Massive thermal fluctuation of massless graphene electrons

Hosang Yoon and Donhee Ham

School of Engineering and Applied Sciences, Harvard University, Cambridge, Massachusetts 02138

(Dated: July 8, 2014)

PACS numbers: 05.30.Fk, 05.40.Ca, 72.70.+m, 72.80.Vp

Whereas thermal current noise $\langle I^2 \rangle$ in typical conductors is proportional to temperature $T$, $\langle I^2 \rangle$ in graphene exhibits a nonlinear $T$ dependence due to the massless nature of individual electrons. This unique $\langle I^2 \rangle$ arising from individually massless electrons is intimately linked to the non-zero collective mass of graphene electrons; namely, $\langle I^2 \rangle$ is set by the equipartition theorem applied to the collective mass’s kinetic energy, with the nonlinear $T$-dependence arising from the $T$-dependence of the collective mass. This link between thermal fluctuation and collective dynamics unifies $\langle I^2 \rangle$ in graphene and typical conductors, while elucidating the uniqueness of the former at the same time.

I. FLUCTUATION: MICROSCOPIC FORMALISM

We first formulate the electron thermal velocity fluctuation $\langle v_f^2 \rangle$ in a general conductor. This formulation is applicable to conductors in any dimensions, but for simplicity, we consider a two-dimensional (2D) conductor, whether it be graphene with massless electrons or 2D conductors with massive $(m \neq 0)$ electrons (e.g., GaAs/AlGaAs quantum well). An electron with a wavevector $k$ assumes an intrinsic velocity of $v_k$ for a massive 2D electron gas, $v_k = \hbar k/m$, where $k \equiv |k|$; for massless electrons in graphene, $v_k = v_F$ (constant).

$\langle v_f^2 \rangle$ is evaluated by considering the intrinsic velocities judiciously together with the Fermi-Dirac distribution, $f_k = 1/[e^{(\varepsilon_k - \mu)/k_B T} + 1]$ ($\varepsilon_k$: single electron energy; $\mu$: chemical potential). Note that $\langle v_f^2 \rangle$ is not the average of $v_k^2$ over all electrons, $(1/n) \int (d^2k/(2\pi)^2)g v_k^2 f_k$ ($g$: spin/valley degeneracy; $n$: electron density). This all-electron average counts many electron pairs moving in opposite directions with the same velocity deep below the Fermi surface, whose velocities cancel and cannot contribute to fluctuations. Its inadequacy is also evident as it does not vanish at $T = 0$, whereas $\langle v_f^2 \rangle$ must.

For $\langle v_f^2 \rangle$, we only consider electrons whose velocities do not cancel. The probability that a $k$-state is occupied and a $-k$-state is not occupied is $f_k(1 - f_{-k})$, and thus,

$$\langle v_f^2 \rangle = \frac{1}{n} \int \frac{d^2k}{(2\pi)^2} g v_k^2 f_k (1 - f_{-k}),$$

where the electron density $n$ is

$$n = \int \frac{d^2k}{(2\pi)^2} g f_k.$$  

With $\varepsilon_k = \varepsilon_{-k}$, we rewrite $f_k(1 - f_{-k})$ as

$$f_k(1 - f_{-k}) = \frac{\partial f_k}{\partial (\mu/k_B T)} = -\frac{\partial f_k}{\partial (\varepsilon_k/k_B T)}.$$  

which we will make use of later. At low $T$, since $f_k(1 - f_{-k})$ in $k$-space peaks around the Fermi surface with a vanishing width for $T \to 0$, $\langle v_f^2 \rangle$ of Eq. (1) vanishes at $T = 0$, as it should.
\[ \langle I^2 \rangle \text{ leads to the total current thermal fluctuation (I^2).} \]

Consider a 2D conductor of width W and length l along the x axis, with \( I^2 \) measured along the length. Only the x-component of \( v_\perp \) or \( v_{\perp,x} \) is relevant to the measured fluctuation. As a single electron contributes a fluctuation current of \( ev_{\perp,x}/l \), and as there are a total of \( nWL \) electrons,

\[ \langle I^2 \rangle = nWL \frac{e^2}{l} \left( \frac{v^2_{\perp,x}}{2} \right) = \frac{ne^2}{l} \frac{W}{T} \langle v^2_{\perp,x} \rangle, \]  

