In order to investigate the reliability of the classical approximation for non-perturbative real time correlation functions at finite temperature we study the two-point correlator for the anharmonic oscillator. For moderately large times the classical limit gives a good approximation of the quantum result but after some time $t_*$ the classical approximation breaks down even at high temperature.

1 Introduction

Real time processes at high temperatures play an important role in the physics of the early universe and of heavy ion collisions. Some of the corresponding physical quantities, like the sphaleron rate, can be related to real time correlation functions in finite temperature field theory. A lot of work on this subject has been done in perturbation theory. However, some physical quantities of interest are non-perturbative and the only known method of calculating them on the lattice is the classical approximation.

The usual argument why the classical approximation should be reliable goes as follows: Typically, the non-perturbative effects are due to field modes with small spatial momenta $|k| \ll T/\hbar$. For these momenta the Bose-Einstein distribution function is large, $n(|k|) \gg 1$. In this limit one expects physics to be classical.

The question I would like to address in this talk is what one can say about the reliability of the classical approximation in quantitative terms. Is there an approximation scheme which tells us how large quantum corrections to the classical approximation are?

There is such an approximation scheme for equal time correlation functions called dimensional reduction. Equal time correlation functions in $(d+1)$
dimensions can be computed in an effective classical theory which is a \(d\)-dimensional Euclidean field theory. Corrections to the effective theory can be estimated and they are small at sufficiently high temperatures.

One possibility to investigate this problem for the case of unequal time correlation functions is to expand the quantum correlation function \(C(t)\) in powers of Planck’s constant \(\hbar\):

\[
C(t) = C_{cl}(t) + \hbar C_1(t) + \hbar^2 C_2(t) + \mathcal{O}(\hbar^3).
\]  

(1)

Each term in this expansion contains all orders in the coupling constant. The reliability of the classical approximation can then be estimated by comparing the classical correlation \(C_{cl}(t)\) function with the first quantum correction.

## 2 The Anharmonic Oscillator

The anharmonic oscillator is the simplest interacting theory and it can be viewed as a \((0+1)\)-dimensional field theory. We consider the finite temperature correlator

\[
C(t) = \left\langle \frac{1}{2} \left[ Q(t)Q(0) + Q(0)Q(t) \right] \right\rangle,
\]

(2)

where \(\langle \cdots \rangle = Z^{-1} \text{tr}[\cdots \exp(-\beta H)]\) denotes the thermal average and \(Z = \text{tr}[\exp(-\beta H)]\). Furthermore, \(Q(t) = \exp(iHt/\hbar)Q(0)\exp(-iHt/\hbar)\) is the position operator at time \(t\). The Hamiltonian is \(H = p^2/2 + U(Q)\) with

\[
U(Q) = \pm \frac{1}{2} \omega^2 Q^2 + \frac{1}{4} g^2 Q^4.
\]

(3)

We refer to the two cases of a positive and of a negative quadratic term as the symmetric and the broken case, respectively.

The classical approximation to Eq. (2) is given by

\[
C_{cl}(t) = Z_{cl}^{-1} \int \frac{dp dq}{2\pi \hbar} e^{-\beta H(p,q)} q \dot{q}(t),
\]

(4)

where \(Z_{cl} = \int \frac{dp dq}{(2\pi \hbar)} e^{-\beta H(p,q)}\) and \(q_c(t)\) is the solution of the classical equations of motion with the initial conditions \(q_c(0) = q, \dot{q}_c(0) = p\).

