AFFINE WREATH PRODUCT ALGEBRAS

ALISTAIR SAVAGE

ABSTRACT. We study the structure and representation theory of affine wreath product algebras and their cyclotomic quotients. These algebras, which appear naturally in Heisenberg categorification, simultaneously unify and generalize many important algebras appearing in the literature. In particular, special cases include degenerate affine Hecke algebras, affine Sergeev algebras (degenerate affine Hecke–Clifford algebras), and wreath Hecke algebras. In some cases, specializing the results of the current paper recovers known results, but with unified and simplified proofs. In other cases, we obtain new results, including proofs of two open conjectures of Kleshchev and Muth.

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1. INTRODUCTION

Affine Hecke algebras and their degenerate analogs play a vital role in the representation theory of Lie algebras. Over the past couple of decades, several modified versions of degenerate affine Hecke algebras have appeared. In particular, a super analog arose in the work of Nazarov [Naz97] studying the projective representation theory of the symmetric group, and Wan and Wang [WW08] introduced the so-called wreath Hecke algebras associated to an arbitrary finite group $G$.

Recently, degenerate affine Hecke algebras and their analogs have appeared in Heisenberg categorification. The degenerate affine Hecke algebra itself appeared in the endomorphism algebras of certain objects in Khovanov’s conjectural categorification of the Heisenberg algebra [Kho14]. Then, a certain $\mathbb{N}$-graded superalgebra appeared in the Heisenberg categorification of Cautis and Licata [CL12, §10.3] satisfying relations similar to those of the degenerate affine Hecke algebra. More generally, Rosso and the author developed a Heisenberg categorification depending on an arbitrary $\mathbb{N}$-graded Frobenius superalgebra $F$. Specializing $F$ recovers the categorifications of Khovanov and Cautis–Licata. Again, in the endomorphism algebras of these categories, one sees algebras depending on $F$, and satisfying relations similar to those of the degenerate affine Hecke algebra. In the special

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case where $F$ is a symmetric algebra and purely even, these algebras have recently been studied by Kleshchev and Muth [KM]. Related algebras were also considered in [CG].

In the current paper, we investigate the aforementioned algebras appearing in the categorification of [RS17]. In particular, to every nonnegative integer $n$ and every $\mathbb{N}$-graded Frobenius superalgebra $F$ over a commutative ring $\mathbb{k}$, we associate an affine wreath product algebra $A_n(F)$, which is an affine version of the wreath product algebra $F^\otimes n \rtimes S_n$. As an $\mathbb{N} \times \mathbb{Z}_2$-graded $\mathbb{k}$-module we have $A_n(F) \cong \mathbb{k}[x_1, \ldots, x_n] \otimes_{\mathbb{k}} (F^\otimes n \rtimes S_n)$. The factors $\mathbb{k}[x_1, \ldots, x_n]$ and $F^\otimes n \rtimes S_n$ are subalgebras, and the relations between these two factors are given in Definition 3.1. In particular, the relations between the $x_i$ and the elements of $F^\otimes n$ involve the Nakayama automorphism of $F$, which is trivial if and only if $F$ is a symmetric algebra. Remarkably, affine wreath product algebras turn out to provide an appropriate setting for the simultaneous unification and generalization of all of the analogs of degenerate affine Hecke algebras mentioned above. In particular, we have the following:

(a) When $F = \mathbb{k}$, $A_n(F)$ is the degenerate affine Hecke algebra.
(b) When $F$ is the two-dimensional Clifford superalgebra $\text{Cl}$, $A_n(F)$ is the affine Sergeev algebra (also called the degenerate affine Hecke–Clifford algebra) of Nazarov [Naz97]. Here the Nakayama automorphism of $F$ is nontrivial.
(c) When $F$ is the group algebra of a group, $A_n(F)$ is the wreath Hecke algebra of Wan and Wang [WW08]. For $n = 2$, this algebra was also defined in [Pus97, §3].
(d) When $F$ is a symmetric algebra (i.e. the Nakayama automorphism is trivial) concentrated in even parity, $A_n(F)$ is the affinized symmetric algebra considered by Kleshchev and Muth [KM, §3]. If $F$ is commutative, closely related algebras were defined in [CG, §4.2]. (See [KM, Rem. 3.6] for the precise relation.)

We show in the current paper that, despite their great level of generality, the setting of wreath product algebras turns out to be very tractable. In particular, it is possible to deduce a great deal of the structure and representation theory of these algebras. Specializing the Frobenius algebra $F$ then recovers known results in some cases while, in other cases, yields new results and proofs of open conjectures.

We now give an overview of the structure of the paper and highlight the main results. After discussing some necessary background on superalgebras, Frobenius algebras, and smash products in Section 2, we give the definition of affine wreath product algebras in Section 3. We also explain there how these algebras recover the special cases mentioned above, and deduce some automorphisms and additional relations that will be used in proofs throughout the paper.

In Section 4, we study the structure theory of the affine wreath product algebras $A_n(F)$. We begin by introducing some operators that can be viewed as Frobenius algebra deformations of divided difference operators. We then describe an explicit basis of $A_n(F)$ in Theorem 4.6. After a brief discussion of a natural filtration on $A_n(F)$ and the associated graded algebra (Section 4.3), we describe the center of $A_n(F)$ in Theorem 4.14. We then introduce analogs of Jucys–Murphy elements (see (4.22)), state a Mackey Theorem for induction/restriction (Theorem 4.22), and introduce intertwining elements (Section 4.7). For certain choices of Frobenius algebra $F$, the results of this section recover results for the above-mentioned special cases of affine wreath product algebras. However, the benefit of the current approach is that the proofs are completely uniform. For example, we are able to unify the treatments of these subjects in Parts I and II of the book [Kle05] by keeping track of the Nakayama automorphism that appears in our general definition.

In Section 5, we turn our attention to representation theory. Inspired by the methods of [WW08], we classify the simple $A_n(F)$-modules when $\mathbb{k}$ is an algebraically closed field (Theorem 5.19) by giving an explicit equivalence of categories that reduces the classification to the known cases of degenerate affine Hecke algebras, affine Sergeev algebras, and wreath product algebras. Outside of cases (a), (b), and (c) above, this classification appears to be new.
In Section 6, we define cyclotomic quotients $\mathcal{A}_n^C(F)$ of the algebras $\mathcal{A}_n(F)$ and find explicit bases of these quotients (Theorem 6.11). In cases (a), (b), and (c) above, Theorem 6.11 recovers known results, but with a unified proof. In other cases, it appears to be new. In particular, in case (d) above, it implies an open conjecture of Kleshchev and Muth (Corollary 6.12). We then prove that the cyclotomic quotients are themselves $\mathbb{N}$-graded Frobenius superalgebras, and give an explicit description of the associated Nakayama automorphism (Theorem 6.15). In case (a), this recovers a known result, but with a more direct proof. When $F$ is the group algebra of a finite cyclic group, Theorem 6.15 also recovers a known result. However, in other cases, even where $F$ is CI (case (b)) or the group algebra of a more general finite group, the result appears to be new. In particular, in case (d), Theorem 6.15 implies another open conjecture of Kleshchev and Muth (Corollary 6.16). We then state a cyclotomic Mackey Theorem (Theorem 6.18). Finally, under some restrictions on the cyclotomic quotient parameter $C$ allowing us to view $\mathcal{A}_n^C(F)$ as a subalgebra of $\mathcal{A}_{n+1}^C(F)$, we prove that $\mathcal{A}_n^C(F)$ is a Frobenius extension of $\mathcal{A}_n^C(F)$ (Theorem 6.19). In particular, induction is both left and right adjoint to restriction up to degree shift (Corollary 6.20)—a property that plays a vital role in categorification.

The fact that affine wreath product algebras are so general, yet one is able to deduce so much of their structure and representation theory explicitly, leads us to believe that these algebras will play an important role in many of the areas where degenerate affine Hecke algebras and their analogs appear. We conclude the paper in Section 7 with a brief discussion of some such future directions. For the benefit of the reader, in Appendix A we summarize the notation used in the paper, including an index of notation.

Note on the arXiv version. For the interested reader, the tex file of the arXiv version of this paper includes hidden details of some straightforward computations and arguments that are omitted in the pdf file. These details can be displayed by switching the details toggle to true in the tex file and recompiling.

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2. Preliminaries

We recall here some basic facts about graded associative superalgebras and graded Frobenius superalgebras. Throughout, $k$ is an arbitrary commutative ring of characteristic not equal to 2, unless otherwise stated.

2.1. Graded superalgebras and their modules. Throughout this paper, we will use the term algebra to mean $\mathbb{Z}$-graded associative $k$-superalgebra. Similarly, the term module will mean $\mathbb{Z}$-graded left supermodule. (In the statement of theorems, etc. we will sometimes include the words "graded" and "super" for reference purposes.) We use the term degree to refer to the $\mathbb{Z}$-grading and parity to refer to the $\mathbb{Z}_2$-grading. Similarly, degree shift refers to a shift in the $\mathbb{Z}$-grading. We assume that algebras are concentrated in nonnegative degree (i.e. they are $\mathbb{N}$-graded). For a homogeneous element $a$ of an algebra or module, we denote its parity by $\bar{a}$ and its degree by $|a|$. Whenever we give definitions involving the parity, it is understood that we extend by linearity to nonhomogeneous elements.

Let $A$ be an algebra. The opposite algebra $A^{op}$ has the same underlying $k$-module structure, but multiplication is given by

$$a \cdot b = (-1)^{\bar{a} \bar{b}} ba, \quad a, b \in A^{op},$$

where juxtaposition denotes the original multiplication in $A$. 


For $A$-modules $M$ and $N$, we define $\text{HOM}_k(M, N)$ to be the space of all $k$-linear maps from $M$ to $N$. For $i \in \mathbb{Z}$ and $\varepsilon \in \mathbb{Z}_2$, we let

$$\text{HOM}_A(M, N)_{i,\varepsilon} := \{\alpha \in \text{HOM}_k(M, N) \mid \alpha(am) = (-1)^{i\varepsilon}a\alpha(m) \forall a \in A, m \in M,$$

$$\alpha(M_{j,\varepsilon}) \subseteq N_{i+j,\varepsilon+\varepsilon'} \forall j \in \mathbb{Z}, \varepsilon' \in \mathbb{Z}_2\}$$

denote the space of all homogeneous $A$-module maps of degree $i$ and parity $\varepsilon$. We then define

$$(2.2) \quad \text{HOM}_A(M, N) := \bigoplus_{i \in \mathbb{Z}, \varepsilon \in \mathbb{Z}_2} \text{HOM}_A(M, N)_{i,\varepsilon},$$

$$(2.3) \quad \text{Hom}_A(M, N) := \text{HOM}_A(M, N)_{0,0}.$$  

The notation $\text{END}$ and $\text{End}$ is defined similarly. We will use the terms homomorphism and isomorphism to mean elements of $\text{HOM}_A(M, N)$. The terms even homomorphism and even isomorphism refer to elements of $\text{HOM}_A(M, N)$. We use the symbol $\cong$ to denote an isomorphism (i.e. an invertible element of $\text{HOM}_A(M, N)$) and $\sim$ to denote an even isomorphism (i.e. an invertible element of $\text{Hom}_A(M, N)$). We let $A$-$\text{mod}$ denote the category of finitely-generated $A$-modules with even homomorphisms.

We have a parity shift functor

$$\Pi: A$-$\text{mod} \rightarrow A$-$\text{mod}, \quad M \mapsto \Pi M,$$

that switches the $\mathbb{Z}_2$-grading, that is, $(\Pi M)_{i,\varepsilon} = M_{i,\varepsilon+1}$. The action of $A$ on $\Pi M$ is given by $a \cdot m = (-1)^i am$, where $am$ is the action on $M$.

We let $S(A)$ denote the set of even isomorphism classes of simple $A$-modules (i.e. isomorphism classes in $A$-$\text{mod}$) up to degree shift. By a slight abuse of notation, we will also use the notation $S(A)$ to denote a set of representatives of these isomorphism classes, where we shift degrees so that each representative has nonzero degree zero piece and is concentrated in nonnegative degree.

Given an $A$-module $V$ and an algebra automorphism $\alpha$ of $A$, we define the twisted module $\alpha V$ to be the $A$-module with underlying $k$-module $V$ and with action

$$(2.4) \quad a \cdot v = \alpha^{-1}(a)v, \quad a \in A, v \in V.$$  

Here and in what follows we use $\cdot$ to denote the twisted action and juxtaposition to denote the original (untwisted) action. Since $\alpha V$ is simple if $V$ is, the twisting induces a permutation of $S(A)$.

A simple $A$-module is said to be of type $\mathbb{M}$ if it is evenly isomorphic to its own parity shift. Otherwise it is said to be of type $\mathbb{Q}$. Schur’s Lemma for superalgebras states that, for simple $A$-modules $M$ and $N$ with $M \not\cong N$, we have $\text{HOM}_A(M, N) = 0$. On the other hand, if $M$ is simple and $k$ is an algebraically closed field, then

$$\text{END}_A(M) \simeq \begin{cases} k & \text{if } M \text{ is of type } \mathbb{M}, \\ \text{Cl} & \text{if } M \text{ is of type } \mathbb{Q}. \end{cases}$$  

Here $\text{Cl}$ is the two-dimensional Clifford algebra with one odd generator $c$ satisfying $c^2 = 1$.

Unadorned tensor products will be understood to be over $k$. Recall that multiplication in the tensor product of algebras $A_1$ and $A_2$ is given by

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{a_2b_1}(a_1a_2 \otimes b_1b_2), \quad a_1, b_1 \in A_1, a_2, b_2 \in A_2.$$  

The outer tensor product of an $A_1$-module $M$ and an $A_2$-module $N$ is denoted $M \boxtimes N$. As $k$-modules, we have $M \boxtimes N = M \otimes N$. The action is given by

$$(a_1 \otimes a_2)(m \otimes n) = (-1)^{a_2m}a_1m \otimes a_2n, \quad a_1 \in A_1, a_2 \in A_2, m \in M, n \in N.$$  

For the remainder of this subsection, we assume $k$ is an algebraically closed field. Suppose $M$ and $N$ are simple modules. If at least one of them is of type $\mathbb{M}$, then $M \boxtimes N$ is a simple $A_1 \otimes A_2$-module.
However, if \( M \) and \( N \) are both of type \( \mathcal{Q} \), then
\[
M \boxtimes N \simeq (M \otimes N) \oplus \Pi(M \otimes N)
\]
for some simple submodule \( M \oplus N \subseteq M \boxtimes N \) of type \( \mathcal{I} \). The simple submodule \( M \oplus N \) is unique up to even isomorphism and parity shift. We will also use the notation \( M \oplus N \) to denote \( M \boxtimes N \) in the case where \( M \) or \( N \) is of type \( \mathcal{II} \). It follows that, if \( M_1, N_1 \) are simple \( A_1 \)-modules, and \( M_2, N_2 \) are simple \( A_2 \)-modules, then for any nonzero even \( A_1 \otimes A_2 \)-module homomorphism \( \psi : M_1 \boxtimes M_2 \rightarrow N_1 \boxtimes N_2 \), we have a nonzero induced even homomorphism \( M_1 \oplus M_2 \rightarrow N_1 \oplus N_2 \), (making a different choice for \( M_1 \oplus M_2 \) or \( N_1 \oplus N_2 \) if necessary). Furthermore, every simple \( A_1 \otimes A_2 \)-module is isomorphic to \( M_1 \oplus M_2 \) for some simple \( A_1 \)-module \( M_1 \) and simple \( A_2 \)-module \( M_2 \).

2.2. Frobenius algebras. Fix an \( \mathbb{N} \)-graded Frobenius \( \mathbb{k} \)-superalgebra \( \mathcal{F} \) with homogeneous (in the \( \mathbb{N} \)-degree) parity-preserving linear trace map \( \text{tr} : \mathcal{F} \rightarrow \mathbb{k} \). In other words, the map
\[
(2.5) \quad \mathcal{F} \rightarrow \text{Hom}_\mathbb{k}(\mathcal{F}, \mathbb{k}), \quad f \mapsto \left( g \mapsto (-1)^{fg} \text{tr}(gf) \right),
\]
is a homogeneous parity-preserving isomorphism. (One could also allow the trace map \( \text{tr} \) to be parity-reversing, but we will not do so in the current paper. See Remark 3.3.) We will assume that \( \mathcal{F} \) is free as a \( \mathbb{k} \)-module. Let \( \psi \) denote the Nakayama automorphism, so that
\[
\text{tr}(fg) = (-1)^{fg} \text{tr}(g\psi(f)) \quad \text{for all } f, g \in \mathcal{F}.
\]
Let \( \delta \in \mathbb{N} \) be the maximum degree of elements of \( \mathcal{F} \), so that \( \text{tr} \) is of degree \( -\delta \). We make the following assumptions:

- The Nakayama automorphism has finite order \( \theta \).
- The characteristic of \( \mathbb{k} \) does not divide \( \theta \).
- The action of the Nakayama automorphism is diagonalizable (i.e. there exists a basis of \( \mathcal{F} \) consisting of eigenvectors of \( \psi \)).

Note that if \( \mathbb{k} \) is an algebraically closed field, the first two assumptions imply the third. (These assumptions are not necessary in the current section, but are needed later in the paper.)

Let \( B \) be a basis of \( \mathcal{F} \) consisting of homogeneous elements, and let \( B^\vee = \{ b^\vee \mid b \in B \} \) be the left dual basis. Thus
\[
\text{tr}(b^\vee c) = \delta_{b,c} \quad b, c \in B.
\]
It follows that, for all \( f \in \mathcal{F} \), we have
\[
(2.6) \quad \sum_{b \in B} b \text{tr}(b^\vee f) = f, \quad (2.7) \quad \sum_{b \in B} \text{tr}(fb)b^\vee = f.
\]
We also have
\[
(2.8) \quad \left(b^\vee\right)^\vee = (-1)^b \psi^{-1}(b).
\]

Remark 2.1. When comparing statements in the current paper to those of [RS17], the reader should be aware that the notation \( \vee \) denotes right duals in [RS17], as opposed to left duals as in the current paper.

Suppose \( \mathcal{F} \) and \( \mathcal{F}' \) are Frobenius algebras with trace maps \( \text{tr} \) and \( \text{tr}' \), respectively. A homomorphism of Frobenius algebras is an algebra homomorphism \( \xi : \mathcal{F} \rightarrow \mathcal{F}' \) such that \( \text{tr} = \text{tr}' \circ \xi \). If \( \xi \) is invertible, we call it an isomorphism of Frobenius algebras. Note that all homomorphisms of Frobenius algebras are injective. If \( \xi : \mathcal{F} \rightarrow \mathcal{F}' \) is an isomorphism of Frobenius algebras, then \( \xi \circ \psi = \psi' \circ \xi \), where \( \psi \) and \( \psi' \) are the Nakayama automorphisms of \( \mathcal{F} \) and \( \mathcal{F}' \), respectively. The opposite algebra \( \mathcal{F}^{\text{op}} \) is also a Frobenius algebra, with trace map \( \text{tr} \) and Nakayama automorphism \( \psi^{-1} \).
Lemma 2.2. Suppose \( \tau : F \to F^{\text{op}} \) is an isomorphism of Frobenius algebras. Then \( \tau \circ \psi = \psi^{-1} \circ \tau \). Furthermore, the left duals of the elements of the basis \{\( \tau(b^\vee) \mid b \in B \)\} of \( F \) are \( \tau(b^\vee)^\vee = (-1)^b \tau(b) \).

Proof. That \( \tau \circ \psi = \psi^{-1} \circ \tau \) follows from the fact that \( \psi^{-1} \) is the Nakayama automorphism of \( F^{\text{op}} \). To prove the second assertion note that, for all \( b, c \in B \), we have

\[
\delta_{b,c} = \text{tr}(b^\vee c) = \text{tr}(\tau(b^\vee c)) = (-1)^{bc} \text{tr}(\tau(c) \tau(b^\vee)),
\]

which implies that \( \tau(b^\vee)^\vee = (-1)^b \tau(b) \).

The following example will help us illustrate some of the concepts to be discussed later in the paper.

Example 2.3 (Taft Hopf algebra). Fix \( q \in \mathbb{N} \) and a primitive \( q \)-th root of unity \( \omega \in \mathbb{C} \). Consider the \( \mathbb{C} \)-algebra \( T_q \) with generators \( g, y \) and relations \( g^q = 1, y^q = 0, \) and \( yg = \omega gy \). This algebra can be given the structure of a Hopf algebra, and is called the Taft Hopf algebra. We can define a grading on \( T_q \) by declaring \( g \) to be even of degree zero and \( y \) to be even of arbitrary degree. The algebra \( T_q \) has basis given by \( y^k g^\ell, 0 \leq k, \ell \leq q - 1 \). It is straightforward to verify that the linear map \( \text{tr} : T_q \to k \) determined by \( \text{tr}(y^k g^\ell) = \delta_{k,q-1} \delta_{\ell,0} \) is nondegenerate and hence makes \( T_q \) a Frobenius algebra. The Nakayama automorphism is given by \( \psi(g) = \omega y \) and \( \psi(y) = y \).

