A Two Term Truncation of the Multiple Ising Model
Coupled to 2d Gravity

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Abstract

We consider a model of $p$ independent Ising spins on a dynamical planar $\phi^3$ graph. Truncating the free energy to two terms yields an exactly solvable model that has a third order phase transition from a pure gravity region ($\gamma_{str} = -1/2$) to a tree-like region ($\gamma_{str} = 1/2$), with $\gamma_{str} = 1/3$ on the critical line. We are able to make an order of magnitude estimate of the value of $p$ above which there exists a branched polymer (i.e., tree-like) phase in the full model, that is, $p \sim 13–23$, which corresponds to a central charge of $c \sim 6–12$.

1 Introduction

There has been much interest in studying models of conformal matter coupled to 2d quantum gravity and in particular in investigating the nature of the $c = 1$ barrier, beyond which KPZ theory breaks down. These models tend to exhibit a branched polymer phase for large enough values of the central charge $c$, and our aim is to shed some light on the extent of this phase for the multiple Ising model.

This letter extends the work of a previous paper in which we studied a model of $p$ independent Ising spins coupled to 2d gravity, the central charge of this model being $c = p/2$. The partition function of the model is

$$Z_N(p) = \sum_{G \in \mathcal{G}} \frac{1}{s_G} (Z_G)^p,$$

where the sum is over some set $\mathcal{G}$ of connected planar $N$-vertex $\phi^3$ graphs. The symmetry factor $s_G$ is equal to the order of the symmetry group of the graph $G$. $Z_G$ is the Ising
partition function for a fixed graph $G$ with a single spin per vertex and coupling constant $\beta$, that is,

$$Z_G = \frac{1}{Z_0} \sum_{\{S\}} \exp \left( \beta \sum_{<ij>} S_i S_j \right),$$

with

$$Z_0 = 2^N (\cosh \beta)^{3N/2}.$$  \hfill (3)

$S_i$ is the spin on the $i$-th vertex ($S_i = \pm 1$) and the sum in the action is over nearest neighbours on the $\phi^3$ graph.

As in our previous paper we define three different versions of the model. For model I the set $G$ consists of all connected planar $N$-vertex $\phi^3$ graphs. In model II the set $G$ is restricted to one-particle irreducible graphs and for model III only two-particle irreducible graphs are used.

It has been proven \cite{2} that in the limit of large $p$ tree-like graphs dominate the partition function, for model I (see fig \ref{fig:tree} for an example of a tree graph). By solving a model in which the free energy of the Ising partition function was truncated to a single term we managed to show how this change to dominance occurs, for small $\beta$. In this approximation the change was continuous and for finite $p$ there was no phase transition to a tree-like region (which we would associate with the branched polymer region occurring in other similar models).

The purpose of this letter is to extend the calculation by including two terms in the truncated free energy. We will show that for this two-term truncation of model I there exists a third order phase transition from a pure gravity region (with $\gamma_{\text{str}} = -1/2$) to a tree-like region (for which $\gamma_{\text{str}} = 1/2$) and that $\gamma_{\text{str}} = 1/3$ on the critical line. The location of this critical line approximates to that in the full untruncated model and allows us to estimate the value of $p$ for the point at which the tree-like, unmagnetized and magnetized phases meet in this model; the result is $p^* \sim 13–23$, corresponding to a central charge of $c \sim 6–12$. In this letter we present the main details of the calculation, further information about the solution can be found in reference \cite{3}.

**Figure 1: Tree-like graph**

![Tree-like graph](image)
2 Definition of the two-term truncated model

For a given \( \phi^3 \) graph \( G \), the Ising partition function can be written in terms of \( t \equiv \tanh \beta \) as a high temperature expansion,

\[
Z_G = 1 + \sum_l n_l t^l,
\]

(4)

where \( n_l \) is the number of closed (but possibly disconnected) loops in the graph, which contain \( l \) links and use no link more than once. Defining \( \mu_G \) by

\[
\mu_G = \lim_{N \to \infty} \frac{1}{N} \log Z_G,
\]

(5)

we have for model I,

\[
\mu_G = \frac{1}{N} \left[ n_1 t + \left( n_2 - \frac{1}{2} n_1^2 \right) t^2 + \cdots \right].
\]

(6)

Defining \( n_2^c \) to be the number of connected loops of length two (hereafter referred to as "2-loops") then \( n_2 = n_2^c + \frac{1}{2} n_1(n_1 - 1) \) and we have

\[
\mu_G = \frac{1}{N} \left[ n_1 t + \left( n_2^c - \frac{1}{2} n_1 \right) t^2 + O(t^3) \right].
\]

