ASYMPTOTICS OF SINGULAR VALUES FOR QUANTUM DERIVATIVES

RUPERT L. FRANK, FEDOR SUKOCHEV, AND DMITRIY ZANIN

Abstract. We obtain Weyl type asymptotics for the quantised derivative $\overline{df}$ of a function $f$ from the homogeneity Sobolev space $W^{1,d}(\mathbb{R}^d)$ on $\mathbb{R}^d$. The asymptotic coefficient $\|\nabla f\|_{L^d(\mathbb{R}^d)}$ is equivalent to the norm of $\overline{df}$ in the principal ideal $L^{d,\infty}$, thus, providing a non-asymptotic, uniform bound on the spectrum of $\overline{df}$. Our methods are based on the $C^\ast$-algebraic notion of the principal symbol mapping on $\mathbb{R}^d$, as developed recently by the last two authors and collaborators.

1. Introduction and main results

The topic of quantum derivatives lies at the intersection of various fields in mathematics, from noncommutative geometry over spectral and operator theory to real and harmonic analysis. Our specific goal in this paper is to derive spectral asymptotics for quantum derivatives, and we will do this under minimal regularity assumptions. These asymptotics will be deduced from spectral asymptotics for a certain class of pseudodifferential operators, which we also prove here and which are of independent interest.

The idea of Connes’ Quantum Calculus \cite{14, 16} is to replace topological spaces with $C^\ast$-algebras and Riemannian manifolds with spectral triples $(\mathcal{A}, \mathcal{H}, D)$. Here, the $\ast$-algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$ should be seen as a generalisation of $C^\infty(X)$, where $X$ is a Riemannian manifold. The operator $D$ should be seen as a non-commutative version of a Dirac operator, which defines the differential calculus on the Riemannian manifold (e.g., differential forms are the linear span of $\left[ D, a_1 \right]$, $a_1, a_2 \in \mathcal{A}$).

However, rough features of the manifold $X$ (e.g., its conformal geometry) do not need the full information about the operator $D$. Namely, Connes links conformal geometry with the sign of $D$; see \cite{14, 15}. In this context, a differential form $a_1[D, a_2]$, $a_1, a_2 \in \mathcal{A}$, is replaced with its so-called quantised form $a_1[\text{sgn}(D), a_2]$. The operator $da = i[\text{sgn}(D), a]$ is called the quantised derivative of $a \in \mathcal{A}$.

Of particular interest is the membership of $da$ to some ideal of compact operators. The compact operators on $\mathcal{H}$ are described by Connes as being analogous to infinitesimals, and the rate of decay of the sequence $\mu(T) = \{\mu(n, T)\}_{n=0}^{\infty}$ of singular values corresponds in some way to the “size” of the infinitesimal $T$; see \cite{16, 17}. In this setting one can quantify the smoothness of an element $a \in \mathcal{A}$ in terms of the rate of decay of $\mu(da)$. Of particular interest are those elements $a \in \mathcal{A}$
that satisfy:

\[ \mu(n, da) = O((n + 1)^{-1/p}), \]  
\[ \sum_{n=0}^{\infty} \mu(n, da)^p < \infty, \]  
\[ \sup_{n \geq 0} \frac{1}{\log(n + 2)} \sum_{k=0}^{n} \mu(k, da)^p < \infty, \]

for some \( p \in (0, \infty) \). The first condition stated above means that \( da \) is in the weak-Schatten ideal \( L_{p,\infty} \), the second condition means that \( da \) is in the Schatten ideal \( L_p \), and the final condition is that \( |da|^p \) is in the Macaev–Dixmier ideal \( M_{1,\infty} \) \cite{10} Chapter 4, Section 2.\beta; see also \cite{32} Example 2.6.10.

Concrete studies of quantised derivatives in various classical settings are available in the literature: for classical Riemannian manifolds we refer to \cite{14} Theorem 3 and for noncommutative tori and for noncommutative Euclidean spaces to \cite{34, 35}. In particular, for compact Riemannian manifolds, Connes established (see \cite{14} Theorem 3) that \( |df|^\text{dim}(X) \) belongs to Dixmier–Macaev ideal for every \( f \in C^\infty(X) \).

Let us now step back from the case of general manifolds and consider Euclidean space \( \mathbb{R}^d \). We ask the following question: “For which functions \( f \) on \( \mathbb{R}^d \), does their quantised derivative belong to a particular Schatten ideal?” This question ties together several themes in operator theory and harmonic analysis. At the moment, the answer is known for the Schatten \( L_p \) and \( L_p,\infty \) ideals and is given in terms of some classical (Sobolev, Besov, etc) function spaces. Below, we recall a few key results.

1.1. Spectral asymptotics for quantum derivatives. Let us briefly set-up our notation. We work on \( \mathbb{R}^d \) with \( d \geq 2 \) and denote variables by \( x = (x_1, \ldots, x_d) \) and derivatives by \( \nabla = (\partial_1, \ldots, \partial_d) \). We also write \( D_j = -i\partial_j \). Let \( N := 2^{[d/2]} \). There are Hermitian \( N \times N \) matrices \( \gamma_1, \ldots, \gamma_d \) such that

\[ \gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{j,k} \quad \text{for all } 1 \leq j, k \leq d. \]

Different choices of these matrices lead to equivalent results and we fix one such choice and define the Dirac operator

\[ D = \sum_{j=1}^{d} \gamma_j \otimes D_j. \]

This is an unbounded linear operator in the Hilbert space \( \mathbb{C}^N \otimes L_2(\mathbb{R}^d) \). We are interested in its sign, defined by the functional calculus or, equivalently, by

\[ \text{sgn} \, D := \sum_{j=1}^{d} \gamma_j \otimes \frac{D_j}{\sqrt{D_j^2 + \cdots + D_d^2}}. \]

Given a measurable scalar function \( f \) on \( \mathbb{R}^d \) we denote by \( M_f \) the linear operator in \( L_2(\mathbb{R}^d) \) of pointwise multiplication by \( f \). We are interested in the operator

\[ df := i \text{[sgn} \, D, 1 \otimes M_f] \]

acting in \( \mathbb{C}^N \otimes L_2(\mathbb{R}^d) \).

While for \( f \in L_\infty(\mathbb{R}^d) \) the operator \( df \) is clearly bounded in \( \mathbb{C}^N \otimes L_2(\mathbb{R}^d) \), it also extends to a bounded operator for a certain class of unbounded functions \( f \).
Indeed, it is a result by Coifman, Rochberg and Weiss [13] that \( df \) extends to a bounded operator in \( C^N \otimes L_2(\mathbb{R}^d) \) if and only if 
\[
  f \in BMO(\mathbb{R}^d).
\]
The latter condition means, by definition, that \( f \in L_{1,\text{loc}}(\mathbb{R}^d) \) and 
\[
  \sup_{a \in \mathbb{R}^d, r > 0} |B_r(a)|^{-1} \int_{B_r(a)} \left| f(x) - |B_r(a)|^{-1} \int_{B_r(a)} f(y) dy \right| dx < \infty.
\]
Here \( B_r(a) := \{ x \in \mathbb{R}^d : |x - a| < r \} \). For a textbook presentation of this result we refer, for instance, to [21 Section 3.5]. Moreover, as shown in [53], \( df \) is compact in \( C^N \otimes L_2(\mathbb{R}^d) \) if and only if \( f \in VMO(\mathbb{R}^d) \) which is, by definition, the closure in \( BMO(\mathbb{R}^d) \) of continuous, compactly supported functions.

In this paper, we are interested in quantitative compactness properties of \( df \), measured in terms of the decay of its singular values. We will introduce our notation for singular values and the Schatten spaces in Subsection 2.2 below. We recall that Janson and Wolff [27] have characterized membership of \( \bar{df} \) for singular values and the Schatten spaces in Subsection 2.2 below. We recall that measured in terms of the decay of its singular values. We will introduce our notation in \( C^f \) and only if what follows). Moreover, at the endpoint case \( p = d \) they showed that \( df \in L_d \) if and only if \( f \) is constant. The latter result was improved by Rochberg and Semmes [41] on the Lorentz scale and then further in [23].

Of particular interest is the endpoint case of membership of \( df \) to the weak trace ideal \( L_{d,\infty} \). To set the stage, we state the following theorem whose history we discuss momentarily. We denote by \( \dot{W}^1_2(\mathbb{R}^d) \) the space of all \( f \in L_{1,\text{loc}}(\mathbb{R}^d) \) whose distributional gradient belongs to \( L_d(\mathbb{R}^d, C^d) \). Note also that, by the Poincaré inequality, \( \dot{W}^1_2(\mathbb{R}^d) \subset BMO(\mathbb{R}^d) \). We refer the reader to the books [1] [28] [52] for further information on Sobolev spaces.

**Theorem 1.1.** Let \( d \geq 2 \) and \( f \in BMO(\mathbb{R}^d) \). Then \( df \in L_{d,\infty} \) if and only if \( f \in \dot{W}^1_2(\mathbb{R}^d) \). Moreover, with two constants \( 0 < c_d \leq C_d < \infty \) depending only on \( d \), 
\[
  c_d \|
g \|
_{L_d(\mathbb{R}^d, C^d)} \leq \|
g \|
_{L_{d,\infty}} \leq C_d \|
g \|
_{L_d(\mathbb{R}^d, C^d)}.
\]

To lighten the notations, in what follows we frequently write \( \|
g \|
 \) instead of \( \|
g \|
_{L_d(\mathbb{R}^d, C^d)} \) or \( \|
g \|
_{L_d(\mathbb{R}^d, C^d)} \). The particular norm is always clear from the context. A similar convention is applied to the operator norms; for instance, we write \( \|
g \|
_{d,\infty} \) instead of \( \|
g \|
_{L_{d,\infty}} \).

Let us discuss precursors of Theorem 1.1. Its statement appears explicitly in the appendix of the paper [18] by Connes, Sullivan and Teleman, written jointly with Semmes. The argument there relies on a deep analysis by Rochberg and Semmes [41] of singular values of a certain class of operators. The novel ingredient in [18] is a certain derivative-free characterization of the Sobolev space \( \dot{W}^1_2(\mathbb{R}^d) \), whose proof they sketched; see also [40] for an earlier discussion of these derivative-free conditions. In the recent paper [23], one of us gave an alternative proof of the characterization in the appendix of [18]. Meanwhile, in [31] two of us proved Theorem 1.1 under the additional assumption \( f \in L_\infty(\mathbb{R}^d) \). The proof there is independent of the work of Rochberg–Semmes and Connes–Sullivan–Teleman–Semmes.
and uses, instead, some tools from operator theory and pseudodifferential operator theory. Our proof of the implication \( f \in \dot{W}^1_d(\mathbb{R}^d) \Rightarrow df \in L_{d,\infty} \) in Theorem 1.1 uses the result of \cite{31} as an input. Our proof of the converse implication \( df \in L_{d,\infty} \Rightarrow f \in \dot{W}^1_d(\mathbb{R}^d) \) is to a large extent independent of \cite{31} and relies on the proof of spectral asymptotics, which we discuss next.

Indeed, our main result concerning \( df \) is the following theorem about its spectral asymptotics.

**Theorem 1.2.** Let \( d \geq 2 \). If \( f \in \dot{W}^1_d(\mathbb{R}^d) \) is real-valued, then

\[
\lim_{t \to \infty} t^d \mu(t, df) = \kappa_d \|\nabla f\|_{L_d(\mathbb{R}^d)},
\]

where

\[
\kappa_d = (2\pi)^{-1} \left( Nd^{-1} \int_{S^{d-1}} (1 - s^2_d)^{\frac{d}{2}} ds \right)^{\frac{1}{d}}.
\]

This result substantially strengthens the main result in \cite{31} concerning singular traces of the operator \( |df|^d \). For a detailed study of singular traces, we refer the reader to the books \cite{32, 33}. Their applications in quantised calculus are given in \cite{30}.

The following corollary is an almost immediate consequence of Theorems 1.1 and 1.2.

**Corollary 1.3.** Let \( d \geq 2 \). If \( f \in BMO(\mathbb{R}^d) \) satisfies

\[
\lim_{t \to \infty} t^d \mu(t, df) = 0,
\]

then \( f \) is constant.

This corollary strengthens the Janson–Wolff and Rochberg–Semmes results mentioned above. Whether the same conclusion also holds under the weaker assumption \( \liminf_{t \to \infty} t^d \mu(t, df) = 0 \) is not known; see \cite{23} for a result in this direction.

Theorems 1.1 and 1.2 give a complete answer to the question about spectral asymptotics for \( df \). They show that, in order to have \( df \in L_{d,\infty} \), it is necessary that \( f \in \dot{W}^1_d(\mathbb{R}^d) \) and, for real-valued \( f \), as soon as this assumption is satisfied, one has power-like spectral asymptotics. In particular, we see that the asymptotic coefficient (namely, \( \|\nabla f\|_{L_d(\mathbb{R}^d)} \)) provides a non-asymptotic, uniform bound on the spectrum.

The problem of deriving spectral asymptotics under minimal regularity assumptions was emphasized by Birman and Solomyak, who also pointed out the important role played by non-asymptotic, uniform bounds. Their work and that of their collaborators is summarized, for instance in \cite{6, 7}.

One motivation for the work of Birman and Solomyak came from the study of eigenvalues of Schrödinger-type operators. The spectral asymptotics in this case are typically referred to as Weyl-type asymptotics. Non-asymptotic, uniform bounds in this area are, for instance, Lieb–Thirring and Cwikel–Lieb–Rozenblum inequalities. Important results in this context were obtained, for instance, in \cite{42, 43, 44, 45}; see also the textbook \cite[Chapter 4]{24} and the references therein, as well as the recent paper \cite{22}. Our results in Theorems 1.1 and 1.2 can be viewed as the analogues for \( df \) of these results for Schrödinger operators. The asymptotics in Theorem 1.2 are also of semi-classical nature, but they are somewhat more subtle because of cancelations in the commutator defining \( df \).

To the best of our knowledge, Theorems 1.1 and 1.2 do not follow directly from the work of Birman and Solomyak and their school. Probably some parts of the
proof of Theorem 1.2 could be abbreviated by referring to [5]. Given the limited availability of the latter paper, we decided to give a complete, rather self-contained proof of Theorem 1.2. Moreover, we widen our perspective and, instead of just studying $df$, we consider a large class of pseudodifferential operators. The connection between this class of pseudodifferential operators and our original object of interest, $df$, is probably not immediately obvious and will be clarified later in the proof of Proposition 5.8.

There is another line of works to which our paper is connected. It concerns commutators with singular integral operators and their higher order analogues. From this huge literature, we only cite [13, 53, 25, 27, 26, 2, 41] and refer to the references therein. The one-dimensional case is somewhat different and has been treated in [37, 12, 38]. These classical works concern almost exclusively non-asymptotic, uniform bounds and do not consider asymptotics. Exceptions are, for instance, [39, 31] where asymptotics are proved in the weaker sense of the existence of a singular trace. We believe that the techniques that we develop in this paper might be useful in some of the above mentioned situations and provide asymptotics there as well.

1.2. Spectral asymptotics for classical pseudodifferential operators. The following $C^*$-algebra $\Pi$ is the closure (in the uniform norm) of the $*$-algebra of all compactly supported classical pseudodifferential operators of order $0$. However, we use an elementary definition of $\Pi$, which does not involve pseudodifferential operators. The idea to consider this closure may be discerned in [3, Proposition 5.2]. For recent developments of this idea we refer to [50, 36].

Definition 1.4. Let $\pi_1 : L_\infty(\mathbb{R}^d) \to B(L_2(\mathbb{R}^d))$, $\pi_2 : L_\infty(S^{d-1}) \to B(L_2(\mathbb{R}^d))$ be defined by setting, for all $f \in L_\infty(\mathbb{R}^d)$ and $g \in L_\infty(S^{d-1})$,

$$\pi_1(f) = Mf \quad \text{and} \quad \pi_2(g) = g\left(\frac{-i\nabla}{\sqrt{-\Delta}}\right).$$

In other words, $\pi_2(g)$ acts on $L_2(\mathbb{R}^d)$ as a (homogeneous) Fourier multiplier

$$\mathcal{F}(\pi_2(g)\xi)(s) = g\left(\frac{s}{|s|}\right)(\mathcal{F}\xi)(s), \quad \xi \in L_2(\mathbb{R}^d), \quad s \in \mathbb{R}^d.$$

Let

$$A_1 = \mathbb{C} + C_0(\mathbb{R}^d) \quad \text{and} \quad A_2 = C(S^{d-1}),$$

and let $\Pi$ be the $C^*$-subalgebra in $B(L_2(\mathbb{R}^d))$ generated by the algebras $\pi_1(A_1)$ and $\pi_2(A_2)$.

In the definition of $A_1$, by $C_0(\mathbb{R}^d)$ we denote the set of continuous functions tending to zero at infinity.

It should be pointed out that the $*$-representations $\pi_1$ and $\pi_2$ are continuous in the strong operator topology. This fact will play a substantial role in our study.

We say that $T \in B(L_2(\mathbb{R}^d))$ is compactly supported from the right if there is a $\phi \in C_0^{\infty}(\mathbb{R}^d)$ such that $T = T\pi_1(\phi)$.

According to [51] (where a much stronger result is given in Theorem 1.2) or [38] (where a very general result is given in Theorem 3.3 and examplified on p. 284), there is a $*$-homomorphism

$$\text{sym} : \Pi \to A_1 \otimes_{\min} A_2 = C(S^{d-1}, C + C_0(\mathbb{R}^d))$$
such that, for all $f \in \mathbb{C} + C_0(\mathbb{R}^d)$ and $g \in C(\mathbb{S}^{d-1})$,
\[
\text{sym}(\pi_1(f)) = f \otimes 1 \quad \text{and} \quad \text{sym}(\pi_2(g)) = 1 \otimes g.
\]
This $\ast$-homomorphism is called a principal symbol mapping. It properly extends
the notion of the principal symbol of the classical pseudodifferential operator.

If $T \in \Pi$ is compactly supported from the right, then $\text{sym}(T) \in C_c(\mathbb{R}^d \times \mathbb{S}^{d-1})$. Indeed, if $T = T\pi_1(\phi)$ with $\phi \in C_c^\infty(\mathbb{R}^d)$, then
\[
\text{sym}(T) = \text{sym}(T) \cdot \text{sym}(\pi_1(\phi)) = \text{sym}(T) \cdot (\phi \otimes 1).
\]
Thus, $\text{sym}(T)$ is supported on $\text{supp}(\phi) \times \mathbb{S}^{d-1}$.

The following is our main result concerning spectral asymptotics for pseudodifferen-
tial operators.

**Theorem 1.5.** Let $d \geq 2$. If $T \in \Pi$ is compactly supported from the right, then
\[
\lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, T(1 - \Delta)^{-\frac{1}{2}}) = d^{-\frac{1}{2}} (2\pi)^{-1} \| \text{sym}(T) \|_{L^1(\mathbb{R}^d \times \mathbb{S}^{d-1})}.
\]

Let us show that these asymptotics are of semiclassical nature. We consider the
special case $T = \pi_1(f)\pi_2(g)$. On the one hand, we have $(\text{sym}(T))(t, s) = f(t)g(s)$. On
the other hand, the function
\[
P(t, p) = f(t)g \left( \frac{p}{|p|} \right) (1 + |p|^2)^{-\frac{d}{2}}, \quad (t, p) \in \mathbb{R}^d \times \mathbb{R}^d,
\]
is the ‘semiclassical analogue’ of the operator $T(1 - \Delta)^{-1}$, and if we denote by $\mu(P)$
its decreasing rearrangement with respect to the measure $m \times (2\pi)^{-d} m$ on $\mathbb{R}^d \times \mathbb{R}^d$,
then we easily find that
\[
\lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, P) = d^{-\frac{1}{2}} (2\pi)^{-1} \| f \|_d \| g \|_d.
\]
Thus,
\[
\lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, T(1 - \Delta)^{-\frac{1}{2}}) = \lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, P),
\]
which, indeed, shows the semiclassical nature of Theorem 1.5.

