LAGRANGIAN COBORDISM IN LEFSCHETZ FIBRATIONS.

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Abstract. Given a symplectic manifold \((M^{2n}, \omega)\) we study Lagrangian cobordisms \(V \subset E\) where \(E\) is the total space of a Lefschetz fibration having \(M\) as generic fiber. We prove a generation result for these cobordisms in the appropriate derived Fukaya category. As a corollary, we analyze the relations among the Lagrangian submanifolds \(L \subset M\) that are induced by these cobordisms. This leads to a unified treatment - and a generalization - of the two types of relations among Lagrangian submanifolds of \(M\) that were previously identified in the literature: those associated to Dehn twists that were discovered by Seidel [Sei2] and the relations induced by cobordisms in trivial symplectic fibrations described in our previous work [BC2, BC3].

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1. Introduction

1.1. Motivation. The derived Fukaya category $D\mathcal{F}uk(N)$ of a symplectic manifold $(N, \omega)$ is a triangulated category whose objects are obtained as the completion of a certain class - here denoted by $\mathcal{L}(N)$ - of Lagrangian submanifolds of $N$. The completion can be summarized as follows. As a set, each Lagrangian $L$ can be described as a collection of sets each consisting of intersection points $L' \cap L$ where $L'$ is a variable Lagrangian transverse to $L$. This family of intersection points can be assembled in a family of vector spaces $\mathbb{Z}_2 \langle - \cap L \rangle$ again with $L'$ viewed as a variable. In the absence of some coherence relations among all these vector spaces this is obviously not a useful description of $L$. However, given some almost complex structure $J$, compatible with $\omega$, there are natural relations among the vector spaces $\mathbb{Z}_2 \langle - \cap L \rangle$ that reflect the existence of $J$-holomorphic curves with Lagrangian boundary conditions along families $L_1, \ldots, L_k \in \mathcal{L}(N)$ and $L$. The formal way to express this is to construct first an $A_\infty$-category $\mathcal{F}uk(N)$ called the Fukaya category of $N$ with objects $\mathcal{L}(N)$, with morphisms the vector spaces $\text{hom}(L', L'') = \mathbb{Z}_2 \langle L' \cap L'' \rangle$ and so that the higher multiplications $\mu_k$ are given by counts of $J$-holomorphic polygons with boundary components along $L_1, L_2, \ldots, L_{k+1}$. In this formalism the family $\mathbb{Z}_2 \langle - \cap L \rangle$ becomes a module over $\mathcal{F}uk(N)$, called the Yoneda module associated to $L$, $\mathcal{Y}(L)$. The modules over an $A_\infty$-category are algebraic structures that behave in ways very similar to chain complexes. In particular, given a morphism between two modules $f : \mathcal{M} \to \mathcal{M}'$, one can take the cone over it $\mathcal{M}'' = \text{cone}(f)$ by a formula similar to the cone over a chain map. The category $D\mathcal{F}uk(N)$ has as objects all the modules that can be obtained by iterated cones from the Yoneda modules. The morphisms in this category
are the homology classes of the module morphisms. The exact triangles are the image of
the module-level cone attachments. We refer to [Sei3] for the detailed construction. We remark
that our variant of the derived Fukaya category is not completed with respect to idempotents,
by contrast to other versions of this notion that are present in the literature.

Two closely inter-related types of results are key from this perspective: decomposition
results, that show that all objects in some class can be decomposed in $D\mathcal{F}uk(-)$ in terms of
simple objects, similarly to the way a $CW$-complex can be decomposed into cells; constructive
results producing exact triangles in $D\mathcal{F}uk(-)$ out of geometric structures or operations.

1.2. Main result. The main aim of this paper it to prove a decomposition result for a class
of Lagrangian submanifolds with cylindrical ends - called cobordisms - that are embedded in
the total space of a Lefschetz fibration $\pi : E \to \mathbb{C}$. We consider here such cobordisms $V$ with
“negative” ends only: outside of a compact subset, the projection of $V$ to $\mathbb{C}$ is a union of rays
of the type $\ell_i = (-\infty, \ell_i] \times \{i\}, \ i \in \mathbb{N}$. Such cobordisms will be called negatively-ended.

We work with uniformly monotone Lagrangians - see §3.1 for the definitions. Let $L^*(E)$
be the class of these cobordisms in $E$. The superscript $-*$ will denote at all times below the
monotonicity constraint imposed on the Lagrangians involved. We denote by $\mathcal{A}$ the universal
Novikov ring over $\mathbb{Z}_2$. We work at all times in this paper in an ungraded context and over the
base field $\mathbb{Z}_2$.

We state here the main decomposition result and refer to §4.1 where the result is restated
after making the various ingredients more precise. Our conventions and notation regarding
iterated cone decompositions are explained in §3.1.1.

**Theorem A.** There exists a Fukaya category with objects the cobordisms in $L^*(E)$. Let
$D\mathcal{F}uk^*(E)$ be the associated derived Fukaya category. Consider one object, $V \in L^*(E)$, fix
points $z_i \in \ell_i$ along the rays associated to $V$ and let $L_i = V \cap \pi^{-1}(z_i)$. Let $T_i$ be the thimbles
associated to the curves $t_i$ as in Figure 1, and let $\gamma_i L_i \subset E$ be obtained by the (union of)
parallel transports of $L_i$ along the curve $\gamma_i$, in the same figure.

There exist finite rank $\mathcal{A}$-modules $E_k$, $1 \leq k \leq m$, and an iterated cone decomposition
taking place in $D\mathcal{F}uk^*(E)$:

$$V \cong (T_1 \otimes E_1 \to T_2 \otimes E_2 \to \ldots \to T_m \otimes E_m \to \gamma_s L_s \to \gamma_{s-1} L_{s-1} \to \ldots \to \gamma_2 L_2) .$$

The precise meaning of the notation in the last formula will be be explained in §3.1.1. The
modules $E_i$ are made explicit in the proof - see (55).

1.3. Some consequences. Cobordisms are of interest not only for their own sake but also
because they can be viewed as relators among their ends, in the sense of the usual cobordism
relation. In this direction, one of the main consequences of Theorem A is that each such
cobordism $V$ produces an iterated cone decomposition inside $D\mathcal{F}uk^*(M)$, with $M = \pi^{-1}(z_1)$, that expresses the end $L_1$ of $V$ as an iterated cone involving the ends $L_i, i \geq 2$ and the vanishing cycles of the singularities of $\pi$ - see §5.1. Thus, cobordisms in $E$ and the triangular decompositions in the fiber are intimately related - see Corollary 5.1.1.

To discuss a further consequence, recall that to any triangulated category $\mathcal{C}$ one can associate a Grothendieck group, $K_0\mathcal{C}$ defined as the quotient of the free abelian group generated by the objects of $\mathcal{C}$ modulo the relations $B = A + C$ associated to each exact triangle $A \to B \to C$. We remark that in this paper we work with ungraded categories, hence our Grothendieck groups will always be 2-torsion (i.e. $2A = 0$ for every $A \in K_0\mathcal{C}$).

Another application of Theorem A - see §5.2 - is to give a description of the Grothendieck group $K_0D\mathcal{F}uk^*(M)$ as an “algebraic” cobordism group. To explain this result we focus here on the case of the trivial fibration $E = \mathbb{C} \times M$ even if we establish the relevant results in more generality in the paper. Recall from [BC3] the definition of the cobordism group $\Omega_{\text{Lag}}^*(M)$. It is the quotient of the free abelian group generated by the objects in $\mathcal{L}^*(M)$ modulo the relations $L_1 + L_2 + \ldots + L_s = 0$ for each negatively-ended cobordism $V \subset \mathbb{C} \times M$ whose ends are $L_1, \ldots, L_s$. The natural restriction to the ends - that associates to a cobordism $V$ its $i$-th end - admits an extension to all the objects of $D\mathcal{F}uk^*(\mathbb{C} \times M)$. The $i$-th end of an object $\mathcal{M}$ in $D\mathcal{F}uk^*(\mathbb{C} \times M)$ is denoted by $[\mathcal{M}]_i \in \text{Ob}(D\mathcal{F}uk^*(M))$. It is natural to define an algebraic cobordism group $\Omega_{\text{Alg}}^*(M)$ as the free abelian group generated by the (isomorphism classes of) objects of $D\mathcal{F}uk^*(M)$ modulo the relations $\sum_i [\mathcal{M}]_i = 0$ for each object $\mathcal{M}$ of $D\mathcal{F}uk^*(\mathbb{C} \times M)$. Equivalently, $\Omega_{\text{Alg}}^*(M)$ is obtained in just the same way as $\Omega_{\text{Lag}}^*(M)$ but by adding to the relations those obtained from the non-geometric objects in $D\mathcal{F}uk^*(\mathbb{C} \times M)$. There is an obvious quotient $q : \Omega_{\text{Lag}}^*(M) \to \Omega_{\text{Alg}}^*(M)$. A consequence of Theorem A, Corollary 5.2.3, is that there exists a group isomorphism

$$\Theta_{\text{Alg}} : \Omega_{\text{Alg}}^*(M) \to K_0(D\mathcal{F}uk^*(M))$$
so that the composition $\Theta_{Alg} \circ q$ coincides with the Lagrangian Thom morphism

$$\Theta : \Omega^*_\text{Lag}(M) \to K_0 D\text{Fuk}^*(M)$$

previously introduced in [BC3]. One of the reasons why this is of interest is that this result should shed some light on the kernel of $\Theta$ which is at present somewhat mysterious. Another implication of the fact that $\Theta_{Alg}$ is an isomorphism appears in Corollary 5.2.4 and claims that the obvious map $\Omega^*_\text{Lag}(M) \to QH_*(M)$ admits an extension to $\Omega^*_{Alg}(M)$.

Finally, we also obtain a periodicity result for $K_0$ - Corollary 5.2.6:

$$K_0(D\text{Fuk}^*(\mathbb{C} \times M)) = \mathbb{Z}_2[t] \otimes K_0(D\text{Fuk}^*(M))$$

1.4. Relation to previous work. Theorem A can be viewed as a simultaneous generalization of the two previously known methods to produce exact triangles in the derived Fukaya category. The first such method is due to Seidel [Sei2], [Sei3] and, in its basic form, it associates an exact triangle of the form:

$$\tau_S L \to L \to S \otimes HF(S, L)$$

to the Dehn twist $\tau_S : M \to M$ corresponding to a Lagrangian sphere $S$ and any $L \in \mathcal{L}^*(M)$ (Seidel works in an exact setting, but as we will prove below, this triangle remains valid in the monotone context too). More general results of Seidel roughly correspond to the statement in Theorem A when the cobordism $V$ has a single end and provide a description of a Fukaya category associated to the Lefschetz fibration $\pi : E \to \mathbb{C}$ that corresponds, in our context, to the full and faithful subcategory of $\mathcal{F}uk^*(E)$ generated by the thimbles $T_i$. This category is related to mirror symmetry questions and, indeed, cobordisms with a single end appear in relation to mirror symmetry, see for instance [HAV]. Cobordisms with multiple ends as well as a category somewhat similar to $\mathcal{F}uk^*(E)$ appear in the recent paper [AS]. The second method appears in a previous paper of ours [BC3]. It is shown there that if $V \subset \mathbb{C} \times M$ is a cobordism, then the ends of $V$ are related by a cone-decomposition in $D\mathcal{F}uk^*(M)$. This decomposition coincides with the one resulting from Theorem A when $E$ is the trivial fibration $\mathbb{C} \times M$. It should be noted, however, that the statement of Theorem A - which concerns decompositions of cobordisms - is new even for the trivial fibration.

The exact triangle associated to a Dehn twist and the exact triangle obtained through the cobordism machinery coincide when there is a single intersection between $S$ and $L$. This can be shown by methods already in the literature. For example, this follows from a combination of the results from [Sei1] and [BC3] (see also [FOOO, Oh3] for an earlier approach). In this case, Seidel’s exact triangle coincides with the surgery exact sequence which is associated to
a specific cobordism (in $\mathbb{C} \times M$) whose ends are $\tau_s L, L, S$. This cobordism is constructed as the trace of the Lagrangian surgery at the intersection point $S \cap L$. Theorem A and its proof go beyond this case and further clarify the interplay between these two constructions.

From a technical standpoint, we rely heavily on Seidel’s work - in particular, the detailed constructions of $DFuk(-)$ and his set-up of Lefschetz fibrations in the symplectic setting in [Sei3]. There are also a variety of other specific points where our work is related to his and these are mentioned along the text. We also make heavy use of the constructions in our previous papers [BC2] and [BC3]. At the same time, in attempt to keep this text readable we will recall several ingredients from [BC2, BC3] that are crucial for the present paper.

1.5. Outline of the paper. Most of the paper is aimed towards the proof of Theorem A. This proof requires two preliminaries. The first is contained in §2. That section contains the general set-up and terminology concerning Lefschetz fibrations. We introduce a special type of such fibrations called tame which are basically Lefschetz fibrations over $\mathbb{C}$ that are symplectically trivial outside a $U$-like region in the plane - see Figure 3. Tame fibrations are much easier to handle in the technical parts of the proof. One of the reasons is that cylindrical ends can be moved around in the trivial region without the need of parallel transport. Additionally, the Fukaya $A_\infty$ category with objects cobordisms in such fibrations can be defined following closely the constructions in [BC3]. In §2.3 we show that any Lefschetz fibration with a finite number of (simple) singularities can be transformed into a tame one. As a consequence, Theorem A follows from the corresponding result - stated as Theorem 4.2.1 - for tame fibrations.

The second preliminary is the construction of the Fukaya category $Fuk^*(E)$. This is described in §3. We first give the main elements of the construction when the Lefschetz fibration $\pi : E \to \mathbb{C}$ is tame. In this case, the construction that appears in [BC3] applies essentially without change and we review the main steps. We then indicate the modifications needed for the general case and we also remark that for the study of the Fukaya category of a general fibration it is enough to analyze the category associated to an appropriate associated tame fibration. It is useful to note for the discussion below that the objects in our categories are cobordisms in $E$ whose projection onto $\mathbb{C}$ is contained in the upper semi-plane and that are cylindrical outside some fixed strip $[-a, a] \times \mathbb{R}$.

