TIED-DOWN OCCUPATION TIMES
OF INFINITE ERGODIC TRANSFORMATIONS*

BY

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Dedicated to Benjamin Weiss on his 80th birthday

ABSTRACT
We prove distributional limit theorems (conditional and integrated) for the
occupation times of certain weakly mixing, pointwise dual ergodic transforma-
tions at “tied-down” times immediately after “excursions”. The lim-
itigating random variables include the local times of $q$-stable Lévy-bridges
($1 < q \leq 2$) and the transformations involved exhibit “tied-down renewal
mixing” properties which refine rational weak mixing. Periodic local limit
theorems for Gibbs–Markov and AFU maps are also established.

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0. Introduction

Let \((X, m, T)\) denote a measure preserving transformation (MPT) \(T\) of the non-atomic, Polish measure space \((X, m)\) (where \(m\) is a \(\sigma\)-finite, non-atomic measure defined on the Borel subsets \(B(X)\) of \(X\)).

The associated transfer operator \(\hat{T} : L^1(m) \rightarrow L^1(m)\) is the predual of \(f \mapsto f \circ T\) \((f \in L^\infty(m))\), that is
\[
\int_X \hat{T}f g \, dm = \int_X fg \circ T \, dm \quad \text{for } f \in L^1(m) \text{ and } g \in L^\infty(m).
\]

The MPT \((X, m, T)\) is called pointwise dual ergodic (PDE) if there is a sequence \(a(n) = a_n(T)\) (the return sequence of \((X, m, T)\)) so that
\[
\left(\text{PDE}\right) \quad \frac{1}{a(n)} \sum_{k=0}^{n-1} \hat{T}^k f \xrightarrow{n \to \infty} \int_X fdm \quad \text{a.e. } \forall f \in L^1(m).
\]

Pointwise dual ergodicity entails
- recurrence (conservativity—no non-trivial wandering sets), and
- ergodicity (no non-trivial invariant sets).

Distributional limits. Let \((X, m, T)\) be pointwise dual ergodic with \(\gamma\)-regularly varying return sequence \(a(n) = a_n(T)\) \((0 < \gamma < 1)\).

By the Darling–Kac theorem ([DK57])\(^1\)
\[
\left(\mathcal{D}\right) \quad \frac{1}{a(n)} S_n(f) \xrightarrow{\text{d}} Y_\gamma m(f) \quad \forall f \in L^1_+.
\]

where \(Y_\gamma\) is the Mittag-Leffler distribution of order \(\gamma\) (see [Fel66, XIII.8]) normalized so that \(\mathbb{E}(Y_\gamma) = 1\), \(m(f) := \int_X f \, dm\) and \(\xrightarrow{\text{d}}\) (on \((X, m))\) denotes convergence in distribution with respect to \(m\)-absolutely continuous probabilities.

Return time process. Let \(\Omega \in \mathcal{F}_+: \{A \in B(X) : 0 < m(A) < \infty\}\). The return time function to \(\Omega\) is \(\varphi = \varphi_\Omega : \Omega \to \mathbb{N}\) defined by
\[
\varphi(\omega) := \min\{n \geq 1 : T^n \omega \in \Omega\} < \infty
\]
a.s. by conservativity.

\(^1\) See also the version in [Aar97, Ch. 3].
The induced transformation on $\Omega$ is $T_\Omega : \Omega \to \Omega$ defined by

$$T_\Omega(\omega) := T^{\varphi(\omega)}(\omega).$$

As shown in [Kak43], it is an ergodic, probability preserving transformation of $(\Omega, m_\Omega)$ (where $m_\Omega(A) := \frac{m(\Omega \cap A)}{m(\Omega)}$).

A standard inversion argument on $(\Omega, m_\Omega, T_\Omega, \varphi_\Omega)$ shows that the return time process $(\Omega, m_\Omega, T_\Omega, \varphi_\Omega)$ on $\Omega$ satisfies the stable limit theorem:

$$\frac{\varphi_n}{a^{-1}(n)} \xrightarrow{\circ} \frac{Z_\gamma}{m(\Omega)^{1/\gamma}}$$

on $(\Omega, m_\Omega)$ where

$$\varphi_n := \sum_{k=0}^{n-1} \varphi \circ T_\Omega^k.$$

As shown in [Fel66, XIII.8], $Z_\gamma := Y_{\gamma^{-1}}$ is the positive, stable random variable of order $\gamma$ with Laplace transform $E(e^{-sZ_\gamma}) = \exp[-\frac{s^{\gamma}}{1(1+\gamma)}]$ $(s > 0)$ whence with characteristic function

$$\Phi_{Z_\gamma}(t) := E(e^{itZ_\gamma}) = \exp \left[ -\frac{|t|\gamma}{\Gamma(1 + \gamma)} \left( \cos \frac{\gamma\pi}{2} - i\text{sgn}(t)\sin \frac{\gamma\pi}{2} \right) \right].$$

The stable limit theorem holds for the return time process of any $\Omega \in \mathcal{F}_+$. By “choosing” $\Omega$ carefully, it is sometimes possible to obtain stronger properties for the return time process and possibly also for the MPT.

Tied-down, renewal mixing properties. In this paper we study additional properties of certain pointwise dual ergodic transformations preserving infinite measures. Analogous properties for invertible MPTs are considered in §6.

Our additional properties are related to “tied-down renewal theory” which studies renewal counts at renewal times as in [Wen64], [Lig70] and [God17] and extend the strong renewal theorem as in [GL63, MT12, Gou11, CD19].

We will consider the following conditional, tied-down renewal mixing properties of a PDE MPT $(X, m, T)$ with $a(n) = a_n(T) \gamma$-regularly varying ($\gamma \in (0, 1)$):

$$\frac{1}{a(N)} \sum_{n=1}^{N} \left| \tilde{T}_n \left( 1_C g \left( \frac{S_n(f)}{a(n)} \right) \right) - m(C)E(g(m(f)W_\gamma))u(n) \right| \xrightarrow{N \to \infty} 0$$

a.e. $\forall C \in \mathcal{F}_+, g \in C_B(\mathbb{R}_+) f \in L_1^+$

where $u(n) \sim \frac{\gamma a(n)}{N}$, $W_\gamma \in \text{RV} (\mathbb{R}_+)$, $E(g(W_\gamma)) = E(Y_\gamma g(Y_\gamma))$. 

\[\]
The convergence mode in \((\mathbf{X})\) is considered in [Aar13, §4] as \(u\)-strong Cesaro convergence. Direct convergence is impossible, e.g., for \(C \in \mathcal{F}_+\) a weakly wandering set as in [HK64].

The limit random variables \(W_\gamma (0 < \gamma < 1)\) appear in [Wen64]. For \(0 < \gamma \leq \frac{1}{2}\) they correspond to the local time at zero up to time 1 of the symmetric \(\frac{1}{1-\gamma}\)-stable bridge and for \(0 < \gamma < 1\) to the local time at zero up to time 1 of the Bessel bridge of dimension \(2 - 2\gamma\) (see [PY97]).

Note that \((\mathbf{X})\) entails conditional rational weak mixing as in [AT20], whence spectral weak mixing and ergodicity of \((X, m, T^N) \forall N \geq 1\). Calculation now shows that if \((X, m, T)\) satisfies \((\mathbf{X})\), then so does \((X, m, T^N) \forall N \geq 1\) with \(a_n(T^N) \sim \frac{a_n(T)}{N}\).

Local limits for asymptotically stable, stationary processes. For \(\gamma \in (0, 1),\) call the positive, ergodic, stationary process \((\Omega, \mu, \tau, \phi)\) (i.e., where \((\Omega, \mu, \tau)\) is an ergodic probability preserving transformation (EPPT) and \(\phi : \Omega \to \mathbb{R}\) is measurable) **asymptotically \(\gamma\)-stable** if

\[
\mu([\phi > t]) \sim \frac{1}{\Gamma(1+\gamma)\Gamma(1-\gamma)} \cdot \frac{1}{a(t)}
\]

with \(a(t)\) \(\gamma\)-regularly varying and \((\Omega, \mu, \tau, \phi)\) satisfies the stable limit theorem as in (SLT).

Let \((\Omega, \mu, \tau, \phi)\) be an asymptotically \(\gamma\)-stable, positive, ergodic, stationary process with \(\phi : X \to G = \mathbb{Z}\) in the lattice case or \(G = \mathbb{R}\) in the continuous case.

Let \(\alpha \subset \mathcal{B}(\Omega)\) be a countable partition so that \((\Omega, \mu, \tau, \alpha)\) is a fibered system as defined in §3.

Denote

\[
\alpha_n := \bigvee_{k=0}^{n-1} \tau^{-k}\alpha \quad \text{and} \quad \mathcal{C}_\alpha := \bigcup_{n \geq 1} \alpha_n,
\]

the collection of **\(\alpha\)-cylinders**.

We will say that \((\Omega, \mu, \tau, \alpha, \phi)\) satisfies

- an **aperiodic local limit theorem** (LLT) if \(\forall A \in \mathcal{C}_\alpha, I \subset G\) a bounded interval,

\[
\begin{align*}
\lim_{n \to \infty} \frac{1}{a_n}\sum_{k_n \in G, k_n \in \mathbb{Z}} f_{Z_\gamma}(x)m_G(I)m(A)
\end{align*}
\]

uniformly in \(x \in [c, d]\) whenever \(0 < c < d < \infty\), where \(m_G\) denotes Haar measure on \(G\) and \(f_{Z_\gamma}\) is the probability density function of \(Z_\gamma\);
• a periodic LLT if $\exists p \in \mathbb{G}, \ p > 0$ (the period) and $\xi \in (0, p) \cap \mathbb{G}$ (the drift) so that $\langle \{ n\xi : n \geq 1 \} + p\mathbb{Z} \rangle = \mathbb{G}$ and $\forall A \in C_{\alpha}, \ I \subseteq \mathbb{G}$ a bounded interval, with $I_J := I \cap [Jp, (J + 1)p)$ ($J \in \mathbb{Z}$), we have

$$a^{-1}(n)\tau^n (1_{A \cap [\phi_n \in k_n + I]}(x)) \approx p f_{Z\gamma}(x) m(A) \sum_{J \in \mathbb{Z}} 1_{p\mathbb{Z} + I_J}(k_n - n\xi)$$

(\text{\#})

as $n \to \infty$, $k_n \in \mathbb{G}$, $\frac{k_n}{a^{-1}(n)} \to x$

uniformly in $x \in [c, d]$ whenever $0 < c < d < \infty$; and

• a generalized LLT (GLLT) if it satisfies either an aperiodic, or a periodic LLT.

Here and throughout, $a_n \approx b_n$ means $a_n - b_n \xrightarrow{n \to \infty} 0$.

Local limits are studied (here) using Fourier theory and throughout the paper, for $\mathbb{H}$ a locally compact, Polish, abelian group, $\hat{\mathbb{H}}$ denotes the multiplicative group of characters, that is

$$\hat{\mathbb{H}} := \{ \chi : \mathbb{H} \to S^1 : \chi \text{ a continuous homomorphism} \}$$

where $S^1 := \{ z \in \mathbb{C} : |z| = 1 \}$.

For example, $\hat{\mathbb{R}}^d \cong \mathbb{R}^d$ via $\chi_t(x) = e((x, t)) \ (x, t \in \mathbb{R}^d)$ where $(x, t) := \sum_{j=1}^d x_j t_j$ and $e(s) := e^{2\pi is} \ (s \in \mathbb{R})$.

LOCAL LIMITS FOR PERIODIC COCYCLES. Let $(\Omega, \mu, \tau)$ be an ergodic, probability preserving transformation and let $\mathbb{G} = \mathbb{Z}$ or $\mathbb{R}$.

The measurable function $\phi : \Omega \to \mathbb{G}$ is called:

• non-arithmetic if for $\chi \in \hat{\mathbb{G}}, \ \chi \not\equiv 1, \ \chi \circ \phi$ is not a $\tau$-coboundary, i.e., the equation $\chi \circ \phi = \frac{g(x)}{g(Tx)}$ where $g : \Omega \to S^1$ is measurable, has no solutions; and

• aperiodic if for $\chi \in \hat{\mathbb{G}}, \ \chi \not\equiv 1, \ \lambda \chi \circ \phi$ is not a $\tau$-coboundary.

It is not hard to check that if $(\Omega, \mu, \tau)$ is weakly mixing, and $\phi : \Omega \to \mathbb{G}$, then

(i) in case the skew product $(\Omega \times \mathbb{G}, \mu \times m_{\mathbb{G}}, \tau_{\phi})$ is ergodic, it is weakly mixing iff $\phi$ is aperiodic; and

(ii) in case $\phi > 0, \ (\Omega, \mu, \tau)^{\phi}$ is weakly mixing iff $\phi$ is non-arithmetic.
Here, in case $G = \mathbb{Z}$, $(\Omega, \mu, \tau)^{\phi}$ denotes the Kakutani skyscraper: the conservative, ergodic MPT (CEMPT) defined in [Kak43] by $(\Omega, \mu, \tau)^{\phi} := (X, m, T)$ where

$$X := \{(\omega, n) \in \Omega \times \mathbb{N} : 1 \leq n \leq \phi(\omega)\}, \quad m := \mu \times |X|$$

$$T(\omega, n) := \begin{cases} 
(\omega, n + 1), & n < \phi(\omega) \\
(\tau(\omega), 1), & n = \phi(\omega)
\end{cases}$$

with return time process to $\Omega \times \{1\} \cong (\Omega, \mu, \tau, \phi)$.

In case $G = \mathbb{R}$, $(\Omega, \mu, \tau)^{\phi}$ denotes the suspended semiflow over $(\Omega, \mu, \tau)$ under the ceiling $\phi$ (see §5).

We will establish in §3 the periodic LLT (Theorem 3.1 below) for $(\Omega, \mu, \tau, \alpha)$ a D-F fibered system (defined in §3) which is Gibbs–Markov or AFU and $\phi : \Omega \to G$ a non-arithmetic, D-F cocycle (§3). This uses the aperiodic LLTs of [AD01, ADSZ04].

Periodic LLTs for independent, identically distributed random variables were established in [She64].

**Local limit sets.** Let $(X, m, T)$ be a CEMPT.

We will say that $\Omega \in \mathcal{B}(X)$, $0 < m(\Omega) < \infty$ is a generalised local limit (GLL) set for $T$ if the return time process $(\Omega, m_\Omega, T_\Omega, \varphi_\Omega)$ satisfies a lattice GLLT with respect to some one-sided, countable generator $\alpha \subset \mathcal{B}(\Omega)$. In §5, we will consider semiflows with sections satisfying continuous GLLTs.

**1. Results and examples**

**Tied-down renewal mixing.** Our main result is

**Theorem A:** Suppose that $(X, m, T)$ is pointwise dual ergodic with $a(n) = a_n(T)$ $\gamma$-regularly varying ($\gamma \in (0, 1)$) and which has a GLL set $\Omega \in \mathcal{F}_+$. Then $(X, m, T)$ satisfies (X) (defined above).

In particular $(X, m, T)$ is conditionally rationally weakly mixing. This latter property also follows, in the aperiodic case, from proposition 3.1 of [AN17].

We will prove Theorem A in §2. In §5 we consider analogous properties for semiflows. Theorem 5.2 is a continuous time version of Theorem A.
Existence of GLL sets. Asymptotically $\gamma$-stable, positive, ergodic, stationary, stochastic processes are considered in §3, where sufficient conditions for their GLLTs are established via:

**Theorem B:** Suppose that $(X, m, T)$ is a pointwise dual ergodic, weakly mixing MPT with $a(n) = a_n(T) \gamma$-regularly varying with $0 < \gamma < 1$.

