A DIRICHLET PROBLEM OF THE FRACTIONAL LAPLACE EQUATION IN THE BOUNDED LIPSCHITZ DOMAIN

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Abstract. In this paper, we study a Dirichlet problem of a fractional Laplace equation in a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Our main result is that for the given data $F \in H^s(\Omega^c)$, $0 < s < 1$, we find the function which satisfies that $\Delta^s u = 0$ in $\Omega$, $u|_{\Omega^c} = F$ and $\|u\|_{H^s(\mathbb{R}^n)} \leq c\|F\|_{H^s(\Omega^c)}$. Furthermore, we represent the solution with an integral operator.

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1. Introduction

In this paper, we study the Dirichlet problem of a fractional Laplace equation for the bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$, $n \geq 2$. For $0 < s < 1$, we define the fractional Laplacian of a function $u : \mathbb{R}^n \to \mathbb{R}$ as

$$\Delta^s u(x) := c(n,s) \int_{\mathbb{R}^n} \frac{u(x + y) - 2u(x) + u(x - y)}{|y|^{n+2s}} \, dy,$$

where $c(n,s)$ is some normalization constant. The fractional Laplacian of $u$ also can be defined as a pseudo-differential operator

$$(-\Delta)^s u(\xi) = (2\pi|\xi|)^{2s} \hat{u}(\xi),$$

where $\hat{u}(\xi) := \int_{\mathbb{R}^n} u(x)e^{-2\pi i \xi \cdot x} \, dx$, $\xi \in \mathbb{R}^n$ is the Fourier transform of $u$ in $\mathbb{R}^n$.

Compared with the classical Laplacian of $u$, $\Delta u = \sum_{1 \leq i \leq n} \frac{\partial^2 u}{\partial x_i \partial x_i}$ is a local property but the fractional Laplacian of $u$ (1.1) or (1.2) is a non-local property. That is, to define the classical Laplacian of $u$ at $x \in \mathbb{R}^n$, we only need the information of $u$ in the neighborhood of $x$, but to define the fractional Laplacian of $u$ at $x$, we also need the information in $\mathbb{R}^n$. So, to establish the Dirichlet problem of the fractional Laplace equation in $\Omega$, we need the condition which is defined in $\Omega^c$.

We introduce the Dirichlet problem of a fractional Laplace equation in the bounded Lipschitz domain; given the function $F$ defined in $\Omega^c$, we find the function $u$ satisfying the
following equation

\[
\begin{aligned}
\Delta^s u(x) &= 0 \quad x \in \Omega, \\
u(x) &= F(x) \quad x \in \Omega^c,
\end{aligned}
\]

(1.3)

The probability theory is a good tool to represent the solution of (1.3). Let \(X_t\) be a 2s-stable process in \(\mathbb{R}^n\) and \(\tau_\Omega = \inf\{t > 0 | X_t \notin \Omega\}\), the first exit time of \(X_t\) in \(\Omega\). Note that \(X_t\) is right continuous and has left limits a.s. Furthermore \(P\{\tau_\Omega \in \partial \Omega\} = 0\) (see [1] and [7]) and so \(X_{\tau_\Omega} \in \Omega^c\), a.s. Let \(F \in C^\infty(\Omega^c)\) with \(\int_{\Omega^c} |F(x)|(1 + |x|)^{-n-2s}dx < \infty\) and we define the function

\[
\begin{aligned}
u(x) &= \begin{cases} 
E_x F(X_{\tau_\Omega}) & x \in \Omega, \\
F(x) & x \in \Omega^c,
\end{cases}
\end{aligned}
\]

(1.4)

where \(E_x\) denotes an expectation with respect to \(P^x\) of the process starting from \(x \in \Omega\). Then, \(\nu\) defined in (1.4) is the solution of (1.3). Moreover, there is a Poisson kernel \(K(x, y)\) defined in \(\Omega \times \Omega^c\) such that

\[
u(x) = \int_{\Omega^c} K(x, y)F(y)dy.
\]

(1.5)

(see [1] and [7]).

In this paper, we study the regularity problem of the equation (1.3). Our main result is the following theorem.

