SOME COMBINATORIAL PROPERTIES OF SPLITTING TREES

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Abstract. We show that splitting forcing does not have the weak Sacks property below any condition, answering a question of Laguzzi, Mildenberger and Stuber-Rousselle. We also show how some partition results for splitting trees hold or fail and we determine the value of cardinal invariants after an \(\omega_2\)-length countable support iteration of splitting forcing.

1. Introduction

We will study some properties of splitting trees and the associated splitting forcing (see Definition 1.2). This is a forcing notion that gives a natural way to add a splitting real (see more below) generating a minimal extension of the ground model (see [9] Corollary 4.21 and also [13]). Splitting trees are part of a more general class of perfect trees that appeared in [11, Definition 2.11]. Recall that a set \(T\) of perfect trees is called a forcing notion consisting of splitting trees ordered by inclusion.

Recall that for \(x, y\), if it is closed under initial segments and that we denote with \([x]^{\omega}\) the set of all infinite branches through \(T\), i.e. there is no \(s \in T\) such that \(t \subseteq s\). The set of terminal nodes of \(T\) is denoted \(\text{term}(T)\). We say that \(t_0, t_1\) are incompatible, or write \(t_0 \perp t_1\), to say that neither \(t_0 \subseteq t_1\) nor \(t_1 \subseteq t_0\). Finally, for any \(t \in T\), we define the restriction of \(T\) to \(t\) as \(T \upharpoonright t = \{s \in T : s \neq t\}\).

Definition 1.1. Let \(a \subseteq \omega\) and \(X \subseteq 2^{<\omega}\). Then we say that \(X\) covers \(a\) if for every \(n \in a\) and \(i \in 2\), there is \(s \in X\) with \(s(n) = i\).

Definition 1.2 (Splitting tree). Let \(T \subseteq 2^{<\omega}\) be a tree. Then \(T\) is a splitting tree if for every \(t \in T\), \(T \upharpoonright t\) covers a cofinite subset of \(\omega\). We write \(\mathbb{S}\) for the forcing notion consisting of splitting trees ordered by inclusion.

Recall that for \(x, y \in [\omega]^\omega\), we say that \(x\) splits \(y\) if \(y \cap x\) as well as \(y \setminus x\) are infinite. A real \(x \in [\omega]^\omega\) is called splitting over a model \(V\) if for every \(y \in V \cap [\omega]^\omega\), \(x\) splits \(y\). We will often identify sets of naturals with their characteristic functions in \(2^\omega\). Thus we may sometimes say things such as \(x \in 2^\omega\) splits \(y \in [\omega]^\omega\), when we mean that \(x^{-1}(1)\) splits \(y\). Then we see that \(\mathbb{S}\) adds a generic splitting real over the ground model \(V\). Namely, if \(y \in [\omega]^\omega\), \(n \in \omega\), \(i \in 2\) and \(T \in \mathbb{S}\) are arbitrary, we find \(s \in T\), such that \(s(m) = i\) for some \(m \in y \setminus n\). Thus, passing to the splitting tree \(T \upharpoonright s \leq T\), we force that the generic real intersects or avoids \(y\) above \(n\), depending on the choice of \(i\).

Shelah showed in [11] that \(\mathbb{S}\) is proper and \(\omega^\omega\)-bounding. Moreover it has the continuous reading of names (see e.g. [14]). This is saying that for any \(\mathbb{S}\) name \(\dot{y}\) for an element of \(\omega^\omega\) and a condition \(T \in \mathbb{S}\), there is \(S \leq T\) and a continuous

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function \( f: [S] \rightarrow \omega^\omega \) such that \( S \models \dot{y} = f(\dot{x}_{\text{gen}}) \), where \( \dot{x}_{\text{gen}} \) is a name for the generic real added by \( \mathbb{SP} \). Together with the minimality of the forcing extension this makes splitting forcing very similar to Sacks forcing. In [7], the authors ask the natural question of whether splitting forcing also has the Sacks property (see Definition 2.1 below). This alone can in fact already be answered using an earlier result of Zapletal (see Remark 4.4). But we will show in Section 2 that \( \mathbb{SP} \) does not even have the weak Sacks property (Definition 2.2) below any of its conditions.

