Disk counting on toric varieties via tropical curves

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DISK COUNTING ON TORIC VARIETIES VIA TROPICAL CURVES

By Takeo Nishinou

Abstract. In this paper, we define two numbers. One is defined by counting tropical curves with a stop, and the other is the number of holomorphic disks in toric varieties with Lagrangian boundary condition. Both of these curves should satisfy some incidence conditions. We show that these numbers coincide. These numbers can be considered as Gromov-Witten type invariants for holomorphic disks, and they have similarities as well as differences to the counting numbers of closed holomorphic curves. We study several aspects of them.

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**Notation.** We always work over \(\mathbb{Z}, \mathbb{Q}, \mathbb{R}\) or \(\mathbb{C}\). We use the same notation as in [18]. In particular, the letter \(N\) denotes a free abelian group of rank greater than or equal to 2, \(N_{\mathbb{Q}}\) is the vector space \(N \otimes_{\mathbb{Z}} \mathbb{Q}\) and \(N_{\mathbb{R}}\) is the vector space \(N \otimes_{\mathbb{Z}} \mathbb{R}\). The letter \(M\) denotes the dual abelian group \(\text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})\) of \(N\). We write by \(D\) the closed unit disk in \(\mathbb{C}\). A toric variety is always regarded as a complex variety, sometimes with a symplectic structure induced from a dual polytope (this symplectic structure can be singular along lower dimensional toric strata, but this does not affect our argument). Let \(G(N) = N \otimes_{\mathbb{Z}} \mathbb{C}^{\ast}\) be the complex torus which acts on the toric varieties. Then \(G_{\mathbb{R}}(N)\) denotes the maximal compact torus of \(G(N)\). A maximal dimensional orbit by the action of \(G_{\mathbb{R}}(N)\) will be identified with a Lagrangian torus fiber of the moment map. If \(\Xi \subset N_{\mathbb{Q}}\) is a subset, then \(L(\Xi) \subset N_{\mathbb{Q}}\) denotes the linear subspace spanned by the differences \(v - w\), where \(v, w \in \Xi\). While \(C(\Xi) \subset N_{\mathbb{Q}} \times \mathbb{Q}\) is the closure of the convex hull of \(Q_{\geq 0} \cdot (\Xi \times \{1\})\). In particular, if \(A\) is an affine subspace, then \(L(A) \subset N_{\mathbb{Q}}\) is the associated linear subspace and \(LC(A) := L(C(A)) \subset N_{\mathbb{Q}} \times \mathbb{Q}\) is the linear closure of \(A \times \{1\}\). The letter \(A\) denotes the set of affine constraints for tropical curves (with or without stop).

1. **Introduction.** In [18], we considered two numbers. One is the counting number of genus zero tropical curves in \(\mathbb{R}^n\), and the other is the counting number of rational curves in an \(n\)-dimensional toric variety. Both curves should satisfy appropriate incidence conditions. The result of [18] is that these numbers coincide and do not depend on the place of incidence conditions.

In Floer homology theory and also in more broader contexts of mirror symmetry, it is important to consider not only closed curves, but also curves with boundary. In [5], certain set of disks with interior marked points plays an important role, and such curves are the object of the study in this paper. Namely, in this paper we extend the arguments of [18] to the case when the curve has a boundary component. More precisely, we count the following objects:

1. **(Holomorphic side)** Stable maps from genus zero prestable bordered Riemann surfaces with one boundary component (see Definition 4.6) to a toric variety, where the boundary is mapped to a fixed Lagrangian torus fiber (or a family of torus fibers) of the moment map.

2. **(Tropical side)** Genus zero tropical curves with one “stop” (see Figure 1). These curves are counted with appropriate weights.

These curves should satisfy appropriate incidence conditions (see Section 5.5). We show that these two numbers coincide (Theorem 9.3). Moreover, they do not change when the incidence conditions are slightly perturbed.

As a result, we obtain Gromov-Witten type invariants for holomorphic disks. These numbers change when the incidence conditions are largely moved, which is the common phenomenon observed when one considers the open curve counting. This is due to the bubbling of disks which corresponds to the boundary of the moduli space of stable maps from curves with boundary (see Section 9.2.1).
In particular, the invariants are well-defined only in the neighborhoods of some degenerations, which correspond to the maximal degeneration limit in the language of mirror symmetry. Due to the bubbling phenomenon mentioned above, when one considers large changes of the moduli (which mean the large moves of the incidence conditions in our case), only the homotopy types of the algebraic structures constructed from these counting make sense, as extensively studied in [5]. In general, calculating these structures is very difficult. One method to perform the calculation is using localization [10, 13, 19]. Our method is based on degeneration. This covers the cases where the localization method is not available yet. Some numerical properties of the numbers of disks are deduced easily via tropical geometry, while it seems quite difficult to deduce these properties purely from complex geometry.

The results and methods in this paper have several applications and extensions. For example, combining with the toric degenerations of (not necessarily toric) Fano varieties [16, 17], we can count holomorphic disks in such varieties. Combining with the methods of [15], the results in this paper can be extended to the cases of curves with arbitrary number of boundary components and genus.

The content of this paper is as follows. In Sections 2 and 3, we give a preparation for tropical curves and introduce tropical curves with stops. In Section 4, we give a preparation for complex curves (with nodes and boundaries). In Section 5, we review toric degenerations and stable maps into them. This is a preparation for Sections 6 and 7. In Sections 6 and 7, we discuss how one can construct stable maps from a disk starting from a tropical curve with a stop. In Section 8, we study the family of stable maps from a disk with Lagrangian boundary condition. In particular, we give the converse of the result of Sections 6 and 7. Namely, we prove that any stable map from a disk into the toric variety satisfying the required conditions is contained in one of the families constructed in Section 7. This establishes the relation between tropical curves and stable maps, and also the invariance of the counting number under perturbation. In Section 9, we prove the main theorem, that
is, the well-definedness of the counting invariants, and provide several examples. In Section 9.2, we consider examples mainly in two dimensional cases. We give examples which exhibit the dependence of the invariant on the large change of the places of the incidence conditions, the relation to the counting number of closed curves, and also give examples where one can calculate the invariant explicitly.

Acknowledgments. This paper is a continuation of [18], and the author benefited from discussions with Bernd Siebert. Most importantly, the idea of using logarithmic deformation theory in the context of tropical geometry is due to him. He also gave many useful remarks on this paper. The author would like to express his deep gratitude to him, and to thank the referee very much for reading the manuscript very carefully, and making many valuable comments (especially those which were essential to correcting Section 8).

2. Tropical curves with stops. Here we give a definition of tropical curves and introduce tropical curves with stops, which is the notion corresponding to holomorphic curves with boundary components.

Let $\Gamma$ be a weighted, connected, finite graph. Its sets of vertices and edges are denoted by $\Gamma^{[0]}$ and $\Gamma^{[1]}$, respectively. Let $w_\Gamma : \Gamma^{[1]} \to \mathbb{N} \setminus \{0\}$ be the weights of the edges. An edge $E \in \Gamma^{[1]}$ has adjacent vertices

$$\partial E = \{V_1, V_2\}.$$  

Let $\Gamma^{[0]}_\infty \subset \Gamma^{[0]}$ be the set of one-valent vertices. Then define the associated non-compact graph $\Gamma$ by

$$\Gamma = \Gamma \setminus \Gamma^{[0]}_\infty.$$  

Non-compact edges of $\Gamma$ are called unbounded edges. Let $\Gamma^{[1]}_\infty$ be the set of unbounded edges. Let $\Gamma^{[0]}$ and $\Gamma^{[1]}$ be the sets of vertices and edges of $\Gamma$, and let $w_\Gamma$ be the weight function, which are induced from those of $\Gamma$ in an obvious way.

Definition 2.1. A parametrized tropical curve in $N_{\mathbb{R}}$ is a proper map

$$h : \Gamma \longrightarrow N_{\mathbb{R}}$$

satisfying the following conditions.

(i) For every edge $E \subset \Gamma$, the restriction $h|_E$ is an embedding with the property that the image $h(E)$ is contained in an affine line with a rational slope.
(ii) For every vertex \( V \in \Gamma^{[0]} \), the following balancing condition holds. Let \( E_1, \ldots, E_m \in \Gamma^{[1]} \) be the edges adjacent to \( V \) and let \( u_i \in N \) be the primitive integral vector emanating from \( h(V) \) in the direction of \( h(E_i) \). Then

\[
\sum_{j=1}^{m} w(E_j) u_j = 0.
\]

An isomorphism of parametrized tropical curves \( h : \Gamma \rightarrow N_R \) and \( h' : \Gamma' \rightarrow N_R \) is a homeomorphism \( \Phi : \Gamma \rightarrow \Gamma' \) respecting the weights such that \( h = h' \circ \Phi \) holds. A tropical curve is an isomorphism class of parametrized tropical curves. (We often do not distinguish parametrized tropical curves and tropical curves.)

**Definition 2.2.** We call a tropical curve \((\Gamma, h)\) immersive if

- \( h \) is an immersion and,
- if \( V \in \Gamma^{[0]} \), then \( h^{-1}(h(V)) = \{V\} \).

Unless noticed, tropical curves in this paper are always immersive, since for genus zero curves (or more generally for non-superabundant curves), it suffices to consider only immersive tropical curves for enumerative problems. See [15].

The set of flags of \( \Gamma \) is the set

\[
F \Gamma = \{(V, E) \mid V \in \partial E \}.
\]

By (i) of Definition 2.1, we have a map

\[
u : F \Gamma \longrightarrow N
\]

sending a flag \((V, E)\) to the primitive integral vector \( u_{(V,E)} \in N \) emanating from \( V \) in the direction of \( h(E) \).

Let \( \Gamma^{[0]}_{\infty} = \{q_1, \ldots, q_l, a_1, \ldots, a_m\} \) be the set of one-valent vertices. Let \( \Gamma_s = \Gamma \setminus \{q_1, \ldots, q_l\} \). Let \( E_1, \ldots, E_m \) be the edges emanating from \( a_1, \ldots, a_m \).

**Definition 2.3.** A tropical curve with stops is the isomorphism class of a proper map \( h : \Gamma_s \rightarrow N_R \) which satisfies the conditions of Definition 2.1, with the isomorphism between two maps defined in an obvious way. We call each \( E_i \) a stopping edge and \( a_i \) a stop. Let \( \Gamma_{s,\text{stop}}^{[1]} \) be the set of stopping edges and \( \Gamma_{s,\infty}^{[1]} \) be the set of unbounded edges. Then obviously \( \Gamma_{s,\infty}^{[1]} = \Gamma_{s,\infty}^{[1]} \setminus \Gamma_{s,\text{stop}}^{[1]} \) holds.

We write by \( \Gamma_s^{[0]} \) and \( \Gamma_s^{[1]} \) the sets of vertices and edges of \( \Gamma_s \), respectively. Let \( \Gamma_{s,\text{stop}}^{[0]} \) be the set of stops. Then \( \Gamma_s^{[0]} = \Gamma_s^{[0]} \sqcup \Gamma_{s,\text{stop}}^{[0]} \) holds. There is a canonical one-to-one correspondence between \( \Gamma_{s}^{[1]} \) and \( \Gamma^{[1]} \).

An \( l \)-marked tropical curve (resp. tropical curve with stops) is a tropical curve \( h : \Gamma \rightarrow N_R \) (resp. \( \Gamma_s \rightarrow N_R \)) together with a choice of \( l \) edges \( E = (E_1, \ldots, E_l) \in (\Gamma^{[1]})^l \) (resp. \( (\Gamma_{s}^{[1]})^l \)).
Remark 2.4. We do not prohibit that $E_i$ equals $E_j$ for some $i \neq j$. But this does not occur in general situations, see Definition 3.5.

The type of an $l$-marked tropical curve $(\Gamma, E, h)$ (resp. an $l$-marked tropical curve with stops $(\Gamma_s, E, h)$) is the marked graph $(\Gamma, E)$ (resp. $(\Gamma_s, E)$) together with the map $u : F\Gamma \to N$ (resp. $u : F\Gamma_s \to N$).

The degree of a type $(\Gamma, E, u)$ (resp. $(\Gamma_s, E, u)$) of a tropical curve (resp. tropical curve with stops) is the function

$$\Delta : N \setminus \{0\} \to \mathbb{N}$$

with finite support defined by

$$\Delta(v) := \# \{(V, E) \in F\Gamma \mid E \in \Gamma\{l\}, w(E)u(V,E) = v\}$$

(resp. $\Delta(v) := \# \{(V, E) \in F\Gamma_s \mid E \in \Gamma_s\{l\}, w(E)u(V,E) = v\}$).

Definition 2.5. The marked total weight of an $l$-marked tropical curve with stops is

$$w(\Gamma_s, E) = \prod_{E \in \Gamma_s\{l\}\setminus \Gamma_s\{\infty\}} \prod_{i=1}^{l} w(E_i).$$

2.1. Examples. In this section, we give basic examples of tropical curves with a stop. Relations to complex curves (in particular, holomorphic disks in toric varieties) are indicated.

Example 1 (half line). The most basic example of a tropical curve with a stop is the half line with a rational slope. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$. Then if the slope of the half line is $e_i$ or $-(e_1 + \cdots + e_n)$, it corresponds to a holomorphic disk in $\mathbb{P}^n$ with the following properties:

- It intersects with a unique toric divisor at one interior point.
- The boundary is mapped to a Lagrangian torus, which is a fiber of the moment map.

In fact, half lines are the images of such holomorphic disks under the moment map. Precisely speaking, we identify $\mathbb{R}^n$ with the interior of the moment polytope via the canonical diffeomorphism between them, see [7]. Sometimes it is called the logarithmic (moment) map. Any holomorphic disk in any toric variety which intersects the toric divisor just once, and whose boundary is mapped to a Lagrangian fiber of the moment map, has this property. This fact follows, for example, from the explicit description of holomorphic disks with Lagrangian boundary condition given in [2], Theorem 5.3.

Example 2 (disks in $\mathbb{P}^2$). Using the explicit presentation in [2] as above, we can draw the image of the logarithmic moment map (i.e., the amoeba) of disks in toric
varieties. For example, the amoeba of a disk with Maslov index four has the shape drawn in Figure 2. The corresponding tropical curve with a stop is also drawn on the right.

One observes that there is a clear resemblance between the amoeba and the tropical curve of a line in $\mathbb{P}^2$.

3. Constraints for tropical curves. Let $\Gamma_s$ be a weighted graph with one stop. Let $a$ be the stop of $\Gamma_s$. Let $E = (E_1, \ldots, E_l)$ be the marking of the edges.

Definition 3.1. For $d = (d_0, \ldots, d_l) \in \mathbb{N}^{l+1}$, a set of affine constraints of codimension $d$ is an $(l+1)$-tuple $A = (A_0, \ldots, A_l)$ of affine subspaces $A_i \subset \mathbb{Q}^n$ with $\dim A_i = n - d_i - 1$. An $l$-marked tropical curve with a stop matches the constraints $A$ if

$$h(E_i) \cap A_i \neq \emptyset, \quad i = 1, \ldots, l,$$

$$h(a) \in A_0.$$ (5)

Remark 3.2. Later, similar conditions are imposed to holomorphic curves. Namely, a holomorphic curve is required to intersect with some set of varieties $Z = (Z_0, \ldots, Z_l)$. We call $A$ constraints and $Z$ incidence conditions, to clarify which curves we are talking, although these might be occasionally abused.

The proofs and results in Section 2 of [18] hold in the case of tropical curves with one stop, because the slope of the stopping edge is uniquely determined by the remaining part of the curve due to the balancing condition. In particular, if we extend the stopping edge to infinity, we obtain a usual tropical curve. Then $A_0$ may be considered as a usual affine constraint for this tropical curve. This gives a $d_0$-dimensional condition. Let $\mathcal{T}_{(\Gamma_s, E, u)}$ be the set of tropical curves with one stop of type $(\Gamma_s, E, u)$.

Proposition 3.3. [18, Proposition 2.1] For any $\Delta \in \text{Map}(\mathbb{N} \setminus \{0\}, \mathbb{N})$ with finite support and any $g \in \mathbb{N}$, there are only finitely many types of tropical curves
with one stop of degree $\Delta$ and genus $g$, that is, the set
\[ \{ (\Gamma_s, u) \mid \Delta(\Gamma_s, u) = \Delta, \ g(\Gamma_s, u) = g, \ \mathcal{I}_{(\Gamma_s, E, u)} \neq \emptyset \} \]
is finite. Here $\Gamma_s$ is a weighted graph with one stop, and $u : F\Gamma_s \to N$.

Remark 3.4. If there are two or more stops, this proposition need not be true. For example, one constructs tropical curves with two stops and one unbounded edge with the slopes of the edges $(k, 1), (-k, 1), (0, 2), k \in \mathbb{Z}_{>0}$. See Figure 3. So there are obviously infinite number of types.

