Diffusive limit for 3-dimensional KPZ equation. (2) Generalized PDE estimates through Hamilton-Jacobi-Bellman formalism

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We study in the present series of articles the Kardar-Parisi-Zhang (KPZ) equation

\[ \partial_t h(t, x) = \nu \Delta h(t, x) + \lambda \vert \nabla h(t, x) \vert^2 + \sqrt{D \eta(t, x)}, \quad x \in \mathbb{R}^d \]

in \( d \geq 3 \) dimensions in the perturbative regime, i.e. for \( \lambda > 0 \) small enough and an initial condition \( h_0 = h(t = 0, \cdot) \) such that \( \lambda \vert \nabla h_0 \vert = o(1) \). The forcing term \( \eta \) in the right-hand side is a regularized white noise. We prove a large-scale diffusive limit for the solution, in particular a heat-kernel behaviour for the covariance in a parabolic scaling. The proof is generally based on perturbative estimates obtained by a multi-scale cluster expansion, and a rigorous implementation of K. Wilson’s renormalization group scheme; it extends to equations in the KPZ universality class,

\[ \partial_t h(t, x) = \nu \Delta h(t, x) + \lambda V(\nabla h(t, x)) + \sqrt{D \eta(t, x)}, \quad x \in \mathbb{R}^d \]

for a large class of convex, isotropic deposition rates \( V \geq 0 \). Our expansion is meant to be a rigorous substitute for the response field formalism whenever the latter makes predictions. An important part of the proof relies however more specifically on a priori bounds for the solutions of the KPZ equation, coupled with large deviation estimates.

The present article is dedicated to a generalization of the PDE estimates obtained in the previous article for the infra-red cut-off equation to the case of a function \( V \) with polynomial growth of arbitrary order at infinity. The main tool here is the representation of \( h \) as the solution of some minimization problem through the Hamilton-Jacobi-Bellman formalism.

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0 Introduction

We consider in this series of articles inhomogeneous, non-linear, viscous Hamilton-Jacobi equations of the form

\[ \partial_t h(t, x) = \nu \Delta h(t, x) - \varepsilon h(t, x) + \lambda V(\nabla h(t, x)) + g(t, x), \quad h(0, x) = h_0(x) \]  \hspace{1cm} (0.1)

on \( \mathbb{R} \times \mathbb{R}^d, \ d \geq 1 \), where \( h_0 \) is the initial condition, \(-\varepsilon h (\varepsilon \geq 0)\) a linear damping term, \( g \) a forcing term in the right-hand side, and \( \lambda > 0 \) a positive constant. The non-negative function \( V : \mathbb{R}^d \rightarrow \mathbb{R}_+ \), called deposition rate, is only assumed here to be convex, besides very general properties (\( C^2 \) regularity, polynomial growth at infinity). Physically, the above equation modelizes the growth of an interface under (i) diffusion; (ii) material deposition at site \( x \) depending only on the gradient of the interface at that point; (iii) a forcing term \( g \) viewed as a noise. By a vertical drift, \( h(t, x) \rightarrow h(t, x) - tV(0) \), and a change of coordinate, \( x \rightarrow x - t\nabla V(0) \), one may (and shall) assume that \( V(0) = 0 \) and \( \nabla V(0) = 0 \), so that \( V(\nabla h) = O(|\nabla h|^2) \) at a site where the interface is locally almost flat (horizontal), i.e. for \( |\nabla h| \) small. For physical reasons (although this condition is by no means necessary for the estimates developed in this article), we shall also assume \( V \) to be isotropic.

Such PDEs are generalizations of the Kardar-Parisi-Zhang (KPZ) equation,

\[ \partial_t h(t, x) = \nu \Delta h(t, x) + \lambda \left( \sqrt{1 + |\nabla h(t, x)|^2} - 1 \right) + g(t, x), \quad h(0, x) = h_0(x) \]  \hspace{1cm} (0.2)

also written (using second-order Taylor expansion around a locally flat interface),

\[ \partial_t h(t, x) = \nu \Delta h(t, x) + \frac{\lambda}{2} |\nabla h(t, x)|^2 + g(t, x), \quad h(0, x) = h_0(x). \]  \hspace{1cm} (0.3)
As explained in details in the introduction of the previous article [23], (1) *PDE estimates*, our general motivation is to study, by a rigorous implementation of K. Wilson’s renormalization scheme, the large-scale limit in dimensions $d \geq 3$ of the noisy KPZ equation, for which $g = \eta$ is a regularized white noise. We are however only concerned in single-scale estimates here; the sum over all scales is deferred to the next article, (3) *The multi-scale expansion*. The scale $j$ infra-red cut-off KPZ equation ($j \geq 0$) is given by (0.1) with $\epsilon = M^{-j}$ for some constant $M > 1$, and a right-hand side $g$ satisfying bounds typical for ‘averaged functions’ of the form $e^{M^{1/2}h}$, roughly speaking, $|\nabla^k g| = O((M^{-j/2})^k)$, as follows from standard parabolic estimates. If $h_0$, $\nabla h_0$, $g$ and $\nabla g$ are bounded, then the maximum and comparison principles apply to solutions of eq. (0.1). As a matter of fact, a lot is known ($L^p$-bounds for $h$ or its gradient, asymptotic long-time behaviour,...) about the solutions of the homogeneous equation ($g = 0$), see e.g. [2], [3], [4], [12], [18]; although solutions are smooth for $t > 0$, the theory of viscosity solutions plays an important rôle in these developments.

We are however typically interested in *unbounded* solutions which arise naturally when $g$ is a space-translation invariant in law, random forcing term. With the application to the noisy KPZ equation in mind, we introduced in [23] new functional spaces $W^{1,\infty}_j \subset L^1([0,T] \times \mathbb{R}^d)$, $W^{1,\infty}_j([0,T]) \subset L^1_{\text{loc}}([0,T] \times \mathbb{R}^d)$ ($T > 0$) having roughly the following properties:

(i) $W^{1,\infty}_j \subset W^{1,\infty}_{j,\lambda} \subset L^1([0,T], W^{1,\infty}_j \subset W^{1,\infty}_{j,\lambda}([0,T]),$ where $W^{1,\infty}_j \subset L^\infty(\mathbb{R}^d)$ is the subspace of functions with bounded generalized gradient; conversely, $\eta \in W^{1,\infty}_{j,\lambda}(\mathbb{R}^d)$.

(ii) $W^{1,\infty}_j$ (and similarly $W^{1,\infty}_{j,\lambda}([0,T])$) is given in terms of a family of *pointwise quasi-norms* $||| \cdot |||_{j,\lambda}(x)$, namely (see [23], section 3.2), $W^{1,\infty}_j := \{f \in L^1_{\text{loc}}(\mathbb{R}^d); \forall x \in \mathbb{R}^d, |||f|||_{j,\lambda}(x) < \infty\}$, and

$$|||f|||_{j,\lambda}(x) \leq |||f|||_{j,\lambda'}(x) \quad (\lambda \leq \lambda'), \quad |||\mu f|||_{j,\lambda}(x) \leq |\mu| |||f|||_{j,\mu|\lambda}(x) \quad (\mu \in \mathbb{R}); \quad (0.4)$$

$$|||f_1 + f_2|||_{j,\lambda}(x) \leq \frac{1}{p_1} |||f_1|||_{j,\lambda}(x) + \frac{1}{p_2} |||f_2|||_{j,\lambda}(x) \quad (p_1, p_2 \geq 1, \frac{1}{p_1} + \frac{1}{p_2} = 1), \quad (0.5)$$

from which it follows in particular that $W^{1,\infty}_j$ is a convex subset of $L^1_{\text{loc}}(\mathbb{R}^d)$.

(iii) A comparison principle holds for solutions in $W^{1,\infty}_j$.

(iv) Eq. (0.1) has a unique viscosity solution $h \in C([0,T] \times \mathbb{R}^d)$ such that $h_t \in W^{1,\infty}_j$ for all $t \in [0,T]$ if $h_0 \in W^{1,\infty}_j \cap C(\mathbb{R}^d)$ and $g \in W^{1,\infty}_j([0,T]) \cap C([0,T] \times \mathbb{R}^d)$. One has an explicit, $t$-independent bound on $|||h_t|||_{W^{1,\infty}_j(x)}$ in terms of $|||h_0|||_{W^{1,\infty}_j(x)}$ and $|||g|||_{W^{1,\infty}_j([0,t])}(x)$.

The general principle underlying the definition of these functional spaces is recalled in subsection 1.2 below.

One of the main ingredients in the proof of these results has been the Cole-Hopf transformation, $h \mapsto e^{\epsilon h}$. This transformation maps a solution of (0.1) into a sub-solution of the linear heat equation provided $V(\nabla h) \leq |\nabla h|^2$ if one chooses $\lambda' = \frac{1}{\epsilon}$. The above pointwise quasi-norms measure *local averages of the Cole-Hopf transform* of their arguments. If $V$ is not quadratically bounded at infinity, essentially all our conclusions in [23] fall down.

We tackle here the same questions from a different perspective, starting from a *Hamilton-Jacobi-Bellman* representation of the solutions of (0.1): roughly speaking (see subsection 2.1 for a thorough
the KPZ equation has all its we make the reasonable assumption that results are proved in section 3. Summarizing sections 2 and 3, and emphasizing the similarities and turn out to be necessary in the sequel. We introduce new functional spaces in section 2; all main but the Hamilton-Jacobi-Bellman approach gives much more latitude for proving bounds which substantial novelties; in general it may be said that the maximum principle yields optimal results, are not excluded a priori. The results are in the same spirit as those of [23], but there are also discussion) the function $h_t$ is obtained as the maximum over an admissible class of random paths $X$ driven by Brownian motion of a functional $\int_0^1 F(s, X_s)ds$. When $V(\nabla h) = |\nabla h|^2$, the same approach for the Cole-Hopf transform of the solution gives rise to the much simpler random polymer model. Contrary to the maximum principle approach followed in the previous article, this approach turns out to be also suitable for functions $V$ which are not quadratically bounded at infinity; for definiteness, we make the reasonable assumption that $V$ is polynomially bounded, though further extensions are not excluded a priori. The results are in the same spirit as those of [23], but there are also substantial novelties; in general it may be said that the maximum principle yields optimal results, whereas the Hamilton-Jacobi-Bellman approach gives much more latitude for proving bounds which turn out to be necessary in the sequel. We introduce new functional spaces in section 2; all main results are proved in section 3. Summarizing sections 2 and 3, and emphasizing the similarities and differences with respect to the approach of [23]:

(i) $W^{1,\infty;4}_j$ and $W^{1,\infty;4}([0, T])$ are to be replaced with similar $W^{d',\infty;4}_j$ and $W^{d',\infty;4}([0, T])$ ($d' = d$ or $d + 1$) involving locally uniform bounds on gradients of order $\leq d'$.

(ii) $W^{d,\infty;4}_j$ and $W^{d,\infty;4}([0, T])$ are also given in terms of a family of pointwise quasi-norms. As in [23], these measure local averages of some convex function $f = f(z)$ of their arguments. The Cole-Hopf transform $f(z) = e^{z^2}$ is one possibility, but polynomials of degree $> 1 + d/2$ are equally well suited, thus defining new spaces $W^{d',\infty;4}_j$ and $W^{d',\infty;4}([0, T])$, with weaker pointwise quasi-norms depending on the choice of $f$. When $f$ is a polynomial, the dependence on $\lambda$ factors out, the associated pointwise quasi-norms are all equivalent and the functional spaces are vector spaces, which makes a large difference with respect to the exponential case. Typically, the solution (and its gradients of arbitrary order) lies in exponential $W^\cdot$-spaces, whereas its derivative with respect to the right-hand side (see discussion below) lies in polynomial $W^\cdot$-spaces, which may therefore be seen as "tangent spaces" to the space of all solutions of all KPZ equations.

(iii) There is no comparison principle in this setting.

(iv) Calling $W^{1,\infty;4}_j$-solution a function in $W^{1,\infty;4}_j$ which is the limit on every compact of $W^{1,\infty}_j$ solutions of modified KPZ equations converging to the original one, eq. (0.1) has a unique $W^{1,\infty;4}_j$-solution. Time-independent bounds for pointwise quasi-norms of the solution are constructed in terms of $\|h_0\|_{W^{d',\infty;4}_j(x)}$ and $\|g\|_{W^{d',\infty;4}([0, T])}$.

When the right-hand side $g$ is a regularized white noise, $g = \eta$, one deduces from the bounds in the $W^{d',\infty;4}_j$-norms and from the results of [23], section 6, that the solution of the infra-red cut-off KPZ equation has all its exponential moments finite. However, we also need – as a crucial ingredient for the proof of the diffusive limit – to prove bounds for the derivatives of the solution with respect to the right-hand side. These satisfy linearized PDEs for which we are unable to prove bounds in the $W^{d',\infty;4}_j$-norms; we prove such bounds for the generalized functional spaces $W^{d',\infty;4}_j$ where $f$ is a polynomial of arbitrary high order, which shows that derivatives of the solution has all its moments finite. This problem is intimately related to the difficulty of getting large-deviation estimates for the supremum of random path $X$ over a finite-time interval: we are only able to prove that this random variable has a power law tail distribution (see Corollary 3.5), a very weak estimate which is however sufficient for our purposes.
The outline of the article is as follows. We introduce the infra-red cut-off KPZ equation in section 1, as well as the \( W \)-functional spaces. Section 2 contains a brief reminder about Hamilton-Jacobi-Bellman equations, straightforward applications to the KPZ equation and an existence and unicity result in \( W \)-functional spaces (Theorem 2.1). PDE estimates are all collected in section 3: a bound in \( W \)-functional spaces for the solution and its gradient, which may be seen as our main result (Theorem 3.1 and Corollary 3.1), followed by bounds for its higher-order derivatives based on Schauder estimates, and bounds for the derivatives of the solutions with respect to the right-hand side (Theorem 3.3), valid only in polynomial \( W \)-functional spaces.

Notations. The scale parameter is an arbitrary constant \( M > 1 \) fixed once and for all. The notation: \( f(u) \lesssim g(u) \), resp. \( f(u) \gtrsim g(u) \) means: \( |f(u)| \leq C|g(u)| \), resp. \( |f(u)| \geq C|g(u)| \), where \( C > 0 \) is an unessential constant (depending only on \( d, \nu, D \) and \( M \)). Similarly, \( f(u) \approx g(u) \) means: \( f(u) \lesssim g(u) \) and \( g(u) \lesssim f(u) \). We denote by \( L^p \), \( p \in [1, \infty] \) the usual Lebesgue spaces with associated norm \( \| \cdot \|_p \), by \( W^{1,\infty} \) the Sobolev space of bounded functions with bounded generalized derivative, and by \( C^{1,2} \) the space of functions which are \( C^1 \) in time and \( C^2 \) in space. The average \( \frac{1}{|\Omega|} \int_{\Omega} f \) of a function \( f \) on a bounded domain \( \Omega \) is denoted by \( \fint_{\Omega} f \).

1 The infra-red cut-off KPZ equation

1.1 First formulation

A KPZ equation is a viscous Hamilton-Jacobi equation,
\[
\partial_t h(t, x) = \nu \Delta h(t, x) + \lambda V(\nabla h(t, x)) + g(t, x),
\]
where \( g \) is a right-hand side (or forcing term) in a suitable functional space. The deposition rate \( V \) satisfies in this article the following assumptions.

Assumption 1.1 The deposition rate \( V \) satisfies the following assumptions,

1. \( V \) is \( C^2 \);
2. \( V \) is isotropic, i.e. \( V(\nabla h) \) is a function of \( y = |\nabla h| \); by abuse of notation we shall consider \( V \) either as a function of \( \nabla h \) or of \( y \);
3. \( V \) is convex;
4. \( V(0) = 0 \) and \( 0 \leq V(y) \leq \max(\frac{y^2}{2}, \frac{1}{2}y^2, \frac{1}{2}y^\beta) \) for all \( y \geq 0 \), where \( \beta \geq 2 \) is called the growth exponent at infinity of \( V \).

Although our assumptions are general enough to define what may be called a universality class for the KPZ equation, we shall fix in the sequel a function \( V \) with the above properties and refer to the KPZ equation.

Compared to the above class of equations, the infra-red cut-off KPZ equation has a supplementary linear damping term. Recall \( M > 1 \) is a constant.

Definition 1.2 (infra-red cut-off equation) Let \( V \) satisfy the above assumptions, and \( j \geq 0 \). Then the infra-red cut-off KPZ equation of scale \( j \) is the following (class of) equation(s),
\[
\partial_t h(t, x) = \nu \Delta h(t, x) - M^{-j} h(t, x) + \lambda V(\nabla h(t, x)) + g(t, x).
\]
The extra term \(-M^{-j}h\) in the right-hand side implies in principle an exponential decay of memory. As discussed in \([23]\), sections 4 and 5, the operator \((\nu \Delta - M^{-j})^{-1}\) is a kind of ersatz for the high-momentum propagator \(G^{j\rightarrow} := \sum_{i=0}^{j} G^i\) with scale \(j\) infra-red cut-off.

### 1.2 Functional spaces

The purpose of this article is to show that the infra-red cut-off KPZ equation of scale \(j\), eq. (1.2), has a single solution \(h_j\) in an adequate functional space under suitable assumptions on the right-hand side \(g^j\), and to give appropriate bounds in suitable norms for \(h^j\) in terms of \(h^0\) and \(g\).

For the applications we have in mind (including the noisy KPZ equation, or more generally viscous Hamilton-Jacobi equations with an extra noise term in the right-hand side), the initial condition \(h_0\) and the right-hand side \(g\) are unbounded, which led us to introduce new functional spaces, \(\mathcal{W}_{j}^{1,\text{loc},1}\) and \(\mathcal{W}_{j}^{1,\text{loc},1}([0, T])\) in the previous article. Compared to \([23]\), our arguments here have larger range of validity but give less precise bounds, which requires some rather minor changes in our definitions. However, the general principles underlying the construction of all these spaces is the same. We now describe them briefly and refer to \([23]\), sections 3 and 4 for more details.

Assume first \(h_0 \in \mathcal{W}^{1,\infty}\) and \(g \in L_{\text{loc}}^1(\mathbb{R}^d, \mathcal{W}^{1,\infty})\). By classical arguments derived from the parabolic maximum principle (see \([23]\), section 2), the associated Cauchy problem (1.1) has a unique, global solution \(h\) which lies in \(\mathcal{W}^{1,\infty}\) for all \(t \geq 0\) and is classical for strictly positive times, that is, \(h \in C([0, +\infty) \times \mathbb{R}^d) \cap C^{1,2}((0, \infty) \times \mathbb{R}^d)\). Furthermore,

\[
\|h_t\|_{\infty} \leq \|h_0\|_{\infty} + \int_0^t \|g_s\|_{\infty} ds. \quad (1.3)
\]

Recall however that the emphasis in this series of articles is in the noisy KPZ equation, for which \(g = \eta\) is a (suitably regularized) white noise. Generally speaking, \(h\) is expected to behave more or less like the solution \(\phi\) of the linearized equation,

\[
\partial_t \phi = \nu \Delta \phi + g, \quad (1.4)
\]

which is simply \(Gg\). If \(g = \eta\), then \(g\) and \(\phi = Gg\) will be space-translation invariant in law, hence are a.s. unbounded. Thus we cannot expect \(h_t\) to lie in \(\mathcal{W}^{1,\infty}\). However, local averages

\[
\frac{\int_{B(x,r)} dy|\eta_t(y)|}{\text{Vol}(B(x,r))} = \frac{\int_{B(x,r)} dy|\eta_0(y)|}{\text{Vol}(B(x,r))}, \quad r > 0 \quad (1.5)
\]

of \(\eta\) are locally uniformly bounded, which amounts to saying that \(\eta \in \mathcal{H}^0(\mathbb{R}^d)\) and \(\eta^* \in L_{\text{loc}}^\infty\), in the sense of the following definition.

**Definition 1.3** Let

\[
\mathcal{H}^0 := \{ f \in L_{\text{loc}}^1(\mathbb{R}^d) \mid \forall x \in \mathbb{R}^d, f^*(x) < \infty \} \quad (1.6)
\]

where

\[
f^*(x) := \sup_{r>0} e^{r \Delta} |f|(x) \in [0, +\infty]. \quad (1.7)
\]

Due to the averaging and scaled decay properties of the heat kernel, it is natural to expect that \(e^{r \Delta}|f|(x)\) may be substituted with \(\int_{B(x,r^2)} dy|f(y)|\) in the above definition; it is actually proven in \([23]\) that \(f^*(x) \approx \sup_{r>0} \int_{B(x,r)} dy|f(y)|\).
With this definition in hand, we may substitute the usual parabolic estimates,

\[ \| \nabla^k e^{r\Delta} f \|_\infty \lesssim r^{k/4} \| f \|_\infty, \quad k \geq 0 \tag{1.8} \]

with the stronger, pointwise parabolic estimates (see [23], section 3),

\[ (\nabla^k e^{\Delta} f)^*(x) \lesssim r^{k/2} f^*(x), \quad k \geq 0 \tag{1.9} \]

including the obvious but fundamental \((e^{\Delta} f)^*(x) \leq f^*(x), t \geq 0.\)

Contrary to what was the case in the preceding article, however, we also need here a control over local suprema of the functions.

**Definition 1.4 (local supremum of order \(j\))** Let \(f : \mathbb{R}^d \to \mathbb{R}\) be a function in \(L^\infty_{\text{loc}}(\mathbb{R}^d).\) Then \(\text{loc sup}_j(f) : \mathbb{R}^d \to \mathbb{R}\) is the function in \(L^\infty_{\text{loc}}(\mathbb{R}^d)\) defined by \(\text{loc sup}_j(f)(x) := \sup_{y \in B(x, M^{1/2})} |f(y)|.\)

We keep the scale \(j\) fixed in the sequel and write simply \(\text{loc sup}(\cdot)\) instead of \(\text{loc sup}_j(\cdot).\) This local supremum operation allows one to discretize space. Let \(D^j\) be the set of all cells of the lattice \(M^{1/2} \mathbb{Z}^d,\) i.e. \(\Delta \in D^j\) if and only if \(\Delta = [M^{1/2}k_1, M^{1/2}(k_1 + 1)] \times \cdots \times [M^{1/2}k_d, M^{1/2}(k_d + 1)]\) for some \((k_1, \ldots, k_d) \in \mathbb{Z}^d.\) Clearly, if \(r \geq 1,

\[ r^{-d} \sum_{\Delta \in D^j; \Delta \subset B(x, r M^{1/2})} \sup_{\Delta} |f| \lesssim \int_{B(x, r M^{1/2})} \text{loc sup}_j(f)(x) \lesssim r^{-d} \sum_{\Delta \in D^j; \Delta \cap B(x, r M^{1/2}) \neq \emptyset} \sup_{\Delta} |f|, \tag{1.10} \]

so

\[ \text{loc sup}_j(f)^*(x) \approx \sup_n n^{-d} \sum_{\Delta \in D^j; \Delta \subset B_n(x, n M^{1/2})} \sup_{\Delta} |f|, \tag{1.11} \]

where \(n\) ranges either over \(\mathbb{N}^*\) or on the set of dyadic integers \(2^k, k \in \mathbb{N}.\) We also get the following important bound on spherical averages,

\[ \int_{\partial B(x, r M^{1/2})} |f| \lesssim r^{-(d-1)} \sum_{\Delta \in D^j; \Delta \cap \partial B(x, r M^{1/2}) \neq \emptyset} \sup_{\Delta} |f| \lesssim r \text{ loc sup}_j(f)^*(x). \tag{1.12} \]

Similary,

\[ \sup_{\Delta} |f| \lesssim d^j(0, \Delta)^d \text{ loc sup}_j(f)^*(0) \tag{1.13} \]

for \(d^j(0, \Delta) = M^{-j/2} \min_{x \in \Delta} |x| \geq 1,\) so a function \(f\) such that \(\text{loc sup}_j(f)^*(0) < \infty\) has at most polynomial growth of order \(d\) at infinity.

Thus we shall consider in the sequel functions \(f\) satisfying conditions of the type \(\text{loc sup}_j(f)^*(x) < \infty, x \in \mathbb{R}^d.\)

Now the general idea is that the initial condition, the right-hand side, and the solution of the KPZ equation are to be considered in functional spaces modeled after \(\mathcal{H}^0\) if one wants to define and solve a natural class of equations including the noisy KPZ equation. Three pieces are however missing. (i) One needs analogous conditions on the gradients. With the tools used here, we actually also need bounds on higher-order derivatives of \(h_0, g,\) up to order \(d\) or \(d + 1\) (see below). (ii) The KPZ equation is non-linear, hence pointwise parabolic estimates do not hold for KPZ solutions; however, by the comparison principle, the Cole-Hopf transformation \(h_t \mapsto e^{ht}\) maps a solution of
the homogeneous KPZ equation \( h_t \) to a subsolution of the linear heat equation \( \text{provided} \ V(y) \leq y^2 \) is quadratically bounded. For such deposition rates, one should hence take the “pull-back” of the functional spaces by the Cole-Hopf transformation. That the Cole-Hopf transformation should play a rôle for the more general rates studied in this article is a priori unexpected. (iii) There remains to take into account the right-hand side \( g \). This is a difficult task, involving the multi-scale analysis developed in the next article, for the full KPZ equation, and relying on arguments valid only for the noisy equation. However, for the infra-red cut-off equation, due to the exponential time-decay, the solution at time \( t \) depends essentially on \( (g_s)_{s \geq t-M^2} \), which makes it possible to treat the initial condition and the right-hand side approximately on the same footing.

