Numerical Approximation of Stationary Distribution for SPDEs

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Abstract

In this paper, we show that the exponential integrator scheme both in spatial discretization and time discretization for a class of stochastic partial differential equations has a unique stationary distribution whenever the stepsize is sufficiently small, and reveal that the weak limit of the law for the exponential integrator scheme is in fact the counterpart for the stochastic partial differential equation considered.

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1 Introduction

The convergence and the stability of numerical schemes for finite-dimensional stochastic differential equations (SDEs) have been extensively investigated, see, e.g., Kloeden and Platen [16] and Schurz [20]. Nowadays, numerical approximate schemes for stochastic partial differential equations (SPDEs) are also becoming more and more popular. There is extensive literature on strong/weak convergence of approximate solutions for SPDEs. For instance, under a dissipative condition, Caraballo and Kloeden [3] showed the pathwise convergence of finite-dimensional approximations for a class of reaction-diffusion equations. Applying the Malliavin calculus approach, Debussche [6] discussed the error of the Euler scheme applied to an SPDE. Greksch and Kloeden [7] investigated the approximation of parabolic SPDEs through eigenfunction argument. Gyöngy [8], Shardlow [21], and Yoo [23] applied finite differences to approximate the mild solutions of parabolic SPDEs driven by space-time white noise. Hausenblas [10, 11] utilized spatial discretization and time discretization, including implicit Euler, explicit Euler scheme and Crank-Nicholson scheme, to approximate quasilinear evolution equations. Higher order pathwise numerical approximations of SPDEs with
additive noise was considered in [13]. For the Taylor approximations of SPDEs, we refer to the monograph [13].

However, there are few results on the asymptotic behavior of numerical solutions for infinite-dimensional SPDEs although the counterpart for the finite-dimensional case has been extensively studied, see, e.g., Schurz [20]. In our present work, we shall investigate the asymptotic behavior of certain numerical scheme for a class of SPDEs. To begin with, we introduce some notation and thus give the framework of our work. Let \((H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)\) be a real separable Hilbert space. Let \(id : H \to H\) be the identity operator, and denote \((\mathcal{L}(H), \| \cdot \|)\) and \((\mathcal{L}_{HS}(H), \| \cdot \|_{HS})\) by the family of bounded linear operators and Hilbert-Schmidt operators from \(H\) into \(H\), respectively. In this paper, we consider an SPDE on the real separable Hilbert space \((H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)\) in the form

\[
(1.1) \quad dX(t) = \{AX(t) + b(X(t))\}dt + \sigma(X(t))dW(t)
\]

with initial value \(X(0) = x \in H\), where \(W(t)\) is an \(H\)-valued cylindrical \(id_H\)–Wiener process defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions, \(b : H \to H\) is a Lipschitz continuous mapping, \(\sigma(x) := \sigma^0 + \sigma^1(x), x \in H\), such that \(\sigma^0 \in \mathcal{L}(H)\) and \(\sigma^1 : H \to \mathcal{L}_{HS}(H)\).

Throughout the paper we impose the following assumptions:

(H1) \((A, \mathcal{D}(A))\) is a self-adjoint operator on \(H\) generating an immediately compact \(C_0\)-semigroup \(\{e^{tA}\}_{t \geq 0}\) such that \(\|e^{tA}\| \leq e^{-\alpha t}\) for some \(\alpha > 0\). In this case, by [13, Theorem 6.26, p.185] and [15, Theorem 6.29, p.187], \(-A\) has discrete spectrum \(\{\lambda_i\}_{i \geq 1}\) such that \(0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots\) and \(\lim_{i \to \infty} \lambda_i = \infty\) with corresponding eigenbasis \(\{e_i\}_{i \geq 1}\) of \(H\).

(H2) There exist \(\theta_1 \in (0, 1)\) and \(\delta_1 \in (0, \infty)\) such that \(\int_0^t \|(-A)^{\theta_1}e^{sA}\sigma^0\|_{HS}^2 ds \leq \delta_1\) for any \(t > 0\), where \((-A)^{\theta_1} := \sum_{k \geq 1} \lambda_k^{\theta_1} (e_k \otimes e_k)\) denotes the fractional power of the operator \(-A\).

(H3) There exist \(L_1, L_2 > 0\) such that

\[
\|b(x) - b(y)\|_H \leq L_1 \|x - y\|_H \text{ and } \|\sigma^1(x) - \sigma^1(y)\|_{HS} \leq L_2 \|x - y\|_H, \quad x, y \in H.
\]

(H4) There exists \(\gamma \in \mathbb{R}\) such that

\[
2\langle x - y, b(x) - b(y) \rangle_H + \|\sigma^1(x) - \sigma^1(y)\|_{HS}^2 \leq -\gamma \|x - y\|_H^2, \quad x, y \in H.
\]

By [4, Theorem 5.3.1, p.66], we know that (H1)-(H3) imply the existence and the uniqueness of the mild solution to (1.1), i.e., there exists a unique \(H\)-valued adapted process \(X_x(t)\) with the initial value \(x \in H\) such that

\[
(1.2) \quad X_x(t) = e^{tA}x + \int_0^t e^{(t-s)A}b(X_x(s))ds + \int_0^t e^{(t-s)A}\sigma(X_x(s))dW(s).
\]
Remark 1.1. In fact, under (H1), (H3) and \( \int_0^t \| e^{sA} \sigma^0 \|_{HS}^2 \, ds \leq \delta_2 \) for any \( t > 0 \) and some \( \delta_2 > 0 \). (1.1) also admits a unique mild solution on \( H \). While (H2) is just imposed for the later numerical analysis. Let \( \sigma^0 = \text{id}_H \), and \( Ax := \partial_x^2 x \) for \( x \in \mathcal{D}(A) := H^2(0, \pi) \cap H^1_0(0, \pi) \). Then \( A \) is a self-adjoint negative operator and \( A e_k = -k^2 e_k \), \( k \in \mathbb{N} \), where \( e_k(\xi) := (2/\pi)^{1/2} \sin k \xi \), \( \xi \in [0, \pi] \), \( k \in \mathbb{N} \). A simple computation shows that

\[
\int_0^t \| (-A)^{\theta_1} e^{sA} \|_{HS}^2 \, ds = \sum_{k=1}^{\infty} (k^2)^{2\theta_1 - 1} \int_0^t e^{-2k^2 s} \, ds \leq \frac{1}{2} \sum_{k=1}^{\infty} (k^2)^{2\theta_1 - 1}.
\]

Then (H2) holds with \( \delta_1 = \frac{1}{2} \sum_{k=1}^{\infty} (k^2)^{2\theta_1 - 1} \) for \( \theta_1 \in (0, 1/4) \).

Remark 1.2. By (H3), it is readily to see that

\[
\| b(x) \|_H^2 + \| \sigma^1(x) \|_{HS}^2 \leq \overline{T}(1 + \| x \|_H^2), \quad x \in H,
\]

where \( \overline{T} := 2((L_1^2 + L_2^2) \vee \mu) \) with \( \mu := \| b(0) \|_H^2 + \| \sigma^1(0) \|_{HS}^2 \). Moreover, by (H4) one has

\[
2 \langle b(x) \rangle_H + \| \sigma^1(x) \|_{HS}^2 = 2 \langle x, b(x) - b(0) \rangle_H + \| \sigma^1(x) - \sigma^1(0) \|_{HS}^2
\]

\[
+ 2 \langle x, b(0) \rangle_H + 2 \langle \sigma^1(x) - \sigma^1(0), \sigma^1(0) \rangle_H + \| \sigma^1(0) \|_{HS}^2
\]

\[
\leq -(\gamma - \epsilon) \| x \|_H^2 + 2(L_2^2 + 1 + \epsilon) \mu \epsilon^{-1}, \quad \epsilon \in (0, 1), \quad x \in H,
\]

where \( \langle T, S \rangle_{HS} := \sum_{i=1}^{\infty} \langle Te_i, Se_i \rangle_H \) for \( S, T \in \mathcal{L}_{HS}(H) \).

Before establishing the numerical scheme, we further need to introduce some notation. For any \( n \in \mathbb{N} \), let \( \pi_n : H \to H_n := \text{span}\{e_1, \ldots, e_n\} \) be the orthogonal projection, i.e.,

\[
\pi_n x = \sum_{i=1}^n \langle x, e_i \rangle_H e_i, \quad x \in H, \quad A_n := \pi_n A \in \mathcal{L}(H_n), \quad b_n := \pi_n b : H_n \to H_n \quad \text{and} \quad \sigma_n := \pi_n \sigma : H_n \to \mathcal{L}_{HS}(H_n).
\]

Moreover, throughout the paper, let \( x_n := \pi_n x \) for arbitrary \( x \in U \), where \( U \) is a bounded subset of \( H \).