The proof of Theorem 1.5 is, in some sense, divided into three main steps. The
first step concerns establishing a priori bounds, and the second one concerns the
proof of asymptotics in a simple model situation. The third step is the proof of
the full result by combining the first two steps. The a priori bounds in the first
step are the topic of Section 4. Here we rely on [29, 35, 31] for some intermediary
results, but the main result, Theorem 4.1, is new and is of independent interest.
As in many problems of a semiclassical character, the spectral estimates of Cwikel,
Kato–Seiler–Simon and Solomyak play an fundamental role in its proof. More
specific to the problem at hand and of crucial importance both here and in [35, 31]
are techniques from double operator integrals; see also [9, 10, 11, 20]. The above-
mentioned second and third steps are the topic of Sections 5, 6 and 7. In their
technical implementation we have found it convenient to not distinguish too strictly
between the second and third step and to take these steps simultaneously. So, the
a priori bounds will already come in when proving the asymptotics in a model
problem on the torus with simpler symbols (see, for instance, Proposition 6.4).
The fact that in our situation a priori bounds play a more important role than
in problems of a similar nature, considered, for instance, in the works of Birman
and Solomyak mentioned above, comes ultimately from the fact that we are dealing
with semiclassical asymptotics whose first term vanishes.
Having described the content of Sections 4-7, let us briefly describe that of the remaining sections. In Section 2, we set up our notation and recall the relevant definitions for spaces of functions and operators. In Section 3, we present, in an abstract setting, the arguments that we will use in the analysis of the model problem on the torus with simple symbols. Those might be of interest in various other problems as well. In the final Section 8 we deduce Theorems 1.1 and 1.2 from Theorem 1.5. This will be relatively straightforward, given the earlier work in [31].

Acknowledgements. Partial support through U.S. National Science Foundation grant DMS-1954995 and through the German Research Foundation grant EXC-2111-390814868 is acknowledged.

2. Preliminaries and notations

2.1. Function spaces. The weak space $L_{p,\infty}(\mathbb{R}^d)$ is defined in the usual way:

$$L_{p,\infty}(\mathbb{R}^d) = \left\{ f : \sup_{t>0} t^\frac{1}{p} \mu(t,f) < \infty \right\},$$

$$\|f\|_{p,\infty} = \sup_{t>0} t^\frac{1}{p} \mu(t,f).$$

Here, $\mu(\cdot, f)$ is the nonincreasing rearrangement of $|f|$.

We denote by $(L_{p,\infty})_0(\mathbb{R}^d)$ the closure of all compactly supported functions in $L_{p,\infty}(\mathbb{R}^d)$. In other words,

$$(L_{p,\infty})_0(\mathbb{R}^d) = \left\{ f \in L_{p,\infty}(\mathbb{R}^d) : \lim_{t \to \infty} t^\frac{1}{p} \mu(t,f) = 0 \right\}.$$

2.2. Operator spaces. The following material is standard; for more details we refer the reader to [47, 8, 33]. Let $H$ be a complex separable Hilbert space, and let $B(H)$ denote the set of all bounded operators on $H$, equipped with the uniform (operator) norm $\| \cdot \|_{\infty}$. Let $\mathcal{K}(H)$ denote the ideal of compact operators on $H$. Given $T \in \mathcal{K}(H)$, the sequence of singular values $\mu(T) = \{ \mu(k,T) \}_{k=0}^\infty$ is defined as:

$$\mu(k,T) = \inf \{ \|T - R\|_{\infty} : \text{rank}(R) \leq k \}.$$ Equivalently, $\mu(\cdot, T)$ is the sequence of eigenvalues of $|T|$ arranged in nonincreasing order with multiplicities.

It is convenient to define a singular value function $t \mapsto \mu(t,T)$, $t > 0$, by the same formula

$$\mu(t,T) = \inf \{ \|T - R\|_{\infty} : \text{rank}(R) \leq t \}.$$ Note that

$$\mu(\cdot, T) = \sum_{k \geq 0} \mu(k,T) \chi_{(k,k+1)}.$$ In what follows, we do not distinguish between the singular value sequence and the singular value function. We often abbreviate $\mu(\cdot, T)$ by $\mu(T)$.

If $(T_k)_{k \geq 0} \subset B(H)$ is a bounded sequence, then $\bigoplus_{k \geq 0} T_k$ is understood as an element in $B(H \oplus H \oplus \cdots)$. This is a convenient notation due to the following fact: if the sequence $(T_k)_{k \geq 0} \subset B(H)$ consists of pairwise orthogonal operators (i.e. $T_k T_l = T_k^* T_l = 0$ whenever $k \neq l$), then

$$\mu(\sum_{k \geq 0} T_k) = \mu(\bigoplus_{k \geq 0} T_k).$$
Let \( p \in (0, \infty) \). The Schatten class \( \mathcal{L}_p \) is the set of operators \( T \) in \( \mathcal{K}(\mathcal{H}) \) such that \( \mu(T) \) is \( p \)-summable, i.e. an element of the sequence space \( \ell_p \). If \( p \geq 1 \), then the \( \mathcal{L}_p \)-norm is defined as:

\[
\|T\|_p := \|\mu(T)\|_p = \left( \sum_{k=0}^{\infty} \mu(k, T)^p \right)^{\frac{1}{p}}.
\]

With this norm \( \mathcal{L}_p \) is a Banach space and an ideal of \( B(\mathcal{H}) \).

The weak Schatten class \( \mathcal{L}_{p,\infty} \) is the set of operators \( T \) such that \( \mu(T) \) is an element of the weak \( L_p \)-space \( \ell_{p,\infty} \), with quasi-norm:

\[
\|T\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} \mu(t, T) < \infty.
\]

It follows from the inequality

\[
\mu(t, T + S) \leq \mu\left(\frac{t}{2}, T\right) + \mu\left(\frac{t}{2}, S\right), \quad t > 0,
\]

that we have the following quasi-triangle inequality in \( \mathcal{L}_{p,\infty} \):

\[
\|T + S\|_{p,\infty} \leq 2^{\frac{1}{p}} (\|T\|_{p,\infty} + \|S\|_{p,\infty}).
\]

The space \( \mathcal{L}_{p,\infty} \) is an ideal of \( B(\mathcal{H}) \). We also have the following form of Hölder’s inequality,

\[(2.1) \quad \|TS\|_{r,\infty} \leq c_{p,q} \|T\|_{p,\infty} \|S\|_{q,\infty}
\]

where \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) for some constant \( c_{p,q} \). For a detailed discussion of the Hölder inequality and the precise value of the optimal constant \( c_{p,q} \) we refer to \([51]\).

We also need the ideal

\[
(\mathcal{L}_{p,\infty})_0 = \left\{ T \in \mathcal{L}_{p,\infty} : \lim_{t \to \infty} t^{\frac{1}{p}} \mu(t, T) = 0 \right\}.
\]

The ideal \( (\mathcal{L}_{p,\infty})_0 \) is the closure of the ideal of all finite rank operators in the norm \( \| \cdot \|_{p,\infty} \).

Note that if \( T \in \mathcal{L}_{p,\infty} \) and \( A \) is compact, then \( TA, AT \in (\mathcal{L}_{p,\infty})_0 \). Similarly, if \( T \in \mathcal{L}_{p,\infty} \) and \( S \in (\mathcal{L}_{q,\infty})_0 \), then \( TS, ST \in (\mathcal{L}_{r,\infty})_0 \), \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \).

The following lemma is well known. We provide a proof for the sake of completeness.

**Lemma 2.1.** Let \( 0 < p < \infty \) and \( A \in \mathcal{L}_{p,\infty} \). Then \( RA, \mathcal{Z}A \in \mathcal{L}_{p,\infty} \) and

\[
\|RA\|_{p,\infty}, \|\mathcal{Z}A\|_{p,\infty} \leq 2^{\frac{1}{p}} \|A\|_{p,\infty}.
\]

Moreover,

\[
\limsup_{t \to \infty} t^{\frac{1}{p}} \mu(t, RA), \limsup_{t \to \infty} t^{\frac{1}{p}} \mu(t, \mathcal{Z}A) \leq 2^{\frac{1}{p}} \limsup_{t \to \infty} t^{\frac{1}{p}} \mu(t, A).
\]

In particular, if \( A \in (\mathcal{L}_{p,\infty})_0 \), then \( RA, \mathcal{Z}A \in (\mathcal{L}_{p,\infty})_0 \).

**Proof.** By the quasi-triangle inequality, we have

\[
\|RA\|_{p,\infty} = \frac{1}{2} \|A + A^*\|_{p,\infty} \leq 2^{\frac{1}{p}} \|A\|_{p,\infty},
\]

\[
\|\mathcal{Z}A\|_{p,\infty} = \frac{1}{2} \|A - A^*\|_{p,\infty} \leq 2^{\frac{1}{p}} \|A\|_{p,\infty}.
\]
Similarly,
\[ \limsup_{t \to \infty} t^{\frac{1}{p}} \mu(t, RA) = \frac{1}{2} \limsup_{t \to \infty} t^{\frac{1}{p}} \mu(t, A + A^*) \]
\[ \leq \frac{1}{2} \limsup_{t \to \infty} t^{\frac{1}{p}} (\mu(t, \frac{A}{2}) + \mu(t, \frac{A^*}{2})) = 2^{\frac{1}{p}} \limsup_{t \to \infty} t^{\frac{1}{p}} \mu(t, A). \]
Similarly, one proves the inequality for \( \Im A \). \( \square \)

3. Abstract limit theorems

3.1. Birman–Solomyak limit lemmas. Throughout this subsection, we fix a parameter \( 0 < p < \infty \). For proofs of the following two lemmas we refer to [8, Section 11.6].

**Lemma 3.1.** Let \( A \in \mathcal{L}_{p,\infty} \) and \( B \in (\mathcal{L}_{p,\infty})_0 \). Then
\[ \limsup_{t \to \infty} t^{\frac{1}{p}} \mu(t, A + B) = \limsup_{t \to \infty} t^{\frac{1}{p}} \mu(t, A) \]
and
\[ \liminf_{t \to \infty} t^{\frac{1}{p}} \mu(t, A + B) = \liminf_{t \to \infty} t^{\frac{1}{p}} \mu(t, A). \]
In particular,
\[ \lim_{t \to \infty} t^{\frac{1}{p}} \mu(t, A + B) = \lim_{t \to \infty} t^{\frac{1}{p}} \mu(t, A), \]
provided that the limit on the right hand side exists.

**Lemma 3.2.** Let \( (A_n)_{n \geq 0} \subset \mathcal{L}_{p,\infty} \) be such that
1. \( A_n \to A \) in \( \mathcal{L}_{p,\infty} \);
2. for every \( n \geq 0 \), the limit
\[ \lim_{t \to \infty} t^{\frac{1}{p}} \mu(t, A_n) = c_n \]
exists.
Then the following limits exist and are equal,
\[ \lim_{t \to \infty} t^{\frac{1}{p}} \mu(t, A) = \lim_{n \to \infty} c_n. \]

The following lemma is elementary to prove, but useful in applications. Recall that the operators \( (A_k)_{k \geq 0} \subset B(H) \) are called pairwise orthogonal if \( A_k A_l = A_k^* A_l = 0 \) whenever \( k \neq l \).

**Lemma 3.3.** Let \( (A_k)_{k=0}^n \subset \mathcal{L}_{p,\infty} \) be a sequence of pairwise orthogonal operators such that for all \( 0 \leq k \leq n \), the limits
\[ \lim_{t \to \infty} t^{\frac{1}{p}} \mu(t, A_k) = c_k \]
exist.
Then
\[ \lim_{t \to \infty} t^{\frac{1}{p}} \mu(t, \sum_{k=0}^n A_k) = \left( \sum_{k=0}^n c_k^p \right)^{\frac{1}{p}}. \]
3.2. New limit lemmas.

**Lemma 3.4.** Let $p > 0$. Let $A \in \mathcal{L}_{p,\infty}$ be such that the limit
\[
\lim_{t \to \infty} t^\frac{1}{p} \mu(t, A)
\]
exists.

Let $(p_l)_{0 \leq l < k}$ be pairwise orthogonal projections in $H$ summing up to 1 such that

1. $[A, p_l] \in (\mathcal{L}_{p,\infty})_0$ for $0 \leq l < k$;
2. the operators $(p_l A p_l)_{0 \leq l < k}$ are pairwise unitarily equivalent.

Then the following limits exist and are equal,
\[
\lim_{t \to \infty} t^\frac{1}{p} \mu(t, A) = \lim_{t \to \infty} t^\frac{1}{p} \mu(t, p_l A p_l) = \sum_{0 \leq l < k} c_l p_l A p_l
\]
where $c_l = \frac{1}{k} \lim_{t \to \infty} t^\frac{1}{p} \mu(t, p_l A p_l)$.

**Proof.** Set $B = \sum_{0 \leq l < k} p_l A p_l$. It follows from the assumption that
\[
B - A = \sum_{0 \leq l < k} p_l [A, p_l] \in (\mathcal{L}_{p,\infty})_0.
\]

By the assumption on $A$ and by Lemma 3.1, we have
\[
\lim_{t \to \infty} t^\frac{1}{p} \mu(t, B) = \lim_{t \to \infty} t^\frac{1}{p} \mu(t, A).
\]

Since the operators $(p_l A p_l)_{0 \leq l < k}$ are pairwise orthogonal and pairwise unitarily equivalent, it follows that
\[
\mu(t, B) = \mu\left(\frac{t}{k}, p_l A p_l\right), \quad t > 0, \quad 0 \leq l < k.
\]

Hence, we have
\[
\lim_{t \to \infty} t^\frac{1}{p} \mu\left(\frac{t}{k}, p_l A p_l\right) = \lim_{t \to \infty} t^\frac{1}{p} \mu(t, A), \quad 0 \leq l < k.
\]

Equivalently, we have
\[
\lim_{t \to \infty} t^\frac{1}{p} \mu(t, p_l A p_l) = k^{-\frac{1}{p}} \lim_{t \to \infty} t^\frac{1}{p} \mu(t, A), \quad 0 \leq l < k.
\]

Since, by assumption, $A p_l - p_l A p_l = [A, p_l] p_l \in (\mathcal{L}_{p,\infty})_0$, it now follows from Lemma 3.1 that the following limits exist and satisfy
\[
\lim_{t \to \infty} t^\frac{1}{p} \mu(t, A) = \lim_{t \to \infty} t^\frac{1}{p} \mu(t, p_l A p_l) = \sum_{0 \leq l < k} c_l p_l A p_l
\]
where $c_l = \frac{1}{k} \lim_{t \to \infty} t^\frac{1}{p} \mu(t, p_l A p_l)$.

This proves the lemma.

**Lemma 3.5.** Let $p > 0$. Let $A \in \mathcal{L}_{p,\infty}$ and let $(p_l)_{0 \leq l < k}$ be pairwise orthogonal projections in $H$ summing up to 1 such that

1. for every $0 \leq l < k$, the limit
\[
\lim_{t \to \infty} t^\frac{1}{p} \mu(t, p_l A) = c_l
\]
exists;
2. $[A, p_l] \in (\mathcal{L}_{p,\infty})_0$ for every $0 \leq l < k$.

Then
\[
\lim_{t \to \infty} t^\frac{1}{p} \mu(t, A) = \sum_{0 \leq l < k} c_l p_l A p_l
\]
Proof. Since \( p_1 A p_l - p_l A = p_l [A, p_l] \in (\mathcal{L}_{p, \infty})_0 \), it follows from the first assumption and Lemma 3.1 that
\[
\lim_{t \to \infty} t^{\frac{n}{p}} \mu(t, p_l A p_l) = c_l, \quad 0 \leq l < k.
\]
Set \( B = \sum_{l=0}^{k-1} p_l A p_l \). Since the operators \((p_l A p_l)_{0 \leq l < k}\) are pairwise orthogonal, it follows, using Lemma 3.3, that
\[
\lim_{t \to \infty} t^{\frac{n}{p}} \mu(t, B) = (\sum_{l=0}^{k-1} c_l^p)^{\frac{1}{p}}.
\]
Since, by assumption, \( B - A = \sum_{l=0}^{k-1} p_l [A, p_l] \in (\mathcal{L}_{p, \infty})_0 \), the assertion follows from Lemma 3.1.

In the following lemma, we deal with subsets \( I \subset [0, 1]^n \). We call such a subset a cube if \( I = [a, b] = [a_1, b_1] \times \cdots \times [a_n, b_n] \) for some \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \) in \([0, 1]^n\) with \( b_n - a_n = l \) for all \( n = 1, \ldots, N \). Two cubes are congruent if they have the same value of \( l \).

Lemma 3.6. Let \( p > 0 \) and \( n \in \mathbb{N} \). Let \( A \in \mathcal{L}_{p, \infty} \) be such that the limit
\[
\lim_{t \to \infty} t^{\frac{n}{p}} \mu(t, A)
\]
exists.

If \( \nu : [0, 1]^n \to B(H) \) is a spectral measure such that
1. for any congruent cubes \( I_1, I_2 \subset [0, 1]^n \), there is a unitary operator \( U \) such that \( A = U^{-1} A U \) and \( \nu(I_1) = U^{-1} \nu(I_2) U \);
2. for every Borel set \( I \subset [0, 1]^n \), we have
   \( [A, \nu(I)] \subset (\mathcal{L}_{p, \infty})_0 \);
3. there is a function \( \psi \) with \( \psi(+0) = 0 \) such that, for every Borel set \( I \subset [0, 1]^n \), we have
   \[
   \limsup_{t \to \infty} t^{\frac{n}{p}} \mu(t, A \nu(I)) \leq \psi(m(I));
   \]
then, for every Borel set \( I \subset [0, 1]^n \), one has
\[
\lim_{t \to \infty} t^{\frac{n}{p}} \mu(t, A \nu(I)) = m(I)^{\frac{1}{p}} \lim_{t \to \infty} t^{\frac{n}{p}} \mu(t, A).
\]

Proof. To lighten the notations, we assume
\[
\lim_{t \to \infty} t^{\frac{n}{p}} \mu(t, A) = 1.
\]
Suppose first \( I = [\frac{k_0}{k}, \frac{k_0+1}{k}] \) for some \( k \in \mathbb{N} \) and for some \( k_0 \in \{0, \ldots, k-1\}^n \). Set \( p_k = \nu([\frac{k_0}{k}, \frac{k_0+1}{k}]), k \in \{0, \ldots, k-1\}^n \). By assumption (1), the operators \( p_k A p_k \), \( k \in \{0, \ldots, k-1\}^n \), are pairwise unitarily equivalent. By assumption (2), \([A, p_k] \in (\mathcal{L}_{p, \infty})_0 \). Hence, the assumptions in Lemma 3.3 hold for the operator \( A \) and for the projections \((p_k)_{0 \leq k < k-1}\). By Lemma 3.3, the assertion follows for such \( I \).

