Out–of–equilibrium dynamics of $\phi^4$ QFT in finite volume

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Abstract

The $\lambda\phi^4$ model in a finite volume is studied in the infinite $N$ limit and within a non–gaussian Hartree–Fock approximation both at equilibrium and out of equilibrium, with particular attention to certain fundamental features of the broken symmetry phase. The numerical solution of the dynamical evolution equations show that the zero–mode quantum fluctuations cannot grow macroscopically large starting from microscopic initial conditions. Thus we conclude that there is no evidence for a dynamical Bose–Einstein condensation. On the other hand, out of equilibrium the long–wavelength fluctuations do scale with the linear size of the system, signalling dynamical infrared properties quite different from the equilibrium ones characteristic of the same approximation schemes. This result suggests the cause, and the possible remedy, of some unlikely features of the application to out–of–equilibrium dynamics of the standard HF factorization scheme, which coincides with the gaussian restriction of our Hartree–Fock approximation.

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I. INTRODUCTION

In the last few years a great deal of attention has been paid to the study of interacting quantum fields out of equilibrium. There are, in fact, many interesting physical situations in which the standard S–matrix approach cannot give sensible information about the behavior of the system, because it evolves through a series of highly excited states (i.e., states of finite energy density).

As an example consider any model of cosmological inflation: it is not possible to extract precise predictions on physical observables without including in the treatment the quantum back–reaction of the field on the space–time geometry and on itself [1–3].

On the side of particle physics, the ultra-relativistic heavy-ion collisions, scheduled in the forthcoming years at CERN–SPS, BNL–RHIC and CERN–LHC, are supposed to produce hadron matter at very high densities and temperatures; in such a regime the usual approach based on particle scattering cannot be considered a good interpretative tool at all. To extract sensible information from the theory new computational schemes are necessary, that go beyond the simple Feynmann diagram expansion. The use of resummation schemes, like the Hartree–Fock (HF) approximation and the large $N$ limit (LN) [4], or the Hard Thermal Loop resummation for systems at finite temperature (HTL) [7], can be considered a first step in this direction. They, in fact, enforce a sum over an infinite subset of Feynmann diagrams that are dominant in a given region of the parameter space, where the simple truncation of the usual perturbative series at finite order cannot give sensible answers.

Quite recently HF and LN have been used in order to clarify some dynamical aspects of the large $N$ $\phi^4$ theory, reaching the conclusion that the non–perturbative and non–linear evolution of the system might eventually produce the onset of a form of non–equilibrium Bose–Einstein condensation of the long–wavelength Goldstone bosons usually present in the broken symmetry phase [8–10] of the model. Another very interesting result in [8] concerns the dynamical Maxwell construction, which reproduces the flat region of the effective potential in case of broken symmetry as asymptotic fixed points of the background evolution.

In this article we present a detailed study, in finite volume, of dynamical evolution out of equilibrium for the $\Phi^4$ scalar field. More precisely, we determine how such dynamics scales with the size of the periodic box containing the system in the case of uniform back–grounds. This is necessary to address questions like out–of–equilibrium symmetry breaking and dynamical Bose–Einstein condensation.

We apply two of the non–perturbative methods mentioned above, namely the Hartree–Fock approximation and the large $N$ expansion.

In section II we define the model in finite volume, giving all the relevant notations and definitions. We also stress the convexity of the effective potential as an exact result, valid for the full renormalized theory in any volume.

In section III we derive the large $N$ approximation of the $O(N)$–invariant version of $\lambda(\phi^2)^2$ model, according to the general rules of ref. [11]. In this derivation it appears evident the essential property of the $N \to \infty$ limit of being a particular type of classical limit, so that it leads to a classical phase space, a classical hamiltonian with associated Hamilton’s equations of motion [see eqs. (3.15), (3.16) and (3.17)]. We then minimize the hamiltonian function(al) and determine the conditions when massless Goldstone bosons (i.e. transverse fluctuations of the field) to form a Bose–Einstein condensate, delocalizing the vacuum field.
expectation value (cfr. also ref. [12]). This necessarily requires that the width of the zero–mode fluctuations becomes macroscopically large, that is of the order of the volume. Only when the background takes one of the extremal values proper of symmetry breaking the width of the zero–mode fluctuations is of order $L^{1/2}$, as typical of a free massless spectrum.

The study of the lowest energy states of the model is needed for comparison with the results of the numerical simulations, which show that the zero–mode width $\sigma_0$ stays microscopic (that is such that $\sigma_0$/volume $\to 0$ when the volume diverges) whenever it starts from initial conditions in which it is microscopic. Our results, in fact, show clearly the presence of a time scale $\tau_L$, proportional to the linear size $L$ of the system, at which finite volume effects start to manifest. The most remarkable consequence of the presence of such a scale is that it prevents the zero mode amplitude to grow macroscopically large. This result contradicts the interpretation of the linear late–time growth of the zero–mode width as a Bose–Einstein condensation of Goldstone bosons [8–10].

On the other hand we do find that the size of the low–lying widths at time $\tau_L$ is of order $L$, to be compared to the equilibrium situation where they would be of order $L^0$ in the massive case or of order $L^{1/2}$ in the massless case. Perhaps the denomination “microscopic” should be reserved to this two possibilities. Therefore, since our initial condition are indeed microscopic in this restricted sense, we do observe in the out–of–equilibrium evolution a rapid transition to a different regime intermediate between the microscopic one and the macroscopic one characteristic of Bose–Einstein condensation.

At any rate, when one considers microscopic initial conditions for the choice of bare mass which corresponds to broken symmetry, the role itself of symmetry breaking is not very clear in the large $N$ description of the out–of–equilibrium dynamics, making equally obscure the issues concerning the so–called quantum phase ordering [8]. This is because the limit $N \to \infty$ is completely saturated by gaussian states, which might signal the onset of symmetry breaking only developing macroscopically large fluctuations. Since such fluctuations do not appear to be there, the meaning itself of symmetry breaking, as something happening as times goes on and accompanied by some kind of phase ordering, is quite unclear. Of course, in this respect the main limitation of our approach, as well as of those of the references mentioned above, is in the assumption of a uniform background. Nonetheless, phenomena like the asymptotic vanishing of the effective mass and the dynamical Maxwell construction, taking place in this contest of a uniform background and large $N$ expansion, are certainly very significant manifestations of symmetry breaking and in particular of the Goldstone theorem which applies when a continuous symmetry is broken.

To gather more information on these matters, we consider in section IV a more elaborate type of time–dependent Hartree–Fock (tdHF) approximation, which generalizes the standard gaussian self–consistent approach to non–gaussian wave–functionals; in fact, one might envisage the possibility that, while gaussian fluctuations never become of the size of the volume, non–gaussian fluctuations do grow in time to a macroscopic size. We derive therefore the mean–field coupled time–dependent Schroedinger equations for the modes of the scalar field, under the assumption of a uniform condensate, see eqs (4.5), (4.6) and (4.7). A significant difference with respect to previous tdHF approaches [14] concerns the renormalization of ultraviolet divergences. In fact, by means of a single proper substitution of the bare coupling constant $\lambda_b$ with the renormalized one $\lambda$ in the Hartree–Fock hamiltonian, we obtain completely cut–off independent equations (apart from the corrections in reverse
power of the cutoff which are there due to the Landau pole). The substitution is introduced by hand, but is justified by simple diagrammatic considerations.

One advantage of not restricting a priori the self-consistent HF approximation to gaussian wave–functionals, is in the possibility of a better description of the structure of the vacuum in case of broken symmetry. In fact we can show quite explicitly that, in any finite volume, the ground state the zero–mode of the \( \phi \) field is concentrated around the two vacua of the broken symmetry, driving the probability distribution for any sufficiently wide smearing of the field into a two peaks shape. This is indeed what one would intuitively expect in case of symmetry breaking. On the other hand none of this appears in a dynamical evolution that starts from a distribution localized around a single value of the field in the spinodal region, confirming what already seen in the large \( N \) approach. More precisely, within a further controlled gaussian approximation of our tdHF approach, one observe that initially microscopic quantum fluctuations never becomes macroscopic, suggesting that also non–gaussian fluctuations cannot reach macroscopic sizes. As a simple confirmation of this fact, consider the completely symmetric initial conditions \( \langle \phi \rangle = \langle \dot{\phi} \rangle = 0 \) for the background: in this case we find that the dynamical equations for initially gaussian field fluctuations are identical to those of large \( N \) [apart for a rescaling of the coupling constant by a factor of three; compare eqs. (3.13) and (4.28)], so that we observe the same asymptotic vanishing of the effective mass. However, this time no interpretation in terms of Goldstone theorem is possible, since the broken symmetry is discrete; rather, if the width of the zero–mode were allowed to evolve into a macroscopic size, then the effective mass would tend to a positive value, since the mass in case of discrete symmetry breaking is indeed larger than zero.

On the other hand, also in the gaussian HF approach, we do find that the size of the low–lying widths at time \( \tau_L \) is of order \( L \). We then discuss why this undermine the self–consistency of the gaussian approximation, imposing the need of further study, both analytical and numerical.

Finally, in section \( \S \) we summarize the results presented in this article and we sketch some interesting open problems that we plan to study in forthcoming works.

II. CUTOFF FIELD THEORY

We consider the scalar field operator \( \phi \) in a \( D \)--dimensional periodic box of size \( L \) and write its Fourier expansion as customary

\[
\phi(x) = L^{-D/2} \sum_k \phi_k e^{i k \cdot x}, \quad \phi_k^\dagger = \phi_{-k}
\]

with the wavevectors \( k \) naturally quantized: \( k = (2\pi/L)n, \ n \in \mathbb{Z}^D \). The canonically conjugated momentum \( \pi \) has a similar expansion

\[
\pi(x) = L^{-D/2} \sum_k \pi_k e^{i k \cdot x}, \quad \pi_k^\dagger = \pi_{-k}
\]

with the commutation rules \( [\phi_k, \pi_{-k'}] = i \delta^{(D)}_{kk'} \). The introduction of a finite volume should be regarded as a regularization of the infrared properties of the model, which allows to “count” the different field modes and is needed especially in the case of broken symmetry.
To regularize also the ultraviolet behavior, we restrict the sums over wavevectors to the points lying within the $D$-dimensional sphere of radius $\Lambda$, that is $k^2 \leq \Lambda^2$, with $N = \Lambda L / 2\pi$ some large integer. Clearly we have reduced the original field-theoretical problem to a quantum–mechanical framework with finitely many (of order $N^{D-1}$) degrees of freedom.

The $\phi^4$ Hamiltonian reads

$$H = \frac{1}{2} \int d^D x \left[ \pi^2 + (\partial \phi)^2 + m_b^2 \phi^2 + \lambda_b \phi^4 \right] = \frac{1}{2} \sum_k \left[ \pi_k \pi_{-k} + (k^2 + m_b^2) \phi_k \phi_{-k} \right] + \frac{\lambda}{4} \sum_{k_1,k_2,k_3,k_4} \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \delta^{(D)}_{k_1+k_2+k_3+k_4,0}$$

where $m_b^2$ and $\lambda_b$ should depend on the UV cutoff $\Lambda$ in such a way to guarantee a finite limit $\Lambda \to \infty$ for all observable quantities. As is known \[13, 14\], this implies triviality (that is vanishing of renormalized vertex functions with more than two external lines) for $D > 3$ and very likely also for $D = 3$. In the latter case triviality is manifest in the one–loop approximation and in large–$N$ limit due to the Landau pole. For this reason we shall keep $\Lambda$ finite and regard the $\phi^4$ model as an effective low–energy theory (here low–energy means practically all energies below Planck’s scale, due to the large value of the Landau pole for renormalized coupling constants of order one or less).

We shall work in the wavefunction representation where $\langle \phi | \Psi \rangle = \Psi(\phi)$ and

$$\langle \phi_0 | \Psi \rangle = \phi_0 \Psi(\phi), \quad \langle \pi_0 | \Psi \rangle = -i \frac{\partial}{\partial \phi_0} \Psi(\phi)$$

while for $k > 0$ (in lexicographic sense)

$$\langle \phi_{\pm k} | \Psi \rangle = \frac{1}{\sqrt{2}} (\varphi_{k} \pm i \varphi_{-k}) \Psi(\phi), \quad \langle \pi_{\pm k} | \Psi \rangle = \frac{1}{\sqrt{2}} \left( -i \frac{\partial}{\partial \varphi_{k}} \pm \frac{\partial}{\partial \varphi_{-k}} \right) \Psi(\phi)$$

Notice that by construction the variables $\varphi_k$ are all real. Of course, when either one of the cutoffs are removed, the wave function $\Psi(\phi)$ acquires infinitely many arguments and becomes what is usually called a wavefunctional.

In practice, the problem of studying the dynamics of the $\phi^4$ field out of equilibrium consists now in trying to solve the time-dependent Schroedinger equation given an initial wavefunction $\Psi(\phi, t = 0)$ that describes a state of the field far away from the vacuum. By this we mean a non–stationary state that, in the infinite volume limit $L \to \infty$, would lay outside the particle Fock space constructed upon the vacuum. This approach could be generalized in a straightforward way to mixtures described by density matrices, as done, for instance, in [16–18]. Here we shall restrict to pure states, for sake of simplicity and because all relevant aspects of the problem are already present in this case.

It is by now well known \[14\] that perturbation theory is not suitable for the purpose stated above. Due to parametric resonances and/or spinodal instabilities there are modes of the field that grow exponentially in time until they produce non–perturbative effects for any coupling constant, no matter how small. On the other hand, only few, by now standard, approximate non–perturbative schemes are available for the $\phi^4$ theory, and to these we have to resort after all. We shall consider here the time-dependent Hartree–Fock (tdHF) approach (an improved version with respect to what is presented, for instance, in \[4\]) and the large $N$
expansion to leading order. In fact these two methods are very closely related, as shown for instance in [13], where several techniques to derive reasonable dynamical evolution equations for non–equilibrium $\phi^4$ are compared. However, before passing to approximations, we would like to stress that the following rigorous result can be immediately established in this model with both UV and IR cutoffs.