(7)

Suppose that we truncate \( \mu_G \) to the first two terms of the series, denoting this new quantity by \( \mu_G^T \) and consider the model with the partition function

\[
Z_N(p) = \sum_{G \in \mathcal{G}} \frac{1}{s_G} e^{\mu_G^T N p},
\]

(8)

which we will refer to as the two-term truncated model. From now on \( \mathcal{G} \) is taken to be the set of model I graphs. In the limit of small \( \beta \) we might expect it to reproduce the behaviour of the full model. For larger \( \beta \) it will be a bad approximation to the original model and in particular it has no magnetized phase. However we can modify the truncated model slightly so that it is a reasonable approximation to the full model for large \( p \) at any value of \( \beta \). The key to doing this is to note that one can factor out the entire contribution to \( Z_G \) resulting from loops of length one ("1-loops") giving

\[
Z_G = (1 + t)^{n_1} Z'_G,
\]

(9)

where \( Z'_G \) is the partition function on the graph \( G' \) for which 1-loops have been cut off (leaving vertices of coordination number two, see fig 3). It is impossible to factorize all the contributions from 2-loops in the same fashion, but suppose that we consider the partition function

\[
Z''_G = (1 + t)^{n_1} (1 + t^2)^{n_2^c} = 1 + \sum_l n_l t^l.
\]

(10)
This corresponds to a model in which we count $n_l'$ the number of (possibly disconnected) loops of length $l$, but restrict each of the connected parts to be of length two or less. This is a good approximation to the full model if $t$ is small or if the dominant graphs only have short loops in them (which is precisely what happens for large $p$ — since we know that tree-like graphs dominate in this limit). Thus we might expect the modified two-term truncated model to give a reasonably accurate indication of the transition to a tree-like phase even for values of $\beta$ which are not small. The corresponding $\mu_G^T$ is

$$\mu_G^T = \frac{1}{N} \left[ n_1 \log(1 + t) + n_2 \log \left( 1 + t^2 \right) \right]. \quad (11)$$

In this letter we will solve exactly this modified truncated model, defined by equations (8) and (11), and relate the results to the untruncated model. For technical reasons it is much easier to solve the model if we use $\phi^3$ graphs which have a “root”; that is, in the terminology of $\lambda\phi^3$ theory, we are going to use 1-legged Green functions rather than vacuum diagrams - this will not change the free energy and should not affect any of our conclusions. In this case we have

$$Z_N(p) = \sum_{G \in \mathcal{G}} \exp p \left[ n_1 \log(1 + t) + n_2 \log \left( 1 + t^2 \right) \right]$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} G_r^{(1)}(N, n_1, n_2) \ v^{n_1} y^{n_2}, \quad (13)$$

where we have defined

$$v \equiv (1 + t)^p, \quad y \equiv \left( 1 + t^2 \right)^p \quad (14)$$

and $G_r^{(1)}(N, n_1, n_2)$ is the number of $N$-vertex rooted model I graphs with $n_1$ 1-loops and $n_2$ 2-loops; the subscript “r” indicates that we are using rooted graphs. The grand canonical partition function is

$$Z_r = \sum_{N=1}^{\infty} \sum_{\text{odd}} e^{-\mu N} Z_N(p) = \sum_{N=1}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} G_r^{(1)}(N, n_1, n_2) \ x^N v^{n_1} y^{n_2}, \quad (15)$$

where we have put $x \equiv \exp(-\mu)$. 
3 One-term truncation of model II

In the calculation that follows we will need the generating function for $G^{(2)}(N, n_2^c)$, the number of $N$-vertex (unrooted) model II graphs with $n_2^c$ 2-loops. Determining this is equivalent to solving the one-term truncation of model II and the calculation follows that given in [3] for models I and III. As in that paper one can derive a recurrence relation valid for $N \geq 4$,

$$\left(\frac{3N}{2} - 2n_2^c\right) G^{(2)}(N, n_2^c) + 2(n_2^c + 1) G^{(2)}(N, n_2^c + 1) = (n_2^c + 1) G^{(2)}(N + 2, n_2^c + 1).$$

The generating function is defined as

$$Z^{(2)}(x, y) = \sum_{N=4 \text{ even}}^{\infty} \sum_{n_2^c=0}^{\infty} G^{(2)}(N, n_2^c) x^N y^{n_2^c}. \tag{17}$$

