CODES ON SUBGROUPS OF WEIGHTED PROJECTIVE TORI

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Abstract. We obtain certain algebraic invariants relevant to study codes on subgroups of weighted projective tori inside an $n$-dimensional weighted projective space. As application, we compute all the main parameters of generalized toric codes on these subgroups of tori lying inside a weighted projective plane of the form $\mathbb{P}(1, 1, a)$.

1. Introduction

Let $\mathbb{P}(w_0, \ldots, w_n)$ be the weighted projective space over an algebraic closure $\mathbb{F}_q$ of a finite field $\mathbb{F}_q$, defined by some positive integers $w_0, \ldots, w_n$. Without loosing generality, we assume that $n$ of these numbers have no common divisor. It is well known that the $\mathbb{F}_q$-rational points of the weighted projective space $\mathbb{P}(w_0, \ldots, w_n)$ can be represented by the Geometric Invariant Theory quotient $(\mathbb{F}_q^{n+1} \setminus \{0\})/G$, where the group $G = \{(\lambda^{w_0}, \ldots, \lambda^{w_n}) : \lambda \in \mathbb{F}_q^\times\}$. Therefore, a point is an orbit of the form $[p_0 : \ldots : p_n] = \{(\lambda^{w_0}p_0, \ldots, \lambda^{w_n}p_n) : \lambda \in \mathbb{F}_q^\times\}$ known as its homogeneous coordinates as in the classical projective case. Every $\mathbb{F}_q$-rational point has a representative from the set $\mathbb{F}_q^{n+1}$ in this correspondence.

For a thorough introduction to and a fairly good account on general properties of these spaces, see [1]. It is known that $X = \mathbb{P}(w_0, \ldots, w_n)$ is smooth if and only if it is the usual projective space $\mathbb{P}^n$, i.e., $w_0 = \cdots = w_n = 1$.

The ring $S = \mathbb{F}_q[x_0, \ldots, x_n]$ over the field $\mathbb{F}_q$ of a weighted projective space $\mathbb{P}(w_0, \ldots, w_n)$ is graded naturally by the numerical semigroup $NW$ generated by $\deg(x_i) = w_i$, for $i = 0, \ldots, n$, where $\mathbb{N}$ denotes the set of natural numbers with 0. Thus, we have the following decomposition:

$$S = \bigoplus_{\alpha \in NW} S_\alpha,$$

where $S_\alpha$ is the vector space spanned by the monomials of degree $\alpha$.

For any $\alpha \in NW$ and any subset $Y = \{P_1, \ldots, P_N\}$ of $\mathbb{F}_q$-rational points, we have the following evaluation map:

$$\text{ev}_Y : S_\alpha \to \mathbb{F}_q^N, \quad F \mapsto (F(P_1), \ldots, F(P_N)).$$

The image $C_{\alpha, Y} = \text{ev}_Y(S_\alpha)$ is a linear code. The three basic parameters of $C_{\alpha, Y}$ are block-length which is $N$, the dimension which is $K = \dim_{\mathbb{F}_q}(C_{\alpha, Y})$, and the minimum distance $\delta = \delta(C_{\alpha, Y})$ which is the minimum of the number of nonzero components of vectors in $C_{\alpha, Y} \setminus \{0\}$. When $Y$ is the full set of $\mathbb{F}_q$-rational points of $\mathbb{P}(w_0, \ldots, w_n)$, the code is known as the weighted Reed-Muller code. These codes are special cases of what is called generalized toric codes, see Section 2 for details.

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Toric codes are introduced by Hansen in [6] for the $F_q$-rational points of the dense torus $T_X$ of a toric variety $X = X_\Sigma$ and examined further in e.g. [7, 12, 2, 19, 8, 3] producing some codes having the best known parameters. The vanishing ideal $I(Y)$ of $Y$ which is generated by homogeneous polynomials vanishing on $Y$ is a key in studying the parameters. This is because, the kernel of $ev_Y$ is nothing but the subspace $I_\alpha(Y) := I(Y) \cap S_\alpha$, and hence the code $C_{\alpha,Y}$ is isomorphic to the vector space $S_\alpha/I_\alpha(Y)$. Therefore, the dimension $K = \dim_{k_q}(C_{\alpha,Y})$ is the value $H_Y(\alpha) = \dim_{k_q} S_\alpha - \dim_{\mathbb{F}_q} I_\alpha(Y)$ of the multigraded Hilbert function $H_Y$ of $Y$, see [16].

In literature, there are a few papers computing the main parameters of codes on weighted projective spaces. The main parameters of weighted Reed-Muller codes are given explicitly in [1] for weighted projective planes $\mathbb{P}(1, w_1, w_2)$ where $\alpha$ is a multiple of the lcm$(w_1, w_2)$. The main parameters have the most beautiful formulas in the special case of the plane $\mathbb{P}(1, a)$. If $Y = T_X(\mathbb{F}_q)$ is the set of $\mathbb{F}_q$-rational points of the projective torus in $\mathbb{P}^{n-1}$ and $\alpha \geq 1$, then the main parameters are given in [18]. On the other hand, [17] studied the degenerate tori

$$Y_Q = \{ [t_1^{a_1}, \ldots, t_n^{a_n}] \mid t_i \in \mathbb{F}_q, \text{ for all } i \in [n] := \{1, \ldots, n\} \}$$

lying in the classical projective space $X = \mathbb{P}^n$, generalizing [18]. This is because, $Y_Q$ becomes the set of $\mathbb{F}_q$-rational points of the projective torus in $\mathbb{P}^n$, once $a_i = 1$, for all $i \in [n]$. The results in that paper shows that $I(Y_Q)$ is a complete intersection of the binomials $x_j^{a_i} - x_i^{a_j}$, for $i \in [n]$, its degree is $|Y_Q| = s_1 \cdots s_n$, and $\alpha$-invariant is $a_Y = s_1 + \cdots + s_n - n - 1$, where $s_i = (q-1)/\gcd(q-1, a_i)$ for all $i \in [n]$. Some nice formulas are given for the other parameters as well.

The present paper considers the analogue of the same parametrization $Y_Q$ but in the weighted projective space $X = \mathbb{P}(1, w_1, \ldots, w_n)$ with $a_i = w_i$ for all $i$. When $w_i = 1$, for all $i$, our $Y_Q$ becomes the $\mathbb{F}_q$-rational points of the projective torus studied in [18], as well. In the next section, we review basic terminology and theory needed in the sequel. We prove that $I(Y_Q)$ is a complete intersection ideal in Proposition 3.3. We give a formula for the Hilbert function $H_{Y_Q}$ and compute the $\alpha$-invariant of $Y_Q$ in Proposition 3.4. Theorem 4.1 gives formulas for the length and dimension of the code $C_{\alpha,Y_Q}$. Final section displays more explicit formulas for the dimension and minimum distance of the codes coming from the weighted projective plane $\mathbb{P}(1, 1, a)$, see Theorem 5.1.

