation factors \( k_0^{f,c} \) drop. Similarly, the neutrino sector corrections obey (replacing \( \hat{M}_f \to \hat{m}_\nu, V_L^I \to V_L^\nu, V_R^I \to V_L^{\nu\ast} \) and \( \Delta P_e \to \Delta P_L \) in the first formula above)

\[
(\Delta W_L^\nu)_{ij,i \neq j} = \frac{i}{\hat{m}_j^\nu - \hat{m}_i^\nu} \left[ (\hat{m}_i^\nu^2 + \hat{m}_j^\nu^2) \left( V_L^\nu \Delta P_L^T \tilde{V}_L^\nu \right)_{ij} + 2\hat{m}_i^\nu \hat{m}_j^\nu \left( \tilde{V}_L^\nu \Delta P_L^T \tilde{V}_L^\nu \right)_{ji} \right].
\]

Due to the assumed hierarchy in the physical spectra, the first terms tend to dominate over the second (thus screening the ambiguity in the unknown structure of the right-handed rotations in the charged sector) and we shall often neglect the latter.

Notice that formulae (2.14), (2.15) provide only the off-diagonal entries of \( \Delta W_L^f \)'s. However, this reflects the three phase ambiguity in defining the diagonalisation matrices \( W \) by means of relations like (2.13). Thus, it is not surprising that three parameters in \( \Delta W_L^f \)'s remain unconstrained and can be in principle chosen arbitrarily with the only constraint coming from the required perturbativity of the \( W_L^f \) matrices (2.12). For simplicity, we shall put the diagonal entries of all \( W_L^f \)'s to zero keeping in mind the possible need for “standard” rephasing of the physical lepton mixing matrix. Another reason is that in the real case \( W \) become orthogonal and thus generated by antisymmetric purely imaginary \( \Delta W_L^f \)'s. Thus, the \( \Delta W_L^f \) matrices can be without loss of generality chosen in the form:

\[
\Delta W_L^f = \begin{pmatrix}
0 & \Delta W_{L12}^f & \Delta W_{L13}^f \\
\Delta W_{L12}^{f\ast} & 0 & \Delta W_{L23}^f \\
\Delta W_{L13}^{f\ast} & \Delta W_{L23}^{f\ast} & 0
\end{pmatrix},
\]

with the off-diagonal entries given by formulae (2.14) and (2.15). With this at hand one can write the physical\(^8\) quark and lepton mixing matrices \( V_{CKM} \) and \( U_{PMNS} \) in term of the original ones \( \hat{V}_{CKM} \) and \( \hat{U}_{PMNS} \) as:

\[
U_{PMNS} = (1 + i\Delta W_L^f) \hat{U}_{PMNS}(1 - i\Delta W_L^{\nu\dagger}) = \hat{U}_{PMNS} + \Delta U_{PMNS},
\]

\[
V_{CKM} = (1 + i\Delta W_L^u) \hat{V}_{CKM}(1 - i\Delta W_L^{d\dagger}) = \hat{V}_{CKM} + \Delta V_{CKM},
\]

with\(^9\)

\[
\Delta U_{PMNS} = i \left( \Delta W_L^f \hat{U}_{PMNS} - \hat{U}_{PMNS} \Delta W_L^{\nu\dagger} \right) + \ldots ,
\]

\[
\Delta V_{CKM} = i \left( \Delta W_L^u \hat{V}_{CKM} - \hat{V}_{CKM} \Delta W_L^{d\dagger} \right) + \ldots .
\]

\(^8\)From now on we shall always choose the free phases in \( \hat{V}_{L,R}^f \) (i.e. work in a particular basis) so that the \( \hat{V}_{CKM} \) and \( \hat{U}_{PMNS} \) matrices are in their 'standard' form \(^{12}\). This, however, need not be the case after the CN corrections are taken into account and we shall comment on the phases later.

\(^9\)Although \( \Delta W_{L,R}^f \) are by definition Hermitean, we shall often keep the dagger in formulae like (2.17), (2.18) to help reader’s orientation in the text.
Recall that in a particular model, all the ingredients are actually at hand - one can easily diagonalise the Hermitian Kähler metric to get the (conventionally) Hermitian $P_f^{-1}$ factors (and from there $\Delta P_f$'s) and the various $\hat{V}_{L,R}^f$ matrices in (2.14), (2.15) can be inferred in the same manner from the underlying model Yukawa couplings.

3. Kähler corrections to the 23 sector CKM mixing

For sake of illustration, let us discuss a quark sector example of Kähler corrections to the $V_{cb}$ CKM entry in the class of potentially realistic $SU(3)$ setting with a large third family expansion parameter, see e.g. [19, 20, 26] and references therein.

3.1 Typical structure of Kähler corrections to $V_{cb}$ in an $SU(3)$ model

For simplicity reasons, we shall focus on a real $2 \times 2$ case for the two heavy states only for a quasi-diagonal LH quark sector Kähler metric along the lines of [19] discussed in great detail in e.g. [20, 26]. Assuming that the expansion parameters in the Kähler sector coincide with those relevant for the superpotential (c.f. section 6.2 for a detailed discussion of this point), the relevant piece of the matter sector Kähler metric can be written as:\(^{10}\)

$$K_Q = k_0^Q \begin{pmatrix} 1 & \varepsilon^2 \\ \varepsilon^2 & 1 + \varepsilon_3^2 \end{pmatrix},$$

(3.1)

which is diagonalised by means of $U_Q K_Q U_Q^\dagger = K_Q^D$, where:

$$U_Q \approx \begin{pmatrix} 1 & \varepsilon^2 \\ \varepsilon^2 & 1 \end{pmatrix} \text{ and } K_Q^D \approx k_0^Q \text{ diag}(1, 1 + \varepsilon_3^2).$$

(3.2)

Adopting the Hermitian convention $\hat{U}_Q = U_Q$, i.e. $P_Q = U_Q^\dagger (K_Q^D)^{-1/2} U_Q / \sqrt{k_0^Q}$, one obtains:\(^{11}\)

$$P_Q \approx \frac{1}{\sqrt{k_0^Q}} \begin{pmatrix} 1 & -\varepsilon_3^2 \\ -\varepsilon^2 & 1 - \varepsilon_3^2 \end{pmatrix}$$

(3.3)

\(^{10}\)For sake of simplicity, we have chosen a particular shape of $K_Q$ so that the numerical factors are simple.

\(^{11}\)Note that for $\hat{U}_Q = I$ one receives $P_Q \approx \frac{1}{\sqrt{k_0^Q}} \begin{pmatrix} 1 & \varepsilon_3^2 \\ -\varepsilon_3^2 & 1 - \varepsilon_3^2 \end{pmatrix}$ instead with enhanced off-diagonal terms with respect to (3.3). As it was pointed out in [17], such a $P_Q$ matrix can induce a potentially large deviation of the physical Yukawa matrices from their defining basis structure. However, as far as physical observables such as the CKM mixings are concerned, the individual relatively large 23 rotations arising in such case in both up and down sectors act against each other and leave only a subleading effect, which becomes almost trivial to infer upon adopting $\hat{U}_Q = U_Q$.

In short, the “Hermitian” convention for $P_Q$’s adopted here does not induce large fake corrections to the off-diagonal Yukawa couplings and the corresponding $V_{L,U,D}^{u,d}$ matrices.
and the physical Yukawas obey (at the leading order)

\[ \hat{Y}_u \approx \left( \varepsilon^2 \varepsilon^2 \right) \rightarrow Y_u = (P_{1Q})^T \hat{Y}_u P_{u^c} \approx \frac{1}{\sqrt{k_0}} \left( \varepsilon^2 \varepsilon^2 - \frac{1}{2} \varepsilon^2 \varepsilon^2 \right) P_{u^c}, \]

\[ \hat{Y}_d \approx \left( \varepsilon^2 \varepsilon^2 \right) \rightarrow Y_d = (P_{1Q})^T \hat{Y}_d P_{d^c} \approx \frac{1}{\sqrt{k_0}} \left( \varepsilon^2 \varepsilon^2 - \frac{1}{2} \varepsilon^2 \varepsilon^2 \right) P_{d^c}, \tag{3.4} \]

indicating non-negligible additive leading order corrections to 23 rotations in \( V_{L}^{u,d} \), that, however, cancel at the leading order in the CKM mixing matrix.

The net effect eventually emerges from the next to leading order ratio of the 23 and 33 entries and can be readily obtained from the perturbative prescription (2.18) together with (2.14) provided:

\[ \Delta P_Q = -\frac{1}{2} \left( \varepsilon^2 \varepsilon^2 \right), \quad \tilde{V}_L^u \approx \left( \frac{1}{\varepsilon^2 \varepsilon^2} \frac{1}{\varepsilon^2} \right), \quad \tilde{V}_L^d \approx \left( \frac{1}{\varepsilon^2 \varepsilon^2} \frac{1}{\varepsilon^2} \right), \tag{3.5} \]

(giving \( \hat{V}_{CKM} \approx \left( \begin{array}{cc} 1 - \varepsilon^2 \varepsilon^2 \varepsilon^2 \varepsilon^2 & 0 \\ 0 & 1 \end{array} \right) \)) and thus \( \hat{V}_{cb} \approx \frac{\varepsilon^2 - \varepsilon^2}{\varepsilon^2} \left( 0 \right) \). Therefore, (2.18) leads to:

\[ \Delta V_{CKM} \approx -i \hat{V}_{CKM} \Delta W_L^{d^\dagger} \approx \left( \begin{array}{c} 0 \varepsilon^2 - \varepsilon^2 \\ \varepsilon^2 - \varepsilon^2 \end{array} \right), \tag{3.6} \]

so the CKM matrix changes after canonical normalisation into:

\[ V_{CKM} \approx \left( \begin{array}{cc} 1 & \frac{\varepsilon^2 - \varepsilon^2}{\varepsilon^2} \left( 1 + \frac{1}{2} \varepsilon^2 \varepsilon^2 \right) \\ \frac{\varepsilon^2 - \varepsilon^2}{\varepsilon^2} \left( 1 + \frac{1}{2} \varepsilon^2 \varepsilon^2 \right) & 1 \end{array} \right). \tag{3.7} \]

The physical value of the 23 quark-sector mixing is then modified to:

\[ V_{cb} = \hat{V}_{cb} \left( 1 + \frac{1}{2} \varepsilon^2 \varepsilon^2 \right) + \ldots. \tag{3.8} \]

As anticipated, there is a relatively large multiplicative correction due to the presence of the large expansion parameter associated to the third family Kähler corrections, that was not appreciated in [18].
4. Kähler corrections to the lepton sector mixing

In order to study the effects of Kähler corrections to a generic bi-large lepton sector mixing, one can not avoid the first generation anymore. Thus, in what follows, we consider the full $3 \times 3$ structure of the relevant mixing matrices as well as the matter sector Kähler metric.

4.1 Kähler corrections due to third family canonical rescaling

Though the generic shape of the relevant piece of the Kähler metric (i.e. namely $K_L$ as far the lepton sector is concerned) is rather complicated, in realistic cases one can expect the dominant effects coming from the leading non-universal contribution (2.2) governed by $\langle \phi_3 \rangle$. Thus, we shall first focus on the simplified setting where only the entries due to (2.2) are taken into account. Later on (in section 6.1), we shall compare the results obtained here with the full-fledged potentially realistic $SU(3)$ model analysis to reveal that this is indeed a very accurate approximation.

