Latent Variable Discovery Using Dependency Patterns

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Abstract—The causal discovery of Bayesian networks is an active and important research area, and it is based upon searching the space of causal models for those which can best explain a pattern of probabilistic dependencies shown in the data. However, some of those dependencies are generated by causal structures involving variables which have not been measured, i.e., latent variables. Some such patterns of dependency “reveal” themselves, in that no model based solely upon the observed variables can explain them as well as a model using a latent variable. That is what latent variable discovery is based upon. Here we did a search for finding them systematically, so that they may be applied in latent variable discovery in a more rigorous fashion.

Index Terms—Bayesian networks, Latent variables, causal discovery, probabilistic dependencies

I. INTRODUCTION

What enables latent variable discovery is the particular probabilistic dependencies between variables, will typically be representable only by a proper subset of the possible causal models over those variables, and therefore provide evidence in favour of those models and against all the remaining models, as can be seen in the Bayes factor. Some dependency structures between observed variables will provide evidence favoring latent variable models over fully observed models because they can explain the dependencies better than any fully observed model. We call such structures “triggers” and did a systematically search for them. The result is a clutch of triggers, many of which have not been reported before to our knowledge. These triggers can be used as a data preprocessing analysis by the main discovery algorithm.

A. Latent Variable Discovery

Latent variable modeling has a long tradition in causal discovery, beginning with Spearman’s work [7] on intelligence testing. Factor analysis and related methods can be used to posit latent variables and measure their hypothetical effects. They do not provide clear means of deciding whether or not latent variables are present in the first place, however, and in consequence there has been some controversy about that status of exploratory versus confirmatory factor analysis. In this regard, causal discovery methods in AI have the advantage.

One way in which discovery algorithms may find evidence confirmatory of a latent variable model is in the greater simplicity of such a model relative to any fully observed model that can represent the data adequately, as Friedman pointed out using the example in Figure 1.

Figure 1: An illustration of how introducing a latent variable \( \hat{H} \) can simplify a model [1].

Another advantage for latent variable models is that they can better encode the actual dependencies and independencies in the data. For example, Figure 2 demonstrates a latent variable model of four observed variables and one latent variable. If the data support the independencies \( W \perp \{Y, Z\} \) and \( Z \perp \{W, X\} \), it is impossible to construct a network in the observed variables alone that reflects both of these independencies while also reflecting the dependencies implied by the d-connections in the latent variable model. It is this kind of structure which can allow us to infer the existence of latent variables, i.e., one which constitutes a trigger for latent variable discovery.
II. Searching Triggers for Latent Variables

In this paper, latent variables are typically considered only in scenarios where they are common causes which having two or more children. As Friedman [1] points out, a latent variable as a leaf or as a root with only one child would marginalize out without affecting the distribution over the remaining variables. So too would a latent variable that mediates only one parent and one child. For simplicity, we also only search for triggers isolated latent variables rather than multiple latent variables.

We start by enumerating all possible fully observed DAGs in a given number of variables (this step is already super exponential [6]). Then it generates all possible d-separating evidence sets. For example, for the four variables $W, X, Y$ and $Z$, there are eleven evidence sets:

$$\phi, \{W\}, \{X\}, \{Y\}, \{Z\}, \{WX\}, \{WY\}, \{WZ\}, \{XY\}, \{XZ\}, \{YZ\}$$

For each fully observed DAG it produces the corresponding dependencies for each evidence set using the d-separation criterion (i.e., for the four variables $W, X, Y$ and $Z$, the search produces eleven dependency matrices). Next, it generates all possible single hidden-variable models whose latent variable is a common cause of two observed variables. It then generates all the dependencies between observed variables for each latent variable model, conditioned upon each evidence set. The set of dependencies of a latent variable model is a trigger if and only if these dependency sets cannot be matched by any fully observed DAG in terms of d-separation.

We ran our search for 3, 4 and 5 observed variables. Any structures with isolated nodes are not be included. As Table I shows, for three observed variables, we find no trigger, meaning the set of dependencies implied by all possible hidden models can also be found in one or more fully observed models. There are two triggers for four observed variables, the corresponding DAGs are shown in Table II together with the corresponding latent variable models. For five observed variables, we find 57 triggers (see Appendix A).

