On a nonstandard two-parametric quantum algebra and its connections with $U_{p,q}(gl(2))$ and $U_{p,q}(gl(1|1))$

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Abstract. A quantum algebra $U_{p,q}(\zeta, H, X_{\pm})$ associated with a nonstandard $R$-matrix with two deformation parameters$(p, q)$ is studied and, in particular, its universal $R$-matrix is derived using Reshetikhin’s method. Explicit construction of the $(p, q)$-dependent nonstandard $R$-matrix is obtained through a coloured generalized boson realization of the universal $R$-matrix of the standard $U_{p,q}(gl(2))$ corresponding to a nongeneric case. General finite dimensional coloured representation of the universal $R$-matrix of $U_{p,q}(gl(2))$ is also derived. This representation, in nongeneric cases, becomes a source for various $(p, q)$-dependent nonstandard $R$-matrices. Superization of $U_{p,q}(\zeta, H, X_{\pm})$ leads to the super-Hopf algebra $U_{p,q}(gl(1|1))$. A contraction procedure then yields a $(p, q)$-deformed super-Heisenberg algebra $U_{p,q}(sh(1))$ and its universal $R$-matrix.

1. Introduction

The single parameter quantization of the universal enveloping algebra of a simple Lie algebra is well-known [1]. The Yang-Baxter equation (YBE), however, also admits nonstandard solutions [2-4] characterizing quasitriangular
Hopf algebras, which are not deformations of classical algebras. The nonstandard quantum algebra $U_q(\zeta, H, X_{\pm})$ associated with the Alexander-Conway solution of the YBE has been studied [2,3] and the relevant universal $R$-matrix has been obtained [4]. Using transmutation theory [5] it has been argued [4] that the superized $U_q(\zeta, H, X_{\pm})$ coincides with a super-Hopf algebra $U_q(gl(1|1))$. Moreover, using a general $q$-boson [6] realization of the Hopf algebra $U_q(gl(2))$, it has been observed [7] that the nonstandard $R$-matrix of $U_q(\zeta, H, X_{\pm})$ may be obtained from the universal $R$-matrix of $U_q(gl(2))$ in a nongeneric limit.

In another development, the constructions and representations of quantum algebras with multiple deformation parameters have been studied extensively [8-15]. For quasitriangular Hopf algebras, Reshetikhin [9] has developed a general formalism to introduce multiple deformation parameters. Following [9] the universal $R$-matrix of the quantum algebra $U_{p,q}(gl(2))$ with two independent parameters $(p, q)$ has been obtained [15]. An identical procedure may be adopted to construct the universal $R$-matrix of the super-Hopf algebra $U_{p,q}(gl(1|1))$; this verifies the known result obtained by direct computation [13]. Here, we study a $(p, q)$-generalization, $U_{p,q}(\zeta, H, X_{\pm})$, of the Alexander-Conway algebra $U_q(\zeta, H, X_{\pm})$.

Following the prescription in [9] the universal $R$-matrix for the quasitriangular Hopf algebra $U_{p,q}(\zeta, H, X_{\pm})$ is obtained in section 2. The nonstandard Hopf algebra $U_{p,q}(\zeta, H, X_{\pm})$ has been previously considered in [16]. These authors, however, have not discussed the universal $R$-matrix for $U_{p,q}(\zeta, H, X_{\pm})$. Parallel to its single deformation parameter analogue, $U_{p,q}(\zeta, H, X_{\pm})$ exhibits close kinships with $U_{p,q}(gl(2))$ and $U_{p,q}(gl(1|1))$. In particular, the nonstandard $R$-matrix of $U_{p,q}(\zeta, H, X_{\pm})$ is obtained in section 3 using a coloured generalized boson representation of the universal $R$-matrix of $U_{p,q}(gl(2))$ in a nongeneric limit. A general recipe for realizing the finite dimensional nonstandard two-parametric coloured $R$-matrices associated with nongeneric representations of $U_{p,q}(gl(2))$ is also presented. In section 4, a map connect-
ing the two-parametric quantum algebras $U_{p,q}(\zeta, H, X_{\pm})$ and $U_{p,q}(gl(1|1))$ via the superization procedure is described. A contraction procedure is used in section 5 to extract a $(p,q)$-deformed quasitriangular super-Heisenberg algebra $U_{p,q}(sh(1))$. We conclude in section 6.

