The Cost of Randomness for Converting a Tripartite Quantum State to be Approximately Recoverable

Eyuri Wakakuwa‡, Akihito Soeda*, Mio Murao*‡

*Department of Physics, Graduate School of Science, The University of Tokyo, Japan
‡Graduate School of Information Systems, The University of Electro-Communications, Japan

Abstract—We introduce and analyze a task in which a tripartite quantum state is transformed to an approximately recoverable state by a randomizing operation on one of the three subsystems. We consider cases where the initial state is a tensor product of $n$ copies of a tripartite state $\rho^{ABC}$, and is transformed by a random unitary operation on $A^n$ to another state which is approximately recoverable from its reduced state on $A^n B^n$ (Case 1) or $B^n C^n$ (Case 2). We analyze the minimum cost of randomness per copy required for the task in an asymptotic limit of infinite copies and vanishingly small error of recovery, mainly focusing on the case of pure states. We prove that the minimum cost in Case 1 is equal to the Markovianizing cost of the state, for which a single-letter formula is known. With an additional requirement on the convergence speed of the recovery error, we prove that the minimum cost in Case 2 is also equal to the Markovianizing cost. Our results have an application for distributed quantum computation.

I. INTRODUCTION

Tripartite quantum states, for which the quantum conditional mutual information (QCMI) is zero, are called quantum Markov chains, or Markov states for short [1]. They have been investigated in several contexts, for example, in analyzing the cost of quantum state redistribution [2, 3], investigating effects of the initial system-environment correlation on the dynamics of quantum states [4], and computing the free energy of quantum many-body systems [5]. A characterization of Markov states is obtained in [1], in which the following three properties are proved to be equivalent:

1) Vanishing QCMI: A tripartite quantum state $\rho^{ABC}$ satisfies

$$I(A : C|B)_\rho = 0. \quad (1)$$

2) Recoverability: $\rho^{ABC}$ is recoverable from its bipartite reduced state on $AB$ and $BC$, that is, there exist quantum operations $\mathcal{R} : B \rightarrow AB$ and $\mathcal{R}' : B \rightarrow BC$ such that

$$\rho^{ABC} = \mathcal{R}(\rho^{BC}) = \mathcal{R}'(\rho^{AB}). \quad (2)$$

3) Decomposability: $\rho^{ABC}$ takes a simple form under a decomposition of system $B$ into three subsystems $b_0$, $b_L$ and $b_R$ as

$$\rho^{ABC} = \sum_j p_j |j\rangle\langle j|^b_0 \otimes \sigma_j^{Ab_L} \otimes \phi_j^{b_RC}. \quad (3)$$

The equivalence among the three properties, however, breaks down when we require that Equalities (1), (2) and (3) hold approximately, instead of requiring exactly. On the one hand, the result by Fawzi and Renner [6] proves that a tripartite state is recoverable with a small error (i.e., approximately recoverable) if QCMI of the state is small (see also [7]–[11]). On the other hand, QCMI of a state can be vanishingly small, even if the state does not fit into any decomposition in the form of (3), up to a small error $\epsilon$. In the latter, the state is supposed to fit into a decomposition of $B^n$ into three subsystems $b_0$, $b_L$, and $b_R$ as (3), up to a small error $\epsilon$. We analyze and compare the minimum cost of randomness per copy required for each task, by considering an asymptotic limit of $\epsilon \rightarrow 0$ and $n \rightarrow \infty$.

Depending on the type of the recovery map to be applied, we consider two cases for M-Rec. In the first case, the state after the transformation is required to be approximately recoverable from its reduced state on $A^n B^n$, whereas in the second case it is supposed to be approximately recoverable from the reduced state on $A^n B^n$. We call the minimum cost of randomness in each case as the Markovianizing cost in terms of recoverability (M-Rec cost), and denote it by $M^n_{ABC}(\rho)$ and $M^n_{A|ABC}(\rho)$, respectively for each case above.

In the previous work [13], we introduced a similar task that we simply call Markovianization, in which $n$ copies of a tripartite state $\rho^{ABC}$ is transformed by a random unitary...
operation on $A^n$ to another state which is $\epsilon$-close to a Markov state conditioned by $B^n$. As we prove later, this version of Markovianization is equivalent to M-Dec, up to a dimension-independent rescaling of $\epsilon$. Consequently, the minimum cost of randomness per copy required for Markovianization in the version of [13] is equal to the one required for M-Dec. We call the latter as the version of $[13]$ is equal to the one required for M-Dec.

In this paper, we mainly focus on cases where the initial state is pure, that is, $\rho^{ABC} = \ketbra{\Psi}{\Psi}$. The main results of this paper are as follows. First, we prove that $M^{D}_{AB|BC}^{R}(\Psi) = M^{D}_{AB|BC}^{\Gamma}(\Psi)$ holds. Second, we prove that $M^{R}_{A|BC}^{\Gamma}(\Psi) = M^{R}_{A|BC}^{\Gamma}(\Psi)$ holds as well, under an additional requirement that the error of recovery converges to zero faster than $1/n$. Thereby we reveal that the gap between approximate recoverability and approximate decomposability disappears in this information theoretical framework, at least in the case of pure states. The obtained results are applied to an analysis of distributed quantum computation in [14].

The structure of this paper is as follows. In Section II, we introduce rigorous definitions of approximate recoverability and approximate decomposability, and analyze relations among these conditions. In Section III, we introduce the formal definitions of Markovianization in terms of recoverability and that in terms of decomposability, and describe the main results. In Section IV, we introduce and analyze an extension of Markovianization into the one induced by a measurement. In Section V, we briefly review applications of the results to distributed quantum computation. Conclusions are given in Section VI. (See Appendices for proofs of the main theorems.)

Notations. We follow the notations introduced in [13].

II. RECOVERABILITY AND DECOMPOSABILITY

In this section, we present rigorous definitions of approximate recoverability and approximate decomposability. We then prove some general relations among these.

Let us first present three equivalent definitions for “exact Markovness” of tripartite quantum states.

Theorem 1 (Theorem 6 in [1]) The following three conditions are equivalent:

1) $\Upsilon^{ABC}$ satisfies $I(A:C|B)_{\Upsilon} = 0$.

2) There exist quantum operations $R : B \to AB$ and $R' : B \to BC$ such that

$$\Upsilon^{ABC} = R(\Upsilon^{B|C}) = R'(\Upsilon^{A|BC}).$$

3) There exists a decomposition of $\mathcal{H}^B$ into a tensor product of three Hilbert spaces $\mathcal{H}^{b_{A}} \otimes \mathcal{H}^{b_{B}} \otimes \mathcal{H}^{b_{C}}$, described by a unitary isomorphism $\Gamma : \mathcal{H}^{B} \to \mathcal{H}^{b_{A}} \otimes \mathcal{H}^{b_{B}} \otimes \mathcal{H}^{b_{C}},$ such that $\Upsilon^{ABC}$ is decomposed as

$$\Gamma^{B} \Upsilon^{ABC} \Gamma^{B} = \sum_{i} q_{i} |i\rangle_{b_{A}} \otimes \sigma_{i}^{A|BC} \otimes \phi_{i}^{b_{B} b_{C}}$$

with some probability distribution $\{q_{i}\}$, orthonormal basis $\{|i\rangle\}_i$ of $\mathcal{H}^{b_{A}}$, states $\sigma_{i} \in \mathcal{S}(\mathcal{H}^{A} \otimes \mathcal{H}^{b_{B}})$ and $\phi_{i} \in \mathcal{S}(\mathcal{H}^{b_{B}} \otimes \mathcal{H}^{C})$.

A tripartite quantum state that satisfies the conditions in the above theorem is called a Markov state conditioned by $B$. When $\Upsilon^{ABC}$ is a Markov state conditioned by $B$, (4) is called a Markov decomposition of $\Upsilon^{ABC}$, and $\Gamma$ in (4) is called a Markov isomorphism on $B$ with respect to $\Upsilon^{ABC}$.

We now introduce four different characterizations of a tripartite quantum state being "approximately Markov”.

Definition 2 A tripartite state $\rho^{ABC}$ is $\epsilon$-QCMI conditioned by $B$ if it satisfies

$$I(A:C|B)_{\rho} \leq \epsilon.$$  

Definition 3 A tripartite state $\rho^{ABC}$ is $\epsilon$-recoverable from $BC$ if there exists a quantum operation $R : B \to AB$ such that

$$\|\rho^{ABC} - R(\rho^{BC})\|_1 \leq \epsilon.$$  

Definition 4 A tripartite state $\rho^{ABC}$ is $\epsilon$-recoverable from $AB$ if there exists a quantum operation $R' : B \to BC$ such that

$$\|\rho^{ABC} - R'(\rho^{AB})\|_1 \leq \epsilon.$$  

Definition 5 A tripartite state $\rho^{ABC}$ is $\epsilon$-decomposable on $B$ if there exists a Markov state $\Upsilon^{ABC}$ conditioned by $B$ such that

$$\|\rho^{ABC} - \Upsilon^{ABC}\|_1 \leq \epsilon.$$  

As we prove in Appendix A, Condition 3 is equivalent to the condition that $\rho^{ABC}$ fits into the best possible choice of the tensor-product decomposition of $B$ into three subsystems as [3], up to a small error $\epsilon$. This fact supports the use of “decomposable” in Definition 5.

The following relations hold among the conditions described above.

Lemma 6 For an arbitrary tripartite state $\rho^{ABC}$:

1) $\rho^{ABC}$ is $\sqrt{\epsilon}$-recoverable from $AB$ and $BC$ if it is $\epsilon$-QCMI conditioned by $B$.

2) $\rho^{ABC}$ is $f(\epsilon,d_{C})$-recoverable from $BC$ if it is $\epsilon$-recoverable from $AB$ and $\epsilon \leq 1$. Here $f(\epsilon,d) := \sqrt{4\epsilon \log d + 2h(\epsilon)}$ and $h(\epsilon)$ is the binary entropy defined by $h(\epsilon) := -\epsilon \log \epsilon - (1 - \epsilon) \log (1 - \epsilon)$.

3) $\rho^{ABC}$ is $2\epsilon$-recoverable from $AB$ and $BC$ if it is $\epsilon$-decomposable on $B$.

