Towards a Topological Formulation of Fundamental Interactions

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A thought experiment is formulated to unify quantum mechanics and general relativity in a topological manner. An analysis of the interactions in Nature is then presented. The universal ground state of the constructed theory derives from the cyclic properties (S¹ homotopy) of the topological manifold $Q = 2T^3 \oplus 3S^1 \times S^2$ which has 23 intrinsic degrees of freedom, discrete $Z_3$ and $Z_2 \times Z_3$ internal groups, an SU(5) or SO(10) gauge group, and leads to an anomalous $U(1)$ symmetry on a lattice. These properties can in principle reproduce the standard model with a stable proton. The general equation of motion for the unified theory is derived up to the Planck energy and leads to a Higgs field with possible inflation. The thermodynamic properties of $Q$ are discussed and yield a consistent amplitude for the cosmic microwave background fluctuations. The manifold $Q$ possesses internal energy scales which are independent of the field theory defined on it, but which constrain the predicted mass hierarchy of such theories. In particular the electron and its neutrino are identified as particle ground states and their masses are predicted. The mass of the electron agrees very well with observations. A heuristic argument for the occurrence and magnitude of CP violation is given. Future extensions of the presented framework are discussed.

§1. Introduction

One of the outstanding questions in quantum cosmology and particle physics is the unification of gravity with the electro-weak and strong interactions. Much effort has been devoted in the past years to formulate a purely geometrical and topological theory for both types of interactions[1,2]. Probably best known are the theories involving super-gravity and superstrings[1]. These theories have been shown recently to be unified in M-theory, although the precise formulation of the latter is not known yet. Three outstanding problems in these approaches are compactification down to four dimensions, the existence of a unique vacuum state, and most importantly the formulation of a guiding Physical Principle to lead the mathematical construction of the theory. This work aims at extending, in a physical manner, the topological dynamics approach presented in [3] (paperI from here on), which derives from the mathematical consistency of multiply connected space-times when they required to be Lorentz invariant and to support quantum mechanical superposition.

The relevant results of paperI are as follows. The properties of space-time topology are governed by homotopically inequivalent loops in the prime manifolds $S^3$, $S^1 \times S^2$, and $T^3 = S^1 \times S^1 \times S^1$. The three-torus and the handle are nuclear, i.e. they bound a Lorentz four-manifold with $SU(2;C)$ spin structure. This is the only set of primes which assures Lorentz invariance and the superposition principle as formulated by the Feynman path integral for all times, if space-time on the Planck
scale can be described by a (3+1)-dimensional topological manifold. The dynamics of the mathematical theory are determined by the loop creation \( (T^\dagger) \) and loop annihilation \( (T) \) operators, which obey \( [T, T^\dagger] = 1 \). Their actions on a manifold \( M \) are \( T^\dagger M = S^1 \times M \) and \( TM = nM' \), with \( n \) the number of loops in \( M \) and \( M' \) the manifold \( M \) with a loop shrunk to a point. On the Planck scale, space-time has the structure of a lattice of Planck size three-tori \( L(T^3) \) with 4 homotopically inequivalent paths joining where the \( T^3 \) are connected through three-ball surgery. The existence of this four-fold symmetry leads to \( SO(n) \) and \( SU(n) \) gauge groups as well as the numerical factor 1/4 in the expression for black hole entropy (but not the quantum states). The number of degrees of freedom of the prime manifolds, referred to as the prime quanta, under the action of \( O = TT^\dagger + T^\dagger T \), are 1, 3, and 7 for the three-sphere, the handle manifold, and the three-torus, respectively. During the Planck epoch, \( S^1 \times S^2 \) handles (mini black holes) can attach themselves to the \( T^3 \) lattice which leads to additional field interactions between the degrees of freedom of the prime manifolds (to be studied in this work). The \( S^1 \times S^2 \) prime manifolds are referred to as charges on the \( T^3 \) lattice from hereon. Finally, it was found that the cosmological constant \( \Lambda \), viewed as the spontaneous creation of mini black holes from the vacuum, is very small and proportional to the number of (macroscopic) black holes at the current epoch. Since mini black holes evaporate as the universe expands, this provides a natural decrease of the late time cosmological constant, but implies that any macroscopic black hole increases in mass \( M \) proportional to \( \Lambda M^3 \).

As mentioned, these results follow from the required mathematical consistency of a theory based on fundamental prime manifolds. They constitute a generalization of Mach’s principle as interpreted by Einstein in which inertial forces are a consequence of the global geometry as well as topology of the universe. The thought experiment given below will provide a physical basis for these results.

This paper is therefore self-contained, and is organized as follows. Section 2 presents a thought experiment which relates forces (including gravity) to the topology of space-time. Section 3 discusses the general properties of a Quantum-gravitational Grand Unified Theory (QGUT) based on the charged \( T^3 \) lattice and presents the derivation of the fundamental equation of motion for the wave function. The equation of motion constitutes the main result of this work. Section 4 discusses the symmetry groups of \( Q \) and their relation to the standard model. Section 5 contains the conclusions and discussion.

§2. Space-time Topology and the Nature of Forces

The existence of the prime manifold \( T^3 \) in a lattice allows the mathematical implementation of the superposition principle as in the Feynman path integral, while the nuclearity of the three-torus and the handle guarantees Lorentz invariance of the Planck scale topological manifold (paperI). Nevertheless, one needs to formulate an independent Physical Principle which leads to \( T^3 \) and \( S^1 \times S^2 \) so that both superposition and Lorentz invariance are derived concepts. To find this Physical Principle, which must accommodate general relativity, the following thought experiment is performed.
2.1. Thought Experiment

Imagine an observer with a measuring rod of accuracy $\ell$. This same observer is located in a zero gravity environment to witness the motion of an object and to record its position. He can conclude from measurements of its trajectory that a force acts on the object. As he performs similar experiments for smaller and smaller scales he notices that a smooth description of the observational data is limited by the accuracy of his measuring rod. The trajectories of various objects still appear to be continuous but derivatives are very uncertain. Any set of measurements therefore leaves room for the conclusion on his part that something prevents the particle trajectories from becoming identical on scales $\leq \ell$, although they very well could be identical within the error bars. In the former case, the physical obstruction responsible is hidden by his inability to measure accurately. As the observer repeats his experiments for all orientations in three-dimensional space, he finds the same result. Therefore, his finite accuracy is consistent with particle trajectories which are separated from one another by the action of what appear to be loops (enclosed regions) when he projects his data onto a hyperplane. These loops are a measure of the possible differences between the particle histories. The observer also cannot determine whether these loops are linked or not, and again concludes that both situations can occur. Therefore, even though the particle trajectories through space-time may very well be topologically trivial, the observer has no way of confirming this possibility.

