I. INTRODUCTION AND MAIN RESULTS

One-dimensional (1D) magnetic systems exhibit a variety of interesting phenomena signifying their quantum many-body nature. They have been, therefore, the subject of intense theoretical and experimental study. A great number of spin chain models has been proposed and investigated, with various interaction ranges, spin representations and anisotropies, as well as with coupling to other degrees of freedom. The nearest neighbor spin - 1/2 Heisenberg model, in particular, plays an important role being exactly integrable, allowing a detailed analysis of its thermodynamic properties. These properties are found to be generic: they do not differ essentially from low energy properties of other (non integrable) short-range spin chains. On the other hand, the Heisenberg model does not provide a generic description of transport properties. Integrability entails the existence of an infinite number of conservation laws, which in turn imply a dissipationless transport of the elementary excitations (spin-1/2 quantum solitons commonly named spinons) even at finite temperatures, . As a result, measurable dc-transport coefficients such as electric and thermal conductivities are expected to be infinite. In realistic systems described by spin chains, e.g. materials consisting of weakly coupled chains (or quasi-1D structures) of magnetically interacting ions, the conservation laws of the ideal model are partially violated, and the transport coefficients become finite. If the violation is soft one may expect an unusually high thermal conductivity in such systems attributed to spinon transport. Indeed, an experimental investigation of heat transport in materials of the (Sr,Ca,La)CuO series have found a considerable enhancement of the thermal conductivity in the direction parallel to the chains. Other types of systems involving quasi-1D, antiferromagnetic spin-1/2 chains such as organic compounds are in principle accessible to such measurements, however at this point not much data is available Heat conductivity by magnetic excitations has also been observed, for example, in the spin-Peierls compound CuGeO$_3$ or in ladder systems.

In a recent experimental study of SrCuO compounds Sologubenko et al. report and analyze specific heat and thermal conductivity data. The excess thermal conductivity along the chains direction (obtained after subtracting the phonon contribution) is identified as a contribution of the spinons, and its dependence on the temperature $T$ is fitted to an empirical formula, designed to account for scattering of the spinons by localized defects as well as Umklapp processes. A phonon Umklapp mechanisms is suggested as an interpretation of the exponential factor $\exp[T^*/T]$ which describes $\kappa_s(T)$ in the range of temperatures $50 K \leq T \leq 200 K$, with a characteristic temperature scale $T^* \sim 200 K$ close to $\Theta_D/2$, where $\Theta_D$ is the Debye temperature. However, this interpretation leaves open some questions: As the system is at zero magnetic field (corresponding, in fermionic language, to half filling) what inhibits the heat current relaxation to proceed by low-energy spinon–spinon Umklapp scattering which is not exponentially suppressed? Why is not the relevant energy scale to the subleading relaxation processes the exchange interaction $J/k_B \sim 2500K$ rather than $\Theta_D/2 \sim 200 K$? Another interesting observation can be gleaned by comparing the plots of $\ln \kappa(T) vs 1/T$ of the pure phonon contribution and of the spinon contribution. We note that the slope of the former is twice that of the latter. As pure phonon Umklapp processes are governed by $G$, the fundamental reciprocal lattice momentum, we need to understand why it is $G/2$ that dominates the spinon processes.

We shall find that a rather subtle interplay of (approximate) conservation laws and quantum dynamics underlies the experimentally observed heat conductivity, and a sophisticated hydrodynamic field theoretic approach is...
necessary to fully account for it. The arguments are rather general. When a system possesses some conserved quantities $P$ these may “protect” the current $J$ from degrading (when the cross-susceptibility $\chi_{JP} \neq 0$) leading to a pure Drude peak and infinite d.c. conductivity. When the conservation of the pseudo-momenta $P$ is softly violated they will, instead, lead to very long time tails in the decay of the current $J$, since states with a finite pseudo-momentum $P$ typically carry also a finite current $J$ (when $\chi_{JP} \neq 0$). Hence the component of the current “parallel” to $P$, $J_{jP} = (\chi_{jP}/\chi_{PP})P$ with $\chi_{jP} = \chi_{j,JP}P$, will therefore decay exponentially slowly. The presence of such approximately conserved quantities leads then to a natural hydrodynamic description of the system where a separation of fast and slowly decaying modes takes place and a consistent scheme of calculation of the slow mode conductivities can be carried out in terms of a matrix of decay rates of these modes.

These features emerge when we study the transport properties of clean 1D spin chains in the framework of a generic model which accounts for the coupling between spin excitations and lattice phonons. The model is analyzed close to its fixed point Hamiltonian - the Luttinger liquid and the thermal conductivity $\kappa(T)$ is evaluated at low but finite $T$. In this regime the transport coefficients are dominated by the slow relaxation of certain approximately conserved currents due to irrelevant corrections to the fixed point Hamiltonian. The most important correction terms of this sort, which are relatively efficient in degrading the conserved currents of the integrable Luttinger model, are found to be associated with Umklapp scattering terms. These include pure spinon as well as spinon–phonon scattering processes. In the absence of a magnetic field, the latter class is shown to be dominant. Our detailed calculation of $\kappa(T)$ originating from Umklapp scattering of spinons by 3D phonons agrees with the experimental results of Ref. [9]. In particular, it explains the origin of the exponential factor $\exp[T^*/T]$ with $T^* \sim \Theta_D/2$.

We then proceed to investigate the effects of a (large) external magnetic field $h$. The induced magnetization and the corresponding change of the wave vector of the spinons strongly modifies the way how various Umklapp processes relax the heat current. This leads to a fractal-like spiky behavior of $\kappa$ when plotted as a function of magnetization at fixed $T$, where the spikes occur at specific commensurate values of the magnetization. Furthermore, the magnetic field induces a linear spinon–phonon coupling tunable by $h$. The coupling alters the nature of the fixed point: the elementary excitations of the system are composite spinon–phonon objects. As a consequence of this mixing, the Umklapp processes are also modified and the relevant energy scale $T^*$ (again of the order of the minimum of $\Theta_D$ and $J$) depends smoothly on $h$. These effects are experimentally accessible in spin chains with relatively low magnetic exchange interaction $J$.

The paper is organized as follows: in Sec. II we derive the low energy model for the spinon system in the presence of coupling to 3D phonons, and discuss the leading irrelevant corrections and their significance for transport. In Sec. III we present the calculation of the conductivity tensor by means of a memory matrix approach, and derive expressions for the thermal conductivity $\kappa$ as a function of the temperature $T$. Sec. IV is devoted to the study of $\kappa(T)$ in a finite magnetic field $h$, where a linear coupling of spinons and 1D phonons is accounted for within a Luttinger model. Our conclusions are summarized in Sec. V. In Appendix A we emphasize that boundary conditions and finite size effects play an important role in the presence of (approximate) conservation laws and discuss what quantities are measured in a typical heat conduction experiment. Details of the calculation of the memory matrix elements are given in Appendix B. For convenience, throughout the paper we adopt units where $\hbar = \mu_B = k_B = 1$.

II. MODEL FOR THE WEAKLY COUPLED SPINON–PHONON SYSTEM

We wish to compute the low-temperature thermal conductivity of a system consisting of a parallel array of long antiferromagnetic spin chains embedded in a 3D lattice, and interacting with the lattice phonons. Let us begin by describing the spin system. A typical spin chain model, with finite range interaction, is given by,

$$
\hat{H} = \frac{1}{2} \sum_{i,j=1}^{N} J_{ij} \left( S_i^+ S_j^- + S_i^- S_j^+ \right) + \sum_{i,j=1}^{N} J^z_{ij} S_i^z S_j^z - h \sum_{i=1}^{N} S_i^z ,
$$

where $S_i^\pm, S_i^z$ are spin-1/2 operators at lattice site $i$, and $h$ is an external magnetic field applied along the $z$-direction. The coupling is antiferromagnetic: $J_{ij} > 0$, translational invariant: $J_{ij} = J_{i-j}$ and of finite range: $J_{i-j} = 0$ for $i-j > p$.

The low energy dynamics of this class of models is described by a Luttinger liquid Hamiltonian. A fully fledged derivation would proceed via repeated RG transformations and yield, in principle, the fixed point Hamiltonian (the Luttinger liquid) as well as all irrelevant operators around it. Instead of following this route, we shall employ a short cut and proceed via the Jordan–Wigner transformation allowing a fermionization of the spin degrees of freedom and subsequent bosonization. To illustrate it we consider the XXZ model corresponding to

$H = H_{XXZ} + H_{ij} \sum_{i,j=1}^{N} J^z_{ij} S_i^z S_j^z - h \sum_{i=1}^{N} S_i^z ,
$
pressed as

\[ S_i^- = \psi_i \exp \left[ i \pi \sum_{j=1}^{i-1} n_j \right], \]
\[ S_i^+ = : n_i : = \psi_i \psi_i^\dagger - \frac{1}{2} \, \]

The Hamiltonian \( H_{xzz} \) is mapped onto a model of interacting spinless fermions in 1D,

\[
H_{xzz} = -\frac{J}{2} \sum_i (\psi_{i+1}^\dagger \psi_i + \psi_i^\dagger \psi_{i+1}) + J_z \sum_i n_i n_{i+1} + h \sum_i n_i \tag{2}
\]

with \( J_z \) determining the interaction strength and \( h \) playing the role of a chemical potential. The first term in \( H_{xzz} \) corresponds to a kinetic energy term

\[
H_k = -J \sum_k \cos(ka) \psi_k^\dagger \psi_k
\]

(where \( a \) is the lattice spacing), whose low energy excitations (the spinons) are dominated by momenta \( k \) in the close vicinity of the two Fermi points \( \pm k_F \) with

\[
k_F = \frac{\pi}{2a} (1 + M) \tag{3}
\]

where \( M = 2(S_z) \approx h/(\pi J) \) is the magnetization of the spin chain (normalized to 1).

In the low energy limit the field operator \( \psi_i \) is approximated as \( \psi(x) = \psi(x) \approx e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x) \) where the right and left moving fields \( \psi_R(x), \psi_L(x) \) describe the low lying excitations, those near \( \pm k_F \) respectively. Using this expression in the Hamiltonian \( H_{xzz} \) and keeping the leading terms only, one finds:

\[
H_{xzz} \approx H_{LL} = -i(Ja) \int dx (\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L) + J_z \int dx (\rho_R^2 + \rho_L^2 + 4 \rho_R \rho_L) \tag{4}
\]

where \( \rho_R/L = \psi_R^\dagger \psi_R/L \). The Luttinger Hamiltonian \( H_{LLL} \) thus obtained is conformally invariant. It is the fixed point of \( H_{xzz} \), but the coupling constant appearing in it are valid only to first order in \( J_z/J \) and \( h/J \).

