On the Euler–Poincaré characteristics of a simply connected rationally elliptic CW-complex

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Abstract
For a simply connected rationally elliptic CW-complex $X$, we show that the cohomology and the homotopy Euler–Poincaré characteristics are related to two new numerical invariants namely $\eta_X$ and $\rho_X$ which we define using the Whitehead exact sequences of the Quillen and the Sullivan models of $X$.

Keywords  Rationally elliptic space · Sullivan model · Quillen model · Euler–Poincaré characteristic · Whitehead exact sequence

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1 introduction
A simply connected CW-complex $X$ is called rationally elliptic if both the graded vector spaces $H^\ast(X; \mathbb{Q})$ and $\pi_\ast(X) \otimes \mathbb{Q}$ are finite dimensional. To such a space $X$ are attached two numerical invariants namely the cohomology Euler characteristic $\chi_H$ and the homotopy Euler characteristic $\chi_\pi$ defined by

$$\chi_H = \sum_{i \geq 0} (-1)^i \dim H^i(X; \mathbb{Q}), \quad \chi_\pi = \sum_{i \geq 0} (-1)^i \dim \pi_i(X) \otimes \mathbb{Q}.$$  

Computing $\chi_H$ and $\chi_\pi$ and studying their properties is a major task in rational homotopy theory. For instance, it is well-known that (see [7], Prop. 32.10, page 444)

- $\chi_H \geq 0$ and $\chi_\pi \leq 0$. 

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\(\chi_H > 0\) if and only if \(\chi_\pi = 0\).

If \(X[i]\) denotes the \(i\)th Postnikov section of \(X\) and \(X^i\) denotes its \(i\)th skeleton, then we introduce two new numerical homotopy invariants namely

\[
\rho_X = 1 + \sum_{i \geq 2} (-1)^i \dim H^i(X[i-2]; \mathbb{Q}) , \quad \eta_X = 1 + \sum_{i \geq 2} (-1)^i \dim \Gamma_i(X),
\]

where \(\Gamma_i(X) = \ker(\pi_i(X^i) \otimes \mathbb{Q} \longrightarrow \pi_i(X^i, X^{i-1}) \otimes \mathbb{Q})\) for \(i \geq 2\).

By exploiting the well-known properties of a rationally elliptic CW-complex \(X\) as well as the virtue of the Whitehead exact sequences associated respectively to the Sullivan model and the Quillen model of \(X\), we prove the following

**Theorem 1** If \(X\) is a rationally elliptic CW-complex \(X\), then

1. \(H^i(X[i-2]; \mathbb{Q}) \cong \Gamma_{i-1}(X)\) for all \(i \geq 2\);
2. \(\rho_X = \eta_X > 0\);
3. Either \(\rho_X = \chi_H\) or \(\rho_X = -\chi_\pi\);
4. If \(X\) is 2–connected, then \(\Gamma_i(X) = \pi_i(X^i) \otimes \mathbb{Q}\) for all \(i \geq 2\).

As a corollary of Theorem 1, we deduce the following result

**Corollary 2** Let \(X\) be a rationally elliptic CW-complex \(X\). We have the following

1. If \(H^2_i(X[i-2]; \mathbb{Q}) = 0\) for every \(i \geq 2\), then \(X\) is rationally an odd-dimensional sphere.
2. If \(X\) is 2-connected and \(\pi_{2i-1}(X^{2i-2}) \otimes \mathbb{Q} = 0\) for every \(i \geq 1\), then \(X\) is rationally an odd-dimensional sphere.

A special class of rationally elliptic CW-complexes is formed by the \(F_0\)-spaces. Such CW-complex is characterised by the fact that its homotopy Euler characteristics vanishes (see Definition 3.14). Thus, from the main theorem of this paper we show

**Corollary 3** If \(X\) is a 2-connected \(F_0\)-space, then

\[
\pi_{2i+1}(X^{2i}) \otimes \mathbb{Q} = H^{2i+2}(X[2i], \mathbb{Q}) = 0 , \quad \forall i \geq 1.
\]

We show our results using standard tools of rational homotopy theory by working algebraically on the models of Quillen and Sullivan of \(X\). We refer to [7] for a general introduction to these techniques. Recall that every simply connected space of finite type has a corresponding differential commutative cochain algebra called the Sullivan model of \(X\), unique up to isomorphism, that encodes the rational homotopy of \(X\). Dually, every simply connected space \(X\) has a differential graded Lie algebra (DGL for short), called the Quillen model of \(X\) and unique up to isomorphism, which determines completely the rational homotopy type of \(X\).

