Effect of indirect dependencies on "A mutual information minimization approach for a class of nonlinear recurrent separating systems"

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Abstract. In a recent paper [4], Duarte and Jutten investigated the Blind Source Separation (BSS) problem, for the nonlinear mixing model that they introduced in that paper. They proposed to solve this problem by using information-theoretic tools, more precisely by minimizing the mutual information (MI) of the outputs of the separating structure. When applying the MI approach to BSS problems, one usually determines the analytical expressions of the derivatives of the MI with respect to the parameters of the considered separating model. In the literature, these calculations were mainly reported for linear mixtures up to now. They are more complex for nonlinear mixtures, due to dependencies between the considered quantities. Moreover, the notations commonly employed by the BSS community in such calculations may become misleading when using them for nonlinear mixtures, due to the above-mentioned dependencies. We claim that the calculations reported in [4] contain an error, because they did not take into account all these dependencies. In this document, we therefore explain this phenomenon, by showing the effect of indirect dependencies on the application of the MI approach to the mixing and separating models considered in [4]. We thus introduce a corrected expression of the gradient of the considered BSS criterion based on MI. This correct gradient may then e.g. be used to optimize the adaptive coefficients of the considered separating system by means of the well-known gradient descent algorithm. As explained hereafter, this investigation has some similarities with an analysis that we previously reported in another arXiv document [3]. However, these two investigations concern different problems, not only in terms of the considered type of mixture and separating structure, but also of the mathematical tools used to develop BSS methods for these configurations (information theory vs maximum likelihood approach).

Keywords. Information theory, mutual information, blind signal separation, independent component analysis, nonlinear mixture, additive-target mixture (ATM), recurrent separating structure, indirect dependency, total derivative, partial derivative, gradient.
1 Data model

Blind source separation (BSS) consists in restoring a vector \( s(t) \) of \( N \) unknown source signals from a vector \( x(t) \) of \( P \) observed signals (most often with \( P = N \)), where \( x(t) \) is derived from \( s(t) \) through an unknown mixing function \( g \), i.e.

\[
x(t) = g(s(t)).
\]  

(1)

Recently, Duarte and Jutten investigated a specific version of this problem [4], which involves \( P = 2 \) observed signals \( x_1(t) \) and \( x_2(t) \), which are derived from \( N = 2 \) source signals \( s_1(t) \) and \( s_2(t) \), through the nonlinear function defined as

\[
x_1(t) = s_1(t) + a_{12}(s_2(t))^k
\]

(2)

\[
x_2(t) = s_2(t) + a_{21}(s_1(t))^\frac{1}{2}.
\]

(3)

This data model is derived from the Nikolsky-Eisenman empirical model for potentiometric-based ion concentration sensors [4]. As in [4], we omit the time index \( t \) in signal notations hereafter, for readability. The mixing model (2)-(3) may then also be expressed in compact form as

\[
x = g(s).
\]  

(4)

In this equation, \( s = [s_1, s_2]^T \) and \( x = [x_1, x_2]^T \), where \( T \) stands for transpose, and the nonlinear mixing function \( g \) has two components \( g_1 \) and \( g_2 \), with \( x_i = g_i(s), \ \forall i \in \{1, 2\} \). These components \( g_i \) are respectively defined by (2) and (3). Eq. (4) focuses on the signals (i.e. sources and observations). It hides the fact that the observations also depend on the parameters of the mixing model, i.e. on \( a_{12} \) and \( a_{21} \) in the model considered here. This additional dependency can be made explicit, by rewriting (4) as

\[
x = g(s, a_{12}, a_{21}).
\]  

(5)

2 Previously reported results for mutual information minimization

2.1 Overview and issue of previous method

As suggested above, the BSS problem associated with the mixing model (2)-(3) consists in retrieving a sequence of unknown source vectors \( s \) from the corresponding sequence of measured observation vectors \( x \) and from the mixing parameters \( a_{12} \) and \( a_{21} \), which are also initially unknown. These mixing parameters should therefore be estimated before proceeding to the source restoration step. Creating an overall BSS method thus consists in defining two items, i.e. i) a "separating structure", which performs the inversion of the mixing equations (2)-(3) for known mixing parameter values, and ii) a procedure for estimating these mixing parameters.

