Data-driven safe gain-scheduling control

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Abstract
Data-based safe gain-scheduling controllers are presented for discrete-time linear parameter-varying systems (LPV) with polytopic models. First, \( \lambda \)-contractivity conditions are provided under which the safety and stability of the LPV systems are unified through Minkowski functions of the safe sets. Then, a data-based representation of the closed-loop LPV system is provided, which requires less restrictive data richness conditions than identifying the system dynamics. This sample-efficient closed-loop data-based representation is leveraged to design data-driven gain-scheduling controllers that guarantee \( \lambda \)-contractivity and, thus, invariance of the safe sets. It is also shown that the problem of designing a data-driven gain-scheduling controller for a polyhedral (ellipsoidal) safe set amounts to a linear program (a semi-definite program). The motivation behind direct learning of a safe controller is that identifying an LPV system requires satisfying the persistence of the excitation (PE) condition. It is shown in this paper, however, that directly learning a safe controller and bypassing the system identification can be achieved without satisfying the PE condition. This data-richness reduction is of vital importance, especially for LPV systems that are open-loop unstable, and collecting rich samples to satisfy the PE condition can jeopardize their safety. A simulation example is provided to show the effectiveness of the presented approach.

KEYWORDS
data-driven control, gain-scheduling control, invariant sets, safe control, set-theoretic methods

1 | INTRODUCTION
Satisfying safety constraints is a fundamental requirement for control systems that must be deployed on safety-critical systems, such as autonomous vehicles and robots. Design of safe controllers using barrier certificates [1–9] and reachability analysis [10–13] has been widely and successfully considered. These methods, however, mostly rely on a high-fidelity model of the system under control. To account for model uncertainties, robust safe control methods [14, 15] design a controller for the worst-case uncertainty realization. Worst-case-based control design, nevertheless, can result in overly conservative control solutions and even infeasibility. On the other hand, adaptive safe control methods [16, 17] are designed to compensate for uncertainties and avoid conservatism. These methods, however, are based on the availability of an adaptive control Barrier function (aCBF), which is challenging to find for nonlinear systems. Safe reinforcement learning (RL) algorithms have also been presented in the literature [18–24] to learn constrained optimal control solutions for systems with uncertain dynamics. Nevertheless, a model-based safety certifier is used in safe RL methods to intervene with the RL actions whenever they are not safe.
The performance and conservativeness of barrier-based safety certifiers and controllers highly depend on the accuracy of the identified system model. However, as shown in van Waarde et al. [25], conditions imposed on the data richness for identifying a linear system model are generally more restrictive than conditions imposed on data richness for directly learning a controller that satisfies a system property (e.g., stability). Therefore, to avoid data-hungry learning developments, it is desired to directly design safe controllers using measured data along the system trajectories. Moreover, if stability is also of concern in these methods, a control Lyapunov function (CLF) based constraint is also typically imposed besides a control barrier function. When there is a conflict between safety and stability, however, the CLF is relaxed, which can result in the convergence of the closed-loop trajectories to an undesired equilibrium point on the boundary of the unsafe set [26]. To avoid this conflict, the concept of contractive sets [27] can be leveraged for linear systems with convex safe sets to unify the safe and stable control design. This idea is leveraged in previous studies [28–30] to directly design data-driven safe controllers for linear time-invariant systems. The data-based safe control design is also considered in Ahmadi et al. [31] using only measured data collected from open-loop system trajectories. Existing results for direct data-driven safe control design are limited to linear time-invariant systems and impose restrictive requirements on data richness.

Designing data-driven safe and stable controllers for general nonlinear systems with general safety constraints is a daunting challenge. However, many nonlinear systems such as aerospace systems [32–35], and a variety of robotic systems [36, 37] can be expressed by linear-parameter-varying (LPV) systems with a set of gain-scheduling parameters that are not known in advance, but can be measured or estimated during operation of the system. Safe and stable gain-scheduling control strategies can then be unified for LPV systems under convex and compact constraint sets [27].

This paper presents data-based safe gain-scheduling controllers for LPV systems with both ellipsoidal and polytopic safe sets. Our approach is inspired by Bisoffi et al. [29], which is presented for linear time-invariant systems with polytopic safe sets, and extends its results to nonlinear LPV systems with both closed and convex polyhedral and ellipsoidal safe sets. Moreover, it is shown here that the data requirement conditions for directly learning a safe controller are actually weaker than the standard persistence of excitation (PE) condition. That is, even when data samples do not satisfy the PE condition and thus identifying the LPV dynamics is not possible, the presented approach can directly learn a safe controller if the non-PE data satisfy a relaxed condition. When exploration to generate rich data is risky, this direct data-driven approach with a lower sample complexity will be highly advantageous to model-based approaches that rely on system identification.

