On the Asymptotic Integration of a System of Linear Differential Equations with Oscillatory Decreasing Coefficients

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Abstract. A system of linear differential equations with oscillatory decreasing coefficients is considered. The coefficients has the form $t^{-\alpha}a(t)$, $\alpha > 0$, where $a(t)$ is trigonometric polynomial with an arbitrary set of frequencies. The asymptotic behavior of the solutions of this system as $t \to \infty$ is studied. We construct an invertible (for sufficiently large $t$) change of variables that takes the original system to a system not containing oscillatory coefficients in its principal part. The study of the asymptotic behavior of the solutions of the transformed system is a simpler problem. As an example, the following equation is considered:

$$\frac{d^2x}{dt^2} + \left(1 + \frac{\sin \lambda t}{t^{\alpha}}\right)x = 0,$$

where $\lambda$ and $\alpha$, $0 < \alpha \leq 1$, are real numbers.

The stability and asymptotic behavior of the solutions of linear systems of differential equations

$$\frac{dx}{dt} = Ax + B(t)x,$$

where $A$ is a constant matrix, and matrix $B(t)$ is small in a certain sense when $t \to \infty$ has been studied by many authors (see [1-6]) as well as the monographs [7-11]).

Shtokalo [12-13] studied of the stability of solutions of the following system of differential equations

$$\frac{dx}{dt} = Ax + \varepsilon B(t)x. \quad (1)$$

Here $\varepsilon > 0$ is a small parameter, $A$ is a constant square matrix, and $B(t)$ is a square matrix whose elements are trigonometric polynomials $b_{kl}(t)$ ($k, l = 1, ..., m$) in the form

$$b_{kl}(t) = \sum_{j=1}^{m} b_{jl}^{kl} e^{i\lambda_j t};$$

Matrices with such elements are called matrices of class $Sigma$. The mean of a matrix from $\Sigma$ is a constant matrix that consists of the constant terms ($\lambda_j = 0$) of the elements of the matrix.

Using the Bogoliubov averaged method, Shtokalo transformed system (1) into a system with constant coefficients depending on the parameter $\varepsilon$ up to terms of any order in smallness in $\varepsilon$.

We also mention that the method of averaging in the first approximation was utilized in [14,16] for studying the asymptotic behavior of solutions of a particular class of systems of equations with oscillatory decreasing coefficients.
In the present paper we adopt the method of Shtokalo for the problem of asymptotic integration of systems of linear differential equations with oscillatory decreasing coefficients.

We consider the following system of differential equations in \( n \)-dimensional space \( \mathbb{R}^n \)

\[
\frac{dx}{dt} = \left\{ A_0 + \sum_{j=1}^{k} \frac{1}{t^{j\alpha}} A_j(t) \right\} x + \frac{1}{t^{(1+\delta)}} F(t) x. 
\]  

(2)

Here, \( A_0 \) is a constant \( n \times n \) matrix, and \( A_1(t), A_2(t), ..., A_k(t) \) are \( n \times n \) matrices that belong to \( \Sigma \). We shall assume that matrix \( A_0 \) is in Jordan canonical form, a real number \( \alpha \) and a positive integer \( k \) satisfy \( 0 < k\alpha \leq 1 < (k+1)\alpha \), \( \delta > 0 \). The square matrix \( F(t) \) satisfies

\[
||F(t)|| \leq C < \infty
\]

for \( t_0 \leq t < \infty \), where \( ||.|| \) is some matrix norm in \( \mathbb{R}^n \).

We are concerned with the behavior of solutions of system (2) when \( t \to \infty \). Let us construct an invertible change of variables (for sufficiently large \( t \), \( t > t^* > t_0 \)) that would transform system (2) into the simpler system

\[
\frac{dy}{dt} = \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} y + \frac{1}{t^{(1+\varepsilon)}} G(t) y, \quad \varepsilon > 0, \quad t > t^*.
\]  

(3)

Here, \( A_0, A_1, ..., A_k \) are constants square matrices (moreover, \( A_0 \) is the same matrix as in (1)), and, the matrix \( G(t) \) has the same properties as the matrix \( F(t) \) in system (2).

