Note

Spinning, precessing, black hole binary spacetime via asymptotic matching

Hiroyuki Nakano\textsuperscript{1,2}, Brennan Ireland\textsuperscript{2}, Manuela Campanelli\textsuperscript{2} and Eric J West\textsuperscript{3}

\textsuperscript{1} Department of Physics, Kyoto University, Kyoto 606-8502, Japan
\textsuperscript{2} Center for Computational Relativity and Gravitation, and School of Mathematical Sciences, Rochester Institute of Technology, Rochester, NY 14623, USA
\textsuperscript{3} Department of Physics and Astronomy, University of Minnesota Duluth, Duluth, MN 55812, USA

E-mail: hinakano@tap.scphys.kyoto-u.ac.jp

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Abstract
We briefly discuss a method to construct a global, analytic, approximate spacetime for precessing, spinning binary black holes. The spacetime construction is broken into three parts: the inner zones are the spacetimes close to each black hole, and are approximated by perturbed Kerr solutions; the near zone is far from the two black holes, and described by the post-Newtonian metric; and finally the wave (far) zone, where retardation effects need to be taken into account, is well modeled by the post-Minkowskian metric. These individual spacetimes are then stitched together using asymptotic matching techniques to obtain a global solution that approximately satisfies the Einstein field equations. Precession effects are introduced into the coordinate transformation from the inner to near zones according to the precessing equations of motion, in a way that is consistent with the global spacetime construction.

Keywords: binary black holes, post-Newtonian approximation, perturbation theory

1. Introduction

In our previous paper [1], we constructed a global, analytic, approximate spacetime for non-precessing, spinning binary black holes (BBHs) in the (quasi-circular) inspiral phase. To obtain this metric, we used the framework discussed in [2–4]. Using this method, the spacetime is divided into various zones, and described by appropriate approximation for each zone. Each zone’s metric is smoothly matched via the techniques of matched-asymptotic-expansions, and
the use of transition functions (see [5]) in the overlapping regions of validity, called the buffer zones (BZs).

The spacetime close to each black hole (BH) is called the inner zone (IZ); it is derived by satisfying the Teukolsky equations [6], which describe linearized perturbations around a background Kerr BH. There are two IZs, IZ1 around BH1 (with mass $m_1$ and dimensionless spin parameter $\chi_1$) and IZ2 around BH2 (with $m_2$ and $\chi_2$). The post-Newtonian (PN) approach accurately describes the weak gravitational field around two BHs, called the near zone (NZ) (see, e.g. [7]). The PN expansion is a Taylor expansion in $\nu/c \ll 1$ (slow motion) or $GM/(rc^2) \ll 1$ (weak fields), where $\nu$ is the characteristic velocity of the BH, $M$ is the mass of the BH, and $r$ is the radial distance coordinate from the BH. The region far away from the BHs (much larger than a gravitational wavelength) is the far zone (FZ) and also referred to as the wave zone. This zone is modeled by the post-Minkowskian formalism (see, e.g. [7, 8]).

Using the above metrics and matched asymptotic expansion, previous work has given the global metrics on a particular spatial hypersurface, i.e. initial data for non-spinning and non-precessing, spinning BBHs, in [4] and [9], respectively. In [10], the above works were extended to be able to describe the dynamical spacetime valid for arbitrary times. The method was generalized further to include aligned and counter-aligned spins in [1]. Here, we extend this construction for precessing, spinning BBHs.

This paper is organized as follows. In section 2, we derive the coordinate transformation between the IZ and NZ metrics in the matched asymptotic expansion. Since the calculation is almost parallel to the derivation discussed in [1], we do not repeat the detailed analysis in the main text, and briefly summarize it in appendix. In section 3, we discuss the PN equations of motion (EOM) briefly as another important ingredient for the dynamical spacetime construction. Section 4 is devoted to discussions. We use the geometric unit system, where $c = G = 1$. Greek and latin letters are used as spacetime and spatial indices, respectively.