To design direct data-driven safe gain-scheduling controllers, first, λ-contractivity conditions are provided for LPV systems with known dynamics under which the safety and stability of the LPV systems are guaranteed for both ellipsoidal and polyhedral safe sets through their related Minkowski functions. Then, to obviate the requirement of knowing the system dynamics, a data-based representation of the closed-loop system is provided, making evident how this parametrization is naturally related to the λ-contractive sets. The set invariance and stability of the LPV systems are then guaranteed through Minkowski functions. It is shown that the problem of designing controllers to enforce a given polyhedral set and an ellipsoidal set to be λ-contractive in the presented data-based framework amounts to a linear program and a semi-definite program, respectively. A motivation example shows that while a safe controller can be learned using available data measurements, the data is not rich enough to identify the LPV model. Finally, a simulation example is provided to show the effectiveness of the presented approach. The same version of this paper is available in arXiv in Modares et al. [38].

1.1 Outline
The paper is organized as follows: Section II formalizes the problem formulation for the safe gain-scheduling control problem. Section III provides a model-based solution of the formulated problem. Section IV presents the data-based representation of the closed-loop LPV systems with the gain-scheduled controller as a free decision variable parameter. Section V presents a data-based solution to the design of the gain-scheduling control problem. The simulation results are stated in Section VI. Finally, concluding remarks are provided in Section VII.

2 NOTATIONS AND PRELIMINARIES

2.1 Notations
Throughout the paper, ⊗ denotes the Kronecker product and ⊙ denotes the Khatri–Roa product, which is a column-wise Kronecker product of two matrices that have an equal number of columns. I denotes the identity matrix with appropriate dimension, 0ₙ denotes the n × n zero matrix and 1 denotes the vector of all ones of appropriate dimension. If A and B are matrices (or vectors) of the same dimensions, then A(≤, ≥)B implies a componentwise inequality, that is, Aᵢⱼ(≤, ≥)Bᵢⱼ for all i and j, where Aᵢⱼ is
the element of the \(i\)-th and \(j\)-th column of \(A\). In the space of symmetric matrices, \(Q \preceq 0\) denotes that \(Q\) is negative semi definite. Moreover, \(A^T\) is the right inverse of the matrix \(A\).

Given a polyhedron \(A\), \(\text{vert}(A)\) denotes the set of its vertices. Given a set \(S\) and a scalar \(\mu \geq 0\), the set \(\mu S\) is defined as \(\mu S := \{\mu x : x \in S\}\).

**Definition 1** (Blanchini and Miani [27]). A convex and compact set that includes the origin as its interior point is called a C-set.

**Definition 2.** The set \(\varepsilon(P, 1)\) is an ellipsoidal C-set and is represented by
\[
\varepsilon(P, 1) = \left\{ x \in \mathbb{R}^n : \sqrt{x^T P x} \leq 1 \right\},
\]
where \(P\) is a positive definite matrix.

**Definition 3.** A polyhedral C-set \(S(F, \bar{I})\) is represented by
\[
S(F, \bar{I}) = \left\{ x \in \mathbb{R}^n : F x \preceq \bar{I} \right\} = \left\{ x \in \mathbb{R}^n : F^T x \leq 1, i = 1, \ldots, q \right\},
\]
where \(F \in \mathbb{R}^{n \times q}\) is a matrix with rows \(F_i, i = 1, \ldots, q\).

### 3 | PROBLEM FORMULATION

This section formulates problems of safe control design for polytopic LPV systems with both polyhedral and ellipsoidal safe sets.

Consider the discrete polytopic LPV system given by [39–41]
\[
x(t + 1) = A(w(t))x(t) + Bu(t),
\]
where \(x(t) \in X\) is the system's state and \(u(t) \in U\) is the control input with \(X\) and \(U\) as constrained sets (e.g., ellipsoidal or polyhedral) containing the origin in their interiors. Moreover, \(B \in \mathbb{R}^{n \times m}\) is the input dynamic and is assumed fixed. The parameter-varying matrix \(A(w(t))\) is known to lie in the following polytope
\[
A(w(t)) = \sum_{i=1}^{s} A_i w_i(t),
\]
where \(A_i \in \mathbb{R}^{n \times n}, i = 1, \ldots, s\) are vertices of the polytope and \(w \in \Omega\) is a scheduling parameter vector. While the scheduling parameter can be measured online (e.g., the velocity of an aircraft), its future values are not known and are supposed to belong to the following polytope.
\[
\Omega = \left\{ w \in \mathbb{R}^s, w_i \geq 0, \sum_{i=1}^{s} w_i = 1 \right\}.
\] The gain-scheduling controller is typically considered as
\[
u(t) = K(w)x(t),
\]
where
\[
K(w) = \sum_{i=1}^{s} K_i w_i.
\] **Assumption 1.** The number of operating modes, that is, \(s\), is known. This can be prior knowledge or the knowledge obtained through clustering of the data samples collected from the system's trajectories, as performed in Sadati and Hooshmand [42].

