Degenerated Calabi–Yau varieties with infinite components, moduli compactifications, and limit toroidal structures

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Abstract
For any degenerating Calabi–Yau family, we introduce a new limit space which we call galaxy, whose dense subspace is the disjoint union of countably infinite open Calabi–Yau varieties, parametrized by the rational points of the Kontsevich–Soibelman’s essential skeleton, while dominated by the Huber adification over the Puiseux series field. Other topics include: projective limits of toroidal compactifications (Sect. 3), locally modelled on limit toric varieties (Sect. 2.4), the way to attach a tropicalized family to a given Calabi–Yau family (Sect. 4), which are weakly related to each other.

Keywords Degenerations · Calabi–Yau varieties · Moduli spaces · Non-Archimedean geometry

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1 Introduction

Degenerations and moduli spaces of polarized Calabi–Yau varieties\(^1\) have been attracting researchers over decades, especially since they lie among many areas of research, such as algebraic geometry, differential geometry, mirror symmetry, arithmetic geometry, non-Archimedean geometry, tropical geometry, and others.

1.1 Partial summary

This paper introduces and discusses various degeneration structures and compactifications, partially motivated by their own beauty and relations with non-Archimedean geometry (and tropical geometry). Topics in each section (including appendices) are only weakly connected, so the interested readers could directly skip to their own concerned part.

Contents of Sect. 2 (Galaxy models)

Our first construction in Sect. 2 is as follows: for any degenerating one parameter family of (klt) Calabi–Yau varieties, we consider a new kind of degeneration as an intricate connected locally ringed space which we call \( \text{galaxy} \). Here is the simplest example:

\textbf{Example 1.1 (Elliptic curve case)} For a minimal degeneration of elliptic curves \( \mathcal{X} \to \Delta \ni 0 \) of \( I_m \)-type \( (m \in \mathbb{Z}_{>0}) \) in the sense of Kodaira, i.e., with the central fiber \( m \)-gon consisting of transversally intersecting \( \mathbb{P}^1 \)'s, take a finite base change \( \Delta' \to \Delta \) ramifying at 0 of degree \( d \) and consider the fiber product \( \mathcal{X} \times_{\Delta} \Delta' \). Then, after the minimal resolutions of \( m A_{d-1} \)-singularities in \( \mathcal{X} \times_{\Delta} \Delta' \), we obtain an \( I_{md} \)-type degeneration of (essentially same) elliptic curves. If we take a sequence of the degree \( d \) diverging in the divisible order, we obtain a projective system of \( I_{md} \) fibers, i.e., \( md \)-gon of \( \mathbb{P}^1 \)'s where \( m \) varies. Its projective limit \( \mathcal{X}_\infty \) with respect to the degree \( d \), which is the easiest example of galaxy, includes infinite \( (\mathbb{P}^1 \setminus \{0, \infty\})'s \) parametrized by \( \mathbb{Q}/\mathbb{Z} = \bigcup_d \frac{\mathbb{Z}}{\mathbb{Z}} \) and closed points bijectively corresponding to the fractional part of irrational numbers, i.e., parametrized by \( (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z} \).

In Sect. 2, we extend the above space “galaxy” to arbitrarily dimensional Calabi–Yau varieties and reveal the basic structures roughly as follows. Here, \( \Delta \ni 0 \) means the germ of a smooth pointed \( k \)-curve, while we base change over the Puiseux series field \( k((t^Q)) \) which is the fraction field of \( \bigcup_d k[[t^{Q}]] := \bigcup_{d \geq 1} k[[t^{1/d}]] \). See details in the later sections.

\textbf{Theorem 1.2 (cf. Corollary 2.13, Definition 2.12, Sect. 2.5)} For any punctured meromorphic family of klt projective Calabi–Yau varieties \( \mathcal{X}^* \to \Delta^* = \Delta \setminus \{0\} \), its base change to \( \text{Spec} \ k((t^Q)) \) has a model called galaxy model \( \mathcal{X}_\infty \), with the central fiber \( \mathcal{X}_\infty \) (we call it galaxy), such that the following hold:

\(^1\) We use this term as an equivalent of K-trivial varieties in the broad sense, i.e., normal projective varieties with numerically trivial canonical divisors.
(i) **There is a canonical continuous surjective map**

\[ f_{\text{tr}} : X_\infty \to \Delta^\text{KS}(\chi_\eta) = B, \]

where \( \Delta^\text{KS}(\chi_\eta) \) is the essential skeleton [84] inside the Berkovich analytification \( \chi^*_\eta \) of the generic fiber \( \chi^*_\eta \).

(ii) **For any point** \( x \in \Delta(X_0) \), whose coordinates with respect to the natural integral affine structure are rational, \( f_{\text{tr}}^{-1}(x) \) includes as an open dense subset the complex analytification of open Calabi–Yau varieties (each of which is unique up to log crepant birational maps).

(iii) **There is a continuous surjective map**

\[ \chi^{*,\text{ad}}_{\eta,k(t^{(\mathbb{Q})})} \to X_\infty \]

from the Huber adification \( \chi^{*,\text{ad}}_{\eta,k(t^{(\mathbb{Q})})} \) (naturally associated adic space in the sense of [71]) of the base change \( \chi^*_\eta \) of the generic fiber \( \chi^*_\eta \) to the Puiseux series field \( k(t^{(\mathbb{Q})}) \).

These galaxy models seen as an analogue of “minimal” models over non-finite type spectra \( k[[t^{(\mathbb{Q})}]] \). Recall that in the elliptic curve case (Example 1.1), the essential skeleton is \( \mathbb{R}/\mathbb{Z} \cong S^1 \) (the tropical elliptic curve), and the open Calabi–Yau variety appearing in the above theorem is \( \mathbb{P}^1 \setminus \{0, \infty\} \). Note that any \( I_{md} \)-degenerate fiber does not admit a continuous map onto \( \mathbb{R}/\mathbb{Z} \) nor even to the subset \( \frac{1}{md}\mathbb{Z}/\mathbb{Z} \), hence the above (i) is an effect of taking the projective limit.

In particular, we observe the following basic structure of the galaxies \( X_\infty = \lim_{\leftarrow i} X_i \).

**Corollary 1.3** (\( = \) Corollary 2.14: Decomposition of galaxies) Any galaxy \( X_\infty \) has a natural decomposition into an open part and closed part:

\[ X_\infty = \bigcup_{a \in B(\mathbb{Q})} \{ \text{open klt log Calabi–Yau variety } U(a) \} \sqcup X^\text{NKLT}_\infty. \]

Here, \( B \) denotes the essential skeleton (as in Theorem 1.2 again) and \( B(\mathbb{Q}) \) means its rational points with respect to the \( \mathbb{Q} \)-affine structure, which does not depend on \( i \) by Theorem 2.1 (iii) and \( X^\text{NKLT}_\infty := \lim_{\leftarrow i} X^\text{NKLT}_i \), where NKLT stands for the non-klt closed loci.

**Contents of Sect. 3 (Limit toroidal compactifications)**

In Sect. 3 over \( k = \mathbb{C} \) in turn, we discuss on the projective limit of toroidal compactifications of [13] and its analogue for more general (moduli) varieties. It sounds somewhat independent from Sect. 2 but here we observe an analogous phenomenon of the above Theorem 1.2, especially the natural continuous maps to their tropical versions.
More precisely, what we do there is as follows. Fix a locally Hermitian symmetric space $M$ of non-compact type, i.e., of the ubiquitous form $M = \Gamma \backslash G / K$ where $G$ is real-valued points of a simple algebraic group over $\mathbb{Q}$, $K$ its maximal compact subgroup with one-dimensional center, $\Gamma$ an arithmetic discrete subgroup of $G$. Some renowned examples are the moduli space of $g$-dimensional principally polarized abelian varieties or that of primitively polarized K3 surfaces with possibly ADE singularities (of fixed genera).

Recall that for certain combinatorial data, i.e., admissible collection of fans $\Sigma = \{ \Sigma(F) \}$, there is an associate toroidal compactification $\overline{M}_{\text{tor}, \Sigma}^\an$ of $M$ constructed in [13], and its complex analytification $\overline{M}_{\text{tor}, \Sigma, \an}^\an$.

Now, we consider and introduce the projective limit of all of its toroidal compactifications as a locally ringed space and call it the limit toroidal compactification:

$$\overline{M}_{\text{tor}, \Sigma, \an}^\an := \lim_{\Sigma} \overline{M}_{\text{tor}, \Sigma, \an}^\an.$$

More precisely, the ingredient $\overline{M}_{\text{tor}, \Sigma, \an}^\an$ of the right-hand side is the complex analytification of the toroidal compactification with respect to the combinatorial data, i.e., the admissible collection of fans $\Sigma$ [13]. On the other hand, we also recall $\overline{M}^\MSBJ$, the minimal Morgan–Shalen–Boucksom–Jonsson compactification ([114, Appendix], [118, Section 2]), i.e., the MSBJ compactification corresponding to the toroidal compactifications [13] (which do not depend on the combinatorial data, i.e., the admissible collection of rational polyhedra as [118, Theorem 2.1] shows).

**Theorem 1.4** (= Theorem 3.6) There is a natural continuous surjective map

$$\phi_{\text{tr}} : \overline{M}_{\text{tor}, \Sigma, \an}^\an \to \overline{M}^\MSBJ$$

which extends the identity map on $M$.

For a general point $x$ of the boundary $\partial \overline{M}^\MSBJ$, the fiber $\phi_{\text{tr}}^{-1}(x)$ coincides with the limit toric variety $\overline{T}_{n-r, \an}^\an$ which we introduce in Sect. 2.4.

Roughly speaking, the limit toric variety above is the projective limit of all proper toric varieties (of the fixed dimension) which we study in Sect. 2.4.

Also, in Sect. 3 we discuss the Zariski–Riemann type compactification and clarify relations with the above compactifications.

**Contents of Sect. 4 (Attaching tropical family to varieties family)**

Until Sect. 3, we discuss degenerations and moduli compactifications independently although they are both connected with tropicalizing phenomena.

The purpose of Sect. 4 is to fill in this gap to some extent by attaching a family of “tropical varieties” to a family of varieties on the complex moduli. A little more precisely, we construct a family, with a connected total space, over the minimal Morgan–Shalen–Boucksom–Jonsson compactification of the moduli spaces of varieties which is a flat family of polarized varieties in the open locus, whereas at
Degenerated Calabi–Yau varieties with infinite... 1109

the boundary it is a family of “corresponding tropical varieties”. We give a rough vague statement as follows, which is essentially known from [1, Section 8], [28], [102, Section 9 B], etc. The precise meanings are left until Sect. 4.

**Proposition 1.5** (see Sect. 4 for precise meanings) Over the minimal MSBJ compactification $\overline{M}^{\text{MSBJ}}$ of the moduli of $M = M_g$ (resp., $M = A_g$), we have a continuous family which is the original family of hyperbolic curves on $M = M_g$ (resp., principally polarized $g$-dimensional abelian varieties on $M = A_g$) $M$ (at stacky level!) and the family of tropical curves (resp., tropical abelian varieties) on the boundary $\partial \overline{M}^{\text{MSBJ}}$.

We hope that our newly introduced structures and problems can be of its own interest in non-Archimedean geometry and tropical geometry among others.

**On the appendices**

Also, as additional notes, Appendix A discusses Morgan–Shalen type compactifications for general Berkovich analytic spaces, their basic properties such as funtoriality with respect to morphisms. These are partially used in a few places in our main contents. Appendix B discusses possible analogues of classical Satake–Baily–Borel compactifications in the more moduli theoretic contexts, in the spirit of Griffiths [56] for period spaces, followed by [55, 74], etc., but with more focus on the relations with degenerations and moduli.

**1.2 Some background and history**

Before going into the main contents, we also review some historical background especially from birational geometry and asymptotic behaviour of Ricci-flat Kähler metrics, although unfortunately differential geometric perspectives have not yet been substantially developed.

**Differential geometric background**

The well-known work of Yau [147] showed the existence of unique Ricci-flat Kähler metric on polarized Calabi–Yau varieties. Since then, a natural question has been to consider asymptotic behaviour of the metric with respect to the variation of the polarized Calabi–Yau varieties in concern.

Sometimes metrics collapse, i.e., roughly speaking the dimension of the limit is smaller, whereas sometimes not. The non-collapsing situation is well-understood by now, due to [37, 141, 145, 148, 149].

In the collapsing cases, the study was initiated in [64, 84], etc. and developed further in [26, 62, 63] among others. The metrics’ behaviour in these two cases is observed to be very different.

However, the above works are for a one parameter family, either along a holomorphic family or adiabatic limits. Such framework has been recently extended to general sequences or equivalently to the whole moduli in the (ongoing) series of works [114,
The above works are mainly concerned with collapsing of the rescaled Ricci-flat Kähler metrics with fixed diameters.

Whereas we expect that our consideration of an analogue of the Satake–Baily–Borel compactifications in Appendix B could be related to the Gromov–Hausdorff compactifications with respect to different rescales, i.e., with fixed volumes of Ricci-flat metrics, in certain sense. See Question B.8.

On the other hand, differential geometric meaning of other compactifications we discuss in Sect. 3 is more unclear and non-trivial as for now. We leave these directions for future research.

**Algebro-geometric background**

Motivated by the projective moduli construction of ample (log-)canonical class varieties with semi-log-canonical singularities (“KSBA moduli”, see [5, 6, 8, 79, 81, 135]), as well as many other explicit moduli spaces (e.g., [7, 11, 97, 98, 100, 101, 134]) and a lot of profound differential geometric works such as [15, 36, 45, 46], Conjecture 5.2 in [106, Section 5] discussed the possibility of constructing a theory of at least partially compact moduli spaces via K-stability for more general polarised varieties as “K-moduli”: moduli of K-(poly)stable varieties. Indeed, there are many ways from different fields to lead to this expectation, and the above expression of review should be still with personal bias of my own approach. Here, we make some precise but certainly weaker versions of the K-moduli conjecture in the case of Calabi–Yau varieties, and confirm some progress which was done.

The currently most recognized approach for more than a decade to compactifying the moduli of polarized Calabi–Yau varieties has been essentially “log KSBA”, i.e., to attach (extra) ample divisors as boundary to the Calabi–Yau varieties in concern, to make natural logarithmic generalization (i.e., with boundary divisors) of the KSBA approach work. Lately, such approach gave various interesting compactifications and their explicit description, see e.g. [7, 9, 10, 86] among others. However, in general, such approach does not give unique or canonical compactification, due to the additional data, i.e., the choice of divisors. In this paper, together with [115], we still seek for canonical “algebro-“geometric compactifications without adding divisors or canonical “algebro-”geometric limits of degenerations.

Before going into details, for comparison, let us review another story from the K-stability perspectives. Especially since 2012, the construction of K-moduli construction in the case of anticanonically polarized $\mathbb{Q}$-Fano varieties has been developed: projective moduli spaces of K-polystable (Kähler–Einstein) $\mathbb{Q}$-Fano varieties constructed under $\mathbb{Q}$-Gorenstein smoothability condition cf. [89, 111, 120, 137], which are étale locally GIT quotients (called “good moduli spaces” by Alper). More recently, there have been also nice developments to algebraize the arguments in the construction, so that it naturally extends to more general singular $\mathbb{Q}$-Fano varieties, we do not try to make complete list of such references here, with apology. In any case, the notable feature in the anti-canonically polarized case is that for any punctured family of Kähler–Einstein K-polystable $\mathbb{Q}$-Fano varieties, we can conjecturally fill in a K-polystable $\mathbb{Q}$-Fano variety to ensure the compactness of the moduli.
On the other hand, the case of our focus in this paper, polarized varieties with numerically trivial log-canonical class ("Calabi–Yau"), K-polystable ones do not form compact moduli unlike $\mathbb{Q}$-Fano case unfortunately: even for elliptic curve case, the nodal minimal degeneration, i.e., $I_v$-type in the sense of Kodaira is not K-polystable. Indeed, recall that

**Theorem 1.6** ([121, 6.3]) Assume $(X, D)$ is a log smooth Calabi–Yau pair, i.e., $K_X + D = 0$ with a polarization $L$. Then, $((X, D), L)$ is log $K$-semistable if and only if coefficients of $D$ are at most 1 but it cannot be log $K$-polystable unless $\lfloor D \rfloor = 0$.

More generally, semi-log-canonical Calabi–Yau polarized pair $((X, D), L)$ is log $K$-semistable but it is log $K$-polystable if and only if the underlying log pair $(X, D)$ has only klt singularities.

Furthermore, as is well-known, for a fixed punctured family of Calabi–Yau varieties, e.g., $K3$ surfaces, $\mathcal{X}^* \to \Delta^*$, the way of compactifying as relative dlt minimal model $\mathcal{X} \to \Delta$ has a complicated indeterminacy caused by flops on $\mathcal{X}$. For the convenience of readers, we enhance the indeterminacy at polarized level as follows, thus clarifying the problem. Or see [75]. Further, as we confirmed in [115, beginning of Section 4], even in polarized setting, flops can easily cause unseparatedness of moduli, without polystability conditions.

Nevertheless, let us first define the following notion of weak $K$-moduli compactifications, just to coin the well-studied notion in our context. We set the scene by fixing a connected Deligne–Mumford (at least generically smooth) moduli stack $\mathcal{M}$ of polarized log terminal Calabi–Yau varieties. We make a caution that in general even connected $\mathcal{M}$ could be a priori only of locally finite type (i.e., non-quasi-compact), due to unsolved problem on the boundedness of singularities (cf. [153, 1.1 (iii), 1.2], [118, Section 9]) hence we simply take $\mathcal{M}$ which satisfies the quasi-compactness. Then, from the work of [146], $\mathcal{M}$ has a quasi-projective coarse moduli variety $\tilde{\mathcal{M}}$.

Note that from the definition, there is a universal family $\pi: (U, L) \to \mathcal{M}$ of the polarized log-terminal Calabi–Yau varieties. Suppose $\mathcal{M}^0 \subset \mathcal{M}$ denotes the open substack which parametrizes smooth polarized Calabi–Yau varieties.

**Definition 1.7** (Weak $K$-moduli and very weak $K$-moduli) For the above setting, we call the following object a weak $K$-moduli stack (resp., very weak $K$-moduli). A proper Deligne–Mumford moduli stack $\overline{\mathcal{M}}$ compactifying $\mathcal{M}$ (resp., $\overline{\mathcal{M}}$ compactifying $\mathcal{M}^0$), with a $\mathbb{Q}$-Gorenstein family of polarized semi-log-canonical, or equivalently, K-semistable $2$) Calabi–Yau varieties $\overline{\pi}: (\overline{U}, \overline{L}) \to \overline{\mathcal{M}}$ which extends $\pi: (U, L) \to \mathcal{M}$ (resp., the restriction of $\pi: (U, L) \to \mathcal{M}$ to $\mathcal{M}^0$). For $\mathcal{M}$ to be weak $K$-moduli (but not for very weak $K$-moduli), we also require that the underlying family of varieties $\overline{U} \to \overline{\mathcal{M}}$ is effective (i.e., no isomorphic varieties occur as fibers at different $k$-rational points).

The notion is clearly motivated by the examples in [7, 11, 99] (cf. also [72, 102] preceding that) for when $\mathcal{M} = A_g$ the moduli of $g$-dimensional principally polarized abelian varieties, and that of (special) polarized $K3$ surfaces by [4, 9, 10, 59, 73, 86].

