The effects of parameter uncertainties on the numerical plastic analysis of non-prismatic reinforced concrete beams

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Abstract. Structural models and parameters are usually considered deterministic in most studies. However, test results revealed that uncertainties exist in most of these cases due to natural variability or lack of awareness. Therefore, uncertainty in structures is taking huge attention as it has an important role in the structures analysis and the calibration of designing code factors where it enhances the prediction of the models’ behavior. Despite this significance, existing knowledge concerning uncertainties and their parameters seems to suffer from a lack of knowledge especially in unique structures like non-prismatic reinforced concrete beams which are considered a special case in structural engineering for having variable depth across beam section. The present paper aims to examine some uncertainties facing structural engineers and their role in the non-prismatic reinforced concrete beam behavior considering different variables and parameters. Concrete Damage Plasticity (CDP) constitutive model is used to calibrate the numerical model according to the data obtained experimentally. Then different uncertainty parameters were studied by applying several simulations to find out their effect on models’ behavior. The results seem to provide a better understanding of the uncertainties and their effect on the non-prismatic reinforced concrete beams. Lastly, finite element simulations are applied using ABAQUS.

1. Introduction
The lack of experimental researches studying the behavior of reinforced concrete (RC) non-prismatic beams leads to challenges in both deterministic and probabilistic analyses. Non-prismatic beams are slender structures with cross-section parameters (like shape, dimensions and centroid position) that could vary along the beam length. Researchers are interested in this type of structures as it is possible to optimize structures shapes according to the design conditions. For instance, if the size of the beam cross-section changes proportionally to the internal stress amount, the beam gains the needed structural strength with a minimum material amount. Consequently, non-prismatic structures are broadly used in several engineering areas, such as buildings and bridges design [1].

The thought of nonlinearity behavior of concrete is more difficult than reinforcement steel behavior in finite element models. Several constitutive models exist to represent the nonlinear response of concrete. However, damage mechanics and plasticity, or the mixture of them together is commonly utilized as the principal concepts of modeling concrete material. Still, the strain is divided into elastic and plastic sections describing the unloading and path dependency well, besides the decrease of the unloading stiffness can’t be described in plasticity models. In opposite, damage mechanics models are built on the concept of a progressive loss of elastic stiffness whilst they are usually incapable of describing permanent deformations. The mixture of plasticity and isotropic damage to enhance their abilities leads to the concrete damage plasticity [2].

Recently, Finite Element (FE) analyses participated in providing remarkably detailed descriptions. Where modern constructions need further critical and complicated designs, the demand for accurate methods to estimate uncertainties in computer models, loadings, material and geometry properties has progressed importantly. Generally, in problems where randomness is comparatively small, it is preferred to use a deterministic model more than a stochastic model. Yet, as the effect of uncertainty is essential,
stochastic methods are required for both analysis and design. Regularly, both load and strength are variables, the values of which are distributed about their mean values and the factor of safety usually is too conservative, causing a higher costly design. As opposed to the deterministic approach which is built on safety factors, the stochastic method enhances design reliability where it provides several benefits to engineers [3].

Kappos et al., [4] studied the strength and ductility of RC beams and columns cross-sections considering different geometries and levels of restriction where material characteristics are considered as random variables and their effect on section behavior is evaluated. Mainly, the ductility of the section is in several states decided to be essentially reliant on the ultimate concrete strain. Also, the used Monte Carlo simulations reveal that a notable quantity of variability exists in both the strength and ductility of restricted concrete sections because of the changes in the properties of the material.

Numerical investigations have become an important common approach for analyzing different engineering systems. Simulation is usually described as the means of replacing the actual world based on a combination of theories and imagined models of reality. Monte Carlo simulation was developed to be an experimentally probabilistic approach to determine complex deterministic difficulties as computers simply simulate a huge number of experimental tests which hold random results [5].

Nicolis [6] showed that in nature, the method of determination, on which the detective interacts with a physical system, is bounded with finite accuracy. As a consequence, the ‘state’ of a system needs to be recognized in reality, not like a point in phase space but preferably like a small area that the finite accuracy of the measuring apparatus is reflected by size. So, the applicability of Monte Carlo simulation in the uncertainty evaluation of various phases of a system appears to give a further realistic method. Remarkable benefits of Monte Carlo simulation have been reported by Basil and Jamieson [7] where the method can manage both small and large uncertainties in the input quantities.

