CONGRUENCE PRESERVING EXPANSIONS OF NILPOTENT ALGEBRAS

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Abstract. We characterize those nilpotent algebras of prime power order and finite type in congruence modular varieties that have infinitely many polynomially inequivalent congruence preserving expansions.

1. The result

Associated with every algebraic structure $A$, there are two clones which we will study in the present note: the clone of polynomial functions $\text{Pol}(A)$, and the clone of congruence preserving functions $\text{Comp}(A)$. We say that an algebra $A'$, defined on the same universe as $A$, is a congruence preserving expansion of $A$ if $\text{Pol}(A) \subseteq \text{Pol}(A') \subseteq \text{Comp}(A)$. For expanded groups, such expansions with unary operations have been studied in \cite{Pet10}. Considering algebras with the same clone of polynomial functions as equivalent, we say that $A$ has finitely many polynomially inequivalent congruence preserving expansions if the set $\{C \mid C$ is a clone with $\text{Pol}(A) \subseteq C \subseteq \text{Comp}(A)\}$ is finite. One extreme case is $\text{Pol}(A) = \text{Comp}(A)$: then $A$ is called affine complete \cite{KP01}, and clearly $A$ then has only one congruence preserving expansion. On the other side, if $A$ has only finitely many fundamental operations (i.e., it is of finite type) and $\text{Comp}(A)$ is not finitely generated, then $A$ has infinitely many inequivalent congruence preserving expansions. For finite $p$-groups $G$, \cite{ALM16} provides a complete characterization when $\text{Comp}(G)$ is finitely generated. However, there are algebras for which $\text{Comp}(A)$ is finitely generated, but $A$ still has infinitely many inequivalent congruence preserving expansions: the cyclic group with 4 elements \cite{Bul02} and the quaternion group with 8 elements are examples of such a behaviour. Our

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characterization uses a condition on the congruence lattice that has been used in [AM13]: We say that a bounded lattice $L$ splits strongly if it is the union of two proper subintervals $\mathbb{I}[0,\delta] \cup \mathbb{I}[\varepsilon,1]$ with nonempty intersection, which can be expressed by

\[(1.1) \quad L \models \exists \delta, \varepsilon \in L: (0 < \varepsilon \leq \delta < 1 \text{ and } (\forall \alpha \in L: \alpha \leq \delta \text{ or } \alpha \geq \varepsilon)).\]

Note that it is claimed that $\varepsilon \neq 0$ and $\delta \neq 1$. We say that a finite algebra $A$ has few subpowers if there is a polynomial $p \in \mathbb{R}[x]$ such that for each $n \in \mathbb{N}$, the algebra $A^n$ has at most $2^{p(n)}$ subalgebras. In [BIM+10], it is proved that such algebras are characterized by having an edge term and that they generate congruence modular varieties.

The following theorem is the main result of the present note.

**Theorem 1.** Let $A$ be a finite algebra of finite type with few subpowers. Then the following are equivalent:

1. $A$ has infinitely many polynomially inequivalent congruence preserving expansions.
2. The interval $C := \{C \mid C$ is a clone with $\text{Pol}(A) \subseteq C \subseteq \text{Comp}(A)\}$ in the lattice of all clones on $A$ is infinite.
3. There exists a clone $C$ with $\text{Pol}(A) \subseteq C \subseteq \text{Comp}(A)$ that is not finitely generated.

If $A$ is furthermore isomorphic to a direct product $A_1 \times \cdots \times A_n$ of nilpotent algebras of prime power order, and if for all $i,j \in \{1,\ldots,n\}$ with $i \neq j$, we have $\gcd(|A_i|,|A_j|) = 1$, then the three conditions (1)–(3) are equivalent to

4. There is $i \in \{1,\ldots,n\}$ such that the congruence lattice of $A_i$ splits strongly.
5. The congruence lattice of $A$ splits strongly.

The condition on the congruence lattice in items (4) and (5) has also appeared in Theorem 1.1 of [AM13], which states that a finite modular lattice that splits strongly allows infinitely many different sequences satisfying the properties of higher commutator operations. Theorem 1 provides a description of those nilpotent groups that have infinitely many polynomially inequivalent expansions.

**Corollary 2.** Let $G$ be a finite nilpotent group. Then $G$ has infinitely many polynomially inequivalent congruence preserving expansions if and only if the lattice of normal subgroups of $G$ splits strongly, i.e., the normal subgroup lattice of $G$ is the union of two proper subintervals that have at least one normal subgroup in common.
One can view the finiteness of the interval $\mathbb{I}[\text{Pol}(A), \text{Comp}(A)]$ as a polynomial completeness property [KP01]. For finite abelian $p$-groups, this property is described in the following corollary.