In the present case, the lepton sector Kähler metric is given at leading order by:

$$K_L = k^L_0 \left( 1 + \frac{k^L_3}{k^L_0} \frac{\langle \phi_3 \rangle^2}{M^2_K} \right) + \ldots,$$

with $k^L_i$ denoting the relevant $O(1)$ Wilson coefficients in (2.2), while $M_K$ stands for a generic Kähler sector messenger mass. In models where the $33$ Yukawa entries are (at least partly) generated by means of $SU(2)_L$-doublet messengers (that in turn enter also the Kähler potential) $M_K$ is around the scale of the relevant Yukawa-sector-active messengers (denoted by $\chi_i$ in section 5.3) and $\langle \phi_3 \rangle_{M_K}$ is of the order of the Yukawa sector parameter $\varepsilon_3$. At the leading order, the lepton sector Kähler metric can be written in a matrix form:

$$K_L \approx k^L_0 \begin{pmatrix} 1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta^K \end{pmatrix} \end{pmatrix}, \quad \text{where} \quad \eta^K = \frac{k^L_3}{k^L_0} \frac{|\langle \phi_3 \rangle|^2}{M^2_K}. \quad (4.2)$$

Therefore, the $P_L$ matrix is just:

$$P_L = \frac{1}{\sqrt{k^L_0}} \left[ 1 - \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta^K \end{pmatrix} \right], \quad \text{and thus} \quad \Delta P_L = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta^K \end{pmatrix}. \quad (4.3)$$

In what follows, we shall consider the Kähler corrections in several variants of (nearly) tri-bimaximal lepton sector models.

---

\footnote{Recall that $\Delta P_{\nu}$ is screened in (2.14) at the leading order and corrections due to $P_{\nu}$ entirely cancel in the seesaw formula, c.f. (2.3).}
4.2 Example 1: Kähler corrections to the exact tri-bimaximal neutrino mixing

Let us denote the exact tri-bimaximal mixing matrix à la Harisson-Perkins-Scott [8] by:

\[ U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot P_M \quad \text{with} \quad P_M = \begin{pmatrix} e^{i\frac{\alpha_1}{2}} & 0 & 0 \\ 0 & e^{i\frac{\alpha_2}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.4} \]

where \( P_M \) is (so far) experimentally undetermined diagonal matrix encoding the two observable Majorana phase differences.

We shall first focus on the simplest setting with tri-bimaximal lepton mixing (before canonical normalisation) coming entirely from the neutrino sector and then extend the analysis for finite corrections from the charged lepton sector. Thus, let us assume that the lepton mixing generated by the underlying family symmetry happens to be exactly tri-bimaximal (in the defining basis), i.e. \( \hat{U}_{PMNS} = \hat{V}_L^\dagger \hat{V}_L^\dagger = U_{TB} \) and comes entirely from the neutrino sector, i.e. \( \hat{V}_{L,R}^\dagger \approx 1 \) while \( \hat{V}_L^\dagger \approx U_{TB} \). In the canonical basis, \( \hat{U}_{PMNS} = U_{TB} \) changes along (2.17) yielding:

\[ U_{PMNS} = U_{TB} + i \left( \Delta W_L^{\dagger} U_{TB} - U_{TB} \Delta W_L^{\dagger} \right) + \ldots, \tag{4.5} \]

and the correction matrices are given by (2.16) provided (2.14), (2.13). Taking into account the screening of the second terms \( \propto \hat{m}_i \hat{m}_j / (\hat{m}_i^2 - \hat{m}_j^2) \) in formulae (2.14) and (2.15) in case of the hierarchical neutrino spectrum, one obtains:\(^{13}\)

\[
\begin{align*}
(\Delta W_L^{\dagger})_{ij, i \neq j} &\approx \frac{i(\hat{m}_i^2 + \hat{m}_j^2)}{\hat{m}_i^2 - \hat{m}_j^2} (\Delta P_L^T)_{ij} = 0, \\
(\Delta W_L^{\nu})_{ij, i < j} &\approx \frac{i(\hat{m}_i^2 + \hat{m}_j^2)}{\hat{m}_i^2 - \hat{m}_j^2} (U_{TB}^\dagger \Delta P_L^T U_{TB})_{ij} \approx i \left( U_{TB}^\dagger \Delta P_L^T U_{TB} \right)_{ij} + \ldots. \tag{4.6}
\end{align*}
\]

that yields at the leading order

\[ \Delta(U_{TB})_{ij, i < j} \approx - (U_{TB})_{ij} \left( U_{TB}^\dagger \Delta P_L^T U_{TB} \right)_{ij} (\text{no summation over } j). \tag{4.7} \]

Remarkably enough, the corrections to the three matrix elements under consideration are (at the leading order) proportional to their values, c.f. formula (4.7) and thus, in particular, the Kähler corrections to the reactor angle are canceled by the 13 zero of \( U_{TB} \). Second, the Majorana phases are irrelevant for the second bracket on the LHS of formula (4.7) and enter only through the first term. Thus, the phase structure of the correction is identical to the phase structure of the original matrix element and there is no need for an additional rephasing.

\(^{13}\)From now on, we shall always assume that the defining basis masses \( \hat{m}_i \) coincide at the leading order with the corresponding physical quantities.
Figure 1: Corrections to the tri-bimaximal neutrino mixing. The $\eta$-parameter stands for $\eta = \eta^K$ from (4.2) for the canonical normalisation effects, $\eta = \eta^{RG}$ from (5.3) for the leading order running effects, or $\eta = \eta^K + \eta^{RG}$ if both CN and RG corrections are taken into account, c.f. (5.7). The shaded regions correspond to values of $|\eta|$ for which one can expect deviations from the leading-order perturbative results. Remarkably enough, the reactor mixing in this setup is rather stable with respect to third family corrections.

Numerically, this leads for example to $(\Delta U_{TB})_{12} \approx \eta^K \frac{1}{\sqrt{6}}$, $(\Delta U_{TB})_{23} \approx \eta^K \frac{1}{4\sqrt{2}}$ and $(\Delta U_{TB})_{13} \approx 0$ (up to irrelevant phase factors). The zero in the 13 correction, however, emerges only from the first term in the approximation (4.6) and gets lifted at the next-to-leading level. Indeed, employing the full-featured formula (2.13) one recovers (for hierarchical case):

$$
\Delta(U_{PMNS})_{13} = 2 \left[ \frac{\hat{m}^\nu_2}{\hat{m}^\nu_3} (U_{TB})_{12} (U_{TB}^\dagger)_{23} + \frac{\hat{m}^\nu_1}{\hat{m}^\nu_3} (U_{TB})_{11} (U_{TB}^\dagger)_{13} \right] (\Delta P_L)_{33} (U_{TB})_{33} \\
= -\frac{1}{3\sqrt{2}} \left( \frac{\hat{m}^\nu_2}{\hat{m}^\nu_3} e^{\frac{i}{2}} - \frac{\hat{m}^\nu_1}{\hat{m}^\nu_3} e^{\frac{i}{2}} \right) \eta^K ,
$$

(4.8)

where the two phase-factors reflect the Majorana nature of the light neutrino masses. The last formula finally yields (assuming the first term in the bracket dominates):

$$
\theta_{13} \approx |\eta^K| \frac{1}{3\sqrt{2}} \sqrt{\frac{\Delta m^2_{31}}{\Delta m^2_{43}}} \approx 4 \times 10^{-2} |\eta^K| .
$$

(4.9)

All together, this gives at the leading order:

$$
U_{PMNS} \approx \begin{pmatrix}
\sqrt{\frac{2}{3}} - \eta^K \frac{1}{6\sqrt{6}} & \frac{1}{\sqrt{6}} + \eta^K \frac{1}{6\sqrt{3}} & 4 \times 10^{-2} |\eta^K| e^{-id} \\
-\frac{1}{\sqrt{6}} + \eta^K \frac{1}{12\sqrt{6}} & \frac{1}{\sqrt{3}} - \eta^K \frac{2}{6\sqrt{3}} & \frac{1}{\sqrt{2}} + \eta^K \frac{1}{4\sqrt{2}} \\
\frac{1}{\sqrt{6}} + \eta^K \frac{5}{12\sqrt{6}} & -\frac{1}{\sqrt{3}} - \eta^K \frac{1}{6\sqrt{3}} & \frac{1}{\sqrt{2}} - \eta^K \frac{1}{4\sqrt{2}} 
\end{pmatrix} . P_M .
$$

(4.10)
Figure 2: Corrections to the tri-bimaximal neutrino mixing (faint dashed lines) from canonical normalization of the kinetic terms as a function of $\eta = \eta^K \equiv \varepsilon^{K}_{\alpha_4} \kappa_{\beta_3}$ in a potentially realistic $SU(3)_f$ flavour model [20] discussed in detail in section 6.1. The displayed curves correspond to the leading order approximate results given by formulae (6.9). Remarkably enough, these results coincide for small $\eta$ with those obtained by perturbative methods in the simplified setup discussed in section 4.2, thus demonstrating the crucial role played by the dominant 33-sector Kähler correction.

It can be easily checked that $U_{PMNS}$ is unitary up to $O(\eta^K)^2$ terms. Exploiting the parametrisation of [11] one gets:

$$s_{12} = \frac{1}{\sqrt{3}}(1 + s), \quad s_{23} = \frac{1}{\sqrt{2}}(1 + a), \quad s_{13} = \frac{1}{\sqrt{2}}r,$$

and (comparing to (4.10)) the ‘TB-deviation’ parameters read:

$$r \approx 6 \times 10^{-2} |\eta^K|, \quad s = \frac{\eta^K}{6} \quad \text{and} \quad a = \frac{\eta^K}{4}. \quad (4.12)$$

We see, in particular, that $\theta_{13}$ is rather stable and that the atmospheric mixing is changing faster than the solar ($a = \eta^K/4$ while $s = \eta^K/6$), c.f. the shape of curves depicted in Fig. 1.

4.3 Example 2: Tri-bimaximal neutrino mixing with charged lepton corrections

Assume that (in the basis in which $\hat{V}_L^\nu = U_{TB}$) there is a finite contribution to the lepton mixing matrix coming from the charged lepton sector, that is actually common to many potentially realistic models of flavour employing unified gauge symmetries like Pati-Salam [27] or $SO(10)$ [28]. The charged lepton sector mixing in such cases tends to copy the

\footnote{Recall that the $13$ mixing can be always without loss of generality made positive by a suitable redefinition of the lepton sector Dirac CP phase.}
structure of \( \hat{V}_L^d \) (up to Clebsches) that leads to a natural assumption about the structure of the \( \hat{V}_L^l \) matrix (before the effects of canonical normalisation are taken into account):

\[
\hat{V}_L^l \approx \begin{pmatrix}
\hat{c}_{12}^l & \hat{s}_{12}^l e^{-i\hat{\rho}} & 0 \\
-\hat{s}_{12}^l e^{i\hat{\rho}} & \hat{c}_{12}^l & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(4.13)

where \( \hat{s}_{12}^l \) is a small Cabibbo-like mixing (typically \( \hat{s}_{12}^l \approx \lambda/3 \)) and \( \hat{\rho} \) is a generic phase. In such a case, the exact tri-bimaximal structure of the (high-scale) lepton mixing matrix is lifted and one is left with (assuming as before \( \hat{V}_L^{\nu \dagger} = U_{TB} \)):

\[
\hat{U}_{PMNS} = \hat{V}_L^l \hat{V}_L^{\nu \dagger} = \hat{V}_L^l U_{TB},
\]

(4.14)

up to a rephasing to the standard PDG form \[12\], which is needed due to the extra phase in (4.13). The charged lepton sector contribution (4.13) has multiple effects, in particular breaks the direct link between the “measured” (up to the renormalisation group running \[14\]) values of the lepton mixing parameters and the underlying purely neutrino sector rotations. However, due to the particular structure of \( \hat{V}_L^l \) above (leading to just a mild alteration of the tri-bimaximal neutrino mixing pattern), a set of simple relations between the underlying neutrino and charged lepton sector mixings and \( \hat{U}_{PMNS} \) can be obtained.