$^2$ The equivalence follows from [109, 110] (also cf. [111, Section 4]).
among others. Both are obtained by explicitly determining logarithmic
generalization of the KSBA compactification by fixing some “natural” ample divisors
in the Calabi–Yau varieties. Also, it often turns out that the compactification or its nor-
malization is obtained as toroidal compactification [13] or semi-toric compactification
[90]. As works mentioned above show, the following has been a classical problem,
certainly well-recognized folklore expectations among experts over decades.

**Conjecture 1.8** (Weak K-moduli (folklore)) For any moduli algebraic stack $M$ of
polarized log terminal Calabi–Yau varieties with its quasi-projective coarse moduli
variety $M$, there is at least one weak $K$-moduli proper stack. Moreover, if $M$ has
uniformization by a Hermitian symmetric domain, then at least one such compact-
ification’s normalization is dominated by one of toroidal compactifications [13] or
semi-toric compactifications [90].

The reason the latter statements only predict domination is that, at least in a log
version (with non-zero boundaries) or subdomain version, i.e., when $M$ does not locally
cover the Kuranishi space, there do exist examples when the Satake–Baily–Borel
compactifications parametrize polarized semi-log-canonical log Calabi–Yau pairs (cf.,
e.g., [32, Section 4.1], [118, Section 7.2.1]). We believe the following certain weak
form of the above Conjecture 1.8 has been essentially known to experts.

**Theorem 1.9** (cf. [2, 3, 19, 20, 50, 67, 83]) For given a $M^0$, at least one very weak
$K$-moduli compactification exists.

The proof follows from the two kinds of deep techniques, by combining them in
a relatively simple manner; applying the weak semistable reduction [2, 3] (or the
classical semistable reduction [76] for one parameter setting) and then running the
relative log minimal model program (established in [20, 50, 67], etc.) after that. Before
these procedures, boundedness of the pairs in concern is needed, which is achieved in
[19] at fairly broad setting including ours. We include the sketch proof of Theorem 1.9
for convenience.

**Outlined proof of Theorem 1.9** For the given $M^0$ parametrising smooth varieties can be
written as $\Gamma\setminus M'$ with $\Gamma$-equivariant flat projective family of polarized slc Calabi–Yau
varieties which we denote as $(U, L) \to M = [\Gamma\setminus M']$ or $(U, L) \to M'$ with $\Gamma$-action
on it.

Then we apply the functorial weak semistable reduction [2, 3] to (an arbitrary
compactification for base direction of) $U \to M'$ so that after replacing $M'$ by its (fur-
ther) finite Galois cover, we can compactify as $M' \subset \overline{M'}$ with toroidal flat morphism
$U \to \overline{M'}$. Then run relative minimal model over $\overline{M'}$ of $U$ and consider an (absolutely)
ample line bundle $L$ which is relatively linear equivalent to $L$ over $M'$. Then, take a
general relative section $D$ of $|mL|$ for $m \gg 0$. Then take its relative lc model by [20,
50, 67]. It becomes a desired flat family of slc Calabi–Yau varieties with ample divisor
(class), extending $(U, mL) \to M'$. The semi-log-canonicity follows from applying
the adjunction inductively.
Now, turning back to weak K-moduli, the mentioned explicit examples (e.g., [7, 9–11, 90, 99, 134, 154]) are all weak K-moduli, giving the affirmative confirmation to the full Conjecture 1.8, hence they are its supporting evidences. Most of the above are obtained as a special case of logarithmic version (i.e., a version with boundary divisors) of KSBA construction, which is also discussed in recent [19, 83]. However, we should keep in mind and emphasize that a priori these could be moduli of the pairs of varieties and their divisors, hence some loci could possibly preserve the ambient varieties while the divisors change.

It turned out that these weak K-moduli compactifications of the moduli of (log-terminal) polarized Calabi–Yau varieties turned out to be at least non-unique. Indeed, there are at least two non-isomorphic examples of weak K-moduli compactification in the case of the moduli $\mathcal{F}_2$ of degree 2 polarized K3 surfaces; one by [134] (cf. also [90]), obtained as a simple Kirwan blow-up of the GIT moduli of sextic curves, and the other more recent one by [10] with much higher Picard number. The moduli construction of [10] is in terms of the polyhedral decompositions of integral affine spheres (“tropical K3 surfaces”) with singularities. The latter family of tropical K3 surfaces on the 19-dimensional cone is rather close to the one constructed in [118, Section 4] on $\mathcal{F}_2(l)$ in the sense they have same radiance obstructions, hence expected to be obtained from each other by moving worms generically. However, the affine structure itself is different as observed by V. Alexeev. We thank him for sharing the observation.

Now we go back to discussing our main contents, and set the notation and conventions before that.

1.3 Notation and conventions

In this paper, we work over an arbitrary algebraically closed field $k$ of characteristic 0 unless otherwise stated, since often we use the minimal model program and resolution of singularities. Some parts which involve analytifications are discussed only over $\mathbb{C}$ (e.g., Appendix 3) or non-Archimedean fields (e.g., Appendix A) as indicated, which means complete valuation fields with non-Archimedean valuations. Many purely algebro-geometric arguments extend over more general fields but we omit specifications.

In this paper, a Calabi–Yau variety or a K-trivial variety both interchangeably mean a klt projective variety with numerically trivial canonical class, and weak Ricci-flat Kähler metric for it (when $k = \mathbb{C}$) means the singular Kähler metric whose existence is established by [38, 147]. Polarization means ample ($\mathbb{Q}$-)line bundle and variety with polarization is said to be polarized, and whenever it is not genuine line bundle but only $\mathbb{Q}$-line bundle we clarify so. We often work over a pointed smooth $k$-curve germ $\Delta \ni 0$ or $\Delta^+ := \Delta \setminus 0$, but many of discussions go verbatim (or with slight refinements) to the complex disk $\{ t \in \mathbb{C} \mid |t| < 1 \}$ (resp., $\{ t \in \mathbb{C} \mid 0 < |t| < 1 \}$) or Spec $k[[t]]$ (resp., Spec $k((t))$).
2 Degenerations and their limit — Galaxy

2.1 Basics of dlt minimal models

This section discusses dlt minimal models which do not necessarily satisfy open K-polystability (of [115]), and their certain limits. In some part, we give technical refinements of [82, 96, 104, 105].

Notation

In the following dlt minimal model (resp., lc minimal model) \( X \to \Delta \) means \( (X, X_0) \) is a dlt pair (resp., an lc pair), so that \( X_0 \) is reduced in particular, and \( K_X + X_0 \) is relatively nef over \( \Delta \), where \( \Delta \) denotes either germ of pointed smooth curve or \( \text{Spec} \ k[[t]] \). A slight difference with other references (e.g., [104]) is that we assume \( X_0 \) is reduced, just for simplicity, which is not essential assumption because the semistable reduction theorem [76] combined with the semistable MMP [49] always allow to pass to this case without essential change (our following arguments also include explanation). Note that the relative nefness implies relative triviality for the case when the generic fiber has trivial canonical class. We note the following points for the convenience, which are easy to prove from the definitions.

Since our paper focuses on the Calabi–Yau varieties and their degenerations, unless otherwise stated, dlt minimal model also requires the general fiber to be klt Calabi–Yau varieties which are \( \mathbb{Q} \)-factorial. Note that in some references, dlt minimal model requires the total models to be \( \mathbb{Q} \)-factorial. However, unfortunately the conditions exclude various important examples, e.g., well-studied Dwork–Fermat family of K3 surfaces, as they are non-\( \mathbb{Q} \)-factorial dlt minimal models, we discuss in our generality.

Theorem 2.1 (Basics of dlt models)

(i) Consider any small birational map \( \varphi : X \to X' \) and a \( \mathbb{Q} \)-divisor \( D = \sum_i a_i D_i \) on \( X \) with \( D' := \varphi_* D \), such that \( (X, D) \) and hence \( (X', D') \) are both log pairs, i.e., \( K_X + D \) and \( K_{X'} + D' \) are \( \mathbb{Q} \)-Cartier. Suppose \( (X, D) \) is dlt and satisfies the following assumption:

For any strata \( Z \) of \( |D| \), i.e., connected component of \( \bigcap_{i \in I} D_i \) with the index set \( I \) satisfying \( a_i = 1 \) for \( i \in I \), \( \varphi \) restricted to \( Z \) still gives a birational map.

Then, \( (X', D') \) is also dlt. (For the converse direction, see below (iv)).

(ii) If \( (X, X_0) \to \Delta \) is a dlt minimal model, then \( X \) is terminal.

(iii) (cf. [75], [104, 3.2.5]) Take two dlt minimal models \( (X_i, X_{i,0}) \) \((i = 1, 2)\) over \( \Delta \) which coincide over \( \Delta \setminus \{0\} \). Then they are connected by flops and the dual intersection complexes of their central fibers \( X_{i,0} \) are canonically homeomorphic.

(iv) For any lc minimal model \( (X, X_0) \to \Delta \) and its log crepant dlt minimal model \( (X^{\text{dlt}}, X_0^{\text{dlt}}) \) which dominates \( X \), the exceptional locus of \( X^{\text{dlt}} \to X \) is a union of some irreducible components of the central fiber \( X_0^{\text{dlt}} \) and hence cannot be small unless isomorphism.

Moreover, if further \( (X, X_0) \) is dlt, then \( X^{\text{dlt}} \to X \) is small and satisfies the assumption of (i).
As an example of (i), the well-studied degeneration of quartic into four hyperplanes is a dlt (non-Q-factorial) model. Note that the claim does not respect Q-factoriality.

**Proof** The claim (i) can be shown as follows. From the definition, \((X', D')\) is also obviously log canonical, at least. Recall that log canonical centers of \((X', D')\) are nothing but the strata of \([D]\) [49, Section 3.9]. Therefore, the assumption implies the assertion.

For proving (ii), we take a non-snc locus \(W\) of \((\mathcal{X}, \mathcal{X}_0)\). Then for any prime divisor over \(\mathcal{X}\) with the center inside \(W \subset \mathcal{X}_0\), the central fiber takes the multiplicity of \(\mathcal{X}_0\) as at least 1, since \(\mathcal{X}_0\) is Cartier (actually principal). Hence the claim.

The statements of (iii) were known (cf. [104, 3.2.5]) at least in terms of skeleta in the Berkovich analytification but an easy (re)proof of (iii) follows now. By the above (ii), [75] implies that \(\mathcal{X}_i\) are connected by flops (log flips, in reality). Because of [49, 3.9], we can restrict to simple normal crossing part (in particular, toroidal) of \(\mathcal{X}_i\) to discuss dual intersection complexes. Then, from the toric basics, the toric flops only change polyhedral decomposition (cf. [58]).

The last item can be proven as follows. As in Step 1 of the proof of Proposition B.4, any of the log canonical centers \(W\) of \((\mathcal{X}^\text{dlt}, \mathcal{X}_0^\text{dlt})\) which are not irreducible components of \(\mathcal{X}_0^\text{dlt}\) (i.e., if \(\text{codim}_{\mathcal{X}_0^\text{dlt}}(W) \geq 2\)) cannot be contracted since semi-log-canonicity of \(\mathcal{X}_0\) implies the \(S_2\) condition. On the other hand, the prime divisor of \(\mathcal{X}_0^\text{dlt}\) contracted in \(\mathcal{X}\) is automatically log canonical centers, hence the claim. The rest of the proof again follows from the \(S_2\) condition of \(\mathcal{X}_0\).

Since the work of Gross–Siebert and Kontsevich–Soibelman on geometric understanding of Strominger–Yau–Zaslow mirror symmetry conjectures (cf., [58, 84], etc.), the dual intersection complexes of the central fibers of the relative minimal models for Calabi–Yau fibration over curves has been intensively studied. Particularly well-studied is the case of maximal degenerations, and in that case, the dual intersection complex is expected to be the sphere \(S^n\) (cf., [58, 64]).

Partial progress of understanding homeomorphic type is discussed in [82, 104]. [104, 4.1.4], which is a consequence of some results in [78, 80], proves that it is a closed manifold at least in (real) codimension 1. [82] makes a progress, proving the expectation of dual complex to be \(S^n\) in \(n = 3, 4\) under certain strictness condition on the Calabi–Yau varieties. Thanks to the inductive structure of the stratification, it is also confirmed in [82] that the concerned dual intersection complex is topologically a manifold up to dimension 4 (and 5 if the degeneration is simple normal crossing).

### 2.2 Minimizing property of dlt model and polarized version

As an easy uniqueness statement of the Calabi–Yau filling, we recall the following in a global projective setting, which provides an algebro-geometric background to non-collapsing limits of Ricci-flat Kähler manifolds with bounded diameters.

---

4 Although not every flop preserves the dual complex such as the classical example of Atiyah flop over cone over quadric (cf. [33, Example 17]). Indeed, in loc.cit, the divisor in concern is only a part of toric divisor so that the setup is different.
Theorem 2.2 ([108, Section 4, discussion after 4.1, 4.2 (i)]) Suppose \((\mathcal{X}, \mathcal{L}) \to C \ni 0\) is a flat projective polarized family, which are n-dimensional slc Calabi–Yau projective varieties over \(C \setminus \{0\}\) (set \(\mathcal{X}^*\) as preimage over \(C \setminus \{0\}\) i.e., \(K_{\mathcal{X}^*/(C \setminus \{0\})} \equiv 0\). We consider all possible birational transforms \((\mathcal{X}', \mathcal{L}')\) along the fibers over (the preimages of) 0, allowing base changes of \(C \ni 0\) as \(b: C' \to C\).

Then the normalized degree of CM line bundle \([41, 46, 123]\)

\[
\frac{\deg(\lambda_{\text{CM}}(\mathcal{X}', \mathcal{L}'))}{\deg(b)} = \frac{(\mathcal{L}'^n, K_{\mathcal{X}'/C'})}{\deg(b)}
\]

attains the minimum (resp., the minimum as the only minimizer) among all

\[
\frac{\deg(\lambda_{\text{CM}}(\mathcal{X}', \mathcal{L}'))}{\deg(b)}
\]

if and only if \(\mathcal{X}'_0\) is semi-log-canonical (resp., klt) and \(K_{\mathcal{X}'_0} \equiv 0\).

Recall that the CM line bundle becomes a positive multiple of the Hodge line bundle in case of Calabi–Yau family. As we discussed in loc.cit.\(^5\) basically the proof of the above Theorem 2.2 is the same as in [109] in the isotrivial case.

Corollary 2.3 ([108, Corollary 4.3], [22]) If a punctured family of polarized log terminal Calabi–Yau varieties \((\mathcal{X}^*, \mathcal{L}^*) \to \Delta^*\) can be completed to \((\mathcal{X}, \mathcal{L}) \to \Delta\) with a log terminal Calabi–Yau filling \((\mathcal{X}_0, \mathcal{L}_0)\), there is no other slc Calabi–Yau filling.

In particular, moduli space of log terminal polarized Calabi–Yau varieties will be automatically separated (Hausdorff).

In contrast to the above uniqueness claim, as an existence-direction claim, we have the following. The result slightly refines the above written case, while ensuring ampleness of the extended polarization, and generalizes the classical work of Shepherd-Barron [136] for degeneration of surfaces.

Proposition 2.4 (Polarized dlt model) Given a punctured family of polarized log terminal Calabi–Yau varieties \((\mathcal{X}^*, \mathcal{L}^*) \to \Delta^*\), there is a filled-in proper flat family \((\mathcal{X}, \mathcal{L}) \to \Delta\) such that \((\mathcal{X}_{0, \text{red}}, \mathcal{L})\) is dlt and \(\mathcal{L}\) is relatively ample. Here, \(\mathcal{X}_{0, \text{red}}\) means the reduced scheme structure on \(\mathcal{X}_0\).

Furthermore, after finite base change, one can assume \(\mathcal{X}_0\) is reduced, i.e., \(\mathcal{X}\) is dlt minimal model with ample \(\mathcal{L}\).

Proof We take a dlt \(\mathbb{Q}\)-factorial minimal model \(\mathcal{X}'\) and extend the polarization \(\mathcal{L}^*\) to \(\mathbb{Q}\)-line bundle \(\mathcal{L}'\) on \(\mathcal{X}'\). Then, take a general section of \(|m\mathcal{L}'|\) and denote it by \(\mathcal{D}'\). We take a relative log canonical model of \((\mathcal{X}', \epsilon\mathcal{D}')\) for \(0 < \epsilon \ll 1\) which gives the desired property. We denote its total space by \(\mathcal{X}_{\text{lc}}\) and the intermediate relative log (dlt) minimal model’s total space by \(\mathcal{X}_{\text{min}}\).

\(^5\) However, the author mistakenly wrote the content of (ii) also in the place of (i) in op.cit., by which the author wanted to mean the above. We recently noticed the mistake but the arithmetic variant’s proof had also been repeated in [112, Theorem 3.4 (i)]. We apologize for the possible confusion it caused.
Note that $X_{lc}$ is still terminal hence Cohen–Macaulay from Theorem 2.1 (ii). Thus its central fiber is again Cohen–Macaulay so that the generic reducedness implies reducedness. The only remaining thing to confirm, for the former statement of this Theorem 2.4 is that none of $lc$ centers of the $X_{\text{min}}$ is contracted by the morphism to $X_{lc}$. This also follows from the Cohen–Macaulay property of $X_{lc,0}$.

After finite base change, the semistable reduction theorem implies one can ensure reducedness of the central fiber of $X'$. After passing to the relative lc model, the central fiber is still reduced because of the dlt property. \qed

We introduce a simpler local version of the CM degree minimization (cf. [108, Section 4]). Although our main interest lies in the Calabi–Yau case for a moment, we discuss in more general setting.

**Definition 2.5** (Log canonical height) For a proper flat ($\mathbb{Q}$-Gorenstein) family of polarized projective varieties over a smooth proper curve $(X, L) \to C$ and a holomorphic section $s: C \to X$, we define the log canonical height (resp., canonical divisor height) of $s$ as

$$h_{lc}(s) := \deg_C s^*(O_X(K_{X/C} + X_{0,\text{red}})),$$

(resp., $h_{c}(s) := \deg_C s^*(O_X(K_{X/C}))$.) Here, $X_{0,\text{red}}$ denotes the reduced divisor fully supported on $X_0$.

This is a variant of weight function by [84, 96, 143], and the result below characterizes its minimization easily, in a similar manner to loc.cit.

**Proposition 2.6** (Dlt minimal models and heights minimization) For a generically log terminal proper family of projective varieties $X^* \to (C \setminus \{0\})$, and fixing the meromorphic section $s|_{C \setminus \{0\}}$, consider fillings, i.e., proper flat $X \to C$ possibly after a finite base change of $C$ of degree $d$.

Then, among such fillings with $s(0)$ lying in only one irreducible component of $X_0$, the normalized lc divisor height

$$\frac{h_{lc}(s)}{d}$$

is minimized if and only if $s(0)$ lies in an irreducible component which appears in the dlt minimal model of a finite base change of $C$ of degree $d$. \qed

We call the above irreducible component lc height minimizing for $s$. \hfill \# Springer
2.3 Subdivision by base change and galaxies

From now on, we consider finite ramified base change of degenerating Calabi–Yau varieties and its “subdividing” effects, following [115, Section 4]. More precisely, we obtain a projective system of reductions as follows. Here, one main point will be to work over the (formal) Puiseux power series ring which means

$$k[[t^Q]] := \bigcup_{m \geq 1} k[[t^{1/m}]],$$

in this paper. We also denote its fraction field as $k((t^Q))$.