Herrador and González [8] introduced the principal boundaries of the Guide to the expression of Uncertainty Measurement (GUM) method for assessing the uncertainty measurement. The benefits of adopting Monte Carlo simulation upon the GUM approach are described and explained and the main propagation of distributions is illustrated.

Papadopoulos and Yeung [9] Confirmed that the Monte Carlo simulation approach is completely agreeable with the conventional uncertainty evaluation techniques for linear systems and systems with small uncertainties. Where Monte Carlo simulation owns the capacity to take into account partially correlated estimation input uncertainties. It likewise checks the uncertainties of the results of some essential manipulations that may or may not be connected.

Kioumarsi et al., [10] investigated the influence of an interference of corrosion pits on nearby reinforcement steel on the probability of bending failure of a rusted RC beam. Where the probability of failure is evaluated by applying Monte Carlo simulation. Moreover, uncertainties in material properties, geometry, loading, and corrosion damage are taken into consideration.

This study aims to investigate the effect of probability failure on the reinforced concrete non-prismatic beams using deterministic and probabilistic finite element analysis. Firstly, validation of the numerical model was made applying the concrete damage plasticity (CDP) constitutive model after getting the required data from the experimental tests. Then, a MATLAB code was developed to achieve the desired aim considering the probability of failure as a bound while the compressive strength ($f'_c$) and the modulus of elasticity ($E_c$) of concrete are considered variables with a 5% standard deviation. The mean values of the variables are therefore specified by ($f'_c = 35$ N/mm$^2$) and ($E_c = 26420$ N/mm$^2$). Besides, an advanced Monte Carlo sampling technique is used to create the samples of the material properties depending on the statistical parameters of the material where the final step includes the calculation of the corresponding load and displacement values.

This paper structure is detailed as the following: Section 1 defines the introduction while Section 2 develops the concrete damage plasticity model. Furthermore, Section 3 introduces the probability analysis; meanwhile, the numerical model validation is illustrated in section 4. Finally, Sections 5 and 6 conclude the discussion of the results and conclusions respectively.
2. Concrete damage plasticity model

To investigate the behaviour of concrete in an inelastic form, the isotropic damaged elasticity, the isotropic tensile and compressive plasticity are exhibited in the concrete damaged plasticity model. The overall strain amount \( \varepsilon \) is consisted of the elastic division \( \varepsilon^{el} \) and the plastic division \( \varepsilon^{pl} \) as simplified below:

\[
\varepsilon = \varepsilon^{el} + \varepsilon^{pl}
\]

\[
\sigma = D^{el} : (\varepsilon - \varepsilon^{pl})
\]

\[
\bar{\sigma} = D_0^{el} : (\varepsilon - \varepsilon^{pl})
\]

\[
D^{el} = (1 - d) D_0^{el}
\]

Hence \( D^{el} \) refers to the elastic degraded stiffness and \( D_0^{el} \) is the initial (undamaged) elastic stiffness of the material, while \( d \) is the variable of the scalar stiffness degradation. Wherever equation (2) can be updated considering the nominal stress with the decreased elastic tensor given in equation (4) providing the resulting equation:

\[
\sigma = (1 - d) D_0^{el} : (\varepsilon - \varepsilon^{pl})
\]

Consequently, the Cauchy stress \( \sigma \) is related to the effective stress \( \bar{\sigma} \) by the scalar reduction relation:

\[
\sigma = (1 - d) \bar{\sigma} \rightarrow \sigma = (1 - d_t) \tilde{\sigma}_t + (1 - d_c) \tilde{\sigma}_c
\]

Hence the variables of the scalar damage, \( d_t \) and \( d_c \), are varying from 0 expressing the undamaged state until 1 expressing a completely damaged state. Generally, the damage model applied on the concrete got into account the failure manner of tensile cracking and compressive crushing. The uniaxial tension and compression response of concrete are deemed to be affected by plasticity damage as revealed in figure 1. The uniaxial tension and compression response of concrete in accordance to the concrete damage plasticity model affected by compression and tension loads are given by:

\[
\sigma_t = (1 - d_t) E_0 (\varepsilon_t - \varepsilon_t^{pl,h})
\]

\[
\sigma_c = (1 - d_c) E_0 (\varepsilon_c - \varepsilon_c^{pl,h})
\]

