The Game of Poker Chips, Dominoes and Survival

Larry Goldstein

Department of Mathematics
University of Southern California

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Abstract

The Game of Poker Chips, Dominoes and Survival was created to foster cohesion in a group setting. Given two colored poker chips, each player needs to secure a domino before time is called in order to ‘survive’, and can make exchanges according to two simple rules. Analysis reveals that the group will be forced to cooperate at a high level in order to succeed, and a simple time complexity computation shows how the game coordinator can choose the initial distribution of poker chips to the players in order to fine tune the game’s difficulty. A simple criteria is given for determining if the game is ‘solvable’ for any given initial chip distribution, that is, if all players can survive if given sufficient time to make exchanges. The best strategies for group survival, that is, those taking the least amount of time, are provided as consequences of simple complexity arguments. In addition to being a lively game to play in management training or classroom settings, the analysis of the game after play can make for an engaging exercise in any basic discrete mathematics course to give a basic introduction to elements of game theory, logical reasoning, induction, recursion, number theory and the computation of algorithmic complexities.

1 Introduction

Team building exercises designed to further group cohesion have long been offered to businesses and corporations as a resource for their managers and employees. Offerings run the gamut between training offered by business schools, on-line tutorials, and popular texts, see e.g. [1] or [2] or [3]. The ‘egg drop’ exercise, for instance, consists of dividing a group into two subgroups, and, given some raw materials, having each subgroup designs a package in which an uncooked egg would survive an eight foot drop. Each team is allotted time to separately discuss their ideas and come to consensus, make a pitch for their final design, and at last, see how their prototype performs in comparison to their counterpart’s. Another exercise is Minefield, where the group gives verbal cues to a blindfolded member in order to navigate around obstacles.

One game that was created to promote cooperation and coordination that is currently making the rounds [4] uses poker chips and dominoes. The poker chips come in three different colors, and at the start of the game everyone is given two of them. During the game, poker
chips can be traded for chips and dominoes, and a player must be holding at least one domino when time is called in order to survive. There are only two rules for making exchanges:

1. A set of three poker chips, one of each color, can be exchanged for one domino and a chip of the player’s own choosing.

2. A set of three dominoes may be exchanged for seven poker chips, all of the player’s own choosing.

Turning a group of size, let’s say 48, loose after explaining the rules and giving them two colored chips apiece and 15 minutes to make the necessary exchanges, their first attempt at success may be a bit rocky, with uncoordinated bartering and bargaining, and perhaps even some chaos and mayhem. Each player, initially with only two chips, are by themselves powerless to invoke either of the two rules. Individuals who have two chips of different colors may look to make a contract with someone having two chips of the color they lack, and via Rule 1, use their pooled resources to gain a domino for the first player, while gaining a needed color for the second player. Due the special properties of the game that we will prove, the group will discover that applying only such ‘local’ exchanges will not lead to the desired state where all players hold a domino when time is called, and that it can only be to their advantage to pool resources. Indeed, whereas Rule 1 can be effected by two players, more coordination is required by Rule 2, which can only be invoked by three players all of whom are already in possession of a domino, and whose survival is therefore guaranteed. As an additional wrinkle, it may not be immediately clear that the use of Rule 2 is to anyone’s advantage, as nine chips, fewer than the seven chips Rule 2 returns, must change hands to recoup the 3 domino investment. Nevertheless, we show in Theorem 3.1 that survival of all members of the group is impossible without invoking Rule 2. Hence, though experimentation and likely failures, the group will learn to value cooperation.

As a practical matter, those who are in charge of the game need to assure that their initial distribution of colors is sufficiently rich so that a solution exists, that is, so that there exists a sequence of exchanges based on the initial set of chips that result in a domino for each group member. Clearly, some color distributions will be insufficient and the game will be futile, for instance, if all the chips distributed are of the same color. However, for five players, say, it is not immediately clear if the game has a solution when the initial distribution is 7 chips of one color, 2 chips of another and 1 chip of the remaining color.

