A generalization of the Lax pair for the pure spinor superstring in $AdS_5 \times S^5$

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**Abstract**

We show that the Lax pair of the pure spinor superstring in $AdS_5 \times S^5$ admits a generalization where the generators of the superconformal algebra are replaced by the generators of some infinite-dimensional Lie superalgebra.
1 Introduction

1.1 Lax operator

The notion of Lax operator is central in the theory of integrable systems. For the pure spinor superstring in $AdS_5 \times S^5$ the Lax pair was constructed in [1]:

$$L_+ = \frac{\partial}{\partial \tau^+} + \left( J^{[mn]}_{0+} - N^{[mn]}_{+} \right) t^{0}_{[mn]} +$$

$$+ \frac{1}{z} J^{\alpha}_{3+} t^{3}_{\alpha} + \frac{1}{z^2} J^{m}_{2+} t^{2}_{m} + \frac{1}{z^3} J^{\dagger}_{1+} t^{1}_{\dot{\alpha}} + \frac{1}{z^4} N^{[mn]}_{+} t^{0}_{[mn]} \quad (1)$$

$$L_- = \frac{\partial}{\partial \tau^-} + \left( J^{[mn]}_{0-} - N^{[mn]}_{-} \right) t^{0}_{[mn]} +$$

$$+ z J^{\dot{\alpha}}_{1-} t^{1}_{\dot{\alpha}} + z^2 J^{m}_{2-} t^{2}_{m} + z^3 J^{\alpha}_{3-} t^{3}_{\alpha} + z^4 N^{[mn]}_{-} t^{0}_{[mn]} \quad (2)$$

where $t^{0}_{[mn]}, t^{1}_{\dot{\alpha}}, t^{2}_{m}, t^{3}_{\alpha}$ are generators of $g = \mathfrak{psl}(4|4)$:

$$g = \mathfrak{psl}(4|4) \quad (3)$$

satisfying the super-commutation relation:\

$$[t^{\dot{a}}_{A}, t^{\dot{b}}_{B}] = f^{C}_{AB} t^{\dot{a}+\dot{b}}_{C} \mod 4 \quad (4)$$

$J$ are currents:

$$J_{k} = -(dgg^{-1})_{k} \quad (5)$$

$z$ is a complex number which is called spectral parameter.

It follows from the equations of motion that $[L_+, L_-] = 0$. This observation can be taken as a starting point of the classical integrability theory.

One can interpret $\frac{1}{z} t^{3}_{\alpha}, \frac{1}{z^2} t^{2}_{m}, \frac{1}{z^3} t^{1}_{\dot{\alpha}}, \ldots$ as generators of the twisted loop superalgebra $Lg$ $\mathfrak{Lg} = L\mathfrak{psl}(4|4)$. This observation allows us to rewrite (1) and

\textsuperscript{1} “Super-commutator” means commutator of even with even or even with odd, and anti-commutator of odd with odd element; the upper index $\dot{a}, \dot{b}, \ldots$ of $t^{\dot{a}}_{A}$ is redundant, it reminds about the $\mathbb{Z}_4$ grading; the notations we use in this paper are the same as in [2] and [3]

\textsuperscript{2} The word “twisted” means that the power of the spectral parameter mod 4 should correlate with the $\mathbb{Z}_4$-grading of the generators
as follows:

\[
L_+ = \frac{\partial}{\partial \tau^+} + \left(J^{\alpha \beta}_{[\alpha \beta]} - N^{[\alpha \beta]}_{[\alpha \beta]}\right) T^{0}_{[\alpha \beta]} + \\
+ J^\alpha_{\beta+} T^{-1}_{\beta+} + J^m_{2m-2} + J^\alpha_{\beta+} T^{-3}_{\beta+} + N^{[\alpha \beta]}_{[\alpha \beta]} T^{-4}_{[\alpha \beta]} \right] T_0^{0}_{[\alpha \beta]} + \\
+ J^\alpha_{\beta+} T^1_{\beta+} + J^m_{2m} T^2_{m} + J^\alpha_{\beta+} T^3_{\beta+} + N^{[\alpha \beta]}_{[\alpha \beta]} T^4_{[\alpha \beta]} \right] T_0^{0}_{[\alpha \beta]} + \\
+ J^\alpha_{\beta+} T^1_{\beta+} + J^m_{2m} T^2_{m} + J^\alpha_{\beta+} T^3_{\beta+} + N^{[\alpha \beta]}_{[\alpha \beta]} T^4_{[\alpha \beta]} \right] T_0^{0}_{[\alpha \beta]} + \\
+ J^\alpha_{\beta+} T^1_{\beta+} + J^m_{2m} T^2_{m} + J^\alpha_{\beta+} T^3_{\beta+} + N^{[\alpha \beta]}_{[\alpha \beta]} T^4_{[\alpha \beta]} \right] T_0^{0}_{[\alpha \beta]} + \\
+ J^\alpha_{\beta+} T^1_{\beta+} + J^m_{2m} T^2_{m} + J^\alpha_{\beta+} T^3_{\beta+} + N^{[\alpha \beta]}_{[\alpha \beta]} T^4_{[\alpha \beta]} \right] T_0^{0}_{[\alpha \beta]} + \\
+ J^\alpha_{\beta+} T^1_{\beta+} + J^m_{2m} T^2_{m} + J^\alpha_{\beta+} T^3_{\beta+} + N^{[\alpha \beta]}_{[\alpha \beta]} T^4_{[\alpha \beta]} \right]
\]

