A Distributed Quaternion Kalman Filter With Applications to Fly-by-Wire Systems

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Abstract—The introduction of automated flight control and management systems have made possible aircraft designs that sacrifice aerodynamic stability in order to incorporate stealth technology into their shape, operate more efficiently, and are highly maneuverable. Therefore, modern flight management systems are reliant on multiple redundant sensors to monitor and control the rotations of the aircraft. To this end, a novel distributed quaternion Kalman filtering algorithm is developed for tracking the rotation and orientation of an aircraft in the three-dimensional space. The algorithm is developed to distribute computation among the sensors in a manner that forces them to consent to a unique solution while being robust to sensor and link failure, a desirable characteristic for flight management systems. In addition, the underlying quaternion-valued state space model allows to avoid problems associated with gimbal lock. The performance of the developed algorithm is verified through simulations.

Index Terms—Quaternion-valued adaptive signal processing, quaternion-valued state space modeling, distributed adaptive filtering, fly-by-wire.

I. INTRODUCTION

The introduction of fully automated flight control and management systems commonly referred to as “fly-by-wire” has lead to a revolution in aircraft design, allowing for aircraft configurations that are highly maneuverable, incorporating stealth technology into their shape, and provide increased aerodynamic efficiency, at a cost of reduced aerodynamic stability [1]. Since these aircraft configurations are aerodynamically unstable, they are reliant on modern flight management systems that monitor and adjust the pitch, roll, and yaw rotations of the aircraft multiple times per second in order to ensure controlled flight. Given the importance of fly-by-wire systems and from a safety perspective, the following characteristics are highly desirable:

1) The system should incorporate multiple redundant sensors to ensure against sensor failure, in a manner that makes all sensors consent to the best solution.
2) The system should be robust to failure of some of its sensors or the links between them.
3) The underlying mathematical frame work should be robust to problems associated with gimbal lock.

In recent years, sensor networks have been used in a variety of applications, such as collaborative target tracking, distributed fault detection, smart grid magnet systems, and automated vehicle guidance technology [2]-[15]. In addition, owing to the low implementation cost and computational efficiency that distributed estimation and tracking techniques offer, as compared to their centralized counterparts, distributed signal processing algorithms, especially diffusion based algorithms, have proven to be suitable for real-time implementation and robust to link or sensor failure [8], [16].

Quaternions have been used in mathematics, physics, and computer graphics for modeling three-dimensional rotations and orientation in a compact and computationally efficient fashion, where their division algebra allows to avoid problems associated with gimbal lock [17], providing accurate and mathematically tractable solutions with fewer constraints than those obtained in the real domain through vector algebras. In addition, the recent introduction of the $\mathbb{H}$-calculus [18]-[19], a framework for calculating the derivatives of quaternion-valued functions, and the augmented second-order statistics of quaternion-valued signals [20]-[21], a framework for exploiting the full second-order statistical information of quaternion-valued signals, has led to the development of quaternion-valued signal processing algorithms, such as the class of quaternion Kalman filters developed in [22], that are shown to outperform their real and complex-valued counter parts in applications including frequency estimation in smart grids [23], kernel learning [24], and spacecraft orientation tracking [25].

In light of the advantages that quaternion-valued signal processing algorithms offer, we expand the framework of quaternion-valued Kalman filtering to the distributed setting in order to develop a novel distributed quaternion Kalman filter applicable for tracking the rotations of an aircraft. The distributed quaternion Kalman filter is developed through decomposing the operations of the centralized quaternion Kalman filter in such a fashion that they can be performed locally by the individual sensors and allows the sensors to detect erroneous measurements through the introduction of an confidence measure. Finally the performance of the developed algorithm is demonstrated through simulations.