2. Preliminaries

Let $\Sigma \subseteq \mathbb{R}^n$ be a complete simplicial fan with rays generated by the lattice vectors $v_1, \ldots, v_r$. Each cone $\sigma \in \Sigma$, defines an affine toric variety $U_\sigma = \text{Spec}(\mathbb{K}[\sigma \cap \mathbb{Z}^n])$. Gluing these affine pieces, we obtain the toric variety $X_\Sigma$ as an abstract variety over an algebraically closed field $\mathbb{K}$. There is nice correspondence between polytopes in real $n$-space and projective toric varieties. Namely, every polytope $P$ gives rise to a so called normal fan $\Sigma_{P}$ whose rays are spanned by the inner normal vectors of $P$. Assuming $X_\Sigma$ has a free class group, the ray generator yields the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}^n \overset{\phi}{\longrightarrow} \mathbb{Z}^r \overset{\beta}{\longrightarrow} \mathbb{Z}^d \longrightarrow 0 ,$$
where $\phi$ is the matrix $[v_1 \cdots v_r]^T$ and $d = r - n$ is the rank of the class group $\text{Cl}(X) \cong \mathbb{Z}^d$. There is an important lattice $L_\beta$ in $\mathbb{Z}^r$ that is isomorphic to $\mathbb{Z}^n$ via $\phi$, and is spanned by the columns $u_1, \ldots, u_n$ of $\phi$.

Applying $\text{Hom}(-, \mathbb{K}^*)$ functor to $\Psi$ gives the following dual short exact sequence:

$$
\begin{array}{c}
\Psi^* : 1 \longrightarrow G \longrightarrow (\mathbb{K}^*)^r \longrightarrow (\mathbb{K}^*)^n \longrightarrow 1,
\end{array}
$$

where $\pi(P) = (x^{u_1}(P), \ldots, x^{u_n}(P))$ and $x^a(P) = p_1^{a_1} \cdots p_r^{a_r}$ for $P = (p_1, \ldots, p_r) \in (\mathbb{K}^*)^r$ and $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$. Since the lattice $L_\beta$ is saturated, it is well known that the lattice ideal

$$
I_{L_\beta} = \langle x^{m^+} - x^{m^-} | m = m^+ - m^- \in L_\beta \rangle
$$

is prime whose zero locus in $(\mathbb{K}^*)^r$ equals the algebraic group $G = \ker(\pi)$, i.e.

$$
V(I_{L_\beta}) \cap (\mathbb{K}^*)^r = \{ P \in (\mathbb{K}^*)^r | (x^{m^+} - x^{m^-})(P) = 0 \text{ for all } m \in L_\beta \}
= \{ P \in (\mathbb{K}^*)^r | x^m(P) = 1 \text{ for all } m \in L_\beta \} = G.
$$

As proved by Cox in [4], the set $X(\mathbb{K})$ of $\mathbb{K}$-rational points of the toric variety $X := X_\Sigma$ is identified with the geometric quotient $[\mathbb{K}^r \setminus V(B)]/G$, where $B$ is the monomial ideal in $\mathbb{K}[x_1, \ldots, x_r]$ generated by the monomials $x^\sigma = \Pi_{i \in \sigma} x_i$ corresponding to cones $\sigma \in \Sigma$. Hence, points of $X(\mathbb{K})$ are orbits $[P] := G \cdot P$, for $P \in \mathbb{K}^r \setminus V(B)$. When $\mathbb{K} = \mathbb{F}_q$ is an algebraic closure of a finite field $\mathbb{F}_q$, the $\mathbb{F}_q$-rational points $[P]$ are represented by points $P$ from the set $\mathbb{F}_q^r \setminus V(B)$.

There is a nice correspondence between subgroups of the torus $T_X(\mathbb{F}_q) \cong (\mathbb{F}_q^*)^r/G$ and $\beta$-graded lattice ideals in the Cox ring $S = \mathbb{F}_q[x_1, \ldots, x_r]$ of $X$ defined by:

$$
I_L = \langle x^{m^+} - x^{m^-} | m = m^+ - m^- \in L \rangle,
$$

where $L$ is a sublattice of $L_\beta$, see [15].

Example 2.1. Let $X = \mathbb{P}(1, 2, 3)$ be the weighted projective space over $\mathbb{F}_3$, which corresponds to the normal fan $\Sigma_P$ depicted in Figure 2 of the polygon $\mathcal{P}$ depicted
in Figure 2. Then, the first sequence above becomes:

\[
\begin{array}{cccc}
\phi & : & 0 & \longrightarrow \mathbb{Z}^2 \\
& & \phi & \longrightarrow \mathbb{Z}^3 \\
& & \beta & \longrightarrow \mathbb{Z} \\
& & \longrightarrow & 0,
\end{array}
\]

where

\[
\phi = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.
\]

The ring \( S = \mathbb{F}_3[x, y, z] \) is multigraded via

\[
\deg_\lambda(x) = 1, \quad \deg_\lambda(y) = 2 \quad \text{and} \quad \deg_\lambda(z) = 3.
\]

Since \( B = (x, y, z) \), we remove the set \( V(B) = V(x, y, z) = \{0\} \) and therefore obtain the quotient representation \( X(\mathbb{F}_3) = (\mathbb{F}_3^3 \setminus 0)/G \), where

\[
G = \{(x, y, z) \in (\mathbb{F}_3^3)| x^{-2}y = x^{-3}z = 1\} = \{(\lambda, \lambda^2, \lambda^3) | \lambda \in \mathbb{F}_3^*\}
\]

is the zero locus in \((\mathbb{F}_3^3)^3\) of the toric ideal:

\[
I_{L_\beta} := \langle x^u - x^v : u, v \in \mathbb{N}^r \text{ and } \beta u = \beta v \rangle = \langle x^2 - y, x^3 - z \rangle.
\]

One needs to be careful about the field over which the group \( G \) is considered. Even though we use representative from the affine space \( \mathbb{F}_3^3 \) recall that the equivalence of points in an orbit is determined via the subgroup \( G \) of \((\mathbb{F}_3^3)^3\). For instance, the points \([0 : 0 : 1]\) and \([0 : 0 : 2]\) are the same as \( \mathbb{F}_3\)-rational points, since there is \( \lambda \in \mathbb{F}_3^* \) such that \( \lambda^2 = 2 \) and thus we have \((0, 0, 2) = (\lambda, \lambda, \lambda^2) \cdot (0, 0, 1) \). But, these two points would be different if we considered equivalence with respect to the existence of \( \lambda \in \mathbb{F}_3^* \) such that \( \lambda^2 = 2 \), since \( \lambda^2 = 1 \) for all \( \lambda \in \mathbb{F}_3^* = \{1, 2\} \).

