RÉNYI ENTROPY AND PATTERN MATCHING FOR RUN-LENGTH ENCODED SEQUENCES

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Abstract. In this note, we studied the asymptotic behaviour of the length of the longest common substring for run-length encoded sequences. When the original sequences are generated by an $\alpha$-mixing process with exponential decay (or $\psi$-mixing with polynomial decay), we proved that this length grows logarithmically with a coefficient depending on the Rényi entropy of the pushforward measure. For Bernoulli processes and Markov chains, this coefficient is computed explicitly.

1. Introduction

Since Big Data seems to be the trending field of (at least) this decade, data compression algorithms have become a fundamental tool for data storage and are in the first lines of the battle between storage costs, computations costs and delays in data availability. For an introduction to data compression we refer the reader to [32] and to the unavoidable Lempel-Ziv algorithms [35, 36].

For sequences with long runs of the same value, Run-Length Encoding (RLE) is a simple and efficient lossless data compression method. More precisely, for a run of the same value, the algorithm stored the value and the length of the run. For example, the following binary sequence

$00001110000000110011111111110000000$

will be compressed as

$(0, 4)(1, 3)(0, 8)(1, 2)(0, 2)(1, 9)(0, 8)$.  

Thus, this sequence of 37 characters will be represented after compression by a sequence of 14 characters.

RLE is typically used for image compression but has also application in image analysis [20], texture analysis of volumetric data [34] and has also been used for data compression of television signals [29] and fax transmission [22].

Since pattern (or string) matching problems are not only highly significant in computer sciences, information theory and probability (see e.g. [23, 24, 15, 3, ?]) but also in biology [33], geology [26] and linguistics (e.g. [14] and references therein) between others, algorithms to solve string matching problems for RLE strings have been developed (see e.g. [5, 13, 21, 4, 10] and references within).

In this note, we will focus on a particular string matching problem: the longest common substring problem (or longest consecutive common subsequence problem). More precisely, we...
will concentrate on the asymptotics of the length of the longest common substring, i.e. for two sequences $x$ and $y$ drawn randomly from the same alphabet, the behaviour of

$$M_n(x, y) = \max \left\{ k : x_{i+k-1} = y_{j+k-1} \text{ for some } 0 \leq i, j \leq n - k \right\}$$

when $n \to \infty$.

In [7], it was proved that for $\alpha$-mixing process with exponential decay $M_n \sim 2 H_2(\mu) \log n$ almost surely, where $H_2(\mu)$ is the Rényi entropy (see Definition 2.3) of the stationary measure $\mu$. Similar results have been proved for more than two sequences [6] and for random sequences in random environment [30].

In [13], the authors wondered if the above mentioned result holds if the sequences are transformed following certain rules of modification. Thus, if $f$ is a measurable function (called an encoder) transforming a sequence $x$ into another sequence $f(x)$, they studied the behaviour of $M_n(f(x), f(y))$ and obtain a relation with the Rényi entropy of the pushforward measure $f_*\mu$.

A natural question would be to ask if we could apply the results presented in [13] when the encoder is a compression algorithm and in particular the run-length encoder. Unfortunately, to obtain their main result, the authors needed that the encoder does not compress too much the sequences, an hypothesis which is not satisfied by the run-length encoder. Thus, we present here a different proof which allows us to prove, in Theorem 2.6, that, when $f$ is the run-length encoder and the original sequences are generated by an $\alpha$-mixing process with exponential decay (or $\psi$-mixing with polynomial decay), almost surely

$$M_n(f(x), f(y)) \sim_{n \to \infty} 2 \frac{H_2(f_*\mu)}{H_2(\mu)} \log n.$$

We apply this result to Bernoulli processes (Example 2.7) and Markov chains (Examples 2.8 and 2.9), and, in these cases, compute explicitly $H_2(f_*\mu)$. We emphasize that for Markov chains the computation are different whether there are two or more than two states.

Other examples of processes satisfying our mixing assumptions are Gibbs states of a Hölder-continuous potential [8, 31], ARMA processes [27], some renewal processes [2] and stationary determinantal process on the integer lattice [17]. We refer the reader to [16, 9] for more examples and deep surveys on strong mixing conditions.

2. LONGEST COMMON SUBSTRING FOR RLE SEQUENCES

We consider a stationary stochastic process $X = (X_n)_{n \in \mathbb{N}}$ over a finite or countable alphabet $\mathcal{A}$, with stationary measure $\mu$. For $k \in \mathbb{N}$, we denote by $\mathcal{A}^k$ the set of cylinders or strings of length $k$. When there is no ambiguity, cylinders of $\mathcal{A}^k$ will be denoted $\omega$. We will use the notation $x_i^{i+k-1}$ if we need to indicate its time of occurrence $i$. Moreover, $\mu(\omega)$ will denote the probability $\mu(X_i^{i+k-1} = \omega)$ (which is independent of $i$ by stationarity).