One may diagonalize \( H_{LL} \) by changing to bosonic variables: the field \( \phi(x) \) and its conjugate \( \Pi(x) \), satisfying \( [\phi(x), \Pi(x')] = i \delta(x' - x) \). The fermion fields are then given by

\[
\psi_{R/L} = \frac{1}{\sqrt{2\pi a}} e^{i|x|} e^{-i \phi/\sqrt{2a}}, \tag{5}
\]

where \( \theta \) is defined by \( \partial_x \theta = \pi \Pi \). The advantage of the bosonic representation is that it allows to diagonalize the interaction in \( H_{xzz} \) by means of a Bogoliubov rotation leading to,

\[
H_{LL} = v \int \frac{dx}{2\pi} \left( K(\pi \Pi)^2 + \frac{1}{K} (\partial_x \phi)^2 \right) \tag{6}
\]

in which (to leading order in \( |h|/J \) and \( |J_z|/J \))

\[
v \approx \left( J + \frac{J_z}{\pi} \right) a, \quad K \approx \frac{1}{1 + \frac{2J_z}{\pi J}} \, . \tag{7}
\]

For arbitrary \( |J_z| \leq J \) the Luttinger model Eq. \( \ref{H_{LLL}} \) still captures the low energy physics of the spin chain, however its derivation is more subtle. The exact Bethe Ansatz solution of the XXZ model yields\,\cite{13} for \( h = 0 \) (for \( h \neq 0 \) see Ref. \[13\])

\[
K = \frac{1}{2(1 - \frac{1}{\pi} \cos^{-1} \left( \frac{J_z}{J} \right) )}. \tag{7}
\]

In particular, in the physically interesting case of an isotropic Heisenberg antiferromagnet \( J = J_z \) yielding \( K = 1/2 \).

As noted above the transport properties of the XXZ model are not generic. The infinite number of conserved charges which assure its integrability also lead to a purely Drude peak and infinite d.c.-conductivity even at finite temperature. However the Luttinger liquid Eq. \( \ref{H_{LLL}} \) also describes (in the long wavelength limit) more complex spin chain structures as ladders, “zigzag” chains or in general chains with any finite range interaction (as long as no spin gap emerges) with the \( J_{ij}, J_{ij}^z \) dependence of the parameters \( v \) and \( K \) given by model-specific combinations of the coupling coefficients. The difference between integrable and non-integrable spin chains in the low energy limit is captured by the structure of the irrelevant operators around the Luttinger fixed point. In the latter case the irrelevant operators appear with generic coefficients. Consideration of the irrelevant operators is of crucial importance for transport properties. Non-integrable models are expected\,\cite{14} to have a finite heat conductivity at \( T < 0 \).\cite{15,16,17} To compute it, however, the fixed point Hamiltonian \( H_{LLL} \) is insufficient by itself: it is translationally invariant and integrable, and therefore all the currents (e.g. spin current, heat current) described by it cannot degrade, leading to an infinite d.c. conductivity. One must add to the fixed point Hamiltonian all irrelevant operators around it, and compute the conductivity from the resulting effective low-energy Hamiltonian. This implies, in passing, that the heat conductivity is a singular function of irrelevant perturbations, requiring us to recast perturbation theory in terms of a memory formalism (see below). We also note that a number of recent studies\,\cite{18,19} have found an infinite heat conductivity in generic non-integrable models. We believe that these claims are a consequence of either numerical problems\,\cite{19} or the neglect of certain classes of relevant perturbations\,\cite{18}.

We proceed to analyze the various irrelevant perturbations. In a generic model, all perturbations allowed by
symmetry are generated when high energy modes are integrated out. The symmetries relevant for the following discussion are spin-rotation $R_z$ around the z-axis, discrete translations by a lattice spacing $T_\alpha$, inversion $P$ and time-reversal $T$. In terms of the spinless fermions, $R_z$ guarantees charge conservation, $T_\alpha$ leads to momentum conservation up to reciprocal lattice vectors $G=2\pi/a$ and the transformation rules under $P$ and $T$ are given by

$$P: \psi_R \to \psi_R, \psi_L \to \psi_L \quad (8)$$
$$T: \psi_L \to \psi_R, \psi_R \to \psi_L^\dagger, i \to -i \quad (9)$$

The operators consistent with the symmetries above will be further divided into two classes, depending on their role in transport phenomena. The first class, $H^U$, consists of Umklapp operators $H^U_{nm}$, describing processes where $n$ spinons are moved from the right to the left Fermi point (and vice versa), possibly picking up $m$ units of lattice momentum $G$. It is these processes that underlie the degrading of the currents leading to finite conductivities. The other class, $H_{irr}$, contains low-energy processes where the number of spinons around each Fermi point remains conserved. This class includes corrections from band curvature, e.g. $\int \psi_R^\dagger \partial^2 \psi_L$, or from finite-range interactions. They do not affect the conductivities directly. A formal way to distinguish between the two classes is as follows. Consider the two operators

$$J_0 = N_R - N_L \quad (10)$$

where $N_R$ and $N_L$ are the total number of right and left moving spinons, respectively, and the spinon translation operator

$$P_T s = \int dx [\psi_R^\dagger (-i \partial_x) \psi_R + \psi_L^\dagger (-i \partial_x) \psi_L]. \quad (11)$$

These are among the infinite number of operators conserved by $H_{LL}$, but play a special role in what follows. As we show in the next section, the conservation of certain linear combinations of $J_0$ and $P_T$ is minimally violated – in comparison with all other currents, the decay rates are exponentially small at low $T$. The class of operators $H_{irr}$ consists of terms in the Hamiltonian which conserve $N_R, N_L$ and are invariant under (continuous) translations, hence commuting with both, $[H_{irr}, J_0] = [H_{irr}, P_T s] = 0$. The first class, $H^U$, includes Umklapp operators which do not commute at least with one of them.

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For even $n$ the leading terms are of the form

$$H^U_{nm} = \frac{g^U_{nm}}{2\pi a} \int dx \left[ e^{i\Delta k_{nm}x} \prod_{j=0}^n \psi_R^\dagger(x+ja) \psi_L(x+ja) + h.c. \right] = \frac{g^U_{nm}}{(2\pi a)^n} \int dx \left[ e^{i\Delta k_{nm}x} e^{i2n\phi(x)} + h.c. \right] \quad (12)$$

where

$$\Delta k_{nm} = n2k_F - mG \quad (13)$$

Note that $\Delta k_{nm}$ depends on the magnetization through $k_F$ and Eq. (3). For odd $n$ and vanishing magnetic field $h$, time reversal invariance does not allow terms of the form (12), therefore leading perturbations are

$$H^U_{nm} = \frac{g^U_{nm}}{(2\pi a)^n} \int dx \left[ e^{i\Delta k_{nm}x} \psi_R^\dagger(x) \psi_R(x) + \psi_L^\dagger(x) \psi_L(x) \right] \prod_{j=1}^{n+1} \psi_R^\dagger(x+ja) \psi_L(x+ja) + h.c. \quad (14)$$

Thus far we discussed in detail the low energy description of a single spin chain. We now turn to consider the complete system (as investigated experimentally e.g. by Sologubenko et al.\cite{3}) consisting of an array of spin chains interacting with 3-dimensional acoustic phonons.

The Hamiltonian of the system is

$$H = H_s + H^{3D}_p + H_{s,p} \quad (15)$$

where $H_s$ describes the spin array, $H^{3D}_p$ is the phonon Hamiltonian, and $H_{s,p}$ the interactions between spins and phonons. The spin array Hamiltonian is simply given by a sum of chains of the form we just discussed

$$H_s = \sum_\alpha H^\alpha_s \quad (16)$$

where $\alpha$ labels the spin chains (parallel to the x-axis) and $H^\alpha_s = H^\alpha_{LL} + H^\alpha_{irr} + \sum_{nm} H^U_{nm,\alpha}$. The Hamiltonian $H^{3D}_p$ describes the system of three dimensional acoustic phonons to which the array of spin chains is coupled. In the following, we will consider
mainly phonons describing deformations of the lattice parallel to the chains which is chosen as the \(x\)-direction. The dynamics of these deformations, to be denoted by \(q\), is described by

\[
H^{3D}_p = \int \frac{d^3x}{2\pi} \left[ (\pi P)^2 + \sum_\mu v_\mu^2 (\partial_\mu q)^2 \right],
\]

where \(\mu\) denote the \(x, y, z\) directions, \(P\) and \(q\) are (appropriately normalized) canonical phonon momentum and coordinate operators and \(v_\mu\) are the sound velocities with \(v_x = v_p\) and \(v_y = v_z = v_\perp\) assuming a tetragonal symmetry. Acoustic phonons describing vibrations \(q_\perp\) perpendicular to the chains are omitted here but can easily be included. As we will neglect in the following the weak phonon induced interactions between different spin-chains, we will need only the propagator of phonons along a single chain \(\int d^2k_G \omega(k) = \sum_\mu v_\mu^2 (\partial_\mu q)^2 \sim \ln[(\omega^2 + v_\perp^2 k^2)/\Theta^2_{D\perp}]\) where \(\Theta_{D\perp}\) is the Debye frequency perpendicular to the chains. The space-time form of the resulting phonon propagator along a chain at \(T = 0\) is \(G_p(t, x) \sim 1/(x^2 + v_\perp^2 t^2)\).

The phononic and spin degrees of freedom couple in a variety of ways which depend on the symmetries of the underlying lattice and on the strength of spin-orbit coupling. Assuming either inversion symmetry or weak spin-orbit coupling, the dominant coupling arises from the dependence of the exchange couplings on the distance \(J_{ij} = J(R_i - R_j) \approx J + a(\partial_\mu q)^2\) on the corresponding chain \(\alpha\). In the presence of a magnetic field, other couplings can become important as will be discussed in section \ref{sec:field}. In analogy to the classification of spinon-spinon interaction, we classify the corresponding spinon-phonon interactions \(H_{s,p} = H_{s,p}^{irr} + \sum_{nm} H_{nm}^{U,s-p}\) (again, generated by integrating out high energy modes) into Umklapp and non-Umklapp operators by their commutation relations with the operators \(J_0\) and the translation operator \(P_T\) of both spinons and phonons

\[
J_0 = \sum_{\alpha} \int dx \{ \psi_{R\alpha}^{\dagger} \psi_{R\alpha} - \psi_{L\alpha}^{\dagger} \psi_{L\alpha} \},
\]

\[
P_T = -\int d^3x P(\partial_\mu q) + \sum_{\alpha} P_{T\alpha}^n,
\]

where \(P_{T\alpha}\) is the translation operator of spinons [Eq. \ref{eq:transop}] on chain \(\alpha\). As we will show below, spinon-phonon couplings \(H_{s,p}^{irr}\) which commute with both \(J_0\) and \(P_T\), for example \(\{ (\partial_\mu q)^2 \partial_\mu q\},\) will not be able to relax the heat current completely. More important are again Umklapp terms \(H_{nm}^{U,s-p}\). For even \(n\), leading contributions (in a given chain) are of the form

\[
H_{nm}^{U,s-p} = \frac{g_{nm}^{U,p}}{(2\pi)^n} \int dx [e^{i\Delta k_{nm} x} e^{2i\phi (\partial_\mu q)} + \text{h.c.}]
\]

while for odd \(n\) one obtains

\[
H_{nm}^{U,s-p} = \frac{g_{nm}^{U,p}}{(2\pi)^n} \int dx [e^{i\Delta k_{nm} x} e^{2i\phi (\partial_\mu q)}(\partial_\mu \phi) + \text{h.c.}]
\]

where \(\phi = \phi_\alpha\), the phonon field \(q(R_\alpha, x)\) is evaluated on the corresponding chain \(\alpha\), and \(\Delta k_{nm}\) is given by \ref{eq:tidepot}. Processes involving multiple phonons are subleading. We would like to emphasize again, that all the spinon-phonon couplings discussed above are irrelevant by power-counting. But we have to keep them if they are the dominating processes to relax the relevant approximate conservation laws. Indeed, as will be shown in Sec. \ref{sec:field} the spinon–spinon scattering terms Eq. \ref{eq:spinons}, \ref{eq:spinons} for which \(\Delta k_{nm}\) is finite are exponentially suppressed with respect to the spinon–phonon terms in the physically relevant case, where the Debye temperature \(\Theta_D\) is much smaller than \(J\).

