Improved Approximation of Dispatchable Region in Radial Distribution Networks via Dual SOCP

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Abstract—The concept of dispatchable region is useful in quantifying how much renewable generation the power system can handle. In this paper, we aim to provide an improved approximation of the dispatchable region in distribution networks. First, based on the nonlinear Dist-Flow model, an optimization problem is formulated to characterize the dispatchable region. The problem is then relaxed to a solvable second-order cone program (SOCP) whose strong dual problem (also an SOCP) is derived. An SOCP-based projection algorithm (Algorithm 1) is developed to construct a convex polytopic approximation. We prove that Algorithm 1 can generate the accurate SOCP-relaxed dispatchable region under certain conditions, which is an improvement over earlier studies that could only generate approximations. Furthermore, a heuristic method (Algorithm 2) is proposed to approximately remove the regions that make the SOCP relaxation inexact. The final region obtained is the difference of several convex sets and can be nonconvex. Thus, the proposed algorithms may provide a better approximation of the actually nonconvex dispatchable region than previous work that could construct convex sets only. Numerical comparisons demonstrate that the proposed method can achieve a better balance between ensuring security and reducing conservatism.

Index Terms—AC power flow, distribution networks, dispatchable region, optimization, second-order cone program.

NOMENCLATURE

Constant Parameters
\[ r_{ij}, \chi_{ij} \] Resistance and reactance of line \( i \rightarrow j \).
\[ P_i, \overline{P}_i \] Controllable active power limit at node \( i \).
\[ q_i, \overline{q}_i \] Controllable reactive power limit at node \( i \).
\[ \ell_{ij} \] Voltage safety limit on line \( i \rightarrow j \).
\[ A_f, B_f, A_s \] Constant matrices in the feasibility problem.
\[ \gamma_f, \gamma_s \] Constant vectors in the feasibility problem.
\[ A_y, b_y, c_q, \gamma_q \] Constant matrices and vectors in the second-order cone program (SOCP) (5).

Variables
\[ p_i, q_i \] Controllable power injection at node \( i \).
\[ w_i \] Active power generation of the renewable generator at node \( i \).
\[ v_i \] Squared voltage magnitude at node \( i \).
\[ \ell_{ij} \] Squared current magnitude on line \( i \rightarrow j \).
\[ P_{ij}, Q_{ij} \] Active and reactive power flows onto line \( i \rightarrow j \).
\[ x \] Vector of state variables defined as \( x := (p, q, v, \ell, P, Q) \).
\[ z_s, z_q, \overline{z}_q \] Vectors of nonnegative slack variables.
\[ y_{ij} \in \mathbb{R}^3 \] Auxiliary variables in (5) for line \( i \rightarrow j \).
\[ \mu_f, \mu_y \] Dual variables for equality constraints.
\[ \lambda_s, \lambda_q \] Dual variables for inequality constraints.

Optimization Problems, Values, Sets
\[ FP(w) \] Feasibility problem for renewable power \( w \).
\[ fp(w) \] Minimum objective value of \( FP(w) \).
\[ \mathcal{W} \] Dispatchable region of \( w \), in which \( fp(w) = 0 \).
\[ FP'(w) \] SOCP relaxation of \( FP(w) \).
\[ fp'(w) \] Minimum objective value of \( FP'(w) \).
\[ \mathcal{W}' \] SOCP-relaxed dispatchable region of \( w \).
\[ DP'(w) \] Dual problem of \( FP'(w) \), also an SOCP.
\[ D_w(\mu, \lambda) \] Dual objective function.
\[ dp'(w) \] Maximum objective value of \( DP'(w) \).
\[ DP'(w, \delta) \] Dual SOCP with feasible set tightened by \( \delta \).
\[ dp''(w, \delta) \] Maximum objective value of \( DP''(w, \delta) \).
\[ \mathcal{W}'_{poly} \] Polytopic approximation of \( \mathcal{W}' \), returned by Algorithm 1.
\[ \mathcal{W}'_{inex} \] SOCP-inexact region of \( w \).
\[ \mathcal{W}'_{poly} \] Polytopic approximation of \( \mathcal{W}'_{inex} \), returned by Algorithm 2.

I. INTRODUCTION

With the benefits of near-zero carbon emissions and low operating costs, distributed renewable power is experiencing tremendous expansion in recent years [1]. Meanwhile, its volatile and intermittent features pose great challenges to electric grid operation, especially the distribution system. Unlike the bulk system, distribution system has few controllable units and a stronger coupling of active and reactive power flows due to its high resistance to reactance ratio [2]. This makes it even
harder to accommodate the fluctuating renewable generations. Therefore, characterizing the renewable power capacities that can be safely hosted by a distribution network prior to its actual operation is vital. This necessitates finding all renewable power outputs that can ensure solvability of the power flow equations and satisfaction of safety limits.

The first requirement is solvability of the power flow equations. For a transmission network modeled by direct-current (DC) power flow, solvability is easy to check since a closed-form solution can be obtained [3]. However, for distribution networks, the lossless DC model is not accurate enough since the distribution lines have higher resistance to reactance ratios. Some literature proved sufficient conditions under which the alternating current (AC) power flow equations are solvable, by utilizing Banach fixed-point theorem for contraction mappings [4], [5] or Brouwer fixed-point theorem for continuous mappings over compact convex sets [6], [7]. Nonetheless, those methods cannot be readily applied to output the dispatchable region since they are based on power flow equations and can hardly deal with inequality safety constraints.

To further take into account the second requirement, i.e., satisfaction of safety limits, optimization-based methods were developed. Two well-known concepts are the do-not-exceed limit (DNEL) [8] and the dispatchable region [9]. The DNEL provides an allowable power interval for each renewable generator based on robust optimization. Data-driven approach [10] and topology control [11] were incorporated to improve the accuracy of DNEL. The correlation between different renewable generators is ignored in DNEL, so the obtained capacity regions can be conservative. The dispatchable region further considers those correlations and provides the exact region consisting of all renewable power outputs that can be accommodated. An adaptive constraint generation algorithm was proposed to generate the dispatchable region [12]. The interaction between different prosumers with renewable generators was considered in [13]. Similarly, dispatchable region can be applied to quantify the allowable variation of loads based on Fourier-Motzkin elimination [14]. The above studies are based on DC power flow models.

As mentioned above, AC power flow model is a must for a distribution network. Reference [15] solved nonlinear programs to get a set of boundary points that each make a different safety limit binding, and then built a dispatchable region heuristically as the convex hull of those boundary points. Linearized models were used to approximate the real dispatchable region under AC power flow [16], [17]. However, there is no guarantee that all scenarios inside the obtained region are feasible. Reference [18] used the intersection of the dispatchable regions generated from two linearized models to output a more accurate approximation. To guarantee feasibility, certified inner approximations of dispatchable regions were solved from convex programs based on a tightened-relaxed second-order cone approximation [19] or refined linear approximations [20], [21] to AC power flow. However, such estimation typically only works for a specific objective function that merely explores the dispatchable region towards a single direction or with a specific shape of the renewable generation vector. Reference [22] offered a convex inner approximation of a feeder’s aggregate DER hosting capacity in a computationally efficient way. All the aforementioned regions are convex, while the actual dispatchable region can be nonconvex due to the AC power flow constraints.

In this paper, we propose an alternative method to complement the literature above. Our main contributions are two-fold:

1) **Accurate Dispatchable Region of the Second-Order Cone (SOC) Relaxed Model.** A nonlinear Dist-Flow model based optimization problem is developed to characterize the dispatchable region, which is hard to solve due to its nonconvexity. Therefore, we first relax the problem to a convex second-order cone program (SOCP). Then, unlike reference [18] that further linearized the SOC constraint using polyhedral approximations [23], we generate the dispatchable region directly without further linearization. To be specific, the dual problem of the SOCP is derived or Brouwer fixed-point theorem for continuous mappings over compact convex sets [6], [7]. Nonetheless, those methods cannot be readily applied to output the dispatchable region since they are based on power flow equations and can hardly deal with inequality safety constraints.

The other inaccuracy lies in the possible inexactness of the SOC relaxation. In fact, the actual dispatchable region may be nonconvex, but the algorithms developed in previous studies can only generate convex regions. Distinctly, we propose a heuristic method to find out the SOC-inexact regions by requiring the corresponding dual variables to be larger than some small positive values. Removing the SOC-inexact regions from the region generated by Algorithm 1 from the SOC relaxed model, we can build a tighter approximation of the actual dispatchable region. The proposed method provides an innovative idea for constructing an accurate dispatchable region as the difference of several convex sets. Numerical results show that the proposed method can approximate the complicated dispatchable region with a simple polytope after moderate computation, while preserving relatively good accuracy. Comparisons with other dispatchable region characterization methods (e.g.,[15],[22]) show that the proposed method can reach a satisfactory balance between ensuring safety and reducing conservatism.

