Unpolarized DIS structure functions in Double-Logarithmic Approximation

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We present description of the DIS structure functions $F_1$ and $F_2$ at small $x$ obtained in double-logarithmic approximation (DLA). First we clarify our previous results on $F_1$ and then obtain explicit expressions for $F_2$. Our calculations confirm our previous result that the small-$x$ asymptotics of $F_1$ is controlled by a new Pomeron that has nothing to do with the BFKL Pomeron, though their intercepts are pretty close. The latter means that studying the small-$x$ dependence of the unpolarized DIS cannot ascertain which of those Pomerons is actually involved. However, we predict a quite different and universal $Q^2$-dependence of $F_{1,2}$ in DLA compared to the approaches involving the both DGLAP and BFKL. On that basis, we construct simple relations between logarithms of $F_{1,2}$, which can be verified with analysis of experimental data. In contrast to $F_1$, the intercept controlling the small-$x$ asymptotics of $F_2$ is very small but positive, which ensures growth of $F_2$ at $x \to 0$.

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I. INTRODUCTION

Theoretical investigation of the structure functions $F_1$ and $F_2$ in the QCD framework includes both fixed-order calculations [1–12] and approaches operating with all-order resummations, which usually either are based on DGLAP [13] (see e.g. Ref. [14]) or combine DGLAP and BFKL [13]. In the latter case, the role of BFKL is providing the structure functions with the necessary growth at small $x$ while the role of DGLAP is describing the $Q^2$ dependence (see e.g. Refs. [10,11]). There also are more involved approaches engaging the dipole model [15]. For example, Ref. [19] involves contribution of the multiple Pomeron exchange to $F_2$. Numerical analysis of HERA data within the dipole model can be found in Ref. [20]. The dipole model was used in the global analysis of experimental data in Ref. [21]. A detailed bibliography on this issue can be found in Ref. [22]. Let us remind that growth of $F_2$ at small $x$ was also suggested in the Regge inspired models [23,24] in the context of Diffractive DIS.

In contrast to the aforecited approaches, we suggest an alternative description of $F_{1,2}$, which is not based on DGLAP and BFKL as well as on their modifications. Namely, we calculate $F_1$ and $F_2$ in Double-Logarithmic Approximation (DLA). In order to calculate and sum DL contributions to $F_{1,2}$ to all orders in $\alpha_s$, we construct and solve Infra-Red Evolution Equations (IREE). This method was initiated by L.N. Lipatov [25] and then it has been applied many times to a wide spectrum of calculations in QCD and Standard Model. The basis of this method was constructed in the pioneer works [26] where it was found that DL contributions come equally from virtual quarks and gluons with small transverse momenta. In the context of the Unpolarized DIS, the IREE method was applied to calculating $F_1$ in Ref. [27] and $F_L$ in Ref. [28]. Applications of IREE to the high-energy spin physics can be found in Ref. [29].

Our paper is organized as follows: In sect. II we introduce our definitions for invariant amplitudes which simplify the structure of the hadronic tensor. In Sect. III we compose IREEs for $F_1$ and $F_2$. Actually, IREEs for $F_1$ were already constructed and solved in Ref. [27] but a part of DL contributions was not accounted for there. This flaw is corrected in Sect. III. Solutions to the IREEs are obtained in Sect. IV. Problem of inputs to the IREEs is considered in Sect. V. After the inputs have been specified, the explicit expressions for $F_{1,2}$ were obtained. The small-$x$ asymptotics of $F_{1,2}$ are obtained in Sect. VI and they are used to derive simple relations between logarithms of $F_{1,2}$. Finally, Sect. VII is for concluding remarks.

II. INVARIANT AMPLITUDES SIMPLIFYING CALCULATION OF THE HADRONIC TENSOR

The unpolarized part of the hadronic tensor $W_{\mu\nu}$ describing DIS off a hadron is conventionally parameterized as follows:

$$W_{\mu\nu}(p,q) = (g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}) F_1 + \frac{1}{pq} \left( p_{\mu} - q_{\mu} \frac{pq}{q^2} \right) \left( p_{\nu} - q_{\nu} \frac{pq}{q^2} \right) F_2$$  \hspace{1cm} (1)
The standard notations $q$ and $p$ in Eq. (1) stand for momenta of the virtual photon and hadron respectively. In the framework of QCD factorization $W_{\mu \nu}$ can be represented through convolutions of perturbative and non-perturbative contributions:

$$W_{\mu \nu} = W_{\mu \nu}^{(q)} \otimes \Phi_q + W_{\mu \nu}^{(g)} \otimes \Phi_g,$$

with $p$ standing for the initial parton momenta. $\Phi_q$ and $\Phi_g$ In Eq. (2) are the initial quark and gluon distributions in the hadron respectively. They accommodate non-perturbative contributions. In contrast, $W_{\mu \nu}^{(q)}$ and $W_{\mu \nu}^{(g)}$ are perturbative objects. They describe DIS off a quark (gluon) respectively. Parametrization of their tensor structure is similar to Eq. (1):

$$W_{\mu \nu}^{(q)}(p,q) = \left( -g_{\mu \nu} + \frac{q_{\mu}q_{\nu}}{q^2} \right) F_1^{(q)} + \frac{1}{pq} \left( p_{\mu} - q_{\mu} \frac{pq}{q^2} \right) \left( p_{\nu} - q_{\nu} \frac{pq}{q^2} \right) F_2^{(q)},$$

$$W_{\mu \nu}^{(g)}(p,q) = \left( -g_{\mu \nu} + \frac{q_{\mu}q_{\nu}}{q^2} \right) F_1^{(g)} + \frac{1}{pq} \left( p_{\mu} - q_{\mu} \frac{pq}{q^2} \right) \left( p_{\nu} - q_{\nu} \frac{pq}{q^2} \right) F_2^{(g)},$$

with $p$ being the initial parton momentum. $F_1^{(q)}$ and $F_1^{(g)}$ in Eq. (3) are calculated by means of perturbative QCD. The standard way is to calculate $F_1^{(q,g)}$ through auxiliary invariant amplitudes $A^{(q,g)}, B^{(q,g)}$ defined as follows:

$$A^{(q,g)} = g_{\mu \nu} W_{\mu \nu}^{(q,g)} = -3F_1^{(q,g)} + \frac{F_2^{(q,g)}}{2x} + O(p^2),$$

$$B^{(q,g)} = \frac{P_\mu P_\nu}{pq} W_{\mu \nu}^{(q,g)} = -\frac{1}{2x} F_1^{(q,g)} + \frac{1}{4x^2} F_2^{(q,g)},$$

where we have used the standard notations $x = Q^2/w$ ($Q^2 = -q^2 > 0$), $w = 2pq$. Eqs. (4,5) yield that

$$F_1^{(q,g)} = -\frac{A^{(q,g)}}{2} + xB^{(q,g)},$$

$$F_2^{(q,g)} = -xA^{(q,g)} + 6x^2 B^{(q,g)} = 2xF_1^{(q,g)} + 4x^2 B^{(q,g)}.$$ 

The longitudinal structure functions $F_L^{(q,g)}$ are related to $B^{(q,g)}$:

$$F_L^{(q,g)} = 4x^2 B^{(q,g)}.$$ 

Eq. (3) reads that the contributions of $B^{(q,g)}$ to $F_1^{(q,g)}$ can be neglected at small $x$ compared to $A^{(q,g)}$ only if $B^{(q,g)}$ is less singular than $x^{-1-\sigma}$, with $\sigma > 0$.