This is given by

$$Z^{(2)} = \frac{1}{6} x^2 \left(h^{-3} - 3y + 2\right) - \frac{1}{2} \log h + \sum_{N=4 \text{ even}}^{\infty} G^{(2)}(N) x^N h^{-3N/2}, \tag{18}$$

where $h \equiv 1 - x^2(y - 1)$ and $G^{(2)}(N)$ is the number of $N$-vertex model II graphs,

$$G^{(2)}(N) = \frac{2^N \left(\frac{3N}{2} - 1\right)!}{\left(\frac{N}{2}\right)! (N + 2)!} \sim e^{\frac{1}{2} \log(\frac{27}{2})} N^{-\frac{7}{2}}. \tag{19}$$

4 Cayley Trees

In the previous section we gave the generating function for $G^{(2)}(N, n_2^c)$, which is the number of one-particle irreducible (denoted “1PI”) graphs with a given number of vertices and 2-loops. If we could generalize this generating function to the case of 1PI graphs with $m + 1$ legs, then connecting together such graphs into Cayley trees would give model I graphs and allow us to keep track of both $n_1$ and $n_2^c$.

Figure 3: (a) 1PI Green function, $G^{1m}_{1m}$ (b) Generating function for trees, $T$ (c) Example of a rooted tree
The generating function for 1PI graphs with a root and $m$ other legs will be denoted by $G'_{1m}$ and drawn as in fig 3a (sometimes the root will be labelled with an “r” to distinguish it from the other legs). It is a sum over such graphs weighted with factors of $x^N y^{n_2}$. Note that although the legs have been drawn on the exterior of the Green function some of them may in fact be attached to links in the interior.

The generating function for trees will be denoted $T$ and is drawn as in fig 3b. Fig 3c gives an example of a tree contributing to $T$. Figure 4 shows that $T$ satisfies the equation

$$T = \sum_{m=0}^{\infty} G'_{1m} T^m.$$  

(20)

It will be convenient to define a slightly different function $G_{1m}$ which generates 1PI graphs with a root and $m$ “exits”. An exit is defined to be a place at which a tree can be hung (in either of two directions) and is drawn as a link with a cross on it. For example fig 3a contributes to $G_{12}$ and there are four corresponding contributions to $G'_{12}$. In general $G'_{1m} = 2^m G_{1m}$, but it should be noted that fig 3b can not be represented in this fashion and must be excluded from $G_{12}$. To compensate an extra term (fig 3c), $x T^2$, must be added to the equation, giving

$$T = \sum_{m=0}^{\infty} G_{1m} (2T)^m + x T^2.$$  

(21)

This formula then correctly distinguishes between graphs which differ only in that a tree has been hung off a link in the opposite direction (these correspond to different triangulated surfaces).

Now define a generating function for the $G_{1m}$,

$$\Gamma(x, y, z) = \sum_{N>0} \sum_{n_2^c=0} \sum_{m=0}^{\infty} G_{1m}(N, n_2^c) x^N y^{n_2^c} z^m,$$  

(22)

where $G_{1m}(N, n_2^c)$ is the number of 1PI graphs with one root, $m$ exits, $N$ vertices and $n_2^c$ 2-loops, that is,

$$G_{1m} = \sum_{N>0} \sum_{n_2^c=0}^{\infty} G_{1m}(N, n_2^c) x^N y^{n_2^c}.$$  

(23)

Figure 4: Equation for $T$
This is what is needed, except for the fact that the 1-loops have been ignored. The only contribution to $(22)$ from 1-loops occurs in $G_{10}$ and is equal to $x$. The 1-loops will be weighted with an extra factor of $v$ and this is achieved by defining

$$\tilde{\Gamma}(x, y, z, v) = \Gamma(x, y, z) - x + xv.$$  \hspace{1cm} (24)

Hence from (21) we have that $T$ must satisfy

$$T = \tilde{\Gamma}(x, y, 2T, v) + xT^2$$  \hspace{1cm} (25)

and then the solution $T(x, y, v)$ is equal to $Z_r$ defined in (15). The next task is to find the function $\tilde{\Gamma}(x, y, z, v)$.

## 5 Determination of $\tilde{\Gamma}(x, y, z, v)$

In this section we are going to determine $\Gamma(x, y, z)$ the generating function for $G_{1m}(N, n^c_2)$. However it is useful to find $G_{10}(N, n^c_2)$ first, as this will provide a boundary condition. Below we show that

$$G_{10}(N-1, n^c_2-1) = 2n^c_2 G^{(2)}(N, n^c_2),$$  \hspace{1cm} (26)

for $N \geq 4$ ($N$ is even).

