JACOB’S LADDERS, RIEMANN’S OSCILLATORS, QUOTIENT OF TWO OSCILLATING MULTIFORMS AND SET OF METAMORPHOSES OF THIS SYSTEM

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Abstract. In this paper we introduce complicated oscillating system, namely quotient of two multiforms based on Riemann-Siegel formula . We prove that there is an infinite set of metamorphoses of this system (=chrysalis) on critical line \( \sigma = \frac{1}{2} \) into a butterfly (=infinite series of Möbius functions in the region of absolute convergence \( \sigma > 1 \)).

To memory of the Hardy’s Pure Mathematics

1. Introduction

1.1. Let us remind the Riemann-Siegel formula

\[
Z(t) = 2 \sum_{n \leq \tau(t)} \frac{1}{\sqrt{n}} \cos \left( \vartheta(t) - t \ln n \right) + \mathcal{O}(t^{-1/4}),
\]

(1.1)

\[
Z(t) = e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right), \quad \tau(t) = \sqrt{\frac{t}{2\pi}},
\]

(see [7], p. 60, comp. [8], p. 79), where

\[
\vartheta(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma \left( \frac{1}{4} + \frac{i}{2} \right),
\]

(see [8], p. 239). Next, we have defined in our paper [6] the following multiform

\[
G(x_1, \ldots, x_k) = \prod_{r=1}^{k} |Z(x_r)| = \prod_{r=1}^{k} \left| \zeta \left( \frac{1}{2} + ix_r \right) \right|,
\]

(1.2)

\[x_r > T > 0, \quad k = 1, 2, \ldots, k_0, \quad k_0 \in \mathbb{N}\]

in connection with the Riemann-Siegel formula (1.1). Furthermore, we have defined the subset of the set of points

\[(x_1, \ldots, x_k) \in [T, +\infty)^k: \quad T < x_1 < x_2 < \cdots < x_k, \quad x_r \neq \gamma : \quad \zeta \left( \frac{1}{2} + i\gamma \right) = 0, \quad r = 1, \ldots, k,\]

and we have proposed the following

Question. Is there in this subset a point

\[(y_1, \ldots, y_k)\]

of metamorphosis of the multiform (1.2) that is also the point of significant change of the structure of the multiform (1.2)?

Key words and phrases. Riemann zeta-function.
1.2. We have obtained the following answer (see [6]): there is an infinite set of elements

\[
\{\alpha_0(T), \alpha_1(T), \ldots, \alpha_k(T)\}, \quad T \in (T_0, +\infty), \quad T_0 > 0,
\]

where \(T_0\) is sufficiently big, such that

\[
\prod_{r=1}^{k} \left| \sum_{n \leq \tau(\alpha_r)} \frac{2}{\sqrt{n}} \cos \{\theta(\alpha_r) - \alpha_r \ln n\} + O(\alpha_r^{-1/4}) \right| \sim \frac{\Lambda}{\sqrt{\sum_{n \leq \tau(\alpha_0)} \frac{2}{\sqrt{n}} \cos \{\theta(\alpha_0) - \alpha_0 \ln n\} + O(\alpha_0^{-1/4})}}
\]

\(T \to \infty,\)

where (see [6], (1.7))

\[
\Lambda = \sqrt{2\pi} \frac{\sqrt{\prod H_k}}{\ln k} T,
\]

i.e. to the infinite subset

\[
\{\alpha_1(T), \ldots, \alpha_k(T)\}, \quad T \in (T_0, +\infty)
\]

an infinite set of metamorphoses of the multiform (1.2) into quite distinct form on the right-hand side of (1.4) corresponds.

**Remark 1.** We shall call the elements of the set (1.3) as a control parameters (functions) of the metamorphosis. The reason to do so is that the parameters

\[
\{\alpha_1(T), \ldots, \alpha_k(T)\}
\]

the set (1.3) plays a similar role as the shem-ha-m’forash in Golem’s metamorphosis.