Proof: Property 1) is proved in [6] (see Inequality (6) therein). As for Property 2), suppose $\rho^{ABC}$ is $\epsilon$-recoverable
from BC. There exists a linear CPTP map \( R : B \to AB \) such that
\[
\| \rho^{ABC} - R(\rho^{BC}) \|_1 \leq \epsilon.
\]

Due to Inequality (8.28) in [9], we have
\[
I(A : C|B)_\rho \leq 4e \log dc + 2h(\epsilon).
\]

Applying Property 1), we obtain 2).

Property 3) is proved as follows. Suppose \( \rho^{ABC} \) is \( \epsilon \)-decomposable on \( B \), and let \( \Upsilon^{ABC} \) be a Markov state conditioned by \( B \) satisfying \([5]\).

There exist quantum operations \( R : B \to AB \) and \( R' : B \to BC \) such that \( \Upsilon^{ABC} = R(\Upsilon^{BC}) = R'(\Upsilon^{AB}) \).

From \([5]\) and the monotonicity of the trace distance, we have
\[
\| R(\rho^{BC}) - \Upsilon^{ABC} \|_1 \leq \epsilon, \quad \| R'(\rho^{AB}) - \Upsilon^{ABC} \|_1 \leq \epsilon.
\]

By the triangle inequality, we obtain
\[
\| \rho^{ABC} - R(\rho^{BC}) \|_1 \leq 2\epsilon, \quad \| \rho^{ABC} - R'(\rho^{AB}) \|_1 \leq 2\epsilon,
\]

which completes the proof of Property 3).

III. MARKOVIANIZING COSTS

In this section, we present a concept of Markovianization, and describe the main results on the Markovianizing costs of tripartite quantum states. Proofs are given in Appendix C and D.

Let us first present Markovianization as formulated in [13].

**Definition 7** (Definition 7 in [13]) A tripartite state \( \rho^{ABC} \) is Markovianized with the randomness cost \( R \) on \( A \), conditioned by \( B \), if the following statement holds. That is, for any \( \epsilon > 0 \), there exists \( n_\epsilon \) such that for any \( n \geq n_\epsilon \), we find a random unitary operation \( \forall_n : \tau \mapsto 2^{-nR} \sum_{k=1}^{2^n} V_k\tau V_k^\dagger \) on \( A^n \), so that \( \forall_n((\rho^{ABC})_n) \) is \( \epsilon \)-recoverable on \( B^nC^n \).

The Markovianizing cost of \( \rho^{ABC} \) in terms of recoverability from \( BC \) is defined as \( M_{A|BC}^R(\rho^{ABC}) := \inf \{ R | \rho^{ABC} \) is Markovianized with the randomness cost \( R \) on \( A \), in terms of recoverability from \( BC \) \}.

**Definition 9** A tripartite state \( \rho^{ABC} \) is Markovianized with the randomness cost \( R \) on \( A \), in terms of recoverability from \( BC \), if the following statement holds. That is, for any \( \epsilon > 0 \), there exists \( n_\epsilon \) such that for any \( n \geq n_\epsilon \), we find a random unitary operation \( \forall_n : \tau \mapsto 2^{-nR} \sum_{k=1}^{2^n} V_k\tau V_k^\dagger \) on \( A^n \), so that \( \forall_n((\rho^{ABC})_n) \) is \( \epsilon \)-recoverable from \( B^nC^n \).

The Markovianizing cost of \( \rho^{ABC} \) in terms of recoverability from \( BC \) is defined as \( M_{A|BC}^R(\rho^{ABC}) := \inf \{ R | \rho^{ABC} \) is Markovianized with the randomness cost \( R \) on \( A \), in terms of recoverability from \( BC \) \}.

The following two theorems are the main results of this paper. The first one (Theorem 10) shows general properties of the \( M \)-Rec costs of an arbitrary (possibly mixed) tripartite state, and the second one (Theorem 11) states that the three types of the Markovianizing cost are equal for pure states. We also present a lemma that plays a central role in the proof of Theorem 11. Proofs are given in Appendix C and D.

**Theorem 10** For any tripartite state \( \rho^{ABC} \), we have
\[
I(A : C|B)_\rho \leq M_{A|BC}^R(\rho^{ABC}) \leq M_{A|B}^D(\rho^{ABC})
\]
and
\[
I(A : C|B)_\rho \leq M_{A|BC}^R(\rho^{ABC}) \leq M_{A|B}(\rho^{ABC}).
\]

**Theorem 11** For any tripartite pure state \( \Psi^{ABC} \), we have
\[
M_{A|BC}^R(\Psi^{ABC}) = M_{A|B}(\Psi^{ABC}).
\]

If we additionally require in Definition 7 that
\[
\lim_{\epsilon \to 0} \epsilon \cdot n_\epsilon = 0,
\]
we also have
\[
M_{A|BC}^R(\Psi^{ABC}) = M_{A|B}(\Psi^{ABC}).
\]

**Lemma 12** Let \( |\Psi^{ABC}\rangle \) be a pure state, and for any \( n \) and \( \epsilon > 0 \), let \( \mathcal{E} \) be a quantum operation on \( A^n \) that satisfy
\[
\left\| (\mathcal{E}A^n \otimes \text{id}C^n)(|\Psi^{AC}\rangle \otimes |\Psi^{CB}\rangle) - (|\Psi^{AC}\rangle \otimes |\Psi^{CB}\rangle) \right\| \leq \epsilon.
\]

Then we have
\[
\frac{1}{n} I(A^n : B^nC^n)_{\mathcal{E}A^n(\Psi^{AC})} \geq M_{A|BC}^D(\Psi^{ABC}) - 2\eta(\zeta_\epsilon) \log (d_A d_B d_C).
\]

Here, \( \zeta_\epsilon(\epsilon) \) is a function that satisfies \( \lim_{\epsilon \to 0} \zeta_\epsilon(\epsilon) = 0 \), and does not depend on \( n \).

It is left open whether Equality (10) holds when we drop Condition 9. An underlying problem is whether we can
eliminate the dimension dependence of the error in Property 2) in Lemma 6. We formulate this problem by the following conjecture.

**Conjecture 13** There exists a nonnegative function $g(\epsilon)$, which is independent of dimensions of quantum systems and satisfies $\lim_{\epsilon \to 0} g(\epsilon) = 0$, such that the following statement holds for an arbitrary tripartite state $\rho^{ABC}$ and $\epsilon > 0$: The state $\rho^{ABC}$ is $g(\epsilon)$-recoverable from $BC$ if it is $\epsilon$-recoverable from $AB$.

Condition 9 in Theorem 11 can be eliminated if the above conjecture is true. The reason is as follows: If $\psi_n((\rho^{ABC})^\otimes n)$ is $\epsilon$-recoverable from $A^n B^n$, the state is $g(\epsilon)$-recoverable from $B^n C^n$. Thus we have $M_{R|AB}^R(\Psi^{ABC}) \leq M_{A|AB}^R(\Psi^{ABC})$, which implies Equality (10) when combined with (7) and (8).

**IV. ALTERNATIVE FORMULATIONS**

In this section, we introduce an extension of M-Rec to that by a measurement (Figure 1), which will be referred to as measurement-induced Markovianization in terms of recoverability. In particular, we consider an extension of the M-Rec cost in Definition 9 to that by a measurement. The result obtained here has a direct application for distributed quantum computation [14]. In the following, we denote systems $A^n$, $B^n$ and $C^n$ by $A$, $B$ and $C$ for simplicity of notation.

Let $|\Psi^{ABC}\rangle$ be a bipartite pure state, and let $|\varrho_n\rangle_{A_0G}$ be a bipartite pure entangled state shared by Alice and George. Consider a state transformation of $(\Psi^{\otimes n})^{ABC} \otimes |\varrho_n\rangle_{A_0G}$ induced by a measurement on $\bar{A} A_0$, which is described by a set of measurement operators $\{M_k^{\bar{A} A_0 \rightarrow \bar{A}'}\}_k$. The probability of obtaining the measurement outcome $k$ is given by

$$p_k = \| M_k^{\bar{A} A_0 \rightarrow \bar{A}'} |\Psi^{\otimes n}\rangle^{ABC} |\varrho_n\rangle_{A_0G} \|^2,$$

and the post-measurement state corresponding to the outcome $k$ is given by

$$|\Psi_k\rangle^{A' \bar{B} \bar{C} G} = \frac{1}{\sqrt{p_k}} M_k^{\bar{A} A_0 \rightarrow \bar{A}'} |\Psi^{\otimes n}\rangle^{ABC} |\varrho_n\rangle_{A_0G}. \hspace{1cm} (12)$$

We require that (i) the measurement does not significantly change the reduced state on $BC$ on average, and (ii) the reduced state of the post-measurement state (12) on $A' \bar{B} \bar{C}$ is approximately recoverable from $A' \bar{B}$ on average. We focus on the minimum amount of a correlation between systems $\bar{B} \bar{C}$ and $G$, which is inevitably generated by the measurements that satisfy the two conditions. A precise definition is given as follows.

**Definition 14** A tripartite pure state $|\Psi\rangle^{ABC}$ is Markovianized with the correlation production $R$ by a measurement on $A$, in terms of recoverability from $AB$, if the following statement holds. That is, for any $\epsilon > 0$, there exists $n_\epsilon$ such that for any $n \geq n_\epsilon$, we find a measurement $\{M_k^{\bar{A} A_0 \rightarrow \bar{A}'}\}_k$ and a pure state $|\varrho_n\rangle_{A_0G}$ satisfying the following conditions:

1) The measurement does not significantly change the reduced state on $BC$ on average, that is,

$$\sum_k p_k \left\| (\Psi^{\otimes n})^{\bar{B} \bar{C}} - \Psi_k^{\bar{B} \bar{C}} \right\|_1 \leq \epsilon. \hspace{1cm} (13)$$

2) There exist quantum operations $\mathcal{R}_{n,k}: B^n \rightarrow B^n C^n (k)$ that satisfy

$$\sum_k p_k \left\| \Psi_k^{A' \bar{B} \bar{C}} - \mathcal{R}_{n,k}(\Psi_k^{A' \bar{B}}) \right\|_1 \leq \epsilon. \hspace{1cm} (14)$$

3) The correlation between $\bar{B} \bar{C}$ and $G$ produced by the measurement is at most $n R$ bits in QMI on average, that is,

$$I(G : \bar{B} \bar{C})_{av} := \sum_k p_k I(G : \bar{B} \bar{C})_{\Psi_k} \leq n R.$$

The measurement-induced Markovianizing cost of $|\Psi\rangle^{ABC}$ in terms of recoverability from $AB$ is defined as $M_{A|AB}^{R,m}(\Psi^{ABC}) := \inf R | \Psi^{ABC} \rangle$ with the correlation production $R$ by a measurement on $A$, in terms of recoverability from $AB$.

