NARROW ORTHOGONALLY ADDITIVE OPERATORS ON LATTICE-NORMED SPACES

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Abstract. The aim of this article is to extend results of M. Popov and second named author about orthogonally additive narrow operators on vector lattices. The main object of our investigations are an orthogonally additive narrow operators between lattice-normed spaces. We prove that every $C$-compact laterally-to-norm continuous orthogonally additive operator from a Banach-Kantorovich space $V$ to a Banach lattice $Y$ is narrow. We also show that every dominated Uryson operator from Banach-Kantorovich space over an atomless Dedekind complete vector lattice $E$ to a sequence Banach lattice $\ell_p(\Gamma)$ or $c_0(\Gamma)$ is narrow. Finally, we prove that if an orthogonally additive dominated operator $T$ from lattice-normed space $(V, E)$ to Banach-Kantorovich space $(W, F)$ is order narrow then the order narrow is its exact dominant $|T|$.

1. Introduction

Narrow operators in framework of the theory of ordered spaces were introduces in [13]. Today the theory of linear narrow operators is a very active area of Functional Analysis (see [22]). Firstly, this class of operators in the framework of the theory of vector lattices was introduced in [13]. In [20] Pliev and Popov have considered a wide class of narrow nonlinear maps which called orthogonally additive operators. This class of operators acting between vector lattices was introduced and studied in 1990 by Mazón and Segura de León [14, 15], and then extended to lattice-normed spaces by Kusraev and the second named author [8, 9, 16]. In the present paper we generalize the main results of [20] on orthogonally additive narrow operators on vector lattices to a wider class which includes nonlinear orthogonally additive operators between vector-valued function spaces. We consider orthogonally additive narrow operators in the framework of lattice-normed spaces. The notion of a lattice-normed space was firstly introduced by Kantorovich in the first part of 20th century [6]. Later, Kusraev and his school had provided a deep theory. Orthogonally additive operators on lattice-normed spaces were investigated by Pliev and collaborates in [2, 3, 8, 9, 10, 17, 18, 21].

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2. Preliminaries

The goal of this section is to introduce some basic definitions and facts. General information on vector lattices, Banach spaces and lattice-normed spaces the reader can find in the books [1, 7, 10, 11, 12].

Consider a vector space $V$ and a real archimedean vector lattice $E$. A map $| \cdot | : V \to E$ is a vector norm if it satisfies the following axioms:

1) $|v| \geq 0; \quad |v| = 0 \iff v = 0; \quad (\forall v \in V)$.
2) $|v_1 + v_2| \leq |v_1| + |v_2|; \quad (v_1, v_2 \in V)$.
3) $|\lambda v| = |\lambda| |v|; \quad (\lambda \in \mathbb{R}, v \in V)$.

A vector norm is called decomposable if

4) for all $e_1, e_2 \in E_+$ and $x \in V$ from $|x| = e_1 + e_2$ it follows that there exist $x_1, x_2 \in V$ such that $x = x_1 + x_2$ and $|x_k| = e_k, (k := 1, 2)$.

A triple $(V, | \cdot |, E)$ (in brief $(V, E), (V, | \cdot |)$ or $V$ with default parameters omitted) is a lattice-normed space if $| \cdot |$ is a $E$-valued vector norm in the vector space $V$. If the norm $| \cdot |$ is decomposable then the space $V$ itself is called decomposable. We say that a net $(v_{\alpha})_{\alpha \in \Delta}$ (bo)-converges to an element $v \in V$ and write $v = \text{bo-lim} v_{\alpha}$ if there exists a decreasing net $(e_{\gamma})_{\gamma \in \Gamma}$ in $E_+$ such that $\inf_{\gamma \in \Gamma}(e_{\gamma}) = 0$ and for every $\gamma \in \Gamma$ there is an index $\alpha(\gamma) \in \Delta$ such that $|v - v_{\alpha(\gamma)}| \leq e_{\gamma}$ for all $\alpha \geq \alpha(\gamma)$. A net $(v_{\alpha})_{\alpha \in \Delta}$ is called (bo)-fundamental if the net $(v_{\alpha} - v_{\beta})_{(\alpha, \beta) \in \Delta \times \Delta}$ (bo)-converges to zero. A lattice-normed space is called (bo)-complete if every (bo)-fundamental net (bo)-converges to an element of this space. Let $e$ be a positive element of a vector lattice $E$. By $[0,e]$ we denote the set $\{v \in V : |v| \leq e\}$. A set $M \subset V$ is called (bo)-bounded if there exists $e \in E_+$ such that $M \subset [0,e]$. Every decomposable (bo)-complete lattice-normed space is called a Banach-Kantorovich space (a BKS for short).

Let $(V, E)$ be a lattice-normed space. A subspace $V_0$ of $V$ is called a (bo)-ideal of $V$ if for $v \in V$ and $u \in V_0$, from $|v| \leq |u|$ it follows that $v \in V_0$. A subspace $V_0$ of a decomposable lattice-normed space $V$ is a (bo)-ideal if and only if $V_0 = \{v \in V : |v| \in L\}$, where $L$ is an order ideal in $E$ ([7], Prop. 2.1.6.1). Let $V$ be a lattice-normed space and $y, x \in V$. If $|x| \perp |y| = 0$ then we call the elements $x, y$ disjoint and write $x \perp y$. The equality $x = \bigoplus_{i=1}^{n} x_i$ means that $x = \sum_{i=1}^{n} x_i$ and $x_i \perp x_j$ if $i \neq j$. An element $z \in V$ is called a component or a fragment of $x \in V$ if $0 \leq |z| \leq |x|$ and $x \perp (x - z)$. Two fragments $x_1, x_2$ of $x$ are called mutually complemented or $MC$, in short, if $x = x_1 + x_2$. The notations $z \subseteq x$ means that $z$ is a fragment of $x$. According to ([1], p.111) an element $e > 0$ of a vector lattice $E$ is called an atom, whenever $0 \leq f_1 \leq e, 0 \leq f_2 \leq e$ and $f_1 \perp f_2$ imply that either $f_1 = 0$ or $f_2 = 0$. A vector lattice $E$ is atomless if there is no atom $e \in E$.