**Definition 2.7** 1. Consider a (polarized) dlt minimal model $(X, L) \to \Delta \ni 0$, and another dlt minimal model $(X', L') \to \Delta' \ni 0$ over a finite ramified cover of $\Delta$ whose complements of the central fiber is identified after the base change. Note that $\varphi$ is automatically birational and log crepant. We call $(X', L')$ (resp., $X'$) **admissibly dominating** $(X, L)$ (resp., $X$) if $\varphi : X' \dashrightarrow X \times_\Delta \Delta'$ is a morphism (i.e., everywhere defined) such that the following conditions hold (resp., the second condition holds):

- $L'$ and $\varphi^* L$ coincide at open strata $X^\text{klt}_0$,
- $\varphi$ is toroidal at the snc (open) locus of $(X, X_0)$.

2. Take a local uniformizer $t$ of $O_{\Delta, 0}$ and consider $O_{\Delta, 0} \hookrightarrow k[[t]] \hookrightarrow k[[t^Q]]$, where the last target denotes the local ring of formal Puiseux power series. If $(X', L') \to \Delta'$ admissibly dominates $(X, L) \to \Delta$ in the above sense 1., we call

$$(X', L') \times_\Delta k[[t^Q]] \to (X, L) \times_\Delta k[[t^Q]]$$

or its restriction to the central fiber $(X'_0, L'_0) \to (X, L)$ also admissibly dominating.

The term **admissibly** comes with reminiscence to that of Raynaud [126], although our main concern is more detailed analysis of very particular case of admissible blow-ups in the sense of *loc.cit*. The base change trick in [115, Section 4.1], whose origin at least goes back to [44] for Kulikov degenerations of K3 surfaces and Kodaira for $I_\nu$-type degenerations of elliptic curve case (cf. also [76], [26, Section 5.9], etc.) shows the following proposition.

**Proposition 2.8** For any dlt model $(X, L) \to \Delta$ as a polarized Calabi–Yau family, and its ramified finite base covering $\Delta' \to \Delta$, we have a dlt model $(X', X'_0)$ as a vertical blow-up of $X \times_\Delta \Delta'$ admissibly dominating $X$.

**Proof** This is essentially proven in [115, the proof of Theorem 4.8], especially Step 2 and Step 3 in *loc.cit*. So we refer to *loc.cit.* and omit the repetition of the arguments, except for pointing out that we should take the regular subdivision to ensure the relative projectivity criterion by Tai ([13, Chapter IV, Section 2, esp., Theorem 2.1]). Note that from the construction, the obtained dlt model is $\mathbb{Q}$-factorial even if the original $X$ is not $\mathbb{Q}$-factorial. \qed

\[ \text{Springer} \]
The above construction of admissibly dominating models does not give their uniqueness, especially for the ambiguity caused by the simple use of the relative minimal model program. Here are basic properties of the admissible dominations.

**Theorem 2.9** For admissibly dominating morphisms between dlt minimal models as above $\psi: \mathcal{X}' \to \mathcal{X} \times_{\Delta} \Delta'$, the following sets of closed subsets of $\mathcal{X}_0$ are the same:

(i) the set of log canonical centers of $(\mathcal{X}, \mathcal{X}_0)$,
(ii) the set of components of intersections of some components of $\mathcal{X}_0$,
(iii) the set of the images of log canonical centers of $(\mathcal{X}', \mathcal{X}'_0)$ by $\psi$, and
(iv) the set of components of intersections of some components of $\mathcal{X}'_0$.

**Proof** Before going to the proof, we set further notations: we denote the ramification degree of $\mathcal{X}' \to \mathcal{X} \times_{\Delta} \Delta'$ at the origin as $N$, $\mathcal{X} \times_{\Delta} \Delta'$ as $\mathcal{X}^{(N)}$. We take a log resolution $\varphi: \widetilde{\mathcal{X}} \to \mathcal{X}$ with a simple normal crossing vertical divisor $D$ of $\widetilde{\mathcal{X}}$ so that $(\widetilde{\mathcal{X}}, D)$ is log crepant to $(\mathcal{X}, \mathcal{X}_0)$, which exists by the definition of log resolution. Here, in this paper, the notion of simple normal crossingness for a divisor means locally normal crossing and any finite intersection of the components are connected and smooth.

Note that the divisor $D$ can be written as $\varphi^{-1}_* \mathcal{X}_0 + E$ where $E$ is supported on $\varphi$-exceptional locus. Now, we set $\widetilde{\mathcal{X}}^{(N)} := \widetilde{\mathcal{X}} \times_{\Delta} \Delta'$ with its vertical divisor $\widetilde{D}^{(N)}$ such that the log smooth pair $(\widetilde{\mathcal{X}}^{(N)}, \widetilde{D}^{(N)})$ is log crepant to $(\widetilde{\mathcal{X}}, D)$ and $(\mathcal{X}, \mathcal{X}_0)$.

From the simple normal crossing property of $D$, $(\widetilde{\mathcal{X}}^{(N)}, \widetilde{D}^{(N)})$ is not only log canonical, but even toroidal due to the local equations of two terms. Furthermore, since $D$ is supposed to have only connected smooth finite intersections, we can take a toroidal log resolution of the pair $(\widetilde{\mathcal{X}}^{(N)}, \widetilde{D}^{(N)})$ as $(\widetilde{\mathcal{X}}^{\approx(N)}, \widetilde{D}^{\approx(N)})$.

In this situation, we confirm the following desired natural identifications of the sets:

$$
\begin{align*}
\text{log canonical centers of } (\mathcal{X}^{(N)}, \mathcal{X}^{(N)}_0) &= \text{log canonical centers of } (\mathcal{X}^{(N)}, \mathcal{D}^{(N)}) \\
&= \text{strata (components of finite intersections) of } [\mathcal{D}^{(N)}] \\
&= \text{strata (components of finite intersections) of } [\mathcal{D}] \\
&= \text{log canonical centers of } (\mathcal{X}, \mathcal{D}).
\end{align*}
$$

The reasons of the above to hold are: (1) follows from the log crepantness, (2) follows from toroidal structures, (3) obviously holds, and (4) follows from the dlt-ness of $(\widetilde{\mathcal{X}}, D)$ thanks to [49, 3.9] for instance.

Now, for the setting of the statements, we can take $\mathcal{X}^{\approx(N)}$ as one that dominates $\mathcal{X}'$. Then, sending the elements (loci of various varieties) to $\mathcal{X}_0$, we obtain the desired identifications. □

It readily follows from the above Theorems 2.1, 2.9 that:

**Corollary 2.10** In the setup of Proposition 2.8 and Theorem 2.9,

(i) For fixed $\mathcal{X} \to \Delta$ and $N$, $\# \{\text{components of } \mathcal{X}'_0\}$ does not depend on the choice of dlt minimal model $\mathcal{X}'$.
for $N \to \infty$, where $m$ denotes the dimension of the dual intersection complex.

**Proof** The proof of (i): it immediately follows from Theorem 2.1 (ii) and [75]. More precisely, we have a canonical identification of the sets for different choices of $X'$ (for fixed $X$ and $N$).

The proof of (ii): Recall from [78] that the minimal log canonical centers all have the same dimensions, hence the dual intersection complex is the union of $m$-dimensional simplices [104, 4.1.4].

Therefore, since $\Delta(X'_0)$ is canonically homeomorphic to $\Delta(X_0)$ with $N$ multiplied affine structures (Theorem 2.1 (iii)), we conclude the proof.

Note that before taking the limit, the above cannot be an equality unless the abelian varieties case.

Recall that for dlt model $X \to \Delta$, with the generic fiber $X_\eta$ and the central fiber $X_0$, we have the essential skeleton $\Delta_{\text{alg}}^\text{an}(X_0) \subset X_\eta^\text{an}$ as introduced in [84], with more refined understanding in [24, 25, 96, 104]. As [84, Section 4.1] inferred indirectly and written explicitly in [96, Section 3.2] (cf. also [24, Section 2.1]), it also admits a natural $\mathbb{Z}$-affine structure. Recall from the basics of Berkovich geometry [16] that, for the degree $N$ ramifying finite map $\Delta' \to \Delta$ and $X^{(N)} := X \times_\Delta \Delta'$ as before, there is a natural map $(X'_\eta)^\text{an} / \text{Gal}(\Delta'/\Delta) \simeq X_\eta^\text{an}$.

**Proposition 2.11** In the setup of Theorem 2.9, the restriction of homeomorphism

$$(X'_\eta)^\text{an} / \text{Gal}(\Delta'/\Delta) \simeq X_\eta^\text{an}$$

gives a natural homeomorphism between the dual intersection complexes of $X'_0$ and $X_0$ as

$$\Delta_{\text{alg}}^\text{an}(X'_0) \simeq \Delta_{\text{alg}}^\text{an}(X_0),$$

with the $\mathbb{Z}$-affine structure simply $N$ multiplied (in the sense of local affine coordinates).

**Proof** If we replace $X'$ by another dlt minimal model (without changing $\Delta' \to \Delta$), the essential skeletons do not change including the affine structures, as proven in [104, Section 3.2] after [96].

Therefore, we can assume that $\Delta'$ is the one constructed in Step 2 of the proof of [115, Theorem 4.4], as used in Proposition 2.8. From the construction, over simple normal crossing locus of $(X, X_0)$ which contains all lc centers, the birational morphism $X' \to X^{(N)} = X \times_\Delta \Delta'$ is toroidal and the exceptional divisors correspond to $1/N$-integral points of the maximal simplices of $\Delta_{\text{alg}}^\text{an}(X_0)$, from the definition of the affine structures in [96, Section 3.2] (cf. also [24, Section 2.1]). Hence, from Theorem 2.9, we conclude with the desired assertion. □
Now we are going to expand the construction of Proposition 2.8 over Puiseux formal power series ring to a field \( k \) as \( k[[t^Q]] \) by base change. Then, the above admissible dominating morphism over such Puiseux formal power series ring is also a finite type blow-up morphism. For simplicity of our notation, we denote the Puiseux formal power series ring over a field \( k \) as \( k[[t^Q]] \), although it is strictly smaller than the whole set of formal power series with rational exponents as it may look. Note that although our base \( \text{Spec}(k[[t^Q]]) \) is non-Noetherian, the blow-up still makes sense for any quasi-coherent ideal of finite type. Generally such blow-up may be not “fppf”, i.e., finitely presented a priori, but nevertheless in our case fppf holds because of the toroidal description above and [115, proof of Theorem 4.4] (compare with [138, 37 0EV4]). Now, we are ready to introduce the notion of (quasi-)galaxies.

**Definition 2.12 ((quasi-) limit dlt model and (quasi-) galaxies)**

(i) We say a locally ringed space \( X_\infty \) over \( k[[t^Q]] \) is quasi-limit dlt model of polarized Calabi–Yau family if there is a projective system of the base changes to \( k[[t^Q]] \) of dlt minimal models with admissibly dominating morphisms between them, whose projective limit is \( X_\infty \to k[[t^Q]] \), which we call quasi-galaxy model (over \( k[[t^Q]] \)). We call such projective system a presentation of \( X_\infty \) and its each dlt minimal model a dlt approximation model.

We call the reduction (at \( t = 0 \)) of limit dlt model a quasi-limit sdlt reduction or quasi-galaxy. For simplicity, from now on we denote the reduction at \( t = 0 \) of \( X_i \) (resp., \( X_\infty \)) as \( X_i \) (resp., \( X_\infty \)). Also, if \( k \) is a valuation field, we put \( X_i^{\text{an}} \) the complex analytification of \( X_i \) and \( X_\infty^{\text{an}} := \varprojlim_i X_i^{\text{an}} \).

(ii) A quasi-limit dlt model or its reduction, i.e., quasi-limit sdlt reduction is said to be open K-polystable if one can take dlt approximation models sequence as all open K-polystable ones (cf. [115]).

(iii) We say a quasi-limit dlt model over Puiseux series in the above sense is complete or limit dlt model if it is represented by a sequence of dlt approximation models \( X_i \) over \( k[[t^{1/N_i}]] \) satisfying: for any positive integer \( N \), there is \( i_0 \) with \( N \mid N_i \) for all \( i > i_0 \). Intuitively, viewing the dual intersection complexes, this means “sufficiently divided” at its combinatorial looking. We call the reduction \( X_\infty \) or \( X_\infty^{\text{an}} \) (at \( t = 0 \)) of limit dlt model a limit sdlt reduction or simply galaxy, while the (quasi-galaxy) model \( X_\infty \) as galaxy model over \( k[[t^Q]] \).

(iv) We call a limit dlt model over Puiseux series in the sense of (i) a limit (dlt K-) polystable model if it is complete in the sense of (iii) and one can take the dlt approximation models as all open K-polystable in the sense of [115].

**Corollary 2.13** For any dlt model \((X, \mathcal{L}) \to \Delta \) as polarized Calabi–Yau family, there is a (complete) limit dlt model over \( k[[t^Q]] \) which dominates \((X, \mathcal{L}) \times_\Delta k[[t^Q]] \).

**Proof** This is a corollary of (the proof of) Proposition 2.8 simply because \( k[[t^Q]] = \bigcup_m k[[t^{1/m}]] \) so that we can take the projective limit of their base changes. \( \square \)

---

6 Intuitively (non-mathematically) speaking, galaxy in this sense (resp., galaxy in normal sense) looks like a totality which, while connected, consists of so many tiny fine pieces, each of which is still rich and beautiful world (log Calabi–Yau variety resp., planetary system). This might also remind some readers of dragonfly’s compound eyes, or either Kumiko or Kiriko, the traditional crafts in Japan.
The next statements give basic structure of galaxies, decomposing into pieces.

**Corollary 2.14** (of Theorem 2.9: decomposition of galaxies) Any complete limit sdlt reduction \(X_{\infty,0} = \lim_{\leftarrow i} X_{i,0}\) has a natural decomposition into an open part and closed part:

\[
X_{\infty,0} = \left( \bigsqcup_{a \in B(\mathbb{Q})} \{ \text{open klt log Calabi–Yau variety } U(a) \} \right) \sqcup X_{\text{NKLT}}^{\infty,0}.
\]

Here, \(B\) denotes the dual intersection complex of \(X_{i,0}\) and \(B(\mathbb{Q})\) means its rational points with respect to the \(\mathbb{Q}\)-affine structure, which does not depend on \(i\) by Theorem 2.1 (iii) and \(X_{\text{NKLT}}^{\infty,0} := \lim_{\leftarrow i} X_{i,0}^{\text{NKLT}}\), where NKLT stands for the non-klt loci.

Note that from Theorem 2.1 (iii), it follows that for each fixed \(a\), birational type of \(U(a)\) is unique. Corollary 2.14 is an immediate consequence of Theorem 2.9. We call the former part klt locus while the latter part non-klt locus, and denote them as \(X^{\text{klt}}_{\infty,0}\) (resp., \(X^{\text{nklt}}_{\infty,0}\)) or \(X^{\text{klt}}_{\infty}\) (resp., \(X^{\text{nklt}}_{\infty}\)) accordingly.

As in [115, Section 4], from differential geometric perspective, an interesting case is maximal degenerations with open K-polystable components of the reduction. At least if the general fibers are abelian varieties, such open K-polystable reduction exists. Indeed, there is the Néron model after finite base change, whose abelian parts of the reduction are parametrized as the limits inside the Satake(–Baily–Borel) compactification of \(A_g\) (cf. [29, 4.4.1]). This is also confirmed in [114, Corollary 2.14]. On the other hand, from the construction of [39, 65, 66] combined with the Delaunay decomposition used in [11, 102], we can relatively compactify to projective family. Hence, the existence of limit sdlt open K-polystable reduction does hold.

Here comes the relation with non-Archimedean geometry of Zariski–Riemann, Fujiwara, Huber type. We leave the details and the proof to Sect. 3.2.

**Proposition 2.15** (Galaxies dominated by Huber analytification) In the above setup, consider the generic fiber of \(X_{\infty}\) which we denote as \(X_{\infty,\eta}\). Then, any of its quasi-galaxy is dominated by the Huber analytification of the \(k((t^{\mathbb{Q}}))\)-variety \(X_{\infty,\eta}\) by a natural continuous surjective map.

### 2.4 Limit toric variety — a local toy model

For (quasi-)limit sdlt reduction, often the irreducible components get blown up infinitely many times. However, the blow-up is certainly log crepant and not arbitrary. In this subsection, we introduce a prototypical example of a toy model for such structure. Toys give us joy. This is prototypical not only to some constructions in the previous subsection but also in Sect. 3.1 for the compactified moduli.

To make the longer story of this subsection short, we consider the projective limit of toric variety. This may be of its own interest in much more general context. To set the scene, recall for a lattice \(N \cong \mathbb{Z}^n\) that the proper toric variety \(T_N \text{ emb } \Sigma\) of dimension \(n\) corresponds to a rational polyhedral decomposition of \(N_{\mathbb{R}} = N \otimes \mathbb{R}\).

Here is the definition of limit toric spaces. Note that projective limit exists in the category of locally ringed spaces (cf., e.g., [52, Section 4.1], [54], [142, Remark 2.1.1]).
Definition 2.16 For a natural number \( n \), we define locally ringed spaces

\[
\mathcal{T}_n := \lim_{\Sigma} T_N \text{ emb } \Sigma,
\]

and if \( k \) is a valuation field, we also set the Berkovich analytification version ([16]):

\[
\mathcal{T}^{an}_n := \lim_{\Sigma} (T_N \text{ emb } \Sigma)^{an}.
\]

Here, the directed set \( \{ \Sigma \} \) is defined as the set of all complete rational polyhedral decompositions of \( \mathbb{N}_\mathbb{R} \) and \( \Sigma' \preceq \Sigma \) if and only if \( \Sigma' \) is a subdivision of \( \Sigma \). We call the above locally ringed spaces the \( n \)-dimensional limit toric space.

Note that this story is totally different from a more ubiquitous toric construction for infinite, but locally finite, fan such as the one used in [13, 94] among others. Here are some basic properties.

Theorem 2.17 Suppose our base field \( k \) is a valuation field.

(i) \( \mathcal{T}^{an}_n \supset T^{an}_N \sqcup (\bigcup l D^o_l) \). Here, \( T^{an}_N \cong (\mathbb{C}^*) \otimes_{\mathbb{Z}} N \) and \( l \) runs over rational half-lines inside \( \mathbb{N}_\mathbb{R} \). By \( D_l \), we mean the torus invariant prime divisor of \( T_N \text{ emb } \Sigma \) with \( \Sigma \ni l \), corresponding to \( l \), and \( D^o_l (\subset D_l) \) denotes its torus invariant open subset.