2. Asymptotic matching

To describe the NZ spacetime, we restrict to the currently available explicit PN expressions published in the literature for the non-spinning and spin–orbit terms in the PN harmonic (PN-H) coordinate system ($x^\alpha = (t, x, y, z)$) [7]. Hence, we use [11] for the spin independent terms up to 2.5PN order, and [12, 13] for the non-vanishing spin terms up to 1.5PN order, and [14] for the next-to-leading-order spin terms. Higher order spin coupling terms have been calculated for the EOM and the equations of the precession of the spins [15, 16], but not the bulk metric. These terms will be added to the metric in the future as they become explicitly available.

The FZ metric is presented in [4, 8, 17], and asymptotically matched to the NZ in the NZ–FZ BZ automatically [9]. Therefore, we focus only on the matching calculation between the IZ and NZ metrics here.

2.1. Coordinate rotation

For simplicity and clarity of presentation, we discuss only IZ1 around BH1. The treatment of BH2 is handled by changing the labels, $I \leftrightarrow 2$. The IZ spacetime is described by the Kerr background metric with the mass $M_1$ and Kerr spin parameter $a_1 = M_1 \chi_1$ and its perturbation [18] under the ingoing radiation gauge condition in Cook-Scheel harmonic (CS-H) coordinates ($X^\alpha = (T, X, Y, Z)$) [19], where the spin is aligned along the Z coordinate direction. This is a preferred coordinate system to describe the IZ spacetime.
On the other hand, for precessing BBHs, we need to treat an arbitrary time-dependent spin direction in the PN-H coordinates where we want to describe the IZ and NZ metrics. The spin direction specifies the $Z$ coordinate direction of the CS-H coordinates. We express the spin as

$$\chi_i = \chi_i (\sin \Theta, \cos \Phi, \sin \Theta, \cos \Phi),$$

(1)

in the spacial PN-H coordinates, $\{x, y, z\}$. Although $\Theta$ and $\Phi$ are functions of time, we may consider the instantaneous spin angles and forget the time dependence in the matching calculation because the time derivative of the spin angles becomes a higher order than the matching order of this note. In [1], we discussed the asymptotic matching between the IZ and NZ metrics for aligned and counter-aligned spin cases, i.e. the spins along the $z$ coordinate direction. Therefore, it is useful to introduce a rotation of the spin as the following. The above spin $\chi_i$ is transformed to $\chi'_i$ along the $z$ coordinate by two rotations; first the rotation through angle $-\Phi$ about the $z$ axis and then the rotation through angle $-\Theta$ about the $y$ axis as

$$\chi'_i = Y(-\Theta)Z(-\Phi)\chi_i = \chi_i(0, 0, 1),$$

(2)

where

$$Z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y(\beta) = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}.$$  

(3)

We will use a notation here $\hat{Z}(\alpha)$ and $\hat{Y}(\beta)$ in which the time-time component $\eta = 1$ and the time-space components $\eta = 0$ are added to $Z(\alpha)$ and $Y(\beta)$, respectively. For conciseness, it is helpful to define and use ‘tensor-like’ notations for the above rotation matrices, i.e. $\hat{R}_{\mu\nu}(\hat{Y}(-\Theta), \hat{Z}(-\Phi)) = \hat{Y}(-\Theta)\hat{Z}(-\Phi)$.

### 2.2. Matching calculation

Now that we have rotated of the spin direction, we follow the procedure described in [1] directly to match the IZ to the NZ. In the BZ between the IZ and NZ, the NZ metric $g^{NZ}_{\alpha\beta}$ is expanded by using $m_1 << r_1 << b$, where $r_1$ denotes the distance from BH1 and $b$ is the orbital separation in the PN-H coordinates:

$$g^{NZ}_{\alpha\beta} = (g^{NZ}_{\alpha\beta})_0 + \frac{m_2}{b} (g^{NZ}_{\alpha\beta})_1 + \left(\frac{m_2}{b}\right) (g^{NZ}_{\alpha\beta})_2 + O(v^2),$$