The term “splitting tree” probably first appeared in [13], where they were introduced to give a topological characterization of analytic \( \omega \)-splitting families, similar to the existing ones, e.g. for unbounded families (see [4]) or dominating families (see [12, 3]). Recall that a splitting family is a set \( S \subseteq [\omega]^\omega \) so that for every \( y \in [\omega]^\omega \) there is \( x \in S \) splitting \( y \). Moreover \( S \subseteq [\omega]^\omega \) is called \( \omega \)-splitting if for every countable \( H \subseteq [\omega]^\omega \), there is \( x \in S \) simultaneously splitting every member of \( H \). Again, identifying subsets of \( \omega \) with characteristic functions, we may talk about subsets of \( 2^\omega \) being splitting or \( \omega \)-splitting. The main result of [13] is the following.

**Theorem 1.3** ([13, Theorem 1.2]). An analytic set \( A \subseteq 2^\omega \) is \( \omega \)-splitting if and only if it contains the branches of a splitting tree.

This is a fundamental property of splitting trees. Essentially, it is showing that splitting forcing is an idealized forcing in the sense of [16] in which closed sets are dense. Namely, let \( I \subseteq \mathcal{P}(2^\omega) \) be the \( \sigma \)-ideal consisting of sets that are not \( \omega \)-splitting. Then Theorem 1.3 shows that \( \mathbb{SP} \) is equivalent in the sense of forcing to the partial orders of \( I \)-positive analytic, Borel or closed subsets of \( 2^\omega \).

Later, in [14], Spinas studied properties of splitting trees related to Borel subset-colorings. The main result\(^2\) of [14] is that

\[ 2^{<\omega} \not\rightarrow_{\text{Borel}} [\mathbb{SP}]_2^{\omega_0}. \]

Here, we adopt the well-known arrow notation from partition calculus. The above is then saying that for any Borel map \( c: [2^\omega]^{<2} \rightarrow 2^\omega \), there is some splitting tree \( T \) such that \( c''([T])^{<2} \subseteq 2^\omega \), i.e. we can avoid at least one color on the branches of a splitting tree. Whenever \( T \subseteq 2^{<\omega} \) is a tree, \( \Gamma \) is some class of pair-colorings on \( [T] \) (such as the continuous, Borel or Baire-measurable colorings) and \( \mathcal{P} \) is a set of trees, we write \( T \rightarrow_\Gamma (\mathcal{P})^2_j \) to say that for every coloring \( c: [[T]]^{<2} \rightarrow j \) that is in \( \Gamma \), there is \( S \subseteq T, S \in \mathcal{P} \) such that \( c \) is constant on \( [[S]]^{<2} \). For example, when \( S \) is Sacks forcing, i.e. the collection of all perfect subtrees of \( 2^{<\omega} \), then

\[ T \rightarrow_{\text{Baire}} (S)^2_j, \]

for \( T \) an arbitrary perfect tree and \( j \in \omega \), is known as Galvin’s Theorem (see [25, Theorem 19.6]). This is generally wrong for splitting trees as we shall see in Section 3. In fact there is a dense set of splitting trees \( T \in \mathbb{SP} \) so that

\[ T \not\rightarrow_{\text{cont}} (\mathbb{SP})^2_2. \]

\(^1\)That \( \mathbb{SP} \) is an idealized forcing in fact already follows from [16, Prop. 2.1.6].

\(^2\)Spinas’ result also follows immediately from Lemma 2.6 below.

\(^3\)The outer pair of brackets in \( [[T]]^{<2} \) corresponds to the usual notation for the collection of two-sized subsets \( [X]^{<2} \) of a set \( X \).
Combined with Theorem 1.3 this is showing that there are continuous pair-colorings on the branches of a splitting tree without a homogeneous $\omega$-splitting subfamily.

On the other hand

$$2^{<\omega} \rightarrow_{\text{Baire}} (\mathbb{S}\mathbb{P})^2_j$$

holds for every $j \in \omega$.

In Section 4, we will study the model obtained by iterating $\mathbb{S}\mathbb{P}$ in a countable support iteration of length $\omega_2$. We will show how to decide the value of the classical cardinal invariants in Cichoń’s diagram (see [1] for a reference) and the combinatorial ones appearing in [15] or [2].

2. The Sacks Property

Let us recall the Sacks property and the weak Sacks property.