With this preparation, the actual proof of Theorem A is contained in §4 and it consists of three main ingredients. The first one deals with decompositions of cobordisms $V'$ - called remote with respect $E$ - that are included in the total space $E'$ of a Lefschetz fibrations that coincides with $E$ over the upper half plane. The defining property of such a $V'$ is that it can be moved inside $E'$ away from the singularities of $E$, so that its only intersection with an object $X$ of $Fuk^*(E)$ occurs in the region where both $V'$ and $X$ are cylindrical. We show in §4.3 that such a remote cobordism viewed as a module over $Fuk^*(E)$ admits a decomposition
just as the one in the statement of Theorem A but without any of terms $T_i \otimes E_i$. The second step, in §4.4, shows how to transform a general cobordism $V$ into a remote one. This is a geometric step, potentially of independent interest. It is done, roughly speaking, by placing $V$ inside a new Lefschetz fibration $E'$ obtained from $E$ by adding singularities in the lower half plane and showing that the cobordism $V' \subset E'$ obtained as an iterated Dehn twist of $V$, $V' = (\tau_{S_m} \circ \ldots \circ \tau_{S_i} \circ \ldots \circ \tau_{S_1})(V)$, where $S_i$ are certain matching cycles in $E'$, is remote with respect to $E$. The third ingredient - in §4.5 - is Seidel’s exact triangle for which we provide a new proof reflecting our cobordism perspective. These ingredients are put together in §4.6. In short, the cobordism $V' = (\tau_{S_m} \circ \ldots \circ \tau_{S_i})(V)$ is remote with respect to $E$ and thus, by the first step, it admits a certain decomposition involving the ends of $V$, but as it is obtained by an iterated Dehn twist from $V$, it can be related to $V$ by another decomposition, involving the matching cycles $S_i$, by using the relevant Seidel exact triangles. The two decompositions combine as in the statement of Theorem A.

The Corollaries of Theorem A described above are proven in §5.

The paper ends with §6 that consists of examples and related discussion. The main part of the section - §6.5 - is focused on a class of Lagrangian cobordisms in real Lefschetz fibrations.

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2. LEFSCHETZ FIBRATIONS

2.1. Basic definitions. Lefschetz fibrations will play a central role in this paper. From the symplectic viewpoint there are several versions of this notion in the literature. Our setup is similar to [Sei3, Sei2] but with some modifications.

We begin with Lefschetz fibrations having a compact fiber.

**Definition 2.1.1.** A Lefschetz fibration with compact fiber consists of the following data:

i. A symplectic manifold $(E, \Omega_E)$ without boundary, endowed with a compatible almost complex structure $J_E$.

ii. A Riemann surface $(S, j)$ (which is generally not assumed to be compact; typically we will have $S = \mathbb{C}$).

iii. A proper $(J_E, j)$-holomorphic map $\pi : E \to S$. (In particular all fibers of $\pi$ are closed manifolds.)

iv. We assume that $\pi$ has a finite number of critical points. Moreover, we assume that every critical value of $\pi$ corresponds to precisely one critical point of $\pi$. We denote the set critical points of $\pi$ by $\text{Crit}(\pi)$ and by $\text{Crit}_v(\pi) \subset S$ the set of critical values of $\pi$.
v. All the critical point of $\pi$ are ordinary double points in the following sense. For every $p \in \text{Crit}(\pi)$ there exist a local $J_E$-holomorphic chart around $p$ and a $j$-holomorphic chart around $\pi(p)$ with respect to which $\pi$ is a holomorphic Morse function.

For $z \in S$ we denote by $E_z = \pi^{-1}(z)$ the fiber over $z$. We will sometimes fix a base-point $z_0 \in S \setminus \text{Crit}_v(\pi)$ and refer to the symplectic manifold $(M := \pi^{-1}(z_0), \omega_M := \Omega_E|_M)$ as “the” fiber of the Lefschetz fibration. We will also use the following notation: for a subset $S \subset S$ we denote $V|_S = \pi^{-1}(S) \cap V$.

Our constructions work for the most part also when the fiber is not compact. To this end we will need some adjustments to the preceding definition as follows. Let $(M, \omega_M)$ be a (non-compact) symplectic manifold which is convex at infinity. We define a Lefschetz fibration $\pi : E \longrightarrow S$ with fiber $(M, \omega_M)$ to be as in Definition 2.1.1 with the following modifications. Firstly, properness in condition iii is removed (thus allowing, in particular, for the fibers to be non-compact). Secondly, the map $\pi : E \setminus \pi^{-1}(\text{Crit}_v(\pi)) \longrightarrow S \setminus \text{Crit}_v(\pi)$ is now explicitly assumed to be a smooth locally trivial fibration. Finally, $E$ is assumed to satisfy the following additional condition.

**Assumption $T_{\infty}$ (Triviality at infinity).** Let $\pi : E \longrightarrow S$ be as above. We say that $E$ is trivial at infinity if there exists a subset $E^0 \subset E$ with the following properties:

1. For every compact subset $K \subset S$, $E^0 \cap \pi^{-1}(K)$ is also compact. (In other words, $\pi|_{E^0} \longrightarrow S$ is a proper map.)

2. Set $E^\infty = E \setminus E^0$ and $E^\infty_{z_0} = E^\infty \cap \pi^{-1}(z_0)$, where $z_0 \in S \setminus \text{Crit}_v(\pi)$ is a fixed base-point. Then there exists a trivialization $\phi : S \times E^\infty_{z_0} \longrightarrow E^\infty$ of $\pi|_{E^\infty} : E^\infty \longrightarrow S$ such that

$$\phi^* \Omega_E = \omega_S \oplus \omega_M|_{E^\infty_{z_0}}, \quad \text{and} \quad \phi^* J_E = j \oplus J_0$$

where $\omega_S$ is a positive (with respect to $j$) symplectic form on $S$ and $J_0$ is a fixed almost complex structure on $M = \pi^{-1}(z_0)$, compatible with $\omega_M$.

This extended definition in fact generalizes the preceding one: if $M$ is compact we take $E^0 = E$ and $E^\infty = \emptyset$. From now on, unless otherwise stated, by a Lefschetz fibration we mean one with compact fiber that satisfies Definition 2.1.1 or, more generally, with a non-compact fiber that is convex at infinity and satisfies the conditions above, including $T_{\infty}$.

**Remark 2.1.2.** a. The assumption that the fiber of $E$ is either closed or symplectically convex was made in order to assure that the fiber is amenable to techniques of symplectic topology such as pseudo-holomorphic curves and Floer theory. Nevertheless in one instance later on in the paper we will drop this assumption and assume instead that $M$ is itself the total space of another Lefschetz fibration.

b. Assumption $T_{\infty}$ is a variant of boundary horizontality that appears in [Sei2] and [Sei3].
2.1.1. Connections, parallel transport and trails of Lagrangians. To a Lefschetz fibration as above we can associate a connection $\Gamma = \Gamma(\Omega_E)$ on $E \setminus \text{Crit} (\pi)$ as follows. The connection $\Gamma$ is defined by setting its horizontal distribution $\mathcal{H} \subset T(E)$ to be the $\Omega_E$-orthogonal complement of the tangent spaces to the fibers. More specifically, for every $x \in E \setminus \text{Crit} (\pi)$ we set

$$\mathcal{H}_x = \{ u \in T_x (E) \mid \Omega_E (\xi, u) = 0 \ \forall \xi \in T^v_x (E) \},$$

where $T^v_x (E)$ stands for the vertical tangent space at $x$.

The connection $\Gamma$ induces parallel transport maps. Let $\lambda : [a, b] \to \mathbb{C} \setminus \text{Crit} (\pi)$ be a smooth path. We denote by $\Pi_\lambda : E_{\lambda (a)} \to E_{\lambda (b)}$ the parallel transport along $\lambda$ with respect to the connection $\Gamma$. Notice that even when the fiber of $E$ is not compact, parallel transport is still well defined. Indeed, thanks to assumption $T_\infty$, the connection $\Gamma$ is trivial at infinity with respect to the trivialization $\phi$. In particular, relatively to the trivialization $\phi$, parallel transport becomes identity at infinity in the sense that $\phi^{-1} \circ \Pi_\lambda \circ \phi (\lambda (a), x) = (\lambda (b), x)$ for every $x \in E_\infty$.

By the results of [MS2, MS1] $\Pi_\lambda$ is a symplectomorphism (we endow here the fibers of $\pi$ with the symplectic structure induced by $\Omega_E$). If $\lambda$ is a loop starting and ending at $z \in \mathbb{C} \setminus \text{Crit} (\pi)$ then the symplectomorphism $\Pi_\lambda : E_z \to E_z$ is also called the holonomy of $\Gamma$ along $\lambda$. If the loop $\lambda$ is contractible (within $\mathbb{C} \setminus \text{Crit} (\pi)$) then the holonomy $\Pi_\lambda$ is in fact a Hamiltonian diffeomorphism of $E_z$ (see §6.4 of [MS1] for the proof).

Let $\lambda : [a, b] \to \mathbb{C} \setminus \text{Crit} (\pi)$ be a smooth embedding and $L \subset E_{\lambda (a)}$ a Lagrangian submanifold. Consider the images of $L$ under the parallel transport along $\lambda$, namely $L_t := \Pi_{\lambda |_{[a, t]}} (L) \subset E_{\lambda (t)}$, $t \in [a, b]$ and set

$$\lambda L := \cup_{t \in [a, b]} L_t.$$

Then $\lambda L$ is a Lagrangian submanifold of $(E, \Omega_E)$. We call $\lambda L$ the trail of $L$ along $\lambda$.

We refer the reader to [Sei3] for the foundations of the symplectic theory of Lefschetz fibrations and to [MS1] (Chapter 6) and [MS2] (Chapter 8) for more details on symplectic fibrations.

2.2. Lagrangians with cylindrical ends. Let $\pi : E \to \mathbb{C}$ be a Lefschetz fibration and $\mathcal{U} \subset \mathbb{C}$ an open subset containing $\text{Crit} (\pi)$. The following terminology is useful. A horizontal ray $\ell \subset \mathbb{C}$ is a half-line of the type $(-\infty, -a_\ell) \times \{ b_\ell \}$ or $[a_\ell, \infty) \times \{ b_\ell \}$ with $a_\ell > 0$, $b_\ell \in \mathbb{R}$. The imaginary coordinate $b_\ell$ is also referred to as the “height” of $\ell$.

**Definition 2.2.1.** A Lagrangian submanifold (without boundary) $V \subset (E, \Omega_E)$ is said to have cylindrical ends outside of $\mathcal{U}$ if the following conditions are satisfied:

i. For every $R > 0$, the subset $V \cap \pi^{-1} ([-R, R] \times \mathbb{R})$ is compact.

ii. $\pi (V) \setminus \mathcal{U}$ consists of a finite union of horizontal rays, $\ell_i \subset \mathbb{C}$, $i = 1, \ldots, r$. Moreover, for every $i$ we have $V |_{\ell_i} = L_i^{(\lambda)}$ for some Lagrangian $L_i \subset E_{\sigma_i}$, where $\sigma_i \in \mathbb{C}$ stands for
the starting point of the ray \( \ell_i \). (Note that we do allow \( r = 0 \), i.e. that \( V \) has no ends at all.)

In case all the heights of the rays \( \ell_i \) are positive integers \( b_i \in \mathbb{N}^* \) the Lagrangian \( V \) is called a *cobordism* in \( E \).

In short, over each one of the rays appearing in \( \pi(V) \setminus U \) the Lagrangian submanifold \( V \) is the trail under parallel transport of \( L_i \) along \( \ell_i \) - see Figure 2.

**Figure 2.** A Lagrangian \( V \) with cylindrical ends outside \( U \) in a Lefschetz fibration \( \pi : E \to \mathbb{C} \) with critical values \( v_i \).

The above notion of cobordism extends the definition of Lagrangian cobordism as given for the trivial fibration \( E \) in [BC2]. Note however that this terminology is slightly imprecise because we have not specified a (topological) trivialization of the fibration \( E \to \mathbb{C} \) at infinity (and in general there is no canonical trivialization). Moreover, even when one fixes such a trivialization the parallel transport along a ray \( \ell_i \) might not be trivial (even not at infinity), hence the actual ends of \( V \) at infinity are not well defined. In view of that, we will often work with a restricted type of Lefschetz fibrations, called tame, where this imprecision is not present and that have a number of additional technical advantages. We will see later on that this does not restrict the generality of our theory.

**Definition 2.2.2.** Let \( \pi : E \to \mathbb{C} \). Let \( U \subset \mathbb{C} \) be a closed subset, let \( z_0 \in \mathbb{C} \setminus U \) be a base point and \( (M, \omega_M) \) be the fiber over \( z_0 \). We say that this Lefschetz fibration is tame outside of \( U \) if there exists a trivialization

\[
\psi_{E, \mathbb{C} \setminus U} : (\mathbb{C} \setminus U) \times M \to E|_{\mathbb{C} \setminus U}
\]

such that \( \psi_{E, \mathbb{C} \setminus U}^*(\Omega_E) = c\omega_{\mathbb{C}} \oplus \omega_M \), where \( \omega_{\mathbb{C}} \) is the standard symplectic structure on \( \mathbb{C} \cong \mathbb{R}^2 \) and \( c > 0 \) is a constant. The manifold \( (M, \omega_M) \) is called the generic fiber of \( \pi \).
It follows from the definition that all the critical values of $\pi$ must be contained inside $U$. Sometimes it will be more natural to fix the complement of $U$, say $W = \mathbb{C} \setminus U$, and say that the fibration is tame over $W$. Given a tame Lefschetz fibration the set $U = U_E$, the point $z_0$ as well as the symplectic trivialization $\psi_{E,\mathbb{C} \setminus B}$ are viewed as part of the fixed data associated to the fibration.

