Linear and non-linear magneto-convection in couple stress fluid with non-classical heat conduction law

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Abstract. A theoretical examination of the thermal convection for a couple stress fluid which is electrically conductive and possessing significant thermal relaxation time with an externally applied magnetic field is carried out. Fourier’s law fails when fluids are subjected to rapid heating or when it is confined and in the case of nano-devices. A frame invariant constitutive equation for heat flux is considered. The linear analysis is carried out implementing a normal mode solution and the non-linear stability of the system is analyzed using a double Fourier series. The analysis of the transfer of heat is determined in terms of the Nusselt number.

1. Introduction
The constitutive law by Fourier explains the phenomena of transfer of heat in various real life scenarios and situations. However, it violates the principle of causality. This is because Fourier’s law along with the energy equation gives a parabolic profile for the temperature field implying that a disturbance in the temperature field is instantaneously noticed everywhere. This, also known as the “paradox of heat conduction”, was pointed out by Maxwell [1]. A thermal disturbance will propagate through a medium with finite speed as it is transferred by molecular interactions (Joseph & Preziosi [2]). A modification of Fourier’s law was put forth by Cattaneo [3]. Cattaneo added an ephemeral term to Fourier’s law to account for the relaxation time. This model was modified and improved by Christov [4] by incorporating an upper convected derivative of Oldroyd’s type in lieu of the Maxwell-Cattaneo’s model time derivative. Stranges et al. [5] examined natural convection of non-Fourier fluids. The neutral stability curve was found to have a Fourier branch and an oscillatory branch for small relaxation time. Khayat et al. [6] reviewed non-Fourier heat transfer for liquid and gases, and examined conditions and configurations of flow under which non-Fourier effects are pronounced. Fluids with perceptible thermal relaxation time were examined for its thermal stability by Stranges et al. [7]. A dynamical system of low order was employed to derive the convection state and the Nusselt deduced was dependent on Rayleigh number as well as Cattaneo number.

Classical continuum theory analyzes the mechanical behaviour of materials as a continuous manifold. This theory considers all material bodies to have continuous mass densities. The laws of motion and the axiom of constitutions are assumed to be valid for every part of the body irrespective of its size. Macroscopic continuum descriptions of matter overlook its
intrinsic and fine structure. Eringen [8] pointed out that the molecular constituents of the body are individually stimulated with an external impact and this excitation should be taken into account to study the impact of an external physical effect on the body. Stokes [9] introduced couple stresses and rotational motion of the continuum and proposed the equations for couple stress theory which generalized the classical theory. The practical applications of fluids which are non-Newtonian in nature have motivated researchers to study thermal instability of such fluids. Hayat et al. [10] considered a three dimensional flow and examined the effects of magnetohydrodynamics under the influence of heat generation in a couple stress fluid. They found that the couple stresses in fluid enhanced the temperature distribution and increased the thickness of the thermal boundary layer. Keshri et al. [11] studied nonlinear heat transfer of an electrically conducting couple stress liquid under a magnetic field modulation with an internal heat source and found that couple stresses increase the heat transfer in the system. Kumar et al. [12] analyzed convective instability of a doubly diffusive incompressible couple stress fluid layer with Coriolis force.

Magneto-convection is an essential area of study in astrophysics. It poses an engaging challenge in applied mathematics and serves as a prototype to understand double-diffusive behaviour in fluid dynamics, in the field of oceanography and laboratory experiments. Stars like the Sun have deep magnetically active outer convection zones. The interactions between convection, rotation and magnetic fields in the interiors of stars result in hydromagnetic dynamo action which maintains their magnetic field, just as a dynamo secures the geomagnetic field in the Earth’s liquid core. Kumar et al. [13] explored the consequences of magnetic field and rotation on the onset of two-component convection in a porous medium filled dust like particles in suspension. The existence of overstability was discussed by obtaining a sufficient condition. Yu et al. [14] numerically analyzed Rayleigh-Bénard flows in liquid Gallium subjected to magnetic fields in different directions and strengths. The magnetic field was found to enhance the critical Rayleigh number.