III. THE THERMAL CONDUCTIVITY FOR ZERO MAGNETIC FIELD

We turn now to the computation of the transport properties of the spin chains, assuming the parameters of the system to be compatible with the SrCuO compounds studied in Ref. \ref{ref:kitaev}. Since the exchange coupling \(J\) in these materials is extremely high (over 2000 K), magnetic field effects are negligible even at the strongest accessible fields (of order a few tens of Tesla). Hence, throughout this section we set \(h = 0\). We note that in other compounds where \(J\) is much smaller (e.g., organic spin chains), interesting magnetic field effects should be observable, as will be shown in Sec. \ref{sec:field}. We assume in addition the presence of inversion symmetry (or the absence of spin–orbit coupling), so that the spinons couple to phonons only via \(H_{s,p} = H_{s,p}^{irr} + \sum_{nm} H_{nm}^{U,s-p}\) with \(H_{nm}^{U,s-p}\) given by Eqs. \ref{eq:transop}, \ref{eq:transop2}.

The transport properties of the spin chains at low temperature - like the charge transport in (quasi-) one-dimensional metals \ref{ref:kitaev,ref:kitaev2,ref:kitaev3} - are governed by the approximate conservation of certain quantities \(P_{nm}\) (to be called below “pseudo-momenta”). The exponentially slow decay of a given \(P_{nm}\) will lead to an exponentially large heat conductivity for low \(T\). As already mentioned earlier, the reason is that states with a finite pseudo-momentum \(P_{nm}\) typically carry also a finite heat current: the component of the heat current “parallel” to \(P_{nm}\) will therefore decay exponentially slowly. \ref{fig:fig1}

We now proceed to identify the pseudo-momenta. Both \(J_0\) and \(P_T\) defined in \ref{eq:transop} decay rather slowly as they commute with all non-Umklapp terms \(H_{LL}, H_{irr}, H_{3D}, H_{s,p}^{irr}\). More important are certain linear combinations

\[
P_{nm} = P_T + \frac{\Delta k_{nm}}{2n} J_0
\]

which we call “pseudo-momenta” (\(P_{n0}\) is the usual momentum operator). The pseudo-momentum \(P_{nm}\) further commutes with all Umklapp terms with the quantum numbers \(n\) and \(m\) (and integer multiples \(kn, km\),
namely:

\[ [P_{nm}, H_{3D}^p + H_{LL}^p + H_{rr} + H_{s,p}^{irr}] = 0 \]

\[ [P_{nm}, H_{n,m}^U + H_{n,m}^{U,sp}] = 0 = [P_{nm}, H_{kn,km}^U + H_{kn,km}^{U,sp}] \]

For example, in the case of vanishing magnetic field considered in this section, where \( k_B = G/4 \) [see Eq. (3)], \( P_{21} = P_T \) obviously commutes with all translationally invariant terms with \( \Delta k_{nm} = 0 \) and therefore with all possible low energy processes.

The fixed point Hamiltonian \( H_{LL} \) being conformally invariant possesses an infinite number of conservation laws. However, compared to our pseudo-momenta these terms decay much faster when generic perturbations are added since they do not commute with all the (low-energy) terms which we have collected in \( H_{rr} \) and \( H_{s,p}^{irr} \). They are therefore not important in the following discussion. The same argument applies to \( H_{LL} + H_2^I \) which is integrable or any other particular combination that happens to possess conserved charges.

Having established the presence of the slowly decaying modes we now turn to the question of how to calculate perturbatively transport coefficients in a situation dominated by a few of these modes. The method of choice is the memory matrix approach, which is based on the idea that while the conductivity is a highly singular function of the various perturbations this is not the case for the matrix of decay-rates of the slowest modes in the system. The memory matrix approach is formulated in a vector space of slowly decaying operators, spanned in our case by \( P_T, J_0 \) and the heat current \( J_Q \) as we want to calculate the heat conductivity. For convenience, we use instead of \( P_T, J_0 \) and \( J_Q \) the operators \( J_T, J_s \) and \( J_Q \) with:

\[ J_T = v^2 P_T \quad \text{and} \quad J_s = vKJ_0, \quad (22) \]

where \( J_s \) is the spin current. In bosonized form

\[ J_T = -v^2 \left[ \int d^3x P \partial_s q + \sum_{\alpha} \int dx \Pi_\alpha \partial_s \phi_\alpha \right] \quad (23) \]

\[ J_s = vK \sum_{\alpha} \int dx \Pi_\alpha . \quad (24) \]

The heat current \( J_Q = \int d^3x J_{Q2} \) (along the chain direction) is determined from the continuity equation \( \partial_\mu J_{Q\mu} + \partial_\nu \mathcal{H} = 0 \) where \( \mathcal{H} \) is the energy density. For low temperatures it is sufficient to include only contributions from the fixed point, \( \int d^3x \mathcal{H} \approx H_{low} \) with

\[ H_{low} = H_{3D}^p + \sum_{\alpha} H_{LL}^\alpha \]

and one obtains

\[ J_Q = -\int d^3x v_p^2 P \partial_s q - \sum_{\alpha} \int dx v^2 \Pi_\alpha \partial_s \phi_\alpha . \quad (25) \]

Adding further contributions e.g. from Unklapp terms or band curvature does not affect results to leading order. Note that the operators \( J_Q \) and \( J_T \) are intimately related—in fact, they differ by the relative weight of the spinon and phonon degrees of freedom associated with the different velocities. However, there is a significant distinction between them: while \( J_T \propto P_T \) remains conserved under all translational invariant corrections to \( H_{low} \) [even those which mix spinons and phonons, like \( \int (\partial \phi)^2 \partial_s q \)], \( J_Q \) does not remain so.

We now set up the memory matrix formalism in the space spanned by the slow modes \( J_T, J_s, J_Q, J_0 \). To do so we follow Ref. 22 and introduce a scalar product \( (A|B) \) on the operators of the theory,

\[ (A(t)|B) = T \int_0^{1/T} d\lambda \langle A(t)| B(i\lambda) \rangle . \quad (26) \]

Then the dynamic correlation function of the operators \( A \) and \( B \) is

\[ C_{AB}(\omega) = \int_0^\infty dt e^{i\omega t} (A(t)|B) \quad (27) \]

\[ = \left( A \left| \frac{i}{\omega - \mathcal{L}} \right| B \right) \quad (28) \]

\[ = \left( IT \int_0^\infty d\omega \ e^{i\omega t} \langle [A(t), B]\rangle - \frac{(A|B)}{i\omega} \right) \quad (29) \]

with the Liouville operator \( \mathcal{L} \) defined by \( \mathcal{L} A = [H, A] \).

In the space spanned by \( J_T, J_s, J_Q \) the matrix of conductivities is therefore given by

\[ \sigma_{pq}(\omega, T) = \frac{1}{TV} C_{J_{p\mu}J_q} \quad (30) \]

where \( V \) is the volume of the system and \( p, q \) are either of \( T, s \) and \( Q \). The heat conductivity \( \kappa \) is given by (c.f. Appendix A)

\[ \kappa(\omega, T) = \frac{1}{TV} \sigma_{QQ}(\omega, T) \quad (31) \]

and \( \sigma_{ss} \) can be identified with the spin conductivity. The matrix of static susceptibilities can be written as

\[ \hat{\chi}_{pq} = \frac{1}{TV} (J_p|J_q) . \quad (32) \]

As argued above, the matrix of conductivities \( \hat{\sigma} \) has no good perturbative expansion. We therefore express it in terms of a memory matrix \( \hat{M} \) defined by

\[ \hat{\sigma}(\omega, T) = \hat{\chi}(T) \left( \hat{M}(\omega, T) - i\omega \hat{\chi}(T) \right)^{-1} \hat{\chi}(T) . \quad (33) \]

and explicitly given as \[ \hat{M}_{pq}(\omega) = \frac{1}{T} \left( \partial_t J_p \left| \frac{Q}{\omega - Q\mathcal{L}Q} \right| \partial_t J_q \right) . \quad (34) \]

Note that in the literature 22 the memory matrix is usually defined as \( \hat{M} \hat{\chi}^{-1} \). The operator \( \mathcal{Q} \) in Eq. (34) is the
projection operator on the space perpendicular to the slowly varying variables $J_p$,
\[
\mathcal{Q} = 1 - \sum_{pq} (J_p) \frac{1}{T} (\hat{\chi}^{-1})_{pq}(J_q) .
\]
\[
(35)
\]
This separation between fast and slow modes underlies the perturbative expansion of $\hat{M}$ to which we now turn.

The perturbative evaluation of $\hat{M}$ is greatly simplified by the observation that since $[H_{\text{low}}, J_k] = 0$ (for $k = s, T, Q$), the operators $\partial J_k$ are already linear in perturbations around the low energy Hamiltonian $H_{\text{low}}$. Hence, when these coupling terms are included to leading order in perturbation theory, one can set $\mathcal{L} = \mathcal{L}_{\text{low}}$ with $\mathcal{L}_{\text{low}} = [H_{\text{low}}, \ldots]$ and $\mathcal{Q} = 1$ in Eq. (35). The expectation values in Eq. (35) are also computed with respect to the low energy Hamiltonian $H_{\text{low}}$. Under these approximations, the expression for the memory matrix Eq. (35) can be written as
\[
\hat{M} = \frac{1}{T} \left[ \sum_{nm} (\hat{M}_{nm} + \hat{M}_{nm,s-p}) \right]
\]
\[
(36)
\]
where $\hat{M}_{nm}$ and $\hat{M}_{nm,s-p}$ are matrices in the space of the slow modes with matrix elements given by,
\[
M_{pq}^{nm} = \frac{\langle F_p; F_q \rangle_\omega \langle F_p; F_q \rangle_\omega = 0}{i\omega},
\]
\[
(37)
\]
\[
M_{nm,s-p}^{pq} = \frac{\langle F_p; F_q \rangle_\omega - \langle F_p; F_q \rangle_\omega = 0}{i\omega}.
\]
\[
(38)
\]
Here $F_p = i[J_p, H^U]$, $\langle F_p; F_q \rangle_\omega$ is the retarded correlation function calculated with respect to $H_{\text{low}}$, and similarly for $F_{p-s} = i[J_p, H^U,s-p]$ (the indices $n, m$ have been omitted for brevity). Note that all the matrices $\hat{M}_{nm}$, $\hat{M}_{nm,s-p}$ are symmetric. The static susceptibility matrix (for $aT \ll v_p$) is given by
\[
\hat{\chi} \approx \begin{pmatrix}
2vK/\pi & 0 & 0 \\
0 & \frac{\pi q^2}{3} & \frac{\pi q^2}{3} \\
0 & \frac{\pi q^2}{3} & \frac{\pi q^2}{3}
\end{pmatrix}
\]
\[
(39)
\]
(\text{where the matrix indices } p, q \text{ take the values } s, T, Q).