The paper is organised as follows. In Sect. 2, we recall the definitions of the Whitehead exact sequences associated with the Quillen and the Sullivan models of a rationally elliptic CW-complex as well as we formulate and prove all the results in an algebraic setting. In Sect. 3, we give a mere transcription of the above results in the topological context.
2 Whitehead exact sequences in rational homotopy theory

2.1 Whitehead exact sequence associated with a DGL

Let $W = (W_{\geq 1})$ be a finite dimensional graded vector space over $\mathbb{Q}$ and let $(\mathbb{L}(W), \delta)$ be a DGL. We define the linear maps

$$j_i : H_i(\mathbb{L}(W_{\leq i})) \rightarrow W_i, \quad b_i : W_i \rightarrow H_{i-1}(\mathbb{L}(W_{\leq i-1})), \quad (1)$$

by setting $j_i([w + y]) = w$ and $b_i(w) = [\delta(w)]$, where $[\delta(w)]$ denotes the homology class of $\delta(w)$ in the sub-Lie algebra $\mathbb{L}_{i-1}(W_{\leq i-1})$. Recall that if $x \in H_i(\mathbb{L}(W_{\leq i}))$, then $x = [w + y]$, where $w \in W_i$, $y \in \mathbb{L}_i(W_{\leq i-1})$ and $\delta(w + y) = 0$.

To every DGL $(\mathbb{L}(W), \delta)$ we can assign of (see [1,2,4,6] for more details) the following long exact sequence

$$\cdots \rightarrow W_{i+1} \stackrel{b_{i+1}}{\rightarrow} \Gamma_i \rightarrow H_i(\mathbb{L}(W)) \rightarrow W_i \stackrel{b_i}{\rightarrow} \cdots \quad (2)$$

called the Whitehead exact sequence of $(\mathbb{L}(W), \delta)$, where

$$\Gamma_i = \ker(j_i : H_i(\mathbb{L}(W_{\leq i})) \rightarrow W_i), \quad i \geq 2. \quad (3)$$

**Remark 2.1** First, it is clear that if $H_i(\mathbb{L}(W_{\leq i})) = 0$, then $\Gamma_i = 0$.

Next, if $(\mathbb{L}(W), \delta)$ satisfies $W_1 = 0$, then

$$\Gamma_i = H_i(\mathbb{L}(W_{\leq i-1})), \quad i \geq 2. \quad (4)$$

Indeed; let us consider the following long exact sequence

$$\cdots \rightarrow H_{i+1}(\mathbb{L}(W_{\leq i})/\mathbb{L}(W_{\leq i-1})) \rightarrow H_i(\mathbb{L}(W_{\leq i-1})) \rightarrow H_i(\mathbb{L}(W_{\leq i})) \rightarrow \cdots$$

which is associated with the following short exact of chain complexes

$$(\mathbb{L}(W_{\leq i-1}), \delta) \hookrightarrow (\mathbb{L}(W_{\leq i}), \delta) \rightarrow (\mathbb{L}(W_{\leq i})/\mathbb{L}(W_{\leq i-1}), \tilde{\delta}),$$

where $(\mathbb{L}(W_{\leq i})/\mathbb{L}(W_{\leq i-1}), \tilde{\delta})$ is the quotient chain complex and the differential $\tilde{\delta}$ is induced by $\delta$. Now clearly, we have $(\mathbb{L}(W_{\leq i})/\mathbb{L}(W_{\leq i-1})) \cong W_i$ and as $W_1 = 0$ it follows that $H_{i+1}(\mathbb{L}(W_{\leq i})/\mathbb{L}(W_{\leq i-1})) = 0$. As a result we obtain (4).

2.2 Whitehead exact sequence associated with a Sullivan algebra

A Sullivan algebra $(\Lambda V, \partial)$ over $\mathbb{Q}$ is simply connected if $V^1 = 0$. Let $(\Lambda(V^{\leq i-2}), \partial)$ be the sub algebra generated by the graded vector apace $V^{\leq i-2}$. For simplicity let us write

$$L^i = H^i(\Lambda(V^{\leq i-2})), \quad i \geq 2. \quad (5)$$
The Whitehead exact sequence of $(\Lambda V, \partial)$ is defined as follows (see [2,4,5] for more details)

\[
\cdots \to H^i(\Lambda V) \to V^i \xrightarrow{b^i} L^{i+1} \to H^{i+1}(\Lambda V) \to V^{i+1} \xrightarrow{b^{i+1}} \cdots
\] (6)

Here $b^i(v) = [\partial(v)]$, where $[\partial(v)]$ is the cohomology class of the cycle $\partial(v)$ in $\Lambda(V^{i-2})$.