In answer to this question, Theorem 4.2 shows that there exist two special sets of chips, each only of size seven, such that no matter how large the size of the group, their survival can occur if, and only if, at least one of these sets is contained in the initial distribution. For emphasis, if only one of these special sets of size seven is distributed in a group of 1,000 players, every individual in the entire group, properly organized, and given sufficient time to make the necessary exchanges, could survive.

The understanding behind the fact that a large group’s survival can be assured by only a small set of chips leads in a natural way to a successful strategy that can be effected by an organized group. It relies on what we call ‘the $D^3$ machine’, as presented Theorem 4.1. In Section 5 we consider the number of exchanges required for a group to survive. We find that the use of $D^3$, which is sometimes necessary, is costly as measured by the number of exchanges, and hence, the time it requires. Depending on the initial color distribution of chips, much less costly, and hence faster, strategies can be applied. From the point of view
of the facilitator, the level of difficulty of the game can be set from easy to impossible by controlling the colors of the chips the group initially receives.

2 Set up and Notation

Given an initial distribution of chips, the game stops immediately unless the initial distribution of chips contains at least one chip of each of the three colors. If it does, then the only possible move is to exchange such a ‘full’ triple using Rule 1. The color requested in the exchange should clearly be the least frequent one in the collection that remains, as that choice gives the most opportunities for future exchanges. The same is true for the colors returned in Rule 2, thinking of the chips returned to the players given one at a time. One can equivalently consider the chips returned in exchanges as ‘joker’ chips that can play the role of any color. Indeed, one way to see this equivalence is to notice that the game is deterministic, and that given any strategy the colors requested for the returned chips will be the ones that can be determined, in advance, to be the ones that yield the maximum advantage for the players. Making this determination in advance has the same effect as making it at the time when the returned chips are used in an exchange, choosing their color at that time to be the one needed. Hence, a chip of any color, a joker. Use of the joker will allow us to avoid cumbersome and unnecessary bookkeeping.

Including the joker, the rules of exchange now become:

1. A set of three poker chips, one of each color, can be exchanged for one domino and a joker.

2. A set of three dominoes may be exchanged for seven jokers.

Let \( X \) represent the joker. A set with \( a, b \) and \( c \) chips of color \( A, B \) and \( C \) respectively, \( d \) dominoes and \( x \) jokers will be denoted by the string \( A^a B^b C^c D^d X^x \), where \( a \geq b \geq c \) by our convention. We make the further convention that order is unimportant and repetitions are allowed; for instance \( A^2 B^2 D^2 = A^2 D^2 B^2 \).

We let \( I \) denote the initial distribution of chips to the group. We may consider arbitrary, but finite, initial distributions of chips, in particular, those not corresponding to the game of \( p \) players, where each receives two colored chips. Naturally though, that special case is of our primary interest.

For strings \( E \) and \( F \) we write \( E \rightarrow F \), read as \( E \) yields \( F \), if \( F \) is obtained from \( E \) by making successive exchanges according to either Rule 1 or Rule 2; we also allow the trivial case of making no exchanges, so \( E \rightarrow E \). We will also say \( E \) achieves \( F \) if \( E \) yields a string of which \( F \) is a substring. Writing the two exchange rules in this notation, we have

1. \( ABC \rightarrow DX \)

2. \( D^3 \rightarrow X^7 \)

As jokers \( X \) can play the role of chips of any color, a consequence of Rule 1 is

3. The strings \( X^3, AX^2, BX^2, CX^2, ABX, ACX \) and \( BCX \) all yield \( DX \).
We say we invoke Rule 1 when using either Rule 1 or its consequence Rule 3.

Due to the presence of the joker, choices may arise when invoking Rule 1. For instance, the string $ABX^2$ can be transformed to $AXD$ by using Rule 1 on $BX^2$, and can also be transformed to $X^2D$, by applying Rule 1 on $ABX$. As $X$ can play the role of color, whatever is achievable by the first result may also be achieved by the second. Hence, as exchanges always return the same number of chips, those that use the least number of jokers are always preferred, leading to the imposition of the following policy that will be in force throughout.

