Approximation Algorithms for the Maximum Carpool Matching Problem

Gilad Kutiel

Department of Computer Science, Technion, Haifa, Israel
gkutiel@cs.technion.ac.il

Abstract. The Maximum Carpool Matching problem is a star packing problem in directed graphs. Formally, given a directed graph $G = (V, A)$, a capacity function $c : V \to \mathbb{N}$, and a weight function $w : A \to \mathbb{R}$, a feasible carpool matching is a triple $(P, D, M)$, where $P$ (passengers) and $D$ (drivers) form a partition of $V$, and $M$ is a subset of $A \cap (P \times D)$, under the constraints that for every vertex $d \in D$, $\deg^M_{\text{in}}(d) \leq c(d)$, and for every vertex $p \in P$, $\deg^M_{\text{out}}(p) \leq 1$. In the Maximum Carpool Matching problem we seek for a matching $(P, D, M)$ that maximizes the total weight of $M$.

The problem arises when designing an online carpool service, such as Zimride [1], that tries to connect between passengers and drivers based on (arbitrary) similarity function. The problem is known to be NP-hard, even for uniform weights and without capacity constraints.

We present a 3-approximation algorithm for the problem and 2-approximation algorithm for the unweighted variant of the problem.

1 Introduction

Carpooling, is the sharing of car journeys so that more than one person travels in a car. Knapen et al. [2] describe an automatic service to match commuting trips. Users of the service register their personal profile and a set of periodically recurring trips, and the service advises registered candidates on how to combine their commuting trips by carpooling. The service acts in two phases.

In the first phase, the service estimates the probability that a person $a$ traveling in person’s $b$ car will be satisfied by the trip. This is done based on personal information and feedback from users on past rides. The second phase is about finding a carpool matching that maximizes the global (total expected) satisfaction.

The second phase can be modeled in terms of graph theory. Given a directed graph $G = (V, A)$. Each vertex $v \in V$ corresponds to a user of the service and an arc $(u, v)$ exists if the user corresponding to vertex $u$ is willing to commute with the user corresponding to vertex $v$. A capacity function $c : V \to \mathbb{N}$ is defined according to the number of passengers each user can drive if she was selected as a driver. A weight function $w : A \to \mathbb{R}$ defines the amount of satisfaction $w(u, v)$, that user $u$ gains when riding with user $v$. 
A feasible carpool matching (matching) is a triple \((P, D, M)\), where \(P\) and \(D\) form a partition of \(V\), and \(M\) is a subset of \(A \cap (P \times D)\), under the constraints that for every driver \(d \in D\), \(\text{deg}^M_{\text{in}}(d) \leq c(d)\), and for every passenger \(p \in P\), \(\text{deg}^M_{\text{out}}(p) \leq 1\). In the Maximum Carpool Matching problem we seek for a matching \((P, D, M)\) that maximizes the total weight of \(M\). In other words, the Maximum Carpool Matching problem is about finding a set of (directed toward the center) vertex disjoint stars that maximizes the total weights on the arcs. Figure 1 is an example of the Maximum Carpool Matching problem.

**Fig. 1.** A carpool matching example: (a) a directed graph with capacities on the vertices and weights on the arcs. (b) a feasible matching with total weight of 26. \(P\) is the set of blue vertices, and \(D\) is the set of red, dashed vertices.

Hartman et al. [5] proved that the Maximum Carpool Matching problem considered in this paper is NP-hard, and that the problem remains NP-hard even for a binary weight function when the capacity function \(c(v) \leq 2\) for every vertex in \(V\). It is also worth mentioning, that in the undirected, uncapacitated, unweighted variant of the problem, the set of drivers in an optimal solution form a minimum dominating set. When the set of drivers is known in advanced, however, the problem becomes tractable and can be solved using a reduction to a flow network problem.

