The role of the input in Roy’s equations for π-π scattering

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Abstract

The Roy equations determine the S- and P-wave π-π phase shifts on a low energy interval. They allow the derivation of threshold parameters from experimental data. We examine the solutions of these equations that are in the neighborhood of a given solution by means of a linearization procedure. An updated survey of known results on the dimension of the manifold of solutions is presented. The solution is unique for a low energy interval with upper end at 800 MeV. We determine its response to small variations of the input: S-wave scattering lengths and absorptive parts above 800 MeV. We confirm the existence of a universal curve of solutions in the plane of the S-wave scattering lengths and provide a control of the decrease of the influence of the input absorptive parts with increasing energy. A general result on the suppression of unphysical singularities at the upper end of the low energy interval is established and its practical implications are discussed.

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1 Introduction

Low-energy π-π scattering is a major testing ground of chiral perturbation theory [1]. Some of its coupling constants are directly related to the π-π threshold parameters. At present this relation is established at the two-loop level [1, 2]. As it is impossible to measure π-π scattering at threshold, this relation cannot be exploited directly. A reliable extrapolation of the available experimental data down to threshold is required. Such an extrapolation is performed presently [3] with the aid of the Roy equations [4, 5]. These equations are
based on the analyticity, the crossing symmetry and the unitarity of the $\pi-\pi$ partial wave amplitudes. The S- and P-wave Roy equations are solved in [3] by means of elaborate numerical methods. In this work we discuss aspects of the problem which allow an analytical approach and our effort is complementary to the work in [3].

The Roy equations contain as input the S-wave scattering lengths, the S- and P-wave absorptive parts above an energy $E_0$ (which will be called the “matching point”) and driving terms coming from the higher partial waves. The Roy equations determine, at fixed elasticities, the S- and P-wave phase shifts below the matching point. This is a difficult, non-linear, problem that cannot be solved analytically. Here we restrict ourselves to questions that can be answered by linearization and which allow a partly analytic treatment. These concern the multiplicity of the solution and its sensitivity to small variations of the input.

Such questions have already been treated in [6, 7] in conjunction with the early phenomenological applications of the Roy equations [8] and we can reduce the discussion of the multiplicity of the solution to the statement of our old results. The matching point used in [7] is at 1.13 GeV whereas the one used nowadays in [3] is at $E_0 = 800$ MeV. The answers to our questions depend on the choice of $E_0$. The solution is non-unique if $E_0 = 1.13$ GeV and becomes unique when $E_0 = 800$ MeV. The response to variations of the input also depends strongly on the position of the matching point and our analysis in [7] has to be updated.

The Roy equations with arbitrarily chosen input make up a well defined mathematical problem. A peculiar feature of this problem is that its solutions exhibit unphysical singularities at the matching point [we exclude throughout matching points coinciding with an inelastic threshold]. The physical input is therefore a special one admitting at least one solution, the physical phase shifts, that is regular at $E_0$. Inputs with solutions regular at $E_0$ have been called “analytic inputs” in [3] in the context of simplified elastic one-channel Roy equations. The discussion of that class of inputs is extended here to the case of the complete coupled inelastic Roy equations. The main conclusion is that an analytic input admits a unique solution that is regular at the matching point. The non-uniqueness problem is thus evacuated.

Although non-uniqueness and singularities at $E_0$ are physically excluded, they show up in practical calculations because one is working with an approximate input which is not exactly an analytic one. An arbitrary variation of an analytic

\[1\] By physical input we denote the input corresponding to scattering amplitudes which would be measured in the absence of isospin violation. Ideally, our physical amplitudes are those provided by QCD; in practice they are given by the available analyses of experimental data assuming isospin symmetry."
input produces a non-analytic one and induces singularities at \( E_0 \) even if the choice of \( E_0 \) guarantees uniqueness.

We find that one may stay close to an analytic input by correlating suitably the variations of two distinct pieces of the input. This comes mainly from the matching point at \( E_0 = 800 \) MeV that is near the \( \rho \)-meson mass. For instance, singularities at \( E_0 \) are largely suppressed by correlating variations of the isospin 0 and 2 S-wave scattering lengths \( a^0_0 \) and \( a^2_0 \). This confirms the existence of a physically acceptable family of solutions along a “universal curve” in the \((a^0_0, a^2_0)\)-plane \([\pi]\). Similar suppressions of singularities take place if a localized variation of an input absorptive part is combined with a variation of one of the scattering lengths, \( a^2_0 \) for instance. The response to such variations provides information on the sensitivity of the phase shifts to the uncertainties on the input absorptive parts. We find a very weak sensitivity to the uncertainties above 1 GeV. All our results are in qualitative and quantitative agreement with those obtained numerically in \([3]\).

The coupling between the S- and P-wave channels built into the Roy equations is a manifestation of crossing symmetry. The practical implications of this symmetry are not well understood and the effects of variations of the input might be expected to provide some insight. We find that this is not really the case. In our framework the response to a change of the input absorptive part in one channel is largest in the same channel but the responses in the other channels are not much smaller. All we may say is that crossing symmetry produces a substantial coupling of the three S- and P-wave channels, but we do not recognize very striking features.

The paper is organized as follows. The linearization procedure developed in \([6, 7]\) is described, and the status of the uniqueness problem is outlined, in Section 2. Section 3 is devoted to the response to variations of the S-wave scattering lengths and the existence of a universal curve. The effects of correlated localized variations of input absorptive parts and variations of a scattering length are presented in Section 4. Variations of the driving terms are also briefly discussed in that section and our conclusions are displayed in Section 5. The fact that an analytic input admits only one solution that is regular at the matching point is a crucial result. We find it convenient to separate its proof from the presentation of phenomenological results and to explain it in Appendix A. Our approximation scheme for the determination of linear responses is described in Appendix B and Appendix C gives a list of the kernels entering the S- and P-wave Roy equations.
2 Solution manifold of the S- and P-wave Roy equations

To set the stage we recall the main features of the S- and P-wave Roy equations [4]. They relate the real and imaginary parts of the S- and P-wave \(\pi-\pi\) scattering amplitudes at low energies, below the matching point \(E_0\):

\[
\Re f_i(s) = (s-4) \sum_{j=0}^{2} \frac{1}{\pi} \int_{s_0}^{s} \frac{dx}{x-4} \left[ \frac{\delta_{ij}}{x-s} + R_{ij}(s, x) \right] \Im f_j(x) + \phi_i(s),
\]

\(i = 0, 1, 2\). To lighten the writing, our notation differs from the standard one: \(f_0\) and \(f_2\) are the isospin \(I = 0\) and \(I = 2\) S-wave amplitudes and \(f_1\) is the isospin \(I = 1\) P-wave. We return to the conventional notation \(f_I\) in the presentation of final results. The variables \(s\) and \(x\) are squared total CM energies in units of \(M^2\pi\) (\(M_\pi =\) pion mass, \(s_0 = (E_0/M_\pi)^2\)). The equations (2.1) contain singular diagonal Cauchy kernels and regular kernels \(R_{ij}\) which are displayed in Appendix C.

The \(\phi_i\) are input functions

\[
\phi_i(s) = a_i + (s-4) \left\{ c_i(2a_0 - 5a_2) + \sum_{j=0}^{2} \frac{1}{\pi} \int_{s_0}^{\infty} \frac{dx}{x-4} \left[ \frac{\delta_{ij}}{x-s} + R_{ij}(s, x) \right] A_j(x) + \psi_i(s) \right\}.
\]

In this equation \(a_0\) and \(a_2\) are the isospin 0 and 2 S-wave scattering lengths, \(a_1 = 0\) here and

\[
c_0 = \frac{1}{12}, \quad c_1 = \frac{1}{72}, \quad c_2 = -\frac{1}{24};
\]

the \(A_i\) are the absorptive parts above the matching point:

\[
A_i(s) = \Im f_i(s), \quad s \geq s_0,
\]

and the \(\psi_i\) are so-called driving terms describing the contributions of the higher partial waves \((l \geq 2)\). They have partial wave expansions converging in \([4, s_0]\) as long as \(s_0 < 125.31\) [3]. The equations (2.1) constrain the S- and P-waves on \([4, s_0]\) at given input \((a_i, A_i, \psi_i)\). Unitarity implies

\[
f_i(s) = \frac{1}{2i\sigma(s)} \left( \eta_i(s)e^{2i\delta_i(s)} - 1 \right), \quad \sigma(s) = \sqrt{1 - \frac{4}{s}},
\]

where \(\delta_i\) is the channel \(i\) phase shift and \(\eta_i\) is the elasticity parameter \((0 \leq \eta_i \leq 1)\) which we incorporate into the input.

At given input the equations (2.1) are coupled non-linear integral equations for the phase shifts \(\delta_i\) on the interval \([4, s_0]\). To be acceptable, a solution of these
equations has to provide absorptive parts below \( s_0 \) that join continuously the inputs \( A_i \) at that point:

\[
\lim_{s \to s_0} \frac{1}{2\sigma(s)} \left( 1 - \eta(s) \cos(2\delta_i(s)) \right) = A_i(s_0).
\] (2.6)

This boundary condition has to be added to the equations (2.1).

The Roy equations being singular, the uniqueness of their solution is by no means guaranteed. We sum up the discussion of that point using the technique developed in [3, 7]. This technique will be our main tool throughout this article.

