AN ASYMMETRIC ZCZ SEQUENCE SET WITH INTER-SUBSET UNCORRELATED PROPERTY AND FLEXIBLE ZCZ LENGTH

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Abstract. In this paper, we propose a novel method for constructing new uncorrelated asymmetric zero correlation zone (UA-ZCZ) sequence sets by interleaving perfect sequences. As a type of ZCZ sequence set, an A-ZCZ sequence set consists of multiple sequence subsets. Different subsets are correlated in conventional A-ZCZ sequence set but uncorrelated in our scheme. In other words, the cross-correlation function (CCF) between two arbitrary sequences which belong to different subsets has quite a large zero cross-correlation zone (ZCCZ). Analytical results demonstrate that the UA-ZCZ sequence set proposed herein is optimal with respect to the upper bound of ZCZ sequence set. Specifically, our scheme enables the flexible selection of ZCZ length, which makes it extremely valuable for designing spreading sequences for quasi-synchronous code-division multiple-access (QS-CDMA) systems.

1. Introduction

Conventional zero correlation zone (C-ZCZ) sequences are often used for eliminating multiple-access interference or multi-path interference in a quasi-synchronous code-division multiple-access (QS-CDMA) system [1], and a number of design methods have been introduced [1, 18, 17, 9, 7, 3, 6]. In particular, the C-ZCZ construction based on interleaved technique has aroused considerable interest in recent year.

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Although the C-ZCZ sequences can be treated as spreading sequences in QS-CDMA systems to eliminate co-channel interference in each cell, the inter-cell interference from adjacent cells cannot be avoided. In order to overcome the above shortcoming, a new class of zero correlation zone (ZCZ) sequence set referred as asymmetric ZCZ (A-ZCZ) sequence set has recently been proposed. An A-ZCZ sequence set consists of multiple C-ZCZ sequence subsets, and the cross-correlation function (CCF) of two sequences which belong to different subsets has a wider ZCZ than that of arbitrary two sequences in the same subset. Therefore, the A-ZCZ sequences can be used for designing spreading sequences in QS-CDMA systems to eliminate both the co-channel interference and inter-cell interference. Particularly, the ZCZ of CCF for any two sequences from different subsets is referred as zero cross-correlation zone (ZCCZ).

In the literature, several construction methods for the A-ZCZ sequences have been proposed. Hayashi et al. improved two types of ternary A-ZCZ sequence sets from given Hadamard matrices and mutually orthogonal complementary sets. Torii et al. presented several types of polyphase A-ZCZ sequence sets from a given perfect sequence and a discrete Fourier transform (DFT) matrix. In our previous works, a type of polyphase A-ZCZ sequence set as well as two class of polyphase A-ZCZ sequence pair set was introduced via interleaving a given perfect sequence. However, all of above schemes have two common shortcomings. First, the ZCZ length can not be flexibly chosen according to the synchronisation deviation of QS-CDMA systems. Second, the ZCCZ length is not up to the maximum, i.e., it’s less than that of sequence in the A-ZCZ sequence set. In this paper, a new method will be proposed.

The contribution of this paper is as follows. We construct new A-ZCZ sequence sets based on interleaved technique and perfect sequence. The derived A-ZCZ sequence sets have the following important properties: the CCFs between any two sequences in the same subset have a ZCZ; the CCFs between two arbitrary sequences in different subsets are zeros at all shifts; the ZCZ length of all subsets can be flexibly specified and thus meets the demands of different synchronisation deviation in QS-CDMA system.

The rest of this paper is organized as follows. Section 2 introduces the notations and the preliminaries. A novel UA-ZCZ sequence set is presented via interleaving perfect sequence in Section 3. Finally, Section 5 concludes the work with some remarks.

2. Notation and preliminary

Let \( x = (x_0, x_1, \cdots, x_{P-1}) \) and \( y = (y_0, y_1, \cdots, y_{P-1}) \) be two complex-valued sequences of period \( P \), then their periodic CCF \( R_{x,y}(\tau) \) can be defined as

\[
R_{x,y}(\tau) = \sum_{j=0}^{P-1} x_j y_{j+\tau}^*, \quad 0 \leq \tau < P,
\]

where \( \tau \) is the time shift, \((\cdot)^*\) is the complex conjugate operation, and \( j + \tau = (j + \tau) \mod P \). Particularly, \( R_{x,y}(\tau) \) is called the periodic autocorrelation function (ACF) of \( x \) if \( x = y \), and can be simply denoted as \( R_x(\tau) \).