While \( \varepsilon_t^{pl,h} \) and \( \varepsilon_c^{pl,h} \) are the equivalent plastic strains in tension and compression, respectively. Therefore, the effective uniaxial compression and tension stresses \( \tilde{\sigma}_t \) and \( \tilde{\sigma}_c \) are concluded as follows:

\[
\tilde{\sigma}_t = \frac{\sigma_t}{1 - d_t} = E_0 (\varepsilon_t - \varepsilon_t^{pl,h})
\]

\[
\tilde{\sigma}_c = \frac{\sigma_c}{1 - d_c} = E_0 (\varepsilon_c - \varepsilon_c^{pl,h})
\]

As tensile strain \( \varepsilon_t \) equals to \( \varepsilon_t^{pl,h} + \varepsilon_t^{el} \), and compressive strain \( \varepsilon_c \) equals to \( \varepsilon_c^{pl,h} + \varepsilon_c^{el} \).

\[\text{Figure 1. Concrete response to uniaxial loading condition}\]
2.1. Probability analysis

By presenting the main concepts of probability analysis and by presuming that $X_R$ expresses the nonnegative limit for $X_S$, the failure problem could be determined by $X_R \leq X_S$. Let $X_R$ and $X_S$ be the independent random variables accompanied by the functions of density probability $f_R(X_R)$ and $f_S(X_S)$. Then the failure probability can be calculated from the following:

$$P_f = P[X_R \leq X_S] = \int_{X_R \leq X_S} f_R(X_R) f_S(X_S) dX_R dX_S$$  \hspace{1cm} (11)

An optional description of the earlier problem is given in terms of the so-called boundary state function identified by:

$$g(X_R, X_S) = X_R - X_S$$  \hspace{1cm} (12)

Seeing that $g \leq 0$ designates the domain of failure $D_f$. Consequently, the failure probability $P_f$ is denoted by:

$$P_f = F_g(0)$$  \hspace{1cm} (13)

Additionally, $P_f$ can be estimated as:

$$P_f = \int_{g(X_R, X_S) \leq 0} f(X) dX = \int_{D_f} f(X) dX$$  \hspace{1cm} (14)

Monte Carlo (MC) technique includes creating realizations $x$ of the random vector $X$ of their collective probability density function $f_X(x)$ and verifying if a delivered realization drives to failure or not. Points number in the domain of failure concerning the entire number of formed points is an estimator of failure probability. This concept may be signified by proposing the next indicator function of the failure domain $D_f$:

$$\chi_{D_f}(x) = \begin{cases} 1 & \text{if } x \in D_f \\ 0 & \text{if } x \notin D_f \end{cases}$$  \hspace{1cm} (15)

Therefore, the probability of failure in equation (14) can be rewritten as:

$$P_f = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \chi_{D_f}(x) f_X(x) dx$$  \hspace{1cm} (16)

Then, the function $\chi_{D_f}(X)$ is accordingly considered as a random variable with two points distribution.

$$\mathbb{P}[\chi_{D_f}(X) = 1] = P_f$$  \hspace{1cm} (17)

$$\mathbb{P}[\chi_{D_f}(X) = 0] = 1 - P_f$$  \hspace{1cm} (18)

As $P_f = \mathbb{P}[X \in D_f]$, so the variance and the mean value of $\chi_{D_f}(X)$ are provided by:

$$\mathbb{E} [\chi_{D_f}(X)] = 1 \cdot P_f + 0 \cdot (1 - P_f) = P_f$$  \hspace{1cm} (19)

$$\text{Var} [\chi_{D_f}(X)] = \mathbb{E} [\chi_{D_f}^2(X)] - (\mathbb{E} [\chi_{D_f}(X)])^2 = P_f - P_f^2 = P_f (1 - P_f)$$  \hspace{1cm} (20)

Furthermore, the following mean value estimator is applied to estimate the failure probability in the MC sampling:
\[ \mathbb{E}[X_{D_f}(X)] = \frac{1}{K} \sum_{k=1}^{K} X_{D_f}(X^{(k)}) = \bar{P}_f \] (21)

As \( X^{(k)} \) are considered independent random vectors (where \( k = 1, \ldots, K \)) with functions of probability of density presented by \( f_X(x) \). Moreover, the fraction of volume is described as a random variable and for determination of effortlessness, it supports the Gaussian distribution with the delivered mean \( \mathbb{E} \) and the variance by \( \text{Var} \). Then, the mean and the variance of the estimator can be simply estimated as follows:

\[ \mathbb{E}[\bar{P}_f] = \frac{1}{K} \sum_{k=1}^{K} \mathbb{E}[X_{D_f}(X^{(k)})] = \frac{1}{K} K P_f = P_f \] (22)

\[ \text{Var}[ar{P}_f] = \frac{1}{K^2} \sum_{k=1}^{K} \text{Var}[X_{D_f}(X^{(k)})] = \frac{1}{K^2} K P_f (1 - P_f) = \frac{1}{K} P_f (1 - P_f) \] (23)