The rest of this paper is organized as follows. Section II introduces the power network model we use. Section III defines the dispatchable region and the optimization problem to characterize it. Section IV elaborates our method to approximate the dispatchable region. Section V reports numerical experiments, and Section VI concludes the paper.

### II. Power Network Model

Consider the single-phase equivalent model of a distribution network, which is a radial graph with a set $N$ of nodes and a set $L$ of lines. Index the nodes as $N = \{0, 1, \ldots, N\}$, where 0 represents the root node (slack bus). For convenience, we treat the lines as directed; for example, if a line connects nodes $i$, $j \in N$, where node $i$ is closer to the root than node $j$, then the line directs from $i$ to $j$ and is denoted by $i \to j$. The power flow in
the network at a particular time instant can be modeled by the classic Dist-Flow equations purely in real numbers [24], [25], elaborated as follows.

At each node \( i \in \mathcal{N} \): let \( v_i \) denote the squared voltage magnitude; aggregate all the controllable power sources and loads into a complex power injection \( p_i + jq_i \); denote the uncontrollable active power generation of a renewable energy source as \( w_i \). Let \( \ell_{ij} \) denote the squared current magnitude through each line \( i \rightarrow j \). Let \( P_{ij} \) and \( Q_{ij} \) denote the net active and net reactive power, respectively, that are sent by node \( i \) onto line \( i \rightarrow j \); they are different from the net power arriving at node \( j \) due to power loss, and are negative if node \( i \) receives power from line \( i \rightarrow j \). Let \( r_{ij}, \chi_{ij} \) denote the constant resistance and reactance of line \( i \rightarrow j \), respectively. The Dist-Flow equations are:

\[
\forall i \rightarrow j: \quad P_{ij} - r_{ij} \ell_{ij} - \sum_{k:j \rightarrow k} P_{jk} + p_j + w_j = 0 \quad (1a)
\]

\[
Q_{ij} - \chi_{ij} \ell_{ij} - \sum_{k:j \rightarrow k} Q_{jk} + q_j = 0 \quad (1b)
\]

\[
v_i - v_j - 2(r_{ij}P_{ij} + \chi_{ij}Q_{ij}) + (r^2_{ij} + \chi^2_{ij}) \ell_{ij} = 0 \quad (1c)
\]

\[
P^2_{ij} + Q^2_{ij} - v_i^w \ell_{ij} = 0. \quad (1d)
\]

Suppose renewable energy sources only exist at a subset of nodes \( \mathcal{N}_w \subseteq \mathcal{N} \setminus \{0\} \), whose cardinality is \( W := |\mathcal{N}_w| \). For simplicity, we assume that the renewable generators are equipped with voltage source converters (VSCs) to maintain a fixed reactive power output under variable active power. This can be achieved by changing the internal voltage magnitude and phase angle [26]. Therefore, the reactive power of renewable sources is considered as a constant term (merged into \( q_j \)) in the model. Nonetheless, the proposed method can also deal with other scenarios, such as renewable generators with constant power factors \( \zeta_j \), by adding \( \zeta_j w_j \) to (1b). For nodes \( i \notin \mathcal{N}_w \), set constant \( w_i \equiv 0 \). The variables in Dist-Flow equations (1) are grouped as follows:

- Renewable power generation \( w := (w_i, i \in \mathcal{N}_w) \in \mathbb{R}^W \), which is treated as input to the system;
- State variables \( x := (p, q, v, \ell, P, Q) \), where each of \( p, q, v, \ell, P, Q \) is a column vector indexed by \( \{1, \ldots, N\} \).

Remark: Without loss of generality, we assume there is only one node, indexed as node 1, connected to the root node 0. In this case, the power exchange between the distribution network and the upper grid at node 0 is \( p_0 + jq_0 = P_{01} + jQ_{01} \), so that it is just considered as part of \( (P, Q) \), not \( (p, q) \). As customary, assume \( v_0 \) is a given constant and thus not in state variable \( v \).

The radial network has \( N \) lines, where each line \( i \rightarrow j \) can be uniquely indexed by its destination node \( j \), so we can index line variables \( \ell, P, Q \) by \( \{1, \ldots, N\} \).

Assume known capacity limits of controllable power:

\[
P_i \leq p_i \leq \overline{p}_i, \quad \forall i = 1, \ldots, N \quad (2a)
\]

\[
q_i \leq q_i \leq \overline{q}_i, \quad \forall i = 1, \ldots, N \quad (2b)
\]

At any node \( i \) where there are only fixed (or zero) power injections, the constant limits can be set as \( p_i = \overline{p}_i \) (\( = 0 \)) and/or \( q_i = \overline{q}_i \) (\( = 0 \)). In addition, power system operations require the following safety limits to be satisfied:

\[
v_i \leq v_i \leq \overline{v}_i, \quad \forall i = 1, \ldots, N \quad (3a)
\]

\[
0 \leq \ell_{ij} \leq \overline{\ell}_{ij}, \quad \forall i \rightarrow j \quad (3b)
\]

where the voltage limits \( v_i, \overline{v}_i \) for all nodes \( i \) and the current limits \( \overline{\ell}_{ij} \) for all lines \( i \rightarrow j \) are given as positive constants.

With the model above, we next define and analyze the dispatchable region of renewable power generation.

### III. Dispatchable Region and Relaxation

One of the operational goals of today’s power systems is to accommodate as much renewable generation as possible without sacrificing system security. Traditionally, we curtail surplus renewable power when the system runs out of flexibility. However, the curtailment actions are taken only after the system has already experienced certain security violations. Thus, the system may still suffer from instability or oscillations as the curtailments are done in an ex-post/correctional manner. To tackle this challenge, an ex-ante/preventive guidance is needed to let the operator know how much renewable generation the system can afford before actual operation. The dispatchable region provides such a guidance. The dispatchable region is the region of renewable power generation \( w \), for which there is a feasible dispatch. It will be utilized for the hourly adequacy/security assessment of an electric grid [27]. Moreover, since the dispatchable region evaluates the system’s ability to accommodate fluctuating renewable generations, so the region characterizes the range of active power only, same as in references [9], [12], [17]. With the dispatchable region, the operator can provide a dispatch guideline for renewable generators, forcing them to produce no more than the set limits [8]; otherwise, they will be penalized. The formal definition of dispatchable region is provided below.

**Definition 1 (Dispatchable region):** A vector of renewable power generation \( w \in \mathbb{R}^W \) has a feasible dispatch if there exists \( x = (p, q, v, \ell, P, Q) \in \mathbb{R}^6N \) such that \( (w, x) \) satisfies power flow equations (1), capacity limits (2), and safety limits (3). The dispatchable region of renewable power generation is defined as:

\[
\mathcal{W} := \{ w \in \mathbb{R}^W \mid \text{w has a feasible dispatch} \}.
\]

Other work proposed concepts related to, but different from, the dispatchable region in Definition 1. For instance, reference [28] derived a convex restriction for the region of robust dispatch strategies, under each there exist feasible AC power flow solutions for all the possible realizations of uncertain renewable generations. Our paper aims to characterize a region of uncertainty renewable generations, under each there exists a feasible AC power flow solution that can be implemented by at least one dispatch strategy. The region characterized by [28] is thus useful for robust dispatch, while the dispatchable region in this paper serves a different purpose—the adequacy/security assessment of a system’s hosting capacity of renewables. For this purpose, the more accurate the obtained region, the better.
Considering that the actual dispatchable region could be nonconvex, the proposed algorithms may be a better approximation since they can generate nonconvex regions.

