Real zeros of \(2F_1\) hypergeometric polynomials

D Dominici\(^1\)

\(\text{Department of Mathematics, State University of New York at New Paltz, 1 Hawk Dr. Suite 9, New Paltz, NY 12561-2443, USA}\)

SJ Johnston

\(\text{Department of Mathematical Sciences, University of South Africa, PO Box 392, UNISA 0003, South Africa}\)

K Jordaan\(^2\)

\(\text{Department of Mathematics and Applied Mathematics, University of Pretoria, Lynnwood Road, Pretoria, 0002, South Africa}\)

Abstract
We use a method based on the division algorithm to determine all the values of the real parameters \(b\) and \(c\) for which the hypergeometric polynomials \(2F_1(-n; b; c; z)\) have \(n\) real, simple zeros. Furthermore, we use the quasi-orthogonality of Jacobi polynomials to determine the intervals on the real line where the zeros are located.

Keywords: Orthogonal polynomials, zeros, hypergeometric polynomials.

2000 MSC: 33C05, 33C45, 42C05.

1. Introduction
The \(2F_1\) hypergeometric function is defined by (cf. [1])

\[
2F_1(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1,
\]

where \(a, b\) and \(c\) are complex parameters, \(-c \notin \mathbb{N}_0 = \{0, 1, 2, \ldots\}\) and

\[
(a)_k = \begin{cases} 
\alpha(\alpha+1)\ldots(\alpha+k-1), & k \in \mathbb{N}, \\
1, & k = 0, \alpha \neq 0
\end{cases}
\]

is Pochhammer’s symbol. This series converges when \(|z| < 1\) and also when \(z = 1\) provided that \(\text{Re}(c-a-b) > 0\) and when \(z = -1\) provided that \(\text{Re}(c-a-b+1) > 0\). When one of the numerator parameters is equal to a nonpositive integer, say \(a = -n, n \in \mathbb{N}_0\), the series terminates and the function is a polynomial of degree \(n\) in \(z\).

The problem of describing the zeros of the polynomials \(2F_1(-n; b; c; z)\) when \(b\) and \(c\) are complex arbitrary parameters, has not been solved. Even when \(b\) and \(c\) are both real, the only cases that have been fully analyzed impose additional restrictions on \(b\) and \(c\). Recent publications (cf. [4], [6], [7], [8], [11] and [13]) considered the zero location of special classes of \(2F_1(-n; b; c; z)\) with restrictions on the parameters \(b\) and \(c\). Results on the asymptotic zero distribution of certain classes of \(2F_1(-n; b; c; z)\) have also appeared (cf. [5], [10], [14], [15] and [27]).

1Research by this author was supported by a Humboldt Research Fellowship for Experienced Researchers from the Alexander von Humboldt Foundation.

2Research by this author was partially supported by the National Research Foundation under grant number 2054423.

Preprint submitted to Journal of Computational and Applied Mathematics December 11, 2013
Different types of $2F_1(-n, b; c; z)$ have well-established connections with classical orthogonal polynomials, notably the Jacobi polynomials and the Gegenbauer or ultraspherical polynomials (cf. [1]). For the ranges of the parameters where these polynomials are orthogonal, information about the zeros of $2F_1(-n, b; c; z)$ follows immediately from classical results (cf. [1], [28]). The asymptotic zero distribution of $2F_1(-n, b; c; z)$ when $b$ and $c$ depend on $n$ can be deduced from recent results by Kuijlaars, Martínez-Finkelshtein, Martínez-González and Orive (cf. [20], [21], [22], [23]) on the asymptotic zero distribution of Jacobi polynomials $P_n^{(a, b)}(x)$, which can also be proved directly using the Rodrigues’ formula for the polynomials extremely valuable.

The orthogonality of the polynomials $2F_1(-n, b; c; z)$ given in the next theorem follows from the orthogonality of the Jacobi polynomials (cf. [25, p. 257-261]) and can also be proved directly using the Rodrigues’ formula for the polynomials $2F_1(-n, b; c; z)$ (cf. [1], p. 99) as was done in [9] and [21].

**Theorem 1** (cf. [9]). Let $n \in \mathbb{N}_0$, $b, c \in \mathbb{R}$ and $-c \notin \mathbb{N}_0$. Then $2F_1(-n, b; c; z)$ is the $n^{th}$ degree orthogonal polynomial for the $n$-dependent positive weight function $|z|^{-1}(1-z)^{b-c-n}$ on the intervals

1. $(-\infty, 0)$ for $c > 0$ and $b < 1 - n$;
2. $(0, 1)$ for $c > 0$ and $b > c + n - 1$;
3. $(1, \infty)$ for $c + n - 1 < b < 1 - n$.

