Collective Excitations 
and Ground State Correlations

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October 1, 2018

Abstract
A generalized RPA formalism is presented which treats pp and ph correlations on an equal footing. The effect of these correlations on the single-particle Green function is discussed and it is demonstrated that a self-consistent treatment of the single-particle Green function is required to obtain stable solutions. A simple approximation scheme is presented which incorporates for this self-consistency requirement and conserves the number of particles. Results of numerical calculations are given for $^{16}$O using a G-matrix interaction derived from a realistic One-Boson-Exchange potential.
1 Introduction

A lot of effort has been made to evaluate the basic properties of nuclei, like binding energies and radii of charge distributions, directly from a realistic model of the nucleon-nucleon (NN) interaction which describes the experimental NN scattering data. Already many years ago it was realized that such investigations require a very careful consideration of the short-range correlations, which are induced by the strong short-range and tensor components of a realistic NN interaction. Various techniques of many-body theory have been developed to account for these particular correlations. Here we would like to mention the Brueckner-Bethe hole line expansion \[1\], the variational approach, in particular the extension to the method of correlated basis functions \[2, 3\], the so-called “exponential S” method \[4\] and the self-consistent Green function approach \[5, 6\].

Such microscopic calculations, however, had rather limited success. As an example we refer to the Brueckner hole-line expansion applied to nuclear matter, for which very detailed studies have been made considering various models for a realistic NN interaction. It turned out that Brueckner-Hartree-Fock (BHF) calculations employing a nuclear interaction with a weak tensor force were able to reproduce the empirical value for the binding energy of nuclear matter but predicted a saturation density which was too large by almost a factor 2. Other interactions with stronger tensor forces lead to satisfying results for the saturation density but underestimate the binding energy by around 5 MeV per nucleon. This result, that microscopic many-body calculations employing realistic NN interactions fail to reproduce binding energy and saturation density, is commonly referred to as the problem of the “Coester band” in nuclear matter \[7\]. Similar results are obtained also for the ground-state properties of finite nuclei \[4, 8, 9\].

Various attempts have been made to cure this problem and some of them even succeeded to determine a saturation point of nuclear matter which is very close to the empirical point. One possibility is to introduce many-body forces \[10\]. Kuo and coworkers demonstrated that the inclusion of particle-particle hole-hole (pphh) ring diagrams tends to shift the saturation point off the Coester band towards the empirical value \[11, 12\]. Another possible solution is the inclusion of relativistic effects within the Dirac-BHF approximation. Motivated by the success of the Walecka model \[13\] various groups tried to incorporate the modification of the Dirac spinor for the nucleon, due to the interaction with the other nucleons, in BHF calculations \[14, 15, 16\]. With the inclusion of the relativistic effects one even obtains results for the saturation properties of nuclear matter which are in agreement with the empirical data \[17\].

The success of the Dirac-BHF approach cannot be extended to the description of finite nuclei. The relativistic features tend to significantly improve the agreement between calculated values for the energies and radii and the corresponding experimental data \[18, 19\]. The remaining discrepancies (around 0.7 MeV per nucleon in energy and around 0.2 fm for the radius of light nuclei like \(^{16}\text{O}\) and \(^{40}\text{Ca}\)), however, are still too large to be accounted for.

This situation encourages the investigation of correlations beyond those included in the Dirac-BHF approximation. Such correlations should have a sizeable effect on the groundstate properties of finite nuclei but should not spoil the success of Dirac-BHF in nuclear matter. One can expect that differences of this kind show up in correlations, which are due to surface
effects in finite nuclei. Such correlations are typically described in terms of particle-hole (ph) excitations. Therefore our studies should incorporate the correlation effects contained in the particle-hole random phase approximation (ph RPA). Furthermore we would also like to account for the effects of pphh ring diagrams. As discussed above, these correlations show very desirable effects in studies of nuclear matter. Therefore it is interesting to investigate these correlations for finite nuclei as well.

A “super-RPA” (SRPA) technique which allows the study of ph, pphh and the possible interference between these kinds of correlations was presented a few years ago [20]. It has been possible to solve the non-linear equations resulting from this SRPA approach for a small model space or by an artificial weakening of the residual interaction. No solution could be obtained for larger model-spaces using realistic interactions. In the present investigation we would like to demonstrate that this failure of the SRPA can be cured by a consistent definition of the single-particle propagator.