2. Quasitriangular Hopf algebra $U_{p,q}(\zeta, H, X_{\pm})$

We study the Hopf algebra associated with the two-parametric nonstandard solution [16] of the YBE

\[
R = \begin{pmatrix}
Q^{-1} & 0 & 0 & 0 \\
0 & \lambda^{-1} & 0 & 0 \\
0 & \sigma & \lambda & 0 \\
0 & 0 & 0 & -Q
\end{pmatrix}, \quad \sigma = Q^{-1} - Q. \tag{2.1}
\]

The defining relation of the quantum inverse scattering method [17]

\[
R(T \otimes \mathbb{I})(\mathbb{I} \otimes T) = (\mathbb{I} \otimes T)(T \otimes \mathbb{I})R, \tag{2.2}
\]

with the $R$-matrix as given by (2.1), describes a transfer matrix

\[
T = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}, \tag{2.3}
\]

whose elements obey the braiding relations

\[
ab = p^{-1}ba, \quad ac = q^{-1}ca, \quad db = -p^{-1}bd, \quad dc = -q^{-1}cd, \\
p^{-1}bc = q^{-1}cb, \quad ad - da = (p^{-1} - q)bc, \quad b^2 = 0, \quad c^2 = 0, \tag{2.4}
\]

where

\[
p = \lambda Q, \quad q = \lambda^{-1}Q. \tag{2.5}
\]

A conjugate $R$-matrix
\[
\tilde{R} = (R^{(+)})^{-1} = \begin{pmatrix}
Q & 0 & 0 & 0 \\
0 & \lambda^{-1} & -\sigma & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & -Q^{-1}
\end{pmatrix}
\]  \hspace{1cm} (2.6)

also fits (2.2) with the elements of \(T\) obeying (2.4). The matrices \(R^{(\pm)}\) are defined by

\[ R^{(+)} = P R P, \quad R^{(-)} = R^{-1}, \] \hspace{1cm} (2.7)

where \(P\) is the permutation matrix, given by

\[ P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \] \hspace{1cm} (2.8)

If \(a\) and \(d\) are invertable, the elements \(\{a, b, c, d, a^{-1}, d^{-1}\}\) generate a Hopf algebra \(A_{p,q}(R)\) (or \(Fun_{p,q}(R)\)) whose coalgebraic structure readily follows. The coproduct(\(\Delta\)), counit(\(\epsilon\)) and antipode(\(S\)) maps are, respectively, given by

\[ \Delta(T) = T \otimes T, \quad \Delta(a^{-1}) = a^{-1} \otimes a^{-1} - a^{-1}ba^{-1} \otimes a^{-1}ca^{-1}, \quad \Delta(d^{-1}) = d^{-1} \otimes d^{-1} - d^{-1}cd^{-1} \otimes d^{-1}bd^{-1}, \] \hspace{1cm} (2.9)

\[ \epsilon(T) = 1, \] \hspace{1cm} (2.10)

\[ S(T) = T^{-1}, \quad S(a^{-1}) = a - bd^{-1}c, \quad S(d^{-1}) = d - ca^{-1}b, \] \hspace{1cm} (2.11)

where
\[ T^{-1} = \begin{pmatrix}
  a^{-1} + a^{-1}bd^{-1}ca^{-1} & -a^{-1}bd^{-1} \\
  -d^{-1}ca^{-1} & d^{-1} + d^{-1}ca^{-1}bd^{-1}
\end{pmatrix} \quad (2.12) \]

and \( \otimes \) denotes the tensor product coupled with usual matrix multiplication.

In the Hopf algebra \( A_{p,q}(R) \) an invertible ‘group-like’ element \( D \) exists:

\[ D = ad^{-1} - bd^{-1}cd^{-1}, \quad D^{-1} = da^{-1} - ba^{-1}ca^{-1}. \quad (2.13) \]

Using (2.4), the commutation relations for \( D \) follow:

\[ [D, a] = 0, \quad [D, d] = 0, \quad \{D, b\} = 0, \quad \{D, c\} = 0. \quad (2.14) \]

The induced coalgebra maps for \( D \) are obtained from the relations (2.9-2.12):

\[ \Delta(D) = D \otimes D, \quad \epsilon(D) = 1, \quad S(D) = D^{-1}. \quad (2.15) \]

