Multiple Axially Asymmetric Solutions to a Mean Field Equation on $S^2$

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Abstract. We study the following mean field equation

$$\Delta_g u + \rho \left( \frac{e^u}{\int_{S^2} e^u d\mu} - \frac{1}{4\pi} \right) = 0 \quad \text{in} \; S^2,$$

where $\rho$ is a real parameter. We obtain the existence of multiple axially asymmetric solutions bifurcating from $u = 0$ at the values $\rho = 4n(n+1)\pi$ for any odd integer $n \geq 3$.

Key Words: Mean field equation, axially asymmetric solutions, bifurcation.

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1 Introduction

In this paper, we consider the mean field equation on the unit sphere

$$\Delta_g u + \rho \left( \frac{e^u}{\int_{S^2} e^u d\mu} - \frac{1}{4\pi} \right) = 0 \quad \text{in} \; S^2, \quad (1.1)$$

where $\rho$ is a real parameter, $\Delta_g$ stands for Laplace-Beltrami operator on $S^2$ associated to the metric $g$ inherited from the ambient Euclidean metric and $d\mu$ is the volume form with respect to $g$. Since the above equation is invariant by adding a constant to a solution, we introduce

$$\mathcal{H} = \left\{ u \in H^2(S^2) \mid \int_{S^2} u d\mu = 0 \right\},$$

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where $H^2(S^2)$ is the classical sobolev space. Note that $\mathcal{H}$, equipped with the $H^2(S^2)$ norm, is a Hilbert space. We write $S^2$ in the following coordinate system

$$
S^2 = \left\{ \left( \sqrt{1-z^2} \cos \theta, \sqrt{1-z^2} \sin \theta, z \right) \bigg| z \in [-1,1], \theta \in [0,2\pi] \right\}.
$$

Note that $\int_{S^2} d\mu = 4\pi$. Clearly equation (1.1) admits solution $u \equiv 0$, so in what follows mentioned existence of solutions of (1.1) means existence of non-trivial solution. The corresponding energy functional of (1.1) is

$$
J_\rho(u) = \frac{1}{2} \int_{S^2} |\nabla u|^2 d\mu - \rho \log \left( \int_{S^2} e^u d\mu \right). \tag{1.2}
$$

The study of existence of solution for mean field equations possess a long history and huge literature. This kind equations arise from Onsager’s vortex model for turbulent Euler flows, see [20]. They also arise from the Chern-Simons-Higgs model vortex when some parameter tends to zero, we refer the reader to [3, 25–27].

For $\rho < 8\pi$, by the Moser-Trudinger inequality, $J_\rho$ is bounded below and coercive, so the proof of the existence of a minimizer of $J_\rho$ is standard. For $\rho = 8\pi$, the existence of a minimizer of (1.2) had been proved in [21] and [7]. For $\rho > 8\pi$, $J_\rho$ is not bounded below. Topological degree theory plays an important role in the solvability of (1.1). Starting with the work of Li in [16], one knows that the solutions of (1.1) are uniformly bounded on any compact subset of $\bigcup_{k=0}^{\infty} (8m\pi, 8(m+1)\pi)$, and the Leray-Schauder topological degree $d_\rho = 1$ for $\rho < 8\pi$. Due to the result of Li and the homotopy invariance of the degree, it is readily checked that $d_\rho$ is constant in each interval $\rho \in (8m\pi, 8(m+1)\pi)$. Further Lin in [17] proved that $d_\rho = -1$ for $8\pi < \rho < 16\pi$, and $d_\rho = 0$ for $16\pi < \rho < 24\pi$. Subsequently Chen and Lin in [4] obtained apriori bound for a sequence $\rho_n$ with $\rho = \rho_n$. Using this apriori bound, they were able to calculate the degree in [5] $d_\rho = 0$ for $\rho \in (8m\pi, 8(m+1)\pi)$, $m \in \mathbb{N}$ with $m \geq 2$. By a more precise topological argument, the author of [8] proved that in this case (1.1) admits a solution for any $\rho \in \mathbb{R}\setminus 8\pi\mathbb{N}$. Dolbeault, Esteban and Tarantello in [9] proved that for all $k \geq 2$ and $\rho > 4k(k+1)\pi$ (so $\rho > 24\pi$), (1.1) admits at least $2(k-2) + 1$ distinct axially symmetric solutions by using bifurcation method. Indeed, they proved that for any $k \geq 2$ there are two continuous unbounded half-branches of solutions of (1.1) bifurcating from the trivial solution at points $\rho = 4k(k+1)\pi$. Note that blow-up solutions could appear only when $\rho \to 8m\pi$, $m \in \mathbb{N}$. Lin in [17] establish the existence of the blow-up solutions to (1.1) when $\rho$ approaches $16\pi$ from above. Recently Gui and Hu [13] proved the existence of a family of blow-up solutions for $\rho$ approaches $32\pi$ by using Lyapunov reduction method. As far as uniqueness is concerned, Lin in [18] showed that the solution to (1.1) is unique for $0 < \rho < 8\pi$, namely (1.1) only admits the trivial solution. Lin in [19] showed that the axially symmetric solution to (1.1) is unique for $8\pi < \rho \leq 16\pi$, namely $u \equiv 0$ is the only axially symmetric solution of (1.1). By developing a "sphere covering inequality", Gui and Moradifam in [14] extend the uniqueness result to a broader parameter range $\rho \in (0, 8\pi) \cup (8\pi, 16\pi)$ for any solutions of (1.1).