**Theorem 1.1.** Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^n\) and \(0 < s < 1\). Then, given \(F \in \dot{H}^s(\Omega^c)\), there is the unique weak solution \(u \in \dot{H}^s(\mathbb{R}^n)\) of the equation (1.3) such that

\[
\|u\|_{\dot{H}^s(\mathbb{R}^n)} \leq c\|F\|_{\dot{H}^s(\Omega^c)}
\]

for some positive constant independent of \(F\).

The function spaces \(\dot{H}^s(\mathbb{R}^n)\) are homogeneous Sobolev spaces (see (2.2)) and \(\dot{H}^s(\Omega^c)\) are restrictions of \(\dot{H}^s(\mathbb{R}^n)\) over \(\Omega^c\) (see (2.9)).

To show the theorem 1.1, we use the Riesz potential. In section 3 we will define the integral operator \(S_s : \dot{H}^{-s}_0(\Omega^c) \to \dot{H}^s(\Omega^c)\) by

\[
S_s \phi = (I_{2s} \tilde{f})|_{\Omega^c}, \quad f \in \dot{H}^{-s}_0(\Omega^c),
\]

where \(\dot{H}^{-s}_0(\Omega^c)\) is a dual space of \(\dot{H}^s(\Omega^c)\), \(I_{2s}\) is the Riesz transform and \(\tilde{\phi}\) is a zero extension of \(\phi \in \dot{H}^{-s}_0(\Omega^c)\) (see (2.1a)). Note that \(u = I_{2s} \tilde{\phi}\) is a weak solution of the equation (1.3) (see section 3). So, the existence of a solution of the equation (1.3) is related with the bijectivity of the operator \(S_s : \dot{H}^{-s}_0(\Omega^c) \to \dot{H}^s(\Omega^c)\). In section 4, we will show that \(S_s : \dot{H}^{-s}_0(\Omega^c) \to \dot{H}^s(\Omega^c)\) is bijective.
Related with the regularity problem of the fractional Laplace equation, T. Chang showed the Dirichlet problem of the fractional Laplace equation whose domain is $\mathbb{R}^n \setminus \partial \Omega$ and boundary is $\partial \Omega$ for a bounded Lipschitz domain $\Omega$. He showed the existence of the solution of a fractional Laplace equation whose boundary data is in $H^{s-\frac{1}{2}}(\partial \Omega)$ and then the solution is in $\dot{H}^s(\mathbb{R}^n)$ such that $|u(x)| = O(|x|^{-n+2s})$ at infinite. To show this, he showed the bijectivity of the boundary integral operator induced from the Riesz potential.

We introduce another equivalent definition of the weak solution of a fractional Laplace equation in $\Omega$. We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is a $2s$-harmonic in $\Omega$ if
\begin{equation}
(1.7) \quad u(x) = E_x u(X_{\tau_V})
\end{equation}
for every bounded open set $V$ whose closure $\bar{V}$ is contained in $\Omega$. We say that $u$ is a regular $2s$-harmonic in $\Omega$ if
\begin{equation}
(1.8) \quad u(x) = E_x u(X_{\tau_\Omega}).
\end{equation}

By the strong Markov property of $X_t$, if $u$ is a $2s$-harmonic function in $\Omega$ and $V$ is a open subset of $\Omega$ such that $\bar{V} \subset \Omega$, then $u$ is a $2s$-regular harmonic in $V$. Moreover, $(1.8)$ implies $(1.7)$, so that regular $2s$-harmonic functions are $2s$-harmonic functions. The converse is not true (see section 3 in [3]). It is well-known that $u$ is $2s$-harmonic in $\Omega$ if and only if $u$ is continuous and $\Delta^s u(x) = 0$ for $x \in \Omega$ (see Theorem 3.9 in [2]).

The second main result of this paper is the following theorem.

**Theorem 1.2.** If $\Omega$ is a bounded $C^{1,1}$ domain and $0 < s < \frac{n}{2(n-1)}$, then the weak solution $u$ of the theorem 1.1 is regular $2s$-harmonic.

This paper is organized as follows. In section 2, we introduce several function spaces and study their properties. In section 3, we introduce integral operators in function spaces defined in section 2. In section 4 and section 5, we prove the main results.

### 2. Function spaces

In this paper, we denote $\Omega$ is a bounded Lipschitz domain. we also denote the letters $x$, $y$, $\xi$ as points in $\mathbb{R}^n$. The letter $c$ denotes a positive constant depending only on $n$, $s$ and $\Omega$.

