BICONVEX POLYTOPES AND TROPICAL LINEAR SPACES

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Dedicated to Bernd Sturmfels on the occasion of his 60th birthday

Abstract. A biconvex polytope is a classical and tropical convex hull of finitely many points. Given a biconvex polytope, for each vertex of it we construct a directed bigraph and a gammoid so that the collection of the base polytopes of those gammoids is a matroid subdivision of the hypersimplex, thereby proving a biconvex polytope arises as a cell of a tropical linear space. Our construction provides manually feasible guidelines for subdividing the hypersimplex into base polytopes, without resorting to computers. We work out the rank-4 case as a demonstration. We also show there is an injection from the vertices of any \((k - 1)\)-dimensional biconvex polytope into the degree-\((k - 1)\) monomials in \(k\) indeterminates.

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INTRODUCTION

Tropical geometry is geometry over exponents of algebraic expressions. It is naturally equipped with a pair of “logarithmized” addition and multiplication which is either \((\min, +)\) or \((\max, +)\). Whichever to choose is a matter of preference, but we need to fix one and our playground will be \(\text{min-plus algebra}\) with \((\min, +)\). In precise terms “logarithmizing” is tropicalizing. Tropicalized notions are delicate and often do not conform to our classical sense. Tropical convexity and tropical linearity are two of such. We deepen our understanding of them by investigating a classical and tropical convex hull of finitely many points, which we call a biconvex polytope.\(^1\)

We assume the reader is familiar with matroid theory. Starting from scratch, we show the biconvexity is a well-defined notion first. Then we conduct face analysis of biconvex polytopes. We define a new graph-theoretic notion of directed bigraph as a directed graph with a specific ordered bipartite structure. So, two same directed graphs with different bipartite structures are distinguished. With this we construct three correspondences as follows:

- a correspondence from the vertices of any biconvex polytope to directed bigraphs,
- a correspondence from directed bigraphs to gammoids, and hence
- a correspondence from the vertices of any biconvex polytope to gammoids.

In the second correspondence, one should a priori choose a pair of a ground set and a partition. To obtain connected matroids as the outcome it suffices to take a little care when choosing such a pair, see Lemma 3.1. Therefore we may assume the outcome is a collection of connected matroids.

For a fixed biconvex polytope, we define a map from the edges of the biconvex polytope to the subsets of the ground set, which we call the combinatorial log map by which the face structure of the base polytope of each gammoid is completely described. We prove the collection of the base polytopes of those gammoids is a matroid subdivision of the hypersimplex. We will then immediately see that any biconvex polytope arises as a cell of a tropical linear space.

This theory is not cohomology-based which leads to its construction being concise and elegant. Nonetheless, one can read off and translate back and forth cohomology-related information. For instance, we show there is an injection from the vertices of any \((k - 1)\)-dimensional biconvex polytope into the degree-\((k - 1)\) monomials in \(k\) indeterminates. When the biconvex polytope has the maximum number of vertices which is equal to the number of those monomials, the injection becomes a bijection.

Our construction provides manually feasible guidelines for subdividing the hypersimplex into base polytopes. As a demonstration we work out the rank-4 case.

All the computations are manually done with pen and paper by using our theory, without resorting to computers.

\(^1\)This is also called a polytrope, but the “r” in it is apt to be blown past and cause unnecessary confusion. We call it a biconvex polytope because not only is it clear, but the name says it all.
Terminological note. For a finite set $S$, we denote by $\mathbb{R}^S$ the Cartesian product of $|S|$ copies of $\mathbb{R}$ that are labelled by the elements of $S$.

The rank-$k$ uniform matroid on $S$ is denoted by $U^k_S$. The base polytope of $U^k_S$ is denoted by $\Delta^k_S$ which is called a hypersimplex. For $S = [n] := \{1, \ldots, n\}$, we write $U^k_n$ for $U^k_{[n]}$ and $\Delta^k_n$ for $\Delta^k_{[n]}$. We will often write $\Delta$ without a superscript nor a subscript unless confusion could arise.

For all $i \in S$ we understand $x_i$ as coordinate functions of $\mathbb{R}^S$ or indeterminates for the coordinates. For a vector $v \in \mathbb{R}^S$ and $i \in S$ we denote by $x_i(v)$ the $i$-th coordinate of $v$. For a nonempty subset $A$ of $S$, we denote $x(A) = \sum_{i \in A} x_i$.

Let $Q$ be a polyhedron in $\mathbb{R}^S$ and $Q$ a set of describing equations and inequalities of it. We frequently write $Q$ for $Q$ (even though there can be different such sets). So, $\{x(S) = k\}$ may denote the polyhedron in an ambient space determined by the equation $x(S) = k$. For instance, $\Delta^k_S = \{x(S) = k\} = [0, 1]^S \cap \{x(S) = k\}$.

We deal with directed graphs in Section 2 for which we use the terms “nodes” and “arrows” instead of “vertices” and “edges” to avoid any possible confusion because we use the latter terms for biconvex polytopes. But, arrows without directions can still be referred to as edges.

We use **boldface** to define terms and *italics* for emphasis.

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1. Face Analysis of Biconvex Polytopes

We need to fix notation first. For $a, b \in \mathbb{R}$, the **tropical sum** of $a$ and $b$ is $a \oplus b = \min \{a, b\}$ and the **tropical product** of $a$ and $b$ is $a \odot b = a + b$. Then, $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ is a semiring which is called **min-plus algebra**. By replacing minimum with maximum, we obtain another semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$ which is called **max-plus algebra**. These two algebras are isomorphic.

1.1. Biconvex polytopes. Let $k$ be a positive integer and $\mathbb{1}$ denote the all-one vector $(1, \ldots, 1) \in \mathbb{R}^k$. For $a \in \mathbb{R}$ and $x \in \mathbb{R}^k$, the **tropical scalar multiplication** $a \odot x$ is

$$a \odot x = a \mathbb{1} + x.$$

For vectors $x_1, \ldots, x_n \in \mathbb{R}^k$, their tropical sum $x_1 \oplus \cdots \oplus x_n$ is entrywise defined, that is, the $j$-th entry of $x_1 \oplus \cdots \oplus x_n$ is $x_{ij} \oplus \cdots \oplus x_{nj}$ where $x_{ij}$ is the $j$-th entry of $x_i$. A **tropical linear sum** or a **tropical linear combination** of $x_1, \ldots, x_n$ is

$$a_1 \odot x_1 \oplus \cdots \oplus a_n \odot x_n$$

for some $a_1, \ldots, a_n \in \mathbb{R}$.

\footnote{We use $\oplus$ to denote tropical sum as reserving $\odot$ for direct sum.}
A subset of $\mathbb{R}^k$ is called **tropically convex** if it is closed under the operation of tropical linear sum. The **tropical convex hull** of a subset $V \subset \mathbb{R}^k$ is the smallest tropically convex subset that contains $V$, denoted by $\text{tconv}(V)$. Here, we say $V$ **generates** $\text{tconv}(V)$.

A tropically convex subset in $\mathbb{R}^k$ is closed under tropical scalar multiplication. Thus, tropical convex hull is well-defined over the quotient space $\mathbb{R}^k/\mathbb{R}$ although an individual tropical linear sum is not.

A tropically convex subset in $\mathbb{R}^k$ is an **unbounded** polyhedron. But, it is **bounded** in $\mathbb{R}^k/\mathbb{R}$ if it is generated by finitely many points. This leads us to the following definition.

**Definition 1.1.** A **biconvex polytope** is a convex polytope in $\mathbb{R}^k/\mathbb{R}$ that is a tropical convex hull of finitely many points.

**Notation 1.2.** For two points $a, b \in \mathbb{R}^k$, we define:

$$a \equiv b \text{ if and only if } a-b = t \cdot \mathbb{R}$$

Then, “$\equiv$” is equality in $\mathbb{R}^k/\mathbb{R}$, that is, equality in $\mathbb{R}^k$ modulo $\mathbb{R}$.

1.2. **Maximal biconvex polytopes.** Given a biconvex polytope, by passing to the tropical projective space of the same dimension if necessary, we may assume it is full-dimensional, cf. [DS04, Proposition 17].

