We consider the single channel Kondo problem with the Kondo coupling between a spin $S$ impurity and conduction electrons with spin $j$. These problems arise as multicritical points in the parameter spaces of two- and higher-level tunneling systems, and some impurity models of heavy fermion compounds. In contrast to the previous Bethe-ansatz conjectures, it turns out that the dynamics of the spin sector is the same as that of a spin $S$ impurity coupled to $k(j)$ channels of spin 1/2 electrons with $k(j) = 2j(j + 1)(2j + 1)/3$. As a result, for $2S < k(j)$, the system shows non-Fermi liquid behavior with the same exponents for the thermodynamic quantities as those of $k(j)$ channel Kondo problem. However, both the finite-size spectrum and the operator content are different due to the presence of the other sectors and can be obtained by conformal field theory techniques.

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In the ordinary single channel Kondo problem, the conduction electrons with spin 1/2 interact with a spin 1/2 impurity [1]. This model has been regarded as a canonical model for the physics of dilute magnetic impurities in metals [1]. The physics of the model can be summarized as the complete screening of the impurity spin by the conduction electrons in the low energy limit, which leads to the Fermi-liquid behavior of the physical quantities [1]. Nozières and Blandin [2] pointed out that the above picture of the single channel Kondo model should be changed when the number of channels, $k$, for the electrons with spin 1/2 is larger than twice the impurity spin $S$. This overscreened multichannel Kondo model has been the subject of intensive research [3] due to its non-Fermi liquid behavior.

The purpose of this paper is to study the single channel Kondo model in which the conduction electrons with spin $j$ interact with the spin $S$ impurity. These models arise as multicritical points in two-level systems proposed by Zawadowski and Vladar [3]. Recently, Moustakas and Fisher consider the more general models, for example, systems with three impurity positions [4]. Our models are realized as some special fixed points, with enhanced symmetry, in their parameter space. According to Kim and Cox, $j = 3/2$ case arises in the context of the theory of heavy fermions with $Ce^{3+}$ impurities in the cubic environment [5].

In the classic paper of Wiegmann and Tsvelik [6] on multichannel Kondo problems, it was conjectured that the $k$-channel Kondo problem is equivalent to the single channel Kondo problem with conduction electron spin $k/2$ [7]. We will see that this is not correct. We believe that the discrepancy is due to the fact that the Bethe-ansatz approach does not take care of the bulk degrees of freedom correctly. So far, the only reliable method for studying these models have been finding fixed points in the limit of a large number of flavors [8]. Our results are not only consistent with what was known, but also provide exact scaling dimensions around these fixed points.

We show that the dynamics of the spin sector in this problem is the same as that of a spin $S$ impurity coupled to $k(j)$ channels of spin 1/2 electrons with $k(j) = 2j(j + 1)(2j + 1)/3$. As a result, for $2S < k(j)$, the system shows non-Fermi liquid behavior with the same exponents for the thermodynamic quantities as those of $k(j)$ channel Kondo problem. However, it turns out that the finite size spectrum of the system is different from that of the $k(j)$ channel Kondo problem due to the presence of the remaining degrees of freedom and can be obtained by conformal field theory techniques.

In the Kondo problem, it is sufficient to consider only the s-wave scattering or the radial motion of the conduction electrons. This one-dimensional problem in half-space ($x \geq 0$ and $x$ is the radial coordinate) can be represented in terms of left-going (chiral) fermions $\psi_{\alpha}(x)$ defined for $-\infty < x < \infty$. The Hamiltonian of the conduction electrons with spin $j$ interacting with the spin $S$ impurity, $H = H_0 + H_1$, can be written as

$$H_0 = iv_F \sum_{\alpha} \int_{-\infty}^{\infty} dx \, \psi_{\alpha}^\dagger(x) \partial_x \psi_{\alpha}(x)$$

$$H_1 = \sum_{a=x,y,z} J S^a J^a(0) ,$$

(1)

where $\alpha = j, j - 1, ..., -(j - 1), -j$, $S^a$ are the impurity spin $S$ operators, and $J^a(x)$ are the conduction electron spin currents or densities:

$$J^a(x) = \sum_{\alpha \beta} \psi_{\alpha}^\dagger(x) t^{a}_{\alpha \beta} \psi_{\beta}(x) ,$$

(2)

where $t^{a}_{\alpha \beta}$ are spin $j$ representations of $SU(2)$ generators.

