D-brane Analyses for BPS Mass Spectra and U-duality

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Abstract

We give a confirmation of U-duality of type II superstring by discussing mass spectrum of the BPS states. We first evaluate the mass spectrum of BPS solitons with one kind of R-R charges. Our analysis is based on the 1-loop effective action of D-brane, which is known as “Dirac-Born-Infeld (DBI) action”, and the fact that BPS states correspond to the SUSY cycles with minimal volumes. We show the mass formula derived in this manner is completely fitted with that given by U-duality.

We further discuss the cases of BPS solitons possessing several kinds of R-R charges. These are cases of “intersecting D-branes”, which cannot be described by simple DBI actions. We claim that, in these cases, higher loop corrections should be incorporated as binding energies between the branes. It is remarkable that the summation of the contributions from all loops reproduces the correct mass formula predicted by U-duality.

PACS codes; 11.10.Kk, 11.25.-w, 11.25.Sq, 11.30.Pb
Keywords; D-brane, BPS state, U-duality, mass formula, M-theory
1 Introduction

D-brane analyses have been found to be very successful methods to describe the solitonic states of the string theory [1]-[13]. They give essential insights into the non-perturbative aspects of the string theory, such as the recent success of the microscopic description of the black hole properties [14]. By the use of string theory, they give a hope that those non-perturbative excitation might be quantized.

In this paper, we focus on the type II superstrings compactified on torus. This theory is conjectured to have the U-duality symmetry [11, 15, 16] which originates from the symmetry of the supergravity theory [17]. Our aim is to present a non-trivial check of U-duality. One of the remarkable successes to this aim is the calculations of degeneracy of BPS states given in the excellent works [12, 13]. However, these calculations are essentially independent of the background moduli. Hence it is still meaningful to study the quantities which strongly depend on the moduli for the confirmation of U-duality. One of the typical objects with this property is the BPS mass spectrum.

Motivated with this fact, we intend to examine the mass formula for BPS solitons with Ramond-Ramond (R-R) charges by means of D-brane techniques. We shall compare it with that predicted by U-duality.

For the simple cases in which only one kind of R-R charges are excited, we shall analyse the mass spectra by using the Dirac-Born-Infeld (DBI) action for the D-brane [2, 7, 8, 10]. This is an extension of the study in our previous publication [18].

It is further stimulating to analyse the more complicated situations with several kinds of R-R charges excited, that is, the R-R solitons described by the “intersecting D-branes” [9, 11, 13, 19, 20]. We will stress that the analysis based on DBI action, which includes only the one-loop contribution, is not sufficient to discuss the D-brane bound states. Under general backgrounds, the configurations of intersecting D-branes break SUSY completely. In those cases, the (stringy) higher loop corrections do not vanish. We will show that the higher loop corrections can be regarded as the binding energies among the intersecting branes and analysis based on these loop corrections give the consistent results with U-duality.

Let us give the plan of this paper. In section 2, we give the BPS mass formula which is invariant under U-duality by extending the well-known BPS spectrum of the fundamental string sector [21].
In section 3 we consider the simple BPS solitons possessing only one kind of R-R charges. The BPS states of this type are realized as the supersymmetric cycles [22, 23], equivalently, geometrical configurations with minimal volumes. We evaluate the D-brane mass directly from the DBI action by constructing such a configuration. It is an important check of U-duality to study the objects depending strongly on the background moduli, because the duality transformations map the moduli in a string theory onto those in a dual string theory non-trivially. We will show that the DBI action produces the mass spectra with correct moduli dependences expected by U-duality. In the type IIA case, we also explain the BPS mass formulae from the M-theory viewpoint. It is based on SUSY algebra [24] and the results confirm the validity of our analysis in the DBI action.

We will also argue on the situations in which the gauge fields on the world volumes of branes have topological charges. It is believed that these charges are induced by sub-branes within other D-branes ("branes within branes" [9]). We confirm that our evaluation of BPS masses based on the DBI action produces the results which are consistent with this interpretation. That is, we show that the calculation under the non-trivial background gauge fields yields the correct masses of the bound states expected by U-duality.

Although this result is very satisfactory, we should still keep in mind the fact that this story is not complete to understand the physics of D-brane bound states. There still exist many bound states which cannot be reduced to the cases of “branes within branes”. It is sufficient to take only an example to illustrate such generic situations; the 1-branes wrapping around 9th-axis and the 3-branes wrapping around 678th-axes (we assume the 6789th-directions correspond to the compactified 4-torus). This situation and its mass formula cannot be studied by the trick of gauge fields.

To defeat this difficulty we investigate, in section 4, the problem of bound states not only in the “branes within branes” cases but also in the situations that some branes “literally” intersect with others. We emphasize the necessity of taking higher loop corrections into account in the latter cases. We discuss relations between the SUSY breaking under generic moduli and the non-vanishing higher (string) loop amplitudes associated with annulus diagrams. We will evaluate the binding energies of intersecting branes as contributions from higher loops, and show a remarkable fact: The summation of all loop corrections reproduces the correct mass formula of bound states predicted by U-duality! In the last, we will try to explain the possibility to describe some (not all) bound states by only DBI actions from the point of view of “geometrization of quantum correction".
The last section is devoted to conclusion and a few comments on the open problems.

2 BPS Mass Formulae with U-duality Invariance

In this section, we shall analyse the BPS mass formulae in the IIB superstring compactified on $T^4$. The NS-NS sector mass formula can be written in a manifestly T-duality invariant form. Then we can derive the BPS mass formula in the R-R sector by using a (special) U-duality transformation.

Let us consider type IIB superstring compactified on $T^4$. The massless states in the type IIB string are metric field $G_{MN}$, second order antisymmetric tensor $B_{MN}$, and dilaton $\phi$ from NS-NS sector. A scalar field (axion) $C^{(0)}$, an antisymmetric tensor $C^{(2)}_{MN}$, and a self-dual fourth order antisymmetric tensor $C^{(4)}_{MNPQ}$ appear in the R-R sector.

After the compactification, the scalar fields which describe the moduli of the theory are given by $G_{ij}$, $B_{ij}$ and dilaton $\phi$ from NS-NS sector. We use indices $i,j$ to run 6, 7, 8, 9 to describe the internal space coordinates and use $\mu, \nu$ for uncompactified dimensions. $G_{ij}$ and $B_{ij}$ give a structure of Grassmannian, $M \in O(4,4)/(O(4) \times O(4))$ to the moduli by a combination,

$$M = (\Omega_B \Omega_e)(\Omega_B \Omega_e)^t, \quad \Omega_e = \begin{pmatrix} e & 0 \\ 0 & (e^t)^{-1} \end{pmatrix}, \quad \Omega_B = \exp \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}. \quad (2.1)$$

The $e$ is the vierbein of the string (sigma model) metric $G$, namely, $G = e \cdot e^t$. $\Omega_e$ belongs to $O(4,4)$ and satisfies a relation

$$\Omega_e J \Omega_e^t = J, \quad J = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}. \quad (2.2)$$

Here $I_4$ is a 4 x 4 unit matrix. The left action of $\varpi \in O(4,4;\mathbb{Z})$ ($\Omega_e^* = \varpi \Omega_e$) represents the T-duality of the system. On the other hand, the Ramond moduli, $C^{(0)}$, $C^{(2)}_{ij}$ and $C^{(4)}_{6789}$ are combined to give an 8 component field $\psi^{(a')} (a' = 1, 2, \cdots, 8)$ in the cospinor representation of $O(4,4)$, which we write as $(\psi^{(a')}) = R_c(\varpi)_{a',b'} \psi^{(b')}$. (Similarly, in the type IIA case, $C^{(1)}_i$ and $C^{(3)}_{ijk}$ are combined into a spinor multiplet of $O(4,4)$.)

The vector fields in 6 dimensions are essentially composed of $G_{i\mu}$ and $B_{i\mu}$. The former is associated with the Kaluza-Klein momentum and the latter is coupled with the
winding number around \(i\)-th direction. The NS vector fields are combined into a single 8 component gauge field \(A^{(a)}_{\mu}(a = 1, 2, \cdots, 8)\) in a vector representation under \(O(4, 4)\), \(A^{(a)}_{\mu} = \varpi_{ab} A^{(b)}_{\mu}\). Vector fields from the Ramond sector \(C^{(2)}_{ij\mu}, C^{(4)}_{ijk\mu}\) count the 1,3-brane charges. They are combined into an 8 component vector field \(K^{(a)}_{\mu}\) which transforms as a spinor under \(O(4, 4)\), \(K^{(a)}_{\mu} \rightarrow \left(R_s(\varpi)\right)_{a\beta} K^{(b)}_{\mu}\). These spinor representation matrices, \(R_s(\varpi)\), satisfy \(R_s(\varpi)JR^t_s(\varpi) = J\). (More precisely, there appear some mixing of \(G_{i\mu}\) and \(B_{i\mu}\) in the definition of \(A^{(a)}_{\mu}\) and that of \(C^{(i)}\)'s in the definition of \(K^{(a)}_{\mu}\) under general backgrounds.)

We write the integral charges \(n^{(a)}, m^{(a)}\) associated with \(A^{(a)}_{\mu}\) and \(K^{(a)}_{\mu}\). These charges transform as vector \((n^{(a)})\) and spinor \((m^{(a)})\) of \(O(4, 4; \mathbb{Z})\). For each set of integers, we can define a stable state called the BPS state.

In the fundamental string spectrum, we have vanishing R-R charges, \(m^{(a)} = 0\). The famous mass formula of the (anti)BPS state in this case is given as [21],

\[
M_{BPS}^2 = n'(M \pm J)n = n'(\Omega_B \Omega_e)\Pi^\pm(\Omega_B \Omega_e)^t n', 
\]

\[
\Pi^\pm = \begin{pmatrix} I_4 & \pm I_4 \\ \pm I_4 & I_4 \end{pmatrix}.
\]

Here we write the \(O(4, 4; \mathbb{Z})\) vector \(n = (n^{(a)}) = \begin{pmatrix} w^i \\ n_i \end{pmatrix}\), \((n_i; KK\text{ momentum}, w^i; \text{winding number})\). The \(n^{(a)}\) couples with the gauge field \(A^{(a)}_{\mu}\) as \(n'A_{\mu}\). The \(\Pi^\pm\) are appropriate operators that project to (anti)BPS states. Needless to say, this mass formula is invariant under the right action of \(\sigma \in O(4) \times O(4)\) since \(\Omega_B \Omega_e\) is transformed into \((\Omega_B \Omega_e)\sigma\) and \(\sigma\Pi^\pm\sigma^t = \Pi^\pm\). The invariance under T-transformation \((O(4, 4; \mathbb{Z})\) left-action) is ensured by the transformation law;

\[
\begin{cases} 
\Omega_B \Omega_e & \rightarrow \varpi(\Omega_B \Omega_e), \\
\ n & \rightarrow (\varpi')^{-1}n(\equiv J\varpi Jn),
\end{cases} \quad (\forall \varpi \in O(4, 4; \mathbb{Z})).
\]

We shall make here one important remark: The mass formula \((2. 3)\) is written with respect to the sigma model metric \(G_{\mu\nu}\). Later we will observe that \(G_{\mu\nu}\) is not invariant under general U-duality transformations (even though it is invariant under all the T-transformations \(O(4, 4; \mathbb{Z})\)). Therefore we should write down the mass formula associated not to the sigma model metric \(G_{\mu\nu}\) but to the 6-dimensional Einstein frame metric \(g^{(6)}_{\mu\nu} \equiv e^{-\Phi} G_{\mu\nu}\) which is U-invariant. Here \(\Phi\) denotes the 6-dimensional dilaton that is invariant under T-
transformations;
\[ \Phi = \phi - \frac{1}{4} \log \det(G_{ij}). \]  
(2.5)

It is easy to observe that the mass formula defined with respect to \( g^{(6)}_{\mu\nu} \) should be related with that for \( G_{\mu\nu} \)
\[ M^{(Einstein)} = e^{\Phi/2} M^{(sigma \ model)} . \]  
(2.6)

Namely, we should rewrite the mass formula for the NS-NS charges (2.3) as
\[ M_{BPS}^2 = e^\Phi n^t (\Omega_{B} \Omega_e) \Pi^\pm (\Omega_{B} \Omega_e)^t n = e^\Phi n^t (M \pm J) n . \]  
(2.7)

Beside \( O(4, 4; \mathbb{Z}) \), the type IIB superstring theory is conjectured to be invariant under the strong-weak \( SL(2; \mathbb{Z}) \) duality (S-duality). This symmetry is combined with T-duality symmetry to give the U-duality symmetry group \( O(5, 5; \mathbb{Z}) \) \[11, 15, 16, 25\].

In order to clarify the structure of U-duality \( O(5, 5; \mathbb{Z}) \), we must suitably arrange the moduli fields from both the NS-NS sectors \( G_{ij}, B_{ij}, \Phi \) and the R-R sectors \( \psi_\alpha \). The \( \Omega_e, \Omega_B \) should be embedded in \( O(5, 5) \)-matrix as follows (this is one of the conventions);
\[ \tilde{\Omega}_{e,B} = \left( \begin{array}{cc} I_2 & 0 \\ 0 & R_c(\Omega_{e,B}) \end{array} \right) . \]  
(2.8)

Here the \( R_s(\Omega) \) and \( R_c(\Omega) \) are spinor and cospinor representations of \( \Omega \in O(4, 4) \).

