Quark-Lepton Complementarity in Unified Theories

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Abstract

As pointed out by many authors, recent observations are consistent with an intriguing relation between the Cabibbo angle $\theta_C$ and the solar neutrino mixing angle $\theta_{12}$, namely $\theta_{12} \simeq \pi/4 - \theta_C$. Such quark-lepton complementarity (QLC) may be a signal of an underlying quark-lepton unification at short distances. We discuss possible ways to realize this relation in realistic quark-lepton unification theories by identifying a minimal set of operators that lead to QLC while remaining consistent with other known data. The purpose of this paper is to present the first elements of a unified model at the GUT scale capable of predicting the QLC relation. A generic prediction of our proposed class of models is the new relation for the lepton mixing angle $\theta_{13} \simeq \theta_C$, which allows these models to be confirmed or excluded by the current generation of neutrino oscillation experiments.

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1 Introduction

The unusual nature of neutrino mixings as compared to the quark mixings has inspired a large amount of speculation regarding symmetries in the quark-lepton world as well as other kinds of new physics beyond the standard model [1]. A particularly intriguing clue is the observation that the leptonic mixing angle $\theta_{12}$ and the Cabibbo angle $\theta_C$ (which provides the quark 1-2 mixing) seem to satisfy a relation,

$$\theta_{12} + \theta_C \approx \frac{\pi}{4}. \quad (1)$$

In the context of inverted hierarchy models it was observed some time ago that the predictions for the neutrino mixing angles of $\theta_{12}^\nu = \pi/4$, $\theta_{13}^\nu = 0$, may receive corrections from the charged lepton mixing angle of order the Cabibbo angle, $\theta_{12}^e \approx \theta_C$, resulting in the lepton mixing angle $\theta_{12}^\nu$ being in the LMA MSW range, with $\theta_{13}^\nu$ close to its current experimental limit (see e.g. [2, 3, 4]). However the possibility of a precise relation such as in Eq. (1) was not considered.

While it is quite possible that Eq. (1) is purely accidental, recent papers [7, 8, 9] have speculated that this could be a signal of some high scale quark-lepton unification. Indeed the structure of the Standard Model (SM) itself may be said to already provide some indirect hints of such a high scale unification, for example from the universality of weak currents, and the SU(2)$_L$ assignment of fermions. However the question of how to extend the SM in such a way as to lead to a rather precise QLC relation in Eq. (1) is far from clear, and this is the subject of this paper.

The general strategy we shall follow here is to start from an inverted hierarchy structure of the neutrino mass matrix, which predicts $\theta_{12}^\nu = \pi/4$, $\theta_{13}^\nu = 0$, then look for corrections from the charged lepton sector which can not only give acceptable corrections to the lepton mixing angles as suggested in [2, 3], but can accurately give rise to the QLC relation in Eq. (1). In order to achieve this, it seems necessary to relate the charged lepton mixing angle $\theta_{12}^e$ to the Cabibbo angle $\theta_C$, and this in turn only seems possible within the framework of a quark-lepton symmetry. As we shall see, it is highly non-trivial to obtain the QLC relation in Eq. (1) for several reasons. On the other hand, if such a relation is in fact realized, then this potentially tells us a great deal about the detailed structure of the unified theory, as we shall see. A smoking gun signature of the class of models that we propose here is the additional QLC relation:

$$\theta_{13} \approx \theta_C, \quad (2)$$

which enables the approach presented here to be verified or excluded by the present generation of neutrino experiments.

In a recent paper [11] a model based on the Pati-Salam gauge group SU(2)$_L \times$ SU(2)$_R \times$ SU(4)$_C$ that unifies quarks and leptons [10] was proposed in an attempt to realize the QLC relation. It was pointed out that an inverted hierarchy model based on
the $L_e - L_\mu - L_\tau$ symmetry imposed on single bi-doublet Pati-Salam model could be a good starting point for realizing QLC in a natural gauge theory framework. It was realized in [11] that obtaining realistic models for both the quark and lepton sector is highly nontrivial in such models and inevitably leads to nontrivial constraints on the nature of the theory. In particular, Ref. [11] could lead only to a modified form of QLC (see discussion below) as well as the scale of quark lepton unification to be at the multi-TeV scale. Furthermore, the relation between the Cabibbo angle and the solar mixing angle obey a complementarity relation that is different from that in Eq. (1) (although it is consistent with observations at $2\sigma$ level).

It is the goal of this paper to discuss models for QLC that are more in the spirit of grand unified theories (GUTs) so that quarks and leptons unify at the GUT scale of $10^{16}$ GeV. Our strategy will be to construct from the see-saw mechanism a form of neutrino mass matrix consistent with an inverted hierarchy, using the approach of [2, 3]. Note that in this approach it is possible that the effective neutrino mass matrix has a form which is invariant under $L_e - L_\mu - L_\tau$ without ever imposing this symmetry in the high energy theory. In fact we shall instead rely on a different $U(1)_X$ family symmetry as playing a leading role in achieving this structure. In order to achieve QLC, it is necessary to achieve this within the framework of quark-lepton unification, which we shall assume here to be provided by a high energy Pati-Salam theory, broken near the unification scale. The presence of the Pati-Salam gauge group is necessary to relate the charged lepton mixing angles to the down-quark mixing angles, which in turn dominate the contribution to the Cabibbo angle, allowing the QLC relation to emerge, albeit in a non-trivial way. Our approach in this paper is to list explicitly a set of higher dimensional operators in a minimal supersymmetric Pati-Salam type theory that after symmetry breaking leads to the required form for the mass matrices. We have not attempted to seek a set of discrete symmetries that would enable us to generate the right set of higher dimensional operators, nor have we addressed the question of the orderings of the index contractions of these operators, since both questions would take us well beyond the scope of this paper. However we have identified a $U(1)_X$ family symmetry which controls the leading operators, which would facilitate the construction of a future theory from the building blocks that we present here. The purpose of this paper is therefore to present the first elements of a unified model at the GUT scale capable of predicting the QLC relation.