**Definition 2.1.** A forcing $\mathbb{P}$ has the *Sacks property* if for every $g: \omega \rightarrow \omega \setminus 1$ such that $g(n) \rightarrow \infty$, every condition $p \in \mathbb{P}$ and every name $\check{x}$ for an element of $\omega^\omega$, there is $q \leq p$ and $H \in \prod_{n \in \omega}[\omega]^{g(n)}$, such that

$$q \Vdash \forall n \in \omega (\check{x}(n) \in H(n)) .$$

**Definition 2.2.** A forcing $\mathbb{P}$ has the *weak Sacks property* if for every $g: \omega \rightarrow \omega$ such that $g(n) \rightarrow \infty$, every condition $p \in \mathbb{P}$ and every name $\check{x}$ for an element of $\omega^\omega$, there is $q \leq p$, $a \in [\omega]^\omega$ and $H \in \prod_{n \in a}[\omega]^{g(n)}$, such that

$$q \Vdash \forall n \in a (\check{x}(n) \in H(n)) .$$

**Definition 2.3.** Let $T$ be a splitting tree. Then we say that $T$ is finitely covering if there is a finite set $X \subseteq [T]$ that covers a cofinite subset of $\omega$.

**Lemma 2.4.** Let $T$ be not finitely covering and $l_0, m \in \omega$. Then there is $l_1 > l_0$ so that no $X \subseteq T \cap 2^{l_1}$ of size $m$ can cover $[l_0, l_1)$.

**Proof.** Suppose to the contrary that for every $l > l_0$ there is $(y^*_j)_{j < m} \in [T]^m$ so that $\{y^*_j: j < m\}$ covers $[l_0, l)$. By compactness, there is $a \in [\omega]^\omega$ so that $\langle(y^*_j)_{j < m}: l \in a \rangle$ converges to $(y_j)_{j < m} \in [T]^m$. Since $T$ is not finitely covering there is $l > l_0$ and $i \in 2$ so that for every $j < m$, $y_j(l) = i$. On the other hand, there is $l' \in a \setminus (l + 1)$ so that $y_j(l') \restriction (l + 1) \subseteq y_j$ for every $j < m$. Thus $y_j(l') = i$ for every $j < m$ and $\{y^*_j: j < m\}$ does not cover $[l_0, l')$. \hfill $\square$

**Lemma 2.5.** Let $T$ be not finitely covering and $g: \omega \rightarrow \omega$ such that $g(n) \rightarrow \infty$. Then there is a continuous function $f: [T] \rightarrow \omega^\omega$ so that for any subtree $S \subseteq T$, $a \in [\omega]^\omega$ and $H \in \prod_{n \in a}[\omega]^{g(n)} \times \omega_\omega$, if

$$f^m[S] \subseteq \prod_{n \in \omega} H(n) ,$$

then $S$ is not a splitting tree.

**Proof.** Using Lemma 2.4 let $\langle l_n: n \in \omega \rangle$ be a sequence so that for every $n \in \omega$, no set $X \subseteq T \cap 2^{l_{n+1}}$ of size $g(n)$ can cover $[l_n, l_{n+1})$. Now simply let $f(x)(n) = m$ iff $x \restriction l_{n+1}$ is the $m$th element of $T \cap 2^{l_{n+1}}$ in lexicographical order. Then, whenever $S \subseteq T$ is such that at most $g(n)$ values can be attained as $f(x)(n)$ for $x \in [S]$, $|S \cap 2^{l_{n+1}}| \leq g(n)$ and $S$ does not cover $[l_n, l_{n+1})$. If infinitely many intervals $[i_n, l_{n+1})$ are not covered then $S$ is not a splitting tree. \hfill $\square$
In order to get a failure of the weak Sacks property it is thus sufficient to find a splitting tree that is not finitely covering. Such a tree is not hard to construct directly, but the next lemma will produce such a tree below any given condition.

**Lemma 2.6.** Let $T$ be a splitting tree, and $T \in M$ for $M$ a countable transitive model of a large enough fragment of ZFC. Then there is $S \leq T$ a splitting tree so that for any pairwise distinct $x_0, \ldots, x_{n-1} \in [S]$,

$$(x_0, \ldots, x_{n-1}) \text{ is } T^n\text{-generic over } M.$$

Here, we view $T^n$ as the forcing ordered by coordinate-wise extension, which is equivalent to Cohen forcing.

**Proof.** This is Proposition 4.16 in [9] when applied to $k = 1$, in combination with Lemma 4.10 (also see Definition 4.6 and 4.9). \qed

We do not want to reprove Lemma 2.6 here since it would result in essentially copying the exact argument from [9], which can be read independently from the rest of the paper. The following is a simple genericity argument.

