On the stability of Einstein manifolds

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Abstract

Certain curvature conditions for stability of Einstein manifolds with respect to the Einstein-Hilbert action are given. These conditions are given in terms of sectional curvature bounds and quantities involving the Weyl tensor and the Bochner tensor. This notion of stability is closely related to the concept of physical stability, which comes from higher dimensional gravity theories. We also give curvature conditions for physical stability.

1 Introduction

Let $M^n$ be a compact manifold of dimension $n \geq 3$ and let $\mathcal{M}$ be the set of smooth Riemannian metrics on it. For any $c > 0$, let $\mathcal{M}_c \subset \mathcal{M}$ be the subset of smooth Riemannian metrics of volume $c$. Ricci-flat metrics can be variationally characterized as critical points of the Einstein-Hilbert action

$$S: \mathcal{M} \to \mathbb{R}, \quad g \mapsto \int_M \text{scal}_g \, dV_g$$

[Hil15]. If the functional is restricted to some $\mathcal{M}_c$, the critical points are precisely the Einstein metrics of volume $c$. It is well-known that Einstein manifolds are neither local maximum nor minimum of the Einstein-Hilbert action on $\mathcal{M}$ [Mut74]. In fact, both index and coindex of $S''$ are infinite on any Einstein space. However, there is a notion of stability which is as follows: We say that an Einstein manifold is stable if $S''(h) \leq 0$ for any $h \in \Gamma(S^2M)$ satisfying $\text{tr} h = 0$ and $\delta h = 0$. Such tensors are called transverse traceless. We call the manifold strictly stable, if $S''(h) < 0$ for all nonzero transverse traceless tensors.

Stability of compact Einstein metrics appears in mathematical general relativity. In [AM11], Andersson and Moncrief prove that the Lorentzian cone over a compact negative Einstein metric is an attractor of the Einstein flow under the assumption that the compact Einstein metric is stable. Stability also appears in the context of the Ricci flow and its analysis close to Einstein metrics [Ye93, CHI04, Ses06, Has12, CH13]. This is because the second variational formulas of Perelman’s entropies on Einstein metrics are closely related to the second variational formula of the Einstein Hilbert action.

Many classes of Einstein spaces are known to be stable. Most symmetric spaces of compact type (including the sphere and the complex projective space) are stable [Koi80, CH13]. Spin manifolds admitting a nonzero parallel spinor are stable [Wan91, DWW05]. Kähler-Einstein manifolds of nonpositive scalar curvature are stable [DWW07], which essentially follows from the work in [Koi83].

On the other hand, many unstable Einstein manifolds can be explicitly constructed [PP84a, PP84b, GM02, CH02, GHP03, Böh05]. All these examples are of positive scalar curvature. No unstable Einstein manifolds of nonpositive scalar curvature are known which naturally leads to the following

Question ([Dai07] p. 65]). Are all compact Einstein manifolds with nonpositive scalar curvature stable?
For the Ricci-flat case, this question was already asked by Kazdan and Warner [KW75, p. 315]. The statement is not true in the noncompact case since the Riemannian Schwarzschild metric is unstable (see [GPY82, Sec. 5]).

Throughout this work, any manifold $M^n$ is compact and $n \geq 3$ unless the contrary is explicitly asserted. We start with considering flat manifolds which are known to be stable. We compute the kernel of $S''$ restricted to transverse traceless tensors.

**Proposition 1.1.** Let $(M = \mathbb{R}^n/G, g)$ be a Bieberbach manifold and let $\rho$ be the canonical representation of the holonomy of $G$ on $\mathbb{R}^n$. Let

$$\rho \cong (\rho_1)^{i_1} \oplus \ldots \oplus (\rho_l)^{i_l}$$

be an irreducible decomposition of $\rho$. Then the dimension of the space of infinitesimal Einstein deformations is equal to

$$\dim(\ker(\Delta_E|_{TT})) = \sum_{j=1}^{l} \frac{i_j(i_j + 1)}{2} - 1.$$

As another example, we consider products of Einstein spaces and we compute the kernel and the coindex of $S''$ restricted to transverse traceless tensors on products of Einstein spaces (Proposition 4.8).

For the study of curvature conditions, we build up on an important theorem by Koiso, which states the following:

**Theorem 1.2** ([Koi78, Theorem 3.3]). Let $(M, g)$ be an Einstein manifold with Einstein constant $\mu$. If the function $r$ satisfies

$$\sup_{p \in M} r(p) \leq \max \left\{ -\mu, \frac{1}{2} \mu \right\},$$

then $(M, g)$ is stable. If the strict inequality holds, then $(M, g)$ is strictly stable.

Here, $r : M \to \mathbb{R}$ is the largest eigenvalue of the curvature tensor acting on traceless symmetric $(0, 2)$-tensors. The proof is based on the Bochner technique. One can estimate $r$ in a purely algebraic way in terms of sectional curvature bounds and one gets the following corollaries as consequences thereof:

**Corollary 1.3** (Bourguignon, unpublished). Let $(M, g)$ be an Einstein manifold such that the sectional curvature lies in the interval $(\frac{n-2}{n}, 1]$. Then $(M, g)$ is strictly stable.

**Corollary 1.4** ([Koi78, Proposition 3.4]). Let $(M, g)$ be an Einstein manifold with sectional curvature $K < 0$. Then $(M, g)$ is strictly stable.

However, Corollary 1.3 is ruled out for dimensions $n \geq 8$ because any Einstein manifold satisfying this condition is isometric to a quotient of the round sphere. This follows from the proof of the differentiable sphere theorem [BS92]. One gets stability by replacing the strict inequalities in the corollaries by weak inequalities. Moreover, we found out that the existence of infinitesimal Einstein deformations imposes strong conditions on the manifold. These are given in Proposition 5.2 and Proposition 5.5.

Since constant curvature metrics are stable, we find it convenient to formulate stability criterions in terms of the Weyl tensor. Let $w : M \to \mathbb{R}$ be the largest eigenvalue of the Weyl tensor as an operator acting on symmetric $(0, 2)$-tensors. From Koiso’s Bochner formulas, we get

**Theorem 1.5.** An Einstein manifold $(M, g)$ with constant $\mu$ is stable if

$$\|w\|_{L^\infty} \leq \max \left\{ \mu \frac{n+1}{2(n-1)}, -\mu \frac{n-2}{n-1} \right\}.$$ 

If the strict inequality holds, then $(M, g)$ is strictly stable.
Using the Sobolev inequality, we find a different criterion involving an integral of this function:

**Theorem 1.6.** Let \((M, g)\) be an Einstein manifold with positive Einstein constant \(\mu\). If
\[
\|w\|_{L^{n/2}} \leq \mu \cdot \text{vol}(M, g)^{2/n} \cdot \left( \frac{n + 1}{2(n - 1)} \frac{4(n - 1)}{(n(n - 2) + 1} \right)^{-1},
\]
then \((M, g)\) is stable. If the strict inequality holds, then \((M, g)\) is strictly stable.

Observe that for large dimensions, the two above conditions are close to each other. Using the previous criterion and the Gauss-Bonnet formula in dimension six, we prove a stability criterion involving the Euler characteristic of the manifold:

**Theorem 1.7.** Let \((M, g)\) be a positive Einstein six-manifold with constant \(\mu\) and \(\text{vol}(M, g) = 1\). If
\[
\frac{1}{25} \left( \frac{144}{5} - \frac{12 \cdot 7^2 \cdot 3^2}{11^2} \right) \mu^3 \leq 384\pi^3 \chi(M) - 48 \int_M \text{tr}(\tilde{W}^3) \, dV,
\]
then \((M, g)\) is strictly stable. Here, \(\tilde{W}\) is the Weyl curvature operator acting on two-forms.

For Kähler-Einstein manifolds, the Bochner tensor plays a similar role as the Weyl tensor for general Einstein manifolds. We prove similar theorems as Theorem 1.5 and Theorem 1.6 for Kähler-Einstein manifolds which involve the Bochner tensor instead of the Weyl tensor. In this context, we correct a small error in [IN05].

In the last section, we consider the notion of physical stability [BF82, GHP03]. An Einstein manifold with positive Einstein constant \(\mu\) is said to be physically stable if we have \(S''(h) \leq \mu \frac{n+1}{2} \|h\|_{L^2}^2\) for all transverse traceless tensors. We state conditions as in Theorem 1.2, Corollary 1.3, Theorem 1.5 and Theorem 1.6 for physical stability.

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## 2 Preliminaries

Let us first fix some notation and conventions. We define the Laplace-Beltrami operator acting on functions by \(\Delta = -\text{tr}\nabla^2\). For the Riemann curvature tensor, we use the sign convention such that \(R_{X,Y}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z\). Given a fixed metric, we equip the bundle of \((r, s)\)-tensor fields (and any subbundle) with the natural scalar product induced by the metric. By \(S^p M\), we denote the bundle of symmetric \((0, p)\)-tensors. Let \(\{e_1, \ldots, e_n\}\) be a local orthonormal frame. The divergence is the map \(\delta : \Gamma(S^p M) \to \Gamma(S^{p-1} M)\), defined by
\[
\delta T(X_1, \ldots, X_{p-1}) = -\sum_{i=1}^n \nabla_{e_i} T(e_i, X_1, \ldots, X_{p-1})
\]
and its adjoint \(\delta^* : \Gamma(S^{p-1} M) \to \Gamma(S^p M)\) with respect to the natural \(L^2\)-scalar product is given by
\[
\delta^* T(X_1, \ldots, X_p) = \frac{1}{p} \sum_{i=0}^{p-1} \nabla_{X_{i+1}} T(X_{2+i}, \ldots, X_{p+i}),
\]
where the sums \(1+i, \ldots, p+i\) are taken modulo \(p\).
The second variation of $S$ at Einstein metrics was studied in [Koi79]. For details, see also [Bes08, Chapter 4]. A useful fact for studying $S''$ is that any compact Einstein metric except the standard sphere admits the decomposition

$$T_gM = \Gamma(S^2 M) = C^\infty(M) \cdot g \oplus \delta_g^*(\Omega^1(M)) \oplus \text{tr}_g^{-1}(0) \cap \delta_g^{-1}(0)$$

(2.1)

and these factors are all infinite-dimensional. It turned out that this splitting is orthogonal with respect to $S''$. Thus, the second variation can be studied separately on each of these factors.

The first factor of (2.1) is the tangent space of the conformal class of $g$. It is known that $S''$ is positive on volume-preserving conformal deformations. This follows from

$$S''(f \cdot g) = \frac{n-2}{2} \int_M \langle f, (n-1)\Delta_g f - n\mu f \rangle dV_g$$

(2.2)

and the following

**Theorem 2.1** ([Oba62, Theorem 1 and Theorem 2]). Let $(M, g)$ a compact Riemannian manifold and let $\lambda$ be the smallest nonzero eigenvalue of the Laplace operator acting on $C^\infty(M)$. Assume there exists $\mu > 0$ such that $\text{Ric}(X, X) \geq \mu |X|^2$ for any vector field $X$. Then $\lambda$ satisfies the estimate

$$\lambda \geq \frac{n}{n-1} \mu,$$

and equality holds if and only if $(M, g)$ is isometric to the standard sphere.

Later on, we refer to this theorem as Obata’s eigenvalue estimate.

The second factor is the tangent space of the orbit of the diffeomorphism group acting on $g$. By diffeomorphism invariance, $S''$ vanishes on this factor. The third factor is the space of non-trivial constant scalar curvature deformations of $g$. The tensors in the third factor are also often called transverse traceless or TT. From now on, we abbreviate $TT_g = \text{tr}_g^{-1}(0) \cap \delta_g^{-1}(0)$. The second variation of $S$ on $TT$-tensors is given by

$$S''(h) = -\frac{1}{2} \int_M \langle h, \nabla^* \nabla h - 2\hat{R}h \rangle dV.$$

Here, $\hat{R}$ is the action of the curvature tensor on symmetric $(0, 2)$-tensors, given by

$$\hat{R}h(X, Y) = \sum_{i=1}^n h(Re_i X, e_i).$$

**Definition 2.2.** We call the operator $\Delta_E : \Gamma(S^2 M) \to \Gamma(S^2 M)$, $\Delta_E h = \nabla^* \nabla h - 2\hat{R}h$ the Einstein operator.

This is a self-adjoint elliptic operator and by compactness of $M$, it has a discrete spectrum. The Einstein operator preserves all components of the splitting (2.1).

**Definition 2.3.** We call a compact Einstein manifold $(M, g)$ stable, if the Einstein operator is nonnegative on $TT$-tensors and strictly stable, if it is positive on $TT$-tensors. We call $(M, g)$ unstable, if the Einstein operator admits negative eigenvalues on $TT$. Furthermore, elements in $\ker(\Delta_E|TT)$ are called infinitesimal Einstein deformations.

**Remark 2.4.** Let $(M, g)$ be a compact Einstein manifold and let $C_g$ be the set of constant scalar curvature metrics admitting the same volume as $g$. Close to $g$, this is a manifold and

$$T_g C_g = \delta_g^*(\Omega^1(M)) \oplus TT_g,$$

so stability precisely means that $S''$ is nonpositive on $T_g C_g$. 

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Remark 2.5. If \( g \) is a nontrivial curve of Einstein metrics through \( g = g_0 \) orthogonal to \( \mathbb{R} \cdot (g \cdot \text{Diff}(M)) \), then \( g_0 \) is an infinitesimal Einstein deformation. Evidently, an Einstein manifold is isolated (or rigid) in the space of Einstein structures if \( \Delta E|_{TT} \) has trivial kernel.

Definition 2.6. An infinitesimal Einstein deformation \( h \) is called integrable if there exists a curve of Einstein metrics tangent to \( h \).

3 Einstein deformations of Bieberbach manifolds

Bieberbach manifolds are flat connected compact manifolds. It is well known that any Bieberbach manifold is isometric to \( \mathbb{R}^n/G \), where \( G \) is a suitable subgroup of the Euclidean motions \( E(n) = O(n) \times \mathbb{R}^n \). We call such groups Bieberbach groups. For every element \( g \in E(n) \), there exist unique \( A \in O(n) \) and \( a \in \mathbb{R}^n \) such that \( gx = Ax + a \) for all \( x \in \mathbb{R}^n \), and we write \( g = (A, a) \). There exist homomorphisms \( r: E(n) \to O(n) \) and \( t: \mathbb{R}^n \to E(n) \), defined by \( r(A, a) = A \) and \( t(a) = (1, a) \). Let \( G \) be a Bieberbach group. The subgroup \( r(G) \subset O(n) \) is called the holonomy of \( G \) since its natural representation on \( \mathbb{R}^n \) is equivalent to the holonomy representation of \( \mathbb{R}^n/G \) (see e.g. [Cha86, pp. 50-52]).

