PSEUDOSPECTRAL AND SPECTRAL BOUNDS FOR THE OSEEN VORTICES OPERATOR

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Abstract. In this paper, we solve Gallay’s conjecture on the spectral lower bound and pseudospectral bound for the linearized operator of the Navier-Stokes equation in $\mathbb{R}^2$ around rapidly rotating Oseen vortices.

1. Introduction

In this paper, we consider the Navier-Stokes equations in $\mathbb{R}^2$

\[
\begin{cases}
\partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla p = 0, \\
\text{div } v = 0, \\
v(0, x) = v_0(x),
\end{cases}
\]

where $v(t, x)$ denotes the velocity, $p(t, x)$ denotes the pressure and $\nu > 0$ is the viscosity coefficient. Let $\omega(t, x) = \partial_2 v^1 - \partial_1 v^2$ be the vorticity. The vorticity formulation of (1.1) takes

\[
\partial_t \omega - \nu \Delta \omega + v \cdot \nabla \omega = 0, \quad \omega(0, x) = \omega_0(x).
\]

Given the vorticity $\omega$, the velocity can be recovered by the Biot-Savart law

\[
v(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^2} \omega(t, y) dy = K_{BS} * \omega.
\]

It is well known that the Navier-Stokes equations (1.2) has a family of self-similar solutions called Lamb-Oseen vortices of the form

\[
\omega(t, x) = \frac{\alpha}{\sqrt{\nu t}} G\left(\frac{x}{\sqrt{\nu t}}\right), \quad v(t, x) = \frac{\alpha}{\sqrt{\nu t}} v^G\left(\frac{x}{\sqrt{\nu t}}\right),
\]

where the vorticity profile and the velocity profile are given by

\[
G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}, \quad v^G(\xi) = \frac{1}{2\pi |\xi|^2} \left(1 - e^{-|\xi|^2/4}\right).
\]

It is easy to see that $\int_{\mathbb{R}^2} \omega(t, x) dx = \alpha$ for any $t > 0$. The parameter $\alpha \in \mathbb{R}$ is called the circulation Reynolds number.

To investigate the long-time behaviour of (1.2), it is convenient to introduce the self-similar variables

\[
\xi = \frac{x}{\sqrt{\nu t}}, \quad \tau = \log t,
\]

and the rescaled vorticity $w$ and the rescaled velocity $u$

\[
\omega(t, x) = \frac{1}{t} w\left(\log t, \frac{x}{\sqrt{\nu t}}\right), \quad v(t, x) = \sqrt{\frac{\nu}{t}} u\left(\log t, \frac{x}{\sqrt{\nu t}}\right).
\]
Then \((w, u)\) satisfies
\[
\partial_\tau w + u \cdot \nabla w = Lw, \tag{1.5}
\]
where the linear operator \(L\) is given by
\[
L = \Delta + \frac{\xi}{2} \cdot \nabla + 1. \tag{1.6}
\]
For any \(\alpha \in \mathbb{R}\), the Lamb-Oseen vortex \(\alpha G(\xi)\) is a steady solution of (1.5). Gallay and Wayne \([11, 12]\) proved that for the integrable initial vorticity, the long-time behaviour of the 2-D Navier-Stokes equations can be described by the Lamb-Oseen vortex. More precisely, for any initial data \(w_0 \in L^1(\mathbb{R}^2)\), the solution of (1.5) satisfies
\[
\lim_{\tau \to +\infty} \|w(\tau) - \alpha G\|_{L^1(\mathbb{R}^2)} = 0, \quad \alpha = \int_{\mathbb{R}^2} w_0(\xi) d\xi.
\]
This result suggests that \(\alpha G\) is a stable equilibrium of (1.5) for any \(\alpha \in \mathbb{R}\). This situation is very similar to the Couette flow \((y, 0)\) in a finite channel, which is stable for any Reynolds number \([8]\). Recently, there are many important works \([1, 2, 3, 16, 25]\) devoted to the study of long-time behaviour of the Navier-Stokes(Euler) equations around the Couette flow.

To study the stability of \(\alpha G\), it is natural to consider the linearized equation around \(\alpha G(\xi)\), which takes as follows
\[
\partial_\tau w = (L - \alpha \Lambda)w, \tag{1.7}
\]
where \(\Lambda\) is a nonlocal linear operator defined by
\[
\Lambda w = vG \cdot \nabla w + u \cdot \nabla G = \Lambda_1 w + \Lambda_2 w, \quad u = K_{BS} * w.
\]
The operator \(L - \alpha \Lambda\) in the weighted space \(Y = L^2(\mathbb{R}^2, G^{-1} dx)\) defined in section 2 has a compact resolvent. Thus, the spectrum of \(L - \alpha \Lambda\) in \(Y\) is a sequence of eigenvalues \(\{\lambda_n(\alpha)\}_{n \in \mathbb{N}}\) satisfying \(\text{Re}\lambda_n(\alpha) \leq 0\) for any \(n, \alpha\). A very important problem is to study how the spectrum changes as \(|\alpha| \to +\infty\), which corresponds to the high Reynolds number limit (the most relevant regime for turbulent flows).

The eigenvalues which correspond to the eigenfunctions in the kernel of \(\Lambda\) do not change as \(\alpha\) varies. We denote by \(L_{\perp}\) and \(\Lambda_{\perp}\) the restriction of the operators \(L\) and \(\Lambda\) to the orthogonal complement of \(\ker \Lambda\) in \(Y\). Then we define the spectral lower bound
\[
\Sigma(\alpha) = \inf \left\{ \text{Re} z : z \in \sigma(-L_{\perp} + \alpha \Lambda_{\perp}) \right\}, \tag{1.9}
\]
and pseudospectral bound
\[
\Psi(\alpha) = \left( \sup_{\lambda \in \mathbb{R}} \left\| (L_{\perp} - \alpha \Lambda_{\perp} - i\lambda)^{-1} \right\|_{Y \to Y} \right)^{-1}. \tag{1.10}
\]
For selfadjoint operators, spectral and pseudospectral bounds are the same. Here \(L - \alpha \Lambda\) is a non-selfadjoint operator. It is easy to see that \(\Sigma(\alpha) \geq \Psi(\alpha)\) for any \(\alpha \in \mathbb{R}\). In fact, \(\Sigma(\alpha)\) and \(\Psi(\alpha)\) are different. Moreover, the pseudo-spectrum plays an important role in the hydrodynamic stability \([22]\), and the spectrum theory of non-selfadjoint operator is also a very active topic \([4, 5, 20, 21]\).

Maekawa \([17]\) proved that \(\Sigma(\alpha)\) and \(\Psi(\alpha)\) tend to infinity as \(|\alpha| \to +\infty\). However, the proof does not provide explicit bounds on \(\Sigma(\alpha)\) and \(\Psi(\alpha)\). Numerical calculations performed by Prochazka and Pullin \([18, 19]\) indicate that \(\Sigma(\alpha) = O(|\alpha|^\frac{1}{2})\) as \(|\alpha| \to +\infty\). Based on the analysis for a model problem, Gallay \([9]\) proposed the following conjecture.
The Oseen Vortices Operator

Conjecture: there exists $C > 0$ independent of $\alpha$ so that as $|\alpha| \to +\infty$,

$$\Sigma(\alpha) \geq C^{-1}|\alpha|^\frac{1}{3}, \quad C^{-1}|\alpha|^\frac{1}{3} \leq \Psi(\alpha) \leq C|\alpha|^\frac{1}{3}.$$ 

If this conjecture is true, then it shows that the linearized operator $L - \alpha \Lambda$ becomes highly non-selfadjoint in the fast rotating limit, and the fast rotation has a strong stabilizing effect on vortices.

To solve this conjecture, Gallagher and Gallay suggested the following model problem (see Villani [24] P. 53 and [23]).

Model problem: identify sufficient condition on $f : \mathbb{R} \to \mathbb{R}$, so that the real parts of the eigenvalues of

$$H_\alpha = -\partial_x^2 + x^2 + i\alpha f(x)$$

in $L^2(\mathbb{R})$ go to infinity as $|\alpha| \to +\infty$, and estimate this rate.

Let $\Sigma(\alpha)$ be the infimum of the real part of $\sigma(H_\alpha)$ and $\Psi(\alpha)^{-1}$ be the supremum of the norm of the resolvent of $H_\alpha$ along the imaginary axis. Under the appropriate conditions on $f$, Gallagher, Gallay and Nier [13] proved that $\Sigma(\alpha)$ and $\Psi(\alpha)$ go to infinity as $|\alpha| \to +\infty$, and presented the precise estimate of the growth rate of $\Psi(\alpha)$. Their proof used the hypocoercive method, localization techniques, and semiclassical subelliptic estimates.

For the simplified linearized operator $L - \alpha \Lambda_1$, Deng [6] proved that $\Psi(\alpha) = O(|\alpha|^\frac{1}{3})$. The same result was proved by Deng [7] for the full linearized operator restricted to a smaller subspace than $\text{ker}(\Lambda)^\perp$. Deng used the multiplier method based on Weyl calculus [15].

The goal of this paper is to give a positive answer on Gallay’s conjecture. The main difficulty comes from the nonlocal operator $\Lambda_2$ so that the hypocoercive method introduced by Villani [24] does not work. In fact, the linearized operator with the nonlocal skew-adjoint operator often appears in the linear stability theory of the incompressible fluids.

In this paper, we develop a method to handle the nonlocal operator. The most key idea is to reduce the nonlocal operator to a model operator by constructing the wave operator. This is motivated by the following simple fact in the scattering theory. Let $A, B$ be two selfadjoint operators in the Hilbert space $H$. Let $U(t) = e^{itA}$ and $V(t) = e^{itB}$ be the strongly continuous groups of unitary operators. The wave operator is defined by

$$W_\pm = \lim_{t \to \pm \infty} W(t), \quad W(t) = U(-t)V(t).$$

Then it holds that

$$AW_\pm = W_\pm B.$$ 

In fact, we have

$$e^{isA}e^{-itA}e^{itB} = e^{-i(t-s)A}e^{i(t-s)B}e^{isB},$$

which gives by taking $t \to \pm \infty$ that

$$e^{isA}W_\pm = W_\pm e^{isB}.$$ 

Then the identity (1.11) follows by taking the derivative in $s$ at $s = 0$.

In a joint work of the last two authors and Zhao [26], we use similar ideas to prove the optimal enhanced dissipation rate for the linearized Navier-Stokes equations in $\mathbb{T}^2$ around the Kolmogorov flow.
2. Spectral Analysis of the Linearized Operator

In this section, we recall some facts about the spectrum of the linearized operator $L - \alpha \Lambda$ from [11, 12, 9, 10]. Although these facts will not be used in our proof, they will be helpful to understand this spectral problem.

Let $\rho(\xi)$ be a nonnegative function. We introduce the weighted $L^2$ space

$$L^2(\mathbb{R}^2, \rho d\xi) = \left\{ w \in L^2(\mathbb{R}^2) : \|w\|_{L^2(\rho)}^2 = \int_{\mathbb{R}^2} |w(\xi)|^2 \rho(\xi) d\xi < +\infty \right\},$$

which is a (real) Hilbert space equipped with the scalar product

$$\langle w_1, w_2 \rangle_{L^2(\rho)} = \int_{\mathbb{R}^2} w_1(\xi) w_2(\xi) \rho(\xi) d\xi.$$

We denote

$$Y = L^2(\mathbb{R}^2, G^{-1} d\xi).$$

Lemma 2.1. It holds that

1. the operator $L$ is selfadjoint in $Y$ with compact resolvent and purely discrete spectrum

$$\sigma(L) = \left\{ -\frac{n}{2} : n = 0, 1, 2, \cdots \right\},$$

2. the operator $\Lambda$ is skew-symmetric in $Y$.

The first fact follows from the following observation:

$$L = -\nabla^2 + \frac{|\xi|^2}{16} - \frac{1}{2}$$

is a two-dimensional harmonic oscillator, which is self-adjoint in $L^2(\mathbb{R}^2)$ with compact resolvent and discrete spectrum given by $-\sigma(L)$. Furthermore, we know that

1. $\lambda_0 = 0$ is a simple eigenvalue of $L$ with the eigenfunction $G$;
2. $\lambda_1 = -\frac{1}{2}$ is an eigenvalue of $L$ of multiplicity two with the eigenfunctions $\partial_1 G$ and $\partial_2 G$;
3. $\lambda_3 = -1$ is an eigenvalue of $L$ of multiplicity three with the eigenfunctions $\Delta G, (\partial_1^2 - \partial_2^2) G$ and $\partial_1 \partial_2 G$.

Now we consider the spectrum of $L - \alpha \Lambda$ in $Y$ for any fixed $\alpha \in \mathbb{R}$. Since $\Lambda$ is a relatively compact perturbation of $L$ in $Y$, $L - \alpha \Lambda$ has a compact resolvent in $Y$ by the classical perturbation theory [14]. So, the spectrum of $L - \alpha \Lambda$ is a sequence of eigenvalues $\{\lambda_n(\alpha)\}_{n \in \mathbb{N}}$.

Using the fact that

$$\Lambda w = 0 \quad \text{for} \quad w = G, \partial_1 G, \partial_2 G, \Delta G,$$

we deduce that $0, -\frac{1}{2}, -1$ are also eigenvalues of $L - \alpha \Lambda$ for any $\alpha \in \mathbb{R}$. Let us introduce the following subspaces of $X$:

$$Y_0 = \left\{ w \in Y : \int_{\mathbb{R}^2} w(\xi) d\xi = 0 \right\} = \left\{ G \right\}^\perp,$$

$$Y_1 = \left\{ w \in Y_0 : \int_{\mathbb{R}^2} \xi w(\xi) d\xi = 0 \right\} = \left\{ G, \partial_1 G, \partial_2 G \right\}^\perp,$$

$$Y_2 = \left\{ w \in Y_1 : \int_{\mathbb{R}^2} |\xi|^2 w(\xi) d\xi = 0 \right\} = \left\{ G, \partial_1 G, \partial_2 G, \Delta G \right\}^\perp.$$

These spaces are invariant under the linear evolution generated by $L - \alpha \Lambda$.

The following proposition shows that the Oseen vortex $\alpha G$ is spectrally stable in $Y$ for any $\alpha \in \mathbb{R}$. 

**Proposition 2.2.** For any $\alpha \in \mathbb{R}$, the spectrum of $L - \alpha \Lambda$ satisfies
\[
\sigma(L - \alpha \Lambda) \subset \{ z \in \mathbb{C} : \text{Re}(z) \leq 0 \} \quad \text{in} \quad Y,
\]
\[
\sigma(L - \alpha \Lambda) \subset \{ z \in \mathbb{C} : \text{Re}(z) \leq -\frac{1}{2} \} \quad \text{in} \quad Y_0,
\]
\[
\sigma(L - \alpha \Lambda) \subset \{ z \in \mathbb{C} : \text{Re}(z) \leq -1 \} \quad \text{in} \quad Y_1,
\]
\[
\sigma(L - \alpha \Lambda) \subset \{ z \in \mathbb{C} : \text{Re}(z) < -1 \} \quad \text{in} \quad Y_2.
\]

The operator $L - \alpha \Lambda$ is invariant under rotations with respect to the origin. Thus, it is natural to introduce the polar coordinates $(r, \theta)$ in $\mathbb{R}^2$. Let us decompose
\[
Y = \oplus_{n \in \mathbb{N}} X_n, \tag{2.3}
\]
where $X_n$ denote the subspace of all $w \in Y$ so that
\[
w(r \cos \theta, r \sin \theta) = a(r) \cos(n\theta) + b(r) \sin(n\theta)
\]
for some radial functions $a, b : \mathbb{R}^+ \to \mathbb{R}$.

**Lemma 2.3.** $\ker \Lambda = X_0 \oplus \{ \alpha \partial_1 G + \beta \partial_2 G \}$. In particular, $\ker \Lambda \perp \oplus_{n>0} X_n$.

### 3. Reduction to One-Dimensional Operators

Following Deng’s work [7], we reduce the linearized operator to a family of one-dimensional operators.