To statement the main results, we introduce several results of harmonic analysis (see, eg., chapter 9 in [8]). Let $\mathcal{S}(\mathbb{R}^n)$ be Schwartz space and $\mathcal{S}'(\mathbb{R}^n)$ be its dual space (space of distribution).
1) (Fourier transform of distribution) We define Fourier transform \( \hat{f} \in \mathcal{S}'(\mathbb{R}^n) \) of \( f \) by
\[
\langle \hat{f}, \phi \rangle := \langle f, \phi \rangle \quad \phi \in \mathcal{S}(\mathbb{R}^n),
\]
where \( \langle \cdot, \cdot \rangle \) is duality pairing between \( \mathcal{S}(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \).

2) (Convolution of distribution) We define the convolution of \( f \in \mathcal{S}'(\mathbb{R}^n) \) and \( \phi \in \mathcal{S}(\mathbb{R}^n) \) by
\[
f * \phi (x) := \langle f, \phi(x - \cdot) \rangle \quad x \in \mathbb{R}^n.
\]
Note that \( f * \phi \) is slowly decay \( C^\infty(\mathbb{R}^n) \) function and so \( f * \phi \in \mathcal{S}'(\mathbb{R}^n) \) is well-defined and
\[
\langle f * \phi, \psi \rangle = \int_{\mathbb{R}^n} f * \phi(x) \psi(x) \, dx = \langle f, \phi \ast \psi \rangle \quad \psi \in \mathcal{S}(\mathbb{R}^n).
\]

3) For \( f \in \mathcal{S}'(\mathbb{R}^n) \) and \( \psi, \phi \in \mathcal{S}(\mathbb{R}^n) \), we have
\[
f * (\psi * \phi) = (f * \psi) * \phi = (f * \phi) * \psi.
\]

Now, we are ready to define the function spaces. Let \( \eta \) be in Schwartz space \( \mathcal{S}(\mathbb{R}^n) \) such that
\[
\begin{cases}
\hat{\eta}(\xi) > 0 & \text{on } 2^{-1} < |\xi| < 2, \\
\hat{\eta}(\xi) = 0 & \text{elsewhere}, \\
\sum_{-\infty < i < \infty} \hat{\eta}(2^{-i} \xi) = 1 & \xi \neq 0.
\end{cases}
\]

We define functions \( \eta_i \in \mathcal{S}(\mathbb{R}^n) \) whose Fourier transforms are written by
\[
(2.1) \quad \hat{\eta}_i(\xi) := \hat{\eta}(2^{-i} \xi) \quad (i = 0, \pm 1, \pm 2, \cdots).
\]

For \( \alpha \in \mathbb{R} \), we define the homogeneous Sobolev space \( \dot{H}^\alpha(\mathbb{R}^n) \) by
\[
(2.2) \quad \dot{H}^\alpha(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \| f \|_{\dot{H}^\alpha(\mathbb{R}^n)} < \infty \}.
\]

Here,
\[
(2.3) \quad \| f \|^2_{\dot{H}^\alpha(\mathbb{R}^n)} := \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\hat{\eta}_i * f(\xi)|^2 \, d\xi,
\]
where * is a convolution in \( \mathbb{R}^n \). Note that \( \| f \|_{\dot{H}^\alpha(\mathbb{R}^n)} = 0 \) if and only if \( \text{supp} \, \hat{f} = \{0\} \), i.e. if and only if \( f \) is a polynomial (see chapter 6.3 in [3]). By 2), \( \eta_i * f \) is slow decay \( C^\infty \) function and so (2.3) is well-defined for \( \alpha < 0 \) also.

For \( \alpha \geq 0 \),
\[
\sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\hat{\eta}_i * \hat{f}(\xi)|^2 \, d\xi = \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\hat{\eta}_i(\xi)|^2 |\hat{f}(\xi)|^2 \, d\xi \approx \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\hat{f}(\xi)|^2 \, d\xi,
\]
where
where "$A \approx B$" means that there are positive constants $c_1$ and $c_2$ such that $c_1 A \leq B \leq c_2 A$. Hence, for $\alpha \geq 0$, we get

\begin{equation}
\dot{H}^\alpha(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\hat{f}(\xi)|^2 d\xi < \infty \}.
\end{equation}

and

$$
\|f\|_{\dot{H}^\alpha(\mathbb{R}^n)} \approx \left( \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.
$$

