Two interacting Ising chains in relative motion

H J Hilhorst

Laboratoire de Physique Théorique, Université Paris-Sud and CNRS,
Bâtiment 210, 91405 Orsay Cedex, France
E-mail: Henk.Hilhorst@th.u-psud.fr

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Abstract. We consider two parallel cyclic Ising chains counter-rotating at a relative velocity \( v \), the motion actually being a succession of discrete steps. There is an in-chain interaction between nearest-neighbor spins and a cross-chain interaction between instantaneously opposite spins. For velocities \( v > 0 \) the system, subject to a suitable Markovian dynamics at a temperature \( T \), can reach only a nonequilibrium steady state. This one-dimensional version of a model introduced by Kadau et al was shown by Hucht to undergo, for \( v = \infty \), a para-ferromagnetic transition, essentially because each chain exerts an effective fluctuating field on the other one. The present study of the \( v = \infty \) case determines the consequences of the fluctuations of this field when the system size \( N \) is finite. We show that, whereas to leading order the system obeys detailed balancing with respect to an effective time-independent Hamiltonian, the higher-order finite-size corrections violate detailed balancing. Expressions are given to various orders in \( N^{-1} \) for the interaction free energy between the chains, the spontaneous magnetization, and the in-chain and cross-chain spin–spin correlations. It is shown how finite-size scaling functions may be derived explicitly. This study was motivated by recent work on a two-lane traffic problem in which a similar phase transition was found.

Keywords: classical phase transitions (theory), finite-size scaling, driven diffusive systems (theory), stationary states
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1. Introduction

Recently Kadau et al [1] and Hucht [2], motivated by the phenomenon of magnetic friction, formulated a nonequilibrium steady state (NESS) Ising model of a new type. We will be interested here in its one-dimensional version [2], which consists of two parallel linear Ising chains having a relative velocity $v$. In addition to a nearest-neighbor interaction in each chain, any pair of spins facing each other on the two chains has an instantaneous interaction. In the case easiest to study each chain is finite and periodic; we will therefore speak of ‘cyclic counter-rotating Ising chains’ (CRIC). In [2] it was shown that this model, subject to a suitable temperature-dependent Markovian dynamics, and when at velocity $v = \infty$, has a para-to ferromagnetic phase transition that, in the limit of infinitely long chains, may be understood in terms of an equivalent equilibrium model.

The CRIC seems to us of the same fundamental importance as Glauber’s [3] original kinetic Ising model. First, it is of interest in its own right as a new member of the class of NESS models. Second, its interest is enhanced in the wider context of recent work on Ising models that in one way or another are driven, dissipate energy, or have some novel type of coupling; such work has appeared in a variety of contexts [4]–[6]. In particular, the work by Kadau et al was extended to Potts variables by Iglói et al [7], who found remarkable nonequilibrium phase transitions. Whereas both Hucht and co-workers [2,1] and Iglói et al [7] also consider higher-dimensional models, we in this paper restrict ourselves to the one-dimensional case and contribute further to the study of the CRIC. Our emphasis is on basic issues, the starting point being the master equation that defines the model and from which all other properties have to be derived.

Hucht’s solution [2] is based on showing that at $v = \infty$ the stationary state dynamics of the CRIC is actually that of an equilibrium Ising chain in an effective magnetic field $H_0$, this field being zero above the transition temperature and nonzero below. This equivalence is valid in the limit where the chain length $N$ tends to infinity (for finite $N$ there is no sharp transition). In this work we show that it is possible to find the stationary state distribution in phase space as an expansion in powers of $N^{-1/2}$. To lowest order we recover the Boltzmann weight of the equivalent equilibrium system found in [2]. To higher orders the fluctuations of the field $H_0$ come into play and appear as finite-size effects. The finite $N$ case is of interest, first of all, on the level of principles, and secondly, for the analysis of simulations. We expect, furthermore, that our approach will help prepare the way for future work on the $v < \infty$ case, which is considerably harder.

The basic master equation is defined by the choice of its transition rates. It is natural to have these satisfy detailed balancing (DB) with respect to the time-dependent Hamiltonian of the moving chains, but that requirement does not fix them uniquely. The first step in the analysis of [2] was to argue heuristically (but correctly) the existence of an equivalent equilibrium system. Only then the question arose by means of which rates to achieve that upon averaging over the fluctuating field (and for $N \to \infty$) the dynamical system become equivalent to this equilibrium model. In answer to this question it was rightly pointed out [2] that not all transition rates, even when satisfying DB, achieve this equivalence, and special so-called multiplicative rates were constructed.

The work by Iglói et al [7] on Potts models in relative motion restricts its attention to the heuristically equivalent equilibrium model, its focus being on the interest of the phase diagram. For the two moving Potts chains this work altogether avoids defining the
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Figure 1. Ladder of spins with an intrachain nearest-neighbor interaction \(J_1 = J\). The two chains constituting the ladder have a relative velocity \(v\), the motion taking place in discrete steps of one lattice unit. There is an interchain nearest-neighbor interaction \(J_2 = \eta J\) between each pair of spins facing each other at any instant in opposite chains.

transition rates, the argument, plausible but not proven, being that changing these rates will affect the results only quantitatively but not qualitatively.

In this paper we adopt a reverse approach: the transition rates postulated at the outset are taken to satisfy DB with respect to the time-dependent Hamiltonian. To leading order in the large-\(N\) expansion [2] there then appear, after averaging over the fluctuating field, effective transition rates that again satisfy DB, but now with respect to the time-independent Hamiltonian of the equivalent equilibrium system. Our analysis reveals, however, that to higher orders in \(N^{-1/2}\) the DB symmetry of the effective rates is broken. We show that the stationary state distribution may nevertheless be found explicitly, at least to the lowest DB-violating order. Knowing this state one can calculate all desired NESS properties.

In section 2 of this paper we define the rules of the Markovian dynamics for relative velocity \(v\) and then specialize to \(v = \infty\). In section 3 we discuss the DB violation that occurs in higher orders of \(N^{-1}\). In section 4 we consider the stationary state distribution to zeroth order, as was already done by Hucht [2]. In section 5 we show how \(N^{-1}\) can be introduced as an expansion parameter and we define a ‘leading order’, composed of the zeroth order and a first-order correction. In section 6 we show how for the stationary state distribution an expansion may be found in powers of \(N^{-1}\). We present the explicit result to next-to-leading order. In section 7 we calculate for various quantities of physical interest their stationary state averages to successive orders in the expansion. In section 8 we briefly discuss the relation of the present model to a two-lane road traffic model studied earlier by ourselves [8] and from which we drew our initial motivation. In section 9 we conclude.

2. Counter-rotating Ising chains

2.1. A stochastic dynamical system

We consider Ising spins on the ladder lattice shown in figure 1. The spins in the upper chain are denoted by \(r_j\), those in the lower chain by \(s_i\), where the integers \(j\) and \(i\) are site indices. There is a nearest-neighbor interaction \(J_1 = J\) inside each chain and an interaction \(J_2 = \eta J\) between each pair of spins facing each other in opposite chains.
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take $J > 0$ and $\eta$ of arbitrary sign. The feature that distinguishes this model from the standard Ising model on a ladder lattice is that the two chains move with respect to one another at a speed $v > 0$. This will mean the following: the time axis is discretized in intervals of duration $\tau = a_0 / v$ (where $a_0$ is the lattice spacing) and at the end of each interval the upper chain is shifted one lattice spacing $a_0$ to the right with respect to the lower one. The Hamiltonian $\mathcal{H}(t)$ of this system is therefore time-dependent and given by

$$
\mathcal{H}(t) = -J \sum_j [r_j r_{j+1} + s_j s_{j+1}] - \eta J \sum_j r_j s_j [1 + v t / a_0],
$$

(2.1)

where $\lfloor x \rfloor$ is the largest integer less than or equal to $x$.

We will consider cyclic boundary conditions\textsuperscript{1}. In this case the chains become counter-rotating loops of length say $N$; the site indices $i$, $j$, and $[j + v t / a_0]$ must then be interpreted modulo $N$. Employing the shorthand notation $r = \{r_j | j = 1, 2, \ldots, N\}$ and $s = \{s_j | j = 1, 2, \ldots, N\}$, we may indicate a spin configuration of the system by $(r, s)$.

We associate with $\mathcal{H}(t)$ a stochastic time evolution of $(r, s)$. Its precise definition requires some caution. We will first define it as a Monte Carlo procedure and then write down the corresponding master equation and pass to analytic considerations. Single-spin reversals are attempted at random instants distributed uniformly on the continuous time axis at a rate of $1 / \tau_0$ for each site\textsuperscript{2}. Each attempt is governed by transition probabilities. Since there are $2N$ sites, there are $2N$ different single-spin flips by which a state $(r, s)$ may be excited. Given that a reversal attempt takes place, we let $(2N)^{-1} W^r_j (r; s; t)$ and $(2N)^{-1} W^s_j (s; r; t)$ be the probabilities that $r_j$ and $s_j$ are flipped, respectively, with $0 \leq W^r_j, W^s_j \leq 1$. We specify the $W^r_j$ and $W^s_j$ in such a way that at any time $t$ the system strives to attain the canonical equilibrium at a prescribed temperature $T$ with respect to the instantaneous Hamiltonian $\mathcal{H}(t)$. The choice is not unique. We choose

$$
W^r_j (r; s; t) = \frac{1}{4} \{ 1 - \frac{1}{2} r_j (r_{j-1} + r_{j+1}) \tanh 2K \} \{ 1 - r_j s_i \tanh \eta K \},
$$

$$
W^s_j (s; r; t) = \frac{1}{4} \{ 1 - \frac{1}{2} s_j (s_{j-1} + s_{j+1}) \tanh 2K \} \{ 1 - s_j r_j \tanh \eta K \},
$$

(2.2)

where we have set $K = J / T$ and where in both equations $i$ and $j$ are related by

$$
i = \lfloor j + v t / a_0 \rfloor \mod N.
$$

(2.3)

Equation (2.2) is different both from the heat bath and the Metropolis transition probability. It is such that the effects of the in-chain interaction (coupling $K$) and the cross-chain interaction (coupling $\eta K$) appear in factorized form, which is why, following [2], we may refer to the $w^r_j (r; s)$ and $w^s_j (s; r)$ as ‘multiplicative rates’. This factorization property was essential in [2] and is similarly essential here. The factor

$$
w^G_j (r) = \frac{1}{2} \{ 1 - \frac{1}{2} r_j (r_{j-1} + r_{j+1}) \tanh 2K \}
$$

(2.4)

and the analogous factor $w^G_j (s)$ that occurs in (2.2), represent the Glauber transition probabilities for an isolated one-dimensional (1D) Ising model. This choice ensures that for $\eta = 0$ we recover the analytically solvable Glauber model [3]. In [2] the

\textsuperscript{1} In connection with the traffic problem, open boundary conditions are certainly also worthy of consideration. These have however the inconvenience of breaking the translational symmetry.