For conciseness, we rewrite the linear part (1a)–(1c) of DistFlow equations as \( A_f x + B_f w + \gamma_f = 0 \) and affine inequalities (2)–(3) as \( A_x + \gamma_s \leq 0 \), where both equality and inequality are element-wise, and constant matrices and vectors \( A_f, B_f, \gamma_f, A_x, \gamma_s \) are provided in Appendix A. Given any \( w \), we introduce the following optimization to check its feasibility.

\[
\begin{align*}
\text{FP}(w) & : \min 1^T z \\
\text{over } x \in P & : (p, q, v, \ell, P, Q) \\
\text{s.t. } & A_f x + B_f w + \gamma_f = 0 \\
& A_x + \gamma_s \leq z_s \quad (4b) \\
& P_{ij}^2 + Q_{ij}^2 - v_i \ell_{ij} \leq z_{q,ij}, \quad \forall i \rightarrow j \quad (4d) \\
& v_i \ell_{ij} - (P_{ij}^2 + Q_{ij}^2) \leq z_{q,ij}, \quad \forall i \rightarrow j \quad (4e)
\end{align*}
\]

where \( 1^T \) in objective (4a) is a row vector of all ones. Any element of the slack variable \( z \) can increase as needed to satisfy the corresponding inequality constraint, but only \( z = 0 \) can guarantee feasibility in terms of (1)–(3). Therefore, denoting the minimum objective value of \( \text{FP}(w) \) as \( \text{fp}(w) \), the dispatchable region in Definition 1 is equivalently:

\[
\mathcal{W} = \{ w \in \mathbb{R}^W | \text{fp}(w) = 0 \}.
\]

Due to the nonconvex quadratic inequality constraint (4e), problem \( \text{FP}(w) \) is nonconvex and thus hard to analyze. By removing (4e) and rewriting (4d), we relax \( \text{FP}(w) \) to a convex second order cone program (SOCP):

\[
\begin{align*}
\text{FP}'(w) & : \min 1^T z \\
\text{over } x, y, z & : (s, z_q, \tilde{z}_q) \geq 0 \\
\text{s.t. } & (4b)–(4c) \\
& y = A_y x + b_y \quad (5b) \\
& \| y_{ij} \| \leq c_{q,ij} x + \gamma_{q,ij} + z_{q,ij}, \quad \forall i \rightarrow j \quad (5c)
\end{align*}
\]

where \( y \in \mathbb{R}^{3N}, A_y \in \mathbb{R}^{(3N) \times (6N)}, \) and \( b_y \in \mathbb{R}^{3N \times 6N} \) vertically stack \( y_{ij} \in \mathbb{R}^3, A_{y,ij} \in \mathbb{R}^{3 \times (6N)}, \) and \( b_{y,ij} \) respectively for all lines \( i \rightarrow j \). Row vector \( c_{q,ij} \in \mathbb{R}^{1 \times (6N)} \) and scalar number \( \gamma_{q,ij} \) are also stacked vertically for all \( i \rightarrow j \) as \( c_q \in \mathbb{R}^{N \times (6N)} \) and \( \gamma_q \in \mathbb{R}^{N} \). The constant matrices and vectors \( A_y, b_y, c_q, \gamma_q \) are provided in Appendix A, which make:

\[
\begin{align*}
A_y x + b_y & = [2P_{ij}, 2Q_{ij}, v_i - \ell_{ij}]^\top, \quad \forall i \rightarrow j \\
c_{q,ij} x + \gamma_{q,ij} & = v_i + \ell_{ij}, \quad \forall i \rightarrow j
\end{align*}
\]

and thus make (5b)–(5c) equivalent to (4d).\(^\dagger\)

Problem \( \text{FP}'(w) \) facilitates the definition of an SOCP-relaxed dispatchable region:

\[
\mathcal{W}' := \{ w \in \mathbb{R}^W | \text{fp}'(w) = 0 \}
\]

\(^\dagger\)Given \( z \), the values of \( z_q \) in (4d) and (5c) are generally not equal, but we do not differentiate notation due to their identical role as slack variables.

where \( \text{fp}'(w) \) is the minimum objective value of \( \text{FP}'(w) \). It is obvious that \( \mathcal{W} \subseteq \mathcal{W}' \), i.e., \( \mathcal{W}' \) is a relaxation of \( \mathcal{W} \).

A common practice to further simplify the dispatchable-region characterization is to outer approximate the second-order cone (5c) with a polytopic cone, which can achieve arbitrary precision by constructing sufficiently many planes tangent to the surface of the second-order cone \( [23, 29] \). Consequently, \( \text{FP}'(w) \) is relaxed to a linear program, and then the algorithm in [9], [12], [13] can be employed to get a convex polytopic outer approximation of \( \mathcal{W}' \). In this work, we propose an alternative method that does not rely on such linearization. Instead, we work directly on the SOCP \( \text{FP}'(w) \) and its dual problem to preserve the intrinsic nonlinearity of the AC power flow model and hence the accuracy of our characterization.

IV. POLYTOPIC APPROXIMATION ALGORITHMS

The dispatchable region is determined prior to the actual operation. Since the realization of uncertain renewable generations is still unknown, we cannot solve the DistFlow-based ACOPF problem directly. Some research proposed to use Monte Carlo simulations to estimate the dispatchable region through generating random renewable power output samples and testing feasibility of the AC power flow problem under those samples. However, a vast number of samples will be required to obtain an assessment with high accuracy, which may be time-consuming. Moreover, we may only get part of the actual dispatchable region if the sampling area is not properly chosen. The following proposed algorithms can overcome the above limitations and return an accurate estimation of the full dispatchable region with moderate computation.

To offer a closed-form approximation of dispatchable region \( \mathcal{W} \), we first develop a convex polytopic approximation of its relaxation \( \mathcal{W}' \) via the dual problem of SOCP \( \text{FP}'(w) \). We then develop a heuristic method to approximately remove the renewable generations that make the SOCP relaxation inexact, resulting in a tighter approximation of \( \mathcal{W} \).

A. Dual SOCP

Let \( \mu := (\mu_f, \mu_y) \) denote the dual variables for the equality constraints in problem \( \text{FP}'(w) \), with \( \mu_f \in \mathbb{R}^{3N} \) for (4b) and \( \mu_y \in \mathbb{R}^{3N} \) for (5b) vertically stacking \( \mu_{y,ij} \in \mathbb{R}^3, \forall i \rightarrow j \). Let \( \lambda := (\lambda_s, \lambda_q) \) denote the dual variables for the inequality constraints, with \( \lambda_s \in \mathbb{R}^{3N} \) for (4c) and \( \lambda_q = (\lambda_{q,ij}, \forall i \rightarrow j) \in \mathbb{R}^N \) for (5c). Then the Lagrangian of \( \text{FP}'(w) \) is:

\[
L_u = 1^T z + \mu_f^\top (A_f x + B_f w + \gamma_f) + \mu_y^\top (y - A_y x - b_y) + \sum_{i \rightarrow j} \lambda_{s,ij} (||y_{ij}||^2 - c_{q,ij} x - \gamma_{q,ij} - z_{q,ij})
\]

\[
= z^T (1 - \lambda) + \sum_{i \rightarrow j} (y_{ij} \mu_{y,ij} + ||y_{ij}||^2 \lambda_{q,ij})
\]

\[
+ x^T (A_f^\top \mu_f + A_y^\top \lambda_s - A_y^\top \mu_y - c_f^\top \lambda_q)
\]

\[
+ \mu_f^\top (B_f w + \gamma_f) + \lambda_s^\top \gamma_s - \mu_y^\top b_y - \lambda_q^\top \gamma_q.
\]

(6)
Through $\min_{z \geq 0, x, y} L_w(x, y, z; \mu, \lambda)$ we can get the dual objective function. By (6), $L_w$ can only attain a finite minimum over $(z \geq 0, x, y)$ when the dual variables satisfy:

\begin{align}
0 \leq \lambda & \leq 1 \quad (7a) \\
A_f^T \mu_f + A_q^T \lambda_q &= A_p^T \mu_y + c^T \lambda_q & (7b) \\
\|\mu_{y,ij}\| &\leq \lambda_{q,ij}, \quad \forall i \rightarrow j \quad (7c)
\end{align}

Note that $\lambda \geq 0$ in (7a) is a general requirement for all the dual variables associated with inequality constraints, and (7c) must hold by noticing

\[ y_{ij}^T \mu_{y,ij} + \|y_{ij}\| \lambda_{q,ij} \geq (\lambda_{q,ij} - \|\mu_{y,ij}\|) \|y_{ij}\|. \]

When (7) is satisfied, all the terms containing $(x, y, z)$ in (6) attain their minimum value zero, and hence we obtain the dual problem for $FP''(w)$, which is also an SOCP:

\[
DP''(w) : \max_{\mu, \lambda} \mu_f^T (B_f w + \gamma_f) + \lambda_q^T \gamma_q - \mu_y^T b_y - \lambda_q^T q_q \\
\text{s.t.} \quad (7).
\]

Let $D_w(\mu, \lambda)$ denote the objective function and $dp'(w)$ denote the maximum objective value of $DP''(w)$. The following result lays the foundation for approximating the SOCP-relaxed dispatchable region $\mathcal{W}$ via the dual SOCP $DP''(w)$.