Agatz et al. [2] outlined the optimization challenges that arise when developing technology to support ride-sharing and survey the related operations research models in the academic literature. Hartman et al. [6] designed several heuristic algorithms for the Maximum Carpool Matching problem and compared their performance on real data. Other heuristic algorithms were developed as well [8]. Arkin et al. [3], considered other variants of capacitated star packing where a capacity vector is given as part of the input and capacities need to be assigned to vertices.

Nguyen et al. [9] considered the spanning star forest problem (the undirected, uncapacitated, unweighted variant of the problem). They proved the
following results: 1. there is a polynomial-time approximation scheme for planner graphs; 2. there is a polynomial-time $\frac{5}{3}$-approximation algorithm for graphs; 3. there is a polynomial-time $\frac{1}{2}$-approximation algorithm for weighted graphs. They also showed how to apply the spanning star forest model to aligning multiple genomic sequences over a tandem duplication region. Chen et al. [4] improved the approximation ratio to 0.71, and also showed that the problem can not be approximated to within a factor of $\frac{31}{32} + \epsilon$ for any $\epsilon > 0$ under the assumption that $P \neq NP$. It is not clear, however, if any of the technique used to address the spanning star forest problem can be generalized to approximate the directed capacitated variant.

In section 3 we present an exact, efficient algorithm for the problem when the set of drivers and passengers is given in advanced. In section 4 we present a 2-approximation local search algorithm for the unweighted variant of the problem. Finally in section 5 we give a 3-approximation algorithm for the problem.

2 Maximum Weight Flow

A flow network is a tuple $N = (G = (V, A), s, t, c)$, Where $G$ is a directed graph, $s \in V$ is a source vertex, $t \in V$ is a target vertex, and $c : A \rightarrow \mathbb{R}$ is a capacity function. A flow $f : A \rightarrow \mathbb{R}$ is a function that has the following properties:

- $f(e) \leq c(e)$, $\forall e \in A$
- $\sum_{(u,v) \in A} f(u, v) = \sum_{(v,w) \in A} f(v, w)$, $\forall v \in V \setminus \{s,t\}$

Given a flow function $f$, and a weight function $w : A \rightarrow \mathbb{R}$, the flow weight is defined to be: $\sum_{e \in A} w(e) f(e)$. A flow with a maximum weight (maximum weight flow) can be efficiently found by adding the arc $(t, s)$, with $c(t, s) = \infty$, and $w(t, s) = 0$ and reducing the problem (by switching the sign of the weights) to the minimum cost circulation problem [10]. When the capacity function $c$ is integral, a maximum weight integral flow can be efficiently found.

3 Fixed Maximum Carpool Matching

In the Fixed Maximum Carpool Matching problem, $P$ and $D$ are given, and the goal is to find $M$ that maximizes the total weight. This variant of the problem can be solved efficiently [1], by reducing it to a maximum weight flow (flow) problem as follow: Let $(G = (V, A), c, w)$ be a Maximum Carpool Matching instance, let $(F, D)$ be a partition of $V$, let $N = (G' = (V', A'), s, t, c')$ be a flow network, and let $w' : A \rightarrow \mathbb{N}$ be a weight function, where

\[ \text{A solution to this variant of the problem was already proposed in [6]. For the sake of completeness, however, we describe a detailed solution for this variant. More importantly, the described solution helps us develop the intuition and understand the basic idea behind the approximation algorithm described in Section 5.} \]
\[ V' = P \cup D \cup \{s, t\} \]
\[ A' = A_{sp} \cup A_{pd} \cup A_{dt} \]
\[ A_{sp} = \{(s, p) : p \in P\} \]
\[ A_{pd} = A \cap (P \times D) \]
\[ A_{dt} = \{(d, t) : d \in D\} \]
\[ c'(u, v) = \begin{cases} c(u) & \text{if } (u, v) \in A_{dt} \\ 1 & \text{otherwise} \end{cases} \]
\[ w'(e) = \begin{cases} w(e) & \text{if } e \in A_{pd} \\ 0 & \text{otherwise} \end{cases} \]

The flow network is described in Figure 2.