**Corollary 3.** Let $p$ be a prime, let $r \in \mathbb{N}$ and let $m_1, \ldots, m_r \in \mathbb{N}$ with $m_1 \geq \cdots \geq m_r$. Then the abelian group $G := \prod_{i=1}^{r} \mathbb{Z}_{p^{m_i}}$ has finitely many polynomially inequivalent congruence preserving expansions if and only if $(r \geq 2$ and $m_1 = m_2)$ or $(r = 1$ and $m_1 = 1)$.

We compare this to known completeness properties: $G = \prod_{i=1}^{r} \mathbb{Z}_{p^{m_i}}$ with $r \in \mathbb{N}$ and $m_1 \geq \cdots \geq m_r \geq 1$ is affine complete if and only if $r \geq 2$ and $m_1 = m_2$ [Nöb76]. By Corollary 3, $G$ has only finitely many polynomially inequivalent congruence preserving expansions if and only if it is affine complete or simple. Finally by [ALM16, Theorem 1.2], the clone of congruence preserving functions of $G$ is finitely generated if and only if $G$ is affine complete or cyclic.

The proofs of the results stated in this introduction will be given in Section 3. Before, Section 2 provides some auxiliary results about clones and direct products.

### 2. Direct products

We need some information on clones acting on direct products. The results contained in this section are for the most part known, or follow quite immediately from existing theory. In the sequel, a vector $(a_1, \ldots, a_n)$ will sometimes be written as $a$.

**Definition 4.** Let $A, B$ be sets, let $n \in \mathbb{N}$, and let $c : A^n \to A$ and $d : B^n \to B$. Then we define the mapping $c \otimes d : (A \times B)^n \to A \times B$ by

$$c \otimes d \ (\left( a_1, \ldots, a_n \right)) := \left( c(a_1), \ldots, c(a_r), d(b) \right)$$

for $a \in A^n$, $b \in B^n$.

For a clone $C$ on the set $A$, we let $C^{[n]}$ be its $n$-ary part $C \cap A^A$.

**Definition 5.** Let $A, B$ be sets, let $C$ be a clone on $A$, and let $D$ be a clone $B$. We define the set $C \otimes D$, which consists of finitary functions on $A \times B$, by

$$C \otimes D := \{ c \otimes d \mid n \in \mathbb{N}, c \in C^{[n]}, d \in D^{[n]} \}.$$ 

**Lemma 6.** Let $A, B$ be sets, let $C$ be a clone on $A$, and let $D$ be a clone $B$. Then the set $C \otimes D$ is a clone on $A \times B$.

**Proof.** $C \otimes D$ contains all projections. For $f \in (C \otimes D)^{[n]}$ and $g_1, \ldots, g_n \in (C \otimes D)^{[m]}$, straightforward calculations show $f(g_1, \ldots, g_n) \in (C \otimes D)^{[m]}$. □
For a set $X$ of finitary functions on $A$, the clone generated by $X$ is denoted by $\text{Clo}_A(X)$.

**Lemma 7.** Let $A, B$ be sets, let $C$ be a clone on $A$ that is generated by $X \subseteq C$, and let $D$ be a clone on $B$ that is generated by $Y \subseteq D$. Let

$$Z := \{(g_b) \mapsto (f(a)) \mid f \in X\} \cup \{(g_b) \mapsto (a_1 g_b) \mid g \in Y\} \cup \{((a_1 b_1), (a_2 b_2)) \mapsto (a_1 b_2)\}.$$  

Then the clone on $A \times B$ that is generated by $Z$ is equal to $C \otimes D$.

**Proof.** We proceed as in the proof of Proposition 4.1 of [ALM16]. We define $\psi_C : C \otimes D \rightarrow C$ by $\psi_C(c \otimes d) = c$ and $\psi_D : C \otimes D \rightarrow D$ by $\psi_D(c \otimes d) = d$. Adopting the viewpoint of [Mal66], we consider clones as function algebras: the idea of this approach is that a clone on $A$ is a subalgebra of $\bigcup_{n \in \mathbb{N}} A^{4^n}$ equipped with the unary operations $\zeta$ (rotation of the arguments), $\tau$ (swapping the first two arguments), $\Delta$ (taking a minor), $\nabla$ (adding an inessential argument), and one binary operation $\circ$ that composes two functions in a certain way. A detailed account of this point of view is given in [PK79] p. 38. Using this approach, we observe that the mapping $\psi_C$ is an epimorphism from the algebra $(C \otimes D, \text{id}_{A \times B}, \zeta, \tau, \Delta, \nabla, \circ)$ to the algebra $(C, \text{id}_A, \zeta, \tau, \Delta, \nabla, \circ)$. Since $X \subseteq \psi_C(Z)$, $\text{Clo}_A(\psi_C(Z)) = C$. Now a basic property on the interaction of homomorphisms and subalgebra generation [BSS11, Theorem II.6.6] yields $\psi_C(\text{Clo}_{A \times B}(Z)) = C$. Similarly, $\psi_D(\text{Clo}_{A \times B}(Z)) = D$. We are now ready to show