### 2.3 Elliptic algebras

A Sullivan algebra $(\Lambda V, \partial)$ is called elliptic if

\[
\dim H^*(\Lambda V) = \sum_{i \geq 0} \dim H^i(\Lambda V) < \infty \quad \text{and} \quad \dim V = \sum_{i \geq 2} \dim V^i < \infty.
\]

Let us call $n = \max \{i : H^i(\Lambda V) \neq 0\}$ the formal dimension of $(\Lambda V, \partial)$. For an elliptic Sullivan algebra $(\Lambda V, \partial)$, we define the following two numbers

\[
\chi_{\Lambda V} = \sum_{i \geq 0} \dim H^{2i}(\Lambda V) - \sum_{i \geq 0} \dim H^{2i+1}(\Lambda V) = H^{\text{even}}(\Lambda V) - H^{\text{odd}}(\Lambda V),
\]

\[
\chi_V = \sum_{i \geq 1} \dim V^{2i} - \sum_{i \geq 1} \dim V^{2i+1} = \dim V^{\text{even}} - \dim V^{\text{odd}}.
\]

The following are some important properties of elliptic Sullivan algebras.

**Theorem 2.2** ([4], Sect. 32) Suppose $(\Lambda V, \partial)$ is elliptic of formal dimension $n$. Then

1. $\dim V^{\text{odd}} \geq \dim V^{\text{even}}$;
2. $V^{i} = 0$, for $i \geq 2n$;
3. $V^{i} = 0$, for $i > n$ and $i$ even;
4. There is only one non-trivial $V^{i}$, for $i > n$ and $i$ odd. Necessary, we have $\dim V^{i} = 1$;
5. $\chi_{\Lambda V} \geq 0$ and $\chi_V \leq 0$. Moreover $\chi_{\Lambda V} = 0 \iff \chi_V < 0$.

**Proposition 2.3** Suppose $(\Lambda V, \partial)$ is elliptic of formal dimension $n$.

1. For every even number $i$ such that $i > n + 1$, we have $L^i = 0$;
2. For every odd number $i$ such that $i > n + 1$ we have $L^i \cong V^{i-1}$;
3. For every $i$ such that $i > 2n$, we have $L^i = 0$.

**Proof** As $(\Lambda V, \partial)$ has formal dimension $n$, it follows that $H^i(\Lambda V) = 0$ for $i > n$. Therefore, by Theorem 2.2 the Whitehead exact sequence of $(\Lambda V, \partial)$ can be written as

\[
H^n(\Lambda V) \to V^n \to L^{n+1} \to 0 \to \cdots \to 0 \to V^{2n-1} \to L^{2n} \to 0 \to V^{2n} \to 0 \to L^{2n+1} \to 0
\]

Consequently, the three assertions of Proposition 2.3 are easily derived. \qed
**Definition 2.4** Let \((\Lambda V, \partial)\) be an elliptic Sullivan algebra. Using (5), we define
\[
\rho_{\Lambda V} = 1 + \dim L^{\text{even}} - \dim L^{\text{odd}},
\]
where
\[
\dim L^{\text{even}} = \dim L^4 + \dim L^6 + \dim L^8 + \ldots
\]
\[
\dim L^{\text{odd}} = \dim L^5 + \dim L^7 + \dim L^9 + \ldots
\]
Note that from (5), we deduce that \(L^i = 0\) for \(i = 2, 3\).

**Proposition 2.5** If \((\Lambda V, \partial)\) is elliptic of formal dimension \(n\), then we have
\[
0 \leq \rho_{\Lambda V} - \sum_{i=4}^{n+1} (-1)^i \dim L^i \leq 2. \tag{7}
\]

**Proof** First by (4) of Proposition 2.3, we have
\[
\rho_{\Lambda V} = 1 + \sum_{i=4}^{2n} (-1)^i \dim L^i = 1 + \sum_{i=4}^{n+1} (-1)^i \dim L^i + \sum_{i=n+2}^{2n} (-1)^i \dim L^i.
\]
Next, let us focus on the integer \(\sum_{i=n+2}^{2n} (-1)^i \dim L^i\). Using the relations (3) and (4) of Theorem 2.2 and (2) of Proposition 2.3, we deduce that
\[
-1 \leq \sum_{i=n+2}^{2n} (-1)^i \dim L^i \leq 1.
\]
Therefore
\[
\sum_{i=4}^{n+1} (-1)^i \dim L^i \leq \rho_{\Lambda V} \leq \sum_{i=4}^{n+1} (-1)^i \dim L^i + 2,
\]
as wanted. \(\square\)

The next result shows the relationship between the \(\rho_{\Lambda V}, \chi_{\Lambda V}\) and \(\chi_V\).