Furthermore, arbitrary two different sequences \( a \) and \( b \) are called uncorrelated sequences if their CCF satisfies \( R_{a,b}(\tau) = 0 \) for any \( \tau \). Meanwhile, a sequence \( a \)
with length \( P \) is referred as a perfect sequence if its ACF satisfies \( R_a(\tau) = 0 \) for any \( \tau \) and \( 0 < |\tau| < P \).

**Definition 2.1.** Suppose that \( \mathcal{A} = \{ A^{(0)}, A^{(1)}, ..., A^{(N-1)} \} \) is a set which consists of \( N \) sequence subsets. Here, a subset \( A^{(n)} \) with \( M \) sequences of length \( P \) can be represented as

\[
A^{(n)} = \left\{ a^{(n,0)}, a^{(n,1)}, ..., a^{(n,m)}, ..., a^{(n,M-1)} \right\},
\]

\[
a^{(n,m)} = \left( a_0^{(n,m)}, a_1^{(n,m)}, ..., a_{p}^{(n,m)}, ..., a_{p-1}^{(n,m)} \right).
\]

If any sequence or sequences pair in \( \mathcal{A} \) has the following property,

\[
R_{a^{(n,m)}, a^{(n',m')}}(\tau) = \begin{cases} 
0, & \text{for } m = m' \text{ and } 0 < |\tau| < z, \\
0, & \text{for } m \neq m' \text{ and } 0 \leq |\tau| < z, \\
0, & \text{for } n \neq n' \text{ and } 0 \leq |\tau| < z_A,
\end{cases}
\]

then set \( \mathcal{A} \) is called an asymmetric ZCZ (A-ZCZ) sequence set with parameters \( Z\mathcal{A}(P, [M, N], [z, z_A]) \), where \( z \) and \( z_A \) are the ZCZ and ZCCZ length of \( A^{(n)} \) and \( \mathcal{A} \), respectively.

In fact, each subset \( A^{(n)} \) is a C-ZCZ sequence set with parameters \( Z\mathcal{A}(P, [M, N], [z]) \) in set \( \mathcal{A} \) and \( z \) is the ZCZ length of subset \( A^{(n)} \). If the CCF of the set \( \mathcal{A} \) satisfies \( R_{a^{(n,m)}, a^{(n',m')}}(\tau) = 0 \) for \( n \neq n' \) and any shift \( \tau \), then \( \mathcal{A} \) is defined as an uncorrelated asymmetric ZCZ (UA-ZCZ) sequence set with parameters \( Z\mathcal{A}(P, [M, N], [z, P]) \).

Meanwhile, if a ZCZ sequence set has the correlation property \( R_{a^i, a^j}(\tau) = 0 \) (\( i \neq j \)) for any \( \tau \), then \( \mathcal{A} \) is called an uncorrelated ZCZ (U-ZCZ) sequence set with parameters \( Z\mathcal{A}(P, [M, z], [P]) \).

The theoretical upper bound has been obtained by Tang et al. in [8], in which any \( Z\mathcal{A}(P, M, z) \) sequence set satisfies \( z \leq \left\lfloor \frac{P}{M} \right\rfloor \). If the parameters \( P, M \) and \( z \) of any \( Z\mathcal{A}(P, M, z) \) sequence set satisfy \( z = \left\lfloor \frac{P}{M} \right\rfloor \) or \( z + 1 = \left\lfloor \frac{P}{M} \right\rfloor \), then \( Z\mathcal{A}(P, M, z) \) is called an optimal or quasi-optimal ZCZ sequence set [8],[12].

**Definition 2.2.** Let \( a = (a_0, a_1, \cdots, a_{P-1}) \) be a complex sequence of period \( P \) and \( e = (e_0, e_1, \cdots, e_{T-1}) \) be a sequence of length \( T \) over \( \mathbb{Z}_p = \{0, 1, \cdots, P-1\} \). Then an \( P \) by \( T \) matrix \( U = (U_{i,j})_{P \times T} \) can be obtained by defining its \((j+1)\)-th column as \( L_{\tau_1}^{(1)}(a) \) for \( 0 \leq j < P \), namely,

\[
U = \left( \begin{array}{cccc}
L_{\tau_0}^{(1)}(a) & L_{\tau_1}^{(1)}(a) & \cdots & L_{\tau_{T-1}}^{(1)}(a)
\end{array} \right),
\]

where \( L_{\tau}^{(\cdot)}(\cdot) \) is the left cyclical shift operator, i.e., \( L_{\tau}^{(\cdot)}(a) = (a_{i}, \cdots, a_{P-1}, a_0, \cdots, a_{i-1}) \).