**Lemma 2.7.** Let $S$ be as in Lemma 2.6 for arbitrary $T$ and $M$. Then $S$ is not finitely covering.

Finally we get:

**Theorem 2.8.** Splitting forcing does not have the weak Sacks property below any condition.

At first sight, Lemma 2.6 seems to be overkill to find a subtree that is not finitely covering. But we are not aware of a simpler proof that is not essentially the same combinatorial argument used in the proof of Lemma 2.6, which revolves around having to deal with arbitrary tuples of nodes, rather than just extending single nodes along the construction of a splitting subtree.

In the next theorem we characterize splitting trees on which a construction such as in Lemma 2.5 is possible and we shall see that finite-covering is an essential idea. Moreover, we will see that it does not matter whether we consider Baire-measurable functions or just continuous ones. For a splitting tree $T$ and a class $\Gamma$ of functions $f: [T] \to \omega^\omega$ we will write $w\text{Sacks}_\Gamma(T)$ to say that the definition of the weak Sacks property holds applied to the condition $p = T$ and a name $\dot{x}$ for $f(\dot{x}_{\text{gen}})$ where $f \in \Gamma$ and $\dot{x}_{\text{gen}}$ is a name for the generic real.

**Theorem 2.9.** Let $T$ be a splitting tree. Then the following are equivalent.

1. $\text{wSacks}_{\text{Baire}}(T)$
2. $\text{wSacks}_{\text{cont}}(T)$
3. $\exists S \leq T \forall s \in S(S \upharpoonright s \text{ is finitely covering})$

**Proof.**

$(3) \to (1):$ Let $g \in \omega^\omega$ be so that $g(n)$ diverges to $\infty$ and let $f: [T] \to \omega^\omega$ be Baire-measurable. Then there are open dense sets $O_n \subseteq [T]$, for $n \in \omega$, so that $f$ is continuous on $X := \bigcap_{n \in \omega} O_n$. By $(3)$ we can assume wlog that $T$ is such that for every $s \in T$, $T \upharpoonright s$ is finitely covering. For every $s \in T$, fix $x_0^s, \ldots, x_{k-1}^s$ in $[T \upharpoonright s]$ and $z^s \in [T \upharpoonright s] \cap X$, all pairwise distinct, $k(s) = k \in \omega$ and $m(s)$ so that $\{x_0^s, \ldots, x_{k-1}^s\}$ covers $[m(s), \omega)$. We are going to construct a sequence $\langle T_n : n \in \omega \rangle$ of finite subtrees of $T$ that is an strictly increasing by end-extension, such that $S := \bigcup_{n \in \omega} T_n$ is a splitting tree witnessing the weak Sacks property for $f$ and $g$.
Along the construction, there will be a map $\sigma$ that maps every $n \in \text{term}(T_n)$ to a dedicated “working node” $\sigma(s) \in \text{term}(T_{n+1})$ and there will be a strictly increasing sequence $\langle i_n : n \in \omega \rangle$ of naturals. The following properties will be satisfied for every $n \in \omega$:

(a) For every $s \in \text{term}(T_{n+1})$, $[T \upharpoonright s] \subseteq O_n$,
(b) $s$ determines $f(x)(i_n)$ for every $x \in [T \upharpoonright s] \cap X$ and $|\text{term}(T_{n+1})| < g(i_n)$.
(c) For every $s \in \text{term}(T_n)$, $T_{n+1} \upharpoonright s$ covers $[m(s), m(\sigma(s))]$.

(a) implies that $[S] \subseteq X$ and (b) then yields that $S$ witnesses wSacks$(T)$ for $f$ and $g$.  (c) on the other hand implies that for every $s \in \text{term}(T_n)$, $S \upharpoonright n$ covers $[m(s), \omega)$ and thus that $S$ is a splitting tree.

The construction is as follows. We start with $T_0 = \emptyset$ and $i_0 = 0$. Next, given $T_n$, we let $K = \sum_{s \in \text{term}(T_n)} (1 + k(s))$. Since the values of $g$ diverge to infinity, there is $i_n > i_{n-1}$ so that $g(i_n) > K$. For each $s \in \text{term}(T_n)$, let $\sigma(s) := z^s \upharpoonright l$ for $l$ large enough such that $[T \upharpoonright (z^s \upharpoonright l)] \subseteq O_n$, $z^s \upharpoonright l$ determines the value of $f(x)(i_n)$ on $X$ and $z^s \upharpoonright l$ is not an initial segment of any of the $x_i^s$, for $i < k(s)$. Let $m = \max\{m(\sigma(s)) : s \in \text{term}(T_n)\}$ and for every $s \in \text{term}(T_n)$ and $i < k(s)$, find $t_i^s$ extending $x_i^s \upharpoonright m$ such that $[T \upharpoonright t_i^s] \subseteq O_n$ and $t_i^s$ decides the value $f(x)(i_n)$ on $X$. Finally let $T_{n+1}$ be the downwards closure of $\bigcup_{s \in \text{term}(T_n)} \{t_0^s, \ldots, t_{k(s)-1}^s\} \cup \{\sigma(s) : s \in \text{term}(T_n)\}$.