As a consequence of orthogonality, we know that for each $n$, the $n$ zeros of $2F_1(-n, b; c; z)$ are real, simple and lie in the interval of orthogonality for the corresponding ranges of the parameters (see, for example, [12], Theorem 4) as illustrated in Figure 1.

![Figure 1: Values of $b$ and $c$ for which $2F_1(-n, b; c; z)$ is orthogonal and has $n$ real simple zeros in the intervals $(0, 1), (-\infty, 0)$ and $(1, \infty)$ are indicated by regions $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ respectively.](image)

In his classical paper (cf. [19]), Felix Klein obtained results on the precise number of zeros of $2F_1(a, b; c; z)$ that lie in each of the intervals $(-\infty, 0), (0, 1)$ and $(1, \infty)$ by generalizing earlier results of Hilbert (cf. [17]). These Hilbert-Klein formulas are valid for hypergeometric functions and not only for polynomials. Szegő recaptured these results for the special case of Jacobi polynomials $P_n^{(a, b)}(x)$, which have a representation as $2F_1(-n, b; c; z)$, in the intervals $(-\infty, -1), (-1, 1)$ and $(1, \infty)$ (cf. [25], p.145, Theorem 6.72). The number and location of the real zeros of $2F_1(-n, b; c; z)$ for $b$ and $c$ real can be deduced as follows.

3
Theorem 2 (cf. [11], Theorem 3.2). Let \( n \in \mathbb{N} \), \( b, c \in \mathbb{R} \) and \( c > 0 \). Then,

(i) For \( b > c + n \), all zeros of \( zF_1(-n,b;c;z) \) are real and lie in the interval \((0,1)\).

(ii) For \( c < b < c + n \), \( c + j - 1 < b < c + j \), \( j = 1,2,\ldots,n \), \( zF_1(-n,b;c;z) \) has \( j \) real zeros in \((0,1)\). The remaining \((n-j)\) zeros of \( zF_1(-n,b;c;z) \) are non-real if \((n-j)\) is even, while if \((n-j)\) is odd, \( zF_1(-n,b;c;z) \) has \((n-j-1)\) non-real zeros and one additional real zero in \((1,\infty)\).

(iii) For \( 0 < b < c \), all the zeros of \( zF_1(-n,b;c;z) \) are non-real if \( n \) is even, while if \( n \) is odd, \( zF_1(-n,b;c;z) \) has one real zero in \((1,\infty)\) and the other \((n-1)\) zeros are non-real.

(iv) For \( -n < b < 0 \), \( -j < b < -j + 1 \), \( j = 1,2,\ldots,n \), \( zF_1(-n,b;c;z) \) has \( j \) real negative zeros. The remaining \((n-j)\) zeros of \( zF_1(-n,b;c;z) \) are non-real if \((n-j)\) is even, while if \((n-j)\) is odd, \( zF_1(-n,b;c;z) \) has \((n-j-1)\) non-real zeros and one additional real zero in \((1,\infty)\).

(v) For \( b < -n \), all zeros of \( zF_1(-n,b;c;z) \) are real and negative.

The values of the parameters \( b \) and \( c \) for which \( zF_1(-n,b;c;z) \) has exactly \( n \) real simple zeros in \((0,1)\) given in Theorem 2(i) and (ii) correspond to those in Theorem 1(ii) while the parameter values in Theorem 1(i) that ensure \((n-j)\) real zeros in \((1,\infty)\) due to Pfaff.

A natural question to ask is whether the parameter ranges in Theorems 1 and 2 are the only values of \( b, c \) for which \( zF_1(-n,b;c;z) \) have \( n \) real simple zeros. In this paper, we use a method that does not rely on orthogonality to determine all the real values of the parameters \( b \) and \( c \) for which \( zF_1(-n,b;c;z) \) have \( n \) real simple zeros. We apply an algorithm which counts the zeros of polynomials with real coefficients and their multiplicities. We also determine the intervals where the real zeros are located for these values of \( b \) and \( c \).

2. The algorithm

Let \( f(x) \) be a real polynomial with \( \deg(f) = n \geq 2 \). Define

\[
 f_0(x) := f(x) \quad \text{and} \quad f_1(x) := f'(x)
\]

and proceed for \( k \in \mathbb{N} \) as follows:

If \( \deg(f_k) > 0 \) perform the division of \( f_{k+1} \) by \( f_k \) to obtain

\[
 f_{k+1}(x) = q_{k+1}(x)f_k(x) - r_k(x).
\]

Define

\[
 f_{k+1}(x) = \begin{cases} r_k(x) & \text{if } r_k(x) \neq 0 \\ f_k'(x) & \text{if } r_k(x) \equiv 0 \end{cases}
\]

and generate the sequence of numbers \( c_1, c_2, \ldots \) where

\[
 c_k = \begin{cases} \frac{\deg(f_{k+1})}{\deg(f_{k-1})} & \text{if } r_k(x) \neq 0 \\ 0 & \text{if } r_k(x) \equiv 0 \end{cases}
\]
When $f_k$ is constant, the algorithm terminates.
Note that the algorithm must terminate, since the degrees of the polynomials $f_k(x)$ decrease on each step.