Figure 6: (a) Rooting a model II graph (b) Special case

Take an $N$-vertex model II graph $G$ with $n^c_2$ 2-loops. It has $2n^c_2$ orientated 2-loops (ie we count each 2-loop twice, once with an arrow going in one direction and once with the arrow reversed). These can be grouped into equivalence classes of size $s_G$; each class consists of 2-loops which can be transformed into each other under the action of the symmetry group. Removing an orientated 2-loop and replacing it with a leg (as shown in fig 5a) produces a graph with $N - 1$ vertices and $(n^c_2 - 1)$ 2-loops. Note that we are
ignoring the special case fig 6b, for which \( n_2^c = 3 \) and \( \Delta n_2^c = -3 \); so the resulting formula will not apply for \( N = 2 \). There is a one-to-one correspondence between equivalence classes of orientated 2-loops and rooted graphs. The number of equivalence classes is

\[
\sum_{G \in \Gamma(N, n_2^c)} 2n_2^c = 2n_2^c G^{(2)}(N, n_2^c),
\]

where the subscript on the summation means that we are summing over model II graphs labelled with a \( G \), which have \( N \) vertices and \( n_2^c \) 2-loops. Equating this with the number of rooted graphs \( G_{10}(N-1, n_2^c-1) \) gives the required result.

Now,

\[
\Gamma(x, y, z = 0) = \sum_{N>0} \sum_{n_2^c=0}^{\infty} G_{10}(N, n_2^c) \ x^N y^{n_2^c}
\]

and substituting (26) into this yields

\[
\Gamma(x, y, z = 0) = \frac{2 \partial Z^{(2)}}{x \partial y} + x,
\]

where \( Z^{(2)}(x, y) \) is given by (18).

Figure 7: Adding an exit to: (a) an ordinary link (b) a 2-loop (c) a 1-loop

\[\begin{array}{ccc}
\text{(a)} & \text{(b)} & \text{(c)} \\
\includegraphics[width=0.3\textwidth]{figure7a} & \includegraphics[width=0.3\textwidth]{figure7b} & \includegraphics[width=0.3\textwidth]{figure7c}
\end{array}\]

Having determined the number of rooted \( N \)-vertex 1PI graphs with \( n_2^c \) 2-loops, we wish to generalize this to the case in which the graphs have \( m \) “exits”. Consider adding an exit to a graph which has \( m - 1 \) exits and \( N - 1 \) vertices; this will produce an \( N \)-vertex graph with \( m \) exits. If the exit is added to an ordinary link then \( n_2^c \) is unchanged (fig 7a), if added to a 2-loop then \( \Delta n_2^c = -1 \) (fig 7b) and if added to a 1-loop then \( \Delta n_2^c = +1 \) (fig 7c). Ignoring the special case (fig 7c) for a moment, then we wish to make \( m \) \( N \)-vertex graphs, with \( m \) exits and \( n_2^c \) 2-loops. Bearing in mind that each such graph can be produced in \( m \) ways (ie any of the \( m \) exits could be the one that we have just added), we have

\[
m \ G_{1m}(N, n_2^c) = L_1 G_{m-1}(N-1, n_2^c) + L_2 G_{m-1}(N-1, n_2^c + 1),
\]

where \( L_1 \) is the number of ordinary links in a graph with \( N - 1 \) vertices, \( m - 1 \) exits and \( n_2^c \) 2-loops, and \( L_2 \) is the number of links lying on a 2-loop in a graph with \( n_2^c + 1 \) such loops. One can easily show that

\[
L_1 = \frac{3}{2} (N - (m + 1)) + m - 2n_2^c,
\]

\[
L_2 = 2(n_2^c + 1).
\]
Thus we have a recurrence relation for the coefficients of $G_{1m}$ in terms of those in $G_{1m-1}$ and a formula for $G_{10}$. Note that (30) does not apply for $N = 2, m = 1$ due to the special case (fig 7c).

Using (30) one can derive the following differential equation for $\Gamma(x, y, z)$,

$$\frac{3}{2} x^2 \frac{\partial \Gamma}{\partial x} + 2x(1 - y) \frac{\partial \Gamma}{\partial y} - \left( \frac{1}{2} xz + 1 \right) \frac{\partial \Gamma}{\partial z} - \frac{1}{2} x \Gamma + x^2(y - 1) = 0. \quad (33)$$

The solution is straightforward and gives $\tilde{\Gamma}$ as

$$\tilde{\Gamma}(x, y, z, v) = \sqrt{1 - xz} \Gamma_0 \left( x(1 - xz)^{-\frac{3}{2}}, 1 + (y - 1)(1 - xz)^2 \right) + x^2 z(y - 1) + x(v - 1), \quad (34)$$

where $\Gamma_0(x', y', z' = 0)$. Using (29) and (18), this is given by

$$\Gamma_0(x', y') = x' \left[ \frac{x'^2}{h^4} + \frac{1}{h} + \frac{3}{h} \sum_{N=4}^{\infty} G^{(2)}(N) x'^N N h^{-3N/2} \right], \quad (35)$$

with $h \equiv 1 - x'^2(y' - 1)$.