Now we turn to the genus zero case. As we have mentioned, the degree $\Delta$ of a tropical curve with a stop uniquely determines the direction $v$ (the primitive integral vector from the vertex which is not the stop, multiplied by the weight) of the stopping edge. Let
\[ \Delta' = \Delta + \{v\} \in \text{Map}(\mathbb{Z}^2 \setminus \{0\}, \mathbb{N}), \]
here $\{v\} \in \text{Map}(\mathbb{Z}^2 \setminus \{0\}, \mathbb{N})$ is the map which takes the value 1 on $v$ and 0 otherwise.

Given an $l$-marked tropical curve $(\Gamma_s, E, h)$ of degree $\Delta$, it uniquely determines an $(l + 1)$-marked tropical curve $(\Gamma, E', h')$, where $(\Gamma, h')$ is the obvious prolongation of $(\Gamma_s, h)$ at the stop, and $E' = E \cup \{E_0\}$, where $E_0 \subset \Gamma$ is the prolongation of the stopping edge. Clearly, the degree of $(\Gamma, h')$ is $\Delta'$. Based on this construction, we give the following definition.

Definition 3.5. Let $\Delta \in \text{Map}(\mathbb{N} \setminus \{0\}, \mathbb{N})$ be a degree and
\[ e := |\Delta| = \#\Gamma_{s, \infty}^{[1]} \]
be the number of unbounded edges. A set of constraints \( A = (A_0, \ldots, A_l) \) of codimension \( d = (d_0, \ldots, d_l) \) is \textit{general} for \( \Delta \) if

- \( \sum_{i=0}^{l} d_i = e + n - 2 \) and,
- \( A \) is general for \((l + 1)\)-marked rational tropical curves of degree \( \Delta' \). That is, if \( (\Gamma, E', h') \) is an \((l + 1)\)-marked tropical curve of genus zero, degree \( \Delta' \) and matching \( A \), the following conditions hold:
  (i) \( \Gamma \) is trivalent at \( \Gamma_0 \).
  (ii) \( h'(\Gamma_0) \cap \bigcup_{i=0}^{l} A_i = \emptyset \).
  (iii) \( h' \) is injective for \( n > 2 \). For \( n = 2 \), it is at least injective on the vertices, and for any \( p \in h'(\Gamma_s) \), \( (h')^{-1}(p) \) is a finite set.
- This is the definition given in [18], Definition 2.3, but here we add one more condition:
  (iv) If \( i \neq j \), then \( h'(E_i) \cap A_j = \emptyset \).

Otherwise it is called \textit{non-general}.

In fact, according to the construction just before the definition, it suffices to consider only the marking \( E' \) which satisfies the condition that \( E' \) contains an unbounded edge of the direction \( v \) (using the notation above), but it makes little difference to later arguments.

The proof of the following result for tropical curves with a stop is the same as the case without a stop proved in [18]. Furthermore, even when the additional condition (iv) in the definition above is imposed, the proof requires little change.

**Proposition 3.6.** [18, Proposition 2.4] Let \( \Delta \in \text{Map}(\mathbb{N} \setminus \{0\}, \mathbb{N}) \) and let \( A = \{A_0, A_1, \ldots, A_l\} \) be a set of affine constraints of codimension \( d = (d_0, \ldots, d_l) \in \mathbb{N}^{l+1} \) with \( \sum_{i=0}^{l} d_i = e + n - 2 \). Denote by \( \mathbb{A} := \prod_{i=0}^{l} \mathbb{N}_Q/L(A_i) \) the space of affine constraints that are parallel to \( A \). Then the subset

\[ \mathcal{Z} := \{ A' \in \mathbb{A} \mid A' \text{ is non-general for } \Delta \} \]

of \( \mathbb{A} \) is nowhere dense.

Moreover, for any \( A' \in \mathbb{A} \setminus \mathcal{Z} \) and any \( l \)-marked type \((\Gamma_s, E, u)\) of genus zero and degree \( \Delta \) there is at most one tropical curve with one stop of type \((\Gamma_s, E, u)\) matching \( A' \).

**Definition 3.7.** For later purpose, we introduce a divalent vertex on every marked edge of \( \Gamma \). If \( E_i \) is a marked edge, then let us write by \( e_i \) this divalent vertex. Then, for an \( l \)-marked tropical curve \((\Gamma, E, h)\) and a general set of constraints \( A \), we impose the condition:

The vertex \( e_i \) is mapped to \( A_i \).

Note that if we write by \( \tilde{\Gamma} \) the abstract graph obtained from \( \Gamma \) by adding divalent vertices as above, and induce weights on the edges of \( \tilde{\Gamma} \) from those of \( \Gamma \) in an
obvious way, then the marked total weight of the original curve (Definition 2.5) equals the usual (i.e., unmarked) weight of $\Gamma$.

4. Bordered Riemann surfaces. We prepare some terminologies for surfaces with boundary. See also [10].

Definition 4.1. Let $A$ and $B$ be nonempty subsets of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) \geq 0\}$. A continuous function $f : A \to B$ is holomorphic on $A$ if it extends to a holomorphic function $\tilde{f} : U \to \mathbb{C}$, where $U$ is an open neighborhood of $A$ in $\mathbb{C}$.

Definition 4.2. A surface is a Hausdorff, connected, topological space $\Sigma$ together with a family $A = \{(U_i, \phi_i) \mid i \in I\}$:
- $\{U_i \mid i \in I\}$ is an open covering of $\Sigma$.
- Each map $\phi_i : U_i \to A_i$ is a homeomorphism onto an open subset $A_i$ of $\mathbb{H}$.

The boundary of $\Sigma$ is the set
$$\partial \Sigma = \{x \in \Sigma \mid \exists i \in I \text{ such that } x \in U_i, \phi_i(x) \in \mathbb{R}\}.$$ By definition, the maps $\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are surjective homeomorphisms, called the transition functions of $A$. The covering $A$ is called a holomorphic atlas if all of its transition functions are holomorphic.

Definition 4.3. A bordered Riemann surface is a compact surface with nonempty boundary equipped with the holomorphic structure induced by a holomorphic atlas on it. A disk is a compact surface which is biholomorphic to the unit disk in the complex plane.

We mainly deal with bordered Riemann surfaces of genus zero having one boundary component and some nodes (see below).

Definition 4.4. A nodal bordered Riemann surface is a compact, Hausdorff, connected, topological space $\Sigma$ with a finite point set $\{s_1, \ldots, s_k\}$ satisfying the following properties. Let $\sigma_1, \ldots, \sigma_l$ be the connected components of $\Sigma \setminus \{s_1, \ldots, s_k\}$ ($l$ must be finite due to the condition at each $s_i$ mentioned below). Then $\sigma_i$ is equipped with a structure of a holomorphic surface in the sense of Definition 4.2. At each $s_i$, there is a neighborhood which is homeomorphic to one of the following.
- (Interior node) A neighborhood of $(0,0) \in \{xy = 0\}$, where $(x,y)$ are the standard coordinates on $\mathbb{C}^2$.
- (Boundary node) A neighborhood of $(0,0) \in \{xy = 0\}/A$, where $A(x,y) = (\overline{x}, \overline{y})$ is the complex conjugation.

Moreover, the holomorphic structures induced on these neighborhoods are compatible with the given holomorphic structures on $\sigma_i$. The points $s_1, \ldots, s_k$ are called nodes.
Definition 4.5. A prestable bordered Riemann surface is a singular bordered Riemann surface (see [11] for singularities of Riemann surfaces) whose singularities are nodes. A pointed prestable bordered Riemann surface is a prestable bordered Riemann surface with a collection of points \( \{p_i\}_{i=1}^n \) on it. No \( p_i \) coincides with a node. A stable bordered Riemann surface is a pointed prestable bordered Riemann surface whose automorphism group is finite.

Definition 4.6. Let \( \Sigma \) be a prestable bordered Riemann surface, let \( (X, J, \omega) \) be a symplectic manifold together with a compatible almost complex structure, and let \( L \subset X \) be a Lagrangian submanifold of \( X \). A prestable map \( f : (\Sigma, \partial \Sigma) \to (X, L) \) is a continuous map which is \( J \)-holomorphic, in the sense of Definition 4.1 with a straightforward modification. A stable map from a pointed prestable Riemann surface \( (\Sigma, \partial \Sigma, \{p_i\}) \) is a prestable map whose automorphism group is finite.

Remark 4.7. We can generalize the boundary condition from a Lagrangian submanifold to a totally real submanifold.

4.1. Log structures on bordered Riemann surfaces. Most of definitions and basic results in log geometry can be formulated in analytic setting. Let \( (X, \partial X) \) be a complex analytic manifold (or more generally a complex space in the sense of [11], Definition 43.2) possibly with boundary. Let \( \mathcal{O}^{\text{an}}_X \) be the sheaf of analytic functions on \( X \).

Definition 4.8. A pre-log structure on \( (X, \partial X) \) is a map of sheaves of monoids

\[
\alpha : \mathcal{M} \longrightarrow \mathcal{O}^{\text{an}}_X
\]  

Definition 4.9. A pre-log structure \( \alpha \) is called a log structure when the restriction

\[
\alpha : \alpha^{-1}\left((\mathcal{O}^{\text{an}}_X)^{\times}\right) \longrightarrow (\mathcal{O}^{\text{an}}_X)^{\times}
\]  

is an isomorphism.

A complex analytic manifold (possibly with boundary) equipped with a log structure is called a log complex analytic manifold and we write it as \( (X, \partial X, \mathcal{M}) \).

Definitions of the log structure associated to a pre-log structure, morphism of log complex analytic manifolds, log smooth morphisms, etc., generalize to this situation in an obvious manner. For more information about log structures, see [8, 9].

5. Pre-log disks in toric degeneration.

5.1. Polyhedral decomposition of \( N_\mathbb{Q} \) and toric degeneration. First we recall the toric degeneration induced by a polyhedral decomposition of \( N_\mathbb{Q} = \mathbb{Q}^n \),
from [18], Section 3. Nothing new is added to [18], but we describe some details of definitions for our readers’ convenience, because of the importance of them in our argument.

A rational polyhedron is the solution set in $\mathbb{N}_Q$ of finitely many linear inequalities

$$\langle m, \cdot \rangle \geq \text{const.},$$

where $m \in M_Q$. Note that a rational polyhedron may not be bounded. A face of a rational polyhedron is a subset where some of the defining inequalities are equalities. A vertex is a zero dimensional face. When $\Xi$ is a rational polyhedron, we write by $F(\Xi)$ the set of faces of $\Xi$. A rational polyhedron is called strongly convex when it has at least one vertex.

**Definition 5.1.** A (finite) polyhedral decomposition of $\mathbb{N}_Q$ is a covering $\mathcal{P} = \{\Xi\}$ of $\mathbb{N}_Q$ by a finite number of strongly convex rational polyhedra satisfying the following properties:

1. If $\Xi \in \mathcal{P}$ and $\Xi' \subset \Xi$ is a face, then $\Xi' \in \mathcal{P}$.
2. If $\Xi, \Xi' \in \mathcal{P}$, then $\Xi \cap \Xi'$ is a common face of $\Xi$ and $\Xi'$.

The asymptotic fan $\Sigma_\mathcal{P}$ of $\mathcal{P}$ is defined to be

$$\Sigma_\mathcal{P} := \{ \lim_{a \to 0} a\Xi \subset \mathbb{N}_Q \mid \Xi \in \mathcal{P} \}.$$

Note that for each $\Xi \in \mathcal{P}$, the scaling limit $\lim_{a \to 0} a\Xi$ exists in Hausdorff sense and it is either a point (when $\Xi$ is bounded) or a cone (otherwise). Moreover, we have the following.

**Lemma 5.2.** [18, Lemma 3.2] $\Sigma_\mathcal{P}$ is a complete fan.

Let us define a cone $C(\Xi)$ in $\mathbb{N}_Q \times \mathbb{Q}$ by

$$C(\Xi) = \overline{\{ a \cdot (n, 1) \mid a \geq 0, n \in \Xi \}}.$$

Here $\overline{\{ \cdots \}}$ means the closure. This is a strongly convex polyhedral cone. We define a fan $\tilde{\Sigma}_\mathcal{P}$ by

$$\tilde{\Sigma}_\mathcal{P} = \{ \sigma \mid \sigma \in F(C(\Xi)), \Xi \in \mathcal{P} \}.$$

Then we have the following:

**Lemma 5.3.** [18, Lemma 3.3] If we identify $\mathbb{N}_Q$ with $\mathbb{N}_Q \times \{0\}$, then

$$\Sigma_\mathcal{P} = \{ \sigma \cap (\mathbb{N}_Q \times \{0\}) \mid \sigma \in \tilde{\Sigma}_\mathcal{P} \}.$$
The second projection $\mathbb{N} \times \mathbb{Q} \to \mathbb{Q}$ defines a map of fans

$$\tilde{\Sigma}_\mathcal{P} \longrightarrow \{0, \mathbb{Q}_{\geq 0}\},$$

here $\{0, \mathbb{Q}_{\geq 0}\}$ is a fan in $\mathbb{Q}$ corresponding to the affine line $\mathbb{A}^1 = \mathbb{C}$. This induces a toric morphism

$$\pi : \tilde{X} \longrightarrow \mathbb{C}.$$  

Here $\tilde{X}$ is the toric variety associated to the fan $\tilde{\Sigma}_\mathcal{P}$. We have the following description of the general fiber.

**Lemma 5.4.** [18, Lemma 3.4] For a closed point $t \in \mathbb{C} \setminus \{0\}$, the fiber $\pi^{-1}(t) \subset \tilde{X}$ with the action of $G(\mathbb{N}) \subset G(\mathbb{N} \times \mathbb{Z})$ is torically isomorphic to $X(\Sigma_\mathcal{P})$. Here $X(\Sigma_\mathcal{P})$ is the toric variety associated to the fan $\Sigma_\mathcal{P}$.

We can also describe the central fiber $X_0$ using $\mathcal{P}$, assuming $\mathcal{P}$ is integral. Namely, at each vertex $v \in \mathcal{P}[0]$, the star of $\mathcal{P}$ at $v$ defines a complete fan:

$$\Sigma_v := \{Q_{\geq 0} \cdot (\Xi - v) \mid \Sigma \in \mathcal{P}, v \in \Xi\}.$$  

More generally, for $\Xi \in \mathcal{P}$, the rays emanating from $\Xi$ through adjacent $\Xi' \in \mathcal{P}$ define a complete fan $\Sigma_{\Xi}$ in $N_{\mathbb{Q}}/L(\Xi)$:

$$\Sigma_{\Xi} := \{Q_{\geq 0} \cdot (\Xi' - \Xi) \subset N_{\mathbb{Q}}/L(\Xi) \mid \Xi' \in \mathcal{P}, \Xi \subset \Xi'\}.$$  

Each of these fans defines the associated toric variety $X_{\Xi} = X(\Sigma_{\Xi})$. Then the central fiber $X_0$ is obtained by gluing $X_v = X(\Sigma_v), v \in \mathcal{P}[0]$ along $X_{\Xi}$, and the gluing rule is read off from the combinatorics of $\mathcal{P}$. See [18], Proposition 3.5 for more details.

As [18], Proposition 3.9, we can fix a polyhedral decomposition of $N_{\mathbb{Q}}$ suited to a tropical curve with a stop.

**Proposition 5.5.** Fix a complete fan $\Sigma$. Let $h : \Gamma_s \to N_{\mathbb{R}}$ be a tropical curve with a stop such that the directions of unbounded edges are contained in $\Sigma^{[1]}$. Then there exists a polyhedral decomposition $\mathcal{P}$ of $N_{\mathbb{Q}}$ with the asymptotic fan $\Sigma$ such that

$$\cup_{b \in \Gamma_s^{[\mu]}} h(b) \subset \cup_{\Xi \in \mathcal{P}^{[\mu]}} \Xi,$$

for $\mu = 0, 1$.

**Proof.** It is easy to see that we can graft a tree to the stopping edge so that the resulting graph becomes a tropical curve such that the directions of unbounded edges are contained in $\Sigma^{[1]}$. Then apply the construction in the proof of [18], Proposition 3.9.  \[\square\]
Remark 5.6. In our situation, we apply this proposition to the tropical curve \((\tilde{\Gamma}_s, h)\), where \(\tilde{\Gamma}_s\) is the abstract graph obtained from \(\Gamma_s\) by adding divalent vertices to marked edges, see Definition 3.7.

5.2. Pre-log disks on the central fiber \(X_0\). We extend the definition of pre-log curves to the case with boundary. First, we specify the ambient space for those curves.