The dilaton and RR moduli are also incorporated as
\[ \tilde{\Omega}_{e,B,\Phi,\psi} = \tilde{\Omega}_\psi \cdot \tilde{\Omega}_\Phi \cdot \tilde{\Omega}_B \cdot \tilde{\Omega}_e , \]
\[ \tilde{M} = \tilde{\Omega}_{e,B,\Phi,\psi} \cdot \tilde{\Omega}^t_{e,B,\Phi,\psi} , \]  
(2.9)

with
\[ \tilde{\Omega}_\Phi = \left( \begin{array}{ccc} e^\Phi & 0 & 0 \\ 0 & e^{-\Phi} & 0 \\ 0 & 0 & I_8 \end{array} \right) , \]  
(2.10)

\[ \tilde{\Omega}_\psi = \exp \left\{ \sum_{\alpha'=1}^{4} \psi_{\alpha'} (E(2, 6 + \alpha') - E(\alpha' + 2, 1)) + \sum_{\alpha'=5}^{8} \psi_{\alpha'} (E(2, \alpha' - 2) - E(\alpha' + 2, 1)) \right\} , \]

where \( E(i, j) \) is a \( 10 \times 10 \) matrix with only one non-zero entry at \( (i, j) \). Obviously \( \tilde{M} \) is a symmetric matrix in \( O(5, 5) \), and hence parametrizes the “extended Teichmüller space” \( O(5, 5)/(O(5) \times O(5)) \).
Now, let us study R-R sector mass formula. In the following discussion we turn off all the R-R moduli $\psi_\alpha = 0$. We first pick up a special U-transformation $S^{(6)} := S^{(10)}T^{6789}S^{(10)}$ ("S-transformation in the sense of 6-dimension") such that $(S^{(6)})^2 = 1$. Here $S^{(10)}$ stands for the 10-dimensional S-duality transformation (which corresponds to the transformation $\tau \to -\frac{1}{\tau}$ of $SL(2; \mathbb{Z})$). Also the $T^{6789}$ is a T-duality transformation along the 6789th directions. Then the $S^{(6)}$ acts on the $G$, $B$ and $\Phi$

$$
S^{(6)} : \begin{cases} 
  G_{ij} \to \sqrt{|G|}(G^{-1})^{ij}, \\
  B_{ij} \to \sqrt{|G|}(\ast B)^{ij} \equiv \frac{1}{2} e^{ijkl}B_{kl}, \\
  e^\Phi \to e^{-\Phi},
\end{cases} \tag{2.11}
$$

and completely exchanges the NS-NS charges $n^{(a)}$ and the R-R charges $m^{(\alpha)}$. The 6-dimensional space-time Einstein metric $g^{(6)}_{\mu\nu}$ is invariant under the $S^{(6)}$, $S^{(10)}$ but the $G_{\mu\nu}$ is not.

It is remarkable that

$$
S^{(6)}(M) = \begin{pmatrix} 
  \sqrt{|G|}(G^{-1} - \ast B \cdot G \cdot \ast B) & (\ast B \cdot G) \\
  -(G \cdot \ast B) & \frac{1}{\sqrt{|G|}} G
\end{pmatrix} \equiv R_s(M), \tag{2.12}
$$

where $R_s(\ast)$ denotes the spinor representation matrix as is already introduced. Hence we can rewrite the transformation law (2.11) as

$$
S^{(6)} : \begin{cases} 
  e^\Phi M \to e^{-\Phi} R_s(M), \\
  e^{-\Phi} R_s(M) \to e^\Phi M. 
\end{cases} \tag{2.13}
$$

In this way, we can easily find out that the mass formula of the R-R sector must take the following form if we claim the invariance of total mass spectrum under the $S^{(6)}$-transformation;

$$
M_{BPS}^2 = e^{-\Phi} m^t R_s(\Omega_{e,B}) \Pi^{\pm} R_s(\Omega_{e,B}^t) m = e^{-\Phi} m^t (R_s(M) \pm J) m. \tag{2.14}
$$

We will compare this formula with the results of D-brane analyses in the later sections.
3 BPS mass formula from Dirac-Born-Infeld action

In the previous section, we conjectured the BPS mass formula by using U-duality. The outcome becomes algebraic and uses the representation theory of $O(5,5;\mathbb{Z})$. In the following, on the other hand, we evaluate the BPS mass geometrically by minimizing the D-brane worldvolume, or more precisely by minimizing the Dirac-Born-Infeld type integral. This condition should be equivalent to the requirement to keep half of the supersymmetries, namely the BPS condition. Such a supersymmetric configuration is called a “supersymmetric $p$-cycle” [22], and at least in the case that Kalb-Ramond moduli $B_{ij}$ is equal to zero, this statement is proved in [22]. In the general case of $B_{ij} \neq 0$, the equivalence of the conditions of minimal volume and supersymmetry is not so clear. But, from the physical point of view this assumption is very plausible, since the BPS states must respectively have the minimal energies in the charged sectors. It is interesting to observe that the mass formula obtained geometrically coincides with algebraic one.

In the subsection 3.3, we also explain the full mass formula in the IIA theory including both NS-NS and R-R sectors from the M-theory viewpoint. The analysis is based on the SUSY algebra in 6-dimension [24]. We will observe that our analysis based on the DBI action is consistent with the M-theory approach.

3.1 Type IIB on $T^4$

First of all, we consider the type IIB case compactified on $T^4$. For this case the situations are somewhat simpler than those for type IIA.

Our starting point is the (one loop) effective action of Dirichlet $p$-brane [2, 3, 10] ($p$ is an odd number for type IIB string);

$$S_{DBI} := \int d^{p+1}\sigma e^{-\phi} \sqrt{\det (G_{\alpha\beta} + F_{\alpha\beta})}, \quad (3.1)$$

$$S_{WZ} := i \int e^{\mathcal{F}} \wedge C, \quad (3.2)$$

$$\mathcal{F} := B + \frac{1}{2\pi} F, \quad C := C^{(4)} + C^{(2)} + C^{(0)}. $$

Here the $G_{\alpha\beta}, B_{\alpha\beta}$ are the induced metric and anti-symmetric field on the world volume.

\footnote{For the embedding of two- (resp. three-) cycle into the Calabi-Yau two- (resp. three-)fold, this condition becomes holomorphic embedding (resp. the special Lagrangian submanifold [23]).}
The $C^{(l)}$s ($l = 0, 2, 4$) are the Ramond-Ramond fields for the type IIB string and $F_{\alpha\beta}$ is the field strength of $U(1)$ gauge field $A$. $S_{DBI}$ is usually called the “Dirac-Born-Infeld action” and $S_{WZ}$ is often called “Wess-Zumino term”. (Several authors also call it “Chern-Simons term”.)

We evaluate the BPS mass for the 6-dimensional space-time theory along the same line of argument developed in the [18]. Consider the $p$-brane with the specific configuration $R \times \Sigma$ ($R$ is the time axis and $\Sigma$ is a $p$-dimensional subspace of the internal $T^4$). In this setup, the effective mass of particle associated with this $p$-brane is directly evaluated by DBI action:

$$S_{DBI} = M_p \int_R \sqrt{g^{(6)}_{00}} \, d\tau$$

$$M_p := e^{-\Phi/2} |G|^{-1/4} \int_{\Sigma} d^p \sigma \sqrt{\det (G_{\alpha\beta} + F_{\alpha\beta})}.$$  \hspace{1cm} (3.3)

The 6-dimensional space-time dilaton $\Phi$ is related with the 10-dimensional dilaton $\phi$ and the torus metric $G_{ij}$ in (2.5), in other words

$$e^{\Phi - \phi} = |G|^{-1/4},$$  \hspace{1cm} (3.4)

where $|G| := \det G_{ij}$. As we already stressed in the previous section, the mass should be evaluated with respect to the 6-dimensional Einstein frame $g^{(6)}_{\mu\nu} = e^{-\Phi} G_{\mu\nu}$. We used the relation; $\sqrt{g^{(6)}_{00}} e^{-\Phi/2} |G|^{-1/4} \equiv \sqrt{G_{00}} e^{-\phi}$ in Eqs.(3.3).

The topological nature of $\Sigma$ (more precisely speaking, the homology class of $\Sigma$, the homotopy class of $X$ and the topological charge of the bundle where $A$ is defined) determines the Ramond-Ramond charge of this particle. We can find this fact by observing the Wess-Zumino term $S_{\text{WZ}}$ (3.2) and will see it explicitly in the following examples. As is already commented, the BPS state should correspond to a pair of configurations of the map $X : \Sigma \to T^4$ and the $U(1)$-gauge field $A$ that minimizes the integral $\{3.3\}$. Hence our problem is reduced to search the minimal configuration of $(X, A)$ under a fixed topological nature of world volume.

If we turn off the R-R fields, the compactified type IIB moduli space is described by $G_{ij}$ and $B_{ij}$ of $T^4$. In the following argument we assume these $G_{ij}$, $B_{ij}$ are all some constants on $T^4$, which is of course possible since $T^4$ is a flat space. The induced metric $G_{\alpha\beta}$ and anti-symmetric tensor $B_{\alpha\beta}$ on the $p$-brane are defined as

$$G_{\alpha\beta} := G_{ij} \partial_\alpha X^i \partial_\beta X^j, \quad B_{\alpha\beta} := B_{ij} \partial_\alpha X^i \partial_\beta X^j,$$

with the coordinates $\{\sigma^\alpha\}$ ($\alpha = 1, 2, \ldots, p$) on the world volume.
1-brane (D-string)

Let us start by taking the 1-brane \((\mathbb{R} \times \Sigma^{(1)})\) case. We easily obtain

\[
\mathcal{M}_1 = e^{-\Phi/2}|G|^{-1/4} \int_{\Sigma^{(1)}} d\sigma \sqrt{G_{\sigma\sigma}} \\
\geq e^{-\Phi/2}|G|^{-1/4} \sqrt{q^i G_{ij} q^j},
\]

(3.5)

where \(q^i := \int_{\Sigma^{(1)}} d\sigma \partial_\sigma X^i\) denotes the winding number of D-string. Obviously, the above inequality (3.5) is saturated when \(\Sigma^{(1)}\) is a straight line in the torus. (We have no constraints for the configuration of \(A\).) Hence the desired BPS mass is obtained as

\[
\mathcal{M}_1 = e^{-\Phi/2}|G|^{-1/4} \sqrt{q^i G_{ij} q^j}.
\]

(3.6)

In order to clarify the meaning of the winding \(q^i\) in (3.5), we take a look at the Wess-Zumino term. The local coordinate of 1-brane \(\mathbb{R} \times \Sigma^{(1)}\) is expressed by \((t, \sigma)\) and the Wess-Zumino term turns into a form,

\[
\int_{\mathbb{R} \times \Sigma^{(1)}} e^{iF} \wedge C = \int_{\mathbb{R} \times \Sigma^{(1)}} \left( C^{(2)} + iFC^{(0)} \right) \\
= q^i \left( \int_{\mathbb{R}} \hat{\mathcal{C}}^{(2)} dt \right),
\]

\[
\hat{\mathcal{C}}^{(2)} := C^{(2)} + iBC^{(0)}.
\]

(3.7)

The \(q^i\) turns out to be a “charge” associated with the gauge field \(\hat{\mathcal{C}}^{(2)}\).

3-brane

Next we consider the 3-brane case. Let \((t, \sigma^1, \sigma^2, \sigma^3)\) be a local coordinate of 3-brane \(\mathbb{R} \times \Sigma^{(3)}\). Then one can read the “charges” coupled to gauge fields \(\hat{\mathcal{C}}\) from the Wess-Zumino term (3.2),

\[
\int_{\mathbb{R} \times \Sigma^{(3)}} e^{iF} \wedge C \\
= \int_{\mathbb{R} \times \Sigma^{(3)}} \hat{\mathcal{C}}^{(4)} + \int_{\mathbb{R} \times \Sigma^{(3)}} \frac{i}{2\pi} F \wedge \hat{\mathcal{C}}^{(2)} + \int_{\mathbb{R} \times \Sigma^{(3)}} -\frac{1}{8\pi^2} F \wedge F \cdot C^{(0)} \\
= W_i \cdot \int_{\mathbb{R}} \frac{1}{6} \varepsilon^{ijk} \hat{\mathcal{C}}^{(4)}_{ijk0} dt + w^i \cdot \int_{\mathbb{R}} \hat{\mathcal{C}}^{(2)}_{i0} dt, \\
\left\{ \begin{array}{l}
\hat{\mathcal{C}}^{(4)} := C^{(4)} + iB \wedge C^{(2)} - \frac{1}{2} B \wedge B \cdot C^{(0)}, \\
\hat{\mathcal{C}}^{(2)} := C^{(2)} + iB \cdot C^{(0)},
\end{array} \right.
\]

(3.8)
The \( W_i \) is the 3-brane winding number. We remark that the non-vanishing expectation value for \( F \) makes the situation a little complicated. The Poincaré dual of \( \frac{i}{2\pi} F \) \( \in H^2(\Sigma^{(3)}; \mathbb{Z}) \) is represented by some 1-branes within our 3-brane \( \Sigma^{(3)} \) (the case “branes within branes” \[9\]). The \( w^i \) is regarded as the “effective winding number” of induced 1-branes. In this way we can describe by DBI action some bound states among 3-branes and 1-branes effectively.