Since the element \( y \) is nilpotent, simple modules for \( F \) are equivalent to simple modules for \( T_q / \langle y \rangle \), which is isomorphic to the group algebra of the cyclic group of order \( q \). Thus, a complete list of simple \( T_q \)-modules, up to isomorphism, is given by \( L_0, \ldots, L_{q-1} \), where, for \( 0 \leq k \leq q - 1 \), we have \( L_k \simeq \mathbb{C} \) as \( \mathbb{C} \)-modules, and

\[
g \cdot z = e^{2k\pi i/q} z, \quad y \cdot z = 0, \quad z \in L_k.
\]

2.3. Smash products and wreath product algebras. Suppose we have a left action of a group \( G \) on a \( k \)-algebra \( A \) via algebra automorphisms. Then we can form the smash product algebra \( A \rtimes G \). As a \( k \)-module, we have

\[
A \rtimes G = A \otimes \mathbb{k}G.
\]

The multiplication is determined by the fact that \( A \) and \( \mathbb{k}G \) are subalgebras, and

\[
(2.9) \quad wa = (w \cdot a)w, \quad w \in \mathbb{k}G, \ a \in A.
\]

As a special case of the above construction, we have the natural action of \( S_n \) on \( F^{\otimes n} \) by superpermuting the factors:

\[
s_i : (f_1 \otimes \cdots \otimes f_n) = (-1)^{j_i f_{i+1} f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \otimes \cdots \otimes f_n},
\]

where \( s_i, 1 \leq i \leq n - 1 \), denotes the simple transposition of \( i \) and \( i + 1 \). We use \( \tau a \) to denote \( \tau \cdot a \) in this case. We call \( F^{\otimes n} \rtimes_{\rho} S_n \) the wreath product algebra associated to \( F \) where, here and in what follows, we use the subscript \( \rho \) when the action is by superpermutations.

The above construction is a special case of the more general notion of a Hopf smash product. Another example that will be important for us is the following. An algebra automorphism \( \alpha \) of an algebra \( A \) gives rise to a natural right action of the polynomial algebra \( \mathbb{k}[x] \) on \( A \). Then we can form the algebra \( \mathbb{k}[x] \rtimes A \). Again, we have

\[
\mathbb{k}[x] \rtimes A = \mathbb{k}[x] \otimes A
\]

as \( k \)-modules. The multiplication is determined by the fact that \( \mathbb{k}[x] \) and \( A \) are subalgebras, and

\[
ax = x\alpha(a), \quad a \in A.
\]

3. Affine wreath product algebras

In this section we introduce our main object of study, the affine wreath product algebras. In this section \( k \) is an arbitrary commutative ring of characteristic not equal to two.
3.1. **Definition.** Recall that $F$ is an $\mathbb{N}$-graded Frobenius superalgebra with basis $B$, Nakayama automorphism $\psi$, and top degree $\delta$. Fix $n \in \mathbb{N}_+$. For $f \in F$ and $1 \leq i \leq n$, we define

\begin{align}
(3.1) & \quad f_i = 1^{\otimes (i-1)} \otimes f \otimes 1^{\otimes (n-i)} \in F^\otimes n, \\
(3.2) & \quad \psi_i = \text{id}^{\otimes (i-1)} \otimes \psi \otimes \text{id}^{\otimes (n-i)}; F^\otimes n \to F^\otimes n.
\end{align}

**Definition 3.1** (Affine wreath product algebra $A_n(F)$). We define the affine wreath product algebra $A_n(F)$ to be the $\mathbb{N}$-graded superalgebra that is the free product of $k$-algebras

$$k[x_1, \ldots, x_n] \ast F^\otimes n \ast kS_n,$$

modulo the relations

\begin{align}
(3.3) & \quad f x_i = x_i \psi_i(f), & 1 \leq i \leq n, \quad f \in F^\otimes n, \\
(3.4) & \quad s_i x_j = x_j s_i, & 1 \leq i \leq n - 1, \quad 1 \leq j \leq n, \quad j \neq i, i + 1, \\
(3.5) & \quad s_i x_i = x_{i+1}s_i - t_{i,i+1}, & 1 \leq i \leq n - 1, \\
(3.6) & \quad \pi f = \pi f \pi, & \pi \in S_n, \quad f \in F^\otimes n,
\end{align}

where

\begin{equation}
(3.7) \quad t_{i,j} := \sum_{b \in B} b_i b_j^\vee \quad \text{for} \quad 1 \leq i, j \leq n, \quad i \neq j,
\end{equation}

and where we always interpret $b_j^\vee$ as $(b^\vee)_j$. The degree and parity on $A_n(F)$ are determined by

\begin{align}
|f_i| = |f|, & \quad \bar{f}_i = \bar{f}, & 1 \leq i \leq n, \quad f \in F, \\
|\pi| = 0, & \quad \bar{\pi} = 0, & \pi \in S_n.
\end{align}

By convention, we set $A_0(F) = k$.

Note that $|t_{i,j}| = \delta$ and $\bar{t}_{i,j} = 0$ for all $i, j$. It is straightforward to verify that $t_{i,j}$ is independent of the choice of basis $B$. Conjugating by $s_i$, we see that (3.5) is equivalent to

\begin{equation}
(3.9) \quad s_i x_{i+1} = x_i s_i + t_{i,i+1}.
\end{equation}

It follows from Definition 3.1 that we have an algebra homomorphism

$$F^\otimes n \times_{\rho} S_n \to A_n(F), \quad f \mapsto f, \quad \pi \mapsto \pi, \quad f \in F^\otimes n, \quad \pi \in S_n.$$

We also have a natural algebra homomorphism

$$k[x_1, \ldots, x_n] \to A_n(F).$$

We will use these homomorphisms to view elements of $F^\otimes n \times_{\rho} S_n$ and $k[x_1, \ldots, x_n]$ as elements of $A_n(F)$. In fact, we will see in Theorem 4.6 that both of the above homomorphisms are injective, allowing us to view $F^\otimes n \times_{\rho} S_n$ and $k[x_1, \ldots, x_n]$ as subalgebras of $A_n(F)$.

**Lemma 3.2.** Up to isomorphism, $A_n(F)$ depends only on the underlying algebra $F$, and not on the trace map $\text{tr}$.

**Proof.** Trace maps for a given algebra differ by multiplication by an invertible element. (For the graded super setting of the current paper, see [PS16, Prop. 4.7].) Thus, if we have another trace map $\text{tr}'$, there exists an invertible $u \in F^\times$ (which must be even since both trace maps are even) such that $\text{tr}'(f) = \text{tr}(fu)$ for all $f \in F$. It is then straightforward to verify that the map determined by

\begin{equation}
(3.10) \quad x_i \mapsto x_i u_i, \quad f \mapsto f, \quad \pi \mapsto \pi, \quad 1 \leq i \leq n, \quad f \in F^\otimes n, \quad \pi \in S_n,
\end{equation}

is an isomorphism of algebras from $A_n(F)$ to $A_n(F')$, where $F'$ is the Frobenius algebra that is the same underlying algebra as $F$, but with trace map $\text{tr}'$. \[\square\]
Remark 3.3. We could allow the trace map of $F$ to be parity-reversing. Then we would define the $x_i$ to be odd and we would require them to anticommute. In addition, (3.3) would involve signs. While it would be interesting to investigate the resulting “odd wreath product algebras”, for simplicity we restrict our attention in the current paper to the case where the trace map of $F$ is parity-preserving.

3.2. Examples. Before investigating the properties of the algebra $A_n(F)$, we first discuss how various choices of the Frobenius algebra $F$ recover well-studied algebras. In its full generality, the algebra $A_n(F)$ first appeared in [RS17], where it occurred naturally in the endomorphism space of the object $Q^n$ of the diagrammatic category $H_F$. More precisely, $A_n(F)$ is isomorphic to the opposite algebra of the algebra $D_n$ defined in [RS17, Def. 8.12], after relabelling indices by interchanging $i$ and $n-i$.

Example 3.4 (Degenerate affine Hecke algebra). The algebra $A_n(k)$ is the usual degenerate affine Hecke algebra. We have $t_{i,j} = 1$ for all $i, j$.

Example 3.5 (Wreath Hecke algebra). Fix a finite group $G$ and consider the group algebra $kG$ (with trivial grading) with trace map $\text{tr}: kG \to k$ given by $\text{tr} \left( \sum_{g \in G} agg \right) = ae$, where $e$ is the identity element of $G$. Then

$$t_{i,j} = \sum_{g \in G} g_i g_j^{-1}, \quad i \neq j,$$

and the Nakayama automorphism $\psi$ is trivial. Thus, $A_n(kG)$ is the wreath Hecke algebra of $[WW08, Def. 2.4]$. Of course, there are other trace maps for $G$, given by projecting onto the coefficients of other elements of the group. However, by Lemma 3.2, these yield isomorphic affine wreath product algebras.

Example 3.6. Let $F = k[z]/(z^2)$, with $|z| = 2$ and $\bar{z} = 0$. This is an $\mathbb{N}$-graded Frobenius superalgebra with trace map $\text{tr}(a + bz) = b$ for $a, b \in k$. The Nakayama automorphism is trivial, $1^\vee = z$, and $z^\vee = 1$. Thus, $t_{i,j} = z_i + z_j$. Then $A_n(F)$ is precisely the algebra $H_n^T$ defined in [CL12, §10.3].

Example 3.7 (Affine Sergeev algebra). Consider the Clifford superalgebra $Cl$ generated by a single odd generator $e$ satisfying $e^2 = 1$. This is a Frobenius superalgebra with trace given by $\text{tr}(c) = 0$, $\text{tr}(1) = 1$. Then the Nakayama automorphism satisfies $\psi(c) = -c$. If we choose $B = \{1, c\}$, then $1^\vee = 1$ and $c^\vee = c$. Thus,

$$t_{i,j} = 1_1 1_i^\vee + c_i c_j^{\vee} = 1 + c_i c_j.$$

It follows that $A_n(Cl)$ is the algebra introduced in [Naz97, §3], where it was called the degenerate affine Sergeev algebra. It is also sometimes called the degenerate affine Hecke–Clifford algebra. Note that [Naz97, §3] uses a slightly different presentation, where $c^2 = -1$. Here we follow the conventions used in [Kle05, §14.1].

Example 3.8 (Affine zigzag algebra). When $F$ is a certain skew-zigzag algebra (see [HK01, §3] and [Cou16, §5]), the algebras $A_n(F)$ appear in the endomorphism algebras of the categories constructed in [CL12] to study Heisenberg categorification and the geometry of Hilbert schemes. They were then also considered in [KM], where they were related to imaginary strata for quiver Hecke algebras (also known as KLR algebras).

3.3. Automorphisms. It is straightforward to verify that we have an algebra automorphism of $A_n(F)$ given by

$$x_i \mapsto x_{n+1-i}, \quad f_i \mapsto f_{n+1-i}, \quad s_j \mapsto -s_{n-j}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n - 1.$$

Any automorphism $\xi: F \to F$ of Frobenius algebras induces an algebra automorphism of $A_n(F)$ given by

$$x_i \mapsto x_i, \quad f \mapsto \xi^\otimes f, \quad s_j \mapsto s_j, \quad 1 \leq i \leq n, \quad f \in F^\otimes n, \quad 1 \leq j \leq n - 1.$$
Lemma 3.9. Suppose \( \tau: F \to F^{\text{op}} \) is an isomorphism of Frobenius algebras. Then \( \hat{\tau}: A_n(F) \to A_n(F)^{\text{op}}, \) 
\[ \hat{\tau}(x_i) = x_i, \quad \hat{\tau}(f) = \tau^{\otimes n}(f), \quad \hat{\tau}(\pi) = \pi^{-1}, \quad 1 \leq i \leq n, \quad f \in F^{\otimes n}, \quad \pi \in S_n, \]
is an isomorphism of algebras.

Proof. It is clear that \( \hat{\tau} \) is a homomorphism when restricted to the subalgebras \( k[x_1, \ldots, x_n], F^{\otimes n}, \) and \( kS_n. \) Using Lemma 2.2, it is straightforward to verify that it preserves relations (3.3), (3.4), and (3.6). To verify that it preserves (3.5), we compute
\[ \hat{\tau}(s_i x_i) = \hat{\tau}(x_i) \hat{\tau}(s_i) = x_i s_i = s_i x_i + t_i, \]
and
\[ \hat{\tau}(x_i s_i - t_i) = s_i x_i - \sum_{b \in B} \tau(b_i) \tau(b^*)_{i+1} = s_i x_i - \sum_{c \in B'} (-1)^{c_i} c_i c_{i+1} = s_i x_i - t_i, \]
where, in the second equality, we introduced the basis \( B' = \{ \tau(b^*) \mid b \in B \} \) of \( F \) and used Lemma 2.2. It follows that \( \hat{\tau} \) is a homomorphism. Since it has inverse \( \tau^{-1}, \) it is an isomorphism. \( \square \)

Remark 3.10. In [KM, Lem. 3.11], which assumes that \( \psi = \text{id} \) (i.e. \( F \) is a symmetric algebra) and \( F \) is purely even, it is asserted that the map \( \hat{\tau} \) is an isomorphism of algebras for any algebra isomorphism \( \tau: F \to F^{\text{op}}, \) without the assumption that \( \tau \) preserves the trace map. However, this appears to be false as one can see by considering the example \( F = \mathbb{C}[z]/(z^2), \) with trace map \( \text{tr}(a + bz) = b, a, b \in \mathbb{C}, \) and \( \tau \) determined by \( \tau(z) = 2z. \)

3.4. Additional relations. We now deduce some relations that will be useful in our computations to follow. For \( k \in \mathbb{N}_+, 1 \leq i, j \leq n, i \neq j, \) define
\[ t^{(k)}_{i,j} := \sum_{b \in B} b_i x_i^k x_j^x - b_j x_j^x, \]
(Note that the rational expression is actually a polynomial in \( x_i \) and \( x_j. \)) Hence \( |t^{(k)}{i,j}| = k \delta, t^{(k)}_{i,j} = 0, \) and \( t^{(1)}_{i,j} = t_{i,j}. \) Note that
\[ t^{(k)}_{i,j} = t^{(k)}_{j,i}, \quad \text{if} \ \psi = \text{id}. \]

Lemma 3.11. For \( k \in \mathbb{N}_+ \) and \( 1 \leq i \leq n - 1, \) we have
\[ s_i x_i^k = x_i x_i s_i - t^{(k)}_{i,i+1}, \]
\[ s_i x_i^k = x_i x_i s_i + t^{(k)}_{i,i+1}. \]

Proof. This follows from (3.5) and (3.9) by a straightforward induction. \( \square \)

Lemma 3.12. For \( 1 \leq i, j \leq n, i \neq j, k \in \mathbb{N}_+, \) and \( f \in F^{\otimes n}, \) we have
\[ f t^{(k)}_{i,j} = t^{(k)}_{i,j} \psi_i^k (s_{i,j} f) = t^{(k)}_{i,j} \psi_{i,j} (f), \]
where \( s_{i,j} \in S_n \) is the transposition of \( i \) and \( j. \)

Proof. It suffices to prove that, for \( 1 \leq i, j \leq n, i \neq j, k \in \mathbb{N}_+, \) and \( f \in F, \) we have
\[ f t_{i,j} = t_{i,j} f, \quad \ell \neq i, j, \]
\[ f t^{(k)}_{i,j} = t^{(k)}_{i,j} f, \]
We have
\[ \psi(3.21) \]

**Lemma 3.13.** For where we have used the fact that the parities of \( \pi t \) (3.25)
\[ F \]

Recall that \( F \) has a basis consisting of eigenvectors for \( \psi \) (see Section 2.2). Thus, to prove (3.20), it suffices to consider \( f \in F \) satisfying \( \psi(f) = \omega f \), where \( \omega \) is a \( \theta \)-th root of unity. Then
\[ f_j t_{i,j}^{(k)} = f_j \sum_{b \in B} b \frac{x_i^k - x_j^k}{x_i - x_j} b_j \]  
(2.6)
\[ = \sum_{b,c \in B} (-1)^{bc} b \frac{x_i^k - \omega x_i x_j^k}{x_i - x_j} \omega x_j c_j f_i = \sum_{c \in B} c_i \frac{x_i^k - x_j^k}{x_i - x_j} c_j f_i = t_{i,j}^{(k)} \psi^k(f)_i. \]
**Proof.** We have
\[ \psi_j^{-1}(t_{j,i}) = \sum_{b \in B} \psi^{-1}(b) b_j \]  
(2.7)
\[ = \sum_{b \in B} (-1)^{b_j} b_i \psi^{-1}(b)_i \]  
(2.8)
\[ = \sum_{b \in B} b_i^{\psi^{-1}(b)} t_{i,j} \]
where we have used the fact that the parities of \( b \) and \( b^{\psi^{-1}} \) are equal (since the trace map is even) in the second equality, and used the fact that the definition of \( t_{i,j} \) is independent of the basis in the final equality (where we are summing over the basis \( \{b^{\psi^{-1}} \mid b \in B\} \)).

**Lemma 3.14.** For \( 1 \leq i, j \leq n, i \neq j, k \in \mathbb{N}_+, \pi \in S_n, \) and \( p \in k[x_1^\theta, \ldots, x_n^\theta] \), we have
\[ x_i t_{i,j}^{(k)} = t_{i,j}^{(k)} x_\ell, \quad \ell \neq i, j, \]
(3.22)
\[ x_i t_{i,j} = t_{j,i} x_i, \]
(3.23)
\[ p t_{i,j} = t_{i,j} p, \]
(3.24)
\[ \pi t_{i,j} = t_{\pi(i),\pi(j)} \pi, \]
(3.25)
\[ s_{i,i+1}^{(k)} = t_{i+1,i}^{(k)} s_i. \]
(3.26)
**Proof.** Relation (3.22) follows immediately from (3.3). For relation (3.23), we have
\[ x_i t_{i,j} = \psi^{-1}(t_{i,j}) x_i = t_{j,i} x_i, \]
Relation (3.24) follows immediately from (3.3) and the fact that \( \theta \) is the order of the Nakayama automorphism \( \psi \). Relation (3.25) follows immediately from (3.6). Finally, (3.26) follows from multiplying (3.15) on the left by \( s_i \), multiplying (3.16) on the right by \( s_i \), and adding the resulting two equations.
4. Structure Theory

In this section we examine the structure theory of affine wreath product algebras. In particular, we give an explicit basis, describe the center, define Jucys–Murphy elements, give a Mackey Theorem, and define intertwining elements. In this section $k$ is an arbitrary commutative ring of characteristic not equal to 2.

4.1. Deformed divided difference operators. Let $P_n = k[x_1, \ldots, x_n]$, and let $P_n(F)$ be the graded superalgebra such that

$$P_n(F) = P_n \otimes F \otimes^n$$

as a $k$-module, where the two factors are subalgebras, and where we impose the relations (3.3). Equivalently, $P_n(F)$ is the free product of $k$-algebras $P_n \ast F \otimes^n$ modulo the relations (3.3). We also have a natural isomorphism of algebras

$$P_n(F) \simeq (k[x] \times F) \otimes^n,$$

and we will often identify the two. The parity and degrees of the $x_i$ are given by (3.8). We define the polynomial degree of an element of $P_n(F)$ to be its total degree as a polynomial in the $x_i$.

For $1 \leq i \leq n - 1$, we define a skew derivation $\Delta_i : P_n(F) \to P_n(F)$ inductively as follows. First, we define $\Delta_i(F \otimes^n) = 0$ and, on elements of $P_n$ of polynomial degree one, we define

$$\Delta_i(x_i) = t_{i,i+1}, \quad \Delta_i(x_{i+1}) = -t_{i+1,i}, \quad \Delta_i(x_j) = 0, \quad j \neq i, i + 1,$$

and extend $k$-linearly. Then we extend $\Delta_i$ to all of $P_n(F)$ by requiring that

$$\Delta_i(a_1a_2) = \Delta_i(a_1)a_2 + s_i a_1 \Delta_i(a_2), \quad a_1, a_2 \in P_n(F).$$

Note, in particular, that

$$(4.3) \quad \Delta_i(fa) = s_i f \Delta_i(a), \quad f \in F \otimes^n, \quad a \in P_n(F).$$

**Lemma 4.1.** For all $a \in P_n(F)$ and $1 \leq i \leq n - 1$, in $A_n(F)$ we have

$$s_i a = s_i a s_i - \Delta_i(a).$$

**Proof.** The result for $a$ of polynomial degree zero and one follows immediately from (3.4)–(3.6) and (3.9). Suppose it holds for $a_1, a_2 \in P_n(F)$. Then

$$s_i(a_1a_2) = s_i a_1 s_i a_2 - \Delta_i(a_1)a_2 = s_i a_1 s_i - s_i a_1 \Delta_i(a_2) - \Delta_i(a_1)a_2 = s_i(a_1a_2) - \Delta_i(a_1a_2),$$

and hence the result follows by induction. \qed

The operators $\Delta_i$ can be thought of as $F$-deformations of divided difference operators. In particular, it follows from Lemmas 4.1 and 3.11 that, for $1 \leq i \leq n - 1$ and $k \in \mathbb{N}_+$, we have

$$(4.4) \quad \Delta_i \left( x_i^k \right) = i^{(k)}_{i,i+1} = \sum_{b \in B} b_i \frac{x_i^k - x_{i+1}^k}{x_i - x_{i+1}} b_{i+1}^k = \sum_{b \in B} b_i \partial_i \left( x_i^k \right) b_{i+1}^k,$$

$$(4.5) \quad \Delta_i \left( x_{i+1}^k \right) = -i^{(k)}_{i+1,i} = \sum_{b \in B} b_{i+1} \frac{x_{i+1}^k - x_i^k}{x_i - x_{i+1}} b_i^k = \sum_{b \in B} b_{i+1} \partial_i \left( x_{i+1}^k \right) b_i^k,$$

where

$$(4.6) \quad \partial_i(p) = \frac{p - s_i p}{x_i - x_{i+1}}, \quad p \in P_n,$$

is the usual divided difference operator. In addition, we have the following result.
Proposition 4.2. We have
\begin{align}
\Delta_i^{(s_j a)} &= s_j \Delta_i(a), & 1 \leq i, j \leq n - 1, \ |i - j| > 1, \ a \in P_n(F), \\
\Delta_i^{(s_i a)} &= -s_i \Delta_i(a), & 1 \leq i \leq n - 1, \ a \in P_n(F), \\
\Delta_i \Delta_j &= \Delta_j \Delta_i, & 1 \leq i, j \leq n - 1, \ |i - j| > 1, \\
\Delta_i^2 &= 0, & 1 \leq i \leq n - 1.
\end{align}

Proof. We prove (4.7) and (4.8) by induction on the polynomial degree of $a$. The results for $a$ of polynomial degree less than or equal to one follow immediately from the definition of $\Delta_i$. Assume the results hold for elements of polynomial degree less than or equal to $k$, and let $a_1, a_2 \in P_n(F)$ have polynomial degree less than or equal to $k$. Then, if $|i - j| > 1$,
\[
\Delta_i^{(s_j a_1a_2)} = \Delta_i^{(s_j (a_1a_2 + a_1s_i a_2))} = \Delta_i^{(s_j a_1) a_2 + s_j a_1 \Delta_i(a_2)} = \Delta_i^{(s_j a_1)} a_2 + s_j a_1 \Delta_i(a_2),
\]
completing the inductive step for (4.7). The proof of the inductive step for (4.8) is similar.