Using the FRT-approach \[17\], the commutation relations for the generators \((\zeta, H, X_{\pm})\) of the Hopf algebra \( U_{p,q}(\zeta, H, X_{\pm}) \), dual to the algebra \( A_{p,q}(R) \), are obtained from the relations

\[ R^{(+)} (L^{(\varepsilon_1)} \otimes \mathbb{1}) (\mathbb{1} \otimes L^{(\varepsilon_2)}) = (\mathbb{1} \otimes L^{(\varepsilon_2)})(L^{(\varepsilon_1)} \otimes \mathbb{1}) R^{(+)}, \quad (2.16) \]

where \((\varepsilon_1, \varepsilon_2) = (+, +), (-, -), (+, -)\) and

\[ L^{(+)} = \begin{pmatrix}
  p^{-H}q^{-\zeta} & \sigma p^{-H-\frac{H}{2}} X_- \\
  0 & gp^{-H}q^{\zeta}
\end{pmatrix}, \quad L^{(-)} = \begin{pmatrix}
  q^H p^{\zeta} & 0 \\
  -\sigma gq^H - \frac{H}{2} X_+ & gq^H p^{-\zeta}
\end{pmatrix}, \quad (2.17) \]

with \( g = (-1)^{\zeta-H} \). The corresponding commutation relations read

\[ X_{\pm}^2 = 0, \quad [H, X_{\pm}] = \pm X_{\pm}, \quad \{X_+, X_-\} = [2\zeta], \quad [\zeta, X] = 0 \quad \forall X = H, X_{\pm}. \quad (2.18) \]
where

\[ [X] = \frac{Q^X - Q^{-X}}{Q - Q^{-1}}. \]  \hspace{1cm} (2.19)

The comultiplication maps for the generators are

\[
\Delta(X_{\pm}) = X_{\pm} \otimes g^{\mp 1}Q^\zeta \lambda^{\pm} + Q^{-\zeta} \lambda^{\mp} \otimes X_{\pm}, \\
\Delta(H) = H \otimes 1 \otimes H, \\
\Delta(\zeta) = \zeta \otimes 1 \otimes \zeta, \] \hspace{1cm} (2.20)

which follow from the relations

\[
\Delta(L_{\pm}) = L_{\pm} \otimes L_{\pm}. \] \hspace{1cm} (2.21)

The counit and the antipode maps are given by

\[
\epsilon(X) = 0, \quad \forall X = \zeta, H, X_{\pm}, \] \hspace{1cm} (2.22)

\[
S(\zeta) = -\zeta, \quad S(H) = -H, \quad S(X_{\pm}) = g^{\pm 1}X_{\pm}. \] \hspace{1cm} (2.23)

In spite of the appearance of anticommutator in (2.18), the Hopf algebra \( U_{p,q}(\zeta, H, X_{\pm}) \) is bosonic as it follows the direct product rule

\[
(A \otimes B)(C \otimes D) = AC \otimes BD, \quad \forall A, B, C, D \in U_{p,q}(\zeta, H, X_{\pm}). \] \hspace{1cm} (2.24)

For the finite dimensional faithful representation [3] of the algebra (2.18)

\[
X_+ = \begin{pmatrix} 0 & [z] \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\zeta = \frac{z}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] \hspace{1cm} (2.25)
with \( z \in \mathbb{C} \), the duality relations between the Hopf algebras \( A_{p,q}(R) \) and \( U_{p,q}(\zeta, H, X_{\pm}) \) assume the form

\[
\langle L^{(\pm)}, T \rangle = R^{(\pm)},
\] (2.26)
for the choice \( z = 1 \).

We now discuss the quasitriangular character of \( U_{p,q}(\zeta, H, X_{\pm}) \) containing a group-like element \( g \), that is, now and henceforth, assumed to satisfy \( g^2 = 1 \). For a quasitriangular Hopf algebra \( U \), the universal \( \mathcal{R} \)-matrix \((\in U \otimes U)\) satisfies the relations

\[
\tau \circ \Delta(X) = \mathcal{R}\Delta(X)\mathcal{R}^{-1}, \quad \tau \circ (X \otimes Y) = Y \otimes X, \quad \forall X, Y \in U,
\]
(\(\Delta \otimes \text{id}\))\(\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}\), \(\text{id} \otimes \Delta)\(\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}\),
\[
(\epsilon \otimes \text{id}) = (\text{id} \otimes \epsilon)\mathcal{R} = \mathbb{I},
\]
(\(S \otimes \text{id}\))\(\mathcal{R} = \mathcal{R}^{-1}, \quad (\text{id} \otimes S)\mathcal{R}^{-1} = \mathcal{R},
\] (2.27)
where the subscripts in \( \mathcal{R}_{ij} \) indicate the embedding of \( \mathcal{R} \) in \( U^{\otimes 3} \). The explicit expression for the universal \( \mathcal{R} \)-matrix of \( U_Q(\zeta, H, X_{\pm}) \) is [4]

\[
\mathcal{R}_{\lambda=1} = (-1)^{(\zeta-H)\otimes(\zeta-H)}Q^{2(\zeta \otimes H + H \otimes \zeta)} \left( \mathbb{I} \otimes \mathbb{I} + \sigma Q^\zeta X_{\pm} \otimes \sigma g Q^{-\zeta} X_{\mp} \right).
\] (2.28)