Suppose \( I \) is a finite union of cubes as in the preceding paragraph. That is, \( I = \cup_{0 \leq l < L} I_l \), where \((I_l)_{0 \leq l < L}\) are pairwise disjoint cubes as in the preceding paragraph. Set \( p_l = \nu(I_l) \), \( 0 \leq l < L \). By the preceding paragraph, the first assumption in Lemma 3.3 holds. By assumption (2), the second assumption in Lemma 3.3 holds. By Lemma 3.3, the assertion for such \( I \) follows.
Let now $I$ be arbitrary. Fix $\epsilon > 0$ and choose $J$ as in the preceding paragraph such that

$$m(J \triangle A) < \epsilon, \quad \psi(m(J \triangle I)) < \epsilon^{1 + \frac{1}{\theta}}.$$  

Recall the inequality

$$\mu(t_1 + t_2, T + S) \leq \mu(t_1, T) + \mu(t_2, S).$$

Since

$$\nu(I), \nu(J) \leq \nu(I \cup J) = \nu(I) + \nu(I \setminus J),$$

it follows that

$$\mu(t(1 + \epsilon), A \nu(J)) \leq \mu(t(1 + \epsilon), A \nu(I \cup J)) \leq \mu(t, A \nu(I \setminus J)) + \mu(t \epsilon, A \nu(I \setminus J)).$$

Thus,

$$(1 + \epsilon)^{-\frac{1}{\theta}} \limsup_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I)) = \limsup_{t \to \infty} t^\frac{1}{\theta} \mu(t(1 + \epsilon), A \nu(I))
\leq \limsup_{t \to \infty} \mu(t, A \nu(I)) + \mu(t \epsilon, A \nu(I \setminus J))
\leq \limsup_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I)) + \limsup_{t \to \infty} t^\frac{1}{\theta} \mu(t \epsilon, A \nu(I \setminus J))
= \limsup_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I)) + \epsilon^{-\frac{1}{\theta}} \limsup_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I \setminus J)).$$

Recall that the assertion is already proved for $J$. By assumption (3), we have

$$(3.1) \quad (1 + \epsilon)^{-\frac{1}{\theta}} \limsup_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I)) \leq \nu(J)^\frac{1}{\theta} + \epsilon^{-\frac{1}{\theta}} \psi(m(I \setminus J)) \leq (\nu(I) + \epsilon)^\frac{1}{\theta} + \epsilon.$$

Similarly,

$$(1 + \epsilon)^{-\frac{1}{\theta}} (m(I) - \epsilon)^\frac{1}{\theta} \leq (1 + \epsilon)^{-\frac{1}{\theta}} m(J)^\frac{1}{\theta}
\leq \liminf_{t \to \infty} t^\frac{1}{\theta} \mu(t(1 + \epsilon), A \nu(I))
\leq \liminf_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I)) + \limsup_{t \to \infty} t^\frac{1}{\theta} \mu(t \epsilon, A \nu(I \setminus J))
\leq \liminf_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I)) + \epsilon^{-\frac{1}{\theta}} \limsup_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I \setminus J)).$$

By assumption (3), we have

$$(3.2) \quad (1 + \epsilon)^{-\frac{1}{\theta}} (m(I) - \epsilon)^\frac{1}{\theta} \leq \liminf_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I)) + \epsilon^{-\frac{1}{\theta}} \psi(m(J \setminus I))
\leq \liminf_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I)) + \epsilon.$$

Since $\epsilon > 0$ in (3.1) and (3.2) is arbitrarily small, it follows that

$$\limsup_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I)) \leq \nu(I)^\frac{1}{\theta} \leq \liminf_{t \to \infty} t^\frac{1}{\theta} \mu(t, A \nu(I)).$$

This proves the assertion for an arbitrary $I$. \qed
4. Commutator estimates

In this section, we prove the following two results about commutators.

**Theorem 4.1.** Let \( f \in L_\infty(\mathbb{R}^d) \) be compactly supported and let \( g \in L_\infty(\mathbb{S}^{d-1}) \). We have

\[
[\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}, \pi_2(g)] \in (L_{d,\infty})_0, \quad [\pi_1(f), \pi_2(g)(1 - \Delta)^{-\frac{1}{2}}] \in (L_{d,\infty})_0.
\]

Our second result is the analogue on the torus \( \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d \).

In the following, we slightly abuse the notation and denote

\[
(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \overset{\text{def}}{=} (-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \cdot (1 - P),
\]

where \( P : L_2(\mathbb{T}^d) \to L_2(\mathbb{T}^d) \) is the orthogonal projection onto the subspace of constants. In particular, \(-i\nabla_{\mathbb{T}^d}(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \) is defined to vanish on constants.

**Theorem 4.2.** Let \( f \in L_\infty(\mathbb{T}^d) \) and \( g \in C(\mathbb{S}^{d-1}) \). We have

\[
[M_f(1 - \Delta)^{-\frac{1}{2}}, g(\frac{-i\nabla_{\mathbb{T}^d}}{\sqrt{-\Delta_{\mathbb{T}^d}}})] \in (L_{d,\infty})_0, \quad [M_f, g(\frac{-i\nabla_{\mathbb{T}^d}}{\sqrt{-\Delta_{\mathbb{T}^d}}})(1 - \Delta)^{-\frac{1}{2}}] \in (L_{d,\infty})_0.
\]

We only prove Theorem 4.1. The proof of Theorem 4.2 is similar (but simpler in many aspects) and is, therefore, omitted.

Let us briefly discuss these results. Their main point is that they express a cancellation coming from the commutator. Indeed, the individual operators \( \pi_1(f)(1 - \Delta)^{-\frac{1}{2}} \pi_2(g) = \pi_1(f)\pi_2(g)(1 - \Delta)^{-\frac{1}{2}} \) and \( \pi_2(g)\pi_1(f)(1 - \Delta)^{-\frac{1}{2}} \pi_2(g) = (1 - \Delta)^{-\frac{1}{2}} \pi_1(f) \) belong \(^2\) to \( L_{d,\infty} \), but not to \( (L_{d,\infty})_0 \) (unless \( f \equiv 0 \) or \( g \equiv 0 \)). According to Theorem 4.1 when taking their difference, there is a significant cancellation and the resulting operator does belong to \( (L_{d,\infty})_0 \).

We also remark that with slightly more effort one can show that the assertion remains valid under weaker conditions on \( f \) and \( g \). The above version, however, is sufficient for our purposes. What is important in our applications of this theorem is that no continuity of \( f \) and \( g \) is required (except for that of \( g \) in Theorem 4.2 which is needed to define the operator).

4.1. Some trace ideal bounds.

**Lemma 4.3.** Let \( d > 2 \). For \( f \in L_\infty(\mathbb{R}^d) \) with compact support and \( g \in L_\infty(\mathbb{S}^{d-1}) \) we have

\[
\|\pi_1(f)\pi_2(g)(-\Delta)^{-\frac{1}{2}}\|_{d,\infty} \leq C_d\|f\|_d\|g\|_d.
\]

**Proof.** Set

\[
h(s) = g(\frac{s}{|s|})|s|^{-1}, \quad s \in \mathbb{R}^d.
\]

We have

\[
\pi_1(f)\pi_2(g)(-\Delta)^{-\frac{1}{2}} = M_fh(-i\nabla).
\]

\(^2\)In the setting of torus, this is obvious. In the setting of Euclidean space the shortest (though not the easiest) way to see this is to use Theorem 4.1. In Euclidean setting, it is of crucial importance that \( f \) is compactly supported — without this condition the operator \( \pi_1(f)\pi_2(g)(1 - \Delta)^{-\frac{1}{2}} \) is not even compact (take \( f \equiv 1 \) and \( g \equiv 1 \)).
By Cwikel’s estimate in \( L_{d,\infty}, d > 2 \), (stated in [37] Theorem 4.2); see also [19] and a more general assertion in [29]) we have
\[
\| \pi_1(f)\pi_2(g)(-\Delta)^{-\frac{\beta}{2}} \|_{d,\infty} \leq c_d \| f \|_d \| h \|_{d,\infty}.
\]
Passing to spherical coordinates, we infer that
\[
\| h \|_{d,\infty} = c'_d \| g \|_d.
\]
The assertion follows by combining the two last equations. \( \square \)

The following theorem is a restatement of the result of Solomyak (see [48] Theorem 3.1)) for \( d = 2 \).

**Theorem 4.4.** If \( f \in (L_2 \log L)(\mathbb{R}^2) \) is supported on a compact set \( K \), then
\[
\| M_f(1 - \Delta)^{-\frac{\beta}{2}} \|_{2,\infty} \leq c(K) \| f \|_{L_2 \log L}.
\]

Here, \( L_2 \log L \) is a common shorthand for the Orlicz space \( L_M \) with \( M(t) = t^4 \log(e + t), t > 0 \).

The next lemma is a substitute for Lemma 4.3 for the case \( d = 2 \).

**Lemma 4.5.** Let \( d = 2 \). For \( f \in C_c^\infty(\mathbb{R}^2) \) and \( g \in L_\infty(\mathbb{S}^1) \) we have
\[
\limsup_{t \to \infty} t^{\frac{\beta}{4}} \mu(t, \pi_1(f)\pi_2(g)(1 - \Delta)^{-\frac{\beta}{2}}) \leq c_{abs} \| f \|_{2} \| g \|_{4}.
\]

**Proof.** There is a decomposition\(^3 \)
\( f = f_1f_2 \) with \( f_1, f_2 \in C_c^\infty(\mathbb{R}^2) \) and such that \( \| f_1 \|_4 \| f_2 \|_4 \leq c_{abs} \| f \|_2 \). We write
\[
\begin{align*}
\pi_1(f)\pi_2(g)(1 - \Delta)^{-\frac{\beta}{2}} &= \pi_1(f_1) \cdot \pi_1(f_2)(1 - \Delta)^{-\frac{\beta}{2}} \cdot \pi_2(g) \\
&= \pi_1(f_1)(1 - \Delta)^{-\frac{\beta}{4}} \cdot \pi_1(f_2)(1 - \Delta)^{-\frac{\beta}{4}} \cdot \pi_2(g) \\
&\quad + \pi_1(f_1) \cdot \left( \pi_1(f_2)(1 - \Delta)^{-\frac{\beta}{4}} \cdot (1 - \Delta)^{-\frac{\beta}{4}} \pi_1(f_2)(1 - \Delta)^{-\frac{\beta}{4}} \right) \cdot \pi_2(g).
\end{align*}
\]

Since the bracket in the last line belongs to \( (L_{2,\infty})_0 \) (Theorem 1.6 in [35] taken with \( \alpha = -\frac{1}{4} \) and \( \beta = \frac{1}{4} \) yields much stronger assertion that this bracket belongs to \( L_{1,\infty} \), it follows from Lemma 3.1 and Hölder’s inequality that
\[
\limsup_{t \to \infty} t^{\frac{\beta}{4}} \mu(t, \pi_1(f)\pi_2(g)(1 - \Delta)^{-\frac{\beta}{2}}) = \limsup_{t \to \infty} t^{\frac{\beta}{4}} \mu(t, \pi_1(f_1)(1 - \Delta)^{-\frac{\beta}{4}} \cdot \pi_1(f_2)(1 - \Delta)^{-\frac{\beta}{4}} \pi_2(g))
\]
\[
\leq \left\| \pi_1(f_1)(1 - \Delta)^{-\frac{\beta}{4}} \cdot \pi_1(f_2)(1 - \Delta)^{-\frac{\beta}{4}} \pi_2(g) \right\|_{2,\infty}
\]
\[
\leq 2^{\frac{\beta}{2}} \left\| \pi_1(f_1)(1 - \Delta)^{-\frac{\beta}{4}} \right\|_{4,\infty} \left\| \pi_1(f_2)(1 - \Delta)^{-\frac{\beta}{4}} \pi_2(g) \right\|_{4,\infty}.
\]

By Cwikel’s estimate in \( L_{4,\infty} \) (stated in [37] Theorem 4.2); see also [19] and a more general assertion in [29]), we have
\[
\left\| \pi_1(f_1)(1 - \Delta)^{-\frac{\beta}{4}} \right\|_{4,\infty} \leq c_{abs} \| f_1 \|_4.
\]

\(^3\)Let \( G(t) = e^{-|t|^2}, t \in \mathbb{R}^2 \). Let \( \phi \in C_c^\infty(\mathbb{R}^2) \) be such that \( \| \phi \|_{\infty} = 1 \) and \( \phi = 1 \) on \( \text{supp}(f) \).
Setting \( f_1 = f(f^2 + \| f \|_2^2 G^2)^{-\frac{\beta}{4}} \) and \( f_2 = \phi(f^2 + \| f \|_2^2 G^2)^{-\frac{\beta}{4}} \), we obtain the required decomposition.
An argument identical to that in Lemma 4.3 yields
\[
\| \pi_1(f_2)(1 - \Delta)^{-\frac{1}{2}} \pi_2(g) \|_{4, \infty} \leq c_{\text{abs}} \| f_2 \|_4 \| g \|_4.
\]
A combination of the three last estimates yields the claim. \qed

4.2. A preliminary commutator estimate. Our goal in this subsection is to prove the commutator estimate in Lemma 4.8. We need some preparations.

Lemma 4.6. If \( f \in C_c^\infty(\mathbb{R}^d) \), then
\[
[(1 - \Delta)^{\frac{1}{2}}, \pi_1(f)](1 - \Delta)^{-1} \in (\mathcal{L}_{d, \infty})_0.
\]

Note that, similarly to the discussion after Theorems 4.1 and 4.2, the main point about this lemma is that it captures a cancellation. Indeed, both operators \( \pi_1(f)(1 - \Delta)^{-\frac{1}{2}} \) and \( (1 - \Delta)^{\frac{1}{2}} \pi_1(f)(1 - \Delta)^{-1} \) belong to \( \mathcal{L}_{d, \infty} \), but not to \( (\mathcal{L}_{d, \infty})_0 \) (unless \( f = 0 \)).

Proof. The proof of this lemma requires some tools from the theory of Double Operator Integrals, for which we refer the reader to [11, 11, 9, 20]. The symbol \( T_\phi^A \) denotes the double operator integral, based on the operator \( A \), with the symbol \( \phi \) as defined in these papers.

Let
\[
\phi(\lambda, \mu) = \frac{\lambda^{\frac{1}{2}} \mu^{\frac{1}{2}}}{\lambda + \mu}, \quad \lambda, \mu > 0.
\]

Note that
\[
[(1 - \Delta)^{\frac{1}{2}}, \pi_1(f)] = T_\phi^{(1 - \Delta)^{\frac{1}{2}}}(1 - \Delta)^{-\frac{1}{2}} [1 - \Delta, \pi_1(f)](1 - \Delta)^{-\frac{1}{2}}.
\]

Thus,
\[
[(1 - \Delta)^{\frac{1}{2}}, \pi_1(f)](1 - \Delta)^{-1} = T_\phi^{(1 - \Delta)^{\frac{1}{2}}}(1 - \Delta)^{-\frac{1}{2}} [1 - \Delta, \pi_1(f)](1 - \Delta)^{-\frac{1}{2}}.
\]

By Lemma 6.2.3 in [30], the operator \( T_\phi^{(1 - \Delta)^{\frac{1}{2}} : (\mathcal{L}_{d, \infty})_0 \rightarrow (\mathcal{L}_{d, \infty})_0 \) is bounded. It, therefore, suffices to establish
\[
(1 - \Delta)^{-\frac{1}{2}} [1 - \Delta, \pi_1(f)](1 - \Delta)^{-\frac{1}{2}} \in \mathcal{L}_d \subset (\mathcal{L}_{d, \infty})_0.
\]

To see the latter assertion, note that
\[
[1 - \Delta, \pi_1(f)] = \sum_{k=1}^d [D_k^2, \pi_1(f)] = \sum_{k=1}^d (D_k \pi_1(D_k f) + \pi_1(D_k f) D_k)
\]
\[
= -\pi_1(\Delta f) + 2 \sum_{k=1}^d \pi_1(D_k f) D_k.
\]

It, therefore, suffices to establish
\[
(1 - \Delta)^{-\frac{1}{2}} \pi_1(D_k f)(1 - \Delta)^{-\frac{1}{2}} \in \mathcal{L}_d, \quad 1 \leq k \leq d, \quad (1 - \Delta)^{-\frac{1}{2}} \pi_1(D_k f)(1 - \Delta)^{-\frac{1}{2}} \in \mathcal{L}_d.
\]

Furthermore, it suffices to show
\[
\pi_1(D_k f)(1 - \Delta)^{-\frac{1}{2}} \in \mathcal{L}_d, \quad 1 \leq k \leq d, \quad \pi_1(D_k f)(1 - \Delta)^{-\frac{1}{2}} \in \mathcal{L}_d.
\]

Both inclusions follow from the Kato–Seiler–Simon inequality (see, e.g., [17, Theorem 4.1]). This completes the proof. \qed
Lemma 4.7. If \( f \in C_c^\infty(\mathbb{R}^d) \), then
\[
\left[ \pi_1(f), \frac{D_k}{(-\Delta)^{\frac{1}{2}}} \right] (1 - \Delta)^{-\frac{1}{2}} \in (\mathcal{L}_{d,\infty})_0, \quad 1 \leq k \leq d.
\]

Proof. Without loss of generality, assume that \( f \) is real-valued. Let
\[
g_k(t) := \frac{t_k}{|t|} \frac{1}{(1 + |t|^2)^{\frac{1}{2}}} = \frac{t_k}{|t|} \frac{1}{(1 + |t|^2)^{\frac{1}{2}}(|t| + (1 + |t|^2)^{\frac{1}{2}})}, \quad t \in \mathbb{R}^d.
\]
We decompose the operator \( \left[ \pi_1(f), \frac{D_k}{(-\Delta)^{\frac{1}{2}}} \right] (1 - \Delta)^{-\frac{1}{2}} \) into four parts as follows:
\[
\left[ \pi_1(f), \frac{D_k}{(-\Delta)^{\frac{1}{2}}} \right] (1 - \Delta)^{-\frac{1}{2}} = I + II + III + IV,
\]
where
\[
I = \frac{D_k}{(1 - \Delta)^{\frac{1}{2}}}[(1 - \Delta)^{\frac{1}{2}}, \pi_1(f)](1 - \Delta)^{-1}, \quad II = [\pi_1(f), D_k](1 - \Delta)^{-\frac{1}{2}},
\]
\[
III = \pi_1(f)g_k(-i\nabla)(1 - \Delta)^{-\frac{1}{2}}, \quad IV = -g_k(-i\nabla)\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}.
\]

It follows from Lemma 4.6 that \( I \in (\mathcal{L}_{d,\infty})_0 \). Since \( [D_k, \pi_1(f)] = \pi_1(D_kf) \), it follows from the Kato–Seiler–Simon inequality (see, e.g., [47, Theorem 4.1]) that II, III, IV \( \in \mathcal{L}_d \subset (\mathcal{L}_{d,\infty})_0 \). A combination of all four inclusions completes the proof. \( \square \)

Lemma 4.8. If \( f \in C_c^\infty(\mathbb{R}^d) \) and if \( g \in C^\infty(\mathbb{R}^{d-1}) \), then
\[
[\pi_1(f), \pi_2(g)](1 - \Delta)^{-\frac{1}{2}} \in (\mathcal{L}_{d,\infty})_0.
\]

Proof. Let \( B = \frac{-i\nabla}{(-\Delta)^{\frac{1}{2}}} = \left\{ \frac{D_k}{(-\Delta)^{\frac{1}{2}}} \right\}_{k=1}^d \). We may extend \( g \) to a Schwartz function \( h \) on \( \mathbb{R}^d \). The Fourier transform of \( h \) is also a Schwartz function. By Definition 1.1, we have
\[
\pi_2(g) = g(B) = h(B) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} (\mathcal{F}h)(t)e^{i(B\cdot t)}dt.
\]
Therefore,
\[
[\pi_1(f), \pi_2(g)](1 - \Delta)^{-\frac{1}{2}} = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} (\mathcal{F}h)(t)[\pi_1(f), e^{i(B\cdot t)}](1 - \Delta)^{-\frac{1}{2}}dt.
\]

An elementary computation yields
\[
[x, e^{iy}] = i \int_0^1 e^{isy}[y, x]e^{i(1-s)y}ds
\]
for all bounded operators \( x \) and self-adjoint bounded operators \( y \). If \( [x, y] \in (\mathcal{L}_{d,\infty})_0 \), then the intergand in (4.3) belongs to \( (\mathcal{L}_{d,\infty})_0 \) and is weakly measurable. Since \( (\mathcal{L}_{d,\infty})_0 \) is a separable Banach space, the integrand is Bochner measurable and the integral can be understood in the Bochner sense in \( (\mathcal{L}_{d,\infty})_0 \). Thus, \( [x, e^{iy}] \in (\mathcal{L}_{d,\infty})_0 \) and
\[
[\|[x, e^{iy}]\|_{d,\infty} \leq ||[y, x]||_{d,\infty}.
\]

\[\text{For example, we may set } h(t) = g(\frac{t}{|t|})\phi_0(|t|), \quad t \in \mathbb{R}^d, \text{ where } \phi_0 \text{ is a Schwartz function on } \mathbb{R} \text{ which vanishes on } (-\infty, \frac{1}{2}) \text{ and such that } \phi_0(1) = 1.\]
Thus, for every $t \in \mathbb{R}^d$,
\[
[\pi_1(f), e^{i(B,t)}](1 - \Delta)^{-\frac{1}{2}} = [\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}, e^{i(B,t)}] \in (L_{d,\infty})_0
\]
and
\[
(4.5) \quad \left\|[\pi_1(f), e^{i(B,t)}](1 - \Delta)^{-\frac{1}{2}}\right\|_{d,\infty} \leq \sum_{k=1}^{d} |t_k| \cdot \left|[\pi_1(f), B_k](1 - \Delta)^{-\frac{1}{2}}\right|_{d,\infty},
\]
where the right hand side in (4.5) is finite by Lemma 4.7.