First we show that the dynamics of the spin sector is the same as that of a spin $S$ impurity coupled to $k(j)$ channels of spin 1/2 electrons with

$$k(j) = 2j(j + 1)(2j + 1)/3 .$$

(3)

This relies on the fact that, in the Kondo perturbation theory, the only quantities which enter are the multipoint...
correlators of the electron spin densities at the impurity site. The long time behavior of these correlators is specified by the level \( k \) of the current algebra satisfied by the spin currents in the equivalent one dimensional problem. The level \( k \) can be determined from the two-point function of the currents:

\[
\langle J^a(x)J^b(y) \rangle = \frac{k/2}{(x-y)^2} \delta^{ab} .
\] (4)

Here \( k \) is given by \( k(j) = 2 \text{ Tr}(t^a t^b) = \frac{2}{3} \sum \text{ Tr}(t^a t^b) = 2j(j+1)(2j+1)/3 \). Therefore, the spin sector is described by \( SU_{k(j)}(2) \) Wess-Zumino-Witten (WZW) model \([3]\). The interaction of the electrons with the impurity spin can be incorporated by the changes in the boundary condition of the \( SU_{k(j)}(2) \) WZW model \([3]\). In the case of the \( k \) channel Kondo model for \( k \geq 2 \), the change in the boundary condition corresponds to the fusion with the spin \( S \) operator of \( SU_k(2) \) \([3]\). Since the Hamiltonian in the spin sector of the present problem looks exactly the same as that of the \( k(j) \) channel Kondo problem, we expect that at the conformal fixed point the boundary condition is provided by the fusion with the same operator. Therefore, for \( S < j(j+1)(2j+1)/3 \), we have overscreening of the impurity spin and non-Fermi liquid behavior even in the single channel problem.

In order to consider the full Hilbert space, we have to consider other degrees of freedom as well as the spin sector. First, we consider the \( U(1) \) charge sector with the corresponding charge current \( J(x) = \sum \alpha \psi_\alpha^\dagger(x) \psi_\alpha(x) \). One can write down \( SU_1(2j+1) \) current algebra for the other degrees of freedom by considering the currents of the form \( J^A = \sum_{\alpha,\beta} \psi_\alpha^\dagger T^A(x) \psi_\beta \), where \( T^A \)'s are general Hermitian traeless \((2j+1) \times (2j+1)\) matrices. Only three of these currents are needed to describe the spin degrees of freedom and form \( SU_{k(j)}(2) \) current algebra. Thus the rest of the degrees of freedom other than spin and charge can be formally described by the coset model \( SU_1(2j+1)/SU_{k(j)}(2) \). Note that the central charge of this coset model is \( c_{\text{coset}} = 2j - 3k(j)/[k(j) + 2] \).

To be more specific, let us consider \( j = 1 \) and \( j = 3/2 \). For \( j = 1 \), \( k(1) = 4 \) (which was discussed in \([3]\)) and the coset model is trivial in the sense that there is no degree of freedom \( c_{\text{coset}} = 0 \). For \( j = 3/2 \), \( k(3/2) = 10 \) and the coset model corresponds to single Majorana fermion or equivalently the critical Ising model \( c_{\text{coset}} = 1/2 \). This can be shown more explicitly by using Abelian bosonization as follows.

