Classification of Static Charged Black Holes in Higher Dimensions

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The uniqueness theorem for static charged higher dimensional black hole containing an asymptotically flat spacelike hypersurface with compact interior and with both degenerate and non-degenerate components of event horizon is proposed. By studies of the near-horizon geometry of degenerate horizons one was able to eliminate the previous restriction concerning the inequality fulfilled by the charges of the adequate components of the aforementioned horizons.

I. INTRODUCTION

The uniqueness theorems of black holes in general relativity is to show that the static electrovac black hole spacetime are described by Reissner-Nordström (RN) spacetime whereas the circular one is diffeomorphic to Kerr-Newman (KN) spacetime. Israel [1] was the first to raise the problem of classification of non-singular black hole solutions. Müller zum Hagen et al. [2] and Robinson [3], were able to weaken Israel’s assumptions while the most complete results were proposed in Refs. [4–8]. In Ref. [9] the classification of static vacuum black hole solutions was finished in [9] (see also the latest refinement of the proof [10]), where the condition of non-degeneracy of the event horizon was removed. In Ref. [11] it was shown that for the static electro-vacuum black holes all degenerate components of the event horizon should have charges of the same signs. Recently, by study of the near-horizon geometry this last restriction was removed [12].

The uniqueness black hole theorem for stationary axisymmetric spacetime was elaborated in Refs. [13], while the complete proof was found by Mazur [14] and Bunting [15] (see for a review of the uniqueness of black hole solutions story see [16] and references therein).

The unification attempts such as M/string theory caused the resurgence of works devoted to the mathematical aspects of the low-energy string theory black holes as well as higher dimensional ones. Namely, the staticity theorem for Einstein-Maxwell axion dilaton (EMAD) gravity was studied in Ref. [17]. Then, the uniqueness of the black hole solutions in dilaton gravity was proved in works [18–20], while the uniqueness of the static dilaton \( U(1)^2 \) black holes being the solution of \( N = 4, \ d = 4 \) supergravity was provided in [21]. The extension of the proof to the theory to allow for the inclusion of \( U(1)^N \) static dilaton black holes was established in Ref. [22].

On the other hand, the uniqueness theorem for \( n \)-dimensional black hole, both in vacuum and charged case were proposed [23–26]. The complete classification of \( n \)-dimensional charged black holes having both degenerate and non-degenerate components of event horizon was provided in Ref. [27]. In Ref. [36], taking into account the both electric and magnetic components of \((n - 2)\)-gauge form \( F_{\mu_1...\mu_{n-2}} \) the uniqueness of static higher dimensional electrically and magnetically charged black hole containing an asymptotically flat hypersurface with compact interior and non-
degenerate components of the event horizon was proved. On the other hand, the staticity theorem for generalized Einstein-Maxwell (EM) system was discussed in [37].

Proving the uniqueness theorem for stationary $n$-dimensional black holes is much more complicated. It turned out that generalization of Kerr metric to arbitrary $n$-dimensions proposed by Myers-Perry [28] is not unique. The counterexample showing that a five-dimensional rotating black hole ring solution with the same angular momentum and mass but the horizon of which was homeomorphic to $S^2 \times S^1$ was presented in [29] (see also Ref. [30]). The existence of black rings is consistent with the generalization of Hawking’s theorem [31]. It turned out that [32] cross sections of event horizons and outer horizons are of positive Yamabe type, i.e., they admit metrics of positive scalar curvature. In Ref. [33] the possible geometric types of event horizons for five and higher dimensional spacetimes were discussed. Recently, it has been established that Myers-Perry solution is the unique black hole in five-dimensions in the class of spherical topology and three commuting Killing vectors [34]. The uniqueness theorem for self-gravitating nonlinear $\sigma$-models in higher dimensional spacetime was obtained in [35].

Our paper will be devoted to the uniqueness of static higher dimensional charged black holes in generalized EM system described by $(n-2)$-gauge form $F_{\mu_1...\mu_{n-2}}$ containing an asymptotically flat spacelike hypersurface with compact interior with both degenerate and non-degenerate components of the event horizon. Studies of the near-horizon geometry of degenerate black hole enables one to eliminate the assumption concerning the charge of the adequate components of the horizons, i.e., to eliminate the last restriction in the complete classification of charged static $n$-dimensional black holes described in [27].

