The leptonic widths of high $\psi$-resonances in unitary coupled-channel model

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The leptonic widths of high $\psi$-resonances are calculated in a coupled-channel model with unitary inelasticity, where analytical expressions for the mixing angles between $(n+1)^3S_1$ and $n^3D_1$ states and probabilities $Z_i$ of the $c\bar{c}$ component are derived. These factors depend on energy (mass) and can be different for $\psi(4040)$ and $\psi(4160)$. However, our calculations give a small difference between the mixing angles, $\theta(\psi(4040)) = (28.4^{+1.2}_{-1.0})^\circ$ and $\theta(\psi(4160)) = (29.5^{+1.4}_{-1.2})^\circ$, and $\sim 10\%$ difference between the probabilities $Z_1(\psi(4040)) = 0.85^{+0.02}_{-0.03}$ and $Z_2(\psi(4160)) = 0.79 \pm 0.01$. It provides the leptonic widths $\Gamma_{ee}(\psi(4040)) = (1.0 \pm 0.1)$ keV, $\Gamma_{ee}(\psi(4160)) = (0.62 \pm 0.07)$ keV in agreement with experiment; for $\psi(4415)$ $\Gamma_{ee}(\psi(4415)) = (0.66 \pm 0.06)$ keV is obtained, while for the missing resonance $\psi(4510)$ we predict its mass, $M(\psi(4500)) = (4512 \pm 2)$ MeV, and $\Gamma_{ee}(\psi(4510)) = (0.68 \pm 0.14)$ keV.

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I. INTRODUCTION

The high $\psi$-resonances, $\psi(4040)$, $\psi(4160)$, and $\psi(4415)$, occur far above the open-charm threshold and their masses, total widths, and leptonic widths (LWs) are known from the total cross section of $e^+e^- \to$ hadrons and exclusive $e^+e^-$ processes. The PDG gives their masses with a good accuracy, better than 10 MeV, however, the discussion on the true values of their leptonic widths continues. In particular, four different solutions of LWs, which equally well describe the BES data, are presented in Ref. Also some parameters, recently extracted from the Belle data on exclusive $e^+e^-$ processes to open-charm decay channels, differ from those given by the PDG. In Table I we summarize the values of the LWs extracted from different experiments, from which one can see that there exists a large uncertainty in the LW of $\psi(4040)$, which can vary from 0.66 keV to 1.6 keV.

|     | $\Gamma_{ee}(\psi(4040))$ | $\Gamma_{ee}(\psi(4160))$ | $\Gamma_{ee}(\psi(4415))$ |
|-----|---------------------------|---------------------------|---------------------------|
| BES | 0.81 ± 0.20               | 0.50 ± 0.27               | 0.37 ± 0.14               |
| BES | 0.66 to 1.40              | 0.42 to 1.09              | 0.45 to 0.77              |
| PDG | 0.86 ± 0.07               | 0.48 ± 0.22               | 0.58 ± 0.07               |
| Belle | 1.6 ± 0.3                | 0.7 ± 0.4                | 1.4 ± 0.3                |

The theory of charmonium properties is developing already for forty years, mostly in different potential models, relativistic and nonrelativistic, where calculations of high excitations are mostly performed in closed-channel approximation, neglecting open decay channels. Surprisingly, the predicted masses appear to be weakly dependent on the model used and mostly agree with each other and the experimental values, within $\pm(20 - 40)$ MeV (see Table III). This result can easily be interpreted. Consider charmonium in a nonrelativistic model with the Cornell potential and then vary the $c$-quark mass and parameters of the $Q\bar{Q}$ potential in a special way. Then even identical spectra can be obtained. However, such freedom in the choice of parameters does not agree with fundamental ideas about the true value of the $c$-quark mass and the $Q\bar{Q}$ static potential, and therefore additional physical restrictions on the parameters must be put using new fundamental results.

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In Table II we give the masses of high $n^3S_1, m^3D_1$ charmonium states, obtained in closed-channel approximation and using a linear confining potential.

Here one can see that with exception of $M(3^3S_1)$, the theoretical values coincide with the experimental masses of the resonances with $\sim (20 - 40)$ MeV accuracy and therefore one may expect that the mass shifts of the $\psi$-resonances due to open channels are not large, $\sim (20 - 50)$ MeV. Such not so large mass shifts were predicted in the $C^3$ model [15]. Notice that if instead of a linear potential, the so-called screened confining potential is used [20, 22], very large mass shifts, $\sim (100 - 150)$ MeV, are obtained, e.g. $M(4^3S_1) = 4273$ MeV, $M(3^3D_1) = 4317$ MeV in Ref. [20], and $M(4^3S_1) = 4389$ MeV, $M(3^3D_1) = 4426$ MeV in Ref. [22]. Specific features of the screened potential will be discussed later.

In the $\psi$-family one resonance, originating from the $3^3D_1$ state, is not observed yet, although its predicted mass, $M(3^3D_1) = (4510 \pm 20)$ MeV, lies in the region which was already studied in different $e^+e^-$ experiments [8, 9, 23]. Here we would like to notice that in exclusive experiments of the Belle Collaboration [1]: $e^+e^- \rightarrow D^+D^-, \bar{D}^+\bar{D}^-$, one can see a wide peak (structure) in the region $4.5 \text{ GeV} < \sqrt{s} < 4.6 \text{ GeV}$. These data were analyzed, using the $\psi(4040)$, $\psi(4160)$, and $\psi(4415)$ resonances [6]. However, in that analysis the mass of $\psi(4415)$, $M(\psi(4415)) = (4515 \pm 18)$ MeV, appears to be $100$ MeV larger than what is found in PDG [3] and other experiments. The interesting point is that this large value of the mass just coincides with that of the missing $\psi(4510)$, predicted in different theoretical models (see Table II). To confirm or exclude the manifestation of $\psi(4510)$ it would be important to analyze these exclusive reactions, taking into account both resonances, $\psi(4415)$ with the mass $\sim 4420$ MeV and $\psi(4510)$. In our calculations the predicted LW of $\psi(4510)$ is not small, $\Gamma_{ee}(\psi(4510)) \sim 0.6$ keV, if the $3^3D_1$ state has rather large admixture of the $4^3S_1$ state [12].

In the present paper we concentrate on the LWs of high $\psi$-resonances and relate their parameters to the many-channel picture. Their values appear to be sensitive to the $c$-quark mass taken and the parameters of the $Q^Q$ static potential. As shown in Ref. [24], the squared wave functions (w.fs.) at the origin of the $3^3S_1$ and $2^3D_1$ states can change several times for different $Q^Q$ potentials, giving very different LWs. Including the asymptotic-freedom behaviour of the strong coupling is also important, decreasing the w.f. at the origin by $\sim 30\%$ [25].

For those reasons we choose here a static potential defined only in terms of fundamental parameters derived in pQCD [18] and the field correlator method [26, 27]. In particular, we pay attention to the fact that the new value of the QCD constant for $n_f = 3$, $\Lambda_{\overline{MS}}(n_f = 3) = (339 \pm 10)$ MeV [18], is rather large and gives rise to a large QCD vector constant, $\Lambda_V$, defining the strong vector coupling $\alpha_V$, since these constants are interrelated, $\Lambda_V(n_f = 3) = 1.4752 \Lambda_{\overline{MS}} = (500 \pm 15)$ MeV. In our previous analysis a smaller $\Lambda_V$ [28] was used, while a larger value of $\Lambda_V = 500$ MeV increases the w.fs. at the origin and LWs of $\psi$ resonances.

