Abstract

Using a novel resummation procedure of thermal loops, real-time correlations in the scalar and pseudo-scalar channels are studied in the $O(4)$ linear $\sigma$ model at finite temperature. A threshold enhancement of the spectral function in the scalar channel is shown to be a noticeable precritical phenomenon of the chiral phase transition.

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Observing the restoration of chiral symmetry at finite-temperature ($T$) is one of the central aims in the future relativistic heavy-ion experiments at RHIC and LHC [1]. Numerical studies of quantum chromodynamics (QCD) on the lattice for two and three flavors are actively pursued to reveal the nature of the chiral transition [2].

From the theoretical point of view, one of the ideal signals of the second-order (or weakly first order) chiral transition is the existence of the soft modes associated with chiral order parameters $\phi(x, t) = (\bar{q}q(x, t), \bar{q}\gamma_5\tau q(x, t))$ [3]. The softening is a dynamical phenomena characterized by the anomalous enhancement of the dynamic susceptibility defined by

$$D^R_\phi(\omega, k) = -i \int d^4x e^{i k x} \theta(t) \langle [\phi(x, t)\phi(0, 0)]_T \rangle. \quad (1)$$

This phenomena at finite $T$ was first discussed in ref. [4] using an effective theory of QCD. Also there are subsequent theoretical and phenomenological studies on this problem [5]. In condensed matter and solid state physics, the soft modes have been studied extensively in neutron and light scattering experiments [6].

The purposes of this paper are twofold. First one is to study the spectra of the soft modes at finite $T$ by taking into account the strong coupling between the scalar and pseudo-scalar channels. Despite the fact that this has a considerable effect on the qualitative behavior of $D^R_\phi$, it has not been studied seriously so far. The second purpose is to present a systematic resummation procedure of thermal loops which is applicable even when the dynamical symmetry breaking takes place. We will show that this procedure is inevitable for studying the first problem.

To demonstrate the above points, we adopt the $O(4)$ linear $\sigma$ model with explicit symmetry breaking. The model has been used to study critical exponents of the chiral transition [7] on the basis of the static and dynamical universality. The Lagrangian density reads

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi_i)^2 + \mu^2 \phi_i^2] - \frac{\lambda}{4!}(\phi_i^2)^2 + h\phi_0 + \text{counter terms},$$

with $\phi_i = (\sigma, \pi)$, and $h\phi_0$ being the explicit symmetry breaking term. All the divergences are removed by counter terms in the symmetric phase ($\mu^2 < 0$) in the $\overline{MS}$ scheme. In this symmetric and mass independent scheme, the renormalization constants are $\mu^2$ independent [8]. The lagrangian after dynamical symmetry breaking ($\mu^2 > 0$) is obtained from (2) by the replacement $\sigma \rightarrow \sigma + \xi$, where $\xi$ is determined by the stationary condition for the effective potential $\partial V(\xi)/\partial \xi = 0$.

Causal meson propagators at $T = 0$ have a general form $D_\phi(q) = [q^2 - m_{\phi}^2 - \Sigma_\phi(q^2) + i\epsilon]^{-1}$, where $m_\phi$ is the tree-level mass, namely $m_{\phi}^2 = -\mu^2 + \lambda \xi^2/2$ and $m_{\phi}^2 = -\mu^2 + \lambda \xi^2/6$. The one-loop calculation of the self-energy $\Sigma_\phi$ is standard [8] and we do not recapitulate it here.

The renormalized couplings $\mu^2$, $\lambda$ and $h$ are determined by the following renormalization conditions: (i) on-shell condition for pion, $D_\pi^{-1}(q^2 = m_\pi^2) = 0$ with $m_\pi = 140$ MeV, (ii) PCAC in one-loop, $f_\pi m_\pi^2 = h\sqrt{Z_\pi}$ with $Z_\pi$ being the wave function renormalization constant for pion and $f_\pi = 93$ MeV, (iii) the peak position of the spectral function $\rho_\sigma = -(1/\pi)\text{Im}D_\sigma$ is chosen to be $550$ MeV; this number corresponds to the value obtained in the recent re-analyses of the $\pi-\pi$ scattering phase shift [9]. We have also taken $750$ MeV and $1$ GeV [4], and checked that our main conclusions do not suffer qualitative change. The arbitrary
renormalization point $\kappa$ is chosen so that $\rho_\sigma(\omega)$ starts from the correct continuum threshold at $\omega = 2m_\pi$: This is achieved by demanding $m_{0\pi} = m_\pi$. The resultant values read $\mu^2 = (283\text{ MeV})^2$, $\lambda = 73.0$, $h = (123\text{ MeV})^3$ and $\kappa = 255\text{ MeV}$.