(2) $\implies$ (3): Assume that (3) fails. For a splitting tree $S$ consider a pruning derivative $S' := \{s \in S : S \upharpoonright s$ is finitely covering\}. Now define $T_0 = T$, $T_{\alpha+1} = T'_\alpha$ and $T_\alpha = \bigcap_{\beta < \alpha} T_\beta$ for every $\alpha < \omega_1$ and $\gamma < \omega_1$ a limit ordinal. Since we assume that (3) fails, we have that for every $\alpha < \omega_1$, if $T_\alpha \neq \emptyset$, then $T_\alpha \setminus T_{\alpha+1} \neq \emptyset$. In particular there is some $\alpha < \omega_1$ so that $T_\alpha = \emptyset$.

**Lemma 2.10.** Let $\langle X_n : n \in \omega \rangle$ be a partition of $2^\omega$ into closed sets and $f_n : X_n \to \omega^\omega$ be continuous, for every $n \in \omega$. Then there is a continuous function $f : 2^\omega \to \omega^\omega$ so that for every $n \in \omega$, $f(x)(m) = f_n(x)(m)$ for all $m \geq n$ and $x \in X_n$.

**Proof.** Let $X_n = [R_n]$ for a perfect tree $R_n$, for every $n \in \omega$. We define a sequence $\langle \varphi_n : n \in \omega \rangle$, where $\varphi_n : 2^{\leq i_n} \to \omega^{\leq n}$ is order preserving for every $n$ and $\langle i_n : n \in \omega \rangle$ is increasing, inductively as follows. Start with $\varphi_0 = \{(\emptyset, 0)\}$ and $i_0 = 0$. Given $\varphi_n$ and $i_n$, let $i_{n+1} > i_n$ be large enough so that $R_i \cap R_j \cap 2^{i_{n+1}} = \emptyset$ and for every $x \in X_i$, $f_i(x)(n)$ is decided by $x \upharpoonright i_{n+1}$, for every $i, j < n$. Let $\varphi_{n+1} \supseteq \varphi_n$, $\varphi_{n+1} : 2^{\leq i_{n+1}} \to \omega^{\leq n}$ be arbitrary so that $\varphi_{n+1}(s) \subseteq f_i(x)$ for $s \in R_i \cap 2^{i_{n+1}}$, $x \in X_i$ and $i < n$. Finally, let $\varphi = \bigcup_{n \in \omega} \varphi_n$ and

\[ f(x) := \bigcup_{n \in \omega} \varphi(x \upharpoonright n). \]

We see that $f$ is as required. \(\square\)

For every $\beta < \alpha$, let $A_\beta$ be the set of minimal $s \in T_\beta \setminus T_{\beta+1}$. For any $s \in A_\beta$, applying Lemma 2.13 we find a continuous function $f_{\beta,s} : [T_\beta \upharpoonright s] \to \omega^\omega$ so that for any splitting tree $S \subseteq T_\beta \upharpoonright s$, for almost all $n \in \omega$, more than $2^n$ values can be obtained as $f_{\beta,s}(x)(n)$ for $x \in [S]$. Apply the previous Lemma to find a continuous function $f : [T] \to \omega^\omega$ so that $f(x) = f_{\beta,s}(x)$ uniformly for all $x \in [T_\beta \upharpoonright s]$. We claim that $f$ is as required. Namely, suppose that there is $S \subseteq T$ so that for infinitely many $n \in \omega$, $|\{f(x)(n) : x \in [S]\}| < 2^n$. Then there is $\beta < \alpha$ and $s \in \omega$ so that $[S] \cap [T_\beta \upharpoonright s] \subseteq \omega^\omega$-splitting and in particular contains the branches of a splitting tree. This is a contradiction.
Theorem 3.1. Let $T$ be a splitting tree and assume that $T \rightarrow_{\text{cont}} (\mathbb{S}P)^2_2$. Then $w\text{Sacks}_{\text{cont}}(T)$ is satisfied.