We conjugate the linearized operator $L - \alpha \Lambda$ with $G^\frac{1}{2}$, and then obtain a linear operator $H_\alpha$ in $L^2(\mathbb{R}^2, d\xi)$:
\[
H_\alpha = -G^{-\frac{1}{2}} L G^\frac{1}{2} + \alpha G^{-\frac{1}{2}} \Lambda G^\frac{1}{2} = \mathcal{L} + \alpha \mathcal{M},
\]
where $\mathcal{L}$ is defined by (2.2) and $\mathcal{M}$ is defined by
\[
\mathcal{M}w = v^G \cdot \nabla w - \frac{1}{2} G^\frac{1}{2} \xi \cdot (K_{BS} * (G^\frac{1}{2} w)).
\]

Let us introduce some notations:
\[
K_k[h] = \frac{1}{2|k|} \int_0^{+\infty} \min\left(\frac{r}{s}, \frac{s}{r}\right)^{|k|} sh(s) ds, \tag{3.2}
\]
\[
\sigma(r) = \frac{1 - e^{-r^2/4}}{r^2/4}, \quad g(r) = e^{-r^2/8}. \tag{3.3}
\]

Then for $w = \sum_{k \in \mathbb{Z}^*} w_k(r)e^{ik\theta}$, we have
\[
(\mathcal{L} - i\lambda)w(r \cos \theta, r \sin \theta) = \sum_{k \in \mathbb{Z}^*} (\mathcal{H}_{\alpha,k,\lambda} w_k(r)) e^{ik\theta},
\]
where the operator $\mathcal{H}_{\alpha,k,\lambda}$ acts on $L^2(\mathbb{R}^+, r dr)$ and is given by
\[
\mathcal{H}_{\alpha,k,\lambda} = -\partial_r^2 - \frac{1}{r} \partial_r + \frac{k^2}{r^2} + \frac{r^2}{16} - \frac{1}{2} + i\beta_k (\sigma(r) - \nu_k) - i\beta_k g K_k[g], \tag{3.4}
\]
where
\[
\beta_k = \frac{\alpha k}{8\pi}, \quad \lambda = \beta_k \nu_k \in \mathbb{R}. \tag{3.5}
\]

Without loss of generality, we assume $|\beta_k| \geq 1$ for any $|k| \leq 1$.

We introduce the operator
\[
\tilde{H}_{\alpha,k,\lambda} = r^{\frac{1}{2}} \mathcal{H}_{\alpha,k,\lambda} r^{-\frac{1}{2}} := \tilde{\mathcal{H}}_k. \tag{3.6}
\]
Then $\tilde{H}_k$ acts on $L^2(\mathbb{R}_+, dr)$ and is given by

\begin{align}
(3.7) \quad \tilde{H}_k &= -\partial^2_r + \frac{k^2 - \frac{1}{2}}{r^2} + \frac{r^2}{16} - \frac{1}{2} + i\beta_k(\sigma(r) - \nu_k) - i\beta_k g\tilde{K}_k[g], \\
(3.8) \quad \tilde{K}_k[h] &= \frac{1}{2|k|} \int_0^{+\infty} \min\left(\frac{r}{s}, \frac{s}{r}\right)|\frac{k}{r}s h(s) ds,
\end{align}

and $C_c^\infty(\mathbb{R}_+)$ is a core of the operator $\tilde{H}_k$ with domain

\begin{align}
(3.9) \quad D(\tilde{H}_k) &= \{ \omega \in H^2_{loc}(\mathbb{R}_+, dr) \cap L^2(\mathbb{R}_+, dr) : \tilde{H}_k \omega \in L^2(\mathbb{R}_+, dr) \}.
\end{align}

That is,

\begin{align*}
D &= D(\tilde{H}_k) = \{ w \in L^2(\mathbb{R}_+, dr) : \partial^2_r w, \frac{w}{r^2}, r^2 w \in L^2(\mathbb{R}_+, dr) \} \quad |k| \geq 2, \\
D_1 &= D(\tilde{H}_k) = \{ w \in L^2(\mathbb{R}_+, dr) : r^{\frac{1}{2}} \partial^2_r (w/r^{\frac{1}{2}}), r^{\frac{1}{2}} \partial_r (w/r^{\frac{3}{2}}), r^2 w \in L^2(\mathbb{R}_+, dr) \} \quad |k| = 1.
\end{align*}

Then the resolvent estimate is reduced to the following estimate

\[ \| \tilde{H}_k u \|_{L^2(\mathbb{R}_+, dr)} \geq |\beta_k|^\frac{1}{2} \| u \|_{L^2(\mathbb{R}_+, dr)}. \]

We also write

\begin{align}
(3.10) \quad \tilde{H}_k &= \tilde{A}_k + i\beta_k \tilde{B}_k - i\lambda,
\end{align}

where

\begin{align*}
\tilde{A}_k &= -\partial^2_r + \frac{k^2 - \frac{1}{2}}{r^2} + \frac{r^2}{16} - \frac{1}{2}, \\
\tilde{B}_k &= \sigma(r) - g\tilde{K}_k[g].
\end{align*}

It is easy to see that

\begin{align}
(3.11) \quad \text{Ker}(\tilde{B}_1) &= \text{span}\{ r^{\frac{3}{2}} g(r) \}, \quad \text{Ker}(\tilde{B}_k) = \{0\} \quad \text{for } |k| \geq 2.
\end{align}

Thus, $L - \alpha\Lambda|_{(\text{Ker} \Lambda)^\perp}$ is unitary equivalent to $\bigoplus_{|k|=1} \tilde{H}_k|_{(\text{Ker} \tilde{B}_1)^\perp} \bigoplus_{|k| \geq 2} \tilde{H}_k$.

In the sequel, we denote by $\langle \cdot, \cdot \rangle$ the $L^2(\mathbb{R}_+, dr)$ inner product, and by $\| \cdot \|$ the norm of $L^2(\mathbb{R}_+, dr)$, $\| \cdot \|_{L^p}$ the norm of $L^p(\mathbb{R}_+, dr)$. The notation $a \gtrsim b$ or $a \lesssim b$ means that there exists a constant $C > 0$ independent of $\alpha, k, \lambda$ so that

\[ a \geq C^{-1} b \quad \text{or} \quad a \leq C b. \]

4. Resolvent estimate of $\tilde{H}_1$

As $\overline{\tilde{H}_1 w} = \tilde{H}_1 \overline{w}$, it is enough to prove the following resolvent estimate for $\tilde{H}_1$.

Theorem 4.1. For any $\lambda \in \mathbb{R}$ and $w \in \{ r^{\frac{3}{2}} g(r) \}^\perp \cap D_1$, we have

\[ \| \tilde{H}_1 w \| \gtrsim |\beta_1|^{\frac{1}{2}} \| w \|. \]

Moreover, there exist $\lambda \in \mathbb{R}$ and $v \in \{ r^{\frac{3}{2}} g(r) \}^\perp \cap D_1$ so that

\[ \| \tilde{H}_1 v \| \lesssim |\beta_1|^{\frac{1}{3}} \| v \|. \]
4.1. **Reduction to the model operator** $\mathcal{L}_1$. Let us introduce the operator $T$ defined by

\begin{equation}
T w (r) = w(r) + \frac{I_1[w](r) g(r)}{\sigma'(r) r^\frac{3}{2}},
\end{equation}

where

\begin{equation}
I_1[w](r) = \int_0^r s^\frac{3}{2} g(s) w(s) ds.
\end{equation}

It is easy to check that $T$ is a bounded linear operator in $L^2(\mathbb{R}_+, dr)$. The adjoint operator $T^*$ is also a bounded linear operator in $L^2(\mathbb{R}_+, dr)$ given by

\begin{equation}
T^* \omega(r) = \omega(r) + r^\frac{3}{2} g(r) \int_r^{+\infty} \frac{\omega(s) g(s)}{s^\frac{3}{2} \sigma'(s)} ds.
\end{equation}

**Lemma 4.2.** It holds that

1. $\|T w\|^2 = \|w\|^2 - \frac{(w \cdot r^\frac{3}{2} g)^2}{\|r^\frac{3}{2} g\|^2}$;
2. $T^* T = P$, where $P$ is the projection to $\{r^\frac{3}{2} g(r)\}^\perp \cap L^2(\mathbb{R}_+, dr)$;
3. $TT^* = I_{L^2(\mathbb{R}_+, dr)}$.

**Proof.** The first one is equivalent to the second one. Thanks to

\[-(r^3 \sigma'(r))' = r^3 g(r)^2 \quad \text{and} \quad I_1[s^\frac{3}{2} g(s)](r) = \int_0^r s^3 g(s)^2 ds = -r^3 \sigma'(r),\]

we find that

\[T(r^\frac{3}{2} g)(r) = r^\frac{3}{2} g(r) + \frac{I_1[s^\frac{3}{2} g(s)](r) g(r)}{\sigma'(r) r^\frac{3}{2}} = 0.\]

Thus, it suffices to check that for any $w \in \{r^\frac{3}{2} g(r)\}^\perp \cap L^2(\mathbb{R}_+, dr)$, $u \in C_0^\infty(\mathbb{R}_+),$

\[\langle T^* T w, u \rangle = \langle w, u \rangle,
\]

which is equivalent to verifying that

\[\left\langle \frac{I_1[w](r) g(r)}{\sigma'(r) r^\frac{3}{2}}, u \right\rangle + \left\langle w, \frac{I_1[w](r) g(r)}{\sigma'(r) r^\frac{3}{2}} \right\rangle + \left\langle \frac{I_1[w](r) g(r)}{\sigma'(r) r^\frac{3}{2}}, \frac{I_1[w](r) g(r)}{\sigma'(r) r^\frac{3}{2}} \right\rangle = 0.
\]

Using the facts that for $w \in \{r^\frac{3}{2} g(r)\}^\perp$,

\[I_1[w](0) = \lim_{r \to +\infty} I_1[w](r) = 0,
\]

and $(r^3 \sigma'(r))' = -r^3 g(r)^2$, we get by integration by parts that

\[\left\langle \frac{I_1[w](r) g(r)}{\sigma'(r) r^\frac{3}{2}}, u \right\rangle = \int_0^{+\infty} \frac{I_1[w]}{\sigma'(r) r^3} dI_1[w] = -\int_0^{+\infty} I_1[w] \left( \frac{I_1[w]}{\sigma'(r) r^3} \right)' dr = -\int_0^{+\infty} I_1[w] \frac{r^\frac{3}{2} g(r) w(r) \sigma'(r) r^3}{(\sigma'(r) r^3)^2} dr = -\int_0^{+\infty} w(r) I_1[w] g(r) dr - \int_0^{+\infty} I_1[w] I_1[w] \frac{g^2(r)}{(\sigma'(r) r^\frac{3}{2})^2} dr.
\]

This shows that $T^* T = P$. 

On the other hand, we have
\[ |T^* w|^2 - |w|^2 = -\partial_r (r^2 \sigma'(r)|f_1|^2), \quad f_1(r) = \int_r^{+\infty} \frac{w(s)g(s)}{s^\frac{3}{2} \sigma'(s)} \, ds, \]
which gives \( |T^* w|^2 = |w|^2 \), thus \( TT^* = I \).

We have the following important relationship between \( T \) and \( \tilde{B}_1 \).

**Lemma 4.3.** It holds that
\[ T \tilde{B}_1 = \sigma(r)T. \]

**Proof.** Direct calculation gives
\[
T \tilde{B}_1 w = \tilde{B}_1 w + \frac{I_1[\tilde{B}_1 w]g(r)}{\sigma'(r)r^\frac{3}{2}}
\]
which gives
\[
\sigma(r)w - g\tilde{K}_1[gw] + I_1[\sigma w]g(r) - I_1[g\tilde{K}_1[gw]]g(r) + \frac{I_1[\sigma w]g(r)}{\sigma'(r)r^\frac{3}{2}} - \frac{I_1[g\tilde{K}_1[gw]]g(r)}{\sigma'(r)r^\frac{3}{2}}.
\]
Thus, it suffices to show that
\[
I_1[\sigma w] = \sigma'(r)r^\frac{3}{2}\tilde{K}_1[gw] + I_1[g\tilde{K}_1[gw]] + \sigma(r)I_1[w].
\]

Direct calculation shows that
\[
\sigma'(r)r^\frac{3}{2}\tilde{K}_1[gw] = r^\frac{3}{2}(e^{-\frac{r^2}{4}} - \sigma(r)) \int_0^{+\infty} \min(\frac{r}{s}, \frac{s}{r}) (r s)^\frac{3}{2} g(s) w(s) ds
\]
\[
= r^\frac{3}{2}(e^{-\frac{r^2}{4}} - \sigma(r)) \int_0^r r^{-\frac{3}{2}} s^\frac{3}{2} g(s) w(s) ds + \int_r^{+\infty} r^\frac{3}{2} s^{-\frac{3}{2}} g(s) w(s) ds
\]
\[
= (e^{-\frac{r^2}{4}} - \sigma(r))I_1[w] + r^2(e^{-\frac{r^2}{4}} - \sigma(r)) \int_r^{+\infty} s^{-\frac{3}{2}} g(s) w(s) ds,
\]
and
\[
I_1[g\tilde{K}_1[gw]](r) = \int_0^r s^\frac{3}{2} g^2(s) \tilde{K}_1[gw](s) ds
\]
\[
= \int_0^r s^\frac{3}{2} g^2(s) ds \int_0^{+\infty} \min(\frac{t}{s}, \frac{s}{t}) t^\frac{3}{2} g(t) w(t) dt
\]
\[
= \frac{1}{2} \int_0^r s^{-\frac{3}{2}} g^2(s) ds \int_0^r t^\frac{3}{2} g(t) w(t) dt + \frac{1}{2} \int_0^r s^\frac{3}{2} e^{-\frac{t^2}{4}} ds \int_s^{+\infty} t^{-\frac{1}{2}} g(t) w(t) dt
\]
\[
= \frac{1}{2} \int_0^r t^\frac{3}{2} g(t) w(t) dt \int_0^r s^{-\frac{3}{2}} g^2(s) ds + \frac{1}{2} \int_0^r s^\frac{3}{2} e^{-\frac{s^2}{4}} ds \int_s^r t^{-\frac{1}{2}} g(t) w(t) dt
\]
\[
+ \frac{1}{2} \int_0^r s^\frac{3}{2} e^{-\frac{s^2}{4}} ds \int_r^{+\infty} t^{-\frac{1}{2}} g(t) w(t) dt
\]
\[
= \int_0^r t^\frac{3}{2} g(t) w(t) (e^{-\frac{t^2}{4}} - e^{-\frac{s^2}{4}}) dt + \frac{1}{2} \int_0^r t^{-\frac{1}{2}} g(t) w(t) dt \int_0^r s^3 e^{-\frac{s^2}{4}} ds
\]
\[
+ [4(1 - e^{-\frac{r^2}{4}}) - r^2 e^{-\frac{r^2}{4}}] \int_r^{+\infty} t^{-\frac{1}{2}} g(t) w(t) dt
\]
\[
= -e^{-\frac{r^2}{4}} I_1[w] + I_1[\sigma w] + r^2(\sigma(r) - e^{-\frac{r^2}{4}}) \int_r^{+\infty} t^{-\frac{1}{2}} g(t) w(t) dt,
\]
which gives \( (4.4) \).
Lemma 4.4. It holds that

\[ [T, \tilde{A}_1]w = T\tilde{A}_1w - \tilde{A}_1Tw = f(r)Tw, \]

where

\[ f(r) = 2\frac{g(r)^4}{(\sigma'(r))^2} + \frac{g(r)^2}{\sigma'(r)}\left(\frac{6}{r} - r\right) \geq 0. \]

Proof. First of all, we have

\[ [T, \tilde{A}_1]w = \frac{I_1[-\partial_r^2 + \frac{3}{4}r^2 + \frac{r^2}{16}]w(r)}{\sigma'(r)r^\frac{3}{2}} - \left( -\partial_r^2 + \frac{3}{4} \frac{1}{r^2} + \frac{r^2}{16} \right) \left( I_1[w]g(r) \right). \]

Using the facts that

\[ I_1[-\partial_r^2] = -r^\frac{3}{2}g(r)w'(r) + (r^\frac{3}{2}g(r))'w(r) - \int_0^r (s^\frac{3}{2}g(s))''w(s)ds, \]

\[ (-\partial_r^2 + \frac{3}{4} \frac{1}{r^2} + \frac{r^2}{16})r^\frac{3}{2}g(r) = r^\frac{3}{2}g(r), \]
we deduce that

\[ I_1[-\partial_r^2 + \frac{3}{4} \frac{1}{r^2} + \frac{r^2}{16}]w = -r^\frac{3}{2}g(r)\omega'(r) + (r^\frac{3}{2}g(r))'\omega(r) + I_1[\omega], \]

Direct calculation gives

\[ \partial_t^2 \left( \frac{I_1[\omega]g(r)}{\sigma'(r)r^\frac{3}{2}} \right) = \omega'(r)\frac{g^2(r)}{\sigma'(r)} + \omega(r) \left( \frac{g^2(r)}{\sigma'(r)} \right)' + r^\frac{3}{2}g(r)\omega(r) \left( \frac{g(r)r^\frac{3}{2}}{\sigma'(r)r^\frac{3}{2}} \right)' + I_1[\omega] \left( \frac{g(r)r^\frac{3}{2}}{\sigma'(r)r^\frac{3}{2}} \right)'' . \]

