Geometrical model of massive spinning particle in four-dimensional Minkowski space

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Abstract. We propose the model of a massive spinning particle traveling in four-dimensional Minkowski space. The equations of motion of the particle are obtained from the requirement that its classical paths lie on a cylinder with the time-like axis in Minkowski space. All the paths on one and the same cylinder are gauge equivalent. The equations of motion are found in implicit form for general time-like paths, and they are non-Lagrangian. The explicit equations of motion are derived for trajectories with small curvature and helices. The momentum and total angular momentum are expressed in terms of characteristics of the path in all the cases. The constructed model of the spinning particle has a geometrical character, with no additional variables in the space of spin states being introduced.

1. Introduction
The classical spinning particle is a point object in space-time that is equipped with the internal degrees of freedom. This concept is a useful tool for the classical and semiclassical description of dynamics of elementary particles and rotating bodies in flat space-time and external fields. Among the numerous applications, we can mention the high energy [1, 2] and accelerator physics [3], and also spintronics [4]. In general relativity, the spinning particle concept is used in the analysis of the Lense-Thirring precession [5] and spin effects in the compact binaries and black holes [6, 7], including the dynamics of so-called extreme-mass-ratio inspirals [8]. The cosmological applications of spinning particles are considered in [9, 10]. For the recent progress of the classical relativistic theory of spin see the articles [11–13] and references therein.

A large variety of spinning particle models is known. In many of them [1–6, 8, 11–17] the particle’s configuration space includes besides the positions in space-time, the positions in the “internal space”. Among the possibilities the vector, tensor, spinor and twistor types of spinning particle models are distinguished [18]. The geometric theories with no “internal space” are best suited for study of general properties of particle’s propagation in space-time. In this class of models, the spin is described in terms of derivatives of classical path. The geometrical models inevitably involve higher derivatives in the Lagrangian. For examples we mention the papers [19–22]. The common feature shared by all the geometrical models is that the massless theories are irreducible\(^\dagger\), while the massive theories describe reducible particles.

\(^\dagger\) The spinning particle model is irreducible if its states are points of the co-orbit of the Poincare group. The irreducible spinning particle models describe the dynamics of elementary particles with spin.
The possible paths of an irreducible massive spinning particle can be described by non-Lagrangian equations of motion. For example, in the papers [23, 24] the partial gauge fixing was applied to propose the spinning particle theory with isotropic paths. The most general approach to construction of geometrical models of spinning particles has been developed in the paper [25]. The idea of the proposed world sheet concept is that the positions of irreducible particle lie on a certain cylindrical surface in Minkowski space. In $d = 3, 4$ the world sheets are $2d$ cylinders with the time-like axis. The position of the cylinder in space is determined by the values of particle’s linear momentum and total angular momentum. Particle’s equations of motion follow from the fact that all the paths on one and the same cylinder are gauge equivalent. The geometrical model of massive spinning particle traveling light-like path in 3d Minkowski space has been developed in the original work [25].

In the present research, we construct a geometrical model of spinning particle in 4d Minkowski space. We solve the following problems: classification of possible world sheets and derivation of particle’s equations of motion. Once the physical space-time is four-dimensional, we believe that the obtained classical motion mimic the propagation of the elementary particles with spin in the real world. The content of the article is based on the paper [26], but more attention is paid to the geometry of particle’s paths. In particular, we consider the cylindrical curves with small curvature, and also helical paths. The idea of derivation of differential equations for particle’s trajectories is borrowed from the papers [29, 30], while the necessary information concerning the differential geometry of curves in non-Euclidean spaces can be found in the books [27, 28].

The rest of the article is organized as follows. In Section 2, we describe the geometry of world sheets of a massive spinning particle in four-dimensional Minkowski space. In Section 3, we derive the equations for general time-like path on the world sheet. The equations are derived in the implicit form. In Section 4, some particular types of cylindrical curves with the explicit equations of motion are considered.