In particular, the original zero reactor angle is lifted by the 12 rotation in the charged lepton sector to:

\[
(\hat{U}_{PMNS})_{13} = -\hat{s}_{23}^\nu \hat{s}_{12}^l e^{-i\hat{\rho}} \Rightarrow \hat{\theta}_{13} \approx \hat{s}_{23}^\nu \hat{s}_{12}^l = \frac{1}{\sqrt{2}} \hat{s}_{12}^l
\]

(4.15)

(no phases enter because we are looking at a magnitude of the 13 term only), which in Georgi-Jarlskog type of unified models \[23\] (where \( \hat{s}_{12}^l = \theta_C/3 \) with \( \theta_C \) denoting the quark-sector Cabibbo mixing) yields:

\[
\hat{\theta}_{13} \approx \frac{\theta_C}{3\sqrt{2}}.
\]

(4.16)

Second, there is an interesting phenomenologically testable sum-rule for the deviation of the solar angle from its exactly tri-bimaximal value \( \theta_{TB} = 35^\circ 16' \) in the form \[2\]

\[
\hat{\theta}_{12} = \theta_{12}^{TB} + \hat{\theta}_{13} \cos \hat{\delta}, \quad \text{i.e.} \quad \hat{s} = \hat{r} \cos \hat{\delta}
\]

(4.17)

where \( \hat{\delta} \) stands for the Dirac CP phase in the lepton sector\(^{15}\). An interested reader can find the derivation of formulae (4.15) and (4.17) in Appendix \[3\].

\(^{15}\)This sum-rule can be easily derived (c.f. Appendix \[3\]) from the magnitude of the 31 entry of \( \hat{U}_{PMNS} = \hat{V}_L^l U_{TB} \) and thus is insensitive to the Majorana phases.
4.4 Canonical normalisation effects in the sum-rules

In view of results of section 4.2, let us discuss the stability of these formulae with respect to the effects of canonical normalisation. We shall again assume the (leading order) 33-sector nonuniversality in the corresponding Kähler metric (4.2). Remarkably enough, though \( \hat{V}_L^t \) is nontrivial, the block-structure of \( K_f \) is such that \( \hat{V}_L^t \) plays essentially no role in the leading-order formula (2.14) and one recovers (4.6) as in the simplest case discussed in the previous section. The canonical normalisation corrections to the lepton mixing matrix then obey:

\[
\Delta U_{PMNS} = -i \hat{U}_{PMNS} \Delta W_{\nu_L}^t,
\]

(4.18)

where \( \hat{U}_{PMNS} \) is not equal to \( U_{TB} \) as in the simplest case, but \( \hat{U}_{PMNS} = \hat{V}_L^t U_{TB} \).

**Corrections to the charged-lepton-sector-induced 13 mixing:**

Let us look first at the CN corrections induced in the simpler formula (4.15). There is no a-priori reason the 13 entry of \( U_{PMNS} \) should vanish as it was the case at the leading order in the purely tri-bimaximal setting (4.10). Indeed, we have:

\[
\Delta(U_{PMNS})_{13} = -i(\hat{U}_{PMNS})_{11}(\Delta W_{\nu_L}^t)_{13} - i(\hat{U}_{PMNS})_{12}(\Delta W_{\nu_L}^t)_{23},
\]

(4.19)

giving at the leading order:

\[
-\sum_{i=1,2} (\hat{U}_{PMNS})_{1i}(\Delta W_{\nu_L}^t)_{i3} = \sum_i \sum_j (\hat{V}_L^t)_{ij}(U_{TB})_{ji} \left( U_{TB}^\dagger \Delta P_L^T U_{TB} \right)_{i3}
- \sum_j (\hat{V}_L^t)_{1j}(U_{TB})_{j3} \left( U_{TB}^\dagger \Delta P_L^T U_{TB} \right)_{33}.
\]

(4.20)

Due to unitarity and the shape (4.3) of \( \Delta P_L \), the first term on the RHS of (4.20) is zero, while the latter yields:

\[
\Delta(U_{PMNS})_{13} \approx (\hat{V}_L^t)_{12}(U_{TB})_{23} \frac{\eta_K^2}{4} = -s_{12} e^{-i\hat{\rho}} \frac{\eta_K}{4\sqrt{2}}.
\]

(4.21)

Notice that the Majorana phase structure of this correction is again the same like the phase structure of the defining basis 13 entry\(^{16}\) in (4.13) and thus the Dirac CP phase is stable under CN effects. Taking into account also the subleading correction \( \propto \sqrt{\Delta m_2^2/\Delta m_A^2} \) (which is of the same order as the term in Eq. (4.21)) of the type (4.8), the last formula is extended to:

\[
\Delta(U_{PMNS})_{13} \approx -\frac{\eta_K}{4\sqrt{2}} \left( e^{-i\hat{\rho}} s_{12} + c_{12} \frac{4}{3} e^{i\hat{\alpha}_2} \sqrt{\frac{\Delta m_2^2}{\Delta m_A^2}} \right),
\]

\(^{16}\)i.e. zero phase in the given global phase convention fixing the shape of \( P_M \) with a real 33 entry.
where, as before, the $\alpha_2$ phase accounts for the extra phase ambiguity due to the Majorana nature of the neutrinos, c.f. discussion of formula (4.8). In order to deduce the Kähler correction to the ‘induced’ 13 mixing (4.15), this result should be added to the RHS of formula (4.15) leading to:

$$s_{13} \approx \frac{1}{\sqrt{2}} \left( 1 + \frac{\eta^K}{4} \right) - \frac{\hat{c}_{12}}{\sqrt{2}} e^{i \frac{\Delta m_{21}^2}{\Delta m_{31}^2}} \sqrt{\frac{\Delta m_{21}^2}{\Delta m_{31}^2}}$$

(4.23)

for the physical 13 mixing in $U_{PMNS}$. Notice that there is a slight ambiguity due to the phase factor in the second term, that can not be neglected with respect to the $\eta^K$-part of the first term therein. However, the smallness of the ‘charged-lepton-sector-induced’ reactor mixing angle $s_{13} \approx \frac{1}{\sqrt{2}} s_{12}$ (which is typically $\theta_{13} \sqrt{2}$ for Georgi-Jarlskog type of flavour models corresponding to the first term above, c.f. formula (4.16) and the discussion around) is not disturbed by the effects of canonical normalisation.

**Canonical normalisation corrections to $\dot{s} = \dot{r} \cos \hat{\delta}$**:

With the information about the $\theta_{13}$ stability at hand, one can infer the leading additive correction to the defining basis formula (4.17), that (swapping all the defining basis quantities for their physical counterparts) should read:

$$\theta_{12}^{TB} = \theta_{12} - \theta_{13} \cos \delta + f(\eta^K).$$

(4.24)

where $f(\eta^K)$ is a linear function of $\eta^K$ vanishing for $\eta^K \rightarrow 0$, i.e. $f(\eta^K) = c \eta^K$ with a real proportionality factor $c$. As we have seen in the previous paragraph, the leading CN correction to the 13-mixing (4.23) is only multiplicative (4.23) and thus all the would-be corrections in (4.24) due to the $\eta^K$-sensitivity in $\theta_{13}$ or $\cos \delta$ are suppressed by $\theta_{13}$. This means that in the $\theta_{13} \rightarrow 0$ limit in (4.24), one should recover the simple leading order $\theta_{12}$ scaling obtained in section 4.2. Thus, one gets $c = -\frac{1}{6\sqrt{2}}$, which gives at the leading order:

$$\theta_{12}^{TB} = \theta_{12} - \theta_{13} \cos \delta - \frac{\eta^K}{6\sqrt{2}}.$$  

(4.25)

Formula (4.25) can be finally recast (using $a = \frac{\eta^K}{4}$ derived in section 4.2) into a sum-rule for *measurable quantities* $a$, $s$ and $r$ and $\delta$ only:\footnote{We note that the sum rule of Eq. (4.26) can be readily generalised to arbitrary $\hat{\theta}_{12}^\nu$ (but keeping $\hat{\theta}_{13}^\nu = 0^\circ$ and $\hat{\theta}_{23}^\nu = 45^\circ$ fixed) using $s_{12}^\nu = \delta_{12}^\nu(1 + \eta^K (\hat{c}_{12}^\nu)^2)/4$ and repeating the derivation of Appendix C.}

$$s = r \cos \delta + \frac{2}{3} a.$$  

(4.26)

This relation identifies a characteristic imprint of the canonical normalisation effects in the popular scheme where the charged leptons contribute in the Georgi-Jarlskog manner (i.e.}
only the 12 sector rotation is non-negligible) while the neutrino sector mixing is exactly tri-bimaximal. Note that in addition to the precision measurements required for testing the original sum rule \cite{4}, testing equation (4.26) requires an accurate measurement of the deviation from maximal atmospheric mixing \cite{30}.

The simple argument above can only be used to fix the shape of the leading order additive corrections in (4.24) and derive the main result (4.26), but does not, in general, provide any information on sub-leading corrections (entering either as multiplicative changes in small parameters or higher order effects in $\eta^K$) to (4.24). An interested reader can find a more detailed explicit derivation of (4.25) with a brief discussion of the shape of such subleading corrections in Appendix C. Remarkably enough, this formula is stable also under radiative corrections due to the RG running (see in section 5), which makes it directly testable at future experimental facilities.

5. Renormalisation group corrections to tri-bimaximal neutrino mixing

The predictions for the Yukawa matrices arise at the scale of flavour symmetry breaking $\Lambda_F$, which we will assume to be close to the GUT scale ($M_{\text{GUT}}$). In order to test such predictions experimentally, the renormalisation group (RG) running between $\Lambda_F$ and the electroweak scale $M_Z$ has to be taken into account. In particular, if tri-bimaximal mixing is realised in the neutrino sector, deviations from this pattern are induced by RG running. The accurate calculation of such corrections requires evolving the effective neutrino mass matrix from $\Lambda_F \approx M_{\text{GUT}}$ to low energy using the $\beta$-functions for the energy ranges above and between the see-saw scales and below the mass scale of the lightest right-handed neutrino \cite{13}. Numerically, this can be done conveniently using the software package REAP \cite{31}.

In what follows, we shall be interested mainly in estimating the size of the RG corrections in the case of a hierarchical neutrino spectrum in the MSSM, for which the running effects are comparatively small and where the leading logarithmic approximation works reasonably well. Note that due to the non-renormalisation theorem, only the radiative wave-function corrections contribute to the $\beta$-functions in supersymmetric theories and the RG corrections can be treated in a very similar fashion to the canonical normalisation corrections, as we will see explicitly below.

Following the spirit of section 4, let us consider the case when the wave-function renormalisation due to the 3rd family dominates. More explicitly, we will assume that the 33-elements govern both $Y^e$ and $Y^\nu$ in the model basis (with diagonal $M_M$). This is the case, for instance, in the classes of non-Abelian flavour models discussed in \cite{20} (and in the example given in section 4.4). Therefore, we will take: $Y^e \approx \text{diag}(0, 0, y_e)$ and $Y^\nu \approx \text{diag}(0, 0, y_{\nu_3})$. Above the mass threshold of the heaviest RH neutrino $M_3$, the $\beta$-function for the effective neutrino mass matrix $m_\nu(\mu) = -v_\nu Y^\nu(\nu) M^{-1}_M(\mu) Y^\nu T(\mu)$ (where
\( \mu \) is the renormalisation scale and \( v_u \) is the VEV of the up-type Higgs doublet) reads:

\[
16 \pi^2 \mu \frac{d}{d\mu} m_\nu = \left( Y^e Y^e + Y^u Y^\nu \right) m_\nu + \frac{1}{16 \pi^2} \left[ 2 T r Y^\nu Y^\nu \ln \frac{M_{\text{GUT}}}{M_3} ight]
\]

where the last term is proportional to the unit matrix in flavour space. Below \( M_3 \), the same \( \beta \)-function applies with \( Y^\nu = 0 \).