Let \( F = r^\frac{3}{2}g(r) \) and \( G(r) = \sigma'(r)r^3 \). We have

\[ F' = \left( \frac{3}{2} \cdot \frac{1}{r} - \frac{r}{4} \right) F, \quad F'' = \left( \frac{3}{4} \cdot \frac{1}{r^2} - 1 + \frac{r^2}{16} \right) F, \quad G' = -F^2 . \]

Summing up, we obtain

\[ [T, \tilde{A}_1]w = \left[ F'F \right] \frac{G'}{G} + \left( \frac{F^2}{G} \right)' + \frac{F}{G} \left( \frac{F^2}{G} \right) + \left( 1 - \frac{3}{4} \cdot \frac{1}{r^2} - \frac{r^2}{16} \right) \frac{I_1[w]F}{G} + I_1[w] \left( \frac{F}{G} \right)'' . \]

On the other hand, we have

\[ \left( \frac{F}{G} \right)'' = \left( \frac{3}{4} \cdot \frac{1}{r^2} - 1 + \frac{r^2}{16} \right) F + \frac{F}{G} \left( \frac{4F'}{G} + \frac{2(G')^2}{G^2} \right), \]

\[ \frac{F'/F}{G} + \left( \frac{F^2}{G} \right)' + \frac{F}{G} \left( \frac{F'}{G} \right) = 4 \frac{F'}{G} + 2 \frac{(G')^2}{G^2} , \]

Then we infer that

\[ [T, \tilde{A}_1]w = \left( 4 \frac{F'}{G} + 2 \frac{(G')^2}{G^2} \right)(w + \frac{I_1[w]F}{G}) \]

\[ = \left( 2 \frac{g^4}{(\sigma')^2} + \frac{g^2}{\sigma'(r)} \frac{6}{r} - r \right) Tw = f(r)Tw . \]

It remains to prove that \( f(r) \geq 0 \). We have

\[ 2 \frac{g^4}{(\sigma')^2} + \frac{g^2}{\sigma'(r)} \frac{6}{r} - r = \frac{r^6 + r^2(6 - r^2)(r^2 + 4r^2 - 4e^2)}{32(r^2 + 1 - e^2)^2} . \]
while by Taylor expansion, we have
\[ r^6 + r^2(6 - r^2)(r^2 + 4 - 4e^2) = 2r^4 + 24r^2 - 24r^2e^2 + 4r^4e^2 \]
\[ = 2r^4 + 2r^2\sum_{n=0}^{+\infty} \frac{1}{n!}(\frac{r^2}{4})^n - 12\sum_{n=1}^{+\infty} \frac{1}{n!}(\frac{r^2}{4})^n \]
\[ = 2r^4 \left[ 8\sum_{n=2}^{+\infty} \frac{1}{(n-1)!}(\frac{r^2}{4})^n - 12\sum_{n=2}^{+\infty} \frac{1}{n!}(\frac{r^2}{4})^n \right] \geq 0. \]
This completes the proof.

It follows from Lemma 4.3 and Lemma 4.2 that for \( w \in \{r^2g(r)\}^\perp \cap D_1 \),
\[ T\tilde{H}_1w = T\tilde{A}_1w + i\beta_1 T\tilde{B}_1w - i\lambda Tw \]
\[ = T\tilde{A}_1 T^* Tw + i\beta_1 \sigma(r) Tw - i\lambda Tw. \]
Lemma 4.2 ensures that \( T : \{r^2g(r)\}^\perp \to L^2(\mathbb{R}_+, dr) \) is invertible and \( T^{-1} = T^* \). Let \( w = T^{-1}u \). We infer from Lemma 4.4 that
\[ T\tilde{H}_1 T^{-1}u = T\tilde{A}_1 T^{-1}u + i\beta_1 \sigma(r) u - i\lambda u \]
\[ = \tilde{A}_1 u + f(r)u + i\beta_1 \sigma(r) u - i\lambda u = \mathcal{L}_1 u, \]
where
\[ f(r) = 2 \frac{g(r)^4}{(\sigma(r))^2} + \frac{g(r)^2}{\sigma(r)} \left( \frac{6}{r} - r \right). \]
So, the operator \( T \) plays a role of wave operator. Let
\[ D(\mathcal{L}_1) = \{ \omega \in H^2_{loc}(\mathbb{R}_+, dr) \cap L^2(\mathbb{R}_+, dr) : \mathcal{L}_1 \omega \in L^2(\mathbb{R}_+, dr) \}. \]
Then \( u \in D(\mathcal{L}_1) \iff T^* u \in D(\mathcal{H}_1) \cap \{r^2g(r)\}^\perp \), and \( D(\mathcal{L}_1) = D(\tilde{H}_3) = D \).
Moreover, we have
\[ \langle \tilde{H}_1 w, w \rangle = \langle \tilde{H}_1 T^{-1} u, T^* u \rangle = \langle \mathcal{L}_1 u, u \rangle. \]
On the other hand, \( \|w\| = \|T w\| = \|u\| \) for any \( w \in \{r^2g(r)\}^\perp \cap D_1 \). Thus, we reduce the resolvent estimate of \( \tilde{H}_1 \) to one of the model operator \( \mathcal{L}_1 \).

4.2. Coercive estimates.

**Lemma 4.5.** The operator \( \tilde{A}_1 \) can be represented as
\[ (\tilde{A}_1 - \frac{1}{2})w = -r^{-\frac{3}{2}}g^{-1}\partial_r [r^3g^2\partial_r (r^{-\frac{3}{2}}g^{-1}w)]. \]
In particular, we have
\[ \tilde{A}_k \geq \frac{1}{2} \quad \text{for} \quad k \geq 1. \]

**Proof.** Let \( F(r) = r^\frac{3}{2}g(r) \). Then we have
\[ -r^{-\frac{3}{2}}g^{-1}\partial_r [r^3g^2\partial_r (r^{-\frac{3}{2}}g^{-1}w)] = -F^{-1}\partial_r [F^2\partial_r (F^{-1}w)] \]
\[ = (-\partial_r^2 + \frac{F''}{F})w = (-\partial_r^2 + \frac{3}{4} \cdot \frac{1}{r^2} - 1 + \frac{r^2}{16})w = (\tilde{A}_1 - \frac{1}{2})w, \]
here we used \( F'' = (\frac{3}{4} \cdot \frac{1}{r^2} - 1 + \frac{r^2}{16})F \).
Then for any \( w \in D \), we have
\[
\langle (\tilde{A} - \frac{1}{2})w, w \rangle = -\langle F^{-1}\partial_{\tau}[F^2\partial_{\tau}(F^{-1}w)], w \rangle = \|F\partial_{\tau}(F^{-1}w)\|^2 \geq 0.
\]
This shows that \( \tilde{A}_k \geq \tilde{A}_1 \geq \frac{1}{4} \).

**Lemma 4.6.** It holds that
\[
(4.9) \quad \tilde{A}_1 + f(r) \geq \frac{1}{r^2} + r^2.
\]

**Proof.** By the proof of Lemma 4.3, we know that
\[
f(r) = r^2 \left\{ \sum_{n=2}^{+\infty} \frac{(\frac{2}{n-1})^n}{4(r^2 + 1 - e^{r^2/4})^2} \right\} \geq r^2 \left\{ \sum_{n=2}^{+\infty} \frac{\frac{1}{n}r^2}{4(r^2 + 1 - e^{r^2/4})^2} \right\}
\]
\[
\geq \frac{r^2}{4(e^{r^2} - 1 - r^2)}.
\]

Let \( h(r) = \frac{3}{16} \cdot \frac{1}{r^2} + \frac{2}{16} - \frac{1}{2} + \frac{u}{4(e^{r^2} - 1 - r^2)} \). Then there exists \( \varepsilon_0 \in (0, 1) \) so that \( h(r) \geq \frac{1}{r^2} \) for \( r < \varepsilon_0 \) and \( h(r) \geq r^2 \) for \( r > \frac{1}{\varepsilon_0} \) and \( h(r) \) can attain its minimum. Thus, if \( h(r) > 0 \), \( h(r) \) has a positive lower bound. For this, let \( u = \frac{r^2}{4} \). Then by Taylor’s expansion, we get
\[
h(r) = \frac{3}{16} \cdot \frac{1}{u} + \frac{1}{4}u - \frac{1}{2} + \frac{u}{e^u - 1 - u}
\]
\[
= \frac{3}{16} \sum_{n=2}^{+\infty} \frac{1}{n}u^n + \frac{1}{4}u^2 \sum_{n=2}^{+\infty} \frac{1}{n}u^n - \frac{1}{2}u \sum_{n=2}^{+\infty} \frac{1}{n}u^n + u^2
\]
\[
= \frac{\sum_{n=2}^{+\infty} a_n u^n}{u(e^u - 1 - u)},
\]
where
\[
a_2 = \frac{35}{32}, \quad a_3 = -\frac{7}{32}, \quad a_4 = \frac{19}{384}, \quad 2\sqrt{a_2 a_4} > |a_3|,
\]
\[
a_n = \frac{1}{n!} \left( \frac{3}{16} + \frac{n(n - 1)}{4} - \frac{n}{2} \right) > 0 (n \geq 5).
\]

Hence, there exists \( c_0 > 0 \) such that \( h(r) \geq c_0 \). So, there exists \( C > 0 \) such that for any \( r \in [\varepsilon_0, \frac{1}{\varepsilon_0}] \), we have \( h(r) \geq C(\frac{1}{r^2} + r^2) \).

Summing up, we conclude that
\[
\tilde{A}_1 + f(r) \geq \frac{3}{4r^2} + \frac{r^2}{16} - \frac{1}{2} + f(r) \geq h(r) \geq \frac{1}{r^2} + r^2.
\]

The proof is completed. \( \square \)

### 4.3. Resolvent estimate of \( L_1 \)

In this subsection, we prove Theorem 4.1. It suffices to show that for any \( u = Tw, w \in \{r^2 g(r)\}^1 \cap D_1 \),
\[
(4.10) \quad \|L_1 u\| \geq \|\beta_1 \|^\frac{1}{3}\|u\|.
\]

The proof is split into three cases.
Using the fact that

we deduce that

\[ \lim_{r \to \infty} (1 - \sigma(r)) = 1. \]

Case 1. \( \nu_1 \geq 1 \)

Be Lemma 1.6, we get

\[ |\langle \mathcal{L}_1 u, u \rangle| \sim |\langle \tilde{A}_1 f \rangle u, u \rangle + |\beta_1| |\langle (\nu_1 - \sigma(r)) u, u \rangle| \]

which shows that for \( \nu_1 \geq 1 \),

\[ 1 - \sigma(r) = 1 - \frac{1 - e^{-\frac{\nu}{4}}}{r^2} \sim r^2 (r \to 0), \quad \lim_{r \to \infty} 1 - \sigma(r) = 1, \]

we deduce that

\[ \int_0^1 \left[ \frac{1}{r^2} + |\beta_1 (1 - \sigma(r))| \right] |u|^2 dr \gtrsim \int_0^1 \left( \frac{1}{r^2} + |\beta_1|^2 \right) |u|^2 dr \gtrsim \int_0^1 |\beta_1|^2 |u|^2 dr, \]

Case 2. \( \nu_1 \leq 0 \)

In this case, we have by Lemma 1.6 that

\[ |\langle \mathcal{L}_1 u, u \rangle| \sim |\langle \tilde{A}_1 f \rangle u, u \rangle + |\beta_1| |\langle (\sigma(r) - \nu_1) u, u \rangle| \]

which show that for \( \nu_1 \geq 1 \),

\[ |\langle \mathcal{L}_1 u, u \rangle| \gtrsim |\beta_1| \|u\|^2. \]

\[ (4.11) \]

Case 3. \( 0 < \nu_1 < 1 \)

Let \( \nu_1 = \sigma(r_1) \) for some \( r_1 > 0 \). We split this case into two subcases:

\[ |\beta_1| \leq \max \left( \frac{1}{r_1^2}, r_1^0 \right) \quad \text{and} \quad |\beta_1| \geq \max \left( \frac{1}{r_1^2}, r_1^0 \right). \]

Lemma 4.7. If \( |\beta_1| \leq \max \left( \frac{1}{r_1^2}, r_1^0 \right) \), then we have

\[ \|\mathcal{L}_1 u\| \gtrsim |\beta_1| \|u\|. \]

Proof. If \( |\beta_1| \leq 1 \), then \( \frac{1}{r^2} + r^2 \geq 1 \geq |\beta_1|^2 \). Lemma 1.6 gives

\[ |\langle \mathcal{L}_1 u, u \rangle| \gtrsim |\langle (r^2 + \frac{1}{r^2}) u, u \rangle| \geq |\beta_1|^2 \|u\|^2. \]
If $1 \leq |\beta_1| \leq \max(\frac{1}{r_1^2}, r_1^6)$, we only need to check the following cases

\[
\begin{align*}
& r_1 \leq 1, \quad 1 \leq |\beta_1| \leq \frac{1}{r_1^2} \implies \|L_1 u\| \gtrsim |\beta_1|^{\frac{1}{2}}\|u\|, \\
& r_1 \geq 1, \quad 1 \leq |\beta_1| \leq r_1^4 \implies \|L_1 u\| \gtrsim |\beta_1|^{\frac{1}{2}}\|u\|, \\
& r_1 \geq 1, \quad r_1^4 \leq |\beta_1| \leq r_1^6 \implies \|L_1 u\| \gtrsim |\beta_1|^{\frac{3}{4}}\|u\|.
\end{align*}
\]

By Lemma 4.6 again, we have

\[
|\langle L_1 u, u \rangle| \gtrsim \langle (r^2 + \frac{1}{r^2})u, u \rangle + |\beta_1|\langle (\nu_1 - \sigma(r))u, u \rangle,
\]

which along with Lemma 7.2 gives our results.

\[\square\]

**Lemma 4.8.** If $|\beta_1| \geq \max(\frac{1}{r_1^2}, r_1^6)$, then we have

\[
\|L_1 u\| \gtrsim |\beta_1|^{\frac{1}{2}}\|u\|.
\]

**Proof.** Let $\delta > 0$ be so that $\delta^3|\beta_1| \min(r_1, r_1^{-3}) = 1$. Thanks to $|\beta_1| \geq \max(\frac{1}{r_1^2}, r_1^6)$, we get

\[
|\beta_1|^{-\frac{1}{2}} \leq \min(r_1^2, \frac{1}{r_1}).
\]

Thus, we have

\[
\delta^3|\beta_1|^\frac{1}{2} \leq r_1 \quad \text{for } r_1 \leq 1, \quad \delta^3|\beta_1|^{\frac{1}{2}} \leq 1 \quad \text{for } r_1 \geq 1,
\]

which in particular give $\delta^2|\beta_1|^{\frac{3}{4}} \leq 1$. Also we have $0 < \delta \leq \min(r_1, \frac{1}{r_1})$. Hence, it suffices to show that

\[
\|u\| \leq \delta^2\|L_1 u\|.
\]

Let us choose $r_- \in (r_1 - \delta, r_1)$ and $r_+ \in (r_1, r_1 + \delta)$ so that

\[
|u'(r_-)|^2 + |u'(r_+)|^2 \leq \frac{\|u\|^2}{\delta}.
\]

We get by integration by parts that

\[
\begin{align*}
\text{Re}(L_1 u, \text{isgn}(\beta_1)(\chi_{(0,r_-)} - \chi_{(r_+,+\infty)}))u)
& = \text{Re}(-\partial_t^2 u + i\beta_1(\sigma - \nu_1)u, \text{isgn}(\beta_1)(\chi_{(0,r_-)} - \chi_{(r_+,+\infty)}))u) \\
& = \text{Re} \left( \int_0^{r_-} (-\text{isgn}(\beta_1)|\partial_t u|^2 + |\beta_1|(\sigma - \nu_1)|u|^2)dr + \text{isgn}(\beta_1)(u'\text{W})(r_-) \right) \\
& \quad + \text{Re} \left( \int_{r_+}^{+\infty} (\text{isgn}(\beta_1)|\partial_t u|^2 + |\beta_1|(\nu_1 - \sigma)|u|^2)dr + \text{isgn}(\beta_1)(u'\text{W})(r_+) \right) \\
& \geq \int_0^{r_-} |\beta_1|(\sigma - \nu_1)|u|^2dr + \int_{r_+}^{+\infty} |\beta_1|(\nu_1 - \sigma)|u|^2dr - |(u'\text{W})(r_-)| - |(u'\text{W})(r_+)|. \\
\end{align*}
\]