Definition 5.7. A toric variety \(X\) defined by a fan \(\Sigma\) is called to be associated to a tropical curve \((\Gamma, h)\) (or a tropical curve with a stop) if the set of the rays of \(\Sigma\) contains the set of the rays spanned by the vectors in \(N\) which are in the support of the degree map \(\Delta : N \setminus \{0\} \rightarrow \mathbb{N}\) of \((\Gamma, h)\).

By Proposition 5.5, given a tropical curve with a stop \((\Gamma_s, h)\), there is a toric degeneration of a toric variety associated to \((\Gamma_s, h)\). We call such a family a degeneration of \(X\) defined respecting \((\Gamma_s, h)\). In this paper, all polyhedral decompositions \(\mathcal{P}\) of \(\mathbb{R}^n\) defining toric degenerations are those constructed in this way.

Definition 5.8. Let \(X\) be a toric variety. A holomorphic curve \(C \subset X\) is torically transverse if the following conditions are satisfied.

1. \(C\) is disjoint from all toric strata of codimension greater than one.
2. No irreducible component of \(C\) is entirely contained in a toric divisor.

A stable map \(\varphi : C \rightarrow X\) is torically transverse if \(\varphi(C) \subset X\) is torically transverse and \(\varphi^{-1}(\text{int } X) \subset C\) is dense. Here \(\text{int } X\) is the complement of the union of the toric divisors.

Remark 5.9. The condition (2) above follows from (1) when \(C\) is a curve without boundary. In fact, this is also the case for the disks with Lagrangian boundary conditions we will treat.

Let \(X(\Sigma)\) be a toric variety defined by a fan \(\Sigma\) in \(N\). The toric prime divisors on \(X(\Sigma)\) are denoted by \(D_v\), where \(v \in N\) is the primitive generator of a ray of \(\Sigma\).

Definition 5.10. Let \(C\) be a closed Riemann surface or a disk. Let \(\varphi : C \rightarrow X(\Sigma)\) be a torically transverse holomorphic map. The degree

\[\delta(\varphi) : N \setminus \{0\} \rightarrow \mathbb{N}\]

of \(\varphi\) is defined as follows. For a primitive vector \(v \in N\) and \(\lambda \in \mathbb{N}\), map \(\lambda \cdot v\) to 0 if \(\mathbb{Q}_{\geq 0} v \not\in \Sigma^{[1]}\), and to the number of points of multiplicity \(\lambda\) in \(\varphi^* D_v\) otherwise.

Definition 5.11. Let \(X_0 = \bigcup_{v \in \mathcal{P} \cap 0} X_v\) be the central fiber of the toric degeneration \(\mathcal{X} \rightarrow \mathbb{C}\) defined by an integral polyhedral decomposition \(\mathcal{P}\) of \(N\). A pre-log disk on \(X_0\) is a stable map \(\phi : C \rightarrow X_0\) from a prestable bordered Riemann surface \(C\) of genus zero with \(\partial C = S^1\) to \(X_0\), satisfying the following properties:
(i) For any $v$, the projection $C \times_{X_0} X_v \to X_v$ is a torically transverse stable map.

(ii) Let a point $P \in C$ map to the singular locus of $X_0$. Then $C$ has a node at $P$, and $\phi$ maps the two branches $(C', P), (C'', P)$ of $C$ at $P$ to different irreducible components $X_{v'}, X_{v''}$ of $X_0$. Moreover, if $w'$ is the intersection index of the image of the restriction $\phi|_{C'} : (C', P) \to (X_{v'}, D')$ with the toric boundary $D' \subset X_{v'}$ at $\phi(P)$, and $w''$ accordingly for $\phi|_{C''} : (C'', P) \to (X_{v''}, D'')$, then $w' = w''$.

For the purpose of this paper, we require the following additional property.

(iii) Let $C'''$ be the irreducible component with boundary. Let $X_v$ be the irreducible component of $X_0$ to which $C'''$ is mapped. Then the boundary $\partial C'''$ is mapped to a Lagrangian fiber of the moment map of $X_v$.

In this paper, when we talk about a pre-log disk, it means a stable map which satisfies all the properties (i), (ii), and (iii) above.

5.3. Relation between pre-log disks and tropical curves. A pre-log curve on $X_0$ gives a tropical curve as the dual intersection graph. See Section 4 of [18]. In our case, the dual intersection graph of a pre-log disk need not give a tropical curve with a stop. But it does give a tropical curve with a stop when the image $\phi(C''')$ of the component $C'''$ (in the notation of Definition 5.11(iii)) intersects the toric boundary of $X_v$ just once. In particular, this is the case for the maximally degenerate disks (see Section 6).

5.4. Boundary condition for disks. The boundary condition for a family of stable maps from a degenerating family of disks (see Remark 7.6 for the construction of the domain for such a family) is fixed as follows. The affine subspace $A_0$ (which is a part of the constraints for tropical curves introduced in Definition 3.1) spans the linear subspace $LC(A_0)$ in the vector space $\mathbb{N}_Q \times \mathbb{Q}$ where the fan $\tilde{\Sigma}_{\emptyset}$ lives. Let $P \in \mathfrak{X}$ be a general point. Let $G_R(N)$ be the maximal compact subgroup of $G(N)$ and let $G_{m,A_0}$ be the subtorus $G(LC(A_0) \cap (N \times \mathbb{Z}))$ of the big torus acting on $\mathfrak{X}$. Let $L$ be the closure of the orbit $(G_R(N)G_{m,A_0}) \cdot P$. This is the family of tori to which the boundary of a disk should be mapped.

5.5. Incidence conditions for disks. Incidence conditions for disks are defined as in [18], Proposition 3.6. Namely, an affine subspace $A \subset \mathbb{N}_Q$ spans the linear subspace $LC(A) \subset \mathbb{N}_Q \times \mathbb{Q}$ as above. For any general closed point $P$ in the big torus of $\mathfrak{X}$, the closure of the orbit $G(LC(A) \cap (N \times \mathbb{Z})) \cdot P$ defines a subvariety $Z \subset \mathfrak{X}$ projecting onto $\mathbb{C}$. Our incidence condition is that the disk should have a non-trivial intersection with $Z$. See Section 3 of [18] for details.

There is one point different from the closed curve case. Namely, we also put an extra marked point, which corresponds to the stop of the tropical curve. Take $A_0 \subset \mathbb{N}_Q$ and $P \in \mathfrak{X}$ as in Section 5.4 (this $P$ must be the same as $P$ in Section 5.4, while in the last paragraph, this is not required). Then the cone $LC(A_0)$ defines
a subgroup $G_{m,A_0}$. We set $Z_0$ to be the closure of the orbit $G_{m,A_0}.P$. Note that $Z_0$ is contained in $\mathcal{L}$.

We will count stable maps from pointed prestable disks satisfying the Lagrangian boundary condition and intersecting the varieties $Z_1,\ldots,Z_k$ in the interior and $Z_0$ at the boundary.

6. **Maximally degenerate disks in toric degeneration.**

6.1. **Maximally degenerate disks.** We extend the notion of maximally degenerate curves to the case with boundary (see also Section 5, [18]).

**Definition 6.1.** Let $X$ be a complete toric variety and $Y \subset X$ be the toric boundary. A line on $X$ is the equivalence class of non-constant, torically transverse maps $\varphi: \mathbb{P}^1 \to X$ such that $\sharp \varphi^{-1}(Y) \leq 3$ (not counted with multiplicity), modulo the automorphisms of $\mathbb{P}^1$.

A half line on $X$ is the equivalence class of non-constant, torically transverse maps $\varphi: D \to X$ such that $\sharp \varphi^{-1}(Y) = 1$ and the boundary $\partial D = S^1$ is mapped to a Lagrangian torus fiber of the moment map, modulo the automorphisms of $D$ (see Example 2.1).

**Definition 6.2.** Let $X_0 = \bigcup_{v \in \mathcal{P}^{[0]}} X_v$ be the central fiber of the toric degeneration defined by an integral polyhedral decomposition $\mathcal{P}$. A pre-log disk (Definition 5.11) $\varphi: C \to X_0$ is called *maximally degenerate* if for any $v \in \mathcal{P}^{[0]}$, the projection $C \times_{X_0} X_v \to X_v$ is a line or a half line. For $n = 2$, the union of two divalent lines intersecting disjoint toric divisors is also allowed. Here a divalent line is a line with $\sharp \varphi^{-1}(Y) = 2$.

Thus a maximally degenerate curve is a collection of lines and a half line, at most one of them (two, when $n = 2$) on each irreducible component of $X_0$, whose components match in the sense that they are glued into a pre-log curve.

We collect several properties of lines and maximally degenerate curves. See Section 5 of [18] for details.

- A line $\varphi: \mathbb{P}^1 \to X$ in a toric variety $X$ determines (non-canonically) a tropical curve with one vertex, which is divalent or trivalent, according to $\sharp \varphi^{-1}(Y)$ in the notation of Definition 6.1.
- Let $N$ be a lattice such that the fan defining the toric variety $X$ lies in $N_Q$. The tropical curve above determines the subspace $E_Q$ of $N_Q$ spanned by the direction vectors of the edges of it. In particular, $E_Q$ is one or two dimensional.
- Similarly, a half line in $X$ determines a half line in $N_Q$. This also determines a one dimensional subspace $E_Q$ of $N_Q$. Let $E = E_Q \cap N$.
- The torus $\mathbb{G}(N)$ acts on the set of (half) lines.
  - When $\sharp \varphi^{-1}(Y) = 3$, then the action is simply transitive.
  - When $\sharp \varphi^{-1}(Y) = 2$, then the action is transitive, and the stabilizer is the subtorus $\mathbb{G}(E)$. 

For the case of a half line, the action is transitive, and the stabilizer is the maximal compact subgroup $S^1 \subset G(E)$.

On the other hand, the abelian group $\mathbb{N}_Q$ acts on the set of tropical curves with one vertex.

- When the tropical curve is trivalent, then the action is simply transitive.
- When the tropical curve is divalent, then the action is transitive, and the stabilizer is the subgroup $E_{\mathbb{Q}}$.
- For the case of a half line, the action is simply transitive.

Since the torus $\mathbb{G}(\mathbb{N})$ is naturally isomorphic to the image of the covering map

$$\mathbb{N}_Q \otimes \mathbb{C} = \mathbb{N}_R \oplus \sqrt{-1} \mathbb{N}_R \longrightarrow (\mathbb{N}_R \oplus \sqrt{-1} \mathbb{N}_R) / \sqrt{-1} \mathbb{N},$$

the space parametrizing the lines in $X$ can be locally identified with the complexification of the space parametrizing the corresponding tropical curves. In the case of a half line, the holomorphic disks are parametrized by

$$(\mathbb{N}_R \oplus \sqrt{-1} \mathbb{N}_R) / (\sqrt{-1} \mathbb{N} + \sqrt{-1} \mathbb{R} \cdot v),$$

where $v \in \mathbb{N}$ is the primitive integral generator of the tropical half line. Thus, in this case the space parametrizing the holomorphic disks is locally the quotient of the complexification of the space parametrizing tropical half lines by the circle $S^1$ generated by $\sqrt{-1}v$.

Given an immersive tropical curve with a stop $(\Gamma_s, h)$, one easily sees that there corresponds (non-canonically) a maximally degenerate pre-log disk in the central fiber $X_0$ of a toric degeneration $X$ of a toric variety associated to $(\Gamma_s, h)$. The above argument shows that the space parametrizing such maximally degenerate pre-log disks are locally isomorphic to the complexification of the space parametrizing the tropical curves, modulo the one dimensional difference of freedom caused by the stop, as mentioned above. See Remark 6.4.

This is the local comparison of the moduli spaces of tropical curves and pre-log curves. Introducing incidence conditions, we can deduce an enumerative comparison result, which we give in the following section.

### 6.2. Counting maximally degenerate curves

We recall from [18] the result which gives the number of maximally degenerate curves on $X_0$ matching the incidence conditions. The proof is the same even in the case with a stop. Let $\mathcal{T}_{(\Gamma_s,E,u)}(\mathcal{A})$ be the set of tropical curves of type $(\Gamma_s, E, u)$ of fixed degree $\Delta$ satisfying the set of constraints $\mathcal{A}$.

**Proposition 6.3.** [18, Proposition 5.7]

1. Let $\Delta \in \text{Map}(\mathbb{N} \setminus \{0\}, \mathbb{N})$ be a degree and $\mathcal{A} = (A_0, \ldots, A_l)$ be a set of affine constraints (including the one for the stop) that is general for $\Delta$ and assume
l ≥ 1. If $\Sigma_{(\Gamma, E, u)}(A) \neq \emptyset$ for an $l$-marked tree with a stop $(\Gamma, E)$, then the map

$$\text{Map}(\Gamma[0], N) \longrightarrow \prod_{E \in \Gamma[0] \setminus \Gamma[\infty]} N/\mathbb{Z}(\partial E, E) \times \prod_{i=0}^{l} N/(\mathbb{Q}(\partial E, E_i) + L(A_i)) \cap N),$$

$$h \longmapsto (h(\partial E) - h(\partial E))_E, (h(\partial E_i) - A_i)_i.$$

is an inclusion of lattices of finite index $\mathcal{D} = \mathcal{D}(\Gamma, E, h, A)$. Here $\partial^\pm : \Gamma[0] \setminus \Gamma[\infty] \rightarrow \Gamma[0]$ is an arbitrarily chosen orientation of the bounded edges, that is, $\partial E = \{\partial E, \partial^+ E\}$ for any $E \in \Gamma[0] \setminus \Gamma[\infty]$. If $E \in \Gamma[\infty]$ then $\partial E$ denotes the unique vertex adjacent to $E$ which is not the stop (recall that the stopping edge is contained in $\Gamma[\infty]$).

(2) Let $(\Gamma, h)$ be a tropical curve with a stop in $\Sigma(\Gamma, E, u)(A)$. Assume in addition that $\mathcal{P}$ is an integral polyhedral decomposition of $\mathcal{N}$ with

$$h(\Gamma[\mu]) \subset \bigcup_{\Xi \in \mathcal{P}[\mu]} \Xi, \quad \mu = 0, 1, \quad \text{and} \quad h(\Gamma) \cap A_j \subset \mathcal{P}[0], \quad j = 0, \ldots, l,$$

and let $\mathfrak{X} \rightarrow \mathbb{C}$ be the associated toric degeneration with the central fiber $X_0$. Let $P_j, j = 0, \ldots, l$ be general closed points in the big torus of $\mathfrak{X}$. Then $\mathcal{D}$ equals the number of isomorphism classes of maximally degenerate curves in $X_0$ which are associated to $(\Gamma, h)$ (see the previous section or Section 4 of [18]) and intersecting

$$Z_i := \overline{G(LC(\Gamma[\mu])) \cdot P_i} \subset \mathfrak{X},$$

the closure of the orbit through $P_i$ for the subgroup $G(LC(\Gamma[\mu])) \subset G(N \times \mathbb{Z})$ acting on $\mathfrak{X}$.

**Remark 6.4.** In the case of closed curves as in [18], the kernel of the part

$$\text{Map}(\Gamma[0], N) \longrightarrow \prod_{E \in \Gamma[0] \setminus \Gamma[\infty]} N/\mathbb{Z}(\partial E, E)$$

of the map in the proposition, tensored by $\mathbb{C}^*$, is the space parametrizing maximally degenerate pre-log curves. When there is a boundary, the parameter space is larger by real dimension one, due to the freedom to move the boundary in the direction of $\mathbb{R}$, in the notation at the last part of the previous section.

However, this degree of freedom necessarily breaks the incidence condition, when the set of constraints $A$ is general, so that the space $A_0$ does not contain a vector parallel to $v$. Thus, the incidence conditions for stable disks give a space larger than $\prod_{i=0}^{l} N/(\mathbb{Q}(\partial E, E_i) + L(A_i)) \cap N)$, tensored by $\mathbb{C}^*$, by real dimension one. These one additional dimensions (one for the moduli and the other for the incidence condition) cancel out, and result in the same formulation as in the closed curve case.
7. Deformation theory of maximally degenerate disks. In this section, we extend the arguments in Section 7 of [18] to the case with boundary. We will work in analytic category.

7.1. Sheaves on bordered Riemann surfaces.

Definition 7.1. A Riemann-Hilbert bundle $(E, E_R)$ of rank $n$ on a bordered Riemann surface $(\Sigma, \partial \Sigma)$ is a pair of a smooth complex vector bundle $E$ on $\Sigma$ which is holomorphic (in the sense of Definition 4.1, obviously modified to this situation. See Definition 3.3.12 of [10]) and a totally real subbundle $E_R \mid E\mid_{\partial \Sigma}$.