## II. MIXING OF THE $n^3S_1$ AND $(n-1)^3D_1$ STATES

The w.fs. at the origin of pure $n^3D_1$ states (defined as $R_{nD}(0) = \frac{R_n'(0)}{2\sqrt{n\alpha}}$) are known to be very small [16, 28] and give rise to small LWs. For example, the LW of pure $1^3D_1$ is about seven times smaller than that of $\psi(3770)$ [29]. To explain such a difference a mixing angle $\theta_1$ between the $2^3S_1$ and $1^3D_1$ states, $\theta_1 = (11 \pm 3)^\circ$, was extracted from the ratio of their LWs, $R_1 = \frac{\Gamma_{ee}(\psi(3770))}{\Gamma_{ee}(\psi(4040))} = 0.48 \pm 0.22$ keV [30, 31]. In the same manner, the mixing angle $\theta_2 = 34^\circ$ between the $3^3S_1$ and $2^3D_1$ states was extracted in Ref. [12], giving $\Gamma_{ee}(4040) = (0.86 \pm 0.07) \text{ keV}$ and $\Gamma_{ee}(\psi(4160)) = 0.83 \pm 0.06 \text{ keV}$ in good agreement with the old experimental data on LWs from PDG (2006) [32]. An almost identical value, $\theta_2 = 37^\circ$, was obtained in Ref. [33]. However, now a smaller LW of $\psi(4160)$ is given by the PDG (2014) [3] (see also the LWs in Table II), while the LW of $\psi(4040)$ remains unchanged,

$$\Gamma_{ee}(\psi(4160)) = (0.48 \pm 0.22) \text{ keV}, \quad \Gamma_{ee}(4040) = (0.86 \pm 0.07) \text{ keV}. \quad (1)$$

| State     | NR [11] | R [6] | R [8] | NR [14] | R (this paper) | exp. |
|-----------|---------|-------|-------|---------|----------------|------|
| $M(3^3S_1)$ | 4110    | 4100  | 4095  | 4100    | 4112           | 4039 ± 1 |
| $M(2^3D_1)$ | 4190    | 4194  | 4191  | 4150    | 4195           | 4191 ± 5  |
| $M(4^3S_1)$ | 4460    | 4450  | 4433  | 4445    | 4467           | 4421 ± 4  |
| $M(3^3D_1)$ | -       | 4520  | 4505  | 4525    | 4527           | absent  |
Their ratio also becomes smaller and has a large experimental error,

$$\eta_2(\text{exp.}) = \frac{\Gamma_{ee}(\psi(4160))}{\Gamma_{ee}(\psi(4040))} = 0.56 \pm 0.30,$$

i.e., $\eta_2$ changes in a wide range, from 0.26 to 0.86, and therefore is not useful in our analysis.

To extract the mixing angle the resonance w.f.s. are usually taken in a simplified form:

$$|\psi(4040)| = |3^3 S_1 \cos \theta_2 - |2^3 D_1| \sin \theta_2, \quad |\psi(4160)| = |3^3 S_1 \sin \theta_2 + |2^3 D_1| \cos \theta_2,$$

with equal angles $\theta_2$ in both w.f.s. This assumption is not supported by results following from coupled-channel models, where resonance w.f.s. are given by more complicated expressions and can schematically be written as

$$\varphi(\psi(4040)) = \sqrt{Z_1}(\varphi(3S) \cos \theta_2 - \varphi(2D) \sin \theta_2) + \sqrt{1 - Z_1} \varphi_{\text{cont}},$$
$$\varphi(\psi(4160)) = \sqrt{Z_2}(\varphi(3S) \sin \theta_2 + \varphi(2D) \cos \theta_2) + \sqrt{1 - Z_2} \varphi_{\text{cont}},$$

and in general contain different mixing angles $\theta_2 = \theta(\psi(4040))$, $\tilde{\theta}_2 = \theta(\psi(4160))$ and different probabilities of the $c\bar{c}$ component, $Z_1 = Z(\psi(4040))$ and $Z_2 = Z(\psi(4160))$. Besides, in these w.f.s. a contribution from a continuum w.f., $\varphi_{\text{cont}}$, is also present. The analytical expressions of $\theta_2$, $\tilde{\theta}_2$, $Z_1$, and $Z_2$ will be derived in Section III using the coupled-channel model with unitary inelasticity (CCUI model) and taking into account five strong decay channels, $DD$, $D^*D^*$, $D^*D_s$, $D_sD_s$, and $D_s^*D_s^*$, while here we present some formulas derived later. First, dynamical calculations give different mixing angles and probabilities, because these quantities are defined at a certain energy, equal to the mass of a given resonance. For example,

$$Z_1 = Z_1(E = M(\psi(4040))), \quad Z_2 = Z_2(E = M(\psi(4160))).$$

This type of probabilities $Z_{c\bar{c}}$ was already calculated in the $C^3$ model with the Cornell potential, where equal values $Z_{c\bar{c}}(\psi(4040)) = Z_{c\bar{c}}(\psi(4160)) = 0.494(3)$ were obtained. Surprisingly, in the $C^3$ model the mixing angle between the $3^3 S_1$ and $2^3 D_1$ states was found to be very small, $\theta_2 \lesssim 4^\circ$, much smaller than the mixing angle, $\sim 16^\circ$, between the $3^3 S_1$ and $2^3 S_1$ states. In our calculations the mixing angles $\theta_2$ and $\tilde{\theta}_2$ are not small and appear to be close to each other (see Section IV).

When the LWs are considered, in the w.f. at the origin a contribution from the continuum (four-quark or meson-meson component) can be neglected, since this contribution is very small. Nevertheless, the influence of decay channels on the w.f.s. is kept through the factors $Z_1$ and $Z_2$ and in general the resonance w.f.s. at the origin contain four parameters,

$$\varphi(\psi(4040), r = 0) = \sqrt{Z_1}(\varphi(3S) \cos \theta_2 - \varphi(2D) \sin \theta_2),$$
$$\varphi(\psi(4160), r = 0) = \sqrt{Z_2}(\varphi(3S) \sin \theta_2 + \varphi(2D) \cos \theta_2),$$

where the w.f.s. at the origin, $\varphi(3S)$ and $\varphi(2D)$, are calculated here using the relativistic string Hamiltonian (RSH) (see Section IV). When the factors $Z_i$ are present, then they have to be included to the standard expression of the LW,

$$G_{ee}(\psi(M_\psi)) = \frac{4e^2a^2}{M_\psi^2} R(\psi(M_\psi), r = 0)^2 Z_i \beta_{\text{QCD}}.$$  

Notice that in the RSH the w.f. of the $n^3 D_1$ state is defined as $R_{nD}(0) = \frac{5\Lambda_{\text{QCD}}(0)^2}{2\sqrt{2}\alpha_s}$, where $\omega_0$ is the quark kinetic energy. In Eq. 7 the QCD radiative correction, $\beta_{\text{QCD}} = 1 - \frac{16}{3\pi} \alpha_s(\mu) \Delta_{\text{MS}}^{(n_f = 4)}$, is taken the same for all vector charmonium states, with numerical value $\beta_{\text{QCD}} = 0.60$: it corresponds to $\alpha_s(n_f = 4, \mu) = 0.235$ at the scale $\mu = 4$ GeV, if $\Delta_{\text{MS}}^{(n_f = 4)} = (296 \pm 10)$ MeV is taken from pQCD. This factor $\beta_{\text{QCD}}$ is cancelled in the ratio of the LWs, but for $\psi(4040)$ and $\psi(4160)$ this ratio cannot be used because of the large experimental error in $\eta_2$, see Eq. (2).