Moreover, we will assume that the set $U = U_E$ is so that there exists $a_U > 0$ sufficiently large with the property that $U$ is disjoint from both quadrants:

$$Q_U^- = (-\infty, -a_U] \times [0, +\infty), \ Q_U^+ = [a_U, \infty) \times [0, +\infty)$$

The cobordism relation, as defined in [BC2], admits an obvious extension in a tame Lefschetz fibration.

**Definition 2.2.3.** Fix a Lefschetz fibration that is tame outside $U \subset \mathbb{C}$ with fiber $(M, \omega)$ over $z_0 \in \mathbb{C} \setminus U$. Let $(L_i)_{1 \leq i \leq k_-}$ and $(L'_j)_{1 \leq j \leq k_+}$ be two families of closed Lagrangian submanifolds of $M$. We say that these two families are Lagrangian cobordant in $E$, if there exists a Lagrangian submanifold $V \subset E$ with the following properties:

i. There is a compact set $K \subset E$ so that $V \cap U \subset V \cap K$ and $V \setminus K \subset \pi^{-1}(Q_U^+ \cup Q_U^-)$.

ii. $V \cap \pi^{-1}(Q_U^+) = \bigsqcup_j ([a_U, +\infty) \times \{j\}) \times L'_j$

iii. $V \cap \pi^{-1}(Q_U^-) = \bigsqcup_i ((-\infty, -a_U] \times \{i\}) \times L_i$

The formulas at ii and iii are written with respect to the trivialization of the fibration over the complement of $U$.

The manifold $V$ is obviously a Lagrangian cobordism in the sense of Definition 2.2.1 and - because of tameness - its ends at $\infty$ are well defined so that we can say that $V$ is a cobordism from the Lagrangian family $(L'_j)$ to the family $(L_i)$. We write $V : (L'_j) \rightsquigarrow (L_i)$ or $(V ; (L_i), (L'_j))$.

### 2.3. From general Lefschetz fibrations to tame ones.

We will now see that it is always possible to pass from a general Lefschetz fibration $\pi : E \longrightarrow \mathbb{C}$, as in §2.1, to a tame one.

**Proposition 2.3.1.** Let $\pi : E \longrightarrow \mathbb{C}$ be a Lefschetz fibration and let $\mathcal{N} \subset \mathbb{C}$ be an open subset that contains all the critical values of $\pi$ and has the shape depicted in Figure 3. Let $\mathcal{W} \subset \mathbb{C}$ be another open subset of the shape depicted in Figure 3 with $\overline{\mathcal{W}} \cap \overline{\mathcal{N}} = \emptyset$ and $\text{dist}(\overline{\mathcal{W}}, \overline{\mathcal{N}}) > 0$. Then there exists a symplectic structure $\Omega' = \Omega'_{E,\mathcal{N},\mathcal{W}}$ on $E$ and a trivialization $\phi : \mathcal{W} \times M \longrightarrow E|_{\mathcal{W}}$ with the following properties:

1. On $\mathcal{W} \times M$ we have $\phi^* \Omega' = c\omega_{\mathcal{W}} \oplus \omega_M$ for some $c > 0$.
2. $\Omega'$ coincides with $\Omega_E$ on all the fibers of $\pi$.
3. $\Omega' = \Omega_E$ on $\pi^{-1}(\mathcal{N})$. 

There exists an $\Omega'$-compatible almost complex structure $J'_E$ on $E$ which coincides with $J_E$ on $\pi^{-1}(\mathcal{N})$ and such that the projection $\pi : E \to \mathbb{C}$ is $(J'_E,i)$-holomorphic. In particular, when endowed with the symplectic structure $\Omega'$, the Lefschetz fibration $\pi : E \to \mathbb{C}$ is tame over $\mathcal{W}$.

Remark 2.3.2. It is easy to pass from a cobordism in a general Lefschetz fibration to a cobordism in a tame fibration.

Indeed, let $\pi : E \to \mathbb{C}$ be a Lefschetz fibration and $V \subset E$ a Lagrangian submanifold with cylindrical ends. Let $\mathcal{N} \subset \mathbb{C}$ be a subset as in Proposition 2.3.1 and assume that $V$ has cylindrical ends outside of $\mathcal{N}'$, where $\mathcal{N}' \subset \mathcal{N}$ is a slightly smaller subset than $\mathcal{N}$ which contains $\text{Critv}(\pi)$ and is of the same shape as $\mathcal{N}$. Denote the horizontal rays corresponding to the ends of $V$ by $\ell_i \subset \mathbb{C}$, $i = 1, \ldots, r$ and by $L_i \subset E_{\sigma_i}$ the corresponding Lagrangians over the starting points of these rays. Let $\mathcal{W} \subset \mathbb{C}$ be a subset as in Proposition 2.3.1 and consider the new symplectic structure $\Omega'$ on $E$ provided by that proposition. By performing parallel transport of the $L_i$'s along the horizontal rays $\ell_i$, but this time with respect to the connection corresponding to $(E,\Omega')$, we obtain a new Lagrangian submanifold $V' \subset (E,\Omega')$ with the following properties:

i. $V'$ coincides with $V$ over $\mathcal{N}$.

ii. $V'$ has cylindrical ends outside of $\mathcal{N}$. 

Figure 3. A Lefschetz fibration $\pi : E \to \mathbb{C}$; the domains $\mathcal{N}$ and $\mathcal{W}$ and, in red, the critical values of $\pi$. 
iii. Over $\mathcal{W}$, $V'$ looks like

$$V'|_{\mathcal{W}} = \bigcup_{i=1}^r \ell'_i \times L'_i,$$

where $\ell'_i = \ell_i \cap \mathcal{W}$ and $L'_i$ is the image of the parallel transport of $L_i$ (with respect to the connection $\Gamma(\Omega')$) along the portion of $\ell_i$ that connects $\mathcal{N}'$ with $\mathcal{W}$.

2.3.1. Preparation for the proof of Proposition 2.3.1. Let $(M, \omega)$ be a symplectic manifold, $Q \subset \mathbb{C}$ an open subset and $f : Q \times M \to \mathbb{R}$ a smooth function. We denote by $z = y_1 + iy_2$ the standard complex coordinate in $\mathbb{C}$. Let $\alpha = \{\alpha\}_{z \in Q}, \beta = \{\beta\}_{z \in Q}$ be two families of 1-forms on $M$, parametrized by $z \in Q$ (alternatively we can view $\alpha, \beta$ as differential forms on $Q \times M$ with $\alpha(\frac{\partial}{\partial y_1}) = \beta(\frac{\partial}{\partial y_1}) = 0$). For $z \in Q$, $p \in M$ we write $\alpha_{z,p}$ for the restriction of $\alpha_z$ to $T_p(M)$ and similarly for $\beta$. We denote by $d^\nu$ the exterior derivative of differential forms on $Q \times M$ in the $M$-direction (i.e. $(d^\nu \alpha)_z = d^M(\alpha_z)$, where $d^M$ is the exterior derivative in $M$.) Below we will abbreviate the partial derivatives $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}$ by $\partial_{y_1}, \partial_{y_2}$.

Consider now the following 2-form on $Q \times M$

$$\Omega^{f,\alpha,\beta} := \omega + f dy_1 \wedge dy_2 + \alpha \wedge dy_1 + \beta \wedge dy_2.$$

A simple calculation shows that:

**Lemma 2.3.3.** $\Omega^{f,\alpha,\beta}$ is closed iff $d^\nu \alpha = d^\nu \beta = 0$ and $d^\nu f = \partial_{y_2} \alpha - \partial_{y_1} \beta$.

Define now two families of vector fields $u_0, v_0$ on $M$ (parametrized by the points of $Q$) as follows. For every $z \in Q$, $p \in M$, define $u_0(z,p), v_0(z,p) \in T_p(M)$ by requiring that for every $\xi \in T_p(M)$ we have:

$$\omega_p(\xi, u_0(z,p)) + \alpha_{z,p}(\xi) = 0, \quad \omega_p(\xi, v_0(z,p)) + \beta_{z,p}(\xi) = 0. \quad (5)$$

Denote by $\mathcal{H} \subset T(Q \times M)$ the following 2-dimensional distribution:

$$\mathcal{H}_{z,p} := \mathbb{R}\left(\frac{\partial}{\partial y_1} + u_0(z,p)\right) + \mathbb{R}\left(\frac{\partial}{\partial y_2} + v_0(z,p)\right). \quad (6)$$

Note that $\mathcal{H}$ depends on $\omega, \alpha, \beta$ but not on $f$.

The following two lemmas can be proved by direct calculation.

**Lemma 2.3.4.** For every $(z,p) \in Q \times M$, $\xi \in T_p(M)$ and $w \in \mathcal{H}_{z,p}$ we have $\Omega^{f,\alpha,\beta}(\xi, w) = 0$. In particular, if $\Omega^{f,\alpha,\beta}$ is non-degenerate then $\mathcal{H}$ is the horizontal distribution of the connection induced by $\Omega^{f,\alpha,\beta}$.

**Lemma 2.3.5.** Assume that $f(z,p) \neq \omega_p(u_0(z,p), v_0(z,p))$ for some $(z,p) \in Q \times M$. Then $\Omega^{f,\alpha,\beta}$ is non-degenerate at $(z,p)$. Moreover, there exists an $\Omega_{z,p}^{f,\alpha,\beta}$-compatible complex structure $J_{z,p}$ on $T_{z,p}(Q \times M)$ such that the projection $Q \times M \to Q$ is $(J_{z,p}, i)$-holomorphic at $(z,p)$ if and only if $f(z,p) > \omega_p(u_0(z,p), v_0(z,p))$. 

2.3.2. Proof of Proposition 2.3.1.

To fix ideas, we first provide the proof in the case of compact fibre.

**Step 1.** Using parallel transport with respect to the connection $\Gamma_{\Omega_E}$ along a system of curves in $\mathbb{C} \setminus \mathcal{N}$ emanating from a fixed point $z_0 \in \mathcal{W}$, and using the fact that $\mathbb{C} \setminus \mathcal{N}$ is contractible we obtain a trivialization

$$\varphi : (\mathbb{C} \setminus \mathcal{N}) \times M \rightarrow E|_{\mathbb{C} \setminus \mathcal{N}}$$

with $M = \pi^{-1}(z_0)$ and with the property that the form $\Omega_1 := \varphi^*\Omega_E$ admits the following form

$$\Omega_1 = f dy_1 \wedge dy_2 + \alpha \wedge dy_1 + \beta \wedge dy_2 + \omega,$$

where $\omega = \Omega|_M$ and $f : (\mathbb{C} \setminus \mathcal{N}) \times M \rightarrow \mathbb{R}$ is a smooth function, and $\alpha, \beta$ are vertical 1-forms on $(\mathbb{C} \setminus \mathcal{N}) \times M$ with the property that for every $z \in \mathbb{C} \setminus \mathcal{N}$ the 1-forms $\alpha_z = \alpha|_{z \times M}$, $\beta_z = \beta|_{z \times M}$ are exact (see § 8.2 of [MS2] and § 6.4 of [MS1] for a proof of that). Fix two functions $F, G : (\mathbb{C} \setminus \mathcal{N}) \times M \rightarrow \mathbb{R}$ such that $\alpha = d^v F$, $\beta = d^v G$.

By Lemma 2.3.3 we have:

$$d^v f = \partial_{y_2} \alpha - \partial_{y_1} \beta.$$

**Step 2.** Apart from $\mathcal{W}$ and $\mathcal{N}$ we will fix three additional open subsets $\mathcal{W}_\epsilon, \mathcal{N}_\epsilon, \mathcal{N}_2\epsilon$ with

$$\overline{\mathcal{W}} \subset \mathcal{W}_\epsilon, \quad \overline{\mathcal{N}} \subset \mathcal{N}_\epsilon, \quad \overline{\mathcal{N}}_\epsilon \subset \mathcal{N}_{2\epsilon},$$

and with shapes as described in Figure 4. To be more precise, consider the curves $\gamma_1, \gamma_2, \gamma_3 \subset \mathbb{C}$ depicted in Figure 4. The domain $\mathcal{N}_\epsilon$ is defined to be the connected component of $\mathbb{C} \setminus \gamma_1$ in which all the points have bounded real coordinate. The domain $\mathcal{N}_{2\epsilon}$ is defined similarly but with the curve $\gamma_1$ replaced by $\gamma_2$. The domain $\mathcal{W}_\epsilon$ is defined as the connected component of $\mathbb{C} \setminus \gamma_3$ in which the real coordinate of the points is unbounded. We also require that $\text{dist}(\overline{\mathcal{W}}_\epsilon, \overline{\mathcal{N}}_{2\epsilon}) > 0$.

**Step 3.** We will modify now the form $\Omega_1$ in the following way. Fix a smooth function $\sigma : \mathbb{C} \rightarrow [0, 1]$ such that:

$$\sigma(z) = \begin{cases} 1 & z \in \mathcal{N}_{2\epsilon}, \\ 0 & z \in \mathcal{W}_\epsilon. \end{cases}$$

Define $g : \mathbb{C} \times M \rightarrow \mathbb{R}$ by

$$g(z, p) = \partial_{y_2}(\sigma) F(z, p) - \partial_{y_1}(\sigma) G(z, p).$$

Then we have:

$$g(z, p) = 0 \quad \forall z \in \mathcal{N}_{2\epsilon} \cup \mathcal{W}_\epsilon.$$
Next, choose a function $A : \mathbb{C} \rightarrow \mathbb{R}$ with the following properties:

(A.1) $A(z) \geq 0$ for every $z \in \mathbb{C}$.
(A.2) $A(z) = 0$ for every $z \in \mathcal{N}_\epsilon$.
(A.3) $A(z) \geq |g(z, p)|$ for every $z \in \mathbb{C}$, $p \in M$.
(A.4) Let $u_0, v_0$ be the vector fields associated to the form $\Omega_1 = \Omega^{f,\alpha,\beta}$ from (7) using the recipe from (5). We require that

$$A(z) > \sigma(z) \left| f(z, p) - \sigma(z) \omega_p(u_0(z, p), v_0(z, p)) \right| + |g(z, p)|$$

for every $z \in \mathbb{C} \setminus \mathcal{N}_{2\epsilon}, p \in M$.
(A.5) $A(z) = C$ for every $z \in \mathcal{W}$, for some constant $C > 0$.