Concatenate the successive rows of matrix \( U \) in Eq. (1), namely,

\[
\begin{align*}
\begin{bmatrix}
I(a) & L_{\tau_0}^{(1)}(a), & \cdots, & L_{\tau_{T-1}}^{(1)}(a)
\end{bmatrix},
\end{align*}
\]

where \( I(\cdot) \) denotes the interleaved operator [9],[2]. Then, \( u \) in Eq. (2) is defined as an interleaved sequence and can be denoted as \( I(a, e) \).

According to [9] and [2], the CCF at shift \( \tau \) (where \( \tau = T \tau_1 + \tau_2, 0 \leq \tau_2 < T \)) between \( u \) and \( v \) can be obtained as follows:

\[
R_{u,v}(\tau) = \sum_{t=0}^{T-\tau_2-1} R_u(f_{t+\tau_2} - e_t + \tau_1) + \sum_{t=T-\tau_2}^{T-1} R_u(f_{t+\tau_2-T} - e_t + \tau_1 + 1),
\]

where \( v = I(a, f) \) is an interleaved sequence with shift sequence \( f = (f_0, f_1, \cdots, f_{T-1}) \).
Finally, let \( h = (h_0, h_1, \ldots, h_{T-1}) \) be a sequence of period \( T \) and \( u = I(a, e) \) be an interleaved sequence, the operation \( h \odot u = I(h_0 L^e(a), h_1 L^e(a), \ldots, h_{T-1} L^e(a)) \) can be defined as follows [9]:

\[
\begin{pmatrix}
    h_0 a_0 + e_0 & h_1 a_0 + e_1 & \ldots & h_{T-1} a_0 + e_{T-1} \\
    h_0 a_1 + e_0 & h_1 a_1 + e_1 & \ldots & h_{T-1} a_1 + e_{T-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    h_0 a_{L-1} + e_0 & h_1 a_{L-1} + e_1 & \ldots & h_{T-1} a_{L-1} + e_{T-1}
\end{pmatrix}
\]

(4)

3. CONSTRUCTION OF A-ZCZ SEQUENCE SET BASED ON INTERLEAVED PERFECT SEQUENCE

In this section, we will present a procedure for the construction of A-ZCZ sequence set with subsets from interleaving perfect sequence.

Construction 1. Design of UA-ZCZ sequence set based on a perfect sequence and an uncorrelated sequence set.

Given a perfect sequence \( a = (a_0, a_1, \ldots, a_{P-1}) \) with period \( P \). Let \( H \) be an uncorrelated sequence set of \( T_2 \) sequences with length \( T_1 \), i.e., \( H = \{h^0, h^1, \ldots, h^{T_2-1}\} \) and \( R_{h^i, h^k}(\tau) = 0 \) for any \( \tau \) if \( i \neq k \). Here \( P \) and \( T_1 \) are relatively prime, i.e., \( \gcd(P, T_1) = 1 \). Then the procedure of constructing UA-ZCZ sequence set with inter-subset uncorrelated and flexible ZCZ length can be summarized as follows.

**Step 1.** Generate an U-ZCZ sequence set \( B = \{b^0, b^1, \ldots, b^{T_2-1}\} \) from a given perfect sequence \( a \) (cf. Construction 2).

**Step 2.** Generate an appropriate shift sequence set \( E \) of size \( M \), written as (cf. Construction 3)

\[
E = \{e_0, e_1, \ldots, e_i, \ldots, e_{M-1}\}, \quad \text{and} \quad e_i = (e_{i,0}, e_{i,1}).
\]

**Step 3.** Perform the interleaved operation on each sequence \( b^t \) to get a sequence subset \( C^{(t)} = \{c_0^{(t)}, c_1^{(t)}, \ldots, c_{2M-1}^{(t)}\} \) as follows:

\[
\begin{align*}
&\begin{cases}
    c_{2m}^{(t)} = I(L_{e_0}^{e_0}(b^t), L_{e_0}^{e_1}(b^t)), \\
    c_{2m+1}^{(t)} = I(L_{e_0}^{e_0}(b^t), -L_{e_0}^{e_1}(b^t)),
\end{cases} \\
&\text{where } 0 \leq t < T_2 \text{ and } 0 \leq m < M.
\end{align*}
\]

**Step 4.** Join all sequence subsets \( C^{(t)}(0 \leq t < T_2) \), then we obtain an union sequence set \( C = \bigcup_{t=0}^{T_2-1} C^{(t)} \).