Hence, the integrand in (4.2) belongs to $(L_{d,\infty})_0$ and is weakly measurable. Since $(L_{d,\infty})_0$ is a separable Banach space, the integrand in (4.2) is Bochner measurable in $(L_{d,\infty})_0$ and the integral in (4.2) can be understood in the Bochner sense in $(L_{d,\infty})_0$. □

4.3. **Proof of Theorem 4.1** We are now ready to prove the main result of this section.

**Proof of Theorem 4.1**. **Step 1.** We prove the first assertion in the theorem under the additional assumption that $f \in C_c^\infty(\mathbb{R}^d)$.

Let $h \in C^\infty(S^{d-1})$ and write
\[
[\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}, \pi_2(g)] = [\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}, \pi_2(g - h)] + [\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}, \pi_2(h)]
\]
\[
= \pi_1(f)\pi_2(g - h)(1 - \Delta)^{-\frac{1}{2}} - (1 - \Delta)^{-\frac{1}{2}}\pi_2(g - h)\pi_1(f)
\]
\[
- \pi_2(g - h) \cdot [\pi_1(f), (1 - \Delta)^{-\frac{1}{2}}]
\]
\[
+ [\pi_1(f), \pi_2(h)](1 - \Delta)^{-\frac{1}{2}}.
\]
Noting that $g - h \in L_\infty(S^{d-1})$, the third summand belongs to $(L_{d,\infty})_0$. Indeed, Theorem 1.6 in [35] taken with $\alpha = -1$ and $\beta = 0$ yields the stronger assertion that this summand belongs to $L_{d,\infty}$. The fourth summand belongs to $(L_{d,\infty})_0$ by Lemma 4.8. Therefore, it follows from Lemma 4.9 that
\[
\limsup_{t \to \infty} t^\frac{1}{2} \mu(t, [\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}, \pi_2(g)]) = \limsup_{t \to \infty} t^\frac{1}{2} \mu(t, [\pi_1(f)\pi_2(g - h)(1 - \Delta)^{-\frac{1}{2}} - (1 - \Delta)^{-\frac{1}{2}}\pi_2(g - h)\pi_1(f)])
\]
\[
\leq c_d \limsup_{t \to \infty} t^\frac{1}{2} \mu(t, [\pi_1(f)\pi_2(g - h)(1 - \Delta)^{-\frac{1}{2}}]) + c_d \limsup_{t \to \infty} t^\frac{1}{2} \mu(t, [\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}\pi_2(g - h)]\pi_1(f))
\]
\[
\leq c_d^p \|f\|_p \|g - h\|_p,
\]
where $p = d$ for $d > 2$ and $p = 4$ for $d = 2$. The last inequality uses Lemmas 4.23 and 4.24 for $d > 2$ and $d = 2$, respectively. Choosing a sequence of $h$’s that converges to $g$ in $L_p(S^{d-1})$, we conclude that
\[
\limsup_{t \to \infty} t^\frac{1}{2} \mu(t, [\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}, \pi_2(g)]) = 0,
\]
which is the claimed assertion.

**Step 2.** We now prove the first assertion in the theorem for general $f \in L_\infty(\mathbb{R}^d)$ with compact support.
By applying dilation and translation, we may assume without loss of generality that $f$ is supported on $[0, 1]^d$.

We choose a sequence $(f_n)_{n \geq 1} \subset C_c^\infty(\mathbb{R}^d)$ supported in $[0, 1]^d$ such that $\|f - f_n\|_d < \frac{1}{n}$ if $d > 2$ and $\|f - f_n\|_{L_2 \log L} < \frac{1}{n}$ if $d = 2$. By the first step, we have

$$[\pi_1(f_n)(1 - \Delta)^{-\frac{1}{2}}, \pi_2(g)] \in (L_{d, \infty})_0.$$ 

On the other hand, by Cwikel’s estimate in $L_{d, \infty}$, $d > 2$, (stated as Theorem 4.2 in [47]; see also [19] and a more general assertion in [29]) and Theorem 4.4 for $d = 2$, we have

$$\|\pi_1(f_n - f)(1 - \Delta)^{-\frac{1}{2}}\|_{d, \infty} \leq c_d n, \quad n \geq 1.$$ 

This implies

$$\|\pi_1(f_n)(1 - \Delta)^{-\frac{1}{2}}\|_{d, \infty} \leq \frac{c_d}{n}, \quad n \geq 1.$$ 

Thus,

$$[\pi_1(f_n)(1 - \Delta)^{-\frac{1}{2}}, \pi_2(g)] \to [\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}, \pi_2(g)], \quad n \to \infty,$$

in $L_{d, \infty}$. Since $(L_{d, \infty})_0$ is closed in $L_{d, \infty}$ by Lemma 3.2, the first assertion follows.

**Step 3.** We finally prove the second assertion in the theorem.

Indeed, it follows from the first one and the identity

$$[\pi_1(f), \pi_2(g)(1 - \Delta)^{-\frac{1}{2}}] = [\pi_1(f)(1 - \Delta)^{-\frac{1}{2}}, \pi_2(g)] + \pi_2(g) \cdot [\pi_1(f), (1 - \Delta)^{-\frac{1}{2}}].$$

This completes the proof of Theorem 4.1. □

5. Proof of Theorem 1.5 in a special case. Case $d > 2$

Our goal in this and the next section is to prove the following theorem, which is a special case of Theorem 1.5 and contains the main difficulty.

**Theorem 5.1.** Let $A_1$ and $A_2$ be as in Definition 1.4. Let $(f_n)_{n=1}^N \subset A_1$ be compactly supported and let $(g_n)_{n=1}^N \subset A_2$. If

$$T = \sum_{n=1}^N \pi_1(f_n) \pi_2(g_n),$$

then

$$\lim_{t \to \infty} t^\frac{1}{2} \mu(t, T(1 - \Delta)^{-\frac{1}{2}}) = d^{-\frac{1}{2}}(2\pi)^{-1}\|\text{sym}(T)\|_d.$$ 

The proof of this theorem is somewhat different in dimensions $d \geq 3$ and $d = 2$. In this section we deal with the former case and in the next one with the latter.
5.1. Asymptotics on the torus. Our goal in this subsection is to prove the following analogue of Theorem 5.1, for \( d \geq 3 \) on the torus \( \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d \). We recall from the discussion before Theorem 4.2, our convention for the operator \((\Delta_{\mathbb{T}^d})^{-\frac{1}{2}}\).

**Theorem 5.2.** Let \( \mathbb{T}^d \) be the torus. Let \( f_n, g_n \in C(\mathbb{T}^d) \) and let \( (g_n)_{n=1}^N \subset C(\mathbb{S}^{d-1}) \). Then

\[
\lim_{t \to \infty} t^\frac{d}{2} \mu \left( t, \frac{1}{\sqrt{-\Delta_{\mathbb{T}^d}}} \right) \left( f_n \otimes g_n \right) = d^{-\frac{1}{2}} (2\pi)^{-1} \sum_{n=1}^{N} f_n \otimes g_n.
\]

We begin by proving the assertion for \( N = 1 \) and \( f_1 = 1 \).

**Lemma 5.3.** Let \( g \in C(\mathbb{S}^{d-1}) \). We have

\[
\lim_{t \to \infty} t^\frac{d}{2} \mu \left( t, \frac{g}{\sqrt{-\Delta_{\mathbb{T}^d}}} \right) \left( \frac{1}{\sqrt{-\Delta_{\mathbb{T}^d}}} \right) = d^{-\frac{1}{2}} \| g \|_d.
\]

**Proof.** Step 1. We define the function \( G_1 \) on \( \mathbb{R}^d \) by setting

\[
G_1(s) = g\left( \frac{s}{\| s \|} \right) |s|^{-1}, \quad s \in \mathbb{R}^d,
\]

and claim that

\[
\mu(t, G_1) = d^{-\frac{1}{2}} \| g \|_d t^{-\frac{1}{2}}, \quad t > 0.
\]

Here, \( \mu(\cdot, G_1) \) is a decreasing rearrangement of \( |G_1| \) (as in the Subsection 2.1).

Consider the measure space \( X = \mathbb{R} \times \mathbb{S}^{d-1} \) equipped with the product measure \( \frac{1}{2}m_{\mathbb{R}} \times m_{\mathbb{S}^{d-1}} \), where \( m_{\mathbb{R}} \) is the Lebesgue measure on \( \mathbb{R} \) and \( m_{\mathbb{S}^{d-1}} \) is the Lebesgue measure on the sphere. The spherical coordinate change

\[
s \to (\| s \|, \frac{s}{\| s \|}), \quad s \in \mathbb{R}^d,
\]

preserves the measure (i.e., transforms Lebesgue measure on \( \mathbb{R}^d \) to the above product measure). Hence, this spherical coordinate change preserves the decreasing rearrangement. The image of \( G_1 \) under the spherical coordinate change is \( z^\frac{1}{2} \otimes g \), where \( z(t) = t^{-1} \), \( t > 0 \). Thus,

\[
\mu_{L_\infty(\mathbb{R}^d)}(G_1) = \mu_{L_\infty(\mathbb{R}_+ \times \mathbb{S}^{d-1}, \frac{1}{2}m_{\mathbb{R}} \times m_{\mathbb{S}^{d-1}})}(z^\frac{1}{2} \otimes g) = d^{-\frac{1}{2}} \| g \|_d z^{-\frac{1}{2}},
\]

where we use the notation \( \mu_{L_\infty(X, \nu)}(x) \) to emphasise that \( x \) is a \( \nu \)-measurable function on a measure space \( (X, \nu) \) and that the decreasing rearrangement of \( |x| \) is taken with respect to the measure \( \nu \). This immediately yields the assertion of Step 1.

Step 2. We define the function \( G_2 \) on \( \mathbb{R}^d \) by setting

\[
G_2(s) = g\left( \frac{s}{\| s \|} \right) |s|^{-1} \chi_{\mathbb{R}^d \setminus \{ 0 \}}(\| s \|), \quad s \in \mathbb{R}^d.
\]

Here, \( |s| \) is a shorthand for the vector \( ([s_1], \ldots, [s_d]) \). We claim that

\[
\lim_{t \to \infty} t^\frac{d}{2} \mu(t, G_2) = d^{-\frac{1}{2}} \| g \|_d.
\]

Indeed, it follows from the continuity of \( g \) that

\[
G_1 - G_2 \in (L_{d, \infty})_0(\mathbb{R}^d),
\]

so, by a simple commutative analogue of Lemma 3.1, the assertion of Step 2 follows from that of Step 1.
Step 3. For the sequence (indexed by \( \mathbb{Z}^d \))

\[
G_3(n) = \left\{ g\left( \frac{n}{|n|} \right) |n|^{-1} \chi_{\mathbb{Z}^d \setminus \{0\}}(n) \right\}_{n \in \mathbb{Z}^d}
\]

it is immediate that

\[
\mu_{L_\infty(\mathbb{R}^d)}(G_2) = \mu_{L_\infty(\mathbb{Z}^d)}(G_3).
\]

It follows from Step 2 that

\[
\lim_{t \to \infty} t^{\frac{d}{d+1}} \mu_{L_\infty(\mathbb{Z}^d)}(t, G_3) = d^{-\frac{d}{d+1}} \|g\|_d.
\]

Since the operator

\[
g\left( \frac{-i\nabla_{\mathbb{T}^d}}{\sqrt{-\Delta_{\mathbb{T}^d}}} \right) (-\Delta_{\mathbb{T}^d})^{-\frac{d}{2}}
\]

is diagonal in the Fourier basis and since the corresponding sequence of diagonal entries is exactly \( G_3 \), the assertion follows. \( \square \)

Next, we prove Theorem 5.2 in the case \( N = 1 \) and with \( f_1 \) being an indicator function.

**Proposition 5.4.** Let \( d \geq 3 \). Let \( I \subset \mathbb{T}^d \) be a Borel subset and let \( g \in C(\mathbb{S}^{d-1}) \). Then

\[
\lim_{t \to \infty} t^{\frac{d}{d+1}} \mu\left( t, M_{X_I} g\left( \frac{-i\nabla_{\mathbb{T}^d}}{\sqrt{-\Delta_{\mathbb{T}^d}}} \right) (-\Delta_{\mathbb{T}^d})^{-\frac{d}{2}} \right) = d^{-\frac{d}{d+1}} (2\pi)^{-1} m(I)^\frac{d}{2} \|g\|_d.
\]

**Proof.** Throughout this proof, \( g \in C(\mathbb{S}^{d-1}) \) is fixed.

We will apply Lemma 5.3 with \( p = d \) and

\[
A = g\left( \frac{-i\nabla_{\mathbb{T}^d}}{\sqrt{-\Delta_{\mathbb{T}^d}}} \right) (-\Delta_{\mathbb{T}^d})^{-\frac{d}{2}}.
\]

Then, by Lemma 5.3, \( A \in \mathcal{L}_{d, \infty} \) and

\[
\lim_{t \to \infty} t^{\frac{d}{d+1}} \mu(t, A) = d^{-\frac{d}{d+1}} \|g\|_d.
\]

Let us define a spectral measure \( \nu \) on \([0, 1)^d\). For a Borel set \( I \subset [0, 1)^d \), we define a Borel set \( \tilde{I} \subset \mathbb{T}^d \) by dilating it by a factor of \( 2\pi \) (resulting in a subset of \([0, 2\pi)^d\)), then extending it periodically to a subset of \( \mathbb{R}^d \) and finally interpreting it as a subset of \( \mathbb{T}^d \). If \( \chi_{\tilde{I}} \) is the indicator function of \( \tilde{I} \), then we set

\[
\nu(I) = M_{\chi_{\tilde{I}}}.
\]

This, indeed, defines a spectral measure.

If \( I_1, I_2 \subset [0, 1)^d \) are congruent cubes, then there is a translation \( U \in B(L_2(\mathbb{T}^d)) \) such that \( \nu(I_1) = U^{-1} \nu(I_2) U \). Clearly, translations commute with functions of the gradient and, therefore, with \( A \). This verifies the first assumption in Lemma 3.6.

The second assumption in Lemma 3.6 follows from Theorem 4.2. The third assumption in Lemma 3.6 follows from the estimate

\[
\limsup_{t \to \infty} t^{\frac{d}{d+1}} \mu\left( t, M_{X_I} g\left( \frac{-i\nabla_{\mathbb{T}^d}}{\sqrt{-\Delta_{\mathbb{T}^d}}} \right) (-\Delta_{\mathbb{T}^d})^{-\frac{d}{2}} \right) \\
\leq \|g\|_\infty \limsup_{t \to \infty} t^{\frac{d}{d+1}} \mu\left( t, M_{X_I} (-\Delta_{\mathbb{T}^d})^{-\frac{d}{2}} \right) \\
\leq \|g\|_\infty \|M_{X_I} (-\Delta_{\mathbb{T}^d})^{-\frac{d}{2}}\|_{d, \infty} \\
\leq C_d \|g\|_\infty m(\tilde{I})^{\frac{d}{2}}.
\]
Here, the last step follows from the abstract Cwikel-type estimate (see [29] Theorem 3.4 or Corollary 3.6).

Hence, one can apply Lemma 3.6 and one obtains
\[ \lim_{t \to \infty} t^{\frac{1}{2}} \mu(t, AM_{\chi_j}) = d^{-\frac{1}{2}} 2\pi m(I)^{\frac{1}{2}} \|g\|_d = d^{-\frac{1}{2}} m(f)^{\frac{1}{2}} \|g\|_d. \]

Because of the second assumption in Lemma 3.6 and Lemma 3.1 this remains valid with \( AM_{\chi_j} \) replaced by \( M_{\chi_j}A \), which is the assertion of the proposition. \( \square \)

We can now turn to the proof of the main result of this subsection.