**Theorem 2.6** If \((\Lambda V, \partial)\) is an elliptic Sullivan algebra, then \(\rho_{\Lambda V} = \chi_{\Lambda V} - \chi_V\).

**Proof** Let us consider the Whitehead exact sequence of \((\Lambda V, \partial)\) given in (6)
\[
\cdots \rightarrow H^i(\Lambda V) \rightarrow V^i \xrightarrow{b^i} L^{i+1} \xrightarrow{s^{i+1}} H^{i+1}(\Lambda V) \rightarrow V^{i+1} \xrightarrow{b^{i+1}} \cdots
\]
Let us write \(H^* = H^*(\Lambda V)\). For every \(i \geq 2\), let
\[
V^{i,1} = \ker b^i, \quad V^{i,2} = \text{im} b^i, \quad L^{i+1,1} = \text{im} s^{i+1}.
\]
The exactness of the above sequence implies that
\[ V^i \cong V^{i,1} \oplus V^{i,2}, \quad L^{i+1} \cong V^{i,2} \oplus L^{i+1,1}, \quad H^{i+1} \cong L^{i+1,1} \oplus V^{i+1,1}. \tag{8} \]
Using the direct summands in (8) we deduce the following
\[ \dim V^3 = \dim H^3 + \dim V^{3,2} \]
\[ = \dim H^3 + \dim L^4 - \dim L^{4,1} \]
\[ = \dim H^3 + \dim L^4 - \dim H^4 + \dim V^4. \]
\[ \implies \dim V^3 = \dim H^3 + \dim L^4 - \dim H^4 + \dim V^4 - \dim V^{4,2}. \]

Iterating the above process and taking into consideration that \((\Lambda V, \partial)\) is elliptic which implies that this process must stop, we get the following formula
\[ \dim V^3 = - \sum_{i \geq 3} (-1)^i \dim H^i + \sum_{i \geq 4} (-1)^i \dim L^i + \sum_{i \geq 4} (-1)^i \dim V^i, \]
which implies
\[ \sum_{i \geq 3} (-1)^i \dim H^i - \sum_{i \geq 3} (-1)^i \dim V^i = \sum_{i \geq 4} (-1)^i \dim L^i. \tag{9} \]
As \((\Lambda V, \partial)\) is simply connected, it follows that \(H^0 \cong \mathbb{Q}, H^1 = 0\) and \(H^2 \cong V^2\). Therefore the relation (9) becomes
\[ (\dim H^{\text{even}} - \dim H^{\text{odd}}) - (\dim V^{\text{even}} - \dim V^{\text{odd}}) = 1 + \sum_{i \geq 4} (-1)^i \dim L^i, \]
implying \(\rho_{\Lambda V} = \chi_{\Lambda V} - \chi_V. \]

**Corollary 2.7** If \((\Lambda V, \partial)\) is an elliptic Sullivan algebra, then \(\rho_{\Lambda V} > 0\). Moreover if \(\chi_{\Lambda V} > 0\), then \(\rho_{\Lambda V} = \chi_{\Lambda V}. \)

**Proof** As \(\rho_{\Lambda V} = \chi_{\Lambda V} - \chi_V\), the relation (5) of Theorem 2.2 implies that \(\rho_{\Lambda V} \geq 0\). Now if we assume that \(\rho_{\Lambda V} = 0\), then we derive that \(\chi_{\Lambda V} = \chi_V\). Again by the relation (5) of Theorem 2.2, on one hand we have \(\chi_{\Lambda V} = \chi_V = 0\) and on the other hand we have \(\chi_V < 0\) which is impossible. Thus, \(\rho_{\Lambda V} > 0\). Now if \(\chi_{\Lambda V} > 0\), then the relation (4) of Theorem 2.2 implies that \(\chi_V = 0\). Hence \(\rho_{\Lambda V} = \chi_{\Lambda V} - \chi_V = \chi_{\Lambda V}. \) \(\square\)

**Corollary 2.8** If \((\Lambda V, \partial)\) is an elliptic Sullivan algebra, then \(L^{\text{even}} \geq L^{\text{odd}}. \) Moreover, if \(L^{\text{even}} = 0\), then \(\dim H^*(\Lambda V) = \dim V^* + 1. \)