II. HIGHER DIMENSIONAL GENERALIZED EINSTEIN-MAXWELL SYSTEM

The action of the generalized Maxwell $(n-2)$-gauge form $F_{\mu_1...\mu_{n-2}}$ in $n$-dimensional spacetime we consider here is given by

$$I = \int d^n x \sqrt{-g} \left[ (n) R - F_{(n-2)}^2 \right],$$

where $g_{\mu\nu}$ is $n$-dimensional metric tensor, $F_{(n-2)} = dA_{(n-3)}$ is $(n-2)$-gauge form field. The energy momentum tensor of the $(n-2)$-gauge form $T_{\mu\nu} = \frac{\delta I}{\delta g^{\mu\nu}}$ yields

$$T_{\mu\nu} = (n-2)F_{\mu i_1...i_{n-2}}F_{\nu}^{i_2...i_{n-2}} - \frac{g_{\mu\nu}}{2}F_{(n-2)}^2.$$  

(2)

We shall confine our attention to the asymptotically flat spacetime. Thus, the spacetime with contain a data set $(\Sigma_{\text{end}}, g_{ij}, K_{ij})$ with gauge field determined by $(n-2)$ gauge form field $F_{i_1...i_{n-2}}$ for which $\Sigma_{\text{end}}$ is diffeomorphic to $\mathbb{R}^{n-1}$ minus a ball. The asymptotical conditions imply

$$|g_{ij} - \delta_{ij}| + r |\partial_a g_{ij}| + ... + r^m |\partial_{a_1...a_m} g_{ij}| + r |K_{ij}| + ... + r^m |\partial_{a_1...a_m} K_{ij}| \leq O \left( \frac{1}{r} \right).$$  

(3)

$$|F_{i_1...i_{n-2}}| + r |\partial_a F_{i_1...i_{n-2}}| + ... + r^m |\partial_{a_1...a_m} F_{i_1...i_{n-2}}| \leq O \left( \frac{1}{r^2} \right).$$  

(4)

First, as in Ref. [27] we introduce the generalization of the definition of electric components for the ordinary Maxwell fields. One can define electric $(n-3)$-form by the expression as follows:
\[ E_{i_1 \ldots i_{n-3}} = F_{i_1 \ldots i_{n-2}} k^{i_{n-2}}. \]  

On the other hand, *magnetic* 1-form may be written as

\[ B_k = \frac{1}{\sqrt{2(n-2)!}} \epsilon_{k \mu i_1 \ldots i_{n-2}} F^{i_1 \ldots i_{n-2}} k^\mu. \]

We introduce also the rotation \((n-3)\)-form of the stationary Killing vector field \( k_\mu \) in the form of

\[ \omega_{j_1 \ldots j_{n-3}} = \frac{(n-2)!}{\sqrt{2(n-2)!}} \epsilon_{j_1 \ldots j_{n-3} \mu \nu \gamma} k^\mu \nabla^\nu k^\gamma. \]

Using the definition (7) and definition of *electric* and *magnetic* forms and having in mind equations of motion for \((n-2)\)-gauge form fields one achieves at Eqs. of motion for *magnetic* 1-form \( B_k \)

\[ \nabla^\mu \left( \frac{B_k}{N} \right) = \frac{E^{i_1 \ldots i_{n-3}} \omega_{i_1 \ldots i_{n-3}}}{N^2}, \]

and similarly for *electric* \((n-3)\)-form \( E_{i_1 \ldots i_{n-3}} \)

\[ \nabla^\mu \left( \frac{E_{i_1 j_2 \ldots j_{n-3}}}{N} \right) = - \frac{B^a \omega_{a j_2 \ldots j_{n-2}}}{N^2}, \]

where we have denoted \( N = k_\mu k^\mu \).

If one applies the relation valid for the Killing vector fields \( \nabla_\alpha \nabla_\beta \xi_\gamma = - R_{\alpha \beta \gamma} \xi_\delta \) and definitions (5) and (6) to equation (7), it can be verified that

\[ \omega^{[j_1 \ldots j_{n-3}; b]} = - \alpha E^{i_1 \ldots j_{n-3}} B_{bi}, \]

where \( \alpha = \frac{s(n-2)^{3/2}}{\sqrt{(n-3)!}} \).