Then we have the following theorem which we will apply to $2F_1(-n, b; c; z)$.

**Theorem 3** (cf. [24], Theorem 10.5.7, p.339). Let $f$ be a polynomial of degree $n$ with real coefficients. Then $f$ has only real zeros if and only if the above algorithm produced $n - 1$ non-negative numbers $c_1, \ldots, c_{n-1}$. Moreover, the zeros of $f$ are all real and simple if and only if the numbers $c_1, \ldots, c_{n-1}$ are all positive.

3. Main results
We shall assume throughout our discussion that $b, c \in \mathbb{R}$ with $b, c \neq 0, -1, \ldots, -n + 1$. The assumption on $b$ is made to ensure that $2F_1(-n, b; c; z)$ is a polynomial of degree $n$.

**Proposition 4.** Let $b, c \in \mathbb{R}$. Then,

1. The zeros of $2F_1(-2, b; c; z)$ are real and simple if and only if either (see Figure 2):
   (i) $c < -1$ and $c < b < 0$.
   (ii) $-1 < c < 0$ and $b > 0$ or $b < c$.
   (iii) $c > 0$ and $b < 0$ or $c < b$.

2. The zeros of $2F_1(-3, b; c; z)$ are real and simple if and only if either (see Figure 3):
   (i) $c < -2$ and $1 + c < b < -1$.
   (ii) $-2 < c < -1$ and $-1 < b < 1 + c$.
   (iii) $c > -1, c \neq 0$ and $b < -1$ or $b > c + 1$.

**Theorem 5.** For any integer $n \geq 4$, the polynomial $2F_1(-n, b; c; z)$ has only real and simple zeros if and only if $(c, b)$ belongs to one of the four $n$-dependent regions $R_1, \ldots, R_4$ defined by

- $R_1 = \{ c + n - 2 < b < 2 - n \}$,
- $R_2 = \{ c > -1, \ b < 2 - n \}$,
- $R_3 = \{ c > -1, \ b > n - 2, \ b > c + n - 2 \}$,
- $R_4 = \{ -1 < c < 0, \ c + n - 2 < b < n - 2 \}$.
The parameter values \((c, b) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4\) described in Theorem 5 for which \(2F_1(-n, b; c; z)\), \(n = 4, 5, \ldots\) has \(n\) real simple zeros are illustrated by the grey and diagonally shaded regions in Figure 4 with the grey regions indicating those parameter values that extend the results in Theorems 1 and 2. For these values of \(c\) the polynomials \(2F_1(-n, b; c; z)\) are quasi-orthogonal of order 1 and, in some cases, order 2 (cf. [3] and [2]). Theorem 3 in [2] and Theorem 6 in [18] yield information on the zero location of quasi-orthogonal polynomials with non-varying weight functions. However, these results cannot be applied to \(2F_1(-n, b; c; z)\) since their weight function depends on \(n\). We use information about the zeros of Jacobi polynomials to obtain the following three results.

Theorem 6. Let \(n \in \mathbb{N}\) and \(b, c \in \mathbb{R}\). Then, \(2F_1(-n, b; c; z)\) has all its zeros real and simple and \((n-2)\) of them lie in

(i) \((0, 1)\) for \(-1 < c < 0\) and \(c + n - 2 < b < c + n - 1\). One of the remaining zeros lies in \((1, \infty)\) and the other one in \((-\infty, 0)\).

(ii) \((1, \infty)\) for \(1 - n < b < 2 - n\) and \(b - n + 1 < c < b - n + 2\). One of the remaining zeros lies in \((-\infty, 0)\) and the other one in \((0, 1)\).

(iii) \((-\infty, 0)\) for \(-1 < c < 0\) and \(1 - n < b < 2 - n\). One of the remaining zeros lies in \((1, \infty)\) and the other one in \((0, 1)\).

Theorem 6 applies to the parameter values illustrated in Figure 5.

Theorem 7. Let \(n \in \mathbb{N}\) and \(b, c \in \mathbb{R}\). Then, \(2F_1(-n, b; c; z)\) has all its zeros real and simple and \((n-1)\) of them lie in

(i) \((0, 1)\) for \(-1 < c < 0\) and \(b > c + n - 1\). The remaining zero is negative.

(ii) \((1, \infty)\) for \(1 - n < b < 2 - n\) and \(c < b - n - 1\). The remaining zero is negative.

(iii) \((-\infty, 0)\) for \(-1 < c < 0\) and \(b < 1 - n\). The remaining zero lies in \((0, 1)\).

