AN EXTREMALITY PROPERTY OF SZEGÖ PROJECTIONS ON HEISENBERG GROUPS

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Abstract. We prove that Heisenberg groups, a.k.a. the boundaries of Siegel domains, minimize the $L^p$ operator norm of the Szegö projection in a large class of weighted CR manifolds of hypersurface type.

1. Introduction

The classical Szegö projection associates to every $L^2$ function on the boundary of a domain in $\mathbb{C}^n$ its orthogonal projection onto the space of boundary values of holomorphic functions, that is, the Hardy space $H^2$. For a variety of pseudoconvex domains, the corresponding Szegö projection is known to be a singular integral operator (see, e.g., [PS77, NRSW89, MS97, CD06]), whose $L^p$ mapping behaviour is therefore of considerable interest from the point of view of harmonic and complex analysis alike.

In this note, we compare the $L^p$ mapping behaviour of the Szegö projection $S$ on a real hypersurface $M \subset \mathbb{C}^{n+1}$ with the $L^p$ mapping behaviour of the Szegö projection $S_n$ on the model strictly pseudoconvex hypersurface of the same dimension, namely the Heisenberg group

$$H^n = \{ z \in \mathbb{C}^{n+1} : \text{Im } z_{n+1} = |\hat{z}|^2 \}, \quad \hat{z} = (z_1, \ldots, z_n).$$

The quantities of interest are the operator norms

$$\|S\|_{p \rightarrow p} := \sup_{f} \frac{\|Sf\|_p}{\|f\|_p}, \quad \|S_n\|_{p \rightarrow p} := \sup_{f} \frac{\|S_nf\|_p}{\|f\|_p} \quad (1 \leq p \leq \infty),$$

where the Szegö projections and the $L^p$ norms $\|\cdot\|_p$ are defined with respect to a fixed background measure, which we always assume to have a smooth positive density with respect to Lebesgue measure. The background measure on the model $\mathbb{H}^n$ is Lebesgue measure in the global coordinates $(\hat{z}, \text{Re } z_{n+1})$.

Our main result shows that the model Szegö projection $S_n$ minimizes the $L^p$ operator norm in a large class of hypersurfaces $M$.

Theorem 1. If $M$ is compact and pseudoconvex, then

$$\|S\|_{p \rightarrow p} \geq \|S_n\|_{p \rightarrow p}.$$ 

The same conclusion holds, more generally, if $M$ is a $(2n+1)$-dimensional CR manifold of hypersurface type satisfying property $C$ at some strongly pseudoconvex point $x_0$.

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The definition of property C is given in Section 2. This property depends both on the CR structure and the choice of background measure, and it is automatically satisfied by every compact pseudoconvex CR submanifold of $\mathbb{C}^N$ (not necessarily a real hypersurface, see Remark 5 below) and by the model itself (see Proposition 8).

The idea of the proof is pretty straightforward: since $M$ is well-approximated by the model $\mathbb{H}^n$ near a strongly pseudoconvex point $x_0$, whenever $f \in L^2(M)$ is sharply localized around $x_0$ its Szegő projection $\tilde{S}f$ can be well-approximated by $S_n\tilde{f}$, where $\tilde{f}$ is a function obtained “transplanting” $f$ on the model.

Yet, there is a difficulty. We know a priori that such an approach cannot work on every, even strongly pseudoconvex, CR manifold. In fact, there are compact strongly pseudoconvex CR structures on the three-dimensional torus $\mathbb{T}^3$ with the property that the only square-integrable CR functions are the constants [Bar88]. For such a pathological CR manifold $M$, the associated Szegő projection is the averaging operator $\tilde{S}f = \frac{1}{\nu(M)} \int_M f \, dv$, whose $L^p$ operator norm is $\|\tilde{S}\|_{p\to p} = 1$ for every $p \in [1, \infty]$. Since $S_n$ is unbounded on $L^1$ (see, e.g., [Ste93] Chapter 12), the conclusion of Theorem 1 cannot hold for $M$. Thus, some additional hypothesis, like our property C, and the ensuing twists appearing in our argument, are indeed necessary.

A few more remarks may be of interest.

(1) We do not know whether a minimizer for the $L^p$ operator norm exists in the restricted class of compact pseudoconvex embeddable CR manifolds of hypersurface type (of a fixed dimension). It is natural to ask whether the standard CR sphere, endowed with a rotation invariant measure, is such a minimizer. Since $\mathbb{H}^n$ and the punctured sphere $S^{2n+1}\setminus\{\ast\}$ are CR isomorphic (see, e.g., [Ste93] Chapter 12) and points have zero capacity in dimension 3 or higher, the Szegő projection on the standard CR sphere w.r.t. a rotation invariant measure is equivalent to the Szegő projection $\tilde{S}_n$ on $\mathbb{H}^n$ w.r.t. a certain (finite) background measure, different from Lebesgue measure.

(2) To the best of our knowledge, the constants $\|S_n\|_{p\to p}$ (except for the “trivial” cases $p = 1, \infty$) are unknown. C. Liu showed [Liu18] that $\|S_n\|_{p\to p} \geq \frac{\Gamma(\frac{n+1}{2})\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{3n+1}{2}\right)}$, where $q$ is the conjugate exponent to $p$, and conjectures that equality holds.

(3) Since $\|S_n\|_{1\to 1} = \infty$, Theorem 1 implies that the Szegő projection is unbounded on $L^1$ for a variety of CR structures and background measures. More flexible arguments are available to investigate the $L^1$ (un)boundedness of projection operators onto spaces of solutions of first-order PDEs, see [Dal22].