Fig. 2. Illustration of a flow network corresponding to a Fixed Maximum Carpool Matching instance.

**Observation 1** For every integral flow \( f \) in \( N \), there is a carpool matching \( M \) on \( G \) with the same weight.

**Proof.** Consider the carpool matching \((P, D, M^f)\), where
\[ M^f = \{(p, d) \in A_{pd} : f(p, d) = 1\} \]
one can verify that this is indeed a matching with the same weight as \( f \).

**Observation 2** For every carpool matching \((P, D, M)\) on \( G \), there exists a flow \( f \) on \( N \) with the same weight.
Proof. Consider the flow function

\[ f(s, p_i) = \deg_{\text{out}}^M(p_i) \]

\[ f(p_i, d_j) = \begin{cases} 1 & \text{if } (p_i, d_j) \in M \\ 0 & \text{otherwise} \end{cases} \]

\[ f(d_j, t) = \deg_{\text{in}}^M(d_j) \]

It is easy to verify that \( f \) is indeed a flow function. Also, observe that by construction, the weight of \( f \) equals to the weight of the matching.

As we mentioned, the maximum weight flow problem can be solved efficiently, and so is the Fixed Maximum Carpool Matching problem. It is worth mentioning, that it is possible that in a maximum weight flow, some of the arcs will have no flow at all, that is, it is possible that in a Fixed Maximum Carpool Matching some of the passengers and drivers will be unmatched.

4 Unweighted Carpool Matching

In this section we present a local search algorithm for the unweighted variant of the problem. We show that the approximation ratio of this algorithm is 2 and give an example to show that our analysis is tight.

Given a directed graph \( G = (V, A) \), and a capacity function \( c : V \rightarrow \mathbb{N} \), in the Unweighted Carpool Matching problem, we seek for a matching that maximizes the size of \( M \).

We now present a simple local search algorithm for the problem. The algorithm maintains a feasible matching through its execution. In every iteration of the algorithm, the size of \( M \) increases. The algorithm terminates, when no further improvement can be made.

Recall that the Fixed Maximum Carpool Matching can be solved efficiently. Let \( M = \text{opt}_{\text{fixed}}(P, D) \) be an optimal solution of the Fixed Maximum Carpool Matching problem. For a given matching \( M \), define the following sets:

- \( P^M = \{ v : \deg_{\text{out}}^M(v) = 1 \} \)
- \( D^M = \{ v : \deg_{\text{in}}^M(v) > 0 \} \)
- \( D^M_s = \{ v : \deg_{\text{in}}^M(v) = c(v) \} \)
- \( F^M = \{ v : \deg_{\text{in}}^M(v) = \deg_{\text{out}}^M(v) = 0 \} \)

We refer to the vertices in these sets as, passenger, driver, saturated driver, and free vertex respectively. The local search algorithm, in every iteration, tries to improve the current matching, by switching a passenger or a free vertex into a driver and compute an optimal fixed matching. The local search algorithm is described in Algorithm 4.

First, observe that the outer loop on line 2 of the local search algorithm can be executed at most \( n \) times, where \( n \) is the total number of vertices, this is because the loop is executed only when there was an improvement, and this
Algorithm 1: Local Search

**Input:** \( G = (V, A), c : V \rightarrow \mathbb{N} \)

**Output:** \( M \)

1. \( M \leftarrow \emptyset \)
2. repeat
3. \( done \leftarrow \text{true} \)
4. for \( v \in (V \setminus D^M) \) do
5. \( D \leftarrow D^M \cup \{v\} \)
6. \( P \leftarrow V \setminus D \)
7. \( M' = \text{opt}_{\text{fixed}}(P, D) \)
8. if \( |M'| > |M| \) then
9. \( M \leftarrow M' \)
10. \( done \leftarrow \text{false} \)
11. end
12. end
13. until \( done \); 
14. return \( M \)

This can happen at most \( n \) times. Also, observe that the body of this loop can be computed in polynomial time, and we can conclude that Algorithm 1 runs in polynomial time.