### III. The Mixing Angles and Probabilities $Z_i$ in the CCUI Model

Here we use the CCUI model, where two sectors are considered: one refers to the charmonium conventional states and another to the heavy-light meson sector. For stationary states, like $3S$ and $2D$, one can use the Green's function in energy representation,

$$G_{QQ}^{(0)}(1, 2, E) = \sum_{n_1} \frac{\Psi_{QQ}^{(n_1)}(1) \Psi_{QQ}^{(n_1)}(2)}{E_{n_1} - E} = \frac{1}{H_0 - E}.$$  

(8)
where in the Green’s function the superscript (0) refers to the bare case, when the heavy-light sector is switched off. The w.f. $\Psi^{n_{1}}_{Q\bar{Q}}$ and $E_{n_{1}}$ are the eigenfunctions and eigenvalues of the relativistic string Hamiltonian (RSH) $[33, 40]$ (see the next Section). In the heavy-light sector we use the Green’s function of the pair $(Q\bar{q})(q\bar{Q})$ and neglect the interaction of the two (color singlet) heavy-light mesons. Then in the c.m. system one can write the Green’s function as

$$G^{(0)}_{Q\bar{q}}(1\overline{2}2; E) = \sum_{n_{2}, n_{3}} \int \frac{\Psi_{n_{2}n_{3}}^{(1)}(1, \overline{1}) \Psi_{n_{2}n_{3}}^{(2)}(2, \overline{2})}{E_{n_{2}n_{3}}(p) - E} d\Gamma(p),$$

where $d\Gamma(p)$ is the phase space factor.

To take into account transitions from the $Q\bar{Q}$ state to the sector of heavy-light mesons (strong decays) the Lagrangian of $^{3}P_{0}$ type is used,

$$\mathcal{L}_{sd} = \int \bar{\psi}_{q} M_{q} \psi_{q} d^{4}x, \quad M_{q} = \text{const},$$

where $M_{q} =$ const. $\approx 0.8$ GeV and $\psi_{q}$ are relativistic w.f.s. of a light quark in the field of a heavy antiquark $\bar{Q}$. It is important that the w.f.s., entering the Green’s functions and the RSH, are considered in the c.m. system, where the time coordinates of all particles are the same. Therefore the vertex $\mathcal{L}_{sd}$ occurs between instantaneous w.f.s. of the $Q\bar{Q}$ system on one side and the product of the $Q\bar{q}$ and $q\bar{Q}$ w.f.s. on the other side, thus defining an overlap integral $J_{123}$: 

$$J_{123} = \frac{1}{\sqrt{N_{c}}} \int y_{123} \bar{\psi}_{Q\bar{q}}(u - v) c^{i}P \psi_{Q\bar{q}}(u - x) \psi_{Q\bar{q}}(x - v) d^{3}x d^{3}(u - v),$$

which defines the self-energy contributions to the mass of a $Q\bar{Q}$ meson, appearing due to heavy-light mesons in the intermediate states:

$$w_{nm}(E) = \int \frac{d^{3}p}{(2\pi)^{3}} \sum_{n_{2} n_{3}} \frac{J_{n_{2}n_{3}}(p) J_{n_{2}n_{3}}^{*}(p)}{E_{n_{2}n_{3}}(p) - E},$$

Then the total Green’s function (in the $Q\bar{Q}$ sector) can be written as a sum over bound states:

$$G^{(1)}_{Q\bar{Q}}(1, 2; E) = \sum_{n} \frac{\Psi_{n}^{(1)}(1) \Psi_{n}^{(2)}(2)}{E_{n} - E} - \sum_{n,m} \frac{\Psi_{n}^{(1)}(1) w_{nm}(E) \Psi_{m}^{(2)}(2)}{(E_{m} - E)(E_{m} - E)} + \ldots$$

and the solutions of the equation,

$$\det (E - \hat{E}^{0} - \hat{\omega}) = 0,$$

define a new spectrum, namely, the masses $E_{R_{1}}$ and $E_{R_{2}}$ of two resonances in the two-channel case. In the case we consider here, the index 1 refers to the resonance $\psi(4040)$ and index 2 to the resonance $\psi(4160)$. In Eq. (15), $\hat{E}^{0}$ is a diagonal matrix, $\delta_{nm}E_{m}^{0}$, while the matrix elements $w_{ik}(E)$ determine the mixing angle between the $^{3}S_{1}$ and $^{2}D_{1}$ states and the mass shifts, $w_{11} = w_{SS}$, $w_{22} = w_{DD}$. We will also use the notations: $E_{11}^{0} = E_{11}(1) + w_{11}(E)$ and $E_{22}^{0} = E_{22}(1) + w_{22}(E)$ using the initial masses $E_{n}^{0}$ of the $^{3}S_{1}$ and $^{2}D_{1}$ states, calculated in closed-channel approximation.

The matrix elements $w_{ik}(E)$, dependent on the energy, are taken at the energy equal to the resonance mass: $E = E_{R_{i}}$ for $\psi(4040)$ and $E = E_{R_{2}}$ for $\psi(4160)$. If the non-diagonal matrix elements are small, then in first approximation the masses and widths of the resonances are $E_{R_{1}} = E_{11}^{0} + \text{Re} w_{11}(E_{11}^{0})$ and $E_{R_{2}} = E_{22}^{0} + \text{Re} w_{22}(E_{22}^{0})$. $\Gamma_{R_{1}} = 2 \text{Im} w_{11}(E_{11}^{0})$ and $\Gamma_{R_{2}} = 2 \text{Im} w_{22}(E_{22}^{0})$. However, for high $\psi$-resonances the values of the non-diagonal m.e.s. $w_{ik}$ are not small,
owing to the decay channels. The same procedure can be applied to the states above the decay thresholds, if one
finds to be a unitary matrix $\hat{U}$. Then the Green’s function acquires the new form:

$$C_{QQ}^{(n)} = \sum_{\lambda} \frac{1}{\Phi_{\lambda} E_{\lambda} - \Phi_{\lambda}}, \quad \Phi_{\lambda} = \sum_{n} \Phi_{QQ}^{(n)} U_{n\lambda}^{\dagger}(E),$$

(17)
i.e., the w.fs. $\Phi_{\lambda}$ ($\lambda = 1, 2$) become new orthogonal states, comprising all effects of the mixture between bound states owing to the decay channels. The same procedure can be applied to the states above the decay thresholds, if one neglects the widths of those states. Here we use just this approximation.

Using Eqs. (16,17) one can express the new w.fs. $\Phi_{\lambda}$ via the w.fs. $\Psi_{n}$, taking into account the following relations,

$$(E - \hat{E}^{\emptyset} - \hat{w})^{-1} = \frac{1}{\text{det}(E - \hat{E}^{\emptyset} - \hat{w})} \begin{pmatrix} E - E_{11}^{\emptyset} & w_{12} \\ w_{12} & E - E_{22}^{\emptyset} \end{pmatrix},$$

(18)

With the notations $E_{1}^{\emptyset} = E_{11}^{\emptyset} + w_{22}(E)$, $E_{2}^{\emptyset} = E_{11}^{\emptyset} + w_{11}(E)$, and $E_{R_{i}} = E_{\lambda_{i}}$, we rewrite $\text{det}(E - \hat{E}^{\emptyset})$ as $\text{det}(E - \hat{E}^{\emptyset}) = (E - E_{1}) (E - E_{2}) - w_{12}(E) w_{21}(E)$, which defines the masses of the two resonances. Now we assume that $E_{\lambda_{1}} < E_{\lambda_{2}}$. Then the mass of the resonance with the smaller mass is

$$E_{\lambda_{1}} = \frac{1}{2} (E_{1}^{\emptyset} + E_{2}^{\emptyset}) - \frac{1}{2} \sqrt{(E_{2}^{\emptyset} - E_{1}^{\emptyset})^{2} + 4 w_{12} w_{21}},$$

(19)

where all matrix elements $w_{12}(E)$, $w_{21}(E)$, and $w_{22}(E)$ inside $E_{n}^{\emptyset}$ are taken at the point $E = E_{\lambda_{1}}$. The mass of the higher resonance, $E_{\lambda_{2}}$ is obtained from the equation,

$$E_{\lambda_{2}} = \frac{1}{2} (E_{1}^{\emptyset} + E_{2}^{\emptyset}) + \frac{1}{2} \sqrt{(E_{2}^{\emptyset} - E_{1}^{\emptyset})^{2} + 4 w_{12} w_{21}},$$

(20)

with all matrix elements $w_{ik}(E)$, taken at the point $E = E_{\lambda_{2}}$.