(4) On a more speculative note, it is reasonable to expect that if $M$ satisfies property C at some weakly pseudoconvex point $x_0$, then $\|S\|_{p\to p} \geq \|\tilde{S}\|_{p\to p}$ for an appropriate model Szegő projection $\tilde{S}$ depending on the nature of the point $x_0$. E.g., if $M$ is three-dimensional, pseudoconvex, and $x_0$ has type $2m \geq 2$, then the appropriate model should be $\{\text{Im} \varphi(2z, \mathbb{T})\} \subset \mathbb{C}^2$, where $\varphi$ is a subharmonic (nonharmonic) homogeneous polynomial of degree $2m$. If $\varphi$ is the projector on this model, a natural follow-up question is whether $\|S\|_{p\to p} > \|S|_{p\to p}$ whenever $\varphi$ has degree 4 or higher. If this were the case, equality in Theorem 1 (in an appropriate class of manifolds) could only be achieved in the strongly pseudoconvex case. This would be a first step in the understanding of possible extremizers, other than the model, of the inequality of Theorem 1.

The paper is organized as follows: in Section 2 we rigorously define Szegő projections on “weighted” CR manifolds and discuss property C, while in Section 3 we prove the main theorem exploiting a couple of preliminary lemmas presented in Section 5.

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2. Szegő projections on weighted CR manifolds

We assume that the reader is familiar with the basics of CR manifolds, for which we refer to [BER99, DT07]. We limit ourselves to recall a few notions, mostly to establish notation.
A pair \((M, T_{1,0}M)\) is said to be a CR manifold of hypersurface type if

i. \(M\) is a \((2n + 1)\)-dimensional connected and orientable real smooth manifold;

ii. \(T_{1,0}M\) is a rank \(n\) vector subbundle of the complexified tangent bundle \(\mathbb{C}TM\) such that \(T_{1,0}M \cap [T_{1,0}M, T_{1,0}M] = 0\) and \([T_{1,0}M, T_{1,0}M] \subseteq T_{1,0}M\), i.e., the commutator of smooth sections of \(T_{1,0}M\) is again a section of \(T_{1,0}M\).

A \((2n + 1)\)-dimensional real smooth submanifold \(M\) of \(\mathbb{C}^N\) with the property that \(T_{1,0}M := \mathbb{C}TM \cap T_{1,0}\mathbb{C}^N\) has rank constantly equal to \(n\) is called a CR submanifold of hypersurface type: the pair \((M, T_{1,0}M)\) is a CR manifold of hypersurface type.

In what follows, we will usually omit the specification “of hypersurface type” and the bundle \(T_{1,0}M\) from the notation, and we will just say that \(M\) is a CR manifold, or a CR submanifold of some \(\mathbb{C}^N\).

The CR manifold \(M\) is pseudoconvex if there exists a global nowhere vanishing purely imaginary one-form \(\theta\) annihilating \(T_{1,0}M \oplus T^*_{1,0}M\) and such that \(d\theta(L, L) \geq 0\) for every section \(L\) of \(T_{1,0}M\). A point \(x_0 \in M\) is said to be strongly pseudoconvex if \(d\theta(L, L)_{x_0} > 0\) for every \(L\) that does not vanish at \(x_0\).

**Example.** In this paper \(\mathbb{H}^n\), which will be simply called the Heisenberg group, is the CR manifold obtained endowing \(\mathbb{C}^n \times \mathbb{R}\) with the CR structure bundle generated by the complex vector fields

\[ L_j := \partial_{z_j} + \overline{z}_j \partial_{\bar{z}_j}, \quad j = 1, \ldots, n. \]

See [Ste93, Chapter 12] for details on the nilpotent Lie group structure of \(\mathbb{H}^n\) and its ties with complex analysis. The Heisenberg group is everywhere strongly pseudoconvex. Our model weighted CR manifold will be \((\mathbb{H}^n, \sigma)\), where \(\sigma = \left(\frac{i}{2}\right)^n dz_1 \wedge \overline{dz}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n \wedge dt\) is Lebesgue measure on \(\mathbb{C}^n \times \mathbb{R}\).

If \(f \in L^1_{loc}(M)\), we say that \(f\) is a CR-function if \(Lf = 0\) in the sense of distributions for every smooth section \(L\) of \(T_{1,0}M\). For the convenience of the reader, we recall what this means in our context.

If \(L\) is a smooth vector field with complex coefficients, that is, a smooth section of the complexified tangent bundle \(\mathbb{C}TM\), then we denote by \(L^\dagger\) the adjoint of \(L\) with respect to the natural pairing of smooth functions with top-degree forms. More precisely, there is a unique first order differential operator \(L^\dagger : \Omega^{2n+1}_c M \to \Omega^{2n+1}_c M\) such that

\[ \int_M Lf \cdot \omega = \int_M f L^\dagger \omega, \quad \forall f \in C^\infty(M), \quad \forall \omega \in \Omega^{2n+1}_c M, \]

where \(\Omega^*_c M\) is the space of smooth compactly supported \((2n+1)\)-forms on \(M\). If \(L = \sum_{j=1}^{2n+1} a_j(x) \partial_{x_j}\) and \(\omega = g(x) dx_1 \wedge \cdots \wedge dx_{2n+1}\) in a local coordinate system, then \(L^\dagger \omega = -\sum_{j=1}^{2n+1} a_j(x) \partial_{x_j} g - \left(\sum_{j=1}^{2n+1} \partial_{x_j} a_j(x)\right) g\). If \(f \in L^1_{loc}(M)\), then one says that \(Lf = 0\) in the sense of distributions if

\[ \int_M f L^\dagger \omega = 0, \quad \forall \omega \in \Omega^{2n+1}_c M. \]

A weighted CR manifold is a pair \((M, \nu)\), where \(M\) is a CR manifold and \(\nu\) is a smooth positive \((2n+1)\)-form on \(M\). Of course, \(\nu\) may be thought of as a Borel measure on \(M\) with smooth positive density with respect to Lebesgue measure in any coordinate chart.