We now prove that the local search algorithm achieves an approximation ratio of 2. Let \( M \) be a matching found by the local search algorithm, and let \( M^* \) be an arbitrary but fixed optimal matching. Observe that every arc in \( M^* \) has at least one end point in \( M \), formally:

**Observation 3** If \( (u, v) \in M^* \), then \( \{u, v\} \cap (P^M \cup D^M) \neq \emptyset \)

*Proof.* If this is not the case, Algorithm 1 can improve \( M \) by adding the arc \((u, v)\).

Now, with respect to \( M \), the optimal solution can not match two free vertices to the same passenger, formally:

**Observation 4** If \( (p, d) \in M^* \), \( f_1, f_2 \in F^M \), and \((f_1, p), (f_2, p) \in M^* \), then \( f_1 = f_2 \).

*Proof.* If this is not the case, Algorithm 1 can improve \( M \) by removing the arc \((p, d)\) and adding the arcs \((f_1, p), (f_2, p)\).

Finally, with respect to \( M \), the optimal solution can not match a free vertex to a driver that is not saturated, formally:

**Observation 5** If \( (f, d) \in M^* \), \( f \in F^M \), and \( d \in D^M \), then \( d \in D^M_{c} \).

*Proof.* If this is not the case, once again, Algorithm 1 can improve \( M \) by adding the arc \((f, d)\).
To show that Algorithm 1 is 2-approximation, consider the charging scheme that is illustrated in Figure 3. Load every arc \((p, d) \in M\) with 2 coins, place one coin on \(p\) and one coin on \(d\). Observe that every vertex \(p \in P^M\) is loaded with one coin, and every vertex \(d \in D^M\) is loaded with \(\text{deg}^M_1(d)\) coins. Now, pay one coin for every \((u, v) \in M^*\), charge \(u\) if \(u \in P^M \cup D^M\), otherwise (\(v \in P^M \cup D^M\)) charge \(v\). Clearly, every arc in \(M^*\) is paid. We claim that no vertex is overcharged.

**Observation 6** If \(u \in P^M\), then \(u\) is not overcharged.

*Proof.* If \(u \in P^M^*\), then it is only charged once, otherwise, if \(u \in D^M^*\), then it is only charged for arcs \((w, u)\) where \(w \in F^M\), and by Observation 3 there is at most one such arc.

**Observation 7** If \(u \in D^M\), then \(u\) is not overcharged.

*Proof.* If \(u \in P^M^*\), then it is only charged once, if \(u \in D^M^*\), then it is only charged for arcs \((w, u)\) where \(w \in P^M\), if such arcs exists, then by observation 6 u is saturated, and can not be overcharged.

**Theorem 1.** Algorithm 1 is 2-approximation

*Proof.* We use a charging scheme where we manage to pay 1 coin for each arc in \(M^*\) by using at most \(2|M|\) coins.

To conclude this section, we show that our analysis is tight. Consider the example given in Figure 3. Assume, in this example, that there are no capacity constraints, if the local search algorithm starts by choosing vertex 3 to be a driver, then the returned matching is the single arc \((2, 3)\). At this point, no

![Charging Scheme](image-url)
further improvement can be done. The optimal matching, on the other hand, is \{(1, 2), (3, 2)\}. The path in the example can be duplicated to form an arbitrary large graph (forest).