In this paper, a linear and nonlinear analysis is implemented to understand the influence of non-dimensional parameters that governs the system and their influence on heat transfer. The importance of study of convection for small time scale is pointed out by Straughan & Franchi [15]. Stranges et al. [7] investigated the non-linear characteristics brought in by non-Fourier effects on Newtonian fluids. Analytical study of non-Fourier effects in a non-Newtonian fluid medium has not received much attention. A study of this kind shed light on important applications such as the manufacture of electrically-conducting non-Newtonian liquid, electro-conductive polymers and nano-devices and has not appeared in the literature thus far.

2. Mathematical Formulation

Two horizontal plates separated by a small distance, \(d\), and couple stress fluid confined between the plates in a coordinate system which is Cartesian, as depicted in Fig. 1, are taken into account. The origin is assumed to be at the lower plate and the vertically upward direction is taken to be the \(z\)-axis. A uniform magnetic field is applied to the system along the positive \(z\)-axis and the gravitational force, \(\vec{g}\), is acting on the system vertically downwards.

The governing equations are given by (Siddheshwar & Pranesh [16] and Stranges et al. [7])

\[
\nabla \cdot \vec{q} = 0, \\
\rho \left[ \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right] = -\nabla P + \rho \vec{g}(t) + \mu \nabla^2 \vec{q} - \mu' \nabla^4 \vec{q} + \mu_m (\vec{H} \cdot \nabla) \vec{H}, \\
\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla) T = -\nabla \cdot \vec{Q}, \\
\tau \left[ \frac{\partial \vec{Q}}{\partial t} + \vec{q} \cdot \nabla \vec{Q} - \vec{Q} \cdot \nabla \vec{q} - \nabla \nabla \cdot \vec{Q} \right] = -\vec{Q} - \kappa \nabla T,
\]
\[
\rho = \rho_0 \left[ 1 - \gamma (T - T_0) \right],
\]
\[
\nabla \cdot \vec{H} = 0,
\]
\[
\frac{\partial \vec{H}}{\partial t} + (\vec{q} \cdot \nabla) \vec{H} = (\vec{H} \cdot \nabla) \vec{q} + \nu_m \nabla^2 \vec{H}.
\]

where \( \vec{q} \) defines the velocity, \( T \) the temperature, \( \Delta T \) the temperature difference, \( T_0 \) the reference temperature, \( \rho \) the density, \( \rho_0 \) the density at \( T = T_0 \), \( P \) the hydromagnetic pressure, \( \vec{Q} \) the heat flux vector, \( \mu' \) the couple stress viscosity, \( \mu \) the dynamic viscosity, \( \tau \) the relaxation time. The thermal conductivity is represented by \( \kappa \), the coefficient of thermal expansion by \( \gamma \), the magnetic field intensity by \( \vec{H} \), the magnetic permeability by \( \mu_m \) and the magnetic viscosity by \( \nu_m \).

### 2.1. Basic state

The fluid layer in its basic state represents a state of motionlessness and hence is represented by

\[
\vec{q}_b(z) = \vec{0}, \quad \vec{H}_b = H_0 \hat{k}, \quad \rho = \rho_b(z), \quad \vec{Q} = \vec{Q}_b(z), \quad P = P_b(z), \quad T = T_b(z).
\]

Substitution of Eq. (8) in Eqs. (1)-(7) results in the following equations for pressure \( P_b \), density \( \rho_b \), heat flux \( \vec{Q}_b \) and temperature \( T_b \):

\[
\frac{\partial P_b}{\partial z} = -\rho_0 g_0, \quad \frac{d^2 T_b}{dz^2} = 0, \quad \rho_b = \rho_0 [1 - \gamma (T_b - T_0)].
\]

### 3. Linear stability analysis

In order to perform linear stability, the basic state is superposed with a thermal perturbation given by \( \vec{q}', \rho', P', T', \vec{Q}' \), which is assumed to be infinitesimal. Hence

\[
\vec{q} = \vec{q}_b + \vec{q}', \quad T = T_b + T', \quad P = P_b + P', \quad \rho = \rho_b + \rho', \quad \vec{Q} = \vec{Q}_b + \vec{Q}', \quad \vec{H} = \vec{H}_b + \vec{H}'.
\]