2. World sheets and world paths

The geometry of the world sheet of massive spinning particle in four-dimensional space-time has been described in ref. [26]. It has been shown that the world surface is determined by two algebraic equations

$$\left( x - y \right)^2 + \left( n, x \right)^2 = R^2, \quad \left( x - y, a \right) = \Delta, \quad (1)$$

where $x^\mu, \mu = 0, 3$ are the coordinates in Minkowski space, and $R, \Delta$ are constants. The vector parameters $n, y, a$, are subjected to the constraints

$$\left( n, n \right) = -1, \quad \left( a, a \right) = 1, \quad \left( n, y \right) = \left( a, y \right) = 0. \quad (2)$$

Hereinafter, the round brackets denote the scalar product in Minkowski space,

$$\left( u, w \right) = u_\mu w^\mu, \quad \forall u, w. \quad (3)$$

We use the mostly positive convention for the signature of the metric throughout the paper. All the world indices are raised and lowered by the metric.

The geometric meaning of the equations (1) is following. The first relation determine the circular hypercylinder with time-like axis in Minkowski space. The tangent vector to the cylinder axis is $n$, and the vector $y$ connects the cylinder axis and origin by the shortest path. The radius of hypercylinder is $R$. The second equation (1) determines the hyperplane with the normal. The hypercylinder axis is tangent to this hyperplane. The distance between hypercylinder axis
and hyperplane is $\Delta$. The intersection of the hypercylinder and hyperplane is two-dimensional cylinder (in the further text - cylinder) with the axis tangent vector $n$. The radius of the cylinder reads

$$r = \sqrt{R^2 - \Delta^2}. \quad (4)$$

It is a real quantity if

$$0 \leq |\Delta| \leq R. \quad (5)$$

Otherwise, the intersection of the hypercylinder and hyperplane is empty. In the special case $r = 0$, the cylinder becomes a straight line. We do not consider this possibility below.

The linear momentum $p = p_\mu dx^\mu$ and total angular momentum $J = J_{\mu \nu} dx^\mu \wedge dx^\nu$ of the particle are determined by the world sheet position in the Minkowski space,

$$p = mn, \quad J = my \wedge n + s\varepsilon_{\mu \nu \rho \sigma} n^\mu a^\nu dx^\rho \wedge dx^\sigma, \quad (6)$$

where $\wedge$ denotes the exterior product, and $\varepsilon_{\mu \nu \rho \sigma}, \varepsilon_{0123} = 1$ is the Levi-Civita symbol. The constants $m, s$ are the mass and spin of the particle. The details of the derivation of the formulas (6) are given in [26]. With account of relations (2), the states of the spinning particle belong to the co-orbit of the Poincare group,

$$p^2 + m^2 = 0, \quad w^2 = m^2 s^2, \quad (7)$$

where $w = sa$ is the Pauli-Lyubanski pseudovector. The spin tensor is determined by the standard rule $S = J - x \wedge p$, form which we find

$$S = m(y - x) \wedge n + s\varepsilon_{\mu \nu \rho \sigma} n^\mu a^\nu dx^\rho \wedge dx^\sigma. \quad (8)$$

By construction,

$$\frac{1}{2} S_{\mu \nu} S^{\mu \nu} = s^2 - m^2 R^2, \quad \frac{1}{8} \varepsilon_{\mu \nu \rho \sigma} S^{\mu \nu} S^{\rho \sigma} = -ms\Delta. \quad (9)$$

The right hand sides of these equalities are constants because of irreducibility conditions of the Poincare group representation [26].

The classical trajectories of spinning particles are curves on the world sheet. The casuality condition $\dot{x}^0 > 0$ is imposed on the physical path. We term the path locally regular if each its small segment lies on a unique representative in set of cylinders. Almost all the cylindrical paths are locally regular. Only locally regular paths are considered in this article.

3. Classification of cylindrical curves

In this section, we obtain the system of equations describing the cylindrical curves in four-dimensional Minkowski space.

The starting point of our consideration is that each cylindrical curve $x = x(\tau)$ lies on a certain cylinder in the set (1). In this case, the equations (1) are identically satisfied for all the values of the parameter $\tau$ on the curve. The differential consequences of these identities have the form

$$\frac{d^k}{d\tau^k} \left( (x - y)^2 + (n, x)^2 - R^2 \right) = 0, \quad \frac{d^k}{d\tau^k} \left( (x - y, a) - \Delta \right) = 0, \quad k = 1, 2, 3, \ldots. \quad (10)$$