The parameter values described in Theorem 7 are illustrated in Figure 6.

Theorem 8. Let \(n \in \mathbb{N}\) and \(b, c \in \mathbb{R}\). Then, \(2F_1(-n, b; c; z)\) has all its zeros real and simple and \((n-1)\) of them lie in

(i) \((0, 1)\) for \(c > 0\) and \(c + n - 2 < b < c + n - 1\). The remaining zero is in the interval \((1, \infty)\).
\[(ii) \ (1, \infty) \text{ for } b < 1 - n \text{ and } c + n - 2 < b < c + n - 1. \text{ The remaining zero lies in } (0, 1). \]

\[(iii) \ (-\infty, 0) \text{ for } c > 0 \text{ and } 1 - n < b < 2 - n. \text{ The remaining zero lies in } (1, \infty). \]

Figure 7 illustrates the range of the parameters \(b\) and \(c\) referred to in Theorem 8.

4. Proofs

Proof of Proposition 4.

1. Since \( _2F_1 (-2, b; c; z) = 1 - \frac{2b}{c} z + \frac{b(b + 1)}{c(c + 1)} z^2 \), we see that \( _2F_1 (-2, b; c; z) = 0 \) if and only if

\[ z = \frac{b(c + 1) \pm \sqrt{b(c + 1)(b - c)}}{b(b + 1)}. \]

Hence, the zeros of \( _2F_1 (-2, b; c; z) \) are real and simple if and only if \( b(c + 1)(b - c) > 0 \).

2. The discriminant of

\[ _2F_1 (-3, b; c; z) = 1 - \frac{3b}{c} z + \frac{3b(b + 1)}{c(c + 1)} z^2 - \frac{b(b + 1)(b + 2)}{c(c + 1)(c + 2)} z^3 \]

is given by \( \Delta_3 = 108 \frac{b^2(b + 1)(b - c - 1)(b - c)^2}{c^2(c + 1)^2(c + 2)^2} \) (cf. \[16\]) and therefore \( _2F_1 (-3, b; c; z) \) has real simple roots if and only if \( \Delta_3 > 0 \).

The following two lemmas will be used in the proof of our main result.
Lemma 9. Let

\[ \alpha_{k,l} = \frac{\binom{k-2}{2}}{\binom{k-1}{2}} \frac{(2k-n-b-4)}{4} \frac{(k-b-c)}{4} \]

and let the sequence \( \theta_k \) be recursively defined by

\[ \theta_{k+1} = \alpha_{k,1} \theta_k, \]

for \( k \in \{2, \ldots, n-2\} \), with

\[ \theta_1 = -nb \quad \text{and} \quad \theta_2 = \frac{(b-c)(n-1)}{c(b+n-1)}. \]

Then,

\[ \theta_{2k} = \frac{(b+1)(n+c)}{c(n+b+1)} \alpha_{1,k}, \quad k = 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor, \]

and

\[ \theta_{2k+1} = -\frac{nb}{c} \alpha_{2,k}, \quad k = 0, 1, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor. \]

Proof of Lemma 9. We prove the result by induction on \( k \). When \( k = 1 \), the right-hand side of (2) is

\[ \frac{(b+1)(n+c)\left(\frac{1-n}{2}\right)\left(-\frac{n+b+1}{2}\right)\left(-\frac{c-b}{2}\right)}{c(n+b+1)\left(-\frac{b+1}{2}\right)\left(-\frac{n+b-1}{2}\right)\left(-\frac{c-b}{2}\right)} = \frac{(b-c)(n-1)}{c(b+n-1)}, \]

which is \( \theta_2 \) as required.

We now assume the result is true for \( k = t \) and prove the result true for \( k = t + 1 \). If we let \( k = t + 1 \) on the
right-hand side of (2), we obtain
\[
RHS = \frac{(b + 1)(n + c)}{c(n + b + 1)} \alpha_{1, 2}^{+1}
\]
\[
= \frac{(b + 1)(n + c)}{c(n + b + 1)} \left( \frac{1 - n}{2} + t \right) \left( \frac{-b}{2} + t \right) \left( \frac{-n + b + 1}{4} + t \right) \alpha_{1, 2}^{+}
\]
\[
= \theta_{2}^{+} \frac{(1 - n + 2t)(c - b + 2t)(-n - b - 1 + 4t)}{(-b - 1 + 2t)(-n - c + 2t)(-n - b + 1 + 4t)}
\]
\[
= \alpha_{2, 1}^{+} \theta_{2}^{+}
\]
\[
= \theta_{2}^{+} \text{ from (1)}
\]
and the result follows by induction.

The second relation (3) may be proved by induction in a similar way.