This paper has been organized as follows: in Sec. 2, we discuss generic problems with realizing QLC relation in models where the $SU(4)_C$ quark-lepton unification scale is close to the GUT scale; in Sec. 3, we review a way of generating inverted hierarchy (different from that in Ref. [11] which uses $L_e - L_\mu - L_\tau$ symmetry); in Sec. 4, we present a the set of operators that enables us to realize the inverted hierarchy for neutrinos and eventually, helps us realize the QLC relation. In Appendix A we specify our conventions, and in Appendix B we briefly discuss an alternative class of QLC models with a lopsided charged lepton mass matrix.
2 Challenges for QLC

We shall see that it is not enough to simply postulate a Pati-Salam symmetry and an inverted hierarchical effective neutrino mass matrix: there are a number of additional hurdles one must clear before the QLC relation will emerge. The purpose of this section is to outline the main challenges facing QLC. These challenges can be regarded as a chain of logic, that must remain unbroken at each stage in order to achieve the desired result of QLC. We shall find that there are non-trivial obstacles facing this approach.

The basic starting point of our approach is to assume that the low energy neutrino mass matrix $m_{\text{LL}}$ accurately predicts $\theta_{\nu}^{12} = \pi/4$. We have already observed that this is possible if the neutrino mass spectrum is inverted, with a Majorana parity for $m_1$ and $m_2$. In the see-saw scenario, $m_{\text{LL}}$ is given via the see-saw formula

$$m_{\text{LL}} = v_u^2 Y_\nu M^{-1}_{\text{RR}} Y^T_\nu,$$

where $Y_\nu$ is the neutrino Yukawa matrix, $M_{\text{RR}}$ the mass matrix of the right-handed neutrinos and $v_u = \langle h^0_u \rangle$ is the vev which leads to masses for the up-type quarks. We will discuss the construction of see-saw models with inverted hierarchy in Sec. 3. For the time being let us assume that nearly maximal neutrino mixing $\theta_{\nu}^{12}$ is generated from $m_{\text{LL}}$, which is then subsequently corrected by charged lepton mixing angles in forming the final leptonic mixing angle $\theta_{12}$. Clearly these corrections coming from the charged lepton sector must have a rather special form, if QLC is to emerge.

In our conventions, which are specified in the appendix, the MNS matrix is given by $U_{\text{MNS}} = U_{\ell L} U^\dagger_{\nu L}$. If we parameterize $U_{\ell L}$ and $U_{\nu L}$ in standard parameterization, we obtain (neglecting the very small mixings $\theta_{e23}$ and $\theta_{e13}$):

$$U_{\text{MNS}} \approx R_{R12}^\dagger R_{R23}^\dagger U_{\nu L}^\dagger R_{L12} U_{\ell L} .$$

In general, small contributions from the charged lepton mixing angles $Y_e$ lead to the following corrections to the mixings from the neutrino sector:

$$s_{23} = s_{23}^{\nu} - \theta c_{23},$$

$$\theta_{13} = \theta_{13}^{\nu} - \theta c_{23} + \theta_{23}^{\nu} c_{23} ,$$

$$s_{12} = s_{12}^{\nu} + \theta c_{23} c_{12} - \theta_{12} c_{23} c_{12} ,$$

where we have neglected phases for simplicity. Assuming $\theta_{23}^{\nu} \approx \pi/4$, we obtain

$$s_{23} \approx s_{23}^{\nu} ,$$

$$\theta_{13} \approx \theta_{13}^{\nu} + \frac{1}{\sqrt{2}} \theta_{12}^{\nu} ,$$

$$s_{12} \approx \frac{1}{\sqrt{2}} - \frac{1}{2} \theta_{12}^{\nu} \rightarrow \theta_{12} \approx \frac{\pi}{4} - \frac{1}{\sqrt{2}} \theta_{12}^{\nu} .$$

We immediately see one problem for realizing QLC: with maximal atmospheric mixing $\theta_{23}^{\nu} = \pi/4$ from $m_{\text{LL}}$, shifting the rotation $R_{12}^\dagger$ to the right side of $R_{23}^\nu$ induces a factor of...
1/\sqrt{2} for the correction to \( \theta_{12} \) coming from the charged leptons. Therefore the charged lepton mixing angle \( \theta_{12}^{e} \) needs to be about \( \sqrt{2} \) larger than \( \theta_{C} \) in order to compensate for this. Note Eq. (6c) implies that the correction to \( \theta_{13} \) from the charged lepton sector is also equal to \( 1/\sqrt{2} \) times the charged lepton angle \( \theta_{12}^{e} \).

In our approach it is necessary that the Cabibbo angle be dominated by the down quark 1-2 mixing angle, \( \theta_{C} \approx \theta_{12}^{d} \), with the up-quark 1-2 mixing angle \( \theta_{12}^{u} \) being essentially irrelevant. This is because we shall subsequently use quark-lepton symmetry to relate \( \theta_{12}^{e} \) to \( \theta_{12}^{d} \). Consider a symmetric texture for the quark Yukawa matrices of the form\(^1\)

\[
Y_u = \begin{pmatrix}
0 & b'\epsilon^3 & c'\epsilon^4 \\
 b'\epsilon^3 & \epsilon^2 & a'\epsilon^2 \\
c'\epsilon^4 & a'\epsilon^2 & 1
\end{pmatrix} y_t, \quad Y_d = \begin{pmatrix}
0 & b\epsilon^3 & c\epsilon^4 \\
 b\epsilon^3 & \epsilon^2 & a\epsilon^2 \\
c\epsilon^4 & a\epsilon^2 & 1
\end{pmatrix} y_b .
\]

With parameters \( \epsilon \) and \( \bar{\epsilon} \) determined from the quark masses of the second and third generation at the unification scale \( M_X \approx 2 \cdot 10^{16} \) GeV,

\[
\frac{m_c}{m_t} = \epsilon^2 \rightarrow \epsilon \approx 0.05 , \quad \frac{m_b}{m_t} = \bar{\epsilon}^2 \rightarrow \bar{\epsilon} \approx 0.15 ,
\]

it allows to fit the quark data. Constraints from the CKM matrix can be satisfied by taking e.g.

\[
b \approx 1.5 , \quad c \approx 3 , \quad a \approx 1.3 , \quad |b'| \approx 1 , \quad \text{Arg}(b') \approx \frac{\pi}{2} .
\]

The other parameters, \( a' \) and \( c' \) are rather unconstrained. The texture zero in the (1,1)-entry of \( Y_u \) and \( Y_d \) leads to the famous GST relation \(^5\) which connects the quark masses and the Cabibbo angle

\[
\theta_{C} = |V_{us}| = \lambda = \left| \sqrt{\frac{m_d}{m_s}} - e^{-i\phi} \sqrt{\frac{m_u}{m_c}} \right| , \quad (10)
\]

with \( \phi = \text{Arg}(b') \). For the above parameter set it is seen to be the case that the Cabibbo angle is numerically approximately given from the down quark sector only,

\[
\theta_{C} \approx \theta_{12}^{d} \approx \sqrt{\frac{m_d}{m_s}} , \quad (11)
\]

which satisfies our necessary condition of QLC.