where $T^{-1}_{\alpha}$ replaces $T^3_{\alpha}$ etc.; operators $T^\alpha_{\beta}$ are generators of the twisted loop superalgebra. With these new notations, the spectral parameter is not present in $L_\pm$. Instead of entering explicitly in $L_\pm$, it now parametrizes a representation of the generators $T^\alpha_{\beta}$.

In this paper we will observe that there is a further generalization. One can generalize $L_g$ to the infinite-dimensional superalgebra $L_{\text{tot}}$ which was introduced in [3].

1.2 Super-Yang-Mills algebra

The super-Yang-Mills algebra is the Lie superalgebra formed by the letters $\nabla_\alpha^L$ which satisfy:

\[
\{\nabla_\alpha^L, \nabla_\beta^L\} \Gamma^{\alpha \beta}_{\gamma \delta} = 0
\]

This is the basic constraint on the covariant derivatives in the $N = 4$ $U(N)$ SYM theory:

\[
\nabla_\alpha^L = \frac{\partial}{\partial \theta^\alpha} + \theta^\beta \Gamma^{\alpha \beta}_{\gamma \delta} \frac{\partial}{\partial \theta^\gamma} + A_\alpha
\]

where $A_\alpha = A_\alpha(x, \theta)$ is an $N \times N$ matrix — the super-vector-potential. The constraints (8) imply all the equations of motion [4].

Let us forget (9) and consider (8) as defining an abstract infinite-dimensional Lie superalgebra. This is called “SYM superalgebra”. This algebra is useful by itself, for example it was conjectured in [5] that the deformations of this algebra are in one-to-one correspondence with the deformations of the maximally supersymmetric Yang-Mills theory. However, in our opinion, the physical meaning of this algebra remains to be understood.

Is there a similar algebra for SUGRA? We do not know the answer to this question. However, in [3] we introduced some infinite-dimensional Lie
superalgebra associated to the Type IIB superstring in $AdS_5 \times S^5$. It was useful for the study of massless vertex operators. In this paper we will explain that this algebra is also naturally related to the structure of the Lax operator.

2 Infinite-dimensional Lie superalgebra associated with $AdS_5 \times S^5$

In this section we will discuss the infinite-dimensional Lie superalgebra $\mathcal{L}_{\text{tot}}$ introduced in [3]. We will show that the definition of the commutator is self-consistent.

2.1 SYM superalgebra

The construction of the SYM superalgebra uses the Koszul duality of quadratic algebras. Consider the algebra $A = \mathbb{C}[\lambda_L]$ which is the algebra of polynomial functions of the pure spinor $\lambda_L$:

$$\Gamma^m_{\alpha \beta} \lambda_L^\alpha \lambda_L^\beta = 0 \quad (10)$$

This is a commutative algebra, therefore the Koszul dual $A'$ of $A$ is the universal enveloping of a Lie superalgebra; this Lie superalgebra is formed by the letters $\nabla^L_\alpha$ which satisfy:

$$\{\nabla^L_\alpha, \nabla^L_\beta\} \Gamma^\alpha\beta_{m_1 \cdots m_5} = 0 \quad (11)$$

By definition $A'$ is the factor-algebra of the tensor algebra, i.e. it is formed by linear combinations of expressions of the form:

$$\nabla_{\alpha_1} \otimes \nabla_{\alpha_2} \otimes \cdots \otimes \nabla_{\alpha_p} \quad (12)$$

modulo the subspace formed by the expressions of the form:

$$\cdots \otimes \Gamma^\alpha_{m_1 \cdots m_5} \nabla_\alpha \otimes \nabla_\beta \otimes \cdots \quad (13)$$

2.2 Gluing together two SYM superalgebras

Consider two copies of the Yang-Mills super algebras, $\mathcal{L}_L$ generated by $\nabla^L_\alpha$, and $\mathcal{L}_R$ generated by $\nabla^R_\dot{\alpha}$ (the use of the dotted spinor indices to enumerate
the generators of $\mathcal{L}_R$ is traditional). Those are Lie superalgebras, they are
generated by all the possible super-commutators of $\nabla^L_\alpha$ and $\nabla^R_\alpha$. As in [3], we
will introduce the Lie superalgebra $\mathcal{L}_{\text{tot}}$ which as a linear space is the direct
sum:

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_L + \mathcal{L}_R + g_0$$

(14)

where $g_0$ is a finite-dimensional Lie algebra:

$$g_0 = sp(2) \oplus sp(2)$$

(15)

We will now define the structure of a Lie superalgebra on $\mathcal{L}_{\text{tot}}$ and prove the
consistency of the definition. The basic commutation relations are given by:

$$\{\nabla^L_\alpha, \nabla^R_\beta\} = f_{\alpha\beta} t^0_{[mn]} t^0_{[mn]}$$

(16)

$$[t^0_{[mn]}, \nabla^L_\alpha] = f_{[mn]a} \nabla^L_\beta$$

(17)

$$[t^0_{[mn]}, \nabla^R_\alpha] = f_{[mn]\dot{a}} \nabla^R_\beta$$

(18)

$$\Gamma^\beta_{\alpha\beta} \{\nabla^L_\alpha, \nabla^L_\beta\} = 0$$

(19)

$$\Gamma^\beta_{\dot{a}\beta} \{\nabla^L_\alpha, \nabla^R_\beta\} = 0$$

(20)

The $\mathcal{L}_{\text{tot}}$ can be considered a universal object in the category of pairs $(\phi, L)$
where $L$ is a Lie algebra and $\phi$ a linear map from the linear space formed
by the letters $\nabla^L, \nabla^R, t^0$ to $L$, consistent with (16) — (20). In the rest of
this subsection we give an actual construction of this universal object. As a
linear space, it is (14). It remains to define the commutator of two arbitrary
elements of $\mathcal{L}_{\text{tot}}$ and show that this prescription is compatible with the Jacobi
identity.

### 2.2.1 Definition of the commutator

**Subalgebras $\mathcal{L}_L$ and $\mathcal{L}_R$** First of all, let us remember the structure of
$\mathcal{L}_L$ and $\mathcal{L}_R$. Let $V_L$ denote the linear space generated by 16 basis elements
$\nabla^L_\alpha$, $\alpha \in \{1, \ldots, 16\}$. Similarly, $V_R$ will stand for the 16-dimensional linear
space generated by $\nabla^R_\alpha$. For any linear space $V$, the tensor algebra $T(V)$
is the free associative algebra generated by the elements of $V$. Its elements $T(V)$
are denoted, as usual, $v_1 \otimes \cdots \otimes v_p$, where $v_i \in V$. Notice that $T(V)$ is a
graded algebra, the degree being the rank of a tensor. Consider the following
elements $\mathcal{R}_{LM \cdots m_5} \in T(V_L)$ and $\mathcal{R}_{RM \cdots m_5} \in T(V_R)$:
We denote \( \langle R_L \rangle \) the ideal of \( T(V_L) \) generated by \( R_{Lm1\ldots m5} \), and similarly \( \langle R_R \rangle \subset T(V_R) \). Introduce the quadratic algebras:

\[
A_L = T(V_L)/\langle R_L \rangle \quad (23)
\]

\[
A_R = T(V_R)/\langle R_R \rangle \quad (24)
\]

It is a general fact about quadratic algebras that \( A_L \) is the universal enveloping of the Yang-Mills algebra \( L_L \), and \( A_R \) is the universal enveloping of \( L_R \). The subspace of primitive elements of \( A_L \) is \( L_L \), and the subspace of primitive elements of \( A_R \) is \( L_R \). Primitive elements are those which can be obtained as nested commutators of the letters \( \nabla_L \) (or \( \nabla_R \)), for example this one:

\[
\nabla^L_\alpha \otimes (\nabla^L_\beta \otimes \nabla^L_\gamma + \nabla^L_\gamma \otimes \nabla^L_\beta) - (\nabla^L_\beta \otimes \nabla^L_\gamma + \nabla^L_\gamma \otimes \nabla^L_\beta) \otimes \nabla^L_\alpha \quad (25)
\]

is \( [\nabla^L_\alpha, \{\nabla^L_\beta, \nabla^L_\gamma\}] \). We therefore identified the linear space (14) as the direct sum of three linear spaces:

- the subspace \( L_L \) of primitive elements of \( A_L = T(V_L)/\langle R_L \rangle \)
- the subspace \( L_R \) of primitive elements of \( A_R = T(V_R)/\langle R_R \rangle \)
- and \( g_0 \)

Commutators: \([g_0, g_0], [g_0, L_L] \text{ and } [g_0, L_R] \) The definition of the commutator of two elements in \( g_0 \subset L_{\text{tot}} \):

\[
[t^0_{[kl]}, t^0_{[mn]}] = f^0_{[kl][mn]} t^0_{[pq]} \quad (26)
\]

where \( f^0_{[kl][mn]} \) are structure constants of \( g_0 \subset g \), as in (1).