Let $\text{tconv}(v_1, \ldots, v_k) \subset \mathbb{R}^k/\mathbb{R}$ be full-dimensional, then it contains a unique full-dimensional cell $P$, and henceforth we may assume:

$$P = \text{tconv}(v_1, \ldots, v_k).$$

Then, $\{v_1, \ldots, v_k\}$ is the unique inclusionwise minimal set of points in $\mathbb{R}^k/\mathbb{R}$ that generates $P$ where the inclusion is set inclusion, [DS04, Proposition 21]. Denote

$$\text{Vert}(P) = \{\text{the vertices of } P\}.$$ 

The cardinality of $\text{Vert}(P)$ is at least $k$ and at most $\binom{2k-2}{k-1}$, [DS04, Proposition 19], and it makes sense to introduce the following notation

$$\text{Vert}^0(P) := \text{Vert}(P) - \{v_1, \ldots, v_k\}.$$ 

**Definition 1.3.** A **maximal** biconvex polytope is a full-dimensional one with the maximum number of vertices.

Unless otherwise stated we assume our biconvex polytope is maximal because a biconvex polytope of lower dimension or with fewer number of vertices is obtained as a tropical degeneration of a maximal biconvex polytope $\text{tconv}(v_1, \ldots, v_k)$ for some integer $k$ as varying the $k$ points $v_1, \ldots, v_k$.

1.3. **Min-plus hyperplanes.** The **min-plus hyperplane** $H^a$ at $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ is defined as the set of points $(x_1, \ldots, x_k)$ such that the minimum

$$a_1 \circ x_1 \oplus \cdots \oplus a_k \circ x_k$$

occurs at least twice, that is, the minimum equals $a_i \circ x_i$ and $a_j \circ x_j$ for some $i$ and $j$ with $i \neq j$. Then, $H^a$ is written as

$$H^a := \bigcup_{i,j \in [k], i \neq j} \bigcap_{l \in [k]-\{i,j\}} \{a_l + x_l \geq a_i + x_i = a_j + x_j\}.$$
Min-plus hyperplanes are sometimes called tropical hyperplanes. We call each summand \( \bigcap_{l \in [k]-\{i,j\}} \{ a_l + x_l \geq a_i + x_i = a_j + x_j \} \) with \( i \neq j \) the \{(i,j)\}-min-branch of the tropical hyperplane. The \( i \)-th min-sector by the tropical hyperplane at \( a \) is defined as
\[
E^a_i := \bigcap_{j \in [k]-\{i\}} \{ a_j + x_j \geq a_i + x_i \}.
\]
Min-plus hyperplanes, min-branches, and min-sectors are tropically convex, cf. [DS04, Proposition 6 and Corollary 7].

1.4. Max-plus hyperplanes. In the same manner, the max-plus hyperplane \( \bar{H}^a \) at \( a = (a_1, \ldots, a_k) \) is defined as the set of points \( (x_1, \ldots, x_k) \) such that the maximum
\[
\max (a_1 + x_1, \ldots, a_k + x_k)
\]
occurs at least twice, which is
\[
\bar{H}^a = \bigcup_{i,j \in [k], i \neq j} \bigcap_{l \in [k]-\{i,j\}} \{ a_l + x_l \leq a_i + x_i = a_j + x_j \}.
\]
We call each summand \( \bigcap_{l \in [k]-\{i,j\}} \{ a_l + x_l \leq a_i + x_i = a_j + x_j \} \) with \( i \neq j \) the \{(i,j)\}-max-branch of the max-plus hyperplane. The \( i \)-th max-sector by the max-plus hyperplane at \( a \) is defined as
\[
\bar{E}^a_i := \bigcap_{j \in [k]-\{i\}} \{ a_j + x_j \leq a_i + x_i \}.
\]
Max-sectors and min-sectors are related as follows: for \( i \in [k] \) and \( a \in \mathbb{R}^k \)
\[
\bar{E}^a_i = -E_i^{(-a)}.
\]
Max-plus hyperplanes are closed under tropical scalar multiplication and so are max-branches and max-sectors.

1.5. Tropical objects over the quotient space \( \mathbb{R}^k/\mathbb{R}1 \). All of min-plus and max-plus hyperplanes, branches, and sectors are well-defined over \( \mathbb{R}^k/\mathbb{R}1 \). Any branch has codimension 1, and the boundary of a sector is a union of branches. Thus we will often call the \{i,j\}-branch of a sector the \{i,j\}-facet of the sector.

1.6. Vertices of biconvex polytopes. Let \( P = t\text{conv} (v_1, \ldots, v_k) \subset \mathbb{R}^k/\mathbb{R}1 \) be a maximal biconvex polytope, then \( \{v_1, \ldots, v_k\} \subset \text{Vert} (P) \) generates \( P \). At each vertex \( v_i \) of \( P \) there is a unique max-sector that contains \( P \), say \( \bar{E}^{v_i}_{\pi(i)} \) for some permutation \( \pi : [k] \rightarrow [k] \). From now on, by rearranging indices if necessary, we may assume \( \pi = \text{id} \) and write
\[
\bar{E}_i := \bar{E}_i^{v_i}.
\]
Then:
\[
P = \bigcap_{i \in [k]} \bar{E}_i.
\]

Definition 1.4. For any vertex \( w \in \text{Vert} (P) \), its index set \( I^w \) is defined as
\[
I^w := \{ i \in [k] : w \in (\text{a facet of } \bar{E}_i) \} \neq \emptyset.
\]
For each index \( i \in I^w \), the exponent set \( C_i^w \) and the span set \( D_i^w \) are defined as
\[
C_i^w := \{ l \in [k] - \{i\} : w \in \{ \{i, l\} \text{ - facet of } \bar{E}_i \} \}, \\
D_i^w := [k] - \{i\} - C_i^w.
\]
Then, \( i \notin C_i^w \) and \( i \notin D_i^w \). In particular, for \( w \in \{v_1, \ldots, v_k\} \), say for \( w = v_j \):
\[
I^w_j = \{ j \}, \\
C_j^w = [k] - \{j\}, \\
D_j^w = \emptyset.
\]
We will often write \( I, C_i \) and \( D_i \) without the superscript \( w \) for simplicity unless confusion could arise.

Let \( e_1, \ldots, e_k \) be the standard basis vectors of \( \mathbb{R}^k \). For two \( i, j \in I \), there are nonnegative real numbers \( a_l \) for \( l \in D_i \) and \( b_m \) for \( m \in D_j \) with
\[
w = v_i + \sum_{l \in D_i} a_l (-e_l) \equiv v_j + \sum_{m \in D_j} b_m (-e_m).
\]
Note that we employ two affine coordinate systems here, one for each of \( v_i \) and \( v_j \). Because the maximum of coordinates of \( w - v_i \) is its \( i \)-th coordinate, \( i \notin D_j \) with \( b_i \neq 0 \), and similarly \( j \notin D_i \) with \( a_j \neq 0 \). Thus, \( I \subseteq D_i \cup \{i\} \) for any \( i \in I \), and
(1.1) \[ I \cap C_i = \emptyset. \]
The vertex \( w \) is an intersection of \( k - 1 \) branches, each of which contains one and only one vertex from \( \{v_i : i \in I\} \) by maximality of \( P \). More specifically:
(1.2) \[ w = \bigcap_{i \in I} \bigcap_{l \in C_i} (\text{the } \{i, l\} \text{- facet of } \bar{E}_i). \]
The operand \( \bigcap_{l \in C_i} (\text{the } \{i, l\} \text{- facet of } \bar{E}_i) \) for \( i \in I \) has codimension \( |C_i| \) and
(1.3) \[ \sum_{i \in I} |C_i| = k - 1. \]
A computation shows:
\[
\bigcup_{i \in I} C_i = [k] - I \quad \text{and} \quad \bigcap_{i \in I} D_i = \emptyset.
\]
If \( w \in \text{Vert}^0(P) \), then \( |I^w| \geq 2 \) and vice versa, and we have
(1.4) \[ 0 \leq \left| \bigcap_{i \in I} C_i \right| \leq 1 \quad \text{and} \quad k - 1 \leq \left| \bigcup_{i \in I} D_i \right| \leq k. \]
Now, for any vertex \( w \in \text{Vert}(P) \) and for all \( i \in [k] - I \) define \( C_i := \emptyset \) and \( D_i := \emptyset \). Then, \( C_i \) and \( D_i \) are defined for all \( i \in [k] \). Note that \( C_i \neq \emptyset \) if and only if \( i \) is an index, that is, \( i \in I \).