Such a function $A$ can be constructed as follows. We start by defining a function $A' : \mathbb{C} \rightarrow \mathbb{R}$ which is positive and satisfies the following condition:

$$A'(z) > \sigma(z) \left| f(z, p) - \sigma(z) \omega_p(u_0(z, p), v_0(z, p)) \right| + |g(z, p)| \quad \forall z \in \mathbb{C} \setminus \mathcal{N}_{2\epsilon}.$$
Such a function obviously exists because $M$ is compact. We then cut $A'$ off to make it 0 on $\mathcal{N}_\varepsilon$ and constant on $\mathcal{W}$, where the cutting off takes place within $\mathcal{N}_{2\varepsilon} - \mathcal{N}_\varepsilon$ and within $\mathcal{W}_\varepsilon - \mathcal{W}$, where the function $g$ is 0 anyway. The function resulting from $A'$ after this procedure can be taken to be the desired function $A$. See Figure 5.

Finally, define:

$$f'(z, p) := \sigma(z) f(z, p) + g(z, p) + A(z),$$

(12)

$$\alpha'_{z, p} := \sigma(z) \alpha_{z, p} = d^v(\sigma(z) F)_{z, p},$$

$$\beta'_{z, p} := \sigma(z) \beta_{z, p} = d^v(\sigma(z) G)_{z, p}.$$  

Consider now the form

$$\Omega_2 := \Omega^{f', \alpha', \beta'} = f' dy_1 \wedge dy_2 + \alpha' \wedge dy_1 + \beta' \wedge dy_2 + \omega.$$
Note that $\Omega_2$ coincides with $\Omega_1$ over a small neighborhood of $\overline{N}$ and therefore $\Omega_2$ gives rise via the trivialization $\varphi$ to a well defined 2-form $\Omega'$ over the entire of $E$. Moreover $\Omega'$ coincides with $\Omega$ on $\pi^{-1}(N)$.

We claim that $\Omega'$ is a symplectic form on $E$ and that it satisfies all the properties claimed by Proposition 2.3.1.

We first show that $\Omega$ coincides with $\Omega'$ over a small neighborhood of $\overline{N}$.

By the construction of the function $A$ we have

$$
\frac{\partial}{\partial y_2} (\sigma_\alpha) = \sigma (\partial_2 (\alpha) - \partial_1 (\beta) - \beta) - \frac{\partial}{\partial y_1} (\sigma_\beta) + \left( \frac{\partial}{\partial g} (\sigma_\beta) - \partial_1 (\sigma_\beta) \right)
$$

Thus by Lemma 2.3.5 we only need to check that:

$$
\forall p \in M, \ z \in \mathbb{C} \setminus \overline{N}.
$$

We denote by $T_1 = \sigma (f(z, p) - \omega_p (u_0(z, p), v_0(z, p)))$ the first term on the last line of (14) and by $T_2 = g(z, p) + A(z)$ the second one.

We first verify (13) over $\pi^{-1}(\mathcal{W}_\epsilon)$. Indeed, when $z \in \mathcal{W}_\epsilon$ we have $\sigma(z) = 0$ hence $T_1 = 0$.

By the construction of the function $A$ we have $T_2 > 0$, hence $T_1 + T_2 > 0$.

Next we check (13) over $\pi^{-1}(\mathcal{N}_2 \setminus \overline{N})$. Let $z \in \mathcal{N}_2 \setminus \overline{N}$ and $p \in M$. Note that $\sigma(z) = 1$ hence $T_1 = f(z, p) - \omega_p (u_0(z, p), v_0(z, p)) > 0$ by Lemma 2.3.5. Since $T_2 \geq 0$ we have $T_1 + T_2 > 0$.

Finally, the inequality (13) for $z \in \mathbb{C} \setminus \left( \mathcal{N}_2 \cup \mathcal{W}_\epsilon \right)$ follows easily from requirement (A.4) in the construction of the function $A$.

To finish the proof, we turn to the case of a non-compact fibre. Thus we assume the conditions in §2.1 and, in particular, assumption $T_\infty$. The proof above applies in this case too, and we will preserve all the notation above, but there are a number of adjustments that we describe below. Recall the set $E^\infty$ that appears in the assumption $T_\infty$ and put $M^\infty = M \cap E^\infty$. Recall also that, as before, $M = \pi^{-1}(z_0)$.

$$
\phi : \mathbb{C} \times M^\infty \to E^\infty
$$
be the trivialization provided by $T_\infty$. Consider also the restriction of this trivialization to $\mathbb{C} \setminus \mathcal{N}$:

\begin{equation}
\phi : (\mathbb{C} \setminus \mathcal{N}) \times M^\infty \to E^\infty|_{\mathbb{C} \setminus \mathcal{N}}
\end{equation}

and put $\phi_0 : M^\infty \to M^\infty$, $\phi_0(p) = \phi(z_0, p)$.

Consider also the map $\varphi$ constructed at the \textbf{Step 1} above and its restriction:

$$
\varphi : (\mathbb{C} \setminus \mathcal{N}) \times M^\infty \to E^\infty|_{\mathbb{C} \setminus \mathcal{N}}
$$

which is well defined due to Assumption $T_\infty$.

Given that the connection associated to $\varphi^* \Omega$ is trivial on $(\mathbb{C} \setminus \mathcal{N}) \times M^\infty$, we deduce that $\varphi(z, p) = \phi(z, \phi_0^{-1}(p))$ for all $z \in \mathbb{C} \setminus \mathcal{N}$, $p \in M^\infty$. Therefore $\varphi^* \Omega|_{(C \setminus N) \times M^\infty} = \omega_C \oplus \omega$.

Recall that over $(\mathbb{C} \setminus \mathcal{N}) \times M$ we also have

$$
\Omega = \omega + \alpha \wedge dy_1 + \beta \wedge dy_2 + fdy_1 \wedge dy_2 .
$$

This means that $\alpha, \beta$ vanish over $(\mathbb{C} \setminus \mathcal{N}) \times M^\infty$ and $f$ is constant there. Therefore, we can choose the functions $F, G$ so that they both vanish on $(\mathbb{C} \setminus \mathcal{N}) \times M^\infty$. Starting from this point the remainder of the proof continues as in the compact fibre case by using the fact that $g(z, p)$, as well as $\alpha', \beta', u_0(z, p), v_0(z, p)$ all vanish over $(\mathbb{C} \setminus \mathcal{N}) \times M^\infty$.

Summing up, the form $\Omega_2$ hence also $\Omega'$ are of the form $\varphi^* \Omega' = B(z)\omega_C \oplus \omega$ over $\mathbb{C} \times M^\infty$, where $B(z)$ is positive and bounded. By adding to $\Omega'$ another term of the form $D(z)\pi^* \omega_C$ we obtain a form that verifies all the properties claimed in Proposition 2.3.1 as well as the assumption $T_\infty$.

\[\square\]

### 3. Fukaya categories

The purpose of this section is to introduce the various Fukaya categories that play a role in the paper. We start with a brief sketch of the construction of the Fukaya category $\mathcal{F}uk^*(M)$ of uniformly monotone, closed Lagrangian submanifolds of a symplectic manifold $(M, \omega)$ which is assumed to be either closed or tame. The full construction in the exact case can be found in [Sei3] (the minor adjustments required in the monotone case are described, for instance, in [BC3]). In §3.2, we pursue with the construction of the Fukaya category $\mathcal{F}uk^*(E)$ of uniformly monotone cobordisms in a tame Lefschetz fibration $\pi : E \to \mathbb{C}$ of generic fiber $(M, \omega)$. This follows closely §3 of [BC3] where this construction is implemented for the trivial fibration $E = \mathbb{C} \times M$. The passage from a trivial fibration to a tame one is quite straightforward but we provide enough details on this construction as required for further arguments later in the paper and also to insure that the notions involved are accessible to a reader without prior detailed knowledge of the techniques in [BC3]. In §3.3 we use the construction in the tame
setting together with the results in §2.3 to define a Fukaya category associated to a general Lefschetz fibration.

In the definition of the various algebraic objects used in the paper there are two coefficient rings of interest, \( \mathbb{Z}_2 \) and the universal Novikov ring \( \mathcal{A} \) over \( \mathbb{Z}_2 \):

\[
\mathcal{A} = \left\{ \sum_{k=0}^{\infty} a_k T^{\lambda_k} : a_k \in \mathbb{Z}_2, \, \lambda_k \in \mathbb{R}, \, \lim_{k \to \infty} \lambda_k \to \infty \right\}.
\]

We work over \( \mathcal{A} \) at all times except if otherwise indicated.

3.1. The Fukaya category of \( M \). The main structures in use in the paper are the Fukaya category, \( \mathcal{F}uk^*(\cdot) \), and the derived Fukaya category, \( D\mathcal{F}uk^*(\cdot) \). Here \( \ast \) encodes a uniform monotonicity constraint imposed to the objects of \( \mathcal{F}uk^*(M) \). This constraint is necessary to define the \( A_{\infty} \)-operations.

The book [Sei3] is a comprehensive reference for the basic definitions of the \( A_{\infty} \) machinery as well as the construction of the Fukaya category and its derived version. Our notation - which is homological, in contrast to Seidel’s which is cohomological - and the notation in use is exactly the same as in [BC3], see in particular the Appendix. There is a single difference with respect to [BC3] which is that we use here the universal Novikov ring \( \mathcal{A} \) in the place of \( \mathbb{Z}_2 \). As we shall see, this is not a matter of choice, rather a requirement for a certain part of our results to hold. We emphasize that in the construction of \( D\mathcal{F}uk^*(\cdot) \) we do not complete with respect to idempotents. Moreover, as in [BC3] we work in an ungraded context.

Fix a symplectic manifold \((M, \omega)\), compact or tame at infinity. Given a Lagrangian submanifold \( L \subset M \) there are two morphisms

\[
\mu : \pi_2(M, L) \to \mathbb{Z}, \, \omega : \pi_2(M, L) \to \mathbb{R}
\]

given, the first, by the Maslov index and, the second, by integration of \( \omega \). We say that \( L \) is monotone if \( \omega(\alpha) = \rho \mu(\alpha) \) for some constant \( \rho > 0 \) and if the number

\[
N_L = \min\{\mu(\alpha) : \alpha \in \pi_2(M, L), \, \omega(\alpha) > 0\}
\]

is at least 2.

For a connected monotone Lagrangian \( L \) and for a generic almost complex structure \( J \) compatible with \( \omega \), the number \((\text{mod} 2)\) of \( J \)-holomorphic disks of Maslov number 2 that pass through a generic point of \( L \) is an invariant (in the sense that it does not depend either on the point or on the choice of \( J \)). It is denoted by \( d_L \) (and is defined in detail, for instance, in [BC1]).
In order to define the Fukaya category of $M$ we first need to specify its underlying class of Lagrangian submanifolds. In what follows we will mainly consider two classes of Lagrangians $\mathcal{L}^{(0)}(M)$ and $\mathcal{L}^{(\rho,1)}$, which are defined as follows:

a. The class $\mathcal{L}^{(0)}(M)$: this class consists of all closed monotone Lagrangians $L \subset M$ with $d_L = 0$. This includes in particular also Lagrangians with $N_L \geq 3$. Note also that weakly exact Lagrangians fall into this class too.

b. Class $\mathcal{L}^{(\rho,1)}(M)$: consists of all the closed monotone Lagrangians $L \subset M$ with $d_L = 1$ and with monotonicity constant $\rho$, where $\rho > 0$ is a prescribed positive real number.

Of course one could restrict also to some subclasses of the above. For example, when $M$ is exact it makes sense to restrict to the subclass $\mathcal{L}^{(\text{ex})}(M) \subset \mathcal{L}^{(0)}(M)$ of exact Lagrangian submanifolds.

To simplify the notation will denote any of these two choices by $\mathcal{L}^*(M)$, where the symbol $*$ stands for either $(0)$ in the first case, or for $(\rho,1)$ in the second case. Lagrangians in the class $\mathcal{L}^*(M)$ will be called *uniformly monotone* of class $*$.

In what follows we will work also with uniformly monotone *negatively-ended* Lagrangian cobordisms in the total space of a Lefschetz fibration $E \rightarrow \mathbb{C}$. Similarly to the Lagrangians in $M$ we will denote the various classes of uniformly monotone Lagrangian cobordisms in $E$ by $\mathcal{L}^*(E)$, where the definition of these classes is the same as above except that the Lagrangians in $E$ are not assumed to be compact.

Floer homology will be taken in this paper with coefficients in the Novikov ring $\mathcal{A}$ and its definition will be shortly reviewed below. It was introduced by Floer in [Flo] and, in this monotone setting, by Oh [Oh1, Oh2].

**Remarks.**

a. In contrast to [BC3] there is no injectivity condition on the inclusions $\pi_1(L) \rightarrow \pi_1(M)$ (this is because the coefficient ring is $\mathcal{A}$ and not $\mathbb{Z}_2$).

b. In case there exists a spherical class $A \in \pi_2(M)$ with $\omega(A) > 0$, the monotonicity constant $\rho$ is determined by the proportionality constant between $[\omega]$ and the first Chern class of the ambient symplectic manifold. Thus in this case there is only one class of the type $\mathcal{L}^{(\rho,1)}$.