We call two Bieberbach manifolds \( M_1 \) and \( M_2 \) affinely equivalent if there exists a diffeomorphism \( F : M_1 \to M_2 \) whose lift to the universal coverings \( \pi_1 : \mathbb{R}^n \to M_1, \pi_2 : \mathbb{R}^n \to M_2 \) is an affine map \( \alpha : \mathbb{R}^n \to \mathbb{R}^n \) such that

\[
\pi_2 \circ \alpha = F \circ \pi_1.
\]

If \( M_1 \) and \( M_2 \) are affinely equivalent, the corresponding Bieberbach groups \( G_1 \) and \( G_2 \) are isomorphic via \( \varphi : G_1 \to G_2 \), \( \varphi(g) = \alpha(g) \alpha^{-1} \). Conversely, if two Bieberbach groups \( G_1 \) and \( G_2 \) are isomorphic, there exists an affine map \( \alpha \) such that the isomorphism is given by \( g \mapsto \alpha g \alpha^{-1} \) (see [Wol11, Theorem 3.2.2]). The map \( \alpha \) descends to a diffeomorphism \( F : M_1 \to M_2 \) and \( M_1 \) and \( M_2 \) are affinely equivalent via \( F \).

Now we want to determine whether a Bieberbach manifold has infinitesimal Einstein deformations. Any Bieberbach manifold is stable since

\[
(\Delta_E h, h)_{L^2} = (\nabla^* \nabla h, h)_{L^2} = \|\nabla h\|_{L^2}^2 \geq 0.
\]

Furthermore, any infinitesimal Einstein deformation is parallel. In the following, we will compute the dimension of the kernel of \( \Delta_E = \nabla^* \nabla \) in terms of the holonomy. The following lemma is a consequence of the holonomy principle.

Lemma 3.1 ([Die13], Proposition 4.2). Let \((M, g)\) be a connected Riemannian manifold and let \( h \) be a symmetric \((0, 2)\)-tensor field. Let \( p \in M \) and let \( T_p M = (E_1)_p \oplus \ldots \oplus (E_k)_p \) be a decomposition into irreducible \( \text{Hol}_p(M, g) \) representations and let

\[
TM = E_1 \oplus \ldots \oplus E_k
\]

be the decomposition of the tangent bundle obtained by parallel transport of the \((E_i)_p\). Then \( \nabla h = 0 \) if and only if \( h = \sum_{i=1}^k \lambda_i g_i \) where \( \lambda_i \in \mathbb{R} \) and \( g_i \) is the metric restricted to \( E_i \).

Proof. Consider \( h \) as an endomorphism on \( TM \) and suppose that \( \nabla h = 0 \). By the holonomy principle, \( h_p \) commutes with the holonomy representation, i.e. \( h_p \circ \rho(g) = \rho(g) \circ h_p \) for all \( g \in \text{Hol}_p(M, g) \).

By Schur’s lemma, \( h_p = \sum_{i=1}^k \lambda_i (pr_i)_p \), where \((pr_i)_p : T_p M \to (E_i)_p \) is the projection map. Let \( pr_i : TM \to E_i \) be the global projection map. It follows that \( h = \sum_{i=1}^k \lambda_i pr_i \), since we obtain \( pr_i \) from \((pr_i)_p\) via parallel transport. The converse is clear. \( \square \)
Corollary 3.2. Let \( (M, g) \) be a connected Riemannian manifold. Then there exists a traceless symmetric \((0, 2)\)-tensor field with \( \nabla h = 0 \) if and only if the holonomy of \( (M, g) \) is reducible.

Proof. This follows from Lemma 3.1 since any traceless symmetric \((0, 2)\)-tensor field admits at least two distinct eigenvalues.

Corollary 3.3. A Bieberbach manifold \( M = \mathbb{R}^n / G \) is strictly stable if and only if the subgroup \( r(G) \subset O(n) \) acts irreducibly on \( \mathbb{R}^n \).

Proof. Since the canonical representation of \( r(G) \) on \( \mathbb{R}^n \) is equivalent to the holonomy representation of \( M \), and any infinitesimal Einstein deformation is parallel, the assertion is immediate from Corollary 3.2.

For the moment, let \( (M, g) \) be an arbitrary Riemannian manifold. We compute the dimension of the space of parallel symmetric \((0, 2)\)-tensors on \( (M, g) \) in terms of the holonomy. Let \( TM = E_1 \oplus \cdots \oplus E_k \) be a parallel orthogonal splitting of the tangent bundle in irreducible components. Then a parallel splitting of the bundle of symmetric \((0, 2)\)-tensors is given by

\[
T^*M \oslash T^*M = \bigoplus_{i,j=1}^k E_i^* \oslash E_j^* = \bigoplus_{i=1}^k \bigoplus_{i<j} E_i^* \oslash E_j^*:
\]

(3.1)

Here, \( E_i^* \) is the image of \( E_i \) under the musical isomorphism and \( \oslash \) denotes the symmetric tensor product. We now search the parallel sections in each of these summands. First suppose that \( h \in \Gamma(\oslash^2 E_i) \) is parallel. Considered as an endomorphism on \( TM \), it induces a parallel endomorphism \( h : E_i \rightarrow E_i \). By the proof of Lemma 3.1, its eigensections form a splitting of the bundle \( E_i \). Since \( E_i \) is irreducible, \( h = \lambda g_i \), where \( \lambda \in \mathbb{R} \) and \( g_i \) is the metric restricted to \( E_i \).

Now we consider the second component of the splitting (3.1). Sections of \( E_i^* \oslash E_j^* \), considered as endomorphisms on \( TM \), are sections of \( \text{End}(E_i \oplus E_j) \) which are of the form

\[
h = \begin{pmatrix}
0 & A^* \\
A & 0
\end{pmatrix},
\]

where \( A \in \Gamma(\text{End}(E_i, E_j)) \) and \( A^* \) is its adjoint. If \( h \) is parallel, \( A \) is also parallel. Therefore, \( \ker(A) \) and \( \text{im}(A) \) are both parallel subbundles of \( E_i, E_j \), respectively. Since \( E_i, E_j \) are irreducible, this shows that \( A \) is an isomorphism if it is nonzero.

Fix a point \( p \) and consider a linear isomorphism \( A_p : (E_i)_p \rightarrow (E_j)_p \). By the holonomy principle, \( A_p \) can be extended to a parallel endomorphism \( A : E_i \rightarrow E_j \) if and only if \( A_p \) commutes with the holonomy representation \( \rho(\text{Hol}_p(M, g)) \subset O(T_pM, g_p) \). This condition precisely means that the restricted standard holonomy representations \( \rho(\text{Hol}_p(M, g))|_{E_i} \) and \( \rho(\text{Hol}_p(M, g))|_{E_j} \) are equivalent via \( A_p \). Since these representations are finite dimensional and irreducible, the space of linear maps \( L : (E_i)_p \rightarrow (E_j)_p \) commuting with them is 1-dimensional. This follows easily from a Lemma from representation theory (see e.g. [NS82, p. 27]).

In summary, we have shown that the dimension of the space of parallel sections in \( E_i^* \oslash E_j^* \) equals 1 if the holonomy representations restricted to \( E_i \) and \( E_j \) are equivalent and zero otherwise. Summing over all components of the splitting (3.1), we obtain

Proposition 3.4. Let \( (M, g) \) be a Riemannian manifold and let \( \text{Hol}(M, g) \) be its holonomy representation. Let

\[
\text{Hol}(M, g) \cong (\rho_1)^{t_1} \oplus \cdots \oplus (\rho_t)^{t_t}
\]
be an irreducible decomposition of $\text{Hol}(M, g)$. Then the dimension of parallel symmetric $(0, 2)$-tensors is equal to

$$\dim(\text{par}(S^2 M)) = \sum_{j=1}^{\ell} i_j (i_j + 1).$$

Let us now go back to the special case of a Bieberbach manifold $(\mathbb{R}^n / G, g)$ and recall that infinitesimal Einstein deformations are precisely the traceless parallel symmetric $(0, 2)$-tensors. Using the fact that the canonical representation $r : G \to O(n)$ is equivalent to the holonomy representation of $M$, we obtain Proposition [11].

**Remark 3.5.** Any infinitesimal Einstein deformations on a Bieberbach manifold, if integrable since $g + th$ is a curve of flat metrics, if $g$ is flat and $h$ is parallel.

Recall that two Bieberbach manifolds $M_1$ and $M_2$ are called affinely equivalent if there exists a diffeomorphism $F : M_1 \to M_2$ whose lift to the universal coverings $\pi_1 : \mathbb{R}^n \to M_1$, $\pi_2 : \mathbb{R}^n \to M_2$ is an affine map $\alpha \in GL(n) \times \mathbb{R}^n$ such that $F \circ \pi_1 = \pi_2 \circ \alpha$. Since $\pi_1, \pi_2$ are local isometries and $\alpha$ is affine, the map $F$ is parallel. Therefore, the induced map $F_* : \Gamma(S^2 M_1) \to \Gamma(S^2 M_2)$ maps parallel tensor fields on $M_1$ isomorphically to parallel tensor fields on $M_2$. It follows that the dimension of infinitesimal Einstein deformations only depends on the affine equivalence class of $M$.

For any $n \in \mathbb{N}$ the number of affine equivalence classes of $n$-dimensional Bieberbach manifolds is finite [Bie12]. In dimension 3, a classification of all Bieberbach manifolds up to affine equivalence is known. In fact, there exist 10 Bieberbach 3-manifolds where six of them are orientable and the others are non-orientable. We describe the corresponding Bieberbach groups in the following. Moreover, we will compute the dimension of infinitesimal Einstein deformations explicitly. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$, let $R(\varphi)$ be the rotation matrix of rotation of $\mathbb{R}^3$ about the $e_1$-axis through $\varphi$ and let $E$ be the reflection matrix at the $e_1$-$e_2$-plane, i.e.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$R(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

Let furthermore $t_i = (1, e_i), i \in \{1, 2, 3\}$ and $I$ be the identity map. Then the Bieberbach groups can be described as follows (see e.g. [Kan6] Lemma 2.1):

| generators of $G_i$ |
|---------------------|
| $G_1$ $t_1, t_2, t_3$ $\alpha = (R_{\pi_1, e_1})$ |
| $G_2$ $t_1, t_2, t_3$ and $\alpha = (R_{\pi_1, e_1})$ |
| $G_3$ $t_1, s_1 = (I, R_{\pi_1, e_2}), s_2 = (I, (R_{\pi_1} e_2))$ and $\alpha = (R_{\pi_1} e_1)$ |
| $G_4$ $t_1, t_2, t_3$ and $\alpha = (R_{\pi_1, e_1})$ |
| $G_5$ $t_1, s_1 = (I, R_{\pi_1, e_2}), s_2 = (R_{\pi_1} e_2, I)$ and $\alpha = (R_{\pi_1} e_1)$ |
| $G_6$ $t_1, t_2, t_3, \alpha = (R_{\pi_1, e_1}), \beta = (-E \cdot R_{\pi_1, e_1})$ and $\gamma = (-E, \frac{1}{2}(e_1 + e_2 + e_3))$ |
| $G_7$ $t_1, t_2, t_3$ and $\alpha = (E, \frac{1}{2} e_1)$ |
| $G_8$ $t_1, s_2 = (I, \frac{1}{2}(e_1 + e_2 + e_3))$ and $\alpha = (E, \frac{1}{2} e_1)$ |
| $G_9$ $t_1, t_2, t_3, \alpha = (R_{\pi_1, e_1})$ and $\beta = (E, \frac{1}{2} e_2)$ |
| $G_{10}$ $t_1, t_2, t_3, \alpha = (R_{\pi_1, e_1})$ and $\beta = (E, \frac{1}{2} (e_2 + e_3))$ |

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The manifolds $M/G_i$ are orientable if $1 \leq i \leq 6$ and non-orientable if $7 \leq i \leq 10$. Now we extract the generators of the holonomy and use Proposition 1.1 to compute the dimension of $\ker(\Delta_E|TT)$:

| $G_i$ | $r(G_i)$ | $\dim(\ker\Delta_E|TT)$ |
|-------|----------|--------------------------|
| $G_1$ | $\Gamma$ | 5                        |
| $G_2$ | $R_\pi$  | 3                        |
| $G_3$ | $R_{2\pi}$ | 1                      |
| $G_4$ | $R_{2\pi}$ | 1                      |
| $G_5$ | $R_{\pi}$  | 1                        |
| $G_6$ | $\{R_\pi, -E \cdot R_\pi, -E\}$ | 2                     |
| $G_7$ | $E$      | 3                        |
| $G_8$ | $\{R_\pi, E\}$ | 2                     |
| $G_9$ | $\{R_\pi, E\}$ | 2                     |
| $G_{10}$ | $\{R_\pi, E\}$ | 2                     |

This table in particular shows that each three-dimensional Bieberbach manifold has infinitesimal Einstein deformations and hence, it is also deformable as an Einstein space by our remark above. In fact, the moduli space of Einstein structures on these manifolds concides with the moduli space of flat structures. An explicit description of these moduli spaces is given in [Kan06, Theorem 4.5].

It seems possible but it is not known if there are Bieberbach manifolds which are isolated as Einstein spaces.

4 The Einstein operator on product manifolds

Let $(M,g_1)$ and $(N,g_2)$ be Einstein manifolds and consider the product manifold $(M \times N, g_1 + g_2)$. It is Einstein if and only if the components have the same Einstein constant $\mu$. In this case, the Einstein constant of the product is also $\mu$. We want to determine if a product Einstein space is stable or not. This was worked out in [AM11] in the case, where the Einstein constant is negative. We study the general case.

In the following, we often lift tensors on the factors $M, N$ to tensors on $M \times N$ by pulling back along the projection maps. In order to avoid notational complications, we drop the explicit reference to the projections throughout the section.

At first, we consider the spectrum of the Einstein operator on the product space.

**Proposition 4.1 ([AM11]).** Let $\Delta_{E}^{M \times N}$ be the Einstein operator with respect to the product metric acting on $\Gamma(S^2(M \times N))$. Then the spectrum of $\Delta_{E}^{M \times N}$ is given by

$$\spec(\Delta_{E}^{M \times N}) = (\spec(\Delta_{E}^{M}) + \spec(\Delta_{0}^{N})) \cup (\spec(\Delta_{0}^{N}) + \spec(\Delta_{E}^{M})) \cup (\spec(\Delta_{1}^{M}) + \spec(\Delta_{1}^{N})).$$

Here, $\Delta_{E}^{M}$, $\Delta_{E}^{N}$, $\Delta_{0}^{M}$, $\Delta_{1}^{N}$ denote the connection Laplacians on functions and 1-forms with respect to the metrics on $M$ and $N$, respectively.