Construction 2. Construction of U-ZCZ sequence set via interleaving a perfect sequence.

Given a perfect sequence \( a \) with length \( P \) and an orthogonal matrix \( H \) of order \( T_1 \). A construction scheme of (almost) optimal ZCZ sequence set has been presented in [9]. In particular, this ZCZ sequence set is optimal for \( \gcd(P, T_1) = 1 \), which has parameters \( Z(P T_1, T_1, P) \). The above construction algorithm can be summarized as follows.
Step 1. Let $0 \leq k < T_1$ and $e = (e_0, e_1, \cdots, e_{T_1-1})$ be the shift sequence of length $T_1$, where $e_k = sk \pmod{P}$ and $s = T_1^{-1} \pmod{P}$.

Step 2. Generate an interleaved sequence $u = I(a,e)$ according to Eq. (2) via interleaving given perfect sequence $a$ of length $P$.

Step 3. Generate an interleaved sequence set $B$ according to Eq. (4) by performing the operation $\odot$ on $u$,

$$B = \{b^0, b^1, \cdots, b^{T_1-1}\} = \{h^0 \odot u, h^1 \odot u, \cdots, h^{T_1-1} \odot u\},$$

where $h^k$ represents the $k$-th sequence of the set $H$.

In [9], the authors are interested in only ZCZ length, which is the minimum values of the zero auto-correlation zone (ZACZ) length and ZCCZ length, but they do not consider the maximum value of ZCCZ length. According to Section 1, the ZCCZ length has important applications in anti interference of CDMA system. Therefore, we study the ZCCZ properties of the sequences generated by Construction 2 as follows.

In Tang’s construction [9], if the orthogonal matrix $H$ is replaced with an uncorrelated set, which consists of $T_2$ sequences with length $T_1$ and $\gcd(P,T_1) = 1$, then we can obtain an interesting conclusion as follows.

Lemma 3.1. Let $H$ be an uncorrelated sequence set of $T_2$ sequences with length $T_1$. If $\gcd(P,T_1) = 1$, then newly produced sequence set $B = \{b^0, b^1, \cdots, b^{T_2-1}\}$ by Construction 2 is an U-ZCZ sequence set, which has parameters $Z(T_1P,T_2, [P,P_1P])$, namely,

$$R_{b^i, b^j}(\tau) = \begin{cases} 0, & 0 \leq i = j < T_2 \text{ and } 0 < |\tau| < P, \\ 0, & 0 \leq i \neq j < T_2 \text{ and } \forall \tau. \end{cases}$$

Proof. Let $\gcd(P,T_1) = 1$, $\tau = T_1\tau_1 + \tau_2$, $0 \leq \tau_2 < T_1$, $e_k = sk \pmod{P}$ and $s = T_1^{-1} \pmod{P}$. According to Eq. (3), the CCF at shift $\tau$ between $b^i$ and $b^j$ can be obtained as follows,

$$R_{b^i, b^j}(\tau) = R_{h^i, h^j}(\tau_2)R_u((s\tau_2 + \tau_1) \pmod{P}).$$

(i) When $i = j$, we have $R_{b^i, b^i}(\tau) = 0$ for $0 < |\tau| < P$ and $R_{b^i, b^i}(0) \neq 0$ because of the uncorrelated property of set $H$ and perfect autocorrelation property of sequence $a$.

(ii) When $i \neq j$, according to the uncorrelated properties of $H$, $R_{b^i, b^j}(\tau) = 0$ for any shift $\tau$ because $R_{h^i, h^j}(\tau) = 0$ for any shift $\tau$, whenever $0 \leq i \neq j < T_2$. Here $h^i$ and $h^j$ are the $i$-th and the $j$-th sequences of the uncorrelated set $H$, respectively.

In fact, we get a similar conclusion with the literature [7].

Construction 3. Construction of Shift Sequences.

