BRST invariant branching functions of G/H coset models.

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Abstract

We compute branching functions of $G/H$ coset models using a BRST invariant branching function formulae, i.e. a branching function that respects a BRST invariance of the model. This ensures that only the coset degrees of freedom will propagate. We consider $G/H$ for rank($G/H$) = 0 models which includes the Kazama-Suzuki construction, and $G_{k_1} \times G_{k_2}/G_{k_1 + k_2}$ models. Our calculations here confirm in part previous results for those models which have been obtained under an assumption in a free field approach. We also consider $G_{k_1} \times H_{k_2}/H_{k_1 + k_2}$, where $H$ is a subgroup of $G$, and $\prod_{a=1}^{m} G_{k_a}/G_{\sum_{a=1}^{n} k_a}$, whose branching functions, to our knowledge, has not been calculated before.

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1 Introduction

Two dimensional conformal field theories (CFT’s) has been thoroughly investigated due to its applications in both string theories and in the study of critical phenomena. The internal degrees of freedom for a string theory are described by a unitary CFT. It is generally believed that all CFT’s are described by the Wess-Zumino-Novikov-Witten (WZNW) model \cite{1}, \cite{2}, \cite{3}, or gauged versions of it \cite{4}, \cite{5}. The action of the WZNW model is constructed from fields that take values in some group $G$. From the stress-energy tensor associated with the group $G$ WZNW model, Goddard-Kent-Olive (GKO) \cite{6} constructed more general stress-energy tensors using a so-called coset construction. On the level of states this essentially means that the states of the state-space should all be primary with respect to the currents of some subgroup $H$ of $G$.

The stress-energy tensor of the GKO construction may also be found from the a BRST analysis of the gauged WZNW model \cite{7}. One essential ingredient in this construction is the introduction of an auxiliary sector $\tilde{H}$. In \cite{8} the equivalence of the two approaches was shown for integrable representations of the group $G$.

We will in this paper concentrate on calculating branching functions of various coset models using the BRST approach. In \cite{8} a branching function formula that respects the BRST invariance was given. The BRST analysis shows \cite{8}, that the remaining degrees of freedom are contained in the coset sector, and having a BRST invariant branching function thus ensures us that only the coset degrees of freedom are propagating. The BRST invariant branching function is decomposable into three parts; a character for the entire group $G$, a character for the auxiliary sector $\tilde{H}$ and one for the Faddev-Popov ghosts. Since the BRST invariant branching function contains the character of the full group $G$, it respects the invariances of this group. This means that we have a branching function which is independent of the decomposition of $G$ in $H$, and this will make it possible to compute the branching functions explicitly in a straightforward manner. This is not the case in for instance \cite{9}, where the Virasoro minimal model branching function, $SU(2)_k \times SU(2)_1/SU(2)_{k+1}$, is calculated.

Branching functions or coset characters\footnote{We will in what follows, somewhat inconsistently, use the notation character instead of branching function. Strictly speaking those are not the same. We are most grateful to Prof. B. Schellekens for clarifying this point to us.} have previously been calculated for a
number of special cases, [9], [10], [11], [12], [13], [14], [15], in general confined to various coset constructions of the groups $SU(2)$ and $U(1)$, as well as for general constructions, [16], [17], [18]. We have for example characters for the Kazama-Suzuki [19] models, calculated in [16]. The constructions of reference [16] are, however, computed using an assumption. Since we are provided with a way to compute characters without assumptions, we can make interesting comparisons. References [17], [18] give exact results but consider only one general model (ref. [17]) or calculate only a few examples (ref. [18]). Also in [17] the branching functions are not given in an explicit form. We will in fact, to some extent, show coincidence with those cases.

We will also give coset characters for models that have not previously been considered in literature, at least not in the general case.

2 Preliminaries

We start by a short review of the background and refer to [3], and references therein for more details. Consider a WZNM model on a Riemann surface $\mathcal{M}$ with fields $g$ taking values in a compact Lie group $G$. The action is [1], [2] and [3]

$$S_k = \frac{k}{16\pi} \int_{\mathcal{M}} d\sigma d\tau Tr(\partial_\mu g^\dagger g^{-1} \partial_\mu g^{-1}) + \frac{k}{24\pi} \int_B d^3y \epsilon^{abc} Tr(g^{-1} \partial_a g^{-1} \partial_b g^{-1} \partial_c g), \quad (1)$$

where $\mathcal{M}$ is the boundary of $B$ on which $g$ is supposed to be well defined. $k$ is referred to as the level of the WZNM model. In general, every simple part of $G$ may have different levels but we take for the moment $G$ simple. The action is invariant under the transformations

$$g(z, \bar{z}) \rightarrow \tilde{\Omega}^{-1}(\bar{z}) g(z, \bar{z}) \Omega(z). \quad (2)$$