In the following definition, \( P, P_-, P_+ : \mathbb{R}_+ \to \mathbb{R}_+ \) are strictly increasing, convex functions, with \( P_- \leq P_+ \). The main examples we have in view are the exponential case, \( P_+(z) = e^{|z|} (\lambda' > 0) \), and the polynomial case, \( P(z) = z^{d'} (d' = 1, 2, \ldots) \). In the exponential case we choose \( P_-(z) = e^{|z|} (\lambda_- \leq \lambda_+) \), in the polynomial case \( P_-(z) = z^{d_-} (d_- \leq d_+) \). We abbreviate \( \mathcal{H}^{p_+}, \mathcal{W}_{d'}^{p_+, \text{loc}, P_+}, \mathcal{W}_{d'}^{p_+, \text{loc}, P_+}([0, T]) \) to \( \mathcal{H}^{d_1}, \mathcal{W}_{d'}^{d_1, \text{loc}, P}, \mathcal{W}_{d'}^{d_1, \text{loc}, P}([0, T]) \) for the sake of simplicity, and also to remain coherent with the notations of \([23]\).

**Definition 1.5 (functional spaces)** Let

(i) \[
\mathcal{H}^p := \{ h_0 \in L^1_{\text{loc}}(\mathbb{R}^d) \mid \forall x \in \mathbb{R}^d, \| h_0 \|_{\mathcal{H}^p}(x) < \infty \}
\] (1.14)

where

\[
\| h_0 \|_{\mathcal{H}^p}(x) := \mathcal{P}^{-1}((P(|h_0|))^\ast)(x).
\] (1.15)

In particular, letting \( P(z) = P_+(z) = e^{|z|} \), we have

\[
\mathcal{H}^{d_1} := \{ h_0 \in L^1_{\text{loc}}(\mathbb{R}^d) \mid \forall x \in \mathbb{R}^d, \| h_0 \|_{\mathcal{H}^{d_1}}(x) < \infty \}
\] (1.16)

where

\[
\| h_0 \|_{\mathcal{H}^{d_1}}(x) := \frac{1}{\lambda} \ln(e^{\mathcal{P}^{-1}}(x));
\] (1.17)

(ii) (functional space of scale \( j \) for the initial condition \( h_0 \)) Let, for \( d' \geq 1 \),

\[
\mathcal{W}^{d', \text{loc}, P}_j := \{ h_0 \in L^1_{\text{loc}}(\mathbb{R}^d) ; \forall x \in \mathbb{R}^d, \| h_0 \|_{\mathcal{W}^{d', \text{loc}, P}_j}(x) < \infty \},
\] (1.18)

where

\[
\| h_0 \|_{\mathcal{W}^{d', \text{loc}, P}_j}(x) := \sum_{k=0}^{d'} \|((M^{1/2})^k \text{loc sup} \nabla^k h_0) \|_{\mathcal{H}^p}(x);
\] (1.19)

(iii) (functional space of scale \( j \) for the right-hand side \( g \)) Let, for \( d' \geq 1 \),

\[
\mathcal{W}^{d', \text{loc}, P_+}_{d'}([0, T]) := \{ g \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d) ; \forall x \in \mathbb{R}^d, t \mapsto \| g \|_{\mathcal{W}^{d', \text{loc}, P_+}_{d'}([0, T])}(x) \in L^\infty_{\text{loc}}([0, T]) \},
\] (1.20)

where

\[
\| g \|_{\mathcal{W}^{d', \text{loc}, P_+}_{d'}([0, T])}(x) := P_+^{-1} \circ P_+ \left( \limsup_{n \to \infty} M^{-j/2} \sum_{k=1}^{n} (1 + (M^{-j}k/n)^{d/2}) e^{-M^{-j}k/2} \right),
\] (1.21)
Remark 1. In the exponential case, \( P_+^{-1} \circ P_+ (z) = \lambda_k z \), so (1.21) is independent of \( P_- \); we may then simply write
\[
\|g\|_{W^{\rho, \infty; 1}_{j, \omega}(\{0, z\})} (x) := \limsup_{n \to \infty} M^{-j} \sum_{k=1}^{n} \left( 1 + (M^{-j} k t / n)^{d/2} \right) \left( e^{-M^{-j} k t / n} \|M^j \int_{-\tau}^{t} g(s) ds\|_{W^{\rho, \infty; 1}_{j, \omega}(x)} \right).
\] (1.22)

In the polynomial case, eq. (1.21) is reminiscent of the double-indexed functional spaces \( L^p_\rho(L^q_\rho) = L^p(\mathbb{R}_+; L^q(\mathbb{R}^d)) \) used in hyperbolic problems (in our notations, \( q = d + p = d + d_+ / d_- \) or else of Besov spaces. Such spaces come naturally out of our computations; however, they imply heavy manipulations. See below, remark 3, for possible simplifications.

Remark 2 (connection to the previous article) Note that \( \mathcal{H}^1 \) is identical to the homonymous functional space defined in [23], eq. (3.39), while \( W^{\rho, \infty; 1}_{j, \omega} \) and \( W^{\rho, \infty; 1}_{j, \omega}(\{0, T\}) \) are strictly included in their counterparts \( W^{\rho, \infty; 1}_{j, \omega} \) resp. \( W^{\rho, \infty; 1}(\{0, T\}) \) defined in [23], eq. (4.19), (4.20) and (4.21). The spaces \( W^{\rho, \infty; 1}_{j, \omega} \), \( W^{\rho, \infty; 1}_{j, \omega}(\{0, T\}) \) for \( P \neq P_A \) not exponential are natural and useful generalizations; note that these functional spaces are convex sets in general, and vector spaces when \( P \) is a polynomial. Compared to the arguments based on the maximum principle used in the previous article, our arguments here require bounds on local suprema of order \( j \) of the data, and also scaled bounds on higher derivatives of \( g, h_0 \) up to order \( d' = d \) or \( d + 1 \) (see results in section 3). These locally uniform bounds make it possible to replace the discrete gradients of eq. (4.19), (4.21) by gradients, since
\[
\|M^{1/2} f(\varepsilon + \cdot) - f(\varepsilon)\|_{\mathcal{H}^1}(x) \leq \frac{1}{A} \sup_{\tau \in [0, 1]} \left( e^{\lambda} \left( e^{A M^{1/2} / \varepsilon} \int_{[0, 1]} \lambda \left( e^{\lambda} \left( e^{A M^{1/2} / \varepsilon} \int_{[0, 1]} \lambda \right) \right) \right) \right)(x)
\]
\[
\leq \frac{1}{A} \sup_{\tau \in [0, 1]} \left( e^{\lambda} \left( e^{A M^{1/2} / \varepsilon} \int_{[0, 1]} \lambda \right) \right)(x) \leq \frac{1}{A} \sup_{\tau \in [0, 1]} \left( e^{\lambda} \left( e^{A M^{1/2} / \varepsilon} \int_{[0, 1]} \lambda \right) \right)(x) \leq \frac{1}{A} \sup_{\tau \in [0, 1]} \left( e^{\lambda} \left( e^{A M^{1/2} / \varepsilon} \int_{[0, 1]} \lambda \right) \right)(x)
\] (1.23)

if \( \varepsilon < M^{1/2} \), by Jensen’s inequality.

Remark 3 (time behaviour). Note the polynomial correction \( (1 + (M^{-j} k t / n)^{d/2}) \) appearing in factor of the right-hand side of (1.21). It can be reabsorbed through an arbitrary small decrease of the exponential decay rate, namely, \( (1 + (M^{-j} k t / n)^{d/2}) e^{-M^{-j} k t / n} \leq e^{-(1 - \varepsilon) M^{-j} k t / n} \) if \( \varepsilon > 0 \). With this stronger quasi-norm, the large-deviation estimates obtained for \( \|\eta^j(x, \rho, x)\|_{L^p} \) in [23], section 6 remain correct, as the reader may check (see in particular subsection 6.4). Also, if the series (1.21)

converges with \( M^j \int_{-(k+1)/n}^{(k+1)/n} g(s) \) replaced by \( M^j \sup_{t \in [-(k+1)/n, (k+1)/n]} |\varepsilon| \), then Lebesgue’s differentiation theorem implies the simpler formula
\[
\|g\|_{W^{\rho, \infty; 1}_{j, \omega}(\{0, t\})} (x) = P_+ e^{-\frac{1}{2} M^{-j} k t / n} o P_+ (\|M^j g(t - s)\|_{W^{\rho, \infty; 1}_{j, \omega}(x)})
\] (1.24)

In particular, letting
\[
\text{loc sup}(f(t)) = \text{loc sup}_{j}(f_j(t)) \equiv \sup_{|t-a| < M^j} \text{loc sup}(f_a)
\] (1.25)
be the natural space-time generalization of the above defined spatial local supremum,

\[
\|g\|_{W^{d,\infty}_{\beta}(\Omega(t), t)}(x) \leq P_+^{-1} \circ P_- \left( \sum_{l=0}^{M/2} \sum_{k=0}^{d'} e^{-l/2} P_-^{-1} \circ P_+ \left( \|\| (M^{j/2})^l \text{loc sup}(|\nabla^k g|)(t - kM^{j/2})\|_{H^p}(x) \right) \right)
\]

\[
\leq \sup_{l \leq [M/2]} P_+^{-1} \circ P_- \left( e^{-l/2} \sum_{k=0}^{d'} P_-^{-1} \circ P_+ \left( \|\| (M^{j/2})^l \text{loc sup}(|\nabla^k g|)(t - kM^{j/2})\|_{H^p}(x) \right) \right).
\]

(1.26)

In subsection 3.4 we particularize to \( P_\pm(z) = e^{dz} \) with \( d_+ / d_- \) bounded, in which case one obtains the simple bound

\[
\|g\|_{W^{d,\infty}_{\beta}(\Omega(t), t)}(x) \leq \|g\|_{W^{d,\infty}_{\beta}(\Omega(t), t)}(x) := \sup_{l \leq [M/2]} e^{-cl} \sum_{k=0}^{d'} \|\| (M^{j/2})^l \text{loc sup}(|\nabla^k g|)(t - kM^{j/2})\|_{H^p}(x)
\]

(1.27)

for some constant \( c > 0 \), defining a new functional space with associated pointwise quasi-norm \( \| \cdot \|_{W^{d,\infty}_{\beta}(\Omega(t), t)}(\cdot) \) which does not depend on \( P_- \).

**Remark 4.** These functional spaces are tailor made to get bounds not only on the value of the solution at some point \((t, x)\), but also on its local averages. Our results in section 3 prove in fact that \( e^{\Delta(h(t,x))} \) is controlled by \((e^{\Delta h(\Omega)}(x)) \) and \((e^{\Delta h(\Omega)}(x)) \), \( 0 < s < t \) for \( c \) large enough, and more generally, for each function \( P_- : \mathbb{R}_+ \to \mathbb{R}_+ \), there exists a function \( P_+ : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( P_-(|h(t,x)|) \) is controlled by \((e^{\Delta h(\Omega)}(x)) \) and \((e^{\Delta h(\Omega)}(x)) \), \( 0 < s < t \). In particular, the supremum over all positive times \( \tau > 0 \) in the definition of \( e^{\Delta h(\Omega)} \) or \( P_+ \) may be replaced by a supremum over \( \tau \in (0, t) \). However, if one wants \( e^{\Delta h(\Omega)} \) or \( P_+ \) to have locally bounded averages, then \( \tau \) must be allowed to range over \( \mathbb{R}_+^\ast \).

**Remark 5.** Regularized white noise \( \eta \) belongs a.s. to all these functional spaces (as shown in the exponential case in [23], section 6), and one has explicit a log-normal deviation formula for the local supremum over time intervals of size \( O(M^j) \) of its pointwise quasi-norms at a given space location \( x \).

## 2 A Hamilton-Jacobi-Bellman reformulation of the KPZ equation

The general purpose of this section is to give a “random path representation” for solutions of the KPZ or infra-red cut-off KPZ equation. This is standard using the Hamilton-Jacobi-Bellman theory. It allows us to give a new notion of unbounded solutions (see Definition 2.3 below), equivalent to that of the generalized, unbounded viscosity solutions introduced in [23], subsection 3.2 when \( \beta \leq 2 \). Section 5 below shows how to extend formally this general philosophy to “single-scale KPZ equations” (not used in the sequel), for which we are however unable to give a priori bounds.

### 2.1 The Hamilton-Jacobi-Bellman theory for viscous Hamilton-Jacobi equations

For this paragraph which does not contain any new result, the reader may consult [10], [17] or [22]. We restrict this introductory and somewhat loose discussion to a subclass of Hamilton-Jacobi equations, including the “universality class” of the KPZ equation,

\[
\partial_t h = \nu \Delta h + V(\nabla h) + g, \quad h(t = 0) = h_0
\]

(2.1)
where \( V : \mathbb{R}^d \to \mathbb{R} \) is assumed to be \( C^2 \) and convex, and \( \frac{V(x)}{|x|^{\alpha}} \to \|v\|_1 \to +\infty \), so that \( V \) has a well-defined Legendre transform \( \tilde{V} \) with the same properties; recall \( \tilde{V}(\alpha) := \sup_{x \in \mathbb{R}^d} (\alpha \cdot x - V(x)) \), \( \alpha \in \mathbb{R}^d \) (see e.g. [9], §3.3.2). Here we need more precisely \( \tilde{V}(-\cdot)(\cdot) = \tilde{V}(-\cdot) \). Let \( \alpha^*(x) := \nabla V(-x) \). The parameter conjugate to \( x \) is \( \nabla(V(-x)) = -\nabla V(-x) \); we let \( \alpha^*(x) = +\nabla V(-x) \). By definition, \( V \) and \( \tilde{V} \) are related by

\[
V(-x) = \sup_{\alpha} (\alpha x - \tilde{V}(-\alpha)) = -\alpha^*(x)x - \tilde{V}(\alpha^*(x)).
\]

Let \( B_t, \ t \geq 0 \) be a \( d \)-dimensional Brownian motion, and consider the following class of stochastic differential equations,

\[
dX^\alpha_t = \alpha_0 ds + \sqrt{\nu} dB_s, \ s \geq t
\]

with initial condition \( X^\alpha_0 = x \), where \( \alpha = (\alpha_0)_s \geq 0 \) is an admissible strategy, i.e. a progressively measurable, \( \mathbb{R}^d \)-valued process with respect to the filtration defined by the Wiener process. We shall sometimes leave the dependence of \( X \) on \( \alpha \) implicit and write \( X \) instead of \( X^\alpha \). Fix a terminal time \( T \geq 0 \), a function \( u_T : \mathbb{R}^d \to \mathbb{R} \) and some function \( f = f(t, x; y) \) on \( \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \). Bellman’s original idea is to try and minimize the cost functional

\[
J(t, x; \alpha) := \mathbb{E}^{\mathbb{L}^x} \left[ \int_t^T f(s, X^\alpha_s; \alpha_s) ds + u_T(X^\alpha_T) \right]
\]

with respect to all admissible strategies. The notation \( \mathbb{E}^{\mathbb{L}^x}[-] \) emphasizes the initial condition \( X^\alpha_0 = x \) for the diffusion. The result,

\[
v(t, x) := \inf_{\alpha} J(t, x; \alpha)
\]

is called the value function.

Now, Bellman’s principle states that, for \( t \leq \tilde{t} \leq T \),

\[
v(t, x) = \inf_{\alpha} \left( \mathbb{E}^{\mathbb{L}^x} \left[ \int_t^{\tilde{t}} f(s, X^\alpha_s; \alpha_s) ds \right] + \mathbb{E}^{\mathbb{L}^x} \left[ v(\tilde{t}, X^\alpha_{\tilde{t}}) \right] \right).
\]

This is essentially straightforward since the choice of the optimal strategy after time \( \tilde{t} \) depends by the Markov property of the Wiener process only on \( X_t \). Let now \( \tilde{t} = t + o(1) \) and apply Itô’s formula. Note that the solution of (2.6) is unique provided one assumes the terminal condition \( v_T = u_T \) (take \( \tilde{t} = T \)). One gets

\[
\mathbb{E}^{\mathbb{L}^x} \left[ \int_t^{\tilde{t}} f(s, X_s; \alpha_s) ds \right] = (\tilde{t} - t)f(t, x; \alpha_t) + \ldots
\]

\[
\mathbb{E}^{\mathbb{L}^x} [v(\tilde{t}, X_{\tilde{t}})] = v(t, x) + (\tilde{t} - t)(\partial_t v(t, x) + \mathcal{L}^{\alpha^*} v(t, x)) + \ldots
\]

where \( \mathcal{L}^{\alpha^*}(t, x) = \nu \Delta + \alpha^*_t \cdot \nabla \) is the generator of the diffusion process (2.3). Taking the limit \( \tilde{t} \to t \) yields Bellman’s differential equation,

\[
\inf_{\alpha} \left[ (\partial_t + \mathcal{L}^{\alpha^*}) v(t, x) + f(t, x; \alpha_t) \right] = 0,
\]

together with the obvious terminal condition, \( v_T = u_T \).

Let us now choose

\[
f(t, x; y) := \bar{V}(y) - g(T - t, x)
\]

(2.10)
One immediately checks that Bellman’s equation is equivalent to
\[( \partial_t + \nu \Delta ) v(t, x) + \inf_{\alpha} \left( \alpha_t \cdot \nabla v(t, x) + \check{V}(\alpha_t) \right) - g(T - t, x) = 0 \] (2.11)
or (letting \( \alpha_t = -\alpha^*(\nabla v(t, x)) \))
\[( \partial_t + \nu \Delta ) v(t, x) - V(\nabla v(t, x)) - g(T - t, x) = 0. \] (2.12)

Thus one sees that
\[ h(t, x) := -v(T - t, x), 0 \leq t \leq T \] (2.13)
satisfies the Hamilton-Jacobi equation (2.1).

If the solution \( h \) of the Hamilton-Jacobi equation (2.1) is unique, then the following Feynman-Kac type formula holds.

**Proposition 2.1** Let

(i) \( h(t, \cdot) = -u(T - t, \cdot), \ t \geq 0 \) solve the Hamilton-Jacobi equation (2.1) with initial condition \( h_0 = -u_T \);

(ii) \( X^\alpha \) be the solution of the stochastic differential
\[ dX^\alpha_s = \alpha^*(\nabla u_s(X^\alpha_s)) ds + \sqrt{\nu} dW_s, \ s \geq t \] (2.14)
with initial condition \( X^\alpha_t = x \);

(iii) \( v(t, x) \) be the function defined as
\[ v(t, x) := \mathbb{E}^t_x \left[ \int_t^T \left( \check{V}(\alpha^*(\nabla u_s(X^\alpha_s))) - g(T - s, X^\alpha_s) \right) ds + u_T(X^\alpha_T) \right]. \] (2.15)

Then \( v \equiv u \).

**Proof.** Clearly \( v_T = u_T \). We write \( X = X^\alpha \) and let
\[ w_{t, s}(\tilde{t}) := \mathbb{E}^t_x \left[ \int_{t}^{\tilde{t}} \left( \check{V}(\alpha^*(\nabla u_s(X_s))) - g(T - s, X_s) \right) ds + u_T(X_{\tilde{t}}) \right], \quad t \leq \tilde{t} \leq T. \] (2.16)

By Itô’s formula,
\[
\frac{d}{dt} w_{t, s}(\tilde{t}) = \mathbb{E}^t_x \left[ \check{V}(\alpha^*(\nabla u_{\tilde{t}}(X_{\tilde{t}}))) + \alpha^*(\nabla u_{\tilde{t}}(X_{\tilde{t}})) \cdot \nabla u_{\tilde{t}}(X_{\tilde{t}}) + (\partial_{\tilde{t}} + \nu \Delta) w_{t, s}(\tilde{t}) - g(T - \tilde{t}, X_{\tilde{t}}) \right]. \] (2.17)

The first two summands in (2.17) sum up by definition of \( \alpha^* \) to \(-V(-\nabla u_{\tilde{t}}(X_{\tilde{t}}))\). Since \(-u(T - t, \cdot)\) solves eq. (2.1), \( \frac{d}{dt} w_{t, s}(\tilde{t}) = 0 \). Thus \( w_{t, s}(T) = w_{t, s}(t) \), or in other terms, \( v = u \). \( \square \)

Summarizing the above discussion:
Proposition 2.2 Assume the solution of the Hamilton-Jacobi equation (2.7) is unique. Then
\[ h(t, x) = \sup_\alpha \mathbb{E}^{0, x} \left[ \int_0^t \left( -\tilde{V}(-\alpha_s) + g(t - s, X_s^\alpha) \right) ds + h_0(X_t^\alpha) \right] \]
\[ = \mathbb{E}^{0, x} \left[ \int_0^\infty \left( -\tilde{V}(-\alpha_s^*) + g(t - s, X_s^{\alpha^*}) \right) ds + h_0(X_t^{\alpha^*}) \right] \quad (2.18) \]
where \( \alpha^*_s := \alpha^*(-\nabla h_{t-s}(X_s^{\alpha^*})) \).

We shall also consider later on right-hand sides \( g = g(\tau) \) smoothly depending on a parameter \( \tau \), and need explicit expressions for the derivative with respect to \( \tau \) of the solution \( h(\tau) \) of the corresponding Hamilton-Jacobi equation with fixed initial condition \( h_0 = h_0(t, x) \). Formally at least, \( h' := \partial_\tau h \) satisfies the linearized PDE
\[ \partial_\tau h' = \nu \Delta h' + V'(\nabla h) \cdot \nabla h' + g' \quad (2.19) \]
with \( g' = \partial_\tau g \) and \( h'(0) = 0 \). The following proposition holds.

Proposition 2.3 We keep the same hypotheses as in Proposition 2.2 and let furthermore

(i) \( h'(\tau; t, \cdot) = -u'(\tau; T - t, \cdot), \ t \geq 0 \) solve the linearized Hamilton-Jacobi equation (2.19) with initial condition \( h'(\tau; 0) = -u'(\tau; T) = 0 \);

(iii) \( v'(\tau; t, x) \) be the function defined as
\[ v'(\tau; t, x) := \mathbb{E}^{t, x} \left[ -\int_t^\infty g'(\tau; T - s, X_s^{\alpha^*}) ds \right]. \quad (2.20) \]

Then \( v' \equiv u' \).

Proof. The proof is totally analogous to that of Proposition 2.1. We let
\[ w_{t, x}(\tau; \bar{i}) := \mathbb{E}^{t, x} \left[ -\int_t^\infty g'(\tau; T - s, X_s^{\alpha^*}) ds + u'(\tau; \bar{i}, X_{\bar{i}}) \right], \quad (2.21) \]
Then
\[ \frac{d}{dt} w_{t, x}(\tau; \bar{i}) = \mathbb{E}^{t, x} \left[ -g'(\tau; T - \bar{i}, X_{\bar{i}}^{\alpha^*}) + (\partial_\tau + \nu \Delta + \alpha^*(\nabla u(\tau; \bar{i}, X_{\bar{i}}^{\alpha^*}) \cdot \nabla) u'(\tau; \bar{i}, X_{\bar{i}}^{\alpha^*}) \right] \]
\[ = \mathbb{E}^{t, x} \left[ -g'(\tau; T - \bar{i}, X_{\bar{i}}^{\alpha^*}) + (\partial_\tau - \nu \Delta) - V'(\nabla h(\tau; t - \bar{i}, X_{\bar{i}}^{\alpha^*}) \cdot \nabla) h'(\tau; T - \bar{i}, X_{\bar{i}}^{\alpha^*}) \right] \]
\[ = 0. \quad (2.22) \]
2.2 Application to the infra-red cut-off KPZ equation

We now consider the infra-red cut-off KPZ equation (1.2), with the supplementary assumption \( V(0, \cdot) \to \infty \) (this supplementary assumption is necessary to define the Legendre transform of \( V \), but we show in section 3 how to get rid of it to get bounds on the solutions valid in whole generality). Compared to (2.1), \( V \) has been changed to \( \lambda V \), and an extra linear term \(-M^{-j}h\) appears in the right-hand side. This accounts for two minor modifications with respect to the above analysis. First, a simple scaling argument yields \( \lambda V(p) = \lambda V(\frac{p}{j}) \). Second, the modified generator \( \mathcal{L}^a_{\text{killed}}(t, x) := \nu \Delta + a_\perp \cdot \nabla - M^{-j} \) corresponds to a diffusion with killing rate \( M^{-j} \). A straightforward extension of the results of the previous paragraph yields

**Lemma 2.4** (Hamilton-Jacobi-Bellman representation of the infra-red cut-off KPZ equation)

Assume the solution \( h \) to eq. (1.2) is unique. Then

\[
h(t, x) = \sup_{\alpha} J_\alpha(t, x),
\]

(2.23)

where

\[
J_\alpha(t, x) := \mathbb{E}^{0,x} \left[ \int_0^\infty e^{-M^{-j} s} \left( -\lambda \tilde{V}(\frac{\alpha_s}{\lambda}) + g(t - s, X^\alpha_s) \right) ds + e^{-M^{-j} t} h_0(X^\alpha_t) \right]
\]

(2.24)

where \( X \) is the solution of the stochastic differential equation, \( dX_s = \alpha_s ds + \sqrt{dW_s} \) with initial condition \( X_0 = x \). An explicit optimal path is given by \( \alpha^*_s = \lambda \alpha^*(-\nabla h_{t-s}(X_s)) \).