Imagine the observer in a satellite orbiting a black hole. To determine the properties of the black hole and to see if his conclusions about particle trajectories differ from the zero gravity case, he first establishes the existence of space-time curvature. The observer then measures the motions of probes in closer and closer orbits and establishes the existence of an event horizon. When he performs measurements within $\ell$ of the Schwarzschild radius, he again concludes that there is an uncertainty as to the probe's actual trajectory. Even though the observer is disturbed by the fact that he cannot rule out that some probes moved into and out of the event horizon, he assumes that higher accuracy measurements will. The observer can conclude that there is a closed surface and only one generic loop associated with single particle trajectories due to the focusing effect of space-time curvature close to the horizon.

Now postulate that the finite accuracy of the observer’s measuring rod is a property of Nature itself when $\ell = \ell_{\text{Planck}}$, i.e. Nature reaches these same conclusions everywhere. The measurements then reflect an intrinsic property rather than an observational external one, and the possible existence of topological obstructions in the form of loops becomes a requirement to explain the possibility of different particle trajectories. Also use a mathematical framework in which Planck scale space-time possesses the property of continuity, and is locally flat when gravitational effects can be ignored. One then arrives at a description where quantum mechanics and general relativity require the presence of the flat three-torus, which introduces a length scale $\ell_{\text{Planck}}$, and the curved handle, which gives in addition a mass $m_{\text{Planck}}$ (black holes can come in all masses and sizes). The non-trivial loop homotopy of these manifolds now facilitates the possible distinct particle trajectories. It follows that the observed loops are dynamical objects which reflect a fundamental uncertainty in Nature. Note
also that this suggests that the maximum combined spatial and temporal dimension of the universe is four since the possible linkage of loops is a topological invariant in three dimensions only.

One can now provide a physical basis for Mach’s principle. The question is how a particle knows which way to move under the influence of inertial forces, if one rejects the notion of absolute space-time. The thought experiment suggests that it is the combined motions of all particles up to the present. Since these motions are an expression of the curved lattice of three-tori with handles, the geometry of space-time as well as its topology determines how particles move. Because the thought experiment applies to any time-like slice and for all of the projected loops therein, it is the global topology (just like it is the global geometry for Einstein gravity) which enters Mach’s principle.

The prime quantum of $T^3$ under $O$ then reflects the fact that in a lattice, any point (where particles are moving) is connected to six others (where other particles are moving), through three-ball surgery. In the case of $S^1 \times S^2$, the topology of the two-sphere only admits two such paths. All these manifestations of the prime manifolds are physical because it is the combined motion of particles which determines their inertial frames. Because the three-tori only possess a scale, their prime quantum represents an effective dimension of seven. For the massive handles one concludes that each induces (irrespective of position and size) the spontaneous creation and evaporation of two mini black holes every Planck time, and that they constitute the cosmological constant on the left hand side in the Einstein equation.

2.2. Mathematical Formulation

After these arguments, one can arrive at the underlying mathematics through the following line of thought. The equivalence principle in general relativity leads to the notion of gravitational accelerations as an expression of space-time curvature. In quantum theory the concept of a force is equivalent to an interaction between fields and the notion of a curvature two-form generated by some gauge potential. Mathematically, a curvature involves the second derivatives of some metric field. The occurrence of a second derivative of any field requires that field to be defined in three points. These three points need not be infinitesimally close together because the coarseness of a derivative only depends on the measurement scale one is interested in. This leads to a natural partition of space into triplets. That is, the concept of a force field in a mathematical model requires the possibility of any triplet of points to be mapped into a curvature.

This yields a formal object $s$ denoted as $s = \{\cdots\} \equiv [123]$, where the three dots indicate the notion of a second derivative defined by three arbitrary points. The topological structure reflected by $s$ follows from the realization that the paths (partial trajectories) connecting $[1]$, $[2]$ and $[3]$, are distinct. This then implies

$$[12] \neq [23] \neq [13],$$

Note that the different trajectories contributing to the Feynman path integral now reflect the topological freedom in space-time itself.
and
\[ [ab] = [ba], \]
where the last relation reflects that there is no preferred orientation. Clearly, these relations imply the homotopic structure of the three-torus with the loops [12], [23] and [13], if one demands that the three pairs of points cannot be taken infinitesimally close together. The handle and three-sphere follow analogously. When these loops are viewed as dynamical objects, their creation and annihilation operators clearly do not commute, \([T, T^\dagger] = 1\).

§3. The Fundamental Manifold and the Equation of Motion

3.1. QGUT Phenomenological Preliminaries

One should first assess which types of interactions the combination of \(T^3\) and \(S^1 \times S^2\) allows from a purely topological point of view. The three-torus has 3 loops and it is easy to see that individual loops yield spin 1 particles and pairs of loops lead to spin 1/2, through the introduction of an internal angle \(\gamma\) which runs between 0 and \(2\pi\) for each loop[1,4]. Clearly, spin 1/2 as a fundamental two-valuedness occurs naturally, and \(T^3\) supports both submanifolds without preference. The handle introduces an additional degree of freedom which can increase or decrease the homotopic complexity of the structure to which it is attached. Since the handle reflects the gravitational interaction, one can conclude immediately from the thought experiment that locally any \(T^3\) loop must be connected to the mouth of a handle if gravitational and non-gravitational forces are unified and the universe is in its ground state, i.e. black holes maximize entropy and each path through space-time can enter an event horizon. The aim is now to find a three-dimensional manifold consisting of three-tori and handles, which reproduces the standard model through this homotopic implementation of unification.