We will be interested in some statistical properties of run-length encoded (RLE) sequences where the original sequences are generated by the stochastic process $X$.

**Definition 2.1.** Let $\mathcal{B} = \{(\alpha, k)\}_{\alpha \in \mathcal{A}, k \in \mathbb{N}}$. We define the run-length encoder $f : \mathcal{A}^\mathbb{N} \to \mathcal{B}^\mathbb{N}$ by

$$f(\underbrace{\alpha_1 ... \alpha_1}_{k_1}, \underbrace{\alpha_2 ... \alpha_2}_{k_2}, ..., \underbrace{\alpha_n ... \alpha_n}_{k_n}) = (\alpha_1, k_1)(\alpha_2, k_2) ... (\alpha_n, k_n) ...$$

We observe that for all $i \in \mathbb{N}$, we consider that $\alpha_{i+1} \neq \alpha_i$.

We will focus our analysis on the length of the longest common substring of RLE sequences:
Definition 2.2. Given two sequences \( x, y \), we define the \( n \)-length of the longest common substring by

\[
M_n(x, y) = \max \left\{ k : x_{i+k-1} = y_{j+k-1} \text{ for some } 0 \leq i, j \leq n - k \right\}.
\]

and we will study the behaviour of the \( n \)-length of the longest common substring of the RLE sequences \( f(x), f(y) \)

\[
M_n^{RLE}(x, y) := M_n(f(x), f(y)).
\]

We will prove that \( M_n^{RLE} \) is linked with the Rényi entropy of the pushforward measure \( f_*\mu \). We recall that \( f_*\mu(.) = \mu(f^{-1}(.) \right) and we observe that \( f_*\mu \) is the law of the stochastic process \( f(X) \) but is in general not stationary. We give now the definition of Rényi entropy:

Definition 2.3. The lower and upper Rényi entropies of a measure \( P \) are defined as

\[
H_\alpha(P) = -\lim_{k \to \infty} \frac{1}{k} \log \sum_{\omega} P(\omega)^{2} \quad \text{and} \quad \bar{H}_\alpha(P) = -\lim_{k \to \infty} \frac{1}{k} \log \sum_{\omega} P(\omega)^{2},
\]

where the sums are taken over all cylinders \( \omega \) of length \( k \). When the limit exists we denote by \( H_\alpha(P) \) the common value.

The existence of the Rényi entropy has not been proved for general stochastic processes. However, it was proved for Bernoulli processes, finite Markov chains (e.g. \cite{28}), infinite Markov chains \cite{11}, Gibbs measures of a Hölder-continuous potential \cite{19}, for \( \phi \)-mixing measures \cite{25}, for weakly \( \psi \)-mixing processes \cite{19} and for \( \psi_\rho \)-regular processes \cite{1}.

Definition 2.4. The process \( X \) with stationary measure \( \mu \) is \( \alpha \)-mixing if there exists a function \( \alpha : \mathbb{N} \to \mathbb{R} \) where \( \alpha(g) \) converges to zero when \( g \) goes to infinity and such that

\[
\sup_{A \in \mathcal{F}^n_0 : B \in \mathcal{F}^\infty_{n+g}} |\mu(A \cap B) - \mu(A)\mu(B)| \leq \alpha(g),
\]

for all \( n \in \mathbb{N} \), where for \( 0 \leq J \leq L \leq \infty \), \( \mathcal{F}^J_k \) denotes the \( \sigma \)-algebra \( \sigma(X_k, J \leq k \leq L) \).

When \( \alpha(g) \) decreases exponentially fast to zero, we say that the process is \( \alpha \)-mixing with exponential decay.

The process is \( \psi \)-mixing if there exists a function \( \psi : \mathbb{N} \to \mathbb{R} \) where \( \psi(g) \) converges to zero when \( g \) goes to infinity and such that

\[
\sup_{A \in \mathcal{F}^n_0 : B \in \mathcal{F}^\infty_{n+g}} \left| \frac{\mu(A \cap B) - \mu(A)\mu(B)}{\mu(A)\mu(B)} \right| \leq \psi(g),
\]

for all \( n \in \mathbb{N} \).

In \cite{13}, an upper bound for the growth rate of the length of the longest common substring for encoded sequences (and thus for RLE sequences) has been proved.