Even after this change of situation we can use the similar argument for the 1-brane case to derive the 3-brane mass bound,

\[
\mathcal{M}_3 = e^{-\Phi/2} |G|^{-1/4} \int_{\Sigma^{(3)}} d^3 \sigma \sqrt{\det (G_{\alpha\beta} + B_{\alpha\beta} + F_{\alpha\beta})} \geq e^{-\Phi/2} \sqrt{\left(\frac{\mathcal{M}_{IIB,T^4} J}{W w}\right) \left(W w\right)},
\]

where we set

\[
(B)^{ij} := \frac{1}{2\sqrt{|G|}} \epsilon^{ijkl} B_{kl},
\]

\[
M_{IIB,T^4} := \left(\sqrt{|G|} \cdot (G^{-1} - B \cdot B)^{ij} \left(\frac{1}{\sqrt{|G|}} B_{lij}ight) \right) J.
\]

It is easy to prove that 8 \times 8 matrix \( M_{IIB,T^4} \) belongs to \( O(4, 4) \) and is also symmetric. The inequality (3.9) is saturated when \( \Sigma^{(3)} \) is a 3-dimensional torus “linearly embedded” in \( T^4 \), that is, the map \( X : \Sigma^{(3)} \rightarrow T^4 \) is a homomorphism of abelian groups (not necessarily injective.) and the gauge field \( A \) has a constant curvature (whose value is uniquely determined by \( w^i \)). Hence the desired mass formula can be written as

\[
\mathcal{M}_3 = e^{-\Phi/2} \sqrt{\left(\frac{\mathcal{M}_{IIB,T^4} J}{W w}\right) \left(W w\right)},
\]

2In the situation here, the “intersection term” \( (W w) J (W w)^t \) is always 0. Nevertheless we wrote this term in (3.12) in order to represent the equivalence with the mass formula (2.14) manifestly. In the next section we will face to the cases that this intersection term essentially appears.
This formula (3.12) coincides with our algebraic formula (2.14)! Actually the “moduli matrix” $M^{IIB,T^4}$ induced from DBI action is equal to the moduli matrix $R_a(M)$ derived from U-duality.

To close this subsection we make a remark: The above description of the systems such that the 3-branes and 1-branes coexist, i.e. the bound states of 3- and 1-branes, is not complete. For example, consider the system of the 3-brane wrapping around the 3-torus along the 678’th axes and 1-brane wrapping around the circle along the 9’th axis when the compactified directions are the 6789’th axes. In this situation one cannot reinterpret the 1-brane charge as the field strength $F$ as above, and cannot derive the correct BPS mass from only the DBI action. The most naive approach for this problem is to start with the simple ansatz for the effective action

$$S = S_{1-brane} + S_{3-brane}.$$ (3.13)

Of course this is not correct, since one must also evaluate the effect of interaction between these branes by taking the sectors of DN (or ND) open string into account. In other words, one must calculate higher loop corrections to the effective action. It may be natural to expect that these loop corrections explain the binding energy fitted to the conjecture of U-duality. We will argue on this problem in section 4.

### 3.2 Type IIA on $T^4$

Next we analyze the mass formulae for the type IIA case. In the same way as in the type IIB case, we start from the D-brane action (3.1), (3.2). The only difference from the type IIB case is that we have the R-R fields with odd degrees; $C^{(p)}$, $(p = 1, 3, 5, 7, 9)$, which leads to the even D-branes. In our analysis we need to treat the three kinds of D-branes ($0$-, $2$-, $4$-branes). We first consider the cases that only the same kind of branes exist, and later discuss the problem of the bound states of branes having different dimensions.

**0-brane**

The 0-brane case is trivial. The moduli dependence of BPS mass only originates from the volume of the internal torus (3.4):

$$\mathcal{M}_0 = e^{-\Phi/2}|G|^{-1/4} n.$$ (3.14)
The $n$ is the number of 0-branes and is identified with the RR charge for $C^{(1)}_{\mu}$.

2-brane

For the 2-brane case, we take the configuration $R \times \Sigma^{(2)}$. As in the analysis of the type IIB case, $R$ is the time axis and $\Sigma^{(2)}$ is wrapped around some 2-cycle of $T^4$. Simple evaluation gives the following inequality (which is essentially the Minkowski inequality);

$$\mathcal{M}_2 \geq e^{-\Phi/2}G^{1/4} \sqrt{\det (G_{\alpha\beta} + F_{\alpha\beta})} \geq e^{-\Phi/2}G^{1/4} \left\{ \sum_{i=1}^{3} \left( \int_{\Sigma^{(2)}} X^* J_i \right)^2 + \left( \int_{\Sigma^{(2)}} X^* B \right)^2 \right\},$$  \hspace{1cm} (3.15)

where $J_1, J_2, J_3 \in H^{2+}(T^4; \mathbb{R})$ denotes an orthonormal basis for the self-dual part of $H^2(T^4; \mathbb{R})$ with a normalization $J_i \cdot J_j (\equiv \int_{T^4} J_i \wedge J_j) = 2|G|^{1/2} \delta_{ij}$. The bound of this inequality (3.15) does not depend on the choice of this basis. Here we assume that $\int_{\Sigma^{(2)}} F = \int_{\Sigma^{(2)}} X^* B$. It is the case that the $U(1)$-gauge field $A$ has no “monopole” charge.

When is this inequality (3.15) saturated? It is satisfied if the collective coordinate $X$ is a holomorphic mapping from $\Sigma^{(2)}$ onto some 2-cycle $S$ determined by a given $\hat{C}^{(3)}_{\mu ij}$ charge. Strictly speaking, the minimality of world-volume does not necessarily mean a holomorphic mapping, rather means a more general harmonic mapping. But it is known [22][13] that the BPS condition (the condition of SUSY 2-cycle) leads to a holomorphic mapping at least in the case of $B_{ij} = 0$. We further comment on the following fact: Fix an arbitrary 2-cycle $S \in H_2(T^4; \mathbb{Z})$. The condition that $S$ is represented by a holomorphic curve in $T^4$ is that the Poincaré dual $\alpha_S$ of $S$ belongs to $H^{1,1}(T^4; \mathbb{R})$. This can be always satisfied if we properly choose the holomorphic structure of $T^4$. (The choice of holomorphic structure compatible with the given metric $G_{ij}$ is parametrized over $O(4)/U(2) \cong S^2$, which is identified with $S^2$ spanned by $J_1, J_2, J_3$. These degrees of freedom are just equal to those needed to make $\alpha_S$ a (1,1)-form for an arbitrary $S$.) So, it is sufficient to take $X$ to be a holomorphic mapping from $\Sigma^{(2)}$ onto some holomorphic curve $S$ in $T^4$ with respect to a properly chosen complex structure. If we write the corresponding Kähler form on $T^4$ as $J_S$ normalized by

---

3For the $K3$-compactification we face to the same situation. We can still use the $S^2$ degrees of freedom for the choice of holomorphic structure compatible with the Ricci flat metric in order to make the given 2-cycle to be algebraic. On the other hand, a case of Calabi-Yau 3-fold forms a striking contrast with it. In this case all the 2-cycles are algebraic from the beginning.
\[ J_S \cdot J_S = 2|G|^{1/2} \], which is a proper linear combination of \( J_1, J_2, J_3 \), we obtain

\[
\sqrt{\sum_{i=1}^{3} \left( \int_{\Sigma^{(2)}} X^* J_i \right)^2} = \int_{\Sigma^{(2)}} X^* J_S .
\] (3.16)

For the \( U(1) \)-gauge field \( A \), the condition for the saturation is somewhat non-trivial. This is because the pull-back \( X^* B \) is not a harmonic 2-form even if all of the components \( B_{ij} \) are constants on \( T^4 \). Nevertheless we can choose \( A \) so that \( \mathcal{F}(\equiv X^* B + dA) = a X^* J_S \), where \( a \) is not a function but merely a complex number. (It is most easily proved by making use of the Hodge decomposition.) The assumption \( \int_{\Sigma^{(2)}} \mathcal{F} = \int_{\Sigma^{(2)}} X^* B \) immediately gives a relation;

\[
a = \frac{\int_{\Sigma^{(2)}} X^* B}{\int_{\Sigma^{(2)}} X^* J_S} = \frac{\int_{\Sigma^{(2)}} X^* B}{\sqrt{\sum_{i=1}^{3} \left( \int_{\Sigma^{(2)}} X^* J_i \right)^2}} .
\] (3.17)

Hence, under this configuration of \( X \) and \( A \), we obtain

\[
\mathcal{M}_2 = e^{-\Phi/2} |G|^{-1/4} \int_{\Sigma^{(2)}} d^2 \sigma \sqrt{\det (G_{\alpha\beta} + \mathcal{F}_{\alpha\beta})}
\]

\[
= e^{-\Phi/2} |G|^{-1/4} \sqrt{1 + a^2} \int_{\Sigma^{(2)}} X^* J_S
\]

\[
= e^{-\Phi/2} |G|^{-1/4} \sqrt{\sum_{i=1}^{3} \left( \int_{\Sigma^{(2)}} X^* J_i \right)^2 + \left( \int_{\Sigma^{(2)}} X^* B \right)^2} .
\] (3.18)

This means the saturation of the inequality (3.15) and gives the desired BPS mass formula.

In the similar manner to the case of type IIB 3-brane, one may consider some extra monopole charge \( n \) for \( A \) by taking the the background \( \int_{\Sigma^{(2)}} \mathcal{F} = \int_{\Sigma^{(2)}} X^* B + n \). By analyzing the Wess-Zumino term, we can find that this charge \( n \) can be identified with the extra contribution from the \( n \) 0-branes within our 2-brane \( \Sigma^{(2)} \). The BPS mass formula can be easily generalized to this case;

\[
\mathcal{M}_2 = e^{-\Phi/2} |G|^{-1/4} \sqrt{\sum_{i=1}^{3} \left( \int_{\Sigma^{(2)}} X^* J_i \right)^2 + \left( \int_{\Sigma^{(2)}} X^* B \right)^2 + 2n \left( \int_{\Sigma^{(2)}} X^* B \right) + n^2} .
\] (3.19)

This is indeed the correct mass formula of the bound states of a 2-brane and 0-branes predicted by U-duality.

**4-brane**

In the 4-brane case, the space-part of world-brane \( \Sigma^{(4)} \) must occupy the full volume of \( T^4 \). Consider the smooth map \( X : \Sigma^{(4)} \to T^4 \) covering \( T^4 \) \( m \)-times. We again assume \( \mathcal{F} = [X^* B] \)
in $H^2(\Sigma^{(4)}; \mathbb{R})$ for the time being. The inequality of mass integral is represented;

\[
\mathcal{M}_4 = e^{-\Phi/2}|G|^{-1/4} \int_{\Sigma^{(4)}} d^4\sigma \sqrt{\det (G_{\alpha\beta} + F_{\alpha\beta})} \\
\geq m e^{-\Phi/2} \sqrt{|G|^{1/2} + \left( \int_{T^4} B \wedge *B \right) + \frac{1}{4|G|^{1/2}} \left( \int_{T^4} B \wedge B \right)^2}.
\tag{3. 20}
\]

What is the condition for saturation? Clearly the value of this integral does not depend on the choice of smooth map $X$ (under fixing $\deg X = m$, of course). However, the condition for the gauge field $A$ is similar to the 2-brane case, but rather complicated, because the field $X^*B$ necessarily has constant components.

It reads:

\[
\mathcal{F} \wedge \mathcal{F} = b \, dv, \quad \mathcal{F} \wedge J_i = b_i \, dv \quad (i = 1, 2, 3),
\tag{3. 21}
\]

for some complex numbers $b$, $b_i$, and $dv$ denotes the volume form of $T^4$. The values of these constants can be easily solved in the same way as (3. 17) and we obtain the BPS mass formula;

\[
\mathcal{M}_4 = m e^{-\Phi/2} \sqrt{|G|^{1/2} + \left( \int_{T^4} B \wedge *B \right) + \frac{1}{4|G|^{1/2}} \left( \int_{T^4} B \wedge B \right)^2}.
\tag{3. 22}
\]

Now let us consider the case that the gauge field $A$ has a non-trivial topological charge. Alternatively one may regard it as an “integer theta-parameter shift” [21], $B_{ij} \rightarrow B_{ij} + \Theta_{ij}$ ($\forall \Theta \in H^2(T^4; \mathbb{Z})$), which is a part of T-duality transformations. By analyzing the Wess-Zumino term again, we can find that $\left[ \frac{i}{2\pi} F \right] \in H^2(T^4; \mathbb{Z})$ is identified with the extra 2-brane charge and $\left[ -\frac{1}{8\pi^2} F \wedge F \right] \in H^4(T^4; \mathbb{Z})$ is identified with the extra 0-brane charge. The corresponding mass formula is immediately calculated. This result is rather complicated, but we can show that it is fitted to the similar formula to (3. 12);

\[
\mathcal{M}_4 = e^{-\Phi/2} \sqrt{Q^T (M^{IIA,T^4} + J) Q},
\tag{3. 23}
\]

where the “charge vector” $Q \in \mathbb{Z}^{8}$ describes the RR charge of 4-branes ($\cong \mathbb{Z}$) and the effectively induced 2-brane ($\cong \mathbb{Z}^{6}$) and 0-brane ($\cong \mathbb{Z}$) charges. $M^{IIA,T^4}$ is also a symmetric matrix valued in $O(4, 4)$. Here we do not present its explicit form, since it is cumbersome compared with the type IIB case.

We only point out the following: Firstly, all of three cases (the 0-brane case (3. 14), 2-brane cases (3. 18) and the cases of bound states of 0-branes and 2-branes (3. 19)) are
also summarized by the same formula (3.23). We only have to set the 4-brane charge to be zero in the vector $Q$.