**Proof of Theorem 5.2.** For every \( K \in \mathbb{N} \) and \( k \in \{0, \ldots, K - 1\}^d \), let
\[ h_{k,K}(t) = \chi_{ \{ 2\pi k, 2\pi(k+1) \} } (t), \quad t \in [0,2\pi)^d \equiv \mathbb{T}^d. \]

Let \( f_{n,K} \) be the conditional expectation on \( L^\infty(\mathbb{T}^d) \) of \( f_n \) onto the subalgebra generated by \( (h_{k,K})_{k \in \{0,\ldots,K-1\}^d} \) and write
\[ f_{n,K} = \sum_{k \in \{0,\ldots,K-1\}^d} c_{n,k,K} h_{k,K}. \]

We set
\[ g_{k,K} = \sum_{n=1}^N c_{n,k,K} g_n \]
and
\[ A_K = \sum_{k \in \{0,\ldots,K-1\}^d} \pi_1(h_{k,K}) \pi_2(g_{k,K})(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \pi_1(h_{k,K}). \]

Since the operators appearing in the sum defining \( A_K \) are pairwise orthogonal, we have
\[ \mu(A_K) = \mu \left( \bigoplus_{k \in \{0,\ldots,K-1\}^d} \pi_1(h_{k,K}) \pi_2(g_{k,K})(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \pi_1(h_{k,K}) \right). \]

By Proposition 5.1 we have
\[ \lim_{t \to \infty} t^{\frac{1}{2}} \mu(t, \pi_1(h_{k,K}) \pi_2(g_{k,K})(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}}) = d^{-\frac{1}{2}} K^{-1} \|g_{k,K}\|_d. \]

On the other hand, by Theorem 1.2
\[
\pi_1(h_{k,K}) \pi_2(g_{k,K})(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \pi_1(h_{k,K}) - \pi_1(h_{k,K}) \pi_2(g_{k,K})(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \\
= \pi_1(h_{k,K})[\pi_2(g_{k,K})(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}}, \pi_1(h_{k,K})] \in (\mathcal{L}_{d,\infty})_0,
\]
so, by Lemma 3.1
\[ \lim_{t \to \infty} t^{\frac{1}{2}} \mu(t, \pi_1(h_{k,K}) \pi_2(g_{k,K})(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \pi_1(h_{k,K})) = d^{-\frac{1}{2}} K^{-1} \|g_{k,K}\|_d. \]

We deduce, by Lemma 3.3
\[
\lim_{t \to \infty} t^{\frac{1}{2}} \mu \left( t, \bigoplus_{k \in \{0,\ldots,K-1\}^d} \pi_1(h_{k,K}) \pi_2(g_{k,K})(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \pi_1(h_{k,K}) \right) \\
= d^{-\frac{1}{2}} \left( \frac{1}{K^d} \sum_{k \in \{0,\ldots,K-1\}^d} \|g_{k,K}\|_d^2 \right)^{\frac{1}{2}}.
\]

(5.1)
Now let us introduce
\[ T_K = \sum_{n=1}^{N} \pi_1(f_{n,K}) \pi_2(g_n) = \sum_{k \in \{0,...,K-1\}^d} \pi_1(h_{k,K}) \pi_2(g_{k,K}). \]

By Theorem 4.2 we have
\[ T_K(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} - A_K \]
\[ = \sum_{k \in \{0,...,K-1\}^d} \pi_1(h_{k,K}) \left[ \pi_1(h_{k,K}), \pi_2(g_{k,K})(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \right] \in (\mathcal{L}_d,\infty)_0. \]

It follows now from (5.1) and Lemma 5.5 that
\[ \lim_{t \to \infty} t^\frac{d}{2} \mu(t, T_K(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}}) = d^{-\frac{1}{2}} \left( \frac{1}{K^d} \sum_{k \in \{0,...,K-1\}^d} \|g_{k,K}\|_d^d \right)^{\frac{1}{2}}. \]

We note that
\[ \frac{1}{K^d} \sum_{k \in \{0,...,K-1\}^d} \|g_{k,K}\|_d^d = (2\pi)^{-d} \left\| \sum_{k \in \{0,...,K-1\}^d} h_{k,K} \otimes g_{k,K} \right\|_d^d. \]

Moreover, it is clear from the expression for \( f_{n,K} \) and from the definition of \( g_{k,K} \) that
\[ \sum_{k \in \{0,...,K-1\}^d} h_{k,K} \otimes g_{k,K} = \sum_{n=1}^{N} f_{n,K} \otimes g_n. \]

Thus, (5.2) is the same as
\[ \lim_{t \to \infty} t^\frac{d}{2} \mu(t, T_K(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}}) = d^{-\frac{1}{2}} (2\pi)^{-1} \left\| \sum_{n=1}^{N} f_{n,K} \otimes g_n \right\|_d. \]

Note that \( f_{n,K} \to f_n \) in the uniform norm as \( K \to \infty \). Hence, \( T_K(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \to T(-\Delta_{\mathbb{T}^d})^{-\frac{1}{2}} \) in \( \mathcal{L}_d,\infty \) as \( K \to \infty \) and \( \sum_{n=1}^{N} f_{n,K} \otimes g_n \to \sum_{n=1}^{N} f_n \otimes g_n \) in \( \mathcal{L}_d(\mathbb{T}^d \times \mathbb{S}^{d-1}) \) as \( K \to \infty \). Theorem 5.2 now follows from (5.3) and Lemma 5.5. \( \square \)

5.2. Passing from the torus to the whole space. In this subsection we consider the discrete and continuous Fourier transforms, defined respectively, by
\[ \hat{a}(t) = \sum_{n \in \mathbb{Z}^d} a(n) e^{i n \cdot t}, \quad t \in [-\pi, \pi]^d, \]
and
\[ F^{-1} h(t) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} h(\xi) e^{i \xi \cdot t} \, d\xi, \quad t \in \mathbb{R}^d. \]

Lemma 5.5. Let \( f_1, f_2, h \in L_\infty(\mathbb{R}^d) \) and suppose that \( f_1 \) and \( f_2 \) are supported in \([0,1]^d\) and that \( F^{-1} h \) is a function. If \( a \in l_\infty(\mathbb{Z}^d) \) satisfies \( \hat{a} = (2\pi)^{\frac{d}{2}} F^{-1} h \) on \([-1,1]^d\), then
\[ \pi_1(f_1) h(-i\nabla) \pi_1(f_2)|_{L_2([0,1]^d)} = |_{L_2([0,1]^d)} M_{f_1} a(-i\nabla_{\mathbb{T}^d}) M_{f_2} |_{L_2([0,1]^d)}, \]
where, on the right side, \( f_j \) is identified with the 2\( \pi \)-periodic extension of \( f_j \) on \([-\pi, \pi]^d \).
Proof. It is clear that $h(-i\nabla)$ is an integral operator on $L_2(\mathbb{R}^d)$ with integral kernel
\[(t,u) \mapsto (2\pi)^{-\frac{d}{2}}(F^{-1}h)(t-u),\quad t,u \in \mathbb{R}^d.
\]
Thus, $\pi_1(f_1)h(-i\nabla)\pi_1(f_2)$ is an integral operator with integral kernel
\[(t,u) \mapsto (2\pi)^{-\frac{d}{2}}f_1(t)f_2(u)(F^{-1}h)(t-u),\quad t,u \in \mathbb{R}^d.
\]

On the other hand, $a(-i\nabla\tau_t)$ is an integral operator on $L_2([-\pi,\pi]^d)$ with integral kernel
\[(t,u) \mapsto (2\pi)^{-d}\sum_{n \in \mathbb{Z}^d} a(n)e^{i\pi\cdot t-n}\cdot F^{-1}(n)a(t-u),\quad t,u \in [-\pi,\pi]^d.
\]
Thus, $M_f, a(-i\nabla\tau_t)M_f$ is an integral operator on $L_2([-\pi,\pi]^d)$ with integral kernel
\[(t,u) \mapsto (2\pi)^{-d}f_1(t)f_2(u)a(t-u),\quad t,u \in [-\pi,\pi]^d.
\]

To prove the equality of those operators, it suffices to establish the equality of their integral kernels. In other words, we need to check
\[(2\pi)^{-\frac{d}{2}}f_1(t)f_2(u)(F^{-1}h)(t-u) = (2\pi)^{-d}f_1(t)f_2(u)a(t-u),\quad t,u \in [0,1]^d.
\]
This equality follows from the assumption on $a$ and the fact that $t-u \in [-1,1]^d$.  □

**Fact 5.6.** If distributions $f_1$ and $f_2$ are continuous (except, possibly, at 0) functions, then so is the distribution $f_1 + f_2$. Furthermore, equality $f_3 = f_1 + f_2$ holds pointwise (except, possibly, at 0).

**Lemma 5.7.** Let $d > 2$. Let $g \in C^\infty(\mathbb{S}^{d-1})$ and set
\[h(t) = g\left(\frac{t}{|t|}\right)|t|^{-2},\quad t \in \mathbb{R}^d.
\]
There is an $a \in l_\infty(\mathbb{Z}^d)$ such that $\hat{a} = (2\pi)^\frac{d}{2}F^{-1}h$ on $[-1,1]^d$ and such that
\[a(n) = h(n) + O(|n|^{-d}),\quad 0 \neq n \in \mathbb{Z}^d.
\]

**Proof.** Let $\phi \in C^\infty_c(\mathbb{R}^d)$ have support in $[-2,2]^2$ and satisfy $\phi = 1$ on $[-1,1]^d$. Set
\[\hat{a}(t) = (2\pi)^\frac{d}{2}\phi(t) \cdot (F^{-1}h)(t),\quad t \in [-\pi,\pi]^d.
\]
We have
\[a(n) = (2\pi)^{-d}\int_{[-\pi,\pi]^d} \hat{a}(t)e^{-i\pi\cdot n}dt = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} \phi(t)(F^{-1}h)(t)e^{-i\pi\cdot n}dt = (F(\phi \cdot F^{-1}h))(n).
\]

Since $h$ is a smooth (except at 0) homogeneous function of degree $-2$, it follows from Proposition 2.4.8 and Exercise 2.3.9 (d) in [21] that $F^{-1}h$ is a smooth (except at 0) homogeneous function of degree $2-d$. By Theorem 2.3.21 in [21], $F(\phi \cdot F^{-1}h)$ is a smooth function. Using the equality $F(\phi \cdot F^{-1}h) = h$, we conclude that
\[F(\phi \cdot F^{-1}h) - h = F((1 - \phi) \cdot F^{-1}h)
\]
in the sense of distributions. Since each term on the left side is a continuous (except, possibly, at 0) function, it follows from Fact [5.6] that the right hand side is also a continuous (except, possibly, at 0) function and the equality holds pointwise.
The function \( \psi = (1 - \phi) \cdot F^{-1} h \) is smooth. It is clear that \( \Delta^2 \psi \in L_1(\mathbb{R}^d) \).

Denote for brevity \( u(t) = |t|^4, \ t \in \mathbb{R}^d \). We have
\[
u \cdot \mathcal{F} \psi = \mathcal{F} (\Delta^2 \psi).
\]
Thus,
\[
\|u \cdot \mathcal{F} \psi\|_{\infty} = \|\mathcal{F} (\Delta^2 \psi)\|_{\infty} \leq \|\Delta^2 \psi\|_1 < \infty.
\]
Since \( u \cdot \mathcal{F} \psi \) is bounded and since \( \mathcal{F} \psi \) is continuous (except, possibly, at 0), it follows that
\[
\mathcal{F} \psi(t) = O(|t|^{-4}), \ t \in \mathbb{R}.
\]
In particular,
\[
(\mathcal{F} \psi)(n) = O(|n|^{-4}), \ 0 \neq n \in \mathbb{Z}^d.
\]
Thus,
\[
(\mathcal{F}(\phi \cdot F^{-1} h))(n) = h(n) - (\mathcal{F} \psi)(n) = h(n) + O(|n|^{-4}), \ 0 \neq n \in \mathbb{Z}^d.
\]
This completes the proof. \( \square \)

In the following lemma we slightly abuse the notation and denote
\[
(-\Delta_{T^d})^{-\frac{1}{2}} \defeq (-\Delta_{\mathbb{S}^d})^{-\frac{1}{2}} \cdot (1 - P),
\]
where \( P : L_2(T^d) \to L_2(T^d) \) is the orthogonal projection onto the subspace of constants.

**Lemma 5.8.** Let \( (f_n)_{n=1}^N \subset L_{\infty}(\mathbb{R}^d) \) be supported on \([0,1]^d\) and let \( (g_n)_{n=1}^N \subset C_{\infty}(\mathbb{S}^{d-1}) \). Then for
\[
T = \sum_{n=1}^N \pi_1(f_n) \pi_2(g_n), \quad S = \sum_{n=1}^N M_{f_n} g_n \left( \frac{-i \nabla \otimes \nabla}{\sqrt{-\Delta_{T^d}}} \right),
\]
one has
\[
\limsup_{t \to \infty} t^{\frac{d}{2}} \mu(t, T (-\Delta)^{-\frac{1}{2}}) = \limsup_{t \to \infty} t^{\frac{d}{2}} \mu(t, S (-\Delta_{T^d})^{-\frac{1}{2}})
\]
and
\[
\liminf_{t \to \infty} t^{\frac{d}{2}} \mu(t, T (-\Delta)^{-\frac{1}{2}}) = \liminf_{t \to \infty} t^{\frac{d}{2}} \mu(t, S (-\Delta_{T^d})^{-\frac{1}{2}}).
\]
**Proof.** Denote for brevity
\[
A = T (-\Delta)^{-\frac{1}{2}}, \quad B = S (-\Delta_{T^d})^{-\frac{1}{2}}.
\]
Then
\[
|A^*|^2 = \sum_{n_1, n_2=1}^N \pi_1(f_{n_1}) h_{n_1, n_2} (-i \nabla) \pi_1(f_{n_2}),
\]
\[
|B^*|^2 = \sum_{n_1, n_2=1}^N M_{f_{n_1}} h_{n_1, n_2} (-i \nabla_{T^d}) M_{f_{n_2}},
\]
where
\[
h_{n_1, n_2}(t) = g_{n_1} \left( \frac{t}{|t|} \right) g_{n_2} \left( \frac{t}{|t|} \right) |t|^{-2}, \quad t \in \mathbb{R}^d.
\]
Let \( a_{n_1, n_2} \in \ell_\infty(\mathbb{Z}^d) \) such that
\[
\hat{a}_{n_1, n_2} = (2\pi)^{\frac{d}{2}} F^{-1} h_{n_1, n_2} \text{ on } [-1,1]^d,
\]
and set

\[ X := \sum_{n_1,n_2=1}^{N} M_{f_{n_1}} a_{n_1,n_2}(-i\nabla_T)d)M_{f_{n_2}}. \]

Then, by Lemma 3.1, we have

\[ |A^*|^2_{L_2([0,1]^d)} = X |L_2([0,1]^d)|, \]

and therefore \( \mu(|A^*|^2_{L_2([0,1]^d)}) = \mu(X|L_2([0,1]^d)) \). Since each \( f_n \) is supported on \([0,1]^d\), we have

\[ \mu^2(A) = \mu(|A^*|^2_{L_2([0,1]^d)}) \quad \mu(X) = \mu(X|L_2([0,1]^d)), \]

and consequently,

\[ \mu^2(A) = \mu(X). \]

By Lemma 5.5, one can choose \( a_{n_1,n_2} \) such that

\[ a_{n_1,n_2}(n) = h_{n_1,n_2}(n) + o_{n_1,n_2}(n), \quad o_{n_1,n_2}(n) = O(|n|^{-4}), \quad 0 \neq n \in \mathbb{Z}^d. \]

Thus, \( X = Y + Z \),

\[ Y := \sum_{n_1,n_2=1}^{N} M_{f_{n_1}} h_{n_1,n_2}(-i\nabla_T)d)M_{f_{n_2}}, \quad Z := \sum_{n_1,n_2=1}^{N} M_{f_{n_1}} o_{n_1,n_2}(-i\nabla_T)d)M_{f_{n_2}}. \]

It is immediate that

\[ o_{n_1,n_2}(-i\nabla_T)d) \in \mathcal{L}_{\frac{d}{2},\infty} \subset (\mathcal{L}_{\frac{d}{2},\infty})_0. \]

Since \( f_n \) is bounded for every \( 1 \leq n \leq N \), it follows that \( Z \in (\mathcal{L}_{\frac{d}{2},\infty})_0 \). Note that \( Y = |B^*|^2 \). Thus,

\[ X - |B^*|^2 \in (\mathcal{L}_{\frac{d}{2},\infty})_0. \]

By Lemma 3.1, the last relation implies

\[ \limsup_{t \to \infty} s^\frac{d}{2} \mu(s, X) = \limsup_{t \to \infty} s^\frac{d}{2} \mu(s, B)^2 \]

and

\[ \liminf_{t \to \infty} s^\frac{d}{2} \mu(s, X) = \liminf_{t \to \infty} s^\frac{d}{2} \mu(s, B)^2. \]

Recalling that \( \mu(s, X) = \mu(s, A)^2 \), we deduce the assertion of the lemma.

\[ \square \]

5.3. **Proof of Theorem 5.1** for \( d > 2 \). Combining the results of the previous two subsections, we are finally in position to prove the main result of this section.

**Proof of Theorem 5.1** for \( d > 2 \). Suppose that \( f_n \in C_c(\mathbb{R}^d) \), \( 1 \leq n \leq N \), are supported on \([0,1]^d\) and let \( g_n \in C^\infty(\mathbb{S}^{d-1}) \), \( 1 \leq n \leq N \). Then, identifying \( f_n \) with the \( 2\pi \)-periodic extension of \( f_n|_{[-\pi,\pi]^{d}} \), we deduce from Theorem 5.2 that

\[ \lim_{t \to \infty} |t|^{\frac{d}{2}} \mu(t, \left( \sum_{n=1}^{N} M_{f_{n}} g_{n} \left( -\frac{i\nabla_Td}{\sqrt{-\Delta_Td}} \right) \right) (-\Delta_Td)^{-\frac{d}{2}}) = d^{-\frac{d}{2}} (2\pi)^{-1} \left\| \sum_{n=1}^{N} f_n \otimes g_n \right\|_d. \]

By Lemma 5.8 and the fact that

\[ \left\| \sum_{n=1}^{N} f_n \otimes g_n \right\|_d = \left\| \text{sym}(T) \right\|_d, \]
Definition 6.1. Let $\mu(\cdot, \cdot)$ be the set of all compactly supported $f \in L_\infty(\mathbb{R}^2)$ such that, for every Borel set $I \subset \mathbb{R}^1$, one has

$$\lim_{t \to \infty} t^\frac{3}{2} \mu\left(t, \left(\sum_{n=1}^{N} \pi_1(f_n)\pi_2(g_n)\right)(-\Delta)^{-\frac{3}{2}}\right) = d^{-\frac{3}{2}}(2\pi)^{-1}\|\text{sym}(T)\|_d.$$  

By applying dilations and translations, we can remove the restriction on the support of $f_n$'s. By Lemmas 4.3 and 3.2 we can remove the restriction on the smoothness of the $g_n$'s.

The mapping

$$t \to |t|^{-1} - (1 + |t|^2)^{-\frac{1}{2}}, \quad t \in \mathbb{R}^d,$$

belongs to $(L_{d,\infty}(\mathbb{R}^d))_0$. By [4, Subsection 5.3] (see also [29, Corollary 1.2]), we have

$$\pi_1(f_n)(-\Delta)^{-\frac{1}{2}} - \pi_1(f_n)(1 - \Delta)^{-\frac{1}{2}} \in (L_{d,\infty})_0, \quad 1 \leq n \leq N,$$

and, therefore,

$$\sum_{n=1}^{N} \pi_1(f_n)\pi_2(g_n)(-\Delta)^{-\frac{1}{2}} - \sum_{n=1}^{N} \pi_1(f_n)\pi_2(g_n)(1 - \Delta)^{-\frac{1}{2}} \in (L_{d,\infty})_0.$$  

Hence, by Lemma 3.1, we can replace $-\Delta$ with $1 - \Delta$. This completes the proof of Theorem 5.1 for $d > 2$.

6. Proof of Theorem 1.3 in a special case. Case $d = 2$

In this section we prove Theorem 5.1 for $d = 2$.

6.1. The set $\mathcal{X}_2$.

**Definition 6.1.** Let $\mathcal{X}_2$ be the set of all compactly supported $f \in L_\infty(\mathbb{R}^2)$ such that, for every Borel set $I \subset \mathbb{R}^1$, one has

$$\lim_{t \to \infty} t^\frac{3}{2} \mu\left(t, \pi_1(f)\pi_2(\chi_I)(1 - \Delta)^{-\frac{1}{2}}\right) = 2^{-\frac{1}{2}}(2\pi)^{-1}m(I)^\frac{1}{2}\|f\|_2.$$  

In this subsection, we frequently use the unitary operator $\text{Shift}_c : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$, $c \in \mathbb{R}^2$, defined by setting

$$(\text{Shift}_c f)(u) = f(u + c), \quad f \in L_2(\mathbb{R}^2).$$

We also frequently use the dilation operator $\sigma_t$, $0 < t \in \mathbb{R}$, defined by setting

$$(\sigma_t f)(u) = f\left(\frac{u}{t}\right), \quad f \in L_2(\mathbb{R}^2).$$

**Lemma 6.2.** Let $(f_i)_{i \geq 0} \in \mathcal{X}_2$.

1. if $(f_i)_{i \geq 0}$ are pairwise disjointly supported, then $\sum_{i=0}^{n} f_i \in \mathcal{X}_2$;  
2. if $c_0 \in \mathbb{R}^2$, then $\text{Shift}_{c_0} f_1 \in \mathcal{X}_2$;  
3. if $t > 0$, then $\sigma_t f_1 \in \mathcal{X}_2$;  
4. if $(f_i)_{i \geq 0}$ are supported on a compact set $K$ and $f_i \to f \in L_\infty(\mathbb{R}^2)$ in $(L_2 \log L)(\mathbb{R}^2)$, then $f \in \mathcal{X}_2$.