Let us recall basics of linear codes. Our alphabet is the finite field \( \mathbb{F}_q \) with \( q \) elements. A linear code is a subspace \( C \subset \mathbb{F}_q^N \) whose elements are referred to as the codewords.

**Definition 2.2.** Parameters of a linear code \( C \subset \mathbb{F}_q^N \) are as follows:

- \( N \) is the length of \( C \),
- \( K = \dim_{\mathbb{F}_q} C \) is the dimension of \( C \) as a subspace (a measure of efficiency),
- \( \delta \) is the minimum distance of \( C \) (a measure of reliability), which is the minimum of all Hamming distances between different codewords in \( C \), where Hamming distance is: \( \text{dist}(c_1, c_2) = \# \text{of non-zero entries in } c_1 - c_2 \). So,

\[
\delta(C) = \min_{c \in C \setminus \{0\}} (\# \text{of non-zero entries in } c).
\]

As in Equation 1.1, we get the so called generalized toric codes by evaluating homogeneous polynomials \( F \in S_\alpha \) of degree \( \alpha \) at some subset \( Y \) of \( \mathbb{F}_q\)-rational points in a toric variety \( X \).

**Definition 2.3.** Let \( Y \subseteq X \) be a subset of a toric variety \( X \). Its vanishing ideal \( I(Y) \) is the (homogeneous) ideal in \( S \) generated by homogeneous polynomials vanishing on \( Y \). The multigraded Hilbert function of \( Y \) is \( H_Y(\alpha) := \dim_k S_\alpha - \dim_k I_\alpha(Y) \).

Since, the kernel of the evaluation map in Equation (1.1) consists of the homogeneous polynomials of degree \( \alpha \) whose image is the point \((0, \ldots, 0) \in \mathbb{F}_q^N\), it follows that the dimension of the code \( C_{\alpha,Y} \) equals the value \( H_Y(\alpha) \) of the Hilbert function.
of \( Y \). When \( Y \subset T_X \), the variables \( x_i \) are all non zero-divisors, and thus the Hilbert function behaves nicer in the following manner:

**Proposition 2.4.** ([16]) Let \( Y \subset T_X \). The dimension \( H_Y(\alpha) \) of \( \mathcal{C}_{\alpha,Y} \) is non-decreasing in the sense that \( H_Y(\alpha) \leq H_Y(\alpha') \) for all \( \alpha \preceq \alpha' \).

On the other hand, the minimum distance behaves the opposite way as the following points out:

**Proposition 2.5.** ([14]) Let \( Y \subset T_X \). The minimum distance of \( \mathcal{C}_{\alpha,Y} \) is non-increasing in the sense that \( \delta(\mathcal{C}_{\alpha,Y}) \geq \delta(\mathcal{C}_{\alpha',Y}) \) for all \( \alpha \preceq \alpha' \).

These two results are not that surprising as we have the following well known relation between these two parameters given by the Singleton’s bound:

\[
\delta(\mathcal{C}_{\alpha,Y}) + K(\mathcal{C}_{\alpha,Y}) \leq N(\mathcal{C}_{\alpha,Y}) + 1.
\]

There is an algebro-geometric invariant of the zero-dimensional subvariety \( Y \subset X \) used to eliminate trivial codes which we introduce now.

**Definition 2.6.** The multigraded regularity of \( Y \), denoted \( \text{reg}(Y) \), is the set of \( \alpha \in \mathbb{N}_\beta \) for which \( H_Y(\alpha) = |Y| \), the length of \( \mathcal{C}_{\alpha,Y} \).

**Proposition 2.7.** If \( \alpha \in \text{reg}(Y) \) then \( \delta(\mathcal{C}_{\alpha,Y}) = 1 \).

**Proof.** Let \( \alpha \in \text{reg}(Y) \). Then, the dimension of the code is nothing but the length. So, the claim follows from the Singleton bound, as we always have \( \delta(\mathcal{C}_{\alpha,Y}) \geq 1 \). \( \square \)

Multigraded regularity set is determined by a number also known as the \( a \)-invariant in the case of a weighted projective space. In order to state the precise result, we first recall some relevant concepts.

When \( I \) is a weighted graded ideal, the quotient ring \( S/I \) inherits this grading as well and has a decomposition \( S/I = \bigoplus_{\alpha \in A} (S/I)_\alpha \), where \( (S/I)_\alpha = S/ \alpha I \) is a finite dimensional vector space spanned by monomials of degree \( \alpha \) in the numerical semigroup \( W = \mathbb{N}\{w_0, \ldots, w_n\} \), which do not belong to \( I \). This gives rise to the weighted Hilbert function and series defined respectively by

\[
H_{S/I}(\alpha) = \dim_k (S/I)_\alpha = \dim_k S/\alpha I - \dim_k I/\alpha
\]

and

\[
HS_{S/I}(t) = \sum_{\alpha \in W} H_{S/I}(\alpha)t^\alpha.
\]

Furthermore, the weighted Hilbert series has a rational function representation, that is, we have

\[
HS_{S/I}(t) = \frac{p_{S/I}(t)}{(1-t^{w_0}) \cdots (1-t^{w_n})},
\]

for a unique polynomial \( p_{S/I}(t) \) with integer coefficients.

**Proposition 2.8.** ([16, Proposition 3.12]) Let \( Y \subset T_X \) for \( X = \mathbb{P}(w_0, \ldots, w_n) \) with \( w_0 = 1 \). Then, there is an integer \( a_Y = \deg(p_{S/I(Y)}(t)) - w_0 - \cdots - w_n \) satisfying \( \text{reg}(Y) = 1 + a_Y + \mathbb{N} \).

A nice formula for the \( a \)-invariant is given for the \( \mathbb{F}_q \)-rational points of the torus \( T_X \) when \( X \) is a weighted projective space.
Proposition 2.9. [5] If \( Y = T_X(\mathbb{P}_q) \) for \( X = \mathbb{P}(w_0, \ldots, w_n) \) and \( g(\mathbb{N}W) \) is the Frobenius number of the numerical semigroup \( \mathbb{N}W = \mathbb{N}\{w_0, \ldots, w_n\} \), then
\[
a_Y = (q - 2)[w_0 + \cdots + w_n + g(\mathbb{N}W)] + g(\mathbb{N}W).
\]

There are subgroups of the torus \( T_X \) referred to as degenerate tori which we briefly discuss now.