Once the differential consequences of sufficiently large order are included, the cylinder parameters can be expressed as functions of derivatives of the path,

$$n = n(x, \dot{x}, \ddot{x}, \ldots), \quad y = y(x, \dot{x}, \ddot{x}, \ldots), \quad a = a(x, \dot{x}, \ddot{x}, \ldots), \quad (11)$$
where the dots denote the derivatives by \( \tau \). The existence of unique solution (11) of the equations (10) uses the regularity assumption for the cylinder path. For singular path, the solution is not necessary unique. That is why the singular cylindrical paths are excluded. The substitution of the cylinder parameters (11) into the original equations gives us the equations for the cylindrical curves. The cylinder parameters are integrals of motion of this equation, which can be treated as the equation of motion of spinning particle. The linear momentum and angular momentum of the particle are defined by the rule (6).

We can proceed with explicit derivation of the equations of cylindrical path. It is sufficient to differentiate equations (1) four times. In this case, the differential consequences (10) have the form

\[
\begin{align*}
&\dot{x}, a = 0, \quad (\dot{x}, a) = 0, \quad (\ddot{x}, a) = 0, \quad (\dddot{x}, a) = 0; \\
&\dot{x}, d = 0, \quad (\dot{x}, d) + (\dot{x}, n)^2 = 0, \quad (\ddot{x}, d) + 3(\dot{x}, n)(\dot{x}, n) = 0, \\
&(\dddot{x}, d) + 4(\dddot{x}, n)(\dddot{x}, n) + 3(\dddot{x}, n)^2 = 0,
\end{align*}
\]

(12)

where the notation is used,

\[
d \equiv x + n(n, x) - y.
\]

(14)

The vector \( d \) connects the current position of the particle and hypercylinder axis by the shortest path. To simplify the formulas, we assume that the path is time-like and the parameter \( \tau \) is natural on the curve,

\[
(\dot{x}, \dot{x}) = -1.
\]

(15)

The relations (12), (13) constitute the system of equations (10) for finding of cylinder parameters \( n, y, a \).

The normalized vector \( a \) can be expressed from equations (12),

\[
a = \pm \frac{[\dot{x}, \ddot{x}, \dddot{x}]}{\sqrt{[\dot{x}, \ddot{x}, \dddot{x}]^2}}, \quad [\dot{x}, \ddot{x}, \dddot{x}] \equiv \varepsilon_{\mu\nu\rho\sigma} \dot{x}^\mu \ddot{x}^\nu \dddot{x}^\rho dx^\sigma,
\]

(16)

where the square brackets denote the vector product of three vectors in Minkowski space. The consistency condition for equations (12) reads

\[
([\dot{x}, \ddot{x}, \dddot{x}], \dddot{x}) \equiv (\dot{x}, \ddot{x}, \dddot{x}) = 0,
\]

(17)

where the round brackets denote the mixed product of four vectors in Minkowski space. From the geometrical viewpoint, the obtained condition implies zero torsion of the curve \( x(\tau) \). It ensures that this curve lies in some hyperplane, with \( a \) being its normal.

Now we consider the hypercylinder condition and its differential consequences. We have nine unused conditions (1), (2), (13) to express two vectors \( n, y \) and find a single consistency condition. Introduce the Frenet moving frame for time-like curves [28],

\[
\begin{align*}
e_1 &= \dot{x}, \quad e_2 = \frac{\ddot{x}}{\sqrt{\dot{x}^2 + \ddot{x}^2}}, \\
e_3 &= \frac{\dot{x}^2 \dot{x} + (\dot{x}, \ddot{x}) \dot{x} - (\dot{x}, \ddot{x}) \ddot{x}}{\sqrt{(\dot{x}, \ddot{x})^2 + (\dot{x}, \ddot{x})^4 - (\dot{x}, \ddot{x})^2 \ddot{x}^2}}, \quad e_4 = \frac{[\dot{x}, \ddot{x}, \dddot{x}]}{\sqrt{(\dot{x}, \ddot{x}, \dddot{x})^2}}.
\end{align*}
\]

(18)

All the basis vectors of Frenet frame are normalized and orthogonal to each other,

\[
(e_1, e_1) = -1, \quad (e_2, e_2) = (e_3, e_3) = (e_4, e_4) = 1, \\
(e_i, e_j) = 0, \quad i \neq j.
\]

