Parallel Chip Firing Game associated with $n$-cube orientations

René Ndoundam$^1$, Maurice Tchuente$^{1,2}$, Claude Tadonki$^3$

$^1$ Department of Computer Science, Faculty of Science, University of Yaoundé I, P.o. Box. 812 Yaoundé, Cameroon

$^2$ Department of Mathematics and Computer Science, Faculty of Science, University of Ngaoundéré

$^3$ University of Paris-Sud 11, CNRS, IEF (AXIS Group)

UMR 8622 - Bât 220 - Centre Scientifique d’Orsay - F91405 Orsay Cedex, France

E.mail: ndoundam@yahoo.com, Maurice.Tchuente@ens-lyon.fr, claude.tadonki@u-psud.fr

February 15, 2022

Abstract

We study the cycles generated by the chip firing game associated with $n$-cube orientations. We show the existence of the cycles generated by parallel evolutions of even lengths from 2 to $2^n$ on $H_n$ ($n \geq 1$), and of odd lengths different from 3 and ranging from 1 to $2^{n-1}-1$ on $H_n$ ($n \geq 4$).

Keywords: Graph, chip firing game, parallel evolution, cycle, transient, $n$-cube orientation.

1 Introduction

Consider a digraph $G = (V, A)$, where $V = \{1, ..., n\}$ is the set of vertices and $A \subseteq V \times V$ is the set of arcs. The out-degree (resp. in-degree) of a vertex $i$,
hereafter denoted by $d^+(i)$ (resp. $d^-(i)$), is the number of vertices $j$ such that $(i, j) \in A$ (resp. $(j, i) \in A$). A vertex with out-degree zero is called a sink. All these notions apply to an undirected graph $G = (V, E)$ by considering an edge $e = [i, j]$ as two opposite arcs $(i, j)$ and $(j, i)$.

In the parallel chip firing game played on $G$, a state is a mapping $x : V \rightarrow \mathbb{N}$ which can be viewed as a distribution of chips onto the vertices of $G$. A vertex is said to be active in a state $x$ if $x(i) \geq d^+(i)$, otherwise it is said to be passive. In a move of the game, a state $x$ is transformed into a new state as follows:

1. If it is not possible, i.e. if $i$ is passive, then it resigns;
2. Otherwise, vertex $i$ is active and sends the chips.

It is easily seen that the number of chips remains constant. Therefore, the evolution is ultimately periodic. More precisely, if $x^t, t \geq 0$, denotes the state of the system at time $t$, then there exists an integer $q$ called transient length and another integer $p$ called period or cycle length such that

$$x^{t+p} = x^t \text{ for } t \geq q, \text{ and } x^{t+p'} \neq x^t \text{ for } p' < p. \quad (1)$$

The sequence $x^0, x^1, ..., x^{q-1}$ is called the transient and every sequence of $p$ consecutive states $x^t, x^{t+1}, ..., x^{t+p-1}$, such that $t \geq q$, is called a cycle of the evolution.

Following Spencer’s introductory paper [SPE 86] which was devoted to the chip firing game on chains, many authors have been interested in this problem. The most interesting questions concern the relationships between the structure of the graph on one hand, and the transients and periods generated by the chip firing game on the other hand. Concerning the period, Bitar and Goles have shown that if $G$ is a tree, then only periods one and two occur [BIT 92]. Later, Prisner has studied a generalization of the game by considering multigraphs, i.e.
digraphs with multiplicities on the arcs. He has then shown that there is a sharp contrast in the behavior for eulerian digraphs (i.e. digraphs where the in-degree of each vertex equals its out-degree). More precisely, he has proved that in every strongly connected eulerian multigraph, any divisor of every cycle length occurs as a period [PRI 94]. He has also shown that there is no polynomial \( h(n) \) such that the periods generated by the chip firing game on digraphs of order \( n \) are bounded by \( h(n) \). Readers interested by other results on periods and transients of the chip firing game may refer to [TAR 88, AND 89, BIT 89, BIO 91, ERI 91, GOL 93]. Readers interested by combinatorial games may refer to [Gol 02, GM 02, Gol 04, Sjo 05, Fra 09].