We prove (4.9) again by induction on the polynomial degree of $a$. Suppose $|i - j| > 1$. For $a$ of polynomial degree less than or equal to one, we have $\Delta_i \Delta_j(a) = 0 = \Delta_j \Delta_i(a)$. Assume the result holds for elements of polynomial degree less than or equal to $k$, and let $a_1, a_2 \in P_n(F)$ have polynomial degree less than or equal to $k$. Then
\[
\Delta_i \Delta_j(a_1a_2) = \Delta_i(\Delta_j(a_1)a_2 + s_i a_1 \Delta_j(a_2)) = \Delta_i(\Delta_j(a_1))a_2 + s_i \Delta_j(a_1) \Delta_i(a_2) + \Delta_i(s_i a_1) \Delta_j(a_2) + s_i a_1 \Delta_i \Delta_j(a_2) = \Delta_j(\Delta_i(a_1)a_2 + s_i a_1 \Delta_i(a_2)) = \Delta_j \Delta_i(a_1a_2).
\]

To prove (4.10), note that $\Delta_i^2(a) = 0$ for $a$ of polynomial degree less than or equal to one. Assume the result holds for elements of polynomial degree less than or equal to $k$, and let $a_1, a_2 \in P_n(F)$ have polynomial degree less than or equal to $k$. Then
\[
\Delta_i^2(a_1a_2) = \Delta_i(\Delta_i(a_1a_2 + s_i a_1 \Delta_i(a_2))) = \Delta_i^2(a_1a_2) + s_i \Delta_i(a_1) \Delta_i(a_2) + \Delta_i(s_i a_1) \Delta_i(a_2) + a_1 \Delta_i^2(a_2) = 0.
\]

Lemma 4.3. If $\psi = \text{id}$ (i.e. the Frobenius algebra $F$ is symmetric), then
\[
\Delta_i(f p) = s_i f t_{i+1} \partial_i(p), \quad f \in F^\otimes n, \ p \in P_n,
\]
where $\partial_i$ is the usual divided difference operator of (4.6).

Proof. By (4.3), it suffices to prove the case where $f = 1$. Since $\psi = \text{id}$, the case where $p$ is of polynomial degree one follows from (3.14) and (4.2). The general result then follows by a straightforward induction. \hfill $\Box$

4.2. Basis theorem. We have a natural action of $S_n$ on $P_n(F)$ by superpermutation of the $x_i$ and the factors of $F^\otimes n$. Recall that we also have a natural algebra homomorphism $P_n(F) \to A_n(F)$ and we use this to view elements of $P_n(F)$ as elements of $A_n(F)$. (We will see in Theorem 4.6 that this homomorphism is injective.) Let $\leq$ denote the strong Bruhat ordering on $S_n$.

Lemma 4.4. For $\pi \in S_n$ and $a \in P_n(F)$, we have
\[
\pi a = \pi a \pi + \sum_{\sigma < \pi} a_{\sigma \sigma} \sigma \quad \text{and} \quad a \pi = \pi (\pi^{-1} a) + \sum_{\sigma < \pi} \sigma a_{\pi \sigma},
\]
for some \(a_\sigma, a'_\sigma \in P_n(F)\) of polynomial degree less than the polynomial degree of \(a\).

**Proof.** This follows from the defining relations (or Lemma 4.1).

**Proposition 4.5.** Let \(V\) denote \(P_n(F) \otimes kS_n\), considered as a graded \(k\)-supermodule. Then \(V\) is an \(A_n(F)\)-module under the action

\[
\begin{align*}
    z \cdot (a \otimes w) &= za \otimes w, \\
    s_i \cdot (a \otimes w) &= s_i a \otimes s_i w - \Delta_i(a) \otimes w,
\end{align*}
\]

for all \(z, a \in P_n(F), w \in kS_n,\) and \(1 \leq i \leq n - 1\).

**Proof.** It is clear that the action satisfies the defining relations of \(P_n(F)\). This includes relation (3.3). For \(1 \leq i \leq n - 1\) and \(f \in F^{\otimes n}\), we have

\[
    s_i \cdot (f \cdot (a \otimes w)) = s_i \cdot (fa \otimes w) = s_i (fa) \otimes s_i w - \Delta_i(fa) \otimes w
\]

Thus, the action satisfies relation (3.6).

For \(1 \leq i \leq n - 1, 1 \leq j \leq n, j \neq i, i + 1\), we have

\[
    s_i \cdot (x_j \cdot (a \otimes w)) = s_i (x_ja \otimes w) = s_i x_ja \otimes s_i w - \Delta_i (x_ja) \otimes w
\]

Thus the action satisfies (3.4). We also have

\[
    s_i \cdot (s_i \cdot (a \otimes w)) = s_i \cdot (s_i a \otimes s_i w - \Delta_i(a) \otimes w)
\]

and so the action satisfies (3.5).

It remains to verify the Coxeter relations of \(kS_n\). For \(1 \leq i \leq n - 1\), we have

\[
    s_i \cdot (s_i \cdot (a \otimes w)) = s_i \cdot (s_i a \otimes s_i w - \Delta_i(a) \otimes w)
\]

where the final equality holds by (4.8) and (4.10). For \(1 \leq i, j \leq n - 1\) with \(|i - j| > 1\), we have

\[
    s_i \cdot (s_j \cdot (a \otimes w)) = s_i \cdot (s_j a \otimes s_j w - \Delta_j(a) \otimes w)
\]

where the last equality uses (4.7), (4.9), and the fact that \(s_is_j = s_js_i\).

Finally, suppose \(1 \leq i \leq n - 2\). We claim that, as operators on \(V\),

\[
    (s_is_{i+1}s_is_i - s_{i+1}s_is_{i+1})x_j = x_{s_is_{i+1}s_is_i(j)}(s_is_{i+1}s_i - s_{i+1}s_is_i), \quad 1 \leq j \leq n.
\]

Indeed, by what has already been shown above, we have

\[
    s_is_{i+1}x_i = s_is_{i+1}x_{i+1}s_i - s_is_{i+1}t_{i,i+1}x_{i+1}s_i = s_ix_{i+1}s_is_{i+1}x_{i+1} = s_is_{i+1}x_{i+1}s_i = s_is_{i+1}x_{i+1}s_i
\]

and

\[
    s_is_{i+1}x_{i+1}s_i = s_is_{i+1}s_is_{i+1}x_{i+1}s_i = s_is_{i+1}s_is_{i+1}x_{i+1}s_i = x_{i+1}s_is_{i+1}s_is_{i+1}x_{i+1}s_i
\]

This proves (4.11) when \(j = i\). The other cases are proved by similar direct computations.
We now prove that $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ as operators on $V$ by induction on the polynomial degree of $a$ in $a \otimes w \in V$. The case where $a$ has polynomial degree zero is immediate. If the claim holds for $a$, then, for $1 \leq i \leq n$, we have

\[(s_is_{i+1}s_i - s_{i+1}s_is_{i+1})(x_i a \otimes w) = ((s_is_{i+1}s_i - s_{i+1}s_is_{i+1})x_i) \cdot (a \otimes w) \] 

\[= (x_is_{i+1}s_i(j)(s_is_{i+1}s_i - s_{i+1}s_is_{i+1})) \cdot (a \otimes w) = 0, \] 

proving the inductive step. \(\square\)

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we let

\[(4.12) \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}. \]

We define the graded dimension $\text{grdim} V$ of a $(\mathbb{Z} \times \mathbb{Z}_2)$-graded $k$-module, with finite-dimensional graded pieces, to be

\[\text{grdim} V = \sum_{i \in \mathbb{Z}, \varepsilon \in \mathbb{Z}_2} (\dim V_{i,\varepsilon}) q^i \varepsilon^2 \in \mathbb{N}[q^{\pm 1}, \varepsilon]/(\varepsilon^2 - 1). \]

**Theorem 4.6 (Basis theorem for $\mathcal{A}_n(F)$).**

(a) The map

\[V = P_n(F) \otimes kS_n \to \mathcal{A}_n(F), \quad a \otimes w \mapsto aw, \]

is an isomorphism of graded $\mathcal{A}_n(F)$-supermodules.

(b) The algebra $\mathcal{A}_n(F)$ is free as a $k$-module, with graded dimension

\[\text{grdim} \mathcal{A}_n(F) = n! \left( \frac{\text{grdim} F}{1 - q^2} \right)^n. \]

**Proof.** Let

\[B_1 = \{ x^\alpha b \otimes \pi \mid \alpha \in \mathbb{N}^n, \ b \in B^{\otimes n}, \ \pi \in S_n \} \subseteq V, \]

\[B_2 = \{ x^\alpha b \pi \mid \alpha \in \mathbb{N}^n, \ b \in B^{\otimes n}, \ \pi \in S_n \} \subseteq \mathcal{A}_n(F). \]

Thus $B_1$ is a basis of $V$. It follows easily from Lemma 4.4 that the elements of $B_2$ span $\mathcal{A}_n(F)$. Furthermore, we have $x^\alpha b \pi \cdot (1 \otimes 1) = x^\alpha b \otimes \pi$, and so the elements of $B_2$ are linearly independent, and hence $B_2$ is a basis for $\mathcal{A}_n(F)$.

Since $V$ is a cyclic module generated by $1 \otimes 1$, there is an $\mathcal{A}_n(F)$-module homomorphism $\mathcal{A}_n(F) \to V$ determined by $1 \mapsto 1 \otimes 1$. This map sends $x^\alpha b \pi \in B_2$ to $x^\alpha b \otimes \pi \in B_1$. Since the map is a bijection on $k$-bases, it is an isomorphism. This proves (a). Part (b) follows from (a). \(\square\)

**Example 4.7.** Specializing $F$ in Theorem 4.6 recovers several results that have appeared in the literature:

(a) When $F = \text{Cl}$ (see Example 3.4) we recover a basis for the degenerate affine Hecke algebra. See [Kle05, Th. 3.2.2].

(b) When $F$ is the group algebra of a finite group (see Example 3.5), we recover [WW08, Th. 2.8].

(c) When $F$ is symmetric (i.e. $\psi = \text{id}$) and purely even, we recover [KM, Th. 3.8]. In fact, our proof closely follows the proof of [KM, Th. 3.8].

**Corollary 4.8.** The sets

\[\{ x^\alpha b \pi \mid \alpha \in \mathbb{N}^n, \ b \in B^{\otimes n}, \ \pi \in S_n \} \quad \text{and} \quad \{ \pi x^\alpha b \mid \alpha \in \mathbb{N}^n, \ b \in B^{\otimes n}, \ \pi \in S_n \}, \]

are $k$-bases for $\mathcal{A}_n(F)$.

**Proof.** It was shown in the proof of Theorem 4.6(a) that the first set is a basis. The fact that the second set is also a basis then follows from Lemma 4.4 using induction on the length of $\pi \in S_n$. \(\square\)

Theorem 4.6 allows us to view $\mathcal{A}_n(F)$ as an affine version of the wreath product algebra.
Corollary 4.9. As \((\mathbb{Z} \times \mathbb{Z}_2)\)-graded \(k\)-modules, we have
\[
A_n(F) = k[x_1, \ldots, x_n] \otimes (F^{\otimes n} \rtimes S_n).
\]
The multiplication is determined by the fact that the factors \(k[x_1, \ldots, x_n]\) and \(F^{\otimes n} \rtimes S_n\) are subalgebras and the relations (3.3)–(3.5).

By Theorem 4.6 and Corollary 4.9, we can identify \([k[x_1, \ldots, x_n], F^{\otimes n}, kS_n, F^{\otimes n} \rtimes S_n, P_n(F)\) as subalgebras of \(A_n(F)\). We will do so in the remainder of the paper.

4.3. A filtration and the associated graded algebra. By Theorem 4.6, we can extend the notion of polynomial degree to all of \(A_n(F)\) in the natural way, and we obtain a filtration on \(A_n(F)\). Note that all the relations of Definition 3.1 are homogeneous except for (3.5). It follows that the associated graded algebra is
\[
\text{gr}A_n(F) = (k[x] \ltimes F)^{\otimes n} \rtimes S_n,
\]
where we recall that the subscript \(\rho\) indicates that the action of \(S_n\) is by superpermutation of the factors.

4.4. Description of the center. Let
\[
F_\psi := \{ f \in F \mid \psi(f) = f \}
\]
be the subalgebra of \(F\) consisting of those elements fixed by the Nakayama automorphism. Then we have
\[
P_n(F_\psi) := k[x_1, \ldots, x_n] \otimes F^{\otimes n}_\psi \subseteq P_n(F)
\]
where the tensor product in the center expression is of graded superalgebras.

For \(k \in \mathbb{Z}\), define
\[
F^{(k)} := \{ f \in F \mid gf = (-1)^{fg} f \psi^k(g) \text{ for all } g \in F \} \quad \text{and} \quad F^{(k)}_\psi := F^{(k)} \cap F_\psi.
\]
Note that, for all \(k \in \mathbb{Z}\), \(F^{(k)} = Z(F)\) is the center of \(F\). In particular, if \(\psi = \text{id}\) (i.e. if \(F\) is symmetric), then \(F^{(k)} = Z(F)\) for all \(k \in \mathbb{Z}\).

It is clear that
\[
F^{(k)}F^{(\ell)} \subseteq F^{(k+\ell)} \quad \text{and} \quad F^{(k)}_\psi F^{(\ell)}_\psi \subseteq F^{(k+\ell)}_\psi, \quad k, \ell \in \mathbb{Z}.
\]

Lemma 4.10. The center of \([k[x] \ltimes F]\) is
\[
Z([k[x] \ltimes F]) = \bigoplus_{k=0}^{\infty} x^k F^{(-k)}_\psi.
\]

Proof. It is clear from the definitions that \(\bigoplus_{k=0}^{\infty} x^k F^{(-k)}_\psi \subseteq Z([k[x] \ltimes F])\). Now let
\[
z = \sum_{k \in \mathbb{N}, \ b \in B} a_{k,b} x^k b \in Z([k[x] \ltimes F]),
\]
where \(a_{k,b} \in k\) for all \(k, b\). Without loss of generality, we may assume that \(B = \{b_1, \ldots, b_m\}\) is a basis of \(F\) consisting of eigenvectors for \(\psi\) (see Section 2.2). Let \(i \in \{1, 2, \ldots, n\}\). Then we have
\[
0 = zx - xz = x \sum_{k \in \mathbb{N}, \ b \in B} a_{k,b} x^k (\psi - \text{id})(b).
\]
It follows that \(a_{k,b} = 0\) for all \(b \in B\) such that \(\psi(b) \neq b\). Hence \(z \in k[x] \otimes F_\psi\).

Thus, we can write
\[
z = \sum_{k \in \mathbb{N}} x^k f_k,
\]
where \( f_k \in F_\psi \) for all \( k \in \mathbb{N} \). Without loss of generality, we may assume that \( z \) is homogeneous in the \( \mathbb{Z}_2 \)-grading, so that \( f_k = \bar{z} \) for all \( k \). Then, for \( g \in F \) homogeneous in the \( \mathbb{Z}_2 \)-grading, we then have
\[
0 = gz - (-1)^{\bar{z}g} zg = \sum_{k \in \mathbb{N}} x^k \left( \psi^k(g) f_k - (-1)^{\bar{z}g} f_k g \right).
\]

It follows from Theorem 4.6 that \( f_k \in F^{(-k)} \). \( \square \)

**Example 4.11.**
(a) When \( F = k \), we have \( Z(k[x] \ltimes F) = k[x] \).
(b) If \( F \) is the Taft Hopf algebra of Example 2.3 and \( m \in \mathbb{N}_+ \), we have \( y^{m-1} \in F_\psi^{(1-m)} \). So \( x^m y^{n-1} \in Z(k[x] \ltimes F) \).

**Lemma 4.12.** The centralizer of \( k[x_1, \ldots, x_n] \) in \( A_n(F) \) is contained in the subalgebra \( P_n(F) \).

**Proof.** Let \( z = \sum_{\pi \in S_n} z_{\pi} \) be an element of the centralizer of \( k[x_1, \ldots, x_n] \) in \( A_n(F) \), where \( z_\pi \in P_n(F) \) for all \( \pi \in S_n \). Let \( \pi \in S_n \) be maximal with respect to the strong Bruhat order such that \( z_\pi \neq 0 \). Assume \( \pi \neq 1 \) and choose \( i \in \{1, 2, \ldots, n\} \) such that \( \pi(i) \neq i \). Then, by Lemma 4.4, we have
\[
x_i^\theta z - z x_i^\theta = \left( x_i^\theta - x_i^{\pi(i)} \right) z_{\pi} + \sum_{\sigma. \sigma \bar{\zeta} \pi} z'_\sigma \sigma,
\]
for some \( z'_\sigma \in P_n(F) \). Thus, by Theorem 4.6, \( z \) is not central, giving a contradiction. Hence the center of \( A_n(F) \) is contained in \( P_n(F) \). \( \square \)

For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \), let
\[
F^{(\alpha)} := F^{(\alpha_1)} \otimes \cdots \otimes F^{(\alpha_n)} \quad \text{and} \quad F_\psi^{(\alpha)} := F_\psi^{(\alpha_1)} \otimes \cdots \otimes F_\psi^{(\alpha_n)}.
\]

It follows that
\[
\bigoplus_{\alpha \in \mathbb{N}^n} x^\alpha F^{(-\alpha)} \quad \text{and} \quad \bigoplus_{\alpha \in \mathbb{N}^n} x^\alpha F_\psi^{(-\alpha)}
\]
are subalgebras of \( A_n(F) \). Note that \( x^\alpha F^{(-\alpha)} = (x^{\alpha_1} F^{(-\alpha_1)}) \otimes \cdots \otimes (x^{\alpha_n} F^{(-\alpha_n)}) \).

**Lemma 4.13.** The centralizer of \( P_n(F) \) in \( A_n(F) \) is the subalgebra \( \bigoplus_{\alpha \in \mathbb{N}^n} x^\alpha F_\psi^{(-\alpha)} \). In particular, the center of \( A_n(F) \) is contained in the subalgebra \( \bigoplus_{\alpha \in \mathbb{N}^n} x^\alpha F^{(-\alpha)} \).

**Proof.** This follows immediately from Lemmas 4.10 and 4.12, together with the fact that \( Z(A_1 \otimes A_2) = Z(A_1) \otimes Z(A_2) \) for \( k \)-algebras \( A_1 \) and \( A_2 \). \( \square \)

**Theorem 4.14.** The center of \( A_n(F) \) consists of those elements of \( \bigoplus_{\alpha \in \mathbb{N}^n} x^\alpha F_\psi^{(-\alpha)} \subseteq P_n(F_\psi) = (k[x] \otimes F_\psi)^{\otimes n} \) that are invariant under the action of \( S_n \) by superpermuting the factors of \( (k[x] \otimes F_\psi)^{\otimes n} \). In other words, the center of \( A_n(F) \) consists of finite sums of the form
\[
\sum_{\alpha \in \mathbb{N}^n} x^\alpha f_\alpha, \quad f_\alpha \in F_\psi^{(-\alpha)},
\]
such that \( f_\pi. \alpha = \pi f_\alpha \) for all \( \alpha \in \mathbb{N}^n \) and \( \pi \in S_n \).

**Proof.** Let \( z \) be a central element of \( A_n(F) \). By Lemma 4.13, we have
\[
z = \sum_{\alpha \in \mathbb{N}^n} x^\alpha f_\alpha \quad \text{for some} \quad f_\alpha \in F_\psi^{(-\alpha)}.
\]

For \( i \in \{1, 2, \ldots, n-1\} \), it follows from Lemma 4.4 that \( s_i z - (s_i z) s_i \in P_n(F) \). Thus
\[
z = s_i z s_i = \sum_{\alpha \in \mathbb{N}^n} x^{s_i. \alpha} (s_i f_\alpha) + a s_i,
\]
where \( a \) is the coefficient of \( s_i \) in \( z \).
for some \( \alpha \in P_n(F) \). It then follows from Theorem 4.6 that \( z \) is invariant under the superpermutation action of \( S_n \).

In remains to prove that elements of the form (4.17) commute with elements of \( S_n \). Let \( 1 \leq i \leq n-1 \). Since \( s_i \) commutes with \( x_j \) and \( f_j \) for all \( j \neq i, i+1 \) and \( f \in F \), it suffices to check that \( s_i \) commutes with elements of the form

\[
x_i^q x_{i+1}^r f + x_i^r x_{i+1}^q s_i f, \quad q, r \in \mathbb{N}, \; f \in F^{(-\alpha)},
\]

where

\[
\alpha = (0, \ldots, 0, q, r, 0, \ldots, 0),
\]

with \( q \) appearing in the \( i \)-th component.