Due to $0 < \delta \leq \min(r_1, \frac{1}{r_1})$, $0 < r_1 - \delta < r_1 + \delta \leq 2r_1$. Then we get by Lemma 7.1 that

\[
\begin{align*}
\sigma(r) - \nu_1 & \geq \sigma(r_1 - \delta) - \sigma(r_1) \gtrsim \delta|\sigma'(r_1)| \quad 0 < r < r_1 - \delta, \\
\nu_1 - \sigma(r) & \geq \sigma(r_1) - \sigma(r_1 + \delta) \gtrsim \delta|\sigma'(r_1)| \quad r > r_1 + \delta,
\end{align*}
\]

from which and (4.14), we infer that

\[
\text{Re}(L_1 u, \text{isgn}(\beta_1)(\chi_{(0,r_-)} - \chi_{(r_+,+\infty)}))u).
\]
\[
    \geq \int_0^{r_-} |\beta_1(\sigma - \nu_1)|u|^2 dr + \int_{r_+}^{+\infty} |\beta_1(\nu_1 - \sigma)|u|^2 dr - |(u'\pi)(r_-)| - |(u'\pi)(r_+)|
\]
\[
    \geq C^{-1}\beta_1\delta\sigma'(r_1)\|u\|_{L^2(\mathbb{R}_+ \setminus (r_1-\delta, r_1+\delta))}^2 - \frac{2}{\delta^2} \|u'\|_{L^2} \|u\|_{L^\infty}.
\]

Thanks to \(\sigma'(r) = \frac{2}{r}(e^{-\frac{r^2}{2}} - \frac{1}{r}e^{-\frac{r^2}{4}})\), we have \(|\sigma'(r)| \sim \frac{1}{r}(r \to \infty)\) and \(|\sigma'(r)| \sim r(r \to 0)\). Thus, \(|\sigma'(r)| \sim \min(r, \frac{1}{r})\). Recall that \(\delta^3|\beta_1|\min(r_1, r_1^{-3}) = 1\). Then \(|\beta_1\delta^3\sigma'(r_1)| \sim 1\). Thus, we obtain
\[
    \|u\|_{L^2(\mathbb{R}_+ \setminus (r_1-\delta, r_1+\delta))}^2 \lesssim \delta^2 \|u\|\|L_1 u\|_{L^2} + \delta^2 \|u'\|\|u\|_{L^\infty}.
\]

On the other hand, it is obvious that
\[
    \|u'\|^2 \leq \|u\|\|L_1 u\|_{L^2}, \quad \|u\|_{L^\infty} \leq \|u\|_{L^2}\|u'\|\frac{3}{2}
\]

Consequently, we deduce that
\[
    \|u\|^2 = \|u\|_{L^2(\mathbb{R}_+ \setminus (r_1-\delta, r_1+\delta))}^2 + \|u\|_{L^2(\mathbb{R}_+ \setminus (r_1-\delta, r_1+\delta)}^2
\]
\[
    \lesssim \delta^2 \|u\|\|L_1 u\| + \delta^2 \|u'\|\|u\|_{L^\infty} + \delta \|u\|_{L^\infty}^2
\]
\[
    \lesssim \delta^2 \|u\|\|L_1 u\| + \delta^2 \|u'\|^2 + \delta \|u\|_{L^\infty}^2
\]
\[
    \leq \delta^2 \|u\|\|L_1 u\| + \delta^2 \|u\|\|L_1 u\| + \delta \|u'\|\|u\|
\]
\[
    \lesssim \|u\|((\delta^2\|L_1 u\|) + \|u\|\frac{3}{2}((\delta^2\|L_1 u\|))\frac{3}{2},
\]

which implies (4.13). \(\square\)

4.4. **Sharpness of pseudospectral bound.** Finally, let us prove the sharpness of the pseudospectral bound of \(\tilde{H}_1\). That is, there exist \(\lambda \in \mathbb{R}\) and \(v \in \{r^{\frac{4}{3}}g(r)\}^\perp \cap D_1\), such that
\[
    \|\tilde{H}_1 v\| \leq C|\beta_1|\|v\|.
\]
Take \(\lambda \in \mathbb{R}\) so that \(|\beta_1| = r_1^\delta \geq 1\). We take \(u(r) = \eta(r_1(r - r_1))\), where \(\eta(r) = r^2(r - 1)^2\) for \(0 < r < 1, \eta(r) = 0\) for \(r(r - 1) \geq 0\). Then we have
\[
    \|u\| = r_1^{-\frac{3}{4}}\|\eta\|, \quad \|\partial_\nu^2 u\| = r_1^\frac{3}{4}\|\partial_\nu^2 \eta\| \leq Cr_1^2\|u\|.
\]

By Lemma 7.1, we have
\[
    |\beta_1(\sigma(r) - \sigma(r_1))| \sim |\beta_1\sigma'(r_1)||r - r_1| \leq C\frac{|\beta_1|}{r_1} \leq Cr_1^2, \quad \text{for } |r - r_1| \leq \frac{1}{r_1},
\]
and we also have that for \(0 < r - r_1 < \frac{1}{r_1}\),
\[
    \left|\left(\frac{3}{4}r^2 - \frac{r^2}{16} + \frac{1}{2} + f\right)u\right| \leq Cr_1^2|u|.
\]
Thus, we can conclude that
\[
    \|L_1 u\| \leq Cr_1^2\|u\|,
\]
which along with Lemma 4.2 gives
\[
    \|\tilde{H}_1 T^* u\| = \|T\tilde{H}_1 T^* u\| = \|L_1 u\| \leq C|\beta_1|\|T^* u\| = C|\beta_1|\|T^* u\|.
\]
This gives (4.15) by taking \(v = T^* u\).
5. Resolvent estimate of $\tilde{H}_k, k \geq 2$

In this section, we will prove the following resolvent estimate for $\tilde{H}_k, k \geq 2$.

**Theorem 5.1.** Let $k \geq 2$. For any $\lambda \in \mathbb{R}$ and $w \in D$, we have
\[
\|\tilde{H}_k w\| \geq |\beta_k|^{\frac{1}{2}}|w|.
\]

5.1. Coercive estimates of $\tilde{A}_k$ and $\tilde{B}_k$.

**Lemma 5.2.** For any $|k| \geq 1$ and $w \in L^2(\mathbb{R}_+; \text{dr})$, we have
\[
\langle (I - \tilde{B}_k)w, w \rangle \geq \int_0^{+\infty} (1 - \sigma(r))|w|^2\text{dr},
\]
\[
\langle \tilde{B}_k w, w \rangle \geq (1 - \frac{1}{|k|}) \int_0^{+\infty} \sigma(r)|w|^2\text{dr}.
\]

**Proof.** Let us first prove that the operator $g\tilde{K}_k[g]$ is nonnegative. For this, we write
\[
\tilde{K}_k[w](r) = \frac{1}{2|k|} \int_0^{+\infty} \min\left(\frac{r}{s}, \frac{s}{r}\right)^{|k|} \frac{1}{2} w(s)ds
\]
\[
= \frac{1}{2|k|} \int_0^{r} \left(\frac{1}{2} - |k|\right)^{|k|/2} w(s)ds + \frac{1}{2|k|} \int_r^{+\infty} \left(\frac{r}{s}\right)^{|k|-1/2} w(s)ds.
\]

Then we find that
\[
(\tilde{K}_k[w](r))' = \frac{1}{2|k|} \int_0^{r} \left(\frac{1}{2} - |k|\right)^{|k|/2} w(s)ds + \frac{1}{2|k|} \int_r^{+\infty} \left(\frac{r}{s}\right)^{|k|-1/2} w(s)ds,
\]
\[
(\tilde{K}_k[w](r))'' = \frac{k^2 - \frac{1}{2}}{r^2} \tilde{K}_k[w](r) - w(r).
\]

In particular, we find that
\[
(\frac{1}{2|k|} + \frac{k^2 - \frac{1}{2}}{r^2}) \tilde{K}_k[w](r) = w(r).
\]

Using the following pointwise estimates of $\tilde{K}_k[w](r)$
\[
|\tilde{K}_k[w](r)| \leq \frac{1}{2|k|} \int_0^{+\infty} \min(r, s)|w(s)|ds \leq \frac{1}{2|k|} \min \left(\|w\|_{L^1}, \|rw\|_{L^1}\right),
\]
\[
|\partial_r(\tilde{K}_k[w](r))| \leq \frac{1}{r} \min\left(\frac{r}{s}, \frac{s}{r}\right) |w(s)|ds \leq \min \left(\|w\|_{L^1}, \frac{1}{r} \|rw\|_{L^1}\right),
\]

we infer that
\[
\left.\tilde{K}_k[w](\tilde{K}_k[w])'\right|_{r=0, +\infty} = 0.
\]

Then we get by using (5.2) and integration by parts that
\[
\langle g\tilde{K}_k[gw], w \rangle = \langle \tilde{K}_k[gw], gw \rangle = \langle \tilde{K}_k[gw], (-\partial_r^2 + \frac{k^2 - \frac{1}{2}}{r^2})gw \rangle
\]
\[
= \|\partial_r(\tilde{K}_k[gw])\|^2 + \left(\frac{k^2 - \frac{1}{2}}{r}\right)\|\tilde{K}_k[gw]\|_{r}^2 \geq 0.
\]

Next we give a upper bound for $g\tilde{K}_k[g]$.
\[
\left|\int_0^{+\infty} g\tilde{K}_k[gw](r)dr\right| \leq \frac{1}{2|k|} \int_0^{+\infty} \int_0^{+\infty} \min\left(\frac{r}{s}, \frac{s}{r}\right)^{|k|} (rs)^{\frac{1}{2}} g(r)w(s)|g(s)w(r)|dsdr
\]
The proof is completed. □

Lemma 5.3. Let

which gives our result.

For the case of \( \nu \) which gives

As a consequence, we deduce that

\[
\langle (1 - \bar{B}_k)w, w \rangle = \langle (1 - \sigma)w + g\bar{K}_k[gw], w \rangle \geq \langle (1 - \sigma)w, w \rangle,
\]

\[
\langle \bar{B}_k w, w \rangle = \langle \sigma w, w \rangle - \langle g\bar{K}_k[gw], w \rangle \geq (1 - \frac{1}{|k|})\langle \sigma w, w \rangle.
\]

The proof is completed. □

The following lemma gives a sharper lower bound of \( \tilde{A}_k \) than Lemma 4.5.

Lemma 5.3. Let \( k \geq 2 \). Then for any \( w \in D \), we have

\[
\langle \tilde{A}_k w, w \rangle \geq \langle \left( \frac{k^2}{r^2} + r^2 \right) w, w \rangle.
\]

Proof. For \( |k| \geq 2 \), we have

\[
\tilde{A}_k \geq \frac{k^2}{r^2} - \frac{1}{4} + \frac{r^2}{16} - \frac{1}{2}
\]

\[
= \left( \frac{2}{3} \cdot \frac{k^2}{r^2} - \frac{1}{4} + \frac{3}{32} \cdot \frac{r^2}{k^2 - \frac{1}{4}} - \frac{1}{2} \right) + \frac{1}{3} \cdot \frac{k^2}{r^2} - \frac{1}{4} + \left( \frac{1}{16} - \frac{3}{32} \cdot \frac{1}{k^2 - \frac{1}{4}} \right) r^2
\]

\[
\geq \frac{1}{3} \cdot \frac{k^2}{r^2} - \frac{1}{4} + \left( \frac{1}{16} - \frac{3}{32} \cdot \frac{1}{k^2 - \frac{1}{4}} \right) r^2 \sim \frac{k^2}{r^2} + r^2,
\]

which gives our result. □

5.2. Resolvent estimate for \( \nu_k \geq 1 \) or \( \nu_k \leq 0 \). In this subsection, we prove Theorem 5.1 for the case of \( \nu_k \geq 1 \) or \( \nu_k \leq 0 \).

First of all, for \( \nu_k \geq 1 \), we infer from Lemma 5.2 that

\[
|\langle \tilde{H}_k w, w \rangle| \sim |\langle \tilde{A}_k w, w \rangle + |\beta_k|\langle (\nu_k - \bar{B}_k)w, w \rangle|
\]

\[
\geq |\langle \tilde{A}_k w, w \rangle + |\beta_k|\langle (1 - \bar{B}_k)w, w \rangle|
\]

\[
\gtrsim |\langle \left( \frac{k^2}{r^2} + r^2 \right) w, w \rangle + |\beta_k|\langle (1 - \sigma(r))w, w \rangle|.
\]

Thanks to \( 1 - \sigma(r) = 1 - \frac{k^2}{r^2} - \frac{1}{2} \sim r^2 (r \to 0) \) and \( \lim_{r \to \infty} 1 - \sigma(r) = 1 \), we get

\[
\int_0^1 \left( \frac{k^2}{r^2} + |\beta_k| |1 - \sigma(r)| \right) |w(r)|^2 dr \gtrsim \int_0^1 \left( \frac{k^2}{r^2} + |\beta_k| r^2 \right) |w(r)|^2 dr \gtrsim \int_0^1 |\beta_k|^2 |w(r)|^2 dr,
\]

\[
\int_1^{+\infty} \left( \frac{k^2}{r^2} + |\beta_k| (1 - \sigma(r)) \right) |w(r)|^2 dr \gtrsim \int_1^{+\infty} (1 + |\beta_k|) |w(r)|^2 dr \gtrsim \int_1^{+\infty} |\beta_k|^2 |w(r)|^2 dr,
\]
which yield that for $\nu_k \geq 1$,

$$
(5.3) \quad |\langle \tilde{H}_k w, w \rangle| \gtrsim \left( \frac{1}{r^2} + r^2 \right) w, w + |\beta_k| ((1 - \sigma(r)) w, w) \gtrsim |\beta_k| \frac{1}{2} \|w\|^2
$$

For $\nu_k \leq 0$, we infer from Lemma 5.2 that

$$
|\langle \tilde{H}_k w, w \rangle| \sim (A_k w, w) + |\beta_k| \langle (B_k - \nu_k) w, w \rangle \\
\gtrsim (A_k w, w) + |\beta_k| \langle B_k w, w \rangle \\
\gtrsim \left( \frac{1}{r^2} + r^2 \right) w, w + |\beta_k| \langle \sigma(r) w, w \rangle.
$$

Thanks to $\lim_{r \to 0} \sigma(r) = 1$ and $\sigma(r) \sim \frac{1}{r^2} (r \to \infty)$, we get

$$
\int_0^1 \left[ \frac{1}{r^2} + r^2 + |\beta_k| \sigma(r) \right] |w(r)|^2 dr \gtrsim \int_0^1 (1 + |\beta_k|) |w(r)|^2 dr \gtrsim \int_0^1 |\beta_k| \frac{1}{2} \|w(r)\|^2 dr,
$$

$$
\int_1^{+\infty} (r^2 + |\beta_k| \sigma(r)) |w(r)|^2 dr \gtrsim \int_1^{+\infty} (r^2 + \frac{1}{r^2} |\beta_k|) |w(r)|^2 dr \gtrsim \int_1^{+\infty} |\beta_k| \frac{1}{2} \|w(r)\|^2 dr,
$$

which show that for $\nu_k \leq 0$

$$
(5.4) \quad |\langle \tilde{H}_k w, w \rangle| \gtrsim \left( \frac{1}{r^2} + r^2 \right) w, w + |\beta_k| \langle \sigma(r) w, w \rangle \gtrsim |\beta_k| \frac{1}{2} \|w\|^2.
$$

5.3. **Resolvent estimate for $0 < \nu_k < 1$**. In this subsection, we prove Theorem 5.1 for the case of $0 < \nu_k < 1$.

Let $\nu_k = \sigma(r_k)$ for some $r_k > 0$. We again divide the proof into two cases:

$$
|\beta_k| \leq \max \left( \frac{|k|^3}{r_k^4}, |k|^3, r_k^6 \right) \quad \text{and} \quad |\beta_k| \geq \max \left( \frac{|k|^3}{r_k^4}, |k|^3, r_k^6 \right).
$$

5.3.1. **Case 1**. $|\beta_k| \leq \max \left( \frac{|k|^3}{r_k^4}, |k|^3, r_k^6 \right)$. By Lemma 5.2 it suffices to prove the following lemma.