Having argued that it is plausible that \( \theta_{C} \approx \theta_{12}^{d} \), the next link in the chain of logic leading to QLC is to relate the down quark mixing angle \( \theta_{12}^{d} \) to the charged lepton mixing angle \( \theta_{12}^{e} \). The well known observation is that the above texture can accommodate the

\(^1\)For a detailed discussion of the this texture see, e.g., \(^{12}\).
charged lepton data as well, if a Clebsch factor (Georgi-Jarlskog factor) of -3 in the (2,2)-entry of \( Y_d \) is introduced. The Yukawa matrix

\[
Y_e = \begin{pmatrix}
0 & b\bar{\epsilon}c^2 & c\bar{\epsilon}\epsilon^4 \\
b\epsilon^3 & -3\bar{\epsilon}^2 & a\bar{\epsilon}^2 \\
c\epsilon^4 & a\epsilon^2 & 1
\end{pmatrix} y_\tau
\]  

leads to correct mass ratios for the charged leptons at low energy. Similar textures have been discussed intensively in the literature and have been used for the construction of successful fermion mass models in the framework of unified theories. Here we see an obstacle for QLC with a charged lepton Yukawa matrix \( Y_e \) similar to the one given in Eq. (12): because of the Georgi-Jarlskog factor of -3 in the (2,2)-entry of \( Y_e \), \( \theta_{e12} \) is not equal to the Cabibbo angle, but appears to be three times smaller. Recall that in order to achieve QLC, we would require \( \theta_{e12} \) to be \( \sqrt{2} \) larger than the Cabibbo angle, whereas we have just found it to be three times smaller! Putting these arguments together we find \( \theta_{e12} \approx \frac{1}{3} \theta_C \) which with Eq. (6c) gives a total correction of

\[
\theta_{12} \approx \frac{\pi}{4} - \frac{1}{\sqrt{2}} \frac{1}{3} \theta_C .
\]

In total, the deviation of the lepton mixing \( \theta_{12} \) from maximal is only \( \frac{1}{\sqrt{2}} \frac{1}{3} \theta_C \approx 3^\circ \).

These, then, are the challenges that must be overcome, if we are to achieve QLC. In fact it is possible to overcome these challenges, and we will see in Sec. 4 how these problems can be solved and how a QLC relation \( \theta_{12} = \frac{\pi}{4} - \theta_C \) can be realized within the framework of quark-lepton-unified models. The lesson, however, is that the QLC relation is non-trivial to achieve, and if it is realized in nature, and is not accidental, then we potentially stand to learn a great deal about the structure of the underlying unified theory.

3 Inverted Hierarchy in See-Saw Models

As noted in [11], a good starting point towards obtaining QLC is a Majorana mass matrix for neutrinos that leads to inverted hierarchy for neutrinos as well as maximal solar mixing angle. The mass matrix ought to have the form:

\[
m_{\nu LL} \approx \begin{pmatrix}
0 & b & c \\
b & 0 & 0 \\
c & 0 & 0
\end{pmatrix} m_0
\]  

One way to get this form for the mass matrix is to have an \( L_e - L_\mu - L_\tau \) symmetry [15]. One may however start with a more general mass matrix, following a procedure introduced in [2, 3], which we now briefly review. It has been shown that naturalness suggests
a pseudo-Dirac structure of the mass matrix $M_{RR}$ of the right-handed neutrinos.\footnote{Such nearly degenerate right-handed neutrinos are also interesting with respect to resonant leptogenesis. Note that the right-handed neutrinos can be re-ordered, which corresponds to a re-ordering of the columns in $m_{LR}$.} We begin by writing $M_{RR}$ and the neutrino Dirac mass matrix $m_{LR} = Y_{\nu} v_u$ as

$$M_{RR} \approx \begin{pmatrix} Y & 0 & 0 \\ 0 & 0 & X \\ 0 & X & 0 \end{pmatrix}, \quad m_{LR} = \begin{pmatrix} d & a' & a \\ e & b' & b \\ f & c' & c \end{pmatrix}. \quad (15)$$

We then impose the condition that the pseudo-Dirac right-handed neutrino pair dominates in the see-saw mechanism \cite{2, 3}:

$$\frac{(d + e + f)^2}{Y} \ll \text{Max} \left( \frac{(a' + b' + c')(a + b + c)}{X} \right). \quad (16)$$

In \cite{2} it was shown that one of the two possibilities

$$a', b, c \gg a, b', c' \quad \text{or} \quad a, b', c' \gg a', b, c \quad (17)$$

then leads to a natural class of inverted hierarchy models. For the example $a', b, c \gg a, b', c'$, leads to an inverted hierarchy with mixing angles,

$$\tan \theta_{23} \approx \frac{c}{b}, \quad \theta_{13} \approx \frac{c'b - b'c}{a'\sqrt{b^2 + c^2}}, \quad \tan \theta_{12} \approx 1, \quad (18)$$

(and analogous, with primed quantities interchanged with non-primed ones, for $a, b', c' \gg a', b, c$). The neutrino mass matrix is given by

$$m_{\nu LL} \approx \begin{pmatrix} 0 & b & c \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \frac{a'v_u^2}{X}. \quad (19)$$

For instance, $M_{RR}$ and $m_{LR}$ of the approximate forms

$$M_{RR} = \begin{pmatrix} Y & 0 & 0 \\ 0 & 0 & X \\ 0 & X & 0 \end{pmatrix}, \quad m_{LR} = \begin{pmatrix} 0 & a' & 0 \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}, \quad (20)$$

would lead to the neutrino mass matrix of Eq. (19). Of course, small perturbations have to be added in both cases in order to generate the required small solar mass splitting \cite{2, 3}. Note that, although the resulting effective neutrino mass matrix respects the symmetry $L_e - L_\mu - L_\tau$, it is not necessary to assume this symmetry in the construction of this matrix. In the framework of Pati-Salam models, we will give a possible choice of $U(1)$-charges which lead to mass matrices similar to Eq. (20).
4 QLC in Models of Quark-Lepton Unification

We will assume that the symmetry group of our theory contains the Pati-Salam gauge group \( G_{422} = SU(4)_C \times SU(2)_L \times SU(2)_R \), plus an additional flavour symmetry group \( F \), which we shall only partly specify, but which contains a family symmetry \( U(1)_X \).