(19)
The curvatures \( k_1, k_2, \) and torsion \( k_3 \) of the curve are determined by the rule
\[
\begin{align*}
k_1 &= \sqrt{\langle \dot{x}, \dot{x} \rangle}, \\
k_2 &= -\frac{1}{\langle \dot{x}, \dot{x} \rangle} \sqrt{\langle \dot{x}, \ddot{x} \rangle \langle \ddot{x}, \dot{x} \rangle - \langle \dot{x}, \ddot{x} \rangle^2 - \langle \dot{x}, \dot{x} \rangle^3}, \\
k_3 &= \frac{\langle \dot{x}, \dot{x}, \dddot{x} \rangle}{\sqrt{\langle \dot{x}, \dot{x} \rangle - \langle \dot{x}, \dot{x} \rangle^2 - \langle \dot{x}, \dot{x} \rangle^3}}. 
\end{align*}
\]

The time derivatives of particle’s position can be expressed as the linear combinations of the Frenet basis vectors (18) with the coefficients depending on the curvatures and torsion of path,
\[
\dot{x} = e_1, \quad \ddot{x} = k_1 e_1, \quad \dddot{x} = k_1^2 e_1 + \dot{k}_1 e_2 + k_1 k_2 e_3, \\
\begin{align*}
\dddot{x} &= 3 \dot{k}_1 k_1 e_1 + (\dot{k}_1 + k_1^3 - k_1 k_2^2) e_2 + (2 k_1 k_2 + k_1 \dot{k}_2) e_3 + k_1 k_2 k_3 e_4, 
\end{align*}
\]

The unknown vectors \( n, d \) admit the following representation:
\[
n = \beta_1 e_1 + \beta_2 (-\alpha_2 e_2 + \alpha_1 e_3), \quad d = r(\alpha_1 e_2 + \alpha_2 e_3) \pm \Delta e_4, 
\]
where \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are new unknowns. Substitution of the expressions (21), (22) into (1), (2), (13) brings us the following system of algebraic equations:
\[
\begin{align*}
\alpha_1^2 + \alpha_2^2 &= 1, \\
\beta_1^2 - \beta_2^2 &= 1, \\
r \dot{k}_1 \alpha_1 + \beta_2^2 &= 0, \\
r(\dot{k}_1 \alpha_1 + k_1 k_2 \alpha_2) + 3 k_1 \alpha_2 \beta_1 \beta_2 &= 0, \\
r^2 (\dot{k}_1 + k_1^3 - k_1 k_2^2) \alpha_1 + (2 k_1 k_2 + k_1 \dot{k}_2) \alpha_2 + 4 (\dot{k}_1 \alpha_2 - k_1 k_2 \alpha_1) \beta_1 \beta_2 - k_1^2 (3 \alpha_1^2 - 7 \beta_2^2) - 3 &= 0.
\end{align*}
\]

The other conditions are automatically satisfied.

The general method of solution of systems of algebraic equations uses techniques of resultants. In this approach, all the variables except one are eliminated, while all the other unknowns are expressed as functions of this single variable from subresultants. For system (23), \( \beta_1 \) is a good candidate for the independent variable. It has the sense of the hyperbolic tangent of the angle \( \theta \) between the element of cylinder and particle’s path,
\[
\beta_1 = \text{ch} \theta. 
\]

Eliminating the variables \( \alpha_1, \alpha_2, \beta_2 \) from the system (23) in two different ways, we get two algebraic equations of order 16 and 24 for \( \beta_1, \)
\[
P_1(\beta_1) = 81 k_1^4 \beta_1^{16} - 162 k_1^2 \beta_1^{14} + r^2 (18 k_1^4 k_2^2 - 18 k_1^2 k_2^2 + 81 k_1^4) \beta_1^{12} + r^2 (18 k_1^4 k_2^2 - 18 k_1^2 k_2^2) \beta_1^{10} + r^4 (k_1^4 k_2^2 + \dot{k}_1^4 + 2 k_1^2 k_2^2 - 18 k_1^4 k_2^2) \beta_1^8 + r^4 (18 k_1^4 k_2^2) \beta_1^6 - r^5 (2 k_1^2 k_2^4 + 2 k_1^2 k_2^4) \beta_1^4 + r^6 k_1^4 k_2^2 = 0, \\
P_2(\beta_1) = 81 k_1^4 \beta_2^{24} + 324 k_1^4 \beta_2^{22} + 260 \text{ terms} = 0. 
\]