\textsuperscript{2} We may scale time such that $\tau_0 = 1$.  

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probabilities (2.2) have extra prefactors $A(r_{j-1}r_{j+1})$ and $A(s_{i-1}s_{i+1})$, respectively, chosen such as to maximize them subject only to $W^r_j, W^s_j \leq 1$. This has the advantage of maximally speeding up the Monte Carlo simulations. In the analytic treatment the factors $A$ may easily be carried along if desired. The $W^r_j$ and $W^s_j$ define a discrete-time Markov process with time-dependent transition probabilities that is easy to simulate.

In the special case $v = 0$, the Hamiltonian $\mathcal{H}(t)$ reduces to the equilibrium Hamiltonian of the ladder lattice. For $v$ arbitrary but $\eta = 0$, it reduces to the equilibrium Hamiltonian of two decoupled chains. In both of these special cases the dynamics obeys DB.

In the general case, since the Hamiltonian is time-dependent, the system will not reach equilibrium but instead enter a NESS. Actually, for generic $v$, because of the periodic discrete shifts, the NESS is a $\tau$-periodic function of time; NESS averages are naturally defined to include an average over this period. In the limiting case $v = \infty$, we have $\tau = 0$ and this complication disappears. The infinite velocity NESS is the subject of our interest in the remaining sections. It is a problem that depends only on the two parameters $K$ and $\eta$.

### 2.2. The limit $v \to \infty$

Let $P(r, s; n)$ be the probability distribution on the configurations $(r, s)$ after $n$ spin reversal attempts. We will write down the formal evolution equation for $P(r, s; n)$ for the case $v = \infty$, where important simplifications occur. When $v = \infty$ there is no relation between the indices $i$ and $j$ and hence the spin $r_j$ has a transition probability $w_j(r; s)$ given by the average of $W^r_j(r; s; t)$ on all $i$, which is now considered as an independent variable. We denote this average by $w_j(r, s)$ and thus have

$$w_j(r; s) = \frac{1}{N} \sum_{i=1}^{N} W^r_j(r; s; t)$$

as well as its counterpart

$$w_j(s; r) = \frac{1}{N} \sum_{i=1}^{N} W^s_j(s; r; t)$$

where

$$\mu(s) = \frac{1}{N} \sum_{i=1}^{N} s_i, \quad \mu(r) = \frac{1}{N} \sum_{i=1}^{N} r_i.$$  

We will write $r^j$ for the configuration obtained from $r$ by reversing $r_j$ and define $s^j$ similarly. Summing on all $2N$ reversals by which it is possible to enter or to exit $(r, s)$ we

---

3 The reversal attempts, that is, the steps of the Markov chain, are Poisson distributed on the time axis. This makes it possible at any time to probabilistically connect the elapsed time $t$ to the number of spin reversal attempts $n$. In the large $t$ limit, of course, $n \approx t/\tau_0$. 

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find that the evolution of \( P(r, s; n) \) is described by the master equation

\[
P(r, s; n + 1) = \frac{1}{2N} \sum_{j=1}^{N} [w_j(r^j; s)P(r^j, s; n) + w_j(s^j; r)P(r, s^j; n) \\
+ (1 - w_j(r; s))P(r, s; n) + (1 - w_j(s; r))P(r, s; n)],
\]

(2.7)

where the second line corresponds to unsuccessful spin reversal attempts. In vector
notation equation (2.7) may be written

\[
P(n + 1) = (1 + W)P(n),
\]

(2.8)

where \( P(n) \) is the \( 2^{2N} \)-dimensional vector of elements \( P(r, s; n) \), the symbol \( 1 \) denotes
the unit matrix, and \( W \) is a matrix composed of entries \( w_j \) for which comparison of (2.7)
and (2.8) yields

\[
W(r, s; r', s') = \delta_{rr'}\delta_{ss'}w_j(r^j, s) + \delta_{rr'}\delta_{ss'}w_j(r^j, s^j) - \delta_{rr'}\delta_{ss'} \sum_{j=1}^{N} [w_j(r; s) + w_j(s; r)].
\]

(2.9)

The discrete-time master equation (2.7), together with the Poisson statistics of the reversal
attempts on the time axis, fully defines the CRIC for \( v = \infty \). Equation (2.7) may
be studied analytically, as is the purpose of this work, or may be implemented in a
Monte Carlo simulation.

3. Detailed balancing and its violation

Henceforth we consider the case \( v = \infty \). Our purpose is to find the stationary state
distribution \( P_{st}(r, s) \) of the evolution equation (2.7). This distribution is the solution of
\( P(r, s; n) = P(r, s; n + 1) = P_{st}(r, s) \), whence by (2.8)

\[
0 = WP_{st}.
\]

(3.1)

Combining equations (3.1) and (2.7) yields the stationary state equation

\[
0 = \sum_{j=1}^{N} [w_j(r^j; s)P_{st}(r^j, s) + w_j(s^j; r)P_{st}(r, s^j) - w_j(r; s)P_{st}(r, s) - w_j(s; r)P_{st}(r, s)].
\]

(3.2)

If the transition probabilities were to satisfy the condition of DB, the solution of (3.2)
could easily be constructed; in case of the contrary, there are no general methods. We
examine therefore first the question of whether equation (2.7) satisfies DB.

A Markov process satisfies DB if and only if its transition probabilities are such that
any loop in configuration space is traversed with equal probability in either direction. To
show that the transition probabilities \( w_j \) fail to obey DB we consider an elementary loop
of four single-spin flips,

\[
(r, s) \rightarrow (r^j, s) \rightarrow (r^j, s^j) \rightarrow (r, s^j) \rightarrow (r, s).
\]

(3.3)

Given the system is in \( (r, s) \), we denote by \( p_+(\eta) \) and \( p_-(\eta) \) the probability that in
the next four attempts it goes through this loop in the forward and in backward direction,
respectively. That is,

\[ p_+(\eta) = w_j(r; s)w_j(s; r)w_j(r^2; s^2)w_j(s^2; r), \]

\[ p_-(\eta) = w_j(s; r)w_j(r; s)w_j(s^2; r^2)w_j(r^2; s). \] (3.4)

For \( \eta = 0 \) the two chains are decoupled, and as discussed below equation (2.3), each of them separately satisfies DB; it is easy indeed to verify explicitly that

\[ p_+(0) = p_-(0) \equiv p(0). \]

For general \( \eta \) we may work out the difference \( p_+(\eta) - p_-(\eta) \) with the aid of (2.5), (2.6), and the relations

\[ \mu(r^2) = \mu(r) - \frac{2r_j}{N}, \quad \mu(s^2) = \mu(s) - \frac{2s_j}{N}, \] (3.5)

which yields

\[ p_+(\eta) - p_-(\eta) = 4N^{-1}p(0) \tanh^2 \eta K \left[ r_j \mu(s) - s_j \mu(r) \right] \times \{ [r_j \mu(r) - s_j \mu(s)] + 2N^{-1} \tanh \eta K \}. \] (3.6)

This shows that DB is violated in the general case of nonzero coupling (\( \eta \neq 0 \)) between the chains, but also that it becomes valid again asymptotically in the limit \( N \to \infty \). We therefore cannot hope to rely on any general methods to construct \( P_{st}(r, s) \) for finite \( N \). Indeed, writing out the stationary state equation (3.1) explicitly in full for \( N = 3, 4 \) (only \( N = 2 \) is trivial) has confirmed the nontriviality of the stationary state but has not provided us with any useful insight.