For we look for the static metric the right-hand sides of Eqs. (8) and (9) are equal to zero and we have

\[ \epsilon_{j_1 \ldots j_{n-3} j_{n-2} i_1 \ldots i_{n-3} k} E_{i_1 \ldots i_{n-3}} B_k = 0. \]

If we choose *electric* \((n-3)\)-form as \( E_{i_1 \ldots i_{n-3}} = \delta^0_{i_1} \delta^0_{i_2} \ldots \delta^m_{i_{n-3}} \nabla_m \phi \) and *magnetic* 1-form \( B_k = \nabla_k \psi \) one can conclude that \( \psi = c \phi \). Using equations of motion for magnetic \( B_k \) 1-form in the static case, i.e.,

\[ \nabla^\mu \left( \frac{B_k}{N} \right) = 0, \]

and multiplying it by \( \delta^0_{i_1} \delta^0_{i_2} \ldots \delta^m_{i_{n-3}} \) it then follows directly that \( c = const. \) Thus, by virtue of the hypothesis of staticity and by means of choosing the special components of \( E_{i_1 \ldots i_{n-3}} \), the *magnetic* field can be made vanish by a duality rotation. Namely, suppose that \( c \) is constant on each connected component of the set \( \Omega \), where \( E_{i_1 \ldots i_{n-3}} \neq 0 \) and suppose moreover that \( \Omega_0 \) be any connected component of the above set. By performing a duality rotation we can obtain \( \psi = 0 \) in \( \Omega_0 \). Because of the fact that \( \Omega_0 \) is open set, having in mind equations of motion and the unique continuation theorem of Ref. [38] one has that \( \psi = 0 \) and therefore \( B_k = 0 \).

From now on we shall consider only the *electric* component of \( F_{i_1 \ldots i_{n-2}} \). As in four-dimensional case the idea of the proof will be to show that degenerate components of the event horizon are only admissible for Majumdar-Papapetrou spacetimes. We prove that the metric of static degenerate horizon is spherical. Then, one can use a conformal
transformation to the electric potential, normalized in such a case that it tends to zero at infinity, to prove that this geometry is possible to occur with $\phi = \pm 1$ on a component of the horizon iff the metric is of Majumdar-Papapetrou type. Then this problem reduces to the one where $|\phi|$ is strictly bounded away from one. It was shown that it leads to the Reissner-Nordström geometry [11,27].

To begin with we shall introduce a Gaussian null coordinates near the event horizon of black hole as in [39,40]. Namely, let $\Sigma$ be a Cauchy hypersurface for the exterior region of the black hole. Let us introduce local coordinates $x^a$, where $a = 1, \ldots, n-2$ on an event horizon. Consider a point $p(x^a)$ on $\mathcal{H}$ through which a future directed geodesic with tangent vector $k_a$ will pass. The integral curves of vector field $k_a$ are the null generators of $\mathcal{H}^+$. On a sufficiently small neighbourhood $\mathcal{S}$ of a point $p$, let $\theta : \mathcal{S} \to \mathcal{H}^+$ be the map which takes $\{p, v\}$ into point of $\mathcal{H}^+$ lying at parameter value $v$ along the integral curve of $k^a$ starting at point $p$. This procedure defines coordinates on a neighbourhood $\mathcal{U} \subset \mathcal{H}^+$ with $k_a = \left( \partial / \partial v \right)_a$. At each point of $\mathcal{H}^+$ let $l^a$ be the unique null vector satisfying the conditions $l^a k_a = 1$ and $l^a X_a = 0$, for all $X_a$ tangent to the surface of constant $v$. Let us consider further the map which takes $\{p, r\}$ into the point of manifold lying at affine parameter $r$ along the null geodesic from $p \in \mathcal{U}$ with the tangent vector $l^a$. We have extended the functions $v$, $x^a$ by requiring that their values to be constant along each null geodesics determined by $l^a = \left( \partial / \partial r \right)^a$. On the event horizon $\mathcal{H}^+$ one has $\mathcal{L}_{k_a} l^a = 0$, moreover $k^a$ is a Killing vector field and hence geodesics are mapped to geodesics under the flow of the vector $k_a$. Since, the vector $l^a$ is everywhere tangent to null geodesics we have that $g_{rr} = 0$. Because of the independence of $r$ coordinate, the metric functions $g_{rv} = 1$, $g_{ra} = 0$. The fact that $\partial / \partial x^a$ is tangent to the surface of constant $v$ and hence orthogonal to $k_a$ at $r = 0$, then $g_{va} = rh_a(r, x^a)$ for some function $h_a$ independent of the $u$ coordinate. Hence, there exists smooth function $\varphi$ $|_{\mathcal{H}^+} = \partial g_{vv} / \partial r \mid_{r=0}$ the spacetime metric implies

$$ds^2 = r^2 dv^2 + 2dvdv + 2rh_a dx^a dv + h_{ij} dx^i dx^j,$$  \hspace{0.5cm} (13)

which $h_{ij}$ is a metric tensor on $(n-2)$-dimensional spacetime.