Substitution of Eq. (10) in Eqs. (1)-(7) and using equations in (8) yield the equations that govern the infinitesimal perturbation state as

\[
\nabla \cdot \vec{q}' = 0,
\]
\[
\rho_0 \left( \frac{\partial \vec{q}'}{\partial t} + (\vec{q} \cdot \nabla) \vec{q}' \right) = -\nabla P' + \mu \nabla^2 \vec{q}' - \rho' g_0 \hat{k} + \mu_m H_0 \frac{\partial \vec{H}'}{\partial z} + \mu_m (\vec{H}' \cdot \nabla) \vec{H}'.
\]
of Eq. (22) in Eqs. (19), (20) and (21) yields

\[ \frac{\partial T'}{\partial t} + (q' \cdot \nabla)T' - w' \frac{\partial T_b}{\partial z} = -\nabla \cdot \vec{Q'}, \tag{13} \]

\[ \tau \left( \frac{\partial \vec{Q'}}{\partial t} - \frac{\kappa}{d} \Delta T \frac{\partial W'}{\partial z} \nabla \vec{Q'} - \vec{q} \nabla \vec{Q'} - \nabla \cdot \vec{q} \right) = -\vec{Q}' - \kappa \nabla T', \tag{14} \]

\[ \rho' = -\rho_0 \gamma T', \tag{15} \]

\[ \nabla \vec{H}' = 0, \tag{16} \]

\[ \frac{\partial \vec{H}'}{\partial t} + (q' \cdot \nabla)\vec{H}' = (\vec{H}', \nabla)q' + H_0 \frac{\partial w'}{\partial z} + \nu_m \nabla^2 \vec{H}'. \tag{17} \]

Curl is operated on Eq. (12) twice to exclude the pressure term and \( \vec{Q}' \) is eliminated between Eqs. (14) and (15). The resulting equations are linearized and then non-dimensionalized using:

\[ (x^*, y^*, z^*) = \left( \frac{x}{d}, \frac{y}{d}, \frac{z}{d} \right); t^* = \frac{t}{(\frac{d^2}{\kappa})}; q^* = \frac{q}{(\frac{d}{\kappa})}; T^* = \frac{T'}{\Delta T}; W^* = \frac{W'}{(\frac{\kappa}{d})}; H^* = \frac{\vec{H}'}{H_0}, \tag{18} \]

to obtain, ignoring the asterisks:

\[ \frac{1}{Pr} \frac{\partial (\nabla^2 W)}{\partial t} = \nabla^4 W - C \nabla^6 W + R \nabla^2 T + \frac{Q}{Pr} \frac{\nabla^2}{\partial z} \vec{H}, \tag{19} \]

\[ (1 + M \frac{\partial}{\partial t}) \frac{\partial T}{\partial t} = -M \nabla^2 W + \nabla^2 T + \left( 1 + M \frac{\partial}{\partial t} \right) W; \tag{20} \]

\[ \frac{\partial \vec{H}}{\partial t} = \frac{\partial W}{\partial z} + \frac{Pr}{Pm} \nabla^2 \vec{H}. \tag{21} \]

where

\[ M = \frac{\tau \kappa}{d^2}, \text{ is the Cattaneo number; } C = \frac{\mu'}{\mu d^2}, \text{ is the couple stress parameter; } Q = \frac{\mu_m H_0^2 d^2}{\nu_m \mu}, \text{ is} \]

the Chandrasekhar number; \( Pr = \frac{\mu}{\rho_0 d k} \), is the Prandtl number; \( R = \frac{\rho_0 \gamma g d^3 \Delta T}{\mu \kappa}, \text{ is the Rayleigh number; } Pm = -\frac{\mu}{\rho_0 \nu_m}, \text{ is} \]

the magnetic Prandtl number; \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \nabla^2 = \nabla^2 + \frac{\partial^2}{\partial z^2}. \]

The perturbed quantities are taken to be dependent on time and space. Analyzing the disturbances into normal modes gives

\[ \begin{bmatrix} W \\ T \\ H \end{bmatrix} = \begin{bmatrix} W(z) \\ T(z) \\ H(z) \end{bmatrix} \exp i(lx + my) + \sigma t \tag{22} \]

where \( l \) is the wave number along the \( x \)-direction and \( m \) is that along the \( y \) direction. Substitution of Eq. (22) in Eqs. (19), (20) and (21) yields

\[ a^2 RT + (D^2 - a^2)^2 W - C(D^2 - a^2)^3 W - Q \frac{Pr}{Pm} (D^2 - a^2) DH - \frac{\sigma}{Pr} (D^2 - a^2) W = 0, \tag{23} \]