We are mainly interested in the d.c. thermal conductivity $\kappa(T)$ (Eq. (38) at $\omega = 0$), which can be obtained from Eqs. (36) and (38) through (39) in the limit $\omega \to 0$. We find,
\[
\kappa(T) \approx \frac{\pi^2 e^{2T^3}}{9} \left[ (\hat{M}^{-1})_{TT} + 2(\hat{M}^{-1})_{QT} + (\hat{M}^{-1})_{QQ} \right]
\]
\[
(40)
\]
with
\[
(\hat{M})_{pq} = \sum_{nm} (g^U_{nml})^2 M_{nm}^{pq}(\Delta k_{nm}, T),
\]
\[
(41)
\]
\[
M_{nl}^{QQ}(\Delta k_{nm}, T) \equiv \lim_{\omega \to 0} M_{nm}^{pq}
\]
For conciseness we introduced the index $l$: $l = 0$ denoting $M_{nl}^{pq}$, and $l = 1$ denoting $M_{nm,s-p}^{pq}$ (and similarly for the coupling constants $g^U$). Note that we have only retained contributions of Umklapp operators in $M_{QQ}^{Q}$. There are further contributions arising from $H_{s,p}^{F}$ which turn out to be subleading for vanishing magnetic field and are therefore omitted here. They are, however, important in the case of a finite magnetization, cf. section IV.

The characteristic Luttinger liquid behavior of the spinon system is reflected by the functional dependence of $M_{nl}^{(T)}(\Delta k, T)$ on $T$ and $\Delta k$. Approximate expressions for these functions can be obtained analytically in the high $T$ limit, $T \gg |\Delta k|$, or the low $T$ limit, $T \ll v_p |\Delta k|$, (see Appendix B for a detailed calculation). We first note that the conservation law (22) implies a trivial relationship between $F^s$ and $F^T$, and consequently for any $T$ and $\Delta k$
\[
M_{nl}^{ST}(\Delta k, T) = - \frac{v\Delta k}{2nK} M_{nl}^{s}(\Delta k, T),
\]
\[
M_{nl}^{TT}(\Delta k, T) = \frac{v^2(\Delta k)^2}{4n^2K^2} M_{nl}^{s}(\Delta k, T),
\]
\[
M_{nl}^{QT}(\Delta k, T) = - \frac{v\Delta k}{2nK} M_{nl}^{s}(\Delta k, T).
\]
\[
(41)
\]
\[
(42)
\]
We therefore need to compute directly only three types of functions: $M_{nl}^{s}$, $M_{nl}^{s}$ and $M_{nl}^{QQ}$. In the high $T$ limit we get
\[
M_{nl}^{s}(\Delta k, T) \sim T^{2(n^2K + 1)-3}
\]
\[
M_{nl}^{Q}(\Delta k, T) \sim \Delta k M_{nl}^{s}(\Delta k, T) \Delta k
\]
\[
M_{nl}^{QQ}(\Delta k, T) \sim \Delta k M_{nl}^{s}(\Delta k, T) \Delta k
\]
\[
(42)
\]
More interesting is the low $T$ limit $T \ll v_p |\Delta k|$, in which we find
\[
M_{nl}^{s}(\Delta k, T) \sim (2nKv)^2 M_{nl}^{s}(\Delta k, T),
\]
\[
M_{nl}^{Q}(\Delta k, T) \sim \Delta k M_{nl}^{s}(\Delta k, T),
\]
\[
M_{nl}^{QQ}(\Delta k, T) \sim \Delta k^2 M_{nl}^{s}(\Delta k, T)
\]
\[
(43)
\]
\[
(44)
\]
where in the last line we have used the additional assumption $(T/v|\Delta k|) \ll (v_p/v)$. The expressions for $M_{nl}(\Delta k, T)$ are the following: For even $n$
\[
M_{n0}(\Delta k, T) \approx e^{-\frac{\Delta k}{a\Delta k}} \frac{a^2 - 2n}{\pi^2 T^2(n^2K^2/2)vT} \left( \frac{a\Delta k}{2} \right)^{n^2K^2-2}
\]
\[
(44)
\]
\[ M_{n1}(\Delta k, T) \approx A \frac{a^2 v}{(2\pi a)^{2n} v_p^2 \rho T (\Delta k)^2} \left( \frac{v_p}{v} \right)^{2(n^2 K - 1)} \left( \frac{aT}{2v} \right)^{2n^2K} \left( \frac{a|\Delta k|}{2} \right)^2 \exp \left[ -\frac{v_p|\Delta k|}{2T} \right] \]  

(45)

\((A\) is a numerical factor). For odd \(n\), \(M_{n1}\) are given by the above expressions multiplied by a factor \(\sim (\Delta k)^2\) (for \(l = 0\)) or \(\sim (T/v)^2\) (for \(l = 1\)), in particular

\[ M_{n1}(\Delta k, T) \approx \tilde{A} \frac{a^2 T}{(2\pi a)^{2n} v_p^2 \rho T (\Delta k)^2} \left( \frac{v_p}{v} \right)^{2(n^2 K - 1)} \left( \frac{aT}{2v} \right)^{2n^2K} \left( \frac{a|\Delta k|}{2} \right)^2 \exp \left[ -\frac{v_p|\Delta k|}{2T} \right] . \]

Note that the above exponential factors are always dictated by the smallest of the velocities involved, and in our case \(v_p = v_{\text{min}} = \min\{v, v_p\}\). The physical origin of this behavior is that the minimal energy cost of a process involving a momentum transfer of \(\Delta k\) is associated with initial and final states of the elementary excitations with energy \(v_{\text{min}} \Delta k/2\) each. Since in the system of interest to us \(v_p \ll v\), the exponential factor in \([44]\) dramatically suppresses the pure spinon contribution to the sum in Eq. \([44]\) (in particular, for \(|\Delta k| \sim 1/a\) the exponent becomes \(\sim -(J/T)\)). However, among the particular Umklapp scattering terms for which \(\Delta k = 0\) (and hence the ‘high \(T\)’ limit \([42]\) applies), the spinon–spinon process dominates as it contributes the leading power of \(T\).

We now focus our attention on the low \(T\) behavior of the thermal conductivity Eq. \([39]\). Eq. \([3]\) implies that for \(h = 0\), \(k_F = \pi/2a = G/4\). This is a particular, commensurate value of the filling \(2k_F/G\), in which case the Umklapp term \(n = 2\), \(m = 1\) does not involve a momentum transfer, i.e. \(\Delta k_{21} = 0\). Due to the exponential factor in Eqs. \([43]\) and \([13]\), the sum in Eq. \([10]\) is strongly dominated by terms with a minimal \(\Delta k\). In particular, the leading contribution to \(M^{ss}\) and \(M^{QQ}\) is the single term \(n = 2\), \(m = 1\), \(l = 0\) corresponding to \(\Delta k_{21} = 0\), where \(M^{ss}_G(0,T)\) and \(M^{QQ}_G(0,T)\) are given by Eq. \([42]\). The other matrix elements vanish for \(\Delta k = 0\). In fact, the vanishing of \(M^{TT}_{21}, M^{Tq}_{21}\) (for any \(p, q = s, T, Q\)) reflects the fact that \(\mathbf{J}_T \propto \mathbf{P}_{21}\) commutes with all low energy terms in the Hamiltonian. Their leading contribution is therefore associated with the next smallest \(\Delta k\), i.e. the term \(n = 2\), \(m = 0\) and \(l = 1\). These are given by the low \(T\) approximation \([44]\) with \(n = 1\), \(\Delta k = G/2\). As a result

\[ \left(\hat{M}^{-1}\right)_{TT} \approx \frac{1}{M^{TT}_G(G/2,T)} \left(\frac{G}{2T}\right)^{2(1-K)} \exp \left[ -\frac{T^*}{T} \right] , \]  

(46)

which is exponentially diverging. In contrast, \((\hat{M}^{-1})_{QQ}\) and \((\hat{M}^{-1})_{QT}\) are inversely proportional to \(M^{QQ}_G\), associated with the relatively fast short-time relaxation rate of the heat current \(J_Q\). As a consequence they depend algebraically on \(T\) and hence are exponentially suppressed compared to \(M^{-1}_{TT}\). Inserting into Eq. \([39]\) this yields

\[ \kappa(h = 0) \approx \kappa_0 \left(\frac{T}{T^*}\right)^{2(1-K)} \exp \left[ -\frac{T^*}{T} \right] , \]  

(47)

with

\[ T^* = \frac{v_p G}{4} \]  

(48)

and \(\kappa_0\) depending on the parameters of the spinon–phonon system and the typical Umklapp scattering strength \(g^2\) (\(\kappa_0 \sim g^{-2}\)).

How does this compare to experiments? An exponential behavior of the spin contribution to the heat conductivity has indeed been observed by Sologubenko et al. in SrCuO and Sr$_2$CuO$_3$ for temperatures above 50 K (below which scattering from defects seems to become important). It was emphasized by the authors that \(T^*\) is of the order \(\Theta_D/2\), where \(\Theta_D\) is the Debye temperature. A precise comparison to our result would require a detailed knowledge of the phonon velocities in these systems. However, if we neglect for simplicity all anisotropies of the phonons, the Debye temperature is given by \(\Theta_D = \frac{v_p}{6\pi^2/a^3}^{1/3} \approx 0.6 v_p G\) and therefore \(T^* \approx 0.4 \Theta_D\) in very good agreement with the experimental observation.

In \([61]\), we have defined \(\kappa\) to be determined from an energy-current correlation function. However, in the presence of exact or approximate conservation laws and for finite systems it is far from obvious that this is the quantity measured in a typical heat transport experiment. For example, if spin is exactly conserved, boundary conditions will imply that no spin-current will flow through the surface of the sample and therefore the heat current in the experiment has to be calculated under the boundary condition of vanishing spin-current. As explained in detail in Appendix \(A\) this implies that \(\sigma_{QQ}\) in Eq. \([61]\) has to be replaced by \(\sigma_{QQ} - \sigma_{QS}^2/\sigma_{ss}\). However, we consider in this paper a different limit, assuming that the sample is much longer than typical length scales on which e.g. the spin does decay. Under these assumptions, Eq. \([61]\) is indeed valid – see Appendix \(A\) for details.

A note added on July 12, 2005: an error in the form of the phonon propagator lead to a wrong power-law prefactor in Eq. \((47)\) – the power \(2(1-K)\) should be replaced by \(-2K\). See Appendix \(C\) for details.
IV. EFFECTS OF A FINITE MAGNETIC FIELD

We now consider the effect of a finite magnetic field $h$ on the heat conductivity of the spin chain. The field will have two main effects: the first effect will be to modify $k_F$ [see Eq. (43)] and hence $\Delta k_{nm}$ leading to a fractal–like structure of the conductivity as a function of the magnetic field: as $h$ is varied the system passes from incommensurable to commensurable values (for which $\Delta k_{nm} = 0$) leading to a strong variation of the conductivity (see below). Clearly for this effect to be measurable, the spin–spin coupling $J$ cannot be too large: to be observable with accessible fields, $J$ needs to be of the order of a few tens degrees Kelvin. The second effect of a finite $h$ is to induce linear phonon–spin coupling by the field, which alters the fixed point Hamiltonian of the system. Such a coupling is possible as the magnetic field breaks time reversal invariance $T$. (A similar coupling can arise as a consequence of spin–orbit interaction in crystals without inversion symmetry, even when $h = 0$). For finite $h$ the linear coupling arises from terms of the form $(\partial_x q) S_i S_j \approx (\partial_x q) M \delta S_z$ where $M = 2 \langle S_z \rangle$ is the magnetization and $\delta S_z = S_z - \langle S_z \rangle$.