(ii) If we denote the minimal Morgan–Shalen–Boucksom–Jonsson compactification of \( T^{an}_N \) (cf. [114, Appendix A.1, esp., A.12] and our Appendix A) as \( T^{\text{an MSBJ}}_N \), the boundary is naturally isomorphic to \( (\mathbb{N}_\mathbb{R} \setminus \{0\})/\mathbb{R}_{>0} \). Further we have a natural surjective continuous map:

\[
\varphi_{\text{tr}} : \mathcal{T}^{an}_n \rightarrow T^{\text{an MSBJ}}_N.
\]

Here, \( \text{tr} \) of \( \varphi_{\text{tr}} \) stands for tropical. Moreover, at the boundary level, we have an algebraic version of the morphism \( \varphi_{\text{tr alg}} : \partial \mathcal{T}^{n-\text{tr}}_n \rightarrow \partial T^{\text{an MSBJ}}_N \) which is also continuous.

(iii) Take a point \( x \in (\mathbb{N}_\mathbb{R} \setminus \{0\})/\mathbb{R}_{>0} = \partial T^{\text{an MSBJ}}_N \) (cf. [114, Appendix A.1, esp., A.12], and our Appendix 2.4). If \( x = (x_1, \ldots, x_n) \) with respect to a basis of \( N \) over \( \mathbb{Z} \), and \( r = \text{rank } \mathbb{Q} \sum_i \mathbb{Q} x_i \),

\[
\varphi_{\text{tr}}^{-1} (\varphi_{\text{tr}} (x)) \cong \mathcal{T}^{n-r, an}_n.
\]

Example 2.18 Even for the case \( n = 1 \), the structure of \( \mathcal{T}^1_\infty \) is already somewhat interesting. For each non-zero \( x \in N \otimes \mathbb{Q} \), there are two points \([x + 0]\) (resp., \([x - 0]\)) in \( \partial \mathcal{T}^1_\infty \) which values at \( D_l \setminus D^o_l \) for any \( \Sigma \) which contains \( l \). This is analogous to Type 5 point of [132, 2.20].

As we see, this limit toric space, as well as our analogues, has some mixed properties of varieties or usual analytic spaces, and adic spaces such as Zariski–Riemann spaces [52, 70, 151].
Proof (i): We only need to consider $\Sigma$ whose ray set $\Sigma(1)$ includes $l$. Then, $T_N \operatorname{emb} \Sigma$ contains $T_N \sqcup D_1^\sigma$ as an open subset. This locus remains the same for any $\Sigma$ while further subdivisions, and for different $l$, $D_1^\sigma$ does not intersect. Hence the proof.

(ii): The map $\varphi_{tr}$ is constructed as follows. Take an arbitrary

$$x = (x_\Sigma) \in \overline{T^n_{\infty}} = \lim_{\Sigma} (T_N \operatorname{emb} \Sigma)^{an}.$$  

For each $\Sigma$, we take $\sigma_{\alpha}(\Sigma) \in \Sigma$ such that its corresponding torus orbit $\operatorname{orb}(\sigma_{\alpha}(\Sigma)) \subset T_N \operatorname{emb} \Sigma$ contains $x_\Sigma$. Then, it follows that $\Sigma' > \Sigma$ implies $\sigma_{\alpha}(\Sigma) \supset \sigma_{\alpha}(\Sigma')$. Since $\Sigma$ runs over all rational subdivisions, there is some $y \in N_{\mathbb{R}} \setminus \emptyset$ such that $\mathbb{R}_{>0} y = \bigcap_{\Sigma} \sigma_{\alpha}(\Sigma)$. We put $[\mathbb{R}_{>0} y] \in \partial \overline{T^n_{\infty}}_{\operatorname{MSBJ}}$ as $\varphi_{tr}(x)$.

Then the continuity of $\varphi_{tr}$ can be confirmed as follows. We denote the natural projection map $\overline{T^n_{\infty}}_{\operatorname{an}} \to T_N \operatorname{emb} \Sigma$ as $p_\Sigma$. Now we take a point $y = \varphi_{tr}(x) \in \partial \overline{T^n_{\infty}}_{\operatorname{MSBJ}}$. For an arbitrary open neighborhood $V$ of $y$, we want to show that there is an open neighborhood $U$ of $x$ such that $\varphi_{tr}(U) \subset V$. We can take such $U$ as follows. First we take a small enough regular rational polyhedral cone $\sigma$ whose image contains $y$. Then take a regular rational polyhedral decomposition of $N_{\mathbb{Q}}$ as $\Sigma$. Then one can consider the complex analytification $U^{an}_\sigma$ of the affine toric variety $U_\sigma$ and set

$$U := (T_N^{an} \cup U^{an}_\sigma).$$

Then, it is immediate from the definition of $\varphi_{tr}$ that $\varphi_{tr}(U) \subset V$. Thus the continuity of $\varphi_{tr}$ is shown, as well as the corresponding map from the (algebraic) boundary since $U$ is the analytification of a Zariski open subset.

(iii): After choosing an appropriate element of $\operatorname{GL}(N) \simeq \operatorname{GL}(n, \mathbb{Z})$, we can assume that vectors $(x_1, \ldots, x_r, x_{r+1}, \ldots, x_n)$ are such that $x_1, \ldots, x_r \in \mathbb{R}$ are rationally independent and $x_i$ for $i > r$ are all a $\mathbb{Q}$-linear combination of $x_1, \ldots, x_r$. Then, the assertion is straightforward and we leave it to the readers.

\[\square\]

Theorem 2.17 (i) is analogous to Corollary 2.14 and Theorem 2.17 (ii) is analogous to Theorem 2.20. For another further analogue in the context of moduli compactification, see Sect. 3.1.

Now we go back to the original context of limit sdlt reduction at the end of Sect. 2.3.

### 2.5 Relation with the Kontsevich–Soibelman essential skeleta

Generally, we can show the following connection of the limit sdlt reduction with essential skeleta proposed by Kontsevich–Soibelman \[84\] which were more clarified in \[96, 104\] in the terminology of recent minimal model program. We first note the following, which e.g., follows from our construction above in Proposition 2.8 (and \[115, \text{proof of Theorem 4.4}\]).

Note that for toric degeneration in the sense of \[58\], we can also make sense of the essential skeleton or the dual intersection complex. In particular, such construction can be applied to any of the historical degenerations for abelian varieties, constructed...
Degenerated Calabi–Yau varieties with infinite... 1125

by \([11, 39, 65, 94]\): which we call a semi-toric degeneration and their complete limit as a limit semi-toric degeneration/reduction.

**Lemma 2.19** Consider a dlt minimal model \(X\) over \(k[\![t]\!]\) and another dlt minimal model \(X'\) over \(k[\![t^{1/N}]\!]\) which is isomorphic to the base change of \(X\) away from \(t = 0\). Then the essential skeleta \(\Delta(X_0)\) of \(X\) and \(\Delta(X'_0)\) of \(X'\) are canonically homeomorphic, preserving the rational affine structures.

Now we observe that the limit sdlt reduction hides the structure of the essential skeleton, by two versions of the statements: the case of abelian varieties with slightly general degenerations, and for general Calabi–Yau varieties case. Below, for the abelian varieties case, we allow limit semi-toric degenerations of abelian varieties that means the projective limit of the central fibers of approximation models constructed by \([39, 94]\) with respect to subdivisions. More precisely, their approximation models are the relatively complete model of Raynaud extension divided by the period group \(Y \cong \mathbb{Z}^r\).

Note that a priori such models are only formal, not necessarily algebraic nor projective, but for a given punctured family, there does exist such a projective model thanks to \([11, 102, 154]\).

**Theorem 2.20** For any limit semi-toric degeneration \(X\) of abelian varieties over \(k[\![t^Q]\!]\), \(X^\text{an}_\infty\) has a natural continuous surjective map to \(\Delta(X_0)\):

\[
f_{tr}: X_\infty \to \Delta(X_0).
\]

**Proof** The core argument of the proof closely follows that of Theorem 2.17 (ii), thanks to the toric nature of the construction of \([39, 94]\). Take a projective system of all semi-toric degenerations constructed by \([39, 65, 94]\) and denote it as

\[
\left\{ X_i^{[N_i]} \to \Delta \xrightarrow{t_i \mapsto t_i^{N_i}} \Delta \right\}_{i}.
\]

Here, the index directed set of \(i\) is the set of pairs of a positive integer \(N_i\) and an \(N_i Y\)-periodic regular fan (cone decomposition) of \(Y_\mathbb{R} \oplus \mathbb{R}_{>0}\). The order \(i < i'\) is defined to hold if and only if \(N_i | N_{i'}\) and the corresponding fan to \(i'\) is a subdivision of that for \(i\).

From the construction in *loc.cit.*, the obtained approximating semi-toric degenerations are quotients by \(N_i Y \cong \mathbb{Z}^r\) of a toric variety fiber bundle determined by the \(N_i Y\)-periodic fan, so that the dual intersection complex of the reductions is canonically homeomorphic to the \(r\)-dimensional real \((Y \otimes \mathbb{Z}_\mathbb{R})/Y\) (cf. also \([114]\)).

Take any \(x = (x_i)_i \in \lim_{\longleftarrow} (X_i^{[N_i]})^\text{an}_0\). For each \(i\), take a semitorus strata (locally closed subset) \(Z_i\) which contains \(x_i\), and consider a cone \(\sigma_{N_i}(x)\) which corresponds to \(Z_i\). This is determined only modulo \(N_i Y\). Nevertheless, considering the semi-toroidal structure of the admissibly dominating morphisms, we can take \(\{\sigma_i(x)\}_i\) so that \(i < i'\) implies \(\sigma_i(x) \supset \sigma_{i'}(x)\). Then, we consider \(\bigcap_i \sigma_i(x)\). Since \(i\) runs over the index set of all admissible regular \(Y_{\mathbb{Q}}\)-rational polyhedral cone decompositions, it easily follows that is a half-line, i.e., there is \(\tilde{f}_{tr}(x) \in Y_{\mathbb{R}}\) such that \(\mathbb{R}_{\geq 0}(1, \tilde{f}_{tr}(x)) = \bigcap_i \sigma_i(x)\). Then we set \(f_{tr}(x) := \tilde{f}_{tr}(x) \mod Y\).
Now we show the continuity of \( f_{tr} \). We take an open neighborhood \( V \) of \( \tilde{f}_{tr} (x) \) inside \( Y_\mathbb{R} \), which is a section of a rational regular cone \( \sigma \) of \( N \oplus \mathbb{R}_z \) at the hyperplane \( z = 1 \). We take a partial compactification of the Raynaud extension corresponding to \( \sigma \), i.e., the affine toric variety \( U_\sigma \)-fiber bundle over the abelian part of \( \mathcal{X}_0 \). Then consider its image after the quotient by \( Y \), and denote it as \( U \). We consider a regular rational \( Y \)-admissible polyhedral fan \( \Sigma \) which contains \( \sigma \). Then we consider the corresponding model \( \mathcal{X}_\Sigma \). We define \( U = U \cap \mathcal{X}_\Sigma |_{t=0} \). Then, from the definition of \( f_{tr} \) and \( U \), it obviously holds that \( f_{tr} (U) \subset V \).

Note that, if we only allow regular fans for the semi-toric degenerations, obviously the obtained models are dlt (snc) models so that the inverse limit is a limit sdlt reduction.

Here is another version of the statements for more general Calabi–Yau varieties.

**Theorem 2.21** For any punctured family of Calabi–Yau varieties \( \mathcal{X}^* \to \Delta^* \), and its any galaxy model \( \mathcal{X} \) over \( k[\{ t^{\mathbb{Q}} \}] \), the central fiber (galaxy) \( X^\infty \) has a natural continuous surjective map to \( \Delta(X_0) \):

\[
f_{tr} : X^\infty \to \Delta(X_0).
\]

**Proof** The proof closely follows that of Theorem 2.20. Thus we briefly describe the proof basically as repetition of similar ideas, while showing some subtle differences.

We again take a projective system of the approximating dlt models constructed and denote it as

\[
\{ X_i^{[N_i]} \to \Delta \xrightarrow{t \mapsto t^{N_i}} \Delta \}_{i}.\]

Here, the index directed set of \( i \) is the set of pairs of a positive integer \( N_i \) and the dlt model after the base change of degree \( N_i \). The order \( i < i' \) is defined to hold if and only if \( N_i | N_i' \) and the corresponding polyhedral decomposition of the essential skeleton for \( i' \) is a subdivision of that for \( i \).

Take any \( x = (x_i)_i \in \varprojlim \{ X_i^{[N_i]} \}_0^{an} \). For each \( i \), take a strata (locally closed) \( Z_i \) of the lc stratification of \( X_i \) which contains \( x_i \), and consider a rational polyhedron \( \tilde{\sigma}_i (x) \) in the essential skeleton \( \Delta(X_i) \), which corresponds to \( Z_i \).

Note that our index set of \( i \) is ordered so that \( i < i' \) implies \( \tilde{\sigma}_i (x) \supset \tilde{\sigma}_{i'} (x) \). Then, we consider \( \bigcap_i \tilde{\sigma}_i (x) \), which we can easily show to be a single point and denote it as \( f_{tr} (x) \).

Now we show the continuity of \( f_{tr} \) similarly to the case of abelian varieties. We take a closed neighborhood \( \tilde{\sigma} \) of \( \tilde{f}_{tr} (x) \), which is a rational simplex inside the essential skeleton, which is one regular piece of the simplicial complex \( \Delta(X_{i_0}) \), corresponding to the log canonical center \( Z_{i_0} \). We define the star \( S(Z_{i_0}) \) of the lc center \( Z_{i_0} \) as the union of the strata of the lc stratification of \( X_{i_0} \) whose closure contains the generic point of \( Z_{i_0} \).

From the definition, it is a Zariski open subset of \( X_{i_0} \). We denote the projection \( X^\infty \to X_{i_0} \) as \( p_{i_0} \). Now we define \( p_{i_0}^{-1} (S(Z_{i_0})) \) as \( U \). Then, from the definition of \( f_{tr} \) and \( U \), it obviously holds that \( f_{tr} (U) \subset \tilde{V} = \tilde{\sigma} \). 

\( \square \)
Remark 2.22 The above may somewhat look resembling the Berkovich retraction [18, 144] (also called “non-Archimedean SYZ fibration” as in [105]) from the Berkovich analytification of Calabi–Yau varieties to its essential skeleton (see also [84]). However, that version was not canonical and changes by flops of the models, while our map $f_{tr}$ is canonically defined.

2.6 Visualizing non-Archimedean Calabi–Yau metric by galaxy models

A natural basic strategy towards understanding the structure of limit sdlt reductions is to compare $X_0$ and $X_i, 0$ which are the central fibers at $t = 0$ of dlt minimal models $X \rightarrow \Delta$ and $X_i \rightarrow \Delta'$, where the bases are connected by a finite morphism $\Delta' \rightarrow \Delta$ with ramifying degrees $N_i$. From Definition 2.12, we have an admissibly dominating morphism $X_i \rightarrow X \times_\Delta \Delta'$. What about the “converse-direction”? We partially show that for a given dlt minimal model $X$, $X_i$ can be taken such that they look locally very similar, refining Proposition 2.8.

Proposition 2.23 (Local identification) Take an arbitrary dlt minimal model $X$ which is a toric degeneration [60, 61], so that in particular the central fiber $X_0$ is a Gorenstein stable toric variety in the sense of [7]. We denote its open subset where $(X, X_0)$ with its projection to $\Delta$ is toroidal as $X_\text{tor} \subset X$ (cf. [60], [49, 3.9]). For any finite branched morphism $\Delta' \rightarrow \Delta$ with any ramifying degree $N$, there is a dlt minimal model $X' \rightarrow \Delta'$ with an admissibly dominating morphism $\psi : X' \rightarrow X \times_\Delta \Delta'$ such that the following holds:

For any point $x' \in \psi^{-1}X_\text{tor} (\subset X')$, there is a point $x \in X_0 \cap \psi^{-1}(x')$ such that $(x \in X)$ and $(x' \in X')$ have isomorphic germs.

Proof As in [60, 61], the degeneration corresponds to the dual intersection complex $B$ with its decomposition $\mathcal{P}$ into a union of polyhedra $\{P_i\}_{i}$ respecting the integral affine structures.

If we keep $(B, \{P_i\})$, while multiplying the coordinates of $P_i$ by $N$, we obtain a natural toric degeneration over the open part $X_\text{tor} \subset X$ and then further apply the construction of $X^{(N)}$ after Proposition 2.8 [115, Theorem 4.8, Step 2], which depends on regular subdivision of the dual intersection complex. Here, we take the regular subdivision by the standard one by dividing by $N - 1$ hyperplanes for each facets direction, which subdivides each simplex by unit simplices, so that the local structure remains. Then, we take a log resolution of $X^{(N)}$ which maps to $X$, and take its relative minimal model $X'$ over $X$. It easily follows from the construction from the locally identical toric description that $X'$ satisfies our desired assertion.

Degenerating abelian varieties case

Now we consider the case of families of principally polarized abelian varieties as it is the simplest instance. The construction of degenerating abelian varieties via toric methods and formal geometry is well-established, cf. [94], [39], [65, Section 6] and

Note that [94] constructed semiabelian reduction by auxiliary “relatively complete model” which is neither unique nor canonical. [65, Section 6] extended over general non-Archimedean fields.
special cases for canonically compactifying $A_g$ are also done in more details by [11, 102] among others.

Proposition 2.23 of “subdividing models with locally same structures” is vividly observed in the case of abelian varieties in the context of describing non-Archimedean canonical Chambert–Loir measures [30, 65, 66].

Indeed, for a given semitoric polarized degeneration of abelian varieties $(X, L)$, Proposition 6.7 more explicitly constructed the admissibly dominating morphisms of any semi-toric degenerations of abelian varieties, which we denote as $X^{(N)}, \text{std} \to X \times \Delta \Delta'$ where “std” stands for “standard” and $\Delta' \to \Delta$ is a degree $N$ finite cover with ramification just at the origin. Further, he showed that the non-Archimedean Calabi–Yau metric (cf. [24]) can be described in terms of such approximation models in e.g., [66, 1.3] (cf. also [88]). More precisely,

**Theorem 2.24** ([65, 66]) For any dlt semitoric polarized degeneration of abelian varieties $(X, L) \to \Delta$, there is a natural polarization $L^{(N)}$ on $X^{(N)}$ and the non-Archimedean CY metric $|\cdot|_{n\text{AMA}}$ for the $(X^{(N)}, L^{(N)})$ is the limit of the model metrics induced by $L^{(N)}$ where $N = l^a$ with $a \to \infty$, for any positive integer $l$. Here $X_\eta$ is the generic fiber of $X \to \Delta$ and $L_\eta$ is the restriction of $L$ to it.

In the proof, the group scheme structure (multiplication maps) of abelian schemes over $\Delta \setminus \{0\}$ is effectively used which of course does not exist for other Calabi–Yau varieties. Comparing with our terminology of this section, we could paraphrase the above:

**Corollary 2.25** (of [65, 66]) For any polarized punctured family of abelian varieties $(X^*, L^*) \to \Delta^* = \Delta \setminus \{0\}$, and any positive integer $l > 1$, there are a quasi-galaxy model $X_\infty$ with dlt approximation models $\{X_i \to \Delta' = \text{Spec}(k[[t^{1/l^i}]]); i = 1, 2, \ldots\}$ and their $\mathbb{Q}$-line bundles $L_i$ satisfying the following:

(i) $L_{i+1} = p_{i,i+1}^* L_i(-E_{i+1})$. Here, $p_{i,i+1} : X_{i+1} \to X_i \times_{k[[t^{1/l^i}]]} k[[t^{1/l^i+1}]]$ is the admissibly dominating morphism and $E_{i+1}$ is a $\mathbb{Q}$-Cartier $p_{i,i+1}$-exceptional divisor.