**Notation 1.5.** For a vertex \( w \in \text{Vert}(P) \), we introduce the following notation
\[
w = v_1^{C_1} \cdots v_k^{C_k} = \prod_{i \in [k]} v_i^{C_i}.
\]
Practically, we remove all $v_i^{C_i}$ with $C_i = \emptyset$ from the notation and write

$$w = \prod_{i \in I} v_i^{C_i}.$$ 

In particular, $v_i = v_i^{[k] - \{i\}}$ for a vertex $v_i$.

**Definition 1.6.** Along the formula (1.4) we define the type of a vertex $w \in \text{Vert}^0(P)$ as follows:

$$\text{type}(w) = \begin{cases} 
0 & \text{if } \bigcap_{i \in I} C_i^w = \emptyset, \\
1 & \text{otherwise}.
\end{cases}$$

**Example 1.7.** If $k \leq 4$, every vertex $w \in \text{Vert}^0(P)$ has type 1.

1.7. **Faces of biconvex polytopes.** Let $Q$ be a face of $P$. For $i \in [k]$, denote

$$\text{Vert}_i(Q) := \{ w \in \text{Vert}(Q) : C_i^w \neq \emptyset \} .$$

**Definition 1.8.** The index set of $Q$ is defined as

$$I^Q := \{ i \in [k] : \text{Vert}_i(Q) \neq \emptyset \} .$$

For each index $i \in I^Q$, the exponent set $C_i^Q$ and the span set $D_i^Q$ are defined as

$$C_i^Q := \bigcap_{v \in \text{Vert}_i(Q)} C_i^w, \\
D_i^Q := [k] - \{ i \} - C_i^Q .$$

Using the above definitions, $Q$ is written as the following:

$$Q = \bigcap_{i \in I^Q} \bigcap_{l \in C_i^Q} (\text{the } \{ i, l \}-\text{facet of } \bar{E}_i) \cap P .$$

Moreover, its codimension is

$$\text{codim}(Q) = \sum_{i \in I^Q} |C_i^Q| .$$

Observe that these are consistent with (1.2) and (1.3). Note that $I^Q \neq \emptyset$ and $C_i^Q \neq \emptyset$ for all $i \in I^Q$.

Similarly as in the vertex case, for all $i \in [k] - I^Q$ define $C_i^Q := \emptyset$ and $D_i^Q := \emptyset$. Then, $C_i^Q$ and $D_i^Q$ are defined for all $i \in [k]$. Note that $C_i^Q \neq \emptyset$ if and only if $i$ is an index, that is, $i \in I^Q$.

Thus, we generalized the notions for vertices. We also generalize Notation 1.5.

**Notation 1.9.** For a face $Q$ of $P$, we denote

$$Q = v_1^{C_1^Q} \cdots v_k^{C_k^Q} = \prod_{i \in [k]} v_i^{C_i^Q} .$$

**Remark 1.10.** Notation 1.9 describes how to obtain a face $Q$ of $P$ from the unique generating set of vertices $\{ v_1, \ldots, v_k \}$. It further generalizes to a biconvex polytope that is not maximal, but then the uniqueness of expression fails.
2. Graphical Model and Monomial Map

2.1. Directed bigraphs and faces of biconvex polytopes. A bipartite graph or a bigraph for short is a graph that does not contain any odd cycle.

Definition 2.1. A directed bigraph $G$ is a bigraph with ordered parts $(I, I^c)$ for a subset $I \subseteq V_G$ such that if $(i, c)$ is an arrow, then $i \in I$ and $c \in I^c$.\footnote{A directed bigraph is a directed graph. We do not allow multiple arrows.}

By definition, a directed bigraph is a directed graph with its own ordered bipartite structure. So, two same directed graphs with different ordered bipartite structures are distinguished. Note that one of $I$ and $I^c$ can be empty (in this case the graph is a set of isolated nodes), but not both of them can.

Let $P = \text{tconv} (v_1, \ldots, v_k) \subset \mathbb{R}^k/\mathbb{R}$ be a maximal biconvex polytope. To each vertex $w$ of $P$, assign the directed graph $G_w$ with node set $[k]$ satisfying that $(i, c)$ is an arrow of $G_w$ if and only if $i \in I_w$ and $c \in C_w^i$. Then, $G_w$ is a directed bigraph with ordered parts $(I_w, [k] - I_w)$ by (1.1). Moreover, $G_w$ is a tree by (1.3).

Let $G$ be a directed bigraph with $V_G = [k]$ that is a forest. If $i \in I$ and $c \in I^c$ are adjacent, then denote by $(i, c)$ the unique arrow that connects $i$ to $c$, and denote by $G(i, c)$ the graph obtained from $G$ by removing $(i, c)$ from it. Then, $G(i, c)$ is a directed bigraph with induced bipartite structure.

Let $G^+(i, c)$ and $G^-(i, c)$ be the connected components of $G(i, c)$ containing $i$ and $c$, respectively, both of which are nonempty and have induced bipartite structure. Again, $G^+(i, c)$ and $G^-(i, c)$ are connected by definition whether or not $G$ is connected, and two node sets $V_{G^+(i, c)}$ and $V_{G^-(i, c)}$ are disjoint, which partition the node set of the connected component of $G$ that contains $(i, c)$.

Let $Q$ be a nonempty proper face of $P$. Define a directed graph $G_Q$ with node set $[k]$ such that

$(i, c)$ is an arrow of $G_Q$ if and only if $i \in I^Q$ and $c \in C^Q_i$.

Given a vertex $w$ of $Q$, we have

$$Q = \prod_{i \in [k]} v_i^{C^w_i - (C^w_i - C^Q_i)}.$$ 

I.e. $G_Q$ is obtained from $G_w$ by deleting arrows $(i, c)$ with $i \in I^w$ and $c \in C^w_i - C^Q_i$. Therefore $G_Q$ is a directed bigraph that is a forest. Its bipartite structure is induced from that of $G_w$, but does not depend on the choice of the vertex. The number of its connected components minus the number of its isolated nodes is

$$\text{dim}(Q) = \sum_{i \in I^w} |C^w_i - C^Q_i|.$$
2.2. Edges of biconvex polytopes. Let $Q$ be an edge of $P$ with vertices $\{w_1, w_2\}$. There are arrows $(i_1, c_1)$ and $(i_2, c_2)$ of $G_{w_1}$ and $G_{w_2}$, respectively, such that

$$Q = \left( \prod_{i \in I^{w_1} \setminus \{i_1\}} v_{i1}^{c_{i1} - 1} \right) _{v_{i1}^{c_{i1}}} = \left( \prod_{i \in I^{w_2} \setminus \{i_2\}} v_{i2}^{c_{i2} - 1} \right) _{v_{i2}^{c_{i2}}}$$

where $i_l \in I^{w_l}$ and $c_l \in C^{w_l}_l$ for $l = 1, 2$. In other words:

$$G_Q = G_{w_1}(i_1, c_1) = G_{w_2}(i_2, c_2).$$

Let $G'_1 = G^+_w(i_1, c_1)$ and $G'_2 = G^-_w(i_1, c_1)$, then $G'_1$ and $G'_2$ are the two connected components of $G_Q$ with

$$G_Q = G'_1 \cup G'_2.$$

We show $i_2 \in V_{G'_2}$. For $l = 1, 2$, let $I_l = I^{w_l} \cap V_{G'_l}$ and $C^l = V_{G'_l} - I_l$, then $G'_1$ and $G'_2$ are directed bigraphs with parts $(I_1, C^1)$ and $(I_2, C^2)$, respectively, and $G_Q$ is a directed bigraph with parts $(I_1 \cup I_2, C^1 \cup C^2)$.