As has been known since the works of Weinberg, and Kirzhnits and Linde \cite{10}, naive perturbation theory breaks down for sufficiently high $T$. In the linear $\sigma$ model, this is easily seen from the behavior of $m_{0\phi}$ defined above which becomes tachyonic as one approaches to the symmetric phase ($\xi \rightarrow 0$). This not only destroys the credibility of the loop expansion at high $T$, but also leads to an unphysical threshold in $\rho_\sigma(\omega)$ even at low $T$. The latter aspect is particularly harmful for the purpose of this paper.

To cure these problems, we developed a method based on a resummation procedure proposed by Banerjee and Mallik \cite{11}. It is a generalized mean-field theory and allows one arbitrary mass $m$ at high $T$ the symmetric phase ($\xi \rightarrow 0$) as that at $\omega = 2\pi$. This not only destroys the credibility of the loop expansion at high $T$, but also leads to an unphysical threshold in $\rho_\sigma(\omega)$ even at low $T$. The latter aspect is particularly harmful for the purpose of this paper.

To cure these problems, we developed a method based on a resummation procedure proposed by Banerjee and Mallik \cite{11}. It is a generalized mean-field theory and allows one to carry out systematic perturbation theory and renormalization at finite $T$ \cite{12}. In high $T$ limit, the method reduces to the resummation of hard thermal loops \cite{13}. We will report the technical details elsewhere \cite{14}, and recapitulate here only the essential parts needed for subsequent discussions.

Let us rewrite eq.(2) by adding and subtracting a chiral invariant mass term with an arbitrary mass $m$;

$$
\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi_i)^2 - m^2 \phi_i^2] - \frac{\lambda}{4!}(\phi_i^2)^2 + h \phi_0 \\
+ \frac{1}{2}(A - 1)(\partial_\mu \phi_i)^2 - \frac{1}{2}Bm^2 \phi_i^2 - \frac{\lambda}{4!}C(\phi_i^2)^2 \\
+ \frac{1}{2}(m^2 + \mu^2)(1 + B)\phi_i^2 + h(\sqrt{A} - 1)\phi_0 .
$$

(3)

$A$, $B$ and $C$ are renormalization constants: In one-loop in the $\overline{MS}$ scheme, $A = 1$, $B = \lambda/16\pi^2 \bar{\epsilon}$ and $C = \lambda/8\pi^2 \bar{\epsilon}$. The loop expansion at finite $T$ can be done in the same way as that at $T = 0$ except that (i) $m^2$ should be used instead of $-\mu^2$, and (ii) a new vertex proportional to $m^2 + \mu^2$ appears. $m^2$ is a $T$ dependent mass parameter to be determined later by the dynamics.

At finite $T$, retarded meson propagators read

$$
D^R_\phi(\omega, k) = [k^2 - m^2_{0\phi}(T) - \Sigma^R_\phi(\omega, k; T)]^{-1},
$$

(4)

where $k^2 = \omega^2 - k^2$ and

$$
m^2_{0\pi}(T) = m^2 + \frac{\lambda \epsilon}{2}, \quad m^2_{0\sigma}(T) = m^2 + \frac{\lambda \epsilon}{6}.
$$

(5)

The self-energy $\Sigma^R_\phi$ is obtained either from the imaginary-time or the real-time formalism. We adopt the latter in which $\text{Re}\Sigma^R_\phi(\omega, k; T) = \text{Re}\{\Sigma^R_\phi(\omega, k) + \Sigma^1_\phi(\omega, k; T)\}$ and $\text{Im}\Sigma^R_\phi(\omega, k; T) = \text{Im}\{\Sigma^R_\phi(\omega, k) + \Sigma^1_\phi(\omega, k; T)\}$. Here $\Sigma^1_\phi(\omega, k) (\Sigma^1_\phi(\omega, k; T))$ is a $T$-independent ($T$-dependent) part of the 11-component of the $2 \times 2$ self-energy in the real-time formalism \cite{15}. Associated diagrams in one-loop with our modified loop-expansion are shown in Fig. 1.