(ii) If we denote the non-Archimedean Calabi–Yau metric [24, 66, 88] as $|\cdot|_{n\text{AMA}}$ and the model metric of $L_\eta^{an}$ induced by $L_i$ as $|\cdot|_{L_i}$, then

$$|\cdot|_{n\text{AMA}} = \lim_{i \to \infty} |\cdot|_{L_i}.$$ 

In the abelian varieties case, the above observation connects:

- the limit open K-polystable sdlt reduction (originally motivated by understanding of bubbles of complex (Kähler) Calabi–Yau metrics), and
- the non-Archimedean Calabi–Yau metric (cf. [24, 66, 88]).

We discuss generalization to other K-trivial varieties in a paper in preparation [117].

**Remark 2.26** It is easy to see that for any quasi-limit dlt minimal models and their polarizations, the CM line bundle can be defined by Deligne pairing in spite of non-Noetherianess of $k[t^{1/\infty}]$, $k[t^{1/\infty}]$ (cf. [152, Sections 1.1, 1.2]).
3 Limit toroidal compactification

Some particular features of the limit toroidal compactification and the limit log minimal compactification of a normal quasi-projective variety $M$ (our particular concern is when $M$ is the moduli of polarized log-terminal Calabi–Yau varieties), both to be introduced in this section, are summarized as follows:

- They are locally ringed spaces, although not algebraic varieties.
- They are dominated by the Zariski–Riemannian compactification (Sect. 3.2) which is “much bigger”.
- Unlike the Zariski–Riemann compactification, limit toroidal (or log minimal) compactification certainly respects and reflects the “minimality” in the sense of the (log) minimal model program.
- For the connected Shimura varieties case at least, the limit toroidal compactification dominates both the Satake–Baily–Borel compactification (variety) and the Morgan–Shalen type compactification (not variety), i.e., there are continuous surjections to them. Compare this fact with that no variety compactification dominates any of the Morgan–Shalen type compactifications.

3.1 Construction and expectation

We set the scene as follows. Consider a class of polarized smooth $K$-trivial varieties with connected moduli $M^0$, and all their $\mathbb{Q}$-polarized Calabi–Yau degenerations i.e., degenerated klt Calabi–Yau variety $X$ with $\mathbb{Q}$-line bundle $L$ in $\frac{1}{N}\text{Pic}(X) \subset \text{Pic}(X) \otimes \mathbb{Q}$ (for some $N \in \mathbb{Z}_{>0}$ depending on $X$), and its coarse moduli algebraic space of them all $M$.

Assumption 3.1 We suppose that all of the following (mutually related) assumptions hold:

(i) $N$ has a uniform upper bound for all $X$’s,
(ii) $M$ is a quasi-projective scheme over $k$,
(iii) more strongly, $M$ is a quasi-projective normal variety over $k$.

Due to a famous result of Viehweg [146], the first assumption (i) implies the second assumption (ii), hence the difficulty we face now is a boundedness type problem. This issue was also raised in [153, 1.1, 1.2], [118, Sections 8, 9].

It has been known for decades that Assumption 3.1 is true for abelian varieties or K3 surfaces with polarizations, although it is non-trivial in general (cf., e.g., [57, 103], [118, Section 9, Step 1]). Recently it was also confirmed for $\mathbb{Q}$-Gorenstein smoothable hyperKähler varieties with polarizations in [118, Section 8.3].

Under the above Assumption 3.1, we can take a finite Galois cover $M'$ which is a normal quasi-projective variety on which there is a polarized family in concern (such a cover exists for arbitrary $M$, cf. [146]). We denote the Galois group as $\Gamma := \text{Gal}(M'/M)$. Further, take a $\Gamma$-equivariant projective compactification $M' \subset \overline{M'}$ such that $\overline{M'} \setminus M'$ is a simple normal crossing divisor $D'$. We also set $\overline{M} := \Gamma \setminus \overline{M'}$, its
boundary divisor $D := \overline{M} \setminus M$. We denote the log canonical model of $(\overline{M}', D')$ as $M'_\text{lc}$.

The branch $\mathbb{Q}$-divisor with the standard coefficients $^8$ in $M$ is denoted by $D_M$.

We consider the following.

**Definition 3.2** If $k = \mathbb{C}$ and $M$ has a structure as a locally Hermitian symmetric space (e.g., the complex analytification of a connected Shimura variety), let us consider the projective limit of all of its toroidal compactifications as locally ringed spaces and call them the *limit toroidal compactifications*:

$$\overline{M}^{\text{tor}}_{\infty} := \lim_{\Sigma} \overline{M}^{\text{tor}, \Sigma},$$

and its analytification

$$\overline{M}^{\text{tor}, \infty, \text{an}} := \lim_{\Sigma} \overline{M}^{\text{tor}, \Sigma, \text{an}}.$$

The above $\overline{M}^{\text{tor}, \Sigma}$ stands for the toroidal compactification with respect to the $\Gamma$-admissible collection of fans $\Sigma = \{ \Sigma(F) \}$ in the sense of [13] and the above $\overline{M}^{\text{tor}, \Sigma, \text{an}}$ means their complex analytifications. Here, $F$ denotes the rational boundary component of the Satake–Baily–Borel compactification of $M$ and $\Sigma(F)$ is the fan $\{ \sigma^F_\alpha \}$ of $C(F)$ in the notation of [13]. Here, we add one more assumption:

**Assumption 3.3** $(\overline{M}, D_M + D)$ is of log general type and has only log canonical singularities.

The latter is always true but we also expect the former to be true in general. See [35, 155] for related partial results. Under the assumption, we set

$$\overline{M}^{\text{min}, \infty} := \lim_{\overline{M}^{\text{log}, \min}} (\Gamma \setminus \overline{M}^{\text{log}, \min}),$$

and its analytification

$$\overline{M}^{\text{min}, \infty, \text{an}} := \lim_{\overline{M}^{\text{log}, \min}} (\Gamma \setminus \overline{M}^{\text{log}, \min, \text{an}})$$

where $\overline{M}^{\text{log}, \min}$ runs over all $\Gamma$-equivariant log dlt minimal models of $(\overline{M}', D')$ and $\overline{M}^{\text{log}, \min, \text{an}}$ denotes their complex analytifications. The projective limits are taken in the category of locally ringed topological spaces and we call these the *limit log minimal compactifications*.

It is easy to see that the following holds. Below, we continue to use the above notation.

**Proposition 3.4** For either of the above two compactifications $\overline{M}^{\text{min}, \infty}$, the boundary $\partial \overline{M}^{\text{min}, \infty}$ contains as an open part, which we denote as $\partial \overline{M}^{\text{min}, \infty, o}$, the union of the plt locus of $(\overline{M}, D_M + D)$, where $(\overline{M}', D')$ runs over all $\Gamma$-equivariant log minimal models.

**Proof** Indeed, since the negativity lemma readily implies the log minimal models are log crepant, the plt locus cannot be blown up.

We denote the union of the above $\partial \overline{M}^{\text{min}, \infty, o}$ and $M$ as $\overline{M}^{\text{min}, \infty, o}$.

---

$^8$ Meaning the usual $1 - \frac{1}{d}$ where $d$ is the ramifying degree, cf., e.g., [6, 3.4]
Proposition 3.5 If the limit log minimal compactification $\overline{M}^{\text{min, } \infty}$ dominates (the analytification of) a proper variety $M \subset \overline{M}$, then there is one of log minimal compactifications $(\Gamma \setminus \overline{M}^{\log, \text{min, } \text{an}})$, such that there is a dominant birational map whose image at least contains an open subset of $\overline{M}$ outside codimension 2 locus.

Proof For each prime boundary divisor $F$ of $\overline{M} \supset M$, from the construction of 3.2, there is a log minimal compactification $\Gamma \setminus \overline{M}(F)$ which contains $M$ and the generic point of $F$. We take such log minimal compactification for each $F$, and take a refined log minimal compactification which dominates all such $\Gamma \setminus \overline{M}(F)$.

The moduli-theoretic meaning of this construction is not clear for the moment, while our first speculation would be that $\partial M^{\text{tor, } \infty}$ could parametrize certain reductions of “limit (dlt open K-)polystable models” to be defined in Sect. 2. A possibly hinting result we obtain is the following.

Theorem 3.6 Suppose $M$ is a locally Hermitian symmetric space $\Gamma \setminus G/K$. Then, there is a natural continuous surjective map

$$\phi_\text{tr}: M^{\text{tor, } \infty}^{\text{an}} \to M^{\text{MSBJ}}$$

which extends the identity map on $M$.

Furthermore, using the notation in [13] and [118, Section 2], if we take a general point $x$ inside the open strata $C(F)/\mathbb{R}_{>0} \subset \partial M^{\text{MSBJ}}$ with a 0-dimensional cusp $F$ of the Satake–Baily–Borel compactification $\overline{M}^{\text{SBB}}$ then the fiber $\phi^{-1}_\text{tr}(x)$ is the limit toric variety $\overline{M}^{\text{tor, } \infty}^{\text{an}}$ (defined in Sect. 2.4) where $r$ denotes the $\mathbb{Q}$-rank of $x$ in $U(F)$, where $U(F)$ is the center of the unipotent radical of the real maximal parabolic subgroup of $G$ fixing $F$ (stratwise).

Here, $M^{\text{MSBJ}}$ denotes the minimal Morgan–Shalen–Boucksom–Jonsson compactification ([114, Appendix], [118, Section 2]), i.e., the MSBJ compactification corresponding to the toroidal compactifications [13], which do not depend on the combinatorial data, i.e., the admissible collection of rational polyhedra.

Proof The proof is very similar to that of Theorem 2.17 (ii) (and Theorem 2.20) and essentially follows from the same arguments. Indeed, first, we take an arbitrary $x \in M^{\text{tor, } \infty}$ and consider the rational boundary strata $F$ of the Satake–Baily–Borel compactification of $M$ which contains the image of $x$, following the notation of [13]. For some regular decompositions $(\Sigma(F))$, we denote the natural projection $M^{\text{tor, } \infty} \to M^{\Sigma}$ as $p_\Sigma$. We consider an open neighborhood of $x$ of the form $p^{-1}_\Sigma(U) \subset M^{\text{tor, } \infty}$ and define $\phi_\text{tr}$ from it, which does not depend on the choice, and glue them together.

We denote the strata of $\partial M^{\text{tor, } \Sigma, \text{an}}$ containing the image $x_\Sigma$ of $x$ as $S_x$. Then take a small enough open neighborhood $U'$ of $x_\Sigma$ such that $U' \cap F' \neq \emptyset$ if and only if $F' \supset S_x$. We replace $U$ by $U' \cap U$ which is possible by the basic property of the stratification. From here, we further use the notation in [13] without reviewing all. We just recall that $U(F)$ is the center of the unipotent radical of $N(F)$ which is a real maximal parabolic subgroup of the corresponding reductive Lie group to $M$, $U(F)_Z = U(F) \cap \Gamma$, and $D$ denotes the covering Hermitian symmetric domain.
Since the action of \((\Gamma \cap N(F))/U(F)\) acts properly discontinuously on the quotient \((D/U(F))\Sigma(F)\), we can lift the point \(x_\Sigma = \rho_\Sigma(x)\) to \((D/U(F))\Sigma(F)\). Suppose \(x_\Sigma\) lies in the toric strata corresponding to a cone \(\sigma_F^{\alpha}(\Sigma) \in \Sigma(F)\). Then consider \(\bigcap_\Sigma \sigma_F^{\alpha}(\Sigma)\) similarly to Theorem 2.17 (ii). Completely similarly, we can show that it is a half-line of the form \(\mathbb{R}v_x\) for \(v_x \in U(F) \setminus \{0\}\). Then we set \(\phi_{tr}(x) := \frac{v_x}{v_x}\). The proof of the continuity of the obtained map \(\phi_{tr}\) (resp., the fiber structure) is identical to the one of Theorem 2.17 (ii) (resp., Theorem 2.17 (iii)), so we avoid further essential repetition.

\[\square\]

**A weak analogue for the moduli of curves \(M_g\)**

If we try to search for an analogue of the limit log minimal compactification for the moduli of hyperbolic curves \(M_g\), moduli of hyperbolic curves of genera \(g > 1\), one would naturally replace the set of all toroidal compactifications above by the single Deligne–Mumford compactification \(M_g \subset M_g^{DM}\) since it is smooth lc model at a stacky level. Therefore, we do not obtain a similar compactification in the same manner but still there is another compactification which is analogous to some extent (compare with Theorem 3.6):

Amini–Nicolussi [12] recently constructed a compactification \(M_g \subset M_g^{hyb}\) on whose boundary they parametrize metric complex [4] with the ordered partition of the edge sets, called “layor”s.

**Proposition 3.7** The compactification of \(M_g \subset M_g^{hyb}\) constructed by Amini–Nicolussi is the least common refinement in the sense of [21, I.16.1, I.16.2] of

- the Deligne–Mumford compactification \(M_g \subset M_g^{DM}\),
- the Morgan–Shalen–Boucksom–Jonsson compactification

\[M_g \subset M_g^{MSBJ}(\overline{M_g^{DM}})\]

for the Deligne–Mumford moduli stack \(\overline{M_g^{DM}}\) ([114, Appendix], see also [113]).

**Proof** Recall that the well-known local structure of \(\mathcal{M}_g\) around the boundary and universal curves over it, says in particular that there is a natural one-to-one order (closure relation) reversing bijection between the set of strata of \(\overline{M_g^{DM}}\) and the strata of \(M_g \subset M_g^{MSBJ}(\overline{M_g^{DM}})\), which is the moduli of tropical curves [1]. Tracing the bijection, the desired assertion follows from the construction of [12, Section 3.2] as follows. For a sequence of \(M_g\) towards the boundary converging to \(t\) in a boundary strata \(D_F = D_{E_t}\) in \(\overline{M_g \setminus M_g}\), which corresponds to the index set (of nodes), the set of possible limits in \(M_g \subset M_g^{MSBJ}(\overline{M_g^{DM}})\) is the closure of the open strata in the dual intersection complex of the stacky snc divisor \(\overline{M_g \setminus M_g}\) from the construction in [26, Section 2], [114, Appendix]. It is nothing but the \(\bigcup_{\pi \in \Pi(F)} \sigma_\pi^0\) in [12, (3.10, 3.11)]. Therefore, the least common refinement of the two compactifications parametrizes (smooth projective hyperbolic curves and) the metrized complex in the effective manner. The \(D^0_\pi = D^0_\sigma\)-part of loc.cit. (3.10) parametrizes the limit stable curves, while
its \( \sigma_{\pi} \)-part parametrizes the graph part of the metrized complexes as [12] shows and also our assertion follows.

A caution is to recall that \( M_g \subset \overline{M}_g^{\text{MSBJ}(\overline{M}_g^{\text{DM}})} \) is not compatible with hyperbolic metric behaviour, i.e., different from the Gromov–Hausdorff compactification of \( M_g \) with respect to hyperbolic metrics on the Riemann surfaces. Indeed, we need to replace “glueing function” to describe the Gromov–Hausdorff compactification. See [113] and [114] for details.

From Proposition 3.7, \( M_g^{\text{hyb}} \) and \( \overline{M}_{\text{tor},\infty} \) for locally Hermitian symmetric space \( M \) can be both understood as the least common refinements of

- log minimal model compactifications (projective varieties), and
- its (their) Morgan–Shalen compactification(s).

The critical difference of the \( M_g \) case and connected Shimura variety case is that, at stacky level, we can take unique log minimal model compactification (actually lc model) by [31] as the standard choice, while the latter admits many log minimal model compactifications as toroidal compactifications by [13]. Hence, \( M_g^{\text{hyb}} \) can be seen as a weak analogue of \( \overline{M}_{\text{tor},\infty} \) and \( \overline{M}_{\text{min},\infty} \) nevertheless of many differences.

### 3.2 Zariski–Riemann compactification and comparison

#### 3.2.1 Review of Zariski–Riemann compactification

Here we review a Zariski–Riemann type compactification which dominates the above limit log minimal compactification. Although obviously the idea goes back to Zariski’s innovative idea [151], the reason of our review is that simply we could not find literature providing results in the form we need. So we hope the following accounts to worth writing as an appendix, and perhaps contains slight improvements of the known results.

Recall that the dual intersection complex is tightly connected to the theory of Berkovich analytic space as its “finite part”. Therefore, it is natural to explore the possible great enrichment of the Morgan–Shalen type compactification through Zariski–Riemann space or Huber adic space, which is the aim of this section.

Suppose \( U \) is a \( k \)-variety. Recall that, as we declared in the introduction, \( k \)-variety for a field \( k \) in this paper means an integral separated finite type scheme over \( k \). Here we introduce another “canonical” compactification of \( U \) with the Zariski topology using the theory of Zariski–Riemann spaces [151].

**Definition 3.8** In the above setup, we set

\[
\text{Val}_k(k(U)) = \{ \text{all (Krull) valuations } v : k(U) \to \Gamma \}/\sim,
\]

where \( k(U) \) denotes the meromorphic function field of \( U \) as usual, \( \Gamma \) runs over totally ordered abelian groups, \( \sim \) denotes the equivalence of (Krull) valuations. Denote

\[
\partial \text{Val}(U) := \{ v : k(U) \to \Gamma \mid c(v) \notin U \},
\]
where \( c(v) \) denotes the center of \( v \) in a projective compactification including \( U \). We equip with the topology (Zariski topology) whose open basis are rational domains \( U(x_1, \ldots, x_m) = \{ v \mid v(x_i) \geq 0 \text{ for all } i \} \).

On the other hand, here is another classical notion.

**Definition 3.9** (Zariski–Riemann space [150, 151]) In the above setup, we set

\[
Z_{R_k}(k(U)) := \lim_{\leftarrow X} X,
\]

(in the category of locally ringed spaces) where \( X \) runs over all proper varieties including \( U \) as an open subset (we fix the inclusion), with birational proper morphisms between them to form a projective system.

We denote the center of \( v \) in \( X \) as \( c_X(v) \) following a standard notation. Then, recall that the reduction (specialization) map gives a homeomorphism:

**Theorem 3.10** (Zariski [151]) *In the above setup, there is a natural homeomorphism*

\[
\text{Val}_k(k(U)) \simeq Z_{R_k}(k(U)),
\]

*which sends \( v \) to the centers \( \{ c_X(v) \}_X \).*

Below is a natural variant where \( U \) is preserved as not blown up.

**Definition 3.11** ([150, 151], cf. also [52, Section 4]) In the above setup, consider

\[
Z_{R}(U) := \lim_{\leftarrow U \subset U} \overline{U},
\]

where \( \overline{U} \supset U \) is a proper variety which includes \( U \) as an open dense subset. As before, we put the topology as (the restriction of) the product topology of the Zariski topology on each scheme \( \overline{U} \).