Without loss of generality, we can assume that all eigenvalues of the matrix \( A_0 \) are real. Indeed, if matrix \( A_0 \) has complex eigenvalues, then we can make a change of variables in (2)

\[
y = e^{iR\alpha} z,
\]

where \( R \) is diagonal matrix composed of the imaginary parts of eigenvalues of the matrix \( A_0 \). This change of variable with coefficients, which are bounded in \( t \), \( t \in (-\infty, \infty) \), transforms the matrix \( A_0 \) into the matrix \( A_0 - iR \), which only has real eigenvalues.

We shall try to choose an invertible change of variables (for sufficiently large \( t \), in the form

\[
x = \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} y, \quad (4)
\]

to transform system (2) into system (3), where \( Y_0(t) = I \) is the identity matrix, and, \( Y_1(t), ..., Y_k(t) \) are \( n \times n \) matrices that belong to \( \Sigma \) and have zero mean value. By substituting (4) into (2), and, replacing \( \frac{dy}{dt} \) by the right hand side of (3) we obtain

\[
\left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y + \frac{1}{t^{(1+\varepsilon)}} \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} G(t) y + \frac{1}{t^{(1+\alpha)}} W(t) y + \left\{ \sum_{j=1}^{k} \frac{1}{t^{j\alpha}} \frac{Y_j(t)}{dt} \right\} y = 0
\]

(5)

where
\[ W(t) = \left\{ -\sum_{j=1}^{k} \frac{j\alpha}{t^{(j-1)\alpha}} Y_j(t) \right\}, \quad (6) \]
and
\[ U(t) = F(t) \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\}. \quad (7) \]

Equating the summands that contain \( t^{-j\alpha} (j = 1, \ldots, k) \) in the left and the right hand sides of (5) yields a system of \( k \) linear matrix differential equation with constant coefficients
\[
\frac{dY_j(t)}{dt} - A_0 Y_j(t) + Y_j(t) A_0 = \sum_{i=0}^{j-1} A_{j-i} Y_i(t) - \sum_{i=0}^{j-1} Y_i(t) A_{j-i}, \quad (j = 1, \ldots, k). \quad (8)
\]

The solvability of system (8) was studied [11]. We represent \( Y_j(t) \) as a finite sum
\[
Y_j(t) = \sum_{\lambda \neq 0} y_{j\lambda} e^{i\lambda t},
\]
where \( y_{j\lambda} \) are constants \( n \times n \) matrices, and, obtain matrix equations
\[
i\lambda y_{j\lambda} - A_0 y_{j\lambda} + y_{j\lambda} A_0 = b_{j\lambda}.
\]
Since all the eigenvalues of \( A \) are real, the matrix equations have unique solutions for \( \lambda \neq 0 \) (see, for instance, [16,17]). On each of the \( k \) steps of the solution process we determine the matrix \( A_j \) from the condition that the right hand side of (8) has a zero mean value. In particular, for \( j = 1 \)
\[
\frac{dY_1(t)}{dt} - A_0 Y_1(t) + Y_1(t) A_0 = A_1(t) - A_1,
\]
where \( A_1 \) is the mean value of the matrix \( A_1(t) \). Relation (5) implies the following result.

**Theorem 1.** System (2), for sufficiently large \( t \), can be transformed using a change of variables (4) into a system
\[
\frac{dy}{dt} = \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y + \frac{1}{t^{(1+\varepsilon)}} G(t)y,
\]
where \( \varepsilon > 0 \), and \( ||G(t)|| \leq C_1 < \infty \).

**Proof.** Substituting (4) into (2), we obtain
\[
\left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} \frac{dy}{dt} = \left\{ A_0 + \sum_{j=1}^{k} \frac{1}{t^{j\alpha}} A_j(t) \right\} \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} y -
\frac{1}{t^{(1+\alpha)}} W(t)y + \frac{1}{t^{(1+\alpha)}} U(t)y,
\]
where \( W(t) \) and \( U(t) \) are defined by (6) and (7) respectively. The last relation can be rewritten as

\[
\left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} \left\{ \frac{dy}{dt} - \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y \right\} = \\
= \left\{ A_0 + \sum_{j=1}^{k} \frac{1}{t^{j\alpha}} A_j(t) \right\} \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} y - \left\{ \sum_{j=1}^{k} \frac{1}{t^{j\alpha}} \frac{dY_j(t)}{dt} \right\} y - \\
- \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y - \frac{1}{t^{(1+\alpha)}} W(t)y + \frac{1}{t^{(1+\tau)}} U(t)y.
\]