By simple calculation, for $f \in \dot{H}^\alpha(\mathbb{R}^n)$, $0 < \alpha < 2$, we obtain

\begin{equation}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x + y) - 2f(x) + f(x - y)|^2}{|y|^{n+2\alpha}} dy dx = C \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\hat{f}(\xi)|^2 d\xi.
\end{equation}

Note that for $\alpha > 0$, $\dot{H}^{-\alpha}(\mathbb{R}^n)$ is dual space of $\dot{H}^\alpha(\mathbb{R}^n)$. That is, $\dot{H}^{-\alpha}(\mathbb{R}^n) = (\dot{H}^\alpha(\mathbb{R}^n))^*$. Note that for $\phi \in \dot{H}^{-\alpha}(\mathbb{R}^n)$ and $f \in \dot{H}^\alpha(\mathbb{R}^n)$,

\begin{equation}
<\phi, f> := \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} \hat{\eta}_i \ast \phi(\xi) \hat{f}(\xi) d\xi.
\end{equation}

Suppose that $\psi \in S(\mathbb{R}^n)$ such that $\text{supp} \ \hat{\psi} \cap \text{supp} \ \hat{\eta}_i = \emptyset$. Then, by 1) and 2), we have

$$
<\hat{\eta}_i \ast \hat{\psi}, \phi> = <\eta_i \ast \hat{\psi}, \phi> = <\phi, \eta_i \ast \hat{\psi}>
$$

But, by Plancherel’s Theorem, we get

$$
\eta_i \ast \hat{\psi}(x) = \int_{\mathbb{R}^n} \eta_i(y) \hat{\psi}(x - y) dy = \int_{\mathbb{R}^n} e^{-2\pi i \xi x} \hat{\eta}_i(\xi) \psi(\xi) d\xi = 0.
$$

This implies that $\text{supp} \ f \ast \eta_i \subset \text{supp} \ \hat{\eta}_i$. For $f \in \dot{H}^\alpha(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, we define the fractional Laplacian of $f$ by

$$
\dot{\Delta} f = (2\pi|\xi|)^{2s} \hat{f}.
$$

If $\alpha > 2s$, then above definition is same with (1.2). Since for $f \in \dot{H}^\alpha(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $f = \sum_{-\infty < i < \infty} f \ast \eta_i$, by 3), we have

$$
\eta_i \ast (\dot{\Delta} f) = \mathcal{F}^{-1} (\eta_i \ast (\dot{\Delta} f))
= \mathcal{F}^{-1} \left( \sum_{-\infty < k < \infty} (2\pi|\xi|)^{2s} \hat{\eta}_i \hat{\eta}_k \ast f \right).
$$
where $\mathcal{F}^{-1}$ be a inverse Fourier transform. Since $\text{supp } \hat{\eta}_i \cap \text{supp } \hat{\eta}_k = \emptyset$ for $|i - k| > 2$ and $\eta_i(\xi) + \eta_i(\xi) + \eta_{i+1}(\xi) = 1$ for $\xi \in \text{supp } \eta_i$, we have

$$
\eta_i \ast (\Delta^s f) = \mathcal{F}^{-1} \left( (2\pi|\xi|)^{2s} \hat{\eta}_i \left( \hat{\eta}_{i-1} \ast f + \hat{\eta}_i \ast f + \hat{\eta}_{i+1} \ast f \right) \right)
$$

$$
= \Delta^s \left( (\eta_{i-1} + \eta_i + \eta_{i+1}) \ast (\eta_i \ast f) \right)
$$

$$
= \Delta^s (\eta_i \ast f).
$$

Hence, we get

$$
\|\Delta^s f\|_{\dot{H}^{\alpha-2s}(\mathbb{R}^n)}^2 = \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} |\xi|^{2(\alpha-2s)} |\eta_i \ast \Delta^s f(\xi)|^2 d\xi
$$

$$
= \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} |\xi|^{2(\alpha-2s)} | \Delta^s (\eta_i \ast f)(\xi)|^2 d\xi
$$

$$
= \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} |\xi|^{2\alpha} |(\eta_i \ast f)(\xi)|^2 d\xi
$$

$$
= \|f\|_{\dot{H}^\alpha(\mathbb{R}^n)}^2.
$$

Hence, $\Delta^s : \dot{H}^\alpha(\mathbb{R}^n) \to \dot{H}^{\alpha-2s}(\mathbb{R}^n)$ is isomorphism.