Obviously, we have a natural decomposition

\[
Z_{R}(U) = U \sqcup \lim_{\leftarrow U} (\overline{U} \setminus U) =: \partial \overline{U},
\]

so we set \( \partial Z_{R}(U) := \lim_{\leftarrow U} \partial \overline{U} \). Then we have:

**Proposition 3.12** (cf. [52, 142, 151]) *In the above setup,\( i \) for the natural morphism*

\[
p_U : Z_{R_k}(k(U)) \to Z_{R}(U),
\]

*its restriction*

\[
p_U|_{p_U^{-1}(\partial Z_{R}(U))} : p_U^{-1}(\partial Z_{R}(U)) \to \partial Z_{R}(U)
\]

*is an isomorphism.*

\( \square \) Springer
(ii) There is a natural homeomorphism
\[ \partial \text{Val}(U) \simeq \partial \text{ZR}(U). \]

**Proof** (i): Consider a point in \( \text{ZR}_k(k(U)) \) which can be identified as a Krull valuation on \( k(U) \) by the classical Theorem 3.10. We consider its image in \( \text{ZR}(U) \), which we suppose to be outside \( U \). We denote its germ as \( \emptyset \) with its maximal ideal \( m \), while \((\emptyset_v, m_v)\) is the valuation ring for \( v \). Note that we have \( \emptyset \subset O \subset k(U) \). What remains to show is \( O = O_v \) as the rest automatically follows. We prove by contradiction, so suppose the contrary. Take
\[ \frac{f}{g} \in O_v \setminus O, \quad (5) \]
where \( f, g \in O \). To make arguments (even) simpler, we prepare a reference compactification variety \( U \subset U_{\text{ref}} \) which is a Cartier divisor. This is possible as otherwise we can take the blow-up of \( \partial U = U \setminus U \). We denote the germ of the image of \( v \) in \( U \) as \( \emptyset'(\subset \emptyset) \), around which the Cartier divisor \( U \setminus U \) of \( U \) is locally generated by a single element \( h \in \emptyset' \). Since \( \emptyset \) is a local ring, \( g \in m \) holds. Suppose \( f \not\in m \). Then, it would imply \( \frac{1}{g} \in O_v \setminus O \) by multiplying (5) by \( \frac{1}{f} \). Combining \( g \in m \) with \( \frac{1}{g} \in O_v \), \( 1 \in m_v \) follows which is absurd. Hence it follows that \( f \in m \). Then, we can take a blow-up of \( \overline{U}_{\text{ref}} \) along (some closure of) the locally closed subscheme cut by the ideal \((f, g, h)\). Then it easily contradicts to (5).

(ii): The construction is just by restricting the map of the more classical Theorem 3.10. First, we confirm the continuity as follows. We take an arbitrary reference compactification variety \( U \subset U_{\text{ref}} \) such that \( U_{\text{ref}} \setminus U \) is a Cartier divisor. This is possible as otherwise we can take the blow-up of \( \partial U = U \setminus U \). We denote the germ of the image of \( v \) in \( U \) as \( \emptyset'(\subset \emptyset) \), around which the Cartier divisor \( U \setminus U \) of \( U \) is locally generated by a single element \( h \in \emptyset' \). Since \( \emptyset \) is a local ring, \( g \in m \) holds. Suppose \( f \not\in m \). Then, it would imply \( \frac{1}{g} \in O_v \setminus O \) by multiplying (5) by \( \frac{1}{f} \). Combining \( g \in m \) with \( \frac{1}{g} \in O_v \), \( 1 \in m_v \) follows which is absurd. Hence it follows that \( f \in m \). Then, we can take a blow-up of \( \overline{U}_{\text{ref}} \) along (some closure of) the locally closed subscheme cut by the ideal \((f, g, h)\). Then it easily contradicts to (5).

Proposition 3.13 (Galaxies dominated by Huber analytification (= Proposition 2.15)) Consider any dlt model \( \mathcal{X} \) over \( \Delta \) which we base change to \( k[[t^\mathbb{Q}]] \) and denote as \( \mathcal{X}_\infty \).
Denote its generic fiber as $X_{\infty, \eta}$. Then, any of its quasi-galaxy $X_{\infty}$ is dominated by the Huber analytification of the $k((t^{Q}))$-variety $X_{\infty, \eta}$, which we denote as $X_{\infty, \eta}^{\text{ad}}$, by a natural continuous surjective map:

$$X_{\infty, \eta}^{\text{ad}} \twoheadrightarrow X_{\infty}.$$ 

**Proof of Proposition 2.15 = 3.13** The proof resembles that of the more classical Theorem 3.10 by Zariski. Take the dlt approximation models $X_i$ with their central fibers $X_i$'s and consider their base change over $k[[t^{Q}]]$. As in the classical setting of Theorem 3.10, for any semi-valuation $v = | - |_v$ in $X_{\infty, \eta}^{\text{ad}}$, we take all the centers $c_{X_i}(v) \in X_i$ for each $i$. This gives a continuous map similarly to Theorem 3.10 (cf. [151]).

On the other hand, the surjectivity follows similarly to [151]. We consider all the normal projective models of $X$ over $k[[t^{1/m}]]$, base changed to $k[[t^{Q}]]$, and denote the models as $Y_j$'s and their reductions as $Y_j$'s. Then take the projective limit $\lim_{\leftarrow j} Y_j$ in an analogous manner to Zariski–Riemann compactification as in Definition 3.11.

Take a point inside $X_{\infty}$ which is the image of $\{y_j\}_{j} \in \lim_{\leftarrow j} Y_j$. We consider the natural injective limit $\lim_{\to j} \mathcal{O}_{Y_j, y_j}$, where $\mathcal{O}_{Y_j, y_j}$ denotes the stalk local ring of $y_j \in Y_j$. We denote it as $\mathcal{O}$ and prove it is a valuation ring as desired. Suppose the contrary and take $r = \frac{f}{g}$ in the fraction field of $\mathcal{O}$, with $f, g \in \mathcal{O}$. We can suppose $f, g$ are both realised in the model $Y_j$ for same $j$ which we fix. Then, we take a blow-up of $Y_j$ with respect to the ideal $(f, g, t)$. Then we obtain another model $Y_k$ and easily see that either $r$ or $r^{-1}$ is in the stalk of $\mathcal{O}_{Y_k}$ at $y_k$, hence the proof. $\square$

### 3.2.2 Berkovich type compactification as a separated quotient

First we re-interpret the construction of [125, Section 3] (also cf. [26, Sections 2 and 4], [114, Appendix]), which compactifies (the complex analytification of) a complex variety $U$ with “non-Archimedean” boundary which is a normalized Berkovich space in the sense of [40]. They denoted it as $U^h \subset U^\sim$. See [125] for details.

We add some remarks to their work. For simplicity, we put smoothness assumption on $U$, but as in [114] or our Appendix A, we should be able to relax the condition without difficulties. Take an arbitrary $\overline{U}$ which is a smooth proper variety containing $U$, with the complement a simple normal crossing divisor. Applying [26, Section 2.2], we obtain

$$\overline{U}^{\text{MSBJ}}(\overline{U}) = U^{\text{an}} \sqcup \Delta^{\text{alg}}(\overline{U} \setminus U)$$

as well as

$$\overline{U}^{\lim, \text{MSBJ}} = \lim_{\overline{U}} \overline{U}^{\text{MSBJ}}(\overline{U}),$$

as in [26, Sections 4.2 and 4.4] both as compact Hausdorff topological spaces. Here $\Delta^{\text{alg}}(-)$ means the (algebraic) dual intersection complex.
Proposition 3.14

(i) $\partial \overline{U}_{\text{lim,MSBJ}} = \lim_{\leftarrow U} \Delta^{\text{alg}}(\overline{U} \setminus U)$ can be naturally identified with the set of $\mathbb{R}$-valued valuations of $k(U)$ with the centers lying outside $U$, modulo multiplication of positive real numbers. In particular, it admits a projectivized affine structure in the sense of our Sect. A.2.

(ii) A natural continuous surjective map $\tau : \partial ZR(U) \to \partial \overline{U}_{\text{lim,MSBJ}}$ exists as a separated quotient.

Proof

(i): This is a close analogue of [23, Theorem 1.13] and [25, Corollary 2.5]. We leave details to the readers.

(ii): Since $\partial ZR(U)$ is a valuative topological space in the sense of [53], we can consider a selfmap $\tau : \partial ZR(U) \to \partial ZR(U)$ by sending to the unique maximal generization, whose image is exactly $\partial \overline{U}_{\text{lim,MSBJ}}$. Hence, what remains is to show the continuity. From the definition of the projective limit topology, induced by the product topology, it is enough to show the following: if we fix a compactification $U \subset \overline{U}$ of the above type, i.e., with simple normal crossing $\overline{U} \setminus U$ and its local coordinate chart with the coordinates $f_1, \ldots, f_m$ such that the local equation of $\overline{U} \setminus U$ is $f_1 \cdots f_m = 0$, then there is a corresponding $m$-simplex inside $\Delta^{\text{alg}}(\overline{U} \setminus U)$. It is enough to show that for the limit set of $\left\{ \log |f_j| / \log |f_i| \in (a, b) \right\}$ inside $\partial \overline{U}_{\text{lim,MSBJ}}$, which we denote as $S_{i,j}(a, b)$, the preimage $\tau^{-1}(S_{i,j}(a, b))$ is open. However, it is easy to see that $\tau^{-1}(S_{i,j}(a, b))$ can be written as

$$
\bigcup_{a < l_1 < m_1 < b} \left( \left\{ v \in \partial ZR(U) \mid \frac{f_{i_1}}{f_{j_1}} \leq 1 \right\} \cap \left\{ v \in \partial ZR(U) \mid \frac{f_{m_1}}{f_{i_2}} \leq 1 \right\} \right),
$$

where $l_i$ and $m_i$ are positive integers. The above subset of $\partial ZR(U)$ is open from the definition of the topology on $\partial ZR(U)$.

Note that in the above, we could only discuss boundaries.

4 Family construction over tropical geometric compactifications

So far, in this paper or previous [119], we have been discussing some canonical moduli compactifications and the degeneration relatively independently. This section can be seen as a first step to connect them in a somewhat more direct manner. In this section, we assume $k = \mathbb{C}$.

The main point is that the procedure of the Morgan–Shalen–Boucksom–Jonsson compactification is “functorial” with respect to morphisms as proven in [114, A.15]. The Morgan–Shalen–Boucksom–Jonsson compactification was introduced only for certain varieties and their compactification varieties (cf. [26, 92], [114, Appendix]), but our Appendix A generalizes it to much more general analytic spaces, for both complex and Berkovich non-Archimedean analytic spaces. Below, first we essentially review [114, A.15].

9 However note that in the proof of [114, A.15], mistakenly, the reasoning for $k \geq l$ is not written in a sufficient manner. The correct reasoning is obtained by considering an injective map from the set of the local
Proposition 4.1 (Functoriality, algebraic case) Let $f : X \to Y$ be a $\mathbb{C}$-morphism between normal complex varieties, $D_X$ be a divisor on $X$, $D_Y$ on $Y$, both satisfying Assumption A.1. Here we further assume [33, Definition 8 (2)], i.e., each intersection of irreducible components of the boundary is irreducible. Then there is a natural map $\Delta^{\text{alg}}(D_X) \to \Delta^{\text{alg}}(D_Y)$.

Proof We take a Zariski open covering of $X$ (resp., $Y$) denoted as $\{U_i\}_i$ (resp., $\{V_i\}_i$) where $U_i$ maps to $V_i$ for fixed $i$ and each $U_i$ or $V_i$ contains only one stratum of biggest codimension in it. We do so, by first taking $V_i$ which contains just one smallest dimension stratum and take $U_i$ as subdivisions of the preimages of $V_i$’s. Then $\Delta^{\text{alg}}(D_X \cap U_i)$ and $\Delta^{\text{alg}}(D_Y \cap V_i)$ are both simplices. Note that for any $i, j$, $\Delta^{\text{alg}}(D_X \cap U_i \cap U_j)$ is a closed subcomplex of both $\Delta^{\text{alg}}(D_X \cap U_i)$ and $\Delta^{\text{alg}}(D_X \cap U_j)$ because the strata inside $\Delta^{\text{alg}}(D_X \cap U_i \cap U_j)$ are closed under specialization. The same for $V_i$’s. As discussed in [114, A.15], $f$ induces a natural simplicial map from $\Delta^{\text{alg}}(D_X \cap U_i)$ to $\Delta^{\text{alg}}(D_Y \cap V_i)$, which glues at the closed subcomplex $\Delta^{\text{alg}}(D_X \cap U_i \cap U_j)$. Therefore, we obtain a continuous stratified map from $\Delta^{\text{alg}}(D_X)$ to $\Delta^{\text{alg}}(D_Y).$ \square

Now, let us observe the following examples of degenerations of affine curves over higher dimensional base:

Example 4.2 An easy example is a small crepant resolution of $V(xy - st) \subset \mathbb{A}^2_{x,y} \times \mathbb{A}^2_{x,t}$ with the natural projection $f$ down to $\mathbb{A}^2_{x,t}$. Recall that there are two such crepant small resolutions, say $X_i$ for $i = 1, 2$, both dominated by the blow-up at $(x, y, s, t) = (0, 0, 0, 0)$ of the quadric cone $V(xy - st) = \emptyset$ and are connected by the well-known Atiyah flop. Here we can assume $X_1$ is the blow-up of $\emptyset$ along $(x, z)$ (or equivalently, that along $(y, w)$) and $X_2$ is the blow-up of $\emptyset$ along $(y, z)$ (or equivalently that along $(x, w)$).

We set $D_{X_i} := V(st) \subset X_i$ and $U_{X_i} := X_i \setminus D_{X_i}$ Then Proposition A.4 applies to yield $(U_{X_i}^{an})_{\text{MSBJ}}(X_i) \to ((\mathbb{A}^2_{x,t} \setminus (st = 0))^{an})_{\text{MSBJ}}(\mathbb{A}^2_{x,t}^{an})$ and the map $\tilde{f}^{\text{MSBJ}}$ between the boundaries (the dual intersection complexes) is $[0, 1]^2$ projecting down to $[0, 1]$. Note its fiber is also a segment, which coincides with the dual intersection complex of fibers.

Example 4.3 If we consider $X := V(xy - s) \subset \mathbb{A}^4_{x,y,s,t}$ with the natural projection $f$ down to $Y = \mathbb{A}^2_{x,t}$ and set $D_Y := (st = 0)$, $D_X$ as its pull-back, then the map $\tilde{f}^{\text{MSBJ}}$ between boundaries can be seen as a triangle mapping to the interval where two vertices map to a same edge point while the other point maps to the other edge point of the interval. The fiber is either an interval or a point.

A natural question that occurs from the above nicely behaved examples is as follows. For simplicity, from here, we come back to ordinary varieties category while leaving discussions for the general analytic setup for the appendix.

Question 4.4 (Fibre compatibility) In the setting of Proposition A.4, when $X, Y$ are both (analytifications of) varieties, we consider any morphism from the spectrum of boundary divisors downstairs, labeled by $\{1, \ldots, l\}$, to the set of local boundary prime divisors upstairs, labeled by $\{1, \ldots, k\}$ by looking at the pullback of generic points of the divisors.
DVR $R$, say $\overline{\varphi_Y}: \text{Spec}(R) \to Y$ which maps the generic point $\iota = \text{Spec}(\text{Frac}(R))$ inside $U_Y$. We write the closed point of $\text{Spec}(R)$ as $p$ which we suppose to map inside $\text{Supp}(D_Y)$. We consider the natural extension of $\varphi|_e$ to
\[\overline{\varphi}^{MSBJ}_Y: \text{Spec}(R) \to U_Y^{MSBJ}.\]

Then under certain appropriate condition, does it hold that
\[\left(\tilde{f}^{MSBJ}\right)^{-1}(\overline{\varphi}^{MSBJ}_Y(p))\]
is canonically homeomorphic to the dual intersection complex of $\tilde{f}^{-1}(\overline{\varphi}_Y(p))$?

Note that Examples 4.2 and 4.3 give affirmative answers. On the other hand, without assumption that $f$ is flat and proper, it is easy to find various counterexamples (which we omit). Also, if $U_X \to U_Y$ is not smooth, then there is the following counterexample.

**Example 4.5** If $X$ is a blow-up of $\mathbb{P}^1_s \times A^2_{s,t}$ along $(t = x = 0) \simeq \mathbb{A}^1$, $f: X \to Y := A^2_{s,t}$ is a natural projection, $D_Y^{(1)} := (s = 0)$, $D_X^{(1)} := f^*D_Y^{(1)}$, $t$ Then obviously we have trivial dual intersection complexes of $D_X^{(1)}$ and $D_Y^{(1)}$, hence at $p = (0, 0)$ the above speculation in Question 4.4 is violated. On the other hand, if we put $D_Y^{(2)}$ as $D_Y$ of Example 4.3, then the situation is identical to Example 4.3 hence the speculation holds.

Therefore, we are naturally led to temporarily assume that $f|_{U_X}$ is smooth (or at least with normal fibers) and ask the same Question 4.4 again. Nevertheless, the assumption is still not enough!

**Example 4.6** (Type II and III K3 degeneration as a counterexample) Here is a much subtler counterexample to the speculation of Question 4.4 for degenerations of K3 surfaces including both Types II and III.

We consider the closed subscheme $\mathcal{Y}'$ of $\mathbb{P}^3_{X,Y,Z,W} \times A^2_{s,t}$ defined by the vanishing of
\[(XZ + sQ_1(X, Y, Z, W) + tQ_2(X, Y, Z, W)) \times (ZW + sQ_3(X, Y, Z, W) + tQ_4(X, Y, Z, W)) + stL_1(X, Y, Z, W)L_2(X, Y, Z, W),\]

where $Q_i$ (resp., $L_i$) are general quadric (resp., linear) homogenous polynomials of variables $X, Y, Z, W$. Via projection, we have a natural morphism $\pi': \mathcal{Y}' \to A^2_{s,t}$. Denote the open subset of $\mathcal{Y}'$ defined by $L_i \neq 0$ as $\mathcal{U}'$ and write the de-homogenization of $Q_i|_{\mathcal{U}'}$ as a polynomial $q_i$ of $\frac{X}{L_i}, \ldots, \frac{W}{L_i}$. Then they naturally give a morphism from $\mathcal{U}'$ to $\mathcal{Y}$ of Example 4.2 defined as $(q_1, q_2, s, t)$. Then we take the fiber product with respect to the small resolution of conifold singularity $X_i \to \mathcal{Y}$ as in Example 4.2 and write $\mathcal{U}_i := \mathcal{U}' \times_{\mathcal{Y}} X_i$ for $i = 1, 2$. Then $\pi'$ gives a morphism $\pi_i: \mathcal{U}_i \to A^2_{s,t}$. 

\[Springer\]
We claim that $\pi_2 : U_2 \to \mathbb{A}^2_{t,s}$ with the boundaries $V(st) \subset \mathbb{A}^2_{t,s}$ and its pull-back $D_{U_2}$ do not satisfy the speculation in Question 4.4. Indeed, it is not hard to see that the dual intersection complex of $D_{U_2}$ is a square homeomorphic to $[0, 1]^2$, given that $U_2 \to Y$ is small so that the exceptional set is of codimension at least 2. The continuous map from it to that of $V(st) \subset \mathbb{A}^2_{t,s}$ is a square mapping down to an interval $[0, 1]$ via the projection. On the other hand, over the closed point $s = t = 0$ in $\mathbb{A}^2_{t,s}$, $Y'$ has, hence so do $U_i$'s, the typical Type III degeneration — more explicitly a Zariski open dense subset (which contains all strata) of the $V(XYZW) \subset \mathbb{P}^3_{X,Y,Z,W}$. Therefore, this gives a counterexample to the speculation of Question 4.4.