Suppose that the return time process on $\Omega \in \mathcal{B}(X)$, $0 < m(\Omega) < \infty$ admits a one-sided generator $\alpha \subset \mathcal{B}(\Omega)$ so that $(\Omega, m_\Omega, T_\Omega, \alpha)$ is a mixing and either a Gibbs–Markov map or an AFU map and $\varphi_\Omega$ is $\alpha$-measurable. Then $\Omega$ is a GLL set for $T$.

**Examples:** Tied-down renewals and the strong renewal theorem. Let $(\Omega, \mu, \tau, \varphi)$ be a positive, $\mathbb{G}$-valued, Bernoulli process with $\mathbb{G} = \mathbb{Z}$ (discrete time case) or $\mathbb{R}$ (continuous time case) which is asymptotically $\gamma$-stable with $\gamma \in (0, 1)$ (i.e., satisfies (Q) as defined above).

Suppose that the support of $\varphi$ generates $\mathbb{G}$ in the sense that

$$\{\{y \in \mathbb{G} : \mu([|\varphi - y| < \epsilon]) > 0 \ \forall \ \epsilon > 0\} \} = \mathbb{G}.$$

The following results introduce the ideas of Theorems A and 5.2, respectively.

**Tied-down renewals: discrete time.** In case $\mathbb{G} = \mathbb{Z}$, for $g \in C_B(\mathbb{R}_+)$ and with $u(n) := \frac{\gamma a(n)}{n}$, there is a set $K \subset \mathbb{N}$ with zero density (\(|K \cap [1, n]| = o(n)\)) so that

\begin{align*}
\frac{1}{u(n)} \sum_{k=1}^{n} g\left(\frac{k}{a(n)}\right) \mu([\varphi_k = n]) & \xrightarrow{n \to \infty, \ n \notin K} E(g(W_\gamma))
\end{align*}

where $W_\gamma$ is as in Theorem A.

**Tied-down renewals: continuous time.** In case $\mathbb{G} = \mathbb{R}$, then for $g \in C_B(\mathbb{R}_+)$ and $I \subset (0, 1)$ an interval; with $u(n) := \frac{\gamma a(n)}{n}$, there is a set $K \subset \mathbb{N}$ with zero density (\(|K \cap [1, n]| = o(n)\)) so that

\begin{align*}
\frac{1}{u(n)} \sum_{k \geq 1} g\left(\frac{k}{a(n)}\right) \mu([\varphi_k \in n + I]) & \xrightarrow{n \to \infty, \ n \notin K} |I| E(g(W_\gamma))
\end{align*}

---

2 i.e., where $(\varphi \circ \tau^n : n \geq 0)$ are iidrvs.
Sketch proof of (χ). By [She64], Ω is a GLL set for \((X, m, T) = (Ω, μ, τ)^φ\), the Kakutani skyscraper as defined above. By Lemma 2.1 below

\[
\frac{1}{a(N)} \sum_{n=1}^{N} \left| \sum_{k=1}^{n} g\left(\frac{k}{a(n)}\right) \mu([φ_k = n]) - E(g(W_γ))u(n) \right| \mathop{\longrightarrow}_{N \to \infty} 0
\]

whence (see, e.g., [GL63]) (χ).

Sketch proof of (ψ). Also by [She64], \((Ω, μ, τ, φ)\) is a GLL section for the suspension semiflow \((X, m, Ψ) = (Ω, μ, τ)^φ\) (see §5). In case it is not a standard section, apply Remark 5.5 to obtain a suitable Bernoulli induced section, which, by [She64], is a GLL section. Theorem 5.2 with \(t = 1\) applies and

\[
\frac{1}{a(N)} \sum_{n=1}^{N} \left| \sum_{k \geq 1} g\left(\frac{k}{a(n)}\right) \mu([φ_k \in n + I]) - u(n) \right| |I|E(g(W_γ)) \mathop{\longrightarrow}_{N \to \infty} 0.
\]

whence (see, e.g., [GL63]) (ψ). ■

Convergence in (ψ) and (ψ) is equivalent to the strong renewal theorem (SRT) as in [CD19].

In the discrete case,

\[
\frac{1}{u(n)} \sum_{k=1}^{n} g\left(\frac{k}{a(n)}\right) \mu([φ_k = n]) \mathop{\longrightarrow}_{n \to \infty} E(g(W_γ)) \quad \forall g \in C_B(\mathbb{R}_+)
\]

(χ)

\[
\iff \sum_{k=1}^{n} \mu([φ_k = n]) \sim u(n);
\]

and in the continuous case,

\[
\frac{1}{u(n)} \sum_{k=1}^{n} g\left(\frac{k}{a(n)}\right) \mu([φ_k = n + I]) \mathop{\longrightarrow}_{n \to \infty} |I|E(g(W_γ)) \quad \forall g \in C_B(\mathbb{R}_+) \text{ and intervals } I \subset (0, 1)
\]

(ψ)

\[
\iff \sum_{k=1}^{n} \mu([φ_k = n + I]) \sim u(n)|I|.
\]

These follow from the “remarks about mixing” in Sections 2 and 5.

As shown in [GL63] and [CD19], (χ)/(ψ) always hold when \(a(n) \gg \sqrt{n}\). Examples where they fail exist whenever \(a(n) \approx \sqrt{n}\).
In the case where \((\Omega, \mu, \tau, \varphi)\) is the excursion time process from 0 generated by an aperiodic, \(Z\)-valued random walk in the domain of attraction of a symmetric, \(p\)-stable law \((1 < p \leq 2)\), the convergence version of (\(\varphi\)) follows from results in [Wen64], [Lig70], [Ver79].

**Examples: Intermittent interval maps.** Consider the following piecewise onto interval maps \(T_\gamma : [0, 1] \to [0, 1], (\gamma > 0)\) defined by

\[
T_\gamma x := \begin{cases} 
  x(1 + (2x)^{1/\gamma}), & 0 \leq x < \frac{1}{2}, \\
  2x - 1, & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

Each \(T_\gamma\) is conservative and exact (with respect to Lebesgue measure) and admits an invariant density \(h_\gamma\) (unique up to constant multiplication) which is continuous on \((0, 1)\) and satisfies \(h_\gamma(x) \propto x^{-1/\gamma}\) as \(x \to 0\) (see [Tha83]).

For \(\gamma > 1\), \(h_\gamma\) is integrable and the maps \(T_\gamma : \gamma > 1\) were studied in [LSV99] as “probabilistic intermittency”.

For \(0 < \gamma \leq 1\), the invariant measure \(d\mu_\gamma(x) = h_\gamma(x)dx\) is infinite and (see, e.g., [Aar86], [Aar97, §4]):

- \([\mu_\gamma, T_\gamma, \gamma] = 0 \leq \gamma \leq 1\) is pointwise dual ergodic with \(a_n(T_\gamma) \propto n^{\gamma}\) when \(0 < \gamma < 1\) and \(a_n(T_1) \propto \frac{n}{\log n}\);
- \(T_{[\frac{1}{2}, 1]}\) is a piecewise onto AFU map with the return time function constant of intervals of monotonicity.

By Theorem B, \([\frac{1}{2}, 1]\) is a GLL set for \(T_\gamma\) and \(\mathbf{X}\) holds by Theorem A.

Now let \(\kappa \geq 2\) and consider the maps \(R_\gamma : 0 < \gamma \leq 1\) defined by

\[
R_\gamma x := \begin{cases} 
  x(1 + (\kappa x)^{1/\gamma}) \mod 1, & 0 \leq x < \frac{1}{2}, \\
  2x - 1, & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

These are (possibly non-Markov) AFN maps as in [Zwe00] and, as shown, there are absolutely continuous, \(\sigma\)-finite. \(R_\gamma\)-invariant measures \(\nu_\gamma\) on \([0, 1]\) \((0 < \gamma \leq 1)\). Moreover, each

- \(([0, 1], \nu_\gamma, R_\gamma)\) is conservative, exact and pointwise dual ergodic with \(a_n(R_\gamma) \propto n^{\gamma}\) when \(0 < \gamma < 1\) and \(a_n(R_1) \propto \frac{n}{\log n}\).

It is also shown in [Zwe00] that a conservative AFN map induces an AFU map on some interval, which in turn induces an exact AFU map on some sub-interval. This yields GLL sets for each \(R_\gamma\) and establishes \(\mathbf{X}\) as above.
Examples: Geodesic flows. In §6, we consider an integrated, tied-down mixing property $\mathcal{Q}$ (see Theorem 6.2) which is satisfied by the natural extensions of transformations satisfying $\mathcal{A}$ (Remark 5.4). In particular, we show that the geodesic flow (on the unit tangent bundle) of a cyclic cover of a compact hyperbolic surface satisfies $\mathcal{Q}$.

2. Proof of Theorem A

Opening Remark. We first note that to establish the conditional, tied-down renewal mixing property, it suffices to consider $(X)$ for some countable sub-collection $\mathcal{A}$ of $C([0, \infty))$ with dense linear span.

To see this, fix $C \in \mathcal{F}_+$ and $f \in L^1_+$ and suppose that $(X)$ holds for all fixed $g \in \mathcal{A}$. For $x \in X$ define the measures $P_{n,x}$ on $\mathbb{R}_+$ by

$$P_{n,x}(g) = \int_X gdP_{n,x} := \frac{1}{m(C)u(n)} \hat{T}^n \left(1_{Cg}\left(\frac{S_n(f)}{a(n)}\right)\right)(x).$$

By [Aar13, prop. 4.2], for each $g \in \mathcal{A}$, $\exists X_g \in B(X)$, $m(X \setminus X_g) = 0$ so that for $x \in X_g$, $\exists K_{g,x} \subset \mathbb{N}$ so that $\sum_{k=1}^n 1_{K_{g,x}}(k)u(k) = o(a(n))$ and so that

$$P_{n,x}(g) \xrightarrow{n \to \infty, n \notin K_{g,x}} \mathbb{E}(g(m(f)W_\gamma)).$$

Set

$$X_\mathcal{A} := \bigcap_{g \in \mathcal{A}} X_g.$$ 

Since $\mathcal{A}$ is countable $m(X \setminus X_\mathcal{A}) = 0$.

Next, for fixed $x \in X_\mathcal{A}$, we “uniformize” the sets $\{K_{g,x} : g \in \mathcal{A}\}$.

By [Aar13, rem. 4.1(iii)], $\exists K_x \subset \mathbb{N}$ so that $\sum_{k=1}^n 1_{K_x}(k)u(k) = o(a(n))$ and so that $\forall g \in \mathcal{A}$, $K_{g,x} \cap [n, \infty) \subset K_x$ for large $n \in \mathbb{N}$.

It follows that

$$P_{n,x}(g) \xrightarrow{n \to \infty, n \notin K_x} \mathbb{E}(g(m(f)W_\gamma)) \quad \forall g \in \mathcal{A},$$

whence by uniform approximation $\forall g \in C([0, \infty])$, and by monotone approximation in $L^1(\text{dist.} W_\gamma)$, $\forall g \in C_B(\mathbb{R}_+)$. Thus, again by [Aar13, prop. 4.2], $(X)$ holds $\forall g \in C_B(\mathbb{R}_+)$. Let $\Omega \in \mathcal{F}$ be a GLL set with accompanying $T_\Omega$-generating partition $\alpha$. Up to isomorphism, $\Omega = \alpha^\mathbb{N}$, $T_\Omega : \Omega \to \Omega$ is the shift and the collection $C_\alpha(T_\Omega)$ of $(\alpha, T_\Omega)$-cylinder sets forms a base of clopen sets for the Polish topology on $\Omega$. 
Lemma 2.1: Let the CEMPT \((X, m, T)\) be pointwise dual ergodic with \(\gamma\)-regularly varying return sequence \(a(n) = a_n(T)\) (\(\gamma \in (0, 1)\)).

Suppose that \((X, m, T)\) has a GLL set \(\Omega \in \mathcal{B}(X)\), \(0 < m(\Omega) < \infty\). Then for \(A \in C_\alpha(T_\Omega)\), \(g \in C_B(\mathbb{R})\), \(g \geq 0\),

\[
\lim_{n \to \infty} \frac{1}{u(n)} \hat{T}^n \left(1_A g \left(\frac{S_n(1\Omega)}{a(n)}\right)\right) \geq m(A)E(g(W_\gamma)) \quad \text{a.e. on } \Omega,
\]

(\(\text{GL}\))

\[
\frac{1}{a(N)} \sum_{n=1}^N \hat{T}^n \left(1_A g \left(\frac{S_n(1\Omega)}{a(n)}\right)\right) \xrightarrow{N \to \infty} m(A)E(g(W_\gamma)) \quad \text{a.e. on } \Omega,
\]

(\(\text{A}\))

where \(\alpha\) is the accompanying \(T_\Omega\)-generating partition and \(u(n) \sim \frac{\gamma a(n)}{n}\).

Proof. We begin with (GL) and consider only the periodic case, the aperiodic case being similar and easier (cf. [GL63, lemma 2.2.1] and [Aar13, lemma 9.2]).

Fix \(A \in C_\alpha(T_\Omega)\) and \(0 < c < d < \infty\) and \(g \in C_B(\mathbb{R}_+)\) \(c,d\)-nice in the sense that \(\log G : [c, d] \to \mathbb{R}\) is smooth where

\[G(x) := g \left(\frac{1}{x^\gamma}\right) f_{Z_\gamma}(x)\]

with \(f_{Z_\gamma}\) the probability density of \(Z_\gamma\).

Writing \(x_{k,n} := \frac{n}{a^{-1}(k)}\) for \(1 \leq k \leq n\), we have by regular variation that

\[k \sim \frac{a(n)}{x_{k,n}} \quad \text{as } k, n \to \infty, \quad x_{k,n} \in [c, d].\]

Using this and the GLL property of \(\Omega\), we have, as \(n \to \infty\),

\[
\hat{T}^n \left(1_A g \left(\frac{S_n(1\Omega)}{a(n)}\right)\right) = \sum_{k=1}^n \hat{T}_\Omega^k \left(1_{A \cap [\varphi_k=n]} g \left(\frac{k}{a(n)}\right)\right)\]

\[\geq \sum_{1 \leq k \leq n, \ x_{k,n} \in [c,d]} \hat{T}_\Omega^k \left(1_{A \cap [\varphi_k=n]} g \left(\frac{k}{a(n)}\right)\right)\]

(\(\text{B}\))

\[
\sim \sum_{1 \leq k \leq n, \ x_{k,n} \in [c,d]} g \left(\frac{1}{x_{k,n}}\right) \hat{T}_\Omega^k (1_{A \cap [\varphi_k=n]})
\]

\[
\sim m(A) \sum_{1 \leq k \leq n, \ x_{k,n} \in [c,d]} g \left(\frac{1}{x_{k,n}}\right) p f_{Z_\gamma}(x_{k,n}) a^{-1}(k) pZ(n - k\xi).
\]
To continue, we will need

**Lemma 2.2:** Suppose that $a(n)$ is $\gamma$-regularly varying with $\gamma \in (0, 1)$ and satisfies

\[
a^{-1}(n + 1) - a^{-1}(n) \sim \frac{a^{-1}(n)}{\gamma n}.
\]

Let $\mathbb{G} = \mathbb{Z}$ or $\mathbb{R}$, and let $\xi, p \in \mathbb{G}$. $0 < \xi < p$ satisfy $\langle \{n\xi : n \geq 1\} + p\mathbb{Z} \rangle = \mathbb{G}$. Then, for $0 < c < d < \infty$, $g \in C_B(\mathbb{R}_+)$ $[c, d]$-nice and $I \subset (0, p) \cap \mathbb{G}$ an interval:

\[
\sum_{1 \leq k \leq n, \ x_k,n \in [c,d]} g\left(\frac{1}{\gamma k_n}\right) p f_{\gamma}(\frac{1}{\gamma k_n}) 1_{I+p\mathbb{Z}}(n - k\xi)
\]

\[\to_{n \to \infty} \mathbb{E}(1_{[c,d]}(Z_\gamma)g(Z_\gamma^\gamma)Z_\gamma^\gamma),
\]

where $u(n) := \frac{n a(n)}{a(k)}$ and $x_{k,n} := \frac{n}{a^{-1}(k)}$ for $1 \leq k \leq n$.