Expanding Eq. (2) in powers of \(\hbar\) one finds that the lowest order term is given by Eq. (4). The term linear in \(\hbar\) vanishes and the first quantum correction to Eq. (4) is quadratic in \(\hbar\). Up to this order

\[
C(t) = Z^{-1} \int \frac{dp dq}{2\pi \hbar} e^{-\beta H(p,q)} \left\{ \left[ 1 - \frac{\hbar^2 \beta^2}{24} U''(q) + \frac{\hbar^2 \beta}{24} \left( \partial_q^2 + U''(q) \partial_p^2 \right) \right] q \dot{q}(t) \right. \\
- \frac{\hbar^2}{24} \int_0^1 dt' U'''(q_c(t')) \{ q_c(t'), q_c(t) \} + \mathcal{O}(\hbar^3),
\]

(5)
Figure 1: (a) The classical correlator $C_{cl}(t)$ and (b) the quantum correction $C_{q}(t)$ in the symmetric case. The thick lines represent the exact numerical results and thin lines represent the analytic approximation. The dimensionless variables $V_0$ and $\epsilon$ are $V_0 = \omega^4/(4g^2)$, $\epsilon = g^2/\omega^3$.

where $\{f, g\} = \partial_p f \partial_q g - \partial_p g \partial_q f$ denotes the Poisson bracket and $\{f, g\}_0 = g$, $\{f, g\}_{n+1} = \{f, \{f, g\}_n\}$. Similarly, the expression for $Z$ to order $\hbar^2$ is

$$Z = \int \frac{dpdq}{2\pi\hbar} e^{-\beta H(p,q)} \left[ 1 - \frac{\hbar^2 \beta^2}{24} U''(q) \right].$$

The solutions of the classical equations of motion $q_c(t)$ in Eqs. (4) and (5) can be written in terms of the Jacobi elliptic functions. The phase space integration in Eq. (6) cannot be done analytically. Computing the quantum corrections in Eq. (5) is even more difficult. Therefore we have evaluated these expressions numerically and also in a large time expansion. The latter can be applied only in the symmetric case. Since we are interested in analytical results we will discuss this case here. The numerical calculations and a detailed discussion of the broken case can be found in Reference 4.

The results for the classical correlator $C_{cl}(t)$ in the symmetric case are shown in Fig. 1(a). $C_{cl}(t)$ is an oscillating function. Its amplitude attenuates since the frequency of $q_c(t)$ is energy dependent and the different $q_c(t)$ in Eq. (4) interfere destructively. For very large times

$$\omega t \gg 1, \quad \omega t \gg \beta \omega^4/g^2$$

(7)
the classical correlator can be approximated by

$$C_{\text{cl}}(t) \approx -\frac{16}{9} \frac{1}{hZ_{\text{cl}}} \frac{\omega^5 \cos \omega t}{g^4 (\omega t)^2}$$  \hspace{1cm} (8)

Now we consider the order $\bar{h}^2$ corrections to $C_{\text{cl}}(t)$ which we write as

$$\bar{h}^2 C_2(t) = C_{\text{cl}}^{(a)}(t) + C_{\text{cl}}^{(b)}(t) + C_{\text{cl}}^{(c)}(t).$$

The term $C_{\text{cl}}^{(a)}(t)$ is a sum of the $\bar{h}^2$ correction to the partition function when it combines with the classical result $C_{\text{cl}}(t)$, and of the term proportional to $U''(q)$ in the numerator of Eq. (5). $C_{\text{cl}}^{(b)}(t)$ contains the terms proportional to $\partial_q^2 q q_c(t)$ and to $\partial_p^2 q q_c(t)$. Finally, $C_{\text{cl}}^{(c)}(t)$ is given by the term in Eq. (5) which contains the Poisson brackets.

For large times $C_{\text{cl}}^{(a)}(t)$ and $C_{\text{cl}}^{(b)}(t)$ are proportional to $\beta/((\omega t)^2$. At sufficiently high temperature these quantum corrections are always small compared to the classical approximation. The results for $C_{\text{cl}}^{(c)}(t)$ are shown in Fig. 1(b).