The Fukaya $A_\infty$-category $\mathcal{F}uk^*(M)$ has as objects the Lagrangians in $\mathcal{L}^*(M)$,

$$\text{Ob}(\mathcal{F}uk^*(M)) = \mathcal{L}^*(M).$$

Let $L, L' \in \mathcal{L}^*(M)$ and assume for the moment that $L$ and $L'$ intersect transversely. In this case, the Floer complex, $(CF(L, L'; J), d)$, associated to $L$ and $L'$ is defined by choosing a regular almost complex structure $J$ compatible with $\omega$ and is a free $\mathcal{A}$-module with generators the intersection points of $L$ and $L'$. In this paper $CF(L, L')$ is a complex without grading.
The differential $d$ is defined in terms of $J$-holomorphic strips $u : \mathbb{R} \times [0,1] \to M$ so that $u(\mathbb{R} \times \{0\}) \subset L$ and $u(\mathbb{R} \times \{1\}) \subset L'$ and so that $\lim_{s \to -\infty} u(s,t) = x \in L \cap L'$, $\lim_{s \to +\infty} u(s,t) = y \in L \cap L'$. We have:

$$d(x) = \sum_y \sum_{u \in \mathcal{M}_0(x,y)} T^{\omega(u)} y$$

where the sum is over all the intersection points $y \in L \cap L'$ and $\mathcal{M}_0(x,y)$ is the 0-dimensional subspace of the moduli space of $J$-strips $u$ joining $x$ to $y$. Uniform monotonicity is used to show that $d^2 = 0$.

The homology of this complex, $HF(L,L')$, is the Floer homology of $L$ and $L'$. It is independent of $J$ as well as of Hamiltonian perturbation of $L$ and of $L'$.

The morphisms in $\mathcal{F}uk^*(M)$ are (basically) $\text{Mor}_{\mathcal{F}uk^*(M)}(L,L') = CF(L,L')$. The $A_\infty$ structural maps are, by the definition of an $A_\infty$-category, multilinear maps

$$\mu_k : CF(L_1,L_2) \otimes CF(L_2,L_3) \otimes \ldots \otimes CF(L_k,L_{k+1}) \to CF(L_1,L_{k+1})$$

that satisfy the relation $\mu \circ \mu = \sum \mu(-,-,\ldots,\mu,\ldots,-,-) = 0$. In our case, these maps are so that $\mu_1 = d =$ the Floer differential and, for $k > 1$, $\mu_k$ is defined by:

$$\mu_k(\gamma_1,\ldots,\gamma_k) = \sum_{\gamma} \sum_{u \in \mathcal{M}_0(\gamma_1,\ldots,\gamma_k;\gamma)} T^{\omega(u)} \gamma$$

Here, at least when the $L_i$’s and $L$ are in generic position, $\gamma_i \in L_i \cap L_{i+1}$, $\gamma \in L_1 \cap L_{k+1}$ and $\mathcal{M}_0(\gamma_1,\ldots,\gamma_k;\gamma)$ is a 0-dimensional moduli space of (perturbed) $J$-holomorphic polygons with $k+1$ sides that have $k$ “inputs” asymptotic - in order - to the intersection points $\gamma_i$ and one “exit” asymptotic to $\gamma$. Monotonicity is used to show that the sums appearing in the definition of the $\mu_k$’s are well defined over $\mathcal{A}$. The relation $\mu \circ \mu = 0$ extends the relation $d^2 = 0$.

This is just a rough summary of the construction as, in particular, the operations $\mu_k$ have to be defined for all families $L_1,\ldots,L_{k+1}$ and not only when $L_i, L_{i+1}$, etc., are transverse; moreover, the regularity of these moduli spaces depends on a number of choices of auxiliary data, basically a coherent system of strip-like ends and coherent perturbation data. We refer to [Sei3] for the actual implementation of the construction which is considerably more involved. Additionally, these notions are made more precise in §3.2 where we discuss in more detail some of the ingredients used in the construction of a Fukaya category $\mathcal{F}uk^*(E)$ with objects certain cobordisms in $E$.

Consider next the category of $A_\infty$-modules over the Fukaya category

$$\text{mod}(\mathcal{F}uk^*(M)) := \text{fun}(\mathcal{F}uk^*(M), Ch^{\text{opp}})$$
where $\text{Ch}^{\text{opp}}$ is the opposite of the dg-category of chain complexes over $\mathcal{A}$. The category of $A_\infty$-modules is an $A_\infty$-category in itself and is triangulated in the $A_\infty$-sense with the triangles being inherited from the triangles in $\text{Ch}$ (where they correspond to the usual cone-construction for chain complexes). There is a Yoneda embedding $Y : \text{Fuk}^*(M) \to \text{mod} (\text{Fuk}^*(M))$, the functor associated to an object $L \in \mathcal{L}^*(M)$ being $CF(-, L)$. The derived Fukaya category $DFuk^*(M)$ is the homology category associated to the triangulated completion of the image of the Yoneda embedding inside $\text{mod} (\text{Fuk}^*(M))$.

3.1.1. *Iterated cone decompositions.* We now briefly fix the notation for writing iterated cone-decompositions in a triangulated category $\mathcal{C}$. Suppose that there are exact triangles:

$$C_{i+1} \to Z_i \to Z_{i+1}$$

with $1 \leq i \leq n$ and with $X = Z_{n+1}$, $Z_0 = C_0$. We write such an iterated cone-decomposition as

$$X = (C_{n+1} \to (C_n \to (C_{n-1} \to \ldots \to C_0)) \ldots) .$$

With this notation

$$Z_k = (C_k \to (C_{k-1} \to \ldots \to C_0)) \ldots) .$$

We also notice that we can in fact omit the parentheses in this notation without ambiguity. This follows from the following equality of the two iterated cones:

$$((A \to B) \to C) = (A \to (B \to C)) .$$

In turn, this follows immediately from the axioms of a triangulated category together with the fact that we work here in an ungraded setting (the formula can also be easily adjusted to the graded case). In short, we will write:

$$X = (C_{n+1} \to C_n \to C_{n-1} \to \ldots \to C_0) .$$

There is a slight abuse of notation in the above formula in that, in the absence of the relevant parentheses, the arrows in the formula do not independently correspond to morphisms in the category $\mathcal{C}$. The formula should be interpreted as saying that $X$ can be expressed as an iterated cone attachment with the objects $C_0, \ldots, C_{n+1}$ as described above.

3.1.2. *The Grothendieck group.* The Grothendieck group of a triangulated category $\mathcal{C}$ is the abelian group generated by the objects of $\mathcal{C}$ modulo the relations generated by $B = A + C$ as soon as

$$A \to B \to C$$
is an exact triangle. We denote the Grothendieck group of $\mathcal{C}$ by $K_0(\mathcal{C})$. Notice that, with our terminology, if

$$L_1 = (L_n \to L_{n-1} \to L_{n-2} \to \ldots \to L_2),$$

then, because we work in an ungraded setting, in $K_0(\mathcal{C})$ we have the relation $L_n + L_{n-1} + \ldots + L_1 = 0$. Notice also that, due to the same reason, our version of $K_0(\mathcal{C})$ is always 2-torsion, i.e. $2A = 0$ for every $A \in K_0(\mathcal{C})$.

The main Grothendieck groups of interest in this paper will be those of derived Fukaya categories $K_0DFuk^*(-)$.

3.2. The Fukaya category of negative ended cobordisms in tame Lefschetz fibrations. We consider a Lefschetz fibration $\pi : E \to \mathbb{C}$ that is tame outside $U \subset \mathbb{C}$ and has as generic fibre the symplectic manifold $(M, \omega)$. We will also assume that

$$(16)\quad U \subset \mathbb{R} \times [0, +\infty)$$

The main object of study in this paper is the Fukaya category $\mathcal{F}uk^*(E)$. It has as objects the cobordisms $V$ as in Definition 2.2.3 so that the following additional conditions are satisfied:

i. $V$ is monotone in the class $\ast$.

ii. $V \subset \pi^{-1}(\mathbb{R} \times \left[\frac{1}{2}, +\infty\right))$

iii. $V$ has only negative ends that all belong to $\mathcal{L}^*(M)$. In particular, with the notation from Definition 2.2.3, $k_+ = 1$ and $L'_1 = \emptyset$.

This family of Lagrangians of $E$ with the properties above will be denoted by $\mathcal{L}^*(E)$. In other words, $\text{Ob}(\mathcal{F}uk^*(E)) = \mathcal{L}^*(E)$. Such an object is represented schematically in Figure 6.

**Figure 6.** The projection on $\mathbb{C}$ of an object $V \in \text{Ob}(\mathcal{F}uk^0_0(E))$ together with the set $U$ outside which $\pi$ is tame.

We call the objects $V \in \mathcal{L}^*(E)$ *negatively-ended cobordisms*: they are cobordisms from the void set to a family $(L_1, \ldots, L_s)$. 
Remark 3.2.1.  

a. We restrict in this paper to negatively-ended cobordisms but this is more a matter of convenience than of necessity. Some of the arguments in the paper are simpler in this setting but the same type of constructions allow the definition of a Fukaya category with both negative and positive ends. Similarly, our decomposition results can also be adapted to this more general setting. We do not require $V$ to be connected. Notice also that every Lagrangian cobordism $V \subset E$ that contains positive ends can be transformed to a negatively-ended cobordism by e.g. bending its positive ends along curves that turn to the left, then go above the singularities of $E$ and continue horizontally to $-\infty$.

b. We remark that our notation $L^*(E)$ and $\mathcal{F}uk^*(E)$ somewhat differ from the one used in [BC3]. In that paper we studied Lagrangian cobordisms in trivial fibrations $E = \mathbb{C} \times M$ and denoted by $\mathcal{CL}_d(\mathbb{C} \times M)$ the collection of monotone Lagrangian cobordisms in $\mathbb{C} \times M$ (with possibly negative and positive ends). The corresponding Fukaya category was denoted by $\mathcal{F}uk^{d,\text{cob}}(\mathbb{C} \times M)$. Thus, in the present paper, we could have denoted our $L^*(E)$ by $\mathcal{CL}^{\text{null}}(\mathbb{C} \times M)$ and $\mathcal{F}uk^*(E)$ by $\mathcal{F}uk^{*,\text{null}}(\mathbb{C} \times M)$, but we have decided to drop the additional decorations in order to keep the notation simpler.

The operations $\mu_k$ of the Fukaya category $\mathcal{F}uk^*(E)$ are defined following closely the construction in [BC3] which is basically a variant of the set-up in Seidel’s book [Sei3]. We review here the technical points that will be needed later in the paper. We will first focus on the case when $M$ is compact and we will discuss the additional modifications required when $M$ is convex at infinity at the end of the construction. There are two structures that need to be added compared to the construction of the category $\mathcal{F}uk^*(M)$: transition functions associated to a system of strip-like ends and profile functions. As always, the operations $\mu_k$ are defined in terms of counting (with coefficients in $\mathcal{A}$) perturbed $J$-holomorphic polygons $u$. The role of the transition functions is to allow such $u$ to be transformed by a change of variables into curves $v$ that project holomorphically onto certain regions of $\mathbb{C}$. The role of the profile functions - and particularly that of their bottlenecks - is to insure compactness at infinity for the Floer complexes $CF(V,V')$ and to further restrict the behavior of the $J$-polygons $u$. We explain this point, which is crucial for the arguments used later in the paper, at the end of §3.2.

3.2.1. Transition functions. We first recall the notion of a consistent choice of strip-like ends from [Sei3]. Fix $k \geq 2$. Let $\text{Conf}_{k+1}(\partial D)$ be the space of configurations of $(k+1)$ distinct points $(z_1, \ldots, z_{k+1})$ on $\partial D$ that are ordered clockwise. Denote by $\text{Aut}(D) \cong \text{PLS}(2,\mathbb{R})$ the group of holomorphic automorphisms of the disk $D$. Let

$$\mathcal{R}^{k+1} = \text{Conf}_{k+1}(\partial D)/\text{Aut}(D) \, , \, \hat{S}^{k+1} = (\text{Conf}_{k+1}(\partial D) \times D)/\text{Aut}(D).$$
The projection $\mathcal{S}^{k+1} \to \mathcal{R}^{k+1}$ has sections $\zeta_i[z_1, \ldots, z_{k+1}] = [(z_1, \ldots, z_{k+1}), z_i], i = 1, \ldots, k + 1$ and let $\mathcal{S}^{k+1} = \mathcal{S}^{k+1} \setminus \bigcup_{i=1}^{k+1} \zeta_i(\mathcal{R}^{k+1})$. The fiber bundle $\mathcal{S}^{k+1} \to \mathcal{R}^{k+1}$ is called a universal family of $(k + 1)$-pointed disks. Its fibers $S_r, r \in \mathcal{R}^{k+1}$, are called $(k + 1)$-pointed (or punctured) disks.

Let $Z^+ = [0, \infty) \times [0, 1], Z^- = (-\infty, 0) \times [0, 1]$ be the two infinite semi-strips and let $S$ be a $(k + 1)$ pointed disk with punctures at $(z_1, \ldots, z_{k+1})$. A choice of strip-like ends for $S$ is a collection of embeddings: $\epsilon_i^S : Z^- \to S, 1 \leq i \leq k, \epsilon_{k+1}^S : Z^+ \to S$ that are proper and holomorphic and

$$\begin{align*}
(\epsilon_i^S)^{-1}(\partial S) &= (-\infty, 0) \times \{0, 1\}, \quad \lim_{s \to \infty} \epsilon_i^S(s, t) = z_i, \quad \forall 1 \leq i \leq k, \\
(\epsilon_{k+1}^S)^{-1}(\partial S) &= (0, \infty) \times \{0, 1\}, \quad \lim_{s \to -\infty} \epsilon_{k+1}^S(s, t) = z_{k+1}.
\end{align*}$$

Thus so that the $\epsilon_i^S$’s have pairwise disjoint images. A universal choice of strip-like ends for $\mathcal{S}^{k+1} \to \mathcal{R}^{k+1}$ is a choice of $k + 1$ proper embeddings $\epsilon_i^S : \mathcal{R}^{k+1} \times Z^- \to \mathcal{S}^{k+1}, i = 1, \ldots, k, \epsilon_{k+1}^S : \mathcal{R}^{k+1} \times Z^+ \to \mathcal{S}^{k+1}$ such that for every $r \in \mathcal{R}^{k+1}$ the restrictions $\epsilon_i^S|_{r \times Z^\pm}$ consists of a choice of strip-like ends for $S_r$. See [Sei3] for more details. In the case $k = 1$, we put $\mathcal{R}^2 = \text{pt}$ and $\mathcal{S}^2 = D \setminus \{-1, 1\}$. We endow $D \setminus \{-1, 1\}$ with strip-like ends by identifying it holomorphically with the strip $\mathbb{R} \times [0, 1]$ endowed with its standard complex structure.