Since this constructed manifold will be the building block for a lattice, one should also consider symmetries generated by three-ball surgery. As one patches the individual three-tori together, there is an arbitrary \(O(2)\) rotation, or twist, one can perform without changing the topological properties of the manifold. This \(U(1)\) gauge freedom in the lattice junctions allows for a supersymmetry because one can perform arbitrary rotations of \(\gamma\) through the junctions in \(L(T^3)\), thus facilitating a change in fermionic or bosonic spin structure. This symmetry will be referred to as a lattice supersymmetry.

In paper I it is shown that the \(T^3\) lattice junctions support \(SO(n)\) and \(SU(n)\) symmetry groups because \(L(T^3)\) naturally yields quadratic and quartic interaction terms and a self-interaction potential \(V = \mu^2\Phi^2 + \lambda\Phi^4\), for constants \(\mu, \lambda\), and a scalar multiplet \(\Phi\). The maximum degree of four in the interaction potential follows from the fact that there are four homotopically inequivalent paths on \(L(T^3)\). The potential \(V\) is even since the field \(\Phi\) is defined on \(S^1\) loops. With the prime quantum of the fundamental manifold \(T^3\) discussed above, it follows that the dimension of the configuration space defined by \(L(T^3)\) is \(D = 7 + 3 + 1 = 11\). The homotopically trivial

\(^*\) The vacuum expectation value of \(\Phi\) must be Lorentz invariant because of the three-torus.
manifold $S^3$, which is needed in the construction of $L(T^3)$, and the macroscopic handle provide the large scale (compactified) topology of the universe. The junctions which connect the three-tori correspond to the propagators in a field theory and are homotopically equivalent to line elements. It follows that the dimension of the junction groups which describe the interactions between the propagators on $L(T^3)$ is then $11 - 1 = 10$ at any time, i.e. $SU(5)$ and $SO(10)$, which are viable candidates for a GUT. The very presence of the three-torus also hints strongly at an underlying link with M-theory and the T-dualities among superstrings.

3.2. QGUT Construction

3.2.1. The Fundamental Topological Manifold $Q$

A manifold $Q = aT^3 \oplus bS^1 \times S^2$ which is built from three-tori and handles should have an odd number of constituents because only odd sums of nuclear primes bound Lorentz manifolds (paper I). Because the unification of forces is implemented homotopically, the number of loops in the $a$ three-tori should be equal to the number of mouths of the $b$ handles, i.e. $3a = 2b$. Finally, any solution which has $a_1 = na$ and $b_1 = nb$ can be considered a multiple of the smallest solution $aT^3 \oplus bS^1 \times S^2$. Since one wants to construct a lattice $L(Q)$, the minimal solution is the desired one. Therefore, the coefficients $a = 2$ and $b = 3$ result. This requires the direct sum of three handle manifolds and two three-tori. In $Q$, each handle is connected to two loops and the ground state manifold can be written as

$$Q = 2T^3 \oplus 3S^1 \times S^2,$$

which assures nuclearity and therefore Lorentz invariance. When the density of charges is large enough to support $Q$, the universe is said to be $Q$-symmetric.

3.2.2. The Equation of Motion for Constant Charge

In order to develop a field theory which involves matter interactions at the Planck energy, the topological object $Q$ is considered to be fundamental. The equation of motion for the wave function should then follow from some continuum limit of the loop algebra acting on the four-manifold $Q \times R$. The natural limit of the loop creation and annihilation operators is that of two differential operators $\partial$ and $\partial^\dagger$ with space-time dimension four. $\partial$ and $\partial^\dagger$ must commute in the continuum limit because $[T, T^\dagger] = 1$ and the 1 on the right hand side reflects the discrete nature of the loop algebra. These differential operators should also be conjugate in order to yield a scalar operator of the form $\Box = \partial \partial^\dagger = \partial^\dagger \partial = \partial_\mu \partial^\mu, \mu = 1..4$. These considerations require the identification $(T, T^\dagger \to \partial_\mu, \partial^\mu)$.

The interactions present in the equation of motion must follow directly from the topological structure of $Q$ and $L(T^3)$ if one accepts the thought experiment. One demands on the left hand side a single index equation because there are four homotopically inequivalent paths on $L(T^3)$; cubic interactions since every loop is attached to a handle and a junction; scalar quadratic interactions due to individual loops; and linear second derivatives (curvature). The right hand side is zero because $Q$ and $L(T^3)$ are compact. This yields for the complex four vector $q_\lambda$

$$q^\mu [\partial^\nu q_\mu, \partial_\nu q_\lambda] = 0,$$

(5a)
where only the commutator form satisfies the right hand side for solutions of the Klein-Gordon equation. One finds

$$q^\mu [q_\lambda \Box q_\mu - q_\mu \Box q_\lambda] = 0. \quad (5b)$$

The square of the absolute value of the wave function, $q^\mu q_\mu \equiv \delta^{\mu\nu} q_\nu q_\mu$, assures a positive definite inner product and a well defined probability. The energy of $Q$ is given by the dimensional number $m_{\text{Planck}}$, generated by the handle triplet $\ast$. The individual components of $q_\lambda$ yield probabilities for each of the four homotopically distinct paths through $L(T^3)$. In Paper I it is shown that the $SO(3,1;R)$ gauge group follows naturally from a nuclear manifold. The theory is therefore manifestly Lorentz invariant with $\partial^\mu = \eta^\mu_\nu \partial_\nu$ and $\Box$ the d’Alambertian.

3.2.3. The Equation of Motion in the Presence of Charge Fluctuations

For large charge densities, $L(T^3)$ is completely interconnected by handles. In this limit $L(T^3) \to L(Q)$ as the Planck scale manifold. Unlike the three-tori, the mini black holes couple directly to the matter degrees of freedom through the process of Hawking radiation. Therefore, even though the charge density is of the order of unity, handles are continuously being created by the global distribution of handles ($A$) and destroyed through evaporation (paper I).