**Maximum Principle:** Rule 1 exchanges should always be done using strings with thefewest number of jokers.

It is not difficult to see that the application of the maximal principle completely determines the players’ choice in any Rule 1 exchange. Indeed, if a Rule 1 exchange can be achieved by a given collection, than letting $r$ be the number of colors represented in the collection, we must exchange $r$ chips of the represented colors, and $3 - r$ jokers, to satisfy the maximum principle.

It is simple to verify that Rule 1 and Rule 2 commute, that is, for a string in which both rules can be applied, the order in which they are is unimportant. By imposing, as we do now, that in such cases Rule 1 should be applied before Rule 2, the entire sequence of exchanges in the game become fixed.

When applied to a group, we use the term ‘survival’ as shorthand for the survival of all players. Hence, in our notation, for a game of $p$ players the desired outcome is a superstring of $D^p$. We say a game of $p$ players is trivial if survival cannot be reached under any initial distribution $I$ of chips.

### 3 Rule 1 Restricted Games

To begin to understand the fuller picture, we will consider the game when only Rule 1 can be invoked. As the number of chips in the initial distribution $I$ is finite, and as any application of Rule 1 strictly decreases the number of chips, it is clear that play using only Rule 1 must terminate, that is, reach a state where no further Rule 1 exchange is possible. We refer to a string at such a final state as a terminal string, and we say it is maximal if it achieves the maximum number of dominoes for all choices of Rule 1 exchanges on $I$. Given the unique path mandated by the maximum principal, and the fact that its application leads to the greatest number of exchange options, the terminal strings obtained will be maximal.

The following result, Theorem 3.1, highlights a deficit of Rule 1. The intuition behind the proof starts with the observation that Rule 1, which gives a domino and a chip in return for three chips, sets the price of a domino at two chips. As survival costs two chips, and each individual is initially given two chips, it would seem at first glance that Rule 1 would suffice for survival, at least for some initial distributions of chips. However, for every individual to be able to exchange their two chips for a domino would require that the exchanges be efficient in the sense that all chips are converted to dominoes. But Rule 1 always returns a chip, implying that at all times at least someone in the group of players holds a chip. Hence, the entire collection of chips can never be entirely converted to dominoes. The following result formalizes, and quantifies, this reasoning.
Theorem 3.1 Let $n \geq 1$ denote the number of chips, of any kind or color, initially distributed in a game in which only Rule 1 may be invoked. With $d$ the number of dominoes and $y$ the number of chips at any point in the game, it holds that

$$n = 2d + y \quad \text{with} \quad y \geq 1, \quad \text{and} \quad y \geq 2 \quad \text{when} \quad n \text{ is even.} \quad (1)$$

No more than $p-1$ dominoes can be obtained under Rule 1 only play. Consequently, survival is not possible without invoking Rule 2, and all games with $p \leq 3$ players are trivial.

Proof: Claim (1) holds at the start of the game with $d = 0$ and $y = n$. Now say (1) holds at some point in the game before Rule 1 is to be invoked. In that case, $y \geq 3$, as Rule 1 requires three chips. At the end of the Rule 1 exchange, these three chips are replaced by one domino and one additional chip. Hence, $y$ is replaced by $y - 2$ and $d$ is replaced by $d + 1$. The new value $y - 2$ is therefore at least one, and the exchange leaves the sum $2d + y$ unchanged. Hence (1), with $y \geq 1$, holds throughout the game. When $n$ is even then $2d + y$ is even, so $y$ must be even. As $y \geq 1$ and even, we must have $y \geq 2$.

In the standard game $n = 2p$ is even. As the minimal value of $y$ is 2 in this case, the maximal value of $d$ is given by the solution to $2p = 2d + 2$, or $d = p - 1$. Hence the maximum number of dominoes that can be obtained under Rule 1 only play is $p - 1$, that is, survival is not possible only invoking Rule 1, or equivalently, when never invoking Rule 2.

To show that all games of $p \leq 3$ players are trivial it suffices to show that those games can never invoke Rule 2. To invoke Rule 2 requires that $d \geq 3$, implying by (1) that $2p = 2d + 2 \geq 8$, that is, that $p \geq 4$.