Lemma 10. Let \( n \geq 4 \). Then, for all \( k \in \{2, \ldots, n - 1\} \),
\[
\frac{(n - k)(n + c - k)(b + 1 - k)(b - c + 1 - k)}{(n + b + 2 - 2k)(n + b - 2k)(n + b + 1 - 2k)^2}
\]
and
\[
\frac{(n - 1)(n + c - 1)(b - c)}{(n + b - 2)(n + b - 1)^2}
\]
are positive if and only if \((c, b) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4\).

Proof of Lemma 10. Since \( n - 1 > 0 \) and \((n + b - 1)^2 > 0\) for all \( n \in \mathbb{N}, b \in \mathbb{R}\), we see that (5) is positive if and only
if

\[(c, b) \in [c < 1 - n, b > c, b < 2 - n] = \mathcal{A}_1 \supset \mathcal{R}_1 \text{ or}
\]
\[\in [c > 1 - n, b < c, b < 2 - n] = \mathcal{A}_2 \supset \mathcal{R}_2 \text{ or}
\]
\[\in [c > 1 - n, b > c, b > 2 - n] = \mathcal{A}_3 \supset (\mathcal{R}_3 \cup \mathcal{R}_4) \text{ or}
\]
\[\in [c < 1 - n, b < c, b > 2 - n] = \emptyset.
\]

Clearly
\[
(n-k)(n-k+1)\left(n+b+1-2k^2\right) > 0 \text{ for all } k \in \{2, \ldots, n-1\}, n \in \mathbb{N} \text{ and } b \in \mathbb{R}, b \neq 3-n, 5-n, \ldots, n-5, n-3.
\]

Furthermore \(b > n - 2\) if and only if \(n + b + 2 - 2k > 0, n + b - 2k > 0\) and \(b + 1 - k > 0\) for all \(k \in \{2, \ldots, n-1\}\). Hence, when \(b > n - 2\), (4) will be positive for all \(k \in \{2, \ldots, n-1\}\) if and only if

\[(c, b) \in \{b > c + k - 1, c + n - k > 0, k = 2, \ldots, n-1\} = \mathcal{B} \supset \mathcal{A}_1 \text{ or}
\]
\[\in \{b < c + k - 1, c + n - k < 0, k = 2, \ldots, n-1\} = \mathcal{B} \supset \mathcal{A}_2 \text{ or}
\]
\[\in \{b > c + k - 1, c + n - k > 0, k = 2, \ldots, l\} \cap \{b > c + k - 1, c + n - k > 0, k = l + 1, \ldots, n-1\} = \emptyset \text{ or}
\]
\[\{b < c + k - 1, c + n - k < 0, k = 2, \ldots, l\} \cap \{b > c + k - 1, c + n - k < 0, k = l + 1, \ldots, n-1\} = \emptyset.
\]

Similarly, \(b < 2 - n\) if and only if \(b + 1 - k < 0, n + b + 2 - 2k < 0\) and \(n + b - 2k < 0\) for \(k \in \{2, \ldots, n-1\}\). Hence, when \(b < 2 - n\), (4) will be positive for all \(k \in \{2, \ldots, n-1\}\) if and only if

\[(c, b) \in \{b > c + k - 1, c + n - k < 0, k = 2, \ldots, n-1\} = \mathcal{B} \supset \mathcal{A}_3 \text{ or}
\]
\[\in \{b < c + k - 1, c + n - k > 0, k = 2, \ldots, n-1\} = \mathcal{B} \supset \mathcal{A}_4 \text{ or}
\]
\[\in \{b > c + k - 1, c + n - k > 0, k = 2, \ldots, l\} \cap \{b < c + k - 1, c + n - k > 0, k = l + 1, \ldots, n-1\} = \emptyset \text{ or}
\]
\[\{b < c + k - 1, c + n - k < 0, k = 2, \ldots, l\} \cap \{b > c + k - 1, c + n - k < 0, k = l + 1, \ldots, n-1\} = \emptyset.
\]

For the remaining case where \(2-n < b < n-2\) or, more specifically, \(-2 < b+n-2k < 0\) with \(-b-k+1 < 0\) for all \(k \in \{2, \ldots, n-1\}\), the only non-empty possibility is that (4) is positive for \(k = n-1\) if and only if \(c > -1, b > c+n-2\) and \(-n-4 < b < n-2\) whereas (4) is positive for \(k \in \{2, 3, \ldots, n-2\}\) if and only if \(c > -1, b > c+n-2\) and \(b > n-3\). Hence, when \(2-n < b < n-2\), (4) is positive for all \(k \in \{2, \ldots, n-1\}\) if and only if \((c, b) \in \mathcal{R}_1\).

Since \(\mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{R}_1\) and \(\mathcal{A}_2 \cap \mathcal{A}_3 = \mathcal{R}_2\), the result follows.