Given a weighted CR manifold \((M, \nu)\), we denote by \(CR^2(M, \nu)\) the space of CR functions that are square-integrable with respect to \(\nu\), with the usual identification of almost everywhere equal functions.

**Proposition 2.** \(CR^2(M, \nu)\) is a closed subspace of the Hilbert space \(L^2(M, \nu)\).

This follows immediately from the definition of CR functions and the fact that if a sequence of functions \(\{f_k\}_k\) converges in \(L^2(M, \nu)\) to a function \(f\), then \(\lim_{k \to +\infty} \int_M f_k \omega = \int_M f \omega\) for every \(\omega \in \Omega^{2n+1}_c M\).

Thus, we may define the main object of study of the present paper.

**Definition 3.** The Szegő projection \(S_{M, \nu}\) of the weighted CR manifold \((M, \nu)\) is defined as the orthogonal projection operator mapping \(L^2(M, \nu)\) onto \(CR^2(M, \nu)\). We denote by \(S_n\) the Szegő projection associated to the Heisenberg group \((\mathbb{H}^n, \sigma)\).
We are interested in the $L^p$-operator norms of Szegö projections, namely the quantities defined by
\[ N_p(M, \nu) := \sup \left\{ \left( \int_M |Sf|^p \, d\nu \right)^{\frac{1}{p}} : f \in L^2(M, \nu) \text{ and } \int_M |f|^p \, d\nu = 1 \right\}, \]
for $p \in [1, +\infty)$, and by
\[ N_\infty(M, \nu) := \sup \{ \|Sf\|_\infty : f \in L^2(M, \nu) \text{ and } \|f\|_\infty = 1 \} . \]

To formulate our main result, we need the notion of Property C (C is for “compactness”). Recall that $\overline{\partial}_b$ is the operator mapping $f \in C^\infty(M)$ into the smooth section of $B_{0,1}M$, the dual bundle of $T_{0,1}M$, defined by $\langle \overline{\partial}_b f, T \rangle = \overline{T} f$ for every section $L$ of $T_{1,0}M$. Given a smooth positive measure $\nu$ on $M$ and a Hermitian metric $h$ on $B_{0,1}M$, we have the associated quadratic form
\[ \mathcal{E}(f) = \int_M |f|^2 \, d\nu + \int_M |\overline{\partial}_b f|^2_h \, d\nu . \]

Given a precompact open set $B \subseteq M$, we denote by $\mathcal{D}(B) \subseteq L^2(B)$ the completion of $C_c^\infty(B)$ with respect to $\mathcal{E}$. Notice that different choices of $\nu$ and $h$ produce isomorphic topological vector spaces, because the corresponding quadratic forms $\mathcal{E}$ are comparable on test functions supported on $B$.

**Definition 4** (Property C). We say that the weighted CR manifold $(M, \nu)$ satisfies property C at $x_0 \in M$ if there exists a precompact open neighborhood $B$ of $x_0$ such that the operator
\[ \mathcal{D}(B) \to L^2(B) \]
\[ f \mapsto 1_B(1 - \mathcal{S}_{M,\nu}) f \]
is compact (by the observation right before this definition, compactness is independent of the choice of metric on $B_{0,1}M$).

**Remark 5.** Property C is a local compactness condition of the kind playing an important role in the theories of the $\overline{\partial}$ and the $\overline{\partial}_b$ problem (see, e.g., [Str12, Chapter 4]). By Sobolev embedding, it holds whenever a subelliptic estimate of the form
\[ \|u\|_{W^r(B)} \leq C\|\overline{\partial}_b u\|_{L^2(M)} \]
holds for every $u \in L^2(M)$ orthogonal to CR$^2(M, \nu)$, where $\epsilon > 0$. Such a subelliptic estimate with $\epsilon = \frac{1}{2}$ is known to hold in a small enough neighborhood $B$ of a strongly pseudoconvex point $x_0 \in M$, under the additional assumptions that $M$ is compact, pseudoconvex, and that the maximal $L^2$ extension of $\overline{\partial}_b$ has closed range. We refer to [Koh83] for this and far more general results on these matters.

We point out that, if $M$ is a compact pseudoconvex CR submanifold of $\mathbb{C}^n$, then the closed range property automatically holds [Bar12a, Bar12b]. Since such a CR submanifold necessarily has a point of strong pseudoconvexity (viz., any point at maximal distance from the origin of $\mathbb{C}^n$), we see that every compact pseudoconvex CR submanifold of $\mathbb{C}^n$ has a strongly pseudoconvex point at which property C holds, that is, satisfies the assumption of Theorem 1’ below.

We can now restate the main theorem in the more precise notation of this section.