**Problem 1.** Given a polyhedral C-set \(S(F, \bar{I})\), find the gain-scheduling controller \(u(t) = K(w)x(t), \) with \(K(w)\) defined in (6), such that it guarantees the following:
1) The set \(S(F, \bar{I})\) remains invariant.
2) The origin is asymptotically stable.

**Problem 2.** Given an ellipsoidal C-set \(\varepsilon(P, 1)\), find the gain-scheduling controller \(u(t) = K(w)x(t), \) with \(K(w)\) defined in (6), such that it guarantees the following:
1) The set \(\varepsilon(P, 1)\) remains invariant.
2) The origin is asymptotically stable.

In Problems 1 and 2, the first property guarantees system safety, and the second property guarantees its stability. In this paper, gain-scheduling controllers are designed to solve Problems 1 and 2 based on only the trajectories of data measurements collected from the system's inputs, states, and scheduling parameters. For both polyhedral and ellipsoidal C-sets, the safety and stability properties of LPV systems can be embedded in the notion of \(\lambda\)-contractivity, defined next. This will significantly simplify designing controllers that are both safe and stable and can avoid converging to an undesired equilibrium solution of the closed-loop system that can arise due to the conflict between safety and stability in barrier-certificate-based approaches [26].

**Definition 4** (Minkowski function). Given a C-set \(S\), its Minkowski function is
\[
\Psi_{S}(x) = \inf \{ \alpha \geq 0 : x \in \alpha S \}.
\]

**Definition 5** (Contractive set). Fix \(\lambda \in [0, 1)\). The C-set \(S\) is \(\lambda\)-contractive for the system \(x(t + 1) = f(x(t), t)\) if and only if \(x(0) \in S\) implies that \(x(t) \in \lambda S, \forall t \geq 0\).

For a \(\lambda\)-contractive set \(S\), the following condition holds [27]:
\[
\Psi_{S}(f(x, t)) \leq \lambda, \forall x \in S,
\]
where \( \Psi_S(x) \) is the Minkowski function of \( S \). The following results show that the Minkowski function is actually a (local/global) shared control Lyapunov function that guarantees both stability and safety of the LPV systems with (constrained/unconstrained) inputs.

**Theorem 1** (Blanchini and Miani [27]). Let \( u \in \mathbb{R}^n \) be unbounded and the C-set \( S \) (e.g., polyhedral or ellipsoidal) be \( \lambda \)-contractive for the closed-loop system (3). Then, the system is both safe and stable. Moreover, let \( \Psi_S(x) \) be the Minkowski function for the set \( S \). Then, \( \Psi_S(x) \) is a global control Lyapunov function.

**Corollary 1** (Blanchini and Miani [27]). Let \( u \in U \) be bounded and the C-set \( S \) (e.g., polyhedral or ellipsoidal) be \( \lambda \)-contractive for the closed-loop system (3). Then, the system is both safe and stable. Moreover, let \( \Psi_S(x) \) be the Minkowski function for the C-set \( S \). Then, \( \Psi_S(x) \) is a control Lyapunov function inside the set \( S \).

## 4 | Model-Based Controller Design for Solving Problems 1 and 2

It was shown in Theorem 1 that to solve Problem 1, it is sufficient to design a controller that guarantees that the set \( S(F, 1) \) is \( \lambda \)-contractive. The next results provide conditions under which the \( \lambda \)-contractiveness is guaranteed for both Polyhedral C-sets and ellipsoidal C-sets.

Before proceeding, the following notations are defined and used throughout the paper for the system (3) and (4).

\[
K_{1,s} = [K_1, K_2, \ldots, K_s], \\
A_{1,s} = [A_1, A_2, \ldots, A_s],
\]

where \( K_i \in \mathbb{R}^{m \times n} \). This gives

\[
A_{1,s} + BK_{1,s} = [A_1 + BK_1, \ldots, A_s + BK_s].
\]

### 4.1 | Model-based solving of Problem 1 for polyhedral sets

We present the next result on \( \lambda \)-contractivity for polytopic models under polyhedral C-set constraints on their states.

**Theorem 2.** Consider the LPV system (3) with \( s \) vertices and a polyhedral C-set \( S(F, 1) \) of the form (2). Let \( u(t) = K(w)x(t) \) with \( K(w) \) defined in (6). Then, the C-set \( S(F, 1) \) is \( \lambda \)-contractive for closed-loop system (3) if and only if there exists a non-negative matrix \( P_{1,s} = [p^1, p^2, \ldots, p^s] \) such that

\[
P_{1,s}(\delta_j \otimes 1) \leq \lambda 1, \ j = 1, \ldots, s \\
P_{1,s}(I \otimes F) = F(A_{1,s} + BK_{1,s}),
\]

where \( K_{1,s} \) and \( A_{1,s} + BK_{1,s} \) are defined in (9) and (10), respectively, and \( \delta_j \in \mathbb{R}^s \) is a vector with all elements zero except its \( j \)-th element, which is one.

