CHOICES, INTERVALS AND EQUIDISTRIBUTION

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Abstract. We show that random sequences generated by a family of “choice” interval splitting processes are equidistributed. This includes the max-2 process as well as more general mixtures, solving an open problem from Maillard and Paquette.

1. Introduction

A sequence in [0, 1] is equidistributed if the limiting proportion of terms in each subinterval is equal to the subinterval’s length. Over a century ago Weyl proved that \( \{ \beta n \mod 1 \}_{n \geq 1} \) is equidistributed for any irrational number \( \beta \) (see [Wey10]). Since then connections have been found in ergodic theory, number theory, complex analysis and computer science ([BM72], [Vau77], [FSZ09], [CKK+07]). See [KN06] for an overview.

More recently attention has been given to equidistribution of random sequences. One way to obtain a random sequence in [0, 1] is to independently choose points uniformly. Call the resulting sequence the uniform process. The law of large numbers guarantees this is equidistributed almost surely.

Another random process known to equidistribute points is the Kakutani interval splitting procedure (introduced in [Kak76]), where at each step a point is added uniformly to the current largest subinterval. Almost sure equidistribution is proven in [Zwe78] and [Loo78] using stopping times. Because points are placed in the largest gaps they ought to spread more evenly than the uniform process. Indeed, [Pyk80] proves the size of the largest interval is asymptotic to \( 2/n \); the same order as the average interval. Compare to \( \log n/n \) in the uniform process (see [Dar53]).

[MP14] introduces a family of interval splitting processes that exhibit a wider range of behavior. The quintessential example is the max-2 process. The dynamics are as follows:

- Partition [0, 1] into subintervals by placing finitely many points in any manner.
- At each step sample two points uniformly from [0, 1]. Each lies in a subinterval formed by the previous configuration.
- Keep the point contained in the larger subinterval and disregard the other point. Break a tie by flipping a fair coin.

A discrete analogue of the max-2 process appears in [ABKU99] where \( n \) balls are placed into \( n \) bins. For each ball two bins are selected uniformly and the ball is placed in the bin with fewer balls. They find that the most-filled bin has \( \approx \log_2 \log n \) balls; significantly less then \( \approx \log n \) if the balls were instead placed uniformly. This is studied in more detail in [MRS00] and [LM05].

In the max-2 process choosing the larger gap should spread points more evenly. Despite our intuition this is difficult to formalize, and equidistribution was a primary open problem from [MP14].

Theorem 1. The max-2 process is equidistributed a.s.
The natural counterpart is the min-2 process where the point contained in the smaller subinterval is kept. Unlike the previous processes, points are prone to clump together. We show that some random mixtures of the max-2, uniform and min-2 processes are equidistributed.

**Theorem 2.** Any mixture of max-2, uniform and min-2 processes with probability $p$ of placing a point uniformly and probability less than or equal to $0.6 - 0.5p$ of placing a point according to the min-2 process is equidistributed a.s.

The formal definition of a mixture is in Section 5. As a corollary we state two examples of equidistributed processes intuitively less spread out than the uniform process.

**Example 3.** The following mixtures are equidistributed a.s.

(i) 60%-min-2 and 40%-max-2.
(ii) 20%-min-2 and 80%-uniform (see Figure 1).

Two generalizations are the max-$k$ process and min-$k$ process. In the max-$k$ process keep the point in the largest subinterval among $k$ uniformly placed points. Alternatively, in the min-$k$ process keep the point in the smallest subinterval.

Our final theorem (Theorem 12 in Section 5) informally states that given any collection of max-$k$ and min-$k$ processes there is an equidistributed mixture which places $\epsilon$ weight on these and the other $1 - \epsilon$ on the uniform process.

**Example 4.** The following mixtures are equidistributed a.s.

(i) $(1/k^2)\%$-min-$k$ for fixed $k$ and otherwise uniform.
(ii) $99.95\%$-uniform and $(5^{-k})\%$-min-$k$ for $k = 2, 3, \ldots$.

Discussion and further questions. Processes in our theorems satisfy the special inequality at (18). The reason our approach works for only certain mixtures is unclear. Numerical methods indicate the inequality fails for other processes, suggesting a different approach is needed. Hopefully, the properties we establish for general mixtures in Proposition 11 will aide further progress.

[MP14] conjectures that any max-$k$ or min-$k$ process is equidistributed. Based on this we suspect that any mixture of these is also equidistributed. The rate of convergence to a uniform placement of points and also the asymptotic size of the largest interval are open problems. More thorough discussion can be found in [MP14].

Overview. A more general family of interval splitting processes is introduced in [MP14]. Their main result is that, when properly scaled, the empirical distribution of subinterval lengths converges to a distribution function. The idea behind our argument is to reproduce parts of [MP14] when restricted to subintervals contained in $[0, \alpha]$. We find that the empirical distribution of subinterval lengths in $[0, \alpha]$ evolves to be essentially the same as the unrestricted version on $[0, 1]$. This sameness is enough to deduce equidistribution.

This article is organized to quickly arrive at the proof of Theorem 1. In Section 2 we describe the evolution of $[0, \alpha]$ and give the major definitions. In Section 3 we state without proof several propositions and then prove Theorem 1. Section 4 contains the proofs for the previous section. In Section 5 we generalize to random mixtures. Finally, in Section 6 we prove that processes captured by our theorems satisfy the inequality at (18).
Figure 1. The empirical density of $10^4$ (left) and $10^6$ (right) points for the 20%-min-2/80%-uniform mixture, the uniform process, and the max-2 process. The interval is discretized into 100 equally sized bins.

2. INTERVALS IN $[0, \alpha]$ 

Leading up to Theorem 1 we frame all of our discussion in terms of max-$k$ and min-$k$ processes. The reason being the majority of our propositions hold for any $k$. Moreover, we will see in Section 5 that this readily generalizes to random mixtures.

We start with a formal definition for a process to be equidistributed. Suppose $n_0$ points are initially placed. After $n$ iterations of an interval splitting process let $N^\alpha_n$ be the number of the first $n_0 + n$ terms smaller than $\alpha$. We say a sequence is equidistributed if $n^{-1}N^\alpha_n \to \alpha$ for all $\alpha \in [0, 1]$. It is convenient to work in continuous time. Following [MP14] we have points arrive as a Poisson process with intensity $e^t$. Formal details are in Section 4.1. So, in continuous time equidistribution is equivalent to $e^{-t}N^\alpha_t \to \alpha$ for all $\alpha \in [0, 1]$.

Fix $k \geq 2$ and $\alpha \in [0, 1]$. We use the convention that a bold face letter represents a process indexed by time (i.e. $\tilde{A} = (\tilde{A}_t)_{t \geq 0}$). Define the joint processes $(\tilde{A}^\alpha, \tilde{A}^{\alpha+}, \tilde{A})$ to be the size-biased empirical distributions restricted to intervals contained in $[0, \alpha]$, $[\alpha, 1]$ and $[0, 1]$, respectively. Formally, letting $I^{\alpha,(t)}_1, \ldots, I^{\alpha,(t)}_{N^\alpha_t}$ be the lengths of subintervals contained in $[0, \alpha]$ at time $t$ we define

$$\tilde{A}^\alpha_t(x) = \sum_{j=1}^{N^\alpha_t} I^{\alpha,(t)}_j \cdot 1\{I^{\alpha,(t)}_j \leq x\},$$

and similarly for $\tilde{A}^{\alpha+}$ and $\tilde{A}$. The spark for the refined analysis comes from the relation

$$\tilde{A}^\alpha_t(x) + \tilde{A}^{\alpha+}_t(x) = \tilde{A}_t(x), \quad \forall t, x \geq 0. \quad (1)$$

To ensure that no intervals are double counted assume the initial set of points placed in $[0, 1]$ always contains $\{\alpha\}$. This assumption is only for convenience. Our proof could be adapted to omit it by running the process until two points $\alpha_1 \leq \alpha \leq \alpha_2$ land sufficiently close to $\alpha$, and then using the bound $N_t^{\alpha_1} \leq N^\alpha_t \leq N_t^{\alpha_2}$. 

