General Rotational Surfaces with Pointwise 1-Type Gauss Map in Pseudo-Euclidean Space $E_2^4$

Ferdag KAHRAMAN AKSOYAK ¹, Yusuf YAYLI ²

¹Erciyes University, Department of Mathematics, Kayseri, Turkey
²Ankara University, Department of Mathematics, Ankara, Turkey

Abstract

In this paper, we study general rotational surfaces in the 4-dimensional pseudo-Euclidean space $E_2^4$ and obtain a characterization of flat general rotation surfaces with pointwise 1-type Gauss map in $E_2^4$ and give an example of such surfaces.

Key words: Rotation surface, Gauss map, Pointwise 1-type Gauss map, pseudo-Euclidean space.

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1 Introduction

A pseudo-Riemannian submanifold $M$ of the $m$-dimensional pseudo-Euclidean space $E^n_m$ is said to be of finite type if its position vector $x$ can be expressed as a finite sum of eigenvectors of the Laplacian $\Delta$ of $M$, that is, $x = x_0 + x_1 + ... x_k$, where $x_0$ is a constant map, $x_1, ..., x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are all different, then $M$ is said to be of $k$-type. This definition was similarly extended to differentiable maps in Euclidean and pseudo-Euclidean space, in particular, to Gauss maps of submanifolds [6].

If a submanifold $M$ of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map $G$, then $G$ satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector $C$. Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map and they proved that a compact hypersurface $M$ of $E^{n+1}$ has 1-type Gauss map if and only if $M$ is a hypersphere in $E^{n+1}$ [6].

However, the Laplacian of the Gauss map of several surfaces and hypersurfaces such as a helicoids of the 1st, 2nd and 3rddkind, conjugate Enneper’s surface of the second kind in 3-dimensional Minkowski space $E_3^3$, generalized catenoids,
spherical $n$-cones, hyperbolical $n$-cones and Enneper’s hypersurfaces in $E^n_1$ take the form namely,
\[ \Delta G = f(G + C) \] for some smooth function $f$ on $M$ and some constant vector $C$. A submanifold $M$ of a pseudo-Euclidean space $E^m_s$ is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1) for some smooth function $f$ on $M$ and some constant vector $C$. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector $C$ in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind.

Surfaces in Euclidean space and in pseudo-Euclidean space with pointwise 1-type Gauss map were recently studied in [5], [7], [8], [11], [12], [13], [14], [15], [17], [18]. Also Dursun and Turgay in [10] gave all general rotational surfaces in $E^4$ with proper pointwise 1-type Gauss map of the first kind and classified minimal rotational surfaces with proper pointwise 1-type Gauss map of the second kind. Arslan et al. in [2] investigated rotational embedded surface with pointwise 1-type Gauss map. Arslan at el. in [3] gave necessary and sufficient conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map. Yoon in [20] showed that flat Vranceanu rotation surface with pointwise 1-type Gauss map is a Clifford torus and in [19] studied rotation surfaces in the 4-dimensional Euclidean space with finite type Gauss map. Kim and Yoon in [16] obtained the complete classification theorems for the flat rotation surfaces with finite type Gauss map and pointwise 1-type Gauss map. The authors in [1] studied flat general rotational surfaces in the 4-dimensional pseudo-Euclidean space $E^4_2$ and obtained a characterization for flat general rotation surfaces with pointwise 1-type Gauss map and give an example of such surfaces.

2 Preliminaries

Let $E^m_s$ be the $m$-dimensional pseudo-Euclidean space with signature $(s, m-s)$. Then the metric tensor $g$ in $E^m_s$ has the form
\[ g = \sum_{i=1}^{m-s} (dx_i)^2 - \sum_{i=m-s+1}^{m} (dx_i)^2 \]
where $(x_1, ..., x_m)$ is a standard rectangular coordinate system in $E^m_s$.