By the doubling construction, there is a holomorphic vector bundle $E_\mathbb{C} \rightarrow \Sigma_\mathbb{C}$ with an anti-holomorphic involution $\tilde{\sigma} : E_\mathbb{C} \rightarrow E_\mathbb{C}$ covering $\sigma : \Sigma_\mathbb{C} \rightarrow \Sigma_\mathbb{C}$, such that $E\mid_{\Sigma} = E$ and the fixed locus of $\tilde{\sigma}$ is $E_R \rightarrow \partial \Sigma$ (see [10], Section 3.3).

Let $(\mathcal{E}, \mathcal{E}_R)$ be the sheaf of sections of $(E, E_R)$. Let $U$ be an open subset of $\Sigma$. Then $\mathcal{U} = U \cup \sigma(U)$ is an open subset of $\Sigma_\mathbb{C}$. The sections of the Riemann-Hilbert bundle are characterized as follows:

$$(\mathcal{E}, \mathcal{E}_R)(U) = \mathcal{E}_\mathbb{C}(\mathcal{U})^{\tilde{\sigma}}.$$

On the other hand, $-\tilde{\sigma}$ gives another involution covering $\sigma$, and

$$f = \frac{1}{2}(1 + \tilde{\sigma})f + \frac{1}{2}(1 - \tilde{\sigma})f, \quad f \in \mathcal{E}_\mathbb{C}(\mathcal{U})$$

gives the decomposition

$$\mathcal{E}_\mathbb{C}(\mathcal{U}) = \mathcal{E}_\mathbb{C}(\mathcal{U})^{\tilde{\sigma}} \oplus \mathcal{E}_\mathbb{C}(\mathcal{U})^{-\tilde{\sigma}}$$

of the space of sections of $\mathcal{E}_\mathbb{C}$. The spaces on the right-hand side are the spaces of sections of Riemann-Hilbert bundles.

The sheaf cohomology groups of $(\mathcal{E}, \mathcal{E}_R)$, obtained as the right derived functors of the global section functor as usual, satisfy the equality

$$\dim_{\mathbb{R}} H^q(\Sigma, \partial \Sigma; E, E_R) = \dim_{\mathbb{C}} H^q(\Sigma_\mathbb{C}, E_\mathbb{C}).$$

See [10], Section 3 for more details.

7.2. Deforming maximally degenerate curves. In this section, tropical curves are the ones mentioned in Definition 3.7. Namely, we write simply by $\Gamma_s$ the curve $\tilde{\Gamma}_s$ in the notation of Definition 3.7.

As in [18], Section 7, unadorned letters denote log spaces or morphisms of log spaces, while underlined letters denote the underlying spaces or morphisms. We put a natural log structure on $\mathfrak{X}$ coming from the toric divisors and let $O_0$ be the standard log point (see [18], Section 7).

We extend the results in [18], Section 7 to the case with boundary. Let $(\Gamma_s, h)$ be an immersive tropical curve with a stop. Let $\varphi_0 : (\mathcal{C}_0, \{x_i\}) \rightarrow \mathcal{X}_0$ be a pair.
of a maximally degenerate pre-log disk associated to \((\Gamma_s, h)\) and the set of marked points. Note that the marked point \(x_i\) is placed on the component of \(C_0\) corresponding to the divalent vertex added to the \(i\)th marked edge (see Definition 3.7).

**Proposition 7.2.** Assume that for every bounded edge (containing the stopping edge) \(E \subset \Gamma_s^{[1]}\), the integral length of \(h(E)\) is an integral multiple of its weight \(w(E)\). Then there are exactly \(w(\Gamma_s, E)\) pairwise non-isomorphic pairs

\[ [\varphi_0 : C_0 \to X_0, \{x_i\}] \]

with the following properties:

- The underlying stable map is isomorphic to \((\overline{C_0}, \{x_i\}, \varphi_0)\).
- The logarithmic morphism \(\varphi_0\) is strict wherever \(\pi_0 : X_0 \to O_0\) is strict.
- The composition \(\pi_0 \circ \varphi_0 : C_0 \to X_0 \to O_0\) is log-smooth and integral.

Here \(w(\Gamma_s, E)\) is the marked total weight (Definition 2.5).

**Sketch of a proof.** The proof of this proposition is almost the same as [18], Proposition 7.1 and we omit it. We just remark that if the stopping edge has the weight larger than one, it contributes to the marked total weight. This is because the disk has two special points, one from the node and the other from the marked point on the boundary, so that it has no continuous automorphism. Keeping this remark in mind, the argument of the proof of Proposition 7.1 in [18] applies completely similarly. □

Let \(T_{X/C}, T_{C_0/O_0}, \text{etc.}\) be the logarithmic tangent bundles and let \(T^*_{X/C}, T^*_{C_0/O_0}, \text{etc.}\) be the logarithmic cotangent bundles. Let

\[ \Theta_{X/C}, \quad \Theta_{C_0/O_0}, \quad \Omega_{X/C}, \quad \Omega_{C_0/O_0} \]

be the sheaves of sections of these bundles. Let

\[ N_{\varphi_0} = \varphi_0^* T_{X/C} / T_{C_0/O_0} \]

be the logarithmic normal bundle and \(N_{\varphi_0}\) be the corresponding sheaf of sections. We remark that the sheaf of sections of the logarithmic tangent bundle of a toric variety \(X\) defined by a fan in \(N_{\mathbb{Q}}\) is naturally isomorphic to the free sheaf \(N \otimes_{\mathbb{Z}} \mathcal{O}_X\). See, for example, [8]. It is also easy to see that the logarithmic tangent sheaf of a nodal curve is locally free (it justifies the above terminology “bundle”).

Let \(J_{X(\Sigma)}, J_{C_0}, \text{etc.}\) be the complex structures on the spaces \(X(\Sigma), C_0, \text{etc.}\), respectively. Let

\[ i_{T^n} : T^n \hookrightarrow X_0 \]

be the Lagrangian torus fiber to which the boundary \(S^1 \subset C_0\) is mapped by \(\varphi_0\). We think of \(T^n\) as the locally ringed space associated to the underlying real analytic manifold. Let \(\mathcal{O}_{T^n}\) be the structure sheaf. \(\mathcal{O}_{T^n}\) is a totally real subsheaf (i.e., the
sheaf of sections of a totally real sub-vector bundle) of the sheaf of sections of the bundle $i^*_T \mathcal{C}_{X_0}$, here $\mathcal{C}_{X_0}$ is the trivial line bundle on $X_0$.

Let

$$O_\infty = \text{Spec } \mathbb{C}[[\epsilon]]$$

be the formal disk and

$$\widehat{\mathcal{X}} = \mathcal{X} \times \mathbb{C} O_\infty$$

be the completion of the central fiber. Then $\Theta_{\widehat{\mathcal{X}}/O_\infty} \simeq N \otimes_{\mathbb{Z}} O_{\widehat{\mathcal{X}}}$ and $i^*_T(\Theta_{\widehat{\mathcal{X}}/O_\infty})$ has a totally real subsheaf $N \otimes_{\mathbb{Z}} i\mathcal{O}_T^n$. We identify this with the sheaf of sections of the tangent bundle $T^n$ of $T^n$. Similarly, we identify $N \otimes_{\mathbb{Z}} O_{T^n}$ with the sheaf of sections of the normal bundle $J_{X_0} T^n$ of $T^n$ in $X_0$.

Remark 7.3. The standard definition of the normal bundle is the quotient

$$i^*_T(T_{\widehat{\mathcal{X}}/O_\infty})/T^n.$$

However, in our case there is a natural isomorphism between this quotient and $J_{X_0} T^n$, and we may define the normal bundle by $J_{X_0} T^n$. This can also be understood by thinking that we are implicitly using the natural symplectic structure on the toric variety so that we can identify the tangent bundle and the cotangent bundle.

This induces the splitting

$$i^*_T T_{\widehat{\mathcal{X}}/O_\infty} \simeq T^n \oplus J_{X_0} T^n$$

as real vector bundles. The tangent bundle of $C_0$ also splits on the boundary:

$$i^*_S T_{C_0} = T_S \oplus J_{C_0} T_S,$$

here

$$i_S : S^1 \hookrightarrow C_0$$

is the inclusion. Then $(N_{\varphi_0}, \varphi^*_0(\Theta_T^n)/\Theta_{S^1})$ and $(N_{\varphi_0}, \varphi^*_0(J_{X_0} \Theta_T^n)/J_{C_0} \Theta_{S^1})$ are (the sheaves of the sections of) Riemann-Hilbert bundles on $(C_0, S^1)$. Here

$$\varphi^*_0(\Theta_T^n) = \varphi_0^{-1} \bigg|_{S^1 \times C_0} \Theta_T^n \otimes \varphi_0^{-1} \bigg|_{S^1 \times C_0} \mathcal{O}_{T^n}$$

and we regard $S^1$ as the locally ringed space associated to the underlying real analytic manifold.

The next result is the local description of the moduli space of smoothings of a maximally degenerate pre-log curve. Since the difficult part (i.e., the existence) of
a smoothing is almost immediate by dimensional reason and rationality, it is more or less a consequence of the formal deformation theory.

Roughly speaking, the smoothings are parametrized by sections of the normal bundle at each order. In our case, since the curve $C_0$ has a boundary, we have to take care of it. First note that there is an injection from the sheaf $(\Theta_{C_0}, \Theta_{S_1})$ of sections of the Riemann-Hilbert bundle $(\mathcal{T}_{C_0}, \mathcal{T}_{S_1})$ to the sheaf $\varphi^{*}_{0} \Theta_{X/\mathbb{C}}$. The sheaf $\varphi^{*}_{0} \Theta_{X/\mathbb{C}}$ contains two sheaves of sections of Riemann-Hilbert bundles:

\[
(\varphi^{*}_{0} \Theta_{X/\mathbb{C}}, \varphi^{*}_{0}(\Theta_{T^n})), \quad (\varphi^{*}_{0} \Theta_{X/\mathbb{C}}, \varphi^{*}_{0}(J_{X_0} \Theta_{T^n})).
\]

The sheaf $(\Theta_{C_0}, \Theta_{S_1})$ is contained in the left one, with the quotient

\[
(N_{\varphi_0}, \varphi^{*}_{0}(\Theta_{T^n}))/\Theta_{S_1}).
\]

**Lemma 7.4.** Let $[\varphi_{k-1} : C_{k-1}/O_{k-1} \to \mathfrak{X}]$ be a lift of $[\varphi_0 : C_0/O_0 \to \mathfrak{X}]$. Then the set of isomorphism classes of lifts $[\varphi_k : C_k/O_k \to \mathfrak{X}]$ restricting to $\varphi_{k-1}$ over $O_{k-1}$ is a torsor under the group

\[
H^0(C_0, \partial C_0; N_{\varphi_0}, \varphi^{*}_{0}(\Theta_{T^n}))/\Theta_{S_1}) \oplus H^0(C_0, \partial C_0; \varphi^{*}_{0} \Theta_{X/\mathbb{C}}, \varphi^{*}_{0}(J_{X_0} \Theta_{T^n})).
\]

**Proof.** Let us assume that we have constructed a deformation $\varphi_{k-1}$ of $\varphi_0$ up to order $k-1$. We fix a general section $s$ of $\mathfrak{X} \to O_\infty$ and take its orbit $G_{\mathbb{R}}(N) \cdot s$ by the maximal compact torus $G_{\mathbb{R}}(N)$. This determines a family of Lagrangian tori $\mathcal{T}$. Since $\mathcal{T}$ is a real manifold, it fibers over the locally ringed space $O_{\infty, \mathbb{R}}$, which is the locally ringed space structure on $O_{\infty}$ whose structure sheaf is given by the sheaf of real sections of the direct sum of the structure sheaf of $O_{\infty}$ and its complex conjugate, written by

\[
\mathbb{C}[[\epsilon]] \oplus \overline{\mathbb{C}[[\epsilon]]},
\]

somewhat informally. Similarly, we define the locally ringed spaces $O_{k, \mathbb{R}}$. Then we define

\[
\mathcal{T}_k = \mathcal{T} \times_{O_{\infty, \mathbb{R}}} O_{k, \mathbb{R}}.
\]

We assume that the boundary of $\varphi_{k-1}$ is mapped to $\mathcal{T}_{k-1}$.

To lift $\varphi_{k-1}$ to the $k$th order, we locally lift $\varphi_{k-1}$ and try to extend it globally. Such a local lift exists by the log-smoothness of $\mathfrak{X} \to \mathbb{C}$, and it is possible to take it so that the boundary is mapped to $\mathcal{T}_k$. If we do not take care of the boundary condition, those local lifts form a torsor under the space of sections of $\varphi^{*}_{k-1} \Theta_{X/\mathbb{C}}$.

On a sufficiently small open subset $U$ of $C_{k-1}$ which intersects the boundary, the space of sections of $\varphi^{*}_{k-1} \Theta_{X/\mathbb{C}}$ splits:

\[
\Gamma(U, \varphi^{*}_{k-1} \Theta_{X/\mathbb{C}}) = \Gamma(U, \partial U; \varphi^{*}_{k-1} \Theta_{X/\mathbb{C}}, \varphi^{*}_{k-1}(\Theta_{T/O_{\infty, \mathbb{R}}})) \\
\oplus \Gamma(U, \partial U; \varphi^{*}_{k-1} \Theta_{X/\mathbb{C}}, \varphi^{*}_{k-1}(J_{X} \Theta_{T/O_{\infty, \mathbb{R}}}))
\]
Thus, for any local lift of $\varphi_{k-1}$ on the open sets of a sufficiently fine covering of $C_{k-1}$, we have two obstruction groups

$$H^1(C_{k-1}, \partial C_{k-1}; \varphi_{k-1}^* \Theta_{X/C}, \varphi_{k-1}^* (\Theta_T/O_{\omega,R}))$$

and

$$H^1(C_{k-1}, \partial C_{k-1}; \varphi_{k-1}^* \Theta_{X/C}, \varphi_{k-1}^* (J_X \Theta_T/O_{\omega,R}))$$

(in fact, if we impose the condition that the boundary of the local lift is mapped to $T_k$, the latter group is irrelevant). Recall that there are natural identifications

$$\Theta_{X/C} \simeq N \otimes \mathbb{Z} O_X$$

and

$$i_{T_k-1}^* (\Theta_{X/C}) \simeq \Theta_{T_k-1/O_{k-1,R}} \oplus J_X \Theta_{T_k-1/O_{k-1,R}}$$

$$\simeq N \otimes \mathbb{Z} \Theta_{T_k-1/O_{k-1,R}} \oplus N \otimes \mathbb{Z} O_{T_k-1/O_{k-1,R}}$$

we noted before Remark 7.3. By these identifications and using the fact that $C_{k-1}$ is a prestable disk, doubling construction of the sheaves of sections of Riemann-Hilbert bundles $(\varphi_{k-1}^* \Theta_{X/C}, \varphi_{k-1}^* (\Theta_T/O_{\omega,R})), (\varphi_{k-1}^* \Theta_{X/C}, \varphi_{k-1}^* (J_X \Theta_T/O_{\omega,R}))$ both give the trivial sheaf on a rational prestable curve. It follows that the above cohomology groups vanish. Thus, there is a lift $\varphi_k$ of $\varphi_{k-1}$ satisfying the boundary condition $T_k$.