**Proof.** The closed-loop polytopic LPV system (3) is \( \lambda \)-contractive if and only if it is \( \lambda \)-contractive in its \( s \) vertices. Therefore, the closed-loop polytopic LPV system (3) with \( u(t) = K(w)x(t) \) is \( \lambda \)-contractive if and only if there exist \( s \) non-negative matrices \( P^i \geq 0 \) and \( K_i \) satisfying [27]

\[
P^i F = F(A_i + BK_i), \ P^i \bar{1} \leq \lambda \bar{1}, \ i = 1, \ldots, s.
\]

Compounding the above inequalities and equalities, respectively, yields

\[
[p^1, p^2, \ldots, p^s] (\delta_j \otimes 1) \leq \lambda 1, \ j = 1, \ldots, s \\
[p^1, p^2, \ldots, p^s] (I \otimes F) = F[A_1 + BK_1, \ldots, A_s + BK_s],
\]

which yields (11) all together using \( A_{1,s} + BK_{1,s} \) in (10). This completes the proof. \( \square \)

### 4.2 | Model-based solving of Problem 2 for ellipsoidal sets

We present the next result on \( \lambda \)-contractivity for polytopic models under ellipsoidal constraints.

**Theorem 3.** Consider the LPV system (3) and an ellipsoidal C-set \( \epsilon(P, 1) \) of the form (1). Let \( u(t) = K(w)x(t) \) with \( K(w) \) defined in (6). Then, the C-set \( \epsilon(P, 1) \) is \( \lambda \)-contractive for closed-loop system (3) if and only if

\[
D_i^T(A_{1,s} + BK_{1,s})P(A_{1,s} + BK_{1,s})D_i = \lambda^2 P \leq 0, \ i = 1, \ldots, s,
\]

where \( D_i \in \mathbb{R}^{l \times n}, i = 1, \ldots, s \) and

\[
D_i^T = \begin{bmatrix}
I_n \\
\vdots \\
1\text{-st non matrix} \\
\vdots \\
I_n \\
\vdots \\
1\text{-th non matrix}
\end{bmatrix}
\]

\[
D_i^T = \begin{bmatrix}
o_n \\
\vdots \\
0_n, \ldots, o_n
\end{bmatrix}.
\]

**Proof.** The LPV system (3) with \( u = K(w)x \) is \( \lambda \)-contractive with respect to C-set \( \epsilon(P, 1) \) if and only if [27]
This richness condition for the collected dataset is investigated next. A rich data set will allow us to design data-based controllers that capture the dependency structure of the matrices of the LPV state-space model on the scheduling variables without requiring an explicit model or declaration of dependencies.

**Remark 2.** A promising data-based safe control design approach is presented in Bisoffi et al. [29] for linear time-invariant systems. However, it is not investigated in Bisoffi et al. [29] how the direct learning of a safe controller can reduce the sample complexity (i.e., the number of samples required to learn) compared to learning a system model first and then designing a model-based safe controller. That is, their developments are based on the assumption that the collected data satisfy the PE requirement. Satisfying the PE requirement for the LPV systems amounts to having the data matrix

\[
\begin{bmatrix}
U_0 \\
X_W
\end{bmatrix},
\]

with full row rank. That is, \(ns+m\) independent samples must be collected. This requires \(T \geq ns + m\) samples. As shown in the next theorem, inspired by van Waarde et al. [25], this condition provides necessary and sufficient for uniquely identifying the LPV system. Once the system is identified, the results of Theorems 2 and 3 can be used to design a model-based controller. However, as shown later, one can learn directly a data-based safe controller using less restrictive data informative conditions that require only \(T \geq ns + 1\). Therefore, it is more desirable to directly learn a safe controller.