Let $M$ be an $n$-dimensional pseudo-Riemannian submanifold of a $m$-dimensional pseudo-Euclidean space $E^m_s$. We denote Levi-Civita connections of $E^m_s$ and $M$ by $\nabla$ and $\nabla$, respectively. Let $e_1, ..., e_n, e_{n+1}, ..., e_m$ be an adapted local orthonormal frame in $E^m_s$ such that $e_1, ..., e_n$ are tangent to $M$ and $e_{n+1}, ..., e_m$ normal to
We use the following convention on the ranges of indices: 1 ≤ i, j, k, ..., ≤ n, n + 1 ≤ r, s, t, ..., ≤ m, 1 ≤ A, B, C, ..., ≤ m.

Let \( \omega_A \) be the dual-1 form of \( e_A \) defined by \( \omega_A(X) = \langle e_A, X \rangle \) and \( \varepsilon_A = \langle e_A, e_A \rangle = \pm 1 \). Also, the connection forms \( \omega_{AB} \) are defined by

\[
d e_A = \sum_B \varepsilon_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0
\]

Then we have

\[
\tilde{\nabla}^e = \sum_{j=1}^n \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=n+1}^m \varepsilon_r h^r_{ik} e_r
\]

and

\[
\tilde{\nabla}^{e_s} = -\sum_{j=1}^n \varepsilon_j h^s_{kj} e_j + \sum_{r=n+1}^m \varepsilon_r \omega_{sr}(e_k) e_r, \quad D^{e_s}_{ik} = \sum_{r=n+1}^m \omega_{sr}(e_k) e_r,
\]

where \( D \) is the normal connection, \( h^r_{ik} \) the coefficients of the second fundamental form \( h \).

If we define a covariant differentiation \( \tilde{\nabla} h \) of the second fundamental form \( h \) on the direct sum of the tangent bundle and the normal bundle \( TM \oplus T^\perp M \) of \( M \) by

\[
\left( \tilde{\nabla}_X h \right)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)
\]

for any vector fields \( X, Y \) and \( Z \) tangent to \( M \). Then we have the Codazzi equation

\[
\left( \tilde{\nabla}_X h \right)(Y, Z) = \left( \tilde{\nabla}_Y h \right)(X, Z)
\]

and the Gauss equation is given by

\[
\langle R(X, Y)Z, W \rangle = \langle h( X, W ) , h( Y, Z ) \rangle - \langle h( X, Z ) , h( Y, W ) \rangle
\]

where the vectors \( X, Y, Z \) and \( W \) are tangent to \( M \) and \( R \) is the curvature tensor associated with \( \nabla \). The curvature tensor \( R \) associated with \( \nabla \) is defined by

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

For any real function \( f \) on \( M \) the Laplacian \( \Delta f \) of \( f \) is given by

\[
\Delta f = -\sum_i \left( \tilde{\nabla}_{e_i}^e \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla^{e_i}_e} f \right)
\]

Let us now define the Gauss map \( G \) of a submanifold \( M \) into \( G(n, m) \) in \( \wedge^n \mathbb{E}^m_s \), where \( G(n, m) \) is the Grassmannian manifold consisting of all oriented \( n \)-planes through the origin of \( \mathbb{E}^m_s \) and \( \wedge^n \mathbb{E}^m_s \) is the vector space obtained by the exterior product of \( n \) vectors in \( \mathbb{E}^m_s \). Let \( e_{i_1} \wedge \ldots \wedge e_{i_n} \) and \( f_{j_1} \wedge \ldots \wedge f_{j_n} \) be two vectors of
where the profile curve of $M$ which carries a point $p$ in $M$.