Pointed disks with strip-like ends can be glued in a natural way. Further, the space $\mathcal{R}^{k+1}$ has a natural compactification $\overline{\mathcal{R}}^{k+1}$ described by parametrizing the elements of $\overline{\mathcal{R}}^{k+1} \setminus \mathcal{R}^{k+1}$ by trees [Sei3]. The family $\mathcal{S}^{k+1} \to \mathcal{R}^{k+1}$ admits a partial compactification $\overline{\mathcal{S}}^{k+1} \to \overline{\mathcal{R}}^{k+1}$ which can be endowed with a smooth structure. Moreover, the fixed choice of universal strip-like ends for $\mathcal{S}^{k+1} \to \mathcal{R}^{k+1}$ admits an extension to $\overline{\mathcal{S}}^{k+1} \to \overline{\mathcal{R}}^{k+1}$. Further, these choices of universal strip-like ends for the spaces $\mathcal{R}^{k+1}$ for different $k$’s can be made in a way consistent with these compactifications (see Lemma 9.3 in [Sei3]).

Our construction requires the additional auxiliary structure of transition functions. This structure can be defined once a choice of universal strip-like ends is fixed. It consists of a smooth function $a^{k+1} : \mathcal{S}^{k+1} \to [0, 1]$ with the following properties. First let $k = 1$. In this case $\mathcal{S}^2 = D \setminus \{-1, 1\} \cong \mathbb{R} \times [0, 1]$ and we define $a_r(s, t) = t$, where $(s, t) \in \mathbb{R} \times [0, 1]$. To describe $a^{k+1}$ for $k \geq 2$ write $a_r := a^{k+1}|_{S_r}, r \in \mathcal{R}^{k+1}$. We require the functions $a_r$ to satisfy the following for every $r \in \mathcal{R}^{k+1}$ - see Figure 7:

i. For each entry strip-like end $\epsilon_i : Z^- \to S_r, 1 \leq i \leq k$, we have:
   a. $a_r \circ \epsilon_i(s, t) = t, \forall (s, t) \in (-\infty, -1] \times [0, 1]$.
   b. $\frac{\partial}{\partial s}(a_r \circ \epsilon_i)(s, 1) \leq 0$ for $s \in [-1, 0]$.
   c. $a_r \circ \epsilon_i(s, t) = 0$ for $(s, t) \in ((-\infty, 0] \times \{0\}) \cup (\{0\} \times [0, 1])$.

ii. For the exit strip-like end $\epsilon_{k+1} : Z^+ \to S_r$ we have:
   a’. $a_r \circ \epsilon_{k+1}(s, t) = t, \forall (s, t) \in [1, \infty) \times [0, 1]$. 


Figure 7. The constraints imposed on a transition function for a domain with three entries and one exit: in the red region the function $a$ equals $(s, t) \to t$; along the blue arcs the function $a$ vanishes; the green region is a transition region. There are no additional constraints in the black region.

b'. $\frac{\partial}{\partial s} (a_r \circ \epsilon_{k+1})(s, 1) \geq 0$ for $s \in [0, 1]$.

c'. $a_r \circ \epsilon_{k+1}(s, t) = 0$ for $(s, t) \in ([0, +\infty) \times \{0\}) \cup (\{0\} \times [0, 1])$.

The total function $a^{k+1} : S^{k+1} \to [0, 1]$ will be called a global transition function. The functions $a^{k+1}$ can be picked consistently for different values of $k$ in the sense that $a$ extends smoothly to $\overline{S}^{k+1}$ and along the boundary $\partial S^{k+1}$ it coincides with the corresponding pairs of functions $a^{k'+1} : S^{k'+1} \to [0, 1]$, $a^{k''+1} : S^{k''+1} \to [0, 1]$ with $k' + k'' = k + 1$, associated to trees of split pointed disks.

3.2.2. Profile function. We now discuss the second special ingredient in our construction: profile functions.

To fix ideas we suppose from now on in this construction that $a_U \leq \frac{1}{2}$. (See (4).) In particular,

$$U \subset [-\frac{1}{2}, \frac{1}{2}] \times [0, \infty).$$

We will use a profile function: $h : \mathbb{R}^2 \to \mathbb{R}$ which, by definition, has the following properties (see Figure 8):

i. The support of $h$ is contained in the union of the sets

$$W_i^+ = [2, \infty) \times [i - \epsilon, i + \epsilon] \quad \text{and} \quad W_i^- = (-\infty, -1] \times [i - \epsilon, i + \epsilon], \quad i \in \mathbb{Z},$$

where $0 < \epsilon < 1/4$. 

ii. The restriction of $h$ to each set $T^+_i = [2, \infty) \times [i - \epsilon/2, i + \epsilon/2]$ and $T^-_i = (-\infty, -1] \times [i - \epsilon/2, i + \epsilon/2]$ is respectively of the form $h(x, y) = h_\pm(x)$, where the smooth functions $h_\pm$ satisfy:

- **a.** $h_-$ : $(-\infty, -1] \rightarrow \mathbb{R}$ has a single critical point in $(-\infty, -1]$ at $-\frac{3}{2}$ and this point is a non-degenerate local maximum. Moreover, for all $x \in (-\infty, -2)$, we have $h_-(x) = \alpha^- x + \beta^-$ for some constants $\alpha^-, \beta^- \in \mathbb{R}$ with $\alpha^- > 0$.
- **b.** $h_+ : [2, \infty) \rightarrow \mathbb{R}$ has a single critical point in $[2, \infty)$ at $\frac{5}{2}$ and this point is also a non-degenerate maximum. Moreover, for all $x \in (3, \infty)$ we have $h_+(x) = \alpha^+ x + \beta^+$ for some constants $\alpha^+, \beta^+ \in \mathbb{R}$ with $\alpha^+ < 0$.

iii. The Hamiltonian isotopy $\phi^h_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ associated to $h$ exists for all $t \in \mathbb{R}$; the derivatives of the functions $h_\pm$ are sufficiently small so that the Hamiltonian isotopy $\phi^h_t$ keeps the sets $[2, \infty) \times \{i\}$ and $(-\infty, -1] \times \{i\}$ inside the respective $T^\pm_i$ for $-1 \leq t \leq 1$.

iv. The Hamiltonian isotopy $\phi^h_t$ preserves the strip $[-\frac{3}{2}, \frac{5}{2}] \times \mathbb{R}$ for all $t$, in other words $\phi^h_t([\frac{-3}{2}, \frac{5}{2}] \times \mathbb{R}) = [\frac{-3}{2}, \frac{5}{2}] \times \mathbb{R}$ for every $t$.

**Figure 8.** The graphs of $h_-$ and $h_+$ and the image of $\mathbb{R}$ by the Hamiltonian diffeomorphism $(\phi^h_t)^{-1}$. The profile of the functions $h_-$ at $-3/2$ and $h_+$ at $5/2$ are the “bottlenecks”.

Such functions $h$ are easy to construct. Their main role is to disjoin the ends corresponding to two (or more) cobordisms at $\pm \infty$. The two critical points $(-3/2, i)$ and $(5/2, i)$ are called bottlenecks.

### 3.2.3. Perturbation data, $J$-holomorphic polygons and $\mu_k$.

At this step we describe the (perturbed) $J$-holomorphic polygons that define the $\mu_k$'s.
The construction of $\mu_k$ starts with $\mu_1$ and the so-called Floer datum. For each pair of cobordisms $V, V' \subset E$ the Floer datum $\mathcal{D}_{V,V'} = (\bar{H}_{V,V'}, J_{V,V'})$ consists of a Hamiltonian $\bar{H}_{V,V'} : [0, 1] \times E \to \mathbb{R}$ and a (possibly time dependent) almost complex structure $J_{V,V'}$ on $E$ which is compatible with $\Omega_E$. We will also assume that each Floer datum $(\bar{H}_{V,V'}, J_{V,V'})$ satisfies the following conditions:

i. $\phi_1^{H_{V,V'}}(V)$ is transverse to $V'$.

ii. Write points of $E \setminus \pi^{-1}(U)$ as $(x, y, p)$ with $x + iy \in \mathbb{C}$, $p \in M$. We require that there exists a compact set $K_{V,V'} \subset (\frac{-5}{4}, \frac{5}{4}) \times \mathbb{R} \subset \mathbb{C}$ so that $\bar{H}_{V,V'}(t, (x, y, p)) = h(x, y) + H_{V,V'}(t, p)$ for $(x + iy, p)$ outside of $\pi^{-1}(K_{V,V'})$, for some $H_{V,V'} : [0, 1] \times M \to \mathbb{R}$.

iii. The projection $\pi : E \to \mathbb{C}$ is $(J_{V,V'}(t), (\phi_t^0, 0))$-holomorphic outside of $\pi^{-1}(K_{V,V'})$ for every $t \in [0, 1]$.

Remark 3.2.2. The almost complex structures appearing in this construction can all be viewed as appropriate perturbations of the almost complex structure $J_E$ that is part of the Lefschetz fibration structure as in Definition 2.1.1.

The time-1 Hamiltonian chords $\mathcal{P}_{H_{V,V'}}$ of $\bar{H}_{V,V'}$ that start on $V$ and end on $V'$, form a finite set.

For a $(k+1)$-pointed disk $S_r$, let $C_i \subset \partial S_r$ the connected components of $\partial S_r$ indexed so that $C_1$ goes from the exit to the first entry, $C_i$ goes from the $(i-1)$-th entry to the $i$, $1 \leq i \leq k$, and $C_{k+1}$ goes from the $k$-th entry to the exit.

Following Seidel’s scheme from [Sei3], we now need to choose additional perturbation data. To every collection of cobordisms $V_i$, $1 \leq i \leq k + 1$ we choose a perturbation datum $\mathcal{D}_{V_1,\ldots,V_{k+1}} = (\Theta, J)$ consisting of:

I. A family $\Theta = \{\Theta^r\}_{r \in \mathbb{R}^{k+1}}$, where $\Theta^r \in \Omega^1(S_r, C^\infty(E))$ is a 1-form on $S_r$ with values in smooth functions on $E$. We write $\Theta^r(\xi) : E \to \mathbb{R}$ for the value of $\Theta^r$ on $\xi \in TS_r$.

II. $J = \{J_z\}_{z \in S^{k+1}}$ is a family of $\Omega_E$-compatible almost complex structure on $E$, parametrized by $z \in S_r$, $r \in \mathbb{R}^{k+1}$.

The forms $\Theta^r$ induce forms $Y^r = Y^{\Theta^r} \in \Omega^1(S_r, C^\infty(TE))$ with values in (Hamiltonian) vector fields on $E$ via the relation $Y(\xi) = X^{\Theta(\xi)}$ for each $\xi \in TS_r$ (in other words, $Y(\xi)$ is the Hamiltonian vector field on $E$ associated to the autonomous Hamiltonian function $\Theta(\xi) : E \to \mathbb{R}$).

The relevant Cauchy-Riemann equation associated to $\mathcal{D}_{V_1,\ldots,V_{k+1}}$ (see [Sei3] for more details):

\begin{equation}
(18) \quad u : S_r \to E, \quad Du + J(z, u) \circ Du \circ j = Y + J(z, u) \circ Y \circ j, \quad u(C_i) \subset V_i.
\end{equation}

Here $j$ stands for the complex structure on $S_r$. The $i$-th entry of $S_r$ is labeled by a time−1 Hamiltonian orbit $\gamma_i \in \mathcal{P}_{\bar{H}_{V_i},V_{i+1}}$ and the exit is labeled by a time−1 Hamiltonian orbit $\gamma_{k+1} \in$
\[ \mathcal{P}_{B_{V_1, V_{k+1}}} \]. The map \( u \) satisfies \( u(C_i) \subset V_i \) and \( u \) is required to be asymptotic - in the usual Floer sense - to the Hamiltonian orbits \( \gamma_i \) on each respective strip-like end.

The perturbation data \( \mathcal{D}_{V_1, \ldots, V_{k+1}} \) are constrained by a number of additional conditions that we now describe.