Due to (8) we get

\[
\left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} \left\{ \frac{dy}{dt} - \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y \right\} = \\
= \frac{1}{t^{(1+\alpha+\alpha)}} S(t)y - \frac{1}{t^{(1+\alpha)}} W(t)y + \frac{1}{t^{(1+\delta)}} U(t)y,
\]

where elements of the matrix \( S(t) \) can be represented as \( t^{-j\alpha} a_j(t) \) \((j = 0, \ldots, k)\), and \( a_j(t) \) are trigonometric polynomials. Therefore,

\[
\frac{1}{t^{(1+k)\alpha}} S(t) - \frac{1}{t^{(1+\alpha)}} W(t) + \frac{1}{t^{(1+\delta)}} U(t) = \frac{1}{t^{(1+\varepsilon)}} R(t),
\]

where \( \varepsilon > 0 \) and \( R(t) \) satisfies

\[
||R(t)|| \leq C_2 < \infty.
\]

The identity (9), for sufficiently large \( t \), can be rewritten as

\[
\frac{dy}{dt} = \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y + \frac{1}{t^{(1+\varepsilon)}} \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\}^{-1} R(t)y.
\]

This relation and (10) yield the assertion of the theorem.

The main part of system (3)

\[
\frac{dy}{dt} = \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y
\]

does not have oscillating coefficients. This makes it simpler than the original system (2). In particular, the Fundamental Theorem of Levinson on asymptotic behavior of solutions of linear systems of differential equations (see [1.2], as well as [8,9]), readily yields the following resolt for system (3).

**Theorem 2.** Let us first nonzero matrix among the matrices \( A_j, j = 0, 1, \ldots, k \) be the matrix \( A_1 \). Let the matrix \( A_1 \) have distinct eigenvalues. Then the fundamental matrix of system (3) has the following form

\[
X(t) = (P + o(1))\exp \int_{t^*}^{t} \Lambda(s)ds, \quad t > t^*, \quad t \to \infty,
\]
where $P$ is a matrix composed of the eigenvectors of the matrix $A_t$, and $\Lambda(t)$ is a diagonal matrix whose elements are eigenvalues of the matrix $\sum_{j=t}^{k} t^{-j} A_j$.

To prove this theorem we just have to observe that the system of differential equations
\[
\frac{dx}{dt} = \frac{1}{t^\alpha} A(t)x + \sum_{j=t+1}^{k} \frac{1}{t^j} A_j x
\]
can be transformed into
\[
\frac{dx}{d\tau} = \frac{1}{1 - \lambda} [A(t) + \sum_{j=t+1}^{k} \frac{1}{t^j} A_j] x
\]
using the change of variables $\tau = t^{1-\lambda}$.

As an example we consider an equation of an adiabatic oscillator
\[
\frac{d^2 y}{dt^2} + (1 + \frac{1}{t^\alpha} \sin \lambda t) y = 0, \tag{11}
\]
where $\lambda$, $\alpha$ are real numbers, and $0 < \alpha \leq 1$. The problem of asymptotic integration of the equation (11) has been studied in [5,6, 18–20]. In particular, asymptotics of solutions for $\frac{1}{2} \leq \alpha \leq 1$ were obtained. The method that we proposed in this chapter can be used to obtain (in a simple manner) all known results on asymptotics of solutions of equation (11) as well as to establish new results.

Let us pass from equation (11) to the system of equations $(x = (x_1, x_2))$ using a change of variables
\[
y = x_1 \cos t + x_2 \sin t, \quad y' = -x_1 \sin t + x_2 \cos t. \tag{12}
\]
We obtain the system
\[
\frac{dx}{dt} = \frac{1}{t^\alpha} A(t)x. \tag{13}
\]
It is convenient to rewrite the matrix $A(t)$ in complex form as
\[
A(t) = a_1 e^{i(\lambda+2)t} + \bar{a}_1 e^{-i(\lambda+2)t} + a_2 e^{i(\lambda-2)t} + \bar{a}_2 e^{-i(\lambda-2)t} + a_3 e^{i\lambda t} + \bar{a}_3 e^{-i\lambda t},
\]
where
\[
a_1 = \frac{1}{8} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}, \quad a_2 = \frac{1}{8} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad a_3 = \frac{1}{8} \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix},
\]
and the matrices $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$ are complex conjugates to the matrices $a_1, a_2, a_3$ respectively.