1.3. Let us notice the following: the set of all metamorphoses that is described by the formula (1.4) is connected with the critical line \(\sigma = \frac{1}{2}\). In this paper we introduce a new complicated oscillatory \(Q\)-system, namely the quotient of two multiforms of the type (1.2). Secondly, we obtain an infinite set of metamorphoses of this system into an infinite set of infinite series of the type

\[
\left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\sigma+n}} \right|, \quad \sigma > 1,
\]

where \(\mu(n)\) is the Möbius function. Consequently, we see that the old system (chrysalis), before metamorphosis, is defined on critical line \(\sigma = \frac{1}{2}\), and the new form (butterfly), after metamorphosis, is defined in the region \(\sigma > 1\). That is, mentioned sets are disconnected each to other (their distance reads 1/2).
2. **Theorem**

2.1. Now we introduce new complicated oscillatory system as a quotient of two multiforms of type (1.2) (the Q-system)

\[ G(x_1, \ldots, x_k; y_1, \ldots, y_k) = \prod_{r=1}^{k} \left| \frac{Z(x_r)}{Z(y_r)} \right| = \]

\[
= \prod_{r=1}^{k} \frac{\sum_{n \leq \tau(x_r)} \frac{2}{\sqrt{n}} \cos\{\vartheta(x_r) - x_r \ln n\} + R(x_r)}{\sum_{n \leq \tau(y_r)} \frac{2}{\sqrt{n}} \cos\{\vartheta(y_r) - y_r \ln n\} + R(y_r)} ,
\]

(2.1)

\((x_1, \ldots, x_k) \in M^1_k, \ (y_1, \ldots, y_k) \in M^2_k,\)

\(R(t) = O(t^{-1/4}),\)

\(k \leq k_0 \in \mathbb{N}\)

\((k_0 \text{ is an arbitrary and fixed number}), \)

\(M^1_k = \{(x_1, \ldots, x_k) \in (T_0, +\infty)^k : T_0 < x_1 < x_2 < \cdots < x_k,\)

\(x_r \neq \gamma : \zeta \left(\frac{1}{2} + i\gamma\right) = 0, \ r = 1, \ldots, k\},\)

(2.2)

\(M^2_k = \{(y_1, \ldots, y_k) \in (T_0, +\infty)^k : T_0 < y_1 < y_2 < \cdots < y_k,\)

\(y_r \neq \gamma : \zeta \left(\frac{1}{2} + i\gamma\right) = 0, \ r = 1, \ldots, k\}.\)

2.2. The following theorem holds true about the set of metamorphoses of the Q-system (2.1)

**Theorem.** Let

\[ [T, T+U] \rightarrow [\overbrace{T, T+U}^{1}, \ldots, \overbrace{T, T+U}^{k}] \]

where

\[ [\overbrace{T, T+U}^{r}], \ r = 1, \ldots, j, \ k \leq k_0\]

\[ U = U(T, \Theta) = \ln \ln T + \Theta \ln T, \ \Theta \in [0, 1]\]

be the \(r\)-th reversely iterated segment corresponding to the first segment in (2.3).

Let furthermore

\[ \sigma \in [1 + \epsilon, +\infty), \ \epsilon > 0\]

(2.4)

where \(\epsilon\) is sufficiently small and fixed number. Then there is a sufficiently big

\[ T_0 = T_0(\epsilon)\]

such that for every \(T > T_0\) and every admissible \(\sigma, \Theta, k, \epsilon\) there are functions

\(\alpha_r = \alpha_r(\sigma, T, \Theta, k, \epsilon), \ r = 0, 1, \ldots, k,\)

\(\alpha_r \neq \gamma : \zeta \left(\frac{1}{2} + i\gamma\right) = 0,\)

(2.5)

\(\beta_r = \beta_r(T, \Theta, k), \ r = 1, \ldots, k,\)

\(\beta_r \neq \gamma : \zeta \left(\frac{1}{2} + i\gamma\right) = 0,\)
such that
\[ \prod_{r=1}^{k} \left| \frac{\sum_{n \leq \tau(\alpha_r)} \frac{2}{\sqrt{n}} \cos \{ \vartheta(\alpha_r) - \alpha_r \ln n \} + R(\alpha_r)}{\sum_{n \leq \tau(\beta_r)} \frac{2}{\sqrt{n}} \cos \{ \vartheta(\beta_r) - \beta_r \ln n \} + R(\beta_r)} \right| \sim \]
\[ \sim \sqrt{\zeta(2\sigma)} \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{r+\eta}} \right|, \quad T \to \infty, \]
where \( \mu(n) \) is the Möbius function. Moreover, the sequences
\[ \{ \alpha_r \}_{r=0}^{k}, \quad \{ \beta_r \}_{r=1}^{k} \]
have the following properties
\[ T < \alpha_0 < \alpha_1 < \cdots < \alpha_k, \]
\[ T < \beta_1 < \beta_2 < \cdots < \beta_k, \]
\[ \alpha_0 \in (T, T+U), \quad \alpha_r, \beta_r \in (T, T+U), \quad r = 1, \ldots, k, \]
\[ \alpha_{r+1} - \alpha_r \sim (1-c)\pi(T), \quad r = 0, 1, \ldots, k-1, \]
\[ \beta_{r+1} - \beta_r \sim (1-c)\pi(T), \quad r = 1, \ldots, k-1, \]
where
\[ \pi(T) \sim \frac{T}{\ln T}, \quad T \to \infty \]
is the prime-counting function and \( c \) is the Euler’s constant.

