EXACT NUMBER OF ERGODIC INVARIANT MEASURES FOR
BRATTELI DIAGRAMS

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Abstract. For a Bratteli diagram $B$, we study the simplex $\mathcal{M}_1(B)$ of probability measures on the path space of $B$ which are invariant with respect to the tail equivalence relation. Equivalently, $\mathcal{M}_1(B)$ is formed by probability measures invariant with respect to a homeomorphism of a Cantor set. We study relations between the number of ergodic measures from $\mathcal{M}_1(B)$ and the structure and properties of the diagram $B$. We prove a criterion and find sufficient conditions of unique ergodicity of a Bratteli diagram, in which case the simplex $\mathcal{M}_1(B)$ is a singleton. For a finite rank $k$ Bratteli diagram $B$ having exactly $l \leq k$ ergodic invariant measures, we explicitly describe the structure of the diagram and find the subdiagrams which support these measures. We find sufficient conditions under which: (i) a Bratteli diagram has a prescribed number (finite or infinite) of ergodic invariant measures, and (ii) the extension of a measure from a uniquely ergodic subdiagram gives a finite ergodic invariant measure. Several examples, including stationary Bratteli diagrams, Pascal-Bratteli diagrams, and Toeplitz flows, are considered.

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1. Introduction

The concept of a Bratteli diagram appeared first in the famous paper by O. Bratteli [Bra72] devoted to the classification of approximately finite $C^*$-algebras (named AF-algebras). Bratteli proved that AF-algebras can be completely described combinatorially by introducing a certain type of graphs, now called Bratteli diagrams.

Though Bratteli diagrams were introduced to answer a challenging problem in the $C^*$-algebra theory, these diagrams have found a surprisingly large number of diverse applications in related fields. Among them are: ergodic theory and symbolic dynamics, representation theory, classification problems, analysis and random walk in network models, etc.

In the present article, we use Bratteli diagrams to study ergodic invariant measures of Cantor dynamical systems $(X, T)$ generated by a single homeomorphism of a Cantor set $X$. Our approach is based on the existence of Bratteli-Vershik models for aperiodic (minimal) homeomorphisms of Cantor sets. Loosely speaking, every aperiodic homeomorphism $T$ of a Cantor set $X$ is conjugate to a homeomorphism $\varphi_B$ (called the Vershik map) of a path space $X_B$ of a Bratteli diagram $B$.

This means that all properties of a dynamical system can be seen in terms of the corresponding Bratteli diagram.

This concept was successfully realized in the series of remarkable papers [Ver81, Ver82, HPS92, GPS95] in the context of ergodic theory and minimal Cantor dynamics. Later on, this approach was extended to Borel dynamical systems and aperiodic homeomorphisms of a Cantor set in [BDK06, Med07].

The idea to study the Vershik map $\varphi_B$ defined on the path space $X_B$ of the corresponding Bratteli diagram proved to be very useful and productive. We recall that, first of all, this approach allowed to classify minimal homeomorphisms of a Cantor set up to orbit equivalence [GPS95, GMPS10]. Furthermore, it turns out that the structure of a Bratteli diagram makes it possible to see distinctly on the diagram several important invariants of a homeomorphism. They are, for example, the set of minimal components, the support of any ergodic measure $\mu$, and the values of $\mu$ on clopen sets. We discussed various aspects of this method in a series of papers [BKMS10, BK11, BKMS13, BH14, BKY14, BJ15, BK16]. Other important references on Bratteli diagrams in Cantor dynamics are [DHS99, Dur10, GJ00, Cor06, HKY11, KW04, Pat10, Ska00].

Let $(X, T)$ be a Cantor dynamical system. The question about a complete (or even partial) description of the simplex $\mathcal{M}_1(T)$ of invariant probability measures for $(X, T)$ is one of the most important in ergodic theory. It has a long history and many remarkable results. By the Kakutani-Markov theorem, this simplex is always non-empty. Ergodic invariant measures form extreme points of the simplex. The
cardinality of the set of ergodic measures is an important invariant of dynamical systems. The study of relations between the properties of the simplex $\mathcal{M}_1(T)$ and those of the dynamical system $(X,T)$ is a hard and intriguing problem. There is an extensive list of references regarding this problem, we mention here only the books [Pic01] [Gla03] and the papers [Dow91] [Dow06] [Dow08] for further citations.

It is worth pointing out two important facts that are constantly used in the paper. Firstly, we consider only Bratteli-Vershik models to study the simplex $\mathcal{M}_1(T)$ of probability invariant measures. More precisely, we work in a more general setting considering probability measures invariant with respect to the *tail equivalence relation* $\mathcal{E}$ on the path space $X_B$ of a Bratteli diagram. We denote this set by $\mathcal{M}_1(B)$. It is not hard to see that such measures are also invariant with respect to the Vershik map acting on $X_B$ [BKMS10] if the diagram $B$ admits a Vershik map (see [Med07], [BKY14], [BY17] for more details). Hence, this approach includes the case of dynamical systems generated by a transformation. Moreover, it can be used not only for homeomorphisms of a Cantor set but for measure preserving automorphisms of a standard measure space. Secondly, the structure of a Bratteli diagram is completely described by the sequence of matrices $(\tilde{F}_n)$ with non-negative integer entries. Such matrices are called *incidence matrices*. Knowing entries of the incidence matrices, one can determine the number $h_v^{(n)}$ of all finite paths between the top of the diagram and any vertex $v$ of the level $n$: they are obtained as entries of the products of incidence matrices. This information can be combined together in the sequence $(F_n)$ of *stochastic incidence matrices* whose entries are

$$ f_{vw}^{(n)} = \frac{\tilde{f}_{vw}^{(n)} h_v^{(n)}}{h_v^{(n+1)}}, $$

where $\tilde{f}_{vw}^{(n)}$ are entries of the incidence matrix $\tilde{F}_n$. It turns out that namely these matrices $(F_n)$ carry the information needed to describe the structure of the simplex $\mathcal{M}_1(B)$.

In this paper, we focus mostly on the following problem: given a Bratteli diagram $B$, determine the exact number of ergodic probability measures on $B$ which are invariant with respect to the tail equivalence relation $\mathcal{E}$. This problem includes also the case of uniquely ergodic homeomorphisms for which there exists just one ergodic probability measure. In our earlier papers, we proved a number of results for stationary (i.e., all incidence matrices are equal) and finite rank (i.e., the size of all incidence matrices is bounded) Bratteli diagrams which give partial answers about the structure of the set $\mathcal{M}_1(B)$, see [BKMS10] [BKMS13] [ABKK17]. Especially important for us is Theorem 2.5 (see below) that was proved in [BKMS10]. We recall that in the case of stationary and finite rank Bratteli diagrams the simplex $\mathcal{M}_1(B)$ is finite-dimensional, and the number of ergodic measures cannot exceed the rank of $B$.

Our main results of this paper are given in Theorems 3.1, 4.9, and 5.4. Theorem 3.1 contains a criterion for unique ergodicity of a Bratteli diagram. This condition is formulated in terms of the stochastic matrices $F_n$ associated with a Bratteli
diagram. The theorem asserts that $B$ is uniquely ergodic if and only if
\[
\lim_{n \to \infty} \max_{v, v' \in V_{n+1}} \left( \sum_{w \in V_n} |f^{(n)}(vw) - f^{(n)}(v'w)| \right) = 0.
\]
Another sufficient condition for unique ergodicity of a diagram $B$ is given in Theorem 5.1. In Theorem 4.9, we consider the case when a finite rank $k$ Bratteli diagram $B$ has exactly $l$, $1 \leq l \leq k$, ergodic measures. It turns out that, in this case, one can clarify the structure of the diagram $B$ and describe subdiagrams that support ergodic measures. Theorem 5.4 focuses on the question about finiteness of measure extension from a subdiagram. More precisely, we find conditions that guarantee the finiteness of measure extension from a “chain subdiagram”. It gives a description of ergodic finite measures on $B$ obtained by extension from uniquely ergodic subdiagrams.

The outline of the paper is as follows. In Section 2, we give necessary definitions and notation. We also prove a few preliminary statements which are based on Theorem 2.5 and the construction of Kakutani-Rokhlin towers related to a Bratteli diagram. Section 3 contains the proof of Theorem 3.1. We also included in this section Example 3.2 which shows the importance of the telescoping procedure for the study of invariant measures. Sections 4 and 5 form the core of our work. The proofs of our main results, Theorems 4.9 and 5.4, are given in these two sections. We should remark that the proofs are rather long and difficult. They are based on the technique developed in our previous papers, see e.g. [BKM13] and [ABKK17].

The last two sections are devoted to some applications. In Section 6, we consider several particular cases. Firstly, we show that, in case of stationary Bratteli diagrams, the description of invariant measures given in [BKMS10] can be included in the scheme elaborated in Sections 5 and 6. In two other subsections of Section 6, we discuss the Pascal-Bratteli diagram and the class of simple Bratteli diagrams with countably many ergodic invariant measures. We remark that it is not hard to see that every Bratteli-Vershik diagram, considered in Subsection 6.3, has zero topological entropy. It follows from the N. Frantzikinakis and B. Host result [FH17] that a class of topological dynamical systems constructed in Subsection 6.3 satisfy the logarithmic Sarnac conjecture. The last section contains an interpretation of our results in terms of symbolic dynamics.

\section*{2. Basics on Bratteli diagrams and invariant measures}

In this section, we give necessary definitions, notation, and prove some results about the structure of the set of invariant measures which are used in the paper below.

\subsection*{2.1. Main definitions and notation.}

**Definition 2.1.** A Bratteli diagram is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i \geq 0} V_i$ and the edge set $E = \bigcup_{i \geq 0} E_i$ are partitioned into disjoint subsets $V_i$ and $E_i$ where
The set of vertices $V_i$ is called the $i$-th level of the diagram $B$. A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a finite or infinite path, respectively. For $m < n$, $v \in V_m$ and $w \in V_n$, let $E(v, w)$ denote the set of all paths $e(v, w) = (e_1, \ldots, e_p)$ with $s(e) = s(e_1) = v$ and $r(e) = r(e_p) = w$.

For a Bratteli diagram $B$, let $X_B$ be the set of all infinite paths beginning at the top vertex $v_0$. We endow $X_B$ with the topology generated by cylinder sets $[e]$ where $e = (e_0, \ldots, e_n)$, $n \in \mathbb{N}$, and $[e] := \{ x \in X_B : x_i = e_i, \ i = 0, \ldots, n \}$. With this topology, $X_B$ is a 0-dimensional compact metric space. By assumption, we will consider only such Bratteli diagrams $B$ for which $X_B$ is a Cantor set, that is $X_B$ has no isolated points.

**Remark 2.2.** We will use the following definitions and notation throughout the paper.

- By $\mathbf{x}$, we denote a (column) vector in $\mathbb{R}^k$. We will always denote the coordinates of $\mathbf{x}$ using the subscript. For instance, the notation $\mathbf{x}(w) = (x_v(w))$ means that we have a set of vectors enumerated by $v$, and the $v$-coordinate of $\mathbf{x}(w)$ is $x_v(w)$. By $||\mathbf{x}||$ we denote the Euclidean norm of $\mathbf{x}$.
- The notation $A^T$ is used for the transpose of $A$, and $(x_1, \ldots, x_k)^T$ means the column vector $\mathbf{x}$ with entries $x_i$ in the standard basis in $\mathbb{R}^k$.
- For $\mathbf{x} = (x_1, \ldots, x_k)^T$, $\mathbf{y} = (y_1, \ldots, y_k)^T \in \mathbb{R}^k$, define the metric
  \[
d^*(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^k |x_i - y_i|.
\]

Let $d$ denote the metric on $\mathbb{R}^k$ generated by the Euclidean norm $|| \cdot ||$. Clearly, the two metrics, $d^*$ and $d$, are equivalent.
- Let $x = (x_n)$ and $y = (y_n)$ be two paths from $X_B$. It is said that $x$ and $y$ are tail equivalent (in symbols, $(x, y) \in \mathcal{E}$) if there exists some $n$ such that $x_i = y_i$ for all $i \geq n$. The diagrams with infinite $\mathcal{E}$-orbits are called aperiodic. Note that a Bratteli diagram is simple if the tail equivalence relation $\mathcal{E}$ is minimal, see Definition 2.3 below.
- For a Bratteli diagram $B = (V, E)$, the incidence matrices $\mathbf{F}_n = (f^{(n)}_{w, v})$, where $v \in V_{n+1}$, $w \in V_n$, are formed by entries that indicate the number of edges between vertices $v \in V_{n+1}$ and $w \in V_n$. We reserve the notation $F_n$ and $f^{(n)}_{w, v}$ for the corresponding stochastic matrices (see below).
- Let $B$ be a Bratteli diagram, and let $n_0 = 0 < n_1 < n_2 < \ldots$ be a strictly increasing sequence of integers. The telescoping of $B$ to $(n_k)$ is the Bratteli diagram $B' = (V', E')$ whose $k$-level vertex set $V_k'$ is $V_{n_k}$ and whose incidence
matrices \((\tilde{F}_k')\) are defined by
\[
\tilde{F}_k' = \tilde{F}_{n_k+1} \cdots \tilde{F}_{n_k},
\]
The operation of telescoping preserves the space \(X_B\), the tail equivalence relation, and the set of invariant measures.

- Denote by \(\tilde{h}^{(n)} = (h_v^{(n)} : v \in V_n)\) the vector of heights of Kakutani-Rokhlin towers \([\text{GPS95}]\) \([\text{HPS92}]\), i.e., \(h_v^{(n)} = |E(v_0, v)|\) where \(|A|\) denotes the cardinality of a set \(A\). Here \(h_v^{(1)}\) is the number of edges between \(v_0\) and \(v \in V_1\).
It follows from the definition of a Bratteli diagram that
\[
\tilde{F}_n \tilde{h}^{(n)} = \tilde{h}^{(n+1)}, \quad n \geq 1,
\]

- We will constantly use the sequence of row stochastic matrices \((F_n)\) whose entries are defined by the formula
\[
f_{vw}^{(n)} = \frac{\tilde{f}_{vw}^{(n)} h_w^{(n)}}{h_v^{(n+1)}}.
\]

**Definition 2.3.** Let \(B = (V, E)\) be a Bratteli diagram.

1. We say that \(B\) has finite rank if there exists some \(k \in \mathbb{N}\) such that \(|V_n| \leq k\) for all \(n \geq 1\).
2. For a finite rank diagram \(B\), we say that \(B\) has rank \(d\) if \(d\) is the smallest integer such that \(|V_n| = d\) for infinitely many \(n\).
3. We say that \(B\) is simple if for any level \(n\) there is \(m > n\) such that \(E(v, w) \neq \emptyset\) for all \(v \in V_n\) and \(w \in V_m\). Otherwise, \(B\) is called non-simple.
4. We say that \(B\) is stationary if \(\tilde{F}_n = \tilde{F}_1\) for all \(n \geq 2\).

2.2. **Invariant measures on Bratteli diagrams.** In this paper, we study only positive Borel measures on \(X_B\) which are invariant with respect to the tail equivalence relation \(\bar{E}\). In this subsection, we show that the set of all probability invariant measures on a Bratteli diagram corresponds to the inverse limit of a decreasing sequence of convex sets. Let \(\mu\) be a Borel probability non-atomic \(\bar{E}\)-invariant measure on \(X_B\) (for brevity, we will use the term “measure on \(B\)” below). We denote the set of all such measures by \(\mathcal{M}_1(B)\). The fact that \(\mu\) is an \(\bar{E}\)-invariant measure means that \(\mu([e]) = \mu([e'])\) for any two cylinder sets \(e, e' \in E(v_0, v)\), where \(v \in V_n\) is an arbitrary vertex, and \(n \geq 1\). Since any measure on \(X_B\) is completely determined by its values on clopen (even cylinder) sets, we conclude that in order to define an \(\bar{E}\)-invariant measure \(\mu\), one needs to know the sequence of vectors \(\bar{p}^{(n)} = (p_v^{(n)} : v \in V_n), n \geq 1\), such that \(p_u^{(n)} = \mu([e(v_0, v)])\) where \(e(v_0, v)\) is a finite path from \(E(v_0, v)\). It is clear that, for \(v \in V_n\),
\[
[e(v_0, v)] = \bigcup_{e(v,w), w \in V_{n+1}} [e(v_0, v), e(v, w)]
\]
so that \([e(v_0, w)] \subset [e(v_0, v)]\).
In other words, we see that the entries of vectors $\mathbf{p}^{(n)}$ can be found by the formula

$$
\mathbf{p}^{(n)}_w = \frac{\mu(X^{(n)}_w)}{h^{(n)}_w}
$$

where

$$
X^{(n)}_w = \bigcup_{e \in E(v_0, w)} [e], \ w \in V_n.
$$

This means that $X^{(n)}_w$ is a tower in the Kakutani-Rokhlin partition that corresponds to the vertex $w$ [HPS92]. The measure of this tower is

$$
\mu(X^{(n)}_w) = h^{(n)}_w p^{(n)}_w =: q^{(n)}_w.
$$

Then relation (2.3) implies that

$$
\hat{\mathbf{F}}_n^T \mathbf{p}^{(n+1)} = \mathbf{p}^{(n)}, \ n \geq 1,
$$

where $\hat{\mathbf{F}}_n^T$ denotes the transpose of the matrix $\hat{\mathbf{F}}_n$.

Because $\mu(X_B) = 1$, we see that, for any $n > 1$,

$$
\sum_{w \in V_n} h^{(n)}_w p^{(n)}_w = 1.
$$

**Lemma 2.4.** Let $\mu$ be a probability measure on the path space $X_B$ of a Bratteli diagram $B$. Let $(F_n)$ be a sequence of corresponding stochastic incidence matrices. Then, for every $n \geq 1$, the vector $\mathbf{q}^{(n)} = (q^{(n)}_v : v \in V_n)$ (see (2.7)) is a probability vector such that

$$
F_n^T \mathbf{q}^{(n+1)} = \mathbf{q}^{(n)}, \ n \geq 1.
$$

**Proof.** This fact follows from the definition of the stochastic matrix $F_n$ and relations (2.1) - (2.6) \hfill \square

We see that the formula in (2.6) is a necessary condition for a sequence of vectors $(\mathbf{p}^{(n)})$ to be defined by an invariant probability measure. It turns out that the converse statement is true, in general. We formulate below Theorem 2.5 (proved in [BKMS10]), where all $\mathcal{E}$-invariant measures are explicitly described.

Using the definition of stochastic incidence matrix $F_n$ (see (2.3)) and Lemma 2.4, we define a decreasing sequence of convex polytopes $\Delta^{(n)}_m$, $n, m \geq 1$, and the limiting convex sets $\Delta^{(n)}_\infty$. They are used to describe the set $\mathcal{M}_1(B)$ of all probability $\mathcal{E}$-invariant measures on $B$. Namely, denote

$$
\Delta^{(n)} := \{(z^{(n)}_w)_{w \in V_n} : \sum_{w \in V_n} z^{(n)}_w = 1 \text{ and } z^{(n)}_w \geq 0, \ w \in V_n\}.
$$

The sets $\Delta^{(n)}$ are standard simplices in the space $\mathbb{R}^{|V_n|}$ with $|V_n|$ vertices $\{(\mathbf{e}^{(n)}_w(w) : w \in V_n\}$, where $\mathbf{e}^{(n)}_w(w) = (0, \ldots, 0, 1, 0, \ldots 0)^T$ is the standard basis vector, i.e. $\mathbf{e}^{(n)}_w(u) = 1$ if and only if $u = w$. Since $F_n$ is a stochastic matrix, we have the obvious property that

$$
F_n^T(\Delta^{(n+1)}) \subset \Delta^{(n)}, \ n \in \mathbb{N}.
$$
Let $\mu$ be a probability $\mathcal{E}$-invariant measure $\mu$ on $X_B$ with values $q_w^{(n)}$ on the towers $X_w^{(n)}$. Then $(q_w^{(n)} : w \in V_n)^T$ lies in the simplex $\Delta^{(n)}$. Set
\[
\Delta_m^{(n)} = F_n^T \cdots F_{n+m-1}^T(\Delta^{(n+m)})
\]
for $m = 1, 2, \ldots$. Then we see that
\[
\Delta^{(n)} \supset \Delta_1^{(n)} \supset \Delta_2^{(n)} \supset \ldots
\]
Denote
\[
\Delta^{(n)}_\infty = \bigcap_{m=1}^{\infty} \Delta_m^{(n)}.
\]
It follows from (2.8) and (2.9) that
\[
F_n^T(\Delta^{(n+1)}_\infty) = \Delta^{(n)}_\infty, \quad n \geq 1.
\]

The next theorem, that was proved in [BKMS10], describes all $\mathcal{E}$-invariant probability measures. We formulate here a slightly stronger version of this statement.

**Theorem 2.5.** Let $B = (V, E)$ be a Bratteli diagram with the sequence of stochastic incidence matrices $(F_n)$, and let $\mathcal{M}_1(B)$ denote the set of $\mathcal{E}$-invariant probability measures on the path space $X_B$.

1. If $\mu \in \mathcal{M}_1(B)$, then the probability vector 
   \[
   \tilde{q}^{(n)} = (\mu(X_w^{(n)}))_{w \in V_n}
   \]
   satisfies the following conditions for $n \geq 1$:
   
   (i) 
   \[
   \tilde{q}^{(n)} \in \Delta^{(n)}_\infty,
   \]
   
   (ii) 
   \[
   F_n^T \tilde{q}^{(n+1)} = \tilde{q}^{(n)},
   \]
   where $X_w^{(n)}$ is defined in (2.7).
   Conversely, suppose that \{\tilde{q}^{(n)}\} is a sequence of non-negative probability vectors such that, for every $\tilde{q}^{(n)} = (q_w^{(n)})_{w \in V_n} \in \Delta^{(n)}_\infty$ ($n \geq 1$), the condition (ii) holds. Then the vectors $\tilde{q}^{(n)}$ belong to $\Delta^{(n)}_\infty$, $n \in \mathbb{N}$, and there exists a uniquely determined $\mathcal{E}$-invariant probability measure $\mu$ such that $\mu(X_w^{(n)}) = q_w^{(n)}$ for $w \in V_n, n \in \mathbb{N}$.