Theorem 2.5 (Theorem 2.4 \cite{13}). If \( H_\alpha(f_*\mu) > 0 \), then for almost every \( x, y \),

\[
\lim_{n \to \infty} \frac{M_n^{RLE}(x, y)}{\log n} \leq \phi \frac{2}{H_\alpha(f_*\mu)}.
\]

Under some mixing assumptions and some assumptions on the encoder, a lower bound was also proved in \cite{13}. Nevertheless, the run-length encoder does not satisfy the necessary assumptions since preimage of cylinders under \( f \) can have arbitrary length. Thus we present a different proof here to obtain the lower bound.
Theorem 2.6. If $H_2(f_\ast \mu) > 0$ and the process is $\alpha$-mixing with an exponential decay (or $\psi$-mixing with $\psi(g) = g^{-a}$ for some $a > 0$), then, for almost every realizations $x, y$,

$$\lim_{n \to \infty} \frac{M_n^{RLE}(x, y)}{\log n} \geq \frac{2}{H_2(f_\ast \mu)}.$$ 

Thus, if the Rényi entropy exists, we get for almost every $x, y$,

$$\lim_{n \to \infty} \frac{M_n^{RLE}(x, y)}{\log n} = \frac{2}{H_2(f_\ast \mu)}.$$

We will now give examples satisfying our assumptions and where the Rényi entropy of the pushforward measure can be explicitly computed. First of all, we will treat the case of Bernoulli processes and then of Markov chains. We emphasize that for Markov chains the situation and the computation are different when working with an alphabet of two symbols or an alphabet of more than two symbols.

Example 2.7 (Bernoulli process). Let us consider the alphabet $A = \{a, b\}$ and the Bernoulli measure $\mu$ such that $\mu(a) = p$ and $\mu(b) = 1 - p$ with $0 < p < 1$. Since this process is $\alpha$-mixing with exponential decay, to apply our main theorem, we need to compute the Rényi entropy of the pushforward measure.

Let $n \in \mathbb{N}$. We assume that $n$ is even (the odd case can be treated similarly). We observe that by definition of the run-length encoder, cylinders of length $n$ can only have two types, i.e. the cylinder if of type 1 and $C_n = (a, k_1)(b, k_2)(a, k_3) \ldots (a, k_{n-1})(b, k_n)$ with $k_1, \ldots, k_n \in \mathbb{N}$ or the cylinder is of type 2 and $C_n = (b, k_1)(a, k_2)(b, k_3) \ldots (b, k_{n-1})(a, k_n)$ with $k_1, \ldots, k_n \in \mathbb{N}$.

It is important to notice that $f^{-1}((a, k_1)(b, k_2) \ldots (b, k_n)) = a \ldots ab \ldots b \ldots ba$. Indeed if the last symbol of $C_n$ is $(b, k_n)$, it does not only inform us that in the preimage we have a concatenation of $k_n$ symbols $b$ but also it imposes that this concatenation must be followed by a symbol $a$, otherwise, if it was a symbol $b$, the last symbol of $C_n$ would not be $(b, k_n)$.

Thus, we have

$$\sum_{C_n} f_\ast \mu(C_n)^2 = \sum_{C_n \text{ of type } 1} f_\ast \mu(C_n)^2 + \sum_{C_n \text{ of type } 2} f_\ast \mu(C_n)^2$$

$$= \sum_{k_1, \ldots, k_n \in \mathbb{N}} \mu(f^{-1}(a, k_1)(b, k_2) \ldots (b, k_n))^2 + \sum_{k_1, \ldots, k_n \in \mathbb{N}} \mu(f^{-1}(b, k_1)(a, k_2) \ldots (a, k_n))^2$$

$$= \sum_{k_1, \ldots, k_n \in \mathbb{N}} \mu(a \cdots ab \cdots b \cdots ba)^2 + \mu(b \cdots ba \cdots a \cdots ab)^2$$

$$= \sum_{k_1, \ldots, k_n \in \mathbb{N}} \left( \mu(a)^{k_1} \mu(b)^{k_2} \cdots \mu(b)^{k_n} \mu(a) \right)^2 + \left( \mu(b)^{k_1} \mu(a)^{k_2} \cdots \mu(a)^{k_n} \mu(b) \right)^2$$

$$= \sum_{k_1, \ldots, k_n \in \mathbb{N}} p^{2k_1} (1-p)^{2k_2} \cdots (1-p)^{2k_n} p^2 + (1-p)^{2k_1} p^{k_2} \cdots p^{2k_n} (1-p)^2$$

$$= \left( p^2 + (1-p)^2 \right) \left( \frac{p^2}{1-p^2} \right)^{n/2} \left( \frac{(1-p)^2}{1-(1-p)^2} \right)^{n/2}.$$ 

This implies that the Rényi entropy of the pushforward measure exists and we have

$$H_2(f_\ast \mu) = - \lim_{k \to \infty} \frac{1}{n} \log \sum_{C_n} f_\ast \mu(C_k)^2 = - \frac{1}{2} \log \left( \frac{p(1-p)}{(1+p)(2-p)} \right).$$
Finally, applying Theorem 2.6, we have for almost every realizations \( x, y \)

\[
\lim_{n \to \infty} \frac{M_{RLE}^n(x, y)}{\log n} = \frac{4}{\log \left( \frac{(1+p)(2-p)}{p(1-p)} \right)}.
\]

**Example 2.8** (Markov chain with two states). Let us consider the alphabet \( A = \{a, b\} \) and the transition matrix \( P = (p_{ij})_{i,j \in A} \) with \( p_{aa} = p \) and \( p_{bb} = q \) where \( 0 < p, q < 1 \). The stationary measure \( \mu \) is \( \alpha \)-mixing with exponential decay (see e.g. [9]). Thus to apply our theorem we will compute the Rényi entropy of the pushforward measure.