**Remark.** Let us give an elementary application of Lemma 2.4, assuming \( t \mapsto ||g(t)||_\infty \) is integrable (in particular, finite a.e.). Taking \( \alpha \equiv 0 \) in eq. (2.24), one gets

\[
h(t, x) \geq e^{-M^{-j} t} \inf h_0 - \int_0^\infty e^{-M^{-j} s} ||g_s||_\infty ds
\]

(2.25)

since \( \tilde{V}(0) = \max_p (-V(p)) = 0 \). On the other hand, \( \tilde{V}(p) \geq -V(0) = 0 \) so

\[
h(t, x) \leq e^{-M^{-j} t} \sup h_0 + \int_0^\infty e^{-M^{-j} s} ||g_s||_\infty ds.
\]

(2.26)

Of course, both inequalities also follow from a direct application of the maximum principle.

The existence and unicity of the solution of the KPZ (or equivalently infra-red cut-off KPZ) equation (as a viscosity solution) follows in the case when \( V \) is quadratically bounded at infinity from an extended comparison principle proved in [23]. It relies strongly on the fact that the Cole-Hopf transform of \( h \) is a subsolution of the linear heat equation (see proof of [23], Theorem 3.1). If the growth exponent at infinity, \( \beta \), is \( > 2 \), then we cannot apply this argument any more. Instead, we rely on bounds for the Hamilton-Jacobi-Bellman solution of the KPZ equation on compact domains, similar to those proved in the next section, to prove unicity of the solution in the following sense.

**Definition 2.5** \((W_j^{1,\infty}(P_-, P_+))-solution\) Choose \( P_+ : \mathbb{R}_+ \to \mathbb{R}_+ \) be two convex, strictly increasing functions. Let \( h_0 \in W_j^{1,\infty}(P_-) \), \( g \in W_j^{1+1,\infty}(0, T) \) and \( h \in C([0, T]; W_j^{1,\infty}P_-) \). The function \( h \) is said to be a \( W_j^{1,\infty}(P_-, P_+)-solution \) of the scale \( j \) infra-red cut-off KPZ equation with right-hand side \( g \) if there exists a sequence of functions \( h_0^{(n)} \in W_j^{1,\infty} \), \( g^{(n)} \in L^\infty([0, T]; W_j^{1,\infty}) \) such that (i) for
every compact \( K \subset \mathbb{R}^d \), \( h_0^{(n)} \to_{n \to \infty} h_0 \) in \( \mathcal{W}^{1,\infty}(K) \) and \( g^{(n)} \to_{n \to \infty} g \) in \( L^\infty([0,T]; \mathcal{W}^{d+1,\infty}(K)) \); (ii) \( h^{(n)} \to_{n \to \infty} h \) in \( C([0,T]; \mathcal{W}^{1,\infty}(K)) \), where \( h^{(n)} \) is the unique classical solution in \( C([0,T]; \mathcal{W}^{1,\infty}) \cap C^{1,2}([0,T] \times \mathbb{R}^d) \) of the KPZ equation
\[
(KPZ_n) : \partial_t h^{(n)} = \nu \Delta h^{(n)} + V(\nabla h^{(n)}) + g^{(n)}, \quad h^{(n)}(t) = h_0^{(n)}. \tag{2.27}
\]

We are really interested in the exponential and polynomial cases. In general, without further precision, we shall also speak of \( \mathcal{W} \)-solutions for short.

**Theorem 2.1**  
(i) (exponential case) There exists a uniform constant \( c = c(d, \nu) > 0 \) such that the following holds. Assume \( h_0 \in \mathcal{W}^{d+1,\infty,\lambda}_{1,\alpha}([0,T]) \) and \( g \in \mathcal{W}^{d+1,\infty,\lambda}_{1,\alpha}([0,T]) \). Then the infra-red cut-off KPZ equation (1.2) has a unique \( \mathcal{W}^{1,\infty,\alpha}_{1,\alpha,\lambda} \)-solution on the time-interval \( \mathbb{R}_+ \), given by the Hamilton-Jacobi-Bellman representation (2.23).

(ii) (polynomial case) Let \( d' > d \), \( d'' > \frac{\beta - 1}{\beta} d \), and \( P_-(z) = z^{d'}, \ P_+(z) = z^{d''} \). Assume \( h_0 \in \mathcal{W}^{1,\infty}_{1,\alpha,\lambda} \) and \( g \in \mathcal{W}^{1,\infty}_{1,\alpha,\lambda}([0,T]) \). Then the infra-red cut-off KPZ equation (1.2) has a unique \( \mathcal{W}^{1,\infty}_{1,\alpha,\lambda} \)-solution on the time-interval \( \mathbb{R}_+ \), given by (2.23).

**Remark.** (i) may be rephrased as (ii) if one lets \( P_-(z) = e^{1/(\alpha - 1)}z \) and \( P_+(z) = e^{1/(\alpha - 1)}z \) since \( P_- \circ P_+ = c \).

**Proof.** We give the proof assuming that \( V(y)/y \to_{y \to \infty} +\infty \) (otherwise one should replace \( V(y) \) by \( V(y) + \epsilon y^2 \), \( \epsilon > 0 \), and take the limit \( \epsilon \to 0 \)). Let \( h_0^{(n)} \in \mathcal{W}^{1,\infty} \), \( g^{(n)} \in L^\infty([0,T]; \mathcal{W}^{d+1,\infty}) \) be a sequence of local \( \mathcal{W}^{1,\infty} \)-approximations as in Definition 2.5 and \( h^{(n)} \) the corresponding sequence of solutions of \( (KPZ)^n \) converging to \( h \). Let \( t \in [0,T], \ x \in \mathbb{R}^d \) and \( r > 0 \). We use a straightforward generalization of Lemma 2.4 (to the bounded domain \( B(x,r) \)) (see e.g. [22] for a discussion),

\[
h^{(n)}(t, x) = \sup_{\theta_r} \mathbb{E}_0^{0,x} \left[ \int_0^t e^{-M^{-1/s}} \left( -\lambda \tilde{V}(\frac{\alpha}{\lambda}) + g^{(n)}(t-s, X_s^\alpha) \right) ds + e^{-M^{-1/\theta_r}} h^{(n)}(t - \theta_r, X_0^\alpha) \right], \tag{2.28}
\]

where \( \theta_r := \inf \{ s \in [0,t] ; X_s \notin B(x,r) \} \) (= \( t \) if \( X_s \in B(x,r) \) for all \( s \in [0,T] \)). When \( n \to \infty \), \( h^{(n)} |_K \) converges in \( C([0,T]; \mathcal{W}^{1,\infty}(K)) \) to the classical solution of the KPZ solution with boundary value \( h |_{\partial B(x,r)} \) given by the Hamilton-Jacobi-Bellman representation (2.23) with \( h, g \) instead of \( h^{(n)}, g^{(n)} \).

Consider another \( \mathcal{W}^{1,\infty}_{1,\alpha,\lambda} \)-solution \( \tilde{h} \) and let \( \epsilon \in (0,1) \). Comparing the two Hamilton-Jacobi representations for \( h \) and \( \tilde{h} \) on \( B(x,r) \) and letting \( \alpha^+ \) maximize the right-hand side (2.28) in the limit \( n \to \infty \), we get

\[
h(t, x) - (1-\epsilon) \tilde{h}(t, x) \leq \mathbb{E}_0^{0,x} \left[ \int_0^t e^{-M^{-1/s}} \left( -\lambda \tilde{V}(\frac{\alpha}{\lambda}) + g(t-s, X_s^\alpha) \right) ds + \mathbb{E}_0^{0,x} \left[ 1_{\theta_r < t} e^{-M^{-1/\theta_r}} (h - (1-\epsilon) \tilde{h})(t - \theta_r, X_0^\alpha) \right] \right]. \tag{2.29}
\]

By Corollary 3.2 (see eq. (3.45)),

\[
h(t, x) - (1-\epsilon) \tilde{h}(t, x) \leq \mathbb{E} \left( e^{-M^{-1/s}} P_- \circ P_+ \left( \| h_0 \|_{\mathcal{W}^{1,\infty}_{1,\alpha,\lambda}}(x) \right) + P_- \circ P_+ \left( \| g \|_{\mathcal{W}^{d+1,\infty}_{1,\alpha,\lambda}}([0,T]) \right) \right) + \mathbb{E}^{0,x} \left[ \theta_r < t \right] P_- \circ P_+ \left( \frac{h - (1-\epsilon) \tilde{h}}{\epsilon} \right). \tag{2.30}
\]
Taking $P_+$ to be a polynomial in the above inequality, we may take the coefficient $\frac{1}{d}$ out of the pointwise quasi-norm. From the Schauder estimates, as proved in section 3.2, $(\text{loc sup} |\nabla^k h|)(x)$, $(\text{loc sup} |\nabla^k \tilde{h}|)(x) < \infty$ for $k = 0, \ldots, d - 1$. Finally, the spherical averages involved in the pointwise quasi-norm $W^d_{\beta,0,\infty} P_+(0, T)$ bring a supplementary large factor $O(r)$ (see (1.12)), which is compensated by the kernel $H(t, r)$. Thus the dependence of the bound on the boundary value $(h - (1 - \epsilon)\tilde{h})|_{\partial B(x, r)}$ disappears in the limit $r \to \infty$ (with $\epsilon$ fixed), leaving

\begin{equation}
\begin{aligned}
    & h(t, x) - (1 - \epsilon)\tilde{h}(t, x) \leq \epsilon \left( e^{-M'|t|} P_-^{-1} \circ P_- \left( ||h_0||_{W^d_{\beta,0,\infty} P_+(0, T)}(x) \right) + P_-^{-1} \circ P_+ \left( ||g||_{W^d_{\beta,0,\infty} P_+(0, T)}(x) \right) \right) \to \epsilon \to 0.
\end{aligned}
\end{equation}

Thus $h(t, x) \leq \tilde{h}(t, x)$. Exchanging the rôles of $h$ and $\tilde{h}$ yields finally $h = \tilde{h}$.

\[ \square \]

## 3 PDE estimates

We now use the above Hamilton-Jacobi-Bellman representation of the infra-red cut-off KPZ equation (1.2) to get a priori estimates on the solutions and its gradient in terms of the pointwise quasi-norms introduced in subsection 1.2. Subsection 3.2 below is devoted to a proof of Schauder estimates in the same spirit. As explained in subsection 3.3, these Schauder estimates enter as one of the ingredients of the proof of differentiability of the solution with respect to the forcing term.

### 3.1 Main result

The following theorem is our main result. It state in particular that, for some admissible classes of pairs $(P_-, P_+)$, the solution $h = h_{h_0, \nu}$ of the infra-red cut-off KPZ equation with initial condition $h_0 \in W^d_{\beta,0,\infty} P_+$ and right-hand side $g \in W^d_{\beta,0,\infty} P_+(0, T)$ belongs to $W^d_{\beta,0,\infty} P_+$. The loss of regularity at infinity when going from the data $(h_0, g)$ to the solution is reflected in the fact that one must choose $P_- < P_+$, in the exponential case and in the polynomial case alike; this is not such a big problem, and we show in subsection 3.4 how to handle this. The essential feature of our result is that estimates are local when expressed in pointwise quasi-norms, thus allowing to prove in the case of the noisy equation that $h(t, x) - \tilde{h}(t, x)$ is a.s. bounded, and even some $W^d$-pointwise quasi-norm of $h$ at $(t, x) - \tilde{h}(t, x)$ is a.s. bounded, and admits explicit large deviations (see [23], section 6).

**Theorem 3.1** (i) (exponential case) There exists a universal constant $c = c(d, \nu) > 0$ such that the following holds. Let $h_0 \in W^d_{\beta,0,\infty} \chi^{1/(\beta - 1)}$, $g \in \mathcal{W}^d_{\beta,0,\infty} \chi^{1/(\beta - 1)}(0, T)$ and $h$ be the function defined in (2.23). Then

\begin{equation}
\ln(e^{1/(\beta - 1)||h||}(x)) \leq \lambda^{1/(\beta - 1)} \left( e^{-M't}\||h_0||_{W^d_{\beta,0,\infty} \chi^{1/(\beta - 1)}(0, T)}(x) \right)\right). \tag{3.1}
\end{equation}

(ii) Let $d' > d$, $d'' > \frac{\beta - 1}{\beta}d$, and $P_-(z) = z^{d'}, P_+(z) = z^{d''}$. Assume $h_0 \in \mathcal{W}^d_{\beta,0,\infty} P_-$, $g \in \mathcal{W}^d_{\beta,0,\infty} P_+(0, T)$, and let $h$ be the function defined in (2.23). Then

\begin{equation}
\begin{aligned}
    & P_-^{-1} \left( (P_-(\lambda^{1/(\beta - 1)||h||}(x)(x)) \right) \leq e^{-M't} P_-^{-1} \circ P_+ \left( ||\lambda^{1/(\beta - 1)}h_0||_{W^d_{\beta,0,\infty} P_+(0, T)}(x) \right) \right) + P_-^{-1} \circ P_+ \left( ||\lambda^{1/(\beta - 1)}g||_{W^d_{\beta,0,\infty} P_+(0, T)}(x) \right). \tag{3.2}
\end{aligned}
\end{equation}

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Proof.
We give the proof in the case when \(V(y)/y \to y \to +\infty\). Otherwise one should simply note that a uniform bound is obtained for \(V\) replaced by \(V_\varepsilon(y) := V(y) + \varepsilon y^2\) and let \(\varepsilon \to 0\).

(i) \(\text{(exponential case)}\). Recall that \(h(t, x) = \sup_\alpha J_\alpha(t, x)\), where \(J_\alpha\) is defined in (2.24). Letting \(\alpha = 0\), one has \(h(t, x) \geq h^0(t, x)\), where \(h^0(t, x)\) is the solution of the linear heat equation \(\partial_t h^0 = \nu \Delta h^0 + g\). The bound (3.1) is trivially true for \(h^0\), so all there remains to do is to get an upper bound for \(e^{\lambda^{1/(\beta-1)}h}\).

By Jensen’s inequality,
\[
e^{\lambda^{1/(\beta-1)}h(t, x)} \leq \sup_\alpha J_\alpha(\lambda; t, x),
\]
where
\[
J_\alpha(\lambda; t, x) := \mathbb{E}[0, x] \left[ \exp \lambda^{1/(\beta-1)} \left( \int_0^t e^{-\Lambda^{1/\lambda}} \left( -\lambda \tilde{V}(\alpha t) + g(t - s, X_s^x) \right) ds + e^{-\Lambda^{1/\lambda}}h_0(X_t) \right) \right]
\]
with \(X_t = X_t^x\). The proof will be divided into three steps.

1. Since
\[
\int_0^t \tilde{V}(\alpha s) e^{-M^{1/\lambda}} ds = M^{-1} \int_0^t \tilde{V}(\alpha s) \cdot \left( \int_s^t e^{-M^{1/\lambda}} du \right) ds = M^{-1} \int_0^t \left( \int_0^u \tilde{V}(\alpha s) ds \right) e^{-M^{1/\lambda}} du,
\]
one gets
\[
J_\alpha(\lambda; t, x) \leq \lim_{n \to \infty} \mathbb{E}[0, x] \exp \left( \sum_{k=1}^n e^{-M^{1/\lambda} t_k} \left( -\frac{1}{2} M^{-1/\lambda} \frac{j^{\beta/(\beta-1)}}{n} \int_0^{k t/n} \tilde{V}(\alpha s) ds + M^{-1/\lambda} \lambda^{1/(\beta-1)} \bar{g}_{k t/n}(X_{k t/n}) \right) + e^{-M^{1/\lambda} t} \left( -\frac{1}{2} \lambda^{1/(\beta-1)} \int_0^t \tilde{V}(\alpha s) ds + \lambda^{1/(\beta-1)} h_0(X_t) \right) \right) \tag{3.6}
\]
where \(\bar{g}_{k t/n}(y) := M^{1/\lambda} \int_{t_{k t/n}}^{(k+1)/n} g(t-u, y) du\). Using the generalized Hölder property, \(\mathbb{E}[Y_1 \ldots Y_n] \leq \prod_{k=1}^n \mathbb{E}[X_k^0]^{1/p_k} \frac{1}{p_1} + \ldots + \frac{1}{p_n} = 1\), \(Y_1, \ldots, Y_n \geq 0\), with (letting \(a_j(t) := 1 - e^{-M^{1/\lambda} t} \in [0, 1]\))
\[
p_k = \left( \sum_{k=1}^n e^{-M^{1/\lambda} t_k} \right)^{-1} \sim \lim_{n \to \infty} M^{1/\lambda} a_j(t) e^{M^{1/\lambda} t_k/n},
\]
one gets
\[
J_\alpha(\lambda; t, x) \leq \lim_{n \to \infty} \prod_{k=1}^n \left( \mathbb{E}[0, x] \left[ \exp 2a_j(t) \left( -\frac{1}{2} \lambda^{1/(\beta-1)} \int_0^{k t/n} \tilde{V}(\alpha s) ds \right) + \lambda^{1/(\beta-1)} \bar{g}_{k t/n}(X_{k t/n}) \right) \right)^{1/2}
\]
\[
\leq \lim_{n \to \infty} \prod_{k=1}^n \left( \mathbb{E}[0, x] \left[ \exp \left( -\lambda^{1/(\beta-1)} \int_0^{k t/n} \tilde{V}(\alpha s) ds + \lambda^{1/(\beta-1)} \bar{g}_{k t/n}(X_{k t/n}) \right) \right] \right)^{1/2}
\]
by Jensen’s inequality.

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2. We shall bound each individual term
\[ J(\lambda; u, x) := \exp \left( -2^{\beta/(\beta-1)} \int_0^u \tilde{V}(\frac{\alpha_s}{\lambda}) d\lambda + \lambda^{1/(\beta-1)} \tilde{g}_u(X_u) \right), \]  

\( u = kt/n \) \((k = 1, \ldots, n)\) in the above expression. The factor depending on the initial condition is identical to \( J(\lambda; t, x) \) except that \( \tilde{g}_t \) is replaced with \( h_0 \), so we do not discuss it any more and assume \( h_0 \equiv 0 \) in the sequel to simplify notations. Note also that the generalized Hölder property used in the previous paragraph may also be applied to \( e^{\tau \Delta} \mathbb{E} [ \cdot ] \); thus \( (e^{\tau \lambda} J_0(\lambda; t))(x) \) is bounded by the product of the \( (e^{\tau \lambda} J_{x,u})(x) \).

Note first that
\[ \tilde{V}(p) = \sup_x (p \cdot x - V(x)) \geq (|p|^2 - V(p)) 1_{|p| \leq 1} + \left( |p|^\beta/(\beta-1) - V(|p|^{1/(\beta-1)}) \right) 1_{|p| > 1} \geq \frac{1}{2} \min(|p|^2, |p|^\beta/(\beta-1)) \]  

hence
\[ \lambda \tilde{V}(p) = \lambda \tilde{V}(\frac{p}{\lambda}) \geq \frac{1}{2} \min \left( \frac{|p|^2}{\lambda}, \left( \frac{|p|^\beta}{\lambda} \right)^{1/(\beta-1)} \right). \]  

Let \( \Omega := \{ s \in [0, u] \mid |\alpha_s| \leq 1 \} \), and \( \tilde{\Omega} := [0, u] \setminus \Omega \). On \( \lambda \tilde{V}(\frac{\alpha_s}{\lambda}) \geq \frac{\alpha^2}{2} \) on \( \tilde{\Omega} \), \( \lambda^{\beta/(\beta-1)} \tilde{V}(\frac{\alpha_s}{\lambda}) \geq \frac{\alpha^2}{2} \). We now distinguish two cases. If \( \int_0^u |\alpha_s| ds \geq \frac{1}{2} \int_0^u |\alpha_s| d\lambda \), then
\[ \lambda^{\beta/(\beta-1)} \int_0^u \tilde{V}(\frac{\alpha_s}{\lambda}) d\lambda \geq \frac{1}{2} \int_0^u |\alpha_s|^{\beta/(\beta-1)} d\lambda \geq \frac{1}{2} \int_0^u |\alpha_s| d\lambda \int_0^u |\alpha_s|^{\beta/(\beta-1)} d\lambda \]  

Otherwise it follows from Hölder’s inequality that
\[ \lambda^{\beta/(\beta-1)} \int_0^u \tilde{V}(\frac{\alpha_s}{\lambda}) d\lambda \geq \frac{1}{2} \int_0^u |\alpha_s|^{\beta/(\beta-1)} d\lambda \geq \frac{1}{2} |\Omega|^{-1/(\beta-1)} \int_0^u |\alpha_s| d\lambda \int_0^u |\alpha_s|^{\beta/(\beta-1)} d\lambda \]  

By definition, \( X_u - x = \int_0^u \alpha_s ds + \sqrt{V}_u \), hence the net outcome of all these computations is
\[ \lambda^{\beta/(\beta-1)} \int_0^u \tilde{V}(\frac{\alpha_s}{\lambda}) d\lambda \geq \frac{1}{52} \min \left( \frac{|X_u - x|^2}{u}, \left( \frac{|X_u - x|^\beta}{u} \right)^{1/(\beta-1)} \right) 1_{|X_u - x| \geq 1} \]  

Hence (using Lemma 5.2)
\[ J(\lambda; u, x) \leq 1 + \mathbb{E}^{0, x} \left[ 1_{|X_u - x| \leq 1} \sqrt{V}_u \left( e^{l/(\beta-1)} \tilde{g}_u(X_u) - 1 \right) \right] \]  

where
\[ I_1 \leq \int_{M^{1/2}}^{+\infty} dr \mathbb{E}^{0, x} \left[ |r - |X_u - x| < 2 \sqrt{V}_u \| \sum_{k=0}^d (M^{-j/2})^{d-k} \int_{S_r} |\nabla_y^k (e^{l/(\beta-1)} \tilde{g}_u(x+y)) - 1 | \sigma^{d-1} d\sigma(y) \right] \]  

(3.15)
\[ I_2 \leq \int_{M^{j/2}}^{r \to \infty} dr \mathbb{P}^0_x[|X_u - x| \geq \max(2 \sqrt{v}|B_u|, r)] \]
\[ \sum_{k=0}^{d} (M^{-j/2})^{d-k} \int_{S_r} |\nabla_k^y (e^{-\frac{r^2}{2\sigma^2}} (e^{1/(\beta-1)} \mathcal{G}_u(x+y) - 1))| r^{d-1} d\sigma_r(y); \quad (3.16) \]

\[ I_3 \leq \int_{M^{j/2}}^{r \to \infty} dr \mathbb{P}^0_x[|X_u - x| \geq \max(2 \sqrt{v}|B_u|, r)] \]
\[ \sum_{k=0}^{d} (M^{-j/2})^{d-k} \int_{S_r} |\nabla_k^y (e^{-\frac{r^2}{2\sigma^2}} (e^{1/(\beta-1)} \mathcal{G}_u(x+y) - 1))| r^{d-1} d\sigma_r(y). \]
\[ (3.17) \]

The last integral involves the generalized heat kernels of exponent \( \beta \) introduced in section 4. Thanks to the exponentially decreasing factors in the bounds found for \( I_2 \) and \( I_3 \), it is safe to replace \( \mathbb{P}^0_x \) by \( 1 \) in (3.16) and (3.17).