**Proposition 1:** For all $w \in \mathbb{R}^w$, strong duality holds between $FP''(w)$ and $DP''(w)$, i.e., their optimal values $fp'(w)$ and $dp'(w)$.  

**Proof:** Consider an arbitrary $w \in \mathbb{R}^w$. Since problem $FP''(w)$ is convex, it is sufficient to prove Slater’s condition [30, Section 5.2.3], i.e., existence of $(z \geq 0, x, y)$ that satisfies affine constraints (4b), (4c), and (5b), and strictly satisfies (5c).

Indeed, it is adequate to find a point $x = (p, q, v, \ell, P, Q)$ to satisfy (4b), i.e., $B_f w + \gamma_f \geq 0$; then one can explicitly determine $y$ by (5b) and always find large enough $z$ to make (4c) and (5c) (strictly) feasible, satisfying the condition. A point $x$ can be easily found as follows: set $p = q = \ell = 0 \in \mathbb{R}^N$; determine $(P, Q)$ backward from the leaves to the root of the radial network, using (1a)–(1b); then determine $v$ forward from the root to the leaves, using (1c). This completes the proof.  

By Proposition 1, the relaxed region $W$ is equivalently:

\[
\mathcal{W} = \{ w \in \mathbb{R}^w | dp''(w) = 0 \} = \{ w \in \mathbb{R}^w | D_w(\mu, \lambda) \leq 0, \forall (\mu, \lambda) \text{ satisfying (7)} \} \quad (8)
\]

where the second equality holds because $D_w(\mu, \lambda) = 0$ can always be attained at the dual feasible point $(\mu, \lambda) = 0$.

**Proposition 2:** $\mathcal{W}$ is a convex set.  

**Proof:** Consider arbitrary $w_1, w_2 \in \mathcal{W}$ and $t \in [0, 1]$. Denote $w_t := tw_1 + (1-t)w_2$. Then for every $(\mu, \lambda)$ satisfying (7), we have:

\[
D_w(\mu, \lambda) = tD_w(\mu, \lambda) + (1-t)D_w(\mu, \lambda) \\
\leq t \cdot 0 + (1-t) \cdot 0 = 0
\]

where the first equality is due to linearity of $D_w(\mu, \lambda)$ with respect to $w$ when $(\mu, \lambda)$ is fixed, and the inequality holds because $w_1, w_2 \in \mathcal{W}$. Therefore $w_t \in \mathcal{W}$. By the definition of a convex set, $\mathcal{W}$ is convex.

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**Algorithm 1. Approximate $\mathcal{W}'$.**

1. **Initialization:** $\mathcal{W}_\text{poly} = \{ w \in \mathbb{R}^w | w \leq w \leq \bar{w} \}$ for sufficiently low $u$ and high $\bar{w}$; $\mathcal{V}_\text{safe} = \emptyset$; $c = 0$.

2. **Update vertex set** $vert(\mathcal{W}_\text{poly})$. Let $dp'_\text{max} = 0$.

   for $w \in vert(\mathcal{W}_\text{poly})$ and $w \notin \mathcal{V}_\text{safe}$ do
   
   solve $DP''(w)$ to obtain an optimal solution $(\mu*, \lambda*)$ and maximum objective value $dp''(w)$;
   
   if $dp''(w) > dp'_\text{max}$ then
   
   $dp'_\text{max} \leftarrow dp''(w)$;
   
   $(\mu_{max}, \lambda_{max}) \leftarrow (\mu*, \lambda*)$;
   
   else if $dp''(w) \leq 0$ then $\mathcal{V}_\text{safe} = \mathcal{V}_\text{safe} \cup \{ w \}$;
   
   end
   
   end

3. **Return** $\mathcal{W}'$.

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**B. Approximating SOCP-Relaxed Dispatchable Region**

We propose Algorithm 1 to approximate $\mathcal{W}$ defined in (8). It starts with an initial region $\mathcal{W}_\text{poly}$, that is large enough to contain $\mathcal{W}$. Then it solves the dual SOCP $DP''(w)$ for every vertex $w$ of polytope $\mathcal{W}_\text{poly}$, records the vertex that most severely violates the condition in (8), and adds a corresponding cutting plane to remove that vertex from $\mathcal{W}_\text{poly}$. Meanwhile, all the vertices that satisfy the condition in (8) are added to $\mathcal{V}_\text{safe}$ and never checked again. As long as the initial set can cover the actual region, Algorithm 1 will return a unique SOCP-relaxed region. Algorithm 2 (introduced later) is performed on the SOCP-relaxed region output by Algorithm 1, which is independent of the initial set. Therefore, the shape of the initial set will not influence the final dispatchable region obtained.

**Proposition 3:** The output $\mathcal{W}'$ in an arbitrary iteration of Algorithm 1 is an outer approximation of $\mathcal{W}''$.

**Proof:** Note the initial $\mathcal{W}_\text{poly}$ contains $\mathcal{W}$. We next prove that any cutting plane added to $\mathcal{W}_\text{poly}$ would not remove any point in $\mathcal{W}'$. To show that, consider an arbitrary $w$ removed by a cutting plane whose coefficients are $(\mu_{max}, \lambda_{max})$. Then there must be $D_w(\mu_{max}, \lambda_{max}) > 0$. Since $(\mu_{max}, \lambda_{max})$ is dual feasible satisfying (7), we have $w \notin \mathcal{W}_\text{poly}$ by (8).

Unlike [9], [13] that based on linear programs, the SOCP-relaxed dispatchable region $\mathcal{W}$ may not be the intersection of a finite number of cutting planes (i.e., a convex polytope). Therefore, Algorithm 1 may not guarantee $dp''(w) = 0$ for all vertices $w \in vert(\mathcal{W}_\text{poly})$ in a finite number of iterations. However, if it does so, as what always happens in our numerical experiments, it will produce a nice result as follows.

**Proposition 4:** If Algorithm 1 terminates with $dp'_\text{max} = 0$ in a finite number of iterations, it returns the accurate SOCP-relaxed dispatchable region, i.e., $\mathcal{W}_\text{poly} = \mathcal{W}'$. 

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Proof: Proposition 3 has shown \( \mathcal{W}' \subseteq \mathcal{W}_{poly}' \). If Algorithm 1 terminates with \( d_{\text{max}}' = 0 \) after adding a finite number of cutting planes, then it returns a convex polytope \( \mathcal{W}_{poly}' \). Moreover, all the vertices \( w \in \text{ver}(\mathcal{W}_{poly}') \) satisfy \( d'(w) = 0 \), therefore, \( w \in \mathcal{W}' \) by (8). This fact, together with the convexity of \( \mathcal{W}' \) shown in Proposition 2, implies \( \mathcal{W}_{poly}' \subseteq \mathcal{W}' \). Thus we have proved \( \mathcal{W}_{poly}' = \mathcal{W}' \).

An immediate corollary of Proposition 4 is that if \( \mathcal{W}' \) is not a polytope, then Algorithm 1 cannot terminate in a finite number of iterations with \( d_{\text{max}}' = 0 \). If that happens, one can terminate Algorithm 1 when reaching the maximum number of iterations \( C_{\text{max}}' \), to obtain a convex polytopic outer approximation of \( \mathcal{W}' \). In this sense, the outcome of Algorithm 1 serves as a posterior indicator of the structure of \( \mathcal{W}' \). As the obtained region will get closer to the actual SOCP-relaxed region through the iterations, the approximation error could be very small if the maximum number of iterations \( C_{\text{max}}' \) is set large enough.

The results in Propositions 3 and 4 in turn justify our choice of the Dist-Flow model (1) for the network. Compared to linear approximate models, such as [31, 32] and the first-order Taylor approximation in [33], our choice retains the full nonlinearity of AC power flow to make the result more accurate. Another advantage of our modeling choice lies in its convenience for implementation. The circular complex solution needed by the Wirtinger calculus for the linear approximation in [33] and the additional term related to the soft power factors to deal with the nonconvex limits in [32] both require augmenting the objective (4a), which would cause the feasibility-checking problem (4) to fail as it can no longer tell whether the sum of the slack variables is zero. The linear approximation in [31] required two reference operating points, which are hard to pick a priori because they depend on the unknown dispatchable region, which itself is what we are looking for.