4. Stationary state \( P_{st}(r, s) \) to zeroth order

The limit \( N \to \infty \) was considered by Hucht [2,1] and we briefly recall the results. One may suppose that in this limit \( \mu(r) \) and \( \mu(s) \) have vanishing fluctuations around an as yet unknown common average to be called \( m_0(K, \eta) \). We denote the \( N \to \infty \) limit of \( w_j \) by \( w_{j,0} \). It then follows from (2.5a) that

\[ w_{j,0}(r) = w_j^G(r) \times \left[ 1 - r_j m_0 \tanh \eta K \right]. \] (4.1)

With the transition probabilities \( w_{j,0}(r) \) and \( w_{j,0}(s) \) the r- and the s-chain decouple. Moreover, the expression for these \( w_{j,0} \) is such that the spin dynamics satisfies DB with respect to an effective Hamiltonian

\[ \mathcal{H}_0(r, s)/T = -K \sum_{j=1}^{N} [r_j r_{j+1} + s_j s_{j+1}] - H_0 \sum_{j=1}^{N} [r_j + s_j] \] (4.2)

where the field \( H_0 \) is defined in terms of \( m_0 \) by

\[ \tanh H_0 = m_0 \tanh \eta K \] (4.3)

and where \( K \) and \( H_0 \) both include a factor \( 1/T \). The Hamiltonian \( \mathcal{H}_0(r, s) \) describes two decoupled Ising chains. Let \( m(K, z) \) denote the magnetization per spin of the 1D Ising chain with coupling \( K \) in a field that we will for convenience denote by \( z \). This quantity...
is well known and given by

\[ m(K, z) = \frac{\sinh z}{\sqrt{\sinh^2 z + e^{-4K}}}. \] (4.4)

Consistency requires that

\[ m_0 = m(K, H_0). \] (4.5)

Upon combining (4.3) with (4.5) one obtains an equation for \( H_0 \) (or equivalently \( m_0 \)). The solution \( H_0 \) is a function of the two system parameters \( K \) and \( \eta \) and given by

\[
\tanh H_0(K, \eta) = \begin{cases} 
\left( \frac{\tanh^2 \eta K - e^{-4K}}{1 - e^{-4K}} \right)^{1/2}, & K > K_c, \\
0, & K \leq K_c,
\end{cases}
\] (4.6)

in which there appears a critical coupling \( K_c = J/T_c \) that is the solution of\(^4\)

\[ \tanh \eta K_c = e^{-2K_c}. \] (4.7)

The magnetization \( m_0(K, \eta) \) follows directly from (4.3) and (4.6). For \( T \to T_c^- \) it vanishes as \( m_0 \propto (T_c - T)^\beta \) with the classical exponent \( \beta = \frac{1}{2} \). For later use it is also worthwhile noting that \( H_0(T) \propto (T - T_c)^{1/2} \) when \( T \leq T_c \).

The DB property found below equation (4.1) now allows us to conclude that for \( N \to \infty \) the stationary state distribution \( P_{st,0}(r, s) \) is the Boltzmann distribution corresponding to (4.2), that is,

\[ P_{st,0}(r, s) = N_0 e^{-H_0(r,s)/T}, \] (4.8)

where \( N_0 \) is the normalization. In [2] several system properties were calculated in this \( N \to \infty \) limit by averaging with respect to \( P_{st,0}(r, s) \).

5. Stationary state \( P_{st}(r, s) \) to first order

As has become clear in section 3, the inverse system size \( 1/N \) is a measure of the degree of DB violation. This will lead us to attempt to find the finite \( N \) stationary state by expanding around the known \( N = \infty \) solution (4.8), which will play the role of the zeroth order result. At the basis of the expansion is the hypothesis that the fluctuations \( \delta \mu \) of the chain magnetizations,

\[ \delta \mu(r) = \mu(r) - m_0, \quad \delta \mu(s) = \mu(s) - m_0, \] (5.1)

are of order \( N^{-1/2} \). This hypothesis is confirmed below by the fact that it allows us to solve the stationary state equation \( WP_{st} = 0 \) of (3.1) by an expansion in ascending powers of \( N^{-1/2} \).

\(^4\) Equation (4.7) may be rewritten as \( \sinh(2J_1/T_c) \sinh(2J_2/T_c) = 1 \), which shows, as was also noticed in [2], that \( T_c \) is exactly (but accidentally) equal to the critical temperature of Onsager’s square Ising model with horizontal and vertical couplings \( J_1 \) and \( J_2 \).
5.1. Preliminary remark

A naive attempt to set up the expansion would be to notice that the transition probability \( w_j(r; s) = w_{j,0}(r) + \bar{w}_j(r; s) \), where \( w_{j,0}(r) \) is given by (4.1) and where \( \bar{w}_j(r; s) = w_j^0(r) \times \left( -\frac{1}{2}r_j \right) \delta \mu(s) \tan \eta K \) is of order \( N^{-1/2} \). One might then think that there exists a corresponding expansion

\[
\mathcal{P}_{st}(r, s) = \mathcal{P}_{st,0}(r, s)[1 + \cdots].
\]

However, the dot terms turn out to be of order \( O(1) \) as \( N \to \infty \), which is a sign that this is not the right way to expand. The reason for this failure is that \( \mathcal{P}_{st} \) is the exponential of the extensive quantity \( \mathcal{H}_0 \); one should therefore ask first if this exponential contains any corrections of less divergent order in \( N \) before attempting to multiply it by a series of type \( [1 + \cdots] \). In section 5.2 we describe how the expansion can be set up successfully.

We will find in the end that to first order in the expansion DB continues to hold, but with respect to a Hamiltonian \( \mathcal{H}^{(1)}(r, s) \) that is equal to \( \mathcal{H}_0(r, s) \) plus a first-order correction. In section 5 we present the solution, to be denoted as \( \mathcal{P}_{st}^{(1)}(r, s) \), of the first-order stationary state. In section 6 we show how higher orders can be calculated and find that DB is violated from the second order on.

5.2. First-order result for \( \mathcal{P}_{st}(r, s) \)

We state here a result whose proof will be given in section 6. The upper index ‘(1)’ is employed to indicate any quantity correct up to first order in the expansion. The result is that the correct expansion for \( \mathcal{P}_{st}(r, s) \) takes the form

\[
\mathcal{P}_{st}(r, s) = \mathcal{P}_{st}^{(1)}(r, s)[1 + q_1(r, s) + q_2(r, s) + \cdots],
\]

where the \( q_k(k = 1, 2, \ldots) \), which we will show how to determine later, are of order \( \mathcal{O}(N^{-k/2}) \) and where \( \mathcal{P}_{st}^{(1)}(r, s) \) is explicitly given by

\[
\mathcal{P}_{st}^{(1)}(r, s) = N^{(1)} \exp \left( -\frac{\mathcal{H}^{(1)}(r, s)}{T} \right),
\]

\[
\mathcal{H}^{(1)}(r, s) = \mathcal{H}_0(r, s) - g_0 N \delta \mu(r) \delta \mu(s),
\]

\[
g_0 = \cosh^2 H_0 \tanh \eta K,
\]

in which \( N^{(1)} \) is the appropriate normalization. The second term on the RHS of (5.3b) is a correction to the zeroth order effective Hamiltonian. It is \( \mathcal{O}(1) \) for \( N \to \infty \) and, since it is proportional to \( g_0 \), it vanishes as expected when \( \eta = 0 \).

In order to arrive at (5.2)–(5.3) we split \( \mathcal{W} \) according to

\[
\mathcal{W} = \mathcal{W}^{(1)} + \sum_{k=2}^{\infty} \mathcal{W}_k,
\]

where we take for \( \mathcal{W}^{(1)} \) the matrix with the multiplicative transition probabilities that ensure DB with respect to \( \mathcal{H}^{(1)} \), the \( \mathcal{W}_k \) remaining to be defined. Expression (5.3) for \( \mathcal{H}^{(1)} \) shows that a spin \( r_j \) is subject to a total field \( H_0 + g_0 \delta \mu(s) \). Hence by analogy to (4.1)
the transition probabilities that enter \( W^{(1)} \) are

\[
  w_j^{(1)}(r; s) = w_j^G(r) \times \frac{1}{2} [1 - r_j \tanh \{H_0 + g_0 \delta \mu(s)]\}.
\]

We then have by construction that

\[
  W^{(1)} P^{(1)} = 0.
\]

The explicit form of this equation may be obtained from (3.2) by the substitutions

\[
  w_j \rightarrow w_j^{(1)} \quad \text{and} \quad P_{st} \rightarrow P_{st}^{(1)}.
\]

Clearly the validity of the expansion procedure hinges on our being able to demonstrate that for correctly defined \( W_k \) the corrections to \( P_{st}^{(1)} \) take effectively the form of the series in (5.2) with the \( q_k \) proportional to increasing powers of \( N^{-1/2} \). The proof of this will be given in section 6.

A remark on terminology is relevant at this point. Since the zeroth and first order will often be combined, we will refer to equations (5.3), (5.5), and (5.6) as describing the ‘leading order’. The terms \( q_1, q_2, \ldots \) in the series (5.2) will be referred to as ‘higher-order’ corrections.

6. Stationary state to higher orders

6.1. The perturbation series (5.4) for \( W \)

In order to show that the higher-order corrections to \( P_{st} \) can be expressed as the series of equation (5.2), we must define the \( W_k \) in equation (5.4). Let us define \( \delta w_j(r; s) \) by

\[
  w_j(r; s) = w_j^{(1)}(r; s) + \delta w_j(r; s).
\]

Starting from (6.1) we employ the explicit expressions (2.5a) and (5.5) for \( w_j \) and \( w_j^{(1)} \), respectively, perform a straightforward Taylor expansion in \( \delta \mu \), and use (4.3) to eliminate \( m_0 \) in favor of \( H_0 \). This leads to

\[
  \delta w_j(r; s) = w_j^G(r) \times \left[ \frac{1}{2} [1 - r_j \mu(s) \tanh \eta K] - \frac{1}{2} [1 - r_j \tanh \{H_0 + g_0 \delta \mu(s)]\} \right]
  = w_j^G(r) \times (-\frac{1}{2} r_j) \sum_{k=2}^{\infty} a_k \delta \mu^k(s)
  = \sum_{k=2}^{\infty} w_{j,k}(r; s),
\]

where the last equality, supposed to hold term by term in \( k \), defines \( w_{j,k} \) and shows that it is of order \( N^{-k/2} \). In the third line of (6.2) the vanishing of the term linear in \( \delta \mu \) has of course been pre-arranged. The first two coefficients \( a_k \) in that line are given by

\[
  a_2 = g_0^2 (1 - \tanh^2 H_0) \tanh H_0,
  a_3 = \frac{1}{3} g_0^3 (1 - \tanh^2 H_0)(1 - 3 \tanh^2 H_0).
\]

It becomes clear now that there is a qualitative difference between the high-temperature regime \( T \geq T_c \) where we have \( H_0 = 0, a_2 = 0, \) and

\[
  a_3 = \frac{1}{3} \tanh^3 \eta K, \quad T \geq T_c,
\]

and the low-temperature regime \( T < T_c \) where \( H_0 > 0, a_2 > 0 \).