We compute

\[
(4.18) \quad s_i x_i^q x_{i+1}^r f s_i = x_i^q x_{i+1}^r f + x_i^r x_{i+1}^q s_i f - x_i^q x_{i+1}^r f - x_i^r x_{i+1}^q s_i f.
\]

By symmetry, we have

\[
(4.19) \quad s_i x_i^q x_{i+1}^r f s_i = x_i^q x_{i+1}^r f + x_i^r x_{i+1}^q s_i f - t_{i,i+1} f s_i - t_{i,i+1} f s_i.
\]

We would like to show that

\[
(4.20) \quad s_i (x_i^q x_{i+1}^r f + x_i^r x_{i+1}^q s_i f) s_i - x_i^q x_{i+1}^r f - x_i^r x_{i+1}^q s_i f
\]

is equal to zero. By (4.18) and (4.19), we see that (4.20) is equal to

\[
(4.21) \quad x_i^q x_{i+1}^r f s_i - t_{i,i+1} f s_i - t_{i,i+1} f s_i - t_{i,i+1} f s_i.
\]

Now

\[
x_i^q x_{i+1}^r f s_i = \sum_{b \in B} \sum_{\ell = 0}^{r-1} x_i^\ell b_{i+1} x_i^{r-\ell-1} x_{i+1} b_{i}^\ell f
\]

\[
= \sum_{b \in B} \sum_{\ell = 0}^{r-1} x_i^\ell b_{i+1} x_i^{r-\ell-1} (b_{i+1}^{\ell}) x_i^{r-\ell-1} f
\]

\[
= \sum_{b \in B} \sum_{\ell = 0}^{r-1} x_i^\ell f x_i^{r-\ell-1} x_i^{r-\ell-1} f
\]

\[
= \sum_{b \in B} \sum_{\ell = 0}^{r-1} x_i^\ell f x_i^{r-\ell-1} x_i^{r-\ell-1} f
\]

\[
= \sum_{b \in B} \sum_{\ell = 0}^{r-1} x_i^\ell f x_i^{r-\ell-1} x_i^{r-\ell-1} f
\]

\[
= \left\{ x_i^{r-\ell-1} x_i^q x_{i+1}^r f s_i - t_{i,i+1} f s_i - t_{i,i+1} f s_i \right\}
\]

where, in the fourth equality, we used the fact that the definition of \( t_{i,j} \) is independent of the choice of basis and so we summed over the basis \( \{ x_i^{r-\ell-1} x_i^q f s_i \} \) (and abused notation by calling this new basis \( B \) again.)

Thus, the first and last terms in (4.21) cancel. The proof that the second and third cancel is similar. \( \square \)
Example 4.15. (a) When $F = \mathbb{k}$, Theorem 4.14 recovers the well-known result that the center of the degenerate affine Hecke algebra consists of all symmetric polynomials in $x_1, \ldots, x_n$ (see, for example, [Kle05, Th. 3.3.1]).

(b) When $F$ is the group algebra of a finite group (see Example 3.5), Theorem 4.14 recovers [WW08, Th. 2.10].

(c) When $F = \text{Cl}$ (see Example 3.7), we have $F^{(k)}_\psi = \mathbb{k}$ for $k$ even and $F^{(k)}_\psi = 0$ for $k$ odd. Thus, Theorem 4.14 recovers [Naz97, Prop. 3.1] (see also [Kle05, Th. 14.3.1]), which states that the center of the affine Sergeev algebra consists of all symmetric polynomials in $x_1^\theta, \ldots, x_n^\theta$.

Corollary 4.16. The center of $\mathcal{A}_n(F)$ contains the ring of symmetric polynomials in $x_1^\theta, \ldots, x_n^\theta$. In particular, $\mathcal{A}_n(F)$ is finitely generated as a module over its center.

Proof. That the center contains the ring of symmetric polynomials in $x_1^\theta, \ldots, x_n^\theta$ follows from Theorem 4.14 and the fact that $\mathbb{k} \subseteq F^{(-k\theta)}_\psi$ for all $k \in \mathbb{N}$. Thus $\mathcal{A}_n(F)$ is finitely generated as a module over its center by Theorem 4.6.

Remark 4.17. If $\mathbb{k}$ is an algebraically closed field, then Corollary 4.16 implies that all simple $\mathcal{A}_n(F)$-modules are finite-dimensional.

Proposition 4.18. Suppose $A$ is a maximal commutative subalgebra of $F_\psi$. Then $\mathbb{k}[x_1, \ldots, x_n]A^{\otimes n}$ is a maximal commutative subalgebra of $\mathcal{A}_n(F)$.

Proof. Suppose $z \in \mathcal{A}_n(F)$ commutes with all elements of $\mathbb{k}[x_1, \ldots, x_n]A^{\otimes n}$. It follows immediately from Lemma 4.12 that $z \in P_n(F)$. Hence we may write $z = \sum_{\alpha \in \mathbb{N}^n} x^\alpha f_\alpha$ for some $f_\alpha \in F^{\otimes n}$ with all but finitely many $f_\alpha$ equal to zero. Then, for $1 \leq i \leq n$, we have, by (3.3),

$$x_iz = zx_i \implies \sum_\alpha x_iz^\alpha f_\alpha = \sum_\alpha x_iz^\alpha \psi_i(f_\alpha).$$

It then follows from Theorem 4.6 that $\psi_i(f_\alpha) = f_\alpha$ for all $\alpha$. Hence $f_\alpha \in F^{\otimes n}$ for all $\alpha$.

Now suppose $g \in F^{\otimes n}$. Then

$$gz = zg \implies \sum_\alpha x^\alpha gf_\alpha = \sum_\alpha x^\alpha f_\alpha g.$$

Thus, by Theorem 4.6, $gf_\alpha = f_\alpha g$. Since $A$ is a maximal commutative subalgebra of $F_\psi$, it follows that $f_\alpha \in A^{\otimes n}$, completing the proof of the proposition.

Example 4.19. (a) When $F = \mathbb{k}$ (see Example 3.4), Proposition 4.18 recovers the well-known fact that $\mathbb{k}[x_1, \ldots, x_n]$ is a maximal commutative subalgebra of the degenerate affine Hecke algebra.

(b) When $F = \text{Cl}$ (see Example 3.7), we have $F^{\otimes n} = \mathbb{k}$. Thus, Proposition 4.18 recovers the fact that $\mathbb{k}[x_1, \ldots, x_n]$ is a maximal commutative subalgebra of the affine Sergeev algebra (see [Naz97, Prop. 3.1]).

4.5. Jucys–Murphy elements. Define the Jucys–Murphy elements

$$(4.22) \quad J_1 = 0, \quad J_k = \sum_{i=1}^{k-1} t_{i,k} s_{i,k}, \quad 2 \leq k \leq n,$$

where $s_{i,k} \in S_n$ is the transposition of $i$ and $k$. (See [RS17, (8.7)].)

Proposition 4.20. We have a surjective algebra homomorphism $\mathcal{A}_n(F) \to F^{\otimes n} \rtimes \rho S_n$ that is the identity on $F^{\otimes n} \rtimes \rho S_n$ and maps $x_k$ to $J_k$ for $1 \leq k \leq n$. 

Proof. The given map is clearly a map of \( \mathbb{k} \)-modules. To show that it is an algebra homomorphism, it suffices to prove that it respects the relations of \( \mathcal{A}_n(F) \). This follows inductively from the fact that 
\[
x_{k+1} = s_k x_k s_k + t_{k,k+1} s_k
\] 
by (3.5) and that 
\[
s_k J_k s_k + t_{k,k+1} s_k = \sum_{i=1}^{k-1} s_k t_{i,k} s_i s_k + t_{k,k+1} s_k = \sum_{i=1}^{k-1} t_{i,k} s_i s_k + t_{k,k+1} s_k = J_{k+1}. 
\]

Example 4.21. (a) When \( F = \mathbb{k} \) (see Example 3.4), the \( J_k \) are the usual Jucys–Murphy elements of the symmetric group.

(b) When \( F \) is the group algebra of a finite group (see Example 3.5), the \( J_k \) are the Jucys–Murphy elements of the wreath Hecke algebra introduced independently in [Pus97, Def. 2(a)] and [Wan04, Def. 3.1].

(c) When \( F = \text{Cl} \) (see Example 3.7), the \( J_k \) are the Jucys–Murphy elements of the Sergeev algebra. See, for example, [Kle05, (13.22)].

4.6. Mackey Theorem. For a composition \( \mu = (\mu_1, \ldots, \mu_r) \) of \( n \), let

(4.23) 
\[
S_\mu \cong S_{\mu_1} \times \cdots \times S_{\mu_r}
\]

denote the corresponding Young subgroup of \( S_n \). Then let \( \mathcal{A}_\mu(F) \) denote the parabolic subalgebra of \( \mathcal{A}_n(F) \) generated by \( F^\otimes n, \mathbb{k}[x_1, \ldots, x_n] \), and \( S_\mu \). So we have an isomorphism of graded superalgebras

\[
\mathcal{A}_\mu(F) \cong \mathcal{A}_{\mu_1}(F) \otimes \cdots \otimes \mathcal{A}_{\mu_r}(F),
\]

and an even isomorphism of \((\mathbb{Z} \times \mathbb{Z}_2)\)-graded \( \mathbb{k} \)-modules

\[
\mathcal{A}_\mu(F) \cong \mathcal{P}_n(F) \otimes \mathbb{k} S_\mu.
\]

Let \( D_{\mu,\nu} \) denote the set of minimal length \((S_\mu, S_\nu)\)-double coset representatives in \( S_n \). By [DJ86, Lem. 1.6(ii)], for \( \pi \in D_{\mu,\nu} \), \( S_\mu \cap \pi S_\nu \pi^{-1} \) and \( \pi^{-1} S_\mu \pi \cap S_\nu \) are Young subgroups of \( S_n \). So we can define compositions \( \mu \cap \pi \nu \) and \( \pi^{-1} \mu \cap \nu \) by

\[
S_\mu \cap \pi S_\nu \pi^{-1} = S_{\mu \cap \pi \nu} \quad \text{and} \quad \pi^{-1} S_\mu \pi \cap S_\nu = S_{\pi^{-1} \mu \cap \nu}.
\]

Furthermore, the map \( w \mapsto \pi^{-1} w \pi \) restricts to a length preserving isomorphism

\[
S_{\mu \cap \pi \nu} \to S_{\pi^{-1} \mu \cap \nu}.
\]

One can verify that, for \( \pi \in D_{\mu,\nu} \) and \( s_i \in S_{\mu \cap \pi \nu} \), we have \( \pi^{-1}(i+1) = \pi^{-1}(i) + 1 \), and hence \( \pi^{-1} s_i \pi = s_{\pi^{-1} i} \). Thus, for each \( \pi \in D_{\mu,\nu} \), we have an algebra isomorphism

\[
\varphi_{\pi^{-1}} : \mathcal{A}_{\mu \cap \pi \nu}(F) \to \mathcal{A}_{\pi^{-1} \mu \cap \nu}(F),
\]

\[
\varphi_{\pi^{-1}}(\sigma) = \pi^{-1} \sigma \pi, \quad \varphi_{\pi^{-1}}(f_i) = f_{\pi^{-1} i}, \quad \varphi_{\pi^{-1}}(x_i) = x_{\pi^{-1} i}, \quad \sigma \in S_{\mu \cap \pi \nu}, \quad f \in F, \quad 1 \leq i \leq n.
\]

If \( N \) is a left \( \mathcal{A}_{\pi^{-1} \mu \cap \nu}(F) \)-module, we let \( \pi N \) denote the left \( \mathcal{A}_{\mu \cap \pi \nu} \)-module with action given by

\[
a \cdot v = \varphi_{\pi^{-1}}(a) v, \quad a \in \mathcal{A}_{\mu \cap \pi \nu}, \quad v \in \pi N = N,
\]

where juxtaposition denotes the original action on \( N \).

The inclusion \( \mathcal{A}_\mu(F) \subseteq \mathcal{A}_n(F) \) gives rise to restriction and induction functors

(4.24) 
\[
\text{Res}_{\mu}^{n} : \mathcal{A}_n(F)\text{-mod} \to \mathcal{A}_\mu(F)\text{-mod}, \quad \text{Ind}^{\mu}_{\nu} : \mathcal{A}_\mu(F)\text{-mod} \to \mathcal{A}_n(F)\text{-mod}.
\]

Theorem 4.22 (Mackey Theorem for \( \mathcal{A}_n(F) \)). Suppose \( M \) is an \( \mathcal{A}_\nu(F) \)-module. Then \( \text{Res}_{\nu}^{\mu} \text{Ind}_{\nu}^{\mu} M \) admits a filtration with subquotients evenly isomorphic to

\[
\text{Ind}_{\mu \cap \pi \nu}^{\mu} \pi(\text{Res}_{\nu^{-1} \mu \cap \nu}^{\nu^{-1}} M),
\]

one for each \( \pi \in D_{\mu,\nu} \). Furthermore, the subquotients can be taken in any order refining the strong Bruhat order on \( D_{\mu,\nu} \). In particular, \( \text{Ind}_{\mu \cap \nu}^{\mu} \text{Res}_{\mu \cap \nu}^{\nu} M \) appears as a submodule.
Proof. The proof is almost identical to the proofs of [Kle05, Th. 3.5.2] and [Kle05, Th. 14.5.2] and hence will be omitted. □

4.7. Intertwining elements. Intertwining elements play a fundamental role in the treatment of integral modules for degenerate affine Hecke algebras (see [Kle05, §3.8]), affine Sergeev algebras (see [Kle05, §14.8]), and wreath Hecke algebras (see [WW08, §5.2]). While the treatment of integral modules for affine wreath product algebras is beyond the scope of the current paper, we introduce here intertwining elements in these algebras and prove that they have properties analogous to those in the aforementioned special cases.

For $1 \leq i < n$, define

$$\Omega_i := x_i^\theta s_i - s_i x_i^\theta + s_i(x_i^\theta - x_i^\theta) + t_i^{(\theta)} = (x_i^\theta - x_i^\theta)s_i - t_i^{(\theta)},$$

where the last two equalities follow from Lemma 3.11.

**Lemma 4.23.** For $1 \leq i, j < n$, we have

\begin{equation}
\Omega_i^2 = \left( t_i^{(\theta)} \right)^2 - \left( x_i^\theta - x_i^\theta \right)^2, \tag{4.25}
\end{equation}

\begin{equation}
\Omega_i f_j = f_{s_i(j)} \Omega_i, \tag{4.26}
\end{equation}

\begin{equation}
\Omega_i x_j = x_{s_i(j)} \Omega_i. \tag{4.27}
\end{equation}

\begin{equation}
\Omega_i \Omega_j = \Omega_j \Omega_i \text{ if } |i - j| > 1. \tag{4.28}
\end{equation}

**Proof.** We have

\[
\Omega_i^2 = s_i(x_i^\theta - x_i^\theta)s_i(x_i^\theta - x_i^\theta) + s_i(x_i^\theta - x_i^\theta)t_i^{(\theta)} + t_i^{(\theta)}s_i(x_i^\theta - x_i^\theta) + \left( t_i^{(\theta)} \right)^2.
\]

Now, using Lemma 3.11, we have

\[
(x_i^\theta - x_i^\theta)s_i = s_i(x_i^\theta - x_i^\theta) - t_i^{(\theta)} - t_i^{(\theta)}.
\]

Using this and (3.26), relation (4.25) follows. Relation (4.26) follows easily from (3.3) and the fact that $\theta$ is the order of $\psi$.

To prove (4.27), we compute

\[
x_{i+1} \Omega_i - \Omega_i x_i = (x_{i+1}s_i - s_i x_i)(x_i^\theta - x_i^\theta) - \sum_{b \in B} b_i(x_i^\theta - x_i^\theta)b_{i+1}^\vee \tag{3.5}
\]

\[
= t_{i,i+1}(x_i^\theta - x_i^\theta) - \sum_{b \in B} b_i(x_i^\theta - x_i^\theta)b_{i+1}^\vee = 0,
\]

and

\[
x_i \Omega_i - \Omega_i x_{i+1} = (x_i^\theta - x_i^\theta)(x_i s_i - s_i x_i) + \sum_{b \in B} b_{i+1}(x_{i+1}^\theta - x_i^\theta)b_{i}^\vee \tag{3.9}
\]

\[
= -(x_{i+1}^\theta - x_i^\theta)t_{i+1,i} + \sum_{b \in B} b_{i+1}(x_{i+1}^\theta - x_i^\theta)b_{i}^\vee = 0.
\]

This completes the proof of (4.27). Relation (4.28) is straightforward. □
5. Classification of simple modules

In this section we assume that \( k \) is an algebraically closed field of characteristic not equal to two. We also continue to assume, except as noted in the first subsection, that its characteristic does not divide the order \( \theta \) of the Nakayama automorphism \( \psi \). Hence all simple \( A_\psi(F) \)-modules are finite-dimensional (see Remark 4.17). In this section we classify these modules. Our approach is inspired by that of [WW08]. However, the fact that our setup is more general (e.g., we allow the Nakayama automorphism to be nontrivial) makes the proofs somewhat more involved. We note that the most important case is when \( \delta = 0 \) (i.e., the \( \mathbb{Z} \)-grading on \( F \) is trivial). See Remark 5.21.

5.1. Simple \( k[x] \ltimes F \)-modules. The results of this subsection are valid for any algebra \( F \) over an algebraically closed field \( k \) of characteristic not equal to two, and any algebra automorphism \( \psi \) of \( F \) of finite order \( \theta \). We continue to use the notation \( F \) and \( \psi \) that were introduced for Frobenius algebras earlier since that will be our main interest.

Fix a simple \( F \)-module \( L \). For \( k \in \mathbb{Z} \), write \( kL \) for \( \psi^k L \) (see (2.4)). Let \( r_L \) be the smallest positive integer such that \( \tau^L \psi^k L \simeq L \), and let \( m_L \) be the smallest positive integer such that \( \psi^m_L(f)(v) = f(v) \) for all \( f \in F \) and \( v \in L \). It follows that \( r_L \) divides \( m_L \) and that \( m_L \) divides \( \theta \).

Example 5.1. (a) If \( F = \text{Cl} \) and \( L \) is its unique simple module, then \( r_L = 1 \) (since there is only one simple module and it is of type \( \text{Q} \)) and \( m_L = \theta = 2 \).

(b) Recall the Taft Hopf algebra of Example 2.3. We have \( \psi^k L_k \simeq L_{k+1} \) if we take \( \omega = e^{2\pi i/q} \). Thus \( r_L = m_L = \theta = q \).

(c) Let \( G \) be a finite group with a noncentral element \( h \). Let \( \psi \) be conjugation by \( h \), and let \( L \) be the one-dimensional trivial representation of \( G \). Then \( r_L = m_L = 1 \), while \( \theta > 1 \).

By Schur’s Lemma and the definition of \( r_L \), there is an even \( F \)-module isomorphism

(5.1) \[\tau_L: L \overset{\simeq}{\to} r_L L,\]

which is unique up to a nonzero scalar. Thus \( \tau(\psi^L(f)v) = f\tau(v) \) for all \( f \in F \) and \( v \in L \). Note that \( \tau^m_L/r_L L \simeq m_L L = L \). Thus, by Schur’s Lemma, \( \tau^m_L/r_L L \) is multiplication by a nonzero scalar. Rescaling \( \tau_L \) if necessary, we may assume that

(5.2) \[\tau^m_L/r_L L = \text{id}.\]

Now, for \( a \in k \) we define an \( F \)-module \( L(a) \) as follows. We let

(5.3) \[L(a) := \bigoplus_{\ell=0}^{r_L-1} \ell L,\]

as \( F \)-modules. The action of \( x \) is given by

\[x(v_0, \ldots, v_{r_L-1}) = (v_1, v_2, \ldots, v_{r_L-1}, a\tau_L(v_0)), \quad v_i \in \ell L, \quad 1 \leq i \leq r_L - 1.\]

It is straightforward to verify that \( L(a) \) is a simple \( k[x] \ltimes F \)-module when \( a \in k^\times \).

Example 5.2. Suppose \( F = \text{Cl} \) and \( \psi \) is the Nakayama automorphism (see Example 3.7). The algebra \( F \) has one simple module \( L \), which arises from the action of \( F \) on itself by left multiplication. Then \( r_L = 1 \) and \( m_L = \theta = 2 \). In the notation of [Kle05, §16.1], choose \( \tau_L \) to be the map given by \( \tau_L(v_1) = v_1 \) and \( \tau_L(v_{r_L-1}) = -v_{r_L-1} \). Then the module \( L \left( \sqrt{q(i)} \right), i \in I \), (notation as in the current paper, with \( I \) defined in [Kle05, (15.2)]) is precisely the module denoted by \( L(i) \) in [Kle05, §16.1].

Lemma 5.3. Suppose \( L \) and \( L' \) are simple \( F \)-modules and \( a, b \in k^\times \). The \( k[x] \ltimes F \)-modules \( L(a) \) and \( L'(b) \) are evenly isomorphic if and only if \( a = b \) and \( L' \simeq \ell L \) as \( F \)-modules for some \( \ell \in \{0, \ldots, r_L-1\} \).
Proof. Suppose $L(a) \simeq L'(b)$ are evenly isomorphic as $\mathbb{k}[x] \ltimes F$-modules. Considering the decomposition as $F$-modules, we see that $L' \simeq \ell L$ for some $\ell \in \{0, \ldots, r_L - 1\}$. Relabelling if necessary, we may assume $L = L'$. Then, since $x^{r_L}$ acts on $L(a)$ as $a\tau_L$ and on $L(b)$ as $b\tau_L$, it follows that $a = b$. The converse statement is clear. \hfill \Box

Remark 5.4. 
(a) Note that the indexing $L(a)$ depends on the choice of $\tau_L$. In light of the condition (5.2), the choice of $\tau_L$ is unique up to multiplication by an $(m_L/r_L)$-th root of unity. Different choices of $\tau_L$ simply shift the parameter $a$ by multiplication by this root of unity.