Let $Q_1$ and $Q_2$ be the faces of $P$ corresponding to two directed bigraphs $G'_1 \cup V_{G'_2}$ and $G'_2 \cup V_{G'_1}$, respectively, then

$$Q = Q_1 \cap Q_2.$$  

Every point of $Q_1$ is contained in the $\{l, c\}$-max-branch of the max-plus hyperplane at $v_l$ for all $l \in I_1$ and $c \in C^1 \cap C^Q$. So, for each $i \in V_{G'_1}$ the $i$-th coordinate of the point is a fixed real number, say $a_i$, and so is the $j$-th coordinate of any point of $Q_2$ for each $j \in V_{G'_2}$, say $b_j$. Note that we are employing two affine coordinate systems. Then, given a point $u$ of $Q$ there are real numbers $x_i$ and $y_j$ for $i \in V_{G'_1}$ and $j \in V_{G'_2}$ with:

$$u = \sum_{i \in V_{G'_1}} a_i e_i + \sum_{j \in V_{G'_2}} y_j e_j = \sum_{i \in V_{G'_1}} x_i e_i + \sum_{j \in V_{G'_2}} b_j e_j.$$  

Thus, there is a fixed real number $t$ with $x_i = a_i + t$ and $y_j = b_j - t$ for all $i \in V_{G'_1}$ and $j \in V_{G'_2}$. Let $s = \sum_{i \in V_{G'_1}} a_i e_i + \sum_{j \in V_{G'_2}} b_j e_j$, then

$$u = s - t \cdot 1^{V_{G'_1}} = s + t \cdot 1^{V_{G'_2}}.$$  

Now, since $i_1 \in V_{G'_1}$, we have $i_2 \in V_{G'_2}$. Moreover, the vector $\overrightarrow{w_1 w_2}$ is a positive multiple of $-1^{V_{G'_2}}$ in $\mathbb{R}^k / \mathbb{R} 1$ and $\overrightarrow{w_2 w_1}$ is that of $-1^{V_{G'_1}}$. In particular:

$$G^+_{w_1}(i_1, c_1) = G^-_{w_2}(i_2, c_2) \quad \text{and} \quad G^-_{w_1}(i_1, c_1) = G^+_{w_2}(i_2, c_2).$$

2.3. Combinatorial log map.

**Definition 2.2.** Let $P = \text{tconv}(v_1, \ldots, v_k) \subset \mathbb{R}^k / \mathbb{R} 1$ be a maximal biconvex polytope. For each vertex $w$ of $P$, there are exactly $k - 1$ edges $Q$ of $P$. For each edge $Q$ with $\text{Vert}(Q) = \{w, v\}$, there is the unique subset $V$ of $[k]$ with:

$$v = w - t \cdot 1^V$$

for a positive number $t$ which also uniquely exists. We define $L^w$ such that

$$L^w : \{k - 1 \text{ edges } Q \text{ of } P \text{ containing } w\} \rightarrow 2^{|V|}.$$  

$$Q \mapsto V.$$
We call $L$ the **combinatorial log map**\(^4\) and $L^w$ the **combinatorial log map at $w$** for $P$.

Note that $\emptyset \neq L^w(Q) \neq [k]$ and $L^w(Q) = [k] - L^w(Q)$.

Let $P'$ be a tropical degeneration of $P$, which is also biconvex, but not necessarily maximal. Because the direction vectors of edges of $P'$ are direction vectors of edges of $P$, the combinatorial log map is defined for $P'$. Thus, the combinatorial log map is defined for any biconvex polytope.

**Example 2.3.** The edge structure of $P$ at a type-1 vertex $w \in \text{Vert}^0(P)$ is particularly nice because if $\bigcap_{i \in I^w} C^w_i = 1$, then all subsets $C^w_j - \bigcap_{i \in I^w} C^w_i$ of $[k]$ with $j \in I^w$ are mutually disjoint. The directed bigraph $G_w$ also has a nice structure, see Figure 2.1.\(^5\) It is easy to find all $k-1$ edges $Q$ connected to $w$, whose images under $L^w$ are

$$L^w(Q) = \begin{cases} C^w_j - \bigcap_{i \in I^w} C^w_i & \text{for } j \in I^w, \\ [k] - \{j\} & \text{for } j \in [k] - \bigcap_{i \in I^w} C^w_i - I^w. \end{cases}$$

\(4\)This is named after the logarithmic map producing amoebas, cf. [GKZ94, Chapter 6.1.B].

\(5\)We draw a directed bigraph such that its parts are $(I = \{\text{lower nodes}\}, I^c = \{\text{upper nodes}\})$.

**Figure 2.1.** The directed bigraph $G_w$ of a type-1 vertex $w$.

2.4. **Monomial map.**

**Definition 2.4.** Let $k \geq 2$ be an integer. Let $P = \text{tconv}(v_1, \ldots, v_k) \subset \mathbb{R}^k/\mathbb{R}$ be any full-dimensional biconvex polytope whether maximal or not. We define a map $\mu$ on the collection of vertices of $P$ such that

$$\mu : \{\text{the vertices of } P\} \to \{\text{the degree-}(k-1) \text{ monomials in } x_1, \ldots, x_k\}$$

$$w = v_1^{C^w_1} \cdots v_k^{C^w_k} \mapsto x_1^{C^w_1} \cdots x_k^{C^w_k}.$$

We call this map the **monomial map**.

**Proposition 2.5.** The monomial map $\mu$ is injective.

**Proof.** We prove by induction on dimension $k-1$. For all 1-dimensional biconvex polytopes the map $\mu$ is injective, and the base case holds.

Suppose that $\mu$ is injective for all biconvex polytopes of dimension $\leq k-1$ for some $k \geq 2$. We may assume $P$ is a maximal biconvex polytope of dimension $k$.

Let $w$ and $v$ be vertices of $P$ with $\mu(w) = \mu(v)$, then $I^w = I^v =: I$.

If $|I| = 1$, clearly $w = v$. 

If $|I| = m > 1$, then $I = \{i_1, \ldots, i_m\}$ and $w = v = v_{i_1}^{C^w_{i_1}} \cdots v_{i_m}^{C^w_{i_m}}$. 

\(\cdots\)
If \(|I| \geq 2\), then in a fixed affine coordinate system, the \(i\)-th coordinate of \(w\) and the \(j\)-th coordinate of \(v\) for all \(i, j \in I\) are the same. So, \(w\) and \(v\) are contained in a proper face of \(P\). This face is a biconvex polytope which has the unique inclusionwise minimal generating set, cf. [DS04, Proposition 21], and inherits its geometry from \(P\). Thus, \(w = v\) by the induction hypothesis. \(\Box\)

**Remark 2.6.** If \(P\) is a maximal biconvex polytope, the monomial map \(\mu\) is a bijection because the maximum number of vertices of \(P\) equals the number of degree-(\(k - 1\)) monomials in \(k\) indeterminates, which is \(\binom{2k-2}{k-1}\).

## 3. gammoids and Matroid Subdivision

All italicized terms not defined herein shall have the same definitions as set forth in Appendix A.

### 3.1. gammoids of our interest.**

Let \(G\) be a directed graph\(^6\) with node set \(V_G = [k]\), and \(S\) be a finite set with a partition \(S = \bigcup_{j \in [k]} S_j\) which we will call an **underlying partition**. Denote by \(T(j)\) the collection of \(l \in [k]\) with \((j, l)\) being an arrow of \(G\), which is possibly empty.

Let \(\tilde{G}\) be a directed graph with node set \(V_{\tilde{G}} = [k] \cup S\) (the disjoint union of \([k]\) and \(S\)) such that \((j, s)\) is an arrow if \(j \in [k]\) and \(s \in S_l\) for some \(l \in \{j\} \cup T(j)\).

We denote by \(\Gamma[G]\) the gammoid obtained from \(\tilde{G}\). Let \(E_j = \bigcup_{l \in \{j\} \cup T(j)} S_l\), then \(\Gamma[G]\) is a rank-\(k\) matroid on \(S\) which is the matroid union of rank-1 uniform matroids on \(E_j\):

\[
\Gamma[G] = \bigvee_{j \in [k]} U^{1}_{E_j}.
\]

This is a transversal matroid. Note that \(\Gamma[G]\) is defined for all directed graphs \(G\).

**Lemma 3.1.** Let \(G\) be a directed bigraph with parts \((I, I^c)\). If \(|S_c| = 1\) for some \(c \in I^c\), then \(\Gamma[G]\) is disconnected. If \(G\) is connected and \(|S_c| \geq 2\) for all \(c \in I^c\), then \(\Gamma[G]\) is connected.

**Proof.** If \(|S_c| = 1\) for some \(c \in I^c\), the singleton \(S_c\) is a coloop of \(\Gamma[G]\), and \(\Gamma[G]\) is disconnected.

If \(G\) is connected and \(|S_c| \geq 2\) for all \(c \in I^c\), we show that there is a \((k + 1)\)-element subset \(A \subset S\) with \(M|_A \simeq U^k_{k+1}\), then by Lemma A.1(1) it follows that \(\Gamma[G]\) is connected. Since \(G\) is a tree, there is a partial order \(<\) on its node set with the smallest node \(i_0\) of degree 1. Let \(c_0\) be a node with \(c_0 \supseteq i_0\), i.e. \(c_0\) covers \(i_0\), then \(c_0 \in I^c\) because \(G\) is a directed bigraph. Denote by \(\text{deg}_G j\) the degree of \(j\) in \(G\).