The measurement-induced Markovianizing cost of pure states defined as above is equal to the Markovianizing cost in terms of random unitary operations, as presented by the following theorem. A proof is given in Appendix 13.

**Theorem 15** For any tripartite pure state $\Psi^{ABC}$, we have

$$M_{A|AB}^{R,m}(\Psi^{ABC}) = M_{A|AB}^D(\Psi^{ABC}), \hspace{1cm} (15)$$

if we additionally require in Definition 14 that

$$\lim_{\epsilon \to 0} \epsilon \cdot n_\epsilon = 0. \hspace{1cm} (16)$$

This additional condition can be eliminated if Conjecture 13 is true.
V. APPLICATIONS

In [14], we analyze implementations of bipartite unitaries by local operations and classical communication (LOCC) assisted by shared entanglement. We consider a scenario in which the two distant parties perform the same bipartite unitary on many pairs of input states generated by a completely random i.i.d. (independent and identically distributed) quantum information source. Under the condition that the error vanishes in the limit of infinite input pairs, we ask what is the minimum cost of resources of entanglement and classical communication per copy.

We prove that the minimum costs of entanglement and classical communication in two-round protocols are given by the Markovianizing cost of a tripartite state associated with the unitary. At the core of the proof is the fact that any successful two-round protocols for implementing a bipartite unitary can be described as a combination of Markovianization of a particular tripartite state and the subsequent state merging ([15], [16]). See [14] for further details.

VI. CONCLUSIONS AND DISCUSSIONS

We have introduced the task of Markovianization in terms of recoverability (M-Rec), and that in terms of decomposability (M-Dec). The latter of which turns out to be equivalent to Markovianization in the version of our previous paper [13]. For pure states, we have proven that the minimum cost of randomness required for M-Rec is equal to the one required for M-Dec, for which a single-letter formula has been known. Our results have applications in analyzing optimal costs of resources in distributed quantum computation. An open question is whether Equalities (10) and (15) holds when we drop Condition (2). Another related question is whether we can eliminate the dimension dependence of the error in Property 2 in Lemma 6.

Note added: After the completion of this work, the authors have been informed about another work [17], in which a task similar to M-Rec in our paper was independently proposed. Their definition of the task is more general than ours, in that they consider “coordinated” random unitary operations over systems $A^n$, $B^n$ and $C^n$ for Markovianizing a state. They independently derived a lower bound on the cost of randomness, from which the first inequalities in (6) and (7) are derived as a corollary.

ACKNOWLEDGMENTS

The authors thank Mario Berta and Tomohiro Ogawa for helpful discussions. EW is grateful for Mark Wilde informing the authors about Ref. [17] and Inequality (8.28) of [9], regarding Property 2) in Lemma 6. This work is supported by the Project for Developing Innovation Systems of MEXT, Japan and JSPS KAKENHI (Grant No. 23540463, No. 23240001, No. 26330006, and No. 15H01677). We also gratefully acknowledge to the ELC project (Grant-in-Aid for Scientific Research on Innovative Areas MEXT KAKENHI (Grant No. 24106009)) for encouraging the research presented in this paper.

REFERENCES

[1] P. Hayden, R. Jozsa, D. Petz, and A. Winter, “Structure of states which satisfy strong subadditivity of quantum entropy with equality,” Comm. Math. Phys., vol. 246, pp. 359–374, 2004.
[2] I. Devetak and J. Yard, “Exact cost of redistributing multipartite quantum states,” Phys. Rev. Lett., vol. 100, p. 230501, 2008.
[3] J. T. Yard and I. Devetak, “Optimal quantum source coding with quantum side information at the encoder and decoder,” IEEE Trans. Inf. Theory, vol. 55, p. 5339, 2009.
[4] F. Buscemi, “Complete positivity, markovianity, and quantum data-processing inequality, in the presence of initial system-environment correlations,” Phys. Rev. Lett., vol. 113, p. 140502, 2014.
[5] D. Poulin and M. B. Hastings, “Markov entropy decomposition: A variational dual for quantum belief propagation,” Phys. Rev. Lett., vol. 106, p. 080403, 2011.
[6] O. Fawzi and R. Renner, “Quantum conditional mutual information and approximate markov chains,” Comm. Math. Phys., vol. 340, pp. 575–611, 2015.
[7] F. G. S. L. Brandão, A. W. Harrow, J. Oppenheim, and S. Strelchuk, “Quantum conditional mutual information, reconstructed states, and state redistribution,” Phys. Rev. Lett., vol. 115, p. 050501, 2015.
[8] M. Koashi and N. Imoto, “Operations that do not disturb partially known quantum states,” e-print arXiv:1504.05805, 2015.
[9] M. Berta, K. P. Seshadreesan, and M. M. Wilde, “Renyi generalizations of the conditional quantum mutual information,” J. Math. Phys., vol. 56, p. 022205, 2015.
[10] M. M. Wilde, “Recoverability in quantum information theory,” Proc. R. Soc. A., vol. 471, p. 2182, 2015.
[11] M. Berta, M. Lemm, and M. M. Wilde, “Monotonicity of quantum relative entropy and recoverability,” Quant. Info. and Comp., vol. 15, pp. 1333–1354, 2015.
[12] B. Ibinson, N. Linden, and A. Winter, “Robustness of quantum markov chains,” Comm. Math. Phys., vol. 277, pp. 289–304, 2008.
[13] E. Wakakiuwa, A. Soeda, and M. Murao, “Markovianizing cost of tripartite quantum states,” e-print arXiv:1504.05805, 2015.
[14] ——, “A coding theorem for distributed quantum computation,” e-print arXiv:1505.04352, 2015, under revision.
[15] M. Horodecki, J. Oppenheim, and A. Winter, “Partial quantum information,” Nature, vol. 436, p. 673, 2005.
[16] ——, “Quantum state merging and negative information,” Comm. Math. Phys., vol. 269, p. 107, 2007.
[17] M. Berta, F. G. S. L. Brandão, and M. Wilde, private communication.
[18] M. Wilde, Quantum Information Theory. Cambridge University Press, 2013.
[19] M. Koashi and N. Imoto, “Operations that do not disturb partially known quantum states,” Phys. Rev. A, vol. 66, p. 022318, 2002.
[20] M. Fannes, “A continuity property of the relative entropy density for spin lattice systems,” Comm. Math. Phys., vol. 31, p. 291, 1973.
[21] R. Alicki and M. Fannes, “Quantum conditional mutual information, reconstructed states, and state redistribution,” Comm. Math. Phys., vol. 269, p. 107, 2007.
[22] H. Araki and E. H. Lieb, “Entropy inequalities,” Comm. Math. Phys., vol. 18, pp. 160–170, 1970.
[23] A. Uhlmann, “The ‘transition probability’ in the state space of a *-algebra,” Rep. Math. Phys., vol. 9, p. 273, 1976.
of $\epsilon$. Thereby we justify referring to Condition \[5\] as “$\epsilon$-decomposability”, and to Markovianization in the version of Definition \[7\] as “Markovianization in terms of decomposability”.

For any decomposition of $H_B$ into a tensor product of three Hilbert spaces, which is described by an isomorphism $\Gamma : H_B \to H_b \otimes H_b \otimes H_b$, and for any orthonormal basis $\{ |i\rangle \}_{i}$ of $H_b$, define a map $T_{\Gamma,\{i\}}$ on $S(H_A \otimes H_B \otimes H_C)$ by

$$T_{\Gamma,\{i\}}(\rho^{ABC}) = \Gamma\left( \sum_i p_i |i\rangle \langle i| \otimes \rho^{Abi} \otimes \rho^{bC} \right),$$

where

$$p_i = \text{Tr}[|i\rangle \langle i| \Gamma \rho \Gamma^{\dagger} |i\rangle \langle i|],$$

$$\rho^{Abi} = p_i^{-1} \text{Tr}_{Abi} \left[ |i\rangle \langle i| \Gamma \rho \Gamma^{\dagger} |i\rangle \langle i| \right],$$

and

$$\rho^{bC} = p_i^{-1} \text{Tr}_{Abi} \left[ |i\rangle \langle i| \Gamma \rho \Gamma^{\dagger} |i\rangle \langle i| \right].$$

Due to Theorem \[2\], $T_{\Gamma,\{i\}}(\rho^{ABC})$ is a Markov state conditioned by $B$ for any $\rho^{ABC}$.

Suppose there exists an isometry $\Gamma : H_B \to H_b \otimes H_b \otimes H_b$ and an orthonormal basis $\{ |i\rangle \}_{i}$ of $H_b$ such that

$$\|\rho^{ABC} - T_{\Gamma,\{i\}}(\rho^{ABC})\|_1 \leq \epsilon.$$ 

It immediately follows that there exists a Markov state $\Upsilon^{ABC}(= T_{\Gamma,\{i\}}(\rho^{ABC}))$ that satisfy Inequality \[5\].