**Theorem 1’.** Let $(M, \nu)$ be a weighted CR manifold of dimension $2n + 1$. If $(M, \nu)$ satisfies property C at a strongly pseudoconvex point $x_0 \in M$, then we have
\[ N_p(M, \nu) \geq N_p(\mathbb{H}^n, \sigma) \quad \forall p \in [1, +\infty] . \]

3. Preliminaries

3.1. Folland–Stein coordinates. Define the one-parameter group of parabolic scalings on $\mathbb{C}^n \times \mathbb{R}$ as follows:
\[ \Phi_\lambda(z, t) := (\lambda z, \lambda^2 t) \quad (\lambda > 0) . \]

We say that a monomial $z^\alpha \overline{\xi}^\beta t^\gamma$, where $\alpha, \beta \in \mathbb{N}^n$ and $\gamma \in \mathbb{N}$, has parabolic weight
\[ \sum_{j=1}^n (\alpha_j + \beta_j) + 2\gamma . \]
and that a smooth function defined in a neighborhood of \((0,0) \in \mathbb{C}^n \times \mathbb{R}\) has parabolic weight \(\geq w\) if every monomial with nonzero coefficient in its Taylor expansion has weight \(\geq w\). The parabolic weight of a vector field \(E = \sum_{j=1}^{n} \{a_j(z,t)\partial_{z_j} + b_j(z,t)\partial_t\} + c(z,t)\partial_t\) is computed assigning weight \(-1\) to \(\partial_{z_j}\) and \(\partial_t\) and weight \(-2\) to \(\partial_t\). In other words, \(E\) have parabolic weight \(\geq w\) if every \(a_j\) and every \(b_j\) has parabolic weight \(\geq w + 1\) and \(c\) has parabolic weight \(\geq w + 2\).

We refer to [FS74] and [DT07] Theorem 3.5] for a proof of the following proposition.

**Lemma 6** (Folland–Stein coordinates). Let \(M\) be a CR manifold and let \(x_0 \in M\) be a strictly pseudoconvex point. Then in a neighborhood of \(x_0\) there exists a local system of coordinates \((z,t) \in \mathbb{C} \times \mathbb{R}\) such that \((z(x_0), t(x_0)) = 0\) and a system of local generators of \(T_{1,0}M\) of the form

\[
\begin{equation}
L_j = \partial_{z_j} + i\xi_j \partial_t + E_j \quad (j = 1, \ldots, n),
\end{equation}
\]

where the error terms \(E_j\) have parabolic weight \(\geq 0\).

### 3.2. A density lemma.

**Lemma 7.** There exists a dense subspace \(\mathcal{D}\) of \(\mathcal{C}^2(\mathbb{H}^n, \sigma)\) such that for every \(h \in \mathcal{D}, \alpha, \beta \in \mathbb{N}^n, \gamma \in \mathbb{N}, \) and \(N \in \mathbb{N}\), we have

\[
|\partial_{z}^{\alpha} \partial_{\xi}^{\beta} h(z,t)| \leq C_N(|z|^2 + |t|)^{-N}.
\]

**Proof.** In view of the invariance under \(t\)-translations of the Heisenberg CR structure, we exploit the partial Fourier transform

\[
\mathcal{F} f(z, \xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z, t) e^{-i\xi t} dt.
\]

The operator \(\mathcal{F}\) is a unitary isometry of \(L^2(\mathbb{H}^n, \sigma)\), and

\[
\mathcal{F} \left((\partial_{z_j} - i\xi_j \partial_t) f\right) = (\partial_{z_j} + \xi_j \partial_t) \mathcal{F} f = e^{-|\xi|^2} \partial_{z_j} \left(e^{i|m|^2} \mathcal{F} f\right).
\]

Thus \(f \in \mathcal{C}^2(\mathbb{H}^n, \sigma)\) if and only if \(g(z, \xi) = e^{i|m|^2} \mathcal{F} f\) is holomorphic for every \(\xi \in \mathbb{R}\). Since \(g \in \mathcal{C}^2(\mathbb{C}^n, e^{-2|\xi|^2})\) for almost every \(\xi\) and there are no nonzero Lebesgue square-integrable holomorphic functions on \(\mathbb{C}^n\), we must have \(g(z, \xi) = 0\) for almost every \(\xi \leq 0\). In conclusion, the operator \(f \mapsto e^{i|m|^2} \mathcal{F} f\) establishes a unitary isomorphism between \(\mathcal{C}^2(\mathbb{H}^n, \sigma)\) and the Hilbert space

\[
\mathcal{H} = \left\{ g: \mathbb{C}^n \times (0, +\infty) \rightarrow \mathbb{C}: g(\cdot, \xi) \text{ is holomorphic } \forall \xi > 0 \text{ and } \int_{\mathbb{C}^n \times (0, +\infty)} |g|^2 e^{-2|\xi|^2} < +\infty \right\}.
\]

A dense subspace \(\mathcal{D}'\) of \(\mathcal{H}\) is given by the linear span of functions of the form \(g(z, \xi) = P(z)\varphi(\xi)\), where \(P(z) = P(z_1, \ldots, z_n)\) is a holomorphic polynomial and \(\varphi \in \mathcal{C}^\infty(\mathbb{R}^+)\). To see this, notice that if \(g \in \mathcal{H}\) is orthogonal to \(\mathcal{D}'\), then \(\varepsilon^{-1} \int_{\mathbb{C}^n} g(z, \xi) P(z) e^{-2|\xi|^2} = 0\) for every \(\alpha, \varepsilon > 0\). Letting \(\varepsilon\) tend to zero, we see that, for almost every \(\alpha > 0\), \(\int_{\mathbb{C}^n} g(z, \alpha) P(z) e^{-2|z|^2} = 0\). By the arbitrariness of \(P\) and the density of polynomials in the Fock space \([\text{Zhu12}]\), we conclude that \(g(z, \alpha) = 0\) for almost every \(\alpha > 0\).