**Proof of Theorem 5.** We apply the algorithm to the polynomial

\[f(z) = \text{}_2F_1\left(-n, b; c; z\right).
\]

We have (cf. [25], p.69, ex.1)

\[f_1(z) = f'(z) = -\frac{nb}{c} \text{}_2F_1\left(-n + 1, b + 1; c + 1; z\right).
\]

Using Raimundas Vidūnas’ Maple package for contiguous relations of \(\text{}_2F_1\) hypergeometric series (cf. [29], [30]), we obtain

\[f_0(z) = \frac{1}{n} \left(z - \frac{c + n - 1}{b + n - 1}\right) f_1(z) - \frac{(b - c)(n - 1)}{c(b + n - 1)} \text{}_2F_1\left(-n + 2, b; c + 1; z\right).
\]

This relation can easily be verified by comparing coefficients. Thus,

\[f_2(z) = \text{r}_1(z) = \frac{(b - c)(n - 1)}{c(b + n - 1)} \text{}_2F_1\left(-n + 2, b; c + 1; z\right).
\]

In the next step \((k = 2)\), we get

\[f_3(z) = q_1(z)f_2(z) - r_2(z),
\]
with

\[ q_1(z) = \frac{n(b+n-1)^2(b+n-2)}{(n-1)(c+n-1)(b-c)} \left[ z + \frac{(n-2)(c+n-2)}{b+n-3} - \frac{(n-1)(c+n-1)}{b+n-1} \right] \]

and

\[ r_2(z) = \frac{(b+n-1)(b-1-c)n(n-2)}{c(c+n-1)(b+n-3)} 2F_1(-n+3,b-1;c+1;z). \]

Setting

\[ f_k(z) = \theta_k 2F_1(-n+k,b+2-k;c+1;z), \quad k \in \{1,\ldots,n-1\}, \]

we see that in general we need a contiguous relation of the form

\[ 2F_1(-n+k-1,b+3-k;c+1;z) = q_{k-1}(z) \frac{\theta_k}{\theta_{k-1}} 2F_1(-n+k,b+2-k;c+1;z) - \frac{\theta_{k+1}}{\theta_{k-1}} 2F_1(-n+k+1,b+1-k;c+1;z), \]

for \(k \in \{2,\ldots,n-2\}\), with

\[ \theta_1 = -\frac{nb}{c}, \quad \theta_2 = \frac{(b-c)(n-1)}{c(b+n-1)}. \]

Using Vidūnas’ package, we obtain (1) for \(k=2,3,\ldots\) and from Lemma 9 we conclude that \(\theta_k\) is well defined and non-zero for all \(k \in \{1,\ldots,n-1\}\) when \(c \neq 0\) and \((c,b) \in R_1 \cup R_2 \cup R_3 \cup R_4\). Thus,

\[ c_k = \frac{lc(f_{k+1})}{lc(f_{k-1})}, \quad k \in \mathbb{N}, \]

which implies that

\[ c_1 = \frac{(b-c)(n-1)(-n-2)_{n-2}(b)_{n-2}(c)n!}{(c+b+1-n)(c+1)_{n-2}(n-2)!(-n)_n(b)_n} = \frac{(n-1)(c+n-1)(b-c)}{(b+n-2)(b+n-1)^2} \]

and, for \(k \in \{2,\ldots,n-1\}\),

\[ c_k = \frac{\theta_{k+1}(-n+k+1)_{n-k-1}(b+1-k)_{n-k-1}(n-k+1)(c+1)_{n-k+1}}{\theta_{k-1}(c+1)_{n-k-1}(n-k+1)(c)_{n-k+1}(n+k-1)(b+3-k)_{n-k+1}} \]

\[ = \frac{(n-2)(b+c)(b-k)(-c+1+b-k)}{(n-2k+b+2)(n-2k+b)(b-2k+1+n)^2}. \]

From Lemma 10 we know that \(c_k > 0\) for all \(k \in \{1,\ldots,n-1\}\) when \((c,b) \in R_1 \cup R_2 \cup R_3 \cup R_4\). The result now follows from Theorem 4.

**Proof of Theorem 4.** From [3], Corollary 4 (i), we know that for \(-1 < \alpha < 0\) and \(-1 < \beta < 0\), the Jacobi polynomials \(P_n^{(\alpha,\beta)}(x)\) have real simple zeros and \(n-2\) of them are in the interval \((-1,1)\). The smallest zero is smaller than \(-1\) and the largest zero is larger than \(-1\). Equivalently, the same is true for the zeros of Jacobi polynomials \(P_n^{(\alpha,\beta)}(x)\) when \(-2 < \alpha < -1\) and \(-2 < \beta < -1\).