$\Omega$ and $\tilde{\Omega}$ are arbitrary analytical group-valued matrices. This symmetry gives rise to an infinite number of conserved currents which are found to be of the affine Lie type. That is, they transform as

$$\delta_\omega J(z) = [J(z), \omega(z)] - \frac{k}{2} \partial_z \omega(z)$$

$$\delta_\omega \bar{J}(\bar{z}) = [\bar{J}(\bar{z}), \bar{\omega}(\bar{z})] - \frac{k}{2} \partial_{\bar{z}} \bar{\omega}(\bar{z}) \quad (3)$$

under the infinitesimal version of eq. (2). Making a Laurent expansion this gives the well known affine Lie algebra, $\hat{g}$, of level $k$,

$$[J^A_m, J^B_n] = if^{AB}_C J^C_{m+n} + \frac{k}{2} m g^{AB} \delta_{m+n,0}. \quad (4)$$
Likewise for $\bar{J}$. $f^{AB}_{\ C}$ are structure constants of the Lie algebra $g$ of $G$. $g^{AB}$ is a non-degenerate metric on $G$. The Sugawara stress-energy tensor splits into holomorphic and anti-holomorphic parts, and in the former case it becomes

$$T(z) = \frac{1}{k + c_G} : J^A(z) J_A(z) :$$

$$= \frac{1}{k + c_G} \sum_{m \in \mathbb{Z}} : J^A_{n-m} J_{A,m} : z^{-n-2},$$

(5)

where $c_G$ is the quadratic Casimir of the adjoint representation of $g$. The corresponding central charge becomes

$$c = \frac{kd_G}{k + c_G},$$

(6)

where $d_G$ is the dimension of $G$.

In reference [6] it was described how to enlarge the range of central charges of models with stress-energy tensors given in the form of eq.(5). Taking a subgroup $H$ of $G$ one may note that the difference of the Virasoro generators of $G$ and $H$ also obeys a Virasoro algebra of central charge

$$c^{tot} = \frac{kd_G}{k + c_G} - \frac{kd_H}{k + c_H},$$

(7)

for a simple subgroup $H$. Since the currents of the subgroup $H$ commutes with the stress-energy tensor it is natural to impose the constraint$^4$

$$J^a_n|_{\text{phys}} = 0, \quad n > 0 \quad \text{or} \quad n = 0 \quad \text{and} \quad a \in \Delta_h^+,$$

(8)

on the physical states. This we will refer to as the GKO coset condition.

In order to gauge the WZNW model we use an anomaly-free subgroup $H$ of $G$. We introduce gauge fields which transforms in the adjoint representation of $H$. The gauged action in light-cone coordinates looks like

$$S_k(g, A) = S_k(g) + \frac{1}{4\pi} \int d^2 \xi Tr(A_+ \partial_- g^{-1} - A_- g^{-1} \partial_+ g + A_+ A_+ g^{-1} A_- g^{-1} - A_- A_+),$$

(9)

$^4$We have adapted the convention that uppercase letters take values in $g$ while lowercase take values in $h$. 

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where the gauge fields may be parametrized as

$$A_+ = \partial_+ hh^{-1}, \quad A_- = \partial_- \tilde{h} \tilde{h}^{-1},$$

and $h, \tilde{h}$ are elements of $H$. The partition function may now, after integrating out the gauge fields, be written as [21],[3]

$$Z = \int [dg][dh][db_+][db_-][dc_+][dc_-] \exp[-kS_k(g)] \exp[-(-k-2c_H)S_{-k-2c_H}(\tilde{h})] \times \exp[-Tr \int d^2 \xi (b_+ \partial_- c_+ + b_- \partial_+ c_-)].$$

We thus find that the gauged WZNW models factorize into three parts, the original model of group $G$ and level $k$, another WZNW model of group $H$ and level $-k-2c_H$ and a ghost sector.

The conformal anomaly of the Sugawara-type stress-energy tensor of the gauged WZNW model coincides with the GKO coset conformal anomaly, as noted in [3]. The total action corresponding to the partition function eq.(11) is invariant under the BRST transformation see e.g. [7], [20]. The BRST charge of the holomorphic sector is found to be

$$Q = \oint \frac{dz}{2i\pi} \left[ : c_a(z)(J^a(z) + \tilde{J}^a(z)) : - \frac{i}{2} f^{ade} : c_a(z)c_d(z)b^e(z) : \right].$$

Using the BRST invariance, we find that the stress-energy tensor of the GKO coset construction $T^{GKO} = T^G - T^H$ coincides with the stress-energy tensor of the gauged WZNW model $T^{\text{tot}} = T^G + T^\tilde{H} + T^{gh}$ up to commutators with the BRST charge, i.e.

$$T^{\text{tot}}(z) = T^{GKO}(z) + \frac{1}{k + c_H} \left[ Q, : b_a(z) \left( J^a(z) - \tilde{J}^a(z) \right) : \right].$$

We then find from the BRST physicality condition

$$Q|_{\text{phys}} = 0$$

that the stress-energy tensors of the two models will coincide on the physical subspace.

In [8] the BRST condition eq.(14) was analyzed, or rather the BRST cohomology was computed with the restriction that we have integrable representations for the currents of the $G$ sector. This integrability condition may be expressed as

$$\alpha \cdot \mu \geq 0 \quad \forall \alpha \in \Delta_g^s,$$
where $\Delta_g^+$ is the set of simple roots of $g$, and

$$\hat{\alpha} \cdot \mu \leq \frac{k}{2},$$

(16)

where $\hat{\alpha}$ is the highest root and $\mu$ is the highest weight of the vacuum representation, $[21]$. It was found in [8] that the GKO coset condition eq.(8) coincides with the BRST condition eq.(14) when we have integrable representations for the $G$ sector and no null vectors in the $\tilde{H}$ sector, (i.e., no states that have vanishing inner products with all other states).