We assume we have a set of phase shifts \( \delta_i \) satisfying the equations (2.1) and (2.6), the amplitudes \( f_i \) being given by (2.5). We ask if these equations have other solutions \( \delta_i' \) with the same input. If the \( \delta_i' \) are infinitesimally close to the \( \delta_i \) the differences \( (\delta_i' - \delta_i) \) obey the linearized coupled equations

\[
\cos(2\delta_i(s))h_i(s) = \sum_j \frac{1}{\pi} \int_4^{s_0} \frac{dx}{x-4} \left[ \frac{\delta_{ij}}{x-s} + R_{ij}(s, x) \right] \sin(2\delta_j(s))h_j(s),
\] (2.7)

where

\[
h_i(s) = \frac{1}{\sigma(s)} \eta_i(s) (\delta_i'(s) - \delta_i(s)).
\] (2.8)

The boundary conditions (2.6) imply

\[
h_i(s_0) = 0,
\] (2.9)

i.e. \( \delta_i'(s_0) = \delta_i(s_0) \). The homogeneous equations (2.7) with boundary conditions (2.9) may have non-trivial solutions because of the presence of Cauchy kernels. The uniqueness or non-uniqueness of the \( \delta_i \) depends on the existence of such solutions.

If the regular kernels \( R_{ij} \) are omitted, the equations (2.7) decouple and one recovers the one-channel problem discussed in [9]. The existence of non-trivial solutions of this problem depends on the value of the phase shift \( \delta_i \) at the matching point \( s_0 \). We assume that \( \delta_i(s_0) > -\pi/2 \). There is no solution if \(-\pi/2 < \delta_i(s_0) < \pi/2 \). If \( \delta_i(s_0) > \pi/2 \), the general solution is

\[
h_i(s) = (s - 4)G_i(s)P_i(s),
\] (2.10)

where

\[
G_i(s) = \left( \frac{s_0}{s_0 - s} \right)^{m_i} \exp \left[ \frac{2}{\pi} \int_4^{s_0} \frac{dx}{x-s} \delta_i(x) \right]
\] (2.11)

with

\[
m_i = \left[ \frac{2}{\pi} \delta_i(s_0) \right].
\] (2.12)
is the greatest integer smaller than \(x\) (as in \([9]\), \(s_0\) is chosen in such a way that \(\delta_i(s_0)\) is not an integral multiple of \(\pi/2\)). The last factor \(P_i\) in the r.h.s. of (2.10) is an arbitrary polynomial of degree \(m_i - 1\).

The general solution of the complete set of coupled equations (2.7) has a form similar to (2.10):

\[
h_i(s) = (s - 4)G_i(s) [P_i(s) + H_i(s)]
\]

with corrections \(H_i\) [9].

The \(P_i\) are again arbitrary polynomials of degree \(m_i - 1\): \(m_i\) is given by (2.12) if \(\delta_i(s_0) > \pi/2\); \(m_i = 0\) and \(P_i = 0\) if \(|\delta_i(s_0)| < \pi/2\). The functions \(H_i\) are regular on \([4, s_0]\) and are solutions of a set of coupled non-singular integral equations:

\[
\delta_{m_i,0} H_i(s) - \frac{1}{\pi} \int_4^{s_0} dx \, G_i(x) \sin(2\delta_i(x)) \frac{H_i(x) - H_i(s)}{x - s} = \sum_j \frac{1}{\pi} \int_4^{s_0} dx \, R_{ij}(s,x) G_j(x) \sin(2\delta_j(x)) [P_j(x) + H_j(x)].
\]

According to definition (2.11) we have

\[
G_i(s) \sim (s_0 - s)^{\gamma_i}
\]

for \(s \sim s_0\) with \(\gamma_i = \frac{2}{\pi} \delta_i(s_0) - m_i\). This shows that \(G_i\) vanishes at \(s_0\) if \(\delta_i(s_0) > 0\) and diverges at that point if \(\delta_i(s_0) < 0\). Due to the regularity of \(H_i\) at \(s_0\) the boundary condition (2.9) is automatically fulfilled if \(\delta_i(s_0) > 0\). If \(-\pi/2 < \delta_i(s_0) < 0\), \(H_i\) has to vanish at \(s_0\) (remember that \(P_i = 0\) in this case).

We now apply these results to the uniqueness problem of the physical \(\pi - \pi\) S- and P-waves as solutions of the Roy equations (2.1). The input \((a_i, A_i, \psi_i, \eta_i)\) is identified with the physical one and we take the physical phase shifts as our master solution \(\delta_i\) of the Roy equations. The physical isospin 0 S-wave and isospin 1 P-wave phase shifts being positive [12], we have \(m_0 \geq 0\), \(m_1 \geq 0\) and \(\gamma_0 > 0\), \(\gamma_1 > 0\). On the other hand, the isospin 2 S-wave phase shift \(\delta_2\) is negative \((-\pi/2 < \delta_2 \leq 0)\), \(m_2 = 0\), \(P_2 = 0\) and \(\gamma_2 < 0\). Consequently, the three boundary conditions (2.3) are satisfied if

\[
H_2(s_0) = 0.
\]

As solutions of the equations (2.14), the \(H_i\) are linear functionals of the polynomials \(P_0\) and \(P_1\). Condition (2.16) gives a homogeneous linear equation

\[
\]
relating the coefficients of these polynomials and reduces by one the number of free parameters. If \( m_0 + m_1 > 1 \) we are left with \( m_0 + m_1 - 1 \) free parameters. There is no non-trivial solution if \( m_0 + m_1 \leq 1 \). If \( m_0 + m_1 > 1 \), the physical phase shifts are embedded in a \( d \)-dimensional manifold of solutions of the Roy equations with \( d = m_0 + m_1 - 1 \). If \( m_0 + m_1 \leq 1 \), they form an isolated solution of these equations.

The actual values of \( m_0 \) and \( m_1 \) depend on the choice of the matching point \( s_0 \). Taking into account the known behaviour of the physical phase shifts [12], one finds four different situations when \( E_0 = \sqrt{s_0}M_\pi \) is lowered from 1.15 GeV to threshold.

1. 1 GeV < \( E_0 < 1.15 \) GeV In that interval, \( \pi < \delta_0(s_0) < 3\pi/2, \pi/2 < \delta_1(s_0) < \pi. \) This gives \( m_0 = 2, m_1 = 1 \) and \( d = 2 \). The physical S- and P-waves are members of a two-parameter family of solutions of the Roy equations at fixed physical input and fixed phase shifts at \( s_0 \). The physical solution can be selected by imposing the physical values of the position and width of the \( \rho \)-meson.

2. 860 MeV < \( E_0 < 1 \) GeV We now have \( \pi/2 < \delta_i(s_0) < \pi, i = 0, 1 \) and \( m_0 = m_1 = 1, d = 1 \). The polynomials \( P_0 \) and \( P_1 \) reduce to constants related by the equations (2.16). The physical amplitudes belong to a one-parameter family of solutions. The position of the \( \rho \)-meson can be used as a parameter.

3. 780 MeV < \( E_0 < 860 \) MeV In this interval \( m_0 = 0, m_1 = 1 \) and \( d = 0 \) because \( 0 < \delta_0(s_0) < \pi/2, \pi/2 < \delta_1(s_0) < \pi. \) The polynomial \( P_0 \) vanishes and \( P_1 \) is a constant which is set equal to zero by condition (2.16). The physical amplitudes form an isolated solution of the Roy equations. Position and shape of the \( \rho \)-resonance are determined by the input.

4. 280 MeV < \( E_0 < 780 \) MeV Here \( 0 < \delta_i(s_0) < \pi/2, i = 0, 1, m_0 = m_1 = 0 \) and both \( P_0 \) and \( P_1 \) vanish. The physical amplitudes again define an isolated solution.

The above results concern the mathematical problem defined by equations (2.1), (2.5) and (2.6). Due to the behavior (2.13) of the \( G_i \) at \( s_0 \), the representation (2.13) implies that if there are solutions \( \delta_i' \) in the neighborhood of \( \delta_i \) they are singular at \( s_0 \) and exhibit cusps at that point. These singularities are unphysical because the choice of \( s_0 \) is arbitrary. In fact, all solutions of the Roy equations with arbitrary input are singular at \( s_0 \). The physical amplitudes being regular at \( s_0 \), the physical input has to be such that the corresponding Roy
equations have at least one solution which is nonsingular at \( s_0 \) and coincides with the physical amplitudes. Among all possible inputs the physical input is a very special one: it is an analytic input in the sense of Ref. [9]. It has been shown there that in simplified one-channel Roy equations with analytic input there is only one solution which is regular at \( s_0 \). This crucial result is extended to the present realistic case in appendix A.

We see that there is no non-uniqueness problem when working with the exact physical input. For instance in case 1 above, one could vary the position and width of the \( \rho \)-resonance in the two-parameter family of solutions. The singularities at \( s_0 \) would disappear at the physical values of these parameters. In practice, however, the physical input is only known approximately and one is not really working with an analytic input. Therefore singularities are present at \( s_0 \) and non-uniqueness cannot be avoided if \( E_0 > 860 \text{ MeV} \). We have to put up with these unpleasant features which are merely consequences of a deficient knowledge of the physical input.

From now on we choose the matching point used in the low-energy extrapolation based on the Roy equations performed in [B]: \( E_0 = 800 \text{ MeV} \) (\( s_0 = 33 \)). Non-uniqueness is avoided but there are unwanted cusps at the matching point. It turns out that some of these cusps are in fact a helpful tool. Their suppression provides insights into the correlations constraining the scattering lengths of an analytic input. This will be illustrated repeatedly in this paper.