C. Removing SOCP-Inexact Renewable Generations

Remember our goal is to characterize the dispatchable region \( \mathcal{W} \) that can ensure solvability, which refers to the satisfaction of AC power flow equations (1), including the exactness as SOC relaxation is applied. So far, the \( \mathcal{W}' \) studied is just a SOCP-relaxation of \( \mathcal{W} \). To overcome this drawback, we design a heuristic (Algorithm 2) to approximately remove the SOCP-inexact region \( \tilde{\mathcal{W}} := \mathcal{W}' \setminus \mathcal{W} \) from \( \mathcal{W}' \), so that we can get an approximation of the SOCP-exact region that meets both safety limit and solvability. The renewable generations \( w \in \mathcal{W}' \) are feasible in terms of the SOCP relaxation \( \text{FP}'(w) \) but infeasible in terms of \( \text{FP}(w) \), as formally defined below.

**Definition 2 (SOCP-inexact region):** A vector of renewable power generation \( w \in \mathcal{W}' \) is SOCP-inexact, if every optimal solution of \( \text{FP}'(w) \) satisfies:

\[
\|y_{ij}\|_2 < c_{q,ij}x + \gamma_{q,ij} \quad \text{for some } i \to j.
\]

The **SOCP-inexact region** of \( w \) is defined as:

\[
\tilde{\mathcal{W}} = \{ w \in \mathcal{W}' \mid w \text{ is SOCP-inexact} \}.
\]

Fig. 1. Relationship between \( \mathcal{W} \) and \( \tilde{\mathcal{W}} \).

Our next focus is to build an approximation of \( \tilde{\mathcal{W}} \). For that, we consider the following set defined on the dual SOCP:

\[
\tilde{\mathcal{W}}_d := \{ w \in \mathcal{W}' \mid \text{Every optimal solution of } \text{DP}'(w) \text{ satisfies } \lambda_{q,ij} = 0 \text{ for some } i \to j \}.
\]

By complementary slackness [30, Section 5.5.2], for every primal-dual optimal of \( \text{FP}'(w) \) and \( \text{DP}'(w) \), there is:

\[
\lambda_{q,ij} (\|y_{ij}\|_2 - c_{q,ij}x - \gamma_{q,ij}) = 0, \quad \forall i \to j.
\]

The relationship between \( \mathcal{W} \) and \( \tilde{\mathcal{W}}_d \) is shown in Fig. 1. For a scenario \( w \in \mathcal{W} \), according to the complementary slackness condition (9), we have \( \lambda_{q,ij} = 0 \) for some \( i \to j \). Hence, \( w \in \tilde{\mathcal{W}}_d \), which implies \( \mathcal{W} \subseteq \tilde{\mathcal{W}}_d \). The reverse is not true: if \( w \in \tilde{\mathcal{W}}_d \), \( w \) may or may not be inside \( \mathcal{W} \). The “may not” happens when for those \( i \to j \) with \( \lambda_{q,ij} \neq 0 \), we have (the rare cases):

\[
\lambda_{q,ij} (\|y_{ij}\|_2 - c_{q,ij}x - \gamma_{q,ij}) = 0.
\]

Such \( w \) is an SOCP-exact case, which is in \( \mathcal{W}_d \) but not \( \mathcal{W} \). In other words, \( \mathcal{W} \) is the set of all SOCP-inexact cases, while \( \tilde{\mathcal{W}}_d \) consists of \( \mathcal{W} \) and a few SOCP-exact cases.

Hence we focus on \( \tilde{\mathcal{W}}_d \) as an approximation of \( \mathcal{W} \).

Given an arbitrary \( w \in \mathcal{W}_d \subseteq \mathcal{W}' \), the maximum objective value of \( \text{DP}'(w) \) is \( \text{dp}'(w) = 0 \) but with some \( \lambda_{q,ij} = 0 \) so the SOC relaxation is inexact (except for some very rare case). To approximate \( \tilde{\mathcal{W}}_d \), first we add the following constraint to tighten the dual feasible set (7):

\[
\lambda_q \geq \delta
\]

where the inequality is element-wise and \( \delta \in \mathbb{R}^{2N} \) is a vector of all strictly positive parameters, whose design will be elaborated later. Consider the tightened dual SOCP:

\[
\text{DP}''(w, \delta) : \max_{\mu, \lambda} \mu^T(B_f w + \gamma_f) + \lambda^T \gamma_s - \mu^T b_y - \lambda^T \gamma_q
\]

s.t. (7), (9)

and let \( \text{dp}''(w, \delta) \) denote its maximum objective value. For \( w \in \tilde{\mathcal{W}}_d \), there must be \( \text{dp}''(w, \delta) < 0 \), because otherwise \( \text{DP}'(w) \) would have an optimal solution that satisfies (9), contradicting the definition of \( \tilde{\mathcal{W}}_d \). Actually \( \text{dp}''(w, \delta) \leq -\eta \) for some \( \eta > 0 \) that depends on \( w \) and \( \delta \).

The idea above inspires us to approximate \( \tilde{\mathcal{W}}_d \) (or \( \mathcal{W} \)) by

\[
\tilde{\mathcal{W}}_d \approx \{ w \in \mathbb{R}^w \mid D_w(\mu, \lambda) \leq -\eta, V(\mu, \lambda) \text{ satisfying (7),(9)} \}
\]

To this end, Algorithm 2 can be designed using a similar procedure to Algorithm 1. Algorithm 2 returns a convex polytope \( \tilde{\mathcal{W}}_{poly} \subseteq \tilde{\mathcal{W}}'_d \) that guarantees \( \text{dp}''(w, \delta) \leq -\eta < 0 \) for all \( w \in \tilde{\mathcal{W}}_{poly} \), which is an approximation of \( \tilde{\mathcal{W}}_d \) (or \( \mathcal{W} \)).
Algorithm 2. Approximate $\mathcal{W}_d$ (or SOCP-inexact $\mathcal{W}$).

1. **Initialization:** $\mathcal{W}_\text{poly} = \mathcal{W}_\text{poly}^d$ returned by Alg. 1.
   Given positive $\delta$, $\eta$, $\eta'$; $\mathcal{V}_\text{safe} = \emptyset$; $c = 0$;
2. Update vertex set $\text{vert}(\mathcal{W}_\text{poly})$. Let $d\mathbf{\pi}_\text{max} = -\eta$;
   for $w \in \text{vert}(\mathcal{W}_\text{poly})$ and $w \notin \mathcal{V}_\text{safe}$ do
     solve DP$^{\text{\prime}}(w, \delta)$ to obtain an optimal solution
     $(\mu^*, \lambda^*)$ and maximum objective value $d\mathbf{\pi}(w, \delta)$;
     if $d\mathbf{\pi}(w, \delta) > d\mathbf{\pi}_\text{max}$ then
       $d\mathbf{\pi}_\text{max} \leftarrow d\mathbf{\pi}(w, \delta)$;
       $(\mu^\text{\prime}, \lambda^\text{\prime}) \leftarrow (\mu^*, \lambda^*)$;
     else if $d\mathbf{\pi}(w, \delta) \leq -\eta$ then
       $\mathcal{V}_\text{safe} = \mathcal{V}_\text{safe} \cup \{w\}$;
   end
   if $d\mathbf{\pi}_\text{max} = -\eta$ or $c = C_{\text{max}}$ then
     return $\mathcal{W}_\text{poly}$.
   else
     add to $\mathcal{W}_\text{poly}$ a cutting plane:
     $\mu^\prime_{\text{min}}(B w + \gamma_f) + \lambda^\prime_{\text{min}} \gamma_s \leq 0$;
     $\mu^\prime_{\text{max}} b g + \lambda^\prime_{\text{max}} q - \eta'$;
     $c \leftarrow c + 1$;
     go back to Line 2;
   end

Removing $\mathcal{W}_\text{poly}$ from $\mathcal{W}_\text{poly}^d$, we can obtain an approximation $\mathcal{W}_\text{poly} = \mathcal{W}_\text{poly}^d \setminus \mathcal{W}_\text{poly}$ of the actual dispatchable region $\mathcal{W}$. To make Algorithm 2 more robust, we may choose $\eta' > \eta$ for the added cutting plane in each iteration.