Let us remark that the author lately found out that [1, Section 8] and [28] had essentially solved the problem for the curve case (their $n = 0$ case).

**Theorem 4.7** ([28, Theorem 2], cf. also [1, 8.2.1]) *For the case of moduli of smooth projective curves of fixed genus* $g \geq 2$, Question 4.4 *has an affirmative answer.*

**Proof** It is well known (probably since [77]) that $M_{g,1} \to M_{g,0} = M_g$ defined by forgetting the marked point, at the Deligne–Mumford algebraic stack level, is nothing but the universal curve. Thus, the assertion readily follows from [28, Theorem 2] restricted to charts of cone complex covering it (recall that [28] works over geometric stack of cone complexes), $M_{g,n}^{\text{trop}}$ in [28] is the incidence complex of the (stack theoretically simple normal crossing) Deligne–Mumford algebraic stack $\overline{M}_{g,n} \setminus M_{g,n}$ by [1].

The essential point is that each “degenerating part” (forming nodes) is independent at the discriminant locus $\overline{M}_{g,n} \setminus M_{g,n}$ (which is the reason why it is snc at smooth charts).

Furthermore, we observe that the abelian varieties analogue also holds.

**Proposition 4.8** *For the Namikawa–Alexeev–Nakamura compactification of moduli stack of principally polarized abelian varieties* $\overline{A}^\text{AN}_g$ [7, 11, 99, 102] *with its universal family of polarized degenerated abelian varieties* [7, 11, 99] [10], Question 4.4 *has an affirmative answer.*

**Proof** This is essentially known (cf. [102, Section 9 B]). Indeed, loc.cit. Definition 9.12 introduces a cone decomposition, called the “mixed decomposition”, of

$$\{(y, z) \mid z \in V \otimes \mathbb{R}, y : \text{inner product on } V \otimes \mathbb{R}\},$$

where $V \simeq \mathbb{Z}^g$, and shows that the fibers over $y$ are the union of the corresponding Delaunay cells. Dividing by the discrete group of affine transformations $GL(V) \ltimes V$, we conclude the desired assertion. □

Motivated by this, we conjecture that Question 4.4 has affirmative answers for more general weak K-moduli spaces.

---

10 Its name originates from the literature mentioned above.
Conjecture 4.9  For a universal family over the various (other) weak $K$-moduli stacks (Conjecture 1.8) such as Shah’s compactification of moduli stack of polarized K3 surfaces of degree $2$ $\tilde{\mathcal{T}}_{2}^{\text{Shah}}$ [134], and Alexeev–Engel–Thompson’s compactification of the same moduli stack [10], Question 4.4 has an affirmative answer.

We further conjecture that the obtained family of tropical varieties coincides with those parametrized in our previous works [28, 113, 114, 119]. We observe another affirmative direction to the above Question 4.4.

Proposition 4.10  In the setup of Proposition A.4 and Question 4.4, if $U_X \subset X$ and $U_Y \subset Y$ are a toroidal pair of varieties, and the morphism $X \to Y$ is also toroidal with respect to the toroidal structures of $(X, X \setminus U_X)$ and $(Y, Y \setminus U_Y)$, Question 4.4 has an affirmative answer.

Hence, in particular, if the examples in Conjecture 4.9 are toroidal families over the moduli stack, the desired assertions would follow.

Proof One can localize the problem to when $(X, X \setminus U_X)$ and $(Y, Y \setminus U_Y)$ are both toric with torus equivariant morphism $X \to Y$ with respect to a morphism of the algebraic tori. Then, the fibers over closed points of $Y \setminus U_Y$ are stable toric varieties in the sense of [7]. This follows from [69, 2.1.11] combined with [6, Section 3]. Indeed, the irreducible components of the fibers are toric varieties for the relative stars [69, 2.1.9] whose natural union is nothing but the fiber of the map between the fans. Hence, Question 4.4 has an affirmative answer for this situation.

Hence the recent deep results on birational geometry of “toroidalization” [2, 3] may well be useful as in the proof of Theorem 1.9 or [83].

Appendix A: Analytic Morgan–Shalen construction and dual intersection complexes

This appendix reviews and extends both the dual intersection complex and the Morgan–Shalen type compactification ([26, 92], [114, Appendix]) especially to analytic (including non-Archimedean) general setting. This is logically used only in a few places of the main contents such as Sect. 2.4, Proposition A.4, among others, but we put it here partially as a preparation for future use, and as a review of the original theory.

The original Morgan–Shalen compactification [92] compactifies a complex variety by attaching a certain subset of cell complex, which [92] applied to the character varieties and study of topology of manifolds. This idea is now being expanded in more modern contexts after inspiration coming from the attempt to understand the mirror symmetry geometrically (cf., e.g. [26, 58, 84]). Here, we review and prepare a further extended version of the Morgan–Shalen type compactification, especially to include the non-Archimedean setting.

11 However, for the $M_g$ case, remember that the Morgan–Shalen type topology does not coincide with the Gromov–Hausdorff topology given on $\overline{M_g}^T$, cf. [113].
We first review the construction in the complex algebraic setting, with some technical improvements, and later we extend to the general Berkovich analytic setting.

A.1 Review of (algebraic) dual intersection complex

Starting from the classical theory of dual graphs of curves in surfaces, followed by the work of Kulikov, Pinkham–Persson [85, 124] on the degeneration of surfaces, there are a lot of works on dual intersection complexes (combinatorial data) for the normal crossing varieties (cf. [26, 33, 58, 82, 84, 85, 124, 139, 144] and [114, Appendix], etc.). Here we consider a more general setting.

We keep the assumption that \( k \) is algebraically closed.

**Assumption A.1** (Algebraic \( \mathbb{Q} \)-Cartier-ness + codimension assumption) \( X \) is a normal \( n \)-dimensional variety and \( D := X \setminus U \) is a finite union of irreducible \( \mathbb{Q} \)-Cartier divisors, i.e., all the components are (algebraically) \( \mathbb{Q} \)-Cartier, and the intersection of those \( k \) irreducible components is of pure codimension \( k \) for any \( 1 \leq k \leq n \).

So far, people restricted their attention to the normal crossing case or dlt (= divisorially log terminal) case. Nevertheless, we can still define the (algebraic) dual intersection complex of \( D \) denoted by \( \Delta_{\text{alg}}(D) \) as [33, Section 2] does. Op.cit. (Definition 8) assumes that the intersection of the irreducible components is irreducible but such requirement is not actually necessary for the definition once we associate an \( l \)-simplex to any irreducible component of \( l \) irreducible components of \( D \) and do the same inductive construction for \( \Delta_{\text{alg}}(D) \). On the other hand, the following easy lemma gives an efficient way to relate to the situation under the control of log discrepancies. (This reminds the author of a more difficult variant in a rather converse direction in [51].)

**Lemma A.2** (Log discrepancy control) Under Assumption A.1, take an irreducible component \( Z \) of the intersection of some irreducible components of \( D \), with maximum possible \( \text{codim}(Z \subset X) \) which we denote as \( l \). Write the generic point of \( Z \) as \( \eta_Z \) and the prime divisors containing \( Z \) as \( D_1, \ldots, D_l \). Then we can take a Zariski open neighborhood \( X' \) of \( \eta_Z \) and equidimensional morphism \( f : X' \to \mathbb{A}^l \) such that \( \text{Supp}(f^*(\text{div}(z_i))) = D_i \cap X' \) for all \( i \).

In particular, if \( Z \) is 0-dimensional (i.e., a closed point) then the germ \( Z \in X' \) is log canonical. In general, if we restrict \( X \) to its Zariski open subset \( X' \) without missing any strata, we obtain that \( (X', D_X|_{X'}) \) is log canonical.

**Proof** Take a set of regular functions \( f_1, \ldots, f_l \) on \( X' \) such that \( f_i \) defines the multiple of \( \mathbb{Q} \)-Cartier divisor \( D_i \) around \( \eta_Z \). Then consider the morphism \( f := (f_1, \ldots, f_l) : X' \to \mathbb{A}^l \) and take a Zariski open subset \( Y' \) of \( \mathbb{A}^l \) where \( f \) is equidimensional and replace \( X' \) by \( f^{-1}(Y') \). Then we get the first assertion.

For the second assertion, it follows from the fact that \( X' \) is log crepant to \( Y' \) with appropriate effective boundary which encodes the ramification of \( f \) in codimension 1. Actually, from the proof, with an appropriate explicit effective divisor \( D_X' \) (with standard coefficients) supported on \( \bigcup D_i \), it follows that \( (X', D_X') \) is log canonical.\( \square \)
Let us see some simple examples.

**Example A.3** (i) Let us set $X = (z^d = xy) \subset \mathbb{A}^3$, with boundaries $D_1 = (x = 0)$, $D_2 = (y = 0)$, which maps via the natural projection to $\mathbb{A}^2$ with coordinates $x$ and $y$. Then the induced morphism between the boundary of the corresponding Morgan–Shalen–Boucksom–Jonsson compactification is a homeomorphism between the segment to the segment.

(ii) Let us set $X = \mathbb{A}^2$, with boundaries $D_1 = (y - xm = 0)$, $D_2 = (y = 0)$, mapping to $\mathbb{A}^2$ with the coordinates $s$, $t$ via $s = y$, $t = y - xm$. Then although the morphism is not an isomorphism, we similarly conclude that the induced morphism between the boundary of the corresponding Morgan–Shalen–Boucksom–Jonsson compactification is a homeomorphism between the segment to the segment.

Now, we re-discuss and generalize the functoriality a little more carefully following [114, Appendix] and Sect. 4.

**Proposition A.4** (Functoriality) Let $k = \mathbb{C}$ and $f : X \to Y$ be a $k$-morphism between $k$-analytic spaces, $D_X$ be a divisor on $X$, $D_Y$ on $Y$, both satisfying the Cartier Assumption A.16. Consider the MSBJ compactification of $U_X := X \setminus \text{Supp}(D_X)$ and $U_Y := Y \setminus \text{Supp}(D_Y)$ accordingly. Then $f|_{U_X}$ continuously extends to a continuous map $\overline{U_X}^{\text{MSBJ}}(X) \to \overline{U_Y}^{\text{MSBJ}}(Y)$, which we denote by $\overline{f}^{\text{MSBJ}}$.

**Proof** The proof is very similar to that of Proposition 4.1 (and [114, A.15]). We take a net of affinoids’ domains of $X$ (resp., $Y$) denoted as $\{U_i\}_i$ (resp., $\{V_i\}_i$) where $U_i$ maps to $V_i$ for a fixed $i$ and each $U_i$ or $V_i$ contains only one strata of biggest codimension in it. Then $\Delta(D_X \cap U_i)$ and $\Delta(D_Y \cap V_i)$ are both closed subset of simplices, from the definition. Note that for any $i, j$, $\Delta(D_X \cap U_i \cap U_j)$ is a closed subcomplex of both $\Delta(D_X \cap U_i)$ and $\Delta_{\text{alg}}(D_X \cap U_i \cap U_j)$ because the strata inside $\Delta_{\text{alg}}(D_X \cap U_i \cap U_j)$ is closed under specialization. The same for $V_i$’s. As discussed in [114, A.15], $f$ induces a natural map from $\Delta(D_X \cap U_i)$ to $\Delta(D_Y \cap V_i)$, which glues continuously to the original continuous map $X \setminus D_X$ to $Y \setminus D_Y$. They glue at the closed subset $\Delta(D_X \cap U_i \cap U_j)$. Therefore, we obtain a continuous map from $\Delta(D_X)$ to $\Delta(D_Y)$ and the whole $\overline{U_X}^{\text{MSBJ}}(X)$ to $\overline{U_Y}^{\text{MSBJ}}(Y)$ as well. □

**Remark A.5** At first sight, Proposition A.4 could look like giving a broader extension of [114, A.15] but note that, by the arguments of our Lemma A.2, at least semi-log-canonicities of dense open subsets of $(X, D_X)$ or $(Y, D_Y)$ are implicitly assumed.

**Remark A.6** For $U \subset X$, which are the analytifications of complex varieties, if we replace $X$ by $X'$ which dominates the original $X$ (while preserving $U$), then we have a continuous map $\overline{U}^{\text{MSBJ}}(X') \to \overline{U}^{\text{MSBJ}}(X)$ preserving $U$. The projective limit of these Morgan–Shalen type compactifications $\overline{U}^{\text{MSBJ}}(X)$, where $X$ runs over all $X$, satisfying Assumption A.16 should coincide with the compactification of [125].
least when $U$ is affine, there is a construction by Favre (unpublished) which the author fortunately had a chance to study\textsuperscript{12}, and the coincidence in this case can be proven similarly to \cite[4.12]{26}. In \cite{125} the authors even put locally ringed space structure on the compactification, discussed coherent sheaves for it and proved a GAGA type theorem.

**Remark A.7** A related result in a special situation is \cite[5.2.1, 6.1.6]{1}.

**Remark A.8** Also there is an analogous observation in the “classical” (embedded) tropical geometry setting, cf., e.g, \cite[Section 6]{129}.

### A.2 Projectivizing affine structure

In usual tropical algebraic geometry, affine structure plays a central role either explicitly or implicitly. Here, we introduce a projectivized version, i.e., an analogue of affine structure obtained by dividing by the action of $\mathbb{R}_{>0}$. It is essentially not so new and has been implicit in the literature in the sense that various examples have appeared, but we make a systematic introduction. For instance, this can be seen as a tropicalized version of normalized Berkovich space in the sense of Fantini \cite{40}.

**Definition A.9** Suppose $k$ is a non-Archimedean field and let $0 \in V \simeq \mathcal{M}(\mathcal{A})$ be a germ of $k$-affinoid with the isomorphism as Banach $k$-algebra $\mathcal{A} \simeq k\{T_1, \ldots, T_N\}/I_V$ with $r_i \in \mathbb{R}_{>0}$ and ideal $I_V$. Equivalently, $V$ can be regarded as a subspace in $\mathbb{A}_k^{N, \text{an}}$. We would call this germ $0 \in V$ with the additional data, framed or embedded germ but we may simply write $0 \in V$ if the rest is obvious from the context. We set the coordinates of $\mathbb{A}_k^N$ as $z_i$'s.

Now we consider a natural subset of the real projective space $\mathbb{P}_{\mathbb{R}_{>0}}^{N-1}$ as $((\mathbb{R}_{>0}^N \setminus \{\vec{0}\})/\mathbb{R}_{>0}$ and call it the projective simplex of dimension $N - 1$. Note that the inclusion

$$
\Delta_1 := \left\{ (x_1, \ldots, x_N) \in \mathbb{R}_{>0}^N \mid \sum_i x_i = 1 \right\} \hookrightarrow (\mathbb{R}_{>0}^N \setminus \vec{0}),
$$

composed with the projection to $\mathbb{P}_{\mathbb{R}_{>0}}^{N-1}$ is a homeomorphism and denote it as

$$
\varphi(1, \ldots, 1): \Delta_1 \xrightarrow{\simeq} \mathbb{P}_{\mathbb{R}_{>0}}^{N-1}.
$$

However, we do not really respect this “artificial” map nor the affine structure on the projective simplex induced by this, in general.

Then we define the projective tropicalization map of framed germ $0 \in V$ as

\textsuperscript{12} The author appreciates S. Boucksom, C. Favre and M. Jonsson for this communication.
Degenerated Calabi–Yau varieties with infinite... 1145

\[ \text{PTrop}_V : V \cap \{|z_i| < 1 \forall i\} \rightarrow \mathbb{P}_{\mathbb{R}_{\geq 0}}^{N-1} \]

\[ x \mapsto [\cdots : -\log |z_i|_x : \cdots]. \]

**Definition A.10** Then we set the projective tropicalization set \( \text{PTrop}(0 \in V) \) as the limit set

\[ \{ \lim_i \text{PTrop}_V(x_i) \in \mathbb{P}_{\mathbb{R}_{\geq 0}}^{N-1} | x_i \in V (i = 1, 2, \ldots), \lim_i x_i = \vec{0} \}. \]

Then we have the following characterizations, which can be seen as a projective analogue of the “fundamental theorem” by Kapranov in tropical geometry with embedded formalism (cf., e.g., [91, 3.2.5]).

**Theorem A.11** (Projective tropicalization) The following subsets of \( \mathbb{P}_{\mathbb{R}_{\geq 0}}^{N-1} \) coincide:

(i) \( \text{PTrop}(0 \in V) \), defined above,

(ii) the quotient of \( \bigcap_{0 \neq f \in I_V} V(\text{trop}(f)) \) by the action of \( \mathbb{R}_{>0} \), where \( \text{trop}(-) \) means the usual (non-Archimedean) tropicalization of the \( k \)-polynomial, \( \text{trop}(-)^{(1)} \) is the degree one term of \( \text{trop}(f) \) (i.e., constants discarded), and \( V(-) \) denotes the corresponding tropical hypersurface in \( \mathbb{R}^N \).

(iii) the set of positive directions in \( \text{Trop}(V) \), i.e.,

\[ \{ \vec{v} (\neq \vec{0}) \in \mathbb{R}_{\geq 0} | \mathbb{R}_{\geq 0} \vec{v} + \vec{w} \subset \text{Trop}(V) \text{ for some } \vec{w} \}. \]

In particular, \( \text{PTrop}(0 \in V) \) is a projectivization of a piecewise linear set.

**Proof** The equivalence of (ii) and (iii) is a standard exercise. (ii) \( \supset \) (iii) is immediate from the definitions and (ii) \( \subset \) (iii) is more non-trivial but it still holds since for each unbounded polyhedron \( P \) in (iii), having \( \mathbb{R}_{\geq 0} v_i \)'s as the edge half-lines, any (tropical) term of \( \text{trop}(f) \) whose degree 1 homogeneous part is not minimized at \( P \), is bigger than \( \text{trop}(f) \) at the region \( P + l \sum_i v_i \) for \( l \gg 0 \). The equivalence of (i) and (ii) (or (iii)) follows from the “Fundamental theorem” [91, Theorem 3.2.5] (originally due to Kapranov). \( \square \)

Roughly speaking, the above says that \( \text{PTrop} \) is the “tangent cone of \( \text{Trop} \) at infinity in positive (first quadrant) direction” as an analogue of the tangent cone at infinity in Riemannian geometry. From Theorem A.11 it easily follows that:

**Proposition A.12** If \( I_V = (f) \), denote the normal fan of the Newton polytope of \( f \) as \( \text{Newt}(f) \) and the set of its rays inside the boundary of the cones as \( \partial \text{Newt}(f) \). Then \( \text{PTrop}(0 \in V) \) is

\[ (\partial \text{Newt}(f) \cap \mathbb{R}_{\geq 0}^N) / \mathbb{R}_{>0}. \]
Example A.13 If $N = 2$, and $I_V = (f)$ with degree $d$ polynomial $f$ of $z_i$’s, then $\text{PTrop}(0 \in V)$ is a finite set with order at most $g(d)$ where $g(1) = g(2) = 1$, $g(3) = g(4) = 2$, $g(5) = g(6) = g(7) = 3$ for instance.