First, denote by \( s_{V_1, \ldots, V_{k+1}} \in \mathbb{N} \) the smallest \( l \in \mathbb{N} \) so that \( \pi(\bigcup V_{k+1}) \subset \mathbb{R} \times (0, l) \). Write \( \bar{h} = h \circ \pi : E \to \mathbb{R} \), where \( h : \mathbb{R}^2 \to \mathbb{R} \) is the profile function fixed before. We also write

\[
U^r_i = \epsilon^S_i \left( (-\infty, -1] \times [0, 1] \right) \subset S_r, \quad i = 1, \ldots, k,
\]

\[
U^r_{k+1} = \epsilon^S_{k+1} \left([1, \infty) \times [0, 1]\right) \subset S_r,
\]

\[
W^r = \bigcup_{i=1}^{k+1} U^r_i.
\]

The conditions on \( \mathcal{D}_{V_1, \ldots, V_{k+1}} \) are the following:

a. **Asymptotic conditions.** For every \( r \in \mathcal{R}^{k+1} \) we have \( \Theta|_{U^r_i} = \bar{H}_{V_i, V_{k+1}} dt, \) \( i = 1, \ldots, k \)

   and \( \Theta|_{U^r_{k+1}} = \bar{H}_{V_i, V_{k+1}} dt. \) (Here \( (s, t) \) are the coordinates parametrizing the strip-like ends.) Moreover, on each \( U^r_i, i = 1, \ldots, k, \) \( J_z \) coincides with \( J_{V_i, V_{k+1}} \) and on \( U^r_{k+1} \) it coincides with \( J_{V_1, V_{k+1}} \), i.e. \( J_{\epsilon^S_i(u, t)}(s, t) = J_{V_i, V_{k+1}}(t) \) and similarly for the exit end. Thus, over the part of the strip-like ends \( W^r \) the perturbation datum \( \mathcal{D}_{V_1, \ldots, V_{k+1}} \) is compatible with the Floer data \( \mathcal{D}_{V_1, V_{k+1}}, i = 1, \ldots, k \) and \( \mathcal{D}_{V_1, V_{k+1}} \).

b. **Special expression for \( \Theta \).** The restriction of \( \Theta \) to \( S_r \) equals

\[
\Theta|_{S_r} = da_r \otimes \bar{h} + \Theta_0
\]

for some \( \Theta_0 \in \Omega^1(S_r, C^\infty(E)) \) which depends smoothly on \( r \in \mathcal{R}^{k+1} \). Here \( a_r : S_r \to \mathbb{R} \)

are the transition functions fixes at the point 1. The form \( \Theta_0 \) is required to satisfy the following two conditions:

1. \( \Theta_0(\xi) = 0 \) for all \( \xi \in TC_i \subset T\partial S_r. \)

2. There exists a compact set \( K_{V_1, \ldots, V_{k+1}} \subset (-\frac{3}{2}, \frac{5}{2}) \times \mathbb{R} \) which is independent of \( r \in \mathcal{R}^{k+1} \) so that \( \pi^{-1}(K_{V_1, \ldots, V_{k+1}}) \) contains all the sets \( K_{V_i, V_j} \) involved in the Floer datum \( \mathcal{D}_{V_i, V_j} \), and with

\[
K_{V_1, \ldots, V_{k+1}} \supset \left( [-\frac{5}{4}, \frac{9}{4}] \times [-s_{V_1, \ldots, V_{k+1}} + s_{V_1, \ldots, V_{k+1}}] \right)
\]

so that outside of \( \pi^{-1}(K_{V_1, \ldots, V_{k+1}}) \) we have \( D\pi(Y_0) = 0 \) for every \( r \), where \( Y_0 = X^{\Theta_0} \).

c. **Outside of \( \pi^{-1}(K_{V_1, \ldots, V_{k+1}}) \) the almost complex structure \( J \) has the property that the projection \( \pi \) is \( (J_z, (\phi^{\bar{h}}_{\psi_{a_r}(z)})(s)(i)) \)-holomorphic for every \( r \in \mathcal{R}^{k+1}, z \in S_r. \)

Using the above choices of data we construct the \( A_\infty \)-category \( \mathcal{F}uk^*(E) \) by the method from [Sei3]. As mentioned before, the objects of this category are Lagrangians cobordisms \( V \subset E \) without positive ends that are uniformly monotone of class \(* \), the morphisms space
between the objects $V$ and $V'$ are $CF(V, V'; \mathcal{P}_{V, V'})$, the $\mathcal{A}$-vector space generated by the Hamiltonian chords $\mathcal{P}_{V, V'}$. The $A_\infty$ structural maps

$$\mu_k : CF(V_1, V_2) \otimes CF(V_2, V_3) \otimes \ldots \otimes CF(V_k, V_{k+1}) \rightarrow CF(V_1, V_{k+1})$$

are defined by summing - with coefficients in $\mathcal{A}$ - pairs $(r, u)$ with $r \in \mathcal{R}^{k+1}$ and $u$ a finite energy solution of (18) that belongs to a 0-dimensional moduli space. The coefficient in front of a perturbed $J$-holomorphic polygon $u$ is $T^{w(u)}$. The Gromov compactness and regularity arguments work just as in [BC3]. In fact, as we work here over the universal Novikov ring the compactness verifications are easier in this case (and we do not require the vanishing of the inclusions $\pi_1(V) \rightarrow \pi_1(E)$ as in [BC3]).

The choice of strip-like ends, transition functions and profile function (in particular, the placement of the bottlenecks) changes the resulting $A_\infty$-category only up to quasi-equivalence.

Once the category $\mathcal{F}uk^*(E)$ is constructed the derived category $\mathcal{D}\mathcal{F}uk^*(E)$ is defined by again considering the $A_\infty$-modules $\text{mod}(\mathcal{F}uk^*(M)) := \text{fun}(\mathcal{F}uk^*(E), Ch_{opp})$ and by letting $\mathcal{D}\mathcal{F}uk^*(E)$ be the homological category associated to the triangulated closure of the image of the Yoneda functor $\mathcal{Y} : \mathcal{F}uk^*(E) \rightarrow \text{mod}(\mathcal{F}uk^*(E))$.

3.2.4. The naturality transformation. Assume that $u : S_r \rightarrow E$ is a solution of (18) and satisfies the conditions discussed at the points $a, b, c$ above. Define $v : S_r \rightarrow E$ by the formula:

$$u(z) = \phi_{a_r(z)}^h(v(z)),$$

where $a_r : S_r \rightarrow [0, 1]$ is the transition function.

The Floer equation (18) for $u$ transforms into the following equation for $v$:

$$Dv + J'(z, v) \circ Dv \circ j = Y' + J'(z, v) \circ Y' \circ j.$$

Here $Y' \in \Omega^1(S_r, C^\infty(TM))$ and $J'$ are defined by:

$$Y = D\phi_{a(z)}^h(Y') + da_r \otimes X^h, \quad J_z = (\phi_{a_r(z)})^*_z J'_z.$$

The map $v$ satisfies the following moving boundary conditions:

$$\forall \ z \in C_i, \ \ v(z) \in (\phi_{a(z)}^h)^{-1}(V_i).$$

The asymptotic conditions for $v$ at the punctures of $S_r$ are as follows. For $i = 1, \ldots, k$, $v(\epsilon_i(s, t))$ tends as $s \rightarrow -\infty$ to a time-1 chord of the flow $(\phi_t^h)^{-1} \circ \phi_t^{H_{V_i, V_{i+1}}}$ starting on $V_i$ and ending on $(\phi_1^h)^{-1}(V_{i+1})$. (Here $\epsilon_i(s, t)$ is the parametrization of the strip-like end at the $i$'th puncture.) Similarly, $v(\epsilon_{k+1}(s, t))$ tends as $s \rightarrow \infty$ to a chord of $(\phi_t^h)^{-1} \circ \phi_t^{H_{V_1, V_{k+1}}}$ starting on $V_1$ and ending on $(\phi_1^h)^{-1}(V_{k+1})$. 


Let now $v' = \pi \circ v : S_r \to \mathbb{C}$. It is then easy to see - as in [BC3] - that $v'$ is holomorphic over $\mathbb{C} \setminus ([-\frac{3}{2} + \delta', \frac{3}{2} - \delta'] \times \mathbb{R})$ for small enough $\delta' > 0$.

As discussed in [BC3], there are many useful consequences of the holomorphicity of $v'$ around a bottleneck and we will see some more later in this paper. To give a typical simple example, assume that the bottleneck in question is $a = (-\frac{3}{2}, 0)$ and that the regions $A$ and $B$ in Figure 9 are unbounded. In this case, the image of $v'$ can not switch from one side to the other of $a$. In other words it is impossible to have that $\text{Image}(v') \cap C \neq \emptyset$ and $\text{Image}(v') \cap D \neq \emptyset$ with the regions $C, D$ as in the picture.

![Figure 9. The bottleneck $a$ and the regions $A$, $B$, $C$ and $D$.](image)

The argument is as follows: assume that $\text{Image}(v')$ intersects both $C$ and $D$ and is disjoint from the interiors of both $A$ and $B$. Let $x_1 \in \text{Image}(v') \cap C$ and $x_2 \in \text{Image}(v') \cap D$. Let $c$ be a curve inside the domain of $v'$ that connects $x_1$ to $x_2$. It follows that $a \in v'(c)$. But as there are infinitely many distinct curves $c$ joining $x_1$ to $x_2$ this means that there are infinitely many interior points $z$ with $v'(z) = a$. But this implies $\text{Image}(v') = a$. Thus $\text{Image}(v')$ has to intersect at least one of $A$ and $B$ and, by the open mapping theorem, this contradicts the fact that the closure of $\text{Image}(v')$ is compact.

This argument is used in [BC3] to show the compactness of the moduli spaces required to define $\mu_k$ as well as those used to show $\mu \circ \mu = 0$ and the other subsequent algebraic relations.

Besides this compactness implication, the holomorphicity of $v'$ has an important role in the proof of the main decomposition result in [BC3] as well as in the main result of the current paper. Both these results are consequences of writing certain $A_\infty$-module structures $\mu_k$ in an “upper triangular” form. In turn, this form is deduced from the fact that the planar projections of the $J$-holomorphic polygons giving the module multiplications are holomorphic (over an appropriate region in $\mathbb{C}$) and a “bottleneck-type” argument is used repeatedly to show the vanishing of the relevant components of the $\mu_k$’s.
3.2.5. The case of a non-compact fibre. We now assume that \((M, \omega)\) is noncompact and convex at infinity and that the Lefschetz fibration verifies the conditions in §2.1. From assumption \(T_\infty\) there we deduce that there is a trivialization \(\phi : \mathbb{C} \times M^\infty \to E^\infty\) with respect to which both the symplectic form and the almost complex structure split so that, in particular, \(\phi^* J_E = j \oplus J_0\) where \(J_0\) is a fixed almost complex structure on \(M\) compatible with \(\omega\) and with the symplectic convexity of \(M\). Recall also that \(E^0 = E \setminus E^\infty\).

The objects of the category \(\mathcal{Fuk}^*(E)\) are the same as before. Notice that, by Definition 2.2.3, any cobordism \(V\) has the property that \(V \cap \pi^{-1}(z)\) is compact for any \(z \in \mathbb{C}\). Furthermore, all the construction of the category \(\mathcal{Fuk}^*(E)\) proceeds exactly in the same fashion as in the compact case with an additional requirement: all the almost complex structures involved are required to coincide with \(J_E\) outside a large enough neighbourhood of \(E^0\). More precisely, for any two objects \(V, V' \in \text{Ob}(\mathcal{Fuk}^*(E))\) we require that \(J_{V,V'}\) coincide with \(J_E\) outside a neighbourhood of \(E^0\) that contains both \(V\) and \(V'\). Similarly, each almost complex structure \(J_z\) in the family \(J\) that is part of the perturbation data associated to the collection of cobordisms \(V_1, \ldots, V_{k+1}\) has to coincide with \(J_E\) outside of a neighbourhood of \(E^0\) that contains all of the \(V_i\)’s.

Finally, notice that as explained in §3.2.4 the actual curves \(u\) that appear in the \(\mu_k\)’s are transformed into curves \(v\) that satisfy equations that are holomorphic with respect to almost complex structures of the form \(J'_z = (\phi^*_n(z))^{-1} J_z\). Due to the splitting provided by the trivialization \(\phi\) and because \(T = h \circ \pi\) these structures are also split at \(\infty\) (along the fibre) and, by using the trivialization \(\phi\), it follows that \(J'_z\) restricted to the fiber direction coincides with \(J_0\) (away from a compact). Therefore, over \(E^\infty\) one can again use \(\phi\) to project such a curve \(v\) on \(M^\infty\) thus getting a new curve \(v'\) that way from a compact is \(J_0\)-holomorphic. The usual compactness arguments for manifolds that are symplectically convex at infinity apply to this \(v'\) and thus compactness is achieved without issues.

Remark 3.2.3. In his work, \[Sei3\] (see also \[Sei4\]), Seidel has introduced a Fukaya category associated to a Lefschetz fibration \(\pi : E \to \mathbb{C}\). By neglecting for a moment some technical points that will be revisited below, the relation between this category and the category \(\mathcal{Fuk}^*(E)\) introduced above is that Seidel’s category is quasi-equivalent to the subcategory of \(\mathcal{Fuk}^*(E)\) with objects the thimbles \(T_i\) covering the curves \(t_i\) in Figure 1. The technical points are that, firstly, we work in a monotone and ungraded setting and Seidel’s work is in the exact and graded case (and the grading plays an important role in his work). Secondly, the type of perturbations at infinity that Seidel uses - see in particular \[Sei4\] - are different from ours. Despite these differences, it is easy to see that Seidel’s approach can also be implemented in the monotone case and that the resulting category is quasi-equivalent to the subcategory of \(\mathcal{Fuk}^*(E)\) as mentioned above. At the same time, in the construction of \(\mathcal{Fuk}^*(E)\) above we use the perturbations employing bottlenecks etc because we found it very convenient to use
3.3. Fukaya categories of negative ended cobordisms in general Lefschetz fibrations. In this section we use the construction in §3.2 to associate a Fukaya \( A_\infty \) -category to a general Lefschetz fibration. Let \( \pi : E \to \mathbb{C} \) be a Lefschetz fibration as in §2.1. The category we intend to construct will depend on a tame Lefschetz fibration \( \pi : E_\tau \to \mathbb{C} \) associated to \( E \) and will be denoted by \( \mathcal{F}uk^*(E; \tau) \). The parameter \( \tau \) indicates the choice of a tame symplectic structure on \( E \) with the properties described in the construction below.

We first fix an additional notation. For two constants \( r < 0 < s \), put \( S_{r,s} = [r, s] \times \mathbb{R} \subset \mathbb{C} \). Assume that all the singularities of the fibration \( E \) are contained in the interior of \( \pi^{-1}(S_{x,y}) \).

The construction is now the following. The objects of the category \( \mathcal{F}uk^*(E; \tau) \) are cobordisms \( V \) in \( E \) - in the sense of Definition 2.2.1 - that are cylindrical outside \( S_{x,y} \) and satisfy the following additional constraints:

i. \( V \subset \pi^{-1}(\mathbb{R} \times [\frac{1}{2}, +\infty)) \)
   
ii. \( V \) has only negative ends belonging to \( \mathcal{L}^*(M) \).

iii. \( V \) is monotone of class \( * \).

Condition ii means in this case that for some point \( z \) along one of the rays \( \ell_i \) associated to the ends of \( V \) we have that the Lagrangian \( V \cap \pi^{-1}(z) \) belongs to \( \mathcal{L}^*(M) \). For a fixed ray \( \ell_i \) it is easy to see that this condition does not depend on the choice of the point \( z \).