### III. IREEs for the Auxiliary Amplitudes $A^{(q,g)}$ and $B^{(q,g)}$

We calculate amplitudes $A^{(q,g)}$ and $B^{(q,g)}$ by constructing and solving IREEs for them. The IREEs are differential, so the first step is to obtain general solutions to them. Then the general solutions should be specified. Up to this point the technology of treating amplitudes $A^{(q,g)}$ and $B^{(q,g)}$ is the same, so we construct and solve IREEs for amplitudes $A^{(q,g)}$ and then extend our results to $B^{(q,g)}$. This similarity ends when specifying the general solutions begins, so at this point we will consider $A^{(q,g)}$ and $B^{(q,g)}$ separately. Throughout the paper we will use the Sudakov parametrization for virtual parton momenta $k_i$, representing them as follows:

$$k_i = \alpha_i q' + \beta_i p' + k_{i \perp},$$

where $q'$ and $p'$ are the massless (light-cone) momenta made of momenta $p$ and $q$:
\[ p' = p - q(p^2/w) \approx p, \quad q' = q - p(q^2/w) = q + xp. \] (9)

In Eq. (9) \( q \) denotes the virtual photon momentum while \( p \) is momentum of the initial parton. We remind that we presume that \( p^2 \) is small, so we will neglect it in what follows.

First step to construct IREEs is to introduce an artificial infrared cut-off \( \mu \). It is necessary procedure in DLA to regulate IR divergences. Since that amplitudes \( A(q,g) \) become \( \mu \)-dependent, which makes possible to trace their evolution with respect to \( \mu \). In order to relate this evolution to evolution in \( x \) and \( Q^2 \), we parameterize amplitudes \( A(q,g) \) as follows: \( A(q,g) = A(q,g)(s/\mu^2, Q^2/\mu^2) \). Value of \( \mu \) is arbitrary save the obvious restriction \( \mu > \Lambda_{QCD} \) to enable applicability of the perturbative QCD. For instance, the factorization scale can be used as the IR cut-off. As we are not interested in studying dependence of \( A_{q,g} \) on masses \( m_\ast \) of involved quarks and their virtualities, we presume that \( \mu > m_\ast \). So as to treat virtual quarks and gluons identically, we prescribe The IREEs include convolutions of amplitudes, so it is convenient to write them in terms of the Mellin transform

\[ A_{q,g}(s/\mu^2, Q^2/\mu^2) = \int_{-\infty}^{\infty} dw/(2\pi)^2 (w/\mu^2)^{\omega} f_{q,g}^{(A)}(\omega, Q^2/\mu^2). \] (10)

Throughout the paper we will address \( f_{q,g}^{(A)} \) as the Mellin amplitudes. In what follows we will use the logarithmic variables \( \rho \) and \( y \):

\[ \rho = \ln(w/\mu^2), \xi = \ln(w/Q^2), y = \ln(Q^2/\mu^2) \] (11)

The transform inverse to (10) is

\[ f_{q,g}^{(A)}(\omega, Q^2/\mu^2) = \frac{1}{\omega} \int_{\mu^2}^{\infty} dw/w (w/\mu^2)^{-\omega} A_{q,g}(w, Q^2). \] (12)

The same form of transforms (10)(12) we will use for amplitudes \( B_{q,g} \), denoting their Mellin amplitudes \( f_{q,g}^{(B)} \).

A. Constructing IREEs for amplitudes \( A_{q,g} \) and \( B_{q,g} \)

We start by constructing IREEs for \( A_{q,g} \). The l.h.s of each IREE is obtained with differentiation of \( A_{q,g} \) over \( \mu \). It follows from Eq. (10) that

\[ -\mu^2 dA_{q,g}/d\mu^2 = \partial A_{q,g}/\partial \rho + \partial A_{q,g}/\partial y = \int_{-\infty}^{\infty} dw/(2\pi)^2 (w/\mu^2)^{\omega} \left[ \omega + \partial f_{q,g}^{(A)}/\partial y \right]. \] (13)

Eq. (13) represents the l.h.s. of the IREEs for \( A_{q,g} \). The guiding principle to obtain the r.h.s. is to look for a \( t \)-channel parton with minimal transverse momentum \( = k_\perp \), so \( \mu \) is the lowest limit of integration over \( k_\perp \). Integration over \( k_\perp \) yields a DL contribution only when there is a two-parton intermediate state in the \( t \)-channel. Such pairs can consist of quarks or gluons. They factorize \( A_{q,g} \) into two amplitudes. Applying to them the operator \( -\mu^2 \partial/\partial \mu^2 \) leads to the following IREEs:

\[ \left[ \partial/\partial y + \omega \right] f_{q}^{(A)}(\omega, y) = 1/(8\pi^2) f_{q}^{(A)}(\omega, y) f_{qq}(\omega) + 1/(8\pi^2) f_{g}^{(A)}(\omega, y) f_{qg}(\omega), \] (14)

\[ \left[ \partial/\partial y + \omega \right] f_{g}^{(A)}(\omega, y) = 1/(8\pi^2) f_{q}^{(A)}(\omega, y) f_{gg}(\omega) + 1/(8\pi^2) f_{g}^{(A)}(\omega, y) f_{gq}(\omega), \]

where amplitudes \( f_{rr'} \ (r, r' = q, g) \) are the parton-parton amplitudes. In order to get rid of the factors \( 1/(8\pi^2) \) in Eq. (14) we replace \( f_{rr'} \) by

\[ h_{rr'} = f_{rr'}/(8\pi^2) \] (15)

and rewrite (14) in the following way:
\[
\frac{\partial f_q^{(A)}(\omega, y)}{\partial y} = \left[-\omega + h_{qq}(\omega)\right] f_q^{(A)}(\omega, y) + f_g^{(A)}(\omega, y) h_{gg}(\omega),
\]
\[
\frac{\partial f_q^{(A)}(\omega, y)}{\partial y} = f_q^{(A)}(\omega, y) h_{gg}(\omega) + [\omega + h_{gg}(\omega)] f_g^{(A)}(\omega, y).
\]