The location of the event horizon is given by the relation $r = 0$. The surface gravity equals to $\kappa = -\partial_r (r\varphi)$. Thus, for the degenerate horizon the surface gravity is zero so it implies the fact that $\varphi = A(r, x^a) \cdot r$. The form of the metric (13) guarantees us the existence of a regular near-horizon geometry which is defined by the limit $r \to \epsilon \cdot r$ and $v \to v / \epsilon$. Consequently, when $\epsilon \to 0$ the metric (13) tends to the metric which can be expressed as

$$ds_{(0)}^2 = r^2 A_{(0)} dv^2 + 2dvdv + 2rh_{(0) a} dx^a dv + h_{(0)ab} dx^a dx^b,$$  \hspace{0.5cm} (14)

with the additional conditions as follows:

$$\partial_r A_{(0)} = \partial_r h_{(0) a} = \partial_r h_{(0) ab} = \partial_r A_{(0)} = \partial_r h_{(0) a} = \partial_r h_{(0) ab} = 0,$$  \hspace{0.5cm} (15)

where we have denoted $A_{(0)} = A \mid_{r=0}$, $h_{(0) a} = h_a \mid_{r=0}$ and $h_{(0)ab} = h_{ab} \mid_{r=0}$. So the line element $ds_{(0)}^2$ encompasses information about the behaviour of $h_{ab}$, $h_a$ and $A$ at the event horizon $\mathcal{H}$. Having in mind the form of the metric near horizon, after tedious calculations it can be shown that $n$-dimensional generalized Einstein-Maxwell equations of motion yield

$$\frac{1}{2} h_{(0)a} h_{(0)b} - (n-2) \nabla_{(a} h_{(b)\gamma} = - (n-2) R_{(0)ab} + T_{ab} + \frac{T}{2 - n} h_{(0)ab},$$  \hspace{0.5cm} (16)
where $R_{(0)ab}$ is the Ricci tensor of metric $h_{(0)ab}$ and $(n-2)\nabla_a$ is a derivative with respect to this metric. As was shown in Ref. [10,12] the staticity of $g_{\mu\nu}$ implies the staticity of $g_{(0)\mu\nu}$, which in turns leads to the fact that at the event horizon $h_a$ is a gradient, namely $h_a \, dz^a = d\lambda$. It helps us to rewrite Eq.(16) as

$$
(n-2)\nabla_a (n-2)\nabla_b \lambda = \frac{1}{2} (n-2)\nabla_a \Lambda (n-2)\nabla_b \lambda = -(n-2) R_{(0)ab} + 2(n-3)^2 |\nabla \phi|^2 h_{(0)ab}.
$$

Further, we introduce $\psi = e^{-\lambda/2}$ so one finally gets the following:

$$
(n-2)\nabla_a (n-2)\nabla_b \psi = \frac{\psi}{2} (n-2) R_{(0)ab} - (n-3)^2 |\nabla \phi|^2 \psi h_{(0)ab}.
$$

Consistently with the above the trace of the expression (18) may be determined by

$$
(n-2)\nabla^2 \psi = \frac{\psi}{2} (n-2) R_{(0)} - (n-2)(n-3)^2 |\nabla \phi|^2 \psi.
$$

After taking the $(n-2)\nabla^b$ derivative of equation (18) and using the relation (19), one gets

$$
(n-2)\nabla \left( \psi^3 (n-2) R - \frac{4}{3} B |\nabla \phi|^2 \psi^3 \right) = 0,
$$

where we have denoted $B = (n-3)[(n-3)^2 + \frac{1}{2}(n-2)]$. This is sufficient to establish that the Ricci scalar curvature may be written in the form

$$
(n-2) R_{(0)} = \frac{3c_1 + 4B |\nabla \phi|^2 \psi^3}{3\psi^3},
$$

where $c_1$ is a constant.

