SCHOTTKY GROUPS CANNOT ACT ON $\mathbb{P}^2_n$ AS SUBGROUPS OF $\text{PSL}(2n + 1, \mathbb{C})$

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Abstract. In this paper we look at a special type of discrete subgroups of $\text{PSL}_{n+1}(\mathbb{C})$ called Schottky groups. We develop some basic properties of these groups and their limit set when $n > 1$, and we prove that Schottky groups only occur in odd dimensions, i.e., they cannot be realized as subgroups of $\text{PSL}_{2n+1}(\mathbb{C})$.

1. Introduction

Schottky groups play a significant role in the theory of classical Kleinian groups and Riemann surfaces (see for instance [3, 4, 5]). Their analogues in higher dimensions were introduced by Nori [8] and Seade-Verjovsky [9], though these groups were also known to N. Hitchin (see the commentary of Nori in in [8]). These are a special type of discrete groups of automorphisms of complex projective spaces having non-empty region of discontinuity, where the action is “free” with compact quotient. Hence they are a rich source for complex compact manifolds equipped canonically with a projective structure. Schottky groups also have very interesting dynamics in their limit set, the complement of the region of discontinuity. Moreover, these groups are neither Fuchsian (i.e., subgroups of $PU(n,1)$) nor affine in general. Thus, if we want to study Kleinian actions on higher dimensional complex projective spaces, Schottky groups provide a very nice starting point.

So far Schottky groups have been studied only for odd-dimensional projective spaces (in [8, 9]). It is thus natural to ask whether Schottky groups exists in even dimensions. In this paper we prove they do not: Schottky groups cannot act by complex automorphism on $\mathbb{P}^2_n$. Hence, in order to construct discrete groups of automorphisms of $\mathbb{P}^2_n$ with a rich underlying geometry and dynamics one must follow different methods. This is done for $\mathbb{P}^2$ in [11, 16, 17].

This paper is divided into four sections. In section 1 we define what Schottky groups are and we state the main result of this article. In section 2 we develop some basic dynamical and algebraic facts about Schottky groups. In section 3 we look at the limit set of infinite cyclic groups; and in section 4 we use the previous information to show that Schottky groups cannot be realized in even dimensions.

2. Notations and the Main Result

We recall that the complex projective space $\mathbb{P}^n_\mathbb{C}$ is defined as:

$$\mathbb{P}^n_\mathbb{C} = (\mathbb{C}^{n+1} - \{0\})/\sim,$$

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where \( \sim \) denotes the equivalence relation given by \( x \sim y \) if and only if \( x = \alpha y \) for some non-zero complex scalar \( \alpha \). We know that \( \mathbb{P}_n^k \) is a compact connected complex \( n \)-dimensional manifold, which is naturally equipped with the Fubini-Study metric (see for instance [2]).

If \( [\cdot]_n : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}_n^k \) represents the quotient map, then a non-empty set \( H \subset \mathbb{P}_n^k \) is said to be a projective subspace of dimension \( k \) (in symbols \( \dim_{\mathbb{C}}(H) = k \)) if there is a \( \mathbb{C} \)-linear subspace \( \tilde{H} \) of dimension \( k + 1 \) (in symbols \( \dim_{\mathbb{C}}(\tilde{H}) = k + 1 \)), such that \( \tilde{H}_{|H} = H \). Given a set of points \( P \) in \( \mathbb{P}_n^k \), we define

\[
\langle P \rangle = \bigcap \{ l \subset \mathbb{P}_n^k \mid l \text{ is a projective subspace and } P \subset l \}.
\]

So that \( \langle P \rangle \) is a projective subspace of \( \mathbb{P}_n^k \), see [2].

From now on, the symbols \( e_1, \ldots, e_{n+1} \) will either denote the elements of the standard basis in \( \mathbb{C}^{n+1} \) or their images under \([\cdot]_n \).

Consider the action of \( \mathbb{Z}_n \) (regarded as the \( n \)-roots of unity) on \( SL(n, \mathbb{C}) \) given by \( \alpha(a_{ij}) = (\alpha a_{ij}) \). The quotient \( PSL_n(\mathbb{C}) = SL(n, \mathbb{C})/\mathbb{Z}_n \) is a Lie Group whose elements are called projective transformations. Every representative \( \tilde{\gamma} \) of the coset \( \gamma = \mathbb{Z}_n \tilde{\gamma} \in PSL_n(\mathbb{C}) \) will be called a lifting of \( \gamma \). Observe that \( \gamma \in PSL_{n+1}(\mathbb{C}) \) acts on \( \mathbb{P}_n^k \) as a biholomorphic map by \( \gamma([w]_n) = [\tilde{\gamma}(w)]_n \), where \( [w]_n \in \mathbb{P}_n^k \) and \( \tilde{\gamma} \) is a lifting of \( \gamma \).

**Definition 2.1.** A subgroup \( \Gamma \leq PSL_{n+1}(\mathbb{C}) \) is called a Schottky group if:

1. There are \( 2g, g \geq 2 \), open sets \( R_1, \ldots, R_g \), \( S_1, \ldots, S_g \) in \( \mathbb{P}_n^k \) with the property that:
   a. each of these open sets is the interior of its closure;
   b. the closures of the 2g open sets are pairwise disjoint.
2. \( \Gamma \) has a generating set \( Gen(\Gamma) = \{ \gamma_1, \ldots, \gamma_g \} \) such that \( \gamma_j(R_j) = \mathbb{P}_n^k - \overline{S_j} \) for all \( 1 \leq j \leq g \), here the bar means topological closure.