Conversely, suppose there exists a Markov state $\Upsilon^{ABC}$ that satisfy \[5\]. Let $\Gamma : H_B^{\otimes 2} \to H_b \otimes H_b \otimes H_b$ be a Markov isomorphism on $B$ with respect to $\Upsilon^{ABC}$, and let

$$\Upsilon_{MK} := \Gamma^{\otimes 2} \Upsilon^{ABC} \Gamma^{\dagger} = \sum_i q_i |i\rangle \langle i| \otimes \sigma^{Abi} \otimes \phi^{bC} \quad \text{(18)}$$

be a Markov decomposition of $\Upsilon^{ABC}$. We first assume $H_B^{\otimes 2} = H_B$ for simplicity, and prove

$$\|\rho^{ABC} - T_{\Gamma,\{i\}}(\rho^{ABC})\|_1 \leq 6\epsilon.$$ 

By the triangle inequality, we have

$$\|\rho^{ABC} - T_{\Gamma,\{i\}}(\rho^{ABC})\|_1 \leq \|\rho^{ABC} - \Upsilon^{ABC}\|_1 + \|\Upsilon^{ABC} - T_{\Gamma,\{i\}}(\rho^{ABC})\|_1 \leq \|T_{\Gamma,\{i\}}(\rho^{ABC}) - \Upsilon^{ABC}\|_1 + \epsilon.$$ 

Next, \[17\] and \[18\] imply that

$$\|T_{\Gamma,\{i\}}(\rho^{ABC}) - \Upsilon^{ABC}\|_1 = \|\Gamma(T_{\Gamma,\{i\}}(\rho^{ABC}))^{\dagger} - \Upsilon_{MK}\|_1 \leq \sum_i \|p_i \rho^{Abi} \otimes \rho^{bC} - q_i \sigma^{Abi} \otimes \phi^{bC}\|_1 \leq \sum_i \|p_i \rho^{Abi} \otimes \rho^{bC} - q_i \sigma^{Abi} \otimes \phi^{bC}\|_1 + \sum_i \|q_i \sigma^{Abi} \otimes \phi^{bC} - q_i \sigma^{Abi} \otimes \phi^{bC}\|_1 \leq \sum_i \|p_i \rho^{Abi} \otimes \rho^{bC} - q_i \sigma^{Abi} \otimes \phi^{bC}\|_1 + \sum_i |p_i - q_i|. \quad \text{(21)}$$

In addition, we have

$$p_i \left\| \rho^{Abi} \otimes \rho^{bC} - \sigma^{Abi} \otimes \phi^{bC} \right\|_1 \leq p_i \left\| \rho^{Abi} \otimes \rho^{bC} - \sigma^{Abi} \otimes \rho^{bC} \right\|_1 + p_i \left\| \sigma^{Abi} \otimes \rho^{bC} - \sigma^{Abi} \otimes \phi^{bC} \right\|_1 \leq 2p_i \left\| \rho^{Abi} - \sigma^{Abi} \right\|_1 + p_i \left\| \rho^{bC} - \phi^{bC} \right\|_1 \leq 2p_i \left\| \rho^{Abi} - \sigma^{Abi} \right\|_1 + 2|p_i - q_i|.$$ 

Consider a state $\hat{\rho}$ defined by

$$\hat{\rho} := \sum_i p_i |i\rangle \langle i| \otimes \rho^{Abi} \otimes \rho^{bC},$$

and let $D_{bC}$ be the completely dephasing operation on $b$ with respect to the basis $|i\rangle$. We have $\hat{\rho} = D_{bC}(\Gamma \rho \Gamma^{\dagger})$, as well as $\Upsilon_{MK} = D_{bC}(\Upsilon_{MK})$ from \[18\]. Therefore, by the monotonicity of the trace distance,

$$\|\hat{\rho} - \Upsilon_{MK}\|_1 \leq \epsilon.$$ 

holds from \[5\], which leads to

$$\sum_i \left\| p_i \rho^{Abi} - q_i \sigma^{Abi} \right\|_1 \leq \epsilon.$$ 

By tracing out $\hat{\rho}_{bC} \in \Upsilon_{MK}$ in \[23\], we obtain

$$\sum_i |p_i - q_i| \leq \epsilon.$$ 

Combining \[20\], \[21\], \[22\], \[24\] and \[25\], we obtain \[19\].

An inequality similar to \[19\] is obtained when $H_B^{\otimes 2} \neq H_B$ as well. Note that $\rho^{ABC}$ is invariant when projected onto the support of $\Upsilon_B$ up to a small error $2\sqrt{\epsilon}$, due to Inequality \[5\] and the gentle measurement lemma (see Lemma 9.4.1 in \[18\]).

**APPENDIX B**

**M-Dec Cost of Pure States**

In this section, we summarize a result obtained in \[13\] regarding a single-letter formula for the M-Dec cost of pure states. Let us first present a decomposition of a Hilbert space called the Koashi-Imoto (KI) decomposition, which is first introduced in \[19\] and is extended in \[1\].

**Theorem 16** (\[1\], \[19\], see also Definition 3 and Lemma 4 in \[13\]) Consider a quantum system $A$ and $A'$ described by a finite dimensional Hilbert space $H^A$ and $H^{A'}$, respectively.
Associated to any bipartite quantum state $\Psi^{AA'} \in S(\mathcal{H}^A \otimes \mathcal{H}^{A'})$, there exists a decomposition of $\mathcal{H}^A := \text{supp}[\Psi^A]$ into a tensor product of three Hilbert spaces $\mathcal{H}^{a_1} \otimes \mathcal{H}^{a_2} \otimes \mathcal{H}^{a_3}$, described by a unitary isomorphism $\Gamma: \mathcal{H}^A \rightarrow \mathcal{H}^{a_1} \otimes \mathcal{H}^{a_2} \otimes \mathcal{H}^{a_3}$, such that the following two properties hold.

1) $\Gamma$ gives

$$
\Psi_k^{AA'} := \Gamma^A \Psi^{AA'} \Gamma^A = \sum_{j \in J} p_j |j\rangle^{a_1} \otimes \omega_j \otimes \phi_j^{AA'}
$$

with some probability distribution $\{p_j\}_{j \in J}$, orthonormal basis $\{|j\rangle\}_{j \in J}$ of $\mathcal{H}^{a_1}$, states $\omega_j \in S(\mathcal{H}^{a_2})$ and $\phi_j \in S(\mathcal{H}^{a_3} \otimes \mathcal{H}^{a_4})$.

2) A quantum operation $\mathcal{E}$ on $S(\mathcal{H}^A)$ leaves $\Psi^{AA'}$ invariant if and only if there exists an isometry $U: \mathcal{H}^A \rightarrow \mathcal{H}^A \otimes \mathcal{H}^E$ such that a Stinespring dilation of $\mathcal{E}$ is given by $\mathcal{E}(\tau) = \text{Tr}_E[U \tau U^\dagger]$, and that $U$ is decomposed by $\Gamma$ as

$$(\Gamma \otimes I^E)U\Gamma^A = \sum_{j \in J} \langle j | \otimes U_{aj} \otimes I_{aj}. \rangle
$$

Here, $I_j$ are the identity operator on $\mathcal{H}^{a_j} := \text{supp} \sum k p_{jk}\}$, and $U_j: \mathcal{H}^{a_j} \rightarrow \mathcal{H}^{a_j} \otimes \mathcal{H}^E$ are isometries that satisfy $\text{Tr}_E[U_j \omega_j U_j^\dagger] = \omega_j$ for all $j$, where $\mathcal{H}^{a_j} := \text{supp} \omega_j$.

We call $\Gamma$ as the $K_I$ isomorphism on system $A$ with respect to $\Psi^{AA'}$, and (26) as the $K_I$ decomposition of $\Psi^{AA'}$ on $A$. The $K_I$ decomposition and the corresponding $K_I$ isomorphism are uniquely determined from $\Psi^{AA'}$, up to trivial exchanges of the basis.

A single-letter formula for the M-Dec cost of tripartite pure states is obtained based on the $K_I$ decomposition.

**Theorem 17** (Theorem 8 in [13]) Let $|\Psi^{ABC}\rangle$ be a pure state, and let

$$
\Psi_k^{AC} := \sum_{j \in J} p_j |j\rangle^{a_1} \otimes \omega_j \otimes \phi_j^{AC}
$$

be the $K_I$ decomposition of $\Psi^{AC}$ on $A$. Then we have

$$M_{AB}(\Psi^{ABC}) = H(\{p_j\}_{j \in J}) + 2 \sum_{j \in J} p_j S(\phi_j^{AC}).$$

As we proved in [13] (see Appendix B-B therein), the error $\epsilon$ vanishes exponentially with $n$, which implies that $n_\epsilon$ can be chosen so that $n_\epsilon = O(\log(1/\epsilon))$. Thus Theorem 17 holds even when we additionally require in Definition 9 that $\lim_{n \to 0} \epsilon \cdot n_\epsilon = 0$.

**APPENDIX C**

**Proof of Theorem 10**

In this section, we present a proof of Theorem 10. Proofs of Inequalities (6) and (7) proceeds almost in parallel. Let us start with a summary of the continuity bounds of quantum entropies and mutual informations.

### A. Continuity of Quantum Entropies

Define

$$
\eta_\alpha(x) := \begin{cases} 
-x \log x & (x \leq 1/\epsilon), \\
1/\epsilon & (x > 1/\epsilon),
\end{cases}
$$

$$\eta(x) = x + \eta_\alpha(x)$$

and $h(x) := \eta_\alpha(x) + \eta_\alpha(1-x)$, where $\epsilon$ is the base of natural logarithm. For two states $\rho$ and $\sigma$ in a d-dimensional quantum system $(d < \infty)$ such that $\|\rho - \sigma\|_1 \leq \epsilon$, we have

$$|S(\rho) - S(\sigma)| \leq \epsilon \log d + \eta_\alpha(\epsilon) \leq \eta_\alpha(\epsilon) \log d,$$

which is called the Fannes inequality [20]. For two bipartite states $\rho, \sigma \in S(\mathcal{H}^A \otimes \mathcal{H}^B)$ such that $\|\rho - \sigma\|_1 \leq \epsilon < 1$, we have

$$|S(\rho) - S(\sigma)| \leq 4 \epsilon \log d_A + 2h(\epsilon) \leq 4 \eta_\alpha(\epsilon) \log d_A,$$

which is called the Aliki-Fannes inequality [21]. Note that the upper bound in (28) does not depend on $d_B$. As a consequence, we have

$$|I(\rho : \sigma) - I(\rho : \sigma)\rangle \leq 4 \epsilon \log d_A + 2h(\epsilon)$$

and

$$|I(\rho : \sigma) - I(\rho : \sigma)\rangle \leq 5 \eta_\alpha(\epsilon) \log d_A.\]$$

In addition

$$|I(\rho : \sigma) - I(\rho : \sigma)\rangle \leq 2 \eta_\alpha(\epsilon) \log (d_A d_B),$$

for an arbitrary $\epsilon > 0$.