The following object will be often used in different constructions below. Let $V$ be a lattice-normed space and $x \in V$. A sequence $(x_n)_{n=1}^{\infty}$ is called a
disjoint tree on $x$ if $x_1 = x$ and $x_n = x_{2n} \prod x_{2n+1}$ for each $n \in \mathbb{N}$. It is clear that all $x_n$ are fragments of $x$. All lattice-normed spaces below we consider to be decomposable.

Consider some important examples of lattice-normed spaces. We begin with simple extreme cases, namely vector lattices and normed spaces. If $V = E$ then the modules of an element can be taken as its lattice norm:

$$|v| := |v| = v \vee (-v); v \in E.$$ Decomposability of this norm easily follows from the Riesz Decomposition Property holding in every vector lattice. If $E = \mathbb{R}$ then $V$ is a normed space.

Let $Q$ be a compact and let $X$ be a Banach space. Let $V := C(Q, X)$ be the space of continuous vector-valued functions from $Q$ to $X$. Assign $E := C(Q, \mathbb{R})$. Given $f \in V$, we define its lattice norm by the relation $|f| : t \mapsto \|f(t)\|_X (t \in Q)$. Then $| \cdot |$ is a decomposable norm ([7], Lemma 2.3.2).

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space, let $E$ be an order-dense ideal in $L_0(\Omega)$ and let $X$ be a Banach space. By $L_0(\Omega, X)$ we denote the space of (equivalence classes ) of Bochner $\mu$-measurable vector functions acting from $\Omega$ to $X$. As usual, vector-functions are equivalent if they have equal values at almost all points of the set $\Omega$. If $\tilde{f}$ is the coset of a measurable vector-function $f : \Omega \to X$ then $t \mapsto \|f(t)\|_X (t \in \Omega)$ is a scalar measurable function whose coset is denoted by the symbol $\left\lfloor \tilde{f} \right\rfloor \in L_0(\mu)$. Assign by definition

$$E(X) := \{ f \in L_0(\mu, X) : |f| \in E \}.$$ Then $(E(X), E)$ is a lattice-normed space with a decomposable norm ([7], Lemma 2.3.7). If $E$ is a Banach lattice then the lattice-normed space $E(X)$ is a Banach space with respect to the norm $\| |f| \| := \|f(\cdot)\|_X_E$.

Let $E$ be a Banach lattice and let $(V, E)$ be a lattice-normed space. By definition, $|x| \in E_+$ for every $x \in V$, and we can introduce some mixed norm in $V$ by the formula

$$|||x||| := \| |x| \| \quad (\forall x \in V).$$ The normed space $(V, ||| \cdot |||)$ is called a space with a mixed norm. In view of the inequality $| |x| - |y| | \leq |x - y|$ and monotonicity of the norm in $E$, we have

$$||x| - |y|| \leq ||x - y|| \quad (\forall x, y \in V),$$ so a vector norm is a norm continuous operator from $(V, ||| \cdot |||)$ to $E$. A lattice-normed space $(V, E)$ is called a Banach space with a mixed norm if the normed space $(V, ||| \cdot |||)$ is complete with respect to the norm convergence.

**Definition 2.1.** Let $E$ be a vector lattice, and let $F$ be a real linear space. An operator $T : E \to F$ is called orthogonally additive if $T(x + y) = T(x) + T(y)$ whenever $x, y \in E$ are disjoint.

It follows from the definition that $T(0) = 0$. It is immediate that the set of all orthogonally additive operators is a real vector space with respect to the natural linear operations.
**Definition 2.2.** Let $E$ and $F$ be vector lattices. An orthogonally additive operator $T : E \to F$ is called:

- **positive** if $Tx \geq 0$ holds in $F$ for all $x \in E$;
- **order bounded** if $T$ maps order bounded sets in $E$ to order bounded sets in $F$.

An orthogonally additive, order bounded operator $T : E \to F$ is called an abstract Uryson operator.

For example, any linear operator $T \in L_+(E, F)$ defines a positive abstract Uryson operator by $G(f) = T|f|$ for each $f \in E$.

The set of all abstract Uryson operators from $E$ to $F$ we denote by $\mathcal{U}(E, F)$. Consider some examples. The most famous one is the nonlinear integral Uryson operator.

Consider the following order in $\mathcal{U}(E, F) : S \leq T$ whenever $T - S$ is a positive operator. Then $\mathcal{U}(E, F)$ becomes an ordered vector space. If a vector lattice $F$ is Dedekind complete we have the following theorem.

**Theorem 2.3.** ([14], Theorem 3.2). Let $E$ and $F$ be a vector lattices, $F$ Dedekind complete. Then $\mathcal{U}(E, F)$ is a Dedekind complete vector lattice.

Moreover for $S, T \in \mathcal{U}(E, F)$ and for $f \in E$ following hold

1. $(T \lor S)(f) := \sup\{Tg + Sh : f = g \lor h\}$.
2. $(T \land S)(f) := \inf\{Tg + Sh : f = g \land h\}$.
3. $(T^+(f) := \sup\{Tg : g \sqsubseteq f\}$.
4. $(T^-)(f) := -\inf\{Tg : g \sqsubseteq f\}$.
5. $|Tf| \leq |T|(f)$.

**Definition 2.4.** Let $E$ be a vector lattice and $X$ a vector space. An orthogonally additive map $T : E \to X$ is called even if $T(x) = T(-x)$ for every $x \in E$. If $E, F$ are vector lattices, the set of all even abstract Uryson operators from $E$ to $F$ we denote by $\mathcal{U}^{ev}(E, F)$.

If $E, F$ are vector lattices with $F$ Dedekind complete, the space $\mathcal{U}^{ev}(E, F)$ is not empty. Indeed, for every $T \in \mathcal{U}(E, F)$ by ([14], Proposition 3.4) there exists an even operator $\tilde{T} \in U^e_+(E, F)$ which is defined by the formula,

$$\tilde{T}f = \sup\{|T|g : |g| \leq |f|\}.$$

**Lemma 2.5.** ([21], Lemma 3.2.) Let $E, F$ be vector lattices with $F$ Dedekind complete. Then $\mathcal{U}^{ev}(E, F)$ is a Dedekind complete sublattice of $\mathcal{U}(E, F)$.