(4)

where \( \langle v^2_{\perp,x} \rangle = \langle v^2_\perp \rangle /2 \) (2 degrees of freedom). \( S_I(f) \) readily follows from \( \langle I^2(t) \rangle = \langle I^2 \rangle e^{-|t|/\tau} \) (\( \tau \): Drude scattering time), because electron scatterings randomize initial momenta at an average rate of \( 1/\tau \). The single-sided power spectral density is then \( S_I(f) = 4 \int_0^\infty dt \langle I(0)I(t) \rangle \cos(\omega t) \) with \( \omega = 2\pi f \), or,

\[ S_I(f) = 4\langle I^2 \rangle \frac{\tau}{1 + \omega^2/\tau^2}. \]

(5)

Before applying this formalism to graphene, we first apply it to a massive 2D electron gas, as the result can be compared to the traditional microscopic approach valid for the massive electron gas. Using \( \varepsilon_k = \hbar^2 k^2/2m \), \( v_k = \hbar k/m \), Eqs. (2) and (3), and \( \langle v^2_{\perp,x} \rangle = \langle v^2_\perp \rangle /2 \) in Eq. (1), we find

\[ \langle v^2_{\perp,x} \rangle = \frac{k_B T}{m} \int_0^\infty d\xi \xi \frac{\partial}{\partial \eta} f(\xi - \eta), \]  

(6)

where \( \xi \equiv \varepsilon_k/k_BT, \eta \equiv \mu/k_BT \), and \( f(\xi) \equiv 1/(e^\xi + 1) \). Using \( \int_0^\infty d\xi \xi^n f(\xi - \eta) = -\Gamma(1+s)\text{Li}_{1+s}(-e^{-\eta}) \), where \( \Gamma(z) \) is the gamma function and \( \text{Li}_{1+s}(x) = \sum_{k=1}^{\infty} x^k/k^n \) is the polylogarithm function, we reduce Eq. (6) to

\[ \langle v^2_{\perp,x} \rangle = \frac{k_B T}{m} \frac{\partial}{\partial \eta} \text{Li}_1(-e^{-\eta}) = \frac{k_B T}{m}, \]  

(7)

where we have used \( (d/d\xi)\text{Li}_n(x) = x^{-1}\text{Li}_{n-1}(x) \). This is consistent with the traditional microscopic approach based on Maxwell-Boltzmann statistics, in which Eq. (7) results from the equipartition theorem. Eq. (4) then yields

\[ \langle I^2 \rangle = \frac{ne^2}{l} \frac{W}{T} k_BT \propto T. \]  

(8)

In sum, for the massive electron gas, our general microscopic approach and the traditional microscopic approach agree; importantly, \( \langle v^2_{\perp,x} \rangle \propto T \) and \( \langle I^2 \rangle \propto T \). Incidentally, Eq. (5) then yields \( S_I(f) = 4k_BT(ne^2T/m)(1 + \omega^2T^2)^{-1}(W/l) \), where the real part of the Drude conductivity \( \sigma = (ne^2T/m)/(1 + i\omega T) \) appears inside the square brackets. As \( G = \text{Re}[\sigma W/l] \), we arrive at \( S_I(f) = 4k_BT G \).