As in the Bernoulli case, assuming that \( n \) is even, cylinders of length \( n \) can only have two forms, i.e \( C_n = (a, k_1)(b, k_2)(a, k_3) \ldots (a, k_{n-1})(b, k_n) \) or \( C_n = (b, k_1)(a, k_2)(b, k_3) \ldots (b, k_{n-1})(a, k_n) \) with \( k_1, \ldots, k_n \in \mathbb{N} \). Thus, we have

\[
\sum_{C_n} f \ast \mu(C_n)^2 = \sum_{k_1, \ldots, k_n \in \mathbb{N}} \mu(a \ldots ab \ldots b \ldots ba)^2 + \mu(b \ldots ba \ldots a \ldots ab)^2
\]

\[
= \sum_{k_1, \ldots, k_n \in \mathbb{N}} \left( \mu(a)p_{aa}^{k_2-1}p_{ab}^{k_2-1} \ldots p_{ba}p_{ba}^{k_n-1}p_{bb} \right)^2 + \left( \mu(b)p_{ba}^{k_1-1}p_{bb}p_{aa}^{k_2-1} \ldots p_{ba}p_{ba}^{k_n-1}p_{bb} \right)^2
\]

\[
= (\mu(a)^2 + \mu(b)^2) p_{ba}^n \left( \frac{1}{1-p_{aa}} \right)^{n/2} \left( \frac{1}{1-p_{ba}} \right)^{n/2}
\]

\[
and the Rényi entropy is
\]

\[
H_2(f \ast \mu) = -\frac{1}{2} \log \left( \frac{(1-p)(1-q)}{(1+p)(1+q)} \right).
\]

Applying Theorem 2.6, we have for almost every realizations \( x, y \)

\[
\lim_{n \to \infty} \frac{M_{RLE}^n(x, y)}{\log n} = \frac{4}{\log \left( \frac{(1+p)(1+q)}{(1-p)(1+q)} \right)}.
\]

To study Markov chains with more than two states, we will use another strategy which cannot be used for two states. The idea is that when the original process \( X \) is a Markov chain with finite alphabet, the process \( f(X) \) is a Markov chain with infinite alphabet. However, when working with only two states, this process is not aperiodic preventing us to use the results of [11] (which are based on Perron-Frobenius Theorem).

**Example 2.9** (Markov chain with more than 2 states). Let us consider the alphabet \( A = \{a_i\}_{1 \leq i \leq N} \) and the transition matrix \( P = (p_{ij})_{1 \leq i, j \leq N} \) with \( 0 \leq p_{ij} < 1 \) for every \( 1 \leq i, j \leq N \). The stationary measure \( \mu \) is \( \alpha \)-mixing with exponential decay (see e.g. [9]). Thus to apply our theorem we will compute the Rényi entropy of the pushforward measure.

First of all, we observe that the process \( f(X) \) is also a Markov chain on the alphabet \( B = \{(a, k)\}_{a \in A, k \in \mathbb{N}} \) with transition matrix \( Q = (q_{(a,k),(a,k)}(\alpha,\ell))_{1 \leq i, j \leq N, k, \ell \in \mathbb{N}} \) and initial distribution \( \nu = (\nu((a, k)))_{1 \leq i, j \leq N, k, \ell \in \mathbb{N}} \). By definition of the run-length encoder, we observe that for all \( 1 \leq i \leq N \) and \( k, \ell \in \mathbb{N} \)

\[
q_{(a_i,k),(a_i,\ell)} = \mathbb{P}(f(X)_{n+1} = (a_i, \ell) | f(X)_n = (a_i,k)) = 0.
\]
Moreover, for \( i \neq j \) and \( k, \ell \in \mathbb{N} \), we have
\[
q_{(\alpha_i,k)(\alpha_j,\ell)} = \mathbb{P}(f(X)_{n+1} = (\alpha_j,\ell) | f(X)_n = (\alpha_i, k))
\]
\[
= \mu(\underbrace{\alpha_i \ldots \alpha_i}_k \ldots \underbrace{\alpha_j \ldots \alpha_j}_\ell)^{-1}
\]
where \( \alpha^c \) can be any symbol in \( A \setminus \{ \alpha \} \). Thus, we obtain
\[
q_{(\alpha_i,k)(\alpha_j,\ell)} = \frac{p_{ij}p_{jj}^{\ell-1}(1 - p_{jj})}{(1 - p_{ii})}.
\]

Then, for all \( 1 \leq i \leq N \) and \( k \in \mathbb{N} \), we have
\[
\nu((\alpha_i, k)) = \mathbb{P}(f(X)_1 = (\alpha_i, k)) = \mu(\underbrace{\alpha_i \ldots \alpha_i}_k) = \mu(\alpha_i)p_{ii}^{k-1}(1 - p_{ii}).
\]

Since, \( f(X) \) is a Markov chain over a countable alphabet, we will compute \( H_2(f_*\mathbb{P}) \) using Theorem 2 in [11]. Thus, we need to verify its assumptions. First, we observe that \( f(X) \) is irreducible and aperiodic since \( 0 < p_{ij} < 1 \) for every \( 1 \leq i, j \leq N \).