2. Let $\Omega$ be the subset of the infinite product $\prod_{n \geq 1} \Delta^{(n)}_\infty$ consisting of sequences \{\tilde{q}^{(n)}\} such that $F_n^T \tilde{q}^{(n+1)} = \tilde{q}^{(n)}$. Then the map 
   \[
   \Phi : \mu \mapsto (\tilde{q}^{(n)}) : \mathcal{M}_1(B) \mapsto \Omega,
   \]
   is an affine isomorphism. Moreover, $\Phi(\mu)$ is an extreme point of $\Omega$ if and only if $\mu$ is ergodic.

3. Let $B$ be a finite rank diagram. Then the number of ergodic invariant measures on $B$ is bounded above by the dimension of the finite-dimensional simplex $\Delta^{(1)}_\infty$. 

Remark 2.6. (a) Recall that the set \( \Delta^{(n)}_{\infty} = \bigcap_{m=1}^{\infty} \Delta^{(n)}_{m} \) is a convex subset of \( \Delta^{(n)} \) for all \( n \geq 1 \). It follows from (2.10) that the following sequence of maps is defined:

\[
\Delta^{(1)}_{\infty} \xrightarrow{F_{n}^{T}} \Delta^{(2)}_{\infty} \xrightarrow{F_{n}^{T}} \Delta^{(3)}_{\infty} \xrightarrow{F_{n}^{T}} \ldots
\]

By Theorem 2.5, the set \( M_{1}(B) \) can be identified with the inverse limit of the sequence \( (F_{n}^{T}, \Delta^{(n)}_{\infty}) \). There is a one-to-one correspondence between measures \( \mu \in M_{1}(B) \) and sequences of vectors \( \vec{q}^{(n)} \in \Delta^{(n)}_{\infty} \) such that \( \vec{q}^{(n)} = F_{n}^{T}(\vec{q}^{(n+1)}) \), \( n = 1, 2, \ldots \).

(b) To shorten our notation, by vertices of a convex subset of \( \mathbb{R}^{n}, n \in \mathbb{N} \), we will mean its extreme points. These two notions will be used as synonyms.

(c) Theorem 2.5 states that in order to find all ergodic invariant measures on a diagram it suffices to know the number of extreme points of \( \Delta^{(n)}_{\infty} \) for every \( n \). Thus, in the statements which are proved in this paper, we will fix any natural number \( n \) and investigate the corresponding convex set \( \Delta^{(n)}_{\infty} \).

(d) In general, the set \( \Delta^{(n)}_{\infty} \) is a convex subset of the \( (|V_{n}| - 1) \)-dimensional simplex \( \Delta^{(n)} \). In some cases, which will be considered in Section 4, the set \( \Delta^{(n)}_{\infty} \) is a finite-dimensional simplex itself.

(e) In Theorem 2.5 part (2), the set \( M_{1}(B) \) can be affinely isomorphic to the set \( \Delta^{(1)}_{\infty} \). For instance, it happens when all stochastic incidence matrices are square non-singular matrices of the same dimension \( K \times K \) for some \( K \in \mathbb{N} \). This case will be considered in Section 4.

(f) The procedure of telescoping defined in Remark 2.2 preserves the set of invariant measures; hence we can apply it when necessary without loss of generality.

Example 2.7 (Equal row sums (ERS) Bratteli diagrams). We illustrate Theorem 2.5 proved result by a particular class of Bratteli diagrams that have the equal row sum (ERS) property. This means that the incidence matrices \( \vec{F}_{n} \) of a Bratteli diagram \( B \) satisfy the condition

\[
\sum_{w \in V_{n}} \vec{f}_{v,w}^{(n)} = r_{n}
\]

for every \( v \in V_{n+1} \). It is known that Bratteli-Vershik systems with the ERS property can serve as models for Toeplitz subshifts (see [GJ00]). In particular, we have \( \vec{F}_{0} = \vec{h}^{(1)} = (r_{0}, \ldots, r_{0})^{T} \). It follows from (2.1) that, for ERS Bratteli diagrams, \( h_{w}^{(n)} = r_{0} \cdots r_{n-1} \) for every \( w \in V_{n} \). Thus, for every probability \( \mathcal{E} \)-invariant measure on \( B \), we obtain

\[
\sum_{w \in V_{n}} p_{w}^{(n)} = \frac{1}{r_{0} \cdots r_{n-1}}
\]
for \( n = 1, 2, \ldots \). The proof of this fact follows easily from (2.6) by induction. Furthermore, in the case of ERS diagrams, we have
\[
f^{(n)}_{vw} = \frac{\bar{f}^{(n)}_{vw}}{r_n}.
\]

2.3. Invariant measures in terms of decreasing sequences of polytopes. In this subsection we study the decreasing sequence of convex sets which correspond to probability invariant measures. We focus on extreme points of these convex sets and on the description of these sets as decreasing sequences of convex polytopes.

Let
\[
G_{(n+m,n)} = F_{n+m} \cdots F_n
\]
for \( m \geq 0 \) and \( n \geq 1 \). Denote the elements of \( G_{(n+m,n)} \) by \( (g_{uw}^{(n+m,n)}) \), where \( u \in V_{n+m+1} \) and \( w \in V_n \); then
\[
g_{uw}^{(n+m,n)} = \sum_{(u_m, \ldots, u_1) \in V_{n+m} \times \cdots \times V_{n+1}} f^{(n+m)}_{u,u_m} \cdot f^{(n+m-1)}_{u_m,u_{m-1}} \cdots f^{(n)}_{u_1,w}.
\]
The sets \( \Delta_m^{(n)}, m \geq 0, \) defined in (2.3), form a decreasing sequence of convex polytopes in \( \Delta^{(n)} \). The vertices of \( \Delta_m^{(n)} \) are some (or all) vectors from the set \( \{\bar{g}^{(n+m,n)}(v) : v \in V_{n+m+1}\} \), where we denote
\[
\bar{g}^{(n+m,n)}(v) = (g_{w,v}^{(n+m,n)}(v))_{w \in V_n} = G^{T}_{(m+n,n)}(\bar{e}^{(n+m+1)}(v)).
\]

Obviously, we have the relation
\[
\bar{g}^{(n+m,n)}(v) = \sum_{w \in V_n} g_{uw}^{(n+m,n)} \bar{e}^{(n)}(w).
\]
Let \( \{\bar{g}^{(n,m)}(v)\} \) be the set of all vertices of \( \Delta_m^{(n)} \). Then \( \bar{g}^{(n,m)}(v) = \bar{g}^{(n+m,n)}(v) \) for \( v \) belonging to some subset \( V_{n+m+1}^{(n)} \) of \( V_{n+m+1} \).

We observe the following fact. Every vector \( \bar{g}^{(n)} \) from the set \( \Delta_\infty^{(n)} \) can be written in the standard basis as
\[
\bar{g}^{(n)} = \sum_{w \in V_n} q^{(n)}_{w} \bar{e}^{(n)}(w).
\]
It turns out that the numbers \( q^{(n+m+1)}_v, v \in V_{n+m+1} \), are the coefficients in the convex decomposition of \( \bar{g}^{(n)} \) with respect to vectors \( \bar{g}^{(n+m,n)}(v) \).

**Proposition 2.8.** Let \( \mu \in \mathcal{M}_1(B) \), and \( q^{(n)}_w = \mu(\bar{X}^{(n)}_w) (w \in V_n) \) for all \( n \in \mathbb{N} \). Then
\[
\bar{g}^{(n)} = \sum_{v \in V_{n+m+1}} q^{(n+m+1)}_v \bar{g}^{(n+m,n)}(v).
\]

In particular,
\[
\bar{g}^{(1)} = \sum_{v \in V_{m+1}} q^{(m+1)}_v \bar{g}^{(m+1,1)}(v).
\]
Proof. The result follows from the following computation:
\[
\overline{\eta}^{(n)} = F_n^T \cdots F_{n+m}^T (\overline{\eta}^{(n+m+1)})
\]
\[
= F_n^T \cdots F_{n+m}^T \left( \sum_{v \in V_{n+m+1}} q_v^{(n+m+1)} \overline{\eta}^{(n+m+1)}(v) \right)
\]
\[
= \sum_{v \in V_{n+m+1}} q_v^{(n+m+1)} F_n^T \cdots F_{n+m}^T \overline{\eta}^{(n+m+1)}(v)
\]
\[
= \sum_{v \in V_{n+m+1}} q_v^{(n+m+1)} \overline{\eta}^{(n+m,n)}(v).
\]

For every \( n \geq 1 \), define
\[
\Delta^{(n),\varepsilon} := \bigcup_{\overline{\eta} \in \Delta^{(n)}_{\infty}} B(\overline{\eta}, \varepsilon),
\]
where \( B(\overline{\eta}, \varepsilon) \) is the ball of radius \( \varepsilon > 0 \) centered at \( \overline{\eta} \in \mathbb{R}^{|V_n|} \). Here the metric is defined by the Euclidean norm \( || \cdot || \) on \( \mathbb{R}^{|V_n|} \).

The following lemma can be proved straightforward, so we omit the proof.

**Lemma 2.9.** Fix any natural numbers \( n \) and \( m \). Let \( \Delta^{(n)}_m \) be defined as above. If \( \overline{\eta}^{(n,m)} \in \Delta^{(n)}_m \) for infinitely many \( m \) and \( \overline{\eta}^{(n,m)} \rightarrow \overline{\eta}^{(n)} \) as \( m \rightarrow \infty \), then \( \overline{\eta}^{(n)} \in \Delta^{(n)}_{\infty} \). Moreover, for every \( \varepsilon > 0 \) there exists \( m_0 = m_0(n, \varepsilon) \) such that \( \Delta^{(n)}_m \subset \Delta^{(n),\varepsilon}_{\infty} \) for all \( m \geq m_0 \).

In the next statement we prove that vertices of the limiting convex set \( \Delta^{(n)}_m \) can be obtained as limits of sequences of vertices of convex polytopes.

**Lemma 2.10.** Fix \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). Let \( \Delta^{(n)}_{\infty}, \Delta^{(n),\varepsilon}_{n+m+1}, V^{(n)}_{n+m+1} \) be defined as above for any \( m \in \mathbb{N} \). Then, for every vertex \( \overline{\eta} \in \Delta^{(n)}_{\infty} \) there exists \( m_0 = m_0(n, \varepsilon) \) such that, for all \( m \geq m_0 \), one can find a vertex \( \overline{\eta}^{(n,m)}(v) \in \Delta^{(n)}_{n+m+1}, v \in V^{(n)}_{n+m+1}, \) satisfying the property
\[
||\overline{\eta} - \overline{\eta}^{(n,m)}(v)|| < \varepsilon.
\]

Proof. We first assume that \( \text{diam}(\Delta^{(n)}_{\infty}) = 0 \). Then \( \text{diam}(\Delta^{(n)}_m) \rightarrow 0 \) as \( m \rightarrow \infty \). Hence \( \Delta^{(n)}_{\infty} \) is a single point, and Lemma 2.10 holds.

Let \( \text{diam}(\Delta^{(n)}_{\infty}) > 0 \). Then we can find \( m_0 \geq 1 \) and \( 1 \leq n_0 \leq |V_n| - 1 \) such that \( \text{dim}(\Delta^{(n)}_m) = n_0 \) for all \( m \geq m_0 \). Without loss of generality we can assume that \( m_0 = 1 \).

Assume that Lemma 2.10 is not true, that is the converse statement holds. Then, by Lemma 2.9, we can find \( \varepsilon_0 > 0 \) and an infinite subset \( M \subset \{1, 2, \ldots\} \) such that \( B(\overline{\eta}, \varepsilon_0) \) contains no vertices of \( \Delta^{(n)}_m \) for \( m \in M \). By Caratheodory’s Theorem
on convex hulls, there is an \( n_0 \)-dimensional simplex with vertices \( \{ \mathbf{f}^{(n,m)}(v), v \in V'_{n+m+1} \subset V''_{n+m+1} \} \) such that \( |V'_{n+m+1}| = n_0 + 1 \) and

\[
\mathbf{y} = \sum_{v \in V'} a_v \mathbf{f}^{(n,m)}(v), \quad m \in M,
\]

where each \( a_v \geq 0 \) and \( \sum_{v \in V'_{n+m+1}} a_v = 1 \). Since \( B(\mathbf{y}, \varepsilon_0) \) contains no vectors \( \mathbf{f}^{(n,m)}(v), v \in V'_{n+m+1} \), there is \( \varepsilon'_0 > 0 \) such that \( 0 \leq a_v \leq 1 - \varepsilon'_0 \) for all \( v \in V'_{n+m+1} \).

As far as \( \sum_{v \in V'_{n+m+1}} a_v = 1 \), we can find two different vertices \( v, v' \in V'_{n+m+1} \) (both \( v \) and \( v' \) depend on \( m \)) such that

\[
a_v, a_{v'} \in [\varepsilon'_0, 1 - \varepsilon'_0],
\]

where \( \varepsilon''_0 = \varepsilon'_0 / |V_n| \).

Relation (2.14) can be rewritten in the form

\[
\mathbf{y} = (a_v + a_{v'} \left( \frac{a_v}{a_v + a_{v'}} \mathbf{f}^{(n,m)}(v) + \frac{a_{v'}}{a_v + a_{v'}} \mathbf{f}^{(n,m)}(v') \right) \\
+ (1 - a_v - a_{v'}) \sum_{u \in V'_{n+m+1} \setminus \{v,v'\}} \frac{a_u}{1 - a_v - a_{v'}} \mathbf{f}^{(n,m)}(u)).
\]

First assume that \( (a_v + a_{v'}) \to 1 \) as \( m \to \infty \). We can find an infinite subset \( M' \subset M \) such that when \( m \in M' \) and \( m \to \infty \) we have

\[
a_v \to \lambda, \quad a_{v'} \to 1 - \lambda,
\]

where \( \frac{1}{2} \varepsilon''_0 \leq \lambda \leq 1 - \frac{1}{2} \varepsilon''_0 \), and

\[
\mathbf{f}^{(n,m)}(v) \to \mathbf{z}(v) \in \Delta^{(n)}_{\infty}, \quad \mathbf{f}^{(n,m)}(v') \to \mathbf{z}(v') \in \Delta^{(n)}_{\infty}.
\]

It follows from (2.15) that

\[
\mathbf{y} = \lambda \mathbf{z}(v) + (1 - \lambda) \mathbf{z}(v').
\]

Clearly, \( \mathbf{z}(v) \neq \mathbf{z}(v') \) because otherwise we would get that \( B(\mathbf{y}, \varepsilon_0) \) contains a vertex \( \mathbf{f}^{(n,m)}(v) \) for some \( m \in M' \). It follows from (2.14) that \( \mathbf{y} \) is not a vertex of \( \Delta^{(n)}_{\infty} \) and we get a contradiction.

If \( a_v + a_{v'} \to 1 \) as \( m \to \infty \), then we can find again an infinite subset \( M'' \subset M \) such that conditions (2.10) are satisfied. Moreover, for some \( \tau \in (0,1) \), we obtain

\[
a_v + a_{v'} \to \tau,
\]

and

\[
\sum_{u \in V'_{n+m+1} \setminus \{v,v'\}} \frac{a_u}{1 - a_v - a_{v'}} \mathbf{f}^{(n,m)}(u) \to \mathbf{z}' \in \Delta^{(n)}_{\infty}
\]

as \( m \in M'' \) and \( m \to \infty \). Then (2.15) implies that

\[
\mathbf{y} = \tau (\lambda \mathbf{z}(v) + (1 - \lambda) \mathbf{z}(v')) + (1 - \tau) \mathbf{z}'.
\]
Therefore, we showed that $\overline{y}$ is not a vertex of $\Delta_{\infty}^{(n)}$. This contradiction proves the lemma. \[\square\]

3. A criterion of unique ergodicity of Bratteli diagrams

In this section, we deal with arbitrary Bratteli diagram $B$ and prove a criterion of the unique ergodicity of $B$, i.e., we discuss the case when the space $\mathcal{M}_1(B)$ is a singleton.

**Theorem 3.1.** A Bratteli diagram $B = (V, E)$ is uniquely ergodic if and only if

\[
\lim_{n \to \infty} \max_{v, v' \in V_{n+1}} \left( \sum_{w \in V_n} \left| f_{vw}^{(n)} - f_{v'w}^{(n)} \right| \right) = 0 \tag{3.1}
\]

after an appropriate telescoping. Here $f_{vw}^{(n)}$ are entries of the stochastic matrix $F_n$ defined by the diagram $B$.

**Proof.** We observe that $B$ is uniquely ergodic if and only if the simplex $\Delta_{\infty}^{(n)}$ is a singleton for all $n = 1, 2, \ldots$

To prove the “if” part, it suffices to show that the diameter $\text{diam}(\Delta_{\infty}^{(n)}) \to 0$ as $m \to \infty$. The polytope $\Delta_{m}^{(n)}$ is the convex hull of the vectors $\{\overline{g}^{(n+m,n)}(v)\}_{v \in V_{n+m+1}}$. According to [2,12], we have

\[
d^*(\overline{g}^{(n+m,n)}(v), \overline{g}^{(n+m,n)}(v')) = \sum_{w \in V_n} \left| g^{(n+m,n)}_{vw} - g^{(n+m,n)}_{wv'} \right|.
\]

Then we obtain

\[
\sum_{w \in V_n} \left| g^{(n+m,n)}_{vw} - g^{(n+m,n)}_{wv'} \right| = \sum_{w \in V_n} \sum_{u \in V_{n+m}} (f_{vu}^{(n+m)} - f_{v'u}^{(n+m)}) g^{(n+m-1,n)}_{uw} \leq \sum_{u \in V_{n+m}} \left| f_{vu}^{(n+m)} - f_{v'u}^{(n+m)} \right| \sum_{w \in V_n} g^{(n+m-1,n)}_{uw} = d^*(\overline{f}^{(n+m)}(v), \overline{f}^{(n+m)}(v')).
\]

The last equality is due to the fact that $\sum_{u \in V_{n+m}} g^{(n+m-1,n)}_{uw} = 1$. Thus, we have

\[
d^*(\overline{g}^{(n+m,n)}(v), \overline{g}^{(n+m,n)}(v')) \leq d^*(\overline{f}^{(n+m)}(v), \overline{f}^{(n+m)}(v')) \to 0
\]
as $m \to \infty$. Hence $\text{diam}(\Delta_{m}^{(n)}) \to 0$ which implies that $\Delta_{\infty}^{(n)}$ is a singleton.

Next, we prove the “only if” part. Since $\Delta_{\infty}^{(n)}$ is a single point, we have $\text{diam}(\Delta_{m}^{(n)}) \to 0$ as $m \to \infty$ for each $n = 1, 2, \ldots$. Then

\[
\max_{v, v' \in V_{n+m+1}} d^*(\overline{g}^{(n+m,n)}(v), \overline{g}^{(n+m,n)}(v')) \leq \text{diam}(\Delta_{m}^{(n)}) \to 0 \text{ as } m \to \infty.
\]
Let \((\varepsilon_n)\) be a sequence converging to 0. For every \(n\), we can take sufficiently large \(m = m(n)\) such that
\[
\max_{v, v' \in V_{n+m+1}} \sum_{w \in V_n} \left| g_{vw}^{(n+m,n)} - g_{v'w}^{(n+m,n)} \right| \leq \varepsilon_n.
\]

Hence, we can find sequences \(\{n_i\}_{i \geq 1}, \{m_i\}_{i \geq 1}\) of positive integers such that
\[
\max_{v, v' \in V_{n_i+1}} \sum_{w \in V_{n_i}} \left| g_{vw}^{(n_i+m_i,n_i)} - g_{v'w}^{(n_i+m_i,n_i)} \right| \leq \varepsilon_{n_i},
\]
and such that
\[1 \leq n_1 < n_1 + m_1 = n_2 < n_2 + m_2 = n_3 < \ldots\]

Telescoping the Bratteli diagram \(B = (V, E)\) with respect to the levels \(n_1 < n_2 < n_3 < \ldots\), we obtain a new Bratteli diagram \(B' = (V', E')\) for which stochastic incidence matrices are \(G^{(n_i+m_i,n_i)}\), \(i = 1, 2, \ldots\) Then (3.2) implies (3.1), and the proof is complete.

We remark that the operation of telescoping and using the stochastic incidence matrix instead of the usual integer-valued incidence matrix are crucial steps for Theorem 3.1. In the following example we show that without telescoping the statement of the theorem is not true.

**Example 3.2.** In this example we illustrate the criterion given in Theorem 3.1. Let \(B_1\) be a Bratteli diagram with incidence matrices
\[
\tilde{F}_{n}^{(1)} = \begin{pmatrix} n & 1 \\ 1 & n \end{pmatrix}, \quad n \in \mathbb{N},
\]
and let \(B_2\) be a Bratteli diagram with incidence matrices
\[
\tilde{F}_{n}^{(2)} = \begin{pmatrix} n^2 & 1 \\ 1 & n^2 \end{pmatrix}, \quad n \in \mathbb{N}.
\]

It is known that the diagram \(B_1\) is uniquely ergodic, and \(B_2\) has exactly two finite-invariant ergodic measures, see details in [BKMS13], [ABKK17] Example 3.6, and [FT09].

We show how Theorem 3.1 works in this case and emphasize the importance of telescoping in its proof.

First notice that the both diagrams have the ERS property, hence the corresponding stochastic matrices are easy to compute (see Example 2.7):
\[
F_n^{(1)} = \begin{pmatrix}
1 - \frac{1}{n+1} & \frac{1}{n+1} \\
\frac{1}{n+1} & 1 - \frac{1}{n+1}
\end{pmatrix}
\]
and

\[ F_n^{(2)} = \begin{pmatrix} 1 - \frac{1}{n^2 + 1} & \frac{1}{n^2 + 1} \\ \frac{1}{n^2 + 1} & 1 - \frac{1}{n^2 + 1} \end{pmatrix}. \]

Obviously, without telescoping, for the both diagrams \( B_1 \) and \( B_2 \) the limit in (3.1) equals 2. However, the telescoping procedure reveals the crucial difference between the diagrams \( B_1 \) and \( B_2 \).