Eq. (16) looks quite similarly to the DGLAP equations, with \( h_{rr} \) being new anomalous dimensions. They accommodate double-logarithmic (DL) contributions to all orders in \( \alpha_s \) and can be calculated in DLA with applying the same method: constructing and solving appropriate IREEs for them. The DL contributions in the \( n^{th} \) order are \( \sim \alpha_s^n \ln^2 n \), i.e. in the Mellin space they are \( \sim \alpha_s^n / \omega^{1+2n} \). They are the most singular terms at \( \omega = 0 \), i.e. at small \( x \), so total resummation of them is important for generalizing DGLAP to the small-\( x \) region. In order to specify a general solution to Eq. (16) we use the matching

\[
A_q(\rho, y)|_{y=0} = \tilde{A}_q(\rho), \quad A_g(\rho, y)|_{y=0} = \tilde{A}_g(\rho),
\]

where \( \tilde{A}_q \) and \( \tilde{A}_g \) are the amplitudes of the same process at \( Q^2 \sim \mu^2 \). We denote \( \tilde{f}_q^{(A)} \) and \( \tilde{f}_g^{(A)} \) the Mellin amplitudes conjugated to them. Amplitudes \( f_{q,g}^{(A)} \) should be found independently. The IREEs for them do not contain derivatives because \( Q^2 = \mu^2 \), so they are algebraic equations but in contrast to Eqs. (14)(16) they are inhomogeneous:

\[
\omega f_q^{(A)}(\omega) = \phi_q^{(A)} + \tilde{f}_q^{(A)}(\omega) h_{qq}(\omega) + \tilde{f}_g^{(A)}(\omega) h_{gg}(\omega),
\]

\[
\omega f_g^{(A)}(\omega) = \phi_g^{(A)} + \tilde{f}_q^{(A)}(\omega) h_{qq}(\omega) + \tilde{f}_g^{(A)}(\omega) h_{gg}(\omega),
\]

with inhomogeneous terms \( \phi_q^{(A)} \) and \( \phi_g^{(A)} \). We will call \( \phi_{q,g}^{(A)} \) inputs. We will specify them later. The technology of composing IREEs for amplitudes \( B_q \) and \( B_g \) is absolutely the same. As a result, the equations for \( f_q^{(B)} \) and \( f_g^{(B)} \) in the region \( Q^2 \gg \mu^2 \) are

\[
\frac{\partial f_q^{(B)}(\omega, y)}{\partial y} = \left[-\omega + h_{qq}(\omega)\right] f_q^{(B)}(\omega, y) + f_g^{(B)}(\omega, y) h_{gg}(\omega),
\]

\[
\frac{\partial f_g^{(B)}(\omega, y)}{\partial y} = f_q^{(B)}(\omega, y) h_{gg}(\omega) + [\omega + h_{gg}(\omega)] f_g^{(B)}(\omega, y)
\]

while \( f_{q,g}^{(B)} \) at \( Q^2 \sim \mu^2 \) obey the following IREEs:

\[
\omega \tilde{f}_q^{(B)}(\omega) = \phi_q^{(B)} + \tilde{f}_q^{(B)}(\omega) h_{qq}(\omega) + \tilde{f}_g^{(B)}(\omega) h_{gg}(\omega),
\]

\[
\omega \tilde{f}_g^{(B)}(\omega) = \phi_g^{(B)} + \tilde{f}_q^{(B)}(\omega) h_{qq}(\omega) + \tilde{f}_g^{(B)}(\omega) h_{gg}(\omega),
\]

where \( \phi_q^{(B)} \) and \( \phi_g^{(B)} \) are the inputs. They differ from the inputs \( \phi_{q,g}^{(A)} \) for amplitudes \( A_q,g \). We will specify them later.

### IV. SOLUTION TO IREEs FOR AMPLITUDES \( A_q \) AND \( A_g \)

By obtaining first general solutions to differential equations in Eqs. (10), then solving algebraic equations (18) and using the matching (17) to specify the general solutions, we arrive\(^1\) at the following expressions for \( A_q \) and \( A_g \):

\[
A_q = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-\omega} \left[ \phi_q^{(A)} \left( C^{(+)}_q e^{\Omega^{(+)}y} + C^{(-)}_q e^{\Omega^{(-)}y} \right) \right. + \left. \phi_g^{(A)} \left( C^{(+)}_g e^{\Omega^{(+)}y} + C^{(-)}_g e^{\Omega^{(-)}y} \right) \right],
\]

\[
A_g = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-\omega} \left[ \phi_q^{(A)} \left( C^{(+)}_q e^{\Omega^{(+)}y} + C^{(-)}_q e^{\Omega^{(-)}y} \right) \right. + \left. \phi_g^{(A)} \left( C^{(+)}_g e^{\Omega^{(+)}y} + C^{(-)}_g e^{\Omega^{(-)}y} \right) \right],
\]

where the anomalous dimensions \( \Omega^{(\pm)} \) are made of \( h_{rr} \):

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\(^1\) Details can be found in Ref. [27].
\[ \Omega_{(\pm)} = \frac{1}{2} \left[ h_{gg} + h_{qq} \pm \sqrt{R} \right], \] (22)

with

\[ R = (h_{gg} + h_{qq})^2 - 4(h_{qq}h_{gg} - h_{gg}h_{qq}) = (h_{gg} - h_{qq})^2 + 4h_{gg}h_{qq}. \] (23)

Coefficient functions \( C^{(\pm)}_{q,g}(\omega) \) and \( \tilde{C}^{(\pm)}_{q,g}(\omega) \) are also made of \( h_{rr} \). However, explicit expressions for them are rather bulky, so we put them in Appendix. The IREEs for amplitudes \( B_{q,g} \) are quite similar to the ones for \( A_{q,g} \). As a result, the expressions for them are alike Eq. (21):

\[
B_q = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-i\omega} \left[ \phi_q^{(B)} \left( C_q^{(+)} e^{\Omega_{(+)} y} + C_q^{(-)} e^{\Omega_{(-)} y} \right) + \phi_q^{(B)} \left( C_g^{(+)} e^{\Omega_{(+)} y} + C_g^{(-)} e^{\Omega_{(-)} y} \right) \right],
\]

\[
B_g = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-i\omega} \left[ \phi_g^{(B)} \left( \tilde{C}_q^{(+)} e^{\Omega_{(+)} y} + \tilde{C}_q^{(-)} e^{\Omega_{(-)} y} \right) + \phi_g^{(B)} \left( \tilde{C}_g^{(+)} e^{\Omega_{(+)} y} + \tilde{C}_g^{(-)} e^{\Omega_{(-)} y} \right) \right],
\] (24)

Eqs. (21) and (24) involve the same coefficient functions and anomalous dimensions for \( A_q \) and \( B_q \) (\( A_g \) and \( B_g \)). The only difference between Eqs. (21) and (24) is the inputs. Now we are going to specify them.