$$\text{Clo}_{A \times B}(Z) = C \otimes D.\tag{2.1}$$

The “$\subseteq$”-inclusion follows from $Z \subseteq C \otimes D$. For “$\supseteq$”, we choose $c \otimes d \in C \otimes D$. Since $\psi_C(\text{Clo}_{A \times B}(Z)) = C$, we find $d' \in D$ such that $c \otimes d' \in \text{Clo}_{A \times B}(Z)$. Similarly, we find $c' \in D$ with $c' \otimes d \in \text{Clo}_{A \times B}(Z)$. If we denote the binary projections by $\pi_1$ and $\pi_2$, we see that the last element listed in the definition of $Z$ is $\pi_1 \otimes \pi_2$. Hence the composition $\pi_1 \otimes \pi_2(c \otimes d', c' \otimes d)$ lies in $\text{Clo}_{A \times B}(Z)$, which implies $(c \otimes d) \in \text{Clo}_{A \times B}(Z)$. \hfill \Box

**Corollary 8.** Let $A, B$ be sets, let $C$ be a clone on $A$, and let $D$ be a clone on $B$. Then $C \otimes D$ is finitely generated if and only if both $C$ and $D$ are finitely generated.

**Proof.** The “if”-direction follows from Lemma 7. For the “only if”-direction, we observe that both function algebras $(C, \text{id}_A, \zeta, \tau, \Delta, \nabla, \circ)$ and $(D, \text{id}_B, \zeta, \tau, \Delta, \nabla, \circ)$ are homomorphic images of the algebra $(C \otimes D, \text{id}_{A \times B}, \zeta, \tau, \Delta, \nabla, \circ)$. \hfill \Box
The polynomial functions on the direct product of two algebras can in general not be determined directly from the polynomial functions on the factors (for finite groups, this phenomenon has been studied in [Sco69]). However, under the additional assumption that the algebras lie in a congruence permutable variety and that all congruences in the direct product are product congruences, a decomposition into the direct factors is possible. Let $A := B \times C$. A congruence $\alpha$ of $A$ is a product congruence if there exist $\beta \in \text{Con}(B)$ and $\gamma \in \text{Con}(C)$ such that

$$\alpha = \{((b_1 c_1), (b_2 c_2)) \mid (b_1, b_2) \in \beta \text{ and } (c_1, c_2) \in \gamma\}.$$

A congruence of $A$ that is not a product congruence is a skew congruence. We say that $A = B \times C$ is a skew-free direct product of $B$ and $C$ if $A$ has no skew congruences.

**Lemma 9** ([KM10]). Let $A$ be an algebra with a Mal'cev term. Suppose that $A = B \times C$ is a skew-free direct product of $B$ and $C$. Then $\text{Pol}(A) = \text{Pol}(B) \otimes \text{Pol}(C)$.

**Proof.** This is essentially Corollary 2 from [KM10]; the claim can also be derived directly from Corollary 6.4 of [AM15].\vspace{12pt}

**Corollary 10.** Let $A$ be an algebra with a Mal'cev term. Suppose that $A = B \times C$ is a skew-free direct product of $B$ and $C$. Then the interval between $\text{Pol}(A)$ and $\text{Comp}(A)$ in the lattice of clones on $A$ is given by

$$\text{(2.2)} \quad \mathbb{I}[\text{Pol}(A), \text{Comp}(A)] = \{E \otimes F \mid E \in \mathbb{I}[\text{Pol}(B), \text{Comp}(B)], F \in \mathbb{I}[\text{Pol}(C), \text{Comp}(C)]\}.$$

**Proof.** For $\subseteq$, let $G$ be a clone with $\text{Pol}(A) \subseteq G \subseteq \text{Comp}(A)$. Then $A' := (A, G)$ has the same congruence lattice as $A$ and is therefore a skew-free direct product of two algebras $B'$ and $C'$. Now we let $E := \text{Pol}(B')$ and $F := \text{Pol}(C')$ and use Lemma 9 to obtain $G = E \otimes F$.

For $\supseteq$, we first observe that $A$ is a skew-free direct product. This implies that every function in $\text{Comp}(B) \otimes \text{Comp}(C)$ is a congruence preserving function on $A$. Now we choose $E \otimes F$ from the right hand side of (2.2). Then clearly $\text{Pol}(A) = \text{Pol}(B) \otimes \text{Pol}(C) \subseteq E \otimes F \subseteq \text{Comp}(B) \otimes \text{Comp}(C) \subseteq \text{Comp}(A)$, and therefore $E \otimes F$ lies in the left hand side of (2.2).\vspace{12pt}

We will now investigate the splitting property of lattices that appears in items (4) and (5) of Theorem 1.