Keeping at leading order all terms (but \( m_\nu \)) on the RHS of (5.1) constant, one can integrate (5.1) analytically, yielding

\[
m_\nu(M_Z) = m_\nu(M_{\text{GUT}}) - m_\nu(M_{\text{GUT}}) \frac{1}{16 \pi^2} \left[ 2 T r Y^\nu Y^\nu \ln \frac{M_{\text{GUT}}}{M_3} ight]
\]

which can be rewritten as (forgetting about the doubly-suppressed mixed terms):

\[
m_\nu(M_Z) \approx P_{\text{RG}}^T m_\nu(M_{\text{GUT}}) P_{\text{RG}} \quad \text{with} \quad P_{\text{RG}} = r 1 + \Delta P_{\text{RG}} + \ldots
\]

where

\[
r = 1 - \frac{1}{16 \pi^2} \left[ T r Y^\nu Y^\nu \ln \frac{M_{\text{GUT}}}{M_3} + 3 \left( -\frac{6}{5} g_1^2 - 6 g_2^2 + 2 T r Y^\nu Y^u \right) \ln \frac{M_{\text{GUT}}}{M_3} \right],
\]

\[
\Delta P_{\text{RG}} = -\frac{1}{16 \pi^2} \left[ Y^{\nu*} Y^{eT} \ln \frac{M_{\text{GUT}}}{M_Z} + Y^{\nu*} Y^{\nu T} \ln \frac{M_{\text{GUT}}}{M_3} \right].
\]

Note that the \( r \)-factor in (5.2) is irrelevant for the lepton mixing, because at the leading order one can rewrite \( P_{\text{RG}} \) in the form

\[
P_{\text{RG}} = r (1 + \Delta P_{\text{RG}}) + \mathcal{O} \left[ \frac{1}{16 \pi^2} \right] \text{ terms},
\]

but overall factors like \( r \) drop in formula (2.15).

As we mentioned, the leading order RG effect (5.2) has exactly the form of Eq. (2.5), so both types of corrections, from RG running, as well as from canonical normalisation, can be treated on the same footing in this approximation. Furthermore, using Eq. (5) and
comparing Eq. (5.2) with Eq. (4.3), we find that there is again a single parameter governing the RG corrections to all the mixing angles given by:

\[ \eta_{\text{RG}} = \frac{y_\tau^2}{8\pi^2} \ln \frac{M_{\text{GUT}}}{M_Z} + \frac{y_\nu^2}{8\pi^2} \ln \frac{M_{\text{GUT}}}{M_3}. \]  

(5.5)

The quantitative predictions of the RG running effects can then be obtained from the relevant formulae for the CN corrections \( (4.8), (4.10), (4.12), (4.23), (4.25) \), upon swapping \( \eta^K \leftrightarrow \eta^{\text{RG}} \). The last contribution \( (5.5) \) would be absent if \( M_3 > M_{\text{GUT}} \).

We have cross-checked these results with the analytic approximations presented in \cite{14} and found a perfect agreement for the considered case. In summary, with tri-bimaximal neutrino mixing at the GUT scale, the low scale parameters are given approximately by:

\[ s_{12}(M_Z) = \frac{1}{\sqrt{3}} \left( 1 + \frac{\eta_{\text{RG}}}{6} \right), \quad s_{23}^\nu(M_Z) = \frac{1}{\sqrt{2}} \left( 1 + \frac{\eta_{\text{RG}}}{4} \right), \quad s_{13}^\nu(M_Z) \propto \eta_{\text{RG}} \frac{m_2}{m_3} \]  

(5.6)

5.1 Combined treatment of RG and canonical normalisation corrections

Finally, one can even subsume the effects of the RG and CN corrections to tri-bimaximal mixing into a single physical parameter:

\[ \eta = \eta_{\text{RG}} + \eta^K, \]  

(5.7)

where \( \eta_{\text{RG}} \) is defined in \( (5.5) \) and \( \eta^K \) is given in section \( 4 \) c.f. Eq. \( (4.2) \).

As a consequence, in the presence of both (3rd family dominated) RG and CN corrections, the change to the neutrino mixing sum rule \( (4.26) \) in the presence of charged lepton corrections \( \cite{2} \) depends only on the total correction parameter: \( \theta_{12}^{TB} \approx \theta_{12} - \theta_{13} \cos \delta - \frac{\eta}{6\sqrt{2}} \).

As in section \( 4.4 \) we can use a measurement of \( \theta_{23} \) (see e.g. \cite{30}) to reconstruct \( \eta \) and arrive at a testable relation: \( s = r \cos \delta + \frac{2}{3}a \). We would like to emphasise that this formula holds in the presence of three of the main theoretical corrections to the tri-bimaximal mixing scheme, namely the lepton sector corrections, the RG running between \( M_{\text{GUT}} \) and \( M_Z \) and canonical normalisation of the kinetic terms. However, let us recall that in addition to the assumptions made for the sum rule without corrections \( \cite{2} \) it has been assumed that the 3rd family effects dominate the RG running and also the canonical normalisation. Moreover, the result \( (4.26) \) holds only at leading order in the small quantities \( s_{12}, \theta_{13}, m_2/m_3, \eta \) and at the level of the leading logarithmic approximation as regards the RG corrections. While the size of the RG effects depends mainly on \( \tan \beta \) (which governs the size of \( y_\tau \)) and on the \( M_3 - M_{\text{GUT}} \) hierarchy, the size of the canonical normalisation corrections depends on the messenger sector, as we discuss in detail in the next section.
6. Situation in potentially realistic flavour models

As discussed in the Introduction, it has been pointed out that the observed close-to tri-bimaximal lepton mixing, along with the main features of the quark and charged lepton sector observables, can be understood in frameworks with no non-Abelian family symmetry \( (\mathcal{F}) \), that is spontaneously broken by the VEVs of three flavons \( \phi_3, \phi_{23}, \phi_{123} \) (transforming as triplets under \( \mathcal{F} \)) pointing in particular directions in the family space. These flavon fields give rise to the Yukawa operators of the shape (in the case of \( SU(3) \) family symmetry, dropping superfield hats) \[6\]:

\[
W_Y \approx f_i^f f_{cj}^j H M_2 f_i \left( y_{f1}^j (\phi_{123})_i (\phi_{23})_j + y_{f2}^j (\phi_{23})_i (\phi_{123})_j + y_{f3}^j (\phi_{3})_i (\phi_{3})_j + y_{f4}^j (\phi_{23})_i (\phi_{23})_j \right). \tag{6.1}
\]

This approach requires the vacuum alignment in the \( SU(3) \) space of the form:

\[
\begin{align*}
\phi_3 & \sim (0, 0, 1), \\
\phi_{23} & \sim (0, 1, 1), \\
\phi_{123} & \sim (1, 1, 1),
\end{align*}
\]

up to phases.

Note also that there is in principle at least two distinct types of messengers entering the formula \( \phi_{ij} \), in particular those transmitting the \( SU(2)_L \) doublet nature of \( f = Q, L \) to the Higgs VEV insertion point (for definiteness let’s call them \( \chi_{Q,L} \)), and the \( SU(2)_L \)-singlets propagating further the remaining \( SU(3)_c \otimes U(1)_Y \) quantum numbers to \( f^c = u^c, d^c, e^c \) and \( \nu^c \) (to be called \( \chi_{u,d,e,\nu} \)), c.f. Figure 4. However, for sake of simplicity, we shall use a generic symbol \( M_f \) for both these classes and come back to this distinction only upon getting to physical implications. Later in this section we shall address the question of topology of the underlying messenger sector Feynman graphs giving rise to the operators under consideration. We shall also discuss the relationship between these messengers and those which appear in the Kähler potential.

6.1 Example 3: Potentially realistic \( SU(3)_f \) flavour model example

As an example, let us focus on the canonical normalisation corrections to the tri-bimaximal neutrino mixing in the classes of \( SU(3)_f \) flavour models considered in [20]. Taking into account the irrelevance of the canonical transformation \( P_{\nu} \) in the see-saw formula for the light neutrino masses, the quantity of our main interest is the leading order Kähler metric for the lepton doublets \( K_L \) obeying:

\[
K_L \approx k_0^L 1 + \begin{pmatrix}
\varepsilon^4_K k_1^L e^{i\phi_1} & \varepsilon^4_K k_1^L e^{i\phi_2} \\
\varepsilon^2_K k_2^L e^{i\phi_3} & \varepsilon^2_K k_2^L e^{i\phi_4} \\
\varepsilon^2_K k_3^L e^{i\phi_5} & \varepsilon^2_K k_3^L e^{i\phi_6}
\end{pmatrix} + \ldots , \tag{6.2}
\]

where the subscript \( K \) in the expansion parameters \( \varepsilon_K \) and \( \varepsilon_K \) indicates that the Kähler metric messenger masses \( M_K \) (entering through e.g. \( \varepsilon_K = \langle \phi_3 \rangle / M_K \) may differ from
those relevant for the Yukawa sector \((6.1)\) and the dotted terms in \((6.2)\) can be reconstructed from hermiticity. The \(P_L^{-1}\) matrix is obtained\(^\text{18}\) to leading order in \(\varepsilon_K,\varepsilon_{K3}\) as:

\[
P_L^{-1} = \begin{pmatrix}
\frac{\sqrt{k_0^L} + \varepsilon_1 k_0^L}{2k_0^L} & \frac{\varepsilon_1 k_0^L e^{i\phi_1}}{2\sqrt{k_0^L}} & \frac{\varepsilon_1 k_0^L e^{i\phi_2}}{\sqrt{k_0^L + k_0^L + k_3^L \varepsilon_{K3}^5}} \\
\frac{\varepsilon_1 k_0^L e^{-i\phi_1}}{2k_0^L} & \frac{\varepsilon_2 k_0^L e^{-i\phi_1}}{2\sqrt{k_0^L}} & \frac{\varepsilon_2 k_0^L e^{-i\phi_3}}{\sqrt{k_0^L + k_0^L + k_3^L \varepsilon_{K3}^5}} \\
\frac{\varepsilon_2 k_0^L e^{-i\phi_3}}{\sqrt{k_0^L + k_3^L \varepsilon_{K3}^5}} & \frac{\varepsilon_3 k_0^L e^{-i\phi_3}}{\sqrt{k_0^L + k_3^L \varepsilon_{K3}^5}} & \frac{\varepsilon_3 k_0^L e^{-i\phi_3}}{\sqrt{k_0^L + k_3^L \varepsilon_{K3}^5}}
\end{pmatrix} + \ldots \quad (6.3)
\]

Notice that due to the relatively large \(\varepsilon_{3K} \sim 0.5\), the naive factorisation \(P = \sqrt{\delta M_0}(1 + \Delta P)\) (with \(|\Delta P| = -\frac{1}{2} \Delta K_L \ll 1\) for \(K_L = k_0^L(1 + \Delta K_L)\)) is violated in the third family due to higher power \(\varepsilon_{K3}\)-effects.

**Charged lepton sector:**

As can be seen from the shape of the charged lepton Yukawa matrices before and after canonical normalisation (Eqs. (2.12) and (2.62) of \([20]\)), the charged lepton mixing angles themselves as well as the CN corrections are small. In particular, \(\theta_{12}^{\ell,L} \approx (y_1^L/y_2^L)^2\) (in the notation of \([20]\)) is unchanged at leading order, so we can still treat the charged lepton mixing angles as only (CKM-like) small corrections to the neutrino sector dominated \(U_{PMNS}\). Therefore, in what follows, we shall first focus on the neutrino sector.