The definition of the commutators \([u, x_L] \) and \([u, x_R] \) where \( u \in g_0, x_L \in L_L \) and \( x_R \in L_R \):

\[
[t^0_{[mn]}, \nabla^L_{\alpha_1} \otimes \cdots \otimes \nabla^L_{\alpha_p}] = f^0_{[mn],[\alpha_1,\ldots,\alpha_p]} \nabla^L_{\alpha_1} \otimes \cdots \otimes \nabla^L_{\alpha_p} + \cdots + f^0_{[mn],[\alpha_1,\ldots,\alpha_p]} \nabla^L_{\alpha_1} \otimes \cdots \otimes \nabla^L_{\alpha_p} \quad (27)
\]
and similar formula for $[t^0_{[mn]}, \nabla^R_{\alpha_1} \otimes \cdots \otimes \nabla^R_{\alpha_p}]$. Again, $f_{[mn]\alpha}^\beta$ and $f_{[mn]\check{\alpha}}^\check{\beta}$ are the structure constants of $\mathfrak{g}$.

The $g_0$-invariance of the subspaces $\mathcal{R}_L \subset V_L \otimes V_L$ and $\mathcal{R}_R \subset V_R \otimes V_R$ implies that $[u, x_L] \in \langle \mathcal{R}_L \rangle$ when $x_L \in \langle \mathcal{R}_L \rangle$, and that $[u, x_R] \in \langle \mathcal{R}_R \rangle$ when $x_R \in \langle \mathcal{R}_R \rangle$. This shows that our definition of the commutation of elements of $g_0$ with elements of $\mathcal{L}_L$ and $\mathcal{L}_R$ is correctly defined (respects the equivalence relations).

**Commutator** $[\mathcal{L}_L, \mathcal{L}_R]$. Now we have to define the commutator of the type $[x_L, x_R]$ where $x_L \in \mathcal{L}_L$ and $x_R \in \mathcal{L}_R$.

We will use the fact that $x_R$ is a nested commutator of the letters $\nabla^R_\alpha$, therefore it is enough to define the commutator $[x_L, \nabla^R_\alpha]$ and then define the commutator $[x_L, x_R]$ by commuting $x_L$ consecutively with the constituents of $x_R$, i.e.:

$$[x_L, \nabla^R_{\alpha_1} \otimes \cdots \otimes \nabla^R_{\alpha_n}] = \ldots [[x_L, \nabla^R_{\alpha_1}], \nabla^R_{\alpha_2}], \ldots, \nabla^R_{\alpha_n}]$$ \hspace{1cm} (28)

This method of defining the commutator \textit{a priori} leads to an asymmetry between $L$ and $R$. But we will later explain that in fact there is no such asymmetry.

It remains to define $[x_L, \nabla^R_\alpha]$. By definition $x_L \in \mathcal{L}_L$ is a linear combination of expressions of the form:

$$\nabla^L_{\alpha_1} \otimes \cdots \otimes \nabla^L_{\alpha_p}$$ \hspace{1cm} (29)

Moreover, it is a linear sum of nested commutators/anticommutators:

$$x_L = \left[ \nabla^L_{\alpha_1}, \{ \nabla^L_{\alpha_2}, [\nabla^L_{\alpha_3}, \ldots \{ \nabla^L_{\alpha_{p-1}}, \nabla^L_{\alpha_p} \} \ldots] \right]$$ \hspace{1cm} (30)

(assuming that $x_L$ is odd). We define:

$$\left\{ \nabla^R_{\check{\alpha}}, \left[ \nabla^L_{\alpha_1}, \{ \nabla^L_{\alpha_2}, [\nabla^L_{\alpha_3}, \ldots \{ \nabla^L_{\alpha_{p-1}}, \nabla^L_{\alpha_p} \} \ldots] \right] \right\} = \\
\left[ \nabla^L_{\alpha_1}, \{ \nabla^L_{\alpha_2}, [\nabla^L_{\alpha_3}, \ldots \{ \nabla^L_{\alpha_{p-1}}, \nabla^L_{\alpha_p} \} \ldots] \right] \right\} - \\
\left\{ \nabla^L_{\check{\alpha_1}}, \left[ \nabla^L_{\check{\alpha_2}}, [\nabla^L_{\check{\alpha_3}}, \ldots \{ \nabla^L_{\check{\alpha_{p-1}}}, \nabla^L_{\check{\alpha_p}} \} \ldots] \right] \right\} + \ldots$$ \hspace{1cm} (31)

We then do commute $t^0_{[mn]}$ with the remaining $\nabla^L$, and we are left with the sum of nested commutators of $p-1 \ \nabla^L$. We similarly define the commutator of $\nabla^R$ with any sequence of commutators of $\nabla^L$, not necessarily nested. There is a special case when $x_L = \nabla^L_\alpha$ (i.e. $p = 1$); in this case we are left with $f_{\check{\alpha} \alpha}^{[mn]} t^0_{[mn]} \in \mathfrak{g}_0$. 