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Density

Position

max-2

uniform

0

20%-min-2

0

1

0

1

0

1

0
For the max-$k$ process define $\Psi(u) = u^k$ and for the min-$k$ process define $\Psi(u) = 1 - (1 - u)^k$. Also, let $\psi(u) = \Psi'(u)$. In [MP14, Section 2] the authors prove that

$$\tilde{A}_t(x) = \tilde{A}_0(x) + \int_0^t e^s x^2 \int_x^\infty \psi(\tilde{A}_s(z)) \frac{d\tilde{A}_s(z)}{z} ds + \tilde{M}_t$$

for some martingale $\tilde{M}_t$. The following proposition shows that $\tilde{A}^\alpha_t$ satisfies a similar equation.

**Proposition 5.** For any max-$k$ or min-$k$ process, the joint processes $(\tilde{A}^\alpha_t, \tilde{A}^\alpha_0, \tilde{A})$ satisfy the equation

$$\tilde{A}^\alpha_t(x) = \tilde{A}^\alpha_0(x) + \int_0^t e^s x^2 \int_x^\infty \psi(\tilde{A}_s(z)) \frac{d\tilde{A}_s(z)}{z} ds + \tilde{M}^\alpha_t(x),$$

with $\tilde{M}^\alpha_t$ a martingale.

The similarity between the semimartingale decompositions of $\tilde{A}_t$ and $\tilde{A}^\alpha_t$ is paramount in obtaining our theorems. However, the details are a bit technical. To keep our momentum we delay the proof until Section 4. What follows are facts and notation essential to our main theorems.

Let non-tilde processes represent the original process scaled by $e^{-t}$ (i.e.

$A_t(x) = \tilde{A}_t(e^{-t} x)$).

In light of Proposition 5 a change of variables gives the relationship

$$(2) \quad \mathbf{A}^\alpha = \mathcal{C}(\mathbf{A}^\alpha, \mathbf{A}) + \mathbf{M}^\alpha,$$

where $\mathcal{C} : \mathcal{X} \times \mathcal{X} \to C([0, \infty), L^1_{loc})$ is defined by

$$\mathcal{C}(\mathbf{F}, \mathbf{G})_t(x) = F_0(e^{-t} x) + \int_0^t (e^{s-t} x)^2 \int_{e^{s-t} x}^\infty \psi(G_z(s)) \frac{dG_z(s)}{z} ds.$$

Here $\mathcal{X} = \mathcal{B}([0, \infty), \mathcal{D})$ where $\mathcal{D} = \{ F : [0, \infty) \to [0, 1], \text{càdlàg, increasing}\}$. The set $\mathcal{X}$ is a subspace of the space $\mathcal{B}([0, \infty), L^1_{loc})$ of measurable maps from $[0, \infty)$ to $L^1_{loc}$ with the topology of locally uniform convergence, which we denote by the symbol $\mathcal{X}'$.

We will use $\hat{F}$ and $F^\Psi$ interchangeably to denote the a.s. pointwise limiting distribution of $A_t$ from [MP14, Theorem 1.1]. Also define the stationary distribution $\hat{F}^*$ so that $\hat{F}^*_t = \hat{F}$ for all $t \geq 0$. With the convergence $A_t \to \hat{F}$ in mind, we consider the operator

$$\mathcal{C}^*(\mathbf{F})_t = \mathcal{C}(\mathbf{F}, \hat{F}^*)_t = F_0(e^{-t} x) + \int_0^t (e^{s-t} x)^2 \int_{e^{s-t} x}^\infty \psi(\hat{F}_s(z)) \frac{dF_s(z)}{z} ds.$$

We will see in the proof of Theorem 1 that the limiting distribution of $\tilde{A}^\alpha_t$ belongs to the set of fixed points

$$\hat{\mathcal{F}}^\alpha = \{ F \in \mathcal{X}_1 : F = \mathcal{C}^*(\mathbf{F}), F_t(+\infty) = \alpha \text{ and } (\frac{1}{\alpha} F_t)_{t \geq 0} \text{ tight} \}.$$
3. Proof of Theorem 1

We delay the proofs of two propositions until the next section. The first gives a sufficient condition for processes in $\mathfrak{F}^\alpha$ to converge to $\alpha\hat{F}$ in $\| \cdot \|_{x^{-3/2}}$. This is paired with a lemma proving the condition is met for the max-2 process.

**Proposition 6.** If for all $z \geq 0$

$$2\|z\psi'(\hat{F}(z))\hat{F}'(z) - \psi(\hat{F}(z))| - \psi(\hat{F}(z)) \leq 0,$$

then $\|F_t - \alpha\hat{F}\|_{x^{-3/2}} \leq 6e^{-t/2}$ for all $F \in \mathfrak{F}^\alpha$.

**Lemma 7.** The function $\hat{F}$ from the max-2 process satisfies (3) for all $z \geq 0$.

**Proof.** This is the case $p_2 = 1$ covered by Lemma 13.

We will also need several properties of max-$k$ and min-$k$ processes. Direct analogues hold for random mixtures (see Proposition 11).

**Proposition 8.** For the max-$k$ and min-$k$ processes:

1. $\| A_t^\alpha \|_{x^{-2}} = e^{-t}N_t^\alpha$ and $\| \alpha\hat{F} \|_{x^{-2}} = \alpha$.
2. The collection of distribution functions $(\frac{1}{\alpha}A_t^\alpha)_{t \geq 0}$ is tight.
3. The family $(A_t^\alpha(n))$ defined by $A_t^\alpha(n) = A_{t+n}^\alpha$ is asymptotically equicontinuous.
4. $M_t^\alpha(n) \to 0$ as $n \to \infty$, where $M_t^\alpha(n)(x) = M_{t+n}^\alpha(x) - M_t^\alpha(x)$ for every $t \geq 0$.
5. Define $A_t(n)$ by $A_t(n) = A_{t+n}^\alpha$. If $F(n) \overset{X}{\to} F$ then $\mathscr{C}(F(n), A_t(n)) \overset{X}{\to} \mathscr{C}^*(F)$.

**Proof of Theorem 1.** All statements are meant to hold almost surely. Also we abbreviate items from Proposition 8 as a roman numeral. In the continuous process points are added as a Poisson process with intensity $e^{d't}dt$. So, it suffices to show $e^{-t}N_t^\alpha \to \alpha$.

By (II), (III) and the version of the Arzelá-Ascoli theorem in [MP14, Lemma 7.3] we may choose a sequence $(A_t^{\alpha,n})$ which converges to a family of (scaled by $\alpha$) distributions $F_t^{\alpha,\infty}$ with $F_t^{\alpha,\infty}(+\infty) = \alpha$ for every $t \geq 0$. Taking limits in the formula at (2) we obtain

$$\mathscr{C}(A_t^{\alpha,n_k}, A_t(n_k)) \overset{X}{\to} \mathscr{C}^*(F_t^{\alpha,\infty}).$$

By (IV) and (V) we have

$$\mathscr{C}(A_t^{\alpha,n_k}, A_t(n_k)) \overset{X}{\to} \mathscr{C}^*(F_t^{\alpha,\infty}).$$

Thus, $F_t^{\alpha,\infty}$ is in $\mathfrak{F}^\alpha$. Proposition 6 and Lemma 7 imply that $\|F_t^{\alpha,\infty} - \alpha\hat{F}\|_{x^{-3/2}} \leq 6e^{-t/2}$.