**Lemma 11.** Let $n \in \mathbb{N}$, and let $L_1, \ldots, L_n$ be bounded lattices, and let $L := L_1 \times \cdots \times L_n$. Then $L$ splits strongly if and only if at least one of the lattices $L_i$ splits strongly.
Proof. For the “if”-direction, assume that $L_i$ splits strongly with witnesses $δ_i, ε_i$. Then $K$ splits strongly with $(1, \ldots, 1, δ_i, 1, \ldots, 1)$ and $(0, \ldots, 0, ε_i, 0, \ldots, 0)$ as witnesses ($δ_i$ and $ε_i$ at position $i$).

For the “only if”-direction, we assume that $K$ splits strongly with witnesses $δ = (δ_1, \ldots, δ_n)$ and $ε = (ε_1, \ldots, ε_n)$. Since $δ \neq 1$, there is $i$ such that $δ_i < 1$. Hence $(0, \ldots, 0, 1, 0, \ldots, 0) \not< δ$ (with 1 at place $i$). By the splitting property, we have $(0, \ldots, 0, 1, 0, \ldots, 0) ≥ε$. Thus for $j \neq i$, we have $ε_j = 0$, and therefore, since $ε \neq 0$, we have $ε_i > 0$. Now we show that $L_i$ splits strongly with witnesses $δ_i$ and $ε_i$: since $δ ≥ ε$, we have $δ_i ≥ ε_i$. Now take any $α ∈ L_i$ with $α ≤ δ_i$. Then $(0, \ldots, 0, α, 0, \ldots, 0) ≤ δ$, hence $(0, \ldots, 0, α, 0, \ldots, 0) ≥ ε$, and thus $α ≥ ε_i$. □

A finite algebra is congruence uniform if for every congruence of $A$, all its congruence classes have the same cardinality. For $α, β, γ, δ ∈ \text{Con}(A)$, we define

$$\#(β : α) = |A/α|/|A/β|$$

and write $I[α, β] \not> I[γ, δ]$ (and also $I[γ, δ] \not< I[α, β]$) if $α = β ∧ γ$ and $β ∨ γ = δ$.

Lemma 12. Let $A$ be a finite congruence uniform algebra in a congruence permutable variety, and let $α, β, γ, δ ∈ \text{Con}(A)$. Then we have:

1. If $α ≤ β$, then every $β$-class is the union of $\#(β : α)$ distinct $α$-classes; put differently, for every $a ∈ A$ we have $|\{x/α \mid x ∈ a/β\}| = \#(β : α)$.
2. If $α ≤ β ≤ γ$, then $\#(γ : α) = \#(γ : β) · \#(β : α)$.
3. If $I[α, β] \not> I[γ, δ]$, then $\#(δ : γ) = \#(β : α)$.

Proof. (1) Each $α$-class contains $|A|/|A/α|$ elements, and each $β$-class contains $|A|/|A/β|$ elements. Since every $β$-class is a disjoint union of $α$-classes, we find that every $β$-class must then consist of exactly $|A|/|A/β|/|A/α| = |A/α|/|A/β| = \#(β : α)$ different $α$-classes. Property (2) follows directly from (2.3). For proving (3), we first choose an $a ∈ A$. By item (1), it is sufficient to show that $\{x/α \mid x ∈ a/β\}$ has the same number of elements as $\{x/γ \mid x ∈ a/δ\}$. To this end, we define $f: \{x/α \mid x ∈ a/β\} → \{x/γ \mid x ∈ a/δ\}$ by $f(x/α) = x/γ$. The function $f$ is well-defined because $α ≤ γ$. For injectivity, let $x, y ∈ a/β$ with $x/γ = y/γ$. Then $(x, y) ∈ β ∧ γ = α$, and thus $x/α = y/α$. For surjectivity, we let $y ∈ a/δ$. By congruence permutability, we have $δ = β ∨ γ = γ ∘ β$, and therefore there exists $z ∈ A$ with $(y, z) ∈ γ$ and $(z, a) ∈ β$. Then $f(z/α) = z/γ = y/γ$. Therefore, $f$ is bijective, which establishes (3). □

Lemma 13. Let $A$ be a finite congruence uniform algebra in a congruence permutable variety that is the direct product of two algebras $B$ and $C$ of coprime order. Then this product is skew-free.
Let $\beta$ and $\gamma$ be the projection kernels of $A$ such that $A/\beta \cong B$ and $A/\gamma \cong C$. By [BS81] Lemma IV.11.6, it is sufficient to prove that each congruence $\alpha$ of $A$ satisfies

\[(\alpha \lor \beta) \land (\alpha \lor \gamma) = \alpha.\]