A. A rigorous result: the effective potential is convex

This is a well known fact in statistical mechanics, being directly related to stability requirements. It would therefore hold also for the field theory in the Euclidean functional formulation. In our quantum–mechanical context we may proceed as follow. Suppose the field $\phi$ is coupled to a uniform external source $J$. Then the ground state energy $E_0(J)$ is a concave function of $J$, as can be inferred from the negativity of the second order term in $\Delta J$ of perturbation around any chosen value of $J$. Moreover, $E_0(J)$ is analytic in a finite neighborhood of $J = 0$, since $J\phi$ is a perturbation “small” compared to the quadratic and quartic terms of the Hamiltonian. As a consequence, this effective potential $V_{\text{eff}}(\bar{\phi}) = E_0(J) - J\bar{\phi}, \bar{\phi} = E_0'(J) = \langle \phi \rangle_0$, that is the Legendre transform of $E_0(J)$, is a convex analytic function in a finite neighborhood of $\bar{\phi} = 0$. In the infrared limit $L \to \infty$, $E_0(J)$ might develop a singularity in $J = 0$ and $V_{\text{eff}}(\bar{\phi})$ might flatten around $\bar{\phi} = 0$. Of course this possibility would apply in case of spontaneous symmetry breaking, that is for a double–well classical potential. This is a subtle and important point that will play a crucial role later on, even if the effective potential is relevant for the static properties of the model rather than the dynamical evolution out of equilibrium that interests us here. In fact such evolution is governed by the CTP effective action [19,20] and one might expect that, although non–local in time, it asymptotically reduces to a multiple of the effective potential for trajectories of $\bar{\phi}(t)$ with a fixed point at infinite time. In such case there should exist a one–to–one correspondence between fixed points and minima of the effective potential. This is one of the topics addressed in this paper.

III. LARGE $N$ EXPANSION AT LEADING ORDER

A. Definitions

In this section we consider a standard non–perturbative approach to the $\phi^4$ model which is applicable also out of equilibrium, namely the large $N$ method as presented in [21]. However we shall follow a different derivation which makes the gaussian nature of the $N \to \infty$ limit more explicit.

In the large $N$ method one generalizes the $\phi^4$ model by promoting the single real scalar field $\phi$ to a $N$–component vector $\phi$ of scalar fields, in such a way to ensure $O(N)$ symmetry. This corresponds to theHamiltonian

$$H = \frac{1}{2} \int d^Dx \left[ \pi^2 + (\partial \phi)^2 + m_b^2 \phi^2 + \lambda_b (\phi^2)^2 \right] =$$

$$= \frac{1}{2} \sum_k \left[ \pi_k \cdot \pi_{-k} + (k^2 + m_b^2) \phi_k \cdot \phi_{-k} \right] + \frac{\lambda_b}{4L^D} \sum_{k_1,k_2,k_3,k_4} (\phi_{k_1} \cdot \phi_{k_2})(\phi_{k_3} \cdot \phi_{k_4}) \delta^{(D)}_{k_1+k_2+k_3+k_4,0}$$
where the space integration and the sum over wavevectors are limited by the infrared and ultraviolet cutoff, respectively, according to the general framework presented in section II.

It is known that this theory is well behaved for large $N$, provided that the quartic coupling constant $\lambda$ is rescaled with $1/N$. For example, it is possible to define a perturbation theory, based on the small expansion parameter $1/N$, in the framework of which one can compute any quantity at any chosen order in $1/N$. From the diagrammatic point of view, this procedure corresponds to a resummation of the usual perturbative series that automatically collects all the graphs of a given order in $1/N$ together \[6\]. Moreover, it has been established since the early 80’s that the leading order approximation (that is the strict limit $N \to \infty$) is actually a classical limit \[11\], in the sense that there exists a classical system (i.e., a classical phase space, a Poisson bracket and a classical Hamiltonian) whose dynamics controls the evolution of all fundamental quantum observables, such as field correlation functions, in the $N \to \infty$ limit. For instance, from the absolute minimum of the classical Hamiltonian one reads the energy of the ground state, while the spectrum is given by the frequencies of small oscillations about this minimum, etc. etc.. We are here interested in finding an efficient and rapid way to compute the quantum evolution equations for some observables in the $N \to \infty$ limit, and we will see that this task is easily accomplished just by deriving the canonical Hamilton equations from the large $N$ classical Hamiltonian.

Following Yaffe \[11\], we write the quantum mechanical hamiltonian as

$$H = Nh(A, C)$$

(3.1)

in terms of the square matrices $A, C$ with operator entries ($\varpi_k$ is the canonical momentum conjugated to the real mode $\varphi_k$)

$$A_{kk'} = \frac{1}{N} \varphi_k \cdot \varphi_{k'}, \quad C_{kk'} = \frac{1}{N} \varpi_k \cdot \varpi_{k'}$$

These are example of “classical” operators, whose two-point correlation functions factorize in the $N \to \infty$ limit. This can be shown by considering the coherent states

$$\Psi_{z,q,p}(\varphi) = C(z) \exp \left[ i \sum_k p_k \cdot \varphi_k - \frac{1}{2N} \sum_{kk'} z_{kk'}(\varphi_k - q_k) \cdot (\varphi_{k'} - q_{k'}) \right]$$

(3.2)

where the complex symmetric matrix $z$ has a positive definite real part while $p_k$ and $q_k$ are real and coincide, respectively, with the coherent state expectation values of $\varpi_k$ and $\varphi_k$. As these parameters take all their possible values, the coherent states form an overcomplete set in the cutoff Hilbert space of the model. The crucial property which ensures factorization is that they become all orthogonal in the $N \to \infty$ limit. Moreover one can show \[11\] that the coherent states parameters form a classical phase space with Poisson brackets

$$\{ q^i_k, p_{k'}^j \}_{\text{P.B.}} = \delta_{kk'} \delta^{ij}, \quad \{ w_{kk'}, v_{qq'} \}_{\text{P.B.}} = \delta_{kk'} \delta_{qq'} + \delta_{kk'} \delta_{qq'}$$

where $w$ and $v$ reparametrize $z$ as $z = \frac{1}{2} w^{-1} + i v$. It is understood that the dimensionality of the vectors $q_k$ and $p_k$ is arbitrary but finite [that is, only a finite number, say $n$, of pairs $(\varphi_k, \varpi_k)$ may take a nonvanishing expectation value as $N \to \infty$].
Once applied to the classical operators $A_{kk'}$ and $C_{kk'}$ the large $N$ factorization allow to obtain the classical hamiltonian by simply replacing $A$ and $C$ in eq. (3.1) by the coherent expectation values

$$
\langle A_{kk'} \rangle = q_k \cdot q_{k'} + w_{kk'}, \quad \langle C_{kk'} \rangle = p_k \cdot p_{k'} + (v w v)_{kk'} + \frac{1}{4}(w^{-1})_{kk'}
$$

In our situation, having assumed a uniform background expectation value for $\varphi$, we have $q_k = p_k = 0$ for all $k \neq 0$; moreover, translation invariance implies that $w$ and $v$ are diagonal matrices, so that we may set

$$
w_{kk'} = \sigma^2_k \delta_{kk'}, \quad v_{kk'} = s_k \sigma_k \delta_{kk'}
$$

in term of the canonical couples $(\sigma_k, s_k)$ which satisfy $\{\sigma_k, s_{k'}\}_{P.B.} = \delta_{kk'}$. Notice that the $\sigma_k$ are just the widths (rescaled by $N^{-1/2}$) of the $O(N)$ symmetric and translation invariant gaussian coherent states.

Thus we find the classical hamiltonian

$$
h_{cl} = \frac{1}{2}(p_0^2 + m_b^2 q_0^2) + \frac{1}{2} \sum_k \left[ s_k^2 + (k^2 + m_b^2) \sigma_k^2 + \frac{1}{4 \sigma_k^2} \right] + \frac{\lambda_b}{4 L^D} \left( q_0^2 + \sum_k \sigma_k^2 \right)^2
$$

where by Hamilton’s equations of motion $p_0 = \dot{q}_0$ and $s_k = \dot{\sigma}_k$. The corresponding conserved energy density $E = L^{-D} h_{cl}$ may be written

$$
E = \mathcal{T} + \mathcal{V}, \quad \mathcal{T} = \frac{1}{2} \dot{\vec{q}}^2 + \frac{1}{2 L^D} \sum_k \dot{\sigma}_k^2 \\
\mathcal{V} = \frac{1}{2 L^D} \sum_k \left( k^2 \sigma_k^2 + \frac{1}{4 \sigma_k^2} \right) + V(\vec{\varphi}^2 + \Sigma), \quad \Sigma = \frac{1}{L^D} \sum_k \sigma_k^2
$$

where $\vec{\varphi} = L^{-D/2} q_0$ and $V$ is the $O(N)$—invariant quartic potential regarded as a function of $\varphi^2$, that is $V(z) = \frac{1}{2} m_b^2 z + \frac{1}{4} \lambda_b z^2$. It is worth noticing that eq. (3.3) would apply as is to generic $V(z)$.

### B. Static properties

Let us consider first the static aspects embodied in the effective potential $V_{eff}(\vec{\varphi})$, that is the minimum of the potential energy $\mathcal{V}$ at fixed $\vec{\varphi}$. We first define in a precise way the unbroken symmetry phase, in this large $N$ context, as the case when $V_{eff}(\vec{\varphi})$ has a unique minimum at $\vec{\varphi} = 0$ in the limit of infinite volume. Minimizing $\mathcal{V}$ w.r.t. $\sigma_k$ yields

$$
\sigma_k^2 = \frac{1}{2 \sqrt{k^2 + M^2}}, \quad M^2 = m_b^2 + 2 V'(\vec{\varphi}^2 + \Sigma) = m_b^2 + \lambda_b \vec{\varphi}^2 + \frac{\lambda_b}{L^D} \sum_k \frac{1}{2 \sqrt{k^2 + M^2}}
$$

that is the widths characteristic of a free theory with self–consistent mass $M$ fixed by the gap equation. By the assumption of unbroken symmetry, when $\vec{\varphi} = 0$ and at infinite volume $M$
coincides with the equilibrium mass $m$ of the theory, that may be regarded as independent scale parameter. Since in the limit $L \to \infty$ sums are replaced by integrals

$$
\Sigma \to \int_{k^2 \leq \Lambda^2} \frac{d^Dk}{(2\pi)^D} \sigma_k^2
$$

we obtain the standard bare mass parameterization

$$
m_b^2 = m^2 - \lambda_b I_D(m^2, \Lambda), \quad I_D(z, \Lambda) \equiv \int_{k^2 \leq \Lambda^2} \frac{d^Dk}{2\sqrt{k^2 + z}} (2\pi)^D \sigma_k^2
$$

and the renormalized gap equation

$$
M^2 = m^2 + \lambda \bar{\phi}^2 + \lambda \left[ I_D(M^2, \Lambda) - I_D(m^2, \Lambda) \right]_{\text{finite}}
$$

which implies, when $D = 3$,

$$
\lambda_b = \lambda \left( 1 - \frac{\lambda}{8\pi^2} \log \frac{2\Lambda}{m\sqrt{e}} \right)^{-1}
$$

with a suitable choice of the finite part. No coupling constant renormalization occurs instead when $D = 1$. The renormalized gap equation (3.6) may also be written quite concisely

$$
\frac{M^2}{\lambda(M)} = \frac{m^2}{\lambda(m)} + \bar{\phi}^2
$$

in terms of the one–loop running couplings constant

$$
\hat{\lambda}(\mu) = \lambda \left[ 1 - \frac{\lambda}{8\pi^2} \log \frac{\mu}{m} \right]^{-1}, \quad \hat{\lambda}(m) = \lambda, \quad \hat{\lambda}(2\Lambda e^{-1/2}) = \lambda_b
$$

It is the Landau pole in $\hat{\lambda}(2\Lambda e^{-1/2})$ that actually forbids the limit $\Lambda \to \infty$. Hence we must keep the cutoff finite and smaller than $\Lambda_{\text{pole}}$, so that the theory does retain a slight inverse–power dependence on it. At any rate, there exists a very wide window where this dependence is indeed very weak for couplings of order one or less, since $\Lambda_{\text{pole}} = \frac{1}{2} m \exp(1/2 + 8\pi^2/\lambda) \gg m$. Moreover, we see from eq. (3.3) that for $\sqrt{\lambda} |\bar{\phi}|$ much smaller than the Landau pole there are two solutions for $M$, one “physical”, always larger than $m$ and of the same order of $m + \sqrt{\lambda} |\bar{\phi}|$, and one “unphysical”, close to the Landau pole. One can now easily verify that the effective potential has indeed a unique minimum in $\bar{\phi} = 0$, as required. In fact, if we assign arbitrary $\bar{\phi}$–dependent values to the widths $\sigma_k$, (minus) the effective force reads

$$
\frac{d}{d\bar{\phi}^i} V(\bar{\phi}, \{\sigma_k(\bar{\phi})\}) = M^2 \bar{\phi}^i + \sum_k \frac{\partial V}{\partial \sigma_k} \frac{d\sigma_k}{d\bar{\phi}^i}
$$

and reduces to $M^2 \bar{\phi}^i$ when the widths are extremal as in eq. (3.4); but $M^2$ is positive for unbroken symmetry and so $\bar{\phi} = 0$ is the unique minimum.