Next we describe the local structure of the moduli space of lifts. Let $\varphi_k$ be a lift of $\varphi_{k-1}$ whose boundary is mapped to $T_k$, and let $\psi_k$ be another lift. At this stage we do not care about the boundary condition for $\psi_k$ (we do not even assume that $\psi_k(\partial S^1)$ is contained in some Lagrangian submanifold). We represent the difference between $\varphi_k$ and $\psi_k$ by local sections on a covering as above. Since we fix $\varphi_{k-1}$, only the $(k-1)$th order term of $\epsilon$ is non-trivial, so that the difference between $\varphi_k$ and $\psi_k$ gives a family of sections of

$$\varphi_{k-1}^* \Theta_{X/C} \otimes \mathbb{C}[\epsilon]/\epsilon \simeq \varphi_0^* \Theta_{X/C}.$$

When an open set $U$ in this covering intersects the boundary, the space of sections splits as above:

$$\Gamma(U, \varphi_0^* \Theta_{X/C}) = \Gamma(U, U \cap \partial C_0; \varphi_0^* \Theta_{X/C}, \varphi_0^* (\Theta_T/O_{\omega,R}))$$

$$\oplus \Gamma(U, U \cap \partial C_0; \varphi_0^* \Theta_{X/C}, \varphi_0^* (J_X \Theta_T/O_{\omega,R})).$$

The sheaf $(\varphi_0^* \Theta_{X/C}, \varphi_0^* (\Theta_T/O_{\omega,R}))$ contains the subsheaf $(\Theta_{C_0}, \Theta_{S^1})$. Two local sections of $(\varphi_0^* \Theta_{X/C}, \varphi_0^* (\Theta_T/O_{\omega,R}))$ give the same lift if and only if they differ by a section of $(\Theta_{C_0}, \Theta_{S^1})$. Thus, since $\varphi_k$ is a global lift, for $\psi_k$ to be also globally defined, the family of local sections above must define a sum of global sections of $(\varphi_0^* \Theta_{X/C}, \varphi_0^* (\Theta_T/O_{\omega,R}))/((\Theta_{C_0}, \Theta_{S^1})$ and $(\varphi_0^* \Theta_{X/C}, \varphi_0^* (J_X \Theta_T/O_{\omega,R})))$. That is, it defines a sum of elements in
\[ H^0(C_0, \partial C_0; (\varphi_0^* \Theta X/\mathbb{C}, \varphi_0^*(\Theta_{T/O_{\infty,R}}))/\Theta C_0, \Theta S^1) \]

and

\[ H^0(C_0, \partial C_0; \varphi_0^* \Theta X/\mathbb{C}, \varphi_0^*(J X_{T/O_{\infty,R}})). \]

The latter is isomorphic to \( N_{\mathbb{R}} \), so in particular an element in it is determined by its value at the boundary. So the intersection of these two spaces is trivial, and \( \psi_k \) gives an element in the direct sum

\[ H^0(C_0, \partial C_0; (\varphi_0^* \Theta X/\mathbb{C}, \varphi_0^*(\Theta_{T/O_{\infty,R}}))/\Theta C_0, \Theta S^1) \oplus H^0(C_0, \partial C_0; \varphi_0^* \Theta X/\mathbb{C}, \varphi_0^*(J X_{T/O_{\infty,R}})). \]

Let us write this space of sections by \( V \). Note that sections in the group \( H^0(C_0, \partial C_0; (\varphi_0^* \Theta X/\mathbb{C}, \varphi_0^*(\Theta_{T/O_{\infty,R}}))/\Theta C_0, \Theta S^1) \) do not change the boundary condition, that is, \( \psi_k \) also satisfies \( T_k \). On the other hand, sections in \( H^0(C_0, \partial C_0; \varphi_0^* \Theta X/\mathbb{C}, \varphi_0^*(J X_{T/O_{\infty,R}})) \) change the boundary condition, but by the natural isomorphism with \( N_{\mathbb{R}} \), these sections are constant at the boundary, so the lift also satisfies an appropriate Lagrangian boundary condition though it is not \( T_k \).

Conversely, suppose that we are given an element in \( V \). When we locally lift it to a section of \( \Gamma(U, \varphi_0^* \Theta X/\mathbb{C}) \) on each open set \( U \) of the covering, the error of these local lifts in defining a global lift gives a one cocycle of the sheaf \( (\Theta C_0, \Theta S^1) \). So this error gives an element of \( H^1(C_0, \partial C_0; \Theta C_0, \Theta S^1) \).

This group is isomorphic to the tangent space of the moduli space of pointed stable disks, but since the obstruction \( H^2(C_0, \partial C_0; \Theta C_0, \Theta S^1) \) vanishes by dimensional reason, it really gives the local model of the moduli space, see Remark 7.6 below. Thus, the error of an element of \( V \) in defining a global \( k \)th order lift perturbing \( \varphi_k \) can be cancelled by deforming the domain curve \( C_k \).

Eventually, the lifts of \( \varphi_{k-1} \) are parametrized by \( H^0(C_0, \partial C_0; N_{\varphi_0}, \varphi_0^*(T^u))/\Theta S^1) \oplus H^0(C_0, \partial C_0; \varphi_0^* \Theta X/\mathbb{C}, \varphi_0^*(J X_{T/O_{\infty,R}})). \)

Remark 7.5. Note that \( (T C_0, T S^1) \) is a logarithmic bundle, so the group \( H^1(C_0, \partial C_0; \Theta C_0, \Theta S^1) \) may not be trivial in general.

Remark 7.6. Since we mentioned the tangent space of the moduli space of pointed stable disks in the proof, we make some remarks on the moduli space.

(1) We do not argue the global aspects of the moduli space. In particular, virtual fundamental classes or cycles are not constructed.

(2) Then the well-definedness of the counting number is assured by:

- Genericity: The solutions are always placed away from the boundary of the moduli space.
• Transversality: The obstruction vanishes, as we saw in the proof above. However, as we mentioned in the introduction, the counting number will change when the incidence conditions are largely changed.

(3) Locally the moduli space of the domain curves of the stable maps is constructed as follows. First note that every component of the curve $C_0$ which contains a marked point is stable, since $C_0$ is the domain of a maximally degenerate pre-log curve. This follows from the fact that in a maximally degenerate curve, each sphere component has at least two special points other than the marked point (intersection with the toric divisors of the components of $X_0$). Note that if this intersection is not a node, it is a special point as a log point. As for the component with boundary, there is a special point in the interior (the intersection with the toric divisor) and on the boundary, because there is an incidence condition $Z_0$ for the boundary.

On the other hand, it is possible that a component of $C_0$ which has a log point (such a component corresponds to a vertex with an unbounded edge on the tropical side) is not stable. Let $B$ be such a component. In this case, there is a unique neighboring component and let us write it by $B'$. Then we contract the component $B$ and think $a$ as a special point of $B'$.

Repeating this process, we can stabilize $C_0$ without losing a marked point. We write the stabilized curve also by $C_0$. By doubling construction, we obtain a rational stable curve $\tilde{C}_0$ from $C_0$, which has an anti-holomorphic involution. Consider a neighborhood $U$ of $\tilde{C}_0$ in the corresponding Deligne-Mumford stack of stable curves in analytic category. There is a real analytic substack in $U$ which parametrizes the curves which have anti-holomorphic involutions. The irreducible component of this substack corresponding to the involution whose quotient gives $C_0$ is the local parameter space of disks containing $C_0$.

Let $x_0 = \{x_0, \ldots, x_l\}$ be the ordered set of marked points on $C_0$. The point $x_0$ lies on the boundary, while the others lie in the interior. Let $Z = \{Z_0, \ldots, Z_l\}$ be the ordered set of incidence varieties (see Section 5.5).

**Proposition 7.7.** Let $[\varphi_{k-1} : C_{k-1}/O_{k-1} \to X, x_{k-1}]$ be a lift of $[\varphi_0 : C_0/O_0 \to X, x_0]$ with $x_{k-1}$ factoring over $Z$. Then up to isomorphism there is a unique lift $[\varphi_k : C_k/O_k \to X, x_k]$ to $O_k$ with $x_k$ factoring over $Z$.

**Proof.** The proof goes along the same line as in [18], Proposition 7.3. By Lemma 7.4, it suffices to prove the ‘transversality’ map

$$ H^0(C_0, \partial C_0; N_{\varphi_0}, \varphi_0^*(\Theta_{T^n})/\Theta_{S^1}) \oplus H^0(C_0, \partial C_0; \varphi_0^*\Theta_{X/C}, \varphi_0^*(J_{X_0}\Theta_{T^n})) $$

$$ \longrightarrow T_{X/C, \varphi_0(x_0)}/(T_{Z_0/C, \varphi_0(x_0)} + D\varphi_0(T_{S^1, x_0})) $$

$$ \oplus \prod_{i=1}^l T_{X/C, \varphi_0(x_i)}/(T_{Z_i/C, \varphi_0(x_i)} + D\varphi_0(T_{C_0/O_0, x_i})) $$

is an isomorphism.
As we noted before, $H^0(C_0, \partial C_0; \varphi_0^* \Theta_{X/\mathbb{C}}, \varphi_0^*(J_{X_0} \Theta_{T^n})) \simeq \mathbb{N}_R$. Clearly its subspace $\mathbb{R} \cdot v$, where $v$ is a nonzero vector along the stopping edge, maps to a subspace of the right-hand side which projects to $D\varphi_0(J_{C_0}T_{S^1,x_0}) \subset \mathbb{T}_X/\mathbb{C}, \varphi_0(x_0)/(T_{Z_0/\mathbb{C}, \varphi_0(x_0)} + D\varphi_0(T_{S^1,x_0}))$ bijectively (recall that $C_0$ is a maximally degenerate curve associated to a tropical curve with a stop $(\Gamma_s, h)$).

So it suffices to prove the map

\begin{equation}
H^0(C_0, \partial C_0; N_{\varphi_0}, \varphi_0^*(\Theta_{T^n})/\Theta_{S^1}) \oplus \mathbb{N}_R/\mathbb{R} \cdot v
\end{equation}

is an isomorphism.

We have to interpret the left-hand side of (9) in terms of toric data. At a vertex $v \in \Gamma_s^{[0]}$, which is not the stop, the map $\varphi_0^* \Theta_{X/\mathbb{C}} \to N_{\varphi_0}$ induces a canonical surjection

$Q_v : N_C = \Gamma(C_v, N \otimes_{\mathbb{Z}} \mathcal{O}_{C_v}) \longrightarrow \Gamma(C_v, N_{\varphi_0} \otimes \mathcal{O}_{C_v})$

as in [18]. Here $C_v$ is the rational component of the maximally degenerate curve corresponding to $v$. Namely, consider the restriction to $C_v$ of the exact sequence

$0 \longrightarrow \Theta_{C_0/O_0} \longrightarrow \varphi_0^* \Theta_{X/\mathbb{C}} \longrightarrow N_{\varphi_0} \longrightarrow 0.$

When $v$ is a trivalent vertex, then

$\Theta_{C_0/O_0}|_{C_v} \cong \mathcal{O}_{C_v}(-1),$ 

so $\ker Q_v = H^0(C_v, \Theta_{C_0/O_0}|_{C_v}) = 0$. Thus, in this case the above map $Q_v$ is an isomorphism.

When $v$ is a divalent vertex, then

$\Theta_{C_0/O_0}|_{C_v} \cong \mathcal{O}_{C_v},$

and $\ker Q_v = H^0(C_v, \Theta_{C_0/O_0}|_{C_v})$ is naturally identified with

$L(h(E)) \otimes \mathbb{C} \subset N_C,$

where $E$ is an edge emanating from $v$ (the space $L(h(E))$ does not depend on the choice of an edge, because $v$ is divalent).

At the stop, there are natural surjections

$i_1 : N_R = \Gamma(D, \partial D; N \otimes_{\mathbb{Z}} (\mathcal{O}_D, \mathcal{O}_{S^1})) \longrightarrow \mathbb{N}_R/\mathbb{R} \cdot v$
and

\[ i_2 : iN_R = \Gamma(D, \partial D; N \otimes \mathbb{Z} (O_D, iO_{S^1})) \rightarrow \Gamma(D, \partial D; N_{\varphi_0} \otimes O_D, \varphi_0^*(\Theta_T)) / \Theta_{S^1}. \]

Here \( O_{S^1} \) is the sheaf of real analytic functions on \( \partial D = S^1 \), and \( (O_D, O_{S^1}) \) is the sheaf of sections of the Riemann-Hilbert bundle \( (C \times D^2, \mathbb{R} \times S^1) \) on \( (D^2, S^1) \). It is easy to see there is a natural isomorphism

\[ \ker i_1 \oplus \ker i_2 \cong L(h(E)) \otimes \mathbb{C} = \mathbb{C} \cdot v \subset N_C, \]

here \( E \) is the edge emanating from the stop.

Write

\[ \Gamma_1 = N_R / \mathbb{R} \cdot v \]

and

\[ \Gamma_2 = \Gamma(D, \partial D; N_{\varphi_0} \otimes O_D, \varphi_0^*(\Theta_T)) / \Theta_{S^1}. \]

Let \( v \) be the vertex adjacent to the stop (this is a trivalent vertex for general constraints). Take

\[ h'_v \in N_C \]

and a representative

\[ h_1 + h_2 \in N_R \oplus iN_R \]

of \( [h_1] + [h_2] \in \Gamma_1 \oplus \Gamma_2. \) Then \( h'_v \) and \( [h_1] + [h_2] \) glue if and only if

\[ h'_v - (h_1 + h_2) \in L(h(E)) \otimes \mathbb{C}. \]

Since \( L(h(E)) \otimes \mathbb{C} \) is isomorphic to \( \ker i_1 \oplus \ker i_2 \), the classes \( [h_1], [h_2] \) in \( \Gamma_1, \Gamma_2 \) are uniquely determined by \( h'_v \).

Similarly, for the adjacent pairs \( (w, w') \) of vertices other than the stop, the representatives

\[ h_w, h_{w'} \in N_C \]

of elements of \( \Gamma(C_w, N_{\varphi_0} \otimes O_{C_w}) \) and \( \Gamma(C_{w'}, N_{\varphi_0} \otimes O_{C_{w'}}) \) define a section of \( N_{\varphi_0} \) over \( C_w \cup C_{w'} \) if and only if

\[ h_w - h_{w'} \in L(h(E)) \otimes \mathbb{C}, \]

here \( E \) is the edge connecting \( w \) and \( w' \). Note that this argument is correct even when one of \( w, w' \) is a divalent vertex.
From this, we see that the left-hand side of (9) is isomorphic to
\[ \ker(\Phi : \text{Map}(\Gamma^0, N_C) \to \prod_{E \in \Gamma^0 \setminus \Gamma^1} N_C / \mathbb{C}u(\partial^+ E, E)) \],
here the map $\Phi$ is defined by
\[ h \mapsto (h(\partial^+ E) - h(\partial^- E))_E, \]
see Proposition 6.3 for the notation. Merging $\Phi$ with (9), we obtain the map of Proposition 6.3, tensored by $\mathbb{C}$. This proves that (9) is an isomorphism.

**COROLLARY 7.8.** For every log structure $\varphi_0 : C_0 \to X_0$ on $C_0$ given in Proposition 7.2, there is a unique family of stable maps from prestable disks satisfying the incidence conditions $Z$, over some neighborhood of the origin of $\mathbb{C}$.

**Proof.** By Proposition 7.7, there is a unique family over $O_{\infty}$ satisfying the incidence conditions. Since these constraints are algebraic, we can refer to Artin’s implicit function theorem [1] to ensure the existence of a family over some neighborhood of $0 \in \mathbb{C}$.

**Remark 7.9.** The argument here applies in general to the so-called non-superabundant cases [14]. Speyer [21] studied the superabundant case with genus one from non-Archimedean view point. In [15], we developed a general theory dealing with superabundant curves. There, the above argument is modified to give a local description of the moduli space of “smoothable” tropical or pre-log curves.

### 8. Classification of disks.

The main result in this section is Proposition 8.3, which claims that any holomorphic disk of the given degree in $X_t$, $t \neq 0$, satisfying the incidence and boundary conditions is contained in some family constructed in Corollary 7.8, when $|t|$ is small. Thus, as well as the maximally degenerate curves in $X_0$, holomorphic disks in $X_t$ for $t \neq 0$ are also classified by tropical curves. This is the key step in the proof of the main theorem (Theorem 9.3).

#### 8.1. Preliminary.

First we define some notions introduced by Ye [23] which are useful for our purpose. Let $\Sigma, \Sigma'$ be prestable bordered Riemann surfaces. A continuous surjective map $\varphi : \Sigma \to \Sigma'$ is called a node map if the following conditions hold.

(i) For each node $x \in \Sigma'$, $\varphi^{-1}(x)$ is either a node, a simple closed curve in the interior which is disjoint from nodes, or a simple arc which is disjoint from nodes and has its endpoints exactly on the boundary.

(ii) $\varphi$ is a diffeomorphism away from the curves or the points which are the preimages of the nodes.

Let $\{X, \omega, J\}$ be a symplectic manifold with a compatible almost complex structure. Let $\Sigma$ and $\widetilde{\Sigma}$ be prestable bordered Riemann surfaces.
Definition 8.1. (The $C^k$-topology on the space of prestable maps) Let $f: \Sigma \to X$ be a prestable map. For each $\epsilon > 0$, a metric on $\Sigma$ and a neighborhood $\tilde{U}$ of the nodes of $\tilde{\Sigma}$, a neighborhood $F$ of $f$ is defined as follows. A prestable map $\tilde{f}: \tilde{\Sigma} \to X$ belongs to $F$ if

1. There is a node map $\varphi: \Sigma \to \tilde{\Sigma}$.
2. $\| f - \tilde{f} \circ \varphi \|_{C^k} < \epsilon$ on $\Sigma$.
3. $\| j - \varphi^* \tilde{j} \|_{C^k} < \epsilon$ on $\varphi^{-1}(\tilde{\Sigma} \setminus \tilde{U})$.
4. $|\text{area}(f(\Sigma)) - \text{area}(\tilde{f}(\tilde{\Sigma}))| < \epsilon$.

Here $j$ and $\tilde{j}$ denote the complex structures of $\Sigma$ and $\tilde{\Sigma}$, respectively. The norms and areas are defined in terms of the given metrics on $X$ and $\Sigma$.