1. Take an element from \(S_c\) for each \(c \in I^c\).
2. Take an element from \(S_i\) for each \(i \in I\) with \(\text{deg}_G i = 1\).
3. For each \(i \in I\) with \(\text{deg}_G i > 1\), take an element from \(S_c\) for a \(c \supseteq i\).
4. Take an element from \(S_{c_0}\).

Thus \(k + 1\) elements are taken from \(S\). Let \(A\) be the set of these elements, then \(|A \cap S_c| \leq 2\) for all \(c \in I^c\) and \(M|_A = U^k_A\). The proof is done. \(\Box\)

\(^6\)We do not allow multiple arrows.
Corollary 3.2. Let $G$ be a directed bigraph with $|S_c| \geq 2$ for all $c \in I^c$. If $G$ has $m$ connected components $G_1, \ldots, G_m$, then $\Gamma [G_1], \ldots, \Gamma [G_m]$ are connected and

$$\Gamma [G] = \Gamma [G_1] \oplus \cdots \oplus \Gamma [G_m].$$

In particular, the number of connected components of $G$ equals that of $\Gamma [G]$.

Remark 3.3. If $G$ is an isolated node, $\Gamma [G]$ is a rank-1 uniform matroid. So, for a set of isolated nodes, we often ignore its bipartite structure.

3.2. Flats of $\Gamma [G]$ for a directed bigraph $G$. For any subgraph $G'$, denote by $S_{V'_G}$ or more simply by $S_{G'}$ the set $\bigcup_{j \in V_{G'}} S_j$. Then, it is immediate that $S_{G^+(i,c)}$ for each arrow $(i,c)$ of $G$ is a flat of $\Gamma [G]$, of rank $|V_{G^+(i,c)}|$. Moreover:

$$\Gamma [G^+(i,c)] = \Gamma [G] |_{S_{G^+(i,c)}}.$$

Observe that any base of

$$\Gamma [G(i,c)] = \Gamma [G^+(i,c)] \oplus \Gamma [G^-(i,c)]$$

is a base of $\Gamma [G]$ whose intersection with $S_{G^+(i,c)}$ is a base of $\Gamma [G^+(i,c)]$, and vice versa. Therefore it is a base of

$$\Gamma [G] |_{S_{G^+(i,c)}} \oplus \Gamma [G]/S_{G^+(i,c)}$$

and the converse holds. Thus we obtain

$$\Gamma [G(i,c)] = \Gamma [G] |_{S_{G^+(i,c)}} \oplus \Gamma [G]/S_{G^+(i,c)}.$$

In particular:

$$\Gamma [G^-(i,c)] = \Gamma [G]/S_{G^+(i,c)}.$$

All this proves $S_{G^+(i,c)}$ is a non-degenerate flat of $\Gamma [G]$.

Since $G$ has at most $k - 1$ arrows, the gammoid $\Gamma [G]$ has at most $k - 1$ minimal non-degenerate flats of the form $S_{G^+(i,c)}$.

Then, because any intersection of flats is a flat, the sets $S_i$ for all $i \in I$ are flats of rank 1. In the same way, $\bigcup_{j \in N[c]} S_j$ for some $c \in C$ is a flat of rank $|N[c]|$ where $N[c]$ denotes the set of $c$ and its adjacent nodes.

Further, for a subset $C \subset I^c$ the set $\bigcup_{c \in C} \bigcup_{j \in N[c]} S_j$ is a flat of rank $|\bigcup_{c \in C} N[c]|$ which is connected. This flat is non-degenerate if and only if $\bigcup_{c \in C} N[c]$ is the node set $V_{G^+(i_0,c_0)}$ of a graph $G^+(i_0,c_0)$ for some arrow $(i_0,c_0)$.

Any singleton $\{s\}$ in $S_c$ for $c \in I^c$ is a rank-1 flat and of course $M|_{\{s\}}$ is connected. Moreover, it is a non-degenerate flat because one can choose a subset $A \subset S$ that contains $s$ with $M|_A \simeq U^k_{k+1}$ as in Lemma 3.1 so that $M/A \{s\} = M/A \{s\} = U^{k-1}_{A \setminus \{s\}} \simeq U^{k-1}_{k}$ which implies that $M/\{s\}$ is connected.

The following lemma says that any non-degenerate flat of $\Gamma [G]$ arises in a way described above.

Lemma 3.4. Let $G$ be a directed bigraph with $|S_c| \geq 2$ for all $c \in I^c$. Then, a minimal non-degenerate flat of $\Gamma [G]$ is either $S_{G^+(i,c)}$ for an arrow $(i,c)$ of $G$ or a singleton contained in $S_c$ for some $c \in I^c$. 

Proof. We may assume \( G \) is connected. Let \( A \) be a non-degenerate flat of \( \Gamma [G] \) that is not obtained by removing an arrow of \( G \).

If \( A \cap S_c = \emptyset \) for all \( c \in I^c \), then \( A \) is a disjoint union of \( A_i \) with \( i \in I \) and is disconnected. So, assume \( A \cap S_c \neq \emptyset \) for some \( c \in I^c \).

Then, again since \( A \) is a connected flat, if \( |A \cap S_c| > 1 \), then \( \bigcup_{j \in N[c]} S_j \subseteq A \).

If \( |A \cap S_c| = 1 \) and \( A \cap \bigcup_{j \in N[c]} S_j \neq \emptyset \), similarly \( \bigcup_{j \in N[c]} S_j \subseteq A \).

Therefore, \( |A \cap S_c| = 1 \) and \( A \cap \bigcup_{j \in N[c]} S_j = \emptyset \). This implies \(|A| = 1 \). \( \square \)

Corollary 3.5. Let \( G \) be a connected directed bigraph with \( |S_c| \geq 2 \) for all \( c \in I^c \). Then, there are precisely \( k - 1 \) facets of the base polytope \( \text{BP}_{\Gamma [G]} \) of \( \Gamma [G] \) that are not contained in the boundary of the hypersimplex \( \Delta = \Delta_S^k \), which are \( \text{BP}_{\Gamma [G(i,c)]} \) for the \( k - 1 \) arrows \((i,c)\) of \( G \).

3.3. Matroid subdivisions dual to biconvex polytopes. Fix an integer \( k \geq 1 \) and an underlying partition \( S = \bigcup_{i \in [k]} S_i \). For any subset \( A \subseteq [k] \), we will denote

\[
S_A = \bigcup_{j \in A} S_j.
\]

We assume \(|S_i| \geq 2 \) for all \( i \in [k] \) throughout this subsection.

The base polytope of a matroid \( M \) on \( S \) with rank function \( r \) is the intersection of \( \{ x(S) = r(S) \} \) and the following half-spaces:

\[
\begin{align*}
\{ x_i \geq 0 \} & \quad \text{for all } i \in S, \\
\{ x(A) \leq r(A) \} & \quad \text{for all subsets } A \subseteq S.
\end{align*}
\]

Here, every vector \( 1^A \) is normal to the face \( \text{BP}_M \cap \{ x(A) = r(A) \} \) of \( \text{BP}_M \), and moreover outward-pointing normal to \( \text{BP}_M \) through the face.

Let \( P = \text{teconv} (v_1, \ldots, v_k) \) in \( \mathbb{R}^k / \mathbb{R}^1 \) be a biconvex polytope. We may assume \( P \) is maximal. For any vertex \( w \) of \( P \), \( \Gamma [G_w] \) is connected by Lemma 3.1, which has precisely \( k - 1 \) non-degenerate flats of the form \( S_{G_w}^+ (i,c) \). Then, \( \text{BP}_{\Gamma [G_w]} \) is full-dimensional, and the vectors \( v_i S_{G_w}^+ (i,c) \) are outward-pointing normal to \( \text{BP}_{\Gamma [G_w]} \).

Let \( Q \) be an edge of \( P \) with \( \text{Vert} (Q) = \{ v, w \} \). Then, \( S_{L_v} (Q) \) and \( S_{L_w} (Q) \) are non-degenerate flats of \( \Gamma [G_v] \) and \( \Gamma [G_w] \), respectively, with \(|k| = L^v (Q) \cup L^w (Q) \) and \( S = S_{L_v} (Q) \cup S_{L_w} (Q) \).