\[ \sigma (1 + M \sigma) T - (1 + M \sigma) W + M(D^2 - a^2) W - (D^2 - a^2) T = 0, \tag{24} \]

\[ \sigma H - DW - \frac{Pr}{Pm} (D^2 - a^2) H = 0. \tag{25} \]
Eqs. (23), (24) and (25) are solved for the boundaries which are stress-free and isothermal. Couple-stresses are assumed to be vanishing at the boundaries. Hence at the lower plate $z = 0$ and at the upper plate $z = 1$, the conditions are given by

$$W = \frac{\partial^2 W}{\partial z^2} = 0, \; T = 0, \; \frac{\partial H}{\partial z} = 0.$$  

(26)

Eliminating $H$ and $T$ from Eqs. (23), (24) and (25), and letting $W = \sin(\pi z)$ gives

$$R = \left(\frac{k_1^2\eta + \frac{\sigma}{Pr_k^2}}{Pr^2}\left(k_1^4 + \sigma + M \sigma^2\right)\right) + \frac{Q}{Pm^2}\pi^2k_1^4\left(k_1^2 + \sigma + M \sigma^2\right)$$

(27)

where $\eta = 1 + C k_1^2, \; k_1^2 = \pi^2 + a^2$.

### 3.1. Stationary convection

Setting $\sigma = 0$ in Eq. (27) yields

$$R_s = \frac{k_1^6 \eta}{a^2(1 + Mk_1^2)} + \frac{Q \pi^2 k_1^2}{a^2(1 + Mk_1^2)}$$

(28)

**Limiting cases:**

When $C = 0, \; M = 0$ and $Q = 0$ in Eq. (28), it reduces to $R = \frac{k_1^6}{a^2}$, which is the critical Rayleigh number for a Newtonian fluid (Chandrasekhar [17]).

When $M = 0$ and $Q = 0$ in Eq. (28), it reduces to $R = \frac{k_1^6(1 + Ck_1^2)}{a^2}$, which is the critical Rayleigh number obtained by Siddheshwar & Pranesh [16].

$M = 0$ in Eq. (28) results in $R = \frac{k_1^6(1 + Ck_1^2)}{a^2} + \frac{Q \pi^2 k_1^2}{a^2}$, the critical Rayleigh number obtained by Pranesh & George [18].

$C = 0$ in Eq. (28) gives $R = \frac{k_1^6}{a^2(1 + Mk^2)} + \frac{Q \pi^2 k_1^2}{a^2(1 + Mk_1^2)}$, the critical Rayleigh number obtained by Pranesh & Kiran [19].

### 3.2. Oscillatiry convection

Setting $\sigma = i\omega$ in Eq. (27) yields

$$R_o = N_1 + i\omega N_2$$

(29)

where

$$N_1 = Q Pr \pi^2 k_1^2 \left[\frac{\lambda(k_1^2 Pr \beta - \omega^2 M) + \omega^2(\beta + \frac{M k_1^2 Pr}{Pm})}{a^2(\beta^2 + \omega^2 M^2)\left(k_1^4 Pr^2 + \omega^2\right)}\right] + \frac{\beta(k_1^2 \eta \lambda - \omega^2 k_1^2 Pr)}{a^2(\beta^2 + \omega^2 M^2)},$$

(30)

$$N_2 = Q Pr \pi^2 k_1^2 \left[\frac{k_1^2 Pr \beta - \omega^2 M - \lambda \beta - \frac{\lambda M k_1^2 Pr}{Pm}}{a^2(\beta^2 + \omega^2 M^2)\left(k_1^4 Pr^2 + \omega^2\right)}\right] + \frac{\beta\left(k_1^2 \lambda Pr + k_1^2 \eta \lambda - \omega^2 k_1^2 Pr\right)}{a^2(\beta^2 + \omega^2 M^2)}.$$
where $\beta = 1 + Mk_1^2$ and $\lambda = k_1^2 - \omega^2M$.