The above topology was introduced to describe a Gromov compactness result for pseudoholomorphic curves with boundary ([23], see also [20] for more standard closed curve case), which claims that a suitable sequence of pseudoholomorphic curves has a subsequence which converges in this topology. We do not need their compactness result, due to the fact that our disks can be analytically continued to maps from a rational curve, and then we can apply the result for rational tropical curves, as we explain below. We only need a claim in the obvious direction: If we have an analytic family $\varphi_t$ of pseudoholomorphic disks in a fixed symplectic manifold parametrized by a neighborhood $U$ of the origin of the complex plane (in other words, the family is a priori partially compactified), then $\varphi_t$ is close to $\varphi_0$ in the above topology, if $|t|$ is sufficiently small.

Now we explain the outline of the proof of Proposition 8.3. Let $\pi: \mathcal{X} \to \mathbb{C}$ be a toric degeneration with the central fiber $X_0$. We torically embed the family $\pi: \mathcal{X} \to \mathbb{C}$ to $\mathbb{P}^d \times \mathbb{C} \to \mathbb{C}$ for some integer $d$ ([22]). Here $\mathbb{P}^d$ is in general a weighted projective space (so is an orbifold), and we fix an orbifold metric on it (or, by blowing up $\mathbb{P}^d$, we can assume it is smooth).

According to [2], Theorem 5.3, any holomorphic disk in a smooth toric variety with Lagrangian boundary condition given by a torus fiber of the moment map can be lifted to $\mathbb{C}^A$ for some integer $A$, and the lift can be written as

$$z_i = c_i \cdot \prod_{j=1}^{\mu_i} \frac{z - \alpha_{i,j}}{1 - \alpha_{i,j} z},$$

with $c_i \in \mathbb{C}^*$, $\alpha_{i,j} \in \{z \in \mathbb{C}; |z| < 1\}$ and $\mu_i \in \mathbb{N}$. Here $z_i$, $i = 1, \ldots, A$ are the standard coordinates of $\mathbb{C}^A$, and $z$ is the standard coordinate of the disk $D$. In particular, such a map can be analytically continued to a map from a Riemann sphere.

Suppose we have a holomorphic disk in $X_t$ which satisfies the incidence and boundary conditions. If we have a rational curve by the analytic continuation as above, the result of [18] applies, which claims that such a rational curve is contained in some family which is the smoothing of a maximally degenerate rational curve in the central fiber.
On the other hand, cutting the maximally degenerate rational curve appropriately, we obtain a maximally degenerate disk with a log structure. By Corollary 7.8, such a disk can be smoothed. Thus, we have two disks in $X_t$:

- The original disk which was analytically continued to a rational curve. This rational curve is contained in a family of rational curves.
- A disk in a family of disks constructed in Corollary 7.8.

In other words, the original disk is contained in a family of rational curves and the latter disk is contained in a family of disks. Of course the latter family of disks can be also analytically continued to a family of rational curves, but since the incidence conditions we use to construct the families are different, we cannot immediately conclude that the original and the latter disks are the same.

But as these disks are deformations of the same maximally degenerate disk, we see that they are close in the $C^k$-topology as we mentioned above. Then an appropriate transversality result allows us to conclude that these disks coincide.

8.2. Classification of disks. Let $U$ be a sufficiently small disk with the center at $0 \in \mathbb{C}$. Let

$$U^* = U \setminus \{0\}$$

be the punctured disk. Take the product

$$C^* = D \times U^*$$

with the closed unit disk and let

$$\pi : C^* \to U^*$$

be the projection. Let $\{x^*_i\}_{i=1}^l$ be holomorphic sections of $\pi$ which do not intersect $\partial D$. Let $x^*_0$ be a real analytic section of $\partial D \times U^* \to U^*$.

Remark 8.2. Since the case when the Maslov index is two (see Example 2.1) is very easy (in fact, there is no degeneration in this case), we assume that the Maslov index is larger than two. This requires $l \geq 1$ to rigidify stable maps from the disk of the given degree.

Let

$$\phi : (C^*, \{x^*_i\}_{i=0}^l) \to \mathcal{X}$$

be an analytic family of stable maps over $U^*$ satisfying the Lagrangian boundary condition and the incidence condition given in Sections 5.4 and 5.5. Note that we always assume that the incidence varieties $\{Z_i\}$ are general. The following is the main result in this section.
Proposition 8.3. The map $\phi$ is the restriction of one of the families constructed in Corollary 7.8.

This is proved as a corollary of Proposition 8.12 below. To prove this, we need an appropriate transversality result. We note that such a transversality result is already proved for maximally degenerate disks and their deformation (the “transversality” map in the proof of Proposition 7.7). In particular, the parameter space for the incidence varieties $Z = \{Z_0, \ldots, Z_l\}$ can be locally identified with the parameter space for the stable maps. Since each $Z_i$ is the closure of a general orbit of a torus action, the parameter space for $Z$ is locally given by a neighborhood $\mathcal{N}$ of the origin of

$$\prod_{l=0}^{l} \mathbb{C}^n/(L(A_i) \otimes_{\mathbb{R}} \mathbb{C}),$$

here $A_i$ is the affine space which gives the constraint (see Section 5.5). We formulate the transversality as follows.

Lemma 8.4. Let $\varphi_0 : C_0 \to X_0$ be a maximally degenerate disk with a log structure satisfying boundary and incidence conditions, and let $\varphi_t : D \to X_t$ be the deformation of $\varphi_0$ given in Corollary 7.8. Then around $\varphi_t$, the space of stable maps from prestable disks in $X_t$ satisfying boundary and incidence conditions is a smooth manifold and an open subset $\mathcal{N}' \subset \mathcal{N}$, which does not depend on $t$ when $|t|$ is sufficiently small, gives a coordinate neighborhood.

The parametrization is valid until when the parameter hits the configuration of incidence varieties at which the maximally degenerate disk exhibits a singular behavior, which corresponds to the change of combinatorial type on the tropical side. In particular, the size of $\mathcal{N}'$ is bounded from below by some fixed constant (measured by fixing a metric on $\prod_{l=0}^{l} \mathbb{C}^n/(L(A_i) \otimes_{\mathbb{R}} \mathbb{C})$) and this ensures the independence of $\mathcal{N}'$ to $t$ in the lemma.

Let $\varphi_0 : C_0 \to X_0$ be a maximally degenerate disk as in the lemma. Let $\{x_0, \ldots, x_l\}$ be the marked points on $C_0$. In the space of maximally degenerate disks and their deformations, we consider a topology analogous to the $C^k$-topology (Definition 8.1). Due to the existence of marked points, we only use those node maps $\varphi : \Sigma \to \tilde{\Sigma}$ which satisfies

$$\varphi(x_i) = x'_i, \quad \forall i,$$

here $\{x_i\}$ and $\{x'_i\}$ are marked points of $\Sigma$ and $\tilde{\Sigma}$, respectively.

Definition 8.5. We call the topology on the space of prestable maps from pointed prestable bordered Riemann surfaces whose neighborhood is defined as in Definition 8.1 under the above restriction to the node map the pointed $C^k$-topology.
In our case, the $C^k$-norm is defined by the metric on $\mathbb{P}^d$ (recall that the family $X$ is embedded in $\mathbb{P}^d \times \mathbb{C}$). Also note that the space of disks in Lemma 8.4 is not the whole space of stable maps from pointed prestable disks, but is the subspace of it consisting of those disks which satisfy given boundary and incidence conditions. The pointed $C^k$-topology is defined on the whole space, and we induce the topology on the relevant subspace.

The transversality of Lemma 8.4, in particular the claim that $\mathcal{N}'$ is a manifold, means the following.

**Corollary 8.6.** There is a small positive constant $\delta$, which does not depend on $t$ when $|t|$ is sufficiently small, with the following property. Let $\varphi_t$ be an element of $\mathcal{N}'$. Take a stable map $\varphi'_t$ from a prestable disk to $X_t$ satisfying given boundary and incidence conditions. Then if $\varphi'_t$ satisfies the estimates of Definition 8.1(i), (ii), and (iii) with the constant $\epsilon$ replaced by $\delta$, $f$ and $\tilde{f}$ replaced by $\varphi_t$ and $\varphi'_t$ respectively, then $\varphi'_t$ is an element of $\mathcal{N}'$. Here the node map should be taken in the pointed sense as in Definition 8.5.

When $t \neq 0$, the domain is a disk $D$ (in particular, there is no node) and it does not have an automorphism as a pointed curve. By taking $x_0 = x'_0 = 1$ and $x_1 = x'_1 = 0$, we can canonically identify the domains of $\varphi_t$ and $\varphi'_t$. In this case, the condition that $\varphi'_t$ is contained in a neighborhood of $\varphi_t$ in the pointed $C^k$-topology reduces to simpler estimates:

**Lemma 8.7.** For $t \neq 0$ with sufficiently small $|t|$, a neighborhood system of $\varphi_t$ in $\mathcal{N}$ for the pointed $C^k$-topology is given by the sets

$$V_{\epsilon, \eta, t} = \{ \varphi'_t : D \to X_t \mid \| \varphi_t - \varphi'_t \|_{C^k} < \epsilon, \ d(x_i, x'_i) < \eta, \forall i \}.$$  

Here $d$ is a metric on $D$ induced from a flat metric on $\mathbb{R}^2$, $\epsilon$ and $\eta$ are small positive constants, and $\{x_i\}$, $\{x'_i\}$ are the marked points of the domain of $\varphi_t$, $\varphi'_t$, respectively. We fix an identification of the domains of $\varphi_t$ and $\varphi'_t$ as remarked above.

**Proof.** It is clear that for any neighborhood $U$ of $\varphi_t$ in the pointed $C^k$-topology, $V_{\epsilon, \eta, t}$ is contained in $U$ when we take $\epsilon$ and $\eta$ sufficiently small.

Conversely, fix $\epsilon$ and $\eta$. We must show that there is a small positive number $\epsilon'$ such that if there is a node map $\varphi : D \to D$ (since there is no node, this is in fact a diffeomorphism) such that $\varphi(x_i) = x'_i$ and

$$\| \varphi_t - \varphi'_t \circ \varphi \|_{C^k} < \epsilon', \quad \| j - \varphi^* j \|_{C^k} < \epsilon',$$

then $\varphi'_t$ belongs to $V_{\epsilon, \eta, t}$. Here $j$ is the complex structure on the disk. It suffices to prove that the $C^k$-norm of $\varphi - Id_{D^2}$ is small. Now the node map $\varphi$ is a diffeomorphism fixing $x_0$ and $x_1$. The equation

$$j - \varphi^* j = 0$$
is the equation for a (pseudo) holomorphic disk with boundary condition given by $S^1 \subset \mathbb{R}^2$, so in particular elliptic. Since we impose the condition fixing $x_0$ and $x_1$, the unique solution is the identity map. Then by standard elliptic estimate, one sees that $\| \varphi - Id_{D^2} \|_{C^k}$ is bounded from above by $\| j - \varphi^* j \|_{C^k}$ times some constant which does not depend on $\varphi$. This proves the lemma.  

\[ \square \]

**COROLLARY 8.8.** For $t \neq 0$ with sufficiently small $|t|$, there is a small positive number $\varepsilon$ such that, when a map $\varphi'_t : D \to X_t$ satisfying the boundary and incidence conditions further has the properties

$$\| \varphi_t - \varphi'_t \|_{C^k} < \varepsilon;$$

and

$$d(x_i, x'_i) < \varepsilon, \quad \forall i,$$

then $\varphi'_t$ is contained in $\mathcal{N}'$. We fix an identification of the domains of $\varphi_t$ and $\varphi'_t$ as in Lemma 8.7

**Proof.** This follows from Corollary 8.6 and Lemma 8.7.  

\[ \square \]

**Remark 8.9.** For this corollary, we only need the obvious direction of Lemma 8.7 (the topology generated by $V_{\varepsilon, \eta, t}$ is stronger than the pointed $C^k$-topology). We use the argument of proof of the another direction in the proof of Proposition 8.12.

**Remark 8.10.** On the set $\mathcal{N}'$, there are two natural topologies. One is the subspace topology induced from $\prod_{l=0}^L \mathbb{C}^n / (L(A_i) \otimes \mathbb{R} \mathbb{C})$, and the other is the restriction of the pointed $C^k$-topology. One can prove these topologies coincide.

**Remark 8.11.** We defined the $C^k$-norm using the metric on $\mathbb{P}^d$. But we can instead induce a metric on each component of $X_0$ from $\mathbb{P}^d$, and define the $C^k$-norm for each component of $C_0$ using this metric. Clearly these two norms give the same topology. A similar remark applies to the $t \neq 0$ case.

Now we return to the family $\phi$. We prove the following:

**PROPOSITION 8.12.** For any $t$ with sufficiently small $|t|$, the map $\phi_t = \phi|_{D \times \{t\}}$ is contained in one of the families constructed in Corollary 7.8.

Proposition 8.3 is an immediate corollary of this: Since the number of families constructed in Corollary 7.8 is finite, for any $t$ with sufficiently small $|t|$, the family to which $\phi_t$ belongs is the same. That is, the family $\phi$ itself is the restriction to a punctured disk of one of these families.

**Proof.** As we mentioned above, we can extend $\phi_t = \phi|_{D \times \{t\}}$ to a stable map $\tilde{\phi}_t : S^2 \to X_t$ from the sphere. We recall a result about transversality of a subset in a toric variety.
PROPOSITION 8.13. [18, Proposition 6.2] Let $X$ be a toric variety and $W \subset X$ be a closed complex analytic subset of codimension $> c$. We assume that no irreducible component of $W$ is contained in the toric boundary. Then there exists a toric blow-up $\Upsilon : \tilde{X} \to X$ such that the strict transform $\tilde{W}$ of $W$ under $\Upsilon$ is disjoint from any toric stratum of dimension $\leq c$.

By this proposition, possibly after blowing-up, we can assume the map $\tilde{\phi}_t$ is torically transverse. This proposition also assures us that we can assume each incidence condition $Z_i$ does not intersect a toric subvariety of $X$ of codimension larger than $\dim Z_i$. Hereafter, we assume this for any incidence condition.

By the blowing-up above, the family $\tilde{X}$ is modified, which we write by $\tilde{X}$. The incidence conditions $\{Z_i\}$ naturally lift and we write them by the same notation. Moreover, adding suitable families of subvarieties $\{Z_i'\} \subset \tilde{X}$ over $\mathbb{C}$ as incidence conditions to rigidify $\tilde{\phi}_t$, the main result of [18] (Theorem 8.3) claims that there is a family of rational curves over $\mathbb{C}$ containing $\tilde{\phi}_t$ which degenerates to a maximally degenerate rational curve in the central fiber $\tilde{X}_0$ of $\tilde{X}$ satisfying the incidence conditions $\{Z_i, Z_i'\}$. We write this by $\tilde{\phi}_0$.

Note that in $\tilde{\phi}_0$, the component incident to $Z_0 \cap \tilde{X}_0$ intersects the toric divisor at two points, by definition of maximally degenerate curve. Namely, this component is the closure of the orbit of a one dimensional subtorus of the torus acting on $\tilde{X}_0$. So the intersection of this component and a Lagrangian torus fiber is (if non-empty) a circle, and this circle splits the maximally degenerate curve into two maximally degenerate disks.

Since $\tilde{\phi}_0$ is contained in a family degenerating to $\tilde{\phi}_0$, $\tilde{\phi}_t$ is in a small neighborhood of $\tilde{\phi}_0$ in the pointed $C^k$-topology. When $t$ is sufficiently small, the $C^k$-distance between them is bounded from above by some quantity governed by $t$, at least away from the inverse image of the nodes under the node map. In particular, we can assume the distance is much smaller than the constant $\epsilon$ in Corollary 8.8. Then one of the maximally degenerate disks in the last paragraph has the same degree as $\phi_t$. This map descends to a maximally degenerate disk in $X_0$, and is incident to $\{Z_i \cap X_0\}$. We write this by $\phi_0$.

Since $\tilde{\phi}_0$ is the degenerate limit of a family, it has a natural log structure. This induces a log structure on $\phi_0$. Moreover, the marked points for $\tilde{\phi}_0$ naturally give rise to the marked points $\{x_{0,i}\}_{i=0}^1$ of the domain $\tilde{C}_0$ of $\phi_0$, as it is a subset of the domain $\tilde{C}_0$ of $\tilde{\phi}_0$, a marked curve.