We observe that if the alphabet \( A \) as only two symbols, \( f(X) \) is periodic of period 2, thus we cannot apply [11].

We will now check Assumptions 1 of [11] to apply their results. Since \( 0 < p_{ij} < 1 \) for every \( 1 \leq i, j \leq N \), we have
\[
\sup_{1 \leq i, j \leq N, k, \ell \in \mathbb{N}} q_{(\alpha_i,k)(\alpha_j,\ell)} = \sup_{1 \leq i, j \leq N, k, \ell \in \mathbb{N}} \frac{p_{ij}p_{jj}^{\ell-1}(1 - p_{jj})}{(1 - p_{ii})} \leq \sup_{1 \leq j \leq N} (1 - p_{jj}) < 1
\]
and Assumption 1.A is satisfied.

Let \( s > 0 \). We have for any \( 1 \leq i, j \leq N \) and \( k \in \mathbb{N} \)
\[
\sum_{\ell \in \mathbb{N}} q^{s}_{(\alpha_i,k)(\alpha_j,\ell)} = \sum_{\ell \in \mathbb{N}} \left( \frac{p_{ij}p_{jj}^{\ell-1}(1 - p_{jj})}{(1 - p_{ii})} \right)^s
\]
\[
= \left( \frac{p_{ij}(1 - p_{jj})}{(1 - p_{ii})} \right)^s \frac{1}{1 - p_{jj}^s}.
\]

Thus,
\[
\sup_{1 \leq i \leq N, k \in \mathbb{N}} \sum_{1 \leq j \leq N, \ell \in \mathbb{N}} q^{s}_{(\alpha_i,k)(\alpha_j,\ell)} = \sup_{1 \leq i \leq N} \sum_{1 \leq j \leq N} \sum_{\ell \in \mathbb{N}} q^{s}_{(\alpha_i,k)(\alpha_j,\ell)}
\]
\[
= \sup_{1 \leq i \leq N} \sum_{1 \leq j \leq N} \left( \frac{p_{ij}(1 - p_{jj})}{(1 - p_{ii})} \right)^s \frac{1}{1 - p_{jj}^s} < +\infty. \quad (1)
\]

Moreover,
\[
\sum_{1 \leq i \leq N, k \in \mathbb{N}} \nu((\alpha_i, k))^s = \sum_{1 \leq i \leq N, k \in \mathbb{N}} \left( \mu(\alpha_i)p_{ii}^{k-1}(1 - p_{ii}) \right)^s
\]
\[
= \sum_{1 \leq i \leq N} \frac{\mu(\alpha_i)^s(1 - p_{ii})^s}{1 - p_{ii}^s} < +\infty. \quad (2)
\]

[1] and (2) imply that Assumption 1.B is satisfied.
Let $\varepsilon > 0$, $s > 0$ and define

$$M = \sup_{1 \leq i \leq N} \sum_{1 \leq j \leq N} \left( \frac{p_{ij}(1-p_{jj})}{1-p_{ii}} \right)^s.$$

Since $0 < p_{ij} < 1$ for every $1 \leq i, j \leq N$, it exists $m \in \mathbb{N}$ such that for all $1 \leq j \leq N$, we have

$$\frac{p_{ij}^m}{1-p_{jj}^m} < \frac{\varepsilon}{M}.$$ 

Let $A = \{(\alpha_i, k)\}_{1 \leq i \leq N, 1 \leq k < m}$. We observe that $A$ has a finite number of elements and that

$$\sup_{1 \leq i \leq N, k \in \mathbb{N}} \sum_{(\alpha_i, k) \in B \setminus A} q^{s}_{(\alpha_i, k)(\alpha_j, \ell)} = \sup_{1 \leq i \leq N} \sum_{1 \leq j \leq N} \sum_{\ell = m}^{\infty} q^{s}_{(\alpha_i, k)(\alpha_j, \ell)}$$

$$= \sup_{1 \leq i \leq N} \sum_{1 \leq j \leq N} \left( \frac{p_{ij}(1-p_{jj})}{1-p_{ii}} \right)^s \frac{p_{ij}^m}{1-p_{jj}^m} < \frac{\varepsilon}{M}.$$ 

Thus, Assumption 1.C is satisfied.