In this section, we study the flat rotation surfaces with pointwise 1-type Gauss map in the 4-dimensional pseudo-Euclidean space $E^4$. Let $M_1$ and $M_2$ be the rotation surfaces in $E^4$ defined by

$$\varphi(t, s) = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & \cosh t & \sinh t & 0 \\ 0 & \sinh t & \cosh t & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix} \begin{pmatrix} 0 \\ x(s) \\ 0 \\ y(s) \end{pmatrix}.$$ (7)

and

$$\varphi(t, s) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix} \begin{pmatrix} x(s) \\ 0 \\ y(s) \\ 0 \end{pmatrix}.$$ (8)

where the profile curve of $M_1$ (resp. the profile curve of $M_2$) is unit speed curve, that is, $(x'(s))^2 - (y'(s))^2 = 1$. We choose a moving frame $e_1, e_2, e_3, e_4$ such that $e_1, e_2$ are tangent to $M_1$ and $e_3, e_4$ are normal to $M_1$ and choose a moving frame $\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4$ such that $\bar{e}_1, \bar{e}_2$ are tangent to $M_2$ and $\bar{e}_3, \bar{e}_4$ are normal to $M_2$ which are given by the following:

$$e_1 = \frac{1}{\sqrt{\varepsilon_1 (y^2(s) - x^2(s))}} (y(s) \cosh t, x(s) \sinh t, x(s) \cosh t, y(s) \sinh t)$$

$$e_2 = (y'(s) \sinh t, x'(s) \cosh t, x'(s) \sinh t, y'(s) \cosh t)$$

$$e_3 = (x'(s) \sinh t, y'(s) \cosh t, y'(s) \sinh t, x'(s) \cosh t)$$

$$e_4 = \frac{1}{\sqrt{\varepsilon_1 (y^2(s) - x^2(s))}} (x(s) \cosh t, y(s) \sinh t, y(s) \cosh t, x(s) \sinh t)$$

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and
\[
\tilde{e}_1 = \frac{1}{\sqrt{\varepsilon_1 (y^2(s) - x^2(s))}} (-x(s) \sin t, x(s) \cos t, -y(s) \sin t, y(s) \cos t)
\]
\[
\tilde{e}_2 = (x'(s) \cos t, x'(s) \sin t, y'(s) \cos t, y'(s) \sin t)
\]
\[
\tilde{e}_3 = (y'(s) \cos t, y'(s) \sin t, x'(s) \cos t, x'(s) \sin t)
\]
\[
\tilde{e}_4 = \frac{1}{\sqrt{\varepsilon_1 (y^2(s) - x^2(s))}} (y(s) \sin t, -y(s) \cos t, x(s) \sin t, -x(s) \cos t)
\]
where \(\varepsilon_1 (y^2(s) - x^2(s)) > 0, \varepsilon_1 = \pm 1\). Then it is easily seen that
\[
\langle e_1, e_1 \rangle = -\langle e_4, e_4 \rangle = \varepsilon_1, \quad \langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = 1
\]
\[
-\langle \tilde{e}_1, \tilde{e}_1 \rangle = \langle \tilde{e}_4, \tilde{e}_4 \rangle = \varepsilon_1, \quad \langle \tilde{e}_2, \tilde{e}_2 \rangle = -\langle \tilde{e}_3, \tilde{e}_3 \rangle = 1
\]
we have the dual 1-forms as:
\[
\omega_1 = \varepsilon_1 \sqrt{\varepsilon_1 (y^2(s) - x^2(s))} dt \quad \text{and} \quad \omega_2 = ds
\]
and
\[
\tilde{\omega}_1 = -\varepsilon_1 \sqrt{\varepsilon_1 (y^2(s) - x^2(s))} dt \quad \text{and} \quad \tilde{\omega}_2 = ds
\]
By a direct computation we have components of the second fundamental form and the connection forms as:
\[
h^3_{11} = b(s), \quad h^3_{12} = 0, \quad h^3_{22} = c(s)
\]
\[
h^4_{11} = 0, \quad h^4_{12} = b(s), \quad h^4_{22} = 0
\]
\[
\tilde{h}^3_{11} = -b(s), \quad \tilde{h}^3_{12} = 0, \quad \tilde{h}^3_{22} = c(s)
\]
\[
\tilde{h}^4_{11} = 0, \quad \tilde{h}^4_{12} = b(s), \quad \tilde{h}^4_{22} = 0
\]
\[
\omega_{12} = \varepsilon_1 a(s) \omega_1, \quad \omega_{13} = \varepsilon_1 b(s) \omega_1, \quad \omega_{14} = b(s) \omega_2
\]
\[
\omega_{23} = c(s) \omega_2, \quad \omega_{24} = \varepsilon_1 b(s) \omega_1, \quad \omega_{34} = \varepsilon_1 a(s) \omega_1
\]
\[
\tilde{\omega}_{12} = \varepsilon_1 a(s) \tilde{\omega}_1, \quad \tilde{\omega}_{13} = \varepsilon_1 b(s) \tilde{\omega}_1, \quad \tilde{\omega}_{14} = b(s) \tilde{\omega}_2
\]
\[
\tilde{\omega}_{23} = c(s) \tilde{\omega}_2, \quad \tilde{\omega}_{24} = -\varepsilon_1 b(s) \tilde{\omega}_1, \quad \tilde{\omega}_{34} = -\varepsilon_1 a(s) \tilde{\omega}_1
\]
By covariant differentiation with respect to \(e_1\) and \(e_2\) (resp. \(\tilde{e}_1\) and \(\tilde{e}_2\)) a straightforward calculation gives:
\[
\tilde{\nabla}_{e_1} e_1 = a(s) e_2 - b(s) e_3
\]
\[
\tilde{\nabla}_{e_2} e_1 = -\varepsilon_1 b(s) e_4
\]
\[
\tilde{\nabla}_{e_1} e_2 = -\varepsilon_1 a(s) e_1 - \varepsilon_1 b(s) e_4
\]
\[
\tilde{\nabla}_{e_2} e_2 = -c(s) e_3
\]
\[
\tilde{\nabla}_{e_1} e_3 = -\varepsilon_1 b(s) e_1 - \varepsilon_1 a(s) e_4
\]
\[
\tilde{\nabla}_{e_2} e_3 = -c(s) e_2
\]
\[
\tilde{\nabla}_{e_1} e_4 = -b(s) e_2 + a(s) e_3
\]
\[
\tilde{\nabla}_{e_2} e_4 = -\varepsilon_1 b(s) e_1
\]
and