Employing the Reshetikhin procedure [9] we obtain the universal \( \mathcal{R} \)-matrix for \( U_{p,q}(\zeta, H, X_{\pm}) \). To this end, we note that the coproduct relations (2.20) for the generators of \( U_{p,q}(\zeta, H, X_{\pm}) \) and the corresponding relations for the generators of \( U_Q(\zeta, H, X_{\pm}) \), obtained in the limit \( \lambda = 1 \), are related by a similarity transformation

\[
\Delta(X) = F(\Delta_{\lambda=1}(X)) F^{-1}, \quad \forall X = \zeta, H, X_{\pm},
\] (2.29)
where
\[ F = \lambda^{(H \otimes \zeta - \zeta \otimes H)}. \]  

(2.30)

Then, the procedure in [9] yields the universal \( R \)-matrix for \( U_{p,q}(\zeta, H, X_\pm) \)

\[ R = F^{-1}R_{\lambda=1}F^{-1} \]  

(2.31)

which, by construction, satisfies the required relations (2.27). Explicitly, the universal \( R \)-matrix of \( U_{p,q}(\zeta, H, X_\pm) \) reads

\[ R = (-1)^{(\zeta - H) \otimes (\zeta - H)} Q^{2(\zeta \otimes H + H \otimes \zeta)} \lambda^{2(\zeta \otimes H - H \otimes \zeta)} \times \left( \mathbb{1} \otimes \mathbb{1} + \sigma Q^\zeta \lambda^{\xi} X_+ \otimes gQ^{-\zeta} \lambda^{\xi} X_- \right). \]  

(2.32)

For the representation (2.25) with the choice \( z = 1 \), \( R \) in (2.32) reduces to the matrix \( \tilde{R} \) in (2.6), thus expressing an aspect of the duality of \( A_{p,q}(R) \) and \( U_{p,q}(\zeta, H, X_\pm) \).

3. Coloured \( R \)-matrices associated with nongeneric representations of \( U_{p,q}(gl(2)) \)

After completing the above construction of the dual Hopf algebras corresponding to the nonstandard solution (2.1) of the YBE, we now relate the \( R \)-matrix (2.1) to the representations of \( U_{p,q}(gl(2)) \) for nongeneric values of \( Q \), namely, roots of unity. Here, we closely follow the treatment of \( U_q(gl(2)) \) by Ge et al. [7]. These authors have constructed a general parameter dependent finite dimensional \( q \)-boson realization of the universal \( R \)-matrix of \( U_q(gl(2)) \); thereby generating the nonstandard \( R \)-matrices at \( q \), a root of unity. When different parameters are chosen for different components of the representation module, the resulting finite dimensional \( R \)-matrices are said to be coloured and they obey a coloured YBE [7].

In the standard Hopf algebraic structure of \( U_{p,q}(gl(2)) \) (see, e.g., [15]) the commutation relations between the generators are
\[ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0], \quad [Z, X] = 0, \quad \forall \ X \in (J_0, J_\pm), \quad (3.1) \]

and the comultiplication rules are

\[
\Delta (X_\pm) = J_\pm \otimes Q^{J_0} \lambda^{\pm Z} + Q^{-J_0} \lambda^{\mp Z} \otimes J_\pm, \\
\Delta (J_0) = J_0 \otimes 1 + 1 \otimes J_0, \quad \Delta (Z) = Z \otimes 1 + 1 \otimes Z. \quad (3.2)
\]

The universal \( \mathcal{R} \)-matrix of \( U_{p,q}(gl(2)) \) reads \[15\]

\[
\mathcal{R} = Q^{2(J_0 \otimes J_0)} \lambda^{2(Z \otimes J_0 - J_0 \otimes Z)} \times \sum_{n=0}^{\infty} \frac{(1 - Q^{-2})^n}{[n]!} Q^{\frac{1}{2}n(n-1)} \left( Q^{J_0} \lambda^Z J_+ \otimes Q^{-J_0} \lambda^Z J_- \right)^n, \quad (3.3)
\]

where \([n]! = [n][n-1] \ldots [2][1]\).