6 Solution of the model

Putting $z = 2T \equiv 2Z_r$ and using (25) gives

$$xz = 2x\tilde{\Gamma}(x, y, z, v) + \frac{1}{2}(xz)^2. \quad (36)$$

As $x$ is increased from zero there is a non-analyticity at some critical value, $x_c$. This will give us the free energy of the model through $\mu_c = -\log x_c$. In order to determine $x_c$ we first need to find an expression for the grand canonical partition function $Z_r$.

Defining $\epsilon \equiv y - 1, \eta \equiv v - 1$ and

$$Y \equiv \sqrt{1 - xz}, \quad Q \equiv \frac{x}{(1 - xz)^{\frac{3}{4}}} = \frac{x}{Y^3}, \quad (37) \quad (38)$$

we can eliminate $x$ and $z$ from (30) in favour of $Y$ and $Q$ giving

$$\left( 1 - Y^2 \right) = \frac{1}{2} \left( 1 - Y^2 \right)^2 + 2Y^3 \left[ Y \Gamma_0 \left( Q, 1 + \epsilon Y^4 \right) + Y^3 Q \left( 1 - Y^2 \right) \epsilon + QY^3 \eta \right]. \quad (39)$$

Now defining

$$H \equiv \frac{Q^2}{1 - \epsilon Q^2 Y^4} = \frac{x^\frac{3}{4}}{1 - xz - \epsilon x^2} = \frac{x^\frac{3}{4}}{Y^2 - \epsilon x^2}, \quad (40)$$

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we have using \((35)\)

\[
\Gamma_0 \left( Q, 1 + \epsilon Y^4 \right) = Q^{\frac{5}{4}} \left[ H^4 + H + 3H \sum_{N=4}^{\infty} \mathcal{G}^{(2)}(N) NH^{\frac{1}{2}} \right] = Q^{\frac{5}{4}} F(H),
\]

where we have defined a function \(F(H)\). Putting this into \((39)\) gives

\[
0 = Y^4 - 1 + 4 \left[ Y Q^{\frac{1}{2}} \right]^4 F(H) + 4 \left[ Y Q^{\frac{1}{2}} \right]^6 \left( \epsilon \left( 1 - Y^2 \right) + \eta \right).
\]

Noting that \(Y Q^{\frac{1}{2}} = x^{\frac{1}{2}}\) we can use \((40)\) to eliminate \(Y\) and write \((42)\) in terms of \(H\) and \(f \equiv x^{\frac{1}{3}}\).

\[
\left(3\epsilon^2\right) f^6 + \frac{2\epsilon}{H} f^4 - 4f^3 (\eta + \epsilon) - f^2 \left( 4F(H) + \frac{1}{H^2} \right) + 1 = 0.
\]

This is a sixth order polynomial in \(f\), giving \(f = f(H)\).

To proceed further we require a more compact form for \(F(H)\). Brézin et al. [4] give a formula for the 2-legged 1PI generating function \(\Gamma_2\) (equation 64 of that paper),

\[
\Gamma_2 = \left( 1 - 2\tau \right)^2 \frac{1}{1 - 3\tau}, \quad g'^2 = \tau(1 - 2\tau)^2
\]

where each vertex has a weight of \(g'\) (note that \(g'\) is equivalent to \(3g\) in their notation). Now if we take each \(N\)-vertex model II graph, pull out an orientated link from each equivalence class (of size \(s_G\)) and cut it, then we get

\[
\sum_{G:II}^{\text{2-legged}} \frac{3N}{8G} = 3NG^{(2)}(N)
\]

2-legged Green functions. Each possible connected 2-legged Green function is generated exactly once, so that denoting the number of such functions by \(G_2(N)\) we have \(G_2(N) = 3NG^{(2)}(N)\). Thus defining a new symbol \(\hat{\Gamma}\),

\[
\hat{\Gamma} \equiv \sum_{N=4}^{\infty} 3NG^{(2)}(N)g'^N = \sum_{N=4}^{\infty} G_2(N)g'^N.
\]