We define the symmetry as broken whenever the infinite volume $V_{\text{eff}}$ has more than one minimum. Of course, as long as $L$ is finite, $V_{\text{eff}}$ has a unique minimum in $\bar{\phi} = 0$, because of
the unicity of the ground state in Quantum Mechanics, as already discussed in section II A. Let us therefore proceed more formally and take the limit $L \to \infty$ directly on the potential energy $\mathcal{V}$. It reads

$$\mathcal{V} = \frac{1}{2} \int_{k^2 \leq \Lambda^2} \frac{d^Dk}{(2\pi)^D} \left( k^2 \sigma_k^2 + \frac{1}{4\sigma_k^2} \right) + V(\bar{\phi}^2 + \Sigma), \quad \Sigma = \int_{k^2 \leq \Lambda^2} \frac{d^Dk}{(2\pi)^D} \sigma_k^2$$

where we write for convenience the tree–level potential $V$ in the positive definite form $V(z) = \frac{1}{4} \lambda_b \left( z + m_b^2 / \lambda_b \right)^2$. $V$ is now the sum of two positive definite terms. Suppose there exists a configuration such that $V(\bar{\phi}^2 + \Sigma) = 0$ and the first term in $V$ is at its minimum. Then this is certainly the absolute minimum of $V$. This configuration indeed exists at infinite volume when $D = 3$:

$$\sigma_k^2 = \frac{1}{2|k|}, \quad \bar{\phi}^2 = v^2, \quad m_b^2 = -\lambda_b \left[ v^2 + I_3(0, \Lambda) \right] \quad (3.10)$$

where the nonnegative $v$ should be regarded as an independent parameter fixing the scale of the symmetry breaking. It replaces the mass parameter $m$ of the unbroken symmetry case: now the theory is massless in accordance with Goldstone theorem. On the contrary, if $D = 1$ this configuration is not allowed due to the infrared divergences caused by the massless nature of the width spectrum. This is just the standard manifestation of Mermin–Wagner–Coleman theorem that forbids continuous symmetry breaking in a two–dimensional space–time \[22\].

At finite volumes we cannot minimize the first term in $V$ since this requires $\sigma_0$ to diverge, making it impossible to keep $V(\bar{\phi}^2 + \Sigma) = 0$. In fact we know that the unicity of the ground state with finitely many degrees of freedom implies the minimization equations (3.4) to hold always true with a $M^2$ strictly positive. Therefore, broken symmetry should manifest itself as the situation in which the equilibrium value of $M^2$ is a positive definite function of $L$ which vanishes in the $L \to \infty$ limit.

We can confirm this qualitative conclusion as follows. We assume that the bare mass has the form given in eq. (3.10) and that $\bar{\phi}^2 = v^2$ too. Minimizing the potential energy leads always to the massive spectrum, eq. (3.4), with the gap equation

$$\frac{M^2}{\lambda_b} = \frac{1}{2L^3 M} + \frac{1}{2L^3} \sum_{k \neq 0} \frac{1}{\sqrt{k^2 + M^2}} - \frac{\Lambda^2}{8\pi^2} \quad (3.11)$$

If $M^2 > 0$ does not vanish too fast for large volumes, or stays even finite, then the sum on the modes has a behavior similar to the corresponding infinite volume integral: there is a quadratic divergence that cancels the infinite volume contribution, and a logarithmic one that renormalizes the bare coupling. The direct computation of the integral would produce a term containing the $M^2 \log(\Lambda / M)$. This can be split into $M^2 \log(\Lambda / v) - \log(M / v)$ by using $v$ as mass scale. The first term renormalizes the coupling correctly, while the second one vanishes if $M^2$ vanishes in the infinite volume limit.

When $L \to \infty$, the asymptotic solution of (3.11) reads

$$M = \left( \frac{\lambda}{2} \right)^{1/3} L^{-1} + \text{h.o.t.}$$
that indeed vanishes in the limit. Note also that the exponent is consistent with the assumption made above that $M$ vanishes slowly enough to approximate the sum over $k \neq 0$ with an integral with the same $M$.

Let us now consider a state whose field expectation value $\bar{\phi}^2$ is different from $v^2$. If $\bar{\phi}^2 > v$, the minimization equations (3.4) leads to a positive squared mass spectrum for the fluctuations, with $M^2$ given self-consistently by the gap equation. On the contrary, as soon as $\bar{\phi}^2 < v$, one immediately see that a positive $M^2$ cannot solve the gap equation

$$M^2 = \lambda_b \left( \bar{\phi}^2 - v^2 + \frac{\sigma_0^2}{L^3} + \frac{1}{2L^3} \sum_{k \neq 0} \frac{1}{\sqrt{k^2 + M^2}} - \frac{\Lambda^2}{8\pi^2} \right)$$

if we insist on the requirement that $\sigma_0$ not be macroscopic. In fact, the r.h.s. of the previous equation is negative, no matter which positive value for the effective mass we choose, at least for $L$ large enough. But nothing prevent us to consider a static configuration for which the amplitude of the zero mode is macroscopically large (i.e. it rescales with the volume $L^3$). Actually, if we choose

$$\frac{\sigma_0^2}{L^3} = v^2 - \bar{\phi}^2 + \frac{1}{2L^3 M}$$

we obtain the same equation as we did before and the same value for the potential, that is the minimum, in the limit $L \to \infty$. Note that at this level the effective mass $M$ needs not to have the same behavior in the $L \to \infty$ limit, but it is free of rescaling with a different power of $L$. We can be even more precise: we isolate the part of the potential that refers to the zero mode width $\sigma_0$ ($\Sigma'$ does not contain the $\sigma_0$ contribution)

$$\frac{1}{2} \left[ m_b^2 + \lambda_b (\bar{\phi}^2 + \Sigma') \right] \frac{\sigma_0^2}{L^3} + \frac{\lambda_b^4}{4} \frac{\sigma_0^4}{L^6} + \frac{1}{8L^3 \sigma_0^2}$$

and we minimize it, keeping $\bar{\phi}^2$ fixed. The minimum is attained at $t = \sigma_0^2 / L^3$ solution of the cubic equation

$$\lambda_b t^3 + \alpha \lambda_b t^2 - \frac{1}{4} L^{-6} = 0$$

where $\alpha = \bar{\phi}^2 - v^2 + \Sigma' - I_3 (0, \Lambda)$. Note that $\lambda_b \alpha$ depends on $L$ and it has a finite limit in infinite volume: $\lambda(\bar{\phi}^2 - v^2)$. The solution of the cubic equation is

$$\lambda_b t = \lambda_b (v^2 - \bar{\phi}^2) + \frac{1}{4} [L^3(v^2 - \bar{\phi}^2)]^{-2} + \text{h.o.t.}$$

from which the effective mass can be identified as proportional to $L^{-3}$. The stability equations for all the other modes can now be solved by a massive spectrum, in a much similar way as before.

Since $\sigma_0$ is now macroscopically large, the infinite volume limit of the $\sigma_k$ distribution (that gives a measure of the transverse fluctuations in the $O(N)$ model) develop a $\delta$—like singularity, signalling a Bose condensation of the Goldstone bosons:

$$\sigma_k^2 = (v^2 - \bar{\phi}^2) \delta^{(D)}(k) + \frac{1}{2k}$$

(3.12)
At the same time it is evident that the minimal potential energy is the same as when $\bar{\phi}^2 = v^2$, that is the effective potential flattens, in accord with the Maxwell construction.

Eq. (3.12) corresponds in configuration space to the 2-point correlation function

$$\lim_{N \to \infty} \frac{\langle \phi(x) \cdot \phi(y) \rangle}{N} = \bar{\phi}^2 + \int \frac{d^Dk}{(2\pi)^D} \sigma_k^2 e^{ik \cdot (x-y)} = C(\bar{\phi}^2) + \Delta_D(x-y)$$

(3.13)

where $\Delta_D(x-y)$ is the massless free-field equal-time correlator, while

$$C(\bar{\phi}^2) = v^2 \Theta(v^2 - \bar{\phi}^2) + \bar{\phi}^2 \Theta(\bar{\phi}^2 - v^2) = \max(v^2, \bar{\phi}^2)$$

(3.14)

This expression can be extended to unbroken symmetry by setting in that case $C(\bar{\phi}^2) = \bar{\phi}^2$.

Quite evidently, when eq. (3.14) holds, symmetry breaking can be inferred from the limit $|x-y| \to \infty$, if clustering is assumed [23,24], since $\Delta_D(x-y)$ vanishes for large separations. Of course this contradicts the infinite volume limit of the finite-volume definition, $\bar{\phi} = \lim_{N \to \infty} N^{-1/2} \langle \phi(x) \rangle$, except at the extremal points $\bar{\phi}^2 = v^2$.

In fact the $L \to \infty$ limit of the finite volume states with $\bar{\phi}^2 < v^2$ do violate clustering, because they are linear superpositions of vectors belonging to superselected sectors and therefore they are indistinguishable from statistical mixtures. We shall return in more detail on this aspects in section IV, where a generalized HF approximation is considered for $N = 1$. For the moment we can give the following intuitive picture for large $N$. Consider any one of the superselected sectors based on a physical vacuum with $\bar{\phi}^2 = v^2$. By condensing a macroscopic number of transverse Goldstone bosons at zero-momentum, one can build coherent states with rotated $\bar{\phi}$. By incoherently averaging over such rotated states one obtains new states with field expectation values shorter than $v$ by any prefixed amount. In the large $N$ approximation this averaging is necessarily uniform and is forced upon us by the residual $O(N-1)$ symmetry.

C. Out–of–equilibrium dynamics

We now turn to the dynamics out of equilibrium in this large $N$ context. It is governed by the equations of motion derived from the total energy density $\mathcal{E}$ in eq. (3.3), that is

$$\frac{d^2 \bar{\phi}}{dt^2} = -M^2 \bar{\phi} \ , \ \frac{d^2 \sigma_k}{dt^2} = -(k^2 + M^2) \sigma_k + \frac{1}{4\sigma_k^2}$$

(3.15)

where the generally time–dependent effective squared mass $M^2$ is given by

$$M^2 = m^2 + \lambda_b \left[ \bar{\phi}^2 + \Sigma - I_D(m^2, \Lambda) \right]$$

(3.16)

in case of unbroken symmetry and

$$M^2 = \lambda_b \left[ \bar{\phi}^2 - v^2 + \Sigma - I_3(0, \Lambda) \right]$$

(3.17)

for broken symmetry in $D = 3$.  

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At time zero, the specific choice of initial conditions for $\sigma_k$ that give the smallest energy contribution, that is

$$\dot{\sigma}_k = 0, \quad \sigma_k^2 = \frac{1}{2\sqrt{k^2 + M^2}} \tag{3.18}$$

turns eq. (3.16) into the usual gap equation (3.4). For any value of $\bar{\phi}$ this equation has one solution smoothly connected to the value $M = m$ at $\bar{\phi} = 0$. Of course other initial conditions are possible. The only requirement is that the corresponding energy must differ from that of the ground state by an ultraviolet finite amount, as it occurs for the choice (3.18). In fact this is guaranteed by the gap equation itself, as evident from eq. (3.9): when the widths $\sigma_k$ are extremal the effective force is finite, and therefore so are all potential energy differences.

This simple argument needs a refinement in two respects.

Firstly, in case of symmetry breaking the formal energy minimization w.r.t. $\sigma_k$ leads always to eqs. (3.18), but these are acceptable initial conditions only if the gap equation that follows from eq. (3.17) in the $L \to \infty$ limit, namely

$$M^2 = \lambda_b \left[ \bar{\phi}^2 - \nu^2 + I_D(M^2, \Lambda) - I_D(0, \Lambda) \right] \tag{3.19}$$

admits a nonnegative, physical solution for $M^2$.

Secondly, ultraviolet finiteness only requires that the sum over $k$ in eq. (3.3) be finite and this follows if eq. (3.18) holds at least for $k$ large enough, solving the issue raised in the first point: negative $M^2$ are allowed by imposing a new form of gap equation

$$M^2 = \lambda_b \left[ \bar{\phi}^2 - \nu^2 + \frac{1}{L_D} \sum_{k^2 < |M^2|} \sigma_k^2 + \frac{1}{L_D} \sum_{k^2 > |M^2|} \frac{1}{2\sqrt{k^2 - |M^2|}} - I_D(0, \Lambda) \right] \tag{3.20}$$

where all $\sigma_k$ with $k^2 < |M^2|$ are kept free (but all by hypothesis microscopic) initial conditions. Of course there is no energy minimization in this case. To determine when this new form is required, we observe that, neglecting the inverse–power corrections in the UV cutoff we may write eq. (3.19) in the following form

$$\frac{M^2}{\lambda(M)} = \bar{\phi}^2 - \nu^2 \tag{3.21}$$

There exists a positive solution $M^2$ smoothly connected to the ground state, $\bar{\phi}^2 = \nu^2$ and $M^2 = 0$, only provided $\bar{\phi}^2 \geq \nu^2$. So, in the large $N$ limit, as soon as we start with $\bar{\phi}^2 \leq \nu^2$, we cannot satisfy the gap equation with a positive value of $M^2$. The situation will be quite different in the case of $N = 1$ in the HF approximation (cfr. section [VIB]).

Once a definite choice of initial conditions is made, the system of differential equations (3.13), (3.16) or (3.17) can be solved numerically with standard integration algorithms. This has been already done by several authors [3914], working directly in infinite volume, with the following general results. In the case of unbroken symmetry it has been established that the $\sigma_k$ corresponding to wavevectors $k$ in the so–called forbidden bands with parametric resonances grow exponentially in time until their growth is shut off by the back–reaction. For broken symmetry it is the region in $k$–space with the spinodal instabilities caused by
an initially negative $M^2$, whose widths grow exponentially before the back–reaction shutoff. After the shutoff time the effective mass tends to a positive constant for unbroken symmetry and to zero for broken symmetry (in D=3), so that the only width with a chance to keep growing indefinitely is $\sigma_0$ for broken symmetry.