To analyze this case, we focus for simplicity on a strictly 1D geometry considering only longitudinal phonons traveling along the chain direction. We note, however, that much of our forthcoming predictions are expected to be qualitatively applicable to spin chains embedded in higher dimensional systems as well.

For 1D phonons the free Hamiltonian Eq. (44) reduces to

$$H_p = v_p \int \frac{dx}{2\pi} [(\pi P)^2 + (\partial_x q)^2] .$$

The normalization of $P$ and $q$ here is chosen differently than in Eq. (17), so that their dimensions are the same as $\Pi$ and $\phi$, respectively. It thus has the form of a Luttinger liquid with velocity $v_p$ and Luttinger parameter $K = 1$.

To this we add a spinon–phonon coupling term of the form

$$H_{s-p} = -u_0 \int \frac{dx}{\pi} \partial_x \phi \partial_x q .$$

At finite magnetic field, $u_0$ grows linearly with the magnetization and as a consequence can be controlled. For $h = 0$ and in the absence of inversion symmetry $P$ similar terms which couple linearly to $P$ rather than $\partial_q$ arise from spin orbit coupling. They also give rise to a mixing of modes (the roles of $P$ and $\partial_q$ can be interchanged in the analysis below).

A term of the form leads to new eigenmodes of mixed spinon–phonon excitations. The Hamiltonian

$$H^* = H_{LL} + H_p + H_{s-p}$$

is still scale invariant and is the fixed point of the coupled spinon-phonon system. We turn to diagonalize it. It is useful to define the free boson fields $\tilde{\phi} = \phi/\sqrt{K}$ and $\tilde{\Pi} = \sqrt{K}\Pi$, in terms of which Eq. (51) can be written as

$$H^* = \int \frac{dx}{2\pi} v_1 [(\pi \tilde{\Pi})^2 + (\partial_x \tilde{\phi})^2] + \int \frac{dx}{2\pi} \left\{ \frac{v_p [(\pi P)^2 + (\partial_x q)^2]}{2} - 2u\partial_x \tilde{\phi} \partial_x q \right\}$$

$$(u = u_0 \sqrt{K})$$. We then diagonalize $H^*$ using the transformation

$$\left( \begin{array}{c} \phi \\ q \end{array} \right) = \left( \begin{array}{c} C \\ (S(v/v_p)^{1/2} - S(v/v_p)^{1/2}) \end{array} \right) \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right)$$

and similarly for the canonical momenta

$$\left( \begin{array}{c} \Pi \\ P \end{array} \right) = \left( \begin{array}{c} C \\ (S(v/v_p)^{1/2} - S(v/v_p)^{1/2}) \end{array} \right) \left( \begin{array}{c} \Pi_1 \\ \Pi_2 \end{array} \right) ,$$

where (assuming $v > v_p$)

$$C = \frac{1}{\sqrt{2}} \left[ 1 + \frac{v^2 - v_p^2}{U^2} \right]^{1/2}$$

$$S = \frac{1}{\sqrt{2}} \left[ 1 - \frac{v^2 - v_p^2}{U^2} \right]^{1/2}$$

$$U^2 \equiv [(v^2 - v_p^2)^2 + u^2 v^2 v_p^2]^{1/2} .$$

The transformation is symplectic in order to preserves the canonical commutators, $[\phi_{\nu}(x), \Pi_{\nu'}(x')] = i\delta_{\nu,\nu'}\delta(x - x')$. The resulting Hamiltonian $H^*$ takes the diagonal form

$$H^* = \int \frac{dx}{2\pi} \sum_{\nu=1,2} v_\nu \left( K_\nu (\pi \Pi_{\nu})^2 + \frac{1}{K_\nu} (\partial_x \phi_{\nu})^2 \right)$$

where

$$\frac{v_1}{K_1} = \frac{1}{2v} \left[ v^2 + v_p^2 + U^2 \right] , \quad \frac{v_2}{K_2} = \frac{1}{2v_p} \left[ v^2 + v_p^2 - U^2 \right]$$

$$v_1 K_1 = v , \quad v_2 K_2 = v_p .$$

The strongest mixing (for a given $u$) occurs when $v \sim v_p$, which is realized in spin chain systems provided the exchange interaction $J$ is not too large, and comparable to $T_D$. We then define $\tilde{v} = (v + v_p)/2$, $\delta v = v - v_p$ where $\delta v \ll \tilde{v}$. Assuming in addition $u \ll \tilde{v}$, we obtain

$$v_1 \approx \tilde{v}[1 + \Delta(h)] , \quad v_2 \approx \tilde{v}[1 - \Delta(h)] ,$$

$$K_1 \approx 1 + \frac{\delta v}{2\tilde{v}} - \Delta(h) , \quad K_2 \approx 1 - \frac{\delta v}{2\tilde{v}} + \Delta(h) ,$$

where

$$\Delta(h) = \frac{[4(\delta v)^2 + u^2]^{1/2}}{4\tilde{v}}$$

which depends on the magnetic field $h$ via the coupling $u$. 
We proceed to evaluate the heat transport coefficient following the memory matrix method described in section 11. The linear coupling modifies the low-energy heat current $J_Q$, which is now defined with respect to the elementary degrees of freedom of $H^*$ rather than $H_{LL} + H_p$.

In terms of the two eigenmodes of $H^*$, $J_Q$ is given by

$$J_Q = - \int dx \sum_{\nu=1,2} \nu^2 \Pi_\nu \partial_x \phi_\nu. \quad (60)$$

In analogy with Eq. 23, $J_T$ (now defined as $J_T = \bar{v}^2 P_T$) is given by

$$J_T = -\bar{v}^2 \int dx \sum_{\nu=1,2} \Pi_\nu \partial_x \phi_\nu. \quad (61)$$

The derivations of the memory matrix and the static susceptibility proceed in the same way as in Sec. 11. In particular, the memory matrix $\hat{M}$ is generally given by Eq. 23 and by Eq. 10 in the $\omega \to 0$ limit. The static susceptibility matrix $\hat{\chi}$ is given by an expression nearly identical to Eq. 23, except that in $\hat{\chi}_{TT}, \hat{\chi}_{TQ}$ and $\hat{\chi}_{QQ}$ the velocity $\bar{v}$ is replaced by $2\bar{v}$ (to leading order in $\Delta(h)$).

The most prominent modification compared to the $\omega \to 0$ limit is that $\hat{M}$ (i.e., the $l = 0$ terms): as a result of the composite nature of the elementary excitations, it is no longer a pure spin contribution, and is dominated at low $T$ by the same exponential factor as the $l = 1$ terms. Since the latter involve a higher power of $T$, $M_{n0}^{\alpha}(\Delta k_{nm}, T) \equiv M_{n0}^{\alpha}$ actually dominates for any $\Delta k_{nm}$. We therefore neglect the $l = 1$ terms, and Eq. 10 becomes

$$\hat{M}(T) = \sum_{nm} (g_{nm}^U)^2 M_n(\Delta k_{nm}, T). \quad (62)$$

Similarly to Eq. 11, we find

$$M_{nT}^{\alpha}(\Delta k, T) = -\bar{\nu}^2 \frac{\Delta k}{2nKv} M_n^{\alpha}(\Delta k, T),$$
$$M_{nT}^{\alpha}(\Delta k, T) = -\bar{\nu}^2 (\Delta k)^2 \frac{2nKv}{(2nKv)^2} M_n^{\alpha}(\Delta k, T), \quad (63)$$
$$M_{nT}^{\alpha Q}(\Delta k, T) = -\frac{\bar{\nu}^2 \Delta k}{2nKv} M_n^{\alpha Q}(\Delta k, T).$$

The calculation of the functions $M_n^{pQ}(\Delta k)$ (for $p, q$ denoting either of $s, Q$) is essentially the same as in the $\omega \to 0$ case (see Appendix E). In the high limit $T \gg v_2 |\Delta k|$ we get

$$M_n^{q}(\Delta k, T) \sim T^{2(\alpha_n + \beta_n) - 3},$$
$$M_n^{Qs}(\Delta k, T) \sim M_n^{\alpha s}(\Delta k, T) \Delta k,$$  
$$M_n^{Qs}(\Delta k, T) \sim M_n^{\alpha s}(\Delta k, T) T^2,$$  

where

$$\alpha_n \equiv n^2 K_{1C} C^2, \quad \beta_n \equiv n^2 K_{2S}^2 (v/v_p). \quad (65)$$

and $C$, $S$ are defined in Eq. 55. The low $T$ limit corresponding to $T \ll v_2 |\Delta k|$ and $T/(v_1 |\Delta k|) \ll S \ll 1$ yields

$$M_n^{\alpha s}(\Delta k, T) = (2nKv)^2 M_n(\Delta k, T),$$
$$M_n^{Qs}(\Delta k, T) \sim \Delta k M_n(\Delta k, T), \quad (66)$$
$$M_n^{Qs}(\Delta k, T) \sim (\Delta k)^2 M_n(\Delta k, T),$$

with

$$M_n(\Delta k, T) \approx \frac{Aa^2}{(2\pi a)^2 vT^2 |\Delta(h)|^2 |\Delta(h)\alpha_n(2\beta_n)^{2\beta_n}} \times \left(\frac{aT}{2\bar{v}}\right)^{2\alpha_n} \left(\frac{2\bar{v} |\Delta k|}{T}\right)^{2\beta_n} \exp\left[-\frac{v_2 |\Delta k|}{2T}\right]. \quad (67)$$

Here $A$ is a numerical factor, $\Delta(h)$ is defined in Eq. 55 and we have used the fact that $v_2 = \min\{v_1, v_2\}$. Note that since time--reversal symmetry is broken at finite $h$, Eq. 12 is the leading Umklapp term for arbitrary $n$, and hence Eq. 67 hold for both odd and even $n$.

Inserting Eqs. 63 through 67 into 62 we get the dominant contributions to most elements of the matrix $\hat{M}(T)$. An exception is $M_{QQ}$, which includes additional corrections neglected in the above approximations: these are associated with irrelevant perturbations such as $\int (\partial \phi)^2 \partial_x g$, which do not commute with the heat current $J_Q$. These lead to contributions to $M_{QQ}$ which are power law in $T$. In the case of a finite magnetization, when all the Umklapp terms are either suppressed exponentially [Eq. 67] or by large powers of $T$, these corrections cannot be neglected. Hence, at low $T$ one always gets $M_{QQ} \gg M_{TT}, M_{TQ}$ and $\kappa$ is dominated by $M_{TT}^{-1}$. Using that $M_{QQ}$ is exponentially larger than the smallest eigenvalue of the matrix $\hat{M}$, we obtain with exponential precision $M_{TT}^{-1} \approx \kappa_s/(M_{ss} M_{TT} - M_{st}^2)$ and therefore for the thermal conductivity $\kappa$

$$\kappa(T) \approx \frac{4\pi^2 \bar{v}^2 T^3}{9} M_{TT}^{-1}(T) \approx \frac{4\pi^2 T^3}{9\bar{v}^2} \sum_{nm} M_{nm}^2 n^2 \quad (68)$$

with

$$D \approx \frac{1}{2} \sum_{nm} \sum_{n'm'} M_{nm} M_{nm'} (n |\Delta n_{n'm'}| - n' |\Delta k_{nm}|)^2 \quad (69)$$

and $M_{nm}$ as an abbreviation for $(g_{nm}^U)^2 M_n(\Delta k_{nm}, T)$.