The unknown \( \beta_1 \) is the positive common root of these polynomials. We can’t express it in the explicit form, but we can argue that it is a function of first and second curvatures \( k_1, k_2 \) with its derivatives,
\[
\beta_1 = \beta_1(k_1, k_2, \dot{k}_1, \ddot{k}_1). 
\]
As the curvatures are functions of the derivatives of the path, this solution involves derivatives of $x$ up to fourth order. The other unknowns are expressed using $\beta_1$,

$$
\alpha_1 = \frac{1 - \beta_1^2}{rk_1}, \quad \alpha_2 = \frac{2r\dot{k}_1 k_1^{-2} k_2 (1 - \beta_1^2)((1 - \beta_1^2)^2 - r^2 k_1^2)}{r^2 k_1^4 (1 - \beta_1^2)^2 + k_1^2 (9\beta_1^2 (1 - \beta_1^2) - r^2 k_2^2)((1 - \beta_1^2)^2 - r^2 k_1^2)},
$$

$$
\beta_2 = -\frac{r^2 k_1^2 (1 - \beta_1^2)^2 + k_1^2 (9\beta_1^2 (1 - \beta_1^2) - r^2 k_2^2)((1 - \beta_1^2)^2 - r^2 k_1^2)}{6r k_1^2 k_2^2 \beta_1 ((1 - \beta_1^2)^2 - r^2 k_1^2)}.
$$

(27)

The momentum and total angular momentum of the particle are expressed from equations (6) and (27),

$$
p = m\left[\beta_1 e_1 + \frac{k_1 (1 - \beta_1^2)}{3r k_1^2 \beta_1} e_2 - \frac{r^2 k_1^2 (1 - \beta_1^2)^3 + k_1^2 (9\beta_1^2 (1 - \beta_1^2) - r^2 k_2^2)((1 - \beta_1^2)^2 - r^2 k_1^2)}{6r^2 k_1^2 k_2 \beta_1 ((1 - \beta_1^2)^2 - r^2 k_1^2)} e_3\right],
$$

$$
J = m(x - d) \wedge n \pm s \left[\beta_1 e_2 \wedge e_3 - \frac{k_1 (1 - \beta_1^2)}{3r k_1^2 \beta_1} e_1 \wedge e_3 - \frac{r^2 k_1^2 (1 - \beta_1^2)^3 + k_1^2 (9\beta_1^2 (1 - \beta_1^2) - r^2 k_2^2)((1 - \beta_1^2)^2 - r^2 k_1^2)}{6r^2 k_1^2 k_2 \beta_1 ((1 - \beta_1^2)^2 - r^2 k_1^2)} e_1 \wedge e_2\right].
$$

(28)

By construction, $p$ and $J$ belong to the co-orbit (7) of the Poincare group. This ensures the irreducibility of the particle that moves the cylindrical path. The equation of hypercylindrical curves is the consistency condition of equations (25). It is given by the resultant of two polynomials,

$$
\text{Res}_{\beta_1} (P_1, P_2) = 0.
$$

(29)

The explicit expression of this resultant is too long and not informative. As a matter of principle, we mention that the resultant involves the derivatives of the path up to fourth order. So the curves on the hypercylinder are described by the fourth-order equation.

The paths of spinning particle lie on the intersection of hypercylinder and hyperplane. Such paths meet both the conditions (17) and (29). This is a system of two equations of fourth order. The model of spinning particle, being described by these equations, has the geometrical character because no additional variables in the spin sector are introduced.

4. Some particular cases

Let us consider some particular cases of cylindrical curves such that the system (23) can be solved explicitly.

At first, suppose that the path of particle is close to the straight line. The curvatures and derivatives are small quantities of same order of magnitude:

$$
r k_1 \sim r k_2 \sim r^2 k_1 \sim r^2 k_2 \sim r^3 \dot{k}_1 \ll 1.
$$

(30)

The parameters $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ are sought in the form

$$
\alpha_2 = -1 + o(\varphi), \quad \alpha_1 = \varphi + o(\varphi^2), \quad \beta_1 = 1 + o(\theta), \quad \beta_2 = \theta + o(\theta^2),
$$