Table I: Number of triggers found

| Observed variables | DAGs  | Connected DAGs | Triggers |
|--------------------|-------|----------------|---------|
| 3                  | 6     | 4             | 0       |
| 4                  | 31    | 24            | 2       |
| 5                  | 302   | 268           | 57      |

Table II: The DAGs of the two triggers found for four observed variables.

All the dependency structures in the observed variables revealed as triggers will be better explained with latent variables than without. While it is not necessary to take triggers into account explicitly in latent variable discovery, since random structural mutations combined with standard metrics may well find them, they can be used to advantage in the discovery process, by focusing it, making it more efficient and more likely to find the right structure.

III. Learning Triggers with Causal Discovery Algorithms

The most popular causal discovery programs in general, come from the Carnegie Mellon group and are incorporated into TETRAD, namely FCI and PC [9]. Hence, they are the natural foil against which to compare anything we might produce. Our ultimate goal is to incorporate latent variable discovery into a metric-based program. As that is a longer project, here we report experimental results using an ad hoc arrangement of running the trigger program as a filter to ordinary PC (Trigger-PC) and comparing that with the unaltered FCI and PC. Our experimental procedure, briefly, was:

1) Generate random networks of a given number of variables, with three categories of dependency: weak, medium and strong.
2) Generate artificial data sets using these networks.
3) Optimize the alpha level of the FCI and PC programs using the above.
4) Experimentally test and compare FCI and PC.

The FCI and PC algorithms do not generally return a single DAG, but a hybrid graph [3]. An arc between two nodes may be undirected ‘—’ or bi-directional ‘→→’, which indicates the presence of a latent common cause. Additionally, the graph produced by FCI may contain ‘o—o’ or ‘o→’. The circle represents an unknown relationship, which means it is not known whether or not an arrowhead occurs at that end of the arc [3]. So, in order to measure how close the models learned by FCI and PC are to the true model, we developed a special version of the edit distance between graphs (see the Appendix B).

Figure 2: One causal structure with four observed variables and one latent variable $H$. 

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2Note that sets of three or more evidence variables leave nothing left over to separate.

3In this search, labels (variable names) are ignored, of course, since all that matters are the dependency structures.
A. Step one: generate networks with different dependency strengths.

Genetic algorithms (GAs) \[5\] are commonly applied as a search algorithm based on an artificial selection process that simulates biological evolution. Here we used a GA algorithm to find good representative, but random, graphs with the three levels of desired dependency strengths between variables: strong, medium and weak. The idea is to test the learning algorithms across different degrees of difficulty in recovering arcs (easy, medium and difficult, respectively). Mutual information \[4\] is used to assess the strengths of individual arcs in networks.

To make the learning process more efficient, we set the arities for all nodes in a network to be the same, either two or three. We randomly initialized all variables’ CPT parameters for each individual graph and used a whole population of 100 individuals. The GA was run 100 generations. We ran the GA for each configuration (number of nodes and arities) three times, the first two to obtain networks with the strongest and weakest dependencies between parents and their children and the third time to obtain networks closest to the average of those two degrees of strength.

B. Step two: generate artificial datasets.

All networks with different arc strength levels were used to generate artificial datasets with sample sizes of 100, 1000 and 10000. We used Netica API \[2\] to generate random cases. The default sampling method is called “Forward Sampling” \[3\] which is what we used.

| Number of observed variables | Structure type | Number of structures | Total number of simulated datasets |
|-----------------------------|---------------|---------------------|-----------------------------------|
| 4                           | Trigger       | 2                   | 36                                |
| 4                           | DAG           | 24                  | 432                               |
| 5                           | Trigger       | 57                  | 1026                              |
| 5                           | DAG           | 268                 | 4824                              |

Table III: Number of simulated datasets

As Table III shows, there are a relatively large number of simulated datasets. This is due to the different state numbers, arc strength levels and data sizes. For example, there are 57 trigger structures for 5 observed variables, so there are \(57 \times 2 \times 3 \times 3 = 1026\) simulated datasets.

C. Step three: optimize alpha to obtain the shortest edit distance from true models

FCI and PC both rely on statistical significance tests to decide whether an arc exists between two variables and on its orientation. They have a default alpha level (of 0.05), but the authors have in the past criticized experimental work using the default and recommended instead optimizing the alpha level for the task at hand, so here we do that. The idea is to give the performance of FCI and PC the benefit of any possible doubt. Given the artificial data sets generated, we can use FCI and PC with different values of alpha to learn networks and compare the results to the models used to generate those data sets. We then used our version of edit distance between the learned and generating models (see Appendix \[3\]) to find the optimal alpha levels for both algorithms.