**Neutrino sector:**

In the class of models under consideration, the Majorana mass matrix \(\hat{M}_M\) originates from operators which involve factors like \(f^{ij} f^{ij}(\phi_{23})_i(\phi_{23})_j\) and \(f^{ij} f^{ij}(\phi_{123})_i(\phi_{123})_j\). The relevant matrix structures read \([6]\):

\[
\hat{Y}_\nu = \begin{pmatrix}
0 & B & C_1 \\
A & B e^{i\phi_1} + A e^{i\phi_1} & C_2 \\
A e^{i\phi_3} & B e^{i\phi_2} + A e^{i(\phi_1 + \phi_3)} & C_3
\end{pmatrix}, \quad \hat{M}_M = \begin{pmatrix}
M_A & M_A e^{i\phi_1} & 0 \\
M_A e^{i\phi_1} & M_A e^{2i\phi_1} + M_B & 0 \\
0 & 0 & M_C
\end{pmatrix}, \quad (6.4)
\]

where the real positive entries in \(\hat{M}_M\) satisfy \(M_A < M_B < M_C\). In terms of the expansion parameters \([20]\), the neutrino Yukawa matrix is given by

\[
\hat{Y}_\nu = \begin{pmatrix}
0 & \varepsilon_3 y_1 \\
\varepsilon_3 y_2 & \varepsilon_3 (y_1 e^{i\phi_1} + y_2 e^{i\phi_1}) \\
\varepsilon_3 y_2 e^{i\phi_3} & \varepsilon_3 (y_1 e^{i\phi_2} + y_2 e^{i(\phi_1 + \phi_2)})
\end{pmatrix} + \ldots, \quad (6.5)
\]

\(^{18}\)Recall that relation \((2.3)\) fixes the \(P\)-matrices only up to a global unitary transformation; as before we adopt the convention \(\hat{U}_L = \hat{U}_L\) so that \(P\)'s are Hermitean, c.f. section \([2.4]\).
which matches Eq. (1.4) with \( A = y_2^E \varepsilon^3 \) and \( B = y_1^E \varepsilon^3 \), \( \varepsilon \approx 0.05 \). The CN transformation (2.3) then yields (since \( P_{\nu e} \) drops off the seesaw formula for the light Majorana neutrinos, we are free to choose for simplicity \( P_{\nu e} = 1 \)) \( M_M = \hat{M}_M \) and:

\[
Y' = \begin{pmatrix}
\mathcal{O}(\varepsilon^7) \\
\varepsilon^3 y_2 \\
\varepsilon^3 y_3 e^{i\phi_3} \\
\sqrt{k_0^L} \left( 1 + \varepsilon^2 K_3 \right)
\end{pmatrix}
\]

In order to extract the mixing angles analytically from these matrices, it is convenient to transform \( Y' \) and \( M_M \) by means of a suitable non-singular matrix \( S \) [10, 20]:

\[
Y' \rightarrow Y'' = Y' S^{-1} \quad M \rightarrow M' = S^{-1} M S^{-1} \quad M^{-1} \rightarrow M''^{-1} = S M^{-1} S^T,
\]

(6.7)

(which again leaves the neutrino mass matrix invariant) to the case of a diagonal \( M_M = \text{diag}(M_A, M_B, M_C) \) which corresponds to:

\[
Y'' = \begin{pmatrix}
\mathcal{O}(\varepsilon^7) \\
\varepsilon^3 y_2 \\
\varepsilon^3 y_3 e^{i\phi_3} \\
\sqrt{k_0^L} \left( 1 + \varepsilon^2 K_3 \right)
\end{pmatrix}
\]

(6.8)

**Formulae for the corrected neutrino mixing angles:**

Since the last transformation brought the neutrino Yukawa and Majorana matrices into a particular form along the lines of the Sequential Dominance setting [23], from Eq. (1.8) we can directly read off the mixing angles (imposing \( \phi_2 - \phi_1 = \phi_3 - \pi \)) at leading order in \( m_2/m_3 \) (making use of the generic formulae given in [24, 25]):

\[
\tan \theta_{23}^\nu \approx \sqrt{1 + \eta^K}, \quad \tan \theta_{12}^\nu \approx \frac{1}{\tan \theta_{23}^\nu + \frac{\eta^K}{\sqrt{1 + \eta^K}}}, \quad \theta_{13}^\nu \approx \frac{m_2}{m_3} \sqrt{\frac{\cos^2 \theta_{12}^\nu}{1 + \eta^K}} \quad (6.9)
\]

with \( \eta^K = k_3^L \varepsilon^2 K_3 / k_0^L \). The \( \eta^K \)-behavior of these relations is illustrated in Fig. 2. We see that this independent calculation confirms the findings of the previous sections for small \( \eta^K \). To give a quantitative example we may take \( \varepsilon K_3 = 0.5 \) and set the \( \mathcal{O}(1) \) coefficients \( k_3^L \) and \( k_0^L \) to 1 yielding \( \eta^K = 0.25 \) and \( \theta_{23}^\nu = 50.8^\circ, \theta_{12}^\nu = 38.3^\circ \) and \( \theta_{13}^\nu = 1.1^\circ \), compared to the tri-bimaximal neutrino mixing predictions \( \theta_{23}^\nu = 45^\circ, \theta_{12}^\nu = 35.26^\circ \) and \( \theta_{13}^\nu = 0^\circ \) before canonical normalisation.
6.2 Heavy $SU(2)_L$ doublet messengers & natural $\eta^K$ suppression

Since the naive estimate of the CN effects above leads to non-negligible deviations from the TB-mixing in the lepton sector (in particular for relatively large $|\eta^K|$), let us sketch in brief the prospects of getting $k_3^L/k_0^L$ (and thus $\eta^K$) naturally suppressed in the class of popular $SU(3)$ and $SO(3)$ flavour models.

Recall first that the $k_0^L$ coefficient governs the “canonical”, i.e. renormalizable contribution in the Kähler $\propto \partial^\mu \tilde{L}^\dagger \partial_\mu \tilde{L}$ (for scalars) while $k_3^L$ emerges at higher order via operators like $\frac{1}{M_K} \partial^\mu \tilde{L}^\dagger \partial_\mu \tilde{L} \phi_3^\dagger \phi_3$ only and therefore is sensitive to the relevant messenger sector masses. Second, due to the self-conjugated structure of this type of operators, any messenger $\psi, \psi^c$ relevant for the Yukawa sector operators, i.e. with simultaneous couplings to flavon and matter superfields (like e.g. $\tilde{L} \phi \tilde{\psi}^c$) necessarily enters the matter sector Kähler metric via effective operators of the form $\frac{1}{M_K} \partial^\mu \tilde{L}^\dagger \partial_\mu \tilde{L} \phi^\dagger \phi$ because no symmetry forbids such structures, c.f. Figure 3.

![Figure 3](image)

**Figure 3:** A typical tree level correction to the Kähler of LH fermions generating the effective $k_3^L$ couplings driving the canonical normalisation corrections to the tri-bimaximal lepton mixing pattern.

Since $SU(2)_L$ must remain intact upon flavour symmetry breaking, the messengers potentially affecting $k_3^L$ must necessarily be $SU(2)_L$-doublets, otherwise they can not couple to $\tilde{L} \phi$. In what follows, we shall namely check whether the $SU(2)_L$-doublet part of the messenger sector (if any) in the popular models can be naturally made heavy compared to the $SU(2)_L$-singlet messenger fields (transforming as $SU(2)_R$-doublet in the PS approach).

**Models with $SU(3)$ family symmetry**

Starting with models based on $SU(3)$ family symmetry (or its discrete subgroups like $\Delta_{27}$), the triplet nature of both matter chiralities $f, f^c$ calls for a pair of antitriplet flavon insertions (up to the singular case of $3_f, 3_{f^c}, 3_{\phi}$-type contractions) so that the simplest Yukawa couplings have the internal structure depicted at Figure 4 (for discussions of the messenger sector of SU(3) models, see e.g. [31]).
case 1:

$$SU(3)_f : 3 \ 3 \ 1 \ 1 \ 1 \ 1 \ 3 \ 3 \ 3$$
$$SU(2)_L : 2 \ 2 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1$$
$$SU(2)_R : 1 \ 2 \ 2 \ 2 \ 1 \ 2 \ 1 \ 1 \ 2$$
$$PS : 4 \ 1 \ 4 \ 4 \ 1 \ 4 \ 4 \ 1 \ 4$$

case 2:

$$SU(3)_f : 3 \ 3 \ 1 \ 1 \ 1 \ 1 \ 3 \ 3 \ 3$$
$$SU(2)_L : 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 1 \ 1 \ 1$$
$$SU(2)_R : 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 1 \ 2 \ 2$$
$$PS : 4 \ 1 \ 4 \ 4 \ 1 \ 4 \ 4 \ 1 \ 4$$

case 3:

$$SU(3)_f : 3 \ 3 \ 1 \ 1 \ 3 \ 3 \ 3 \ 1 \ 3$$
$$SU(2)_L : 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 2 \ 1 \ 2$$
$$SU(2)_R : 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 2$$
$$PS : 4 \ 1 \ 4 \ 4 \ 1 \ 4 \ 4 \ 1 \ 4$$

Figure 4: The three basic configurations of the messenger sector leading to the lowest order Yukawa sector effective operators in a typical $SU(3)$ flavour model. The position the SM Higgs VEV enters determines the $SU(2)_L \otimes SU(2)_R$ quantum numbers of the underlying messenger sector. We have used $\psi_i$ for the $SU(2)_L$ doublets while $\chi_i$ for the $SU(2)_R$ doublets respectively.

The usual strategy in order to keep the particle content of a model minimal is to exploit just some of these topologies for all the Yukawa sector entries. Typically, the first alternative is chosen, because in such a case the spectra of the $\chi$-type of messengers are sensitive to the large scale $SU(2)_R$ breaking providing for a bit more freedom in the Yukawa sector construction. This actually works rather well for all but the 33 Yukawa entries, that are preferred close to each other, at odds with the scaling properties of the other Yukawa entries (driven by expansion factors $\varepsilon^2$ or $\overline{\varepsilon}^2$ with $\varepsilon \sim 0.05$ and $\overline{\varepsilon} \sim 0.15$ for the up- and down-type sector respectively) and thus calling for extra contributions.

Such terms can then come from either an extra $\phi_3$-type flavon entering the graph of the same type (i.e. case 1 in Figure 1) which has been exploited e.g. in the $SU(3)$ model by Varzielas-Ross [6] by means of the particular $SU(2)_R$-structure of $\phi_3 = 1 \oplus 3$ (c.f. Figure 2), or from a more complicated messenger sector with a left-handed “$\psi$-type” messengers admitting the other (case 2,3 in Figure 4) contributions to the 33 Yukawa
Figure 5: Generating a pair of extra contributions to the 33 Yukawa entries by means of an extra $SU(2)_R$ triplet of $\phi_3$-type flavon fields $\phi_3'$. Notice that the messenger sector is $SU(2)_L$-singlet and thus does not trigger a potentially large violation of tri-bimaximal mixing in the lepton sector.

coupling. However, with the latter choice, a relatively light “left-handed” messenger must be postulated, leading to the instability of the tri-bimaximal lepton sector mixing generated by a potentially large deviation from universality in the $K_L$ part of the Kähler metric.