**Remark:** The parameters $\delta$ and $\eta$ are essential for Algorithm 2. A general guideline is that (1) given $\delta$, choosing a smaller $\eta$ and (2) given $\eta$, choosing a bigger $\delta$ will both make $\mathcal{W}_\text{poly}$ bigger and lead to a smaller (more conservative) approximation of $\mathcal{W} = \mathcal{V}_\text{poly} \setminus \mathcal{W}$. Moreover, sometime it is difficult for Algorithm 2 to use a single convex polytope $\mathcal{W}_\text{poly}$ to accurately approximate the most likely nonconvex $\mathcal{W}$. To deal with this difficulty, we propose to run Algorithm 2 multiple times with different vectors $\delta$. As a result, we obtain multiple convex polytopes whose union serves as a better approximation of $\mathcal{V}$. Those vectors $\delta$ can be selected in the following way. We traverse the vertices of $\mathcal{W}_\text{poly}$, select one vertex $w$, and solve the dual SOCP $\text{DP}(w)$ to get an optimal solution $(\mu^*, \lambda^*)$. Then $\delta$ is constructed by keeping all the strictly positive elements of $\lambda^*$ as they are, and add a small positive perturbation to all the zero elements.

**D. Some Discussions**

1) **About the computational complexity.** Computing the dispatchable region is an NP-hard problem [12], so a theoretical analysis of the computational complexity is difficult. Most existing work [9], [12], [17] did not provide such an analysis even on a linear approximation model. Furthermore, the proposed algorithms are based on the dual SOCP to better preserve the nonlinear feature of the original problem than those linear approximations. The more sophisticated nature of an SOCP further complicates the analysis. In this paper, though without a computational complexity analysis, the scalability and efficiency of the proposed algorithms have been illustrated by the numerical studies in Section V.

2) **About the approximation error.** The characterization of dispatchable region involves two steps:

- **Step 1:** Apply Algorithm 1 to approximate the SOCP-relaxed dispatchable region. For this step, first, deriving the dual SOCP introduces no error because strong duality holds. Second, we have proved in Proposition 3 that if Algorithm 1 terminates with $d\mathbf{\pi}_\text{max} = 0$ in a finite number of iterations, it returns the accurate SOCP-relaxed dispatchable region. Numerical results show that it typically takes a small number of iterations for the approximation error of this step to diminish to zero.

- **Step 2:** Apply Algorithm 2 to remove the SOCP-inexact region. It may introduce some errors that are hard to characterize theoretically because of the heuristic nature of Step 2. However, it is worth mentioning that most previous work only provides an approximation for the SOCP-relaxed region (i.e., a result comparable to Step 1 only), while Step 2 in this paper further removes some SOCP-inexact regions. Therefore, the proposed method is expected to provide a better approximation of the actual dispatchable region even without formal analysis of the errors in Step 2. 3) **About the possible extension.** The proposed method can be extended to characterize the flexibility of active and reactive power adjustment at the any node (say node 1). In that case, we assume that the variation ranges of renewable generators are given and project the feasible region of (1)–(3) to the $(p_1, q_1)$ space. To be specific, the constraint (2) for $(p_1, q_1)$ is removed from (4) and the renewable range $w$ is added. Then the flexibility at bus 1 is defined as

$$\{(p_1, q_1) \in \mathbb{R}^2 \mid d\mathbf{\pi}(p_1, q_1) = 0\}$$

(10)

Algorithms 1-2 can be applied with minor adaptations to approximate this region.

**Summary.** The proposed dispatchable region contains all the renewable output scenarios $w \in \mathbb{R}^W$ that fulfill solvability and safety limits. Whether a scenario $w$ satisfies the two conditions or not is deterministic. Therefore, the proposed dispatchable region is a deterministic set. Our algorithms return a single set-valued approximation to this set itself, rather than finding a number of possible subsets in this set as in [22]. The relationship of different regions mentioned in this paper is summarized in Table I. As discussed, the dispatchable region $\mathcal{W} = \mathcal{W}_\text{poly} \setminus \mathcal{W}$, where $\mathcal{W}$ is the SOCP-relaxed dispatchable region and $\mathcal{W}$ is the SOCP-inexact region. We develop Algorithm 1 to get $\mathcal{W}_\text{poly}$, a convex polytopic
approximation of \( \mathcal{W}' \); and Algorithm 2 to get \( \mathcal{W}'_{\text{poly}} \), a convex polytopic approximation of \( \mathcal{W} \). Algorithm 2 can run multiple times to obtain a more accurate approximation of nonconvex \( \mathcal{W}_d \) (or \( \mathcal{W} \)). The outputs of multiple runs of Algorithm 2 are then removed from \( \mathcal{W}'_{\text{poly}} \) to obtain a generally nonconvex polytopic approximation of \( \mathcal{W} \). The final dispatchable region obtained is the difference between \( \mathcal{W}'_{\text{poly}} \) and \( \mathcal{W}'_{\text{poly}} \), which can be nonconvex. Therefore, the proposed method can return a dispatchable region with “holes” within it. The obtained nonconvex region can be decomposed into the union of several convex sets [34], as shown in Appendix B.

V. CASE STUDIES

In this section, we conduct numerical experiments on the IEEE 33-bus and 141-bus systems. The impact of several factors is tested and the proposed method is compared with other approaches to show its advantages.

A. Benchmark

The topology of the IEEE-33 bus system is in Fig. 2. There are 5 controllable generators whose parameters are given in Table II. Two renewable generators \((w_1, w_2)\) are located at nodes 13 and 29, respectively. For comparison, the actual SOCP-relaxed region \( \mathcal{W}' \) and the actual dispatchable region without relaxation \( \mathcal{W} \) are generated as in Fig. 3. This can be done by checking the feasibility of a nonlinear optimization with (1)-(3) as its constraints, over sample points \( w \) in the \((w_1, w_2)\) space using the nonlinear solver IPOPT. As we can see from Fig. 3, the actual dispatchable region \( \mathcal{W} \) can be nonconvex and the SOCP-relaxed region is not accurate enough. In the following, we apply the proposed algorithms to output a more accurate region.

First, we test the performance of Algorithm 1. We observe that the algorithm terminates with \( d_{\text{max}} = 0 \) in 25 iterations, taking about 289.84 s. The output regions \( \mathcal{W}'_{\text{poly}} \) in the 2nd, 5th, 10th, and final iterations are given in Fig. 4. The Algorithm 1 removes the nondispatchable regions iteratively (the blue region is becoming smaller), and finally returns a convex polytope \( \mathcal{W}'_{\text{poly}} \) exactly the same as the actual SOCP-relaxed region \( \mathcal{W}' \) (dashed line). This validates Proposition 4.

Even though Algorithm 1 can output the accurate SOCP-relaxed dispatchable region, as we can see in Fig. 3, there is still a gap between the actual dispatchable region \( \mathcal{W} \) and the relaxed one \( \mathcal{W}' \). If the renewable generator output \((w_1, w_2)\)
lies in the gap area, there is actually no feasible dispatch that satisfies power flow (1) and safety limits (2)-(3). Thus, using the SOCP-relaxed region as a guidance will threaten power system security. In this paper, Algorithm 2 is developed to further remove the nondispatchable points. As in Fig. 5, the \( \tilde{W}_{\text{poly}} \) (white area) generated by Algorithm 2 is removed and the resulting region \( W_{\text{poly}} \) (grey area) is closer to the actual region (red dash line). This shows the great potential of the proposed algorithm in improving the accuracy of dispatchable region in a distribution system. The operational risk under the obtained region \( W_{\text{poly}} \) and the SOCP-relaxed region \( W' \) will be compared later in Table IV.

### B. Impact of Different Factors

In the following, we test the impact of two factors (adjustable capability of controllable generators \([p_i^l, p_i^u]\), \(\forall i \) and current limit \(\ell\)) on the shape of the dispatchable region and the performance of the proposed algorithm.

To show how \([p_i^l, p_i^u]\) influences the dispatchable region, we test three cases: (1) Benchmark, which has the same setting as in Section V-A. (2) Case L, where the generators have less adjustable capability than the benchmark. (3) Case H, where the generators have more adjustable capability than the benchmark. The parameters of the Cases L and H are given in Table III. The regions returned by Algorithm 1 (\(W'_{\text{poly}}\)) and Algorithm 2 (\(W_{\text{poly}}\)) are given in Fig. 6. Subfigures (a), (b) are for Case L and subfigures (c), (d) are for Case H. The changes of \(d_{\text{max}}'\) under three cases are recorded in Fig. 7.

![Fig. 5](image-url)  
**Fig. 5.** The gray polytope \(W_{\text{poly}}\) is an approximation of \(W\) (red dash line). It is obtained by removing the output \(W_{\text{poly}}\) of Algorithm 2 (white polytope) from the output \(W'_{\text{poly}}\) of Algorithm 1 (the outside blue dash line).