However \(\Gamma_2\) generates 1PI 2-legged Green functions and thus the right-hand side of the above equation is given by \(\Gamma_2^{-1} - g'^2\) (see fig 8); the \(g'^2\) subtracts off the \(N = 2\) case. The critical value of \(g'\) is \(g'_c = \sqrt{\frac{2}{27}}\) and the corresponding critical value of \(\tau\) is \(\tau_c = 1/6\). Hence,

\[
F(H) = H^4 + H + H\hat{\Gamma}(g' = H^{3/2}) = H\Gamma_2^{-1}(g' = H^{3/2}).
\]

We are lead to define the variable \(\tau\) by

\[
H^3 = \tau(1 - 2\tau)^2
\]
and thus 

\[ 4H^2 F(H) + 1 = (1 + 6\tau)(1 - 2\tau). \]  

(50)

Now defining \( k \equiv f\sqrt{H/\tau} \) we can rewrite (43) as

\[ 3\epsilon^2 k^6 \tau^2 + 2\epsilon k^4 \tau - 4k^3(\eta + \epsilon)\tau(1 - 2\tau) - k^2(1 + 6\tau)(1 - 2\tau) + (1 - 2\tau)^2 = 0. \]  

(51)

This is a quadratic in \( \tau \) and can be rewritten as

\[ A(k)\tau^2 + B(k)\tau + C(k) = 0 \]  

(52)

with

\[ A(k) = 3\epsilon^2 k^6 + 8k^3(\eta + \epsilon) + 12k^2 + 4 \]  

(53)

\[ B(k) = 2\epsilon k^4 - 4k^3(\eta + \epsilon) - 4k^2 - 4 \]  

(54)

\[ C(k) = 1 - k^2. \]  

(55)

The aim of this calculation is to find the closest singularity to the origin for the grand canonical partition function, \( Z_r \) (note that \( Z_r = z/2 \)) and thus we wish to express \( Z_r \) in terms of the new variables and hence in terms of \( x \). From (41) we have that

\[ Z_r = \frac{1}{2x} \left[ 1 - \epsilon x^2 - \frac{f}{H} \right] \]  

(56)

and any non-analyticity in \( Z_r \) is due to the singular behaviour of \( f/H \), which is given by

\[ \frac{f}{H} = \frac{k}{1 - 2\tau}. \]  

(57)

Since the term in the square brackets is a function of \( x^2 \) it will be convenient to define a variable \( q = x^2 \) (bear in mind that we have already defined \( f = x^{2/3} \), so that \( q = f^3 = x^2 \)). Given the definition of \( k \),

\[ k^3 = f^3 \frac{H^{\frac{2}{3}}}{\tau^2} = \frac{q}{\tau}(1 - 2\tau), \]  

(58)

so that

\[ \tau = \frac{q}{k^3 + 2q} \]  

(59)

and substituting this into (52) yields

\[ \left( \frac{q}{k^3} \right)^2 (A + 2B + 4C) + \left( \frac{q}{k^3} \right)(B + 4C) + C = 0. \]  

(60)
Using equations (53) to (55) this gives
\[ k^4 - 2\epsilon q k^3 - k^2 \left[ 3\epsilon^2 q^2 - 4q(\eta + \epsilon) + 1 \right] + 8qk - 4\epsilon q^2 = 0, \] (61)
which is quartic in \( k \), so that we can in principle solve for \( k(q) \). This leaves the question of which root should be chosen. In the limit \( q \to 0 \), we have from (56) that \( f/H \to 1 \), \( H \to 0 \) and \( \tau \to 0 \). Thus from (57) we require \( k \to 1 \), looking at (61) we see that at \( q = 0 \) it becomes \( k^2 (k^2 - 1) = 0 \) and the root we want is the one for which \( k(q = 0) = 1 \).

Using (57) and (59) we can express \( f/H \) in terms of \( k \) and \( q \). Thus finally we have a formula for the grand canonical partition function,
\[ Z_r = \frac{1}{2x} \left[ 1 - \epsilon x^2 - k - \frac{2x^2}{k^2} \right], \] (62)
where \( k(x^2) \) is a calculable function given by solving (61).

At this point the simplest course of action was to write a computer program, which could evaluate (62) at different values of \( x \) and, by slowly increasing \( x \) from zero, locate the closest singularity to the origin. The non-analyticity occurs when \( k(x^2) \), which is real for small real \( x \), becomes complex as \( x \) is increased.