(b) It is important to note that Lemma 5.3 is a statement about even isomorphisms. In general, it is possible for $L(a) \cong L(b)$, via an odd isomorphism, for $a \neq b$. In particular, when $F = \mathbb{C}$, we have $\Pi L(a) \cong L(-a)$, and so $L(a)$ is isomorphic to $L(-a)$ via an odd isomorphism. See [Kle05, §16.1].

We have an algebra homomorphism

$$ \mathbb{k}[x] \ltimes F \to F, \quad x \mapsto 0, \quad f \mapsto f, \quad f \in F. $$

For an $F$-module $V$, let $V(0)$ denote the $\mathbb{k}[x] \ltimes F$-module obtained by inflating via the homomorphism (5.4). Clearly $V(0)$ is simple if $V$ is, and $V(0) \cong V'(0)$ as $\mathbb{k}[x] \ltimes F$-modules if and only if $V \cong V'$ as $F$-modules.

Proposition 5.5. The modules

$$ L(a), \quad L \in \mathcal{S}(F), \quad a \in \mathbb{k}, $$

are a complete list, up to even isomorphism and degree shift, of simple $\mathbb{k}[x] \ltimes F$-modules. Furthermore,

- if $a \neq b$, then $L(a) \neq L(b)$,
- $L(0) \simeq L'(0)$ if and only if $L \simeq L'$, and
- for $a \neq 0$, $L(a) \simeq L'(a)$ if and only if $L' \simeq \ell L$ for some $\ell \in \mathbb{Z}$.

Proof. It remains to prove that every simple $\mathbb{k}[x] \ltimes F$-module is evenly isomorphic to $L(a)$ for some simple $F$-module $L$ and $a \in \mathbb{k}$.

Let $V$ be a simple $\mathbb{k}[x] \ltimes F$-module. Shifting the degree if necessary, we assume that $V$ is concentrated in nonnegative degree with nonzero degree zero piece. We first prove that $V$ is semisimple as an $F$-module. Since the center of $\mathbb{k}[x] \ltimes F$ contains $\mathbb{k}[x^0]$, and $\mathbb{k}[x] \ltimes F$ is of finite rank over $\mathbb{k}[x^0]$, it follows that $V$ is finite dimensional. Now let $L$ be a simple $F$-submodule of $V$. Then $x^k L$ is either zero or is a simple $F$-submodule of $V$ eventually isomorphic to $kL$. Since $\sum_{k=0}^{\infty} x^k L$ is a $\mathbb{k}[x] \ltimes F$-submodule of $V$ and $V$ is simple as a $\mathbb{k}[x] \ltimes F$-module, we have $V = \sum_{k=0}^{\infty} x^k L$ (with the sum actually being finite by the finite-dimensionality of $V$), and so $V$ is semisimple as an $F$-module.

It follows that

$$ V = \bigoplus_{\ell=0}^{r_L-1} V_{\ell}, $$

where $V_{\ell}$ is the $\ell L$-isotypic component of $V$.

Now, if $x$ acts by zero on any $F$-submodule of a summand $V_{\ell}$ in (5.5), that submodule would be a $\mathbb{k}[x] \ltimes F$-submodule, and hence equal to all of $V$. Furthermore, simplicity of $V$ would imply that $V_{\ell} \simeq \ell L$. Thus $V \simeq \ell L(0)$ and we are done.

Now assume that $x$ does not act by zero on any $F$-submodule of a summand in (5.5). It follows from Schur’s Lemma that the action of $x$ induces linear isomorphisms $V_{\ell} \to V_{\ell+1}$ for all $0 \leq \ell \leq r_L - 1$. In addition, $x^{r_L}$ induces a linear automorphism of $V_0$. Let $v \in V_0$ be an eigenvector of $x^{r_L}$. It follows from the relation $x f = \psi^{-1}(f)x$, $f \in F$, that $x^{r_L}$ leaves $Fv$ invariant. Then $\bigoplus_{\ell=0}^{r_L-1} x^\ell Fv$ is a $\mathbb{k}[x] \ltimes F$-submodule of $V$, with $x^\ell Fv \subseteq V_{\ell}$. Since $V$ is simple, it follows that $Fv = V_0$. So $V_0$ is simple and hence isomorphic to $L$. 


It follows from the definition of \( r_L \) that the action of \( x^{r_L} \) induces an even isomorphism \( L \xrightarrow{\sim} r_L L \). Thus there is some nonzero scalar \( a \in k^\times \) such that \( x^{r_L} \) acts by \( a r_L \). Then clearly \( V \xrightarrow{\sim} L(a) \). \(\square\)

5.2. **Action of** \( t_{k,\ell} \). For \( \mathbb{Z} \)-graded super vector spaces \( V_1 \) and \( V_2 \) define the linear map

\[
\text{flip}: V_1 \otimes V_2 \to V_2 \otimes V_1, \quad \text{flip}(v_1 \otimes v_2) = (-1)^{v_1v_2} v_2 \otimes v_1.
\]

In particular, \( \text{flip}: F^\otimes 2 \to F^\otimes 2 \) is an algebra homomorphism, and for \( F \)-modules \( V_1 \) and \( V_2 \),

\[
\text{flip}: \text{flip}(V_1 \otimes V_2) \to V_2 \otimes V_1
\]

is an isomorphism of \( F^\otimes 2 \)-modules. As explained in Section 2.1, if \( V_1 \) and \( V_2 \) are simple, we have an induced even isomorphism

\[
\text{flip}: \text{flip}(V_1 \otimes V_2) \to V_2 \otimes V_1.
\]

**Lemma 5.6.** Suppose \( L \) and \( L' \) are simple \( F \)-modules.

(a) The element \( t_{1,2} \in F^\otimes 2 \) acts as zero on the \( F^\otimes 2 \)-module \( L \otimes L' \) unless \( \psi L \xrightarrow{\sim} L \cong L' \) and \( \delta = 0 \) (i.e. \( F \) is concentrated in degree zero).

(b) If \( \psi L \xrightarrow{\sim} L \) as \( F \)-modules and \( \delta = 0 \), then \( t_{1,2} \) acts on \( L \otimes L' \) either as zero or as \( \text{flip} \circ (\tau \otimes \text{id}) \), where \( \tau: L \xrightarrow{\sim} \psi L \) is an even isomorphism of \( F \)-modules.

**Proof.** By Lemma 3.12, we have

\[
(f \otimes g)t_{1,2} = (-1)^{fg} t_{1,2}(\psi(g) \otimes f), \quad f, g \in F.
\]

Thus, \( t_{1,2} \) induces an even homomorphism (since \( t_{1,2} \) is even) of \( F^\otimes 2 \)-modules

\[
L \otimes L' \to \text{flip}(L \otimes \psi L')
\]

By (5.7), we then have an even homomorphism of \( F^\otimes 2 \)-modules

\[
\text{flip} \circ t_{1,2}: L \otimes L' \to \psi L' \otimes L.
\]

Thus, by Schur’s Lemma, \( t_{1,2} \) acts as zero unless \( L \otimes L' \xrightarrow{\sim} \psi L' \otimes L \) as \( F^\otimes 2 \)-modules or, equivalently, unless \( \psi L \xrightarrow{\sim} L \cong L' \). Furthermore, since \( |t_{1,2}| = \delta \), it follows that \( t_{1,2} \) must act as zero if \( \delta > 0 \) (since isomorphisms live in degree zero). This proves part (a).

Now suppose \( \tau: L \to \psi L \) is an even isomorphism of \( F \)-modules and \( \delta = 0 \). Then

\[
(\tau^{-1} \otimes \text{id}) \circ \text{flip} \circ t_{1,2}: L \otimes L \to L \otimes L
\]

is an even homomorphism of \( F^\otimes 2 \)-modules and thus must be multiplication by a scalar, by Schur’s Lemma. If this scalar is nonzero, we may rescale \( \tau \) so that this scalar is equal to one. This completes the proof of part (b). \(\square\)

If \( L_1, \ldots, L_n \) are simple \( F \)-modules with \( L_k = L_\ell \), we have an even \( F \)-module homomorphism

\[
\rho_{k,\ell}: L_1 \otimes \cdots \otimes L_n \to L_1 \otimes \cdots \otimes L_n
\]

that superpermutes the \( k \)-th and \( \ell \)-th factors.

**Corollary 5.7.** Suppose \( k, \ell \in \{1, \ldots, n\} \) and \( k \neq \ell \). If \( L_k = L_\ell \) and \( \delta = 0 \), then \( t_{k,\ell} \) acts on \( L_1 \otimes \cdots \otimes L_n \) as zero or as

\[
\rho_{k,\ell} \circ \left( \text{id} \otimes (k-1) \right. \otimes \tau \otimes \text{id} \otimes (n-k)
\]

where \( \tau: L_k \to \psi L_k \) is an even isomorphism of \( F \)-modules. Otherwise, \( t_{k,\ell} \) acts as zero.

**Proof.** This follows from Lemma 5.6. \(\square\)
Example 5.8.  (a) In the setting of Example 3.5, the Nakayama automorphism $\psi$ is the identity and the content of Lemma 5.6 is contained in [WW08, Lem. 3.1].

(b) If $F = \mathbb{C}$, then $F$ has one simple module $L$, $\psi$ has order 2, $\delta = 0$, and $\psi : L \xrightarrow{\cong} L$. Then $t_{1,2}$ acts on $L \oplus L$ as a nonzero scalar multiple of flip $\circ (\psi \otimes \text{id})$.

5.3. Associated algebras. Recall that $\psi$ acts on the set $S(F)$ by twisting (see (2.4)). Let $N$ be the number of $\psi$-orbits in $S(F)$, and let

$$L_1, L_2, \ldots, L_N$$

be a set of representatives of these orbits. For $k \in \{1, 2, \ldots, N\}$, let $r_k := r_{L_k}$, so that $r_k$ is the smallest positive integer such that $r_k L_k \simeq L_k$. Hence

$$L_1^{r_1}, L_2^{r_2}, \ldots, L_N^{r_N},$$

is an enumeration of the elements of $S(F)$.

Let $A_n(F)$ denote the full subcategory of $A_n(F)$-modules that are semisimple as $F^{\otimes n}$-modules. Let

$$C_n := \{\mu = (\mu_1, \ldots, \mu_N) \mid \mu_1, \ldots, \mu_N \in \mathbb{N}, \mu_1 + \cdots + \mu_N = n\}$$

denote the set of compositions of $n$ of length at most $N$. Recall the parabolic subalgebras $A_{\mu}(F)$ of $A_n(F)$ introduced in Section 4.6. We let $A_{\mu}(F)$-mod denote the full subcategory of $A_{\mu}(F)$-modules consisting of finite-dimensional $A_{\mu}(F)$-modules that are semisimple as $F^{\otimes n}$-modules.

For $\mu \in C_n$ and $1 \leq k \leq n$, let $\ell_k$ denote the unique integer such that

$$\mu_1 + \cdots + \mu_{\ell_k-1} < k \leq \mu_1 + \cdots + \mu_{\ell_k}.$$

For $1 \leq k \leq N$, let $\tau_k = \tau_{L_k}$ (see (5.1)). If $t_{1,2}$ acts as zero on $L_k \oplus L_k$, let $a_k = 1$. (This choice is not crucial; we choose $a_k = 1$ for simplicity.) Otherwise, choose $a_k \in \mathbb{K}^\times$ such that $t_{1,2}$ acts on $L_k \oplus L_k$ as flip $\circ (a_k \tau_k \otimes \text{id})$ (see Lemma 5.6).

Let

$$L(\mu) := L_1^{\otimes \mu_1} \otimes \cdots \otimes L_N^{\otimes \mu_N}.$$  

Thus $L(\mu)$ is a $P_n(F)$-module on which the action of $x_k$ is invertible for all $1 \leq k \leq n$. By a slight abuse of notation, we will write $x_k^{-1}$ for the endomorphism of $L(\mu)$ that is the inverse of the action of $x_k$. If $\ell_k = \ell_{k+1}$ (i.e. $s_k \in S_{\mu}$), it follows from our choices that, as an operator on $L(\mu)$, we have

$$t_{k,k+1} = \begin{cases} 
0 & \text{if } t_{1,2} \text{ acts as zero on } L_{\ell_k} \oplus L_{\ell_k}, \\
\rho_{k,k+1} x_k & \text{if } t_{1,2} \text{ does not act as zero on } L_{\ell_k} \oplus L_{\ell_k}.
\end{cases}$$

We let

$$E(\mu) = \text{END}_{F^{\otimes n}}^\text{op} L(\mu) \simeq \left( \text{END}_{F}^\text{op} L_1(a_1) \right)^{\otimes \mu_1} \otimes \cdots \otimes \left( \text{END}_{F}^\text{op} L_N(a_N) \right)^{\otimes \mu_N},$$

so that $L(\mu)$ is naturally a right $E(\mu)$-module via the action

$$vz = (-1)^{\mu_2} zv, \quad z \in E(\mu), \quad v \in L(\mu),$$

where, on the right side, we view $z$ as an element of $\text{END}_{F^{\otimes n}} L(\mu)$.

Note that, for $1 \leq k \leq N$,

$$\text{END}_{F} L_k(a_k) \simeq (\text{END}_{F} L_k)^{\oplus r_k}.$$  

Now, for $1 \leq k \leq N$, we have a natural right action of $\mathbb{K}[y]$ on $\text{END}_{F}^{\text{op}} L_k(a_k)$ given by

$$z \cdot y = x^{-1} z x, \quad z \in \text{END}_{F}^{\text{op}} L_k(a_k),$$

where, on the right side, $x$ denotes the endomorphism of $L_k(a_k)$ given by the action of $x$ and, since this action is invertible, $x^{-1}$ denotes its inverse. Thus we can form the smash product $\mathbb{K}[y] \ltimes \text{END}_{F}^{\text{op}} L_k(a_k)$. 

For $1 \leq \ell \leq N$ and $n \in \mathbb{N}$, define
\begin{equation}
\mathcal{H}^\ell_n = \begin{cases} 
\langle k[y] \times \text{END}^{op}_F L_\ell(a_\ell) \rangle \rtimes_n \chi_\rho S_n & \text{if } t_{1,2} \text{ acts as zero on } L_\ell \otimes L_\ell, \\
\mathcal{A}_n (\text{END}^{op}_F L_\ell) & \text{if } t_{1,2} \text{ does not act as zero on } L_\ell \otimes L_\ell.
\end{cases}
\end{equation}
(Recall that we use the notation $\rtimes$ to denote a smash product with $kS_n$, where the action is via superpermutations.) Note that $\mathcal{H}^\ell_n$ depends on $F$ and our ordering of the simple $F$-modules. It is also important to note that $r_\ell = 1$ whenever $t_{1,2}$ does not act as zero on $L_\ell \otimes L_\ell$. In this case, we have
\[
\text{END}^{op}_F L_\ell = \text{END}^{op}_F L_\ell(a_\ell) \simeq \begin{cases} 
k & \text{if } L_\ell \text{ is of type } \mathcal{M}, \\
\mathcal{C} & \text{if } L_\ell \text{ is of type } \mathcal{Q}.
\end{cases}
\]

Define
\begin{equation}
\mathcal{R}_n = \bigoplus_{\mu \in \mathcal{C}_n} \mathcal{R}_\mu, \quad \mathcal{R}_\mu = \mathcal{H}^{\mu_1}_1 \otimes \cdots \otimes \mathcal{H}^{\mu_N}_N.
\end{equation}

We will denote the polynomial generators in $\mathcal{R}_n$ by $y_1, \ldots, y_n$ to avoid confusion with the generators $x_1, \ldots, x_n \in \mathcal{A}_n(F)$. Note that $E(\mu)$ can naturally be viewed as a subalgebra of $\mathcal{R}_\mu$. Then $\mathcal{R}_\mu$ is generated as an algebra by $E(\mu)$, $y_1, \ldots, y_n$, and $S_\mu$.

### 5.4. An equivalence of categories.

If $A_1$ and $A_2$ are algebras, $V$ is an $(A_1, A_2)$-bimodule, and $W$ is an $A_1$-module, then $\text{HOM}_{A_1}(V, W)$ is naturally a (left) $A_2$-module under the action
\[
(a\alpha)(v) = (-1)^{\delta(\alpha,v)}\alpha(va), \quad a \in A_2, \quad \alpha \in \text{HOM}_{A_1}(V, W), \quad v \in V.
\]

**Lemma 5.9.** Suppose $A$ is a graded superalgebra and $V$ is a finite-dimensional semisimple $A$-module. Let $\mathcal{C}_V$ denote the full subcategory of $A$-mod whose objects are evenly isomorphic to finite direct sums of degree shifts of simple submodules of $V$. Then, viewing $V$ as an $(A, \text{END}^{op}_A V)$-bimodule, the functors
\[
V \otimes_{\text{END}^{op}_A V} - : (\text{END}^{op}_A V)\text{-mod} \to \mathcal{C}_V, \quad \text{HOM}_A(V, -) : \mathcal{C}_V \to (\text{END}^{op}_A V)\text{-mod},
\]
yield an equivalence of categories between $\mathcal{C}_V$ and $(\text{END}^{op}_A V)\text{-mod}$.

**Proof.** Suppose $U \in (\text{END}^{op}_A V)\text{-mod}$. Then
\[
\text{HOM}_A(V, V \otimes_{\text{END}^{op}_A V} U) \simeq (\text{END}^{op}_A V) \otimes_{\text{END}^{op}_A V} U \simeq U,
\]
and these isomorphisms are natural in $U$. (We use the fact that, for any algebra $R$, $R \simeq R^{op}$ as $(R, R)$-bimodules.)

On the other hand, for $W \in \mathcal{C}_V$, we have an even homomorphism of $A$-modules
\[
V \otimes_{\text{END}^{op}_A V} \text{HOM}_A(V, W) \to W, \quad v \otimes \phi \mapsto \phi(v).
\]
This homomorphism is surjective by the definition of $\mathcal{C}_V$ and Schur’s Lemma. It is then injective since its domain and codomain have the same dimension by the double centralizer property. It is clearly natural in $W$. \[\]

For $V \in \mathcal{A}_n(F)\text{-mod}^\mathbb{N}$ and $\mu \in \mathcal{C}_n$, let $I_\mu V$ be the sum of all $F^\otimes n$-submodules of $V$ evenly isomorphic to degree shifts of submodules of $L(\mu)$. Then define
\begin{equation}
V_\mu := \sum_{\pi \in S_n} \pi(I_\mu V).
\end{equation}

**Lemma 5.10.** Suppose $\mu \in \mathcal{C}_n$ and $V \in \mathcal{A}_n(F)\text{-mod}^\mathbb{N}$. Then $I_\mu V$ is an $A_\mu(F)$-submodule of $V$ and $V_\mu$ is an $\mathcal{A}_n(F)$-submodule of $V$. Furthermore, $V_\mu \simeq \text{Ind}^n_{\mu}(I_\mu V)$.
Proof. As discussed in Section 5.1, it follows from (3.3) that \( x_i \) maps a simple \( F^{\otimes n} \)-submodule \( W \) of \( V \) either to zero or to a submodule evenly isomorphic to \( ^{\psi_1} W \). Hence, it follows from the definition of \( L(\mu) \) that \( I_\mu V \) is invariant under the action of \( P_n(F) \). It also follows from (3.6) that \( \pi \in S_\mu \) maps a simple \( F^{\otimes n} \)-submodule of \( V \) evenly isomorphic to degree shift of a simple summand of \( L(\mu) \) to one evenly isomorphic to another degree shift of a simple summand of \( L(\mu) \). Thus \( I_\mu V \) is an \( A_\mu(F) \)-submodule of \( V \). It then follows immediately from the definition that \( V_\mu \) is an \( A_\mu(F) \)-submodule of \( V \).

If \( I_\mu V = 0 \), the isomorphism asserted in the lemma is trivially true. So we assume that \( I_\mu V \neq 0 \). Then the inclusion \( I_\mu V \hookrightarrow V_\mu \) is a nonzero even homomorphism of \( A_\mu(F) \)-modules. Since induction is left adjoint to restriction, we have a nonzero even homomorphism \( \phi : \text{Ind}_{\mu}^{n}(I_\mu V) \rightarrow V_\mu \) of \( A_n(F) \)-modules. It follows from the definition of \( V_\mu \) that \( \phi \) is surjective. Furthermore, if \( X \) is a complete set of representatives of left cosets of \( S_\mu \) in \( S_n \), we have

\[
\text{Ind}_{\mu}^{n}(I_\mu V) = \bigoplus_{\pi \in X} \pi \otimes I_\mu V \quad \text{and} \quad V_\mu = \bigoplus_{\pi \in X} \pi(I_\mu V).
\]

Therefore, \( \text{Ind}_{\mu}^{n}(I_\mu V) \) and \( V_\mu \) have the same dimension, and so \( \phi \) is injective. \qed

Lemma 5.11. For \( V \in A_n(F)\text{-mod}^\delta \), we have the decomposition \( V = \bigoplus_{\mu \in \mathcal{C}_n} V_\mu \) in \( A_n(F)\text{-mod}^\delta \).

Proof. Let \( V \in A_n(F)\text{-mod}^\delta \). By definition, \( V \) is semisimple as an \( F^{\otimes n} \)-module. For \( \mu \in \mathcal{C}_n, V_\mu \) is the direct sum of the isotypic components corresponding to simple \( F^{\otimes n} \)-modules containing, for each \( 1 \leq i \leq N \), exactly \( \mu_i \) tensor factors evenly isomorphic to degree shifts of \( ^{\ell} L_i \) for some \( 0 \leq \ell \leq r_i \). The lemma follows. \qed

Note that \( S_\mu \) acts on \( L(\mu) \) by superpermuting the factors. For \( \pi \in S_\mu \), we will denote this action on \( L(\mu) \) by \( \rho_{\pi} \), to avoid confusion with the action of \( \pi \) on \( A_n(F)\text{-mod}^\delta \).