\[ \tilde{\nabla}_{\tilde{e}_1} \tilde{e}_1 = -a(s)\tilde{e}_2 + b(s)\tilde{e}_3 \]
\[ \tilde{\nabla}_{\tilde{e}_2} \tilde{e}_1 = \varepsilon_1 b(s)\tilde{e}_4 \]
\[ \tilde{\nabla}_{\tilde{e}_1} \tilde{e}_2 = -\varepsilon_1 a(s)\tilde{e}_1 + \varepsilon_1 b(s)\tilde{e}_4 \]
\[ \tilde{\nabla}_{\tilde{e}_2} \tilde{e}_2 = -c(s)\tilde{e}_3 \]
\[ \tilde{\nabla}_{\tilde{e}_1} \tilde{e}_3 = -\varepsilon_1 b(s)\tilde{e}_1 + \varepsilon_1 a(s)\tilde{e}_4 \]
\[ \tilde{\nabla}_{\tilde{e}_2} \tilde{e}_3 = -c(s)\tilde{e}_2 \]
\[ \tilde{\nabla}_{\tilde{e}_1} \tilde{e}_4 = -b(s)\tilde{e}_2 + a(s)\tilde{e}_3 \]
\[ \tilde{\nabla}_{\tilde{e}_2} \tilde{e}_4 = \varepsilon_1 b(s)\tilde{e}_1 \]

where

\[ a(s) = \frac{x(s)x'(s) - y(s)y'(s)}{\varepsilon_1 (y^2(s) - x^2(s))} \] (17)
\[ b(s) = \frac{x(s)y'(s) - y(s)x'(s)}{\varepsilon_1 (y^2(s) - x^2(s))} \] (18)
\[ c(s) = x''(s)y'(s) - x'(s)y''(s) \] (19)