Examination of the data for \( \mu_c \) indicates that there is a third order phase transition (see fig 9, which plots various third derivatives of \( \mu_c \)). The location of the transition is most clearly seen in fig 10, which plots the quantity \( 1 - 6\tau \) evaluated at \( x_c \); the axes used are \( t \) and the product \( pt \). There are two regions one for which \( \tau_c = 1/6 \) and the other for which \( 0 < \tau_c < 1/6 \).

7 The critical line

In this section we are going to derive an equation for the critical line in the two-term model and determine the exponent \( \gamma_{s\text{tr}} \) for the different regions of the phase diagram.

It will be convenient to rewrite (61) as
\[ k^4 + a_3 k^3 + a_2 k^2 + a_1 k + a_0 = 0, \] (63)
where we have defined the coefficients \( a_0, \ldots, a_3 \). Away from the critical line the closest singularity to the origin is due to a double root in this equation and thus, evaluating everything at \( x_c \),
\[ 4k^3 + 3a_3 k^2 + 2a_2 k + a_1 = 0. \] (64)
On the critical line we have a triple root and hence
\[ 6k^2 + 3a_3 k + a_2 = 0. \] (65)
Figure 9: Plot of (a) $\frac{\partial^3 \mu_c}{\partial (pt)^3}$; (b) $\frac{\partial^3 \mu_c}{\partial (pt)^2 \partial t}$ for a range of $pt$ at $t = 1$, taking $\mu_c = \mu_c(pt, t)$.

Figure 10: Plot of $(1 - 6\tau_c)$ for the two-term model
Now we have three equations in four unknowns \((k, q, \epsilon, \eta)\) giving a one parameter solution. In general it would be difficult to solve these equations, but in this case there is a short cut; we know that on the critical line \(\tau_c = 1/6\) and so \(4q = k^3\) (from (59)). Substituting this into (54) to eliminate \(a_1\) yields

\[
6k^2 + 3a_3k + 2a_2 = 0. \tag{66}
\]

Comparing this with (65) we have \(a_2 = 0\) and \(2k + a_3 = 0\). Introducing a parameter \(e\) gives

\[
\epsilon = \frac{4}{e^2}, \quad k = e, \quad (\eta + \epsilon) = \frac{1}{e^3} \left(1 + 3e^2\right), \quad q = \frac{1}{4}e^3. \tag{67}
\]

It is easy to check that (63) is satisfied and to derive a formula for the critical line,

\[
\eta = \frac{\sqrt{e}}{2} \left(\frac{e}{4} + 3\right) - \epsilon. \tag{68}
\]

At \(t = 1, \epsilon = \eta = 2p - 1\); the above equation gives \(\sqrt{e} = 2(4 + \sqrt{13})\) and hence the critical line intercepts the \(t = 1\) axis at \(p \approx 7.860327\). For \(p \to \infty, \eta \sim e^{3/2}\) and the intercept with the line \(p = \infty\) occurs at the root of

\[
(1 + t) = \left(1 + t^2\right)^{\frac{3}{2}}, \tag{69}
\]

which is \(t \approx 0.6124088\) (note that \(t = 0\) is not the correct root).

In order to calculate \(\gamma_{str}\) it is necessary to find out how the derivatives of \(Z\) diverge as \(q \to q_c\). Putting \(\overline{Z} \equiv 2xZ_r\),

\[
\frac{\partial \overline{Z}}{\partial q} = -\epsilon - \frac{2}{k^2} - \frac{1}{q} \left[1 - \frac{4q}{k^3}\right], \tag{70}
\]

where \(q' \equiv \frac{\partial q}{\partial k}\); note that \(q'(q_c) = 0\) and \(q''(q_c) \neq 0\) in general, but that on the critical line \(q''(q_c) = 0\). For the region at large \(p\) and \(t\), \(\tau_c \neq 1/6\) and hence

\[
\frac{\partial \overline{Z}}{\partial q} \sim \frac{1}{q'} \sim \frac{1}{\sqrt{q - q_c}}. \tag{71}
\]