**Lemma 5.4.** If $|\beta_k| \leq \max \left( \frac{|k|^3}{r_k^4}, |k|^3, r_k^6 \right)$, then we have

$$
\|\tilde{H}_k w\| \gtrsim |\beta_k| \frac{1}{2} \|w\|.
$$

**Proof.** If $|\beta_k| \leq |k|^3$, then $k^2 + r^2 \geq |k| \geq |\beta_k| \frac{1}{2}$. Thus,

$$
|\langle \tilde{H}_k w, w \rangle| \gtrsim \langle (k^2 \frac{1}{r^2} + r^2) w, w \rangle \geq |\beta_k| \frac{1}{2} \|w\|^2.
$$

If $k^3 \leq |\beta_k| \leq \max \left( \frac{|k|^3}{r_k^4}, r_k^6 \right)$, we only need to check the following cases

$$
r_k \leq 1, \quad |k|^3 \leq |\beta_k| \leq \frac{|k|^3}{r_k^4} \implies \|\tilde{H}_k w\| \gtrsim |\beta_k| \frac{1}{2} \|w\|,
$$

$$
r_k \geq \sqrt{k}, \quad |k|^3 \leq |\beta_k| \leq r_k^4 \implies \|\tilde{H}_k w\| \gtrsim |\beta_k| \frac{1}{2} \|w\|,
$$

$$
r_k \geq \sqrt{k}, \quad r_k^4 \leq |\beta_k| \leq r_k^6 \implies \|\tilde{H}_k w\| \gtrsim |\beta_k| \frac{1}{2} \|w\|,
$$

which can be deduced from the following fact

$$
|\langle \tilde{H}_k w, w \rangle| \gtrsim \langle (r^2 + \frac{k^2}{r^2}) w, w \rangle + |\beta_k| \max(\langle (\nu_k - \sigma(r)) w, w \rangle, \langle (\sigma(r)/2 - \nu_k) w, w \rangle, 0)
$$

and Lemma 5.3.
5.3.2. Case 2. $|\beta_k| \geq \max \left( \frac{|k|^3}{r_k}, |k|^3, r_k^6 \right)$. Let us introduce the operator

\begin{equation}
(5.5) \quad \overline{K}^{(r_k)}(r) = \int_0^{r_k} \overline{K}^{(r_k)}(r, s)f(s)ds,
\end{equation}

where for $0 \leq r, s \leq r_k$,

\[ \overline{K}^{(r_k)}(r, s) = \frac{1}{2|k|} \min \left( \frac{r}{s}, \frac{s}{r} \right)^{|k|(r, s)} - \frac{1}{2|k|} \left( \frac{r^2}{s^2} |k|(r, s) \right) \geq 0. \]

Let $u(r) = \overline{K}^{(r_k)}[w](r)$. Then $u \in H^1_0(0, r_k)$ is the unique solution to

\[ \left( -\partial_r^2 + \frac{k^2}{r^2} - \frac{1}{r^2} \right) u = w \quad \text{in} \quad (0, r_k). \]

**Lemma 5.5.** It holds that

\[ \text{Re} \int_0^{r_k} g(r)\overline{K}^{(r_k)}[gw](r)f(r)dr \leq \frac{2}{|k| + 1} \int_0^{+\infty} (\sigma(s) - \nu_k)|w(s)|^2ds. \]

**Proof.** As $(r^3\sigma'(r))' = -r^3g(r)^2$ and $g^2$ is decreasing, we have $-r^3\sigma'(r) \geq r^4g^2(r)/4$, and

\[ \left( -\partial_r^2 + \frac{k^2}{r^2} - \frac{1}{r^2} \right) \left( r^{\frac{|k|}{2}}(\sigma(r) - \nu_k) \right) = r^{|k|+\frac{3}{2}}(-\partial_r^2\sigma - (2|k|+1)r^{-1}\partial_r\sigma) \]

\[ = r^{|k|+\frac{3}{2}} \left( g^2 - (2|k|-2)(2r^{-1}\partial_r\sigma) \right) \]

\[ \geq r^{|k|+\frac{3}{2}} \left( g^2 + \frac{|k|}{2}g^2 \right) \]

\[ = r^{|k|+\frac{3}{2}} \frac{|k|+1}{2}g^2, \]

which implies that

\[ r^{\frac{|k|}{2}}(\sigma - \nu_k) = \overline{K}^{(r_k)} \left[ r^{\frac{|k|}{2}}(g^2 - (2|k|-2)\partial_r\sigma) \right] \]

\[ \geq \frac{|k|+1}{2} \overline{K}^{(r_k)} \left[ r^{\frac{|k|}{2}}g^2 \right]. \]

Therefore, we obtain

\[ \text{Re} \int_0^{r_k} g\overline{K}^{(r_k)}[gw](r)\widehat{v}(r)dr \]

\[ \leq \int_0^{r_k} \int_0^{r_k} \overline{K}^{(r_k)}(r, s)|g(r)|w(s)||g(s)|w(r)|drds \]

\[ \leq \frac{1}{2} \int_0^{r_k} \int_0^{r_k} \overline{K}^{(r_k)}(r, s) \left( \left( \frac{r}{s} \right)^{|k|+\frac{1}{2}} g^2(r)|w(s)|^2 + \left( \frac{s}{r} \right)^{|k|+\frac{1}{2}} g^2(s)|f(r)|^2 \right) drds \]

\[ = \int_0^{r_k} \overline{K}^{(r_k)} \left[ r^{\frac{|k|}{2}}g^2 \right] \int_0^{r_k} \frac{|w(s)|^2}{s^{|k|+\frac{1}{2}}} ds \leq \frac{2}{|k| + 1} \int_0^{r_k} (\sigma(s) - \nu_k)|w(s)|^2ds. \]

This completes the proof. \qed

To proceed, we introduce the following decomposition: let $\psi = \overline{K}[gw]$ and decompose

\begin{equation}
(5.6) \quad \psi(r) = \psi_1(r) + \psi_2(r),
\end{equation}
where \( \psi_2(r) \) is given by
\[
\psi_2(r) = \begin{cases} 
\left( \frac{r}{r_k} \right)^{|k|+\frac{1}{2}} \psi(r_k), & 0 < r < r_k, \\
\left( \frac{r}{r_k} \right)^{-|k|+\frac{1}{2}} \psi(r_k), & r > r_k.
\end{cases}
\]

Then we find that \( \psi_1(r) \in H^1_0(0, r_k) \) and solves
\[
(5.7) \quad \left( -\partial_r^2 + \frac{k^2 - 1}{r^2} \right) \psi_1 = g_w, \quad r \in \mathbb{R}_+ \setminus \{r_k\}.
\]

Thus, \( \psi_1(r) = \tilde{K}_k^{(r_k)}[g_w](r) \) in \((0, r_k)\).

Let \( \delta > 0 \) be such that
\[
\delta^3 |\beta_k| \min(r_k, r_k^{-3}) = 1.
\]
Due to \( |\beta_k| \geq \max\left(\frac{k^2}{r_k^2}, |k|^3, r_k^6\right) \), we have
\[
(5.8) \quad 0 < \delta \leq \min\left(\frac{r_k}{|k|}, \frac{1}{r_k}\right),
\]
Due to \( |\sigma'(r)| \sim \min(r, \frac{1}{r^2}) \), we also have
\[
(5.9) \quad |\beta_k \delta^3 \sigma'(r_k)| \sim 1.
\]

We denote
\[
\mathcal{E}(w) = \left\| \frac{w'}{\sqrt{\beta_k \delta^2 \sigma'(r_k)}} \right\|_{L^\infty} + \left\| \tilde{H}_k w \right\|_{L^\infty} + \left\| \frac{w'w}{\beta_k \delta^2 \sigma'(r_k)} \right\|_{L^\infty} + \delta \left\| w \right\|^2_{L^\infty}
\]
\[
+ \frac{|\psi(r_k)J(r_k)|}{\min(1, r_k^2)} + \frac{g(r_k)^2 |\psi(r_k)|^2}{\delta |\sigma'(r_k)|^2} + \frac{|\psi(r_k)|^2}{r_k^3 + 1} = \mathcal{E}_1(w) + \cdots + \mathcal{E}_7(w),
\]
where
\[
J(r) = \int_0^r \left( \frac{s}{r} \right)^{|k|+\frac{3}{2}} g(s)w(s)ds - \int_r^{+\infty} \left( \frac{r}{s} \right)^{|k|+\frac{3}{2}} g(s)w(s)ds.
\]
It is easy to see that
\[
(5.10) \quad \partial_r \psi = -\frac{J}{2} + \frac{\psi}{4|k|r}.
\]

**Lemma 5.6.** It holds that for any \( w \in D \),
\[
\|w\|^2 \leq C\mathcal{E}(w).
\]

**Proof.** Recall that
\[
\tilde{H}_k w = -\partial_r^2 w + \left( \frac{k^2 - 1/4}{r^2} + \frac{r^2}{16} - \frac{1}{2} \right) w + i\beta_k((\sigma - v_k)w - g\psi).
\]
Then we get by integration by parts that
\[
\text{Re}(\tilde{H}_k w, \text{isgn}(\beta_k)(\chi_{(0, r_k)} - \chi_{(r_k, +\infty)}))w)
\]
\[
= \text{Re}\left( -\partial_r^2 w + i\beta_k((\sigma - v_k)w - g\psi), \text{isgn}(\beta_k)(\chi_{(0, r_k)} - \chi_{(r_k, +\infty)}))w \right)
\]
\[
= \text{Re}\left( \int_0^{r_k} (-\text{isgn}(\beta_k)|\partial_r w|^2 + |\beta_k|(\sigma - v_k)|w|^2 - |\beta_k|g\psi w)dr + \text{isgn}(\beta_k)\omega w(r_k) \right)
\]
Then we conclude that

\[
+ \text{Re} \left( \int_{r_k}^{+\infty} (\text{sgn}(\beta_k)|\partial_r w|^2 + |\beta_k|(|\nu_k - \sigma)|w|^2 + |\beta_k|g\psi \overline{\omega}) dr + i\text{sgn}(\beta_k)u\overline{\omega}(r_k) \right)
\]

\[
\geq |\beta_k| \int_{0}^{+\infty} |\sigma - \nu_k||\omega|^2 dr - |\beta_k|\text{Re} \left( \int_{r_k}^{+\infty} g\psi \overline{\omega} dr - \int_{r_k}^{+\infty} g\psi \overline{\omega} dr \right) - 2|w\overline{\omega}(r_k)|.
\]

Using (5.6), we write

\[
\int_{0}^{r_k} g\psi \overline{\omega} dr - \int_{r_k}^{+\infty} g\psi \overline{\omega} dr
\]

\[
= \int_{0}^{r_k} g\psi \overline{\omega} dr - \int_{r_k}^{+\infty} g\psi \overline{\omega} dr + \int_{0}^{r_k} g\psi \overline{\omega} dr - \int_{r_k}^{+\infty} g\psi \overline{\omega} dr
\]

\[
= \int_{0}^{r_k} g\psi \overline{\omega} dr - \int_{r_k}^{+\infty} \psi_1(-\partial^2_r + (k^2 - 1/4)r^{-2})\psi_1 dr
\]

\[
+ \psi(r_k) \int_{0}^{r_k} \left( \frac{r}{r_k} \right)^{|k|+\frac{1}{2}} g\psi \overline{\omega} dr - \psi(r_k) \int_{r_k}^{+\infty} \left( \frac{r}{r_k} \right)^{-|k|+\frac{1}{2}} g\psi \overline{\omega} dr
\]

\[
= \int_{r_k}^{r_k} g\psi \overline{\omega} dr - \int_{r_k}^{+\infty} \left( |\partial_r \psi|^2 + (k^2 - 1/4)r^{-2} |\psi|^2 \right) dr + \psi(r_k)J(r_k).
\]

By (5.7), we have

\[
\int_{0}^{r_k} g\psi \overline{\omega} dr = \int_{0}^{r_k} \psi_1(-\partial^2_r + \frac{k^2 - 1/4}{r^2})\psi_1 dr = \int_{0}^{r_k} \left( |\partial_r \psi|^2 + \frac{k^2 - 1/4}{r^2} |\psi|^2 \right) dr.
\]

We get by Lemma 5.5 that

\[
\int_{0}^{r_k} g\psi \overline{\omega} dr = \int_{0}^{r_k} g\overline{\psi} (r) |\psi| |\omega| dr \leq \frac{2}{|k| + 1} \int_{0}^{r_k} (\sigma(s) - \nu_k)|w(s)|^2 ds.
\]

Then we conclude that

\[
\text{Re}(\overline{\mathcal{H}_k}w, \text{sgn}(\beta_k)(\chi_{(0,r_k)} - \chi_{(r_k, +\infty)})w)
\]

\[
\geq |\beta_k| \left( \int_{0}^{r_k} \overline{\omega} |\partial_r \psi|^2 + \frac{k^2 - 1/4}{r^2} |\psi|^2 dr \right)
\]

\[
- |\beta_k|\psi(r_k)J(r_k) - 2|w\overline{\omega}(r_k)|
\]

\[
= |\beta_k| \left( \int_{0}^{r_k} \overline{\omega} |\partial_r \psi|^2 + \frac{k^2 - 1/4}{r^2} |\psi|^2 dr \right) - |\beta_k|\psi(r_k)J(r_k) - 2|w\overline{\omega}(r_k)|
\]

\[
= |\beta_k| \left( \frac{|k| - 1}{2} \int_{0}^{r_k} \overline{\omega} |\partial_r \psi|^2 + \frac{k^2 - 1/4}{r^2} |\psi|^2 dr \right) + \int_{r_k}^{+\infty} \left( |\partial_r \psi|^2 + \frac{k^2 - 1/4}{r^2} |\psi|^2 dr \right)
\]

\[
- |\beta_k|\psi(r_k)J(r_k) - 2|w\overline{\omega}(r_k)|
\]

\[
\geq |\beta_k| \left( \frac{A_1}{2} - |\beta_k|\psi(r_k)J(r_k) \right) - 2|w\overline{\omega}(r_k)|,
\]

where

\[
A_1 = \int_{0}^{+\infty} \left( |\partial_r \psi|^2 + (k^2 - 1/4)r^{-2} |\psi|^2 \right) dr.
\]

This shows that

\[
(5.11) \quad A_1 \leq 2|\psi(r_k)J(r_k)| + \frac{1}{|\beta_k|} \left( 4|w\overline{\omega}(r_k)| + 2||\overline{\mathcal{H}_k}w|| \right).
\]
Similarly, we have
\[
\text{Re}(\tilde{\mathcal{H}}_k w, \text{isgn}(\beta_k) \frac{\chi_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)}}{\sigma - \nu_k}) w)
\]
\[
= \text{Re}(-\delta_k^2 w + i\beta_k((\sigma - \nu_k) w - g\psi), \text{isgn}(\beta_k) \frac{\chi_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)}}{\sigma - \nu_k}) w)
\]
\[
= \text{Re} \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} \left(-i \frac{\text{sgn}(\beta_k) |w|^2}{\sigma - \nu_k} + i \frac{\text{sgn}(\beta_k) w' \bar{w} \sigma'}{\sigma - \nu_k} + |\beta_k||w|^2 - \frac{|\beta_k| |g\psi w|}{\sigma - \nu_k} \right) \, dr
\]
\[
+ \text{Re} \left( \text{isgn}(\beta_k) \frac{w' \bar{w}}{\sigma - \nu_k} (r_k - \delta) - \text{isgn}(\beta_k) \frac{w' \bar{w}}{\sigma - \nu_k} (r_k + \delta) \right)
\]
\[
\geq -\|w'\| \|w\|_{L^\infty} \left( \frac{\sigma'}{2} \right) + |\beta_k| \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} |w|^2 \, dr
\]
\[
- \frac{|\beta_k|}{2} \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} \left( |w|^2 + \frac{g^2|\psi|^2}{(\sigma - \nu_k)^2} \right) \, dr - \|w'\|_{L^\infty} \|w\|_{L^\infty} \left( \frac{1}{\sigma(r_k - \delta) - \nu_k} + \frac{1}{\nu_k - \sigma(r_k + \delta)} \right)
\]
\[
= \frac{|\beta_k|}{2} \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} \left( |w|^2 - \frac{g^2|\psi|^2}{(\sigma - \nu_k)^2} \right) \, dr - \|w'\|_{L^2} \|w\|_{L^\infty} \left( \frac{\sigma'}{2} \right) \left( \frac{1}{\sigma(r_k - \delta) - \nu_k} + \frac{1}{\nu_k - \sigma(r_k + \delta)} \right)
\]
which gives
\[
\|w\|^2_{L^2(\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta))} \leq \frac{2}{|\beta_k|^2} \|\tilde{\mathcal{H}}_k w\| \|w\| \left( \frac{1}{\sigma - \nu_k} \right) \|w\|_{L^\infty(\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta))}
\]
\[
+ \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} \frac{g^2|\psi|^2}{(\sigma - \nu_k)^2} \, dr + \frac{2}{|\beta_k|^2} \|w'\| \|w\|_{L^\infty} \left( \frac{\sigma'}{2} \right) \left( \frac{1}{\sigma - \nu_k} \right) \|w\|_{L^\infty(\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta))}
\]
\[
+ \frac{2}{|\beta_k|^2} \|w'\|_{L^\infty} \left( \frac{1}{\sigma(r_k - \delta) - \nu_k} + \frac{1}{\nu_k - \sigma(r_k + \delta)} \right).
\]