These quantum perturbations in the local number of handles lead to the generation of an additional field. This field is envisaged to reflect changes (the topology of $Q$ is not altered) in the wave amplitudes $q_\lambda$, flowing through $L(T^3)$. The fundamental object to solve for on $L(Q)$ is therefore $\Omega_\lambda \equiv e^{2\pi i \phi} q_\lambda$, with $\phi$ a function of time and position. This phase transformation leads to the full QGUT equation of motion

$$4\pi i \partial_\nu \phi [(\partial^\nu q_\lambda) q_\mu - (\partial^\mu q_\lambda) q_\nu q_\lambda] = q_\lambda q^\mu \Box q_\mu - q^\mu q_\mu \Box q_\lambda, \quad (6)$$

with an additional scalar constraint

$$q^\mu q_\mu = \text{cst}, \quad (7)$$

which becomes void when $Q$-symmetry is broken. The scalar constraint signifies the fact that, due to the continuous creation and destruction of handle manifolds, it is possible to travel from one point along a homotopic path to any other point along a different homotopic path in $Q$. Therefore, from the perspective of the wave amplitudes, any point in $Q$ becomes indistinguishable from any other, while the topology of the thought experiment persists. Equation (7) thus expresses that the total probability to be somewhere in $Q$ is the same for all points in $Q$.

Because the evolution of $\phi$ is driven by the handles, it follows that on the neutral $T^3$ lattice the field $\phi$ obeys the limiting condition $\partial_\nu \phi = 0$, and is effectively frozen in at a value which does not have to be zero a priori. In the zero charge limit, the numerical value of the field $\phi$ must correspond to a constant, not necessarily zero, Lorentz invariant vacuum expectation value, i.e. $\langle 0 | \bar{\Phi} | 0 \rangle$ under the junction potential $V$ on $L(T^3)$. A non-zero vacuum expectation value of $\Phi$ would require $\mu^2 < 0$ in $V$ and can lead to spontaneous symmetry breaking, as first suggested by

\(^*) The third dimensional number in the theory is the speed of light.
Nambu and co-workers. The additional scalar \( \phi \) therefore induces a Higgs field \( \Phi \), although the number of Higgses is not constrained. In Section 4, the topology of \( Q \) will be used to fix the end value of \( \phi \) as well as the vacuum energy associated with \( V \).

3.2.4. The Thermodynamics of \( Q \)

The number of degrees of freedom of \( Q \) under the action of the loop algebra operator \( O \) is defined as

\[
N_Q = (TT^\dagger + T^\dagger T)(2T^3 \oplus 3S^1 \times S^2) = 23Q.
\] (8)

In the loop homotopic approach adopted here, these degrees of freedom are all distinct and they reflect the different ways in which the loop creation and annihilation operators can act on \( Q \), i.e. induce the different possible topological obstructions from the thought experiment. As discussed above, the weight of 23 which \( O \) assigns to \( Q \), represents the actual number of Planck scale realizations of the manifold at any given time. Because \( Q \) is the topological expression of a mass, a scale, and a gauge group, the degrees of freedom of \( Q \) are identified with different particle states.

For the neutral submanifold \( P = T^3 \oplus T^3 \), one has \( N_P = 14 \). The latent heat associated with the evaporation of the handle triplet \( \Theta = 3S^1 \times S^2 \) is therefore

\[
H = (N_Q - N_P)m_{\text{Planck}}/N_Q = 9m_{\text{Planck}}/23.
\] (9)

This number is uniquely determined by the homotopic structure of space-time and the Planck mass. Since the two three-tori in the structure \( Q \) are identical objects, the specific heat per three-torus is given by

\[
h = H/2 = 9m_{\text{Planck}}/46.
\] (10)

3.2.5. The QGUT Phase and Large Scale Structure

During the \( Q \)-symmetry phase the 23 degrees of freedom of \( Q \) are accessible to the wave amplitudes \( \Omega_\lambda \) traveling through \( L(Q) \). As these currents self-interact, they do so cubically as in equation (6). The amplitudes carry the mass-energy of the universe and their interactions determine the perturbations in that mass-energy, which is distributed over the 23 realizations of \( Q \). The equation of motion (6) is invariant under a global scale transformation \( q_\lambda \rightarrow Aq_\lambda \), and the topology of \( Q \) then fixes \( A = N_Q^{-1} \).

The magnitude scale of the perturbations in the mass-energy is thus given by

\[
\delta \rho/\rho(Q) = N_Q^{-3} = 8.2 \times 10^{-5},
\] (11a)

where the degrees of freedom are considered equally accessible thermodynamically. For adiabatic perturbations, the fluctuations in the Cosmic Microwave Background (CMB) temperature are 1/3 of \( \delta \rho/\rho(Q) \) and one finds a 1 \( \sigma \) Gaussian (by assumption) CMB amplitude

\[
\delta T/T = \frac{1}{3} \delta \rho/\rho(E) = 2.7 \times 10^{-5}.
\] (12)

This value is consistent with recent COBE and ground based measurements, although it does not include any physics after QGUT symmetry-breaking. Analogously, one finds that the characteristic dispersion of \( T^3 \) is \( 7^{-3} \).
§4. The Symmetries of $Q$ and $L(T^3)$ in the Standard Model

With the fundamental manifold and the equation of motion for the wave function which lives on it in place, one now needs to isolate the symmetry properties of $Q$ and $L(T^3)$ in order to establish a link with the standard model, and to provide an interpretation for the equation of motion in terms of matter properties on $L(T^3)$.