**Definition 2.6.** Let $(V, E)$ and $(W, F)$ be lattice-normed spaces. A map $T : V \to W$ is called orthogonally additive if $T(u + v) = Tu + Tv$ for every $u, v \in V$, $u \perp v$. An orthogonally additive map $T : V \to W$ is called a dominated Uryson operator if there exists $S \in \mathcal{U}^{ev}(E, F)$ such that $|Tv| \leq S |v|$ for every $v \in V$. In this case we say that $S$ is a dominant for $T$. The set of all dominants of the operator $T$ is denoted by $\text{Domin}(T)$. If there is the least element in $\text{Domin}(T)$ with respect to the order induced by $\mathcal{U}^{ev}(E, F)$ then it is called the least or the exact dominant of $T$ and is denoted by
$|T|$ . The set of all dominated Uryson operators from $V$ to $W$ is denoted by $\mathcal{D}_U(V, W)$.

**Example 2.7.** Let $X, Y$ be normed spaces. Consider the lattice-normed spaces $(X, \mathbb{R})$ and $(Y, \mathbb{R})$. Then a given map $T : X \to Y$ is an element of $\mathcal{D}_U(X, Y)$ if and only if there exists an even function $f : \mathbb{R} \to \mathbb{R}_+$ such that $f(0) = 0$, the set $f(D)$ is bounded for every bounded subset $D \subset \mathbb{R}$ and the inequality $\|Tx\| \leq f(|x|)$ holds for every $x \in X$.

**Example 2.8.** Let $E, F$ be vector lattices with $F$ Dedekind complete. Consider the lattice-normed spaces $(E, E)$ and $(F, F)$ where the lattice valued norms coincide with the modules. We may show that the vector space $\mathcal{D}_U(E, F)$ coincide with $\mathcal{U}(E, F)$. Indeed, if $T \in \mathcal{D}_U(E, F)$, then there exists $S \in \mathcal{U}_+(E, F)$ such that $|Tx| \leq S|x|$ for every $x \in E$. Thus, $T$ is order bounded. If $T \in \mathcal{U}(E, F)$ then by [14], Proposition 3.4 there exists $S \in \mathcal{U}_+(E, F)$, so that $|Tf| \leq S(f) \leq S(|f|)$ and therefore $T \in \mathcal{D}_U(E, F)$.

**Example 2.9.** Let $(A, \Sigma, \mu)$ be a finite complete measure space, $E$ an order dense ideal in $L_0(\mu)$ and $X$ a Banach space. Let $N : A \times X \to X$ be a function satisfying the following conditions:

(C0) $N(t, 0) = 0$ for $\mu$-almost all $t \in A$;

(C1) $N(\cdot, x)$ is Bochner $\mu$-measurable for all $x \in X$;

(C2) $N(t, \cdot)$ is continuous with respect to the norm of $X$ for $\mu$-almost all $t \in A$.

(C3) There exists a measurable function $M : A \times \mathbb{R} \to \mathbb{R}_+$, so that $M(t, \cdot)$ is increasing and $M(t, r) = M(t, -r)$ for $\mu$-almost all $t \in A$, $r \in \mathbb{R}$ and

$$\sup_{\|x\| \leq r} \|N(t, x)\| \leq M(t, r) \text{ for } \mu\text{-almost all } t \in A, r \in \mathbb{R}.$$ 

By $\text{Dom}(N)$ we denote the set of the Bochner $\mu$-measurable vector-function $f : A \to X$, so that $N(\cdot, f(\cdot)) \in L_1(\mu, X)$. If $E(X) \subset \text{Dom}(N)$ and $M(\cdot, g(\cdot)) \in L_1(\mu)$ for every $g \in E$, then is defined the orthogonally additive operator $T : E(X) \to X$ by the formula

$$Tf := \int_A N(t, f(t)) \, d\mu(t).$$

Let us show that $T \in \mathcal{D}_U(E(X), X)$. Indeed

$$|Tf| = \|Tf\| = \left\| \int_A N(t, f(t)) \, d\mu(t) \right\| \leq \int_A \|N(t, f(t))\| \, d\mu(t) \leq$$

$$\leq \int_A M(t, |f(t)|) \, d\mu(t) = S |f|,$$

where $S : E \to \mathbb{R}_+$ is the integral Uryson operator, $Se = \int_A M(t, e(t)) \, d\mu(t)$ and $S$ is a dominant for $T$. 

3. Definition and Some Properties of Orthogonally Additive Narrow Operators

In this section we introduce a new class of nonlinear operators in lattice-normed spaces and describe some of their properties.

**Definition 3.1.** Let \((V, E)\) be a lattice-normed space over an atomless vector lattice \(E\) and \(X\) be a Banach space. A map \(T : V \to X\) is called:

- **narrow** if for every \(v \in V\) and \(\varepsilon > 0\) there exist \(MC\) fragments \(v_1, v_2\) of \(v\) such that \(\|Tv_1 - Tv_2\| < \varepsilon\);

- **strictly narrow** if for every \(v \in V\) there exist \(MC\) fragments \(v_1, v_2\) of \(v\) such that \(Tv_1 = Tv_2\).

Next is the corresponding new definition of an order narrow operator.

**Definition 3.2.** Let \((V, E)\) and \((W, F)\) be lattice-normed spaces with \(E\) atomless. A map \(T : V \to W\) is called

- **order narrow** if for every \(v \in V\) there exists a net of decompositions \(v = u_\alpha^1 \sqcup u_\alpha^2\) such that \(\langle Tu_\alpha^1 - Tu_\alpha^2 \rangle \overset{(bo)}{\to} 0\).

The set of all orthogonally additive narrow operators from a lattice-normed space \((V, E)\) to a Banach space \(X\) we denote by \(\mathcal{NOA}(V, X)\).

**Lemma 3.3.** Let \((V, E)\) be a lattice-normed space and let \((W, F)\) be a Banach space with a mixed norm. Then every \(T \in \mathcal{NOA}(V, W)\) is order narrow.

**Proof.** Take an arbitrary element \(u \in V\). Let \(\varepsilon_n := \frac{1}{2^n}\) and let \(u = u_n^1 \sqcup u_n^2\), \(n \in \mathbb{N}\) be a sequence of decomposition of the element \(u\), such that

\[
\|Tu_n^1 - Tu_n^2\| = \|Tu_n^1 - Tu_n^2\| \leq \varepsilon_n.
\]

We set \(f_n = \sum_{k=n}^{\infty} \|Tu_k^1 - Tu_k^2\|\), \(f_n \in F_+, f_n \downarrow 0\) and the following estimate holds \(\|Tu_n^1 - Tu_n^2\| \leq f_n\). Thus, \(\langle Tu_n^1 - Tu_n^2 \rangle \overset{(bo)}{\to} 0\). \(\square\)

The sets of orthogonally additive narrow and order narrow operators coincide if a vector lattice \(F\) is good enough.

