Radiation dominated era and the power of general relativity

Christian Corda

May 1, 2014

Institute for Theoretical Physics and Advanced Mathematics Einstein-Galilei,
Via Santa Gonda 14, 59100 Prato, Italy

and

Inter-University Centre Engineering of life and Environment, LIUM
University, Via Lugano 2, 6500 Bellinzona, Switzerland

E-mail addresses: cordac.galilei@gmail.com

Abstract

An analysis in the framework of the radiation dominated era permits to put bounds on the weak modification of general relativity which arises from the Lagrangian $R^{1+\epsilon}$. Such a theory has been recently discussed in various papers in the literature. The new bounds together with previous ones in the literature rule out this theory in an ultimate way.

It is well known that Einstein’s General Relativity Theory (GRT) can be adopted to describe various astrophysical observations and, at scales of the Solar System, it results consistent with many, very accurately precise, astrophysical measurements, such as the gravitational bending of light, the perihelion precession of Mercury and the Shapiro time delay [1, 2]. On the other hand, at larger scales, several shortcomings are present, like the famous Dark Energy [3] and Dark Matter [4] problems.

An alternative approach consists in assuming that gravitational interaction could act in different way at large scales [5]. This different framework does not require to find out candidates for dark energy and dark matter at fundamental level (not detected up to now), but takes into account only the observed ingredients (i.e. curvature, radiation and baryon matter), changing the left hand side of the field equations [6]–[7]. In this way, a room for alternative theories can be introduced and the most popular Dark Energy and Dark Matter models can be, in principle, achieved by considering Extended Theories of Gravity (ETG), i.e $f(R)$ theories of gravity, where $R$ is the Ricci curvature scalar, see [8]–[10].
and references within, and Scalar Tensor Theories [11][12][13], which are generalizations of the Jordan-Fierz-Brans-Dicke Theory [14][15][16].

An ultimate endorsement for the approach of ETG should be the realization of a consistent gravitational wave astronomy [11]. In fact, in the case of ETG, gravitational waves generate different oscillations of test masses with respect to gravitational waves in standard GRT. Thus, an accurate analysis of such a motion can be used in order to discriminate among various theories, see [11] for details.

Another key point is that Solar System tests imply that modifications of GRT in the sense of ETG have to be very weak [5][11]. In other words, such theories have to be viable. In the framework of viable ETG, the theory arising from the action (in this paper we work with $8\pi G = 1$, $c = 1$ and $\hbar = 1$)

$$S = \frac{1}{2} \int d^4x \left( \sqrt{-g} f_0 R^{1+\epsilon} + S_m \right),$$

where $f_0 > 0$ has the dimensions of a mass squared and $\epsilon$ is a small real number, has been discussed in various papers in the literature [17]-[27] and [33]. Equation (1) is a particular choice in $f(R)$ theories of gravity [6]-[10], [31][32] with respect to the well known canonical one of General Relativity (the Einstein - Hilbert action [28]) which is

$$S = \frac{1}{2} \int d^4x \left( \sqrt{-g} R + S_m \right).$$

Various observational constraints set the limits

$$0 \leq \epsilon \leq 7.2 \times 10^{-19}$$

on the parameter $\epsilon$ [17][18], while the recent work [25] obtained a lower limit

$$0 \leq \epsilon \leq 5 \times 10^{-30}.$$

Gravitational waves in this particular theory have been discussed in [26]. In [27], a spherically symmetric and stationary universe has been analyzed in the tapestry of this theory.

In order to discuss this particular theory in the framework of the radiation dominated era, the well known Friedman-Robertson-Walker cosmological line - element has to be used [1][28], and, for the sake of simplicity, we will consider the flat case, because the WMAP data are in agreement with it [29]

$$ds^2 = -dt^2 + a^2 (dz^2 + dx^2 + dy^2).$$

We also recall that in the radiation dominated era the equation of state is [1]

$$p = \frac{1}{3} \rho$$

and the the energy density is given by [1]
\[ \rho = \frac{f\pi^2}{120}k^4T^4, \quad (7) \]

where \( k \) is the Boltzmann constant and \( f \) is a parameter depending from the particular radiation, for example \( f = 8 \) for electromagnetic radiation, \( f = 7 \) for neutrinos, etc., see [1] for details.