We observe that $\mathbb{I}[\alpha, \alpha \lor \beta] \not\subseteq \mathbb{I}[\alpha \land \beta, \beta]$. Since every congruence permutable variety is congruence modular, we can use the modular law to obtain $\alpha \land \beta = (\alpha \land \beta) \lor 0 = (\alpha \land \beta) \lor (\gamma \land \beta) = ((\alpha \land \beta) \lor \gamma) \land \beta$. Therefore $\mathbb{I}[\alpha \land \beta, \beta] \not\ni \mathbb{I}[(\alpha \land \beta) \lor \gamma, 1_A]$. Applying item (3) of Lemma 13, we obtain $\#(\alpha \lor \beta : \alpha) = \#(1_A : (\alpha \land \beta) \lor \gamma)$, which by item (2) of the same lemma divides $\#(1_A : \gamma) = |A/\gamma|$. Hence, using (2) again, we have $\#((\alpha \lor \beta \land (\alpha \lor \gamma) : \alpha) \mid |A/\gamma|$. Changing the roles of $\beta$ and $\gamma$, we obtain $\#((\alpha \lor \beta \land (\alpha \lor \gamma) : \alpha) \mid |A/\beta|$. Now since $|A/\beta|$ and $|A/\gamma|$ are coprime, we obtain $\#((\alpha \lor \beta \land (\alpha \lor \gamma) : \alpha) = 1$, which implies (2.4).

3. Proof of the main results

Proof of Theorem 1. The items (1) and (2) are equivalent by definition.

(2) $\Rightarrow$ (3): We assume that that the interval $C = \mathbb{I}[\text{Pol}(A), \text{Comp}(A)]$ in the clone lattice is infinite. By [Aic10, Theorem 5.3], the set $(C, \subseteq)$ satisfies the descending chain condition, and therefore, there is $C \in C$ which is minimal such that $\mathbb{I}[\text{Pol}(A), C]$ is infinite. We prove that $C$ is not finitely generated. Seeking a contradiction, assume that $C$ is finitely generated. We call a clone $D$ a sub-cover of $C$ if $D \subseteq C$ and there is no clone $D'$ with $D \subset D' \subseteq C$. Then by [PK79, Charakterisierungssatz 4.1.3(i)$\Rightarrow$(iii)], $C$ has only finitely many subcovers $D_i$, $i \in I$, and for each clone $E$ on $A$ with $E \subseteq C$ there is $i \in I$ with $E \subseteq D_i$. Let $J := \{i \in I \mid \text{Pol}(A) \subseteq D_j\}$. Then $\mathbb{I}[\text{Pol}(A), C] = \{C\} \cup \bigcup_{j \in J} \mathbb{I}[\text{Pol}(A), D_j]$. Hence one interval $\mathbb{I}[\text{Pol}(A), D_j]$ must be infinite, contradicting the minimality of $C$.

(3) $\Rightarrow$ (2): Let $m$ be the maximal arity of the fundamental operations on $A$, and let $C$ be a nonfinitely generated clone with $\text{Pol}(A) \subseteq C \subseteq \text{Comp}(A)$. For $n \geq m$, let $C_n$ be the subclone of $C$ generated by its $n$-ary members. Then $C_m \subseteq C_{m+1} \subseteq \cdots$ and $\bigcup_{n \geq m} C_n = C$. Since $C$ is not finitely generated, we have $C_n \subset C$ for all $n \geq m$, and therefore the set $\{C_n \mid n \in \mathbb{N}, n \geq m\}$ is infinite.

Before proving the equivalence with (1) and (5), we additionally assume that $A$ is isomorphic to $A_1 \times \cdots \times A_n$, and we also assume that for each $i \in \{1, \ldots, n\}$, $A_i$ is nilpotent and $|A_i|$ is a prime power, and that for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we have $\text{gcd}(|A_i|, |A_j|) = 1$. Since $A$ has few subpowers, $A$ generates a
congruence modular variety \([\text{BLM}^{+}10, \text{Theorem } 4.2]\). Thus all the algebras \(A_i\) are nilpotent algebras in a congruence modular variety. Representing the congruence \(1_A\) of \(A\) as the join of the projection kernels and using the join distributivity of the binary commutator \([\text{FM}87, \text{Proposition } 4.3]\) to compute the lower central series of \(A\), we see that then \(A\) is nilpotent, too. Hence by Theorem 6.2 of \([\text{FM}87]\), \(A\) has a Mal’cev term, which we will denote by \(\delta\). By \([\text{BM}87, \text{Corollary } 7.5]\) \(A\) and its homomorphic images \(A_1, \ldots, A_n\) are all congruence uniform. Now Lemma \(13\) implies that for every \(i \in \{1, \ldots, n\}\), \(A\) is a skew-free product \(A_i \times C\) with \(C := \prod_{j \in \{1, \ldots, n\}\setminus\{i\}} A_j\).