The generators of the deformed boson algebra \[6\] satisfy

\[
[N, a_\pm] = \pm a_\pm, \quad a_+ a_- = [N], \quad a_- a_+ = [N + 1], \quad (3.4)
\]

and the map \[7\]

\[
J_+ = a_+ [\omega - N], \quad J_- = a_-, \quad J_0 = N - \frac{\omega}{2}, \quad \omega \in \mathbb{C}, \quad (3.5)
\]

provides an infinite dimensional representation of the algebra \((3.1)\) in the Fock space \( \mathcal{F}_z \{ |m z\rangle = a_+^m |0 z\rangle \mid a_- |0 z\rangle = 0, N |0 z\rangle = 0, m \in \mathbb{Z}^+, z \in \mathbb{C} \} \):  

\[
J_+ |m z\rangle = [\omega - m] |m + 1 z\rangle, \quad J_- |m z\rangle = [m] |m - 1 z\rangle, \\
J_0 |m z\rangle = \left( m - \frac{\omega}{2} \right) |m z\rangle, \quad Z |m z\rangle = z |m z\rangle. \quad (3.6)
\]
The parameter \( \omega \) is called colour and provides the key to obtain the non-standard \( R \)-matrix (2.1) starting from the universal \( R \)-matrix of \( U_{p,q}(gl(2)) \), namely \( \tilde{R} \) in (3.3).

For the nongeneric cases \( \{ Q^n = \pm 1, n \in \mathbb{Z} \} \) the identity \([\alpha n] = 0\) holds for \( \alpha \in \mathbb{Z}^+ \) suggesting the existence of an extremal vector \( \{|\alpha n \rangle | \langle \alpha n | = 0\} \). The corresponding invariant subspace that renders the representation (3.6) reducible is \( V_{\alpha z} \{ |\alpha n + m \rangle | m \in \mathbb{Z}^+ \} \). Then, on the quotient space \( V_{Jz} = \mathcal{F}_z/V_{\alpha z} \{ |JM \rangle (= |m \rangle) | m = 0, 1, \ldots, (\alpha n - 1) = 2J, M = m - J \} \) a finite dimensional representation for algebra (3.1) holds:

\[
\begin{align*}
J_+|JM\rangle &= [\omega - J - M]|JM + 1\rangle \theta(J - 1 - M), \\
J_-|JM\rangle &= [J + M]|JM - 1\rangle, \\
J_0|JM\rangle &= (J + M - \omega)|JM\rangle, \\
Z|JM\rangle &= z|JM\rangle,
\end{align*}
\]

where \( \theta(x) = 1 (0) \) for \( x \geq 0 (< 0) \). For \( \alpha = 1 \), the representation (3.7) is irreducible and for \( \alpha \geq 2 \), it is indecomposable [7]. Now, using the representation (3.7), \( \tilde{R} \) in (3.3) may be written in the matrix form

\[
\tilde{R} |J_1 M_1 z_1\rangle \otimes |J_2 M_2 z_2\rangle = \sum_{M_1' M_2'} (\tilde{R}^{J_1 z_1 \omega_1, J_2 z_2 \omega_2})^{M_1' M_2'}_{M_1 M_2} |J_1 M_1' z_1\rangle \otimes |J_2 M_2' z_2\rangle
\]

(3.8)

where different representations and colour parameters are chosen in the two sectors of the tensor product space. Explicitly we have

\[
(\tilde{R}^{J_1 z_1 \omega_1, J_2 z_2 \omega_2})^{M_1' M_2'}_{M_1 M_2} = Q^{2(J_1+M_1'-\frac{\omega_1}{2})} (J_2+M_2'-\frac{\omega_2}{2}) \lambda^2 (z_1(J_2+M_2'-\frac{\omega_2}{2})-z_2(J_1+M_1'-\frac{\omega_1}{2})) \sum_{n=0}^{\infty} \frac{(1-Q^{-2})^n}{[n]!} Q^{-\frac{1}{2} n(n-1)} Q^n (J_1-J_2+M_1'-M_2'-\frac{\omega}{2}(\omega_1-\omega_2)) \lambda^n(z_1+z_2)
\]

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\[\Pi_{l=1}^{n} [\omega_1 - J_1 - M_1' + l] [J_2 + M_2' + l] \delta^{M_1'}_{M_1+n} \delta^{M_2'}_{M_2-n} \]  

(3.9)

It should be noted that the presence of the second deformation parameter \( \lambda \) in (3.9) will produce many new \( R \)-matrices. For nongeneric values of \( Q \), the matrix representation (3.9) acts as a source for obtaining the finite dimensional two parametric \((p, q)\) nonstandard coloured \( R \)-matrices. Let us consider the simplest example \((J_1 = J_2 = \frac{1}{2}, z_1 = z_2 = z)\) with different colour parameters in the two sectors of the Hilbert space. For \( Q^2 = -1 \), we get

\[
\bar{R}_{\frac{1}{2}z, \frac{1}{2}z}^{\omega_1, \omega_2} = c \begin{pmatrix} t_1 & 0 & 0 & 0 \\
0 & s^{-1} & w & 0 \\
0 & 0 & st_1t_2^{-1} & 0 \\
0 & 0 & 0 & -t_2^{-1} \end{pmatrix},
\]