From now on, \( Int(A) \) will denote the topological interior and \( \partial(A) \) the topological boundary of the set \( A \) and for each \( 1 \leq j \leq g \), \( R_j \) and \( S_j \) will be denoted by \( R^*_j, S^*_j \) respectively.

**Examples 2.2.**

1. Every classical Schottky group of Möbius transformations (see [3] [4] [5]) is Schottky in the sense of definition 2.1. Moreover by the characterization of Schottky groups acting on the Riemann sphere given by Maskit [4], it is not hard to prove that every group of Möbius transformations which is Schottky in the sense of definition 2.1 is a Schottky group in \( PSL_2(\mathbb{C}) \).
2. In [8] Nori gave the following construction of the higher-dimensional analogues of the classical Schottky groups: let \( n = 2k + 1 \), \( k > 1 \) and \( g \geq 1 \). Choose \( 2g \) mutually disjoint projective subspaces \( L_1, \ldots, L_{2g} \) of dimension \( k \) in \( \mathbb{P}_n^k \) and \( 0 < \alpha < \frac{1}{2} \). For every integer \( 1 \leq j \leq g \) choose a basis of \( \mathbb{C}^{n+1} \) so that \( L_j = \{ [z_0, \ldots, z_k = 0] - \{0\} \}_{n} \) and \( L_{g+j} = \{ [z_{k+1}, \ldots, z_n = 0] - \{0\} \}_{n} \). Define \( \phi_j : \mathbb{P}_n^k \to \mathbb{R} \) by the formula \( \phi_j[z_0, \ldots, z_n] = \frac{|z_0|^2 + \cdots + |z_k|^2}{|z_0|^2 + \cdots + |z_n|^2} \) and consider the open neighborhoods \( V_j = \{ x \in \mathbb{P}_n^k : \phi_j(x) < \alpha \} \) and \( V_{g+j} = \{ x \in \mathbb{P}_n^k : \phi_j(x) > \alpha \} \) of \( L_j \) and \( L_{g+j} \) respectively. Consider the automorphism \( \gamma_j \) of \( \mathbb{P}_n^k \) given by \( \gamma_j[z_0, \ldots, z_n] = [\lambda z_0, \ldots, \lambda z_k, z_{k+1}, \ldots, z_n] \) where \( \lambda \in \mathbb{C} \).
Lemma 2.7. For a subgroup $\Gamma$ generated by $\gamma_1, \ldots, \gamma_g$ is a Schottky group.

Theorem 1. If $\Gamma \subseteq \text{PSL}_{2n+1}(\mathbb{C})$ is a discrete subgroup, then $\Gamma$ cannot be a Schottky group acting on $\mathbb{P}^n_{\mathbb{C}}$.

2.1. Basic Properties of Schottky Groups.

Definition 2.3. For a subgroup $\Gamma \subseteq \text{PSL}_n(\mathbb{C})$ satisfying Definition 2.1 we define:

1. $F(\Gamma) = \mathbb{P}^n_{\mathbb{C}} - (\bigcup_{\gamma \in \text{Gen}(\Gamma)} R^*_{\gamma} \cup S^*_\gamma)$.
2. $\Omega(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma(F(\Gamma))$.

Example 2.4. If $\Gamma \subseteq \text{PSL}_{2n}(\mathbb{C})$ is any of the groups of the example 2.2 then $\mathbb{P}^n_{\mathbb{C}} - \Omega(\Gamma)$ is homeomorphic to $\mathbb{P}^n_{\mathbb{C}} \times \mathcal{C}$, where $\mathcal{C}$ is a Cantor set, see [5, 8, 9].

Proposition 2.5. If $\Gamma$ is a Schottky group, then:

1. $\Gamma$ is a free group generated by $\text{Gen}(\Gamma)$.
2. $\Omega(\Gamma)/\Gamma$ is a compact complex $n$-manifold and $\text{Int}(F(\Gamma))$ is a fundamental domain for the action of $\Gamma$.

Before we prove this result we state a definition and prove a technical lemma.

Definition 2.6. Let $\Gamma \subseteq \text{PSL}_n(\mathbb{C})$ be a subgroup. For an infinite subset $H \subset \Gamma$ and a non-empty, $\Gamma$-invariant open set $\Omega \subset \mathbb{P}^n_{\mathbb{C}}$, we define $\text{Ac}(H, \Omega)$ to be the closure of the set of cluster points of $HK$, where $K$ runs over all the compact subsets of $\Omega$. Recall that $p$ is a cluster point of $HK$ if there is a sequence $(g_n)_{n \in \mathbb{N}} \subset H$ of different elements and $(x_n)_{n \in \mathbb{N}} \subset K$ such that $g_n(x_n) \xrightarrow{n \to \infty} p$.