(4)

where $(\cdot)_i$ denotes the $i$th order quantity, and

$$(g^{NZ}_{\alpha\beta})_0 = \eta_{\alpha\beta}, \quad (g^{NZ}_{\alpha\beta})_1 = 0,$$

$$(g^{NZ}_{\alpha\beta})_2 = \left[ \frac{2m_1}{m_2} \frac{b}{(r_1)_0} + 2 - \frac{2}{b} (\langle r_1 \rangle_0 \cdot (\hat{b})_0) + \frac{1}{b^2} [3(\langle r_1 \rangle_0 \cdot (\hat{b})_0)^2 - \langle (r_1)\rangle_0] \right] \Delta_{\alpha\beta}.$$  

(5)

Here, $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric, $\Delta_{\alpha\beta} = \text{diag}(1, 1, 1, 1)$, and $\langle (\hat{b})_0 \rangle$ is the unit vector from BH2 to BH1. We will also use a notation $\beta^0 = \{0, n_{12}\}$ later. The difference from [1] due to the precession is in $\langle (\hat{b})_0 \rangle$, which is in the $x$–$y$ plane for the non-precessing case [1], but in this work is in an arbitrary direction.

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4 We will discuss the time dependence in section 4 later.
As in the non-precessing case, we carry the asymptotic matching up to $O(m_2/b)^3$. For the NZ and IZ metrics, the asymptotic matching is implemented order by order with respect to $(m_2/b)^{1/2}$, based on the relation between two metrics,

$$
g_{NZ}^{ij} = \frac{\partial X^\alpha}{\partial x^\alpha} \frac{\partial X^\beta}{\partial x^\beta} g_{IZ}^{\alpha \beta}, (6)$$

Here, we consider the matching of the IZ metric $g_{IZ}^{\gamma \delta}$ to the NZ metric for an arbitrary spin direction. Since the spin terms do not enter into the matching calculation up to $O(m_2/b)^3$, except for the spin direction, the result can be easily obtained as described below.

The equations to derive the coordinate transformation are only slightly modified from [1] (see appendix for the detailed calculation). In the main text, we present only the schematic derivation. The coordinate transformation up to first order $(O(m_2/b)^{1/2})$, (A.12), is rewritten as

$$
(X^{\alpha})_{11} = R^{\alpha \beta} (\tilde{Z}(\Phi_1), \tilde{Y}(\Theta_1)) \left[ \tilde{x}^\beta - \sqrt{\frac{m_2}{b}} \sqrt{\frac{m_2}{m}} \tilde{y}_C \tilde{t}^\beta \right], (7)
$$

where we have used \{1\} to describe the leading + first order quantity, the total mass $m = m_1 + m_2$, and $\tilde{x}^i, \tilde{y}_C, \tilde{t}^\beta$ are defined in (A.4), (A.11) and (A.6), respectively. It is noted that the expression in the bracket of the above equation is same as (20) in [1].

In what follows, we derive the second order coordinate transformation for precessing, spinning, BBHs. In doing so, we first generalize equation (24) in [1] to include the rotation of the spin direction due to the precession (see A.16). The solution to (A.16) is given by applying the rotation $R^{\alpha \beta} (\tilde{Z}(\Phi_1), \tilde{Y}(\Theta_1))$, to the solution derived in (28) of [1]:

$$
(X^{\alpha})_{2} = R^{\alpha \beta} (\tilde{Z}(\Phi_1), \tilde{Y}(\Theta_1)) \eta^{\beta \gamma}(X_{2,\text{nonP}}^{\gamma}), (8)
$$

where

$$
(X_{2,\text{nonP}}^{\alpha}) = \left( 1 + \frac{m_2}{2m} \right) (\tilde{x}^{2, l_0}^{\alpha}) \tilde{t}_0 + \left( 1 - \frac{\tilde{y}_C}{\tilde{x}} \right) \Delta_{l\alpha} \tilde{x}^\beta + \frac{\Delta_{l\beta} \tilde{x}^\beta}{\tilde{x}} \tilde{y}_C \tilde{t}_0 + \frac{m_2}{2m} \tilde{y}_C \tilde{t}_0
$$

$$
- \frac{1}{b^2} (r_1 \tilde{y}_C \tilde{t}_1 - \tilde{y}_C \tilde{t}_1 r_1) \tilde{t}_0 + \frac{1}{3b^2} (r_1 \tilde{y}_C (r_1 \tilde{t}_0) \tilde{t}_0), (9)
$$

where $\tilde{x}_C$ and $\tilde{t}_0$ are defined in (A.15) and (A.7), respectively.