### B. Proof of Inequality (6)

We prove the first inequality in (6) by showing that any $R$ satisfying $R > M_{AB}^{R^BC}(\rho^{ABC})$ also satisfies $R \geq I(\rho : C|B)$. By definition, for an arbitrary $R > M_{AB}^{R^BC}(\rho^{ABC})$, $\epsilon > 0$ and sufficiently large $n_\epsilon$ there exists a random unitary operation $V_n: \tau \mapsto 2^{-nR} \sum_{k=1}^{2^n} V_k \tau V_k^\dagger$ on $A^n$ and a quantum operation $\mathcal{R}_n: B^n \rightarrow A^n B^n$ that satisfy

$$\|V_n((\rho^{ABC})^{\otimes n}) - \mathcal{R}_n((\rho^{BC})^{\otimes n})\|_1 \leq \epsilon.$$

Let $|\psi^{ABC}_{\rho}\rangle$ be a purification of $\rho^{ABC}$, $E$ be a quantum system with dimension $2^n R$, and let $\{|k\rangle\}_{k=1}^{2^n}$ be an orthonormal basis of $\mathcal{H}_k$. Defining an isometry $W: A^n \rightarrow EA^n$ by

$$W = \sum_{k=1}^{2^n} |k\rangle^E \otimes V_k^{\otimes n},$$

a Stinespring dilation of $\mathcal{R}_n$ is given by $V_n(\tau) = \text{Tr}_E[W \tau W^\dagger]$. Then a purification of $\rho^{ABC} := V_n((\rho^{ABC})^{\otimes n})$ is given by $|\psi^{ABC}_{\rho}\rangle^{EA^n B^n C^n D^n} := W(|\psi^{ABC}_{\rho}\rangle^{ABC}_{\otimes n})$. For this state, we have

$$nR \geq S(E)_{\psi^{ABC}_{\rho}} = S(A^n B^n C^n D^n)_{\psi^{ABC}_{\rho}} \geq S(A^n B^n C^n)_{\psi^{ABC}_{\rho}} - S(D^n)_{\psi^{ABC}_{\rho}} = S(A^n B^n C^n)_{\rho^{ABC}} - S(D^n)_{\rho^{ABC}} = S(A^n B^n C^n)_{\rho^{ABC}} - nS(ABC)_{\rho}.$$
where the third line follows by the Araki-Lieb inequality [22]. The first term satisfies

$$S(A^n B^n C^n)\rho_n^\epsilon = S(C^n|A^n B^n)\rho_n^\epsilon + S(A^n B^n)\rho_n^\epsilon \geq S(C^n|A^n B^n)\rho_n^\epsilon + S(A^n B^n)\rho_n^\epsilon = S(C^n|A^n B^n)\rho_n^\epsilon + nS(AB)\rho,$$

\[ (34) \]

where the third line follows because the von Neumann entropy is nondecreasing under random unitary operations. Define $\rho_n^{ABC} := \mathcal{R}_n((\rho^{BC})^{\otimes n})$. Note also that

$$\rho_n^{B^n C^n} = \text{Tr}_{\mathcal{A}n}\mathcal{V}_n((\rho^{ABC})^{\otimes n}) = (\rho^{BC})^{\otimes n}.\nonumber$$

From (33), (34) and (35), we obtain

$$R \geq I(A : C|B) - 2\eta(\epsilon)\log(d_AD_Bd_C).\nonumber$$

Since this relation holds for any $R > M^{R}_{A|AB}(\rho^{ABC})$ and $\epsilon > 0$, we have the first inequality in (6).

The proof for the second inequality is as follows. For any $R > M^{D}_{A|B}(\rho^{ABC})$, $\epsilon > 0$ and sufficiently large $n$, there exists a random unitary operation $\mathcal{V}_n : \tau \rightarrow 2^{-nR}\sum_{k=1}^{2^nR} V_k V_k^\dagger$ on $A^n$ and a Markov state $\mathcal{Y}^{A^n B^n C^n}$ conditioned by $B^n$ that satisfy

$$\|\mathcal{V}_n(\rho^{\otimes n}) - \mathcal{Y}^{A^n B^n C^n}\|_1 \leq \frac{\epsilon}{2}.\nonumber$$

(36)

Let $\mathcal{R}_n : B^n \rightarrow A^n B^n$ be a quantum operation that satisfy

$$\mathcal{Y}^{A^n B^n C^n} = \mathcal{R}_n(\mathcal{Y}^{B^n C^n}).\nonumber$$

By tracing out $A^n$ in (36), we have

$$\|((\rho^{BC})^{\otimes n} - \mathcal{Y}^{B^n C^n})\|_1 \leq \frac{\epsilon}{2},\nonumber$$

and consequently,

$$\|\mathcal{R}_n((\rho^{BC})^{\otimes n}) - \mathcal{Y}^{A^n B^n C^n}\|_1 \leq \frac{\epsilon}{2}.\nonumber$$

Therefore, by the triangle inequality, we obtain

$$\|\mathcal{V}_n(\rho^{\otimes n}) - \mathcal{R}_n((\rho^{BC})^{\otimes n})\|_1 \leq \epsilon,\nonumber$$

\[ (37) \]

which implies $R \geq M^{R}_{A|ABC}(\rho^{ABC})$. Thus we have the second inequality in (7).

C. Proof of Inequality (7)

For an arbitrary $R > M^{R}_{A|AB}(\rho^{ABC})$, $\epsilon > 0$ and sufficiently large $n$, there exists a random unitary operation $\mathcal{V}_n : \tau \rightarrow 2^{-nR}\sum_{k=1}^{2^nR} V_k V_k^\dagger$ on $A^n$ and a linear CPTP map $\mathcal{R}_n : B^n \rightarrow B^n C^n$ that satisfy

$$\|\mathcal{V}_n((\rho^{ABC})^{\otimes n}) - (\mathcal{V}_n \otimes \mathcal{R}_n)((\rho^{AB})^{\otimes n})\|_1 \leq \epsilon.\nonumber$$

(38)

Define states $|\psi\rangle^{ABC}_D$, $\rho_n^{ABC}$ and $|\psi'_n\rangle^{EA^n B^n C^n D^n}$ in the same way as in Appendix C-B. For these states, in addition to (33), we have

$$S(A^n B^n C^n)\rho_n^\epsilon = S(A^n|B^n C^n)\rho_n^\epsilon + S(B^n C^n)\rho_n^\epsilon = S(A^n|B^n C^n)\rho_n^\epsilon + nS(BC)\rho_n^\epsilon,\nonumber$$

\[ (39) \]

where the third line follows from $\rho_n^{B^n C^n} = (\rho^{BC})^{\otimes n}$. Using (38), it holds that

$$S(A^n B^n C^n)\rho_n^\epsilon \geq S(A^n|B^n C^n)\rho_n^\epsilon - 2n\eta(\epsilon)\log(d_AD_Bd_C) \geq S(A^n|B^n)\rho_n^\epsilon - 2n\eta(\epsilon)\log(d_AD_Bd_C) \geq S(A^n B^n)\rho_n^\epsilon - S(B^n)\rho_n^\epsilon - 2n\eta(\epsilon)\log(d_AD_Bd_C) = n(S(AB)\rho_n^\epsilon - S(B)\rho_n^\epsilon) - 2n\eta(\epsilon)\log(d_AD_Bd_C).\nonumber$$

(40)

Here, the second line follows by the Fannes [20] inequality; the third line by the data processing inequality; and the fifth line by the von Neumann entropy being nondecreasing under random unitary operations, in addition to $\rho_n^{B^n C^n} = (\rho^{BC})^{\otimes n}$. From (33), (39) and (40), we obtain

$$R \geq I(A : C|B) - 2\eta(\epsilon)\log(d_AD_Bd_C),\nonumber$$

which concludes the proof for the first inequality in (7).

The second inequality is proved as follows. Consider Inequality (36), and let $\mathcal{R}_n : B^n \rightarrow B^n C^n$ be a linear CPTP map that satisfy

$$\mathcal{Y}^{A^n B^n C^n} = \mathcal{R}_n(\mathcal{Y}^{A^n B^n}).\nonumber$$

By tracing out $C^n$ in (36), we have

$$\|\mathcal{V}_n((\rho^{AB})^{\otimes n}) - \mathcal{Y}^{A^n B^n}\|_1 \leq \frac{\epsilon}{2},\nonumber$$

which implies

$$\|\mathcal{V}_n \otimes \mathcal{R}_n((\rho^{AB})^{\otimes n}) - \mathcal{Y}^{A^n B^n}\|_1 \leq \frac{\epsilon}{2}.\nonumber$$

Therefore, by the triangle inequality, we obtain

$$\|\mathcal{V}_n((\rho^{ABC})^{\otimes n}) - (\mathcal{V}_n \otimes \mathcal{R}_n)((\rho^{AB})^{\otimes n})\|_1 \leq \epsilon,\nonumber$$

\[ (41) \]

which implies $R \geq M^{R}_{A|ABC}(\rho^{ABC})$. Thus we have the second inequality in (7).
Appendix D
Proof of Theorem 11

In this Appendix, we provide a rigorous proof of Theorem 11. We first prove Lemma 12. We then prove Equalities (8) and (10) by using the obtained result. In the following, we denote systems $A^n$, $B^n$ and $C^n$ by $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ for simplicity of notation. We informally denote the composite systems $a_0d_Ld_R$ by $\tilde{A}$ and $b_0b_Lb_R$ by $\tilde{B}$, when there is no fear of confusion.