Now, we consider a bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$, $n \geq 2$.

**Definition 2.1.** Let $0 < s < 1$. We say that $v \in \dot{H}^s(\mathbb{R}^n)$ is a weak solution of a fractional Laplace equation $\Delta^s$ in $\Omega$ if $\Delta^s v = 0$ in $\Omega$. That is, $v$ satisfies

$$
< \Delta^s v, \psi > = 0 \quad \text{for all } \psi \in \dot{H}^s(\mathbb{R}^n) \text{ whose support is contained in } \Omega.
$$

**Remark 2.2.** (1) By the definition of the fractional Laplacian (1.2) and (2.6), for $f, \psi \in \dot{H}^s(\mathbb{R}^n)$ we have

$$
< \Delta^s f, \psi > = \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} \eta_i \ast \Delta^s f(\xi) \hat{\psi}(\xi) d\xi
$$

$$
= \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} \Delta^s (\eta_i \ast f)(\xi) \hat{\psi}(\xi) d\xi
$$

$$
= \int_{\mathbb{R}^n} (2\pi|\xi|)^{2s} \hat{f}(\xi) \hat{\psi}(\xi) d\xi.
$$

(2) In fact, $v$ is continuous function in $\Omega$ and satisfies

$$
\Delta^s v(x) = 0 \quad \text{for } x \in \Omega.
$$

(see Theorem 3.9 in [2]).
For \( \alpha > 0 \), we define the function spaces
\[
\tilde{H}^\alpha(\Omega^c) := \{ f|_{\Omega^c} \mid f \in \tilde{H}^\alpha(\mathbb{R}^n) \},
\]
(3.9)
\[
\tilde{H}_0^\alpha(\Omega^c) := \{ f \in L^2_{\text{loc}}(\Omega^c) \mid \tilde{f} \in \tilde{H}^\alpha(\mathbb{R}^n) \},
\]
where \( \tilde{f} \) is zero extension of \( f \) to \( \mathbb{R}^n \). That is,
\[
\tilde{f}(x) = \begin{cases} f(x) & x \in \Omega, \\ 0 & \text{otherwise}. \end{cases}
\]
The norms are
\[
\| f \|_{\tilde{H}^\alpha(\Omega^c)} := \inf_{\tilde{F} \in \tilde{H}^\alpha(\mathbb{R}^n), F|_{\Omega^c} = f} \| \tilde{F} \|_{\tilde{H}^\alpha(\mathbb{R}^n)},
\]
\[
\| f \|_{\tilde{H}_0^\alpha(\Omega^c)} := \| \tilde{f} \|_{\tilde{H}^\alpha(\mathbb{R}^n)}.
\]
We also define \( \tilde{H}^{-\alpha}(\Omega^c), \tilde{H}_0^{-\alpha}(\Omega^c) \) are dual spaces of \( \tilde{H}_0^\alpha(\Omega^c) \) and \( \tilde{H}^\alpha(\Omega^c) \), respectively. For \( f \in \tilde{H}_0^{-\alpha}(\Omega^c) \), we define \( \tilde{f} \in \tilde{H}^{-\alpha}(\mathbb{R}^n) \) by
\[
< \tilde{f}, \phi > := < f, \phi|_{\Omega^c} > \quad \phi \in \tilde{H}^\alpha(\mathbb{R}^n).
\]
Note that
\[
(\alpha)(\tilde{\phi})_{\tilde{H}^{-\alpha}(\mathbb{R}^n)} \approx \| \tilde{\phi} \|_{\tilde{H}_0^{-\alpha}(\Omega^c)}.
\]

3. Integral Operators

For \( 0 < s < n \), we define Riesz transform in \( \tilde{H}^\alpha(\mathbb{R}^n), \alpha \in \mathbb{R} \) by
\[
I_s : \tilde{H}^\alpha(\mathbb{R}^n) \to \tilde{H}^{\alpha+s}(\mathbb{R}^n), \quad \widehat{I_s f} = (2\pi |\xi|)^{-s} \hat{f}, \quad f \in \tilde{H}^\alpha(\mathbb{R}^n).
\]
As the same method of the case of \( \Delta^s \), we can induce the result \( \eta_i * (I_s f) = I_s(\eta_i * f) \).