We first estimate \( I_1 \). One has \( \mathbb{P}^0_x[r < |X_u - x| < 2 \sqrt{v}|B_u|] \leq \mathbb{P}^0_x[|B_u| > r/2 \sqrt{v}] \); if \( r \geq \sqrt{u} \) this is \( O \left( u^{-d/2} \int_{r/2}^{\infty} \sqrt{s} d\rho \rho^{d-1} e^{-r^2/2\rho} \right) \leq \left( \frac{r^2}{u} \right)^{d-1} e^{-r^2/8u} (1 + O(\frac{r^2}{u})) \leq e^{-r^2/16u} \). Thus
\[ I_1 \leq \sum_{k=0}^{d} (M^{-j/2})^{d-k} \int_{M^{j/2}}^{r \to \infty} dr \ e^{-r^2/16uv} \left( \int_{S_r} |\nabla_k^y (e^{1/(\beta-1)} \mathcal{G}_u(x+y) - 1)| r^{d-1} d\sigma_r(y) \right) \]
\[ \leq (M^{-j} u)^{d/2} \sum_{k=0}^{d} (M^{j/2})^k \|\nabla_k^y (e^{1/(\beta-1)} \mathcal{G}_u - 1)\|_{\mathcal{H}(x)} . \]
\[ (3.18) \]

A bound for the integrals \( I_2 \) follows immediately: since
\[ |\nabla_k^y (e^{-\frac{r^2}{2\sigma^2}} (e^{1/(\beta-1)} \mathcal{G}_u(x+y) - 1))| \leq e^{-\frac{r^2}{2u}} \sum_{l=0}^{k} u^{-l/2} |\nabla_l^y (e^{1/(\beta-1)} \mathcal{G}_u(x+y) - 1)| , \quad |y| = r \]
\[ (3.19) \]

one gets
\[ I_2 \leq \sum_{k=0}^{d} \sum_{l=0}^{k} (M^{-j} u)^{(d-l)/2} (M^{j/2})^{l-1} \|\nabla_l^y (e^{1/(\beta-1)} \mathcal{G}_u - 1)\|_{\mathcal{H}(x)} . \]
\[ (3.20) \]

As for \( I_3 \), we get the following bounds for the derivatives of the generalized heat kernels \( \Phi_u^\beta(y) = e^{-|y|^{1/(\beta-1)}} \) (see section 4) for \( y \geq M^{j/2} \),
\[ |\nabla \Phi_u^\beta(y)| \leq (|y|/u)^{1/(\beta-1)} \Phi_u^\beta(y) \leq u^{-1/\beta} \Phi_u^\beta(y); \]
\[ (3.21) \]
\[ |\nabla^2 \Phi_u^\beta(y)| \leq \left( \frac{(|y|/u)^{1/(\beta-1)}}{|y|} + (|y|/u)^{2/(\beta-1)} \right) \Phi_u^\beta(y) \]
\[ \leq (M^{-j} u)^{-1/\beta} + u^{-2/\beta} \Phi_u^\beta(y) \leq u^{-2/\beta}(1 + (M^{-j} u)^{1/\beta}) \Phi_u^\beta(y) \]
\[ (3.22) \]
and more generally

$$|\nabla_j^l \Phi^\beta_k(y)| \lesssim u^{-l/\beta}(1 + (M^{-l}u)^{(l-1)/\beta}) \Phi^\beta_k(y), \quad l \geq 1 \quad (3.23)$$

so (letting $|y| = r$)

$$\left| \nabla_y^k \left( e^{-1/(\beta u^{1/(\beta-1)})} (e^{1/(\beta-1)\bar{g}_u(x+y)} - 1) \right) \right| \lesssim e^{-1/(\beta u^{1/(\beta-1)})} \sum_{l=0}^{k} (1 + (M^{-l}u)^{(l-1)/\beta}) u^{-l/\beta} |\nabla_y^{k-l} \left( e^{1/(\beta-1)\bar{g}_u(x+y)} - 1 \right) |. \quad (3.24)$$

Hence

$$I_3 \lesssim \sum_{k=0}^{d} \sum_{l=0}^{k} (M^{-l/2})^{d-k} (1 + (M^{-l}u)^{(l-1)/\beta}) u^{-l/\beta} \cdot \int_{0}^{+\infty} \int_{S_y} e^{-1/(\beta u^{1/(\beta-1)})} |\nabla_y^{k-l} \left( e^{1/(\beta-1)\bar{g}_u(x+y)} - 1 \right) | r^{d-1} d\sigma_r(y). \quad (3.25)$$

The above integral is bounded by Lemma 4.1 by a constant times $u^{d/\beta} |||\nabla^{k-l} (e^{1/(\beta-1)\bar{g}_u} - 1) |||_H^0(x)$, so, collecting all power-counting factors,

$$I_3 \lesssim \sum_{k=0}^{d} \sum_{l=0}^{k} (1 + (M^{-l}u)^{(l-1)/\beta})(M^{-l}u)^{(d-l)/(\beta-1)} (M^{l/2})^{k-l} |||\nabla^{k-l} (e^{1/(\beta-1)\bar{g}_u} - 1) |||_H^0(x) \leq \sum_{k=0}^{d} \sum_{l=0}^{k} (1 + (M^{-l}u)^{(l-1)/\beta})(M^{l/2})^{k-l} |||\nabla^{k-l} (e^{1/(\beta-1)\bar{g}_u} - 1) |||_H^0(x). \quad (3.26)$$

To proceed further we must bound the above $H^0$-norms in terms of the $W_{j,\infty}^{1,1}$-norm:

$$|||e^{1/(\beta-1)\bar{g}_u} - 1 |||_H^0(x) = |||e^{1/(\beta-1)\bar{g}_u} - 1 |||_H^{1/(\beta-1)}(x) - 1 = e^{1/(\beta-1)\bar{g}_u} - 1; \quad (3.27)$$

$$M^{l/2} |||\nabla (e^{1/(\beta-1)\bar{g}_u} - 1) |||_H^0(x) = |||1^{1/(\beta-1)} M^{l/2} \bar{g}_u e^{1/(\beta-1)\bar{g}_u} |||_H^0(x) \leq |||e^{1/(\beta-1)(M^{l/2} \nabla \bar{g}_u + [\bar{g}_u, \bar{g}_u])} - 1 |||_H^0(x) \quad \text{using } xe^y \leq e^{x+y} - 1 \quad (x, y \geq 0) \quad (3.28)$$

by Jensen’s inequality, and similarly for higher-order derivatives. We leave it to the reader to check that, summing all terms, one gets

$$\tilde{J}(\lambda; u, x) \leq 1 + c(1 + (M^{-l}u)^{d/2}) \left\{ e^{1/(\beta-1)\bar{g}_u} |||W_{j,\infty}^{1,1} |||_H^{1/(\beta-1)}(x) - 1 \right\} \quad (3.29)$$

for some constant $c = c(d) > 0$. The same bound applies to $(e^T \tilde{J}(\lambda; u))(x)$. 

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3. Combining (3.7), (3.8) and (3.29), one gets

\[ \ln(J_\alpha(\lambda; t))^n(x) \leq \lim_{n \to \infty} M^{-j/t/n} \sum_{k=1}^{n} e^{-M^{-j}k/n} \ln \left( 1 + c(1 + (M^{-j}kt/n)^{d/2}) f(kt/n) \right), \quad (3.30) \]

where \( f(u) := e^{\lambda^{1/\beta-1}||g||_W^{d',(1)}(x)} - 1 \).

By definition, \( M^{-j/t/n} \sum_{k=1}^{n} e^{-M^{-j}k/n} \ln(1 + f(kt/n)) \leq \frac{1}{\epsilon} \lambda^{1/\beta-1}||g||_{W_j^{d',(1)}}(0,t)(x) \). We must still take into account the polynomial correction \( c(1 + (M^{-j}u)^{d/2}) \) in the right-hand side of (3.30). Let \( \Omega_1 := \{ u \in [0, t] \mid f(u) \gg 1 \} \) and \( \Omega_2 := [0, t] \setminus \Omega_1 \). On \( \Omega_1 \), \( e^{-M^{-j}u} \ln(1 + c(1 + (M^{-j}u)^{d/2}) f(u)) \leq e^{-M^{-j}u} \ln(1 + (M^{-j}u)^{d/2}) + \ln(1 + f(u)) \leq (1 + (M^{-j}u)^{d/2}) e^{-M^{-j}u} \ln(1 + f(u)) \). Similarly, on \( \Omega_2 \), \( e^{-M^{-j}u} \ln(1 + c(1 + (M^{-j}u)^{d/2}) f(u)) \leq (1 + (M^{-j}u)^{d/2}) e^{-M^{-j}u} f(u) \leq (1 + (M^{-j}u)^{d/2}) e^{-M^{-j}u} \ln(1 + f(u)) \). Now the resulting factor \( (1 + (M^{-j}u)^{d/2}) \) has been included into the definition of \( || \cdot ||_{W_j^{d',(1)}}(0,t) \).

(ii) (polynomial case).

The proof is very similar and we only emphasize the differences. We start from \( J_\alpha(t, x) \) as in (i) and use the following immediate consequence of the generalized Hölder property for real-valued random variables \( (Y_k)_k \),

\[ \left( E \left[ \left( \sum_k \mu_k (Y_k)_x \right)^{d'} \right] \right)^{1/d'} \leq \sum_k \mu_k \left( E [(Y_k)_{x,x}] \right)^{1/d'} \quad (3.31) \]

with \( \mu_k \approx M^{-j/L} e^{-M^{-j}k/t/n} \) normalized so that \( \sum_k \mu_k = 1 \), and \( Y_k = \lambda^{1/\beta-1} \left( - \lambda \int_0^{kt/n} \overline{F}( \frac{\nu}{\alpha} ) \, d s + \tilde{g}^{k/n}(X_{kt/n}) \right) \). Hence (compare with (3.7) and (3.8)), extending \( P_- \) to zero on \( \mathbb{R}_- \),

\[ P_-(E[\lambda^{1/\beta-1} J_\alpha(t, x)]) \leq E[P_-(\lambda^{1/\beta-1} J_\alpha(t, x))] \leq P_-(\limsup_{n \to \infty} M^{-j/t} \frac{1}{n} \sum_{k=1}^{n} e^{-M^{-j}k/n} P_-^{1/2}(J_\alpha; (k/t,n,x))) \].

(3.32)

Instead of (3.14), one obtains

\[ J(P_-; u, x) \leq E^{0,x} \left[ 1_{|X_u-x|^2 \leq 2 \sqrt{\tilde{g}_u}} P_-(\lambda^{1/\beta-1} \tilde{g}_u(X_u)) \right] + E^{0,x} \left[ 1_{|X_u-x|^2 \geq 2 \sqrt{\tilde{g}_u}} P_- \left( \lambda^{1/\beta-1} |\tilde{g}_u(X_u) - \frac{|X_u-x|^2}{32u}| \right) \right] \]

\[ + E^{0,x} \left[ 1_{|X_u-x|^2 \leq 2 \sqrt{\tilde{g}_u}} P_- \left( \lambda^{1/\beta-1} |\tilde{g}_u(X_u) - \frac{(X_u-x)^{1/\beta}}{32u} | \right) \right] \]

\[ \leq \sup_{B(x, M^{d/2})} P_-(\lambda^{1/\beta-1} \tilde{g}_u(X_u)) + I_1 + I_2 + I_3 \]

(3.33)

where \( I_1, I_2, I_3 \) are the same as the quantities in (3.15), (3.16), (3.17), except that \( e^{\lambda^{1/\beta-1} \tilde{g}_u - 1} \) is replaced by \( P_-(\lambda^{1/\beta-1} \tilde{g}_u) \), and \( I_1 \) is estimated as in (i).

After these rather general considerations, we now fix \( P_-(z) = z^{d'}, d' > d \) and estimate \( I_2 \) by some specific arguments. Instead of (3.19), we have \( \nabla^{k} \left( P_- \left( \lambda^{1/\beta-1} \tilde{g}_u(x+y) - \frac{y^2}{32u} \right) \right) = 0 \).
if \( \lambda^{1/\beta-1} \g_u(x + y) \leq \frac{y^2}{32u} \), otherwise

\[
\left| \nabla_x \left( \lambda^{1/\beta-1} \g_u(x + y) - \frac{y^2}{32u} \right) \right| \leq \sum_{l=0}^{k} \sum_{l_1 + l_2 = l} \frac{\lambda^{l-1} y^l}{u^{l_1 + l_2}} \left| \nabla_{y'} \left( \lambda^{1/\beta-1} \g_u(x + y') - \frac{y^2}{32u} \right) \right|^{d-l_1-l_2}
\]

\[
\leq \sum_{l=0}^{k} \sum_{l_1 + l_2 = l} \frac{\lambda^{l-1} y^l}{u^{l_1 + l_2}} \sup_{\sum_i q_i = k-l} \left( \lambda^{1/\beta-1} \g_u(x + y) - \frac{y^2}{32u} \right)^{d-l_1-l_2} \left| \nabla_{y'} \left( \lambda^{1/\beta-1} \g_u(x + y') \right) \right|^{d-l_1-l_2}.
\]

By hypothesis, the exponent \( d' - l_1 - l_2 - \sum_{i=1}^{d'} q_i \geq d' - l_1 - (k-l) \geq d' - d \geq 1 \). We now use the trivial key inequality \( (a - b)^{d-(l_1+l_2)-\sum q_i} \leq \frac{(a - b)^{d-(l_1+l_2)-\sum q_i}}{b^{l_1+l_2+d'}} (a \geq b > 0) \) and bound the general term of the above expression by

\[
\frac{\lambda^{l-1} y^l}{u^{l_1 + l_2}} \sup_{\sum_i q_i = k-l} \left( \lambda^{1/\beta-1} \g_u(x + y) - \frac{y^2}{32u} \right)^{d-l_1-l_2} \left| \nabla_{y'} \left( \lambda^{1/\beta-1} \g_u(x + y') \right) \right|^{d-l_1-l_2}.
\]

The kernel \( \Phi(y) = (M^{l/2})_{l_1 + l_2 + d'} \left( y^2 \right)^{(l_1 + l_2 + d')/2} \) is integrable at infinity since (by hypothesis) \( d' > d/2 \), and satisfies the hypotheses of Lemma 4.1 (up to a normalization constant of order 1). Hence

\[
I_2 \leq u^{d'} \sum_{l \leq k-l} \sup_{\sum_i q_i = k-l} \left( \lambda^{l-1/2} (\g_u + M^{l/2} \nabla \g_u + \ldots + M^{l/2} (\nabla^d \g_u)) \right)^{d'+d''} \left| \nabla_{y'} \left( \lambda^{1/\beta-1} \g_u \right) \right|^2.
\]

The same arguments apply to \( I_3 \) except that the replacement of \( \frac{y^2}{u} \) by \( \left( \frac{y}{u} \right)^{1/\beta} \) yields an integrable kernel \( \Phi(y) \) which behaves at infinity (up to scaling factors) like \( \left( \frac{y}{u} \right)^{l-i(l_1+l_2)-l} \left( \frac{y}{u} \right)^{(l_1+l_2+d')/2} = O(\left( \frac{y}{u} \right)^{d'}) \), still integrable by hypothesis.

By increasing if necessary the value of the constant \( c \), one gets an analogous bound on the gradient of the solution.

**Corollary 3.1** Let \( h_0 \in \mathcal{W}^{d+1,\infty;P_+}_j \), \( g \in \mathcal{W}^{d+1,\infty;P_+}_j \) and \( h \) be the function defined in (2.23). Then \( h_t \in \mathcal{W}^{1,\infty;P_+}_j \) for all \( t \geq T \), and

\[
P^{-1}_+ \left( \left( \lambda^{1/\beta-1} \right)^{d/2} \nabla h_t \right) \leq e^{-M^{l/2}} P^{-1}_+ \left( \left\| h_0 \right\|_{\mathcal{W}^{d+1,\infty;P_+}_j} \right) + P^{-1}_+ \left( \left\| g \right\|_{\mathcal{W}^{d+1,\infty;P_+}_j} \right). \]

**Proof.**

(i) (gradient of the solution) Let \( \lambda^+ \) maximize the right-hand side of (2.23), \( \epsilon \in \mathbb{R}^d \setminus \{0\} \) and \( \delta_\epsilon h(t, x) := h(t, x + \epsilon) - (1 - M^{-l/2}) h(t, x) \), \( \delta_\epsilon g(t, x) := g(t, x + \epsilon) - (1 - M^{-l/2}) g(t, x) \) (see [23], section 4.3 for a similar proof). Then

\[
\delta_\epsilon h(t, x) \leq \mathbb{E}^{0,x} \left[ \int_0^t e^{-M^{-l/2}} s \lambda^{-\epsilon} \left( \frac{\epsilon_0^+}{\lambda^+} + \delta_\epsilon g(t-s, X^\epsilon_s) \right) ds + e^{-M^{-l/2}} \delta_\epsilon h(0, X^\epsilon_0) \right].
\]

(3.37)

Proceeding as in the proof of Theorem 3.1 and using \( \delta_\epsilon h(t, \cdot) = (h(t, \cdot + \epsilon) - h(t, \cdot) - M^{-l/2}) \| \epsilon g(t, \cdot) \), \( \delta_\epsilon g(t, \cdot) = (g(t, \cdot + \epsilon) - g(t, \cdot)) - M^{-l/2} \| \epsilon g(t, \cdot) \), one gets (3.36).
isotropic and eq. (3.38) yields essentially a weighted integral over time of the spherical averages

Also, we are only interested in bulk estimates; viewed from the bulk, the boundary is essentially

where

Corollary 3.2 (generalization to a case of a bounded domain) There exists a universal constant

c = c(d, ν) > 0 such that the following holds. Let \( Ω = B(0, r) \) be a ball, with boundary \( ∂Ω = S_r \), and

\[ \mathcal{W}^d,∞_{j, S_r}([0, T] × ∂Ω) := \{ v ∈ L^1_{loc}([0, T] × ∂Ω; ∀t ∈ [0, T], ∀x ∈ Ω, ||v||_{W^d,∞_{j,S_r}(B(0, r) × ∂Ω)}(t, x) < ∞, \right\} \]

where

\[ ||v||_{W^d,∞_{j,S_r}(B(0, r) × ∂Ω)}(t, x) := P^{-1}_+ \circ P_-(\sup_{k=0, ..., d'} e^{-\frac{1}{2}M^{-1/t}} P^{-1}_1(\Phi(t, r) \int_{S_r} r^k P_+(α^{1/(β-1)}|\nabla^k v(t)|))) \]

+ \( \sup_{k=0, ..., d'} \int_0^t dθ e^{-\frac{1}{2}M^{-1/θ}} P^{-1}_1(\Phi(θ, r) \int_{S_r} r^k P_+(α^{1/(β-1)}|\partial_θ \nabla^k v(θ)|)) \),

with \( \Phi(θ, r) = \max(\Phi^2(θ, cr), \Phi^2(θ, cr)) \), see (4.3) in the exponential case, and \( \Phi(θ, r) = \max( \frac{1}{(r^2/θ)^α}, \frac{1}{(r^2/θ)^β(β-1)} ) \)
in the polynomial case. Then

\[ h(t, x) := \sup_a \left[ \int_0^s e^{-\frac{1}{2}M^{-1/t}} \left( -\lambda \tilde{V}(\frac{α}{λ}) + g(t - s, X^α_s) \right) ds + e^{-M^{-1/θt}} h(t - θt, X^α_{θt}) \right] \]

where \( θ_1 := \inf\{ s ∈ [0, t] \mid X_s ∈ Ω \} = t \) if \( X_s ∈ Ω \) for all \( s ∈ [0, t] \), satisfies when \( x ∈ B(0, r/2) \) is in the ”bulk” the same kind of estimate as in Theorem 3.3

\[ P^{-1}_-(P_+(α^{1/(β-1)}|h(t)|))^s(x) ≤ e^{-\frac{1}{2}M^{-1/t}} P^{-1}_1 \circ P_-(||h||_{W^d,∞_{j,S_r}(B(0, r/2) × ∂Ω)}(x)) \]

+ \( P^{-1}_+ \{ Θ < t \} P^{-1}_1 \circ P_+(||v||_{W^d,∞_{j,S_r}(B(0, r/2) × ∂Ω)}(t, x)) \).
Remark. It is proved in Corollary 3.3 that \( \mathbb{P}^{0, \lambda}[\theta_{\Omega < t}] = O(r^{-\alpha}) \) for some \( \alpha > 0 \) when \( v \) is the boundary value of the solution on the whole space.

Proof.

The strategy is the same as for Theorem 3.1. The contribution of the boundary value to 
\( P^{-1}_\Omega \left( \left( P_\Omega \left( 1^{(1/\beta - 1)}[h_i] \right) \right)^* \right) \) vanishes unless \( \theta_{\Omega < t} \). To estimate it, we replace \( \mathbb{E}^{0, \lambda}[1_{\theta_{\Omega < t}}] \) by \( \mathbb{P}^{0, \lambda} \), where \( \mathbb{E}^{0, \lambda}[\cdot] := \mathbb{E}^{0, \lambda}[\cdot | \theta_{\Omega < t}] \) is the conditional expectation. The reader may check (either replacing \( p_{\theta_{\Omega}} \) by \( p_{\theta_{\Omega}}/\mathbb{P}^{0, \lambda}[\theta_{\Omega < t}] \) in Hölder’s inequality in the exponential case, see 3.5), or \( \mu_{\theta_{\Omega}} \) by \( \mu_{\theta_{\Omega}}/\mathbb{P}^{0, \lambda}[\theta_{\Omega < t}] \) in eq. (3.3) in the polynomial case) that the coefficient \( \mathbb{P}[\theta_{\Omega < t}] \) may be turned into a multiplicative factor in front of the contribution of the boundary value. Then we bound

\[
\tilde{J}(P; \theta_{\Omega}, x) := \mathbb{E}^{0, \lambda}[P_\Omega \left( -A^{(\beta)}(\theta_{\Omega}, X_{\theta_{\Omega}}) \right)]
\]

by \( I_1 + I_2 + I_3 \), where (using first an integration by parts with respect to time to get rid of \( \theta_{\Omega} \) as in (4.8), then a multi-dimensional integration by parts on the boundary as in Lemma 4.3)

\[
I_1 \leq \mathbb{E}^{0, \lambda}[2 \sqrt{V} |B_r| > r] \sum_{k=0}^{d-1} r^k \int_{S_r} |\nabla^k v(t)| + \int_0^t d\theta \mathbb{E}^{0, \lambda}[\mathbf{2} \sqrt{V} |B_r| > r] \sum_{k=0}^{d-1} r^k \int_{S_r} |\nabla^k \partial_\theta P_{\theta_{\Omega}} A^{(\beta)}(\theta_{\Omega}, y) |
\]

and \( I_2, I_3 \) are given by similar expressions, with \( A^{(\beta)}(\theta_{\Omega}, y) \) replaced by \( A^{(\beta)}(\theta_{\Omega}, y) - y^2/32\theta \) or \( A^{(\beta)}(\theta_{\Omega}, y) - \frac{1}{32} y^3 \theta^{1/3} \). Details are left to the reader.

Note that if \( \nabla^k \theta_{\Omega}(x, k) = 0, 1, \ldots, d-1 \) are bounded in time, then the time integration by parts is not necessary, and (3.42) holds with an alternative pointwise quasi-norm \( \| v \|_{W^{d-1, \infty, \rho}_{(0, T) \times \partial \Omega}} \), where now

\[
\| v \|_{W^{d-1, \infty, \rho}_{(0, T) \times \partial \Omega}}(t, x) = P_{\rho}^{-1} \left( \Phi(t, r) \sup_{k=0, \ldots, d-1} \int_{S_r} r^k \sup_{\theta_{\Omega}[0, T]} (\nabla^k v(t)) \right).
\]

3.2 Schauder estimates

We present in this paragraph Schauder estimates for arbitrary \( W^{1, \infty, \lambda} \)-solutions \( h \) of the infra-red cut-off KPZ equation, from which we derive bounds for higher-order derivatives of \( h \). Schauder estimates for all types of elliptic or parabolic PDEs, linear or nonlinear, are scattered in many articles and textbooks. We cite a precise Schauder-type bound for linear parabolic PDEs as found in [24].

Proposition 3.3 Let \( v \) solve the linear parabolic PDE \( (\partial_t - \kappa \Delta)u(t, x) = b(t, x) \cdot \nabla u(t, x) + f(t, x) \) on the "parabolic ball" \( Q^{(j)}(t_0, x_0) := \{ (t, x) \in \mathbb{R} \times \mathbb{R}^d ; t_0 - M^j \leq t \leq t_0, x \in B(x_0, M^{j/2}) \} \), with given initial-boundary value \( u_{\partial \Omega}^{(j)} = v \), where \( \partial \Omega Q^{(j)}(t_0, x_0) := \{ (t_0 - M^j) \times B(x_0, M^{j/2}) \} \). If \( v \) (hence \( u \)) is bounded and

\[
\| f \|_{L^2, Q^{(j)}(t_0, x_0)} := \sup_{(t, x)} \frac{|f(t, x) - f(t', x')|}{|x - x'|^\alpha + |t - t'|^{\beta/2}} < \infty
\]

for some \( \alpha \in (0, 1) \), and similarity \( \| b \|_{L^2, Q^{(j)}(t_0, x_0)} < \infty \), then

\[
\sup_{Q^{(j)}(t_0, x_0)} |\nabla u| \leq M^{j/2} \left( M^{j/2} M^{j/2} + (M^{j/2} M^{j/2} + M^{-1} R^{-1} b) \sup_{Q^{(j)}(t_0, x_0)} \| u \| \right),
\]
\[ \sup_{Q^{(j)}(t_0, x_0)} \| \nabla^2 u \| \leq M^j \alpha/2 \| f \| + (M^j \alpha) \sup_{Q^{(j)}(t_0, x_0)} |u|; \quad (3.48) \]

\[ \| \nabla^2 u \|_{Q^{(j)}(t_0, x_0)} \leq M^j \alpha/2 \left( (M^j \alpha) \| f \| + (M^j \alpha) \| b \| + M^{-j} R^{-1/2} \right) \sup_{Q^{(j)}(t_0, x_0)} |u| \quad (3.49) \]

where \( R_b := \left( 1 + M^{j/2} b(0, 0) \right)^{-1} \).