Using the notation (A.1), two matroids \( \Gamma [G_v] (S_{L_v} (Q)) \) and \( \Gamma [G_w] (S_{L_w} (Q)) \) are face matroids of \( \Gamma [G_v] \) and \( \Gamma [G_w] \), respectively, which are the same. And we have

\[
\text{BP}_{\Gamma [G_v]} (S_{L_v} (Q)) = \text{BP}_{\Gamma [G_w]} (S_{L_w} (Q)) = \text{BP}_{\Gamma [G_v]} \cap \text{BP}_{\Gamma [G_w]}
\]

which is the common facet of \( \text{BP}_{\Gamma [G_v]} \) and \( \text{BP}_{\Gamma [G_w]} \). Moreover, for \( V = [k] \):

\[
\bigcap_{w \in \text{Vert} (P)} \text{BP}_{\Gamma [G_w]} = \text{BP}_{\Gamma [V]}.
\]

Therefore

\[
\Sigma = \{ \text{BP}_{\Gamma [G_w]} : w \in \text{Vert} (P) \}
\]

is a matroid tiling, that is, a face-fitting collection of base polytopes that is connected in codimension 1. By Corollary 3.5, its support \( \Sigma \) has no facets that are not contained in the boundary of \( \Delta \), which means that \(|\Sigma| = \Delta \) and \( \Sigma \) is a matroid subdivision of \( \Delta \).
Let $\pi : \mathbb{R}^S \rightarrow \mathbb{R}^k$ be a linear map defined by

$$\mathbf{x} = \sum_{s \in S} x_s \mathbf{e}_s \mapsto \mathbf{y} = \sum_{i \in [k]} \left( \sum_{s \in S_i} x_s \right) \mathbf{e}_i.$$

Then, the following hold.

- The hyperplane $\{ x (S_i) = 1 \}$ of $\mathbb{R}^S$ maps to the hyperplane $\{ y_i = 1 \}$ of $\mathbb{R}^k$.
- The hyperplane $\{ x (S) = k \}$ maps to $\{ y (V) = k \}$ for $V = [k]$.
- If $Q = \text{BP} \Gamma | \mathcal{V} | = \text{BP} \Theta | \mathcal{V} | \cup \chi$, then $\pi (Q) = \{ \mathbb{I} \}$.
- If $| S_i | \geq k$ for all $i \in [k]$, then $\pi (\Delta_S^k)$ is a regular $(k - 1)$-simplex.

For any polytope $P \subseteq \Delta_S^k$, let us call its image $\pi (P)$ a **quotient polytope**. Denote

$$\pi (\Sigma) = \{ \pi (P) : P \in \Sigma \}$$

which we call a **quotient tiling** if $\Sigma$ is a tiling, and a **quotient subdivision** if $\Sigma$ is a subdivision.

Because it is convenient to have $\pi (\Delta_S^k)$ being a regular $(k - 1)$-simplex, let us assume $| S_i | \geq k$ for all $i \in [k]$.

For a $\text{BP} \Gamma | \mathcal{G}_w | \in \Sigma$, its facets not contained in the boundary of $\Delta_S^k$ are precisely $\text{BP} \Gamma | \mathcal{G}_w (i , c) |$ for the $k - 1$ arrows $(i, c)$ of $\mathcal{G}_w$. The vector $1^{\mathcal{V} \mathcal{G}_w (i , c)}$ is outward-pointing normal to the quotient polytope $\pi (\text{BP} \Gamma | \mathcal{G}_w |)$.

Now, denote $P^* = \mathbb{I} - P$ for a polytope $P$, which is an involution of $P$. Let $\Sigma^*$ be the collection of those involutions of all members of $\Sigma$:

$$\Sigma^* = \{ P^* \subseteq \Delta_S^{|S| - k} : P \in \Sigma \}$$

which is a polyhedral subdivision of the hypersimplex $(\Delta_S^k)^* = \Delta_S^{|S| - k}$.

For each $\text{BP} \Gamma | \mathcal{G}_w | \in \Sigma$, we have $\text{BP} \Gamma | \mathcal{G}_w |^{*} = \mathbb{I} - \text{BP} \Gamma | \mathcal{G}_w | \in \Sigma^*$ and vice versa, where $\Gamma | \mathcal{G}_w |^{*}$ is the dual matroid of $\Gamma | \mathcal{G}_w |$. So, $\Sigma^*$ is a matroid subdivision, and the vector $-1^{\mathcal{V} \mathcal{G}_w (i , c)}$ is outward-pointing normal to $\pi (\text{BP} \Gamma | \mathcal{G}_w |^{*})$.

Thus, the biconvex polytope $P$ is the bounded part of a polyhedral complex that is dual to $\pi (\Sigma^*)$.

**Remark 3.6.** The polyhedral subdivisions $\pi (\Sigma)$ of $\Delta$, $\pi (\Sigma^*)$ of $\Delta^*$, and the matroid subdivisions $\Sigma$ of $\Delta$ and $\Sigma^*$ of $\Delta^*$ are all regular.

### 3.4 Face-fitting directed bigraphs

Let $G_1$ and $G_2$ be two directed bigraphs with the same node set $V_{G_1} = V_{G_2} = [k]$, and fix an underlying partition. Then, $\text{BP} \Gamma | V_{G_1} |$ is a common face of $\text{BP} \Gamma | G_1 |$ and $\text{BP} \Gamma | G_2 |$.

Suppose that none of $\text{BP} \Gamma | G_1 |$ and $\text{BP} \Gamma | G_2 |$ contains the other, and that $\text{BP} \Gamma | G_1 | \cap \text{BP} \Gamma | G_2 |$ is a common face of them. Then, there is a maximum common subgraph $G'$ of $G_1$ and $G_2$ with $\text{BP} \Gamma | G_1 | \cap \text{BP} \Gamma | G_2 | = \text{BP} \Gamma | G' |$. Moreover, there are subgraphs $G'_1$ and $G'_2$ of $G_1$ and $G_2$, respectively, with

$$G'_1 (e_1) = G'_2 (e_2)$$

$$G'_1 (e_1) = (G'_2)^- (e_2)$$

$$G'_1 (e_1) = (G'_2)^+ (e_2)$$

for some arrows $e_1$ and $e_2$ of $G'_1$ and $G'_2$, respectively.
Conversely, if there is a maximum common subgraph $G'$ of $G_1$ and $G_2$ with subgraphs $G'_1$ and $G'_2$, respectively, satisfying (3.1), then $BP_{Γ[G_1]} \cap BP_{Γ[G_2]}$ is a common proper face of $BP_{Γ[G_1]}$ and $BP_{Γ[G_2]}$. In this case, we say that the two directed bigraphs $G_1$ and $G_2$ are face-fitting. We say that multiple directed bigraphs with the same node set are face-fitting if they are pairwise face-fitting.

**Example 3.7.** The directed bigraphs $G_1$ and $G_2$ of Figure 3.1 have parts $(\{1\}, \{2, 3, 4\})$ and $(\{1, 4\}, \{2, 3\})$, respectively, and are face-fitting since $G_1(1, 4) = G_2(3, 4)$ with $G_1^+(1, 4) = G_2^-(3, 4)$ and $G_1^-(1, 4) = G_2^+(3, 4)$. Then $BP_{Γ[G_1]} \cap BP_{Γ[G_2]} = BP_{Γ[G_1(1, 4)]} = BP_{Γ[G_2(3, 4)]}$ is a codimension-1 common face of $BP_{Γ[G_1]}$ and $BP_{Γ[G_2]}$, and $\{BP_{Γ[G_1]}, BP_{Γ[G_2]}\}$ is a matroid tiling.

**Example 3.8.** The 6 directed bigraphs $G_1, \ldots, G_6$ of Figure 3.2 are face-fitting, and $Σ = \{BP_{Γ[G_l]} : l ∈ [6]\}$ is a matroid tiling with 5 connecting facets, i.e. 5 common facets of two polytopes of $Σ$, whose matroids are:

- $Γ[G_2(4, 2)] = Γ[G_1(1, 4)]$,
- $Γ[G_2(1, 3)] = Γ[G_5(3, 2)]$,
- $Γ[G_3(1, 3)] = Γ[G_4(4, 1)]$,
- $Γ[G_3(4, 2)] = Γ[G_6(2, 3)]$,
- $Γ[G_2(1, 2)] = Γ[G_3(4, 3)]$.