To find the explicit expressions of the matrix elements of the unitary matrix $U_{ik}$ we assume that the imaginary parts of $w_{ik}$ are small and can be omitted. Then the inverse matrix in Eq. (15) is written as

$$(E - \hat{E}^{\emptyset})_{11}^{-1} = \sum_{\lambda=1,2} U_{1\lambda}^{\dagger}(E) \frac{1}{E - E_{\lambda}} U_{\lambda 1}(E) = \frac{1}{E - E_{\lambda_{1}}} + \frac{E_{\lambda_{2}} - E_{\lambda_{1}}^{\emptyset}}{E_{\lambda_{1}} - E_{\lambda_{2}}} \left( \frac{1}{E - E_{\lambda_{1}}} - \frac{1}{E - E_{\lambda_{2}}} \right).$$

(21)

Therefore the products of the matrix elements are

$$U_{11}^{\dagger} U_{11} = 1 + \frac{E_{\lambda_{2}} - E_{\lambda_{1}}^{\emptyset}}{E_{\lambda_{1}} - E_{\lambda_{2}}}, \quad U_{21}^{\dagger} U_{21} = - \frac{E_{\lambda_{2}} - E_{\lambda_{1}}^{\emptyset}}{E_{\lambda_{1}} - E_{\lambda_{2}}}$$

(22)

Notice that these matrix elements satisfy the property of unitarity: $U_{11}^{\dagger} U_{11} + U_{12}^{\dagger} U_{21} = 1$. In the same way we find the product of the other matrix elements,

$$U_{11}^{\dagger} U_{12} = \frac{E_{\lambda_{1}} - E_{\lambda_{2}}^{\emptyset}}{E_{\lambda_{1}} - E_{\lambda_{2}}}, \quad U_{22}^{\dagger} U_{22} = 1 - \frac{E_{\lambda_{1}} - E_{\lambda_{2}}^{\emptyset}}{E_{\lambda_{1}} - E_{\lambda_{2}}},$$

(23)

which satisfy the condition $U_{21}^{\dagger} U_{12} + U_{22}^{\dagger} U_{22} = 1$. From the relations (22) and (23) the diagonal matrix elements are found to be

$$U_{11} = \sqrt{\frac{1 + E_{\lambda_{2}} - E_{\lambda_{1}}^{\emptyset}}{E_{\lambda_{1}} - E_{\lambda_{2}}}},$$

$$U_{22} = \sqrt{\frac{1 - E_{\lambda_{1}} - E_{\lambda_{2}}^{\emptyset}}{E_{\lambda_{1}} - E_{\lambda_{2}}}}.$$
The non-diagonal matrix elements are given by
\[ U_{12} = \frac{w_{21}}{(E_{\lambda_1} - E_{\lambda_2})U_{11}}, \quad U_{12}^\dagger = \frac{w_{21}}{(E_{\lambda_1} - E_{\lambda_2})U_{22}}, \]
and satisfy the condition: \[ U_{21}^\dagger U_{11} + U_{22}^\dagger U_{21} = 0. \] From the obtained expressions the w.f. \( \Phi_1 \) of the lower resonance with the mass \( E_{\lambda_1} \) and the w.f. \( \Phi_2 \) of the upper resonance with the mass \( E_{\lambda_2} \) \( (E_{\lambda_1} < E_{\lambda_2}) \), can be written as
\[ \Phi_1(\psi(4040)) = U_{11} \left( \Psi_1 - \Psi_2 \frac{w_{12}}{E_2^* - E_{\lambda_1}} \right) = \sqrt{Z_1}(\Psi_1 \cos \theta_1 + \Psi_2 \sin \theta_1), \]
where we have taken into account that \( w_{12} = -|w_{12}| \) and introduced
\[ \tan \theta_1 = \frac{|w_{12}|}{E_2^* - E_{\lambda_1}}, \quad \sin \theta_1 = \frac{|w_{12}|}{\sqrt{(E_2^* - E_{\lambda_1})^2 + w_{12}^2}}. \]
The probability \( Z_1 \), given by
\[ Z_1 = \frac{(E_2^* - E_{\lambda_1})^2 + w_{12}^2}{(E_{\lambda_2} - E_{\lambda_1})(E_2^* - E_{\lambda_1})}, \]
determines the weight of the state 1 \( (3^3 S_1) \) in the resonance w.f. of \( \psi(4040) \) where all matrix elements are taken at \( E = E_{\lambda_1} \). Its value is close to unity if the transition matrix element \( |w_{12}| \) is small, \( w_{12}^2 \ll (E_2^* - E_{\lambda_1})^2 \). However, in a realistic situation where \( |w_{12}| \sim (30-50) \text{ MeV} \), it can be of the same order as the mass difference, \( E_2^* - E_{\lambda_1} \sim 100 \text{ MeV} \).

For the w.f. of the upper resonance an expression similar to Eq. (27) applies,
\[ \Phi_2(\psi(4160)) = \sqrt{Z_2}(-\Psi_1 \sin \tilde{\theta}_2 + \Psi_2 \cos \tilde{\theta}_2), \]
with
\[ Z_2 = \frac{(E_{\lambda_2} - E_1^*)^2 + w_{21}^2}{(E_{\lambda_2} - E_{\lambda_1})(E_{\lambda_2} - E_1^*)}, \]
and
\[ \sin \tilde{\theta}_2 = \frac{|w_{21}(E)|}{\sqrt{(E_{\lambda_2} - E_1^*)^2 + w_{21}^2}}. \]

In Eqs. (29)\( (31) \) all matrix elements are taken at the energy \( E = E_{\lambda_2} \).

Thus for mixed \( (n+1)^3 S_1 \) and \( n^3 D_1 \) states analytical expressions were derived, which allow to calculate the mixing angles and the probabilities in the w.f.s of the \( 4040 \) and \( 4160 \) resonances and to understand dynamical effects, produced by the five decay channels \( D\bar{D}, D\bar{D}^*, D\bar{D}^* \), \( D_s\bar{D}_s \), and \( D_s\bar{D}_s^* \). These relations and the condition \( Z_i < 1.0 \) establish important correlations between the different mass shifts.

Also we would like to notice that in our calculations the sign of the w.f. at the origin is taken with the factor \( (-1)^{n+1} = (-1)^{n_r} \) \( (n_r = 0, 1, ...) \) as in simple harmonic oscillator (SHO) functions.