Proof. Let $f: [T] \rightarrow \omega^\omega$ be continuous and $g: \omega \rightarrow \omega$ diverge to $\infty$. Define an increasing sequence $(i_n : n \in \omega)$ recursively such that for every $n \in \omega$, there is $m > i_n$ with $g(m) > 2^{i_n}$ and $f(x)(m)$ is decided by $x \upharpoonright i_{n+1}$ for every $x \in [T]$. Define a coloring $c: [T]^2 \rightarrow 2$ such that

$$c(x, y) = \begin{cases} 0 & \text{if } \exists k \in \omega(\Delta(x, y) \in [i_{2k}, i_{2k+1}]) \\ 1 & \text{if } \exists k \in \omega(\Delta(x, y) \in [i_{2k+1}, i_{2k+2}]) \end{cases},$$

where $\Delta(x, y) = \min\{l \in \omega : x(l) \neq y(l)\}$. Now let $S \leq T$ be such that $[S]$ is $c$-homogeneous, say with constant color 0. Then we see that for every $k \in \omega$, $|S \cap 2^{\leq 2k+2}| \leq 2^{2k+1}$. Thus there is $m > i_{2k+1}$ such that at most $2^{2k+1} < g(m)$ many values can be attained as $f(x)(m)$ for $x \in [S]$. Similarly for color 1. □

From Theorem 3.1 and the results of the last section we immediately find that:

Theorem 3.2. Let $T$ be a splitting tree. Then there is a splitting tree $S \leq T$ so that

$$S \nleftrightarrow_{\text{cont}} (\mathbb{S}P)^2_2.$$

Our motivation behind proving this theorem were the results of [10] which use Galvin’s Theorem for Sacks forcing and an analogue for its countable support iterations (see [10] Theorem 3.17]) in a crucial way to get certain canonization results for binary relations. These results were vastly generalized in [9] using a technique based on Lemma 2.6 to be applied, among other things, to splitting forcing. The above theorem shows that the method of [10] does not apply in a similar way to splitting forcing. In its simplest form, the canonization result says that for any analytic equivalence relation $E$ on $2^\omega$ and any perfect tree $T$, there is a perfect subtree $S \subseteq T$ so that $E$ is canonical on $[S]$, i.e. either $[S]$ consists of pairwise $E$-inequivalent reals or $[S]$ is contained in an $E$-equivalence-class. This is immediate from Galvin’s Theorem. For splitting trees on the other hand, we apply Lemma 2.6 to $T$ and a model $M$ containing a code of $E$ to get $S \leq T$ consisting of mutual $T$-generics over $M$. If $x, y \in [S]$ are $E$-equivalent, then there is $s \subseteq x$ forcing over $M[y]$ that $x Ey$. In particular, for every $x \in [S \upharpoonright s]$, $x Ey$ and $S \upharpoonright s$ is contained in an $E$-class.

Theorem 3.3. Let $c: [2^\omega]^2 \rightarrow j$ be Baire-measurable, $j \in \omega$. Then there is a splitting tree $T$, so that $[T]$ is $c$-homogeneous.

Proof. In the following, we will identify $c$ with the corresponding symmetric function on $(2^\omega)^2$. Since $c$ is Baire-measurable, there is a decreasing sequence $(O_n)_{n \in \omega}$ of dense open subsets of $(2^\omega)^2$ such that $c$ is continuous on $X = \bigcap_{n \in \omega} O_n$. Fix $i \in j$ and $s \in 2^{<\omega}$ so that for any $t \in 2^{<\omega}$, $s \subseteq t$, there are incompatible $t_0, t_1$ extending $t$ such that $c$ is constant with color $i$ on $(|t_0| \times |t_1|) \cap X$. Without loss of generality we may assume that $s = \emptyset$. We will construct an increasing sequence $(T_n)_{n \in \omega}$ of
finite subtrees of $2^{<\omega}$ and an increasing sequence $(m_n)_{n\in\omega}$ of natural numbers with the following properties for every $n \in \omega$.

1. All terminal nodes of $T_n$ have length $m_n$,
2. for any terminal nodes $t_0 \neq t_1$ in $T_n$, $c$ has constant value $i$ on $([t_0] \times [t_1]) \cap X$ and $[t_0] \times [t_1] \subseteq O_n$,
3. every $t \in T_n$ has at least 4 pairwise incompatible extensions in $T_{n+1}$,
4. for every $t \in T_n$, $\{ t' \in T_{n+2} : t \subseteq t' \}$ covers $[m_{n+1}, m_{n+2})$.