4.1. General Properties of $Q$

4.1.1. Discrete Groups Generated by $P$ and $\Theta$

The handle manifolds which are created as quantum fluctuations form triplets on $L(Q)$. Therefore, the effective action $s^3$, of the triplet as a whole, obeys

$$s^3 = 1. \quad (13)$$

That is, a round trip along the manifold $\Theta$ necessarily picks up three phases, which should add up to $2\pi$ since the loop algebra satisfies $[T, T^\dagger] = 1$. Because all 3 handles are identical, this implies a $Z_3$ invariance for the individual quantum fields in the theory defined on $Q$, with angles $\theta_i = \{0, \pm 2\pi/3\}$. From the same arguments it follows that the submanifold $P = T^3 \oplus T^3$ in $Q$ generates a $Z_2 \times Z_3$ symmetry because one cannot distinguish either three-torus in $P$.

4.1.2. $T^3$ Junctions and $U(1)$ Symmetries

The important distinction between $T^3$ and $L(T^3)$, or $L(Q)$ for that matter, is the presence of junctions which connect the individual three-tori through three-ball surgery and create a lattice. In paperI it was suggested that the presence of a lattice facilitates a geometric description of gravitational effects because the three-dimensional junctions can bend according to some curvature tensor. Indeed, on scales much larger than $\ell_{\text{Planck}}$ the neutral lattice $L(T^3)$ appears as a smooth manifold. On the Planck scale on the other hand, the existence of junctions between the three-tori generates an additional $U(1)$ symmetry as discussed above. Obviously, there is only one twist per $Q$ manifold, but each individual three-torus in the lattice formally has six of them. Such additional $U(1)$ factors have been proposed as a possible resolution of the doublet-triplet splitting problem[5], and will be studied below.

4.1.3. Particle Sectors on $Q$

The homotopic properties of $Q$, the $SU(5)$ or $SO(10)$ gauge group on the junction, and the $Z_3$ and $Z_2 \times Z_3$ cyclic symmetries of $\Theta$ and $P$ should lead to specific particle sectors. In this, the photon and graviton are not viewed as being generated through the homotopic structure of $Q$, but result from the junction degrees of freedom, i.e. the $U(1)$ twist and $GL(4)$ curvature of the lattice. Furthermore, there is a Higgs field $\Phi$ which lives on the lattice and can cause symmetry breaking, leading to Higgs bosons.

The number of degrees of freedom $N_Q$ is the eigenvalue of the operator $O = TT^\dagger + T^\dagger T = A + B$ acting on $Q$. There is then a natural division of the 23 degrees of freedom under $AQ = 14Q$ and $BQ = 9Q$. Furthermore, the decomposition $OQ = O(P \oplus \Theta)$ has the same distribution of degrees of freedom under $A$ and $B$ and
leads to the further divisions

\[ AQ = OP = (8 + 6)P, \]  
with eight plus six particles and

\[ BQ = O\Theta = (3 + 6)\Theta, \]  
with three plus six particles. The number of loops plus the number of pairs of loops tells us that there are \((6+6)\) elementary particles. The number of field particles then follows naturally from the decomposition. The junction potential on \(L(T^3)\) or \(L(Q)\) supports the symmetry group \(SU(5)\) or \(SO(10)\). The \(P\) and \(\Theta\) sectors decompose \(Q\) and are therefore associated with subgroups. These subgroups can only contain \(SU(n < 5)\) and \(U(1)\) because of the junction potential \(V\) and twist. For \(SU(5) \sim SU(3) \times [SU(2) \times U(1)]\) these constraints are satisfied, because \(SU(3)\) contains 8 \((P)\) field particles and \(SU(2)\) only 3 \((\Theta)\). Below it will be shown that the field particles must be bosonic. Therefore, \(Q\) defines a ground state which corresponds to the standard model with only one \(U(1)\) factor.

4.1.4. Supersymmetry Breaking and the Pauli Exclusion Principle

A priori, both fermionic and bosonic sectors exist for the equivalence classes identified above. That is, because the form of \(Q\) is motivated by Lorentz invariance and unification, the identified equivalence classes can be both fermionic and bosonic in nature. When \(Q\)-symmetry is broken, the lattice supersymmetric structure \(L(T^3)\) becomes the fundamental (Lorentz invariant) Planck scale object. Subsequently, interactions are mediated by field particles which travel along the 6 junctions surrounding any \(T^3\). That is, it is the discrete three-torus with its seven degrees of freedom under the operator \(O\) which supports a field and its quanta. Any field dynamics on \(L(T^3)\), after lattice supersymmetry has been broken, therefore requires the interaction of two (identical) field particles on a three-torus. If these field particles are manifestly fermionic, this violates the Pauli exclusion principle. Thus, only bosonic field particles can carry the strong and electro-weak force, and satisfy the Pauli exclusion principle on \(L(T^3)\), after supersymmetry has been broken.

The origin of the Pauli exclusion principle actually follows from the homotopic structure of \(T^3\) and the fact that a spin \(1/2\) particle requires two loops on a three-torus for its support. For two identical fermions one finds the general relation (see §2.2 above)

\[ [ac][cb] = [ca][ab], \]  
which yields

\[ [cb] = [ab]. \]  

This indicates that because one \(S^1\) loop is a part of both fermions, the other two are collapsed to one. The consequence is that the three-torus becomes indistinguishable from the prime manifold \(S^1 \times R_4\), with \(R_4\) a Riemann surface of genus one. This manifold is not nuclear and breaks Lorentz invariance. Therefore, one finds a topological restriction which precludes interactions mediated by fermionic field particles. Any super-partners of fundamental particles are therefore rendered inert, except for gravitational interactions.
4.2. Interpretation of the Equation of Motion

Equations (5), (6) and (7) describe the quantum-mechanical interactions of matter in full, i.e. including quantum gravity, without any need to know the specific properties of the particles in the ultimate field theory. The boundary conditions for the solutions to these equations follow from the cyclic properties of $L(T^3)$. The topology of $T^3$ requires the solutions $O_\lambda(x, y, z, t)$ on the lattice to be periodic on a scale $L_i = n_i \ell_{\text{Planck}}$ for positive integers $n_i$, $i = 1..3$, and at every time $t$,

$$O_\lambda(x, y, z, t) = O_\lambda(x + L_1, y + L_2, z + L_3, t).$$  

(17)

The $Z_3$ group of the handle triplet then yields a solution which limits to $|\phi| = 1/3$ or 0 when $Q$-symmetry is broken. This phase relation is consistent with the fact that the equation of motion (6) is invariant under the global transformation $\phi \rightarrow \phi + \alpha$. This freedom is thus fixed by the underlying topology of $Q$. The initial conditions at $t = 0$ for the solutions of (6) can then be taken as $q_\lambda(0) = \text{cst}$, derivatives $\partial_t q_\lambda(0) = 1$ in Planck units, and $\phi(0) = 0$.