To make the rest of the paper more accessible we give the affine Lie algebra in the Cartan-Weyl basis,

$$[J^i_{m}, J^j_{n}] = \frac{k}{2} \epsilon^{ij} \delta_{m,-n}$$

$$[J^i_{m}, J^a_{n}] = \alpha^i J^a_{m+n}$$

$$[J^a_{m}, J^b_{n}] = \begin{cases} 
\epsilon(\alpha, \beta) J^{\alpha + \beta}_{m+n} & \text{if } \alpha + \beta \text{ is a root} \\
\frac{1}{\alpha} (\alpha_i J^i_{m+n} + \frac{k}{2} m \delta_{m,-n}) & \text{if } \alpha = -\beta \\
0 & \text{otherwise.}
\end{cases}$$

(17)

We also denote $\Delta^+_g = \text{the set of positive roots of } g$. We introduce vacuum states of the three sectors defined to obey

$$J^A_n|0; R\rangle = 0 \quad n > 0 \quad \text{or} \quad n = 0 \quad \text{and} \quad A \in \Delta^+_g$$

$$J^a_n|0; \tilde{R}\rangle = 0 \quad n > 0 \quad \text{or} \quad n = 0 \quad \text{and} \quad a \in \Delta^+_h$$

$$b^a_n|0\rangle_{gh} = 0 \quad n \geq 0$$

$$c^a_n|0\rangle_{gh} = 0 \quad n \geq 1.$$  

(18)

An arbitrary state is then given by \( |s\rangle = |s_G\rangle \times |s_{\tilde{R}}\rangle \times |s_{gh}\rangle \) where

$$|s_G\rangle = \sum_R \prod_{A,n} J^A_{-n}|0; R\rangle, \quad |s_{\tilde{R}}\rangle = \sum_{\tilde{R}} \prod_{a,n} J^a_{-n}|0; \tilde{R}\rangle,$$

(19)

and \( |s_{gh}\rangle \) is a sum over states of the form

$$\prod_{a_1,n_1} \prod_{a_2,n_2} b^{a_1}_{-n_1} c^{a_2}_{-n_2} |0\rangle_{gh}.$$ 

(20)

The state \( |0\rangle_{gh} \) is the \( SL(2, \mathbb{R}) \) invariant ghost vacuum, which is required to be annihilated by \( L_n^{gh} \) for \( n = 0, \pm 1 \). The state \( |0; R\rangle \) is a highest weight primary with
respect to the affine Lie algebra \( \hat{g} \), and transforms in some representation \( R \) of the Lie algebra \( g \). Likewise for the auxiliary sector. This means that the product in (19) is taken over \( n > 0 \) or \( n = 0 \) and negative roots. For integrable representations the \( G \) sector may always be decomposed into states of the type

\[
|s_G\rangle = \sum_{\Phi} \prod_{a,n} J_{-\alpha(n)}^a |\Phi\rangle,
\]

(21)

where \( |\Phi\rangle \) are primary with respect to the currents of the subgroup \( H \).

It is convenient to first study the problem on the relative state space defined as

\[
b_{0,i}|s\rangle = 0 \quad i = 1, \ldots, r_h,
\]

(22)

where \( r_h \) is the rank of the Lie algebra \( h \). This means that we can split the BRST charge as

\[
Q = \hat{Q} + M_i b_i^0 + c_{0,i} J_{0}^{\gamma,i}.
\]

(23)

Since \( J_{m\gamma}^{\gamma,i} \) is a BRST commutator we find that states in the relative cohomology must have zero eigenvalue of \( J_{0}^{\gamma,i} \) and that on the relative state space the \( Q \) cohomology coincides with the \( \hat{Q} \) cohomology.

When we analyze the relative cohomology we will find that there are no states at ghost number different from zero. To be more specific, if we restrict the choices of representations of the \( \hat{H} \) sector in an analogous manner to eq.(17) and eq.(16) namely

\[
\alpha \cdot (\hat{\mu} + \rho_h) \leq 0,
\]

(24)

and

\[
\hat{\alpha} \cdot \hat{\mu} \geq \frac{\hat{k}}{2} + 1,
\]

(25)

where \( \alpha \in \Delta_h^* \), \( \hat{\alpha} \) is the highest root and \( \hat{k} = -k - 2c_H \), then the relative cohomology contains only states of ghost number zero. Furthermore, those states have no \( \hat{J}_a \) excitations and they satisfy

\[
J_{m}^{a}|\phi\rangle = b_{0}^{a}|\phi\rangle = c_{n}^{a}|\phi\rangle = 0
\]

(26)

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for \( n > 0 \) or \( n = 0 \) and \( a \in \Delta_h^+ \). We should also remark that this choice of representations (24), (25) is the one that excludes all null vectors in the auxiliary sector. It also means that \(-\tilde{\mu} - \rho_h\) is an integrable weight, a fact which will be important in the next section. This is another way of finding the appropriate representation restrictions (24), (25). For the case of a \( U(1) \) group, we have that the eigenvalue of the Cartan generator of the auxiliary sector take values in \( \mathbb{Z}/2 \).

We thus conclude that for this choice of representation the relative cohomology is completely specified.

Turning to the cohomology of the full space we will only comment that the absolute cohomology contains more states as can be seen from the \( 2^h \) degeneracy of the ghost vacua. This degeneracy is removed by requiring the physical states to be in the relative state-space.