Example A.14 For any homogeneous $k$-polynomial $f$ of degree $d$, it is easy to see that $\varphi_1^{-1}(\text{PTrop}(0 \in V = V(f))) \subset \Delta_1$ is the set of rays of a fan with the center $(1, \ldots, 1)$ in any way.

A.3 General analytic (Berkovich) setting

Now we move on to analytic extension. Let us consider the following examples.

Example A.15 Let $X$ be $\mathbb{P}^2$ and $D$ be the nodal rational cubic curve. Then $U := X \setminus D$ is an easy typical example of the so-called log Calabi–Yau surface. In this case, the algebraic dual intersection complex whose definition was briefly recalled in the previous subsection, is just a single closed point. However, it is sometimes more natural to consider a loop, i.e., topologically $S^1$ with one vertex and one edge, as the natural candidate for a “more correct” dual intersection complex.

To remedy the above problem, related to monodromy, we introduce the following analytic extension. We still keep the assumption that $k$ is a non-Archimedean field.

Recall that the sheaf of Cartier divisor over any locally ringed space is defined (similarly to the case of schemes), simply as the sheaf of meromorphic functions divided by that of invertible holomorphic functions. In particular, the case of complex manifolds, the theory of divisors is of course well-established and frequently used. Here, in the setting of Berkovich analytic spaces, we use the same theory of Cartier divisors with respect to the (weak) topology, not G-topology.

How about Weil divisors? For general complex spaces, definitions are made in a similar manner (cf., e.g., [34, 6.7]). In the non-Archimedean literature, the case when $X$ is smooth and one-dimensional (often just an analytification of a smooth curve) is treated (cf. [17] and later) and we can similarly think of formal linear combination of codimension 1 closed irreducible analytic spaces following the notion of (irreducible) closed analytic subspace in [16, Section 3.1]. In this paper, we say a closed analytic subspace $Z$ of $X$ is a Cartier divisor, if the corresponding coherent ideal sheaf $I_Z$ is invertible, i.e., for any $x \in Z$, there is a small enough open neighborhood of $x$, say, $U \subset X$, such that $I_Z|_U$ is generated by a regular element of $\Gamma(U, \mathcal{O}_X)$.

Assumption A.16 (Analytic $\mathbb{Q}$-Cartier-ness + codimension assumption) $X$ is a normal (irreducible) $n$-dimensional $k$-strict analytic space and $D := X \setminus U$ is a finite union of (supports of) irreducible Cartier divisors $D_i$’s, i.e., there is a covering by strict affinoid domains $\{V \cong \mathcal{M}(A_V)\}_V$ and $z_{1,V}, \ldots, z_{k_V,V} \in A_V$ such that

$$D_i \cap V = \{y \in V \mid |z_{i,V}(y)| = 0\}.$$

Furthermore, we assume that the intersection of any $k$ among $D_i$’s has pure codimension $k$ for all $1 \leq k \leq n$. © Springer
In this setting, we will define a (generalized) Morgan–Shalen–Boucksom–Jonsson partial compactification of \( U \) as a topological space

\[
(U \subset) \overline{U}^{\text{MSBJ}}(X),
\]
as well as its boundary — the dual intersection complex \( \Delta^{\text{an}}(D) = \Delta(D) \). Our construction generalizes the previous ones: [92] (affine case), [26] (smooth with normal crossing boundary case), [114, Appendix] (divisorially log terminal case). As in [114, Appendix], we can also easily generalize it to Deligne–Mumford stacks whose étale charts (covering) and the pull-back of those boundaries satisfy the above condition A.16. Leaving such verbatim stacky extension to the readers, we give the definition when \( X \) is an analytic space (under Assumption A.16) similarly to [26, 92] and [114, Appendix]. The construction is as follows:

**Construction A.17**  
**Step 1 (Local ambient space of boundary)** For a strict \( k \)-affinoid domain \( V \cong M(A_V) \) of \( X \) such that \( \emptyset \neq (V \setminus U) = \bigcup_{1 \leq i \leq k_V} (z_i, V = 0) \) where \( z_i, V \) are elements of \( A_V \) and each \( z_i, V = 0 \) gives a distinct irreducible divisor of \( V \) with multiplicity one, this is possible because of the above Assumption A.16. Then we consider

\[
\widetilde{\Delta}_V(U \subset X) := \mathbb{P}^{k_V}_{\mathbb{R}_{>0}} = (\mathbb{R}^{k_V}_{>0} \setminus \{0\})/\mathbb{R}_{>0},
\]
(or simply \( \widetilde{\Delta}_V \) when \( U, X \) are obvious from the context) where each \( i \)-th real (projectivized) coordinate will be connected to the local function \( z_i, V \). An important remark is that \( \widetilde{\Delta}_V \) does not have natural affine structures, but can only be regarded as a subset of real projective space, i.e., the projective structure in the sense of Sect. A.2.

**Step 2 (Tropicalization and local boundary)** Under the above setting, we define the tropicalization map as usual as

\[
\operatorname{Trop}_V: (V \cap U) \cap \{ |z_i, V| < 1 \text{ for all } i \} \rightarrow \widetilde{\Delta}_V(U \subset X)
\]

\[
\psi \ x \quad \mapsto \quad [\cdots : -\log |z_i, V|_x : \cdots],
\]

and \( \Delta_V(U \subset X) \subset \widetilde{\Delta}_V(U \subset X) \) as

\[
\left\{ \lim \operatorname{Trop}_V(x_i) \in \widetilde{\Delta}_V(U \subset X) \mid x_i \in (V \cap U) \ (i = 1, 2, \ldots), \lim_i x_i \in V \cap D \right\}.
\]

From diagonal arguments, \( \Delta_V(U \subset X) \subset \widetilde{\Delta}_V(U \subset X) \) is a closed subset since \( V \) is compact. It is easy to see that \( \Delta_V(U \subset X) \) is the projective tropicalization set \( P\operatorname{Trop}(V \cap U) \) we introduced in Sect. A.2. Note that this construction does not depend on \( z_i, V \)'s since its replacement only changes \( z_i, V \) by unit, hence the change of \( -\log |z_i, V|_x \) is bounded above by a constant depending on \( V \).

**Step 3 (Global ambient space of boundary)** For affinoid subdomains \( V_i \subset X \ (i = 1, 2) \) both satisfying above requirements and \( V_1 \cap V_2 \neq \emptyset \), we can naturally consider the
projectivization of a natural real linear map:

$$\tilde{\Delta}_{V_1 \cap V_2}(U \subset X) \to \tilde{\Delta}_V(U \subset X)$$

which maps a vertex $\Delta_{V_1 \cap V_2}(U \subset X)$ corresponding to a prime analytic divisor $D_j \cap V_1 \cap V_2$ to the vertex of $\Delta_V(U \subset X)$ corresponding to the same prime divisor $D_j \cap V_1$. We call this map a gluing map and denote it as $\varphi = \varphi_{V_1 \cap V_2, V}$. It is easy to see $\varphi$ maps $\Delta_{V_1 \cap V_2}(U \subset X)$ into $\Delta_V(U \subset X)$ from the definition. Then we consider the equivalence relation $\sim$ on $\bigcup_{V \subset X: \text{affinoid domains}} \Delta_V(U \subset X)$ generated by the identification of the source point and the target point of gluing maps $\varphi$’s. Then we set

$$\Delta(U \subset X) := \left( \bigcup_{V \subset X: \text{affinoid domains}} \Delta_V(U \subset X) \right) / \sim.$$ 

**Step 4 (Topology at local level)** For each $V \subset X$ satisfying above requirement, we put topology on

$$(V \cap U)_{\text{MSBJ}}(V) := (V \cap U) \cup (\Delta_V(U \subset X)),$$

in the same way as in [92, p. 415] or [26, Section 2.2, Definition 2.3] using $\text{Trop}_V$. It is easy to see that the construction does not depend on the choices of local defining equations $z_i, V$’s.

**Step 5 (Topology at global level)** We define the Morgan–Shalen–Boucksom–Jonsson partial compactification of $X$, which we write as $\overline{U}_{\text{MSBJ}}(X)$, as the colimit topological space of $(V \cap U)_{\text{MSBJ}}(V) := (V \cap U) \cup (\Delta_V(U \subset X))$ for all affinoid subdomains $V \subset X$’s. From the construction, it has a tautological open subset which can be identified with $U$. Then, the analytic dual intersection complex is

$$\Delta(D) := \partial \overline{U}_{\text{MSBJ}}(X) := \overline{U}_{\text{MSBJ}}(X) \setminus U.$$

As in [114, Appendix], the above construction also naturally extends to Deligne–Mumford stack pair $(\mathcal{X}, \mathcal{D})$ which satisfies the same Assumption A.16. We omit the details as it is verbatim.

A difference with Archimedean situation is that the topological dimension of $\Delta(D)$ is at most $\dim(U) - 1$ (which indeed is attained if $D$ is “maximally degenerate”, e.g. normal crossing divisor with 0-dimensional strata).

**Proposition A.18 ($\Delta^\text{alg}$ vs. $\Delta$)** For a given $U \subset X$, we have a natural continuous surjection $\Delta(X \setminus U) \to \Delta^\text{alg}(X \setminus U)$.

**Proof** This easily follows from the definitions, since given any Zariski open covering of $X$, we can refine the covering of $X^\text{an}$ by analytifying all open subsets, to (fine enough, if necessary) strict affinoid subdomains. We leave the details to readers (or myself in future). \qed

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For instance, in Example A.15, a loop shrinks to a point by the above map in Proposition A.18. In general, if $X$ is compact, e.g., analytification of proper scheme [16, 3.4.8 (cf. also 3.5.3)], then $\overline{U}^{MSBJ}(X)$ becomes compact by the above construction.

**Remark A.19 (Non-Archimedean symmetric space)** Recall that [118, Section 2] proved that in the setting over complex numbers, Satake compactification of adjoint type for locally Hermitian symmetric space coincides with Morgan–Shalen type compactification. It is natural to seek for its non-Archimedean analogue. However, the right formulation seems to be more non-trivial to the author as we observe in what follows.

Recall that Berkovich made a compactification of Bruhat–Tits building $\mathcal{B}(G, k)$ for a semisimple algebraic group $G$ over non-Archimedean field, i.e., closure inside the analytification of flag variety along [16, Theorem 5.5.1], and its extension by [127, 128] which they call an analogue of Satake compactification. Restricting to an apartment of $\mathcal{B}(G, k)$, we observe that its closure inside the compactification of [16, 127, 128] is of Kajiwara–Payne type compactification hence not compatible with any Morgan–Shalen–Boucksom–Jonsson compactification of $G^{an}$.

**Appendix B: Towards Satake–Baily–Borel type compactification**

Here we briefly discuss the possible analogue of the Satake–Baily–Borel type compactification [14, 130, 131] for moduli of Calabi–Yau varieties which do not necessarily have a structure of locally Hermitian symmetric domain. The discussions in this section do not contain substantial results but rather only propose conjectures and some observations, which is the reason of presentation as short notes in the appendix.

Recall that the algebro-geometric meanings of the Satake(–Baily–Borel) compactification [130] of the moduli $A_g$ of principally polarized abelian varieties are well understood (cf., e.g., [27]) which can be more geometrized by the Grothendieck semi-abelian reduction theorem (cf., e.g., [29, 4.4.1]). Similar results are also known for K3 surfaces in somewhat weaker sense, cf. e.g., [44], i.e., the boundary parametrizes the graded sum of the weight filtration of the limit mixed Hodge structures and the only non-trivial boundaries, the modular curves, parametrize elliptic curves which are the minimal log canonical centers of the Kulikov type II degenerations.

We also discuss possible differential geometric meaning, for $k = \mathbb{C}$ case, via Ricci-flat Kähler metrics, as parametrizing the limits while volumes fixed, in Question B.8 (in the short appendix) which we hope to explore more in future.

Recently, there was also made an attempt of generalizing the Satake–Baily–Borel compactification [55] for the image of periods map from the moduli. Hoping to complement the geometric perspective somewhat to their works, we would like to discuss another generalization in the following manner. The connection is unclear yet, mainly due to the lack of general Torelli-type theorem for general K-trivial varieties which we hope to clarify in future.

**Conjecture B.1** For any moduli $M$ of polarized log-terminal Calabi–Yau varieties, under Assumptions 3.1, 3.3, we also have another compactification $M \subset \overline{M}^{SBB}$ as follows:
(i) (Baily–Borel compactification) $\overline{M}_{\text{SBB}}$ is the log canonical model $M_{\text{lc}}$ of $M$. More precisely, $M \subset \overline{M}_{\text{SBB}}$ is isomorphic to the quotient by $\Gamma$ of the log canonical model $M_{\text{lc}}'$ of $(\overline{M}', D')$ in the setting of Sect. 3.1. Note that it does not depend on $(\overline{M}', D')$.

In particular, if $M$ is uniformized by a Hermitian symmetric domain (e.g., when $X$ is abelian varieties or holomorphic symplectic manifold), then $\overline{M}_{\text{SBB}}$ is nothing but the Satake–Baily–Borel compactification of $M$ by [95].

(ii) For any weak $K$-moduli compactification $\overline{M}$, there is a natural morphism $\overline{M} \to \overline{M}_{\text{SBB}}$.

(iii) (Hodge-CM line bundle) The CM line bundle on $\overline{M}$ (which is automatically a positive rational multiple of the Hodge line bundle) is a pullback of an ample $\mathbb{Q}$-line bundle on $\overline{M}_{\text{SBB}}$, by the morphism (ii) above.

(iv) In the case of moduli of strict Calabi–Yau varieties i.e., general fibers $X$ satisfy $H^i(\mathcal{O}_X) = 0$ for all $0 < i < \dim(X)$, $\overline{M}_{\text{SBB}}$ has a natural finite morphism to the conjectural compactification of [55, Conjecture 1.2], as an extension of the period map from $M$, and that the extended Hodge line bundle pulls back to the CM line bundle modulo taking positive tensor powers.

Remark B.2 In the case of polarized abelian varieties, K3 surfaces and (compact) hyper-Kähler manifolds and their irreducible symplectic degenerations, the above conjectures are known to experts as fully confirmed (cf., e.g., [39, 44, 48, 133], [118, Section 8], [55]).

Remark B.3 Note that if [55, Conjecture 1.2] is true, [55, Theorem 1.3.10] implies that the above (iii) for the strict Calabi–Yau case would follow from (iv).

To go into the depth of the above Conjecture B.1, we assign the concept of minimal log canonical center, the role of a key player. They are “canonical” at least in some weak sense, for degenerating Calabi–Yau family, as we partially show. More precisely, at the moment we have the following, which refines the $\mathbb{P}^1$-linking theorem of Kollár [78] to a generalized setup without specifying models, unlike loc.cit., by using previous lemmas.

Proposition B.4 (Birational uniquenss of minimal log canonical centers) Fix a polarized klt Calabi–Yau family $(\mathcal{X}^*, L^*) \to \Delta^* = \Delta \setminus \{0\}$ (as in Sect. 2) and its base changes for the $N$-th ramifying morphism $\Delta \to \Delta$ as $(\mathcal{X}^{[N]*}, L^{[N]*})$. Minimal log canonical centers of the central fibers of dlt minimal models $(\mathcal{X}^{[N]}, L^{[N]}) \to \Delta$ are all birational (we allow to chang $N$ and the models).

Proof We divide the arguments into three steps.

Step 1 (Q-factorization) We consider dlt minimal model $\mathcal{X}'$ and its crepant $\mathbb{Q}$-factorialization $\mathcal{X}$. Since $\mathcal{X}$ is terminal by Lemma 2.1 (ii), it is Cohen–Macaulay and in any case $\mathcal{X}_0$ satisfies the Serre $S_2$ condition. Therefore, the strata of $\mathcal{X}$ cannot be an exceptional locus (contratable) to $\mathcal{X}'_0$.

13 The assumption ensures the coincidence of the augmented Hodge line bundle and the Hodge line bundle in [55].
Step 2 (Flops) For any two ℚ-factorial dlt minimal models \( X_1, X_2 \), they are connected by flops due to [75]. Then we apply the same arguments as above Step 1 to the flopping contraction to check the assertion.

For fixed \( X \), the assertion follows from the \( \mathbb{P}^1 \)-linking theorem [78].

Step 3 (Base change effect) We show that for a dlt minimal model \( X \to \Delta \) and any positive integer \( N \), there is a dlt minimal model \( X^{[N]} \) admissibly dominating the base change of \( X \) which satisfies the assertion. This follows directly from the actual construction of [115, Section 4] as partially reviewed in Proposition 2.8. \( \square \)

Accordingly, we expect parametrization of minimal lc centers on the boundary of the Satake–Baily–Borel type compactification of Conjecture B.1. We introduce the following terminology:

Definition B.5 We define a set

\[
\mathcal{M}_{\text{CY}}(d) := \{ d\text{-dimensional klt log Calabi–Yau pairs} \}/\sim,
\]

where in the right hand side \( \sim \) means the equivalence relation generated by log crepant birational maps ("B-birational map" in [47]).

Conjecture B.6 (Parametrising minimal log canonical centers) For any of the conjectural compactification \( M \subset \overline{M}^{\text{SBB}} \) from Conjecture B.1, there is a natural map

\[
\psi_{\text{mlcc}} : \partial \overline{M}^{\text{SBB}} \to \bigsqcup_{0 \leq d < n} \mathcal{M}_{\text{CY}}(d)
\]

such that the following holds:

Take a polarized dlt minimal model \( (X, L) \to C \not
\}
\to M \), which corresponds to \( \varphi^\circ : C \setminus \{0\} \to M \), and consider the holomorphic extension \( \varphi : C \to \overline{M}^{\text{SBB}} \). Then \( \psi_{\text{mlcc}}(\varphi(0)) \) is represented (in the sense of Definition B.5) by the minimal log canonical center of \( (X, X_0) \), which makes sense by Proposition B.4).

Also recall a fact in the abelian varieties case from [114]:

Theorem B.7 (SBB compactification as a GH compactification [114, Section 2.5.3, Corollary 2.14]) Let us consider the moduli \( A_g \) of \( g \)-dimensional principally polarized abelian varieties \( (X, L) \) over \( k = \mathbb{C} \), and associate the flat Kähler metrics on \( X \) with the Kähler classes \( 2\pi c_1(L) \) so that the volume is always 1. Then, the pointed Gromov–Hausdorff limits of those are (trivial) \( \mathbb{R}^r \)-fibrations over the minimal log canonical centers which are \( (g - r) \)-dimensional principally polarized abelian varieties, with the natural flat metrics, where \( r \) denotes the torus rank of the degenerations.

In particular, those pointed Gromov–Hausdorff limits are parametrized by the Satake–(Baily–Borel) compactification \( \overline{A}_g^{\text{SBB}} \) [130].

Naturally this leads to the following question.

Question B.8 (Satake–Baily–Borel compactification as a Gromov–Hausdorff compactification?) For the above situation, when \( k = \mathbb{C} \), does \( \partial \overline{M}^{\text{SBB}} \) parametrize certain information on the (pointed) Gromov–Hausdorff limits of Ricci-flat Kähler spaces \( (X, \omega_X) \) where \( (X, L) \in M, [\omega_X] = c_1(L) \), and \( \text{diam}(-) \) denotes the diameter?
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