The values of $\alpha$ and $\lambda$ significantly affect the behavior of solutions of system (13). We denote by $R(t)$ a $2 \times 2$ matrix that satisfies
\[
||R(t)|| \leq C_3 < \infty.
\]
for all $t$. 
First, assume $\frac{1}{2} < \alpha \leq 1$. For $\lambda \neq \pm 2$ system (3) becomes

$$\frac{dy}{dt} = \frac{1}{t^{1+\varepsilon}} R(t)y, \quad \varepsilon > 0.$$ 

Therefore, it is easy to see (taking into consideration change (12)), that the fundamental system of solutions of equation (11) for $\frac{1}{2} < \alpha \leq 1$, $\lambda \neq \pm 2$ as $t \to \infty$ has the form

$$x_1 = \cos t + o(1), \quad x_2 = \sin t + o(1),$$

$$x'_1 = -\sin t + o(1), \quad x'_2 = \cos t + o(1).$$

Therefore, it is easy to see (taking into consideration change (12)), that the fundamental system of solutions of equation (11) for $1/2 < \alpha \leq 1$, $\lambda \neq \pm 2$ as $t \to \infty$ has the form

$$x_1 = \cos t + o(1), \quad x_2 = \sin t + o(1),$$

$$x'_1 = -\sin t + o(1), \quad x'_2 = \cos t + o(1).$$

We shall represent the fundamental system of solutions of equation (11) as a matrix with rows $x_1, x_2$ and $x'_1, x'_2$.

Now assume $\lambda = \pm 2$. More specifically let $\lambda = 2$. Then system (3) becomes

$$\frac{dy}{dt} = \frac{1}{t^\alpha} A_1 y + \frac{1}{t^{1+\varepsilon}} R(t)y, \quad \varepsilon > 0.$$ 

Here

$$a_2 + \bar{a}_2 = A_1 = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Theorem 2 implies that for $t \to \infty$ the fundamental matrix of system (11) has the following form

$$Y(t) = \begin{pmatrix} \exp \left( \int_t^\infty \frac{1}{4s} ds \right) & 0 \\ 0 & \exp \left( -\int_t^\infty \frac{1}{4s} ds \right) \end{pmatrix} [I + o(1)].$$

Therefore, for $\alpha = 1$, $\lambda = 2$ we obtain the fundamental system of solutions of equation (11) as

$$\begin{pmatrix} t^{1/4} \cos t & t^{-1/4} \sin t \\ -t^{1/4} \sin t & t^{-1/4} \cos t \end{pmatrix} [I + o(1)],$$

while for $1/2 < \alpha < 1$, $\lambda = 2$ for $t \to \infty$ we get

$$\begin{pmatrix} \exp \left( \frac{t^{1-\alpha}}{4(1-\alpha)} \right) \cos t & \exp \left( -\frac{t^{1-\alpha}}{4(1-\alpha)} \right) \sin t \\ -\exp \left( \frac{t^{1-\alpha}}{4(1-\alpha)} \right) \sin t & \exp \left( -\frac{t^{1-\alpha}}{4(1-\alpha)} \right) \cos t \end{pmatrix} [I + o(1)].$$

We note that, for $\lambda = \pm 2$, and $\frac{1}{2} < \alpha \leq 1$, equation (11) has unbounded solutions. Moreover, for $\alpha = 1$, the solutions have a polynomial growth, while, for $\alpha \neq 1$, they grow exponentially.

We now assume $\frac{1}{3} < \alpha \leq \frac{1}{2}$. In this case a change of variables (4) transforms (13) into

$$\frac{dy}{dt} = \frac{1}{t^\alpha} A_1 y + \frac{1}{t^{2\alpha}} A_2 y + \frac{1}{t^{1+\varepsilon}} R(t)y, \quad \varepsilon > 0.$$
If \( \lambda \neq \pm 2, \pm 1 \), then the matrix \( A_1 \) is zero, and matrix \( A_2 \) has the form

\[
A_2 = i \left[ \frac{1}{\lambda + 2}(a_1 \bar{a}_1 - \bar{a}_1 a_1) + \frac{1}{\lambda - 2}(a_2 \bar{a}_2 - \bar{a}_2 a_2) + \frac{1}{\lambda}(a_3 \bar{a}_3 - \bar{a}_3 a_3) \right]. \tag{14}
\]