The explicit expressions for the coordinate transformation are also derived in a similar fashion:

$$
X^{\alpha} = R^{\alpha \beta} (\tilde{Z}(\Phi_1), \tilde{Y}(\Theta_1)) X_{\text{nonP}}^{\beta}, (10)
$$

where $X_{\text{nonP}}^{\alpha}$ are given as

$$
T_{\text{nonP}} = t - \frac{m_2}{r_12} \frac{m_2}{m} \tilde{y}_C + \frac{m_2}{r_12} \left( \frac{r_1}{3} - \frac{3 \tilde{y}_C \tilde{x}_l}{r_1 \tilde{x}_l} \right) \tilde{t}_0 + \frac{5}{384} \frac{(2m + m_2)r_1^3 - r_1(0)^3}{m^2 m_1},
$$

$$
X_{\text{nonP}}^{i} = \tilde{x}^i + \frac{m_2}{r_12} \left( \frac{1 - \tilde{y}_C}{r_12} \tilde{x}_l^i + \frac{1}{2} \frac{\tilde{t}_0^2}{r_12} n_{12}^i + \frac{1}{2} \frac{m_2 \tilde{y}_C \lambda_{12}}{m} \tilde{t}_0^i - \frac{\tilde{x}_l^i \lambda_{12}}{r_12^2} n_{12}^i - \tilde{y}_C \tilde{x}_l^i \tilde{t}_0^i \right). (11)
$$

where $T_{\text{nonP}} \equiv X_{\text{nonP}}^{0}$, and $\lambda_{12}^i$ is defined in (A.7). Here, we have used the evolution of the orbital separation $b = r_{12} = r_{1a}(t)$, and introduced $\tilde{t}_0 = \sqrt{\tilde{x}^i \tilde{t}_0} (= (r_1 \tilde{t}_0)$. In [1], the orbital plane is always in the $x–y$ plane. For precessing BBHs, the orbit is fully 3 dimensional. Therefore, $X_{\text{nonP}}^{i}$ has a vector form rather than that specified only by the equatorial orbit.
3. Post-Newtonian equations of motion

Another important ingredient to complete the dynamical spacetime construction for our precessing, spinning BBHs is the introduction of the PN EOM in the harmonic gauge. Up to 3.5PN order, and for maximal spin (|χ_s| = 1), spin–orbit effects contribute to the EOM at 1.5PN, 2.5PN, and 3.5PN. Spin–spin effects contribute at 2PN and 3PN. Cubic-in-spin effects start from 3.5PN. Quartic- and higher order-in-spin effects are beyond 3.5PN. Following [7, 20, 21], we use the Tulczyjew spin supplementary condition (SSC) to define a spin vector with conserved Euclidean norm. For this SSC, the higher order spin-orbit terms have been derived in [13, 14, 21, 22], see also [7]. The next-to-leading order spin–spin terms were derived in [16], and the leading order cubic-in-spin terms were derived in [15]. To date, leading order quartic- and quintic-in-spin contributions to the EOM have not been derived in PN-H coordinates with this SSC. The full EOM will be shown in a future paper [23], where we present the PN EOM in a ready-to-use form.