Note that $\varnothing$ can be ensured by possibly passing to an asymptotically equivalent function.

We will prove Lemma 2.2 after the proof of Lemma 2.1.

**Proof of (GL) given Lemma 2.2.** By Lemma 2.2, for $g \in C_B(\mathbb{R}_+)$ $[c, d]$-nice:

\[
\sum_{1 \leq k \leq n, \ x_k,n \in [c,d]} g\left(\frac{1}{\gamma k_n}\right) p f_{\gamma}(\frac{1}{\gamma k_n}) 1_{I+p\mathbb{Z}}(n - k\xi)
\]

\[\to_{n \to \infty} \mathbb{E}(1_{[c,d]}(Z_\gamma)g(Z_\gamma^\gamma)Z_\gamma^\gamma).
\]

Now

\[\mathbb{E}(1_{[c,d]}(Z_\gamma)g(Z_\gamma^\gamma)Z_\gamma^\gamma) \to_{c \to 0, \ d \to \infty} \mathbb{E}(g(Y_\gamma)Y_\gamma) = \mathbb{E}(g(W_\gamma)),
\]

\[\therefore \mathbb{T}^n(1_A g(S_n(1_\Omega) \frac{a(n)}{a(\Omega)})) \geq u(n) m(A) \mathbb{E}(g(W_\gamma)). \]

**Proof of (\varnothing) given (GL).** It suffices to show that a.e.,

\[\lim_{N \to \infty} \frac{1}{a(N)} \sum_{n=1}^{N} \sum_{1 \leq k \leq n, \ x_k,n \not\in [c,d]} \mathbb{T}_\Omega^k(1_{A \cap \{\varphi_k=n\}} g\left(\frac{k}{a(\Omega)}\right)) \to_{c \to 0, \ d \to \infty} 0.
\]

To this end, note that

\[\sum_{1 \leq k \leq n, \ x_k,n \not\in [c,d]} \mathbb{T}_\Omega^k(1_{A \cap \{\varphi_k=n\}} g\left(\frac{k}{a(\Omega)}\right)) \leq \|g\|_{\infty} \sum_{1 \leq k \leq n, \ x_k,n \not\in [c,d]} \mathbb{T}_\Omega^k(1_{A \cap \{\varphi_k=n\}})
\]
and
\[
\sum_{n=1}^{N} \sum_{1 \leq k \leq n, \ x_{k,n} \notin [c,d]} \hat{T}_k^\Omega(1_{A \cap [\varphi_k = n]})
\]
\[
= \sum_{n=1}^{N} \sum_{1 \leq k \leq n} \hat{T}_k^\Omega(1_{A \cap [\varphi_k = n]}) - \sum_{n=1}^{N} \sum_{1 \leq k \leq n, \ x_{k,n} \in [c,d]} \hat{T}_k^\Omega(1_{A \cap [\varphi_k = n]})
\]
\[
= \sum_{n=1}^{N} \hat{T}^n 1_A - \sum_{n=1}^{N} \sum_{1 \leq k \leq n, \ x_{k,n} \in [c,d]} \hat{T}_k^\Omega(1_{A \cap [\varphi_k = n]})
\]
\[
= \Delta_N(c, d).
\]

Now by (GL) with \( g \equiv 1 \),
\[
\frac{\Delta_N(c, d)}{a(N)} \lesssim m(A)(1 - \mathbb{E}(1_{[c,d]}(Z_\gamma Z_\gamma^{-1}))) \xrightarrow{c \to 0, \ d \to \infty} 0. \quad \blacksquare
\]

The proof of Lemma 2.2 makes use of the following standard

**Equidistribution Lemma:** Suppose that \( K \) is a compact abelian group and that \( \xi \in K, \{ n\xi : n \geq 1 \} = K \).

If \( u_n^{(\nu)} \geq 0 \ (n, \nu \geq 1) \) satisfies
\[
\begin{align*}
(i) & \quad \sum_{n \geq 1} u_n^{(\nu)} \xrightarrow{\nu \to \infty} C \in \mathbb{R}_+; \\
(ii) & \quad \sum_{n \geq 1} |u_n^{(\nu)} - u_{n+1}^{(\nu)}| \xrightarrow{\nu \to \infty} 0,
\end{align*}
\]
then
\[
\sum_{n \geq 1} u_n^{(\nu)} 1_U(x + n\xi) \xrightarrow{\nu \to 0} Cm_K(U) \quad \forall U \in \mathcal{B}(K), \ m_K(\partial U) = 0
\]
uniformly in \( x \in K \), where \( m_K \) denotes normalized Haar measure on \( K \).

**Proof skicth.** Define \( r : K \to K \) by \( r(x) = x + \xi \). Since \( \{ n\xi : n \geq 1 \} = K \), the only \( r \)-invariant probability on \( K \) is the normalized Haar measure \( m_K \).

Define probabilities \( p_{\nu,x} \in \mathcal{P}(K) \ (x \in K, \ \nu \geq 1) \) by
\[
p_{\nu,x}(U) := \frac{1}{C_{\nu}} \sum_{n \geq 1} u_n^{(\nu)} 1_U(x + n\xi)
\]
where
\[
C_{\nu} := \sum_{n \geq 1} u_n^{(\nu)}.
\]
If (I) fails, \( \exists \epsilon > 0, x_k \in X, n_k \to \infty \) and \( f \in C(K) \) so that
\[
|p_{n_k,x_k}(f) - m_K(f)| \geq \epsilon \quad \forall k \geq 1.
\]
By compactness, by passing to a subsequence, it can be ensured that
\[
x_k \xrightarrow{k \to \infty} x \in K \quad \text{and} \quad p_{n_k,x_k} \xrightarrow{k \to \infty} P \in \mathcal{P}(K)
\]
weakly, whence \( |P(f) - m_K(f)| \geq \epsilon \).

On the other hand, it follows from (ii) that
\[
p_{n_k,x_k}(g) - p_{n_k,x_k}(g \circ r) \xrightarrow{k \to \infty} 0 \quad \forall g \in C(K)
\]
whence \( P = m_K \) contradicting failure of (I).

**Proof of Lemma 2.2.** We establish \( \Delta \).

By \( \mathscr{A} \),
\[
x_{k,n} - x_{k+1,n} = \frac{n}{a^{-1}(k)} - \frac{n}{a^{-1}(k+1)} \sim \frac{n}{\gamma k a^{-1}(k)}
\]
as \( k, n \to \infty \), \( x_{k,n} \in [c,d] \).

Also
\[
a(n) = a(x_{k,n}a^{-1}(k)) \sim x_{k,n}^\gamma a(a^{-1}(k)) \sim x_{k,n}^\gamma k
\]
as \( k, n \to \infty \), \( x_{k,n} \in [c,d] \) by the uniform convergence theorem for regularly varying functions. Thus:
\[
\frac{1}{a^{-1}(k)} \sim \frac{\gamma k}{n} \cdot (x_{k,n} - x_{k+1,n}) \sim \frac{\gamma a(n)}{n} \cdot \frac{x_{k,n} - x_{k+1,n}}{x_{k,n}^\gamma}
\]
(\( \delta \))
\[
= u(n) \cdot \frac{x_{k,n} - x_{k+1,n}}{x_{k,n}^\gamma}
\]
whence, as \( n \to \infty \),
\[
\sum_{1 \leq k \leq n, x_{k,n} \in [c,d]} g \left( \frac{1}{x_{k,n}^\gamma} \right) \frac{p f_{Z_\gamma}(x_{k,n})}{a^{-1}(k)} 1_{I + pZ}(n - k \xi)
\]
\[
\sim p u(n) \sum_{1 \leq k \leq n, x_{k,n} \in [c,d]} g \left( \frac{1}{x_{k,n}^\gamma} \right) \frac{(x_{k,n} - x_{k+1,n})}{x_{k,n}^\gamma} f_{Z_\gamma}(x_{k,n}) 1_{I + pZ}(n - k \xi).
\]

In order to use the Equidistribution Lemma, define
\[
v_k^{(n)} := g \left( \frac{1}{x_{k,n}^\gamma} \right) \frac{p f_{Z_\gamma}(x_{k,n})}{a^{-1}(k)} 1_{[x_{k,n} \in [c,d]]} = \frac{p H(x_{k,n})}{n} 1_{[x_{k,n} \in [c,d]]}
\]
with \( H(x) := x g \left( \frac{1}{x^\gamma} \right) f_{Z_\gamma}(x) \).
By (δ), as \( k, n \to \infty \), \( x_{k,n} \in [c, d] \),
\[
v_k^{(n)} \sim u(n) g \left( \frac{1}{x_{k,n}^{\gamma}} \right) \frac{p f_{Z_{\gamma}}(x_{k,n})(x_{k,n} - x_{k+1,n})}{x_{k,n}^{\gamma}} 1_{\{x_{k,n} \in [c, d]\}}.
\]

Thus, using the convergence of Riemann sums to the Riemann integral,
\[
\sum_{k \geq 1} v_k^{(n)} \sim p u(n) \mathbb{E}(1_{[c,d]}(Z_{\gamma})g(Z_{\gamma}^{-\gamma})Z_{\gamma}^{-\gamma}).
\]

Next suppose that \( x_{k_0-1,n} < c \leq x_{k_0,n} < x_{k_1,n} \leq d < x_{k_1+1,n} \). For \( k \in [k_0, k_1) \),
\[
|v_k^{(n)} - v_{k+1}^{(n)}| = \frac{p}{n} |H(x_{k,n}) - H(x_{k+1,n})|
\sim \frac{p}{n} |H'(x_{k,n})|(x_{k,n} - x_{k+1,n})
\leq M \frac{p}{n} (x_{k,n} - x_{k+1,n})
\]
where \( M := \sup_{x \in [c, d]} |H'(x)| \) and
\[
\sum_{k \geq 1} |v_k^{(n)} - v_{k+1}^{(n)}| \leq \frac{pM}{n} \sum_{k=k_0}^{k_1-1} (x_{k,n} - x_{k+1,n}) + |v_{k_0}^{(n)}| + |v_{k_1}^{(n)}|
= \frac{pM}{n} (x_{k_0,n} - x_{k_1,n}) + \frac{2pK}{n} \quad \text{with} \quad K := \sup_{x \in [c, d]} |H(x)|
\leq \frac{R}{n} \quad \text{with} \quad R := p(M(d - c) + 2K)
= o(u(n)) \quad \text{as} \quad n \to \infty.
\]

Thus, by (1) with \( u_k^{(n)} := \frac{v_k^{(n)}}{u(n)} \),
\[
\sum_{k \geq 1, x_{k,n} \in [c, d]} g \left( \frac{1}{x_{k,n}^{\gamma}} \right) \frac{p f_{Z_{\gamma}}(x_{k,n})}{a^{-1}(n)} 1_{I + pZ}(n - k\xi)
= \sum_{k \geq 1} v_k^{(n)} 1_{I + pZ}(n - k\xi)
\sim pu(n) \mathbb{E}(1_{[c,d]}(Z_{\gamma})g(Z_{\gamma}^{-\gamma})Z_{\gamma}^{-\gamma}) \cdot m_{\mathbb{E}/pZ}(I + pZ)
= u(n)m_{\mathbb{E}}(I) \mathbb{E}(1_{[c,d]}(Z_{\gamma})g(Z_{\gamma}^{-\gamma})Z_{\gamma}^{-\gamma}).
\]

Remark: The aperiodic form of Lemma 2.2 is * with \( p = 1 \).
Remarks about mixing. It follows from (GL) and (GL) that for \((X,m,T)\) pointwise, dual ergodic MPT with \(a(n) = a_n(T)\) \(\gamma\)-regularly varying \((\gamma \in (0,1))\), and a GLLT set \(\Omega \in \mathcal{B}(X), 0 < m(\Omega) < \infty\), the following are equivalent:

(i) \[
\frac{1}{u(n)} \hat{T}^n(1_A g\left(\frac{S_{n-1}(1\Omega)}{a(n)}\right)) \xrightarrow{n \to \infty} m(A) E(g(W_{\gamma})) \text{ a.e. on } \Omega, \forall A \in C_{\alpha}(T_{\Omega}), g \in C_B(\mathbb{R}), g \geq 0;
\]

(ii) \[
\lim_{n \to \infty} \frac{1}{u(n)} \sum_{1 \leq k \leq n, x_{k,n} \notin [c,d]} \hat{T}^{k}_{\Omega}(1_{A \cap [\varphi_k = n]} \sim \hat{T}^{n-1}_{\Omega} 1_{\Omega} \sim u(n) \text{ a.s. on } \Omega.
\]

By [MT12, theorem 2.1], (iii) holds (whence also (i)) if \(\gamma \in \left(\frac{1}{2}, 1\right)\), \(T_{\Omega}\) is mixing and there exists a partition \(\alpha \subset B(\Omega)\) so that \((\Omega, m_{\Omega}, T_{\Omega}, \alpha)\) is either a Gibbs–Markov or an AFU map with \(\varphi : \Omega \to \mathbb{N}\) non-arithmetic and \(\alpha\)-measurable.

Proof of Theorem A: finish. It suffices to prove \((\mathfrak{X})\) for \(g > 0\) smooth, with

\[
x \mapsto \log g(e^x)
\]

uniformly continuous on \([-\infty, \infty]\) (see the opening remark at the beginning of the section).

We first fix \(f = 1_{\Omega}\) and \(g > 0\) as above; and show that \((\mathfrak{X})\) holds for \(C \in F_+\).

To this end, let

\[
\mathfrak{R} = \mathfrak{R}(g) := \{A \in F_+: (\mathfrak{X}) \text{ holds for } 1_{\Omega}, g \text{ and } A\}.
\]

We will show that \(\mathfrak{R} = F_+,\) and first show that \(\mathfrak{R}\) contains a dense, hereditary ring.

By Lemma 2.1, we see, using [Aar13, prop. 4.2] as in the proof of [AN17, prop. 3.1], that

\[
\mathfrak{B}(\Omega) \subset \mathfrak{R}.
\]

Next, we claim that \(T^{-1}\mathfrak{R} \subset \mathfrak{R}\).

To see this, let \(C \in \mathfrak{R}\). Then

\[
\hat{T}^n\left(1_{T^{-1}C} g\left(\frac{S_{n}(1\Omega)}{a(n)}\right)\right) = \hat{T}^n\left(1_{C} g\left(\frac{1\Omega+S_{n-1}(1\Omega)}{a(n)}\right)\right)
\]

\[
\sim \hat{T}^n\left(1_{C} g\left(\frac{S_{n-1}(1\Omega)}{a(n-1)}\right)\right) \text{ by u.c. of } x \mapsto \log g(e^x)
\]

\[
= \hat{T}^{n-1}\left(1_{C} g\left(\frac{S_{n-1}(1\Omega)}{a(n-1)}\right)\right)
\]

and \(T^{-1}C \in \mathfrak{R}\).
Thus \( \mathcal{R} \) contains the dense, hereditary ring

\[
\mathcal{R} := \left\{ A \in \mathcal{F}_+ : \exists \ n \geq 0, \ A \subset \bigcup_{k=0}^{n} T^{-k}\Omega \right\}.
\]

Now fix \( C \in \mathcal{F}_+ \) and let \( D \in \mathcal{R} \) so that \( D \subset C \).