Finally, since $f(X)$ satisfies all the assumptions of Theorem 2 in [11], $H_2(f_\ast \mu)$ exists and

$$H_2(f_\ast \mu) = -\log \lambda$$

where $\lambda$ is the largest positive eigenvalue of the matrix $Q_2 = \left( q^{s}_{(\alpha_i, k)(\alpha_j, \ell)} \right)_{1 \leq i, j \leq N, k, \ell \in \mathbb{N}}$.

Applying Theorem 2.6, we have for almost every realizations $x, y$

$$\lim_{n \to \infty} \frac{M_{n}^{RLE}(x, y)}{\log n} = \frac{2}{-\log \lambda}.$$ 

We will now prove our main result.

Proof of Theorem 2.6. Let $\varepsilon > 0$ and define

$$k_n = \left\lfloor \frac{2 \log n + b \log \log n}{H_2(f_\ast \mu) + \varepsilon} \right\rfloor$$

where $b$ is a constant to be chosen.

For $0 \leq i, j \leq n - 1$ we define the following event

$$A_{i, j} = \{ f(X)^{i+k_n}_{i+1} = f(Y)^{j+k_n}_{j+1} \}$$

and the following random variable

$$S_n = \sum_{i, j = 0, \ldots, n - 1} \mathbb{1}_{A_{i, j}}.$$ 

First of all, we observe that

$$\mathbb{P}(M_{n}^{RLE} < k_n) = \mathbb{P}(S_n = 0)$$

thus, by Chebyshev’s inequality

$$\mathbb{P} \left( M_{n}^{RLE} < k_n \right) \leq \frac{\text{var}(S_n)}{\mathbb{E}(S_n)^2}.$$
We use the stationarity of $\mu$ to obtain

$$E(S_n) = \sum_{0 \leq i,j \leq n-1} \sum_{\omega \in \mathcal{A}^{kn}} \mathbb{P}(f(X)^{i+k}_{i+1} = f(Y)^{j+k}_{j+1} = \omega)$$

$$= \sum_{0 \leq i,j \leq n-1} \sum_{\omega \in \mathcal{A}^{kn}} \mu(f(X)^{i+k}_{i+1} = \omega) \mu(f(Y)^{j+k}_{j+1} = \omega)$$

$$= n^2 \sum_{\omega \in \mathcal{A}^{kn}} \mu(f^{-1}\omega)^2 = n^2 \sum_{\omega \in \mathcal{A}^{kn}} f_*\mu(\omega)^2$$

For the variance of $S_n$, we observe that

$$\text{var}(S_n) = \sum_{0 \leq i,i',j,j' \leq n-1} E(1_{A_{i,j}} 1_{A_{i',j'}}) - E(S_n)^2.$$

Let $g = g(n) = (\log n)^{\beta}$, for some $\beta > 0$ to be defined later.

Firstly, we assume that $i' - i > g + k_n$ and $j' - j > g + k_n$ (the case $i - i' > g + k_n$ and $j - j' > g + k_n$ can be treated identically), then we have

$$E(1_{A_{i,j}} 1_{A_{i',j'}}) = \sum_{\omega,\omega' \in \mathcal{A}^{kn}} \mathbb{P}(f(X)^{i+k}_{i+1} = f(Y)^{j+k}_{j+1} = \omega, f(X)^{i'+k}_{i'+1} = f(Y)^{j'+k}_{j'+1} = \omega')$$

$$= \sum_{\omega,\omega' \in \mathcal{A}^{kn}} \mu(f(X)^{i+k}_{i+1} = \omega, f(X)^{i'+k}_{i'+1} = \omega') \mu(f(Y)^{j+k}_{j+1} = \omega, f(Y)^{j'+k}_{j'+1} = \omega')$$

$$= \sum_{\omega,\omega' \in \mathcal{A}^{kn}} \mu(f(X)^{k}_{i} = \omega, f(X)^{k}_{i'-i+1} = \omega') \mu(f(Y)^{k}_{j} = \omega, f(Y)^{k}_{j'-j+1} = \omega')$$

by stationarity of $\mu$.

To use the mixing property, we need to work with cylinders whose preimage under $f$ does not have a length too large so that the gap is preserved. Thus, we define the set

$$\mathcal{Z}_n = \{ \omega \in \mathcal{A}^{kn} : |f^{-1}\omega| \leq k_n^2 \}$$

and, using the $\alpha$-mixing, we obtain
Thus, (3) together with (4) gives us that when