### 3 Characters

We are now ready to define the BRST invariant character of the gauged WZNW model. In general the affine character is defined as

\[
\chi(\tau, \theta) = \text{Tr} \left( e^{2i\pi \tau (L^G_0 + L^H_0 + L^g_0)} e^{i(\theta, J^{\text{tot},i} + \theta_I' J^{I'}_0)} (-1)^{N_{gh}} \right),
\]

where we have omitted the usual factor \( e^{-2i\pi \tau c/24} \) for convenience. Here \( i \) take labels in \( h \) and \( I' \) in \( g/h \). We assume the existence of a basis such that this decomposition is possible. Since the physical states live on the relative state space, we do not include a summation over ghost vacua in the trace. We exclude this part of the trace to remove the ghost vacuum degeneracy. On the physical state space we have in addition that \( J^{\text{tot},i}_0 |_{\text{phys}} = 0 \). This follows from the definition of the relative state space \( b_0^i \phi = 0 \), the BRST condition \( Q|_{\text{phys}} = 0 \) and the fact that \( J^{\text{tot},i}_0 = \{Q, b_0^i\} \). To only have physical degrees of freedom propagating we must, therefore, implement this condition in the character. This can be done by integrating over the \( \theta \)-components associated with \( J^{\text{tot},i}_0 \), i.e. we define

\[
\chi^{G/H}(\tau, \theta) = \int \prod \frac{d\theta_i}{2\pi} \text{Tr} \left( e^{2i\pi \tau (L^G_0 + L^H_0 + L^g_0)} e^{i(\theta, J^{\text{tot},i} + \theta_I' J^{I'}_0)} (-1)^{N_{gh}} \right).
\]

This definition of the character now respects the BRST symmetry of the model, and we are then assured that only states in the coset will propagate. In what follows we will instead of \( \tau \) use \( q = e^{2\pi i \tau} \). In order to make the formulae more transparent we
choose to display characters only for simply-laced algebras.

The BRST invariant character (28), is a product of three different terms $\chi^G$, $\chi^\tilde{H}$ and $\chi^{gh}$. $\chi^G$ is given by the Kac-Weyl formula [24],

$$\chi^G(q, \theta) = \sum_{t \in \Lambda^e} q^{(\lambda + \rho/2 + t)^2 - \rho^2/4} \sum_{\sigma \in W(g)} \epsilon(\sigma) e^{i(\sigma(\lambda + \rho/2 + t) - \rho/2) \cdot \theta} R^{-1}_G(q, \theta) ,$$

(29)

where $\Lambda^e$ denotes a lattice which is spanned by $t = n(k + c_G)\alpha \psi^2/\alpha^2$ where $\alpha \in \Delta^s_g$, $n \in \mathbb{Z}$ and $\psi$ is a long root, ( which we have chosen to normalize as $\psi^2 = 1$ ). $t$ is thus proportional to the co-roots of $g$. $\sigma$ are elements of the Weyl group $W(g)$, and

$$R_G(q, \theta) = \prod_{n=1}^{\infty} (1 - q^n)^{r_g} \prod_{\alpha \in \Delta^+_g} (1 - q^n e^{i\alpha \cdot \theta})(1 - q^{n-1} e^{-i\alpha \cdot \theta}).$$

(30)

$r_g = \text{rank}(g)$ and $\Delta^+_g$ is the set of positive roots of $g$.

$\chi^\tilde{H}$ is straightforward to determine, since we know that there are no null vectors in the auxiliary sector. A moment’s reflection yields us

$$\chi^{\tilde{H}^+}(q, \theta) = e^{i\theta \cdot \tilde{\mu}} q^{-\tilde{\mu}(\lambda + \rho)_{\tilde{H}}} R^{-1}_H(q, \theta) ,$$

(31)

for highest weight representations. The ghost contribution may be constructed for each pair $b^\alpha_n$ and $c^\alpha_n$. They contribute as does conformal ghosts, see eg. [25], with the exception of the twist according to the eigenvalue of $J^{gh,i}_0$. For the zero modes $b^\alpha_0$, $c^\alpha_0$ we find $e^{i\rho \cdot \theta} \prod_{\alpha > 0} (1 - e^{-i\alpha \theta})^2$. This leaves us with

$$\chi^{gh}(q, \theta) = e^{i\rho \cdot \theta} R^2_H(q, \theta)$$

(32)

for the ghost sector.

In ref. [26] it is shown that the BRST invariant character coincides with the usual definition of a branching function,

$$\chi^G = \sum_{\Lambda} \chi^{G/H}_{\Lambda, \Lambda} \chi^{H}_{\Lambda} ,$$

(33)

where we take the sum over integrable highest weights $\lambda$ of $H$.

Furthermore, in reference [26], the branching function of $G/(U(1))^x$ models are calculated, for $1 \leq x \leq r_g$ where $r_g$ is the rank of the Lie algebra $g$. A general formula for string functions for any group $G$ is also displayed.
In [8] the coincidence between the BRST invariant character and conventional character calculations were shown for the parafermion model $SU(2)_k/U(1)$ previously obtained in [11], and minimal Virasoro models $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$ known from [9].

Here we will first compute the characters of $G/H$ cosets such that rank($G/H$) = 0 as well as ($G \times G$)/$G$ models. Special cases have occurred in several publications. Apart from those mentioned above we have for instance $SU_k(N) \times SU_l(N)/SU_{k+l}(N)$ in [12], [14] and [15].

Furthermore, all these models are all computed in ref. [16] but only under certain assumptions. The ($G \times G$)/$G$ model is also calculated in ref. [17] using a different method than [16]. Characters of these kinds are also computed in [18] in a general context. Reference [18] is, however, restricted to a few explicit examples.