Lemma 2.7. For a subgroup $\Gamma \subseteq \text{PSL}_n(\mathbb{C})$ satisfying Definition 2.1 one has:

1. For each reduced word $w = z^n_{\epsilon_1} \cdots z^{\epsilon_2} z^1_{\epsilon_1} \in \Gamma$ one has:
   (a) If $\epsilon_n = 1$ then $w(\text{Int}(F(\Gamma))) \subset S^*_{z^n_{\epsilon_1}}$.
   (b) If $\epsilon_n = -1$ then $w(\text{Int}(F(\Gamma))) \subset R^*_{z^n_{\epsilon_1}}$.
2. Let $\gamma \in \text{Gen}(\Gamma)$. Then $R(\gamma) = \bigcap_{k \in \mathbb{N}\cup\{0\}} \gamma^k(R^*_{\gamma})$ and $S(\gamma) = \bigcap_{k \in \mathbb{N}\cup\{0\}} \gamma^k(S^*_{\gamma})$ are closed disjoint sets contained in $\mathbb{P}^n_{\mathbb{C}} - \Omega(\Gamma)$.
(3) Let \( F_k = \{ \gamma(f) : f \in F(\Gamma) \text{ and } \gamma \in \Gamma \text{ is a reduced word of Length at most } k \} \). Then \( F(\Gamma) \subset F_1(\Gamma) \subset \ldots \subset F_k(\Gamma) \subset \ldots \) and
\[
\Omega(\Gamma) = \bigcup_{k \in \mathbb{N} \cup \{0\}} \text{Int}(F_k(\Gamma)).
\]

(4) For each \( \gamma \in \text{Gen}(\Gamma) \) one has that \( \emptyset \neq \text{Ac}(\{\gamma^n\}_{n \in \mathbb{N}}, \Omega(\Gamma)) \subset S(\gamma) \) and \( \emptyset \neq \text{Ac}(\{\gamma^{-n}\}_{n \in \mathbb{N}}, \Omega(\Gamma)) \subset R(\gamma) \).

**Proof.** Let us proceed by induction on the length of the reduced words. Clearly the case \( k = 1 \) is done by the definition of Schottky group. Now assume we have proven the statement for \( j = k \). Let \( w = z_{k+1}^{\epsilon_k+1} \ldots z_1^{\epsilon_1} \) be a reduced word and \( x \in \text{Int}(F(\Gamma)) \). By the inductive hypothesis we deduce that \( z_{k+1}^{\epsilon_k+1}w(x) \in \mathbb{P}_C^n - R_{z_{k+1}} \) if \( \epsilon_{k+1} = 1 \) and \( z_{k+1}^{\epsilon_k+1}w(x) \in \mathbb{P}_C^n - R_{z_{k+1}} \) if \( \epsilon_{k+1} = -1 \). Now the proof follows by the definition of Schottky group.

Let \( \gamma \in \text{Gen}(\Gamma) \). Since \( \gamma^m(S_\gamma) \subset \gamma^{m-1}(S_\gamma) \) we deduce that \( \bigcap_{m \in \mathbb{N}} \gamma^m(S_\gamma) \subset \bigcap_{m \in \mathbb{N}} \gamma^{m-1}(S_\gamma) = S(\gamma) \). To conclude observe that:
\[
S(\gamma) = \bigcap_{m \in \mathbb{N}} \gamma^{m-1}(S_\gamma) \subset \bigcap_{m \in \mathbb{N}} \gamma^{m-2}(S_\gamma) \subset \bigcap_{m \in \mathbb{N}} \gamma^{m-3}(S_\gamma).
\]

We will prove that that \( F(\Gamma) \subset \text{Int}(F_1(\Gamma)) \). Let \( x \in \partial(F(\Gamma)) \), then there is a reduced word \( \gamma_0 \in \text{Gen}(\Gamma) \) such that \( x \in \partial S_{\gamma_0} \). Define \( r_1 = \min\{d(x, \gamma_0(S_{\gamma_0})) : \gamma \in \text{Gen}(\Gamma)\} \), \( r_2 = \min\{d(x, R_{\gamma_0}) : \gamma \in \text{Gen}(\Gamma)\} \), \( r_3 = \min\{d(x, R_{\gamma_0}) : \gamma \in \text{Gen}(\Gamma) - \{\gamma_0\}\} \), \( r_4 = \min\{d(x, S_{\gamma_0}) : \gamma \in \text{Gen}(\Gamma) - \{\gamma_0\}\} \) and \( r = \min\{r_1, r_2, r_3, r_4\} \) (here \( d \) denotes the Fubini-Study metric). Clearly \( r > 0 \). Now, let \( y \in B_{r/4}(x) \cap \mathbb{P}_C^n \) and by the definition of \( r \) we have that \( y \in F(\Gamma) \cup \gamma(\Gamma) \). If \( y \in B_{r/2}(x) \cap \mathbb{P}_C^n \) then by definition of \( r \) we deduce \( y \in F(\Gamma) \). In other words, we have shown \( F(\Gamma) \subset \text{Int}(F_1(\Gamma)) \). Therefore:
\[
F_k(\Gamma) \subset \{ \gamma(f) : \gamma \text{ is a reduced word of length at most } k \} \text{ and } \text{Int}(F_1(\Gamma)) \subset F_{k+1}(\Gamma)
\]
i.e., \( F_k(\Gamma) \subset \text{Int}(F_{k+1}(\Gamma)) \). To conclude the proof observe that:
\[
\Omega(\Gamma) = \bigcup_{k \in \mathbb{N} \cup \{0\}} F_k(\Gamma) \subset \bigcup_{k \in \mathbb{N} \cup \{0\}} \text{Int}(F_{k+1}(\Gamma)) \subset \bigcup_{k \in \mathbb{N} \cup \{0\}} \text{Int}(F_k(\Gamma)).
\]