IV. THE STATIC POTENTIAL

From the analytical expressions Eqs. (29)\( (31) \) one can see that the parameters of the resonances explicitly depend on the bare masses \( E_0 \) and \( E_0^* \), defined by the static potential \( V_0(r) \). In our approach this potential contains only fundamental quantities, established in pQCD \( [18] \) and the field correlator method \( [26, 27] \), and owing to the so-called Casimir scaling \( V_0(r) \) has to be the sum of the confining and gluon-exchange terms \( [27] \). In the confining term the string tension \( \sigma_0 = (0.18 \pm 0.02) \text{ GeV}^2 \) is fixed by the slope of the leading Regge trajectory of light mesons, while the asymptotic freedom behaviour of the vector coupling is defined by the QCD vector constant \( \Lambda_V(n_f = 3) \) in full agreement with the value \( \Lambda_{\text{MS}}(n_f = 3) = (339 \pm 10) \text{ MeV} \) from pQCD \( [18] \), since they are interrelated. Namely, \( \Lambda_V(n_f = 3) = 1.4753 \Lambda_{\text{MS}}(n_f = 3) = (500 \pm 15) \text{ MeV} \).

At small momenta in two-loop vector coupling \( \alpha_V(q^2) \) we take the value of the infrared regulator \( M_B \), which enters the logarithm \( \ln \left( \frac{q^2 + M_B^2}{\Lambda_5^2} \right) \), from Ref. \( [11] \), to be \( M_B = \sqrt{2\pi\sigma_0} \), which is not an extra parameter, since it is expressed through the same string tension \( \sigma_0 \) as occurs in the leading Regge trajectory of light mesons.
It is of interest to notice that for $\Lambda_V = (480 \pm 20)\text{ MeV}$ and $M_B = (1.10 \pm 0.05)\text{ GeV}$ the critical (asymptotic) vector coupling, $\alpha_{\text{crit}} = \alpha(q^2 = 0) = 0.60 \pm 0.04$ has the value close to $\alpha_{\text{crit}} = 0.60$ in the Godfrey-Isgur model [3].

In closed-channel approximation we use the RSH [39], where in the kinetic term the pole $c$-quark mass, $m_c(pole) = (1.440 \pm 0.015)\text{ GeV}$ corresponds to the conventional current quark mass, $\overline{m}_c = (1.267 \pm 0.011)\text{ GeV}$ [3]. Then for the $Q\bar{Q}$ potential,

$$V_0(r) = \sigma_0 r - \frac{4\alpha_V(r)}{3r}, \quad (32)$$

the centroid masses $M_{\text{cog}}(nl)$ coincide with the eigenvalues of the spinless Salpeter equation (SSE):

$$\left(2\sqrt{p^2 + m_Q^2} + V_0(r)\right)\psi(r) = M_{\text{cog}}(nl)\psi(r). \quad (33)$$

Thus the charmonium spectrum is defined without fitting parameters. In Eq. (32) the two-loop coupling $\alpha_V(r)$ in coordinate space is expressed via the two-loop vector coupling in momentum space,

$$\alpha_V(r) = \frac{2}{\pi} \int_0^{\infty} dq \frac{\sin(qr)}{q} \alpha_V(q^2), \quad (34)$$

and its properties were studied in detail in Ref. [19]. Here we take the following set of the parameters,

$$\sigma_0 = 0.18\text{ GeV}^2, \quad \Lambda_V(n_f = 3) = 500 \text{ MeV} \text{ or } \Lambda_{\text{SSS}}(n_f = 3) = 339 \text{ MeV},$$

$$M_B = 1.15\text{ GeV}, \quad \alpha_{\text{crit}} = 0.635, \quad m_c = 1.440\text{ GeV}. \quad (35)$$

In the CCUI model the masses of the resonances and the mass shifts are determined via the “unperturbed” masses. They include spin corrections and in Section III were denoted as $E_1^0 = M(3^3S_1) = M_{\text{cog}}(3S) + 1/4\Delta_{\text{hf}},$ with $\Delta_{\text{hf}}(3S)$ being the hyperfine shift of the $3^3S_1$ state, and $E_2^0 = M(2^3D_1) = M_{\text{cog}}(2D) - \Delta_{\text{fs}},$ where $\Delta_{\text{fs}}$ is a shift of the $2^3D_1$ state due to the fine-structure interaction.

In $V_0(r),$ Eq. (32), we use a linear confining potential (without flattening or screening effects), in order to escape double counting in the coupled-channel calculations. About the screened potential,

$$V_{\text{scr}} = \lambda \frac{1 - \exp(-\mu r)}{\mu r}, \quad (36)$$

it is worth to notice that it is going to a constant at not so large distances, $R \sim 2\text{ fm}$: $V_{\text{scr}}(r) \rightarrow \text{const.} = \frac{\lambda}{\mu} = 2.145\text{ GeV}$ [20]. Such an asymptotic behavior violates the boundary conditions of the relativistic SSE as well as the Schrödinger equation, and makes the gluon-exchange potential dominant even at large distances. Besides, in a high charmonium state with the mass, $M(nl) \geq (2m_c + 2.145) \sim (5.0 - 5.2)\text{ GeV},$ its constituents, a quark and an antiquark, are not confined but can be liberated.

In Table III we give the w.fs. at the origin, calculated in closed-channel approximation with the static potential $V_0(r),$ which are needed for further coupled-channel analysis, and also the IWs of pure $n^3S_1,$ $m^3D_1$ states, taking the factor $\beta_{QCD} = 0.60$ for all states. The unperturbed masses $E_i^0$ are also given.

The w.fs. $R_{nS}(0)$ are calculated here in two ways, because the original form of the string Hamiltonian [19, 28, 39, 40],

$$H_{\text{str}} = \omega + m_Q^2/\omega + p^2/\omega + V_0(r), \quad (37)$$

has to be supplemented by the extremum condition, which can be of two different kinds. In the first case the condition $\partial H_{\text{str}}/\partial \omega = 0$ is used, which allows to reduce $H_{\text{str}}$ to the Hamiltonian $H_0,$ present in the SSE, Eq. (33). This equation, when solved numerically [10], is very convenient, since it gives simultaneously the whole meson spectrum, the quark kinetic energies $\omega_{\text{nl}},$ all matrix elements, etc. However, for the SSE the radial w.fs. $R_{nS}(0)$ have an unpleasant feature – they diverge near the origin for any potential $V_0(r)$ with Coulomb-type term. Therefore, a regularization of $R_{nS}(r)$ at small distances, which can produce additional fitting parameters, is needed. Here we use a procedure which allows to escape the introduction of new parameters doing regularization.

For the SSE at small $r$ the derivatives $R_{nS}'(r)$ increase, starting to grow at a critical distance $r_{\text{crit}} \sim 0.07\text{ fm}.$ On the contrary, the eigenfunctions of the $H_{\text{str}}$ are regular in so-called einbein approximation [28, 39, 40] (as well as those of the Schrödinger equation) and their derivatives decrease for small $r,$ approaching zero at $r \rightarrow 0.$ In the einbein approximation to $H_{\text{str}},$ Eq. (37), the extremum condition is put on the meson mass $M_{\text{nl}} : \partial M_{\text{nl}}(\omega_{\text{nl}})/\partial \omega_{\text{nl}} = 0,$ where the mass is $M_{\text{nl}} = \omega_{\text{nl}} + m_Q^2/\omega_{\text{nl}} + E_{\text{nl}}$ and this extremum condition defines the quark kinetic energy $\omega_{\text{nl}}$ as

$$\omega_{\text{nl}}^2 = m_Q^2 + \omega_{\text{nl}}^2 \partial E_{\text{nl}}/\partial \omega_{\text{nl}}, \quad (38)$$
while the eigenvalues are the solutions of the equation,

\[
(p^2/\omega + V_0(r))\psi_{nl}(r) = E_{nl}\psi_{nl}(r).
\]

(39)

Thus in the einbein approximation one needs to solve in a consistent way two equations, Eqs. (38 and 39). The numerical calculations of the eigenvalues has an accuracy of ~ 1 MeV. Moreover, the w.f. at the origin \( R^E_{nSR}(0) \) can also be defined with the use of the relation \( S \):

\[
|R^E_{nS}(0)|^2 = \omega_{nS}(dV_0/dr)_{nS} = \omega_{nS}(\sigma_0 + 4/3(\alpha(r)/r^2)_{nS} - 4/3(\alpha'(r)/r)_{nS}),
\]

(40)

where all matrix elements are calculated with great accuracy. Notice that the relation \( 10 \) contains the quark kinetic energy \( \omega_{n1} \), which depends on the quantum numbers, while in the nonrelativistic case, instead, the quark mass \( m_Q \) enters for all states.