Suppose we have an ERS diagram with \( 2 \times 2 \) stochastic incidence matrices

\[ F_n = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix}. \]

As before, let \( G_{(n,n+m)} = (g_{vw}^{(n+m,n)}) \) be the corresponding product matrix. It can be easily proved by induction that, for arbitrary \( n,m \in \mathbb{N} \), the following formula holds:

\[ S(n,m) = \sum_{w \in V_n} \left| g_{vw}^{(n+m,n)} - g_{v'w}^{(n+m,n)} \right| = 2 \prod_{i=0}^{m} |(a_{n+i} - b_{n+i})|. \]

In the case of the diagram \( B_1 \), we obtain

\[ S(n,m) = 2 \prod_{i=0}^{m} \left( 1 - \frac{2}{n + i + 1} \right), \quad n,m \in \mathbb{N}. \]

Since the series \( \sum_{n=1}^{\infty} n^{-1} \) diverges, we see that

\[ S(n,m) \to 0, \quad m \to \infty. \]

Choose a decreasing sequence \( (\varepsilon_k) \) such that \( \varepsilon_k \to 0 \) as \( k \to \infty \). For \( n = n_1 \) and \( \varepsilon_1 \), find \( m_1 \) such that \( S(n_1, m_1) < \varepsilon_1 \). Set \( n_2 = n_1 + m_1 \). For \( \varepsilon_2 \), find \( m_2 \) such that \( S(n_2, m_2) < \varepsilon_2 \). Set \( n_3 = n_2 + m_2 \). Continuing in the same manner, we construct a sequence \( (n_k) \) such that \( S(n_k, n_{k+1} - n_k) < \varepsilon_k \). Telescope the diagram with respect to the levels \( (n_k) \). By Theorem 3.1, we conclude that the diagram \( B_1 \) is uniquely ergodic.

In the case of diagram \( B_2 \), the convergence of the series \( \sum_{n=1}^{\infty} n^{-2} \), does not allow us to conclude that \( S(n,m) \to 0 \) as \( m \to \infty \). Hence, the criterion is not applicable for the diagram \( B_2 \).

The following example shows that using stochastic incidence matrices is an important part of Theorem 3.1. If we use usual integer-valued incidence matrices instead of stochastic ones, then Theorem 3.1 will be not true.

**Example 3.3.** Let \( B \) be the stationary Bratteli diagram defined by the incidence matrices

\[ \tilde{F}_n = \tilde{F} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \]

for every \( n \in \mathbb{N} \). It is well known that \( B \) has a unique finite ergodic measure supported by the 3-odometer (see e.g. [BKMS10]). We show that \( B \) satisfies the
condition of unique ergodicity formulated in Theorem 3.1. It is easy to check that
the \( n \)-th power of \( \tilde{F} \) is

\[
\tilde{F}^n = \begin{pmatrix}
3^n & 0 \\
3^n - 2^n & 2^n
\end{pmatrix}.
\]

Hence the entries of the matrix \( \tilde{F} \) do not satisfy (3.1) even after taking products
of these matrices (which corresponds to telescoping of \( B \)). Notice that \( B \) has the
ERS property. For any \( n \in \mathbb{N} \) and a vertex \( w \in V_n \), we have \( h^{(n)}_w = 3^n \). Therefore,
the corresponding stochastic incidence matrix and its \( n \)-th power are

\[
F = \begin{pmatrix}
1 & 0 \\
\frac{1}{3} & \frac{2}{3}
\end{pmatrix}
\]

and

\[
F^n = \begin{pmatrix}
1 & 0 \\
1 - \frac{2^n}{3^n} & \frac{2^n}{3^n}
\end{pmatrix}.
\]

Hence, we see that \( B \) satisfies (3.1).

4. Ergodic invariant measures for finite rank Bratteli diagrams

In this section, we study ergodic invariant measures on finite rank Bratteli di-
agrams. Let \( B = (V, E) \) be a Bratteli diagram of rank \( k \geq 2 \), i.e., \( |V_n| = k \) for
\( n = 1, 2, \ldots \) Then, for any \( n \), the incidence matrices \( \tilde{F}_n = (\tilde{f}^{(n)}_{vw})_{v \in V_{n+1}, w \in V_n} \) and

\[
F^n = \begin{pmatrix}
1 & 0 \\
1 - \frac{2^n}{3^n} & \frac{2^n}{3^n}
\end{pmatrix}.
\]

Hence, we see that \( B \) satisfies (3.1).

4.1. Invariant measures in terms of decreasing sequences of simplices. Under the made assumption, the number of probability ergodic invariant measures
on a Bratteli diagram equals the number of vertices of a simplex \( \Delta_1^{(1)} \), which, in
turn, is the limit of a decreasing sequence of simplices \( \{\Delta_m^{(1)}\}_{m=1}^{\infty} \). Indeed, since the
square matrices \( F_T^n \) are invertible, one can restore the whole sequence of vectors
\( (\gamma^{(n)})_{n=1}^{\infty} \) from the vector \( \gamma^{(1)} \), see Theorem 2.5. Below we explicitly define these
simplices, using the incidence matrices of the Bratteli diagram, and study the way
these simplices are embedded one into another. This study will allow us to derive
Remark 4.1. (1) Because of the assumption formulated above, it is easily seen that, for any \( m, n \in \mathbb{N} \), the sets \( \Delta_m^{(n)} \) are simplices with exactly \( k \) vertices. Then the simplices \( \Delta^{(n)} (= \Delta^{(1)}) \) do not depend on the level \( V_n \). The index \( n \) points out only that \( \Delta^{(n)} \) has the basis enumerated by vertices of the level \( V_n \).

(2) Since all incidence matrices are non-singular, the intersection \( \Delta^{(1)}_\infty = \bigcap_{m=1}^\infty \Delta^{(1)}_m \) is a simplex which we can identify with the set \( \mathcal{M}_1(B) \) of all probability \( \mathcal{E} \)-invariant Borel measures on \( X_B \). The simplex \( \Delta^{(1)}_\infty \) has at most \( k \) vertices (see the proof in [BKMS13]) which is based on [Phe01, Pul71], and [BKMS10]). It is not hard to see that the dimension of \( \Delta^{(1)}_\infty \) can be any natural number \( l \), \( 1 \leq l \leq k \). In [BKMS13], the reader can find various examples of Bratteli diagrams where the number of ergodic \( \mathcal{E} \)-invariant measures varies from 1 (this is the case of uniquely ergodic diagrams) to \( k \).

(3) If \( \Delta^{(1)}_\infty \) is a simplex with \( l \) vertices then there are exactly \( l \) ergodic \( \mathcal{E} \)-invariant Borel probability measures on \( B \). In the case of finite rank Bratteli diagrams with nonsingular incidence matrices, the simplex \( \Delta^{(n)}_\infty \) also has exactly \( l \) vertices for every \( n \).

(4) We recall that \( \Delta^{(m)} \) denotes the standard simplex in \( \mathbb{R}^{|V_m|} \), and the entries of a vector from this simplex are enumerated by vertices from \( V_m \). In the case when \( |V_m| = k \), these simplices \( \Delta^{(m)} \) are essentially the same. Since the incidence matrices of \( B \) are nonsingular, we obtain that the vectors

\[
\mathbf{y}^{(m)}(w) = F_1^T \cdots F_{m-1}^T (\mathbf{e}^{(m)}(w))
\]  

are extreme in \( \Delta^{(1)}_m \) and form a basis in the simplex \( \Delta^{(1)}_m, m \geq 1 \). Hence, there is a one-to-one correspondence between vertices \( w \) of \( V_m \) and the extreme vectors \( \mathbf{y}^{(m)}(w) \) from \( \Delta^{(1)}_m \).

Lemma 4.2 shows explicitly how the simplex \( \Delta^{(1)}_n \) is embedded into \( \Delta^{(1)}_n \).

Lemma 4.2. Let \( \mathbf{y}^{(n)}(w) = (y_v^{(n)}(w)) : v \in V_1^T \) be the extreme vector from the simplex \( \Delta^{(1)}_n \) which corresponds to a vertex \( w \in V_n \), see (4.1). Then, for any \( v \in V_{n+1} \),

\[
\mathbf{y}^{(n+1)}(v) = \sum_{w \in V_n} f_{vw}^{(n)} \mathbf{y}^{(n)}(w).
\]  

Proof. We observe that the vector \( F_n^T (\mathbf{e}^{(n+1)}(v)) \) coincides with the \( v \)-th row of \( F_n \), and it can be written as a linear combination of vectors from the standard basis.
\{\overline{\eta}^{(n)}(w) : w \in V_n\}. Hence, similarly to Proposition 2.8 we have
\[
\overline{\eta}^{(n+1)}(v) = F_1^T \cdots F_{n-1}^T F_n^T \overline{\eta}^{(n)}(v)
\]
\[
= (F_1^T \cdots F_{n-1}^T) \left( \sum_{w \in V_n} f_{vw}^{(n)} \overline{\eta}^{(n)}(w) \right)
\]
\[
= \sum_{w \in V_n} f_{vw}^{(n)} \overline{\eta}^{(n)}(w).
\]
This proves the lemma. □

Theorem 2.5 states that, under the made assumption, the set \(\mathcal{M}_1(B)\) can be identified with the simplex \(\Delta^{(1)}_\infty\). In fact, since \(\det(F_n) \neq 0\) for \(n \geq 1\), we have
\[
\Delta^{(1)}_\infty = F_1^T \cdots F_{n-1}^T (\Delta^{(n)}_\infty).
\]
Therefore, each \(\Delta^{(n)}_\infty\) is a simplex with the same number of vertices. The inverse limit \(\lim\limits_{\xi} (F_n^T, \Delta^{(n)}_\infty)\) can be identified with \(\Delta^{(1)}_\infty\).

We recall that, in this case, every vector \(\overline{\eta}^{(1)}\) from the simplex \(\Delta^{(1)}_\infty\) corresponds to an \(E\)-invariant measure \(\mu\). The measure \(\mu\) is defined by a sequence of vectors \(\{\overline{\eta}^{(n)}\}\) such that \(\overline{\eta}^{(n)} = F_n^T \overline{\eta}^{(n+1)}\). The entries of \(\overline{\eta}^{(n)}\) in the standard basis \(\{\overline{\eta}^{(n)}(w)\}\) are the numbers \((q_w^{(n)})\) which give the values of measure \(\mu\) on the towers \(\{X_w^{(n)} : w \in V_n\}\). On the other hand, since \(\overline{\eta}^{(1)}\) belongs to every simplex \(\Delta^{(1)}_n\) (recall that \(F_n\) is nonsingular for every \(n\)), this vector can be represented in the basis formed by the extreme vectors \(\{\overline{\eta}^{(n)}(w) : w \in V_n\}\).

In the next result, we establish, using the same argument as in Proposition 2.8, the relation between the vector \(\overline{\eta}^{(1)}\) and the vectors \(\{\overline{\eta}^{(n)}(w) : w \in V_n\}\).

**Lemma 4.3.** Let \(\mu, \overline{\eta}^{(n)}\) and \(\{\overline{\eta}^{(n)}(w) : w \in V_n\}, n \in \mathbb{N}\) be as above. Then for every \(n > 1\), the entries \((q_w^{(n)} : w \in V_n)\) of \(\overline{\eta}^{(n)}\) are the barycentric coordinates of the vector \(\overline{\eta}^{(1)}\) in \(\Delta^{(1)}_\infty\).

**Proof.** Indeed, we have the following chain of equalities that proves the lemma.
\[
\overline{\eta}^{(1)} = (F_1^T \cdots F_{n-1}^T)(\overline{\eta}^{(n)})
\]
\[
= (F_1^T \cdots F_{n-1}^T) \left( \sum_{w \in V_n} q_w^{(n)} \overline{\eta}^{(n)}(w) \right)
\]
\[
= \sum_{w \in V_n} q_w^{(n)} (F_1^T \cdots F_{n-1}^T)(\overline{\eta}^{(n)}(w))
\]
\[
= \sum_{w \in V_n} q_w^{(n)} \overline{\eta}^{(n)}(w).
\]

Observe that relation (4.3) uniquely defines the numbers \(q_w^{(n)}\) by the vector \(\overline{\eta}^{(1)} \in \Delta^{(1)}_\infty\).
4.2. Subdiagrams of Bratteli diagrams and ergodic invariant measures. In this subsection, we study connections between vertex subdiagrams of a Bratteli diagram $B$ and ergodic invariant measures on $B$.

By a Bratteli subdiagram, we mean a Bratteli diagram $B'$ that can be obtained from $B = (V, E)$ by removing some vertices and edges from each level of $B$. Then $X_B' \subset X_B$ (for more details see e.g. [ABKK17]). In this paper, we will consider only the case of vertex subdiagrams. To define such a subdiagram, we start with a sequence $W = \{W_n\}_{n \geq 0}$ of proper subsets $W_n$ of $V_n$ for all $n$. The $n$-th level of the vertex subdiagram $B' = (W, E)$ is formed by the vertices from $W_n$. To define the edge set $E = \{E_n\}$ we take the restriction of $E_n$ on the set of edges connecting vertices from $W_n$ and $W_{n+1}$, $n \in \mathbb{N}$. This means that $e \in E_n$ if the source and range of $e$ are in $W_n$ and $W_{n+1}$, respectively. Thus, the incidence matrix $F_n'$ of $B'$ has the size $|W_{n+1}| \times |W_n|$, and it can be seen as a submatrix of $F_n$ corresponding to the vertices from $W_n$ and $W_{n+1}$. We say, in this case, that $W = (W_n)$ is the support of $B'$.

In the base of our approach to finite rank Bratteli diagrams is the following result proved in Theorem 2.1 in [Pul71]. We include its formulation for the reader’s convenience.

**Theorem 4.4 ([Pul71]).** Suppose $\{C_1, C_2, \ldots, C_n, \ldots\}$ is a sequence of finitely generated cones which is nested downward by inclusion (i.e. $C_{n+1} \subseteq C_n$ for all $n \geq 1$). If, for some $m \geq 0$, the size of each $C_n$ is $m$, then the intersection of the $C_n$ is also a finitely generated cone whose size is at most $m$.

If a cone $C$ is finitely generated, then there is some minimal number of points necessary to generate the cone. This number is called the size of $C$.

The following theorem will play an important role in our study of ergodic measures and their supports. For a finite rank Bratteli diagram $B$, it describes how vertices of $\Delta^{(1)}_\infty$ determine subdiagrams of $B$.

**Theorem 4.5.** Let $B$ be a Bratteli diagram of rank $k$, and let $B$ have $l$ probability ergodic invariant measures, $1 \leq l \leq k$. Let $\{\overline{y}_1, \ldots, \overline{y}_l\}$ denote the extreme vectors in $\Delta^{(1)}_\infty$. Then, after telescoping and renumbering vertices, there exist exactly $l$ disjoint subdiagrams $B_i$ with the corresponding sets of vertices $\{V_{n,i}\}_{n=0}^\infty$ such that

(a) $|V_{n,i}| = k_i > 0$ for every $n \geq 1$ and $i = 1, \ldots, l$;

(b) for any choice of $v_n \in V_{n,i}$, the extreme vectors $\overline{y}^{(n)}(v_n) \in \Delta^{(1)}_n$ converge to the extreme vector $\overline{y}_i \in \Delta^{(1)}_\infty$.

In general, the diagram $B$ can have up to $k - l$ disjoint subdiagrams $B'_j$ with vertices $\{V'_{n,j}\}_{n=0}^\infty$ such that they are also disjoint with subdiagrams $B_i$ and for any $w_n \in V'_{n,j}$, the extreme vectors $\overline{y}^{(n)}(w_n) \in \Delta^{(1)}_n$ converge to a non-extreme vector $\overline{y} \in \Delta^{(1)}_\infty$.

**Proof.** In the proof, we describe the process of forming the vertex sets $\{V_{n,i}\}_{n=1}^\infty$ and the corresponding subdiagrams $B_i$ for $i = 0, \ldots, l$. We observe that the way
of assigning a vertex to a subdiagram is not necessary unique and depends on the telescoping of the diagram.

For every \( i = 1, \ldots, l \), we can take, by Lemma 2.10 a sequence of vectors \( \overline{y}^{(n)}(w_i) \in \Delta^{(1)}_n \) (and corresponding vertices \( w_i = w_i(n) \in V_n \)) such that \( || \overline{y}^{(n)}(w_i) - \overline{y}_i || \to 0 \) as \( n \to \infty \). If \( n \) is sufficiently large, then the vectors \( \overline{y}^{(n)}(w_i) \}_{i=1}^l \) (and hence the vertices \( \{ w_i(n) \}_{i=1}^l \)) are disjoint. Therefore, we can choose the beginnings of the sequences \( \{ w_i(n) \}_{n=1}^{\infty} \) in such a way that all vertices \( \{ w_i(n) \}_{i=1}^l \) are disjoint for all \( n \) and the limits of the sequences \( \{ \overline{y}^{(n)}(w_i) \}_{n=1}^{\infty} \) stay unchanged. We set \( w_i(n) \in V_{n,i} \). Thus, at this stage of construction, each set \( V_{n,i}, i = 1, \ldots, l \) consists of a single point.

Now for every \( n \), pick up any vertex \( u = u(n) \) in \( V_n \setminus \{ w_i(n) \}_{i=1}^l \). Since the decreasing sequence of simplices \( \Delta^{(1)}_n \) lies in the compact set \( \Delta^{(1)}_1 \), the sequence \( \{ \overline{y}^{(n)}(u) \}_{n=1}^{\infty} \) has a convergent subsequence \( \{ \overline{y}^{(n_k)}(u) \}_{k=1}^{\infty} \). We recall that simplices \( \Delta^{(1)}_0 \) are decreasing, so that we may work only with the simplices \( \{ \Delta^{(1)}_{n_k} \}_{k=1}^{\infty} \) and the limiting simplex \( \Delta^{(1)}_\infty \). For the corresponding Bratteli diagram, this means that we apply telescoping with respect to the levels \( \{ n_k \}_{k=1}^{\infty} \). We note that the operation of telescoping does not change the limits of the converging sequences \( \overline{y}^{(n_k)}(w_i(n)) \), since the telescoping corresponds to the picking a subsequence \( \overline{y}^{(n_k)}(w_i(n_k)) \). Hence, after telescoping, we have the sequence of vertices \( u(n) \in V_n \setminus \{ w_i(n) \}_{i=1}^l \) such that there exists

\[
\lim_{n \to \infty} \overline{y}^{(n)}(u(n)) = \overline{y},
\]

where \( \overline{y} \in \Delta^{(1)}_{\infty} \) (by Lemma 2.10). If \( \overline{y} = \overline{y}_i \) for some \( i = 1, \ldots, l \), then we set \( u(n) \in V_{n,i} \). Otherwise, we set \( u(n) \in V_{n,0} \). Now, for every \( n \), pick a vertex \( u' \in V_n \setminus \{ w_i \}_{i=1}^l \cup \{ u \} \) and apply the same procedure again. Since \( B \) is of finite rank \( k \), we will use at most \( k - l \) telescopings to finally form the sets \( \{ V_{n,i} \}_{i=0}^l \). In general, the set \( V_{n,0} \) may be empty. If \( V_{n,0} \) is not empty, we may also divide it into disjoint subsets \( \{ V_{n,j} \} \) depending on the limits of \( \{ \overline{y}^{(n)}(u(n)) \}_{n=1}^{\infty} \), though we will not use such a partition later. The theorem is proved. \( \square \)

Renumber the vertices of the diagram such that in every \( V_n \) first come the vertices of \( V_{n,1} \), then the vertices of \( V_{n,2} , \ldots, V_{n,l} \) and, at last, the vertices of \( V_{n,0} \). Then \( F_n \) have blocks \( V_{n+1,j} \times V_{n,j} \) on the diagonal for \( j = 0, 1, \ldots, n \). Thus, after telescoping and renumbering vertices, the stochastic incidence matrices of the Bratteli diagram \( B \) from Theorem 4.15 will look as follows

\[
F_n = \begin{pmatrix}
F_{B_1}^{(n)} & * & \cdots & * & * \\
* & F_{B_2}^{(n)} & \cdots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & \cdots & F_{B_l}^{(n)} & * \\
* & * & \cdots & * & F_{B_0}^{(n)}
\end{pmatrix}, \quad (4.4)
\]
where $F^{(n)}_{B_i}$ are submatrices of the matrix $F_n$ of the size $|V_{n+1,i}| \times |V_{n,i}|$ for $i = 0, \ldots, l$ and symbol * represents other elements of the matrix.

The following theorem describes the properties of the submatrices $F^{(n)}_{B_i}$ for $i = 1, \ldots, l$. We use here notation of Theorem 4.5.

**Theorem 4.6.** Suppose that the Bratteli diagram $B$ can be written in the form $\Delta F$ or $u$ where $\delta$ for every $\sum w \in V_{n,i}$ tends to zero as $\lim_{n \to \infty}$. As far as $\sum w \in V_{n,i}$ we recall what we know about the vectors $\{\bar{y}^{(n)}(w)\}_{w \in V_n}$ (vertices of the simplex $\Delta^{(1)}_n$):

(a) they form a basis of $\mathbb{R}^k$ for any $n \in \mathbb{N}$,
(b) $\bar{y}^{(n)}(w) \to \bar{y}_j$ s $n \to \infty$, where $w \in V_{n,i}$, $i = 1, \ldots, l$,
(c) $\lim_{n \to \infty} \bar{y}^{(n)}(w)$ does not belong to the set $\{\bar{y}_i\}_{i=1}^l$ as $w \in V_{n,0}$.

For every $j = 1, \ldots, l$, we represent $\bar{y}_j \in \Delta^{(1)}_n$ as a barycentric combination of $\bar{y}^{(n)}(w)$:

$$\bar{y}_j = \sum_{w \in V_n} \alpha_w^{(n)} \bar{y}^{(n)}(w), \quad (4.5)$$

where $\sum_{w \in V_n} \alpha_w = 1$ and $\alpha_w \geq 0$ for all $w$.