For a game with $p$ players, Theorem 3.1 gives the upper bound of $p - 1$ on the maximum number of dominoes that can be obtained using only Rule 1. The following result show that this upper bound is achieved when $I$ consists of only jokers. We will see at the end of this section that the upper bound is also achieved for some fortuitous distributions of colored chips.

Theorem 3.2 In a game where only Rule 1 may be applied, for all $x \geq 1$

$$X^x \rightarrow D^{d(x)}X^{1+\phi(x)} \quad \text{where} \quad d(x) = \frac{x - (1 + \phi(x))}{2} \quad \text{and} \quad \phi(x) = \begin{cases} 0 & \text{for } x \text{ odd} \\ 1 & \text{for } x \text{ even.} \end{cases}$$

This transformation of $X^x$ is accomplished with $d(x)$ applications of Rule 1. When $2p$ jokers are distributed, $p - 1$ dominoes result.

Proof: It is clear from the definitions that $d(1) = d(2) = 0$ and $d(3) = 1$, and we see easily see by the structure of Rule 1 that the claim of the theorem holds for these values; for instance, starting with two or fewer jokers, no exchanges can be made. Assume, as an induction hypotheses, that the theorem holds for all positive integers strictly less than some $x \geq 4$.

Uniquely write $x = 3m + r$ for $m \geq 1$ and $r \in \{0, 1, 2\}$. Applying Rule 1 a total of $m$ times yields

$$X^x = X^{3m+r} \rightarrow D^mX^{m+r}.$$ 

As $m + r < 3m + r = x$, the induction hypothesis yields

$$D^mX^{m+r} \rightarrow D^{m+d(m+r)}X^{1+\phi(m+r)}.$$
As the value of \( \phi(x) \) is unchanged when adding any even number to \( x \), we have \( \phi(m + r) = \phi(3m + r) = \phi(x) \), and hence also that

\[
m + d(m + r) = m + \frac{m + r - (1 + \phi(m + r))}{2} = \frac{x - (1 + \phi(x))}{2} = d(x),
\]
completing the induction step.

As each domino is achieved by a single application of Rule 1, the total number of times Rule 1 is invoked must equal \( d(x) \), the number of dominoes achieved. That \( d(2p) = p - 1 \) is immediate from the form of \( d(x) \). \( \square \)

To see that \( p - 1 \) is in fact achievable in the standard game where \( I \) consists of only colored chips, consider, for instance the case where \( 2p = 3m + r \) for \( m \geq 1 \) odd. Since \( 2p \) is even we must have \( r = 1 \), and we take \( I = A^{m+1}B^mC^m \). Applying Rule 1 a total of \( m \) times, followed by Theorem 3.2, we obtain the upper bound \( p - 1 \) via

\[
A^{m+1}B^mC^m \rightarrow AD^{m}X^m \rightarrow AD^{m+d(m)}X = AD^{(3m-1)/2}X = AD^{p-1}X.
\]
We leave it to the interested reader to characterize all starting strings that achieve the maximal possible number \( p - 1 \) of dominoes making exchanges only using Rule 1, and more generally, to determine all terminal sequences obtained from a given string, using only Rule 1. Theorem 3.2, that solves this problem for the string \( X^x \), gives a starting point for the exercise.

4 The \( D^3 \) Machine, and Minimal Sufficient Sequences

Section 3 shows how applying Rule 1 alone cannot lead to success. Along with Rule 2 though, the ‘\( D^3 \) machine’ given in Theorem 4.1 can be powered, which has the surprising feature of begin able to produce a collection of dominoes of arbitrary size. At its core lie the exchanges in (2). We answer the question of when it can be applied in Theorem 4.2, that shows that, if given sufficient time, the survival of the group can occur if and only if at least one of two special sets of chips is a substring of the initial configuration \( I \).