**Proof.** Let $\{\alpha_i\}, \{\omega_i\}, \{h_i\}$ be complete orthonormal systems of symmetric $p$-eigentensors ($p = 0, 1, 2$) of the operators $\Delta_{0}^{M}$, $\Delta_{1}^{M}$, $\Delta_{E}^{M}$, respectively. Let $\lambda_{i}^{(0)}, \lambda_{i}^{(1)}, \lambda_{i}^{(2)}$ be the corresponding eigenvalues. Let $\{\beta_i\}, \{\phi_i\}, \{k_i\}$ be complete orthonormal systems of symmetric $(0,p)$-eigentensors ($p = 0, 1, 2$) of the operators $\Delta_{0}^{N}$, $\Delta_{1}^{N}$, $\Delta_{E}^{N}$, respectively. Let $\kappa_{i}^{(0)}, \kappa_{i}^{(1)}, \kappa_{i}^{(2)}$ be their eigenvalues. By [AM11] Lemma 3.1, the tensor products $\alpha_i h_j$, $\beta_i \phi_j$, $\omega_i \otimes \phi_j$ form a complete orthonormal system in $\Gamma(S^2(M \times N))$. 

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Straightforward calculations show that
\[
\begin{align*}
\Delta_E^{M \times N}(\alpha_i k_j) &= (\lambda^{(0)}_i + \kappa_j^{(2)})\alpha_i k_j, \\
\Delta_E^{M \times N}(\omega_i \otimes \phi_j) &= (\lambda^{(1)}_i + \kappa_j^{(1)})\omega_i \otimes \phi_j, \\
\Delta_E^{M \times N}(\beta_i h_j) &= (\kappa_i^{(0)} + \lambda_j^{(2)})\beta_i h_j,
\end{align*}
\]
from which the assertion follows. \(\square\)

Another operator closely related to the Einstein operator is the Lichnerowicz Laplacian acting on \(\Gamma(S^2 M)\), defined by
\[
\Delta_L h = \nabla^* \nabla h + \text{Ric} \circ h + h \circ \text{Ric} - 2\tilde{R} h.
\]
It satisfies some useful properties:

**Lemma 4.2.** Let \((M, g)\) be a Riemannian manifold. Then
\[
\begin{align*}
\Delta_L (f \cdot g) &= (\Delta f) \cdot g, \quad (4.2) \\
\text{tr}(\Delta_L h) &= \Delta(\text{tr} h) \quad (4.3)
\end{align*}
\]
for all \(f \in C^\infty(M), h \in \Gamma(S^2 M)\). Moreover, if Ric is parallel,
\[
\begin{align*}
\Delta_L (\delta^* \omega) &= \delta^* (\Delta_H \omega), \quad (4.4) \\
\delta(\Delta_L h) &= \Delta_H (\delta h), \quad (4.5) \\
\Delta_L (\nabla^2 f) &= \nabla^2 (\Delta f) \quad (4.6)
\end{align*}
\]
for all \(f \in C^\infty(M), \omega \in \Omega^1(M), h \in \Gamma(S^2 M)\). Here, \(\Delta_H = \nabla^* \nabla - \text{Ric}\) is the Hodge Laplacian on 1-forms.

**Proof.** Formula (4.2) follows from an easy calculation. Formula (4.6) follows from (4.4) and the well-known formula \(\Delta_H (\nabla f) = \nabla(\Delta f)\). For a proof of the other formulas, see e.g. [Lic61, pp. 28-29]. \(\square\)

Observe that on Einstein manifolds, we have the relation \(\Delta_L = \Delta_E + 2\mu \cdot \text{id}\) where \(\mu\) is the Einstein constant.

**Lemma 4.3.** Let \((M, g)\) be an Einstein manifold with constant \(\mu\). Then the spectrum of \(\Delta_E\) on \(\Gamma(S^2 M)\) can be decomposed as
\[
\text{spec}(\Delta_E) = \text{spec}(\Delta_0 - 2\mu \cdot \text{id}) \cup \text{spec}_+(((\Delta_1 - \mu \cdot \text{id})|_W)) \cup \text{spec}(\Delta_{E|TT}),
\]
where \(W = \{\omega \in \Omega^1(M) \mid \delta \omega = 0\}\).

**Proof.** If \((M, g)\) is not the standard sphere, we consider the decomposition
\[
\Gamma(S^2 M) = C^\infty(M) \cdot g \oplus \delta_g^*(\Omega^1(M)) \oplus TT_g.
\]
Let \(\{f_i\}, i \in \mathbb{N}_0\) be an eigenbasis of \(\Delta_0\) to the eigenvalues \(\lambda_i^{(0)}\), where \(f_0\) is the constant eigenfunction. Let \(\{\omega_i\}, i \in \mathbb{N}\), be an eigenbasis of \(\Delta_1 = \Delta_H - \mu\) acting on \(W\) with eigenvalues \(\lambda_i^{(1)}\). Let \(\{h_i\}_{i \in \mathbb{N}}\) be an eigenbasis of \(\Delta_{E|TT}\) with eigenvalues \(\lambda_i^{(2)}\). Then \(\{\nabla f_i\}, i \in \mathbb{N}, \{\omega_i\}, i \in \mathbb{N}\) form an eigenbasis of \(\Delta_1\) on all 1-forms and \(\{f_i \cdot g\}, i \in \mathbb{N}_0, \{\nabla^2 f_i\}, i \in \mathbb{N}, \{\delta^* \omega_i\}, i \in \mathbb{N}\) and \(\{h_i\}, i \in \mathbb{N}\) form a basis of \(\Gamma(S^2 M)\). On the round sphere, we have
\[
\Gamma(S^2 M) = [C^\infty(M) \cdot g + \delta_g^*(\Omega^1(M))] \oplus \text{tr}_g^{-1}(0) \cap \delta_g^{-1}(0).
\]
and

\[ C^\infty(S^n) \cdot g_{st} \cap \delta g_{st}(\Omega^1(S^n)) = \{ f \cdot g_{st} \in C^\infty(S^n) : g_{st} | \Delta f = n \cdot f \}, \]

where \( n \) is the first nonzero eigenvalue of the Laplacian (see [Res08, Lemma 4.57] and [Oba02, Theorem A]). If \((M, g) = (S^n, g_{sp})\), we therefore have a basis, if we remove from \( \{ \nabla^2 f_i \} \) the \( f_i \) which are the eigenfunctions to the first nonzero eigenvalue of the Laplacian. By the relation \( \Delta_E = \Delta_L - 2\mu \cdot \text{id} \) and Lemma 4.2 we have

\[
\Delta_E(f_i \cdot g) = (\lambda_i^{(0)} - 2\mu)f_i \cdot g, \\
\Delta_E(\nabla^2 f_i) = (\lambda_i^{(0)} - 2\mu)\nabla^2 f_i, \\
\Delta_E(\delta^* \omega_i) = (\lambda_i^{(1)} - \mu)\delta^* \omega_i,
\]

which shows that we have obtained a basis of eigentensors of \( \Delta_E \). By Lemma 4.4 below, \( \lambda_i^{(1)} - \mu \geq 0 \) and equality holds if and only if \( \delta^* \omega_i = 0 \). This finishes the proof of the lemma. \( \square \)

**Lemma 4.4.** Let \((M, g)\) be an Einstein manifold with constant \( \mu \) and \( W \) as in Lemma 4.3 above. Then

\[ \|\nabla \omega\|^2_{L^2} = 2 \|\delta^* \omega\|^2 + \mu \|\omega\|^2_{L^2} \]

for any \( \omega \in W \). In particular, \( \text{spec}((\Delta_1 - \mu \cdot \text{id})|W) \) is nonnegative.

**Proof.** Let \( \{e_1, \ldots, e_n\} \) be a local orthonormal frame. Then

\[
\|\nabla \omega\|^2_{L^2} = \int_M \sum_{i,j} (\nabla_{e_i} \omega(e_j))^2 \, dV \\
= \frac{1}{2} \int_M \sum_{i,j} (\nabla_{e_i} \omega(e_j) + \nabla_{e_j} \omega(e_i))^2 - 2(\nabla_{e_i} \omega(e_j) \nabla_{e_j} \omega(e_i)) \, dV \\
= 2 \|\delta^* \omega\|^2 + \int_M \sum_{i,j} \omega(e_j) \nabla_{e_i} \omega(e_i) \, dV \\
= 2 \|\delta^* \omega\|^2 + \int_M \sum_{i,j} \omega(e_j) \mathcal{R}_{e_i} \omega(e_i) \, dV \\
= 2 \|\delta^* \omega\|^2 + \int_M \sum_{i,j} \omega(e_j) (\omega \circ \mathcal{R})(e_i) \, dV \\
= 2 \|\delta^* \omega\|^2 + \mu \|\omega\|^2_{L^2}
\]

and if \( \mu \) is nonnegative, the nonnegativity of \( \Delta_1 - \mu \cdot \text{id} = \nabla^* \nabla - \mu \cdot \text{id} \) follows. \( \square \)

**Proposition 4.5.** If \((M, g_1)\) and \((N, g_2)\) are two stable Einstein metrics with \( \mu \leq 0 \), the product manifold \((M \times N, g + h)\) is also stable.

**Proof.** By Lemma 4.3 and since \( \mu \leq 0 \), the operators \( \Delta_E^M, \Delta_E^N \) are nonnegative on all of \( \Gamma(S^2 M) \) if and only if their restriction to \( TT \)-tensors is, respectively. By Proposition 4.1 \( \Delta_E^{M \times N} \) is nonnegative since the sum of the spectra does not contain negative elements. \( \square \)

If \((M, g)\) and \((N, g_2)\) are stable Einstein manifolds with constant \( \mu < 0 \), it is also quite immediate that

\[ \ker(\Delta_E^{M \times N}|_{TT}) \cong \ker(\Delta_E^M|_{TT}) \oplus \ker(\Delta_E^N|_{TT}) \]

(see [AM11, Lemma 3.2]). We show that if \( \mu = 0 \), the situation is slightly more subtle.
Proposition 4.6. Let $(M^{n_1}, g_1)$ and $(N^{n_2}, g_2)$ be stable Ricci-flat manifolds. Then
\[ \ker(\Delta_{E}^{M \times N}|_{TT}) \cong \mathbb{R}(n_2 \cdot g_1 - n_1 \cdot g_2) \oplus (\text{par}(\Omega^1(M)) \circ \text{par}(\Omega^1(N))) \oplus \ker(\Delta_{E}^{M}|_{TT}) \oplus \ker(\Delta_{E}^{N}|_{TT}). \]
Here, par$(\Omega^1(M))$, par$(\Omega^1(N))$ denote the spaces of parallel 1-forms on $M, N$ respectively. If all infinitesimal Einstein deformations of $M$ and $N$ are integrable, then all infinitesimal Einstein deformations of $M \times N$ are integrable.

Proof. By the proof of Proposition 4.1, the kernel of $\Delta_{E}^{M \times N}$ is spanned by tensors of the form $\alpha_i k_j, \beta_i h_j, \omega_i \circ \phi_j$ where $\alpha_i, \omega_i, h_i$ and $\beta_i, \phi_i, k_i$ are eigentensors of $\Delta_0, \Delta_1, \Delta_E$ on $M$ and $N$, respectively. By Lemma 4.3, these operators are nonnegative, so the eigentensors have to lie in the kernel of the corresponding operators. This shows
\[ \ker(\Delta_{E}^{M \times N}) \cong \mathbb{R} \cdot g_1 \oplus \mathbb{R} \cdot g_2 \oplus (\text{par}(\Omega^1(M)) \circ \text{par}(\Omega^1(N))) \oplus \ker(\Delta_{E}^{M}|_{TT}) \oplus \ker(\Delta_{E}^{N}|_{TT}). \]

The first assertion follows from restricting $\Delta_{E}^{M \times N}$ to $TT$-tensors. Any deformation $h \in \mathbb{R}(n_2 \cdot g_1 - n_1 \cdot g_2)$ is integrable since it can be integrated to a curve of metrics of the form $(g_1) + (g_2)t$, where $(g_1)$ and $(g_2)$ are just rescalings of $g_1$ and $g_2$. This of course does not affect the Ricci-flatness of $M \times N$.

Now, consider the situation where $h \in (\text{par}(\Omega^1(M)) \circ \text{par}(\Omega^1(N)))$. Let $\omega_1, \ldots, \omega_{m_1}$ be a basis of par$(\Omega^1(M))$ and $\phi_1, \ldots, \phi_{m_2}$ be a basis of par$(\Omega^1(N))$. Suppose for simplicity that all these forms have constant length 1. Then
\[ h = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \alpha_{ij} \omega_i \circ \phi_j. \]

We show that $h$ is integrable. By the holonomy principle, we have parallel decompositions
\[ TM = E \oplus \bigoplus_{i=1}^{m_1}(\mathbb{R} \cdot \omega^i), \quad TN = F \oplus \bigoplus_{j=1}^{m_2}(\mathbb{R} \cdot \phi^j), \]
and the metrics split as $g_1 = \tilde{g}_1 + \sum_{i=1}^{m_1} \omega_i \otimes \omega_i, g_2 = \tilde{g}_2 + \sum_{j=1}^{m_2} \phi_j \otimes \phi_j$. The metrics $\tilde{g}_1$ and $\tilde{g}_2$ are also Ricci-flat. The tangent bundle of the product manifold obviously splits as
\[ T(M \times N) = E \oplus F \oplus \bigoplus_{i=1}^{m_1}(\mathbb{R} \cdot \omega^i) \oplus \bigoplus_{j=1}^{m_2}(\mathbb{R} \cdot \phi^j). \]

Observe that $g_1 + g_2$ is flat when restricted to
\[ G = \bigoplus_{i=1}^{m_1}(\mathbb{R} \cdot \omega^i) \oplus \bigoplus_{j=1}^{m_2}(\mathbb{R} \cdot \phi^j). \]

Consider the curve of metrics $t \mapsto g_t = g_1 + g_2 + th$ on $M \times N$. The metric restricted $E \oplus F$ does not change and stays flat if we restrict to $G$. Thus, $g_t$ is a curve of Ricci-flat metrics, so $h$ is integrable.