Let \( K \subset \Omega(\Gamma) \) be a compact set and \( x \) a cluster point of \( \{\gamma^m(K)\}_{m \in \mathbb{N}} \). Then there is a subsequence \( (n_m)_{m \in \mathbb{N}} \subset (m)_{m \in \mathbb{N}} \) and a sequence \( (x_m)_{m \in \mathbb{N}} \subset K \) such that \( \gamma^{n_m}(x_m) \xrightarrow{m} x \). In case \( x \notin S(\gamma) \) it is deduced that there is \( k_0 \in \mathbb{N} \) such that \( x \notin \gamma^{k_0}(S_\gamma) \). Taking \( r = d(x, \gamma^{k_0}(S_\gamma)) \) we have that:
\[
B_{r/2}(x) \cap \gamma^{k_0}(S_\gamma) = \emptyset.
\]

On the other hand, observe that since \( K \) is compact, by part (3) of the present lemma there is \( l_0 \in \mathbb{N} \) such that \( K \subset F_{l_0}(\Gamma) \); also observe that since \( (n_m)_{m \in \mathbb{N}} \) is an strictly increasing sequence, there is \( k_1 \in \mathbb{N} \) such that \( n_m > l_0 + 1 + k_0 \) for \( m > k_1 \). With these facts in mind we deduce \( \gamma^{k_0+1}(K) \subset S_\gamma \) and therefore:
\[
\gamma^{n_m}(x_m) \in \gamma^{n_m-l_0-1}(S_\gamma) \subset \gamma^{k_0}(S_\gamma) \text{ for } m > k_1.
\]
Hence \( x \notin \gamma^{k_0}(S_\gamma) \), which contradicts (2.1). Thus \( \emptyset \neq \text{Ac}(\{\gamma^n\}_{n \in \mathbb{N}}, \Omega(\Gamma)) \subset S(\gamma) \). Observe that similar arguments prove also \( \emptyset \neq \text{Ac}(\{\gamma^{-n}\}_{n \in \mathbb{N}}, \Omega(\Gamma)) \subset R(\gamma) \).
Proof of proposition 2.7

(1) Assume there is a reduced word $h$ with length $> 0$ such that $h = Id$. Now, let $x \in \text{Int}(F(\Gamma))$, then by part (1) of lemma 2.7, $x = h(x) \in \bigcup_{\gamma \in \text{Gen}(\Gamma)} (R_{\gamma}^* \cup S_{\gamma}^*)$, which contradicts the choice of $x$. Therefore $\Gamma$ is free.

(2) Let $K \subset \Omega(\Gamma)$ be a compact set, then by part (3) of lemma 2.7, there is $k \in \mathbb{N}$ such that $K \subset F_k(\Gamma)$. Assume there is a word $w$ with length $\geq 2k + 2$ such that $w(F_k(\Gamma)) \cap F_k(\Gamma) \neq \emptyset$. So there are $x_1, x_2 \in F(\Gamma)$ and words $w_1, w_2$ of length at most $k$ such that $x_1 = w_1^{-1}w_2^{-1}w_2x_2$. On the other hand $w_1^{-1}w_2^{-1}w_2$ is a word with length $\geq 2$. By (1) of lemma 2.7, $x_1 = w_1^{-1}w_2^{-1}(w_2(x_2)) \in \bigcup_{g \in \text{Gen}(\Gamma)} S_j^g \cup R_j^g$, but this contradicts the choice of $x_1$. Therefore $\Gamma$ acts properly discontinuously and freely on $\Omega(\Gamma)$. \hfill \square

Remark 2. All the results in this section remain valid if we change $\mathbb{P}_c^n$ for $\mathbb{P}_\mathbb{R}^n$.

3. Dynamics of Projective Transformations

Lemma 3.1. Let $V$ be a $\mathbb{C}$-linear space with $\text{dim}_\mathbb{C}(V) = n$, $T : V \to V$ an invertible linear transformation and $\lambda \in \mathbb{C}$ such that $|\alpha| < |\lambda|$ for every eigenvalue $\alpha$ of $T$.

For every $l \in \mathbb{N}$ we have uniform convergence $\lambda^{-m} \left( \begin{array}{c} m \\ l \end{array} \right) T^m \xrightarrow{m \to \infty} 0$ on compact subsets of $V$.

Here $\left( \begin{array}{c} m \\ l \end{array} \right)$ denotes the number of sets with $l$ elements from a set with $m$ elements.