At small \( r \) the derivatives \( R'_{nS}(EA, r) \) are very small, approaching zero, and their values can be used for regularization of the SSE w.f.s.. Using this procedure one obtains the regularized w.f.s. \( R^E_{nS}(0) \) of the SSE, which values appear to be very close to \( R^E_{nS}(0) \), calculated in einbein approximation. Due to this fact and to escape additional uncertainties coming from the regularization, we use here \( R^E_{nS}(0) \) as the w.f. at the origin of SSE, taking the proper value of the quark kinetic energy \( \omega_{nS} \).

TABLE III: The w.f.s. \( R_{nl}(0) \) (in GeV\(^{3/2} \)), the masses \( M(nl) \) (in MeV), and the leptonic widths (in keV) of pure \( n^3S_1 \) and \( m^3D_1 \) charmonium states for the static potential \( V_0(r) \) and \( m_c = 1.44 \) GeV, \( \beta_{QCD} = 0.60 \)

| state  | \( R_{nl}(0) \) | \( \Gamma_{ee}(th.) \) | \( \Gamma_{ee}(exp.\mid 3 \) | \( M_{V}(nl) \) | experiment |
|--------|----------------|-----------------|----------------|----------------|--------------|
| \( 1\,^3S_1 \) | 0.961 | 5.47 | 5.55 ± 0.14 | 3093 | 3096.90 ± 0.01 |
| \( 2\,^3S_1 \) | -0.801 | 2.68 | 2.34 ± 0.04 | 3689 | 3686.10 ± 0.03 |
| \( 1\,^3D_1 \) | 0.096 | 0.037 | 0.262 ± 0.018 | 3800 | 3773.13 ± 0.35 |
| \( 3\,^3S_1 \) | 0.752 | 1.97 | 0.86 ± 0.07 | 4121 | 4039 ± 1 |
| \( 2\,^3D_1 \) | -0.146 | 0.069 | 0.48 ± 0.22 | 4195 | 4191 ± 5 |
| \( 4\,^3S_1 \) | -0.738 | 1.58 | 0.58 ± 0.07 | 4467 | 4421 ± 4 |
| \( 3\,^3D_1 \) | 0.170 | 0.080 | - | 4527 | - |

As seen from Table III, good agreement with experiment is obtained only for the LW of \( J/\psi \). For \( \psi(3686) \) the LW is 15% larger, while the LWs of the other \( n^3S_1 \) states are about two times larger than the experimental numbers. On the other hand, the LWs of pure \( n^3D_1 \) (\( n = 1, 2 \)) states are about seven times smaller than the experimental values of \( \Gamma_{ee}(\psi(3773)) \) and \( \Gamma_{ee}(\psi(4190)) \). It is of interest to notice that for the gluon-exchange potential with large vector constant, \( \Lambda_V = 500 \) MeV, the w.f.s. at the origin are relatively large and owing to that, \( \Gamma_{ee}(J/\psi) = 5.47 \) keV is found to be in good agreement with the experimental value.

For the set of parameters, Eq. (35), the unperturbed masses are calculated numerically with accuracy ~ 1 MeV (see also Table II),

\[
E_1^0 = M(3\,^3S_1) = 4112 \text{ MeV}; \quad E_2^0 = M(3\,^3D_1) = 4195 \text{ MeV}.
\]

(41)

and for the \( 4\,^3S_1 \) and \( 3\,^3D_1 \) states we find

\[
E_1^0(4\,^3S_1) = M(4\,^3S_1) = 4467 \text{ MeV}, \quad E_2^0(3\,^3D_1) = M(3\,^3D_1) = 4527 \text{ MeV}.
\]

(42)

These values are used further in the CCUI model.

Now we compare the parameters we have chosen, Eq. (35), with those from Ref. 28, where in the vector coupling \( \alpha_V \) a smaller \( \Lambda_V(n_f = 4) = 360 \) MeV is used. Notice, that this value of \( \Lambda_V \) corresponds to \( \Lambda_{\overline{MS}}(n_f = 4) = 253 \) MeV, which is ~ 17% smaller than \( \Lambda_{\overline{MS}}(n_f = 4) = (296 ± 10) \) MeV, accepted now in pQCD 3, 18. Therefore, this QCD constant has to be considered as a fitting parameter. Also a significant difference takes place between the value of the QCD factor, \( \beta_{QCD} = 0.72 \) in Ref. 28 and \( \beta_{QCD} = 0.60 \), used in the present analysis for all states.

In Table IV we compare the LWs, calculated here and in Ref. 28, and give also the values of the radial w.f.s. at the origin for low-lying charmonium states. For \( \psi(3686) \) and \( \psi(3773) \) the \( 2S - 1D \) mixing angle, \( \theta = 11^\circ \), is taken here, while in Ref. 28 \( \theta = 15^\circ \); also for \( \Lambda_V = 360 \) MeV and \( M_B = 1.0 \) GeV, taken in 28, the critical (frozen) \( \alpha_{crit} = 0.547 \) is smaller than in our present case, where \( \alpha_{crit} = 0.635 \).

Nevertheless, in Ref. 28 the values of \( R_{nS}(0) \) appeared to be only ~ (5 – 7)% smaller than in our case. For the \( 1S, 2S, \) and \( 1D \) states their values (in GeV\(^{3/2} \)) are 0.905, 0.767, 0.094 in Ref. 28 and 0.962, 0.801, 0.096.
in our calculations. Therefore, if the same $\beta_{QCD}$ is taken, then the LWs in Ref. 28 would be $10 - 15\%$ smaller than in our case. For that reason, to fit the experimental LWs a larger $\beta_{QCD} = 0.72$ is taken in Ref. 28. This choice of $\beta_{QCD} = 1 - \frac{4\pi}{3}\alpha_s(\mu) = 0.72$ cannot be considered as a good one, since it corresponds to a very small $\alpha_s(\mu)(n_f = 4) = 0.165$, or to a very large scale $\mu > 8.0$ GeV. In our calculations $\beta_{QCD} = 0.60$ is smaller and corresponds to $\alpha_s(\mu)(n_f = 4) = 0.235$ with a reasonable value of the scale $\mu \sim (3.7 - 4.0)$ GeV, which is in agreement with pQCD.

In our calculations of the LWs, presented in Table IV, their values are defined by Eq. (7), while in 28 the LWs contain an additional relativistic factor $\xi_R = \frac{m^2 + \omega^2 + P^2/2}{2\omega}$, originating from the vector decay constant expression 42. Therefore, for a comparison it is convenient to divide the LWs of $J/\psi$, $\psi(3686)$, and $\psi(3773)$ from Ref. 28 by the values of $\xi_R$, equal to 0.929, 0.910, 0.910, respectively.