A set of shift sequences $E = \{e_0, e_1, \cdots, e_{M-1}\}$ has been obtained by Zhou et al. [18], where $e^{(i)} = (e_{i,0}, e_{i,1})$, $e_{i,j} \in \{0, 1, \cdots, z-1\}$, and $j = 0, 1$. Let $q$ and $Z$ be two nonnegative integers such that $z = (qZ + r) > Z \geq 2$ with $0 \leq r < Z$. There are two cases:

Case 1. $Z$ is even. Let $M = \lfloor \frac{z-2}{2} \rfloor$. Then $e^{(i)}(0 \leq i < M)$ are obtained by

$$e^{(i)} = (e_{i,0}, e_{i,1}) = \begin{cases} \left(\frac{z}{2}i, z - 1 - \frac{z}{2}(i+1)\right), & Z|z-1, \\ \left(\frac{z}{2}i, z - \frac{z}{2}(i+1)\right), & otherwise. \end{cases}$$
Case 2. Z is odd. Let $M = \left\lfloor \frac{z-1}{2} \right\rfloor$. Then $e^{(i)} (0 \leq i < M)$ are obtained by

i) $Z \mid z$

$$e^{(i)} = (e_{i,0}, e_{i,1}) = \begin{cases} \left( \frac{Z}{2} i, z - \frac{Z}{2} - \frac{Z}{2} i \right), & \text{if } i \text{ is even,} \\ \left( z - \frac{Z}{2} (i + 1), \frac{Z+1}{2} \right), & \text{if } i \text{ is odd.} \end{cases}$$

(8)

ii) $Z \nmid z$

$$e^{(i)} = (e_{i,0}, e_{i,1}) = \begin{cases} \left( \frac{Z}{2} i, z - \frac{Z}{2} - \frac{Z}{2} i \right), & \text{if } i \text{ is even,} \\ \left( z - \frac{Z}{2} (i + 1), \frac{Z+1}{2} \right), & \text{if } i \text{ is odd.} \end{cases}$$

(9)

For the shift sequences defined by Eqs. (7)-(9) for $0 \leq i, j < M$, the following inequalities hold [18]:

$$\begin{align*}
\min_{e^{(i)} \in E, e^{(j)} \neq e^{(i)}} \{e_{i,0} - e_{j,0}, e_{i,1} - e_{j,1}\} & > \frac{Z}{2}, \\
\min_{e^{(i)} \in E} \{e_{i,0} - e_{j,1}, e_{i,1} - e_{j,0} - 1\} & > \frac{(Z - 1)}{2}.
\end{align*}$$

(10)

In fact, in the above construction of shift sequences, the parameters $P$ and $Z$ are referred to as the length of a perfect sequence and the ZCZ length of the C-ZCZ sequence set [18], respectively. Since the perfect sequence can be regarded as a special ZCZ sequence, we can generalize the parameter $P$ and $Z$ to $z$ and $z_A$ of the A-ZCZ sequence set $Z_A(P, [M, N], [z, z_A])$. Then, we have the following Lemma 3.2 to provide a novel construction of A-ZCZ sequence.

**Lemma 3.2.** Let $C^{(t)}$ be any subset in Step 3 of Construction 1, and $E$ be a shift sequence set with size $M$ defined in Construction 3, then $C^{(t)}$ is a ZCZ sequence set with parameters $(2T, P, 2M, Z)$.

**Proof.** Let $d_0 = e_{i,0} - e_{j,0}$, $d_1 = e_{i,1} - e_{j,1}$, $d_2 = e_{i,0} - e_{j,1}$, $d_3 = e_{i,1} - e_{j,0} - 1$, $\tau = 2\tau_1 + \tau_2$ and $0 \leq \tau_2 < 2$. According to Eq. (3) and Eq. (10), the CCF of sequence $c^{(t)}_i$ and $c^{(t)}_j$ for fixed $t$ and $0 \leq i, j < 2M$ can be calculated as follows.

(1) When $\tau_2 = 0$.

**Case 1.** $i = 2m$ and $j = 2m + 1$ or $(i = 2m + 1$ and $j = 2m)$ for $0 \leq m < M$.

$$R_{c^{(t)}_i, c^{(t)}_j} (\tau) = R_{b^{(t)}_i} (\tau_1) - R_{b^{(t)}_j} (\tau_1) = 0.$$  

**Case 2.** $i = 2m$ and $j = 2k$ for $0 \leq m \neq k < M$.

$$R_{c^{(t)}_i, c^{(t)}_j} (\tau) = R_{b^{(t)}_i} (\tau_1 - d_0) + R_{b^{(t)}_j} (\tau_1 - d_1) = 0,$$

for $|\tau_1| < Z/2$.