II. THERMAL FLUCTUATION IN GRAPHENE

We now apply the formalism to graphene with \( c_k = \pm \hbar v_F k \) and \( v_F = v_F \). The constant \( v_F \), arising from the massless nature of individual electrons and holes, will yield a nonlinear \( T \)-dependency of \( \langle v^2_{\perp,x} \rangle \) and \( \langle I^2 \rangle \), sharply contrasting the linear \( T \)-dependency of the massive case. The electron/hole coexistence will further enrich the thermal fluctuation behavior. \( \langle v^2_{\perp,x} \rangle \) and \( n \) of Eqs. (1) and (2) are calculated separately for electrons in the conduction band and holes in the valence band:

\[ \langle v^2_{\perp,x} \rangle_c = \frac{v_F^2}{2} \text{Li}_1(-e^{-\eta}) \quad \langle v^2_{\perp,x} \rangle_v = \frac{v_F^2}{2} \text{Li}_2(-e^{-\eta}), \]

\[ n_c = \frac{-g(k_BT)^2}{2\pi(hv_F)^2} \text{Li}_2(-e^{-\eta}); \quad n_h = \frac{-g(k_BT)^2}{2\pi(hv_F)^2} \text{Li}_2(-e^{-\eta}), \]

where subscripts ‘c’ and ‘h’ indicate electrons and holes, and for the hole case, we have used \( f_k = f(\xi + \eta) \) and \( f_k(1 - f_k) = -(\partial / \partial \eta)f_k = -(\partial / \partial \xi)f_k \).

To first see the massless effect without the complication from the electron-hole coexistence, consider a fictitious graphene with the conduction band only (electrons only) with \( c_k = hv_F k \). The \( T \)-dependency depends on whether the chemical potential \( \mu \) or electron density \( n_e \) is fixed for varying \( T \). We consider the constant \( n_e \), as it is practically achieved with electrostatic gating. Then \( n_e = \text{constant condition} \ [\text{Eq. (10)} \text{ determines } \mu(T)] \) with \( \mu(0) = \varepsilon_F = h v_F \sqrt{4 \pi n_e / g} > 0 \) [Table 1]. With this particular \( \mu(T) \), \( \langle v^2_{\perp,x} \rangle_c \) first grows linearly with \( T \) just as in the massive case, but eventually saturates to \( v_F^2/2 \), deviating from the persistent linear \( T \)-dependence of the massive case [Fig. 1].

This low-\( T \) similarity, high-\( T \) difference between the massless and massive case can be explained with Eq. (7). For \( k_BT \ll \varepsilon_F \), \( f_k(1 - f_k) \) peaks sharply around the Fermi surface, so \( v_k = v_F \) for graphene coincides with \( v_k \approx v_F \) for the massive case, while this peak’s width grows linearly with \( T \). So Eq. (1) is linear to \( T \) in both massless and massive cases. For \( k_BT > \varepsilon_F \), \( \mu \rightarrow -\infty \) [Table 1] (the same holds for a massive gas), in the conduction band, \( f_k(1 - f_k) \approx f_k \approx e^{-(\varepsilon_F-\mu)/k_BT} \) is the fast tail of the Fermi-Dirac distribution. So \( v_k^2 = v_F^2 \) (massless) and \( v_k^2 \propto k^2 \) (massive) makes a difference in Eq. (1); in the former, \( \langle v^2_{\perp,x} \rangle \) saturates; in the latter, \( \langle v^2_{\perp,x} \rangle \propto T \) persists.

| Bands | Held constant | \( T = 0 \) | \( T \rightarrow \infty \) |
|-------|---------------|-------------|----------------|
| Conduction only | \( n_e \) | \( \varepsilon_F \) | 0 |
| Valence only | \( n_h \) | \( -\varepsilon_F \) | \( \infty \) |
| Conduction and valence | \( n_e - n_h \) | \( \varepsilon_F \) | 0 |
We can also consider a fictitious graphene with the valence band only (holes only) with \( \epsilon_k = -\hbar v_F k \). In this case, \( n_h = \text{constant} \) [Eq. (10)] determines \( \mu(T) \) with \( \mu(0) = -\epsilon_F < 0 \) [Table I]. The resulting \( T \)-dependence of \( \langle v_{hf,x}^2 \rangle \) is exactly the same as that of \( \langle v_{ef,x}^2 \rangle \) [Fig. 1].