Of course, in all these approaches the integration over modes in the back–reaction $\Sigma$ cannot be done exactly and is always replaced by a discrete sum of a certain type, depending on the details of the algorithms. Hence there exists always an effective infrared cutoff, albeit too small to be detectable in the numerical outputs. A possible troublesome aspect of this is the proper identification of the zero–mode width $\sigma_0$. Even if a (rather arbitrary) choice of discretization is made where a $\sigma_0$ appears, it is not really possible to determine whether during the exponential growth or after such width becomes of the order of the volume. Our aim is just to answer this question and therefore we perform our numerical evolution in finite volumes of several growing sizes. Remanding to the appendix for the details of our method, we summarize our results in the next subsection.

D. Numerical results

After a careful study in $D = 3$ of the scaling behavior of the dynamics with respect to different values of $L$, the linear size of the system, we reached the following conclusion: there exist a $L$–dependent time, that we denote by $\tau_L$, that splits the evolution in two parts; for $t \leq \tau_L$, the behavior of the system does not differ appreciably from its counterpart at infinite volume, while finite volume effects abruptly alter the evolution as soon as $t$ exceeds $\tau_L$; in particular

- $\tau_L$ is proportional to the linear size of the box $L$ and so it rescales as the cubic root of the volume.
- $\tau_L$ does not depend on the value of the quartic coupling constant $\lambda$, at least in a first approximation.

The figures show the behavior of the width of the zero mode $\sigma_0$ (see Fig. 1), of the squared effective mass $M^2$ (see Fig. 2) and of the back–reaction $\Sigma$ (see Fig. 3), in the more interesting case of broken symmetry. The initial conditions are chosen in several different ways (see the appendix for details), but correspond to a negative $M^2$ at early times with the initial widths all microscopic, that is at most of order $L^{1/2}$. This is particularly relevant for the zero–mode width $\sigma_0$, which is instead macroscopic in the lowest energy state when $\bar{\phi}^2 < v^2$, as discussed above. As for the background, the figures are relative to the simplest case $\bar{\phi} = 0 = \dot{\bar{\phi}}$, but we have considered also initial conditions with $\bar{\phi} > 0$, reproducing the “dynamical Maxwell construction” observed in ref. [8]. At any rate, for the purposes of this work, above all it is important to observe that, due to the quantum back–reaction, $M^2$ rapidly becomes positive, within the so–called spinodal time [8][9][14], and then, for times before $\tau_L$, the weakly dissipative regime takes place where $M^2$ oscillates around zero with amplitude decreasing as $t^{-1}$ and a frequency fixed by the largest spinodal wavevector, in complete agreement with the infinite–volume results [5]. Correspondingly, after exponential grow until the spinodal time, the width of the zero–mode grows on average linearly with time, reaching a maximum for $t \approx \tau_L$. Precisely, $\sigma_0$ performs small amplitude oscillations with the
same frequency of $M^2$ around a linear function of the form $A + Bt$, where $A, B \approx \lambda^{-1/2}$ (see Fig. [4]), confirming what already found in refs. [3][9]; then quite suddenly it turns down and enters long irregular Poincaré–like cycles. Since the spinodal oscillation frequency does not depend appreciably on $L$, the curves of $\sigma_0$ at different values of $L$ are practically identical for $t < \tau_L$. After a certain number of complete oscillations, a number that scales linearly with $L$, a small change in the behavior of $M^2$ (see Fig. [3]) determines an inversion in $\sigma_0$ (see Fig. [3]), evidently because of a phase crossover between the two oscillation patterns. Shortly after $\tau_L$ dissipation practically stops as the oscillations of $M^2$ stop decreasing in amplitude and become more and more irregular, reflecting the same irregularity in the evolution of the widths.

The main consequence of this numerical scenario is that the linear growth of the zero–mode width at infinite volume cannot be consistently interpreted as a form of Bose–Einstein Condensation (BEC) [8]. If a macroscopic condensation were really there, the zero mode would develop a $\delta$ function in infinite volume, that would be announced by a width of the zero mode growing to values $O(L^{3/2})$ at any given size $L$. Now, while it is surely true that when we push $L$ to infinity, also the time $\tau_L$ tends to infinity, allowing the zero mode to grow indefinitely, it is also true that, at any fixed though arbitrarily large volume, the zero mode never reaches a width $O(L^{3/2})$, just because $\tau_L \propto L$. In other words, if we start from initial conditions where $\sigma_0$ is microscopic, then it never becomes macroscopic later on.

On the other hand, looking at the behavior of the mode functions of momenta $k = (2\pi/L)n$ for $n$ fixed but for different values of $L$, one realizes that they obey a scaling similar to that observed for the zero–mode: they oscillate in time with an amplitude and a period that are $O(L)$ (see fig. [4] and [5]). Thus, each mode shows a behavior that is exactly half a way between a macroscopic amplitude [i.e. $O(L^{3/2})$] and a usual microscopic one [i.e. at most $O(L^{1/2})$]. This means that the spectrum of the quantum fluctuations at times of the order of the diverging volume can be interpreted as a massless spectrum of interacting Goldstone modes, because their power spectrum develops in the limit a $1/k^2$ singularity, rather than the $1/k$ pole typical of free massless modes. As a consequence the equal–time field correlation function [see eq. (3.13)] will fall off as $|x - y|^{-1}$ for large separations smaller only than the diverging elapsed time. This is in accord with what found in [3], where the same conclusion where reached after a study of the correlation function for the scalar field in infinite volume.

The fact that each mode never becomes macroscopic, if it started microscopic, might be regarded as a manifestation of unitarity in the large $N$ approximation: an initial gaussian state with only microscopic widths satisfies clustering and clustering cannot be spoiled by a unitary time evolution. As a consequence, in the infinite–volume late–time dynamics, the zero–mode width $\sigma_0$ does not play any special role and only the behavior of $\sigma_k$ as $k \to 0$ is relevant. As already stated above, it turns out from our numerics as well as from refs. [3][4][5][6] that this behavior is of a novel type characteristic of the out–of–equilibrium dynamics, with $\sigma_k \propto 1/k$.

IV. TIME-DEPENDENT HARTREE–FOCK

The main limitation of the large $N$ approximation, as far as the evolution of the widths $\sigma_k$ is concerned, is in its intrinsic gaussian nature. In fact, one might envisage a scenario
in which, while gaussian fluctuations stay microscopic, non–gaussian fluctuations grow in time to a macroscopic size. In this section we therefore consider a time–dependent HF approximation capable in principle of describing the dynamics of non–gaussian fluctuation of a single scalar field with $\phi^4$ interaction.

As anticipated in section I, we examine in this work only states in which the scalar field has a uniform, albeit possibly time–dependent expectation value. In a tdHF approach we may then start from a wavefunction of the factorized form (which would be exact for free fields)

$$\Psi(\varphi) = \psi_0(\varphi_0) \prod_{k>0} \psi_k(\varphi_k, \varphi_{-k})$$  (4.1)

The dependence of $\psi_k$ on its two arguments cannot be assumed to factorize in general since space translations act as $SO(2)$ rotations on $\varphi_k$ and $\varphi_{-k}$ (hence in case of translation invariance $\psi_k$ depends only on $\varphi_k^2 + \varphi_{-k}^2$). The approximation consists in assuming this form as valid at all times and imposing the stationarity condition on the action

$$\delta \int dt \langle i\partial_t - H \rangle = 0, \quad \langle \cdot \rangle \equiv \langle \Psi(t) | \cdot | \Psi(t) \rangle$$  (4.2)

with respect to variations of the functions $\psi_k$. To enforce a uniform expectation value of $\phi$ we should add a Lagrange multiplier term linear in the single modes expectations $\langle \varphi_k \rangle$ for $k \neq 0$. The multiplier is then fixed at the end to obtain $\langle \varphi_k \rangle = 0$ for all $k \neq 0$. Actually one may verify that this is equivalent to the simpler approach in which $\langle \varphi_k \rangle$ is set to vanish for all $k \neq 0$ before any variation. Then the only non trivial expectation value in the Hamiltonian, namely that of the quartic term, assumes the form

$$\int d^Dx \langle \phi(x)^4 \rangle = \frac{1}{L^D} \left[ \langle \varphi_0^4 \rangle - 3 \langle \varphi_0^2 \rangle^2 \right] + \frac{3}{2L^D} \sum_{k>0} \left[ \langle (\varphi_k^2 + \varphi_{-k}^2) \rangle - 2 (\langle \varphi_k \rangle + \langle \varphi_{-k} \rangle)^2 \right]$$

$$+ \frac{3}{L^D} \left( \sum_k \langle \varphi_k^2 \rangle \right)^2$$  (4.3)

Notice that the terms in the first row would cancel completely out for gaussian wavefunctions $\psi_k$ with zero mean value. The last term, where the sum extends to all wavevectors $k$, corresponds instead to the standard mean field replacement $\langle \phi^4 \rangle \rightarrow 3\langle \phi^2 \rangle^2$. The total energy of our trial state now reads

$$E = \langle H \rangle = \frac{1}{2} \sum_k \left\langle \frac{\partial^2}{\partial \varphi_k^2} + (k^2 + m_0^2) \varphi_k^2 \right\rangle + \frac{\lambda b}{4} \int d^Dx \langle \phi(x)^4 \rangle$$  (4.4)

and from the variational principle (4.2) we obtain a set of simple Schroedinger equations

$$i\partial_t \psi_k = H_k \psi_k$$  (4.5)

$$H_0 = -\frac{1}{2} \frac{\partial^2}{\partial \varphi_0^2} + \frac{1}{2} \omega_0^2 \varphi_0^2 + \frac{\lambda b}{4L^D} \varphi_0^4$$

$$H_k = -\frac{1}{2} \left( \frac{\partial^2}{\partial \varphi_k^2} + \frac{\partial^2}{\partial \varphi_{-k}^2} \right) + \frac{1}{2} \omega_k^2 (\varphi_k^2 + \varphi_{-k}^2) + \frac{3\lambda b}{8L^D} (\varphi_k^2 + \varphi_{-k}^2)^2$$  (4.6)
which are coupled in a mean–field way only through

\[ \omega_k^2 = k^2 + m_b^2 + 3\lambda_b \Sigma_k, \quad \Sigma_k = \frac{1}{L^D} \sum_{q^2 \leq \Lambda^2} \langle \varphi_q^2 \rangle \]  

(4.7)

and define the HF time evolution for the theory. By construction this evolution conserves the total energy \( E \) of eq. (4.4).

It should be stressed that in this particular tdHF approximation, beside the mean–field back–reaction term \( \Sigma_k \) of all other modes on \( \omega_k^2 \), we keep also the contribution of the diagonal scattering through the diagonal quartic terms. In fact this is why \( \Sigma_k \) has no contribution from the \( k \)–mode itself: in a gaussian approximation for the trial wavefunctions \( \psi_k \) the Hamiltonians \( H_k \) would turn out to be harmonic, the quartic terms being absent in favor of a complete back–reaction

\[ \Sigma = \Sigma_k + \frac{\langle \varphi_k^2 \rangle + \langle \varphi_{-k}^2 \rangle}{L^D} = \frac{1}{L^D} \sum_k \langle \varphi_k^2 \rangle \]  

(4.8)

Of course the quartic self–interaction of the modes as well as the difference between \( \Sigma \) and \( \Sigma_k \) are suppressed by a volume effect and could be neglected in the infrared limit, provided all wavefunctions \( \psi_k \) stays concentrated on mode amplitudes \( \varphi_k \) of order smaller than \( L^{D/2} \). This is the typical situation when all modes remain microscopic and the volume in the denominators is compensated only through the summation over a number of modes proportional to the volume itself, so that in the limit \( L \to \infty \) sums are replaced by integrals

\[ \Sigma_k \to \Sigma \to \int_{k^2 \leq \Lambda^2} \frac{d^Dk}{(2\pi)^D} \langle \varphi_k^2 \rangle \]

Indeed we shall apply this picture to all modes with \( k \neq 0 \), while we do expect exceptions for the zero–mode wavefunction \( \psi_0 \).

The treatment of ultraviolet divergences requires particular care, since the HF approximation typically messes things up (see, for instance, remarks in [26]). Following the same login of the large \( N \) approximation, we could take as renormalization condition the requirement that the frequencies \( \omega_k^2 \) are independent of \( \Lambda \), assuming that \( m_b^2 \) and \( \lambda_b \) are functions of \( \Lambda \) itself and of renormalized \( \Lambda \)–independent parameters \( m^2 \) and \( \lambda \) such that

\[ \omega_k^2 = k^2 + m^2 + 3\lambda [\Sigma_k]_{\text{finite}} \]  

(4.9)

where by \([\cdot]_{\text{finite}}\) we mean the (scheme–dependent) finite part of some possibly ultraviolet divergent quantity. Unfortunately this would not be enough to make the spectrum of energy differences cutoff–independent, because of the bare coupling constant \( \lambda_b \) in front of the quartic terms in \( H_k \) and the difference between \( \Sigma \) and \( \Sigma_k \) [such problem does not exist in large \( N \) because that is a purely gaussian approximation]. Again this would not be a problem whenever these terms become negligible as \( L \to \infty \). At any rate, to be ready to handle the cases when this is not actually true and to define an ultraviolet–finite model also at finite volume, we shall by hand modify eq. (4.3) as follows:

\[ \lambda_b \int d^Dx \langle \phi(x)^4 \rangle = \lambda L^{-D} \left\{ \langle \varphi_0^4 \rangle - 3\langle \varphi_0^2 \rangle^2 + \frac{3}{2} \sum_{k>0} \left[ \langle (\varphi_k^2 + \varphi_{-k}^2)^2 \rangle - 2 \left( \langle \varphi_k^2 \rangle + \langle \varphi_{-k}^2 \rangle \right)^2 \right] \right\} \]

\[ + 3\lambda_b L^D \Sigma^2 \]
We keep the bare coupling constant in front of the term containing $\Sigma^2$ because that part of the hamiltonian is properly renormalized by means of the usual cactus resummation which corresponds to the standard HF approximation. On the other hand, within the same approximation, it is not possible to renormalize the part in curly brackets of the equation above, because of the factorized form that we have assumed for the wavefunction of the system. In fact, the 4-legs vertices in the curly brackets are diagonal in momentum space; when we contract two or more vertices of this type, no sum over internal loop momenta is produced, so that all higher order perturbation terms are suppressed by volume effects. However, we know that in the complete theory, the wavefunction is not factorized and loops contain all values of momentum. This suggests that, in order to get a finite hamiltonian, we need to introduce in the definition of our model some extra resummation of Feynmann diagrams, that is not automatically contained in this self–consistent HF approach. The only choice consistent with the cactus resummation performed in the two–point function by the HF scheme is the resummation of the 1-loop fish diagram in the four–point function. This amounts to the change from $\lambda_b$ to $\lambda$ and it is enough to guarantee the ultraviolet finiteness of the hamiltonian through the redefinition

$$H_0 \rightarrow H_0 + \frac{\lambda - \lambda_b}{4L^D} \varphi_0^4, \quad H_k \rightarrow H_k + \frac{3(\lambda - \lambda_b)}{8L^D} \left( \varphi_k^2 + \varphi_{-k}^2 \right)^2$$ (4.10)

At the same time the frequencies are now related to the widths $\langle \varphi_{\pm k}^2 \rangle$ by

$$\omega_k^2 = k^2 + M^2 - 3\lambda L^{-D} \left( \langle \varphi_k^2 \rangle + \langle \varphi_{-k}^2 \rangle \right), \quad k > 0$$

$$M^2 \equiv \omega_0^2 + 3\lambda L^{-D} \langle \varphi_0^2 \rangle = m_b^2 + 3\lambda \Sigma$$ (4.11)

Apart for $O(L^{-D})$ corrections, $M$ plays the role of time–dependent mass for modes with $k \neq 0$, in the harmonic approximation.