Theorem 4.1 (The \( D^3 \) machine) For \( r \in \{0, 1\} \), the string \( D^3X^r \) yields \( D^3X^{r+1} \) in four exchanges, and \( D^4X \) in nine exchanges when \( r = 1 \). For all \( p \geq 4 \) we have \( D^3 \rightarrow D^pX \), and a group of size \( p \) can achieve \( D^p \) if and only if \( I \) achieves \( D^3 \).

Proof: Applying Rule 2 on \( D^3X^r \) for the first exchange, and then Rule 1 for the remaining three exchanges, yields

\[
D^3X^r \rightarrow X^{7+r} \rightarrow DX^{5+r} \rightarrow D^2X^{3+r} \rightarrow D^3X^{1+r},
\]
showing the first claim. Applying this transformation twice, at a cost of 8 exchanges, we obtain

\[
D^3X^r \rightarrow D^3X^{r+1} \rightarrow D^3X^{r+2}.
\]

When \( r = 1 \) applying Rule 1 once to the result of (3) yields, for a total of 9 exchanges, the sequence

\[
D^3X \rightarrow D^3X^3 \rightarrow D^4X,
\]
as claimed.

Applying the exchanges in (2) with \( r = 0 \) three times, followed by another application of Rule 1 gives,

\[ D^3 \to D^3X \to D^4X. \]  

(4)

Hence \( D^3 \to D^pX \) for \( p = 4 \). Assuming this same claim holds for some \( p \geq 4 \), by arguing similarly as in (4), we see that it also holds for \( p + 1 \) by starting with \( D^pX \) and applying (2) with \( r = 1 \) twice.

For the final claim, we have already shown that if \( p \geq 4 \) and \( I \) achieves \( D^3 \) then it can achieve \( D^p \). Conversely, if \( I \) does not achieve \( D^3 \), then Rule 2 can never be invoked, and Theorem 3.1 shows that survival is not possible. \( \square \)

The steps in (2) for \( r = 0 \) could well be called ‘\( X \)-mining’, as a joker seems to be produced from nowhere, taking as input \( D^3 \) and giving the output \( D^3X \). However, if we recall that the cost of a domino, as fixed by Rule 1, is two chips, the exchange of three dominoes from seven chips reveals the source of the benefit.

We now turn to the question of which initial strings lead to the survival of all group members, given sufficient time to make exchanges. We say that the string \( E \) is sufficient for \( D^3 \), or just sufficient, if \( E \) achieves \( D^3 \). We say \( E \) is minimal sufficient if \( E \) is sufficient and no substring of \( E \) achieves \( D^3 \).

**Theorem 4.2** There are exactly two minimal sufficient strings, \( A^3B^3C \) and \( A^3B^2C^2 \). For a group given sufficient time to make exchanges, survival can occur if and only if at least one of these strings is a substring of \( I \). Games with \( p \) players are non-trivial if and only if \( p \geq 4 \).

**Proof:** Let \( I = A^aB^bC^c \) be an arbitrary initial distribution, where by convention \( a \geq b \geq c \). No exchanges can be made unless \( c \geq 1 \), in which case only Rule 1 may be invoked, leading to

\[ A^aB^bC^c \to A^{a-1}B^{b-1}C^{c-1}DX. \]  

(5)

Now consider the case \( c = 1 \). No exchanges can be made from (5) unless \( b \geq 2 \), and then the only option available for continuation is to apply Rule 1 to obtain

\[ A^{a-1}B^{b-1}DX \to A^{a-2}B^{b-2}D^2X. \]

Now we must impose the condition that \( b \geq 3 \) in order to continue, in which case we obtain, again using Rule 1,

\[ A^{a-2}B^{b-2}D^2X \to A^{a-3}B^{b-3}D^3X. \]

Hence, the minimal string that can achieve \( D^3 \) when \( c = 1 \) is therefore the one with \( a = 3, b = 3 \), that is, the string \( M = A^3B^3C \).