The separating structure used in [4] was derived by Duarte and Jutten from the structure for linear-quadratic mixtures proposed by Hosseini and Deville in [5], [6], [1], [2]. The structure in [4] belongs to the general class of structures proposed by Deville and Hosseini in [2] for the ATM class of mixing models, which includes the specific model (2)-(3).

As for the estimation of the mixing parameters, Duarte and Jutten developed a procedure based on information-theoretic tools, more precisely on the minimization of the
mutual information (MI) of the outputs of the separating structure. However, we here claim that this procedure contains an error, which is due to a difficulty encountered with nonlinear mixing models in general, for different classes of BSS methods. This difficulty is somewhat similar to the one that we highlighted in another arXiv document [3]: unlike the method considered hereafter, the BSS approach described in [3] is not based on information theoretic tools, but on the maximum likelihood framework. Moreover, it concerns a different class of nonlinear mixtures. However, similar quantities appear in the calculations performed for both methods, and they deserve special care in both of them.

The current document therefore aims at explaining and correcting the error which was made in [4]. We thus show how the BSS method of [4] should be modified so as to actually achieve mutual information minimization. Before focusing on the issue faced in [4], we now summarize the features of that approach which are of importance hereafter.

2.2 Description of previous method

The considered separating structure has internal adaptive coefficients $w_{12}$ and $w_{21}$. For each time $t$, this structure determines and output vector $y = [y_1, y_2]^T$ from its current internal coefficients and from the current observation vector $x$. To this end, it iteratively updates its output according to

$$y_1(n+1) = x_1 - w_{12}(y_2(n))^k$$
$$y_2(n+1) = x_2 - w_{21}(y_1(n))^{1/2}.$$  

The convergence of this recurrence therefore corresponds to a state such that

$$y_1 = x_1 - w_{12}y_2^k$$
$$y_2 = x_2 - w_{21}y_1^{1/2}.$$  

For a given time $t$, we denote as $Y_1$ and $Y_2$ the random variables respectively associated with the output signal samples $y_1$ and $y_2$ obtained after the above recurrence has converged. We also define the corresponding output random vector as $Y = [Y_1, Y_2]^T$.

The optimum values of $w_{12}$ and $w_{21}$ are defined as those which minimize the mutual information of $Y_1$ and $Y_2$, which is denoted $I(Y)$. Equivalently, they are those which minimize a quantity $C(Y)$. This quantity is equal to $I(Y)$, up to an additive term which only depends on the observations and which therefore does not depend on $w_{12}$ and $w_{21}$. That quantity reads

$$C(Y) = \left( \sum_{i=1}^{2} H(Y_i) \right) - E\{\ln |J_h|\}$$  

where $H(Y_i)$ is the differential entropy of $Y_i$ while $E\{\cdot\}$ stands for expectation and $J_h$ is the Jacobian\footnote{The quantities to be respectively considered in these two methods depend on different signals (source signals vs outputs of separating system) and functions (mixing function vs separating function). However, these signals and functions yield similar phenomena concerning the topic addressed in this document.\footnote{For the sake of readability, we use the same notation, i.e. $J_h$, for (i) the sample value of this Jacobian associated to sample values $y_1$ and $y_2$ (see e.g. [11]) and (ii) the random variable defined by this quantity when considered as a function of the random variables $Y_1$ and $Y_2$ (see e.g. [12]). To know whether we are considering the sample value of $J_h$ or the associated random variable in an equation, one just has to check whether that equation involves the sample values $y_1$ and $y_2$ or the associated random variables $Y_1$ and $Y_2$: see e.g. [11] and [12].}} of the separating function $h = g^{-1}$ achieved by the considered separating
structure, i.e. \(J_h\) is the determinant of the Jacobian matrix of \(h\). For the function \(h\) considered in this investigation, the authors show that

\[
J_h = \frac{1}{1 - w_{12}w_{21}y_1^k y_2^k}.
\]

(11)