Note here that $\phi$ is driven by the $\Theta$ manifold, and can lead to inflation through the vacuum energy associated with the interaction potential $V$ if it limits to a non-zero end value. Most importantly, the possible final values of $\phi$ after $Q$-symmetry breaking are determined by the $Z_3$ symmetry of $\Theta$ and lead to different solutions of the equation of motion, thereby naturally including the influence of this vacuum energy. Below it is shown that this vacuum energy (the GUT scale) is uniquely determined by the topology of $Q$, so that the equation of motion knows about it. This provides a crucial link between the mass-energy wave function and the observational characteristics of the low energy field theory.

To follow the evolution of the wave function after the handles have evaporated, one should solve equation (5) with the end solution of (6) as initial conditions. During this phase, the characteristic amplitude of the fluctuations remains as computed above because all the original 23 degrees of freedom of $Q$ (later to become particles) are above the GUT unification scale (see its computation below). Once the GUT is broken at some energy, the Einstein equation describes the later time evolution of the mass-energy distribution (the expectation value of $q^{\mu} q_{\mu}$), as the universe expands. The fact that general relativity does not constrain the topology of space-time thus appears to follow from the fact that it is only valid if one can ignore the topology of space-time. Nevertheless, Einstein gravity is still a part of the QGUT through the presence of the handles.

A question which can be addressed through solutions to (5) and (6) is the nature of the statistics of mass-energy fluctuations during the radiation-dominated era, after the GUT is broken. Furthermore, the solutions of (5) on $L(T^3)$, with the dispersion $7^{-3}$, then provide the possible probability distributions for the rest masses of particles. Finally, the periodicity condition (17) renders $O_\lambda$ identical on each $T^3$ prior to GUT breaking. Therefore, if inflation increases the size of a Planck scale region to no more than the local horizon scale, then periodicities in the matter distribution could be present at the current epoch.

$^*)$ If merging contributes significantly, then a lot of primordial black holes may be formed.
4.3. Predictions for the Standard Model

4.3.1. Unification Energies on $L(Q)$ and $L(T^3)$

A fundamental problem which requires a resolution in GUT is the specific form of the fermion mass hierarchy. The popular approach is to use the anomalous $U(1)$ gauge symmetry as a horizontal symmetry\cite{5}. The motivation is that the $U(1)_A$ symmetry breaking scale is given by the square root of the Fayet-Iliopoulos term $\xi$\cite{5}. The ratio of $\xi$ to the Planck mass is of the order of the fermion mass ratios in neighboring families. The precise value of $M_A = \sqrt{\xi/q}$, with $q$ the anomalous $U(1)$ charge, depends on the value of $\text{tr}Q=\text{tr}Q_{\text{obs}}+\text{tr}Q_{\text{hid}}$, with contributions from the observable and hidden matter singlet\cite{5}. A second mass scale corresponds to $M_{\text{GUT}}$, which marks the energy at which the $SU(5)$ or $SO(10)$ theory is broken down to the $SU(3) \times [SU(2) \times U(1)]$ symmetry of the standard model. Obviously, both energy scales should have a common origin.

When the handles evaporate, the value of $|\phi|$ can be $1/3$ which breaks the $SU(5)$ or $SO(10)$ symmetry through $V$. Since $Q$ is the ground state for a grand unified theory, the GUT energy scale should correspond to $1/3$ ($Z_3$ introduces three branches) times the energy of a degree of freedom per three-torus ($P = 2T^3$) on $Q$. Because the 23 degrees of freedom on $Q$ are equally favorable one expects equipartition of energy. Therefore, one finds $M_{\text{GUT}} = 1/3m_{\text{Planck}}/2N_Q = 8.8 \times 10^{16}$ GeV, with $m_{\text{Planck}} = 1/\sqrt{G} = 1.22 \times 10^{19}$ GeV in units with $\hbar = c = 1$. Above it was shown that the latent heat per $T^3$ associated with breaking of $Q$-symmetry, which yields $L(T^3)$, equals $h = 9/46m_{\text{Planck}}$. Therefore, one finds $M_A = 1/3h = 8.0 \times 10^{17}$ GeV, for the energy above which lattice supersymmetry operates\cite{6}.

Finally, the energy per degree of freedom on $Q$ is $m_{\text{Planck}}/23$ whereas it is $m_{\text{Planck}}/7$ on the three-torus because $OT^3 = 7T^3$. Therefore, when the handles evaporate the energy per degree of freedom on $L(T^3)$ increases by a factor 23/7. It follows that $Q$ is the ground state in an entropy as well as energy sense. From the above one then finds that the transition $L(Q) \rightarrow L(T^3)$ leaves the universe in an excited state associated with the supersymmetric extension of $SU(5)$ or $SO(10)$ through $U(1)_A$, whose decay can leave behind new particle states.