There is a particular case where the chip firing game is related to graph orientations. Indeed, let us consider an undirected graph \( G = (V, E) \), and let us assume that initially, the edges of \( G \) can be oriented in such a way that the number of chips of every vertex equals the in-degree of that vertex. If this property is true in the initial configuration, then it remains true throughout the game. One step of the game then consists in reversing the orientations of all edges going into sinks. Goles and Prisner [GOL 00] have studied gardens of Eden, i.e. states that can appear only at time \( t = 0 \). They have also studied the relationships between graph orientations and evolutions induced by states with \( |E| \) chips. Moreover, Kiwi, Ndoundam, Tchuente and Goles [KIW 94] have exhibited cycles of exponential length \( e^{\Omega(n \log n)} \) generated by the chip firing game associated with the orientations of cascades of rings. Other results on this particular case may be found in [ERI 94].

In this paper, we study the dynamics generated by the chip firing game associated with \( n\)-cube orientations. More precisely, using a recurrent approach, we show that for \( n \geq 4 \), there exists cycles of even lengths from 2 to \( 2^n \) on \( H_n \) \((n \geq 1)\), and of odd lengths different from 3 and ranging from 1 to \( 2^{n-1} - 1 \) on \( H_n \) \((n \geq 4)\).

The remainder of this paper is organized as follows. In the next section,
we present some basic notations and definitions related to \( n \)-cubes. Section 3 is devoted to the recurrent construction of left cyclic partitions and possible period lengths whereas section 4 presents some concluding remarks.

## 2 Basic notations and definitions

An \( n \)-dimensional hypercube (or \( n \)-cube) is an undirected graph \( H_n = (V, E) \), where \( V = \{0, 1\}^n \) is the set of vertices and two nodes \( u = (u_0, u_1, ..., u_{n-1}) \) and \( v = (v_0, v_1, ..., v_{n-1}) \) are neighbors if and only if they differ in only one bit in their binary representations, i.e. there is an integer \( i \) such that \( u_i \neq v_i \) and \( u_j = v_j \) for \( j \neq i \). One can define recursively the \( n \)-cube as follows:

- The 0-cube is reduced to one vertex;
- \( H_{n+1} \) is obtained by taking two copies of \( H_n \) and connecting all pairs of equivalent vertices.

Fig. 1 illustrates this constructions for \( n = 0, 1, 2, 3 \).

**Figure 1:** \( n \)-cubes for \( n \leq 3 \).

Hereafter, given a set \( W \) and a boolean value \( x \), we denote \( xW = \{ xu : u \in W \} \). With this notation, we can write \( H_{n+1} = 0H_n \cup 1H_n \).

**Definition 1.** A block-sequential evolution of the chip firing game associated with graph orientations and played on an \( n \)-cube is obtained as follows. Consider a sequence of non empty subsets \( \{ W_i : i \geq 0 \} \) of \( \{0, 1\}^n \). At time \( t \), every vertex \( u \) of \( W_t \) is considered. If \( u \) is a sink then the orientation of all its in-going arcs are reversed, otherwise no action is undertaken. Hereafter, we say
that a vertex fires at time $t$ if it belongs to $W_t$ and is a sink at time $t$.

The parallel evolution is therefore a particular case of the general scheme described above. Another classical evolution scheme is the sequential evolution where $W_t$ is reduced to one vertex (i.e. $|W_t| = 1$) and there is a permutation $\sigma$ of $\{0, ..., 2^n-1\}$ such that $\{W_i; i \geq 0\}$ is periodic of period $W_{\sigma(0)}, W_{\sigma(1)}, ..., W_{\sigma(2^n-1)}$. Both parallel and sequential evolutions are particular cases of the so-called serial-parallel evolutions where the sequence $\{W_i; i \geq 0\}$ is periodic of period $W_0, W_1, ..., W_{k-1}$, with the constraint that $W_0 \cup W_1 \cup ... \cup W_{k-1}$ is a partition of $\{0, 1, ..., 2^n-1\}$.

**Definition 2.** A partition $S_0 \cup S_1 \cup ... \cup S_{k-1}$ of the vertices of an $n$-cube is called a left cyclic partition if the two following statements hold.

- For all $i$ from $0$ to $k-1$, every vertex of $S_i$ has a neighbor in $S_{i-1}$, where index operations are performed modulo $k$.

- For all $i$ from $0$ to $k-1$, there is no edge between two vertices of $S_i$.

**Comment 2.** Canonical decompositions defined in [GOL 00] for acyclic digraphs are obtained from left cyclic partitions by orienting the edges such that all arcs from the set $S_i$ go to sets $S_j$ such that $j > i$. On the other hand, left cyclic partitions are more restrictive than the partitions introduced in [PRI 94] since we do not allow edges joining two vertices of the same subset. Indeed, in the chip firing game associated with graph orientations, two neighbors cannot fire simultaneously, whereas this situation is possible for the general chip firing game.