Proof. Decomposing $T$ into one or more Jordan blocks according to Jordan’s Normal Form Theorem we reduce the problem to the case where there is $0 < |\lambda| < 1$ and an ordered basis $\beta = \{v_1, \ldots, v_n\}$, $n \geq 2$, such that the matrix of $T$ with respect to $\beta$ (in symbols $[T]_{\beta}$) satisfies:

$$[T]_{\beta} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$ 

An inductive argument shows that for all $m > n$:

$$[T^m]_{\beta} = \begin{pmatrix} \lambda^m & \left( m \atop 1 \right) \lambda^{m-1} & \left( m \atop 2 \right) \lambda^{m-2} & \cdots & \left( m \atop n-1 \right) \lambda \alpha^{m+1-n} \\ 0 & \lambda^m & \left( m \atop 1 \right) \lambda^{m-1} & \cdots & \left( m \atop n-2 \right) \lambda \alpha^{m+2-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^m \end{pmatrix}.$$ 

For a compact subset $K \subset V$ set $\sigma(K) = \sup \{ \sum_{j=1}^n |\alpha_j| : \sum_{j=1}^n \alpha_j v_j \in K \}$. Let $z \in K$, $z = \sum_{j=1}^n \alpha_j v_j$, then by equation (3.1) we deduce:

$$|T^m(z)| \leq \sigma(K) \max \{|v_j| : 1 \leq j \leq n\} \sum_{j=1}^n \sum_{k=0}^{j-1} \left( \begin{array}{c} m \\ k \end{array} \right) |\alpha^{m-k}|.$$
Hence it is sufficient to observe that:
\[
\left| \begin{pmatrix} m \\ l \\ k \end{pmatrix} \right| \alpha^{m-k} \leq m^{2\max\{k,l\}} |\alpha|^{m-k} \quad \text{as } m \to \infty.
\]
\[\square\]

**Definition 3.2.** Let \( V \) be a \( \mathbb{C} \)-linear space with \( \dim_{\mathbb{C}}(V) = n \) and let \( T : V \to V \) be a \( \mathbb{C} \)-linear transformation. We define \( \text{Eve}(T) = \{(v \in V : v \text{ is an eigenvector of } T)\} \).

Where \( \langle \{v \in V : v \text{ is an eigenvector of } T\} \rangle \) will denote the linear subspace generated by the eigenvectors of \( T \).

**Lemma 3.3.** Let \( l, k \in \mathbb{N} \cup \{0\} \) with \( l < k \). Then
\[
\left( \begin{pmatrix} m \\ l \\ k \end{pmatrix} \right)^{-1} \quad \text{as } m \to \infty \quad 0.
\]

**Proof.**
\[
\left( \begin{pmatrix} m \\ l \\ k \end{pmatrix} \right)^{-1} = \prod_{j=l}^{k-1} \left( \begin{pmatrix} m \\ j \\ m-j \end{pmatrix} \right) \leq \left( \begin{pmatrix} k \\ m \end{pmatrix} \right) \quad \text{as } m \to \infty \quad 0.
\]

**Lemma 3.4.** Let \( V \) be a \( \mathbb{C} \)-linear space with \( \dim_{\mathbb{C}}(V) = n > 1 \) and let \( T : V \to V \) be an invertible linear transformation such that there are \( \lambda \in \mathbb{C} \), with \( |\lambda| = 1 \), and an ordered basis \( \beta = \{v_1, \ldots, v_n\} \) for which:
\[
[T]_\beta = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \lambda
\end{pmatrix}
\]

that is \( [T]_\beta \) is a \( n \times n \)-Jordan block. Then for every \( v \in V - \{0\} \) there is a unique \( k(v, T) \in \mathbb{N} \cup \{0\} \) such that the set of cluster points of \( \left\{ \left( \begin{pmatrix} m \\ k(v, T) \end{pmatrix} \right)^{-1} T^m(v) \right\} \)\( m \in \mathbb{N} \)
lies in \( \langle \{v_1\} \rangle - \{0\} \).

**Proof.** Let \( z = \sum_{j=0}^{n} \alpha_j v_j \) and \( k(z, T) = \max\{1 \leq j \leq n : \alpha_j \neq 0\} - 1 \), then we have that:
\[
\left( \begin{pmatrix} m \\ k(v, T) \end{pmatrix} \right)^{-1} T^m(z) = \sum_{j=1}^{n} \left( \begin{pmatrix} m \\ k \end{pmatrix} \right) \left( \begin{pmatrix} m \\ k(v, T) \end{pmatrix} \right)^{-1} \lambda^{m-k} \alpha_{k+j} v_j
\]
The result now follows from lemma 3.3. \[\square\]

**Corollary 3.5.** Let \( V \) be a \( \mathbb{C} \)-linear space with \( \dim_{\mathbb{C}}(V) = n \) and let \( T : V \to V \) be a linear transformation such that there are \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \), with \( |\alpha_j| = 1 \) for each \( 0 \leq j \leq 0 \), and an ordered basis \( \beta = \{v_1, \ldots, v_n\} \) for which \( T(\sum_{j=0}^{n} \beta_j v_j) = \sum_{j=0}^{n} \alpha_j \beta_j v_j \). Then \( k(v, T) = 0 \) is the unique positive integer for which the set of cluster points of \( \left\{ \left( \begin{pmatrix} m \\ k(v, T) \end{pmatrix} \right)^{-1} T^m(v) \right\} \)\( m \in \mathbb{N} \)
lies on \( V - \{0\} \).