We first tried simulated annealing to search for an optimal alpha, but in the end simply generated sufficiently many random values from the uniform distribution over the range of \([0.0, 0.5]\). We evaluated alpha values for the datasets with 2 states and 3 states separately.

As shown in the following graphs, the average edit distances between the learned and true models approximate a parabola with a minimum around 0.1 to 0.2. The results below are specific to the exact datasets and networks we developed for this experimental work.

1) FCI algorithm

Results for FCI were broadly similar. In summary, the optimal alphas found for the above cases (in the same order) were: 0.12206, 0.19397, 0.20862 and 0.12627.

- Number of observed variables: 4
  - Datasets: 2 state DAG structure simulated dataset
  - Number of datasets: 24*9 = 216

Results:
Minimum average edit distance: 14.85185
Maximum average edit distance: 15.82870
Mean average edit distance: 15.37168
Best Alpha: 0.12206
95% confidence level: 0.03242
95% confidence interval: (15.37168-0.03242, 15.37168+0.03242)

- Number of observed variables: 4
  - Datasets: 3 state DAG structure simulated dataset
  - Number of datasets: 24*9 = 216
Results:
Minimum average edit distance: 11.42593
Maximum average edit distance: 13.43981
Mean average edit distance: 12.09355
Best Alpha: 0.19397
95% confidence level: 0.07259
95% confidence interval: (12.09355-0.07259, 12.09355+0.07259)

- Number of observed variables: 5
- Datasets: 2 state DAG structure simulated dataset
- Number of datasets: 268*9 = 2412

Results:
Minimum average edit distance: 23.72844
Maximum average edit distance: 24.98466
Mean average edit distance: 24.08980
Best Alpha: 0.20862
95% confidence level: 0.03880
95% confidence interval: (24.08980-0.03880, 24.08980-0.03880)

2) PC algorithm
Results for PC were quite similar. In summary, the optimal alphas found for the above cases (in the same order) were:
0.12268, 0.20160, 0.20676 and 0.13636.

D. Step four: compare the learned models with true model.

Finally, we were ready to test FCI and PC on the artificial datasets of trigger (i.e., latent variable) and DAG structures. Artificial datasets generated by trigger structures were used to determine True Positive (TP) and False Negative (FN) results (i.e., finding the real latent and missing the real latent, respectively), while the datasets of (fully observed) DAG structures were used for False Positive (FP) and True Negative (TN) results. Assume the latent variable in every trigger structure is the parent of node A and B, we used the following definitions:

- TP: The learned model has a bi-directional arc between A and B.
- FN: The learned model lacks a bi-directional arc between A and B.
- TN: The learned model has no bi-directional arcs.
- FP: The learned model has one or more bi-directional arcs.

We tested the FCI and PC algorithms on different datasets with their corresponding optimized alphas. We do not report confidence intervals or significance tests between different algorithms under different conditions, since the cumulative results over 6,318 datasets suffices to tell the comparative story.

The following tables show the confusion matrix summing over all datasets (see Appendix C for more detailed results):
With corresponding optimal alpha, the FCI’s predictive accuracy was 0.71 (rounding off), its precision 0.19, its recall 0.22 and its false positive rate 0.19. The predictive accuracy for PC was 0.74, the precision was 0.22, its recall 0.21 and the false positive rate was 0.16.

We also did the same tests using the default alpha of 0.05. The results are shown as follow (see Appendix C for more detailed results):

|        | Latent | No Latent | Latent | No Latent |
|--------|--------|-----------|--------|-----------|
| Positive | 211    | 767       | 205    | 615       |
| Negative | 851    | 4489      | 857    | 4641      |

With alpha of 0.05, the FCI’s predictive accuracy was 0.74, its precision 0.22, its recall 0.19 and its false positive rate 0.15. PC’s predictive accuracy was 0.77, the precision was 0.25, its recall 0.19 and the false positive rate was 0.12.