How does the conductivity $\kappa$ depend on $T$ and $h$? As discussed above, a magnetic field $h$ leads to a linear spinon-phonon coupling and therefore tunes the parameters $\Delta(h)$ and $v_2$ in Eq. 67. However, for large fields, of the order $h \sim J$, another effect is even more important: the finite magnetization $M$ induces a shift of the Fermi momentum $k_F$ according to Eq. 8. The filling $2k_F/G = 1 + M$ is set to an arbitrary, generally irrational value, and can be tuned continuously by varying $h$. Upon tuning $k_F$, the characteristic momentum transfer $|\Delta k_{nm}|$ associated with an Umklapp process $H'_n$ is modified accordingly. As Umklapp processes at low $T$ are
suppressed exponentially by $e^{-v_2 \Delta k_{nm} / (2T)}$, a change in $\Delta k_{nm}$ modifies exponentially the contribution of $H_{nm}^U$ to the various relaxation rates.

We first analyze $\kappa$ for low $T$ and an almost commensurate magnetization with $M \approx 2 \frac{m_0}{n_0} - 1$ or $k_F \approx \frac{G}{2 \pi n_0}$ where $m_0$ and $n_0$ are (small) integer numbers. In this situation, $H_{nm_{0m_0}}^U$ is the strongest Umklapp term as, according to Fig. 11. $\Delta k_{nm_{0m_0}} \approx 0$. However, due to the approximately conservation of the pseudo-momentum $\hat{I}_{nm_{0m_0}}$, this Umklapp cannot relax the heat current alone, i.e. $\mathcal{D}$ in Eq. (68) vanishes if only $n_0$ and $m_0$ are included in the sum. Hence $\kappa$ is determined by the second strongest Umklapp $H_{nm_m0}^U$ with the smallest possible (but finite) momentum transfer, $\Delta k_{nm_m0} = \pm G/n_0$ (the corresponding values of $n_0'$, $m_0'$ depend strongly on $n_0$, $m_0$). We therefore obtain for the heat conductivity close to commensurability,

$$\kappa_{com} \sim T^{4 - 2 \alpha_m} \exp \left[ \frac{v_2 G}{2 \pi n_0 T} \right], \quad (70)$$

where $\alpha_n$ is defined in Eq. (69). The expression for $\kappa_{com}$ is valid as long as $k_F$ is sufficiently close to the commensurate value so that $\Delta k_{nm_{0m_0}} \ll T/v_1$ (in which case the “high $T$” approximation holds for $M_{pq}(\Delta k_{nm_{0m_0}}, T)$).

Note that the conductivity $\kappa$ is largest close to a commensurate magnetization $M$ where $n_0$ is small in apparent contradiction to the expectation that Umklapp is most efficient for commensurate fillings. The reason is simple: while the strongest Umklapp is enhanced for commensurate $M$, the second-strongest which determines the size of $\kappa$ is suppressed.

How large is the conductivity $\kappa$ for a typical incommensurate magnetization or for temperatures where the asymptotic behavior (70) is not yet reached? Umklapp processes $H_{nm}^U$ with small $n$ are suppressed at low $T$ because $\Delta k_{nm}$ is large while contributions with large $n$ are suppressed by algebraic prefactors with large exponents $\alpha_n$ and $\beta_n$ [see Eqs. (64) and (67)]. We therefore estimate $\kappa(T)$ using Eq. (68) as follows: first, the summation over $m$ is performed for a given $n$, noting that the dominant term (which maximizes $M_{nm}$) is $m = m_0$, where $m_0/n$ is the closest rational approximation of $2k_F/G$. The second-strongest Umklapp (for this $n$) is therefore characterized by a momentum transfer $\Delta k \approx G/N$ where $N = n\alpha$ (with $\alpha$ of order unity). We then evaluate the remaining sum over $n$ in a saddle point approximation. This yields

$$\kappa_{typ}(T) \sim \exp \left[ \left( \frac{T^*}{T} \right)^{2/3} \right], \quad (71)$$

where $T^* = C[\ln(v_2 G/T)]^{1/2}v_2 G$ and $C$ is a constant of order unity.

In Figs. 11 and 2 we show schematically the dependence of $\ln[\kappa]$ on the magnetization. For a precise quantitative prediction of $\kappa(M, T)$ the RG flow of all Umklapp terms has to be calculated. While this is in principle possible e.g. for a highly anisotropic spin chain (small $J_z$) this is beyond the scope of this paper. To obtain the schematic picture of Figs. 4 and 2 we have set all $g_{nm}^U$ to unity and used the asymptotic expressions (65)–(67) for the memory matrix in (68). As expected, the $M$ dependence of $\kappa(M)$ becomes more and more “spiky” towards lower temperatures with maxima close to commensurate magnetizations, e.g. for $M = 0, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \ldots$. Unfortunately, the multiple peaks in $\kappa(M)$ occur at extremely high values of $\kappa$ where in most experimental systems the heat transport will be dominated by impurities or sample boundaries. Therefore this effect, while being an amusing theoretical prediction, is difficult to be observed experimentally. At high temperatures, however, a pronounced minimum in $\kappa$ should be experimentally observable (see Fig. 2) in a regime where inelastic scattering still dominates transport. The precise position of this minimum depends on the temperature and on details of the system under consideration. We also would like to point out that a picture similar to Figs. 4 and 2 would emerge even in the absence of spinon-phonon coupling, only the relevant energy scales would be different and set by $J$ rather than the minimum of $\Theta_D$ and $J$.

Note that the position of the maxima in Fig. 11 seems to be shifted away from the commensurate positions. Consider for example the situation when $M$ is close to $\frac{1}{9}$, $M = \frac{1}{9} + \delta M$. As $|\Delta k_{nm}| = G|n\alpha M - m|$ according to Eqs. (3) and (11), the dominating scattering process is $H_{32}^U$ with $|\delta k_{32}| = \frac{2\pi}{G} G$ being very small. Therefore the heat transport is dominated by the relaxation of $P_{32}$ by the second strongest Umklapp $H_{11}^U$ with $|\Delta k_{11}| = G(\frac{1}{9} + \delta M)$ and $\ln \kappa \approx \frac{v_2}{2} G(\frac{1}{9} + \delta M)$ for small $\delta M$ and low $T$. The position of the local maximum to the left of $M = 1/3$ is determined by the competition with $H_{21}^U$ with $|\Delta k_{21}| = G(\frac{1}{3} + \delta M)$. While the exponential factors in Eq. (67) favor $H_{21}^U$ for $\delta M < 0$ as
spinon-phonon process considered by us the relevant momentum transfer is $G - 2k_F = G/2$ rather than $G$. Indeed, by comparing the plots of $\ln \kappa(T)$ vs. $1/T$ of the phonon and the spinon contribution in Sr$_2$CuO$_{2-x}$O$_x$ one can identify slopes differing by this factor of 2.

In the presence of a magnetic field, spinon and phonon modes start to mix. Furthermore, the Umklapp processes depend exponentially on the magnetization. This leads to a fractal-like spiky dependence of $\kappa$ on magnetization. Surprisingly, $\kappa$ is largest for commensurate magnetizations. This is again a consequence of the approximately conserved pseudo-momenta: it is not the strongest, but the second strongest Umklapp process which determines the thermal transport.

**V. CONCLUSIONS**

In this paper we have studied the thermal conductivity $\kappa$ of clean spin-chains coupled to phonons. The heat transport in the absence of defects is strongly influenced by approximately conserved pseudo-momenta. Due to their presence, low energy processes cannot relax the heat current and therefore $\kappa$ is exponentially large at low $T$. The exponent is determined by the slowest mode in the system, i.e. in most materials by the phonon velocity. In semi-quantitative agreement with experiments we find of at commensurate fillings, while the corresponding overlap relevant pseudo-momentum and the heat current is finite.

Due to their presence, low energy processes cannot relax heat transport in the absence of defects is strongly influenced by approximately conserved pseudo-momenta. In comparison to Ref. [15] it is interesting to note that the heat conductivity $\kappa$ is suppressed for exactly commensurate fillings, this is therefore only prevail for sufficient large $\Delta M < 0$ or sufficient low $T$.

In comparison to Ref. [15] it is interesting to note that while the electrical (or spin-) conductivity in 1d systems is suppressed for exactly commensurate fillings, while the corresponding overlap of $P_{nm}$ and spin or charge currents vanishes with exponential precision — therefore also the Wiedemann-Franz law will be violated exponentially for commensurate fillings.

**APPENDIX A: CONDUCTIVITY, APPROXIMATE CONSERVATION LAWS AND BOUNDARY CONDITIONS**

In this appendix we discuss on general grounds, how boundary conditions and approximate conservation laws influence transport measurements. Let us assume that in an experiment the transport of a conserved charge $q_1$ (e.g. the energy density) is measured using a 4-point probe. We want do investigate the influence of a second approximately conserved charge $q_2$ (e.g. the spin-density) on the experiment. The arguments given below can easily be generalized to include further exact or approximate conserved charges. The relevant continuity equations read

$$\partial_t q_1 + \partial_x j_1 = 0 \quad (A1)$$
$$\partial_t q_2 + \partial_x j_2 = -q_2/\tau_2, \quad (A2)$$

where $\tau_2$ describes phenomenologically the (slow) relaxation of $q_2$ (assuming that $q_2 = 0$ in equilibrium by definition). To set up a hydrodynamic description of the measurement, we assume that currents are driven by the force $F_1$ (e.g. $\partial_x/T$) and by gradients of $q_2$

$$j_1 = \sigma_{11} F_1 - D_{12} \partial_x q_2 \quad (A3)$$
$$j_2 = \sigma_{21} F_1 - D_{22} \partial_x q_2, \quad (A4)$$

where the conductivity $\sigma_{12}$ describes e.g. the spin-analog of thermopower. In steady state, $\partial_t q_1 = 0$, one obtains easily from Eqs. (A1)–(A4)

$$j_2 = j_1 \sigma_{21}/\sigma_{11} - D_{22} \partial_x q_2 \quad (A5)$$
$$\tilde{D}_{22} \partial_x^2 q_2 = q_2/\tau_2 \quad (A6)$$

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with $j_1(x) = \text{const.}$ and $\tilde{D}_{22} = D_{22} - D_{12}\sigma_{21}/\sigma_{11}$. These equations have to be solved with the appropriate boundary conditions. For our example the experimentally relevant boundary conditions are $j_2(\pm L/2) = 0$ (no spin-current flowing out of the sample) where $L$ is the size of the sample. From (A6) one therefore obtains for $-L/2 \leq x \leq L/2$

$$q_2 = q_2^0 \left( \exp \left[ -\frac{x + L/2}{l_2} \right] - \exp \left[ -\frac{x - L/2}{l_2} \right] \right)$$

(A7)

where $q_2^0$ is determined from the boundary conditions and $l_2 = \sqrt{D_{22}\tau}$ is the (spin-) diffusion length.

Obviously, one has to distinguish two different situations. If $L \gg l_2$ both $q_2$ and $\partial_x q_2$ vanishes exponentially in the sample. Therefore according to (A3) the (heat) transport of $q_1$ inside the sample is determined by $\sigma_{11}$.

$$j_1 = \sigma_{11} F_1 \quad \text{for} \quad L \gg l_2.$$  

(A8)

This is the situation considered in this paper (formally, within our Hamiltonian $l_2 = \infty$ as $S_z$ is conserved, but in a real material $S_z$ will decay e.g. due to spin-orbit scattering from impurities). Note that Eq. (A4) implies that the (spin-) current $j_2$ inside the sample is finite for $\sigma_{12} \neq 0$, i.e. for $h \neq 0$. This current, however, decays close to the sample boundaries in a width of order $l_2$.