**Theorem 4.** The LPV system (3) can be uniquely identified if the matrix (22) is full row rank. Moreover, under this condition, it has the following equivalent data-based representation

\[
x(t + 1) = X_1 \begin{bmatrix} U_0 \\ X_W \end{bmatrix}^\dagger \begin{bmatrix} u(t) \\ w(t) \otimes x(t) \end{bmatrix}.
\]  

**Proof.** Based on (3), the data collected in (17)–(21) satisfy

\[
X_1 = A_{1,s} \begin{bmatrix} w_d(0) \otimes x_d(0), \ldots, w_d(T-1) \otimes x_d(T-1) \end{bmatrix} + B \begin{bmatrix} u_d(0), \ldots, u_d(T-1) \end{bmatrix} = \begin{bmatrix} U_0 \\ X_W \end{bmatrix}
\]

where \(A_{1,s}\) is defined in (9). There exists a right inverse \([V_1 V_2]\) such that

\[
\begin{bmatrix} U_0 \\ X_W \end{bmatrix} [V_1 V_2] = I,
\]
if and only if the matrix (22) is full row rank. In this case, multiplying both sides of (24) by $[V_1 \ V_2]$, one can uniquely find $A_{1,s}$ and $B$ as $A_{1,s} = X_1 V_2$ and $B = X_1 V_1$. We now show that (23) holds. Based on (3), one has

$$x(t + 1) = \begin{bmatrix} B \\ A_{1,s} \end{bmatrix} \begin{bmatrix} u(t) \\ w(t) \otimes x(t) \end{bmatrix}. \tag{26}$$

On the other hand,

$$\begin{bmatrix} u \\ w \otimes x \end{bmatrix} = \begin{bmatrix} U_0 \\ X_W \end{bmatrix} g, \tag{27}$$

admits a solution $g$ given by

$$g = [V_1 \ V_2] \begin{bmatrix} u \\ w \otimes x \end{bmatrix} + \left( I - [V_1 \ V_2] \begin{bmatrix} U_0 \\ X_W \end{bmatrix} \right) d, \tag{28}$$

for any $d \in R^T$, where $\left( I - [V_1 \ V_2] \begin{bmatrix} U_0 \\ X_W \end{bmatrix} \right)$ is the orthogonal projector onto the kernel of $\begin{bmatrix} U_0 \\ X_W \end{bmatrix}$. Using (27) in (26), one has

$$x(t + 1) = \begin{bmatrix} B \\ A_{1,s} \end{bmatrix} \begin{bmatrix} U_0 \\ X_W \end{bmatrix} g(t). \tag{29}$$

Using (24) and (28) this becomes

$$x(t + 1) = X_1 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{bmatrix} u \\ w \otimes x \end{bmatrix} + \left( I - [V_1 \ V_2] \begin{bmatrix} U_0 \\ X_W \end{bmatrix} \right) d, \tag{30}$$

where

$$X_1 \left( I - [V_1 \ V_2] \begin{bmatrix} U_0 \\ X_W \end{bmatrix} \right) = 0. \tag{31}$$

which completes the proof. \qed

**Remark 3.** Theorem 4 provides a data-based representation that predicts the system’s state for any given input. However, in the safe control design, one only needs the data-based closed-loop representation of the system for a state feedback controller that must be designed to assure safety. Therefore, instead of requiring both $B$ and $A_{1,s}$ to be implicitly known, only $A_{1,s} + BK_{1,s}$ must be implicitly known through data for a specific data-dependent $K_{1,s}$. Therefore, the rank condition requirement in Theorem 4 can be relaxed for the data-based closed-loop representation with a data-dependent $K_{1,s}$. The data richness requirement for the safe control design under which the gain $K_{1,s}$ can be obtained from the closed-loop representation is presented next.

**Assumption 2.** The matrix $X_W$ has full row rank. Note that since $X_W \in \mathbb{R}^{n_x(T-1)}$, satisfying the full row rank condition of Assumption 2 requires $T - 1 \geq ns$, or, equivalently, $T \geq ns + 1$.

**Theorem 5.** Let Assumption 2 hold. Then, the closed-loop system (3) with the gain-scheduling controller $u(t) = K(w)x(t)$, where $K(w)$ is defined in (6), has the following representation

$$x(t + 1) = X_1 G_{K_{1,s}}(w(t) \otimes x(t)), \tag{32}$$

or equivalently

$$A_{1,s} + BK_{1,s} = X_1 G_{K_{1,s}}, \tag{33}$$

where $G_{K_{1,s}} \in R^{n_x(n)}$ satisfies

$$\begin{bmatrix} K_{1,s} \\ I \end{bmatrix} = \begin{bmatrix} U_0 \\ X_W \end{bmatrix} G_{K_{1,s}}, \tag{34}$$

where $K_{1,s}$ is defined in (9).