Consider finite-dimensional approximation associated with (1.1) on \( H_n \simeq \mathbb{R}^n \)

\[
\begin{cases}
\mathrm{d}X^n(t) = \{A_n X^n(t) + b_n(X^n(t))\} \, \mathrm{d}t + \sigma_n(X^n(t)) \, \mathrm{d}W(t), \\
X^n(0) = x_n.
\end{cases}
\]

The spatial approximation (1.5) is also called the Galerkin approximation of (1.1). Due to

\[
\pi_n Ax = \pi_n A \left( \sum_{i=1}^n \langle x, e_i \rangle_H e_i \right) = - \sum_{i=1}^n \langle x, e_i \rangle_H \lambda_i e_i, \quad x \in H_n,
\]

it follows that

\[
A_n x = Ax, \quad e^{tA_n} x = e^{tA} x \quad \text{and} \quad \langle x, b_n(y) \rangle_H = \langle x, b(y) \rangle_H
\]

for all \( x, y \in H_n \). By (H3) and the property of the projection operator \( \pi_n \), we have

\[
\| A_n (x - y) + b_n(x) - b_n(y) \|_H^2 + \| \sigma_n^1(x) - \sigma_n^1(y) \|_{HS}^2
\]

\[
\leq 2 \| A_n (x - y) \|_H^2 + 2 \| b_n(x) - b_n(y) \|_H^2 + \| \sigma_n^1(x) - \sigma_n^1(y) \|_{HS}^2
\]

\[
\leq 2(\lambda_n^2 + L_1^2 + L_2^2) \| x - y \|_H^2, \quad x, y \in H_n.
\]
Hence, under (H1) and (H3), (1.5) admits a unique strong solution \( \{X^n_x(t)\}_{t \geq 0} \) with the starting point \( x_n \in H_n \).

Next we introduce a time-discretization scheme for (1.5). For a stepsize \( \triangle \in (0, 1) \) and each integer \( k \geq 0 \), compute the discrete Exponential Integrator (EI) scheme \( Y_{x_n}^{n, \triangle}((k+1)\triangle) \approx X^n_x(k\triangle) \) by setting \( Y_{x_n}^{n, \triangle}(0) := x_n \) and forming

\[
Y_{x_n}^{n, \triangle}((k+1)\triangle) := e^{\triangle A_n} \{Y_{x_n}^{n, \triangle}(k\triangle) + b_n(Y_{x_n}^{n, \triangle})(\triangle) + \sigma_n(Y_{x_n}^{n, \triangle}(k\triangle))\triangle W_k\},
\]

where \( \triangle W_k := W((k+1)\triangle) - W(k\triangle) \), and define the continuous EI scheme associated with (1.5) by

\[
Y_{x_n}^{n, \triangle}(t) := e^{tA_n} x_n + \int_0^t e^{(t-s)A_n} b_n(Y_{x_n}^{n, \triangle}([s]))ds
+ \int_0^t e^{(t-s)A_n} \sigma_n(Y_{x_n}^{n, \triangle}([s]))dW(s)
\]

(1.8)

due to (1.6), where \([t] := \lfloor t/\triangle \rfloor \triangle \) with \([t/\triangle] \) standing for the integer part of \( t/\triangle \). It is easy to see from (1.8) that

\[
Y_{x_n}^{n, \triangle}(t) = e^{(t-s)A_n} Y_{x_n}^{n, \triangle}(s) + \int_s^t e^{(t-[r])A_n} b_n(Y_{x_n}^{n, \triangle}([r]))dr
+ \int_s^t e^{(t-[r])A_n} \sigma_n(Y_{x_n}^{n, \triangle}([r]))dW(r), \quad 0 \leq s \leq t.
\]

(1.9)

By \( Y_{x_n}^{n, \triangle}(0) = Y_{x_n}^{n, \triangle}(0) \), we deduce from (1.7) and (1.9) that \( Y_{x_n}^{n, \triangle}(k\triangle) = Y_{x_n}^{n, \triangle}(k\triangle) \), i.e., \( Y_{x_n}^{n, \triangle}(t) \) coincides with the discrete EI approximate solution at the gridpoints.

Remark 1.3. For the finite-dimensional SDEs, the discrete Euler-Maruyama (EM) scheme and the continuous EM scheme are standard, e.g., [18, p.113]. While the roots of constructing the schemes (1.8) and (1.9) go back to, e.g., [5, 17].

For the discrete EI scheme (1.7), in this paper we are concerned with the following two questions:

- Given \( n \in \mathbb{N} \), for what choices of the stepsize \( \triangle \in (0, 1) \) does the EI scheme have a unique stationary distribution?
- Will the stationary distribution of the EI scheme converge weakly to some probability measure? If so, what’s the weak limit probability measure?
In what follows, we shall give the positive answers to these two questions one-by-one. It is also worth pointing out that, for the finite-dimensional case, Yuan and Mao [24] studied the invariant measure of EM numerical solutions for a class of SDEs, and Yevik and Zhao [22] discussed by the global attractor approach the existence of stationary distribution of EM scheme for SDEs which generate random dynamical systems. Comparing the EI scheme (1.7) with the EM scheme for the finite-dimensional case, e.g., [18, p.113], we note that the explicit EI schemes (1.7) is based not only on the spatial discretization but also on the time discretization. Moreover, in (1.1), the linear operator \( A \) is generally unbounded, and the diffusion coefficient is not Hilbert-Schmidt, which leads to be unavailable of the Itô formula. Therefore, our approaches are different from those of [22, 24]. What’s more, Bréhier [2] investigated the existence of invariant measure for semi-implicit Euler scheme (in time), and discussed the numerical approximation of the invariant measure for a class of parabolic SPDEs driven by additive noise, where the drift coefficient is assumed to be bounded.

The organization of this paper goes as follows: In Section 2, for a given \( n \in \mathbb{N} \) and a sufficiently small stepsize \( \Delta \in (0, 1) \), we show that the EI approximate solution \( \{Y_{x_n}^{n,\Delta}(k\Delta)\}_{k \geq 0, x_n \in H_n} \) admits a unique stationary distribution under the properties (P1) and (P2); Section 3 is devoting to providing some sufficient conditions such that (P1) and (P2) hold; In the last section, we reveal that the weak limit of the law for the EI approximate solution \( \{Y_{x_n}^{n,\Delta}(k\Delta)\}_{k \geq 0, x_n \in H_n} \) is in fact the counterpart for (1.1).

### 2 Stationary Distribution for the EI Scheme

For fixed integer \( n \in \mathbb{N} \), arbitrary integer \( k \geq 0 \) and \( \Gamma \in \mathcal{B}(H_n) \), define the k-step transition probability kernel for the discrete EI approximate solution \( Y_{x_n}^{n,\Delta}(k\Delta) \) by

\[
P^n_{\Delta}(x_n, \Gamma) := \mathbb{P}(Y_{x_n}^{n,\Delta}(k\Delta) \in \Gamma).
\]

Following the argument of that of [24 Theorem 1.2], we deduce that

**Lemma 2.1.** \( \{Y_{x_n}^{n,\Delta}(k\Delta)\}_{k \geq 0} \) is a homogeneous Markov process.

We still need to introduce some additional notation and notions. For a real separable Hilbert space \((K, \| \cdot \|_K)\), let \( \mathcal{P}(K) \) stand for the collection of all probability measures on \( K \). For \( P_1, P_2 \in \mathcal{P}(K) \), define the metric \( d_L \) as follows:

\[
d_L(P_1, P_2) := \sup_{f \in L} \left| \int_K f(u)P_1(du) - \int_K f(u)P_2(du) \right|
\]

where \( L := \{ f : K \to \mathbb{R} : |f(u) - f(v)| \leq \|u - v\|_K \text{ and } |f(\cdot)| \leq 1 \} \).

**Remark 2.1.** It is known that the weak convergence of probability measures is a metric concept, see, e.g., [12 Proposition 2.5, p.6]. In other words, a sequence of probability measures \( \{P_k\}_{k \geq 1} \in \mathcal{P}(K) \) converges weakly to a probability measure \( P_0 \in \mathcal{P}(K) \) if and only if \( \lim_{k \to \infty} d_L(P_k, P_0) = 0 \).
**Definition 2.1.** For a given \( n \in \mathbb{N} \) and a given stepsize \( \triangle \), \( \{Y_{xn}^{n,\triangle}(k\triangle)\}_{k \geq 0, x_n \in H_n} \) is said to have a stationary distribution \( \pi_{n,\triangle} \in \mathcal{P}(H_n) \) if \( \lim_{k \to \infty} d_L(\mathbb{P}_{k}^{n,\triangle}(x_n, \cdot), \pi_{n,\triangle}(\cdot)) = 0 \) for every \( x_n \in H_n \).