Mathematical notations: Scalars, column vectors, and matrices are denoted by lower case, bold lower case, and bold upper case letters respectively. The transpose and Hermitian transpose operators are denoted by $\cdot^T$ and $\cdot^H$, whereas $E[\cdot]$ denotes the statistical expectation operator. Finally, the real and quaternion domains are denoted by $\mathbb{R}$ and $\mathbb{H}$. 
II. QUATERNION ALGEBRA AND STATISTICS

Quaternions are a non-commutative, associative division algebra, where a quaternion variable $q \in \mathbb{H}$ is consisted of a real part, $\Re(q)$, and a three-dimensional imaginary part or pure quaternion, $\Im(q)$, comprised of the three imaginary components, $\Im_i(q)$, $\Im_j(q)$, and $\Im_k(q)$. Hence, $q$ can be expressed as

$$q = \Re(q) + \Im(q) = \Re(q) + \Im_i(q) + \Im_j(q) + \Im_k(q)$$

where $q_r, q_i, q_j, q_k \in \mathbb{F}$, while $i$, $j$, and $k$ are imaginary units obeying the following product rules

$$ij = k, jk = i, ki = j, i^2 = j^2 = k^2 = ijk = -1$$

whereas the conjugate and norm of $q \in \mathbb{H}$ are given by $q^* = \Re(q) - \Im(q)$ and $|q| = \sqrt{qq^*}$ respectively. A quaternion $q \in \mathbb{H}$ can alternatively be expressed by its polar presentation, given by [27]

$$q = |q|e^{\xi \theta} = |q|(\cos(\theta) + \xi \sin(\theta))$$

where

$$\xi = \frac{\Im(q)}{|\Im(q)|}, \quad \theta = \arctan\left(\frac{\Im(q)}{\Re(q)}\right)$$

The involution of $q \in \mathbb{H}$ around $\mu \in \mathbb{H}$ is defined as $q^\mu = \mu q \mu^{-1}$ [28] and can be seen as the quaternion equivalent of the complex conjugate, as the real-valued components of a quaternion number, $q \in \mathbb{H}$, can be expressed using involutions as [18]-[19], [20]

$$q_r = \frac{1}{4} (q + q^i + q^j + q^k), \quad q_i = \frac{1}{4i} (q + q^i - q^j - q^k)$$

$$q_j = \frac{1}{4j} (q - q^i + q^j - q^k), \quad q_k = \frac{1}{4k} (q - q^i - q^j + q^k).$$

Moreover, quaternion involutions have seen extensive use in expressing three-dimensional rotations. In this setting, a three-dimensional vector expressed in the Cartesian coordinates as $\xi \in \mathbb{R}^3$ and containing three-dimensional rotations. In this setting, a three-dimensional rotation can be performed through the use of a rotation matrix $A = e^{\xi \theta}$, where $A$ is a 3D rotation matrix and $\theta$ denotes the rotation angle.

The augmented quaternion vector $q^a = [q, q^i, q^j, q^k]^T \in \mathbb{H}^4$ and the augmented quaternion statistics in conjunction with their pseudo-covariances have also been instrumental in the development of augmented quaternion Kalman filters [22]. For example, the evolution of the augmented quaternion Kalman filters can be expressed in real-time through observations

$$y_n = H_n x_n^a + \omega_n$$

where $y_n$ and $H_n$ are the augmented observation vector and observation matrix at time instant $n$, while $\omega_n$ is the augmented measurement noise sequence. The objective is to track $x_n^a$ in real-time through observations

$$\hat{x}_n^a = A_n^a \hat{x}_{n-1}^a + \nu_{n-1}^a$$

The augmented quaternion statistics in conjunction with the $\mathbb{H}$-calculus have led to the development of a class of augmented quaternion Kalman filters [22]. For example, consider the evolution of the quaternion-valued augmented state vector sequence $\{x_n^a, n = 0, 1, 2, \ldots\}$, given by

$$x_n^a = A_n^a x_{n-1}^a + \nu_{n-1}^a$$

where $A_n^a$ is the state evolution matrix at time instant $n$ and $\nu_n^a$ is the augmented state evolution noise sequence. The objective is to track $x_n^a$ in real-time through observations