Both [16] and [17] uses free field techniques, although they choose different approaches. Bouwknegt, McCarthy and Pilch, [16], uses free field Fock spaces to obtain resolutions of highest weight modules for $G/H$ coset models. This is done using techniques of double complexes. Firstly one utilizes an operator first introduced by Felder in the context of minimal models [27], and Bernard and Felder [28], for $SL(2,\mathbb{R})$, and secondly a BRST operator. The assumption made is that the cohomologies of the Felder-like as well as the BRST operator are constrained to ghost number zero. This has so far only been proven for the case of minimal models [27], and for $G = SL(2,\mathbb{R})$ [28], for the Felder-like operator. The result of the cohomology of the BRST operator is later proven in [29] but only under the the assumption that the Felder-like operator cohomology is non-trivial only for ghost number zero.

Christie and Ravaninni [17] decompose $G_{k_1} \times G_{k_2}/G_{k_1+k_2}$ in parafermions and free fields. They are then able to write down a branching function in terms of string functions [22]. Those branching functions are proven to be in an explicit form in [8].

### Characters of $G/H$ for rank($G/H$) = 0

We first assume that $H$ is a direct sum of simple subgroups $H_a$. For the $G$ sector we find the character, [29],

$$
\chi^G_{\lambda}(q, \theta) = \sum_{t \in \Lambda^\nu} q^{(\lambda + \rho/2 + t)/2} \varepsilon(\sigma) e^{i\sigma(\lambda + \rho/2 + t - \rho/2) \cdot \theta}
$$
For each subgroup we find from eq. (31) and eq. (32) the $\tilde{H}$ character and the ghost contribution to be

$$\chi^{\tilde{H}}_{\tilde{\mu}_a}(q, \theta) = e^{i\tilde{\theta}_a \cdot \tilde{\mu}_a} \frac{\prod_{n=1}^{\infty} (1 - q^n)^{r_g} \prod_{\alpha \in \Delta^+_g} (1 - q^n e^{i\alpha \cdot \theta})(1 - q^n e^{-i\alpha \cdot \theta})}{\prod_{n=1}^{\infty} \prod_{\alpha \in \Delta^+_h} (1 - q^n e^{i\alpha \cdot \theta})(1 - q^n e^{-i\alpha \cdot \theta})}^{-1}$$

(35)

and

$$\chi^{gh}_a(q, \theta) = e^{i\tilde{\theta}_a \cdot \rho_a} \frac{\prod_{n=1}^{\infty} (1 - q^n)^{r_g} \prod_{\alpha \in \Delta^+_g} (1 - q^n e^{i\alpha \cdot \theta})(1 - q^n e^{-i\alpha \cdot \theta})}{\prod_{n=1}^{\infty} \prod_{\alpha \in \Delta^+_h} (1 - q^n e^{i\alpha \cdot \theta})(1 - q^n e^{-i\alpha \cdot \theta})}^2$$

(36)

respectively. The range of permissible weights $\tilde{\mu}_a$ is given from the choice of representations (24), (25) and we have that $-\tilde{\mu}_a - \rho_a$ take values in the set of integrable representations.

We now insert those ingredients into the integral

$$\chi^{G/H}(\tau, \theta) = \int \prod_i \frac{d\tilde{\theta}_i}{2\pi} \chi^{G} \chi^{\tilde{H}} \chi^{gh}.$$ (37)

Under the assumption of the embedding of $h$ in $g$, i.e. that there exists a basis for $g$ such that the Cartan sub-algebra of $h$ is just a subset of the Cartan sub-algebra of $g$, we have that $\alpha_a \cdot \theta_a = \alpha \cdot \theta$ for some $\alpha_a \in \Delta^+_h$ and $\alpha \in \Delta^+_g$. In order to perform the integration, we use the identity due to Thorn [30]

$$\left( \prod_{n=1}^{\infty} \prod_{\alpha \in \Delta^+_g} (1 - q^n e^{i\alpha \cdot \theta})(1 - q^n e^{-i\alpha \cdot \theta}) \right)^{-1} = \prod_{n=1}^{\infty} (1 - q^n)^{-2|\Delta^+_g|} \prod_{j=1}^{2|\Delta^+_g|} \sum_{p_j \in \mathbb{Z}} \sum_{s_j=0}^{\infty} (-1)^{s_j} q^{|\Delta^+_g|} \frac{1}{2} (q^{s_j+1/2} - q^{-1/2}) e^{ip_j \theta \cdot \alpha_j}$$ (38)

where $|\Delta^+_g|$ denotes the number of positive roots of $g$. Performing the integration we will find

$$\chi^{G/H}(q, \theta) = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^{2|\Delta^+_g/h|}} \sum_{\sigma \in W(g)} \sum_{t \in \Lambda^+_g}.$$
\[ \epsilon(\sigma)q^{\frac{(\lambda+\rho/2+t)^2-\rho^2/4}{k+G}}q^{-\sum_a \tilde{\mu}_a (\rho a + \rho)} \]

\[ \prod_{j=1}^{\left|\Delta_{/\mathcal{H}}^+\right|} \sum_{p_j \in \mathbb{Z}} \sum_{s_j = 0}^{\infty} (-1)^{s_j} q^{\frac{1}{2}(s_j-p_j+1/2)^2-\frac{1}{2}(p_j-1/2)^2}. \]  

(39)

The restriction on the sum of \( p_j \), indicated by the prime on the sum, should be taken as

\[ \sigma(\lambda + \frac{\rho}{2} + t) - \frac{\rho}{2} + \tilde{\mu} + \rho + \sum_{j=1}^{\left|\Delta_{/\mathcal{H}}^+\right|} p_j \alpha_j = 0. \]  

(40)

Here \( \tilde{\mu} \) is an \( r_g \) dimensional vector such that \( \sum_a \tilde{\mu}_a \cdot \theta_a = \tilde{\mu} \cdot \theta \).