Now consider the actual graphene with both the conduction and valence bands. Let graphene be doped at \( T = 0 \) and the total charge density \( \propto n_e(T) - n_h(T) \) be fixed via electrostatic gating. \( \mu(0) = \epsilon_F = \hbar v_F \sqrt{4 \pi n_e(0)/g} > 0 \), \( n_h(0) = 0 \), and \( n_e(T) - n_h(T) = n_e(0) \) for any \( T \). Using \( n_e(T) \) and \( n_h(T) \) from Eq. (10), this last expression can be rewritten as

\[
g(k_B T)^2 / 2 \pi \epsilon_F = -\text{Li}_2(-\epsilon^2) + \text{Li}_2(-e^{-\epsilon}) = n_e(0). \tag{11}
\]

Eq. (11) determines \( \mu(T) \) [Table I]. \( \mu \to 0 \) for \( T \to \infty \) contrasts the electron- or hole-only case; this is because \( n_e \) and \( n_h \) grow increasingly similar \( (n_e/n_h \to 1) \) with \( T \) despite their fixed difference. \( \langle v_{ef,x}^2 \rangle \) and \( \langle v_{hf,x}^2 \rangle \) are still given by Eq. (9), but due to the new \( \mu(T) \), \( T \)-dependence of \( \langle v_{ef,x}^2 \rangle \) and \( \langle v_{hf,x}^2 \rangle \) [Fig. 2] now deviates from Fig. 1.

\( \langle v_{ef,x}^2 \rangle \) is still linear to small \( T \), as the actual electron-doped graphene in this regime is different from the fictitious, electron-only graphene. For \( T \to \infty \), \( \langle v_{ef,x}^2 \rangle \) also saturates, but to \( (6 \ln(2)/\pi^2) \epsilon_F^2 \) instead of \( \epsilon_F^2 / 2 \), because \( \mu(T \to \infty) = 0 \) now, while \( \mu(T \to \infty) \to -\infty \) in the electron-only graphene. \( \langle v_{hf,x}^2 \rangle \) in Fig. 2 more drastically differs from Fig. 1 as we start from an electron-doped graphene. The small number of holes in the valence band at low \( T \) are at the far tail of the Fermi-Dirac distribution (similar to the \( T \to \infty \) case of Fig. 1), so \( \langle v_{hf,x}^2 \rangle \to \epsilon_F^2 / 2 \) for low \( T \). For large \( T \), \( \mu \to 0 \), so \( \langle v_{hf,x}^2 \rangle \) approaches \( (6 \ln(2)/\pi^2) \epsilon_F^2 \) just like \( \langle v_{ef,x}^2 \rangle \).

These behaviors of \( \langle v_{ef,x}^2 \rangle \) and \( \langle v_{hf,x}^2 \rangle \) lead to a complex nonlinear \( T \)-dependence of \( \langle I^2 \rangle \). Considering both the electron and hole current fluctuations, Eq. 4 is now

\[
\langle I^2 \rangle = \frac{g e^2 W}{4 \pi \hbar^2 l} (k_B T)^2 \left[ -\text{Li}_1(-\epsilon^2) - \text{Li}_1(-e^{-\epsilon}) \right]. \tag{12}
\]
Incidentally, graphene intraband conductivity is
\[
\sigma = \frac{-ge^2k_BT}{4\pi\hbar^2(\tau^{-1} + \omega^2)} \int_0^\infty d\xi \xi \left( \frac{\partial f(\xi - \eta)}{\partial \xi} - \frac{\partial f(-\xi - \eta)}{\partial \xi} \right)
= \frac{ge^2k_BT}{4\pi\hbar^2(\tau^{-1} + \omega^2)} [ -L_1(-e\eta) - L_1(-e^{-\eta}) ], \tag{13}
\]
where the conduction and valence band contributions are separated. Comparing the real part of the above with Eq. (12) and using \( G = \text{Re}[\sigma W/l] \), we attain \( \langle I^2 \rangle = k_BT G(1 + \omega^2/\tau^2)/\tau \). By plugging this into Eq. (5), we arrive at \( S_I(f) = 4k_BT G \). That is, despite the nonlinear \( T \)-dependence of \( \langle I^2 \rangle \), as \( \sigma \) shows the same nonlinear \( T \)-dependence except for the extra \( k_BT \) factor, the Johnson noise still holds. This is how the fluctuation-dissipation relation manifests in graphene.