$$y_n^a = H_n^a x_n^a + \omega_n^a$$

III. THE DISTRIBUTED AUGMENTED QUATERNION KALMAN FILTER

Consider a set of sensors denoted by $\mathcal{N}$ interconnected in a network where the neighborhood of a sensor is referred to the isomorphism between $\mathbb{R}^4$ and $\mathbb{H}$, a relation is established between the derivatives taken in $\mathbb{R}^4$ and those taken directly in $\mathbb{H}$, allowing for a unified framework for calculating the derivatives of quaternion-valued functions directly in $\mathbb{H}$.
as the subset of sensors that communicate with it, including self-communication. Organizing all observations made by the sensors of such a network in the column vector

\[ y_{\text{col},n} = [y_{1,n}, \ldots, y_{|\mathcal{N}|,n}]^T \]

where \( y_{n} \) represents the augmented observation vector at node \( m \) at time \( n \) and \( |\mathcal{N}| \) denotes the number of nodes in the network, allows the augmented state vector sequence to be estimated by the centralized augmented quaternion Kalman filter (CAQKF) given in Algorithm 2 where

\[ H_{\text{col},n} = [H_{1,n}^T, \ldots, H_{|\mathcal{N}|,n}^T]^T \]

is the column block matrix of the augmented observation functions with \( H_{m,n} \) representing the observation function at node \( m \) and at time instant \( n \), while \( \omega_{\text{col},n} \) is the augmented covariance matrix of the column vector of the combined augmented observational noises given by

\[ \omega_{\text{col},n} = [\omega_1^T, \ldots, \omega_{|\mathcal{N}|}^T]^T \]

with \( \omega_{m,n} \) denoting the observational noise at node \( m \) and at time instant \( n \).

**Algorithm 2.** Centralized Augmented Quaternion Kalman Filter (CAQKF)

**Initialize with:**

\[ \hat{x}_{0|0}^a = E[x_0^a] \]
\[ \hat{M}_{0|0}^a = E[(x_0^a - E[x_0^a])(x_0^a - E[x_0^a])^H] \]

**Model update:**

\[ \hat{x}_{n|n-1}^a = A_{n-1}^a \hat{x}_{n-1|n-1} \]
\[ \hat{M}_{n|n-1}^a = A_{n-1}^a \hat{M}_{n-1|n-1}^a A_{n-1}^{AH} + C_{\nu}^a \]

**Measurement update:**

\[ \hat{M}_{n|n}^a = \hat{M}_{n|n-1}^a + H_{\text{col},n}^a C_{\omega_{\text{col},n}}^{-1} H_{\text{col},n}^a \]
\[ G_n = \hat{M}_{n|n}^{-1} H_{\text{col},n}^a C_{\omega_{\text{col},n}}^{-1} \]
\[ \hat{x}_{n|n}^a = \hat{x}_{n|n-1} + G_n (y_{\text{col},n} - H_{\text{col},n} \hat{x}_{n|n-1}) \]

Although the CAQKF is optimal in the sense that it incorporates all the available information in the network, its operation is reliant on a central processing unit and therefore vulnerable to its failure. However, it is reasonable to assume that the observational noise at one sensor is uncorrelated with the observational noise at other sensors in the network, which leads to \( C_{\omega_{\text{col},n}} \) having a block diagonal structure. Therefore, the \( a \text{ posteriori} \) augmented state vector estimate can be expressed as

\[ \hat{x}_{n|n}^a = \hat{x}_{n|n-1} + \sum_{\forall l \in \mathcal{N}} \hat{M}_{n|n}^a H_{l,n}^{AH} C_{\omega_{l,n}}^{-1} \left( y_{l,n}^a - H_{l,n}^a \hat{x}_{l,n|n-1} \right) \]

(2)

Therefore, the \( a \text{ posteriori} \) augmented state vector estimate in (2) can be alternatively calculated by the summation

\[ \hat{x}_{n|n}^a = \hat{x}_{n|n-1} + \sum_{\forall l \in \mathcal{N}} \Delta \hat{x}_{l,n}^a \]

(3)

where

\[ \Delta \hat{x}_{l,n}^a = \hat{M}_{n|n}^a H_{l,n}^{AH} C_{\omega_{l,n}}^{-1} \left( y_{l,n}^a - H_{l,n}^a \hat{x}_{l,n|n-1} \right) \]