If $h \in \ker(\Delta_{E}^{M}|_{TT})$, then there exists a curve of Einstein metrics $(g_1)$ on $M$ tangent to $h$ by assumption. Consequently, the curve $(g_1) \oplus g_2$ is a curve of Einstein metrics on $M \times N$ tangent to $h$, so $h$ is integrable (considered as an infinitesimal Einstein deformation on $M \times N$). If $h \in \ker(\Delta_{E}^{N}|_{TT})$, an analogous argument shows the integrability of $h$. \hfill \square

Now, let us turn to the case where the Einstein constant is positive.
Lemma 4.7. Let $(M, g)$ be a positive Einstein manifold with constant $\mu$. Then
\[
\dim(\ker\Delta_E) = 2 \cdot \text{mult}_{\Delta_0}(2\mu) + \dim(\ker\Delta_E|_{TT}),
\]
\[
\text{ind}(\Delta_E) = 1 + \sum_{\lambda \in (\frac{1}{2n-1}\mu, 2\mu]} 2 \cdot \text{mult}_{\Delta_0}(\lambda) + \text{ind}(\Delta_E|_{TT}),
\]
where $\text{mult}_{\Delta_0}(\lambda)$ is the multiplicity of $\lambda$ as an eigenvalue of $\Delta_0$ and $\text{ind}(\Delta_E)$ is the index of the quadratic form $h \mapsto (\Delta_E h, h)|_{L^2}$.

Proof. This follows immediately from the proof of Lemma 4.3 and Obata’s theorem (Theorem 2.1).

Proposition 4.8. Let $(M^{m_1}, g_1), (N^{m_2}, g_2)$ be stable Einstein manifolds with constant $\mu > 0$. Then
\[
\dim(\ker\Delta_E|_{TT}) = \dim(\ker\Delta_M|_{TT}) + \dim(\ker\Delta_N|_{TT}) + \text{mult}_{\Delta_0}(2\mu) + \text{mult}_{\Delta_0}(2\mu),
\]
\[
\text{ind}(\Delta_E|_{TT}) = 1 + \sum_{\lambda \in (\frac{1}{2n-1}\mu, 2\mu]} \text{mult}_{\Delta_0}(\lambda) + \sum_{\lambda \in (\frac{1}{2n-1}\mu, 2\mu]} \text{mult}_{\Delta_0}(\lambda).
\]

Proof. We now prove the first assertion. By Lemma 4.4, $\Delta_M$ and $\Delta_N$ are positive. Thus by Proposition 4.1 we have to count the number of eigenvalues (with their multiplicity) $\lambda_i^{(0)} \in \text{spec}(\Delta_M)$, $\lambda_i^{(2)} \in \text{spec}(\Delta_M^2)$, $\kappa_i^{(0)} \in \text{spec}(\Delta_N)$, $\kappa_i^{(2)} \in \text{spec}(\Delta_N^2)$ such that $\lambda_i^{(0)} + \kappa_i^{(2)} = 0$ and $\lambda_i^{(2)} + \kappa_i^{(0)} = 0$.

Consider the first equation. If $\lambda_i^{(0)} = \lambda_i^{(2)} = 0$, then also $\kappa_i^{(2)} = 0$ and the multiplicity of $\kappa_i^{(2)}$ is given in Lemma 4.7. If $\lambda_i^{(0)} > 0$, then $\kappa_i^{(2)} < 0$. By Lemma 4.3, Lemma 4.4 and since $(M, g_1)$ is stable, $\kappa_i^{(2)} + 2\mu = \kappa_i^{(0)} \in \text{spec}(\Delta_N)$. We thus have to find $\kappa_i^{(0)}$ such that $\lambda_i^{(0)} + \kappa_i^{(0)} = 2\mu$ for $\lambda_i^{(0)} > 0$. By Obata’s eigenvalue estimate, we have a lower bound $\lambda_i^{(0)}, \kappa_i^{(0)} \geq \frac{n}{n-1}\mu$ for nonzero eigenvalues of the Laplacian. Therefore, the only situation which remains possible is that $\lambda_i^{(0)} = 2\mu$ and $\kappa_i^{(0)} = \kappa_i^{(0)} = 0$. Since eigenvalue zero has always multiplicity 1, $\kappa_i^{(2)} = \kappa_i^{(0)} - 2\mu = -2\mu$ is of multiplicity 1. Now we do the same game for the equation $\lambda_i^{(2)} + \kappa_i^{(0)} = 0$. We obtain, after summing up both cases,
\[
\dim(\ker\Delta_E|_{TT}) = \dim(\ker\Delta_M|_{TT}) + \dim(\ker\Delta_N|_{TT}) + 3\text{mult}_{\Delta_0}(2\mu) + 3\text{mult}_{\Delta_0}(2\mu).
\]

By the formula
\[
\text{mult}_{\Delta_0}(\tau) = \sum_{\lambda + \kappa = \tau} \text{mult}_{\Delta_0}(\lambda) \cdot \text{mult}_{\Delta_0}(\kappa)
\]
(4.7)
and by Obata’s eigenvalue estimate,
\[
\text{mult}_{\Delta_0}(2\mu) = \text{mult}_{\Delta_0}(2\mu) + \text{mult}_{\Delta_0}(2\mu).
\]

From Lemma 4.7 we get the dimension of $\ker\Delta_E|_{TT}$.

To show the second assertion, we compute the number of eigenvalues (with multiplicity) satisfying $\lambda_i^{(0)} + \kappa_i^{(2)} < 0$ or $\lambda_i^{(2)} + \kappa_i^{(0)} < 0$. Consider the first inequality. If $\lambda_i^{(0)} = \lambda_i^{(2)} = 0$, then $\kappa_i^{(2)} < 0$ and the number of such eigenvalues (with multiplicity) is given by Lemma 4.7. If $\lambda_i^{(0)} > 0$, then $\lambda_i^{(2)} \geq \frac{n}{n-1}\mu$ and $\kappa_i^{(2)} < -\frac{n}{n-1}\mu$. By Lemma 4.3, $\kappa_i^{(2)} + 2\mu = \kappa_i^{(0)} \in \text{spec}(\Delta_N)$ and $\kappa_i^{(0)} < \frac{n-2}{n-1}\mu$. By Obata’s eigenvalue estimate, $\kappa_i^{(0)} = \kappa_i^{(0)} = 0$ and $\kappa_i^{(2)} = -2\mu$ appears with multiplicity 1. This also implies that $\lambda_i^{(0)} < 2\mu$. 

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Similarly, we deal with the inequality \( \lambda^{(2)}_i + \kappa^{(0)}_i < 0 \). Summing up over both cases, we obtain

\[
\text{ind}(\Delta_{E}^{M \times N}) = 2 + 3 \sum_{\lambda \in (n_1^2 - 1, 2\mu)} \text{mult}_{\Delta^M_0}(\lambda) + 3 \sum_{\lambda \in (2, 2\mu)} \text{mult}_{\Delta^N_0}(\lambda) + 2 \cdot \text{mult}_{\Delta^M_0}\left(\frac{n_1}{n_1 - 1}\mu\right) + 2 \cdot \text{mult}_{\Delta^N_0}\left(\frac{n_2}{n_2 - 1}\mu\right).
\]

By (4.7) and by Obata’s eigenvalue estimate,

\[
\sum_{\lambda \in (0, 2\mu)} \text{mult}_{\Delta_{E}^{M \times N}}(\lambda) = \sum_{\lambda \in (0, 2\mu)} \text{mult}_{\Delta^M_0}(\lambda) + \sum_{\lambda \in (0, 2\mu)} \text{mult}_{\Delta^N_0}(\lambda)
\]

and the second assertion follows from Lemma 4.7.

**Corollary 4.9.** Any product of positive Einstein metrics is unstable.

**Remark 4.10.** We also see that small eigenvalues of the Laplacian enlarge the index of the form

\[ TT \ni h \mapsto (\Delta_{E} h, h)_{L^2}. \]

If \( 2\mu \) is an eigenvalue of the Laplace-Beltrami operator on \( M \) or \( N \) (this holds e.g. for the complex projective space) then the product metric has infinitesimal Einstein deformations. Such infinitesimal Einstein deformations were studied on \( \mathbb{C}P^{2n} \times S^2 \) by Koiso [Koi82]. He showed that they are not integrable.

### 5 Stability under sectional curvature bounds

Recall Theorem 1.2, which is a first attempt to relate stability of Einstein manifolds to curvature assumptions. Because we also work with his methods later on, we will sketch Koiso’s proof of the theorem. Let \( S^2 g M \) be the vector bundle of symmetric \((0, 2)\)-tensors whose trace with respect to \( g \) vanishes. We define a function \( r : M \to \mathbb{R} \) by

\[
r(p) = \sup \left\{ \frac{(Rg, \eta)_p}{|\eta|^2_p} \mid \eta \in (S^2 g M)_p \right\}.
\]

We now use the Bochner technique. Let the two differential operators \( D_1 \) and \( D_2 \) be defined by

\[
D_1 h(X, Y, Z) = \frac{1}{\sqrt{3}}(\nabla_X h(Y, Z) + \nabla_Y h(Z, X) + \nabla_Z h(X, Y)),
\]

\[
D_2 h(X, Y, Z) = \frac{1}{\sqrt{2}}(\nabla_X h(Y, Z) - \nabla_Y h(Z, X)).
\]

For the Einstein operator, we have the Bochner formulas

\[
(\Delta_{E} h, h)_{L^2} = \|D_1 h\|_{L^2}^2 + 2\mu \|h\|_{L^2}^2 - 4(\mathring{R} h, h)_{L^2} - 2 \|\delta h\|_{L^2}^2,
\]

\[
(\Delta_{E} h, h)_{L^2} = \|D_2 h\|_{L^2}^2 - \mu \|h\|_{L^2}^2 - (\mathring{R} h, h)_{L^2} + \|\delta h\|_{L^2}^2,
\]

see [Koi78] p. 428 or [Bes08] p. 355 for more details. Because of the bounds on \( r \) and \( \delta h = 0 \), we obtain either \( (\Delta_{E} h, h)_{L^2} \geq 0 \) or \( (\Delta_{E} h, h)_{L^2} > 0 \) for \( TT \)-tensors by (5.2) or (5.3). This proves Theorem 1.2.
Lemma 5.1. Let \((M, g)\) be Einstein and \(p \in M\). Let \(K_{\min}\) and \(K_{\max}\) be the minimum and maximum of its sectional curvature at \(p\), then

\[
r(p) \leq \min \left\{ (n-2)K_{\max} - \mu, \mu - nK_{\min} \right\}.
\]  
(5.4)

If equality holds, i.e.

\[
r(p) = (n-2)K_{\max} - \mu = \mu - nK_{\min},
\]  
(5.5)

then \((M, g)\) is even-dimensional. Let \(\eta \in (S^2g)^p\) be such that \(\hat{R}\eta = r(p)\eta\). Then \(\eta\) has only two eigenvalues \(\lambda, -\lambda\) and the eigenspaces \(E(\lambda), E(-\lambda)\) are both of dimension \(m = n/2\). Moreover, \(K(P) = K_{\max}\) for each plane \(P\) lying in either \(E(\lambda)\) or \(E(-\lambda)\) and \(K(P) = K_{\min}\) if \(P\) is spanned by one vector in \(E(\lambda)\) and one in \(E(-\lambda)\).

Proof. Estimate (5.4) was already proven in [Fuj79]. We redo the proof of [Bes08 Lemma 12.71]. Choose \(\eta\) such that \(\hat{R}\eta = r(p)\eta\). Let \(\{e_1, \ldots, e_n\}\) be an orthonormal basis in which \(\eta\) is diagonal with eigenvalues \(\lambda_1, \ldots, \lambda_n\) such that \(\lambda_i = \sup |\lambda_i|\) and \(\sum \lambda_i = 0\). Then

\[
r(p)\lambda_i = (\hat{R}\eta)(e_i, e_i) = \sum_i K_{i1}\lambda_i,
\]

where \(K_{i1}\) is the sectional curvature of the plane spanned by \(e_i\) and \(e_1\). Thus,

\[
r(p)\lambda_i = \sum_{i \neq 1} K_{\max}\lambda_i - \sum_{i \neq 1} (K_{\max} - K_{i1})\lambda_i \\
\leq -\lambda_1 K_{\max} + \lambda_1 \sum_{i \neq 1} (K_{\max} - K_{i1}) \\
= ((n-2)K_{\max} - \mu)\lambda_1.
\]  
(5.6)

On the other hand,

\[
r(p)\lambda_i = \sum_{i \neq 1} K_{\min}\lambda_i + \sum_{i \neq 1} (K_{i1} - K_{\min})\lambda_i \\
\leq -\lambda_1 K_{\min} + \lambda_1 \sum_{i \neq 1} (K_{i1} - K_{\min}) \\
= (-nK_{\min} + \mu)\lambda_i,
\]  
(5.7)

so we have proven the first assertion. Suppose now that (5.5) holds, then equality must hold both in (5.6) and (5.7). From (5.6), we get that either \(\lambda_i = -\lambda_1\) or \(K_{i1} = K_{\max}\) whereas (5.7) implies \(\lambda_i = \lambda_1\) or \(K_{i1} = K_{\min}\) for each \(i\). Thus there only exist two eigenvalues \(\lambda\) and \(-\lambda\) which are of same multiplicity since the trace of \(\eta\) vanishes. In particular, \((M, g)\) is even-dimensional.

Let \(P \subset T_pM\) be a plane which satisfies one of the assumptions of the lemma. We then may assume that \(P\) is spanned by two vectors of the eigenbasis we have chosen. If \(P \subset E(\lambda)\) or \(P\) is spanned by two vectors in \(E(\lambda), E(-\lambda)\), respectively, we may assume \(e_1 \in P\). Then the assertions follow from the above. If \(P \subset E(-\lambda)\), we may replace \(\eta\) by \(-\eta\) and the roles of \(E(\lambda)\) and \(E(-\lambda)\) interchange. \(\square\)

From Theorem 1.2 and the first part of Lemma 5.1, the Corollaries 1.3 and 1.4 are consequences. We now prove refinements of these corollaries.