$N_2 = 0$ in Eq. (29) gives a relation for dispersion of the form

$$B_2(\omega^2)^2 + B_1\omega^2 + B_0 = 0.$$  \hspace{1cm} (32)

where

$$B_2 = k_1^2\left(\frac{1}{Pr}(M - 1) + M^2\eta\right),$$  \hspace{1cm} (33)

$$B_1 = k_1^2\left[\frac{Mk_1^4Pr}{Pm^2} (1 + Pr\eta) - Q\pi^2MPr(1 - \frac{k_1^2Pr}{Pm}) + \beta\left(\frac{k_1^2}{Pr} + \eta - \frac{M\omega^2k_1^4Pr}{Pm}\right) - Mk_1^2\eta\right].$$  \hspace{1cm} (34)

$$B_0 = k_1^2\left[\frac{\beta k_1^4Pr}{Pm^2}(k_1^2 + \eta Pr) + \frac{QPr^2\pi^2k_1^2}{Pm}\left(\beta - Mk_1^2\right) - \frac{Mk_1^6Pr^2\eta}{Pm^2} - QPr^2k_1^2\beta\right].$$  \hspace{1cm} (35)

Eq. (32) results in no positive values of $\omega^2$. Hence the possibility of the occurrence of oscillatory convection does not exist in the case of couple stress fluid.

Elimination of $H$ between Eqs. (23) and (25) after putting $\sigma = 0$ in Eqs. (23), (24) and (25) gives

$$-a^2RT + [(D^2 - a^2)^2 - C(D^2 - a^2)^3 - QD^2]W = 0, \hspace{1cm} (36)$$

$$(D^2 - a^2)T + [1 - M((D^2 - a^2))]W = 0. \hspace{1cm} (37)$$

Thus an eigenvalue problem is formed with Eqs. (36) and (37) together with boundary conditions given by Eq. (26) in which eigenvalue is the thermal Rayleigh number $R$. Consequently $W$ and $T$ could be expressed as

$$W(z) = \sum_{i=1}^{n} p_i W_i(z), \hspace{1cm} T(z) = \sum_{i=1}^{n} q_i T_i(z) \hspace{1cm} (38)$$

where $p_i$ and $q_i$ are coefficients that are to be evaluated. The choice of $W_i(z)$ and $T_i(z)$ in Eq. (38) are made so that the corresponding boundary conditions are satisfied. Substituting Eq. (38) for $i = 1$ in Eqs. (36) and (37) and multiplying Eq. (36) by $W$ and Eq. (37) by $T$ and carrying out integration by parts with respect to $z$ between $z = 0$ and $z = 1$ and the resulting equation when solved for $R$ yields

$$R = \frac{\langle W(D^2 - a^2)^2W \rangle - C\langle W(D^2 - a^2)^3W \rangle - Q\langle WD^2W \rangle \langle T(D^2 - a^2)T \rangle}{a^2\langle WT \rangle [M\langle T(D^2 - a^2)W \rangle - \langle WT \rangle]} \hspace{1cm} (39)$$

where $\langle gh \rangle = \int_{0}^{1} gh \ dz$.

The trial functions satisfying the boundary conditions in Eq. (26) are

$$W = z^6 - 3z^5 + 5z^3 - 3z \hspace{1cm} (40)$$

$$T = z(1 - z) \hspace{1cm} (41)$$

Substitution of Eqs. (40) and (41) in Eq. (39) and on integration yields

$$R = \frac{X_1 - CX_2 - Qc \frac{691}{154} \left[ \frac{1}{3} - \frac{a^2}{30} \right]}{a^2X_3} \hspace{1cm} (42)$$
where
\[ X_1 = \frac{310}{7} + \frac{691}{77} a^2 - \frac{5461}{12012} a^4; \quad X_2 = -\frac{3060}{7} - \frac{930}{7} a^2 - \frac{2073}{154} a^4 - \frac{5461}{12012} a^6; \]
\[ X_3 = M \left( \frac{17}{14} + \frac{31}{252} a^2 \right) + \frac{31}{252}. \]

4. Non-linear analysis
This study restricts itself to two-dimensional disturbances. Hence the stream function \( \Psi \) is taken to be \((u', w') = \left( \frac{\partial \Psi}{\partial z}, -\frac{\partial \Psi}{\partial x} \right)\) and magnetic potential \( \Phi \) as \((H_x', H_z') = \left( \frac{\partial \Phi}{\partial z}, -\frac{\partial \Phi}{\partial x} \right)\).