6
2.2.2 Consistency

Jacobi identity  We have to verify that the commutator which we defined satisfies the Jacobi identity. There are two cases to verify:

1. \[
[[x_L, y_L], \nabla^R_{\dot{\alpha}}] = [x_L, [y_L, \nabla^R_{\dot{\alpha}}]] \pm [[x_L, \nabla^R_{\dot{\alpha}}], y_L]
\]

2. \[
[t_{[mn]}^0, [x_L, \nabla^R_{\dot{\alpha}}]] = [[t_{[mn]}^0, x_L], \nabla^R_{\dot{\alpha}}] + [x_L, [t_{[mn]}^0, \nabla^R_{\dot{\alpha}}]]
\]

Both follow immediately from the definitions.

In particular, our definition does not depend on the choice of presentation of \(x_L\) as a sum of nested commutators of \(\nabla^L\).

Consistency with the quadratic relations  There are two things to prove. First, we have to prove:

\[
[\langle R_{Lm_1...m_5} \rangle , \nabla^R_{\dot{\alpha}}] \subset \langle R_L \rangle \tag{32}
\]

This follows from:

\[
[R_{Lm_1...m_5} , \nabla^R_{\dot{\alpha}}] = 0 \tag{33}
\]

The proof of (33) is formally indistinguishable from the verification of the Jacobi identity for \(psl(4|4)\):

\[
[t^3_{\dot{\alpha}}, \{t^{1}_{\dot{\beta}}, t^{1}_{\dot{\gamma}}\}] = \{[t^3_{\dot{\alpha}}, t^{1}_{\dot{\beta}}], t^{1}_{\dot{\gamma}}\} - \{[t^3_{\dot{\beta}}, t^{1}_{\dot{\alpha}}], t^{1}_{\dot{\gamma}}\} \tag{34}
\]

Second, we have to prove:

\[
[x_L , R_{Rm_1...m_5}] = 0 \tag{35}
\]

Since \(x_L\) is a nested commutator of \(\nabla^L\), the Jacobi identity implies that it is enough to prove (35) for \(x_L = \nabla^L_{\dot{\alpha}}\). For any pure spinor \(\lambda^R_{\dot{\alpha}}\) we should prove that \([\nabla^L_{\dot{\alpha}}, \{\lambda_R, \lambda_R\}] = 0\), where \(\lambda_R = \lambda^R_{\dot{\alpha}} \nabla^R_{\dot{\alpha}}\). Indeed:

\[
[\nabla^L_{\dot{\alpha}}, \{\lambda_R, \lambda_R\}] = 2[\{\nabla^L_{\dot{\alpha}}, \lambda_R\}, \lambda_R] = 2\lambda_{\dot{\alpha}} \lambda_{\dot{\beta}} f_{\alpha \beta} [^{mn}] f_{[mn]} [^{\dot{\gamma}}] \nabla^R_{\dot{\alpha}} \tag{36}
\]

This is zero since by the \(psl(4|4)\) Jacobi identity \(f_{\alpha \beta} [^{mn}] f_{[mn]} [^{\dot{\gamma}}] = -2f_{\beta [^m} f_{\dot{\gamma} n]}\), and \(\lambda_{\dot{\alpha}} \lambda_{\dot{\beta}} f_{\beta [^m} f_{\dot{\gamma} n]} = 0\) because \(\lambda_R\) is a pure spinor.
Commutator of two primitive elements is a primitive element. Consider a commutator \([x_L, x_R]\) where \(x_L \in \mathcal{L}_L\) and \(x_R \in \mathcal{L}_R\). We have defined \(\mathcal{L}_L \subset A_L\) and \(\mathcal{L}_R \subset A_R\) as the subsets of primitive elements. We have to prove that \([x_L, x_R]\) is either an element of \(g_0\) or belongs to either \(A_L\) or \(A_R\) and is primitive. If \(\deg x_L \geq \deg x_R\), then this follows immediately. If \(\deg x_L < \deg x_R\), then the result of the commutator falls into \(A_R\), and we have to show that it is a primitive element of \(A_R\). Let us proceed by induction in \(\deg x_R\). Start with \(\deg x_R = 1\), i.e. \(x_R = \nabla^R_\alpha\). In this case either \(\deg x_L \geq \deg x_R\) or \(x_L \in g_0\). In both cases the statement follows immediately. Now suppose that \(\deg x_R = n > 1\). Then \(x_R\) is a linear combination of elements of the form \([\nabla^R_\alpha, y_R]\) where \(\deg y_R = n - 1\). We have:

\[
[x_R, x_L] = [\nabla^R_\alpha, [y_R, x_L]] - [y_R, [\nabla^R_\alpha, x_L]]
\] (37)

Both terms on the right hand side are primitive by the assumption of the induction.