By Lemma \[\text{Lemma 1.1}\] we have
\[
\left\| \frac{1}{\sigma - \nu_k} \right\|_{L^\infty(\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta))} \leq \frac{C}{|\sigma'(r_k)| \delta},
\]
and
\[
\left\| \frac{\sigma'}{(\sigma - \nu_k)^2} \right\|_{L^2(\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta))} \leq \left\| \frac{\sigma'}{\sigma - \nu_k} \right\|_{L^\infty(\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta))} \left\| \frac{\sigma'}{(\sigma - \nu_k)^3} \right\|_{L^1(\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta))}
\]
\[
\leq \frac{C}{\delta} \left( \frac{1}{(\sigma(r_k - \delta) - \nu_k)^2} + \frac{1}{(\nu_k - \sigma(r_k + \delta))^2} \right)
\]
\[
\leq \frac{C}{\delta^3(\sigma'(r_k) \delta)^2}.
\]
Thus, we obtain
\[
\|w\| \leq \|w\|^2_{L^2(\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta))} + \|w\|^2_{L^2(\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta))}
\]
\[
\leq \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} \frac{g^2|\psi|^2}{(\sigma - \nu_k)^2} \, dr + \left[ C \|w'\| \|w\|_{L^\infty} \left( \frac{1}{|\beta_k|^2} \frac{\delta^2(\sigma'(r_k))}{\sigma'(r_k) \delta^2} \right) \right].
\]
by Hardy's inequality that
\[ \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} \frac{g^2|\psi|^2}{(\sigma - \nu_k)^2} dr \leq 2 \int_{\mathbb{R}^+} \frac{g^2|\psi_1|^2}{(\sigma - \nu_k)^2} dr + 2 \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} \frac{g^2|\psi_2|^2}{(\sigma - \nu_k)^2} dr.
\]

It remains to estimate the first term, which is bounded by
\[ \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} \frac{g^2|\psi_1|^2}{(\sigma - \nu_k)^2} dr \leq 2 \int_{\mathbb{R}^+} \frac{g^2|\psi_1|^2}{(\sigma - \nu_k)^2} dr + 2 \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} \frac{g^2|\psi_2|^2}{(\sigma - \nu_k)^2} dr.
\]

Let us first consider the case of \(0 < r_k < 1\). By Lemma 7.1, we have
\[ |\sigma(r) - \nu_k| \geq C^{-1}|r - r_k|\sigma(r_k) \geq C^{-1}|r - r_k|r_k, \quad 0 < r < r_k + 1,
\]
\[ |\sigma(r) - \nu_k| \geq C^{-1}, \quad r \geq r_k + 1.
\]

Due to \(\psi_1(r_k) = 0\), we get by Hardy’s inequality that
\[ \int_0^{r_k + 1} \frac{g^2|\psi_1|^2}{(\sigma - \nu_k)^2} dr \leq C \int_0^{r_k + 1} \frac{|\psi_1|^2}{r^2} dr \leq C \int_0^{r_k + 1} \frac{|\partial_r \psi_1|^2}{r^2} dr \leq \frac{CA_1}{r_k},
\]
and by Lemma 7.1,
\[ \int_{r_k + 1}^{+\infty} \frac{g^2|\psi_1|^2}{(\sigma - \nu_k)^2} dr \leq C \int_{r_k + 1}^{+\infty} \frac{g^2|\psi_1|^2}{r^2} dr \leq C \int_{r_k + 1}^{+\infty} \frac{|\psi_1|^2}{r^2} dr \leq CA_1.
\]

Thanks to \(0 < \delta < r_k\) and \(|\psi_2(r)| \leq |\psi(r_k)|\), we have
\[ \int_{(0, r_k + 1) \setminus (r_k - \delta, r_k + \delta)} \frac{g^2|\psi_2|^2}{(\sigma - \nu_k)^2} dr \leq C \int_{(0, r_k + 1) \setminus (r_k - \delta, r_k + \delta)} \frac{|\psi(r_k)|^2}{r^2} dr \leq C \frac{|\psi(r_k)|^2}{r_k^2 \delta},
\]
and
\[ \int_{r_k + 1}^{+\infty} \frac{g^2|\psi_2|^2}{(\sigma - \nu_k)^2} dr \leq C |\psi(r_k)|^2 \int_{r_k + 1}^{+\infty} \frac{g^2(r)}{r^2} dr \leq C |\psi(r_k)|^2.
\]

Therefore, we obtain
\[ \int_{\mathbb{R}^+ \setminus (r_k - \delta, r_k + \delta)} \frac{g^2|\psi|^2}{(\sigma - \nu_k)^2} dr \leq \frac{CA_1}{r_k} + \frac{C|\psi(r_k)|^2}{r_k^2 \delta} \leq CE(w),
\]
where we used (5.11) and the facts that \(|\sigma'(r_k)| \sim r_k, g(r_k) \geq C^{-1}\) and \(|\delta \sigma'(r_k)| \leq C r_k^2\).

Next we consider the case of \(r_k \geq 1\). By Lemma 7.1, we have
\[ |\sigma(r) - \nu_k| \geq C^{-1}|r - r_k|\sigma(r_k) \geq C^{-1}|r - r_k|r_k^{-3}, \quad |r - r_k| < 1,
\]
\[ |\sigma(r) - \nu_k| \geq C^{-1}(1 + r)^{-1}, \quad |r - r_k| \geq 1/r_k,
\]
and \(g(r) \leq C g(r_k)\) for \(|r - r_k| < 1/r_k\). Thanks to \(\psi_1(r_k) = 0\) and \(g(r_k - 1)^2 r_k^6 \leq C\), we get by Hardy’s inequality that
\[ \int_{r_k - 1}^{r_k + 1} \frac{g^2|\psi_1|^2}{(\sigma - \nu_k)^2} dr \leq C \int_{r_k - 1}^{r_k + 1} \frac{g^2|\psi_1|^2 r_k^6}{r^2} dr \leq C \int_{r_k - 1}^{r_k + 1} \frac{|\partial_r \psi_1|^2}{r^2} dr \leq CA_1,
\]
and
\[ \int_{\mathbb{R}^+ \setminus B(r_k, 1)} \frac{g^2|\psi_1|^2}{(\sigma - \nu_k)^2} dr \leq C \int_{\mathbb{R}^+} \frac{g^2(r)(1 + r)^8 |\psi_1|^2}{r^2} dr \leq C \int_{\mathbb{R}^+} \frac{|\psi_1|^2}{r^2} dr \leq CA_1,
\]
where we denote \( B(a, b) = (a - b, a + b) \). Since \( 0 < \delta < 1/r_k \) and \( |\psi_2(r)| \leq |\psi(r_k)| \), we have
\[
\int_{B(r_k, 1/r_k) \setminus B(r_k, \delta)} \frac{g^2(r^2) |\psi_2(r)|^2}{(\sigma - \nu_k)^2} dr \leq C \int_{B(r_k, 1/r_k) \setminus B(r_k, \delta)} \frac{g^2(r_k) |\psi(r_k)|^2}{r - r_k} dr \leq \frac{Cg^2(r_k) |\psi(r_k)|^2}{\sigma'(r_k)^2 \delta},
\]
and due to \( |\psi_2(r)| \leq \left( \frac{r}{r_k} \right)^{\frac{\beta}{2}} |\psi(r_k)| \), we have
\[
\int_{\mathbb{R}_+ \setminus B(r_k, 1/r_k)} \frac{g^2(r^2) |\psi_2(r)|^2}{(\sigma - \nu_k)^2} dr \leq C \int_{\mathbb{R}_+} \left( \frac{r}{r_k} \right)^5 g^2(r) |\psi(r_k)|^2 (1 + r)^8 dr \leq \frac{C |\psi(r_k)|^2}{r_k^5}.
\]
Therefore, we obtain
\[
\int_{\mathbb{R}_+ \setminus B(r_k, \delta)} \frac{g^2(r^2) |\psi_2(r)|^2}{(\sigma - \nu_k)^2} dr \leq CA_1 + \frac{Cg^2(r_k) |\psi(r_k)|^2}{\sigma'(r_k)^2 \delta} + \frac{C |\psi(r_k)|^2}{r_k^2} \leq CE(w),
\]
where we used (5.11) and the facts that \( C^{-1}r_k^3 < |\sigma'(r_k)| \leq Cr_k^{-3} \) and \( |\delta \sigma'(r_k)| \leq C \).

This completes the proof of the lemma.

Now we are in a position to show that for \( |\beta_k| \geq \max \left( \frac{1}{\delta} \frac{k^2}{r_k^4}, k^3, r_k^6 \right) \),
\[
(5.12) \quad \|\tilde{H}_k w\| \geq |\beta_k| \frac{1}{2} \|w\|.
\]

By Lemma 5.6 we have
\[
\|w\| \leq CE(w) = C(\mathcal{E}_1(w) + \cdots + \mathcal{E}_7(w)).
\]
In the following, we handle each \( \mathcal{E}_i(w) \). Using the fact that
\[
(5.13) \quad |J(r)| \leq \int_0^r |w(s)| ds + \int_0^{\infty} \left( \frac{r}{s} \right)^2 |w(s)| ds \leq Cr\|w\|_{L^\infty},
\]
\[
(5.14) \quad |J(r)| \leq \int_0^{\infty} \left( \frac{r}{s} \right)^{\frac{\alpha}{2}} |gw|(s) ds \leq Cr^{-\frac{\alpha}{2}} \min(\|w\|_{L^\infty}, \|w\|),
\]
we deduce that
\[
(5.15) \quad \mathcal{E}_5(w) \leq \frac{C}{r_k^4} |\psi(r_k)| \|w\|_{L^\infty} \leq \mathcal{E}_4(w) + C \frac{1}{\delta} \frac{|\psi(r_k)|^2}{r_k^2} \leq \mathcal{E}_4(w) + C\mathcal{E}_6(w), \quad 0 < r_k \leq 1,
\]
\[
(5.16) \quad \mathcal{E}_5(w) \leq \frac{C}{r_k^{5/2}} |\psi(r_k)| \|w\| \leq C\mathcal{E}_7(w) \frac{1}{2} \|w\|, \quad r_k \geq 1.
\]
Using the fact that
\[
(5.17) \quad |\psi(r)| \leq r \int_0^r |w(s)| ds + \int_0^{\infty} \left( \frac{r}{s} \right)^2 (rs)^{\frac{\alpha}{2}} |w(s)| ds \leq Cr^2 \|w\|_{L^\infty},
\]
\[
(5.18) \quad |\psi(r)| \leq \int_0^{\infty} \left( \frac{s}{r} \right)^2 (rs)^{\frac{\alpha}{2}} |gw|(s) ds \leq Cr^{-\frac{3}{2}} \min(\|w\|_{L^\infty}, \|w\|),
\]
we deduce that
\[
(5.19) \quad \mathcal{E}_7(w) \leq Cr_k^{-8} \|w\|^2,
\]
As \( \delta < 1 \), \( |\sigma'(r_k)| < C \), we also have
\[
(5.20) \quad \mathcal{E}_7(w) \leq C\mathcal{E}_6(w) g(r_k)^{-2}.
\]
We introduce
\[
(5.21) \quad F(w) = \delta \|w\|_{L^\infty}^2 + \delta^2 \|w'\|^2 + \delta^2 \|w\| \|\tilde{H}_k w\| + \delta^4 \|\tilde{H}_k w\|^2.
\]
It is easy to see that
\begin{equation}
E_4(w) \leq F(w),
\end{equation}
and by (5.9), we have
\begin{equation}
E_1(w) + E_2(w) \leq CF(w).
\end{equation}

To proceed, we need the following $L^\infty$ estimate of $w'$ and $\psi$.

**Lemma 5.7.** It holds that
\[
\delta^3 \|w'\|^2_{L^\infty(B(r_k, \delta))} + \frac{1}{(\sigma'(r_k))^2 \delta} \|g\psi\|^2_{L^\infty(B(r_k, \delta))} \leq CF(w).
\]

**Proof.** Let
\[
u = \mathcal{H}_kw, \quad u_1 = g\psi, \quad u_2 = \left(\frac{k^2 - 1/4}{r^2} + \frac{r^2}{16} - \frac{1}{2}\right)w + i\beta_k(\sigma - \nu_k)w.
\]

Due to $0 < \delta \leq \min\left(\frac{1}{|r_k|}, \frac{1}{r_k}\right)$ and Lemma 5.1, we have
\[
\|u_2\|_{L^\infty(B(r_k, \delta))} \leq \left\|\frac{k^2 - 1/4}{r^2} + \frac{r^2}{16} - \frac{1}{2}\right\|_{L^\infty(B(r_k, \delta))} \|w\|_{L^\infty} + \|\beta_k(\sigma - \nu_k)\|_{L^\infty(B(r_k, \delta))} \|w\|_{L^\infty}
\leq C \left(\frac{k^2}{r_k^2} + \frac{r^2}{r_k^2} + |\beta_k\sigma'(r_k)|\right) \|w\|_{L^\infty}
\]
\[
\leq C(\delta^{-2} + |\beta_k\sigma'(r_k)|) \|w\|_{L^\infty} \leq C\delta^{-2} \|w\|_{L^\infty}.
\]

By (5.17), (5.13) and (5.10), we get
\[
|\partial_r\psi| \leq C\|w\|_{L^\infty},
\]
which gives
\[
|\partial_r u_1| \leq g|\partial_r\psi| + |\partial_r g||\psi| \leq g(|\partial_r\psi| + r|\psi|) \leq Cg(r)^{1/2}(r + r^3)\|w\|_{L^\infty} \leq C\sigma'(r)\|w\|_{L^\infty}.
\]

In particular, for $r, s \in B(r_k, \delta)$,
\begin{equation}
|u_1(r) - u_1(s)| \leq C\delta|\sigma'(r_k)|\|\omega\|_{L^\infty}.
\end{equation}

Choose $r_* \in (r_k - \delta, r_k)$ such that
\[
|w'(r_*)|^2 + |w'(r_* + \delta)|^2 \leq \frac{1}{\delta} \|w'||^2,
\]
which along with (5.24) gives
\[
\int_{r_*}^{r_* + \delta} (u_2 - i\beta_k u_1 - u)dr = |w'(r_* + \delta) - w'(r_*)| \leq 2\delta^{-\frac{1}{2}} \|w'||,
\]
from which and (5.25), we infer that
\[
\int_{r_*}^{r_* + \delta} u_1 dr \leq |\beta_k|^{-1}(\|u_2\|_{L^1(r_*, r_* + \delta)} + \|u\|_{L^1(r_*, r_* + \delta)} + 2\delta^{-\frac{1}{2}} \|w'||)
\leq |\beta_k|^{-1}(\delta \|u_2\|_{L^\infty(r_*, r_* + \delta)} + \delta^{\frac{1}{2}} \|u\| + 2\delta^{-\frac{1}{2}} \|w'||)
\leq |\beta_k|^{-1}(C\delta^{-1} \|w\|_{L^\infty} + \delta^{\frac{1}{2}} \|\mathcal{H}_k w\| + 2\delta^{-\frac{1}{2}} \|w'||)
For $s \in B(r_k, \delta)$, we get by (5.26) that
\[
\left| \delta u_1(s) - \int_{r_s}^{r_s+\delta} u_1 \, dr \right| \leq \int_{r_s}^{r_s+\delta} |u_1(s) - u_1(r)| \, dr \leq C \delta^2 |\sigma'(r_k)||w||L^\infty,
\]
which gives
\[
(5.27) \quad |u_1(s)| \leq C(\delta^{\frac{3}{2}} |\sigma'(r_k)| + |\delta_k|^{-1} \delta^{-\frac{3}{2}}) A^\frac{1}{2} \leq C \delta^{\frac{3}{2}} |\sigma'(r_k)||F(w)^{-\frac{1}{2}},
\]
that is,
\[
\|g\psi\|_{L^\infty(B(r_k, \delta))} = \|u_1\|_{L^\infty(B(r_k, \delta))} \leq C \delta^{\frac{3}{2}} |\sigma'(r_k)||F(w)^{-\frac{1}{2}}.
\]
Using (5.24), (5.25) and (5.27), we infer that
\[
\|w'\|_{L^\infty(B(r_k, \delta))} \leq |w'(r_s)| + \|w''\|_{L^1(B(r_k, \delta))} \leq \delta^{-\frac{1}{2}} \|w'\| + \|u_2\|_{L^1(B(r_k, \delta))} + |\delta_k| |u_1||L^1(B(r_k, \delta)) + \|u||L^1(B(r_k, \delta)) \leq \delta^{-\frac{1}{2}} \|w'\| + 2\|u_2\|_{L^\infty(B(r_k, \delta))} + 2|\delta_k| |u_1||L^\infty(B(r_k, \delta)) + 2\delta^{\frac{1}{2}} \|u|| \leq \delta^{-\frac{1}{2}} \|w'\| + C\delta^{-1} \|w\|_{L^\infty} + C|\delta_k| \delta^{\frac{1}{2}} |\sigma'(r_k)||F(w)^{-\frac{1}{2}} + 2\delta^{\frac{1}{2}} \|u|| \leq C \delta^{-\frac{1}{2}} F(w)^{-\frac{1}{2}}.
\]
This completes the proof of the lemma.