\[ \sum_{\omega' \in A^{kn}} \mu \left( f(X)^{kn}_1 = \omega, f(X)^{j'-i+k_n}_i = \omega' \right) \mu \left( f(Y)^{kn}_1 = \omega, f(Y)^{j'-j+k_n}_j = \omega' \right) \leq \sum_{\omega' \in A^{kn}} \left[ \mu \left( f(X)^{kn}_1 = \omega \right) \mu \left( f(X)^{kn}_1 = \omega' \right) + \alpha(g + k_n - k^2_n) \right] \times \mu \left( f(Y)^{kn}_1 = \omega, f(Y)^{j'-j+k_n}_j = \omega' \right) \leq \sum_{\omega' \in A^{kn}} \mu \left( f(X)^{kn}_1 = \omega \right) \left[ \sum_{\omega' \in A^{kn}} \mu \left( f(Y)^{kn}_1 = \omega, f(Y)^{j'-j+k_n}_j = \omega' \right) \right] + \sum_{\omega' \in A^{kn}} \alpha(g + k_n - k^2_n) \mu \left( f(Y)^{j'-j+k_n}_j = \omega' \right) \leq 2 \alpha(g + k_n - k^2_n) + \left[ \sum_{\omega \in A^{kn}} f_s \mu (\omega)^2 \right]^2 . \]  

(3)

When the length of the preimage cylinders is too large (i.e. \( \omega / \notin Z_n \)), we cannot use the mixing property, however, we observe that

\[
\sum_{\omega \in Z_n} \mu \left( f(X)^{kn}_1 = \omega, f(X)^{j'-i+k_n}_i = \omega' \right) \mu \left( f(Y)^{kn}_1 = \omega, f(Y)^{j'-j+k_n}_j = \omega' \right) \leq \sum_{\omega \in Z_n} \mu \left( f(X)^{kn}_1 = \omega \right) \sum_{\omega' \in A^{kn}} \mu \left( f(Y)^{kn}_1 = \omega, f(Y)^{j'-j+k_n}_j = \omega' \right) \leq \sum_{\omega \in Z_n} \mu \left( f(X)^{kn}_1 = \omega \right) \mu \left( f(Y)^{kn}_1 = \omega \right) \leq \sum_{\omega \in A^{kn}} \mu (\omega)^2 . \]  

(4)

Thus, (3) together with (4) gives us that when \( i' - i > g + k_n \) and \( j' - j > g + k_n \)

\[
\mathbb{E} \left( \mathbb{1}_{A_{i,j}} \mathbb{1}_{A_{i',j'}} \right) \leq 2 \alpha(g + k_n - k^2_n) + \left[ \sum_{\omega \in A^{kn}} f_s \mu (\omega)^2 \right]^2 + \sum_{\omega \in A^{kn}} \mu (\omega)^2 . \]  

(5)

We observe that when \( i' - i > g + k_n \) and \( j - j' > g + k_n \) (the case \( i' - i > g + k_n \) and \( j' - j > g + k_n \) can be treated identically) then we can obtain (3) only if we restrict our sum to \( \omega' \in Z_n \), thus the estimate for \( \mathbb{E} \left( \mathbb{1}_{A_{i,j}} \mathbb{1}_{A_{i',j'}} \right) \) will be slightly different and we will have

\[
\mathbb{E} \left( \mathbb{1}_{A_{i,j}} \mathbb{1}_{A_{i',j'}} \right) \leq 2 \alpha(g + k_n - k^2_n) + \left[ \sum_{\omega \in A^{kn}} f_s \mu (\omega)^2 \right]^2 + 2 \sum_{\omega \in A^{kn}} \mu (\omega)^2 . \]  

(6)

Now, we assume that \( i' - i > g + k_n \) and \( 0 \leq j' - j \leq g + k_n \) (the other cases such that \( |i' - i| > g + k_n \) and \( |j' - j| \leq g + k_n \) can be treated identically), using the mixing property
as in (3) and then, using Hölder’s inequality, we have
\[
\sum_{\omega \in Z_n, \omega \in A^{k_n}} \mu \left( f(X)_1^{k_n} = \omega, f(X)_{j'-i+1}^{j'-i+k_n} = \omega' \right) \mu \left( f(Y)_1^{k_n} = \omega, f(Y)_{j'-j+1}^{j'-j+k_n} = \omega' \right)
\]
\[
\leq \alpha(g + k_n - k_n^2) + \sum_{\omega \in Z_n, \omega \in A^{k_n}} \mu \left( f(X)_1^{k_n} = \omega \right) \mu \left( f(Y)_1^{k_n} = \omega \right) \mu \left( f(X)_{j'-j+1}^{j'-j+k_n} = \omega' \right)
\]
\[
\leq \alpha(g + k_n - k_n^2) + \int \mu \left( f(X)_1^{k_n} = x_1^{k_n} \right) \mu \left( f(X)_{j'-j+1}^{j'-j+k_n} = x_{j'-j+1}^{j'-j+k_n} \right) d\mu(x)
\]
\[
\leq \alpha(g + k_n - k_n^2) + \left[ \int \mu \left( f(X)_1^{k_n} = x_1^{k_n} \right)^2 d\mu(x) \right]^{1/2} \left[ \int \mu \left( f(X)_{j'-j+1}^{j'-j+k_n} = x_{j'-j+1}^{j'-j+k_n} \right)^2 d\mu(x) \right]^{1/2}
\]
\[
= \alpha(g + k_n - k_n^2) + \sum_{\omega \in A^{k_n}} f_s \mu(\omega)^3
\]
where the last inequality comes from the subaditivity of the map \( x \mapsto x^{3/2} \).