### 3 Varying the S-wave scattering lengths: universal curve

We come now to our main topic, the linear response to small variations of the input. We proceed along the same lines as in the previous section. Starting from the solution \( \delta_i \) with input \((a_i, A_i, \psi_i, \eta_i)\), we determine the solution \( \delta'_i \) produced by a slightly modified input in linear approximation. To obtain quantitative results, we need a model for the \( \delta_i \) which provides an acceptable representation of the physical phase shifts. We use the Schenk parametrization [13]:

\[
\delta(s) = \tan^{-1} \left\{ \frac{\sigma(s)}{s-z_i} \left[ a_i + b_i q^2 + c_i q^4 \right] f_i(s) \right\},
\]

where

\[
q^2 = \frac{s}{4} - 1, \quad f_i = \begin{cases} 1 & \text{for } i = 0, 2, \\ q^2 & \text{for } i = 1. \end{cases}
\]
The values of the parameters are given in Table 1 and the phase shifts are shown in Fig. 1.

| i | $a_i$ | $b_i$ | $c_i$ | $z_i$ |
|---|---|---|---|---|
| 0 | 0.200 | 0.245 | -0.0177 | 39.3 |
| 1 | 0.035 | $2.76 \cdot 10^{-4}$ | $-6.9 \cdot 10^{-5}$ | 31.1 |
| 2 | -0.041 | -0.0730 | $-3.2 \cdot 10^{-4}$ | -37.3 |

Table 1: Values of the coefficients in the parametrization (3.1) of the physical S- and P-wave phase shifts.

The elasticities $\eta_i$ are very close to 1 below our matching point and will be set equal to 1 in all our numerical results.

In the present section, we vary only the S-wave scattering lengths:

$$a_i \rightarrow a'_i = a_i + \delta a_i, \quad i = 0, 2. \quad (3.3)$$

One of the main goals of the low-energy extrapolation of the experimental data lies in the determination of these scattering lengths. This cannot be achieved directly by solving the Roy equations because the scattering lengths enter into the input of these equations. However, they can be predicted in an indirect way because the physical input is an analytic one. Consequently, the scattering lengths are not independent of the other pieces of the input. If we know the physical $A_i$, $\psi_i$ and $\eta_i$ we may solve the Roy equations for arbitrary scattering lengths $a_i$. According to Proposition 1 in Appendix A, their physical values are obtained by varying these $a_i$ until one arrives at a solution which is regular at $s_0$. In practice, when working with an approximation of the physical $A_i$, $\psi_i$ and $\eta_i$, the scattering lengths have to be varied until the corresponding solution of the Roy equations can be declared a good approximation of the solution of the problem with exact input. This is precisely the procedure used in [3] and it is instructive to have an explicit control of the response to the variations (3.3).

Our task is to determine functions $h_i$ defined as in equation (2.8). In the linearized scheme the representation (2.13) is replaced by

$$h_i(s) = G_i(s) \left\{ \frac{\delta a_i}{G_i(4)} + (s - 4) \left[ (-s)^m c_i (2\delta a_0 - 5\delta a_2) + p \delta_i + H_i(s) \right] \right\}. \quad (3.4)$$
We recall that \( m_0 = m_2 = 0, m_1 = 1, P_0 = P_2 = 0 \) and \( P_1 \) is a constant which we call \( p \). The functions \( H_i \) are the solutions of inhomogeneous extensions of equations (2.14). They have the form of equations (B.1) with

\[
Z_i = \sum_{j=0}^{2} Z_{ij},
\]

\[
Z_{ij}(s) = \frac{1}{\pi} \int_{4}^{s_0} dx R_{ij}(s, x) G_j(x) \sin(2\delta_j(x)) \times \left\{ \frac{1}{(x-4)} \frac{\delta a_j}{G_j(4)} + \left[ (-x)^{\gamma_j} c_i (2\delta a_0 - 5\delta a_2) + p \delta_{j,1} \right] \right\}. \tag{3.5}
\]

The solutions \( H_i \) depend linearly on \( p \) and this constant is fixed in such a way that \( h_2(s_0) = 0 \), i.e.

\[
\frac{\delta a_2}{G_2(4)} + (s_0 - 4) [c_2 (2\delta a_0 - 5\delta a_2) + H_2(s_0)] = 0. \tag{3.6}
\]

We refer the reader to Ref. [7] for a derivation of equations (3.4) and (B.1).

To obtain the \( h_i \) we have to evaluate the molulating functions \( G_i \) on the right-hand side of (3.4) and find the solution \( H_i \) of (B.1) and (3.5). The functions \( G_i \) defined in (2.11) and obtained from the model (B.1) are shown in Fig. 2. The exponents \( \gamma_i \) appearing in (2.15) are

\[
\gamma_0 = 0.89, \quad \gamma_1 = 0.19, \quad \gamma_2 = -0.20. \tag{3.7}
\]
The small value of $\gamma_1$ comes from the fact that $s_0$ is close to $z_1$, the position of the $\rho$-resonance. This leads to the spectacular cusp of $G_1$ at $s_0$, seen in Fig. 2. As $h_2$ behaves as $(s_0 - s)G_2$ at $s_0$, it is this product which is relevant and shown in Fig. 2. The exponents $\gamma_0$ and $\gamma_2 + 1$ are close to 1 and the cusps of $G_0$ and $(s_0 - s)G_2$ are not visible in the figure.

![Figure 2: The functions $G_i$ defined in (2.11) appearing as factors in the responses to variations of the input.](image)

The $H_i$ are slowly varying and (3.4) tells us that $h_1$ has a sharp cusp at $s_0$. The $\delta_i$ defined in (3.1) are models of the physical phase shifts: they are regular at $s_0$ and meant to be produced by an analytic input $(a_i, A_i, \psi_i, \eta_i)$. We see that $\delta'_1$ obtained from (2.8) has a sharp cusp at $s_0$. This singular behavior is a visible signal that the modified input $(a_i + \delta a_i, A_i, \psi_i, \eta_i)$ is no longer an analytic one. The differences $(\delta'_i - \delta_i)$ are linear in $\delta a_0$ and $\delta a_2$:

$$\delta'_i(s) - \delta_i(s) = G_i(s) \left[ f_{i0}(s) \delta a_0 + f_{i2}(s) \delta a_2 \right], \quad (3.8)$$

where $f_{i0}$ and $f_{i2}$ are regular at $s_0$. We see that the cusp of $\delta'_1$ is suppressed if

$$f_{i0}(s_0) \delta a_0 + f_{i2}(s_0) \delta a_2 = 0. \quad (3.9)$$

There is a direction in the $(a_0, a_2)$-plane along which $\delta'_1$ has no cusp. There is still a singularity at $s_0$ but no infinite slope. This indicates that the input $(a_i + \delta a_i, A_i, \psi_i, \eta_i)$ is close to an analytic input if condition (3.9) is satisfied. An analytic input is not an isolated one: it is transformed into new analytic
inputs by suitably correlated variations of its ingredients. Our finding and the fact that \( \delta'_0 \) and \( \delta'_2 \) have no visible cusps show that the physical input is transformed into nearly analytic inputs by variations of the scattering lengths obeying (3.3). One can show that variations of the scattering lengths alone cannot transform an analytic input exactly into an analytic one. A movement along a direction in the \((a_0, a_2)\)-plane has to be accompanied by modifications of the remaining pieces of the input if one wants to keep it exactly analytic. Our results show that these modifications are small and we confirm at the local level the existence of a one-parameter family of nearly analytic inputs along a universal curve, \( a_2 = a_2(a_0) \) in the \((a_0, a_2)\)-plane [11].

To go beyond qualitative results, equations (3.1) have to be solved and we apply the approximation scheme described in Appendix B. In this scheme, the \( H_i \) have the form

\[
H_i(s) \simeq \left( \frac{s}{s_0} \right)^{m_i} \left( \hat{H}_{i,0}(s) \delta a_0 + \hat{H}_{i,2}(s) \delta a_2 + \hat{H}_{i,3}(s)p \right),
\]

(3.10)

where the \( \hat{H}_{i,k} \) are second-degree polynomials. Once we have determined \( \hat{H}_{i,k} \) and when \( p \) has been fixed by (3.6), we find that the ratio \( \delta a_2/\delta a_0 \) defined in (B.9), for which the cusp in \( \delta'_1 \) is suppressed, is equal to 0.197. In fact the ratio \( f_1,0(s)/f_1,2(s) \) is nearly constant and equal to its value at \( s_0 \) on the whole interval \([4, s_0] \). We identify the ratio 0.197 with the slope of the universal curve at \((a_0, a_2)\): it coincides with the slope found in [3]. There is a strong compensation of the two terms in the right-hand side of equation (3.8) when \( i = 1 \) if one moves along the universal curve. This compensation is maximally removed in the orthogonal direction where \( \delta a_2/\delta a_0 = -5.08 \). This is illustrated in Fig. 3 which displays the relative phase-shift differences \( (\delta'_i - \delta_i)_\parallel/\delta_i \) and \( (\delta'_i - \delta_i)_\perp/\delta_i \). The differences \( (\delta'_i - \delta_i)_\parallel \) are obtained when the point \((a_0, a_2)\) moves along the universal curve

\[
\delta a_0 = \delta a_\parallel \cos \theta_\parallel, \quad \delta a_2 = \delta a_\parallel \sin \theta_\parallel,
\]

(3.11)

with \( \theta_\parallel = \tan^{-1} 0.197 = 11^\circ \). The differences \( (\delta'_i - \delta_i)_\perp \) are obtained in response to a displacement \((\delta a_0, \delta a_2)\) normal to the universal curve, \( \delta a_0 \) and \( \delta a_2 \) being given by (3.11) with \( \delta a_\parallel \) replaced by \( \delta a_\perp \) and \( \theta_\parallel \) replaced by \( \theta_\perp = 101^\circ \).
Figure 3: Relative responses \((\delta_l' - \delta_l)/\delta_l\) to variations of the S-wave scattering lengths. (a) displacement (B.11) along the universal curve, \(\delta a_{\parallel} = 0.05a_0^2\). (b) displacement orthogonal to the universal curve, \(\delta a_{\perp} = 0.05|a_0^2|\).