We define the matrices \( W_k \) in expansion (5.4) in terms of the \( w_{j,k} \) by analogy to (2.9). Hence for \( T \geq T_c \) we have \( W_2 = 0 \).

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6.2. Higher-order equations for $P_{st}(r, s)$

The leading order equation (5.6) being satisfied, we turn to the higher orders. Substitution of (5.4) in (3.1) and use of (5.6) leads to an expansion of which in the low-temperature phase the first term is

\[ \mathcal{W}^{(1)} P_{st}^{(1)} q_1 + \mathcal{W}_2 P_{st}^{(1)} = 0, \quad T < T_c. \]  

(6.5a)

In the high-temperature phase the fact that $\mathcal{W}_2 = 0$ implies that $q_1 = 0$ and therefore (6.5a) is replaced by the next term in the expansion,

\[ \mathcal{W}^{(1)} P_{st}^{(1)} q_2 + \mathcal{W}_3 P_{st}^{(1)} = 0, \quad T \geq T_c. \]  

(6.5b)

Either will be referred to as the ‘next-to-leading order’ equation. One obtains all higher-order equations in explicit form by inserting in the full stationary state equation (3.2) the expansion (5.2) for $P_{st}(r, s)$ and expressions (5.5), (6.1), and (6.2) for $w_j(r; s)$.

6.3. Equation for $q_1(r, s)$ when $T < T_c$

By the procedure indicated above we obtain for the next-to-leading order equation (6.5a) the explicit form

\[ 0 = \sum_j [w_{j,2}(r^j; s) P_{st}^{(1)}(r^j, s) - w_{j,2}(r; s) P_{st}^{(1)}(r, s) \\
+ w_{j,2}(s^j; r) P_{st}^{(1)}(r, s^j) - w_{j,2}(s; r) P_{st}^{(1)}(r, s) \\
+ w_j^{(1)}(r^j; s) P_{st}^{(1)}(r^j, s) q_1(r^j, s) - w_j^{(1)}(r; s) P_{st}^{(1)}(r, s) q_1(r, s) \\
+ w_j^{(1)}(s^j; r) P_{st}^{(1)}(r, s^j) q_1(r, s^j) - w_j^{(1)}(s; r) P_{st}^{(1)}(r, s) q_1(r, s)]. \]  

(6.6)

We wish to divide (6.6) by $P_{st}^{(1)}(r, s)$, and therefore have to compute

\[ \frac{P_{st}^{(1)}(r^j, s)}{P_{st}^{(1)}(r, s)} \equiv e^{-2R_j(r, s)}. \]  

(6.7)

We easily find

\[ 2R_j(r; s) = [\mathcal{H}^{(1)}(r^j, s) - \mathcal{H}^{(1)}(r, s)]/T \\
= [\mathcal{H}_0(r^j, s) - \mathcal{H}_0(r, s)]/T - g_0N\delta\mu(s) [\delta\mu(r^j) - \delta\mu(r)] \\
= -2K(r_{j-1} + r_{j+1}) - 2r_j \{\mathcal{H}_0 + g_0\delta\mu(s)\}, \]

(6.8)

where we used (5.3) and (4.2). Detailed balancing says that

\[ w_j^{(1)}(r^j; s) P_{st}^{(1)}(r^j, s)(r^j, s) = w_j^{(1)}(r; s) P_{st}^{(1)}(r, s). \]  

(6.9)

Using (6.7) and (6.8) in the first two lines and (6.9) in the last two lines of (6.6) we obtain

\[ 0 = \sum_j [w_{j,2}(r^j; s)e^{-2R_j(r, s)} - w_{j,2}(r; s) + w_{j,2}(s^j; r)e^{-2R_j(s, r)} - w_{j,2}(s, r) \\
+ w_j^{(1)}(r; s)\{q_1(r^j, s) - q_1(r, s)\} + w_j^{(1)}(r; s)\{q_1(r, s^j) - q_1(r, s)\}]. \]  

(6.10)
The expression in the first line of (6.10) may be rewritten as

\[ w_{j,2}(r; s)e^{-2R_j(r; s)} - w_{j,2}(r; s) = w_j^G(r) \times \frac{1}{2}[1 - r_j \tanh H_0] \times 4r_j\delta \mu^2(s)g_0^2 \tanh H_0, \]  

(6.11)

of which the first two factors on the RHS are again exactly \( w_{j,0}. \) In (6.10), \( w_j^{(1)} \) is of order \( N^0 \) but contains corrections of higher order in \( N^{-1/2}. \) When we substitute (6.11) in (6.10) and apply to \( w_j^{(1)} \) the \( N \to \infty \) limit (that is, replace it by \( w_{j,0} \) of equation (4.1), which amounts to neglecting contributions of higher order), we obtain the final form of the equations for the next-to-leading order correction to the stationary state,

\[ 0 = \sum_j [w_{j,0}(r) \times 4r_jg_0^2 \tanh H_0 \delta \mu^2(s) + w_{j,0}(s) \times 4s_jg_0^2 \tanh H_0 \delta \mu^2(r) \]

\[ + w_{j,0}(r)\{q_1(r, s) - q_1(r, s)\} + w_{j,0}(s)\{q_1(r, s') - q_1(r, s)\}]. \]  

(6.12)

The solution of (6.12) will be studied in section 6.6.

### 6.4. Equation for \( q_2(r, s) \) when \( T \geq T_c \)

For \( T \geq T_c \) we should consider equation (6.5b). When rendered explicit, it leads to expressions that are identical successively to (6.6), (6.10), and (6.12), apart from the substitutions \( q_1 \rightarrow q_2 \) and \( w_{j,2} \rightarrow w_{j,3}. \) In this case \( w_{j,0}(r) = \frac{1}{2}w_j^G(r) \), where \( w_j^G(r) \) is given by (2.4), and \( \delta \mu = \mu \) since \( H_0 = m_0 = 0. \) Hence instead of (6.12) we get

\[ 0 = \sum_j [w_j^G(r) \times 4r_ja_3\mu^3(s) + w_j^G(s) \times 4s_ja_3\mu^3(r) \]

\[ + w_j^G(r)\{q_2(r, s) - q_2(r, s)\} + w_j^G(s)\{q_2(r, s') - q_2(r, s)\}]. \]  

(6.13)

Finding the solutions of (6.12) and (6.13) will be the subject of the next two subsections. We will first consider the easier case of \( T \geq T_c \) and then the case \( T < T_c. \)

### 6.5. Solution for \( q_2(r, s) \) when \( T \geq T_c \)

In the high-temperature phase, equation (6.13) applies. DB would be satisfied if inside the brackets in that equation the four terms were pairwise zero, that is, if we had

\[ q_2(r, s) - q_2(r, s) = -4a_3r_j\mu^3(s), \quad q_2(r, s') - q_2(r, s) = -4a_3s_j\mu^3(r) \]  

(6.14)

for all \( j. \) It can easily be shown that it is impossible to satisfy these equations. However, they suggest that we look for a solution \( q_2 \) of the form

\[ q_2(r, s) = NC_2a_3[\mu(r)\mu^3(s) + \mu(s)\mu^3(r)], \]  

(6.15)

where only the constant \( C_2 \) is still adjustable. The difference \( q_2(r, s) - q_2(r, s) \) is easy to calculate. We are interested only in its leading order and find from (6.15) and (3.5) that

\[ q_2(r, s) - q_2(r, s) = -2C_2a_3r_j[\mu^3(s) + 3\mu(s)\mu^2(r)] + \mathcal{O}(N^{-2}), \]  

(6.16a)

\[ q_2(r, s') - q_2(r, s) = -2C_2a_3s_j[\mu^3(r) + 3\mu(r)\mu^2(s)] + \mathcal{O}(N^{-2}). \]  

(6.16b)
Two interacting Ising chains in relative motion

We recall that \( \mu \) is of order \( N^{-1/2} \) for \( T > T_c \). It may then be noted that whereas (6.15) is of order \( N^{-1} \), the differences (6.16) are of order \( N^{-3/2} \). We now need

\[
\sum_j w_j^G(r)\{q_2(r, s) - q_2(r, s)\} = -2C_2a_3 \left( \sum_j w_j^G(r)r_j \right) [\mu^3(s) + 3\mu(s)\mu^2(r)]. \tag{6.17}
\]

With the aid of the explicit expression (2.4) for \( w_j^G(r) \) one evaluates readily

\[
\sum_j w_j^G(r)r_j = \frac{1}{4}(1 - \gamma)N\mu(r). \tag{6.18}
\]

After substituting (6.17) and (6.18) in (6.13) we see that equation is satisfied for \( C_2 = \frac{1}{8} \).

Hence from (6.15) we get

\[
q_2(r, s) = \frac{1}{24}N(\tanh^3 \eta K)[\mu(r)\mu^3(s) + \mu(s)\mu^3(r)]. \tag{6.19}
\]

Again taking into account that \( \mu \) is of order \( N^{-1/2} \) we see that this final result for \( q_2(r, s) \) is of order \( \tilde{N}^{-1} \), as anticipated.