2.2. Numerical model validation

In this study, the experimental results considered are presented by Ibrahim and Rad [11]. According to that, reinforced concrete non-prismatic beams were experimentally built with an overall length of 2000 mm, depth at supports (h_s) of 250 mm while the width was about 150 mm, and 9° haunched angle (\( \alpha \)) value which creates a mid-span depth (h_m) of 150 mm, figure 2 displays beams geometry and testing conditions as beams were tested under monotonic loading up to failure. Furthermore, in order to obtain shear failure, beams have been provided with steel reinforcement bars that have different diameters as illustrated in figure 3 while the properties of materials obtained experimentally are presented in table 1.

![Figure 2. Beams geometry and loading condition.](image)

![Figure 3. Reinforcement details.](image)

| Material | Specifications | Yield strength (N/mm²) | Compressive strength, \( f'_c \) (N/mm²) | Ultimate tensile strength (N/mm²) | Flexural strength (N/mm²) | Elastic modulus, \( E_c \) (N/mm²) |
|----------|----------------|------------------------|------------------------------------------|----------------------------------|--------------------------|---------------------------------|
| Steel    | \( \phi = 16 \) mm | 490                    | 558.58                                   | 632                              | 629                      | 210000                          |
| Steel    | \( \phi = 8 \) mm  | 556                    |                                          | 632                              | 629                      | 210000                          |
| Steel    | \( \phi = 4 \) mm  | 548                    |                                          |                                  |                          | 210000                          |
| Concrete |                | 35                     | 3.27                                     | 3.6                              |                          | 26420                           |
Further, Finite element (FE) analysis was conducted to model the nonlinear behaviour of the non-prismatic reinforced concrete beams considering three different hunched angles ($\alpha = 11.7^\circ, \alpha = 9^\circ, 6.18^\circ$) applying damage plasticity model. Additionally, the boundary conditions were chosen to resemble experimental conditions as a pin is set at the first support to generate rotation while a roller is set at the other support to generate rotation and horizontal movement. Figure 4 shows the meshed numerical model while figure 5 illustrates the element types used to build up the model.

![Figure 4. The numerical model.](image)

![Figure 5. The numerical model elements.](image)

3. Results and discussions

Basically, ABAQUS is used to validate the numerical model using the data collected from experimental tests. Then, a MATLAB program is written and connected to ABAQUS to start an analysis considering the probability of failure as bound and concrete properties ($f'_c$ and $E_c$) as variables with a 5% standard deviation. The mean values for theses variables are then specified by ($f'_c = 35$ N/mm$^2$) and ($E_c = 26420$ N/mm$^2$). Monte Carlo simulation method is used to analyse the samples under different material properties to study the effect of the probability of failure on the behaviour of the beams considering a specific number of sample point ($K = 1.5 \times 10^3$).

For illustration, tables 2,3 and 4 show some iterations results for three different hunched angles ($\alpha = 11.7^\circ, \alpha = 9^\circ, 6.18^\circ$) values. Clearly, at a specific failure probability value ($P_f$), displacement value ($U$) is changing according to the geometry of the beams whereas $\alpha$ value decreases, the displacement values are decreasing too demonstrating that beams are acting less elastically for smaller $\alpha$ values. Also, it can be observed that slimmer beams (with larger $\alpha$ values) are exposed to fail under lower loads. Besides, the effect of $P_f$ is obvious on the results as it works as a bound alongside the varied material properties to give the corresponding loads ($F$) and displacements ($U$).

Sensibly, adopting higher $P_f$ values will produce higher loads, thus higher displacements, so a high $P_f$ value will require more load to be achieved. However, having variable material properties with a variance will cause the operation to provide random material properties for each iteration and this is how the uncertainty part is taken into consideration in this study, and as a result, unexpected lower loads are produced for higher $P_f$ values when the randomly chosen concrete properties are small and cause the beam to fail earlier due to its weakness.
| Table 2. $\alpha = 11.7^\circ$ beam analysis results. |
|--------------------------------------------------|
| $P_f$ | $f'_c$ (N/mm$^2$) | $E_c$ (N/mm$^2$) | $U$ (mm) | $F$ (N) |
|-------|------------------|------------------|--------|--------|
| 0.00066 | 34.26720889 | 27122.79498 | 20.83770457 | 23900 |
| 0.00330 | 34.21558567 | 26263.07422 | 22.23618428 | 24400 |
| 0.00530 | 34.15450458 | 27140.87577 | 20.63089823 | 24100 |