The pressure term in Eq. (12) is eliminated and \( \vec{Q}' \) is eliminated between Eqs. (13) and (14) by operating divergence on Eq. (14). These equations are then non-dimensionalized using Eq. (18) and \( \Psi^* = \frac{\Psi}{\kappa}; \quad \Phi^* = \frac{\Phi}{dH_0}; \) (43) to obtain, ignoring the asterisks,
\[
\frac{1}{Pr} \frac{\partial}{\partial t} \left( \nabla^2 \Psi \right) - \frac{1}{Pr} J \left( \nabla^2 \Psi, \Psi \right) = \nabla^4 \Psi - RT_x + QPm \frac{\partial}{\partial t} \left( \nabla^2 \Phi \right) - QPm J \left( \nabla^2 \Phi, \Phi \right),
\]
(44)
\[
M \left[ \frac{\partial^2 T}{\partial t^2} - 2J \left( \Psi, T_t \right) - J \left( \Psi_t, T \right) + \Psi_x J \left( \Psi, \Psi_x \right) - \Psi_x J \left( \Psi_x, T \right) + \Psi_x J \left( \Psi_x, T \right) + \Psi_x J \left( \Psi_x, T \right) \right] + T_t + \Psi_x - J \left( \Psi, T \right) = \nabla^2 T,
\]
(45)
\[
\Phi_t - J \left( \Psi, \Phi \right) = \Psi_z + Pm \nabla^2 \Phi \quad (46)
\]
The boundaries considered are same as that for linear analysis. Hence
\[
\Psi = D^2 \Psi = 0, \quad D \Phi = 0, \quad T = 0 \text{ at } z = 0 \text{ and } z = 1.
\]
(47)
Eqs. (44), (45) and (46) are solved for the conditions given by Eq. (47).

4.1. A weak nonlinear theory
In order to carry out the nonlinear analysis, the finite amplitude free convection is described by a minimal Fourier series. The series are hence taken to be
\[
\Psi = a(t) \sin(\pi \alpha x) \sin(\pi z)
\]
(48)
\[
\Phi = f(t) \sin(\pi \alpha x) \cos(\pi z) + g(t) \sin(2\pi \alpha x)
\]
(49)
\[
T = h(t) \cos(\pi \alpha x) \sin(\pi z) + e(t) \sin(2\pi z)
\]
(50)
where \( a(t), e(t), f(t), g(t) \) and \( h(t) \) represent the amplitudes. The dynamics of the system yield the amplitudes.

Substitution of Eqs. (48), (49) and (50) in Eqs. (44), (45) and (46) results in
\[
\dot{a}(t) = -Pr k_t^2 \eta a(t) - \pi \alpha R \frac{Pr}{k_t^2} h(t) - \pi Q \frac{Pr^2}{F_m} f(t)
\]
(51)
\[
\dot{h}(t) = x(t)
\]
(52)
\begin{equation}
\dot{x}(t) = \pi \alpha Pr k_1^2 \eta a(t) + \pi^2 \alpha^2 R Pr k_1 h(t) + Q \pi^2 \alpha \frac{Pr^2}{Pm} f(t) - \frac{1}{2M} x(t) - \frac{\pi \alpha}{2M} a(t) - \frac{k_1^2}{2M} h(t) \quad (53)
\end{equation}

\begin{equation}
\dot{c}(t) = y(t) \quad (54)
\end{equation}

\begin{equation}
\dot{y}(t) = 2\pi^2 \alpha a(t)x(t) - \pi^2 \alpha Pr k_1^2 \eta a(t) h(t) - \pi^2 \alpha^2 R \frac{Pr}{k_1^2} h(t)^2 - Q \pi^3 \alpha \frac{Pr^2}{Pm} f(t) h(t) + \pi^3 \alpha^2 a(t) - \frac{1}{M} y(t) + \frac{\pi^2 \alpha}{2M} a(t) h(t) - \frac{4\pi^4}{M} e(t) \quad (55)
\end{equation}

\begin{equation}
\dot{f}(t) = -2\pi^2 \alpha a(t) g(t) + \pi a(t) - k_1^2 \frac{Pr}{Pm} f(t) \quad (56)
\end{equation}

\begin{equation}
\dot{g}(t) = -\frac{\pi^2 \alpha}{2} a(t) f(t) - 4\pi^2 \alpha^2 \frac{Pr}{Pm} g(t) \quad (57)
\end{equation}