To determine the values of \(w_{12}\) and \(w_{21}\) which minimize \(C(Y)\), the authors then consider the gradient of \(C(Y)\) with respect to the vector composed of \(w_{12}\) and \(w_{21}\). Each component of this gradient is equal to the derivative of \(C(Y)\) with respect to one of the parameters \(w_{k\ell}\). In [4], the authors denoted this gradient by using the notation most often employed in the BSS community (see e.g. [7]), i.e. each of its components reads \(\frac{\partial C(Y)}{\partial w_{k\ell}}\). We keep this notation in this section, in order to clearly refer to the equations available in [4], but in Section 3 we will show that it may be misleading and we will therefore introduce another notation. So, in [4], it was showed that these derivatives read

\[
\frac{\partial C(Y)}{\partial w_{k\ell}} = \left( \sum_{i=1}^2 E\{\psi_i(Y_i) \frac{\partial Y_i}{\partial w_{k\ell}} \} \right) - E\left\{ \frac{1}{J_h} \frac{\partial J_h}{\partial w_{k\ell}} \right\}
\]

(12)

where

\[
\psi_i(u) = -\frac{d \ln f_{Y_i}(u)}{du} \quad \forall i \in \{1, 2\}
\]

(13)

are the score functions of the output signals, denoting \(f_{Y_i}(.)\) the probability density functions of these signals.

The last stage of this investigation consists in deriving the expressions of all the terms of the right-hand side of (12). In Equation (26) of [4], an explicit expression is provided and it is stated that it is equal to (the vector form of) the term \(E\left\{ \frac{1}{J_h} \frac{\partial J_h}{\partial w_{k\ell}} \right\}\) which appears in (12). We claim that this is not true, because the expression whose expectation is provided in the right-hand side of Equation (26) of [4] is only one of the terms which compose the complete expression to be then used in (12) as the term misleadingly denoted \(\frac{1}{J_h} \frac{\partial J_h}{\partial w_{k\ell}}\) in (12). In the following section of the current document, we clarify this point and we determine the complete expression of the term denoted \(\frac{1}{J_h} \frac{\partial J_h}{\partial w_{k\ell}}\) in (12). We also comment about the other terms of (12).

3 New results for mutual information minimization: corrected expression of gradient

When determining the values of \(w_{12}\) and \(w_{21}\) which minimize \(C(Y)\), that function \(C(Y)\) is considered for the fixed set of observed vectors. The only independent variable in this approach is the set of parameters to be estimated, i.e. \(w_{12}\) and \(w_{21}\). The outputs \(y_1\) and \(y_2\) of the separating system are dependent variables, here linked to the observations and to \(w_{12}\) and \(w_{21}\) by (8)-(9). The overall variations of \(C(Y)\) with respect to \(w_{12}\) and \(w_{21}\) result from two types of terms contained in the expression of \(C(Y)\), i.e. (i) the terms involving \(w_{12}\) and \(w_{21}\) themselves and (ii) the terms involving the output random variables \(Y_1\) and \(Y_2\), which are here considered as functions of \(w_{12}\) and \(w_{21}\) and which may therefore be denoted as \(Y_1(w_{12}, w_{21})\) and \(Y_2(w_{12}, w_{21})\) for the sake of clarity.

This approach should be kept in mind when interpreting all equations in [4], which were partly gathered in Section 2 of the current document. Especially, the function \(C(Y)\) itself, which appears in the left-hand side of (10), may be denoted as
C(w_{12}, w_{21}, Y_1(w_{12}, w_{21}), Y_2(w_{12}, w_{21})) for the sake of clarity. In order to determine the location of the minimum of this function, one should then consider the total derivatives of
\[ C(w_{12}, w_{21}, Y_1(w_{12}, w_{21}), Y_2(w_{12}, w_{21})) \] with respect to \( w_{12} \) and \( w_{21} \). The notations with partial derivatives in (12) may therefore be misleading, as confirmed below. Therefore, (12) should preferably be rewritten as:

\[
\frac{dC(Y)}{dw_{k\ell}} = \left( \sum_{i=1}^{2} E\{\psi_i(Y_i) \frac{dY_i}{dw_{k\ell}}\} \right) - \frac{1}{J_h} \frac{dJ_h}{dw_{k\ell}}.
\] (14)

still with (13). The term \( \frac{dJ_h}{dw_{k\ell}} \) in (14) then deserves some care because, as shown by (11), the Jacobian \( J_h \) contains the above-defined two types of dependencies with respect to \( w_{12} \) and \( w_{21} \), i.e. (i) direct dependencies due to the factors in (11) which explicitly contain \( w_{12} \) and \( w_{21} \) and (ii) indirect dependencies due to the factors in (11) which depend on \( y_1 \) and \( y_2 \), which themselves depend on \( w_{12} \) and \( w_{21} \) in this approach. We here have to consider the total derivative \( \frac{dJ_h}{dw_{k\ell}} \), which takes into account both types of dependencies, and which therefore reads

\[
\frac{dJ_h}{dw_{k\ell}} = \frac{\partial J_h}{\partial w_{k\ell}} + \sum_{i=1}^{2} \frac{\partial J_h}{\partial y_i} \frac{dy_i}{dw_{k\ell}}.
\] (15)