**Definition 2.2.** For a given \( n \in \mathbb{N} \) and a given stepsize \( \triangle \), \( \{Y_{xn}^{n,\triangle}(k\triangle)\}_{k \geq 0, x_n \in H_n} \) is said to have Property \((P_1)\) if
\[
\sup_{k \geq 0} \sup_{x_n \in U} \mathbb{E}\|Y_{xn}^{n,\triangle}(k\triangle)\|_H^2 < \infty
\]
while it is said to have Property \((P_2)\) if
\[
\lim_{k \to \infty} \sup_{x_n, y_n \in U} \mathbb{E}\|Y_{xn}^{n,\triangle}(k\triangle) - Y_{yn}^{n,\triangle}(k\triangle)\|_H^2 = 0,
\]
where \( U \) is a bounded subset of \( H_n \).

Our main result in this section is stated as follows.

**Theorem 2.2.** Assume that \((P_1)\) and \((P_2)\) hold. Then, for a given \( n \in \mathbb{N} \) and a given stepsize \( \triangle \), \( \{Y_{xn}^{n,\triangle}(k\triangle)\}_{k \geq 0, x_n \in H_n} \) has a unique stationary distribution \( \pi_{n,\triangle} \in \mathcal{P}(H_n) \).

**Proof.** For fixed \( n \in \mathbb{N} \), we note that \( H_n \simeq \mathbb{R}^n \) is finite-dimensional, and choose a bounded subset \( U \subseteq H_n \) such that \( x_n, y_n \in U \). Following the argument to derive [24, Lemma 2.4 and Lemma 2.6], we deduce that
\[
\lim_{k \to \infty} \sup_{x_n, y_n \in U} d_L(\mathbb{P}_{k}^{n,\triangle}(x_n, \cdot), \pi_{n,\triangle}(\cdot)) = 0,
\]
and that, together with Lemma 2.1, there exists \( \pi_{n,\triangle} \in \mathcal{P}(H_n) \) such that
\[
\lim_{k \to \infty} d_L(\mathbb{P}_{k}^{n,\triangle}(0, \cdot), \pi_{n,\triangle}(\cdot)) = 0.
\]
Then the desired assertion follows from (2.2), (2.3) and the triangle inequality
\[
d_L(\mathbb{P}_{k}^{n,\triangle}(x_n, \cdot), \pi_{n,\triangle}(\cdot)) \leq d_L(\mathbb{P}_{k}^{n,\triangle}(x_n, \cdot), \mathbb{P}_{k}^{n,\triangle}(0, \cdot)) + d_L(\mathbb{P}_{k}^{n,\triangle}(0, \cdot), \pi_{n,\triangle}(\cdot)).
\]

\( \square \)

### 3 Sufficient Conditions for Properties \((P_1)\) and \((P_2)\)

To make Theorem 2.2 more applicable, in this section we intend to give some sufficient conditions such that \((P_1)\) and \((P_2)\) hold. In what follows, \( C > 0 \) is a generic constant whose values may change from line to line. For notational simplicity, let
\[
Z_{xn}^{n,\triangle}(t) := \int_0^t e^{(t-[s])A} \sigma_n^0 dW(s) \quad \text{and} \quad \tilde{Y}_{xn}^{n,\triangle}(t) := Y_{xn}^{n,\triangle}(t) - Z_{xn}^{n,\triangle}(t).
\]
Lemma 3.1. Under (H1)-(H3), \n
\begin{equation}
\mathbb{E}\|\tilde{Y}^n_{x_n}(t) - \bar{Y}^n_{x_n}([t])\|_H^2 \leq \beta_1 \Delta (1 + \mathbb{E}\|\tilde{Y}^n_{x_n}([t])\|_H^2), \quad t \geq 0,
\end{equation}

where \(\beta_1 := 3\{(\lambda_n^2 + 2T) \vee (2T(1 + \|(-A)^{-\theta_1}\|_H^2))\} \).

**Proof.** Observe from (1.3) that

\begin{equation}
\tilde{Y}^n_{x_n}(t) = e^{tA}x_n + \int_0^t e^{(t-s)A}b_n(Y^n_{x_n}([s]))ds + \int_0^t e^{(t-s)A}\sigma^n(Y^n_{x_n}([s]))dW(s).
\end{equation}

This further gives

\[
\tilde{Y}^n_{x_n}(t) = e^{(t-[t])A}\tilde{Y}^n_{x_n}([t]) + \int_{[t]}^{t} e^{(t-s)A}b_n(Y^n_{x_n}([s]))ds \\
+ \int_{[t]}^{t} e^{(t-s)A}\sigma^n(Y^n_{x_n}([s]))dW(s).
\]

Then, by the Hölder inequality, the Itô isometry and (H1), one has

\begin{align}
&\mathbb{E}\|\tilde{Y}^n_{x_n}(t) - \bar{Y}^n_{x_n}([t])\|_H^2 \\
&\leq 3\{ \mathbb{E}\|e^{(t-[t])A} - \text{id}_H\bar{Y}^n_{x_n}([t])\|_H^2 + \mathbb{E}\int_{[t]}^{t} \|b(Y^n_{x_n}([s]))\|_H^2 ds \\
&+ \mathbb{E}\int_{[t]}^{t} \|\sigma^n(Y^n_{x_n}([s]))\|_{HS}^2 ds \} \\
&=: 3\{I_1(t) + I_2(t) + I_3(t)\}.
\end{align}

Recalling the fundamental inequality \(1 - e^{-y} \leq y, y > 0\), we obtain from (H1) that

\begin{align}
&\|e^{(t-[t])A} - \text{id}_Hu\|_H^2 = \left\|\sum_{i=1}^n (e^{-\lambda_i(t-[t])} - 1)\langle u, e_i\rangle_H e_i\right\|_H^2 \\
&\leq (1 - e^{-\lambda_n(t-[t])})^2\|u\|_H^2 \\
&\leq \lambda_n^2 \Delta^2\|u\|_H^2, \quad u \in H_n.
\end{align}

Thus we arrive at

\begin{equation}
I_1(t) \leq \lambda_n^2 \Delta^2\mathbb{E}\|\tilde{Y}^n_{x_n}([t])\|_H^2.
\end{equation}

Note from the Itô isometry, (H1) and (H2) that

\begin{align}
&\mathbb{E}\|Z^n_{x_n}(t)\|_H^2 = \int_0^t \|e^{(s-[s])A}e^{(t-s)A}\sigma^n\|_{HS}^2 ds \\
&\leq \|(-A)^{-\theta_1}\|_H^2 \int_0^t \|(-A)^{\theta_1}e^{(t-s)A}\sigma^n\|_{HS}^2 ds \\
&\leq \|(-A)^{-\theta_1}\|_H^2 \int_0^t \|(-A)^{\theta_1}e^{(t-s)A}\sigma^n\|_{HS}^2 ds \\
&\leq \|(-A)^{-\theta_1}\|_H^2 \int_0^t \|(-A)^{\theta_1}e^{(t-s)A}\sigma^n\|_{HS}^2 ds \leq \|(-A)^{-\theta_1}\|_H^2 \|\sigma^n\|_{HS}.
\end{align}
Thus, by (1.3) and (3.6) it follows that

$$I_2(t) + I_3(t) \leq \Delta\mathbb{E}\{\|b(Y_{x_n}^{\Delta}(\lfloor t \rfloor))\|^2_H + \|\sigma_{1}^1(Y_{x_n}^{\Delta}(\lfloor t \rfloor))\|_{HS}^2\}$$

(3.7)

$$\leq 2\Delta\{1 + \mathbb{E}\|\tilde{Y}_{x_n}^{\Delta}(\lfloor t \rfloor)\|_H^2 + \mathbb{E}\|Z_{x_n}^{\Delta}(\lfloor t \rfloor)\|_H^2\}$$

$$\leq 2\Delta\{1 + \|(-A)^{-\delta_1}\|^2\delta_1 + \mathbb{E}\|\tilde{Y}_{x_n}^{\Delta}(\lfloor t \rfloor)\|_H^2\}.$$  

As a result, (3.1) follows by substituting (3.5) and (3.7) into (3.3).

**Theorem 3.2.** Let (H1)-(H4) hold and assume further that $2\alpha + \gamma > 0$. If $\Delta < \min\{1, (2\alpha + \gamma)^2/(4\beta_1^2)\}$, then

$$\sup_{t \geq 0} \sup_{x_n \in U} \mathbb{E}\|Y_{x_n}^{\Delta}(t)\|^2_H < \infty,$$

where $\rho_1 := 2 + (|14\alpha - \gamma|^2/64 + 2\mathbb{T} + |14\alpha - \gamma|/8)\beta_1 + (1 + \beta_1 + \lambda^2_1\mathbb{T})$ and $U$ is a bounded subset of $H_n$. Hence Property (P1) holds whenever the stepsize $\Delta$ is sufficiently small.