Proposition 5.2. Let \((M, g)\) be an Einstein manifold with constant \(\mu\) such that the sectional curvature lies in the interval \([\mu - nK_{\max}, 0] \cdot K_{\max}\). Then \((M, g)\) is stable. If \(\ker \Delta_{E|TT}\) is nontrivial, \(M^n\) is
even-dimensional. Furthermore, there exists an orthogonal splitting \( TM = E \oplus F \) into two subbundles of dimension \( n/2 \). The two \( C^\infty(M) \)-bilinear maps

\[
I : \Gamma(E) \times \Gamma(E) \to \Gamma(F), \quad (X, Y) \mapsto pr_F(\nabla_XY)
\]

and

\[
II : \Gamma(F) \times \Gamma(F) \to \Gamma(E), \quad (X, Y) \mapsto pr_E(\nabla_XY)
\]

are both antisymmetric in \( X \) and \( Y \). Moreover, the sectional curvature of a plane \( P \) is equal to \( K_{\text{max}} \) if \( P \) either lies in \( E \) or \( F \). If \( P = \text{span}(e, f) \) with \( e \in E \) and \( f \in F \), then \( K(P) = K_{\text{min}} \).

**Proof.** Because of the curvature assumptions, \( \mu \geq \frac{2}{3}(n-2)K_{\text{max}} \) or \( \mu \leq 2nK_{\text{min}} \) at each point. In both cases, the function \( r \) from Lemma 5.1 satisfies \( r \leq \frac{1}{2} \mu \). Thus, \( r_0 \leq \frac{1}{2} \mu \) and Theorem 1.2 implies that \( (M,g) \) is stable. Suppose now there exists \( h \in \ker \Delta_E|TT, h \neq 0 \). Then by (5.3),

\[
0 = (\Delta_E h, h)_{L^2} = ||D_1h||^2_{L^2} + 2\mu \|h\|^2_{L^2} - 4(h, \hat{\nabla}h)_{L^2} \geq 0 + 2\mu \|h\|^2_{L^2} - 2\mu \|h\|^2_{L^2} = 0.
\]

Therefore, \( D_1h \equiv 0 \) and \( (\hat{\nabla}h, h)_p \equiv \frac{\mu}{2} \|h\|^2_p \) for all \( p \in M \). The second equality implies that

\[
\mu = \frac{2}{3}(n-2)K_{\text{max}} = 2nK_{\text{min}}
\]

and

\[
r(p) = (n-2)K_{\text{max}} - \mu = -nK_{\text{min}}.
\]

Thus, Lemma 5.1 applies and at each point where \( h \neq 0 \), the tangent space splits into the two eigenspaces of \( h \), i.e. \( T_pM = E_p(\lambda) \oplus E_p(-\lambda) \). Since \( D_1h \equiv 0 \), we have

\[
g(\nabla_{e_i}h(e_j), e_k) + g(\nabla_{e_i}h(e_k), e_i) + g(\nabla_{e_i}h(e_i), e_j) = 0 \quad (5.8)
\]

for any local orthonormal frame \( \{e_1, \ldots, e_n\} \). Here, we considered \( h \) as an endomorphism \( h : TM \to TM \). Choose a local eigenframe of \( h \) around some \( p \) outside the zero set of \( h \). A straightforward calculation shows

\[
(\nabla_{e_i}h(e_j), e_k) = (\nabla_{e_i}\lambda_j)\delta_{jk} + \lambda_j \Gamma^k_{ij} - \lambda_k \Gamma^k_{ij}, \quad (5.9)
\]

where \( \lambda_j \) is the eigenvalue of \( e_j \). Now we rewrite (5.8) as

\[
(\lambda_j - \lambda_k) \Gamma^k_{ij} + (\lambda_k - \lambda_i) \Gamma^i_{jk} + (\lambda_i - \lambda_j) \Gamma^j_{ki} = -(\nabla_{e_i}\lambda_j)\delta_{jk} - (\nabla_{e_j}\lambda_k)\delta_{ki} - (\nabla_{e_k}\lambda_i)\delta_{ij}. \quad (5.10)
\]

If we choose \( i = j = k \), we obtain

\[
0 = -3(\nabla_{e_i}\lambda_i).
\]

Since \( \lambda_i = \pm \lambda \), it is immediate that \( \lambda \) is constant and it is nonzero. Thus, we obtain a global splitting \( TM = E \oplus F \) where the two distributions are defined by

\[
E = \bigcup_{p \in M} E_p(\lambda), \quad F = \bigcup_{p \in M} E_p(-\lambda).
\]

By Lemma 5.1, the assertion about the sectional curvatures is immediate. To finish the proof, it just remains to show the antisymmetry of the maps \( I, II \), respectively.
Let \( \{e_1, \ldots, e_n\} \) be the eigenframe from before. We may assume that \( e_1, \ldots, e_{n/2} \) are local sections in \( E \) and \( e_{n/2+1}, \ldots, e_n \) are local sections in \( F \). Choose \( i, j \in \{1, \ldots, n/2\}, k \in \{n/2 + 1, \ldots, n\} \). Then \( \lambda_i = \lambda_j = \lambda, \lambda_k = -\lambda \) and (5.10) yields
\[
0 = 2\lambda (\Gamma^k_{ij} + \Gamma^k_{ji}),
\]
since the right-hand side of (5.10) vanishes for any \( i, j, k \). Now consider the map \( I \). We have
\[
I(e_i, e_j) = \text{pr}_E(\nabla_{e_i} e_j) = \sum_{k=n/2+1}^n \Gamma^k_{ij} e_k,
\]
and by (5.11), we immediately get \( I(e_i, e_j) = -I(e_j, e_i) \). Similarly, antisymmetry is shown for \( II \). \( \square \)

Now let us turn to the case of nonpositive sectional curvature.

**Definition 5.3.** Let \( (M, g) \) be a Riemannian manifold and let \( \{e_1, \ldots, e_n\} \) be an orthonormal frame at \( p \in M \). Then \( K_{ij} = R_{ijji} \) is the sectional curvature of the plane spanned by \( e_i \) and \( e_j \) if \( i \neq j \) and is zero if \( i = j \). We count the number of \( j \) such that \( K_{ij} = 0 \) for a given \( i \) and call the maximum of such numbers over all orthonormal frames at \( p \) the flat dimension of \( M \) at \( p \), denoted by \( \text{fd}(M)_p \). The number \( \text{fd}(M) = \sup_{p \in M} \text{fd}(M)_p \) is called the flat dimension of \( M \).

**Proposition 5.4 (Koč 78 Proposition 3.4).** Let \( (M, g) \) be a non-flat Einstein manifold with nonpositive sectional curvature. Then \( (M, g) \) is stable. If \( \ker(\Delta_E|TT) \) is nontrivial, the flat dimension of \( M \) satisfies \( \text{fd}(M)_p \geq \left\lceil \frac{n}{2} \right\rceil \) at each \( p \in M \).

If in addition, a lower bound on the sectional curvature is assumed, we obtain stronger consequences of the existence of infinitesimal Einstein deformations:

**Proposition 5.5.** Let \( (M, g) \) a non-flat Einstein manifold with nonpositive sectional curvature and Einstein constant \( \mu \). If \( K_{\text{min}} > \frac{2}{n} \mu \), then \( (M, g) \) is strictly stable. If \( K_{\text{min}} \geq \frac{2}{n} \mu \), then \( (M, g) \) is stable. If \( \ker(\Delta_E|TT) \) is nontrivial, then \( M \) is even-dimensional and we have an orthogonal splitting \( TM = E \oplus F \). Both subbundles are of dimension \( n/2 \). The \( C^\infty(M) \)-bilinear maps
\[
I : \Gamma(E) \times \Gamma(E) \to \Gamma(F), \quad (X, Y) \mapsto \text{pr}_E(\nabla_X Y)
\]
and
\[
II : \Gamma(F) \times \Gamma(F) \to \Gamma(E), \quad (X, Y) \mapsto \text{pr}_E(\nabla_X Y)
\]
are symmetric. Moreover, \( K(P) = 0 \) for any plane lying in \( E \) or \( F \).

**Proof.** Since the sectional curvature is nonpositive but not identically zero, the Einstein constant is negative. Now we follow the same strategy as in the proof of Proposition 5.2. If \( K_{\text{min}} > \frac{2}{n} \mu \), then \( r_p < -\mu \) and by Proposition 1.2 \( (M, g) \) is strictly stable. If \( K_{\text{min}} \geq \frac{2}{n} \mu \) and \( h \in \ker(\Delta_E|TT) \), we obtain from (5.3) that
\[
0 = (\Delta_E h, h)_{L^2} = \|D_2 h\|^2_{L^2} - \mu \|h\|^2_{L^2} - (h, R h)_{L^2} \geq -\mu \|h\|^2_{L^2} + \mu \|h\|^2_{L^2} = 0.
\]
Consequently, \( D_2 h \equiv 0 \) and \( r(p) = K_{\text{max}} - \mu = \mu - nK_{\text{min}} \). Again by Lemma 5.3 there is a splitting \( T_p M = E_p(\lambda) \oplus E_p(-\lambda) \) at each point \( p \in M \) where \( h \neq 0 \) and \( E_p(\pm \lambda) \) is the \( n/2 \)-dimensional eigenspaces of \( h \) to the eigenvalue \( \pm \lambda \), respectively. Evidently, \( (M, g) \) is even-dimensional. We will now show that \( \lambda \) is constant in \( p \). Let \( \{e_1, \ldots, e_n\} \) be a local eigenframe of \( h \) such that \( e_1, \ldots, e_{n/2} \in E(\lambda) \) and
and $e_{n/2+1}, \ldots, e_n \in E(-\lambda)$ and let $\lambda_1 \equiv \ldots \equiv \lambda_{n/2}$ and $\lambda_{n/2+1} \equiv \ldots \equiv \lambda_n$ be the corresponding eigenfunctions. Since $D_2 h \equiv 0$, (5.9) yields
\[
(\lambda_j - \lambda_k) \Gamma^k_{ij} - (\lambda_i - \lambda_k) \Gamma^k_{jk} = - (\nabla e_i, \lambda_j) \delta_{jk} + (\nabla e_j, \lambda_i) \delta_{ik} = 0
\]  
(5.13)
for $1 \leq i, j, k \leq n$. Choose $i \neq j$ and $j = k$ such that $e_i, e_j, e_k$ lie in the same eigenspace. Then by (5.13),
\[
0 = - \nabla e_i, \lambda_j
\]
and since $\lambda_j$ equals either $\lambda$ or $-\lambda$, the eigenvalues of $h$ are constant in $p$. A splitting of the tangent bundle is obtained by $TM = E \oplus F$ where the two distributions are defined by
\[
E = \bigcup_{\lambda \in \mathbb{R}} E_\lambda, \quad F = \bigcup_{\lambda \in \mathbb{R}} E_{-\lambda}.
\]
The flatness of planes in $E$ and $F$ follows from Lemma 5.1. It remains to show the symmetry of $I$ and $II$. Let $\{e_1, \ldots, e_n\}$ an orthonormal frame such that $e_1, \ldots, e_{n/2}$ are local sections in $E$ and $e_{n/2+1}, \ldots, e_n$ are local sections in $F$. Let $i, j \in \{1, \ldots, n/2\}$ and $k \in \{n/2+1, \ldots, n\}$. By (5.13),
\[
2 \lambda \Gamma^k_{ij} = 2 \lambda \Gamma^k_{ji} = 0.
\]
and since
\[
I(e_i, e_j) = \sum_{k=n/2+1}^n \Gamma^k_{ij} e_k,
\]  
(5.14)
$I$ is symmetric. The symmetry of $II$ is shown by the same arguments. It is furthermore easy to see that both maps are $C^\infty(M)$-bi-linear. 

Remark 5.6. By symmetry of the operators $I$ and $II$, the map $(X, Y) \mapsto [X, Y]$ preserves the splitting $TM = E \oplus F$. Thus, both distributions are integrable by the Frobenius theorem.

It is not known whether the pinching assumptions of Proposition 5.2 can be further improved. We conclude this section with some eigenvalue estimates for the Einstein operator.

Proposition 5.7. Let $(M, g)$ be a Riemannian manifold of constant curvature $K$. Then the smallest eigenvalue of $\Delta_E|_{TT}$ satisfies the estimate
\[
\lambda \geq \max \{2(n+1)K, -(n-2)K\}.
\]
In particular, $(M, g)$ is stable and if $K \neq 0$, $(M, g)$ is strictly stable.

Proof. For constant curvature metrics, the Riemann curvature tensor is given by
\[
R_{X,Y,Z} = K(g(Y,Z)X - g(X,Z)Y)
\]
and we have $\mu = K(n-1)$ for the Einstein constant. The action of the curvature tensor on traceless tensors is given by $\hat{R}h(X, Y) = -K h(X, Y)$. Now, Bochner formula (5.2) yields
\[
(\Delta_E h)_{L^2} = \|D_1 h\|^2_{L^2} + 2 \mu \|h\|^2_{L^2} - 4(h, \hat{R}h)_{L^2} \geq 2(n+1)K \|h\|^2_{L^2},
\]
and from (5.3), we obtain
\[
(\Delta_E h)_{L^2} = \|D_2 h\|^2_{L^2} - \mu \|h\|^2_{L^2} - (h, \hat{R}h)_{L^2} \geq -(n-2)K \|h\|^2_{L^2}.
\]  
\[\square\]
Remark 5.8. For nonnegative \(K\), this lower bound is optimal. For the flat case, see Section 3. The bound is also achieved on the round sphere. The spectrum of the Lichnerowicz Laplacian (from which we obtain the spectrum of the Einstein operator) on \(TT\)-tensors on the round sphere was explicitly computed in [Bou99, Theorem 3.2]. For hyperbolic spaces, it is not known if this inequality is optimal.

**Proposition 5.9.** Let \((M, g)\) an Einstein manifold with constant \(\mu\) and sectional curvature \(K \geq 0\). Then the smallest eigenvalue of \(\Delta_E|TT\) satisfies

\[
\lambda \geq -2\mu.
\]

Moreover, equality holds if and only if the holonomy of \((M, g)\) is reducible.

**Proof.** By curvature assumptions and Lemma 5.1, \(\|r\|_{L^\infty} \leq \mu\), where \(r\) is defined in (5.1), respectively. Therefore,

\[
(\Delta_E h, h) = \|\nabla h\|_{L^2}^2 - 2(\hat{R}h, h)_{L^2} \geq -2\mu \|h\|_{L^2}^2,
\]

and equality implies that \(h\) is parallel. It follows from Lemma 5.1 that \((M, g)\) has reducible holonomy. Conversely, if \((M, g)\) has reducible holonomy, the metric splits as \(g = g_1 + g_2\) and any tracefree linear combination \(\alpha g_1 + \beta g_2\) is an eigentensor of \(\Delta_E|TT\) to the eigenvalue \(-2\mu\).