2.2.3 L↔R symmetry

Our construction of the commutator \([\mathcal{L}_L, \mathcal{L}_R]\) is a priori asymmetric under L↔R. But in fact it is L↔R symmetric. Notice that our construction implies the basic commutation relations (16), (17), (18), (19), (20). At the same time, it can be in fact derived from these relations. These relations are L↔R symmetric, and our algebra \(\mathcal{L}_{\text{tot}}\) is characterized by them, therefore \(\mathcal{L}_{\text{tot}}\) enjoys the L↔R symmetry.

3 Generalization of the Lax pair

The basic relations (11) imply:

\[
\{\nabla^L_\alpha, \nabla^L_\beta\} = f_{\alpha \beta}^m A^L_m
\] (38)

\[
[\nabla^L_\alpha, A^L_m] = f_{\alpha m}^\beta P^\gamma_{\beta \gamma} W^\gamma_L
\] (39)

and similar equations for the commutators of \(\nabla^R_\beta\). Here \(P^\gamma_{\beta \gamma}\) is the constant \(P^\gamma_{\bar{\alpha} \alpha}\) bispinor corresponding to the background RR field strength.
We propose the following generalization of the Lax pair:

\[
L_+ = \left( \frac{\partial}{\partial \tau^+} + J_0^{[mn]} t_{[mn]}^0 \right) + J_{3+}^a \nabla^L_a + J_{2+}^m A_m^L + (J_{1+})_a W_L^a + \\
+ \lambda^\alpha_L w^L_{\beta+} \left( \{ \nabla^L_\alpha, W^\beta_L \} - f^m_{\beta[mn]} t_{[mn]}^0 \right)
\]

\[
L_- = \left( \frac{\partial}{\partial \tau^-} + J_0^{[mn]} t_{[mn]}^0 \right) + J_{1-}^{\dot{\alpha}} \nabla^R_{\dot{\alpha}} + J_{2-}^m A_m^R + (J_{3-})_{\dot{\alpha}} W_R^\alpha + \\
+ \lambda_{\dot{\alpha}} R w^R_{\dot{\beta}-} \left( \{ \nabla^R_{\dot{\alpha}}, W^\dot{\beta}_R \} - f^m_{\dot{\beta}[mn]} t_{[mn]}^0 \right)
\]

Here the currents \( J_{\pm} \) are defined in the same way as in the “standard” Lax pair (1), (2) except for \((J_{1+})_a\) and \((J_{3-})_{\dot{\alpha}}\) which are related to the \( J_{1+}^a \) and \( J_{3-}^{\alpha} \) of (1), (2) by lowering the indices with \( P_{a\dot{\alpha}} \):

\[
(J_{1+})_a = P_{a\dot{\alpha}} J_{1+}^{\dot{\alpha}}, \quad (J_{3-})_{\dot{\alpha}} = P_{\alpha\dot{\alpha}} J_{3-}^\alpha
\]

The purpose of introducing this \( P_{a\dot{\alpha}} \) (which cancels with the \( P_{\alpha\dot{\alpha}} \) of (39)) is to keep notations for generators as in the flat space YM algebra.

In the rest of this section we will verify various properties of the generalized Lax pair.

### 3.1 Gauge invariance under \( sp(2) \oplus sp(2) \)

The construction preserves the \( g_0 \) gauge invariance, in the following sense. For a \( \tau^\pm \)-dependent “gauge parameter” \( \xi_0(\tau^+, \tau^-) \), consider the transformation \( \delta_{\xi_0} \) acting on the currents (5) and ghosts \( w_{\pm}, \lambda_L, \lambda_R \) as follows:

\[
\delta_{\xi_0} g = \xi_0 g \tag{43}
\]

\[
\delta_{\xi_0} \lambda_L = [\xi_0, \lambda_L] \tag{44}
\]

\[
\delta_{\xi_0} w_{\pm} = [\xi_0, w_{\pm}] \tag{45}
\]

It results in the “covariant” transformation of \( L_{\pm} \), i.e.:

\[
\delta_{\xi_0} L_{\pm} = [\xi_0, L_{\pm}] \quad \text{for} \; \xi_0 \in g_0 \tag{46}
\]

### 3.2 Zero curvature equations

The proof of \( [L_+, L_-] = 0 \) is straightforward using the equations of motion. The list of equations of motion can be found e.g. in Section 2.2 of [2]. When
\([L_+, L_-] = 0\) is verified in \([1]\) for (6) and (7), only some (but not all) of the commutation relations of the twisted loop superalgebra are used. The definition of \(L_{\text{tot}}\) is such that those commutation relations which are really used to verify \([L_+, L_-] = 0\) are identical to the commutations relations in the twisted loop algebra. The following observations are useful:

### 3.2.1 Grading of \(L_{\text{tot}}\)

The algebra \(L_{\text{tot}}\) is \(\mathbb{Z}\)-graded, with \(\deg(\nabla^L_a) = 1\) and \(\deg(\nabla^R_a) = -1\). Obviously, all elements of \(L_L \subset L_{\text{tot}}\) are of positive grade, while all elements of \(L_R \subset L_{\text{tot}}\) are of negative grade. Let \(L_{\text{tot}}^{[m,n]}\) denote the subspace of \(L_{\text{tot}}\) consisting of elements \(x\) such that \(m \leq \deg x \leq n\). The twisted loop algebra \(Lg\) also has grading, \(\deg z = -1\), with similar notations \(Lg^{[m,n]}\).

### 3.2.2 Structure of \(L_{\text{tot}}^{[-3,3]}\)

Notice that \(L_{\text{tot}}^{[1,3]}\), as a linear space, coincides with \(g_3 + g_2 + g_1\). Namely, \(\nabla^L_a\) corresponds to \(t_3^a\), then \(\{\nabla^L_a, \nabla^L_b\} = f_{\alpha \beta}^m A^L_m\) and \(A^L_m\) corresponds to \(t_2^m\), then \([\nabla^L_a, A^L_m]\) = \(f_{\alpha \beta}^m P_{\alpha \beta} W^\beta\) and \(P_{\alpha \beta} W^\beta\) corresponds to \(t_1^1\). Consider now \(L_{\text{tot}}^{[-3,3]}\). Observation:

- As a linear space \(L_{\text{tot}}^{[-3,3]}\) coincides with the twisted loop algebra \(Lg\) in degrees from \(-3\) to \(3\)

- The commutators of elements of \(L_{\text{tot}}^{[-3,3]}\) coincide with the commutators of the corresponding elements of the twisted loop algebra, but only as long as they do not lead outside the degree range \([-3, 3]\)

### 3.2.3 Commutator of \(L_{\text{tot}}^{[4,4]}\) with \(L_{\text{tot}}^{[-3,1]}\)

As a linear space \(L_{\text{tot}}^{[4,4]}\) is a direct sum of \(g_0\) and some complementary space \(N_L\):

\[
L_{\text{tot}}^{[4,4]} = g_0 \oplus N_L
\]

As a linear space \(N_L \simeq (C^5 \otimes C^5)\), it is generated by the elements of the form \([A^L_m, A^L_n]\) for \(m \in \{0, \ldots, 4\}\) and \(n \in \{5, \ldots, 9\}\).

- The commutator of the elements of \(N\) with the elements of \(L_{\text{tot}}^{[-3,1]}\) is zero.
• The commutators of $g_0 \subset \mathcal{L}_{\text{tot}}^{[4,4]}$ with the elements of $\mathcal{L}_{\text{tot}}^{[-3,-1]}$ can be described by identifying, as a linear space, $\mathcal{L}_{\text{tot}}^{[-3,-1]}$ with $Lg_{[-3,-1]}$. The subspace $g_0 \subset \mathcal{L}_{\text{tot}}^{[4,4]}$ can be identified as a linear space with $Lg_{[-3,-1]}^{[4,4]}$. With these identifications, the commutator of an element of $Lg_{[-3,-1]}^{[1,3]} \simeq \mathcal{L}_{\text{tot}}^{[1,3]}$.

3.3 Gauge transformation of $w_\pm$

The definition of $L_\pm$ is correct w.r.to the gauge transformations of $w_\pm$:

$$\delta_A w_{\alpha+} = \Gamma^m_{\alpha\beta} \Lambda_{m+} \lambda^\beta$$

This follows from the existence $F_{mn}^L$ such that $\{\nabla^L_\alpha, W_\beta^L\} = F_{mn}^L (\Gamma^m)^\beta_\alpha$ — see Appendix B of [6].

3.4 BRST transformation

We still have the standard BRST transformation rule:

$$Q_{BRST} L_\pm = [L_\pm, (\lambda^L_\alpha \nabla^L_\alpha + \lambda^R_\alpha \nabla^R_\alpha)]$$

3.5 Comment about the coupling of ghosts

Notice that the coupling to $\lambda_L w_+$ and $\lambda_R w_-$ involves the full so(10) Lorentz currents, not only the so(1, 4)⊕so(5) part. Indeed, the term $\lambda^L_\beta w_{\beta+}^L \{\nabla^L_\alpha, W_\beta^L\}$ in (40) can be split into two parts according to (47):

$$\lambda^L_\beta w_{\beta+}^L \{\nabla^L_\alpha, W_\beta^L\} = \lambda^L_\beta w_{\beta+}^L \left( \{\nabla^L_\alpha, W_\beta^L\}_{g_0} + \{\nabla^L_\alpha, W_\beta^L\}_{N_L} \right)$$