Proposition 5.12. Suppose \( \mu \in \mathcal{C}_n \) and \( V \in A_n(F)\text{-mod}^\delta \). Then \( \text{HOM}_{F^{\otimes n}}(A_n(F), V) \) is an \( R_\mu \text{-module under the action} \)

\[
\pi \circ \phi = \pi \phi \rho_{\pi}^{-1}, \quad \pi \in S_\mu, \\
y_k \circ \phi = x_k \phi x_k^{-1}, \quad 1 \leq k \leq n, \\
z \circ \phi = (-1)^{\bar{\psi} z} \phi z, \quad z \in E(\mu),
\]

for \( \phi \in \text{HOM}_{F^{\otimes n}}(A_n(F), V) \). Thus, we have a functor

\[
\text{HOM}_{F^{\otimes n}}(A_n(F), -) : A_n(F)\text{-mod}^\delta \rightarrow R_\mu\text{-mod}.
\]

Proof. It is straightforward to verify that \( \pi \circ \phi, y_k \circ \phi, \) and \( z \circ \phi \) are \( F^{\otimes n} \)-module homomorphisms.

Now suppose \( 1 \leq k \leq n - 1 \) such that \( \ell_k = \ell_{k+1} \) (i.e. such that \( s_k \in S_\mu \)). Then we have

\[
(s_k y_k) \circ \phi = s_k x_k \phi x_k^{-1} \rho_{\ell_k, k+1} \quad \text{and} \quad (y_{k+1} s_k) \circ \phi = x_{k+1} s_k \phi \rho_{\ell_k, k+1} x_{k+1}^{-1} = x_{k+1} s_k \phi x_k^{-1} \rho_{\ell_k, k+1}.
\]

Thus

\[
(s_k y_k - y_{k+1} s_k) \circ \phi = (s_k x_k - x_{k+1} s_k) \phi x_k^{-1} \rho_{\ell_k, k+1} \quad \text{and} \quad -t_{\ell_k, k+1} \phi x_k^{-1} \rho_{\ell_k, k+1} = 0 \quad \text{if} \quad t_{\ell_k, 1, 2} \text{ acts as zero on } L_{\ell_k} \bigoplus L_{\ell_k}, \\
-\phi \quad \text{if} \quad t_{\ell_k, 1, 2} \text{ does not act as zero on } L_{\ell_k} \bigoplus L_{\ell_k}.
\]

For \( z \in E(\mu) \), we also have

\[
(z y_k) \circ \phi = (-1)^{\bar{\psi} z} x_k \phi x_k^{-1} z = (-1)^{\bar{\psi} z} x_k \phi(z \cdot y_k) x_k^{-1} = (y_k(z \cdot y_k)) \circ \phi.
\]
The remainder of the relations, involving only elements of $S_\mu$, only elements of $E(\mu)$, or only the $y_k$, are straightforward to verify.

**Proposition 5.13.** Suppose $M$ is an $R_\mu$-module. Then $L(\mu) \otimes_{E(\mu)} M$ is an $A_\mu(F)$-module under the action

$$f \ast (w \otimes v) = fw \otimes v, \quad f \in F^\otimes n,$$

$$\pi \ast (w \otimes v) = \rho_\pi w \otimes \pi v, \quad \pi \in S_\mu,$$

$$x_k \ast (w \otimes v) = x_kw \otimes y_kv,$$

for $w \in L(\mu), v \in M$.

**Proof.** It is straightforward to verify that the given actions of $f$, $\pi$, and $x_k$ are well-defined on the tensor product, that is, that they are balanced with respect to the tensor product over $E(\mu)$.

The relations involving only elements of $F^\otimes n$, only elements of $S_\mu$, or only the $x_k$ are clear. If $1 \leq k \leq n$, $f \in F$, $w \in L(\mu)$, and $v \in M$, then

$$(f x_k) \ast (w \otimes v) = fx_kw \otimes y_kv = x_k \psi_k(f)w \otimes y_kv = (x_k \psi_k(f)) \ast (w \otimes v).$$

If $1 \leq k \leq n - 1$ and $\ell_k = \ell_{k+1}$ (i.e. $s_k \in S_\mu$), we also have

$$(s_k x_k) \ast (w \otimes v) = \rho_{k,k+1} x_k w \otimes s_k y_kv,$$

$$x_{k+1} s_k \ast (w \otimes v) = x_{k+1} \rho_{k,k+1} w \otimes y_{k+1} s_k v = \rho_{k,k+1} x_k w \otimes y_{k+1} s_k v.$$

So we have

$$(s_k x_k - x_{k+1} s_k) \ast (w \otimes v) = \rho_{k,k+1} x_k w \otimes (s_k y_kv - y_{k+1} s_k) v = -x_{k+1} * (w \otimes v).$$

**Proposition 5.14.** Let $M$ be an $R_\mu$-module. Then

$$\Phi: M \to HOM_{F^\otimes n} (L(\mu), L(\mu) \otimes_{E(\mu)} M), \quad \Phi(v)(w) = (-1)^{\bar{v}\bar{w}} w \otimes v, \quad v \in M, \quad w \in L(\mu),$$

is an even isomorphism of $R_\mu$-modules. Furthermore, $L(\mu) \otimes_{E(\mu)} M$ is a simple $A_\mu(F)$-module if and only if $M$ is a simple $R_\mu$-module.

**Proof.** It is clear that $\Phi(v) \in HOM_{F^\otimes n}(L(\mu), L(\mu) \otimes_{E(\mu)} M)$ for all $v \in M$. It is also straightforward to verify that $\Phi$ is a homomorphism of $R_\mu$-modules. By Lemma 5.9, $\Phi$ is an isomorphism of $E(\mu)$-modules. Therefore, it is bijective and hence an isomorphism of $R_\mu$-modules.

Suppose that $L(\mu) \otimes_{E(\mu)} M$ is a simple $A_\mu(F)$-module and $W$ is a nonzero $R_\mu$-submodule of $M$. Then $L(\mu) \otimes_{E(\mu)} W$ is an $A_\mu(F)$-submodule of $L(\mu) \otimes_{E(\mu)} M$. Hence $W = M$. So $M$ is simple.

Conversely, suppose that $M$ is a simple $R_\mu$-module and that $W$ is a nonzero $A_\mu(F)$-submodule of $L(\mu) \otimes_{E(\mu)} M$. Then, by Proposition 5.12, $HOM_{F^\otimes n}(L(\mu), W)$ is a nonzero $R_\mu$-submodule of $HOM_{F^\otimes n}(L(\mu), L(\mu) \otimes_{E(\mu)} M) \simeq M$, which is simple. Hence $HOM_{F^\otimes n}(L(\mu), W) \simeq M$. Then, as an $F^\otimes n$-module, $W \simeq L(\mu) \otimes_{E(\mu)} M$ by Lemma 5.9. So $L(\mu) \otimes_{E(\mu)} M$ is simple.

**Proposition 5.15.** Suppose $V \in A_n(F)$-mod. Then

$$\Upsilon: L(\mu) \otimes_{E(\mu)} HOM_{F^\otimes n}(L(\mu), I_\mu V) \to I_\mu V, \quad v \otimes \varphi \mapsto (-1)^{\bar{v}\bar{\varphi}} \varphi(v),$$

defines an even isomorphism of $A_\mu(F)$-modules.

**Proof.** By Lemma 5.10, $I_\mu V$ is an $A_\mu(F)$-module. Then, by Propositions 5.12 and 5.13, $L(\mu) \otimes_{E(\mu)} HOM_{F^\otimes n}(L(\mu), I_\mu V)$ is an $A_n(F)$-module. It is straightforward to verify that $\Upsilon$ is a homomorphism of $A_\mu(F)$-modules. Now, as an $F^\otimes n$-module, $I_\mu V$ is isomorphic to a finite direct sum of modules evenly isomorphic to degree shifts of simple summands of $L(\mu)$. It follows from Lemma 5.9 that $\Upsilon$ is bijective.
The following theorem is a generalization of [WW08, Th. 3.9], which treats the case where $F$ is the group algebra of a finite group (see Example 3.5).

**Theorem 5.16.** The functor $F: \mathcal{A}_n(F)\text{-mod}^a \to \mathcal{R}_n\text{-mod}$ defined by

$$F(V) = \bigoplus_{\mu \in C_n} \text{HOM}_{F^{\otimes n}}(L(\mu), I_\mu V)$$

is an equivalence of categories with inverse $G: \mathcal{R}_n\text{-mod} \to \mathcal{A}_n(F)\text{-mod}^a$ given by

$$G \left( \bigoplus_{\mu \in C_n} M_\mu \right) = \bigoplus_{\mu \in C_n} \text{Ind}_\mu^n \left( L(\mu) \otimes E(\mu) \right) M_\mu.$$

**Proof.** The map $\Phi$ of Proposition 5.14 is natural in $M$, and the map $\Upsilon$ of Proposition 5.15 is natural in $V$. Then, using Lemmas 5.10 and 5.11 and Propositions 5.12–5.15, it is straightforward to verify that $FG \cong \text{id}$ and $GF \cong \text{id}$. \qed

**Remark 5.17.** Note that if $F$ is semisimple, then $\mathcal{A}_n(F)\text{-mod} = \mathcal{A}_n(F)\text{-mod}^a$ and thus Theorem 5.16 implies that $\mathcal{A}_n(F)$ is Morita equivalent to $\mathcal{R}_n$.

5.5. **Simple $\mathcal{A}_n(F)\text{-modules.** We can now classify the simple $\mathcal{A}_n(F)$-modules.

**Proposition 5.18.** Every simple $\mathcal{A}_n(F)$-module is semisimple as an $F^{\otimes n}$-module. In particular, the categories $\mathcal{A}_n(F)\text{-mod}^a$ and $\mathcal{A}_n(F)\text{-mod}$ have the same class of simple modules.

**Proof.** Suppose $V$ is a simple $\mathcal{A}_n(F)$-module. Let $W$ be a simple $P_n(F)$-submodule of $V$. Then $\sum_{\pi \in S_n} \pi W$ is an $\mathcal{A}_n(F)$-submodule of $V$ (e.g., by Lemma 4.4) and hence $V = \sum_{\pi \in S_n} \pi W$ since $V$ is simple. By Proposition 5.5, $W \simeq L_1(a_1) \otimes \cdots \otimes L_n(a_n)$, where $L_1, \ldots, L_n$ are simple $F$-modules and $a_1, \ldots, a_n \in \mathbb{k}$. Thus each $\pi W$ is semisimple as an $F^{\otimes n}$-module, and so $V$ is semisimple as an $F^{\otimes n}$-module. \qed

The following theorem is a generalization of [WW08, Th. 4.4], which treats the case where $F$ is the group algebra of a finite group (see Example 3.5).

**Theorem 5.19.** Every simple $\mathcal{A}_n(F)$-module is evenly isomorphic to a module of the form

$$\text{Ind}_\mu^n \left( L(\mu) \otimes E(\mu) \right) (V_1 \oplus \cdots \oplus V_N),$$

where $\mu = (\mu_1, \ldots, \mu_N) \in C_n$, and $V_\ell$ is a simple $H^{\ell}_{n_\ell}$-module for $1 \leq \ell \leq N$. Furthermore, the above modules (over all $\mu \in C_n$ and $V_\ell$, $1 \leq \ell \leq N$, ranging over a set of representatives of even isomorphism classes, up to degree shift) form a complete set of pairwise not evenly-isomorphic simple $\mathcal{A}_n(F)$-modules, up to degree shift.

**Proof.** This follows immediately from Proposition 5.18 and Theorem 5.16. \qed

**Remark 5.20.** Theorem 5.19 reduces the study of simple $\mathcal{A}_n(F)$-modules to the study of simple modules for the $H^{\ell}_{n_\ell}$. If $t_{1,2}$ does not act as zero on $L_\ell \otimes L_\ell$, then $H^{\ell}_{n_\ell}$ is either a degenerate affine Hecke algebra (when $L_\ell$ is of type $\mathcal{H}$) or an affine Sergeev algebra (when $L_\ell$ is of type $Q$), as explained in Examples 3.4 and 3.7. The simple modules for these algebras have been classified. See, for example, [Kle05]. On the other hand, if $t_{1,2}$ acts as zero on $L_\ell \otimes L_\ell$, then $H^{\ell}_{m}$ is a wreath product algebra of the form $(\mathbb{k}[y] \ltimes A)^{\otimes m} \otimes S_m$, for a finite-dimensional algebra $A$. Modules for such wreath product algebras can be classified using the results of Section 5.1 and Clifford theory (e.g., see [RS17, §4] for the characteristic zero case). In this way, Theorem 5.19 provides a complete classification of the simple $\mathcal{A}_n(F)$-modules.
Remark 5.21 (Nontrivial \( \mathbb{Z} \)-gradings). The case where \( \delta > 0 \) (i.e. the \( \mathbb{Z} \)-grading on \( F \) is nontrivial) is of particular interest in Heisenberg categorification. In particular, it is exactly this property that allows one to conclude in [CL12, RS17] that the Grothendieck groups of the categories defined there are isomorphic to Heisenberg algebras. For the original Heisenberg category of [Kho14], where the grading is trivial, Khovanov proves that the Heisenberg algebra embeds into the Grothendieck group, and it is still an open conjecture that one has equality.

For affine wreath product algebras, the study of simple modules simplifies considerably in the presence of nontrivial \( \mathbb{Z} \)-gradings. Since \( |x_i| = \delta > 0 \), the \( x_i \) act as zero on any simple module. Similarly, the \( t_{i,j} \), which also have degree \( \delta \), act as zero on simple modules. Thus, the study of simple \( A_\pi(F) \)-modules reduces to the study of simple modules for wreath product algebras, which can be classified using Clifford theory. However, the full representation theory (i.e. the study of modules that are not necessarily simple) remains much more intricate in general.

6. Cyclotomic quotients

In this section we introduce and study cyclotomic quotients of affine wreath product algebras. These simultaneously unify and generalize cyclotomic quotients of degenerate affine Hecke algebras (see, e.g., [Kle05, §7.3]), wreath Hecke algebras (see [WW08, §5]) and affine Sergeev algebras (see, e.g., [Kle05, §15.3]). Choosing particular Frobenius algebras \( F \) will recover known results as well as proofs of open conjectures (Corollaries 6.12 and 6.16). In this section \( k \) is an arbitrary commutative ring of characteristic not equal to 2.

6.1. Shifting homomorphisms. For \( 1 \leq i \leq n \) and \( k \in \mathbb{Z} \), define

\[
F_i^{(k)} := \left( F_\psi^{(i-1,k,0^{n-i})} \right)^{S_i^k} \subseteq F^{\otimes n} \subseteq A_\pi(F), \quad \text{where } S_i^k = \{ \pi \in S_n | \pi i = i \},
\]

and \((0^{i-1},k,0^{n-i}) = (0,\ldots,0,k,0,\ldots,0)\), where the \( k \) appears in the \( i \)-th place. Intuitively, one should think of \( F_i^{(k)} \) as the subspace of \( F^{\otimes n} \) consisting of those elements that commute with elements of \( A_\pi(F) \) just as \( x_i^k \) does.

Lemma 6.1. For \( 1 \leq i \leq n \) and \( \pi \in S_n \), we have \( \pi F_i^{(k)} \pi^{-1} = F_i^{(k)} \).

Proof. It suffices to show that \( \pi F_i^{(k)} \pi^{-1} \subseteq F_i^{(k)} \), since the reverse inclusion then follows by considering \( \pi^{-1} \). First, it is clear that

\[
\pi F_i^{(k)} \pi^{-1} \subseteq \pi F_i^{(k)} \subseteq F_i^{(k)} \subseteq F_i^{(0^{i-1},k,0^{n-i})},
\]

where the last inclusion follows from the fact that \( \pi \left( F_\psi^{(\alpha)} \right) = F_\psi^{(\pi \alpha)} \) for \( \alpha \in \mathbb{Z}^n \). Now suppose \( f \in F_i^{(k)} \).

For \( \pi_1 \in S_{\pi_1}^n \), we have

\[
\pi_1 (\pi_1 f \pi_1^{-1}) = \pi_1 \pi_1 \pi_1 f \pi_1^{-1} \pi_1^{-1} = \pi \left( \pi_1 \pi_1 \pi_1 \pi_1 f \right) \pi_1^{-1} \pi_1^{-1} = \pi \left( \pi_1 \pi_1 \pi_1 \pi_1 f \right) \pi_1^{-1} = \pi f \pi_1^{-1},
\]

where the last equality follows from the fact that \( \pi_1 \pi_1 \pi_1 \pi_1 \in S_i^k \).

Proposition 6.2. Suppose \( c \in F_i^{(1)} \) is even of degree \( \delta \). Then

\[
x_i \mapsto x_i + s_{i-1} \cdots s_1 c s_1 \cdots s_{i-1}, \quad f \mapsto f, \quad \pi \mapsto \pi, \quad 1 \leq i \leq n, \quad f \in F^{\otimes n}, \quad \pi \in S_n,
\]

determines an algebra automorphism of \( A_\pi(F) \).

Proof. Let \( \zeta \) denote the given map. It suffices to show that \( \zeta \) is an algebra homomorphism, since it is then clearly invertible, with inverse given by the same map with \( c \) replaced by \( -c \).

For \( 1 \leq i \leq n \), define

\[
c^{(i)} = s_{i-1} \cdots s_1 c s_1 \cdots s_{i-1}.
\]
By Lemma 6.1, we have $c^{(i)} \in F_i^{(1)}$. It follows that, for all $1 \leq i, j \leq n$ and $f \in F^{\otimes n}$, we have

$$c^{(i)}c^{(j)} = c^{(j)}c^{(i)}, \quad x_ic^{(j)} = c^{(j)}x_i, \quad fc^{(i)} = c^{(i)}\psi_i(f),$$

and that

$$s_ic^{(j)} = c^{(j)}s_i, \quad 1 \leq i \leq n - 1, \quad 1 \leq j \leq n, \quad j \neq i, i + 1.$$

It is clear that $\zeta$ is a homomorphism when restricted to $F^{\otimes n}$ and $\mathbb{k}S_n$. It follows easily from the above relations that $\zeta(x_i)\zeta(x_j) = \zeta(x_j)\zeta(x_i)$ for all $1 \leq i, j \leq n$. Hence $\zeta$ is also a homomorphism when restricted to $\mathbb{k}[x_1, \ldots, x_n]$.

To prove that $\zeta$ preserves (3.5), we compute

$$\zeta(s_ix_i) = s_ix_i + s_ic^{(i)} = x_i+1s_i - t_{i,i+1} + s_ic^{(i)} = x_i+1s_i + c^{(i+1)}s_i - t_{i,i+1} = \zeta(x_i+1s_i - t_{i,i+1}).$$

The remaining relations (3.3), (3.4), and (3.6) are straightforward to verify.

6.2. Cyclotomic wreath product algebras. For $1 \leq k \leq \theta$, choose $e_k \in \mathbb{N}$ and degree $k\delta$ elements

$$c^{(k,1)}, \ldots, c^{(k,e_k)} \in F_i^{(k)}.$$

Define

$$C = \left( c^{(1,1)}, \ldots, c^{(1,e_1)}, \ldots, c^{(\theta,1)}, \ldots, c^{(\theta,e_\theta)} \right).$$

Let $J_C$ be the two-sided ideal in $A_n(F)$ generated by the homogeneous element

$$(6.2) \quad \chi_C = \prod_{k=1}^{\theta} \prod_{j=1}^{e_k} \left( x_1^k - c^{(k,j)} \right).$$

Note that $\chi_C$ is independent of the order of the factors in (6.2) by the definition of $F_i^{(k)}$. In fact, this was the essential motivation for the definitions (4.14), (4.16), and (6.1). We choose the elements $c^{(k,j)}$ so that they commute with elements of $A_n(F)$ in the same way that $x_1^k$ does.

We define the cyclotomic wreath product algebra to be the quotient

$$(6.3) \quad A^C_n(F) := A_n(F)/J_C.$$  

By convention, we set $A^C_n(F) = k$. Since $\chi_C$ is homogeneous, $A^C_n(F)$ inherits the structure of a graded superalgebra. We define the level of $C$ and the corresponding algebra $A^C_n(F)$ to be the polynomial degree of $\chi_C$, which we denote by $d_C$. Thus

$$(6.4) \quad d = d_C = \sum_{k=1}^{\theta} ke_k.$$

For any element of $A_n(F)$, we will denote its canonical image in $A^C_n(F)$ by the same symbol.

**Proposition 6.3.** If $\delta > 0$ or $\mathbb{k}$ is an algebraically closed field, then every finite-dimensional $A_n(F)$-module is the inflation of an $A^C_n(F)$-module for some $C$.

**Proof.** Let $V$ be a finite-dimensional $A_n(F)$-module. If $\delta > 0$, then the action of $x_1$ on $V$ is nilpotent by degree considerations and the result is clear. On the other hand, when $\delta = 0$, we simply take $\chi_C$ to be the minimal polynomial of $x_1$ on $V$, which factors in the form (6.2) when $\mathbb{k}$ is algebraically closed.
6.3. **Basis theorem.** We now describe an explicit basis for $\mathcal{A}_n^\mathcal{C}(F)$. Our approach is inspired by the methods of [Kle05, §7.5, §15.4].