**Proof.** Since $X_W$ has full row rank, there exits a right inverse $G_{K_{1,s}}$ such that

$$X_W G_{K_{1,s}} = I. \tag{35}$$

By applying the input sequence (17) and the scheduling sequence (20) to the LPV system (3), one has

$$X_1 = A_{1,s}X_W + BU_0. \tag{36}$$

Multiplying $G_{K_{1,s}}$ to both sides of (36) from right gives

$$X_1 G_{K_{1,s}} = A_{1,s} + BU_0 G_{K_{1,s}}. \tag{37}$$

On the other hand, from (34), the control gain is $K_{1,s} = U_0 G_{K_{1,s}}$. Therefore, (37) becomes

$$X_1 G_{K_{1,s}} = A_{1,s} + BK_{1,s}, \tag{38}$$

which is equivalent to (33). Moreover, the system (3) with the gain-scheduling control law $u = \sum_{i=1}^{s} K_i w_i \otimes x$ transforms to

$$x(t + 1) = \sum_{i=1}^{s} (A_i + BK_i) w_i \otimes x(t) \tag{39}$$

Using (38) in (39) gives (32). This completes the proof. \qed
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Theorem 5 showed that the closed-loop gain-scheduling system is parameterized through data via (32) and (34). Since the matrix \( G_{K_{1,s}} \) in the closed-loop representation of Theorem 5 is not unique, the next results will treat it as a decision variable to design data-based safe controllers.

6.1 Data-based safe gain scheduling for polyhedral sets

In this subsection, the data-based closed-loop representation provided in Theorem 5 is leveraged to directly design safe gain-scheduling controllers for a given polyhedral set.

\textbf{Theorem 6.} Consider the data collected in (17)–(21). Let Assumptions 1 and 2 hold. Let \( u \in \mathbb{R}^m \). Then, Problem 1 is solved if and only if there exist decision variables \( G_{K_{1,s}} \) and \( P_{1,s} \geq 0 \) such that

\begin{align}
\tag{40a} P_{1,s} (\delta_j \otimes \mathbb{1}) & \leq \lambda \mathbb{1}, \quad j = 1, \ldots, s \\
\tag{40b} P_{1,s} (I \otimes F) & = FX_1 G_{K_{1,s}} \\
\tag{40c} X_W G_{K_{1,s}} & = I.
\end{align}

Moreover, \( K_{1,s} = U_0 G_{K_{1,s}} \), and thus, the control gains that solve the problem are obtained as \( K_i = U_0 G_{K_{i,s}} D_i \).

\textbf{Proof.} We first prove the “if” (i.e., sufficiency) condition. It was shown in Theorem 2 that to solve Problem 1, the gain matrix \( K_{1,s} \) must satisfy (11). Therefore, the sufficiency proof is completed if one shows that satisfying (40a)–(40c) implies satisfying (11) with \( K_{1,s} = U_0 G_{K_{1,s}} \). The inequalities in both equations are identical. Using (40c) and \( K_{1,s} = U_0 G_{K_{1,s}} \), one has (34), which has a solution under Assumption 2, based on Theorem 5. Comparing the second equation of (11) and (40b), the sufficiency proof is completed if we show that the term \( A_{1,s} + BK_{1,s} \) in (11) is equal to \( X_1 G_{K_{1,s}} \). This is shown in Theorem 5 under (34). To prove the “only if” (i.e., necessary) condition, one needs to show that if (11) is satisfied for a control gain \( K_{1,s} \), then there exists a solution \( G_{K_{1,s}} \) to (40a)–(40c) under Assumption 2. For the control gain \( K_{1,s} \), under Assumption 2, there exists a solution \( G_{K_{1,s}} \) to \( X_1 G_{K_{1,s}} = A_{1,s} + BK_{1,s} \). Therefore, (40a)–(40c) are feasible. This completes the proof. \( \square \)

\textbf{Remark 5.} By Theorem 1, if there exist decision variables \( P_{1,s} \) and \( G_{K_{1,s}} \) that satisfy (27), then the closed-loop LPV system is globally asymptotically stable, and Minkowski function of the polyhedral set \( \max_{i=1,\ldots,s} (F^T x) \) is a global Lyapunov function. Note that if the solution \( K_{1,s} \) to (11) is unique, then, based on Theorem 6, the same control input is learned by solving (40a)–(40c) for \( G_{K_{1,s}} \), and finding the gain \( K_{1,s} = U_0 G_{K_{1,s}} \). However, the solution to (11) is generally not unique [27]. Therefore, as the data set \( X_W \) changes, a different solution \( G_{K_{1,s}} \) might lead to a different control gain \( K_{1,s} \) that satisfies \( X_1 G_{K_{1,s}} = A_{1,s} + BK_{1,s} \). That is, a different closed-loop system is learned. However, any data set that satisfies Assumption 2 and regardless of the control gain corresponding to it, the resulting closed-loop system remains safe and stable because of the conditions imposed by (40a)–(40c).