As we can see from the results, the performance of FCI and PC are quite similar. Neither are finding the majority of latent variables actually there, but both are at least showing moderate false positive rates. Arguably, false positives are a worse offense than false negatives, since false negatives leave the causal discovery process no worse off than an algorithm that ignores latents, whereas a false positive will positively mislead the causal discovery process.

IV. APPLYING TRIGGERS IN CAUSAL DISCOVERY

A. An extension of PC algorithm (Trigger-PC)

We implemented triggers as a data filter into PC, yielding Trigger-PC, and see how well it would work. If Trigger-PC finds a trigger pattern in the data, then it returns that trigger structure, otherwise it returns whatever structure the PC algorithm returns, while replacing any incorrect bi-directed arcs by undirected arcs. So the Trigger-PC algorithm (see Algorithm 1) gives us a more specific latent model structure and, as we shall see, has fewer false positives.

We tested our Trigger-PC algorithm with the alpha optimized for the PC algorithm. The resultant confusion matrix was (see Appendix C for more details):

|        | Latent | No Latent |
|--------|--------|-----------|
| Positive | 30     | 3         |
| Negative | 1032   | 5253      |

Trigger-PC’s predictive accuracy was 0.84, its precision 0.91, its recall 0.03 and the false positive rate 0.0006. We can see that Trigger-PC is finding far fewer latents than either PC or FCI, but when it asserts their existence we can have much greater confidence in the claim. As we indicated above, avoiding false positives, while having at least some true positives, appears to be the more important goal in latent variable discovery.

As before, we also tried the default 0.05 alpha in Trigger-PC, with the results (see more details in Appendix C):

|        | Latent | No Latent |
|--------|--------|-----------|
| Positive | 35     | 4         |
| Negative | 1027   | 5252      |

And, again, these results are only slightly different. With alpha of 0.05, the trigger-PC’s predictive accuracy was 0.84,
its precision 0.90, its recall 0.03 and the false positive rate 0.0008.

V. CONCLUSION

We have presented the first systematic search algorithm to discover and report latent variable triggers: conditional probability structures that are better explained by latent variable models than by any DAG constructed from the observed variables alone. For simplicity and efficiency, we have limited this to looking for single latent variables at a time, although that restriction can be removed. We have also applied this latent discovery algorithm directly in an existing causal discovery algorithm and compared the results to existing algorithms which discover latents using different methods. The results are certainly different and arguably superior. Our next step will be to implement this approach within a metric-based causal discovery program.

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APPENDIX A
57 TRIGGERS FOR LATENT VARIABLE MODELS FOUND FOR FIVE OBSERVED VARIABLES
APPENDIX B
EDIT DISTANCE USED IN CAUSAL MODEL RESULTS

Edit distance between different type of arcs (between variable A and B) produced by PC algorithm and true arc type:

| True arc | A   | B   | Learned arc | A   | B   | Distance |
|----------|-----|-----|-------------|-----|-----|----------|
| A → B    | Tail| Arrow| A—B         | Tail| Tail| 2        |
| A ← B    | Tail| Arrow| A → B       | Tail| Arrow| 0       |
| A ↔ B    | Arrow| Tail| A ← B       | Arrow| Tail| 4       |
| A ↔ B    | Arrow| Arrow| A ↔ B      | Arrow| Arrow| 2       |
| null     | null| null| null        | null| null| 6       |

Edit distance between different type of arcs (between variable A and B) produced by FCI algorithm and true arc type:

| True arc | A   | B   | Learned arc | A   | B   | Distance |
|----------|-----|-----|-------------|-----|-----|----------|
| A → B    | Tail| Arrow| A—B         | Tail| Tail| 4        |
| A ← B    | Tail| Arrow| A → B       | Tail| Arrow| 2       |
| A ↔ B    | Arrow| Tail| A ← B       | Arrow| Tail| 2       |
| A ↔ B    | Arrow| Arrow| A ↔ B      | Arrow| Arrow| 0       |
| null     | null| null| null        | null| null| 6       |

null     null     null     A—B         Tail| Tail| 6
A → B    Tail| Arrow| 6
A ← B    Arrow| Tail| 6
A ↔ B    Arrow| Arrow| 6
null     null| null| 0
### APPENDIX C