Definition 2.10. The following subgroup
\[\{[t_1^{a_1}: \cdots : t_r^{a_r}] : t_i \in \mathbb{F}_q^*\}\]
of the torus \( T_X \) is called a degenerate torus, lying inside a toric variety \( X \).

If \( \mathbb{P}_q^* = \langle \eta \rangle \), every \( t_i \in \mathbb{P}_q^* \) is of the form \( t_i = \eta^{k_i} \), for some \( 0 \leq k_i \leq q - 2 \). Let \( d_i = |\eta^{k_i}| \) and \( D = diag(d_1, \ldots, d_r) \).

Proposition 2.11. [9, Corollary 3.13 (ii)] If \( Y = T_A \) is a complete intersection in \( X = \mathbb{P}^{r-1} \) and \( g := \gcd(d_1, \ldots, d_r) \) so that \( d_i' = d_1/g, \ldots, d_r' = d_r/g \) generate a numerical semigroup \( \mathbb{N}D' \) with the Frobenius number \( g(\mathbb{N}D') \). Then
\[1 + a_Y = g \cdot g(\mathbb{N}D') + d_1 + \cdots + d_r - (r - 1).
\]

Notice that when \( a_i = 1 \) and \( w_j = 1 \), for all \( i \) and \( j \), we have \( d_i = q - 1 \), and so \( d_i' = 1 \). The greatest integer not belonging to the numerical semigroup \( \mathbb{N}W = \mathbb{N}D' = \mathbb{N} \) is \( g(\mathbb{N}W) = g(\mathbb{N}D') = -1 \) so both formulas in Proposition 2.9 and Proposition 2.11 yield \( a_Y = n(q - 2) - 1 \), for the torus \( Y = T_X(\mathbb{P}_q) \) in the projective space \( X = \mathbb{P}^n \).

Definition 2.12. A binomial is of the form \( x^a - x^b \), and \( J \) is called a binomial ideal if it is generated by binomials. \( J \) is called a complete intersection if it is generated by height(\( J \)) many binomials.

Definition 2.13. For a lattice \( L \subset \mathbb{Z}^r \), the lattice ideal \( I_L \) is the binomial ideal generated by special binomials \( x^a - x^b \) for all \( a - b \in L \). Therefore,
\[I_L = \langle x^a - x^b \mid a - b \in L \rangle \subset S.
\]

Theorem 2.14. [13, Theorem 4.5] If \( Y = Y_A \) then \( I(Y) = I_L \) for \( L = D(L_{BD}) \).

If \( a_i = 1 \), for all \( i \), then \( Y_A = T_X(\mathbb{P}_q) \) and \( d_i = q - 1 \), for all \( i \), so that the matrix \( D \) is just \( q - 1 \) times the identity matrix yielding the following:

Corollary 2.15. [13] If \( Y = T_X(\mathbb{P}_q) \) then \( I(Y) = I_L \) for \( L = (q - 1)L_{BD} \).

Proposition 2.16. [13, Proposition 4.12] A generating system of binomials for \( I(Y_A) \) is obtained from that of \( I_{L_{BD}} \) by replacing \( x_i \) with \( x_i^d \). \( I(Y_A) \) is a complete intersection if and only if so is the toric ideal \( I_{L_{BD}} \). In this case, a minimal generating system is obtained from a minimal generating system of \( I_{L_{BD}} \) this way.

3. Degenerate Tori on Weighted Projective Spaces

In this section, we explore properties of some degenerate tori on a weighted projective space. To start with, we prove that they are complete intersections of special type of binomial hypersurfaces.

We focus on a weighted projective space \( X = \mathbb{P}(w_0, \ldots, w_n) \) and use the notation \( S = \mathbb{F}_q[x_0, \ldots, x_n] \) for the Cox ring of \( X \). Set
\[\tilde{w}_i = \frac{w_i}{\gcd(q - 1, w_i)}, \quad d_i = \frac{q - 1}{\gcd(q - 1, w_i)} \quad i = 0, 1, \ldots, n.
\]
The following concept is very helpful to see when a lattice ideal is complete intersection.

**Definition 3.1.** If each column of a matrix has both a positive and a negative entry we say that it is mixed. If it does not have a square mixed submatrix, then it is called dominating.

**Theorem 3.2.** [11, Theorem 3.9] Let $L \subseteq \mathbb{Z}^r$ be a lattice with the property that $L \cap \mathbb{N}^r = 0$. Then, $I_L$ is complete intersection $\iff$ $L$ has a basis $m_1, \ldots, m_k$ such that the matrix $[m_1 \cdots m_k]$ is mixed dominating. In the affirmative case, we have

$$I_L = (x^{m_1^+} - x^{m_1^-}, \ldots, x^{m_k^+} - x^{m_k^-}).$$

**Proposition 3.3.** Let $Q = \text{diag}(w_0, \ldots, w_n)$ and $Y_Q = \{(t_0 \cdots t_n) | t_i \in \mathbb{F}_q^*\}$ be the corresponding subgroup of $T_X$ for $X = \mathbb{P}(w_0, \ldots, w_n)$. If $w_0 \mid q - 1$ and $F_i = x_i^{d_i} - x_i^{d_i}$, $i = 1, 2, \ldots, n$, then, the vanishing ideal of $Y_Q$ is the following complete intersection lattice ideal:

$$I(Y_Q) = \langle F_1, F_2, \ldots, F_n \rangle.$$

**Proof.** Since $D = \text{diag}(d_0, \ldots, d_n)$ and $\beta = [w_0 \cdots w_n]$, it follows that their product is $\beta D = [w_0 d_0 \cdots w_n d_n]$. It is clear that $\tilde{w}_i(q - 1) = w_i d_i$, and so

$$\gcd(w_0 d_0, \ldots, w_n d_n) = (q - 1) \gcd(\tilde{w}_0, \ldots, \tilde{w}_n).$$

Therefore, we have the equality of the lattices $L_{\beta D} = L_{\tilde{W}}$, where $\tilde{W}$ is the matrix with columns $\tilde{w}_i$, for $i = 0, \ldots, n$.