V. SPECIFYING THE INPUTS

By definition, inputs in evolution equations stand for the starting points of the evolution. They are considered elementary and cannot be obtained through evolution.

A. Inputs for amplitudes \( A_{q,g} \)

Evolution of amplitude \( A_q \) starts from the Born contribution which is given by the following expression:

\[
A_q^{(\text{Born})} = \frac{1}{\pi g_{\mu\nu}} 3 \left[ \frac{1}{2} \frac{\bar{u}(p)\gamma_{\mu} (\hat{p} + \hat{q}) \gamma_{\nu} u(p)}{w(1-x) - \mu^2 + i\epsilon} \right] = -2 \delta(1-x - \lambda),
\] (25)

where we have denoted \( \lambda = \mu^2/w \). In Eq. (25) we dropped the quark electric charge and introduced the IR cut-off \( \mu \).

It is clear that \( \lambda \) can be neglected compared to \( x \) in the kinematics \( Q^2 \gg \mu^2 \), so \( A_q^{(\text{Born})} \) is \( \mu \)-independent in this region and vanishes after differentiation over \( \mu \). It explains why the quark inhomogeneous term is absent in Eqs. (13-16). In contrast, \( A_q^{(\text{Born})} \) depends on \( \mu \) in the region \( Q^2 \sim \mu^2 \) and appears in Eq. (18). Dropping \( x \) in Eq. (25) compared to \( \lambda \) in this region and applying the transform of Eq. (12), we obtain the input \( \phi_q^{(A)} \). Remembering that there is no Born contribution to \( A_g \) at any \( Q^2 \), we arrive at the following expressions for the inputs:

\[
\phi_q^{(A)} = -2,
\]

\[
\phi_g^{(A)} = 0.
\] (26)

B. Inputs for amplitudes \( B_{q,g} \)

The situation with specifying inputs \( \phi_q^{(B)} \) is more involved. In the Born approximation we have

\[
B_q^{(\text{Born})} = \frac{1}{\pi} \frac{p_{\mu} p_{\nu}}{pq} 3 \left[ \frac{1}{2} \frac{\bar{u}(p)\gamma_{\mu} (\hat{p} + \hat{q}) \gamma_{\nu} u(p)}{w(1-x) - \mu^2 + i\epsilon} \right] \sim p^2 \approx 0.
\] (27)
In addition, \( B_{g}^{(\text{Born})} = 0 \). Thus, the both Born amplitudes are zeros, which excludes using the Born approximation as the starting point of the evolution. They are non-zeros in the first loop:

\[
B_{q}^{(1)} = \frac{\alpha_{s}}{2\pi} C_{F}, \quad B_{q}^{(1)} = \frac{\alpha_{s}}{\pi} n_{f}(1 - x),
\]

where we have used the standard notations \( C_{F} = (N^{2} - 1)/2N = 4/3 \) and \( n_{f} \) is the number of flavours. Therefore, \( B_{q,g}^{(1)} \) could be used as inputs in IREEs for \( B_{q,g} \). In this case, the result of evolving \( B_{q,g} \) to the order \( \alpha_{s}^{2} \) would be \( \sim \alpha_{s}^{2} \ln^{2}(w/\mu^{2}) \). However, the most important second-loop contributions to \( B_{q,g} \) are proportional to \( \alpha_{s}^{2}/x \) and they cannot be obtained with evolving the inputs \( B_{q,g}^{(1)} \). The only way to include them in IREEs is to choose them as the inputs and evolve them with IREEs. We remind that calculations of \( F_{1,2} \) in the first and second loops can be found in Refs. [1] - [11] whereas Ref. [12] contains the third-loop calculations. In the present paper we will use the leading second-loop contributions \( B_{q}^{(2)}(B_{g}^{(2)}) \) to \( B_{q} \) \( B_{g} \) obtained in Ref. [28]. They are given by the following expressions:

\[
\begin{align*}
B_{q}^{(2)} &= C_{q}^{(2)}(2) \gamma_{(2)}^{(2)} \rho x^{-1}, \\
B_{g}^{(2)} &= C_{g}^{(2)}(2) \gamma_{(2)}^{(2)} \rho x^{-1},
\end{align*}
\]

with

\[
\gamma_{(2)}^{(2)} = \alpha_{s}^{2} \ln^{2} 2/2\pi
\]

and the color factors \( C_{q}^{(2)} = C_{F}^{2} + C_{F} n_{f} \) and \( C_{g}^{(2)} = N n_{f} + C_{F} n_{f} \). We use \( B_{q,g}^{(2)} \) as the starting point of IR evolution. Combining Eqs. (12) and (29), we obtain the following expressions for the inputs \( \phi_{q,g}^{(B)} \):

\[
\begin{align*}
\phi_{q}^{(B)} &= x^{-1} \gamma_{(2)}^{(2)} \rho C_{q}^{(2)}, \\
\phi_{g}^{(B)} &= x^{-1} \gamma_{(2)}^{(2)} \rho C_{g}^{(2)}.
\end{align*}
\]

The factor \( \rho \) in Eq. (29) is not replaced by \( 1/\omega^{2} \) in Eq. (31) in contrast to Ref. [28] by the following reason: \( \rho \) was not obtained through the IR evolution, so it does not participate in the Mellin transform. This point is the main difference between the present paper and our previous paper Ref. [28].