It follows that

\[
\left| \hat{T}^n \left( 1_C g \left( \frac{S_n(1\Omega)}{a(n)} \right) \right) - u(n)m(C)\mathbb{E}(g(W_\gamma)) \right|
\leq \hat{T}^n \left( 1_{C \setminus D} g \left( \frac{S_n(1\Omega)}{a(n)} \right) \right) + \hat{T}^n \left( 1_D g \left( \frac{S_n(1\Omega)}{a(n)} \right) \right) - u(n)m(D)\mathbb{E}(g(W_\gamma))
\]
\[
+ u(n)m(C \setminus D)\mathbb{E}(g(W_\gamma))
\]
\[
\leq \left| \hat{T}^n \left( 1_D g \left( \frac{S_n(1\Omega)}{a(n)} \right) \right) - u(n)m(D)\mathbb{E}(g(W_\gamma)) \right|
\]
\[
+ \|g\|_{C_{B}} \left( \hat{T}^n 1_{C \setminus D} + u(n)m(C \setminus D) \right),
\]

whence using \( D \in \mathcal{R} \subseteq \mathcal{R} \),

\[
\lim_{N \to \infty} \frac{1}{a(N)} \sum_{n=1}^{N} \left| \hat{T}^n \left( 1_C g \left( \frac{S_n(1\Omega)}{a(n)} \right) \right) - u(n)m(C)\mathbb{E}(g(W_\gamma)) \right|
\leq \|g\|_{C_{B}} \lim_{N \to \infty} \frac{1}{a(N)} \sum_{n=1}^{N} \left( \hat{T}^n 1_{C \setminus D} + u(n)m(C \setminus D) \right)
\]
\[
= 2\|g\|_{C_{B}} m(C \setminus D)
\]
a.e. by PDE of \( T \)

\[
\xrightarrow{D \uparrow C, D \in \mathcal{R}} 0
\]

and \( \mathcal{R} = \mathcal{F}_+ \).

To complete the proof of (\( \mathcal{X} \)), fix \( F \in L^1_+ \) and \( g > 0 \) smooth, with \( x \mapsto \log g(e^x) \) uniformly continuous on \([ -\infty, \infty ] \).

By the ratio ergodic theorem

\[
\frac{S_n(F)}{S_n(1\Omega)} \xrightarrow{n \to \infty} m(F) \quad \text{a.e.}
\]

Suppose the convergence is uniform on \( C \in \mathcal{F}_+ \). Then \( \exists \ \epsilon_N \downarrow 0 \) so that

\[
g \left( \frac{S_n(F)}{a(n)} \right) = (1 \pm \epsilon_n)g \left( \frac{m(F)S_n(1\Omega)}{a(n)} \right) \quad \text{on } C \forall n \geq 1.
\]
Thus
\[
\left| \hat{T}^n \left( 1_C g \left( \frac{S_n(F)}{a(n)} \right) \right) - m(C) \mathbb{E}(g(m(F)W_\gamma)) u(n) \right| \\
\leq \left| \hat{T}^n \left( 1_C \left( g \left( \frac{S_n(F)}{a(n)} \right) - g \left( \frac{m(F)S_n(1_\Omega)}{a(n)} \right) \right) \right) \right| \\
+ \left| \hat{T}^n \left( 1_C g \left( \frac{m(F)S_n(1_\Omega)}{a(n)} \right) \right) - m(C) \mathbb{E}(g(m(F)W_\gamma)) u(n) \right| \\
\leq \epsilon_n \|g\|_\infty \hat{T}^n 1_C + \left| \hat{T}^n \left( 1_C g \left( \frac{m(F)S_n(1_\Omega)}{a(n)} \right) \right) - m(C) \mathbb{E}(g(m(F)W_\gamma)) u(n) \right|
\]
and using the case \( f = 1_\Omega \),
\[
\frac{1}{a(N)} \sum_{n=1}^{N} \left| \hat{T}^n \left( 1_C g \left( \frac{S_n(F)}{a(n)} \right) \right) - m(C) \mathbb{E}(g(m(F)W_\gamma)) u(n) \right| \\
\leq \frac{1}{a(N)} \sum_{n=1}^{N} \left( \epsilon_n \|g\|_\infty \hat{T}^n 1_C \right. \\
\left. + \left| \hat{T}^n \left( 1_C g \left( \frac{m(F)S_n(1_\Omega)}{a(n)} \right) \right) - m(C) \mathbb{E}(g(m(F)W_\gamma)) u(n) \right| \right)
\xrightarrow{N \to \infty} 0 \ a.e.
\]

Now fix \( D \in \mathcal{F}_+ \). By Egorov’s theorem \( \exists C_\nu \in \mathcal{F}_+ \), \( C_\nu \uparrow D \ mod \ m \) so that
\[
\frac{S_n(F)}{S_n(1_\Omega)} \xrightarrow{n \to \infty} m(F) \quad \text{uniformly on each } C_\nu.
\]

Thus, a.e., \( \forall \nu \geq 1 \), as \( N \to \infty \),
\[
\frac{1}{a(N)} \sum_{n=1}^{N} \left| \hat{T}^n \left( 1_D g \left( \frac{S_n(F)}{a(n)} \right) \right) - m(D) \mathbb{E}(g(m(F)W_\gamma)) u(n) \right| \\
\leq \frac{\|g\|_\infty}{a(N)} \sum_{n=1}^{N} (\hat{T}^n 1_{D \setminus C_\nu} + m(D \setminus C_\nu) u(n)) \\
\xrightarrow{N \to \infty} 2 \|g\|_\infty m(D \setminus C_\nu) \xrightarrow{\nu \to \infty} 0.
\]
3. Periodic LLTs

In this section, we give conditions for an asymptotically $\gamma$-stable, positive, ergodic, stationary process $(\Omega, \mu, \tau, \phi)$ to satisfy a periodic LLT.

**Fibered systems.** We assume first that there is a countable, generating partition $\alpha \subset B(\Omega)$ so that $\tau : a \to \tau a$ is invertible and nonsingular for $a \in \alpha$. Such $(\Omega, \mu, \tau, \alpha)$ is called a fibered system.

Up to isomorphism, a fibered system $(\Omega, \mu, \tau, \alpha)$ has the form:

$$\Omega \subset S^\mathbb{N}$$

where $S$ is a finite or countable state space and $\Omega$ is a subshift (i.e., shift invariant and closed with respect to the polish product topology on $S^\mathbb{N}$) and

$$\alpha = \{ [s] = \{ (x_1, x_2, \ldots) \in X : x_1 = s \} : s \in S \}.$$ 

Thus the fibered system $(\Omega, \mu, \tau, \alpha)$ can be considered as a continuous map of a polish space.

If $(\Omega, \mu, \tau, \alpha)$ is a fibered system, then for $n \geq 1$, so is $(\Omega, \mu, \tau^n, \alpha_n)$ where

$$\alpha_n := \bigvee_{k=0}^{n-1} \tau^{-k} \alpha.$$ 

**Inverse branches and the transfer operator.** For $n \geq 1$, there are $\mu$-nonsingular inverse branches of $\tau^n$ denoted $v_a : \tau^n a \to a$ ($a \in \alpha_n$) with Radon–Nikodym derivatives

$$v'_a := \frac{d\mu \circ v_a}{d\mu} : \tau^n a \to \mathbb{R}_+.$$ 

The transfer operator is given by

$$\hat{\tau}(f) := \sum_{a \in \alpha} 1_{\tau a} v'_a f \circ v_a.$$ 

**Doeblin–Fortet operators.** Let $\mathcal{L} \subset L^1(\mu)$ be a Banach space so that $(L^1(\mu), \mathcal{L})$ is an adapted pair in the sense that

$$\mathcal{L} \subset L^1(\mu), \quad \| \cdot \|_{L^1(\mu)} \leq \| \cdot \|_\mathcal{L}, \quad \mathcal{L} L^1(\mu) = L^1(\mu),$$

and $\mathcal{L}$-closed, $\mathcal{L}$-bounded sets are $L^1(\mu)$-compact.

---

3 See [AD01] and references therein, also [Nor72, HH01].
We say that an operator $P \in \text{hom}(\mathcal{L}, \mathcal{L}) \cap \text{hom}(L^1(\mu), L^1(\mu))$ is Doeblin–Fortet (D-F) on $(L^1(\mu), \mathcal{L})$ if

$$\text{DF}(i) \|P^n f\|_1 \leq H\|f\|_1 \quad \forall n \in \mathbb{N}, f \in L^1(\mu)$$

$$\text{DF}(ii) \|Pf\|_\mathcal{L} \leq \theta\|f\|_\mathcal{L} + R\|f\|_1 \quad \forall f \in \mathcal{L},$$

where $R, H \in \mathbb{R}_+$ and $\theta \in (0, 1)$.

It shown in [ITM50] that a D-F operator $P \in \text{hom}(\mathcal{L}, \mathcal{L})$ has spectral radius $\rho(P) \leq 1$ and that, if $\rho(P) = 1$, then $P$ is quasicompact in the sense that

$$\exists A = A(P) \in \text{hom}(\mathcal{L}, \mathcal{L}) \quad \text{of the form}$$

$$A = \sum_{k=1}^N \lambda_k E_k$$

with $N \geq 1$, $E_1, \ldots, E_N \in \text{hom}(\mathcal{L}, \mathcal{L})$ finite dimensional projections,

$$\lambda_1, \ldots, \lambda_N \in S^1 := \{z \in \mathbb{C} : |z| = 1\}$$

so that $\rho(P - A) < 1$.

In particular, for $\rho(P - A) < \theta < 1$, $\exists M > 0$ so that

$$\|P^n f - A^n f\|_\mathcal{L} \leq M\theta^n\|f\|_\mathcal{L} \quad \forall n \geq 1, f \in \mathcal{L}.$$}

Moreover, the spectral radius of a D-F operator $P \in \text{hom}(\mathcal{L}, \mathcal{L})$ satisfies $\rho(P) \leq 1$ with equality iff $\exists \lambda \in S^1, g \in L^1(\mu)$ so that $Pg = \lambda g$, in which case $|g|$ is constant, $g \in \mathcal{L}$ and

$$\|g\|_\mathcal{L} \leq \frac{RH}{1 - \theta}\|g\|_1$$

with $\theta, R, H$ as in DF(ii).

For $P \in \text{hom}(\mathcal{L}, \mathcal{L})$ D-F with $\rho(P) = 1$ we will write $\dim P := \dim A(P)\mathcal{L}$.

If $\dim P = 1$, then $A(P) = \lambda(P)N(P)$ where $\lambda(P) \in \mathbb{C}$, $|\lambda(P)| \leq 1$ and $N(P) \in \text{hom}(\mathcal{L}, \mathcal{L})$ is a projection onto a one dimensional subspace. Let

$$\text{DF}(1) := \{P \in \text{hom}(\mathcal{L}, \mathcal{L}) : \text{D-F, } \rho(P) = 1 \text{ and } \dim P = 1\}.$$ 

The mixing of $\tau$ ensures that its transfer operator $\hat{\tau}$ is D-F, then $\hat{\tau} \in \text{DF}(1)$ with $\lambda = 1$ and $N(\hat{\tau}) f = \int_\Omega f d\mu$. We will be interested in fibered systems for which the transfer operator is DF on some $(L^1(\mu), \mathcal{L})$.

**DOEBLIN–FORTET FIBERED SYSTEMS.** We will call the probability preserving, mixing fibered system $(\Omega, \mu, \tau, \alpha)$ a **Doeblin–Fortet (D-F) fibered system** if there is a Banach space $L_\tau$ so that $(L^1(\mu), L_\tau)$ is an adapted pair, $1_A \in L_\tau \forall A \in \alpha$ and the transfer operator $\hat{\tau}$ is DF on $(L^1(\mu), L_\tau)$. 


EXAMPLE 1: AFU maps. As in [Zwe98], an AFU map is an interval map \((Ω, μ, τ, α)\) where for each \(a ∈ α\), \(τ\)|\(_a\) is (the restriction of) a \(C^2\) diffeomorphism \(τ : \overline{a} → \overline{τa}\) satisfying in addition:

\[(A) \sup_X |τ''| (τ')^2 < ∞,\]
\[(F) τα := \{τA : A ∈ α\} \text{ is finite},\]
\[(U) \inf_{x∈a∈α} |τ'(x)| > 1.\]

If \((Ω, μ, τ, α)\) is an AFU map, then so is \((Ω, μ, τ^N, α^N)\) for \(N ≥ 2\) with \(α^N := \bigvee_{k=0}^{N-1} τ^{-k}α\).

It follows from [Ryc83, Zwe98] that:

- a totally ergodic AFU map \((Ω, μ, τ, α)\) is exact, and a D-F fibered system with \(L_τ = BV\), the space of functions on the interval \(Ω\) with bounded variation equipped with the norm

\[\|f\|_{L_τ} := \|f\|_1 + \sqrt{Ω f};\]

- any AFU map induces an exact AFU map on some interval.

EXAMPLE 2: Gibbs–Markov maps. The fibered system \((Ω, μ, τ, α)\) is a Markov map if \(τa ∈ σ(α) \mod μ \ ∀ a ∈ α\), and a Gibbs–Markov (G-M) map if, in addition, \(\inf_{a∈α} μ(τa) > 0\) and, for some \(θ ∈ (0, 1)\),

\[\sup_{n≥1, a∈α_0^{-1}} \frac{1}{gt(x,y)} \left| \frac{v'_{a}(x)}{v'_{a}(y)} - 1 \right| < ∞\]

with

\[t(x, y) = \min\{n ≥ 1 : α_n(x) ≠ α_n(y)\} ≤ ∞.\]

Note that a Markov AFU map is Gibbs–Markov.

As shown in [AD01, §1], a G-M map \((Ω, μ, τ, α)\) is a D-F fibered system with \(L_τ\), the space of \((α, θ)\)-Hölder functions on \(Ω\); that is

\[\{f : Ω → R : D_{Ω,θ,α}(f) := \sup_{x,y∈Ω} |f(x) - f(y)| \frac{1}{gt(x,y)} < ∞\}\]

equipped with the norm

\[\|f\|_{L_τ} := \|f\|_1 + D_{Ω,θ,α}(f).\]
DOEBLIN–FORTET COCYCLES. Let \((\Omega, \mu, \tau, \alpha)\) be a D-F fibered system.

Let \(\mathbb{G} \leq \mathbb{R}^d\) be a subgroup of full dimension. For \(\phi : \Omega \to \mathbb{G}\) measurable and \(\chi \in \hat{\mathbb{G}}\), define \(P_{\phi, \chi} \in \text{hom}(L^1(\mu), L^1(\mu))\) by

\[
P_{\phi, \chi}(f) := \hat{\tau}(\chi(\phi)f).
\]

Call the measurable function (cocycle) \(\phi = (\phi^{(1)}, \ldots, \phi^{(d)}) : \Omega \to \mathbb{G}\)

- **Doeblin–Fortet** (D-F) if it is locally \(L_\tau\) in the sense that \(\exists M > 0\) so that \(\forall 1 \leq k \leq d\) and \(A \in \alpha\),

\[
1_A \phi^{(k)} \in L_\tau \quad \text{and} \quad \|1_A \phi^{(k)} - \int_A \phi^{(k)} d\mu\|_{L_\tau} \leq M;
\]

- **admissible** if

  (i) \(P_{\phi, \chi}\) is a D-F operator \(\forall \chi \in \hat{\mathbb{G}}\) and

  (ii) \(\chi \mapsto P_{\phi, \chi}\) is continuous \((\hat{\mathbb{G}} \to \text{hom}(L_\tau, L_\tau))\).

It follows from [AD01] and [ADSZ04] (respectively) that a D-F cocycle over a G-M-/AFU-map is admissible.