We will show that $\alpha_w^{(n)} \to 0$ as $n \to \infty$ for $w \in V_n \setminus V_{n,j}$. Indeed, relation (4.5) can be written in the form

$$\bar{y}_j - \sum_{w \in V_n, j} \alpha_w^{(n)} \bar{y}^{(n)}(w) = \sum_{w \in V_n, j} \alpha_w^{(n)} \bar{y}^{(n)}(u).$$

Since $\sum_{w \in V_n} \alpha_w^{(n)} = 1$, we get

$$\sum_{w \in V_n, j} \alpha_w^{(n)} (\bar{y}_j - \bar{y}^{(n)}(w)) = \sum_{i \neq j} \sum_{w \in V_n, i} \alpha_w^{(n)} (\bar{y}^{(n)}(u) - \bar{y}_j). \quad (4.6)$$

As far as $\lim_{n \to \infty} \bar{y}^{(n)}(w) = \bar{y}_j$ for $w \in V_{n,j}$, the left-hand side of relation (4.6) tends to zero as $n \to \infty$. In the right-hand side of (4.6), we obtain that, for $u \in V_{n,i}$ and $i = \{1, \ldots, l\} \setminus \{j\}$,

$$\bar{y}^{(n)}(u) \to \bar{y}_j - \bar{y}_i, \quad n \to \infty.$$

For $u \in V_{n,0}$, we have $\bar{y}^{(n)}(u) \to \bar{z}(u)$, where $\bar{z}(u) \in \Delta^{(1)}_\infty$, and $\bar{z}(u)$ does not coincide with any of the vertices $\{\bar{y}_i\}_{i=1}^l$. Hence $\bar{z}(u) = \sum_{i=1}^l \beta_i \bar{y}_i$, where all $\beta_i$ are
non-negative, \( \sum_{i=1}^{l} \beta_i = 1 \), and at least two of coefficients from the set \( \{ \beta_i \}_{i=1}^{l} \) are strictly positive. Thus,
\[
\overline{z}(u) - \overline{y}_j = \sum_{i=1}^{l} \beta_i \overline{y}_i - \sum_{i=1}^{l} \beta_i \overline{y}_j = \sum_{i \neq j} \beta_i (\overline{y}_i - \overline{y}_j) \neq 0.
\]

Therefore the right-hand side of (4.6) is a linear combination of vectors which converge to linearly independent non-zero vectors \( \{ \overline{y}_i - \overline{y}_j \}_{i \neq j} \). It follows that all the coefficients \( \{ \alpha_{u}^{(n)} \}_{u \notin V_{n,j}} \) converge to zero as \( n \to \infty \). This proves that
\[
\sum_{w \in V_{n,j}} \alpha_{w}^{(n)} \to 1 \text{ as } n \to \infty.
\]

Since \( \overline{y}_{n+1}(v) \) approaches arbitrary close to \( \overline{y}_j \) for \( v \in V_{n+1,j} \) and \( n \) and sufficiently large \( n \), the same result holds for the coefficients of barycentric combination for \( \overline{y}_{n+1}(v) \in \Delta_{n+1}^{(1)} \). By (4.2), the corresponding combination for \( \overline{y}_{n+1}(v) \) has coefficients \( f_{vw}^{(n)} \). Hence, for every \( v \in V_{n+1,j} \) we have
\[
\sum_{w \in V_{n,j}} f_{vw}^{(n)} \to 1,
\]
and \( f_{vw}^{(n)} \to 0 \) as \( n \to \infty \) and \( w \in V_n \setminus V_{n,j} \). This means that
\[
\frac{1}{|V_{n+1,j}|} \sum_{v \in V_{n+1,j}} \sum_{w \in V_{n,j}} f_{vw}^{(n)} \to \delta_{ij} \text{ as } n \to \infty,
\]
and the proof is complete. \( \square \)

For the matrices \( G^{(n+m,n)} = F_{n+m} \cdots F_n \), the equality
\[
\overline{y}_{n+1}^{(n+m+1)}(u) = \sum_{w \in V_n} g_{uw}^{(n+m,n)} \overline{y}_{w}^{(n)}(w)
\]
can be established similarly to (4.2). Moreover, by the same argument as in Theorem 4.6 we can prove parts (1) and (2) of the following proposition (we leave the details to the reader). We note that statement (3) of this proposition is a straightforward corollary of Theorem 4.5.

**Corollary 4.7.** In notation of Theorem 4.6, the following properties hold: for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for every \( n > N \), any \( j = 1, \ldots, l \), and any \( m = 0, 1, \ldots \)
\[
\begin{align*}
(1) & \quad \min_{u \in V_{n+m+1,j}} \sum_{w \in V_{n,j}} g_{uw}^{(n+m,n)} \geq 1 - \varepsilon; \\
(2) & \quad \max_{u \in V_{n+m+1,j}} \sum_{w \in V_n \setminus V_{n,j}} g_{uw}^{(n+m,n)} \leq \varepsilon,
\end{align*}
\]

where \( g_{uw}^{(n+m,n)} \) are entries of \( G_{(n+m,n)} \);
there exist $C > 0$ and $N \in \mathbb{N}$ such that for every $n > N$ and every $j = 1, \ldots, l$ we have
\[ d(\overline{y}^{(n)}(w), \overline{y}_j) \geq C, \quad w \in V_{n,0}, \]
and
\[ \max_{w \in V_{n,0}} d(\overline{y}^{(n)}(w), \Delta_{\infty}^{(1)}) \to 0 \]
as $n \to \infty$.

4.3. The main theorem I. One of our main results, Theorem [4.9], is proved in this subsection. This result holds for a class of Bratteli diagrams which satisfy the following condition.

**Definition 4.8.** For a Bratteli diagram $B$, we say that a sequence of subsets $U_n \subset V_n$ defines blocks of vanishing weights (or vanishing blocks) in the stochastic incidence matrices $F_n$ if
\[ \sum_{w \in U_n, v \in U_{n+1}} f_{vw}^{(n)} \to 0, \quad n \to \infty \]
where $U_n^c = V_n \setminus U_n$.

If additionally, for every sequence of vanishing blocks $(U_n)$, there exists a constant $0 < C_1 < 1$ such that, for sufficiently large $n$,
\[ \min_{v \in U_{n+1}} \sum_{u \in U_n^c} f_{vu}^{(n)} \geq C_1 \max_{v \in U_{n+1}} \sum_{u \in U_n^c} f_{vu}^{(n)}, \quad (4.8) \]
then we say that the stochastic incidence matrices $F_n$ of $B$ have regularly vanishing blocks.

For the next theorem, we will use the following notation. Set
\[ \overline{d}_j^{(n)} = \frac{1}{|V_{n,j}|} \sum_{w \in V_{n,j}} \overline{y}^{(n)}(w), \quad j = 0, 1, \ldots, l, \]
where the subsets $V_{n,j}$ are defined as in Theorem 4.5. Then $\overline{d}_j^{(n)} \in \Delta_{n,j}^{(1)} := \text{Conv}\{\overline{y}^{(n)}(w), w \in V_{n,j}\}$, the convex hull of the set $\{\overline{y}^{(n)}(w), w \in V_{n,j}\}$. The sets $\Delta_{n,j}^{(1)}$ are subsimplices of $\Delta_n^{(1)}$, $j = 0, 1, \ldots, l$. We observe that, by Lemma 2.9,
\[ \max_{\overline{u} \in \Delta_n^{(1)}} \text{dist}(\overline{u}, \Delta_{\infty}^{(1)}) \to 0 \]
as $n \to \infty$, where $\Delta_{\infty}^{(1)} = \bigcap_{n=1}^{\infty} \Delta_n^{(1)}$.

**Theorem 4.9.** Let $B$ be a Bratteli diagram of rank $k$ such that the incidence matrices $F_n$ have the property of regularly vanishing blocks, see Definition 4.8. If $B$ has exactly $l$ ($1 \leq l \leq k$) ergodic invariant probability measures, then, after telescoping, the set $V_n$ can be partitioned into subsets $\{V_{n,1}, \ldots, V_{n,l}, V_{n,0}\}$ such that
(a) $V_{n,i} \neq \emptyset$ for $i = 1, \ldots, l$;
(b) $|V_{n,i}|$ does not depend on $n$, i.e., $|V_{n,i}| = k_i$ for $i = 0, 1, \ldots, l$ and $n \geq 1$;
(c) for \( j = 1, \ldots, l, \)
\[
\sum_{n=1}^{\infty} \left( 1 - \min_{v \in V_{n+1,j}} \sum_{w \in V_{n,j}} f_{vw}^{(n)} \right) < \infty;
\]

(d) for \( j = 1, \ldots, l, \)
\[
\max_{v, v' \in V_{n+1,j}} \sum_{w \in V_{n,j}} |f_{vw}^{(n)} - f_{v'w}^{(n)}| \to 0
\]
as \( n \to \infty; \)

(e1) for every \( w \in V_{n,0} \)
\[
\text{vol}_1 S(\overline{a}_1^{(n)}, \ldots, \overline{a}_l^{(n)}, \overline{g}^{(n)}(w)) \to 0
\]
as \( n \to \infty, \) where \( S \) is a simplex with vertices \( \overline{a}_1^{(n)}, \ldots, \overline{a}_l^{(n)}, \overline{g}^{(n)}(w), \) and \( \text{vol}_1(S) \) stands for the volume of \( S; \)

(e2) for every \( v \in V_{n+1,0} \) and for sufficiently large \( n, \) there exists some \( C > 0 \) such that, for every \( j = 1, \ldots, l, \)
\[
F_v^{(n,j)} = \sum_{w \in V_{n,j}} f_{vw}^{(n)} < 1 - C.
\]

Proof. First, assume that \( l = 1, \) i.e., the diagram \( B \) is uniquely ergodic. In this case we can set \( V_{n,1} = V_n \) for every \( n \geq 1. \) Then conditions (a) - (c) are obviously true. Condition (d) coincides with (3.1) (see Proposition 3.1), and we have \( V_{n,0} = \emptyset \) for every \( n. \)

Proof of (a) and (b). Consider now the case when \( l > 1. \) We construct partitions \( \{V_{n,1}, \ldots, V_{n,l}, V_{n,0}\} \) as in Theorem 4.5. It follows that conditions (a) and (b) are satisfied because they are proved in Theorem 4.3.

Proof of (c). Fix any \( \varepsilon > 0. \) Then, by property (2) of Corollary 4.7, we obtain that for every sufficiently large \( n, \) any \( m = 0, 1, \ldots, \) any \( j \in \{1, \ldots, l\}, \) and any \( v \in V_{n+m+2,j}, \) the following estimate holds:
\[
\varepsilon \geq \sum_{w \in V_n \setminus V_{n,j}} g_{v(w)}^{(n+m,n)}
\]
\[
= \sum_{u \in V_{n+m+1}} f_{vu}^{(n+m+1)} \sum_{w \in V_{n,j}} g_{uw}^{(n+m,n)}
\]
\[
= \sum_{u \in V_{n+m+1,j}} f_{vu}^{(n+m+1)} \sum_{w \in V_{n,j}} g_{uw}^{(n+m,n)} + \sum_{u \in V_{n+m+1,j}} f_{vu}^{(n+m+1)} \sum_{w \in V_{n,j}} g_{uw}^{(n+m,n)}
\]

(4.9)
Claim 1. For sufficiently large $n$ and for every $m = 0, 1, \ldots$, there exists a constant $C > 0$ such that
\[
\sum_{w \notin V_{n,j}} g_{uw}^{(n+m,n)} \geq C
\] (4.10)
whenever $u \notin V_{n+m+1,j}$.

Proof of Claim 1. If $u \in V_{n+m+1,i}$ for some $i = 1, \ldots, l$ and $i \neq j$, then Claim 1 is satisfied in virtue of Property (1) of Corollary 4.7.

Suppose that $u \in V_{n+m+1,0}$. Assume for contrary that $\sum_{w \notin V_{n,j}} g_{uw}^{(n+m,n)} \to 0$ for an infinite subsequence $(n_k)$. We telescope the diagram with respect to these levels. Then
\[
\sum_{w \in V_{n,j}} g_{uw}^{(n+m,n)} \to 1
\]
as $n \to \infty$, and by Theorem 4.6, we conclude that $d(\nu^{(n+m+1)}(u), \nu_j) \to 0$ as $n \to \infty$. This contradiction proves Claim 1.

We obtain from (4.9), (4.10)
\[
\varepsilon \geq C \sum_{u \notin V_{n+m+1,j}} f_{vu}^{(n+m+1)} + \sum_{u \in V_{n+m+1,j}} f_{vu}^{(n+m+1)} \sum_{w \notin V_{n,j}} g_{uw}^{(n+m,n)}.
\]
For $u \in V_{n+m+1,j}$, we have
\[
\sum_{w \notin V_{n,j}} g_{uw}^{(n+m,n)} = \sum_{v' \in V_{n+m}} f_{uv'}^{(n+m)} \sum_{w \notin V_{n,j}} g_{v'w}^{(n+m-1,n)}
\]
\[
= \sum_{v' \notin V_{n+m,j}} f_{uv'}^{(n+m)} \sum_{w \notin V_{n,j}} g_{v'w}^{(n+m-1,n)}
\]
\[
+ \sum_{v' \in V_{n+m,j}} f_{uv'}^{(n+m)} \sum_{w \notin V_{n,j}} g_{v'w}^{(n+m-1,n)}.
\]
Using the same arguments as above, we get
\[
\varepsilon \geq C \sum_{u \notin V_{n+m+1,j}} f_{vu}^{(n+m+1)} + C \sum_{u \in V_{n+m+1,j}} f_{vu}^{(n+m+1)} \sum_{v' \notin V_{n+m,j}} f_{uv'}^{(n+m)}
\]
\[
+ \sum_{u \in V_{n+m+1,j}} f_{vu}^{(n+m+1)} \sum_{v' \in V_{n+m,j}} f_{uv'}^{(n+m)} \sum_{w \notin V_{n,j}} g_{v'w}^{(n+m-1,n)}.
\]
A similar computation allows us to deduce
\[
\varepsilon \geq C \left[ \sum_{u \notin V_{n+m+1,j}} f_{vu}^{(n+m+1)} + \sum_{u_1 \in V_{n+m+1,j}, u_2 \notin V_{n+m,j}} f_{vu_1}^{(n+m+1)} f_{u_1u_2}^{(n+m)} + \ldots + \right.
\]
\[
\left. + \sum_{(u_1, \ldots, u_m) \in V_{n+m+1,j} \times \ldots \times V_{n+1,j}, u_{m+1} \notin V_{n,j}} f_{vu_1}^{(n+m+1)} f_{u_1u_2}^{(n+m)} \ldots f_{umu_{m+1}}^{(n)} \right]
\]
\[
= C \cdot M_v^{(n+m+1,n)},
\]
where by $M_{v}^{(n+m+1,n)}$ we denote the expression in quadratic brackets from the inequality above. Thus, we proved that for sufficiently large $n$, every $m = 0, 1, \ldots$, and every $v \in V_{n+m+2,j}$,

$$M_{v}^{(n+m+1,n)} \leq \frac{\varepsilon}{C}. \quad (4.11)$$

Denote

$$s_{v}^{(n+m+1)} = \sum_{u \in V_{n+m+1,j}} f_{vu}^{(n+m+1)}.$$  

Then, by definition of $M_{v}^{(n+m+1,n)}$, we have

$$M_{v}^{(n+m+1,n)} = s_{v}^{(n+m+1)} + \sum_{u \in V_{n+m+1,j}} f_{vu}^{(n+m+1)} M_{u}^{(n+m,n)}$$

$$= s_{v}^{(n+m+1)} + (1 - s_{v}^{(n+m+1)}) \sum_{u \in V_{n+m+1,j}} \frac{f_{vu}^{(n+m+1)}}{1 - s_{v}^{(n+m+1)}} M_{u}^{(n+m,n)}.$$

If we denote

$$r_{vu}^{(n+m+1)} = \frac{f_{vu}^{(n+m+1)}}{1 - s_{v}^{(n+m+1)}},$$  

then we see that

$$\sum_{u \in V_{n+m+1,j}} r_{vu}^{(n+m+1)} = 1. \quad (4.12)$$

Recall that $v$ is a vertex from $V_{n+m+2,j}$. Using (4.11), we get

$$M_{v}^{(n+m+1,n)} = s_{v}^{(n+m+1)} \left(1 - \sum_{u \in V_{n+m+1,j}} r_{vu}^{(n+m+1)} M_{u}^{(n+m,n)}\right)$$

$$+ \sum_{u \in V_{n+m+1,j}} r_{vu}^{(n+m+1)} M_{u}^{(n+m,n)}$$

$$\geq \frac{1}{2} s_{v}^{(n+m+1)} + \sum_{u \in V_{n+m+1,j}} r_{vu}^{(n+m+1)} M_{u}^{(n+m,n)}$$

$$\geq \frac{1}{2} \min_{v \in V_{n+m+2,j}} \sum_{u \in V_{n+m+1,j}} f_{vu}^{(n+m+1)} + \sum_{v \in V_{n+1,j}} \sum_{u \in V_{n,j}} f_{vu}^{(n)}.$$

Applying the same reasoning to $M_{u}^{(n+m,n)}$, $M_{u}^{(n+m-1,n)}$, $\ldots$, $M_{u}^{(n,n)}$ and using (4.12), we obtain

$$\frac{\varepsilon}{C} \geq \frac{1}{2} \left(\min_{v \in V_{n+m+2,j}} \sum_{u \in V_{n+m+1,j}} f_{vu}^{(n+m+1)} + \ldots + \min_{v \in V_{n+1,j}} \sum_{u \in V_{n,j}} f_{vu}^{(n)} \right). \quad (4.13)$$
Since $B$ is a diagram with regularly vanishing blocks, we can take $U_n = V_{n,j}$, $U_c = V_n \setminus V_{n,j}$ and apply Definition 4.8. It follows that
\[
\min_{v \in V_{n+1,j}} \sum_{u \notin V_{n,j}} f^{(n)}_{vu} \geq C_1 \max_{v \in V_{n+1,j}} \sum_{u \notin V_{n,j}} f^{(n)}_{vu}
\]
(4.14)
for sufficiently large $n$.

Applying (4.13) and (4.14), we obtain that for every $\varepsilon > 0$ there is some $N$ such that for $n > N$, every $m$, and every $j = 1, \ldots, l$,
\[
\max_{v \in V_{n+m+2,j}} \sum_{u \notin V_{n+m+1,j}} f^{(n+m+1)}_{vu} + \ldots + \max_{v \in V_{n+1,j}} \sum_{u \notin V_{n,j}} f^{(n)}_{vu} \leq \frac{2\varepsilon}{C \cdot C_1}.
\]
Hence, we finally obtain
\[
\sum_{n=1}^{\infty} \left( 1 - \min_{v \in V_{n+1,j}} \sum_{u \in V_{n,j}} f^{(n)}_{vu} \right) = \sum_{n=1}^{\infty} \max_{v \in V_{n+1,j}} \sum_{u \in V_{n,j}} f^{(n)}_{vu} < \infty.
\]
Thus, condition $(c)$ is proved.

Proof of $(d)$. The proof is based on an application of the property
\[
\max_{u, u' \in V_{n,j}} ||\bar{y}^{(n)}(u) - \bar{y}^{(n)}(u')|| \to 0 \text{ as } n \to \infty
\]
where $j = 1, \ldots, l$.

Recall that $d$ is the metric on $\mathbb{R}^k$ generated by the Euclidean norm $|| \cdot ||$, and the metric $d^*(\overline{x}, \overline{y}) = \sum_{i=1}^{k} |x_i - y_i|$ ($\overline{x}, \overline{y} \in \mathbb{R}^k$) is equivalent to $d$. Then, for the standard basis $\{e_i\}_{i=1}^{k}$, we have
\[
d^* \left( \bar{y}^{(n)}(u), \bar{y}^{(n)}(u') \right) = \sum_{w \in V_1} \left| g^{(n_1)}_{uw} - g^{(n_1)}_{uw'} \right|.
\]
Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers converging to zero.

Claim 2. There exist two sequences of natural numbers, $\{n_i\}_{i=0}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$, such that
\[
1 = n_0 < n_1 < n_2 = n_1 + m_1 < n_3 = n_2 + m_2 < \ldots
\]
and
\[
\sum_{w \in V_{n_i}} \left| g^{(n_i+m_i, n_i)}_{uw} - g^{(n_i+m_i, n_i)}_{uw'} \right| < \varepsilon_i
\]
for $u, u' \in V_{n_i+m_i,j}$.

Proof of Claim 2. We first choose $n_1 > 1$ such that
\[
\sum_{w \in V_1} \left| g^{(n_1)}_{uw} - g^{(n_1)}_{uw'} \right| < \varepsilon_1
\]
for all \( n \geq n_1 \) and all \( u, u' \in V_{n,j} \). This is possible since both \( \vec{y}^{(n)}(u) \) and \( \vec{y}^{(n)}(u') \) tend to \( \vec{y}_j \) as \( n \to \infty \) and \( u, u' \in V_{n,j} \). We also have the relation

\[
\vec{y}^{(n+1)}(u) = \sum_{w \in V_{n_1}} g_{uw}^{(n+1)} \vec{y}^{(n)}(w). \tag{4.15}
\]

There exists \( n_2 > n_1 \) such that for all \( n \geq n_2 \) and all \( u, u' \in V_{n,j} \) the following inequality holds:

\[
d^*(\vec{y}^{(n+n_1)}(u), \vec{y}^{(n+n_1)}(u')) < \varepsilon_2.
\]

Since the metrics \( d^*(E_1) \) and \( d^*(E_2) \), computed with respect to two different bases \( E_1 \) and \( E_2 \) in \( \mathbb{R}^k \), are equivalent, we can work with the basis \( \{\vec{y}^{(n_1)}_{w}\}_{w \in V_{n_1}} \). Hence, for \( n_2 \) sufficiently large, we get

\[
\sum_{w \in V_{n_1}} \left| g_{uw}^{(n_1+m_1,n_1)} - g_{u'w}^{(n_1+m_1,n_1)} \right| < \varepsilon_2
\]

for \( u, u' \in V_{n_2,j} \) and \( m_1 = n_2 - n_1 \). Next, we find \( n_3 > n_2 \) such that for all \( n \geq n_3 \) and all \( u, u' \in V_{n_3,j} \)

\[
\sum_{w \in V_{n_2}} \left| g_{uw}^{(n_2+m_2,n_2)} - g_{u'w}^{(n_2+m_2,n_2)} \right| < \varepsilon_3,
\]

where \( m_2 = n_3 - n_2 \). We continue the procedure and construct the sequences \( \{n_i\}_{i=0}^\infty \) and \( \{m_i\}_{i=1}^\infty \) which satisfy the statement of Claim 2.