**Case 3.** $i = 2m + 1$ and $j = 2k + 1$ for $0 \leq m \neq k < M$. This case is similar to Case 2.

(2) When $\tau_2 = 1$.

**Case 1.** $i = 2m$ and $j = 2m + 1$ or $(i = 2m + 1$ and $j = 2m)$ for $0 \leq m < M$.

$$R_{c^{(t)}_i, c^{(t)}_j} (\tau) = R_{b^{(t)}_i} (\tau_1 - d_2) - R_{b^{(t)}_j} (\tau_1 - d_3) = 0,$$

for $|\tau_1| < Z/2$.

**Case 2.** $i = 2m$ and $j = 2k$ for $0 \leq m \neq k < M$.

$$R_{c^{(t)}_i, c^{(t)}_j} (\tau) = R_{b^{(t)}_i} (\tau_1 - d_2) + R_{b^{(t)}_j} (\tau_1 - d_3) = 0,$$
for $|\tau_1| < Z/2$.

**Case 3.** $i = 2m + 1$ and $j = 2k + 1$ for $0 \leq m \neq k < M$. This case is similar to Case 2.

Based on the above analysis, it is clear that $C^{(i)}$ is a ZCZ sequence set with parameters $(2T_1 P, 2M, Z)$.

**Lemma 3.3.** When $0 \leq n_1 \neq n_2 < T_2$, let $C^{(n_1)}$ and $C^{(n_2)}$ be two arbitrary subsets in Step 3 of Construction 1, then the CCF between $c_{m}^{(n_1)} \in C^{(n_1)}$ and $c_{k}^{(n_2)} \in C^{(n_2)}$ has property $R_{c_{m}^{(n_1)} c_{k}^{(n_2)}}(\tau) = 0$ for any shift $\tau$, where $0 \leq m, k < 2M$.

According to Lemma 3.1 and Lemma 3.2, the conclusion of Lemma 3.3 can be obtained immediately.

Thus, we have the following **Theorem 3.4.**

**Theorem 3.4.** Resulting sequence set $U = \bigcup_{n=0}^{N-1} U^{(n)}$ from Construction 1 is an UA-ZCZ sequence set with parameters $Z_A(2T_1 P, [2M, T_2], [Z, 2T_1 P])$. Furthermore, when $T = T_1 = T_2$, $U$ is optimal if one of the following two conditions holds:

1. $Z$ is even, $1 < r < \frac{Z}{2T}$ and $4T < Z$;
2. $Z$ is odd, $0 < r < \frac{Z}{2T}$ and $2T < Z$.

**Proof.** According to Lemma 3.1, Lemma 3.2 and Lemma 3.3, $U = \bigcup_{n=0}^{N-1} U^{(n)}$ from Construction 1 is an UA-ZCZ sequence set with parameters $Z_A(2T_1 P, [2M, T_2], [Z, 2T_1 P])$. We denote the optimal size from the theoretical bound and the family actual size as $M_o$ and $M_a = 2MT$, respectively. Then

$$M_o = \left\lfloor \frac{2TP}{Z} \right\rfloor = \left\lfloor \frac{2TqZ + 2Tr}{Z} \right\rfloor = 2Tq + \left\lfloor \frac{2Tr}{Z} \right\rfloor.$$

Note that

$$M = \left\lfloor \frac{P - 2}{Z} \right\rfloor = \begin{cases} q, & 1 < r < Z; \\ q - 1, & r = 0, 1. \end{cases}$$

if $Z$ is even, and otherwise

$$M = \left\lfloor \frac{P - 1}{Z} \right\rfloor = \begin{cases} q, & 0 < r < Z; \\ q - 1, & r = 0. \end{cases}$$

**Case 1.** $Z$ is even. Combining (11) with (12), we obtain

$$M_o - 2MT = \begin{cases} 0, & 1 < r < \frac{Z}{2T}; \\ 1, & \frac{Z}{2T} \leq r < \frac{Z}{T}; \\ \vdots \end{cases}$$

**Case 2.** $Z$ is odd. Similarly, from (11) with (12), we have

$$M_o - 2MT = \begin{cases} 0, & 0 < r < \frac{Z}{2T}; \\ 1, & \frac{Z}{2T} \leq r < \frac{Z}{T}; \\ \vdots \end{cases}$$