\textbf{Remark 6.} One way to collect \( ns + m \) independent samples to satisfy the rank condition in (22) is to ensure that the input signal is persistently exciting of order \( n + 1 \) [43]. Note that the persistency of excitation of order \( n + 1 \) requires at least \( T \geq mns + ns + m \) data samples. The PE requirement, however, is only sufficient. An approach based on the input design is presented in van Waarde [44] that only requires \( T \geq ns + m \) samples, which is achieved by selecting the control input at each step on the basis of inputs and states that have been collected at previous time steps. Note that to satisfy Assumption 2, which is required in the presented approach, using the method of van Waarde [44], only \( ns \) inputs must be designed instead of \( ns + m \) inputs, which is highly advantageous for safety-critical systems that must commit to making safe decisions using a minimum number of collected data and before safety is jeopardized. Extension of the approach in van Waarde [44] to safe data collection is a future research direction. Since Theorem 6 relies on the closed-loop representation provided in Theorem 5, which requires Assumption 2 to be satisfied on data, it only requires \( T \geq ns + 1 \) samples to find a solution. The following example shows how the data can be leveraged to design a safe controller directly. \textbf{Example} Consider a polytopic LPV system in the form of (23) with

\begin{equation}
A_1 = \begin{bmatrix} .4 & 0 \\ 0 & .1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 0 \\ 1 & .1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{equation}

Let \( U_0 = [.63 \quad .812 \quad -.75 \quad .83 \quad .265] \) and the initial condition be \( x_0 = [1 \quad 1]^T \). Then, the collected data is

\begin{equation}
X_0 = \begin{bmatrix} 1 \cdot .2659 \cdot .0538 \cdot .0058 \cdot .0002 \\ 1.6709 \cdot .7033 \cdot .675 \cdot .8208 \end{bmatrix}, \quad X_1 = \begin{bmatrix} -.2659 \cdot .0538 \cdot .0058 \cdot .0002 \cdot 0 \\ 1.6709 \cdot .7033 \cdot .675 \cdot .8208 \cdot .2675 \end{bmatrix}.
\end{equation}
FIGURE 1 Input sequences of data as in (17) with $T = 6$.

FIGURE 2 State sequences of data as in (18) and (19), with $T = 6$.

FIGURE 3 gain-scheduling sequences of data as in (17)–(20) with $T = 6$.

with the gain-scheduling data

$$W_0 = \begin{bmatrix} .0488 & .1392 & .2734 & .4788 & .4824 \\ .9512 & .8608 & .7266 & .5212 & .5176 \end{bmatrix}. \quad (43)$$

which gives

$$X_W = \begin{bmatrix} .0488 & -.037 & .0147 & -.0028 & -.0001 \\ .0488 & .2327 & .1923 & -.3232 & .396 \\ .9512 & -.2288 & .0391 & -.003 & -.0001 \\ .9512 & 1.4382 & .511 & -.3519 & .4248 \end{bmatrix}. \quad (44)$$

The number of samples is $T = 5$, which satisfies Assumption 2 that requires $T \geq ns + 1$. Let now design a safe controller for the polyhedral set in form (2) with the matrix $F$ as

$$F = \begin{bmatrix} 1/5 & 2/5 \\ -1/5 & -2/5 \\ -3/20 & 1/5 \\ 3/20 & -1/5 \end{bmatrix}. \quad (45)$$

Using Theorem 6, a safe gain-scheduling controller is learned as

$$K_{1s} = U_0G_{K_{1s}} = \begin{bmatrix} -5.63 & -0.744 & 1.2847 & 0.745 \\ -12.76 & -0.5156 & 0.2298 & 0.4952 \\ 67.57 & -1.315 & -4.784 & 1.143 \\ 67.88 & -1.4725 & -5.7252 & 0.7227 \\ 30.78 & 2.274 & -2.642 & 0.2248 \end{bmatrix}. \quad (46)$$

Therefore, while it is impossible to learn the system dynamics using the collected data, a safe controller is directly learned.

The following result solves Problem 1 for the case in which the control input is constrained.

**Corollary 2.** Consider the data collected in (17)–(21). Let Assumptions 1 and 2 hold. Let $u \in U'$ where

$$U' = \{u \in R^m : Uu \leq \bar{1}\}. \quad (48)$$

Then, Problem 1 is solved if there exist decision variables $G_{K_{1s}}$ and $P_{1s} \geq 0$, such that

$$\begin{align*}
P_{1s}(\bar{\delta}_j \otimes \bar{1}) & \leq \bar{1}, j = 1, \ldots, s \\
P_{1s}(U_{ss} \otimes F) & = FX_1G_{K_{1s}} \\
X_WG_{K_{1s}} & = I \\
UU_0G_{K_{1s}}D_k & \leq \bar{1}, i = 1, \ldots, s \text{ and } \forall k \in \text{ vert}(F, \bar{1}) \\
\end{align*} \quad (49)$$

Moreover, $K_{1s} = U_0G_{K_{1s}}$, and the control gains that solve the problem are obtained as $K_i = U_0G_{K_{1s}}D_i$. 

FIGURE 4 The safe set $S(F, \bar{1})$ and solutions arising from the gain-scheduling state feedback law.
Data-based safe gain-scheduling for ellipsoidal sets

The next result provides a data-based control design procedure for LPV systems for which their safe set is described by an ellipsoidal set.