**Proof.** By simple parabolic scaling we may assume that \( j = 0 \). The following bound for \( \| \nabla^2 u \| \) follows straightforwardly from [24], section 2,

\[ \| \nabla^2 u \| \leq \| \nabla^2 u_0 \| + \| f \| + \| b \| \sup_{Q^{(0)}(t_0, x_0)} |\nabla u|, \quad (3.50) \]

where \( u_0 \) solves the "frozen" equation \((\partial_t - \nu \Delta) u(t, x) = f(0, 0)\). By exactly the same arguments one proves the same bound for \( \sup_{Q^{(j)}(t_0, x_0)} |\nabla u| \), resp. \( \sup_{Q^{(j)}(t_0, x_0)} \| \nabla^2 u \| \), with \( \| \nabla^2 u_0 \| \) replaced by the corresponding norm \( \sup_{Q^{(j)}(t_0, x_0)} |\nabla u_0| \), resp. \( \sup_{Q^{(j)}(t_0, x_0)} \| \nabla^2 u_0 \| \). If \( |b(0, 0)| \leq 1 \) then one has the standard inequality \( \sup_{Q^{(j)}(t_0, x_0)} |\nabla^2 u_0|, \sup_{Q^{(j)}(t_0, x_0)} |\nabla u_0| \leq \sup_{Q^{(0)}(t_0, x_0)} |u| \). If \( |b(0, 0)| > 1 \) then one makes the Galilean transformation \( x \mapsto x - b(0, 0) t \) to get rid of the drift, after which the boundary of \( Q^{(0)}(t_0, x_0) \) lies at distance \( R = O(1/|b(0, 0)|) \) instead of \( O(1) \) of \((t_0, x_0)\); this accounts for the factors \( R^{-j/2}, R^{-1}, R^{-1/2} \) appearing in the formulas. Finally, by standard Hölder interpolation inequalities [19],

\[ \sup_{Q^{(0)}(t_0, x_0)} |\nabla u| \leq e^{\| \nabla^2 u \|_{Q^{(0)}(t_0, x_0)}^{1/2 \alpha}} \sup_{Q^{(0)}(t_0, x_0)} |u|^{(1+\alpha)/(2+\alpha)} \leq e^{2+\alpha} \| \nabla^2 u \|_{Q^{(0)}(t_0, x_0)} + e^{-(2+\alpha)/(1+\alpha)} \sup_{Q^{(0)}(t_0, x_0)} |u| (3.51) \]

for every \( \epsilon > 0 \). Choosing \( e^{2+\alpha} \approx 1/\| b \| \) yields [3.47 3.48 3.49]. \( \Box \)

**Theorem 3.2 (Schauder estimates)** Let \( h_0 \) and \( g \) be as in Corollary [3.7] and \( h \) be a \( W^{1,\infty}_j(P, P) \)-solution of the infra-red cut-off KPZ equation (1.2). Fix \((t, x) \in [M^j, +\infty) \times \mathbb{R}^d \), let \( n = 0, 1, \ldots \) and assume furthermore if \( n \geq 2 \) that

\[ |V'(y)| \leq \max(y, y^2), \quad |V^{(k)}(y)| \leq \max(1, y^2) \quad (k = 2, \ldots, n-1), \quad y \geq 0 \quad (3.52) \]

and that

\[ N_{t,x}(\nabla^k g) := (M^{j/2})^k M^j \sup_{x \in [M^{j-1}, t]} \sup_{y \in B(x, M^{j/2})} \max \left( M^{j/2} |\nabla^{k+1} g(x)|, M^{j/2} |\partial_y \nabla^k g(y)| \right) < \infty \quad (3.53) \]

for all \( k = 0, \ldots, n-2 \). Then \( \nabla^k h, k = 0, \ldots, n \) is well-defined and Hölder continuous on \( Q^{(j)}(t, x) = [t - M^{j-1}, t] \times B(x, M^{j-1/2}) \), and the Hölder norms \( \| \nabla^k h \|_{Q^{(j)}(t,x)} \) may be given explicitly in terms of \( \| h_0 \|_{W^{1,\infty}_j(P, P)} \), \( \| g \|_{W^{1,\infty}_j(P, P)} \), \( N_{t,x}(\nabla^k g) \), \( k = 0, \ldots, n-2 \) and the constants implicit in (3.52).

Precise bounds for \( \| \nabla^k h \|_{Q^{(j)}(t,x)} \) are given at the end of the proof. Constants depend on \( n \) in a way which may be quantified, but depends in a non-trivial way on the constants in (3.52) and (3.53). In particular, each iteration of the Schauder estimates leads to shrink the domain of validity of the estimates, or to consider local suprema over domains growing linearly with \( n \). Which procedure is better depends a priori on \( g \).
Proof.

Recall $G(t, x) = \frac{e^{-|x|^2/2t}}{(2\pi t)^{d/2}}$ is the Green kernel of the Laplacian. Our theorem is true for $n = 0$ and $n = 1$, as follows from the much more precise estimates proved in the previous paragraph. Let us give a proof for $n \geq 2$. The condition $t \geq M^j$ will allow us to apply Proposition 3.3. We abbreviate $Q^{(j)}(t, x)$ to $Q^{(j)}$ and $N_{t,x}(\nabla^k g)$ to $N(\nabla^k g)$, and define

$$N_h := \sup_{Q^{(j)}} (M^{j/2}|\nabla h| + (M^{j/2}|\nabla h|)^2 + \ldots + (M^{j/2}|\nabla h|)^n)^{\ast} (t, x) + M^{j \ast} \sup_{Q^{(j)}} g^\ast,$$

where in general $\sup_{Q^{(j)}} u^\ast$ is defined as $\sup_{(t', x') \in Q^{(j)}} (u^\ast)(x')$, and

$$N^{(n)} := \max \left\{ N_{h, s}, \text{ max}_{k=0, \ldots, n} N(\nabla^k g) \right\}.$$

(i) (Hölder continuity in $x$ for the linear equation) Let $f$ be a function in $\mathcal{H}^0$. Using the elementary Taylor estimates $\nabla G(t, x) \leq t^{-(1+d/2)}|x| e^{-|x|^2/2t}$, $|\nabla G(t, x) - \nabla G(t, x + \delta x)| \leq t^{-(1+d/2)}|\delta x| e^{-|\delta x|^2/4t}$ ($|\delta x| \ll |x|$), one derives, for $\delta x \in \mathbb{R}^d$ such that $|\delta x| \leq \sqrt{t}$,

$$|\nabla e^{\delta x} f(0) - \nabla e^{\delta x} f(\delta x)| \leq t^{-(1+d/2)} \left( |\delta x| \int_{|x| \leq |\delta x|} |f(x)|dx + |\delta x| \int_{|x| \geq |\delta x|} e^{-|x|^2/4t}|f(x)|dx \right)$$

$$\leq f^\ast(0) t^{-(1+d/2)} \left( |\delta x| d + d^{1/2} |\delta x| \right)$$

$$\leq f^\ast(0) t^{-(1-e)/2} |\delta x|^{1-e}$$

for every $e \in [0, 1]$.

Similarly, under the same hypothesis $|\delta x| \leq \sqrt{t}$,

$$|e^{\delta x} f(0) - e^{\delta x} f(\delta x)| \leq f^\ast(0) t^{-(1-e)/2} |\delta x|^{1-e}.$$

(ii) (Hölder continuity in $t$ for the linear equation) Assume $0 \leq \delta t \ll t$. Using this time $|\nabla G(t, x) - \nabla G(t + \delta t, x)| \leq t^{-(3+d)/4} \delta t e^{-|\delta x|^2/4t}$, we get

$$|\nabla e^{\delta t} f(0) - \nabla e^{\delta t} f(\delta t)| \leq t^{-(1+d/2)} \sqrt{\delta t} \int_{|x| \leq \sqrt{\delta t}} |f(x)| dx + t^{-(3+d)/4} \delta t \int_{|x| \geq \sqrt{\delta t}} e^{-|x|^2/4t} |f(x)| dx$$

$$\leq f^\ast(0) \left( |\delta t|^{d+1/2} + t^{-3/2} \delta t \right)$$

$$\leq f^\ast(0) t^{-(1-e)/2} (|\delta t|^{1-e}/2)$$

for every $e \in [0, 1]$, and similarly

$$|e^{\delta t} f(0) - e^{\delta t} f(\delta t)| \leq f^\ast(0) t^{-(1-e)/2} (|\delta t|^{1-e}/2).$$

(iii) (Hölder continuity for the gradient of the solution) Let $t \gg M^j$ and $|\delta x| \leq M^{j/2}$, $\delta t \leq M^j$.

From the integral form of the infra-red cut-off KPZ equation,

$$g_t = e^{i(v\Delta - M^j)} g_t - M^j + \lambda \int_0^M e^{i(v\Delta - M^j)} (V(\nabla h_{t-s}) + g_{t-s})ds,$$

$$h_t = e^{i(v\Delta - M^j)} h_t - M^j + \lambda \int_0^M e^{i(v\Delta - M^j)} (V(\nabla h_{t-s}) + g_{t-s})ds,$$

$$h_t = e^{i(v\Delta - M^j)} h_t - M^j + \lambda \int_0^M e^{i(v\Delta - M^j)} (V(\nabla h_{t-s}) + g_{t-s})ds,$$
it follows for $\varepsilon \in (0, 1)$, using (i) and (ii),
\[
|\nabla h_t(x) - \nabla h_t(x + \delta x)| \leq (M^{-j/2})^{1-\varepsilon} |\delta x|^{1-\varepsilon} |\nabla h_{t-M^{\delta}}|^{\lambda}(x)
\]
\[+
\lambda \left( |\delta x|^{1-\varepsilon} \int_0^{M^\delta} ds \, s^{-(1-\varepsilon)/2} + \int_0^{M^\delta} ds \, s^{1/2} \right) \left( \left( |\nabla h_{t-M^{\delta}}|^{\lambda}\right)(x) \right) \right]^{\lambda}(x)
\]
\[\leq N_h M^{-j/2}(M^{-j/2})^{1-\varepsilon} |\delta x|^{1-\varepsilon}
\]
and similarly,
\[
|\nabla h_t(x) - \nabla h_{t-\delta t}(x)| \leq (M^{-j/2})^{1-\varepsilon} (\delta t)^{(1-\varepsilon)/2} |\nabla h_{t-M^{\delta}}|^{\lambda}(x)
\]
\[+
\lambda \left( (\delta t)^{(1-\varepsilon)/2} \int_0^{\delta t} ds \, s^{-(1-\varepsilon)/2} + \int_0^{\delta t} ds \, s^{1/2} \right) \left( \left( |\nabla h_{t-M^{\delta}}|^{\lambda}\right)(x) \right) \right]^{\lambda}(x)
\]
\[\leq N_h M^{-j/2}(M^{-j/2})^{1-\varepsilon} (\delta t)^{(1-\varepsilon)/2}.
\]

Using a Taylor expansion, \(V(\nabla h_t(x)) - V(\nabla h_t(x + \delta x)) = V'(1 - \tau)\nabla h_t(x) + \tau \nabla h_t(x + \delta x)) \cdot (\nabla h_t(x) - \nabla h_t(x + \delta x)) \quad (\tau \in [0, 1])\), one deduces:
\[
|V(\nabla h_t(x)) - V(\nabla h_t(x + \delta x))| \leq N_h^2 M^{-j}(M^{-j/2})^{1-\varepsilon} |\delta x|^{1-\varepsilon}
\]
and similarly
\[
|V(\nabla h_t(x)) - V(\nabla h_{t-\delta t}(x))| \leq N_h^2 M^{-j}(M^{-j/2})^{1-\varepsilon} (\delta t)^{(1-\varepsilon)/2}.
\]

We shall also need the following elementary Hölder bound on \(g\),
\[
|g_t(x) - g_t(x + \delta x)| \leq N(g) M^{-j}(M^{-j/2})^{1-\varepsilon} |\delta x|^{1-\varepsilon}, \quad |g_t(x) - g_{t-\delta t}(x)| \leq N(g) M^{-j}(M^{-j/2})^{1-\varepsilon} (\delta t)^{(1-\varepsilon)/2}.
\]

(iv) (Schauder estimates, \(n = 2\)) Applying Proposition 3.3 with \(b := 0\) and right-hand side \(f := \lambda V(\nabla h) + g\), and using eqs. (3.61, 3.62, 3.63) we obtain
\[
|\nabla^2 h(t, x)| \leq \lambda (M^{j/2})^{1-\varepsilon} \sup_{Q_j} |h| + M^{-j} \sup_{Q_j} |h|
\]
\[\leq M^{-j} \left( N_h^2 + N(g) \sup_{Q_j} |h| \right)
\]
\[\leq M^{-j} \max(N^{(0)}, (N^{(0)})^2)
\]
and similarly
\[
\|\nabla^2 h\|_{L^1(Q_j)} \leq (M^{-j/2})^{1-\varepsilon} M^{-j} \max(N^{(0)}, (N^{(0)})^2).
\]

(v) (Schauder estimates, \(n \geq 3\))

We proceed by induction on \(n\) and differentiate the KPZ equation \(n\) times,
\[
(\partial_t - \nu \Delta) \nabla h = \lambda V'(\nabla h) \cdot \nabla^2 h + \nabla g
\]
\[\quad (\partial_t - \nu \Delta) \nabla^2 h = \lambda \left( V''(\nabla h) \cdot (\nabla^2 h) + V'(\nabla h) \cdot \nabla^3 h \right) + \nabla^2 g \ldots
\]
and derive in a straightforward way Hölder estimates on the non-linear term using Hölder estimates for $\nabla h$ and Schauder estimates found in previous stages for lower-order gradients of the solution. Combinatorics due to iterated differentiation of $V(\nabla h)$ are represented by planar graphs, producing in the bounds a prefactor bounded by the Catalan number, $\frac{2n}{n+1} \leq 4^n$.

On the other hand, the domain of validity of the bounds, $Q^{(0)}(t,x) \supset Q_1 \supset \ldots Q_{n+1}$ restricts at each step, yielding potentially factorials to some power for $n$ arbitrarily large. We shall not enter this discussion and simply let the constants $C_n, C'_n, \ldots$ in the following lines depend on $n$. Assume by induction that

$$(M^{-j/2})^{1-\epsilon}(M^{-i/2}n)\|\nabla h\|_{1-\epsilon, Q_{n+1}}, (M^{-j/2})^n \sup_{Q_n} |\nabla^m h| \leq C_n \max \left( N(n-1), (N(n-1))^{2(n-1)} \right)$$

(3.70)

for $m = 2, \ldots, n + 1$. The PDE satisfied by $\nabla^n h$ involves in the right-hand side products of terms of the form $A(k) := V^{(k)}(\nabla h)(\nabla^2 h)(\nabla^3 h) \ldots$, with $k_1 \geq 1$ and $k_2 + 2k_3 + \ldots = n$. Similarly to (3.61), (3.62), we obtain

$$||V^{(k)}(\nabla h)||_{1-\epsilon, Q_n} \leq (1 + N_h)N_h M^{-j/2}(M^{-i/2})^{1-\epsilon}$$

(3.71)

while simply sup_{Q_n} |V'(\nabla h)| \leq N_h M^{-j/2}, sup_{Q_n} |V^{(k)}(\nabla h)| \leq 1 + N_h (k \geq 2). Using Proposition 3.3 together with our induction hypothesis, one obtains (see (3.66), (3.67))

$$(M^{-j/2})^{-1-\epsilon}(M^{-i/2})^{n+2}\|\nabla^{n+2} h\|_{1-\epsilon, Q_{n+2}}, (M^{-j/2})^{n+2} \sup_{Q_{n+2}} |\nabla^{n+2} h| \leq C_n \max \left( N(n), (N(n))^{2(n+2)} \right).$$

(3.72)

\hfill \square

### 3.3 Non-explosion for the optimal path

The results of the subsection 3.1 do not entail the existence of an optimal path $X^{a^x}$. As shown in Lemma 2.4, the SDE

$$dX^a = V'(\nabla h_{t-s}(X^a_s)) + \sqrt{v} dB_s$$

(3.73)

defines in principle such an optimal path. However, the driving vector field $V'(\nabla h_{t-s}(\cdot)) \cdot \nabla$ is not uniformly Lipschitz in general, hence the SDE is a priori defined only till some random explosion time. On the other hand, note that provided $h_0 \in W^{1,\infty}$ and $g \in L^\infty([0, T], W^{1,\infty})$, the vector field $y \mapsto V'(\nabla h_{t-s}(y))$ is indeed uniformly Lipschitz, uniformly in time, as follows from the Schauder estimates proved in the previous paragraph, so the solution $X^{a^x}$ of (3.73) is well-defined.

We show in this subsection that optimal paths a.s. do not explode under the assumptions of Theorem 3.1. The general idea is the following: the solution $h$ of the KPZ equation in $\mathbb{R}^d$, resp. in a ball $\Omega \subset \mathbb{R}^d$, is represented as the sum of three, resp. four terms, viz.

$$h(t, x) = \sup_a \mathbb{E}_{x}^{0, t} \left[ \int_0^t e^{-M^{-j/2}} \left( -\Delta \tilde{V} \left( \frac{\partial_s}{\sigma_s} \right) + g(t-s, X_s) \right) ds + e^{-M^{-j/1}} h_0(X_t) \right],$$

(3.74)

or

$$h(t, x) = \sup_a \mathbb{E}_{x}^{0, t} \left[ \int_0^{t_1} e^{-M^{-j/2}} \left( -\Delta \tilde{V} \left( \frac{\partial_s}{\sigma_s} \right) + g(t-s, X_s) \right) ds + 1_{t_1 \leq t} e^{-M^{-j/1}} h_0(X_t) \
+ 1_{t_1 < t} e^{-M^{-j/1}} h_{t-t_1}(X_{t_1}) \right].$$

(3.75)
where \( \theta_\Omega := \inf \{ s \in [0, t] \mid X_s \notin \Omega \} \) (\( \pi \) if \( X_s \in \Omega \) for all \( s \in [0, t] \)). We must now prove bounds for each of these terms. It is convenient in the following lemmas to denote by (i) \( h_{h_0, g} \), resp. (ii) \( h_{h_0, g,v} \) the solution on \( \mathbb{R}^d \), resp. \( \Omega \) of the infra-red cut-off KPZ equation with (i,ii) initial condition \( h_0 \), force term \( g \) and (ii) boundary condition \( h|_{\partial \Omega} = v \). In particular, \( h_{h_0, g} \) and \( h_{h_0, g,v} \) trivially coincide on \( \tilde{\Omega} \) and have the same random path representation using \( X^{a^*} \).

**Lemma 3.4** Assume \( 2h_0 \in \mathcal{W}^{d,\infty, p_x} \), \( 2g \in \mathcal{W}^{d,\infty, p_x}([0, T]) \) and let \( X = X^{a^*} \) be the solution of the SDE (3.73), where \( h = h_{h_0, g} \). Let also \( f \) be a function such that \( 2f \in \mathcal{W}^{d,\infty, p_x}([0, t]) \).

(i) (bounds on \( \mathbb{R}^d \))

\[
\mathbb{E}^{0,x} \left[ \int_0^t e^{-M^{-j}s} f_{t-s}(X_s) ds \right] \leq e^{-M^{-j}t} \int_0^t e^{-M^{-j}s} f_{t-s}(X_s) ds \leq e^{-M^{-j}t} P_{\alpha}^{-1} \circ P_{\alpha} \left( \|h_0\|_{\mathcal{W}^{d,\infty, p_x}}(x) \right) + P_{\alpha}^{-1} \circ P_{\alpha} \left( \|g + f\|_{\mathcal{W}^{d,\infty, p_x}}([0, t]) (x) \right) + \|x\|_{\mathcal{W}^{d,\infty, p_x}}([0, t]) (x) \right) \right).
\]

(3.76)

in particular, letting \( f = \pm|g| \), one has an upper bound for \( \mathbb{E}^{0,x} \left[ \int_0^t e^{-M^{-j}s} f_{t-s}(X_s) ds \right] \):

\[
\mathbb{E}^{0,x} \left[ e^{-M^{-j}t} |h_0(X_t)| \right] \leq e^{-M^{-j}t} P_{\alpha}^{-1} \circ P_{\alpha} \left( \|2h_0\|_{\mathcal{W}^{d,\infty, p_x}}(x) \right) + P_{\alpha}^{-1} \circ P_{\alpha} \left( \|2g\|_{\mathcal{W}^{d,\infty, p_x}}([0, t]) (x) \right) \right).
\]

(3.77)

(ii) (bounds on a ball \( \Omega \)) Let \( v \in \mathcal{W}^{d,\infty, p_x}([0, T] \times \partial \Omega) \). Then

\[
0 \leq \mathbb{E}^{0,x} \left[ \int_0^\tau e^{-M^{-j}s} \lambda \tilde{V} \left( \frac{\tilde{\alpha}}{\lambda} \right) ds \right] \leq e^{-M^{-j}t} P_{\alpha}^{-1} \circ P_{\alpha} \left( \|2h_0\|_{\mathcal{W}^{d,\infty, p_x}}(x) \right) + P_{\alpha}^{-1} \circ P_{\alpha} \left( \|2g\|_{\mathcal{W}^{d,\infty, p_x}}([0, t]) (x) \right) \right).
\]

(3.79)

\[
0 \leq \mathbb{E}^{0,x} \left[ \int_0^\tau e^{-M^{-j}s} f_{t-s}(X_s) ds \right] \leq e^{-M^{-j}t} P_{\alpha}^{-1} \circ P_{\alpha} \left( \|2h_0\|_{\mathcal{W}^{d,\infty, p_x}}(x) \right) + P_{\alpha}^{-1} \circ P_{\alpha} \left( \|2g\|_{\mathcal{W}^{d,\infty, p_x}}([0, t]) (x) \right) \right).
\]

(3.80)

\[
0 \leq \mathbb{E}^{0,x} \left[ \int_0^\tau e^{-M^{-j}s} f_{t-s}(X_s) ds \right] \leq e^{-M^{-j}t} P_{\alpha}^{-1} \circ P_{\alpha} \left( \|2h_0\|_{\mathcal{W}^{d,\infty, p_x}}(x) \right) + P_{\alpha}^{-1} \circ P_{\alpha} \left( \|2g\|_{\mathcal{W}^{d,\infty, p_x}}([0, t]) (x) \right) \right).
\]

(3.81)

\[
0 \leq \mathbb{E}^{0,x} \left[ \int_0^\tau e^{-M^{-j}s} f_{t-s}(X_s) ds \right] \leq e^{-M^{-j}t} P_{\alpha}^{-1} \circ P_{\alpha} \left( \|2h_0\|_{\mathcal{W}^{d,\infty, p_x}}(x) \right) + P_{\alpha}^{-1} \circ P_{\alpha} \left( \|2g\|_{\mathcal{W}^{d,\infty, p_x}}([0, t]) (x) \right) \right).
\]

(3.82)
Remark. As follows from the proof and a simple application of Jensen’s inequality, bounds (3.76, 3.77, 3.78, 3.79, 3.80) actually hold for the $W_d^{j,\infty}P\_-$-pointwise quasi-norm of their left-hand sides, where $P\_-(z) := \sqrt{P\_-(z)} = e^{1/(p-1)}z$ in the exponential case $P\_-(z) = e^{1/(p-1)}z$, and $P\_-(z) = P\_-(z)$ in the polynomial case.

Proof.