One sees that the amplitude does not decrease with time and therefore $C_{\text{cl}}^{(c)}(t)$ will become larger than $C_{\text{cl}}(t)$ at sufficiently large times. In the limit

$$C_{\text{cl}}^{(c)}(t) \approx -\frac{1}{12} \frac{1}{hZ_{\text{cl}}} \frac{\bar{h}^2 \cos \omega t}{\omega}$$  \hspace{1cm} (9)

Comparing Eqs. (8) and (9) we see that $C_{\text{cl}}^{(c)}(t)$ becomes as large as the classical correlator for $t \sim t^*$ where

$$\omega t^* = \frac{\omega^3}{h g^2},$$  \hspace{1cm} (10)

and for $t > t^*$ the classical approximation breaks down.

So far we have considered the classical limit of Eq. (2) without modifying the parameters of the Hamiltonian. One may wonder whether one can “improve” the classical approximation by using an effective theory in which the parameters are determined by dimensional reduction. This would correspond to a resummation of certain quantum corrections.

To order $\bar{h}^2$ one can easily calculate the “mass” parameter of the effective theory:

$$\omega_{\text{eff}}^2 = \omega^2 + 3g^2 T \sum_{n \neq 0} \frac{1}{(2\pi n T/\bar{h})^2} = \omega^2 + \frac{1}{4} g^2 \bar{h}^2 \beta$$  \hspace{1cm} (11)

The change in the coupling constant is of order $\bar{h}^4$ and thus does not contribute in the present $\bar{h}^2$-calculation.
Now consider Eq. (5) were we have \( U'' = \omega^2 + 3g^2q^2 \). The term \( \omega^2 \) is canceled by the corresponding term in \( Z^{-1} \). The term \( 3g^2q^2 \) in Eq. (6) can be reproduced by calculating the classical partition function \( Z_{cl} \) with \( \omega_{eff} \) instead of \( \omega \). Similarly, the term \( 3g^2q^2 \) in Eq. (5) is accounted for by using \( \omega_{eff} \). Thus \( C^{(a)}_{\hbar^2}(t) \) can be taken into account by changing the parameters of the classical theory.

There remain the terms \( C^{(b)}_{\hbar^2}(t) \) and \( C^{(c)}_{\hbar^2}(t) \). We have seen that the latter is responsible for the breakdown of the classical approximation. The question is whether it can be taken into account using \( \omega_{eff} \) in the classical equations of motion which determine \( q_c(t) \). To answer this question we can simply replace \( \omega \rightarrow \omega_{eff} \) in Eq. (4) and expand in \( \hbar \) to obtain

\[
C_{cl}^{\text{eff}}(t) \approx \left[ 1 + b_1 \frac{g^2\hbar^2\beta}{\omega^2} \right] C_{cl}(t) + \frac{2}{9} \frac{1}{\hbar Z_{cl}} \frac{\hbar^2\beta \omega^3 \sin \omega t}{g^2} \omega t + \mathcal{O}(\hbar^3),
\]

where \( b_1 \) is some number. Now we see that for large times \( C_{cl}^{\text{eff}}(t) \) has a different functional behavior than \( C_{\hbar^2}^{(c)}(t) \). Therefore the large quantum corrections due to \( C_{\hbar^2}^{(c)}(t) \) cannot be resummed in this way.

3 Summary

We have studied the classical finite temperature real time two-point correlation function and its first quantum corrections for the anharmonic oscillator. The expansion around the classical limit is made in powers of \( \hbar \), so that each order contains all orders in the coupling constant \( g^2 \).

As long as \( t \ll t_* = \omega^2/(\hbar g^2) \) the quantum corrections are small compared to the classical correlation function. From this we would expect that in this region the classical limit gives a good approximation for the full quantum correlation function. The expansion parameter for the quantum corrections in this region is not just the naive \( (\beta \hbar \omega) \). Instead, also the factors \( \epsilon(\beta \hbar \omega) \) and \( \epsilon^2 \), where \( \epsilon = \hbar g^2/\omega^3 \), appear.

The semiclassical expansion breaks down at \( t \sim t_* \) when the quantum corrections become as large as the classical result. Moreover, we found that these large corrections cannot be resummed by modifying the parameters of the classical theory.

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