A similar argument as the conclusion of the proof of [MP14, Theorem 7.1] gives almost sure pointwise convergence $A_t^\alpha \to \alpha\hat{F}$. [MP14, Theorem 1.1] states that $A_t \to \hat{F}$ pointwise. We can then deduce from (1) that $A_t^{\alpha,p} \to (1-\alpha)\hat{F}$. Combining pointwise convergence, (1) and Fatou’s lemma we deduce that $\|A_t^\alpha\|_{x^{-2}} \to \|\alpha\hat{F}\|_{x^{-2}}$. Indeed,

$$\liminf_{t} \|A_t^\alpha\|_{x^{-2}} \geq \|\alpha\hat{F}\|_{x^{-2}},$$

$$\limsup_{t} \|A_t^\alpha\|_{x^{-2}} = 1 - \liminf_{t} \|A_t^{\alpha,p}\|_{x^{-2}} \leq 1 - (1 - \alpha) = \|\alpha\hat{F}\|_{x^{-2}}.$$

This finishes the proof since (I) states that $\|A_t^\alpha\|_{x^{-2}} = e^{-t}N_t^\alpha$ and $\|\alpha\hat{F}\|_{x^{-2}} = \alpha$. \qed
4. Technical Proofs

4.1. Proposition 5. The idea is to compare subintervals selected from $[0, \alpha]$ against subintervals from $[\alpha, 1]$. For example, in an iteration of the max-$k$ process we consider the event that $j$ of the $k$ candidate points land in $[0, \alpha]$. The other $k-j$ must land in $[\alpha, 1]$. We obtain an interval selected from $[0, \alpha]$ according to a max-$j$ process and another selected from $[\alpha, 1]$ according to a max-$(k-j)$ process. If the interval from $[0, \alpha]$ is larger, then the point is kept and accounted for by $A^\alpha_i$. The relationship at (1) and a combinatorial identity then yield the desired formula.

Before giving the proof we first build up some necessary definitions. Our construction is for the max-$k$ process. The definitions for the min-$k$ process are similar. Define

$$\Psi_j^\alpha(u) = (u/\alpha)^j$$

to be the distribution function for the maximum and minimum of $j$ independent Uniform$[0, \alpha]$ random variables. We use the convention that $\psi_j = \Psi_j^1(u)$ and $\psi(u) = \Psi_j^\epsilon(u)$.

Let $\xi \sim \text{Bin}(k, \alpha)$ with $q_j = \mathbb{P}[\xi = j] = \binom{k}{j} \alpha^j (1-\alpha)^{k-j}$. Define a Poisson random measure $\Pi$ on

$$[0, \infty) \times [0,1] \times \{0,1, \ldots, k\} \times [0,\alpha]^k \times [0,1-\alpha]^k$$

with intensity

$$e^t dt \otimes dv \otimes d\xi \bigotimes_{j=1}^k (d\Psi_j^\alpha \otimes d\Psi_{k-j}^{1-\alpha}).$$

Denote points in $\Pi$ by $\pi_s = (s, v, \xi, u, w)$ where $u = u_1, \ldots, u_k, w = w_0, \ldots, w_{k-1}$.

Let $\ell_s^\alpha(u) = (\hat{A}_s^\alpha)^{-1}(u)$ and $\ell_s^{\alpha+}(w) = (\hat{A}_s^{\alpha+})^{-1}(w)$ be intervals sampled from $[0, \alpha]$ and $[\alpha, 1]$, respectively. Also, let $h(v, \ell, x) = v\mathbb{1}\{\ell \leq x\} + (1-v)\mathbb{1}\{\ell(1-v) \leq x\}$. With this define

$$\bar{B}^\alpha(\pi_s, x) = \sum_{j=1}^k \mathbb{1}\{\xi = j\} \ell_s^\alpha(u_j) \mathbb{1}\{\ell_s^\alpha(u_j) > x\} \mathbb{1}\{\ell_s^\alpha(u_j) > \ell_s^{\alpha+}(w_{k-j})\} h(v, \ell_s^\alpha(u_j), x),$$

so that $\hat{A}_k^\alpha(x) = A_0^\alpha(x) + \sum_{\pi_s \in \Pi, s \leq t} \bar{B}^\alpha(\pi_s, x)$. To help clarify we give a brief explanation for each term:

- $\mathbb{1}\{\xi = j\}$ accounts for how many of the $k$ points land in $[0, \alpha]$.
- $\ell_s^\alpha(u_j)$ is the initial length of the interval that potentially will be added to $A_i^\alpha$.
- $\mathbb{1}\{\ell_s^\alpha(u_j) > x\}$ is zero if the selected interval is smaller than $x$ and is already being counted by $A_i^\alpha$.
- $\mathbb{1}\{\ell_s^\alpha(u_j) > \ell_s^{\alpha+}(w_{k-j})\}$ indicates whether the interval from $[0, \alpha]$ is larger than that from $[\alpha, 1]$. The inequality would be reversed for the min-$k$ process.
- $h(v, \ell_s^\alpha(u_j), x)$ “cuts” the interval $\ell_s^\alpha$ and detects whether the resulting subintervals are smaller than $x$ and so, should be added to $A_i^\alpha$.

Proof of Proposition 5. Our proof is for the max-$k$ process, the argument for the min-$k$ process is similar (the functions $\Psi$ would change as would the bounds of the inside integral at (6)). We will obtain the semi-martingale decomposition of $\hat{A}_k^\alpha(x) = A_0^\alpha(x) + \sum_{\pi_s \in \Pi, s \leq t} \bar{B}^\alpha(\pi_s, x)$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ where $\mathcal{F}_t = \sigma(\Pi_{t}]0,1]\times\{0,1,\ldots,k\}\times[0,\alpha]^k \times [0,1-\alpha]^k$.

We start by computing the integral

$$\int_0^\alpha \int_0^{1-\alpha} \int_0^1 \int_0^{\bar{B}^\alpha(\pi_s, x)} d\xi \, dv \, d\Psi_{k-j}^{1-\alpha}(w_{k-j}) \, d\Psi_j^\alpha(u_j).$$
Using the fact that \( \int_0^1 h(v, \ell, x)dv = (x/\ell)^2 \), we first integrate with respect to \( \xi \) and \( v \) to write as

\[
\sum_{j=1}^{k} q_j x^2 \int_0^\alpha \int_0^{1-\alpha} \frac{1}{\ell_s^\alpha(u_j)^2} \mathbf{1}\{\ell_s^\alpha(u_j) > x\} \mathbf{1}\{\ell_s^\alpha(u_j) > \ell_s^{alpha}(w_{k-j})\} d\Psi_{k-j}^{1-\alpha}(w_{k-j}) d\Psi_s^\alpha(u_j).
\]

Integrating one step further and normalizing the \( \Psi_s^\alpha \) and \( \Psi_{k-j}^{1-\alpha} \) to \( \Psi_j \) and \( \Psi_{k-j} \) we obtain factors of \( \alpha^{-j}(1-\alpha)^{-(k-j)} \). This lets us cancel all but the binomial coefficients from the \( q_j \) terms and obtain

\[
(5) \quad \sum_{j=1}^{k} x^2 \int_0^\alpha \int_0^{1-\alpha} \frac{1}{\ell_s^\alpha(u_j)^2} \mathbf{1}\{\ell_s^\alpha(u_j) > x\} \mathbf{1}\{\ell_s^\alpha(u_j) > \ell_s^{alpha}(w_{k-j})\} d\Psi_{k-j}(w_{k-j}) d\Psi_j(u_j).
\]

Make the change of variables \( y = \ell^\alpha(w_{k-j}) \) so that \( \tilde{A}_s^\alpha(y) = w_{j-k} \). Hence the above can be written as

\[
(6) \quad \sum_{j=1}^{k} x^2 \int_0^\alpha \int_0^{1-\alpha} \frac{1}{\ell_s^\alpha(u_j)^2} \mathbf{1}\{\ell_s^\alpha(u_j) > x\} d\Psi_{k-j}(\tilde{A}_s^\alpha(y)) d\Psi_j(u_j).
\]

Integrate one step further and use the fact that \( \Psi_k(0) = 0 \) to obtain

\[
\sum_{j=1}^{k} x^2 \int_0^\alpha \int_0^{1-\alpha} \frac{1}{\ell_s^\alpha(u_j)^2} \mathbf{1}\{\ell_s^\alpha(u_j) > x\} \Psi_{k-j}(\tilde{A}_s^\alpha(\ell_s^\alpha(u_j))) d\Psi_j(u_j).
\]

Now apply the change of variables \( z = \ell_s^\alpha(u_j) \) and so \( \tilde{A}_s^\alpha(z) = u_j \) to rewrite the above as

\[
x^2 \int_x^\infty \sum_{j=1}^{k} \binom{k}{j} \frac{1}{\ell_s^\alpha(u_j)} \frac{\Psi_{k-j}(\tilde{A}_s^\alpha(z))}{z} d\Psi_{k-j}(\tilde{A}_s^\alpha(z)).
\]