**Example 3.9.** Replace $G_2$ and $G_3$ of Figure 3.2 with $G'_2$ and $G'_3$ of Figure 3.3, then $G_1, G_4, G_5, G_6, G'_2, G'_3$ are face-fitting and the corresponding matroid tiling with 5 connecting facets whose matroids are:

- $Γ[G'_2(4, 3)] = Γ[G_1(1, 4)]$,
- $Γ[G'_2(1, 2)] = Γ[G_6(2, 3)]$,
- $Γ[G'_3(1, 2)] = Γ[G_4(4, 1)]$,
- $Γ[G'_3(4, 3)] = Γ[G_5(3, 2)]$,
- $Γ[G'_2(1, 3)] = Γ[G'_3(4, 2)]$. 

**Figure 3.1.** Face-Fitting Directed Bigraphs, I
Figure 3.2. Face-Fitting Directed Bigraphs, II

Figure 3.3. Face-Fitting Directed Bigraphs, III

Remark 3.10. \( \{G_2, G_3\} \) and \( \{G_2', G_3'\} \) are two unique pairs of directed bigraphs that can be added to \( \{G_1, G_4, G_5, G_6\} \) in order to extend \( \{\operatorname{BP}_{l[G_l]} : l = 1, 4, 5, 6\} \) to a matroid tiling.
4. Biconvex Polytope as Cell of Tropical Linear Space

In this short section, we prove a biconvex polytope arises as a cell of a tropical linear space. Before doing so, we give a brief review of the tropical linear spaces. Readers are referred to [MS15] for more.

Let $M$ be a rank-$k$ connected matroid on $S$ with $n = |S|$, and $B$ its base collection. The Dressian\(^7\) $D_M$ of $M$ is the intersection of the tropical hypersurfaces in $\mathbb{R}^B / \mathbb{R}^1$ defined by the tropicalized Plücker relations for $M$. Here a tropicalized Plücker relation for $M$ is a tropical polynomial $\boxplus_{j \in \tau'} z_{\sigma \cup \{j\}} \odot z_{\tau - \{j\}}$ for a coordinate vector $z = (z_B)_{B \in B}$ in $\mathbb{R}^B / \mathbb{R}^1$. Now, define

$$L_{z} := \bigcap_{\tau} L_{\tau}(z)$$

which is a $(k - 1)$-dimensional balanced contractible polyhedral complex in $\mathbb{R}^S / \mathbb{R}^1$, and called a tropical linear space.

Every $z \in \mathbb{R}^B / \mathbb{R}^1$ induces a regular polyhedral subdivision of the base polytope $\text{BP}_M$. Furthermore:

**Proposition 4.1.** A point $z$ of $\mathbb{R}^B / \mathbb{R}^1$ is contained in $D_M$ if and only if the regular subdivision of $\text{BP}_M$ that $z$ induces is a matroid subdivision.

Now, let $P = \text{tconv}(v_1, \ldots, v_k) \subset \mathbb{R}^k / \mathbb{R}^1$ be any maximal biconvex polytope. Then, it immediately follows that for the regular matroid subdivision $\Sigma^*$ of $\Delta_S^{[S] - k}$ that is constructed in Subsection 3.3, $P$ is tropically isomorphic to the bounded part of a tropical linear space that is a polyhedral complex dual to $\Sigma^*$.

The above statement holds for any biconvex polytope because every non-maximal biconvex polytope is obtained as a tropical degeneration of a maximal one.

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\(^7\)This is named after Andreas Dress due to his original work on “valuated matroids” with Walter Wenzel, [DW92]. One might think this should be called the “tropical Grassmannian”. However, “being generated” for ideals is not transferred via tropicalization, and the tropical Grassmannian is defined as the intersection of all tropical hypersurfaces coming from tropicalized elements of the Plücker ideal.
5. Subdividing Hypersimplices: Rank-4 Case

We employ our theory to manually conduct all the computations with pen and paper without resorting to computers, which reflects the power of our theory. Fix an underlying partition $S = \bigcup_{i \in [k]} S_i$ with $|S_i| \geq k$ for all $i \in [k]$. We first borrow necessary tools from [Shi19], and then work out the rank-4 case.

Lemma 5.1. Let $S$ be a finite set and $k$ an integer with $1 \leq k < |S|$.

(1) [Shi19, Lemma 4.9] Let $F \subset S$ be a proper subset of size $\geq 2$ and $\rho$ a positive integer with $k + |F| - |S| < \rho < \min\{k, |F|\}$.

Then, $P_1 = \Delta^k_S \cap \{x(F) \leq \rho\}$ and $P_2 = \Delta^k_S \cap \{x(S - F) \leq k - \rho\}$ are full-dimensional base polytopes that are face-fitting, which form a matroid subdivision of $\Delta^k_S$.

(2) [Shi19, Lemma 4.9] Let $P_1 = BP_{M_1}$ and $P_2 = BP_{M_2}$, then $F$ and $S - F$ are the unique non-degenerate flats of size $\geq 2$ of $M_1$ and $M_2$, respectively, with ranks $\rho$ and $k - \rho$.

(3) [Shi19, Corollary 4.8] For a partition $S = \bigcup_{j \in [k]} S_j$, cutting $\Delta^k_S$ with all $k$ hyperplanes of the form $\{x(S_j) = 1\}$ produces a matroid subdivision.

Example 5.2. Let $k = 4$. By cutting $\Delta^4_S$ with all 4 hyperplanes $\{x(S_i) = 1\}$ we get a matroid subdivision $\Sigma$. See Figure 5.1 for the quotient subdivision $\pi(\Sigma)$. We have $\Gamma[V] = \bigoplus_{i \in [4]} U^3_i$ for $V = [4]$, and

$$Q = BP \bigoplus_{i \in [4]} U^3_i$$

is the maximum common face of all members of $\Sigma$ with $\pi(Q) = (1, 1, 1, 1)$. See Figure 5.2 for the typical 3 (quotient) polytopes of $\Sigma$ and the corresponding directed graphs where the graphs of Figure 5.2 are from Figure 3.2.

![Figure 5.1. A quotient matroid subdivision of rank 4.](image-url)
Observe that in Example 5.2, by cutting \( \Delta_S^4 \) with 4 hyperplanes \( \{ x(S - S_i) = 3 \} \) instead, we obtain the same matroid subdivision because:
\[
\{ x(S - S_i) = 3 \} = \{ x(S_i) = 1 \}.
\]

Observe also that the 4 polytopes
\[
\bigcap_{j \in [4] - \{i\}} \{ x(S_j) \geq 1 \}
\]
whose quotients are parallelepipeds positioned at the 4 corners of the tetrahedron of Figure 5.1, cannot be further cut into full-dimensional base polytopes, and neither can the following 4 polytopes whose quotients are smaller tetrahedra positioned at the centers of the 4 facets of the tetrahedron
\[
\bigcap_{j \in [4] - \{i\}} \{ x(S_j) \leq 1 \}.
\]

Now, we want to obtain a matroid subdivision of \( \Delta_S^4 \) with \( Q = BP \bigoplus_{i \in [4]} U_{S_i}^1 \) being a common cell that is finer than the matroid subdivision of Example 5.2. Then, we know that we have to cut each of its 6 polytopes whose quotients are positioned in the middle of the 6 edges of the tetrahedron, with hyperplanes of the form
\[
\{ x(A) = 2 \} \quad \text{for a subset } A \subset S
\]
because we have already cut \( \Delta_S^4 \) with all hyperplanes of the form \( \{ x(A) = 1 \} \) or \( \{ x(A) = 3 \} \) and all 6 of those polytopes contain \( Q \), cf. Lemma A.1(2). Further, each \( A \) is a rank-2 flat of the matroid \( \bigoplus_{i \in [4]} U_{S_i}^1 \), and is a union of precisely two of \( S_1, S_2, S_3, \) and \( S_4 \). Thus the hyperplanes we cut with have the following form
\[
(5.1) \quad \{ x(S_J) = 2 \} \quad \text{for a size-2 subset } J \subset [4].
\]
Moreover, by the following lemma, the number of such cutting hyperplanes cannot exceed 1, and hence is 1.
Lemma 5.3. Let $M$ be a rank-4 connected matroid with a rank-2 non-degenerate flat $F$. If $L$ is a non-degenerate flat such that $BP_{M(F)} \cap BP_{M(L)}$ is a codimension-2 face of $BP_M$ that is not contained in a coordinate hyperplane, then $r(L) \neq 2$.