Quarks and leptons are unified in the \( SU(4)_C \)-quartets \( F_f \) and \( F_C^f \) of \( SU(4)_C \), which are doublets of \( SU(2)_L \) and \( SU(2)_R \), respectively,

\[
F_i = \begin{pmatrix} u_i & u_i & u_i & \nu_i \\ d_i & d_i & d_i & e_i \end{pmatrix}, \quad F_C^j = \begin{pmatrix} u_C^j & u_C^j & u_C^j & \nu_C^j \\ d_C^j & d_C^j & d_C^j & e_C^j \end{pmatrix}.
\]

(21)

\( i \) and \( j \) are family indices. The field content of the types of models we are considering in this section is summarized in Tab. 1. \( \theta \) and \( \bar{\theta} \) are flavon fields which we will use in Sec. 4.1. They are \( G_{422} \)-singlets but may be charged under the family symmetry \( U(1)_X \subset F \), which we will introduce below and which will lead to inverted hierarchy in the neutrino sector.

| field       | \( F_1 \) | \( F_2 \) | \( F_3 \) | \( F_C^1 \) | \( F_C^2 \) | \( F_C^3 \) | \( h \) | \( H \) | \( \bar{H} \) | \( \theta \) | \( \bar{\theta} \) |
|-------------|----------|----------|----------|------------|------------|------------|------|------|--------|--------|--------|
| \( SU(4)_C \) | 4        | 4        | 4        | 4          | 4          | 4          | 1    | 4    | 1      | 1      | 1      |
| \( SU(2)_L \) | 2        | 2        | 2        | 2          | 1          | 1          | 2    | 1    | 1      | 1      | 1      |
| \( SU(2)_R \) | 1        | 1        | 1        | 2          | 2          | 2          | 2    | 2    | 1      | 1      | 1      |

Table 1: Field content assumed in this section.

The Yukawa matrices and the mass matrix \( M_{RR} \) for the right-handed neutrinos will receive contributions from operators of the form \[13, 14\]:

\[
(Y_{l})_{ij} : F_i F_C^j h \left( \frac{H \bar{H}}{M_V^2} \right)^n \left( \frac{\theta}{M_V} \right)^{\bar{p}_{ij}} \left( \frac{\bar{\theta}}{M_V} \right)^{\bar{q}_{ij}},
\]

(22a)

\[
(M_{RR})_{ij} : F_C^i F_C^j \frac{HH}{\Lambda} \left( \frac{H \bar{H}}{M_V^2} \right)^m \left( \frac{\theta}{M_V} \right)^{n_{ij}} \left( \frac{\bar{\theta}}{M_V} \right)^{\bar{q}_{ij}},
\]

(22b)

where all indices apart from family indices have been suppressed. If the operators contain \( H \bar{H} \) to some power, various contractions of the indices are possible. After the Higgs fields \( H, \bar{H} \) and \( h \) acquire their vevs, which breaks \( G_{422} \) to the SM gauge group \( G_{321} \) and \( G_{321} \) to \( SU(3)_C \times U(1)_{el} \), this can lead to different contributions to the Yukawa couplings of up-type quarks, down-type quarks, charged leptons and neutrinos,

\[
a_{ij} \left[ (x_u)_{ij} u_i u_j^C h_u^0 + (x_d)_{ij} d_i d_j^C h_d^0 + (x_e)_{ij} e_i e_j^C h_e^0 + (x_{\nu})_{ij} \nu_i \nu_j^C h_{\nu}^0 \right] \delta^{m} e^{p_{ij} + \bar{q}_{ij}}.
\]

(23)
$(x_u)_{ij}$, $(x_d)_{ij}$, $(x_e)_{ij}$ and $(x_\nu)_{ij}$ are the Clebsch factors for a given operator $O$. Some operators, which will be used in the remainder, are listed in Tab. 4 and the corresponding Clebsch factors are given in Tab. 5 for convenience. An extensive list of the operators and the possible Clebsch factors can be found, e.g., in [13]. We will give an explicit example in Eq. (27). $a_{ij}$ is an $O(1)$-factor which arises from generating the effective operator. $h_0^u$ and $h_0^d$ are the electrically neutral components of the bi-doublet Higgs $h$. Of course, operators can be forbidden, e.g. by the horizontal symmetry $F$ or if there is no appropriate massive field in the full theory for generating this operator. $G_{422}$ is broken to the SM via vevs $\langle \mathcal{H} \rangle = \langle \overline{\mathcal{H}} \rangle$ of the quartet-Higgses $\mathcal{H}$ and $\overline{\mathcal{H}}$,

\[
H = \begin{pmatrix} 0 & 0 & 0 & v_H \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 0 & 0 & 0 & v_H \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

In the following, we will assume a single expansion parameter $\lambda$, where

\[
e := \frac{\langle \theta \rangle}{M_V} = \frac{\langle \overline{\theta} \rangle}{M_V} = \lambda, \quad \delta := \frac{\langle H \rangle \langle \overline{H} \rangle}{M_V^2} = \lambda,
\]

with $\lambda \approx 0.22$ being the Cabibbo angle.

### 4.1 A Minimal set of Operators that Leads to QLC

We now present a minimal set of operators, which will lead to QLC while remaining in agreement with other fermion masses and mixings. Ideally, one would like to have the choice of operators dictated entirely by some symmetry group $F$, which will then provide more fundamental insight into the nature of high scale physics. Our goal in this note is more modest and we remain content with simply writing down the minimal set of operators that fulfill our requirements within an SU(4)$_C \times$ SU(2)$_L \times$ SU(2)$_R \times$ U(1)$_X$ model, where U(1)$_X \subset F$. The U(1)$_X$ family symmetry will lead to inverted hierarchy in the neutrino sector and, with charge assignments\(^3\) as in Tab. 2, it will serve as a starting point for the construction of the set of operators. For the set of operators presented in this subsection, maximal lepton mixing $\theta_{12}$ as well as large lepton mixing $\theta_{23}$ shall originate from the neutrino sector. An alternative route for model building with QLC will be discussed in Appendix B.