For \( j = 3/2 \), the four fermion operators can be bosonized as \([13]\)

\[
\psi_\alpha(x) = \frac{1}{\sqrt{2\pi a}} e^{-i\Phi_\alpha(x)} ,
\] (5)

where \( \alpha = 3/2, 1/2, -1/2, -3/2 \) and \( a \) is a short-distance cutoff of the order of the lattice constant. Let us consider

\[
\Phi_\epsilon = \frac{1}{2} (3\Phi_{3/2} + \Phi_{1/2} + \Phi_{-1/2} + \Phi_{-3/2})
\]

\[
\Phi_s = \frac{1}{2\sqrt{5}} (3\Phi_{3/2} + \Phi_{1/2} - \Phi_{-1/2} - 3\Phi_{-3/2})
\]

\[
\Phi_t = \frac{1}{2\sqrt{5}} (\Phi_{3/2} - 3\Phi_{1/2} + 3\Phi_{-1/2} - \Phi_{-3/2})
\]

\[
\Phi_b = \frac{1}{2} (\Phi_{3/2} - \Phi_{1/2} - \Phi_{-1/2} + \Phi_{-3/2}) .
\] (6)

The spin currents can be written as

\[
J^s = \sqrt{5} \partial_x \Phi_s ,
\]

\[
J^t = \frac{1}{\pi a} (\sqrt{3} \cos \Phi_b e^{i(\Phi_s + 2\Phi_t)/\sqrt{5}} + e^{i(\Phi_s - 3\Phi_t)/\sqrt{5}}) ,
\] (7)

where \( J^s = J^x \pm i J^y \). Note that the spin currents involve only \( \Phi_s, \Phi_t \), and a Majorana fermion \( \chi_1 = \cos \Phi_b \). The boson \( \Phi_c \) and the Majorana fermion \( \chi_2 = \sin \Phi_b \) decouple from the spin sector. The first three degrees of freedom provide the central charge \( c = 5/2 \) as expected for \( SU_{10}(2) \). \( \Phi_c \) corresponds to the charge sector and the remaining Majorana fermion \( \chi_2 \) is the coset degree of freedom refereed previously.

In conformal field theory, there is one-to-one correspondence between operators and states. Each of these operators can be considered as a product of operators coming from three sectors described earlier. This is called the conformal embedding \([3]\). The above mentioned fusion will affect only the operators in the spin sector through the formal operator product expansions. The new operators found after fusion can be mapped to a well defined set of states of the same conformal field theory. These states form the Hilbert space of the interacting problem.

As the first step, one constructs the states in the finite-size free electron Hilbert space and writes down the corresponding operators in terms of operators coming from the three sectors. Let us consider the free electrons moving on a finite-size one-dimensional lattice, which can be regarded as the regularization of the continuum model. In the absence of any potential, this model has the particle-hole symmetry. Let us be at the half-filling and preserve the symmetry. In this case, for even number, \( 2m \), of sites, the chemical potential, \( \mu \), lies exactly in the middle of \( m \)th and \((m + 1)\)th levels and all other levels are symmetrically placed around \( \mu \). On the other hand, if we had odd number, \( 2m + 1 \), of sites, the chemical potential would lie exactly on the \((m + 1)\)th level and the remaining levels would be placed symmetrically around it.

Let us consider the continuum model given by \( H_0 \) with \( x \) restricted to the interval \(-L/2 < x < L/2 \). If the boundary condition is \( \psi(L/2) = e^{i\delta} \psi(-L/2) \), all the energy levels with respect to the chemical potential are quantized as \( E_n = v_F k_n = \frac{2\pi \delta}{L}(n + \frac{1}{2}) \). For the particle-hole symmetric case, \( \delta \) is either 0 or \( \pi/2 \). Note that \( \delta = 0 \) corresponds to the case of odd number of sites in the lattice problem while \( \delta = \pi/2 \), to the case of even number of sites. We will call the Hilbert spaces corresponding to these two cases as the periodic sector (P) and the anti-periodic sector (AP). In principle, the energy spectra of
P and AP sectors can be compared with those of the odd and the even iterations in numerical renormalization group calculation.