In the remaining case \( c \geq 2 \), and starting from the right hand side of (5), the maximal principle dictates the exchange

\[ A^{a-1}B^{b-1}C^{c-1}DX \to A^{a-2}B^{b-2}C^{c-2}D^2X^2. \]
At this point, we must impose the condition that $a \geq 3$ for the process to continue. Strings with $b \geq 3$ will contain $A^3B^3C$ as a substring, so will achieve $D^3$, but will not be minimal. Hence, we are left with the constraints $a \geq 3, b = 2, c = 2$, for which we may make the exchange

$$A^{a-2}D^2X^2 \rightarrow A^{a-3}D^3X.$$ 

Hence the minimal string that achieves $D^3$ when $c \geq 2$ is $N = A^3B^2C^2$. As the cases considered are exhaustive, $\{M, N\}$ is the set of minimal sufficient strings.

When at least one of $M$ or $N$ is a substring of $I$, then $I$ achieves $D^3$ and survival can occur by Theorem 4.1. Conversely, if neither $M$ nor $N$ is a substring of $I$, and survival occurs, then $I$ achieves $D^3$ by Theorem 3.1. In that case, $I$, or some substring of $I$, is a minimal sufficient string that does not contain $M$ or $N$ as a substring, which is a contradiction.

For the final claim on non-triviality, Theorem 3.1 gives that games with $p \leq 3$ players are trivial. The converse obtains noting that the minimal sufficient sets of chips have size 7, and hence can be distributed to games having $p \geq 4$ players. □

Given these results, one strategy for survival is for the group to locate one of the minimal sufficient sets in their distribution; if they find neither, it is game over. When this set is $A^3B^3C$, or $A^3B^2C^2$, they should respectively make the three Rule 1 exchanges

$$A^3B^3C \rightarrow A^2B^3DX \rightarrow ABD^2X \rightarrow D^3X \quad \text{or} \quad A^3B^2C^2 \rightarrow A^2B^2CDX \rightarrow AD^2X^2 \rightarrow D^3X,$$

and then fire up the $D^3$ machine, making the moves specified in Theorem 4.1. How much work is in store from them is answered in the next section, where we find that, depending on $I$, some of the use of the $D^3$ machine may be replaced by a more efficient, and thus faster, plan.

## 5 Complexity

Here we consider the issue of computing $e(p)$, the number of exchanges required for a group of size $p \geq 4$ to achieve success. As $e(p)$ depends on the initial distribution of chips, we focus on the two extreme cases, the least favorable one where the group starts from a minimal sufficient string and the remaining chips of a single color, and the most favorable case, where $I$ contains as many ‘full strings’ $ABC$ as possible. The order of growth of $e(p)$ in $p$ tells us how complex the problem is to resolve in practice.

For the first case, start with either minimal sufficient string of Theorem 4.2, both of size 7, and the remaining $2p - 7$ chips of a single color. It will not be difficult to see from what follows that the amount of work does not depend on the color of these remaining chips, and we take them to be $A^{2p-7}$. As was shown at the end of the previous section, 3 exchanges are needed to convert either minimal sufficient string into $D^3X$. Next, by Theorem 4.1 for $r = 1$, four exchanges are needed to convert $A^{2p-7}D^3X$ to $A^{2p-7}D^3X^2$ and a single further Rule 1 exchange to obtain $A^{2p-8}D^4X$, for a total of five. As these five exchanges need to be run $p - 3$ times to achieve $D^p$, we have

$$e(p) = 5(p - 3) + 3 = 5p - 12 \quad \text{for } p \geq 4. \quad (6)$$

Hence the complexity of the problem is linear in its size. Nevertheless, in practice this plan may be cumbersome to implement. For instance, the group of 48 players considered...
in the introduction who were given 15 minutes to play, having only the worst case string $I$ at their disposal, are required to make $e(48) = 228$ exchanges. Even if exchanges could be made every 5 seconds with no errors or delays, performing this many would take 19 minutes. And exchanges at this pace may be unrealistic, especially in the version of the game without a joker, where the players need to keep track of the current least frequent color.