Then Corollary 7.8 uniquely determines a deformation of $\phi_0$. Let $\phi'_t : D \to X_t$ be the disk in this family with parameter $t$. Since the log structures we use to deform the curves are the same, there is a diffeomorphism $G$ of $\mathbb{P}^d$ such that

$$G(\phi_t(D)) = \phi'_t(D), \quad \|G - I_d\|_{C^k} < \epsilon'',$$

here $\epsilon'$ is a constant which becomes zero when $|t|$ becomes zero, and is much smaller than the constant $\epsilon$ in Corollary 8.8. Since $G$ has small $C^k$-norm, the push
forward of the complex structure on $\phi_t(D)$ is nearly integrable. That is, there is a diffeomorphism $h$ of $\phi_t'(D)$ (or a diffeomorphism of the normalization of it when $n = 2$) such that

$$h \circ G \circ \phi_t : D \longrightarrow \mathbb{P}^d$$

is a holomorphic map. Arguing as in the proof of Lemma 8.7, we can take the $C^k$-norm of $h - Id_{D^2}$ very small. Since $h \circ G \circ \phi_t$ has the same image as $\phi_t'$, if we forget the marked points, $h \circ G \circ \phi_t$ defines the same stable map as $\phi_t'$.

Since $G$ and $h$ have small $C^k$-norm, and marked points are away from the nodes, there is a unique point $y_i$ in $Z_i \cap \phi_t'(D)$ such that

$$d(x_i, (h \circ G \circ \phi_t)^{-1}(y_i)) < \epsilon'$$

holds for each $i = 0, \ldots, l$. If we take $(h \circ G \circ \phi_t)^{-1}(y_i)$ as the marked point for $h \circ G \circ \phi_t$, by construction the stable map

$$h \circ G \circ \phi_t : (D, (h \circ G \circ \phi_t)^{-1}(y_0), \ldots, (h \circ G \circ \phi_t)^{-1}(y_l)) \longrightarrow (X_t, Z_t)$$

must be the same as

$$\phi_t : (D, x'_0, \ldots, x'_l) \longrightarrow (X_t, Z_t).$$

Then by definition of $G$ and $h$, we have

$$\| \phi_t - \phi_t' \|_{C^k} < \epsilon.$$ 

Also, we have

$$d(x_i, x'_i) < \epsilon.$$ 

Thus, the conditions of Corollary 8.8 hold, so $\phi_t$ is contained in $\mathcal{M}'$. Since both $\phi_t$ and $\phi_t'$ satisfy the incidence condition $Z_t$, we conclude that they coincide. \[\square\]

9. Counting invariant of disks. We will define two numbers by counting two objects. One is the number of tropical curves of genus zero with one stop and given degree $\Delta$, which match a set of constraints $A$ (we count these with weights). The other is the number of holomorphic disks in a toric variety which intersect the toric divisors in a way specified by the degree $\Delta$ (Definition 5.10), and also intersect incidence varieties $Z = \{Z_0, \ldots, Z_l\}$ (see Section 5.5). We claim that these numbers coincide.

9.1. Main theorem. We will count the number of tropical curves of genus zero with one stop with the following properties.

(i) The degree is $\Delta$.

(ii) They match the set of constraints $A$ of codimension $d = (d_0, \ldots, d_l)$ with $\sum_{i=0}^l d_i = n + |\Delta| - 2$. 

Tropical curves must be counted with weights. The weight is given by

$$W(\Gamma_s, E, h, A) = w(\Gamma_s, E) \cdot D(\Gamma_s, E, h, A) \cdot \prod_{i=0}^{l} \delta_i.$$  

(10)

Here $\delta_i$ is the index of the lattice

$$\mathbb{Z}u(\partial - E_i, E_i) + L(A_i) \cap N$$

in

$$\left(\mathbb{Q}u(\partial - E_i, E_i) + L(A_i)\right) \cap N$$

(see Remark 5.8 of [18]).

Remark 9.1. The integer $\delta_i$ is characterized by the property that the number of intersection points between the incidence variety $Z_i$ and the corresponding rational component of $C_0$ is $w_i \delta_i$ (counted with multiplicity). Here $w_i = w(E_i)$ is the weight of the corresponding edge of the tropical curve.

In general, this counting number depends on the choice of $A$. However, we have the following.

Proposition 9.2. For any general set of constraints $A$, there is a neighborhood $W$ of $A$ in $\mathbb{A}$ (see Proposition 3.6) such that the number of genus zero tropical curves with one stop (counted with the weight $W(\Gamma_s, E, h, A)$), whose degree is $\Delta$, and which match the given constraints, is constant on $W$.

Proof. Extend the stopping edge of a genus zero tropical curve with one stop, whose degree is $\Delta$ and which matches $A$. We obtain a rational tropical curve satisfying the set of constraints $A$. By Definition 3.5, $A$ is general as a set of constraints for rational tropical curves. By the proof of [18], Proposition 2.4, the type of the tropical curves matching $A$ will not change when we move $A$ slightly. This, combined with the invariance of the counting numbers of the rational tropical curves ([18], Corollary 8.4), proves the proposition. \qed

Let $\mathcal{P}$ be an integral polyhedral decomposition of $N_{\mathbb{Q}}$ which contains all the tropical curves with one stop, matching $A$. (If $\mathcal{P}$ is not integral, we can make it so by replacing $N$ by $\frac{1}{d}N$ for some integer $d$. This corresponds to a base change in the toric degeneration.) Let $\Sigma_{\mathcal{P}}$ be the asymptotic fan. Let $\pi : \mathcal{X} \rightarrow \mathbb{C}$ be the toric degeneration defined by the fan $\tilde{\Sigma}_{\mathcal{P}}$.

Theorem 9.3. There is a neighborhood $U$ of $0 \in \mathbb{C}$ such that the following two numbers are equal.

1. The weighted number of genus zero tropical curves with one stop and the fixed degree $\Delta$, which match the set of constraints $A$. Here the weight of a tropical curve is given by $W(\Gamma_s, E, h, A)$ defined above.
The number of torically transverse holomorphic disks in the fibers of \( \pi^{-1}(U \setminus \{0\}) \), of the given degree, satisfying the incidence condition \( Z \).

**Proof.** As in the proof of [18], Theorem 8.3, step 3, we want to prove that both numbers are equal to the number of maximally degenerate disks satisfying the incidence condition and equipped with log structures.

In Section 7, we constructed maximally degenerate disks from tropical curves with one stop, and also constructed logarithmic structures on them, and we saw that the number of compatible log structures equaled the marked total weight \( w(\Gamma_s, E) \) of the tropical curve (Proposition 7.2).

On the other hand, there are \( \mathcal{D} \) maximally degenerate disks associated to the tropical curve (Proposition 6.3), and for each maximally degenerate disk there are \( w_i \delta_i \) intersections with the incidence variety \( Z_i \), as in Remark 9.1.

The product of these weights gives \( w(\Gamma_s, E) \cdot \mathcal{D} \cdot \prod_{i=0}^{l} w_i \delta_i \), but taking the automorphisms into account, we have \( \mathcal{W}(\Gamma_s, E, h, A) \) isomorphism classes of maximally degenerate disks.

We have also proved that each of these logarithmic curves can be deformed to give a family of stable maps defined over a neighborhood \( U \) of \( 0 \in \mathbb{C} \) (Corollary 7.8).

These results give

\[
\sum \mathcal{W}(\Gamma_s, E, h, A)
\]

families of torically transverse stable maps from a disk. Here the sum is taken over the isomorphism classes of genus zero \( l \)-marked tropical curves of degree \( \Delta \) with one stop matching \( A \).

By Proposition 8.3, any disk satisfying the conditions is contained in one of these families, for sufficiently small \( U \). So the theorem is proved. \( \square \)

**Definition 9.4.** Let \( N_D(\Delta, A) \) be the number determined by Theorem 9.2. We call this as the disk counting number of type \( (\Delta, A) \).

On the tropical side, this number is invariant under small perturbation of the set of constraints \( A \) (Proposition 9.1). Let \( O \) be a neighborhood of \( A \) in the space \( \mathbb{A} \) of constraints. On the holomorphic disk side, this number is invariant in some neighborhood of the origin \( U \subset \mathbb{C} \), with the incidence conditions \( Z' \) determined by any \( A' \in O \). It is also invariant under small changes of \( Z \) by the actions of \( \mathbb{G}(N) \), which is the same as changing \( P_j, j = 0, \ldots, l \) in Proposition 6.3.

**9.2. Examples of calculations of the invariant.**

**9.2.1. Disks in \( \mathbb{C}P^2 \) of Maslov index four.** This is a very basic example, exhibiting the dependence of the invariant on the places of the constraints. The constraints are given by two points, both on \( \mathbb{R}^2 \) (tropical side) and on \( \mathbb{C}P^2 \) (complex
curve side). One is for the boundary marked point (or the stop), and the other is for the interior marked points. On the tropical side, it looks like the following picture (Figure 4).

The black point $p_0$ is the stop and the circle $a$ is the constraint for the interior marked point. When the circle is sufficiently high above the black point, then there is one trivalent tropical curve with one stop. This means that there is a unique smooth disk satisfying the corresponding incidence conditions.

When $a$ moves downwards, and intersects the half line of slope $(1, 1)$ emanating from the stop, then

$$A = \{ p_0, a \}$$

is not general in the sense of Definition 3.5. In this case, the stopping edge degenerates as in the figure. On the complex curve side, a disk bubble occurs on the central fiber, and the existence of a disk satisfying the incidence conditions near the central fiber depends on the places of the points $x, p \in X$ which are used to determine the incidence variety and the boundary condition (Sections 5.4 and 5.5).

When $a$ moves downwards further, tropical and holomorphic curves satisfying the constraints vanish.

**9.2.2. Count by lattice paths.** When the constraints are placed in a specific position, we can compute the invariant by Mikhalkin’s lattice path count. Recall that Mikhalkin’s lattice path count is applied to count the number of tropical curves with the constraints

$$\{ p_1, p_2, \ldots \}$$

placed on a line of general slope, with

$$d(p_1, p_2) \ll d(p_2, p_3) \ll \cdots,$$

where $d(p, q)$ is the Euclidean distance between $p, q \in \mathbb{R}^2$. 
In our case too, we consider the constraints satisfying this condition, but note that the constraint for the stop \((A_0\) in the notation of Definition 3.1) is also included.

Let \((\Gamma_s, h)\) be a tropical curve with one stop \(P\), satisfying the constraints above. Let \(E\) be the edge of \(\Gamma_s\) containing the stop. Let \(v \in \mathbb{Z}^2 \cong N\) be the primitive integral vector in the direction of \(h(E)\) emanating from \(h(P)\). Let \[
\Delta' = \Delta + \{-v\}
\]
be the map from \(\mathbb{Z}^2 \setminus \{0\}\) to \(N\) which is the sum of \(\Delta\) and \(\{-v\}\) \(\in \text{Map}(\mathbb{Z}^2 \setminus \{0\}, N)\). Here \(\{-v\}\) means the map \(\mathbb{Z}^2 \setminus \{0\} \to N\) which takes the value 1 on \(-v\) and 0 otherwise.

We count the number of lattice paths on the polytope \(\mathcal{P}\) dual to the degree \(\Delta'\), with the following condition. Let \(p_1, \ldots, p_r\) be the constraints, and suppose that \(A_0 = p_i\) is the constraint for the stop. The direction \(-v\) of the stopping edge determines the dual edge \(\hat{-v}\) of \(\mathcal{P}\). Then only the lattice paths whose \(i\)th step is on \(\hat{-v}\) contribute to \(N_D(\Delta, A)\). So we have the following:

**PROPOSITION 9.5.** In the situation above, \(N_D(\Delta, A)\) is given by the number of lattice paths on the polytope \(\mathcal{P}\) dual to the degree \(\Delta'\), counted with the weight defined in [14], but these paths should satisfy the condition that the \(i\)th step is on \(\hat{-v}\).

**9.2.3. Relation to closed curves.** Let \((\Gamma_s, h)\) be a tropical curve with a stop. Let \(p \in N\) be the image of the stop. Take \(E\) and \(\Delta'\) as above. Extending \(h(E)\) to infinity, we obtain a tropical curve of genus zero of degree \(\Delta'\) without a stop. Let \(\tilde{E}\) be the extended edge and \(\tilde{\Gamma}\) be the extended tropical curve. The following is clear from the construction.

**PROPOSITION 9.6.** \(N_D(\Delta, A) \leq N^{alg}(\mathbf{L})\)

Here \(N^{alg}(\mathbf{L})\) is the number of rational curves of degree \(\Delta'\) which satisfy the incidence conditions defined by \(\mathbf{L}\) (= linear subspaces of \(N_\mathbb{Q}\) parallel to \(A\)) introduced in Definition 8.2 of [18]. Note that this proposition depends heavily on tropical geometry. It seems difficult to directly find such a bound for the number of holomorphic disks.

More generally, consider a rational tropical curve \((\Gamma, h)\) in \(\mathbb{R}^2\) and let \(E \in \Gamma^{[1]}\) be the \(i\)th marked edge. Let \(r \in E\) be the divalent vertex which is the inverse image of the intersection \(h(E) \cap p_i\), here \(p_i\) is the \(i\)th constraint. Splitting \(\Gamma\) at \(r\), we obtain two tropical curves with a stop \((\Gamma_{s,1}, h|_{\Gamma_{s,1}})\) and \((\Gamma_{s,2}, h|_{\Gamma_{s,2}})\).

Conversely, given two tropical curves of genus zero with one stop whose stopping edges \(E_1, E_2\) have the opposite directions, we can glue them so that we obtain a rational tropical curve without a stop.

The argument in this paper shows that we can perform a similar process for holomorphic disks to obtain closed curves, though in this case we cannot glue two disks directly.
In Figure 5, we start from the upper left, which is the picture of the amoebas of two holomorphic disks $\phi_1, \phi_2$, with the boundary mapped to the same Lagrangian torus fiber $\pi^{-1}(p)$. Here $\pi : X \to \Delta$ is the moment map of a toric surface $X$, and $p$ is a point in the interior of $\Delta$. Moreover, we assume that their degrees $\delta_1, \delta_2$ sum up to the opposite directions:

$$
\sum_{v \in N \setminus \{0\}} \delta_1(v) v = - \sum_{v \in N \setminus \{0\}} \delta_2(v) v,
$$

as vectors in $N \setminus \{0\}$. Apparently we cannot always glue such disks directly. But through tropical geometry, we can justify this naive gluing process as follows.

Namely, we “tropicalize” them, recording the logarithmic structures they have (the lower left picture). In the language of complex curves, this corresponds to considering the maximally degenerate curves (with a logarithmic structure). In this tropicalized situation, two disks can be readily glued into one rational maximally degenerate curve with a logarithmic structure (the lower right picture).

Finally, we smooth back the maximally degenerate curve using the logarithmic structure, and obtain a rational holomorphic curve (the upper right picture).

These rational curves have the degree which is the sum of the degrees of the two disks and satisfy the incidence conditions which are the sum of the incidence conditions for the two disks. All the rational curves associated to the tropical curve in the lower right picture of Figure 5 are obtained in this way (over the open set $U$ of Theorem 9.2).
Although we argued in two dimensional ambient space here, this process can be done in any dimensional ambient space.

9.2.4. Constraints in asymptotic position. Let \( p \in \mathbb{R}^n \) be a fixed point. Let

\[
\ell = \{ \ell(a) = p + av \mid a \in \mathbb{R}_{\geq 0} \}
\]

be the half line emanating from \( p \) in the direction \( v \in \mathbb{Q}^n \). Let

\[
Y_{a,\theta}, \quad a \in \mathbb{R}_{\geq 0}, \quad \theta \in \left(0, \frac{\pi}{2}\right)
\]

be the cone defined by

\[
Y_{a,\theta} = \{ x \in \mathbb{R}^n \mid 0 \leq \angle(x\ell(a)\ell(a+1)) \leq \theta \}.
\]

We consider constraints satisfying the following condition.

**Definition 9.7.** We say that a set of constraints \( A = (A_0, \ldots, A_l) \) satisfies the condition \((a, \theta)\) if \( A_0 = p \), and for each \( i > 0 \), \( A_i \cap Y_{a,\theta} \neq \emptyset \).

We specialize to the two dimensional case. Then \( A \) is a sequence of points. Let \( \langle , \rangle \) be the Euclidean inner product on \( \mathbb{R}^2 \).

**Proposition 9.8.** Suppose that the stopping edge has the direction \(-v\) (emanating from the vertex other than the stop). Suppose also that any direction \( w \) of the other unbounded edges (emanating from their unique adjacent vertices) satisfies \( w = -v \), or the condition \( \langle v, w \rangle \geq 0 \). Then there are positive numbers \( \eta > 0 \) and \( \epsilon \ll 1 \) such that if a general set of constraints \( A \) satisfies the condition \((a, \theta)\) with \( a \geq \eta \) and \( \theta \leq \epsilon \), then the number of genus zero tropical curves of degree \( \Delta \) with one stop which match \( A \) is independent of the choice of \( A \).

**Proof.** By the results of [14, 18], the number of rational tropical curves of fixed degree which match the set of constraints \( A \) does not depend on the choice of \( A \). So, to prove the proposition, it suffices to prove the following claim.