We present an important property of left cyclic partitions on an $n$-cube.

**Theorem 1** If a partition $S_0 \cup S_1 \cup ... \cup S_{k-1}$ of the vertices of an $n$-cube $H_n$
is a left cyclic partition then there is a cyclic evolution of the chip firing game associated with graph orientations and played on $H_n$, such that for every $t \geq 0$, $S_t$ is the set of vertices which are fired at time $t$.

**Proof.** Let $S_0, S_1, ..., S_k$ be a left cyclic partition. Consider an orientation where every edge $e = [u, v]$ such that $u \in S_i$ and $v \in S_j, i < j$, is oriented from $v$ to $u$. It is easily seen that in the parallel chip firing game starting with such a configuration, the subsets of vertices which fire at successive steps correspond to a periodic sequence of period $S_0, S_1, ..., S_k$.

$\square$

### 3 Recurrent construction of left cyclic partitions

In this section, we first present the construction of left cyclic partitions of even lengths.

**Lemma 1** An $n$-cube admits left cyclic partitions of all even lengths from 2 to $2^n$.

**Proof.** Let $H_n = (V, E)$ be an $n$-cube an let $p$ be an even integer between 2 and $2^n$. It is well known that, since $p$ is even, there is a cycle $[x_0, x_1, ..., x_{p-1}, x_0]$ of length $p$ in $H_n$. Now, for every vertex $u$, let $\Gamma(u)$ denote the set of all neighbors of $u$ in $H_n$. This notation is naturally extended to a set of vertices. A left cyclic partition of order $p$ is obtained as follows.

For $i = 0, ..., p - 1$

1. $S_i \leftarrow \{x_i\}$

endfor

$S = V - \{x_0, x_1, ..., x_{p-1}\}$

while ($S \neq \emptyset$) do

For $i \leftarrow 0$ to $p - 1$

1. $S_{i+1} \leftarrow S_{i+1} \cup (\Gamma(S_i) \cap S)$

endfor
S ← S − (Γ(S_i) ∩ S)
endfor
endwhile

It is obvious that S_0, ..., S_{p−1} is a partition of V and that every vertex in S_i has at least one neighbor in S_{i−1}. So we just need to show that two vertices of the same subset S_i cannot be neighbors. Let a and b be two vertices of S_i.

- There is a path from a to x_0 of length \ell_1 such that \ell_1 = i \mod p,
- There is a path from b to x_0 of length \ell_2 such that \ell_2 = i \mod p,

Since p is even, it follows that \ell_1 = \ell_2 \mod 2. Hence, if a and b were neighbors, there would exist a cyclic path of odd length \ell_1 + \ell_2 + 1 joining a and b in H_n, which is not possible since H_n is a bipartite graph. This shows that two vertices of the same subset cannot be neighbors.

\[\square\]

The following figure displays the partition of order 4 in H_3 obtained by the previous procedure starting with the cycle [000, 001, 011, 010].

![Figure 2: Left cyclic partition of orders 4 generated in H_3.](image)

Let us now turn to the construction of left cyclic partitions of odd lengths.

**Lemma 2** If S_0, S_1, S_2 is a left cyclic partition of H_n, n ≥ 2, then every vertex of S_i has at least two neighbors in S_{i−1} for i = 0, 1, 2.

**Proof.** Because of symmetry considerations, we can assume that i = 2. So let x be a vertex of S_2. From the definition of left cyclic partitions,

- x has a neighbor x ⊕ e_j in S_1, where ⊕ is the XOR operator and e_j is a
vector of the canonical basis.

- similarly, \( x \oplus e_j \) has a neighbor \( x \oplus e_j \oplus e_k \) in \( S_0 \).

Now consider the vertex \( x \oplus e_k \).

- It is a neighbor of \( x \), hence it does not belong to \( S_2 \).
- It is a neighbor of \( x \oplus e_j \oplus e_k \), hence it does not belong to \( S_0 \).

It then follows that \( x \oplus e_k \) belongs to \( S_1 \), hence \( x \) admits two neighbors \( x \oplus e_j \) and \( x \oplus e_k \) which are both in \( S_1 \).

\[ \square \]

**Lemma 3** If \( H_n, n \geq 3 \) admits a left cyclic partition of order 3, then \( H_{n-1} \) admits a left cyclic partition of order 3.