When $w_0 \mid q - 1$, we have $\tilde{w}_0 = 1$ and thus the lattice $L_{\tilde{W}}$ has the following basis

$$\{(-\tilde{w}_1, e_1), \ldots, (-\tilde{w}_n, e_n)\},$$

where $e_i$ form the standard basis for $\mathbb{Z}^n$. Form the matrix $M$ whose columns are the basis vectors of $L_{\tilde{W}}$ given above. Since the matrix $M$ is mixed-dominating, it follows from Theorem 3.2 that the lattice ideal of $L_{\tilde{W}}$ is a complete intersection generated by the binomials $x_i - x_i^{\tilde{w}_i}$, $i = 1, 2, \ldots, n$.

By Theorem 2.14, the vanishing ideal $I(Y_Q) = I_L$ for the lattice $L = D(L_{\beta D})$, whose generators are obtained substituting $x_i^{d_i}$ for $x_i$ in the binomials above generating the lattice ideal of $L_{\tilde{W}}$, by Proposition 2.16. Therefore, the vanishing ideal $I(Y_Q)$ is a complete intersection generated by the binomials $F_1, F_2, \ldots, F_n$. □

**Proposition 3.4.** Let $Q = \text{diag}(w_0, \ldots, w_n)$ and $Y_Q = \{(t_0 \cdots t_n) | t_i \in \mathbb{F}_q^*\}$ be the corresponding subgroup of $T_X$ for $X = \mathbb{P}(w_0, \ldots, w_n)$. If $w_0 \mid q - 1$ then, for any $\alpha \in \text{NW}$ we have

$$H_{Y_Q}(\alpha) = \sum_{s=0}^{n} (-1)^s \sum_{I \subseteq [n], |I| = s} \dim_k S_{\alpha - \alpha_I},$$

where $\alpha_I = \sum_{i \in I} \alpha_i$. Moreover, the $\alpha$-invariant of $Y_Q$ is given by the formula

$$a(Y_Q) = (d_0 - 1)w_0 + \cdots + (d_n - 1)w_n - w_0.$$

**Proof.** Since $I(Y_Q)$ is a complete intersection by Proposition 2.16 generated by binomials of degrees $\alpha_1 = d_1 w_1, \ldots, \alpha_n = d_n w_n$, its minimal free resolution is given by the Koszul complex. As in the proof of [16, Proposition 3.13] we have the following exact sequence

$$0 \to W_n \to \cdots \to W_s \to \cdots \to W_1 \to S_\alpha \to (S/I(Y_Q))_\alpha \to 0,$$
where, for every $s = 1, \ldots, n$, the vector space $W_s$ is given by
\[
W_s = \bigoplus_{I \subseteq [n], |I| = s} S(-\alpha_I) = \bigoplus_{I \subseteq [n], |I| = s} S_{\alpha - \alpha_I}.
\]

Therefore, we obtain:
\[
H_{Y_Q}(\alpha) = \dim_k S_\alpha + \sum_{s=1}^{n} (-1)^s \dim_k W_s
\]
\[= \sum_{s=0}^{n} (-1)^s \sum_{I \subseteq [n], |I| = s} \dim_k S_{\alpha - \alpha_I}, \tag{3.1}
\]

where $\alpha_I = \sum_{i \in I} \alpha_i$. By Proposition 8.23 in [10], the numerator of the Hilbert series in Equation 2.1 is as follows:
\[
p_{S/I(Y_Q)} = \sum_{s=0}^{n} (-1)^s \sum_{I \subseteq [n], |I| = s} t^{\alpha_I}.
\]

Hence, $p_{S/I(Y_Q)}$ has degree $\alpha_1 + \cdots + \alpha_n = d_1 w_1 + \cdots + d_n w_n$, and thus
\[
a(Y_Q) = (d_1 - 1)w_1 + \cdots + (d_n - 1)w_n - w_0
\]
by Proposition 2.8. \hfill \sqcup \sqcap

**Example 3.5.** Let $X = \mathbb{P}(1,1,2)$. Consider the matrix $Q = \text{diag}(1,1,2)$ and $Y_Q = \{[t_0 : t_1 : t_2^2] | t_0, t_1, t_2 \in \mathbb{F}_q^*\}$. Assume that $q$ is odd. So, we have
\[
(d_0, d_1, d_2) = (q - 1, q - 1, (q - 1)/2) \quad \text{and} \quad (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2) = (1, 1, 1).
\]

Thus, $I(Y_Q) = (F_1, F_2) = (x_1^{q-1} - x_0^{q-1}, x_2^{(q-1)/2} - x_0^{(q-1)/2})$. As the degrees of the generators are $\alpha_1 = q - 1$ and $\alpha_2 = q - 1$, a graded minimal free resolution of $I(Y_Q)$ is given by:
\[
0 \to S_{\alpha - \alpha_1 - \alpha_2} \xrightarrow{[-F_2, F_1]^T} S_{\alpha - \alpha_1} \oplus S_{\alpha - \alpha_2} \xrightarrow{[F_1 F_2]} S_\alpha \to (S/I(Y_Q))_\alpha \to 0.
\]

Therefore, the Hilbert function is computed to be
\[
H_{Y_Q}(\alpha) = \dim_k S_\alpha - \dim_k S_{\alpha - \alpha_1} - \dim_k S_{\alpha - \alpha_2} + \dim_k S_{\alpha - \alpha_1 - \alpha_2} = \dim_k S_\alpha - 2 \dim_k S_{\alpha - (q-1)} + \dim_k S_{\alpha - 2(q-1)}.
\]

We first notice the following
\[
\dim_k S_\alpha = \begin{cases} 
(\alpha_0 + 1)^2 & \text{if } \alpha = 2\alpha_0 \\
(\alpha_0 + 1)(\alpha_0 + 2) & \text{if } \alpha = 2\alpha_0 + 1.
\end{cases}
\]

Thus, if $0 \leq \alpha \leq q - 2$, then $\dim_k S_{\alpha - (q-1)} = \dim_k S_{\alpha - 2(q-1)} = 0$. Hence,
\[
H_{Y_Q}(\alpha) = \begin{cases} 
(\alpha_0 + 1)^2 & \text{if } \alpha = 2\alpha_0 \\
(\alpha_0 + 1)(\alpha_0 + 2) & \text{if } \alpha = 2\alpha_0 + 1.
\end{cases}
\]