**Proof.** Note that (3.2) can be rewritten in the differential form

(3.9)  

$$d\tilde{Y}_{x_n}^{\Delta}(t) = \{A\tilde{Y}_{x_n}^{\Delta}(t) + e^{(t-[t])A}b_n(Y_{x_n}^{\Delta}(\lfloor t \rfloor))\}dt + e^{(t-[t])A}\sigma_{1}^1(Y_{x_n}^{\Delta}(\lfloor t \rfloor))dW(t)$$

with $\tilde{Y}_{x_n}^{\Delta}(0) = x_n$. For any $\nu > 0$, by the Itô formula we derive from (3.9) and (H1) that

$$\mathbb{E}(e^{\nu t}\|\tilde{Y}_{x_n}^{\Delta}(t)\|^2_H)$$

$$\leq \|x\|^2_H + \mathbb{E}\int_0^t e^{\nu s}\{\nu\|\tilde{Y}_{x_n}^{\Delta}(s)\|^2_H + 2\langle\tilde{Y}_{x_n}^{\Delta}(s), A\tilde{Y}_{x_n}^{\Delta}(s)\rangle\}ds$$

$$\leq \|x\|^2_H + \mathbb{E}\int_0^t e^{\nu s}\{(2\alpha - \nu)\|\tilde{Y}_{x_n}^{\Delta}(s)\|^2_H$$

$$+ 2\langle\tilde{Y}_{x_n}^{\Delta}(s), e^{(s-[s])A}b_n(Y_{x_n}^{\Delta}(\lfloor s \rfloor))\rangle\}H + \|\sigma_{1}^1(Y_{x_n}^{\Delta}(\lfloor s \rfloor))\|^2_{HS}\}ds.$$  

(3.10)

Since

$$\|\tilde{Y}_{x_n}^{\Delta}(t)\|^2_H = \|\tilde{Y}_{x_n}^{\Delta}(\lfloor t \rfloor)\|^2_H + 2\langle\tilde{Y}_{x_n}^{\Delta}(\lfloor t \rfloor), \tilde{Y}_{x_n}^{\Delta}(t) - \tilde{Y}_{x_n}^{\Delta}(\lfloor t \rfloor)\rangle_H$$

$$+ \|\tilde{Y}_{x_n}^{\Delta}(t) - \tilde{Y}_{x_n}^{\Delta}(\lfloor t \rfloor)\|^2_H,$$

(3.11)

and

$$\langle\tilde{Y}_{x_n}^{\Delta}(t), e^{(t-[t])A}b_n(Y_{x_n}^{\Delta}(\lfloor t \rfloor))\rangle_H$$

$$= \langle Y_{x_n}^{\Delta}(\lfloor t \rfloor), b(Y_{x_n}^{\Delta}(\lfloor t \rfloor))\rangle_H + \langle \tilde{Y}_{x_n}^{\Delta}(t) - \tilde{Y}_{x_n}^{\Delta}(\lfloor t \rfloor), b(Y_{x_n}^{\Delta}(\lfloor t \rfloor))\rangle_H$$

$$- \langle Z_{x_n}^{\Delta}(\lfloor t \rfloor), b(Y_{x_n}^{\Delta}(\lfloor t \rfloor))\rangle_H + \langle \tilde{Y}_{x_n}^{\Delta}(t), e^{(t-[t])A} - \text{id}_H\rangle b_n(Y_{x_n}^{\Delta}(\lfloor t \rfloor))\rangle_H,$$
it follows from (3.10) that

\[ \mathbb{E}(e^{\nu t} \| \widetilde{Y}_{x_n}(t) \|^2_H) \leq \| x \|^2_H + \mathbb{E} \int_0^t e^{\nu s} \left\{ -2(2\alpha - \nu) \| \widetilde{Y}_{x_n}(s) \|^2_H + \| \sigma^1(Y_{x_n}(s)) \|^2_{HS} \right\} ds \\
+ 2 \langle Y_{x_n}(s), b(Y_{x_n}(s)) \rangle_H \\
- 2(2\alpha - \nu) \langle \widetilde{Y}_{x_n}(s), \widetilde{Y}_{x_n}(s) - \widetilde{Y}_{x_n}(s) \rangle_H \\
- (2\alpha - \nu) \| \widetilde{Y}_{x_n}(s) - \widetilde{Y}_{x_n}(s) \|^2_H \\
+ 2 \langle \widetilde{Y}_{x_n}(s) - \widetilde{Y}_{x_n}(s), b(Y_{x_n}(s)) \rangle_H \\
+ 2 \langle \widetilde{Y}_{x_n}(s), (e^{(s-s)A} - \text{id}_H) b_n(Y_{x_n}(s)) \rangle_H \} ds. \]

This, together with (1.4), yields that

\[ \mathbb{E}(e^{\nu t} \| \widetilde{Y}_{x_n}(t) \|^2_H) \leq \| x \|^2_H - (2\alpha + \gamma - \epsilon - \nu) \mathbb{E} \int_0^t e^{\nu s} \| \widetilde{Y}_{x_n}(s) \|^2_H ds \\
+ \mathbb{E} \int_0^t e^{\nu s} \left\{ -2(2\alpha - \nu) \langle \widetilde{Y}_{x_n}(s), \widetilde{Y}_{x_n}(s) - \widetilde{Y}_{x_n}(s) \rangle_H \\
- (2\alpha - \nu) \| \widetilde{Y}_{x_n}(s) - \widetilde{Y}_{x_n}(s) \|^2_H \\
+ 2 \langle \widetilde{Y}_{x_n}(s) - \widetilde{Y}_{x_n}(s), b(Y_{x_n}(s)) \rangle_H \} ds \\
+ 2 \mathbb{E} \int_0^t e^{\nu s} \langle \widetilde{Y}_{x_n}(s), (e^{(s-s)A} - \text{id}_H) b_n(Y_{x_n}(s)) \rangle_H ds \\
+ \mathbb{E} \int_0^t e^{\nu s} \left\{ 2(L_2^2 + 1 + \epsilon^{-1}) \mu e^{-1} - 2 \langle Z_{n, \Delta}(s), b(Y_{x_n}(s)) \rangle_H \\
- 2(\gamma - \epsilon) \| Z_{n, \Delta}(s) \|^2_H \right\} ds \]

(3.12)

= J_1(t) + J_2(t) + J_3(t) + J_4(t). \]

By the elemental inequality: \(2ab \leq \kappa a^2 + b^2/\kappa, a, b \in \mathbb{R}, \kappa > 0,\) and (3.1), we arrive at

\[ J_2(t) \leq \mathbb{E} \int_0^t e^{\nu s} \left\{ \Delta^\frac{1}{2} \| \widetilde{Y}_{x_n}(s) \|^2_H + 2^{-1} \Delta^\frac{1}{2} \| b(Y_{x_n}(s)) \|^2_H \right\} ds \\
+ \{ (|2\alpha - \nu|^2 + 2L) \Delta^{-\frac{1}{2}} + |2\alpha - \nu| \} \| \widetilde{Y}_{x_n}(s) - \widetilde{Y}_{x_n}(s) \|^2_H \}
\]

\[ \leq \mathbb{E} \int_0^t e^{\nu s} \left\{ 2 \Delta^\frac{1}{2} \| \widetilde{Y}_{x_n}(s) \|^2_H + 2^{-1} \Delta^\frac{1}{2} + \Delta^\frac{1}{2} \| Z_{n, \Delta}(s) \|^2_H \right\} ds \\
+ \{ (|2\alpha - \nu|^2 + 2L) \Delta^{-\frac{1}{2}} + |2\alpha - \nu| \} \| \widetilde{Y}_{x_n}(s) - \widetilde{Y}_{x_n}(s) \|^2_H \}
\]

where in the last step we have used (1.3). Combining (3.1) with (3.6), we thus obtain that

\[ J_2(t) \leq \int_0^t e^{\nu s} \left\{ 2 + (|2\alpha - \nu|^2 + 2L + |2\alpha - \nu|) \beta_1 \right\} \Delta^\frac{1}{2} \mathbb{E} \| \widetilde{Y}_{x_n}(t) \|^2_H \\
+ \{ 1 + \| (\beta A)^{-\delta_1} \|^2 \delta_1 + (|2\alpha - \nu|^2 + 2L + |2\alpha - \nu|) \beta_1 \} \Delta^\frac{1}{2} \} \}
\]

(3.13)
On the other hand, we deduce from (1.3), (3.4), (3.6) and (3.11) that

\[ J_3(t) \leq \mathbb{E} \int_0^t e^{\nu s} \left\{ 2(L_2^2 + 1 + \epsilon^{-1}) \mu \epsilon^{-1} + 2 \right\} \| Y_{x_0}^{n,\Delta}([s]) \|_H \| b(Y_{x_0}^{n,\Delta}([s])) \|_H \\
+ \| Y_{x_0}^{n,\Delta}([s]) \|_H \| \tilde{Y}_{x_0}^{n,\Delta}([s]) \|_H + \| \gamma - \epsilon \| \| Z^{n,\Delta}([s]) \|_H^2 \right\} ds \]

(3.14)