Let \(\mathcal{D} := \mathcal{F}^{-1}(\mathcal{D}')\). By what we just proved, \(\mathcal{D}\) is dense in \(\mathcal{C}^2(\mathbb{H}^n)\). It is generated by elements of the form

\[
P(z) \int_{\mathbb{R}} \varphi(\xi) e^{-|\xi|^2} e^{i\xi t} d\xi
\]

with \(P\) and \(\varphi\) as above. If \(\varphi\) is supported on \([a, +\infty)\), then a standard integration by parts shows that

\[
\left|(|z|^2 - it)^N \int \varphi(\xi) e^{-|\xi|^2} e^{i\xi t} d\xi\right| \leq e^{-a|z|^2} \int |\varphi(\xi)| d\xi.
\]

Thus, every element of \(\mathcal{D}\) decays faster than any negative power of \(|z|^2 + |t|\). A similar argument proves that the same holds for every derivative. \(\square\)
3.3. Models satisfy property C. The next result is not strictly needed for the proof, but shows that the noncompact model \( (\mathbb{H}^n, \sigma) \) is in the class of CR manifolds to which the theorem applies.

**Proposition 8.** The model \( (\mathbb{H}^n, \sigma) \) satisfies property C at every point.

**Proof.** Let \( \mathcal{F} \) be the partial Fourier transform as in the proof of Lemma 7. By (3), we have the identity

\[
\mathcal{F}(S_n f)(\cdot, \xi) = e^{-|\xi|^2} B_\xi \left( e^{\xi \cdot \cdot} \mathcal{F} f(\cdot, \xi) \right),
\]

where \( B_\xi \) is the Bergman projection of the Fock space \((\mathbb{C}^n, e^{-2|z|^2})\), that is, the orthogonal projection onto the closed subspace of entire holomorphic functions that are square integrable with respect to the Gaussian measure with density \( e^{-|z|^2} \). In particular, \( B_\xi = 0 \) when \( \xi \leq 0 \). The following estimate is well-known:

\[
(4) \quad \int_{\mathbb{C}^n} |(1 - B_\xi) u|^2 e^{-2|z|^2} dA(z) \leq 2\xi^{-1} \int_{\mathbb{C}^n} \left( \sum_j |\partial z_j u|^2 \right) e^{-2|z|^2} dA(z) \quad \forall u \in C_\infty^2(\mathbb{C}^n), \xi > 0.
\]

In fact, \( g = (1 - B_\xi) u \) is the solution of minimal Gaussian \( L^2 \) norm of the equation \( \overline{\partial} g = \overline{\partial} u \), and one may apply [Hör90, Lemma 4.4.1].

Let \( f \in C_\infty^2(\mathbb{H}^n) \). Define the projection operator \( P_{\leq T} \) \((T \in \mathbb{R})\) by the identity

\[
\mathcal{F}(P_{\leq T} f)(\cdot, \xi) = 1_{(-\infty, T]}(\xi) \mathcal{F} f(\cdot, \xi),
\]

and put \( P_{> T} := I - P_{\leq T} \).

Applying (4) to \( e^{\xi^2} \mathcal{F} f(\cdot, \xi) \) and integrating in \( \xi \in (T, +\infty) \), we find

\[
(5) \quad \int_{\mathbb{H}^n} |P_{> T} f - S_n P_{> T} f|^2 d\sigma \leq \frac{2}{T} \int_{\mathbb{H}^n} \left( \sum_j |\overline{\partial}_j f|^2 \right) d\sigma.
\]

Notice that we used the fact that \( \overline{\partial}_j \) and \( S_n \) commute with \( P_{> T} \). We treat the low-frequency component using the standard commutator formula

\[
\int_{\mathbb{H}^n} |\overline{\partial}_j f|^2 d\sigma = \int_{\mathbb{H}^n} |\overline{\partial}_j f|^2 d\sigma + \int_{\mathbb{H}^n} |\overline{\partial}_j \overline{\partial}_j f|^2 d\sigma.
\]

Since \( [\overline{\partial}_j, \overline{\partial}_j] = -2i\partial_t \), this immediately yields

\[
(6) \quad \int_{\mathbb{H}^n} |\overline{\partial}_j P_{\leq T} f|^2 d\sigma \leq \int_{\mathbb{H}^n} |\overline{\partial}_j P_{\leq T} f|^2 d\sigma + 2T \int_{\mathbb{H}^n} |P_{\leq T} f|^2 d\sigma.
\]

The identity \( [\overline{\partial}_j, \overline{\partial}_j] = -2i\partial_t \) also implies that \( \{\mathbb{N}(L)_j, \mathbb{N}(L)_j\}_j \) is a system of vector fields satisfying Hörmander’s bracket condition of order 2. Hence (see, e.g., [Koh73]), denoting by \( \chi \) and \( \chi' \) two test functions with \( \chi' = 1 \) on a neighborhood of the support of \( \chi \), we have

\[
(7) \quad \|f\|^2_{W^{1/2}(\mathbb{H}^n)} \leq C_B \left( \|f\|^2_{L^2} + \sum_j (\|L_j f\|^2_{L^2} + \|\overline{\partial}_j f\|^2_{L^2}) \right) \quad \forall f \in C_\infty^2(\mathbb{H}^n),
\]

where \( B \subset \mathbb{C}^n \) is a ball (or any, say, smooth open set) and \( W^{1/2} \) denotes the fractional Sobolev norm of order 1/2. Putting (6) and (7) together (and using again \([\overline{\partial}_j, P_{\leq T}] = 0\) ), we obtain

\[
(8) \quad \|P_{\leq T} f\|^2_{W^{1/2}} \leq C_B (1 + 2T) \|f\|^2_{L^2} + 2C_B \sum_j \|L_j f\|^2_{L^2}.
\]