To be more precise, by the assumption that $f_i$ and $f_{i'}$ are pairwise disjointly supported we mean that

$$m(\text{supp} f_i \cap \text{supp} f_{i'}) = 0.$$
Proof. Fix a Borel set \( I \subset S^1 \).

For the proof of the first assertion we want to apply Lemma 3.5 and set

\[
A = \pi_1 \left( \sum_{l=0}^{n} f_l \pi_2 (\chi_l) (1 - \Delta)^{-\frac{d}{2}} \right).
\]

Since \( \pi_2 (\chi_l) \) commutes with \((1 - \Delta)^{-\frac{d}{2}}\) and \( \sum_{l=0}^{n} f_l \) is a bounded, compactly supported function, we deduce from Theorem 4.4 that \( A \in \mathcal{L}_{2,\infty} \). Let

\[
p_l = \pi_1 (\chi_{\text{supp}(f_l)}), \quad 0 \leq l \leq n,
\]

and let \( p_{-1} = 1 - \sum_{0 \leq l < k} p_l \). The first assumption in Lemma 3.5 follows from the equality

\[
p_l A = \pi_1 (f_l) \pi_2 (\chi_l) (1 - \Delta)^{-\frac{d}{2}}, \quad 0 \leq l \leq n,
\]

and the assumption \( f_l \in \mathcal{X}_2 \), \( 0 \leq l \leq n \), as well as \( p_{-1} A = 0 \). Since \( \pi_1 (\sum_{k=0}^{n} f_k) \) commutes with \( p_l \), \( 1 \leq l \leq n \), the second assumption in Lemma 3.5 follows from Theorem 4.1. Applying Lemma 3.5, we complete the proof of the first assertion.

It is immediate that

\[
\pi_1 (\text{Shift}_{c_l} (f_l)) \pi_2 (\chi_l) (1 - \Delta)^{-\frac{d}{2}} = \text{Shift}_{c_l} \cdot \pi_1 (f_l) \pi_2 (\chi_l) (1 - \Delta)^{-\frac{d}{2}} \cdot \text{Shift}_{-c_l}.
\]

Therefore,

\[
\mu (\pi_1 (\text{Shift}_{c_l} (f_l)) \pi_2 (\chi_l) (1 - \Delta)^{-\frac{d}{2}}) = \mu (\pi_1 (f_l) \pi_2 (\chi_l) (1 - \Delta)^{-\frac{d}{2}})
\]

The second assertion follows from the latter equality and the assumption \( f_l \in \mathcal{X}_2 \).

To see the third assertion, note that

\[
\pi_1 (\sigma_l f_l) \pi_2 (\chi_l) (1 - \Delta)^{-\frac{d}{2}} = \sigma_l \cdot \pi_1 (f_l) \pi_2 (\chi_l) (1 - t^{-2} \Delta)^{-\frac{d}{2}} \cdot \sigma_l^{-1}.
\]

Thus,

\[
\mu (\pi_1 (\sigma_l f_l) \pi_2 (\chi_l) (1 - \Delta)^{-\frac{d}{2}}) = \mu (\pi_1 (f_l) \pi_2 (\chi_l) (1 - t^{-2} \Delta)^{-\frac{d}{2}}).
\]

Note that, by a straightforward evaluation of the Hilbert–Schmidt norm,

\[
\pi_1 (f_l) \pi_2 (\chi_l) (1 - t^{-2} \Delta)^{-\frac{d}{2}} - t \pi_1 (f_l) \pi_2 (\chi_l) (1 - \Delta)^{-\frac{d}{2}} \in \mathcal{L}_2 \subset (\mathcal{L}_{2,\infty})_0.
\]

Using \( f_l \in \mathcal{X}_2 \) and Lemma 3.5, we deduce

\[
\lim_{t \to \infty} t^{\frac{d}{2}} \mu (\pi_1 (\sigma_l f_l) \pi_2 (\chi_l) (1 - \Delta)^{-\frac{d}{2}}) = t \cdot m (I)^{\frac{d}{2}} \| f_l \|_2 = m (I)^{\frac{d}{2}} \| \sigma_l f_l \|_2.
\]

The fourth assertion follows from Theorem 4.4 and Lemma 3.5.\( \square \)

The following lemma contains the main idea of the proof of Theorem 5.1 for \( d = 2 \). The rest of the argument consists either of verification of its conditions or of its applications.

Lemma 6.3. If \( f \in \mathcal{L}_\infty (\mathbb{R}^2) \) is compactly supported and rotation invariant, then \( f \in \mathcal{X}_2 \).

Proof. Set

\[
A = \pi_1 (f) (1 - \Delta)^{-\frac{d}{2}}.
\]

We claim that

\[
\lim_{t \to \infty} t^{\frac{d}{2}} \mu (t, A) = (4\pi)^{-\frac{d}{2}} \| f \|_2.
\]

Indeed, for \( f \in C_c (\mathbb{R}^2) \) (not necessarily rotation invariant) this follows by the Birman–Schwinger principle and simple Dirichlet–Neumann bracketing for the resulting Schrödinger operator \(-\Delta + 1 - \alpha f^2\) as in the proof of [24, Theorem 4.28].
Using Theorem 4.4 and Lemma 3.2 the asymptotics extend to all compactly supported \( f \in (L_2 \log L)(\mathbb{R}^2) \), in particular, to those in the statement of the lemma.

To deduce the assertion of the lemma we want to apply Lemma 3.6 with \( p = 2 \).

We define a spectral measure \( \nu \) on the Borel \( \sigma \)-algebra on \([0, 1)\) by setting

\[
\nu(I) = \pi^2 (\chi_{2\pi I \mod 2\pi}), \quad I \subset [0, 1)\.
\]

That \( \nu \) is indeed a spectral measure follows from the continuity of \( \pi^2 \) in the strong operator topology.

Let \( R_\theta \) be a rotation operator on \( L_2(\mathbb{R}^2) \) by the angle \( \theta \in (0, 2\pi) \). If \( I_1 \) and \( I_2 \) are sub-intervals in \([0, 1)\) of equal length, then there is a \( \theta \) such that

\[
\nu(I_2) = R_\theta^{-1}\nu(I_1)R_\theta.
\]

Since \( f \) is rotation invariant, it follows that

\[
A = R_\theta^{-1}AR_\theta.
\]

This verifies the first assumption on \( \nu \) in Lemma 3.6.

The second assumption on \( \nu \) in Lemma 3.6 follows from Theorem 4.1. The third assumption on \( \nu \) in Lemma 3.6 follows from Lemma 4.5 with \( g = \chi_{2\pi I \mod 2\pi} \) and with \( \psi(t) = Ct^{\frac{1}{4}}, \ t \in (0, 1) \). Thus, all the assumptions in Lemma 3.6 are met.

Applying this lemma, we complete the proof. \( \square \)

**Lemma 6.4.** Every \( f \in C_c(\mathbb{R}^2) \) belongs to \( \mathcal{X}_2 \).

**Proof.** By applying dilation and translation, we may assume without loss of generality that \( f \) is supported on \([0, 1]^2\).

The idea is to approximate \( f \) in \( (L_2 \log L)(\mathbb{R}^2) \) by a sum of pairwise disjointly supported (shifted) rotation invariant functions.

For \( l \in \mathbb{N} \), choose a sequence \( \{B_{r_m,l}(c_{m,l})\}_{m \geq 0} \) of non-intersecting balls such that each radius \( r_{m,l} < \frac{1}{l} \) and such that

\[
\bigcup_{m \geq 0} B_{r_m,l}(c_{m,l}) = [0, 1]^2
\]

modulo a set of measure 0.

Set

\[
f_{m,l} = \chi_{B_{r_m,l}(c_{m,l})} \cdot \frac{1}{\operatorname{Vol}(B_{r_m,l}(c_{m,l}))} \int_{B_{r_m,l}(c_{m,l})} f.
\]

Clearly, \( \text{Shift}_{-c_{m,l}} f_{m,l} \) is bounded, compactly supported and rotation invariant, so, by Lemma 6.3, \( \text{Shift}_{-c_{m,l}} f_{m,l} \in \mathcal{X}_2 \). Thus, by Lemma 6.2 (2), \( f_{m,l} \in \mathcal{X}_2 \). For a fixed \( l \in \mathbb{N} \), the functions \( (f_{m,l})_{m \geq 0} \) are pairwise disjointly supported. By Lemma 6.2 (1), we have \( \sum_{m=0}^{M} f_{m,l} \in \mathcal{X}_2 \) for every \( M \geq 0 \). By Lemma 6.2 (4), we have \( \sum_{m \geq 0} f_{m,l} \in \mathcal{X}_2 \) for every fixed \( l \in \mathbb{N} \).

Since \( f \in C([0, 1]^2) \), it follows that

\[
\sum_{m \geq 0} f_{m,l} \to f, \quad l \to \infty,
\]

in the uniform norm. Applying again Lemma 6.2 (4), we complete the proof. \( \square \)
6.2. Proof of Theorem 5.1 for \( d = 2 \). As a consequence of the results in the previous subsection, we are finally in position to complete the proof of Theorem 5.1 in the remaining case \( d = 2 \).

Proof of Theorem 5.1 for \( d = 2 \). For every \( K \in \mathbb{N} \) and \( 0 \leq k < K \), let \( h_{k,K}(s) = \chi_{[\frac{2\pi k}{K}, \frac{2\pi (k+1)}{K})} \) (Arg \((s)) \), \( s \in S^1 \). Let \( g_{n,K} \) be the conditional expectation on \( L_\infty(S^1) \) of \( g_n \) onto the subalgebra generated by \( (h_{k,K})_{0 \leq k < K} \) and write \( g_{n,K} = \sum_{k=0}^{K-1} c_{n,k,K} h_{k,K} \). We set \( f_{k,K} = \sum_{n=1}^{N} c_{n,k,K} f_n \) and \( S_K = \sum_{k=0}^{K-1} \pi_2(h_{k,K}) \pi_1(f_{k,K}) \pi_2(h_{k,K}) \).

The operators \( \pi_2(h_{k,K}) \pi_1(f_{k,K}) \pi_2(h_{k,K})(1 - \Delta)^{-\frac{1}{2}}, \ 0 \leq k < K \), are pairwise orthogonal. Thus, \( \mu(S_K(1 - \Delta)^{-\frac{1}{2}}) = \mu\left( \bigoplus_{k=0}^{K-1} \pi_2(h_{k,K}) \pi_1(f_{k,K}) \pi_2(h_{k,K})(1 - \Delta)^{-\frac{1}{2}} \right) \).

By Lemma 6.4 and Definition 6.1 we have
\[
\lim_{t \to \infty} t^{\frac{1}{2}} \mu(t, \pi_1(f_{k,K}) \pi_2(h_{k,K})(1 - \Delta)^{-\frac{1}{2}}) = (4\pi)^{-\frac{1}{2}} K^{-\frac{1}{2}} \|f_{k,K}\|_2.
\]

Meanwhile, by Theorem 4.1
\[
\pi_2(h_{k,K}) \pi_1(f_{k,K}) \pi_2(h_{k,K})(1 - \Delta)^{-\frac{1}{2}} - \pi_1(f_{k,K}) \pi_2(h_{k,K})(1 - \Delta)^{-\frac{1}{2}} = [\pi_2(h_{k,K}) \pi_1(f_{k,K})(1 - \Delta)^{-\frac{1}{2}}] \pi_2(h_{k,K}) \in (L_2, \infty)_{0},
\]
so, by Lemma 3.1
\[
\lim_{t \to \infty} t^{\frac{1}{2}} \mu(t, \pi_2(h_{k,K}) \pi_1(f_{k,K}) \pi_2(h_{k,K})(1 - \Delta)^{-\frac{1}{2}}) = (4\pi)^{-\frac{1}{2}} K^{-\frac{1}{2}} \|f_{k,K}\|_2.
\]

We deduce, by Lemma 3.3
\[
\lim_{t \to \infty} t^{\frac{1}{2}} \mu(t, \sum_{k=0}^{K-1} \pi_2(h_{k,K}) \pi_1(f_{k,K}) \pi_2(h_{k,K})(1 - \Delta)^{-\frac{1}{2}}) = (4\pi)^{-\frac{1}{2}} \left( \frac{1}{K} \sum_{k=0}^{K-1} \|f_{k,K}\|_2^2 \right)^{\frac{1}{2}}.
\]

In other words, we have
\[
\lim_{t \to \infty} t^{\frac{1}{2}} \mu(S_K(1 - \Delta)^{-\frac{1}{2}}) = (4\pi)^{-\frac{1}{2}} \left( \frac{1}{K} \sum_{k=0}^{K-1} \|f_{k,K}\|_2^2 \right)^{\frac{1}{2}}.
\]
Now let us introduce
\[ T_K = \sum_{n=1}^{N} \pi_1(f_n)\pi_2(g_{n,K}) = \sum_{k=0}^{K-1} \pi_1(f_{k,K})\pi_2(h_{k,K}). \]
By Theorem 4.1 we have
\[ (T_K-S_K)(1-\Delta)^{-\frac{1}{2}} = \sum_{k=0}^{K-1} [\pi_1(f_{k,K})(1-\Delta)^{-\frac{1}{2}}, \pi_2(h_{k,K})] \cdot \pi_2(h_{k,K}) \in (L_{2,\infty})_0. \]
It follows now from (6.1) and Lemma 3.1 that
\[ \lim_{t \to \infty} t^{\frac{1}{2}}\mu(t, T_K(1-\Delta)^{-\frac{1}{2}}) = (4\pi)^{-\frac{1}{2}} \left( \frac{1}{K} \sum_{k=0}^{K-1} \| f_{k,K} \|_2^2 \right)^{\frac{1}{2}}. \]
We note that
\[ \frac{1}{K} \sum_{k=0}^{K-1} \| f_{k,K} \|_2^2 = (2\pi)^{-1} \left\| \sum_{k=0}^{K-1} f_{k,K} \otimes h_{k,K} \right\|_2^2. \]
Moreover, it is clear from the expression for \( g_{n,K} \) and from the definition of \( f_{k,K} \)
that
\[ \sum_{k=0}^{K-1} f_{k,K} \otimes h_{k,K} = \sum_{k=0}^{K-1} \sum_{n=1}^{N} c_{n,k,K} f_n \otimes h_{k,K} = \sum_{n=1}^{N} f_n \otimes g_{n,K}. \]
Thus, (6.2) is the same as
\[ \lim_{t \to \infty} t^{\frac{1}{2}}\mu(T_K(1-\Delta)^{-\frac{1}{2}}) = 2^{-\frac{1}{2}}(2\pi)^{-1} \left\| \sum_{n=1}^{N} f_n \otimes g_{n,K} \right\|_2. \]
Note that \( g_{n,K} \to g_n \) in the uniform norm as \( K \to \infty \). Hence, \( T_K(1-\Delta)^{-\frac{1}{2}} \to T(1-\Delta)^{-\frac{1}{2}} \) in \( L_{2,\infty} \) as \( K \to \infty \) and \( \sum_{n=1}^{N} f_n \otimes g_{n,K} \to \sum_{n=1}^{N} f_n \otimes g_n \) in \( L_2(\mathbb{R}^2 \times S^1) \) as \( K \to \infty \). The assertion follows now from (6.3) and Lemma 4.2.

7. Proof of Theorem 1.5

In this section, we prove our main result on spectral asymptotics, Theorem 1.5. A crucial ingredient will be Theorem 5.1. In order to deduce the former from the latter, we will use the following two simple lemmas.

**Lemma 7.1.** If \( T_1, T_2 \in \Pi \), then
\[ [T_1, T_2] \in \mathcal{K}(L_2(\mathbb{R}^d)). \]

**Proof.** Let \( q : B(L_2(\mathbb{R}^d)) \to B(L_2(\mathbb{R}^d))/\mathcal{K}(L_2(\mathbb{R}^d)) \) be the canonical quotient map. Recall (see the proof of [5], Theorem 3.3]) that sym is constructed as a composition
\[ \text{sym} = \theta^{-1} \circ q, \]
where \( \theta : A_1 \otimes_{\min} A_2 \to q(\Pi) \) is some *-isomorphism (its definition and properties are irrelevant at the current proof). Since sym is a *-homomorphism, it follows that
\[ \text{sym}([T_1, T_2]) = [\text{sym}(T_1), \text{sym}(T_2)] = 0, \quad T_1, T_2 \in \Pi. \]
Thus,
\[ q([T_1, T_2]) = \theta(\text{sym}([T_1, T_2])) = 0, \]
which yields the assertion.

\[ \square \]
Lemma 7.2. Let \((f_k)_{k=1}^m \subset A_1\) and \((g_k)_{k=1}^m \subset A_2\). Then
\[
\prod_{k=1}^m \pi_1(f_k)\pi_2(g_k) \in \pi_1(\prod_{k=1}^m f_k)\pi_2(\prod_{k=1}^m g_k) + K(L_2(\mathbb{R}^d)).
\]

Proof. We prove the assertion by induction on \(m\). For \(m = 1\), there is nothing to prove. Let us prove the assertion for \(m = 2\). We have
\[
\pi_1(f_1)\pi_2(g_1)\pi_1(f_2)\pi_2(g_2) = \pi_2(g_1, \pi_1(f_1, f_2)) \cdot \pi_2(g_2) + [\pi_1(f_1), \pi_2(g_1)] \cdot \pi_1(f_2)\pi_2(g_2) + \pi_1(f_1, f_2)\pi_2(g_1, g_2).
\]
By Lemma 7.1, we have
\[
[\pi_1(f_1), \pi_2(g_1)], [\pi_2(g_1), \pi_1(f_1, f_2)] \in K(L_2(\mathbb{R}^d)).
\]
Therefore,
\[
\pi_1(f_1)\pi_2(g_1)\pi_1(f_2)\pi_2(g_2) \in \pi_1(f_1, f_2)\pi_2(g_1, g_2) + K(L_2(\mathbb{R}^d)).
\]
This proves the assertion for \(m = 2\).

It remains to prove the step of induction. Suppose the assertion holds for \(m \geq 2\) and let us prove it for \(m + 1\). Since
\[
\prod_{k=1}^{m+1} \pi_1(f_k)\pi_2(g_k) = \pi_1(f_1)\pi_2(g_1) \cdot \prod_{k=2}^{m+1} \pi_1(f_k)\pi_2(g_k),
\]
using the inductive assumption, we obtain
\[
\prod_{k=1}^{m+1} \pi_1(f_k)\pi_2(g_k) \in \pi_1(f_1)\pi_2(g_1) \cdot \prod_{k=2}^{m+1} \pi_1(f_k)\pi_2(g_k) + K(L_2(\mathbb{R}^d)).
\]
Using the assertion for \(m = 2\), we obtain
\[
\pi_1(f_1)\pi_2(g_1) \cdot \prod_{k=2}^{m+1} \pi_1(f_k)\pi_2(g_k) \in \pi_1(f_1)\pi_2(g_1) \cdot \prod_{k=2}^{m+1} \pi_1(f_k)\pi_2(g_k) + K(L_2(\mathbb{R}^d)).
\]
Combining the last two equations, we obtain
\[
\prod_{k=1}^{m+1} \pi_1(f_k)\pi_2(g_k) \in \pi_1(f_1)\pi_2(g_1) \cdot \prod_{k=1}^{m+1} \pi_1(f_k)\pi_2(g_k) + K(L_2(\mathbb{R}^d)).
\]
This establishes the step of induction and completes the proof of the lemma. □

We are finally in position to prove our main result on spectral asymptotics.