VI. EXPRESSIONS FOR \( F_{1} \) AND \( F_{2} \)

Using Eqs. (26,31) to specify inputs in Eqs. (21,24), we obtain explicit expressions for \( A_{q,g} \) and \( B_{q,g} \). Combining these results with Eq. (6), we arrive at explicit expressions for \( F_{1}^{(q,g)} \) and \( F_{2}^{(q,g)} \):

\[
\begin{align*}
F_{1}^{(q)} &= (1 + \gamma_{(2)}^{(2)} \rho C_{q}^{(2)}) I_{q} + \gamma_{(2)}^{(2)} \rho C_{g}^{(2)} I_{g}, \\
F_{1}^{(g)} &= (1 + \gamma_{(2)}^{(2)} \rho C_{g}^{(2)}) \tilde{I}_{g} + \gamma_{(2)}^{(2)} \rho C_{g}^{(2)} \tilde{I}_{g}, \\
F_{2}^{(q)} &= 2x \left[ (1 + 3\gamma_{(2)}^{(2)} \rho C_{q}^{(2)}) I_{q} + 3\gamma_{(2)}^{(2)} \rho C_{g}^{(2)} \rho I_{g} \right], \\
F_{2}^{(g)} &= 2x \left[ (1 + 3\gamma_{(2)}^{(2)} \rho C_{q}^{(2)}) \tilde{I}_{g} + 3\gamma_{(2)}^{(2)} \rho C_{g}^{(2)} \tilde{I}_{g} \right],
\end{align*}
\]

with \( I_{q,g} \) and \( \tilde{I}_{q,g} \) being defined as follows:
\[ I_q = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-\omega} \left( C_q^{(+)} e^{\Omega(+)y} + C_q^{(-)} e^{\Omega(-)y} \right), \tag{34} \]

\[ I_g = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-\omega} \left( C_g^{(+)} e^{\Omega(+)y} + C_g^{(-)} e^{\Omega(-)y} \right), \]

\[ \bar{I}_q = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-\omega} \left( \bar{C}_q^{(+)} e^{\Omega(+)y} + \bar{C}_q^{(-)} e^{\Omega(-)y} \right), \]

\[ \bar{I}_g = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-\omega} \left( \bar{C}_g^{(+)} e^{\Omega(+)y} + \bar{C}_g^{(-)} e^{\Omega(-)y} \right). \]

Comparison of Eqs. (32) and (33) yields that \( F_{L}^{(q,g)} \) are given by the following expressions

\[ F_{L}^{(q)} = 4x\gamma(2)\rho \left[ C_q^{(2)} I_q + C_g^{(2)} I_g \right], \]

\[ F_{L}^{(g)} = 4x\gamma(2)\rho \left[ C_g^{(2)} \bar{I}_q + C_q^{(2)} \bar{I}_g \right]. \tag{35} \]

We remind that explicit expressions for \( C_q^{(\pm)}, C_g^{(\pm)}, \) and \( \bar{C}_q^{(\pm)} \) can be found in Appendix A. Eqs. (32, 33, 35) are valid at \( Q^2 > \mu^2 \) (\( \mu \approx 1 \text{ GeV} \), see Ref. [29] for detail) but it is easy to generalize them for smaller \( Q^2 \). The prescription is obtained in Ref. [33]. Eqs. (32, 33, 35) can be used at arbitrary \( Q^2 \) providing that \( Q^2 \) is replaced by \( Q^2 = Q^2 + \mu^2 \). It leads to replacing \( x, y \) and \( \xi \) by \( \tilde{x}, \tilde{y}, \tilde{\xi} \) respectively:

\[ \tilde{x} = \left( Q^2 + \mu^2 \right) / w, \quad \tilde{y} = \left( Q^2 + \mu^2 \right) / \mu^2, \quad \tilde{\xi} = \ln \left[ \left( Q^2 + \mu^2 \right) / w \right]. \tag{36} \]

Convolving Eqs. (32, 33, 35) with the parton distributions \( \Phi_q \) and \( \Phi_g \), we obtain expressions for the structure functions \( F_1, F_2, F_L \) of the unpolarized DIS, which can be used at small \( x \) and arbitrary \( Q^2 \):

\[ F_1 = \left( 1 + \gamma(2)\rho \right) \left( J_q + \bar{J}_q \right) + \gamma(2)\rho \left( J_g + \bar{J}_g \right), \]

\[ F_2 = 2x \left[ \left( 1 + 3\gamma(2)\rho \right) C_q^{(2)} \left( J_q + \bar{J}_q \right) + 3\gamma(2)\rho C_g^{(2)} \left( J_g + \bar{J}_g \right) \right], \]

\[ F_L = 4x\gamma(2)\rho \left[ C_q^{(2)} \left( J_q + \bar{J}_q \right) + C_g^{(2)} \left( J_g + \bar{J}_g \right) \right]. \tag{37} \]

Each of \( J_{g,2} \) and \( \bar{J}_{g,2} \) includes both the perturbative integrands of Eq. (34) and the non-perturbative parton distributions \( \hat{\Phi}_{q,g}(\omega) \):

\[ J_q = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \tilde{x}^{-\omega} \left( C_q^{(+)} e^{\Omega(+)\tilde{y}} + C_q^{(-)} e^{\Omega(-)\tilde{y}} \right) \hat{\Phi}_q(\omega), \tag{38} \]

\[ J_g = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \tilde{x}^{-\omega} \left( C_g^{(+)} e^{\Omega(+)\tilde{y}} + C_g^{(-)} e^{\Omega(-)\tilde{y}} \right) \hat{\Phi}_g(\omega), \]

\[ \bar{J}_q = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \tilde{x}^{-\omega} \left( \bar{C}_q^{(+)} e^{\Omega(+)\tilde{y}} + \bar{C}_q^{(-)} e^{\Omega(-)\tilde{y}} \right) \hat{\Phi}_q(\omega), \]

\[ \bar{J}_g = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \tilde{x}^{-\omega} \left( \bar{C}_g^{(+)} e^{\Omega(+)\tilde{y}} + \bar{C}_g^{(-)} e^{\Omega(-)\tilde{y}} \right) \hat{\Phi}_g(\omega). \]

The notations \( \hat{\Phi}_{q,g}(\omega) \) stand for the parton distributions in the \( \omega \)-space. They are related to the parton distributions \( \Phi_{q,g} \) in the \( x \)-space by the transform [110]. At this point we stress once more that, in contrast to DGLAP, the parton distributions \( \Phi_{q,g} \) should not involve factors \( x^{-a} \). The role of such factors is to ensure the structure functions with fast growth at \( x \to 0 \). However, the total resummation of DL contributions in Eq. (34) automatically leads to the Regge asymptotics at small \( x \), which we will demonstrate in the next Sect.
VII. SMALL-\(x\) ASYMPTOTICS OF THE UNPOLARIZED STRUCTURE FUNCTIONS

Eq. (22) reads that \(\Omega_{(+)} > \Omega_{(-)}\). Because of that we drop the terms comprising \(\Omega_{(-)}\) in Eq. (38), when calculate the small-\(x\) asymptotics of \(J_{q,g}\) and \(\bar{J}_{q,g}\). Then we represent Eq. (38) in the following exponential form:

\[
J_{q,g} \approx \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} e^{\omega \xi + \Psi_{q,g}(\omega)},
\]

\[
\bar{J}_{q,g} \approx \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} e^{\omega \xi + \bar{\Psi}_{q,g}(\omega)},
\]

with

\[
\Psi_{q,g} = \Omega_{(+)} + \ln C_{q,g}^{(+)};
\]

\[
\bar{\Psi}_{q,g} = \Omega_{(+)} + \ln \bar{C}_{q,g}^{(+)}.
\]