A. Proof of Lemma 12

Let $\Gamma$ be the KI isomorphism on $A$ with respect to $\Psi^{AC}$, and suppose the KI decomposition of $\Psi^{AC}$ on $A$ is given by

$$\Psi_{KI}^{AC} := \Gamma^A \Psi^{AC} \Gamma^A |A\rangle = \sum_{j \in J} P_j |j\rangle_{a_0}^a \otimes \omega_j^{a_c} \otimes \varphi_j^{a_C}.$$ 

As we prove in [13] (see Lemma 10 therein), there exists a unitary isomorphism $\Gamma' : \mathcal{H}^B \rightarrow \mathcal{H}^{b^0} \otimes \mathcal{H}^{b_2} \otimes \mathcal{H}^{b_0}$ such that $|\Psi\rangle^{ABC}$ is decomposed as

$$|\Psi\rangle := (\Gamma' \otimes \Gamma^{B'}) |\Psi\rangle^{ABC} = \sum_{j \in J} \sqrt{P_j} |j\rangle_{a_0}^a \otimes |\omega_j\rangle^{a_1} \otimes |\varphi_j\rangle^{a_2} \otimes \varphi_j^{a_C},$$

where $|\omega_j\rangle^{a_1} = \sum_l a^{a_1} \otimes \varphi_j^{a_2}$ and $|\varphi_j\rangle^{a_2} \otimes \varphi_j^{a_C}$ are purifications of $\omega_j^{a_1}$ and $\varphi_j^{a_C}$, respectively, and $|\omega_j\rangle^{a_1} = \delta_{j_0}$. Let $A_i$ denote the $i$-th copy of $A$ in $A^n$. For $\mathcal{E}$ that satisfies \[1\], define a quantum channel on $A_1 (1 \leq i \leq n)$ by

$$\mathcal{E}_1(\tau^{A_1}) := Tr_{\tilde{A} \setminus A_1} \left[ \mathcal{E} (\Psi^{A_1} \otimes \cdots \otimes \Psi^{A_{i-1}} \otimes \tau^{A_i} \otimes \Psi^{A_{i+1}} \otimes \cdots \otimes \Psi^{A_n}) \right],$$

where $Tr_{\tilde{A} \setminus A_1}$ denotes the partial trace over $A_1 \cdots A_{i-1} A_{i+1} \cdots A_n$. From \[1\], we have

$$\left\| \mathcal{E}_1(\Psi^{A_i}) - \Psi^{A_i} \right\|_1 \leq \epsilon$$

for any $1 \leq i \leq n$.

Define a quantum operation $\mathcal{F}$ on $S(\mathcal{H}^B)$ and a state $\tilde{\Psi}^{ABC}$ by

$$\mathcal{F}(\tau) = \Gamma'^T \left( \sum_{j \in J} |j\rangle_{a_0}^a \otimes Tr_{\tilde{B}_{[0,1]} \setminus \tau^{B_1}} |j\rangle_{a_0}^{b_{0_1}} \otimes \omega_j^{b_0} \right),$$

and $\tilde{\Psi}^{ABC} := \mathcal{F}(\tilde{\Psi}^{ABC})$. It immediately follows from \[42\] that

$$\tilde{\Psi}_{KI}^{ABC} := (\Gamma^A \otimes \Gamma'^B) \tilde{\Psi}^{ABC} (\Gamma^A \otimes \Gamma'^B)^T = \sum_{j \in J} P_j |j\rangle_{a_0}^a \otimes |\omega_j\rangle^{a_1} \otimes |\varphi_j\rangle^{a_2} \otimes \varphi_j^{a_C} \otimes \omega_j^{b_0}.$$ 

Define a function $\zeta_{\epsilon}(\epsilon)$ by

$$\zeta_{\epsilon}(\epsilon) := \sup \left\{ \left\| \mathcal{G} (\tilde{\Psi}^{ABC}) - \tilde{\Psi}^{ABC} \right\|_1 \left| \mathcal{G} (\Psi^{AC}) - \Psi^{AC} \right\|_1 \leq \epsilon \right\},$$

where the supremum is taken over quantum operations $\mathcal{G}$ on $A$. As we proved in [13] (see Appendix B-E therein), this function satisfies $\lim_{\epsilon \to 0} \zeta_{\epsilon}(\epsilon) = 0$.

From \[43\], we have

$$\left\| \mathcal{E}_i (\Psi_{KI}^{A_i B C}) - \tilde{\Psi}_{KI}^{A_i B C} \right\|_1 \leq \zeta_{\epsilon}(\epsilon)$$

for any $1 \leq i \leq n$. By the Fannes inequality, it follows that

$$I(\tilde{A} : BC)_{\tilde{\Psi}} - I(A_i : A_i B C_i)_{\tilde{\Psi}(\epsilon)} \leq 2\eta(\zeta_{\epsilon}(\epsilon)) \log (d_A d_B d_C),$$

and consequently, that

$$n I(A : BC)_{\tilde{\Psi}} - \sum_{i=1}^n I(A_i : A_i B C_i)_{\tilde{\Psi}(\epsilon)} \leq 2n \eta(\zeta_{\epsilon}(\epsilon)) \log (d_A d_B d_C). \tag{45}$$

We also have

$$I(\tilde{A} : \tilde{B} \tilde{C})_{\tilde{\Psi}(\epsilon)} = S(B \tilde{C})_{\tilde{\Psi}(\epsilon)} - S(B \tilde{C} | \tilde{A})_{\tilde{\Psi}(\epsilon)} = S(B \tilde{C})_{\tilde{\Psi}(\epsilon)} - n \sum_{i=1}^n S(B_i | A_i)_{\tilde{\Psi}(\epsilon)}$$

$$\geq n \sum_{i=1}^n S(B_i | A_i) - S(B_i | A_i)_{\tilde{\Psi}(\epsilon)} = \sum_{i=1}^n I(A_i : B_i)_{\tilde{\Psi}(\epsilon)} = \sum_{i=1}^n I(A : B_i)_{\tilde{\Psi}(\epsilon)} \tag{46}.$$

Here, we used the fact that $\mathcal{E}$ on $\tilde{A}$ does not change the reduced state on $BC$, and that

$$Tr_{\tilde{A} \setminus A_1 \setminus B \setminus C} \left[ \mathcal{E} (\tilde{\Psi})^{\epsilon} \right] = \mathcal{E}_i (\tilde{\Psi}^{A_i B C})_{\tilde{\Psi}(\epsilon)},$$

because of $\tilde{\Psi}^{A_i} = \Psi^{A_i}$. Combining \[45\] and \[46\], we obtain

$$n I(A : BC)_{\tilde{\Psi}} \leq I(\tilde{A} : \tilde{B} \tilde{C})_{\tilde{\Psi}(\epsilon)} \log (d_A d_B d_C).$$

The L.H.S. in this inequality is computed from \[44\] as

$$I(A : BC)_{\tilde{\Psi}} = H(\{p_j\}_{j \in J}) + 2 \sum_{j \in J} p_j S(\varphi_j^{a_2}) \geq M_{A \mid B}(\Psi^{ABC}).$$

The data processing inequality yields

$$I(\tilde{A} : \tilde{B} \tilde{C})_{\tilde{\Psi}(\epsilon)} \leq I(\tilde{A} : \tilde{B} \tilde{C})_{\tilde{\Psi}(\epsilon)} \leq I(\tilde{A} : \tilde{B} \tilde{C})_{\tilde{\Psi}(\epsilon)} \log (d_A d_B d_C).$$

The various expressions have been obtained from \[44\]. Thus we obtain

$$\frac{1}{n} I(\tilde{A} : \tilde{B} \tilde{C})_{\tilde{\Psi}(\epsilon)} \geq M_{A \mid B}(\Psi^{ABC}) - 2\eta(\zeta_{\epsilon}(\epsilon)) \log (d_A d_B d_C).$$

This proves Lemma 12 since $M_{A \mid B}(\Psi) = M^D_{A \mid B}(\Psi)$ due to Theorem 10.
B. Proof of Equality [5]

We prove $M_{R_{A|BC}}^R(\Psi^{ABC}) \geq M_{A|B}^D(\Psi^{ABC})$, which, together with Inequality (4), implies Equality (3). The proof presented here also provides an alternative proof for the converse part of Theorem 8 in [13].

For an arbitrary $R > M_{R_{A|BC}}^R(\Psi^{ABC})$, $\epsilon > 0$ and sufficiently large $n$, there exist a random unitary operation $\mathcal{V}_n : \tau \mapsto 2^{-nR} \sum_{k=1}^{2^nR} V_k \tau V_k^\dagger$ on $A^n$ and a quantum operation $\mathcal{R}_n : B^n \rightarrow A^n B^n$ that satisfy

$$\left\| \mathcal{V}_n((\Psi^{ABC})^\otimes n) - \mathcal{R}_n((\Psi^{ABC})^\otimes n) \right\|_1 \leq \epsilon. \quad (47)$$

Define an isometry $U_1 : A \rightarrow AG$ by

$$U_1 := \frac{1}{\sqrt{2^nR}} \sum_{k=1}^{2^nR} |k\rangle \otimes V_k^A,$$

where $\{|k\rangle\}_{k=1}^{2^nR}$ is an orthonormal basis of $H^G$. A purification of $\mathcal{V}_n((\Psi^{ABC})^\otimes n)$ is then given by

$$|\Psi_{\mathcal{V}_n}\rangle^{\bar{A}BCG} := U_1 |\Psi^{\otimes n}\rangle^{\bar{A}BC} = \frac{1}{\sqrt{2^nR}} \sum_{k=1}^{2^nR} |k\rangle \otimes V_k |\Psi^{\otimes n}\rangle^{\bar{A}BC}. \quad (48)$$

Let $E$ be an ancillary system with a sufficiently large dimension, $W : B \rightarrow ABE$ be an isometry such that a Stinespring dilation of $\mathcal{R}_n$ is given by $\mathcal{R}_n(\tau) = \text{Tr}_E[W \tau W^\dagger]$, and let $A_c$ be a system which is identical to $A$. Then a purification of $\mathcal{R}_n((\Psi^{ABC})^\otimes n)$ is given by

$$|\Psi_{\mathcal{R}_n}\rangle^{\bar{A}BCA_cE} := W |\Psi^{\otimes n}\rangle_{\bar{A}cBC}. \quad (49)$$

From (47), (48), (49) and Uhlmann’s theorem [23], there exists an isometry $U_2 : G \rightarrow A_c$ such that

$$\left\| U_2 U_1 |\Psi^{\otimes n}\rangle^{\bar{A}BC} U_1^\dagger U_2^\dagger - W |\Psi^{\otimes n}\rangle^{\bar{A}BC} W^\dagger \right\|_1 \leq \epsilon. \quad (50)$$