Secondly, in the similar manner to the type IIB case, the moduli matrix $M^{IIA,T^4}$ can be identified with the cospinor representation matrix $R_c(M)$ of $O(4,4)$. This means the consistency of our mass formulae (3.23) with the prediction of U-duality. This result is also expected from the fact that the replacement $M^{IIA,T^4} \leftrightarrow M^{IIB,T^4}$ by T-duality should occur. But the check of this claim is not so self-evident, because the moduli matrices become rather complicated under general background. In this sense, we may also say our results support the consistency of DBI action under T-duality.

Lastly, we comment on the same difficulty as that in the analysis of type IIB case. One must understand that generic bound states of 0-, 2-, and 4-branes cannot be described by the recipe of the branes within branes [9]. We can at most analyze, by using only the DBI action, the cases which can be connected by the T-duality transformations $B_{ij} \rightarrow B_{ij} + \Theta_{ij}$ with the cases that 4-branes alone exist. Of course, one cannot describe all the bound states by making use of these T-duality transformations. In order to complete our analyses we need to argue seriously on the binding energies of branes. In section 4 we will return to this problem.

\footnote{Here we only consider the charges of “$U(1)$-sectors” of gauge fields. We will present in the next section some comments on the possibility to develop our analysis to that based on the “non-abelian Born-Infeld action (NBI)” proposed by Tseytlin [26].}
3.3 Considerations from M-theory

In this subsection we reconsider the case of IIA on $T^4$ by the M-theory approach. The discussion here is based on that of [24].

The M-theory is believed to exist in 11 dimensional space-time, whose massless multiplet is a 11 dim. supergravity one, $\hat{g}_{MN}$, $c_{KLM}$, and $\psi_\alpha^M$. Here the $\hat{g}_{MN}$ is the 11 dimensional graviton and $c_{KLM}$ is the rank 3 anti-symmetric field. The spinor index “$\alpha$” of the gravitino $\psi_\alpha^M$ labels a 32 component Majorana fermion. The indices “$K,L,M,\cdots$” represent space-time coordinates and run from 0 to 10.

Let us consider a compactification of this theory on the 5-dimensional torus $T^5$. The 6-dimensional space-time is extended in the $(X^0,X^1,\cdots,X^5)$-directions and the remaining coordinates $(X^m)$ ($m = 6,7,8,9,10$) represent the torus $T^5$. Especially the 5th coordinate “$X^{10\alpha}$” of the $T^5$ is the “longitudinal” one in this 11 dimensional theory. The $T^5$ is reexpressed as a product $T^5 = T^4 \times S^1$ of a transverse torus $T^4$ with $(X^\bar{m})$ $(\bar{m} = 6,7,8,9)$ and a circle $S^1$ parametrized by the $X^{10}$. (We use indices “$\mu,\nu,\lambda,\cdots$” for the 6 dimensional space time.)

This compactified theory $M/T^5$ on $T^5(=T^4 \times S^1)$ is equivalent to the IIA string on $T^4$ because of the string duality “$M/S^1 \cong IIA$”.

The moduli space of this is expressed by a 6 dimensional dilaton $\Phi$ and 25 scalars including $\hat{g}_{mn}$’s and $c_{mn}$’s. These 25 scalars parametrize the homogeneous space $O(5,5)/(O(5) \times O(5))$ (U-duality group). In the context of the IIA string, the NS-NS scalar part contains 17 scalars (i.e. $10 \; \hat{g}_{\bar{m}m}, \; 6 \; B_{\bar{m}m} = c_{\bar{m}m10}$ and a $\hat{g}_{1010}$). The remaining eight scalars, i.e. $4 \; C_{m}^{(1)} = \hat{g}_{m10}$ and $4 \; C_{\bar{m} \bar{m}}^{(3)} = c_{\bar{m} \bar{m}}$, are combined into a R-R moduli part.

The BPS states are characterised by their charges. There are 16 charges in the $M/T^5$-theory. Vectors $\hat{g}_{\mu m}$ couple to a charge vector $r_m$ and $c_{\mu mn}$ and its dual $c^{*}_{\mu klmnr}$ similarly couple to 10 charges $s^{mn}$ and one charge $q$ respectively. From the point of view of the string theory (IIA/$T^4$-theory), the momenta $p_m$, winding numbers $w^n$ in the NS sector are identified with $r_m$, $s^{n10}$ respectively. Also we can identify D0-, D4-brane charges $q_4$, $q_0$ with $q_4 = q$, $q_0 = -r_{10}$ and six D2-brane charges $s^{\bar{m} \bar{n}}$ are combined into a rank 2 anti-symmetric form $q_2 := \frac{1}{4} \epsilon_{\bar{k} \bar{l} \bar{m} \bar{n}} s^{\bar{k} \bar{l}} dX^\bar{m} \wedge dX^\bar{n}$. It is useful to note that $q_2$ is the Poincaré dual of the homology cycle around which the D2-brane is wrapping; $\Sigma := s^{k \bar{l}} \Sigma_{k \bar{l}}$, where $\Sigma_{k \bar{l}}$ ($k < \bar{l}$) denote the bases of $H_2(T^4)$ defined by the relation $\int_{\Sigma_{k \bar{l}}} dX^\bar{m} \wedge dX^\bar{n} = \delta^{\bar{m}}_{[k} \delta^\bar{n}]_{\bar{l}]}. $

We summarize these scalars, vectors and charges of the $M/T^5$-theory (equivalently IIA/$T^4$-theory) in the tables [24].
The M/T⁵ theory has scalars $\hat{g}_{mn}$, $c_{lmm}$ derived from the graviton, rank 3 anti-symmetric field in 11 dimensional SUGRA. The $\Phi$ is a six dimensional dilaton combined with the volume of $T^5$ and 10 dimensional dilaton $\phi$. In NS-NS sector of the IIA/T⁴ side, $\hat{g}_{\bar{m}\bar{n}}$, $B_{\bar{m}\bar{n}}$ are scalar fields and $\hat{g}_{10,10}$ is essentially ten dimensional dilaton. There are RR 1-form $C^{(1)}$ and RR 3-form $C^{(3)}$ in the RR-sector. Also numbers “♯” in parentheses (♯) are degrees of freedom of associated scalars.

| M/T⁵     | IIA/T⁴           |
|-----------|------------------|
| NS-NS     | (16)             |
| R-R       | (8)              |
| $\hat{g}_{mn}$ (15) | $\hat{g}_{\bar{m}\bar{n}}$ (10) | $C_{\bar{m}}^{(1)} = \hat{g}_{\bar{m}10}$ (4) |
| $c_{lmm}$ (10) | $B_{\bar{m}\bar{n}} = c_{m\bar{n}10}$ (6) | $C_{\bar{m}\bar{n}}^{(3)}$ (4) |
| $\Phi$ (1) | $\hat{g}_{10,10}$ (1) |                           |

Table 1: Scalar fields of M/T⁵ and IIA/T⁴
There are three kinds of vector fields $\hat{g}_{\mu m}$, $c_{\mu mn}$, and its dual $c^{\ast}_{\mu klmn}$ in the M/T5 theory. They can couple to charges $r_m$, $s_{mn}$ and $q$ respectively. In the context of the IIA string, eight vectors $\hat{g}_{\mu m}$, $c_{\mu m10}$ are combined into a NS sector multiplet. Their associated charges $r_{\bar{m}}$, $s_{\bar{m}10}$ are called momenta $p_{\bar{m}}$, winding numbers $w_{\bar{m}}$ respectively. In the R-sector, the vector multiplet consists of the remaining eight vectors $\hat{g}_{\mu 10}$, $C^{(3)}$ and $C^{(6)}$. Numbers of parentheses represent the degrees of freedom of the corresponding charges.

Now let us consider BPS mass formulae from supersymmetric algebra. Firstly the resulting theory $M/T^5 = IIA/T^4$ has a space-time 6-dimensional $(2,2)$ supersymmetry\(^5\). The SUSY algebra has an internal $SO(5) \cong USp(4)$ R-symmetry and there is a symplectic structure $\omega_{ab} (= - \omega_{ba})$ ($a, b = 1, 2, 3, 4$), and we denote $\omega^{ab} := \omega_{ab}^{-1} (= - \omega_{ab})$. Super charges $Q^a_{\alpha}$, $\bar{Q}^b_{\bar{\beta}}$ are 4-component $USp(4)$-(pseudo)Majorana Weyl spinors. The Latin indices “$a, b$” label the R-symmetry $USp(4)$. The Greek indices “$\alpha, \bar{\beta}$” ($\alpha, \bar{\beta} = 1, 2, 3, 4$) represent 4-component Weyl spinors in 6 dimension. The Latin indices should be raised and lowered by the “charge conjugation matrices” $\omega^{ab}$ and $\omega_{ab}$. For example, $\omega_{ab}Q^b_{\alpha} = Q^a_{\alpha\alpha}$.

These charges satisfy the following SUSY algebra

$$\{Q^a_{\alpha}, Q^b_{\beta}\} = \omega^{ab}\gamma^\mu_{\alpha\beta}p_\mu, \quad \{Q^a_{\alpha}, \bar{Q}^b_{\bar{\beta}}\} = \delta_{\alpha\bar{\beta}}Z^{ab},$$

where $\gamma^\mu_{\alpha\beta}$’s are 6 dim. space-time $\gamma$-matrices and $p_\mu$ is the space-time momentum. The central charge $Z^a_b$ can be decomposed by internal $SO(5) \gamma$-matrices $\Gamma^m_{ab}$ ($m = 6, 7, 8, 9, 10$),

$$Z^a_b = \left(q\sqrt{\det \hat{g}} + r_m\Gamma^m + \frac{1}{2}s_{mn}\Gamma_{mn}\right)_b^a, \quad \Gamma_{mn} := \Gamma_{[m}\Gamma_{n]}, \quad \{\Gamma_m, \Gamma_n\} = 2\hat{g}_{mn}. \quad (3.24)$$

\(^5\)This 6-dimensional SUSY is often called “(4,4)” in some literature.
When we turn off the Kalb-Ramond field \( B_{m\bar{n}} \) and RR fields \( C^{(1)}_l \), \( C^{(3)}_{k\bar{l}m} \) for simplicity, the coefficients \( q, r, s \) of the decomposition can be identified with the previous charges of the M/MT\(^5\)-theory. In the following, we abbreviate space-time spinor indices \( \alpha, \bar{\beta} \).

The 1/4-susy condition of BPS states can be written

\[
(\epsilon_a Q^a + \bar{\epsilon}_b \bar{Q}^b) |BPS\rangle = 0.
\]

This condition is transformed into eigenvalue problems of operators \( ZZ^\dagger, Z^\dagger Z \) with eigenvectors \( \epsilon, \bar{\epsilon} \)

\[
(ZZ^\dagger)_a^b \epsilon^b = ((m_0^2 + 2(K_m + W_m)\Gamma^m)_a^b) \epsilon^b = m_{BPS}^2 \epsilon^a,
\]

\[
(Z^\dagger Z)_a^b \bar{\epsilon}^b = ((m_0^2 + 2(K_m - W_m)\Gamma^m)_a^b) \bar{\epsilon}^b = m_{BPS}^2 \bar{\epsilon}^a,
\]

\[
m_0^2 := q^2 \det \hat{g} + \hat{g}^{mn} r_m r_n + \frac{1}{2} \hat{g}_{mn} \hat{g}_{k\bar{l}m} \hat{g}_{k\bar{l}n},
\]

\[
K_s := (qr_s - \frac{1}{8} s^{kl} s^{mn} \epsilon_{k\bar{m}l\bar{n}}) \sqrt{\det \hat{g}},
\]

\[
W^s := r_l s^{ls},
\]

\[
Z^\dagger := \omega Z \omega^T, \quad \omega^T = -\omega.
\]

Here the \( m_{BPS}^2 := -p^2 \) is the square of BPS mass and is given explicitly

\[
m_{BPS}^2 = m_0^2 \pm 2 \sqrt{\hat{g}_{mn} W^m W^n + \hat{g}^{mn} K_m K_n}.
\]

The ambiguity of the signature in Eq.(3.26) depends on whether the state is a BPS or anti-BPS state. In the following we restrict calculations to the BPS case and choose the “+” sign in Eq.(3.26).

Next we will rewrite this BPS mass formula in the IIA/T\(^4\) side. We have switched off all the RR scalars \( \hat{g}_{10i} = C^{(1)}_i = 0 \) (RR 1-form), and \( C^{(3)}_{k\bar{l}m} = 0 \) (RR 3-form) and the Kalb-Ramond field \( B_{k\bar{l}} = e_{k\bar{l}i0} \). The charges in the NS sector are momenta \( p_m(\equiv r_m) \) and winding numbers \( w^m(\equiv s^{m10}) \). On the other hand, remember that the RR-sector charges are classified into three types; \( q_4 \equiv q \) (4-brane charge), \( q_2 \equiv \frac{1}{4} \epsilon_{k\bar{m}l\bar{n}} s^{kl} dX^m \wedge dX^n \) (2-brane charge), and \( q_0 \equiv -r_{10} \) (0-brane charge).