**Lemma 3.4.** Let \((V, E)\) and \((W, F)\) be the same as in Lemma 3.3 and let \(F\) be a Banach lattice with order continuous norm. Then orthogonally additive operator \(T : V \to W\) is order narrow if and only if \(T\) is narrow.

**Proof.** Let \(T\) be an order narrow operator. Then for every \(u \in V\) there exist a net of decompositions \(u = u_\alpha^1 \sqcup u_\alpha^2\), such that \(\langle Tu_\alpha^1 - Tu_\alpha^2 \rangle \overset{(bo)}{\to} 0\). Fix any \(\varepsilon > 0\). Using the fact that the norm in \(F\) is order continuous we can find \(\alpha_0 \in \Lambda\) such that \(\|Tu_\alpha^1 - Tu_\alpha^2\| < \varepsilon\) for every \(\alpha \geq \alpha_0\). In view of Lemma 3.3, the converse is true. \(\square\)
4. C-compact operators and operators to a sequence vector lattices

In this section we investigate connections between narrow and C-compact orthogonally additive operators. Firstly we give a definitions.

Recall that a net \((x_\alpha)\) in a lattice-normed space \((V, E)\) laterally converges to \(x \in V\) if \(x_\alpha \subseteq x_\beta \subseteq x\) for all indices \(\alpha < \beta\) and \(x_\alpha \xrightarrow{(bo)} x\). In this case we write \(x_\alpha \xrightarrow{\text{lat}} x\).

**Definition 4.1.** Let \((V, E)\) be a lattice-normed space and \(F\) be a Banach space. The orthogonally additive operator \(T : V \to F\) is called

- **laterally-to-norm** continuous provided \(T\) sends laterally convergent nets in \((V, E)\) to norm convergent nets in \(F\);
- **generalized AM-compact** (or GAM-compact for short), if \(T(M)\) are precompact in \(F\) for any \((bo)\)-bounded set \(M \subset V\);
- **C-compact** if the sets \(\{Tg : v \in F_v\}\) are precompact in \(F\) for every \(v \in V\).

The set of all C-compact dominated Uryson operators from \(V\) to \(F\) is denoted by \(\mathcal{CD}(V, F)\).

**Example 4.2.** Let \((V, E)\) be a lattice-normed space, \(F\) be a Banach space. Since \(F_v\) is \((bo)\)-bounded by the element \(|v|\) for any \(v \in V\), an every GAM-compact orthogonally additive operator \(T : V \to F\) is the C-compact.

**Example 4.3.** Let \(((0, 1], \Sigma_1, \mu)\) and \(((0, 1], \Sigma_2, \nu)\) be two measure spaces, \(E = C[0, 1]\), which is a sublattice of \(L_\infty(\mu)\), and \(F = L_\infty(\nu)\). Consider the integral Uryson operator \(T : E \to F\) with the kernel \(K(s, t, r) = 1_{[0, 1]}(t)1_{[0, 1]}(s)\). Since the interval \([0, 1]\) is connected, every numerical continuous function \(f : [0, 1] \to \mathbb{R}\) is an atom, that is, \(f\) has no nonzero fragment and therefore \(\{Tg : g \subseteq f\}\) is a relatively compact set in \(F\) for every \(f \in E\). Take \(u(t) = 1_{[0, 1]}(t)\) and consider the order bounded set \(D = \{f \in E : |f| \leq u\}\) in \(E\). Then we have

\[
T(f)(s) = \int_0^1 1_{[0, 1]}(t)1_{[0, 1]}(s)f(t)d\mu(t) = 1_{[0, 1]}(s)\int_0^1 |f(t)|d\mu(t).
\]

Observe that \(T(D)\) is not relatively compact in \(F\). Therefore \(T\) is C-compact, but not AM-compact.

Now, we need the following known property of atoms in vector lattices.

**Proposition 4.4.** ([12], Theorem 26.4 (ii)) For any two atoms \(u, v\) in a vector lattice \(E\), either \(u \perp v\), or \(v = \lambda u\) for some \(0 \neq \lambda \in \mathbb{R}\).

We say that a vector lattice \(E\) is **discrete** if there is a collection \((u_i)_{i \in I}\) of atoms in \(E^+\), called a generating collection of atoms, such that \(u_i \perp u_j\) for \(i \neq j\) and for every \(x \in E\) if \(|x| \wedge u_i = 0\) for each \(i \in I\) then \(x = 0\).

Let \(X\) be a Banach space, \(E = \mathbb{R}^n\) for some \(n \in \mathbb{N}\), and \(E(X)\) be a Köthe-Bochner space of finite \(X\)-valued sequences. Observe that \(E(X)\) is a lattice-normed space with the vector norm defined by the formula \(|f| := \|f(\cdot)\|_X\).
for every \( f \in E(X) \), and \( E(X) \) is also a Banach space with a mixed norm \( \|f\|_{E(X)} := \|\|f\|\|_E \).

**Example 4.5.** The identity operator \( I : E(X) \to E(X) \) is the \( C \)-compact. Indeed, \( E \) is finite-dimensional atomic Banach lattice, and therefore the set \( \mathcal{F}_{\|f\|} \) and consequently the set \( \mathcal{F}_f \) has a finite number of the nonzero fragments for every \( f \in E(X) \). Hence the \( \mathcal{F}_f \) is the precompact set in \( E(X) \).

Remark that a \( C \)-compact abstract Uryson operator \( T : E \to F \) between Banach lattices \( E, F \) with \( F \) \( \sigma \)-Dedekind complete is \( AM \)-compact if, in addition, \( T \) is uniformly continuous on order bounded subsets of \( E \) [13, Theorem 3.4].

Now we need a some auxiliary lemmas. The following lemma is known as the lemma on rounding off coefficients ([5], p. 14).