**Proof.** Let \( S_0, S_1, S_2 \) be a left cyclic partition of order 3 of \( H_n \). Let \( x \) be a vertex of \( S_i \). We can assume without loss of generality that \( x = 1a \). Since \( H_n = 0H_{n-1} \cup 1H_{n-1} \), \( y = 0a \) is the unique neighbor of \( x \) in \( 0H_{n-1} \). Consequently, from lemma 2, \( x \) admits a neighbor in \( 1H_{n-1} \cap S_i \). This shows that the subgraph \( 1H_{n-1} \) which is isomorphic to \( H_{n-1} \), contains a left cyclic partition of order 3.

\[ \square \]

**Proposition 1** \( n \)-cubes do not admit left cyclic partitions of order 3.

**Proof.** An \( n \)-cube with \( n \leq 1 \) has less than 3 vertices and cannot admit a left cyclic partition of order 3. On the other hand, from lemma 2, if \( S_0, S_1, S_2 \) is a left cyclic partition of an \( n \)-cube, \( n \geq 2 \), then every \( S_i \) contains at least two elements (i.e. \( |S_i| \geq 2 \)). Consequently, the 2-cube \( H_2 \) which is of cardinality 4 cannot admit a left cyclic partition of order 3. By application of lemma 3, we deduce that no \( n \)-cube, \( n \geq 3 \) admits a left cyclic partition of order 3.

\[ \square \]

Proposition 1 gives the lower bound for left cyclic partitions of odd lengths. Let now study the upper bound.
Proposition 2 If $S_0, ..., S_{p-1}$ is a left cyclic partition of odd order $p$ of $H_n$, then $p \leq 2^{n-1} - 1$.

Proof. We just have to show that in such a case, $|S_i| \geq 2$ for $i = 0, ..., p - 1$.

Indeed, starting from a vertex $a_{p-1} \in S_{p-1}$, we construct a chain $[a_{p-1}, a_{p-2}, ..., a_0, b_{p-1}, b_{p-2}, ..., b_0]$ such that $a_i, b_i \in S_i$ for $i = 0, ..., p-1$. It is clear that $a_i \neq b_i, i = 0, ..., p-1$, otherwise we would have displayed a closed path of odd length in $H_n$ which is not possible.

Now that we have established lower and upper bounds for left cyclic partitions of odd lengths, let us show that all intermediate lengths are admissible.

Lemma 4 If $H_n$ admits a left cyclic partition of order $p$, then $H_{n+1}$ admits left cyclic partition of order $p$.

Proof. If $S_0, ..., S_{p-1}$ is a left cyclic partition of order $p$ in $H_n$, then it is easily checked that $1S_i \cup 0S_{i-1}, i = 0, ..., p - 1$ is a left cyclic partition of order $p$ in $H_{n+1}$.

Lemma 5 If $H_n$ admits a left cyclic partition of odd order $p$, $p \geq 5$ then $H_{n+1}$ admits a left cyclic partition of order $2p - 1$. Moreover, if $p \geq 7$, then $H_{n+1}$ admits a left cyclic partition of order $2p - 3$.

Proof. Let $S_0, S_1, ..., S_{p-1}$ be a left cyclic partition of odd order $p$.

• Case $p \geq 5$

The following sequence is a left cyclic partition of order $2p - 1$ in $H_{n+1}$.

$0S_0, 1S_0 \cup 0S_1, 1S_1, 1S_2, 0S_2, 0S_3, 1S_3, ..., 1S_{2t}, 0S_{2t}, 0S_{2t+1}, 1S_{2t+1}, ..., 1S_{p-3}, 0S_{p-3}, 0S_{p-2}, 1S_{p-2}, 1S_{p-1}, 0S_{p-1}$.

• Case $p \geq 7$

A left cyclic partition of order $2p - 3$ in $H_{n+1}$ is obtained from the left cyclic...
Parallel Chip firing game on hypercube

Partition exhibited in the case $p \geq 5$ by replacing the subsequence $1S_2, 0S_3, 1S_3, 1S_4, 0S_4, 0S_5, 1S_5$ by $1S_2, 0S_2 \cup 1S_3, 0S_3, 1S_4 \cup 0S_5, 1S_5$.

Lemma 6 $H_4$ admits left cyclic partitions of orders 5 and 7.