The Gaussian curvature \( K \) of \( M_1 \) and \( \bar{K} \) that of \( M_2 \) are respectively given by

\[ K = \varepsilon_1 b^2(s) - b(s)c(s) \] (20)

and

\[ \bar{K} = b(s)c(s) - \varepsilon_1 b^2(s) \] (21)

If the surfaces \( M_1 \) or \( M_2 \) is flat, then (20) and (21) imply

\[ b(s)c(s) - b^2(s) = 0. \] (22)

Furthermore, after some computations we obtain Gauss and Codazzi equations for both surfaces \( M_1 \) and \( M_2 \)

\[ \varepsilon_1 a^2(s) - a'(s) = b(s)c(s) - \varepsilon_1 b^2(s) \] (23)

and

\[ b'(s) = 2\varepsilon_1 a(s)b(s) - a(s)c(s) \] (24)

respectively.

By using (6), (15), (16) and straightforward computation, the Laplacians \( \Delta G \) and \( \bar{\Delta} G \) of the Gauss map \( G \) and \( \bar{G} \) can be expressed as

\[ \Delta G = -\left(3b^2(s) + c^2(s)\right) (e_1 \wedge e_2) + (2a(s)b(s) - \varepsilon_1 a(s)c(s) + c'(s))(e_1 \wedge e_3) \]
\[ + (3a(s)b(s) - \varepsilon_1 b'(s))(e_2 \wedge e_4) + 2 \left(\varepsilon_1 b(s)c(s) - b^2(s)\right)(e_3 \wedge e_4) \] (25)
\[
\Delta \bar{G} = - (3b^2(s) + c^2(s)) (e_1 \wedge e_2) + (2a(s)b(s) - \varepsilon_1 a(s)c(s) + c'(s)) (e_1 \wedge e_3) \\
+ (-3a(s)b(s) + \varepsilon_1 b'(s)) (e_2 \wedge e_4) + 2 (b^2(s) - \varepsilon_1 b(s)c(s)) (e_3 \wedge e_4)
\] (26)

Now we investigate the flat rotation surfaces in \(E^4_2\) with the pointwise 1-type Gauss map satisfying (1).

Suppose that the rotation surface \(M_1\) given by the parametrization (7) is a flat rotation surface. From (20), we obtain that \(b(s) = 0\) or \(\varepsilon_1 b(s) - c(s) = 0\). We assume that \(\varepsilon_1 b(s) - c(s) \neq 0\). Then \(b(s)\) is equal to zero and (24) implies that \(a(s)c(s) = 0\). Since \(\varepsilon_1 b(s) - c(s) \neq 0\), it implies that \(c(s)\) is not equal to zero. Then we obtain as \(a(s) = 0\). In that case, by using (17) and (18) we obtain that \(\alpha(s) = 0, x(s), 0, y(s)\) is a constant vector. This is a contradiction. Therefore \(\varepsilon_1 b(s) = c(s)\) for all \(s\).

From (14), we get

\[
\varepsilon_1 a^2(s) - a'(s) = 0
\] (27)

whose the trivial solution and non-trivial solution

\[
a(s) = 0
\]

and

\[
a(s) = \frac{1}{-\varepsilon_1 s + c'},
\]

respectively. We assume that \(a(s) = 0\). By (24) \(b(b_0)\) is a constant and \(c = \varepsilon_1 b_0\).