From the above discussions, it is easy to see that the new ZCZ sequence set is optimal if and only if ($Z$ is even, $1 < r < \frac{Z}{2T}$ and $4T < Z$) or ($Z$ is odd, $0 < r < \frac{Z}{2T}$ and $2T < Z$) with respect to the theoretical bound.
Example 1. Let \( a = ( + + 0 + 0 - ) \) be a ternary perfect sequence of length 7, where the symbols “+”, “-” and “0” represent “+1”, “-1” and “0”, respectively. Let \( H = \{ h^0, h^1, h^2, h^3 \} \) be a quaternary uncorrelated sequence set, where \( h^0 = (1, 1, 1, 1), h^1 = (1, -i, -1, i), h^2 = (1, i, -1, -i), h^3 = (1, -1, 1, -1) \) and \( i = \sqrt{-1} \).

Firstly, let \( P = 7, T = 4 \). We can obtained the shift sequence \( e = (0, 2, 4, 6) \). Then an U-ZCZ sequence set \( B = \{ b^0, b^1, b^2, b^3 \} \) with the parameters \( Z(28, 4, [7, 28]) \) can be obtained by construction 2 as follows.

\[
\begin{align*}
b^0 & = (1, 0, 0, -1, 1, 1, 0, 1, 0, 0, -1, 1, 1, 0, 1, 0, 0, -1, 1, 1, 0, 0, -1, 1, 1, 0) \\
b^1 & = (1, 0, 0, -i, 1, -i, 0, i, 0, 1, i, 1, 0, -1, 0, 0, i, -1, i, 0, -i, 0, 0, -1, -i, -1, 0) \\
b^2 & = (1, 0, 0, i, 1, i, 0, -i, 0, 0, 1, -i, 1, 0, -1, 0, 0, -i, -1, -i, 0, 0, 0, -1, i, -1, 0) \\
b^3 & = (1, 0, 0, 1, 1, -1, 0, -1, -1, 1, 0, 1, 0, 0, 1, 1, -1, 0, -1, 0, 0, -1, -1, 1, 0).
\end{align*}
\]

The CCF of sequences \( b^j \) and \( b^k \) has the following periodic correlation properties in Eq. (13).

\[
R_{b^j, b^k}(\tau) = \begin{cases} 
16, & 0 \leq l = k < 4 \text{ and } \tau = 0 \text{(mod 7)}, \\
0, & 0 \leq l = k < 4 \text{ and } \tau \neq 0 \text{(mod 7)}, \\
0, & 0 \leq l \neq k < 4 \text{ and } \forall \tau.
\end{cases}
\]

(13)

For example, the ACFs of \( b^0 \) (Fig. 1) can be obtained as follows.

\[R_{b^0}(\tau) = (16, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).\]

The other sequence in this U-ZCZ sequence set \( B \) has also AFC ZCZ length 7. On the other hand, the CCFS between \( b^j \) and \( b^k \) (Fig. 1) can be obtained as follows.

\[|R_{b^j, b^k}(\tau)| = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).\]

In fact, any two different sequences in this U-ZCZ sequence set \( B \) is uncorrelated.

\[\text{Figure 1. Periodic correlation properties of U-ZCZ sequence set } B.\]

Secondly, let \( 7 = z = qZ + r = 2 \times 3 + 1 \), here \( q = 2, Z = 3 \) and \( r = 2 \). Then we can construction a shift sequence set \( E = \{ e_0, e_1 \} \) according to construction 3, where \( e_0 = (0, 6) \) and \( e_1 = (4, 2) \).
Finally, an UA-ZCZ sequence set $C = \{C^{(0)}, C^{(1)}, C^{(2)}, C^{(3)}\}$ can be obtained by construction 1 and as shown in Table 1.