In this new setup the conserved energy reads

$$E = \sum_{k \geq 0} \langle H_k \rangle - \frac{3}{4} \lambda_b L^D \Sigma^2 + \frac{3}{4} \lambda L^{-D} \left[ \langle \varphi_0^2 \rangle^2 + \sum_{k > 0} \left( \langle \varphi_k^2 \rangle + \langle \varphi_{-k}^2 \rangle \right)^2 \right]$$ (4.12)

Since the gap–like equations (4.11) are state–dependent, we have to perform the renormalization first for some reference quantum state, that is for some specific collection of wavefunctions $\psi_k$; as soon as $m_b^2$ and $\lambda_b$ are determined as functions $\Lambda$, ultraviolet finiteness will hold for the entire class of states with the same ultraviolet properties of the reference state. Then an obvious consistency check for our HF approximation is that this class is closed under time evolution.

Rather than a single state, we choose as reference the family of gaussian states parametrized by the uniform expectation value $\langle \phi(x) \rangle = L^{-D/2} \langle \varphi_0 \rangle = \bar{\phi}$ (recall that we have $\langle \varphi_k \rangle = 0$ when $k \neq 0$ by assumption) and such the HF energy $E$ is as small as possible for fixed $\bar{\phi}$. Then, apart from a translation by $L^{D/2} \bar{\phi}$ on $\varphi_0$, these gaussian $\psi_k$ are ground state eigenfunctions of the harmonic Hamiltonians obtained from $H_k$ by dropping the quartic terms. Because of the $k^2$ in the frequencies we expect these gaussian states to dominate in the ultraviolet limit also at finite volume (as discussed above they should dominate in the infinite–volume limit for any $k \neq 0$). Moreover, since now

$$\langle \varphi_0^2 \rangle = L^D \bar{\phi}^2 + \frac{1}{2\omega_0}, \quad \langle \varphi_{\pm k}^2 \rangle = \frac{1}{2\omega_k}, \quad k \neq 0$$ (4.13)
the relation $(4.11)$ between frequencies and widths turn into the single gap equation

$$M^2 = m_b^2 + 3 \lambda_b \left( \bar{\phi}^2 + \frac{1}{2 L^D} \sum_{q^2 \leq \Lambda^2} \frac{1}{\sqrt{k^2 + M^2}} \right)$$  \hspace{1cm} (4.14)$$

fixing the self-consistent value of $M$ as a function of $\bar{\phi}$. It should be stressed that $(4.11)$ turns through eq. $(4.13)$ into the gap equation only because of the requirement of energy minimization. Generic $\psi_k$, regarded as initial conditions for the Schrödinger equations $(4.5)$, are in principle not subject to any gap equation.

The treatment now follows closely that in the large $N$ section, the only difference being in the value of the coupling, now three times larger. In fact, in case of $O(N)$ symmetry, the quantum fluctuations over a given background $\langle \phi(x) \rangle = \phi$ decompose for each $k$ into one longitudinal mode, parallel to $\phi$, and $N - 1$ transverse modes orthogonal to it; by boson combinatorics the longitudinal mode couples to $\phi$ with strength $3 \lambda_b / N$ and decouple in the $N \to \infty$ limit, while the transverse modes couple to $\phi$ with strength $(N - 1) \lambda_b / N \to \lambda_b$; when $N = 1$ only the longitudinal mode is there.

As $L \to \infty$, $\omega_k \to k^2 + M^2$ and $M$ is exactly the physical mass gap. Hence it must be $\Lambda$–independent. At finite $L$ we cannot use this request to determine $m_b^2$ and $\lambda_b$, since, unlike $M$, they cannot depend on the size $L$. At infinite volume we obtain

$$M^2 = m_b^2 + 3 \lambda_b [\bar{\phi}^2 + I_D(M^2, \Lambda)] \ , \quad I_D(z, \Lambda) \equiv \int_{k^2 \leq \Lambda^2} \frac{d^Dk}{(2\pi)^D} \frac{1}{2\sqrt{k^2 + z}}$$  \hspace{1cm} (4.15)$$

When $\bar{\phi} = 0$ this equation fixes the bare mass to be

$$m_b^2 = m^2 - 3 \lambda_b I_D(m^2, \Lambda)$$  \hspace{1cm} (4.16)$$

where $m = M(\bar{\phi} = 0)$ may be identified with the equilibrium physical mass of the scalar particles of the infinite–volume Fock space without symmetry breaking (see below). Now, as in large $N$, the coupling constant renormalization follows from the equalities

$$M^2 = m^2 + 3 \lambda_b [\bar{\phi}^2 + I_D(M^2, \Lambda) - I_D(m^2, \Lambda)]$$

$$= m^2 + 3 \lambda \bar{\phi}^2 + 3 \lambda \left[ I_D(M^2, \Lambda) - I_D(m^2, \Lambda) \right]_{\text{finite}}$$  \hspace{1cm} (4.17)$$

and reads when $D = 3$

$$\frac{\lambda}{\lambda_b} = 1 - \frac{3\lambda}{8\pi^2} \log \frac{2\Lambda}{m\sqrt{\epsilon}}$$  \hspace{1cm} (4.18)$$

that is the standard result of the one–loop renormalization group [23]. When $D = 1$, that is a $1 + 1$–dimensional quantum field theory, $I_D(M^2, \Lambda) - I_D(m^2, \Lambda)$ is already finite and the dimensionfull couplings constant is not renormalized, $\lambda_b = \lambda$.

Again, the Landau pole in $\lambda_b$ prevents the actual UV limit $\Lambda \to \infty$. Nonetheless, neglecting all inverse powers of the UV cutoff when $D = 3$, it is possible to rewrite the gap equation $(4.17)$ as in eq. $(3.8)$:

$$\frac{M^2}{\lambda(M)} = \frac{m^2}{\lambda(m)} + 3 \bar{\phi}^2$$  \hspace{1cm} (4.19)$$
in terms of the one–loop running couplings constant

\[ \hat{\lambda}(\mu) = \lambda \left[ 1 - \frac{3\lambda}{8\pi^2} \log \frac{\mu}{m} \right]^{-1} \]

It is quite clear that the HF states for which the renormalization just defined is sufficient are all those that are gaussian–dominated in the ultraviolet, so that we have [compare to eq. (4.13)]

\[ \langle \varphi_{\pm k}^2 \rangle \sim \frac{1}{2\omega_k}, \quad k^2 \sim \Lambda^2, \quad \Lambda \rightarrow \infty \]  

(4.20)

If this property holds at a certain time, then it should hold at all times, since the Schrödinger equations (4.3) are indeed dominated by the quadratic term for large \( \omega_k \) and \( \omega_k^2 \sim k^2 + \text{const} + O(k^{-1}) \) as evident from eq. (4.9). Thus this class of states is indeed closed under time evolution and the parameterizations (4.16) and (4.18) make our tdHF approximation ultraviolet finite. Notice that the requirement (4.20) effectively always imposes a gap equation similar to eq. (4.14) in the deep ultraviolet.

Another simple check of the self–consistency of our approach, including the change in selected places from \( \lambda_b \) to \( \lambda \), as discussed above, follows from the energy calculation for the gaussian states with \( \langle \phi(x) \rangle = \bar{\phi} \) introduced above. Using eq. (4.4) and the standard replacement of sums by integrals in the infinite volume limit, we find

\[ \mathcal{E}(\bar{\phi}) = \lim_{L \rightarrow \infty} \frac{E}{L^D} = \frac{1}{2} \bar{\phi}^2 (M^2 - \lambda \bar{\phi}^2) + \frac{1}{2} \int_{k^2 \leq \Lambda^2} \frac{d^Dk}{(2\pi)^D} \sqrt{k^2 + M^2} - \frac{3}{4} \lambda_b \left[ \bar{\phi}^2 + I_D(M^2, \Lambda) \right]^2 \]

where \( M = M(\bar{\phi}) \) depends on \( \bar{\phi} \) through the gap equation (4.17). The explicit calculation of the integrals involved shows that the energy density difference \( \mathcal{E}(\bar{\phi}) - \mathcal{E}(0) \) [which for unbroken symmetry is nothing but the effective potential \( V_{\text{eff}}(\bar{\phi}) \)], is indeed finite in the limit \( \Lambda \rightarrow \infty \), as required by a correct renormalization scheme. Notice that the finiteness of the energy density difference can be showed also by a simpler and more elegant argument, as presented below in section 11V.3. This check would fail instead when \( D = 3 \) if only the bare coupling constant \( \lambda_b \) would appear in the last formula.

The tdHF approximation derived above represents a huge simplification with respect to the original problem, but its exact solution still poses itself as a considerable challenge. As a matter of fact, a numerical approach is perfectly possible within the capabilities of modern computers, provided the number of equations (4.3) is kept in the range of few thousands. As will become clear later on, even this numerical workout will turn out not to be really necessary in the form just alluded to, at least for the purposes of this paper.

A. On symmetry breaking

Quite obviously, in a finite volume and with a UV cutoff there cannot be any symmetry breaking, since the ground state is necessarily unique and symmetric when the number of degrees of freedom is finite [23]. However, we may handily envisage the situation which would imply symmetry breaking when the volume diverges.
Let us first consider the case that we would call of unbroken symmetry. In this case the HF ground state is very close to the member with $\phi = 0$ of the family of gaussian states introduced before. The difference is entirely due to the quartic terms in $H_k$. This correction vanish when $L \to \infty$, since all wavefunctions $\psi_k$ have $L$–independent widths, so that one directly obtains the symmetric vacuum state with all the right properties of the vacuum (translation invariance, unicity, etc.) upon which a standard scalar massive particle Fock space can be based. The HF approximation then turns out to be equivalent to the resummation of all “cactus diagrams” for the particle self–energy $\Sigma_0$. In a finite volume, the crucial property of this symmetric vacuum is that all frequencies $\omega_k^2$ are strictly positive. The generalization to non–equilibrium initial states with $\bar{\omega}$ amounts to a shift by $\omega$ volume, the crucial property of this symmetric vacuum is that all frequencies $\omega_k^2$ are strictly positive. The generalization to non–equilibrium initial states with $\phi \neq 0$ is rather trivial: it amounts to a shift by $L^{D/2} \omega$ on $\psi_0(\varphi_0)$. In the limit $L \to \infty$ we should express $\psi_0$ as a function of $\xi = L^{-D/2} \varphi_0$ so that, $|\psi_0(\xi)|^2 \to \delta(\xi - \bar{\phi})$, while all other wavefunctions $\psi_k$ will reconstruct the gaussian wavefunctional corresponding to the vacuum $|0,M\rangle$ of a free massive scalar theory whose mass $M = M(\phi)$ solves the gap equation (1.17). The absence of $\psi_0$ in $|0,M\rangle$ is irrelevant in the infinite volume limit, since $\langle \varphi_0^2 \rangle = L^D \bar{\phi}^2 + \text{terms of order } L^0$. The effective potential $V_{\text{eff}}(\bar{\phi}) = \mathcal{E}(\bar{\phi}) - \mathcal{E}(0)$, where $\mathcal{E}(\bar{\phi})$ is the lowest energy density at fixed $\bar{\phi}$ and infinite volume, is manifestly a convex function with a unique minimum in $\bar{\phi} = 0$.

Now let us consider a different situation in which one or more of the $\omega_k^2$ are negative. Quite evidently, this might happen only for $k$ small enough, due to the $k^2$ in the gap equation [thus eq. (1.20) remains valid and the ultraviolet renormalization is the same as for unbroken symmetry]. Actually we assume here that only $\omega_0^2 < 0$, postponing the general analysis. Now the quartic term in $H_0$ cannot be neglected as $L \to \infty$, since in the ground state $\psi_0$ is symmetrically concentrated around the two minima of the potential $\frac{1}{2} \omega_0^2 \varphi_0^2 + \frac{\lambda}{2} \varphi_4^4$, that is $\varphi_0 = \pm \left(-\frac{\omega_0^2 L^D}{\lambda}\right)^{1/2}$. If we scale $\varphi_0$ as $\varphi_0 = L^{D/2} \xi$ then $H_0$ becomes

$$H_0 = -\frac{1}{2L^D} \frac{\partial^2}{\partial \xi^2} + \frac{L^D}{2} \left(\omega_0^2 \xi^2 + \frac{\lambda}{2} \xi^4\right) \tag{4.21}$$

so that the larger $L$ grows the narrower $\psi_0(\xi)$ becomes around the two minima $\xi = \pm \left(-\omega_0^2 / \lambda\right)^{1/2}$. In particular $\langle \xi^2 \rangle \to -\omega_0^2 / \lambda$ when $L \to \infty$ and $\langle \varphi_0^2 \rangle \approx L^D \langle \xi^2 \rangle$. Moreover, the energy gap between the ground state of $H_0$ and its first, odd excited state as well as difference between the relative probability distributions for $\xi$ vanish exponentially fast in the volume $L^D$.