The strategy is the following: The choice of the U(1)$_X$-charges determines the basic common structure of the Yukawa matrices, i.e. the suppression of the entries by powers of $\epsilon$ (cf. Eq. (25)). Obviously, this structure has to be modified in order to be consistent with the charged lepton and quark data at low energy. This can be done by forbidding some of the leading operators compatible with U(1)$_X$ symmetry, yielding additional

\(^3\)Note that in Pati-Salam models any set of U(1)-charges can be transformed to an anomaly-free set of charges leading to the same model. Thus, anomaly cancellation for the U(1)-symmetry does not restrict model building in the Pati-Salam framework.
Table 2: $U(1)_X$-charges (anomaly-free after GS anomaly cancellation) which help to realize the desired inverted hierarchy in the neutrino sector. It leads to $M_{RR}$ and $Y_\nu$ consistent with the requirements of Sec. However, as explained in the text, $U(1)_X$ should be understood as being part of a larger set of family symmetries $F$.

suppression of these entries of the Yukawa matrices by factors of $\delta$. Where entries of the Yukawa matrices are allowed at the same order in $\epsilon$ and in $\delta$, the freedom of choosing the operator which gives these entries will be used in some cases to introduce Clebsch factors, as given in Tab. Using these tools, we arrive at a model with $Y_u, Y_d, Y_e, Y_\nu$ and $M_{RR}$ given in Tab. with the operators defined in Tab. We will now discuss how it solves the obstacles for QLC found in Sec. and investigate its additional predictions.

4.1.1 Realizing QLC

Before we discuss the predictions in detail, let us point out the main properties of the set of operators given in Tab. which allow to realize QLC in model of quark-lepton unification. Omitting common $O(1)$-coefficients and treating tiny elements as zero, $Y_e, Y_d$ and $m_{LL}$ are given by (cf. Tab.)

\[
Y_d \sim \begin{pmatrix} 0 & \lambda^4 & \lambda^4 \\ * & \lambda^3 & \lambda^2 \\ * & * & 1 \end{pmatrix}, \quad Y_e \sim \begin{pmatrix} 0 & -3\lambda^4 & \lambda^4 \\ * & 2\lambda^3 & \lambda^2 \\ * & * & 1 \end{pmatrix}, \quad m_{LL} \sim \begin{pmatrix} 0 & m & m' \\ m & 0 & 0 \\ m' & 0 & 0 \end{pmatrix},
\]

where we remind the reader that $\lambda \approx \theta_C$. For QLC, the relation between $Y_e$ and $Y_d$ is crucial. Here we have chosen the following operators for the $(1,2)$- and $(2,2)$-entries in $Y_e$ and $Y_d$:

\[
(Y_{e,d})_{12} : b_{12} O_W \epsilon^3 \leftrightarrow \frac{b_{12}}{M_V^4} (F_1 F_2^C)_{15} h_{a}^{y} y_{a} \epsilon^{zz} (H^w H_z)_{15} \left[ (H^z H_z)_{1} \right]^2 \theta,
\]

\[
(Y_{e,d})_{22} : a_{22} O_G \epsilon^2 \delta \leftrightarrow \frac{a_{22}}{M_V^4} (F_2^C H^w)_{10} h_{a}^{y} y_{a} \epsilon^{zz} (F_2^C H_z)_{10} \bar{\theta}^2,
\]

where only indices of SU(2)$_L$ and SU(2)$_R$ are shown explicitly and where brackets $(XY)_R$ indicates the product of the fields such that the expression transforms in the representation R of SU(4)$_C$. After symmetry breaking of $G_{422}$ to the SM, the operators lead to Clebsch factors $-3$ for $(Y_{e,d})_{12} / (Y_{e,d})_{12}$ and $2$ for $(Y_{e,d})_{22} / (Y_{e,d})_{22}$ (cf. Tab.).

\footnote{The product is normalized such that the resulting Clebsch factors are in the convention used in Tab. i.e. $\sum_i x_i^2 = 4$. More details can be found, e.g. in Appendix 1 of.}
Table 3: Complete minimal set of operators for $Y_u$, $Y_d$, $Y_e$, $Y_\nu$, and $M_{RR}$ that lead to QLC as discussed in Sec. 4. $\mathcal{O}$ specifies the type of operator which yields this contribution, using the convention given in Appendix 1 of [13]. The form of the used operators is given explicitly in Tab. 4. The powers of $\delta$ and $\epsilon$ indicate the order in $H\tilde{H}/M^2_V$ and $\theta/M_V$ or $\bar{\theta}/M_V$, respectively. The resulting Clebsch factors are summarized in Tab. 4.
Table 4: Explicit form of the operators used within the set of operators which leads to QLC (Tab. 3). Only indices of SU(2)_{L} and SU(2)_{R} and family indices i, j are shown explicitly and brackets (XY)_{R} indicates the product of the fields such that the expression transforms in the representation R of SU(4)_{C}. The complete set of possible index contractions for the operators of Eq. (22a) can be found, e.g., in Appendix 1 of [13]. After symmetry breaking of G_{122} to the SM, the operators lead to Clebsch factors given in Tab. 5.