Hence, for \( f \in \tilde{H}^\alpha(\mathbb{R}^n), \alpha \in \mathbb{R} \),
\[
\| I_s f \|_{\tilde{H}^{\alpha+s}(\mathbb{R}^n)}^2 = \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} |\xi|^{2(\alpha+s)} |\eta_i \ast I_s f(\xi)|^2 d\xi
\]
\[
= \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} |\xi|^{2(\alpha+s)} |I_s(\eta_i \ast f)(\xi)|^2 d\xi
\]
\[
= \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\eta_i \ast f(\xi)|^2 d\xi
\]
\[
= \| f \|_{\tilde{H}^\alpha(\mathbb{R}^n)}^2.
\]
Hence, \( I_s : \tilde{H}^\alpha(\mathbb{R}^n) \to \tilde{H}^{\alpha+s}(\mathbb{R}^n) \) is isomorphism. By the definition of fractional Laplacian of \( I_{2s} f \), we have \( \Delta^s I_{2s} f = f \) with distribution sense.

If \( 0 < s < n \) and \( f \in \tilde{H}^\alpha(\mathbb{R}^n), \alpha \geq 0, \) then \( I_s f \) is represented by
\[
I_s f(x) = \int_{\mathbb{R}^n} \Gamma_s(x - y)f(y)dy \quad x \in \mathbb{R}^n,
\]
(3.1)
where

\[ \Gamma_s(x) = c(n, s) \frac{1}{|x|^{n-s}} \]

is the Riesz potential of order \( s \) in \( \mathbb{R}^n \) (see section 4 in [9]). Since \( I_{2s} f \in \dot{H}^s(\mathbb{R}^n) \) for \( f \in \dot{H}^{-s}(\mathbb{R}^n) \), by (2.6), we have

\[
< f, I_{2s} f > = \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} \hat{\eta}_i \ast \hat{f}(\xi) \hat{I}_{2s} f(\xi) d\xi
\]

\[
= \sum_{-\infty < i < \infty} \sum_{-\infty < k < \infty} \int_{\mathbb{R}^n} \hat{\eta}_i \ast \hat{f}(\xi) \hat{\eta}_k \ast \hat{I}_{2s} f(\xi) d\xi
\]

\[
= \sum_{-\infty < i < \infty} \sum_{-\infty < k < \infty} \int_{\mathbb{R}^n} \hat{\eta}_i \ast \hat{f}(\xi)|2\pi \xi|^{-2s} \hat{\eta}_k \ast \hat{f}(\xi) d\xi
\]

\[
= \sum_{-\infty < i < \infty} \int_{\mathbb{R}^n} |\xi|^{-2s} |\hat{\eta}_i \ast \hat{f}(\xi)|^2 d\xi
\]

\[
\approx \| f \|_{H^{-s}(\mathbb{R}^n)}. \tag{3.3}
\]

For \( \phi \in \dot{H}_0^{-s}(\Omega^c) \), let us \( u(x) := I_{2s} \tilde{\phi}(x) \). Note that \( \Delta^s u = \tilde{\phi} \). Hence, \( u \) is weak solution of (1.3) and by (2) of remark 2.2, we have

\[ \Delta^s u(x) = 0, \quad x \in \Omega. \]

We define bounded operator \( S_s : \dot{H}_0^{-s}(\Omega^c) \rightarrow \dot{H}^s(\Omega^c) \) by

\[ S_s \phi := (I_{2s} \tilde{\phi})|_{\Omega^c}, \quad \phi \in \dot{H}_0^{-s}(\Omega^c). \]

To prove theorem 1.1 we prove the following theorem.

**Theorem 3.1.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \) and \( 0 < s < 1 \). Then

\[ S_s : \dot{H}_0^{-s}(\Omega^c) \rightarrow \dot{H}^s(\Omega^c) \]

is bijective.

**4. Proof of Theorem 3.1**

To prove Theorem 3.1 we need several following lemmas.