Thus, in order to avoid the potentially dangerous light $SU(2)_L$-doublet messengers $\psi$ one should base the effective Yukawa sector on the topologies of type 1 in Fig. 4 that, however, comes for the price of extending the third-family flavon sector along the lines of $\phi_3$ and further complication in the vacuum alignment mechanism.

**Models with $SO(3)$ family symmetry**

The situation in models based on $SO(3)$ is slightly simplified by the fact that the basic nontrivial singlet structure can be built out of two rather than three triplets without complex conjugation. Thus, in order to get realistic Yukawa patterns, only one chiral component (typically $f$) should transform as a triplet while the other as an $SO(3)$-singlet $[33, 24, 7]$. At the lowest level (in number of flavon insertions), we are left with only two basic options depicted in Fig. 6.

Again, one can utilise the right-handed messengers (i.e. doublets of $SU(2)_R$) to obtain most of the desired Yukawa structures, however, the above mentioned “irregularity” in the 33 entries calls for an extra contribution as in the $SU(3)$ case. Again, the basic options are either adding a left-handed (i.e. $SU(2)_L$ doublet ) messenger sector fields along case 2 indicated at Figure 6, c.f. 6, with potential impact on the left-handed Kähler corrections, or employ an extra $\phi_3$-type flavon, c.f. Figure 7.

To conclude, the popular $SU(3)_f$-based flavour models à la Ross and Varzielas 11, do not in general suffer from large Kähler corrections to the lepton sector tri-bimaximal mixing pattern due to the mere absence of the potentially dangerous $SU(2)_L$ doublets in

\[^{19}\]This, however, leads to problems with universality of the right-handed soft masses in SUSY, that in potentially realistic setups must be addressed by further assumptions.
case 1: 

\[
\begin{array}{cccc}
SO(3)_f & : & 3 & 1 & 3 \\
SU(2)_L & : & 2 & 2 & 1 \\
SU(2)_R & : & 1 & 2 & 2 \\
PS & : & 4 & 1 & 4
\end{array}
\]

Figure 6: The two basic configurations of the messenger sector leading to the lowest order Yukawa sector effective operators in a typical SU(3) flavour model. The position of the SM Higgs VEV determines the SU(2)_L \otimes SU(2)_R quantum numbers of the underlying messenger fields. As before, we have used \(\psi_i\) for the SU(2)_L doublets while \(\chi_i \) for the SU(2)_R doublets respectively. Note that the \(\psi_1\) messenger in case 2 is usually “flavon-specific” and it is possible to forbid all the unwanted \(\phi_{123}, \phi_{23}\) type of insertions by just a proper choice of the messenger sector quantum numbers.

\[
\begin{array}{cccc}
SO(3)_f & : & 3 & 3 & 1 \\
SU(2)_L & : & 2 & 1 & 2 \\
SU(2)_R & : & 1 & 1 & 1 \\
PS & : & 4 & 1 & 4
\end{array}
\]

the messenger sector. On the other hand, the SO(3)-class of models in versions [7] can lead to substantial Kähler corrections because of employing a relatively light SU(2)_L doublet in the messenger sector to resolve the 33 Yukawa issue. However, these models can be cured easily by invoking instead the “extra \(\phi_3\)-type flavon solution” with only SU(2)_R-doublet light messengers entering the Kähler metric along the lines sketched above.

7. Summary and discussion

In summary, we have analysed the effect on quark and lepton mixing of third family wavefunction corrections. Such third family wavefunction effects can arise from either the canonical normalisation of the kinetic terms or renormalisation group running effects. At leading order we have shown that both sorts of corrections can be subsumed into a single universal parameter. With hierarchical neutrinos and tri-bimaximal mixing in the neutrino sector, and Cabibbo-like mixing in the charged lepton sector, we showed that the solar mixing angle deviates from its tri-bimaximal value due to the effect of both charged lepton mixing and third family wavefunction corrections, leading to the lepton mixing sum...
rule \( s = r \cos \delta + \frac{2}{7}a \) (where \( s, r, a \) describe the deviations of solar, reactor and atmospheric mixing angles from their tri-bimaximal values, and \( \delta \) is the observable Dirac CP phase).

Where a specific framework was useful or necessary in order to quantify the effects, we have considered realistic SUSY models based on non-Abelian family symmetry. In particular we have focussed on classes of models capable of providing a successful description of quark and lepton masses and mixing including neutrino masses and tri-bimaximal mixing. We have focussed on models with non-Abelian family symmetries spanning all three generations such as \( SU(3) \) or \( SO(3) \), or their discrete subgroups \( \Delta_{27} \) or \( A_4 \), in which tri-bimaximal neutrino mixing can naturally emerge. In such models all Yukawa couplings vanish in the family symmetry limit and become generated only after the family symmetry gets broken, due to flavon \( \phi \) VEVs, leading to effective Yukawa couplings expressed in terms of expansion parameters \( \varepsilon \sim \langle \phi \rangle / M \) (with \( M \) denoting the typical underlying messenger sector mass). For the third family, such expansion parameters appearing in the superpotential are rather large. If such large expansion parameters also enter the matter sector Kähler metric it necessitates a large canonical normalization of the third family superfields leading to potentially large corrections to the quark sector \( V_{cb} \) mixing and the tri-bimaximal mixing in the lepton sector. Such effects are of particular relevance in the lepton sector, because the physical solar and atmospheric mixing angles are large, and the canonical corrections are generally multiplicative in nature.

According to such arguments, the effects of the third family rescaling are expected to dominate over the effects in the other families, and one can parametrise all the effects in the physical mixing matrices by means of a single parameter \( \eta^K \) proportional to the non-universalities in the LL-part of the matter sector Kähler metric in the quark and lepton sectors respectively. We developed a general perturbative formalism which enables the CN effects in both quark and lepton sectors to be estimated. In the (2,3) quark sector, we found that such rescaling effects can account for a relatively large change in \( V_{cb} \), that (although still only multiplicative) could be much larger than the estimates given previously in the literature where the presence of a large third family expansion parameter was not considered. Concerning leptons, we found the same kind of multiplicative corrections for all three mixing angles, c.f. Eq. (4.7). The physical effect, however, is strongly amplified compared to the quark sector, because of the tri-bimaximality of the solar and atmospheric mixings. On the other hand, the reactor angle receives only sub-leading corrections and remains stable. Our results receive a particularly simple form in terms of the \( r, s \) and \( a \) parameters quantifying the departure from the exact tri-bimaximal setting, in particular \( r \approx 6 \times 10^{-2} \eta^K \), \( s \approx \frac{2}{8} \eta^K \), and \( a \approx \frac{3}{4} \eta^K \), see Eq. (4.12) and Fig. [1]. We have also independently checked the results obtained by means of the perturbative method by considering the situation in a particular potentially realistic \( SU(3) \) model, that in certain situations allows a more straightforward estimation of the CN effects in the lepton sector. Paying particular
attention to the sub-leading corrections due to first and second generation effects, we have shown that the approximations used in the general estimate are very accurate and the higher order terms as well as the light sector effects can be in most cases neglected, see Fig.2 in comparison to Fig.1.

Certain classes of unified theories of flavour, including those considered explicitly here, predict very accurate tri-bimaximal neutrino mixing which is subject to charged lepton corrections that are Cabibbo-like, i.e. dominated by the 12 sector mixing. As discussed, this leads to a testable sum rule relating the solar and reactor angle deviations from trimaximality \( s, r \) and the lepton sector Dirac CP phase \( \delta \), namely \( s = r \cos \delta \). We studied in detail how such a relation is affected by the third family rescaling corrections. Since the leading order effects in the solar and atmospheric mixings are both governed by \( \eta^K \), this leads to a generalised sum rule of the form \( s = r \cos \delta + \frac{2}{3}a \), where \( a \) stands for the deviation of the atmospheric mixing from maximality. Remarkably enough, this sum-rule is also stable under leading order renormalisation group running effects. This is a consequence of the fact that both the CN and RG corrections are dominated by third generation effects, and thus both effects can be accounted for simultaneously by a redefinition of the value of the canonical parameter \( \eta^K \rightarrow \eta^K + \eta^{RG} \) where the separate contributions are indistinguishable in the leading formulae under consideration.

We have discussed in some detail the question of the magnitude of the canonical corrections that are expected in realistic SUSY models based on non-Abelian family symmetry. The quantitative significance of the effects described in this paper rests on the key question to what extent the large Yukawa sector expansion parameter is mirrored to the left chirality piece of the matter sector Kähler metric, i.e. whether the third family rescaling parameter \( \eta \) is necessarily numerically large. In order to address this we have provided a detailed discussion of the messenger sectors responsible for both the Kähler potential and the superpotential corrections in the class of \( SU(3) \) and \( SO(3) \) flavour models and its derivates. We argue that the relevant piece of the Kähler metric is controlled by the same \( SU(2)_L \)-doublet messenger fields \( \psi \) that on the superpotential side give rise to a particular class of effective Yukawa operators involving direct couplings of the matter sector \( SU(2)_L \)-doublets \( Q, L \) to \( \psi \). The magnitude of \( \eta^K \) descending from the Kähler potential then may or may not be related to the large expansion parameter in the Yukawa operators, depending on whether these are governed by diagrams in which the lightness of \( \psi \) plays a crucial role. In many cases, for example the \( SU(3) \) or \( \Delta_{27} \) models, the left-handed messengers sector essentially decouples from the effective Yukawa couplings and the Kähler metric for the left-chirality matter fields is only subject to small corrections leading to \( \eta^K \sim 0 \) and thus negligible CN effects. On the other hand, in the \( SO(3) \) or \( A_4 \) models, the left-handed messengers \( \psi \) have been assumed to be quite light, in which case the wavefunction effects of third family rescaling described in this paper are expected to be large with \( \eta^K \sim \mathcal{O}(1) \).
In conclusion, third family wavefunction corrections arising from either renormalisation group running or canonical normalization effects (or in practice both together) can give important corrections to the physical predictions of realistic SUSY models based on non-Abelian family symmetry, where such models are interesting since they are capable of predicting the entire quark and lepton spectrum including tri-bimaximal neutrino mixing. Although the precise magnitude of the canonical normalization effect is model dependent, it is possible to derive a theoretically stable lepton mixing sum rule $s = r \cos \delta + \frac{2}{3}a$ (where $s, r, a$ describe the deviations of solar, reactor and atmospheric mixing angles from their tri-bimaximal values, and $\delta$ is the observable Dirac CP phase) which is insensitive to the canonical and renormalisation group effects in leading order, independent of their magnitude. Such a theoretically reliable sum rule can be tested in future high precision neutrino experiments.

Acknowledgements

We acknowledge partial support from the following grants: PPARC Rolling Grant PPA/G/S/2003/00096; EU Network MRTN-CT-2004-503369; EU ILLIS RII3-CT-2004-506222; NATO grant PST.CLG.980066.

Appendices

A. Conventions - CKM & PMNS mixing matrices

In general, the mixing matrix in the lepton sector, the PMNS matrix $U_{PMNS}$, is defined as the matrix appearing in the charged electroweak currents expressed in terms of lepton mass eigenstates. Denoting the charged lepton mass matrix by $M_l$ and the light neutrino mass matrix by $m_\nu$, the mass part of the matter sector lagrangian reads:

$$L = -\bar{L}_L M_l l_R - \frac{1}{2}\bar{\nu}_L m_\nu \nu_L^c + H.c.$$  \hspace{1cm} (A.1)

Performing the transformation from flavour to mass basis by

$$V_L^T M_l V_R = \text{diag}(m_e, m_\mu, m_\tau), \quad V_L^\nu m_\nu V_L^{\nu T} = \text{diag}(m_1, m_2, m_3),$$ \hspace{1cm} (A.2)

the PMNS matrix is given by

$$U_{PMNS} = V_L^T \nu_L^{\nu T}.$$ \hspace{1cm} (A.3)

Here it is assumed implicitly that unphysical phases are removed by field redefinitions, and $U_{PMNS}$ contains one Dirac phase and two Majorana phases\textsuperscript{20}.