This is in agreement with the result of reference [16]. Since our result is derived without any assumptions whatsoever this is a verification of the result of Bouwknegt, McCarthy and Pilch [16]. For the choice \( G = SU(3) \) and \( H = SU(2) \times U(1) \) this is also found to agree with the result of Huito, Nemeshansky and Yankielowicz [18].

\( G_{k_1} \times G_{k_2}/G_{k_1+k_2} \) models.

In those models one usually chooses to divide out the diagonal subgroup, a prescription which we also chose to follow.

Our starting point is as usual the BRST invariant character

\[ \chi^{G/H}(\tau, \theta) = \int \prod_i \frac{d\theta_i}{2\pi} \chi^G \chi^\mathcal{H} \chi^\mathcal{H}. \]  

(41)

The characters of the two groups \( G_{k_1} \) and \( G_{k_2} \) are given from

\[ \chi^G_{\lambda}(q, \theta) = \sum_{\nu \in \Delta_g^+} q^{\frac{(\lambda+\rho/2+t)^2-\rho^2/4}{k+G}} \sum_{\sigma \in W(\nu)} \epsilon(\sigma)e^{i\sigma(\lambda+\rho/2+t)-\rho/2} \cdot \left( \prod_{n=1}^{\infty} (1 - q^n)^{r_g} \prod_{\alpha \in \Delta_g^+} (1 - q^n e^{i\alpha}) (1 - q^{n-1} e^{-i\alpha}) \right)^{-1}, \]  

(42)

with labels 1 and 2 respectively i.e. \( \chi^G_{\lambda_1, \lambda_2} = \chi^G_{\lambda_1} \chi^G_{\lambda_2} \).

For the auxiliary sector, in which the range of weights \( \tilde{\lambda}_3 \) are restricted according to the our choice \([24],[25]\), we find the character

\[ \chi^G_{\tilde{\lambda}_3}(q, \theta) = e^{i\tilde{\lambda}_3 \cdot \theta} q^{\frac{\tilde{\lambda}_3 (\lambda_3 + \rho)}{k+G}} \left( \prod_{n=1}^{\infty} \prod_{\alpha \in \Delta_g^+} (1 - q^n)^{r_g} (1 - q^n e^{i\alpha}) (1 - q^{n-1} e^{-i\alpha}) \right)^{-1}. \]  

(43)
In order to compare our result with other authors, we note that \(-\bar{\lambda}_3 - \rho\) is an integrable weight.

As for the ghosts we have

\[
\chi^{gh}(q, \theta) = e^{i\theta \rho} \left( \prod_{n=1}^{\infty} \prod_{\alpha \in \Delta^+} (1 - q^n)^{3n}(1 - q^n e^{i\alpha \cdot \theta})(1 - q^n e^{-i\alpha \cdot \theta}) \right)^2.
\] (44)

The integration is carried out using identity (38), and the result becomes

\[
\chi^{G/H}_{\lambda_1, \lambda_2, \bar{\lambda}_3}(q, \theta) = \frac{1}{\prod_{n=1}^{\infty}(1 - q^n)^{2|\Delta^+_g| + r_g}} \sum_{\sigma_1 \in W(g)} \sum_{\sigma_2 \in W(g)} \sum_{t_1 \in \Delta^g_{\nu, g}} \sum_{t_2 \in \Delta^g_{\nu, g}} \epsilon(\sigma_1) \epsilon(\sigma_2) q^{(\lambda_1 + \rho/2 + t_1)^2 - \rho/4} q^{(\lambda_2 + \rho/2 + t_2)^2 - \rho/4} q^{\bar{\lambda}_3 (\lambda_3 + \rho)} \\
\cdot \prod_{j=1}^{r} \sum_{p_j \in \mathbb{Z}} \sum_{s_j = 0}^{\infty} (-1)^{s_j} q^{\frac{1}{2}(s_j - p_j + 1/2)^2 - \frac{1}{2}(p_j - 1/2)^2}.
\] (45)

The restriction of the primed sum should be taken as

\[
\sigma_1(\lambda_1 + \rho/2 + t_1) - \rho/2 + \sigma_2(\lambda_2 + \rho/2 + t_2) - \rho/2 + \bar{\lambda}_3 + \rho + \sum_{j=1}^{r} p_j \alpha_j = 0.
\] (46)

In eq. (45) we have suppressed the non-diagonal \(\theta\) behavior partly for clarity, and partly since this is standard in literature. In general, \(\chi^{G/H}\) of course depends on the \(\theta\) components that we do not integrate over, but we may choose to write the character in the point of moduli-space where the \(\theta\) dependence vanishes. In analyzing the modular properties things would, however, be more transparent if the \(\theta\) dependence was kept intact.

Our result (45), (46) essentially agree with the result of ref. [16]. There is a difference of a constant factor

\[
\frac{\rho^2}{4} \frac{k_1}{(k_2 + c_G)(k_1 + k_2 + c_G)}.
\] (47)

For the case of \(G = SU(2)\) this is found to agree with previous calculations [14] and [15]. These models are also discussed in [12] but no explicit formula for the branching function is given.