Now we infer from Lemma 5.7 that
\[
E_0(w) \leq C F(w),
\]
and by (5.9),
\[
E_3(w) \leq \frac{1}{|\delta_k| |u_1||L^\infty(B(r_k, \delta))||w||L^\infty
\]
\[
\leq C \delta^{\frac{1}{2}} \|w'\|_{L^\infty(B(r_k, \delta))}\|w\|_{L^\infty} \leq C F(w).
\]
If $0 < r_k \leq 1$, we deduce from (5.24), (5.25), (5.22), (5.16), (5.28), (5.20) that
\[
\|w\|^2 \leq C F(w).
\]
If $r_k \geq 1$, we similarly have
\[
\|w\|^2 \leq C(F(w) + E_7(w)) \leq C F(w) + E_7(w).
\]
Now if $\|w\|^2 < 2C_0 E_7(w)$, we get by (5.19) that
\[
\|w\|^2 < 2 C r_k^{-\frac{3}{2}} \|w||L^2
\]
which implies that $r_k \leq C$, thus, $g(r_k)^{-2} \leq C$. Hence,
\[
E_7(w) \leq C E_0(w) \leq CA \implies \|w\|^2 \leq C F(w)
\]
While, if $\|w\|^2 \geq 2C_0 E_7(w)$, we have
\[
\|w\|^2 \leq 2 C F(w).
\]
Thanks to $\|w\|^2_{L^\infty} \leq \|w'\|\|w\|$ and $\|w''\|^2 \leq \|w||\|H_k w\|$, we have
\[
F(w) \leq \delta^2 \|w||w|| + 2\delta^2 \|w||\|H_k w\|^2 \leq \|w||^\frac{3}{2} (\delta^2 \|H_k w\|)^{\frac{1}{2}} + 2 \|w||_{L^2} (\delta^2 \|H_k w\|) + ((\delta^2 \|H_k w\|))^2,
\]
which along with \( \|w\|^2 \leq CF(w) \) implies that
\[
\|w\| \leq C\delta^2 \|\widetilde{H}_kw\|.
\]
Due to the choice of \( \delta \), we obtain
\[
\|\widetilde{H}_kw\| \gtrsim |\beta_k|^1/2\|w\|.
\]

6. Spectral lower bound

Recall that
\[
\widetilde{H}_{\alpha,k,0} = -\partial_r^2 + \frac{k^2}{r^2} - \frac{1}{2} + \frac{r^2}{16} - \frac{1}{2} + i\beta_k\sigma(r) - i\beta_k g \tilde{K}_k[g].
\]
We know that for \( |k| = 1 \), \( \widetilde{H}_{\alpha,k,0} \) in \( \{r^2 g(r)\} \perp L^2(\mathbb{R}_+,dr) \) is isometric with \( T\widetilde{H}_{\alpha,k,0}T^{-1} = -\partial_r^2 + \frac{1}{4r^2} + \frac{r^2}{16} - \frac{1}{2} + f(r) + i\beta_k\sigma(r) \) in \( L^2(\mathbb{R}_+,dr) \). Hence, we just consider \( \widetilde{H}_{\alpha,k} \) in the form
\[
\widetilde{H}_{\alpha,k} = \left\{ \begin{array}{l}
-\partial_r^2 + \frac{3}{4r^2} + \frac{r^2}{16} - \frac{1}{2} + f(r) + i\beta_k\sigma(r), \quad |k| = 1, \\
-\partial_r^2 + \frac{k^2 - 1/4}{4r^2} + \frac{r^2}{16} - \frac{1}{2} + i\beta_k\sigma(r) - i\beta_k g \tilde{K}_k[g], \quad |k| \geq 2.
\end{array} \right.
\]
Notice that
\[
\sigma(L - \alpha\Lambda|_{\ker\Lambda}) = \bigcup_{k \in \mathbb{Z}\setminus\{0\}} \sigma(\widetilde{H}_{\alpha,k}).
\]
Then we define
\[
\Sigma(\alpha,k) = \inf \text{Re} \sigma(\widetilde{H}_{\alpha,k}), \quad \Sigma(\alpha) = \inf_{k \in \mathbb{Z}\setminus\{0\}} \Sigma(\alpha,k).
\]

Our main result is the following spectral lower bound.

**Theorem 6.1.** For any \( |k| \geq 1 \), we have
\[
\Sigma(\alpha,k) \geq C^{-1}|\beta_k|^{1/2}, \quad \Sigma(\alpha) \geq C^{-1}|\alpha|^{1/2}.
\]

Motivated by [3], we will use the complex deformation method.

6.1. Complex deformation. We introduce the group of dilations
\[
(U_\theta \omega)(r) = e^{\theta/2}\omega(e^\theta r),
\]
which are unitary operators for \( \theta \in \mathbb{R} \). We consider
\[
(6.1) \quad \widetilde{H}_{\alpha,k}^{(\theta)} = U_\theta \widetilde{H}_{\alpha,k} U_{\theta}^{-1}.
\]
Then we have
\[
\widetilde{H}_{\alpha,k}^{(\theta)} = \left\{ \begin{array}{l}
-e^{-2\theta}\partial_r^2 w + \left(\frac{3e^{-2\theta}}{4r^2} + \frac{r^2 e^{2\theta}}{16} - \frac{1}{2} + f(e^\theta)\right) + i\beta_k\sigma(e^\theta), \quad |k| = 1, \\
-e^{-2\theta}\partial_r^2 + \left(\frac{k^2 - 1/4}{4r^2 e^{2\theta}} + \frac{r^2 e^{2\theta}}{16} - \frac{1}{2}\right) + i\beta_k(\sigma(e^\theta) - e^{2\theta} g(\tilde{K}_k[g(e^\theta)])), \quad |k| \geq 2.
\end{array} \right.
\]
Now we consider the analytic continuation of \( \widetilde{H}_{\alpha,k}^{(\theta)} \). For this, we first consider the analytic continuation of the functions \( f, \sigma, g \). Let
\[
F_0(z) = e^z - z - 1, \quad F_1(z) = (1 - e^{-z})/z,
\]
Thus
\[ F_2(z) = e^{z/2}, \quad F_3(z) = \left( \frac{z^2}{F_0(z)} - 6 + 4z \right) \frac{z/2}{F_0(z)}. \]

Then \( F_0, F_1, F_2 \) are holomorphic in \( \mathbb{C} \) (0 is a removable singularity of \( F_1 \)) and \( F_3 \) is meromorphic in \( \mathbb{C} \), and we have
\[
 f(r) = F_3(r^2/4), \quad \sigma(r) = F_1(r^2/4), \quad g(r) = F_2(r^2/4).
\]

The poles of \( F_3 \) are the zeros of \( F_0 \). If \( F_0(z) = 0, \quad z = x + iy, \quad x, y \in \mathbb{R}, \quad x > 0 \), then
\[
 e^{2x} = |e^z| = |1 + z|^2 = (1 + x)^2 + y^2,
\]
\[
 y^2 = e^{2x} - (1 + x)^2 > 1 + 2x + (2x)^2/2 - (1 + x)^2 = x^2 \implies |y| > x,
\]
hence, \( F_3(z) \) is holomorphic in a neighbourhood of \( \Gamma \), which is defined as
\[
 \Gamma = \{ x + iy | x > 0, -x \leq y \leq x \} = \{ re^{i\theta} | r > 0, -\pi/4 \leq \theta \leq \pi/4 \}.
\]

Let \( F_4(z) = F_3(z) - \frac{8}{z} \). As \( \lim_{z \to 0} zF_3(z) = 8 \), \( F_4(z) \) is holomorphic in a neighbourhood of \( \Gamma \cup \{0\} \). We have
\[
 |F_0(z)| \geq |e^z| - |z| - 1 \geq e^{|z|/2} - |z| - 1, \quad z \in \Gamma,
\]
and
\[
 \lim_{z \to \infty, z \in \Gamma} \frac{z^2}{F_0(z)} = 0, \quad \lim_{z \to \infty, z \in \Gamma} F_3(z) = 0, \quad \lim_{z \to \infty, z \in \Gamma} F_4(z) = 0,
\]
thus \( |F_4(z)| \leq C \) in \( \Gamma \). We also have
\[
 F_2(z) \leq C(1 + |z|)^{-1}.
\]

Now we rewrite \( \tilde{\mathcal{H}}_{\alpha,k}^{(\theta)} \omega \) as follows, for \(|k| = 1\),
\[
 \tilde{\mathcal{H}}_{\alpha,k}^{(\theta)} \omega = -e^{-2\theta} \partial_r^2 \omega + \left( \frac{35 e^{-2\theta}}{4r^2} + \frac{r^2 e^{2\theta}}{16} - \frac{1}{2} + F_4 \frac{r^2 e^{2\theta}}{4} \right) w + i\beta_k F_1 \frac{r^2 e^{2\theta}}{4} w,
\]
and for \(|k| \geq 2\),
\[
 \tilde{\mathcal{H}}_{\alpha,k}^{(\theta)} \omega = -e^{-2\theta} \partial_r^2 \omega + \left( \frac{k^2 - 1}{4r^2 e^{2\theta}} + \frac{r^2 e^{2\theta}}{16} - \frac{1}{2} \right) w + i\beta_k F_1 \frac{r^2 e^{2\theta}}{4} w
\]
\[
 - e^{2\theta} F_2 \frac{r^2 e^{2\theta}}{4} \tilde{K}_k \langle F_2 \frac{r^2 e^{2\theta}}{4} w \rangle.
\]

Thanks to the properties of \( F_i(z)(i = 0, 1, \cdots, 4) \) which are shown above, \( \{ \tilde{\mathcal{H}}_{\alpha,k}^{(\theta)} \} \) are defined as an analytic family of type (A) in the strip \( \Gamma_1 = \{ \theta \in \mathbb{C} | |3\theta| < \frac{\pi}{8} \} \) with common domain \( D = \{ \omega \in H^2(\mathbb{R}_+) | \omega/r^2, r^2 \omega \in L^2(\mathbb{R}_+) \} \). In particular, the spectrum of \( \tilde{\mathcal{H}}_{\alpha,k}^{(\theta)} \) is always discrete and depends holomorphically on \( \theta \). Since the eigenvalues of \( \tilde{\mathcal{H}}_{\alpha,k}^{(\theta)} \) are constant for \( \theta \in \mathbb{R} \), they are also constant for \( \theta \in \Gamma_1 \).

Now we have
\[
 (6.2) \quad \Sigma(\alpha, k) = \inf \text{Re} \sigma(\tilde{\mathcal{H}}_{\alpha,k}^{(\theta)}) \geq \inf_{w \in D, \|w\| = 1} \text{Re} \langle \tilde{\mathcal{H}}_{\alpha,k}^{(\theta)} w, w \rangle.
\]
6.2. **Proof of Theorem 6.1** We need the following lemma.

**Lemma 6.2.** For $r > 0, 0 < \theta < \frac{\pi}{4}$, we have

$$-\text{Im} F_1(r^{i\theta}) \geq C^{-1} \sin \theta \min\left(r, \frac{1}{r}\right).$$

**Proof.** Thanks to

$$F_1(r^{i\theta}) = \frac{1 - e^{-re^{i\theta}}}{re^{i\theta}} = \frac{e^{-r} - e^{-re^{i\theta}} - i\theta}{r},$$

we have

$$-\text{Im} F_1(r^{i\theta}) = \frac{F_5(r, \theta)}{r},$$

where

$$F_5(r, \theta) = -\text{Im}(e^{-i\theta} - e^{-re^{i\theta}} - i\theta)$$
$$= \sin \theta - e^{-r \cos \theta} \sin(r \sin \theta + \theta).$$

Using the inequality

$$|\sin(r \sin \theta + \theta)| \leq |\sin(r \sin \theta) \cos \theta| + |\cos(r \sin \theta) \sin \theta|$$
$$\leq r \sin \theta \cos \theta + \sin \theta,$$

we get

$$(6.3) \quad F_5(r, \theta) \geq \sin \theta (1 - e^{-r \cos \theta} (1 + r \cos \theta)).$$

This shows that

$$F_5(r, \theta) \geq C^{-1} \min((r \cos \theta)^2, 1) \sin \theta \geq C^{-1} \min(r^2, 1) \sin \theta,$$

thus,

$$-\text{Im} F_1(r^{i\theta}) = \frac{F_5(r, \theta)}{r} \geq C^{-1} \min(r, \frac{1}{r}) \sin \theta.$$

This completes the proof. \(\square\)

Now we are in a position to prove Theorem 6.1.

Let us first consider the case of $|k| = 1$. It follows from Lemma 4.5 that

$$\text{Re} \langle \tilde{\mathcal{H}}_{\alpha, k} w, w \rangle \geq \frac{||w||^2}{2},$$

which gives

$$(6.4) \quad \Sigma(\alpha, k) \geq 1/2.$$

For $\theta \in (-\pi/8, \pi/8)$, $\text{sgn} \theta = \text{sgn} \beta_k$, we have

$$\text{Im} F_1\left(\frac{r^2 e^{2i\theta}}{4}\right) = \text{sgn} \theta \text{Im} F_1\left(\frac{r^2 e^{2i|\theta|}}{4}\right),$$

from which and Lemma 6.2, we infer that

$$\text{Re} \langle \tilde{\mathcal{H}}_{\alpha, k}^{(i\theta)} w, w \rangle = \int_{\mathbb{R}^+} \text{Re} \left(35e^{-2i\theta} + \frac{r^2 e^{2i\theta}}{16} - \frac{1}{2} + F_4\left(\frac{r^2 e^{2i\theta}}{4}\right) + i\beta_k F_1\left(\frac{r^2 e^{2i|\theta|}}{4}\right)\right) |w|^2 dr + \cos 2\theta \|\partial_r w\|^2$$
$$\geq \int_{\mathbb{R}^+} \left(35 \cos(2\theta) + \frac{r^2 \cos(2\theta)}{16} - \frac{1}{2} - C - |\beta_k| \text{Im} F_1\left(\frac{r^2 e^{2i|\theta|}}{4}\right)\right) |w|^2 dr$$

$$\geq \int_{\mathbb{R}^+} \left(35 \cos(2\theta) + \frac{r^2 \cos(2\theta)}{16} - \frac{1}{2} - C - |\beta_k| \text{Im} F_1\left(\frac{r^2 e^{2i|\theta|}}{4}\right)\right) |w|^2 dr$$
Thus, we obtain
\[ K \] and here we used the fact that
\[ \sin(\theta) \geq \sin(\frac{\pi}{2}) = 1 \]
which shows that
\[ \Sigma(\alpha, k) \geq C^{-1} |\beta_k \sin \theta|^{\frac{1}{2}} - C \geq C^{-1} |\beta_k|^{\frac{1}{2}} - C, \]
if we take \( \theta = (\text{sgn} \beta_k)^\frac{\pi}{12} \). Then by (6.4), we get
\[ \Sigma(\alpha, k) \geq \max(C^{-1} |\beta_k|^{\frac{1}{2}} - C, 1/2) \geq C^{-1} |\beta_k|^{\frac{1}{2}}. \]