#### CONFUSION MATRIX

1) Results of FCI algorithm (with optimized alpha)
- 4 observed variables with 2 state each simulated data

| Alpha: 0.12206 |

#### Maximum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 0         | 1      | 0         | 0      | 0         |
| Negative         | 2      | 24        | 1      | 24        | 2      | 24        |

#### Medium arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 0         | 2      | 2         | 1      | 0         |
| Negative         | 2      | 24        | 0      | 22        | 1      | 24        |

#### Minimum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 0         | 0      | 0         | 0      | 2         |
| Negative         | 2      | 24        | 2      | 24        | 2      | 22        |

- 4 observed variables with 3 state each simulated data

Alpha: 0.19397

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Maximum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 0         | 1      | 0         | 2      | 1         |
| Negative         | 2      | 24        | 1      | 24        | 0      | 23        |

Medium arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 4         | 1      | 1         | 2      | 0         |
| Negative         | 2      | 20        | 1      | 23        | 0      | 20        |

Minimum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 1         | 0      | 2         | 0      | 4         |
| Negative         | 2      | 23        | 2      | 22        | 2      | 20        |

- 5 observed variables with 2 state each simulated data

Alpha: 0.20862

Maximum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 1      | 38        | 3      | 43        | 12     | 30        |
| Negative         | 56     | 230       | 54     | 225       | 45     | 238       |

Medium arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 1      | 52        | 16     | 72        | 34     | 57        |
| Negative         | 56     | 216       | 41     | 196       | 23     | 211       |

Minimum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 7         | 0      | 26        | 0      | 41        |
| Negative         | 57     | 261       | 57     | 242       | 57     | 227       |

- 5 observed variables with 3 state each simulated data

Alpha: 0.12627

Maximum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 6      | 82        | 26     | 59        | 51     | 16        |
| Negative         | 51     | 186       | 31     | 209       | 6      | 252       |

Medium arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 1      | 104       | 19     | 98        | 46     | 15        |
| Negative         | 56     | 164       | 38     | 170       | 11     | 233       |
Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
|                  | Latent | No latent | Latent | No latent | Latent | No latent |
| Positive         | 0 | 6 | 1 | 80 | 8 | 138 |
| Negative         | 57 | 262 | 56 | 168 | 49 | 130 |

2) Results of PC algorithm (with optimized alpha)
- 4 observed variables with 2 state each simulated data

Alpha: 0.12268

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
|                  | Latent | No latent | Latent | No latent | Latent | No latent |
| Positive         | 0 | 0 | 0 | 0 | 0 | 0 |
| Negative         | 2 | 24 | 2 | 24 | 2 | 24 |

Medium arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
|                  | Latent | No latent | Latent | No latent | Latent | No latent |
| Positive         | 0 | 2 | 1 | 1 | 2 | 0 |
| Negative         | 2 | 24 | 0 | 23 | 1 | 24 |

Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
|                  | Latent | No latent | Latent | No latent | Latent | No latent |
| Positive         | 0 | 0 | 0 | 0 | 2 | 1 |
| Negative         | 2 | 24 | 2 | 24 | 2 | 24 |

- 4 observed variables with 3 state each simulated data

Alpha: 0.20160

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
|                  | Latent | No latent | Latent | No latent | Latent | No latent |
| Positive         | 0 | 2 | 1 | 1 | 2 | 0 |
| Negative         | 2 | 22 | 1 | 23 | 0 | 24 |

Medium arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
|                  | Latent | No latent | Latent | No latent | Latent | No latent |
| Positive         | 0 | 2 | 1 | 1 | 2 | 0 |
| Negative         | 2 | 22 | 1 | 23 | 0 | 24 |

Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
|                  | Latent | No latent | Latent | No latent | Latent | No latent |
| Positive         | 0 | 0 | 0 | 2 | 0 | 4 |
| Negative         | 2 | 24 | 2 | 22 | 2 | 20 |

- 5 observed variables with 2 state each simulated data
Alpha: 0.20676

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 1 | 38 | 4 | 29 | 11 | 16 |
| Negative | 56 | 230 | 53 | 239 | 46 | 252 |

Medium arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 1 | 49 | 15 | 52 | 30 | 40 |
| Negative | 56 | 219 | 42 | 216 | 27 | 228 |

Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 0 | 7 | 0 | 23 | 0 | 41 |
| Negative | 57 | 261 | 57 | 245 | 57 | 227 |