| operator | abbreviation in Tab. 3 |
|----------|-------------------------|
| O^{A}_{ij}+\bar{O}^{+}_{ij} \delta^{n+1} | \frac{1}{M^{ij}_{A}} \left( F_{ij}^{a} h_{a}^{y} F_{ij}^{C} C_{y}^{j} \right)_{1} \theta_{ij} \bar{\theta}_{ij} \left[ (H^{x} H^{z})_{1} \right]^{n+1} |
| O^{W}_{ij}+\bar{O}^{+}_{ij} \delta^{n+1} | \frac{1}{M^{ij}_{W}} \left( F_{ij}^{a} F_{ij}^{C} \right)_{15} h_{a}^{y} y_{w} y_{w} \varepsilon^{x z} \left( H^{w} H_{z}^{x} \right)_{15} \theta_{ij} \bar{\theta}_{ij} \left[ (H^{z} H_{z})_{1} \right]^{n} |
| O^{G}_{ij}+\bar{O}^{+}_{ij} \delta^{n+1} | \frac{1}{M^{ij}_{G}} \left( F_{ij}^{a} F_{ij}^{C} \right)_{15} h_{a}^{y} y_{w} y_{w} \varepsilon^{x z} \left( F_{ij}^{C} H_{z}^{j} \right)_{10} \theta_{ij} \bar{\theta}_{ij} \left[ (H^{z} H_{z})_{1} \right]^{n} |
| O^{B}_{ij}+\bar{O}^{+}_{ij} \delta^{n+1} | \frac{1}{M^{ij}_{B}} \left( F_{ij}^{a} F_{ij}^{C} \right)_{15} h_{a}^{y} y_{w} y_{w} \varepsilon^{x z} \left( F_{ij}^{C} H_{z}^{j} \right)_{10} \theta_{ij} \bar{\theta}_{ij} \left[ (H^{z} H_{z})_{1} \right]^{n} |
| O^{V}_{ij}+\bar{O}^{+}_{ij} \delta^{n+1} | \frac{1}{M^{ij}_{V}} \left( F_{ij}^{a} F_{ij}^{C} \right)_{15} h_{a}^{y} y_{w} y_{w} \varepsilon^{x z} \left( F_{ij}^{C} H_{z}^{j} \right)_{10} \theta_{ij} \bar{\theta}_{ij} \left[ (H^{z} H_{z})_{1} \right]^{n} |
| O^{M}_{ij}+\bar{O}^{+}_{ij} \delta^{n+1} | \frac{1}{M^{ij}_{M}} \left( F_{ij}^{a} F_{ij}^{C} \right)_{15} h_{a}^{y} y_{w} y_{w} \varepsilon^{x z} \left( F_{ij}^{C} H_{z}^{j} \right)_{10} \theta_{ij} \bar{\theta}_{ij} \left[ (H^{z} H_{z})_{1} \right]^{n} |
| O^{I}_{ij}+\bar{O}^{+}_{ij} \delta^{n+1} | \frac{1}{M^{ij}_{I}} \left( F_{ij}^{a} F_{ij}^{C} \right)_{15} h_{a}^{y} y_{w} y_{w} \varepsilon^{x z} \left( F_{ij}^{C} H_{z}^{j} \right)_{10} \theta_{ij} \bar{\theta}_{ij} \left[ (H^{z} H_{z})_{1} \right]^{n} |

Table 5: Clebsch factors $x_{i}$ for the operators defined in Tab. 4 normalized to $\sum_{i} x_{i}^{2} = 4$. A complete list of operators and Clebsch factors can be found in the appendix of [13].
Table 6: Accommodating the estimated quark data and the charged lepton masses at $M_X \approx 10^{16}$ GeV (from [14], for large $\tan \beta \approx 40$) by determining the $\hat{O}(1)$-coefficients for the model defined in Tab. 3.

In addition, the model predicts $m_\mu/m_\tau \approx 2$ at $M_X$, as discussed in the text – and of course the desired QLC relation $\theta_{12} \approx \frac{\pi}{4} - \theta_C$ and additionally $\theta_{13} = \lambda_C$ at $M_X$. We have ignored the quark sector CP phase $\delta$.

choice of Clebsch factors is a crucial feature of the model with respect to QLC. It solves the problems found in Sec. 3 and in particular leads to a charged lepton mixing angle approximately $\sqrt{2}$ larger than the Cabibbo angle: $\theta_{12}^c = \frac{3}{2} \theta_C \approx \sqrt{2} \theta_C$. Using Eq. (6c), this leads to the desired relation $\theta_{12} \approx \frac{\pi}{4} - \theta_C$ at high energy $M_X$.

We also see immediately that the Clebsch factor 2 for $(Y_c)_{22}/(Y_d)_{22}$ leads to the prediction $m_\mu/m_\tau \approx 2$ at $M_X$. Furthermore, from Eq. (6b), we see that an additional prediction is $\theta_{13} \approx \theta_C$ at $M_X$. Contrary to these main features, some details of the model such as the choice of some of the other operators are quite arbitrary.

4.1.2 Predictions at $M_X$

The model given in Tab. 3 can accommodate the estimated parameters of the quark and charged sector at the unification scale $M_X = 10^{16}$ GeV (e.g. from [14]) by choosing appropriate values for the $\hat{O}(1)$-coefficients. This is demonstrated analytically in Tab. 6. In addition, we obtain a prediction for $m_\mu$, which we will discuss separately below.

Let us now consider the lepton sector. In the neutrino sector, $Y_\nu$ and $M_{RR}$ in Tab. 3 have the approximate forms of the matrices in Eq. (20), with the condition that the pseudo-Dirac right-handed neutrino pair dominates, and so predicts the required effective neutrino mass matrix in Eq. (13). This leads to an inverted neutrino mass hierarchy with almost maximal $\theta_{12}^\nu \approx \pi/4$ due to the approximate Pseudo-Dirac structure in the 1-2 submatrix of $m_{LL}$ in Eq. (13). Furthermore, $\theta_{23}^\nu$ is large and $\theta_{13}^\nu$ is generated by sub-
leading entries in $m_{LL}$ (not displayed in Eq. (26)) and is thus very small. The resulting mass matrix $m_{LL}$ of the light neutrinos is given by

$$m_{LL} = \begin{pmatrix}
\mathcal{O}((\lambda^4)) & 4a_{12}a_{23}\lambda^2 & 2a_{12}a_{33}\lambda \\
4a_{12}a_{23}\lambda^2 & \mathcal{O}((\lambda^3)) & \mathcal{O}((\lambda^3)) \\
2a_{12}a_{33}\lambda & \mathcal{O}((\lambda^3)) & \mathcal{O}((\lambda^3))
\end{pmatrix} \frac{\lambda_v^2}{r_{23}M_R}. \quad (28)$$

$\theta_{12}$ is very close to maximal as desired, $\theta_{13}^e$ is tiny and arctan $\theta_{23}^\nu \approx a_{33}/(2a_{23}\lambda)$. Thus, $a_{33} \approx 2a_{23}\lambda$ and with $a_{33} \approx 0.68$ from Tab. 5, $a_{23} \approx 1.5$ is required for an approximately maximal mixing $\theta_{23}^\nu$. Furthermore, in order to have $2a_{12}a_{33}\lambda$ of order 1, we choose $a_{12} \approx 2$. All three right-handed neutrinos have masses around $M_R \approx 10^{14}$ for generating the atmospheric mass squared difference $\Delta m^2_{31}$ of the right order. The solar mass squared difference $\Delta m^2_{21}$ is generated from the corrections to the leading order structure in $m_{LL}$. With $\mathcal{O}((\lambda^3))$ entries in the sub-leading elements of $m_{LL}$, it is roughly of the order $2\Delta m^2_{31}\lambda^3$ and a value consistent with experiment is produced for many choices of yet undetermined $\mathcal{O}(1)$-coefficients.\(^6\) For the corrections from the charged lepton sector, let us consider $Y_d$ and $Y_e$ (Tab. 3 or Eq. (26)). Diagonalizing $Y_d$ by $U_{dL}U_{dR}^\dagger$ with $U_{dL}^\dagger = R_{23}U_{13}R_{12} \approx U_{CKM}$, we read off $\theta_{12}^q = \theta_C \approx \lambda$, $\theta_{13}^q \sim \lambda^2$ and $\theta_{23}^q \sim \lambda^2$. Diagonalizing $Y_e$ by $U_{eL}U_{eR}^\dagger$ with $U_{eL}^\dagger = R_{23}U_{13}R_{12}$, we obtain

$$\theta_{12}^q \approx \frac{3}{2}\lambda \approx \frac{3}{2}\theta_C, \quad \theta_{13}^q \sim \lambda^2, \quad \theta_{23}^q \sim \lambda^2. \quad (29)$$