After this introduction we summarize in section 2 of this paper the basic ingredients of the SRPA and recall the equations to calculate expectation values for the correlation energy and the occupation probabilities. The self-consistent definition of the single-particle propagator is discussed in section 3. As an example for an application of the SRPA we present in section 4 results on $^{16}O$ employing various realistic One-Boson-Exchange (OBE) interactions in different model-spaces. The main conclusions are summarized in the final section.

## 2 Ground-State Correlations in SRPA

In this section we will briefly review the SRPA formalism, treating particle-hole (ph), particle-particle (pp) and hole-hole (hh) on the same level. We shall work in a basis which is not angular-momentum coupled, since this is more transparent. More details and also the equations for states coupled to angular momenta are given in ref.[20]. As usual the Hamiltonian is broken into an unperturbed part and a perturbation, $H = H_0 + V$, where $H_0$ has eigensolutions

$$H_0 |A⟩ = ε_A |A⟩ . \quad (2.1)$$

We restrict our attention to a closed shell system for which we can use standard non-degenerate perturbation theory, see e.g. Brandow [21]. Denoting the unperturbed closed shell wavefunction by $|φ_0⟩$ and noting that we are interested in 2p-2h correlations, the true wavefunction can be written in the form

$$|ψ⟩ = \exp \left[ \sum_{αβ} t_{αβ} a_α^+ a_β^+ a_β a_α \right] |φ_0⟩ , \quad (2.2)$$

where we use latin (greek) letters to designate states which are unoccupied (occupied) in $|φ_0⟩$. A single letter here is used to represent all the quantum numbers needed to specify an orbital.

We are then required to set up equations for the coefficients $t$. The notation for the basic interactions $V$ that we have at hand are standard: those that create (or destroy) a 2p-2h state we denote by $B$, those that scatter a pair of particles (or holes) we label $A$ and those
that scatter a p-h pair we label $A$. We can set up a multiple scattering series for $t$ in the form

$$t_{\alpha\beta} = \frac{1}{\epsilon_{\alpha\beta}} \sum \text{repeated indices} \left\{ \frac{1}{4} B_{\alpha\beta} + \frac{1}{2} A_{\alpha\beta} + \frac{1}{2} t_{\alpha\beta} \right\},$$

where the energy denominator $\epsilon_{\alpha\beta} = \epsilon_\alpha + \epsilon_\beta - \epsilon_a - \epsilon_b$. Consider iterating this equation. The lowest order contribution is simply the first term in the braces which creates the 2p-2h state. Subsequent iterations build in the processes shown in Fig. 1 (a) — assuming $t$ is given in order $n$, order $(n+1)$ is obtained by allowing an additional pp interaction (term 2 in braces), or hh interaction (term 3), or one of the four possible ph interactions (term 4).

In addition we must allow for “backward-going” $B$-vertices, as shown in Fig. 1(b), which may be of ph type (term 5) or pp-hh type (term 6). Here we have used the factorization theorem [21] to render the energy denominators of the left and right boxes independent. Thus the wavefunction diagrams generated are of the general type indicated in Fig. 1 (c) and, as advertised, we include diagrams with all orientations of arrows on the lines subject to the requirement that $V$ does not create particles, i.e., two arrows point towards and two away from each vertex.

Because of the antisymmetry of fermions, the only linear combination of coefficients $t$ that enters is

$$t_{\alpha\beta} - t_{\beta\alpha} = 2K_{\alpha\beta}.$$

(2.4)

It is useful to exploit this in actual calculations in order to cut down storage requirements and it is simple to recast Eqs. (2.2) and (2.3) in terms of $K$:

$$|\psi\rangle = \exp\left[\frac{1}{2} \sum_{\alpha\beta} K_{\alpha\beta} a_\alpha^\dagger a_\beta a_\beta^\dagger a_\alpha\right]|\phi_0\rangle,$$

(2.5)

$$K_{\alpha\beta} = \frac{1}{\epsilon_{\alpha\beta}} \sum \text{repeated indices} \left\{ \frac{1}{4} B_{\alpha\beta} + \frac{1}{2} A_{\alpha\beta} + \frac{1}{2} t_{\alpha\beta} \right\},$$

where

$$A_{\alpha\beta} = \sum_{\gamma\delta} A_{\gamma\delta} = \sum_{\gamma\delta} A_{\gamma\delta} = \sum_{\gamma\delta} A_{\gamma\delta}.$$
For the norm of the correlated ground state wavefunction and the single particle occupation probabilities, we generalize the approach discussed in Ref. \[22\]. The norm of $|\psi\rangle$ can be written

$$
\langle \psi | \psi \rangle = \exp \left\{ \sum_{aba, N} \frac{L_{aba}(N)}{N} K_{aba} \right\},
$$