Let $\chi_1 := \chi_\mathcal{C}$ and, for $i = 2, \ldots, n$, define

$$\chi_i = s_{i-1} \cdots s_1 \chi_1 s_1 \cdots s_{i-1}.$$  

**Lemma 6.4.** For $f \in F$ and $1 \leq i, j \leq n$, we have

$$f_i \chi_j = \begin{cases} 
\chi_j f_i & \text{if } i \neq j, \\
\chi_j \psi^d(f_i) & \text{if } i = j.
\end{cases}$$

In particular,

$$(6.5)$$

$$F^\otimes n \chi_j = \chi_j F^\otimes n, \quad 1 \leq j \leq n.$$  

**Proof.** The case $j = 1$ follows immediately from the definition of $\chi_\mathcal{C}$. The result for general $j$ then follows from a straightforward calculation using the definition of $\chi_j$. \hfill \Box

**Lemma 6.5.** We have $x_1 \chi_1 = \chi_1 x_1$. For $1 \leq i < j \leq n$, we also have $x_j \chi_i = \chi_i x_j$.

**Proof.** The first statement follows from the fact that $c^{(k)}(i) \in F^\otimes n$ for all $k$. For $1 \leq i < j \leq n$, we have

$$x_j \chi_i = (3.4) s_1 \cdots s_j x_j \chi_1 s_1 \cdots s_{i-1} = s_1 \cdots s_j (3.4) \chi_1 x_j s_1 \cdots s_{i-1} = \chi_i x_j.$$  

**Lemma 6.6.** For $i = 1, \ldots, n$, we have

$$\chi_i - x_i^d = \sum_{e=0}^{d-1} P_{i-1} x_i^e F^\otimes i S_i.$$  

**Proof.** The case $i = 1$ is immediate. Assuming the result for some $1 \leq i < n$, we have

$$\chi_{i+1} - x_{i+1}^d = (3.15) s_i \chi_i s_i - s_i x_i^d s_i - t_{i,i+1}^{(d)} s_i = s_i \left( \chi_i - x_i^d \right) s_i - t_{i,i+1}^{(d)} s_i$$

$$\leq s_i \sum_{e=0}^{d-1} P_{i-1} x_i^e F^\otimes i S_i s_i - t_{i,i+1}^{(d)} s_i \leq \sum_{e=0}^{d-1} P_{i-1} x_i^e F^\otimes (i+1) S_{i+1}. \quad \Box$$

For $Z = \{z_1 \prec \cdots \prec z_k\} \subseteq \{1, \ldots, n\}$, let

$$\chi_Z := \chi_{z_1} \chi_{z_2} \cdots \chi_{z_k} \in \mathcal{A}_n(F).$$

We also define

$$\Pi_n := \{ (\alpha, Z) \mid Z \subseteq \{1, \ldots, n\}, \quad \alpha \in \mathbb{N}^n, \quad \alpha_i < d \text{ whenever } i \notin Z \},$$

$$\Pi_n^+ := \{ (\alpha, Z) \in \Pi_n \mid Z \neq \emptyset \}.$$

**Lemma 6.7.** We have that $\mathcal{A}_n(F)$ is a free right $F^\otimes n \rtimes \mathfrak{S}_n$-module on the basis

$$\{ x^\alpha \chi_Z \mid (\alpha, Z) \in \Pi_n \}.$$  

**Proof.** Consider the ordering $\prec$ on $\mathbb{N}^n$ given by $\alpha \prec \alpha'$ if and only if

$$\alpha_n = \alpha'_n, \ldots, \alpha_{n-k} = \alpha'_{n-k}, \quad \alpha_k < \alpha'_k$$

for some $k \in \{1, \ldots, n\}$. Define a function

$$\gamma : \Pi_n \to \mathbb{N}^n, \quad \gamma(\alpha, Z) := (\gamma_1, \ldots, \gamma_n), \quad \text{where } \gamma_i = \begin{cases} 
\alpha_i & \text{if } i \notin Z, \\
\alpha_i + d & \text{if } i \in Z.
\end{cases}$$
Using induction on \( n \) and Lemma 6.6, we see that, for \((\alpha, Z) \in \Pi_n\), we have
\[
\alpha^\alpha \chi_n - \gamma(\alpha, Z) \in \sum_{\beta < \gamma(\alpha, Z)} \beta^\beta F^{\otimes n} S_n.
\]
Now, \( \gamma : \Pi_n \to \mathbb{N}^n \) is a bijection and, by Theorem 4.6, \( \{\alpha^\alpha \mid \alpha \in \mathbb{N}^n\} \) is a basis for \( \mathcal{A}_n(F) \) viewed as a right \( F^{\otimes n} \times \beta S_n \)-module. Thus, the lemma follows from (6.6).

\[ \square \]

**Lemma 6.8.** For \( n > 1 \), we have \( F^{\otimes (n-1)} S_{n-1} \chi_n F^{\otimes n} S_n = \chi_n F^{\otimes n} S_n \).

**Proof.** It follows from the definition of \( \chi_n \) and the relations in \( \mathcal{A}_n(F) \) that multiplication by \( s_j, 1 \leq j \leq n - 2 \), leaves the space \( \chi_n F^{\otimes n} S_n \) invariant. That multiplication by \( F^{\otimes (n-1)} \) also leaves this space invariant follows from (6.5).

\[ \square \]

**Lemma 6.9.** We have \( J_C = \sum_{i=1}^n P_n \chi_i F^{\otimes n} S_n \).

**Proof.** We have
\[
J_C = A_n(F) \chi_1 A_n(F) = A_n(F) \chi_1 P_n F^{\otimes n} S_n = A_n(F) \chi_1 F^{\otimes n} S_n
\]
\[
= P_n F^{\otimes n} S_n \chi_1 F^{\otimes n} S_n = \sum_{i=1}^n \sum_{u \in S(n, n-1)} P_n F^{\otimes n} s_{i-1} \cdots s_1 u \chi_1 F^{\otimes n} S_n
\]
\[
= \sum_{i=1}^n P_n F^{\otimes n} s_{i-1} \cdots s_1 \chi_1 F^{\otimes n} S_n = \sum_{i=1}^n P_n F^{\otimes n} \chi_i F^{\otimes n} S_n = \sum_{i=1}^n P_n \chi_i F^{\otimes n} S_n,
\]
where the third equality uses Lemma 6.5, and the final equality uses (6.5).

\[ \square \]

**Lemma 6.10.** For \( d > 0 \), we have \( J_C = \sum_{(\alpha, Z) \in \Pi_n^+} x^{\alpha} \chi_n F^{\otimes n} S_n \).

**Proof.** We proceed by induction on \( n \). When \( n = 1 \), the right side of the equality in the statement of the lemma is
\[
\sum_{k \in \mathbb{N}} x_1^k \chi_1 F = P_1 \chi_1 F = J_C,
\]
where, in the last equality, we use (6.5) and Lemma 6.5.

Now suppose \( n > 1 \). Let \( J_C' := A_{n-1}(F) \chi_1 A_{n-1}(F) \), so that
\[
J_C' = \sum_{(\alpha', Z') \in \Pi_{n-1}^+} x^{\alpha'} \chi_{Z'} F^{\otimes (n-1)} S_{n-1}
\]
by the induction hypothesis. Let \( J = \sum_{(\alpha, Z) \in \Pi_n^+} x^{\alpha} \chi_n F^{\otimes n} S_n \). It is clear that \( J \subseteq J_C \). Therefore, by Lemma 6.9, it suffices to prove that \( x^{\alpha} \chi_i F^{\otimes n} S_n \subseteq J \) for all \( \alpha \in \mathbb{N}^n \) and \( 1 \leq i \leq n \).

First consider \( x^{\alpha} \chi_n F^{\otimes n} S_n \) for \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \). Let \( \beta = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}^{n-1} \), so that \( x^{\alpha} = x_n^{\alpha_n} x^{\beta} \). Expanding \( x^{\beta} \) in terms of the basis of \( \mathcal{A}_{n-1}(F) \) from Lemma 6.7, we see that
\[
x^{\alpha} \chi_n F^{\otimes n} S_n \subseteq \sum_{(\alpha', Z') \in \Pi_{n-1}} x_n^{\alpha_n} x^{\alpha'} \chi_{Z'} F^{\otimes (n-1)} S_{n-1} \chi_n F^{\otimes n} S_n
\]
\[
\subseteq \sum_{(\alpha', Z') \in \Pi_{n-1}} x_n^{\alpha_n} x^{\alpha'} \chi_{Z'} \chi_n F^{\otimes n} S_n \subseteq J,
\]
where the second inclusion follows from Lemma 6.8.

Now suppose \( 1 \leq i < n \) and consider \( x^{\alpha} \chi_i F^{\otimes n} S_n \). Again, let \( \beta = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}^{n-1} \), so that \( x^{\alpha} = x_n^{\alpha_n} x^{\beta} \). By the induction hypothesis, we have
\[
x^{\alpha} \chi_i F^{\otimes n} S_n = x_n^{\alpha_n} x^{\beta} \chi_i F^{\otimes n} S_n \subseteq \sum_{(\alpha', Z') \in \Pi_{n-1}} x_n^{\alpha_n} x^{\alpha'} \chi_{Z'} F^{\otimes n} S_n.
\]

\[ \square \]
We now show by induction on \( \alpha_n \) that \( x^{\alpha_n}x^{\alpha'} F^{\otimes n} S_n \subseteq J \) for all \((\alpha', Z') \in \Pi_n^+\). This follows immediately from the definition of \( \Pi_n^+ \) if \( \alpha_n < d \). So suppose \( \alpha_n \geq d \). By Lemmas 6.5 and 6.6, we have

\[
x^{\alpha_n}x^{\alpha'} Z\chi x F^{\otimes n} S_n = x^{\alpha_n - d}x^{\alpha'} Z\chi x F^{\otimes n} S_n \subseteq x^{\alpha_n - d}x^{\alpha'} Z\chi F^{\otimes n} S_n + \sum_{e=0}^{d-1} x^{\alpha_n - d + e} J_{C, F}^{\otimes n} S_n.
\]

By the definition of \( J \), we have \( x^{\alpha_n - d}x^{\alpha'} Z\chi F^{\otimes n} S_n \subseteq J \). Now, by (6.7), for \( 0 \leq e < d \), we have

\[
x^{\alpha_n - d + e} J_{C, F}^{\otimes n} S_n \subseteq \sum_{(\alpha', Z') \in \Pi_n^+} x^{\alpha_n - d + e, Z\chi} F^{\otimes n} S_n.
\]

Since \( 0 \leq \alpha_n - d + e < \alpha_n \), each term in the above sum is contained in \( J \) by induction. This completes the proof.

\[
\square
\]

**Theorem 6.11 (Basis theorem for cyclotomic quotients).** The canonical images of the elements

\[
\{ x^{\alpha}b, \pi \mid \alpha \in \mathbb{N}^n, b \in B^{\otimes n}, \pi \in S_n \}
\]

form a basis for \( \mathcal{A}_n^C(F) \).

**Proof.** By Lemmas 6.7 and 6.10, the elements \( \{ x^{\alpha} Z \chi \mid (a, Z) \in \Pi_n^+ \} \) form a basis for \( J_C \) viewed as a right \( F^{\otimes n} \rtimes_{\rho} S_n \)-module. Thus Lemma 6.7 implies that

\[
\{ x^{\alpha} \mid \alpha \in \mathbb{N}^n, \alpha_1, \ldots, \alpha_n < d \}
\]

is a basis for a complement to \( J_C \) in \( \mathcal{A}_n(F) \), viewed as a right \( F^{\otimes n} \rtimes_{\rho} S_n \)-module. The theorem follows.

\[
\square
\]

When \( F = \mathbb{k} \) or \( F = \text{Cl} \), Theorem 6.11 recovers known results. (See, e.g., [Kle05, Th. 7.5.6 and Th. 15.4.6].) When \( F \) is the group algebra of a group (see Example 3.5), the result is stated without proof in [WW08, Prop. 5.5]. In other cases, Theorem 6.11 seems to be new. In particular, as noted in the introduction, when \( F \) is a symmetric algebra, concentrated in even parity, \( \mathcal{A}_n(F) \) is the affinized symmetric algebra considered by Kleshchev and Muth [KM, §3]. Under these additional assumptions, those authors prove that the elements given in Theorem 6.11 are a spanning set and then conjecture that they are a basis [KM, Conj. 3.21].

**Corollary 6.12.** Conjecture 3.21 of [KM] holds.

**Proof.** This follows immediately from the special case of Theorem 6.11 where \( F \) is purely even and symmetric (i.e. \( \psi = \text{id} \)), and \( C \in \langle Z(F)^{\otimes n} \rangle S_n \).

\[
\square
\]

**Corollary 6.13.** Every level one cyclotomic wreath product algebra is isomorphic to \( \mathcal{A}_n^C(F) \rtimes_{\rho} S_n \).

**Proof.** If \( \chi_C = x_1 \), then \( \mathcal{A}_n^C(F) \simeq F^{\otimes n} \rtimes_{\rho} S_n \). The projection \( \mathcal{A}_n(F) \rightarrow \mathcal{A}_n^C(F) \) is precisely the homomorphism described in Proposition 4.20. The general result then follows from Proposition 6.2.

\[
\square
\]

**Remark 6.14.** When \( F = \mathbb{k} \), Brundan proved in [Bru08, Th. 1] that the centers of the cyclotomic quotients consist of symmetric polynomials in \( x_1, \ldots, x_n \). However, this is not true in general (i.e. for arbitrary \( F \)). Indeed, the \( x_1 \) have degree \( \delta \). Thus, if \( \delta > 0 \), but \( Z(F) \cap F_{\psi} \neq \mathbb{k} \), there are elements of the center of \( \mathcal{A}_n^C(F) \) that cannot possibly be expressed as symmetric polynomials in \( x_1, \ldots, x_n \) for degree reasons.
6.4. Frobenius algebra structure. By Theorem 6.11, we can define an even linear map $\text{tr}_C : A_n(F) \to k$ by defining
\begin{equation}
\text{tr}_C(x^\alpha f) = \delta_{\alpha,(d-1,d-1)} \text{tr}^{\otimes n}(f) \delta_{\pi,1}, \quad f \in F^{\otimes n}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \quad \alpha_1, \ldots, \alpha_n < d_C.
\end{equation}
and extending by linearity.

**Theorem 6.15.** The cyclotomic quotient $A_n^C(F)$ is an $\mathbb{N}$-graded Frobenius superalgebra with trace map $\text{tr}_C$ and Nakayama automorphism given by
\[
x_i \mapsto x_i, \quad f \mapsto \left(\psi^{dC}\right)^{\otimes n}(f), \quad \pi \mapsto \pi, \quad 1 \leq i \leq n, \quad f \in F^{\otimes n}, \quad \pi \in S_n.
\]
It follows that $A_n(F)$ is a symmetric algebra if the level $d_C$ is a multiple of the order of the Nakayama automorphism $\psi$ of $F$. In particular, if $F$ is a symmetric algebra (i.e. $\psi = \text{id}$), then so is $A_n^C(F)$.

**Proof.** To check that $\text{tr}_C$ satisfies the required property that (2.5) is an isomorphism, it is enough to verify that the basis of $A_n^C(F)$ given in Theorem 6.11 has a left dual basis with respect to $\text{tr}_C$. Consider the strong Bruhat order on $S_n$ and the total order on $\mathbb{N}^n$ given by
\begin{equation}
\alpha < \beta \iff \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i \text{ for some } 1 \leq i \leq n.
\end{equation}
It suffices to prove that, for each basis element $x^\alpha b\pi$, we can find $z \in A_n^C(F)$ such that $\text{tr}_C(z x^\alpha b\pi) = 1$ and $\text{tr}_C(z x^{\alpha'} b' \pi') = 0$ for all
\[
\pi' \not\leq \pi \text{ or } (\pi' = \pi, \alpha' < \alpha) \text{ or } (\pi = \pi', \alpha = \alpha', b' \neq b).
\]
Indeed, if this is true, one can find a dual basis by inverting a unitriangular matrix.

Fix a basis element $x^\alpha b\pi$. By Lemma 4.4, we have
\[
\pi^{-1} x^\alpha b\pi = \pi^{-1} (x^\alpha b) + \sum_{\pi \neq 1} P_n(F)\pi.
\]
Hence, without loss of generality, we may assume that $\pi = 1$. Now, let
\[
g = (\psi^{-\alpha_1} \otimes \psi^{-\alpha_2} \otimes \cdots \otimes \psi^{-\alpha_n})(b^\vee), \quad \beta = (d-1-\alpha_1, \ldots, d-1-\alpha_n).
\]
Then we have
\[
\text{tr}_C(x^\beta gx^\alpha b) = \text{tr}_C(x_1^{d-1} \cdots x_n^{d-1} b^\vee b) = 1
\]
It also follows from Lemma 6.6 that $\text{tr}_C(x^\beta gx^{\alpha'} b') = 0$ when $\alpha' < \alpha$ or when $\alpha' = \alpha$ and $b' \neq b$.

This completes the proof that $\text{tr}_C$ satisfies the defining property of a trace map.

That the Nakayama automorphism is the given map on $F^{\otimes n}$ and $S_n$ is verified via a straightforward direct computation using Theorem 6.11.

Now suppose that the Nakayama automorphism maps $x_i$ to $x_i$ for some $1 \leq i \leq n - 1$. Then, by (3.5), it maps $x_{i+1}$ to
\[
s_i x_i s_i + (\psi^d)^{\otimes n}(t_{i,i+1}) = s_i x_i s_i + t_{i,i+1} = x_{i+1},
\]
where the second equality follows from the fact that the definition of $t_{i,i+1}$ is independent of the chosen basis. Therefore, to complete the proof of the theorem, it suffices to show that the Nakayama automorphism associated to $\text{tr}_C$ maps $x_1$ to $x_1$. That is, it remains to show that
\begin{equation}
\text{tr}_C(x^\alpha f \pi x_1) = \text{tr}_C(x_1 x^\alpha f \pi)
\end{equation}
for all $f \in F^{\otimes n}, \pi \in S_n$, and $\alpha \in \mathbb{N}^n$ with $\alpha_1, \ldots, \alpha_n \leq d - 1$.

First suppose $\pi(1) = 1$. If $\alpha_1 < d - 1$, then
\[
\text{tr}_C(x^\alpha f \pi x_1) = \text{tr}_C(x_1 x^\alpha \psi_1(f) \pi) = \delta_{\alpha_1,d-2} \delta_{\alpha_2,d-1} \cdots \delta_{\alpha_n,d-1} \delta_{\pi,1} \text{tr}^{\otimes n}(\psi_1(f)) = \delta_{\alpha_1,d-2} \delta_{\alpha_2,d-1} \cdots \delta_{\alpha_n,d-1} \delta_{\pi,1} \text{tr}^{\otimes n}(f) = \text{tr}_C(x_1 x^\alpha f \pi).
\]
Now assume $\alpha_1 = d - 1$. It follows from the definition of $\chi_C$ that we can write
\begin{equation}
6.11 \quad x_1^d = \sum_{i=0}^{d-1} f(i)x_1^i, \quad f(i) \in F_{\psi}^\otimes n, \quad 0 \leq i \leq d - 1.
\end{equation}

Then, if $\alpha' = (0, \alpha_2, \alpha_3, \ldots, \alpha_n)$, we have
\[
\text{tr}_C (x^\alpha f \pi x_1) = \text{tr}_C (x_1 x^\alpha \psi_1(f) \pi)
\]
\[
= \sum_{i=0}^{d-1} \text{tr}_C (x_1^i x^\alpha f(i) \psi_1(f) \pi)
\]
\[
= \delta_{\alpha_2,d} \cdot \ldots \cdot \delta_{\alpha_n,d} \cdot \delta_{\pi,1} \cdot \text{tr}^\otimes n (f(d-1) \psi_1(f)) \tag{6.11}
\]
\[
= \delta_{\alpha_2,d} \cdot \ldots \cdot \delta_{\alpha_n,d} \cdot \delta_{\pi,1} \cdot \text{tr}^\otimes n (\psi_1(f(d-1)f)) \quad \text{(since $f(d-1) \in F_{\psi}^\otimes n$)}
\]
\[
= \sum_{i=0}^{d-1} \text{tr}_C (x_1^i x^\alpha f(i) \pi)
\]
\[
= \text{tr}_C (x_1 x^\alpha f \pi).
\]

This proves (6.10) in the case that $\pi(1) = 1$.

Next we consider the case where $\pi = s_1 \pi'$ for some $\pi' \in S_n$ with $\pi'(1) = 1$. Using (6.11) and the fact that $s_1 \pi' \neq 1$, we see that
\[
\text{tr}_C (x_1 x^\alpha f s_1 \pi') = 0.
\]

On the other hand,
\begin{equation}
6.12 \quad \text{tr}_C (x^\alpha f s_1 \pi' x_1) = \text{tr}_C (x^\alpha f s_1 x_1 \pi') \quad \text{(3.5)}
\end{equation}
\[
= \text{tr}_C (x^\alpha f x_2 s_1 \pi') - \text{tr}_C (x^\alpha f t_{1,2} \pi').
\]

This is clearly equal to zero if $\alpha \neq (d - 1, \ldots, d - 1)$, so we now assume $\alpha = (d - 1, \ldots, d - 1)$. Now,
\[
x_2^d = (3.15) \quad x_2^d = s_1 x_1^d + t_{1,2}^{(d)}
\]
\[
= \sum_{i=0}^{d-1} s_1 f(i)x_1^i + t_{1,2}^{(d)}
\]
\[
= \sum_{i=0}^{d-1} s_1 f(i) s_1 x_1^i + t_{1,2}^{(d)}
\]
\[
= \sum_{i=0}^{d-1} s_1 f(i) \left( x_1^i - t_{1,2}^{(i)} \right) + t_{1,2}^{(d)}.
\]

Thus, if $\beta = (d - 1, 0, d - 1, \ldots, d - 1)$, we have
\[
\text{tr}_C (x^\alpha f x_2 s_1 \pi') = \text{tr}_C (x^\alpha f x_2 \psi_2(f) s_1 \pi')
\]
\[
= \sum_{i=0}^{d-1} \text{tr}_C \left( x^\beta \left( s_1 f(i) \right) \left( x_2^i - t_{1,2}^{(i)} \right) \psi_2(f) s_1 \pi' \right) + \text{tr}_C \left( x^\beta t_{1,2}^{(d)} s_1 \psi_2(f) s_1 \pi' \right)
\]
\[
= \text{tr}_C \left( x^\alpha s_1 f(d-1) \psi_2(f) s_1 \pi' \right) + \text{tr}_C \left( x^\beta t_{1,2}^{(d)} s_1 \psi_2(f) s_1 \pi' \right)
\]
\[
= \text{tr}_C \left( x^\alpha s_1 \psi_2(f) \pi' \right) + \text{tr}_C \left( x^\alpha t_{1,2}^{(d)} s_1 \psi_2(f) \pi' \right)
\]
\[
= \text{tr}_C \left( x^\alpha f t_{1,2} \pi' \right),
\]
where the third equality follows from consideration of the powers of $x_2$ and the fact that the highest power of $x_2$ appearing in $t_{1,2}^{(k)}$ is $x_2^{2^{-1}}$, and the fourth equality follows from the fact that $s_1\pi' \neq 1$ and $t_{1,2}^{(d)} = x_2^{d-1}t_{1,2}$ up to lower order terms in $x_2$. Combined with (6.12), this proves that

(6.13) \[ \text{tr}_C \left( x^0 f_{s_1 \pi' x_1} \right) = 0 \quad \text{when } \pi'(1) = 1. \]

This completes the proof of (6.10) in the case that $\pi = s_1 \pi'$, where $\pi'(1) = 1$.