**Proof** By combining Definition 2.4 and Corollary 2.7, we obtain
\[ \rho_{\Lambda V} = 1 + \dim L^{\text{even}} - \dim L^{\text{odd}} > 0 \implies L^{\text{even}} \geq L^{\text{odd}}. \]
Thus, if $L^{\text{even}} = 0$, then $L^{\text{odd}} = 0$ implying that $L^i = 0$, $\forall i \geq 2$. Finally, from the Whitehead exact sequence (6) of $(\Lambda V, \partial)$, we deduce that

$$H^i(\Lambda V) \cong V^i, \quad \forall i \geq 2. \quad (10)$$

Consequently, if $n$ is the formal dimension of $(\Lambda V, \partial)$, then $V^i = 0$, $\forall i > n$ and we obtain

$$\dim H^*(\Lambda V) = \dim H^0(\Lambda V) + \dim H^1(\Lambda V) + \dim H^2(\Lambda V) + \cdots + \dim H^n(\Lambda V) = 1 + \dim V^2 + \dim V^3 + \cdots + \dim V^n = 1 + \dim V^*$$

as desired. \hfill \Box

### 3 Topological applications

Let $X$ be a simply connected CW-complex. For every $i \geq 2$, we define the vector space

$$\Gamma_i(X) = \ker(\pi_i(X^i) \otimes \mathbb{Q} \longrightarrow \pi_i(X^i, X^{i-1}) \otimes \mathbb{Q}). \quad (11)$$

Here $X^i$ denotes the $i$th-skeleton of $X$.

Thus, if $(\mathbb{L}(W), \delta)$ denotes the Quillen model of $X$, then by virtue of the properties of this model, we obtain the following identifications (valid for any $i \geq 2$)

$$\pi_i(X) \otimes \mathbb{Q} \cong H_{i-1}(\mathbb{L}(W)) \quad , \quad H_i(X; \mathbb{Q}) \cong W_{i-1} \quad , \quad \Gamma_i(X) \cong \Gamma_{i-1} \quad , \quad (12)$$

where $\Gamma_i$ is defined in (3). Therefore the Whitehead exact sequence (2) of this model can be written as

$$\cdots \longrightarrow H_{i+1}(X; \mathbb{Q}) \xrightarrow{b_{i+1}} \Gamma_i(X) \longrightarrow \pi_i(X) \otimes \mathbb{Q} \xrightarrow{h_i} H_i(X; \mathbb{Q}) \xrightarrow{b_i} \cdots \quad (13)$$

where $h_i$ is the Hurewicz homomorphism.

**Remark 3.1** Firstly, since $(\mathbb{L}(W_{\leq i-1}), \delta)$ can be chosen as the Quillen model of the skeleton $X^i$ (see [7], pp. 323) it follows that

$$\pi_i(X^i) \otimes \mathbb{Q} \cong H_{i-1}(\mathbb{L}(W_{\leq i-1})), \quad \forall i \geq 2$$

and based on Remark 2.1 and the identifications (12), it follows that

$$\pi_i(X) \otimes \mathbb{Q} = 0 \implies \Gamma_i(X) = 0, \quad \forall i \geq 2 \quad (14)$$

Secondly, if the space $X$ is 2-connected, then by Remark 2.1 we deduce that

$$\Gamma_i \cong H_i(\mathbb{L}(W_{\leq i-1})) \cong \pi_{i+1}(X^i) \otimes \mathbb{Q}, \quad i \geq 2,$$
and the Whitehead exact sequence (13) becomes

\[ \cdots \to H_{i+1}(X; \mathbb{Q}) \xrightarrow{b_{i+1}} \pi_i(X^{i-1}) \otimes \mathbb{Q} \xrightarrow{i_i} \pi_i(X) \otimes \mathbb{Q} \to H_i(X; \mathbb{Q}) \xrightarrow{b_i} \cdots \]

Dually, if \((\Lambda V, \partial)\) is the Sullivan model of \(X\), then we have

\[ H^i(X; \mathbb{Q}) \cong H^i(\Lambda V), \quad H^{i+1}(X^{i-1}; \mathbb{Q}) \cong H^{i+1}(\Lambda V \leq i-1), \]

\[ V^i \cong \text{Hom}(\pi_i(X) \otimes \mathbb{Q}; \mathbb{Q}), \tag{15} \]

where \(X^{[i]}\) denotes the \(i\)th Postnikov section of \(X\). Therefore, the Whitehead exact sequence of this model can be written as

\[ \cdots \to \text{Hom}(\pi_i(X) \otimes \mathbb{Q}, \mathbb{Q}) \xrightarrow{h^i} H^{i+1}(X^{i-1}; \mathbb{Q}) \to H^{i+1}(X; \mathbb{Q}) \to \text{Hom}(\pi_{i+1}(X) \otimes \mathbb{Q}, \mathbb{Q}) \to \cdots \tag{16} \]