Proof. Since $BP_{M(F)} \cap BP_{M(L)} = BP_{M(F) \cap M(L)}$ is nonempty, $\{F, L\}$ is a modular pair by Lemma A.1(3). Also, $M(F) \cap M(L)$ is a loopless face matroid. To prove by contrapositive, suppose $r(L) = 2$. Then, $r(L \cap F) < 2$ since $L \neq F$. Since $\{F, L\}$ is a modular pair, we have either $r(F \cap L) = 0$ and $r(F \cup L) = 4$, or $r(F \cap L) = 1$ and $r(F \cup L) = 3$. In the former case, $F \cap L = \emptyset$ which implies that $F \cup L$ is a non-flat of rank 4 by Lemma A.1(4), but then $M(F) \cap M(L) = M(F \cup L)$ has a loop, a contradiction. In the latter case, $F \cap L \neq \emptyset$ and $F \cup L \neq S$. Again by Lemma A.1(4), $F \subsetneq L$ or $L \subsetneq F$, which contradicts $r(F) = r(L) = 2$. Thus we conclude $r(L) \neq 2$. $\square$

By Lemmas 5.1 and 5.3, for any matroid subdivision of our interest, the number of its members does not exceed $4 + 4 + 2 \cdot 6 = 20$ which is the maximum number of vertices of a 3-dimensional biconvex polytope. Moreover, if we cut each of those 6 polytopes with a hyperplane of the form (5.1), then we obtain a matroid subdivision with 20 members, see Figures 5.3 and 5.4; hence the maximum is attained.

Appendix A. Definitions and Lemmas

Throughout the section, $M$ is a matroid on a finite set $S$ with rank function $r$.

A pair $\{A, B\}$ of subsets of $S$ is called a modular pair of $M$ if equality holds in the submodular inequality $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$, that is:

$$r(A) + r(B) = r(A \cup B) + r(A \cap B).$$

A subset $A \subseteq S$ is called a separator of $M$ if $\{A, S - A\}$ is a modular pair. Both $S$ and $\emptyset$ are separators which are called trivial separators. Then, $M$ and its dual matroid $M^*$ have the same set of separators.

The matroid $M$ is called connected if it has no nontrivial separators, and disconnected otherwise. A subset $A \subseteq S$ is called connected if the restriction matroid $M|_A$ is, and disconnected otherwise.

We denote by $\kappa(M)$ the number of all nonempty inclusionwise minimal separators of $M$ where the inclusion is set inclusion. Let $S_1, \ldots, S_{\kappa(M)}$ be all nonempty inclusionwise minimal separators of $M$, then:

$$M = M|_{S_1} \oplus \cdots \oplus M|_{S_{\kappa(M)}}.$$  

Here, $M|_{S_i}$ are called the connected components of $M$.

For a subset $A \subseteq S$, we denote

$$(A.1) \quad M(A) : = M|_A \oplus M/A.$$

The subset $A$ is called non-degenerate\(^8\) if

$$\kappa(M(A)) = \kappa(M) + 1$$

and degenerate otherwise. Every separator is degenerate. If $A$ is a non-degenerate subset of $M$, then $S - A$ is a non-degenerate subset of its dual matroid $M^*$.

---

\(^8\)The definition of non-degenerate subsets was originally given in [GS87] for connected matroids, and generalized to the current form in [Shi19].
If $M$ is a disconnected matroid, there can be different non-degenerate subsets $A_1, \ldots, A_n$ with $M(A_1) = \cdots = M(A_n)$, but there exists the smallest such.

The **indicator vector** of $A \subseteq S$ is the vector $v \in \mathbb{R}^S$ such that $x_i(v)$ is 1 if $i \in A$ and 0 otherwise, which we denote by $1^A$.

The set of the bases of $M$ is denoted by $B(M)$. The **matroid base polytope** or simply the **base polytope** of $M$ is the convex hull of all indicator vectors $1^B$ of $B \in B(M)$, which we denote by $BP_M$. Its dimension is $\dim BP_M = |S| - \kappa(M)$.

Using inequalities, $BP_M$ is written as

$$BP_M = \{x \geq 0 : i \in S\} \cap \{x(A) \leq r(A) : A \in 2^S\} \cap \{x(S) = r(S)\}.$$  

The correspondence between matroids and base polytopes is one-to-one. The base polytope $BP_M$ is full-dimensional if and only if $M$ is connected.

A **face matroid** of $M$ is the matroid of a face of $BP_M$.

Two polytopes are called **face-fitting** if their intersection is a common face of both, empty or not.

A $(k, S)$-**tiling** or simply a **tiling** is a finite face-fitting collection of polytopes in $\Delta_k^S$ that is connected in codimension 1. The **support** $|\Sigma|$ of a tiling $\Sigma$ is the union of its members. The **dimension** of $\Sigma$ is the dimension of $|\Sigma|$. Throughout the paper, a tiling is assumed equidimensional, i.e. all of its members have the same dimension.

A tiling induced by a convex or concave function is called **regular**.

When mentioning **cells** of $\Sigma$, we identify $\Sigma$ with the polyhedral complex that its polytopes generate with intersections. A nonempty cell of $\Sigma$ is called a **common cell** if it is a face of all members of $\Sigma$.

A **matroid tiling** is a tiling whose members are base polytopes, which is well defined because every face of a base polytope is again a base polytope.

A **matroid subdivision** is a matroid tiling whose support is a base polytope.

The **base intersection** of two matroids $M_1$ and $M_2$ is the intersection of the base collections of $M_1$ and $M_2$, which we denote by $M_1 \cap M_2$.

When $M_1 \cap M_2$ is the base collection of a matroid, by abuse of notation, we denote the matroid by $M_1 \cap M_2$.

For a subcollection $\mathcal{A}$ of the power set $2^S$ of $S$, let $P_\mathcal{A}$ be the convex hull of the indicator vectors $1^A$ for all $A \in \mathcal{A}$. Then:

$$BP_{M_1} \cap BP_{M_2} = P_{M_1 \cap M_2}.$$  

**Lemma A.1.** Let $M$ be a rank-$k$ matroid on $S$ with rank function $r$.

1. [Shi19, Lemma 4.2] If there is a subset $A \subseteq S$ of size $k + 1$ with $M|_A = U_A^k$, then $M \setminus \emptyset_M$ is a connected matroid where $\emptyset_M$ denotes the set of loops of $M$.

2. [Shi19, Lemma 2.30] Suppose $M = M|_{S_1} \oplus \cdots \oplus M|_{S_{\kappa(M)}}$ is loopless. Then, its base polytope $BP_M$ is determined by $\kappa(M)$ equations $x(S_i) = r(S_i)$ and the following inequalities:

$$\begin{cases} x_i \geq 0 & \text{for all } i \in S, \\ x(F) \leq r(F) & \text{for all minimal non-degenerate flats } F \text{ of } M. \end{cases}$$
(3) [Shi19, Lemma 2.24] The pair \( \{F, L\} \) of subsets of \( S \) is modular if and only if \( M(F) \cap M(L) \neq \emptyset \).

(4) [Shi19, Lemma 2.38] Suppose \( M \) is connected with \( k \geq 3 \). Let \( \{F, L\} \) be a modular pair of non-degenerate flats with \( \text{codim} BP_{M(F) \cap M(L)} = 2 \). Then, precisely one of the following 4 cases happens.

| \( F \cap L = \emptyset \) | \( M(F) \cap M(L) = M(F \cup L) \) with \( M|_{F \cup L} = M|_F \oplus M|_L \) |
| \( F \cup L = S \) | \( M(F) \cap M(L) = M(F \cap L) \) with \( M/(F \cap L) = M/F \oplus M/L \) |
| \( F \supseteq L \) | \( M(F) \cap M(L) = M/F \oplus M|_F/L \oplus M/L \) |
| \( F \subseteq L \) | \( M(F) \cap M(L) = M/L \oplus M|_{L/F} \oplus M|_F \) |

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Figure 5.3. A matroid subdivision of $P$ with the corresponding directed bigraphs, I.
Figure 5.4. A matroid subdivision of $P$ with the corresponding directed bigraphs, II.