As an example of overscreened Kondo problem, we calculate the finite-size spectrum for the case of \( j = 3/2 \) and \( S = 1/2 \). Let us start with AP sector. After the conformal embedding, some of the low lying states for \( j = 3/2 \) and their quantum numbers are given in the Table I. Here \( j_{\text{tot}} \) refers to the total spin of each state. In case that this state is a descendent in the spin current algebra, we indicate it by putting prime (’). In the second column, we indicate the conformal primary from the Ising sector (Majorana fermion) \([5]\). \( q \) refers to the total charge and the prime again indicates the descendent in the charge current algebra. The last two columns provide the excitation energy (in units of \( 2\pi v_F/L \)) with respect to the ground state energy \( \Delta_0 \) and the corresponding degeneracy of the state.

The spin \( j_{\text{tot}} \) primaries of \( SU(k/3/2)(2) \) have dimension \( j_{\text{tot}}(j_{\text{tot}} + 1)/|k(3/2) + 2| = j_{\text{tot}}(j_{\text{tot}} + 1)/12 \). In the Ising sector, the dimensions of \( \epsilon \) and \( \sigma \) are 1/2 and 1/16 respectively. The charge sector primaries with charge \( q \) have dimension \( q^2/8 \). In each sector, the descendents can be obtained by repeatedly applying Fourier modes of the spin current \( J^a(x) \), the charge current \( J(x) \), and the stress tensor \( T(x) \) of the Ising sector respectively. Applying the \( n \)th Fourier mode increases the dimension by \( n \). The total energy (in units of \( 2\pi v_F/L \)) of the state is given by the sum of the dimensions from the three sectors.

Now we introduce the interaction with the impurity spin \( S \). At the new fixed point, after the interaction is included, the states in the Hilbert space of the interacting problem can be also written in terms of the constituent degrees of freedom from the spin, the charge, and the coset sectors. In order to get the Hilbert space of the interaction problem, we use the general algebraic procedure suggested by Ref. [14], which is to fuse the operator in the spin sector with the spin \( S \) operator of \( SU(k/3/2)(2) \). If the operator from the spin sector has spin \( j' \), then the fusion generates operators of \( |j' - S|, |j' - S| + 1, \ldots, \min(j' + S, k(j) - j' - S) \). Following this procedure in the case of \( j = 3/2 \) and \( S = 1/2 \), we get the Table II from the Table I and it represents the spectrum of the interacting problem. For convenience, we list only states with excitation energy below \( 2\pi v_F/L \) (this corresponds to \( \Delta_{\text{tot}} - \Delta_0 < 1 \)). Note that for none of the tables we try to list all the primary operators in the spectrum, which can be done in a straightforward manner.

For example, the second row of Table I has a \( j_{\text{tot}} = 3/2 \) primary in the spin sector. Fusing with the spin 1/2 primary of \( SU(1)(2) \) gives rise to spin 1 and spin 2 primary operators. These provide the second and the third row of the Table II, where only the spin quantum numbers are changed.

For the case of P sector, following the same procedure, we obtain the Table III and the Table IV which correspond to the noninteracting and the interacting spectrum respectively. Here we also list only the first few levels in the interacting spectrum.

All the singular behaviors in the thermodynamic quantities are due to the leading irrelevant boundary operator, which can be added to the Hamiltonian. Here this leading irrelevant operator is exactly the same as that in the \( k(j) \) channel Kondo problem, i.e., the first descendent of the spin \( j_{\text{tot}} = 1 \) operator \([5]\). As a result, it has the dimension \( 1 + \Delta = 1 + \frac{2}{k(3/2) + 2} \), which is \( 7/6 \) for \( j = 3/2 \). Here \( \Delta = \frac{2}{k(3/2) + 2} \) is the dimension of the spin \( j_{\text{tot}} = 1 \) operator. Accordingly the susceptibility and the specific heat coefficient diverge as \( T^{2\Delta - 1} = T^{-2/3} \) as in the 10 channel Kondo model \([3]\).