**Corollary 3.6.** Let \( V \) be a \( \mathbb{C} \)-linear space with \( \dim_{\mathbb{C}}(V) = n \) and let \( T : V \to V \) be an invertible linear transformation such that each of its eigenvalues is a unitary complex number. Then for every \( v \in V - \{0\} \) there is a unique \( k(v, T) \in \mathbb{N} \cup \{0\} \) for which the set of cluster points of \( \left\{ \left( \begin{pmatrix} m \\ k(v, T) \end{pmatrix} \right)^{-1} T^m(v) \right\} \)\( m \in \mathbb{N} \)
lies in \( \text{Eve}(T) - \{0\} \).
Proof. By the Jordan’s Normal Form Theorem there are \( k \in \mathbb{N}; V_1, \ldots, V_k \subset V \) linear subspaces and \( T_i : V_i \rightarrow V_i, 1 \leq i \leq k \) such that:

1. \( \bigoplus_{j=1}^{k} V_j = V \).
2. For each \( 1 \leq i \leq k \), \( T_i \) is a non-zero \( \mathbb{C} \)-linear map whose eigenvalues are unitary complex numbers.
3. \( \bigoplus_{j=1}^{k} T_j = T \).
4. For each \( 1 \leq i \leq k \), \( T_i \) is either diagonalizable or \( n_i = dim_{\mathbb{C}} > 1 \); \( V_i \) contains an ordered basis \( \beta_i \) for which \( [T]_{\beta} \) is a \( n_i \times n_i \)-Jordan block.

Let \( v \in V - \{0\} \) then there is a non empty finite set \( W \subset \bigcup_{j=1}^{k} V_j - \{0\} \) such that \( v = \sum_{w \in W} w \). Now, take \( i : W \rightarrow \mathbb{N} \) where \( i(w) \) is the unique element in \( \{1, \ldots, k\} \) such that \( w \in V_{i(w)} \), \( k(v, T) = \max\{k(w, T_{i(w)}): w \in W\}, W_1 = \{w \in W: k(w, T_{i(w)}) < k(v, T)\} \) and \( W_2 = W - W_1 \) then:

\[
T^m(v) = \sum_{w \in W_1} \left( \frac{m}{k(v, T)} \right) T^m(w)_{i(w)} + \sum_{w \in W_2} \left( \frac{m}{k(w, T_{i(w)})} \right) T^m(w)_{i(w)}.
\]

The result now follows from equation (3.2), lemmas 3.3 and 3.4 and corollary 3.5. \( \square \)

**Definition 3.7.** Let \( \gamma \in PSL_n(\mathbb{C}) \) be an element of infinite order and let \( \tilde{\gamma} \) be a lifting of \( \gamma \). Then we define:

1. \( |Eva(\gamma)| = \{|\lambda| \in \mathbb{R}: \lambda \) is an eigenvalue of \( \tilde{\gamma}\} \)
2. \( L_r(\gamma) = \langle \{v \in \mathbb{C}^n: v \) is an eigenvector of \( \tilde{\gamma} \) and \( |\tilde{\gamma}(v)| = r \) \rangle \).
3. \( L(\gamma) \) as the closure of accumulation points of \( \{\gamma^m(z)\}_{m \in \mathbb{Z}} \) where \( z \in \mathbb{P}^n_\mathbb{C} \).

Clearly parts 1 and 2 of this definition do not depend on the choice of \( \tilde{\gamma} \).

**Proposition 3.8.** Let \( \gamma \in PSL_{n+1}(\mathbb{C}) \) be an element of infinite order, then:

\[ L(\gamma) = \bigcup_{r \in |Eva(\Gamma)|} L_r(\gamma). \]

**Proof.** Since \( \bigcup_{r \in |Eva(\Gamma)|} L_r(\gamma) \subset L(\Gamma) \) is trivially verified, it is enough to check that \( L(\gamma) \subset \bigcup_{r \in |Eva(\Gamma)|} L_r(\gamma) \). Let \( \tilde{\gamma} \) be a lifting of \( \gamma \), then by the Jordan’s Normal Form Theorem there are \( k \in \mathbb{N}; V_1, \ldots, V_k \subset \mathbb{C}^{n+1} \) linear subspaces; \( \gamma_i : V_i \rightarrow V_i, 1 \leq i \leq k \) and \( r_1, \ldots, r_k \in \mathbb{R} \) which satisfy:

1. \( \bigoplus_{j=1}^{k} V_j = \mathbb{C}^{n+1} \).
2. For each \( 1 \leq i \leq k \), \( \gamma_i \) is a non-zero \( \mathbb{C} \)-linear map whose eigenvalues are unitary complex numbers.
3. \( 0 < r_1 < r_2 < \ldots, < r_k \).
4. \( \bigoplus_{j=1}^{k} r_j \gamma_j = \tilde{\gamma} \).