The BPS mass formula (3.26) is evaluated by the Einstein metric \( \hat{g}_{mn} \) in 11 dimension. However, we want to compare this formula with the results in the previous subsection, and it was calculated in the 6 dimensional Einstein frame. Hence we will write down the relations between the 11D metric \( \hat{g}_{mn} \), 10D string metric \( G_{mn} \), and 6D Einstein metric \( \hat{g}_{mn}^{(6)} \)

\[
\hat{g}_{10,10} := e^{\frac{4}{3} \phi},
\]
\[
\hat{g}_{\hat{m}\hat{n}} = e^{-\frac{2}{3}\phi} G_{\hat{m}\hat{n}} \quad (\hat{m}, \hat{n} = 0, 1, \ldots, 9), \\
G_{\mu\nu} = e^\Phi g_{\mu\nu}^{(6)} \quad (\mu, \nu = 0, 1, \ldots, 5).
\]

The \( \Phi \) is the 6 dimensional dilaton and is associated with the 10 dim. dilaton \( \phi \) and the volume of the \( T^4 \) as
\[
e^\Phi = e^{\phi |G|^{-1/4}}, \\
|G| := \det(G_{ij}) \quad (i, j = 6, 7, 8, 9).
\]

Note a relation;
\[
\sqrt{\hat{g}_{00}} = e^{-\frac{1}{3}\phi} \sqrt{G_{00}} \\
= e^{-\frac{1}{3}\phi} e^{\frac{1}{2}\Phi} \sqrt{g_{00}^{(6)}}.
\]

Thus we have to multiply a factor \( e^{-\frac{1}{3}\phi} e^{\frac{1}{2}\Phi} \) to results obtained from Eq.(3.26) in order to evaluate BPS mass \( m_{BPS}^{(6)} \) in the 6 dimensional Einstein frame.

As a first case, we concentrate on the mass formula in the NS-NS sector with no D-brane charges. We can evaluate its BPS mass from the Eq.(3.26)
\[
(m_{BPS}^{(6)})^2 = e^\Phi \left(G^{\hat{m}\hat{n}} p_\hat{m} p_\hat{n} + w^k G_{\hat{i}\hat{k}} w^{\hat{k}} + 2 p_\hat{m} \delta^{\hat{m}}_{\hat{k}} w^{\hat{k}}\right). \tag{3.27}
\]

As a second case, the mass formula with only D-brane charges can be calculated as
\[
(m_{BPS}^{(6)})^2 = e^{-\Phi} |G|^{-1/2} \left[(q_4 |G|^{1/2} + q_0)^2 + 2 |G|^{1/2} (q_2^+ \cdot q_2^+)\right], \tag{3.28}
\]

where \( * \) is a Hodge dual in the \( T^4 \) and \( q_2^+ \) corresponds to the self-dual part of \( q_2 \). For arbitrary 2-forms \( A_1, A_2 \) on \( T^4 \), the inner product \( (A_1 \cdot A_2) \) is defined as
\[
(A_1 \cdot A_2) := \int_{T^4} A_1 \wedge A_2.
\]

If we want to incorporate further the Kalb-Ramond moduli \( B_{kl} \neq 0 \), we only have to perform a suitable \( O(5, 5) \)-rotation on the charge vectors \( q, r_l, s^{kl} \) (corresponding to the element \( \Omega_B \) of \( O(4, 4) \)-subgroup in Eq.(2.1)). Namely, we should replace \( q, r_l, s^{kl} \) in the expressions of \( m_0, K_s \) and \( W^s \) in Eqs.(3.25) by the following \( \tilde{q}, \tilde{r}_l \) and \( \tilde{s}^{kl} \).

\[
\tilde{q} := q, \\
\tilde{r}_l := r_l + \frac{1}{2} s^{mn} c_{mnl} + \frac{1}{4} q \sqrt{\det \hat{g}} \, (\ast_5 c)^{mn} c_{mnl}, \\
\tilde{s}^{kl} := s^{kl} + q \sqrt{\det \hat{g}} \, (\ast_5 c)^{kl}.
\]
Here the symbol "\(*_5\)" means the Hodge dual in the $T^5$. Then the BPS mass formula is expressed in the same form as that of Eq. (3.26) with these replacements.

The NS-NS sector BPS mass can be written as

\[
(m_{BPS}^{(6)})^2 = e^{-\Phi} \left[ |G|^{-1/2} q_0^2 + q_4^2 \left\{ |G|^{1/2} + (B \cdot *B) + \frac{1}{4|G|^{1/2}} (B \cdot B)^2 \right\} + |G|^{-1/2} \left\{ (q_2 \cdot B)^2 + 2|G|^{1/2} (q_2^+ \cdot q_2^-) \right\} + q_0 q_4 \left\{ 2 - |G|^{-1/2} (B \cdot B) \right\} - 2q_0 |G|^{-1/2} (q_2 \cdot B) + q_4 \left\{ |G|^{-1/2} (B \cdot B) (q_2 \cdot B) + 2(q_2 \cdot *B) \right\} \right].
\]  

(3.29)

It reproduces the well-known BPS mass formula of fundamental string excitations. Similarly we obtain RR sector mass formula

\[
(m_{BPS}^{(6)})^2 = e^{-\Phi} \left[ |G|^{-1/2} q_0^2 + q_4^2 \left\{ |G|^{1/2} + (B \cdot *B) + \frac{1}{4|G|^{1/2}} (B \cdot B)^2 \right\} + |G|^{-1/2} \left\{ (q_2 \cdot B)^2 + 2|G|^{1/2} (q_2^+ \cdot q_2^-) \right\} + q_0 q_4 \left\{ 2 - |G|^{-1/2} (B \cdot B) \right\} - 2q_0 |G|^{-1/2} (q_2 \cdot B) + q_4 \left\{ |G|^{-1/2} (B \cdot B) (q_2 \cdot B) + 2(q_2 \cdot *B) \right\} \right].
\]  

(3.30)

As special cases when there are only one kind of RR charges, we write down results;

- **0-brane**
  \[
  (m_{BPS}^{(6)})^2 = e^{-\Phi} |G|^{-1/2} q_0^2,
  \]  

(3.31)

- **2-brane**
  \[
  (m_{BPS}^{(6)})^2 = e^{-\Phi} |G|^{-1/2} \left\{ (q_2 \cdot B)^2 + 2|G|^{1/2} (q_2^+ \cdot q_2^-) \right\},
  \]  

(3.32)

- **4-brane**
  \[
  (m_{BPS}^{(6)})^2 = e^{-\Phi} q_4^2 \left\{ |G|^{1/2} + (B \cdot *B) + \frac{1}{4|G|^{1/2}} (B \cdot B)^2 \right\}.
  \]  

(3.33)

These results can be compared with evaluations from D-brane techniques in the previous subsection and there are completely agreements between them. (It is useful to remark the simple relations $2|G|^{1/2} (q_2^+ \cdot q_2^-) = \sum_{i=1}^{3} \left( \int_{\Sigma} J_i \right)^2$, $q_2 \cdot B = \int_{\Sigma} B$, where $\Sigma \in H_2(T^4)$ denotes the Poincaré dual of $q_2$.) Also the mass formula of D0-D4 system is written down

\[
(m_{BPS}^{(6)})^2 = e^{-\Phi} \left( |G|^{-1/4} q_0 + q_4 \left\{ |G|^{1/2} + (B \cdot *B) + \frac{1}{4|G|^{1/2}} (B \cdot B)^2 \right\} \right)^2 - 2e^{-\Phi} \left\{ 1 + |G|^{-1/2} (B \cdot *B) + \frac{1}{4|G|} (B \cdot B)^2 \right\} - \left( 1 - \frac{(B \cdot B)}{2|G|^{1/2}} \right) q_0 q_4.
\]  

(3.34)
These results Eqs. (3.32), (3.34) will be compared to the binding energy calculations in the next section.

3.4 Type IIA on $K3$

The type IIA string compactified on $K3$ is very similar to the case of type IIA over $T^4$. This is because $K3$ has orbifold limits described by $T^4/Z_2$. But one needs to take account of the extra 64 moduli and 16 gauge fields associated with the degrees of freedom of blow-ups of 16 fixed points (which correspond to the 16 matter multiplets in the sense of 6D $N = (1, 1)$ theory [27] and to the twisted sectors in the language of orbifold CFT).

Our above analysis of D-brane masses is also applicable to this case. The calculation is almost parallel to the case of type IIA over $T^4$. We can summarize this result as follows;

$$M = e^{-\Phi/2} \sqrt{Q^T (M^{IIA,K3} + L) Q}.$$  \hspace{1cm} (3.35)

Here the charge vector $Q$ has 24($=8 + 16$) components and the moduli matrix $M^{IIA,K3}$ is an $O(4, 20)$ symmetric matrix which parametrizes the K3-moduli space $O(4, 20)/(O(4) \times O(20))$ [28]. The “intersection matrix” $L$ can be expressed as $L = (E_8(-1))^\oplus 2 \oplus \sigma_1^\oplus 4$ in the standard basis.

This formula (3.35) is manifestly consistent with the famous conjecture of type IIA - heterotic duality in 6-dimension [15], since the duality transformation maps the K3-moduli matrix $M^{IIA,K3}$ into the matrix of Narain moduli space $M^{het,T4}$ of heterotic string.

However, there is a crucial difference from the case of $T^4$, which is due to the fact that $K3$ is a curved manifold. It is known [13, 28] that the 4-brane in the $K3$ case has the extra 0-brane charge $-1$ due to the 1st Pontrjagin number of $K3$. This leads to unexpected assignments of R-R charges to the configurations of D-branes, and gives a serious contradiction to our analysis of the BPS mass spectrum of the R-R solitons. In order to get over this difficulty we will have to take account of the extra degrees of freedom that are absent in the $T^4$-case, the twisted sectors in $T^4/Z_2 \cong K3$. If the contributions from the twisted sectors to the effective action (or the equations of motion) can be interpreted as bound states of 4-branes and effective 0-branes, which perhaps reside at the fixed points of $Z_2$-action, the similar analyses to those in the next section might yield the correct results. However, our study
on this problem is still far from the complete solution. We would like to present a further discussion elsewhere.

4 Higher-Loop Corrections to the Mass Formula as the Binding Energies among Intersecting Branes

In this section we shall further develop our D-brane analysis for BPS mass spectra. In our previous analyses in sections 3.1, 3.2 we only used the 1-loop effective actions of D-branes (DBI action). We already mentioned the limits of 1-loop analysis. We will face to the problems:

1. Calculations of the binding energies of the bound states of D-branes which cannot be evaluated from the DBI action.

2. Calculations of the binding energies of the bound states of D-branes and fundamental excitations of string (in the cases of type IIA and type IIB over $T^4$).

3. Evaluation of the dependence of the BPS masses on the R-R moduli (in the cases of type IIA and type IIB over $T^4$).

4. Evaluation of the correction to the mass formula from the extra 0-brane charge due to the 1st Pontrjagin number (in the case of type IIA over K3).

These four problems share an important point; they all may be resolved by analyzing $N(\geq 2)$–loop corrections to the D-brane actions, or equivalently, to the $\beta$-functions. In this section we present a higher loop analysis and show that this statement is indeed true for the 1st problem. The 2nd, 3rd, 4th problems are more challenging, but we believe that the higher loop analyses will also give the correct answers. We would like to discuss these problems elsewhere.

To solve the 1st problem, we will discuss an important relationship between the binding energies of the bound states and some SUSY breaking. This statement may sound strange, since we should now consider the mass spectra of BPS solitons which should preserve a part of SUSY! But this is not a contradiction. One must carefully understand the term “BPS”. This
should be used in the framework of the 6-dimensional supergravity theory which is the low energy effective theory compactified over the 4-torus. On the other hand, if we interpret the BPS solitons with RR charges as D-branes, we must treat the full 10-dimensional superstring theory. Of course, the states with some unbroken SUSY in the sense of 10-dimensional theory are also supersymmetric in the sense of 6-dimensional effective theory. But the inverse is not correct. Actually, we will later focus on some brane configurations which break SUSY in the sense of 10-dimensional string theory but should correspond to the BPS states in the sense of 6-dimensional SUGRA. We can expect that even if these states have no higher loop corrections in the framework of 6-dimensional SUGRA, they can have stringy loop corrections in the framework of 10-dimensional superstring. In this sense we may say our calculation of binding energies will give a non-trivial check of U-duality in the level of quantum string theory.