In the light of what has been shown before, it can be noted that one can rewrite Eq.(19) and (18) as follows:

$$
(n-2)\nabla^2 \psi = \frac{3c_1 + 2(n-3)\{n(n-2) - n(n-3)\}}{6\psi^2} |\nabla \phi|^2 \psi^3,
$$

and

$$
(n-2)\nabla_a (n-2)\nabla_b \psi = \frac{\left[ 3c_1 + 4(n-3)\{(n-3)(n-6) + 2(n-2)\} \right] |\nabla \phi|^2 \psi^3}{12\psi^2} h_{(0)ab}.
$$

One can also find that

$$
(n-2)\nabla^m \psi (n-2)\nabla_m \psi = c_2 - \frac{c_1}{2\psi} + \frac{1}{6} K |\nabla \phi|^2 \psi^2,
$$

where $K = 2(n-3)\{(5-n) + (n-3)^2\}/6$ and $c_2$ is some constant.

Now we shall take up the question about the critical set of $\psi$. We denote by $a = \inf \psi$ and $b = \sup \psi$. Suppose further, that $a \neq b$. Because of the fact that $\psi$ is smooth and positive we have $0 < a \leq b < \infty$. By virtue of the fact that for any point $p$ such that $\psi(p) = b$, one has that $(n-2)\nabla^m \psi(p) = 0$ and $(n-2)\nabla^m \psi (n-2)\nabla_m \psi(p) < 0$ examination of the relation (22) reveals the following:

$$
c_1 < -\frac{2(n-3)}{3} \left[ (n-2) - n(n-3) \right] |\nabla \phi|^2 b^3,
$$

for $\nabla \phi \neq 0$. On further simplification, Eq.(23) can be brought to the form revealing the condition for $c_2$, i.e.,
from

The other condition for \(c\) is given by

\[
c_2 = \frac{(n-3)^2}{18}(n-4) \mid \nabla \phi \mid^2 b^2.
\]

Let \(p_{\text{min}}\) be a minimum of \(\psi\), then \(\psi \circ \gamma\) is a solution of the Cauchy problem for differential equation

\[
d^2 \psi = \left[\frac{3c_1 + 4(n-3)[(n-3)(n-6) + 2(n-2)]}{12\psi^2}\right] \mid \nabla \phi \mid^2 \psi^3,
\]

with the conditions \(\psi(0) = p_{\text{min}}\) and \(\frac{d\psi}{ds}(0) = 0\). As in Ref. [23] we conclude that \(\psi\) depends only on geodesic distance from \(p_{\text{min}}\) and not on the direction of the geodesics. This leads to the conclusion that the level sets of \(\psi\) coincide with the geodesic sphere centered at \(p_{\text{min}}\) (within the injectivity radius of \(p_{\text{min}}\)). The same considerations take place for the case of \(p_{\text{max}}\), where \(p_{\text{max}}\) is the maximum of \(\psi\).

On the set \(\Omega = \{(n-2)\nabla \psi \neq 0\}\), \(\psi\) may be locally used as a coordinate. Thus, the metric may be written in the form as

\[
d\psi = \frac{d\psi}{F^2(\psi)} + H^2(\psi, \xi) d\xi^2,
\]

where \(\xi\) is a local coordinate on the level set of \(\psi\), while \(F^2(\psi)\) is given by

\[
F^2(\psi) = \frac{(n-2)\nabla \psi \mid^2 = c_2 - \frac{c_1}{2\psi} + \frac{1}{6} K \mid \nabla \phi \mid^2}{2}.
\]

Inside the radius of injectivity of \(p_{\text{min}}\), one has that \(\frac{d\psi}{d\rho} = \rho\), where \(\rho\) is the distance function from \(p_{\text{min}}\). Normalization of \(\xi\) to run from zero to \(2\pi\) and by redefinition \(\xi \rightarrow \lambda \xi\) provides the condition that \(F^2(\psi) = \lambda^2 \rho^2 + O(\rho^2)\). This is sufficient to establish a diffeomorphism among the level sets of \(\psi\) within \(\Omega\).