To finish the proof of (d), we telescope \( B \) with respect to levels \( \{n_i\}_{i=0}^\infty \), and we are done.

**Proof of (e1) and (e2).** We observe that formulas (e1) and (e2) are consequences of condition (3) of Corollary 4.7. We note that condition (e2) has been already proved in Claim 1 above. Here we prove (e1).

By Lemma 2.9 we have, for any \( w \in V_{n,0} \),

\[
d^*(\vec{y}^{(n)}(w), \Delta^{(1)}_\infty) \to 0 \text{ as } n \to \infty.
\]

Denote by \( D \) the distance of \( \vec{y}^{(n)}(w) \) to the \((l-1)\)-dimensional subspace containing the simplex \( \Delta^{(1)}_\infty \). Then

\[
\operatorname{vol}_l S(\vec{y}_1, \ldots, \vec{y}_l, \vec{y}^{(n)}(w)) = \frac{1}{l} D \operatorname{vol}_{l-1} S(\vec{y}_1, \ldots, \vec{y}_l) \\
\leq \frac{1}{l} d(\vec{y}^{(n)}(w), \Delta^{(1)}_\infty) \operatorname{vol}_{l-1} S(\vec{y}_1, \ldots, \vec{y}_l) \\
\to 0
\]

as \( n \to \infty \). The above property and the conditions \( \vec{y}^{(n)}_j \to \vec{y}_j \) as \( n \to \infty \) imply (e1). \( \square \)
5. Bratteli diagrams of arbitrary rank and ergodic invariant measures

The main result of this section is Theorem 5.4. This theorem can be viewed as a converse statement to Theorem 4.9. We note that Theorem 5.4 holds for a wider set of Bratteli diagrams than Theorem 4.9.

5.1. A sufficient condition of unique ergodicity. In this subsection, we focus on condition (d) of Theorem 4.9. It states that after telescoping of a Bratteli diagram \(B\), which satisfies Theorem 4.9, one has

\[
\max_{v,v' \in V_{n+1,j}} \sum_{w \in V_n} \left| f^{(n)}_{vw} - f^{(n)}_{v'w} \right| \to 0
\]

as \(n \to \infty\) for \(j = 1, \ldots, l\). As was mentioned above, this condition is a consequence of the fact that

\[
d^*(\gamma^{(n)}(u),\gamma^{(n)}(u')) \to 0
\]

as \(n \to \infty\) whenever \(u, u' \in V_{n,j}\) for some \(j = 1, \ldots, l\).

It can be seen that relation (5.1) is equivalent to the following:

\[
\max_{u,u' \in V_{n,j}} \sum_{w \in V_1} \left| g^{(n,1)}_{uw} - g^{(n,1)}_{u'w} \right| \to 0
\]

(5.2)

as \(n \to \infty\).

Now we can formulate and prove an assertion in terms of matrices \(F_n\) (without using telescoping) which implies (5.2). We keep the same notation that was used in the previous section.

Let

\[
m_{n,j} = \min_{v \in V_{n+1,j}, w \in V_n} f^{(n)}_{vw}, \quad M_{n,j} = \max_{v \in V_{n+1,j}, w \in V_n} f^{(n)}_{vw},
\]

and

\[
m_{w}^{(n,j)} = \min_{v \in V_{n+1,j}} g^{(n,1,j)}_{vw}, \quad M_{w}^{(n,j)} = \max_{v \in V_{n+1,j}} g^{(n,1,j)}_{vw}, \quad w \in V_{1,j},
\]

where \(G_{(n,1,j)} = (g^{(n,1,j)}_{vw})_{v \in V_{n+1,j}, w \in V_{1,j}}\) is a matrix obtained by multiplication of only those blocks of incidence matrices \(F_n, \ldots, F_1\) which correspond to the vertices from \(\{V_{m,j}\}\) for \(m = 1, \ldots, n\). In other words, we have

\[
g^{(n+1,1,j)}_{vw} = \sum_{u \in V_{n+1,j}} f^{(n+1)}_{vu} g^{(n,1,j)}_{uw}
\]

(5.3)

where \(v \in V_{n+1,j}, w \in V_{1,j}\).

**Theorem 5.1.** Let \(B\) be a Bratteli diagram of rank \(k\) which satisfies conditions (a) – (c) from Theorem 4.9. If

\[
\sum_{n=1}^{\infty} m_{n,j} = \infty,
\]

then condition (5.2) holds, and subdiagram \(B_j\) corresponding to the vertices \(\{V_{n,j}\}_{n=1}^{\infty}\) is uniquely ergodic.
Proof. First, we remark that if \(|V_{n,j}| = 1\), then (5.2) is true. So, we assume, without loss of generality, that \(|V_{n,j}| \geq 2\). Further, if \(M_{w}^{(n,j)} = m_{w}^{(n,j)}\) for infinitely many \(n\) and some \(w \in V_{n}\) then, by (c), condition (5.2) automatically holds. Thus, we assume that \(m_{w}^{(n,j)} < M_{w}^{(n,j)}\) for every \(n \geq 1\) and every \(w \in V_{n}\). It follows that if \(m_{w}^{(n,j)} = g_{vw}^{(n,1,j)}\) and \(M_{w}^{(n,j)} = g_{vw'}^{(n,1,j)}\) for some \(v, v' \in V_{n+1,j}\), then \(v \neq v'\). For every \(v \in V_{n+1,j}\), denote

\[
S_{v}^{(n,j)} = \sum_{u \in V_{n,j}} f_{vu}^{(n)},
\]

It follows from condition (c) of Theorem 4.9 that \(S_{v}^{(n,j)} > 0\) where \(v \in V_{n+1,j}\) and \(n\) is large enough. In the following part of the proof we will implicitly assume that \(n\) is already chosen sufficiently large. Using (5.3), we obtain that, for every \(v \in V_{n+2,j}\) and \(w \in V_{1,j}\),

\[
M_{w}^{(n,j)} - g_{vw}^{(n+1,1,j)} = \sum_{u \in V_{n+1,j}} f_{vu}^{(n+1)} S_{v}^{(n+1,j)} - \sum_{u \in V_{n+1,j}} f_{vu}^{(n+1)} g_{uw}^{(n+1,1,j)}
\]

\[
= \sum_{u \in V_{n+1,j}} f_{vu}^{(n+1)} \frac{S_{v}^{(n+1,j)}}{S_{v}^{(n+1,j)}} M_{w}^{(n,j)} - S_{v}^{(n+1,j)} \sum_{u \in V_{n+1,j}} f_{vu}^{(n+1)} g_{uw}^{(n+1,1,j)}
\]

\[
= \sum_{u \in V_{n+1,j}} f_{vu}^{(n+1)} \frac{S_{v}^{(n+1,j)}}{S_{v}^{(n+1,j)}} (M_{w}^{(n,j)} - g_{uw}^{(n,1,j)})
\]

\[
+ (1 - S_{v}^{(n+1,j)}) \sum_{u \in V_{n+1,j}} f_{vu}^{(n+1)} \frac{S_{v}^{(n+1,j)}}{S_{v}^{(n+1,j)}} g_{uw}^{(n,1,j)}
\]

\[
\geq \frac{m_{n+1,j}}{S_{v}^{(n+1,j)}} (M_{w}^{(n,j)} - m_{w}^{(n,j)}) + (1 - S_{v}^{(n+1,j)}) \sum_{u \in V_{n+1,j}} f_{vu}^{(n+1)} \frac{S_{v}^{(n+1,j)}}{S_{v}^{(n+1,j)}} g_{uw}^{(n,1,j)}
\]

\[
\geq m_{n+1,j} (M_{w}^{(n,j)} - m_{w}^{(n,j)}) + (1 - S_{v}^{(n+1,j)}) \sum_{u \in V_{n+1,j}} f_{vu}^{(n+1)} \frac{S_{v}^{(n+1,j)}}{S_{v}^{(n+1,j)}} g_{uw}^{(n,1,j)}.
\]

Similarly, we can prove that

\[
g_{vw}^{(n+1,1,j)} - m_{w}^{(n,j)} \geq m_{n+1,j} (M_{w}^{(n,j)} - m_{w}^{(n,j)}) - (1 - S_{v}^{(n+1,j)}) \sum_{u \in V_{n+1,j}} f_{vu}^{(n+1)} \frac{S_{v}^{(n+1,j)}}{S_{v}^{(n+1,j)}} g_{uw}^{(n,1,j)}.
\]
It follows from the above inequalities that
\[ m_w^{(n,j)} + m_{n+1,j}(M_w^{(n,j)} - m_w^{(n,j)}) - (1 - S_v^{(n+1,j)}) \sum_{u \in V_{n+1,j}} \frac{f_{vu}^{(n+1)}}{S_v^{(n+1,j)}} g_{uw}^{(n+1,j)} \leq \]
\[ \leq g_{w,1,1}^{(n+1,j)} \leq \]
\[ \leq M_w^{(n,j)} - m_{n+1,j}(M_w^{(n,j)} - m_w^{(n,j)}) - (1 - S_v^{(n+1,j)}) \sum_{u \in V_{n+1,j}} \frac{f_{vu}^{(n+1)}}{S_v^{(n+1,j)}} g_{uw}^{(n+1,j)}. \]
Therefore,
\[ m_w^{(n,j)} + m_{n+1,j}(M_w^{(n,j)} - m_w^{(n,j)}) - (1 - S_v^{(n+1,j)}) \sum_{u \in V_{n+1,j}} \frac{f_{vu}^{(n+1)}}{S_v^{(n+1,j)}} g_{uw}^{(n+1,j)} \leq \]
\[ \leq m_{w,1,1}^{(n+1,j)} \leq M_{w}^{(n+1,j)} \leq \]
\[ \leq M_w^{(n,j)} - m_{n+1,j}(M_w^{(n,j)} - m_w^{(n,j)}) - (1 - S_v^{(n+1,j)}) \sum_{u \in V_{n+1,j}} \frac{f_{vu}^{(n+1)}}{S_v^{(n+1,j)}} g_{uw}^{(n+1,j)}. \]
Hence we have
\[ M_{w}^{(n+1,j)} - m_{w}^{(n+1,j)} \leq (M_w^{(n,j)} - m_w^{(n,j)}) (1 - 2m_{n+1,j}). \tag{5.4} \]
Applying (5.4) finitely many times, we finally obtain
\[ M_{w}^{(n+1,j)} - m_{w}^{(n+1,j)} \leq (M_w^{(1,j)} - m_w^{(1,j)}) \prod_{s=2}^{n+1} (1 - 2m_{s,j}). \]
The condition \( \sum_{n=1}^{\infty} m_{n,j} = \infty \) implies that \( \prod_{s=1}^{\infty} (1 - m_{s,j}) = 0. \) This means that
\[ \lim_{n \to \infty} (M_w^{(n,j)} - m_w^{(n,j)}) = 0 \]
for each \( w \in V_{1,j}. \) The last condition is equivalent to (5.2). The unique ergodicity of \( B_j \) follows from Theorem 5.1 and the proposition is proved.

The following criterion of unique ergodicity is an immediate corollary of Theorem 5.1. It is important to observe that it is true for an arbitrary Bratteli diagram.

**Corollary 5.2.** Let \( B \) be a Bratteli diagram with stochastic incidence matrices \( F_n \) and
\[ m_n = \min_{v \in V_{n+1}, w \in V_n} f_{vw}^{(n)}. \]
If
\[ \sum_{n=1}^{\infty} m_n = \infty, \]
then \( B \) is uniquely ergodic.
5.2. The main theorem II. In this subsection, we define a class of Bratteli diagrams that generalizes, in some sense, the class of Bratteli diagrams of finite rank. Then we prove the converse of Theorem 4.9.

We recall that a (vertex) subdiagram \( B' \) of a Bratteli diagram \( B \) is defined by a sequence of proper subsets of vertices \( W_n \subset V_n \) and by the corresponding sequence of incidence matrices \( F'_n = (f_{vw}^{(n)})_{v \in W_{n+1}, w \in W_n} \). Denote by \( X_{B'} \) the set of all infinite paths of \( B' \). Then \( X_{B'} \subset X_B \) is a closed subset of \( X_B \). Let \( \hat{X}_{B'} \) be the subset of paths in \( \hat{X}_B \) which are equivalent to paths from \( X_{B'} \). Let \( \mu' \) be any tail invariant probability measure on \( B' \). Then \( \mu' \) can be uniquely extended to an invariant (finite or infinite) measure \( \hat{\mu}' \) on \( \hat{X}_{B'} \) such that \( \hat{\mu}'|_{X_{B'}} = \mu'|_{X_{B'}} \) (for more details see [ABKK17]).

We will need the following result from [ABKK17, Theorem 2.2] which holds for arbitrary Bratteli diagram.

**Proposition 5.3 ([ABKK17]).** Let \( B \) be a Bratteli diagram, and let \( B' \) be its subdiagram defined by a sequence of vertices \( W_n \). If

\[
\sum_{n=1}^{\infty} \max_{v \in W_{n+1}} \left( \sum_{w \notin W_n} f_{vw}^{(n)} \right) < \infty,
\]

then any tail invariant probability measure \( \mu' \) on \( X_{B'} \) extends to a finite invariant measure \( \hat{\mu}' \) on \( \hat{X}_{B'} \).

In what follows, we will assume that (after telescoping) a Bratteli diagram \( B \) admits a partition

\[ V_n = \bigcup_{i=0}^{l_n} V_{n,i}, \quad n = 1, 2, \ldots, \]

into disjoint subsets \( V_{n,i} \) such that \( V_{n,i} \neq \emptyset \), for \( i = 1, \ldots, l_n \), and \( l_n \geq 1 \). Moreover, let

\[ L_{n+1} = \{1, \ldots, l_{n+1}\} = \bigcup_{i=1}^{l_n} L_{n+1}^{(i)}, \]

where \( L_{n+1}^{(i)} \neq \emptyset \) and \( L_{n+1}^{(i)} \cap L_{n+1}^{(j)} = \emptyset \) for \( i \neq j, i, j = 1, \ldots, l_n \). Hence, for every \( j = 1, \ldots, l_{n+1} \), there exists a unique \( i = i(j) \in \{1, \ldots, l_n\} \) such that \( j \in L_{n+1}^{(i)} \). Denote

\[ V_{n+1}^{(i)} = \bigcup_{j \in L_{n+1}^{(i)}} V_{n+1,j} \]

for \( 1 \leq i \leq l_n \).

Now we formulate conditions (c1), (d1), (e1) which are analogues of conditions (c), (d), (e) used in Theorem 4.9.
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\[ \sum_{n=1}^{\infty} \left( \max_{i \in L_n} \max_{v \in V_{n+1}^{(i)}} \sum_{w \notin V_{n,i}} f_{vw}^{(n)} \right) < \infty; \]

\[ \max_{j \in L_{n+1}} \max_{v,v' \in V_{n+1,j}} \sum_{w \in V_n} |f_{vw}^{(n)} - f_{v'w}^{(n)}| \to 0 \text{ as } n \to \infty; \]

\[ \text{for every } v \in V_{n+1,0}, \]

\[ \sum_{w \in V_n \setminus V_{n,0}} f_{vw}^{(n)} \to 1 \text{ as } n \to \infty; \]

\[ \text{there exists } C > 0 \text{ such that } F_{vi}^{(n)} \leq 1 - C \text{ for every } i = 1, \ldots, l, \text{ where} \]

\[ F_{vi}^{(n)} = \sum_{w \in V_{n,i}} f_{vw}^{(n)}. \]

We can interpret the sets \( L_n^{(i)} \), defined above, in terms of subdiagrams. For this, select a sequence \( \bar{i} = (i_1, i_2, \ldots) \) such that \( i_1 \in L_1, i_2 \in L_2^{(i_1)} \), \( i_3 \in L_3^{(i_2)} \), and define a subdiagram \( B_{\bar{i}} = (V, E) \), where

\[ V = \bigcup_{n=1}^{\infty} V_{n,i_n} \cup \{v_0\}. \]

Let

\[ \mathcal{L} = \{ \bar{i} = (i_1, i_2, \ldots) : i_1 \in L_1, i_{n+1} \in L_{n+1}^{(i_n)}, n = 1, 2, \ldots \}. \]

We call such a sequence \( \bar{i} \in \mathcal{L} \) a chain. A finite chain \( \bar{i}(m,n) \) is a sequence \( \{i_{n+m+1}, \ldots, i_n\} \) such that \( i_{n+s} \in L_{n+s+1}^{(i_{n+s-1})}, s = 1, \ldots, m + 1 \). We remark that a Bratteli diagram \( B = (V, E) \) of finite rank has the form described in this subsection (see Corollary 5.6 below).

**Theorem 5.4.** Let \( B = (V, E) \) be a Bratteli diagram satisfying the conditions (c1), (d1), (e1). Then:

1. for each \( \bar{i} \in \mathcal{L} \), any measure \( \mu_{\bar{i}} \) defined on \( B_{\bar{i}} \) has a finite extension \( \hat{\mu}_{\bar{i}} \) on \( B \),
2. each subdiagram \( B_{\bar{i}}, \bar{i} \in \mathcal{L} \), is uniquely ergodic,
3. after normalization, the measures \( \hat{\mu}_{\bar{i}}, \bar{i} \in \mathcal{L} \), form the set of all probability ergodic invariant measures on \( B \).

**Proof.** (1) To prove finiteness of the measure extension \( \hat{\mu}_{\bar{i}} \) from \( B_{\bar{i}} \), we apply Proposition 5.3 with \( W_n = V_{n,i_n} \) for every \( n \). Then we have

\[ \sum_{n=1}^{\infty} \max_{v \in V_{n+1,i_{n+1}}} \left( \sum_{w \notin V_{n,i_n}} f_{vw}^{(n)} \right) \leq \sum_{n=1}^{\infty} \max_{i \in L_n} \max_{v \in V_{n+1}^{(i)}} \left( \sum_{w \notin V_{n,i}} f_{vw}^{(n)} \right) < \infty. \]
Therefore, any invariant probability measure $\mu$ on $X_{B_{\tau}}$ extends to a finite measure $\hat{\mu}$ on $\hat{X}_{B_{\tau}}$.

(2) We show that there exists a unique probability invariant measure concentrated on the set $\hat{X}_{B_{\tau}}$. Let $\mu = \hat{\mu}$ be a probability measure on $X_B$ which can be obtained as the extension of a measure defined on the subdiagram $B_{\tau}$. Let $\{\overline{q}^{(n)} = (q^{(n)}_w)_{w \in V_n}\}$ be a sequence of vectors such that $\mu(X^{(n)}_w) = q^{(n)}_w$. The condition $\mu(\hat{X}_{B_{\tau}}) = 1$ implies that

$$Q^{(n)}_1 = \sum_{w \in V_{n,m}} q^{(n)}_w \rightarrow 1$$

as $n \to \infty$. Fix $n \geq 1$ and apply (2.19). Denoting $i = i_{n+m+1}$, we obtain

$$\overline{q}^{(n)} = \sum_{v \in V_{n+m+1,i}} q^{(n+m+1)}_v \overline{g}^{(n+m,n)}_{v} + \sum_{v \notin V_{n+m+1,i}} q^{(n+m+1)}_v \overline{g}^{(n+m,n)}_{v}$$

$$= Q^{n+m+1}_{1} \sum_{v \in V_{n+m+1,i}} q^{(n+m+1)}_v \overline{g}^{(n+m,n)}_{v} + (1 - Q^{n+m+1}_{1}) \sum_{v \notin V_{n+m+1,i}} q^{(n+m+1)}_v \overline{g}^{(n+m,n)}_{v}$$

$$= Q^{n+m+1}_{1} \overline{y}^{(n+m+1)}_{1} + (1 - Q^{n+m+1}_{1}) \overline{y}^{(n+m+1)}_{2}$$

where $\overline{y}^{(n+m+1)}_{1}$ and $\overline{y}^{(n+m+1)}_{2}$ denote the corresponding sums in the third equality above.

Let $\Delta^{(n)}_{m,i}$ be the convex polytope in $\Delta^{(n)}$ spanned by the vectors $\overline{g}^{(n+m,n)}_{v}$, $v \in V_{n+m+1,i}$. Then $\overline{y}^{(n+m+1)}_{1} \in \Delta^{(n)}_{m,i}$. Since $Q^{n+m+1}_{1} \rightarrow 1$ as $m \rightarrow \infty$, we have $\overline{y}^{(n+m+1)}_{1} \rightarrow \overline{q}^{(n)}$ as $m \rightarrow \infty$, hence

$$d^*(\overline{q}^{(n)}, \Delta^{(n)}_{m,i}) \rightarrow 0$$

as $m \rightarrow \infty$. If $\mu'$ is another probability measure such that $\mu'(\hat{X}_{B_{\tau}}) = 1$ and $q^{(n)}_w = \mu(X^{(n)}_w)$, then $d^*(\overline{q}^{(n)}, \Delta^{(n)}_{m,i}) \rightarrow 0$ as $m \rightarrow \infty$. Using the same arguments as in the proof of Theorem 3.1 we see that condition (d1) implies that $\text{diam}(\Delta^{(n)}_{m,i}) \rightarrow 0$ as $m \rightarrow \infty$. Thus, $\overline{q}^{(n)} = \overline{q}^{(n)}$ for every $n = 1,2, \ldots$ and this implies $\mu = \mu'$. This equality means that $B_{\tau}$ is uniquely ergodic.

(3) Before passing to the proof of the property (3), we need to prove some additional properties of the matrices $F_{n}$’s and $G_{(n+m,n)}$’s. The following lemma is an analogue of part (1) of Corollary 4.7.

**Lemma 5.5.** Let $1 \leq i_{n+m+1} \leq l_{n+m+1}$, and let $l(m,n)$ be the finite chain determined by $i_{n+m+1}$. Then, for every $v \in V_{n+m+1,i'}$, $i' = i_{n+m+1}$, the following
relation holds

\[ \sum_{w \in V_{n,i}} g_{v_{m,n}}^{(n+m,n)} \geq C_n \to 1, \quad n \to \infty, \]

where \( i = i_n \) is the last element of the chain \( T(m,n) \).