4.1 Higher Loop Corrections to $\beta$-Functions and SUSY Breaking

In the sequel we consider the type II (A or B) string over $T^4$. Let us start with the loop corrected equations of motion of string\footnote{The “loop” means the sum of world sheet genera and boundary loops. We evaluate the contributions to the $\beta$ function from these world sheet loops. From the point of view of the sigma model perturbation, we calculate the quantity in the first order of the $\alpha'$-expansion.}. Throughout this section we take a convention $\mu, \nu = 0, \ldots, 5$ (6-dimensional space-time), $i, j = 6, \ldots, 9$ (the internal torus). We also use the notation $|G| = \det G_{ij}$ (the square of volume of internal torus). Recall that $G_{\mu\nu} = e^\Phi g_{\mu\nu}^{(6)} \equiv e^\phi |G|^{-1/4} g_{\mu\nu}^{(6)}$, where $g_{\mu\nu}^{(6)}$ denotes the 6-dimensional Einstein frame metric. Set $g_{\mu\nu}^{(6)} = \eta_{\mu\nu} + h_{\mu\nu}$. The (linearized) equation of motion for $h_{00}$ can be written as

$$
\beta_{h_{00}} \equiv -\lambda^{-2} \Box h_{00}(x) + c_{(1)}(x) + c_{(2)}(x) + \cdots + c_{(n)}(x) + \cdots = 0. \quad (4.1)
$$

Here $\lambda$ is the string coupling constant $\lambda \equiv e^\phi$ and $c_{(n)}$ denotes the $n$-loop contribution to $\beta_{h_{00}}$ ($n := \sharp \{\text{open string loops}\} + 2\sharp \{\text{closed string loops}\}$). Of course $c_{(n)}$ should have the $\lambda$-dependence $\sim \lambda^{n-2}$. In our setting, both the $h_{00}(x)$ and $c_{(n)}(x)$ do not depend on the coordinates along the internal torus $x^6, \ldots, x^9$ (and also we assume that they do not depend on the time $x^0$, since we are now considering a static problem), so the equation of motion (4.1) is reduced to that in the 6-dimensional space-time.
Consider a Dirichlet $p$-brane $D$ wrapping around an internal $p$-cycle so as to be observed as a rest particle in our 6-dimensional space-time. We can rewrite the equation of motion (4.1) as

$$\lambda^{-2} \Box h_{00}(x) = \sum_{n \geq 1} c(n)(x) . \quad (4.2)$$

The L.H.S. is a (linearised) Ricci tensor and the R.H.S. can be interpreted as the “matter” terms. Then it is easy to see that, in general, $c(n)(x) \sim -M(n)\delta^{(5)}(x - X)$ for some constants $M(n)$, where $x^i = X^i$ ($i = 1, \ldots, 5$) express the position of the rest particle. We thus find that $M(n)$ can read as the $n$-loop correction to the rest mass of our particle. In this way we can directly evaluate the mass of D-branes from the $\beta$-functions.

Especially, it is easy to calculate the 1-loop contribution $c(1)$: Consider a 1-loop (disk) amplitude

$$A_D = \int_0^\infty dT \langle 0 | e^{-TH} | D \rangle , \quad (4.3)$$

where $|D\rangle$ denotes the suitable boundary states corresponding to the D-brane $D$. The divergence of moduli integral for $A_D$ has its origin in the massless components of $D$. Under the natural assumption for the backgrounds $h_{0\mu} = B_{0\mu} = 0$, the divergent part of $\frac{\delta}{\delta h_{00}(x)} A_D$ is easily calculated as

$$\frac{\delta}{\delta h_{00}(x)} A_D \sim \langle 0 | V^{h_{00}}_{(-1,-1)}(x) | D \rangle^{(0)}_{NS-NS}$$

$$= -\eta_{00} \delta^{(5)}(x - X) \int d^p \sigma \sqrt{\det (X^*G + X^*B)} , \quad (4.4)$$

where $V^{h_{00}}_{(-1,-1)}(x)$ is the graviton emission vertex in the $(-1, -1)$-picture, and the superscript “(0)” indicates the massless sector of the boundary state $|D\rangle$. We should notice that the position integrals along the Neumann directions are left (on the other hand, the momentum integrals do not exist for these directions). Clearly the boundary state of R-R sector does not contribute to the above calculation. Recall the relation $e^\Phi = e^\phi |G|^{-1/4}$, $G_{\mu\nu} = e^\Phi g_{\mu\nu}^{(6)}$. We can also approximate $g_{\mu\nu}^{(6)}$ by the Minkowski metric $\eta_{\mu\nu}$ in its R.H.S., because the deviation of metric $h_{\mu\nu}$ should be a quantity of the same order as the string coupling. We can easily obtain the 1-loop equation of motion

$$\Box h_{00}(x) = e^{-\Phi/2} |G|^{-1/4} \int d^p \sigma \sqrt{\det (X^*G + X^*B)} \delta^{(5)}(x - X) . \quad (4.5)$$

Needless to say, this leads to the same results as those in the previous sections

$$M_p^{(1)} \equiv e^{-\Phi/2} |G|^{-1/4} \int d^p \sigma \sqrt{\det (X^*G + X^*B)} . \quad (4.6)$$
It is no other than the 1-loop mass derived in the previous section.

For higher loop discussions, it is important to consider moduli dependent D-brane configurations and investigate the space-time SUSY breaking from the point of view of the boundary states. We now focus on only the open string loops, since we will later observe that closed string loops are negligible in the relevant region for our analysis of D-brane masses.

It is useful to study first the 2-loop case. Let us take two (the same or different kinds of) D-branes $\mathcal{D}, \mathcal{D}'$. The 2-loop amplitude $A_{\mathcal{D}\mathcal{D}'}$ is nothing but a cylinder. If there remains any unbroken supersymmetries in the open string channel $[4]$, this amplitude vanishes. Therefore the problem if the beta function (4.1) has higher loop corrections is reduced to the discussion of the unbroken space-time supersymmetries. In the following arguments, it is convenient to express the boundary states as

$$|\mathcal{D}\rangle = \sum_{s+s'=-2} |\mathcal{D}\rangle_{s,s'} , \quad |\mathcal{D}'\rangle = \sum_{s+s'=-2} |\mathcal{D}'\rangle_{s,s'} .$$

The subscripts $s, s'$ express the picture (the ghost charge of bosonic ghosts) and the terms with $s \in \mathbb{Z}$, $s \in \frac{1}{2} + \mathbb{Z}$ belong to the NS-NS sector, the R-R sector respectively.

Let us take a cylinder amplitude with one graviton emission vertex operator $V_{^{(0,0)}}^{h\mu\nu}(0,0)$ in the $(0,0)$-picture, $\langle \mathcal{D}|V_{^{(0,0)}}^{h\mu\nu}|\mathcal{D}'\rangle$. It is trivial to extend to the cases with other pictures. Recall a relation of the graviton vertex operator $V_{^{(0,0)}}^{h\mu\nu}(0,0)$ and a photon vertex operator $V_{^{(0)}}^{\mu}$(0)

$$V_{^{(0,0)}}^{h\mu\nu} = V_{^{(0)}}^{\nu} \hat{V}_{^{(0)}}^{\mu},
= u_A \bar{u}_A v_B \tilde{v}_B \left\{ Q_{1/2}^A, \left[ \tilde{Q}_{1/2}^A, V_{1/2}^B \tilde{V}_{1/2}^B \right] \right\},
(4.8)$$

$$u_A \left( -\frac{i}{\sqrt{2}} (C\gamma_\alpha)^{AB} \right) v_B = \delta_\alpha^0 ,
(4.9)$$

$$(C; \text{charge conjugation matrix with } C^{-1} \gamma^\alpha T C = -\gamma^\alpha).$$

Here the $Q_{1/2}^A, \tilde{Q}_{1/2}^A$ are supercharges in the 1/2-picture and we used fermion vertex operators $V_{-1/2}^B, \tilde{V}_{-1/2}^B$ in the $-1/2$-picture. Let us introduce the abbreviated notations such as $u \cdot Q \overset{\text{def}}{=} u_A Q_A^B, v \cdot V \overset{\text{def}}{=} v_B V_B^B$ and so on. So the above cylinder amplitude can be re-expressed as

$$\langle \mathcal{D} \left\{ u \cdot Q, \left[ \bar{u} \cdot \tilde{Q}, (v \cdot V) (\bar{v} \cdot \tilde{V}) \right] \right\} |\mathcal{D}'\rangle$$
and we can rewrite it

$$\langle \mathcal{D} \left\{ u \cdot Q, \left[ \bar{u} \cdot \tilde{Q}, (v \cdot V) (\bar{v} \cdot \tilde{V}) \right] \right\} |\mathcal{D}'\rangle
= -\frac{1}{2} \langle \mathcal{D} \left\{ u \cdot Q_+ (N), \left[ (N^{-1} \bar{u}) \cdot Q_- (N), v \cdot V (N^{-1} \bar{v}) \cdot \tilde{V} \right] \right\} |\mathcal{D}'\rangle ,
(4.10)$$

$$Q_\pm^A (N) := Q^A \pm N_B A \tilde{Q}^B ,$$

26
for some suitable matrix $N$. For special matrices $M_B^A$, $M'_B^A$ depending on the moduli fields, the boundary states satisfy relations \[ [30, 6] \]:

\[
Q_+(M)|D\rangle = 0, \\
Q_+(M')|D'\rangle = 0 .
\] (4. 11)

For a $p$-brane with Neumann coordinates $\{X^\mu\}$, $(\mu = 0, 1, 2, \cdots, p)$ with background fields $G$ and $\mathcal{F}$, the matrix $M$ is written as \[ [30, 6] \]

\[
M_B^A = i \left\{ \gamma_{01 \cdots p} \exp \left( \frac{1}{2} \gamma^{\alpha\beta} \mathcal{F}_{\alpha\beta} \right) \right\}_B^A \cdot \{ \det(G + \mathcal{F}) \}^{-1/2} .
\]

Here the gamma matrices $\gamma_\alpha$, $\gamma^\alpha$ are normalized as

\[
\{ \gamma_\alpha, \gamma_\beta \} = 2G_{\alpha\beta}^1, \quad \{ \gamma^\alpha, \gamma^\beta \} = 2\gamma^{\alpha\beta}^1,
\] (4. 12)

and we introduced the notations for the anti-symmetrized gamma matrices

\[
\gamma_\alpha \cdots \gamma_\beta := \gamma_\alpha \cdots \gamma_\beta, \quad \gamma^\alpha \cdots \gamma^\beta := \gamma^{\alpha \cdots \beta} .
\] (4. 13)

It is well-known that unbroken supersymmetries exist along the directions $u_A$ such that $M'^{-1}Mu = u$ (i.e. the eigenvectors of the matrix $M'^{-1}M$ with eigenvalue 1): $u \cdot Q_+(M)|D\rangle = u \cdot Q_+(M')|D'\rangle = 0$, or equivalently $u \cdot Q_+(M')|D\rangle = u \cdot Q_+(M')|D'\rangle = 0$ are satisfied. If there are suitable number of these eigenvectors $u$ so as to satisfy the relation \[ [1, 9] \), the cylinder amplitude $\langle D|V^{\text{hoo}}|D'\rangle$ vanishes because of the identity \[ [4, 10] \), or equivalently by the cancellation of NS-NS and R-R sectors. It follows that the 2-loop corrections to $\beta_{\text{hoo}}$ should vanish.

On the other hand, in the case that the supersymmetries on the D-branes are broken, the cancellation between NS-NS and R-R sectors in the amplitudes is not complete and higher loop corrections to the $\beta$-function may exist.

Now we study separately the aspects of SUSY breaking in the following two cases:

1. Only one kind of D-branes $\mathcal{D}$ exist and their relative configurations are all parallel. In this case, all loops of open string boundaries are the same type of D-branes $\mathcal{D}$.

2. Different kinds of D-branes coexist.
**One kind of parallel D-branes**

In this case there remains one half of the supersymmetry on the D-brane in arbitrary background moduli and the value of the 2 loop amplitude is zero. This is because $M^{-1}M = 1$ and so one half the supersymmetry is trivially preserved.

How about the higher loop cases? The higher loop corrections to the $\beta$-function originate from the divergence of the amplitudes $A_{D\cdots D}$ when all the Teichmüller parameters go to infinity. In this situation the higher loop amplitudes may be factorized to the products of the “pants diagrams” (or the cylinder amplitudes with an insertion of suitable massless vertex). Hence the problem is reduced to the 2-loop case. We can conclude that all the higher loop corrections to the equations of motion vanish thanks to the cancellation by the supersymmetry. Therefore the results are exact in one loop level, that is, the analyses based on the DBI action give rigorous mass spectra! (Of course we here neglect the effects of closed string loops, and so this observation may be not correct in the case when effects of the closed string loops are not small.)

**Some Different kinds of D-branes**

Let us take the two different kinds of D-branes $D$, $D'$. (The dimensions of the world volumes of $D$ and $D'$ are different or $D$ and $D'$ are not placed in parallel if they have the same dimension.) Here we shall only consider the configurations such that the branes touches with each other, or at least, the branes placed very closely. This is because we are interested in only bound states of branes and otherwise, the description by the coordinates of the center of mass would lose its meaning. In this “intersecting D-brane” case [4, 11, 13, 19, 20], we will find out that the cancellation between NS-NS and R-R sectors is not complete for general background.

For example, consider a 3-brane wrapping around the 678-th axes of $T^4$ and a 1-brane wrapping around the 9-th axis (Fig.1), which is the case we will later analyze in detail. It is well-known that this configuration is supersymmetric (so-called “short multiplet”, in which 1/4 of space-time SUSY are unbroken) in the special background;

$$B_{ij} = 0, \quad G_{ij} = \begin{pmatrix}
  0 & G_{6\leq i,j\leq 8} & 0 \\
  0 & 0 & G_{99} \\
  0 & 0 & 0
\end{pmatrix}. \quad (4.14)$$
(The 1-brane intersects the 3-brane perpendicularly, and the background Kalb-Ramond field is set zero.) However, if we put general background moduli, we have no SUSY any longer. In fact, consider the unbroken SUSY charges $Q^A(M)$ and $Q^A(M')$ respectively associated with boundary states $|D\rangle$ and $|D'\rangle$. In the special background (4.14), $M'^{-1}M$ indeed has 1 as an eigenvalue. But it is not the case for generic moduli. It follows that the higher loop corrections to $\beta_{h00}$ no longer vanish. This aspect sharply contrasts with the case of one kind of branes, in which we always have some unbroken SUSY independent of the VEV of moduli fields.

We have observed that the higher loop corrections to D-brane mass is inevitable in the broken SUSY cases. We will evaluate these amounts in detail in the next subsection.