In the radius of injectivity of \(p_{\text{max}}\) the arguments are the same, leading to the condition \(F^2(\psi) = \lambda^2 \rho^2 + O(\rho^2)\), where now \(\hat{\rho}\) is a distance function from \(p_{\text{max}}\). Eliminating \(\lambda\) one concludes that the following implies

\[
3c_1(b^2 + a^2) = -2(n-3)^2 \mid \nabla \phi \mid^2 a^2b^2(a + b).
\]

The other condition for \(c_1\) we may obtain from the relation \(F(a) = F(b) = 0\), i.e.,

\[
c_1 = -\frac{1}{3}(n-3)^2 \mid \nabla \phi \mid^2 ab(a + b).
\]

Substituting \(c_1\) from (31) to relation (30) we get that \((a-b)^2 = 0\), i.e., \(a = b\) which leads to a contradiction. In the light of the above considerations, one can see that the regularity of the metric requires the condition that \(\mid (n-2)\nabla \psi \mid = 0\). This provides the fact that \(\lambda = \text{const}\). It implies from Eq.(17) that \(h_{(0)ab}\) is the metric of \(S^{n-2}\) sphere.

Our next step is to define on the hypersurface \(\Sigma\) the orbit metric \(\gamma_{ab}\) defined as follows:

\[
\gamma(Y, Z) = g(Y, Z) - \frac{g(X, Y) g(X, Z) g(X, X)}{g(X, X)},
\]

where by \(X\) we have denoted the Killing vector which asymptotes \(\frac{\partial}{\partial \psi}\) in the asymptotic region. It also fulfills the staticity condition. As in Ref. [23]-[25] we shall consider conformal transformation of the form \(g_{\mu\nu} = \Omega_{\pm}^2 \gamma_{\mu\nu}\), where

\[
\Omega_{\pm} = \left[\frac{(1 \pm V)^2 - \phi^2}{4}\right]^\frac{1}{n-3}.
\]

As in our previous work [27] \(\phi\) satisfies the condition

\[
c_2 < \frac{(n-3)^2}{18}(n-4) \mid \nabla \phi \mid^2 b^2.
\]
One can see that the conformal factors $\Omega_{\pm}$ are nonnegative. The inequalities are strict except when the metric is locally of Majumdar-Papapetrou type [11,27]. The possibility that $|\phi|$ is strictly bounded away from one gives us the Reissner-Nordström solutions.

On the contrary, we assume a contradiction. Namely, we suppose that there exists a component of the event horizon $H$ on which $\phi = \pm 1$ and in this case $H$ is degenerate. Eq.(34) provides that $\phi = \pm (1 - V + O(r^{n-2}))$. Taking into account the fact that $V = |g_{uu}|^{1/2} = \sqrt{A_{(0)} r + O(r^{n-2})}$, one gets the following expressions for the conformal factors:

\begin{align}
\Omega_+ &= A_{(0)}^{\beta} r^\beta + O(r^{n-2+\beta}), \\
\Omega_- &= O(r^{\beta/n-3}),
\end{align}

where $\beta = 1/(n - 3)$.

Thus, it leads to the following form of the conformal metric

\[ ds^2_\pm = \Omega_\pm^2 g_{uu} \gamma_{\mu\nu} = A_{(0)}^{\beta - 1} r^{2(\beta - 1)} \left[ 1 + O(r^\beta) \right] [dr^2 + O(r^2)dx^a dr] + r^2(1 + O(r))A_{(0)} h_{(0)ab} dx^a dx^b + O(r^3)dx^a dx^b. \]

The coordinate $r$ can be thought as a radial coordinate, As was shown $h_{(0)ab}$ is the metric of the sphere, just $A_{(0)}h_{(0)ab}$ is the unit $S^{n-2}$ sphere. One has the following:

\[ r^2 A_{(0)} h_{(0)ab} dx^a dx^b = dx_m dx^m - dr^2. \]

This part of the metric in Eq.(38) combines with the leading part of the term $dr^2$ in relation (38). The term $O(r^\beta)dr^2$ gives a contribution which in coordinates $x^m$ vanishes at the origin of coordinates as $O(x^i)$. On the other hand, the similar reasoning as presented in Ref. [12] shows that $O(r^2)dx^a dr$ gives the contribution in $x_i$ coordinates of the form $r(g(x_a/r)) dx^i dx^j$, where $g(x_a/r)$ is a smooth function. The analysis of $dx^m dx^n$ terms reveals that $ds^2_\pm$ can be rewritten as

\[ ds^2_\pm = A_{(0)}^{\beta - 1} r^{2(\beta - 1)} \left[ \delta_{ij} + O(x^i) \right] dx^i dx^j. \]

One can see that in four-dimensional case we get the limit treated in Ref. [12], namely $A_{(0)}^{\beta - 1} r^{2(\beta - 1)} = 1$.