Proof. We have

\[
\sum_{w \in V_{n,i}} g_{v_{m,n}}^{(n+m,n)} = \sum_{(u_n, \ldots, u_1) \in V_{n+m} \times \cdots \times V_{n+1}} f_{v_{1,u_n}}^{(n+m)} f_{u_{n,u_{n-1}}}^{(n+m-1)} \cdots f_{u_{2,u_1}}^{(n)} \sum_{w \in V_{n,i}} f_{u_1}^{(n)}
\geq \left( \min_{u_1 \in V_{n+1}} \sum_{w \in V_{n,i}} f_{u_1}^{(n)} \right) \cdots \left( \min_{u_{n+1} \in V_{n+m}, u_n \in V_{n+m,i_n+1}} \sum_{w \in V_{n+m,i_n+1}} f_{u_{n+1}u_n}^{(n)} \right)
\geq \prod_{s=1}^{\infty} \left( \min_{u_s \in V_{n+i_s+1}} \sum_{1 \leq l_s \leq l_s} f_{u_s}^{(s)} \right)
= C_n \to 1 \quad \text{as} \quad n \to \infty.
\]

Thus, Lemma 5.5 is proved. \( \square \)

We continue the proof of (3). Let \( \mu \) be an ergodic invariant probability measure on \( B \). We find a chain \( \tilde{T} \) such that the corresponding subdiagram \( B_{\tilde{T}} \) supports \( \mu \). Denote \( q_{v_{m,n}}^{(n)} = \mu(X_{v_{m,n}}^{(n)}) \) and consider the sequence of vectors \( \{ \tilde{T}_{v_{m,n}}^{(n)} = (q_{v_{m,n}}^{(n)})_{w \in V_{n,i}} \} \), where each vector \( \tilde{T}_{v_{m,n}}^{(n)} \in \Delta_\infty^{(n)} \) is considered as a vertex of the convex set \( \Delta_\infty^{(n)} \), \( n = 1, 2, \ldots \).

According to Lemma 2.10, we can find a sequence of vectors \( \{ \tilde{v}_{v_{m,n}}^{(n,m)}(v_{m}) \} \), \( v_m = v_{m,n} \in V_{n+m+1} \) of \( \Delta_\infty^{(n)} \) such that \( \tilde{v}_{v_{m,n}}^{(n,m)}(v_{m}) \to \tilde{T}_{v_{m,n}}^{(n)} \) as \( m \to \infty \). Recall that

\[
\tilde{v}_{v_{m,n}}^{(n,m)}(v_{m}) = F_{n}^{T} \cdots F_{n+1}^{T} n_m \mu (v_{m}) (v_{m+1}) = \tilde{v}_{v_{m,n}}^{(n+m,n)}(v_{m}).
\]

Assume first that \( v_m \notin V_{n+m+1,0} \) for infinitely many \( m \). We telescope the diagram so that \( v_m \notin V_{n+m+1,0} \) for all \( m \). We will show that in this case \( \mu \) coincides with the measure \( \mu_i \) supported by subdiagram \( B_{\tilde{T}_i} \), where \( B_{\tilde{T}_i} \) is determined by vectors \( \{ \tilde{v}_{v_{m,n}}^{(n,m)}(v_{m}) \} \). First, we notice that there exists a unique \( 1 \leq i_{n+m+1} \leq l_{n+m+1} \) such that \( v_m \in V_{n+m+1,i_{n+m+1}} \). The number \( i_{n+m+1} \) determines the finite chain \( \{ i_{n+m+1}, i_{n+m}, \ldots, i_n \} \).

Then, by Lemma 5.5, we have

\[
\sum_{w \in V_{n,i_n}} g_{v_{m,n}}^{(n+m,n)} \geq C_n \to 1 \quad (5.5)
\]

as \( n \to \infty \). We choose \( m_n \) such that, for \( m \geq m_n \), we have

\[
d^*(\tilde{v}_{v_{m,n}}^{(n+m,n)}, q^{(n)}) \leq \varepsilon_n \quad (5.6)
\]
where $\varepsilon_n \to 0$. This inequality implies that
\[
d^*(g^{(n+m,n)}_{v_m}, g^{(n+s,n)}_{v_s}) \leq 2\varepsilon_n
\]
for $m, s \geq m_n$. Therefore,
\[
\sum_{w \in V_n} \left| g^{(n+m,n)}_{v_m w} - g^{(n+s,n)}_{v_s w} \right| \leq 2\varepsilon_n
\]
for $m, s \geq m_n$.

We show that all vertices $v_m$ belong to the same chain. Indeed, let
\[
\{i_n, i_{n+1}, \ldots, i_{n+s-1}, i_{n+s+1} \}
\]
be the chain determined by $i_n$ as above, $v_s \in V_{n+s+1,i_{n+s+1}}$. Then condition (c1) and relation (5.7) imply that $i_n = i_n$. The property (5.6) implies that
\[
g^{(n+m,n)}_{v_m w} \to q^{(n)}_w
\]
as $m \to \infty$ for every $w \in V_n$. Further, we have from (5.5) that
\[
\sum_{w \in V_n i_n} q^{(n)}_w = \lim_{m \to \infty} \sum_{w \in V_n i_n} g^{(n+m,n)}_{v_m w} \geq C_n.
\]
Since $C_n \to 1$, we obtain that
\[
\mu \left( \bigcup_{w \in V_n i_n} X^{(n)}_w \right) \to 1
\]
as $n \to \infty$. In this way we have found numbers $1 \leq i_n \leq l_n$ such that (5.8) holds. Using the same reasoning as above, we prove that $i_{n+1} \in I^{(i_n)}_{n+1}$ for every $n \geq 1$, so $\tilde{\tau} = (i_1, i_2, \ldots)$ forms an infinite chain. Therefore, we determined the subdiagram $B^{(n)}_{\tilde{\tau}}$ corresponding to the chain $\tilde{\tau}$, which supports measure $\mu$. This allows us to conclude that $\mu = \mu_{\tau}$.

To finish the proof of property (3), we consider the case when $g^{(n+m,n)}_{v_m} \to \overline{g}^{(n)}_{v_m}$ as $m \to \infty$ and $v_m \in V_{n+m+1,0}$ for infinitely many $m$, or, equivalently, for all $m \geq 1$. Using (e1), we can find, for every $v \in V_{n+1,0}$, a vector
\[
\overline{b}^{(n)}(v) = \overline{b}^{(n)} = \sum_{s \in V_{n+1,0}} B^{(n)}_s \overline{\tau}^{(n)}_s,
\]
where $\sum_{s \notin V_{n+1,0}} B^{(n)}_s = 1$ and $B^{(n)}_s \geq 0$ for all $s, n$, and such that
\[
d^*(\overline{\tau}^{(n)}_v, \overline{b}^{(n)}) \leq \varepsilon_n,
\]
where $\varepsilon_n \to 0$ as $n \to \infty$. We will show that, for every $m \geq 1$ and for every $v \in V_{n+m+1,0}$, there exists a vector
\[
\overline{b}^{(n+m,n)}(v) = \overline{b}^{(n+m,n)} \in \text{Conv}(g^{(n+m,n)}_s, s \in V_{n+m+1} \setminus V_{n+m+1,0})
\]
such that
\[
d^*(g^{(n+m,n)}_v, \overline{b}^{(n+m,n)}) \leq \varepsilon_n \to 0, \quad n \to \infty.
\]
It follows from (5.9) and (5.10) that, taking $n + m$ instead of $n$, we can find a vector

$$\mathbf{v}^{(n+m)} = \sum_{s \notin V_{n+m+1,0}} B_s^{(n+m)}\mathbf{f}_s^{(n+m)}$$

where $\sum_{s \notin V_{n+m+1,0}} B_s^{(n+m)} = 1$ and $B_s^{(n+m)} \geq 0$, which satisfies the relation

$$\sum_{u \in V_{n+m}} \left| f_{vu}^{(n+m)} - \sum_{s \notin V_{n+m+1,0}} B_s^{(n+m)} f_{su}^{(n+m)} \right| \leq \varepsilon_{n+m}.$$ 

Define

$$\mathbf{v}^{(n+m,n)}(v) := \sum_{s \notin V_{n+m+1,0}} B_s^{(n+1)}\mathbf{f}_s^{(n+m,n)},$$

where

$$g_v^{(n+m,n)} = \sum_{u \in V_{n+m}} f_u^{(n+m)} g_u^{(n+m-1,n)}.$$ 

Then we obtain

$$\mathbf{v}^{(n+m,n)}(v) = \sum_{u \in V_{n+m}} \left( \sum_{s \notin V_{n+m+1,0}} B_s^{(n+1)} f_{su}^{(n+m)} \right) g_u^{(n+m-1,n)}.$$
Combining the proved relations, we can finally obtain the following result:

\[
d^*(\overline{g}_{m,n}^{(n+m,n)}, \overline{b}_{m,n}^{(n+m,n)}(v)) = \sum_{w \in V_n} |g_{m,n}(n+m-1,n) - b_{m,n}(n+m,n)(v)|
\]

\[
= \sum_{w \in V_n} \left| \sum_{v \in V_{n+m}} f_{vu}(n+m) g_{uw}(n+m-1,n) - \sum_{s \in V_{n+m+1,0}} B_s^{(n+1)} f_{su}(n+m +1,n) \right|
\]

\[
= \sum_{w \in V_n} \sum_{v \in V_{n+m}} g_{uw}(n+m-1,n) \times \left( f_{vu}(n+m) - \sum_{s \in V_{n+m+1,0}} B_s^{(n+1)} f_{su}(n+m) \right) \times \left( f_{vu}(n+m) - \sum_{s \in V_{n+m+1,0}} B_s^{(n+1)} f_{su}(n+m) \right)
\]

\[
\leq \sum_{w \in V_n} \sum_{v \in V_{n+m}} g_{uw}(n+m-1,n) \times B_s^{(n+1)} \overline{f}_{s}^{(n+m)}
\]

\[
\leq \varepsilon_{n+m} \leq \sup_{m \geq 1} \varepsilon_{n+m} = \varepsilon_n \to 0.
\]

Here we used the fact that \( \sum_{w \in V_n} g_{uw}(n+m-1,n) = 1. \)

On the other hand, we have \( \overline{b}_{m,n}^{(n+1,n)}(v) \in \text{Conv}(\overline{g}_{m,n}^{(n+m,n)}, s \notin V_{n+m+1,0}). \) Thus, we conclude that \( \overline{b}_{m,n}^{(n+m,n)}(v_m) \to \overline{f}_{m}^{(n)} \) as \( m \to \infty. \) The vector \( \overline{f}_{m}^{(n)} \), considered as a vertex of \( \Delta_m^{(n)} \), is the limit vector of the polytope \( \Delta_m^{(n)} = \text{Conv}(\overline{g}_{m,n}^{(n+m,n)}, s \notin V_{n+m+1,0}). \)

Repeating the same arguments as in Lemma 2.10, we obtain that \( \overline{b}_{m,n}^{(n+m,n)}(v_m) \) must be one of the vertices of polytope \( \Delta_m^{(n)} \), i.e. \( \overline{b}_{m,n}^{(n+m,n)}(v_m) = \overline{g}_{v_m}^{(n+m,n)} \) for some \( v_m' \notin V_{n+m+1,0}. \) Therefore, \( v = v' \) for a chain \( \overline{f} \), and this proves (3).

Now we can prove the converse of Theorem 1.9 which we obtain as a corollary of Theorem 5.4. We remark that the following result does not require the stochastic form of incidence matrices of a Brattel diagram in order to have the property of regularly vanishing blocks.

**Corollary 5.6.** Let \( B \) be a Bratteli diagram of finite rank \( k \geq 2 \) with nonsingular stochastic incidence matrices \( (F_n) \). Suppose that after telescoping \( B \) satisfies
conditions (a) – (e2) of Theorem 4.9. Then \( B \) has \( l \) ergodic probability invariant measures.

**Proof.** Since \( B \) has a finite rank, we can identify the sets \( V_n = V \) for \( n = 1, 2, \ldots \) and we have \( l_n = l \), \( L_n = \{1, \ldots, l\} \), \( L_n^{(s)} = \{s\} \) for every \( n = 1, 2, \ldots \) and \( s = 1, \ldots, l \). The chains \( \bar{s} \) reduce to the form \( (s, s, \ldots) \). Notice that conditions (a) – (e2) imply conditions (e1), (d1), (e1.2), while condition (e1.1) is stronger than condition (e1) and does not follow from (a) – (e2). We define subdiagrams \( B^{(s)}_{\bar{s}} \) as before and repeat the proof of properties (1) and (2) in the same way as in Theorem 5.4 (we do not use the property (e1.1) in this part of the proof). To complete the proof, it suffices to show that the vertices \( \bar{y}_1, \ldots, \bar{y}_l \) form the set of all vertices of \( \Delta_{\infty}^{(1)} \) (the notations were defined in Subsection 4.4.1). Assume that a vector \( \bar{z} \) is an additional vertex of \( \Delta_{\infty}^{(1)} \). Then we can find a sequence of vertices \( \bar{y}^{(n)}(w) \) of \( \Delta_{n}^{(1)} \), \( w \in V_n \), such that \( \bar{y}^{(n)}(w) \to \bar{z} \) as \( n \to \infty \). Condition (e1) guarantees that \( \bar{z} \) is a vector of the plane generated by \( \bar{y}_1, \ldots, \bar{y}_l \). The dimension of the simplex \( S(\bar{y}_1, \ldots, \bar{y}_l) \) equals to the dimension of the above plane. If \( \bar{z} \) does not belong to \( S(\bar{y}_1, \ldots, \bar{y}_l) \) then \( \Delta_{\infty}^{(1)} \) is not a simplex. Therefore, \( \bar{z} \in S(\bar{y}_1, \ldots, \bar{y}_l) \) and this implies Property (3) of Theorem 5.4. Hence, the number of the probability ergodic invariant measures on \( B \) equals \( l \). \( \square \)

Let \( B = (V, E) \) be a Bratteli diagram of finite rank \( k \). Suppose that, for \( n \in \mathbb{N} \), we have a subset \( W_n \subset V_n \) such that \( |W_n| = k' = \text{const} \), where \( 1 \leq k' < k \). Let \( B' = (\overline{W}, E) \) be the subdiagram of rank \( k' \) generated by \( \{W_n\}_{n=1}^{\infty} \). Then, after renumbering vertices, \( B' \) can be viewed as a “vertical” subdiagram of \( B = (V, E) \).

An infinite \( \sigma \)-finite measure \( \mu \) on \( X_B \) is called regular infinite if there exists a clopen set such that \( \mu \) takes a finite (non-zero) value on this set.

In [BKMS13, Theorem 3.3] the following result describing the structure of all ergodic invariant measures on a finite rank diagram \( B = (V, E) \) was proved.

**Proposition 5.7** ([BKMS13]). (I) Each ergodic measure \( \mu \) (finite or infinite) on \( X_B \) is obtained as an extension of finite ergodic measure from some vertical subdiagram \( B_{\mu} = (V_{\mu}, E_{\mu}) \);

(II) The number of finite or regular infinite ergodic invariant measures is not greater than \( k \);

(III) One can telescope the diagram \( B \) in such a way that \( V_{\mu} \cap V_{\nu} = \emptyset \) for different ergodic measures \( \mu \) and \( \nu \);

(IV) Given a probability ergodic invariant measure \( \mu \), there exists a constant \( \delta > 0 \) such that for any \( v \in V_{\mu} \) and any level \( n \) one has \( \mu(X_v^{(n)}) \geq \delta \);

(V) Each subdiagram \( B_{\mu} = (V_{\mu}, E_{\mu}) \) is simple and uniquely ergodic;

(VI) For all \( v \notin V_{\mu} \), one has

\[
\lim_{n \to \infty} \mu(X_v^{(n)}) = 0.
\]

In the following proposition, we will show how the results of Sections 4 and 5 are related to Proposition 5.7. Suppose that the matrices \( F_n \)'s of a Bratteli diagram \( B \)
of rank $k$ are nonsingular and satisfy conditions (a) – (c2) of Theorem 4.9. Then, by Corollary 5.6, $B$ has exactly $l$ ergodic invariant probability measures $\{\hat{\mu}_j\}_{j=1}^l$. After renumbering each $V_n = V$, we can think of the corresponding subdiagrams $B_j = B_j^*$ as of vertical subdiagrams. Each measure $\hat{\mu}_j$ is an extension of the unique ergodic measure $\mu_j$ on the subdiagram $B_j$.

**Proposition 5.8.** Let $\hat{\mu}_j$, $j = 1, \ldots, l$ be as above. Then the measures $\hat{\mu}_j$, $j = 1, \ldots, l$, coincide with the ergodic measures, described in Proposition 5.7.

**Proof.** Indeed, it is clear that the measures $\hat{\mu}_j$, $j = 1, \ldots, l$, satisfy conditions (I) - (III) and the condition of unique ergodicity in (V) of Proposition 5.7. We need to prove that the measures and the corresponding subdiagrams satisfy conditions (IV), (VI), and the condition of simplicity in (V). Recall that any $\mathcal{E}$-invariant probability measure $\mu$ on $B$ is determined by a vector $\overline{v}^{(1)} \in \Delta_\infty^{(1)}$. This vector defines a sequence of vectors $\overline{v}^{(n)} = (q_w^{(n)}) = (\mu(V_w^{(n)}))$, $w \in V_n$, such that, for $n = 1, 2, \ldots$ and $m = 0, 1, \ldots$, we have

$$q_w^{(n)} = \sum_{v \in V_{n+m+1}} g_{vw}^{(n+m,n)} q_v^{(n+m+1)}.$$

The vectors $\overline{v}^{(n)} = (q_w^{(n)})_{w \in V_n}$ also satisfy (4.3).

Without loss of generality, we can assume that the measure $\hat{\mu}_j$ is the ergodic invariant measure on $B$ defined by $\overline{v}^{(1)} = \overline{v}_j$ for some $j = 1, \ldots, l$. Then we have $\overline{v}_j = \lim_{n \to \infty} \overline{v}_w^{(n)}$ whenever $w \in V_n$. To find the vectors $\overline{v}^{(n)}$ corresponding to measure $\mu_j$, fix $n \geq 1$, we represent $\overline{v}_j$ as follows:

$$\overline{v}_j = \lim_{m \to \infty} \overline{v}_w^{(n+m+1)} = \lim_{m \to \infty} \sum_{w \in V_n} g_{vw}^{(n+m,n)} \overline{v}_w^{(n)} = \sum_{w \in V_n} \overline{v}_w^{(n)} \lim_{m \to \infty} g_{vw}^{(n+m,n)}.$$

It follows from (4.3) that

$$q_w^{(n)} = \lim_{m \to \infty} g_{vw}^{(n+m,n)}$$

for $v \in V_{n+m+1,j}$.

To assure that the conditions (IV) and (V) are satisfied, we must replace the sets $V_{n,j}$, $j = 1, \ldots, l$ by some subsets $V_{n,j}'$. Indeed, to prove that condition (IV) holds, we should show that $\lim_{m \to \infty} g_{vw}^{(n+m,n)} \geq \delta$ for some $\delta > 0$ and any $j = 1, \ldots, l$, and any $v \in V_{n+m,j}$ and $w \in V_{n,j}$. By condition (d) of Theorem 4.9, matrices $F_n$'s satisfy the condition

$$\max_{v \in V_{n+1,j}} f_{nv}^{(n)} - \min_{v \in V_{n+1,j}} f_{nv}^{(n)} \to 0$$

for every $w \in V_{n,j}$. Assume that for some $w \in V_{n,j}$, $n = 1, 2, \ldots$ we have $\max f_{vw}^{(n)} \to 0$ as $n \to \infty$. Set $V_{n,j}' = V_{n,j} \setminus \{w\}$ and $V_{n,0}' = V_{n,0} \cup \{w\}$. Since $\mu_j$ satisfies condition (c) of Theorem 4.9, it is clear that the set $V_{n,j}'$ is non-empty. Moreover, for $v \in V_{n+m,j}'$,
the following holds:
\[
\sum_{w \not\in V'_{n,j}} g_{vw}^{(n+m,n)} = \sum_{w \not\in V'_{n,j}} \sum_{u \in V_{n+1}} g_{vu}^{(n+m,n+1)} f_{uw}^{(n)}
\]
\[
= \sum_{w \in V_{n+1,j}} \sum_{u \not\in V'_{n,j}} g_{vu}^{(n+m,n+1)} f_{uw}^{(n)} + \sum_{w \not\in V'_{n,j}} \sum_{u \in V_{n+1,j}} g_{vu}^{(n+m,n+1)} f_{uw}^{(n)}
\]
\[
\leq \max_{w \in V_{n+1,j}} \sum_{u \not\in V'_{n,j}} f_{uw}^{(n)} + \max_{v \in V_{n+m+1,j}} \sum_{u \not\in V'_{n,j}} g_{vu}^{(n+m,n+1)}
\]
\[
\to 0 \text{ as } n \to \infty
\]
uniformly with respect to \(m\).

Now one can repeat the same reasoning as in the proof of condition (c) of Theorem 4.9 and get
\[
\sum_{n=1}^{\infty} \left( 1 - \min_{v \in V_{n+1,j}} \sum_{w \in V'_{n,j}} f_{vw}^{(n)} \right) < \infty.
\]

We construct the sets \(V_{n,j}, j = 1, \ldots, l\) and \(V_{n,0}\) as in the proof of Theorem 4.9.

If there are infinitely many levels \(n_k\) such that \(\max_{w \in V_{n+k,j}} f_{vw}^{(n_k)} \geq \delta > 0\) for \(k = 1, 2, \ldots\) and \(w \in V_{n_k,j}\), then we telescope the diagram with respect to \(n_k\) and condition (IV) of Proposition 5.7 is proved. Otherwise, if \(\max_{w \in V_{n+1,j}} f_{vw}^{(n)} \to 0\) as \(n \to \infty\) for some \(w \in V_{n,j}\), then we replace \(V_{n,j}\) with \(V_{n,j}' = V_{n,j} \setminus \{w\}\). Repeating this procedure finitely many times, we find non-empty sets \(V_{n,j}\) satisfying condition (IV) of Proposition 5.7. The described reduction of the sets \(V_{n,j}, j = 1, \ldots, l\) to the sets \(V_{n,j}', j = 1, \ldots, l\) also implies that the subdiagrams \(B'_j\) are simple.