**Lemma 4.1.**

\[ S_s : \dot{H}_0^{-s}(\Omega^c) \rightarrow \dot{H}^s(\Omega^c) \]

is one-to-one.
Proof. Suppose that $S_s f = 0$ for $f \in H_0^{-s}(\Omega^c)$. By definition $\tilde{f}$, we have $\tilde{f} \in H^{-s}(\mathbb{R}^n)$. Then, by (3.3), we have

$$\langle f, S_s f \rangle_{\Omega^c} = \langle \tilde{f}, I_{2s} \tilde{f} \rangle \approx \| \tilde{f} \|^2_{H^{-s}(\mathbb{R}^n)}. \quad (4.1)$$

Hence, we have $\tilde{f} = 0$ in $H^{-s}(\mathbb{R}^n)$ and so $\tilde{f}$ is polynomial. Since $\tilde{f} = 0$ in $\Omega$, we get $\tilde{f} = 0$ in $\mathbb{R}^n$. This implies that $f = 0$. Hence, $S_s : H_0^{-s}(\Omega^c) \to \dot{H}^s(\Omega^c)$ is one-to-one. □

Lemma 4.2.

$$S_s : \dot{H}_0^{-s}(\Omega^c) \to \dot{H}^s(\Omega^c)$$

has closed range.

Proof. From (4.1) and (2.11), we have

$$\| f \|^2_{\dot{H}_0^{-s}(\Omega^c)} \leq c \| \tilde{f} \|^2_{\dot{H}^{-s}(\mathbb{R}^n)} \leq c \| f \|_{\dot{H}_0^{-s}(\Omega^c)} \| S_s f \|_{\dot{H}^s(\Omega^c)}. \quad \text{This implies that} \quad S_s : \dot{H}_0^{-s}(\Omega^c) \to \dot{H}^s(\Omega^c) \quad \text{has closed range.} \quad \Box$$

For the proof of bijectivity of $S_s$, it remains only to show $S_s$ is onto. Because of lemma 4.2, we have only to show that $S_s$ has dense range. Let $S_s^*$ be a dual operator of $S_s$, $\text{Ker}(S_s^*)$ be the kernel of $S_s^*$, $R(S_s)$ be the range of $S_s$ and $R(S_s)^\perp$ be an orthogonal complement of $R(S_s)$. Then there is a relation

$$\text{Ker}(S_s^*) = R(S_s)^\perp = R(S_s)^\perp.$$

Hence $S_s$ has dense range if and only if $S_s^*$ is one-to-one. Suppose that $S_s^* \phi = 0$ for some $\phi \in (\dot{H}^s(\Omega^c))^* = \dot{H}_0^{-s}(\Omega^c)$. Then, we have

$$0 = \langle S_s^* \phi, \phi \rangle = \langle \phi, S_s \phi \rangle \approx \| \tilde{\phi} \|^2_{H^{-s}(\mathbb{R}^n)}. \quad \text{This implies that} \quad \tilde{\phi} = 0 \quad \text{and hence} \quad \phi = 0. \quad \text{Therefore,} \quad S_s^* : (\dot{H}^s(\Omega^c))^* \to (\dot{H}_0^{-s}(\Omega^c))^* \quad \text{is one-to-one.}$$

Uniqueness of solution. Suppose that $u \in \dot{H}^s(\mathbb{R}^n)$ is a weak solution of (1.3) such that $u = 0$ in $\Omega^c$. Then, by (2.7), we have

$$\int_{\mathbb{R}^n} |\Delta^s u(x)|^2 \, dx = \int_{\mathbb{R}^n} (2\pi|\xi|)^{2s} |\hat{u}(\xi)|^2 \, d\xi = 0.$$

This means that $\text{supp} u = \{0\}$, that is, $u$ is polynomial. Since $u = 0$ in $\Omega^c$, we have $u \equiv 0$ and so we proved the uniqueness of solution. □
5. Proof of Theorem 1.2

To prove the theorem 1.2, we need the following proposition and lemma.

**Proposition 5.1.** Let $\Omega$ be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ and $K$ be a potential defined in (1.5). Then, there are positive constants $c_1$, $c_2 > 0$ such that for $x \in \Omega$ and $y \in \Omega^c$

$$c_1 \frac{\delta_s(x)}{\delta_s(y)(\delta(y) + 1)^s|x - y|^n} \leq K(x, y) \leq c_2 \frac{\delta_s(x)}{\delta_s(y)(\delta(y) + 1)^s|x - y|^n},$$

where $\delta(z) = \text{dist}(z, \partial \Omega)$ for $z \in \mathbb{R}^n \setminus \partial \Omega$.