Now we push \(\bar{x} \to 0\) (i.e. \(\xi \to \infty\)) and apply the Saddle-Point method to each expression in Eq. (39). Handling any of \(J_{q,g}\) and \(\bar{J}_{q,g}\) is the same. This method states that the small-\(x\) asymptotics of the structure functions is given by the following expressions (see Appendix C for detail):

\[
F_1 \sim \frac{\chi_1}{\xi^{3/2}} \bar{x}^{-\omega_0} \left(\bar{Q}^2/\mu^2\right)^{\omega_0/2},
\]

\[
F_2 \sim \frac{\chi_2}{\xi^{3/2}} \bar{x}^{-\omega_0 + 1} \left(\bar{Q}^2/\mu^2\right)^{\omega_0/2},
\]

\[
F_L \sim \frac{\chi_L}{\xi^{3/2}} \bar{x}^{-\omega_0 + 1} \left(\bar{Q}^2/\mu^2\right)^{\omega_0/2},
\]

where \(\bar{x}\) and \(\bar{Q}^2\) are defined in Eq. (36), so Eq. (41) is valid for large and small \(Q^2\), including \(Q^2 = 0\). The non-exponential factors \(\chi_{1,2,L}(\omega_0)\) are defined as follows:

\[
\chi_1 = \left(1 + \gamma^{(2)} \rho C_g^{(2)} \right) C_q^{(+)}(\omega_0) \lambda_q(\omega_0) + \gamma^{(2)} \rho C_q^{(2)} C_g^{(+)}(\omega_0) \lambda_g(\omega_0),
\]

\[
\chi_2 = 2 \left(1 + 3 \gamma^{(2)} \rho C_q^{(2)} \right) C_q^{(+)}(\omega_0) \lambda_q(\omega_0) + 3 \gamma^{(2)} \rho C_g^{(2)} C_q^{(+)}(\omega_0) \lambda_g(\omega_0),
\]

\[
\chi_L = 4 \gamma^{(2)} \rho \left(C_q^{(2)} C_g^{(+)}(\omega_0) \lambda_q(\omega_0) + C_g^{(2)} C_q^{(+)}(\omega_0) \lambda_g(\omega_0) \right).
\]

Obviously,

\[
\left(\bar{Q}^2/\mu^2\right)^{\omega_0/2} \approx \left(Q^2/\mu^2\right)^{\omega_0/2}
\]

at \(Q^2 \gg \mu^2\) (we remind that \(\mu \approx 1\) GeV, see for detail Ref. [29]) while in the region of small \(Q^2\), at \(Q^2 \ll \mu^2\),

\[
\left(\bar{Q}^2/\mu^2\right)^{\omega_0/2} \approx 1 + \left(\omega_0/2\right) \left(Q^2/\mu^2\right).
\]

The notation \(\omega_0\) in Eq. (41) stands for the rightmost singularities of the perturbative factors \(\Psi_{q,g}\) and \(\bar{\Psi}_{q,g}\) in Eq. (10). It turns out that the rightmost singularity of any them is the rightmost root of the equation \(W = 0\), with \(W\) being defined in Eq. (61). Explicitly, this equation is

\[
(\omega^2 - 2(b_{qq} + b_{gg}))^2 - 4(b_{qq} - b_{gg})^2 - 16b_{gg}b_{qq} = 0,
\]

with \(b_{ik}\) defined in Eq. (62). As \(W\) takes place in expressions for each structure function, \(\omega_0\) is the same for any of them. The factors \(\chi_{1,2,3}\) include as numerical factors of the perturbative origin as the parton distributions \(\Phi_{q,g}(\omega)\) at
\( \omega = \omega_0 \). Those factors are different for different structure functions. Eq. (45) can be solved analytically for the fixed \( \alpha_s \) approximation only\(^2\). When \( \alpha_s \) runs, Eq. (45) has to be solved numerically, which leads to the value

\[
\omega_0 = 1.07. \tag{46}
\]

In contrast to the \( x \)-dependence, the asymptotics of \( F_{1,2,L} \) in Eq. (41) exhibit the almost identical \( Q^2 \)-dependence. It is generated by the term \( e^{\Omega_+}(\omega) \) which participates in each \( \Psi_r \). Using explicit expressions for \( h_{rr} \), in Appendix B, it is easy to show that \( \Omega_+(\omega_0) = \omega_0/2 \).

Eq. (41) demonstrates that asymptotics of all structure functions are of the Regge type. The Saddle-Point method turns the total sum of the terms \( \sim (\alpha_s \ln^2(1/x))^n \) into the Regge power factor \( x^{-a} \). It grows steeply at small \( x \), which makes redundant factors \( x^{-a} \) in the parton distributions \( \Phi_{q,g} \). The intercept of the Reggeon controlling \( F_1 \) exceeds unity, so it is a new (soft) Pomeron. Although it has nothing in common with the BFKL Pomeron, its intercept is surprisingly close to the one of the BFKL Pomeron in NLO. This issue was considered in detail in Ref. [27]. In contrast, the intercepts of the other Reggeons in Eq. (41) are much smaller than unity but nevertheless they predict the slow growth of \( F_2 \) and \( F_L \) when \( x \) decreases. Below we briefly consider some corollaries of Eq. (41).

### A. Asymptotic scaling

The asymptotics in Eq. (41) at \( Q^2 \gg \mu^2 \) can approximately be represented as follows:

\[
F_1 \sim \zeta^{-1.07},
\]

\[
F_2 \sim x\zeta^{-1.07},
\]

\[
F_L \sim x\zeta^{-1.07},
\]

with \( \zeta = x \sqrt{\mu^2/Q^2} \), so that \( F_1 \) as well as \( F_2/x \) and \( F_L/x \) at \( x \ll 1 \) and \( Q^2 \gg \mu^2 \) depend on the argument \( \zeta \) save the logarithmic factors dropped in Eq. (47). Such scaling of the asymptotics of the structure functions has not been predicted by any other approach.

### B. Ratio \( F_L/F_2 \) at small \( x \)

It follows from Eq. (41) that the ratio \( F_L/F_2 \) is given by the following expression:

\[
R_{2L} \equiv F_L/F_2 = \frac{2\gamma^{(2)}(2)\rho}{1 + 3\gamma^{(2)}(2)\rho} \frac{C_q^{(2)}(\omega_0)\lambda_q(\omega_0) + C_g^{(2)}(\omega_0)\lambda_g(\omega_0)}{C_q^{(2)}(\omega_0)\lambda_q(\omega_0) + 3C_g^{(2)}(\omega_0)\lambda_g(\omega_0)}. \tag{48}
\]

We remind that \( \gamma^{(2)} \) is defined in Eq. (30) and \( \rho = \xi + y \approx \xi \) at \( x \ll 1 \). Obviously, \( F_L/F_2 \approx \gamma^{(2)}\rho \approx 0.006\rho \) at \( \rho \ll 1/\gamma^{(2)} \) which corresponds to the energy scale presently available at experiment. In the opposite case, i.e. at \( \rho \gg 1/\gamma^{(2)} \), the ratio \( F_L/F_2 \sim 2/3 \), though this limit can be achieved at really asymptotic energies.