The non-linear autonomous system given Eqs. (51)-(57) is the generalized Lorenz model. Also,

\begin{equation}
\frac{\partial \dot{a}}{\partial a} + \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{f}}{\partial f} + \frac{\partial \dot{g}}{\partial g} = -\left( Pr k_1^2 \eta + \frac{1}{2M} + \frac{1}{M} + k_1^2 \frac{Pr}{Pm} + 4\pi^2 \alpha^2 \frac{Pr}{Pm} \right). \quad (58)
\end{equation}

Eq. (58) is always negative. The system is thus inferred to be bounded as well as dissipative.

4.2. Heat transport

The evaluation of the characteristics of the heat transport is crucial in the investigation of convection because of the notable effect Rayleigh number has on the transfer of heat. If \( J \) gives transport of heat per unit area, then

\begin{equation}
J = -\kappa \left\langle \frac{\partial T_{\text{total}}}{\partial z} \right\rangle_{z=0}, \quad (59)
\end{equation}

where

\begin{equation}
T_{\text{total}} = T_0 - \Delta T \frac{z}{d} + T(x, z, t). \quad (60)
\end{equation}

and the angular brackets represents the horizontal average. Eq. (50) substituted in Eq. (60) and the resulting in Eq. (59) to obtain

\begin{equation}
J = \frac{\kappa \Delta T}{d} - \frac{\kappa \Delta T}{d} 2\pi e(t). \quad (61)
\end{equation}

Hence the expression for Nusselt number is derived to be

\begin{equation}
Nu = \frac{J}{\kappa \Delta T/d} = 1 - 2\pi e(t). \quad (62)
\end{equation}

5. Results and discussion

This work analyzed the impact of relaxation time or non-Fourier effects and couple stress in magneto-convection. The analysis is carried out by investigating the impact of various governing parameters of the system. The non-Fourier effect is characterized by Cattaneo number, \( M \), the particles in suspension is quantified by the couple stress parameter, \( C \) and the characteristics brought in by the imposed magnetic field on convection is accounted by Chandrasekhar number, \( Q \).
The thermal diffusion time is the ratio of the square of the distance between the plates to 
the thermal conductivity of the fluid, i.e., $d^2/\kappa$. Therefore Cattaneo number, given by $M = \frac{\tau \kappa}{d^2}$, 
can be considered as the ratio of the thermal relaxation time and the thermal diffusion time. 
The non-Fourier effects are more prominent as the value of $d$ decreases and hence this study is 
significant for nanodevices dealing with heat transport and flow (Stone et al. [20]).

Fig. 2 depicts the characteristics of $M$ on the critical Rayleigh number, $R_c$ for a range of 
values of $C$. As $M$ increases, $R_c$ decreases for the all the plots in Fig. 2. An increase in $M$ 
results in the contraction of the convective cells and reduction in the wave number as $M$ varies 
inversely with square of the characteristic length, $d$. Hence it advances the onset of convection. 
However, there is an increase in $R_c$ as $C$ increases. An explication for this feature of $C$ is given 
in discussion for Fig. 4. In the absence of Cattaneo number and Chandrasekhar number, the 
results agree with that of Siddheshwar & Pranesh [16].

In Fig. 3, a similar characteristics as in the case of Fig. 2 is observed. As $Q$ increases, $R_c$ 
increases. An explanation of characteristics brought in $Q$ is given in the discussion for Fig. 6. 
However each of the three cases depicted, an increase in $M$ results in the lowering of the value 
of $R_c$ as discussed for Fig. 2.

The influence of $C$ on $R_c$ is shown in Figs. 4 and 5. As $C$ increases, $R_c$ increases. $C$ 
corresponds to the amount of suspended particles in the fluid layer. More amount of energy 
is required for the onset convection due to the presence of these particles as they increase the 
viscosity of the fluid. Thus there is an enhancement in the stability of the system. As discussed 
for Fig. 2 an increase in $M$ leads to a lowering of the $R_c$. The results obtained in this study 
agree with the ones obtained by Pranesh & Kiran [19], for the case when $C = 0$.