Noting that for rooted graphs

\[
\frac{\partial \overline{Z}}{\partial q} \sim (q - q_c)^{-\gamma_{str}}, \tag{72}
\]

we have \(\gamma_{str} = 1/2\) (that is, this is a tree-like region). For the region at small \(p\) and \(t\), \(\tau_c = 1/6\) and the first derivative of \(\overline{Z}\) is finite. Differentiating (70) with respect to \(q\) we find that

\[
\frac{\partial^2 \overline{Z}}{\partial q^2} \sim \frac{1}{\sqrt{q - q_c}} \sim (q - q_c)^{-1(\gamma_{str} + 1)} \tag{73}
\]
and hence $\gamma_{str} = -1/2$ (the pure gravity value). On the critical line

$$\frac{\partial Z}{\partial q} \sim \frac{1}{(q - q_c)^{3/2}},$$

(74)
giving $\gamma_{str} = 1/3$. Thus the third order phase transition in the two-term model is a transition from an unmagnetized non-tree-like region to a tree-like phase. This provides good evidence for the existence of such a transition in the full model. Presumably as more terms are included in the free energy, the critical line will move somewhat and it is not entirely obvious whether the intercept with the line $p = \infty$ will occur at non-zero $t$ or $t = 0$ in the full model.

The original aim of solving this model was to determine the extent of the tree-like region, in particular we are interested in finding the point at which the tree-like, magnetized and unmagnetized phases meet (assuming that such a point exists); the value of $p$ at this point is denoted $p^*$. In previous work we made an estimate of the location of the tree-like region based on the region for which $\mu_c$ was approximately linear. It will be useful to compare the location of the critical line with the boundary of the approximately linear region (referred to as the “knee” of the solution); the linear region can be defined to be that for which $\left| \frac{\partial \mu_c}{\partial (pt)} \right| < \delta$. We will take $\delta = 0.0168$ giving $pt \approx 6 + 2.7t + O(t^2)$ at the knee; this corresponds to our previous estimate that the graphs were tree-like for $pt \gtrsim 6$ at small $t$.\[2\]

In fig 11 is plotted the line of the phase transition in the $p-t$ plane and also the location of the knee. The tree-like phase is inside the linear region and the two lines are quite close in the area of interest (near $t \approx 0.73$). Using the phase line as an approximation to the location of the phase transition in the full model we can estimate the point at which it intercepts the boundary of the magnetized region. Although we do not know exactly where the critical line separating the magnetized and unmagnetized phases lies, we have that $t_c \approx 0.73$ for the single spin Ising model \[4\] and we know from Monte Carlo simulations \[1, 2, 3\] that $t_c$ only increases slowly with $p$. Taking $t = 0.73$ gives an estimate of $p^* \approx 23$, but this is likely to be an over-estimate since the phase line changes rapidly with $t$ and the value of $t$ that we are using is too small (it corresponds to the critical coupling for $p = 1$). An estimate using our old criterion based on the knee would give a value of $p^* \approx 13$. Overall then we conclude that $p^* \sim 13 - 23$ and that the corresponding value of the central charge, above which there exists a tree-like phase in the full model, is $c \sim 6 - 12$.\[15\]
Figure 11: Phase diagram for the two-term model: (a) Phase line (b) Knee ($\delta = 0.0168$) (c) The line $t = 0.73$
8 Conclusion

In fig 12 we have drawn some possible forms for the phase diagram of model I. The magnetized region is labelled M, the tree-like region T and the remaining unmagnetized non-tree-like region U. If the critical line in the two-term model were an accurate representation of that in the full model then the correct phase diagram would be fig 12a. However, one expects that as more terms are included in the truncated model the location of the critical line will change and it is not entirely clear where point A will move to. It is possible that A tends towards the $t = 0$ axis as more terms are added in which case fig 12b would be the correct diagram. Other possibilities, such as fig 12c or the absence of a tree-like region, can not be entirely ruled out, but seem unlikely given the results presented in this letter and those due to Wexler [9, 10]. The two-term model has a third order transition with $\gamma_{str} = \frac{1}{3}$ on the critical line AC and the full model may well show the same behaviour. It is interesting to note that the model in reference [11] also has $\gamma_{str} = \frac{1}{3}$ on the line separating the branched polymer and unmagnetized phases (in this model

Figure 12: Possible phase diagrams for model I
\( \gamma_{str} = \frac{1}{4} \) at the point where the three phases meet). The order of magnitude estimate of \( p^* \sim 13-23 \) (point C in fig.12) is similar to the largest value of \( p \) (ie \( p = 16 \)) used in simulations so far. It is to be hoped that future simulations will manage to locate the onset of the branched polymer phase. One should note that our estimate of \( p^* \) is not sufficiently accurate to exclude the possibility that \( p^* = 2 \), although numerical simulations seem to rule this out. In conclusion then, significant progress has been made in elucidating the nature of the phase diagram for the multiple Ising model on dynamical planar \( \phi^3 \) graphs, in particular in the large \( p \) limit, but much work remains to be done before the nature of the \( \epsilon = 1 \) barrier is fully understood.

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