In what follows \( (\gamma_i, \{V_i\}_{i=1}^{k}, \{r_i\}_{i=1}^{k}) \) will be called a decomposition for \( \gamma \). Now let \( [v]_n \in \mathbb{P}^n_\mathbb{C}, \) thus \( v = \sum_{j=1}^{k} v_j \) where \( v_j \in V_j \). Set \( j_0 = \max\{1 \leq j \leq k: v_j \neq 0\} \). One has:

\[
\left( \frac{m}{k(v_{j_0}, T_{j_0})} \right)^{-1} \frac{\tilde{\gamma}^m(v)}{r^m_{j_0}} = \sum_{j=1}^{k} \left( \frac{m}{k(v_{j}, T_{j})} \right)^{-1} \frac{r^m_{j} \gamma_j^m(v_j)}{r^m_{j_0}}.
\]
By equation 3.3 lemma 3.3 and corollary 3.6 we conclude that the set of cluster points of \( \{ \gamma^m(v) \}_{m \in \mathbb{Z}} \) lies in \( [\text{Eve}(\gamma_{j_0}) - 0]_n = L_{r_{j_0}}(\gamma) \).

\[ \square \]

4. Proof of the Main Theorem

**Lemma 4.1.** Let \( \Gamma \leq PSL_{2n+1}(\mathbb{C}) \) be a group and \( \Omega \) a non-empty, \( \Gamma \)-invariant open set where \( \Gamma \) acts properly discontinuously and such that whenever \( l \) is a projective subspace contained in \( \mathbb{P}_n - \Omega \) then \( \dim_C(l) < n \). Then for every \( \gamma \in \Gamma \) with infinite order there is a connected set \( L(\gamma) \subset Ac(\{ \gamma \}_{m \in \mathbb{Z}}, \Omega) \cup L(\gamma) \) such that \( L(\gamma) \subset L(\gamma) \).

**Proof.** Let \( \gamma \in \Gamma \) be an element with infinite order, and choose a decomposition \( (\tilde{\gamma}, k, \{ V_i \}_{i=1}^k, \{ \gamma_i \}_{i=1}^k, \{ r_i \}_{i=1}^k) \) for \( \gamma \). Take \( j_0 = \min\{1 \leq j \leq k : \sum_{i=1}^j \dim_C(V_i) \geq n + 1\} \). From proposition 3.8 we can assume that \( k \geq 2 \). For the moment let us assume that \( j_0 \neq 1, k \). Observe that since \( \sum_{i=1}^{j_0} \dim_C(V_i) \geq n + 1 \) we conclude that there is \( w = \sum_{i=1}^{j_0} w_j \in \bigoplus_{j=1}^{j_0} V_j \) non-zero, where \( w_j \in V_j \), such that \( [w]_{2n} \in \Omega \) and since \( \Omega \) is open we can assume that \( w_{j_0} \) is non-zero. Now, let \( z \in \bigoplus_{j>j_0} V_j - \{0\} \) then by lemma 3.1

\[
w_m(z) = \left[w + \left(\frac{m}{k(w, \gamma_{j_0})}\right) \sum_{j>j_0} \frac{r_{j_0}}{r_j} \gamma_j^{-m}(z)\right]_{2n} \rightarrow_{m \rightarrow \infty} [w]_{2n}\]

thus for \( m(z) \) large \( (w_m(z))_{m \geq m(z)} \subset \Omega \). On the other hand, by corollary 3.6 there is an strictly increasing sequence \( (n_m)_{m \in N} \subset N \) and \( w_0 \in \text{Eve}(\gamma_{j_0}) - \{0\} \) such that:

\[
\left(\frac{n_m}{k(w_{j_0}, \gamma_{j_0})}\right)^{-1} \gamma_{j_0}^{n_m}(w_0) \rightarrow_{m \rightarrow \infty} w_0.
\]

From here and lemma 3.1 we deduce that:

\[
\gamma^{n_m}(w_{m}) = \left[\left(\frac{n_m}{k(w_{j_0}, \gamma_{j_0})}\right)^{-1} \sum_{j \leq j_0} \frac{r_j}{r_{j_0}} \gamma_j^{n_m}(w_j) + z\right]_{2n} \rightarrow_{m \rightarrow \infty} [w_0 + z]_{2n}.
\]

From here it follows that:

\[
\bigcup_{j > j_0} L_{r_j}(\gamma) \subset ([w_0]_{2n}, \bigoplus_{j > j_0} V_j - \{0\}]_{2n}) \subset Ac(\{ \gamma^m \}_{m \in \mathbb{Z}}, \Omega) \cup L(\gamma).
\]

To conclude consider the following observations:

**Obs. 1** Observe that in the previous argument, the assumption \( j \neq k \) is not crucial, so for the case \( j = 1 \) it is verified that there is \( w_1 \in L(r_1) \) such that

\[
\bigcup_{j > 1} L(r_j) \subset ([w_1, \bigoplus_{j > 1} V_j - \{0\}]_{2n}) \subset Ac(\{ \gamma^m \}_{m \in \mathbb{Z}}, \Omega) \cup L(\gamma)
\]

thus in case \( j = 1 \) it is enough to take

\[
L(\gamma) = ([w_1, \bigoplus_{j > 1} V_j - \{0\}]_{2n}) \cup L_{r_1}(\gamma).
\]