6.6. Solution for \( q_1(r, s) \) when \( T < T_c \)

In the low-temperature regime equation (6.12) applies. In order to solve this equation we postulate

\[
q_1(r, s) = NC_1b_2[\delta \mu^3(r) + \delta \mu^3(s)] \tag{6.20}
\]

where \( C_1 \) is an adjustable constant and

\[
b_2 = 4a_2/(1 - \tanh^2 H_0) = 4\tilde{g}_0^2 \tanh H_0. \tag{6.21}
\]

Expression (6.20) is of order \( N^{-1/2} \). Instead of (6.16) we now have

\[
q_1(r, s) = -6C_1b_2r_j\delta \mu^2(r) + O(N^{-3/2}) \tag{6.22}
\]

which is of order \( N^{-1} \). The first two lines of (6.12) require that we evaluate

\[
\sum_j w_j,0(r)r_j = \frac{1}{4}N \sum_j [1 - \frac{1}{2}r_j(r_j-1 - r_j+1)][1 - r_j \tanh H_0]r_j. \tag{6.23}
\]

Unlike the sum in (6.17), this is not a sum of zero-average random terms. It will produce a result of order \( N \), which we may replace by its average. This yields

\[
\sum_j w_j,0(r)r_j = \frac{1}{4}N\{[1 - \gamma)m_0 - (1 - \gamma a_H)\tanh H_0\} \equiv NG, \tag{6.24}
\]

where the last equality defines \( G \) and where \( a_H \) is the nearest-neighbor spin–spin correlation \( \langle r_jr_{j+1} \rangle \) of the 1D Ising chain in a field described by \( H_0 \) (equation (4.2)).

Expression (6.24), contrary to its \( T \geq T_c \) counterpart (6.18), has no spin dependence and is therefore equal for the \( r- \) and \( s- \) spines. The first two lines of (6.12), to be denoted \( S_1 \), become

\[
S_1 = 4NG\tilde{g}_0^2 \tanh H_0[\delta \mu^2(s) + \delta \mu^2(r)]. \tag{6.25}
\]
We use (6.22) to write the last two lines of (6.12) as
\[ S_2 = -6C_1b_2 \left( \sum_j w_j,0(r) r_j \right) \delta \mu^2(r) + \left( \sum_j w_j,0(s) s_j \right) \delta \mu^2(s) \]
\[ = -6NGC_1b_2 [\delta \mu^2(r) + \delta \mu^2(s)]. \]  
(6.26)
The stationary state equation (6.12) may then be written as
\[ S_1 + S_2 = 0, \]
and we see that it is satisfied for
\[ C_1 = \frac{1}{6}, \]
whence
\[ q_1(r,s) = \frac{2}{3}Ng_0^2 \tanh H_0 [\delta \mu^3(r) + \delta \mu^3(s)]. \]  
(6.27)
This completes the solution of the higher-order corrections to the stationary state distribution in phase space.

6.7. Section summary

We have studied in the preceding subsections the large-$N$ expansion of the stationary state distribution $P_{st}(r,s)$ of the infinite velocity CRIC defined in section 2. We have shown, for $T < T_c$ and $T \geq T_c$ separately, the existence of a series of correction terms $q_k$ that multiplies the leading order result $P_{st}^{(1)}$, which itself is composed of a zeroth and a first-order contribution. The existence of this expansion also furnishes the proof that the prefactor $P_{st}^{(1)}$ represents indeed the ‘leading order’ behavior. We have determined explicitly the first nonzero correction term in the expansion: $q_1$ for $T \geq T_c$ and $q_2$ for $T < T_c$.

When looking ahead beyond this leading order correction, it appears that the $q_k$ (for $k \geq 2$ when $T < T_c$ and for $k \geq 3$ when $T \geq T_c$) involve not only $\delta \mu(r)$ and $\delta \mu(s)$, but also energy fluctuations such as $N^{-1} \sum_j (r_j r_{j+1} - a_H)$, if not longer-range correlations. Therefore, even though on the basis of the results of this section one might be tempted to postulate a general solution of the simple type $P_{st}(r,s) = P_{st}^{(1)}(r,s)Q(\delta \mu(r), \delta \mu(s))$, it is unlikely that the true $P_{st}(r,s)$ is of this form.

7. Stationary state averages

Stationary state averages $\langle A \rangle$ of observables $A(r,s)$ are averages with respect to $P_{st}(r,s)$, so that using (5.2) and (5.3a) we have
\[ \langle A \rangle = \frac{\sum_{r,s} A(r,s) e^{-H^{(1)}(r,s)/T} [1 + q_1(r,s) + q_2(r,s) + \cdots]}{\sum_{r,s} e^{-H^{(1)}(r,s)/T} [1 + q_1(r,s) + q_2(r,s) + \cdots]} \]
\[ = \langle A \rangle^{(1)} + [\langle Aq_1 \rangle^{(1)} - \langle A \rangle^{(1)} \langle q_1 \rangle^{(1)}] + \cdots, \]  
(7.1)
where $\langle \cdots \rangle^{(1)}$ indicates an average with weight $P_{st}^{(1)}(r,s)$ (equation (5.3)), the second line results from a straightforward expansion, and
\[ \ell = \begin{cases} 2, & T \geq T_c, \\ 1, & T < T_c, \end{cases} \]  
(7.2)
for the lowest order nonzero term in the expansion. Although the $q_k$ are accompanied by increasing powers of $N^{-1/2}$, the order in $N^{-1/2}$ of each of the terms in the series (7.1) must be analyzed for each observable $A$ separately.
7.1. Integral representation of the partition function

The denominator in the first line of (7.1) is a normalization factor to which we may refer (although slightly improperly) as the partition function $Z$. In order to find expressions for the averages $\langle \cdots \rangle^{(1)}$ in the second line of (7.1), we begin by evaluating $Z$ to leading order,

$$Z^{(1)}(K, H_0, g_0) = \sum_{r, s} e^{-H^{(1)}(r, s)/T},$$

with $H^{(1)}$ given by (5.3b), in which one should substitute (4.2) and (5.1). Formally, $Z^{(1)}$ is a partition function, namely the trace of a Boltzmann factor, but it should be remembered that $H^{(1)}$ depends nonlinearly on $1/T$ through the field $H_0$. The notation $Z^{(1)}(K, H_0, g_0)$ is meant to indicate that we wish to consider $Z^{(1)}$ as a function of three independent parameters, ignoring for the moment expression (5.3c) for $g_0$. The $r$- and $s$-spins in (7.3) may be decoupled by the integral representation

$$Z^{(1)} = \frac{N}{\pi g_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-g_0^{-1} N(x^2 + y^2) - 2x m_0 |\zeta(K, H_0 + x + iy)|^2},$$

in which $m_0 = m(K, H_0)$ follows from (4.3) and (4.6). The two factors in brackets in (7.4) are seen to be the partition functions $\zeta(K, H_0 + x \pm iy)$ of independent Ising chains in magnetic fields $H_0 + x \pm iy$. Hence

$$Z^{(1)} = \frac{N}{\pi g_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-g_0^{-1} N(x^2 + y^2) - 2x m_0 |\zeta(K, H_0 + x + iy)|^2}. (7.5)$$

We recall that

$$\zeta(K, B) \equiv \lambda^N + \lambda^N,$$

where

$$\lambda_{\pm}(K, B) = e^{K[\cosh B \pm \sqrt{\sinh^2 B + e^{-4K}}]},$$

are the transfer matrix eigenvalues.

7.2. Stationary point and fluctuations

The $x$ and $y$ integrals in (7.5) are easily evaluated by the saddle point method. In the limit of large $N$, we may neglect in (7.6) the exponentially small corrections due to $\lambda_-$ and get from (7.5)

$$Z^{(1)} \simeq \frac{N}{\pi g_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-N \mathcal{F}(x, y)},$$

where

$$\mathcal{F}(x, y) = \mathcal{F}(x, -y) = \frac{g_0^{-1} N(x^2 + y^2)}{2} + \frac{m_0}{2} |\zeta(K, H_0 + x + iy)|^2.$$
where
\[ F(x, y) = g_0^{-1}(x^2 + y^2) + 2xm_0 - \log |\lambda_+(K, H_0 + x + iy)|^2. \] (7.9)

Let \((x^*, y^*)\) denote the stationary point of the integration in (7.8). The stationary point equations \(F_x = F_y = 0\) can be expressed as
\[ g_0^{-1}(x^* \pm iy^*) = m(K, H_0 \mp iy^*) - m_0, \] (7.10)
with the magnetization \(m(K, B) = \lambda_+^{-1}(K, B)\partial \log \lambda_+(K, B)/\partial B\) given by (4.4). For reasons of symmetry the stationary point must have \(y^* = 0\). This reduces (7.10) to the single real equation
\[ g_0^{-1}x^* = m(K, H_0 + x^*) - m(K, H_0), \] (7.11)
where we used that \(m_0 = m(K, H_0)\) (equation (4.5)). Equation (7.11) has for all \(H_0\) the obvious solution \(x^* = 0\). We investigate the stability of the stationary point \((x^*, y^*)\) by calculating the matrix of second derivatives, which is given by
\[ F^{\ast}_{xx} = \begin{cases} 2(e^{-2K} \mp \tanh \eta K) & T \geq T_c, \\ e^{-2K} \tanh \eta K & T < T_c, \end{cases} \] (7.14)
in which the upper (lower) sign refers to the \(xx\) (to the \(yy\)) derivative. It can be seen that \(F^{\ast}_{xx}\) is positive for all temperatures, but that \(F^{\ast}_{yy}\), which is positive in both the high- and the low-temperature phase, vanishes as \(T \to T_c\). Hence for all \(T \neq T_c\) the stability of the stationary point is ensured by the quadratic terms in the expansion of \(F(x, y)\).