Since by hypothesis all $\omega_k^2$ with $k \neq 0$ are strictly positive, the ground state $\psi_k$ with $k \neq 0$ are asymptotically gaussian when $L \to \infty$ and the relations (4.11) tend to the form

$$\omega_k^2 = k^2 + M^2 \equiv k^2 + m^2$$

$$M^2 = -2\omega_0^2 = m_0^2 + 3\lambda b(L^{-D}\langle \varphi_0^2 \rangle + \Sigma_0) = m_0^2 + 3\lambda b \omega_0^2 + 3\lambda b I_D(m^2, \Lambda)$$

This implies the identification $\omega_0^2 = -m^2/2$ and the bare mass parameterization

$$m_0^2 = \left(1 - \frac{3}{2} \frac{\lambda b}{\lambda}\right) m^2 - 3\lambda b I_D(m^2, \Lambda) \tag{4.22}$$

characteristic of a negative $\omega_0^2$ [compare to eq. (4.16)], with $m$ the physical equilibrium mass of the scalar particle, as in the unbroken symmetry case. The coupling constant
renormalization is the same as in eq. (4.18) as may be verified by generalizing to the minimum energy states with given field expectation value \( \bar{\phi} \); this minimum energy is nothing but the effective potential \( V_{\text{eff}}(\bar{\phi}) \) of course, since \( \psi_0 \) is no longer asymptotically gaussian, we cannot simply shift it by \( L^{D/2} \bar{\phi} \) but, due to the concentration of \( \psi_0 \) on classical minima as \( L \to \infty \), one readily finds that \( V_{\text{eff}}(\bar{\phi}) \) is the convex envelope of the classical potential, that is its Maxwell construction. Hence we find

\[
\langle \varphi_0^2 \rangle_{L \to \infty} \sim \begin{cases} -L^D \omega_0^2 / \lambda, & \lambda \bar{\phi}^2 \leq -\omega_0^2 \\ L^D \bar{\phi}^2, & \lambda \bar{\phi}^2 > -\omega_0^2 \end{cases}
\]

and the gap equation for the \( \bar{\phi} \)-dependent mass \( M \) can be written, in terms of the step function \( \Theta \) and the extremal ground state field expectation value \( v = m/\sqrt{2\lambda} \),

\[
M^2 = m^2 + 3\lambda_b(\bar{\phi}^2 - v^2) \Theta(\bar{\phi}^2 - v^2) + 3\lambda_b \left[ I_D(M^2, \Lambda) - I_D(m^2, \Lambda) \right] \tag{4.23}
\]

We see that the specific bare mass parameterization (4.22) guarantees the non-renormalization of the tree-level relation \( v^2 = m^2 / 2\lambda \) ensuing from the typical symmetry breaking classical potential \( V(\phi) = \frac{1}{4}\lambda(\phi^2 - v^2)^2 \). With the same finite part prescription as in eq. (4.17), the gap equation (4.23) leads to the standard coupling constant renormalization (4.18) when \( D = 3 \).

In terms of the probability distributions \( |\psi_0(\xi)|^2 \) for the scaled amplitude \( \xi = L^{-D/2} \varphi_0 \), the Maxwell construction corresponds to the limiting form

\[
|\psi_0(\xi)|^2_{L \to \infty} \sim \begin{cases} \frac{1}{2}(1 - \bar{\phi}/v) \delta(\xi - v) + \frac{1}{2}(1 - \bar{\phi}/v) \delta(\xi + v), & \bar{\phi}^2 \leq v^2 \\ \delta(\xi - \bar{\phi}), & \bar{\phi}^2 > v^2 \end{cases} \tag{4.24}
\]

On the other hand, if \( \omega_0^2 \) is indeed the only negative squared frequency, the \( k \neq 0 \) part of this minimum energy state with arbitrary \( \bar{\phi} = \langle \phi(x) \rangle \) is better and better approximated as \( \Lambda \to \infty \) by the same gaussian state \( |0, M \rangle \) of the unbroken symmetry state. Only the effective mass \( M \) has a different dependence \( M(\bar{\phi}) \), as given by the gap equation (4.23) proper of broken symmetry.

As in the large \( N \) approach, at infinite volume we may write

\[
\langle \varphi_k^2 \rangle = C(\bar{\phi}) \delta^{(D)}(k) + \frac{1}{2\sqrt{k^2 + M^2}}
\]

where \( C(\bar{\phi}) = \bar{\phi}^2 \) in case of unbroken symmetry (that is \( \omega_0^2 > 0 \)), while \( C(\bar{\phi}) = \max(v^2, \bar{\phi}^2) \) when \( \omega_0^2 < 0 \). This corresponds to the field correlation in space

\[
\langle \phi(x)\phi(y) \rangle = \int \frac{d^Dk}{(2\pi)^D} \varphi_k^2 e^{ik \cdot (x-y)} = C(\bar{\phi}) + \Delta_D(x - y, M)
\]

where \( \Delta_D(x - y, M) \) is the massive free field equal-time two points function in \( D \) space dimensions, with self-consistent mass \( M \). As in large \( N \), the requirement of clustering

\[
\langle \phi(x)\phi(y) \rangle \to \langle \phi(x) \rangle^2 = v^2
\]
contradicts the infinite volume limit of

$$\langle \phi(x) \rangle = L^{-D/2} \sum_k \langle \phi_k \rangle e^{ik \cdot x} = \langle \varphi_0 \rangle = \bar{\phi}$$

except at the (now only two) extremal points \( \bar{\phi} = \pm v \). In fact we know that the \( L \to \infty \) limit of the finite volume states with \( \bar{\phi}^2 < v^2 \) violate clustering, because the two peaks of \( \psi_0(\xi) \) have vanishing overlap in the limit and the first excited state becomes degenerate with the vacuum: this implies that the relative Hilbert space splits into two orthogonal Fock sectors each exhibiting symmetry breaking, \( \langle \phi(x) \rangle = \pm v \), and corresponding to the two independent equal weight linear combinations of the two degenerate vacuum states. The true vacuum is either one of these symmetry broken states. Since the two Fock sectors are not only orthogonal, but also superselected (no local observable interpolates between them), linear combinations of any pair of vectors from the two sectors are not distinguishable from mixtures of states and clustering cannot hold in non–pure phases. It is perhaps worth noticing also that the Maxwell construction for the effective potential, in the infinite volume limit, is just a straightforward manifestation of this fact and holds true, as such, beyond the HF approximation.

To further clarify this point and in view of subsequent applications, let us consider the probability distribution for the smeared field \( \phi_f = f d^D x \phi(x) \), where

\[
f(x) = f(-x) = \frac{1}{L^D} \sum_k f_k e^{ik \cdot x} \sim \int \frac{d^D k}{(2\pi)^D} \tilde{f}(k) e^{ik \cdot x}
\]

is a smooth real function with \( \int d^D x f(x) = 1 \) (i.e. \( f_0 = 1 \)) localized around the origin (which is good as any other point owing to translation invariance). Neglecting in the infinite volume limit the quartic corrections for all modes with \( k \neq 0 \), so that the corresponding ground state wavefunctions are asymptotically gaussian, this probability distribution evaluates to

\[
\Pr(u < \phi_f < u + du) = \frac{du}{(2\pi \Sigma_f)^{1/2}} \int_{-\infty}^{+\infty} d\xi |\psi_0(\xi)|^2 \exp \left\{ \frac{-(u - \xi)^2}{2\Sigma_f} \right\}
\]

where

\[
\Sigma_f = \sum_{k \neq 0} \langle \phi_k^2 \rangle f_k^2 \sim \int \frac{d^D k}{(2\pi)^D} \frac{\tilde{f}(k)^2}{2\sqrt{k^2 + m^2}}
\]

In the unbroken symmetry case we have \( |\psi_0(\xi)|^2 \sim \delta(\xi - \bar{\phi}) \) as \( L \to \infty \), while the limiting form \((4.24)\) holds for broken symmetry. Thus we obtain

\[
\Pr(u < \phi_f < u + du) = p_f(u - \bar{\phi}) du , \quad p_f(u) = (2\pi \Sigma_f)^{-1/2} \exp \left( \frac{-u^2}{2\Sigma_f} \right)
\]

for unbroken symmetry and

\[
\Pr(u < \phi_f < u + du) = \begin{cases} \frac{1}{2} (1 + \bar{\phi}/v) p_f(u - v) du + \frac{1}{2} (1 - \bar{\phi}/v) p_f(u + v) du , & \bar{\phi}^2 \leq v^2 \\ p_f(u - \bar{\phi}) du , & \bar{\phi}^2 > v^2 \end{cases}
\]
for broken symmetry. Notice that the momentum integration in the expression for \( \Sigma_f \) needs no longer an ultraviolet cutoff; of course in the limit of delta–like test function \( f(x) \), \( \Sigma_f \) diverges and \( p_f(u) \) flattens down to zero. The important observation is that \( \Pr(u < \phi_f < u + du) \) has always a single peak centered in \( u = \tilde{\phi} \) for unbroken symmetry, while for broken symmetry it shows two peaks for \( \tilde{\phi}^2 < v^2 \) and \( \Sigma_f \) small enough. For instance, if \( \tilde{\phi} = 0 \), then there are two peaks for \( \Sigma_f < v^2 \) [implying that \( f(k) \) has a significant support only up to wavevector \( k \) of order \( v \), when \( D = 3 \), or \( m \exp(\text{const } v^2) \) when \( D = 1 \)].

To end the discussion on symmetry breaking, we may now verify the validity of the assumption that only \( \omega_0^2 \) is negative. In fact, to any squared frequency \( \omega_k^2 \) (with \( k \neq 0 \)) that stays strictly negative as \( L \to \infty \) there corresponds a wavefunction \( \psi_k \) that concentrates on \( \varphi_k^2 + \varphi_{-k}^2 = -\omega_k^2 L^D / \lambda \); then eqs. (4.11) implies \(-2\omega_k^2 = k^2 + m^2 \) for such frequencies, while \( \omega_k^2 = k^2 + m^2 \) for all frequencies with positive squares; if there is a macroscopic number of negative \( \omega_k^2 \) (that is a number of order \( L^D \)), then the expression for \( \omega_0^2 \) in eq. (4.11) will contain a positive term of order \( L^D \) in the r.h.s., clearly incompatible with the requirements that \( \omega_0^2 < 0 \) and \( m_0^2 \) be independent of \( L \); if the number of negative \( \omega_k^2 \) is not macroscopic, then the largest wavevector with a negative squared frequency tends to zero as \( L \to \infty \) (the negative \( \omega_k^2 \) clearly pile in the infrared) and the situation is equivalent, if not identical, to that discussed above with only \( \omega_0^2 < 0 \).

### B. Out–of–equilibrium dynamics

We considered above the lowest energy states with a predefinite uniform field expectation value, \( \langle \phi(x) \rangle = \tilde{\phi} \), and established how they drastically simplify in the infinite volume limit. For generic \( \tilde{\phi} \) these states are not stationary and will evolve in time. By hypothesis \( \psi_k \) is the ground state eigenfunction of \( H_k \) when \( k > 0 \), and therefore \( |\psi_k|^2 \) would be stationary for constant \( \omega_k \), but \( \psi_0 \) is not an eigenfunction of \( H_0 \) unless \( \tilde{\phi} = 0 \). As soon as \( |\psi_k|^2 \) starts changing, \( \langle \varphi_0^2 \rangle \) changes and so do all frequencies \( \omega_k \) which are coupled to it by the eqs. (4.11). Thus the change propagates to all wavefunctions. The difficult task of studying this dynamics can be simplified with the following scheme, that we might call gaussian approximation. We first describe it and discuss its validity later on.

Let us assume the usual gaussian form for the initial state [see eq. (4.13) and the discussion following it]. We know that it is a good approximation to the lowest energy state with given \( \langle \varphi_0 \rangle \) for unbroken symmetry, while it fails to be so for broken symmetry, only as far as \( \psi_0 \) is concerned, unless \( \tilde{\phi}^2 \geq v^2 \). At any rate this is an acceptable initial state: the question is about its time evolution. Suppose we adopt the harmonic approximation for all \( H_k \) with \( k > 0 \) by dropping the quartic term. This approximation will turn out to be valid only if the width of \( \psi_k \) do not grow up to the order \( L^D \) (by symmetry the center will stay in the origin). In practice we are now dealing with a collection of harmonic oscillators with time–dependent frequencies and the treatment is quite elementary: consider the simplest example of one quantum degree of freedom described by the gaussian wavefunction

\[
\psi(q,t) = \frac{1}{(2\pi \sigma^2)^{1/4}} \exp \left[ -\frac{1}{2} \left( \frac{1}{2\sigma^2} - i \frac{s}{\sigma} \right) q^2 \right]
\]

where \( s \) and \( \sigma \) are time–dependent. If the dynamics is determined by the time–dependent harmonic hamiltonian \( \frac{1}{2}[-\partial_q^2 + \omega(t)^2 q^2] \), then the Schroedinger equation is solved exactly
provided that $s$ and $\sigma$ satisfy the classical Hamilton equations

$$\dot{s} = s, \quad \dot{\sigma} = -\omega^2 \sigma + \frac{1}{4\sigma^3}$$

It is not difficult to trace the “centrifugal” force $(4\sigma)^{-3}$ which prevent the vanishing of $\sigma$ to Heisenberg uncertainty principle [12,16].