In this expression, \( \frac{\partial J_h}{\partial w_{k\ell}} \) is the partial derivative of \( J_h \) with respect to \( w_{k\ell} \), calculated by considering that the signals \( y_1 \) and \( y_2 \) are constant (in addition to the fact that the other internal coefficient \( w_{ij} \) of the separating system is also constant). This partial derivative is the quantity that is taken into account in the right-hand side of (26) of [4]. However, let us insist again that this partial derivative is first to be added with the other terms in the right-hand side of (15), in order to obtain the overall total derivative \( \frac{dJ_h}{dw_{k\ell}} \) defined by (15).

What should eventually be used in the last term of (12) or (14) is this total derivative.

So, starting from the expression of \( J_h \) provided in (11), one easily derives all its partial derivatives involved in (15). They read as follows

\[
\frac{\partial J_h}{\partial w_{12}} = \frac{w_{21} y_1^{\frac{k-1}{k}} y_2^{\frac{1}{k}}}{[1 - w_{12} w_{21}]^{\frac{k}{k}}} - \frac{w_{12} y_1^{\frac{k-1}{k}} y_2^{\frac{k-1}{k}}}{[1 - w_{12} w_{21}]^{\frac{k}{k}}},
\] (16)

\[
\frac{\partial J_h}{\partial w_{21}} = \frac{w_{12} y_1^{\frac{k-1}{k}} y_2^{\frac{k-1}{k}}}{[1 - w_{12} w_{21}]^{\frac{k}{k}}} - \frac{w_{21} y_1^{\frac{k}{k}} y_2^{\frac{1}{k}}}{[1 - w_{12} w_{21}]^{\frac{k}{k}}}.
\] (17)

\[\text{Each derivative } \frac{dC(Y)}{dw_{k\ell}} \text{ is "total" only with respect to the considered coefficient } w_{k\ell} \text{ (which is one of the two coefficients } w_{12} \text{ and } w_{21}, \text{ i.e. it takes into account all variations of } C(y) \text{ with respect to that coefficient } w_{k\ell} \text{ while the other coefficient, i.e. } w_{\ell k}, \text{ is kept constant. For the sake of clarity, we could therefore denote that derivative } \frac{dC(Y)}{dw_{k\ell}} \text{ with respect to that coefficient in } (15), \text{ to show that } w_{k\ell} \text{ is constant. However, this would decrease readability. Therefore, in all this paper we omit the notation } (\ )_{w_{k\ell}}, \text{ but it should be kept in mind that each considered derivative with respect to } w_{k\ell} \text{ is calculated with } w_{\ell k} \text{ constant. Then, in this framework, what we have to distinguish are: (i) the total derivative due to the variations of } w_{k\ell}, Y_1 \text{ and } Y_2 \text{ and (ii) the partial derivative only due to } w_{k\ell}. \text{ We then have to use two different notations for these two types of derivatives, such as } \frac{dJ_h}{dw_{k\ell}} \text{ and } \frac{\partial J_h}{\partial w_{k\ell}} \text{ in (15). This type of notations is commonly used in the literature for functions which depend (i) on a single independent variable, i.e. time, and (ii) on other variables which themselves depend on time, such as coordinate variables: see e.g. [http://en.wikipedia.org/wiki/Total_derivative]. We here extend this concept to a configuration which involves several independent variables, i.e. } w_{12} \text{ and } w_{21} \text{ (and, again, other variables which themselves depend on the independent variables, i.e. } Y_1 \text{ and } Y_2). \text{ We keep the same type of notations as in the standard case involving a single independent variable.} \]
\[
\frac{\partial J_h}{\partial y_1} = \frac{w_{12}w_{21}}{[1 - w_{12}y_2^{-1}y_1^{-1}]^2} \left( \frac{1}{k} - 1 \right) \frac{1}{2} y_2^{-\frac{1-1}{2}} y_2^{-k-1} y_1^{-\frac{1}{2}}
\]
(18)

\[
\frac{\partial J_h}{\partial y_2} = \frac{w_{12}w_{21}}{[1 - w_{12}y_2^{-1}y_1^{-1}]^2} \left( k - 1 \right) \frac{1}{2} y_2^{-\frac{1-1}{2}} y_2^{-k-2} y_1^{-\frac{1}{2}}
\]
(19)