Proof.
- A left cyclic partition of order 5 in $H_4$ is the following:
  \{0000, 1101\}, \{0001, 1100, 0010, 1111\}, \{0110, 1011\}, \{0100, 0111, 1001, 1010\},
  \{0111, 0101, 1000, 1110\}.
- A left cyclic partition of order 7 in $H_4$ is the following:
  \{0000, 1101\}, \{0001, 1100\}, \{0011, 1110\}, \{0010, 1111\}, \{0110, 1011\},
  \{0100, 0111, 1001, 1010\}, \{0101, 1000\}.

Fig. 3 displays the partitions.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{left_cyclic_partitions}
\caption{Left cyclic partitions of orders 5 and 7 in $H_4$.}
\end{figure}

\end{document}
Parallel Chip firing game on hypercube

11

x, and symbol / denotes integer division. It can be easily checked that this sequence corresponds to a **hamiltonian cycle** in $H_{n-1}$. Now, let us denote $v_i = u_i \oplus 1 \oplus 2^{n-2}$ (i.e. $v_i$ is obtained from $u_i$ by changing the first and last bits) and $N = 2^n$. It is also easy to check that \{v_i; 0 \leq i \leq 2^{n-1} - 1\} is a **hamiltonian cycle** of $H_{n-1}$. Let us now consider the following sets

$$\{0u_0, 1v_0\}, \ldots, \{0u_{N-4}, 1v_{N-4}\}, \{0u_{N-3}, 0u_{N-1}, 1v_{N-3}, 1v_{N-1}\}, \{0u_{N-2}, 1v_{N-2}\}. \quad (2)$$

At this step, it is important to recall that two vertices referenced by $i$ and $j$ are neighbors in the hypercube if and only if there is an integer $k$ such that $i \oplus j = 2^k$. Observe that $0u_i \oplus 1v_i = 2^{n-1} \oplus (u_i \oplus v_i) = 2^{n-1} \oplus 1 \oplus 2^{n-2}$. Hence, $0u_i$ and $1v_i$ are not neighbors in the hypercube $H_n$. On the other hand, $u_{N-4} = 100\ldots010$, $u_{N-2} = 10\ldots01$, $u_{N-1} = 10\ldots0$ and $v_0 = u_0 \oplus 1 \oplus 2^{n-2} = 10\ldots01 = u_{N-2}$. Hence,

$$0u_{N-4}, 0u_{N-1}, 0u_{N-2}, 1v_0$$

is a chain of $H_n$. \quad (3)

Moreover, $v_{N-4} = 0\ldots011$, $v_{N-2} = 0\ldots0 = u_0$ and $v_{N-1} = 0\ldots01 = u_1$. Hence

$$1v_{N-4}, 1v_{N-1}, 1v_{N-2}, 0u_0$$

is a chain of $H_n$. \quad (4)

Properties 3 and 4 together with the fact that \{u_i; 0 \leq i \leq 2^{n-1}-1\} and \{v_i; 0 \leq i \leq 2^{n-1}-1\} are both **hamiltonian cycles** of $H_n$, imply that the partition exhibited in (2) is a **left cyclic partition**.

\[ \square \]

**Proposition 3** $H_n$, $n \geq 4$, admits left cyclic partitions of all odd orders from 5 to $2^{n-1} - 1$. 
Proof. We proceed by induction on \( n \). For \( n = 4 \) the result follows from lemma 6.

Assuming that the result holds for \( n \geq 4 \), let us consider an \((n+1)\)-cube together with an odd integer \( p \in [5, 2^n - 1] \).

- Case 1: \( 5 \leq p \leq 2^{n-1} - 1 \). The result follows from the induction hypothesis by application of lemma 4.

- Case 2: \( 2^{n-1} - 1 < p < 2^n - 1 \). There is an odd integer \( q \) such that \( 7 < q < 2^{n-1} - 1 \), such that \( p = 2q - 1 \) or \( p = 2q - 3 \). The result follows from the induction hypothesis by application of lemma 5.

- Case 3: \( p = 2^n - 1 \). The result follows from lemma 7.

\( \square \)

We are now ready to state the main theorem.

**Theorem 2** There exists cycles generated by the parallel chip firing game associated with \( n \)-cube orientations, \( n \geq 4 \), are of even lengths from 2 to \( 2^n \), and of odd lengths different from 3 and ranging from 1 to \( 2^{n-1} - 1 \).

**Proof.** We just need to show that this property holds for left cyclic partitions of vertices of \( n \)-cubes. The existence of left cyclic partitions of all even lengths from 2 to \( 2^n \) follows from lemma 1. Let us turn to odd periods \( p \).