Provided we have constructed such a sequence, we shall check that $T = \bigcup_{n \in \omega} T_n$ is as required. It is clear from (2) that $[T]$ is $c$-homogeneous with color $i$. To show that $T$ is a splitting tree, let $t \in T$ be arbitrary, say without loss of generality that $t$ is a terminal node of $T_n$ for some $n \in \omega$. Moreover let $l \in [m_k, m_{k+1})$ for some $k \geq n + 1$. Then we may extend $t$ to $t^+$ in $T_{k-1}$ and by (4) there are $t_0, t_1$ extending $t^+$ in $T$ so that $t_0(l) = 0$ and $t_1(l) = 1$. Thus we have shown that $T_i$ covers $[m_{n+1}, \omega)$.

Now let us construct the sequence recursively. We start with $T_0$ and $m_0 \in \omega$ so that (1) and (2) are satisfied. To find such $T_0$ and $m_0$, let $t_0 \perp t_1$ be arbitrary such that $c$ has constant color $i$ on $([t_0] \times [t_1]) \cap X$. Next extend $t_0$ to $t_0^+$ and $t_1^+$ such that $t_0^+ \perp t_1^+$ and $c$ has constant color $i$ on $([t_0^+] \times [t_1^+]) \cap X$ and similarly find $t_0^+, t_1^+$. Finally, extend $t_0^+, t_1^+, t_0^i$ and $t_1^i$ to nodes of the same length $m_0$ such that the open subsets of $(2^\omega)^2$ determined by any pair of these nodes are contained in $O_n$. Finally let $T_0$ be the tree generated by these nodes. Given $T_n$ and $m_n$, we proceed as follows. We construct an increasing sequence of finite trees $(R_k)_{k<K}$ of some unspecified finite length $K$, starting with $R_0 = T_n$. For every $k < K$, all terminal nodes of $R_k$ will have the same length. In each step $k < K$, there is a working node $t \in \text{term}(R_k)$, that has exactly two incompatible extensions $t_0, t_1$ which are terminal nodes in $R_{k+1}$ such that $c$ has color $i$ on $([t_0] \times [t_1]) \cap X$. All other terminal nodes $t'$ of $R_k$ are extended uniquely to terminal nodes of the form $t' \perp 0 \ldots \perp 0$ or $t' \perp 1 \ldots \perp 1$ in $R_{k+1}$. This is done in a way that for every $r \in T_{n-1}$, there is one such $t'$ extending $r$ which is extended by $0$'s and another one that is extended by $1$'s. This is possible since every $r \in T_{n-1}$ is extended by at least 3 (in fact 4) pairwise incompatible nodes in $T_n$, so at most 1 of them is the working node $t$. In case $n = 0$, this step is irrelevant. By varying the working nodes we can easily ensure in finitely many steps $k < K \in \omega$ that all terminal nodes of $T_n$ are extended by at least 4 pairwise incompatible nodes in $R_{K-1}$. Next we construct an increasing sequence $(S_l)_{l<L}$ of unspecified finite length $L$, starting with $S_0 = R_{K-1}$. Again all terminal nodes of $S_l$ have the same length for each $l < S$. This time, in every step $l$ there is a working pair $(u, v) \in \text{term}(S_l)^2$. In term$(S_{l+1})$, $u$ and $v$ are extended uniquely to $u'$ and $v'$ respectively such that $[u'] \times [v'] \subseteq O_{n+1}$. All other terminal nodes of $S_l$ are extended by $0$'s only or by $1$'s only in the same way as before, now using that at least 4 pairwise incompatible nodes extend each $r \in T_{n-1}$. After finitely many steps, we can ensure that every $(u, v) \in \text{term}(S_0)$ is extended by $u'$ and $v'$ respectively such that $[u'] \times [v'] \subseteq O_{n+1}$. Finally, we let $T_{n+1} = S_{L-1}$. □

4. The splitting model

**Theorem 4.1.** Let $\mathbb{P}$ be the $\omega_2$-length countable support iteration of $\mathbb{S}$. Then, in $V^\mathbb{P}$, $N_1 = \text{cof}(M) = a < r = \text{non}(N) = c = N_2$. 
Lemma 4.2. Let \( P \) be a countable support iteration of \( S P \), \( p \in P \) and \( \dot{y} \) a \( P \)-name for a real. Moreover let \( A \in V \) be arbitrary. Then there is a countable elementary submodel \( M \preceq H(\theta) \), for large enough \( \theta \), with \( A \in M \), \( q \leq p \) and a name \( \dot{c} \), so that 
\[ q \models \text{“} \dot{c} \text{ is a Cohen real over } M \text{ and } \dot{y} \in M[\dot{c}] \text{”}. \]

Proof. This follows from a much stronger result proved in [9] Lemma 4.22]. See also the start of Section 4.3 and Lemma 2.2 in [9].