4. Discussion

We have derived here a new global, analytic, approximate spacetime for precessing BBHs. The construction follows closely the methods employed in [1]. In [1], we tested the validity of the global metric for non-precessing, spinning BBHs by using the Ricci scalar to estimate the violations to the Einstein vacuum field equations, and the relative Kretschmann invariant to discuss a normalized violation. The difference between [1] and this paper is only the precession due to misaligned spins. The leading order effect of the precession on the violations arises from the time derivative of (A.3) where Θ_t and Φ_t are the time-dependent spin angles, and the order is estimated as O(⟨n_2⟩^2) (see, e.g. (A6) of [24]). On the other hand, the error in the matching is O(⟨n_2⟩^3/2). Therefore, the precession effect is higher order than the matching error, and the violations of the new approximate spacetime will be similar to those evaluated in [1]. It should be noted that in higher order matching we will have completely different transformations which cannot be described by a simple extension of the nonspinning case, due to the precession of spins.

Our expectation is that this new spacetime can be used directly in general relativistic magneto-hydrodynamic and hydrodynamic simulations to study the circumbinary disk around the BBH and individual mini disks around each BH for long time evolutions in the inspiral regime without back reaction (see, e.g. circumbinary disk [25], mini disks [26]).

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Appendix. Details on the matching calculation

The matching calculation in the BZ between the IZ and NZ is almost parallel to the discussion given in [1].
A.1. Zeroth-order matching: $O((m_2/b)^0)$

As the zeroth order, we may consider the matching as

$$\left( g^{\mathrm{NZ}}_{\alpha\beta} \right)_0 = (A_{\alpha}^\gamma)_0 (A_{\beta}^\delta)_0 (g^{\mathrm{IZ}}_{\gamma\delta})_0. \quad (A.1)$$

Here, $A_{\alpha}^\beta = \partial_\alpha \chi^\beta$ (where $\partial_\alpha = \partial/\partial x^\alpha$), and $\left( g^{\mathrm{IZ}}_{\alpha\beta} \right)_0 = \eta_{\alpha\beta}$. Using

$$R^{\alpha\beta}_{\gamma\delta} (\Phi_1), \widetilde{Y}(\Theta_1)) R^{\gamma\delta}_{\alpha\beta} (\Phi_1), \widetilde{Y}(\Theta_1)) \eta^{\alpha\beta} = \eta_{\mu\nu}, \quad (A.2)$$

and taking into account the position of BH1 and the spin direction, we have

$$(X^0)_0 = R^{\alpha\beta}_{\gamma\delta} (\Phi_1), \widetilde{Y}(\Theta_1)) \tilde{x}^{\beta}, \quad (A.3)$$

where $R^{\alpha\beta}_{\gamma\delta} (\Phi_1), \widetilde{Y}(\Theta_1))$ denotes the inverse transformation of $R^{\alpha\beta}_{\gamma\delta} (\Phi_1), \widetilde{Y}(\Theta_1))$, and

$$\tilde{x}^{\alpha} = x^{\alpha} - \frac{m_2}{b} \beta^{\alpha}, \quad (A.4)$$

where $(\tilde{r}^\alpha)_0 = \tilde{x}^{\alpha}$ in the NZ metric of (5). Here, we have considered the position of BH1 by $\tilde{x}^{\beta}$ and the spin direction by $R^{\alpha\beta}_{\gamma\delta} (\Phi_1), \widetilde{Y}(\Theta_1))$ separately to use the analysis in [1]. The expression in (A.3) for the leading order coordinate transformation leads to a simple modification of the analysis given in [1].

Since $\beta^{\alpha}$ has a time dependence, the time derivative of (A.3) becomes

$$\partial_t (X^0)_0 = \dot{t}^\alpha - \frac{m_2}{\sqrt{b}} [\frac{m_2}{m} R^{\alpha\beta}_{\gamma\delta} (\Phi_1), \widetilde{Y}(\Theta_1)) \dot{\beta}^\beta], \quad (A.5)$$

in our current analysis of the matching calculation. Here, we have defined

$$\dot{t}^\alpha = \{ 1, 0, 0, 0 \}, \quad (A.6)$$

and

$$\dot{\beta}^\alpha = \{ 0, \lambda_{\gamma\delta} \} = \partial_\beta \beta^{\alpha} / \Omega. \quad (A.7)$$

Raising and lowering tensor indices are done by the Minkowski metric, e.g. $\dot{t}^\alpha = \eta_{\alpha\beta} \dot{t}^\beta = \{-1, 0, 0, 0 \}$. Here, the time derivative of $\Theta_1$ and $\Phi_1$ gives $O(\sqrt{\chi_2}(m_2/b)^2)$ in $\partial_t (X^0)_0$ (see, e.g. (A6) of [24]), and ignored safely in the current matching calculation.