DISCRETE D-F COCYCLES. For \((\Omega, \mu, \tau, \alpha)\) a G-M map, a cocycle \(\phi : \Omega \to \mathbb{Z}^d\) is D-F iff \(\exists N \in \mathbb{N}, M > 0, \alpha_N := \sqrt{N-1}^{-k} \alpha\)-measurable functions \(g_A : A \to \mathbb{Z}^d\) \((A \in \alpha)\) with \(\|g_A\|_{\infty} \leq M\) and \(H : \alpha \to \mathbb{Z}^d\) so that

\[
\phi|_A = H(A) + g_A \quad \forall A \in \alpha.
\]

For \((\Omega, \mu, \tau, \alpha)\) an AFU map, a cocycle \(\phi : \Omega \to \mathbb{Z}^d\) is D-F iff \(\exists M > 0\), step functions \(g_A : A \to \mathbb{Z}^d\) \((A \in \alpha)\) with \(\|g_A\|_{\infty} \leq M\) and \(H : \alpha \to \mathbb{Z}^d\) so that

\[
\phi|_A = H(A) + g_A \quad \forall A \in \alpha.
\]

REDUCTION OF COCYCLES. It is standard that for an EPPT \((\Omega, \mu, \tau)\), \(\phi : \Omega \to \mathbb{G}\) measurable, \(\chi \in \hat{\mathbb{G}}, \lambda \in \mathbb{S}^1\) and \(g \in L^1(\mu)\):

\[
P_{\phi, \chi}(g) = \lambda g \iff \chi(\phi)g = \lambda g \circ \tau
\]

and in this case, \(|g|\) is constant.

If, in addition, \((\Omega, \mu, \tau, \alpha)\) is a D-F fibered system and \(\phi : \Omega \to \mathbb{G}\) is admissible, this situation is characterized by

\[
\rho(P_{\phi, \chi}) = 1
\]

and entails \(g \in L_\tau\).
Thus for $\phi : \Omega \to \mathbb{G}$ admissible over a D-F fibered system:

- $\phi$ is non-arithmetic iff for $\chi \in \hat{\mathbb{G}} \setminus \{1\}$, 1 is not an eigenvalue of $P_{\phi,\chi}$, and
- $\phi$ is aperiodic iff for $\chi \in \hat{\mathbb{G}} \setminus \{1\}$, $P_{\phi,\chi}$ has no eigenvalues on the unit circle.

As in [AD01, §3], let

$$\mathcal{Q}_\phi := \{\chi \in \hat{\mathbb{G}} : \rho(P_{\phi,\chi}) = 1\} = \{\chi \in \hat{\mathbb{G}} : \exists \lambda \in \mathbb{S}^1, g \in L^1, \chi(\phi)g = \lambda g \circ \tau\}.$$

Then

- [AD01, Proposition 3.8]: $\mathcal{Q}_\phi$ is a closed subgroup of $\hat{\mathbb{G}}$;
- $\phi$ is aperiodic iff $\mathcal{Q}_\phi = \{1\}$.

Let

$$\mathcal{K}_\phi = \mathcal{Q}_\phi^\perp := \{x \in \mathbb{G} : \chi(x) = 1 \forall \chi \in \mathcal{Q}_\phi\}$$

be a closed subgroup of $\mathbb{G}$.

**Cocycle reduction formula:** Let $(\Omega, \mu, \tau, \alpha)$ be a DF fibered system, either G-M or AFU.

Suppose that $\phi : \Omega \to \mathbb{G} = \mathbb{R}$ or $\mathbb{Z}$ is a non-arithmetic, D-F cocycle. Then

$$\phi = \xi - g + g \circ \tau + \psi$$

where $\xi \in \mathbb{G}$, $Z\xi + \mathcal{K}_\phi = \mathbb{G}$, $g \in L_\tau$ and $\psi : \Omega \to \mathcal{K}_\phi$ is locally $L_\tau$ and aperiodic.

We will call the formula (a) the \textbf{reduction} of $\phi$.

In case $(\Omega, \mu, \tau, \alpha)$ is G-M, (a) follows from the topological Markov property of $\tau$ and is a version of Livsic’s cohomology theorem ([Liv72]). See also [ANS06, lemma 4.3].

**Proof.** If $\phi$ is aperiodic, then $\mathcal{Q}_\phi = \{1\}$, $\mathcal{K}_\phi = \mathbb{G}$ and we may set $\xi = 0$, $g \equiv 0$ and $\psi \equiv \phi$.

Otherwise, by non-arithmeticity,

$$\mathcal{Q}_\phi^\perp \cong \mathcal{K}_\phi \cong \mathbb{Z}.$$

By [AD01, prop. 3.8], $\exists \xi \in \hat{\mathcal{Q}}_\phi \subseteq \mathbb{G}$ and $G_{\chi} \in L_\tau$ ($\chi \in \Omega$) satisfying

$$P_{\phi,\chi}(G_{\chi}) = \xi(\chi)G_{\chi}.$$

Possibly renormalizing the $G_{\chi}$, we can ensure that, in addition, $|G_{\chi}| = 1$ and $G_{\chi+\chi'} = G_{\chi}G_{\chi'}$. 
Thus $\exists g : \Omega \to \mathbb{G}$ so that $G_{\chi} = \chi(g)$. Standard lifting theory now shows that $g \in L_{\tau}$.

Lastly, define $\psi : \Omega \to \mathbb{G}$ by

$$\psi := \phi - \xi + g - g \circ \tau.$$  

Evidently $\psi$ is locally $L_{\tau}$.

Moreover, $\chi(\psi) \equiv 1 \ \forall \chi \in \mathbb{G}_{\phi}$, whence $\psi : \Omega \to \mathbb{K}_{\phi}$.

To see that $\mathbb{Z}\xi + \mathbb{K}_{\phi} = \mathbb{G}$, suppose otherwise. Then $\exists \chi \in \mathbb{G} \setminus \{1\}$ so that

$$\chi(n\xi + k) = 1 \ \forall n \in \mathbb{Z}, k \in \mathbb{K}_{\phi}.$$  

It follows that

$$\chi(\phi)\chi(g) = \chi(\xi + g \circ \tau + \psi) = \chi(g \circ \tau)$$

contradicting non-arithmeticity.

To see that $\psi$ is aperiodic suppose that $\chi \in \mathbb{G}_{\phi}$, $\lambda \in S^1$ and $H \in L^1(\mu)$ satisfy $\chi(\psi)H = \lambda H \circ \tau$. Then

$$\chi(\phi)H\chi(g) = \chi(\psi + \xi + g \circ \tau)$$

$$= \chi(\psi)Hg\chi(\xi)\frac{\chi(g \circ \tau)}{\chi(g)}$$

$$= \lambda\chi(\xi)H \circ \tau\chi(g \circ \tau)$$

whence $\chi \in \mathbb{G}_{\phi}$ and $\lambda\chi(\xi) = \chi(\xi)$ entailing $\lambda = 1$.  

If $\phi$ is aperiodic, the aperiodic LLTs follow from [AD01, §6] (G-M case) and [ADSZ04, §5] (AFU case).

Our proof of Theorem B is based on this and our advertised

**Theorem 3.1 (Periodic LLT):** Suppose that $(\Omega, \mu, \tau, \alpha)$ is a $D$-$F$ fibered system, G-M or AFU.

Let $\phi : \Omega \to \mathbb{G} = \mathbb{R}$ or $\mathbb{Z}$ be a positive $D$-$F$ cocycle so that $(\Omega, \mu, \tau, \phi)$ is an asymptotically $\gamma$-stable, positive, ergodic, stationary, stochastic process with $0 < \gamma < 1$.

If $\phi : \Omega \to \mathbb{G}$ is non-arithmetic but not aperiodic, then $(\Omega, \mu, \tau, \alpha, \phi)$ satisfies a periodic LLT.

The theorem for independent, identically distributed random variables is [She64, theorem 3].
The rest of this section is devoted to the

**Proof of Theorem 3.1.** By Nagaev’s theorem ([Nag57]) (see also [AD01, theorem 4.1]): There are constants $\epsilon > 0$, $K > 0$ and $\theta \in (0, 1)$; and continuous functions $\lambda = \lambda_\phi : B(0, \epsilon) \to B_C(0, 1)$, $N = N_\phi : B(0, \epsilon) \to \text{hom}(L_\tau, L_\tau)$ such that

$$\|P_{n,t} h - \lambda(t)^n N(t) h\|_{L_\tau} \leq K \theta^n \|h\|_{L_\tau} \quad \forall |t| < \epsilon, n \geq 1, h \in L_\tau.$$ 

Here $P_{\phi,t} := P_{\phi,\chi_t}$ with $\chi_t(s) = e(t s)$ is as defined in the introduction and for $|t| < \epsilon$, $N(t)$ is a projection onto a one-dimensional subspace (spanned by $g(t) := N(t)\mathbb{1}$).

In view of (\(\Psi\)), [AD01, theorem 5.1] applies in the G-M case and [ADSZ04, theorem 8] applies in the AFU case to show that

$$\left| \lambda_\phi(t) - \int_{\Omega} \chi_t(\phi) d\mu \right| = o\left( \frac{1}{\alpha(|t|)} \right) \text{ as } t \to 0.$$ 

Consequently, \(\Psi\)

$$\lambda_\phi\left( \frac{t}{a^{-1}(n)} \right)^n \xrightarrow{n \to \infty} \mathbb{E}(\chi_t(Z_\gamma)).$$

**Proof of theorem 3.1 in the continuous case.** Possibly renormalizing, we assume without loss of generality that $\mathcal{K}_\phi = \mathbb{Z}$ in the cocycle reduction formula and $\phi$ has reduction

$$\phi = \xi - F + F \circ \tau + \psi,$$

where $\xi \in (0, 1)$, $\mathbb{Z} \xi + \mathbb{Z} = \mathbb{R}$, $F \in L_\tau$ and $\psi : \Omega \to \mathbb{Z}$ is locally $L_\tau$ and aperiodic.

It suffices to show, for fixed $h \in L_\tau$, $g \in L^1(\mathbb{R})$ with Fourier transform $\hat{g} \in C^\infty(\mathbb{R})$ and $k_n \in \mathbb{R}_+$, $\frac{k_n}{a^{-1}(n)} \xrightarrow{n \to \infty} \kappa \in \mathbb{R}_+$, that

$$a^{-1}(n) \hat{g}^n(h g(\phi_n - k_n)) \approx G_g(n \xi - k_n) \mathbb{E}(h) f_{Z_\gamma}(\kappa)$$

where

$$G_g(y) := \sum_{J \in \mathbb{Z}} g(y + J)$$

and $f_{Z_\gamma}$ is the probability density of $Z_\gamma$. 


Writing \( e(t) := e^{2\pi it} \), we have

\[
\hat{\tau}^n(hg(\phi_n - k_n)) = \int_\mathbb{R} \hat{g}(z)e(k_nz)\hat{\tau}^n(e(-z\phi_n)h)dz = \int_\mathbb{R} \hat{g}(z)e(k_nz)\hat{\tau}^n(e(-z(\psi_n + n\xi + F - F \circ \tau^n))h)dz = \sum_{J \in \mathbb{Z}} \int_{\left(-\frac{1}{2}, \frac{1}{2}\right]} \hat{g}(z+J)e((k_n - n\xi)(z+J))\hat{\tau}^n(e(-(z+J)(\psi_n + F - F \circ \tau^n))h)dz.
\]

For \( J \in \mathbb{Z}, z \in \left(-\frac{1}{2}, \frac{1}{2}\right) \),

\[
\hat{g}(z + J)e((k_n - n\xi)(z + J))\hat{\tau}^n(e(-(z + J)(\psi_n + F - F \circ \tau^n))h) = \hat{g}(z + J)e((k_n - n\xi)J)e((k_n - n\xi)z)e(-zF)P_{\psi}^n e(-zF)h.
\]

Set

\[
\Gamma_{n,g}(z) := \sum_{J \in \mathbb{Z}} \hat{g}(z + J)e((k_n - n\xi)J).
\]

Then by the Poisson summation formula,

\[
\Gamma_{g,n}(0) = G_g(k_n - n\xi),
\]

and

\[
\hat{\tau}^n(hg(\phi_n - k_n)) = \int_{\left(-\frac{1}{2}, \frac{1}{2}\right]} \Gamma_{n,g}(z)e((k_n - n\xi)z)e(-zF)P_{\psi}^n e(-zF)h)dz = \int_{-\delta}^{\delta} e(z(-F + n\xi - k_n))\Gamma_{n,g}(z)\lambda_{\psi}(z)\n N_{\psi}(z)(he(zF))dz + O(\rho^n) \quad \text{in } L_\tau
\]

\[
= \mathbb{E}(h) \int_{-\delta}^{\delta} e(z(n\xi - k_n))\Gamma_{n,g}(z)\lambda_{\psi}(z)dz + \mathcal{E}_n + O(\rho^n)
\]

where

\[
\mathcal{E}_n := \int_{-\delta}^{\delta} \Gamma_{n,g}(z)e^{-izk_n}\lambda_{\phi}(z)(e(-zF)N_{\psi}(z)(he(zF)) - \mathbb{E}(h))dz.
\]
We claim that
\[(\mathcal{E}_n)_{L^\tau} = O\left(\frac{1}{a^{-1}(n)^2}\right) \text{ as } n \to \infty.\]  

To see this, set 
\[n(z) := e(-zF)N\psi(z)(he(zF)).\]

Then by [AD01, theorem 2.4] (G-M case) and [ADSZ04, prop. 5] (AFU case), for some constant \(K > 0\)
\[
\|\mathcal{E}_n\|_{L^\tau} \leq K\|\mathcal{E}_n - \mathbb{E}(h)\|_{L^\tau} \leq K\|\mathcal{E}_n - \mathbb{E}(h)\|_{L^\tau},
\]
Thus
\[
\|\mathcal{E}_n\|_{L^\tau} \leq \|g\|_1 \int_{-\delta}^\delta |\lambda_\phi(z)|^n |n(z) - \mathbb{E}(h)|_{L^\tau} dz
\]
and
\[
\int_{-\delta}^\delta |\lambda_\phi(z)|^n |z| dz = \frac{1}{a^{-1}(n)^2} \int_{-\delta a^{-1}(n)}^{\delta a^{-1}(n)} |x| |\lambda_\phi(x)\left(\frac{x}{a^{-1}(n)}\right)|^n dx
\]
\[
\sim \frac{1}{a^{-1}(n)^2} \int_{\mathbb{R}} |x| e^{-C|x|^\gamma} dx \quad \text{with } C := \frac{(2\pi)^\gamma \cos(\frac{\pi \gamma}{2})}{\Gamma(1 + \gamma)} \quad \text{(by (3))}.
\]

Thus,
\[
a^{-1}(n)\tau^n(hg(\phi_n - k_n))
\]
\[
= a^{-1}(n)\mathbb{E}(h) \int_{-\delta}^\delta e(z(n\xi - k_n))\Gamma_{n,g}(z)\lambda_\psi(z)^n dz + O\left(\frac{1}{a^{-1}(n)}\right) \quad \text{in } L^\tau,
\]
\[
= \mathbb{E}(h) \int_{-\delta a^{-1}(n)}^{\delta a^{-1}(n)} e\left(\frac{z(n\xi - k_n)}{a^{-1}(n)}\right)\Gamma_{n,g}\left(\frac{z}{a^{-1}(n)}\right)\lambda_\psi\left(\frac{z}{a^{-1}(n)}\right)^n dz
\]
\[
\approx \Gamma_{n,g}(0)\mathbb{E}(h) \int_{\mathbb{R}} e(-z\kappa)\hat{f}_{Z_\gamma}(z) dz
\]
\[
= G_g(k_n - n\xi)\mathbb{E}(h)f_{Z_\gamma}(\kappa) \quad \text{(by the aperiodic LLT)}. \]
Proof of theorem 3.1 in the lattice case. Suppose that \( \frac{k_n}{a^{-1}(n)} \xrightarrow{n \to \infty} \kappa \). We will show that

\[
a^{-1}(n)\hat{\tau}^n(h1_{\phi_n=k_n}) \approx p1_{p\mathbb{Z}}(n\xi - k_n)\mathbb{E}(h)f_{Z,\gamma}(\kappa) \quad \text{as } n \to \infty.
\]