When, $q - 1 < \alpha < q - 1 + 2(d_2 - 1) = 2(q - 1) - 2$, we have $\dim_k S_{\alpha - 2(q-1)} = 0$. It is easy to see that
\[
\dim_k S_{\alpha - (q-1)} = \begin{cases} 
(\alpha_0 + 1 - (q - 1)/2)^2 & \text{if } \alpha = 2\alpha_0 \\
(\alpha_0 + 1 - (q - 1)/2)(\alpha_0 + 2 - (q - 1)/2) & \text{if } \alpha = 2\alpha_0 + 1.
\end{cases}
\]
Finally, when \( \alpha \)

\[
\begin{cases}
(\alpha_0 + 1)^2 - 2(\alpha_0 + 1 - (q-1)/2)^2 & \text{if } \alpha = 2\alpha_0 \\
(\alpha_0 + 1)(\alpha_0 + 2) - 2(\alpha_0 + 1 - (q-1)/2)(\alpha_0 + 2 - (q-1)/2) & \text{if } \alpha = 2\alpha_0 + 1.
\end{cases}
\]

Finally, when \( \alpha \geq 2(q-1) \), we get

\[
\dim_{\mathbb{K}} S_{\alpha-2(q-1)} = \begin{cases}
(\alpha_0 + 1 - (q-1)/2)^2 & \text{if } \alpha = 2\alpha_0 \\
(\alpha_0 + 1 - (q-1))(\alpha_0 + 2 - (q-1)) & \text{if } \alpha = 2\alpha_0 + 1.
\end{cases}
\]

Therefore, we have

\[
H_{Y_Q}(\alpha) = \begin{cases}
(q-1)^2/2 & \text{if } \alpha = 2\alpha_0 \\
(q-1)^2/2 & \text{if } \alpha = 2\alpha_0 + 1.
\end{cases}
\]

4. LENGTH AND DIMENSION WHEN \( X = \mathbb{P}(1, w_1, \ldots, w_n) \)

Let \( \mathbb{F}_q^w = \langle \eta \rangle \) so that the order of the generator \( \eta \) is \( q-1 \). It follows easily that the order of \( \eta_i := \eta^{w_i} \) is

\[
d_i = \frac{q-1}{\gcd(q-1, w_i)} \quad i = 1, \ldots, n.
\]

By using \( I(Y_Q) \), length and the dimension of \( C_{\alpha,Y_Q} \) is computed as follows.

**Theorem 4.1.** Let \( X = \mathbb{P}(1, w_1, \ldots, w_n) \) be a weighted projective space over the field \( \mathbb{F}_q \). Consider \( Q = \text{diag}(1, w_1, \ldots, w_n) \) and the subgroup it defines in \( X(\mathbb{F}_q) \):

\[Y_Q = \{[t_0 : t_1^{w_1} : \cdots : t_n^{w_n}] | t_i \in \mathbb{F}_q^*, \text{ for all } i = 0, \ldots, n\}.\]

Then, the length of \( C_{\alpha,Y_Q} \) is \( |Y_Q| = d_1 \cdots d_n \) and the dimension is

\[
\dim(C_{\alpha,Y_Q}) = \sum_{m_1=0}^{\min\{\lfloor \frac{\alpha}{w_1} \rfloor, d_1-1\}} \sum_{m_2=0}^{\min\{\lfloor \frac{\alpha-m_1 w_1}{w_2} \rfloor, d_2-1\}} \cdots \sum_{m_n=0}^{\min\{\lfloor \frac{\alpha-m_1 w_1-\cdots-m_{n-1} w_{n-1}}{w_n} \rfloor, d_n-1\}} 1.
\]

Moreover, the \( a \)-invariant is given by

\[a(Y_Q) = (d_1 - 1)w_1 + \cdots + (d_n - 1)w_n - 1.\]

**Proof.** We first prove that

\[
Y_Q = \langle [1 : \eta_1 : 1 : \cdots : 1] \rangle \times \cdots \times \langle [1 : \cdots : 1 : \eta_n] \rangle.
\]

Multiplying by \( [\lambda : \lambda^{w_1} : \cdots : \lambda^{w_n}] \) does not change an equivalence class for every \( \lambda \in \mathbb{F}_q^\times \). So, we have the equality of the following points:

\[t_0 : t_1^{w_1} : \cdots : t_n^{w_n} = [1 : (t_1/t_0)^{w_1} : \cdots : (t_n/t_0)^{w_n}].\]

Hence, we have

\[Y_Q = \{[1 : s_1^{w_1} : \cdots : s_n^{w_n}] | s_i \in \mathbb{F}_q^*, \text{ for all } i = 1, \ldots, n\}.
\]

Since \( s_i = \eta^{k_i} \), for some \( k_i \in \mathbb{N} \), it is clear that \( s_i^{w_i} = \eta_i^{k_i} \) and thus

\[Y_Q = \{[1 : \eta_1^{l_1} : \cdots : \eta_n^{l_n}] | 0 \leq l_1 \leq d_1, \ldots, 0 \leq l_n \leq d_n\},
\]

from which the claim in (4.1) is deduced, and thus \(|Y_Q| = d_1 \cdots d_n\).

If \( w_0 = 1 \), then \( d_0 = q - 1 \) and so the vanishing ideal of \( Y_Q \) is generated by the binomials \( F_i = x_i^{d_i} - x_0^{d_i w_0} \), for \( i = 1, 2, \ldots, n \). With respect to any term order making \( x_0 \) the smallest variable, the leading monomial of \( F_i \) is clearly \( x_i^{d_i} \). Since the monomials \( x_i^{d_i} \) and \( x_j^{d_j} \) are relatively prime for different \( i \) and \( j \), it readily
follows that the binomials $F_1, \ldots, F_n$ form a Groebner basis for the vanishing ideal $I(Y_Q)$. It is well-known then that a basis for the vector space $S_{\alpha}/I_{\alpha}(Y_Q)$ is given by the monomials $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ of degree $\alpha$ that can not be divided by the leading monomials $x_i^{d_i}$ of $F_i$, for all $i = 1, 2, \ldots, n$ and for

$$\alpha = m_0 + m_1 w_1 + \cdots + m_n w_n \in \mathbb{N} = \{1, w_1, \ldots, w_n\}.$$  

Therefore, a basis for $S_{\alpha}/I_{\alpha}(Y_Q)$ corresponds to the set of tuples $(m_0, m_1, \ldots, m_n)$ satisfying $\alpha = m_0 + m_1 w_1 + \cdots + m_n w_n$ and $m_i \leq d_i - 1$, for all $i = 1, 2, \ldots, n$. The elements of this set can be identified step by step as we explain now. We start first by choosing an integer $m_n$ between 0 and $\min\{\lfloor \frac{\alpha}{w_n} \rfloor, d_n - 1\}$ and observe that the elements of the set in question can be partitioned into subsets for every choice of $m_n$ in the aforementioned range. More precisely, for each fixed $m_n$, we have a subset consisting of tuples $(m_0, m_1, \ldots, m_n)$ satisfying

$$m_0 + m_1 w_1 + \cdots + m_{n-1} w_{n-1} = \alpha - m_n w_n \text{ and } m_i \leq d_i - 1, \text{ for all } i = 1, 2, \ldots, n-1.$$  