**Lemma 4.6.** Let \((x_i)_{i=1}^n\) be a finite collection of vectors in a finite dimensional normed space \( X \) and let \((\lambda_i)_{i=1}^n\) be a collection of reals with \( 0 \leq \lambda_i \leq 1 \) for each \( i \). Then there exists a collection \((\theta_i)_{i=1}^n\) of numbers \( \theta_i \in \{0,1\} \) such that

\[
\left\| \sum_{i=1}^n (\lambda_i - \theta_i) x_i \right\| \leq \frac{\dim X}{2} \max_i \|x_i\|.
\]

**Lemma 4.7.** Let \((V,E)\) and \( F \) be the same as in the Theorem [4.1] and \( T : V \to F \) be an orthogonally additive laterally-to-norm continuous operator. If \( e \in E_+ \), \( |v_n| \leq e \) and \( v_n \perp v_m \) for each integers \( n \neq m \) then \( \lim_{n \to \infty} \|T(v_n)\| = 0 \).

**Proof.** Since \( V \) is Banach-Kantorovich space, the sequence \( u_n = \sum_{k=1}^n v_k \) laterally converges to \( u = \sum_{k=1}^\infty v_k \). Then the laterally-to-norm continuity of \( T \) implies that \( Tu_n \) converges to \( Tu \) in \( F \). The sequence \((Tu_n)_{n=1}^\infty\) is fundamental, that is, \( \lim_{n,m \to \infty} \|Tu_n - Tu_m\| = 0 \), hence

\[
\|Tu_n - Tu_{n-1}\| = \|T\left(\sum_{k=1}^n v_k\right) - T\left(\sum_{k=1}^{n-1} v_k\right)\| = \|Tu_n\|
\]

implies \( \lim_{n \to \infty} \|Tu_n\| = 0 \). \( \square \)

**Lemma 4.8.** Let \((V,E)\) be a Banach-Kantorovich space over atomless Dedekind complete vector lattice \( E \), \( F \) be a finite dimensional Banach space, \( T : V \to F \) be an orthogonally additive laterally-to-norm continuous operator and \( v \in V \). Then there exist \( MC \) fragments \( v_1, v_2 \) of \( v \) such that \( \|T(v_1)\| = \|T(v_2)\| \).

**Proof.** Fix any \( MC \) fragments \( v_1, v_2 \) of \( v \). If \( \|T(v_1)\| = \|T(v_2)\| \) then there is nothing to prove. With no loss of generality we may and do assume that \( \|T(v_1)\| - \|T(v_2)\| > 0 \). Consider the partially ordered set

\[
D = \{w \subseteq v_1 : \|T(v_1 - w)\| - \|T(v_2 + w)\| \geq 0\}
\]
Lemma 4.9. Let \( (V, E) \), \( F, T : V \to F \) be the same as in the Lemma 4.8, \( v \in V \) and \( (v_n)_{n=1}^\infty \) be a disjoint tree on \( v \). If \( \|T(v_{2n})\| = \|T(v_{2n+1})\| \) for every \( n \geq 1 \) then
\[
\lim_{m \to \infty} \gamma_m = 0, \text{ where } \gamma_m = \max_{2^m \leq i < 2^{m+1}} \|T(v_i)\|.
\]

Proof. Set \( \varepsilon = \limsup_{m \to \infty} \gamma_m \) and prove that \( \varepsilon = 0 \), which will be enough for the proof. Suppose on the contrary that \( \varepsilon > 0 \). Then for each \( n \in \mathbb{N} \) we set
\[
\varepsilon_n = \limsup_{m \to \infty} \max_{2^m \leq i < 2^{m+1}, v_i \subseteq v_n} \|T(v_i)\|.
\]
Hence, for each \( m \in \mathbb{N} \) one has
\[
\max_{2^m \leq i < 2^{m+1}} \varepsilon_i = \varepsilon. \quad (\ast)
\]

Now we are going to construct a sequence of mutually disjoint elements \( (v_n)_{j=1}^\infty \) such that \( \|T(v_{n_j})\| \geq \frac{\varepsilon}{2} \), that is impossible by Lemma 4.7. At the first step we choose \( m_1 \) so that \( \max_{2^{m_1} \leq i < 2^{m_1+1}} \|T(v_i)\| \geq \frac{\varepsilon}{2} \). By (\ast), we choose \( i_1, 2^{m_1} \leq i_1 < 2^{m_1+1} \) so that \( \varepsilon_{i_1} = \varepsilon \). Using \( \|T(v_{2n})\| = \|T(v_{2n+1})\| \), we choose \( n_1 \neq i_1, 2^{m_1} \leq n_1 < 2^{m_1+1} \) so that \( \|T(v_{n_1})\| \geq \frac{\varepsilon}{2} \). At the second step we choose \( m_2 > m_1 \) so that
\[
\max_{2^{m_2} \leq i < 2^{m_2+1}, v_i \subseteq v_{n_1}} \|T(v_i)\| \geq \frac{\varepsilon}{2}.
\]
By (\ast), we choose \( i_2, 2^{m_2} \leq i_2 < 2^{m_2+1} \) so that \( \varepsilon_{i_2} = \varepsilon \). Then we choose \( m_2 \neq i_2, 2^{m_2} \leq i_2 < 2^{m_2+1} \) so that \( \|T(v_{m_2})\| \geq \frac{\varepsilon}{2} \). Proceeding further, we construct the desired sequence. Indeed, \( \|T(v_{m_j})\| \geq \frac{\varepsilon}{2} \) by the construction and the mutual disjointness for \( v_{m_1}, v_{m_j}, j \neq i \) is guaranteed by the condition \( m_j \neq i_j \), because the elements \( v_{m_{j+1}} \) are fragments of \( v_{i_j} \) which are disjoint to \( v_{m_j} \).
Lemma 4.10. Let \((V,E)\) be a Banach-Kantorovich space over atomless Dedekind complete vector lattice \(E\) and \(F\) be a finite dimensional Banach space. Then every orthogonally additive laterally-to-norm continuous \(C\)-compact operator \(T : V \to F\) is narrow.