| Table 3. $\alpha = 9^\circ$ beam analysis results. |
|--------------------------------------------------|
| $P_f$ | $f'_c$ (N/mm$^2$) | $E_c$ (N/mm$^2$) | $U$ (mm) | $F$ (N) |
|-------|------------------|------------------|--------|--------|
| 0.00066 | 35.43147989 | 26495.14933 | 14.50823663 | 37600 |
| 0.00330 | 32.27323598 | 27915.95509 | 14.93224562 | 37900 |
| 0.00530 | 33.10731313 | 25562.37959 | 15.39884225 | 37800 |

| Table 4. $\alpha = 6.18^\circ$ beam analysis results. |
|--------------------------------------------------|
| $P_f$ | $f'_c$ (N/mm$^2$) | $E_c$ (N/mm$^2$) | $U$ (mm) | $F$ (N) |
|-------|------------------|------------------|--------|--------|
| 0.00066 | 34.22 | 26263 | 13.13 | 45900 |
| 0.00330 | 35.84 | 26663 | 13.73 | 46300 |
| 0.00530 | 34.71 | 26257 | 13.25 | 46100 |

Moreover, the variation of $f'_c$ and $E_c$, illustrated in tables (2-4), proves the performance of 5% standard deviation applied on these two values and the results are varied accordance where in all the tables above the material properties are affecting directly the displacement and load values cooperating with the inserted $P_f$ values.

On the other hand, another comparison is made to clarify the damage pattern of different models under different conditions as shown in table 5. As three different models having different $\alpha$ values were considered, taking into consideration two cases for each, the first case shows probabilistic results and damage patterns of models at $P_f = 0.002$ while the second case shows the deterministic results considering ultimate load ($F_u$) and the corresponding displacement ($U$). Obviously, damaged areas represented by red colour are more intensive in the deterministic cases compared to the probabilistic cases. In this way, the $P_f$ is working as a limit for a safe design controlling the damage in the models, where the intensity of the damaged areas is graded from maximum red to minimum blue as seen in table 5.

| Table 5. Damage pattern of different models under different conditions. |
|--------------------------------------------------|
| $\alpha$ Value | Properties | Damage pattern |
|----------------|------------|----------------|
| $\alpha = 11.7^\circ$ | $F = 24000$ N, $U = 20.85$ mm, $P_f = 0.002$ |
| $\alpha = 9^\circ$ | $F_u = 26210$ N, $U = 23.666$mm |
| $\alpha = 6.18^\circ$ | $F = 38300$ N, $U = 13.67$ mm, $P_f = 0.002$ |
4. Conclusions
In this research, the effect of probability failure on the reinforced concrete non-prismatic beams is investigated using deterministic and probabilistic finite element analysis. Validation of the numerical model was done using the concrete damage plasticity (CDP) constitutive model after collecting the required data from the experimental tests. Then, a MATLAB code was developed to achieve the desired purpose considering the probability failure as a bound while the compressive strength and the modulus of elasticity of concrete are considered variables with a 5% standard deviation. The mean values for the variables are then specified by ($f'_c = 35$ N/mm$^2$) and ($E_c = 26420$ N/mm$^2$). Therefore, based on the above, the following conclusions can be drawn:

- The effect of $P_f$ is obvious on the results where it is working as a bound alongside the varied material properties to give the corresponding loads ($F$) and displacements ($U$).
- At a specific probability of failure value ($P_f$), displacement value ($U$) is changing according to the geometry of the beams whereas $\alpha$ value decreases, the displacement values are decreasing too demonstrating that beams are acting less elastically for smaller $\alpha$ values.
- Slimmer beams (with larger $\alpha$ values) are exposed to failure under lower loads.
- Choosing higher $P_f$ values will produce higher loads, thus higher displacements. However, having variable material properties with a specific variance will cause the operation to provide random material properties for each iteration, and as a result, unexpected lower loads are formed for higher $P_f$ values that the randomly chosen concrete properties are small and cause the beam to fail earlier because of its weakness.
- The 5% standard deviation of $f'_c$ and $E_c$ values causes random varied results where the material properties are affecting directly the displacement and load values cooperating with the inserted $P_f$ values.
- Damaged areas are more intensive in the deterministic cases compared to the probabilistic cases, thus the $P_f$ is working as a limit parameter controlling the damage in the models.

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