Furthermore, due to (1.3) and (3.6), for arbitrary \( \kappa > 0 \) one has

\[ J_4(t) \leq \mathbb{E} \int_0^t e^{\nu s} \left\{ 2(L_2^2 + 1 + \epsilon^{-1}) \mu \epsilon^{-1} + 2 \right\} \| Z^{n,\Delta}([s]) \|_H \| b(Y_{x_0}^{n,\Delta}([s])) \|_H \\
+ \| Y_{x_0}^{n,\Delta}([s]) \|_H \| \tilde{Y}_{x_0}^{n,\Delta}([s]) \|_H + \| \gamma - \epsilon \| \| Z^{n,\Delta}([s]) \|_H^2 \right\} ds \]

(3.15)

In particular, taking \( \epsilon = \nu = (2\alpha + \gamma)/8 \) and \( \kappa = (2\alpha + \gamma)/(4(1 + 2\bar{T})) \) yields that

Putting (3.13)-(3.15) into (3.12), we deduce that

\[ \mathbb{E}(e^{\nu t} \| \tilde{Y}_{x_0}^{n,\Delta}(t) \|_H^2) \leq \| x \|_H^2 + C \int_0^t e^{\nu s} ds \]

(3.16)

For \( \Delta < (2\alpha + \gamma)^2/(4\rho_1^2) \), it is trivial to see that \( 2\alpha + \gamma - 2\rho_1 \Delta^{1/2} > 0 \). Thus we have

\[ \sup_{t \geq 0} \sup_{x_n \in U} \mathbb{E}(\| \tilde{Y}_{x_0}^{n,\Delta}(t) \|_H^2) < \infty. \]

Finally, (3.8) follows by recalling \( \tilde{Y}_{x_0}^{n,\Delta}(t) = Y_{x_0}^{n,\Delta}(t) - Z^{n,\Delta}(t) \) and (3.6).
**Theorem 3.3.** Let the assumptions of Lemma 3.2 hold. If \( \Delta < \min \{ (2\alpha + \gamma)^2 / (4\rho^2) \} \), then

\[
\lim_{t \to \infty} \sup_{x_n, y_n \in U} \mathbb{E}\| Y^{n, \Delta}_{x_n}(t) - Y^{n, \Delta}_{y_n}(t) \|^2_H = 0,
\]

where \( \rho_2 := (6(\lambda_n^2 + \overline{L})(2\alpha - \gamma) + 1) + 7\overline{L} + \lambda_n^2 \overline{L} + 6\lambda_n^2 \) and \( U \) is a bounded subset of \( H_n \). Hence Property (P2) holds whenever the stepsize \( \Delta \) is sufficiently small.

**Proof.** Let

\[
Z^{n, \Delta}_{x_n, y_n}(t) := Y^{n, \Delta}_{x_n}(t) - Y^{n, \Delta}_{y_n}(t).
\]

Note from (1.3) that

\[
Z^{n, \Delta}_{x_n, y_n}(t) - Z^{n, \Delta}_{x_n, y_n}([t]) = (e^{(t-[t])A} - \operatorname{id}_H)Z^{n, \Delta}_{x_n, y_n}([t]) \\
+ \int_t^t e^{(t-[s])A}(b_n(Y^{n, \Delta}_{x_n}([s])) - b_n(Y^{n, \Delta}_{y_n}([s]))) ds \\
+ \int_t^t e^{(t-[s])A}(\sigma_n(Y^{n, \Delta}_{x_n}([s])) - \sigma_n(Y^{n, \Delta}_{y_n}([s]))) dW(s).
\]

Following the argument of that of (3.1), we derive that

\[
\mathbb{E}\| Z^{n, \Delta}_{x_n, y_n}(t) - Z^{n, \Delta}_{x_n, y_n}([t]) \|^2_H \leq 3(\lambda_n^2 + \overline{L})\Delta \mathbb{E}\| Z^{n, \Delta}_{x_n, y_n}([t]) \|^2_H.
\]

For \( \nu := (2\alpha + \gamma)/2 \), by the Itô formula it follows from (1.3), (H1) and (H4) that

\[
\mathbb{E}(e^{\nu t}\| Z^{n, \Delta}_{x_n, y_n}(t) \|^2_H) \leq \| x - y \|^2_H + \nu \mathbb{E} \int_0^t e^{\nu s}\| Z^{n, \Delta}_{x_n, y_n}(s) \|^2_H ds \\
+ \mathbb{E} \int_0^t e^{\nu s}\{ 2\langle Z^{n, \Delta}_{x_n, y_n}(s), AZ^{n, \Delta}_{x_n, y_n}(s) \rangle_H \\
+ 2\langle Z^{n, \Delta}_{x_n, y_n}([s]), b_n(Y^{n, \Delta}_{x_n}([s])) - b_n(Y^{n, \Delta}_{y_n}([s])) \rangle_H \\
+ \| \sigma_n(Y^{n, \Delta}_{x_n}([s])) - \sigma_n(Y^{n, \Delta}_{y_n}([s])) \|^2_H \\
+ 2\| Z^{n, \Delta}_{x_n, y_n}(s) - Z^{n, \Delta}_{x_n, y_n}([s]), b_n(Y^{n, \Delta}_{x_n}([s])) - b_n(Y^{n, \Delta}_{y_n}([s])) \rangle_H \\
+ 2\langle Z^{n, \Delta}_{x_n, y_n}(s), (e^{(s-[s])A} - \operatorname{id}_H)(b_n(Y^{n, \Delta}_{x_n}([s])) - b_n(Y^{n, \Delta}_{y_n}([s]))) \rangle_H \} ds \\
\leq \| x - y \|^2_H + (2\alpha + \gamma - \nu) \mathbb{E} \int_0^t e^{\nu s}\| Z^{n, \Delta}_{x_n, y_n}([s]) \|^2_H ds \\
+ \mathbb{E} \int_0^t e^{\nu s}\{-2(2\alpha - \nu)\langle Z^{n, \Delta}_{x_n, y_n}([s]), Z^{n, \Delta}_{x_n, y_n}(s) - Z^{n, \Delta}_{x_n, y_n}([s]) \rangle_H \\
- (2\alpha - \nu)\| Z^{n, \Delta}_{x_n, y_n}(s) - Z^{n, \Delta}_{x_n, y_n}([s]) \|^2_H \\
+ 2\| Z^{n, \Delta}_{x_n, y_n}(s) - Z^{n, \Delta}_{x_n, y_n}([s]), b_n(Y^{n, \Delta}_{x_n}([s])) - b_n(Y^{n, \Delta}_{y_n}([s])) \rangle_H \} ds \\
+ 2\mathbb{E} \int_0^t e^{\nu s}\langle Z^{n, \Delta}_{x_n, y_n}(s), (e^{(s-[s])A} - \operatorname{id}_H)(b_n(Y^{n, \Delta}_{x_n}([s])) - b_n(Y^{n, \Delta}_{y_n}([s]))) \rangle_H ds \\
=: J_1(t) + J_2(t) + J_3(t).
\]
where we have also used the (3.11) with \( \tilde{Y}_{x_n}^\Delta(t) \) replaced by \( Z_{x_n,y_n}^\Delta(t) \). By (H3) and (3.18), one has

\[
\mathcal{J}_2(t) \leq \mathbb{E} \int_0^t e^{\nu s} \left\{ \Delta \int_0^s \left[ \frac{1}{2} \| Z_{x_n,y_n}^\Delta(s) \|_H^2 + \frac{3}{2} \right] b(Y_{x_n}^\Delta(s)) - b(Y_{y_n}^\Delta(s)) \right\}^2 ds + \{2\alpha - \nu + (2\alpha - \nu + 1)\lambda_2 \Delta^2 \} \| Z_{x_n,y_n}^\Delta(s) - Z_{x_n,y_n}^\Delta(s) \|_H^2 ds \
\leq \{6(\lambda_2^2 + 1)(2\alpha - \nu + 1) + 1 + \lambda_2 \} \Delta^2 \mathbb{E} \int_0^t e^{\nu s} \| Z_{x_n,y_n}^\Delta(s) \|_H^2 ds.
\]

On the other hand, carrying out a similar argument to that of (3.14) leads to

\[
\mathcal{J}_3(t) \leq 2\mathbb{E} \int_0^t e^{\nu s} \left\{ \Delta \int_0^s \left[ \frac{1}{2} \| Z_{x_n,y_n}^\Delta(s) - Z_{x_n,y_n}^\Delta(s) \|_H^2 + \frac{3}{2} \right] (Z_{x_n,y_n}^\Delta(s) - Z_{x_n,y_n}^\Delta(s)) \right\} H ds + \| e^{(s - \nu)\Delta} - \text{id}_H \| (b_n(Y_{x_n}^\Delta(s)) - b_n(Y_{y_n}^\Delta(s))) \|_H^2 ds \leq (2 + \lambda_2^2 \lambda_2 + 6\lambda_2^2) \Delta^2 \mathbb{E} \int_0^t e^{\nu s} \| Z_{x_n,y_n}^\Delta(s) \|_H^2 ds.
\]

Hence we arrive at

\[
\mathbb{E}(e^{\nu t} \| Z_{x_n,y_n}^\Delta(t) \|_H^2) \leq \| x - y \|_H^2 - \frac{2\alpha + \gamma - 2\lambda_2 \Delta^2}{2} \mathbb{E} \int_0^t e^{\nu s} \| Z_{x_n,y_n}^\Delta(s) \|_H^2 ds,
\]

and then the desired assertion (3.17) follows by \( \Delta \leq \min\{1, (2\alpha + \gamma)^2/(4\lambda_2^2)\} \).

\[
\hfill \Box
\]

## 4 Weak Limit Distribution

In the previous section, we give some sufficient conditions such that (1.7) has a unique stationary distribution \( \pi^{n,\Delta} \in \mathcal{P}(H_n) \) for a fixed \( n \) and a sufficiently small stepsize \( \Delta \in (0, 1) \). In this section we proceed to discuss the weak limit behavior of \( \pi^{n,\Delta} \in \mathcal{P}(H_n) \) and give positive answers to the following questions:

- Will the stationary distribution \( \pi^{n,\Delta} \) converge weakly to some probability measure in \( \mathcal{P}(H) \) whenever \( n \to \infty \) and \( \Delta \to 0 \) ?