Fig. 5 shows similar behaviour of graphs as seen in Fig. 4. As $Q$ increases, the value of 
$R_c$ increases. In addition, it is observed that in each of the three cases depicted in Fig. 4, an 
increase in $C$ also results in an increase in $R_c$ as explained for Fig 4.

Fig. 6 shows that an increment in $Q$ produces an advancement in $R_c$. When a fluid layer is 
subjected to an external magnetic field, a current is induced in the system. The combination 
of this current and the magnetic field results in Lorenz force. The direction of Lorenz force is 
opposite to that of the velocity, increasing the viscosity of the fluid layer. Hence $Q$ leads to an 
advancement in the outset of convection, thereby stabilizing the system. However, a lowering 
in the critical value of $R_c$ is seen as the value of $M$ increases, explanation for which is given in 
Fig. 2.

A similar characteristics as in the case of Fig. 6 is observed in Fig. 7. As $Q$ increases, $R_c$ 
increases. In addition to this, in each of the three cases depicted, an increase in $C$ results in an 
increase in $R_c$. An explanation of this behaviour is given in the discussion for Fig. 4. 
Figs. 8, 9, 10, 11 and 12 depict the transient behaviour of the Nuseelt number with respect 
to time, $t$. Nusselt number is derived from the nonlinear system of Eqs. (51)-(57) employing 
the Runge-Kutta-Gill method and by considering the appropriate initial conditions. It can be 
noticed in Figs. 8-12 that the oscillation of the Nusselt number, observed as time progresses 
from $t = 0$, eventually reaches a steady state.

In Fig. 8, an increase in Nusselt number as $M$ increases is observed. As explained for Fig. 2, 
the setting in of convection is advanced with an increase in $M$ and hence raises the transfer of 
heat. Fig. 9 depicts that the oscillations of Nusselt number dampen for couple stress parameter. 
This characteristic is because $C$ defer the outset of convection resulting in an increase in $R_c$ 
as seen in the explanation for Fig. 4. Fig. 10 shows the dampening of Nusselt number as $Q$ 
increases. An increment in $Q$ inhibits the outset of convection as explained for Fig. 6 and hence 
reduces the heat transfer. Fig. 11 shows the characteristics brought in by the Prandtl number, 
$Pr$, on heat transfer. A rise in $Pr$ results in a decrease in the Nusselt number. Hence the heat 
transfer is diminished. Prandtl number varies inversely with the thermal diffusivity and thus
large Prandtl number implies less thermal diffusivity. This reduces the temperature profile as well as the thickness of the thermal boundary layer, resulting in reduced heat transfer. The influence of magnetic Prandtl number, $Pm$, on Nusselt number is seen in Fig. 12. It is seen that the effect of $Pm$ on heat transfer is insignificant.
Figure 8. Change in Nusselt number, $Nu$, with time, $t$, for distinct Cattaneo number.

Figure 9. Change in $Nu$ with $t$ for distinct couple stress parameter.

Figure 10. Change in $Nu$ with $t$ for distinct Chandrasekhar number.

Figure 11. Change in $Nu$ with $t$ for distinct Prandtl number.

Figure 12. Change in $Nu$ with $t$ for distinct magnetic Prandtl number.

Figs. 13–18 depict streamlines for $t = 0.05, 0.07, 0.1, 0.2, 0.3, 0.4$ respectively where $M = 0.01$, $C = 0.3$, $Q = 10$, $Pr = 5$ and $Pm = 15$. In Fig. 13, it can be seen that the magnitude of the streamlines is small for small time $t$. As the time $t$ progress, there is an increase in the magnitude of streamlines which indicate that convection is taking place. Convection is seen to become faster on further increasing of time $t$. However, there is no change in the magnitude of the streamlines after $t = 0.3$. Hence the system is observed to have achieved steady state after $t = 0.3$. Streamlines attain their maximum size beyond $t = 0.3$. 
6. Conclusion
This paper attempts to understand the impact of relaxation time or non-Fourier effects and couple stress in magneto-convection. Cattaneo number is observed to destabilize the system by advancing the occurrence of convection. Hence the transport of heat in the system is enhanced. The impact of couple stress parameter and Chandrasekhar number is to stabilize the system and hence reduces the transport of heat. The Nusselt number is observed to decrease with increasing Prandtl number and hence transport of heat is diminished. The role played of magnetic Prandtl number on heat transfer is found to be insignificant.
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