**Proof.** Fix any \(v \in V\) and \(\varepsilon > 0\). Using Lemma 4.8 we construct a disjoint tree \((v_n)\) on \(v\) with \(|T(v_{2n})| = |T(v_{2n+1})|\) for all \(n \in \mathbb{N}\). By lemma 4.9 we choose \(m\) so that \(\gamma_m \dim F < \varepsilon\). Then using Lemma 4.6, we choose numbers \(\lambda_i \in \{0, 1\}\) for \(i = 2^m, \ldots, 2^m+1 - 1\) so that

\[
\|2 \sum_{i=2^m}^{2^m+1-1} \left(\frac{1}{2} - \lambda_i\right) T(v_i)\| \leq \dim F \max_{2^m \leq i < 2^m+1} \|T(v_i)\| = \gamma_m \dim F < \varepsilon.
\]

(4.1)

Observe that for \(I_1 = \{i = 2^m, \ldots, 2^m+1 - 1 : \lambda_i = 0\}\) and \(I_2 = \{i = 2^m, \ldots, 2^m+1 - 1 : \lambda_i = 1\}\) the vectors \(w_j = \sum_{i \in I_j} v_i\), \(j = 1, 2\) are MC fragments of \(v\) and by (4.1),

\[
\|T(w_1) - T(w_2)\| = \|\sum_{i=2^m}^{2^m+1-1} (1 - 2\lambda_i) T(v_i)\| < \varepsilon.
\]

□

The following theorem is the first main result of the section.

**Theorem 4.11.** Let \((V,E)\) be a Banach-Kantorovich space over atomless Dedekind complete vector lattice \(E\) and \(F\) be a Banach space. Then every orthogonally additive laterally-to-norm continuous \(C\)-compact operator \(T : V \to F\) is narrow.

**Proof.** We may consider \(F\) as a subspace of some \(l_\infty(D)\) space

\[
F \hookrightarrow F^{**} \hookrightarrow l_\infty(B_{F^*}) = l_\infty(D) = W.
\]

By the notation \(\hookrightarrow\) we mean an isometric embedding. It is well known that if \(H\) is a relatively compact subset of \(l_\infty(D)\) for some infinite set \(D\) and \(\varepsilon > 0\) then there exists a finite rank operator \(S \in l_\infty(D)\) such that \(|x - Sx| \leq \varepsilon\) for every \(x \in H\) [22] Lemma 10.25]. Fix any \(v \in V\) and \(\varepsilon > 0\). Since \(T\) is a \(C\)-compact operator, \(K = \{T(u) : u\) is a fragment of \(w\}\) is relatively compact in \(X\) and hence, in \(W\). By the above, there exists a finite rank linear operator \(S \in \mathcal{L}(W)\) such that \(|w - Sw| \leq \frac{\varepsilon}{4}\) for every \(w \in K\). Then \(R = S \circ T\) is an orthogonally additive laterally-norm continuous finite rank operator. By Lemma 4.10 there exist MC fragments \(v_1, v_2\) of \(v\) such that
\[ \|R(v_1) - R(v_2)\| < \frac{\varepsilon}{2}. \] Thus,
\[
\|T(v_1) - T(v_2)\| = \|T(v_1) - T(v_2) + S(T(v_1)) - S(T(v_2)) - S(T(v_1)) + S(T(v_2))\| \\
= \|R(v_1) - R(v_2)\| + \|T(v_1) - S(T(v_1)) - (T(v_2) - S(T(v_2)))\| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Now we present the second main result of this section.

**Theorem 4.12.** Let \((V, E)\) be a Banach-Kantorovich space over atomless Dedekind complete vector lattice \(E\) and \(\Gamma\) any set. Let \(X = X(\Gamma)\) denote one of the Banach lattices \(c_0(\Gamma)\) or \(\ell_p(\Gamma)\) with \(1 \leq p < \infty\). Then every laterally-to-norm continuous dominated Uryson operator \(T : V \to X\) is narrow.

**Proof.** Let \(T : V \to X\) be a dominated Uryson operator, \(v \in V\) and \(\varepsilon > 0\). Now we may write
\[
|Tu| \leq |T| |u| \leq |T| |v|
\]
for all \(u \subseteq v\). Choose \(x \in X^+\) so that \(|T(u)| \leq x\) for all \(u \subseteq v\). Then we choose a finite subset \(\Gamma_0 \subseteq \Gamma\) so that
\[
(1) \ |x(\gamma)| \leq \varepsilon/4 \text{ for all } \gamma \in \Gamma \setminus \Gamma_0 \text{ if } X = c_0(\Gamma) \text{ and } \quad (2) \ \sum_{\gamma \in \Gamma \setminus \Gamma_0} |x(\gamma)|^p \leq (\varepsilon/4)^p \text{ if } X = \ell_p(\Gamma).
\]
Let \(P\) be the projection of \(X(\Gamma_0)\) onto \(X(\Gamma \setminus \Gamma_0)\) and \(Q = I - P\) the orthogonal projection. Obviously, both \(P\) and \(Q\) are positive linear bounded operators. Since \(S = P \circ T : V \to X(\Gamma_0)\) is a finite rank laterally-to-norm continuous operator, by Lemma 4.11, \(S\) is narrow, and hence, there are \(MC\) fragments \(v_1, v_2\) of \(v\) with \(\|S(v_1) - S(v_2)\| < \varepsilon/2\). Since \(|T(v_i)| \leq x\), by the positivity of \(Q\) we have that \(Q(T(v_i)) \leq Qx\), and thus, \(\|Q(T(v_i))\| \leq \|Q(x)\|\) for \(i = 1, 2\). Moreover, by (1) and (2), \(\|Q(x)\| \leq \varepsilon/4\). Then
\[
\|T(v_1) - T(v_2)\| = \|S(v_1) + Q(T(v_1)) - S(v_2) - Q(T(v_2))\| \\
\leq \|S(v_1) - S(v_2)\| + \|Q(T(v_1))\| + \|Q(T(v_2))\| \\
< \frac{\varepsilon}{2} + \|Q(x)\| + \|Q(x)\| \leq \varepsilon.
\]

The idea used in the proof of Theorem 4.12 could be generalized as follows.

**Definition 4.13.** Let \(F\) be ordered vector space. We say that a linear operator \(G : F \to F\) is quasi-monotone with a constant \(M > 0\) if for each \(x, y \in F^+\) the inequality \(x \leq y\) implies \(Gx \leq MGy\). An operator \(G : F \to F\) is said to be quasi-monotone if it is quasi-monotone with some constant \(M > 0\).
If $G \neq 0$ in the above definition, we easily obtain $M \geq 1$. Observe also that the quasi-monotone operators with constant $M = 1$ exactly are the positive operators.