- If yes, what is the weak limit probability measure ?

Denote \( \{X_x(t)\}_{t \geq 0, x \in H} \) by the mild solution of (1.1) starting from the point \( x \) at time \( t = 0 \), which is a homogenous Markov process. For any subset \( \Gamma \subset \mathcal{B}(H) \) and arbitrary \( t \geq 0 \), let \( \mathbb{P}_t(x, \Gamma) := \mathbb{P}(X_x(t) \in \Gamma) \).

**Definition 4.1.** \( \{X_x(t)\}_{t \geq 0, x \in H} \) is said to have a stationary distribution \( \pi(\cdot) \in \mathcal{P}(H) \) if

\[
\lim_{t \to \infty} d_L(\mathbb{P}_t(x, \cdot), \pi(\cdot)) = 0.
\]

To reveal the limit behavior of \( \pi^{n,\Delta}(\cdot) \), we first give several auxiliary lemmas.
Lemma 4.1. Let (H1)-(H4) hold and assume further that $2\alpha + \gamma > 0$. Then the mild solution $\{X_x(t)\}_{t \geq 0, x \in H}$ of (1.1) has a unique stationary distribution $\pi(\cdot) \in \mathcal{P}(H)$.

Proof. We remark that [1, Theorem 3.1] investigates the stationary distribution of (1.1) with $\sigma_0 = 0$, i.e., the diffusion coefficient there is a Hilbert-Schmidt operator. For $\sigma_0 \neq 0$, note that $\sigma$ is not Hilbert-Schmidt. Therefore [1, Theorem 3.1] is unavailable for (1.1). Let

$$Z(t) := \int_0^t e^{(t-s)A}\sigma^0 dW(s)$$

and

$$\overline{X}_x(t) := X_x(t) - Z(t).$$

Then (1.1) can be rewritten in the form

$$d\overline{X}_x(t) = \{AX_x(t) + b(X_x(t))\}dt + \sigma^1(X_x(t))dW(t).$$

To be precise, (4.2) is first meant in the mild sense. But under (H1)-(H3) it also has a unique variation solution, and therefore the Itô formula applies to $\|\overline{X}_x(t)\|_H^2$. Carrying out similar arguments to those of Theorem 3.2 and Theorem 3.3 respectively, for some bounded subset $U \subseteq H$ we deduce that

$$\sup_{t \geq 0} \sup_{x \in U} \mathbb{E}\|X_x(t)\|_H^2 < \infty$$

and

$$\lim_{t \to \infty} \sup_{x, y \in U} \mathbb{E}\|X_x(t) - X_y(t)\|_H^2 = 0.$$ 

Then, following the argument of that of [1, Theorem 3.1] yields the desired assertion. □

Lemma 4.2. Let (H1) and (H2) hold and assume further that there exists $\delta_2 > 0$ and $\theta_2 \in (0, 1)$ such that

$$\int_0^\Delta \|e^{sA}\sigma^0\|_{HS}^2 ds \leq \delta_2 \Delta^{\theta_2}.$$ 

Then

$$\sup_{t \geq 0} \mathbb{E}\|\overline{Z}(t) - \overline{Z}([t])\|_H^2 \leq C\Delta^{\theta_1 \wedge \theta_2},$$

where $C > 0$ is a constant independent of $\Delta$.

Proof. Recall from [19, Theorem 6.13, p.74] that there exists $C_1 > 0$ such that

$$\|(-A)^{\alpha_1} e^{tA}\| \leq C_1 t^{-\alpha_1}, \quad \|(-A)^{-\alpha_2}(1 - e^{tA})\| \leq C_1 t^{\alpha_2},$$

for arbitrary $\alpha_1 \geq 0$, $\alpha_2 \in [0, 1]$, and that

$$(-A)^{\alpha_3 + \alpha_4} x = (-A)^{\alpha_3} (-A)^{\alpha_4} x, \quad x \in \mathcal{D}((-A)^\gamma),$$

for arbitrary $\alpha_3, \alpha_4 \geq 0$. Therefore

$$\int_0^\Delta \|e^{sA}\sigma^0\|_{HS}^2 ds \leq \delta_2 \Delta^{\theta_2}.$$
for any $\alpha_3, \alpha_4 \in \mathbb{R}$, where $\gamma := \max\{\alpha_3, \alpha_4, \alpha_3 + \alpha_4\}$. In the light of the independent increment of Wiener process and the Itô’s isometry,

$$\mathbb{E}\|\mathcal{Z}(t) - \mathcal{Z}(\lfloor t \rfloor)\|^2_H = \int_0^{\lfloor t \rfloor} \|(e^{(t-[t])A} - \text{id}_H)e^{([t]-s)A}\sigma^0\|^2_H ds + \int_{[t]}^t \|(e^{(t-s)A}\sigma^0\|^2_H ds.$$

This, combining (H2), (4.4), (4.6) with (4.7), yields that

$$\mathbb{E}\|\mathcal{Z}(t) - \mathcal{Z}(\lfloor t \rfloor)\|^2_H \leq \int_0^{\lfloor t \rfloor} \|(-A)^{-\theta_1}(e^{(t-[t])A} - \text{id}_H)\|^2 \cdot \|(-A)^{\theta_1}e^{([t]-s)A}\sigma^0\|^2_H ds + \int_{[t]}^t \|e^{sA}\sigma^0\|^2_H ds + 2\delta_2 \Delta \theta_2$$

and therefore the desired assertion follows.

**Remark 4.1.** Let $\sigma^0 = \text{id}_H$ and $A$ be the Laplace operator defined in Remark 1.1. A straightforward computation shows that

$$\int_0^{\Delta} \|e^{sA}\|^2_H ds = \frac{1}{2} \sum_{k=1}^\infty \frac{1}{k^2} (1 - e^{-2k^2 \Delta}).$$

Recall that for arbitrary $\delta \in (0, 1)$ and $x, y \geq 0$

$$|e^{-x} - e^{-y}| \leq |x - y|^{\delta}.$$ 

It then follows from (4.8) and (4.9) that

$$\int_0^{\Delta} \|e^{sA}\|^2_H ds \leq 2^{\delta - 1} \Delta \sum_{k=1}^\infty \frac{1}{k^2(1-\delta)}.$$ 

Hence, (4.4) holds with $\delta_2 = 2^{\delta - 1} \sum_{k=1}^\infty \frac{1}{k^2(1-\delta)}$ and $\theta_2 = \delta \in (0, 1/2)$.

**Lemma 4.3.** Let the assumptions of Lemma 4.1 hold and

$$\tau := \alpha^{-1}L_1 + (2\alpha)^{-1/2}L_2 \in (0, 1).$$

Then

$$\sup_{t \geq 0} \mathbb{E}\|X_x(t) - Y_n^{\tau}(t)\|^2_H \leq C\{\lambda_n^{-(\theta_1, \tau/2)} + \Delta^{\theta_1, \theta_2}\},$$

where $C > 0$ is a constant dependent on $x \in H$ but independent of $n$ and $\Delta$. 