**Remark 5.10.** If the holonomy is reducible, \((M, g)\) is locally isometric to a product \((M_1, g_1) \times (M_2, g_2)\) [Bau09, Satz 5.6]. In particular, the sectional curvature cannot be positive in this case.

Similarly, we can prove the following assertion for the Lichnerowicz Laplacian (Recall (4.1) for the definition):

**Proposition 5.11.** Let \((M, g)\) be a Riemannian manifold with nonnegative sectional curvature. Then the Lichnerowicz Laplacian is positive semidefinite on \(\Gamma(S^2 M)\). For any \(h \in \Gamma(S^2 M)\), \(\Delta_L h = 0\) if and only if \(\nabla h = 0\).

**Proof.** As a consequence of [Bar93, Lemma 2.4], the quadratic form

\[
h \mapsto (h, \text{Ric} \circ h + h \circ \text{Ric} - 2\hat{R}h)_{L^2}
\]

is nonnegative if the sectional curvature is nonnegative. Thus, the Lichnerowicz-Laplacian is positive semidefinite and any \(h \in \ker(\Delta_L)\) must be parallel.

On the other hand, if \(h\) is parallel, Lemma 3.1 implies that it is of the form \(h = \sum \lambda_i g_i\) where \(\lambda_i\) are the eigenvalues of \(h\) (which are constant on \(M\)) and \(g_i\) is the metric restricted to the eigenspace of \(\lambda_i\). It is straightforward to check that \(\Delta_L h = 0\) if \(h\) is of this form.

**Remark 5.12.** By Proposition 3.4, the dimension of \(\ker(\Delta_L)\) can be explicitly computed in terms of the holonomy, if the curvature is nonnegative.

### 6 Stability and Weyl curvature

We have seen that constant curvature metrics and sufficiently pinched Einstein manifolds are stable. This motivates to prove stability theorems in terms of the Weyl tensor which measures the deviation of an Einstein manifold of being of constant curvature. Recall that on Einstein manifolds, the curvature tensor decomposes as

\[
R = W + \frac{\mu}{2(n-1)} (g \otimes g),
\]

(6.1)
where $\mu$ is the Einstein constant of $g$. The tensor $W$ is the Weyl curvature tensor and $\otimes$ denotes the Kulkarni-Nomizu product of symmetric $(0, 2)$-tensors, given by

$$(h \otimes k)(X, Y, Z, W) = h(X, W)k(Y, Z) + h(Y, Z)k(X, W) - h(X, Z)k(Y, W) - h(Y, W)k(X, Z).$$

The Weyl tensor acts naturally on symmetric $(0, 2)$-tensors by

$$\tilde{W}h(X, Y) = \sum_{i,j=1}^{n} W(e_i, X, Y, e_j)h(e_j, e_i),$$

and a straightforward calculation shows that the action of the Riemann tensor decomposes as

$$\hat{R}h(X, Y) = \tilde{W}h(X, Y) + \frac{\mu}{n-1} \{g(X, Y)tr h - h(X, Y)\}.$$

**Lemma 6.1.** Let $(M, g)$ be any Riemannian manifold and let $p \in M$. The operator $\tilde{W} : (S^2M)_p \to (S^2M)_p$ is trace-free. It is indefinite as long as $W_p \neq 0$.

**Proof.** First we compute the trace of $\tilde{W}$ acting on all symmetric $(0, 2)$-tensors. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_pM$. Then an orthonormal basis of $(S^2M)_p$ is given by

$$\eta_{ij} = \frac{1}{\sqrt{2}} e^*_i \otimes e^*_j, \quad 1 \leq i \leq j \leq n,$$

where $\otimes$ denotes the symmetric tensor product. Simple calculations yield

$$\langle \tilde{W}\eta_{ij}, \eta_{ij} \rangle = -W_{ijji}.$$

Thus,

$$\text{tr} \tilde{W} = \sum_{1 \leq i \leq j \leq n} \langle \tilde{W}\eta_{ij}, \eta_{ij} \rangle = -\sum_{1 \leq i \leq j \leq n} W_{ijji} = -\frac{1}{2} \sum_{i,j=1}^{n} W_{ijji} = 0$$

because the Weyl tensor has vanishing trace. Suppose now that the operator $\tilde{W}$ vanishes, then all $W_{ijji}$ vanish. By the symmetries of the Weyl tensor, this already implies that $W_p$ vanishes. \qed

To study the behavior of this operator, we define a function $w : M \to \mathbb{R}$ by

$$w(p) = \sup \left\{ \frac{\langle \tilde{W}\eta, \eta \rangle_p}{|\eta|^2_p} \middle| \eta \in (S^2M)_p \right\}.$$

(6.2)

Thus, $w(p)$ is the largest eigenvalue of the action $\tilde{W} : (S^2M)_p \to (S^2M)_p$. Lemma 6.1 implies that the function $w$ is nonnegative. Since $\tilde{W}g = 0$, $w(p)$ is also the largest eigenvalue of $W$ restricted to $(S^2M)_p$.

The decomposition of $\hat{R}$ allows us to estimate the smallest eigenvalue of $\Delta_E$ acting on $TT$-tensors in terms of the function $w$. From (5.2), we obtain

$$\Delta_E h, h = \|D_1 h\|_{L^2}^2 + 2\mu \|h\|_{L^2}^2 - 4(\hat{R}h, h) \geq 2\mu \|h\|_{L^2}^2 + \frac{\mu}{n-1} \|h\|_{L^2}^2 - 4(\tilde{W}h, h) \geq \left[ 2\mu \frac{n+1}{n-1} - 4 \|w\|_{\infty} \right] \|h\|_{L^2}^2.$$
and similarly from (5.3),
\[
(\Delta_E h, h) = \|D_2 h\|^2_{L^2} - \mu \|h\|^2_{L^2} - (\check{R} h, h)
\geq -\mu \|h\|^2_{L^2} + \frac{\mu}{n-1} \|h\|^2_{L^2} - (\check{W} h, h)
\geq \left[ -\mu \frac{n-2}{n-1} \|w\|_{\infty} \right] \|h\|^2_{L^2}.
\]

**Proposition 6.2.** Let \((M, g)\) be Einstein with constant \(\mu\) and let \(\lambda\) be the smallest eigenvalue of \(\Delta_E|TT\). Then
\[
\lambda \geq \max \left\{ 2\mu \frac{n+1}{n-1} - 4 \|w\|_{\infty}, -\mu \frac{n-2}{n-1} \|w\|_{\infty} \right\}.
\]

Now, Theorem 1.5 is an obvious consequence. This is similar to the work in [IN05]. Let \(\overline{w}(p)\) be the largest eigenvalue of the Weyl curvature operator at \(p \in M\).

**Theorem 6.3** ([IN05, Theorem 1]). Let \((M, g)\) be a compact, connected oriented Einstein manifold with negative Einstein constant \(\mu\). If
\[
\sup_{p \in M} \overline{w}(p) < -\frac{\mu}{n-1}, \tag{6.3}
\]
then \((M, g)\) is strictly stable.

However, this condition is equivalent to the condition that the Riemann curvature operator on \((M, g)\) is negative. By Corollary 1.4 strict stability holds under the weaker condition of negative sectional curvature.

It seems not convenient to formulate stability criterions in terms of the curvature operator. In the proof of the above theorem, the very rough estimate \(\max W_{ijkl} \leq \overline{w}(p)\) is used and we find no other way to estimate \(w(p)\) in terms of \(\overline{w}(p)\).

We now give a different stability criterion involving an integral of the function \(w\). The main tool we use here is the Sobolev inequality which holds for Yamabe metrics. Recall that a metric is called Yamabe if it realizes the Yamabe metric in its conformal class, given by
\[
Y([g]) = \inf_{\tilde{g} \in [g]} \text{vol}(M, \tilde{g})^{(2-n)/n} \int_M \text{scal}_{\tilde{g}} dV_{\tilde{g}}.
\]

**Proposition 6.4** (Sobolev inequality). Let \((M, g)\) be a Yamabe metric in a conformal class and suppose that \(\text{vol}(M, g) = 1\). Then for any \(f \in H^1(M)\),
\[
\frac{4n-1}{n-2} \|\nabla f\|^2_{L^2} \geq \text{scal} \left\{ \|f\|^2_{L^p} - \|f\|^2_{L^2} \right\}, \tag{6.4}
\]
where \(p = 2n/(n-2)\).

**Proof.** This follows easily from the definition of Yamabe metrics, see e.g. [IS02, p. 140].

**Remark 6.5.** The inequality holds if \(f\) is replaced by any tensor \(T\) because of Kato’s inequality
\[
|\nabla[T]| \leq |\nabla T|. \tag{6.5}
\]
Since any Einstein metric is Yamabe (see e.g. [LeB99, p. 329]), the Sobolev inequality holds in this case. Now we are ready to prove Theorem 1.6.
Proof of Theorem 1.6. The sides of the inequality in the statement are scale-invariant, see Lemma 6.6 below. Therefore, we may assume vol(M, g) = 1 from now on. First, we estimate the largest eigenvalue of the Weyl tensor action by

\[
(W h, h)_L^2 \leq \int_M |w||h|^2 \, dV \leq \|w\|_{L^{n/2}} \|h\|_{L^{2n/(n-2)}}^2 \leq \|w\|_{L^{n/2}} \left(4\frac{n-1}{\mu \mu(n-2)} \|\nabla h\|^2_{L^2} + \|h\|^2_{L^2}\right).
\]

We used the Hölder inequality and the Sobolev inequality. With the estimate obtained, we can proceed as follows:

\[
(\Delta E_h, h)_L^2 = \|\nabla h\|^2_{L^2} - 2(\hat{\Delta} h, h)_L^2
= \|\nabla h\|^2_{L^2} + 2\frac{\mu}{n-1} \|h\|^2_{L^2} - 2(\hat{W} h, h)_L^2
\geq \|\nabla h\|^2_{L^2} - 2 \|w\|_{L^{n/2}}^2 \left(4\frac{n-1}{\mu \mu(n-2)} \|\nabla h\|^2_{L^2} + \|h\|^2_{L^2}\right)
= \left(1 - 8\frac{n-1}{\mu \mu(n-2)} \|w\|_{L^{n/2}}\right) \|\nabla h\|^2_{L^2} + 2\left(\frac{\mu}{n-1} - \|w\|_{L^{n/2}}\right) |h|_{L^2}.
\]

The first term on the right hand side is nonnegative by the assumption on w. It remains to estimate \(\|\nabla h\|^2_{L^2}\). This can be done by using (5.2). We have

\[
\|\nabla h\|^2_{L^2} = \|D_1 h\|^2_{L^2} + 2\mu \|h\|^2_{L^2} - 2(\hat{\Delta} h, h)_L^2
\geq 2\mu \|h\|^2_{L^2} + 8\frac{n-1}{\mu \mu(n-2)} \|w\|_{L^{n/2}} \|\nabla h\|^2_{L^2} + \|h\|^2_{L^2}
\geq 2\mu \frac{n}{n-1} \|h\|^2_{L^2} - 8\frac{n-1}{\mu \mu(n-2)} \|w\|_{L^{n/2}} \|\nabla h\|^2_{L^2},
\]

and therefore, \(\|\nabla h\|^2_{L^2}\) can be estimated by

\[
\|\nabla h\|^2_{L^2} \geq 2\left(\frac{\mu}{n-1} - \|w\|_{L^{n/2}}\right) \left(1 + 8\frac{n-1}{\mu \mu(n-2)} \|w\|_{L^{n/2}}\right)^{-1} \|h\|^2_{L^2}.
\]

Combining these arguments, we obtain

\[
(\Delta E_h, h)_L^2 \geq \left\{2\left(1 - 8\frac{n-1}{\mu \mu(n-2)} \|w\|_{L^{n/2}}\right) \left(\frac{\mu}{n-1} - \|w\|_{L^{n/2}}\right) \right\} \|h\|^2_{L^2}.
\]

The manifold \((M, g)\) is stable if the right-hand side of this inequality is nonnegative. It is elementary to check that this is equivalent to

\[
\|w\|_{L^{n/2}} \leq \mu \frac{n+1}{2(n-1)} \left(4\frac{n-1}{n(n-2)} + 1\right)^{-1}.
\]

The assertion about strict stability is also immediate. \(\Box\)

Lemma 6.6. The \(L^{n/2}\)-Norm of the function \(w\) is conformally invariant.
Proof. Let $g, \tilde{g}$ be conformally equivalent, i.e. $\tilde{g} = f \cdot g$ for a smooth positive function $f$. Let $W$ and $\tilde{W}$ be the Weyl tensors of the metrics $g$ and $\tilde{g}$, respectively. It is well-known that $\tilde{W} = f \cdot W$ when considered as $(0,4)$-tensors. Therefore,

$$(\tilde{W} h, h)_{\tilde{g}} = f^{-3} (W h, h)_{g}.$$ 

Furthermore, we have

$$|h|^2_{\tilde{g}} = f^{-2} |h|^2_{g}, \quad dV_{\tilde{g}} = f^{n/2} dV_g.$$ 

We now see that $\tilde{w} = f^{-1} w$ and

$$\|\tilde{w}\|_{L^{n/2}(\tilde{g})} = \|w\|_{L^{n/2}(g)}.$$ 

which shows the lemma.

Corollary 6.7. Let $(M, g)$ be a Riemannian manifold and let $Y([g])$ be the Yamabe constant of the conformal class of $g$. If

$$\|w\|_{L^{n/2}(g)} \leq Y([g]) \cdot n + 1 \cdot \frac{4(n-1)}{n(n-2) + 1}, \quad (6.7)$$

any Einstein metric in the conformal class of $g$ is stable.

Proof. Suppose that $\tilde{g} \in [g]$ is Einstein. We know that $\tilde{g}$ is a Yamabe metric in the conformal class of $g$. By the definition of the Yamabe constant, the Einstein constant of $\tilde{g}$ equals

$$\mu = \frac{1}{n} \cdot Y([g]) \cdot \text{vol}(M, g)^{2/n},$$

and Lemma 6.6 yields

$$\|\tilde{w}\|_{L^{n/2}(\tilde{g})} = \|w\|_{L^{n/2}(g)} \leq \mu \cdot \text{vol}(M, g)^{2/n} \cdot \frac{n + 1}{2(n-1)} \cdot \left( \frac{4(n-1)}{n(n-2) + 1} \right)^{-1}.$$ 

The assertion now follows from Theorem 1.6.