(i) By definition,

$$
\mathbb{E}^{0,x} \left[ \int_0^\infty e^{-M^+s} f_{\tilde{\nu}}(X_s) ds \right] = \mathbb{E}^{0,x} \left[ \int_0^\infty e^{-M^+s} \left( -\tilde{\lambda} \tilde{V} \left( \frac{\alpha}{\lambda} s \right) + (f + g)(t - s, X_s) \right) ds 
+ e^{-M^+s} h_0(X_t) \right] - h(t, x) \leq h_{\text{lo},f+g}(t, x) - h(t, x)
$$

which gives the first bound. Varying the initial condition instead, one gets for a general $h_0 \in W_d^{j,\infty}P\_-$

$$
\mathbb{E}^{0,x} (h_0 - h_0)(X_t) = \mathbb{E}^{0,x} \left[ \int_0^\infty e^{-M^+s} \left( -\tilde{\lambda} \tilde{V} \left( \frac{\alpha}{\lambda} s \right) + g(t - s, X_s) \right) ds + e^{-M^+s} h_0(X_t) \right] - h(t, x)
\leq h_{\text{lo},g}(t, x) - h(t, x).
$$

Letting $\tilde{h}_0 = h_0 + |h_0|$ gives an estimate for $\mathbb{E}^{0,x} |h_0(X_t)|$. Thus one also has a bound for

$$
\mathbb{E}^{0,x} \left[ \int_0^\infty e^{-M^+s} \tilde{\lambda} \tilde{V} \left( \frac{\alpha}{\lambda} s \right) ds \right] = -h(t, x) + \mathbb{E}^{0,x} \left[ \int_0^\infty e^{-M^+s} g_{\text{lo}}(X_s) ds \right] + h_{\text{lo},g}(X_t).
$$

(ii) Eq. (3.79), resp. (3.80) is a trivial consequence of (3.78), resp. (3.76). As for (3.81), if follows simply from the equation (3.75) and the previous bounds (3.77, 3.79, 3.80). Finally,

$$
\mathbb{E}^{0,x} \left[ (v - h)(t - \theta_{\Omega_0}, X_{\theta_{\Omega_0}}) \right] = \mathbb{E}^{0,x} \left[ \int_0^{\theta_{\Omega_0}} e^{-M^+s} \left( -\tilde{\lambda} \tilde{V} \left( \frac{\alpha}{\lambda} s \right) + g(t - s, X_s) \right) ds 
+ 1_{\theta_{\Omega_0} \geq t} e^{-M^+s} h_0(X_t) \right] - h(t, x) \leq h_{\text{lo},g,v}(t, x) - h(t, x)
$$

whence (3.82).

□

Corollary 3.5 Optimal paths a.s. do not explode.

Proof. Let $\Omega_n = B(x, r_n), r_n \to_{n \to \infty} \infty$, and choose $v_n : \partial \Omega_n \to \mathbb{R}$ to be the constant function equal to $v_n := \varepsilon \min \left( \frac{r_n^2}{t}, (r_n^2/t)^{1/(p-1)} \right)$ for $\varepsilon > 0$ small enough in the exponential case, and $v_n := \min \left( \frac{r_n^2}{t}, (r_n^2/t)^{1/(p-1)} \right)^{d''/(d' + d'')} \infty$ in the polynomial case (see Theorem 3.1 (ii)). Then, as follows from (3.45), $\|v_n\|_{W_d^{j,\infty}} \leq 1$. On the other hand, $\mathbb{E}^{0,x} \left[ 1_{\theta_{\Omega_0} < t} v_n(t - \theta_{\Omega_0}, X_{\theta_{\Omega_0}}) \right] = v_n \mathbb{P} [\theta_{\Omega_0} < t]$ and $\mathbb{E}^{0,x} \left[ 1_{\theta_{\Omega_0} < t} h(t - \theta_{\Omega_0}, X_{\theta_{\Omega_0}}) \right] < \infty$ by (3.81). Hence (3.82) implies that $\mathbb{P} [\theta_{\Omega_0} < t] = \mathbb{P} [\sup_{s \in [0,t]} X_s > r_n] = O(1/v_n) \to_{n \to \infty} 0$. □
3.4 Differentiability with respect to the forcing term

As a key argument for the proof of the diffusive limit of KPZ$_3$ equation (see next article), we need to understand the dependence of the solution on the forcing term. Here we prove differentiability to any arbitrary order with precise bounds; because of Lemma 4.2, we shall require Schauder estimates up to order $n = d$ for the linearized equation. We fix a pair of convex functions $(P_+, P_-)$ such that the solution $h = h_{h_0, g}$ of the KPZ equation with initial condition $h_0$ and right-hand side $g$ in $W$-spaces with parameters $P_+$ belongs to the corresponding $V$-spaces with parameter $P_-; for instance, those appearing in Theorem 3.1.

Definition 3.6 Let $(g_\tau)_{\tau \in [0, 1]}$ be a family of functions in $W^d,\infty;P_+([0, T]), T \in (0, +\infty]$, such that:

(i) $\sup_{\tau} \|g_\tau\|_{W^d,\infty;P_+([0,T])}(x) < \infty$ for every $x \in \mathbb{R}^d$;

(ii) $\tau \mapsto g_\tau(t, x)$ is differentiable for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and its derivative, $g'_\tau := \partial_t g_\tau$, satisfies $\sup_{\tau} \|g'_\tau\|_{W^d,\infty;P_+([0,T])}(x) < \infty$ for every $x$.

We denote by $h(\tau)$ the solution in $W^d,\infty;P_- \ -\text{of the infra-red cut-off KPZ equation (1.2)}$ with fixed initial condition $h_0 := h_0(t, x)$ and varying forcing term $g = g(\tau)$.

By Theorem (3.1), $(\tau, t) \mapsto \|h_\tau(t)\|_{W^d;\infty;P_-}$ is uniformly bounded in $\tau \in [0, 1]$ and $t \in [0, T]$ for $x$ fixed. The Schauder estimates proved above show more generally that $(\tau, t) \mapsto \|h_\tau(t)\|_{W^d,\infty;P_-}(x)$ is uniformly bounded provided $\sup_{\tau} \|g_\tau\|_{W^d;\infty;P_-([0,T])}(x) < \infty$.

Definition 3.7 ($\mathcal{W}$-continuity) Let $\mathcal{W} = W^{d + d', \infty;P_+([0, T])}$ for some parameters $d', P_+$. The family $(g_\tau)_{\tau \in [0, 1]}$ is “$\mathcal{W}$-continuous at $\tau$ if there exists a neighbourhood $\Omega$ of $\tau$ and a “dominating” positive function $\tilde{g} = \tilde{g}(t, x)$ in $\mathcal{W}$ such that $\sup_{\tau (t, x) \in \mathbb{R}^d} \|\frac{\tilde{g}_\tau - \tilde{g}_{\tau'}}{\tau' - \tau}\| \leq 1, \tau' \in \Omega$, for some positive function $c$ such that $c(\tau') \to \tau' \to 0$. It is uniformly $\mathcal{W}$-continuous if one may choose a dominating function $h$ and a positive function $c = c(\tau, \tau')$ such that $\sup_{\tau (t, x) \in \mathbb{R}^d} \|\tilde{g}_\tau - \tilde{g}_{\tau'}\| \leq 1$ for all $\tau, \tau'$ and $c(\tau, \tau') \to |\tau' - \tau| \to 0$.

Remark. If $P_+$ are polynomials, then the above conditions are equivalent to requiring that $|\frac{\tilde{g}_\tau - \tilde{g}_{\tau'}}{\tilde{g}}|$ is bounded, and that its supremum goes to 0 when $\tau' \to \tau$. However, the latter conditions are weaker than those of the definition in the exponential case, and insufficient to obtain the following lemma.

Lemma 3.8 (continuity) Let $\mathcal{W} = W^{d;P_+}$ and $\mathcal{W} = W^{d;P_+}$, $W([0, T]) = W^{d;P_+}([0, T])$. If $h_0 \in \mathcal{W}$ and $(g_\tau)_{\tau \in [0, 1]} is uniformly $\mathcal{W}([0, T])$-continuous, then $\|h_\tau(t) - h_\tau(t)||_{\mathcal{W}^-} \to 0$ locally uniformly in $t$, where $\mathcal{W}^-$ is a $\mathcal{W}$-functional space with parameter $P_-$ defined $P_-(z) = e^{\lambda_{\mathcal{W}^-} \rho - \rho^2}$ with $\lambda < 1$ if $P_-(z) = e^{\lambda_{\mathcal{W}^-} \rho - \rho^2}$ (exponential case), and $\mathcal{W}^-$ is $\mathcal{W}$ in the polynomial case.

Proof. Let $\alpha = \alpha'(\tau)$ be an optimal strategy for $h_\tau$, and $X = X^\rho$ the corresponding random path. Then, for all $\varepsilon \in (0, 1)$,

$$h_\tau(t, x) - (1 - \varepsilon) h_\tau(t, x) \leq \varepsilon \mathbb{E} \left[ \int_0^\tau e^{-\lambda s} \left( -\lambda \tilde{V}(g_\tau - g_\tau(t - s, X_s)) + \frac{1}{\varepsilon}(g_\tau(t - s, X_s) - g_\tau(t - s, X_s)) + h_\tau(X_s) \right) ds \right].$$

(3.86)
Thus
\[
\|h_\tau(t) - (1 - \varepsilon)h_{\tau'}(t)\|_{W^\infty} \leq \varepsilon \left( P^{-1}_\tau \circ P_+ \left( \|\frac{g_\tau - g_{\tau'}}{\varepsilon}\|_{W^1([0,T])} \right) + \sup_\tau P^{-1}_\tau \circ P_+ \left( \|g_\tau\|_{W^1([0,T])} \right) \right) + e^{-M^{-1}} \sup_\tau P^{-1}_\tau \circ \left( \|h_\tau\|_{W^1} \right).
\] (3.87)

Letting $\varepsilon = c(\tau, \tau')$, one gets (using Jensen’s formula and only for $\tau'$ sufficient close to $\tau$ in the exponential case) $\|h_\tau(t) - h_{\tau'}(t)\|_{W^\infty} = O(c(\tau, \tau')) \to \tau' \to \tau 0$. Exchanging the rôles of $\tau, \tau'$ and using continuity at $\tau'$ yields an analogous lower bound.

In the sequel we assume $\text{loc sup} \ g < \infty$ and use the simplified single-index ‘$W$’-functional space $W_j^{d; 0; 0; p^N, p}(0, T)$ defined in subsection 1.2, Remark 3. We therefore consider the obvious modification of Definition 3.6 obtained by replacing ‘$W_j^{d; 0; 0; p}(0, T)$’ by ‘$W_j^{d; 0; 0; 0; p}(0, T)$’.

The main result we want to prove is the following. We write $g^{(q)}(\tau) = \partial_\tau^q g, \partial_\tau h$.

**Theorem 3.3 (dependence on the forcing term)** Let $N \geq 0$ and $P_-(z) = z^d$, $P_+(z) = z^d$, such that $d_+ - 1$ is small enough (depending on $N$). Assume
\[
h_0 \in W_j^{d + N, 0; 0; p^N, 0}, \sup_{q = 0, \ldots, N} \sup_{\tau} \|s^{(q)}\|_{W_j^{d + N, 0; 0; p^N, 0}(0, T)}(x) < \infty \quad (x \in \mathbb{R}^d).
\] (3.88)

Assume conditions (3.52) are satisfied. Then, for all $q \leq N$ and $p = 0, \ldots, d$, $\nabla^p \partial_\tau^q h(\tau; t)$ is well-defined and Hölder continuous on $Q_j^{(j - 1)}(t, x) = [t - M^{j - 1}, t] \times B(x, M^{(j - 1)/2})$. Furthermore, one has explicit bounds on the Hölder norms $\|\nabla^p \partial_\tau^q h(\tau; t)\|_{W_j^{d + N, 0; 0; p^N, 0}}(t, x)$ and on the pointwise quasi-norms $\|\partial_\tau^q h(\tau; t)\|_{W_j^{d + N, 0; 0; p^N, 0}}(t, x)$ in terms of $\|h_0\|_{W_j^{d + N, 0; 0; p^N, 0}(0, T)}(x)$ and the constants implicit in (3.52).

As in Theorem 3.2, the bounds involve an $N$-dependent coefficient which may grow rapidly with $N$ due to the fact that each application of the Schauder estimates makes the domain of validity of the estimates shrink. Note that another possibility would be to consider space-time local suprema over domains with a size which grows linearly with $N$. Another simpler problem is the growth of the coefficients $|V^{(k)}(y)|$ with the number of derivatives, and, more interestingly, of the pointwise quasi-norms $\|s^{(q)}\|_{W_j^{d + N, 0; 0; p^N, 0}(0, T)}(x)$ with $N$. Our proof requires $d_+ \geq CN$ for a large enough universal constant $C = C(d, N).

We start with a lemma before we prove the theorem.

**Lemma 3.9**

1. Assume $(g_\tau)_{\tau \in [0, 1]}$ satisfies the hypothesis of Definition 3.6. Then the function $\tau \mapsto h(\tau; t, x)$ is differentiable for every $(t, x) \in [0, T] \times \mathbb{R}^d$, and its derivative $h'\tau) := \frac{\partial \tau}{\partial \tau}$ is the unique solution with polynomial growth at infinity of the linearized PDE
\[
\partial_\tau h' = \nu \Delta h' + \lambda V'(\nabla h) \cdot \nabla h' + g'
\] (3.89)

Furthermore, $h'$ admits the following random path representation,
\[
h'(\tau; t, x) := \mathbb{E}^{0^d} \left[ \int_0^\tau e^{-M^{-1} s} g'(\tau; t - s, X_s) ds \right],
\] (3.90)
where $X_{\alpha}^\tau$ is the $\tau$-dependent optimal path defined in Proposition 2.1 and there exists a universal constant $c = c(\nu, d) > 0$ such that

$$||h_j||_{\mathcal{W}^{j,\infty}_{\nu, P}(x)} \leq e^{-cM^{-j}}P_+ \left(||h_0||_{\mathcal{W}^{j,\infty}_{\nu, P}(x)} + P_-^1 \left(||g||_{\mathcal{W}^{j,1,\infty}_{\nu, P}(\{0, t\})} + ||g'||_{\mathcal{W}^{j,1,\infty}_{\nu, P}(\{0, t\})} \right) \right).$$

(3.91)

2. (Schauder estimates) (see Theorem 3.2 for notations). Assume conditions (3.52) are satisfied, and $N_{t,x}(\nabla^k g') < \infty$ for all $k = 0, \ldots, n - 2$. Then $\nabla^k h$, $h' = 0, \ldots, n$ are well-defined and Hölder continuous on $Q^{(j-1)}(t, x)$, and the Hölder norms may be given explicitly in terms of $||h_0||_{\mathcal{W}^{j,1,\infty}_{\nu, P}(x)}$, $||g||_{\mathcal{W}^{j,1,\infty}_{\nu, P}(\{0, t\})}$, $N_{t,x}(\nabla^k g')$, $k = 0, \ldots, n - 2$ and the constants implicit in (3.52).

Proof.

1. Note that the solution of (3.89), if it exists, is clearly unique (see e.g. [11], chap. 1, section 9). Hence we must prove that it exists and that it is given by (3.90). By Lemma 3.4 the right-hand side of (3.90) belongs for each $t, \tau$ to $\mathcal{W}^{j,\infty}_{\nu, P}$ if $g' \in \mathcal{W}^{j,1,\infty}_{\nu, P}(\{0, t\})$. Let $\Omega_n := B(x, nM^{1/2})$. Thus $h(n, \tau, t, x) \in \mathcal{W}^{0,\infty}_{0, \nu}(0, t)$, where $\theta_n := \inf\{s \in [0, t]; X_{\alpha}^\tau \not\in \Omega_n\}$ (with the usual convention $\theta_n := t$ if this set is empty) is uniformly bounded in $n$, and $h'(n, \tau, t, x) \rightarrow_{n \rightarrow \infty} h' \in \mathcal{W}^{0,\infty}_{0, \nu}(0, t)$ for each fixed $(t, x)$. On the other hand, as shown using the Hamilton-Jacobi-Bellman principle, $h'(n, \tau, t, x)$ solves equation (3.89) with Dirichlet boundary condition on $\partial \Omega_n$. By the Schauder estimates, $n \mapsto h'(n, \tau)$ is locally bounded in $C^{2+\alpha}$ norm for every $\alpha < 1$, hence there exists a subsequence $h'(\phi(n), \tau)$ which converges to a function satisfying (3.89). Thus $h'$ satisfies (3.89).

2. The proof is modeled on that of Theorem 3.2 from which we borrow the notations. We apply Proposition 3.3 with $b := V'(\nabla h)$ this time. From the proof of Theorem 3.2 we know that, for every $\alpha < 1$,

$$||b||_{\alpha, Q^{(j)}} \leq (1 + N_h)N_h M^{-j/2}(M^{-j/2})^{\alpha/2}, \quad \sup_{Q^{(j)}} |b| \leq N_h M^{-j/2}. \quad (3.92)$$

We introduce new 'norms' more adapted to the present problem,

$$\bar{N}(f) := \max \left(\bar{N}_0(f), (\bar{N}_0(f))^{d_x/d_{\alpha}}, (\bar{N}_0(f))^{d_y/d_{\alpha}} \right) \quad (3.93)$$

with $\bar{N}_0(f) := e^{-cM^{-\alpha}||h_0||_{\mathcal{W}^{j,1,\infty}_{\nu, P}(x)} + ||g||_{\mathcal{W}^{j,1,\infty}_{\nu, P}(\{0, t\})} + ||f||_{\mathcal{W}^{j,1,\infty}_{\nu, P}(\{0, t\})}$, and $\bar{N}^{(j)} := \max_{k=0,\ldots,j}(\bar{N}(\nabla^k g'))$. Note that $\bar{N}(0) \geq \max(\bar{N}_h, \bar{N}_h^d)$ provided $d \geq \beta$, and that $\bar{N}(g') \leq \bar{N}(g')$ trivially. Thus, by the Schauder estimates,

$$||\nabla^2 h'||_{\alpha, Q^{(j)}} \leq ||g||_{\alpha} + (M^{-j})^{\alpha/2}M^{-j} \left(1 + N_h^d N_h^2 + (1 + N_h)^{1+\alpha/2} \sup_{Q^{(j)}} |h'| \right) \leq (M^{-j})^{\alpha/2}M^{-j} \left(N(g') + (1 + \bar{N}(0))\bar{N}(g') \right) \leq (M^{-j/2})^\alpha M^{-j} \max(\bar{N}(0), (\bar{N}^{(j)})^2). \quad (3.94)$$

The same bound holds for $\sup_{Q^{(j)}} |\nabla^2 h'|$, resp. $\sup_{Q^{(j)}} |\nabla h'|$, resp. $||\nabla h'||_{\alpha, Q^{(j)}}$ without a prefactor $M^{-j}$, resp. $M^{-j/2}$, resp. $M^{-j/2}$ instead of $(M^{-j/2})^\alpha M^{-j}$. 33
Consider now the equations obtained by differentiating,

\[
(\partial_t - \nu \Delta - \lambda V'(\nabla h)) \nabla h' = \lambda V''(\nabla h) \cdot (\nabla^2 h \otimes \nabla h') + \nabla g'
\]  
(3.95)

\[
(\partial_t - \nu \Delta - \lambda V'(\nabla h)) \nabla^2 h' = \lambda \left\{ V''(\nabla h) \cdot (2 \nabla^2 h \otimes \nabla^2 h' + \nabla^3 h \otimes \nabla h') + V'''(\nabla h)(\nabla^2 h)^2 \nabla h' \right\} + \nabla^2 g'
\]  
(3.96)

and so on. Assume by induction that

\[
(M^{j/2})^a (M^{j/2})^a \sup_{Q_n} |\nabla^m h'| \leq C_n \max \left\{ \tilde{N}^{(n-1)}(0), (\tilde{N}^{(n-1)})^{2(m-1)} \right\} \]  
(3.97)

for \( m = 2, \ldots, n + 1 \). Note that the same bound holds for \( \|\nabla^m h\|_{\alpha, Q_n} \) and \( \sup_{Q_n} |\nabla^m h'| \), as we already know from (3.70). The PDE satisfied by \( \nabla^m h' \) has an \( n \)-independent drift term \( b \cdot \nabla = \lambda V'(\nabla h) \cdot \nabla \), and a right-hand side which is a sum of terms of the form \( A_j(k) := V(k_2)(\nabla h) \nabla^l (\nabla^2 h)^k (\nabla^3 h)^k \cdots \), with \( k_2 \geq 2, 1 \leq l \leq n \) and \( (l-1) + 2k_2 + 2k_3 + \cdots = n \), plus \( \nabla^g g' \). Collecting all factors, one gets by the induction hypothesis

\[
\|A_j(k)\|_{\alpha, Q_n} \leq (M^{-j/2})^a M^{-j} M^{-n/2} \max \left\{ \tilde{N}^{(n-1)}(0), (\tilde{N}^{(n-1)})^{2(n+2)} \right\}.
\]  
(3.98)

Thus

\[
\sup_{Q_{n+1}} |\nabla^{n+2} h'| \leq M^{-j} M^{-n/2} \max \left\{ \tilde{N}^{(n)}(0), (\tilde{N}^{(n)})^{2(n+2)} \right\}
\]  
(3.99)

and similarly for \( \|\nabla^{n+2} h'\|_{\alpha, Q_{n+1}} \) with a supplementary \( M^{-j\nu/2} \) prefactor.

\[\Box\]

**Proof of Theorem 3.3**

We combine the proofs of Theorem 3.2 and Lemma 3.9, to which the reader is referred for notations and further details. We write for brevity’s sake \( \|\|f\||_{\nu} \) instead of \( \|\|f\||_{W_j^{p,\infty,p,\nu}(0,T)} \) where \( P_{p,\nu}(z) = e^{z} \nu \), and define new norms

\[
\mathcal{N}_\alpha(g) := \max_{k=0, \ldots, d} \max_{l=0, \ldots, n} \left( \|\!(\!(M^{j/2})^k M^{j/2} \nabla^l g(0))\!\!\|_{d}, \|\!(\!(M^{j/2})^k M^{j/2} \nabla^l g(0))\!\!\|_{d} \right)^{8\delta d, (2n-1)}
\]  
(3.100)

To take into account the initial condition, one must simply replace \( \|\!(\!(M^{j/2})^k M^{j/2} \nabla^l g(0))\!\!\|_{d} \) with \( \|\!(\!(M^{j/2})^k M^{j/2} \nabla^l g(0))\!\!\|_{d} + e^{-\epsilon M^{-j}} \|\!(\!(M^{j/2})^k M^{j/2} \nabla^l g(0))\!\!\|_{W^{n,\infty,p,\nu}} \) in the above expression, with \( \epsilon \) small enough.

Our induction hypothesis is the following: for \( q = 1, \ldots, n - 1 \),

\[
\sup_{\tilde{Q}_{n+1}} (M^{j/2})^a \|\!(\!(M^{j/2})^k M^{j/2} \nabla^l g(0))\!\!\|_{d}, (M^{j/2})^a (M^{j/2})^a \|\!(\!(M^{j/2})^k M^{j/2} \nabla^l g(0))\!\!\|_{d} \leq C_n \mathcal{N}_\alpha(g), \quad p = 0, \ldots, d.
\]  
(3.101)

Consider the PDE satisfied by \( \nabla^m h(n) (n \geq 2) \); it is of the form \( \dot{h} - \nu \Delta - b(t,x) \cdot \nabla h = F \), where \( b = V'(\nabla h) \) is independent of \( m, n \), and the right-hand side \( F \) is a sum of terms of the form

\[
A_{m,n}(k) = V^{(k_1)}(\nabla h) \prod_{p,q} (\nabla^p h(q))^{\delta_{p,q}}, \quad 1 \leq p \leq d, 1 \leq q \leq n,
\]  
(3.102)
with $k_1 \geq 2$, $\sum_{p,q} (p-1)k_{p,q} = m$, $\sum_{p,q} qk_{p,q} = n$, $\sum_{p} k_{p,n} = 1$. From the induction hypothesis one gets

$$
\|A_{m,n}(k)\|_{Q_{n-1}} \leq M^{-j}(M^{-1/2})^m \prod_{p,q} \max \left( ||| (M^{1/2})^p M^j \nabla^p g^{(l)} |||_{d_r}, ||| (M^{1/2})^p M^j \nabla^p g^{(l)} |||_{d_{(2q-1)}} \right).
$$

(3.103)

The sum of the exponents in the previous expression is \( \leq \bar{k} = 8\beta d_+ (2n - 1) - 8\beta d_+ \). We now state the following immediate consequence of the generalized Hölder inequality, where it is assumed that \( k_1 + \ldots + k_r \leq \bar{k} \) and \( kd_r \geq \max(d_1, \ldots, d_r) \),

$$
\left\| \left( \left\| f_1 \right\|_{d_1} \cdots \left\| f_r \right\|_{d_r} \right) \right\|_{kd_r} \leq \left\| \left( \left\| f_1 \right\|_{d_1} \right) \left\| f_{kd_1} \right\|_{kd_1} \cdots \left( \left\| f_r \right\|_{d_r} \right) \left\| f_{kd_r} \right\|_{kd_r} \right\|_{kd_r} \leq \left\| \left\| f_1 \right\|_{kd_1} \cdots \left\| f_{kd_r} \right\|_{kd_r} \right\|_{kd_r},
$$

(3.104)

from which we are able to bound the \( \| \cdot \|_{d_r} \)-pointwise quasi-norm of the right-hand side of (3.103).