2.3. Now, we give the following remarks.

Remark 2. The asymptotic behavior of the following sets
\[ \{ \alpha_r \}_{r=0}^{k}, \quad \{ \beta_r \}_{r=1}^{k} \]
is as follows (see (2.8); if \( T \to \infty \) then the points of every set in (2.9) recede unboundedly each from other and all these points together recede to infinity. Hence, at \( T \to \infty \) each set of (2.9) behaves as one-dimensional Friedmann-Hubble universe.

Remark 3. In this Theorem we have obtained three resp. two parametric sets of control functions (=Golem’s shem) for fixed and admissible \( k, \epsilon \)
\[ \{ \alpha_0(\sigma, T, \Theta), \alpha_1(\sigma, T, \Theta), \ldots, \alpha_k(\sigma, T, \Theta) \}, \]
\[ \{ \beta_1(\sigma, T, \Theta), \ldots, \beta_k(\sigma, T, \Theta) \}, \]
\( \sigma \in [1 + \epsilon, +\infty), \quad T \in (T_0, +\infty), \quad \Theta \in [0, 1], \)
of the metamorphoses (2.4), (comp. Remark 1 and [6]).

Remark 4. The mechanism of metamorphosis is as follows. Let (comp. (2.2), (2.10))
\[ M_3^k = \{ \alpha_1(\sigma, T, \Theta), \ldots, \alpha_k(\sigma, T, \Theta) \}, \]
\[ M_4^k = \{ \beta_1(\sigma, T, \Theta), \ldots, \beta_k(\sigma, T, \Theta) \}, \]
\( \sigma \in [1 + \epsilon, +\infty), \quad T \in (T_0, +\infty), \quad \Theta \in [0, 1], \)
where, of course,
\[ M_3^k \subset M_1^k \subset (T_0, +\infty)^k, \]
\[ M_4^k \subset M_2^k \subset (T_0, +\infty)^k. \]
Now, if we obtain after random sampling (say) of the points 

\[(x_1, \ldots, x_k), (y_1, \ldots, y_k)\]

(see conditions (2.2) on these) that

\[(x_1, \ldots, x_k) = (\alpha_1(\sigma, T, \Theta), \ldots, \alpha_k(\sigma, T, \Theta)) \in M_3^k(\epsilon),\]

\[(y_1, \ldots, y_k) = (\beta_1(T, \Theta), \ldots, \beta_k(T, \Theta)) \in M_4^k\]

(see (2.11), (2.12)) then - at the points (2.13) - change (see (2.6)) the Q-system (2.1) its old form (=chrysalis) into a new form (=butterfly) and the last is controlled by the function \(\alpha_0(\sigma, T, \Theta)\).

3. **RIEMANN’S OSCILLATORS AS A BASIS OF THE Q-SYSTEM (2.1)**

3.1. The following local variant of the Riemann-Siegel formula holds true.

**Spectral Formula.**

\[
Z(t) = 2 \sum_{n \leq \tau(x_r)} \frac{1}{\sqrt{n}} \cos \left\{ t \ln \frac{\tau(x_r)}{n} - \frac{\tau(x_r)}{2} - \frac{\pi}{8} \right\} + O(x_r^{-1/4}),
\]

\[t \in [x_r, x_r + H], \quad H \in (0, \sqrt{x_r}],\]

where (comp. (2.2))

\[T_0 < x_r, \quad r = 1, \ldots, k.\]

**Proof.** We will make the following transformations of the Riemann-Siegel formula

\[
Z(t) = 2 \sum_{n \leq \tau(t)} \frac{1}{\sqrt{n}} \cos \{ \theta(t) - t \ln n \} + O(t^{-1/4}).
\]