![Fig. 6](image-url)  
**Fig. 6.** Left: the \(W'_{\text{poly}}\) returned by Algorithm 1 for Case L (subfigure (a)) and Case H (subfigure (c)). Right: the \(W_{\text{poly}}\) returned by Algorithm 2 for Case L (subfigure (b)) and Case H (subfigure (d)).

![Fig. 7](image-url)  
**Fig. 7.** The change of \(d_{\text{max}}'\) over iterations of Algorithm 1.

As shown in Figs. 4 and 6, Algorithm 1 can always output the accurate SOCP-relaxed dispatchable region, i.e., \(W'_{\text{poly}} = W'\). The final dispatchable regions (grey area) returned by the proposed algorithms are much closer to the actual ones compared with the SOCP-relaxed regions. In addition, as the adjustable capability of generators decreases, the system’s ability to accommodate volatile renewable power becomes weaker, and thus, the dispatchable region becomes smaller. We also find that with a weaker adjustable capability, the actual dispatchable region \(W\) is more likely to be nonconvex and to differ more from the SOCP-relaxed region. The difference between the red dash line and the blue dash line in Fig. 6(b) is more significant than that in Fig. 6(d). In the future power systems, more renewable generators are replacing the controllable generators, so the use of an SOCP-relaxed dispatchable region is not accurate enough.

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### TABLE IV

| Case      | \(\text{FR}(W')\) | \(\text{FR}(W'_{\text{poly}})\) | Reduction | Time(s) |
|-----------|-------------------|-------------------|----------|---------|
| Benchmark | 10.4%             | 4.5%              | 56.73%   | 289.84  |
| Case L    | 15.7%             | 8.7%              | 44.59%   | 291.01  |
| Case H    | 3.5%              | 2.5%              | 28.57%   | 724.60  |

As shown in Figs. 4 and 6, Algorithm 1 can always output the accurate SOCP-relaxed dispatchable region, i.e., \(W'_{\text{poly}} = W'\). The final dispatchable regions (grey area) returned by the proposed algorithms are much closer to the actual ones compared with the SOCP-relaxed regions. In addition, as the adjustable capability of generators decreases, the system’s ability to accommodate volatile renewable power becomes weaker, and thus, the dispatchable region becomes smaller. We also find that with a weaker adjustable capability, the actual dispatchable region \(W\) is more likely to be nonconvex and to differ more from the SOCP-relaxed region. The difference between the red dash line and the blue dash line in Fig. 6(b) is more significant than that in Fig. 6(d). In the future power systems, more renewable generators are replacing the controllable generators, so the use of an SOCP-relaxed dispatchable region is not accurate enough.
Therefore, the proposed Algorithms 1-2 to remove the nondispatchable points will be helpful.

In Fig. 7, the \( \delta p_{\text{max}}^{\prime} \) under all three cases decrease towards zero when Algorithm 1 terminates. The computational times are 289.84 s (Benchmark), 291.01 s (Case L), and 724.60 s (Case H), respectively, showing that our algorithm is efficient. Moreover, we randomly generate 2000 points \((w_1, w_2)\) in the SOCP-relaxed dispatchable region \(W'\) and the final obtained region \(W_{\text{poly}} = W_{\text{poly}}' \setminus \tilde{W}_{\text{poly}}\), and calculate the failure rate defined as

\[
FR(S) = \frac{\text{No. of points } w \in S \text{ that is nondispatchable}}{\text{No. of points } w \in S} \quad (11)
\]

The failure rates under three cases are summarized in Table IV. In all three cases, the proposed method can greatly reduce the failure rate, and the reduction is more than 50% under benchmark. This can help better ensure system security. Moreover, we can find that in a system with relatively small adjustable capability, the reduction is more significant.

Furthermore, we test the impact of current limit by running two other cases where we halve and double the \(\ell\), respectively. The obtained region are shown in Fig. 8. The computation times are both less than 500 s, which is acceptable. A more stringent line-flow limit results in a smaller dispatchable region and also a greater deviation between the relaxation region \(W'\) and the exact region \(W\).

### C. Comparison With Other Methods

We then compare the performance of the proposed algorithms with three other approaches based on (1) Linearized DistFlow model [24] (denoted as LinDistFlow); (2) The rectangular inner approximation in [22] (denoted as InnerApprox); and (3) Polyhedral approximation of the SOCP-relaxed model [29] (denoted as SOCP-Linear). The LinDistFlow and SOCP-Linear models lead to linear programs to replace (4), on which the adaptive constraint generation algorithm in [12] can generate approximate dispatchable regions. The comparison results are shown in Fig. 9. Theoretically, the LinDistFlow region can be an inner/outer approximation or a region that intersects the actual dispatchable region. In the benchmark case, the LinDistFlow region is very small and conservative. The SOCP-Linear region is always an outer approximation of the actual region. We can see that it is very close to the SOCP-relaxed region \(W'\). To better illustrate the results under different approaches, we calculate the failure rate (11) and the missing rate (MR) defined below.

\[
MR(S) = \frac{\text{No. of points } w \in W \text{ but } \notin S}{\text{No. of points } w \in W} \quad (12)
\]

The failure rate and missing rate under different approaches are compared in Table V. We can find that, in this simulation case, the LinDistFlow region is an inner approximation so its failure rate is zero. However, it has a very high missing rate, meaning that the region is too conservative. Similarly, the InnerApprox might be conservative: the result under our selected parameters (the shaded rectangle) misses 67.0% of the feasible points in the actual region. Even if we were able to obtain the maximum-area rectangular inner approximation (the empty rectangle) using idealized parameters, there is still a 32.6% missing rate. The SOCP-Linear region is always an outer approximation so its missing rate is zero, but its failure rate is high. Compared to them, our method can achieve a better balance between security and conservatism, with failure and missing rates both small enough for applications.

| Method               | \(FR(W_{\text{poly}}')\) | \(MR(W_{\text{poly}}')\) |
|----------------------|---------------------------|---------------------------|
| LinDistFlow          | 0%                        | 91.1%                     |
| InnerApprox          | 0%                        | 67.0%                     |
| MaxInnerApprox       | 0%                        | 32.6%                     |
| Proposed Method      | 4.5%                      | 2.7%                      |
| SOCP-Linear          | 11.8%                     | 0%                        |

The failure rate and missing rate under different approaches are compared in Table V. We can find that, in this simulation case, the LinDistFlow region is an inner approximation so its failure rate is zero. However, it has a very high missing rate, meaning that the region is too conservative. Similarly, the InnerApprox might be conservative: the result under our selected parameters (the shaded rectangle) misses 67.0% of the feasible points in the actual region. Even if we were able to obtain the maximum-area rectangular inner approximation (the empty rectangle) using idealized parameters, there is still a 32.6% missing rate. The SOCP-Linear region is always an outer approximation so its missing rate is zero, but its failure rate is high. Compared to them, our method can achieve a better balance between security and conservatism, with failure and missing rates both small enough for applications.
TABLE VI
TOTAL ITERATIONS AND COMPUTATIONAL TIME WITH DIFFERENT NUMBERS OF RENEWABLES

| No. of renewable sources | 2   | 3   | 4   | 6   |
|-------------------------|-----|-----|-----|-----|
| iterations of Algorithm 1 | 10  | 11  | 20  | 13  |
| iterations of Algorithm 2 | 3   | 4   | 4   | 3   |
| Computational time (s)   | 127.78 | 196.68 | 1297.15 | 3123.46 |

Fig. 10. Change of $dp_{\text{max}}'$ under different numbers of renewable sources.

D. Scalability

We further show the scalability of the proposed algorithms. First, we use the IEEE 33-bus system to test the impact of the number of renewable sources. The error tolerance is set as 0.3 to achieve a better balance between accuracy and efficiency. The total iteration numbers and computational times are recorded in Table VI. We can find that the number of iterations needed does not necessarily grow with additional renewable sources, but rather it is more related to the shape of the dispatchable region. The computational time increases due to a larger number of vertices $w \in \text{vert}(W_{\text{poly}})$ to be examined in each iteration. Under all four cases, the $dp_{\text{max}}'$ drops to zero quickly, as shown in Fig. 10, which validates scalability of the proposed algorithm. Since the proposed method will be utilized for the hourly adequacy/security assessment of an electric grid, the computational time is acceptable.