A.2. First-order matching: $O((m_2/b)^{1/2})$

The matching equation at the first order is written as

$$\left( g^{\mathrm{NZ}}_{\alpha\beta} \right)_1 = (A_{\alpha}^\gamma)_0 (A_{\beta}^\delta)_0 g^{\mathrm{IZ}}_{\gamma\delta} + 2 (A_{\alpha}^\gamma)_0 (A_{\beta}^\delta)_0 (g^{\mathrm{IZ}}_{\gamma\delta})_1, \quad (A.8)$$

where $T_{\alpha\beta} = (T_{\alpha\beta} + T_{\beta\alpha})/2$ denotes a symmetric tensor, and $(g^{\mathrm{IZ}}_{\alpha\beta})_1 = (g^{\mathrm{IZ}}_{\alpha\beta})_1 = 0$. Using (A.5), we have $\partial_t (X^1)_0 = 0$ and $\partial_t (X^0)_0 = R^{\alpha\beta}_{\gamma\delta} (\Phi_1), \widetilde{Y}(\Theta_1)) \dot{\beta}^\beta$, the above equation becomes

$$(A_{\alpha}^\gamma)_0 = \partial_\beta (X^0)_0 = R^{\alpha\beta}_{\gamma\delta} (\Phi_1), \widetilde{Y}(\Theta_1)) + [\frac{m_2}{b} \frac{m_2}{m} \dot{t}^\gamma R^{\alpha\beta}_{\gamma\delta} (\Phi_1), \widetilde{Y}(\Theta_1)) \dot{\beta}^\beta].$$
\[ \eta_{\beta} R^\gamma_{\alpha}(\hat{Z}(\Phi_{1}), \hat{Y}(\Theta_{1}))(A_{\beta})_{\gamma} + \frac{m_{2}}{m} \hat{\iota}_{\alpha} R^\gamma_{\beta} = 0, \]  

(A.9)

where the second term of the left hand side arises from the zeroth order coordinate transformation. The solution is obtained as

\[ (X^\alpha)_{1} = - \frac{m_{2}}{m} \hat{\gamma}_{C} R^\alpha_{\beta}(\hat{Z}(\Phi_{1}), \hat{Y}(\Theta_{1}))\hat{t}^\beta = - \frac{m_{2}}{m} \hat{\gamma}_{C} \hat{t}^\alpha, \]  

(A.10)

where we have defined

\[ \hat{\gamma}_{C} = \hat{\iota}_{\alpha} \hat{x}^\alpha. \]  

(A.11)

Since \((X^\alpha)_{1}\) has only the time component, there is no effect due to the precession, i.e. \(R^\alpha_{\beta}(\hat{Z}(\Phi_{1}), \hat{Y}(\Theta_{1}))\hat{t}^\beta = \hat{t}^\alpha\).

A.3. Second-order matching: \(O(m_{2}/b)\)

In a similar analysis given in [9], we obtain \((M)_{0} = m_{1}\) from the divergent part in \((r_{1})_{0} = |\vec{r}| \to 0\). And using the zeroth-order matching, the tidal field \((\hat{E}_{y})_{0}\) which is the perturbation around the Kerr BH in the IZ calculation, is derived as \((\hat{E}_{y})_{0} = \delta_{y} - 3 \hat{\beta}_{y}\beta_{y}\), where \(\hat{\beta}_{y}\) denotes the spatial components of \(\hat{\beta}_{y} = \eta_{\alpha\beta} \beta_{\alpha}^{\beta} = \eta_{\alpha\beta} R^\beta_{\gamma}(\hat{Z}(\Phi_{1}), \hat{Y}(\Theta_{1}))\beta_{\gamma}^{\beta}\). This \((\hat{E}_{y})_{0}\) is used to evaluate the tidal tensor in (12) of [1].