By varying the action (1) with respect to \( g_{\mu\nu} \) (see [26] for a detailed computation) the field equations are obtained

\[ G_{\mu\nu} = \frac{1}{(1 + \varepsilon)f_0R^\varepsilon}\left\{-\frac{1}{2}g_{\mu\nu}f_0R^{1+\varepsilon}\right\} \]

\[ + \left[(1 + \varepsilon)f_0R^{\varepsilon}\right]_{\mu\nu} - g_{\mu\nu}\square[(1 + \varepsilon)f_0R^\varepsilon]\right\} + T_{\mu\nu}, \quad (8) \]

where

\[ T_{\mu\nu} = \begin{vmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{vmatrix} \quad (9) \]

is the well known stress-energy tensor of the matter [1, 28]. Taking the trace of the field equations (8) one gets

\[ 3\Box(1 + \varepsilon)f_0R^\varepsilon = (1 - \varepsilon)f_0R^{1+\varepsilon} + T, \quad (10) \]

where \( T = \rho - 3p \) is the trace of the stress-energy tensor [9] [28].

Following [1], if one computes the components of eqs. (8) and (10) by using the line element (5) three independent Friedman equations are obtained

\[ 3\dot{R}^2(1 - \varepsilon^2)f_0R^{-2} + 3\dot{R}(1 + \varepsilon)f_0R^{\varepsilon-1} + 9\frac{\dot{a}}{a}\dot{R}(1 + \varepsilon)f_0R^{\varepsilon-1} - 2f_0R^{1+\varepsilon} = 0 \]

\[ 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)(1 + \varepsilon)f_0R^\varepsilon + 3f_0R^{1+\varepsilon} - 12\frac{\ddot{a}}{a}\dot{R}(1 + \varepsilon)f_0R^{\varepsilon-1} + 6\dot{R}^2(1 - \varepsilon^2)f_0R^{-2} - 6\dot{R}(1 + \varepsilon)f_0R^{\varepsilon-1} = \rho \]

\[ 6\dot{a}(1 + \varepsilon)f_0R^\varepsilon + f_0R^{1+\varepsilon} - 6\frac{\ddot{a}}{a}\dot{R}(1 + \varepsilon)f_0R^{\varepsilon-1} = \rho. \quad (11) \]

We recall that the Ricci scalar is given by [30]

\[ R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \quad (12) \]

One can also use the Bianchi identities [1] to get another independent equation

\[ a\dot{\rho} = -4\rho\dot{a}. \quad (13) \]

In standard general relativity, during the radiation dominated era, the scale factor is [1]

\[ a \sim t^{\frac{1}{2}}. \quad (14) \]

Hence, in the theory which arises from the action [1] one assumes
\[ a \sim t^{\left( \frac{1}{2} + \delta \right)} . \]  

(15)

By using the second and the third of eqs. (11) and eq. (13) one gets

\[ \varepsilon = 2\delta . \]  

(16)

By deriving eq. (12) and by using eq. (16) we write

\[ \dot{R} = \frac{6 \varepsilon (1 + \varepsilon)}{t^3} . \]  

(17)

Considering the third of eqs. (11) together with eq. (7) one obtains

\[ T = \frac{40}{\pi^3} \int_\varepsilon \sqrt{6(1+\varepsilon)(1+\varepsilon)} \frac{1}{4} \left[ 8(1+\varepsilon) - 5(1+\varepsilon)^2 - 2 \right] (-\varepsilon)^\varepsilon \frac{1}{kt^{1+\varepsilon}} . \]  

(18)

Putting

\[ F(\varepsilon) \equiv 6(1+\varepsilon)(1+\varepsilon) \frac{1}{4} \left[ 8(1+\varepsilon) - 5(1+\varepsilon)^2 - 2 \right] (-\varepsilon)^\varepsilon , \]  

(19)

we need

\[ F(\varepsilon) \geq 0 \]  

(20)

in order \( T \) to be a real value. The constrain (20) is satisfied for

\[ -0.69 \leq \varepsilon \leq 0 , \]  

(21)

see figure 1.