By Theorem 1.6 and the Cauchy-Schwarz inequality, any positive Einstein manifold of unit volume is stable, if

$$\|W\|_{L^{n/2}} \leq \mu \cdot \frac{n + 1}{2(n-1)} \left( \frac{4(n-1)}{n(n-2) + 1} \right)^{-1}.$$ 

(6.8)

On the other hand, we have the following isolation theorem for the Weyl tensor:

**Theorem 6.8 (IS02, Main theorem).** Let $(M, g)$ be a compact connected, oriented Einstein-manifold, $n \geq 4$, with positive Einstein constant $\mu$ and of unit-volume. Then there exists a constant $C(n)$, depending only on $n$, such that if the inequality $\|W\|_{L^{n/2}} < C(n)\mu$ holds, then $W = 0$ so that $(M, g)$ is a finite isometric quotient of the sphere.

A careful investigation of the proof shows that $W$ vanishes if

$$\|W\|_{L^{n/2}} \leq \begin{cases} \frac{n(n-2)}{2(n-1)} \mu & \text{if } 4 \leq n \leq 9, \\ \frac{1}{2} \mu & \text{if } n \geq 10. \end{cases}$$ 

(6.9)

A comparison of the last two inequalities shows that (6.8) is not ruled out by the above isolation theorem. In dimension 4, we have another isolation theorem, proven with different techniques:
Theorem 6.9 ([CL99, Theorem 1]). Let \((M, g)\) be a compact oriented Einstein 4-manifold with constant \(\mu > 0\) and let \(W^+\) be the self-dual part of the Weyl tensor. If \(W^+ \neq 0\), then
\[
\int_M |W^+|^2 \, dV \geq \int_M \frac{8\mu^2}{3} \, dV,
\]
with equality if and only if \(\nabla W^+ \equiv 0\).

Obviously, the same gap theorem holds for the whole Weyl tensor. By passing to the orientation covering, we see that the same gap also holds for the Weyl tensor on non-orientable manifolds. Unfortunately, this theorem rules (6.8) out.

7 Six-dimensional Einstein manifolds

In this section, we compute an explicit representation of the Gauss-Bonnet formula for six-dimensional Einstein manifolds. We use this representation to show a stability criterion for Einstein manifolds involving the Euler characteristic.

The generalized Gauss-Bonnet formula for a compact Riemannian manifold \((M, g)\) of dimension \(n = 2m\) is
\[
\chi(M) = \frac{(-1)^m}{2^{3m}\pi^m m!} \int_M \Psi_g \, dV.
\]
The function \(\Psi_g\) is defined as
\[
\Psi_g = \sum_{\sigma, \tau \in S_m} \text{sgn}(\sigma)\text{sgn}(\tau) R_{\sigma(1)\sigma(2)\tau(1)\tau(2)} \cdots R_{\sigma(n-1)\sigma(n)\tau(n-1)\tau(n)},
\]
where the coefficients are taken with respect to an orthonormal basis (see e.g. [Zhu00, Theorem 4.1]). In dimension four, this yields the nice formula
\[
\chi(M) = \frac{1}{32\pi^2} \int_M (|W|^2 + |Sc|^2 - |U|^2) \, dV \quad (7.1)
\]
(see also [Bes08, p. 161]). Here, \(Sc = \frac{\text{scal}}{2n(n-1)} g \otimes g\) is the scalar part and \(U = \frac{1}{n-2} \text{Ric}^0 \otimes g\) is the traceless Ricci part of the curvature tensor. Due to different conventions for the norm of curvature tensors, formula (7.1) often appears with the factor \(\frac{1}{8\pi^2}\) instead of \(\frac{1}{32\pi^2}\). On Einstein manifolds, we have \(U = 0\) and the Gauss-Bonnet formula simplifies to
\[
\chi(M) = \frac{1}{32\pi^2} \int_M \left( |W|^2 + \frac{8}{3} \mu^2 \right) \, dV \quad (7.2)
\]
where \(\mu\) is the Einstein constant. As a nice consequence, we obtain a topological condition for the existence of Einstein metrics:

Theorem 7.1 ([Ber65, p. 41]). Every compact 4-manifold carrying an Einstein metric \(g\) satisfies the inequality
\[
\chi(M) \geq 0.
\]
Moreover, \(\chi(M) = 0\) if and only if \((M, g)\) is flat.
Another consequence of (7.2) is the following: Let \((M, g)\) be of unit volume. Then there exists a constant \(C > 0\) such that, if \(\mu \geq C \cdot \sqrt{\chi(M)}\), the Weyl curvature satisfies \(\|W\|_{L^2} \leq \frac{C}{\mu}\). This implies stability by Theorem 1.6. Unfortunately, the same condition on the Weyl tensor already implies that it vanishes, as we discussed in the last section.

In dimension six, an explicit representation of the Gauss-Bonnet formula is given by

\[
\chi(M) = \frac{1}{384\pi^3} \int_M \left\{ \text{scal}^3 - 12\text{scal}|R|^2 + 3\text{scal}|\nabla R|^2 + 16\langle \text{Ric}, \text{Ric} \rangle - 24\text{Ric}^i \text{Ric}^k R_{ikkl} - 24\text{Ric}^i R^{klmn} R_{ijkl} + 8R^i_{ijkl} R_{imkn} R_{jlm}^n - 2R^i_{ijkl} R_{ij}^m R_{klmn} \right\} dV
\]

(see [Sak71] Lemma 5.5). When \((M, g)\) is Einstein, this integral is equal to

\[
\chi(M) = \frac{1}{384\pi^3} \int_M \left\{ 24\mu^3 - 6\mu|R|^2 + 8R_{ijkl} R_{imkn} R_{jlm}^n - 2R^i_{ijkl} R_{ij}^m R_{klmn} \right\} dV.
\]  

(Lemma 7.2)

\[
\|\nabla R\|^2_{L^2} = -\int_M \left\{ 4R^i_{jkl} R_{i}^{mn} R_{jmn} + 2R^i_{ijkl} R_{ij}^m R_{klmn} + 2\mu|R|^2 \right\} dV.
\]

Proof. This is [Sak71] (2.15)) in the special case of Einstein metrics. \(\square\)

Note that we translated the formulas from [Sak71] to our sign convention for the curvature tensor.

Proposition 7.3. Let \((M, g)\) be an Einstein six-manifold with constant \(\mu\). Then

\[
\chi(M) = \frac{1}{384\pi^3} \int_M \left\{ -\frac{14}{5} \mu|W|^2 - 2|\nabla W|^2 + \frac{144}{25} \mu^3 + 48\text{tr}(\hat{W}^3) \right\} dV.
\]

Here, \(\hat{W}^3 = \hat{W} \circ \hat{W} \circ \hat{W}\), where \(\hat{W}\) is the Weyl curvature operator acting on 2-forms.

Proof. By Lemma 7.2 and 7.3, we can rewrite as

\[
384\pi^3 \chi(M) = \int_M \left\{ 24\mu^3 - 10\mu|R|^2 - 2|\nabla R|^2 - 6R^i_{ijkl} R_{ij}^m R_{klmn} \right\} dV.
\]

Moreover, \(\nabla W = \nabla R\) because the difference \(R - W = S\) is a parallel tensor. Thus,

\[
384\pi^3 \chi(M) = \int_M \left\{ 24\mu^3 - 10\mu|R|^2 - 2|\nabla W|^2 - 6R^i_{ijkl} R_{ij}^m R_{klmn} \right\} dV
\]

\[
= \int_M \left\{ 24\mu^3 - 10\mu \left( \frac{12\mu^2}{5} + |W|^2 \right) - 2|\nabla W|^2 - 6R^i_{ijkl} R_{ij}^m R_{klmn} \right\} dV
\]

\[
= \int_M \left\{ -10\mu|W|^2 - 2|\nabla W|^2 - 6R^i_{ijkl} R_{ij}^m R_{klmn} \right\} dV.
\]

Now we analyse the last term on the right hand side. Recall that the Riemann curvature operator \(\hat{R}\) and the Weyl curvature operator \(W\) are defined by

\[
(\hat{R}(X \wedge Y), Z \wedge V) = R(Y, X, Z, V),
\]

\[
(\hat{W}(X \wedge Y), Z \wedge V) = W(Y, X, Z, V).
\]

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Let \( \{e_1, \ldots, e_n\} \) be a local orthonormal frame of \( TM \). Then \( \{e_i \wedge e_j\}, i < j \) is a local orthonormal frame of \( \Lambda^2 M \). A straightforward calculation shows
\[
-6 \sum_{i,j,k,l,m,n} R_{ijkl} R_{jkmn} R_{klnm} = -48 \sum_{i<j, k<l, m<n} R_{ijkl} R_{ijmn} R_{klmn} = 48 \text{tr} \hat{R}^3,
\]
where the coefficients of \( R \) are taken with respect to the orthonormal frame. The decomposition (6.1) of the 4-curvature tensor induces the decomposition \( \hat{R} = \hat{W} + \frac{\mu}{5} \text{id} \Lambda^2 M \). This yields
\[
48 \text{tr} \hat{R}^3 = 48 \left\{ \text{tr}(\hat{W}^3) + \frac{3}{5} \mu \text{tr}(\hat{W}^2) + \frac{3}{25} \mu^2 \text{tr} \text{id} \Lambda^2 M \right\}
= 48 \text{tr}(\hat{W}^3) + \frac{36}{5} \mu |W|^2 + \frac{144}{25} \mu^3.
\]
Inserting this in the above formula finishes the proof.

Proof of Theorem 1.7. By the Sobolev inequality,
\[
\|W\|_{L^3}^2 \leq \frac{5}{6\mu} \|\nabla W\|_{L^2}^2 + \|W\|_{L^2}^2.
\]
Therefore we have, by Proposition 7.3
\[
384\pi^3 \chi(M) < -\frac{12}{5} \mu \|W\|_{L^3}^2 - 2 \|\nabla W\|_{L^2}^2 + \frac{144}{25} \mu^3 + 48 \int_M \text{tr}(\hat{W}^3) \ dV
\leq -\frac{12}{5} \mu \|W\|_{L^3}^2 + \frac{144}{25} \mu^3 + 48 \int_M \text{tr}(\hat{W}^3) \ dV.
\]
Now if \( \mu \) satisfies the estimate of the statement in the theorem, we obtain
\[
\frac{12}{5} \mu \|W\|_{L^3}^2 < \frac{144}{25} \mu^3 - 384\pi^3 \chi(M) + 48 \int_M \text{tr}(\hat{W}^3) \ dV
\leq \frac{144}{25} \mu^3 - \frac{1}{25} \left( 144 - \frac{12 \cdot 7^2 \cdot 3^2}{5 \cdot 11^2} \right) \mu^3 = \frac{12}{5} \mu \cdot 7^2 \cdot 3^2 \mu^2,
\]
which is equivalent to
\[
\|W\|_{L^3} < \frac{7 \cdot 3}{5 \cdot 11} \mu.
\]
By Theorem 1.6 and the Cauchy-Schwarz inequality, \((M, g)\) is strictly stable.

8 Kähler manifolds

Here, we prove stability criterions for Kähler-Einstein manifolds in terms of the Bochner curvature tensor, which is an analogue of the Weyl tensor.

Definition 8.1. Let \((M, g)\) be a Riemannian manifold of even dimension. An almost complex structure on \( M \) is an endorphism \( J : TM \to TM \) such that \( J^2 = -\text{id}_{TM} \). If \( J \) is parallel and \( g \) is hermitian, i.e. \( g(JX, JY) = g(X, Y) \), we call the triple \((M, g, J)\) a Kähler manifold. If \((M, g)\) is Einstein, we call \((M, g, J)\) Einstein-Kähler.
The bundle of traceless symmetric \((0,2)\)-tensors splits into hermitian and skew-hermitian ones, i.e. we have \(S^2 \overline{\mathbb{g}} = H_1 \oplus H_2\), where

\[
H_1 = \{ h \in S^2 \overline{\mathbb{g}} \mid h(X,Y) = h(JX, JY) \}, \\
H_2 = \{ h \in S^2 \overline{\mathbb{g}} \mid h(X,Y) = -h(JX, KY) \}.
\]

Stability of Kähler-Einstein manifolds was studied in [Koi83, IN05, DWW07]. We sketch the ideas of [Koi83] in the following. It turns out that the Einstein operator preserves the splitting \(\Gamma(H_1) \oplus \Gamma(H_2)\). Therefore to show that a Kähler-Einstein manifold is stable it is sufficient to show that the restriction of \(\Delta \overline{E}\) to the subspaces \(\Gamma(H_1)\) and \(\Gamma(H_2)\) is positive semidefinite, respectively. In fact, we can use the Kähler structure to conjugate the Einstein operator to other operators. If \(h_1 \in H_1\), we define a 2-form by

\[
\phi(X,Y) = h_1 \circ J(X,Y) = h_1(X, J(Y)).
\]

We have

\[
\Delta_H \phi = (\Delta_E h_1) \circ J + 2\mu \phi, \tag{8.1}
\]

where \(\Delta_H\) is the Hodge Laplacian on 2-forms and \(\mu\) is the Einstein constant. Since \(\Delta_H\) is nonnegative, \(\Delta_E\) is nonnegative on \(\Gamma(H_1)\), if \(\mu \leq 0\). For \(h_2 \in H_2\), we define a symmetric endomorphism \(I : TM \to TM\) by

\[
g \circ I = h_2 \circ J,
\]

and since \(IJ + JI = 0\), we may consider \(I\) as a \(T^{1,0} M\)-valued 1-form of type \((0,1)\). We have the formula

\[
g \circ (\Delta_C I) = (\Delta_E h_2) \circ J, \tag{8.2}
\]

where \(\Delta_C\) is the complex Laplacian. Thus, the restriction of the Einstein operator to \(\Gamma(H_2)\) is always nonnegative, since \(\Delta_C\) is. As a consequence, we have

**Corollary 8.2** ([DWW07, Corollary 1.2]). *Any compact Kähler-Einstein manifold with nonpositive Einstein constant is stable.*

Using (8.1) and (8.2), \(\dim(\ker \Delta_E|_{TT})\) can be expressed in terms of certain cohomology classes (see [Koi83, Corollary 9.4] or [Bes08, Proposition 12.98]). Moreover, integrability of infinitesimal Einstein deformations can be related to integrability of infinitesimal complex deformations ([Koi83, Proposition 10.1] and [IN05, Theorem 3]).

We discuss conditions under which a Kähler-Einstein manifold is strictly stable in the nonpositive case and stable in the positive case. This can be described in terms of the Bochner curvature tensor which has similar properties as the Weyl tensor.