The situation is different in the second case when $L \ll l_2$ or if $q_2$ is exactly conserved. Then $j_2$ vanishes inside the sample due to the boundary conditions and plugging (A4) into (A3) one obtains

$$j_1 = \left( \frac{\sigma_{11} - D_{12}\sigma_{21}}{D_{22}} \right) F_1 \quad \text{for} \quad L \ll l_2.$$  

(A9)

In passing, we note that in general the presence of approximate conservation laws implies the existence of large length scales like $l_2$ on which transport is inhomogeneous. This may be related to the experimental observation\cite{22} that heat transport in spin chains is often extremely inhomogeneous.

**APPENDIX B: THE CALCULATIONS**

To evaluate the matrix elements $M_{pq}^{nm}$, we first find explicit expressions for the operators $F_{nm}^q(t) = \langle j_p, H_{nm}^q \rangle$ ($l = 0, 1$), where $H_{nm}^0 = H_n^m, H_{nm}^1 = H_{nm}^{s-p}$ and the currents are given by Eqs. (24), (25). We restrict our calculation to the even $n$ terms (12), (13). The extension to odd $n$ [Eqs. (14) and (26)] is straightforward: due to the extra factors of $\partial_x \phi$, it will essentially amount to trading $C_{ss}$ by $C_{dd}$ below [see Eqs. (15), (16)]. We also define all operators below for a single chain. To leading order in $(v_p/v) \ll 1$, we find

$$F_{nml}^s(t) = -\frac{i g_n^{lU} 2nK_v}{(2\pi a)^n} \int dx (e^{i\Delta_k z x} e^{i2n\phi(x,t)} b_l(x) - \text{h.c.}),$$

(B1)

$$F_{nml}^Q(t) \approx \frac{i g_n^{lQ} 2nK_v}{(2\pi a)^n}$$

$$\times \int dx (e^{i\Delta_k z x} e^{i2n\phi(x,t)} \partial_x \phi b_l(x) - \text{h.c.}),$$

where for abbreviation we have omitted the indices $n, m$ from $\Delta_k$ and introduced the definition

$$b_l(x) \equiv (\partial_x q)^l.$$  

(B2)

We then find that the retarded correlation function $\langle F_{nm}^p; F_{nm}^q \rangle$ for a single chain is given by\cite{24}

$$\langle F_{nm}^p; F_{nm}^q \rangle = 2A_pq \int_{-\infty}^\infty dx \int_0^\infty dt \, e^{i(\omega t - \Delta_k x)} \text{Im} \{ C_{pq}(x,t) \},$$

where

$$C_{ss}(\xi) = \langle \text{exp}[i2n\phi(\xi)] \text{exp}[-i2n\phi(0)] \rangle G_t(\xi)$$

(B4)

and

$$C_{QQ}(\xi) = \langle \partial_x \phi(\xi) \partial_x \phi(0) \rangle G_t(\xi),$$

(B5)

$$C_{dd}(\xi) = \langle \partial_x \phi(\xi) \partial_x \phi(0) \rangle.$$  

(B6)

$(\xi)$ is an abbreviation for $(x,t)$;

$$G_t(\xi) \equiv \langle b_l^\dagger(\xi)b_l(0) \rangle$$

(B7)

$$A_{ss} = \frac{4(g_n^{lU})^2 n^2 K_v}{(2\pi a)^{2n}},$$

(A8)

$$A_{ss} = \frac{4(g_n^{lQ})^2 n^2 K_v}{(2\pi a)^{2n}},$$

(B8)

Eq. (83) yields

$$C_{ss}(x,t) = e^{4n^2 \phi(x,t)} G_t(x,t),$$

(B9)

$$G_\phi(x,t) \equiv \langle \phi(x,t) \phi(0,0) \rangle$$

where at finite $T$ the Green’s function $G_\phi(x,t)$ is given by\cite{24}

$$G_\phi(x,t) = \frac{K}{4} \ln \left[ \frac{\pi a T/v}{\text{sinh} \left( \pi T(x - vt + ia)/v \right)} \right]$$

(B10)

$$+ \frac{K}{4} \ln \left[ \frac{\pi a T/v}{\text{sinh} \left( \pi T(x + vt - ia)/v \right)} \right].$$
\[ G_0(\xi) = 1, \text{ and the phonon propagator } G_1(\xi) = G_p(\xi) \text{ (at finite } T \text{) has the form} \]

\[ G_p(x, t) \approx B \left[ \frac{\pi aT/v_p}{\sinh\{\pi T(x - v_p t + i a)/v_p\}} \right] \times \left[ \frac{\pi aT/v_p}{\sinh\{\pi T(x + v_p t - i a)/v_p\}} \right] \]  

(B11)

Substituting in Eq. (B11) and using the notation \( B_t \), where \( B_0 = 1 \) and \( B_1 = B \), we obtain

\[ C_{ss}(x, t) = B_t \left( \frac{\pi aT}{v} \right)^2 \frac{K}{4} \left( \frac{\pi aT}{v_p} \right)^2 \frac{K}{4} \times \left[ \sinh\{\pi T(x - vt + i a)/v\} \sinh\{\pi T(x + vt - i a)/v\} \right]^{nK} \]

\[ \times \left[ \sinh\{\pi T(x - v_p t + i a)/v_p\} \sinh\{\pi T(x + v_p t - i a)/v_p\} \right]^{-l}. \]  

(B12)

Using the identity

\[ \partial_x \phi(x, t) = \lim_{\gamma \to 0} \lim_{y \to x} (i \gamma)^{-1} \partial_y \exp[i \gamma \phi(y, t)] \]

we can also express \( C_d, C_{dd} \) in terms of the function \( G_\phi(x, t) \) and its derivatives:

\[ C_d(x, t) = -2n \partial_x G_\phi(x, t) C_{ss}(x, t) \]  

(B13)

\[ C_{dd}(x, t) = C_{ss}(x, t) \left[ 4n^2(\partial_x G_\phi(x, t))^2 + \partial_x^2 G_\phi(x, t) \right], \]  

(B14)

where

\[ \partial_x G_\phi(x, t) = \frac{K}{4} \left( \frac{\pi aT}{v} \right)^2 \left[ \coth(\pi T(x - vt + i a)/v) + \coth(\pi T(x + vt - i a)/v) \right], \]  

(B15)

and

\[ \partial_x^2 G_\phi(x, t) = \frac{K}{4} \left( \frac{\pi aT}{v} \right)^2 \left[ \sinh^{-2}\{\pi T(x - vt + i a)/v\} + \sinh^{-2}\{\pi T(x + vt - i a)/v\} \right]. \]  

(B16)

We now recall Eq. (37) for \( f \) and consider the limit \( \omega \to 0 \) where the correlation functions \( \langle F^p_{nml}; F^q_{nml} \rangle_0 \) given by Eq. (B8) are expanded to linear order in \( \omega \). We then get

\[ M_{nl}^{pq}(\Delta k, T) = \lim_{\omega \to 0} M_{nml}^{pq} = A_{pq} \int_{\infty}^{\infty} dx e^{-i \Delta ks} \int_{-\infty}^{\infty} dt \text{Im}\{C_{pq}(x, t)\}, \]  

(B17)

where we have used the fact that \( C^*_{pq}(x, t) = C_{pq}(x, -t) \) [see Eqs. (B8) through (B10) and (B12)] and hence the function \( \text{Im}\{C_{pq}(x, t)\} \) is symmetric with respect to \( t \to -t \). The matrix element \( M_{nml}^{pq}(\Delta k, T) \) can be computed by substituting Eq. (B12) in (B17) and performing the integrals. To compute \( M_{nml}^{pq}(\Delta k, T) \), we insert Eqs. (B8), (B12), (B15) and (B16) into (B13), (B14). This yields

\[ A_{sQ} C_{sQ}(x, t) \approx \frac{2(g_{nm}^U)^2 (nv)^3 K^2 B_l}{(2\pi a)^2} \left( \frac{\pi aT}{v} \right)^{2nK+1} \left( \frac{\pi aT}{v_p} \right)^{2l} f_{sQ}(x, t), \]  

(B18)

\[ f_{sQ}(x, t) \equiv \left[ \coth\{\pi T(x - vt + i a)/v\} + \coth\{\pi T(x + vt - i a)/v\} \right] \left[ \sinh\{\pi T(x - vt + i a)/v\} \sinh\{\pi T(x + vt - i a)/v\} \right]^{-nK} \]

\[ \times \left[ \sinh\{\pi T(x - v_p t + i a)/v_p\} \sinh\{\pi T(x + v_p t - i a)/v_p\} \right]^{-l}, \]

and

\[ A_{QQ} C_{QQ}(x, t) \approx \frac{(g_{nm}^U)^2 (v a)^4 K^2 B_l}{(2\pi a)^2} \left( \frac{\pi aT}{v} \right)^{2nK+2} \left( \frac{\pi aT}{v_p} \right)^{2l} f_{QQ}(x, t), \]  

(B19)

\[ f_{QQ}(x, t) \equiv \left[ \pi^2 K (\coth\{\pi T(x - vt + i a)/v\} + \coth\{\pi T(x + vt - i a)/v\})^2 + \sinh^{-2}\{\pi T(x - vt + i a)/v\} \right. \]

\[ + \left. \sinh^{-2}\{\pi T(x + vt - i a)/v\} \right] \left[ \sinh\{\pi T(x - vt + i a)/v\} \sinh\{\pi T(x + vt - i a)/v\} \right]^{-nK} \]

\[ \times \left[ \sinh\{\pi T(x - v_p t + i a)/v_p\} \sinh\{\pi T(x + v_p t - i a)/v_p\} \right]^{-l}. \]
is then recast in the form
\[ M_{ss}^{nq}(\Delta k, T) = T^{\eta_{ss} + 2(n^2K + t) - 3} \]
\[ \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dsds e^{-i\text{sgn}(\Delta k) s(s + s')/2} \mathcal{F}_{nn}^{pq}(s, s') \]
where \( \eta_{ss} = 0, \eta_{QQ} = 1, \eta_{QQ} = 2 \) and the functions \( \mathcal{F}_{nn}^{pq}(s, s') \) do not contain any dependence on \( \Delta k \) and \( T \). Hence the double integral depends on them only through the parameters \( \lambda \) and \( \text{sgn}(\Delta k) \) in the exponential factor. In particular, the high and low \( T \) limit cases are distinguished by \( \lambda < 1 \) and \( \lambda > 1 \), respectively. In the high \( T \) limit the exponential factor is expanded up to first order in \( \lambda \). The leading contribution to \( M_{ss}^{nq} \) and \( M_{QQ}^{QQ} \) comes from the 0'th order, i.e., the integration results in a constant independent of \( \Delta k \) and \( T \). However, since \( C_{ss}(x, t) \) is an odd function of \( x \), the leading contribution to \( M_{ss}^{nq} \) comes from the first order leading to an overall factor of \( \sqrt{\Delta k}/T \). These approximations yield Eq. \( 12 \).