\(\mathbf{3} \Rightarrow \mathbf{5}\): We proceed by contraposition. We assume that the congruence lattice \(\text{Con}(A)\) does not split strongly and show that every clone in \(\mathbb{I}[\text{Pol}(A), \text{Comp}(A)]\) is finitely generated. We will need another notion of splitting: we say that a lattice “splits strongly” in that “splitting” does not claim that the subintervals intersect \([\text{AM}13, \text{p. } 861]\). Hence \(\mathbb{L}\) splits if

\[
(3.1) \quad \mathbb{L} \models \exists \delta, \varepsilon \in \mathbb{L} : (0 < \varepsilon \text{ and } \delta < 1 \text{ and } (\forall \alpha \in \mathbb{L} : \alpha \leq \delta \text{ or } \alpha \geq \varepsilon)).
\]

Let \(C\) be a clone with \(\text{Pol}(A) \subseteq C \subseteq \text{Comp}(A)\), and let \(A' := (A, C)\) be the corresponding congruence preserving expansion of \(A\). Since the lattice \(\text{Con}(A)\) does not split strongly, \([\text{AM}13, \text{Corollary } 3.4(2)]\) yields that \(A'\) is isomorphic to a direct product \(B' \times C'_1 \times \ldots \times C'_n\) such that \(\text{Con}(B')\) does not split, \(n \in \mathbb{N}_0\), and each \(C'_i\) is simple. We will now show that each of these direct factors has a finitely generated clone of polynomial functions. Let us first examine \(B'\). The congruence lattice of \(B'\) does not split, hence by Propositions 3.7 and 3.8 of \([\text{ALM}16]\), \(\text{Pol}(B')\) is finitely generated. Examining the factors \(C'_i\), we let \(i \in \{1, \ldots, n\}\) and observe that \(C'_i\) is a simple finite algebra with Mal’cev term. If \(C'_i\) is abelian, \(C'_i\) is polynomially equivalent to a module over a ring. Hence its clone of polynomial functions is generated by its binary members. If \(C'_i\) is nonabelian, then \(\text{Pol}(C'_i)\) consists of all finitary operations on \(C'_i\) (this follows, e. g., from \([\text{HH}82, \text{Corollary } 3.5]\)) and is therefore generated by its binary members by \([\text{Pos}21, \text{p. } 180]\) (cf. \([\text{Sie}45]\)). Since by \([\text{AM}13, \text{Corollary } 3.4]\), the direct product \(B' \times C'_1 \times \ldots \times C'_n\) has no skew congruences, we may use Lemma \(9\) \(n\) times to obtain \(\text{Pol}(B' \times C'_1 \times \ldots \times C'_n) = \text{Pol}(B') \otimes \text{Pol}(C'_1) \otimes \ldots \otimes \text{Pol}(C'_n)\). Now Lemma \(7\) implies that \(\text{Pol}(B' \times C'_1 \times \ldots \times C'_n)\) is finitely generated, and thus also \(\text{Pol}(A')\) is finitely generated. Since \(\text{Pol}(A') = C, C\) is finitely generated.

\(\mathbf{3} \Rightarrow \mathbf{4}\): From Lemma \(13\) we obtain that for each \(i \in \{1, \ldots, n - 1\}\), the direct product \(B \times C\) with \(B := A_i\) and \(C := \prod_{j=i+1}^n A_j\) is skew-free. Hence we obtain
that \( \text{Con}(\mathbb{A}) \) is isomorphic to the lattice \( \bigwedge_{i=1}^n \text{Con}(\mathbb{A}_i) \). Now Lemma (1) yields that there is \( i \in \{1, \ldots, n\} \) such that \( \text{Con}(\mathbb{A}_i) \) splits strongly.

\((1) \Rightarrow (3)\): Let \( i \in \{1, \ldots, n\} \) be such that \( \text{Con}(\mathbb{A}_i) \) splits strongly. The first part of the proof will produce a nonfinitely generated clone \( D \) between \( \text{Pol}(\mathbb{A}_i) \) and \( \text{Comp}(\mathbb{A}_i) \). From \( D \), it will then be easy to produce a nonfinitely generated clone between \( \text{Pol}(\mathbb{A}) \) and \( \text{Comp}(\mathbb{A}) \).

In order to produce such a clone \( D \), we let \( \mathbb{B} := \mathbb{A}_i \) and \( \mathbb{C} := \prod_{j \neq i} \mathbb{A}_j \). Let \( \delta, \varepsilon \in \text{Con}(\mathbb{B}) \) be two congruences witnessing that \( \text{Con}(\mathbb{B}) \) splits strongly as in \((1)\); we may choose them in such a way that \( \varepsilon \) is an atom of \( \text{Con}(\mathbb{B}) \). Let \( (a, b) \in \varepsilon \) with \( a \neq b \), and for every \( n \in \mathbb{N} \), let \( f_n : \mathbb{B}^n \rightarrow \mathbb{B} \) be defined by

\[
\begin{align*}
    f_n(x) &= b \text{ if } x \in (B \setminus (a/\delta))^n, \text{ and } \\
    f_n(x) &= a \text{ if there exists an } i \in \{1, \ldots, n\} \text{ such that } x_i \in a/\delta.
\end{align*}
\]