In that case by using (17), (18) and (19), \(x\) and \(y\) satisfy the following differential equations

\[
x^2(s) - y^2(s) = \mu \quad \mu \text{ is a constant,}
\] (28)

\[
x(s)y'(s) - x'(s)y(s) = -\varepsilon_1 b_0 \mu,
\] (29)

\[
x''y'(s) - x'(s)y'' = \varepsilon_1 b_0.
\] (30)

From (28) we may put

\[
x(s) = \frac{1}{2} \varepsilon \left( \mu_2 e^{\theta(s)} + \mu_1 e^{-\theta(s)} \right), \quad y(s) = \frac{1}{2} \varepsilon \left( \mu_2 e^{\theta(s)} - \mu_1 e^{-\theta(s)} \right),
\] (31)

where \(\theta(s)\) is some smooth function, \(\varepsilon = \pm 1\) and \(\mu = \mu_1 \mu_2\). Differentiating (31) with respect to \(s\), we have

\[
x'(s) = \theta'(s)y(s), \quad y'(s) = \theta'(s)x(s)
\] (32)

By substituting (31) and (32) into (19), we get

\[
\theta(s) = -\varepsilon_1 b_0 s + d, \quad d = \text{const.}
\]

And since the curve \(\alpha\) is a unit speed curve, we have

\[
b_0^2 \mu = -1.
\]
Since $\mu = -\frac{1}{b_0}$, $y^2(s) - x^2(s) > 0$. In that case we obtain that $\varepsilon_1 = 1$. Then we can write components of the curve $\alpha$ as:

\begin{align*}
x(s) &= \frac{1}{2} \varepsilon \left( \mu_2 e^{(-b_0 s + d)} + \mu_1 e^{(-b_0 s + d)} \right), \\
y(s) &= \frac{1}{2} \varepsilon \left( \mu_2 e^{(-b_0 s + d)} - \mu_1 e^{(-b_0 s + d)} \right), \quad \mu_1 \mu_2 = -\frac{1}{b_0^2}
\end{align*}

(33)

On the other hand, by using (25) we can rewrite the Laplacian of the Gauss map $G$ with $a(s) = 0$ and $b = c = b_0$ as follows:

$$\Delta G = -4b_0^2 (e_1 \wedge e_2)$$

that is, the flat surface $M$ is pointwise 1-type Gauss map with the function $f = 4b_0^2$ and $C = 0$. Even if it is a pointwise 1-type Gauss map of the first kind.

Now we assume that $a(s) = \frac{1}{-\varepsilon_1 s + c}$. By using $c(s) = \varepsilon_1 b(s)$ and (24) we get

$$b'(s) = \varepsilon_1 a(s) b(s)$$

(34)

or we can write

$$\frac{b'(s)}{b(s)} = \frac{\varepsilon_1}{-\varepsilon_1 s + c},$$

whose the solution

$$b(s) = \frac{\lambda}{| -\varepsilon_1 s + c |}, \quad \lambda \text{ is a constant.}$$

(35)

By using (25) we can rewrite the Laplacian of the Gauss map $G$ with the equations $c(s) = \varepsilon_1 b(s)$, $b'(s) = \varepsilon_1 a(s) b(s)$ and $a'(s) = \varepsilon_1 a^2(s)$

$$\Delta G = -4b_0^2 (e_1 \wedge e_2) + 2a(s)b(s) (e_1 \wedge e_3) + 2a(s)b(s) (e_2 \wedge e_4).$$

(36)

We suppose that the flat rotational surface $M_1$ has pointwise 1-type Gauss map. From (1) and (36), we get

\begin{align*}
-4\varepsilon_1 b^2(s) &= f \varepsilon_1 + f \langle C, e_1 \wedge e_2 \rangle \\
-2\varepsilon_1 a(s)b(s) &= f \langle C, e_1 \wedge e_3 \rangle \\
-2\varepsilon_1 a(s)b(s) &= f \langle C, e_2 \wedge e_4 \rangle
\end{align*}

(37) (38) (39)

Then, we have

$$\langle C, e_1 \wedge e_4 \rangle = 0, \quad \langle C, e_2 \wedge e_3 \rangle = 0, \quad \langle C, e_3 \wedge e_4 \rangle = 0$$

(40)