Here we can choose \( \bar{k} = 8\beta d_+ (2n - 1) - 8\beta d_+ \). Formula (3.91) gives a bound of \( \sup_{Q_{n-1}} ||| h^{(n)} \| \| (x), \) and more precisely of \( \sup_{Q_{n-1}} ||| h^{(n)} \| \|_{W^{\beta;D} \Phi_r(x)} , \) in terms of the \( \| A_{m,n}(k) \|_{d_r}, m = 0, \ldots, n \), and \( ||| g^{(n)} \| \|_{W^{\beta;D} \Phi_r(x)} \), raised to the power \( d_+ / d_+ \). Recall from Theorem 3.1 that we must have \( d_+ - d_+ > \beta \). Choosing \( d_+ / N \) large enough and \( d_+ - d_+ = d \), the above condition is verified and furthermore, \( kd_+ / d_+ \leq 8\beta d_+ (2n - 1) - 4\beta d_+ \). Thus

$$
\sup_{Q_{n-1}} ||| h^{(n)} \| \| \leq \max_{k=0,\ldots,d} \max_{l=1,\ldots,n} \left( \left\| \left( M^{1/2} \right)^k M^j \nabla^k g^{(l)} \right\|_{d_r}, \left\| \left( M^{1/2} \right)^k M^j \nabla^k g^{(l)} \right\|_{d_{(2q-1)}} \right).
$$

(3.105)

Following now the proof of Theorem 3.2 to bound \( \nabla^m h^{(n)} \), \( m = 1, \ldots, d \), one sees that each gradient cost essentially a multiplicative factor \( O(N^{d}_{\Phi}) \), with \( N_{\Phi} \leq 2 + \sup_{m=0,\ldots,d} \| \nabla^m g \|_{d_+}^{2\beta} \leq 1 + \sup_{m=0,\ldots,d} \| \nabla^m g \|_{d_+}^{2\beta} \), altogether \( O \left( 1 + \sup_{m=0,\ldots,d} \| \nabla^m g \|_{d_+}^{2\beta} \right) \) for the highest-order derivative \( \nabla^d h^{(n)} \). Since \( d_+ > d/2 \), \( 8\beta d_+ \leq 4\beta d_+ \), which adds up to the previous exponent \( kd_+ / d_+ \) to give a total exponent \( \leq 8\beta d_+ (2n - 1) \).

\[\square\]

### 4 Integration lemmas

**Lemma 4.1** Let \( \Phi : \mathbb{R}^d \rightarrow \mathbb{R}_+ \) be a rotationally invariant, positive, smooth function such that \( \int \Phi(x)dx = 1 \). Then

$$
\int dy \Phi(x-y) |f(y)| \leq C |f|^\ast(x)
$$

(4.1)

for some universal constant \( C \) (depending only on \( d \)).

**Proof.** By abuse of notation, we write \( \Phi(r) \) for the value of \( \Phi \) on the sphere \( S_r = \partial B(0, r) \). Then

$$
\int dy \Phi(x-y) |f(y)| = \int dr \Phi(r) \int_{S_r} dy |f(y)| = \int dr \Phi'(r) \int_{B(x,r)} dy |f(y)|
$$

\( \leq f^\ast(x) \int dr \Phi'(r) \int_{B(x,r)} dy = f^\ast(x) \int dy \Phi(x-y) = f^\ast(x), \)

(4.2)

where \( f^\ast(x) = \sup_{r > 0} \frac{1}{\int_{B(x,r)}} |f| \) is the supremum of the local averages of \( |f| \) around \( x \) (see Introduction or [23], section 3.1). We conclude using [23], Lemma 3.2. \[\square\]

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The above lemma applies to the generalized heat kernels with exponent $\beta$,
\[
\Phi^\beta_t(x-y) := e^{-((x-y)^\beta/|t|^{1/(\beta-1)})},
\]  
(4.3)
where $c^{-1} = c^{-1}(\beta) := \int \Phi^\beta_t(y)dy$ is a normalization constant. Note also the following elementary truncated integral estimates,
\[
\int_{|y| > A} dy \Phi(y) |f(x-y)|dy \leq f^\delta(x) \int_{|y| > A} dy \Phi(y).
\]  
(4.4)
In particular, an integration by parts yields for $A \gg t^{1/\beta}$
\[
\int_{|y| > A} dy \Phi^\delta_t(x-y) |f(x-y)|dy \leq f^\delta(x)A^{d-1}(t/A)^{1/(\beta-1)} \Phi^\beta_t(A)
\]
\[
\leq f^\delta(x)e^{-\frac{1}{4}(\Phi^\beta_t)^{1/(\beta-1)}}.
\]  
(4.5)

**Lemma 4.2** Let $X$ be an $\mathbb{R}^d$-valued random variable, $r^{d-1}d\sigma_r$ the surface measure on the sphere $S_r = \{|x| = r\} \subset \mathbb{R}^d$ of radius $r$, and $u : \mathbb{R}^d \to \mathbb{R}$ a $d$-times continuously differentiable function. Then
\[
\mathbb{E}[u(X)] \leq \sup_{y \in B(0, M^{1/2})} u(y) + \sum_{k=0}^d (M^{-j/2}r^{d-k}) \int_0^{+\infty} dr \mathbb{P}[|X| > r] \int_{S_r} |\nabla^k u(x)| r^{d-1} d\sigma_r(x).
\]  
(4.6)

**Proof.** We use a finite partition of unity, $1 = \chi_0 + \chi_1 + \chi_2$ such that: $\chi_0, \chi_1, \chi_2 \geq 0$; supp($\chi_0$) $\subset B(0, M^{1/2})$, supp($\chi_i$) $= \{rx, x \in \Omega; r \geq M^{1/2-1}\}$, $i = 1, 2$ where $\Omega_1$, resp. $\Omega_2$ are closed subsets of the unit sphere containing the northern, resp. southern hemisphere. We choose two smooth diffeomorphisms (e.g. spherical coordinates up to normalization) $\phi_i : \Omega_i \to [0,1]^{d-1}$, $i = 1, 2$, let $\Phi_i : \text{supp}(\chi_i) \to \mathbb{R}_+ \times [0,1]^{d-1}$, $rx \mapsto (r, \phi_i(x))$ be its "suspension" with inverse $\Phi_i^{-1}(r, y) = r\phi_i^{-1}(y)$, and assume $\tilde{x}_i := \chi_i \circ \Phi_i^{-1}$ satisfy the natural hypotheses $||\Phi_i \nabla \tilde{x}_i||_{C^k} = O(M^{-j/2}r^{k})$ for arbitrary $k \geq 0$ and multi-index $l$. Finally, we may assume by a density argument that $X$ has a smooth density $f$, and let $\tilde{f}_i := |J|f \circ \Phi_i^{-1}$ be the densities of the transferred variables $\tilde{X}_i := X \circ \Phi_i^{-1}$, where $J(r, y) := |\frac{\partial \Phi_i(r, y)}{\partial (r, y)}|$ is the Jacobian; similarly, we let $\tilde{u}_i := u \circ \Phi_i^{-1}$.

We first write $\mathbb{E}[u(X)] \leq \sup_{B(0, M^{1/2})} u + \sum_{i=1,2} \mathbb{E}[u(X)|\chi_i(X)]$ and
\[
\mathbb{E}[u(X)|\chi_i(X)] = \int \chi_i(x) u(x) f(x) dx = \int \tilde{X}_i(r, y) \tilde{u}_i(r, y) \tilde{f}_i(r, y) dr dy.
\]  
(4.7)
Inspired by the one-dimensional integration-by-parts formula,
\[
\mathbb{E}[u(Y)] = u(0) + \int_0^{+\infty} u'(y) \mathbb{P}[Y > y]dy \quad (Y \geq 0),
\]  
(4.8)
we integrate by parts with respect to each of the $d$ coordinates in $\mathbb{R}_+ \times [0,1]^{d-1}$. Since there are no boundary terms, one gets (letting $\tilde{X}_i^1, \ldots, \tilde{X}_i^d$ be the coordinates of $\tilde{X}_i$ and $y = (y_2, \ldots, y_d)$)
\[
\int \tilde{X}_i(r, y) \tilde{u}_i(r, y) \tilde{f}_i(r, y) dr dy = \int \left[ \prod_{i=2}^d \left( \frac{1}{r} \partial_{y_i} \right) \partial_r (\tilde{X}_i \tilde{u}_i) \right] (r, y) \mathbb{P}[\tilde{X}_i^1 > r, \tilde{X}_i^2 > y_2, \ldots, \tilde{X}_i^d > y_d] r^{d-1} dr dy.
\]  
(4.9)
Now \( \mathbb{P}[\tilde{X}_i^1 > r, \tilde{X}_i^2 > y_2, \ldots, \tilde{X}_i^d > y_d] \leq \mathbb{P}[X > r] \); the derivative with respect to \( r \) produces a factor \( O(M^{-j/2}) \), resp. \( O(1) \) when applied to the cut-off \( \tilde{X}_i \), resp. to the function \( \tilde{u}_i \), while normalized angular derivatives \( \frac{1}{r} \partial_{y_m} \) yield factors \( O(r^{-1}) \leq M^{-j/2} \), resp. \( O(1) \). All together one gets the result.

One has a similar lemma for functions supported on a ball (note that extensions to more general bounded domains with \( c^{d-1} \) boundary would require scaled bounds on the curvature tensor and its derivatives which may be worked out by looking at the following proof).

**Lemma 4.3** Let \( X \) be an \( \mathbb{R}^d \)-valued random variable supported on the sphere \( S_r \), and \( u : S_r \to \mathbb{R} \) a \((d-1)\) times continuously differentiable function. Then

\[
\mathbb{E}[u(X)] \leq \sum_{k=0}^{d-1} r^k \int |\nabla^k u(x)| d\sigma_r(x). \tag{4.10}
\]

**Proof.**

The proof is essentially the same. We define \( \Omega_i, \phi_i \) (\( i = 1, 2 \)) as in Lemma 4.2 and introduce a partition of unity, \( 1_{S_r} = \chi_1 + \chi_2 \), with \( \text{supp}(\chi_i) = r\Omega_i \). We now have two maps \( \Phi_i : \text{supp}(\chi_i) \to [0,1]^{d-1}, r\chi \mapsto \phi_i(\chi) \) with inverse \( \Phi_i^{-1}(y) = r\phi_i^{-1}(y) \), and may assume that \( ||\nabla^k \Phi_i||_{\infty} = O(r^{-k}) \), with \( \tilde{X}_i = \chi_1 \circ \Phi_i^{-1} \). Then

\[
\mathbb{E}[u(X)] = \sum_{i=1,2} \mathbb{E}[u(X_i)(X_i)] = \sum_{i=1,2} \int \tilde{X}_i(y) \tilde{u}_i(y) \tilde{\phi}_i(y) dy
\]

\[
= \sum_{i=1,2} \int \left( \prod_{m=2}^{d} \frac{1}{r} \partial_{y_m} \right)(\tilde{X}_i \tilde{u}_i)(y) \mathbb{P}[\tilde{X}_i^2 > y_2, \ldots, \tilde{X}_i^d > y_d] r^{d-1} dr dy. \tag{4.11}
\]

Angular derivatives \( \frac{1}{r} \partial_{y_m} \) yield factors \( O(r^{-1}) \), resp. \( O(1) \) when applied to \( \tilde{X}_i \), resp. \( \tilde{u}_i \).  

5 Dead ends: how not to cut into scales

The present section (provided for arXiv only) contains a natural but inconclusive approach to the problem of multi-scale decomposition for the noisy KPZ equation. We introduce a \textit{scale-j KPZ equation} and show how to extend in some sense the Hamilton-Jacobi-Bellman formalism (let us rather say random path representation) to this equation. However the diffusion process \( X^\alpha \) becomes a \textit{signed jump process}, and the presence of signed probabilities prevented us from getting a priori estimates like those proved in the previous section. We felt however free to include this section because the original ideas it contains may prove useful for other problems, and also because it is good to know that this approach is a dead end for the present problem.

As explained in [23], the infra-red cut-off in eq. (1.2) is somewhat artificial for the original problem we have in view, that is, the analysis of the full noisy KPZ equation, in the sense that the solution of the full equation has no simple representation in terms of 'superposition' of the solutions of the scale \( j \) infra-red cut-off equations (in the next article we actually use more brutal cut-offs, for which the bounds of section 3 hold, and do not cut the original noise into scales). If one would follow the natural strategy for this problem, in connection with a renormalization à la Wilson, as done with
success e.g. in the alternative proof of KAM’s theorem by Bricmont, Gawedzki and Kupiainen [6], one would first rewrite equation (1.1) as the integral equation,

\[ h_t(x) = \int_0^t ds \int_E dy G(t - s; x - y)(V(\nabla h_s(y)) + g_s(y)) \]  

(5.1)

or simply

\[ h = G(V(\nabla h) + g), \]  

(5.2)

where \( G = (\partial_t - \nu \Delta)^{-1} \) is the Green kernel of the linear heat equation, explicitly, \( G(t; x) = (2\pi \nu t)^{-d/2} e^{-x^2 / 2\nu t} \). Then one would decompose \( G \) and \( g \) into scales, i.e. we write \( G = \sum_{j=0}^{+\infty} G^j \) and \( g = \sum_{j=0}^{+\infty} g^j \), where \( G^j \), \( g^j \) essentially pick up the fluctuations of \( G \), \( g \) at time-distances \( O(M^j) \) and space-distances \( O(M^{1/2}) \), and set about to define a scale \( j \) KPZ equation. Since \( G = (\partial_t - \nu \Delta)^{-1} \), it is natural to define \( g^j = (\partial_t - \nu \Delta) G^j g \), as suggested in [23]. We choose \( G^j \) as in [23], Definition 5.2, which we recall here to satisfy the bounds,

\[ \chi^0(\cdot) := \sum_{n\geq 0} \tilde{\chi}(M^n), \quad \tilde{\chi}^j(\cdot) := \tilde{\chi}(M^{-j} \cdot) \quad (j \geq 1) \]

(5.3)

form a partition of unity, i.e. \( \sum_{j \geq 0} \tilde{\chi}^j \equiv 1 \) on \( \mathbb{R}_+ \).

**Definition 5.1 (non-local cut-off)** Let \( G^j \) be the operator

\[ (G^j g)(t) := \int \tilde{\chi}^j(s)e^{\nu \Delta} g(t - s) ds, \quad j \geq 0 \]

(5.4)

The kernels \( G^j \), \( j \geq 0 \) are causal, i.e. \( G^0(t - s; x - y) = 0 \) if \( t < s \); furthermore, \( G^j(t - s) \equiv 0 \) unless \( M^{-j-1} < t - s < M^{j+1} \) \((j \geq 1)\), \( t - s < M \) \((j = 0)\). From [23], Lemma 5.6, we know them to satisfy the following bounds,

\[ |\nabla_n^p \partial_t^q \nabla_{x'}^p \partial_t^q G^j(t, x, t', x')| \leq M^{-jd/2} M^{-\frac{j}{2}(p+p')-\frac{j}{2}(q+q')} 1_{|t-t'|} e^{-cM^{j-1/2}|x-x'|}, \quad j \geq 1 \]

(5.5)

and

\[ |\nabla_n^p \partial_t^q \nabla_{x'}^p \partial_t^q G^j(t, x, t', x')| \leq 1_{|t-t'|} (t - t')^{-d/2 - \frac{j}{2}(p+p')-\frac{j}{2}(q+q')} e^{-c(t-t')^{-1/2}|x-x'|}. \]

(5.6)

**Definition 5.2 (single-scale KPZ equation, first formulation)** Let \( h^j : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \).

(i) The scale \( j \) KPZ equation or single-scale KPZ equation is the integral equation

\[ h^j_t(x) = \int_0^\infty ds \int_{\mathbb{R}^d} dy G^j(t - s; x - y)(V(\nabla h^j_s(y)) + g^j_s(y)), t \in \mathbb{R} \]

(5.7)

or simply

\[ h^j = G^j(V(\nabla h^j) + g) \]

(5.8)

(ii) (generalized Cauchy problem) Let \( u : \mathbb{R}_- \times \mathbb{R}^d \to \mathbb{R} \) solve (5.7) for \( t \leq 0 \), and \( h^j : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) solve (5.7) for \( t \in \mathbb{R} \) with \( h^j|_{t \leq 0} = u^j \). Then one says that \( h^j \) solves the generalized Cauchy problem for eq. (5.8) with generalized initial condition \( u \).
The interest in this series of articles is in the \textit{noisy scale j KPZ equation}, for which \( g = \eta^j = (\partial_t - \nu \Delta) G^j \eta \) is the scale \( j \) component of the regularized white noise \( \eta \) (see [23], Definition 5.1). However, there is nothing very particular about this specific equation in the limited scope of discussion chosen for this article, so the reader need not worry about probabilistic properties of white noise, already discussed at length in the previous article.

We shall refer to the particular case \( g = 0 \) as the \textit{homogeneous single-scale KPZ equation}.

We now give a second formulation of the integral equation (5.8).

\textbf{Definition 5.3} Let \( K^j := (\partial_t - \nu \Delta + M^{-j}) G^j \).

Since \( (\partial_t - \nu \Delta)(G^j g)(t) = \int (\tilde{\chi}^j)'(s) e^{\nu \Delta} g(t-s) ds, \ K^j(t-s) \equiv 0 \) unless \( M^{-j-1} < t-s < M^{-j+1} \) \((j \geq 1), \ t-s < M \) \((j = 0)\), just like for \( G^j \). From eq. (5.5) and (5.6),

\[
|\nabla_x^p \partial_x^q \nabla_{t'}^{p'} \partial_{t'}^{q'} K^j(t, x; t', x')| \leq (M^{-j})^{1+d/2} M^{-j(2-p+p')-j(q+q')} |_{t \leq t'} e^{-cM^{j}(t-t')-cM^{-j}|x-x'|}, \quad j \geq 1
\]

and

\[
|\nabla_x^p \partial_x^q \nabla_{t'}^{p'} \partial_{t'}^{q'} K^0(t, x; t', x')| \leq |_{t \leq t'} (t-t')^{-1-d/2-\frac{1}{2}(2-p+p')-j(q+q')} e^{-c(t-t')-c|x-x'|},
\]

(5.9)

\textbf{Definition 5.4 (KPZ equation)} Let \( \bar{h}^j \) solve the integro-differential equation

\[
(\partial_t - \nu \Delta + M^{-j}) \bar{h}^j(x) = \int_0^\infty ds \int_{\mathbb{R}^d} dy \nabla K^j(t-s, x-y) \bar{h}^j(y) + g(t, x), \quad t \in \mathbb{R}
\]

or simply

\[
(\partial_t - \nu \Delta + M^{-j}) \bar{h}^j = \nabla(\nabla K^j \bar{h}^j) + g.
\]

(5.11)

The KPZ equation looks like a viscous Hamilton-Jacobi equation, except that it features a non-local deposition rate \( \nabla(\nabla K^j \bar{h}^j) \) which takes into account values of the interface \( \bar{h}^j \) at different space locations and earlier times. As for the single-scale KPZ equation, see Definition 5.2 one may consider the associated generalized Cauchy problem with generalized initial condition \((\bar{u}_t)_{t \leq 0}\) satisfying (5.11) for \( t \leq 0 \).

The KPZ equation is simply a second reformulation of the single-scale KPZ equation. Namely, let \( \bar{h}^j \) solve (5.11). Then \( h^j := K^j \bar{h}^j \) solves the equation

\[
(\partial_t - \nu \Delta + M^{-j}) h^j = K^j(\nabla h^j) + g,
\]

(5.12)

which is equivalent to (5.8). The correspondence goes through for the Cauchy problems in the following sense. Assume \((\bar{u}^j_t)_{t \leq 0}\) is a generalized initial condition for the KPZ equation, i.e. \( \bar{u}^j_t \) satisfies (5.11) for \( t \leq 0 \), and let \( u^j_t = (K^j \bar{u}^j)_t, \ t \leq 0 \). Then \((u^j_t)_{t \leq 0}\) is a generalized initial condition for the single-scale KPZ equation. So, letting \( h^j \) be a solution of the KPZ equation with generalized initial condition \( \bar{u}^j \), \( h^j := K^j \bar{h}^j \) is a solution of the single-scale KPZ equation with initial condition \( u^j \).

It is not clear from the previous considerations that \textit{any} generalized initial condition for the single-scale KPZ equation is of the above form \((K^j \bar{u}^j)_{t \leq 0}\). We shall not consider this problem in this article since we shall always consider generalized initial conditions of this form. Let us however briefly discuss how this could be done formally. Choose some reference time \( T_0 \) instead of 0, and
assume \( g_{t \leq T_0} = 0 \). Then \( u_t^j = 0, \ t \leq T_0 \) is obviously a generalized condition of the above form, with \( \bar{u}_t^j = 0, \ t \leq T_0 \). This allows us to construct iteratively solutions \( (h_t^j)_{t \leq T_n} \) with generalized initial condition \( u_t^j = K^j \bar{u}_t^j, t \leq T_n \) by the above procedure, with \( (T_n)_{n \geq 0} \) increasing to infinity. Finally one may take the limit \( T_0 \to -\infty \) to get a solution valid for an arbitrary forcing term \( g \) and for arbitrary \( t \in \mathbb{R} \).

5.1 Signed probabilities

Despite the formal analogy with eq. (2.1), the operator \( \bar{h}^j \mapsto \nabla K^j \bar{h}^j \) appearing in the right-hand side of (5.12) is very different from the drift-generating operator \( h \mapsto \nabla h \) in the right-hand side of (1.1).

Seeking a Bellman type formula for the solution would lead to introduce signed measures. However, the expectation \( \mathbb{E}^{t,x} \) is not positive any more, so \( \mathbb{E}^{t,x} \) and \( \inf_{\alpha} \) do not commute and Bellman’s principle (2.6) is wrong. On the other hand, the Feynman-Kac formula (2.15) may be generalized. Again, the most natural generalization of Proposition 2.1 (with \( \alpha^* \) given as above) leads to a signed stochastic process \( X_t \). The major drawback is that \( \mathbb{E}[h(X_t)] \) is not bounded by \( \|h\|_{\infty} \) in general if \( \mathbb{E} \) is the expectation with respect to a signed measure; the associated \( C_0 \) semi-group is not a contraction semi-group, and the \( C_0 \) norm increases exponentially with a rate proportional to \( \alpha \), thus making it impossible to get a priori bounds for the solution.

Let us be a little more specific. The analogue of the diffusion operator \( L^\alpha \) of the previous section is the generator of a somewhat complicated signed process with jumps. The following lines are dedicated to some general considerations on signed probabilities and to the construction of \( L^\alpha \).

**Definition 5.5** A signed probability space is a measurable set \( \Omega \) equipped with a signed measure \( \mu \) such that \( \int |d\mu| < \infty \).

In the sequel, we only consider topological spaces with their Borel \( \sigma \)-fields, and Borel signed measures. The set of all Borel signed measures \( \mu \) on \( \Omega \) such that \( \int |d\mu| < \infty \) will be denoted by \( \text{Meas}(\Omega) \).

**Definition 5.6**  
(i) A signed random variable is a measurable function \( X : \Omega \to \mathbb{R}^d \) on a signed probability space.

(ii) A signed stochastic process \( (X_t)_{t \in \mathbb{T}}, T \subset \mathbb{R} \) is a time-indexed family of signed random variables.

**Definition 5.7** (signed transition kernels and Markov semi-groups) A signed (time-homogeneous, Feller) transition kernel on \( \mathbb{R}^d \) is a family of functions \( p_t : \mathbb{R}^d \times \mathbb{B}(\mathbb{R}^d) \to \mathbb{R}, (x,A) \mapsto p_t(x,A) \), \( t > 0 \), such that \( x \mapsto p_t(x,A) \) is measurable for \( A \) fixed; \( A \mapsto p_t(x,A) \) is a signed measure for \( x \) fixed; \( P_t : f \mapsto (x \mapsto \int p_t(x,dy)f(y)) \) defines a family of bounded operators \( P_t : C_0 \to C_0 \); and

(i) (semi-group property) \( P_s \circ P_t = P_{s+t} (s,t > 0) \);

(ii) \( \exists C > 0, \forall t > 0, \forall x \in \mathbb{R}^d, \int |p_t(x,dy)| \leq e^{Ct} \);

(iii) (probability preservation) for all \( t > 0 \) and \( x \in \mathbb{R}^d \), \( \int p_t(x,dy) = 1 \); and

(iv) 
\[
\forall \varepsilon > 0, \int_{\mathbb{R}^d \setminus B(x,\varepsilon)} |p_t(x,dy)| \to_{t \to 0} 0
\]  
(5.14) 
uniformly in \( x \).
Property (ii) is equivalent to saying that $\|P_t\| \leq e^{C_t}$, where $\| \cdot \|$ is the operator norm. Property (iii) and (iv) imply immediately that $P_t f \to_{t \to 0} f$ uniformly for every $f \in C_0$.