To get the low \( T \) limit expressions for \( M_{ss}^{nq}(\Delta k, T) \), we evaluate the integrals in Eq. \( 12 \) in a saddle point approximation where the large parameter is \( \lambda \). As implied by Eqs. \( 12, \ 11 \), and \( 10 \), the functions \( C_{pq}(x, t) \) and hence \( \mathcal{F}_{nn}^{pq}(s, s') \) have branch-cut singularities along the imaginary axis (this is for the generic case where \( K \) is not an integer), one of which is close to the real axis. The integration over \( s' \) is performed first, with a slight deformation of the real axis to include the single saddle point
\[ s_0' = -\frac{2}{\lambda} \approx 0^+ \]  
(21)

Then, for \( \text{sgn}(\Delta k) < 0 \) (sgn(\( \Delta k \)) > 0), the contour of integration over \( s \) is deformed from the real axis to the upper (lower) imaginary axis. The series of saddle points \( \{s_i\}_{i=0} \) dominating this integral are close to the zeros of the sinh functions in \( 12 \) up to a correction of order \( 1/\lambda \) \( \rightarrow 0 \). Since the contribution of each saddle point involves an exponential factor \( e^{-\lambda |s_i|/2} \), the overall result will be dominated by the minimal \( s_i \) for which the functions \( \mathcal{F}_{nn}^{pq}(s, s') \) do not vanish by symmetry. The latter requirement excludes the contribution of \( s_0 = s_0' \). The leading contribution therefore originates from the saddle point
\[ s_1 = \pm i \]  
(22)

where \( \text{sgn}(\Delta k) = \mp \). Noting once again that \( v_p < v \), we obtain the low \( T \) expression for \( M_{ss}(\Delta k, T) \) in Eq. \( 15 \).

The matrix element \( M_{ss}^{nq}(\Delta k, T) \) is then given directly up to a prefactor. To get the other matrix elements in \( 15 \), we use scaling arguments noting that as long as \( (T/v|\Delta k|) \ll (v_p/v) \), the dominant momentum scale is \( \Delta k \).

The derivation of \( M_{ss}^{nq}(\Delta k, T) \) [Eqs. \( 6, \ 7 \), through \( 7 \)] in the case where spinons and phonons couple linearly involves essentially the same calculation. In this case, however, the memory matrix is dominated by the contribution of the \( l = 0 \) Umklapp terms Eqs. \( 12, \ 11 \).

In addition, the spinon–phonon mixing introduces the fields \( \phi_1, \phi_2 \) (the eigenmodes of the Hamiltonian \( H^* \)), in terms of which the spinon field \( \phi \) can be written as
\[ \phi = \sqrt{R} \left( C\phi_1 - \frac{v}{v_p} \phi_2 \right) \]  
(23)

where we have used the definition \( \phi = \sqrt{R} \phi \) and the transformation Eq. \( 29 \). As a consequence the operators \( F_{nm}^{pq} = i[J_p, H_{nm}^{pq}] \) for even \( n \) are given by
\[ F_{nm}^{pq}(t) = -i\frac{g_{nm}^{U}}{(2\pi a)^2} \int dx(e^{i\Delta k x} e^{i2n\phi(x, t) - h.c.}) \]
\[ \times \left( v_1^2 C\partial_x \phi_1 - v_2^2 S\left( \frac{v}{v_p} \right) \partial_x \phi_2 \right) \]  
(24)

The correlators \( \langle F_{nm}^{pq}F_{nm}^{pq} \rangle^0 \) are again written in terms of the functions \( C_{pq}(x, t) \) [Eq. \( 13 \)]. Here,
\[ C_{ss}(\xi) = \langle \exp[i2n\phi(\xi)] \exp[-i2n\phi(0)] \rangle^0 \]  
(25)

and
\[ C_{sQ}(\xi) = C_{Qs}(\xi) = v_1^2 C C_1(\xi) - v_2^2 S\left( \frac{v}{v_p} \right) \]  
(26)

\[ C_{QQ}(\xi) = v_1^2 C^2 C_1(\xi) - 2(v_1 v_2^2) C S\left( \frac{v}{v_p} \right) \]  
(27)

\[ + (v_2)^4 S^2 v^2 \]  
(28)

\[ C_{v}(\xi) = \langle \partial_x \phi_v(\xi) \exp[i2n\phi(\xi)] \exp[-i2n\phi(0)] \rangle^0 \]  
(29)

\[ C_{vv}(\xi) = \langle \partial_x \phi_v(\xi) \partial_x \phi_v(0) \exp[i2n\phi(\xi)] \exp[-i2n\phi(0)] \rangle^0 \]  
(30)

and the coefficients \( A_{pq} \) are given by
\[ A_{ss} = \frac{4(g_{nm}^{U})^2(nK^2)}{(2\pi a)^2} \]  
(31)

\[ A_{QQ} = \frac{4(g_{nm}^{U})^2 n^2K^2}{(2\pi a)^2} \]  
(32)

Inserting Eq. \( 29 \) into \( 29 \) we get
\[ G_{ss}(x, t) = e^{4\pi^2 K C^2 g_1(x, t) + 4\pi^2 K S^2 (v/v_p) g_2(x, t)} \]  
(33)

\[ G_{v}(x, t) = \langle \phi_v(x, t) \phi_v(0, 0) \rangle^0 \]  
(34)

where similarly to Eq. \( 12 \)
\[ G_{v}(x, t) = \frac{K_{\nu}}{4} \left[ \frac{\pi \nu T/v_{\nu}}{\sinh{\pi T(x - v_{\nu}t + ia)/v_{\nu}}} \right] \]  
(35)

\[ + \frac{K_{\nu}}{4} \left[ \frac{\pi \nu T/v_{\nu}}{\sinh{\pi T(x + v_{\nu}t - ia)/v_{\nu}}} \right] \]  
(36)
Also, similarly to the derivation of Eqs. (B13), (B14) we obtain expressions for $C_{tt}$, $C_{tv}$ in terms of the functions $G_\nu(x,t)$ and their derivatives:

\[ C_1(x,t) = -2n\sqrt{KC} \partial_x G_1(x,t) C_1(x,t) \]  
\[ C_2(x,t) = -2n\sqrt{KS} \left( \frac{v}{v_p} \right)^{1/2} \partial_x G_2(x,t) C_2(x,t), \]

\[ C_{11}(x,t) = C_{ss}(x,t) \left[ 4n^2 KC^2 (\partial_x G_1(x,t))^2 + \partial_x^2 G_1(x,t) \right] \]
\[ C_{12}(x,t) = 4n^2 KC S \left( \frac{v}{v_p} \right)^{1/2} \times \]
\[ \partial_x G_1(x,t) \partial_x G_2(x,t) C_{ss}(x,t) \]
\[ C_{22}(x,t) = C_{ss}(x,t) \]
\[ \times \left[ 4n^2 KS^2 \frac{v}{v_p} (\partial_x G_2(x,t))^2 + \partial_x^2 G_2(x,t) \right]. \]

The explicit dependence on $x$ and $t$ is obtained from Eqs. (B20), (B28), which yield

\[ C_{ss}(x,t) = \left( \frac{\pi a T}{v_1} \right)^{2\alpha_n} \left( \frac{\pi a T}{v_2} \right)^{2\beta_n} \times \]
\[ [\sinh{\pi T(x-v_1t+ia)/v_1}\sinh{\pi T(x+v_1t-ia)/v_1}]^{-\alpha_n} \]
\[ [\sinh{\pi T(x-v_2t+ia)/v_2}\sinh{\pi T(x+v_2t-ia)/v_2}]^{-\beta_n}. \]

**(APPENDIX C: ERRATUM)**

During the preparation of a subsequent article, we have spotted an error in the derivation of the effective propagator of 3-dimensional phonons $G_\rho(t,x)$, obtained after integration over the momenta in the direction perpendicular to the spin chains. The revised calculation of the thermal conductivity $\kappa(h=0)$ as a function of $T$ [Eq. (47)] yields a modified power-law prefactor, however, our main result – the exponential behavior with the characteristic temperature scale $T^*$ [Eq. (48)] – is unchanged.

The effective single-phonon propagator $G_\rho(t,x) = \langle \partial_x g(x,t) \partial_x g(0,0) \rangle$ (where $g$ is the displacement field in a particular chain) can be derived by an inverse Fourier transformation of $G_\rho(k,t) = k^2 \int d^2k_\perp G^{3D}(k,t)$, where $k$ is the component of $k$ along the chain and $G^{3D}(k,t)$ the correlator $\langle q_k(t)q_k(0) \rangle$ of free (isotropic) 3D-phonons at finite $T$:

\[ G^{3D}(k,t > 0) = \text{const} \times \frac{1}{|k|} \left[ n_k e^{iv_p |k|t} + (1 + n_k) e^{-iv_p |k|t} \right], \]

where $n_k = (e^{v_p |k|/T} - 1)^{-1}$ is the phonon occupation. We obtain the following expression, to replace Eq. (B11) in Appendix B:

\[ G_\rho(t,x,t) = B \left( \frac{1}{v_p t} \left( \frac{1}{(x+v_p t)^3} - \frac{1}{(x-v_p t)^3} \right) \right) \]
\[ + \left( \frac{T}{v_p} \right)^4 \sum_{n=1}^{\infty} \sum_{\sigma = \pm} \left( \frac{1}{(n+\sigma it)(n+\sigma(t-x/v_p)T)^4} + \frac{1}{(n+\sigma it)(n+\sigma(t+x/v_p)T)^4} \right), \]

with $B$ a numerical constant.

To find the contribution $M_{\nu}^\rho(\Delta k, T)$ to the memory matrix (where $l \neq 0$ denotes the number of phonons involved in the Umklapp process), we use $G_l(\xi) = [G_\rho(\xi)]^l$ in Eq. (B9) with $G_\rho$ given by Eq. (B22) above. At low $T$, the single-phonon process $l = 1$ yields a subdominant contribution $\sim e^{-v_p |\Delta k|/T}$. The physical reason for that is that the energy cost of the process involves an excitation of a single phonon that carries the entire momentum transfer $\Delta k$, i.e. $v_p |\Delta k|$. It is energetically favorable to distribute the momentum equally between the initial and final state using $\alpha_n = n^2 KK_1 C^2$ and $\beta_n = n^2 KK_2 S(v/v_p)$. Note that this expression is essentially the same function as (B12) with different parameters: $v \rightarrow v_1$, $v_p \rightarrow v_2$, $n^2 K \rightarrow \alpha_n$ and $l \rightarrow \beta_n$. As a result, the integrals in Eq. (B3) can be evaluated in the same manner, yielding Eqs. (B4) through (B7).
a two–phonon process \((l = 2)\): a phonon with momentum \(\Delta k/2\) is scattered to a final state with momentum \(-\Delta k/2\). This yields the leading contribution to the matrix element \(M_{l2}^{TT}\):

\[
M_{l2}^{TT} \sim T^{2K+3} \exp\left[-\frac{v_p|\Delta k|}{2T}\right],
\]

which should replace \(M_{l2}^{TT}\) in Eq. (46) (with \(\Delta k = G/2\)). Higher order processes \((l > 2)\) give the same exponential dependence but are suppressed by power law prefactors due to phase space restrictions. The resulting expression for the thermal conductivity is

\[
\kappa(h = 0) \approx \kappa_0 \left(\frac{T^*}{T}\right)^{2K} \exp\left[\frac{T^*}{T}\right],
\]

where \(T^* = \frac{v_p G}{4}\). This should replace Eq. (47).

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