It is convenient to rescale $g_{\mu\nu(+)}$ in such a way that its scalar curvature vanishes. Just we shall look for a conformal transformation of the metric $g_{\mu\nu(+)}$ such that the scalar curvature vanishes. We construct $\psi^{2\alpha} g_{\mu\nu(+)}$ to be the solution of the following relation:

\[ \psi^{2\alpha + 1} R \left( \psi^{2\alpha} g_{\mu\nu(+)} \right) = R \left( g_{\mu\nu(+)} \right) \psi - 2g_{(+)} \nabla^\mu g_{(+)} \nabla_\mu \psi = 0, \]

where $\alpha = \frac{1}{n-1}$.

Then, one can search for $\psi$ in the form as $\psi = 1 + u$, where $u$ vanishes on a coordinate sphere of radius $b$. It implies

\[ -2g_{(+)} \nabla^\mu g_{(+)} \nabla_\mu u + R(g_{(+)} + R(g_{\mu\nu(+)})) u = -R(g_{\mu\nu(+)}). \]
For some constant the scalar curvature $R\left(g_{\mu\nu}(+)\right)$ satisfies the inequality $R\left(g_{\mu\nu}(+)\right) \leq C/|x|$. It can be seen from Eq.(39) and the fact that the first and the second derivative of $g_{\mu\nu}(+)$ is bounded. Using the similar arguments as presented in Ref. [10] one can show the existence of a solution $u_\epsilon$ to the equation (41) vanishing both on the coordinate sphere of radius one as well as on radius $\epsilon$.

In the case of $g_{\mu\nu}(-)$ one can write the following:

$$g_{\mu\nu}(-) = \left(\frac{\Omega^2}{\Omega^+}\right)^{2\alpha}\tilde{g}_{\mu\nu}(+) = \hat{\psi}^{2\alpha}\hat{g}_{\mu\nu}(+),$$

where $\hat{g}_{\mu\nu}(+) = \psi^{2\alpha}g_{\mu\nu}(+)$. It can be checked that under the conformal transformation the scalar curvature tensor has the form as

$$\hat{\psi}^{2\alpha+1}R\left(g_{\mu\nu}(-)\right) = R\left(\hat{g}_{\mu\nu}(+)\right)\psi - 2\hat{g}_{\mu\nu}(+)\nabla\mu\nabla\nu\hat{\psi}. \quad (43)$$

The first term of the right-hand side of Eq.(43) vanishes due to our previous construction. Applying the maximum principle and the Serin removable singularity theorem [41] as well as the comparison principle one gets that $\hat{\psi}$ at the origin of coordinates $x^i$ is greater than zero (see for the details Ref. [12]). On the other hand, having in mind the form of conformal factors $\Omega_{\pm}$, one concludes that this contradicts the fact that $\Omega_-/\Omega_+ \to 0$ as we approach the origin of the coordinates. Then, one obtains that $\phi$ is equal $\pm 1$ at the origin of coordinates. This situation is only possible for Majumdar-Papapetrou solutions. This fact leads as to the main conclusion of the work, namely:

**Theorem:**

Let us consider a static solution to the $n$-dimensional Einstein-Maxwell equations of motion with an asymptotically timelike Killing vector field $k_{\mu}$. Suppose, further that the manifold under consideration consists of a connected and simply connected spacelike hypersurface $\Sigma$ to which $k_{\mu}$ is orthogonal. The topological boundary $\partial \Sigma$ of $\Sigma$ is a nonempty topological manifold with $h_{ij}k^ik^j = 0$ on $\partial \Sigma$. Then, one arrives at the following conclusions:

1. If $\partial \Sigma$ is connected, then there exist a neighbourhood of $\Sigma$ which is isometrically diffeomorphic to an open subset of $n$-dimensional Reissner-Nordström spacetime (extreme or non-extreme).

2. If $\partial \Sigma$ is not connected then there is an open neighbourhood of $\Sigma$ which is isometrically diffeomorphic to $n$-dimensional Majumdar-Papapetrou spacetime.

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