On the other hand, the group may be given an initial distribution of chips with many copies of the ‘full’ string $ABC$ that can be traded for a domino and a joker, in one move. Obtaining a domino in this way is more economical than using the $X$-mining steps from the $D^3$ machine at a cost of 5. To find the maximum amount of advantage that would be gained by making use of full strings, consider the case where $p$ is some multiple of three, so that we may write $2^p = 3^m$, and consider the ‘best case’ initial distribution that has the maximum number of full strings, $A^m B^m C^m$; other cases where the maximum number of full strings are initially distributed is left for the reader. Here, for non-triviality, we must take $p \geq 6$, the smallest multiple of 3 that is at least 4.

Applying $m$ Rule 1 exchanges, followed by Theorem 3.2, and the observation that $m$ must be even, yields

$$A^m B^m C^m \rightarrow D^m X^m \rightarrow D^{m+d(m)} X^{1+\phi(m)} = D^{p-1} X^2.$$  \hfill (7)

By Theorem 3.2 the cost of reaching this state is $m + d(m) = p - 1$. To continue we must invoke Rule 2, which is possible as $p - 1 \geq 3$. To obtain the final domino we then apply Theorem 3.2 again, yielding the sequence

$$D^{p-1} X^2 \rightarrow D^{p-4} X^9 \rightarrow D^{p-4+d(9)} X^{1+\phi(9)} = D^p X.$$ \hfill (8)

The Rule 2 exchange is followed by $d(9) = 4$ applications of Rule 1, by Theorem 3.2, for a total of 5 exchanges. Hence, adding in the cost of $p - 1$ for (7) we obtain

$$e(p) = p + 4.$$  

The order $e(p)$ is still linear in $p$, but the multiplier has dropped from 5 to 1, which is quite a savings. Returning to our group of $p = 48$ players, satisfying $2p = 3m$ for $m = 32$, we see that with this most favorable $I$, the work decreases from 228 exchanges to 52, and the time required, at 5 seconds per exchange, from 19 minutes to just over 4. For a game fixed at 15 minutes, the facilitator can make the game anywhere from near impossible to completely easy.

A rough computation shows one can expect this type of savings quite generally in the case where $I$ is most favorable. To avoid the handling of cumbersome ‘boundary cases’ we let $m \geq 3$, so that $p \geq 6$. Using only the substring $D^{m+d(m)} X$ produced in (7), the $D^3$ machine can be used to produce the remaining needed $p - (m + d(m))$ dominoes, by Theorem 4.1, at 9 exchanges per use. Writing $2p = 3m + r$, for $r \in \{0, 1, 2\}$ we obtain the upper bound

$$e(p) \leq m + d(m) + 9(p - m - d(m)) = 9p - 8(m + d(m)) \leq 9p - 8(3m/2 - 1) = 9p - 8(p - r/2 - 1) = p + 4r + 8 \leq p + 16,$$

using $d(m) \geq m/2 - 1$ for the first inequality, that $m = (2p - r)/3$, and then $r \leq 2$ for the final inequality. Comparing this upper bound with the precise value of $e(p)$ given in (6) in
one special case gives some reason to hope that this ‘rough’ bound isn’t very off from the truth in general.

By considering these two extremes we see that, in all cases, the facilitator has a great deal of control in the amount of work demanded of the group, ranging from an order that grows as 5 times the group size to a case where it grows no more than the size of the group, plus a fixed, reasonably sized number.

6 Conclusion

The game of chips, dominoes and survival is an exciting team building exercise that, due to Theorem 3.1, enforces that the group cooperate at a high level. The existence of two small sets sufficient for stoking the $D^3$ machine and, guaranteeing the survival of a group of any size, given sufficient time to make exchanges, may be somewhat surprising. The group can take advantage of the full sets in their initial distribution, but only to the extent that a facilitator can control. In cases where the least favorable set of chips is distributed, the abundance of chips in larger games may contribute more to disorder than to quick resolution.

This game and its analysis, though on the one hand simple enough, contains glimmerings of a much wider landscape, including cooperative game theory, logical reasoning, induction, recursion, number theory and the computation of algorithmic complexities. We hope readers not only have fun playing and facilitating this game, but also in dreaming up extensions with higher level pieces and more complex exchange rules, and seeing what new and surprising properties these may have in store.

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