TABLE IV: Comparison of the radial w.fs. at the origin (in GeV$^{3/2}$, first three rows) and the leptonic widths (in keV) for low-lying charmonium states. The mixing angle $\theta = 11^\circ$ in Ref. 28 and $\theta = 11.5^\circ$ in the present paper.

| State          | [28] | this paper | experiment, [3] |
|----------------|------|------------|-----------------|
| $J/\psi$       | 0.905| 0.961      |                 |
| $\psi(3686)$   | 0.735| 0.764      |                 |
| $\psi(3773)$   | 0.238| 0.246      |                 |
| $\Gamma_{ee}(J/\psi)$ | 5.82 | 5.47       | 5.55 $\pm$ 0.14 Ref. [14] |
| $\Gamma_{ee}(\psi(3686))$ | 2.71 | 2.44       | 2.34 $\pm$ 0.04 Ref. [4] |
| $\Gamma_{ee}(\psi(3773))$ | 0.27 | 0.242      | 0.262 $\pm$ 0.018 Ref. [18] |

A comparison of the LWs with the experimental data 3 shows that for our set of parameters the LWs of low-lying states are obtained in good agreement with experiment.

Notice that the existing uncertainty in the value of the QCD constant $\Lambda_V(n_f = 3) = 500\pm15$ MeV does not change $R_{ss}(0)$ by more than 1%. There is also an uncertainty in the value of the infrared regulator, $M_\beta = (1.07\pm0.08)$ GeV, and here we fix $M_\beta = 1.15$ GeV according to the analysis of the bottomonium spectrum in Ref. 19.

It is important to stress that the vector coupling in coordinate space, Eq. (34), is taken with $n_f = 3$, while in momentum space the regions with different $q^2$ are described by the strong coupling with different numbers of flavours $n_f$. In coordinate space the situation is different, because the "exact" (or combined) coupling $\alpha_C(r)$, defined by Eq. (34), coincides with $\alpha_V(n_f = 3, r)$ for all distances with the exception of very small $r < 0.06$ fm 19, so that the use of $\alpha_V(n_f = 3, r)$ in the whole region provides high accuracy in the charmonium masses and w.fs.. On the contrary, a choice of the vector coupling with $n_f = 4$ and $\Lambda_V(n_f = 4) = 360$ MeV in coordinate space 28 has no fundamental grounds.

V. RESULTS

In the CCUI, the calculation of the matrix elements $w_{ik}$, defined by the overlap integral $J_{nn'\eta\eta'}$, Eq. 14, is the most important part of the numerical calculations. In this overlap integral we approximate the exact w.fs., expanding them in a series of simple harmonic oscillator (SHO) functions and take five terms for the charmonium w.fs. and one SHO function for heavy-light mesons. All parameters of these SHO functions are given in Ref. 33. The accuracy of the numerical calculations is estimated to be $\sim 10\%$.

Since the matrix elements $w_{ik}(E)$ depend on the energy, they differ at the points $E = E_{\lambda_1} = M(\psi(4040)) = 4056$ MeV and $E = E_{\lambda_2} = M(\psi(4160)) = 4190$ MeV. In Table V we give the values of $w_{ik}$ with the errors arising from numerical calculations. Notice that the agreement with the experimental LWs is reached not for the central values, but for the maximal values of $|w_{ik}|$. We also introduce new notations: $w_{11} = w_{SS}$, $w_{12} = w_{SD}$, $w_{22} = w_{DD}$.

Then from Eqs. (13) (20) the masses of the resonances are as follows,

$$M(\psi(4040)) = (4056 \pm 8) \text{ MeV}, \quad M(\psi(4160)) = (4190^{+3}_{-1}) \text{ MeV},$$

i.e., the central value of $M(\psi(4040))$ is obtained to be 56 MeV lower than the initial mass of the $3^3S_1$ state. The situation is different for the higher solution of Eq. (20), when this mass almost coincides with the mass of the $2^3D_1$ state; it happens because this mass decreases due to the self-energy shift $w_{DD}$ but increases owing to the non-diagonal matrix element $w_{SD}$.
Using Eqs. (41) and (44) and $w_{ik}$ from Table VI one obtains that the mixing angles $\theta_2$ and $\tilde{\theta}_2$ have rather close values,

$$\theta_2(\psi(4040)) = (28^{+1}_{-2})^\circ; \quad \tilde{\theta}_2(\psi(4160)) = (29.5^{+2.0}_{-3.0})^\circ.$$

Although the difference between these angles is small, the use of them gives rise to LWs in better agreement with the experimental values. Using these mixing angles and the w.f.s. at the origin from Table III, one obtains the following radial w.f.s. at the origin of the resonances: $R(\psi(4040), r = 0) = (0.593^{+0.006}_{-0.008})$ GeV$^{3/2}$ and $R(\psi(4160), r = 0) = (-0.496^{+0.023}_{-0.031})$ GeV$^{3/2}$. Then the probabilities $Z_i$ and the LWs are

$$Z_1(\psi(4040)) = 0.86^{+0.04}_{-0.03} \quad \Gamma_{ee}(\psi(4040)) = Z_1(1.21^{+0.02}_{-0.03}) \text{ keV} = (1.04^{+0.07}_{-0.06}) \text{ keV}. \quad (45)$$

Here the central value of the LW is $\sim 10\%$ larger than the upper limit of the experimental value $\Gamma_{ee}(\psi(4040)) = (0.86 \pm 0.07)$ keV, given by the PDG [3], but smaller than the LW, obtained in the analysis of the BES data [7] and the Belle data [6] (see Table III). For the $\psi(4160)$ resonance we have found

$$Z_2 = (0.79 \pm 0.01), \quad \Gamma_{ee} = Z_2(0.80^{+0.07}_{-0.06}) \text{ keV} = (0.62 \pm 0.07) \text{ keV}. \quad (46)$$

We have also checked the sensitivity of our results to the choices of the $c$-quark mass and of $\Lambda_V$, varying them in a very narrow range, since the pole $c$-quark mass and $\Lambda_V$ are known with $\pm 20$ MeV accuracy in pQCD. To describe the masses of low-lying charmonium states with a good accuracy (with the smaller $c$-quark mass, $m_c = 1.425$ GeV), one needs to use the value of $\Lambda_V = 465$ MeV, which is smaller than that accepted in pQCD, $\Lambda_V(n_f = 3) = (500 \pm 15)$ MeV. Also in this case the matrix elements $|w_{ik}|$ are a bit smaller, than those in Table VI giving the smaller $\theta_2 = 23^\circ$, $M(\psi(4040)) = 4034$ MeV, and $Z_1 = 0.97$, so that the calculated value $\Gamma_{ee}(\psi(4040)) = Z_1 1.30 = 1.26$ keV is larger compared to the value in Eq. (45). For the higher resonance the mass $M(\psi(4160)) = 4160$ MeV is also $30$ MeV smaller than in Eq. (44), while $Z_2 = 0.78$ and the mixing angle, $\tilde{\theta}_2 = 30^\circ$, is not changed, giving the same value of $\Gamma_{ee}(\psi(4160)) = Z_2 0.81 \text{ keV} = 0.63 \text{ keV},$ as in Eq. (46).