(a) Since

\[
\sum_{\tau(x_r) < n \leq \tau(x_r + H)} 1 = O(\tau(x_r + H) - \tau(x_r)) = O\left( \sqrt{\frac{x_r + H}{2\pi}} - \sqrt{\frac{x_r}{2\pi}} \right) + O\left( \frac{H}{\sqrt{x_r}} \right),
\]

then

\[
\sum_{\tau(x_r) < n \leq \tau(x_r + H)} \frac{2}{\sqrt{n}} \cos \{ \theta(t) - t \ln n \} = O\left( \frac{H}{\sqrt{x_r}} \sqrt{\tau(x_r)} \right) = O\left( \frac{H}{x_r^{3/2}} \right) = O(x_r^{-1/2}).
\]

Consequently, we have (see (3.2)) that

\[
Z(t) = 2 \sum_{n \leq \tau(x_r)} \frac{1}{\sqrt{n}} \cos \{ \theta(t) - t \ln n \} + O(x_r^{-1/4}),
\]

\[t \in [x_r, x_r + H].\]
(b) Next, we use the following formula

$$\vartheta(t) = \vartheta(x_r) + \vartheta'(x_r)(t-x_r) + \frac{1}{2} \vartheta''(x_r)(t-x_r)^2,$$

$$\xi_r \in (x_r, x_r+H), \ t \in [x_r, x_r+H].$$

Since (see \cite{8}, pp. 221, 329)

$$\vartheta(t) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O(t^{-1}),$$

$$\vartheta'(t) = \frac{1}{2} \ln \frac{t}{2\pi} + O(t^{-1}),$$

$$\vartheta''(t) \sim \frac{1}{2t},$$

then

$$\vartheta(x_r) = x_r \ln \tau(x_r) - \frac{x_r}{2} - \frac{\pi}{8} + O\left(\frac{1}{x_r}\right),$$

and

$$\vartheta'(x_r)(t-x_r) = \left\{ \frac{1}{2} \ln \frac{x_r}{2\pi} + O\left(\frac{1}{x_r}\right) \right\} (t-x_r) =$$

$$= \left\{ \ln \tau(x_r) + O\left(\frac{1}{x_r}\right) \right\} (t-x_r) =$$

$$= t \ln \tau(x_r) - x_r \ln \tau(x_r) + O\left(\frac{t-x_r}{x_r}\right) =$$

$$= t \ln \tau(x_r) - x_r \ln \tau(x_r) + O\left(\frac{H}{x_r}\right).$$

Hence (see (3.3)–(3.6))

$$\vartheta(t) - t \ln n = t \ln \frac{\tau(x_r)}{n} - \frac{x_r}{2} - \frac{\pi}{8} + O\left(\frac{1+H}{x_r}\right) =$$

$$= t \ln \frac{\tau(x_r)}{n} - \frac{x_r}{2} - \frac{\pi}{8} + O(x_r^{-1/4}).$$

Putting (3.7) into (3.3) we obtain the result (3.1).

3.2. It is natural to introduce the following terminology (based on the formulae (3.1), (3.2))

Remark 5. We shall call:

(a) the formula (3.1) as the local spectral form of the Riemann-Siegel formula (3.2)

(b) the sets

$$\{\omega_n(x_r)\}_{n \leq \tau(x_r)}, \ \omega_n(x_r) = \ln \frac{\tau(x_r)}{n},$$

$$\{\omega_n(y_r)\}_{n \leq \tau(y_r)}, \ \omega_n(y_r) = \ln \frac{\tau(y_r)}{n},$$

$$r = 1, \ldots, k$$

as the local spectrum of the cyclic frequencies of the Q-system (2.1).
(c) the expressions

$$\frac{2}{\sqrt{n}} \cos \left\{ t \omega_n(x_r) - \frac{x_r}{2} - \frac{\pi}{8} \right\}, \ldots$$

$$t \in [x_r, x_r + H], \; 1 \leq n \leq \tau(x_r), \ldots$$

as the local Riemann’s oscillators with the set of incoherent local phase constants

$$\left\{ -\frac{x_r}{2} - \frac{\pi}{8} \right\}, \left\{ -\frac{y_r}{2} - \frac{\pi}{8} \right\},$$

and, moreover, with the set of non-synchronized local times $t = t(x_r)$.