(4)

and represents the expected update to \( \hat{x}_{n|n} \) given the observation made at node \( l \) and time instant \( n \). Moreover, from Algorithm 2 assuming uncorrelated observation noise throughout the network we have

\[ \hat{M}_{n|n}^{-1} = \hat{M}_{n|n-1}^{-1} + \sum_{\forall l \in \mathcal{N}} H_{l,n}^{AH} C_{\omega_{l,n}}^{-1} H_{l,n}^a \]

(5)

Notice that \( \hat{M}_{n|n}^a \) in the formulation given in (5) can be calculated locally by the sensors in the network through the diffusion of the parameters \( H_{l,n}^{AH} C_{\omega_{l,n}}^{-1} H_{l,n}^a \), where it should be noted that this is only required in occasions when the parameter changes at a sensor. Now, given that \( \hat{M}_{n|n}^a \) and \( \hat{M}_{n|n-1}^a \) can be calculated locally through (5) and the state space model respectively, \( \Delta \hat{x}_{l,n}^a \) in the formulation given in (6) can be calculated by the individual sensors of the network allowing the \( a \text{ posteriori} \) estimate of the augmented state vector, \( \hat{x}_{l,n}^a \), to also be calculated locally through diffusing \( \Delta \hat{x}_{l,n}^a \) over the network. Therefore, the operations of the CAQKF can be mirrored in a distributed fashion through implementing the distributed augmented quaternion Kalman filter (DAQKF) in Algorithm 3 where \( \mathcal{N}_l \) denotes the set of nodes in the neighborhood of node \( l \).

**Algorithm 3.** Distributed Augmented Quaternion Kalman Filter (DAQKF)

**For node \( l = \{1, \ldots, |\mathcal{N}|\} \):**

**Initialize with:**

\[ \hat{x}_{l,0|0} = E[x_0^a] \]
\[ \hat{M}_{l,0|0}^a = E[(x_0^a - E[x_0^a])(x_0^a - E[x_0^a])^H] \]

**Model update:**

\[ \hat{x}_{l,n|n-1} = A_{l,n-1}^a \hat{x}_{l,n-1|n-1} \]
\[ \hat{M}_{l,n|n-1}^a = A_{l,n-1}^a \hat{M}_{l,n-1|n-1}^a A_{l,n-1}^{AH} + C_{\nu}^a \]

**Measurement update:**

\[ \hat{M}_{l,n|n}^a = \hat{M}_{l,n|n-1}^a + \sum_{\forall m \in \mathcal{N}_l} \left( H_{m,n}^a C_{\omega_{m,n}}^{-1} H_{m,n}^a \right) \]
\[ \Delta \hat{x}_{l,n}^a = \hat{M}_{l,n|n}^{-1} H_{l,n}^{AH} C_{\omega_{l,n}}^{-1} \left( y_{l,n}^a - H_{l,n}^a \hat{x}_{l,n|n-1} \right) \]
\[ \hat{x}_{l,n|n}^a = \hat{x}_{l,n|n-1} + \sum_{\forall m \in \mathcal{N}_l} \Delta \hat{x}_{m,n}^a \]

Given that \( y_{l,n} = H_{l,n}^a x_{l,n} + \omega_{l,n} \), from Algorithm 3 we have

\[ \Delta \hat{x}_{l,n}^a = \hat{M}_{l,n|n}^{-1} H_{l,n}^{AH} C_{\omega_{l,n}}^{-1} H_{l,n}^a \left( x_{l,n} - \hat{x}_{l,n|n-1} \right) \]

(6)