### 7.3. Free energy

By the ‘free energy’ per spin (divided by \(T\)) of this driven system we will denote the quantity \(F = -N^{-1} \log Z\), where \(Z\) is the denominator in the first line of (7.1). We are now in a position to calculate various quantities of interest. The first one will be the contribution \(F^{\ast}_{\text{int}}\) to \(F\) coming from the interaction between the two chains. It will turn out to have an expansion
\[ F^{\ast}_{\text{int}} = F^{(0)}_{\text{int}} + N^{-1}f_{\text{int}} + \cdots, \quad T \neq T_c. \] (7.15)
To show this we pursue the calculation of $Z^{(1)}$ begun in (7.8). We substitute in that equation the expansion

$$\mathcal{F}(x, y) = \mathcal{F}^* + \frac{1}{2}\mathcal{F}^*_{xx}x^2 + \frac{1}{2}\mathcal{F}^*_{yy}y^2 + \cdots.$$  

(7.16)

We can then carry out the integrations in (7.8) by the saddle point method and find that for $T \neq T_c$ only the quadratic terms in (7.16) contribute. The result has the form

$$Z^{(1)} \simeq e^{-NF^* - f_{\text{int}}[1 + \mathcal{O}(N^{-1})]}.$$  

(7.17)

where

$$NF^* = -2N \log \lambda_+(K, H_0)$$  

(7.18)

and

$$f_{\text{int}}(K, \eta) = \frac{1}{2} \log[1 - g_0^2 \chi^2(K, H_0)].$$  

(7.19)

We will comment in turn on the two terms (7.18) and (7.19). First, $\mathcal{F}^*$ is the free energy (divided by $T$) of two independent Ising chains in an effective field $H_0$. Since $H_0$ is proportional to the coupling $\eta K$ between the chains, the field-dependent part of $\mathcal{F}^*$ actually represents the bulk interaction free energy $NF^{(0)}_{\text{int}}$ between the chains, that is,

$$NF^{(0)}_{\text{int}}(K, \eta) = \begin{cases} 0, & T \geq T_c, \\ -2N \log \left(\frac{\lambda_+(K, H_0)}{\lambda_+(K, 0)}\right), & T < T_c. \end{cases}$$  

(7.20)

It appears that $f_{\text{int}}$ has a linear cusp at $T = T_c$, as happens for equilibrium systems at a first-order transition, and which corresponds to a critical exponent $\alpha = 1$. Secondly, $f_{\text{int}}(K, \eta)$ is a residual interaction free energy between the two chains which remains of order $N^0$ as $N \rightarrow \infty$. We easily obtain $f_{\text{int}}$ explicitly in terms of the two system parameters $K$ and $\eta$ by substituting in (7.19) the expressions for $g_0$ and $\chi$ given in (5.3c) and (7.13), respectively, and (when $T < T_c$) eliminating $H_0$. The result is that

$$f_{\text{int}}(K, \eta) = \begin{cases} \frac{1}{2} \log (1 - e^{4K} \tanh^2 \eta K), & T > T_c, \\ \frac{1}{2} \log (1 - e^{-8K} \tanh^{-4} \eta K), & T < T_c. \end{cases}$$  

(7.21)

This shows that $f_{\text{int}}$ diverges logarithmically for $T \rightarrow T_c$. This—interesting—weak divergence of the finite-size correction $f_{\text{int}}$ renders the free energy nonconvex as a function of temperature. One should now remember that the usual convexity requirement applies to thermodynamic equilibrium, and that in spite of the formal equivalence to an equilibrium system expressed by (7.3), the present model is nevertheless a NESS.

### 7.4. Finite-size scaling of the free energy near $T_c$

We will show how our approach allows finite-size scaling functions to be found. By way of an example we consider the singular part of the free energy. As shown by (7.21), for $T \rightarrow T_c$ the quantity $f_{\text{int}}$ diverges, and this is due to the vanishing of the second order derivative $\mathcal{F}^*_{xx}$. In order for integral (7.8) combined with (7.16) to converge at $T = T_c$, we have to include higher-order terms in expansion (7.16). We will write

$$\mathcal{F}(x, y) = \mathcal{F}^* + \frac{1}{2}\mathcal{F}^*_{xx}x^2 + \frac{1}{2}\mathcal{F}^*_{yy}y^2 + \frac{1}{6}\mathcal{F}^*_{xxx}x^3 + \frac{1}{24}\mathcal{F}^*_{xxxx}x^4 + \cdots$$  

(7.22)

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and will argue below that near $T_c$ the terms not exhibited explicitly in this series are of higher order\textsuperscript{5}. In order to find the coefficients in (7.22) we straightforwardly differentiate (7.9) and set $x^* = y^* = 0$. We define
\begin{equation}
\epsilon = \frac{T - T_c}{T_c} = \frac{K - K_c}{K_c},
\end{equation}
which, in the vicinity of $T_c$, leads to
\begin{equation}
H_0 = B_{\pm} \epsilon^{1/2} + \mathcal{O}(\epsilon)
\end{equation}
where from (4.6) we have
\begin{equation}
B^2_{\pm} = \begin{cases} 0, & T > T_c, \\ 2e^{-2K_c}(\eta + 1/ \sinh 2K_c)K_c, & T < T_c. \end{cases}
\end{equation}
When using (7.24) in the coefficients found above we obtain
\begin{align}
\mathcal{F}^{*}_{xx} &= a_{\pm} \epsilon + \mathcal{O}(\epsilon^2), & \mathcal{F}^{*}_{yy} &= 4e^{2K_c} + \mathcal{O}(\epsilon), \\
\mathcal{F}^{*}_{xxx} &= b_{\pm}(-\epsilon)^{1/2} + \mathcal{O}(\epsilon^{3/2}), & \mathcal{F}^{*}_{xxxx} &= c + \mathcal{O}(\epsilon),
\end{align}
where
\begin{align}
a_{\pm} &= \begin{cases} 2/4 & (e^{4K_c} - 1)(\eta + 1/ \sinh 2K_c)K_c, & T > T_c, \\ 0, & T < T_c. \end{cases} \\
b^2_{\pm} &= \begin{cases} 0, & T > T_c, \\ 4(3e^{4K_c} - 1)(\eta + 1/ \sinh 2K_c)K_c, & T < T_c. \end{cases} \\
c &= 6e^{2K_c}.
\end{align}
We substitute the explicit expressions (7.26) in (7.22) and use that expansion in the integral (7.8). When we introduce the scaled variables of integration $u$ and $v$ defined by
\begin{equation}
x = N^{-1/4}u, \quad y = N^{-1/2}v,
\end{equation}
as well as the scaling variable
\begin{equation}
\tau = \epsilon N^{1/2},
\end{equation}
the factor $N$ disappears from the exponential. After carrying out the Gaussian integration on $v$ we get
\begin{equation}
Z^{(1)} \simeq e^{-N\mathcal{F}^*} \times \frac{N^{1/4}e^{K_c}}{\sqrt{2\pi}} \mathcal{Z}(\tau),
\end{equation}
valid in the scaling limit $N \to \infty$, $T \to T_c$ with $\tau$ fixed, and where $\mathcal{Z}$ is the scaling function
\begin{equation}
\mathcal{Z}(\tau) = \int_{-\infty}^{\infty} du \exp[-\frac{1}{2}a_{\pm}|\tau|u^2 - \frac{1}{6}b_{\pm}(-\tau)^{1/2}u^3 - \frac{1}{24}cu^4].
\end{equation}
It is of a type that occurs standardly in problems with mean-field-type critical behavior;\textsuperscript{5}

\textsuperscript{5} Terms with an odd number of $y$ derivations vanish by symmetry.
they have been studied recently by Grüneberg and Hucht [9]. We have the limiting behavior

\[
Z(\tau) \simeq \begin{cases} 
(3/2c)^{1/4} \Gamma(\frac{1}{4}), & \tau \to 0, \\
\left(\frac{2\pi}{a_\pm|\tau|}\right)^{1/2}, & \tau \to \pm\infty.
\end{cases}
\]  

(7.32)

Upon combining (7.17) and (7.30) we find that

\[
f_{\text{int}}(K, \eta) = -\frac{1}{4} \log N - \log Z(\epsilon N^{1/2}) - \frac{1}{2} \log \left(\frac{e^{2K_c}}{2\pi}\right) + \cdots,
\]

(7.33)

again valid in the scaling limit, and where the dots stand for terms that vanish as \(N \to \infty\).

It follows, in particular, that equation (7.21) may now be completed by

\[
f_{\text{int}}(K_c, \eta) \simeq \frac{1}{4} \log N + \log Z(0) + \cdots, \quad T = T_c,
\]

(7.34)

where the dots stand for terms that vanish as \(N \to \infty\).

**7.5. Susceptibilities**

Of primary interest are the correlations between the fluctuations of the magnetizations in the two chains. We set as before \(\delta \mu = \mu - m_0\). The general expression that we will study here is

\[
\chi_{k\ell} \equiv \langle \delta \mu^k(r) \delta \mu^\ell(s) \rangle
\]

\[
= \langle \delta \mu^k(r) \delta \mu^\ell(s) \rangle^{(1)} + \cdots,
\]

(7.35)

where the dots in the last line, obtained according to (7.1), represent higher-order terms. The special cases of (7.35) that we will consider are the cross-chain susceptibility \(\chi_{\text{int}}\) and the single-chain susceptibility \(\chi_{\text{sin}}\), defined as

\[
\chi_{\text{int}} = N\chi_{11} = N\langle \delta \mu(r) \delta \mu(s) \rangle,
\]

(7.36a)

\[
\chi_{\text{sin}} = N\chi_{20} = N\langle \delta \mu^2(r) \rangle,
\]

(7.36b)

with symmetry dictating that \(\chi_{20} = \chi_{02}\).