Proof. By (1.3) and (4.3), it follows that

\[
(4.12) \quad \sup_{t \geq 0} \mathbb{E}\|b(X_x(t))\|_H^2 + \sup_{t \geq 0} \mathbb{E}\|\sigma^1(X_x(t))\|_{HS}^2 \leq C.
\]

Note that \((\mathbb{E}\| \cdot \|_H^2)^{1/2}\) is a norm and recall from [9] Theorem 202] the Minkowski integral inequality:

\[
\left( \mathbb{E} \left[ \int_0^t F(s)ds \right]^2 \right)^{1/2} \leq \int_0^t \left( \mathbb{E}\|F(s)\|^2 \right)^{1/2} ds, \quad t \geq 0,
\]

where \(F : [0, \infty) \times \Omega \to R\) is measurable and locally integrable. Then, applying the Itô isometry and using (H1), we obtain from (1.2) that

\[
(4.13) \quad \sum_{k=2}^5 F_k(t) = F_1(t) + F_2(t) + F_3(t) + F_4(t) + F_5(t).
\]

Let \(\rho := (\theta_1 \wedge \theta_2)/2\). In view of (4.6), (4.7), (H1) and the boundedness of \((-A)^{-(1-\rho)/2}\), one has

\[
F_1(t) = \|(-A)^{-(1-\rho)/2}e^{\theta(t)}(-A)^{-\rho/2}b(X_x(t))\|_H^2 \leq \|(-A)^{-(1-\rho)/2}e^{\theta(t)}\|^2 \cdot \|(-A)^{-\rho/2}b(X_x(t))\|_H^2 \leq C\|(-A)^{-(1-\rho/2)}\|^2 \cdot \|Ax\|_H^2 \Delta^\rho.
\]

Also, by (4.6) and (4.7), we obtain from (4.12) that for \(\tilde{\theta} \in (0, 1)\)

\[
\sum_{k=2}^5 F_k(t) \leq C\Delta^{1/2} + C \int_0^t \|(-A)^{\rho}e^{\tilde{\theta}(t-s)}A\| \cdot \|e^{(1-\tilde{\theta})(t-s)}A\| \cdot \|(-A)^{-\rho}b(-A)^{-(1-\rho/2)}A\| ds
\]

\[
+ C \left( \int_0^t \|(-A)^{\rho}e^{\tilde{\theta}(t-s)}A\|^2 \cdot \|e^{(1-\tilde{\theta})(t-s)}A\|^2 \cdot \|(-A)^{-\rho}b(-A)^{-(1-\rho/2)}A\|^2 ds \right)^{1/2}
\]

\[
\leq C\Delta^{1/2} + C\Delta^\rho \left( \int_0^t (\tilde{\theta}s)^{-\rho}e^{-\alpha(1-\tilde{\theta})s}ds + C\Delta^\rho \left( \int_0^t (\tilde{\theta}s)^{-2\rho}e^{-2\alpha(1-\tilde{\theta})s}ds \right)^{1/2} \right).
\]

Observe that

\[
\int_0^t s^{-\rho}e^{-\alpha(1-\tilde{\theta})s}ds \leq (\alpha(1-\tilde{\theta}))^{\rho-1} \int_0^\infty s^{-\rho}e^{-s}ds = (\alpha(1-\tilde{\theta}))^{\rho-1}\Gamma(1-\rho),
\]

...
and similarly
\[
\int_{0}^{[t]} s^{-2\rho} e^{-2\alpha(1-\tilde{\theta})s} ds \leq (2\alpha(1-\tilde{\theta}))^{2\rho-1}\Gamma(1-2\rho),
\]
where \(\Gamma(\cdot)\) is the Gamma function. Hence
\[
\sum_{k=2}^{4} F_k(t) \leq C \Delta^{(\theta_1 \wedge \theta_2)/2}.
\]
This, together with the estimate of \(F_1(t)\), gives that
\[
\sup_{t \geq 0} \mathbb{E} \|\overline{X}_x(t) - \overline{X}_x([t])\|^2_H \leq C \Delta^{\theta_1 \wedge \theta_2}.
\]
Noting that \(\overline{X}_x(t) = X_x(t) - Z(t)\) and utilizing (4.5), one has
\[
(4.15) \quad \sup_{t \geq 0} \mathbb{E} \|X_x(t) - X_x([t])\|^2_H \leq C \Delta^{\theta_1 \wedge \theta_2}.
\]
Since
\[
\|(\text{id}_H - \pi_n)(-A)^{-\theta_1}u\|^2_H = \left\| \sum_{k=n+1}^{\infty} \lambda_k^{-\theta_1} \langle u, e_k \rangle_H e_k \right\|^2_H \leq \lambda_n^{-2\theta_1} \|u\|^2_H, \quad u \in H,
\]
we arrive at
\[
(4.16) \quad \|(\text{id}_H - \pi_n)(-A)^{-\theta_1}\|^2 \leq \lambda_n^{-2\theta_1}.
\]
By virtue of the Itô isometry, (H2), (4.16), (4.6) and (4.7), it follows that
\[
\mathbb{E} \|Z(t) - Z^n,\Delta(t)\|^2_H \leq 2 \int_{0}^{t} \|e^{sA}(\text{id}_H - \pi_n)\sigma^0\|^2_{HS} ds
\]
\[
+ 2 \int_{0}^{t} \|(A)^{-\theta_1}(\text{id}_H - e^{(s-[s])A})(-A)^{\theta_1}e^{(t-s)A}\sigma^0\|^2_{HS} ds
\]
\[
\leq 2 \|(\text{id}_H - \pi_n)(-A)^{-\theta_1}\|^2 \int_{0}^{t} \|(A)\theta_1 e^{sA}\sigma^0\|^2_{HS} ds
\]
\[
+ C \Delta^{2\theta_1} \int_{0}^{t} \|(A)\theta_1 e^{sA}\sigma^0\|^2_{HS} ds
\]
\[
\leq C(\|(\text{id}_H - \pi_n)(-A)^{-\theta_1}\|^2 + \Delta^{2\theta_1}) \int_{0}^{t} \|(A)\theta_1 e^{sA}\sigma^0\|^2_{HS} ds
\]
\[
\leq C(\lambda_n^{-2\theta_1} + \Delta^{2\theta_1}).
\]
Following the argument of (4.13), we have

\[
(\mathbb{E}\|X_n(t) - Y_{x_n}^{\Delta}(t)\|_{H}^2)^{1/2} 
\leq \|e^{tA}(\text{id}_H - \pi_n)x\|_H
\]

\[
+ \int_0^t \|e^{(t-s)A}(\text{id}_H - \pi_n)||b(X_x(s))\|_{H}^2\mathrm{ds}^{1/2}
\]

\[
+ \left( \int_0^t \|e^{(t-s)A}(\text{id}_H - \pi_n)||\sigma_1^\| b_n(X_x(s)) - b_n(X_x([s]))\|_{H}^2\mathrm{ds}^{1/2}
\]

\[
+ \left( \int_0^t \|e^{(t-s)A}\|\|b_n(X_x([s])) - b_n(Y_{x_n}^{\Delta}([s]))\|_{H}^2\mathrm{ds}^{1/2}
\]

\[
+ \left( \int_0^t \|e^{(t-s)A}\|\|\sigma_1^b(X_x([s])) - \sigma_1^b(Y_{x_n}^{\Delta}([s]))\|_{H}^2\mathrm{ds}^{1/2}
\]

\[
+ \left( \int_0^t \|e^{(t-s)A}\{\text{id}_H - e^{(s-[s])A}\}||b(Y_{x_n}^{\Delta}([s]))\|_{H}^2\mathrm{ds}^{1/2}
\]

\[
+ \left( \int_0^t \|e^{(t-s)A}\{\text{id}_H - e^{(s-[s])A}\}||\sigma^1(Y_{x_n}^{\Delta}([s]))\|_{H}^2\mathrm{ds}^{1/2}
\]

\[
= : \sum_{i=1}^9 G_i(t).
\]

A straightforward computation shows that

\[
\|e^{tA}(\text{id}_H - \pi_n)u\|_{H}^2 = \sum_{i=n+1}^{\infty} e^{-\lambda_i t}\langle u, e_i \rangle_{H}^2, \quad u \in H.
\]

This further gives that

\[
(4.19) \quad \|e^{tA}(\text{id}_H - \pi_n)\|_{H}^2 \leq e^{-\lambda_n t}
\]

and that

\[
(4.20) \quad G_1(t) \leq \left( \sum_{i=n+1}^{\infty} \frac{e^{-\lambda_i t}}{\lambda_i} \lambda_i^2 \langle x, e_i \rangle_{H}^2 \right)^{1/2} \leq \lambda_n^{-1} \|Ax\|_H
\]

by recalling that \(\{\lambda_i\}_{i \geq 1}\) is a nondecreasing sequence. By (4.12) and (4.19), one has

\[
G_2(t) + G_3(t) \leq C \int_0^t \|e^{(t-s)A}(\text{id}_H - \pi_n)\|_{H}^2\mathrm{ds} + C\left( \int_0^t \|e^{(t-s)A}(\text{id}_H - \pi_n)\|_{H}^2\mathrm{ds} \right)^{1/2}
\]

\[
\leq C \int_0^t e^{-\lambda_n (t-s)}\mathrm{ds} + C\left( \int_0^t e^{-2\lambda_n (t-s)}\mathrm{ds} \right)^{1/2} \leq C(\lambda_n^{-1} + \lambda_n^{-1/2}).
\]
Taking (H1), (H3) and (4.15) into account gives that

\[
G_4(t) + G_5(t) \leq C \Delta^{(\theta_1,\theta_2)/2} \left\{ \int_0^t \|e^{(t-s)A}\|ds + \left( \int_0^t \|e^{(t-s)A}\|^2ds \right)^{1/2} \right\}
\]

\[
\leq C \Delta^{(\theta_1,\theta_2)/2}.
\]