(i) One of the connections between Jacobi polynomials and the polynomials \(2F_1(-n,b;c;z)\) is given by (cf. [25], p. 254, eq. 3)

\[ P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(1+\beta)_n}{n!} 2F_1\left(-n,1+\alpha+\beta+n;1+\beta;\frac{x+1}{2}\right) \quad (6) \]

where \(\alpha = b+n-c\) and \(\beta = c+1\). The conditions \(-2 < \beta < -1\) and \(-2 < \alpha < -1\) are equivalent to \(-1 < c < 0\) and \(c+n-2 < b < c+n-1\). Furthermore, the intervals \((-1,1)\), \((1,\infty)\) and \((-\infty,-1)\) are transformed to \((0,1)\), \((1,\infty)\) and \((-\infty,-1)\) respectively under the linear mapping \(x = 2z-1\). Thus, when \(c \in (-1,0)\) and \(b \in (c+n-2,c+n-1)\), \(2F_1(-n,b;c;z)\) has \(n-2\) real, simple zeros in the interval \((0,1)\), one zero in \((1,\infty)\) and one zero in \((-\infty,0)\) for each \(n \in \mathbb{N}\).
(ii) The representation (\cite{25}, p. 255, eq. 9)

\[
P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha + \beta)_n}{n!} \binom{x + 1}{n} \frac{1}{2} \binom{-n, -\beta - n; -\alpha - \beta - 2n; \frac{2}{x + 1}}{F_1}(n, -\beta - n; -\alpha - \beta - 2n, \frac{2}{x + 1})
\]

where \(\alpha = b - c - n\) and \(\beta = -b - n\) yields the stated result, since the restrictions \(-2 < \alpha < -1\) and \(-2 < \beta < -1\) will correspond to \(1 + b - n < c < 2 + b - n\) and \(1 - n < b < 2 - n\) while the intervals \(-1 < x < 1, x > 1\) and \(x < -1\) are mapped to \(z > 1, 0 < z < 1\) and \(z < 0\) respectively under the fractional transformation \(z = \frac{2}{x + 1}\).

(iii) For this case we use the representation (cf. \cite{25}, p. 255, eq. 8)

\[
P_n^{(\alpha, \beta)}(x) = \frac{(1 + \beta)_n}{n!} \binom{x + 1}{n} \frac{1}{2} \binom{-n, -\alpha - n; \beta + 1; \frac{x + 1}{x - 1}}{F_1}(n, -\alpha - n; \beta + 1, \frac{x + 1}{x - 1})
\]

where \(\alpha = -b - n\) and \(\beta = c - 1\). Under the transformation \(z = \frac{x + 1}{x - 1}\), the interval \(-1 < x < 1\) is mapped to the negative real line while the intervals \(x > 1\) and \(x < -1\) are mapped to \(z = 1 + \frac{2}{2 - z} > 1\) and \(0 < z < 1\) respectively. Also, since \(-2 < \alpha < -1\) and \(-2 < \beta < -1\) correspond to \(1 - n < b < 2 - n\) and \(-1 < c < 0\), the result follows.

**Proof of Theorem 7** From \cite{3}, Corollary 4 (ii) (a), we know that for \(-1 < \alpha < -1\) and \(-1 < \beta < 0\), the zeros of the Jacobi polynomial \(P_n^{(\alpha, \beta)}(x)\) are all real and simple, and \((n - 1)\) of them lie in the interval \((-1, 1)\). The smallest zero is less than \(-1\). We may equivalently say the same about the zeros of the Jacobi polynomial \(P_n^{(\alpha, \beta)}(x)\) for \(\alpha > -1\) and \(-2 < \beta < -1\). Using transformations \(6\), \(7\) and \(8\) the results can be proven following the same reasoning as in the proof of Theorem 6.

**Proof of Theorem 8** From \cite{2}, Corollary 4 (ii) (b), we know that for \(-1 < \alpha < 0\) and \(-1 < \beta < 1\), the zeros of the Jacobi polynomial \(P_n^{(\alpha, \beta)}(x)\) are all real and simple, and \((n - 1)\) of them lie in the interval \((-1, 1)\). Moreover, the largest zero is greater than 1. We may equivalently say the same about the zeros of the Jacobi polynomial \(P_n^{(\alpha, \beta)}(x)\) for \(-2 < \alpha < -1\) and \(\beta > -1\). The results now follow from \(6\), \(7\) and \(8\) as before.

**References**

[1] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, volume 71 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1999.

[2] C. Brezinski, K. A. Driver and M. Redivo-Zaglia, Quasi-orthogonality with applications to some families of classical orthogonal polynomials, *Appl. Numer. Math.* 48 (2):157-168, 2004.

[3] T.S. Chihara, On quasi-orthogonal polynomials, *Proceedings of the American mathematical society*. 8:765—767, 1957.