Proof of Theorem 4.1. Note first, that we can assume wlog that \( V \models CH \) since this is forced in the first \( \omega_1 \)-many steps of any csi of nontrivial forcings. Then it is clear that \( V^p \models \sigma = \omega_1 \) as \( S P \) is \( \omega_\omega \)-bounding. Next, we show that \( V \cap 2^\omega \) is non-meager in \( V^p \). Suppose to the contrary that \( V \cap 2^\omega \) is forced to be contained in a meager \( F_\sigma \) set \( X \) by a condition \( p \in P \). \( X \) is going to be coded by a real \( \dot{y} \) and by the previous lemma we find \( q \leq p, M \preceq H(\theta) \) and \( \dot{c} \) so that 
\[ q \models \text{“} \dot{c} \text{ is a Cohen real over } M \text{ and } \dot{y} \in M[\dot{c}] \text{”}. \]

But Cohen forcing preserves the set of ground model reals to be non-meager. In particular, letting \( G \) be \( P \)-generic over \( V \) with \( q \in G \), we have that 
\[ M[\dot{c}|G]\models \text{“} \text{The } F_\sigma \text{ set coded by } \dot{y} \text{ does not cover the reals of } M \text{”}. \]

By an absoluteness argument between \( M[\dot{c}|G]\) and \( V[G]\), we find that there is a real \( x \in M \subseteq V \) so that \( x \notin X[G] \), yielding a contradiction. Now that we have that \( \sigma = \text{non}(M) = \omega_1 \) in \( V^p \), we may also follow that \( \text{cof}(M) = \omega_1 \) since \( \text{cof}(M) = \max(0, \text{non}(M)) \) (see e.g. [1] Theorem 2.2.11).

The argument for showing that \( a = \omega_1 \) is similar to the one we just made for \( \text{non}(M) \). Namely, we show that any Cohen-indestructible mad family is also \( P \)-indestructible. To this end, let \( A \) be a mad family so that \( \models_C \text{“} A \text{ is mad} \text{”} \). Towards a contradiction, suppose that \( \dot{y} \) is a \( P \)-name for an infinite subset of \( \omega \) and that \( p \models \forall x \in A \langle |x \cap y| < \omega \rangle \). Then, once again, there is an elementary enough \( M \), this time with \( A \in M \), a name \( \dot{c} \) and \( q \leq p \) so that \( q \) forces that \( \dot{c} \) is Cohen generic over \( M \) and that \( M[\dot{c}] \) contains \( \dot{y} \). But then there must be \( x \in A \cap M \) so that \( |y \cap x| = \omega \). Cohen-indestructible mad families exist under \( CH \) (see [6] Exercise IV.4.12).

The reaping number \( r \) is large in \( V^p \) as \( S P \) adds splitting reals. Finally, it was shown in [16] Proposition 4.1.29 that there is a condition \( p \in S P \) forcing that the set of ground model reals is null. This easily implies that in \( V^p \), \( \text{non}(\mathcal{N}) = 8_2 \).

Remark 4.3. The arguments for \( \text{non}(M) \) and \( a \) above are an instance of a more general result that shows that \( \text{very tame invariants} \) (see [10] Definition 6.1.9) that Cohen forcing keeps small, stay small after forcing with \( S P \). More generally, it is not hard to apply the theory developed in [16] 6.1 to the \( \sigma \)-ideal of non \( \omega \)-splitting sets of reals to get similar results for other forcing notions adding \( \omega \)-splitting reals.

Remark 4.4. The fact that some condition in \( S P \) forces the ground model reals to be null already shows that below this condition \( S P \) cannot have the Sacks property. This is because the Sacks property keeps the ground model reals non-null (in fact it keeps \( \text{cof}(\mathcal{N}) \) small, see [1] 2.3)).

Other than the results above, we can also show that \( P \)-points exist in \( V^p \) ([8] Section 4.5.2)]. This uses the technique developed in [9] in an essential way. The countable support iteration of Sacks forcing, for instance, outright preserves ground model \( P \)-points, but this is an impossible task for a forcing adding splitting reals.
Rather, $\mathbb{P}$ can preserve the union of $\aleph_1$ many ground model Borel sets to be reinterpreted as a $\mathbb{P}$-point in the extension. To our best knowledge, this is the only existence result for $\mathbb{P}$-points of this kind.

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