### C. Relations between logarithmic derivatives of the structure functions

Logarithmic derivatives, i.e. \( \partial \ln F_r/\partial \ln Q^2 = (1/F_r)\partial F_r/\partial \ln Q^2 \), with \( r = 1,2,L \), were already discussed in the literature in the context of DGLAP and the dipole model (see e.g. Refs. [14, 30]). It motivates us to construct analogous relations for \( F_r \) in DLA at the small-\( x \) by differentiating Eq. (41). First of all, there are relations for the \( Q^2 \)-dependence of the structure functions:

\(^2\) see Ref. [27] for detail
\[ \frac{\partial \ln F_1}{\partial y} = \frac{\partial \ln F_2}{\partial y} \approx \frac{\partial \ln F_L}{\partial y}. \]  

(49)

Then, the relations involving the \( x \)- and \( Q^2 \)-dependence of \( F \):

\[ \frac{\partial \ln F_1}{\partial \xi} - 2 \frac{\partial \ln F_1}{\partial y} \sim 0, \]
\[ \frac{\partial \ln F_2}{\partial \xi} - 2 \frac{\partial \ln F_2}{\partial y} \sim -1, \]
\[ \frac{\partial \ln F_L}{\partial \xi} - 2 \frac{\partial \ln F_L}{\partial y} \sim -1 \]

at \( x \ll 1 \) and \( Q^2 > \mu^2 \approx 1 \text{ GeV}^2 \). We stress that the relations (50) differ a lot from the results in all approaches based on BFKL and DGLAP or on their modifications (see e.g. Refs. [14, 19]). The difference between our results and the results (see e.g. Ref. [31]) obtained in the Regge inspired models [23, 24] is even greater: the intercepts in our approach do not depend on \( Q^2 \).

D. Remark on Soft and Hard Pomerons

One of results obtained in Ref. [27] is the estimate of the region of applicability of the Regge asymptotics: the expressions for the small-\( x \) asymptotics in Eq. (41) are reliable at \( x \leq 10^{-6} \). The straightforward way to describe \( F_{1,2} \) and \( F_L \) at larger \( x \) is to apply the parent expressions of Eqs. (37) despite their complexity. The same should be done, when BFKL is applied. However, there is a tendency in the literature to use the Regge asymptotics at \( x \gg 10^{-6} \), which inevitably leads to introducing phenomenological Pomerons with intercepts much greater than 1.07. In order to simplify our explanation we use the generic notation \( F \) for any of \( F_{1,2}, F_L \) and denote \( \Gamma \) their small-\( x \) asymptotics.

The ratio

\[ R_{\text{as}} \equiv \Gamma / F \]  

(51)

depends on both \( Q^2 \) and \( x \), i.e. \( R_{\text{as}} = R_{\text{as}}(x, Q^2) \). To begin with, we put \( Q^2 = \mu^2 \) and study dependence of \( R_{\text{as}}(x, \mu^2) \) on \( x \). It turns out that at \( x_0 = 10^{-6}, x_1 = 10^{-4}, x_2 = 10^{-3} \) the values of \( R_{\text{as}}(x, \mu^2) \) are as follows:

\[ R_{\text{as}}(x_0, \mu^2) = 0.9, \]
\[ R_{\text{as}}(x_1, \mu^2) = 0.67, \]
\[ R_{\text{as}}(x_2, \mu^2) = 0.5. \]  

(52)

Eq. (52) explains why \( x_0 \) was chosen in Ref. [27] as the border of applicability region of the asymptotics. Numerical estimates show that \( R_{\text{as}} \) decreases when \( Q^2 \) grows. On the other hand, in practice the Regge asymptotics are used at \( x > x_0 \). In this case a new Reggeon is supposed to mimic the structure functions and as a result it should equate the Regge factor of Eq. (41):

\[ x_0^{-\omega_0} = x^{-a}. \]  

(53)

For example, if the asymptotics is used at \( x_1 = 10^{-4} \), the intercept \( a_1 \) of the phenomenological Reggeon is

\[ a_1 = \omega_0 \frac{\ln x_0}{\ln x_1} = \frac{6}{4} \omega_0 = 1.6. \]  

(54)

This estimate demonstrates that approximating parent amplitudes by their asymptotics beyond the applicability regions inevitably leads to introducing phenomenological hard Pomerons.
VIII. SUMMARY

In the present paper, we have calculate the structure functions $F_1$ and $F_2$ in DLA. By doing so, we first correct our previous results on $F_1$ obtained in Ref. [27] and then calculate $F_2$. We find that the contributions to $F_1$ we neglected in Ref. [27] are sizable but do not change qualitatively basic features of $F_1$. The instrument we use in order to sum DL contributions to all orders in $\alpha_s$ is the IREE method. As a result, we obtain explicit expressions for $F_1$ and $F_2$. Then we use the Saddle-Point Method to calculate the small-$x$ asymptotics of $F_{1,2}$. These asymptotics prove to be of the Regge type, though controlled by different Reggeons. The intercept of the Reggeon governing $F_1$ is greater than unity, so it is a new contribution to Pomeron. In contrast, the intercept of the Reggeon governing $F_2$ is small but positive, which leads to slow growth of $F_1$ when $x$ is decreasing. We demonstrate that DLA predicts identical $Q^2$-dependence of $F_1$ and $F_2$ and explain the reason to it. The asymptotics $F_{1,2}$ are used to obtain several differential relations between logarithms of $F_1$ and $F_2$, which are absent in all other approaches available in the literature. It would be interesting to check these relations with analysis of experimental data.