Fig. 4. Local Search - Worst Case Example

5 Maximum Carpool Matching

5.1 Super Matching

A super-matching is a relaxed variant of the Maximum Carpool Matching problem where every node can act both as a driver and as a passenger. Formally, given a directed graph \( G = (V, A) \), a capacity function \( c : V \rightarrow \mathbb{N} \), and a weight function \( w : A \rightarrow \mathbb{R} \), a super-matching is a set \( M \subseteq A \), under the constraint that \( \forall v \in V, \deg^M_{\text{in}}(v) \leq c(v) \), and \( \deg^M_{\text{out}}(v) \leq 1 \). Clearly, the following observation holds:

**Observation 8** Every matching \((P, V, M)\) is a super-matching \(M\)

A maximum super matching can be found efficiently by the following reduction to a maximum weight flow problem: Let \( N = (G', s, t, c', w') \) be a flow network, where

\[
\begin{align*}
G' &= (P \cup D \cup \{s, t\}, A_{sp} \cup A_{pd} \cup A_{dt}) \\
P &= \{p_v : v \in V\} \\
D &= \{d_v : v \in V\} \\
A_{sp} &= \{(s, p_v) : p_v \in P\} \\
A_{pd} &= \{((p_u, d_v)) : (u, v) \in A\} \\
A_{dt} &= \{(d_v, t) : d_v \in D\} \\
c'(s, p_v) &= c'(p_u, d_v) = 1 \\
c'(d_v, t) &= c(v) \\
w'(p_u, d_v) &= \begin{cases} w(u, v) & \text{if } (p_u, d_v) \in A_{pd} \\ 0 & \text{otherwise} \end{cases}
\end{align*}
\]

That is, we construct a bipartite graph where the left side represents each vertex in \( V \) being a passenger, and the right side represents each vertex in \( V \) being a driver. Figure 5 illustrates this flow network. One can verify that this is indeed a (integral) flow network and that there is a straightforward translation between a flow and a super matching with the same weight.
5.2 3-approximation

We now present a 3-approximation algorithm for the MAXIMUM CARPOOL MATCHING problem. This algorithm acts in two phases. In the first phase it computes a maximum super-matching of $G$, in the second phase it decomposes the super-matching into 3 feasible carpool matching and output the best of them.

We now describe how a super-matching can be decomposed into 3 feasible carpool matching. First, consider the graph obtained by an optimal super-matching. Recall that in a super matching the out degree of every vertex is at most 1, that is, the graph obtained by an optimal super matching is a pseudoforest - every connected component has at most one cycle. We now eliminate cycles from the super-matching by removing one edge from every connected component. It is easy to see that the resulting graph is a forest of anti-arborescences. Each of these anti-arborescences can be, in turn, decomposed into two disjoint feasible carpool matching. This can be done, for example, by coloring each such anti-arborescences with two colors, say red and green, and then consider the two solutions: one where the green nodes are the drivers, and the other where the red nodes are the drivers. We describe the algorithm in Algorithm 2 and illustrate it in Figure 6.
Fig. 6. Illustration of the SuperMatching algorithm: (a) a directed graph, (b) a maximum super-matching, (c) an anti-arborescence: $M_1$ is the set of arcs exiting red, dashed vertices, and $M_2$ is the set of arcs exiting blue vertices, (d) a feasible carpool matching with total value of 6.

Theorem 2. Algorithm 2 achieves a 3-approximation ratio.

Proof. Let $M_a = \bigcup_i \{a_i\}$ be the set of all removed arcs in the cycle elimination phase. Let $M_1 = \bigcup_i M_1^i$, and $M_2 = \bigcup_i M_2^i$. Clearly, $M_a \cup M_1 \cup M_2 = A'$, and that $\max(w(M_a), w(M_1), w(M_2)) \geq \frac{w(A')}{3}$. The observation that the weight of a maximum super-matching is an upper bound on the weight of a maximum carpool matching finishes the proof.

To see that our analysis is tight, consider the example in Figure 7. Assume, for the given graph in the figure, that all weights are 1 and that there is no capacity constraint. The maximum matching, then, is 3 (\{(1, 4), (2, 4), (3, 4)\}), but the algorithm can return the super matching \{(1, 2), (2, 3), (3, 1)\} from which only one arc can survive.

Fig. 7. Super Matching Algorithm, worst case example

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