To assess the quality of the results displayed in Fig. 3, Table 2 gives the values of the \(\chi_i\) defined in (B.10) in the parallel and orthogonal directions. The values of the \(\chi_i^\perp\) are acceptable whereas those for \(\chi_i^\parallel\) are surprisingly small.
The relative variations of the phase shifts in the direction of the universal curve are decreasing functions of \( s \). The S-waves have peaks at threshold, whose sizes are dictated by the values of \( \delta a_0 \) and \( \delta a_2 \). The pattern in the orthogonal direction is different and more complicated. The effects of the variation of the scattering length spread over the whole interval \([4, s_0]\) and \((\delta_1' - \delta_1)_{\perp}/\delta_1\) has a cusp which cannot be overlooked.

The overall size of the variations \((\delta_1' - \delta_1)_{\perp}\) is significantly larger than that of the corresponding \((\delta_1' - \delta_1)_{\parallel}\). To characterize this fact quantitatively we evaluate the mean values of the absolute ratios over the interval \([4, s_0 - 2]\) \((s_0 - 2 \) instead of \( s_0 \) as upper limit to avoid effects of the cusp in \((\delta_1' - \delta_1)_{\perp}\))

\[
\rho_i = \left\langle \left| \frac{\delta_1' - \delta_1}_{\perp} \right| \frac{\delta_1' - \delta_1}_{\parallel} \right\rangle.
\]

(3.12)

We take \( \delta a_{\perp} = \delta a_{\parallel} \), i.e. we assume that the same distance is covered along and perpendicularly to the universal curve, and find

\[
\rho_0 = 15.7, \quad \rho_1 = 217, \quad \rho_2 = 18.6. \quad (3.13)
\]

These large values reflect the sharp definition of the universal curve obtained in [3].

### 4 Combined variations of input absorptive parts, driving terms and scattering length

This section is mainly devoted to variations \( \delta A_i \) of the absorptive parts \( A_i \) above the matching point. As shown in [7] the response is obtained from the following Ansatz for the functions \( h_i \) in (2.8):

\[
h_i(s) = G_i(s)(s - 4)[p \delta_{i,1} + F_i(s) + H_i(s)],
\]

(4.1)
where
\[ F_i(s) = \frac{s^{m_i}}{\pi} \int_{s_0}^{\infty} dx \frac{1}{x^{m_i}} \frac{1}{x - 4 - s} \frac{1}{G_i(x)} \]  
(4.2)

The functions \( H_i \) on the right-hand side of eq. (4.1) are solutions of the equations (B.1) with
\[ Z_i(s) = \sum_{j=0}^{2} Z_{ij}(s), \]  
(4.3)

where
\[ Z_{ij}(s) = Y_{ij}(s) + \frac{1}{\pi} \int_{s_0}^{s} dx R_{ij}(s, x) \sin(2\delta_j(x))G_j(x) \left[ \frac{1}{p\delta_j + F_j(x)} \right], \]  
(4.4)

and
\[ Y_{ij}(s) = \frac{1}{\pi} \int_{s_0}^{\infty} dx \frac{1}{x - 4} R_{ij}(s, x) \delta A_j(x). \]  
(4.5)

Equations (4.2)-(4.5) have been derived in [7]. For simplicity we assume that the \( \delta A_i \) vanish at \( s_0 \) and the boundary condition (2.9) remains unchanged: \( h_i(s_0) = 0. \)

Using the analyticity properties of the kernels \( R_{ij} \) referred to in Appendix B, the integral in the right-hand side of eq. (4.4) can be transformed to give
\[ Z_{ij}(s) = p Q_i(s)\delta_{j,1} + \int_{s_0}^{\infty} du M_{ij}(s, u) \frac{\delta A_j(u)}{u - 4 G_j(u)}, \]  
(4.6)

where \( Q_i \) is a known polynomial and
\[ M_{ij}(s, u) = -\frac{1}{2\pi} \oint_{\Gamma'} dx R_{ij}(s, x)G_j(x) \left( \frac{x}{u} \right)^{m_i} \frac{1}{x - u}. \]  
(4.7)

The contour \( \Gamma' \) encircles the segment \( -(s - 4), 0 \). Formula (4.7) affords an explicit evaluation of \( M_{ij} \) once the \( G_j \) have been approximated by polynomials, as in appendix B. The functions \( H_i \) are determined by applying the method of that Appendix. The condition \( h_2(s_0) = 0 \) fixes \( p \) as a linear functional of the \( \delta A_i \) and the differences \( \delta_i' - \delta_i \) resulting from (2.8) and (1.1) can be written as
\[ \delta_i'(s) - \delta_i(s) = \sum_{j=0}^{2} \int_{s_0}^{\infty} du K_{ij}(s, u)\delta A_j(u). \]  
(4.8)

The kernel \( K_{ij}(u) \) gives the effect on the channel \( i \) phase shift of a variation of the channel \( j \) absorptive part at point \( u \) \((u > s_0)\). It follows from (1.1) that \( K_{ij}(s, u) \) is proportional to \( G_i(s) \) and \( K_{1,j} \) exhibits, as a function of \( s \), a sharp cusp at \( s = s_0 \). An arbitrary variation of the input absorptive parts transforms an analytic input into a non-analytic one. We correct this partly and stay in the vicinity of an analytic input by modifying simultaneously the scattering.
length \(a_2\) and choosing \(\delta a_2\) in such a way that the cusp in \(K_{1,j}\) is suppressed. This variation \(\delta a_2\) is a linear functional of the \(\delta A_i\):

\[
\delta a_2 = \sum_{i=0}^{2} \int_{s_0}^{\infty} du \kappa_i(u) \delta A_i(u)
\] (4.9)

and Section 3 tells us that equations of the form (4.8) are still valid if the \(K_{ij}\) are replaced by new kernels \(\hat{K}_{ij}\).

The kernels \(K_{ij}\) and \(\hat{K}_{ij}\) have been evaluated as functions of \(s\) for 3 values of \(u\):

\(u_1 = 35\) (\(E_1 = 828\) MeV) close to the matching point, \(u_2 = 51\) (\(E_2 = 1\) GeV) and \(u_3 = 100\) (\(E_3 = 1.4\) GeV).

![Figure 4: The kernels \(K_{11}\) and \(\hat{K}_{11}\) at \(u = 35\) as functions of \(s\). \(\hat{K}_{11}\) includes the effect of a variation of \(a_0^2\) suppressing the cusp in \(K_{11}\).](image)

The passage from \(K_{11}\) to \(\hat{K}_{11}\) at \(u = u_1 = 35\), slightly above the matching point, is illustrated in Fig. 4. As must be the case, the large cusp in \(K_{11}\) has disappeared in \(\hat{K}_{11}\). The effect of the induced variation of \(a_2\) (\(\kappa_1(u_1) = -0.0085\)) dominates the response to the variation of \(A_1\) outside the neighborhood of \(s_0\). The fact that \(\hat{K}_{11}\) is larger than \(K_{11}\) is peculiar: \(\hat{K}_{10}\) and \(\hat{K}_{12}\) are much smaller than \(K_{10}\) and \(K_{12}\).

The values of the kernels \(\hat{K}_{ij}\) at \(u = u_1\) determine the responses to small variations \(\delta A_j\) concentrated around that point. If \(\delta A_j\) is sufficiently small and narrow the equations (4.8) and (4.9) give

\[
\delta_i(s) - \delta_i(s) \simeq \sum_j \hat{K}_{ij}(s, u_1) \Delta A_j, \quad \delta a_2 \simeq \sum_i \kappa_i(u_1) \Delta A_i,
\] (4.10)
with
$$\Delta A_i = \int du \delta A_i(u).$$  \hspace{1cm} (4.11)

The relative phase shift differences produced by such variations of the input absorptive parts with corresponding variations of the scattering length $a_2$ are displayed in Figs. 5, 6 and 7. The $\Delta A_i$ have been chosen in such a way that the responses are of the order of a few percent. Our linearization should be reliable under these circumstances. To describe the situation in physical terms we can imagine that the $\Delta A_j$ are produced by the insertion of fictitious narrow elastic resonances of width $\Gamma_j$ at $u_1$. The values of the $\Delta A_j$ used in Figs. 5, 6 and 7 correspond to $\Gamma_0 = 0.76$ MeV, $\Gamma_1 = 1.56$ MeV, $\Gamma_2 = 0.92$ MeV. The effects of these resonances sitting just above the matching point spread over the whole interval $[4, s_0]$. The induced variation of $a_2$ produces a modest peaking of $(\delta_2' - \delta_2)/\delta_2$ at threshold. The responses to a variation $\Delta A_0$ in the isospin 0 S-wave are globally smaller than the effects of variations $\Delta A_1$ and $\Delta A_2$ of the same size in the other channels. A variation $\Delta A_i$ in channel $i$ produces a response in the same channel that is enhanced near $s_0$ and dominates the responses in the other channels. This dominance is significant but not very strong in the case of $\Delta A_2$. Apart from these observations we do not discover any striking feature characterizing qualitatively the coupling of the S- and P-waves.