$m^2$ is a fictitious parameter and physical quantities should not depend on it. This leads us to several procedures to choose optimal $m^2$ in perturbation theory, such as the principle of minimal sensitivity (PMS), the fastest apparent convergence (FAC) and so on \cite{16}. We
find that a variance of FAC is most suited for the purpose of this paper: A condition for
making the loop-correction to the real part of the pion propagator as small as possible reads
\[ \omega^2 - m_{0\pi}^2 - \text{Re}[\Sigma_{\pi}^{11}(\omega, 0) + \Sigma_{\pi}^{11}(0, 0; T)] |_{\omega=m_{0\pi}} = 0. \] (6)
\(\omega\) in \(\Sigma_{\pi}^{11}(\omega, 0; T)\) is chosen to be zero by a technical reason for getting a continuous solution
of \(m^2\) as a function of \(T\) [14]. Our procedure naturally gives \(m^2 \rightarrow -\mu^2\) as \(T \rightarrow 0\) and
\(m^2 \rightarrow \lambda T^2/12\) as \(T \rightarrow \infty\). The former guarantees that the loop-expansion at \(T = 0\) is not
spoiled, and the latter guarantees that thermal tadpole diagrams are resummed correctly.

In Fig.2(A), the chiral condensate \(\xi(T)\) obtained by minimizing the effective potential
is shown for \(m_\pi(T = 0) = 140\) MeV as well as for \(m_\pi(T = 0) = 50\) MeV. The latter
-corresponds to a fairly small quark mass compared to the physical value \(m_q^{\text{phys}}: m_q/m_q^{\text{phys}} \simeq
(50\text{MeV}/140\text{MeV})^2 = 0.13\) [17].

In Fig.2(B), \(m_{0\pi}(T)\), \(m_{0\sigma}(T)\) and \(m^2(T)\) are shown. Because of our resummation pro-
cedure, tree-level masses \(m_{0\pi}(T)\) and \(m_{0\sigma}(T)\) do not show tachyonic behavior and both
approach to the classical plasma limit \(\lambda T^2/12\) at high \(T\). This behavior is important to have
physical continuum threshold in the spectral functions, because the threshold is dictated by
the particle masses running inside the loops (see Fig.1).

Let us now turn to the discussion on the spectral functions (imaginary part of the dy-
namical susceptibility) defined by \(\rho_\phi = -(1/\pi)\text{Im}D_{\phi}^R:\)
\[ \rho_\phi(\omega, k) = -\frac{1}{\pi} \frac{\text{Im} \Sigma_{\phi}^R}{(k^2 - m_{0\phi}^2 - \text{Re} \Sigma_{\phi}^R)^2 + (\text{Im} \Sigma_{\phi}^R)^2}. \] (7)
In Fig.3(A),(B), \(\rho_\phi(\omega, k = 0)\) is shown in \(\pi\) and \(\sigma\) channels for \(T = 0, 120, 145\) MeV. In
the \(\pi\) channel at \(T = 0\), there is a distinct pion pole as well as a continuum starting from
\(m_{0\pi} + m_{0\pi} \simeq 690\) MeV. In the \(\sigma\) channel at \(T = 0\), continuum starts at \(2m_{0\pi} = 280\) MeV.
Also, there is a broad peak centered at \(\omega = 550\) MeV with a total width of 260 MeV. This
corresponds to a \(\sigma\)-pole located far from the real axis in the complex \(\omega\) plane. The large
width is due to a strong coupling of \(\sigma\) with \(2\pi\) in the linear \(\sigma\) model. If we choose parameters
so that the peak position of \(\sigma\) are 750 MeV (1 GeV), corresponding width reads 657 MeV
(995 MeV).

In the \(\pi\)-channel for \(T \neq 0\), a continuum arises in \(0 < \omega < m_{0\sigma} - m_{0\pi}\). This is caused
by the induced “decay” through the process \(\pi + \pi \rightarrow \sigma\). Besides this, the pion still has its
-quasi-particle feature with no appreciable modification of the mass. This is in accordance
with the Nambu-Goldstone nature of the pion.