• 5 observed variables with 3 state each simulated data

Alpha: 0.13636

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 6 | 73 | 24 | 43 | 49 | 8 |
| Negative | 51 | 195 | 33 | 225 | 8 | 260 |

Medium arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 1 | 85 | 19 | 74 | 46 | 13 |
| Negative | 56 | 183 | 38 | 194 | 11 | 255 |

Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 0 | 7 | 1 | 83 | 8 | 125 |
| Negative | 57 | 261 | 56 | 185 | 49 | 143 |

3) Results of trigger-PC algorithm (with optimized alpha)
• 4 observed variables with 2 state each simulated data

Alpha: 0.12268

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 0 | 0 | 0 | 0 | 0 | 0 |
| Negative | 2 | 24 | 2 | 24 | 2 | 24 |

Medium arc strength simulated data:
Minimum arc strength simulated data:

Data case number | 100 | 1000 | 10000
---|---|---|---
Latent | No latent | Latent | No latent | Latent | No latent |
Positive | 0 | 0 | 0 | 0 | 0 |
Negative | 2 | 24 | 2 | 24 | 2 | 24 |

- 4 observed variables with 3 state each simulated data

Alpha: 0.20160

Maximum arc strength simulated data:

Data case number | 100 | 1000 | 10000
---|---|---|---
Latent | No latent | Latent | No latent | Latent | No latent |
Positive | 0 | 0 | 0 | 0 | 0 |
Negative | 2 | 24 | 2 | 24 | 2 | 24 |

Medium arc strength simulated data:

Data case number | 100 | 1000 | 10000
---|---|---|---
Latent | No latent | Latent | No latent | Latent | No latent |
Positive | 0 | 0 | 0 | 0 | 0 |
Negative | 2 | 24 | 2 | 24 | 2 | 24 |

- 5 observed variables with 2 state each simulated data

Alpha: 0.20676

Maximum arc strength simulated data:

Data case number | 100 | 1000 | 10000
---|---|---|---
Latent | No latent | Latent | No latent | Latent | No latent |
Positive | 0 | 0 | 0 | 0 | 0 |
Negative | 57 | 268 | 57 | 268 | 57 | 268 |

Medium arc strength simulated data:

Data case number | 100 | 1000 | 10000
---|---|---|---
Latent | No latent | Latent | No latent | Latent | No latent |
Positive | 0 | 0 | 0 | 0 | 0 |
Negative | 57 | 268 | 57 | 266 | 57 | 268 |

Minimum arc strength simulated data:

Data case number | 100 | 1000 | 10000
---|---|---|---
Latent | No latent | Latent | No latent | Latent | No latent |
Positive | 0 | 0 | 0 | 0 | 0 |
Negative | 57 | 268 | 57 | 268 | 57 | 268 |
5 observed variables with 3 state each simulated data

Alpha: 0.13636

Maximum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 0         | 1      | 1         | 13     | 0         |
| Negative         | 57     | 268       | 56     | 267       | 44     | 268       |

Medium arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 0         | 0      | 0         | 9      | 0         |
| Negative         | 57     | 268       | 57     | 268       | 48     | 268       |

Minimum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 0         | 0      | 0         | 0      | 0         |
| Negative         | 57     | 268       | 57     | 268       | 56     | 268       |

4) Results of FCI algorithm (with alpha of 0.05)

4 observed variables with 2 state each simulated data

Maximum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 0         | 1      | 1         | 0      | 0         |
| Negative         | 2      | 24        | 1      | 24        | 2      | 24        |

Medium arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 1         | 2      | 1         | 2      | 0         |
| Negative         | 2      | 23        | 0      | 23        | 0      | 24        |

Minimum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 0         | 0      | 0         | 0      | 0         |
| Negative         | 2      | 24        | 2      | 24        | 2      | 24        |

4 observed variables with 3 state each simulated data

Maximum arc strength simulated data:

| Data case number | Latent | No latent | Latent | No latent | Latent | No latent |
|------------------|--------|-----------|--------|-----------|--------|-----------|
| Positive         | 0      | 0         | 1      | 0         | 1      | 0         |
| Negative         | 2      | 24        | 1      | 24        | 1      | 24        |