Figure 5: Relative responses $(\delta_i' - \delta_i)/\delta_i$ to a variation of $A_0$ concentrated on $u = 35$, $\Delta A_0 = -0.1$.  
- $(\delta_0' - \delta_0)/\delta_0$,  
- $(\delta_1' - \delta_1)/\delta_1$,  
- $(\delta_2' - \delta_2)/\delta_2$.  

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Figure 6: Relative responses \( (\delta_i' - \delta_i')/\delta_i \) to a variation of \( A_1 \) concentrated on \( u = 35, \Delta A_1 = -0.1 \). \( (\delta_0'/\delta_0), (\delta_1'/\delta_1), (\delta_2'/\delta_2). \)

The \( \hat{K}_{ij} \) are decreasing functions of \( u \) without significant change in their shape as functions of \( s \). The decrease is rapid just above the matching point. For instance, the \( \hat{K}_{i0} \) are scaled down at \( u = 36 \) to 70% of their values at \( u = 35 \).
To characterize the decrease of the responses when variations $\Delta A_j$ are shifted to higher energies, we compute averages $\rho_{ij}$ of the absolute values of the relative phase shift differences at $u_1 = 35$, $u_2 = 51$ and $u_3 = 100$. According to (4.10), these are given by

$$\rho_{ij}(s) = \frac{1}{s_0 - 4} \int_{s_0}^{s_4} ds \left| \frac{K_{ij}(s, u_k)}{\delta_i(s)} \Delta A_j(s) \right|.$$  

(4.12)

Approximate values of the $\rho_{ij}(u_1)$ are given in Table 3. The ratios $\rho_{ij}(u_2)/\rho_{ij}(u_1)$ and $\rho_{ij}(u_3)/\rho_{ij}(u_1)$ show the decrease of the responses at higher energies. None of the mean responses to variations located at $u_2$ exceed 11% of the corresponding responses at $u_1$. This percentage is reduced to 1.2% when $u_2$ is replaced by $u_3$. Table 4 gives the values of the variations $\delta a_2$ coming from (4.10).

| $(i, j)$ | $\rho_{ij}(u_1)$ | $\rho_{ij}(u_2)/\rho_{ij}(u_1)$ | $\rho_{ij}(u_3)/\rho_{ij}(u_1)$ |
|---------|-----------------|---------------------------------|---------------------------------|
| (0, 0)  | $7.5 \cdot 10^{-3}$ | 0.061                           | 0.0049                          |
| (1, 0)  | $4.2 \cdot 10^{-4}$ | 0.10                            | 0.010                           |
| (2, 0)  | $1.1 \cdot 10^{-3}$ | 0.10                            | 0.010                           |
| (0, 1)  | $2.4 \cdot 10^{-3}$ | 0.10                            | 0.012                           |
| (1, 1)  | $2.6 \cdot 10^{-2}$ | 0.073                           | 0.0069                          |
| (2, 1)  | $6.8 \cdot 10^{-3}$ | 0.11                            | 0.012                           |
| (0, 2)  | $8.3 \cdot 10^{-3}$ | 0.080                           | 0.0069                          |
| (1, 2)  | $7.5 \cdot 10^{-3}$ | 0.079                           | 0.0069                          |
| (2, 2)  | $2.2 \cdot 10^{-2}$ | 0.052                           | 0.0035                          |

Table 3: Mean relative responses $\rho_{ij}(u_1)$ defined in (4.12) for $|\Delta A_j| = 0.1$ and ratios of mean relative responses at $u_2$ and $u_3$ vs. responses at $u_1$, $\sqrt{u_1} M_\pi = 828$ MeV, $\sqrt{u_2} M_\pi = 1$ GeV, $\sqrt{u_3} M_\pi = 1.4$ GeV.

We conclude that the solution of the Roy equations is quite insensitive to the errors on the input absorptive parts above $E_3 = \sqrt{u_3} M_\pi = 1.4$ GeV. The solution of the Roy equations is most sensitive to the input absorptive parts close to the matching point. According to Table 4, the uncertainty in $a_2$, associated in our scheme with an error on the absorptive parts at $u_3$, is less than 1% of the uncertainty due to the same error at $u_1$. 

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Table 4: Relative variations of the scattering length $a_2$ induced according to (4.10) by variations of the input absorptive parts $A_j$ at $u_1$, $|\Delta A_j| = 0.1$ and ratios of variations at $u_2$ and $u_3$ vs. variations at $u_1$, $\sqrt{u_1} M_\pi = 828$ MeV, $\sqrt{u_2} M_\pi = 1$ GeV, $\sqrt{u_3} M_\pi = 1.4$ GeV.

| $j$ | $\frac{\delta a_2(u_1)}{a_2}$ | $\frac{\delta a_2(u_2)}{\delta a_2(u_1)}$ | $\frac{\delta a_2(u_3)}{\delta a_2(u_1)}$ |
|-----|-------------------------------|--------------------------------------|--------------------------------------|
| 0   | $-4.8 \cdot 10^{-3}$           | 0.09                                 | 0.009                                 |
| 1   | $-2.1 \cdot 10^{-2}$           | 0.10                                 | 0.011                                 |
| 2   | $3.6 \cdot 10^{-2}$            | 0.07                                 | 0.006                                 |

We close the discussion of variations of the input absorptive parts with an assessment of the accuracy of our results. The errors come from our functions $H_i$. These form an approximate solution of equations (3.1) with inhomogeneous terms $Z_i$ containing a component (4.3) coming from variations of the $A_j$ at $u_k$ and a component (3.5) due to the corresponding variation of $a_2$. Let $\chi_j(u_k)$ be the total discrepancy between left- and right-hand sides of eq. (3.1) defined in (B.11). These quantities are listed in Table 5. All equations (3.1) are verified at least at the percent level, which is sufficient for our purpose.

Table 5: Total discrepancies $\chi_j$ defined in (B.11) to variations of the input absorptive part $A_j$ at $u_k$ and the correlated variation of the scattering length $a_2$, $\sqrt{u_1} M_\pi = 828$ MeV, $\sqrt{u_2} M_\pi = 1$ GeV, $\sqrt{u_3} M_\pi = 1.4$ GeV.

|       | $u_1$ | $u_2$ | $u_3$ |
|-------|-------|-------|-------|
| $\chi_0$ | 0.012 | 0.012 | 0.005 |
| $\chi_1$ | 0.024 | 0.057 | 0.016 |
| $\chi_2$ | 0.017 | 0.019 | 0.014 |

We close this section with a survey of the response to variations of the driving terms $\psi_i$ in equation (2.2). The $\psi_i$ are small and approximated by polynomials on $[4, s_0]$ in [3]. We consider variations of these polynomials. The Ansatz for the functions $h_i$ defined in (2.8) becomes

\[
h_i(s) = (s - 4)G_i(s) (p \delta_{i,1} + H_i(s)),
\]

(4.13)
where \( p \) is a constant and the \( H_i \) form a solution of the equations (B.1) with

\[
Z_i(s) = \delta \psi_i(s) + \delta_{i,1} p \int_4^{s_0} dx \, R_{i1}(s, x) \sin(2\delta_1(x)) G_1(x).
\] (4.14)

As before, the variations of the driving terms are combined with variations of \( a_2 \) such that \( h_1 \) has no cusp at \( s_0 \). The result shows that large relative variations of the driving terms affect only weakly the phase shifts below \( s_0 \). For instance, a reduction of the size of \( \psi_0 \) or \( \psi_2 \) by 50% changes the \( \delta_i \) by less than 5%. In the case of a 50% reduction of \( \psi_1 \) the response is smaller than 0.5%.

5 Summary and conclusions

We have developed an approximation scheme to determine the linear response of the solution of the S- and P-wave Roy equations with matching points \( s_0 = 33 \) to small variations of their input (S-wave scattering lengths, S- and P-wave absorptive parts above \( s_0 \), and driving terms). Our results are precise at the percent level, which is sufficient for a qualitative insight. Our problem has been solved long ago, in a different way, for a higher matching point \( s_0 = 70 \) in [7]. At \( s_0 = 33 \) the solution of the Roy equations is unique, entirely determined by their input.

An arbitrary input leads to a solution that is singular at \( s_0 \). As the physical amplitudes are regular at \( s_0 \), the physical input belongs to the restricted class of our analytic inputs producing a solution that is non-singular at \( s_0 \). We prove that under legitimate assumptions an analytic input has in fact only one solution regular at \( s_0 \) (Appendix A).

An arbitrary variation of the input transforms an analytic input into a non-analytic one and induces responses that are singular at \( s_0 \). Due to the fact that our \( s_0 \) is close to \( M^2_{\rho} \) (\( M_\rho = \rho \)-meson mass), the sharpest singularities show up as cusps in the isospin 1 P-wave responses. These cusps are suppressed by correlating suitably the variations of two pieces of the input. We choose to associate in this way variations of the isospin 2 S-wave scattering length \( a_0^2 \) to arbitrary variations of other components of the input.

It is instructive to compare our strategy with the procedure used in [4] when solving the Roy equations themselves. The solution is parametrized in [4] by an Ansatz that is regular at \( s_0 \). As one is working with a non-analytic approximation of the physical input, the solution is singular at \( s_0 \) and cannot be fitted exactly by the Ansatz. An approximate solution is constructed by a least square procedure tuning simultaneously the parameters in the Ansatz.

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and the scattering length \(a_0^2\) in the input. In this way the input is brought close
to an analytic one and the Ansatz gives a model of the corresponding solution.
In some of its features this machinery resembles our simple strategy. In fact,
their equivalence for the computation of responses to small variations of the
input has been checked in the case of the variation of the isospin 0 S-wave
absorptive parts displayed in Fig. 5. The response obtained by solving the full
Roy equations coincides with our result within a few percents. This confirms
that the main factor tuning \(a_0^2\) in [3] is the avoidance of a cusp in the isospin 1
P-wave phase shift.