The case when \( k = 1 \) deserves a comment. As shown by (2)-(3), the mixing model then becomes linear. Besides, as shown by (18)-(19), we then have

\[
\frac{\partial J_h}{\partial y_1} = 0
\]
(20)

\[
\frac{\partial J_h}{\partial y_2} = 0,
\]
(21)

so that the total derivative \( \frac{dJ}{dw_{kl}} \) in (15) becomes equal to the partial derivative \( \frac{\partial J}{\partial w_{kl}} \) in (15). This clearly shows that the problems due to the distinction between these two derivatives, that we address in this paper, concern nonlinear mixtures.

The last terms which are required to obtain the complete expressions in (14) and (15) are all four derivatives \( \frac{dy}{dw_{kl}} \). For the sake of clarity, we now show how they may be considered, when taking into account the above comments about total and partial derivatives. Here again, \( w_{12} \) and \( w_{21} \) should be considered as the independent variables, while \( y_1 \) and \( y_2 \) are functions of them and the observations are constant. All these parameters are linked by (8)-(9). By first computing the total derivatives of the latter equations with respect to \( w_{12} \), one gets

\[
\frac{dy_1}{dw_{12}} = -(y_2^k + w_{12}y_2^{k-1} \frac{dy_2}{dw_{12}})
\]
(22)

\[
\frac{dy_2}{dw_{12}} = -w_{21} \frac{y_2^{-1} \frac{1}{k} y_1^{k-1}}{1 - w_{12}y_2^{-1}y_1^{-1}}
\]
(23)

Inserting (23) in (22), one derives

\[
\frac{dy_1}{dw_{12}} = -\frac{-y_2^k}{1 - w_{12}w_{21}y_1^{-1}y_2^{-k-1}}
\]
(24)

Then inserting (24) in (23), one obtains

\[
\frac{dy_2}{dw_{12}} = \frac{w_{21} y_1^{-1} y_2^k}{1 - w_{12}w_{21}y_1^{-1}y_2^{-k-1}}
\]
(25)

Similarly, computing the total derivatives of (8)-(9) with respect to \( w_{21} \) eventually yields

\[
\frac{dy_1}{dw_{21}} = \frac{w_{12}y_1^{-1} y_2^{-k-1}}{1 - w_{12}w_{21}y_1^{-1}y_2^{-k-1}}
\]
(26)

\[
\frac{dy_2}{dw_{21}} = \frac{-y_1^{-1}}{1 - w_{12}w_{21}y_1^{-1}y_2^{-k-1}}
\]
(27)

---

4 Eq. (13) is obtained by taking the derivative of (10) with respect to \( w_{kl} \). It thus relies on the fact that \( \frac{dH(Y_i)}{dw_{kl}} = E\{\psi_i(Y_i) \frac{dy}{dw_{kl}}\} \). In [4], this result was borrowed from [8]. Considering the problems due to indirect dependencies in nonlinear mixtures found in [4], one may wonder whether the relationship \( \frac{dH(Y_i)}{dw_{kl}} = E\{\psi_i(Y_i) \frac{dy}{dw_{kl}}\} \) still holds for the nonlinear mixing model studied in [4]. We claim that it does hold.
The expressions of all four derivatives \(\frac{d\mu}{dw_{k\ell}}\) obtained with this approach remain equal to the expressions (30)-(33) of [4], except that all partial derivative notations \(\frac{\partial \mu}{\partial w_{k\ell}}\) in [4] are here replaced by total derivative notations \(\frac{d\mu}{dw_{k\ell}}\).

Gathering all above expressions then makes it possible to determine the total derivative \(\frac{dJ}{dw_{k\ell}}\) in (15), and then the overall gradient components in (14). This yields the correct expression of the gradient of the considered BSS criterion based on mutual information.

This correct gradient expression may eventually be used to optimize the adaptive coefficients \(w_{12}\) and \(w_{21}\), e.g. using the well-known gradient descent algorithm.

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