4.2 Evaluation of Higher-Loop Corrections as the Binding Energies

We explain the method to calculate the contributions from higher loops to the $\beta$-functions concretely. The contribution for a fixed diagram comes from the divergent part of the amplitude when the Teichmüller parameters of the world sheet simultaneously go to large values. Then the size of all the boundary loops shrink to zero simultaneously. In this limit only the massless modes are relevant for our computations.

First of all, we stress the following fact: We must assume that D-brane mass of 1-loop level, which we evaluated in the previous section based on the DBI action, is sufficiently heavy for the validity to treat the D-branes as static backgrounds. In other words, we must consider the cases with large amounts of R-R charges. Otherwise, we would have to take account of fluctuations of the D-brane configurations [31]. This assumption is also necessary so that the perturbative expansion is applicable. It is not difficult to show that, under this assumption, the diagrams with no closed string loops are dominant for a fixed number of loops ($L := \sharp \{\text{open string loops}\} + 2\sharp \{\text{closed string loops}\}$). This is because each of the open string boundaries is assigned a very large factor $\sim 1$-loop mass. In this way we can conclude that only the open string loop diagrams with no genera are relevant for our calculation.

It is convenient to sum up all the “tadpole diagrams” (Fig.2) first. We evaluate the tadpole corrections to the pants-type diagram. Each wavy-line connecting a pants and one
Figure 1: In the perpendicularly intersecting case with $F_{ij} = 0$, there remains a $1/4$ SUSY. In contrast, the SUSY is completely broken in the slantwise intersecting configuration or in the non-vanishing $F_{ij}$ case.
Figure 2: The bare string coupling is improved by insertions of gravitons emitted from boundary states.

boundary state expresses the graviton and it leads to a factor $-\lambda \hat{M}^{(1)}$, which is essentially the normalization factor of NS-NS boundary state. Here we define the $\hat{M}^{(1)}$ as

$$\hat{M}^{(1)} := e^{\Phi/2} |G|^{1/4} M^{(1)},$$

by using the one-loop mass $M^{(1)}$ evaluated by the DBI action in the previous section. The reason why it includes $\lambda$ is clear: Each tadpole gains one open string loop.

We comment on a combinatorial factor for these tadpole diagrams. When we add one boundary operator to diagrams with $(N - 1)$ boundary operators, there are $N$ cases of possibilities to connect them. So there are $N!$ cases of possibilities to connect $N$ boundary operators and a bare coupling pants diagram each other. But there is a symmetry factor $1/N!$ for these diagrams because of the invariance under the permutation of $N$ boundaries, and this cancels the factor $N!$.

In this way the summation of all the tadpole diagrams leads to the following factor;

$$1 + (-\lambda \hat{M}^{(1)}) + (-\lambda \hat{M}^{(1)})^2 + (-\lambda \hat{M}^{(1)})^3 + \cdots = \frac{1}{1 + \lambda M^{(1)}} \sim \frac{1}{\lambda M^{(1)}}. \quad (4.16)$$

In the last line, we used our assumption that the one-loop D-brane mass is very large. We may also reinterpret this correction as follows: The string coupling $\lambda$ should be replaced with an “effective string coupling” $\lambda_{\text{eff}} \equiv \frac{\lambda}{\lambda \hat{M}^{(1)}} = \frac{1}{\hat{M}^{(1)}}$, by taking account of all the tadpole diagrams. However one should notice a fact: Generally, in the limit of large Teichmüller parameters, any $L$-loop diagram can be factorized into $(L-1)$ pants diagrams. On the other hand, we know an $L$-loop diagram should include a factor of $\lambda^{L-2}$. So, we find that any $L$-loop diagram should include a factor

$$\lambda^{L-2} \left( \frac{1}{\lambda \hat{M}^{(1)}} \right)^{L-1} = \lambda^{-1} \left( \frac{1}{\hat{M}^{(1)}} \right)^{L-1}. \quad (4.17)$$

We make one remark here: When we consider an arbitrary diagram with odd-number of open boundary loops, the amplitude necessarily contains tadpole type amplitudes. But
we have already included all the contributions from tadpole type diagrams (Fig.3) into the factor (4. 17) and it will be an over counting if we take these diagrams into account.

Collecting all the above observations, we can conclude that only the diagrams with the following two properties can contribute to the calculations:

1. A diagram with no closed string loops.

2. A diagram with no tadpole, which is inevitably a diagram with only even-number of open string loops. The contributions of the tadpole type diagrams are already included in the factor (4. 17).

As a result, there are contributions to the $\beta$-function from diagrams composed of only cylinder type diagrams whose two boundaries are put on different D-branes (Fig.4).

Now, we arrive at the stage to evaluate concretely the higher loop corrections to the D-brane masses. As a simple example, let us consider the type IIB case compactified on $T^4$
with coordinates \((x^6, x^7, x^8, x^9)\) and take an “Intersecting D-brane” configuration of Dirichlet 1-branes and D3-branes. Consider \(n\) D1-branes (\(\mathcal{D}\)) wrapping around the 9th axis and \(n'\) D3-branes (\(\mathcal{D}'\)) wrapping around the 678th axes. The center of mass of this system is specified by \((X^1, X^2, \ldots, X^5)\).

The one-loop mass \(\mathcal{M}^{(1)}_{1-3}\) was calculated from the disk amplitude \(c^{(1)}\) in section 3

\[
\mathcal{M}^{(1)}_{1-3} \equiv e^{-\Phi/2} |G|^{-1/4} \tilde{\mathcal{M}}^{(1)}_{1-3},
\]

\[
\tilde{\mathcal{M}}^{(1)}_{1-3} = n \sqrt{G_{99}} + n' \sqrt{|G|} \left\{ G^{-1} - (\ast B) G (\ast B) \right\}^{99},
\]

\[
(\ast B)^{ij} := \frac{1}{\sqrt{|G|}} \frac{1}{2} \varepsilon^{ijkl} B_{kl}.
\]

First we take a 2-loop cylinder diagram whose boundaries are put on the D1-brane \(\mathcal{D}\), the D3-brane \(\mathcal{D}'\), respectively. As we already pointed out, for the computation of \(\beta\)-functions we only have to evaluate the value of amplitude in the limit of large Teichmüller parameters, which is the IR limit in the closed string channel (or equivalently, UV limit in the open string channel). In this limit only the massless components of boundary states can contribute

\[
c^{(2)} \sim \langle \mathcal{D}|V^{h_{00}}|\mathcal{D}' \rangle^{(0)},
\]

where the superscript \((0)\) indicates the massless part. However, there is one subtle point. The massless condition indicates that the momenta of particles which are exchanged between the
boundaries must be zero. (Under the Neumann boundary condition, one can generally obtain non-zero winding. In our case, because the 0-th direction is Neumann and non-compact, the winding should be also 0. So, we obtain \( p_L^0 = p_R^0 = 0 \).) If considering on the conservation of the ghost charge, we can find that a closed string propagator should be inserted. It is the result of an integral about the position of an inserted vertex \( V^{hoo} \). Naively the insertion of this propagator seems to lead to a divergence because of the zero-momentum. But an appropriate choice of picture of \( V^{hoo} \) gives a momentum factor which cancels the pole of the propagator. More precisely, we should put some infinitesimal momenta along the 1∼5th directions to the vertex operator \( V^{hoo} \). After the calculation they should be set zero. (Notice that for the compact 6,7,8,9th directions the momenta are quantized and the 0th direction is always Neumann.) In the calculation of this type, we obtain contributions only from the “contact terms” among \( V^{hoo} \) and the shrunked open boundaries on D-branes.

It is convenient to make use of the same technique as \((4.10)\):

\[
(0) \langle D | V^{hoo} | D' \rangle^{(0)} = (0) \langle D | \{ u \cdot Q, [\tilde{u} \cdot \tilde{Q}, v \cdot V \tilde{v} \cdot \tilde{V}] \} | D' \rangle^{(0)}
\]

\[
= -\frac{1}{2} (0) \langle D | \{ u \cdot Q_+(M), [(M^{-1} \tilde{u}) \cdot Q_-(M), v \cdot V (M^{-1} v) \cdot \tilde{V}] \} | D' \rangle^{(0)},
\]

\[
M := i \frac{\gamma_0 \gamma_9}{\sqrt{-G_{00}G_{99}}},
\]

where \( M \) is chosen so that \( Q_+(M) | D \rangle^{(0)} = 0 \). Since \( (0) \langle D | Q_+(M) = 0 \), the most non-trivial part of the calculation is to evaluate the term \( Q_+(M) | D' \rangle^{(0)} \). Making use of the identity

\[
Q_+(M') | D' \rangle^{(0)} = 0, \quad M' := i \frac{e^{\frac{1}{2} \tilde{F}_{\alpha \beta} \gamma_{\alpha \beta}}}{\sqrt{-G_{00} \sqrt{\det(G + \tilde{F})}}} \gamma_0 \gamma_{678},
\]

\[
\tilde{G} \equiv G_{6 \leq i,j \leq 8}, \quad \tilde{F} \equiv F_{6 \leq i,j \leq 8},
\]

and taking the spinor \( u \) with a property:

\[
\frac{1}{\sqrt{|G|}} \gamma_{6789} u = -u,
\]

we obtain

\[
u \cdot Q_+(M) | D' \rangle^{(0)} = \frac{\sqrt{|G| \triangle}}{\mathcal{N} \mathcal{N}'} u \cdot Q | D' \rangle^{(0)} + \cdots,
\]

\[
\triangle := \sqrt{G_{99}} \sqrt{G^{-1} - (\ast B) G (\ast B)^9} = \left( 1 - G_{96} \cdot (\ast B)^9 \right),
\]

\[
\mathcal{N} := \sqrt{G_{99}},
\]

\[
\mathcal{N}' := \sqrt{\det(G + \tilde{F})}
\]

\[
\equiv \sqrt{\frac{\sqrt{G} \{ G^{-1} - (\ast B) G (\ast B) \}^{99}}{|G|}}.
\]
In the R.H.S. of above identity (4.24), “…” denotes the unimportant terms that do not contribute to our calculation. \( \mathcal{N}, \mathcal{N}' \) are no other than the normalizations of the boundary states in NS-NS sector \(|D\rangle, |D'\rangle\). The spinor \( u \) satisfying (4.23) actually exists, since

\[
\left( \frac{1}{\sqrt{|G|}} \gamma_{6789} \right)^2 = 1 \quad \text{holds, and we should remark the fact that} \quad u \text{ does not depend on any moduli.}
\]

It is further worth while to note that \( u \cdot Q_+(M) \) under (4.23) is no other than the unbroken SUSY charge in the supersymmetric configuration of branes (4.14). Under (4.14), we obtain \( u \cdot Q_+(M)|D'\rangle = 0 \), and especially \( \Delta = 0 \). This choice of \( u \) is appropriate to our calculations, because we are now interested in the perturbative expansions around the supersymmetric vacuum\(^7\). It may be natural to regard \( \Delta \) as a value characterizing SUSY violation.

Now it is not hard to write down the 2-loop correction to \( \beta_{h_{00}} \). After including all the tadpole corrections (4.17)(4.21)(4.24), we can obtain

\[
c^{(2)}(x) = \frac{1}{2} G_{00} \lambda^{-1} \frac{1}{\mathcal{M}_{1-3}^{(1)}} (2nn' \sqrt{|G|}\Delta) \frac{\delta^{(5)}(x - X)}{\sqrt{\det G_{\mu\nu} \sqrt{|G|}}}.
\]

\[
= \eta_{00} \frac{nn' \Delta}{(|G|^{-1/4}\mathcal{M}_{1-3}^{(1)})} \frac{\delta^{(5)}(x - X)}{\sqrt{\det G_{\mu\nu} \sqrt{|G|}}}.
\]

(4.26)

In the first line of (4.26), the factor 2 before \( nn' \Delta \) corresponds to the existence of contact terms between the graviton vertex and 2 boundaries, one of which resides on the 1-branes \( D \) and the other of which does on the 3-branes \( D' \). The factor \( \frac{1}{2} \) is nothing but a symmetry factor due to the exchange of the boundaries of cylinder. Thus the 2-loop correction to the mass can be read as follows;

\[
\mathcal{M}_{1-3}^{(2)} = -e^{-\Phi/2} \frac{nn' \Delta}{(|G|^{-1/4}\mathcal{M}_{1-3}^{(1)})}.
\]

(4.27)

It implies that we may naturally regard \( \Delta \) as the binding energy between two types of D-branes \( D, D' \). As is already observed, \( \Delta \) vanishes under the supersymmetric brane configuration (4.14), but does not for the general non-supersymmetric backgrounds. We can conclude that the binding energy among the branes has its origin in the SUSY violation.