Our technique shows that one stays in the vicinity of an analytic input when
moving infinitesimally along a given direction of the \((a_0, a_0^2)\) plane without
changing the other pieces of the input. This confirms the existence of a so-
called universal curve at the linear response level.

We have determined the response to localized variations of the input absorptive
parts above the matching point. It spreads over the whole interval \([4, s_0]\) and
illustrates the intricate coupling of the S- and P-waves produced by crossing
symmetry. It shows that the sensitivity to the errors in the input absorptive
parts decreases rapidly with increasing energy.

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Appendix A. Analytic input and uniqueness

An analytic input \((a_i, A_i, \psi_i, \eta_i)\) is defined as an input admitting at least one
solution of the Roy equations which is regular at the matching point. A precise
definition is given below. In any case it is an indirect definition: as shown at
the end of this appendix, we know how to construct analytic inputs but we
are unable to identify an analytic input by direct inspection. Its components
are correlated: in particular, the scattering lengths depend on the \(A_i, \psi_i\) and
\(\eta_i\). Analytic inputs are relevant objects because the physical input belongs to
that class. The aim of this appendix is to prove that an analytic input has
only one solution regular at \(s_0\). The requirement of regularity at \(s_0\) eliminates
in principle the uniqueness problem. This result has already been established
in [3] for simplified one-channel Roy equations.

To establish our result we need general analyticity properties of the partial
wave amplitudes \(f_i\). Let \(f\) be one of them. It is known to be the boundary
value of an analytic function \(F\) on the interval \([4, 125.31]\) [10]. This function is
holomorphic in a complex domain \(\Delta\) extending on the real axis from \(s_L = -28\) to \(s_R = 125.31\) and provided with a left-hand cut \([s_L, 0]\) and a right-hand cut \([4, s_R]\). We have

\[
f(s) = \lim_{\epsilon \to 0} F(s + i\epsilon), \quad s \in [4, s_R]. \tag{A.1}
\]

Our matching point \(s_0\) being above the first inelastic threshold \(i_1 = 16\), we need properties characterizing the elasticity parameters \(\eta\) which enter into an analytic input. According to (2.5), \(\eta\) is equal to the modulus of the S-matrix element \((1 + 2i\sigma f)\) which is the boundary value of

\[
S(z) = 1 - 2 \sqrt{\frac{4 - z}{z}} F(z). \tag{A.2}
\]

This function is regular in \(\Delta\). Using the relation \(\bar{S}(z) = S(\bar{z})\) we write

\[
\eta^2(s) = \lim_{\epsilon \to 0} S(s + i\epsilon) S(s + i\epsilon) = \lim_{\epsilon \to 0} S(s + i\epsilon) S(s - i\epsilon), \quad s \in [i_1, s_R]. \tag{A.3}
\]

Although it cannot be derived from first principles [16], it is legitimate to assume that the inelastic thresholds \(i_k (k = 1, 2, \ldots)\) are the only singularities of \(f\) on \([4, s_R]\) and that \(S\) has an analytic continuation \(S_{II}\) into the sheet reached by crossing the cut \([4, s_R]\) from below between two successive inelastic thresholds \(i_k\) and \(i_{k+1}\) \([S_{II}\) depends on the pair \((i_k, i_{k+1})]\). Equation (A.3) gives

\[
\eta^2(s) = \lim_{\epsilon \to 0} S(s + i\epsilon) S_{II}(s + i\epsilon) \quad s \in (i_k, i_{k+1}), \tag{A.4}
\]

as long as \(i_{k+1} < s_R\).

We assume that \(S_{II}\) is regular in the upper half-plane, in a neighborhood \(D\) of the segment \((i_k, i_{k+1})\). Equation (A.4) tells us that the real-valued \(\eta^2\) is the boundary value on \((i_k, i_{k+1})\) of a function holomorphic in \(D\) and we apply the following general result.

**Lemma 1** Let \(w\) be a real-valued function defined on the interval \((i_k, i_{k+1})\). If \(w\) is the boundary value of an analytic function \(W\) holomorphic in \(D\), it is the restriction to \((i_k, i_{k+1})\) of a function regular in the domain \(D \cup \bar{D}\) where \(\bar{D}\) is the mirror domain of \(D\): \(\bar{D} = \{z | \bar{z} \in D\}\).

A proof of this Lemma is given at the end of this appendix. It implies that \(\eta^2\) has an analytic continuation regular in a complex neighborhood of \((i_k, i_{k+1})\). We assume that the possible complex zeros of \(\eta^2\) are at a finite distance from \((i_k, i_{k+1})\). We choose \(D\) sufficiently narrow so that \(\eta^2\) is non-vanishing on \(D\) and we have
Lemma 2: If the above conditions are fulfilled $\eta$ has a holomorphic continuation from each interval $(i_k, i_{k+1})$ with $i_{k+1} < s_R$ into a complex neighborhood of that interval with $i_k$ and $i_{k+1}$ on its boundary.

We turn now to properties of the full amplitude $f$ and establish

Lemma 3: The real and imaginary parts of $f$ are separately holomorphic in a complex neighborhood of each interval $(i_k, i_{k+1})$ with $i_{k+1} < s_R$ and $i_k$ and $i_{k+1}$ on its boundary. Here $k = 0, 1, 2, \ldots$ with $i_0 = 4$.

This is a well known result in the case of the interval $[4, i_1]$ [17]. For any interval we define the function

$$V = \frac{1}{i\sigma} \frac{1 - \eta + 2i\sigma f}{1 + \eta + 2i\sigma f} \quad \text{(A.5)}$$

on $(i_k, i_{k+1})$ [$\eta = 1$ on $(4, i_1)$]. According to Lemma 2, $V$ has a regular analytic continuation into a domain $N$ in the upper half plane - $[i_k, i_{k+1}]$ belongs to the boundary of $N$ - except for poles at the possible zeros of the denominator. Using unitarity

$$\text{Im } f = \sigma |f|^2 + \frac{1}{4\sigma} (1 - \eta^2) \quad \text{(A.6)}$$

we find that $\text{Im } V = 0$ on $(i_k, i_{k+1})$. Lemma 1 is easily extended to the case of meromorphic functions and one concludes that $V$ has a meromorphic continuation into $N \cup \bar{N}$. The definition (A.5) gives

$$\text{Re } f = \frac{\eta V}{1 + \sigma^2 V^2}, \quad \text{Im } f = \frac{\sigma \eta V^2}{1 + \sigma^2 V^2} + \frac{1}{2\sigma} (1 - \eta). \quad \text{(A.7)}$$

We assume again that the zeros of the denominators are at a finite distance from the real axis and discover that $\text{Re } f$ and $\text{Im } f$ are indeed separately holomorphic in a neighborhood of $(i_k, i_{k+1})$ contained in $N \cup \bar{N}$. The phase shift $\delta$ is also regular in a neighborhood of each $(i_k, i_{k+1})$.

We close our preliminaries with the structure of $f$ at an inelastic threshold $i_k$ with square root singularity. There are four functions $a$, $b$, $c$ and $d$ that are regular in a circle $C_k$ with center $i_k$ and radius $\rho$ such that

$$\eta_\geq(s) = \exp \left[ -2 \left( a(s) + \sqrt{s - i_k} b(s) \right) \right]$$

$$\delta_\geq(s) = c(s) + \sqrt{s - i_k} d(s) \quad \text{(A.8)}$$

for $s \in (i_k, i_{k+1})$: $\eta_\geq$ and $\delta_\geq$ designate respectively the elasticity parameter and the phase shift above $i_k$. The amplitude $f$ can be written on $(i_k, i_{k+1})$ in
terms of a complex phase shift $\tilde{\delta}_>$ as
\begin{equation}
  f = \frac{1}{2i\sigma} \left( e^{2i\tilde{\delta}_>} - 1 \right),
  \tilde{\delta}_>(s) = c(s) + ia(s) + \sqrt{s - i_k} (d(s) + ib(s)).
\end{equation}

The value of $f$ below $i_k$, on $(i_k - \rho, i_k)$, is obtained through analytic continuation of the expression (A.9) in the upper half-plane along curves contained in $C_k$. The outcome is determined by a complex phase shift
\begin{equation}
  \tilde{\delta}_<(s) = c(s) + ia(s) - \sqrt{i_k - s} (b(s) - id(s)).
\end{equation}

The elasticity parameter $\eta_<$ and the phase shift $\delta_<$ below $i_k$ are given by
\begin{equation}
  \eta_<(s) = \exp \left[ -2 \left( a(s) + \sqrt{i_k - s} d(s) \right) \right],
  \delta_<(s) = c(s) - \sqrt{i_k - s} b(s),
\end{equation}
on $(i_k - \rho, i_k)$. The functions $b$ and $d$ interchange their roles when we cross $i_k$. Below $i_1$, $\eta_<$ is equal to 1. This implies
\begin{equation}
  a = d = 0
\end{equation}
in the case $k = 1$.

We summarize our findings in

**Lemma 4** The structure of $f$ at a square root inelastic threshold $i_k$ is described by formulas (A.8) and (A.11). Equation (A.12) holds at $i_1$.

After this lengthy preparation we are ready for a complete definition of an analytic input.

**Definition 1** An analytic input $(a_i, A_i, \psi_i, \eta_i)$ contains elasticities fulfilling Lemma 2 on $(4, s_0)$. It admits at least one solution $f_i$, $i = 0, 1, 2$, of the S- and P-wave Roy equations with Re$f_i$ regular at $s_0$ in the sense that they are holomorphic in a circle $C_{s_0} : |s - s_0| < \epsilon$. These $f_i$ satisfy Lemmas 3 and 4.