For the terms with \( \omega \notin Z_n \) we will use the estimate (4). Thus, for \( |j' - i| > g + k_n \) and \( |j' - j| > g + k_n \) we have
\[
\mathbb{E} \left( 1_{A_{i,j}} 1_{A_{i',j'}} \right) \leq \alpha(g + k_n - k_n^2) + \left[ \sum_{\omega \in A^{k_n}} f_s \mu(\omega)^2 \right]^{3/2} + \sum_{\omega \in A^{k_n}} \mu(\omega)^2.
\]
(7)

Finally, when \( |j' - i| \leq g + k_n \) and \( |j' - j| \leq g + k_n \), we just observe that
\[
\mathbb{E} \left( 1_{A_{i,j}} 1_{A_{i',j'}} \right) \leq \mathbb{E} \left( 1_{A_{i,j}} \right) = \sum_{\omega \in A^{k_n}} f_s \mu(\omega)^2.
\]
(8)

Combining together the estimates (6), (7), (7) and (8), we obtain
\[
\text{var}(S_n) \leq n^4 \left[ 2\alpha(g + k_n - k_n^2) + 2 \sum_{\omega \in A^{k_n^2}} \mu(\omega)^2 \right]
\]
\[
+ 4n^3(g + k_n) \left[ \alpha(g + k_n - k_n^2) + \left[ \sum_{\omega \in A^{k_n}} f_s \mu(\omega)^2 \right]^{3/2} + \sum_{\omega \in A^{k_n^2}} \mu(\omega)^2 \right]
\]
\[
+ 4n^2(g + k_n)^2 \sum_{\omega \in A^{k_n}} f_s \mu(\omega)^2.
\]

By definition of \( k_n \), for \( n \) large enough we have \( k_n^2(\mathcal{H}_2(\mu) - \epsilon) \geq 5 \log n \), thus by definition of the Rényi entropy, we obtain
\[
\sum_{\omega \in A^{k_n^2}} \mu(\omega)^2 \leq e^{-k_n^2(\mathcal{H}_2(\mu) - \epsilon)} \leq n^{-5}.
\]
Moreover, recalling that $g = (\log n)^\beta$, one could choose $\beta$ large enough such that
\[2n^4 \alpha(n + k_n - k_n^2) = \mathcal{O}(n^{-1}).\]
and such that
\[4n^3 (g + k_n) \alpha(n + k_n - k_n^2) = \mathcal{O}(n^{-1}).\]
Thus,
\[
P\left(M^{RLE}_n < k_n\right) \leq \frac{\text{var}(S_n)}{\mathbb{E}(S_n)^2} \leq \frac{4(g + k_n)}{\mathbb{E}(S_n)^{1/2}} + \frac{4(g + k_n)^2}{\mathbb{E}(S_n)} + \mathcal{O}(n^{-1}).
\]
By definition of $k_n$ and the Rényi entropy, we have
\[
\mathbb{E}(S_n) = n^2 \sum_{\omega \in \mathcal{A}^{kn}} f_\ast \mu(\omega)^2 \geq n^2 e^{-k_n (H_2(f_\ast \mu) + \epsilon)} \geq (\log n) - b
\]
and choosing $b \ll -1$, we obtain
\[
P\left(M^{RLE}_n < k_n\right) \leq \mathcal{O}\left((\log n)^{-1}\right).
\]
Choosing a subsequence $(n_\ell)_{\ell \in \mathbb{N}}$ such that $n_\ell = [e^{\ell}]$ we have that $\sum_\ell \mathbb{P}\left(M^{RLE}_n < k_{n_\ell}\right) < +\infty$. Thus, by the Borel-Cantelli lemma, we have almost surely, if $\ell$ is large enough,
\[
M^{RLE}_{n_\ell} \geq k_{n_\ell}
\]
and then
\[
\frac{M^{RLE}_{n_\ell}}{\log n_\ell} \geq \frac{1}{H_2(f_\ast \mu) + \epsilon} \left(2 + \log \frac{\log n_\ell}{\log n_\ell} + 1 + \log \log \log n_\ell\right).
\]
Taking the limit superior in this inequality and observing that $(M^{RLE}_n)_n$ and $(n_\ell)_\ell$ are increasing and that $\lim_{\ell \to \infty} \log n_\ell = 1$, we obtain almost surely
\[
\lim_{n \to \infty} \frac{M^{RLE}_n}{\log n} = \lim_{\ell \to \infty} \frac{M^{RLE}_{n_\ell}}{\log n_\ell} \geq \frac{2}{H_2(f_\ast \mu) + \epsilon}.
\]
And the theorem is proved since $\epsilon$ can be chosen arbitrarily small.

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□
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12

JÉRÔME ROUSSEAU

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