**Definition 8.3** (Bochner curvature tensor). Let \((M, g, J)\) be a Kähler manifold and let \(\omega(X,Y) = g(J(X), Y)\) be the Kähler form. The Bochner curvature tensor is defined by

\[
B = R + \frac{\text{scal}}{2(n+2)(n+4)} \left\{ g \otimes g + \omega \otimes \omega - 4\omega \otimes \omega \right\} \\
- \frac{1}{n+4} \left\{ \text{Ric} \otimes g + (\text{Ric} \circ J) \otimes \omega - 2(\text{Ric} \circ J) \otimes \omega - 2\omega \otimes (\text{Ric} \circ J) \right\}
\]

(see e.g. [IK04, p. 229]).
The Bochner curvature tensor possesses the same symmetries as the Riemann tensor and in addition, any of its traces vanishes. If \((M, g)\) is Kähler-Einstein, the Bochner tensor is

\[ B = R - \frac{\mu}{2(n+2)} \{g \circ g + \omega \circ \omega - 4\omega \circ \omega\} , \]

where \(\mu\) is the Einstein constant (see e.g. [IK04, p. 229] and mind the different sign convention for the curvature tensor). The Bochner tensor acts naturally on symmetric \((0,2)\)-tensors by

\[ \hat{B}h(X,Y) = \sum_{i,j=1}^{n} B(e_i, X, Y, e_j) h(e_i, e_j) , \]

where \(\{e_1, \ldots, e_n\}\) is an orthonormal basis. Let

\[ b^+(p) = \left\{ \frac{(\hat{B} \eta, \eta)}{|\eta|^2} \middle| \eta \in \mathcal{H}_1 \right\} . \]

For Kähler-Einstein manifolds with negative Einstein constant, it was proven by M. Itoh and T. Nakagawa that they are strictly stable if the Bochner tensor is small.

**Theorem 8.4 ([IN05 Theorem 4.1]).** Let \((M, g, J)\) be a compact Kähler-Einstein manifold with negative Einstein constant \(\mu\). If the Bochner curvature tensor satisfies

\[ \|b^+\|_{L^\infty} < -\frac{\mu}{n+2} \] (8.3)

then \(g\) is strictly stable.

However, an error occurred in the calculations and the result is slightly different. Therefore, we redo the proof. By straightforward calculation,

\[ (\hat{R}h, h) = (\hat{B}h, h) - \frac{\mu}{n+2} \{|h|^2 - 3 \sum_{i,j} h(e_i, e_j) h(J(e_i), J(e_j))\} . \] (8.4)

In particular,

\[ (\hat{R}h_1, h_1) = (\hat{B}h_1, h_1) + 2 \frac{\mu}{n+2} |h_1|^2 \]

for \(h_1 \in H_1\) and

\[ (\hat{R}h_2, h_2) = (\hat{B}h_2, h_2) - 4 \frac{\mu}{n+2} |h_2|^2 \]

for \(h_2 \in H_2\). By [8.1], \(\Delta_E\) is positive definite on \(\Gamma(H_1)\) so it remains to consider \(\Gamma(H_2)\). By [5.3],

\[ \langle \Delta_E h_2, h_2 \rangle_{L^2} = \|D_2 h_2\|_{L^2}^2 - \mu \|h_2\|_{L^2}^2 - (h_2, \hat{B}h_2)_{L^2} + \|\delta h_2\|_{L^2}^2 . \]

\[ \geq -\mu \|h_2\|_{L^2}^2 - (h_2, \hat{B}h_2)_{L^2} + 4 \frac{\mu}{n+2} \|h_2\|_{L^2}^2 . \]

\[ \geq -\frac{n-2}{n+2} \|h_2\|_{L^2}^2 - \|b^+\|_{L^\infty} \|h_2\|_{L^2}^2 . \]

**Remark 8.5.** Theorem 8.4 is true if we replace (8.3) by

\[ \|b^+\|_{L^\infty} < -\frac{\mu}{n+2} \] (8.5)
Now, let us turn to positive Kähler-Einstein manifolds. We will use Bochner formula (5.2). Unfortunately, we cannot make use of the vector bundle splitting \( S^2 M = H_1 \oplus H_2 \). In order to apply (5.2), we need the condition \( \delta h = 0 \), which is not preserved by the splitting into hermitian and skew-hermitian tensors. Let
\[
 b(p) = \sup \left\{ \frac{\langle \hat{B}h, \eta \rangle}{|\eta|^2} \mid \eta \in (S^2 M)_p \right\}.
\] (8.6)
Since the trace of the Bochner tensor vanishes, \( \hat{B} : (S^2 M)_p \to (S^2 M)_p \) has also vanishing trace (this follows from the same arguments as used in the proof of Lemma 6.1). Thus, \( b \) is nonnegative.

**Theorem 8.6.** Let \((M, g, J)\) be Kähler-Einstein with positive Einstein constant \( \mu \). If
\[
\|b\|_{L^\infty} \leq \frac{\mu(n - 2)}{2(n + 2)},
\]
then \((M, g)\) is stable.

**Proof.** Let \( h \in TT \). By (8.4) and the Cauchy-Schwarz inequality,
\[
(\hat{R}h, h) \leq \langle \hat{B}h, h \rangle + 2 \frac{\mu}{n + 2} |h|^2.
\]
Using (5.2), we therefore obtain
\[
(\Delta_E h, h)_{L^2} = \|D_1 h\|^2_{L^2} + 2\mu \|h\|^2_{L^2} - 4(h, \hat{R}h)_{L^2}
\]
\[
\geq 2\mu \|h\|^2_{L^2} - 4(h, \hat{B}h)_{L^2} - 8 \frac{\mu}{n + 2} \|h\|^2_{L^2}
\]
\[
\geq 2\mu \left( \frac{n - 2}{n + 2} \right) \|h\|^2_{L^2} - 4 \|b\|_{L^\infty} \|h\|^2_{L^2}.
\]
Under the assumptions of the theorem, \( \Delta_E|_{TT} \) is nonnegative.

We also prove a stability criterion involving the \( L^{n/2} \)-norm of \( b \):

**Theorem 8.7.** Let \((M, g, J)\) be a positive Kähler-Einstein manifold with constant \( \mu \). If the function \( b \) satisfies
\[
\|b\|_{L^{n/2}} \leq \mu \cdot \text{vol}(M, g)^{2/n} \cdot \frac{(n - 2)}{2(n + 2)} \left( \frac{4(n - 1)}{\mu(n - 2)} + 1 \right)^{-1},
\]
then \((M, g)\) is stable.

**Proof.** The proof is very similar to that of Theorem 1.6. Let \( h \in TT \). By assumption, \((M, g)\) is a Yamabe metric. Thus, we can use the Sobolev inequality and we get
\[
(\hat{B}h, h)_{L^2} \leq \int_M |b|^2 \, dV \leq \|b\|_{L^{n/2}} \|h\|^2_{L^{2n/(n-2)}} \leq \|b\|_{L^{n/2}} \left( \frac{4(n - 1)}{\mu(n - 2)} \|\nabla h\|^2_{L^2} + \|h\|^2_{L^2} \right).
\]
By the above,
\[
(\Delta_E h, h)_{L^2} = \|\nabla h\|^2_{L^2} - 2(\hat{R}h, h)_{L^2}
\]
\[
\geq \|\nabla h\|^2_{L^2} - 2(\hat{B}h, h)_{L^2} - \frac{4\mu}{n + 2} \|h\|^2_{L^2}
\]
\[
\geq \|\nabla h\|^2_{L^2} - 2 \|b\|_{L^{n/2}} \left( \frac{4(n - 1)}{\mu(n - 2)} \|\nabla h\|^2_{L^2} + \|h\|^2_{L^2} \right) - \frac{4\mu}{n + 2} \|h\|^2_{L^2}
\]
\[
= \left( 1 - \frac{8(n - 1)}{\mu(n - 2)} \|b\|_{L^{n/2}} \right) \|\nabla h\|^2_{L^2} - 2 \|b\|_{L^{n/2}} \|h\|^2_{L^2} - \frac{4\mu}{n + 2} \|h\|^2_{L^2}.
\]

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The first term on the right hand side is nonnegative by the assumption on \( b \). To estimate \( \| \nabla h \|_{L^2}^2 \), we rewrite (5.2) to get
\[
\| \nabla h \|_{L^2}^2 = \| D_1 h \|_{L^2}^2 + 2 \mu \| h \|_{L^2}^2 - 2(h, \tilde{R}h)_{L^2} \\
\geq 2 \mu \frac{n}{n+2} \| h \|_{L^2}^2 - 2(h, \tilde{B}h)_{L^2} \\
\geq 2 \mu \frac{n}{n+2} \| h \|_{L^2}^2 - 2 \| b \|_{L^{n/2}} \left( \frac{4(n-1)}{\mu n(n-2)} \| \nabla h \|_{L^2}^2 + \| h \|_{L^2}^2 \right).
\]
Thus,
\[
\| \nabla h \|_{L^2}^2 \geq 2 \left( \mu \frac{n}{n+2} - \| b \|_{L^{n/2}} \right) \left( 1 + \frac{8(n-1)}{\mu n(n-2)} \| b \|_{L^{n/2}} \right)^{-1} \| h \|_{L^2}^2.
\]
By combining these arguments,
\[
(\Delta_E h, h)_{L^2} \geq 2 \left( \mu \frac{n}{n+2} - \| b \|_{L^{n/2}} \right) \left( 1 + \frac{8(n-1)}{\mu n(n-2)} \| b \|_{L^{n/2}} \right)^{-1} \| h \|_{L^2}^2 \frac{4 \mu}{n+2} \| b \|_{L^{n/2}} - 2 \| b \|_{L^{n/2}} - \| b \|_{L^{n/2}}^2,
\]
and the right-hand side is nonnegative if the assumption of the theorem holds.

**Remark 8.8.** By the Cauchy-Schwarz inequality, we clearly have
\[
b(p) \leq |B|_p.
\]

**Remark 8.9.** As for the Weyl tensor, there also exist isolation results for the \( L^{n/2} \)-norm of the Bochner tensor, see [IK04, Theorem A]. The methods are similar to those used in [IS02] and for the constant \( C_n \) appearing in formula (24) of [IK04], the value 1/6 seems to be not too far away from the optimum. A criterion combining Theorem 8.7 and (8.7) is not ruled out by these results, if \( n \geq 5 \). If \( n = 4 \), \( B = W^- \) (see [IK04, p. 232]). Then Theorem 6.9 applies and this criterion is ruled out.

# 9 Physical stability

In higher-dimensional gravity theories, spacetime models are manifolds of the form \( B \times M \) where \( B \) is noncompact and \( M \) is compact. One considers metrics on the product which are Lorentzian on the fibers \( B \times \{p\} \) and Riemannian on the fibers \( \{q\} \times M \). Given compact positive Riemannian Einstein manifolds \((M, g)\), one can build two particular models on them. These are the product Anti-de-Sitter spaces
\[
(AdS \times M, g_{st} + g),
\]
where \((AdS, g_{st})\) is the standard Anti-de-Sitter space and the Generalised Schwarzschild-Tangherlini spacetimes
\[
\left( \mathbb{R} \times \mathbb{R}_+ \times M, -\left( 1 - \left( \frac{m}{r} \right)^{n-1} \right) dt \otimes dt + \left( 1 - \left( \frac{m}{r} \right)^{n-1} \right)^{-1} dr \otimes dr + r^2 g \right),
\]
where \( n = \dim(M) \) and \( m \) is the mass of the black hole. Observe that one recovers the standard Schwarzschild metric if \((M, g) = (S^2, g_{st})\). On both models, an eigentensor of the Lichnerowicz Laplacian on \((M, g)\) correspond to a scalar field on the whole spacetime admitting a certain mass resp. energy
[BF82, GHP03], see also [Die13, Section 2.5] for details. If the eigenvalue is too small, the scalar field
is unstable in a certain sense. It turned out that in both models, the same condition on the Lichnerowicz
Laplace spectrum prohibits the existence of such scalar fields [GHP03, Section 2.2 and 2.3]. It is given
by

**Definition 9.1.** An Einstein manifold \((M, g)\) with positive Einstein constant \(\mu\) is said to be physically
stable if the smallest eigenvalue of the Einstein operator on \(TT\)-tensors satisfies the estimate

\[
\lambda \geq -\frac{n-1}{4} \mu
\]
or equivalently, the smallest eigenvalue of the Lichnerowicz Laplacian on \(TT\)-tensors satisfies

\[
\lambda \geq \frac{\mu}{n-1} \left( 4 - \frac{1}{4} (n-5)^2 \right).
\]

**Remark 9.2.** Obviously, a positive Einstein metric is physically stable if it is stable in the sense of Defi-
nition 2.3.

Any positive Einstein manifold admitting a non-zero Killing spinor is physically stable [GHP03,
Section 4.3]. With the methods used in the previous sections, we find curvature conditions which ensure
physical stability. Using (5.2), we derive

**Theorem 9.3.** An Einstein metric \((M, g)\) with constant \(\mu > 0\) is physically stable if

\[
\sup_{p \in M} r(p) \leq \frac{n + 7}{16} \mu,
\]
where \(r : M \to \mathbb{R}\) is the function defined in (5.1).

Combining this with Lemma 5.1, we obtain

**Corollary 9.4.** Let \((M, g)\) be a positive Einstein manifold and let \(K_{\text{max}} > 0\) be its maximal sectional
curvature. If the minimal sectional curvature satisfies

\[
K_{\text{min}} \geq \frac{(9-n)(n-2)}{(n+23)n} K_{\text{max}},
\]
then \((M, g)\) is physically stable.

By Proposition 6.2, we have

**Theorem 9.5.** A positive Einstein manifold \((M, g)\) with constant \(\mu\) is physically stable if

\[
\|w\|_{L^{n/2}} \leq \mu \cdot \frac{1}{16} \frac{(n+3)^2}{n-1} \cdot \frac{\text{vol}(M, g)^{2/n}}{n-1} \left( \frac{(9-n)(n-1)}{2n(n-2)} + 1 \right)^{-1},
\]
where \(w : M \to \mathbb{R}\) is the function defined in (6.2).

By (6.6), we also get an \(L^{n/2}\)-criterion for physical stability:

**Theorem 9.6.** A positive Einstein manifold \((M, g)\) with constant \(\mu\) is physically stable if

\[
\|w\|_{L^{n/2}} \leq \mu \cdot \text{vol}(M, g)^{2/n} \cdot \frac{1}{16} \frac{(n+3)^2}{n-1} \left( \frac{(9-n)(n-1)}{2n(n-2)} + 1 \right)^{-1}.
\]
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