Now

\[
\hat{\tau}^n(h1_{\phi_n=k_n})(x) = \int_{\mathbb{T}} e(-k_nt)\hat{\tau}^n(e(t\phi_n)h)(x)dt
\]

\[= \int_{\mathbb{T}} e(-k_nt)\hat{\tau}^n(e(t(p\psi_n + n\xi + F - F \circ \tau^n))h)(x)dt
\]

\[= \int_{\mathbb{T}} e(t(n\xi - k_n))\hat{\tau}^n(e(p\psi_n)e(t(F - F \circ \tau^n))h)(x)dt
\]

\[= \cdot \sum_{q=0}^{p-1} e\left(\frac{Kq}{p}\right) = 1_{p\mathbb{Z}}(K) \text{ and } F \circ \tau^n = F \text{ mod } p;
\]

\[= \int_{\mathbb{T}} \left( \sum_{q=0}^{p-1} e\left(\frac{t+q}{p}(n\xi - k_n)\right)\hat{\tau}^n(e(t\psi_n)e\left(\frac{t+q}{p}(F - F \circ \tau^n)\right)h(x)) \right) dt
\]

\[= p1_{p\mathbb{Z}}(n\xi - k_n) \int_{\mathbb{T}} e\left(\frac{t}{p}(n\xi - k_n)\right)\hat{\tau}^n\left(e(t\psi_n)\left(\frac{t}{p}(F - F \circ \tau^n)\right)h(x)\right) dt
\]

\[= p1_{p\mathbb{Z}}(n\xi - k_n) \int_{\mathbb{T}} e\left(\frac{t}{p}(n\xi - k_n - F(x))\right)P^n_{\psi,t}\left(e\left(\frac{tF}{p}\right)h\right)(x)dt.
\]

As in the proof of the continuous case, using the aperiodic LLT,

\[
a^{-1}(n)\int_{\mathbb{T}} e\left(\frac{t}{p}(n\xi - k_n - F(x))\right)P^n_{\psi,t}\left(e\left(\frac{tF}{p}\right)h\right)(x)dt \xrightarrow{n \to \infty} \mathbb{E}(h)f_{Z,\gamma}(\kappa).
\]

4. Proof of Theorem B

UNIFORM SETS FOR T. Let the MPT \((X, m, T)\) be pointwise dual ergodic. The set \( \Omega \in \mathcal{B}(X) \), \( 0 < m(\Omega) < \infty \) is called

- a **uniform set** for \( T \) if for some \( f \in L^1(m)_+ \), the convergence in (PDE) is uniform on \( \Omega \); and

- a **Darling–Kac set** if this is the case for \( f = 1_\Omega \). By Egorov’s theorem, every pointwise dual ergodic MPT \((X, m, T)\) has uniform sets, which form a dense hereditary collection denoted by \( \mathcal{U}(T) \).
If \((X, m, T)\) is pointwise dual ergodic \(\gamma\)-regularly varying return sequence \(a(n)\) \((0 < \gamma < 1)\), then as in [Aar97, ch. 3], the return time process to a uniform set is asymptotically \(\gamma\)-stable.

**Proof of Theorem B.** Write \(\mu = m_\Omega\), \(\varphi = \varphi_\Omega\) and \(\tau = T_\Omega\) and let \(\alpha \subset B(\Omega)\) be the partition for which \((\Omega, \mu, \tau, \alpha)\) is a mixing G-M/AFU map.

If \((\Omega, \mu, \tau, \alpha)\) is G-M, then as in [AD01, §1], \((\Omega, \mu, \tau, \varphi)\) is continued fraction mixing and, by the Main Lemma in [Aar86], \(\Omega\) is a Darling–Kac set for \(T\). By the asymptotic renewal equation (see, e.g., [Aar97, chapter 3]) we have \((\mathcal{A})\).

It is shown in [AN05] that a mixing AFU map \((\Omega, \mu, \tau, \alpha)\) is exponentially reverse \(\phi\)-mixing. Thus, by [AZ14, theorem 2.2], the pointwise dual ergodicity of \((\Omega, \mu, \tau)\) with \(\gamma\)-regularly varying return sequence ensures \((\mathcal{A})\).

By our remarks on periodicity of cocycles, \(\varphi : \Omega \to \mathbb{Z}\) is non-arithmetic. The required GLLT follows from Theorem 3.1.

**Example: Random walk skew products.** Let \((\Omega, \mu, T, \alpha)\) be a mixing PP Gibbs–Markov map and let \(\phi : \Omega \to \mathbb{Z}\) be \(\alpha\)-measurable and aperiodic with 
\[ \mu - \text{dist}(\phi) \in \text{DA} (S q S) \text{ with } 1 < q \leq 2, \]
then \((\Omega \times \mathcal{G}, \mu \times \# , T_{\phi})\) is: conservative, exact and pointwise dual ergodic with \(a(n) \gamma = 1 - \frac{1}{q}\)-regularly varying and \(\Omega \times \{0\}\) is a uniform set for \(T_{\phi}\). See [AD01].

**Proposition C:** \(\Omega \times \{0\}\) is a GLLT set for \(T_{\phi}\).

**Proof.** We will apply Theorem B. To this end, since \((\Omega \times \mathcal{G}, \mu \times \# , T_{\phi})\) is exact, hence weakly mixing, it suffices to exhibit a suitable Markov partition for the return time process to \(\Omega \times \{0\}\).

Evidently \(\varphi_{\Omega \times \{0\}} : \Omega \times \{0\} \to \mathbb{N}\) is given by
\[
\varphi_{\Omega \times \{0\}}(x, 0) := \varphi(x) := \min \{ n \geq 1 : \phi_n(x) = 0 \}, \\
T_{\Omega \times \{0\}}(x, 0) = (\tau(x), 0) := (T^{\varphi(x)}(x), 0).
\]

Evidently \([\varphi = n]\) is \(\alpha_n\)-measurable. Write \([\varphi = n] = \bigcup_{a \in \mathcal{B}_n} a\) where \(\mathcal{B}_n \subset \alpha_n\) and let \(\beta \subset B(\Omega)\) be the partition defined by
\[
\beta := \bigcup_{k \geq 1} \mathcal{B}_k.
\]

It follows that \((\Omega, \mu, \tau, \beta)\) is a mixing Gibbs–Markov map and \(\varphi : \Omega \to \mathbb{N}\) is \(\beta\)-measurable.

The Proposition now follows from Theorem B. ■
5. Continuous time

A measure preserving semiflow is a continuous semigroup homomorphism \( \Psi : \mathbb{R}_+ \to \text{MPT}(X, m) \), where \((X, m)\) is a \(\sigma\)-finite, polish measure space and \text{MPT}(X, m)\) is the topological semigroup of measure preserving transformations equipped with the weak operator topology.

Let \((\Omega, \mu, \tau)\) be a probability preserving transformation and let \(r : \Omega \to \mathbb{R}_+\) be measurable.

Define the suspended semiflow \((\Omega, \mu, \tau, r) = (X, m, \Psi)\) by
\[
X := \{(x, s) \in \Omega \times \mathbb{R}_+ : 0 \leq s < r(x)\}, \quad m := \mu \times \text{Leb}
\]
and define \(\Psi : \mathbb{R}_+ \to \text{MPT}(X, m)\) by
\[
\Psi_t(x, y) = (\tau^n(x), y + t - r_n(x)),
\]
where \(n = n_t(x, y)\) is so that
\[
0 \leq y + t - r_n(x) < r(\tau^n x) \quad \text{i.e.,} \quad y + t \in [r_n(x), r_{n+1}(x)).
\]

In this case, we call \((\Omega, \mu, \tau, r)\) a section of the semiflow \((X, m, \Psi)\), which in turn is aka the suspension of \((\Omega, \mu, \tau)\) (under the ceiling \(r)\).

We will call the section \((\Omega, \mu, \tau, r)\) standard if the ceiling is bounded below in the sense that \(r \geq \Delta\) a.s. for some \(\Delta =: \min r > 0\).

**D-K sections for pointwise dual ergodic semiflows.** Call the semiflow \(\Psi\) pointwise dual ergodic if each \text{MPT} \(\Psi_t\) \((t > 0)\) is pointwise dual ergodic. In this case,
\[
a_t(n) := a_n(\Psi_t) \sim \frac{a(nt)}{t}
\]
with
\[
a(n) := a_1(n).
\]

We will call a standard section \((\Omega, \mu, \tau, r)\) a Darling–Kac (D-K) section for \((X, m, \Psi) = (\Omega, \mu, \tau)^r\) if \(\Omega \times [0, t]\) is a Darling–Kac set for \(\Psi_t \forall 0 < t < \min r\).
Proposition 5.1: Suppose that \((\Omega, \mu, \tau, r)\) is a D-K section for \(\Psi\) and that \(a_n(\Psi_1) := a(n)\) is \(\gamma\)-r.v. \((\gamma \in (0, 1))\). Then
\[
\mu([r > t]) \sim \frac{1}{\Gamma(1 + \gamma)\Gamma(1 - \gamma)} \cdot \frac{1}{a(t)}.
\]

Proof. Fix \(0 < \Delta < \min \tau\). Then \(\Omega_\Delta := \Omega \times [0, \Delta]\) is a Darling–Kac set for \(\Psi_\Delta\). As shown above, the return sequence
\[
a_\Delta(n) := a_n(\Psi_\Delta) \sim \frac{a(n\Delta)}{\Delta}.
\]

Let \(\varphi = \varphi_{\Omega_\Delta} : \Omega_\Delta \to \mathbb{N}\) denote the first return time to \(\Omega_\Delta\) under iterates of \(\Psi_\Delta\). Then for \((x, y) \in \Omega_\Delta = \Omega \times [0, \Delta]\),
\[
\varphi(x, y) = \left\lceil \frac{\tau(x) - y}{\Delta} \right\rceil.
\]

By the asymptotic renewal equation (see, e.g., [Aar97, ch. 3])
\[
\int_{\Omega_\Delta} (\varphi \wedge t)dm \sim \frac{1}{\Gamma(1 + \gamma)\Gamma(2 - \gamma)} \cdot \frac{t}{a_\Delta(t)},
\]
and as \(t \to \infty\), whence by Karamata’s differentiation theorem,
\[
m([\varphi > t]) \sim \frac{d}{dt} \int_{\Omega} (\varphi \wedge t)dm \sim \frac{C_\gamma}{a_\Delta(t)}
\]
with
\[
C_\gamma := \frac{1}{\Gamma(1 - \gamma)\Gamma(1 + \gamma)}.
\]

Using this and the sandwich principle
\[
\left[ \varphi > \frac{t}{\Delta} \right] \supset [r > t] \times [0, \Delta] \supset \{(x, y) \in \Omega_\Delta : \tau(x) - y > t\} \supset \left[ \varphi > \frac{t}{\Delta} + 1 \right],
\]
we have
\[
\frac{C_\gamma}{a(t)} \sim \frac{C_\gamma}{\Delta a_\Delta(t)} \sim \frac{1}{\Delta} m([\varphi > \frac{t}{\Delta}]) \sim \mu([r > t]).
\]

Local limit sections for pointwise dual ergodic semiflows. A generalized local limit (GLL) section for \((X, m, \Psi)\) is a quintuple \((\Omega, \mu, \tau, \alpha, r)\) where \((X, m, \Psi) = (\Omega, \mu, \tau)^r\), \((\Omega, \mu, \tau, \alpha)\) is a fibered system and \((\Omega, \mu, \tau, \alpha, r)\) satisfies a GLLT as defined in the introduction.
Theorem 5.2: Suppose that \((X, m, \Psi)\) is pointwise dual ergodic with \(\gamma\)-regularly varying return sequence \((\gamma \in (0, 1))\) and which has a D-K, GLL section, then for \(t > 0\),

\[
\frac{1}{a(N)} \sum_{n=1}^{N} |\tilde{\Psi}_t^n - \left(1, Cg \left(\frac{S_n^{(\Psi_t)}}{a_t(n)} \right) \right)| - m(C)\mathbb{E}(g(m(f)W_\gamma))u_t(n) \quad \lim_{N \to \infty} \rightarrow 0
\]

a.e. \(\forall C \in F_+\), \(g \in C_B(\mathbb{R}_+)\) and \(f \in L^1_+\)

where \(a_t(n) := a_n(\Psi_t), u_t(n) \sim \gamma a_t(n), W_\gamma \in RV(\mathbb{R}_+), \mathbb{E}(g(W_\gamma)) = \mathbb{E}(Y_\gamma g(Y_\gamma))\).

The property (\(\Theta\)) \((\forall t > 0)\) is a semiflow analogue of \((X)\) and is required for \((\square)\). See Remark 5.4 for a "time-free" version of \((\Theta)\).

The proof of Theorem 5.2 mirrors that of Theorem A, beginning with the continuous time version of Lemma 2.1:

Lemma 5.3: Suppose that \((X, m, \Psi) = (\Omega, \mu, \tau)\) is pointwise dual ergodic with \(\gamma\)-regularly varying return sequence \(a_n(\Psi_1) =: a(n) \ (\gamma \in (0, 1))\) with \((\Omega, \mu, \tau, \alpha, \gamma)\) a periodic, LLT, D-K section.

Suppose also that \(a\) satisfies \((\mathcal{G})\). Then for \(0 < \Delta < \min \tau\), \(A \times I \subset X\) with \(A \in C_\alpha\) and \(I \subset \mathbb{R}_+\) an interval,

\[
(\mathcal{GL}^*) \quad \lim_{n \to \infty} \frac{1}{u_\Delta(n)} \tilde{\Psi}_\Delta^n \left(1, A \times I \mathbb{E}(g(\Delta W_\gamma))\right) \geq \mu(A)\mathbb{E}(g(\Delta W_\gamma)).
\]

Proof of \((\mathcal{GL}^*)\). Fix \(A \in C_\alpha(\tau), I \subset \mathbb{R}_+\) and \(g \in C_B(\mathbb{R}_+)\) as above and fix \(0 < c < d < \infty\). Assume without loss of generality that \(I \subset [p k, p(k + 1)]\) for some \(k \in \mathbb{Z}\) where \(p\) is the period of the LLT of the section.