**Theorem 7.** Consider the data collected in (17)–(21). Let Assumptions 1 and 2 hold. Let \( n \in R^m \). Then, Problem 2 is solved if and only if there exist decision variables \( G_{K,s} \), such that

\[
\begin{cases}
\lambda^2 P (X_1 G_{K,s} D_1) (X_1 G_{K,s} D_1)^T P^{-1} > 0 & i = 1, \ldots, s \\
X_W G_{K,s} = I.
\end{cases}
\]

(50)

**Proof.** The \( \lambda \)-contractivity condition for the LPV systems with ellipsoidal safe sets is shown in (14). Therefore, the sufficiency proof is completed if one shows that the satisfaction of (50) implies the satisfaction of (14) with \( K_{1,s} \). Using the Schur complement, (50) is equivalent to

\[
D_1^T (X_1 G_{K,s})^T P (X_1 G_{K,s}) D_1 - \lambda^2 P \leq 0 \quad i = 1, \ldots, s.
\]

(51)

Using the equality in (50) and \( K_{1,s} = U_0 G_{K,s} \), (34) is obtained with \( A_{1,s} + BK_{1,s} = X_1 G_{K,s} \), based on Theorem 5. Comparing (50) with (14) and using \( A_{1,s} + BK_{1,s} = X_1 G_{K,s} \), completes the sufficiency proof. The necessary proof is similar to that of Theorem 6. \( \square \)

**Remark 8.** The results of Theorems 5–7 showed that direct learning of a safe controller for an LPV system can be advantageous over model-based safe control design that relies on system identification.

**Remark 9.** Note that (50) corresponds to solving a semi-definite program in the decision variables \( G_{K,s} \) and \( P_{1,s} \), hence it is numerically appealing. Compared with polyhedral sets, however, it is computationally more demanding. To verify our results, the following simulation example is considered. The safe set \( S(F, I) \) is defined as in (2) with \( F \) defined in (45), and the set \( U \) specifies the condition \(-8 \leq u \leq 8\). The contractivity level is chosen as \( \lambda = .84 \). The data used for learning the controller are collected from an open-loop experiment, as shown in Figures 1–3, for the control input, the state and the gain-scheduling variable, respectively. The control inputs are chosen from a random variable uniformly distributed on \([-1, 1]\). The matrices \( A_1, A_2 \) and \( B \) generating these data are

\[
A_1 = \begin{bmatrix} 1 & 2/3 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4/5 & 2/5 & 6/5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\
1 \end{bmatrix}.
\]

(52)

The linear optimization problem in Theorem 6 is solved in the variables \( G_{K,s} \) and \( P_{1,s} \), and the resulting \( K_{1,s} \) is \( K_{1,s} = [.3056, -.3889, .2389, -.5889] \). Only for illustrative purposes, we also solve the model-based safe control design conditions (11) and obtain a gain matrix \( K_{1,2,3,s} = [.2680, -.8398, .4722, -.4556 \) . The safe set is shown with a green solid line in Figure 4. Figure 4 shows the states of the system for different initial conditions resulting from both the data-based controller (orange) and the model-based controller based on the classical model-based approach (blue). The set \( S \) (green, solid) and the sets \( \lambda S, \lambda^2 S, \lambda^3 S, \ldots \) (green, dotted) are also shown. This shows that safety is guaranteed as the states only evolve in the safe set. Figure 5 also certifies that the control signal satisfies the constraints given by \( U \).
7 | CONCLUSION

This paper presents a data-based solution to the safe gain-scheduling control problem. The presented solution finds a safe controller for nonlinear systems represented in LPV form while only relying on measured data, and it is shown that it enforces not only stability but also invariance with a given polyhedral or ellipsoidal set. The presented data-based solution results in a numerically efficient linear program for polyhedral sets and a semi-definite program for ellipsoidal sets. Moreover, it is shown that the number of samples required to learn a safe controller directly is much less than the number of samples required to learn a system model. A simulation example is provided to verify the effectiveness of the presented data-based approach. A future work direction is to learn piecewise affine controllers inspired by the explicit model-predictive control design presented in Chen et al. [45] to ensure the existence of a solution to Problems 1 and 2 for a larger class of pre-specified safe sets. Another future direction is to extend these results to data-based robust safe control design algorithms LPV systems under disturbances and noises.

AUTHOR CONTRIBUTIONS
Amir Modares: Conceptualization (lead); writing—original draft (lead); methodology. Nasser Sadati: Conceptualization (supporting); review and editing (equal). Hamidreza Modares: Conceptualization (supporting); review and editing (equal).

CONFLICT OF INTEREST STATEMENT
The authors declare no potential conflict of interests.

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