(3.10)

where \( c = Q^{-\omega_2(1-\frac{\lambda}{\omega_2})} \lambda^{\omega_1-\omega_2} \), \( t_1 = -Q^{-\omega_1} \), \( t_2 = -Q^{-\omega_2} \), \( s = \lambda^{2z} \) and \( w = \left(t_1 - t_1^{-1}\right)t_1^2t_2^{-2} \). This is an example of coloured \( R \)-matrix and may be viewed as a generalization of the result obtained in [7]. When \( \omega_1 = \omega_2 = \omega \) the \( R \)-matrix (3.10) reduces to

\[
\bar{R}_{\frac{1}{2}z, \frac{1}{2}z}^{\omega, \omega} \sim \begin{pmatrix} t & 0 & 0 & 0 \\
0 & s^{-1} & t - t^{-1} & 0 \\
0 & 0 & s & 0 \\
0 & 0 & 0 & -t^{-1} \end{pmatrix}.
\]

(3.11)

Apart from a scale factor, the matrix in (3.11) agrees with the nonstandard \( R \)-matrix \( \bar{R} \) in (2.6) after the replacement \( Q \rightarrow t \), \( \lambda \rightarrow s \). This completes our discussion of the connection between the nonstandard two-parameter \( R \)-matrix (2.1) with the universal \( R \)-matrix of \( U_{p,q}(gl(2)) \).

4. Superization of \( U_{p,q}(\zeta, H, X_{\pm}) \)

Using a superization procedure [4] we now discuss the connection between \( U_{p,q}(\zeta, H, X_{\pm}) \) and the super-Hopf algebra \( U_{p,q}(gl(1|1)) \). In [4] it is argued that if \( \mathcal{H} \) is a Hopf algebra containing a group-like element \( g \) such that \( g^2 \)
1, then, there exists a super-Hopf algebra \( \hat{H} \) with identical algebraic and counit structures while the coproduct, antipode and the universal \( R \)-matrix of \( H \), \( \{ \Delta(h) \ (= \sum_k x_k \otimes y_k), S(h), R \ (= \sum_k X_k \otimes Y_k) \mid (h, x_k, y_k, X_k, Y_k) \in H \} \), are modified to the corresponding quantities of \( \hat{H} \):

\[
\Delta(\hat{h}) = \sum_k x_k \otimes g^{\text{deg}(x_k)} y_k, \quad S(\hat{h}) = S(h)g^{\text{deg}(h)},
\]

\[
\hat{R} = R_g \sum_k X_k \otimes g^{\text{deg}(X_k)} Y_k,
\]

\[
R_g = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g).
\]

The map \( \{ h \mapsto \hat{h} \mid h \in H, \hat{h} \in \hat{H} \} \) preserves the algebraic structure and it is understood that, in the right hand side of (4.1) the elements of \( H \), after simplification, are mapped to their hatted superpartners in \( \hat{H} \). The \( \text{deg}(h) \), \( \forall h \in H \), is given by \( ghg^{-1} = \text{deg}(h)h \). We exhibit these properties in \((p, q)\)-deformed examples.

The \( R \)-matrix

\[
\hat{R} = \begin{pmatrix}
Q^{-1} & 0 & 0 & 0 \\
0 & \lambda^{-1} & 0 & 0 \\
0 & \sigma & \lambda & 0 \\
0 & 0 & 0 & Q
\end{pmatrix}
\]

satisfies the super-YBE and is known [11] to describe the ‘function’ Hopf algebra \( \text{Fun}_{p,q}(GL(1\mid 1)) \) (or \( GL_{p,q}(1\mid 1) \)). The universal \( R \)-matrix \( \hat{R} \) for the dually paired enveloping algebra \( U_{p,q}(gl(1\mid 1)) \) [12] has been obtained by direct computation [13]. We can derive this \( \hat{R} \) by applying Reshetikhin’s technique [9] for introducing multiple deformation parameters as follows. The commutation relations for the generators \( (\hat{\zeta}, \hat{H}, \hat{X}_\pm) \) of \( U_{p,q}(gl(1\mid 1)) \) read

\[
\hat{X}_\pm^2 = 0, \quad [\hat{H}, \hat{X}_\pm] = \pm \hat{X}_\pm, \quad \{ \hat{X}_+, \hat{X}_- \} = [2\hat{\zeta}^2],
\]

\[
[\hat{\zeta}, X] = 0, \quad \forall X \in (\hat{H}, \hat{X}_\pm),
\]