By using (38) and (39) we obtain

$$\langle C, e_1 \wedge e_3 \rangle = \langle C, e_2 \wedge e_4 \rangle$$

(41)
By differentiating the first equation in (41) with respect to \( e_1 \) and by using the third equation in (41) and (42), we get

\[
2a(s)\langle C, e_1 \wedge e_3 \rangle - b(s)\langle C, e_1 \wedge e_2 \rangle = 0
\]  

(42)

Combining (38), (39) and (42) we then have

\[
f = 4 \left( a^2(s) - b^2(s) \right)
\]

that is, a smooth function \( f \) depends only on \( s \). By differentiating \( f \) with respect to \( s \) and by using (35) and (27), we get

\[
f' = 2\varepsilon_1 a(s)f
\]

(43)

By differentiating (38) with respect to \( s \) and by using (15), (27), (35), (36) and (37) we have

\[a^2b = 0\]

or from (35) we can write

\[\lambda a^3 = 0\]

Since \( a(s) \neq 0 \), it follows that \( \lambda = 0 \). Then we obtain that \( b = c = 0 \). Then the surface \( M_1 \) is a part of plane.

Thus we can give the following theorems.

**Theorem 1.** Let \( M_1 \) be the flat rotation surface given by the parametrization (7). Then \( M_1 \) has pointwise 1-type Gauss map if and only if \( M \) is either totally geodesic or parametrized by

\[
\varphi(t, s) = \begin{pmatrix}
\frac{1}{2}\varepsilon \left( \mu_2 e^{(-b_0s+d)} - \mu_1 e^{(-b_0s+d)} \right) \sinh t, \\
\frac{1}{2}\varepsilon \left( \mu_2 e^{(-b_0s+d)} + \mu_1 e^{(-b_0s+d)} \right) \cosh t,
\end{pmatrix}, \quad \mu_1\mu_2 = -\frac{1}{b_0^2}
\]

(44)

where \( b_0, \mu_1, \mu_2 \) and \( d \) are real constants.

**Example 1.** Let \( M_1 \) be the flat rotation surface with pointwise 1-type Gauss map given by the parametrization (44). If we take as \( b_0 = -1, \mu_1 = -1, \mu_2 = 1, d = 0 \) and \( \varepsilon = 1 \), then we obtain a surface as follows:

\[
\varphi(t, s) = (\cosh s \sinh t, \sinh s \cosh t, \sinh s \sinh t, \cosh s \cosh t).
\]

This surface is the product of two plane hyperbolas.

**Theorem 2.** Let \( M_2 \) be the flat rotation surface given by the parametrization (8). Then \( M_2 \) has pointwise 1-type Gauss map if and only if \( M_2 \) is either totally geodesic or parametrized by

\[
\varphi(t, s) = \begin{pmatrix}
\frac{1}{2}\varepsilon \left( \mu_2 e^{(-b_0s+d)} + \mu_1 e^{(-b_0s+d)} \right) \cos t, \\
\frac{1}{2}\varepsilon \left( \mu_2 e^{(-b_0s+d)} + \mu_1 e^{(-b_0s+d)} \right) \sin t,
\end{pmatrix}, \quad \mu_1\mu_2 = -\frac{1}{b_0^2}
\]

(45)
Example 2. Let $M_1$ be the flat rotation surface with pointwise 1-type Gauss map given by the parametrization (44). If we take as $b_0 = -1$, $\mu_1 = -1$, $\mu_2 = 1$, $d = 0$ and $\varepsilon = 1$, then we obtain a surface as follows:

$$\varphi(t, s) = \varphi(t, s) = (\cosh s \cos t, \cosh s \sin t, \cosh s \cos t, \cosh s \sin t).$$

This surface is the product of a plane circle and a plane hyperbola.

Corollary 1. Let $M$ be flat general rotation surface given by the parametrization (7) or (8). If $M$ has pointwise 1-type Gauss map then the Gauss map $G$ on $M$ is of 1-type.

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