In the \(\sigma\)-channel for \(0 < T < 145\) MeV, there are two noticeable modification of the
spectral function. One is the shift of the \(\sigma\)-peak toward the low mass region. The other is
the sharpening of the spectral function just above the continuum threshold starting from
\(\omega = 2m_{0\pi}(T)\).

These features are actually related with each other and can be understood in the following
way. Because of the partial restoration of chiral symmetry at finite \(T\) together with the strong
\(\sigma\)-2\(\pi\) coupling, the real part of \((D_\sigma^R)^{-1}\) (which appears in the first term of the denominator
of eq. (7)) approaches zero at \(\omega = 2m_{0\pi}\) for \(T \sim 145\) MeV as shown in Fig.3(C). (The
cusp structure in this figure originates from the pion-loop contribution to \(\text{Re} \Sigma_{\sigma}^R\)). In this
situation, \(\rho_\sigma(\omega \sim 2m_{0\pi}) \sim 1/\text{Im} \Sigma_{\sigma}^R \propto \theta(\omega - 2m_{0\pi})/(\sqrt{1 - (2m_{0\pi}/\omega)^2})\), which shows a singular
behavior just above the threshold.
In other words, the threshold enhancement in Fig. 3(B), although it occurs at relatively low $T$, is caused by a combined effect of the partial restoration of chiral symmetry and the strong $\sigma$-$2\pi$ coupling. Note also that, near the chiral limit, the continuum threshold $2m_{0\pi}$ approaches to zero and the threshold enhancement occurs at the critical temperature where the chiral transition takes place.

The spectral functions of $\pi$ ($\sigma$) for $T > 165$ (145) MeV behave in a standard way as expected from the previous analyses: Simple $\sigma$ and $\pi$ poles without width appear, because the decay ($\sigma \rightarrow 2\pi$) and induced decay ($\pi + \pi \rightarrow \sigma$) are forbidden kinematically. As $T$ increases, these simple poles gradually merge into a degenerate (chiral symmetric) state. For sufficiently high $T$, the system is supposed to be in the deconfined phase and the decay ($\sigma, \pi \rightarrow q\bar{q}$) starts to occur. This is not taken into account in the present linear $\sigma$ model. A calculation based on the Nambu-Jona-Lasinio model shows, however, that there is still a chance for collective modes to survive as far as $T/T_c$ is not so far from unity.

In summary, we have studied the spectral function of the soft modes associated with chiral transition on the basis of a special resummation method at finite $T$. An enhancement of the continuum threshold in the scalar channel are shown as a typical signal of the partial restoration of chiral symmetry. Detectability of this phenomenon in experiments through the decays such as $\sigma \rightarrow 2\pi, 2\gamma, e^+e^-$ remains as an interesting future problem.

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FIGURES

FIG. 1. One-loop self-energy $\Sigma^{11}$ for $\sigma$ and $\pi$ in the modified loop-expansion at finite $T$.

FIG. 2. (A) $\xi(T)$ for $m_\pi = 140\text{MeV}$ and 50 MeV. (B) Masses in the tree-level $m_{0\pi}(T)$ and $m_{0\sigma}(T)$ shown with left vertical scale, and the mass parameter $m^2(T)$ with right vertical scale.

FIG. 3. Spectral function in $\pi$ channel (A) and in $\sigma$ channel (B) for $T = 0, 120, 145\text{MeV}$. The real part of $(D^R_\sigma(\omega, 0; T))^{-1}$ as a function of $\omega$ is shown in (C).
\[-i \Sigma_{\sigma}^{11}(\omega, k) - i \Sigma_{\sigma}^{11}(\omega, k; T) = \]

\[
\begin{align*}
&\frac{\sigma}{\sigma} + \frac{\pi}{\sigma} + \pi(m^2 + \mu^2) \\
+ \frac{\sigma}{\sigma} + \text{CO}
\end{align*}
\]

\[-i \Sigma_{\pi}^{11}(\omega, k) - i \Sigma_{\pi}^{11}(\omega, k; T) = \]

\[
\begin{align*}
&\frac{\pi}{\pi} + \frac{\sigma}{\pi} + \pi(m^2 + \mu^2) \\
+ \frac{\pi}{\pi} + \text{CO}
\end{align*}
\]