(4.3)
and the coalgebraic structure is

\[
\Delta \left( \hat{X}_\pm \right) = \hat{X}_\pm \otimes Q^\pm \lambda^\pm \hat{\zeta} \pm Q^{-\hat{\zeta}} \lambda^{\mp} \hat{\zeta} \otimes \hat{X}_\pm,
\]

\[
\Delta \left( \hat{H} \right) = \hat{H} \otimes 1 + 1 \otimes \hat{H}, \quad \Delta \left( \hat{\zeta} \right) = \hat{\zeta} \otimes 1 + 1 \otimes \hat{\zeta},
\]

\[
\epsilon \left( X \right) = 0, \quad S \left( X \right) = -X, \quad \forall X \in \left( \hat{\zeta}, \hat{H}, \hat{X}_\pm \right).
\] (4.4)

The coproduct relations (4.4) reveal that a structure similar to (2.29) holds with the choice

\[
\hat{F} = \lambda^{\left( \hat{H} \otimes \hat{\zeta} - \hat{\zeta} \otimes \hat{H} \right)}.
\] (4.5)

Now, using the known universal \( R \)-matrix of \( U_Q(gl(1|1)) \) \cite{1}

\[
\hat{R}_{\lambda=1} = Q^{2(\hat{\zeta} \otimes \hat{H} + \hat{H} \otimes \hat{\zeta})} \left( 1 \otimes 1 + \sigma Q^\hat{\zeta} \hat{X}_+ \otimes Q^{-\hat{\zeta}} \hat{X}_- \right)
\] (4.6)

and the prescription (2.31), we obtain the universal \( R \)-matrix for \( U_{p,q}(gl(1|1)) \) as

\[
\hat{R} = Q^{2(\hat{\zeta} \otimes \hat{H} + \hat{H} \otimes \hat{\zeta})} \lambda^{2(\hat{\zeta} \otimes \hat{H} - \hat{H} \otimes \hat{\zeta})} \left( 1 \otimes 1 + \sigma Q^\hat{\zeta} \lambda^\hat{\zeta} \hat{X}_+ \otimes Q^{-\hat{\zeta}} \lambda^{\hat{\zeta}} \hat{X}_- \right)
\] (4.7)

which satisfies the super-YBE with the composition rule for the graded operators \( (A, B, C, D) \in U_{p,q}(gl(1|1)) \)

\[
(A \otimes B)(C \otimes D) = (-1)^{\deg(B)\deg(C)}(AC \otimes BD).
\] (4.8)

Our expression for \( \hat{R} \) in (4.7) is seen to agree with the result obtained in \cite{13} by direct computation.

Comparing (2.18) and (2.22) with (4.3) and (4.4), while introducing the map \( (\zeta, H, X_\pm) \to (\hat{\zeta}, \hat{H}, \hat{X}_\pm) \), it follows that the commutation relations and the counit maps for the super-Hopf algebra \( U_{p,q}(gl(1|1)) \) are identical to those of \( U_{p,q}(\zeta, H, X_\pm) \). The coproduct (4.4), antipode (4.4) and the universal
\( R \)-matrix (4.7) for \( U_{p,q}(gl(1|1)) \) follow from the analogous formulae, (2.20), (2.23) and (2.32), for \( U_{p,q}(\hat{\zeta}, \hat{H}, \hat{X}_\pm) \) according to the prescription (4.1). The quantity \((-1)^{(\hat{\zeta}-\hat{H})\otimes(\hat{\zeta}-\hat{H})}R_g \) is central in nature as may be observed by direct computation. It may, therefore, be dropped while mapping \( R \) to \( \hat{R} \).