(2.8)

where the quantity in braces gives the sum of all linked normalization diagrams \[21\] and the exponential form follows from considering all possible products of linked diagrams. For $N = 1$ we have

$$
L_{aba}(1) = K_{aba}.
$$

(2.9)

This clearly arises from expanding the exponential form given for $|\psi\rangle$ in Eq. (2.5) and taking the contribution to $\langle \psi | \psi \rangle$ which contains 2 $K$-coefficients. To include the linked contribution from 4, 6, … $K$-coefficients, corresponding to $N = 2, 3, \ldots$, we define an iterative equation for $L$:

$$
L_{aba}(N) = \sum_{\text{repeated indices}} \left\{ L_{ab\gamma}(N-1) K_{cd\gamma} K_{ed\alpha} ight. \\
+2L_{ada}(N-1) K_{cd\gamma} K_{bc\beta} - 2L_{bda}(N-1) K_{cd\gamma} K_{ac\beta} \\
\left. -2L_{ad\beta}(N-1) K_{cd\gamma} K_{bc\alpha} + 2L_{bd\beta}(N-1) K_{cd\gamma} K_{ac\alpha} \right\}. 
$$

(2.10)

Here $L(N)$ is built by joining $L(N-1)$ to one $K$ coming from the bra and another coming from the ket. Thus $L(N)$ corresponds to a chain of $(2N - 1)$ coefficients $K$ which is linked together in a manner consistent with our previous diagrams. The chain is finally closed off by taking $LK$ in the braces of Eq. (2.8). The first term on the right in Eq. (2.10) connects via pp and hh pairs; each $K$ carries a factor of $\frac{1}{2}$, but each pair $(cd$ or $\gamma\delta)$ can be contracted in two ways, hence the net factor is unity. The remaining four terms connect via ph pairs; the 4 ways of picking $aba\beta$ are shown explicitly. We may also interchange the particle and/or hole labels on the $K$’s which, by symmetry, gives a factor of 16; against this must be set the factor of $\frac{1}{2}$ carried by each $K$ and a further factor of $\frac{1}{2}$ because our interchanges implicitly include $KKL$ as well as $LKK$ and these are equivalent. Thus there is a net factor of 2. We should also comment on the factor of $N^{-1}$ in Eq. (2.8). In joining together the $K$’s, there are $N!$ permutations in the bra, but $(N-1)!$ permutations in the ket to yield distinct (and identical) results. Combining with the factor of $(N!)^{-2}$ arising from the expansion of the exponentials, we obtain a net factor of $N^{-1}$.

The single particle occupation probabilities can be obtained by evaluating the expectation value of the operator

$$
\mathcal{N}_P = a_P^\dagger a_P ,
$$

(2.11)

where the orbital $P$ is unoccupied in $|\phi_0\rangle$. This operator will pick out the particular orbital $P$ from the chain of $K$-coefficients discussed above and since there are $2N$ possibilities, each of which gives the same result, we obtain

$$
n_P \equiv \frac{\langle \psi | \mathcal{N}_P | \psi \rangle}{\langle \psi | \psi \rangle} = 2 \sum_{ba, \beta} L_{Pba\beta}(N) K_{Pba\beta}.
$$

(2.12)
A completely analogous expression can be given for the hole occupation probabilities.

More details on the SRPA equations, the embedding of various approximations and the coupling to states of good angular momentum are given in ref. [20].

3 Correlations and Single-Particle Properties

The correlations considered in the SRPA formalism presented in the preceding section are reflected in occupation probabilities which deviate from the occupation numbers one and zero of the independent particle model. This implies that the Lehmann representation of the single-particle Green function is not any longer given by only one pole but must be presented in the general form (see e.g. [6])

$$g(\alpha, \omega) = \int_{-\infty}^{\epsilon_F} d\omega' \frac{S_h(\alpha, \omega')}{\omega - \omega' + i\eta} + \int_{\epsilon_F}^{+\infty} d\omega' \frac{S_p(\alpha, \omega')}{\omega - \omega' - i\eta}. \quad (3.13)$$