Computing \( A_2 \) yields

\[
A_2 = \frac{1}{4(\lambda^2 - 4)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The system

\[
\frac{dy}{dt} = \frac{1}{t^{2\alpha}} A_2 y
\]

can be integrated. We obtain that, for \( t \to \infty \), the fundamental system of solutions of equation (11) with \( \alpha = \frac{1}{2} \), \( \lambda \neq \pm 2, \pm 1 \), has the form

\[
\begin{pmatrix} \cos(t + \gamma \ln t) & \sin(t + \gamma \ln t) \\ -\sin(t + \gamma \ln t) & \cos(t + \gamma \ln t) \end{pmatrix} [I + o(1)],
\]

where \( \gamma = \frac{1}{4(\lambda^2 - 4)} \). For \( \frac{1}{3} < \alpha < \frac{1}{2} \), and \( \lambda \neq \pm 2, \pm 1 \), the fundamental system of solutions of equation (11) has the form

\[
\begin{pmatrix} \cos \left( \frac{t^{1-2\alpha}}{4(1-2\alpha)(\lambda^2 - 4)} \right) & \sin \left( \frac{t^{1-2\alpha}}{4(1-2\alpha)(\lambda^2 - 4)} \right) \\ -\sin \left( \frac{t^{1-2\alpha}}{4(1-2\alpha)(\lambda^2 - 4)} \right) & \cos \left( \frac{t^{1-2\alpha}}{4(1-2\alpha)(\lambda^2 - 4)} \right) \end{pmatrix} [I + o(1)]
\]

as \( t \to \infty \).

We now assume \( \alpha = \frac{1}{2}, \lambda = 1 \). In this case \( A_1 \) is zero, and \( A_2 \) is determined by

\[
iA_2 = -\frac{1}{3} a_1 \bar{a}_1 + \frac{1}{3} \bar{a}_1 a_1 - \bar{a}_2 a_2 + a_2 \bar{a}_2 - a_3 \bar{a}_3 + \bar{a}_3 a_3 + a_2 a_3 + a_3 a_2 - \bar{a}_2 \bar{a}_3 + \bar{a}_3 \bar{a}_2.
\]

A simple calculation yields

\[
A_2 = \frac{1}{24} \begin{pmatrix} 0 & -5 \\ -1 & 0 \end{pmatrix}.
\]

The corresponding system (3) has the following form

\[
\frac{dy}{dt} = \frac{1}{t} A_2 y + \frac{1}{t^{1+\varepsilon}} R(t)y, \quad \varepsilon > 0.
\]

By integrating the system

\[
\frac{dy}{dt} = \frac{1}{t^{1+\varepsilon}} A_2 y,
\]

we obtain its fundamental matrix

\[
Y(t) = \begin{pmatrix} -\sqrt{5}t^{\theta} & \sqrt{5}t^{-\theta} \\ t^{\theta} & t^{-\theta} \end{pmatrix},
\]
where \( \varrho = \frac{\sqrt{5}}{24} \). Then the fundamental system of solutions of equation (11) for \( \alpha = \frac{1}{2} \), \( \lambda = 1 \), and \( t \to \infty \), has the form

\[
\begin{pmatrix}
t^\varrho \sin(t - \beta) & t^{-\varrho} \sin(t + \beta) \\
t^\varrho \cos(t - \beta) & t^{-\varrho} \cos(t + \beta)
\end{pmatrix}
[I + o(1)],
\]

where

\[
\varrho = \frac{\sqrt{5}}{24}, \quad \beta = \arctan \sqrt{5}, \quad 0 < \beta < \frac{\pi}{2}.
\] (15)

If \( \frac{1}{3} < \alpha < \frac{1}{2} \) and \( \lambda = 1 \) we have the system

\[
\frac{dy}{dt} = \frac{1}{t^\alpha} A_2 y + \frac{1}{t^{1+\varepsilon}} R(t)y, \quad \varepsilon > 0.
\]