As a second step, we fix $m_{n-1}$ between 0 and $\min\{\lfloor \frac{\alpha - m_n w_n}{w_{n-1}} \rfloor, d_{n-1} - 1\}$, and look for the solutions $(m_0, m_1, \ldots, m_{n-2})$ satisfying

$$m_0 + m_1 w_1 + \cdots + m_{n-2} w_{n-2} = \alpha - m_n w_n - m_{n-1} w_{n-1} \text{ and } m_i \leq d_i - 1,$$

for all $i = 1, 2, \ldots, n - 2$. Continuing inductively, we end up with a unique $m_0$ satisfying

$$m_0 = \alpha - m_n w_n - m_{n-1} w_{n-1} - \cdots - m_1 w_1.$$  

Hence, the dimension of the code, which is nothing but the dimension of the vector space $S_{\alpha}/I_{\alpha}(Y_Q)$, is exactly the sum given by the formula

$$\dim(C_{\alpha,Y_Q}) = \sum_{m_n = 0}^{\min\{\lfloor \frac{\alpha}{w_n} \rfloor, d_n - 1\}} \sum_{m_{n-1} = 0}^{\min\{\lfloor \frac{\alpha - m_n w_n}{w_{n-1}} \rfloor, d_{n-1} - 1\}} \cdots \sum_{m_1 = 0}^{\min\{\lfloor \frac{\alpha - m_n w_n - \cdots - m_2 w_2}{w_1} \rfloor, d_1 - 1\}} 1.$$  

The $a-$invariant can be obtained from Proposition 3.4, by subtituting $w_0 = 1$. □

5. Codes on $Y_Q \subset \mathbb{P}(1, 1, a)$

For any positive integer $a$, we compute the basic parameters of the code $C_{\alpha,Y_Q}$, for the subgroup $Y_Q = \{[t_1 : t_2 : t_3^q] \mid t_1, t_2, t_3 \in \mathbb{F}_q^*\}$ of $T_X(\mathbb{F}_q)$ for the weighted projective space $X = \mathbb{P}(1, 1, a)$.

**Theorem 5.1.** Let $d_2 = \frac{q-1}{\gcd(q-1, a)}$, $k = \lfloor \frac{\alpha - (q-2)}{a} \rfloor$ and $\mu_2 = \min\{\lfloor \frac{\alpha}{a} \rfloor, d_2 - 1\}$. Then, the length of $C_{\alpha,Y_Q}$ is $N = |Y_Q| = (q-1)d_2$. Its dimension $K(C_{\alpha,Y_Q})$ is

$$\begin{align*}
(\mu_2 + 1)(\alpha + 1 - \mu_2 a/2), & \text{ if } 0 \leq \alpha \leq q - 2 \\
(q - 1)(k + 1) + (\mu_2 - k)[\alpha + 1 - (\mu_2 + k + 1)a/2], & \text{ if } 0 < \alpha - (q-2) < (d_2 - 1)a \\
N & \text{ otherwise.}
\end{align*}$$

and the minimum distance of $C_{\alpha,Y_Q}$ is:

$$\delta(C_{\alpha,Y_Q}) = \begin{cases} 
\frac{d_2(q-1-\alpha)}{2} & \text{ if } 0 \leq \alpha \leq q - 2 \\
\frac{d_2 - k}{2} & \text{ if } q - 2 < \alpha < (q-2) + (d_2 - 1)a \\
1 & \text{ otherwise.}
\end{cases}$$
Proof. Since \(w_1 = 1\), we have \(d_1 = q - 1\). It follows from Equation 4.1 that
\[
Y_Q = \{[1 : \eta_1^{i_1} : \eta_2^{i_2}] \mid 0 \leq i_1 \leq d_1 \text{ and } 0 \leq i_2 \leq d_2\},
\]
so the length of the code is \(d_1d_2 = (q - 1)d_2\).

When \(0 \leq \alpha \leq q - 2\), the dimension formula in Theorem 4.1 specializes to
\[
\dim(C_{\alpha,Y_Q}) = \sum_{m_2=0}^{\mu_2} \min\{\alpha - m_2a, q - 2\} \sum_{m_1=0}^{\alpha - m_2a} 1 = \sum_{m_2=0}^{\mu_2} \sum_{m_1=0}^{\alpha - m_2a} 1
= \sum_{m_2=0}^{\mu_2} (\alpha - m_2a + 1) = (\mu_2 + 1)(\alpha + 1) - a \sum_{m_2=0}^{\mu_2} m_2
= (\mu_2 + 1)(\alpha + 1) - \frac{a\mu_2(\mu_2 + 1)}{2}.
\]

If \(q - 2 < \alpha < (q - 2) + (d_2 - 1)a\), then using the formula in Theorem 4.1 again, we get
\[
\dim(C_{\alpha,Y_Q}) = \sum_{m_2=0}^{\mu_2} \min\{\alpha - m_2a, q - 2\} \sum_{m_1=0}^{\alpha - m_2a} 1
= \sum_{m_2=0}^{k} \sum_{m_1=0}^{q - 2} 1 + \sum_{m_2=k+1}^{\mu_2} \sum_{m_1=0}^{\alpha - m_2a} 1
= (q - 1)(k + 1) + \sum_{m_2=k+1}^{\mu_2} (\alpha - m_2a + 1)
= (q - 1)(k + 1) + (\mu_2 - k)(\alpha + 1) - a \sum_{m_2=k+1}^{\mu_2} m_2
= (q - 1)(k + 1) + (\mu_2 - k)(\alpha + 1) - \frac{a\mu_2(\mu_2 + 1) - k(k + 1)}{2}.
\]