Since the commutator of $L_R$ with $N_L$ is zero, the zero curvature equation $[L_+, L_-] = 0$ would hold even if we drop the coupling to $\{\nabla^L_\alpha, W_\beta^L\}_{N_L}$. The requirement of $g_0$-gauge invariance, as well as the invariance under the gauge transformation of $w_\pm$, would be also satisfied. The only thing that would
break is the BRST covariance \((49)\). Indeed, for \(L_+\) to be BRST-covariant, the terms of \(L_+\) (and similarly \(L_-\)) should satisfy the chain of identities:

\[
\left[ \epsilon \lambda_\alpha^\alpha \nabla^\alpha_L , \left( \frac{\partial}{\partial \tau^+} + \left( J_{0+}^{[mn]} - \lambda_\alpha^\alpha w_{\beta+}^L f_{\alpha \beta}^{[mn]} \right) t_{[mm]}^0 \right) \right] = \epsilon Q_L J_{3+}^\alpha \nabla^\alpha_L \\
\left[ \epsilon \lambda_\alpha^\alpha \nabla^\alpha_L , J_{3+}^\beta \nabla^\beta_L \right] = \epsilon Q_L J_{2+}^m A_m^L \\
\left[ \epsilon \lambda_\alpha^\alpha \nabla^\alpha_L , J_{2+}^m A_m^L \right] = \epsilon Q_L (J_{1+})_\alpha W_L^\alpha \\
\left[ \epsilon \lambda_\alpha^\alpha \nabla^\alpha_L , (J_{1+})_\beta W_L^\beta \right] = \lambda_\alpha^\alpha \epsilon Q_L w_{\beta+}^L \times \\
\times \left\{ \nabla^\alpha_L , W_L^\beta \right\} \\
\left[ \epsilon \lambda_\alpha^\gamma \nabla^\gamma_L , \lambda_\alpha^\alpha w_{\beta+}^L \{ \nabla^\alpha_L , W_L^\beta \} \right] = 0
\]

If we drop the coupling to \(\{ \nabla^\alpha_L, W_L^\beta \} \big|_{\mathcal{N}_L}\), or modify its coefficient, then Eq. \(54\) will break.

Generally speaking, BRST invariance should imply the zero curvature conditions, along the lines of [7]. But the presented example shows that the zero curvature condition does not necessarily imply the BRST invariance.

Notice that \(\mathcal{N}_L\) generates an ideal \(I(\mathcal{N}_L)\) of \(\mathcal{L}_{\text{tot}}\) which is a subset of \(\mathcal{L}_L\):

\[ I(\mathcal{N}_L) \subset \mathcal{L}_L \]

and similarly \(\mathcal{N}_R\). It seems plausible that the factorialgebra \(\mathcal{L}_{\text{tot}}/(I(\mathcal{N}_L) + I(\mathcal{N}_R))\) is \(L_g\), but we have not looked into that.

### 4 Open questions

Given the Lax pair \(L_+, L_-\), of the form:

\[ L_\pm = \frac{\partial}{\partial \tau^\pm} + A_\pm \]

one can construct the transfer-matrix:

\[ T(\tau_{\text{fin}}, \tau_{\text{in}}) = P \exp \left( - \int_{\tau_{\text{in}}}^{\tau_{\text{fin}}} (A_+ d\tau^+ + A_- d\tau^-) \right) \]

This is not \(g_0\)-gauge invariant. The variation of \(T(\tau_{\text{fin}}, \tau_{\text{in}})\) under the gauge transformation of Section 3.1 is:

\[ \delta_\xi T(\tau_{\text{fin}}, \tau_{\text{in}}) = \xi(\tau_{\text{fin}}) T(\tau_{\text{fin}}, \tau_{\text{in}}) - T(\tau_{\text{fin}}, \tau_{\text{in}}) \xi(\tau_{\text{in}}) \]
One could say that the transfer matrix is gauge invariant “up to the boundary terms”. These boundary terms could be cancelled by considering the trace of the transfer matrix over the closed contour. Alternatively, one could imagine some networks of Wilson lines corresponding to different representations, with $g_0$-invariant vertices. In any case, there is a potential difficulty. It is not clear for which representations of $L_{\text{tot}}$ the trace of $T$ is well-defined. It may be the case, that the trace is only well-defined in those representations which are representations of the twisted loop algebra. If this is true, then our construction does not give any new conserved charges.

In any case, it would be interesting to understand the representations of $L_{\text{tot}}$. Moreover, even the representations of the Yang-Mills algebra $L_L$ are not well-understood. Physically, what do they correspond to?

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