[4] D. Dominici, K. Driver and K. Jordaan, Polynomial solutions of differential-difference equations, *J. Approx. Theory*. 163:41-48, 2011.

[5] K. Driver and P. Duren, Asymptotic zero distribution of hypergeometric polynomials, *Numer. Algor.* 21:147-156, 1999.

[6] K. Driver and P. Duren, Zeros of the hypergeometric polynomials \(F(-n, b; 2b; z)\), *Indag. Math.* 11(1):43-51, 2000.

[7] K. Driver and P. Duren, Trajectories of the zeros of Hypergeometric Polynomials \(F(-n, b; 2b; z)\) for \(b < -\frac{1}{2}\), *Constr. Approx.* 17:169-179, 2001.

[8] K. Driver and P. Duren, Zeros of ultraspherical polynomials and the Hilbert-Klein formulas, *J. Comput. and Appl. Math.* 135:293-301, 2001.

[9] K. Driver and S. J. Johnston, Quasi-orthogonality and zeros of some \(F_2\) hypergeometric polynomials, *Quaest. Math.* 27:365-373, 2004.

[10] K. Driver and S. J. Johnston, Asymptotic zero distribution of a class of hypergeometric polynomials, *Quaest. Math.* 30:219-230, 2007.

[11] K. Driver and K. Jordaan, Zeros of the hypergeometric polynomial \(F(-n, b; c; z)\), *Proceedings of Algorithms for Approximations IV* 436-445, 2002.

[12] K. Driver and K. Jordaan, Convergence of ray sequences of Padé approximants for \(F_n(a, 1; c; z), c > a > 0\), *Quaest. Math.* 25:1-7, 2002.

[13] K. Driver and M. Möller, Quadratic and cubic transformations and the zeros of hypergeometric polynomials, *J. Comput. Appl. Math.* 142:411-417, 2002.

[14] K. Driver and M. Möller, Zeros of the hypergeometric polynomials \(F(-n, b; -2n; z)\), *J. Approx. Theory*. 110(1):74-87, 2001.

[15] P.L. Duren and B.J. Guillou, Asymptotic properties of zeros of hypergeometric polynomial, *J. Approx. Theory*. 111(2):329-343, 2001.

[16] L.M. Gelfand, M.M. Karpman and A.V. Zelevinsky, Discriminants, resultants, and multidimensional determinants. Mathematics: Theory & Applications, Birkauser Boston, Inc., Boston, MA, 1994.

[17] D. Hilbert, Über die Diskriminanten der im Endlichen abbrechenden hypergeometrischen Reihe, *J. Reine Angew. Math.* 103:337-345, 1888.

[18] H. Joulak, A contribution to quasi-orthogonal polynomials and associated polynomials, *Appl. Numer. Math.* 54(1):65-78, 2005.
[19] F. Klein, Über die Nullstellen der hypergeometrischen Reihe, Math. Ann. 37:573-590, 1890.
[20] A. B. J. Kuijlaars and A. Martínez-Finkelshtein, Strong asymptotics for Jacobi polynomials with varying nonstandard parameters, J. Anal. Math. 94:195-234, 2004.
[21] A. B. J. Kuijlaars, A. Martínez-Finkelshtein and R. Orive, Orthogonality of Jacobi polynomials with general parameters, Electron. Trans. Numer. Anal. 19:1–17, 2005.
[22] A. Martínez-Finkelshtein, P. Martínez-González, and R. Orive, Zeros of Jacobi polynomials with varying non-classical parameters, Special functions (Hong Kong, 1999), pages 98—113. World Sci. Publishing, River Edge, NJ, 2000.
[23] A. Martínez-Finkelshtein and R. Orive, Riemann-Hilbert analysis of Jacobi polynomials orthogonal on a single contour, J. Approx. Theory 134(2):137—170, 2005.
[24] Q. I. Rahman and G. Schmeisser, Analytic theory of polynomials, Oxford University Press, 2002.
[25] E. D. Rainville, Special Functions, The Macmillan Company, New York, 1960.
[26] G. Schmeisser, A real symmetric tridiagonal matrix with a given characteristic polynomial, Linear Algebra Appl. 193:11-18, 1993.
[27] H. M. Srivastava, J Zhou and Z Wang, Asymptotic distributions of the zeros of certain classes of hypergeometric functions and polynomials, Math. Comput. 80:1769-1784, 2011.
[28] G. Szegö, Orthogonal Polynomials, American Mathematical Society, New York, 1959.
[29] R. Vidunas and T. Koornwinder, Webpage of the NWO project, Algorithmic methods for special functions by computer algebra, http://www.science.uva.nl/~thk/specref/compalg.html 2000.
[30] R. Vidunas, Contiguous relations of hypergeometric series, J. Comput. Appl. Math. 153: 507-519, 2003.