Now, based on our Remark 5 we give the following

**Remark 6.** We see that the Q-system

$$G(x_1, \ldots, x_k; y_1, \ldots, y_k)$$

(see (2.1)) expresses the complicated oscillating process that is generated by oscillations of a big number of the local Riemann’s oscillators (3.8). Just for this oscillating Q-system we have the oscillators (3.8). Just for this oscillating Q-system we have obtained the infinite set of metamorphoses described by the formula (2.6). For example, if

$$k = F_4 = 2^{2^4} + 1 = 65537$$

where $F_4$ is the Fermat-Gauss prime, then the above mentioned big number of Riemann’s oscillators (interacting oscillators) is bigger than

$$2 \left( \sqrt{\frac{T_0}{2\pi}} \right)^{65537}.$$  

4. PROOF OF THE THEOREM

4.1. Let us remind the formula (see [4], (2.1))

$$\int_T^{T+U} |\zeta(\sigma + it)|^2 \, dt = \zeta(2\sigma) U + \mathcal{O}(1).$$

(4.1)

This formula holds true uniformly for

$$T, U > 0, \; \sigma \geq 1 + \epsilon, \; \epsilon > 0,$$

where $\epsilon$ is arbitrary small fixed number and the $\mathcal{O}$-constant depends on the choice of that $\epsilon$. Since

$$\zeta(2\sigma) U + \mathcal{O}(1) = \zeta(2\sigma) U \left\{ 1 + \mathcal{O} \left( \frac{1}{T} \right) \right\}$$

then the formula (4.1) is asymptotic one. For example, in the case

$$U \geq \ln \ln T, \; \sigma \in [1 + \epsilon, +\infty).$$

We use in this paper the following local version of the formula (4.1)

$$\int_T^{T+U(T, \Theta)} |\zeta(\sigma + it)|^2 \, dt \sim \zeta(2\sigma) U(T, \Theta),$$

(4.2)

$$U(T, \Theta) = \ln \ln T + \Theta \ln T,$$

$$\sigma \in [1 + \epsilon, +\infty), \; \Theta \in [0, 1].$$

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4.2. Since (see (4.2))

\[ U(T, \Theta) = o \left( \frac{T}{\ln T} \right), \]

then we have by our lemma (see [5], (7.1), (7.2)) that

\[ \int_{T}^{T+U(T,\Theta)} |\zeta(\sigma + it)|^2 dt = \]

\[ = \int_{T}^{T+U} |\zeta[\sigma + i\varphi^k_1(t)]|^2 \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi^r_1(t)]dt \]

i.e. (see (4.2), (4.4))

\[ \int_{T}^{T+U} |\zeta[\sigma + i\varphi^k_1(t)]|^2 \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi^r_1(t)]dt \sim \zeta(2\sigma)U. \]

Next, we obtain by using the mean-value theorem in (4.5) that

\[ \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi^r_1(t)] \sim \zeta(2\sigma) \frac{U}{T + U - T} \frac{1}{|\zeta[\sigma + i\varphi^k_1(t)]|^2}, \]

\[ d = d(\sigma, T, \Theta, k, \epsilon) \in (T, T + U). \]

4.3. Further, we have (comp. [5], Property 2, (6.4) and [6], (5.4)) that

\[ d \in (T, T + U) \Rightarrow \varphi^r_1(d) \in (T, T + U), \quad r = 0, 1, \ldots, k, \]

i.e.

\[ \varphi^0_1(d) = \alpha_k \in (T, T + U), \]
\[ \varphi^1_1(d) = \alpha_{k-1} \in (T, T + U), \]
\[ \vdots \]
\[ \varphi^{k-2}_1(d) = \alpha_2 \in (T, T + U), \]
\[ \varphi^{k-1}_1(d) = \alpha_1 \in (T, T + U), \]
\[ \varphi^k_1(d) = \alpha_0 \in (T, T + U). \]

Consequently, from (4.6) by (4.7) the formula

\[ \prod_{l=1}^{k} \tilde{Z}^2(\alpha_l) \sim \zeta(2\sigma) \frac{U}{T + U - T} \frac{1}{|\zeta[\sigma + i\alpha_0]|^2}, \]

\[ \alpha_r = \alpha_r(\sigma, T, \Theta, k, \epsilon), \quad r = 0, 1, \ldots, k \]

follows.

\[ \text{Remark 7. We see that corresponding inclusions for } \alpha_r \text{ in (2.7) follows from (4.7).} \]
4.4. Next, we introduce the infinite collection of the following disconnected sets

\[ \Delta(T, \Theta, k) = \bigcup_{r=0}^{k} [T, T + U(T, \Theta)], \]

(comp. (4.2)). Let us remind the following properties of (4.9) (see (4.3), comp. [5], (2.5) – (2.7), (2.9)): since

\[ U(t, \Theta) = o\left( \frac{T}{\ln T} \right) \]

then

\[ |[rT, T + U]| = T + U - rT = o\left( \frac{T}{\ln T} \right), \quad r = 1, \ldots, k, \]

\[ \frac{1}{r-1} |[T + U, rT]| \sim (1 - c)\pi(T), \]

\[ [T, T + U] \prec \left[ \frac{1}{T}, \frac{1}{T + U} \right] \prec \cdots \prec \left[ \frac{k}{T}, \frac{k}{T + U} \right]. \]

Consequently, the properties (2.7), (2.8) follows from (4.7), (4.10) immediately.