Recall that a sequence of elements $(f_n)_{n=1}^{\infty}$ (resp., of finite dimensional subspaces $(F_n)_{n=1}^{\infty}$ of a Banach space $F$ is called a basis (resp., a finite dimensional decomposition, or FDD, in short) if for every $f \in F$ there exists a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ (resp. sequence $(u_n)_{n=1}^{\infty}$ of elements $u_n \in F_n$) such that $f = \sum_{n=1}^{\infty} a_n f_n$ (resp., $e = \sum_{n=1}^{\infty} u_n$). Every basis $(f_n)$ generates the FDD $F_n = \{\lambda f_n : \lambda \in \mathbb{R}\}$. Any basis $(f_n)$ (resp., any FDD $(F_n)$) of a Banach space generates the corresponding basis projections $(P_n)$ defined by

$$P_n\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=1}^{n} a_k f_k \quad \text{(resp., } P_n\left(\sum_{k=1}^{\infty} u_k\right) = \sum_{k=1}^{n} u_k\text{),}$$

which are uniformly bounded. For more details about these notions we refer the reader to [10]. The orthogonal projections to $P_n$’s defined by $Q_n = \text{Id} - P_n$, where $\text{Id}$ is the identity operator on $F$, we will call the residual projections associated with the basis $(f_n)_{n=1}^{\infty}$ (resp., to the FDD $(F_n)_{n=1}^{\infty}$).

**Definition 4.14.** A basis $(f_n)$ (resp., an FDD $(F_n)$) of a Banach lattice $F$ is called residually quasi-monotone if there is a constant $M > 0$ such that the corresponding residual projections are quasi-monotone with constant $M$.

In other words, an FDD $(F_n)$ of $F$ is residually quasi-monotone if the corresponding approximation of smaller in modulus elements is better, up to some constant multiple: if $x, y \in F$ with $|x| \leq |y|$ then $\|x - P_n x\| \leq M \|y - P_n y\|$ for all $n$ (observe that $\|z - P_n z\| \to 0$ as $n \to \infty$ for all $z \in F$).

**Theorem 4.15.** Let $(V, E)$ be a Banach-Kantorovich space over atomless Dedekind complete vector lattice $E$ and $F$ be a Banach lattice with a residually quasi-monotone basis or, more general, a residually quasi-monotone FDD. Then every dominated Uryson operator $T : V \to F$ is narrow.

**Proof.** Let $(F_n)$ be an FDD of $F$ with the corresponding projections $(P_n)$, and let $M > 0$ be such that for every $n \in \mathbb{N}$ the operator $Q_n = \text{Id} - P_n$ is quasi-monotone with constant $M$. Let $T : V \to F$ be a dominated Uryson operator, $v \in V$ and $\varepsilon > 0$. Choose $f \in F_+$ so that $|Tu| \leq f$ for all $u \subseteq v$. Since $\lim_{n \to \infty} P_n f = f$, we have that $\lim_{n \to \infty} Q_n f = 0$. Choose $n$ so that

$$\|Q_n f\| \leq \frac{\varepsilon}{4M}.\tag{4.2}$$

Since $S = P_n \circ T : V \to E_n$ is a finite rank dominated Uryson operator, by Lemma [11,10] $S$ is narrow, and hence, there are MC fragments $v_1, v_2$ of $v$ such that $\|Sv_1 - Sv_2\| < \varepsilon/2$. Since $|Tv_1| \leq f$, by the quasi-monotonicity of
Proof. Firstly, fix any \( e \in E_+ \) and \( \varepsilon > 0 \). Since
\[
\left\{ \sum_{i=1}^{n} |Tv_i| : \sum_{i=1}^{n} |v_i| = e ; |v_i| \perp |v_j| ; i \neq j ; n \in \mathbb{N} \right\}
\]
is an increasing net, there exits a net of finite collections \( \{v_1^\alpha, \ldots, v_{n_\alpha}^\alpha\} \subset V, \alpha \in \Lambda \) with
\[
e = \bigcup_{i=1}^{n_\alpha} |v_i^\alpha| , \alpha \in \Lambda
\]
and
\[
\left( |T| (e) - \sum_{i=1}^{n_\alpha} |Tv_i^\alpha| \right) \leq y_\alpha \xrightarrow{\rho} 0 ,
\]
where \( 0 \leq y_\alpha, \alpha \in \Lambda \) is an decreasing net and \( \inf(y_\alpha)_{\alpha \in \Lambda} = 0 \). Fix some \( \alpha \in \Lambda \). Since \( T \) is order narrow operator we may assume that there exist a finite set of a nets of a decompositions \( v_i^\alpha = u_i^{\beta_i} \sqcup w_i^{\beta_i}, i \in \{1, \ldots, n_\alpha\} \) which
\[
Q_n \text{ we have that } \|Q_n(Tv_i)\| \leq M\|Q_nf\| \text{ for } i = 1, 2. \text{ Then by } [4.2],
\]
\[
\|Tv_1 - Tv_2\| = \|Sv_1 + Q(Tv_1) - Sv_2 - Q(Tv_2)\| \\
\leq \|Sv_1 - Sv_2\| + \|Q(Tv_1)\| + \|Q(Tv_2)\| \\
< \frac{\varepsilon}{2} + M\|Qf\| + M\|Qf\| \leq \varepsilon.
\]

\[\square\]

5. Domination problem for narrow operators

In this section we consider a domination problem for the exact dominant of dominated Uryson operators. Firstly, define the following set
\[
\widetilde{E}_+ = \{ e \in E_+ : e = \bigcap_{i=1}^{n} |v_i| ; v_i \in V; n \in \mathbb{N} \}.
\]

**Theorem 5.1.** ([21], Theor. 3.4., 3.7.) Let \((V, E), (W, F)\) be lattice-normed spaces, with \( V \) decomposable and \( F \) Dedekind complete. Then every dominated Uryson operator \( T : V \to W \) has an exact dominant \( |T| \) and it can be calculated by the following formulas:

1. \( |T| (e) = \sup \left\{ \sum_{i=1}^{n} |Tu_i| : \prod_{i=1}^{n} |u_i| = e, n \in \mathbb{N} \right\} (e \in \widetilde{E}_+) ; \)
2. \( |T| (e) = \sup \left\{ |T| (e_0) : e_0 \in \widetilde{E}_+, e_0 \subseteq e \right\} (e \in E_+) \)
3. \( |T| (e) = |T| (e_+) + |T| (e_-), e \in E. \)