Writing out \( \Psi_{k-j}(u) = u^{k-j} \) and \( \psi_j(u) = j u^{j-1} \) and using the equality \( \tilde{A}_s^\alpha(z) = \tilde{A}_s(z) - \tilde{A}_s^\alpha(z) \) from (1) we can rewrite the above as

\[
x^2 \int_x^\infty \sum_{j=1}^{k} \binom{k}{j} \frac{1}{\ell_s^\alpha(u_j)} \frac{\tilde{A}_s(z) - \tilde{A}_s^\alpha(z)}{z} d\Psi_{k-j}(\tilde{A}_s^\alpha(z)).
\]

The identity \( \sum_{j=1}^{k} \binom{k}{j} (a-b)^{k-j} b^{j-1} = ka^{k-1} = \psi(a) \) (derived by applying the binomial theorem to \( a^k = ((a-b)+b)^k \)) then differentiating both sides with respect to \( a \) gives (4) is equal to

\[
x^2 \int_x^\infty \psi(\tilde{A}_s(z)) d\tilde{A}_s^\alpha(z).
\]

Finish by multiplying by \( e^s \) and integrating from 0 to \( t \). \( \square \)

4.2. Proposition 6. The proof of Proposition 6 proceeds analogously to [MP14, Lemma 4.1 and Proposition 3.4]. A significant difference is that they apply integration by parts to

\[
\frac{1}{z} d\Psi(F_s(z)),
\]

whereas our operator \( \mathcal{E}^* \) requires applying integration by parts to

\[
\frac{\psi(F(z))}{z} dF_s(z).
\]
The requirement at (3) arises from the extra term $\psi(\bar{F}(z))$. Also, note that we work in the norm $\| \cdot \|_{x-3/2}$ to obtain the constant $2/3$ in (3). We will need this factor to prove an inequality similar to (3) holds for processes biased towards min-2. In [MP14] they use the norm $\| \cdot \|_{x-2}$. This change of norms does not significantly alter the argument. In fact, we could equally well work with any norm $\| \cdot \|_{x-1-\delta}$ with $0 < \delta < 1$.

**Proof of Proposition 6.** Let $F \in \mathcal{F}^\alpha$. We consider the rescaled processes $\tilde{F}_t(x) = F(e^t x)$, $\tilde{F}_t^\Psi(x) = \tilde{F}(e^t x)$. It then holds that $\tilde{F} = \mathcal{G}(\tilde{F})$ where

$$\tilde{G}(\tilde{F})_t(x) = \bar{F}_0(x) + \int_0^t e^s x^2 \int_x^\infty \frac{\psi(\tilde{F}(z))}{z} dF_s(z) ds.$$

We seek to prove the distance between $\tilde{F}$ and $\alpha \tilde{F}^\Psi$ is decreasing in $t$:  

$$(7) \quad \partial_t \| \tilde{F}_t - \alpha \tilde{F}_t^\Psi \|_{x-3/2} = \int_0^\infty x^{-3/2} \partial_t |\tilde{F}_t(x) - \alpha \tilde{F}_t^\Psi(x)| dx \leq 0.$$

Starting from the calculation

$$\partial_t \tilde{G}(\tilde{F})_t(x) = e^t x^2 \int_x^\infty \frac{\psi(\tilde{F}(z))}{z} d\tilde{F}_t(z)$$

we write for each $x \geq 0$ the dynamics for the difference $\tilde{F}_t(x) - \alpha \tilde{F}_t^\Psi(x)$ as

$$\partial_t (\tilde{F}_t(x) - \alpha \tilde{F}_t^\Psi(x)) = e^t x^2 I_t(x),$$

$$I_t(x) = \int_x^\infty \frac{\psi(\tilde{F}(z))}{z} d\tilde{F}_t(z) - \alpha \tilde{F}_t^\Psi(z) dz.$$

Multiply both sides by $\text{sgn}(\tilde{F}_t - \alpha \tilde{F}_t^\Psi)$ to obtain

$$e^{-t} \partial_t [\tilde{F}_t(x) - \alpha \tilde{F}_t^\Psi(x)] = x^2 \left\{ \begin{array}{ll} \text{sgn}(\tilde{F}_t(x) - \alpha \tilde{F}_t^\Psi(x)) I_t(x), & \tilde{F}_t(x) \neq \alpha \tilde{F}_t^\Psi(x) \\
|I_t(x)|, & \tilde{F}_t(x) = \alpha \tilde{F}_t^\Psi(x). \end{array} \right.$$

Let $\tilde{f}(z) = z \psi'(\tilde{F}(z)) \tilde{F}'(z) - \psi(\tilde{F}(z))$. An application of integration by parts to the integral gives

$$I_t(x) = -\frac{\psi(\tilde{F}(x))}{x} (\tilde{F}_t(x) - \alpha \tilde{F}_t^\Psi(x)) + \int_x^\infty \frac{\tilde{f}(z)}{z^2} (\tilde{F}_t(z) - \alpha \tilde{F}_t^\Psi(z)) dz.$$

The previous two equations therefore yield

$$e^{-t} \partial_t [\tilde{F}_t(x) - \alpha \tilde{F}_t^\Psi(x)] \leq -x \psi(\tilde{F}(x)) |\tilde{F}_t(x) - \alpha \tilde{F}_t^\Psi(x)| + x^2 \int_x^\infty |\tilde{f}(z)| \frac{|\tilde{F}_t(z) - \alpha \tilde{F}_t^\Psi(z)|}{z^2} dz.$$

We next multiply both sides by $x^{-3/2}$ and integrate with respect to $x$ from 0 to infinity to obtain the bound

$$e^{-t} \int_0^\infty x^{-3/2} \partial_t [\tilde{F}_t(x) - \alpha \tilde{F}_t^\Psi(x)] dx \leq \int_0^\infty -x \psi(\tilde{F}(x)) \frac{|\tilde{F}_t(x) - \alpha \tilde{F}_t^\Psi(x)|}{x^{1/2}} dx$$

$$+ \int_0^\infty x^{1/2} \int_x^\infty |\tilde{f}(z)| \frac{|\tilde{F}_t(z) - \alpha \tilde{F}_t^\Psi(z)|}{z^2} dz dx.$$
An application of Fubini’s theorem lets us rewrite the second integral as
\[
\int_0^\infty x^{1/2} \int_x^\infty |\tilde{f}(z)| \frac{|\tilde{F}_1(z) - \alpha \tilde{F}^\Psi(z)|}{z^2} dz dx = \int_0^\infty |\tilde{f}(z)| \frac{|\tilde{F}_1(z) - \alpha \tilde{F}^\Psi(z)|}{z^2} \int_0^z x^{1/2} dx dz = \int_0^\infty \frac{2}{3} |\tilde{f}(z)| \frac{|\tilde{F}_1(z) - \alpha \tilde{F}^\Psi(z)|}{z^{1/2}} dz.
\]
Hence we can combine the integrals to obtain the bound
\[
e^{-t} \int_0^\infty x^{-2} \partial_t |\tilde{F}_1(x) - \alpha \tilde{F}^\Psi(x)| dx \leq \int_0^\infty \left( \frac{2}{3} |\tilde{f}(z)| - \psi(\tilde{F}(z)) \right) \frac{|\tilde{F}_1(z) - \alpha \tilde{F}^\Psi(z)|}{z^{1/2}} dz.
\]
The above is less than or equal to zero by our hypothesis (3). This establishes that
\[
\|\tilde{F}_1 - \alpha \tilde{F}^\Psi\|_{x^{-2}} \leq \|\tilde{F}_0 - \alpha \tilde{F}^\Psi\|_{x^{-3/2}} = \|F_0 - \alpha \tilde{F}\|_{x^{-3/2}}.
\]
A change of variables \(x = e^{-tz}\) gives
\[
\|F_1 - \alpha \tilde{F}\|_{x^{-3/2}} = \int_0^\infty x^{-3/2} |F_1(x) - \alpha \tilde{F}(x)| dx
\]
\[
= e^{-t/2} \int_0^\infty x^{-3/2} |\tilde{F}_1(z) - \alpha \tilde{F}^\Psi(z)| dz
\]
\[
= e^{-t/2} \|\tilde{F}_1 - \alpha \tilde{F}^\Psi\|_{x^{-3/2}}
\]
\[
\leq e^{-t/2} \|F_0 - \alpha \tilde{F}\|_{x^{-3/2}},
\]
where at the last line we apply (8).