Recall that a simply connected CW-complex \(X\) is rationally elliptic if

\[ \dim H^*(X; \mathbb{Q}) = \sum_{i \geq 0} \dim H^i(X; \mathbb{Q}) < \infty \quad \text{and} \quad \dim \pi_*(X) \otimes \mathbb{Q} = \sum_{i \geq 2} \dim \pi_i(X) \otimes \mathbb{Q} < \infty \]

Note that, by virtue of the Sullivan model, \(X\) is rationally elliptic if and only if its Sullivan model is elliptic. In this case the formal dimension of \(X\) is the formal dimension of its Sullivan model.

**Definition 3.2** Let \(X\) be a rationally elliptic CW-complex. We define

\[ \eta_X = 1 + \dim \Gamma_{\text{even}}(X) - \dim \Gamma_{\text{odd}}(X) \]

where \(\Gamma_*(X)\) is defined in (11)

**Definition 3.3** If \(X\) is a rationally elliptic CW-complex and \((\Lambda V, \partial)\) its Sullivan model, then we define \(\rho_X = \rho_{\Lambda V}\), where \(\rho_{\Lambda V}\) is given in Definition 2.4.

Subsequently, we need the following proposition showing that \(H^{i+1}(X^{i-1}, \mathbb{Q})\) and \(V^i\) are isomorphic, as vector spaces, for every \(i \geq 2\).

**Proposition 3.4** If \(X\) is a rationally elliptic CW-complex, then

\[ H^{i+1}(X^{i-1}; \mathbb{Q}) \cong \text{Hom}(\Gamma_i(X), \mathbb{Q}), \quad i \geq 2. \tag{17} \]

**Proof** Applying the exact functor \(\text{Hom}(., \mathbb{Q})\) to the exact sequence (13) we obtain

\[ \cdots \leftarrow H^{i+1}(X; \mathbb{Q}) \leftarrow \text{Hom}(\Gamma_i(X), \mathbb{Q}) \leftarrow \text{Hom}(\pi_i(X) \otimes \mathbb{Q}, \mathbb{Q}) \leftarrow H^i(X; \mathbb{Q}) \leftarrow \cdots \tag{18} \]

Taking in account that

\[ \begin{array}{c}
\end{array} \]
• All the vector spaces involved are finite dimensional.
• The two maps \( H^i(X; \mathbb{Q}) \rightarrow \text{Hom}(\pi_i(X) \otimes \mathbb{Q}, \mathbb{Q}) \) appearing in (16) and (18) are the same morphism for all \( i \geq 2 \).
• \( \text{Hom}(H_\ast(X; \mathbb{Q}); \mathbb{Q}) \cong H^\ast(X; \mathbb{Q}) \).

and by comparing the sequences (16), (18), we get (17).

**Remark 3.5** By virtue of the properties of the Sullivan model \((\Lambda V, \partial)\) of the space \( X \), it is important to mention that for all \( i \geq 2 \), the linear map

\[
b^i : V^i \longrightarrow H^{i+1}(\Lambda(V^{\leq i-1})),
\]

induced by the differential \( \partial \), is the dual of the \( i \)-invariant

\[
k_i \in H^{i+1}(X^{[i-1]}; \pi_i(X)) = \text{Hom}(H_{i+1}(X^{[i-1]}), \pi_i(X)),
\]

which is the dual of the map \( \Gamma_i(X) \longrightarrow \pi_i(X) \otimes \mathbb{Q} \) given in (13).

A mere transcription into the topological context of the above results obtained in Sect. 2 provides the following applications.

**Corollary 3.6** If \( X \) is a rationally elliptic CW-complex of formal dimension \( n \), then

1. For every even number \( i \) such that \( i > n + 1 \), we have

\[
H^i(X^{[i-2]}, \mathbb{Q}) = \Gamma_{i-1}(X) = 0.
\]

2. For every \( i \) such that \( i > 2n \), we have

\[
H^i(X^{[i-2]}, \mathbb{Q}) = \Gamma_{i-1}(X) = 0.
\]

3. If \( X \) is 2-connected, then \( \Gamma_i(X) \cong \pi_i(X^{[i-1]}) \otimes \mathbb{Q} \) for every \( i \geq 2 \).

**Proof** It follows from Remark 2.1, Propositions 2.3 and 3.4.

The next result establishes that the two numerical invariants \( \eta_X \) and \( \rho_Y \) are equal although they are defined differently.