The extension to our case with many degrees of freedom is straightforward and we find the following system of equations

$$i \frac{\partial}{\partial t} \psi_0 = H_0 \psi_0, \quad \frac{d^2 \sigma_k}{dt^2} = -\omega_k^2 \sigma_k + \frac{1}{4\sigma_k^3}, \quad k > 0$$

(4.25)

coupled in a mean–field way by the relations (4.11), which now read

$$\omega_k^2 = k^2 + M^2 - 6\lambda L^{-D} \sigma_k^2, \quad k > 0$$

$$M^2 = m_b^2 + 3\lambda_b \left( L^{-D} \langle \phi_0^2 \rangle + \Sigma_0 \right), \quad \Sigma_0 = \frac{1}{L^D} \sum_{k \neq 0} \sigma_k^2$$

(4.26)

This stage of a truly quantum zero–mode and classical modes with $k > 0$ does not appear fully consistent, since for large volumes some type of classical or gaussian approximation should be considered for $\phi_0$ too. We may proceed in two (soon to be proven equivalent) ways:

1. We shift $\phi_0 = L^{D/2} \tilde{\phi} + \eta_0$ and then deal with the quantum mode $\eta_0$ in the gaussian approximation, taking into account that we must have $\langle \eta_0 \rangle = 0$ at all times. This is most easily accomplished in the Heisenberg picture rather than in the Schroedinger one adopted above. In any case we find that the quantum dynamics of $\phi_0$ is equivalent to the classical dynamics of $\hat{\phi}$ and $\sigma_0 \equiv \langle \eta_0^2 \rangle^{1/2}$ described by the ordinary differential equations

$$\frac{d^2 \tilde{\phi}}{dt^2} = -\omega_0^2 \tilde{\phi} - \lambda \tilde{\phi}^3, \quad \frac{d^2 \sigma_0}{dt^2} = -\omega_0^2 \sigma_0 + \frac{1}{4\sigma_0^3}$$

(4.27)

where $\omega_0^2 = M^2 - 3\lambda L^{-D} \langle \phi_0^2 \rangle$ and $\langle \phi_0^2 \rangle = L^D \langle \phi_0^2 \rangle + \sigma_0^2$.

2. We rescale $\phi_0 = L^{D/2} \xi$ right away, so that $H_0$ takes the form of eq. (4.21). Then $L \to \infty$ is the classical limit such that $\psi_0(\xi)$ concentrates on $\xi = \tilde{\phi}$ which evolves according to the first of the classical equations in (4.27). Since now there is no width associated to the zero–mode, $\tilde{\phi}$ is coupled only to the widths $\sigma_k$ with $k \neq 0$ by $\omega_0^2 = M^2 - 3\lambda \tilde{\phi}^2$, while $M^2 = m_b^2 + 3\lambda_b (\tilde{\phi}^2 + \Sigma_0)$.

It is quite evident that these two approaches are completely equivalent in the infinite volume limit, and both are good approximation to the original tdHF Schroedinger equations, at least provided that $\sigma_0^2$ stays such that $L^{-D} \sigma_0^2$ vanishes in the limit for any time. In this case we have the evolution equations

$$\frac{d^2 \tilde{\phi}}{dt^2} = (2\lambda \tilde{\phi}^2 - M^2) \tilde{\phi}, \quad \frac{d^2 \sigma_k}{dt^2} = -(k^2 + M^2) \sigma_k + \frac{1}{4\sigma_k^3}$$

(4.28)
mean–field coupled by the $L \to \infty$ limit of eqs. (4.26), namely

$$M^2 = m^2 + 3\lambda_b \left( \bar{\phi}^2 + \Sigma - I_D(m^2, \Lambda) \right)$$

(4.29)

for unbroken symmetry [that is $m^2_b$ as in eq. (4.16)] or

$$M^2 = m^2 + 3\lambda_b \left( \bar{\phi}^2 - v^2 + \Sigma - I_D(m^2, \Lambda) \right), \quad m^2 = 2\lambda v^2$$

(4.30)

for broken symmetry [that is $m^2_b$ as in eq. (4.22)]. In any case we define

$$\Sigma = \frac{1}{L^D} \sum_k \sigma_k^2 \sim \int_{k^2 \leq \Lambda^2} \frac{d^D k}{(2\pi)^D} \sigma_k^2$$

as the sum, or integral, over all microscopic gaussian widths [N.B.:this definition differs from that given before in eq. (4.8) by the classical term $\bar{\phi}^2$].

The conserved HF energy (density) corresponding to these equations of motion reads

$$\mathcal{E} = \mathcal{T} + \mathcal{V}, \quad \mathcal{T} = \frac{1}{2} \langle \dot{\bar{\phi}} \rangle^2 + \frac{1}{2} L^D \sum_k \dot{\sigma}_k^2$$

$$\mathcal{V} = \frac{1}{2L^D} \sum_k \left( k^2 \sigma_k^2 + \frac{1}{4\sigma_k^2} \right) + \frac{1}{2} m^2_b (\bar{\phi}^2 + \Sigma) + \frac{3}{4} \lambda_b (\bar{\phi}^2 + \Sigma)^2 - \frac{1}{4} \lambda \bar{\phi}^4$$

(4.31)

Up to additive constants and terms vanishing in the infinite volume limit, this expression agrees with the general HF energy of eq. (4.12) for gaussian wavefunctions. It holds both for unbroken and broken symmetry, the only difference being in the parameterization of the bare mass in terms of UV cutoff and physical mass, eqs. (4.16) and (4.22). The similarity to the energy functional of the large $N$ approach, eq. (4.31), is evident; the only difference, apart from the obvious fact that $\bar{\phi}$ is a $O(n)$ vector rather than a single scalar, is in the mean–field coupling $\sigma_k - \bar{\phi}$ and $\sigma_k - \Sigma$, due to different coupling strength of transverse and longitudinal modes.

This difference between the HF approach for discrete symmetry (i.e. $N = 1$) and the large $N$ method for the continuous $O(N)$-symmetry is not very relevant if the symmetry is unbroken [it does imply however a significantly slower dissipation to the modes of the background energy density]. On the other hand it has a drastic consequence on the equilibrium properties and on the out–of–equilibrium dynamics in case of broken symmetry (see below), since massless Goldstone bosons appear in the large $N$ approach, while the HF treatment of the discrete symmetry case must exhibits a mass also in the broken symmetry phase.

The analysis of physically viable initial conditions proceeds exactly as in the large $N$ approach and will not be repeated here, except for an important observation in case of broken symmetry. The formal energy minimization w.r.t. $\sigma_k$ at fixed $\bar{\phi}$ leads again to eqs. (3.18), and again these are acceptable initial conditions only if the gap equation that follows from eq. (4.30) in the $L \to \infty$ limit, namely

$$M^2 = m^2 + 3\lambda_b \left[ \bar{\phi}^2 - v^2 + I_D(M^2, \Lambda) - I_D(m^2, \Lambda) \right]$$

(4.32)

admits a nonnegative, physical solution for $M^2$. Notice that there is no step function in eq. (4.32), unlike the static case of eq. (4.23), because $\sigma_0^2$ was assumed to be microscopic, so
that the infinite volume $\sigma_k^2$ has no delta–like singularity in $k = 0$. Hence $M = m$ solves eq. (4.32) only at the extremal points $\phi = \pm v$, while it was the solution of the static gap equation (4.23) throughout the Maxwell region $-v \leq \phi \leq v$. The important observation is that eq. (4.32) admits a positive solution for $M^2$ also within the Maxwell region. In fact it can be written, neglecting as usual the inverse–power corrections in the UV cutoff

$$\frac{M^2}{\lambda(M)} = \frac{m^2}{\lambda} + 3(\bar{\phi}^2 - v^2) = 3\bar{\phi}^2 - v^2$$

(4.33)

and there exists indeed a positive solution $M^2$ smoothly connected to the ground state, $\bar{\phi}^2 = v^2$ and $M^2 = m^2$, whenever $\bar{\phi}^2 \geq v^2/3$. The two intervals $v^2 \geq \bar{\phi}^2 \geq v^2/3$ correspond indeed to the metastability regions, while $\bar{\phi}^2 < v^2/3$ is the spinodal region, associated to a classical potential proportional to $(\bar{\phi}^2 - v^2)^2$. This is another effect of the different coupling of transverse and longitudinal modes: in the large $N$ approach there are no metastability regions and the spinodal region coincides with the Maxwell one. As in the large $N$ approach in the spinodal interval there is no energy minimization possible, at fixed background and for microscopic widths, so that a modified form of the gap equation like eq. (3.20) should be applied to determine ultraviolet–finite initial conditions.

The main question now is: how will the gaussian widths $\sigma_k$ grow with time, and in particular how will $\sigma_0$ grow in case of method 1 above, when we start from initial conditions where all widths are microscopic? For the gaussian approximation to remain valid through time, all $\sigma_k$, and in particular $\sigma_0$, must at least not become macroscopic. In fact we have already positively answered this question in the large $N$ approach and the HF equations (4.28) do not differ so much to expect the contrary now. In particular, if we consider the special initial condition $\bar{\phi} = \dot{\bar{\phi}} = 0$, the dynamics of the widths is identical to that in the large $N$ approach, apart from the rescaling by a factor of three of the coupling constant. Thus our HF approximation confirms the large $N$ approach in the following sense: even if one considers in the variational ansatz the possibility of non–gaussian wavefunctionals, the time evolution from gaussian and microscopic initial conditions is effectively restricted for large volumes to non–macroscopic gaussians.

Strictly speaking, however, this might well not be enough, since the infrared fluctuations do grow beyond the microscopic size to become of order $L$. Then the quartic term in the low–$k$ Hamiltonians $H_k$ is of order $L$ and therefore it is not negligible by itself in the $L \to \infty$ limit, but only when compared to the quadratic term, which for a fixed $\omega_k^2$ of order 1 would be of order $L^2$. But we know that, when $\bar{\phi} = 0$, after the spinodal time and before the $\tau_L$, the effective squared mass $M^2$ oscillates around zero with amplitude decreasing as $t^{-1}$ and a frequency fixed by the largest spinodal wavevector. In practice it is “zero on average” and this reflect itself in the average linear growth of the zero–mode fluctuations and, more generally, in the average harmonic motion of the other widths with non–zero wavevectors. In particular the modes with small wavevectors of order $L^{-1}$ feel an average harmonic potential with $\omega_k^2$ of order $L^{-2}$. This completely compensate the amplitude of the mode itself, so that the quadratic term in the low–$k$ Hamiltonians $H_k$ is of order $L^0$, much smaller than the quartic term that was neglected beforehand in the gaussians approximation. Clearly the approximation itself no longer appear fully justified and a more delicate analysis is required. We intend to return on this issue in a future work, restricting in the next section our observations on some rather peculiar consequences of the gaussians approximation that
provide further evidence for its internal inconsistency.

C. Effective potential and late–time evolution

By definition, the gaussian approximation of the effective potential $V_{\text{eff}}(\bar{\phi})$ coincides with the infinite–volume limit of the potential energy $V(\bar{\phi}, \{\sigma_k\})$ of eq. (4.31) when the widths are of the $\bar{\phi}$–dependent, energy–minimizing form (3.18) with the gap equation for $M^2$ admitting a nonnegative solution. As we have seen, this holds true in the unbroken symmetry case for any value of the background $\bar{\phi}$, so that the gaussian $V_{\text{eff}}$ is identical to the HF one, since all wavefunctions $\psi_k$ are asymptotically gaussians as $L \to \infty$. In the presence of symmetry breaking instead, this agreement holds true only for $\bar{\phi}^2 \geq v^2$; for $v^2/3 \leq \bar{\phi}^2 < v^2$ the gaussian $V_{\text{eff}}$ exists but is larger than the HF potential, which is already flat. In fact, for any $\bar{\phi}^2 \geq v^2/3$, we may write the gaussian $V_{\text{eff}}$ as

$$V_{\text{eff}}(\bar{\phi}) = V_{\text{eff}}(-\bar{\phi}) = V_{\text{eff}}(v) + \int_{v}^{\bar{\phi}} du u [M(u)^2 - 2\lambda u^2]$$

where $M(u)^2$ solves the gap equation (4.33), namely $M(u)^2 = \hat{\lambda}(M(u))(3u^2 - v^2)$. In each of the two disjoint regions of definition this potential is smooth and convex, with unique minima in $+v$ and $-v$, respectively. Its HF counterpart is identical for $\bar{\phi}^2 \geq v^2$, while it takes the constant value $V_{\text{eff}}(v)$ throughout the internal region $\bar{\phi}^2 < v^2$. On the other hand the gaussian $V_{\text{eff}}$ cannot be defined in the spinodal region $\bar{\phi}^2 < v^2/3$, where the gap equation does not admit a nonnegative solution in the physical region far away from the Landau pole.

Let us first compare this HF situation with that of large $N$. There the different coupling of the transverse modes, three time smaller than the HF longitudinal coupling, has two main consequences at the static level: the gap equation (3.8) does not admit nonnegative solutions for $\bar{\phi}^2 < v^2$, so that the spinodal region coincides with the region in which the effective potential is flat, and the physical mass vanishes. The out–of–equilibrium counterpart of this is the dynamical Maxwell construction: when the initial conditions are such that $\bar{\phi}^2$ has a limit for $t \to \infty$, all possible asymptotic values exactly span the flatness region (and the effective mass vanishes in the limit). In practice this means that $|\bar{\phi}|$ is not the true dynamical order parameter, whose large time limit coincides with $v$, the equilibrium field expectation value in a pure phase. Rather, one should consider as order parameter the renormalized local (squared) width

$$\lim_{N \to \infty} \frac{\langle \phi(x) \cdot \phi(x) \rangle_R}{N} = \bar{\phi}^2 + \Sigma_R = v^2 + \frac{M^2}{\lambda}$$

where the last equality follows from the definition itself of the effective mass $M$ [see eq. (3.17)]. Since $M$ vanishes as $t \to \infty$ when $\bar{\phi}^2$ tends to a limit within the flatness region, we find the renormalized local width tends to the correct value $v$ which characterizes the broken symmetry phase, that is the bottom of the classical potential.