Furthermore, a larger 141-bus system is tested to show the scalability of the proposed algorithms. The topology of the system is given in Fig. 11. Algorithm 1 takes 337.51 seconds to converge while Algorithm 2 takes 38.63 seconds. In this case, Algorithm 2 outputs an empty set, meaning that there is no SOCP-inexact region to be removed. The change of $dp_{\text{max}}'$, the final obtained region, and the actual region are shown in Fig. 12. We can observe that the obtained dispatchable region is very close to the actual region.

VI. CONCLUSION

In this paper, we develop an improved approximation of the renewable generation dispatchable region in radial distribution networks. First, a nonconvex optimization problem is formulated to describe the dispatchable region. The nonconvex problem is then relaxed to a convex SOCP. An SOCP-based projection algorithm (Algorithm 1) is proposed to generate the accurate SOCP-relaxed dispatchable region under certain conditions. In addition, a heuristic method (Algorithm 2) is developed to remove the SOCP-inexact region from the region obtained above. Therefore, the final region can better approximate the actual nonconvex dispatchable region. Our main findings are:

- The proposed method can reduce the operational risk (quantified by failure rate) by more than 50% compared with the SOCP-relaxed region.
- The proposed method has a greater potential in the future power system with fewer controllable units and thus weaker adjustable capability.
- Compared with existing approaches (LinDistFlow [24], InnerApprox [22], and SOCP-Linear [29]), the proposed method achieves a better tradeoff between security and conservatism.

This paper provides an innovative perspective for constructing the dispatchable region: While the existing literature can only generate convex regions, the proposed algorithm can generate nonconvex approximations. For future work, we aim to improve the accuracy of the proposed algorithms by properly setting the initial points for heuristic searching.

APPENDIX A

CONSTANT PARAMETERS

This appendix provides in full detail the constant matrices, vectors, and numbers used in Section IV.

A. Equation (4): $A_f, B_f, \gamma_f, A_s, \gamma_s$

The vector $x = (p, q, v, \ell, P, Q)$ is arranged in the order explained in Section II. Let $C \in \{-1, 0, 1\}^{(N+1) \times N}$ be the
incidence matrix of the radial network, with its element at the $k$-th row, $j$-th column:

$$C_{kj} = \begin{cases} 1, & \text{if } k = i \text{ for line } i \rightarrow j \\ -1, & \text{if } k = j \text{ for line } i \rightarrow j \\ 0, & \text{otherwise.} \end{cases}$$

Removing the first row of $C$, we get the reduced incidence matrix $\bar{C} \in \{-1, 0, 1\}^{N \times N}$. Define diagonal matrices $R := \text{diag}(r_{ij}, \forall i \rightarrow j)$ and $X := \text{diag}(x_{ij}, \forall i \rightarrow j)$. Denote the $N \times N$ all-zero matrix as $O_{N}$, identity matrix as $I_{N}$, and $N$-dimensional all-zero column vector as $v_{0}$. We have:

$$A_{f} = \begin{bmatrix} I_{N} & O_{N} & O_{N} & -R & -\bar{C} & O_{N} \\
O_{N} & I_{N} & O_{N} & -X & O_{N} & -\bar{C} \\
O_{N} & O_{N} & \bar{C}^{T} & (R^2 + X^2) & -2R & -2X \end{bmatrix}$$

$$\gamma_{f} = [0^T_{N}, 0^T_{N}, v_{0}, 0^T_{N-1}]^T.$$ 

Moreover, we define:

$$B_{f} = [I_{N}, O_{N}, O_{N}]^T$$

and let $B_{f}$ be a submatrix of $B_{f}$ that contains only the columns corresponding to the nodes $i$ with nonzero renewable generation $w_{i}$. Define column vectors $\bar{v} := (v_{i}, \forall i = 1, \ldots, N)$, $\bar{w} := (w_{i}, \forall i = 1, \ldots, N)$, similarly $\bar{p}, \bar{p}, \bar{q}$, and $\bar{t} = (t_{ij}, \forall i \rightarrow j)$. To write inequalities (2)(3) as $A_{s}x + \gamma_{s} \leq 0$, we need:

$$A_{s} = \begin{bmatrix} I_{N} & O_{N} & O_{N} & O_{N} & O_{N} & O_{N} \\
-I_{N} & O_{N} & O_{N} & O_{N} & O_{N} & O_{N} \\
O_{N} & I_{N} & O_{N} & O_{N} & O_{N} & O_{N} \\
O_{N} & O_{N} & -I_{N} & O_{N} & O_{N} & O_{N} \\
O_{N} & O_{N} & O_{N} & I_{N} & O_{N} & O_{N} \\
O_{N} & O_{N} & O_{N} & -I_{N} & O_{N} & O_{N} \end{bmatrix},$$

$$\gamma_{s} = \begin{bmatrix} -\bar{p} \\
\bar{p} \\
-\bar{q} \\
\bar{q} \\
-\bar{t} \\
0_{N-1} \end{bmatrix},$$

B. Equation (5): $A_{p, b, c, q, \gamma_{q}}$

To make (5b)–(5c) the same as:

$$\begin{bmatrix} 2P_{ij} \\
2Q_{ij} \\
v_{i} - \ell_{ij} \end{bmatrix} \leq v_{i} + \ell_{ij} + z_{q,ij}, \quad \forall i \rightarrow j$$

we need $A_{p, b, c, q, \gamma_{q}}$ as follows:

- For all $i \rightarrow j$, $A_{p, b, c, q, \gamma_{q}}$ is $3 \times (6N)$ sparse matrix with all elements zero except its element at the first row, $(4N + j)$-th column equal to 2; at the second row, $(5N + j)$-th column equal to 2; at the third row, $(2N + i)$-th column equal to 1 (if $i \neq 0$), and $(3N + j)$-th column equal to $-1$.
- For all $i \rightarrow j$ except $0 \rightarrow 1$, $b_{q,ij}$ is a three-dimensional column vector of all zeros; $b_{q,01} = [0, 0, v_{0}]^T$.
- For all $i \rightarrow j$, $c_{q,ij}$ is a $(6N)$-dimensional row vector of all zeros except its $(2N + i)$-th (if $i \neq 0$) and $(3N + j)$-th elements both equal to 1.
- $\gamma_{q,ij} = 0$ for all $i \rightarrow j$ except $0 \rightarrow 1$; $\gamma_{q,01} = v_{0}$.

**APPENDIX B**

**REMOVAL OF $\tilde{W}_{poly}$ FROM $W_{poly}$**

We use the following example in Fig. 13 to illustrate how to remove $\tilde{W}_{poly}$ from $W_{poly}$ and how to represent the obtained set as a union of several convex sets.

First, both $W'_{poly}$ and $\tilde{W}_{poly}$ are polytopes. Suppose

$$W_{poly} = \{ w \mid a_{1} w \leq b_{1}, a_{2} w \leq b_{2}, a_{3} w \leq b_{3} \}$$

$$\tilde{W}_{poly} = \{ w \mid c_{1} w \leq d_{1}, c_{2} w \leq d_{2}, c_{3} w \leq d_{3} \}$$

Then $W'_{poly} \cap \tilde{W}_{poly} = W'_{poly} \cap \tilde{W}_{poly}$, where $\tilde{W}_{poly}$ is the complementary set of $\tilde{W}_{poly}$ and can be represented as

$$\tilde{W}_{poly} = \{ w \mid c_{1} w > d_{1} \} \cup \{ w \mid c_{2} w > d_{2} \} \cup \{ w \mid c_{3} w > d_{3} \}$$

Moreover, since

$$W_{poly} \cap \{ w \mid c_{1} w > d_{1} \} \cup \{ w \mid c_{2} w > d_{2} \} \cup \{ w \mid c_{3} w > d_{3} \} = \{ w \mid c_{1} w > d_{1} \} \cup \{ w \mid c_{2} w > d_{2} \} \cup \{ w \mid c_{3} w > d_{3} \}$$

Therefore, we have

$$W'_{poly} \cap \tilde{W}_{poly} = U_{1} \cup U_{2} \cup U_{3}$$
where
\[ U_1 = \{ w \mid a_1 w \leq b_1, a_2 w \leq b_2, a_3 w \leq b_3, c_1 w > d_1 \} \]
\[ U_2 = \{ w \mid a_1 w \leq b_1, a_2 w \leq b_2, a_3 w \leq b_3, c_2 w > d_2 \} \]
\[ U_3 = \{ w \mid a_1 w \leq b_1, a_2 w \leq b_2, a_3 w \leq b_3, c_3 w > d_3 \} \]

\( U_1, U_2, \) and \( U_3 \) are all convex sets.

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