(see theorem 1.5 in [7]).

**Lemma 5.2.** Let $0 < s < \frac{n}{2(n-1)}$ and $u$ be $2s$-harmonic in bounded $C^{1,1}$ domain $\Omega$. If $u \in L^{\frac{2n}{n-2s}}(\Omega)$, then $u$ is regular $2s$-harmonic.

**Proof.** Let $\Omega_k$, $1 \leq k < \infty$ be subset of $\Omega$ satisfying that $\Omega_k$ are $C^{1,1}$ domains for all $k$ such that $\bar{\Omega}_k \subset \Omega_{k+1}$ and $\cup_{1 \leq k < \infty} \Omega_k = \Omega$. Let $x \in \Omega$. Then for some $k_0$ such that $x \in \Omega_{k_0}$.

Note that $u$ is regular $\alpha$-harmonic in $\Omega_k$. Hence, for $k \geq k_0$, we have

$$u(x) = E_x(u(X_{\tau_k})) = E_x(u(X_{\tau_k} \setminus \Omega_k) + E_x(u(X_{\tau_k} \cap \Omega_k))$$

$$= E_x(u(X_{\tau_k} \cap \Omega_k)) = E_x(u(X_{\tau_k} \cap \Omega_k) \setminus \Omega_k) + E_x(u(X_{\tau_k} \cap \Omega_k) \setminus \Omega_k)$$

Here, the semicolon above means as usual that the integration is over the subsequent set. Since $\tau_{\Omega_k} \rightarrow \tau_\Omega$ a.s and $\lim_{k \rightarrow \infty} P^x(X_{\tau_{\Omega_k}} = X_{\tau_\Omega}) = 1$ (see (5.40) in [1]), the second term goes to

$$E_x[u(X_{\tau_\Omega}) \setminus \Omega_k].$$

For the first term, we denote $\delta_k(x) = \text{dist}(x, \partial \Omega_k)$, $r_k = \text{dist}(\partial \Omega_k, \partial \Omega)$ and $K_k$ is potential defined in (1.5) replaced by $\Omega_k \times (\Omega_k \setminus \partial \Omega)$. Using the representation of $2s$-harmonic function, proposition 5.1 and Holder inequality, the first term is

(5.1)

$$\int_{\Omega \setminus \Omega_k} K_k(x, y)u(y)dy \leq c \int_{\Omega \setminus \Omega_k} \frac{\delta_k^s(x)}{\delta_k^s(y)(\delta_k(y) + 1)^s|x - y|^{n+2s}}|u(y)|dy$$

$$\leq c \delta_k^s(x)||u||_{L^{\frac{2n}{n-2s}}(\Omega \setminus \Omega_k)} \left(\int_{\Omega \setminus \Omega_k} \frac{1}{\delta_k^s(y)(\delta_k(y) + 1)^s|x - y|^{n+2s}}dy\right)\frac{n+2s}{2n}$$

$$\leq c \delta_k^{n+2s}(x)||u||_{L^{\frac{2n}{n-2s}}(\Omega)} \left(\int_0^{r_k} r^{-\frac{2n}{n-2s}||u||_{L^{\frac{2n}{n-2s}}(\Omega)}}dr\right)^{-1} \frac{n+2s}{2n}$$

$$\leq c \delta_k^{n+2s}(x)||u||_{L^{\frac{2n}{n-2s}}(\Omega)} r_k^{-1} \frac{n+2s}{2n}.$$
Since $\delta_k(x) \to \delta(x)$, $r_k \to 0$ and $-1 + \frac{n+2s}{2ns} > 0$, the last term goes to zero. Hence, for $x \in \Omega$, we get

$$u(x) = E_x(u(X_{\tau_\Omega}))$$

and so $u$ is regular $2s$-harmonic in $\Omega$.

\[ \square \]

**Proof of Theorem 1.2.** In (5.1), using the Sobolev inequality, we have

$$\left| \int_{\Omega \setminus \Omega_k} K_k(x,y)u(y)dy \right| \leq c\delta_k^{-n+s}(x)\|u\|_{H^s(\Omega)}r_k^{-1+\frac{n+2s}{2ns}} \to 0 \quad \text{as} \quad k \to \infty.$$  

Hence, by the process of the proof of Lemma 5.2 we get $u$ is regular $2s$-harmonic. \[ \square \]

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