III. FLUCTUATION AND COLLECTIVE DYNAMICS

The unique thermal fluctuation behavior \( \langle I^2 \rangle \) has a fundamental connection to the massive collective dynamics of individually massless graphene electrons. To explain this, we first briefly discuss the collective motion of graphene electrons,\(^3\) while setting aside the fluctuation problem. Let graphene electrons collectively move by a voltage \( V \) along the \( x \) axis. Individual electron velocity \( v_E \) remains constant, but their wavevectors change along the \( x \) axis; let this change be \( \Delta \) (same for all electrons) at a certain time. The total kinetic energy of the electron gas then grows by a certain amount \( E_c \); the larger the \( \Delta \), the larger the \( E_c \) whether \( \Delta > 0 \) or \( \Delta < 0 \). So \( E_c \) assumes a (smooth)\(^3\) minimum at \( \Delta = 0 \), thus \( E_c \propto \Delta^2 \) for small \( \Delta \). On the other hand, the collective momentum follows \( P_c \propto \Delta \). So \( E_c \propto P_c^2 \) and the collective motion exhibits a mass \( M_c \) satisfying \( E_c = P_c^2/(2M_c) \), while individual electrons are massless.

Thus in the collective motion, the voltage \( V \) accelerates \( M_c \) according to the Newton’s second law, increasing its velocity \( V_{cc} \equiv P_c/M_c \). The frequency-domain equation of motion is \(-e(n_eW)/(eV/l) = i\omega M_c V_{cc} \). As the current is \( I_e = -neW V_{cc} \), \( V/I_e = i\omega M_c/(en_eW)^2 \equiv i\omega L_{ek} \), where the kinetic inductance \( L_{ek} \) emerges as another manifestation of the collective inertia \( M_c \):

\[
L_{ek} = (e^2n_e^2W^2)^{-1}M_c, \quad L_{hh} = (e^2n_h^2W^2)^{-1}M_h. \tag{14}
\]

Here we have also written the same relation for holes. In sum, while graphene electrons are individually massless, their collective motion is of massive nature and described by \( M_c \) (\( M_h \)) or equivalently by \( L_{ek} \) (\( L_{hh} \)). Note that \( E_c = M_c V_{cc}^2/2 = L_{ek} I_e^2/2 \) and \( E_h = M_h V_{hc}^2/2 = L_{hh} I_h^2/2 \). We can find the expressions of \( L_{ek} \) and \( L_{hh} \) in graphene from Eq. (13). As \( \omega L_k = \text{Im}[l/\sigma W] \), we have

\[
L_k = \frac{4\pi\hbar^2}{ge^2k_BT} \frac{1}{\big| -L_1(-e\eta) - L_1(-e^{-\eta}) \big| W}. \tag{15}
\]

This is the overall kinetic inductance combining \( L_{ek} \) and \( L_{hh} \) in parallel as \( L_k^{-1} = L_{ek}^{-1} + L_{hh}^{-1} \), with

\[
L_{ek} = -\frac{4\pi\hbar^2}{ge^2k_BT} \frac{l/W}{L_1(-e\eta)}, \quad L_{hh} = -\frac{4\pi\hbar^2}{ge^2k_BT} \frac{l/W}{L_1(-e^{-\eta})}. \tag{16}
\]

We now return to the fluctuation problem and find its deep-seated connection to the collective dynamics. By inspection of Eqs. (12) and (15), we see that

\[
\frac{1}{2} L_k \langle I^2 \rangle = \frac{1}{2} k_BT. \tag{17}
\]