We can now complete the proof. Given \( \{f_k\}_k \subset C_\infty^2(B) \) with bounded CR energy, the sequence \( \{P_{\leq T} f_k\}_k \), and a fortiori \( \{(1 - S_n)P_{\leq T} f_k\}_k \), has a subsequence of \( L^2 \)-diameter \( \leq \varepsilon \). This follows from (8) and the compactness of the Sobolev embedding \( W^{1/2}(B) \hookrightarrow L^2(B) \). If \( T \geq 2\varepsilon^{-2} \), by virtue of (5) the whole sequence \( \{(1 - S_n)P_{> T} f_k\}_k \) has \( L^2 \)-diameter \( \leq \varepsilon \) too. The conclusion of the proposition follows immediately. \( \square \)
4. Proof of Theorem 1

Write $S = S_{M,n}$ for simplicity. Fix a coordinate patch $\Omega$ around $x_0$ equipped with a system of Folland–Stein coordinates $(z,t)$, and $p \in [1, +\infty)$ (notice that $S$ is self-adjoint, so the case $p = \infty$ of the theorem follows by duality from the $p = 1$ case).

We identify $\Omega$ with an open neighborhood of $(0, 0)$ in $\mathbb{H}^n \equiv \mathbb{C}^n \times \mathbb{R}$ via these coordinates.

Fix $f \in C_0^\infty(\mathbb{H}^n)$. If $\lambda > 0$ is large enough, depending on the support of $f$, $f \circ \Phi_{\lambda}$ is supported on $\Omega$, and hence it may be thought of as a test function on $M$. Without loss of generality, we may assume that $\frac{d\rho}{d\sigma}(x_0) = 1$, where $\sigma$ is Lebesgue measure in the Folland–Stein coordinates.

Thus $S(f \circ \Phi_{\lambda})$ is a well-defined element of $L^p(M, \nu)$. Let $\rho \in C_0^\infty(\mathbb{H}^n, [0, 1])$ be a cut-off function identically equal to 1 in a neighborhood of the origin, i.e., of $x_0$. We set $\rho_\mu := \rho \circ \Phi_{\mu}$ ($\mu > 0$). Notice that the support of $\rho_\mu$ shrinks to $x_0$ as $\mu$ tends to $\infty$, and that $S(f \circ \Phi_{\lambda})\rho_\mu$ is supported on $\Omega$ for every large $\mu$. Thus, it can be thought of as a function on $\mathbb{H}^n$. Put

$$g_\lambda := (S(f \circ \Phi_{\lambda}) \rho_\lambda) \circ \Phi_{\lambda^{-1}} \in L^2(\mathbb{H}^n, \sigma).$$

**Remark 9.** It will be clear soon that $\sqrt{\lambda}$ could be replaced by any function $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{\lambda \to +\infty} F(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda \to +\infty} \frac{F(\lambda)}{\lambda} = 0.$$  

In other words, what turns out to be crucial is localizing the Szegő projection of $f \circ \Phi_{\lambda}$, which is supported at the parabolic infinitesimal scale $\lambda^{-} \lambda$, at an infinitesimal scale much larger than $\lambda^{-} \lambda$.

It is clear that for every $\lambda$ large

$$\int_{\mathbb{H}^n} |g_\lambda|^p d\sigma = \int_{\Omega} |S(f \circ \Phi_{\lambda}) \rho_\lambda|^p d\sigma$$

$$= (1 + o(1)) \lambda^{2n+2} \int_{\Omega} |S(f \circ \Phi_{\lambda}) \rho_\lambda|^p d\nu$$

$$\leq (1 + o(1)) \lambda^{2n+2} N_p(M, \nu)^p \int_{\Omega} |f \circ \Phi_{\lambda}|^p d\nu$$

$$= (1 + o(1)) \lambda^{2n+2} N_p(M, \nu)^p \int_{\Omega} |f \circ \Phi_{\lambda}|^p d\sigma$$

$$= (1 + o(1)) N_p(M, \nu)^p \int_{\mathbb{H}^n} |f|^p d\sigma,$$

where we used a couple of times the fact that $\frac{d\rho}{d\sigma}(x_0) = 1$.

By Banach–Alaoglu, along a diverging subsequence of $\lambda$s, $g_\lambda$ has a weak limit both in $L^2(\mathbb{H}^n, \sigma)$ and $L^p(\mathbb{H}^n, \nu)$, for any fixed $p > 1$. Denoting by $g \in L^2 \cap L^p(\mathbb{H}^n, \sigma)$ this limit, we clearly have

$$\int_{\mathbb{H}^n} |g|^p d\sigma \leq N_p(M, \nu)^p \int_{\mathbb{H}^n} |f|^p d\sigma. \quad (9)$$

From now on, limits in $\lambda$ are always along appropriate diverging subsequences.