4.3.2. Q Symmetry Groups and Doublet-Triplet Splitting

Another fundamental problem in supersymmetric GUT is the doublet-triplet splitting problem which results from the unavoidable mixing of Higgs doublets $H, \bar{H}$ with their colored triplet partners $T, \bar{T}$. This also leads to an unacceptably rapid proton decay. In \cite{6} it was suggested that there is no need for the heavy triplet if its Yukawa coupling constant is strongly suppressed with respect to the one of the doublet. This mechanism requires an $SO(10)$ invariant operator with tensor indices $i, k$

$$\frac{Y_{\alpha,\beta}}{M_{\text{GUT}}}10_{i}45_{k}16^{\alpha}16^{\beta}$$

(18)

Note that on $L(T^3)$ the GUT and $U(1)_A$ symmetries are broken at an energy per degree of freedom which is a factor of 7 smaller than $M_{\text{GUT}}$ and $M_A$.\footnote{Note that on $L(T^3)$ the GUT and $U(1)_A$ symmetries are broken at an energy per degree of freedom which is a factor of 7 smaller than $M_{\text{GUT}}$ and $M_A$.}
in which $16^\alpha (\alpha = 1..3)$ are three families of matter fermions, $10_i (i = 1..10)$ is the multiplet with $H, \bar{H}$ ($i = 7..10$) and $T, \bar{T}$ ($i = 1..6$). The $45$ is the GUT Higgs in the adjoint presentation of SO(10), $Y_{\alpha,\beta}$ is the coupling constant matrix, and the $\gamma_i$ denote the matrices of the SO(10) Clifford algebra. To realize this effective operator, the 10-plet must transform under the symmetry group $Z_2 \times Z_3$ so that it does not couple to the GUT Higgses and is allowed to interact with $16^\alpha$ only in combination with the 45-plet\[6\].

A possible resolution of the doublet-triplet splitting problem in the standard model thus naturally involves the cyclic group $Z_2 \times Z_3$ associated with $P$. If one now introduces a light gauge singlet superfield $N$, then the triple interactions on $L(T^3)$ determine the associated interaction potential to be of the form

$$W_\mu = \lambda_1 N 10^2 + \lambda_2 N^3. \quad (19)$$

This potential is automatically invariant under $Z_2 \times Z_3$ because the singlet $N$ has an $Z_2$ invariance on $P$. Both $N$ and $10$ do not transform under $Z_3$ due to $\Theta$, and therefore they decouple from the heavy GUT Higgs fields. This provides a natural resolution of the $\mu$ problem in terms of the homotopy of space-time through the symmetry groups anticipated in \[6\], if one accepts the existence of an additional gauge singlet. Because the triplets have no coupling at all, it follows that the proton is essentially stable even if the decoupled triplet is as light as its doublet partner. That is, its decay rate is suppressed by a factor which is no larger than $(M_W/M_{\text{GUT}})^2$, with $M_W$ the mass of the weak scale. Specific models based on SO(10) are constructed in \[6\].

The existence of the supersinglet should be associated with a dynamical symmetry, just like $\phi$ reflects $Q$-symmetry. The only other symmetry related to the $L(Q) \rightarrow L(T^3)$ transition is the anomalous $U(1)$ symmetry, i.e. lattice supersymmetry, which is suppressed in $L(Q)$ but emerges on $L(T^3)$. The additional singlet is therefore associated with $U(1)_A$ and is supersymmetric. Its characteristic energy breaking scale is the topological number $M_A$ computed in \S 4.3.1.

4.3.3. The Absolute Scale of the Mass Ground State

It will now be shown that, just like the energy scales of the fields generated by $Q$ are uniquely determined by the homotopic theory, so is the low mass end of the lepton mass hierarchy in the electro-weak $\Theta$ sector. Conversely, no such limits can be placed on the QCD sector, except through direct relations between different generations in terms of the horizontal $U(1)$ symmetry on $L(T^3)$.

Before $Q$-symmetry is broken, the 23 degrees of freedom of $Q$ are in equilibrium and there are $X_Q = 23!$ different configurations on $Q$. The mass of the particle ground state is thus $m_{\text{Planck}}/X_Q$. The particle should be charged because $Q$ contains a $U(1)$ sector. It follows that

$$m_e^0 = m_{\text{Planck}}/X_Q = 0.47 \quad \text{MeV}, \quad (20)$$

determines the electron mass.

\[1\) Recall that (6) has a fourth order interaction term on the left and a third order interaction term on the right hand side.
For the neutral submanifold $P$ one has $X_P = 14!$, which fixes the neutral ground state on $L(T^3)$. Because the neutrinos have no charge, they cannot be distinguished on $P$, unlike the charged leptons which couple to the $U(1)$ sector on the junction of $Q$. The total number of configurations is now $X_Q X_P$. The mass of the neutral (electron neutrino) ground state thus follows from

$$m^0_{\nu_e} = \frac{m_{\text{Planck}}}{X_Q X_P} = 5.4 \times 10^{-6} \text{ eV.} \quad (21)$$

The photon and graviton are massless because they do not depend on the homotopy of $Q$. The gluons remain massless because they are associated with $P$. The masses of the vector bosons are determined by the breaking of the $SU(2) \times U(1)$ gauge symmetry through the $|\phi| = 1/3$ solutions, which requires direct intervention by $\Phi$.

4.3.4. Corrections to the Mass Ground State

If $s_Q = 23^{-3}$ describes the dispersion of the probability distribution on $Q$ (computed above for the CMB), then one can ask with what accuracy $A$ the properties of $P \subset Q$ can be determined, given that it has a dispersion $s_P = 14^{-3}$. The uncertainty relation then yields $s_Q = A s_P$. This question is relevant to the particle mass in the charged and neutral state since the handles occupy only a part of the total number of degrees of freedom on $Q$. The relative uncertainty in the ground state masses is therefore $A = (14/23)^3 = 0.23$. This uncertainty reflects an upward shift in the mass because the finite accuracy $A$ implies an intrinsic lack of information. From the $Z_3$ symmetry of $\Theta$, one finds that the magnitude of the shift is $A/3$ which yields

$$m_e = (1 + A/3) m^0_e = 0.51 \text{ MeV.} \quad (22)$$

This is in remarkable agreement (error < 0.7%) with the measured value of 0.511 MeV, and lends support to the notion that the 23 degrees of freedom of $Q$ are truly fundamental. Analogously, one finds that

$$m_{\nu_e} = (1 + A/3) m^0_{\nu_e} = 5.8 \times 10^{-6} \text{ eV.} \quad (23)$$

The fact that the electron neutrino has a mass is a unique prediction of the model.