In addition, it is reasonable to assume that for two neighboring nodes at convergence \( \hat{M}_{n|n}^a \approx \hat{M}_{l,n|n}^a \approx \hat{M}_{m,n|n}^a \) and \( \hat{x}_{l,n|n} \approx \hat{x}_{m,n|n} \approx \hat{x}_{n|n} \); then, after some mathematical
manipulations, from the expression in (9), we have
\[ r_{l,m} = \Delta x_{l,n} - \Delta x_{m,n} \approx M_{n|n} P_{l,n} \left( x_n - \hat{x}_{n|n-1} \right) 
+ M_{n|n} H_{l,n}^H C_{\omega_{l,n}^a} \omega_{l,n} - M_{n|n} H_{m,n}^H C_{\omega_{m,n}^a} \omega_{m,n} \]
where \( P_{l,n} = H_{l,n}^H C_{\omega_{l,n}^a}^{-1} H_{l,n}^H - H_{m,n}^H C_{\omega_{m,n}^a}^{-1} H_{m,n}^H \). Given that \( \hat{x}_{n|n-1} \) is an unbiased estimate of \( x_n \), then, \( r_{l,m} \) in the formulation given in (7) is a zero mean quaternion-valued Gaussian random variable with augmented covariance matrix
\[ C_{r_{l,m}} = M_{n|n} P_{l,n} M_{n|n}^H P_{l,n}^H M_{n|n}^H \]
\[ + M_{n|n} H_{l,n}^H C_{\omega_{l,n}^a}^{-1} H_{l,n}^H M_{n|n} C_{\omega_{m,n}^a}^{-1} H_{m,n}^H M_{n|n}^H \]
Therefore, the measure \( r_{l,m}^H C_{r_{l,m}}^{-1} r_{l,m} \) can be used to indicate the likelihood that the update from sensor \( m \) has resulted form an erroneous observation. In this setting, each sensor will stop cooperation with neighboring sensors that are detected as producing erroneous updates effectively isolating faulty nodes.

IV. APPLICATION

One of the most important tasks of fly-by-wire systems is to track the rotations of the aircraft with respect to an fixed inertial coordinate system in real-time. To this end, accelerometers are used to measure the three Euler angles \( \alpha, \beta, \) and \( \gamma \), as shown in Fig. 1 that respectively represent roll, pitch, and yaw angles that are within the range \([ -\pi, \pi ]\). The total rotation of the aircraft is now fully characterized by the pure quaternion
\[ \kappa = \ln \left( e^{i\alpha} e^{j\beta} e^{k\gamma} \right) \]
where \( \ln(\cdot) \) represents the natural logarithm, \( \kappa/|\kappa| \) gives the rotation axis, with the rotation angle given by \( |\kappa| \).

In order to track three-dimensional rotations in real-time, the state vector \( x_n = [\kappa, \partial \kappa / \partial t]^T \) with state evolution function
\[ x_n = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Delta T \ x_{n-1} + \begin{bmatrix} \frac{1}{2} (\Delta T)^2 \\ \Delta T \end{bmatrix} \nu_n \]
was considered in its augmented formulation, where \( \partial \kappa / \partial t \) indicates the first-order rate of change of \( \kappa \), with its second-order rate of change modeled as state evolution noise \( \nu \), whereas \( \Delta T \) denotes the sampling interval.

The sensor network shown in Fig. 2 was used to follow synthetically generated three-dimensional rotations, where 60 s after the experiment started the sensors marked in red suffered a fault leading them to only present the observational noise as measurements; in addition, the links marked in red failed. The sampling interval was set to \( \Delta T = 0.04 \) s, whereas the state noise sequence was a zero-mean unit variance quaternion Gaussian variable with all its pseudo-covariances equal to 0.33 and for all nodes in the network the observational noise was selected as a zero-mean quaternion Gaussian variable with variance equal to 0.009 and with all pseudo-covariances equal to 0.007. The estimate of the rotation parameters is shown in Fig. 3. Observe that the proposed algorithm accurately tracked the three-dimensional rotations even after two of its nodes had stopped operating correctly.

Figure 2. The sensor network used for simulations. The sensors and links marked in red failed 60 seconds after the simulation started.

Figure 3. The real-valued components of the quaternion rotation parameter \( \kappa \).

V. CONCLUSION

A novel distributed augmented quaternion Kalman filter has been developed for tracking three-dimensional rotations using sensor networks. This has been achieved through decomposing the operations of the centralized augmented quaternion Kalman filter in such a fashion that they can be performed by individual sensors in the network so that the final state vector estimate can be obtained by diffusing local update vectors obtained at each sensor. In addition a confidence measure is introduced to detect faulty sensors in the network.
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