Now we prove condition (VI). If \(w \notin V_{n,j}\) then \(\max_{v \in V_{n+m,j}} g_{vw}^{(n+m,n)} \to 0\) as \(m \to \infty\) for \(n = 1, 2, \ldots\) and
\[
\lim_{n \to \infty} f_{w}^{(n)} = \lim_{n \to \infty} \mu_j(X_w^{(n)}) = 0.
\]
Thus, condition (VI) holds.

\[\square\]

6. Examples

6.1. Stationary Bratteli diagrams. A Bratteli diagram of finite rank \(B = (V, E)\) is called stationary, if \(\bar{F}_n = \bar{F}\) for \(n = 1, 2, \ldots\) The paper [BKMS10] contains an explicit description of all ergodic invariant probability measures on \(B\). Using the main results of [BKMS10], we can indicate the sets \(V_{n,j}\) and the ergodic measures \(\tilde{\mu}_j, j = 1, \ldots, l\).

For the reader’s convenience, we recall the necessary definitions and results from [BKMS10]. In this subsection, by \(\mathbf{u}\) we denote a vector, either column or row one, it will be either mentioned explicitly, or understood from the context.

The incidence matrix \(\bar{F} = (\bar{f}_{vw})_{v,w \in V}\) defines a directed graph \(G(\bar{F})\) in a following way: the set of the vertices of \(G(\bar{F})\) is equal to \(V\) and there is a directed edge from a vertex \(v\) to a vertex \(w\) if and only if \(\bar{f}_{vw} > 0\). The vertices \(v\) and \(w\) are
equivalent (we write \( v \sim w \)) if either \( v = w \) or there is a path in \( G(\tilde{F}) \) from \( v \) to \( w \) and also a path from \( w \) to \( v \). Let \( \mathcal{E}_1, \ldots, \mathcal{E}_m \) denote all equivalence classes in \( G(F) \).

We will also identify \( \mathcal{E}_\alpha \) with the corresponding subsets of \( V \). We write \( \mathcal{E}_\alpha \succeq \mathcal{E}_\beta \) if either \( \mathcal{E}_\alpha = \mathcal{E}_\beta \) or there is a path in \( G(\tilde{F}) \) from a vertex of \( \mathcal{E}_\alpha \) to a vertex of \( \mathcal{E}_\beta \). We write \( \mathcal{E}_\alpha > \mathcal{E}_\beta \) if \( \mathcal{E}_\alpha \succeq \mathcal{E}_\beta \) and \( \mathcal{E}_\alpha \neq \mathcal{E}_\beta \). Every class \( \mathcal{E}_\alpha, \alpha = 1, \ldots, m \), defines an irreducible submatrix \( \tilde{F}_\alpha \) of \( \tilde{F} \) obtained by restricting \( \tilde{F} \) to the set of vertices from \( \mathcal{E}_\alpha \). Let \( \rho_\alpha \) be the spectral radius of \( \tilde{F}_\alpha \), i.e.

\[
\rho_\alpha = \max\{|\lambda| : \lambda \in \text{spec}(\tilde{F}_\alpha)\},
\]

where by \( \text{spec}(\tilde{F}_\alpha) \) we mean the set of all complex numbers \( \lambda \) such that there exists a non-zero vector \( \vec{x} = (x_v)_{v \in \mathcal{E}_\alpha} \) satisfying \( \tilde{F}_\alpha \vec{x} = \lambda \vec{x} \).

A class \( \mathcal{E}_\alpha \) is called distinguished if

\[
\rho_\alpha > \rho_\beta \text{ whenever } \mathcal{E}_\alpha > \mathcal{E}_\beta
\]

(6.1)

(in [BKMS10] the notion of being distinguished is defined in an opposite way because it is based on the matrix transpose to the incidence matrix).

The real number \( \lambda \) is called a distinguished eigenvalue if there exists a non-negative left-eigenvector \( \vec{x} = (x_v)_{v \in V} \) such that \( \vec{x} \tilde{F} = \lambda \vec{x} \). It is known (Frobenius theorem) that \( \lambda \) is a distinguished eigenvalue if and only if \( \lambda = \rho_\alpha \) for some distinguished class \( \mathcal{E}_\alpha \). Moreover, there is a unique (up to scaling) non-negative eigenvector \( \vec{x}(\alpha) = (x_v)_{v \in V}, \vec{x}(\alpha) \tilde{F} = \rho_\alpha \vec{x}(\alpha) \) such that \( x_v > 0 \) if and only if there is a path from a vertex of \( \mathcal{E}_\alpha \) to the vertex \( v \). The distinguished class \( \alpha \) defines a measure \( \mu_\alpha \) on \( B = (V, E) \) as follows:

\[
\mu_\alpha(X_v^{(n)}) = \frac{x_v}{\rho_\alpha^{n-1}} h_v^{(n)}, \quad v \in V_n = V.
\]

The main result of [BKMS10] says that the set \( \{\mu_\alpha\} \), where \( \alpha \) runs over all distinguished vertex classes, generates the simplex of all \( \mathcal{E} \)-invariant probability measures on the stationary Bratteli diagram \( B = (V, E) \).

In the next proposition, we relate the distinguished classes to the subsets \( V_{n,j} \) of \( V \) defined in Section 4.

**Proposition 6.1.** Let \( B = (V, E) \) be a stationary Bratteli diagram and \( V_{n,j}, \ j = 1, \ldots, l \) be subsets of vertices defined in Section 4. Then the distinguished classes \( \alpha \) (as subsets of \( V \)) coincide with the sets \( V_{n,j}, \ j = 1, \ldots, l \).
Proof. To show this, we represent the matrix $\tilde{F}$ in the Frobenius normal form (similarly to the way it was done in [BKMS10]):

$$F = \begin{pmatrix}
\tilde{F}_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \tilde{F}_2 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \tilde{F}_s & 0 & \cdots & 0 \\
Y_{s+1,1} & Y_{s+1,2} & \cdots & Y_{s+1,s} & \tilde{F}_{s+1} & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
Y_{m,1} & Y_{m,2} & \cdots & Y_{m,s} & Y_{m,s+1} & \cdots & \tilde{F}_m
\end{pmatrix}$$

where all $\{\tilde{F}_i\}_{i=1}^m$ are irreducible square matrices, and for any $j = s + 1, \ldots, m$, at least one of the matrices $Y_{j,u}$ is non-zero. All classes $\{E_\alpha\}_{\alpha=1}^s$ $(s \geq 1)$, are distinguished (there is no $\beta$ such that $\alpha > \beta$). For every $\alpha \geq s + 1$ such that $E_\alpha$ is a distinguished class and for every $1 \leq \beta < \alpha$ we have either $E_\beta \prec E_\alpha$ and $\rho_\beta < \rho_\alpha$, or there is no relation between $E_\alpha$ and $E_\beta$.

We note that, for a stationary Bratteli diagram, the entries of the stochastic matrix $F_n$ are

$$f_{vw}^{(n)} = \tilde{f}_{vw} h_{w}^{(n+1)} \quad v \in V_{n+1}, w \in V_n. \quad (6.2)$$

We use the notation introduced in Sections 4 and 5 and denote by $\tilde{G}^{(n)} = \{\tilde{g}_{vw}^{(n)}\}_{v \in V_{n+1}, w \in V_1}$ the $n$-th power $\tilde{F}^n$ of the incidence matrix $\tilde{F}$. By Perron-Frobenius theorem the following relation holds:

$$a_{vw} := \lim_{n \to \infty} \frac{\tilde{g}_{vw}^{(n)}}{h_{w}^{(n+1)}} > 0 \quad (6.3)$$

whenever $v \in E_\alpha$, $w \in E_\beta$, and $E_\alpha \geq E_\beta$ (see [BKMS10] Section 4).

Now, consider the distinguished class $E_\alpha$, and let $v \in E_\alpha$. We have

$$h_{v}^{(n+1)} = \sum_{w \in V_1} \tilde{g}_{vw}^{(n)} h_{w}^{(1)} = \sum_{\epsilon_\beta \leq E_\alpha} \left( \sum_{w \in E_\beta} \tilde{g}_{vw}^{(n)} h_{w}^{(1)} \right).$$

It follows from (6.3) that

$$\lim_{n \to \infty} \frac{h_{v}^{(n+1)}}{\epsilon_\alpha} = \sum_{\epsilon_\beta \leq E_\alpha} \left( \sum_{w \in E_\beta} a_{vw} h_{w}^{(1)} \right). \quad (6.4)$$
is a constant depending on \(\alpha\). Further, using (6.4), we compute
\[
\sum_{w \notin E_\alpha} f_{vw}^{(n)} = \sum_{w \notin E_\alpha} \tilde{f}_{vw} \frac{h_w^{(n)}}{h_v^{(n+1)}} = \sum_{E_{\beta} \prec E_\alpha} \left( \sum_{w \in E_{\beta}} \tilde{f}_{vw} \frac{h_w^{(n)}}{h_v^{(n+1)}} \right) \geq \sum_{E_{\beta} \prec E_\alpha} \left( \sum_{w \in E_{\beta}} \tilde{f}_{vw} h_w^{(n)} \rho_{\beta}^{n} \frac{h_v^{(n+1)}}{\rho_{\alpha}} \right) \geq C \left( \frac{\max_{E_{\beta} \prec E_\alpha} \rho_{\beta}}{\rho_{\alpha}} \right)^n,
\]
where \(C\) is a positive constant. Thus, we conclude using (6.1) that
\[
\sum_{n=1}^{\infty} \left( \max_{v \in E_\alpha} \sum_{w \notin E_\alpha} f_{vw}^{(n)} \right) < \infty.
\]
In other words, we have shown that the distinguished classes (sets of vertices) \(E_\alpha\) satisfy condition \((c)\) of Theorem 4.9. Show that these classes satisfy also condition \((d)\) of Theorem 4.9. Indeed, it follows from (6.4) that, for \(v,w \in E_\alpha\),
\[
\lim_{n \to \infty} f_{vw}^{(n)} = \lim_{n \to \infty} \tilde{f}_{vw} \frac{h_w^{(n)}}{h_v^{(n+1)}} = \tilde{f}_{vw} \frac{1}{\rho_{\alpha}}
\]
We can assume, without loss of generality, that \(\tilde{f}_{vw} \geq 1\) for all \(v,w \in E_\alpha\) since \(\tilde{F}_\alpha\) is an irreducible matrix. Then we apply Theorem 5.1 to get condition \((d)\).

Next, we show that the non-distinguished classes \(E_\alpha\) do not satisfy condition \((c)\). We can find a class \(E_{\beta_0} \prec E_\alpha\) such that \(\rho_{\beta_0} \geq \rho_{\alpha}\). By (6.4), for any \(v \in E_\alpha\) we have
\[
\sum_{w \notin E_\alpha} f_{vw}^{(n)} \geq \sum_{w \in E_{\beta_0}} f_{vw}^{(n)} = \sum_{w \in E_{\beta_0}} \tilde{f}_{vw} \frac{h_w^{(n)}}{h_v^{(n+1)}} \geq C' \left( \frac{\rho_{\beta_0}}{\rho_{\alpha}} \right)^n \geq C'
\]
for some positive constant \(C'\). Thus,
\[
\sum_{n=1}^{\infty} \left( \max_{v \in E_\alpha} \sum_{w \notin E_\alpha} f_{vw}^{(n)} \right) = \infty
\]
and condition \((c)\) is not satisfied.

To finish the proof that the sets \(V_{n,j}, j = 1, \ldots, l\), coincide with the distinguished classes \(E_\alpha\), it remains to show that each set \(V_{n,j}\) is contained in some equivalence class \(E_\alpha\). Let \(v,w \in V_{n,j}\). Condition \((c)\) implies that \(f_{vw}^{(n)} > 0\) for sufficiently large \(n\) (the sets \(V_{n,j}\) are minimal sets satisfying Theorem 4.9). By (6.2), we see that \(f_{vw}^{(n)} > 0\) implies that \(\tilde{f}_{vw}^{(n)} > 0\) for all \(v,w \in V_{n,j}\), i.e., \(v \sim w\) and \(V_{n,j} \subset E_\alpha\) for some \(\alpha = 1, \ldots, m\). \(\square\)
6.2. **Pascal-Bratteli diagrams.** Theorem 5.4 defines a class of Bratteli diagrams \( B = (V, E) \) such that the set \( \mathcal{M}_1(B) \) of all invariant probability measures coincides with the set \( \mathcal{L} \) of all infinite chains. Each ergodic probability invariant measure \( \hat{\mu} \) is an extension of a unique invariant measure \( \mu \) from the subdiagram \( B_i \), and the sets \( X_{B_i} \) are pairwise disjoint.

Consider the Pascal-Bratteli diagram \( B_p = (V, E) \), see e.g. [MP05] or [Ver11, Ver14, FPS17] for more information. Our goal is to show that the set \( \mathcal{M}_1(B_p) \) has different structure than that of the set \( \mathcal{L} \). Recall that, for the Pascal-Bratteli diagram, we have \( V_n = \{0, 1, \ldots, n\} \) for \( n = 0, 1, \ldots \), and the entries \( \tilde{f}_{ki}^{(n)} \) of the incidence matrix \( \tilde{F}_n \) are of the form

\[
\tilde{f}_{ki}^{(n)} = \begin{cases} 
1, & \text{if } k = i = 0, \\
1, & \text{if } k = n + 1 \text{ and } i = n, \\
1, & \text{if } i = k \text{ or } k - 1 \text{ for } 0 < k < n + 1, \\
0, & \text{otherwise.}
\end{cases}
\]

where \( k = 0, \ldots, n + 1, i = 0, \ldots, n \). Moreover,

\[
h_i^{(n)} = \binom{n}{i},
\]

for \( i = 0, \ldots, n \). Then we find the entries of the stochastic matrix \( F_n \):

\[
f_{ki}^{(n)} = \begin{cases} 
1, & \text{if } k = i = 0, \\
1, & \text{if } k = n + 1 \text{ and } i = n, \\
\frac{k}{n + 1}, & \text{if } i = k - 1 \text{ and } 0 < k < n + 1, \\
1 - \frac{k}{n + 1}, & \text{if } i = k \text{ and } 0 < k < n + 1, \\
0, & \text{otherwise.}
\end{cases}
\]  

(6.5)

It is known (see e.g. [MP05]) that each ergodic invariant probability measure has the form \( \mu_p, 0 < p < 1 \), where

\[
\mu_p \left( X_i^{(n)} \right) = \binom{n}{i} p^i (1 - p)^{n-i}, \quad i = 0, \ldots, n.
\]

**Proposition 6.2.** For the Pascal-Bratteli diagram, the set \( \mathcal{L} \) of all infinite chains \( i \) is empty.

*Proof.* Assume that there exists a sequence of partitions \( \{V_{n, 0}, V_{n, i}, i = 1, \ldots, l_n\} \) of \( V_n \) satisfying conditions (c1), (d1), (e1) defined in Subsection 5.2. Recall that we denote by \( f_j^{(n)} \) the vector \( (f_{ji}^{(n)})_i \), \( i \in V_n \). It is easy to see that (6.5) implies

\[
d^t \left( f_j^{(n)}, f_{j'}^{(n)} \right) = \sum_{i=0}^{n} |f_{ji}^{(n)} - f_{j'i}^{(n)}| \geq \frac{1}{2}
\]

whenever \( j \neq j', j, j' \in V_{n+1} \). Thus, it follows from condition (d1) that \( V_{n, i} \) is a single point in the set \( V_n \) whenever \( i = 1, \ldots, l_n \).
Suppose that $1 \leq s \leq n$ is an element of $V_{n+1}$ such that $s \in L_{n+1}$. By condition (c1), there exists a unique element $r = r(s) \in \{1, \ldots, l_n\}$ such that

$$\sum_{n=1}^{\infty} \sum_{u \neq V_{n,r}} f^*(n) < \infty. \quad (6.6)$$

Conditions (6.5) and (6.6) imply that $s - 1, s \in V_{n,r}$. Hence, $V_{n,r}$ contains at least two elements of $V_n$ for some $r \in L_n$, and we get a contradiction.

Therefore, we have $V_{n,0} \supset\{1, \ldots, n-1\}$. Then, by (e1.1), we have

$$\sum_{i=1}^{n-1} f^*(n) \to 0$$

for every $j = 1, \ldots, n - 1$. But this fact contradicts (6.5). Thus, we have shown that there is no sequence of partitions $\{V_{n,0}, V_{n,i}, i = 1, \ldots, l_n\}$ of $V_n$ satisfying conditions (c1), (d1), (e1).

6.3. A class of Bratteli diagrams with countably many ergodic invariant measures. In this subsection, we present a class of Bratteli diagrams with countably infinite set of ergodic invariant measures.

To construct such diagrams, we let $V_n = \{0,1,\ldots,n\}$ for $n = 0,1,\ldots$, and let $\{a_n\}_{n=0}^\infty$ be a sequence of natural numbers such that

$$\sum_{n=0}^{\infty} \frac{n}{a_n + n} < \infty. \quad (6.7)$$

To define the edge set $r^{-1}(w)$ for every vertex $w$, we use the following procedure. For $w \in V_{n+1}$ such that $w \neq n + 1$, the set $r^{-1}(w)$ consists of $a_n$ (vertical) edges connecting $w \in V_{n+1}$ with the vertex $w \in V_n$ and by one edge connecting $w \in V_{n+1}$ with every vertex $u \in V_n$, $u \neq w$. For $w = n + 1$, let $r^{-1}(w)$ contain $a_n$ edges connecting $w$ with the vertex $n$ on level $V_n$ and by one edge connecting $w$ with all other vertices $u = 0,1,\ldots,n-1$ of $V_n$. Then

$$|r^{-1}(w)| = a_n + n$$

for every $w \in V_{n+1}$ and every $n = 0,1,\ldots$.

We observe that the Bratteli diagram defined above admits an order generating the Bratteli-Vershik homeomorphism, see [HPS92], [GPS95], or [BKY14], [BK16] for further details. In particular, we can use a so called consecutive ordering. To introduce a consecutive ordering, we define a linear order $\leq$ on every set $r^{-1}(w)$, $w \in V_{n+1}, n \geq 0$ in such a way that the edges from $r^{-1}(w)$ are enumerated from left to right as they appear in the diagram. Then, for every $e_1, e_2, e_3 \in r^{-1}(w)$ such that $e_1 \leq e_2 \leq e_3$ and $s(e_1) = s(e_3) = u$, we have $s(e_2) = u$; and this order is consecutive by definition (see e.g. [Dur10]). In particular, we will obtain that the minimal edge is always some edge between $w$ and vertex $0 \in V_n$ and the maximal edge is some edge between $w$ and the vertex $n \in V_n$. Then, it is easy to see that $X_B$ has the unique minimal infinite path passing through the vertices $0 \in V_n$, $n \geq 0$ and the unique maximal infinite path passing through the vertices $n \in V_n$, $n \geq 0$. Thus,
a Vershik map $\varphi: X_B \to X_B$ exists and it is minimal. Figure 1 below shows an example of such a Bratteli diagram. It is known that all minimal Bratteli-Vershik systems with a consecutive ordering have entropy zero (see e.g. [Dur10]) hence the system that we describe in this subsection has zero entropy.

Denote by $B_i = (W(i), E(i))$, $i = 0, 1, \ldots, \infty$, the subdiagrams of $B$ determined by the following sequences of vertices (taken consecutively from $V_0$, $V_1$, $\ldots$): for $B_0$, $W(0) = (0, 0, 0, \ldots)$; for $B_i$, $W(i) = (0, 1, \ldots, i-1, i, i, \ldots)$ for $i = 1, 2, \ldots$, and for $B_\infty$, $W(\infty) = (0, 1, 2, \ldots)$. Then each $B_i$ is an odometer and $E(i)$ is the set of all edges from $B$ that belong to $B_i$.

Let $\mu_i$ be the unique invariant (hence ergodic) probability measure on the odometer $B_i$. Then, by (6.7), each measure $\mu_i$ can be extended to a finite invariant measure $\hat{\mu}_i$ on the diagram $B$ and it is supported by the set $\hat{X}_B$. The problem about finiteness of measure extension was discussed in detail in the papers [BKMS13], [ABKK17]. We use the same symbol $\hat{\mu}_i$ to denote the normalized (probability) measure obtained from the extension of $\mu_i$ for $i = 0, 1, \ldots, \infty$.

**Proposition 6.3.** The measures $\hat{\mu}_i$, $i = 0, 1, \ldots, \infty$, form a set of all ergodic probability invariant measures on the Bratteli diagram $B = (V, E)$ defined above.

**Proof.** In the same way as in Theorem 5.4 one can show that each $\hat{\mu}_i$ is the unique invariant measure on the set $\hat{X}_B$. Therefore $\hat{\mu}_i$, $i = 0, 1, \ldots, \infty$, is a probability ergodic measure on $B$. Thus, the proof will be complete if we show that any ergodic invariant probability measure $\mu$ on $B$ coincides with one of the measures.
\[\tilde{\mu}_i, i = 0, 1, \ldots, \infty.\] The incidence matrices \(\tilde{F}_n\) of \(B\) have the following form

\[
\tilde{F}_n = \begin{pmatrix}
a_n & 1 & 1 & \ldots & 1 & 1 \\
1 & a_n & 1 & \ldots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & 1 & a_n \\
1 & 1 & 1 & \ldots & 1 & a_n
\end{pmatrix}
\]

for \(n = 1, 2, \ldots\) Therefore we see that

\[
\sum_{w=0}^{n} \tilde{f}^{(n)}_{vw} = a_n + n, \quad v = 0, 1, \ldots, n + 1,
\]

and \(B\) is an ERS Bratteli diagram with \(r_n = a_n + n\) for \(n \geq 0\). Hence,

\[
h^{(n+1)}_v = a_0(a_1 + 1) \cdots (a_n + n), \quad v = 0, 1, \ldots, n + 1, n \geq 0,
\]

and

\[
\frac{h^{(n)}_v}{h^{(n+1)}_v} = \frac{1}{a_n + n}
\]

where \(w \in V_n, v \in V_{n+1}\) and \(n = 1, 2, \ldots\). Furthermore, the entries of the stochastic matrices \(F_n, n \geq 0\), are

\[
f^{(n)}_{vw} = \frac{\tilde{f}^{(n)}_{vw}}{a_n + n} = \begin{cases} 
\frac{a_n}{a_n + n}, & \text{if } v = w, w \in \{0, 1, \ldots, n\}, \text{ or } v = n + 1, w = n, \\
\frac{1}{a_n + n}, & \text{otherwise}.
\end{cases}
\]

Consider the matrices \(G^{(n+m,n)}_{(n+m)} = F_{n+m} \cdots F_n\) for \(n \geq 1\) and \(m \geq 1\). Using (6.7) and repeating the same arguments as in Lemma 5.5 we get

\[
g^{(n+m,n)}_{vw} \geq \prod_{s=0}^{n} \frac{a_s}{a_s + s} = C_n \to 1 \text{ as } n \to \infty, \quad (6.8)
\]

in the case when \(v = w, w \in \{0, 1, \ldots, n\}\), or in the case when \(v = n + 1, \ldots, n + m + 1, w = n\) and \(m = 1, 2, \ldots\).