We establish the following

**Proposition 1** Let $f_i$, $i = 0, 1, 2$, form a solution of the S- and P-wave Roy equations with analytic input $(a_i, A_i, \psi_i, \eta_i)$ that is regular at $s_0$ and verifies Lemmas 3 and 4. A second solution $f_i'$ of these equations, $f_i' \neq f_i$, is singular at $s_0$. 25
This proposition is an extension of Proposition 4 in [9] to the realistic situation. In the following proof we assume $i_1 < s_0 < i_2$, which is true for our $s_0 = 33$.

The proof of Proposition 4 is based on Lemmas 2, 3 and 4. To make sure that the Roy equations (2.1) guarantee the required analyticity of the $f_i$ we rewrite these equations as follows

$$\text{Re } \Phi_i(s) = (s - 4) \frac{1}{\pi} \int_4^\infty \frac{dx}{(x - 4)(x - s)} \text{Im } \Phi_i(x), \quad (A.13)$$

where

$$\Phi_i(s) = f_i(s) - a_i - (s - 4) \left\{ c_i(2a_0 - 5a_2) + \sum_{j=0}^2 \frac{1}{\pi} \int_4^\infty dx R_{ij}(s, x) \text{Im } f_j(x) + \psi_i(s) \right\}. \quad (A.14)$$

In these equations $\text{Im } f_i(s) = A_i(s)$ for $s \geq s_0$, and the $\psi_i$ are the driving terms appearing in (2.2). The fact that $\text{Im } \Phi_i = \text{Im } f_i$ on $[4, \infty)$ has been used. The equations (A.13) ensure that the $\Phi_i$ are boundary values of analytic functions holomorphic in $C \setminus [4, \infty)$. For $x \in [4, \infty)$ the kernels $R_{ij}(s, x)$ are holomorphic functions of $s$ in $C \setminus (-\infty, 0]$ and the driving terms are regular in the domain $\Delta$ without right-hand cut [10]. Taking all this into account, (A.14) provides a representation of the $f_i$ ensuring that they are indeed boundary values of functions, holomorphic in the domain $\Delta$, with right- and left-hand cuts. The same conclusion holds for the $f_i'$. To establish Proposition 1, we show that the $f_i'$ have to coincide with the $f_i$ if the $\text{Re } f_i'$ are regular at $s_0$. Inversion of the dispersion relations (A.13) gives

$$\text{Im } \Phi_i(s) = \text{Im } f_i(s) = -(s - 4) \frac{1}{\pi} \int_{\Re} \frac{dx}{x - 4} \frac{\text{Re } \Phi_i(x)}{x - s}, \quad s \in [4, \infty). \quad (A.15)$$

The regularity of $\text{Re } f_i$ at $s_0$ implies that $\text{Re } \Phi_i$ is regular at that point and it follows from (A.13) that $\text{Im } f_i$ is holomorphic in $C_{s_0}$. The relation (A.15) holds true for $\Phi_i'$ defined by $f_i'$ and the assumed regularity of $\text{Re } f_i'$ at $s_0$ implies the analyticity of $\text{Im } f_i'$ in $C_{s_0}$. According to Definition 1, $\text{Im } f_i$ and $\text{Im } f_i'$ have analytic continuations holomorphic in a complex neighborhood $N'$ of the interval $[s, s_0]$ shown in Fig. 8. We conclude that $\text{Im } f_i$ and $\text{Im } f_i'$ are holomorphic in $N' \cup C_{s_0}$. As

$$\text{Im } f_i'(s) = \text{Im } f_i(s) = A_i(s) \quad \text{for } s \in [s_0, s_0 + \epsilon), \quad (A.16)$$

$\text{Im } f_i'$ and $\text{Im } f$ coincide in $N' \cup C_{s_0}$ and

$$\text{Im } f_i(s) = \text{Im } f_i'(s) \quad \text{for } s \in (i_1, s_0]. \quad (A.17)$$

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To complete our proof, we have to extend the equality (A.16) below the inelastic threshold $i_1$. The discussion of Lemma 4 shows that phase shifts are required to go through $i_1$. The equality of imaginary parts above $i_1$ at fixed $\eta_i$ implies

$$\delta'_i(s) = \pm \delta_i(s) \mod \pi, \quad s \in (i_1, s_0].$$  \hfill (A.18)

We show that the minus sign must be rejected. If, for a given $i$, $\delta'_i = -\delta_i \mod \pi$, (A.8) gives

$$\delta'_{i>}(s) = c'(s) + \sqrt{s - i_1} d'(s)$$  \hfill (A.19)

with $c'(s) = -c(s) \mod \pi$, $d'(s) = -d(s)$. According to (A.11) this would produce an elasticity $\eta'_{i<}$ below $i_1$ that would differ from the input elasticity $\eta_{i<}$.

In terms of the functions $b$ and $c$ appearing at $i_1$, we now have

$$c'_i = c_i \mod \pi$$  \hfill (A.20)

whereas $b'_i = b_i$ is given, $\eta_i$ being a member of the input. Equations (A.10) and (A.12) give

$$\delta'_{i<}(s) = \left(c_i(s) - \sqrt{i_k - s} b_i(s) \right) \mod \pi, \quad s \in (i_1 - \rho, i_1).$$  \hfill (A.21)

Therefore we have

$$\text{Im } f'_i(s) = \text{Im } f_i(s), \quad s \in (i_1 - \rho, i_1).$$  \hfill (A.22)

As both sides of this equations have analytic extensions regular in a neighborhood $N''$ of $(4, i_1)$ the equality (A.22) extends to $(4, i_1)$. Thus we have established that $f'_i$ and $f_i$ have the same imaginary parts on $[4, s_0]$ if $i_1 < s_0 < i_2$ and the Roy equations imply the full equality of these two amplitudes. This result extends to arbitrary choices of the matching point. The proof becomes easier if $s_0 < i_1$; it requires more steps if $s_0 > i_2$.
For completeness we prove Lemma 1. We define a function $\hat{W}$ by $\hat{W}(z) = \bar{W}(\bar{z})$. It is holomorphic in the mirror domain $\bar{D}$ and, $w$ being real, we have $w(s) = \lim_{\epsilon \searrow 0} \hat{W}(s - i\epsilon)$, $s \in [i_k, i_{k+1}]$. We write for $z \in D$

$$W(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{W(x)}{x-z} \, dx + \frac{1}{2\pi i} \oint_{\partial \bar{D}} \frac{\hat{W}(x)}{x-z} \, dx \quad \text{(A.23)}$$

The first term is the Cauchy representation of $W$ and the second integral vanishes because $z \not\in \bar{D}$. The contributions of the segment $[i_k, i_{k+1}]$ to both integrals cancel and one is left with the Cauchy representation of a function holomorphic in $D \cup \bar{D}$.

Three remarks close this appendix.

1. If $\sqrt{s_0}M_\pi = 800$ MeV, we know, according to Section 2, that the physical solution of the Roy equations is an isolated one. The relevance of Proposition 1 comes from the possible existence of other solutions with $\delta_i'(s_0) = \delta_i(s_0) + n_i\pi$ resulting from CDD-pole ambiguities \[19\]. These solutions are singular at $s_0$.

2. The proof of Proposition 1 tells us that the absorptive parts $A_i$ of an analytic input are regular on some interval $[s_0, s'_0)$ above the matching point $(s'_0 \geq s_0 + \epsilon)$ and are the analytic continuation of Im $f_i$ below $s_0$ on that interval. The Roy equations \[2.1\] define real parts Re $f_i$ above $s_0$. On $[s_0, s'_0)$ they are the analytic continuations of the Re $f_i$ below $s_0$. As the interval $[s_0, s'_0)$ cannot contain an inelastic threshold, all the ingredients of the unitarity condition \[A.6\] have analytic continuations from below $s_0$ onto $[s_0, s'_0)$. This implies that \[A.6\] holds on $[s_0, s'_0)$: Re $f_i$ and $A_i$ are the real and imaginary parts of amplitudes verifying unitarity on that interval. This means that they fulfill, at least on $[s_0, s'_0)$, a consistency condition discussed in \[3\].

3. Although we have no direct way of checking whether a given input is an analytic one, we have a recipe for the construction of such inputs. Take a matching point $s'_0$ above $s_0$ ($s'_0 < 125.31$) and choose arbitrarily an input $(a_i', A_i', \psi_i', \eta_i')$. Let $f_i'$ be a solution of the Roy equations with that input, verifying Lemmas 2, 3 and 4. These $f_i'$ are expected to be singular at $s'_0$ but they are regular at $s_0$. Define a new Ansatz $(a_i, A_i, \psi_i, \eta_i)$ with matching point $s_0$:

$$a_i = a_i', \quad \psi_i = \psi_i', \quad \eta_i = \eta_i' \quad \text{on } [4, s_0],$$

$$A_i(s) = \begin{cases} 
\text{Im } f_i'(s) & \text{for } s_0 \leq s \leq s'_0, \\
A_i'(s) & \text{for } s > s'_0. 
\end{cases} \quad \text{(A.24)}$$
The $f_i'$ define a solution $f_i$ of this new problem,

$$f_i(s) = f_i'(s) \text{ for } 4 \leq s \leq s_0. \quad (A.25)$$

This solution is regular at $s_0$ and the Ansatz $(a_i, A_i, \psi_i, \eta_i)$ is an analytic one.