**Claim 9.9.** There is no rational tropical curve of degree \( \Delta' = \Delta \cup \{-v\} \), in the notation of Section 9.2.2, which matches \( A \), with the extra condition that an unbounded edge of the direction other than \(-v\) or a bounded edge matches \( p \).

**Proof.** Suppose that there is a tropical curve which matches \( A \) and some unbounded edge \( l \) passes \( p \), whose direction \( \vec{l} \) satisfies

\[
\langle v, \vec{l} \rangle \geq 0.
\]
Let $p_1$ be the vertex adjacent to $l$. Let $H$ be the line through $p$, perpendicular to $v$. Let

$$\mathbb{R}^2 = H^+ \cup H^-$$

be the decomposition of the plane into half planes induced by $H$.

By the assumption, $A$ is contained in $H^+$ or $H^-$. Assume $A$ is contained in $H^+$. Then at least one of the edges other than $l$, which emanates from $p_1$, is contained in the interior of $H^-$. In other words, it has the direction $w_1$ with

$$\langle v, w_1 \rangle < 0.$$

Let $l_1$ be this edge and if $l_1$ is not an unbounded edge, let $p_2$ be the vertex adjacent to $l_1$ other than $p_1$. Again one of the edges emanating from $p_2$ has the direction $w_2$ which satisfies

$$\langle v, w_2 \rangle < 0.$$

Repeating this process, we have to meet an unbounded edge whose direction $w$ satisfies

$$\langle v, w \rangle < 0.$$

By the assumption, this edge must have the direction $-v$, and matches no constraint. In particular, this edge is unmarked. Denote this edge by $F$. □

Thus, we have shown that when an unbounded edge $l$ with $\langle v, \vec{l} \rangle \geq 0$ matches $p$, then the tropical curve has the following property:

There is an unmarked, unbounded edge $F$ of the direction $-v$.

Let $E$ be the set of unbounded edges with this property.

CLAIM 9.10. A tropical curve in $\mathbb{R}^2$ which has a part like this is not rigid. That is, there is a continuous deformation keeping the matchings.

Proof. Suppose that $F$ is placed on the left of $p$. Then let $F_0$ be the leftmost edge in the set $E$. Let $q_1$ be the vertex adjacent to $F_0$. There are two edges, other than $F_0$, emanating from $q_1$. One of these edges, say $F_1$, has the direction to the left of $p$. That is, when we decompose the plane by the affine line containing $F_0$, which we write as

$$\mathbb{R}^2 = H^+_{F_0} \cup H^-_{F_0},$$

then $F_1$ is contained in the half plane which does not contain $p$.

Such an edge never satisfies $\langle v, w_1 \rangle < 0$ where $w_1$ is the direction (from $q_1$) of the edge $F_1$. This is because if there is such an edge, it is easy to see that there is an unmarked, unbounded edge of direction $-v$ on the left of $F_0$. 

Similarly, one of the edges adjacent to $F_1$, say $F_2$, has the direction to the left of $p$ in the sense as above. If $\theta$ is sufficiently small, $F_2$ is contained in $\mathbb{R}^2 \setminus Y_{a, \theta}$. The direction $w_2$ of the edge $F_2$ will never satisfy $\langle v, w_2 \rangle < 0$, again by the same reason as above. Continuing this process, we have a family of unmarked edges $(F_0, F_1, \ldots, F_k)$ with $F_0$ and $F_k$ unbounded and the others bounded. We can deform this tropical curve by parallel transforming $(F_0, F_1, \ldots, F_k)$ without changing the matchings (see Figure 6). This means that this tropical curve is not rigid.

When $F$ is to the right to $p$, do the same by replacing all “left” by “right” in the above argument. When $F$ is directly below $p$, do the same by using either left or right. \hfill \square

Since the set of constraints $\mathcal{A}$ is general, there is no non-rigid tropical curve matching $\mathcal{A}$, by [18], Proposition 2.4. So there is no tropical curve of degree $\Delta'$ matching $\mathcal{A}$ such that an unbounded edge of the direction other than $-v$ matches $p$. For the case when a bounded edge matches $p$, we can do the same by using one of the vertices adjacent to this edge. \hfill \square

**Corollary 9.11.** Suppose that the set of constraints $\mathcal{A}$ satisfies the condition $(a, \theta)$. Then, if the degree $\Delta$ and $v$ satisfy the assumption of Proposition 9.8, the equality

$$N_D(\Delta, \mathcal{A}) = N_{alg}(L)$$

holds.

When we assume that the rational tropical curves of degree $\Delta'$ satisfying the set of constraints $\mathcal{A}$ do not have self-intersection (in fact, this condition does not depend on the place of $\mathcal{A}$, and if one of the tropical curves matching $\mathcal{A}$ satisfies this condition, then all the tropical curves satisfy it), we can prove a stronger result. The proof below owes to the referee, replacing the original longer and messier proof.
PROPOSITION 9.12. Assume that the rational tropical curves of degree $\Delta'$ matching $A$ do not have a self-intersection. Then if the set of constraints $A$ satisfies $(a, \theta)$ with $a \geq \eta$ and $\theta \leq \epsilon$ as in Proposition 9.8, then the equality

$$N_D(\Delta, A) = N_{alg}(L)$$

holds. Note that the condition $\langle w, v \rangle > 0$, assumed in Proposition 9.8, is not imposed here.

Proof. It suffices to show the following: Any rational tropical curve of degree $\Delta'$ matching $A$ satisfies the property that the edge matching $A_0 = p$ is an unbounded edge of the direction $-v$.

Note that since the degree is $\Delta'$, there is at least one unbounded edge whose direction is $-v$. Since there are only finite number of types of the given degree (Proposition 3.3), we can assume that the directions of the edges, including bounded edges, satisfy one of the following:

- Parallel to $\pm v$.
- Let $C_{\theta}$ be the cone in $\mathbb{R}^2$ defined by

$$C_{\theta} = \{ x \in \mathbb{R}^2 \mid 0 \leq |\angle(xov)| \leq \theta \},$$

where $o \in \mathbb{R}^2$ is the origin. If the direction is not parallel to $\pm v$, then it is not contained in $\pm C_{\theta}$.

Let $(\Gamma, h)$ be a rational tropical curve of degree $\Delta'$ matching $A$. Call a connected component of $\mathbb{R}^2 \setminus h(\Gamma)$ negatively unbounded if it is preserved under translation in the direction of $-v$. It is easy to see the following:

CLAIM 9.13. (1) Any unbounded edge in the direction of $-v$ separates two such components, so there are at least two negatively unbounded components.

(2) Conversely, any negatively unbounded component contains an unbounded edge of direction $-v$ in its boundary.

The latter claim follows from the existence of an unbounded edge of direction $-v$ in $h(\Gamma)$.

Take one of the negatively unbounded components. The boundary of such a component consists of a chain of edges, the first and last of which are unbounded, and at least one of them points downwards, that is,

$$\langle w, v \rangle < 0,$$

where $w$ is the direction of the unbounded edge. Note that any vertex in this chain is a vertex of $h(\Gamma)$, that is, not the transverse intersection of the edges, because of the assumption that there is no self-intersection. Let $e$ be such an unbounded edge. Tracing the chain from the infinity of $e$, the directions of all the bounded
edges have the same sign with respect to \((1,0)\): For the direction \(w'\) of any of the bounded edges, simultaneously
\[
\langle w', (1,0) \rangle > 0,
\]
or
\[
\langle w', (1,0) \rangle < 0
\]
holds.

If none of \(A\) lies on the boundary chain, then one can deform \(h\) as in Figure 6, so the tropical curve is not rigid, a contradiction.

So assume that one of the constraints \(A_i\) lies on the boundary chain of a negatively unbounded component. Then by definition of \(C_\theta\), any edge of \(h(\Gamma)\) which intersects the interior of the downward cone
\[
A_i + (-C_\theta)
\]
must be an unbounded edge of direction \(-v\), since otherwise there will be a self-intersection.

There are two cases:
- \(i = 0\), so \(A_i = p\).
- \(i \neq 0\). In this case, since \(A\) satisfies the assumption \((a, \theta)\), \(A_0 = p\) lies within \(A_i + (-C_\theta)\).

In the second case, \(p\) must lie on an unbounded edge of direction \(-v\) by the above remark. So in this case the proposition is proved.

In the first case, suppose moreover \(p\) does not lie on an unbounded edge of direction \(-v\). Then by definition of \(C_\theta\) and the above inequality concerning the direction of the bounded edges of the boundary chain, it is easy to see that the unbounded edge of direction \(-v\) of Claim 9.13 (any one of them, if there are two) of the chosen negatively unbounded component does not intersect \(A\). Similarly, the closure of the neighboring negatively unbounded component sharing the unbounded edge of direction \(-v\) with the chosen component does not intersect \(A\), which makes the tropical curve non-rigid. This proves the proposition. 

\[\square\]

Remark 9.14. The arguments in Sections 9.2.3 and 9.2.4 are valid for a tropical curve of any genus. So if we establish a correspondence between genus \(g\) tropical curves with a stop and genus \(g\) holomorphic curves with one boundary component, the claims as Proposition 9.6 and Corollary 9.11 are also valid. We hope to investigate this case, together with the cases when there are multiple boundary components somewhere else.
9.2.5. An example of counting in higher dimension. Let us take general points
\[ p \in \mathbb{Q}^2, \ p_1, \ldots, p_l \in \mathbb{Q}^2, \]
a degree function
\[ \Delta_0 : \mathbb{Z}^2 \setminus \{0\} \rightarrow \mathbb{N}, \]
and a vector
\[ v \in \mathbb{Z}^2 \setminus \{0\}, \]
so that the point set \( \{p, p_1, \ldots, p_l\} \) satisfies the condition \((a, \theta)\), and \( \Delta_0 \) and \( v \) satisfy the assumption of Proposition 9.8 or 9.12 (by taking \( \Delta' = \Delta_0 + \{-v\} \) in the notation there. This \( \Delta' \) has nothing to do with the one just below).

Let \( \Delta' \) be any degree function \( \Delta' : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N} \) on a rank \( n \) lattice \( N \). Fix a basis \( \{e_1, \ldots, e_n\} \) of \( N \). Let
\[ \pi : N \rightarrow \mathbb{Z}^2 \]
be the surjection of lattices induced by the decomposition
\[ N = \mathbb{Z}\langle e_1, e_2 \rangle \oplus \mathbb{Z}\langle e_3, \ldots, e_n \rangle \]
and \( \pi_Q : N_Q \rightarrow \mathbb{Q}^2 \) be its extension of scalars.

Let \( w = mu \) be a non-zero vector in \( N \), where \( u \) is primitive and \( m \) is a positive integer. The image \( \pi(u) \in \mathbb{Z}^2 = \mathbb{Z}\langle e_1, e_2 \rangle \) may not be primitive, and write it as \( \pi(u) = n(u)\bar{u} \), here \( \bar{u} \in \mathbb{Z}^2 \) is primitive and \( n(u) \) is a positive integer. So for \( w = mu \), the image by \( \pi \) is given by \( \pi(w) = mn(u)\bar{u} \).

Let \((\Gamma, h)\) be a tropical curve in \( N_Q \) and let \((V, E)\) be a flag of \( \Gamma \). Recall that we write by \( u_{(V, E)} \) the primitive integral vector in the direction of the image of the flag \((V, E)\). The following is clear from the definition.

**Lemma 9.15.** Consider the map
\[ \bar{h} = \pi_Q \circ h : \Gamma \rightarrow \mathbb{Q}^2. \]
Let \( E \) be an edge of \( \Gamma \) with weight \( w_E \). Define the integer \( \bar{w}_E \) by
\[ \bar{w}_E = w_E \cdot n(u_{(V, E)}), \]
where \((V, E)\) is a flag containing \( E \). Then, the pair \((\Gamma, \bar{h})\) with the weight of the edges of \( \Gamma \) given by \( \bar{w}_E \), is a tropical curve.
In particular, the map $\Delta'$ induces a degree function $\Delta^\pi$ on $\mathbb{Z}^2 \setminus \{0\}$ by

$$\Delta^\pi(mu) = \sum_{\pi(w) = mu} \Delta'(w).$$

Here the sum is taken over all vectors $w \in N \setminus \{0\}$.

In general, the tropical curve $(\Gamma, \bar{h}, \bar{w})$ can be highly degenerate. For example, the image $\bar{h}(\Gamma)$ can be a point. So we make use of the following construction.

Take $\Delta'$ so that:

- $\Delta'$ satisfies $|\Delta'| = l + 2$,

- $\Delta^\pi$ satisfies

$$\Delta^\pi = \Delta_0 + \{-v\}.$$ 

Let $\{p, p_1, \ldots, p_l\}$ be the point set as above. Let $A_0$ be a general point in $\pi_Q^{-1}(p)$ and take $A_i = \pi^{-1}(p_i)$, $i = 1, \ldots, l$. Then

$$A = \{A_0, \ldots, A_l\}$$

defines a set of constraints of codimension $n + l - 1$ in $N_Q$. It is easy to see the following.

**Lemma 9.16.** Let $\{p, p_1, \ldots, p_l\}$ be general for $\Delta^\pi$ (see [18], Definition 2.3 or Definition 3.5 of this paper) and let $(\Gamma, h)$ be a rational tropical curve in $N_Q$ of degree $\Delta'$, matching the constraint $A$. Then the following statements hold.

- For each edge $E \subset \Gamma$, the restriction $\bar{h}|_E$ is an embedding.
- For different edges $E, E' \subset \Gamma$, $\bar{h}(E) \cap \bar{h}(E')$ is at most one point.

Clearly, the same holds for $h$.

**Proof.** This follows from [18], Proposition 2.4. \qed

**Corollary 9.17.** The graph $\Gamma$ and the images $h(\Gamma)$ and $\bar{h}(\Gamma)$, all have the same number of edges, and the number of unbounded edges is also the same.

When $\Delta_0$, $v$, and $p, p_1, \ldots, p_l$ satisfy the assumption of Proposition 9.8, then there is a unique vector $w$ in the support of $\Delta'$ with

$$\pi(w) = -v.$$ 

Let

$$\Delta : N \setminus \{0\} \to \mathbb{N}$$
be the degree function uniquely determined by the condition
\[ \Delta' = \Delta + \{w\}. \]

We consider tropical disks in \( \mathbb{N}_Q \) of degree \( \Delta \), which match the constraint \( A \). By Proposition 3.6, the number of these tropical disks is finite. Then, by Proposition 9.8, we have the following.

**Proposition 9.18.** Suppose \( \Delta_0, v, \) and \( p, p_1, \ldots, p_l \) satisfy the assumption of Proposition 9.8. Then the equality
\[ N_D(\Delta, A) = N_{0, \Delta'}(L) \]
holds.

**Proof.** In [18], Theorem 8.3, it is shown that the number \( N_{0, \Delta'}(L) \) is the same as the weighted count \( N_{0, \Delta'}^{\text{trop}}(A) \) of rational tropical curves of degree \( \Delta' \), matching \( A \). By Lemma 9.15, such a tropical curve projects to a rational tropical curve in \( \mathbb{Q}^2 \), which satisfies:
- The degree is \( \Delta_\pi \).
- It matches the constraint \( \{p, p_1, \ldots, p_l\} \).

By Proposition 9.8, the constraint \( p \) is necessarily matched by the unique unbounded edge of the direction \( -v \). So, in \( \mathbb{N}_Q \), the constraint \( A_0 \) is matched by the unique edge of the direction \( w \). This shows \( N_{0, \Delta'}^{\text{trop}}(A) = N_D(\Delta, A) \), hence the proposition. \( \square \)

When \( \Delta_0, v, \) and \( p, p_1, \ldots, p_l \) satisfy the assumption of Proposition 9.12, a similar result holds, but we have to notice that edges of different directions can have the same direction when projected by \( \pi_Q \). Let \( \{w_1, \ldots, w_a\} \) be the set of vectors in \( \mathbb{N} \) such that:
- \( w_i \in \text{Supp}(\Delta') \).
- \( \pi(w_i) = -v \).

Let
\[ \Delta_i : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}, \quad i = 1, \ldots, a \]
be the degree function uniquely determined by
\[ \Delta' = \Delta_i + \{w_i\}. \]

Note that all \( \Delta_i, i = 1, \ldots, a \) satisfy
\[ \Delta_\pi = \Delta_0. \]
**Proposition 9.19.** Suppose $\Delta_0$, $v$, and $p, p_1, \ldots, p_l$ satisfy the assumption of Proposition 9.12. Then the equality

$$\sum_{i=1}^{a} N_D(\Delta_i, A) = N_{alg}^0(\Delta')(L)$$

holds.

We can construct more elaborate examples by repeating this process.

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