Next we consider the contributions from \( 2k \) \((k \geq 2)\) loop \( c^{(2k)} \). As is already commented, in the limit that all the areas of open string boundaries shrink, only the diagrams of the type

\[\text{If we consider the bound states of D-branes and anti-D-branes, namely, in the case} \quad u \cdot Q_+(M)|D\rangle = 0, \quad u \cdot Q_-(M')|D'\rangle = 0, \quad \text{we should take} \quad u \quad \text{such that} \quad \frac{1}{\sqrt{|G|}} \gamma_{6789} u = u \quad \text{for correct perturbative calculations.}\]
Figure 5: The cylinder amplitude $\triangle$ appears in the 2-loop beta function calculation. In the “Intersecting D-brane” case, the cancellation between the NS-NS and the R-R does not exist.

in figure 4 survive, and they are factorized into the $k$ products of cylinder amplitudes of the forms $\langle D | V^{h_{\mu \nu}} | D' \rangle^{(0)}$. Hence we obtain

$$c^{(2k)}(x) = \text{combinatorial factor} \times G_{00} \lambda^{-1} \frac{(2nn'\sqrt{|G|\triangle})^k}{(\mathcal{M}_{1-3}^{(1)})^{2k-1}} \frac{\delta^{(5)}(x-X)}{\sqrt{\det G_{\mu \nu} \sqrt{|G|}}}$$

$$= \text{combinatorial factor} \times \eta_{00} \frac{(2nn'\triangle)^k}{(|G|^{-1/4}\mathcal{M}_{1-3}^{(1)})^{2k-1}} \frac{\delta^{(5)}(x-X)}{\sqrt{\det G_{\mu \nu} \sqrt{|G|}}} \quad (4. 28)$$

Let us evaluate the combinatorial factor of the diagram. When we add one cylinder $\sim \langle D | V^{h_{\mu \nu}} | D' \rangle^{(0)}$ to diagrams with $2(k-1)$ open string loops, there are $(2k-3)$ cases of possibilities to connect them. So there are $(2k-3)!!$ diagrams with a fixed loop number $2k$.

(Note that adding one cylinder $\sim \langle D | V^{h_{\mu \nu}} | D' \rangle^{(0)}$ needs another pants diagram connecting to the original diagram.) But one more symmetry factor $\frac{1}{2^k k!}$ for these diagrams appears. These lead to a correct (moduli-independent) numerical coefficient $\frac{(2k-3)!!}{2^k k!}$. So, the 2k-loop correction to the mass $\mathcal{M}_{1-3}^{(2k)}$ is given by

$$\mathcal{M}_{1-3}^{(2k)} = -\frac{(2k-3)!!}{k!} e^{-\Phi/2} \frac{(nn'\triangle)^k}{(|G|^{-1/4}\mathcal{M}_{1-3}^{(1)})^{2k-1}} \quad (4. 29)$$

Collecting the results (4. 18), (4. 27), (4. 29), we can finally get the mass formula for this bound state of the intersecting $n$ D1-branes and $n'$ D3-branes including all the corrections of higher loops;

$$\mathcal{M}_{1-3} = \mathcal{M}_{1-3}^{(1)} \left[ 1 - (nn'\triangle) \left( \frac{1}{|G|^{-1/4}\mathcal{M}_{1-3}^{(1)}} \right)^2 - \sum_{k=2}^{\infty} \frac{(2k-3)!!}{k!} (nn'\triangle)^k \left( \frac{1}{|G|^{-1/4}\mathcal{M}_{1-3}^{(1)}} \right)^{2k} \right]$$
This result exactly reproduces the BPS mass formula predicted by U-duality in section 2!

Until now we only consider a bound state of the intersecting D1- and D3-branes in the type IIB string compactified over $T^4$. The applications to other bound states are straightforward. For a bound state of $n$ Dirichlet 0-branes and $n'$ D4-branes in the $T^4$-compactified type IIA theory, we only have to replace the value of $\triangle$ with

$$\triangle = \left|G\right|^{-1/2} \left( \int_{T^4} B \wedge *B \right) + \frac{1}{4|G|} \left( \int_{T^4} B \wedge B \right)^2 - \left( 1 - \frac{1}{2|G|^{1/2}} \int_{T^4} B \wedge B \right) \right].$$

This result is also consistent with the prediction of M-theory in Eq.(3.34).

Next let $\mathcal{D}$, $\mathcal{D}'$ be respectively, $n$ Dirichlet 2-brane wrapping around 67th directions and $n'$ D2-brane wrapping around 89th axes in the type IIA theory compactified on the $T^4$. For a bound state of $\mathcal{D}$ and $\mathcal{D}'$, we obtain its binding energy

$$\triangle = |G|^{-1/2} \sqrt{\sum_{i=1}^{3} \left( \int_{\Sigma_1} X^*J_i \right)^2 + \left( \int_{\Sigma_1} X^*B \right)^2} \sqrt{\sum_{i=1}^{3} \left( \int_{\Sigma_2} X^*J_i \right)^2 + \left( \int_{\Sigma_2} X^*B \right)^2} - \left\{ 1 - \left( G_{68}G_{79} - G_{69}G_{78} \right) - B_{67}B_{89} \right\}, \tag{4.31}$$

where the $\Sigma_1$ and $\Sigma_2$ are respectively the world volumes of the D2-branes $\mathcal{D}$ and $\mathcal{D}'$.

We make one remark for this intersecting D2-brane case. Let us assume $n = n'$. As we showed in section 3, by considering general 2-cycle $S \in H_2(T^4; \mathbb{Z})$ (perhaps, having a higher genus), the analysis based on the DBI action gives a correct mass formula for 2-branes including general bound states. Why did it give a correct answer in spite of the restricted analysis only in the one-loop we have done there? In the study in this section, we took the constant background field and discussed higher loop corrections. In contrast, in section 3 we considered holomorphic embeddings with the maps $\{X\}$'s directly. Then the induced background fields $X^*G_{ij}, X^*B_{ij}$ on the world volume are not constant in general. These non-constant holomorphic embeddings are known as supersymmetric cycles and guarantee the cancellation of higher (more than one) loop corrections to the $\beta$-function. That is the reason why we obtained the correct mass formula only in the one-loop analysis for the D2-brane case.

In summary, in order to treat the bound states of the two kinds of D2 branes, one half of which is wrapping around $T^{67}$ and the other half of which is wrapping around $T^{89}$, we
The bound state of intersecting D2-branes can be represented as a SUSY cycle (holomorphic curve) with genus two.

The dimensions of the D3-brane and the D1-brane are different and we cannot represent its bound state as a single world volume.

can incorporate all the quantum corrections (perturbative loop corrections) into a single Dirac-Born-Infeld action with a genus “two” world volume. That is a homological sum of $n$ holomorphic curves (supersymmetric cycles) with genus two (Fig.6). It may be plausible to interpret this aspect as a “geometrization of quantum corrections”.

On the other hand, for a bound state of the D1-branes and D3-branes, the dimensions of these two kinds of branes are different and we cannot describe the state by a single DBI action (Fig.7).

Another interesting example of the “geometrization” is the situation of the branes within branes [9]. Let us again consider the bound states of D1-, D3-branes in IIB string. But, this time we make $n$ D1-branes wrap around the 6th-axis and $n$ D3-branes wrap around
the 678th-axes. We further assume that each 1-brane is contained in each 3-brane. Clearly this configuration of branes breaks SUSY completely (in any background!), and we have a non-zero binding energy

\[ \Delta = \sqrt{G_{66}} \sqrt{\{G^{-1} - (\ast B)G(\ast B)\}}^{99} + G_{6i} \cdot (\ast B)^{16}. \]  

(4. 32)

However, we already knew this case can be also described by a single DBI action with a suitable choice of a background gauge field:

\[ \mathcal{F} \equiv F + B, \quad [F] = \text{Poincaré dual of the 1-brane in the world volume of the 3-brane}. \]

This is a manifestly supersymmetric treatment. It is straightforward to check that these two treatments yield the same mass formula. Therefore, we can find out a remarkable fact: In the case of branes within branes, the non-supersymmetric calculation with non-zero higher loop corrections is equivalent to the supersymmetric treatment based on the single DBI action with suitable background gauge fields. That is, all loop corrections can be transmuted into a charge of the gauge field! One may say this is another example of the geometrizations of quantum corrections.

In the relation to this subject, it may be also meaningful to discuss the non-abelian extension when the gauge symmetry enhancement occurs on D-branes. In section 3, we only considered the charges of background gauge fields along the “U(1)-sectors”. This approach was limited in the sense that we can only realize as the “U(1)-charges” the brane configurations that can be reduced to one kind of branes by T-duality. However, at least in the level of naive observation, if the gauge theory on branes becomes non-abelian, we can interpret more general configurations, which is not necessarily reduced to one kind of branes by T-duality, as the characteristic classes composed of the field strength. The non-abelian extension of DBI action is proposed by Tseytlin [26], in which the symmetrized traces of the products of field strength appear. It may be interesting to check the consistency between the two general descriptions of bound states, one of which is based on the non-abelian Born-Infeld action and the other of which is based on the string loop analysis given in the present section.

To close this section we again emphasize that the DBI action (even the non-abelian DBI) is not sufficient to describe all the bound states. We must inevitably perform the higher loop analysis to complete our studies.
5 Conclusions and Discussions

In this paper, we investigated the mass spectra of R-R solitons by making use of the D-brane techniques in order to confirm the U-duality. We would like to emphasize that our results are obtained under the completely general backgrounds. Especially it is remarkable that the form of the DBI action is perfectly fitted to the moduli dependence of the masses of BPS solitons with (one kind of) the R-R charges. Moreover, the masses of some bound states - branes within branes - can be also evaluated by the DBI action by incorporating suitable charges of gauge fields. In other words, we have shown that the DBI action is consistent with one of the T-duality transformations - “integer theta parameter shift” $B_{ij} \rightarrow B_{ij} + \Theta_{ij}$.

It is a challenging task to analyse more general bound states. We argued on these states, emphasizing the relation with the SUSY violation, and observed that the quantity $\Delta$ which characterizes SUSY violation can be interpreted as a binding energy of D-branes. The characteristic $\Delta$ is essentially a sum of contributions from annulus amplitudes in both NS-NS sector and R-R sector. When cancellations between two sectors in these amplitude are not complete, the space-time SUSY is broken and there exist binding energies among branes. In the case that the only open string loop configurations are dominant, we evaluated the contributions to the binding energy from all these loops. We have shown that the summation of all loop corrections yields the consistent results with U-duality.

However, there are still some open problems. In order to complete our D-brane analysis of BPS mass spectra, we have to further consider three open problems which we presented in the opening of section 4; The first is the analysis of the bound states of D-branes and the fundamental excitations. The second is the evaluation of the dependence of BPS masses on the R-R moduli. The problem about the extra R-R charge originated from the 1st Pontrjagin number of K3 is the third one. We believe that these problems can be also solved by higher loop analyses to $\beta$-functions. In section 2 we discussed the BPS mass formulae based on some part of U-duality invariance. However, according to the analysis in M-theory, we can write down more complete mass formula Eq. (3.26) (with the suitable replacements for the charges $q, r_l, s^{kl}$ by $\tilde{q}, \tilde{r}_l, \tilde{s}^{kl}$). This has the full U-invariance $O(5, 5; \mathbb{Z})$ and remarkably, it includes some interaction terms between the fundamental excitations and R-R solitons. Therefore, the first problem is especially significant in order to confirm the full U-duality from the viewpoints of the D-brane analysis and also to check the consistency between the D-brane calculations and the M-theory approach. We wish to present a more detailed study...
on this subject in future. In the last problem, we will have to treat carefully the open string
loops in the twisted sectors of the K3 orbifold.

It is also worth while remarking on the closed string loops, which we neglected in the
discussion of section 4. Our analysis with only open string loops is valid in the cases when
the D-branes are very heavy and there are no recoils between them. In other words, these
are the cases that R-R charges $N$ assigned to the D-branes are very large and we treat them
(semi)classically (i.e. we do not consider the quantum fluctuations of the branes). This large
$N$ case is the situation that M(atrix) theory [32, 33, 34] has its mean as a M-theory in the
infinite momentum (light-cone) frame. This M(atrix) theory is realized as a large $N$ limit of
a SUSY Yang-Mills theory.

Now, there is a curious point to be mentioned in the relation with the recent studies
about M(atrix) theory. In the works [35, 36] the calculations of quantum corrections (loop
corrections and instanton corrections) in the SUSY Yang-Mills on the world brane are com-
pared with the tree calculation in (11D) bulk SUGRA and they claim these are equivalent.
However, there is a naive question: How about the quantum corrections in SUGRA? If we
assume the description by the M(atrix) theory is completely valid, the consistency of the
computations in [35, 36] will demand that, in the large $N$-limit, the tree level of bulk SUGRA
should be exact. As a result, this classical SUGRA will become equivalent to the quantum
SYM in this limit.

On the other hand, in this paper we evaluated the BPS mass formulae from the open
stringy loop corrections under the D-brane backgrounds and compared the results with the
mass formulae obtained by the classical SUGRA (U-duality). We have actually observed
that the closed string loop corrections can be neglected in the limit of large R-R charges.
Recalling the fact that the loop corrections in open string theory correspond to those in
SYM and the closed string loops correspond to those in SUGRA in the low energy limit, our
results seem to support the validity of M(atrix) theory! Our analysis is still limited, but we
hope it will give some insights to the studies of M(atrix) theory in future.

Although the above consideration is satisfactory, it may be still meaningful to ask whether
the closed string loop corrections exactly vanish, because the U-duality should be valid even
if the amount of R-R charges $N$ is a small value. One possibility that the contributions
from closed string loops do not break our analysis even in the cases with small R-R charges
is a Fischler-Susskind type mechanism [37]. Namely, all the closed string loop corrections
might contribute to only the renormalization of dilaton (= string coupling constant), and
hence the mass formulae might be kept essentially unchanged. However, it remains an open problem for a long whether this mechanism can apply to supersymmetric theories in higher loop order when supersymmetry is broken by boundary conditions. Anyway, we will have to treat carefully the quantum fluctuations of D-branes as in the discussions in [31] in order to work properly in the region where R-R charges are not large.

Acknowledgement

We are especially grateful to H. Ishikawa and Y. Matsuo who participated in the early stage of this work. Y.S. also thanks to T. Eguchi for helpful discussions. K.S. thanks to K. Ezawa for useful comments.

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