4.5. Next, we have (see [2], (3.9), [3], (9.1), (9.2), comp. [6], (4.1), (4.2)) that

\[ \tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt}, \quad \varphi_1(t) = \frac{1}{2} \varphi(t), \]

where

\[ \tilde{Z}^2(t) = \frac{Z^2(t)}{2\Phi_\varphi[\varphi(t)]} = \frac{\zeta\left( \frac{1}{2} + it \right)^2}{\omega(t)}, \]

\[ \omega(t) = \left\{ 1 + O\left( \frac{\ln t}{\ln \ln t} \right) \right\} \ln t. \]

We call the function \( \varphi_1(t) \) the Jacob’s ladder (see our papers [2], [3]) according to Jacob’s dream in Chumash, Bereishis, 28:12. Further we have (comp. [5], (4.3))

\[ \ln t \sim \ln T, \quad \forall t \in (T, T + U), \]

i.e. we have (see (4.8), (4.11), (4.12)) that

\[ \tilde{Z}^2(\alpha_r) \sim \frac{Z^2(\alpha_r)}{\ln T}, \quad T \to \infty. \]

Consequently, the following formula

\[ \prod_{r=1}^{k} Z^2(\alpha_r) \sim \]

\[ \sim \zeta(2\sigma) \frac{U}{T + U - k} \ln^{k} T \frac{1}{|\zeta(\sigma + i\alpha_0)|^2} \]

(see (4.8), (4.13)) holds true.
4.6. Next, let us remind the following formula (see conditions (4.3) and [3], (9.5), comp. [5], (7.1), (7.2))

$$\int_T^{T + U} f(t) dt = \int_T^{T + U} f[\varphi_1^k(t)] \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt$$

holds true. If we put

$$f(t) = 1$$

then we obtain

$$U = \int_T^{T + U} \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt.$$

Further, we have by making use of the mean-value theorem in (4.15)

$$\prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(e)] = \frac{U}{T + U - \frac{k}{T}},$$

(4.16)

$$e = e(T, \Theta, k) \in (\tilde{T}, T + U).$$

Since

$$e \in (\tilde{T}, T + U) \Rightarrow \varphi_1^r(e) \in (\tilde{T}, T + U), \quad r = 0, 1, \ldots, k - 1,$$

then we have (similarly to (4.17)) that

$$\varphi_1^0(e) = \beta_k \in (\tilde{T}, T + U),$$

$$\varphi_1^1(e) = \beta_{k-1} \in (\tilde{T}, T + U),$$

(4.17)

$$\vdots$$

$$\varphi_1^{k-2}(e) = \beta_2 \in (\tilde{T}, T + U),$$

$$\varphi_1^{k-1}(e) = \beta_1 \in (\tilde{T}, T + U).$$

Now, we obtain from (4.16) by (4.17), (comp. (4.13)) the following formula

$$\prod_{r=1}^{k} \tilde{Z}^2(\beta_r) \sim \frac{U}{\frac{k}{T + U} - \frac{k}{T}} \ln^{k} T, \quad T \to \infty,$$

(4.18)

$$\beta_r = \beta_r(T, \Theta, k), \quad r = 1, \ldots, k.$$

Remark 8. Of course, our sequence \(\{\beta_r\}\), similarly to the sequence \(\{\alpha_r\}\) has the properties listen in (2.7), (2.8).

Consequently, the following factorization formula (comp. [6]), (see (4.14), (4.18))

$$\prod_{r=1}^{k} \left| \tilde{Z}[\alpha_r(\sigma)] \right| \tilde{Z}(\beta_r) \sim \frac{\sqrt{\zeta(2\sigma)}}{[\zeta(\sigma + i\alpha_0(\sigma))]!} T \to \infty$$

(4.19)

holds true. Finally, the formula (2.6) follows from (4.19) (see (1.5), (2.1)).
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