Proof of Theorem 13.3. By the definition of the \(C^*\)-algebra \(\mathcal{A}\), for every \(T \in \mathcal{A}\), there is a sequence \((T_n)_{n \geq 1}\) in the \(\ast\)-algebra generated by \(\pi_1(A_1)\) and \(\pi_2(A_2)\) such that \(T_n \to T\) in the uniform norm. We can write
\[
T_n = \sum_{l=1}^{l_n} \prod_{k=1}^{k_n} \pi_1(f_{n,k,l})\pi_2(g_{n,k,l})
\]
with \(f_{n,k,l} \in A_1\) and \(g_{n,k,l} \in A_2\). Let us abbreviate
\[
f_{n,l} = \prod_{k=1}^{k_n} f_{n,k,l} \in A_1, \quad g_{n,l} = \prod_{k=1}^{k_n} g_{n,k,l} \in A_2.
\]
Then, by Lemma 7.2, we have
\[ S_n = T_n - \sum_{l=1}^{n} \pi_1(f_{n,l})\pi_2(g_{n,l}) \in K(L_2(\mathbb{R}^d)). \]

By assumption in Theorem 4.1, the operator \( T \) is compactly supported. Hence, \( T = T\pi_1(\phi) \) for some \( \phi \in C_c^\infty(\mathbb{R}^d) \). We decompose
\[ T_n\pi_1(\phi) = A_n + B_n + C_n, \]
where
\[ A_n = \sum_{l=1}^{n} \pi_1(\phi \cdot f_{n,l})\pi_2(g_{n,l}), \]
\[ B_n = \sum_{l=1}^{n} \pi_1(f_{n,l}) \cdot [\pi_2(g_{n,l}), \pi_1(\phi)], \]
\[ C_n = S_n\pi_1(\phi). \]

By Theorem 5.1 we have
\[ \lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, A_n(1 - \Delta)^{-\frac{1}{2}}) = d^{-\frac{d}{2}}(2\pi)^{-1}||\text{sym}(A_n)||_d. \]

By Theorem 4.1 we have \( B_n(1 - \Delta)^{-\frac{1}{2}} \in (\mathcal{L}_{d,\infty})_0 \). Since \( S_n \) is compact and since \( \pi_1(\phi)(1 - \Delta)^{-\frac{1}{2}} \in \mathcal{L}_{d,\infty} \), it follows that \( C_n(1 - \Delta)^{-\frac{1}{2}} \in (\mathcal{L}_{d,\infty})_0 \). Consequently, \( (B_n + C_n)(1 - \Delta)^{-\frac{1}{2}} \in (\mathcal{L}_{d,\infty})_0 \) and, by Lemma 3.1,
\[ \lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, (A_n + B_n + C_n)(1 - \Delta)^{-\frac{1}{2}}) = \lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, A_n(1 - \Delta)^{-\frac{1}{2}}). \]

On the other hand, since \( B_n \) and \( C_n \) are compact (the former by Lemma 7.1) and since \( \text{sym} \) vanishes on every compact operator in \( \Pi \) (see the proof of Lemma 7.1), it follows that
\[ \text{sym}(A_n) = \text{sym}(A_n + B_n + C_n). \]

Combining all these assertions, we find that
\[ \lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, A_n + B_n + C_n) = d^{-\frac{d}{2}}(2\pi)^{-1}||\text{sym}(A_n + B_n + C_n)||_d. \]

Taking into account that \( A_n + B_n + C_n = T_n\pi_1(\phi) \), we rewrite these asymptotics as
\[ (7.1) \quad \lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, T_n\pi_1(\phi)) = d^{-\frac{d}{2}}(2\pi)^{-1}||\text{sym}(T_n\pi_1(\phi))||_d. \]

Since \( T_n\pi_1(\phi) \to T\pi_1(\phi) = T \) in the uniform norm, we deduce, on the one hand, using dominated convergence that
\[ ||\text{sym}(T_n\pi_1(\phi))||_d \to ||\text{sym}(T)||_d \]
and, on the other hand, using the fact that \( \pi_1(\phi)(1 - \Delta)^{-\frac{1}{2}} \in \mathcal{L}_{d,\infty} \), that
\[ T_n\pi_1(\phi)(1 - \Delta)^{-\frac{1}{2}} \to T(1 - \Delta)^{-\frac{1}{2}} \quad \text{in } \mathcal{L}_{d,\infty}. \]
The assertion follows now from (7.1) and Lemma 3.2.  

\footnote{For \( d = 2 \) this follows from Theorem 4.1. For \( d > 2 \), this follows from Cwikel’s estimate in \( \mathcal{L}_{d,\infty} \).}
8. Proof of Theorems 1.1 and 1.2

Our goal in this section is to prove Theorems 1.1 and 1.2 concerning
\[ df = i[\text{sgn } \mathcal{D}, 1 \otimes M_f]. \]

We begin with one implication in Theorem 1.1, which is a simple consequence of previous work in [31].

Proposition 8.1. If \( d \geq 2 \) and if \( f \in \dot{W}^1_d(\mathbb{R}^d) \), then \( \bar{df} \in L^d, \infty \) with
\[
\| \bar{df} \|_{d, \infty} \leq C_d \| \nabla f \|_d.
\]

Proof. We know [28, Theorem 11.43] that there is a sequence \( (f_n) \subset C_\infty^\infty(\mathbb{R}^d) \) with \( \nabla f_n \rightarrow \nabla f \) in \( L^d \). Applying [31, Theorem 1] we infer that
\[
\| \bar{df}_n \|_{d, \infty} \leq C_d \| \nabla f_n \|_d.
\]
The continuity of the embedding \( \dot{W}^1_d \subset BMO \) implies that \( f_n \rightarrow f \) in \( BMO \) and therefore, by [13, Theorem I], \( \bar{df}_n \rightarrow \bar{df} \) in uniform norm. By the Fatou property, this implies \( \bar{df} \in L^d, \infty \) and
\[
\| \bar{df} \|_{d, \infty} \leq \liminf_{n \rightarrow \infty} \| \bar{df}_n \|_{d, \infty} \leq C_d \liminf_{n \rightarrow \infty} \| \nabla f_n \|_d = C_d \| \nabla f \|_d,
\]
proving the claim. \( \square \)

The converse direction in Theorem 1.1 is harder. We begin by computing the asymptotics in the smooth case. We make use of the following simple lemma.

Lemma 8.2. Let \( d \geq 2 \), let \( f \in C_\infty^\infty(\mathbb{R}^d) \) be real-valued and set
\[
A_k = M_{\partial_k f} - \sum_{j=1}^d \frac{D_k D_j}{-\Delta} M_{\partial_j f}, \quad 1 \leq k \leq d.
\]
Then one has
\begin{equation}
(\Im A_k)(1 - \Delta)^{-\frac{k}{d}} \in (L^d, \infty)_0, \quad 1 \leq k \leq d,
\end{equation}
\begin{equation}
(1 - \Delta)^{-\frac{k}{d}} [A_k, A_l](1 - \Delta)^{-\frac{l}{d}} \in (L^d, \infty)_0, \quad 1 \leq k, l \leq d.
\end{equation}

Proof. Since \( f \) is real-valued, it follows that
\[
A_k^* = M_{\partial_k f} - \sum_{j=1}^d M_{\partial_j f} \frac{D_k D_j}{-\Delta}, \quad 1 \leq k \leq d.
\]
Thus,
\[
\Im A_k = -\sum_{j=1}^d \left[ \frac{D_k D_j}{-\Delta}, M_{\partial_j f} \right], \quad 1 \leq k \leq d.
\]
The first inclusion follows now from Theorem 4.1 (we apply it to the function \( g(s) = s_k s_j, s \in \mathbb{S}^{d-1} \)).
Next,
\[
[A_k, A_l] = - \sum_{j_2=1}^{d} [M_{\partial_{k} f}, \frac{D_l D_{j_2}^2}{-\Delta} M_{\partial_{j_2} f}] - \sum_{j_1=1}^{d} \frac{D_k D_{j_1}^2}{-\Delta} M_{\partial_{j_1} f}, M_{\partial_{l} f}]
+ \sum_{j_1, j_2=1}^{d} \frac{D_k D_{j_1}^2}{-\Delta} M_{\partial_{j_1} f}, \frac{D_l D_{j_2}^2}{-\Delta} M_{\partial_{j_2} f}
= - \sum_{j_2=1}^{d} [M_{\partial_{k} f}, \frac{D_l D_{j_2}^2}{-\Delta} M_{\partial_{j_2} f} - \sum_{j_1=1}^{d} \frac{D_k D_{j_1}^2}{-\Delta} M_{\partial_{j_1} f}, M_{\partial_{j_2} f}
+ \sum_{j_1, j_2=1}^{d} \frac{D_k D_{j_1}^2}{-\Delta} [\frac{D_l D_{j_2}^2}{-\Delta} M_{\partial_{j_2} f} - \sum_{j_1=1}^{d} \frac{D_k D_{j_1}^2}{-\Delta} M_{\partial_{j_1} f}, M_{\partial_{j_2} f}
+ \sum_{j_1, j_2=1}^{d} \frac{D_l D_{j_2}^2}{-\Delta} [\frac{D_k D_{j_1}^2}{-\Delta} M_{\partial_{j_1} f} M_{\partial_{j_2} f} - \sum_{j_1=1}^{d} \frac{D_k D_{j_1}^2}{-\Delta} M_{\partial_{j_1} f}, M_{\partial_{j_2} f}.
\]

Therefore,
\[
(1 - \Delta)^{-\frac{1}{2}} [A_k, A_l](1 - \Delta)^{-\frac{1}{2}}
= - \sum_{j_2=1}^{d} [(1 - \Delta)^{-\frac{1}{2}} M_{\partial_{k} f}, \frac{D_l D_{j_2}^2}{-\Delta} M_{\partial_{j_2} f}]
+ \sum_{j_1=1}^{d} \frac{D_k D_{j_1}^2}{-\Delta} [(1 - \Delta)^{-\frac{1}{2}} M_{\partial_{j_1} f}, \frac{D_l D_{j_2}^2}{-\Delta} M_{\partial_{j_2} f}]
+ \sum_{j_1, j_2=1}^{d} \frac{D_k D_{j_1}^2}{-\Delta} [\frac{D_l D_{j_2}^2}{-\Delta} M_{\partial_{j_2} f}]
- \sum_{j_1=1}^{d} \frac{D_k D_{j_1}^2}{-\Delta} [\frac{D_l D_{j_2}^2}{-\Delta} M_{\partial_{j_2} f}]
+ \sum_{j_1, j_2=1}^{d} \frac{D_l D_{j_2}^2}{-\Delta} [\frac{D_k D_{j_1}^2}{-\Delta} M_{\partial_{j_1} f} M_{\partial_{j_2} f}]
- \sum_{j_1=1}^{d} \frac{D_k D_{j_1}^2}{-\Delta} M_{\partial_{j_1} f}, M_{\partial_{j_2} f}.
\]

By Theorem 4.1, each commutator factor in the above formula belongs to \((\mathcal{L}_{d, \infty})_0\).
Since \(M_{\partial_{j_2} f}(1 - \Delta)^{-\frac{1}{2}} \in \mathcal{L}_{d, \infty}\), \(1 \leq j \leq d\), the assertion follows now from Hölder’s inequality. \(\square\)

The following important lemma explains the role of the operator \(A\).

**Lemma 8.3.** Let \(d \geq 2\) and let \(f \in C_c^\infty(\mathbb{R}^d)\) be real-valued. Then, with \(A_k\) defined in Lemma 8.2, we have
\[
|df|^2 - 1 \otimes (1 - \Delta)^{-\frac{1}{2}} (\sum_{k=1}^{d} |A_k|^2)(1 - \Delta)^{-\frac{1}{2}} \in (\mathcal{L}_{\frac{d}{2}, \infty})_0.
\]

**Proof.** We have (see [30, Theorem 6.3.1])
\[
df - A(1 + D^2)^{-\frac{1}{2}} \in (\mathcal{L}_{d, \infty})_0, \quad A = \sum_{k=1}^{d} \gamma_k \otimes A_k.
\]
This, together with the fact that \(\bar{df} \in L^\infty_d\), (see Proposition \[5.1\]) and the algebraic identity

\[
|X|^2 - |Y|^2 = -|X - Y|^2 + X^* (X - Y) + (X - Y)^* X, \tag{8.3}
\]

implies that

\[
|d f|^2 - |A(1 + D^2)^{-\frac{1}{2}}|^2 \in (L^d_\infty)_0. \]

Clearly,

\[
|A(1 + D^2)^{-\frac{1}{2}}|^2 = \sum_{k,l=1}^d \gamma_k \gamma_l \otimes (1 - \Delta)^{-\frac{1}{2}} A_k^* A_l (1 - \Delta)^{-\frac{1}{2}}.
\]

Using the anticommutation relations of the gamma matrices, we write

\[
|A(1 + D^2)^{-\frac{1}{2}}|^2 - 1 \otimes (1 - \Delta)^{-\frac{1}{2}} (\sum_{k=1}^d |A_k|^2) (1 - \Delta)^{-\frac{1}{2}}
\]

\[
= \sum_{1 \leq k < l \leq d} \gamma_k \gamma_l \otimes (1 - \Delta)^{-\frac{1}{2}} (A_k^* A_l - A_l^* A_k) (1 - \Delta)^{-\frac{1}{2}}
\]

\[
= -2i \sum_{1 \leq k < l \leq d} \gamma_k \gamma_l \otimes (1 - \Delta)^{-\frac{1}{2}} (\Im A_k) \cdot A_l (1 - \Delta)^{-\frac{1}{2}}
\]

\[
+ 2i \sum_{1 \leq k < l \leq d} \gamma_k \gamma_l \otimes (1 - \Delta)^{-\frac{1}{2}} (\Im A_l) \cdot A_k (1 - \Delta)^{-\frac{1}{2}}
\]

\[
+ \sum_{1 \leq k < l \leq d} \gamma_k \gamma_l \otimes (1 - \Delta)^{-\frac{1}{2}} [A_k, A_l] (1 - \Delta)^{-\frac{1}{2}}.
\]

The assertion now follows from Lemma \[8.2\] Cwikel’s estimate and Hölder’s inequality. \(\square\)

**Lemma 8.4.** For \(f \in C^\infty_c(R^d), \chi \in (C_c + \mathbb{C})(R^d),\) set

\[
T = \left(\sum_{k=1}^d |A_k|^2\right)^{\frac{1}{2}} M_\chi.
\]

Then

\[
\lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, T(1 - \Delta)^{-\frac{1}{2}}) = \kappa'_d \left(\int_{R^d} |\chi|^d |\nabla f|^d dx\right)^{\frac{1}{2}}
\]

where

\[
\kappa'_d = (2\pi)^{-1} \left(d^{-1} \int_{S^{d-1}} (1 - s^2_d)^{\frac{1}{2}} ds\right)^{\frac{1}{2}}.
\]

**Proof.** We will derive the claimed asymptotics by applying Theorem \[1.5\] to the operator \(T\). Note that \(T \in \Pi\) (since \(\Pi\) is a \(C^*\) algebra, or more explicitly, by approximating the square root in the definition of \(T\) by polynomials) and that \(T\) is compactly supported from the right (indeed, in the decomposition \(L^2(\mathbb{R}^d) = L^2(\text{supp} \nabla f) \oplus L^2(\mathbb{R}^d \setminus \text{supp} \nabla f)\) the operator \(\sum |A_k|^2\) acts nontrivially only in the first term, and so does its square root). Applying Theorem \[1.5\] we deduce that

\[
\lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, T(1 - \Delta)^{-\frac{1}{2}}) = d^{\frac{d}{2}} (2\pi)^{-1} \|\text{sym}(T)\|_{d}.\]
It remains to compute the right side. Since \( \text{sym} : \Pi \to A_1 \otimes_{\min} A_2 \) is a \( * \)-homomorphism, it follows that
\[
|\text{sym}(T)| = (|\chi| \otimes 1) \cdot \left( \sum_{k=1}^{d} |\text{sym}(A_k)|^2 \right)^{\frac{1}{2}}.
\]
For \( t \in \mathbb{R}^d \) and \( s \in S^{d-1} \), we have
\[
\text{sym}(A_k)(t, s) = (\partial_k f)(t) - s_k \cdot \langle s, (\nabla f)(t) \rangle.
\]
Thus,
\[
|\text{sym}(T)(t, s)| = |\chi(t)| \left| (\nabla f)(t) - s \cdot (s, (\nabla f)(t)) \right|
= |\chi(t)| \cdot \left| (\nabla f)(t) - (s, (\nabla f)(t)) \right|^2 \frac{1}{2}
\]
and
\[
\|\text{sym}(T)\|_d^2 = \int_{\mathbb{R}^d \times S^{d-1}} |\chi(t)|^d \left| (\nabla f)(t) - (s, (\nabla f)(t)) \right|^2 \frac{1}{2} dt ds
= \int_{\mathbb{R}^d} |\chi(t)|^d \left| (\nabla f)(t) \right|^d \left( \int_{S^{d-1}} \left| e(t) \right|^2 - |\langle s, e(t) \rangle|^2 \right)^{\frac{1}{2}} ds dt,
\]
where
\[
e(t) = \frac{(\nabla f)(t)}{|(\nabla f)(t)|}, \quad t \in \mathbb{R}^d.
\]
By rotation invariance,
\[
\int_{S^{d-1}} \left| e(t) \right|^2 - |\langle s, e(t) \rangle|^2 \frac{1}{2} ds = \int_{S^{d-1}} (1 - |\langle s, e(t) \rangle|^2) \frac{1}{2} ds
= \int_{S^{d-1}} (1 - |\langle s, e_d \rangle|^2) \frac{1}{2} ds,
\]
which concludes the proof.

\[\Box\]

**Remark 8.5.** On can express the constant \( \kappa_d' \) in terms of gamma functions. Indeed, changing coordinates \( s = ((\sin \theta)s', \cos \theta) \) with \( s' \in S^{d-2} \) and using \( ds = ds' (\sin \theta)^{d-2} d\theta \), one finds
\[
\int_{S^{d-1}} (1 - s_d^2) \frac{1}{2} ds = \text{Vol}(S^{d-2}) \int_0^\pi \sin^{2d-2} \theta d\theta.
\]
Using
\[
\text{Vol}(S^{d-2}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)},
\]
and the beta function identity
\[
\int_0^\pi \sin^{2d-2} \theta d\theta = B\left(\frac{1}{2}, d - \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma(d - \frac{1}{2})}{(d - 1)!},
\]
we obtain
\[
\kappa_d' = (2\pi)^{-1} \left( \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \frac{\Gamma(d - \frac{1}{2})}{d!} \right)^{\frac{1}{2}}.
\]
Similarly,
\[
(8.4) \quad \kappa_d = N^{\frac{d}{2}} \kappa_d' = (2\pi)^{-1} \left( N \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \frac{\Gamma(d - \frac{1}{2})}{d!} \right)^{\frac{1}{2}}.
\]
We are now in position to compute the asymptotics of $df$. We recall that the constant $\kappa_d$ is defined in Theorem 1.2; see also (8.4).