4.3.5. CP Violation

There is one more degree of freedom on $L(T^3)$. The lattice junction of a pair of three-tori can vibrate. This vibration is such that no curvature is induced, like the motion of a piston, motivated by the thought experiment. The energy $F$, corrected for $A/3$, of the excitation is

$$F = (1 + A/3) \frac{m_{\text{Planck}}}{X_Q X_P} \frac{H}{X_Q X_P} = 3.54 \times 10^{-6} \text{ eV.} \quad (24)$$

The denominator reflects the total number of neutral configurations on $Q$. Until the mini black holes evaporate and the latent heat $H$ is released, an energy $m_{\text{Planck}} - H$ is confined to the internal degrees of freedom of the three-tori. Of course, the presence of this state reflects the conceptual difference between the $\Theta$ and $P$ sector on $Q$. This vibration, if excited, is a special one in that it violates T invariance. That is,
the Planck time vibrations result from a homeomorphism, not a diffeomorphism, of space-time.

To excite this internal degree of freedom one requires a system of two neutral particles interacting through a common decay route. The energy difference between the rest masses of their superposition states should be equal to a positive integer multiple $m$ of $F$ (resonance) in order to excite the vibration. It is well known that the difference in weak self-energy determined by the superposition states

$$K_S \leftrightarrow 2\pi \leftrightarrow K_S$$

and

$$K_L \leftrightarrow 3\pi \leftrightarrow K_L,$$

for the decay of the $K^0$ and $\bar{K}^0$ mesons, is extremely small and equal to $f = 3.52 \times 10^{-6}$ eV. Indeed, $f \approx F$ to less than one percent. It is this remarkable coincidence which can allow the CP violating K meson decay processes to occur through $K_L \rightarrow K_S \rightarrow 2\pi$. This conclusion demands the validity of CPT invariance. The possibility of a CP violating process results from a vibration in space-time itself, which therefore violates T invariance. Now if CPT is preserved, then T violation in fact requires the CP violating decay of the K meson. Although it is not clear whether CPT should be preserved under all circumstances (it is for $K$ decay), the point of view is taken here that it always is.

The level of violation (experimentally $0.227\%$ for two-pion decay) is not predicted by this argument, but is related to the excess in $F$ over $f$. If $F$ is increased by 0.63$\%$ to yield perfect agreement with the measured electron rest mass, i.e. our surroundings are characteristic of a $\sim 2\sigma$ perturbation in the wave function, then the relative energy difference between $F$ and $f$ is $E = 1.29\%$. For the intrinsic dispersion $d = 7^{-3}$ of the $T^3$ lattice vibration and an exponential decay rate $\Sigma_n[7^{-m}\exp^{-E/d}]^n$ per total number of realizations of the three-torus and summed over all resonances, one finds $R_{CP} = 0.17\%$ for $m = 1$. This is in reasonable agreement with experiment, given the use of a simple exponent for the barrier penetration.

§5. Conclusions, Discussion, and Future Prospects

A thought experiment has been proposed which leads to the notion of three-tori and handles as true fundamental objects on the Planck scale, embodying the interplay between general relativity and quantum mechanics. Together these prime manifolds form a fundamental topological manifold which is Lorentz invariant and provides a natural mechanism for symmetry breaking. The general equation of motion has been derived for a possible QGUT on this manifold $Q = 2T^3 \oplus 3S^1 \times S^2$, which naturally leads to a Higgs field (driving inflation) and the amplitude of the primordial density fluctuations. The manifold $Q$ contains the necessary symmetry groups to reproduce the standard model as a ground state with a stable proton, and possesses intrinsic energy scales which determine the masses of the lightest leptons. An attractive feature of the constructed theory is the natural $(3+1)$ dimensions, without the need for compactification. The presented model can be falsified immediately
by measurement of the electron neutrino mass, CP violation, and by comparison of solutions to the equation of motion with (cosmological) observations.

The model as it stands is not complete since it does not select a particular supersymmetric extension of the standard model. It does provide a framework within which all physical interactions can be accommodated and can be reduced to an underlying topological structure, which is richer than this first investigation indicates. Additional avenues to explore are the lattice structure of the anomalous $U(1)$ symmetry, and its use as a horizontal symmetry for the mass and mixing hierarchy[5].

In fact, the very occurrence of symmetry breaking on a lattice of three-tori suggests that the junctions must possess distinct senses of parity. A closer comparison with recent results on black hole entropy is also warranted, since the solutions of the equation of motion (5) should describe the black hole quantum states. In the boundary condition \( L = n L_{\text{Planck}} \) on \( L(T^3) \), the Schwarzschild radius in Planck units now equals \( n \), which fixes the black hole temperature and the longest wavelength modes travelling along any of the six junctions of a three-torus. According to the interpretation of (5), the black hole singularity represents a region in space where all particle rest masses are of the order of the Planck mass, and should be described mathematically by a soliton.

The presented results seem to lead to the notion that the combined description of the standard model and gravity, is a “unification without unification”. That is, space-time topology (not fixed by general relativity) leads to the standard model. Conversely, the Planck scale equation of motion in the presence of strong gravitational effects requires no knowledge of particle species and their interactions. Still, the link between them is provided by the presence of both the (flat) \( T^3 \) and the (curved) \( S^1 \times S^2 \) prime manifold. Each represents a distinct branch of the unification. Both hinge on the existence of a fundamental smallest scale. It is in this respect that the topology of \( Q \) and \( L(T^3) \) provides a conceptually different view of fundamental interactions.

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References

1) S. Kaku, *Introduction to Superstrings* (Springer-Verlag, 1990).
2) S.W. Hawking and A. Strominger, *Quantum Cosmology and Baby Universes*, eds. S. Coleman, J.B. Hartle, T. Piran & S. Weinberg, (World Scientific, 1991) p. 245, p. 272.
3) M. Spaans, Nuc. Phys. B 492 (1997) 526.
4) J.A. Wheeler, *Quantum Cosmology*, eds. L.Z. Fang & R. Ruffini, (World Scientific, 1987) p. 27.
5) Z. Berezhiani and Z. Tavartkiladze, Phys. Lett. B 396 (1997) 150.
6) G. Dvali, Phys. Lett. B 372 (1996) 113.