(31)

where the parameters $\varphi, \theta$ have the same order of magnitude as (30).
Substituting the ansatz (31) into the equations (23) and accounting (30), we get the following equations:

\[ rk_1\dot{\varphi} + \vartheta^2 = 0, \quad r\dot{k}_1\varphi - 3k_1\theta = rk_1k_2, \quad r\ddot{k}_1\varphi - 4\dot{k}_1\theta = r(2k_1k_2 + k_1\dot{k}_2) - 3k_1^2. \]  (32)

This system includes three equations for two unknowns \( \varphi, \theta \). These quantities can be expressed from the linear equations,

\[
\varphi = \frac{3rk_1(2k_1k_2 + k_1\dot{k}_2) - 4rk_1k_2\dot{k}_1 - 3k_1^2}{3k_1k_1 - 4\dot{k}_1^2}, \quad \theta = \frac{rk_1(2\dot{k}_1k_2 + k_1\ddot{k}_2) - 3k_1^2\dot{k}_1}{3k_1k_1 - 4\dot{k}_1^2}.
\]  (33)

This solution exists if the regularity condition for the curve is satisfied,

\[ 3k_1^2k_1 - 4\dot{k}_1^2 \neq 0. \]  (34)

The consistency condition for the equations of motion (32) has the form

\[ rk_1^2[3r(2\dot{k}_1k_2 + k_1\ddot{k}_2) - 4r=k_2\dot{k}_1 - 3k_1^2] + [r(2k_1k_2 + k_1\dot{k}_2) - r\dot{k}_1k_2\dot{k}_1 - 3k_1^2\dot{k}_1]^2 = 0. \]  (35)

This forth order equation for the hypercylindrical curves with small curvature.

The momentum and angular momentum of the particle moving the path with small curvature have the form

\[
p = me_1 + o(1), \quad J = m(x + re_2) \land e_1 \pm se_2 \land e_3 + o(1),
\]  (36)

where the terms labeled \( o(1) \) vanish in the straight line approximation. The obtained formulas are analogue of the formulas (28) in the small curvature limit. The system of equations (17), (35) is a system of two equations of motion of the massive spinning particle. We can explicitly see that both equations have the fourth order in derivatives of the path.

The other type of cylindrical curves are helices. The helices on 2\( d \) cylinder are defined by the constant curvature conditions

\[ k_1, k_2 = \text{const}. \]  (37)

In this case, the system of equations (23) takes the form

\[
\alpha_1^2 + \alpha_2^2 = 1, \quad \beta_1^2 - \beta_2^2 = 1, \quad rk_1\alpha_1 + \beta_2^2 = 0, \quad rk_1k_2\alpha_2 + 3k_1\alpha_2\beta_1\beta_2 = 0,
\]

\[ r(k_1^3 - k_1k_2^2)\alpha_1 - 4k_1k_2\alpha_1\beta_1\beta_2 - k_1^2(3\alpha_1^2\beta_2^2 - 7\beta_2^2 - 3) = 0. \]  (38)

The solution to this system reads

\[ \alpha_1 = -1, \quad \alpha_2 = 0, \quad \beta_1 = \frac{k_2}{\sqrt{k_2^2 - k_1^2}}, \quad \beta_2 = \frac{-k_1}{\sqrt{k_2^2 - k_1^2}}. \]  (39)

The consistency condition for this system is the relation between the curvatures of the helix and its radius,

\[ r = \frac{k_1}{k_2^2 - k_1^2}. \]  (40)

In this way, we see that the helices are particular solutions to the equations of cylindrical curves (17), (29).
The linear momentum and total angular momentum of the spinning particle traveling the helical path have the form

\[
p = m \left( \frac{k_2}{\sqrt{k_2^2 - k_1^2}} e_1 - \frac{k_1}{\sqrt{k_2^2 - k_1^2}} e_3 \right),
\]

\[
J = m(x + re_2) \wedge \left( \frac{k_2}{\sqrt{k_2^2 - k_1^2}} e_1 - \frac{k_1}{\sqrt{k_2^2 - k_1^2}} e_3 \right) \pm s \left( \frac{k_2}{\sqrt{k_2^2 - k_1^2}} e_2 - \frac{k_1}{\sqrt{k_2^2 - k_1^2}} e_1 \right) \wedge e_3.
\]

(41)

By construction, the momentum and total angular momentum of the particle lie on the co-orbit of the Poincare group.

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