Now consider the general situation where $\pi(1) \neq 1$. Then we can write $\pi = \pi_1 s_1 \pi_2$, where $\pi_1, \pi_2 \in S_n$ satisfy $\pi_1(1) = \pi_2(1) = 1$. Using (6.11) and the fact that $\pi_1 s_1 \pi_2 \neq 1$, we see that

\[ \text{tr}_C \left( x_1 x^0 f_{s_1 \pi_1 s_1 \pi_2} \right) = 0. \]

On the other hand,

\[
\text{tr}_C \left( x^0 f_{\pi_1 s_1 \pi_2 x_1} \right) = \text{tr}_C \left( x^0 \pi_1 \pi_1 ' f_{s_1 \pi_2 x_1} \right) \\
= \text{tr}_C \left( \pi_1 x^{\pi^{-1}(\alpha) \pi_1 ' f_{s_1 \pi_2 x_1}} \right) + \sum_{\sigma < \pi_1} \text{tr}_C \left( \sigma p_{\sigma} \pi_1 ' f_{s_1 \pi_2 x_1} \right) \quad \text{(by Lemma 4.4)} \\
= \text{tr}_C \left( x^{\pi^{-1}(\alpha) \pi_1 ' f_{s_1 \pi_2 x_1}} \right) + \sum_{\sigma < \pi_1} \text{tr}_C \left( \sigma p_{\sigma} \pi_1 ' f_{s_1 \pi_2 x_1} \right) \\
= \text{tr}_C \left( x^{\pi^{-1}(\alpha) \pi_1 ' f_{s_1 \pi_2 x_1}} \right) + \sum_{\sigma < \pi_1} \text{tr}_C \left( \sigma p_{\sigma} \pi_1 ' f_{s_1 \pi_2 x_1} \right) \\
= \sum_{\sigma < \pi_1} \text{tr}_C \left( f_{s_1 \pi_2 x_1} \right) = 0, \\
\]

where the third equality follows from the fact that the Nakayama automorphism associated to $\text{tr}_C$ fixes elements of $S_n$, the fourth equality follows from the fact that $\sigma < \pi_1$ implies $\sigma(1) = 1$, and the fifth equality follows from taking $\pi'$ in (6.13) equal to $\pi_2 \pi_1$ and $\pi_2 \sigma$. This completes the proof of (6.10) in the remaining case that $\pi(1) \neq 1$.

When $F = k$, Theorem 6.15 recovers the result that degenerate cyclotomic Hecke algebras are symmetric algebras, a fact that follows from [HM10, Cor. 6.18] together with [BK09, Th. 1.1]. However, the argument given above is more direct. In the case where $F$ is the group algebra of a finite cyclic group, we recover [Cui, Th. 2.4]. In other cases, the result seems to be new. Even in the $F = \text{Cl}$ case, when $A_n(\text{Cl})$ is the affine Sergeev algebra, while it was known that $A_n(\text{Cl})$ is a Frobenius algebra (see, e.g., [Kle05, Cor. 15.6.4]), the precise form of the Nakayama automorphism does not seem to have appeared in the literature. In addition, Conjecture 3.22 of [KM] can be reformulated as the assertion that $A_n(F)$ is a symmetric algebra when $F$ is a symmetric algebra concentrated in even parity.

**Corollary 6.16.** Conjecture 3.22 of [KM] holds.

**Proof.** This follows immediately from the special case of Theorem 6.15 where $F$ is purely even and symmetric (i.e. $\psi = \text{id}$), and $C \in (Z(F)^{\otimes n})^{S_n}$. \qed

### 6.5. Cyclotomic Mackey Theorem

For the remainder of this section, we assume

(6.14) \[ c^{(k,j)}(F) \in F_\psi^{(k)} \quad \text{for all } 1 \leq k \leq \theta, \ 1 \leq j \leq e_k, \]

where we identify $F$ with the first factor of $F \otimes k^{\otimes (n-1)} \subseteq F^{\otimes n}$ for each $n \geq 1$. Hence $A_n^C(F)$ is defined for all $n \in \mathbb{N}$.

Theorem 6.11 implies that the subalgebra of $A_n^C(F)$ generated by $x_1, \ldots, x_n, F^{\otimes n} \otimes 1$, and $S_n$ is isomorphic to $A_n^C(F)$. Thus we have induction and restriction functors $C_{\text{Ind}}^{n+1}$ and $C_{\text{Res}}^{n+1}$

**Proposition 6.17.** Recall that $d = d_C$ is the level of $C$ (see (6.4)).
(a) $A_{n+1}^C(F)$ is a free right $A_n^C(F)$-module with basis
\[ \{x_j^{a}b_j^{s_j} \cdots s_n \mid 0 \leq a < d, \ b \in B, \ 1 \leq j \leq n + 1 \}. \]

(b) We have a decomposition of $(A_n^C(F), A_n^C(F))$-bimodules
\[ A_{n+1}^C(F) = A_n^C(F) s_n A_n^C(F) \oplus \bigoplus_{0 \leq a < d, \ b \in B} x_{n+1}^a b_{n+1} A_n^C(F). \]

(c) Suppose $0 \leq a < d$ and $f \in F$. Then we have isomorphisms of $(A_n^C(F), A_n^C(F))$-bimodules
\[ A_n^C(F) s_n A_n^C(F) \cong A_n^C(F) \otimes A_{n-1}^C(F) A_n^C(F), \]
\[ x_{n+1}^a f_{n+1} A_n^C(F) \cong \Pi f A_n^C(F). \]

Proof. The proof is almost identical to those of [Kle05, Lem. 7.6.1] and [Kle05, Lem. 15.5.1] and so will be omitted. \qed

**Theorem 6.18** (Cyclotomic Mackey Theorem). Recall that $d = d_C$ is the level of $C$ (see (6.4)). For $n \in \mathbb{N}_+$, we have a natural isomorphism of functors
\[ C \text{Res}_{n+1}^n C \text{Ind}_{n+1}^n \cong \text{id} \oplus d \dim F_{\text{even}} \oplus \Pi \oplus d \dim F_{\text{odd}} \oplus C \text{Ind}_{n-1}^n C \text{Res}_{n-1}^n, \]
where $F_{\text{even}}$ and $F_{\text{odd}}$ are the even and odd parts of $F$, respectively.

Proof. This follows from Proposition 6.17. \qed

When $F = k$ or $F = \text{Cl}$, Theorem 6.18 recovers the Cyclotomic Mackey Theorem for the degenerate cyclotomic Hecke algebras and cyclotomic Sergeev algebras, respectively (see [Kle05, Th. 7.6.2 and Th. 15.5.2]).

### 6.6. Frobenius extension structure

We continue to make the assumption (6.14). Let
\[ F_{n+1} = 1 \otimes_k F = \text{Span}_k \{ f_{n+1} \mid f \in F \} \subseteq F^{\otimes(n+1)}. \]

By Proposition 6.17(b), we have a decomposition of $(A_n^C(F), A_n^C(F))$-bimodules
\[ A_{n+1}^C(F) = x_{n+1}^{d-1} F_{n+1} A_n^C(F) \oplus \bigoplus_{a=0}^{d-2} x_{n+1}^a F_{n+1} A_n^C(F) \oplus A_n^C(F) s_n A_n^C(F). \]

Using Proposition 6.17(c) and the trace map of $F$, we have a homomorphism of $(A_n^C(F), A_n^C(F))$-bimodules
\[ x_{n+1}^{d-1} F_{n+1} A_n^C(F) \to A_n^C(F), \quad x_{n+1}^{d-1} f_{n+1} y \mapsto \text{tr}(f)y, \quad f \in F, \ y \in A_n^C(F). \]

Let
\[ \text{tr}_{n+1}^C : A_{n+1}^C(F) \to A_n^C(F) \]
be homomorphism of $(A_n^C(F), A_n^C(F))$-bimodules given by the projection onto the first summand in (6.15) followed by the homomorphism (6.16). Note that the degree of $\text{tr}_{n+1}^C$ is $-d$. Since, in the current setup, the notation $\text{tr}_{C}^C$ is ambiguous, let $\text{tr}_{C}^C : A_n(F) \to k$ denote the trace map (6.8). Then
\[ \text{tr}_{n+1}^C = \text{tr}_1^C \circ \text{tr}_2^C \circ \cdots \circ \text{tr}_{n+1}^C = \text{tr}_{C}^n \circ \text{tr}_{n+1}^C. \]

**Theorem 6.19.** The cyclotomic quotient $A_{n+1}^C(F)$ is a Frobenius extension of $A_n^C(F)$ with trace map $\text{tr}_{n+1}^C$. 
Proof. Let $B$ be a basis of $A_n(F)$. By Theorem 6.15, [PS16, Cor. 7.4], and [PS16, Cor. 3.6], $A_{n+1}(F)$ is a Frobenius extension of $A_n(F)$ with trace map

$$z \mapsto \sum_{b \in B} \text{tr}_{C}^{n+1}(b^\vee z)b.$$  

(Note that $b^\vee$ denotes the right dual of $b$ in [PS16], whereas it denotes the left dual in the current paper.) Now

$$\sum_{b \in B} \text{tr}_{C}^{n+1}(b^\vee z)b = \sum_{b \in B} \text{tr}_{C}^{n}(\text{tr}_{C}^{n+1}(b^\vee z))b = \sum_{b \in B} \text{tr}_{C}^{n}(b^\vee \text{tr}_{C}^{n+1}(z))b \overset{(2.6)}{=} \text{tr}_{C}^{n+1}(z),$$

completing the proof. □

Corollary 6.20. The exact functor $C_{\text{Ind}}^{n+1}$ is left adjoint to $C_{\text{Res}}^{n+1}$ and right adjoint to $C_{\text{Res}}^{n+1}$ up to degree shift.

Proof. Induction is always left adjoint to restriction. That induction is right adjoint to restriction is a standard property of Frobenius extensions. In the graded super setting considered in the current paper see, for example, [PS16, Th. 6.2]. (Note that the twistings $\alpha$ and $\beta$ of [PS16] are trivial in the setting of the current paper.) □

7. Future directions

As mentioned in the introduction, we expect that many of the results for and applications of degenerate affine Hecke algebras and their analogs have natural extensions to the setting of affine wreath product algebras. We conclude this paper with a brief discussion of some such potential directions of future research.

7.1. Double affine versions and $q$-deformations. Degenerate affine Hecke algebras (Example 3.4), affine Sergeev algebras (Example 3.7), and wreath Hecke algebras corresponding to cyclic groups (Example 3.5) can be viewed as degenerations of affine Hecke algebras, affine Hecke–Clifford algebras, and affine Yokonuma–Hecke algebras, respectively. It would be interesting to try to construct a $q$-deformation of affine wreath product algebras. Similarly, it is natural to wonder if there exist natural double affine versions of wreath product algebras generalizing double affine Hecke algebras (also known as Cherednik algebras) and their various degenerations. This should be related to the algebras introduced in [CG, §4.2].

7.2. Heisenberg categorification. In [Kho14], Khovanov gave a conjectural categorification of the Heisenberg algebra based on a graphical category motivated by the representation theory of the symmetric group. This procedure was generalized in [RS17] (see also [RS15, §7]), where the group algebra of the symmetric group was replaced by wreath product algebras. Provided that the Frobenius algebra $F$ is not concentrated in degree zero, it was proved that the corresponding graphical categories do indeed categorify the associated lattice Heisenberg algebras. On the other hand, Khovanov’s construction was generalized in a different direction in [MS], where group algebras of symmetric groups were replaced by degenerate cyclotomic Hecke algebras, yielding a conjectural categorification of higher level Heisenberg algebras. We expect that the cyclotomic wreath product algebras introduced in Section 6 provide a framework for unifying these two generalizations. In particular, one should be able to define a graphical category based on these cyclotomic quotients that specializes to the categories of [MS] when $F = \mathbb{k}$ and to the categories of [RS17] when the quotient is of level one. More ambitiously, $q$-deformations as in §7.1 might allow one to also incorporate the $q$-deformed Heisenberg category of [LS13]. Considering trace decategorifications of such categories should lead to generalizations of the results of [CLLS, CLL, LRS].

¹Since the writing of the current paper, this category has been defined in [Sav].
7.3. **Branching rules.** It would be interesting to investigate branching rules for affine wreath product algebras. For the special cases of degenerate affine Hecke algebras and affine Sergeev algebras, we refer the reader to the exposition in [Kle05]. When \( F \) is the group algebra of a finite group (Example 3.5), branching rules were obtained in [WW08]. When the characteristic of the field \( k \) does not divide the order of the group, the branching rules were identified with crystal graphs of integrable modules for quantum affine algebras (see [WW08, §5.5]). This assumption on the characteristic of \( k \) implies that \( F \) is semisimple, and we expect that the arguments of [WW08] should generalize to the setting where \( F \) is an arbitrary semisimple Frobenius algebra. However, the situation where \( F \) is not semisimple would likely be more involved.

**APPENDIX A. NOTATION**

We let \( \mathbb{N} \) denote the set of nonnegative integers, and let \( \mathbb{N}_+ \) denote the set of positive integers. We assume that \( k \) is a ring whose characteristic is not equal to two. Any additional assumptions on \( k \) are stated at the beginning of each section. We also assume that the action of the Nakayama automorphism on \( F \) is diagonalizable and that the characteristic of \( k \) does not divide the order of the Nakayama automorphism. We let \( k^\times \) denote the group of invertible (under multiplication) elements of \( k \). For the convenience of the reader, we list here some of the most important notation used throughout the paper, in order of appearance.

| Notation | Meaning | Definition |
|----------|---------|------------|
| \( \bar{a} \) | parity of \( a \) | p. 3 |
| \( |a| \) | \( \mathbb{Z} \)-degree of \( a \) | p. 3 |
| \( A^{op} \) | opposite algebra of \( A \) | (2.1) |
| \( \text{HOM}_A(M,N) \) | graded hom space | (2.2) |
| \( \text{Hom}_A(M,N) \) | homogeneous hom space | (2.3) |
| \( \simeq \) | even isomorphism | p. 4 |
| \( \cong \) | isomorphism (not nec. preserving degree or parity) | p. 4 |
| \( \text{Cl} \) | 2-dimensional Clifford algebra | p. 4 |
| \( \otimes \) | outer tensor product of modules | p. 4 |
| \( \odot \) | simple tensor product | p. 5 |
| \( S(A) \) | set of even isom. classes of \( A \)-modules up to degree shift | p. 4 |
| \( \alpha V \) | twisted module | (2.4) |
| \( F \) | graded Frobenius superalgebra | p. 5 |
| \( \text{tr} \) | trace map of \( F \) | p. 5 |
| \( \psi \) | Nakayama automorphism of \( F \) | p. 5 |
| \( \delta \) | maximum degree of \( F \) | p. 5 |
| \( \theta \) | order of \( \psi \) | p. 5 |
| \( B \) | basis of \( F \) | p. 5 |
| \( b^{\lor} \) | left dual basis element | p. 5 |
| \( \ltimes, \rtimes \) | smash product, wreath product algebra | §2.3 |
| \( \pi a \) | action of \( \pi \in S_n \) on \( a \) by superpermutation | p. 6 |
| \( \ltimes_{\rho} \) | smash product with action given by superpermutation | p. 6 |
| \( f_i, f \in F \) | \( 1^\otimes (i-1) \otimes f \otimes 1^\otimes (n-i) \) | (3.1) |
| \( \psi_i \) | \( \text{id}^\otimes (i-1) \otimes \psi \otimes \text{id}^\otimes (n-i) \) | (3.2) |
| \( A_n(F) \) | affine wreath product algebra | Def. 3.1 |
| \( t_{i,j} \) | \( \sum_{b \in B} b_ib_j^\lor \) | (3.7) |
| Notation          | Meaning                                                                 | Definition                      |
|------------------|-------------------------------------------------------------------------|---------------------------------|
| $t^{(k)}_{i,j}$   | $\sum_{b \in B} \frac{x^b}{x^i-x^j} b^\gamma_j$                      | (3.13)                          |
| $P_n$            | $k[x_1, \ldots, x_n]$                                                  | p. 11                           |
| $P_n(F)$         | $(k[x] \ltimes F)^{\otimes n}$                                        | (4.1)                           |
| $\Delta_i$       | deformed divided difference operator                                    | §4.1                            |
| $x^\alpha$       | $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$               | (4.12)                          |
| $F_\psi$         | $\{ f \in F \mid \psi(f) = f \}$                                      | (4.13)                          |
| $F(k), F^{(k)}_{\psi}$ | see definition                                                        | (4.14)                          |
| $\mathbf{P}(\alpha)$ | $F(\alpha_1) \otimes \cdots \otimes F(\alpha_n)$                   | (4.16)                          |
| $\mathbf{F}(\psi)$ | $F(\alpha_1) \otimes \cdots \otimes F(\psi)^{(\alpha_n)}$          | (4.16)                          |
| $S_\mu$          | Young subgroup of $S_n$                                                | (4.23)                          |
| $A_\mu(F)$       | parabolic subalgebra of $A_n(F)$                                       | p. 19                           |
| $\text{Ind}_\mu^n, \text{Res}_\mu^n$ | induction and restriction functors for parabolic subalgebras         | (4.24)                          |
| $kL$             | twisted module $\psi^k L$                                              | p. 21, (2.4)                    |
| $r_k$            | smallest positive integer such that $r_k L \simeq L$                  | p. 21                           |
| $m_k$            | smallest positive integer such that $\psi^{m_k}(f)(v) \forall f \in F, v \in L$ | p. 21                           |
| $\tau_k$         | even $F$-module isomorphism $L \xrightarrow{\psi} r_k L$             | (5.1)                           |
| $\text{flip}$    | $\text{flip}(v_1 \otimes v_2) = (-1)^{(v_1 v_2 \otimes v_1}$   | (5.6)                           |
| $L(\alpha)$      | see definition                                                         | (5.3)                           |
| $\rho_{k,\ell}$  | superpermutation of $k$-th and $\ell$-th factors                       | (5.8)                           |
| $N$              | number of $\psi$-orbits in $S(F)$                                      | p. 24                           |
| $L_1, \ldots, L_N$ | representatives of $\psi$-orbits in $S(F)$                             | (5.9)                           |
| $r_k$            | $r_k$, smallest positive integer such that $r_k L_k \simeq L_k$       | p. 24                           |
| $A_n(F)$-mod$^k$ | category of f.d. $A_n(F)$-modules semisimple as $F^{\otimes n}$-modules | p. 24                           |
| $\mathcal{C}_n$  | set of compositions of $n$ of length at most $N$                       | (5.10)                          |
| $\ell$           | unique integer such that $\ell_1 + \cdots + \ell_{k-1} < k \leq \ell_1 + \cdots + \ell_k$ | (5.11)                          |
| $\tau_k$         | $\tau_k$, even $F$-module isomorphism $L_k \xrightarrow{\psi} r_k L_k$ | p. 24                           |
| $a_k$            | see definition                                                         | p. 24                           |
| $L(\mu)$         | $L_1(a_1)^{\otimes \mu_1} \otimes \cdots \otimes L_n(a_n)^{\otimes \mu_n}$ | (5.12)                          |
| $E(\mu)$         | $\text{END}_{F^{\otimes n}} L(\mu)$                                   | (5.14)                          |
| $\mathcal{H}_\mu^n$ | see definition                                                       | (5.16)                          |
| $\mathcal{R}_\mu, \mathcal{R}_n$ | $\mathcal{R}_\mu = \mathcal{H}_\mu^1 \otimes \cdots \otimes \mathcal{H}_\mu^N, \mathcal{R}_n = \bigoplus_{\mu \in \mathcal{C}_n} \mathcal{R}_\mu$ | (5.17)                          |
| $y_1, \ldots, y_n$ | polynomial generators of $\mathcal{R}_n$                             | p. 25                           |
| $I_\mu V$        | see definition                                                         | p. 25                           |
| $V_\mu$          | $\sum_{\pi \in S_n} \pi(I_\mu V)$                                   | (5.18)                          |
| $\mathbf{F}^{(k)}_\mu$ | see definition                                                      | (6.1)                           |
| $J_C, \chi_C$    | see definition                                                         | (6.2)                           |
| $A_n^C(F)$       | cyclotomic wreath product algebra                                      | (6.3)                           |
| $d = d_C$        | level of $C$                                                           | (6.4)                           |
| $\text{tr}_C$    | trace map for $A_n^C(F)$                                               | (6.8)                           |
| $C\text{Ind}_n^{n+1}, C\text{Res}_n^{n+1}$ | induction and restriction functors for cyclotomic quotients     | p. 36                           |
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Department of Mathematics and Statistics, University of Ottawa
URL: alistairsavage.ca, ORCID: orcid.org/0000-0002-2859-0239
E-mail address: alistair.savage@uottawa.ca