The function \( f_n \) is congruence preserving; to this end, let \( x, y \in \mathbb{B}^n \) and let \( \alpha \) be a congruence of \( \mathbb{B} \) such that for all \( i \), \( (x_i, y_i) \in \alpha \). If \( \alpha \leq \delta \), then \( f_n(x) = f_n(y) \), and therefore \( (f_n(x), f_n(y)) \in \alpha \). If \( \alpha \lesssim \delta \), then by the splitting property, \( \alpha \preceq \varepsilon \). Since \( (f_n(x), f_n(y)) \in \{(a, a), (a, b), (b, a), (b, b)\} \subseteq \varepsilon \), we obtain \( (f_n(x), f_n(y)) \in \alpha \). Hence \( f_n \) is indeed congruence preserving. Now we define \( D \). To this end, let \( \mathbb{B}' \) be the expansion of \( \mathbb{B} \) with the operations \( \{f_i \mid i \in \mathbb{N}\} \), and let \( D := \text{Pol}(\mathbb{B}') \).

Our goal is to show that \( D \) is not finitely generated. To this end, we first show that

\[
(3.2) \quad \text{B'} \text{ is nilpotent.}
\]

For this purpose, we show that \( \mathbb{B}'/\varepsilon \) is nilpotent, and that \( \varepsilon \) is central in \( \mathbb{B}' \).

For the first claim, we observe that \( \mathbb{B}'/\varepsilon \) is an expansion of \( \mathbb{B}/\varepsilon \) with constant operations because all \( f_i \) have their range contained in one single \( \varepsilon \)-class and are therefore constant modulo \( \varepsilon \). This implies \( \text{Pol}(\mathbb{B}'/\varepsilon) = \text{Pol}(\mathbb{B}/\varepsilon) \). Since \( \mathbb{B}/\varepsilon \) is nilpotent, then so is \( \mathbb{B}'/\varepsilon \).

For proving the centrality of \( \varepsilon \), we use the relational description of centrality given in [AM07, Proposition 2.3 and Lemma 2.4], which goes back to Theorem 3.2(iii) of [Kis92]. From these results, we see that \( \varepsilon \) is central in \( \mathbb{B}' \) if and only if all fundamental operations of \( \mathbb{B}' \) preserve the relation

\[
\rho := \{(x_1, x_2, x_3, x_4) \in B^4 \mid (x_1, x_2) \in \varepsilon, d(x_1, x_2, x_3) = x_4\},
\]

where \( d \) is the Mal’cev term of \( \mathbb{B} \) that we produced before proving the implication \((3) \Rightarrow (5)\). We will first show that all \( f_n \) preserve \( \rho \). To this end, let \( n \in \mathbb{N} \) and let \( \left((x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)}) \mid i \in \{1, \ldots, n\}\right) \in \rho^n \), and for \( i \in \{1, \ldots, 4\} \), set
$y_i := f(x^{(1)}_i, \ldots, x^{(n)}_i)$. We have to show $(y_1, y_2, y_3, y_4) \in \rho$. Since $f$ is congruence preserving, we have $(y_1, y_2) \in \varepsilon$. The second property that we have to show is $d(y_1, y_2, y_3) = y_4$. We first observe that for all $i \in \{1, \ldots, n\}$, we have $(x^{(i)}_1, x^{(i)}_2) \in \varepsilon$ and therefore, since $\varepsilon \leq \delta$, also $(x^{(i)}_1, x^{(i)}_2) \in \delta$. Thus $f_n(x^{(1)}_1, \ldots, x^{(n)}_n) = f_n(x^{(1)}_2, \ldots, x^{(n)}_n)$. Hence $y_1 = y_2$ and therefore

$$d(y_1, y_2, y_3) = y_3.$$  

Since for each $i$, $x^{(i)}_3 = d(x^{(i)}_2, x^{(i)}_2, x^{(i)}_3) \equiv \delta d(x^{(i)}_1, x^{(i)}_2, x^{(i)}_3)$, and since $f_n$ is constant on $\delta$-classes, we have $y_3 = f_n(x^{(1)}_3, \ldots, x^{(n)}_3) = f_n(\langle d(x^{(i)}_1, x^{(i)}_2, x^{(i)}_3) | i \in \{1, \ldots, n\} \rangle) = f_n(x^{(1)}_4, \ldots, x^{(n)}_4) = y_4$. Therefore, $f_n$ preserves $\rho$. In $B$, the commutator $[\varepsilon, 1]$ is $0_B$ because $B$ is nilpotent and $\varepsilon$ is a minimal congruence of $B$. Therefore, the relational description of centrality implies that every fundamental operation of $B$ preserves $\rho$. Hence every fundamental operation of $B'$ preserves $\rho$; this implies that $\varepsilon$ is central in $B'$. Since $B'/\varepsilon$ nilpotent and $\varepsilon$ is central, $B'$ is nilpotent, which concludes the proof of (3.2).