Consider a direct-sum decomposition of $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^E$ as

$$\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^E = \mathcal{H}_{R_A} \oplus \mathcal{H}_\bot,$$

where $\mathcal{H}_{R_A}$ is the support of $\Psi^{\bar{A}BC}$ and $\mathcal{H}_\bot$ is its orthogonal complement. Letting $I_\bot$ be the identity operator on $\mathcal{H}_\bot$, define a unitary isomorphism $\tilde{W} : \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^E \rightarrow \mathcal{H}^B \otimes \mathcal{H}_\bot$ by $\tilde{W} := W^\dagger \otimes I_\bot$. Equality (50) then implies

$$\left\| \tilde{W} U_1 |\Psi^{\otimes n}\rangle^{\bar{A}BC} U_1^\dagger U_2^\dagger W - |\Psi^{\otimes n}\rangle^{\bar{A}BC} \right\|_1 \leq \epsilon. \quad (51)$$

as depicted in Figure 2. Define linear CPTP maps $\mathcal{E}_1 : \tilde{A} \rightarrow G$ and $\mathcal{E}_2 : G \rightarrow A_c$ by

$$\mathcal{E}_1(\cdot) = \text{Tr}_{\bar{A}}[U_1(\cdot) U_1^\dagger], \quad \mathcal{E}_2(\cdot) = \text{Tr}_{\bar{A}}[U_2(\cdot) U_2^\dagger]. \quad (52)$$

By taking the partial trace in (51) so that the remaining system is $\bar{A}_c \bar{C}$ (see Figure 2), we obtain

$$\left\| (\mathcal{E}_2 \circ \mathcal{E}_1)(|\Psi^{\otimes n}\rangle^{\bar{A}C}) - (|\Psi^{\otimes n}\rangle^{\bar{A}C}) \right\|_1 \leq \epsilon.$$

Therefore, we see from Lemma 12 that

$$\frac{1}{n} I(A_c : B\bar{C})(\mathcal{E}_2 \circ \mathcal{E}_1)(|\Psi^{\otimes n}\rangle) \geq M_{A|B}^D(\Psi^{ABC}) - 2\eta(\zeta_R(\epsilon)) \log (d_{A}d_{B}d_C),$$

and also

$$\frac{1}{n} I(G : B\bar{C})(\mathcal{E}_2)(|\Psi^{\otimes n}\rangle) \geq M_{A|B}^D(\Psi^{ABC}) - 2\eta(\zeta_R(\epsilon)) \log (d_{A}d_{B}d_C) \quad (53)$$

by the monotonicity of the quantum mutual information.

Let $|\psi\rangle^{A_0G}$ be a $2^nR$-dimensional maximally entangled state shared between Alice and George. Suppose Alice also has the $A^n$ part of $|\Psi^{\otimes n}\rangle$. Consider the following one-way LOCC protocol.

1) Alice performs a measurement on $\bar{A}A_0$ described by measurement operators

$$M_{\bar{A}A_0}^{\bar{A}0} = \frac{1}{\sqrt{2^nR}} \sum_{j=1}^{2^nR} \exp \left( i \frac{2\pi jk}{2^nR} \right) |j\rangle^{A_0} \otimes V_j^\bar{A} \quad (k \in [1, 2^nR])$$

where $\{|j\rangle\}_{j=1}^{2^nR}$ is an orthonormal basis of $\mathcal{H}^{A_0}$.

2) Alice sends the measurement result to George, which costs $nR$ bits of classical communication.
3) George performs a generalized phase gate on $G$ defined by

$$Z_k^G = \frac{1}{2^nR} \sum_{j=1}^{2^nR} \exp \left( -i \frac{2\pi j k}{2^n R} \right) |j\rangle^G \quad (k \in [1, 2^n R]).$$

It is straightforward to verify that the state obtained after this protocol is equal to $|\Psi_{\nu_n}^{ABC}\rangle$, for which we have

$$I(G : \hat{B}C)_{\nu_n} = I(G : \hat{B}C)_{\nu_n}^{(\hat{G} \otimes n)}$$

$$\geq nM^D_{AB}(\Psi^{ABC}) - 2\eta(\zeta_k(\epsilon)) \log(d_{ADB} d_{Dc}),$$

from (53). We also have $nR \geq I(G : \hat{B}C)_{\nu_n}$, because (i) there is initially no correlation between $BC$ and $G$, and (ii) $nR$ bits of classical correlation can increase the mutual information by at most $nR$ bits. Thus we finally obtain

$$R \geq M^D_{A|B}(\Psi^{ABC}) - 2\eta(\zeta_k(\epsilon)) \log(d_{ADB} d_{Dc}),$$

which completes the proof by taking the limit of $\epsilon \to 0$ and noting Inequality (6).

C. Proof of Equality (10)

First we prove $M^R_{A|B}(\Psi^{ABC}) \leq M^D_{A|B}(\Psi^{ABC})$. By definition, for any $R > M^D_{A|B}(\Psi^{ABC})$ and $\epsilon > 0$, there exists $n_{\epsilon/2}$ such that for any $n \geq n_{\epsilon/2}$, we find a random unitary operation $V_n : \tau \mapsto 2^{-nR} \sum_{k=1}^{2^nR} V_k \tau V_k^\dagger$ on $\hat{A}$, so that $V_n(\Psi^{\otimes n}) = (\epsilon/2)$-decomposable on $B^n$. From Lemma 6 it follows that $V_n(\Psi^{\otimes n})$ is $\epsilon$-recoverable from $AB$. In addition, it is proved in [13] that $n_{\epsilon/2}$ can be chosen so that

$$\lim_{\epsilon \to 0} \epsilon \cdot n_{\epsilon/2} = 2 \lim_{\epsilon \to 0} \epsilon' \cdot n_{\epsilon'} = 0.$$

Since this relation holds for any $R > M^D_{A|B}(\Psi^{ABC})$, we obtain $M^R_{A|B}(\Psi^{ABC}) \leq M^D_{A|B}(\Psi^{ABC})$.

Second we prove $M^R_{A|B}(\Psi^{ABC}) \geq M^D_{A|B}(\Psi^{ABC})$. Fix arbitrary $R > M^D_{A|B}(\Psi^{ABC})$ and $\delta \in (0, 1]$. There exist $\epsilon > 0$ and $n_\epsilon$ such that $\epsilon \cdot n_\epsilon \leq \delta$ and that for any $n \geq n_\epsilon$, we find a random unitary operation $V_n : \tau \mapsto 2^{-nR} \sum_{k=1}^{2^nR} V_k \tau V_k^\dagger$ on $A^n$ such that $V_n(\Psi^{ABC \otimes n})$ is $\epsilon$-recoverable from $AB$. From Lemma 6 it follows that $V_n(\Psi^{ABC \otimes n})$ is $\epsilon(\delta, d_{c})$-recoverable from $BC$. For $n = n_\epsilon$, we have

$$f(\epsilon, d_{c}) = \frac{\sqrt{4\epsilon \log d_{c} + 2\epsilon n_\epsilon \log d_{c} + 2\epsilon n_\epsilon}}{\sqrt{\epsilon \log d_{c} + 4\epsilon n_\epsilon \log d_{c} + 2\epsilon n_\epsilon}}$$

which implies that $V_n(\Psi^{ABC \otimes n})$ is $g(\delta, d_{c})$-recoverable from $BC$. Since this relation holds for any $R > M^R_{A|B}(\Psi^{ABC})$ and $\delta > 0$, and since we have $\lim_{\epsilon \to 0} \epsilon(\delta, d_{c}) = 0$, we obtain $M^R_{A|B}(\Psi^{ABC}) \geq M^D_{A|B}(\Psi^{ABC})$. 

Appendix E

Proof of Theorem 15

First we prove $M^R_{A|B}(\Psi^{ABC}) \leq M^D_{A|B}(\Psi^{ABC})$. For any $R > M^R_{A|B}(\Psi^{ABC})$, there exists $\epsilon > 0$ and $n_\epsilon$ such that for any $n \geq n_\epsilon$, we find a random unitary operation $V_n : \tau \mapsto 2^{-nR} \sum_{k=1}^{2^nR} V_k \tau V_k^\dagger$ on $\hat{A}$, so that $V_n(\Psi^{\otimes n})$ is $\epsilon$-recoverable from $A^nB^n$. Let $|\varphi_n^{ABG}\rangle$ be a maximally entangled state with Schmidt rank $2^n R$. Using $V_k$ in $V_n$, construct Alice’s measurement $M_\epsilon = \{M_{k\epsilon}^{A_\epsilon \rightarrow A}\}_{k=1}^{2^n R}$ as

$$M_{k\epsilon}^{A_\epsilon \rightarrow A} = \frac{1}{\sqrt{2^n R}} \sum_{j=1}^{2^n R} \exp \left( i \frac{2\pi j k}{2^n R} \right) |j\rangle_A \otimes V_j^\dagger \quad (k \in [1, 2^n R]).$$

It is straightforward to verify that, for any $k$, the post-measurement state $\Psi_k$ satisfies $\Psi_k^{BC} = V_n(\Psi^{\otimes n})$, which implies that Condition 2) is satisfied. Condition 1) is met as well, since random unitary operations on $\hat{A}$ does not change the reduced state on $BC$ at all. Condition 3) is also satisfied, because we have

$$I(G : \hat{B}C)_{\Psi_k} = S(\Psi_k) + S(BC)_{\Psi_k} - S(\hat{B}\hat{C}G)_{\Psi_k}$$

$$= S(\Psi_k) + nS(BC)_{\Psi_k} - S(\hat{A})_{\Psi_k}$$

$$\leq S(\Psi_k) + nS(BC)_{\Psi_k} - nS(A)_{\Psi_k}$$

where the third line follows by the monotonicity of the von Neumann entropy under random unitary operations. In addition, $n_\epsilon$ can be chosen so that $\lim_{\epsilon \to 0} \epsilon \cdot n_\epsilon = 0$. Since this relation holds for any $R > M^R_{A|B}(\Psi^{ABC})$, we obtain $M^R_{A|B}(\Psi^{ABC}) \leq M^D_{A|B}(\Psi^{ABC})$.