Using Theorem 2 we obtain the asymptotics of the fundamental matrix of this system, and then, using the change (12), the asymptotics of the fundamental system of solutions of equation (1.7.12) for \( \frac{1}{3} < \alpha < \frac{1}{2} \), \( \lambda = 1 \), and \( t \to \infty \):

\[
\begin{pmatrix}
\exp(\varrho \frac{1}{1-2\alpha}) \sin(t - \beta) & \exp(-\varrho \frac{1}{1-2\alpha}) \sin(t + \beta) \\
\exp(\varrho \frac{1}{1-2\alpha}) \cos(t - \beta) & \exp(-\varrho \frac{1}{1-2\alpha}) \cos(t + \beta)
\end{pmatrix}
[I + o(1)],
\]

where \( \varrho \) are \( \beta \) are defined by (15). Thus, for \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \) we observe a polynomial growth of solutions, while for \( \frac{1}{3} < \alpha < \frac{1}{2} \) and \( \lambda = 1 \) the solutions grow exponentially.

Now let \( \alpha = \frac{1}{2} \) and \( \lambda = 2 \). Simple calculations show that

\[
A_1 = \begin{pmatrix}
\frac{1}{4} & 0 \\
0 & -\frac{1}{4}
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & -\frac{1}{64} \\
\frac{1}{64} & 0
\end{pmatrix}.
\]

Therefore, we get a system

\[
\frac{dy}{dt} = \frac{1}{t^{\frac{3}{2}}} A_1 y + \frac{1}{t} A_2 y + \frac{1}{t^{1+\varepsilon}} R(t)y, \quad \varepsilon > 0.
\] (16)

We compute the eigenvalues of the matrix

\[
\frac{1}{t^{\frac{3}{2}}} A_1 + \frac{1}{t} A_2,
\]

integrate them, and, using Theorem 2, we obtain the asymptotics of the fundamental matrix of system (16). Next we find the fundamental system of solutions of equation (11) for \( \alpha = \frac{1}{2} \), \( \lambda = 2 \) and \( t \to \infty \):

\[
\begin{pmatrix}
\exp(\phi(t)) \cos t & \exp(-\phi(t)) \sin t \\
-\exp(\phi(t)) \sin t & \exp(-\phi(t)) \cos t
\end{pmatrix}
[I + o(1)],
\]

where \( \phi(t) = \frac{1}{2} \sqrt{t} \).
For $\frac{1}{3} < \alpha < \frac{1}{2}$, $\lambda = 2$ instead of (16) we get a system
\[
\frac{dy}{dt} = \frac{1}{t^\alpha} A_1 y + \frac{1}{t^{2\alpha}} A_2 y + \frac{1}{t^{1+\varepsilon}} R(t) y,
\]
where $\varepsilon > 0$, with the same matrices $A_1$, $A_2$. Therefore, it is straightforward to write the asymptotics of the fundamental system of solutions of equation (11) for $\frac{1}{3} < \alpha < \frac{1}{2}$, $\lambda = 2$, and $t \to \infty$.

Finally, let $\alpha = \frac{1}{3}$, $\lambda \neq \pm 1$, and $\lambda \neq \pm 2$. Then, it turns out that $A_1$ is zero, and $A_2$ is defined by (14). Matrix $A_3$ differs from zero only if $\lambda = \pm \frac{2}{3}$. Assume $\lambda = \frac{2}{3}$. System (3) then becomes
\[
\frac{dy}{dt} = \frac{1}{t^{\frac{1}{3}}} A_2 y + \frac{1}{t} A_3 y + \frac{1}{t^{1+\varepsilon}} R(t) y,
\]
where $\varepsilon > 0$ and the matrices $A_2$ and $A_3$ are defined by
\[
A_2 = \begin{pmatrix} 0 & -\frac{9}{128} \\ \frac{9}{128} & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -\frac{27}{1024} & 0 \\ 0 & \frac{27}{1024} \end{pmatrix}.
\]

We compute the eigenvalues of the matrix
\[
\frac{1}{t^{\frac{1}{3}}} A_2 + \frac{1}{t} A_3.
\]
These eigenvalues have zero real parts, for sufficiently large $t$. Further, using the same scheme as before we find the asymptotics of the fundamental system of solutions of equation (11). We only note that the solutions of equation (11) are bounded for $\alpha = \frac{1}{3}$, $\lambda = \pm \frac{2}{3}$ as $t \to \infty$.

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