In this equation the index \( \alpha \) refers to the symmetry quantum numbers of the single-particle state under consideration and \( \omega \) is the energy variable in the single-particle Green function \( g(\alpha, \omega) \). The hole spectral function \( S_h(\alpha, \omega) \) is related to the occupation number of the single-particle state \( \alpha \) in the ground state of the nucleus by

$$n_\alpha = \frac{<\psi|a_\alpha^\dagger a_\alpha|\psi>}{<\psi|\psi>} = \int_{-\infty}^{\epsilon_F} d\omega S_h(\alpha, \omega) \quad (3.14)$$

where the integral is restricted to energies \( \omega \) below the Fermi energy \( \epsilon_F \). Within the framework of the Green function formalism the single-particle Green function is usually determined by considering a certain approximation for the self-energy of the nucleons with quantum numbers \( \alpha \) and solving a Dyson equation using this approximation for the self-energy. A problem of this Green function approach is that only very specific approximations for the self-energy lead to a number conserving approach, namely

$$\sum_\alpha (2j_\alpha + 1)n_\alpha = N \quad (3.15)$$

which means that the sum on all occupation probabilities multiplied by the degeneracy of the state \( \alpha \) (here \( 2j_\alpha + 1 \)) yields the required particle-number \( N \). It turns out that rather sophisticated self-consistency requirements have to be fulfilled for any approximation to the self-energy beyond the Hartree-Fock approach [23].

The hole spectral functions \( S_h(\alpha, \omega) \) and particle spectral functions \( S_p(\alpha, \omega) \) can be used to define mean single-particle energies below

$$\epsilon_{<,\alpha} = \frac{1}{n_\alpha} \int_{-\infty}^{\epsilon_F} d\omega \omega S_h(\alpha, \omega) \quad (3.16)$$

and above the Fermi energy

$$\epsilon_{>,\alpha} = \frac{1}{1 - n_\alpha} \int_{\epsilon_F}^{+\infty} d\omega \omega S_p(\alpha, \omega). \quad (3.17)$$
Furthermore it can be shown that for a very general approximation in the self-energy these mean values are related to the Hartree-Fock single-particle energy $\epsilon_{\alpha}^{HF}$ by

$$n_{\alpha}\epsilon_{<,\alpha} + (1 - n_{\alpha})\epsilon_{>,\alpha} = \epsilon_{\alpha}^{HF}.$$  \hspace{1cm} (3.18)

Using the nomenclature introduced so far we can now define an approximation for the single-particle Green function which is a first step beyond the Hartree-Fock approximation by simplifying eq.(3.13) to

$$g^{(1)}(\alpha, \omega) = \frac{n_{\alpha}}{\omega - \epsilon_{<,\alpha} + i\eta} + \frac{1 - n_{\alpha}}{\omega - \epsilon_{>,\alpha} - i\eta}.$$  \hspace{1cm} (3.19)

This means that a nucleon in a single-particle state $\alpha$ is propagating as a hole with probability $n_{\alpha}$ at an energy $\epsilon_{<,\alpha}$ and propagating as a particle with probability $(1 - n_{\alpha})$ at an energy $\epsilon_{>,\alpha}$.

In our investigation we want to consider the interplay between the correlations as calculated in the SRPA approach discussed in the previous section and the single-particle propagator using the approximation of eq.(3.19). This means that we employ the occupation numbers calculated in SRPA according to eq.(2.12). These occupation numbers $n_{\alpha}$ and the relation (3.18) are used to determine the energies $\epsilon_{<,\alpha}$ and $\epsilon_{>,\alpha}$. This is done in the following way: For states $\alpha$, which are mainly occupied ($n_{\alpha} > 0.5$) we assume for the energy of the particle component $\epsilon_{>,\alpha}$ (which is of minor importance for this orbit), the value determined by the BAGEL technique of ref.[25] for the corresponding nucleus and hamiltonian. The energy of the dominant hole component $\epsilon_{<,\alpha}$ is then determined according to (3.18). For orbits which are predominantly particle states ($n_{\alpha} < 0.5$) we proceed in the analogous way and determine the energy for the propagation of the hole state from ref.[25], while the energy of the particle part ($\epsilon_{>,\alpha}$) is derived from (3.18).