For the set of the parameters from Eq. (35) our results are summarized in Table VI.

| state       | $E^0$  | $M_R$    | $\theta_i$ | $Z_i$    | $\Gamma_{ee}$ | exp. |
|-------------|--------|----------|------------|----------|---------------|------|
| $\psi(4040)$| 4112 ± 1| 4056 ± 8 | $(28^{+1}_{-2})^\circ$ | $0.80^{+0.04}_{-0.03}$ | 1.0 ± 0.1 | 0.86 ± 0.07 |
| $\psi(4160)$| 4195 ± 1| 4190 ± 3| $(29^{+2}_{-3})^\circ$ | $0.79 \pm 0.01$ | 0.62 ± 0.07 | 0.48 ± 0.22 |

From Table VI one can see that for the chosen $Q\bar{Q}$ potential, Eq. (36) with the set of the parameters Eq. (35) and the pole mass $m_c(\text{pole}) = 1.44$ GeV, the calculated LWs are found to be in better agreement with the experimental values of BES [2] and PDG [3], than in case of the smaller $m_c = 1.425$ GeV, where the central values are

$$\Gamma_{ee}(\psi(4040)) = Z_1 1.30 = 1.26 \text{ keV}, \quad \Gamma_{ee}(\psi(4160)) = Z_2 0.78 = 0.63 \text{ keV}. \quad (47)$$

In this case $\Gamma_{ee}(\psi(4040)) \sim 1.3$ keV appears to be close to the LW value extracted from the Belle [6] and BES experimental data [7]. Thus the existing disagreement between the experimental data on the LWs of $\psi(4040)$ does not allow to fix the values of the $c$-quark mass and the QCD constant $\Lambda_V(n_f = 3)$ with high accuracy.
Calculation of the LW of $\psi(4415)$ with a good accuracy is a more difficult task, first, because $\psi(4510)$ is not observed yet, and secondly, because the mass difference, $M(4510) - M(4420) = 90$ MeV is rather small and therefore all matrix elements $w_{ik}$ and $Z_i$ have to be determined with great accuracy. Our calculations (with accuracy $\sim 10\%$) give the following $w_{ik}$ at the point $E = 4421$ MeV: $w_{11} = (-23 \pm 2)$ MeV, $w_{12} = (-35 \pm 5)$ MeV, $w_{22} = (-53 \pm 5)$ MeV, and

$$\theta_3(\psi(4415)) = (33 \pm 4)^\circ, \quad \bar{\theta}_3 = (34 \pm 3)^\circ, \quad Z_3(\psi(4415)) = 0.83^{+0.06}_{-0.03}, \quad Z_4(\psi(4510)) = 0.85^{+0.08}_{-0.06}. \tag{48}$$

$$R(\psi(4415), r = 0) = (-0.52 \pm 0.04) \text{ GeV}^{3/2}, \quad R(\psi(4510), r = 0) = (0.53 \pm 0.04) \text{ GeV}^{3/2},$$

to obtain the values

$$\Gamma_{ee}(\psi(4415)) = Z_3(0.79 \pm 0.02) = (0.66^{+0.06}_{-0.03}) \text{ keV} \quad \Gamma_{ee}(\psi(4510)) = Z_4(0.80 \pm 0.10) \text{ keV} = (0.68 \pm 0.14) \text{ keV}, \tag{49}$$

i.e. for $\psi(4415)$ and $\psi(4510)$ the LWs are almost equal.

Thus we conclude that the $4S - 3D$ mixing, occurring via open $DD, DD^*$, and $D^*D^*$ channels, may not be small, with the mixing angle $\sim 33^\circ$, which is very close to the mixing angle $\theta_{ph} = 34.5^\circ$, used in a phenomenological approach [12]. Then for the missing state $\psi(4510)$ the LW (0.68$\pm$0.10) keV is obtained, which is almost equal to that of $\psi(4415)$.

Our calculations were done in a simplified two-channel model, where the coupling to meson-meson channels and also to $c\bar{c}$ vector channels, like $2\frac{1}{2}S_1$ and $1\frac{3}{2}D_1$, is not taken into account. Such many channel considerations are very complex within the analytical CCUI model. However, our dynamical calculations allow us to define mixing angles, which are usually taken as fitting parameters, and also the probabilities of the $c\bar{c}$ components $Z_i$, which partly suppress the values of the LWs. Although these factors depend on energy, for the resonances $\psi(4040)$ and $\psi(4160)$ the mixing angles appear to be almost equal, while the probabilities $Z_i$ in $\psi(4040)$ and $\psi(4160)$ can differ by up to 15%.

In the CCUI model the analysis of LWs is of special importance, because the w.f.s. at the origin do not depend on the admixture of a meson-meson (a multiquark) component, which nevertheless could decrease the probabilities $Z_1$ and $Z_2$ (or $Z_3$ and $Z_4$).

## VI. SUMMARY

We have used a coupled-channel model with unitary inelasticity to describe the mixing of $n\frac{3}{2}S_1$ and $(n-1)\frac{3}{2}D_1$ states, which occurs due to transitions to decay channels. In this model analytical expressions for the resonance masses, mixing angles, and probabilities, needed for understanding the physical picture, are obtained. For the $\psi$-resonances the following masses are calculated: $M(\psi(4040)) = (4056 \pm 8)$ MeV, $M(\psi(4160)) = (4190^{+2}_{-1})$ MeV, $M(\psi(4415)) = (4241^{+7}_{-8})$ MeV, and $M(\psi(4510)) = (4512 \pm 2)$ MeV.

At present there is no consensus about the precise value of the LWs of $\psi(4040)$, $\psi(4160)$ and in different analyses of the experimental data on inclusive and exclusive $e^+e^-$ processes two possibilities are presented: a relatively large LW $\Gamma_{ee}(\psi(4040)) \approx 1.2$ keV in Refs. [2, 3] and a smaller value of $\Gamma_{ee}(\psi(4040)) = 0.86(7)$ keV in Refs. [2, 3]. In our model $\psi(4040)$ and $\psi(4160)$ have almost equal mixing angles (they coincide within errors): $\theta(\psi(4040)) \equiv \theta(\psi(4160)) \equiv (29 \pm 2)^\circ$, nevertheless, a small difference between the mixing angles provides better agreement with the experimental data.

An important suppression of the LWs is possible due to the probabilities $Z_i$, which for $\psi(4040)$ and $\psi(4160)$ differ by only $\sim 10\%$. However,

these values are sensitive to the transition matrix elements and can vary in a wide range, from 0.72 to 0.93 within accuracy of calculations. For the LWs our calculations give $\Gamma_{ee}(\psi(4040)) = (1.0 \pm 0.1)$ keV, and $\Gamma_{ee}(\psi(4160)) = (0.62 \pm 0.07)$ keV.

We have also considered the mixing between $4\frac{3}{2}S_1$ and $3\frac{3}{2}D_1$ via decay channels. It appears to be sufficiently strong, producing a rather large mixing angle: $\theta \sim (33 \pm 4)^\circ$, so that the LWs of $\psi(4415)$ and the missing resonance $\psi(4510)$ have almost equal LWs: $\Gamma_{ee}(\psi(4415)) = (0.66 \pm 0.06)$ keV and $\Gamma_{ee}(\psi(4510)) = (0.68 \pm 0.14)$ keV.

| state   | $E^0$ | $M_R$ | $\theta_i$ | $Z_i$      | $\Gamma_{ee}$ | exp. |
|---------|-------|-------|------------|-----------|---------------|------|
| $\psi(4415)$ | 4467  | $4421^{+7}_{-8}$ | $(33 \pm 4)^\circ$ | $0.83^{+0.06}_{-0.03}$ | $0.66^{+0.06}_{-0.05}$ | $0.58 \pm 0.07$ |
| $\psi(4510)$ | 4527  | $4512 \pm 2$     | $(34 \pm 3)^\circ$ | $0.85^{+0.08}_{-0.06}$ | $0.68 \pm 0.14$ | -    |
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