\textsuperscript{20}The latter are physical only in the case of Majorana neutrinos, for Dirac neutrinos the two Majorana phases can be absorbed as well.
The standard PDG parameterisation of the PMNS matrix (see e.g. [12]) is:

$$U_{PMNS} = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
  -c_{13}s_{23} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & s_{23}c_{13} \\
  s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -c_{23}s_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{23}c_{13}
\end{pmatrix} P_M , \quad (A.4)$$

which is used in most analyses of neutrino oscillation experiments. Here $\delta$ is the Dirac CP violating phase which is in principle measurable in neutrino oscillation experiments, and $P_M = \text{diag}(e^{i\frac{\alpha_1}{2}}, e^{i\frac{\alpha_2}{2}}, 1)$ contains the two measurable Majorana phase differences $\alpha_1, \alpha_2$. In the body of this manuscript we use this standard parameterisation also for $V_{\nu L}^\dagger$ and denote the corresponding mixing angles by $\theta_{\nu ij}^\nu$, while the mixing angles $\theta_{ij}$ without superscript refer to the PMNS matrix.

B. Derivation of the lepton mixing sum-rules

Let us recapitulate here the derivation of the sum-rules of our interest along the lines they were originally obtained in [3]. We shall for the moment forget about the canonical normalisation effects and drop all the hats in what follows. Later on in Appendix C we shall reiterate the same procedure carefully with all the potential sources of deviations due to canonical normalisation taken into account.

Perhaps the simplest method to obtain (4.15) and (4.17) consists in looking at particular elements of the lepton mixing matrix:

$$U_{PMNS} = V_{\nu L}^\dagger V_{\nu L}^\nu = V_{\nu L}^\dagger U_{TB} \quad (B.1)$$

and exploiting the fact that the particular shape of $V_{\nu L}^\dagger$ exposes unaltered the third row of the tri-bimaximal neutrino sector mixing in $U_{TB}$ and also the 13 entry of $U_{PMNS}$ receives a particularly simple form. Indeed, one easily obtains $|U_{PMNS}|_{3i} = |U_{TB}|_{3i}$ that (upon employing the standard parametrisation (A.4)) gives in particular:

$$31 \text{ entry : } \left(s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}\right) e^{i\frac{\alpha_1}{2}} = s_{12}^\nu s_{23}^\nu \quad \text{up to a global phase}, \quad (B.2)$$

and also:

$$13 \text{ entry : } s_{13}e^{-i\delta} = -s_{12}^\nu e^{-i\rho} s_{23}^\nu \quad \text{up to a global phase}. \quad (B.3)$$

First, notice that a nonzero $\theta_{13}$ mixing is generated from a conspiracy between the 12 charged lepton sector mixing and $\theta_{23}^\nu$. In a wide class of models with a built-in Georgi-Jarlskog mechanism (leading typically to $\theta_{12}^\nu \approx \theta_C/3$ with $\theta_C \approx \lambda \approx 0.2$ denoting the

---

21 Since we shall be looking on the magnitude of the matrix elements, the particular phase convention employed here is immaterial, but clearly must not be altered during the computation.
Cabibbo CKM mixing governed by the down-type quark sector) one gets\[ \theta_{13} \approx \frac{\theta_{12}}{\sqrt{2}}.\] Second, formula (B.2) subsequently leads to
\[ s_{12}s_{23} - c_{12}c_{23}s_{13}\cos\delta = s_{12}'s_{23}' . \] (B.4)

Since the 23 sector mixing is stable under the perturbation (4.13), one can trade \( s_{23} \) and \( c_{23} \) in (B.4) for their TB values \( \frac{1}{\sqrt{2}} \) while the RHS gives \( \frac{1}{\sqrt{6}} \). Expanding the left-hand side of (B.4) for small \( s_{13} \approx \theta_{13} \) one gets:
\[ s_{12} - c_{12}\theta_{13}\cos\delta = \frac{1}{\sqrt{3}} . \] (B.5)

The last step is to expand the physical \( \theta_{12} \) around the tri-bimaximal value \( \theta_{12} = \theta_{12}^{TB} + \Delta \theta_{12} \) which yields \( s_{12} = \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\Delta \theta_{12} \) and \( c_{12} = \frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{3}}\Delta \theta_{12} \), leading to:
\[ \sqrt{\frac{2}{3}}\Delta \theta_{12} - \theta_{13}\left(\frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{3}}\Delta \theta_{12}\right)\cos\delta = 0 . \] (B.6)

Forgetting about the doubly-suppressed \( \theta_{13}\Delta \theta_{12} \) term on the LHS of (B.6), we get:
\[ \Delta \theta_{12} \approx \theta_{13}\cos\delta \quad \text{yielding} \quad \theta_{12} = \theta_{12}^{TB} + \theta_{13}\cos\delta , \] (B.7)

providing a simple estimate for the deviation of the solar mixing angle \( \theta_{12} \) from its tri-bimaximal value \( \theta_{12}^{TB} = 35^\circ 16' \) in terms of two other lepton sector measurables, namely the reactor mixing angle \( \theta_{13} \) and the Dirac CP phase \( \delta \).

C. Corrections to the sum-rule \( s = r \cos\delta \)

Suppose the assumptions made in Appendix B, leading in particular to formula (B.7), hold at the underlying flavour-model level, i.e. in the defining basis only. Thus, for sake of consistency with the notation used in the body of the manuscript, we shall re-equip all the relevant quantities therein with hats obtaining \( \hat{\theta}_{12} = \theta_{12}^{TB} + \hat{\theta}_{13}\cos\hat{\delta} \) as only the leading order approximation to the physical (i.e. corrected) sum-rule, that should be written in terms of only unhatted quantities. The scope of this section is to see what happens once the effects of RG running and canonical normalisation are turned on.

Along similar lines as in section 4.4 one obtains first (utilizing the perturbative procedure of section 2.3 for \( \eta = \eta^{K} + \eta^{RG} \), c.f. equations (4.19)-(4.22):
\[ \Delta(U_{PMNS})_{31} = \frac{\eta}{2}(U_{TB})_{31} \left[1 - |(U_{TB})_{31}|^2\right] , \] (C.1)
and thus (up to the Majorana phase associated to the \((U_{TB})_{31}\) entry):

\[
\Delta(U_{PMNS})_{31} = \frac{\eta}{2} s_{12} s_{23} s_{13} \left[ 1 - |s_{12} s_{23}|^2 \right] \equiv s_{12} s_{23} \Delta, \tag{C.2}
\]

where \(\Delta \equiv \frac{\eta}{2} \left( 1 - |s_{12} s_{23}|^2 \right) = \frac{5}{12} \eta\). Notice that due to the phase structure of the \(\Delta(U_{PMNS})_{31}\) correction \((\ref{eq:B2})\), and in particular the \((U_{TB})_{31}\) term therein, the overall phase of the RHS of \((\ref{eq:B2})\) derived from \((\ref{eq:B1})\) and the phase of \(\Delta(U_{PMNS})_{31}\) coincide. This admits to write the analogue of relation \((\ref{eq:B2})\) (derived now from \(U_{PMNS} = \hat{U}_{PMNS} + \Delta U_{PMNS}\)) in a simple form:

\[
\left( s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} \right) e^{i\alpha_{1}/2} = s_{12} s_{23} (1 + \Delta) \quad \text{up to a global phase}. \tag{C.3}
\]

The next step (leading to \((\ref{eq:B5})\) in the ‘unperturbed’ case) would be to trade the 23 rotations for their tri-bimaximal values \(s_{23} = c_{23} = 1/\sqrt{2}\), that is completely plausible if there were no Kähler or RG corrections around, because the neutrino part of the 23-sector rotation in \(U_{PMNS} = V^L \bar{V}_L^T\) formula (provided both \(V^L \bar{V}_L^T\) are written as \(U_{23} U_{13} U_{12}\) along the lines of \([34]\)), hits the small charged-lepton correction only (upon being grouped together with 23-rotation in \(V^L \bar{V}_L^T\)), with just a negligible effect on the resulting physical 23 lepton sector mixing. However, turning \(\eta\) on, \(\theta_{23}\) becomes actually quite \(\eta\)-sensitive even in the simplest case (c.f. section 4.2), and thus putting \(s_{23} = c_{23} = 1/\sqrt{2}\) is not good enough.

Rather than that, we shall exploit the information\(^{22}\) obtained in section 4.2, see e.g. formula \((4.10)\), to write (at the leading order):

\[
s_{23} = \frac{1}{\sqrt{2}} (1 + a) = \frac{1}{\sqrt{2}} \left( 1 + \frac{\eta}{4} \right) \quad \text{and thus} \quad c_{23} = \frac{1}{\sqrt{2}} (1 - a) = \frac{1}{\sqrt{2}} \left( 1 - \frac{\eta}{4} \right), \tag{C.4}
\]

and from \((\ref{eq:C3})\) then (since \(\Delta\) is real):

\[
s_{12} \frac{1}{\sqrt{2}} (1 + a) - c_{12} \frac{1}{\sqrt{2}} (1 - a) \theta_{13} \cos \delta = \frac{1}{\sqrt{6}} (1 + \Delta). \tag{C.5}
\]

Expanding again the physical \(\theta_{12}\) around the tri-bimaximal value \(\theta_{12} = \theta_{12}^{TB} + \Delta \theta_{12}\), i.e. \(s_{12} = \frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}} \Delta \theta_{12}\) and \(c_{12} = \sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}} \Delta \theta_{12}\) and neglecting the higher order terms in \(a\), \(\Delta \theta_{12}\) and \(\theta_{13}\), one receives:

\[
\Delta \theta_{12} = \theta_{13} \cos \delta + \frac{1}{\sqrt{2}} (\Delta - a) = \theta_{13} \cos \delta + \frac{\eta}{6 \sqrt{2}}, \tag{C.6}
\]

\(^{22}\)Those results, though being obtained for zero \(\hat{\theta}_{12}\), provide a good leading order estimate of the atmospheric mixing \(\eta\)-behaviour and since the error due to the nonzero charged-lepton 12-sector mixing (hitting such a corrected 23 mixing) is the same (at the leading order) as in the “pure” case (i.e. without Kähler effects), it can be neglected as far as one looks for the deviations from the original sum-rule \((\ref{eq:B3})\).
which is an analogue of formula (B.7). The sum-rule with the Kähler corrections taken into account then reads:

\[
\theta_{12}^{TB} = \theta_{12} - \theta_{13} \cos \delta - \frac{\eta}{6\sqrt{2}}.
\]  

(C.7)

Notice that in the \(\theta_{12} \to 0\) limit (causing \(\theta_{13} \to 0\) and thus due to (1.23) also \(\theta_{13} \to 0\)) one indeed reveals the leading order effect (1.12) in the solar mixing \(s_{12} = \frac{1}{\sqrt{2}}(1 + \frac{\eta}{6})\) obtained in section 4.3, that in turn provides a non-trivial consistency check of relation (C.7).

References

[1] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530 (2002) 167 
[arXiv:hep-ph/0202074]; P. F. Harrison and W. G. Scott, Phys. Lett. B 535 (2002) 163 
[arXiv:hep-ph/0203209]; P. F. Harrison and W. G. Scott, Phys. Lett. B 557 (2003) 76 
[arXiv:hep-ph/0302025]; an earlier related ansatz was proposed by: L. Wolfenstein, Phys. Rev. D 18 (1978) 958.