Our result should also agree with [17] where the branching functions are given in terms of string functions. (An explicit form of those string functions is proven in [8].)
Not previously considered models.

We will now compute the obvious generalization of the $G_{k_1} \times G_{k_2}/G_{k_1+k_2}$ case namely $\prod_{a=1}^m G_{k_a}/G_{\sum_{a=1}^m k_a}$ models. This construction is the natural building block of any model of the type $G^m/G^n$, (the notation should be obvious), since $G/G$ is essentially unity.

We will also consider $G_{k_1} \times H_{k_2}/H_{k_1+k_2}$, where $H$ is a subgroup of $G$. For the case of $SU_k(2) \times U(1)/U(1)$ this has been calculated in [13].

The $\prod_{a=1}^m G_{k_a}/G_{\sum_{a=1}^m k_a}$ models.

We will here follow the prescription given for the case $m=2$ given above and choose to divide out the diagonal group. For the numerator we here use $m$ copies of

$$
\chi^G_{\lambda_a}(q, \theta) = \sum_{t_a \in \Lambda^+} q^{(\lambda_a + \rho/2 + t_a)^2 - \rho^2/4} \sum_{\sigma_a \in W(g)} \epsilon(\sigma_a) e^{i(\sigma_a(\lambda_a + \rho/2 + t_a) - \rho/2) \cdot \theta}
$$

$$
\cdot \left( \prod_{n=1}^{\infty} (1 - q^n)^r_g \prod_{\alpha \in \Delta^+_g} (1 - q^n e^{i\alpha \cdot \theta})(1 - q^{n-1} e^{-i\alpha \cdot \theta}) \right)^{-1},
$$

where of course $1 \leq a \leq m$.

The character for the auxiliary sector is found to be

$$
\chi^{\tilde{H}}_{\tilde{\mu}}(q, \theta) = e^{i\theta \cdot \tilde{\rho}} q^{\tilde{\mu} \cdot (\tilde{\rho} + \tilde{\rho})} \left( \prod_{n=1}^{\infty} \prod_{\alpha \in \Delta^+_g} (1 - q^n)^r_g (1 - q^n e^{i\alpha \cdot \theta})(1 - q^{n-1} e^{-i\alpha \cdot \theta}) \right)^{-1},
$$

where we have introduced

$$
K \equiv \sum_{a=1}^m k_a.
$$

The set of $\tilde{H}$ weights $\tilde{\mu}$ is restricted according to our choice (24, 25), which means that $-\tilde{\mu} - \rho$ take integrable representations. We also find for the ghosts

$$
\chi^{gh}(q, \theta) = e^{i\theta \cdot \rho} \left( \prod_{n=1}^{\infty} \prod_{\alpha \in \Delta^+_g} (1 - q^n)^r_g (1 - q^n e^{i\alpha \cdot \theta})(1 - q^{n-1} e^{-i\alpha \cdot \theta}) \right)^2.
$$

When we perform the integration

$$
\chi^{G/H}(\tau, \theta) = \int \prod_i \frac{d\theta_i}{2\pi} \chi^G \chi^{\tilde{H}} \chi^{gh}.
$$
we may as usual benefit from the identity due to Thorn [30]

\[
\left( \prod_{n=1}^{\infty} \prod_{\alpha \in \Delta_{q}^{+}} (1 - q^{n} e^{i\alpha \cdot \theta})(1 - q^{-1} e^{-i\alpha \cdot \theta}) \right)^{-1}
= \prod_{n=1}^{\infty} (1 - q^{n})^{-2|\Delta_{q}^{+}|} \prod_{j=1}^{\infty} \sum_{p_{j} \in \mathbb{Z}} \sum_{s_{j}=0}^{\infty} (-1)^{s_{j}} q^{l_{j} = 1} (s_{j} - p_{j} + 1/2)^{2} - \frac{1}{2} (p_{j} - 1/2)^{2} \cdot e^{ip_{j} \theta \cdot \alpha_{j}}
\]

and accordingly find

\[
\chi_{G/H}^{G/H}(q, \theta) = \frac{1}{\prod_{n=1}^{\infty} (1 - q^{n})^{2(m-1)|\Delta_{q}^{+}|+(m-1)r_{q}} \sum_{\sigma_{1} \in W(g)} \cdots \sum_{\sigma_{m} \in W(g)} \sum_{t_{1} \in \Lambda_{r_{q}}^{+}} \cdots \sum_{t_{m} \in \Lambda_{r_{q}}^{+}} \cdot \epsilon(\sigma_{1}) \cdots \epsilon(\sigma_{m}) q^{m_{-1} \Delta_{q}^{+}}/(\Lambda_{r_{q}}^{+})^{2} \cdot \epsilon(\sigma) e^{i(\sigma(\Lambda + \rho/2 + t_{q})) - \rho/2 \cdot \theta} R_{G}^{-1}(q, \theta),
\]

where again $K$ is given by [54]. The restriction of the sum of $p_{j}$'s is

\[
\sum_{a=1}^{m_{-1} \Delta_{q}^{+}} (\sigma_{a} (\lambda_{a} + \rho/2 + t_{a})) - m_{-1} \rho + \rho + \sum_{a=1}^{m_{-1} \Delta_{q}^{+}} \sum_{j_{a}=1}^{t_{a} \in \mathbb{Z}} s_{j_{a}} = 0.
\]

As in the case of $G^{2}/G$ we have chosen to suppress the non-diagonal $\theta$ behavior in eq. [54].