Our recipe is of no practical use because it requires the explicit resolution of the Roy equations with matching point $s'_0$. The important point is that we recognize that an analytic input with matching point $s_0$ is unconstrained above some $s'_0$, $s'_0 > s_0$. It is the behavior of the $A_i$ on $[s_0, s'_0]$ which is constrained and $s'_0$ can be close to $s_0$.

In our definition an input is analytic with respect to its matching point $s_0$.

The physical input is special because it generates inputs with matching points $s'_0 > s_0$ ($s'_0 < 125.31$) that are analytic with respect to $s'_0$.

### Appendix B. Approximation scheme

We write the equations we have to solve in Sections 3 and 4 in the following way.

$$\sum_{j=0}^{2} X_{ij}[H_j](s) = Z_i(s). \quad (B.1)$$

Each $X_{ij}$ is a linear and homogeneous functional of the unknown $H_j$,

$$X_{ij}[H_j](s) = \delta_{i,j} \left\{ \delta_{m_j,0} H_j(s) - \frac{1}{\pi} \int_{4}^{s_0} dx G_j(x) \sin(2\delta_j(x)) \frac{H_j(x) - H_j(s)}{x - s} \right\}$$

$$- \frac{1}{\pi} \int_{4}^{s_0} dx R_{ij}(s,x) G_j(x) \sin(2\delta_j(x)) H_j(x). \quad (B.2)$$

The $Z_i$ are known functions determined by the variation of the input under consideration. The unknown $H_i$ are regular and slowly varying on $[4, s_0]$ and we approximate them by polynomials

$$H_i(s) = s^{m_i} \sum_{n=0}^{N} c_{i,n} s^n. \quad (B.3)$$

We have to determine the coefficients $c_{i,n}$. The $X_{ij}$ become

$$X_{ij}[H_j](s) = \sum_{n=0}^{N} X_{ij}^{(n)}(s) c_{j,n} \quad (B.4)$$

where the $X_{ij}^{(n)}$ are known functions obtained by replacing $H_j(x)$ by $x^{(m_j+n)}$ in the right-hand side of Eq. (B.2).
To evaluate these functions we define auxiliary analytic functions $\overline{G}_i$, holomorphic in $\mathbb{C}\setminus[4, s_0]$

$$\overline{G}_i(z) = \left(\frac{s_0}{s_0 - z}\right)^{m_i} \exp\left[\frac{2}{\pi} \int_4^{s_0} \frac{\delta_i(x)}{x - z} \, dx\right]. \quad (B.5)$$

They are related to the $G_i$ defined in (2.11) by their discontinuity $\text{Disc} \, \overline{G}_i$ across the cut $[4, s_0]$,

$$\frac{1}{2i} \text{Disc} \, \overline{G}_i(s) = G_i(s) \sin(2\delta_i(s)), \quad 4 \leq s \leq s_0. \quad (B.6)$$

The contribution to $X^{(n)}_{ij}$ coming from the first integral in the right-hand side of (B.2) is transformed into a sum of integrals along a closed contour $\Gamma$ surrounding the segment $[4, s_0]$:

$$-\frac{1}{2i \pi} \sum_{m=0}^{n-1} \oint_{\Gamma} dz \, \overline{G}_i(z) z^m s^{n-m-1} = -\sum_{m=0}^{n-1} g_{i,m+1} s^{n-m-1} \quad (B.7)$$

where the $g_{i,p}$ are the coefficients of the Laurent series of $\overline{G}_i$,

$$\overline{G}_i(z) = \sum_{p=0}^{\infty} g_{i,p} \frac{1}{z^p}. \quad (B.8)$$

The second integral in the right-hand side of (B.2) is evaluated in a similar way by exploiting the analyticity properties of the kernels $R_{ij}$. At fixed real $s$, $s \geq 4$, these are analytic functions of $x$, holomorphic in $\mathbb{C}\setminus[-(s - 4), 0]$. Deforming the contour $\Gamma$, we get

$$\int_4^{s_0} dx \, R_{ij}(s, x) \overline{G}_j(x) \sin(2\delta_j(x)) x^n =$$

$$-\frac{1}{2i \pi} \int_{-(s-4)}^{0} dx \, \text{Disc} \, R_{ij}(s, x) \overline{G}_j(x) x^n + \text{polynomial}. \quad (B.9)$$

The polynomial is determined by the asymptotic behavior in $z$ of the product $R_{ij}(s, z) G_j(z) z^n$. The discontinuity $\text{Disc} \, R_{ij}$ of $R_{ij}$ across $[-(s - 4), 0]$ being known, we need the $\overline{G}_j$ on that interval. These smooth functions are approximated by third degree polynomials at a level smaller than 1%. This allows the explicit evaluation of the integral in the right-hand side of Eq. (B.9) and the result is a polynomial in $s$. The $X^{(n)}_{ij}$ are thus approximated by polynomial $\tilde{X}^{(n)}_{ij}$ of degree $\leq 6$.

Evaluated along the same lines, the inhomogeneous terms $Z_i$ in Eqs. (B.1) become known functions $\tilde{Z}_i$. According to the equations (B.1) the $\tilde{X}_i$ and $\tilde{Z}_i$ have to be made approximately equal on $[4, s_0]$ by adjusting the $3(N + 1)$
coefficients $c_{n,i}$ in Eq. (B.4) $\tilde{X}_i$ is obtained by substituting $\tilde{X}_{ij}^{(n)}$ for $X_{ij}^{(n)}$ in (B.4) and inserting the result into (B.3). We keep our calculations simple by using polynomials of low degree for the $H_i$ in (B.3) and choose $N = 2$. To determine the 9 coefficients $c_{i,n}$, the $\tilde{X}_i$ and $\tilde{Z}_i$ are approximated on $[4, s_0]$ by second degree polynomials using a $\chi^2$ technique, and these polynomials are set equal. This gives 9 equations for the 9 unknowns (in $\tilde{X}_i$, each $\tilde{X}_{ij}^{(n)}$ is replaced by a polynomial of degree 2). The whole procedure is legitimate because the $\tilde{X}_i$ and $\tilde{Z}_i$ are slowly varying.

The $\tilde{X}_{ij}^{(n)}$ and $\tilde{Z}_i$ are close to the $X_{ij}^{(n)}$ and $Z_i$, the differences coming only from the replacement of the $\tilde{G}_i$ by third degree polynomials on $[-(s - 4), 0]$. Thus, in view of equations (B.3), the $c_{i,n}$ we obtain must be such that $\tilde{X}_i$ and $\tilde{Z}_i$ are close to each other on $[4, s_0]$. This can be checked by evaluating the mean relative quadratic discrepancies of $\tilde{X}_i$ and $\tilde{Z}_i$

$$\chi_i = \left[ \frac{1}{(s_0 - 4)} \int_4^{s_0} ds \frac{\tilde{X}_i(s) - \tilde{Z}_i(s))^2}{(\tilde{Z}_i(s))^2} \right]^{\frac{1}{2}}. \quad \text{(B.10)}$$

We can also define a total discrepancy

$$\chi = \left[ \frac{1}{3} \sum_{i=0}^{2} \chi_i^2 \right]^\frac{1}{2}. \quad \text{(B.11)}$$

The various values we obtain for these quantities are quoted in Sections 3 and 4.

**Appendix C. The kernels $R_{ij}$**

Our technique makes extensive use of the analyticity properties of the regular kernels $R_{ij}$ in equations (2.1) and (2.2). It is therefore convenient to display them explicitly. They are obtained from 4 functions $L_k, k = 1, \ldots, 4$:

$$L_1(s, x) = \frac{1}{x(s - 4)} \left[ \frac{1}{2} s - x + 2 + (x - 4) \frac{x}{s - 4} \ln \left( 1 + \frac{s - 4}{x} \right) \right],$$

$$L_2(s, x) = \frac{1}{x(s - 4)} \left[ -\frac{3}{2} s - x + 2 + (2s + x - 4) \frac{s}{x} \ln \left( 1 + \frac{s - 4}{x} \right) \right],$$

$$L_3(s, x) = \frac{1}{x(s - 4)^2} \left[ -\frac{1}{6} \left[ s^2 - 8s + 4(3x^2 - 12x + 4) \right. \right. \right.

$$\left. \left. + (2s + x - 4)(x - 4) \frac{x}{s - 4} \ln \left( 1 + \frac{s - 4}{x} \right) \right] \right]. \quad \text{(C.1)}$$
\[ L_4(s, x) = \frac{1}{x(s-4)^2} \left\{ -\frac{1}{6} \left[ s^2 + 8s(3x - 1) + 4(3x^2 - 12x + 4) \right] \right. \\
\left. + (2x + s - 4)(2s + x - 4)(x - 4) \frac{x}{s - 4} \ln \left( 1 + \frac{s - 4}{x} \right) \right\}. \]

The \( R_{ij} \) are given by

\[ R_{00}(s, x) = \frac{2}{3} L_1(s, x) - \frac{1}{x}, \quad R_{02}(s, x) = \frac{10}{3} L_1(s, x), \]
\[ R_{20}(s, x) = \frac{2}{3} L_1(s, x), \quad R_{22}(s, x) = \frac{1}{3} L_1(s, x) - \frac{1}{x}, \]
\[ R_{01}(s, x) = 6L_2(s, x), \quad R_{21}(s, x) = -3L_2(s, x), \] \hspace{1cm} (C.2)
\[ R_{10}(s, x) = \frac{2}{3} L_3(s, x), \quad R_{12}(s, x) = -\frac{5}{3} L_3(s, x), \]
\[ R_{11}(s, x) = 3L_4(s, x) - \frac{1}{x}. \]

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