Now suppose that $D$ is finitely generated by some finite subset $X$ of $D$. Then the algebra $B'' := (B, X)$ satisfies $\text{Pol}(B'') = D = \text{Pol}(B')$. Therefore, $B''$ is nilpotent, of finite type and of prime power order. Hence, using [Kea99 Theorem 3.14(3)⇒(4)], we obtain that there is a $k \in \mathbb{N}$ such that every commutator term of $B''$ is of rank at most $k$. However,

$$w(x_1, \ldots, x_{k+2}) := d(f_{k+1}(d(x_1, x_{k+2}, a), d(x_2, x_{k+2}, a), \ldots d(x_{k+1}, x_{k+2}, a)), a, x_{k+2})$$

lies in $\text{Pol}(B') = D = \text{Clo}_B(X) = \text{Clo}(B'')$. Since $w(z, x_2, \ldots, x_k, x_{k+1}, z) = \cdots = w(x_1, x_2, \ldots, x_k, z, z) = z$ for all $(x, z) \in B^{k+2}$, $w$ is a commutator term of $B''$. Let $c \not\in a/\delta$. Then $w(c, \ldots, c, a) = d(b, a, a) = b$, and therefore $w$ is not the projection to the last component. Hence $w$ is a nontrivial commutator term of rank $k + 1$ in the sense of [Kea99]. This contradiction proves that $D$ is not finitely generated.

From this clone $D$ on $A_i$, we will now produce a clone $E$ on $A_i$. In order to do this, we let $E$ be the clone $D \otimes \text{Pol}(C)$ on $A_i$. Since $A_i$ is a skew-free product, Lemma 9 implies $\text{Pol}(A) = \text{Pol}(B) \otimes \text{Pol}(C) \subseteq D \otimes \text{Pol}(C) \subseteq \text{Comp}(B) \otimes \text{Comp}(C)$. Since $A_i$ is skew-free, $\text{Comp}(B) \otimes \text{Comp}(C) = \text{Comp}(B \times C) = \text{Comp}(A)$. Hence $E$ is a clone in the interval $[\text{Pol}(A), \text{Comp}(A)]$. Since $D$ is not finitely generated, Corollary 8 implies that $E$ is not finitely generated.
Proof of Corollary 2. Since a finite nilpotent group is the direct product of its Sylow-subgroups, the result is an instance of the equivalence (1)⇔(5) from Theorem 1. □

Proof of Corollary 3. From Theorem 1 of [BC05], we know that the subgroup lattice splits (as defined in (3.1)) iff $r = 1$ or $(r \geq 2$ and $m_1 > m_2)$.

For the “if”-direction, let us assume that $(r \geq 2$ and $m_1 = m_2)$ or $(r = 1$ and $m_1 = 1)$. In the case that $r \geq 2$ and $m_1 = m_2$, the description above tells that $S$ does not split strongly. In the case $r = 1$ and $m_1 = 1$, $S$ is a two element chain, which does not split strongly, either. Hence the implication (1)⇒(5) of Theorem 1 yields the result.

For the “only if”-direction, we assume that $G$ has finitely many polynomially inequivalent expansions. We use Theorem 1 and obtain that $S$ does not split strongly. In the case $r = 1$ we obtain that $S$ is a chain with $m_1+1$ elements. Since $S$ does not split strongly, we then must have $m_1 = 1$. We now consider the case $r \geq 2$. Seeking a contradiction, we assume $m_1 > m_2$. By [BC05], $S$ then splits. Lemma 2.1 from [AM13] describes lattices that do split, but not strongly. This lemma yields that $S$ is isomorphic to a direct product $M \times L$ of two lattices such that $M$ does not split, and $L$ is a Boolean lattice. Since $S$ splits and $M$ does not split, we have $|L| > 1$. Also $|M| > 1$: if $M$ has one element, then $S$ is Boolean. But since $r \geq 2$, $G$ has a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, and the subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ form a nondistributive lattice, contradicting that $S$ is Boolean. From the lattice isomorphism $\gamma : S \to M \times L$ we obtain that $G$ is isomorphic to the direct product $H \times K$ of its two non-trivial groups $H = \gamma^{-1}(1_M,0_L)$ and $K = \gamma^{-1}(0_M,1_L)$ and that $H \times K$ is a skew-free product of $H$ and $K$, meaning that for every subgroup $I$ of $H \times K$, we have

\begin{equation}
I = (I \cap (H \times \{0\})) + (I \cap (\{0\} \times K)).
\end{equation}

(3.3)

Taking minimal subgroups $H_1$ of $H$ and $K_1$ of $K$, we see that $H_1 \times K_1$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, and therefore $H_1 \times K_1$ contains $p-1$ skew subgroups $I$ of $H \times K$ that do not satisfy (3.3). This contradicts the fact that $H \times K$ is a skew-free product of $H$ and $K$. Hence the assumption $m_1 > m_2$ leads to a contradiction, proving that $m_1 = m_2$. □

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