**7.5.1. Cross-susceptibility.** We consider the correlations between the fluctuating magnetizations of the two chains. The cross-susceptibility \(\chi_{\text{int}}\) is the quantity most characteristic of these correlations. From equations (5.3a)–(5.3b) and (7.3) it is clear that \(\chi_{\text{int}} = \partial \log Z^{(1)} / \partial g_0\), where the derivative has to be evaluated at fixed \(K\) and \(H_0\), considering \(g_0\) as an independent parameter. Doing the calculation for \(Z^{(1)}\) given by (7.17), (7.18), and (7.19), we observe that \(F^*\) is independent of \(g_0\), so that

\[
\chi_{\text{int}}(K, \eta) = \frac{\partial f_{\text{int}}}{\partial g_0} = \frac{g_0 \chi \tanh \eta K}{1 - g_0 \chi^2}
\]

\[
= \begin{cases} 
e^{-4K} \tanh^2 \eta K, & T > T_c, \\
\frac{e^{-8K}(1 - \tanh^2 \eta K)}{(\tanh \eta K)(1 - e^{-4K})(\tanh^4 \eta K - e^{-8K})}, & T < T_c.
\end{cases}
\]

(7.37)

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For $T \to T_c$ this quantity diverges as $|T - T_c|^{-\gamma}$ with the classical critical exponent $\gamma = 1$. For $\eta = 0$ (whence $T_c = 0$) $\chi_{\text{int}}$ vanishes as expected. Near criticality a scaling function for $\chi_{\text{int}}$ may be derived from the one for $f_{\text{int}}$ given in section 7.4. It shows that at $T = T_c$ the susceptibility $\chi_{\text{int}}(K, \eta)$ diverges as $\sim N^{1/2}$ with the system size.

Since at speed $v = \infty$ all index pairs $(i, j)$ are equivalent, the correlations between the $r$- and the $s$-spins are given by

$$\langle r_i s_j \rangle - m_0^2 = N^{-1} \chi_{\text{int}}(K, \eta)$$

(7.38)

for all values of the temperature $T$.

7.5.2. Single-chain chain susceptibility. The single-chain susceptibility $\chi_{\text{sin}}$ is defined in equation (7.36b). Let us now consider the general expression (7.35) for $\chi_{kl}$, for which the appropriate approach differs slightly from that used for $\chi_{11}$ in the preceding subsection. One may generate insertions $\delta \mu^k(r)$ (or $\delta \mu^l(s)$) in the integral (7.5) by passing from $x$ and $y$ to the two independent variables $z = x + iy$ and $\bar{z} = x - iy$ and letting $N^{-k} \partial^k / \partial z^k$ (or $N^{-l} \partial^l / \partial \bar{z}^l$) act on $e^{-z N m_0} \zeta(K, H_0 + z)$ (or on $e^{-2z N m_0} \zeta(K, H_0 + \bar{z})$). We find, using (7.6) and neglecting again the effect of $\lambda$, which is exponentially small in $N$,

$$N^{-k} \frac{\partial^k}{\partial z^k} [e^{-z N m_0} \zeta(K, H_0 + z)] = J_k(z) \zeta(K, H_0 + z),$$

(7.39)

in which

$$J_0(z) = 1, \quad J_1(z) = \tilde{m} - m_0, \quad J_2(z) = (\tilde{m} - m_0)^2 + N^{-1} \tilde{\chi},$$

$$J_3(z) = (\tilde{m} - m_0)^3 + 3N^{-1}(\tilde{m} - m_0)\tilde{\chi} + N^{-2} \tilde{\chi}',$$

$$J_4(z) = (\tilde{m} - m_0)^4 + 6N^{-1}(\tilde{m} - m_0)^2 \tilde{\chi} + 4N^{-2}(\tilde{m} - m_0)\tilde{\chi}' + 3N^{-2} \tilde{\chi}'' + N^{-3} \tilde{\chi}''',$$

(7.40)

where, in this formula, we abbreviated $\tilde{m} = m(K, H_0 + z)$ and $\tilde{\chi} = \chi(K, H_0 + z)$ (which are given in turn by equations (4.4) and (7.13)) in order to emphasize the $z$ dependence of these quantities, and where the primes on $\chi$ stand for differentiations with respect to $H_0$. Equations (7.39) and (7.40) have counterparts obtained by substituting $r \leftrightarrow s$, $k \leftrightarrow \ell$ and $z \leftrightarrow \bar{z}$. By generating the insertions with the aid of (7.39) we obtain from (7.35) the expression

$$\chi_{kl} = \langle J_k(z) J_\ell(\bar{z}) \rangle^{(1)} + \cdots,$$

(7.41)

where the dots stand for higher-order terms in the $N^{-1}$ expansion. By virtue of equations (7.41) and (7.40) it follows that

$$\chi_{20} = \langle J_2(z) \rangle^{(1)} = \langle (m(K, H_0 + z) - m_0)^2 \rangle^{(1)} + N^{-1} \langle \chi(K, H_0 + z) \rangle^{(1)}.$$

(7.42)

We now expand $m$ and $\chi$ for small $z$. After multiplication by $N$ this gives

$$\chi_{\text{sin}}(K, \eta) = N \chi^2(K, H_0) \langle z^2 \rangle_G + \chi(K, H_0) + O(N^{-1}),$$

(7.43)

which should be accompanied by the following comments. First of all, we anticipate that each factor $z$ or $\bar{z}$, upon integration with weight $\exp(-N^2)$, will produce a factor $N^{-1/2}$; hence both terms exhibited explicitly on the RHS in (7.43) are of order $N^{-1}$. Second, in passing from (7.42) to (7.43) we have replaced the averages $\langle \cdots \rangle^{(1)}$, which are with respect
to \(\exp[-N\mathcal{F}(x,y)]\), by averages \(\langle \cdots \rangle_G^{(1)}\) in which \(\mathcal{F}(x,y)\) of equation (7.9) is restricted to the Gaussian terms in its expansion. This amounts to neglecting in (7.43) terms of higher order in \(N^{-1/2}\).

Upon using in (7.43) the explicit evaluation
\[
\langle z^2 \rangle_G^{(1)} = \langle x^2 \rangle_G^{(1)} - \langle y^2 \rangle_G^{(1)} = \frac{1}{N} \left( \frac{1}{F_{xx}} - \frac{1}{F_{yy}} \right) = \frac{g_0^2 \chi}{N(1 - g_0^2 \chi^2)}, \quad T \neq T_c, \tag{7.44}
\]
we arrive at
\[
\chi_{\text{sin}}(K, \eta) = \frac{\chi}{1 - g_0^2 \chi^2}, \quad T \neq T_c, \tag{7.45}
\]
valid in the limit \(N \to \infty\). Hence the single-chain susceptibility \(\chi_{\text{sin}}\) is equal to the susceptibility of the 1D Ising model enhanced by a factor \((1 - g_0^2 \chi^2)^{-1}\) due to the presence of the other chain.

Using expressions (5.3c) and (7.13) for \(g_0\) and \(\chi\), respectively, we may render (7.45) explicit in terms of \(K\) and \(\eta\) and get
\[
\chi_{\text{sin}}(K, \eta) = \begin{cases} 
\frac{e^{-2K}}{1 - e^{-4K} \tanh^2 \eta K}, & T > T_c, \\
\frac{e^{-4K} (\tanh \eta K)(1 - \tanh^2 \eta K)}{(1 - e^{-4K})(\tanh^4 \eta K - e^{-8K})}, & T < T_c.
\end{cases} \tag{7.46}
\]
For \(T \to T_c\) the susceptibility \(\chi_{\text{sin}}\) diverges as \((T - T_c)^{-\gamma}\) with, again, \(\gamma = 1\). For \(\eta = 0\) (whence \(T_c = 0\)) the first of equations (7.46) reduces to the standard susceptibility of the zero field 1D Ising chain. Finally, in agreement with the symmetry of the problem, \(\chi_{\text{int}}\) is odd and \(\chi_{\text{sin}}\) is even in \(\eta\).

At the critical point, \(T = T_c\), equation (7.42) still holds but in (7.43) we should keep \(\langle z^2 \rangle^{(1)}\) (instead of \(\langle z^2 \rangle_G^{(1)}\)) and evaluate this quantity taking into account the fourth-order terms in \(x\) in the expansion (7.16). This leads to the conclusion that \(\chi_{\text{sin}}(\eta, K)\) diverges with system size as \(\sim N^{1/2}\) at \(T = T_c\).

Both above and below \(T_c\) one easily verifies that in agreement with Schwarz’s inequality we have \(\chi_{\text{int}}/\chi_{\text{sin}} \leq 1\).

### 7.6. Spontaneous magnetization

For \(T \geq T_c\), symmetry dictates that the magnetizations \(\langle \mu(r) \rangle\) and \(\langle \mu(s) \rangle\) are zero to all orders in powers of \(N^{-1/2}\). For \(T < T_c\), the magnetization \(\mu(r) = N^{-1} \sum_{j=1}^N r_j\) has, to leading order, a Gaussian probability distribution of width \(N^{-1/2}\) around \(m_0(K, H_0)\). As a consequence \(\langle \delta \mu(r) \rangle\) vanishes to order \(N^{-1/2}\). However, to order \(N^{-1}\) there appear nonzero corrections terms to \(\langle \mu(r) \rangle\). As an application of equation (7.1) we calculate in this subsection these correction terms.