Considering the bound (21) together with the bounds (3) and (4) one gets immediately \( \varepsilon = 0 \), i.e. general relativity is recovered and the theory which arises from the action (1) is ultimately ruled out.

In summary, in this work we realized an analysis in the framework of the radiation dominated era in order to put bounds on the weak modification of general relativity which arises from the action (1). The new bounds together with previous ones in the literature rule out this theory in a definitive way.

Acknowledgements

I thank a reviewer for useful comments.

Riferimenti bibliografici

[1] C. W. Misner , K. S. Thorne, J. A. Wheeler, “Gravitation”, Feeman and Company (1973).
[2] C. M. Will, Living Reviews in Relativity, 9, 3 (2006).
[3] S. Perlmutter et al., Nature 391, 51 (1998).
The function $F(\varepsilon)$ is plotted. The x axis represents the variable $\varepsilon$, the y axis the variable $F(\varepsilon)$. We see that the condition $F(\varepsilon) \geq 0$ is satisfied for $-0.69 \leq \varepsilon \leq 0$. 
[4] S. M. Carroll, W. H. Press, and E. L. Turner, ARA&A, 30, 499 (1992).
[5] S. Nojiri and S.D. Odintsov, Phys. Rept. 505, 59-144 (2011).
[6] G. Cognola, E. Elizalde, S. D. Odintsov, P. Tretyakov, and S. Zerbini, Phys. Rev. D 79, 044001 (2009).
[7] K. Bamba, S. Nojiri and S. D. Odintsov, J. Cosmol. Astropart. Phys. JCAP10(2008)045.
[8] S. Nojiri and S. D. Odintsov, Phys. Lett. B, 657, 238 (2008).
[9] K. Bamba, S. Nojiri and S. D. Odintsov, Phys. Lett. B 698, 451-456 (2011).
[10] E. Elizalde, S. Nojiri, S.D. Odintsov, L. Sebastiani, S. Zerbini, Phys. Rev. D 83, 086006 (2011).
[11] C. Corda, Int. Journ. Mod. Phys. D, 18, 14, 2275 (2009, Honorable Mention at Gravity Research Foundation).
[12] E. Elizalde, D. Sáez-Gómez, Phys. Rev. D 79, 065023 (2009).
[13] C. Corda, Phys. Rev. D 83, 062002 (2011).
[14] P. Jordan, Naturwiss. 26, 417 (1938).
[15] M. Fierz, Helv. Phys. Acta 29, 128 (1956).
[16] C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).
[17] T. Clifton, Class. Quantum Grav. 23, 7445 (2006).
[18] T. Clifton and J. D. Barrow, Phys. Rev. D 72, 103005 (2005).
[19] J. D. Barrow and T. Clifton, Class. Quantum Grav. 23, L1 (2006).
[20] T. Clifton and J. D. Barrow, Class. Quantum Grav. 23, 2951 (2006).
[21] A. F. Zakharov, A. A. Nucita, F. De Paolis and G. Ingrosso, Phys. Rev. D 74, 107101 (2006).
[22] A. D. Dolgov and M. Kawasaki Phys. Lett. B, 573, 1 (2003).
[23] V. Faraoni, Phys. Rev. D 74 104017 (2006).
[24] V. Faraoni, Phys. Rev. D 75, 067302 (2007).
[25] V. Faraoni, Phys. Rev. D 83, 124044 (2011).
[26] C. Corda, Astropart. Phys. 30, 209 (2008).
[27] C. Corda and H. J. Mosquera Cuesta, Europhys. Lett. 86, 20004 (2009).
[28] L. Landau and E. Lifsits, *Classical Theory of Fields* (3rd ed.), London: Pergamon (1971).

[29] C. L. Bennet et al., Ap. J. Suppl. Ser. **148**, 15 (2003).

[30] C. Corda and H. J. Mosquera Cuesta, Astropart. Phys. **34**, 587 (2011).

[31] S. Capozziello, M. De Laurentis, Phys. Rep. 509, 167 (2011).

[32] S. Nojiri, S.D. Odintsov, Phys. Rep. 505, 59 (2011).

[33] S. Capozziello, M. De Laurentis, M. Francaviglia, Astrop. Phys. 29, 125 (2008).