Let $x \in \mathbb{R}^d$. With such a kernel, one may easily defined a signed stochastic process, $(X^x_t)_{t \in \mathcal{T}}$, where $\mathcal{T} = \{0 < t_1 < \ldots < t_n\}$ is an arbitrary subset of $\mathbb{R}_+$, by equipping the set of trajectories $\Omega_{\mathcal{T}} = \{x \times (X^x_t)_{t \in \mathcal{T} \setminus \{0\}} \} \simeq \mathbb{R}^{ad}$ with the signed measure $\mu_{\mathcal{T}}$,

$$
\mu_{\mathcal{T}}(x_1, \ldots, x_n) = \int_{A_1 \times \cdots \times A_n} p_{t_1}(x, dx_1)p_{t_2-t_1}(x_1, dx_2) \cdots p_{t_n-t_{n-1}}(x_{n-1}, dx_n).
$$

(5.15)

Note that $\int |d\mu_{\mathcal{T}}| \leq e^{C_n}$.

The whole construction is very similar to that of a conventional Markov process, except that the techniques based on inequalities (in particular martingale techniques) are missing because of the signs. The Kolmogorov extension theorem, see e.g. [8], pp. 604 and sq. (an immediate consequence of the measure extension theorem [21], which extends straightforwardly to bounded signed measures) shows the existence and unicity of a stochastic process $(X^x_t)_{t \in \mathcal{T}}$ on a signed probability space $\Omega_{[0,T]}$ which identifies with the space of trajectories. Unfortunately we do not know if a suitable modification $X^x$ has càdlàg (right-continuous with left limits) trajectories (the proof we know [20] relies strongly on martingale techniques). In the examples considered below, we give an explicit realization of the process $X^x$ on the Skorokhod space $\mathcal{D}([0,T], \mathbb{R}^d)$ of càdlàg functions, called canonical version of the process. By construction, the law of $(X^x_t)_{t \in \mathcal{T}}$ coincides with $\mu_{\mathcal{T}}$ for every finite subset $\mathcal{T} \subset [0,T]$.

Under a condition that we shall presently introduce, the signed Markov process $(X^x_t)_{t \geq 0}$ is a pure jump process.

**Definition 5.8** A pure jump trajectory $f : [0, T] \to \mathbb{R}^d$ is a càdlàg trajectory such that $f' = 0$ a.e. Such a trajectory is called regular if $f$ has a finite number of discontinuities (in other words, if $f$ is a step function). Denote by $\mathcal{D}_{\text{pure jump}}([0,T])$ the set of all regular pure jump trajectories.

If $f$ is a pure jump trajectory, there exists an at most denumerable set

$$
\Sigma := \{t_i, i \in I\} \subset [0,T]
$$

(5.16)

(the jumping time set) such that $\lim_{t \downarrow t_i} f(t') \neq f(t), t \in \Sigma$ and $f$ is constant on some open interval $[t - \epsilon, t + \epsilon]$, $\epsilon > 0$ if $t \notin \Sigma$. If, moreover, $f \in \mathcal{D}_{\text{pure jump}}([0,T])$, then $\Sigma = \{t_1 < \ldots < t_n\}$ is a finite set, and $f$ is constant over each interval $[t_i, t_{i+1})$.

**Definition 5.9** (pure jump Markov kernel) A pure jump Markov kernel is a signed Markov transition kernel $(p_t)_{t \geq 0}$ such that

$$
p_t(x, \{x\}) \to_{t \to 0} 1, \quad \int_{\mathbb{R}^d \setminus \{x\}} |p_t(x, dy)| = O_{t \to 0}(t)
$$

(5.17)

uniformly in $x$.

The second condition (useless for a conventional Markov transition kernel) seems to be necessary in general in order to obtain a well-behaved generator.
Theorem 5.1 (jump rates and generator of a jump process) Let \((p_t)_{t \geq 0}\) be a pure jump Markov kernel, and \(\mu^x \in \text{Meas}(\Omega[0,T])\) the law of the canonical version of \((X^x_t)_{t \in [0,T]}\). Then:

(i) The limits \(\lambda_x(A) := \lim_{t \to 0} \mathbb{P}_t \frac{p_t(x,A)}{t} \in \mathbb{R} \) \((A \in \mathcal{B}(\mathbb{R}^d), \ x \notin A)\) and \(\lambda_x := \lim_{t \to 0} \mathbb{P}_t \frac{1-p_t(x,x)}{t} \in \mathbb{R}\) exist, and \(\lambda_x = \int \lambda_x(dy)\). Furthermore, \(\lambda_x()\) is a bounded signed measure on \(\mathbb{R}^d \setminus \{x\}\).

(ii) (generator) The generator \(\mathcal{L} : C_0 \to C_0, \ f \mapsto \mathcal{L}f := \frac{d}{dt}\bigg|_{t=0}P_t f\) has full domain and

\[
\mathcal{L}f(x) = -\lambda_x f(x) + \int \lambda_x(dy)f(y). \tag{5.18}
\]

(iii) The trajectory \(t \mapsto X^x_t\) belongs a.s. to \(\mathcal{D}_{\text{pure jump}}([0,T], \mathbb{R}^d)\).

(iv) Denote by \(\Sigma\) the jumping time set of the random trajectory \(t \mapsto X^x_t\). Assume \(\Sigma = \{T_1 < \ldots < T_n\}\) is finite. Then, for \(0 < t_1 < \ldots < t_n < T\),

\[
\mu^x (\Sigma) = n, T_1, T_2 \in \{t_1 + dt_1, \ldots, T_n \in \{t_n + dt_n\}, X_T \in A_1, \ldots, X_{T_n} \in A_N\}
\]

\[
= \int_{x_1 \in A_1, \ldots, x_n \in A_n} e^{-\lambda_{x_1}(t_1 - t_0)} \lambda_{x_1}(dx_1) e^{-\lambda_{x_2}(t_2 - t_1)} \lambda_{x_1}(dx_2) \cdots e^{-\lambda_{x_{n-1}}(T - T_{n-1})} \lambda_{x_{n-1}}(dx_{n-1}) e^{-\lambda_{x_n}(T - t_n)} dt_1 \ldots dt_n. \tag{5.19}
\]

Proof.

(i) (jump rates) We adapt the proof of Theorem 2.2 in [8]. First, letting \(\alpha, \delta > 0\) and \(n = [\alpha/\delta] + 1\),

\[
\mu^x(X_t = x, 0 \leq t \leq \alpha) = \lim_{\delta \to 0} \mu^x(X_{\alpha} = x, j = 1, \ldots, n) = \lim_{\delta \to 0} \mu^x(x, \{x\})^\delta. \tag{5.20}
\]

For \(\alpha\) small enough, \(p_0(x, \{x\}) > 0\), so one may take the logarithm of the above expressions,

\[
\log \mu^x(X_t = x, 0 \leq t \leq \alpha) = \alpha \lim_{\delta \to 0} \log \frac{\mu^x(x, \{x\})}{\delta} = -\alpha \lim_{\delta \to 0} \frac{1 - p_0(x, \{x\})}{\delta} \in [-\infty, +\infty). \tag{5.21}
\]

This yields the existence of \(\lambda_x \in (-\infty, +\infty]\), together with the equality

\[
\mu^x(X_t = x, 0 \leq t \leq \alpha) = e^{-\lambda_x t}. \tag{5.22}
\]

Let us prove by absurd that \(\lambda_x \not= +\infty\). Let \(0 < \varepsilon < \frac{1}{2}\). For \(\alpha\) small enough, \(\inf_{x \in \mathbb{R}^d, \delta \leq \alpha} p_0(x, \{x\}) \geq 1 - \varepsilon\) and \(\sup_{x \in \mathbb{R}^d, \delta \leq \alpha} \int p_0(x, dy) \leq \varepsilon\). Hence, for every \(n \geq 1\),

\[
1 - \varepsilon \leq p_0(x, \{x\}) \leq \sum_{y \not= x} \int_{y \not= x} \mu^x \left( X_{\frac{n\alpha}{2}} = y, x, j = 1, \ldots, n; X_{\frac{n\alpha}{2}} \in dy \right) \mu^x \left( X_{\frac{n\alpha}{2}} = x, j = 1, \ldots, n; X_{\frac{n\alpha}{2}} \in dy \right)
\]

\[
\leq \mu^x \left( X_{\frac{n\alpha}{2}} = x, j = 1, \ldots, n \right) + \varepsilon \sum_{y \not= x} \int_{y \not= x} \mu^x \left( X_{\frac{n\alpha}{2}} = y, x, j = 1, \ldots, n; X_{\frac{n\alpha}{2}} \in dy \right)
\]

\[
\leq \mu^x \left( X_{\frac{n\alpha}{2}} = x, j = 1, \ldots, n \right) + \varepsilon \left\{ \int_{y \not= x} \mu^x \left( X_{\frac{n\alpha}{2}} = y, x, j = 1, \ldots, n - 1 \right) \right\}
\]

\[
\leq \mu^x \left( X_{\frac{n\alpha}{2}} = x, j = 1, \ldots, n \right) + \varepsilon \left\{ e^{Ca} - \mu^x \left( X_{\frac{n\alpha}{2}} = x, j = 1, \ldots, n - 1 \right) \right\} \tag{5.23}
\]
If \( \lambda_x = +\infty \) then \( \lim_{n \to \infty} \mu^x(X_{\bar{X}_n} = x, j = 1, \ldots, n) = \lim_{n \to \infty} |\mu^x| \left( X_{\bar{X}_n} = x, j = 1, \ldots, n - 1 \right) = 0 \), which is contradictory with (5.23) for \( \alpha \) small enough.

Similarly, if \( x \notin A \),

\[
p_\alpha(x, A) = \sum_{n=0}^{\infty} \int_{y \in X} \mu^x(X_{\bar{X}_n} = x, j = 1, \ldots, n; X_{\frac{n+1}{\alpha}} \in dy) p_{\frac{n+1}{\alpha}}(y, A)
\]

\[
= C_1 + C_2 + C_3,
\]

with

\[
C_1 = \sum_{n=0}^{\infty} \int_{y \in A} \mu^x(X_{\bar{X}_n} = x, j = 1, \ldots, n; X_{\frac{n+1}{\alpha}} \in dy)
\]

\[
= \frac{1 - p_\alpha^\alpha(x, \{x\})}{1 - p_\alpha^\alpha(x, \{x\})} p_\alpha(x, A)
\]

\[
\sim_{n \to \infty} \frac{1 - e^{-\lambda_x \alpha}}{\lambda_x} \frac{p_{\alpha/n}(x, A)}{\alpha/n} \sim_{n \to \alpha} \alpha \frac{p_{\alpha/n}(x, A)}{\alpha/n};
\]

\[
C_2 = -\sum_{n=0}^{\infty} \int_{y \notin A} \mu^x(X_{\bar{X}_n} = x, j = 1, \ldots, n; X_{\frac{n+1}{\alpha}} \in dy) (1 - p_{\frac{n+1}{\alpha}}(y, \{y\})) ;
\]

\[
C_3 = \sum_{n=0}^{\infty} \int_{y \notin A} \mu^x(X_{\bar{X}_n} = x, j = 1, \ldots, n; X_{\frac{n+1}{\alpha}} \in dy) p_{\frac{n+1}{\alpha}}(y, A \setminus \{y\}).
\]

By hypothesis, \( |1 - p_{\frac{n+1}{\alpha}}(y, \{y\})|, |p_{\frac{n+1}{\alpha}}(y, A \setminus \{y\})| \leq \varepsilon \) for \( \alpha \) small enough. In order to use this inequality, one must consider \( |\mu^x| \) instead of \( \mu^x \). Redoing the same computations starting from \( |p_\alpha| \) \((x, A) \) and putting in absolute values everywhere, one obtains \( \frac{|p_{\alpha/n}(x, A)|}{\alpha/n} \leq c_n (1 + 2\varepsilon) \), \( c_n \to_{n \to \infty} 1 \). Thus (taking \( n \to \infty \) and then \( \alpha \to 0 \)) \( \lim_{\alpha \to 0} \frac{|p_{\alpha/n}(x, A)|}{\alpha} \leq \lim_{\alpha \to 0} \frac{\alpha}{\alpha/n} \lambda_x(A) \), which is \( < \infty \) by hypothesis. In turn this yields \( \frac{p_{\alpha/n}(x, A)}{\alpha} \to_{\alpha \to 0} \lambda_x(A) \), hence (taking \( \varepsilon \to 0 \)) \( \frac{p_{\alpha/n}(x, A)}{\alpha} \to_{\alpha \to 0} \lambda_x(A) \in \mathbb{R} \).

(ii) is merely a reformulation of (i).

(iii),(iv) Let \( \tilde{\mu}^x \) be the measure defined in (5.19). We leave it to the reader to check that \( \tilde{\mu}^x \) is the measure of a Markov process \( (\tilde{X}_t)_{t \geq 0} \). Denote by \( \tilde{p}_t \), resp. \( \tilde{L} \) the transition kernel, resp. generator of \( \tilde{X} \). We now prove that \( \tilde{L} = L \), hence \( \tilde{X} = X \). Let \( x \notin A \). Then (the following arguments are inspired from [13])

\[
\tilde{p}_t(x, A) = \tilde{\mu}^x(T_1 \leq t < T_2; X_{T_1} \in A) + \tilde{\mu}^x(T_2 \leq t, X_t \in A)
\]

hence

\[
|\tilde{p}_t(x, A) - \tilde{\mu}^x(T_1 \leq t, X_{T_1} \in A)| = |\tilde{p}_t(x, A) - \frac{1 - e^{-\lambda_x t}}{\lambda_x} \lambda_x(A)| \leq |\tilde{\mu}^x|(T_2 \leq t)
\]
and
\[
|\tilde{\mu}^i(T_2 \leq t)| \leq \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_{y_1 \neq x} \int_{y_2 \neq y_1} e^{-\lambda_{y_1} t_1} |\lambda_x| (dy_1) e^{-\lambda_{y_1} (t_2-t_1)} |\lambda_y| (dy_2) = O(t^2),
\]
so \([d]_{t=0} \tilde{p}_t(x, A) = \lambda_x(A)\). 

\[\square\]

We must also consider \textit{mixed} (signed, Markovian) processes including jumps, diffusion and annihilation (i.e. jumps to a cemetery conventionally denoted by \(\zeta\) for the jumping time set of a general trajectory. We define in the next lemma the canonical version of trajectories of the canonical version of a stochastic process \(X\), started from \(x\) at time \(s\) and conditioned to end at \(y\) at time \(t\).

\textbf{Lemma 5.10} Consider the following measure on \(\mathcal{D}([0, T], \mathbb{R}^d \cup \{\infty\})\), identified with the law of the trajectories of the canonical version of a stochastic process \(X\),

\[
\tilde{\mu}^i \left( \zeta > T, |\Sigma| = n, T_1 \in [t_1, t_1 + dt_1], \ldots, T_n \in [t_n, t_n + dt_n], \tilde{X}_{T_1} \in A_1, \tilde{X}_{T_2} \in A_2, \ldots, \tilde{X}_{T_n} \in A_n \right) = e^{-\lambda T_1} \int_{y_1 \in A_1} \cdots \int_{y_n \in A_n} \lambda_{y_1} (dX_1) \lambda_{y_2} (dX_2) \cdots \lambda_{y_n} (dX_n) \exp \left( -\int_0^{T_1} \lambda_{\tilde{X}, i(s)} ds - \cdots - \int_0^{T_n} \lambda_{\tilde{X}, i(s)} ds \right) \mu_{0}^{(0, x)} (T_1, T_2, \ldots, T_n, y_1, \ldots, y_n, x_1, \ldots, x_n),
\]

where \(\mu_{0}^{(x, s)}\) is the law of the diffusion started from \(x\) at time \(s\), and \(\mu_{0}^{(x, t, y)}\) the law of the diffusion started from \(x\) at time \(s\) and conditioned to end at \(y\) at time \(t\).

Then \(\tilde{\mu}^i\) is the law of a Markov process \(\tilde{X}^i\) with generator \(\mathcal{L} + \mathcal{L}_{\text{diff}} - \lambda\).

\textbf{Proof.}\n
The above lemma generalizes easily to time-inhomogeneous signed Markov processes with time-dependent transition rates \(\lambda_{i,x}, \lambda_{i,x}(dy)\), provided one substitutes \(\lambda_{T_i, j}(dX_i)\), resp. \(\lambda_{i,x}(\tilde{X}_i)\) to \(\lambda_{j,y}(dx_i)\), resp. \(\lambda_{i,x}(\tilde{X}_i)\) in the right-hand side of (5.31).

\[\square\]

5.2 Signed Markov process for the random path representation

We define in this paragraph an auxiliary signed Markov process \(Z_\theta = (T_\theta, X_\theta), \theta \geq 0\) on the state space \(((t, x) | t \in \mathbb{R}_+, x \in \mathbb{R}^d)\). The variable \(\theta\) is an auxiliary time-variable. Recall the kernel \(K^i\) from Definition 5.3. The time-reversed kernel is denoted by \(\tilde{K}^i\) in the sequel, namely, \(\tilde{K}^i(t, x; t', x') = K^i(t - T, x; T - t', x')\). Note that \(\tilde{K}^i\) is anti-causal, i.e. \(\tilde{K}^i(t, \cdot; t', \cdot) \equiv 0\) if \(t > t'\).

Let \(\alpha : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) be a bounded, continuous function, and \(\alpha \cdot \nabla \tilde{K}^i\) the operator with kernel
\[
(\alpha \cdot \nabla \tilde{K}^i)(t, x; t', x') = \alpha(t, x) \cdot \nabla_x \tilde{K}^i(t, x; t', x').
\]

For simplicity we abbreviate \(C_0(\mathbb{R} \times \mathbb{R}^d)\) to \(C_0\) in the discussion. The following properties hold:
1. $\nabla \tilde{K}^j : C_0(\mathbb{R} \times \mathbb{R}^d) \rightarrow C_0(\mathbb{R} \times \mathbb{R}^d)$ is a bounded operator, with

$$|||\alpha \cdot \nabla \tilde{K}^j||| \leq |||\alpha|||_\infty \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^d} \int (\nabla \tilde{K}^j(t, x; t', x'))dt' dx' \leq |||\alpha|||_\infty M^{-j/2}(M^{-j})^{1+d/2} \cdot M^d \frac{d^j}{|||\alpha|||_\infty M^{-j/2}}.$$  

(5.33)

2. The operator $\alpha \cdot \nabla \tilde{K}^j - \gamma \text{Id} : C_0 \rightarrow C_0$ is invertible for $\gamma > |||\alpha \cdot \nabla \tilde{K}^j|||$, with $|||(\alpha \cdot \nabla \tilde{K}^j - \gamma \text{Id})^{-1}||| \leq \frac{1}{\gamma - |||\alpha \cdot \nabla \tilde{K}^j|||}$.

Hence, by the Hille-Yosida theorem (or as follows from direct exponentiation in this very simple case) $\nabla \tilde{K}^j$ generates an $|||\alpha \cdot \nabla \tilde{K}^j|||$-contractive, strongly continuous semi-group, i.e. there exists a family of bounded operators $P_\theta : C_0 \rightarrow C_0 (t \geq 0)$ such that $P_0 = \text{Id}$, $P_\theta \circ P_\eta = P_{\eta + \theta}$ ($\sigma, \theta \geq 0$), and for $f \in C_0$, $P_\theta f \rightarrow \theta \rightarrow 0 f$,

$$|||P_\theta f|||_\infty \leq e^{|||\alpha \cdot \nabla \tilde{K}^j|||\theta} |||f|||_\infty \text{ and } \frac{P_\theta f - f}{\theta} \rightarrow_{\theta \rightarrow 0} \nabla \tilde{K}^j f$$ uniformly.

Thus we know that $\nabla \tilde{K}^j$ is a signed transition kernel in the sense of Definition 5.7, provided we

Choose an annihilation rate $\lambda = M^{-j}$ in [5.31]. The construction in Lemma 5.10 yields an inhomogeneous, signed Markov process $\tilde{Z}^\alpha$ with time-dependent generator $\tilde{L}_\alpha^\theta := L_{\text{diff}} + L_{\text{diff}}^\alpha - M^{-j}$.

Now comes the main result of this section. By construction, the time variable $T_\theta$ satisfies $dT_\theta = d\theta$ a.e. and undergoes positive jumps, $\theta_0 - T_\theta \in [M^{-j-1}, M^{-j}]$ ($j \geq 1$) or $(0, M]$ ($j = 0$) when $\theta \in \Sigma$; clearly, $T_\theta \geq \theta$. Consider the right-continuous inverse $\theta_\tau := \inf \{ \theta : T_\theta > t \}$ (see [20], chap. V).

Consider a bounded, continuous function $u : (t, x) \mapsto u(t, x)$, $t \geq T, x \in \mathbb{R}$, playing the rôle of a generalized initial condition for the $\text{KPZ}$ equation after time reversal.

Lemma 5.12 Let

(i) $h(t, \cdot) := -u(T-t, \cdot)$, $t \in \mathbb{R}$ solve the $\overline{\text{KPZ}}$ equation with generalized initial condition $(h_t)_{t \leq 0} = (-u_{T-t})_{t \leq 0}$;

(ii) $\alpha_\theta(t, x) := \alpha^*(\nabla \tilde{K}^j u(t, x))$, where $\alpha^*(y) = \nabla V(-y)$ as in subsection 2.1;

(iii) $\tilde{Z} := \tilde{Z}^\alpha$, with $\alpha$ as in (ii), be the above defined process with initial condition $\tilde{Z}_0 = (t, x)$, and $\mathbb{E}^{(t,x)} = \mathbb{E}^{0, (t,x)}$ be the expectation with respect to the measure on the trajectories of $\tilde{Z}$;

(iv) $\nu(t, x)$ be the function defined as

$$\nu(t, x) := \mathbb{E}^{(t,x)} \left[ \int_0^{\theta_\tau} \left( \tilde{V}(\alpha_\theta) - \tilde{g} (\tilde{Z}_\theta) \right) d\theta' + u(\tilde{Z}_{\theta_\tau}) \right]$$  

(5.35)

with $\tilde{g}(t, \cdot) := g(T-t, \cdot)$.  

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Then \( v \equiv u \).

Before proving the lemma note the following. If \( t \geq T \) then \( \theta_T = 0 \) and \( v(t, x) = u(t, x) \). Otherwise \( T \leq \theta_T \leq T + M^{j+1} \) and \( \sigma_\theta(t, x) \) is a function of \( (u_r)_{r \leq T} \); hence \( v(t, x) \) depends on the generalized initial condition \( (u_r)_{r \leq T} \) only through \( (u_r)_{T \leq r \leq T + M^{j+1}} \).

**Proof.** The proof is very similar to that of Proposition 2.1. Let

\[
 w^{(t,x)}(\bar{t}) := E^{(t,x)} \left[ \int_0^{\theta_{\bar{t}}} \left( \bar{V}(\alpha_{\theta'}) - g(\bar{Z}_{\theta'}) \right) d\theta' + u(\bar{Z}_{\theta_{\bar{t}}}) \right], \quad t \leq \bar{t} \leq T. \tag{5.36}
\]

By definition, \( w^{(t,x)}(t) = u(t, x) \) and \( w^{(t,x)}(T) = v(t, x) \), so

\[
v(t, x) - u(t, x) = \int_t^T d\bar{t} \mathbb{E}^{t,x} \left\{ \frac{d\theta_{\bar{t}}}{d\bar{t}} \cdot \int_0^{\theta_{\bar{t}}} \left( \bar{V}(\alpha_{\theta'}) - g(\bar{Z}_{\theta'}) \right) d\theta' + u(\bar{Z}_{\theta_{\bar{t}}}) \right\}.
\]

The process \( \frac{d\theta_{\bar{t}}}{d\bar{t}} \) is everywhere equal to 0 or 1. By Itô’s formula,

\[
du(\bar{Z}_{\theta_{\bar{t}}}) = \left( \alpha_{\theta_{\bar{t}}} \cdot \nabla \bar{K} \right) u(\bar{Z}_{\theta_{\bar{t}}}) + (\partial_t + \nu \Delta) u(\bar{Z}_{\theta_{\bar{t}}}) \cdot d\theta_{\bar{t}} + dM_{\theta_{\bar{t}}}, \tag{5.38}
\]

where \( M \) is a martingale. Using the definition of \( \alpha \) and the KPZ equation for \( u \) yields \( v - u \equiv 0 \). \( \square \)

Assuming the initial condition of the KPZ equation and its gradient are uniformly bounded in \([ -M^{j+1}, 0 ] \times \mathbb{R}^d\), and the right-hand side satisfies analogous properties, the quantities in the above lemma may be shown to be well-defined for \( t \) small enough. However this is not more than one would have obtained by a fixed-point theorem starting from the integral form of the equation, and we get no a priori estimates for the solution allowing to extend it to larger time.

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