Upon using (7.1) for the special case \(A = \delta \mu(r)\) and inserting in it the explicit expression (6.27) for \(q_1\) we obtain
\[
\langle \delta \mu(r) \rangle = \langle \delta \mu(r) \rangle^{(1)}(1) + \frac{2}{3} N g_0^2 \text{tanh} H_0 [\langle J_4(z) \rangle^{(1)}(1) + \langle J_1(z) J_3(\bar{z}) \rangle^{(1)}]. \tag{7.47}
\]
When substituting (7.40) in the second term of (7.47) we see that we need
\[
\langle J_4(z) \rangle^{(1)} = \chi^4 \langle z^4 \rangle_G^{(1)} + 6N^{-1} \chi^3 \langle z^2 \rangle_G^{(1)} + 3N^{-2} \chi^2 + O(N^{-5/2}),
\]
\[
\langle J_1(z) J_3(\bar{z}) \rangle^{(1)} = \chi^4 \langle z \bar{z}^3 \rangle_G^{(1)} + 3N^{-1} \chi^3 \langle z \bar{z} \rangle_G^{(1)} + O(N^{-5/2}).
\]
(7.48)

We have replaced the averages \(\langle \cdots \rangle^{(1)}\) by averages \(\langle \cdots \rangle_G^{(1)}\) for the same reasons as in the preceding subsection. Taking into account again that each factor \(z\) or \(\bar{z}\) brings in a power \(N^{-1/2}\), we see that all terms explicitly exhibited on the right-hand sides of equations (7.48) are of the same order in \(N\), namely \(O(N^{-2})\). The Gaussian averages are easily calculated and we are led to
\[
\langle J_4(z) \rangle^{(1)} + \langle J_1(z) J_3(\bar{z}) \rangle^{(1)} = \frac{3\chi^3(\chi + g_0)}{N^2(1 - g_0^2\chi^2)^2} + O(N^{-5/2}).
\]
(7.49)

We should now evaluate the first term on the right-hand side of (7.47), namely
\[
\langle \delta \mu(r) \rangle^{(1)} = \langle J_1(z) \rangle^{(1)} = \chi \langle z \rangle.
\]
(7.50)

The Gaussian average \(\langle z \rangle_G^{(1)}\) vanishes on account of symmetry. However, to calculate \(\langle z \rangle_G^{(1)}\) we must retain the third-order terms, not shown explicitly, in the Taylor expansion (7.16) of \(F(x, y)\). Upon expanding these one gets, after a straightforward calculation that we will not reproduce here,
\[
\langle \delta \mu(r) \rangle^{(1)} = \frac{1}{2} N\chi \chi' \langle x^4 \rangle_G^{(1)} - \frac{3}{2} \langle x^2 y \rangle_G^{(1)}
\]
\[
= N\chi \chi' \langle x^2 \rangle_G^{(1)} \langle \langle x^2 \rangle_G^{(1)} - \langle y \rangle_G^{(1)} \rangle + O(N^{-2})
\]
\[
= \frac{g_0^3 \chi \chi'}{2N(1 - g_0\chi)^2(1 + g_0\chi)} + O(N^{-2}).
\]
(7.51)

The final result for \(\langle \delta \mu(r) \rangle\) is obtained by substitution of (7.51) and (7.49) in (7.47). We see that \(\langle \delta \mu(r) \rangle\) has two contributions of order \(N^{-1}\). The contribution \(\langle \delta \mu(r) \rangle^{(1)}\) comes from the effective leading order Hamiltonian \(\mathcal{H}^{(1)}\). The second contribution accompanies the violation of DB symmetry and is therefore essentially a non-thermodynamic effect.

### 7.7. Pair correlation function

It is of interest to study the pair correlation
\[
g_N(\ell) \equiv \langle r_j r_{j+\ell} \rangle
\]
(7.52)
in a single chain. To that end we consider again expansion (7.1), now with \(A = r_j r_{j+\ell}\). Its first term may be written
\[
g_N^{(1)}(\ell) = Z_\ell^{(1)}/Z^{(1)},
\]
(7.53)
where \(Z_\ell^{(1)}\) is given by (7.4) but with an insertion \(r_j r_{j+\ell}\) in the sum on \(r\). Equivalently, \(Z_\ell^{(1)}\) is given by the same integral as (7.8) but with an insertion \(g_N^{(1)}(\ell; K, H_0 + z)\), this quantity being the pair correlation of the 1D Ising chain in a field \(H_0 + x + iy\). Evaluation by means of the standard transfer matrix method yields
\[
g_N^{(1)}(\ell; K, H_0 + z) = m^2(K, H_0 + z) + \frac{e^{-4K} \tilde{N}(K, H_0 + z)}{\sinh^2(H_0 + z) + e^{-4K}},
\]
(7.54)
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well known in the case \( z = 0 \), in which we defined \( \tilde{\Lambda} = \lambda_+ / \lambda_- \), where the tilde serves as a reminder of the \( z \) dependence, and where contributions exponentially small in \( N \) have again been neglected. In order to obtain the desired physical correlation function \( g_N(\ell) \) of this system we now have to average (7.54) with an appropriately normalized weight \( \exp[-N\mathcal{F}(x, y)] \).

We will consider this quantity in the high-temperature regime \( T > T_c \) where \( H_0 = 0 \). Knowing that \( z \) is of order \( N^{-1/2} \) we expand (7.54) for small \( z \), which gives

\[
g^{(1)}_N(\ell; K, H_0 + z) = e^{4Kz^2} + (\tanh K)(1 - e^{4Kz^2}) \exp[-(e^{-4K} + e^{2K\ell})z^2] + \mathcal{O}(N^{-2}).
\]

(7.55)

To leading order the average on \( z \) may be carried out with the weight \( \exp[-N\mathcal{F}(x, y)] \), in which the expansion \( \mathcal{F} \) is restricted to its quadratic terms. Straightforward calculation yields

\[
g_N(\ell) = \tanh K + (1 - \tanh K) \frac{g_0^2 \chi^3}{1 - g_0^2 \chi^2} N^{-1} + \mathcal{O}(N^{-2}), \quad T > T_c,
\]

(7.56)

valid for \( N \to \infty \) at fixed \( \ell \), where again \( \chi \) stands for the susceptibility \( \chi(K, 0) = e^{2K} \) of the 1D Ising chain and where \( g_0 = \tanh \eta K \). In the scaling limit \( \ell, N \to \infty \) with a fixed ratio one obtains

\[
g_N(\ell) \simeq (\tanh K) \phi(\ell N^{-1}) + \frac{g_0^2 \chi^3}{1 - g_0^2 \chi^2} N^{-1}, \quad T > T_c,
\]

(7.57)

in which each of the two terms is valid up to corrections of relative order \( N^{-1} \) and in which \( \phi \) is the scaling function defined by

\[
\phi^2(x) = \frac{1 - g_0^2 \chi^2}{1 - g_0^2 \chi^2 + 2g_0^2 \chi^2 x}.
\]

(7.58)

Equation (7.56) shows that there is a noncommutativity,

\[
\lim_{N \to \infty} \sum_{\ell=-N/2+1}^{N/2} g_N(\ell) \neq \sum_{\ell=-\infty}^{\infty} \lim_{N \to \infty} g_N(\ell),
\]

(7.59)

the right-hand side of this inequality being equal to \( \chi(K, 0) \), whereas the left-hand side is equal to \( \chi(K, 0) + \chi_{\text{int}}(K, \eta) \).

We conclude by noting that the pair correlation function may also be studied to higher order in \( N^{-1} \) in the low-temperature regime. For \( T < T_c \), the fluctuations of the magnetic field \( z \) are asymmetric and greater care is required. We will not include such a calculation here.

8. Traffic model

Motivated by an interest very different from that of [2,1] we recently introduced a new traffic model describing vehicles that may overtake each other on a road with two opposite lanes [8]. That work shows the appearance of a phase transition when the traffic intensity, taken equal on the two lanes, attains a critical value. Above the critical intensity the symmetry between the two traffic lanes is broken: one lane has dense and slow traffic, the

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other one dilute and fast traffic. The study of [8] invoked a mean-field-type assumption that couples the velocity of a vehicle in a given lane to the average of the vehicle velocities in the opposite lane. This assumption was justified by the argument that a vehicle in one lane encounters, in the course of time, all vehicles in the opposite lane. Although there is no one-to-one correspondence between the two models, they share essentially the same features, as may be seen as follows. For $J_2 < 0$ the two chains of the CRIC studied here have opposite spontaneous magnetizations; up-spins may then be regarded as the vehicles of the traffic problem; they will be denser in one chain (traffic lane) than in the other. The CRIC is more amenable to analysis than the traffic model. It was shown [2] that the CRIC phase transition disappears when $v$ is finite. Our simulations [10] of the traffic model have shown, nevertheless, that this problem is close to the critical point $v = \infty$. This explains the critical-point-like phenomena that we observed in the traffic problem, namely fluctuations that last longer than the simulation time.

9. Conclusion

We have considered in this paper the nonequilibrium steady state (NESS) of a model consisting of two counter-rotating interacting Ising chains introduced by Kadau et al [1] and by Hucht [2]. The model is related to a road traffic model studied earlier by ourselves [8]. Its dynamics is governed by a master equation parametrized by two interaction constants $J/T$ and $\eta$. At infinite relative speed $v$, the model has a phase transition, known to be of mean-field-type, at a critical temperature $T = T_c$.

Starting from the master equation we have shown that in the limiting case of a relative velocity $v = \infty$ the stationary state distribution $P_{st}$ may be studied in an expansion in powers of the inverse system size $N^{-1}$. Knowing this distribution we have calculated, also as expansions in $N^{-1}$, averages of physical interest: the interaction free energy between the chains, the single-chain and cross-chain susceptibilities, the correlation function (for $T > T_c$), and the spontaneous magnetization (for $T < T_c$). Near criticality we have shown how scaling functions may be explicitly calculated.

Whereas to leading order the force exerted by one chain on the other is that of an effective magnetic field $H_0$, the $N^{-1}$ expansion requires that we take into account the fluctuations of this field around its average. It then appears that to leading order the dynamics obeys detailed balancing (DB) with respect to an effective equilibrium Hamiltonian, as was found by Hucht [2], but that to higher order in the expansion DB is violated.

In this work we have addressed many different, albeit interrelated, aspects of the finite-size CRIC. We have not tried to be exhaustive and have not considered, for example, energy dissipation. Similarly, the parallel problem with open boundary conditions has been left aside. We hope that the results of this work will be helpful in guiding the study, which we believe to be worthwhile, of the finite velocity ($v < \infty$) version of the model.

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