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X. APPENDIX

A. Expressions for $C_{q,g}^{(\pm)}(\omega)$ and $\tilde{C}_{q,g}^{(\pm)}(\omega)$

\begin{align}
C_{q}^{(+)} &= \frac{(h_{gg} - \omega) \left( h_{gg} - h_{qq} - \sqrt{R} \right) + 2h_{gg}h_{qq}}{2\Delta \sqrt{R}}, \\
C_{q}^{(-)} &= \frac{-(h_{qq} - \omega) \left( h_{gg} - h_{qq} + \sqrt{R} \right) - 2h_{gg}h_{qq}}{2\Delta \sqrt{R}}, \\
C_{g}^{(+)} &= \frac{-h_{gg} \left( h_{gg} - h_{qq} - \sqrt{R} \right) - 2h_{gg}(h_{qq} - \omega)}{2\Delta \sqrt{R}}, \\
C_{g}^{(-)} &= \frac{h_{gg} \left( h_{gg} - h_{qq} + \sqrt{R} \right) + 2h_{gg}(h_{qq} - \omega)}{2\Delta \sqrt{R}}.
\end{align}

Coefficient functions $\tilde{C}_{q}^{(\pm)}$ and $\tilde{C}_{g}^{(\pm)}$ are related with $C_{q,g}^{(\pm)}$:

\begin{align}
\tilde{C}_{q}^{(+)} &= C_{q}^{(+)} X^{(+)} , \quad \tilde{C}_{g}^{(+)} = C_{g}^{(+)} X^{(+)} , \\
\tilde{C}_{q}^{(-)} &= C_{q}^{(-)} X^{(-)} , \quad \tilde{C}_{g}^{(-)} = C_{g}^{(-)} X^{(-)} ,
\end{align}

where

\begin{equation}
X^{(\pm)} = \frac{h_{gg} - h_{qq} \pm \sqrt{R}}{2h_{gg}}.
\end{equation}

Thus we have expressed all coefficient functions in Eqs. [32 - 35] through $h_{rr'}$. 
B. Expressions for $h_{ik}$

$$h_{qq} = \frac{1}{2} \left[ \omega - Z - \frac{b_{gg} - b_{qq}}{Z} \right], \quad h_{gg} = \frac{b_{gg}}{Z},$$

$$h_{gg} = \frac{1}{2} \left[ \omega + Z + \frac{b_{gg} - b_{qq}}{Z} \right], \quad h_{gq} = \frac{b_{gq}}{Z},$$

where

$$Z = \frac{1}{\sqrt{2}} \sqrt{Y + W},$$

with

$$Y = \omega^2 - 2(b_{qq} + b_{gg})$$

and

$$W = \sqrt{(\omega^2 - 2(b_{qq} + b_{gg}))^2 - 4(b_{qq} - b_{gg})^2 - 16b_{gq}b_{qg}}.$$  

where the terms $b_{rr}$ include the Born factors $a_{rr'}$ and contributions of non-ladder graphs $V_{rr'}$:

$$b_{rr'} = a_{rr'} + V_{rr'}.$$

The Born factors are (see Ref. [29] for detail):

$$a_{qq} = \frac{A(\omega) C_F}{2 \pi}, \quad a_{gg} = \frac{A'(\omega) C_F}{\pi}, \quad a_{gq} = -\frac{A'(\omega) n_f}{2 \pi}, \quad a_{gq} = \frac{2N A(\omega)}{\pi},$$

where $A$ and $A'$ stand for the running QCD couplings as shown in Ref. [32]:

$$A = \frac{1}{b} \left[ \frac{\eta}{\eta^2 + \pi^2} - \int_0^\infty \frac{dze^{-\omega z}}{(z + \eta)^2 + \pi^2} \right], \quad A' = \frac{1}{b} \left[ \frac{1}{\eta} - \int_0^\infty \frac{dze^{-\omega z}}{(z + \eta)^2} \right],$$

with $\eta = \ln \left( \mu^2 / \Lambda_{QCD}^2 \right)$ and $b$ being the first coefficient of the Gell-Mann- Low function. When the running effects for the QCD coupling are neglected, $A(\omega)$ and $A'(\omega)$ are replaced by $\alpha_s$. The terms $V_{rr'}$ represent the impact of non-ladder graphs on $h_{rr'}$ (see Ref. [29] for detail):

$$V_{rr'} = \frac{m_{rr'}}{\pi^2} D(\omega),$$

with

$$m_{qq} = \frac{C_F}{2N}, \quad m_{gg} = -2N^2, \quad m_{gq} = n_f \frac{N}{2}, \quad m_{qg} = -NC_F,$$

and

$$D(\omega) = \frac{1}{2b^2} \int_0^\infty dze^{-\omega z} \ln \left( (z + \eta) / \eta \right) \left[ \frac{z + \eta}{(z + \eta)^2 + \pi^2} - \frac{1}{z + \eta} \right].$$

C. Basics of Saddle-Point Method

Any expression in Eq. (38) can be genericly represented as follows:

$$J = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{i\omega \xi + \Psi(\omega)},$$
where $\Psi$ stands for any of $\Psi_{q,g}$, $\tilde{\Psi}_{q,g}$. Let us expand $\Psi$ in the series in vicinity of the extremum point $\omega_0$, retaining three terms only. Then the exponent in Eq. (68) is

$$\xi \omega + \Psi(\omega) \approx \xi \omega_0 + \Psi(\omega_0) + [\xi + \Psi'(\omega_0)] (\omega - \omega_0) + (1/2) \Psi''(\omega_0)(\omega - \omega_0)^2$$

and there is extremum at $\omega = \omega_0$, then

$$\xi + \Psi'(\omega_0) = 0,$$  \hspace{1cm} (70)

Now push $\xi \to \infty$ and notice the in order to equate $\xi$ in (70), $\Psi'$ should be singular at $\omega = \omega_0$. $\Psi$ has many singularities but the rightmost singularity corresponds to $W = 0$. Then

$$\Psi'(\omega_0) \approx \frac{\partial \Psi}{\partial W} \frac{dW}{d\omega} = \left[ \frac{\partial \Psi}{\partial W} \omega_0 - b_{qq} - b_{gg} \right] \frac{1}{W} \equiv \frac{\kappa_1}{W}.$$  \hspace{1cm} (71)

Combining it with Eq. (70) yields

$$W = \frac{\kappa_1}{\xi}.$$  \hspace{1cm} (72)

When the most singular contribution is accounted for,

$$\Psi''(\omega_0) \approx - \left[ \frac{\partial \Psi}{\partial W} \omega_0^2 (\omega_0 - b_{qq} - b_{gg})^2 \right] \frac{1}{W^3} \equiv \frac{\kappa_2}{W^3} \Rightarrow \frac{\kappa_2}{\kappa_1} \xi^3 = \lambda \xi^3$$

and therefore

$$J \sim e^{\omega_0 \xi + \Psi(\omega_0)} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-(1/2)(\kappa_1/\kappa_2)\xi^3 (\omega - \omega_0)^2} = e^{\omega_0 \xi + \Psi(\omega_0)} \sqrt{\frac{2\pi \kappa_2}{\kappa_1}} \frac{\lambda}{\xi^{3/2}} e^{\omega_0 \xi}.$$  \hspace{1cm} (74)
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