The modified single-particle Green function of eq.(3.19) is then used in the SRPA equations. This means that all summations in e.g. eq.(2.6) which were restricted to particle states now have to consider all orbits $\alpha$ with a weight $(1 - n_{\alpha})$ and an energy of $\epsilon_{>,\alpha}$. The corresponding statement holds for the hole orbits. This solution of the modified SRPA equations leads to new occupation probabilities, which give rise to different single-particle energies. This procedure is repeated until a self-consistent solution is obtained.

It is evident that this self-consistent determination of the single-particle Green function yields a stabilizing effect for the solution of the SRPA equations: If correlation effects are large, the occupation probabilities will deviate strongly from the values of the independent particle model. According to eq.(3.18) this gives rise to hole-energies $\epsilon_{<,\alpha}$ which are significantly below and particle-energies $\epsilon_{>,\alpha}$ significantly above the corresponding Hartree-Fock value $\epsilon_{\alpha}^{HF}$. Therefore one obtains an additional gap between particle- and hole-energies. This change in the single-particle spectrum tends to reduce the correlation effects, calculated in SRPA. This is what we call the stabilizing effect of the single-particle Green function.

4 Results and Discussion
4.1 SRPA and Independent Particle Model

As a first step in our discussion of results we would like to recall some of the features and the problems of the SRPA approach already discussed in ref. [20]. In this first part we consider the SRPA introduced in section 2, i.e. assuming single-particle propagators of the kind of the independent particle model (IPM). This means that the different single-particle orbits of our model space are either hole-states and completely occupied or particle-states (completely unoccupied).

As in [20] we will consider the nucleus $^{16}$O assuming for the first part of this discussion a very small model space. The summation on hole-states shall be restricted to the states of the p-shell while the orbits of the 1s0d shell are considered for the particle states. The matrix elements of the NN interaction are evaluated in a basis of oscillator states (oscillator parameter $b=1.76$ fm) by solving the Bethe-Goldstone equation [26] for the OBE potentials “$A''$, “$B''$ and “$C''$ defined in table A.1 of [14]. More details on the evaluation of these matrix elements and a modification of the interaction which ensures that the chosen oscillator basis is identical to the self-consistent Hartree-Fock (HF) basis are discussed in [24].

One can determine the HF single-particle energies for these three sets of G-matrix elements and try to solve the non-linear SRPA equations by an iterative procedure. It turns out, however, that such an iterative scheme does not converge. The SRPA equations can be simplified to account only for correlations obtained already in the conventional particle-hole (ph) RPA or the particle-particle hole-hole (pphh) RPA approach (see detailed discussion in [20]). The iterative scheme for the solution of these reduced SRPA equations converges and yields the same result for the correlation energy as can be obtained by other techniques [27, 28, 29].

In the case of a model space, which contains an inert core (like the $^4$He core in the model space presently considered, the HF choice for the single-particle energies may not be appropriate. Therefore in [20] we have employed a set of single-particle energies, which is not really self-consistent. Using the same set of single-particle energies we obtain results for the correlation energies using various approximations and interactions as displayed in table 1. These results are very similar to results already discussed in [20], the differences to the results reported before are due to a different choice of the Pauli operator and starting energy in the solution of the Bethe-Goldstone equation.

The total correlation energy is split into the contribution due to the term of second order in the residual interaction, which is part of the correlation energy in all the approaches under consideration and the correlation energy due to the terms of third and higher order in the residual interaction. In table 1 we compare correlation energies resulting from the ph-RPA (with inclusion of exchange terms, see [20]), the pp-hh RPA and the SRPA. As it has already been discussed in [20], one can draw the following conclusions:

- The correlation energy obtained in SRPA is larger than the sum of correlation energies resulting from ph-RPA and pp-hh RPA. This additional energy is due to terms which contain a residual interaction between ph excitations but also between 2 particle or 2 hole states.

- The correlation energy due to the terms of third and higher order are essentially as large as the correlation energy of the second-order term.
The residual interaction is decreasing from OBE potential $A$ to $B$ to $C$. This can be deduced from the terms of second order but also from the total correlation energies. This is in line with the fact that the potential $A$ also yields the largest binding energy in BHF calculations of nuclear matter \[14\] or finite nuclei \[9\].

At first sight all these results seem to be reasonable. A more detailed discussion and investigation, however, gives rise to some doubts:

(i) It has already been mentioned that problems arise if HF single-particle energies are used in the SRPA approach.