**$G_{k_{1}} \times H_{k_{2}} / H_{k_{1} + k_{2}}$ where $H \subset G$.**

We take the embedding of $H_{k_{1} + k_{2}}$ to be the diagonal one. To make the formulae simpler we take $H$ to be simple. The generalization to non-simple $H$ is straightforward.

The character of $G_{k_{1}}$ is given by the Kac-Weyl formula [29]

\[
\chi_{G}^{G}(q, \theta) = \sum_{t_{g} \in \Lambda_{r_{g}}^{+}} q^{(\Lambda + \rho/2 + t_{q})_{G}^{2} - \rho/2, \theta} R_{G}^{-1}(q, \theta),
\]

where we as usual have

\[
R_{G}(q, \theta) = \prod_{n=1}^{\infty} (1 - q^{n})^{r_{q}} \prod_{\alpha \in \Delta_{q}^{+}} (1 - q^{n} e^{i\alpha \cdot \theta})(1 - q^{n-1} e^{-i\alpha \cdot \theta}).
\]

For $H_{k_{2}}$ we also find from [56]

\[
\chi_{G}^{H}(q, \theta) = \sum_{t_{h} \in \Lambda_{r_{h}}^{+}} q^{(\lambda + \rho_{h}/2 + t_{h})_{H}^{2} - \rho/2, \theta} R_{H}^{-1}(q, \theta),
\]

\[
\chi_{H}^{H}(q, \theta) = \sum_{t_{h} \in \Lambda_{r_{h}}^{+}} q^{(\lambda + \rho_{h}/2 + t_{h})_{H}^{2} - \rho/2, \theta} R_{H}^{-1}(q, \theta).
\]
For the auxiliary sector the set of allowed representation is restricted such that we have integrable representations for \(-\bar{\mu} - \rho_h\), cf. equations (24) and (25). This gives us the character
\[
\chi_{\bar{\mu}}^H(q, \theta) = e^{i \bar{\mu} \cdot \theta} q^{-\frac{\bar{\mu} \cdot (\rho + \rho_h)}{k_1 + k_2 + H}} R_H^{-1}(q, \theta),
\]
for the auxiliary sector, and we find for the ghost sector
\[
\chi^{gh}(q, \theta) = e^{i \rho_h \cdot \theta} R_H^2(\tau, \theta),
\]
\(R_H\) is of course given by (57) with labels \(H\) and \(h\) instead of \(G\) and \(g\). With \(h\) embedded in \(g\) under the usual assumption we can take \(\alpha_h \cdot \theta = \alpha \cdot \theta\), for some \(\alpha_h \in \Delta_h^+\) and \(\alpha \in \Delta_g^+\). We will in the following suppress the non-diagonal \(\theta\) behavior.

We now split \(\theta\) into a direct sum of the \(r_h\) first components \(\theta_h\), and the last \(r_g - r_h\) components \(\theta'\), that is \(\theta = \theta_h \oplus \theta'\). We will then integrate over \(\theta_h\) while the remaining diagonal \(\theta\) behavior will be displayed by \(\theta'\). We should then perform the integral
\[
\chi^{G/H}(\tau, \theta) = \int \frac{d\theta_h}{2\pi} \chi^{G/H}(\tau, \theta_h) \chi^{gh}.
\]
Using Thorn’s trick, (38) we find the character to be
\[
\chi^{G/H}(q, \theta) = \prod_{n=1}^{\infty} (1 - q^n)^r \prod_{\alpha \in \Delta_G} (1 - q^n e^{i \alpha \cdot \theta})(1 - q^{n-1} e^{-i \alpha \cdot \theta})^{-1}.
\]
(58)

where the restriction on the sum over \(p_j\)’s is
\[
\left( \sigma_g(\Lambda + \frac{\rho_g}{2} + t_g) - \rho_g \frac{2}{2} + \sigma_h(\Lambda + \frac{\rho_h}{2} + t_h) - \rho_h \frac{2}{2} + \bar{\mu} + \rho_h + \sum_{j=1}^{k_1^1} p_j \alpha_j \right)_i = 0.
\]
(63)
The index \(i\) on the bracket is in the range \(1 \leq i \leq r_h\) since we only require the first \(r_h\) components of this vector to vanish.
4 Concluding remarks.

We have here calculated branching-functions or coset characters for a number of general coset models. We hope that we have displayed how simple and straightforward the application of the BRST invariant character is, to most general coset constructions. Among those examples we have, at least in part, verified results of other authors, [16]. They have under an assumption obtained their branching functions.

In the case of ref. [16] the assumption used is about the cohomology of a Felder-like operator [27], [28] as well as the cohomology of the BRST-like operator. In a later publication [29] it is, however, proven that if the assumption of the Felder-like operator holds then the cohomology of the BRST-like operator may be computed. It would certainly be interesting to know if the remaining assumption is completely validated due to the correctness of the character or if there is still some freedom left.

An obvious step in a more general understanding of coset models would be to analyze non-compact groups. In order to do this one would have to re-analyze the cohomology which does not necessarily follow in a straightforward way from [3]. One of the problems is to choose unitary representations for the group $G$. Those representations are not known for a general non-compact group. What is also important is that for affine algebras we must construct cosets that allow unitary representations, one essentially must divide out the compact directions c.f. ref. [31]. If we choose $G = SU(1, 1)$, where the unitary representations are known, the analysis should probably be straightforward.

Another open problem is constructions with arbitrary rational levels.

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