It remains to prove that \(\|F_0 - \alpha \tilde{F}\|_{x^{-3/2}} \leq 6\). By assumption, \(F \in X_1\) and therefore \(\|F_0\|_{x^{-2}} \leq 1\). As \(0 \leq F_0(x) \leq 1\) we can break up the integral and use integrability of \(x^{-3/2}1\{x > 1\}\):
\[
\int_0^\infty x^{-3/2} F_0(x) dx \leq \int_0^1 x^{-2} F_0(x) dx + \int_1^\infty x^{-3/2} dx \leq \|F_0\|_{x^{-2}} + 2 \leq 3.
\]
Similarly, \(\|\alpha \tilde{F}\|_{x^{-3/2}} \leq 3\). Apply the triangle inequality to conclude \(\|F_0 - \alpha \tilde{F}\|_{x^{-3/2}} \leq \|F_0\|_{x^{-3/2}} + \|\alpha \tilde{F}\|_{x^{-3/2}} \leq 6\).

4.3. Proposition 8. In Proposition 8 we prove that \(A^\alpha_t\) and \(A_t\) have similar properties. Each statement requires some manipulation. Fortunately [MP14] contains much of the ‘heavy-lifting’. We make one remark concerning the proof of (V). In [MP14] they prove continuity of an operator \(\mathcal{G}^\Psi\) with domain \(X\). Our operator \(\mathcal{G}\) has domain \(X \times X\). This makes the proof more involved, and also restricts us to proving continuity in sequences of the form \((F^{(n)}, A^{(n)})\).

Proof of (I). The equality \(\|\alpha \tilde{F}\|_{x^{-2}} = \alpha\) is [MP14, Lemma 3.5]. For the other equality, take \(f_j^{\alpha,(t)}\) to be the length of an interval in \([0, \alpha]\). Define the measure \(\mu_\alpha = e^{-t} \sum_{i=1}^{N_\alpha} \delta_{x_i} f_j^{\alpha,(t)}\). This gives \(\mu_\alpha\) is the empirical distribution of rescaled interval lengths. We can then write
\[
A^\alpha_t(x) = \int_0^x y \mu_\alpha(dy).
\]
Applying Fubini’s theorem shows that
\[
\|A^\alpha_t\|_{x^{-2}} = \int_0^\infty x^{-2} \int_0^x y \mu_\alpha(dy) = \int_0^\infty \mu_\alpha(dy) = e^{-t} N^\alpha_t.
\]
Proof of (II). Recall that a family of distributions \((F_t)_{t \geq 0}\) is tight if for all \(\epsilon > 0\) there exists \(N\) such that \(F_t(N) \geq 1 - \epsilon\) for all \(t \geq 0\). [MP14, Proposition 6.3] implies \((A_t)_{t \geq 0}\) is tight. Fix \(\epsilon > 0\) and let \(N\) be such that \(A_t(N) \geq 1 - \alpha\epsilon\) for all \(t \geq 0\). The relationship at (1) ensures \(A_t^0(N) + A_t^{\alpha+}(N) \geq 1 - \alpha\). As \(A_t^0 \leq \alpha\) and \(A_t^{\alpha+} \leq 1 - \alpha\), this inequality could only hold if \(A_t^0(N) \geq \alpha - \alpha\epsilon\) for all \(t \geq 0\). Hence, \((\frac{1}{\alpha}A_t^0)_{t \geq 0}\) is tight.

Proof of (III). We say that a family of functions \((F^{(n)}(x))_{n \in \mathbb{N}}\) in \(X\) is asymptotically equicontinuous if for every compact \(K \subset [0, \infty)\),

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{|s-t| \leq \delta} \int_K |F^{(n)}_s(x) - F^{(n)}_t(x)|dx = 0.
\]

The proof is similar to [MP14, Lemma 7.5]. We omit the details and just remark that for any \(\delta > 0\) the number of points kept in \([0, \alpha]\) from time \(t\) to \(t + \delta\) is bounded by the number of points added to \([0, 1]\) in that same time interval. Formally, for any \(\delta > 0\) we have \(N^\alpha_{t+\delta} - N^\alpha_t \leq N^1_{t+\delta} - N^1_t\). Thus we use the same bounds.

Proof of (IV). The proof is similar to the decay of the noise subsection in [MP14, Section 7]. The same bounds apply because points are added to \([0, \alpha]\) no faster than to \([0, 1]\). This ensures that \(B^\alpha(s, u, v, x) \leq B(s, u, v, x)\). Here \(B(s, u, v, x)\) is the function defined at [MP14, (3)].

Proof of (V). Suppose that \(F^{(n)}(x) \xrightarrow{\mathcal{D}} F\). An equivalent notion of convergence in the topology of local uniform convergence is that \(F^{(n)}(x) \xrightarrow{\mathcal{L}} F\) if and only if for all compact \(K \subset [0, \infty)\)

\[
\lim_{n \to \infty} \sup_{0 \leq s \leq t} \int_K |F^{(n)}_s(x) - F_s(x)|dx = 0.
\]

[MP14, Theorem 7.1] implies \(A^{(n)}(x) \xrightarrow{\mathcal{L}} F^\ast\). Thus it suffices to prove for any fixed \(T > 0\) and \(K > 0\)

\[
\int_0^K |\mathcal{G}(F^{(n)}, F^\ast)_t(x) - \mathcal{G}(F^{(n)}, A^{(n)})_t(x)|dx \to 0
\]

uniformly for \(t \leq T\). For fixed \(n\) we can write

\[
\mathcal{G}(F^{(n)}, A^{(n)})_t(x) = F^{(n)}_0(x) + \int_0^t (e^{s-t}x)^2 \int_{e^{s-t}x}^\infty \frac{\psi(A^{(n)}_s(z))}{z} dF^{(n)}_s(z)ds.
\]

If we write \(\psi(A^{(n)}_s(z)) = \psi(\hat{F}(z)) + \psi(A^{(n)}_s(z)) - \psi(\hat{F}(z))\) the above becomes

\[
\mathcal{G}(F^{(n)}, A^{(n)})_t(x) = \mathcal{G}(F^{(n)}, F^\ast)_t(x) + \int_0^t (e^{s-t}x)^2 \int_{e^{s-t}x}^\infty \frac{\psi(A^{(n)}_s(z)) - \psi(\hat{F}(z))}{z} dF^{(n)}_s(z)ds.
\]

We can then bound the left side of (10) by

\[
\int_0^K |\mathcal{G}(F^{(n)}, F^\ast)_t(x) - \mathcal{G}(F^{(n)}, F^\ast)_t(x)|dx
\]

\[
+ \int_0^K \int_0^t (e^{s-t}x)^2 \int_{e^{s-t}x}^\infty \frac{|\psi(A^{(n)}_s(z)) - \psi(\hat{F}(z))|}{z} dF^{(n)}_s(z)dsdx.
\]

It suffices to show that as \(n \to \infty\) each summand converges to zero uniformly for \(t \leq T\).
First summand: Start by bounding the summand at (11) by
\[
\int_0^K |F_0(e^{-t}x) - F_0^{(n)}(e^{-t}x)| dx + \int_0^K \int_0^t (e^{s-t}x)^2 \left| \int_{e^{s-t}x}^{\infty} \frac{\psi(\hat{F}(z))}{z} d(F_s(z) - F_s^{(n)}(z)) \right| ds dx.
\]

The first quantity goes to zero uniformly for \( t \leq T \) by the definition of \( F^{(n)} \to F \) since a change of variables gives
\[
\int_0^K |F_0(e^{-t}x) - F_0^{(n)}(e^{-t}x)| dx \leq e^t \int_0^K |F_0(x) - F_0^{(n)}(x)| dx.
\]

Expand the interior of the second quantity with integration by parts and take the absolute value signs inside to bound it by
\[
\frac{\psi(\hat{F}(e^{s-t}x))}{e^{s-t}x} |F_s(e^{s-t}x)| dx - F_s^{(n)}(e^{s-t}x)|ds dx. 
\]