The CCF of sequences $c_{k}^{(t)}$ and $c_{l}^{(t')}$ has the following periodic correlation properties in Eq. (14):

$$R_{c_{k}^{(t)}, c_{l}^{(t')}}(\tau) = \begin{cases} 0, & \text{for } 0 \leq t = t' < 4, 0 \leq k = l < 4 \text{ and } 0 < |\tau| < 3, \\
0, & \text{for } 0 \leq t = t' < 4, 0 \leq k \neq l < 4 \text{ and } 0 \leq |\tau| < 3, \\
0, & \text{for } 0 \leq t \neq t' < 4 \text{ and } \forall \tau. \end{cases}$$
For example, the ACF of $c_1^{(0)}$ (Fig. 2) can be obtained as follows.

$$|R_{c_1^{(0)}}(\tau)| = (32, 0, 0, 16, \cdots, 16, 0, 0).$$

At the same time, the absolute value of the CCF between $c_1^{(0)}$ and $c_3^{(0)}$ (Fig. 2) can be obtained as follows.

$$|R_{c_1^{(0)}, c_3^{(0)}}(\tau)| = (0, 0, 0, 16, \cdots, 16, 0, 0).$$

Additionally, any two sequences which belong to the different sequence subsets of this UA-ZCZ sequence set $C$ is uncorrelated, i.e.,

$$R_{c_1^{(t_1)}, c_k^{(t_2)}}(\tau) = (0, \cdots, 0).$$

**Figure 2.** Periodic correlation properties of UA-ZCZ sequence set $C$.

4. Discussion on the parameters of A-ZCZ sequences

Table 2 lists the comparison of known constructions of A-ZCZ sequence sets. From Table 2, the different subsets of Theorem 1 are not completely uncorrelated and ZCZ length cannot be chosen flexibly in [12]. Although other two classes of sequence sets in [16] and [12] can not choose the ZCZ length flexibly, but the different subsets are uncorrelated. In particular, our method in this work can not only flexibly select the ZCZ length, but also the different subsets are uncorrelated.

Since the design method makes full use of the uncorrelated property of the uncorrelated sequence set, the other three kinds of sequences in Table 2 have inter-subset uncorrelated property. However, the construction of Theorem 1 in [12] only uses the orthogonal property of the orthogonal sequences so that the obtained sequence sets can not have the inter-subset uncorrelated property. In addition, the set of shift sequences can be flexibly selected in our method according to the ZCZ length of desired sequence set, so the ZCZ length of the sequence set is flexible.
Table 2. Comparison of Different Families of A-ZCZ Sequence Sets

| Constructions | Parameters                                      | Uncorrelated or not | Flexible ZCZ or not |
|---------------|-------------------------------------------------|---------------------|---------------------|
| Theorem*1 in [12] | $Z_A(LP; [L, N], [M - 1, 2M - 1])$ | No                  | No                  |
| Theorem†2 in [12]  | $Z_A(TL; [T, N], [M, TL])$                     | Yes                 | No                  |
| Theorem‡2 in [16]  | $Z_A(TLP; [L, T], [P, TLP])$ or $Z_A(TLP; [L, T], [P - 1, TLP])$ | Yes                 | No                  |
| Theorem♯3.4       | $Z_A(2TP; [2M, T], [Z, 2TP])$                  | Yes                 | Yes                 |

* $L$ is the order of orthogonal matrix $O_L$, $P$ is length of perfect sequence, and $L = KM$, $N = \left\lceil \frac{T}{M} \right\rceil > 1$, $K > 1$, $M > 1$.
† $T$ is the order of DFT matrix $H_T$, $L$ is the order of orthogonal matrix $O_L$, and $L = KM$, $N = \left\lceil \frac{T}{M} \right\rceil > 1$, $K > 1$, $M > 1$.
‡ $T$ is the order of DFT matrix $H_T$, $L$ is the order of orthogonal matrix $O_L$, $P$ is length of perfect sequence, and $\gcd(T, P) = 1$, $\gcd(L, P) = 1$ (or $L|P$ or $P|L$).
♯ $T$ is the order of DFT matrix $H_T$, $P$ is length of perfect sequence, and $Z \leq 2$, $M = \left\lfloor \frac{P-2}{Z} \right\rfloor$ or $M = \left\lfloor \frac{P-1}{Z} \right\rfloor$.

5. Conclusion

In this paper, a novel construction of A-ZCZ sequence sets (i.e., UA-ZCZ sequence set) has been presented based on interleaving any given perfect sequence and an uncorrelated sequence set. In this set, the CCF between two arbitrary sequences from different subsets is zero at any shift. Moreover, the ZCZ length of new UA-ZCZ sequence set can be flexibly chosen according to the synchronisation deviation of QS-CDMA systems. Furthermore, our construction can obtain optimal UA-ZCZ sequence set with respect to the theoretical upper bound. The proposed UA-ZCZ sequence set is shown to be beneficial in terms of reducing or avoiding inter-cell interference, especially when the uncorrelated sequence subsets are assigned to adjacent cells.

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