Next, we calculate the second order coordinate transformation. The leading and first order matching gave

\[ (X^\alpha)_{11} = R^\alpha_{\beta}(\hat{Z}(\Phi_{1}), \hat{Y}(\Theta_{1}))\hat{t}^\beta - \frac{m_{2}}{m} \frac{m_{2}}{m} \hat{\gamma}_{C} \hat{t}^\alpha. \]  

(A.12)

The formal expression for the second order matching is written as

\[ (g_{\alpha\beta})^{NZ}_{(2)} = (A_{\alpha}^{\gamma} g_{(2)}^{\gamma\delta}(g_{\gamma\rho})^{\delta\rho}_{(2)})_{(2)}, \]

(A.13)

where \(\{2\}\) denotes the leading + first order + second order quantity. \((A_{\alpha}^{\gamma})_{(2)}\) includes not-yet-determined \((X^\gamma)_{2}\) as

\[
\begin{aligned}
(A_{\alpha}^{\gamma})_{(2)} &= \partial_{\nu}(X^\gamma)_{(2)} = R^\gamma_{\alpha}(\hat{Z}(\Phi_{1}), \hat{Y}(\Theta_{1})) \\
&+ \frac{m_{2}}{m} \left[ \frac{m_{2}}{m} \hat{\iota}_{\gamma} R^\gamma_{\rho}(\hat{Z}(\Phi_{1}), \hat{Y}(\Theta_{1}))\hat{t}^{\rho} - \frac{m_{2}}{m} \hat{\iota}_{\gamma} R^\gamma_{\rho}(\hat{Z}(\Phi_{1}), \hat{Y}(\Theta_{1}))\hat{t}^{\rho} \right] \\
&+ \frac{m_{2}}{m} \left[ - \frac{1}{b} \left( \hat{\gamma}_{C} + \frac{m_{2}}{m} b \right) \hat{\iota}_{\gamma} R^\gamma_{\rho}(\hat{Z}(\Phi_{1}), \hat{Y}(\Theta_{1}))\hat{t}^{\rho} + \partial_{\nu}(X^\gamma)_{2} \right],
\end{aligned}
\]

(A.14)

where

\[ \hat{\gamma}_{C} = \hat{\beta}_{y} \hat{x}^{\alpha}. \]  

(A.15)

Although the above expression is slightly more complicated than that in [1], we can see the relation, \(R^\gamma_{\alpha}(\hat{Z}(\Phi_{1}), \hat{Y}(\Theta_{1}))\) in this paper \(\leftrightarrow \delta^{\gamma}_{\alpha}\) in [1]. Finally, we may solve
\[ 2R^\gamma_{\alpha\beta}(\hat{z}(\Phi_1), \hat{y}(\Theta_1)) \left( A_{\beta\gamma} \right)_2 = \left[ \left( 2 - \frac{2}{b} \tilde{x}_C \right) \Delta_{\alpha\beta} + \frac{2}{b} \tilde{x}_C \tilde{x}_C \delta_{\beta\gamma} + \frac{m_2}{m} \delta_{\alpha\beta} + \frac{m_2}{m} \tilde{u}_C \tilde{u}_C \right] + \left[ \delta_{\beta\gamma} \frac{2}{b} \left( \tilde{x}_C \delta_{\beta\gamma} - (r_0) \tilde{x}_C \tilde{x}_C \delta_{\beta\gamma} \right) \right] + \left[ \delta_{\beta\gamma} \delta_{\alpha\beta} + \delta_{\beta\gamma} \delta_{\alpha\beta} \frac{1}{3b^2} \left( 3(r_0) \tilde{x}_C \tilde{x}_C \delta_{\beta\gamma} - \frac{3}{(r_0) \tilde{x}_C \tilde{x}_C \delta_{\beta\gamma} - 6(r_0) \tilde{x}_C \tilde{x}_C \delta_{\beta\gamma} \right) \left( A.16 \right) \right. \]

The above solution is given in (8).

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