For \(N \geq 1\),

\[
\tilde{\Psi}_\Delta^N \left(1, A \times I \mathbb{E}(g(\Delta W_\gamma))\right)(x, y)
\]

\[
= \sum_{n \geq 1} \tilde{\tau}^n \left(1, A \cap \tau_n \in N \Delta - y + I \mathbb{E}(g(\Delta W_\gamma))\right)
\]

\[
= \sum_{n \geq 1} \tilde{\tau}^n \left(1, A \cap \tau_n \in N \Delta - y + I \mathbb{E}(\frac{n}{a_\Delta(N)})\right).
\]
Writing $x_{n,N} := \frac{N\Delta}{a^{-1}(n)}$, we have by regular variation that

$$n \sim \frac{\Delta^\gamma a(N)}{x_{n,N}^\gamma} \quad \text{and} \quad \frac{1}{a^{-1}(n)} \sim \left(\frac{1}{x_{n,N}}\right)^\gamma \cdot (x_{n,N} - x_{n+1,N}) \cdot \frac{u(N)}{\Delta^{1-\gamma}}$$

as $N, n \to \infty$, $x_{n,N} \in [c, d]$. Let

$$K_N := \{n \geq 1, x_{n,N} \in [c, d]\}.$$ 

Using the LLT property of $\Omega$, ($\xi$) and Lemma 2.2, we have, as $N \to \infty$,

$$\hat{\Psi}_N^1 \left(1_{A \times I} \left(g \left( \frac{S_N^{(\Psi_\Delta)}}{a_\Delta(N)} \right) \right) \right)(x, y)$$

$$\geq \sum_{n \in K_N} g \left( \frac{n}{a_\Delta(N)} \right) \hat{T}^n \left(1_{A \cap [r_n \in N\Delta - y + I]} \right)$$

$$\sim \sum_{n \in K_N} g \left( \frac{\Delta}{x_{n,N}^\gamma} \right) \hat{T}^n \left(1_{A \cap [r_n \in N\Delta - y + I]} \right)$$

$$\sim \sum_{n \in K_N} g \left( \frac{\Delta}{x_{n,N}^\gamma} \right) \frac{p f_{\Delta \gamma}(x_{n,N}) \mu(A)}{a^{-1}(n)} 1_{pZ + I}(N\Delta - n\xi)$$

$$\sim \frac{u(N)\mu(A)}{\Delta^{1-\gamma}} \sum_{n \in K_N} g \left( \frac{\Delta}{x_{n,N}^\gamma} \right) \frac{p f_{\Delta \gamma}(x_{n,N})}{x_{n,N}^{\gamma}} (x_{n,N} - x_{n+1,N}) 1_{pZ + I}(N\Delta - n\xi)$$

$$\sim \frac{u(N)\mu(A)}{\Delta^{1-\gamma}} \mathbb{E}(1_{[c,d]}(Z_{\gamma}) g(\Delta Z_{\gamma}^{-\gamma}) Z_{\gamma}^{-\gamma}) \cdot pm_{\mathbb{R}/pZ}(I + pZ)$$

$$= u_\Delta(N) \mu(A) |I| \mathbb{E}(1_{[c,d]}(Z_{\gamma}) g(\Delta Z_{\gamma}^{-\gamma}) Z_{\gamma}^{-\gamma}).$$

\textbf{Completion of the proof Theorem 5.2.} The proof now proceeds as the completion of the proof of theorem A on p. 16 to obtain $\Theta$ for $0 < t < \min \tau$. This suffices as if $\Psi_t$ satisfies ($\Theta$), then so does $\Psi_{Nt} = \Psi_t^N$ $\forall N \geq 1$. 

\textbf{Remark 5.4 (A “time-free” consequence of ($\Theta$)):

$$(\xi) \quad \frac{1}{a(N)} \int_1^N \left| \hat{\Psi}_t \left(1_{c \cdot f} \left( \frac{\Xi_t^{(\Psi)}}{a(t)} \right) \right) - m(C) \mathbb{E}(g(m(f)W_{\gamma})) u(t) \right| dt \xrightarrow{N \to \infty} 0$$

a.e. $\forall C \in \mathcal{F}_+$, $g \in C_B(\mathbb{R}^+)$ and $f \in L^1_+$

where $\Xi^{(\Psi)}_t(f) := \int_0^t f \circ \Psi_s ds$.}
Proof sketch. The idea here is to use the identity:

$$\Xi_n^{(\Psi)}(f) := \int_0^n f \circ \Psi_s ds = S_n^{(\Psi_1)}(f)$$

with $$\Xi_n^{(\Psi)}(f) := \int_0^1 f \circ \Psi_s ds.$$ We have, for $$f \in L_+^1 \cap L^\infty$$, log $$g \in C([0, \infty])_+$$, $$n \geq 1$$ and $$r \in (0, 1)$$

$$\Xi_{n+r}^{(\Psi)}(f) = \Xi_n^{(\Psi)}(f) \circ \Psi_r - \int_0^r f \circ \Psi_s ds$$

$$= S_{n}^{(\Psi_1)}(f) \circ \Psi_r \pm \bar{f}.$$ Thus, uniformly as $$n \to \infty$$,

$$g \left( \frac{\Xi_{n+r}^{(\Psi)}(f)}{a(n+r)} \right) \sim g \left( \frac{S_{n}^{(\Psi_1)}(f)}{a(n)} \right) \circ \Psi_r$$

and for $$C \in L_+^1 \cap L^\infty$$,

$$\int_n^{n+1} \hat{\Psi}_t \left( Cg \left( \frac{\Xi_t^{(\Psi)}(f)}{a(t)} \right) \right) dt = \int_0^1 \hat{\Psi}_{n+r} \left( Cg \left( \frac{\Xi_{n+r}^{(\Psi)}(f)}{a(n+r)} \right) \right) dr$$

$$\sim \int_0^1 \hat{\Psi}_{n+r} \left( Cg \left( \frac{S_{n}^{(\Psi_1)}(f) \circ \Psi_r}{a(n)} \right) \right) dr$$

$$= \hat{\Psi}_n \left( \tilde{C} \cdot g \left( \frac{S_{n}^{(\Psi_1)}(f)}{a(n)} \right) \right)$$

with $$\tilde{C} := \int_0^1 \hat{\Psi}_r(C)dr.$$ To use this, extend (with a similar proof) (\Theta) with $$t = 1$$ to:

$$\frac{1}{a(N)} \sum_{n=1}^{N} \hat{\Psi}_1^n \left( Cg \left( \frac{S_{n}^{(\Psi_1)}(f)}{a(n)} \right) \right) - m(C)E(g(m(f)W_\gamma))u(n) \big|_{N \to \infty} = 0$$

a.e. $$\forall C \in L_+^1$$, $$g \in C_B(R_+)$$ and $$f \in L_+^1$$

where $$m(C) := \int_X Cdm$$ for $$C \in L_+^1$$.

By (\Theta) and (\Phi) for a.e. $$(x, y) \in X$$, there is a set $$K_{(x,y)} \subset \mathbb{N}$$ of full density so that

$$\frac{1}{u(n)} \int_n^{n+1} \hat{\Psi}_t \left( Cg \left( \frac{\Xi_t^{(\Psi)}(f)}{a(t)} \right) \right)(x, y) dt$$

$$\approx \frac{1}{u(n)} \hat{\Psi}_n \left( \tilde{C} \cdot g \left( \frac{S_{n}^{(\Psi_1)}(f)}{a(n)} \right) \right)(x, y) \xrightarrow{n \to \infty, \ n \in K_{(x,y)}} m(C)E(g(m(f)W_\gamma)),$$

whence $\Phi$. 

Remark 5.5: In certain cases, it is possible to treat non-standard sections in Theorem 5.2 by obtaining a standard induced section.

Let \((X,m,T) = (\Omega,\mu,\tau)^r\) be pointwise dual ergodic with \(\gamma\)-regularly varying return sequence \(a(n) (\gamma \in (0,1),\) and suppose that \(r \geq \Delta > 0\) on \(A \in \mathcal{B}(\Omega)\). It is well known that \((X,\mu(A)^{-1}m,T)\) is a factor semiflow of \((A,\mu_A,\tau_A)^\hat{r}\) where

\[
\hat{r}(x) := \sum_{k=0}^{\varphi_A(x)-1} r(\tau^k x)
\]

with \((A,\mu_A,T_A,\varphi_A)\) the return time process to \(A\).

Moreover \((A,\mu_A,\tau_A,\hat{r})\) is a standard section of \((A,\mu_A,\tau_A)^\hat{r}\).

If \((\Omega,\mu,\tau,\alpha)\) is a positive, Bernoulli process, then so is \((A,\mu_A,\tau_A,\hat{r})\). The latter is also standard, whence a D-K, GLLT section.

Similarly, if \((\Omega,\mu,\tau,\alpha)\) is a G-M map, \(r: \Omega \to \mathbb{R}_+\) is locally \(L\) and \(A \in \alpha\), then \((\Omega,\mu_A,\tau_A,\beta)\) is also a G-M map with

\[
\beta := \bigcup_{n=1}^{\infty} \{ a \in \alpha_n : \varphi|_a \equiv n \}
\]

and \(\hat{r}\) is locally \(L_{\tau_A}\). Again, it follows that \((A,\mu_A,\tau_A,\hat{r})\) is a D-K, GLLT section.

Remarks about mixing. Let \(\Psi = (\Omega,\mu,\tau)^r\) be a pointwise, dual ergodic semiflow with \(a(n) = a_n(\Psi_1)\) \(\gamma\)-regularly varying \((\gamma \in (0,1))\). Suppose that \((\Omega,\mu,\tau,\alpha,\tau)\) is a GLLT section.

In a similar manner to the remarks about mixing in §2, it follows from (GL*) and (⋆) that the following are equivalent:

(i) \(\frac{1}{u_{\Delta}(N)} \hat{\Psi}_N^N(1_A \times IG \left( \frac{S_N^{(\Psi_\Delta)}}{a_N(\Delta)} \right) \left( 1_{A \times [0,\Delta]} \right) ) \stackrel{N \to \infty}{\longrightarrow} \mu(A) \left| I \right| \mathbb{E}(g(\Delta W_\gamma))\)

a.e. on \(\Omega \forall A \in C_\alpha(\tau), I \subset (0,\Delta)\) an interval, \(g \in C_B(\mathbb{R}), g \geq 0\);

(ii) \(\lim_{N \to \infty} \frac{1}{u_{\Delta}(N)} \sum_{n \geq 1, x_{n,N} \notin [c,d]} \hat{\tau}_n(1_A \cap [r_n \in x_n,n a^{-1}(n) - y + [0,\Delta]) \stackrel{e \to 0+, d \to \infty}{\longrightarrow} 0;\)

(iii) \(\frac{1}{u_{\Delta}(N)} \hat{\Psi}_N^N(1_{0 \times [0,\Delta]}) \stackrel{t \to \infty}{\longrightarrow} \Delta.\)

Random walk semiflows. Let \((\Omega,\mu,T,\alpha)\) be a Gibbs–Markov map, let \(\mathfrak{h}: \Omega \to \mathbb{R}_+\) be \((\alpha,\theta)\)-Hölder, and let \(\phi: \Omega \to \mathbb{Z}\) be with \(\alpha\)-measurable and aperiodic with \(\mu - \text{dist}(\phi) \in DA(SqS)\) with \(1 < q \leq 2\).
Consider the measure preserving semiflow 

$$(X, m, \Psi) = (\Omega \times \mathbb{Z}, \mu \times \#, T_\phi)$$

where:

- $(\Omega \times \mathbb{Z}, \mu \times \#, T_\phi)$ is the corresponding random walk skew product; and
- $\overline{h}(x, z) = h(x)$.

Recall that $(\Omega \times \mathbb{Z}, \mu \times \#, T_\phi)$ is exact, pointwise dual ergodic with $a(n) = a_n(T)$ $\gamma$-regularly varying with $\gamma = 1 - \frac{1}{q}$, conditional RWM with rate $u(n) = \frac{\gamma a(n)}{n}$ and $\Omega \times \{0\}$ is a GLL set for $T_\phi$.

We show that, under certain conditions, $\Psi$ is pointwise dual ergodic and has a GLL section, whence it has the tied-down $\gamma$-renewal mixing property ($\Theta$).

Suppose that $(h, \phi) : \Omega \rightarrow \mathbb{R} \times \mathbb{Z}$ is aperiodic. Then:

- $\Psi$ is conditionally RWM with rate $\propto u(t)$ ([AT20, theorem 2]);
- $\Omega \times [0, \Delta)$ is a uniform set for $\Psi_\Delta$ for $0 < \Delta < \min \overline{h}$.

We conclude by showing that $\Psi$ satisfies ($\Theta$) in Theorem 5.2. This is done by finding a GLL section for $\Psi$.

As in the proof of Proposition C, $(\Omega, \mu, \tau, \beta)$ is a mixing Gibbs–Markov map where:

- $\varphi : \Omega \rightarrow \mathbb{N}$ by $\varphi(x) := \min\{n \geq 1 : \phi_n(x) = 0\}$;
- $\tau : \Omega \rightarrow \Omega$ by $\tau(x) := T^\varphi(x)(x)$;
- $\beta \subset \mathcal{B}(\Omega)$ a partition by $\beta = \{a \in \alpha_n : \varphi|_a \equiv n\}$ and the induced map of $(T_\phi)_{\Omega \times \{0\}} \cong \tau$:

$$((T_\phi)_{\Omega \times \{0\}}(x), 0) = (\tau(x), 0).$$

As above, $(\Omega, \mu, \tau, \beta)$ is a mixing Gibbs–Markov map.

The required section for $\Psi$ will be

$$(\Omega, \mu, \tau, \beta, r)$$

where

$$r(x) := \sum_{k=0}^{\varphi(x)-1} \overline{h}(T^k x).$$

Calculation shows that $D_{\beta, 0}(r) < \infty$ and the weak mixing of $\Psi$ ensures that $r$ is non-arithmetic. It follows as above that $(\Omega, \mu, \tau, \beta, r)$ is a GLL section for $\Psi$, and the claim follows from Theorem 5.2.
6. Integrated properties

Weak rational ergodicity. The CEMPT \((X,m,T)\) is called \textbf{weakly rationally ergodic} (WRE) ([Aar77]) if there exists \(F \in \mathcal{F}_+\) so that

\[
\left( \bigstar \right) \quad \frac{1}{a_n(F)} \sum_{k=0}^{n-1} m(B \cap T^{-k}C) \to \text{as } n \to \infty \quad m(B)m(C) \quad \forall B, C \in \mathcal{B} \cap F
\]

where

\[
a_n(F) := \frac{1}{m(F)^2} \sum_{k=0}^{n-1} m(F \cap T^{-k}F).
\]

By theorem 3.3 in [FL72], \(F \in \mathcal{F}_+\) satisfies \(\left( \bigstar \right)\) if and only if

\[
\{S_n^{(T)}(1_F) : n \in \mathbb{N}\}
\]

is uniformly integrable on \(F\).

A useful sufficient condition for this ([Aar77], [Aar97, §3.3]) is

\[
\sup_{n \ge 1} \frac{1}{a_n(F)^2} \int_F S_n(1_F)^2 dm < \infty
\]

and \((X,m,T)\) is called \textbf{rationally ergodic} if there exists such \(F \in \mathcal{F}_+\).

In case \(T\) is weakly rationally ergodic:

- the collection of sets \(R(T)\) satisfying \(\left( \bigstar \right)\) is a hereditary ring;
- \(\exists a_n(T)\) (the return sequence) such that

\[
\frac{a_n(A)}{a_n(T)} \to 1 \quad \forall A \in R(T);
\]

- for conservative, ergodic \(T\), \(R(T) = \mathcal{F}\) only when \(m(X) < \infty\).

By [Aar97, prop. 3.7.1] a pointwise dual ergodic transformation is rationally ergodic with the same return sequence.

Rational weak mixing. As in [Aar13], we call the CEMPT \((X,m,T)\) \textbf{rationally weakly mixing} (RWM) if there exist rates \(u_n > 0\) and \(F \in \mathcal{F}_+\) so that

\[
\left( \bigstar \right) \quad \frac{1}{a(n)} \sum_{k=0}^{n-1} |m(A \cap T^{-k}B) - u_k(F)m(A)m(B)| \to 0 \quad \forall A, B \in \mathcal{B} \cap F
\]

where

\[
a(n) := \sum_{k=0}^{n-1} u_k.
\]

It is shown in [Aar13] that RWM entails WRE with \(\left( \bigstar \right)\) holding \(\forall F \in R(T)\).
PROPOSITION 6.1: If \((X, m, T)\) is conditionally RWM, then it is RWM.

Proposition 6.1 is implicit in [AN17]. The method of proof is the same as that of the next result.