(ii) A closer inspection of the wavefunction shows that the correlation determined by SRPA are very strong, leading to occupation probabilities for the hole states which are small already for the OBE potential $C$ (0.814 and 0.822 for $p_{3/2}$ and $p_{1/2}$ shell, respectively) and extremely small for OBE $A$ (0.276 and 0.375).

(iii) Enlarging the model space to include all hole states of $^{16}$O and the 1s0d plus 1p0f shells for the particle states leads to SRPA equations, which cannot be solved neither by employing the single-particle energies of \[20\] nor using the HF single-particle energies.

It is worth noting that the iterative solution of the SRPA equations converges also in the large model-space if we assume HF single-particle energies and reduce the SRPA equations to the ph-RPA or pp-hh RPA approach. The correlation energy is the same as the one obtained from a diagonalisation of the corresponding RPA equation. If, however, we keep the single-particle energies fixed and continuously enhance the residual interaction by multiplying the matrix elements with a constant ($\lambda > 1$), we observe that at a critical value of $\lambda$ the iteration of the non-linear equation does not converge any more. At the same value of $\lambda$ the diagonalisation of the corresponding RPA equations yields complex eigenvalues.

Therefore we conclude that the HF solution is stable with respect to ph-RPA and pp-hh RPA correlations. This is reflected in our calculations by the fact that RPA calculations using self-consistent single-particle energies yield real energies for the RPA phonons and iterative solutions for the corresponding non-linear equations and, of course, a feature of the HF approximation \[30\]. The fact that the complete SRPA equations do not show an iterative solution can be interpreted to indicate that the HF state is not stable with respect to the excitations of the SRPA kind. The SRPA calls for a better approximation of the single-particle Green function than the one derived from HF.

A very pragmatic way to obtain solution of the full SRPA equations is to introduce a gap between the energies of particle and hole-states. Indeed, shifting e.g. all energies for particle states by a constant of around 3 MeV leads to stable solutions even for the large model space. This shift is rather arbitrary. Therefore we prefer to investigate the modification of the single-particle Green function introduced in section 3, which establishes a self-consistent link between the evaluation of the correlations and the definition of the single-particle properties.
4.2 “Self-Consistent” SRPA

As we expected from our discussion in section 3, the “self-consistent” choice of the single-particle propagator according eq. (3.19) yields a stabilization of the SRPA equations. With this choice we obtain solutions of the non-linear SRPA equations for the small and large model-spaces for all interactions considered. Results for the correlation energies are displayed in table 2 considering the same approximations, interactions and model-spaces as in table 1.

The occupation numbers for the single-particle orbits \( n_\alpha \) depend on the approximation considered. In table 3 occupation numbers are listed, which were calculated for the OBE potential using different approximations and model-spaces. Considerable deviations from the occupations of the IPM are observed. These deviations are slightly larger if one uses the interaction OBE \( A \) and slightly smaller for the interaction \( C \). As the occupation numbers depend on the method considered, also the single-particle propagators defined in eq. (3.19) will be different. Consequently for these self-consistent calculations also the contribution of the second order term will be different for the different approaches. In table 2 we give for each approximation the contribution of this second order term calculated with the self-consistent single-particle propagator and separately the sum of all higher order terms.

The stabilization of the correlation effects due to the self-consistent choice of the single-particle propagator is well documented by the contribution to the energy of second order in the residual interaction. For approximations which yield large correlation effects, like the SRPA, one obtains sizeable deviations of the occupation probabilities from the IPM (see table 3) and contributions to the binding energy from the second order term, which are considerably smaller than the second order terms calculated without readjustment of the single-particle features (see table 1). For approaches like RPA, which yield weaker correlation effects, the quenching of the second order term due to the single-particle propagator is weaker. Therefore the more correlations are taken into account (SRPA as compared to RPA) the smaller the contribution of the second order term. Also the contributions of the terms of third and higher order in the interaction are reduced due to the self-consistent single-particle propagator. The propagator effect, however, is generally weaker than for the second order terms.