**Theorem 3.7** If \( X \) is a rationally elliptic CW-complex, then \( \eta_X = \rho_X \).

**Proof** The result follows from the Definitions 2.4 and 3.2 and by applying Proposition 3.4 after taking into account the identifications (12).

**Corollary 3.8** If \( X \) is a rationally elliptic CW-complex of formal dimension \( n \), then

\[
0 \leq \rho_X - \sum_{i=4}^{n+1} (-1)^i \dim H^i(X^{[i-2]}; \mathbb{Q}) \leq 2, \quad 0 \leq \eta_X - \sum_{i=3}^{n} (-1)^i \dim \Gamma_i(X) \leq 2
\]

**Proof** It follows from Propositions 2.5, 3.4 and Theorem 3.7.
**Proposition 3.9** If $X$ is a rationally elliptic CW-complex such that

$$H^{2i}(X^{[2i-2]}; \mathbb{Q}) = 0, \quad \forall i \geq 2,$$

then we have

1. $H^{2i+1}(X^{[2i-1]}; \mathbb{Q}) = 0$ for every $i \geq 2$;
2. $\Gamma_i(X) = 0$ for every $i \geq 2$;
3. $\dim H^*(X; \mathbb{Q}) = \dim \pi_*(X) \otimes \mathbb{Q} + 1$.

**Proof** All the assertions are a consequence of Corollary 2.8 and the formula (17). □

**Theorem 3.10** Let $X$ be a rationally elliptic CW-complex $X$. If $H^{2i}(X^{[2i-2]}; \mathbb{Q}) = 0$ for every $i \geq 2$, then $X$ is rationally an odd-dimensional sphere.

**Proof** Let $(\mathbb{L}(W), \delta)$ denote the Quillen model of $X$. First, if $H^{2i}(X^{[2i-2]}; \mathbb{Q}) = 0$ for every $i \geq 2$, then by (5) and the identification (15), we deduce that $L^{\text{even}} = 0$ implying $L^{\text{odd}} = 0$ according to Corollary 3.8. Next, by Proposition 17, we get $\Gamma_i = 0$ for every $i \geq 2$, where $\Gamma_i = 0$ is defined in (3). Therefore, the linear map $b_{i+1} : W_{i+1} \to \Gamma_i$, given in (2), is trivial for $i \geq 2$. Now using the relation (1), it follows that $b_i(w) = [\delta(w)] = 0$, for $i \geq 2$ and every $w \in W_{i+1}$.

Consequently, using the properties of the Quillen model (see [7], pp. 323), it follows that all the attaching maps of the space $X$ are rationally trivial. Consequently the Quillen model of $X$ can be chosen as $(\mathbb{L}(W), 0)$ (with trivial differential). Moreover, and as the space $X$ is rationally elliptic, we must have $W = W_{2k} \equiv \mathbb{Q}$ for a certain $k \geq 1$. Thus, the Quillen model of $X$ has the form $(\mathbb{L}(W_{2i}), 0)$ implying that $X$ is rationally an odd-dimensional sphere. □

**Corollary 3.11** Let $X$ be a rationally elliptic CW-complex $X$. If $\pi_{2i-1}(X^{[2i-1]}) \otimes \mathbb{Q} = 0$ for every $i \geq 1$, then $X$ is rationally an odd-dimensional sphere.

**Proof** If $\pi_{2i-1}(X^{[2i-1]}) \otimes \mathbb{Q} = 0$, $\forall i \geq 2$, then by the implication (14) and Corollary 3.6 we deduce that $H^{2i}(X^{[2i-2]}; \mathbb{Q}) = 0$, $\forall i \geq 2$. Then we apply Theorem 3.10 □

**Corollary 3.12** Let $X$ be a rationally elliptic CW-complex $X$. If $X$ is $2$-connected and $\pi_{2i-1}(X^{[2i-2]}) \otimes \mathbb{Q} = 0$ for every $i \geq 1$, then $X$ is rationally an odd-dimensional sphere.