Assuming $\theta_{23}^q \approx \pi/4$, $\theta_{13}^q \approx 0$ and treating $\theta_{13}^q$ as $\approx 0$, this gives the high energy predictions (using Eqs. (5) and (29))

$$s_{23} \approx s_{23}^\nu - \frac{1}{\sqrt{2}}\theta_{23}^\nu \quad \rightarrow \quad \theta_{23} \approx \theta_{23}^\nu - \theta_{23}^q, \quad (30a)$$
$$\theta_{13} \approx \frac{1}{\sqrt{2}}\theta_{12}^e \quad \rightarrow \quad \theta_{13} \approx 1.06 \theta_C, \quad (30b)$$
$$s_{12} \approx \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\theta_{12}^e \quad \rightarrow \quad \theta_{12} \approx \frac{\pi}{4} - \frac{1}{\sqrt{2}}\theta_{12}^e \approx \frac{\pi}{4} - 1.06 \theta_C, \quad (30c)$$

where we have written $\frac{3}{2}\theta_C = 1.06$. Therefore the approximate QLC relation $\theta_{12} \approx \pi/4 - 1.06 \theta_C$ is realized due to the choice of the operators which give the (1,2)- and (2,2)-entries in $Y_e$ and $Y_d$, yielding Clebsch factors $-3$ for $(Y_e)_{12}/(Y_d)_{12}$ and $2$ for $(Y_e)_{22}/(Y_d)_{22}$.

### 4.1.3 RG Modification of QLC and other Corrections

The RG analysis, including the running of the neutrino parameters with successively integrating out the right-handed neutrinos, can be performed conveniently using the software packages REAP/MPT introduced in \(^7\) The RG corrections to the mixing
angles in our model are (assuming $M_{\text{SUSY}} \approx 1$ TeV, $\tan \beta \approx 40$):

\begin{align}
\theta_{12}(M_X) - \theta_{12}(M_Z) &\approx 0.8^\circ, \\
\theta_{13}(M_X) - \theta_{13}(M_Z) &\approx 0.5^\circ, \\
\theta_{23}(M_X) - \theta_{23}(M_Z) &\approx 2.9^\circ.
\end{align}

(31a) (31b) (31c)

In particular, the QLC relation is slightly modified by $\approx 0.8^\circ$ due to the RG correction. In addition, there are corrections within the model itself, which can lead to a deviation from QLC. However, the latter corrections can give deviations in both directions (+/-), whereas the RG running shifts the prediction to smaller values. The predictions of the model at $M_X$ and at $M_Z$ are summarized in Tab. 7.

| quantity | $\theta_{12}$ | $\theta_{13}$ |
|----------|---------------|---------------|
| prediction at $M_X$ | $\frac{\pi}{4} - 1.06 \theta_C$ | $1.06 \theta_C$ |
| prediction at $M_Z$ | $\frac{\pi}{4} - 1.06 \theta_C - 0.8^\circ \approx 30.5^\circ$ | $1.06 \theta_C - 0.5^\circ \approx 13.2^\circ$ |

Table 7: Predictions for the lepton mixing angles at the unification scale $M_X \approx 10^{16}$ GeV and at low energy $M_Z$ for the model discussed in Sec. 4.1. RG corrections shift the QLC prediction to slightly smaller values. The factor 1.06 is equal to $1.5/\sqrt{2}$. We have used $\sin(\theta_C) = 0.224$. In addition to the corrections shown in the table, there are corrections from the model itself, e.g. from sub-leading higher-dimensional operators.

5 Conclusions

We have studied possible ways to realize the QLC relation $\theta_{12} \simeq \pi/4 - \theta_C$ between the Cabibbo angle and the solar mixing angle in realistic quark-lepton unification theories based on the Pati-Salam gauge group $\text{SU}(2)_L \times \text{SU}(2)_R \times \text{SU}(4)_C$. This represents the first attempt at a unified model at the GUT scale capable of predicting the QLC relation. The Pati-Salam gauge group is necessary to relate the charged lepton mixing angles to the down-quark mixing angles, which in turn dominate the contribution to the Cabibbo angle, allowing the QLC relation to emerge, albeit in a non-trivial way. If such a QLC relation, which is consistent with recent observations, is in fact realized and if it is not accidental, this would point towards quark-lepton unification and could tell us a great deal about the detailed structure of the unified theory.

One necessary ingredient for QLC is maximal mixing $\theta_{12}^\nu = \pi/4$ from the neutrino sector. We have therefore started from an inverted hierarchy structure of the neutrino mass matrix, constructed via the see-saw mechanism, which predicts $\theta_{12}^\nu = \pi/4$ and $\theta_{13}^\nu = 0$. Although the effective neutrino mass matrix has a form which is invariant
under $L_e - L_{\mu} - L_\tau$, we have not imposed this symmetry in the high energy theory. In fact we have instead relied on a different $U(1)_X$ family symmetry as playing a leading role in achieving the inverted hierarchy structure.

The QLC relation then has to be realized from the contribution to the lepton mixing matrix of the charged leptons. We have analyzed the generic problems with realizing QLC in models where the $SU(4)_C$ quark-lepton unification scale is close to the GUT scale. In particular, if $\theta_{12}' = \pi/4$ and $\theta_{23}' \approx \pi/4$ are generated by the neutrino mass matrix $m_{\nu LL}$, $\theta_{12}'$ has to be about $\sqrt{2}$ larger than $\theta_C$ in order to lead to the QLC relation. Furthermore, in models of quark-lepton unification, realizing the correct ratio of the strange mass to the muon mass at high energy typically requires an operator which introduces a group theoretical Clebsch factor (Georgi-Jarlskog factor) at the (2,2)-entry of the charge lepton Yukawa matrix. Such a Georgi-Jarlskog factor also affects the relation between $\theta_{12}'$ and the Cabibbo angle, making $\theta_{12}'$ three times larger than $\theta_C$.