It is interesting to note that the sum of second order plus all higher order terms yields a correction to the binding energy, which is almost independent on the approximation considered. For the small model-space we obtain around 9 MeV, while the calculations in the large model-space yield around 19 MeV. This does not imply that correlations beyond RPA are negligible if the single-particle propagator is chosen in a self-consistent manner. Indeed the contribution to the binding energy originating from third and higher order terms is increasing going from RPA via pp-hh RPA to SRPA. This enhancement of the higher order terms is partly canceled by reduction of the second order terms as already discussed above. The importance of correlations beyond RPA is also reflected by the occupation probabilities displayed in table 3. The deviations from the Hartree-Fock values observed in the SRPA are much larger than those obtained in RPA (see also Fig. 2). This is true in particular for the 2 shells close to the Fermi level (the \( 0p \) and the \( 1s0d \) shells). In this case the deviations obtained in SRPA are larger than the sum of the deviations produced by RPA and pp-hh RPA correlations.
The inspection of the correlation energies for the various model-spaces (9 MeV for the small and 19 MeV for the large model-space) may cast doubt on the convergence of the calculated correlation energy with respect to the size of the model-space. It should be noted, however, that the occupation probabilities obtained for the shells away from the Fermi level are not very sensitive to the approximation considered. The RPA and also the pp-hh RPA yields results for the 0s and 1p0f shells, which are very similar as those obtained in SRPA. This indicates that the interplay between the various correlations contained in SRPA is of dominant importance only for the shells close to the Fermi energy. Therefore the pp-hh RPA, which is the summation of all particle-particle and hole-hole ladders, should provide a good approximation for the correlations of these high energy particle-hole excitations.

One possible way to account for these pp-hh correlations would be the use of the self-consistent Green function method [6]. The Green function approach, if considered beyond the Hartree-Fock approximation, requires a very sophisticated self-consistency between the approximation used for the 2-particle Green function and the single-particle Green function to guarantee number conservation. The approximation for the single-particle propagator, which we are using here, is similar to the Green function approach (see discussion in 3.2). The present scheme, however, has the advantage that it always yields the correct particle number.

Employing the self-consistent single-particle Green functions defined in eq. (3.19) one can calculate the expectation value for single-particle operators like the radius of the nucleon distribution. The radii obtained for the various approximations, using the OBE potentials A, B and C in the large model-space are listed in table 4 and compared to the result obtained in the Hartree-Fock approximation. As we discussed already in the beginning of this section, the interactions have been modified to guarantee a Hartree-Fock solution with a radius close to the experimental value [24]. Therefore we should only discuss the modifications in the calculated radius due to the correlation effects. One finds a small but non-negligible enhancement for the calculated radius. As to be expected also for this observable the effect of correlations is larger for the SRPA than for the 2 RPA approximations. Also for the calculation of the radii, the effects of correlations is slightly increasing going from potential A to C.

It is remarkable that the inclusion of SRPA correlation increases the binding energy but also the value for the calculated radius. This implies that calculated ground-state properties are moved off the “Coester-band” towards the experimental point. This effect is not very large but together with the features of the Dirac-BHF approach it may be sufficient to yield results for the ground-state of finite nuclei, which are in good agreement with the empirical data.

5 Conclusions

A method is discussed which allows the non-perturbative evaluation of correlation effects beyond the conventional particle-hole (ph) and particle-particle hole-hole (pp-hh) Random-Phase approximations. It turns out that the resulting non-linear equations of this SRPA yield solutions only, if the single-particle propagator or single-particle Green functions are evaluated in a self-consistent way beyond the Hartree-Fock approximation. A self-consistent
scheme is developed, which conserves the particle number, i.e. the solutions for the single-particle Green function yield the required particle number.

It turns out that the interference between ph and pp-hh correlations, as described by SRPA, is rather important for correlation effects within the major shells close to the Fermi-level. The correlation effects due to configurations involving shells of deep-lying hole states or high-energy particle states seem to be described well within the RPA. The correlations increase the value for the calculated binding energy and lead to an enhancement for the calculated radius. Therefore correlations of the SRPA type combined with the features of the Dirac-BHF approximation [18] may produce results for the ground-state properties of finite nuclei, which are in good agreement with the experimental data.

This investigation has been done within the “Graduiertenkolleg Struktur und Wechselwirkung von Hadronen und Kernen”. The financial support by the Deutsche Forschungsgemeinschaft (DFG Mu 705/5) and the support of the “King Fahd University of Petroleum and Minerals” is acknowledged. We are grateful to Professor Paul Ellis for useful discussions.