As for the minimum distance, we first give an upper bound on the number \(|V_{Y_Q}(F)|\) of zeroes on \(Y_Q\) of a homogeneous polynomial \(F\) of degree \(\alpha\) and demonstrate a specific polynomial attaining that bound. Let \([d_2]\) denote the set of non-negative integers smaller than \(d_2\), and set
\[
J_F := \{j \in [d_2] \mid x_2 - \eta_2^{j}x_0^{w_2} \text{ divides } F\}.
\]
We claim that \(|V_{Y_Q}(F)| \leq d_1|J_F| + (d_2 - |J_F|) \deg_{x_1}(F)\), where \(\deg_{x_1}(F)\) is the usual degree of \(F\) in the variable \(x_1\). The polynomial \(f_j(x_1) := F(1, x_1, \eta_2^{j}) \in \mathbb{F}_q[x_1]\) vanishes at the points \([1 : \eta_1^i : \eta_2^{j}]\), for every \(i \in [d_1]\), when \(j \in J_F\). Thus, there are \(d_1|J_F|\) such roots of \(F\). On the other hand, \(f_j\) is not a zero polynomial when \(j \notin J_F\), and in this case it can have at most its degree many zeroes, giving rise to \((d_2 - |J_F|) \deg_{x_1}(F)\) many roots of \(F\), completing the proof of the claim. Since we always have
\[
F = \prod_{j=1}^{|J_F|} (x_2 - \eta_2^{j}x_0^{w_2})F',
\]
it follows that \( \deg_{x_1}(F) = \deg_{x_1}(F') \leq \alpha - \lfloor |J_F|w_2 \rfloor \). Thus, we have
\[
|V_{Y_Q}(F)| \leq d_1 |J_F| + (d_2 - |J_F|)(\alpha - |J_F|w_2)
\]
\[
\leq d_2 \alpha + |J_F|((d_1 - \alpha - w_2)(d_2 - |J_F|)).
\]
(5.1)

Notice that the number in the parenthesis above is
\[
d_1 - \alpha - w_2(d_2 - |J_F|) = d_1 - \alpha - w_2d_2 + w_2|J_F| = d_1 - (q - 1)w_2 - \alpha + w_2|J_F|
\]
which is non-positive since \( d_1 \leq q - 1 \leq (q - 1)w_2 \) and \( |J_F|w_2 \leq \deg_{x_2}(F) \leq \alpha \).

Hence, altogether, we have the upper bound
\[
|V_{Y_Q}(F)| \leq d_2 \alpha.
\]
(5.2)

Consider now the following polynomial:
\[
F_0 = \prod_{i=1}^{\alpha}(x_1 - \eta_i x_0)
\]
which vanish at points \([1 : \eta_i : \eta_i^2]\), for every \( i \in [\alpha] \) and \( j \in [d_2] \), saying that
\[
|V_{Y_Q}(F_0)| = d_2 \alpha.
\]
As the weight of the codeword \( ev_{Y_Q}(F_0) \) is clearly
\[
|Y_Q| - |V_{Y_Q}(F_0)| = d_2(q - 1) - d_2 \alpha
\]
and that of a general codeword \( ev_{Y_Q}(F) \) is
\[
|Y_Q| - |V_{Y_Q}(F)| \geq d_2(q - 1) - d_2 \alpha,
\]
it follows that the minimum distance of the code is \( d_2(q - 1 - \alpha) \), when \( \alpha < q - 1 \).

When \( \alpha \geq a_Y + 1 = (q - 2) + (d_2 - 1)a \), the code is trivial, so \( \delta(C_{\alpha,Y_Q}) = 1 \).

From now on, assume that \( q - 2 \leq \alpha < a_Y + 1 = (q - 2) + (d_2 - 1)a \). Let \( k \) be the quotient and \( r_0 \) be the remainder of the division of \( \alpha - (q - 2) \) by \( w_2 = a \), i.e.
\[
\alpha - (q - 2) = ka + r_0 \text{ where } 0 \leq k := \left\lfloor \frac{\alpha - (q - 2)}{a} \right\rfloor \leq d_2 - 2 \text{ and } 0 \leq r_0 \leq a - 1.
\]
In this case, we consider the set
\[
I_F := \{i \in [q - 1] \mid x_1 - \eta_i x_0 \text{ divides } F\}.
\]
As in the previous case, we have \( |V_{Y_Q}(F)| \leq d_2 |I_F| + (q - 1 - |I_F|) \deg_{x_2}(F) \), where \( \deg_{x_2}(F) \) is the usual degree of \( F \) in the variable \( x_2 \).

When \( |I_F| = q - 1 \), \( F \) vanishes on \( Y_Q \), so \( F \) gives a codeword with zero weight. Thus, we suppose \( |I_F| \leq q - 2 \). In order to get more zeroes, we assume \( \deg_{x_2}(F) \leq k \), yielding that
\[
|V_{Y_Q}(F)| \leq d_2 |I_F| + (q - 1 - |I_F|)k = |I_F|(d_2 - k) + (q - 1)k
\]
(5.3)
\[
\leq (q - 2)(d_2 - k) + (q - 1)k = (q - 2)d_2 + k.
\]

We consider the following homogeneous polynomial of degree \( \alpha \) now:
\[
G_0 = x_0^r \prod_{i=1}^{q-2} (x_1 - \eta_i x_0) \prod_{j=1}^{k}(x_2 - \eta_j x_0^2)
\]
which vanish at the points \([1 : \eta_i : \eta_i^2]\), for every \( i \in [q - 2] \) and \( j \in [d_2] \), together with the points \([1 : \eta_i : \eta_i^2]\), for \( i = q - 1 \) and \( j \in [k] \). Therefore, the number of roots is \( |V_{Y_Q}(G_0)| = (q - 2)d_2 + k \). It readily follows that the minimum distance \( \delta(C_{\alpha,Y_Q}) \) of the code is the weight \( (q - 1)d_2 - (q - 2)d_2 - k = d_2 - k \) of the codeword corresponding to \( G_0 \).
Example 5.2. Take $a = 2$, $q = 7$. So, $d_2 = 3$ and length is $d_2(q-1) = 18$. Table 1 exhibits the main parameters of the code $C_{a,Y_Q}$ for $\alpha$ in the first column.

Table 1. $a=2$ and $q=7$

| $\alpha$ | $[d, k, n]$  |
|----------|--------------|
| 0        | [18,1,18]    |
| 1        | [18,2,15]    |
| 2        | [18,4,12]    |
| 3        | [18,6,9]     |
| 4        | [18,9,6]     |
| 5        | [18,10,3]    |

Example 5.3. Similarly, we take $a = 3$ and $q = 5$ so that $d_2 = 4$ and length is $d_2(q-1) = 4.4 = 16$. Table 2 gives the main parameters of the corresponding codes:

Table 2. $a=3$ and $q=5$

| $\alpha$ | $[d, k, n]$  |
|----------|--------------|
| 0        | [16,1,16]    |
| 1        | [16,2,12]    |
| 2        | [16,3,8]     |
| 3        | [16,5,4]     |
| 4        | [16,6,4]     |
| 5        | [16,7,4]     |
| 6        | [16,9,3]     |
| 7        | [16,10,3]    |
| 8        | [16,11,3]    |
| 9        | [16,13,2]    |
| 10       | [16,14,2]    |
| 11       | [16,15,2]    |

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