Next, note from (H1) and (H3) that

\[
G_6(t) + G_7(t)
\]

\[
\leq \sup_{0 \leq s \leq t} (E \|b(X_x([s])) - b(Y_{x_n}^n([s]))\|^2_A^{1/2} \int_0^t \|e^{(t-s)A}\|ds
\]

\[
+ \sup_{0 \leq s \leq t} (E \|\sigma(X_x([s])) - \sigma(Y_{x_n}^n([s]))\|^2_A^{1/2} \left( \int_0^t \|e^{(t-s)A}\|^2ds \right)^{1/2}
\]

\[
\leq \alpha^{-1} \sup_{0 \leq s \leq t} (E \|b(X_x([s])) - b(Y_{x_n}^n([s]))\|^2_A^{1/2}
\]

\[
+ (2\alpha)^{-1/2} \sup_{0 \leq s \leq t} (E \|\sigma(X_x([s])) - \sigma(Y_{x_n}^n([s]))\|^2_A^{1/2})
\]

\[
\leq \tau \sup_{0 \leq s \leq t} (E \|X_x(s) - Y_{x_n}^n(s)\|^2_A^{1/2}
\]

\[
\leq \tau \sup_{0 \leq s \leq t} (E \|X_x(s) - Y_{x_n}^n(s)\|^2_A^{1/2} + \tau \sup_{0 \leq s \leq t} (E \|X(s) - Z_{x_n}^n(s)\|^2_A^{1/2}),
\]

where \( \tau \in (0, 1) \) is defined by (4.10). Following the argument of (4.14) leads to

\[
G_8(t) + G_9(t) \leq C \Delta^{(\theta_1,\theta_2)/2}.
\]

Substituting (4.20)-(4.24) into (4.18) yields that

\[
\sup_{t \geq 0} (E \|X_x(t) - Y_{x_n}^n(t)\|^2_A^{1/2}) \leq C(\lambda_n^{-1/2} + \Delta^{(\theta_1,\theta_2)/2})
\]

due to \( \tau \in (0, 1) \). Consequently the desired assertion follows from (4.17). \( \square \)

**Theorem 4.4.** Assume that (H1)-(H4), (4.3) and (4.10) hold. Then, there exists a \( \Delta_n \) such that \( \lim_{n \to \infty} \Delta_n = 0 \)

\[
\lim_{n \to \infty} d_L(\pi_{\Delta_n}^n(\cdot), \pi(\cdot)) = 0.
\]

**Proof.** Fix \( x \in H \) and let \( \epsilon > 0 \) be arbitrary. By Lemma 4.3 there exist a sufficiently large \( n \in \mathbb{N} \) and a \( \Delta_n \) sufficiently small such that such that

\[
d_L(\mathbb{P}_{k_{\Delta_n}^n}(x, \cdot), \mathbb{P}_{n_{\Delta_n}^n}^n(x_n, \cdot)) \leq \epsilon/3.
\]

For the previous \( n \in \mathbb{N} \), by Theorem 2.2 there exist a sufficiently small \( \Delta_n \) and \( T_1 > 0 \) such that

\[
d_L(\mathbb{P}_{n_{\Delta_n}^n}(x_n, \cdot), \pi_{\Delta_n}^n(\cdot)) \leq \epsilon/3
\]

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whenever $k\tilde{\Delta}_n \geq T_1$. Furthermore, due to Lemma 4.1 there exists $T_2 > 0$ such that
\[ d_L(\mathbb{P}_t(x, \cdot), \pi(\cdot)) \leq \epsilon, \quad t \geq T_2. \]

Let $T := T_1 \lor T_2$, \( \triangle_n = \overline{\triangle}_n \land \tilde{\triangle}_n \) and $k = [T/\triangle_n] + 1$. Then the desired assertion follows from the triangle inequality
\[ d_L(\pi^{n,\triangle}(\cdot), \pi(\cdot)) \leq d_L(\mathbb{P}_{k\triangle}(x, \cdot), \pi(\cdot)) + d_L(\mathbb{P}_{k\triangle}(x, \cdot), \mathbb{P}_{k\triangle}^{n}(x_n, \cdot)) \]
\[ + d_L(\mathbb{P}_{k\triangle}^{n}(x_n, \cdot), \pi^{n,\triangle}(\cdot)). \]

Remark 4.2. For the finite-dimensional case, finite-time convergence of numerical scheme is enough to discuss the limit of stationary distribution of numerical solution [18, Theorem 6.23, p.266]. While for the infinite-dimensional case, we need the uniform convergence of EI scheme (1.7) to reveal the limit behavior of $\pi^{n,\triangle}$, which is quite different from the finite-dimensional cases, and therefore (1.10) is imposed. On the other hand, for the finite-time convergence of EM scheme (1.8), condition (1.10) can be deleted by checking the argument of Lemma 4.3 and combining with the Gronwall inequality.

Remark 4.3. By following the procedure of this paper, numerical approximation of stationary distribution of SPDEs with jumps can also be discussed, which will be reported in forthcoming paper.

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References

[1] Bao, J., Hou, Z., Yuan, C., Stability in distribution of mild solutions to stochastic partial differential equations, Proc. Amer. Math. Soc., 138 (2010), 2169–2180.

[2] Bréhier, C.-É., Approximation of the invariant measure with a Euler scheme for Stochastic PDE’s driven by Space-Time White Noise, http://arxiv.org/abs/1202.2707.

[3] Caraballo, T., Kloeden, P.E., The Pathwise Numerical Approximation of Stationary Solutions of Semilinear Stochastic Evolution Equations, Appl. Math. Optim., 54 (2006), 401–415.

[4] Da Prato, G., Zabczyk, J., Ergodicity for Infinite Dimensional Systems, Cambridge University Press, 1996.

[5] Da Prato G., Jentzen, A. and Röckner M., A mild Itô formula for SPDEs, arXiv:1009.3526v3.

[6] Debussche, A., Weak approximation of stochastic partial differential equations :the non-linear case, arXiv:0804.1304v1.
[7] Grecksch, W., Kloeden, P.E., Time-discretised Galerkin approximations of parabolic stochastic PDEs, *Bull. Austral. Math. Soc.*, **54** (1996), 79–85.

[8] Gyöngy, I., Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise: I, *Potential Anal.*, **9** (1998), 1–25; II, *Potential Anal.*, **11** (1999), 1–37.

[9] Hardy, G. H., Littlewood, J. E., Pólya, G., *Inequalities*, Cambridge University Press, 1952.

[10] Hausenblas, E., Numerical analysis of semilinear stochastic evolution equations in Banach spaces, *J. Comput. Appl. Math.*, **147** (2002), 485–516.

[11] Hausenblas, E., Approximation for Semilinear Stochastic Evolution Equations, *Potential Anal.*, **18** (2003), 141–186.

[12] Ikeda, N., Watanabe, S., *Stochastic Differential Equations and Diffusion Processes* (Amsterdam: North-Holland), 1981.

[13] Jentzen, A., Kloeden, P.E., *Taylor approximations for stochastic partial differential equations*, CBMS-NSF Regional Conference Series in Applied Mathematics, 2011.

[14] Jentzen, A., Higher order pathwise numerical approximations of SPDEs with additive noise, *SIAM J. Numer. Anal.*, **49** (2011), 642–667.

[15] Kato, T. *Perturbation Theory for Linear Operators*, Springer, New York, 1966.

[16] Kloeden, P.E., Platen, E., *Numerical solution of stochastic differential equations*, Springer-Verlag, Berlin, 1992.

[17] Kloeden, P.E., Lord, G.J., Neuenkirch, A., Shardlow, T., The exponential integrator scheme for stochastic partial differential equations: pathwise error bounds, *J. Comput. Appl. Math.*, **235** (2011), 1245–1260.

[18] Mao, X., Yuan, C., *Stochastic Differential Equations with Markovian Switching*, Imperial College, London, 2006.

[19] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, No. 44, Springer Verlag, New York, 1983.

[20] Schurz, H., *Stability, stationarity, and boundedness of some implicit numerical methods for stochastic differential equations and applications*, Berlin, 1997.

[21] Shardlow, T., Numerical methods for stochastic parabolic PDEs, *Numer. Fund. Anal. Optim.*, **20** (1999), 121–145.

[22] Yevik, A., Zhao, H., Numerical approximations to the stationary solutions of stochastic differential equations, *SIAM J. Numer. Anal.*, **49** (2011), 1397–1416.
[23] Yoo, H., Semi-discretization of stochastic partial differential equations on $\mathbb{R}^1$ by a finite-difference method, *Math. Comput.*, **69** (2000), 653–666.

[24] Yuan, C., Mao, X., Stability in Distribution of Numerical Solutions for Stochastic Differential Equations, *Stoch. Anal. Appl.*, **25** (2004), 1133–1150.