**Obs. 2** Applying the same argument to \( \gamma^{-1} \) in the case \( j_0 \neq 1, k \), it is deduced that there is \( v \in L(r_{j_0}) \) such that:

\[
\bigcup_{j < j_0} L(r_j) \subset ([v, \bigoplus_{j < j_0} V_j - \{0\}]_{2n}) \subset Ac(\{ \gamma^m \}_{m \in \mathbb{Z}}, \Omega).
\]
Therefore in this case it is enough to take

\[ \mathcal{L}(\gamma) = \langle v, \bigoplus_{j < j_0} V_j - \{0\}\rangle_2 > \cup < [w_0]_{2n}, \bigoplus_{j > j_0} V_j - \{0\}\rangle_2 > \cup L_{r_j} (\gamma). \]

Obs. 3 To obtain the result in the case \( j = k \) it is enough to apply the same argument used in Obs. 1 to \( \gamma^{-1} \).

\[ \square \]

**Lemma 4.2.** If \( \Gamma \leq PSL_{2n+1}(\mathbb{C}) \) is a Schottky group then \( \mathbb{P}^n_c - \Omega(\Gamma) \) does not contain a projective subspace \( V \) with \( dim_C(V) \geq n \).

**Proof.** If \( V \subset \mathbb{P}^n_c - \Omega(\Gamma) \) is a projective subspace with \( dim_C(V) \geq n \), then:

\[ V \subset \mathbb{P}^n_c - \Omega(\Gamma) = \mathbb{P}^n_c - \bigcup_{\gamma \in \Gamma} \gamma(F(\Gamma)) \subset \mathbb{P}^n_c - F(\Gamma) = \bigcup_{\gamma \in Gen(\Gamma)} \gamma_*(R^* \cup \Sigma^*_\gamma). \]

Since \( V \) is connected and \( (V \cup \bigcup_{\gamma \in Gen(\Gamma)} R^*_\gamma, V \cup \bigcup_{\gamma \in Gen(\Gamma)} \Sigma^*_\gamma) \) is a disconnection for \( V \) we deduce that \( V \subset \bigcup_{\gamma \in Gen(\Gamma)} R^*_\gamma \) or \( V \subset \bigcup_{\gamma \in Gen(\Gamma)} \Sigma^*_\gamma \). Moreover by an inductive argument we deduce that there is \( \gamma_0 \in Gen(\Gamma) \) such that \( V \subset R^*_{\gamma_0} \) or \( V \subset \Sigma^*_{\gamma_0} \). For simplicity let us assume that \( V \subset \Sigma^*_{\gamma_0} \). Taking \( \sigma \in Gen(\Gamma) \setminus \{\gamma_0\} \) we have:

\[ (4.1) \quad \sigma^{-1}(V) \subset \sigma^{-1}(\Sigma^*_{\gamma_0}) \subset \sigma^{-1}(\mathbb{P}^n_c - \Sigma^*_{\sigma_0}) = R^*_\sigma. \]

Observe that \( V \) and \( \sigma^{-1}V \) are projective subspaces with \( dim_C(V) + dim_C(\sigma^{-1}V) \geq 2n \) then \( V \cap \sigma^{-1}(V) \neq \emptyset \). However, this is a contradiction since by equation \( 4.1 \) we have that \( V \cap \sigma^{-1}(V) \subset R^*_\sigma \cap \Sigma^*_{\gamma_0} = \emptyset \)

\[ \square \]

**Proof of Theorem 7.** Assume that there is a group \( \Gamma \leq PSL_{2n+1}(\mathbb{C}) \) which is a Schottky group and let \( \gamma \in Gen(\Gamma) \). By lemma \( 4.1 \) there is a connected set \( \mathcal{L}(\gamma) \) such that \( L(\gamma) \subset L(\gamma) \subset \mathcal{L}(\gamma) \subset \mathcal{L}(\gamma) \) \( \subset \mathcal{L}(\gamma) \subset R(\gamma) \). On the other hand by \( 4.1 \) of lemma \( 2.7 \) we have \( Ac(\gamma, \Omega(\Gamma)) \subset S(\gamma) \cup R(\gamma) \). Since \( (R(\gamma) \cap \mathcal{L}(\gamma), S(\gamma) \cap \mathcal{L}(\gamma)) \) is a disconnection for \( \mathcal{L}(\gamma) \) we deduce \( \mathcal{L}(\gamma) \subset R(\gamma) \) or \( \mathcal{L}(\gamma) \subset S(\gamma) \). This implies \( L(\gamma) \cap S(\gamma) = \emptyset \) or \( L(\gamma) \cap R(\gamma) = \emptyset \). However this contradicts \( 4.1 \) of lemma \( 2.7 \) Therefore \( \Gamma \) cannot be a Schottky group.

\[ \square \]

**Remark 3.** (1) If in definition \( 2.1 \) we allow that \( R_j = S_j \) and \( \gamma_j^2 = Id \) for \( 1 \leq j \leq g \), the resulting group is a type of Complex Kleinian Group (see \( 9 \)), and by means of theorem \( 11 \) is not hard to see that for \( g \geq 3 \) this type of groups cannot be realized as subgroups of \( PSL_{2n+1}(\mathbb{C}) \).

(2) Theorem \( 11 \) remains valid if we change \( \mathbb{C} \) by \( \mathbb{R} \).

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