**Proof** If $X$ is $2$-connected and $\pi_{2i-1}(X^{[2i-2]}) \otimes \mathbb{Q} = 0$ for every $i \geq 1$, then by Corollary 3.6 we deduce that $H^{2i}(X^{[2i-2]}; \mathbb{Q}) = 0$, $\forall i \geq 2$. Then we apply Theorem 3.10. □

**Example 3.13** The purpose of this example is to compute the numerical invariants $\rho_{\mathbb{C}P^n_0}$ and $\eta_{\mathbb{C}P^n_0}$, for the space $\mathbb{C}P^n_0$ which is the rationalised complex projective space $\mathbb{C}P^n_0$.

Indeed; the Sullivan model of $\mathbb{C}P^n_0$ is $(\Lambda(V^2 \oplus V^{2n+1}), \delta)$ with $V^2 = \langle x \rangle$, $V^{2n+1} = \langle y \rangle$ and $\partial y = x^{n+1}$ (see [7], example 5, page 333). As we have $L^1 = H^1(\Lambda(V^{2i-2}))$, it follows that $L^{2n+3} = 0$. Moreover it is easy to see that $\dim L^i = 1$ for $i$ even and $\dim L^i = 0$ for $i$ odd. Therefore $\rho_{\mathbb{C}P^n_0} = 1 + L^{\text{even}} = 1 + n$ and since $\chi_\pi(\mathbb{C}P^n_0) = 2$ Springer
dim $V^2 - \dim V^{2n+1} = 0$, we deduce that $\rho_{\mathbb{C}P^n} = \chi_{\mathbb{C}P^n} = 1 + n$. As a result we get $\eta_{\mathbb{C}P^n} = 1 + n$. Notice that the Quillen model of $\mathbb{C}P^n_0$ is given by

$$L(W_1, \ldots, W_{2n-1}) \ , \ W_i = \langle w_i \rangle \ , \ \delta(w_k) = \frac{1}{2} \sum_{i+j=k} [w_i, w_j].$$

Hence, for $\Gamma_i = H_i(L(W_{\leq i-2}))$, we deduce that $\eta_{\mathbb{C}P^n_0} = 1 + \Gamma_{\text{even}} - \Gamma_{\text{odd}} = 1 + n$.

### 3.1 $F_0$-spaces

**Definition 3.14** An $F_0$-space is a rationally elliptic CW-complex $X$ whose rational cohomology algebra is given by $\mathbb{Q}[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$, where the polynomials $P_1, \ldots, P_n$ form a regular sequence in $\mathbb{Q}[x_1, \ldots, x_n]$, i.e., $P_1 \neq 0$ and for every $i \geq 1$, $P_i$ is not a zero divisor in $\mathbb{Q}[x_1, \ldots, x_n]/(P_1, \ldots, P_{i-1})$.

For instance, products of even spheres, complex Grassmannian manifolds and homogeneous spaces $G/H$ such that rank $G = \text{rank } H$ are $F_0$-spaces.

**Proposition 3.15** ([7] Proposition 32.10) Let $X$ be a rationally elliptic CW-complex and $(\Lambda V, \partial)$ its Sullivan model. The following statements are equivalent

1. $X$ is an $F_0$-space;
2. $H^{\text{odd}}(\Lambda V) = 0$;
3. $\dim V^{\text{even}} - \dim V^{\text{odd}} = 0$ and $(\Lambda V, \partial)$ is pure, i.e., $\partial(V^{\text{even}}) = 0$ and $\partial(V^{\text{odd}}) \subseteq \Lambda V^{\text{even}}$.

**Theorem 3.16** If $X$ is an $F_0$-space, then

$$\Gamma_{2i+1}(X) = H^{2i+2}(X^{[2i]}, \mathbb{Q}) = 0 \ , \ \forall i \geq 1.$$

Moreover, if $X$ is 2-connected, then $\pi_{2i+1}(X^{2i}) \otimes \mathbb{Q} = 0$ for every $i \geq 1$.

**Proof** By (6), the Whitehead exact sequence of $(\Lambda V, \partial)$ can be written as

$$\cdots \rightarrow H^{2i}(\Lambda V) \rightarrow V^{2i} \xrightarrow{b^{2i}} L^{2i+1} \rightarrow H^{2i+1}(\Lambda V) \rightarrow V^{2i+1} \xrightarrow{b^{2i+1}} \cdots$$

As $X$ is an $F_0$-space, then by Proposition 3.15, the Sullivan model $(\Lambda V, \partial)$ of $X$ satisfies $H^{\text{odd}}(\Lambda V) = 0$ and $\partial(V^{\text{even}}) = 0$, it follows that the maps $b^{\text{even}} \equiv 0$. Consequently, $L^{\text{odd}} = 0$. So, taking into account the identifications (12) and (15), the result follows from the relation (17). □

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