**Theorem 5.2.** Let \((V, E)\) be a lattice-normed space with \( E \) atomless, \((W, F)\) be a Banach-Kantorovich space, \( F \) be Dedekind complete and \( T \) be an order narrow dominated Uryson operator from \( V \) to \( W \). Then the exact dominant \( |T| : E \to F \) is order narrow.
depends of \( \alpha \), indexed by the same set \( \Delta \), such that \( \left| Tu^\alpha_i - Tw^\alpha_i \right| \xrightarrow{(o)} 0 \), \( i \in \{1, \ldots, n\} \). Let \( f^\alpha = \prod_{i=1}^{n} |u^\alpha_i| \) and \( g^\alpha = \prod_{i=1}^{n} |w^\alpha_i| \). Then we have

\[
0 \leq |T| (f^\alpha) - \sum_{i=1}^{n} \left( |Tu^\alpha_i| \right) \leq |T| (\varepsilon) - \sum_{i=1}^{n} |Tv^\alpha_i| ; \\
0 \leq |T| (g^\alpha) - \sum_{i=1}^{n} \left( |Tw^\alpha_i| \right) \leq |T| (\varepsilon) - \sum_{i=1}^{n} |Tv^\alpha_i| .
\]

Now we may write

\[
|T| f^\alpha - |T| g^\alpha = |T| f^\alpha - \sum_{i=1}^{n} |Tu^\alpha_i| + \sum_{i=1}^{n} |Tu^\alpha_i| - \sum_{i=1}^{n} |Tw^\alpha_i| + \sum_{i=1}^{n} |Tw^\alpha_i| - |T| g^\alpha \leq \\
|T| f^\alpha - \sum_{i=1}^{n} |Tu^\alpha_i| + |T| g^\alpha - \sum_{i=1}^{n} |Tv^\alpha_i| + \sum_{i=1}^{n} |Tu^\alpha_i| - \sum_{i=1}^{n} |Tv^\alpha_i| \leq \\
2 \left( |T| (\varepsilon) - \sum_{i=1}^{n} |Tv^\alpha_i| \right) + \sum_{i=1}^{n} \left| Tu^\alpha_i \right| - \left| Tw^\alpha_i \right| \leq \\
\left( 2 |T| (\varepsilon) - 2 \sum_{i=1}^{n} |Tv^\alpha_i| + \sum_{i=1}^{n} \left| Tu^\alpha_i - Tw^\alpha_i \right| \right) \xrightarrow{(o)} 0.
\]

Therefore \( e = f^\alpha \sqcup g^\alpha \), \( \alpha \in \Lambda \), \( \beta_\alpha \in \Delta \) is a desirable net of decompositions. Now, let \( e \in E_+ \). Observe that \( D = \{ f \subseteq e : f \in \tilde{E}_+ \} \) is a directed set, where \( f_1 \leq f_2 \) mean that \( f_1 \subseteq f_2 \). Indeed, let \( f_1 = \prod_{i=1}^{k} |u_i| , f_1 \subseteq e, f_2 = \prod_{j=1}^{n} |w_j| ; f_2 \subseteq e, u_i, w_j \in V, 1 \leq i \leq k, 1 \leq j \leq n \). Then by decomposability of the vector norm there exists a set of mutually disjoint elements \((v_{ij})\), \( 1 \leq i \leq k, 1 \leq j \leq n \), such that \( u_i = \prod_{j=1}^{n} v_{ij} \) for every \( 1 \leq i \leq k \) and \( w_j = \prod_{i=1}^{k} v_{ij} \) for every \( 1 \leq j \leq n \). Let \( f = \prod |v_{ij}| \). It is clear that \( |T| f_i \leq |T| f \), \( i \in \{1, 2\} \). Let \((e_\alpha)_{\alpha \in \Lambda}, e_\alpha \in D \) be a net, where \( |T| = \sup_{\alpha} |T| e_\alpha \). Fix \( \alpha \in \Lambda \), then for \( e_\alpha \in D \) there exists a net of
decompositions \( e_\alpha = f_\alpha^\beta \sqcup g_\alpha^\beta, \beta \in \Delta \), such that \[ |T| f_\alpha^\beta - |T| g_\alpha^\beta \overset{}{\to} 0. \]

Thus we have
\[
\left| |T| (e - e_\alpha + f_\alpha^\beta) - |T| g_\alpha^\beta \right| = \\
\left| |T| (e - e_\alpha) + |T| f_\alpha^\beta - |T| g_\alpha^\beta \right| \leq \\
\left( |T| e - |T| e_\alpha + |T| f_\alpha^\beta - |T| g_\alpha^\beta \right) \overset{}{\to} 0.
\]

Hence \( e = (e - e_\alpha \sqcup f_\alpha^\beta) \sqcup g_\alpha^\beta \) is a desirable net of decompositions. Since \( |T| \in \mathcal{U}_0^w(E, F) \), then \( |T| (e) = |T| (-e) \) for every \( e \in (-E_+) \) and if \( e = f_\alpha \sqcup g_\alpha \) is a necessary net of decompositions for \( e \), then \( -e = (-f_\alpha) \sqcup (-g_\alpha) \) is a same. Finally for arbitrary element \( e \in E \) we have \( e = e_+ - e_- \) and by \( \text{(5.1)} \) we have \( |T| (e) = |T| (e_+) + |T| (e_-) \). Thus, if \( e_+ = f_1^\alpha \sqcup f_2^\alpha \) and \( e_- = g_1^\alpha \sqcup g_2^\alpha \) are necessary nets of decompositions, then
\[
\left| |T| (f_1^\alpha + g_1^\alpha) - |T| (f_2^\alpha + g_2^\alpha) \right| = \\
\left| |T| (f_1^\alpha - |T| (f_2^\alpha + |T| g_1^\alpha)) - |T| g_2^\alpha \right| \leq \\
\left( |T| f_1^\alpha - |T| f_2^\alpha \right) + \left( |T| g_1^\alpha - |T| g_2^\alpha \right) \overset{}{\to} 0
\]
and \( e = (f_1^\alpha + g_1^\alpha) \sqcup (f_2^\alpha + g_2^\alpha) \) is a desirable net of decompositions. \( \square \)

**Remark 5.3.** This is an open question. Does the order narrowness of the operator \( |T| \) implies the order narrowness of the \( T \)? A particular case was proved in \( \text{(20, Theorem 4.1)} \).

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