Medium arc strength simulated data:
### Minimum arc strength simulated data:

| Data case number | 100  | 1000 | 10000 |
|------------------|------|------|-------|
| Latent           | 0    | 0    | 2     |
| No latent        | 0    | 6    | 16    |

- 5 observed variables with 2 state each simulated data

### Maximum arc strength simulated data:

| Data case number | 100  | 1000 | 10000 |
|------------------|------|------|-------|
| Latent           | 2    | 32   | 2      |
| No latent        | 55   | 236  | 57     |

- 5 observed variables with 3 state each simulated data

### Medium arc strength simulated data:

| Data case number | 100  | 1000 | 10000 |
|------------------|------|------|-------|
| Latent           | 1    | 29   | 33    |
| No latent        | 56   | 239  | 57     |

- 5 observed variables with 3 state each simulated data

### Maximum arc strength simulated data:

| Data case number | 100  | 1000 | 10000 |
|------------------|------|------|-------|
| Latent           | 2    | 64   | 48    |
| No latent        | 55   | 204  | 9      |

### Medium arc strength simulated data:

| Data case number | 100  | 1000 | 10000 |
|------------------|------|------|-------|
| Latent           | 1    | 65   | 48    |
| No latent        | 56   | 203  | 9      |

### Minimum arc strength simulated data:

| Data case number | 100  | 1000 | 10000 |
|------------------|------|------|-------|
| Latent           | 0    | 2    | 3     |
| No latent        | 57   | 266  | 54    |

5) Results of PC algorithm (with alpha of 0.05)

- 4 observed variables with 2 state each simulated data
Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent           | 0   | 0    | 0     |
| No latent        | 2   | 24   | 2     |

Medium arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent           | 0   | 0    | 0     |
| No latent        | 2   | 23   | 2     |

Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent           | 0   | 0    | 0     |
| No latent        | 2   | 24   | 2     |

- 4 observed variables with 3 state each simulated data

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent           | 0   | 0    | 0     |
| No latent        | 2   | 24   | 2     |

Medium arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent           | 0   | 0    | 0     |
| No latent        | 2   | 23   | 2     |

Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent           | 0   | 0    | 0     |
| No latent        | 2   | 24   | 2     |

- 5 observed variables with 2 state each simulated data

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent           | 1   | 25   | 10    |
| No latent        | 56  | 243  | 47    |

Medium arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent           | 1   | 18   | 32    |
| No latent        | 56  | 250  | 25    |

Minimum arc strength simulated data:
- 5 observed variables with 3 state each simulated data

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 0 | 0 | 0 | 2 | 0 | 6 |
| Negative | 57 | 268 | 57 | 266 | 57 | 262 |

Medium arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 2 | 56 | 24 | 48 | 47 | 7 |
| Negative | 55 | 212 | 33 | 220 | 10 | 261 |

Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 0 | 56 | 16 | 95 | 48 | 9 |
| Negative | 57 | 212 | 41 | 173 | 9 | 259 |

6) Results of trigger-PC algorithm (with alpha of 0.05)

- 4 observed variables with 2 state each simulated data

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 0 | 0 | 0 | 0 | 0 | 0 |
| Negative | 2 | 24 | 2 | 24 | 2 | 24 |

Medium arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 0 | 0 | 1 | 0 | 2 | 0 |
| Negative | 2 | 24 | 1 | 24 | 0 | 24 |

Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 0 | 0 | 0 | 0 | 0 | 0 |
| Negative | 2 | 24 | 2 | 24 | 2 | 24 |

- 4 observed variables with 3 state each simulated data

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Latent | No latent | Latent | No latent | Latent | No latent |
| Positive | 0 | 0 | 0 | 0 | 0 | 0 |
| Negative | 2 | 24 | 2 | 24 | 2 | 24 |

Medium arc strength simulated data:
Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Positive         | 0   | 0    | 0     |
| Negative         | 2   | 24   | 0     |

- 5 observed variables with 2 state each simulated data

Maximum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Positive         | 0   | 0    | 0     |
| Negative         | 57  | 268  | 57    |

Medium arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Positive         | 0   | 0    | 0     |
| Negative         | 57  | 268  | 57    |

Minimum arc strength simulated data:

| Data case number | 100 | 1000 | 10000 |
|------------------|-----|------|-------|
| Positive         | 0   | 0    | 0     |
| Negative         | 57  | 268  | 56    |