Multiply term one by \((e^{s-t}x)^2\) and integrate so it becomes
\[
\int_0^K \int_0^t (e^{s-t}x) \psi(\hat{F}(e^{s-t}x)) |F_s(e^{s-t}x) - F_s^{(n)}(e^{s-t}x)| ds dx.
\]

Notice that \( \psi(u) \leq k \) for all \( u \in [0,1] \). Thus, the above is bounded by
\[
k \int_0^K \int_0^t (e^{s-t}x) |F_s(e^{s-t}x) - F_s^{(n)}(e^{s-t}x)| dx.
\]

This puts us in the case of \( I_1 \) from [MP14, Lemma 3.3] and so converges to zero uniformly for \( t \leq T \). As for term two, we differentiate to rewrite it as
\[
(13) \quad \int_{e^{s-t}x}^{\infty} \frac{|z\psi'(\hat{F}(z))\hat{F}'(z) - \psi(\hat{F}(z))|}{z^2} |F_s(z) - F_s^{(n)}(z)| dz.
\]

By Lemma 15 we know that \( zF'(z) \) is bounded. Since \( \psi \) and \( \psi' \) are also bounded we have \( C = \sup_{0 \leq z \leq \infty} |zF'(z)\psi'(\hat{F}(z)) - \psi(\hat{F}(z))| < \infty \). Therefore, (13) is less than
\[
(14) \quad C \int_{e^{s-t}x}^{\infty} \frac{1}{z^2} |F_s(z) - F_s^{(n)}(z)| dz.
\]

Finally, we are in the position of \( I_2 \) from [MP14, Lemma 3.3] and can conclude that (14) goes to zero uniformly for \( t \leq T \).

Summand 2: Fix \( M > 0 \) and for any function \( f : [0, \infty) \to [0,1] \) define \( f^M = f|_{[0,M]} \) to be the restriction to the domain \([0,M] \). We have in [MP14, Theorem 7.1] that \( A_t^M \) converges pointwise to \( \hat{F}^M \). Observe that each \( A_t^M \) is an increasing function with compact domain, and \( \hat{F}^M \) is continuous by [MP14, Lemma 3.5]. Together these imply (see [Rud76, exercise 7.13]) that for any \( \epsilon > 0 \) there exists \( t_\epsilon \) such that for all \( z \in [0,M] \)
\[
\sup_{t \geq t_\epsilon} |A_t^M(z) - \hat{F}_t^M(z)| < \epsilon.
\]

Because the functions \( A_t^{(n)} \) are translates of \( A_t \) it follows that for all \( n > t_\epsilon \) we have
\[
\sup_{t \geq 0} |A_t^{(n),M}(z) - \hat{F}_t^M(z)| \leq \sup_{t \geq t_\epsilon} |A_t^M(z) - \hat{F}_t^M(z)| < \epsilon.
\]
Thus, (16) can be made arbitrarily small.

We truncate the integral then apply (15) to bound the absolute value of (12) by

(16) \[ \epsilon \int_0^K \int_0^t (e^{s-t} x)^2 \int_{e^{-t} x}^M \frac{1}{z} dF_s^{(n)}(z) ds dx \]

+ \[ \int_0^K \int_0^t (e^{s-t} x)^2 \int_{e^{-t} x}^\infty \frac{1}{z} \left| \psi(A_s^{(n)}(z)) - \psi(\hat{F}(z)) \right| dF_s^{(n)}(z) ds dx. \]

Integrate the inside integral of (16) by parts to obtain,

\[ \frac{F_s^{(n)}(M)}{M} - \frac{F_s^{(n)}(e^{s-t} x)}{e^{s-t} x} + \int_{e^{-t} x}^M \frac{F_s^{(n)}(z)}{z^2} dz. \]

Using the fact that \( F_s^{(n)}(z) \leq 1 \) this is bounded by \( \frac{2}{M} \). Multiplying by \( (e^{s-t} x)^2 \) and integrating gives the following bound on (16)

\[ \epsilon \int_0^K \int_0^t (e^{s-t} x)^2 \int_{e^{-t} x}^M \frac{1}{z} dF_s^{(n)}(z) ds dx \leq \epsilon \int_0^K \int_0^t (e^{s-t} x)^2 \frac{2}{M} ds dx \]

\[ = \epsilon \int_0^K \frac{2x^2}{M} (1 - e^{-t}) dx \]

\[ \leq \epsilon \frac{K^3}{M}. \]

Thus, (16) can be made arbitrarily small.

Lastly we consider (17). Since \( \psi(u) \leq k \) we start with the bound

\[ \int_0^K \int_0^t (e^{s-t} x)^2 \int_{e^{-t} x}^\infty \frac{1}{z} \left| \psi(A_s^{(n)}(z)) - \psi(\hat{F}(z)) \right| dF_s^{(n)}(z) ds dx \]

\[ \leq \int_0^K \int_0^t (e^{s-t} x)^2 \int_{e^{-t} x}^\infty \frac{2k}{z} dF_s^{(n)}(z) ds dx. \]

Integrate by parts so the inside becomes

\[ -\frac{2kF_s^{(n)}(M)}{M} + \int_M^\infty \frac{2kF_s^{(n)}(z)}{z^2} dz. \]

Once more using the bound \( F_s^{(n)}(z) \leq 1 \) we conclude that for \( M \) large the above becomes arbitrarily small. Therefore, the absolute value of (12) can be bounded by any \( \epsilon > 0 \) uniformly for \( t \leq T \). \[\square\]

5. Random Mixtures

We conclude by generalizing to a family of random mixtures of max-\( k \), uniform and min-\( k \) processes. Let \( \kappa \) be a random variable supported in \( \mathbb{Z} \setminus \{-1, 0\} \). We let \( P[\kappa = k] = p_k \).

Define the dynamics of the \textit{mix-\( \kappa \) process} as follows:

- Start by partitioning \([0, 1]\) into subintervals by placing finitely many points in any manner.
- Sample an i.i.d. copy \( \kappa_n \sim \kappa \) and add the nth point according to the
  - min-\(|\kappa_n|\) process if \( \kappa_n \leq -2 \),
  - uniform process if \( \kappa_n = 1 \),
max-\kappa_\alpha process if \kappa_\alpha \geq 2.

We follow the same outline as the proof of Theorem 1; first stating the analogous definitions and propositions and then sketching the proofs of our main theorems.

Let $\tilde{A}_{t}^{\alpha,\kappa}$, $\tilde{A}_{t}^{\alpha,\kappa+}$ and $\tilde{A}_{t}^{\alpha}$ be the size-biased empirical interval distribution for subintervals of $[0, \alpha]$, $[\alpha, 1]$ and $[0, 1]$ in the mix-\kappa process. Define

$$
\Psi_{\kappa}(u) = p_{1}u + \sum_{k \geq 2}p_{k}u^{k} + p_{-k}(1 - (1 - u)^{k}).
$$

Also let $\psi_{\kappa} = \Psi_{\kappa}'$. To ensure our processes are well defined we only consider $\kappa$ such that $\psi_{\kappa}'$ is bounded. The evolution of $\tilde{A}_{t}^{\alpha,\kappa}$ is essentially identical to that of $\tilde{A}_{t}^{\alpha}$.

**Proposition 9.** For the mix-\kappa process the joint processes $(\tilde{A}_{t}^{\alpha,\kappa}, \tilde{A}_{t}^{\alpha,\kappa+}, \tilde{A}_{t}^{\alpha})$ satisfy the equation

$$
\tilde{A}_{t}^{\alpha,\kappa}(x) = \tilde{A}_{0}^{\alpha,\kappa}(x) + \int_{0}^{t} e^{s}x^{2} \int_{x}^{\infty} \frac{\psi_{\kappa}(\tilde{A}_{s}^{\alpha}(z))}{z} d\tilde{A}_{s}^{\alpha}(z) ds + \tilde{M}_{t}^{\alpha,\kappa}(x),
$$

with $\tilde{M}_{t}^{\alpha,\kappa}$ a martingale.