We claim that the limit $g$ is CR with respect to the model structure, i.e., that $g \in CR^2(\mathbb{H}^n, \sigma)$. To see this, we start by noticing that by (2),

$$\overline{L}_j (S(f \circ \Phi_{\lambda}) \rho_\lambda) \circ \Phi_{\lambda^{-1}} = \lambda(\partial_{z_j} + i\overline{z_j} \partial_t) g_\lambda + \lambda E_j^\lambda g_\lambda,$$

in the sense of distributions. If $E_j = \sum_{k=1}^n \{ a_{j,k} \partial_{z_k} + b_{j,k} \partial_{\overline{z}_k} \} + c \partial_t$, the rescaled error terms above are given by

$$E_j^\lambda = \sum_{k=1}^n \{ a_{j,k} \circ \Phi_{\lambda^{-1}} \cdot \partial_{z_k} + b_{j,k} \circ \Phi_{\lambda^{-1}} \cdot \partial_{\overline{z}_k} \} + \lambda c \circ \Phi_{\lambda^{-1}} \cdot \partial_t.$$

Notice that the formal adjoint of $E_j^\lambda$ (w.r.t. Lebesgue measure $d\sigma$) is $-\overline{E}_j^\lambda - \lambda^{-1} e_j \circ \Phi_{\lambda^{-1}}$, where $e_j(z, t)$ is a smooth function.
Since $\rho$ is identically 1 on a neighborhood $V$ of the origin, the LHS of (10) vanishes on $\Phi_\Lambda(V)$. Since $\bigcup_\Lambda \Phi_\Lambda(V) = \mathbb{H}^n \times \mathbb{R}$, for any $\varphi \in C_0^\infty(\mathbb{H}^n)$ and $\lambda$ large, we have

$$
\int_{\mathbb{H}^n \times \mathbb{R}} \left( \partial_{x_j} + i \tau_j \partial_t \right) g_\lambda \cdot \varphi \, d\sigma = - \int_{\mathbb{H}^n \times \mathbb{R}} E^\lambda_\varphi g_\lambda \cdot \varphi \, d\sigma = \int_{\mathbb{H}^n \times \mathbb{R}} g_\lambda \cdot \left( E^\lambda_\varphi + \lambda^{-1} \epsilon_j \circ \Phi^{-1}_\lambda \right) \varphi \, d\sigma.
$$

By Lemma 6, $a_{j,k}$ and $b_{j,k}$ have parabolic weight $\geq 1$ and $c$ has weight $\geq 2$, and therefore $\| (E^\lambda_\varphi + \lambda^{-1} \epsilon_j \circ \Phi^{-1}_\lambda) \varphi \|_\infty = O(\lambda^{-1})$. Since $g_\lambda$ are uniformly in $L^2(\mathbb{H}^n \times \mathbb{R})$, we conclude that $(\partial_{x_j} + i \tau_j \partial_t) g = 0$ and the claim is proved.

We are left with the proof that $g = S_\nu(f)$. In fact, if we show this then (10) and the arbitrariness of $f$ entail $N_\rho(\mathbb{H}^n, \sigma) \leq N_\rho(M, \nu)$, as we wanted.

Since we already know that $g$ is CR, to prove that $g = S_\nu(f)$ it is enough to verify this identity for $h$ in the dense subspace $\mathcal{D}$ of Lemma 7. By the self-adjunction of $S$ in $L^2(M, \nu)$, we have

$$
\int_{\mathbb{H}^n} g_\lambda \overline{\nu} \, d\sigma = \int_{\mathbb{H}^n} (S(f \circ \Phi_\lambda) \rho_\lambda \overline{\nu}) \circ \Phi^{-1}_\lambda \cdot \overline{\nu} \, d\sigma = \lambda^{2n+2} \int_M S(f \circ \Phi_\lambda) \rho_\lambda \overline{\nu} \cdot h \circ \Phi_\lambda \, d\nu \, d\nu = \lambda^{2n+2} \int_{\Omega} f \circ \Phi_\lambda \cdot S\left( \rho_\lambda \overline{\nu} \cdot h \circ \Phi_\lambda \right) \, d\nu \, d\sigma = \int_{\mathbb{H}^n} f \cdot \left( S\left( \rho_\lambda \overline{\nu} \cdot h \circ \Phi_\lambda \right) \frac{d\nu}{d\sigma} \right) \circ \Phi^{-1}_\lambda \, d\sigma.
$$

Notice that the various passages from $d\sigma$ to $d\nu$ are meaningful, because for $\lambda$ large, $\rho_\lambda$ and $f \circ \Phi_\lambda$ are supported in $\Omega$.

Thus, our task is reduced to proving that

$$
\lim_{\lambda \to +\infty} \left\{ S\left( \rho_\lambda \frac{d\nu}{d\sigma} \cdot h \circ \Phi_\lambda \right) \frac{d\nu}{d\sigma} \right\} \circ \Phi^{-1}_\lambda = h \quad \forall h \in \mathcal{D}
$$

in the sense of distributions.

Notice that if we remove $S$ from the expression in the limit, we get $h \rho_\lambda^{-\frac{1}{2}}$, which clearly $L^2$-converges to $h$. Thus, setting

$$
u_\lambda := (I - S) \left( \rho_\lambda \frac{d\nu}{d\sigma}, h \circ \Phi_\lambda \right),
$$

it is enough to see that $\lim_{\lambda \to +\infty} (u_\lambda \frac{d\nu}{d\sigma}) \circ \Phi^{-1}_\lambda = 0$ in the sense of distributions, i.e.,

$$
\int_{\mathbb{H}^n} u_\lambda \frac{d\nu}{d\sigma} \lambda^{2n+2} \varphi \circ \Phi_\lambda \, d\sigma = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{H}^n).
$$

Let $B$ be the compact neighborhood of $x_0$ appearing in the definition of Property C. We claim that:

$$
(11) \quad 1_B \lambda^{n+1} u_\lambda \text{ has a strong } L^2 \text{ limit } u, \text{ along an appropriate subsequence.}
$$