[2] S. F. King, JHEP 0508 (2005) 105 [arXiv:hep-ph/0506297]; I. Masina, Phys. Lett. B 633 (2006) 134 [arXiv:hep-ph/0508031]; S. Antusch and S. F. King, Phys. Lett. B 631 (2005) 42 [arXiv:hep-ph/0508044]; S. Antusch, P. Huber, S. F. King and T. Schwetz, JHEP 0704 (2007) 060 [arXiv:hep-ph/0702286].

[3] P. H. Frampton, S. T. Petcov and W. Rodejohann, Nucl. Phys. B 687 (2004) 31 
[arXiv:hep-ph/0401206]; A. Dighe, S. Goswami and W. Rodejohann, Phys. Rev. D 75 (2007) 073023 [arXiv:hep-ph/0612328]; F. Plentinger and W. Rodejohann, Phys. Lett. B 625 (2005) 264 [arXiv:hep-ph/0507143]; R. N. Mohapatra and W. Rodejohann, Phys. Rev. D 72 (2005) 053001 [arXiv:hep-ph/0507312]; K. A. Hochmuth, S. T. Petcov and W. Rodejohann, arXiv:0706.2975 [hep-ph].

[4] G. Altarelli, F. Feruglio and Y. Lin, Nucl. Phys. B 775 (2007) 31 [arXiv:hep-ph/0610165]; G. Altarelli and F. Feruglio, Nucl. Phys. B 741 (2006) 215 [arXiv:hep-ph/0512103]; G. Altarelli and F. Feruglio, Nucl. Phys. B 720 (2005) 64 [arXiv:hep-ph/0504165]; F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B 775 (2007) 120 [arXiv:hep-ph/0702194]; C. Luhn, S. Nasri and P. Ramond, Phys. Lett. B 652 (2007) 27 [arXiv:0706.2341 [hep-ph]].

[5] E. Ma, arXiv:0708.0585 [hep-ph]; E. Ma, arXiv:hep-ph/0701016; E. Ma, Mod. Phys. Lett. A 22 (2007) 101 [arXiv:hep-ph/0610342]; E. Ma, Mod. Phys. Lett. A 21 (2006) 2931 [arXiv:hep-ph/0607190]; E. Ma, Mod. Phys. Lett. A 21 (2006) 1917 [arXiv:hep-ph/0607056]; E. Ma, H. Sawanaka and M. Tanimoto, Phys. Lett. B 641 (2006) 301 [arXiv:hep-ph/0606103]; E. Ma, Phys. Rev. D 73 (2006) 057304; B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma and M. K. Parida, Phys. Lett. B 638 (2006) 345 [arXiv:hep-ph/0603059]; E. Ma, Mod. Phys. Lett. A 20 (2005) 2601 [arXiv:hep-ph/0508099]; E. Ma, Phys. Rev. D 72 (2005) 037301 [arXiv:hep-ph/0505209]; S. L. Chen, M. Frigerio and E. Ma, Nucl. Phys. B 724 (2005) 423 [arXiv:hep-ph/0504181]; E. Ma, Phys. Rev. D 70 (2004) 031901 [arXiv:hep-ph/0404199].
[6] I. de Medeiros Varzielas and G. G. Ross, Nucl. Phys. B 733 (2006) 31
[arXiv:hep-ph/0507176]; I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B 644 (2007) 153 [arXiv:hep-ph/0512313]; I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B 648 (2007) 201 [arXiv:hep-ph/0607045];

[7] S. F. King and M. Malinsky, Phys. Lett. B 645 (2007) 351 [arXiv:hep-ph/0610250];
S. F. King and M. Malinsky, JHEP 0611 (2006) 071 [arXiv:hep-ph/0608021];

[8] P. F. Harrison and W. G. Scott, Phys. Lett. B 557 (2003) 76 [arXiv:hep-ph/0302025];
P. F. Harrison and W. G. Scott, Phys. Lett. B 535 (2002) 163 [arXiv:hep-ph/0203209];
R. N. Mohapatra, S. Nasri and H. B. Yu, Phys. Lett. B 639 (2006) 318
[arXiv:hep-ph/0605020]; R. N. Mohapatra and H. B. Yu, Phys. Lett. B 644 (2007) 346
[arXiv:hep-ph/0610023]; M. C. Chen and K. T. Mahanthappa, Phys. Lett. B 652 (2007) 34
[arXiv:0705.0714 [hep-ph]]; C. I. Low and R. R. Volkas, Phys. Rev. D 68 (2003) 033007
[arXiv:hep-ph/0305243]; X. G. He, Nucl. Phys. Proc. Suppl. 168 (2007) 350
[arXiv:hep-ph/0612080]; A. Aranda, arXiv:0707.3661 [hep-ph].

[9] A. H. Chan, H. Fritzsch and Z. z. Xing, arXiv:0704.3153 [hep-ph]; Z. z. Xing, Phys. Lett. B 618 (2005) 141 [arXiv:hep-ph/0503200];
Z. z. Xing, H. Zhang and S. Zhou, Phys. Lett. B 641 (2006) 189 [arXiv:hep-ph/0607091];
S. K. Kang, Z. z. Xing and S. Zhou, Phys. Rev. D 73 (2006) 013001 [arXiv:hep-ph/0511157];
S. Luo and Z. z. Xing, Phys. Lett. B 632 (2006) 341 [arXiv:hep-ph/0509065];
M. Hirsch, E. Ma, J. C. Romao, J. W. F. Valle and A. Villanova del Moral, Phys. Rev. D 75 (2007) 053006 [arXiv:hep-ph/0606082];
N. N. Singh, M. Rajkhowa and A. Borah, arXiv:hep-ph/0603189;
X. G. He and A. Zee, Phys. Lett. B 645 (2007) 427 [arXiv:hep-ph/0607163];
N. Haba, A. Watanabe and K. Yoshioka, Phys. Rev. Lett. 97 (2006) 041601 [arXiv:hep-ph/0603116].

[10] The precise meaning of the statement that the TB mixing comes entirely from the neutrino sector is discussed for example in:
S. F. King, Nucl. Phys. B 786 (2007) 52.

[11] S. F. King, arXiv:0710.0530 [hep-ph].

[12] Particle Data Group Collaboration, W. M. Yao et al., Review of particle physics, J. Phys. G33 (2006) 1–1232.

[13] P. H. Chankowski and Z. Pluciennik, Phys. Lett. B316 (1993), 312–317, hep-ph/9306333;
K. S. Babu, C. N. Leung, and J. Panteleone, hep-ph/9309223;
S. F. King and N. N. Singh, Nucl. Phys. B 591 (2000) 3, hep-ph/0006229;
S. Antusch, M. Drees, J. Kersten, M. Lindner, and M. Ratz, Phys. Lett. B519 (2001), 238–242, hep-ph/0108005;
S. Antusch, M. Drees, J. Kersten, M. Lindner, and M. Ratz, Phys. Lett. B 525 (2002) 130 [arXiv:hep-ph/0110366];
S. Antusch, J. Kersten, M. Lindner, and M. Ratz, Phys. Lett. B538 (2002), 87–95, hep-ph/0203233;
S. Antusch and M. Ratz, JHEP 07 (2002), 059, hep-ph/0203027.

[14] S. Antusch, J. Kersten, M. Lindner, M. Ratz and M. A. Schmidt, JHEP 0503 (2005) 024
[arXiv:hep-ph/0501272]; S. Antusch, J. Kersten, M. Lindner and M. Ratz, Nucl. Phys. B 674 (2003) 401 [arXiv:hep-ph/0305273].

[15] For a recent analysis, see: A. Dighe, S. Goswami and W. Rodejohann, Phys. Rev. D 75 (2007) 073023 [arXiv:hep-ph/0612328];
A. Dighe, S. Goswami and P. Roy, Phys. Rev. D 76 (2007) 096005 [arXiv:0704.3735 [hep-ph]].
[16] M. Leurer, Y. Nir and N. Seiberg, Nucl. Phys. B 420 (1994) 468 [arXiv:hep-ph/9310320];
Phys. Lett. B 356 (1995) 45 [arXiv:hep-ph/9504292];
E. Dudas, S. Pokorski and C. A. Savoy, Phys. Lett. B 369 (1996) 255
[arXiv:hep-ph/9509410]; Phys. Rev. D 69 (2004) 053002 [arXiv:hep-ph/0305270]; Phys. Lett. B 580 (2004) 72 [arXiv:hep-ph/0309165].

[17] S. F. King and I. N. R. Peddie, Phys. Lett. B 586 (2004) 83 [arXiv:hep-ph/0312237].

[18] S. F. King, I. N. R. Peddie, G. G. Ross, L. Velasco-Sevilla and O. Vives, JHEP 0507 (2005) 049 [arXiv:hep-ph/0407012].

[19] I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B 648 (2007) 201
[arXiv:hep-ph/0607045].

[20] S. Antusch, S. F. King, and M. Malinsky, arXiv:0708.1282 [hep-ph].

[21] S. Antusch, S. F. King and M. Malinsky, arXiv:0711.4727 [hep-ph].

[22] The ISS Physics Working Group, “Physics at a future Neutrino Factory and super-beam facility,” arXiv:0710.4947 [hep-ph].

[23] P. Minkowski, Phys. Lett. B 67 (1977) 421; M. Gell-Mann, P. Ramond and R. Slansky in Sanibel Talk, CALT-68-709, Feb 1979, and in Supergravity (North Holland, Amsterdam 1979); T. Yanagida in Proc. of the Workshop on Unified Theory and Baryon Number of the Universe, KEK, Japan, 1979; S.L.Glashow, Cargese Lectures (1979); R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. 44 (1980) 912; J. Schechter and J. W. Valle, Phys. Rev. D 25 (1982) 774.

[24] S. F. King, JHEP 0508 (2005) 105 [arXiv:hep-ph/0506297].

[25] S. F. King, Phys. Lett. B 439 (1998) 350 [arXiv:hep-ph/9806440]; S. F. King, Nucl. Phys. B 562 (1999) 57 [arXiv:hep-ph/9904210]; S. F. King, Nucl. Phys. B 576 (2000) 85 [arXiv:hep-ph/9912492]; S. F. King, JHEP 0209 (2002) 011 [arXiv:hep-ph/0204360]. For a review, see: S. Antusch and S. F. King, New J. Phys. 6 (2004) 110 [arXiv:hep-ph/0405272].

[26] M. Malinsky, arXiv:0710.2430 [hep-ph].

[27] J. C. Pati and A. Salam, Phys. Rev. D8 (1973), 1240.

[28] H. Georgi, Particles and fields, (edited by Carlson, C. E.), A.I.P., 1975, p. 575; H. Fritzsch and P. Minkowski, Ann. Phys. 93 (1975), 193–266.

[29] H. Georgi and C. Jarlskog, Phys. Lett. B 86 (1979) 297.

[30] S. Antusch, P. Huber, J. Kersten, T. Schwetz and W. Winter, Phys. Rev. D 70 (2004) 097302.

[31] S. Antusch, J. Kersten, M. Lindner, M. Ratz and M. A. Schmidt, JHEP 0503 (2005) 024 [arXiv:hep-ph/0501272].

[32] S. F. King and G. G. Ross, Phys. Lett. B 520 (2001) 243 [arXiv:hep-ph/0108112]; S. F. King and G. G. Ross, Phys. Lett. B 574 (2003) 239 [arXiv:hep-ph/0307190].

[33] S. Antusch and S. F. King, Nucl. Phys. B 705 (2005) 239 [arXiv:hep-ph/0402121].

[34] S. Antusch and S. F. King, Phys. Lett. B 631 (2005) 42 [arXiv:hep-ph/0508044].