Let us investigate various perturbations around the fixed point. In particular, we are interested in \( SU(2) \) symmetry breaking interactions between the impurity spin and the electrons, which is of the form

\[
H_{\text{pert}} = \sum_{a=x,y,z} S^a \psi_\alpha^\dagger \Lambda_{a\beta}^\epsilon \psi_\beta^\dagger \psi_\beta. \tag{8}
\]

Here \( \Lambda_{a\beta}^\epsilon \) are also 4x4 traceless Hermitian matrices which are orthogonal to spin \( j \) matrices in the sense that \( \text{Tr}(\Lambda_{a\beta}^\epsilon b) = 0 \).

In order to classify possible relevant boundary operators (which correspond to adding various interaction terms at the impurity site), we perform double fusion \([5,16]\) and look at the operator content. We need to consider only operators which are bosonic, charge conserving, and have dimension less than one. There are three such operators, which are \( O^a, O^{(ab)}, \text{ and } O^a \times \epsilon \). Here \( O^a \) is a spin 1 operator which has dimension 1/6 and couples to an external magnetic field. \( O^{(ab)} \) is a traceless symmetric matrix and corresponds to a spin 2 operator which has dimension 1/2. The last operator \( O^a \times \epsilon \) involves the energy operator \( \epsilon \) of the Ising sector so that it has the total dimension 2/3. In the absence of the external magnetic field, but in the presence of \( SU(2) \) breaking terms which are of the form in Eq. 8, the second operator can be generated. Since this is a relevant operator, this fixed point will be unstable against such a perturbation.

However, as we consider the case of \( S > 1/2 \) and \( j > 1 \), we find that there is an \( SU(2) \)-invariant charge conserving relevant operator. Which discrete symmetries can rule out this operator is not clear to us. This operator, if present, can take us to a new \( SU(2) \)-invariant fixed point.

In summary, we study the single channel Kondo problem in which the conduction electron with spin \( j \) interact with a spin \( S \) impurity. It is shown that the dynamics of the spin sector is the same as that of \( k(j) = 2j(j + 1)(2j + 1)/3 \) channel Kondo problem in which conduction electrons with spin 1/2 of \( k(j) \) channels interact.
with a spin $S$ impurity. Thus, for $2S < k(j)$, the system shows non-Fermi liquid behavior. As a result, the exponents of the specific heat coefficient and the impurity susceptibility should be the same as those of the $k(j)$ channel Kondo problem. However, it is also pointed out that the finite-size spectrum should be different due to the presence of the remaining degrees of freedom. As an example, we show the low-lying states in finite-size spectrum of the system in the case of $j = 3/2$ and $S = 1/2$ for the periodic and the anti-periodic sectors. We also analyze the possible relevant perturbations around the fixed point.

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Note Added: After this work was completed, we were informed that T.-S. Kim, L. Oliveira, and D. L. Cox studied the model with the conduction electron spin $3/2$ using the numerical renormalization group. [17] We also received a preprint by Fabrizio and Zarand [18], which arrived at the same conclusion as ours.