5. A \((p,q)\)-deformed super-Heisenberg algebra

Finally, we use the contraction technique à la Celeghini et al. [18,19] to extract a two-parametric deformed super-Heisenberg algebra \( U_{p,q}(sh(1)) \) as a limiting case of \( U_{p,q}(gl(1|1)) \). To this end, let us scale the generators and the quantization parameters as

\[
\hat{H} = \frac{1}{2\varepsilon}h - N, \quad \hat{\zeta} = \frac{\xi}{2\varepsilon}, \quad \hat{X}_\pm = \frac{1}{\sqrt{\varepsilon}}c_\pm, \quad Q = e^{\varepsilon\Omega}, \quad \lambda = e^{\varepsilon\nu}. \tag{5.1}
\]

The commutation relations and the coproduct rules for \( U_{p,q}(sh(1)) \), with \( \{\xi, h, N, c_\pm\} \) as the generators, are obtained by studying the \( \varepsilon \to 0 \) limit of the corresponding structures of the algebra \( U_{p,q}(gl(1|1)) \) in (4.3) and (4.4) respectively. The result is

\[
c_\pm^2 = 0, \quad [N, c_\pm] = \pm c_\pm, \quad \{c_+, c_-\} = \Omega^{-1}\sinh \Omega \xi,
\]

\[
[h, X] = 0, \quad [\xi, X] = 0, \quad \forall X \in U_{p,q}(sh(1)), \tag{5.2}
\]

and

\[
\Delta(c_\pm) = c_\pm \otimes e^{\frac{\Omega}{2}(\Omega \mp \nu)\xi} + e^{-\frac{\Omega}{2}(\Omega \mp \nu)\xi} \otimes c_\pm,
\]

\[
\Delta(\xi) = \xi \otimes \mathbb{I} + \mathbb{I} \otimes \xi, \quad \Delta(h) = h \otimes \mathbb{I} + \mathbb{I} \otimes h,
\]

\[
\Delta(N) = N \otimes \mathbb{I} + \mathbb{I} \otimes N. \tag{5.3}
\]

In the contraction limit, the universal \( R \)-matrix (4.7) of \( U_{p,q}(gl(1|1)) \) yields, after the removal of a constant singular factor, the universal \( R \)-matrix of \( U_{p,q}(sh(1)) \):
\[ R^{sh} = e^{\nu(N \otimes \xi - \xi \otimes N) - \Omega(N \otimes \xi + \xi \otimes N)} \left( \mathbb{1} \otimes \mathbb{1} - 2\Omega e^{\frac{1}{2}(\Omega + \nu)\xi} c_- \otimes e^{-\frac{1}{2}(\Omega - \nu)\xi} c_+ \right) . \tag{5.4} \]

A spectral parameter dependent \( R \)-matrix may be obtained from the universal \( R \)-matrix (5.4). To this end, following [18], we define the map

\[
T_x c_\pm = x^{\mp 1} c_\pm , \quad T_x h = h , \quad T_x \xi = \xi , \quad T_x N = N , \tag{5.5}
\]

and let

\[
R^{sh}(x) = (T_x \otimes \mathbb{1}) R^{sh} = e^{\nu(N \otimes \xi - \xi \otimes N) - \Omega(N \otimes \xi + \xi \otimes N)} \times \left( \mathbb{1} \otimes \mathbb{1} - 2\Omega x e^{\frac{1}{2}(\Omega + \nu)\xi} c_- \otimes e^{-\frac{1}{2}(\Omega - \nu)\xi} c_+ \right) . \tag{5.6}
\]

A direct calculation then proves that the matrix \( R^{sh}(x) \) satisfies the spectral parameter dependent YBE

\[
R^{sh}_{12}(x) R^{sh}_{13}(xy) R^{sh}_{23}(y) = R^{sh}_{23}(y) R^{sh}_{13}(xy) R^{sh}_{12}(x) . \tag{5.7}
\]

6. Conclusion

To conclude, we have studied the dually paired Hopf algebras \( A_{p,q}(R) \) and \( U_{p,q}(\zeta, H, X_\pm) \) associated with the nonstandard \( R \)-matrix (2.1) involving two independent parameters \( p \) and \( q \). The universal \( R \)-matrix of \( U_{p,q}(\zeta, H, X_\pm) \) has been obtained. We have demonstrated an explicit construction of the nonstandard \( R \)-matrix through a coloured generalized boson representation of the universal \( R \)-matrix of \( U_{p,q}(gl(2)) \) corresponding to the nongeneric case \( pq = Q^2 = -1 \). In this example, by choosing different colour parameters for the two sectors of nongeneric representations of \( U_{p,q}(gl(2)) \) with a dimension 2, we have obtained a two-parametric coloured \( R \)-matrix (3.10), which
satisfies a coloured YBE. More importantly, the finite dimensional representation (3.9) of the $\mathcal{R}$-matrix gives a recipe for obtaining nonstandard two-parametric coloured $R$-matrices for nongeneric values of $Q$. Superization process describes a map between $U_{p,q}(\zeta, H, X_\pm)$ and $U_{p,q}(gl(1|1))$. A contraction procedure has been used to obtain a $(p, q)$-deformed quasitriangular super-Heisenberg algebra $U_{p,q}(sh(1))$.

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