Second we prove $M^R_{A|B}(\Psi^{ABC}) \geq M^D_{A|B}(\Psi^{ABC})$. For any $R > M^R_{A|B}(\Psi^{ABC})$ and $\delta \in (0, 1]$, there exists $\epsilon > 0$ and $n_\epsilon$ such that $\epsilon \cdot n_\epsilon \leq \delta$, and that for any $n \geq n_\epsilon$, we find a measurement $M_{k\epsilon}^{A_\epsilon \rightarrow A'}_{k\epsilon} \in K$, a pure state $|\varphi_{n_\epsilon}^{ABG}\rangle$ and quantum operations $\mathcal{R}_{n_\epsilon, k} : B^n \rightarrow B^n C^n$ satisfying the conditions in Definition 14. Define

$$\delta_k := \left\| (\Psi^{\otimes n})^{BC}_\delta - \Psi^{\otimes n} \right\|_1,$$

$$\epsilon'_k := \left\| (\Psi^{\otimes n})^{AB}_\delta - \mathcal{R}_{n_\epsilon, k}(\Psi^{\otimes n})^{AB}_\delta \right\|_1.$$

Fix one $k$ for the moment, and assume $\epsilon_k \leq 1/4$, $\epsilon'_k \leq 1$. Lemma 5 and (52) imply there exist quantum operations $\mathcal{R}'_{n_\epsilon, k} : B \rightarrow A'B$ such that

$$\left\| \Psi^{\otimes n}^{\widehat{B}C} - \mathcal{R}_{n_\epsilon, k}(\Psi^{\otimes n}^{\widehat{B}C}) \right\|_1 \leq f(\epsilon_k, d_{c})$$

From (54) and the monotonicity of the trace distance, we have

$$\left\| \mathcal{R}'_{n_\epsilon, k}(\Psi^{\otimes n}^{\widehat{B}C}) \right\|_1 \leq \epsilon_k.$$

By the triangle inequality, we obtain

$$\left\| \Psi^{\otimes n}^{\widehat{B}C} - \mathcal{R}'_{n_\epsilon, k}(\Psi^{\otimes n}^{\widehat{B}C}) \right\|_1 \leq \epsilon_k.$$

(56)
Let $W : B \rightarrow A'B^cE$ be an isometry such that a Stinespring dilation of $\mathcal{R}'_{\eta,k}$ is given by $\mathcal{R}'_{\eta,k}(\tau) = \Tr_{\bar{k}}[\tau W^\dagger W]$. Then a purification of $\mathcal{R}'_{\eta,k}(\Psi^{\otimes n})$ is given by
\begin{equation}
|\Psi_W\rangle^{A'B^c\bar{k}} := W|\Psi^{\otimes n}\rangle\bar{A'}B'.
\end{equation}
Due to (50) and Uhlmann's theorem, there exists an isometry $U_2 : G \rightarrow \bar{A}E$ such that
\begin{equation}
\left\|U_2|\Psi_k\rangle|\Psi_k\rangle U_2^\dagger - |\Psi_W\rangle|\Psi_W\rangle\right\|_1 \leq 2\sqrt{\epsilon_k + f(\epsilon_k', d_C^D)}. (58)
\end{equation}
On the other hand, (54) implies there exists another isometry $U_1 : \bar{A} \rightarrow AG$ such that
\begin{equation}
\left\|U_1|\Psi_k\rangle^{\otimes n}|\Psi_k\rangle U_1^\dagger - |\Psi_k\rangle^{\otimes n}|\Psi_k\rangle^{\bar{A}'\bar{B}'\bar{C}'G}\right\|_1 \leq 2\sqrt{\epsilon_k}. (59)
\end{equation}
From (57), (58) and (59), we obtain
\begin{equation}
\left\|U_2U_1|\Psi_k\rangle^{\otimes n}|\Psi_k\rangle^{\bar{A}'\bar{B}'\bar{C}'G} U_1^\dagger U_2^\dagger - W|\Psi_k\rangle^{\otimes n}|\Psi_k\rangle^{\bar{A}'\bar{B}'\bar{C}'G} W^\dagger\right\|_1 \leq 2\sqrt{\epsilon_k + 2\sqrt{\epsilon_k + f(\epsilon_k', d_C^D)}}. (60)
\end{equation}
Define linear CPTP maps $\mathcal{E}_1 : \bar{A} \rightarrow G$ and $\mathcal{E}_2 : G \rightarrow \bar{A}$ by
$\mathcal{E}_1(\cdot) = \Tr_{A'}[U_1(\cdot)U_1^\dagger]$, $\mathcal{E}_2(\cdot) = \Tr_{\bar{E}}[U_2(\cdot)U_2^\dagger]$. By tracing out $A'$ in (59), we have
\begin{equation}
\left\|\mathcal{E}_1(|\Psi_k\rangle|\Psi_k\rangle^{\otimes n}) - |\Psi_k\rangle^{\bar{A}'\bar{B}'\bar{C}'G}\right\|_1 \leq 2\sqrt{\epsilon_k}.
\end{equation}
Thus we obtain
\begin{align*}
I(G : \bar{B}\bar{C})_{av} &\geq -2(\eta \circ \zeta)(2\sqrt{\epsilon_k} + 2\sqrt{\epsilon_k'+ f(\epsilon_k', d_C^D)}) \log (d_Ad_Bd_C), (61)
\end{align*}
from (60) due to Lemma 12. Thus we obtain
\begin{align*}
I(G : \bar{B}\bar{C})_{av} &\geq -n\xi_k \log (d_Ad_Bd_C),
\end{align*}
where we defined
\begin{align*}
\xi_k := 5\eta(2\sqrt{\epsilon_k} + 2(\eta \circ \zeta)(2\sqrt{\epsilon_k} + 2\sqrt{f(\epsilon_k', d_C^D)}))
\end{align*}
for $k$ such that $\epsilon_k \leq 1/4$ and $\epsilon_k' \leq 1$. Without loss of generality, we assume that $\supp \Psi^A = \mathcal{H}^A$. Since $\Psi^{ABC}$ is a pure state, we see that
\begin{equation}
M^{D}_{A|B}(\Psi^{ABC}) \leq 2\log d_A \leq \log (d_Ad_Bd_C). (62)
\end{equation}
Consider an arbitrary $k \in \mathcal{K}$ and define
\begin{align*}
\xi_k' := \begin{cases} 
\min\{\xi_k, 1\} & \text{if } \epsilon_k \leq 1/4 \text{ and } \epsilon_k' \leq 1, \\
1 & \text{otherwise},
\end{cases}
\end{align*}
Combining (61) and (62),
\begin{align*}
I(G : \bar{B}\bar{C})_{av} &\geq -n\xi_k \log (d_Ad_Bd_C)
\end{align*}
for any $k \in \mathcal{K}$. Thus we obtain
\begin{align*}
I(G : \bar{B}\bar{C})_{av} &\geq nM^{D}_{A|B}(\Psi^{ABC}) - n\xi_{av} \log (d_Ad_Bd_C), (64)
\end{align*}
where $\xi_{av} := \sum k p_k \xi_k$. Let us now evaluate $\xi_{av}$. For any $\lambda > 0$, define two sets $\mathcal{K}_{\inf}(\lambda) \subseteq \mathcal{K}$ and $\mathcal{K}_{\rec}(\lambda) \subseteq \mathcal{K}$ by
\begin{align*}
\mathcal{K}_{\inf}(\lambda) &:= \{ k \in \mathcal{K} \mid \|\Psi_k^{\otimes n} - \Psi_k^{\bar{B}'\bar{C}'G}\|_1 \leq \lambda \}, \\
\mathcal{K}_{\rec}(\lambda) &:= \{ k \in \mathcal{K} \mid \|\Psi_k^{\bar{A}'\bar{B}'\bar{C}'G} - \mathcal{R}_{n,k}(\Psi^{ABC}_k)\|_1 \leq \lambda \}.
\end{align*}
From Conditions (13) and (14), for any $t \geq 1$ we have
\begin{align*}
\sum_{k \notin \mathcal{K}_{\inf}(te)} p_k &\leq \frac{1}{t} \sum_{k \notin \mathcal{K}_{\inf}(te)} p_k, \\
&\leq \frac{1}{t} \sum_{k \notin \mathcal{K}} p_k \leq \frac{2}{t},
\end{align*}
and similarly, have
\begin{align*}
\sum_{k \notin \mathcal{K}_{\rec}(te)} p_k &\leq \frac{1}{t} \sum_{k \notin \mathcal{K}_{\rec}(te)} p_k,
\end{align*}
which leads to
\begin{align*}
\sum_{k \notin \mathcal{K}_{\inf}(te) \cup \mathcal{K}_{\rec}(te)} p_k &\leq \frac{1}{t} \sum_{k \notin \mathcal{K}_{\inf}(te)} p_k + \frac{1}{t} \sum_{k \notin \mathcal{K}_{\rec}(te)} p_k \leq \frac{2}{t}.
\end{align*}
Due to (63), this shows that
\begin{align*}
\xi_{av} &\leq \xi(\delta) \leq 5\eta(2\sqrt{\delta}) + 2(\eta \circ \zeta)(2\sqrt{\delta} + \sqrt{f(\delta, d_C)}) + 2\sqrt{\delta},
\end{align*}
when $te \leq 1/4$. Let $t = 1/\sqrt{\delta}$ and $n = n_t$. Then
\begin{align*}
\sqrt{\delta} &\leq 4\delta, \quad \sqrt{f(\delta, d_C)} \leq \sqrt{\delta},
\end{align*}
and therefore
\begin{align*}
\xi_{av} &\leq \xi(\delta) \leq 5\eta(2\sqrt{\delta}) + 2(\eta \circ \zeta)(2\sqrt{\delta} + \sqrt{f(\delta, d_C)}) + 2\sqrt{\delta},
\end{align*}
for any $\delta \in (0, 1/4]$. Finally, from (64), we obtain
\begin{align*}
I(G : \bar{B}\bar{C})_{av} &\geq nM^{D}_{A|B}(\Psi^{ABC}) - n\xi(\delta) \log (d_Ad_Bd_C). (66)
\end{align*}
Since $\delta > 0$ can be arbitrarily small and $\lim_{\delta \rightarrow 0} \xi(\delta) = 0$, we obtain $M^{R,\inf}_{A|B}(\Psi^{ABC}) \geq M^{D}_{A|B}(\Psi^{ABC})$.}

Remark: If Conjecture 13 is true, $f(\delta, d_C^D)$ in (65) is replaced by $g(\delta)$. Then $\lim_{\delta \rightarrow 0} \xi_{av} = 0$ holds for $t = 1/\sqrt{\delta}$, regardless of $\delta$. Thus we have $M^{R,\inf}_{A|B}(\Psi^{ABC}) \geq M^{D}_{A|B}(\Psi^{ABC})$ irrespective of Condition (16).