Notice that $\nu$ and $\sigma$ are comparable on $B$ and we do not need to specify in which $L^2$ norm the convergence happens.
Let us show that the claim allows to conclude. We write
\[
\int_{\mathbb{H}^n} u_\lambda \frac{dv}{d\sigma} \lambda^{n+2} \varphi \circ \Phi_\lambda \, d\sigma = \int_{\mathbb{H}^n} (\lambda^{n+1} u_\lambda - v_0) \frac{dv}{d\sigma} \lambda^{n+1} \varphi \circ \Phi_\lambda \, d\sigma \\
+ \int_{\mathbb{H}^n} (v - v_0) \frac{dv}{d\sigma} \lambda^{n+1} \varphi \circ \Phi_\lambda \, d\sigma + \lambda^{-n-1} \int_{\mathbb{H}^n} v_0 \frac{dv}{d\sigma} \lambda^{n+2} \varphi \circ \Phi_\lambda \, d\sigma,
\]
where \( v_0 \in C_c(\mathbb{H}^n) \) is such that \( \|v - v_0\|_{L^2} < \varepsilon \), with \( \varepsilon \) small. The first term vanishes in the limit, simply because \( \lambda^{n+1} \varphi \circ \Phi_\lambda \) is uniformly bounded in \( L^2 \) norm, and the second is \( O(\varepsilon) \) for the same reason. Since the last one is asymptotic to \( \lambda^{-n-1} v_0(0, h) \) \( \varphi \, d\sigma \), the conclusion follows by the arbitrariness of \( \varepsilon \).

We now prove claim (11). In virtue of property C, it is enough to prove that the family of test functions \( \{ \lambda^{n+1} \rho \sqrt{\lambda \nabla} \cdot h \circ \Phi_\lambda \lambda \} \) is bounded w.r.t. the “CR energy” (1). The \( L^2 \) norm is clearly bounded uniformly in \( \lambda \). We compute, for \( L_j \) as in Lemma 6
\[
L_j \left( \rho \sqrt{\lambda \nabla} \cdot h \circ \Phi_\lambda \right) = \left[ \nabla \left( \left( \partial_{\bar{z}_j} + \bar{\iota}_{\bar{z}_j} \partial_\bar{h} \right) \rho + E_j^{\nabla} \rho \right) \circ \Phi_\lambda \right) + \rho \sqrt{\lambda \nabla} L_j \left( h \circ \Phi_\lambda \right)
\]
\[= \left[ \left( \sqrt{\lambda} \left( \left( \partial_{\bar{z}_j} + \bar{\iota}_{\bar{z}_j} \partial_\bar{h} \right) \rho + E_j^{\nabla} \rho \right) \circ \Phi_\lambda \right) + \rho \sqrt{\lambda \nabla} L_j \left( h \circ \Phi_\lambda \right) + \rho \sqrt{\lambda \nabla} \right] \left( \lambda \right) \circ \Phi_\lambda \]
where we used the fact that \( h \) is CR with respect to the Heisenberg structure. The easiest term to deal with is
\[
\int_M \left| h \circ \Phi_\lambda, \rho \sqrt{\lambda \nabla} L_j \left( \frac{d\sigma}{dv} \right) \right|^2 \frac{dv}{d\sigma} = (1 + o(1)) \left| \frac{d\sigma}{dv} \right|^2 \lambda^{-2n-2} \int_{\mathbb{H}^n} |h|^2 \, d\sigma.
\]
Next,
\[
\int_M \left| \sqrt{\lambda} \left( \left( \partial_{\bar{z}_j} + \bar{\iota}_{\bar{z}_j} \partial_\bar{h} \right) \rho + E_j^{\nabla} \rho \right) \circ \Phi_\lambda \right|^2 \frac{dv}{d\sigma} \leq C(1 + o(1)) \lambda^{-n-1} \lambda \int_{\mathbb{H}^n} \left| \left( \partial_{\bar{z}_j} + \bar{\iota}_{\bar{z}_j} \partial_\bar{h} \right) \rho + E_j^{\nabla} \rho \right|^2 \, d\sigma \\
\leq C1 \lambda^{-n-1} \lambda \int_{\mathbb{H}^n} \left| h \circ \Phi_\lambda \right|^2 \, d\sigma,
\]
where we used the fact that \( \rho = 1 \) in the neighborhood \( V \) of the origin and that the coefficients of \( E_j^{\nabla} \) are bounded uniformly in \( \lambda \). Here we take advantage of the fact that \( h \) is in \( D \). By Lemma 7 the quantity above may be estimated by \( |h \circ \Phi_\lambda| \leq C_N \lambda^{-N} (|z|^2 + |\lambda|)^{-N} \) for any \( N \in \mathbb{N} \). Choosing \( N \) large enough, we obtain the desired estimate.

Finally,
\[
\int_M \left| \rho \sqrt{\lambda \nabla} \left( E_j^\lambda \right) \circ \Phi_\lambda \right|^2 \frac{dv}{d\sigma} = (1 + o(1)) \lambda^{-2n-2} \int_{\mathbb{H}^n} \lambda \left( E_j^\lambda \right) \rho \circ \Phi_\lambda \, d\sigma.
\]
The vector field \( \lambda E_j^\lambda \) has coefficients bounded uniformly in \( \lambda \). Therefore, to bound the last term we need \( \sum_{j=1}^n (|\partial_{\bar{z}_j} h|^2 + |\partial_\bar{h} h|^2) + |\partial h|^2 < +\infty \), which certainly holds for \( h \in D \). This completes the proof of the boundedness of the CR energy of \( \{ \lambda^{n+1} \rho \sqrt{\lambda \nabla} \cdot h \circ \Phi_\lambda \} \), and hence of the theorem.

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