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Table 1: Correlation energies for $^{16}O$ obtained with fixed single-particle energies as defined in ref.[20]. The correlation energy for a specific approach is the sum of the term of second-order in the residual interaction (first line labeled 2.order) plus the contribution of all higher order terms which are listed in the lines identified by the various approaches. Results are listed for the OBE potentials $A$, $B$ and $C$ defined in table A.1 of ref.[14] considering a small (0p and 1s0d shells) and a large model space (all shells up to 1p0f shell). All energies are given in MeV.

|                  | small space | large space |
|------------------|-------------|-------------|
|                  | OBE $A$     | OBE $B$     | OBE $C$     | OBE $A$     | OBE $B$     | OBE $C$     |
| 2.order          | -9.68       | -9.23       | -8.95       | -21.45      | -20.77      | -20.41      |
| ph RPA           | -1.79       | -1.46       | -1.27       | -3.53       | -2.44       | -1.97       |
| pp-hh RPA        | -3.36       | -2.93       | -2.71       | -8.65       | -7.56       | -7.06       |
| SRPA             | -8.83       | -6.98       | -6.05       | div         | div         | div         |
Table 2: Correlation energies for $^{16}$O obtained with single-particle propagators as defined in eq.(3.19). For each approximation the first line gives the contribution of the terms of second order, whereas the second line shows the sum of all terms of higher order in the residual interaction. Further details see table 1.

| Approximation | small space | large space |
|---------------|-------------|-------------|
|               | OBE A | OBE B | OBE C | OBE A | OBE B | OBE C |
| 2.order ph RPA | -6.75 | -6.67 | -6.61 | -14.60 | -14.67 | -14.73 |
|                | -1.68 | -1.46 | -1.33 | -3.41  | -2.92  | -2.68  |
| 2.order pp-hh RPA | -6.34 | -6.27 | -6.21 | -12.76 | -12.85 | -12.88 |
|                | -2.96 | -2.67 | -2.53 | -7.79  | -7.22  | -7.00  |
| 2.order SRPA   | -4.77 | -4.85 | -4.90 | -8.87  | -9.51  | -9.85  |
|                | -5.15 | -4.68 | -4.41 | -11.14 | -10.20 | -9.73  |
Table 3: Occupation probabilities for the single-particle orbits listed in column 1, calculated from the particle numbers of eq.(2.12). Results are presented for the NN interaction “B”, for various approximations and model spaces (see caption of table 1). The last column lists occupation probabilities, which are obtained if for the SRPA approach in the large model space the sum on the right hand side of eq.(2.12) is restricted to $N = 1$.

| Orbit  | small space | large space | Approx |
|--------|-------------|-------------|--------|
|        | SRPA | ph RPA | pp-hh RPA | SRPA |
| $s_{1/2}$ | - | 0.976 | 0.966 | 0.964 | 0.966 |
| $p_{3/2}$ | 0.918 | 0.942 | 0.935 | 0.842 | 0.894 |
| $p_{1/2}$ | 0.925 | 0.944 | 0.922 | 0.863 | 0.889 |
| $d_{5/2}$ | 0.045 | 0.027 | 0.028 | 0.078 | 0.051 |
| $d_{3/2}$ | 0.019 | 0.017 | 0.027 | 0.045 | 0.036 |
| $f_{7/2}$ | - | 0.002 | 0.003 | 0.003 | 0.003 |
| $1p_{3/2}$ | - | 0.008 | 0.010 | 0.011 | 0.011 |
| $f_{5/2}$ | - | 0.004 | 0.007 | 0.006 | 0.006 |
| $1p_{1/2}$ | - | 0.007 | 0.009 | 0.010 | 0.009 |

Table 4: Results for the radius (in [fm]) of $^{16}$O calculated in various approximations are presented using the three different OBE potentials A, B and C. The interactions have been adjusted to obtain the same result in the Hartree-Fock (HF) approximation. Correlation effects have been calculated in the large model-space.

| Approx | A  | B  | C  |
|--------|----|----|----|
| HF     | 2.634 | 2.634 | 2.634 |
| ph RPA | 2.673 | 2.671 | 2.671 |
| pp-hh RPA | 2.684 | 2.683 | 2.682 |
| SRPA   | 2.732 | 2.718 | 2.712 |
Figure Caption

Figure 1: Iteration of Eq.(2.3) including (a) forward-going and (b) backward-going vertices. In (c) we show a general diagram for the wavefunction.

Figure 2: The occupation probabilities in $^{16}$O for the orbits in the $0s$ to $1s0d$ shell are represented in terms of columns. The results obtained by various approximations are distinguished by the shading of the bars. All results were obtained for OBE potential $C$. 
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