**Proposition 8.6.** Let $d \geq 2$, let $f \in C_c^\infty(\mathbb{R}^d)$ be real-valued. For every $\chi \in \mathbb{C} + C_c(\mathbb{R}^d)$, we have

$$\lim_{t \to \infty} t^\frac{d}{2} \mu(t, df \cdot (1 \otimes M\chi)) = \kappa_d \left( \int_{\mathbb{R}^d} |\chi|^d |\nabla f|^d \, dx \right)^{\frac{1}{d}}$$

and

$$\lim_{t \to \infty} t^\frac{d}{2} \mu(t, (1 \otimes M\bar{\chi}) \cdot df \cdot (1 \otimes M\chi)) = \kappa_d \left( \int_{\mathbb{R}^d} |\chi|^d |\nabla f|^d \, dx \right)^{\frac{1}{d}}.$$

**Proof.** Set

$$S = \left( \sum_{k=1}^{d} |Ak|^2 \right)^{\frac{1}{2}} (1 - \Delta)^{-\frac{1}{2}} M\chi$$

and let $T$ be as in Lemma 8.4. By Theorem 4.1, we have

$$S - T (1 - \Delta)^{-\frac{1}{2}} = \left( \sum_{k=1}^{d} |Ak|^2 \right)^{\frac{1}{2}} (1 - \Delta)^{-\frac{1}{2}} M\chi \in (L_{d,\infty},0).$$

By Lemma 8.3 and Lemma 3.1, one has

$$\lim_{t \to \infty} t^\frac{d}{2} \mu(t, S) = \kappa'_d \left( \int_{\mathbb{R}^d} |\chi|^d |\nabla f|^d \, dx \right)^{\frac{1}{d}}.$$

In other words,

$$(8.5) \quad \lim_{t \to \infty} t^\frac{d}{2} \mu(t, |S|^2) = (\kappa'_d)^2 \left( \int_{\mathbb{R}^d} |\chi|^d |\nabla f|^d \, dx \right)^{\frac{2}{d}}.$$

By Lemma 8.3, we have

$$|df \cdot (1 \otimes M\chi)|^2 - 1 \otimes |S|^2 \in (L_{d,\infty},0).$$

It follows now from (8.5) and Lemma 3.1 that one has

$$\lim_{t \to \infty} t^\frac{d}{2} \mu(t, |df \cdot (1 \otimes M\chi)|^2) = N^d (\kappa'_d)^2 \left( \int_{\mathbb{R}^d} |\chi|^d |\nabla f|^d \, dx \right)^{\frac{2}{d}}.$$

Since $N^d \kappa'_d = \kappa_d$, we obtain the first assertion.

To prove the second one, we use again the fact [30, Theorem 6.3.1] that

$$B = df - A(1 + D^2)^{-\frac{1}{2}} + B \in (L_{d,\infty},0).$$

Thus,

$$\begin{align*}
(1 \otimes M\bar{\chi}) \cdot df \cdot (1 \otimes M\chi) & - df \cdot (1 \otimes M|\chi|^2)
= [(1 \otimes M\bar{\chi}), df] \cdot (1 \otimes M\chi)
= [(1 \otimes M\bar{\chi}), B] \cdot (1 \otimes M\chi) + \sum_{k=1}^{d} (\gamma_k \otimes [M\bar{\chi}, Ak]) \cdot (1 \otimes M\chi)
= [(1 \otimes M\bar{\chi}), B] \cdot (1 \otimes M\chi) + \sum_{k,j=1}^{d} (\gamma_k \otimes [M\bar{\chi}, D_k D_j \Delta]) \cdot (1 \otimes M_{\partial_j f \chi}).
\end{align*}$$
The first summand on the right hand side belongs to \((L_{d,\infty})_0\) since so does \(B\). The second summand on the right hand side belongs to \((L_{d,\infty})_0\) by Theorem \ref{thm:main-thm}. Hence,

\[(1 \otimes M \chi) \cdot df \cdot (1 \otimes M \chi) - df \cdot (1 \otimes M |\chi|^2) \in (L_{d,\infty})_0.\]

The second assertion now follows from the first one. \qed

**Corollary 8.7.** Let \(d \geq 2\), let \(f \in C_c^\infty(\mathbb{R}^d)\) and \(\chi \in C_c(\mathbb{R}^d)\). Then we have

\[
\| (1 \otimes M \chi) \cdot df \cdot (1 \otimes M \chi) \|_{d,\infty} \geq c_d \left( \int_{\mathbb{R}^d} |\chi|^{2d} |\nabla f|^d \, dx \right)^{\frac{1}{d}}.
\]

**Proof.** If \(f\) is real-valued, the claimed bound is an immediate consequence of Proposition \ref{prop:main-prop}. For general, complex-valued \(f\) we have

\[
\Re((1 \otimes M \chi) \cdot df \cdot (1 \otimes M \chi)) = (1 \otimes M \chi) \cdot d\Re f \cdot (1 \otimes M \chi),
\]

\[
\Im((1 \otimes M \chi) \cdot df \cdot (1 \otimes M \chi)) = (1 \otimes M \chi) \cdot d\Im f \cdot (1 \otimes M \chi).
\]

We now deduce from Lemma \ref{lem:complex-lemma} that

\[
\| (1 \otimes M \chi) \cdot df \cdot (1 \otimes M \chi) \|_{d,\infty} + \| (1 \otimes M \bar{\chi}) \cdot d\bar{\Im} f \cdot (1 \otimes M \chi) \|_{d,\infty} \leq C_d \| (1 \otimes M \chi) \cdot df \cdot (1 \otimes M \chi) \|_{d,\infty}.
\]

Meanwhile, by the triangle inequality in \(L_d\), we have

\[
\left( \int_{\mathbb{R}^d} |\chi|^{2d} |\nabla f|^d \, dx \right)^{\frac{1}{d}} \leq \left( \int_{\mathbb{R}^d} |\chi|^{2d} |\nabla \Re f|^d \, dx \right)^{\frac{1}{d}} + \left( \int_{\mathbb{R}^d} |\chi|^{2d} |\nabla \Im f|^d \, dx \right)^{\frac{1}{d}}.
\]

Thus, the bound in the complex case follows from that in the real case. \qed

We are now in a position to prove the second part of Theorem \ref{thm:main-thm}.

**Proposition 8.8.** Let \(d \geq 2\) and \(f \in BMO(\mathbb{R}^d)\) with \(df \in L_{d,\infty}\). Then \(f \in \dot{W}_d^1(\mathbb{R}^d)\) with

\[
\| df \|_{d,\infty} \geq c_d \| \nabla f \|_d.
\]

**Proof.** Let \(0 \leq \Phi \in C_c^\infty(\mathbb{R}^d)\) with \(\| \Phi \|_1 = 1\) and set \(\Phi_t(t) := t^{-d}\Phi(t^{-1}t)\). We have \(\Phi_t \ast f \in C_c^\infty(\mathbb{R}^d)\). Let \(\chi \in C_c^\infty(\mathbb{R}^d)\) with \(|\chi| \leq 1\). Given \(\chi\) we choose a \(\tilde{\chi} \in C_c^\infty(\mathbb{R}^d)\) with \(\tilde{\chi} = \chi\). We have \(\tilde{\chi} \ast (\Phi_t \ast f) \in C_c^\infty(\mathbb{R}^d)\) and

\[
(1 \otimes M \tilde{\chi}) \cdot d(\Phi_t \ast f) \cdot (1 \otimes M \chi) = (1 \otimes M \tilde{\chi}) \cdot d(\tilde{\chi} \cdot (\Phi_t \ast f)) \cdot (1 \otimes M \chi).
\]

By a majorization argument (see \cite[Lemma 18]{lem:complex-lemma}), we have \(d(\Phi_t \ast f) \in L_{d,\infty}\) and

\[
\| d(\Phi_t \ast f) \|_{d,\infty} \leq C_d \| df \|_{d,\infty}.
\]

Using \ref{eq:main-prop}, we obtain

\[
\| (1 \otimes M \tilde{\chi}) \cdot d(\tilde{\chi} \cdot (\Phi_t \ast f)) \cdot (1 \otimes M \chi) \| \leq C_d \| df \|_{d,\infty}.
\]

Applying Corollary \ref{cor:main-cor} to \(\tilde{\chi} \cdot (\Phi_t \ast f) \in C_c^\infty(\mathbb{R}^d)\), we obtain

\[
\| (1 \otimes M \tilde{\chi}) \cdot d(\tilde{\chi} \cdot (\Phi_t \ast f)) \cdot (1 \otimes M \chi) \|_{d,\infty} \geq c_d \| |\chi|^2 \cdot \nabla (\tilde{\chi} \cdot (\Phi_t \ast f)) \|_d.
\]

Combining \ref{eq:main-prop} and \ref{eq:main-cor} and taking into account that \(\nabla \tilde{\chi} = 0\) near the support of \(\chi\), we obtain

\[
C_d \| df \|_{d,\infty} \geq c_d \| |\chi|^2 \cdot \nabla (\Phi_t \ast f) \|_d.
\]

Taking the supremum over all \(\chi \in C_c^\infty(\mathbb{R}^d)\) with \(|\chi| \leq 1\), we deduce that \(\nabla (\Phi_t \ast f) \in L_d(\mathbb{R}^d)\) with

\[
C_d \| df \|_{d,\infty} \geq c_d \| \nabla (\Phi_t \ast f) \|_d.
\]
Since $\Phi_t \ast f \to f$ in $L_{1,\text{loc}}(\mathbb{R}^d)$, this implies by a standard argument that $f$ is weakly differentiable with $\nabla f \in L_d(\mathbb{R}^d)$ and

$$C_d \|df\|_{d,\infty} \geq c_d \|\nabla f\|_d,$$

as claimed. □

It remains to prove the validity of the asymptotics under the minimal regularity assumption.

**Proof of Theorem 1.2** By [28, Theorem 11.43], there is a sequence $(f_n)_{n \geq 0} \subset C_c^\infty(\mathbb{R}^d)$ with $\nabla f_n \to \nabla f$ in $L_d$. By Proposition 8.1, we have $df_n \to df$ in $L_d,\infty$ as $n \to \infty$. By Proposition 8.6 (with $\chi = 1$), we have

$$\lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, df_n) = \kappa_d \|\nabla f_n\|_d, \quad n \geq 0.$$

The assertion now follows from Lemma 3.2. □

**Proof of Corollary 1.3** Let $f \in \text{BMO}(\mathbb{R}^d)$ be such that

$$\lim_{t \to \infty} t^{\frac{d}{2}} \mu(t, df) = 0.$$

In particular, we have $df \in L_{d,\infty}$. By Theorem 1.1, we have $f \in W^{1,1}_d(\mathbb{R}^d)$.

If $f$ is real-valued, we deduce then from Theorem 1.2 that $\|\nabla f\|_d = 0$, that is, $f$ is constant. For complex-valued $f$, we apply Lemma 2.1 and deduce that

$$\limsup_{t \to \infty} t^{\frac{d}{2}} \mu(t, d\Re f), \quad \limsup_{t \to \infty} t^{\frac{d}{2}} \mu(t, d\Im f) \leq 2 \limsup_{t \to \infty} t^{\frac{d}{2}} \mu(t, df) = 0,$$

so the assertion follows from that in the real-valued case. □

**References**

[1] Adams R. *Sobolev spaces*. Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[2] Arazy J., Fisher S. D., Peetre J. *Hankel operators on weighted Bergman spaces*. Amer. J. Math. 110 (1988), no. 6, 989–1053.

[3] Atiyah M., Singer I. *The index of elliptic operators. I*. Ann. of Math. (2) 87 (1968), 484–530.

[4] Birman M. Sh., Karadzhov G. E., Solomyak M. Z. *Boundedness conditions and spectrum estimates for the operators $b(X)a(D)$ and their analogs*. In: Estimates and asymptotics for discrete spectra of integral and differential equations (Leningrad, 1989–90), 85–106, Adv. Soviet Math., 7, Amer. Math. Soc., Providence, RI, 1991.

[5] Birman M. Sh., Solomjak M. Z. *Asymptotics of the spectrum of weakly polar integral operators*. Izv. Akad. Nauk SSSR Ser. Mat. 34 (1970), 1142–1158.

[6] Birman M. Sh., Solomjak M. Z. *Estimates for the singular numbers of integral operators*. Uspehi Mat. Nauk 32 (1977), no. 1 (193), 17–84, 271.

[7] Birman M. Sh., Solomjak M. Z. *Quantitative analysis in Sobolev imbedding theorems and applications to spectral theory*. Amer. Math. Soc. Transl., Series 2, 114. American Mathematical Society, Providence, R.I., 1980.

[8] Birman M. Sh., Solomjak M. Z. *Spectral theory of selfadjoint operators in Hilbert space*. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.

[9] Caspers M., Potapov D., Sukochev F., Zanin D. *Weak type estimates for the absolute value mapping*. J. Operator Theory 73 (2015), no. 2, 361–384.

[10] Caspers M., Potapov D., Sukochev F., Zanin D. *Weak type commutator and Lipschitz estimates: resolution of the Nazarov-Peller conjecture*. Amer. J. Math. 141 (2019), no. 3, 593–610.

[11] Caspers M., Sukochev F., Zanin D. *Weak type operator Lipschitz and commutator estimates for commuting tuples*. Ann. Inst. Fourier (Grenoble) 68 (2018), no. 4, 1643–1669.
[12] Coifman R. R., Rochberg R. Representation theorems for holomorphic and harmonic functions in $L^p$. In: Representation theorems for Hardy spaces, pp. 11–66, Astérisque, 77, Soc. Math. France, Paris, 1980.
[13] Coifman R. R., Rochberg R., Weiss G. Factorization theorems for Hardy spaces in several variables. Ann. of Math. (2) 103 (1976), no. 3, 611–635.
[14] Connes A. Noncommutative differential geometry. Inst. Hautes Études Sci. Publ. Math. 62, 257–360 (1985).
[15] Connes A. The action functional in noncommutative geometry. Comm. Math. Phys. 117 (1988), no. 4, 673–683.
[16] Connes A. Noncommutative geometry. Academic Press, San Diego (1994).
[17] Connes A. Cyclic cohomology, noncommutative geometry and quantum group symmetries. Noncommutative geometry, Lecture Notes in Math., 1831, Fond. CIME/CIME Found. Subser., Springer, Berlin, 2004, 1–71.
[18] Cwikel M. Weak type estimates for singular values and the number of bound states of Schrödinger operators. Ann. of Math. (2) 106 (1977), no. 1, 93–100.
[19] Dodds P., Dodds T., Sukochev F., Zanin D. Arithmetic-geometric mean and related sub-majorisation and norm inequalities for $\tau$-measurable operators: Part II. Integral Equations Operator Theory 92 (2020), no. 4, Paper No. 32, 60 pp.
[20] Grafakos L. Classical Fourier analysis. Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2008.
[21] Frank R. L. Weyl’s law under minimal assumptions. Preprint (2022), arXiv:2202.00323.
[22] Frank R. L. A characterization of $\dot{W}^{1,p}(\mathbb{R}^d)$. Preprint (2022), arXiv:2203.01001.
[23] Frank R. L., Laptev A., Weidl T. Schrödinger Operators: Eigenvalues and Lieb–Thirring Inequalities. Cambridge University Press, Cambridge, to appear.
[24] Janson S. Mean oscillation and commutators of singular integral operators. Ark. Mat. 16 (1978), no. 2, 263–270.
[25] Janson S., Wolff T. H. Schatten classes and commutators of singular integral operators. Ark. Mat. 20 (1982), no. 2, 301–310.
[26] Janson S., Peetre J. Paracommutators—boundedness and Schatten-von Neumann properties. Trans. Amer. Math. Soc. 305 (1988), no. 2, 467–504.
[27] Lord S., McDonald E., Sukochev F., Zanin D. Singular Traces: Theory and Applications, vol. 46, Walter de Gruyter, Berlin (2012).
[28] Lord S., Sukochev F., Zanin D. Singular Traces: Theory and Applications, vol. 46/1, Walter de Gruyter, Berlin (2021).
[29] McDonald E., Sukochev F., Xiong X. Quantum differentiability on quantum tori. Comm. Math. Phys. 371 (2019), no. 3, 1231–1260.
[30] McDonald E., Sukochev F., Xiong X. Quantum differentiability on noncommutative Euclidean spaces. Comm. Math. Phys. 379 (2020), no. 2, 491–542.
ASYMPTOTICS OF SINGULAR VALUES FOR QUANTUM DERIVATIVES

[40] Rochberg R., Semmes S. End point results for estimates of singular values of singular integral operators. In: Contributions to operator theory and its applications (Mesa, AZ, 1987), 217–231, Oper. Theory Adv. Appl. 35, Birkhäuser, Basel, 1988.

[41] Rochberg R., Semmes S. Nearly weakly orthonormal sequences, singular value estimates, and Calderon–Zygmund operators. J. Funct. Anal. 86 (1989), no. 2, 237–306.

[42] Rozenbljum G. V. On the distribution of eigenvalues of the first boundary value problem in unbounded regions. Dokl. Akad. Nauk SSSR 200 (1971), no. 5, 1034–1036.

[43] Rozenbljum G. V. The eigenvalues of the first boundary value problem in unbounded domains. Mat. Sb. (N.S.) 139 (1972), no. 2, 234–247. English translation: Math. USSR Sb. 18 (1972), 235–248.

[44] Rozenbljum G. V. Distribution of the discrete spectrum of singular differential operators. Dokl. Akad. Nauk SSSR 202 (1972), 1012–1015. English translation: Soviet Math. Dokl. 13 (1972), 245–249.

[45] Rozenbljum G. V. Distribution of the discrete spectrum of singular differential operators. Izv. Vysš. Učebn. Zaved. Matematika (1976), no. 1 (164), 75–86. English translation: Soviet Math. (Iz. VUZ) 20 (1976), no. 1, 63–71.

[46] Sarason D. Functions of vanishing mean oscillation. Trans. Amer. Math. Soc. 207 (1975), 391–405.

[47] Simon B. Trace ideals and their applications. Second edition. Mathematical Surveys and Monographs, 120. American Mathematical Society, Providence, RI, 2005.

[48] Solomyak M. Spectral problems related to the critical exponent in the Sobolev embedding theorem. Proc. London Math. Soc. (3) 71 (1995), no. 1, 53–75.

[49] Stein E. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[50] Sukochev F., Zanin D. A C∗-algebraic approach to the principal symbol. I. J. Operator Theory 80 (2018), no. 2, 481–522.

[51] Sukochev F., Zanin D. Optimal constants in non-commutative Hölder inequality for quasi-norms. Proc. Amer. Math. Soc. 149 (2021), no. 9, 3813–3817.

[52] Taylor M. Partial differential equations I. Basic theory. Second edition. Applied Mathematical Sciences, 115. Springer, New York, 2011.

[53] Uchiyama A. On the compactness of operators of Hankel type. Tohoku Math. J. (2) 30 (1978), no. 1, 163–171.

(Rupert L. Frank) Mathematisches Institut, Ludwig-Maximilians Universität München, Theresienstr. 39, 80333 München, Germany, and Munich Center for Quantum Science and Technology, Schellingstr. 4, 80799 München, Germany, and Mathematics 253-37, Caltech, Pasadena, CA 91125, USA

Email address: r.frank@lmu.de

(Fedor Sukochev) School of Mathematics and Statistics, University of New South Wales, Kensington, 2052, NSW, Australia.

Email address: f.sukochev@unsw.edu.au

(Dmitriy Zanin) School of Mathematics and Statistics, University of New South Wales, Kensington, 2052, NSW, Australia.

Email address: d.zanin@unsw.edu.au