We recall that the convex polytope \(\Delta^{(n)}\) is spanned by the probability vectors \(\overline{g}_{v}^{(n+m,n)} = (g^{(n+m,n)}_{vw})_{w=0}^{n+m+1}, v = 0, 1, \ldots, n + m + 1\). Inequalities (6.8) imply that

\[
d^* (\overline{g}_{v}^{(n+m,n)}, \overline{v}_{v}^{(n+1)}) \leq 2(1 - C_n), \quad v = 0, 1, \ldots, n
\]

and

\[
d^* (\overline{g}_{v}^{(n+m,n)}, \overline{v}_{v}^{(n+1)}) \leq 2(1 - C_n), \quad v = n + 1, \ldots, n + m + 1
\]

where \(m = 1, 2, \ldots\). Therefore the set \(\Delta^{(n)}\) has exactly \((n + 1)\) vertices \(\overline{v}_{v}^{(n)}\), where

\[
\overline{v}_{v}^{(n)} = \lim_{m \to \infty} \overline{g}_{v}^{(n+m,n)}.
\]

Moreover,

\[
d^* (\overline{g}_{v}^{(n)}, \overline{v}_{v}^{(n+1)}) \leq 2(1 - C_n).
\]
In fact, $\Delta_{\infty}^{(n)}$ is a simplex in $\mathbb{R}^{n+1}$ for $n$ large enough. Further, we have

$$
\begin{align*}
\mathcal{Q}_{n}^{(n)} &= F_n^T (\mathcal{Q}_n^{(n+1)}) \quad \text{for } v = 0, 1, \ldots, n, \\
\mathcal{Q}_n^{(n)} &= F_n^T (\mathcal{Q}_{n+1}^{(n+1)}).
\end{align*}
$$

Observe that

$$
d^* (\mathcal{Q}_{v}^{(n)}, \mathcal{Q}_{w}^{(n)}) \geq (2C_n - 1) \to 1 \text{ as } n \to \infty \tag{6.9}
$$

whenever $v \neq w, v, w \in \{0, 1, \ldots, n\}$.

Let $\mu$ be an ergodic invariant probability measure on $B = (V, E)$, and let

$$
\mu (X_{w}^{(n)}) = q_{w}^{(n)}, \quad w = 0, 1, \ldots, n.
$$

Denote $\mathcal{Q}^{(n)} = (q_{w}^{(n)})_{w=0}^{n}$. Then $\mathcal{Q}^{(n)}$ is a vertex of $\Delta_{\infty}^{(n)}$, i.e. $\mathcal{Q}^{(n)} = \mathcal{Q}_{w_0}^{(n)}$ for some $0 \leq w_0 \leq n$ and each $n = 1, 2, \ldots$ Moreover, we have

$$
\mathcal{Q}_{w_0}^{(n)} = F_n^T (\mathcal{Q}_{w_{n+1}}^{(n)}).
$$

Then (except for some initial part) the sequence $(w_0, w_1, w_2, \ldots)$ is one of the following sequences: $\mathcal{W}_0 = (0, 0, 0, \ldots)$, $\mathcal{W}_1 = (0, 1, 1, \ldots)$, $\mathcal{W}_2 = (0, 1, 2, 2, \ldots)$, $\ldots$, $\mathcal{W}_{\infty} = (0, 1, 2, 3, \ldots)$. Assume that $w_n = 0$ for every $n = 0, 1, \ldots$ Then

$$
\mu (X_{0}^{(n)}) = q_{0}^{(n)} \geq C_n \not\to 1 \text{ as } n \to \infty
$$

and

$$
\mu \left( \bigcup_{w \neq 0} X_{w}^{(n)} \right) \leq (1 - C_n) \not\to 0 \text{ as } n \to \infty.
$$

The above inequalities imply that $\mu (\hat{X}_{B_0}) = 1$ hence $\mu = \hat{\mu}_0$. In the same way we show that $\mu = \hat{\mu}_i$, $i = 1, 2, \ldots, \infty$ if $(w_0, w_1, w_2, \ldots) = \mathcal{W}^{(i)}$ for $i = 1, 2, \ldots, \infty$. \hfill \Box

7. Interpretation of the Main Theorems in Terms of Symbolic Dynamics

Theorems 4.9 and 5.4 can be interpreted in terms of symbolic dynamics. In order to define a dynamical system (which is known by the name of “Vershik map”) on a Bratteli diagram, one needs to use a partial order on the set of edges. The reader can find details about the definition of dynamical systems on Bratteli diagrams in [HPS92], [GPS95], or in [Dur10], [BK16]. More advanced study of ordered Bratteli diagrams and the existence of corresponding dynamics are discussed in [BKY14] and [BY17].

For simplicity, throughout this section we will consider only simple properly ordered Bratteli diagrams of finite rank which correspond to Cantor minimal systems of topological rank $K > 1$. Recall that a Cantor minimal system has topological rank $K$ if it admits a Bratteli-Vershik representation with $|V_i| = K$ for every $i \geq 1$ and $K$ is the smallest such integer. In [DM08], T. Downarowicz and A. Maass proved that every Cantor minimal system of topological finite rank $K > 1$ is expansive. This result was generalized in [BKM09] to aperiodic Cantor dynamical systems of topological finite rank. Due to Hedlund [Hed69], every expansive Cantor"
dynamical system is conjugate to a subshift. Our goal is to describe explicitly, how to code a Vershik map on a simple properly ordered Bratteli diagram to obtain a conjugacy with a minimal subshift.

7.1. From Bratteli diagrams to subshifts. Let \( \varphi_B : X_B \to X_B \) be a Vershik map on a simple properly ordered Bratteli diagram \( B \). Set \( s_n = \sum_{\omega \in V_n} h^{(n)}_\omega \) and \( S_n = \{0, \ldots, s_n - 1\} \). Let \( \{\overline{0}, \ldots, \overline{s_{n-1}}\} \) be a set of all finite paths between \( v_0 \) and vertices from \( V_n \). Denote by \( X^{(n)} \) the partition of \( X_B \) into the corresponding cylinder sets \( \{X^{(n)}(\overline{s})\}^{s_n-1}_{s=0} \). Then

\[
(\psi_n(x))_i = s \iff \varphi_B^i(x) \in X^{(n)}(\overline{s}), \quad s \in S_n, \quad i \in \mathbb{Z}.
\]  

(7.1)
is a factor map \( \psi_n : X_B \to S^n \). Denote \( Y_n = \psi_n(X_B) \), and let \( \sigma \) be the shift map. Then \( (Y_n, \sigma) \) is a subshift of the full shift \( (S^n, \sigma) \). We note also that since the sequence of partitions \( X^{(n)} \) is nested, the system \( (Y_n, \sigma) \) is a factor of the system \( (Y_{n+1}, \sigma) \) for every \( n \). Moreover, \( (X_B, \varphi_B) \) is conjugate to the inverse limit of the systems \( (Y_n, \sigma) \) for any ordered Bratteli diagram that admits a continuous Vershik map. If \( X^{(n)} \) is a generating partition, then \( \psi_n \) is a conjugacy between \( (X_B, \varphi_B) \) and \( (Y_n, \sigma) \).

Let \( (X_B, \varphi_B) \) have topological finite rank \( K > 1 \), and let \( \delta \) be the expansivity constant for \( (X_B, \varphi_B) \). Then, by [Hed69], every partition of \( X_B \) into clopen sets with diameter less than \( \delta \) is a generating partition. Hence, there exists a natural number \( n_0 \) such that for all \( n \geq n_0 \) the systems \( (X_B, \varphi_B) \) and \( (Y_n, \sigma) \) are conjugate.

Remark 7.1. In general, the rank of the diagram may be bigger than the rank of the corresponding dynamical system. In [FPS17], the following criterion was given for an ordered Bratteli diagram to be isomorphic to an odometer. It is said that the level \( V_n \) of an ordered Bratteli diagram is uniformly ordered if there exists a word \( \omega \) over the alphabet \( V_{n-1} \) such that the coding of each vertex from \( V_n \) in terms of \( V_{n-1} \) is a concatenation of \( \omega \).

Theorem 7.2 ([FPS17]). A simple properly ordered Bratteli diagram is topologically conjugate to an odometer if and only if it has a telescoping for which there are infinitely many uniformly ordered levels.

Let \( B = (V, E) \) be a simple properly ordered Bratteli diagram of topological rank \( K > 1 \) such that the Vershik map \( \varphi_B \) exists. We can assume that \( |V_n| = K \) for every \( n \geq 1 \). Choose \( n_0 \) such that \( (X_B, \varphi_B) \) is topologically conjugate to \( (Y_n, \sigma) \) for all \( n \geq n_0 \). To describe the symbolic shift \( (Y_{n_0}, \sigma) \), we will need a sequence \( \{A_n\}_{n \geq n_0} \) of families of blocks over \( S_{n_0} \). Each family \( A_n \) consists of \( K \) blocks \( A_w^{(n)} \), \( w \in V_{n_0} \). We define \( A_n \) inductively. First, to define blocks \( A_w^{(n_0)} \), \( w \in V_{n_0} \), we write the set \( X_w^{(n_0)} \) in the form \( \{\overline{t}_{i_1} < \overline{t}_{i_2} < \ldots < \overline{t}_{i_w}\} \), where \( \overline{t}_{i_1}, \overline{t}_{i_2}, \ldots, \overline{t}_{i_w} \) are all paths between \( v_0 \) and \( w \) which are compared with respect to the lexicographical order. Set \( A_w^{(n_0)} = (i_1, \ldots, i_w) \). Note that \( i_1, \ldots, i_w \) are symbols from \( S_{n_0} \). Assume that \( A_w^{(n)} \) are already defined for some \( n \geq n_0 \) and for all \( w \in V_n \). Take \( v \in V_{n+1} \) and consider \( r^{-1}(v) = \{e_1 < e_2 < \ldots < e_{|r^{-1}(v)|}\} \). Let \( w_i = s(e_i), i = 1, \ldots, |r^{-1}(v)| \) be
the vertices of \( V_n \) determined by the set \( r^{-1}(v) \). Then we define a block \( A_v^{(n+1)} \) as the concatenation of the blocks \( A_{w_1}^{(n)}, A_{w_2}^{(n)}, \ldots, A_{w_{|r^{-1}(v)|}}^{(n)} \), i.e.

\[
A_v^{(n+1)} = A_{w_1}^{(n)} \cdots A_{w_{|r^{-1}(v)|}}^{(n)}.
\]

(7.2)

In this way we construct the sequence \( \{A_n\}_{n \geq n_0} \). Now we define the language \( L(A_n) \) as the set of all words which appear as factors of \( A_w^{(n)} \) for \( w \in V_n, n \geq n_0 \). Then we set

\[
Y_{n_0} = \{y \in \mathbb{S}_{n_0}^Z : y[-n, n] \in L(\{A_n\}) \text{ for any } n \geq 1 \}.
\]

**Remark 7.3.** We assume that \( f^{(n)}_{vw} \geq 1 \) for each \( v \in V_{n+1}, w \in V_n, \) and \( n \geq 1 \). This obviously implies that \( (X_B, \varphi_B) \) is minimal. However, this does not guarantee that \( B = (V, E) \) is proper, i.e. it has a unique maximal path and a unique minimal path (of course, there exists a properly ordered Bratteli diagram \( B' \) such that the Vershik maps \( (X_B, \varphi_B) \) and \( (X_{B'}, \varphi_{B'}) \) are topologically conjugate, see [HPS92]).

It follows from Proposition 3.2 in [BKY14], that \( B = (V, E) \) has the same number (say \( k \leq K \)) of minimal and maximal paths. Denote them by \( e^{(\text{min}, i)} \) and \( e^{(\text{max}, j)} \) correspondingly, for \( i = 1, \ldots, k \). There exists a permutation \( \rho \) of the set \( \{1, \ldots, k\} \) such that \( \rho(i) = j \) if and only if \( \varphi_B(e^{(\text{max}, i)}) = e^{(\text{min}, j)} \). Minimality of \( (X_B, \varphi_B) \) implies that

\[
X_B = \{\varphi_B^s(e^{(\text{min}, i)}), s \in \mathbb{Z}\}
\]

and

\[
X_{B'} = \{\varphi_B^s(e^{(\text{max}, j)}), s \in \mathbb{Z}\}
\]

for every \( i = 1, \ldots, k \). Thus the trajectories (with respect to the shift \( \sigma \)) of \( y_{n_0}^{(i)} = \psi_{n_0}^{(i)}(e^{(\text{min}, i)}) \) and \( z_{n_0}^{(i)} = \psi_{n_0}^{(i)}(e^{(\text{max}, j)}) \) are dense in \( Y_{n_0} \) for \( i = 1, \ldots, k \). To see how the sequences \( y_{n_0}^{(i)} \) and \( z_{n_0}^{(i)} \) look like, we write

\[
e^{(\text{min}, i)} = (e_1^{(\text{min}, i)}, e_2^{(\text{min}, i)}, \ldots),
\]

\[
e^{(\text{max}, j)} = (e_1^{(\text{max}, i)}, e_2^{(\text{max}, i)}, \ldots),
\]

and let \( r(e^{(\text{min}, i)}) = w_s^{(\text{min}, i)} \), \( r(e^{(\text{max}, j)}) = w_s^{(\text{max}, i)} \) for \( s = 1, 2, \ldots \). Then \( w_s^{(\text{min}, i)} \) and \( w_s^{(\text{max}, i)} \) are the consecutive vertices of the paths \( e^{(\text{min}, i)} \) and \( e^{(\text{max}, j)} \), \( i = 1, \ldots, k \) respectively. It follows from the definition of \( \psi_{n_0} \) (see (7.1)) that

\[
y_{n_0}^{(j)} \left[ h_s^{(\text{max}, i)}, h_s^{(\text{min}, j)} - 1 \right] = A_{w_s^{(\text{max}, i)}}^{(s)} A_{w_s^{(\text{min}, j)}}^{(s)}
\]

and

\[
z_{n_0}^{(i)} = \sigma^{-1}(y_{n_0}^{(j)}),
\]

where \( j = \rho(i) \) and \( h_s^{(\text{min}, j)} = |A_{w_s^{(\text{min}, j)}}^{(s)}|, h_s^{(\text{max}, j)} = |A_{w_s^{(\text{max}, i)}}^{(s)}|, s = n_0, n_0 + 1, \ldots \)
Remark 7.4. We can reformulate Theorems 4.9 and 5.1 using the families of blocks $A^{(n)}_w$, $w \in V_n$, $n \geq 1$. It follows from the above considerations that the elements of the incidence matrices $\tilde{F}_n$ are the numbers of occurrences of the blocks $A^{(n)}_w$ inside the blocks $A^{(n+1)}_v$, $w \in V_n$, $v \in V_{n+1}$. More precisely, we have

$$\tilde{f}^{(n)}_{vw} = \text{card} \left\{ 1 \leq i \leq |r^{-1}(v)| : A^{(n)}_{w_i} = A^{(n)}_w \right\},$$

where $A^{(n)}_{w_i}$ are the blocks appearing in the concatenation (7.2). In this case we have $h^{(n)}_w = |A^{(n)}_w|$, $w \in V_n$. Then Theorem 4.9 reveals the structure of the set of all invariant measures of $(Y_{n_0}, \sigma)$, while Theorem 5.1 allows us to construct symbolic systems with uncountably many probability ergodic invariant measures.

7.2. Toeplitz flows. The details about Toeplitz dynamical systems can be found in [Wil84, DLK95, BK90, Dow90]. A Toeplitz dynamical system is defined by a Toeplitz sequence $\omega$ over a set of symbols $S$, $|S| \geq 2$. The sequence $\omega$ can be obtained as a limit of periodic sequences $\omega_n$ over the alphabet $S \cup \emptyset$, where $\emptyset$ denotes the empty symbol. Take a sequence of natural numbers $\{\lambda_0, \lambda_1, \ldots\}$ such that $\lambda_n \geq 2$ for all $n \geq 0$. Set $p_n = \lambda_0 \lambda_1 \cdots \lambda_n$, $n \geq 0$. By induction we define a sequence of blocks $\{A_n, n \geq 0\}$ over the alphabet $S \cup \emptyset$ with $|A_n| = p_n$. Each block $A_n$ should have some positions filled by the elements of $S$ (these positions are called the filling positions) and some positions occupied by the empty symbol $\emptyset$. We take any block $A_0$ with $|A_0| = p_0 \geq 3$ such that $A_0[0]$ and $A_0[p_0 - 1]$ belong to $S$, and there is at least one symbol $\emptyset$ among the symbols on the positions $\{1, \ldots, p_0 - 1\}$.

Assume that $A_n$ is defined. To define $A_{n+1}$, we first consider a concatenation of $\lambda_{n+1}$ copies of $A_n$ and then fill some but not all of the empty positions in this concatenation. Denote by $l_n$ and $k_n$ the first and the last position in $A_n$, where the empty symbol $\emptyset$ occurs. Then $l_n \leq k_n$ and $A_n[i] \in S$ whenever $0 \leq i \leq l_n - 1$ and $k_n + 1 \leq i \leq p_n - 1$. We require that $l_n$ and $(p_n - k_n)$ tend to infinity as $n$ tends to infinity.

For every $n$, we define a sequence $\omega_n$ as an infinite concatenation of $A_n$ such that the block $A_n$ starts at a zero position. Thus, each $\omega_n$ is a periodic two-sided sequence over $S \cup \emptyset$ with the period $p_n$. Notice that the block $\omega_n[-k_n + 1, l_n - 1]$ is filled with the symbols from $S$. Now define a sequence $\omega$ over the symbols $S$ in such a way that

$$\omega[-k_n + 1, l_n - 1] = \omega_n[-k_n + 1, l_n - 1]$$

(7.3)

for $n = 0, 1, \ldots$. Let us remark that $\omega$ is well defined since $\omega_{n+1}[-k_n + 1, l_n - 1] = \omega_n[-k_n + 1, l_n - 1]$ and $l_n, k_n \to \infty$ as $n \to \infty$.

A sequence $\omega$ defined by (7.3) is called a Toeplitz sequence over $S$ whenever $\omega$ is not periodic. By a Toeplitz flow we mean a topological dynamical system $(\overline{O}(\omega), \sigma)$, where

$$\overline{O}(\omega) = \{\sigma^i(\omega), i \in \mathbb{Z}\} \subset S^\mathbb{Z}.$$

It is known how one can characterize Bratteli-Vershik systems associated to Toeplitz flows. The following theorem was proved in [GJ00].
Theorem 7.5. The family of expansive Bratteli-Vershik systems associated to simple Bratteli diagrams with the ERS property coincides with the family of Toeplitz flows up to conjugacy.

Indeed, in order to construct a Bratteli diagram for a Toeplitz flow \((\mathcal{O}(\omega), \sigma)\), we need a sequence \(\{A_n\}\) of families of blocks over \(S\) as follows:

\[ A_n = \{ \omega[mp_n, (m+1)p_n - 1], m \in \mathbb{Z} \}. \]

Observe that each \(A_n\) is a finite family of blocks \(\{A_1^{(n)}, \ldots, A_{s_n}^{(n)}\}\) and \(|A_i^{(n)}| = p_n\) for every \(i = 1, \ldots, s_n\). We will call the blocks \(A_i^{(n)}\) the \(n\)-symbols. Then every \((n+1)\)-symbol \(A_j^{(n+1)}\) is a concatenation of \(\lambda_{n+1}\) \(n\)-symbols, i.e.

\[ A_j^{(n+1)} = A_j^{(n)} \ldots A_j^{(n)\lambda}, \quad (7.4) \]

where \(\lambda = \lambda_n\).

The families \(A_n\) of \(n\)-symbols allow us to construct a Bratteli diagram \(B_\omega = (V_\omega, E_\omega)\) as follows. Set \(V_n = \{1, \ldots, s_n\}\) and \(n \geq 1\) and \(V_0 = \{v_0\}\). For any \(j \in V_{n+1}\), we set \(r^{-1}(j) = \{ j_1 < j_2 < \ldots < j_\lambda \}\), where the vertices \(j_1, \ldots, j_\lambda \in V_n\) come from \(7.4\) and "<" means an order in \(r^{-1}(j)\). Then the incidence matrices \(\tilde{F}_n\) are the matrices of appearances of \(n\)-symbols inside the \((n+1)\)-symbols, i.e.

\[ \tilde{f}_{ji}^{(n)} = \text{card}\{1 \leq r \leq \lambda_n : A_j^{(n)} = A_j^{(n)r}\}, \]

where \(A_j^{(n)}\) come from \(7.4\). For the Toeplitz-Bratteli diagram \(B_\omega = (V_\omega, E_\omega)\) we have

\[ h_i^{(n)} = p_n \]

for \(i = 1, 2, \ldots, s_n\). Then

\[ f_{ji}^{(n)} = \frac{h_i^{(n)}}{h_{ji+1}^{(n)}} \tilde{f}_{ji}^{(n)} = \frac{1}{\lambda_{n+1}} \tilde{f}_{ji}^{(n)} = f^{(n)}(A_i^{(n)}, A_j^{(n+1)}). \]

Now again we can formulate Theorem 4.9 using the frequency matrices \(F_n\) and describe the set of all invariant ergodic measures of a Toeplitz flow \((\mathcal{O}(\omega), \sigma)\). Using Theorem 5.4 we can construct Toeplitz flows with uncountably many ergodic invariant measures.

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