This can be broken into electron and hole contributions,

\[
\frac{1}{2} L_{ek} \langle I_e^2 \rangle = \frac{1}{2} k_BT, \quad \frac{1}{2} L_{hh} \langle I_h^2 \rangle = \frac{1}{2} k_BT, \tag{18}
\]

as \( \langle I^2 \rangle = \langle I_e^2 \rangle + \langle I_h^2 \rangle \) and \( L_k^{-1} = L_{ek}^{-1} + L_{hh}^{-1} \). Or equivalently, in terms of \( M_e \) and \( M_h \), and their thermal velocity fluctuations \( \{V_{ce}^2\} \) and \( \{V_{hc}^2\} \),

\[
\frac{1}{2} M_e \langle V_{ce}^2 \rangle = \frac{1}{2} k_BT, \quad \frac{1}{2} M_h \langle V_{hc}^2 \rangle = \frac{1}{2} k_BT. \tag{19}
\]

Eqs. (17)–(19) are the same statement on the intimate relation between thermal fluctuations and collective dynamics. Although individual graphene electrons and holes act as massless relativistic particles, their thermal fluctuations are governed by the classical kinetic energies of the collective electron mass \( M_e \) and of the collective hole mass \( M_h \), with each receiving a thermal energy of \( k_BT/2 \), satisfying the equipartition theorem [Eq. (19)], thus determining the collective velocity thermal fluctuations \( \{V_{ce}^2\} \) and \( \{V_{hc}^2\} \) [Eq. (18)]. Eq. (17) expresses this most concisely; the total current thermal fluctuation \( \langle I^2 \rangle \) is determined by the total kinetic inductance storing an average collective kinetic energy of \( k_BT/2 \).

The relationship between thermal fluctuation and collective dynamics captured by Eq. (17) also holds for the massive electron gas, as one can see from Eq. (8), where \( L_k = (m/ne^2)(l/\sigma W) \) is the kinetic inductance of the massive electron gas. However, this massive case is less surprising, as each electron already follows equipartition and the collective mass is their simple aggregate \( M = nWlm \). The more interesting, and unifying, observation is that even in graphene with massless electrons, \( \langle I^2 \rangle \) arises from their non-zero collective mass storing an averaged kinetic energy of \( k_BT/2 \). As much as the relation \( \langle I^2 \rangle = k_BT/L_k \) [Eq. (17)] offers a unified picture for the thermal fluctuation in the massless and massive electron gas, it also directly explains the unique nonlinear \( T \)-dependence of \( \langle I^2 \rangle \) in graphene, as \( L_k \) is decisively temperature dependent in graphene [Eq. (15)], whereas the total electron gas \( L_k \) is constant and thus \( \langle I^2 \rangle \propto T \).

Interrogation of the collective (plasmonic) dynamics of graphene electrons via noise measurement based upon
this study may be an interesting point of future investigation. In addition, the present study may in the future be expanded to take into account the quantum radiation regime.

**ACKNOWLEDGMENTS**

We thank the support of this research by the Air Force Office of Scientific Research (FA9550-13-1-0211) and Office of Naval Research (N00014-13-1-0806).

---

1. J. B. Johnson, Phys. Rev. 32, 97 (1928)
2. H. Nyquist, Phys. Rev. 32, 110 (1928)
3. R. Kubo, Rep. Prog. Phys. 29, 255 (1966)
4. K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Nature 438, 197 (2005)
5. A. C. Betz, F. Vialla, D. Brunel, C. Voisin, M. Picher, A. Cavanna, A. Madouri, G. Fève, J.-M. Berroir, B. Plaçais, and E. Pallecchi, Phys. Rev. Lett. 109, 056805 (2012)
6. A. C. Betz, S. H. Jhang, E. Pallecchi, R. Ferreira, G. Fève, J.-M. Berroir, and B. Plaçais, Nat. Phys. 9, 109 (2013)
7. R. K. Pathria and P. D. Beale, *Statistical mechanics*, 3rd ed. (Elsevier, Oxford, 2011)
8. H. Yoon, C. Forsythe, L. Wang, N. Tombros, K. Watanabe, T. Taniguchi, J. Hone, P. Kim, and D. Ham, *Nature Nanotechnology* advance online publication (2014), 10.1038/nnano.2014.112
9. L. A. Falkovsky, Phys. Usp. 51, 887 (2008)