**Proof.** We obtain a semimartingale decomposition by integrating each variable. The first step is to integrate over the possible values of $\kappa$. This results in a weighted sum of terms corresponding to either a max-\kappa, uniform or min-\kappa process. Handle each term separately as in Proposition 5 to arrive at the decomposition

$$
\tilde{A}_{t}^{\alpha,\kappa}(x) = \tilde{A}_{0}^{\alpha,\kappa}(x) + \sum_{k} p_{k} \int_{0}^{t} e^{s}x^{2} \int_{x}^{\infty} \frac{\psi_{\kappa}(\tilde{A}_{s}^{\alpha}(z))}{z} d\tilde{A}_{s}^{\alpha}(z) ds + \tilde{M}_{t}^{\alpha,\kappa}.
$$

Using linearity of the integral and the fact that $\psi_{\kappa}(u) = \sum_{k} p_{k} \psi_{\kappa}(u)$ gives the desired decomposition. \hfill \square

[MP14, Theorem 1.1] implies $A_{t}^{\alpha} \rightarrow F^{\Psi_{\kappa}}$ a.s. for any mix-\kappa process. Again let $\tilde{F}^{\kappa} = (F^{\Psi_{\kappa}})_{t \geq 0}$. With this convergence in mind, it is natural to consider the analogues of $\mathcal{E}$ and $\mathfrak{S}^{\alpha}$ from Section 2. We define

$$
\mathcal{E}^{\kappa}(F, G)_{t}(x) = F_{0}(e^{-t}x) + \int_{0}^{t} (e^{s-t}x)^{2} \int_{e^{-s}x}^{\infty} \frac{\psi_{\kappa}(G_{s}(z))}{z} dG_{s}(z) ds,
$$

$$
\mathfrak{S}^{\kappa,\alpha}(F)_{t}(x) = F_{0}(e^{-t}x) + \int_{0}^{t} (e^{s-t}x)^{2} \int_{e^{-s}x}^{\infty} \frac{\psi_{\kappa}(F^{\Psi_{\kappa}}(z))}{z} dF_{s}(z) ds,
$$

$$
\mathfrak{S}^{\alpha,\kappa} = \{ F \in \mathcal{L}^{1}_{t} : F = \mathcal{E}^{\kappa,\alpha}(F), \forall t \geq 0 : F_{t}(+\infty) = \alpha \text{ and } (\frac{1}{\alpha} F_{t})_{t \geq 0} \text{ tight} \}.
$$

The corresponding versions of Proposition 6 and Proposition 8 continue to hold for mix-\kappa processes. The proofs are very similar to as before and we omit them.

**Proposition 10.** Let $F = F^{\Psi_{\kappa}}$ and $\psi = \psi_{\kappa}$. If for all $z \geq 0$

$$
\mathfrak{S}^{\alpha,\kappa} \leq 6e^{-t/2} \text{ for } G \in \mathfrak{S}^{\alpha,\kappa}.
$$

**Proposition 11.** For any mix-\kappa process it holds that

(i) $\| A_{t}^{\alpha,\kappa} \|_{x \rightarrow x} = e^{-t}N_{t}^{\alpha} \text{ and } \| A_{0}^{\alpha,\kappa} \|_{x \rightarrow x} = \alpha$.

(ii) If $\mathbf{F}^{(n)} \overset{X}{\rightarrow} \mathbf{F}$ then $\mathcal{E}^{\kappa}(\mathbf{F}^{(n)}, A_{t}^{\alpha,\kappa}(n)) \overset{X}{\rightarrow} \mathcal{E}^{\kappa,\alpha}(\mathbf{F})$.

(iii) The collection of distribution functions $(\frac{1}{\alpha} A_{t}^{\alpha,\kappa})_{t \geq 0}$ is tight.

(iv) The family $(A_{t}^{\alpha,\kappa}(n))$ defined by $A_{t}^{\alpha,\kappa}(n) = A_{t+n}^{\alpha,\kappa}$ is asymptotically equicontinuous.
We break into two cases: Let Theorem 12.

Proof of Theorem 2. Lemma 16 and Remark 17 imply that (18) holds so long as \( p_2 + p_1 + p_{-2} = 1 \) and \( p_{-2} \leq .6 - 5p_1 \). Thus, Proposition 10 and Proposition 11 hold. From here an identical argument as the proof of Theorem 1 gives the desired convergence.

Theorem 12. Let \( C_\kappa = \sum_{k \geq 2} k(k - 1)[p_k + p_{-k}] \). If \( \kappa \) is such that \( C_\kappa \leq 1/4 \) then the mix-\( \kappa \) process is equidistributed a.s.

Proof. Lemma 20 implies that (18) holds so long as \( C_\kappa \leq 1/4 \). Thus, Proposition 10 and Proposition 11 hold. From here an identical argument as the proof of Theorem 1 gives the desired convergence.

6. INEQUALITIES

For this entire section we will let \( F \) denote \( F^{\Psi_\kappa} \), \( \Psi \) denote \( \Psi_\kappa \) and \( \psi \) denote \( \psi_\kappa \). Our goal is to show that the mixtures in Theorem 2 and Theorem 12 satisfy (18). Key to establishing the inequality is the differential equation

\[
(M^{\alpha,\kappa}(n))_t^X \to 0 \text{ as } n \to \infty, \text{ where } M^{\alpha,\kappa}(n)(x) = M^{\alpha,\kappa}_t(x) - M^{\alpha,\kappa}_n(e^{-t}x).
\]

\[
\text{Proof.}
\]

\[
\text{Proof of Theorem 6. MATTHEW JUNGE}
\]

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\[
\frac{d}{dz} \left( zF''(z) - F'(z) + z\psi(F(z))F'(z) \right) = 0
\]

\[
\text{Inequalities for Theorem 2. Our first lemma establishes (18) holds for any mixture of max-2, uniform and min-2 processes so long as } p_{-2} \leq p_2.
\]

Lemma 13. If \( p_2 + p_1 + p_{-2} = 1 \) and \( p_{-2} \leq p_2 \) then the inequality at (18) holds.

Proof. Since \( p_{-2} + p_1 + p_2 = 1 \) it is easily checked that \( \psi'(u) = 2(p_2 - p_{-2}) \). Dropping the constant \( \frac{2}{3} \) from the left side of (18) it suffices to prove that

\[
|\psi(F(z)) - z\psi'(F(z))F'(z)| \leq \psi(F(z)).
\]

We break into two cases:

- First suppose \( \psi(F(z)) \geq z\psi'(F(z))F'(z) \) so that (20) reduces to proving that

\[
-z\psi'(F(z))F'(z) \leq 0.
\]

As \( F \) is increasing we know \( F'(z) \geq 0 \). The hypothesis \( p_{-2} \leq p_2 \) guarantees that \( \psi'(F(z)) \geq 0 \). Thus, the inequality is satisfied.

- Next, suppose \( \psi(F(z)) \leq z\psi'(F(z))F'(z) \). Rearranging (20) we seek to show

\[
2(p_2 - p_{-2})F'(z) \leq 2\psi(F(z)).
\]

Note that both sides are zero at \( z = 0 \). By the fundamental theorem of calculus it then suffices to prove the above inequality holds for the derivatives. Differentiating and again using the fact that \( \psi'(F(z)) = 2(p_2 - p_{-2}) \) reduces the problem to establishing

\[
2(p_2 - p_{-2})(zF''(z) + F'(z)) \leq 4(p_2 - p_{-2})F'(z).
\]

After some algebra this is equivalent to

\[
zF''(z) \leq F'(z).
\]
From (19) we know that \( zF''(z) = F'(z) - z\psi(F(z))F'(z) \). Substitute this into (21) and we have a sufficient condition that

\[
F'(z) - 2z\psi(F(z))F'(z) \leq F'(z).
\]

This holds as \( F'(z) \) and \( \psi(F(z)) \) are nonnegative.