- Case 1: \( p = 1 \). Consider and orientation which contains a hamiltonian cycle. Clearly, such a configuration is a fixed point for the chip firing game associated with graph orientations.

- Case 2: \( p = 3 \). The non existence of period 3 follows from proposition 1.

- Case 3: \( 5 \leq p \leq 2^{n-1} - 1 \). The existence of this period follows from proposition 3.

\( \square \)
4 Conclusion

We show in the particular case of parallel evolutions on \( n \)-cube, the existence of cycles of even lengths from 2 to \( 2^n \), and of odd lengths different from 3 and ranging from 1 to \(-1 + 2^{n-1}\). In case of parallel evolutions on \( n \)-cube, the existence of cycles of lengths greater than \( 2^n \) remains an open question.

Acknowledgement. This work was supported by the French Agency Aire développement through the project Calcul Parallèle and by the Project NTIC of the University of Ngaoundéré.

References

[AND 89] R.J. Anderson, L. Lowasz, P. W. Shor, J. Spencer, E. Tardos, S. Winograd, Disks, balls and walls : analysis of a combinatorical game, American Mathematical Monthly, vol.96, pp. 481-493, 1989.

[BIT 89] J. Bitar, Juegos Combinatoricas en redos automatas, Tésis de ingenieros Matematico, Fac de Cs. Fisicas y Matematicas, U. de Chile, Santiago, Chile 1989.

[BIT 92] J. Bitar and E. Goles, Parallel chip firing games on graphs, Theoretical Computer Science, 92, pp. 291-300, 1992.

[BJO 91] A. Bjorner, L. Lovasz, P. W. Shor, Chip firing game on graphs, European J. Combin. 12, pp. 283-291, 1991.

[ERI 91] K. Erikson, No polynomial bound for the chip firing game on directed graphs, Proc. Amer. Math. Soc. 112, pp. 1203-1205, 1991.

[ERI 94] H. Erikson and K. Erikson, Chip firing game and coxeter elements, Proceedings of FPSAC 94, 1994.
Parallel Chip firing game on hypercube

[Fra 09] Aviezri S. Fraenkel, Combinatorial Games: Selected Bibliography with a Succint Gourmet Introduction, The Electronic Journal of Combinatorics (2009).

[GOL 93] E. Goles, M. A. Kiwi, Games on line graphs and sand piles, Theoretical Computer Science, 115, pp. 321-349, 1993.

[GOL 00] E. Goles, Erich Prisner Source Reversal and Chip Firing Game, Theoretical Computer Science, Volume 233, pages 287-295 2000.

[Gol 02] E. Goles, M. Morvan, H. Duong Phan, Sandpiles and order structure of integer partitions, Discrete Applied Mathematics, Volume 117, 2002, pages 51-54

[GM 02] E. Goles, M. Morvan, H. Duong Phan, The structure of a linear chip firing game and related models, Theoretical Computer Science, Volume 270, 2002, pages 827-841.

[Gol 04] E. Goles, M. Latapy, C. Magnien, M. Morvan, H. Duong Phan, Sandpile models and lattices : a comprehensive survey. Theoretical Computer Science, Volume 322, 2004, pages 383-407

[KIW 94] M. A. Kiwi, R. Ndoundam, M. Tchuente and E. Goles No polynomial bound for the period of the parallel chip firing game on graphs, Theoretical Computer Science, 136, pp. 527-532, 1994.

[PRI 94] E. Prisner, Parallel chip firing on digraphs, Complex Systems, No. 8, pp. 367-383, 1994.

[NDO 94] R. Ndoundam and M. Tchuente, Comportement dynamique d'un réseau d'automates associé aux orientations d'un graphe, Acte du CARF’94, Ouagadougou, pp. 495-505, Octobre 1994.

[NDO 95] R. Ndoundam, Analyse et Synthèse de certains reseaux d’automates, Thèse de Doctorat de 3ème cycle, Université de Yaoundé I, 1995.
[Sjo 05] J. Sjöstrand, *The cover pebbling theorem*, The Electronic Journal of Combinatorics 12 (2005).

[SPE 86] J. Spencer, *Balancing vectors in the max norm*, Combinatorica, 6, pp. 55-66, 1986.

[TAR 88] G.Tardos, *Polynomial bound for the chip firing game on graphs*, SIAM Journal of Discrete Mathematics, 1, 3, pp. 397-398, 1988.