References:
[1] J. Kondo, Prog. Theor. Phys. 32, 37 (1964); P. W. Anderson, J. Phys. C 3, 2346 (1970); P. Nozières, J. Low Temp. Phys. 17, 31 (1974).
[2] P. Nozières and Blandin, J. Phys. (Paris) 41, 193 (1980).
[3] N. Andrei and C. Destri, Phys. Rev. Lett. 52, 364 (1984).
[4] P. B. Wiegmann and A. M. Tsvelik, Z. Phys. B 54, 201 (1985).
[5] I. Affleck, Nucl. Phys. B336, 517 (1990); I. Affleck and A. W. W. Ludwig, Nucl. Phys. B352, 849 (1991); B360, 641 (1991); Phys. Rev. B 48, 7297 (1993).
[6] V. J. Emery and S. Kivelson, Phys. Rev. B 46, 10812 (1992); A. M. Sengupta and A. Georges, Phys. Rev. B 49, 10020 (1994).
[7] M. Fabrizio and A. O. Gogolin, Phys. Rev. B 50, 17732 (1994).
[8] K. Vladar and A. Zawadowski, Phys. Rev. B 28, 1564 (1983); 28, 1582 (1983).
[9] See A. Zawadowski and K. Vladar, in Quantum Tunneling in Condensed Media, edited by Yu. Kagan and A. J. Legget (North Holland, Amsterdam, 1992).
[10] A. Moustakas and D. S. Fisher, APS March Meeting Proceeding, St. Louis (1996).
[11] T.-S. Kim and D. Cox, preprint cond-mat/9603034.
[12] P. D. Sacramento and P. Schlottmann, Phys. Rev. B 43, 13294 (1991); Adv. Phys. 42, 641 (1993).
[13] G. Zarand, Phys. Rev. B 51, 273 (1995).
[14] V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B247, 83 (1984).
[15] See, for example, V. J. Emery, in Highly Conducting One-Dimensional Solids, edited by J. T. Devreese, R. P. Evrard, and V. E. van Doren (Plenum, New York, 1979).
[16] J. Cardy, Nucl. Phys. B324, 581 (1989); Nucl. Phys.

B270, 186 (1986).
[17] D. Cox, private communication.
[18] M. Fabrizio and G. Zarand, preprint.

TABLE I. The energy spectrum for non-interacting electrons with spin $j = 3/2$ in AP sector.

| $j_{tot}$ | Ising | $q$ | $\Delta_{tot} - \Delta_0$ | Deg. |
|----------|-------|----|----------------|------|
| 0        | 1     | 0  | 0              | 1    |
| 3/2      | $\sigma$ | $\pm 1$ | 1/2 | 8    |
| 2        | 1     | $\pm 2$ | 1   | 10   |
| 0        | $\epsilon$ | $\pm 2$ | 1   | 2    |
| 2        | $\epsilon$ | 0     | 1   | 5    |
| 3        | 1     | 0    | 1   | 7    |
| 1'       | 1     | 0    | 1   | 3    |
| 0        | 1     | 0'   | 1   | 1    |

TABLE II. The energy spectrum for interacting system with $j = 3/2$ in AP sector.

| $j_{tot}$ | Ising | $q$ | $\Delta_{tot} - \Delta_0$ | Deg. |
|----------|-------|----|----------------|------|
| 1/2      | 1     | 0  | 0              | 2    |
| 1        | $\sigma$ | $\pm 1$ | 7/24 | 6    |
| 2        | $\sigma$ | $\pm 1$ | 5/8 | 10   |
| 5/2      | 1     | 0   | 2/3 | 6    |
| 3/2      | 1     | $\pm 2$ | 3/4 | 8    |
| 3/2      | $\epsilon$ | 0     | 3/4 | 4    |

TABLE III. The energy spectrum for non-interacting electrons with spin $j = 3/2$ in P sector.

| $j_{tot}$ | Ising | $q$ | $\Delta_{tot} - \Delta_0$ | Deg. |
|----------|-------|----|----------------|------|
| 0        | 1     | $\pm 2$ | 0   | 2    |
| 3/2      | $\sigma$ | $\pm 1$ | 0   | 8    |
| 0        | $\epsilon$ | 0     | 0   | 1    |
| 2        | 1     | 0    | 0   | 5    |

TABLE IV. The energy spectrum for interacting system with $j = 3/2$ in P sector.

| $j_{tot}$ | Ising | $q$ | $\Delta_{tot} - \Delta_0$ | Deg. |
|----------|-------|----|----------------|------|
| 3/2      | 1     | 0  | 0              | 4    |
| 1        | $\sigma$ | $\pm 1$ | 1/2 | 6    |
| 1/2      | 1     | $\pm 2$ | 1/4 | 4    |
| 1/2      | $\epsilon$ | 0     | 1/4 | 2    |
| 2        | $\sigma$ | $\pm 1$ | 3/8 | 10   |