To prove (18) holds when \( p_{-2} > p_2 \) requires a different analysis of the differential equation at (19). Lemma 15 shows \( zF'(z) \) can be bounded in terms of the underlying \( p_k \). We use both (19) and the integro-differential characterization of \( F' \) from [MP14, Lemma 3.5]:

(22)

\[
F'(z) = z \int_z^\infty \frac{1}{y} d\psi(F(y)).
\]

**Lemma 14.** Suppose \( \kappa \) is such that \( p_2 + p_1 + p_{-2} = 1 \) and \( p_{-2} > p_2 \). It holds that

\[
\lim_{\epsilon \to 0} \frac{F'(\epsilon)}{\epsilon} \leq 2.
\]

**Proof.** Starting from the formula at (22) then integrating by parts gives

(23)

\[
\lim_{\epsilon \to 0} \frac{F'(\epsilon)}{\epsilon} = \int_0^\infty \frac{1}{y} d\psi(F(y)) = \|\psi \circ F\|_{x^{-2}}.
\]

Because \( p_2 + p_1 + p_{-2} = 1 \) we have

\[
\Psi(F(y)) = p_1 F(y) + p_2 F(y)^2 + p_{-2}(1 - (1 - F(y))^2) = F(y)[p_1 + (p_2 - p_{-2})F(y) + 2p_{-2}],
\]

The hypothesis \( p_2 < p_{-2} \) means an upper bound for the above is

(24)

\[
\Psi(F(y)) \leq F(y)[p_1 + 2p_{-2}] \leq 2F(y).
\]

Proposition 8 (I) implies that \( \|F\|_{x^{-2}} = 1 \). It follows from (23) and (25) that

\[
\lim_{\epsilon \to 0} \frac{F'(\epsilon)}{\epsilon} \leq 2\|F\|_{x^{-2}} = 2.
\]

\[\Box\]

**Lemma 15.** Suppose \( \kappa \) is such that \( p_2 + p_1 + p_{-2} = 1 \) and \( p_{-2} > p_2 \). It holds that

\[
zF'(z) \leq 2 \left( \frac{2}{p_1 + 2p_{-2}} \right)^2 e^{-2}.
\]

**Proof.** Integrate (19) as in [MP14, Proposition 8.1] so that for any \( \epsilon > 0 \)

\[
F'(z) = \frac{F'(\epsilon)}{\epsilon - z} \exp \left( - \int_z^\epsilon \psi(F(y))dy \right).
\]

Taking \( \epsilon \to 0 \) and applying Lemma 14 gives

(26)

\[
F'(z) \leq 2z \exp \left( - \int_0^z \psi(F(y))dy \right)
\]

Our hypothesis \( p_2 + p_1 + p_{-2} = 1 \) lets us write \( \psi(F(y)) = p_1 + 2p_{-2} + 2(p_2 - p_{-2})F(y) \). The hypothesis \( p_{-2} > p_2 \) and the fact \( F(y) \leq 1 \) then give the bound

(27)

\[
\psi(F(y)) \geq p_1 + 2p_2.
\]

Applying this to (26) and multiplying by \( z \) gives

\[
zF'(z) \leq 2z^2 e^{-(p_1 + 2p_2)z}.
\]
The maximum of $z^2e^{-(p_1+2p_2)z}$ is at $z = 2/(p_1 + 2p_2)$. Plug this in above to obtain the claimed bound.

**Lemma 16.** If $\kappa$ is such that $p_2 + p_1 + p_\kappa = 1$ and $p_\kappa < p_{-\kappa} \leq 0.57 - 0.5p_1$ then (18) holds.

**Proof.** Using the triangle inequality we can bound the left side of (18) by

$$\frac{2}{4}|z\psi'(F(z))F'(z) - \psi(F(z))| - \psi(F(z)) \leq \frac{2}{4}z|\psi'(F(z))|F'(z) - \frac{1}{4}\psi(F(z)).$$

Our goal is to prove the right-hand side is nonpositive. This is equivalent to showing that

$$zF'(z) \leq \frac{\psi(F(z))}{2|\psi'(F(z))|}, \quad \text{for } z \geq 0.$$

We have from (27) that $\psi(u) \geq p_1 + 2p_2$ and can compute $|\psi'(u)| = 2|p_2 - p_{-\kappa}|$. It then suffices to prove

$$zF'(z) \leq \frac{p_1 + 2p_2}{4|p_2 - p_{-\kappa}|}.$$

By Lemma 15 it suffices to choose $p_2, p_1$ and $p_{-\kappa}$ so that

$$2 \left(\frac{2}{p_1 + 2p_2}\right)^2 e^{-2} \leq \frac{p_1 + 2p_2}{4|p_2 - p_{-\kappa}|}.$$

Combining with our hypotheses we have an underdetermined system of equations:

(28)

\[ 32e^{-2}|p_2 - p_{-\kappa}| \leq (p_1 + 2p_2)^3, \]

\[ p_2 + p_1 + p_{-\kappa} = 1, \]

\[ p_2 < p_{-\kappa}. \]

Solving numerically gives $p_{-\kappa} \leq 0.57 - 0.5p_1$ lies in the solution set.

**Remark 17.** It is possible to extend to a slightly larger set of solutions. The constant $\frac{2}{4}$ in (18) can be reduced to $1/(2 - \delta)$ if we instead work in the norm $\| \cdot \|_{x-1-s}$. For simplicity we have held $\delta = \frac{1}{4}$ fixed. If we were to let $\delta$ tend to zero then the constant 32 in (28) would be halved to 16. This new system extends to all mixtures with $p_{-\kappa} \leq 0.6 - 0.5p_1$.

### 6.2. Inequalities for Theorem 12.

We will reprove versions of the previous three lemmas for more general mixtures. Recall that $C_\kappa = \sum_{k \geq 2} k(k-1)(p_k + p_{-k})$.

**Lemma 18.** For all $z \geq 0$ it holds that $F'(z) \leq 1$.

**Proof.** This follows from a simple bound on (22):

$$F'(z) = z \int_z^\infty \frac{\psi(F(y))}{y} F'(y)dy \leq z \cdot \frac{1}{z} \int_z^\infty \psi(F(y))F'(y)dy = \Psi(1) - \Psi(F(z)).$$

Since $\Psi(1) = 1$ we conclude that $F'(z) \leq 1$.

**Lemma 19.** Suppose that $p_1 > 0$. It holds that

$$zF'(z) \leq \left(\frac{2}{e \cdot p_1}\right)^2,$$
Proof. Integrate (19) as in [MP14, Proposition 8.1] so that for any $\epsilon > 0$

$$F'(z) = \frac{F'(\epsilon)}{\epsilon} z \exp \left( -\int_{\epsilon}^z \psi(F(y)) dy \right).$$

Taking $\epsilon = 1$ and applying Lemma 18 gives

(29) $$F'(z) \leq z \exp \left( -\int_1^z \psi(F(y)) dy \right).$$

Notice that

(30) $$\psi(u) = p_1 + \sum_{k \geq 2} k[p_k u^{k-1} + p_{-k} (1 - u)^{k-1}] \geq p_1.$$ 

Apply this to (29) then multiply by $z$ to obtain the bound

$$z F'(z) \leq e^{p_1} z^2 e^{-p_1 z} \leq z^2 e^{-p_1 z}.$$ 

The maximum of $z^2 e^{-p_1 z}$ is at $z = 2/p_1$. Plug this in above to obtain the claimed bound. $\square$

Lemma 20. If $C_\kappa \leq \frac{1}{4}$ then (18) holds.

Proof. As in Lemma 16 we can bound the left side of (18) by

$$\frac{4}{3} |\psi'(F(z))| F'(z) - \frac{1}{3} \psi(F(z)).$$

Our goal is to prove the above is nonpositive. This is equivalent to showing that for $z \geq 0$

$$z F'(z) \leq \frac{\psi(F(z))}{2 |\psi'(F(z))|}.$$ 

We have from (30) that $\psi(u) \geq p_1$ and can compute

$$|\psi'(u)| \leq \sum_{k \geq 2} k(k-1)|u^{k-2} - (1-u)^{k-2}| \leq C_\kappa.$$ 

It then suffices to prove

(31) $$z F'(z) \leq \frac{p_1}{2C_\kappa}.$$ 

By Lemma 19 and the hypothesis $C_\kappa \leq 1/4$ it suffices to choose the $p_k$ so that

$$\left( \frac{2}{e \cdot p_1} \right)^2 \leq 2p_1.$$ 

Rewriting we have $4 \leq e^2 \cdot p_1^3$. It is straightforward to check that if $C_\kappa < 1/4$ then $p_1 \geq 7/8$. Since $4 < e^2(7/8)^3 \approx 4.9$ the inequality holds. $\square$

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