Next we consider the case of \(|k| \geq 2\). We still assume \( \theta \in (-\frac{\pi}{8}, \frac{\pi}{8}) \), \( \sin \theta = \text{sgn} \beta_k \). Then we have
\[
\Re(\tilde{\mathcal{H}}_{\alpha, k} \langle w, w \rangle) = \cos 2\theta \| \partial_r w \|^2_{L^2} + \int_{\mathbb{R}^+} \Re \left( \frac{k^2 - 1/4}{r^2 e^{2i\theta}} + \frac{r^2 e^{2i\theta}}{16} - \frac{1}{2} + i\beta_k F_1\left(\frac{r^2 e^{2i\theta}}{4}\right) \right) |w|^2 dr
\]

\[ \geq \int_{\mathbb{R}^+} \left( C^{-1} |\beta_k \sin \theta|^{\frac{1}{2}} - C \right) |w|^2 dr = \left( C^{-1} |\beta_k \sin \theta|^{\frac{1}{2}} - C \right) \|w\|^2, \]

here \( K_k(r, s) = \frac{1}{2|k|} \min\left( \frac{r}{s}, \frac{s}{r} \right) \) \( (rs)^{\frac{1}{2}} \). Notice that
\[
\Re \left( -i\beta_k e^{2i\theta} F_2\left(\frac{r^2 e^{2i\theta}}{4}\right) F_2\left(\frac{s^2 e^{2i\theta}}{4}\right) \right) = \Re \left( -i\beta_k e^{2i\theta} e^{-(r^2+s^2)\frac{2i\theta}{8}} \right)
\]
\[ = \beta_k e^{-(r^2+s^2)\frac{\cos(2\theta)}{8}} \sin(2\theta) \frac{(r^2 + s^2) \sin(2\theta)}{8} \]
\[ = \frac{\beta_k}{\sin(2\theta)} e^{-(r^2+s^2)\frac{\cos(2\theta)}{8}} \left( \sin(2\theta) - \frac{r^2 \sin(2\theta)}{8} \sin(2\theta) - \frac{s^2 \sin(2\theta)}{8} \right)
\]
\[ = \frac{|\beta_k|}{\sin(2\theta)} \left( g_2(r) g_2(s) - g_3(r) g_3(s) \right), \]
where
\[ g_2(r) = e^{-r^2\frac{\cos(2\theta)}{8}} \sin \left( 2\theta - \frac{r^2 \sin(2\theta)}{8} \right), \quad g_3(r) = e^{-r^2\frac{\cos(2\theta)}{8}} \sin \left( \frac{r^2 \sin(2\theta)}{8} \right), \]
and here we used the fact that
\[ \sin(a - b - c) \sin a = \sin(a - b) \sin(a - c) - \sin b \sin c. \]
Thus, we obtain
\[
\Re(\tilde{\mathcal{H}}_{\alpha, k} \langle w, w \rangle) = \cos 2\theta \| \partial_r w \|^2_{L^2} + \int_{\mathbb{R}^+} \left( \frac{k^2 - 1/4}{r^2} \cos(2\theta) + \frac{r^2}{16} \cos(2\theta) - \frac{1}{2} - |\beta_k| \Im F_1\left(\frac{r^2 e^{2i\theta}}{4}\right) \right) |w|^2 dr
\]
\[ + \frac{|\beta_k|}{\sin(2\theta)} (\langle \tilde{K}_k [g_2 w], g_2 w \rangle + \langle \tilde{K}_k [g_3 w], g_3 w \rangle). \]
By the proof of Lemma 5.2, we know that
\[ \langle \tilde{K}_k [g_2 w], g_2 w \rangle \geq 0. \]
Due to $0 < K_k(r, s) \leq K_2(r, s)$, we have
\[
\langle \tilde{K}_k[g_3 w], g_3 w \rangle \leq \int_0^{+\infty} \int_0^{+\infty} K_k(r, s) g_3 w(s) \|g_3 w(r)\| ds dr
\leq \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} K_2(r, s) \left( \left( \frac{r}{s} \right)^{1/2} g_3(r)^2 |w(s)|^2 + \left( \frac{s}{r} \right)^{1/2} g_3(s)^2 |w(r)|^2 \right) ds dr
= \int_0^{+\infty} \tilde{K}_2 \left[ r^{1/2} g_3^2 \right](s) \frac{|w(s)|^2}{s^{1/2}} ds.
\]
Therefore, we obtain
\[
\text{Re}(\tilde{\mathcal{H}}_{\alpha, k} w, w) \geq \int_{\mathbb{R}^+} \left( \frac{k^2 - 1/4}{r^2} \cos(2\theta) + \frac{r^2}{16} \cos(2\theta) - \frac{1}{2} \right) |w|^2 dr
+ |\beta_k| \int_{\mathbb{R}^+} \left( -\text{Im} F_1 \left( \frac{\rho^2 e^{2|\theta|}}{4} \right) - \frac{\tilde{K}_2 \left[ r^{1/2} g_3^2 \right]}{|\sin(2\theta)| r^{1/2}} \right) |w|^2 dr.
\]
Due to $0 \leq g_3^2(r) \leq e^{-r^2 \cos(2\theta)/4} |r^2 \sin(2\theta)/8|^2$, we have
\[
\tilde{K}_2 \left[ r^{1/2} g_3^2 \right] \leq \frac{\sin(2\theta)}{64 r^2} \tilde{K}_2 \left[ r^{1/2} e^{-r^2 \cos(2\theta)/4} \right]
\leq \frac{\sin(2\theta)}{64 r^2} \left( \int_0^r \left( \frac{s}{r} \right)^{2} (r s) \frac{9}{4} e^{-s^2 \cos(2\theta)/4} ds + \int_r^{+\infty} \left( \frac{r}{s} \right)^{2} (r s) \frac{9}{4} e^{-s^2 \cos(2\theta)/4} ds \right)
= \frac{\sin(2\theta)}{256 r^2} \left( \int_0^r s^7 e^{-s^2 \cos(2\theta)/4} ds + r^4 \int_r^{+\infty} s^3 e^{-s^2 \cos(2\theta)/4} ds \right)
\leq \frac{\sin(2\theta)}{2 r^2 |\cos(2\theta)|^4} \left( \int_0^a \rho^3 e^{-\rho} d\rho + r^4 \left( \frac{\cos(2\theta)}{4} \right)^2 \int_a^{+\infty} \rho e^{-\rho} d\rho \right)
\leq \frac{\sin(2\theta)}{2 r^2 |\cos(2\theta)|^4} \left( 6 - (6 + 6a + 3a^2 + a^3)e^{-a} + a^2(1 + a)e^{-a} \right)
= \frac{\sin(2\theta)}{r^2 |\cos(2\theta)|^4} \left( 3 - (3 + 3a + a^2)e^{-a} \right).
\]
Here $a = \frac{r^2 \cos(2\theta)}{4}$ and we used the change of variable $\rho = a^2 \cos(2\theta)$. On the other hand, thanks to $-\text{Im} F_1 \left( \frac{\rho^2 e^{2|\theta|}}{4} \right) = \frac{F_5 \left( \frac{r^2 \cos(2\theta)}{4} \right)}{r^2}$, we get by (6.3) that
\[
-\text{Im} F_1 \left( \frac{\rho^2 e^{2|\theta|}}{4} \right) \geq \sin(2\theta) \left( \frac{1}{4} - e^{-r^2 \cos(2\theta)/4} (1 + r^2 \cos(2\theta)/4) \right)
= 4 \sin(2\theta) \left( \frac{1 - e^{-a}(1 + a)}{r^2} \right).
\]
Then we have
\[
\frac{\tilde{K}_2 \left[ r^{1/2} g_3^2 \right]}{|\sin(2\theta)| r^{1/2}} \leq \frac{-3 \text{Im} F_1 \left( \frac{\rho^2 e^{2|\theta|}}{4} \right)}{4 |\cos(2\theta)|^4}.
\]
Now we take $\theta = (\text{sgn} \beta_k) \frac{\pi}{2}$, then we have $|\cos(2\theta)|^4 > 3/4$, and
\[
\frac{k^2 - 1/4}{r^2} \cos(2\theta) + \frac{r^2}{16} \cos(2\theta) - \frac{1}{2} \geq C^{-1} \left( \frac{1}{r^2} + r^2 \right).
\]
Then we conclude that
\[
\Re(\tilde{\mathcal{H}}_{\alpha,k}^{(i\theta)}w, w) \geq C^{-1} \int_{\mathbb{R}_+^2} \left( \frac{1}{r^2} + r^2 \right) |w|^2 \, dr \\
+ |\beta_k| \int_{\mathbb{R}_+^2} \left( -\Im F_1 \left( \frac{r^2 e^{2i\theta}}{4} \right) + \frac{3 \Im F_1 \left( \frac{r^2 e^{2i\theta}}{4} \right)}{4 \cos(2\theta)^2} \right) |w|^2 \, dr \\
\geq C^{-1} \int_{\mathbb{R}_+^2} \left( \frac{1}{r^2} + r^2 - |\beta_k| \Im F_1 \left( \frac{r^2 e^{2i\theta}}{4} \right) \right) |w|^2 \, dr \\
\geq C^{-1} \int_{\mathbb{R}_+^2} \left( \frac{1}{r^2} + r^2 + |\beta_k| \sin(2\theta) \min(r^2, \frac{1}{r^2}) \right) |w|^2 \, dr \\
\geq C^{-1} \int_{\mathbb{R}_+^2} |\beta_k|^\frac{1}{2} |w|^2 \, dr = C^{-1} |\beta_k|^\frac{1}{2} \|w\|^2,
\]
which shows that for $|k| \geq 2$,
\[
\Sigma(\alpha, k) \geq C^{-1} |\beta_k|^\frac{1}{2}.
\]

7. **Appendix**

In this appendix, let us present some properties of the function $\sigma(r) = \frac{1-e^{-r^2/4}}{r^2/4}$.

**Lemma 7.1.** It holds that

1. for any $r_0 > 0$,
   \[
   |\sigma'(r)| \sim |\sigma'(r_0)|, \quad \frac{r_0}{2} \leq r \leq 2r_0,
   \]
   \[
   |\sigma(r) - \sigma(r_0)| \gtrsim |r - r_0||\sigma'(r_0)|, \quad 0 < r \leq 2r_0;
   \]

2. for $0 < r_0 < 1$ and $r_0/2 < r \leq 2r_0 + 1$,
   \[
   |\sigma'(r)| \gtrsim |\sigma'(r_0)|, \quad |\sigma(r) - \sigma(r_0)| \gtrsim |r - r_0||\sigma'(r_0)|;
   \]

3. for $r_0 \geq 1$ and $|r - r_0| \geq \frac{1}{r_0}$,
   \[
   |\sigma(r) - \sigma(r_0)| \gtrsim \frac{1}{(1 + r)^4}.
   \]

Here $a \sim b$ means $ca \leq a \leq c^{-1}b$ and $a \gtrsim b$ means $a \leq Cb$, where $c$ and $C$ are constants independent of $r_0$.

**Proof.** Let us prove the first property. Thanks to $\sigma'(r) = \frac{2}{r} \left( e^{-\frac{r^2}{4}} - \frac{1-e^{-\frac{r^2}{4}}}{r^2/4} \right)$, we have

(7.1) \[
|\sigma'(r)| \sim \min \left( r, \frac{1}{r_0} \right),
\]
which shows that if $r_0 \geq 1$, i.e. $r \geq \frac{1}{2}$, then
\[
|\sigma'(r)| \sim \frac{1}{r_3} \sim \frac{1}{r_0^3} \sim |\sigma'(r_0)|,
\]
and if $r_0 \leq 1$, i.e. $r \leq 2$, then
\[
|\sigma'(r)| \sim r \sim r_0 \sim |\sigma'(r_0)|.
\]
Thus, $|\sigma'(r)| \sim |\sigma'(r_0)|$ for $\frac{r_0}{2} \leq r \leq 2r_0$. If $\frac{r_0}{2} \leq r \leq 2r_0$, $|\sigma'(\theta r + (1 - \theta)r_0)| \sim |\sigma'(r_0)|(0 \leq \theta \leq 1)$. Thus, for some $\theta \in (0, 1)$,

$$|\sigma(r) - \sigma(r_0)| = |r - r_0||\sigma'(\theta r + (1 - \theta)r_0)| \sim |r - r_0||\sigma'(r_0)|.$$ 

While, if $0 < r \leq \frac{r_0}{2}$, we get by $\sigma'(r) < 0$ that

$$|\sigma(r) - \sigma(r_0)| \geq \sigma\left(\frac{r_0}{2}\right) - \sigma(r_0) \sim \frac{r_0}{2}|\sigma'(r_0)| \sim |r - r_0||\sigma'(r_0)|.$$ 

The second property could be proved similarly.

Now we prove the third property. If $r_0 \geq 1$ and $r \geq r_0 + \frac{1}{r_0}$, we get by (7.1) that

$$|\sigma(r) - \sigma(r_0)| \geq |\sigma(r_0) - \sigma(r_0 + \frac{1}{r_0})| \sim \frac{1}{r_0^3} \geq \frac{1}{(1 + r)^4},$$

and if $r_0 \geq 1$, $0 < r \leq r_0 - 1$, then

$$|\sigma(r) - \sigma(r_0)| \geq |\sigma(r) - \sigma(r + 1)| \sim |\sigma'(r + 1)| \sim \frac{1}{(1 + r)^4},$$

and if $r_0 \geq 1$, $r_0 - 1 < r \leq r_0 - \frac{1}{r_0}$, then

$$|\sigma(r) - \sigma(r_0)| \geq |\sigma(r_0 - \frac{1}{r_0}) - \sigma(r_0)| \geq \frac{1}{r_0^3} |\sigma'(r_0)| \sim \frac{1}{r_0^3} \geq \frac{1}{(1 + r)^4}.$$ 

This shows the third property of $\sigma(r)$. □

**Lemma 7.2.** Let $0 < \nu_1 < 1$ and $\sigma(r_1) = \nu_1$. There exist constants $c_i \sim 1 (i = 1, \cdots, 4)$, such that for any $r > 0$, we have

1. if $r_1 \leq 1$ and $1 \leq |\beta_1| \leq \frac{1}{r_1}$, then

$$c_1 \left(\frac{1}{r^2} + c_2|\beta_1|(\nu_1 - \sigma(r))\right) \geq |\beta_1|^{\frac{1}{2}};$$

2. if $r_1 \geq 1$ and $1 \leq |\beta_1| \leq r_1^4$, then

$$c_3(1 + r^2) + c_4|\beta_1|(\sigma(r) - \nu_1) \geq |\beta_1|^{\frac{1}{2}};$$

3. if $r_1 \geq 1$ and $r_1^4 \leq |\beta_1| \leq r_1^6$, then

$$c_3(1 + r^2) + c_4r_1^4(\sigma(r) - \nu_1) \geq |\beta_1|^{\frac{1}{2}}.$$ 

**Proof.** We consider the first case. Let $F(r) = c_1 \frac{1}{r^2} + c_2|\beta_1|(\sigma(r_1) - \sigma(r))$, $r_0 = |\beta_1|^{\frac{1}{2}}$. Then we have $r_1 \leq r_0 \leq 1$ and

$$F'(r) = -c_2|\beta_1|\sigma'(r) - \frac{2c_1}{r^3}.$$ 

If we choose $2c_1 = -c_2|\beta_1|\sigma'(r_0)r_0^3$, due to $-\sigma'(r) \sim \min\left(r, \frac{1}{r} \right)$, we have $c_1 \sim c_2$ and $F'(r_0) = 0$. As $-(r^3\sigma'(r))^' = r^3g^2(r) > 0$, we conclude that

$$F'(r) < 0 \text{ for } 0 < r < r_0, \quad F'(r) > 0 \text{ for } r > r_0,$$

which imply that

$$\min_{r > 0} F(r) = F(r_0) \geq \frac{c_1}{r_0^2} = c_1|\beta_1|\frac{1}{2}.$$ 

That is, for $c_1 = 1$ we have,

$$F(r) \geq |\beta_1|^{\frac{1}{2}}.$$
Next we consider the second case. Let $G(r) = c_3(1 + r^2) + c_4|\beta_1|(\sigma(r) - \sigma(r_1))$. Then

$$G(r) \geq c_3(1 + r^2) + \frac{c_4|\beta_1|}{C(1 + r^2)} - \frac{Cc_4|\beta_1|}{r_1^2} \geq C^{-1}(c_3c_4|\beta_1|)^{\frac{1}{2}} - \frac{Cc_4|\beta_1|}{|\beta_1|^{\frac{3}{2}}}.$$  

We can choose constants $c_3, c_4 > 0$ such that $C^{-1}(c_3c_4)^{\frac{1}{2}} - Cc_4 = 1$. Then

$$G(r) \geq |\beta_1|^{\frac{3}{2}}.$$  

Finally, we prove the third case. Let $H(r) = c_3(1 + r^2) + c_4r_1^4(\sigma(r) - \nu_1)$. Then we have

$$H(r) \geq c_3(1 + r^2) + \frac{c_4r_1^4}{C(1 + r^2)} - \frac{Cc_4r_1^4}{r_1^2} \geq C^{-1}(c_3c_4)^{\frac{1}{2}}r_1^2 - Cc_4r_1^2 = r_1^2 \geq |\beta_1|^{\frac{3}{2}}.$$  

The proof is finished. \hfill $\Box$

Similar to Lemma 7.2 we have

**Lemma 7.3.** Let $0 < \nu_k < 1$ and $\sigma(r_k) = \nu_k$. There exist constants $c_i \sim 1 (i = 1, \cdots, 4)$ such that for any $r > 0$, we have

1. if $r_k \leq 1$ and $|k|^3 \leq |\beta_k| \leq \frac{|k|^3}{r_k^4}$, then
   $$c_1\frac{k^2}{r^2} + c_2|\beta_k|(|\nu_k - \sigma(r)|) \geq |\beta_k|^{\frac{1}{2}};$$

2. if $r_k \geq \sqrt{k}$ and $|k|^3 \leq |\beta_k| \leq r_k^4$, then
   $$c_3(1 + r^2) + c_4|\beta_k|(\sigma(r)/2 - \nu_k) \geq |\beta_k|^{\frac{1}{2}};$$

3. if $r_k \geq \sqrt{k}$ and $r_k^4 \leq |\beta_k| \leq r_k^0$, then
   $$c_3(1 + r^2) + c_4r_k^4(\sigma(r)/2 - \nu_k) \geq |\beta_k|^{\frac{1}{2}}.$$

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