Let us now examine what happens instead in the dynamics of the HF approximation, where at the static level the spinodal region $\bar{\phi}^2 < v^2/3$ is smaller than the flatness region $\bar{\phi}^2 < v^2$. Our (preliminary) numerical evidence shows that a dynamical Maxwell construction take place as in the large $N$ approximation, but it covers only the spinodal region [see for
instance figs. 9 and 10. If the background $\bar{\phi}$ starts with zero velocity inside the spinodal interval, then it tends to a limit within the same interval, the asymptotic force vanishing because $M^2 = 2\lambda \bar{\phi}^2$ in the limit [see eq. (4.28)]. Hence we find that the order parameter

$$\langle \phi(x)^2 \rangle_R = \bar{\phi}^2 + \Sigma_R = \frac{v^2}{3} + \frac{M^2 - 2\lambda \bar{\phi}^2}{3\lambda}$$

[the relation (4.30) was used in the last equality] tends to $v^2/3$ not $v^2$. It “stops at the spinodal line”. This fact is at the basis of the so-called spinodal inflation [27]. Even without a conspicuous numerical evidence for the full dynamical Maxwell construction, this results is manifestly true when initially (and therefore at any time) $\bar{\phi} = 0 = \dot{\bar{\phi}}$, because the equations of motion for the widths are the same as in large $N$ and the effective mass only differs by the factor of three in front of the quantum back–reaction $\Sigma$. Since practically all the back–reaction takes place during the exponential growth of the unstable spinodal modes, one could say that, in the gaussian HF approximation, this back–reaction is to strong and prevents the quantum fluctuations from sampling the true minima of the classical potential. A side effect of this is that the effective mass $M$ does not tend to its correct equilibrium value $m$ as $t \to \infty$, unlike in the large $N$ approach.

In the previous section we have discussed the possible origin of the problem: in our(tdHF approach the initial gaussian wavefunctions are allowed to evolve into non–gaussian forms, but they simply do not do it in a macroscopic way, within a further harmonic approximation for the evolution, so that in the infinite–volume limit they are indistinguishable from gaussians at all times. But when $M^2$ is on average not or order $L^0$, but much less, as it happens for suitable initial conditions, infrared modes of order $L$ will be dominated by the quartic term in our Schroedinger equations (4.5), showing a possible internal inconsistency of the gaussians approximation.

Another interesting point concerns the dynamical Maxwell construction itself, within the gaussians approximation. In fact it is not at all trivial that the effective potential, in any of the approximation previously discussed, does bear relevance on the asymptotic behavior of the infinite–volume system whenever a fixed point is approached. Strictly speaking in fact, even in such a special case the effective potential has little to say on the dynamics, since it is obtained from a static minimization and the energy is not at its minimum at the initial time and is exactly conserved in the evolution. On the other hand, if a solution of the equations of motion (1.28) exists in which the background $\bar{\phi}$ tends to a constant $\bar{\phi}_\infty$ as $t \to \infty$, one might expect that the effective action (which however is nonlocal in time) somehow reduces to a (infinite) multiple the effective potential, so that $\bar{\phi}_\infty$ should be an extremal of the effective potential. This is still an open question that deserve further analytic studies and perhaps some further numerical confirmation.

V. CONCLUSIONS AND PERSPECTIVES

In this work we have presented a rather detailed study of the dynamical evolution out of equilibrium, in finite volume (a cubic box of size $L$ in 3D), for the $\phi^4$ QFT. For comparison, we have also analyzed some static characteristics of the theory both in unbroken and broken symmetry phases. We have worked in two non–perturbative approximation schemes, namely...
the large $N$ expansion in the leading order and a generalized time dependent Hartree–Fock approximation.

We have reached two main conclusions.

The first, based on strong numerical evidence, is that the linear growth of the zero–mode quantum fluctuations, observed already in the large $N$ approach of refs. [8–10], cannot be consistently interpreted as a new form of Bose–Einstein condensation. In fact, in finite volume, $\sigma_0$ never grows to $O(L^{3/2})$ if it starts from a microscopic value, that is at most of order $L^{1/2}$. On the other hand all long–wavelength fluctuations rapidly become of order $L$, signalling a novel infrared behavior quite different from free massless fields at equilibrium [recall that the large $N$ or HF approximations are of mean field type, with no direct interaction among particle excitations]. This is in agreement with the properties of the two–point function determined in [8].

The second point concerns the HF approach to the out–of–equilibrium $\phi^4$ QFT. We have shown that, within a slightly enlarged tdHF approach that allows for non–gaussian wavefunctions, one might recover the usual gaussians HF approximation in a more controlled way, realizing that the growth of long–wavelength fluctuations to order $L$ in fact undermines the self–consistency of the gaussians approximation itself. The first manifestation of this weakness is the curious “stopping at the spinodal line” of the width of the gaussian quantum fluctuations. This does not happen in the large $N$ approach because of different coupling of transverse mode (the only ones that survive in the $N \to \infty$ limit) with respect to the longitudinal modes of the $N = 1$ case in the HF approach.

Clearly further study, both analytical and numerical, is needed in our tdHF approach to better understand the dynamical evolution of quantum fluctuations in the broken symmetry phase. Another interesting direction is the investigation, in our HF approximation, of the case of finite $N$, in order to interpolate smoothly the results for $N = 1$ to those of the $1/N$ approach. One first question concerns whether or not the Goldstone theorem is respected in the HF approximation [26]. It would be interesting also to study the dynamical realization of the Goldstone paradigm, namely the asymptotic vanishing of the effective mass in the broken symmetry phases, in different models; this issue needs further study in the $2D$ case [12], where it is known that the Goldstone theorem is not valid.

Another open question concerns the connection between the minima of the effective potential and the asymptotic values for the evolution of the background, within the simplest gaussian approximation. As already pointed out in [8], a dynamical Maxwell construction occurs for the $O(N)$ model in infinite volume and at leading order in $1/N$ in case of broken symmetry, in the sense that any value of the background within the spinodal region can be obtained as large time limit of the evolution starting from suitable initial conditions. Preliminary numerical evidence suggests that something similar occurs also in the Hartree approximation for a single field, but a more thorough and detailed analysis is needed.

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APPENDIX A: DETAILS OF THE NUMERICAL ANALYSIS

We present here the precise form of the evolution equations for the field background and the quantum mode widths, which control the out–of–equilibrium dynamics of the $\phi^4$ model in finite volume at the leading order in the $1/N$ approach and in the tdHF approach, as described in sections [III C] and [IV B]. We restrict here our attention to the tridimensional case.

Let us begin by noticing that each eigenvalue of the Laplacian operator in a 3D finite volume is of the form $k^2_n = \left(\frac{2\pi}{L}\right)^2 n$, where $n$ is a non–negative integer obtained as the sum of three squared integers, $n = n_x^2 + n_y^2 + n_z^2$. Then we associate a degeneracy factor $g_n$ to each eigenvalue, representing the number of different ordered triples $(n_x, n_y, n_z)$ yielding the same $n$. One may verify that $g_n$ takes on the continuum value of $4\pi k^2$ in the infinite volume limit, where $k = \left(\frac{2\pi}{L}\right)^2 n$ is kept fixed when $L \to \infty$.

Now, the system of coupled ordinary differential equations is, in case of the large $N$ approach,

$$\left[\frac{d^2}{dt^2} + M^2\right] \phi = 0 , \quad \left[\frac{d^2}{dt^2} + \left(\frac{2\pi}{L}\right)^2 n + M^2\right] \sigma_n - \frac{1}{4\sigma_n^3} = 0 \quad (A1)$$

where the index $n$ ranges from 0 to $N^2$, $N = \Lambda L/2\pi$ and $M^2(t)$ is defined by the eq. (3.16) in case of unbroken symmetry and by eq. (3.17) in case of broken symmetry. The back–reaction $\Sigma$ reads, in the notations of this appendix

$$\Sigma = \frac{1}{L^D} \sum_{n=0}^{N^2} g_n \sigma_n^2$$

Technically it is simpler to treat an equivalent set of equations, which are formally linear and do not contain the singular Heisenberg term $\sigma_n^{-3}$. This is done by introducing the complex mode amplitudes $z_n = \sigma_n \exp(i\theta_n)$, where the phases $\theta_n$ satisfy $\sigma_n^2 \theta_n = 1$. Then we find a discrete version of the equations studied for instance in ref [14],

$$\left[\frac{d^2}{dt^2} + \left(\frac{2\pi}{L}\right)^2 n + M^2\right] z_n = 0 , \quad \Sigma = \frac{1}{L^D} \sum_{n=0}^{N^2} g_n |z_n|^2 \quad (A2)$$

subject to the Wronskian condition

$$z_n \dot{z}_n - \bar{z}_n \dot{\bar{z}}_n = -i$$

One realizes that the Heisenberg term in $\sigma_n$ corresponds to the centrifugal potential for the motion in the complex plane of $z_n$.

Let us now come back to the equations (A2). To solve these evolution equations, we have to choose suitable initial conditions respecting the Wronskian condition. In case of unbroken symmetry, once we have fixed the value of $\phi$ and its first time derivative at initial
time, the most natural way of fixing the initial conditions for the $z_n$ is to require that they minimize the energy at $t = 0$. We can obviously fix the arbitrary phase in such a way to have a real initial value for the complex mode functions

$$z_n(0) = \frac{1}{\sqrt{2\Omega_n}} \quad \frac{dz_n}{dt}(0) = i\sqrt{\frac{\Omega_n}{2}}$$

where $\Omega_n = \sqrt{k_n^2 + M^2(0)}$. The initial squared effective mass $M^2(t = 0)$, has to be determined self-consistently, by means of its definition (3.10).

In case of broken symmetry, the gap equation is a viable mean for fixing the initial conditions only when $\phi$ lies outside the spinodal region [cfr. eq (3.21)]; otherwise, the gap equation does not admit a positive solution for the squared effective mass. In that case, we have to resort to other methods, in order to choose the initial conditions. Following the discussion presented in III C, one possible choice is to set $\sigma_k^2 = \frac{1}{2\sqrt{k^2 + |M|^2}}$ for $k^2 < |M|^2$ in eq. (3.20) and then solve the corresponding gap equation (3.20). An other acceptable choice would be to solve the gap equation (3.20), once we have set a massless spectrum for all the spinodal modes but the zero mode, which is started from an arbitrary, albeit microscopic, value.

There is actually a third possibility, that is in some sense half a way between the unbroken and broken symmetry case. We could allow for a time dependent bare mass, in such a way to simulate a sort of cooling down of the system. In order to do that, we could start with a unbroken symmetry bare potential (which fixes initial conditions naturally via the gap equation) and then turn to a broken symmetry one after a short interval of time. This evolution is achieved by a proper interpolation in time of the two inequivalent parameterizations of the bare mass, eqs. (3.5) and (3.10).

We looked for the influence this different choices could produce in the results and indeed they depend very little and only quantitatively from the choice of initial condition we make.

As far as the numerical algorithm is concerned, we used a 4th order Runge-Kutta algorithm to solve the coupled differential equations (A2), performing the computations in boxes of linear size ranging from $L = 10$ to $L = 200$ and verifying the conservation of the Wronskian to order $10^{-5}$. Typically, we have chosen values of $\mathcal{N}$ corresponding to the UV cutoff $\Lambda$ equal to small multiples of $m$ for unbroken symmetry and of $\sqrt{\lambda}$ for broken symmetry. In fact, the dynamics is very weakly sensitive to the presence of the ultraviolet modes, once the proper subtractions are performed. This is because only the modes inside the unstable (forbidden or spinodal) band grow exponentially fast, reaching soon non perturbative amplitudes (i.e. $\approx \lambda^{-1/2}$), while the modes lying outside the unstable band remains perturbative, contributing very little to the quantum back–reaction and weakly affecting the overall dynamics. The unique precaution to take is that the initial conditions be such that the unstable band lay well within the cutoff.
FIG. 1. Zero-mode amplitude evolution for different values of the size \( L = 20, 40, 60, 80, 100 \), for \( \lambda = 0.1 \) and broken symmetry, with \( \bar{\phi} = 0 \).

FIG. 2. Time evolution of the squared effective mass \( M^2 \) in broken symmetry, for \( L = 100 \) and \( \lambda = 0.1 \).
FIG. 3. The quantum back-reaction $\Sigma$, with the parameters as in Fig. 2.

FIG. 4. Zero-mode amplitude evolution for different value of the renormalized coupling constant $\lambda = 0.01, 0.1, 1$, for $L = 100$ and broken symmetry, with $\bar{\phi} = 0$. 
FIG. 5. Detail of $M^2$ near $t = \tau_L$ for $L = 40$ (dotted line). The case $L = 80$ is plotted for comparison (solid line).

FIG. 6. Detail of $\sigma_0$ near $t = \tau_L$ for $L = 40$ (dotted line). The case $L = 80$ is plotted for comparison (solid line).
FIG. 7. Next-to-zero mode ($k = 2\pi/L$) amplitude evolution for different values of the size $L = 20, 40, 60, 80, 100$, for $\lambda = 0.1$ and broken symmetry, with $\bar{\phi} = 0$.

FIG. 8. Next-to-zero mode ($k = 2\pi/L$) amplitude evolution for different value of the renormalized coupling constant $\lambda = 0.01, 0.1, 1$, for $L = 100$ and broken symmetry, with $\bar{\phi} = 0$. 
FIG. 9. Evolution of the background for two different initial conditions within the spinodal interval, in the tdHF approximation: $\phi(t = 0) = 0.1$ (dotted line) and $\phi(t = 0) = 0.4$ (solid line).

FIG. 10. Evolution of $M^2$ for the two initial conditions of fig.
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