In this letter we have identified a minimal set of operators within the framework of Pati-Salam models which can lead to approximate QLC while remaining consistent with other known data. A direct prediction of our approach is that the neutrino mass hierarchy is of the inverted type. It is instructive to note that in this model there are significant corrections to exact QLC coming from a variety of different sources. In order to compensate for the $1/\sqrt{2}$ suppression referred to above, we have enhanced the charged lepton mixing angle by a factor of $3/2$, leading to an approximate QLC relation $\theta_{12} \approx \pi/4 - 1.06 \theta_C$. In addition RG corrections reduce the prediction by a further $0.8^\circ$, resulting in the final prediction $\theta_{12} \approx 30.5^\circ$, with some theoretical error. A generic prediction of our approach is $\theta_{13} \approx \theta_C$, where in this case the factor of 1.06 enhancement approximately cancels the RG correction leading to the stated prediction. These predictions allow these models to be confirmed or excluded by the current generation of neutrino oscillation experiments. We have also commented in an Appendix on alternative classes of QLC models with a lopsided charged lepton mass matrix, which lead to somewhat different predictions.

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Appendix

A Our Conventions

For the mass matrix of the charged leptons $m_E = Y_e v_d$, where $v_d = \langle h_0^0 \rangle$, defined by

$$L_e = -(m_E) \overline{f_g} e_R + \text{h.c.},$$

and for the neutrino mass matrix $m_{\nu}^L L$, defined by

$$L_\nu = (m_{\nu}^L) \overline{f_\nu} \nu_L + \text{h.c.},$$

the change from flavour basis to mass eigenbasis can be performed with the unitary diagonalization matrices $U_{eL}, U_{eR}$ and $U_{\nu L}$ by

$$U_{eL} m_E U_{eR}^\dagger = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad U_{\nu L} m_{\nu}^L U_{\nu L}^T = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \quad (A.1)$$

The MNS matrix is then given by

$$U_{\text{MNS}} = U_{eL} U_{\nu L}^\dagger. \quad (A.2)$$

We use the parameterization $U_{\text{MNS}} = R_{23} U_{13} R_{12} P_0$ with $R_{23}, U_{13}, R_{12}$ and $P_0$ being defined as

$$R_{12} := \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{13} := \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i \delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i \delta} & 0 & c_{13} \end{pmatrix},$$

$$R_{23} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad P_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i \beta_2} & 0 \\ 0 & 0 & e^{i \beta_3} \end{pmatrix}. \quad (A.3)$$

and where $s_{ij}$ and $c_{ij}$ stand for $\sin(\theta_{ij})$ and $\cos(\theta_{ij})$, respectively. The matrix $P_0$ contains the possible Majorana phases $\beta_2$ and $\beta_3$. $\delta$ is the Dirac CP phase relevant for neutrino oscillations.

B Alternative Approach: Lopsided $Y_e$

Let us now discuss an alternative route to models with QLC, where nearly maximal lepton mixing $\theta_{23}$ stems from the charged lepton sector. Single maximal lepton mixing $\theta_{12}$ is then generated by $m_{LL}$. This has the following advantage for constructing models with QLC: with $U_{eL} \approx R_{23}^e (\pi/4) R_{12}^e (\theta_C)$, the rotations are already in the correct order for the MNS matrix,

$$U_{\text{MNS}} \approx R_{23}^e (\frac{\pi}{4}) U_{13}^e R_{12}^e (\theta_C) R_{12}^\nu (\frac{\pi}{4}) P_0^\nu \approx R_{23}^e (\frac{\pi}{4}) U_{13}^e R_{12}^e (\frac{\pi}{4} - \theta_C) P_0^\nu. \quad (B.4)$$

The very small neutrino mixings $\theta_{23}^\nu$ and $\theta_{13}^\nu$ are treated as 0. With respect to QLC in lopsided models, it is convenient to perform the 1-2 rotation $R_{12}^e$ first when diagonalizing $Y_e$, followed by the rotations $U_{13}^e$ and $R_{23}^e$. Note that this differs somewhat from the usual convention.
In particular, we do not obtain an unwanted factor of $1/\sqrt{2}$ for the relation between $\theta_C$ and the correction to maximal lepton mixing $\theta_{12}$ (cf. Eq. (6c)). Omitting common $O(1)$-coefficients and treating very small elements as zero, QLC can be achieved with $Y_e$, $Y_d$ and $m_{LL}$ of the form

$$
Y_d = \begin{pmatrix}
0 & 1\lambda^4 & \lambda^4 \\
* & 1\lambda^3 & \lambda^2 \\
* & * & 1
\end{pmatrix}, \quad Y_e = \begin{pmatrix}
0 & -3\lambda^4 & \lambda \\
* & -3\lambda^3 & 1 \\
* & 0 & 1
\end{pmatrix}, \quad m_{LL} \approx \begin{pmatrix}
0 & m & 0 \\
m & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

(B.5)

Some comments are in order: Since we have chosen $(Y_e)_{32}$ to be very small (say $\ll \lambda^4$), $\theta_{12}^e$ is given by $\theta_{12}^e \approx (Y_e)_{12}/(Y_e)_{22} \approx \lambda$. The rotation by $R_{i3}^e$ then has to generate a 0-entry in $(Y_e)_{13}$, which typically leads to $\theta_{13}^e = \theta(\lambda)$, however it can well be smaller due to cancellations. Of course, $\theta_{23}^e$ close to maximal follows from the lopsided structure of $Y_e$. The operators for the (1,2)- and (2,2)-entries in $Y_e$ and $Y_d$ have to be chosen such that we obtain Clebsch factors $-3$ for $(Y_e)_{12}/(Y_d)_{12}$ and for $(Y_e)_{22}/(Y_d)_{22}$. As discussed above, for a lopsided $Y_e$ the equal Clebsch factors are required for realizing the desired relation $\theta_{12} \approx \frac{\pi}{4} - \theta_C$ at $M_X$. For the quark sector, models of this type predict $m_\mu/m_s = 3 \cos(\theta_{23}^e)$ at $M_X$. In contrast to the model in Sec. 4.1 with large $\theta_{23}$ originating from the neutrino sector, such lopsided QLC models do not firmly predict a large $\theta_{13}$ equal to the Cabibbo angle.

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