THEOREM 6.2: Suppose that \((X, m, T)\) satisfies (\(\mathcal{X}\)), then \((X, m, T)\) is WRE and

\[
\mathcal{M} \quad \frac{1}{a(N)} \sum_{n=1}^{N} \left| \int_{B \cap T^{-n} C} g \left( \frac{S_n(f)}{a(n)} \right) dm - m(B) m(C) \mathbb{E}(g(m(f) W_\gamma)) u(n) \right| \xrightarrow{N \to \infty} 0
\]

\(\forall B, C \in \mathcal{R} \cap R(T), \ g \in C_B(\mathbb{R}_+) \ \text{and} \ \ f \in L^1_+\)

where \(a(n) = a_n(T)\).

The integrated, tied-down renewal mixing property \((\mathcal{M})\) of the MPT \((X, m, T)\) is a strengthening of RWM (take \(g \equiv 1\) in \((\mathcal{M})\)).

The proof of Theorem 6.2 uses a standard approximation technique embodied in

LEMMA 6.3: Suppose that \((X, m, T)\) is WRE, \(f \in L^1_+, \ g \in C_B(\mathbb{R}_+)\), \(\Omega \in R(T)\) and that \((\mathcal{M})\) holds for \(A \in \mathcal{R}, \ B \in \mathcal{S}\) where both \(\mathcal{R}, \ \mathcal{S} \subset \mathcal{B}(\Omega)\) are dense in \(\mathcal{B}(\Omega)\). Then \((\mathcal{M})\) holds \(\forall A, B \in \mathcal{B}(\Omega)\).

Proof. We claim first that \((\mathcal{M})\) holds for \(A \in \mathcal{B}(\Omega)\) and \(b \in \mathcal{S}\).

Indeed for \(A \in \mathcal{B}(\Omega)\) and \(\epsilon > 0\), there exists \(a \in \mathcal{R}\) so that \(m(a \Delta A) < \epsilon\) whence

\[
\left| \int_{A \cap T^{-k} b} g \left( \frac{S_n(f)}{a(n)} \right) dm - u_k m(A) m(b) \mathbb{E}(m(f) W_\gamma) \right|
\leq \left| \int_{a \cap T^{-k} b} g \left( \frac{S_n(f)}{a(n)} \right) dm - u_k m(a) m(b) \mathbb{E}(m(f) W_\gamma) \right|
+ \|g\|_{C_B} (m(a \Delta A \cap T^{-k} b) + u_k m(a \Delta A) m(b)).
\]

Using \((\mathcal{M})\) for \(a \in \mathcal{R}, \ b \in \mathcal{S}\) and \(\Omega \in R(T)\), we have as \(n \to \infty\)

\[
\frac{1}{a(n)} \sum_{k=0}^{n-1} \left| \int_{A \cap T^{-k} b} g \left( \frac{S_n(f)}{a(n)} \right) dm - u_k m(A) m(b) \mathbb{E}(m(f) W_\gamma) \right|
\leq \|g\|_{B_B} \frac{1}{a(n)} \sum_{k=0}^{n-1} (m(a \Delta A \cap T^{-k} b) + u_k m(a \Delta A) m(b))
\sim 2m(a \Delta A) m(b) < 2\epsilon.
\]

The extension of \((\mathcal{M})\) to \(A, B \in \mathcal{B}(\Omega)\) is similar. \(\blacksquare\)
Proof of Theorem 6.2. Fix $\Omega \in \mathcal{R}_0 := R(T) \cap \mathcal{R}$. Then $a_n(\Omega) \sim a(n)$. Fix $f \in L^1(m)_+$. Using Egorov’s theorem, for fixed $A \in \mathcal{B}(\Omega)$ and $\epsilon > 0$, $\exists U = U_A \in \mathcal{B}(\Omega)$ so that $m(\Omega \setminus U) < \epsilon$ and so that

$$
\frac{1}{a(n)} \sum_{k=1}^{n} \left| \hat{T}^k \left( 1_A g \left( \frac{S_n(f)}{a(n)} \right) \right) - u_k m(A) \mathbb{E}(m(f)W_\gamma) \right| \xrightarrow{n \to \infty} 0 \text{ uniformly on } U,
$$

whence for $B \in \mathcal{B}(U)$,

$$
\frac{1}{a(n)} \sum_{k=0}^{n-1} \int_{A \cap T^{-k}B} g \left( \frac{S_n(f)}{a(n)} \right) dm - u_k m(A)m(B) \mathbb{E}(m(f)W_\gamma) \xrightarrow{n \to \infty} 0.
$$

Now fix a countable dense algebra $\mathcal{R} \subset \mathcal{B}(\Omega)$.

By the above, there exist $\{U_{A,n} : n \geq 1, A \in \mathcal{R}\} \subset \mathcal{B}(\Omega)$ so that $m(V_n) > m(\Omega) - \frac{1}{n}$ where $V_n := \bigcap_{A \in \mathcal{R}} U_{A,n}$ and so that

$$
\frac{1}{a(n)} \sum_{k=0}^{n-1} \int_{a \cap T^{-k}b} g \left( \frac{S_n(f)}{a(n)} \right) dm - u_k m(a)m(b) \mathbb{E}(m(f)W_\gamma) \xrightarrow{n \to \infty} 0
$$

(\forall a \in \mathcal{R} \text{ and } b \in \mathcal{S} := \bigcup_{n \geq 1} \mathcal{B}(V_n)).

The extension of (\forall) to $A, B \in \mathcal{B}(\Omega)$ follows from Lemma 6.3.

Natural extensions. Suppose that $(X, m, T)$ is a measure preserving transformation of a standard, $\sigma$-finite measure space. Rokhlin’s natural extension $(X^N, \tilde{m}, \tilde{T})$ the minimal, invertible extension of $(X, m, T)$ (which is unique up to isomorphism), is given (as in [Roh60]) by

$$
\tilde{T}(x_1, x_2, \ldots) := (Tx_1, x_1, \ldots),
$$

$$
\tilde{m}([A_1, \ldots, A_n]) = m \left( \bigcap_{k=1}^{n} T^{-(n-k)} A_k \right) \text{ where }
$$

$$
[A_1, \ldots, A_n] = \{(x_1, x_2, \ldots) \in X^N : x_k \in A_k \forall 1 \leq k \leq n\}, \quad A_1, \ldots, A_n \in \mathcal{B}(X).
$$

This is an MPT extending $(X, m, T)$ via $\pi : (x_1, x_2, \ldots) \mapsto x_1$. 

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It is invertible on \( \tilde{X} = \{(x_1, x_2, \ldots) \in X^\mathbb{N} : T x_{n+1} = x_n \forall n \in \mathbb{N}\} \) which has full measure; with inverse \( \tilde{T}^{-1}(x_1, x_2, \ldots) = (x_2, x_3, \ldots) \). Thus \((\tilde{X}, \tilde{m}, \tilde{T})\) is an invertible MPT.

The extension is minimal since \( \bigvee_{n=0}^{\infty} \tilde{T}^n \pi^{-1} B(X) = B(\tilde{X}) \mod \tilde{m} \).

We say that a property \( \mathcal{P} \) (of MPTs) lifts (to the natural extension) if \((X, m, T)\) has property \( \mathcal{P} \Rightarrow (\tilde{X}, \tilde{m}, \tilde{T})\) has property \( \mathcal{P} \).

Parry showed in [Par65] that conservativity and ergodicity lift. It follows (see [Aar97]) that rational ergodicity and weak rational ergodicity both lift and it is shown in [Aar13] that rational weak mixing lifts. The following shows that the integrated tied-down renewal mixing property \((\mathcal{P})\) also lifts.

**Theorem 6.4:** Suppose that \((X, m, T)\) is WRE with \( a(n) = a_n(T) \) \( \gamma \)-regularly varying with \( \gamma \in (0, 1) \) and satisfies \((\mathcal{P})\), then so does \((\tilde{X}, \tilde{m}, \tilde{T})\).

**Proof.** Let \( \pi : (\tilde{X}, \tilde{m}, \tilde{T}) \to (X, m, T) \) be the extension map and write

\[
a(n) = a_n(T) \sim a_n(\tilde{T}).
\]

Let \( \mathcal{R} \) be the hereditary ring in the \( \gamma \)-tied-down renewal mixing of \((X, m, T)\) and set

\[
\mathcal{R}^* := \bigcup_{n \in \mathbb{Z}} \tilde{T}^n \pi^{-1} \mathcal{R}.
\]

We have that \( \mathcal{R}^* \subset R(\tilde{T}) \) and that for \( \Omega \in \mathcal{R}^* \), \( \mathcal{R}^* \cap B(\Omega) \) is dense in \( B(\Omega) \).

Also, for fixed \( f \in L^1(m)_+, g \in C_B(\mathbb{R}_+)_+ \),

\[
(\mathcal{P}) \quad \frac{1}{a(N)} \sum_{n=1}^{N} \int_{B \cap \tilde{T}^{-n} C} g \left( \frac{S_n(f \circ \pi)}{a(n)} \right) \tilde{m}(B) \tilde{m}(C) \mathbb{E}(g(\tilde{m}(f) W_\gamma)) u(n) \left. \right| _{N \to \infty} \to 0
\]

\[ \forall B, C \in \mathcal{R}^*. \]

Thus by Lemma 6.2, for each \( \Omega \in \mathcal{R}^* \), \((\mathcal{P})\) holds \( \forall B, C \in B(\Omega) \) and, putting things together, we see that \((\mathcal{P})\) holds \( \forall f = p \circ \pi, p \in L^1(m)_+, g \in C_B(\mathbb{R}_+) \) and \( B, C \in \mathcal{R} := \bigcup_{\Omega \in \mathcal{R}^*} B(\Omega) \).

We now extend this to general \( F \in L^1(\tilde{m})_+ \). The method is similar to the proof of Theorem A.

Again, it suffices to consider \( g > 0 \) with \( x \mapsto \log g(e^x) \) uniformly continuous on \( \mathbb{R} \).
Fix $F, f \in L^1_+(\tilde{m})_+, f = p \circ \pi, p \in L^1(m)_+, m(p) = 1, g > 0$ as above and $\Omega \in \mathcal{R}^*$. 

By the ratio ergodic theorem
\[
\frac{S_n(F)}{S_n(f)} \xrightarrow{n \to \infty} m(F) \text{ a.e.}
\]

By Egorov’s theorem
\[
\mathcal{U} := \{ C \in \mathcal{B}(\Omega) : \text{convergence uniform on } C \}
\]
is dense in $\mathcal{B}(\Omega)$. 

Let $B \in \mathcal{U}$. Then There exist $\epsilon_N \downarrow 0$ so that
\[
g\left(\frac{S_n(F)}{a(n)}\right) = (1 \pm \epsilon_n)g\left(\frac{m(F)S_n(f)}{a(n)}\right) \text{ on } \forall n \geq 1.
\]

Thus for $C \in \mathcal{B}(\Omega)$,
\[
\left| \int_{B \cap \tilde{T}^{-n}C} g\left(\frac{S_n(F)}{a(n)}\right) d\tilde{m} - \tilde{m}(B)\tilde{m}(C)\mathbb{E}(g(\tilde{m}(F)W_\gamma))u(n) \right|
\]
\[
\leq \left| \int_{B \cap \tilde{T}^{-n}C} \left( \left| g\left(\frac{S_n(F)}{a(n)}\right) - g\left(\frac{m(F)S_n(f)}{a(n)}\right) \right| \right) d\tilde{m}
\]
\[
+ \left| \int_{B \cap \tilde{T}^{-n}C} \left( g\left(\frac{m(F)S_n(f)}{a(n)}\right) \right) d\tilde{m} - \tilde{m}(C)\mathbb{E}(g(\tilde{m}(F)W_\gamma))u(n) \right|
\]
\[
\leq \epsilon_n \|g\|_{\infty}\tilde{m}(B \cap \tilde{T}^{-n}C) + 
\]
\[
+ \left| \int_{B \cap \tilde{T}^{-n}C} \left( g\left(\frac{\tilde{m}(F)S_n(f)}{a(n)}\right) \right) d\tilde{m} - \tilde{m}(C)\mathbb{E}(g(\tilde{m}(F)W_\gamma))u(n) \right|
\]

Now
\[
\frac{1}{a(N)} \sum_{n=1}^{N} \left| \int_{B \cap \tilde{T}^{-n}C} g\left(\frac{\tilde{m}(F)S_n(f)}{a(n)}\right) - \tilde{m}(C)\mathbb{E}(g(\tilde{m}(F)W_\gamma))u(n) \right| \xrightarrow{N \to \infty} 0,
\]
so
\[
\frac{1}{a(N)} \sum_{n=1}^{N} \left| \int_{B \cap \tilde{T}^{-n}C} g\left(\frac{S_n(F)}{a(n)}\right) - \tilde{m}(B)\tilde{m}(C)\mathbb{E}(g(\tilde{m}(F)W_\gamma))u(n) \right|
\]
\[
\leq \frac{1}{a(N)} \sum_{n=1}^{N} \epsilon_n \|g\|_{\infty}\tilde{m}(B \cap \tilde{T}^{-n}C) \xrightarrow{N \to \infty} 0.
\]

Thus (i) holds for all $B \in \mathcal{U}, C \in \mathcal{B}(\Omega)$. Lemma 6.2 now extends (i) to hold for all $B, C \in \mathcal{B}(\Omega)$. \qed
**Example: Geodesic flows on cyclic covers.** We denote by \((U(M), \Lambda, g)\), the geodesic flow on the unit tangent bundle \(U(M)\) of the hyperbolic surface \(M\) equipped with the hyperbolic measure \(\Lambda\) (for definitions see [Hop71], [Ree81], [AN17]).

The hyperbolic surface \(V\) is a **cyclic cover** of the compact hyperbolic surface \(M\) if there is a covering map \(p: V \rightarrow M\) and a monomorphism \(\gamma: \mathbb{Z} \rightarrow \text{Isom}(V)\) (hyperbolic isometries of \(V\)), so that for \(y \in V\),

\[
p^{-1}\{p(y)\} = \{\gamma(n)y : n \in \mathbb{Z}\}.
\]

Symbolic dynamics for the geodesic flow on a compact hyperbolic surface is described in [Bow73] and it is shown in [Ree81] (see also [AN17]) that if \(V\) is a cyclic cover of a compact hyperbolic surface, then \((U(M), \Lambda, g)\) is isomorphic to the natural extension of a semiflow of the form \((X \times \mathbb{Z}, m \times \#_\alpha, T_\varphi)^\mathbb{R}\) where \((X, T, m, \alpha)\) is a mixing Gibbs–Markov map with \(#_\alpha < \infty\) (also known as a subshift of finite type equipped with a Gibbs measure) and \((\phi, r): X \rightarrow \mathbb{Z} \times \mathbb{R}\) is \(\alpha\)-Hölder and ([Sol01]) aperiodic. It was shown in [AN17] that such \((U(M), \Lambda, g)\) is rationally weakly mixing. For more information, see [AT20].

We claim here that

\[\oplus\text{ such (}U(M), \Lambda, g\text{) satisfies (\(\mathfrak{R}\)) in Theorem 6.2 with } a(n) \propto \sqrt{n}.\]

**Proof of \(\oplus\).** As shown in the random walk semiflow example in §5, \((X \times \mathbb{Z}, m \times \#_\alpha, T_\varphi)^\mathbb{R}\) satisfies (\(\mathfrak{R}\)) with \(a(n) \propto \sqrt{n}\). \(\oplus\) now follows from Theorems 6.2 and 6.4. \(\square\)

**Updates.** A functional version of (\(\mathfrak{R}\)) is established in [AS21] under stronger assumptions, as is a \(u\)-weak Cesaro (as in [Aar13, §4]) version for pointwise dual ergodic transformations with \(\gamma\)-regularly varying return sequences \((0 < \gamma \leq 1)\). This latter gives versions of (\(\mathfrak{Q}\)) and (\(\mathfrak{D}\)) when \(\gamma = 1\).

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