To define the morphisms and the operations \( \mu_k \) we proceed as follows. We fix a Lefschetz fibration \( \pi : E_\tau \to \mathbb{C} \) that is tame outside a set \( U \) that contains \((x - 4, y + 4) \times (-1, \infty)\) and coincides with \( E \) over \([x - 3, y + 3] \times [-\frac{1}{2}, \infty)\). Such a fibration exists due to the results from §2.3. Recall from §3.2 the construction of the category \( \mathcal{F}uk^*(E_\tau) \). Each object \( V \in \text{Ob}(\mathcal{F}uk^*(E; \tau)) \) corresponds to an object \( \overline{V} \in \text{Ob}(\mathcal{F}uk^*(E_\tau)) \) that is obtained, as in Remark 2.3.2, by cutting off the ends of \( V \) along the line \( \{x - 3\} \times \mathbb{R} \subset \mathbb{C} \) and extending them horizontally by parallel transport in the fibration \( E_\tau \). It is easy to see that the subcategory of \( \mathcal{F}uk^*(E_\tau) \) that consists of all the objects \( \overline{V} \) obtained in this way is quasi-equivalent to \( \mathcal{F}uk^*(E_\tau) \) itself because each object of this larger category is quasi-isomorphic to one of the \( \overline{V} \)'s. We now put \( \text{Mor}_{\mathcal{F}uk^*(E; \tau)}(V, V') = \text{Mor}_{\mathcal{F}uk^*(E_\tau)}(\overline{V}, \overline{V}') \) and similarly we define all operations in \( \mathcal{F}uk^*(E; \tau) \) associated to \( V_1, \ldots, V_{k+1} \) by means of the corresponding operations associated to \( \overline{V}_1, \ldots, \overline{V}_{k+1} \) in \( \mathcal{F}uk^*(E_\tau) \).

It is clear, by construction, that there is an inclusion:

\[ \mathcal{F}uk^*(E; \tau) \to \mathcal{F}uk^*(E_\tau) \]

that is a quasi-equivalence.
The $A_{\infty}$ category in the statement of Theorem A can be taken to be any of the categories $\mathcal{F}uk^*(E; \tau)$ described above. We will see later in the paper that the derived category $D\mathcal{F}uk^*(E; \tau)$ is independent of $\tau$ up to equivalence. Therefore, the omission of $\tau$ in the statement of Theorem A is justified.

**Remark 3.3.1.** We believe that any two $A_{\infty}$ categories $\mathcal{F}uk^*(E; \tau)$ and $\mathcal{F}uk^*(E; \tau')$ are quasi-equivalent. Indeed, we expect that our construction of the Fukaya category of a tame fibration adapts to the case of a general Lefschetz fibration and the resulting fibration $\mathcal{F}uk^*(E)$ is expected to be quasi-equivalent to $\mathcal{F}uk^*(E; \tau)$ for all $\tau$. The technical ingredients required in the definition of $\mathcal{F}uk^*(E)$ go beyond the construction in the tame case so that we prefer not to further explore this issue here. In a different direction, we also expect that there is a derived Fukaya category of cobordisms with ends of arbitrary heights $\in R^+$ and not only with integral heights, as described in this paper. First, given any infinite sequence of strictly increasing positive reals $S = \{a_1, \ldots, a_n, \ldots\}$ there is a Fukaya category of cobordisms with ends in $S$ that is defined just as in the case of $S = \mathbb{N}^*$. The sets $S$ are ordered by inclusion in an obvious way and this order implies the existence of comparison maps among the corresponding categories. The category in question is expected to be defined as an appropriate limit over $S$. Again, we do not pursue this construction here as it is not significant for the purpose of this paper.

### 4. Decomposing cobordisms

Fix a Lefschetz fibration $\pi : E \to C$ and a Fukaya category $\mathcal{F}uk^*(E; \tau)$ as defined in §3.3. This section contains the main result of the paper. It claims that each object $V$ of $D\mathcal{F}uk^*(E; \tau)$ admits an iterated cone decomposition in terms of simpler objects. We will also see later in the paper that $D\mathcal{F}uk^*(E; \tau)$ is independent of $\tau$.

#### 4.1. Statement of the main result

We will restate here Theorem A after providing more precise definitions of the objects involved.

To fix ideas, we assume that $\pi$ has $m$ critical points $x_k \in E$, $k = 1, \ldots, m$ of corresponding critical values $v_k = (k, \frac{3}{2}) \in C$. Consider a Fukaya category $\mathcal{F}uk^*(E; \tau)$ of uniformly monotone negative ended cobordisms $V \subset E$ that are cylindrical outside $\pi^{-1}(S_{x,y})$ with $x < 0 < y$ and so that all the singularities of $\pi$ are contained in $\pi^{-1}(S_{x,y})$. See §3.3 for the definition. In particular, $\tau$ indicates that the morphisms and operations in $\mathcal{F}uk^*(E; \tau)$ are defined by means of the Fukaya $A_{\infty}$-category $\mathcal{F}uk^*(E_\tau)$ associated to a tame Lefschetz fibration $\pi : E_\tau \to C$ that agrees with $E$ over $[x - 3, y + 3] \times [-\frac{1}{2}, \infty)$.

The objects of $\mathcal{F}uk^*(E; \tau)$ are collected in the set $\mathcal{L}^*(E)$.

#### 4.1.1. The “atoms” of the decomposition

Our first task is to describe the simpler objects that form the basic pieces of our decomposition.
We will make use of two types of smooth curves in the plane.

(I) These curves are denoted by \( \gamma_i, \ i \geq 2 \) and are so that \( \gamma_i : \mathbb{R} \to \mathbb{C} \) is a smooth embedding with

\[
\gamma_i(\mathbb{R}) \subset (-\infty, x) \times \left[ \frac{1}{2}, +\infty \right), \ \gamma_i(-1,1) \subset [x-2,x-1] \times [1,i]
\]

and:

\[
\gamma_i((-\infty,-1]) = (-\infty, x-2] \times \{1\}, \ \gamma_i([1,+\infty)) = (-\infty, x-2] \times \{i\}.
\]

(II) The second type of curve is denoted by \( t_k \). For \( 1 \leq k \leq m \) the curve \( t_k \) is given by smooth functions \( t_k : (-\infty,0] \to \mathbb{C} \) so that we have

\[
t_k(0) = v_k, \ t_k((-\infty,-2]) = (-\infty, x-2] \times \{1\}, \ t_k((-\infty,0)) \subset (-\infty,2m) \times [1,3]
\]

and \( t_k \) turns once around all the points \( v_{k+1}, v_{k+2}, \ldots, v_m \).

Both types of curves are pictured in Figure 10.

![Figure 10. The special curves \( \gamma_3 \) and \( t_1, t_2, t_3 \) for a fibration \( E \) with three critical points.](image)

Let \( x-3 < a < x-2 \) and fix the points \( z_i = (a,i) \in \mathbb{R}^2 \approx \mathbb{C}, \ i \in \mathbb{N} \). Set also \( z_a = (a,1) \in \mathbb{R}^2 \). Let \( (M_{z_i}, \omega_{z_i}) \) be the fiber of \( \pi \) over the point \( z_i \). There are two associated families of Lagrangian cobordisms in \( \mathcal{L}^*(E) \).

(\text{I}) For each Lagrangian in \( L \in \mathcal{L}^*(M_{z_i}) \) we consider the trail \( \gamma_k L \) of \( L \) along the curve \( \gamma_k \).

This is a well-defined Lagrangian in \( E \) and, further, \( \gamma_k L \in \mathcal{L}^*(E) \).

(\text{II}) Denote by \( T_i \) the thimble associated to the singularity \( x_i \) and the curve \( t_i \). Denote by \( S_i \subset M_{z_i} \) the vanishing sphere associated to the singularity \( x_i \) such that \( T_i \) is the trail of \( S_i \) along \( t_i \).
4.1.2. The decomposition. We now reformulate Theorem A in the setting and notation above. Recall that we use the Novikov ring $A$ as coefficients at all times.

**Theorem 4.1.1** (Theorem A reformulated). Let $V \in \mathcal{L}^*(E)$ and let $L_i = V|_{z_i}$. There exist finite rank $A$-modules $E_k$, $1 \leq k \leq m$, and an iterated cone decomposition taking place in $D\mathcal{F}uk^*(E; \tau)$:

$$V \cong (T_1 \otimes E_1 \to T_2 \otimes E_2 \to \ldots \to T_m \otimes E_m \to \gamma_s L_s \to \gamma_{s-1} L_{s-1} \to \ldots \to \gamma_2 L_2).$$

Moreover, the category $D\mathcal{F}uk^*(E; \tau)$ is independent of $\tau$ (up to equivalence).

The proof of Theorem 4.1.1 follows from an analogue result - Theorem 4.2.1, stated in the first subsection below - which applies to tame Lefschetz fibrations. The three subsequent subsections §4.3 - §4.5 form the technical heart of the paper. They provide the arguments that are put together in §4.6 to show Theorem 4.2.1. The decomposition in the statement of Theorem 4.1.1 follows directly from that provided by Theorem 4.2.1. The modules $E_i$ are explicitly identified along the proof - see equation (55). The independence of $D\mathcal{F}uk^*(E; \tau)$ with respect to $\tau$ is postponed to §5 as it is an immediate consequence of Corollary 5.1.3 which is itself deduced from Theorem 4.2.1.

4.2. Decomposition of cobordisms in tame fibrations. Assume now that the Lefschetz fibration $\pi : E \to \mathbb{C}$ is tame outside the set $U$ - as in Definition 2.2.2 - and is so that:

i. the set $U$ contains $[0, n+1] \times [\frac{1}{2}, K]$ and, as in (16), $U \subset \mathbb{R} \times [0, +\infty)$.

ii. as before, $\pi$ has $m$ critical points $x_k \in E$ of corresponding critical values $v_k = (k, \frac{3}{2})$.

iii. as in (4), there exists $a_U > 0$ sufficiently large so that $U$ is disjoint from both quadrants $Q^-_U = (-\infty, -a_U] \times [0, +\infty)$, $Q^+_U = [a_U, \infty) \times [0, +\infty)$.

In this setting we specialize the choices for the curves $\gamma_i$ and $t_j$. They are defined as at the points (I) and (II) in §4.1.1 but for a special choice of $x$ that we require to be $x = -a_U$. As a consequence, the position of these curves relative to the set $U$ is as in Figure 10. With this definition we notice that the Lagrangian $\gamma_k L$ defined at (I') in §4.1.1 is a product $\gamma_k L = \gamma_k \times L$. This is because the fibration is trivial over the complement of $U$ and $\gamma_k$ is entirely contained in this complement.

We reformulate again Theorem A in this context:

**Theorem 4.2.1.** Let $V \in \mathcal{L}^*(E)$, $V : \emptyset \to (L_1, \ldots, L_s)$. There exist finite rank $A$-modules $E_k$, $1 \leq k \leq m$, and an iterated cone decomposition taking place in $D\mathcal{F}uk^*(E)$:

$$V \cong (T_1 \otimes E_1 \to T_2 \otimes E_2 \to \ldots \to T_m \otimes E_m \to \gamma_s \times L_s \to \gamma_{s-1} \times L_{s-1} \to \ldots \to \gamma_2 \times L_2).$$
4.3. Decomposition of remote Yoneda modules. In this subsection we assume the “tame” setting of §4.2 and we consider a particular class of $A_\infty$-modules over $\mathcal{F}uk^*(E)$ associated to certain cobordisms $W$ included in Lefschetz fibrations that extend $E$.

More precisely, fix a large constant $K > 0$ and consider a Lefschetz fibration $\pi : \hat{E} \to \mathbb{C}$ so that:

i. $\hat{\pi}$ is tame outside $\hat{U}$, with $U \subset \hat{U}$ and is so that condition (4) is satisfied for some constant $a_{\hat{U}} > a_U$.

ii. $\hat{U} \subset \mathbb{R} \times [-K, +\infty)$.

iii. $\hat{E}|_{\mathbb{R} \times [-\frac{1}{2}, +\infty)} = E|_{\mathbb{R} \times [-\frac{1}{2}, +\infty)}$ including their symplectic structures.

Similarly to the definition of the category $\mathcal{F}uk^*(E)$ in §3.2 we consider a Fukaya category $\mathcal{F}uk^*(\hat{E})$ whose objects are cobordisms $W \subset \hat{E}$ as in Definition 2.2.3 so that $W : \emptyset \leadsto (L_1, \ldots, L_s)$ is monotone of monotonicity class $\ast = (\rho, d)$, $W$ has only negative ends ($\in \mathcal{L}^*(M)$) and, similarly to ii in §3.2,

$$W \subset \hat{\pi}^{-1}(\mathbb{R} \times [-K + \frac{1}{2}, \infty)) .$$

Notice that, following Definition 2.2.3, the cobordism $W$ is cylindrical outside the set $\hat{U}$ which is larger than $U$. Moreover, the ends of $W$ project to rays of the form $(-\infty, a_k] \times \{k\}$ with $k \in \mathbb{N}^*$.

A cobordism $W$ as before is called remote relative to $E$ if, in addition,

$$(23) \quad W \subset \hat{\pi}^{-1}(\mathbb{R} \times (-\infty, 0] \cup Q_U) .$$

In this case, we deduce, in particular, that $W \cap \pi^{-1}(U) = \emptyset$ (this explains the terminology, in the sense that $W$ is remote from all the singularities of $\pi$). See Figure 11. It is important to note that because $\hat{U}$ can contain an unbounded region disjoint from the upper half plane (in the figure this region goes through the third quadrant), the conditions i,ii,iii allow for $\hat{E}$ to have more singularities than $E$.

Given property ii from §3.2, it is clear that such remote cobordisms $W$ are not objects of $\mathcal{F}uk^*(E)$. On the other hand, each object of $\mathcal{F}uk^*(E)$ is an object of $\mathcal{F}uk^*(\hat{E})$. Moreover, by a simple application of the open mapping theorem, we see that there is an inclusion of $A_\infty$-categories

$$(24) \quad J^{E,\hat{E}} : \mathcal{F}uk^*(E) \to \mathcal{F}uk^*(\hat{E}) .$$

The relevant argument is roughly as follows. All objects of $\mathcal{F}uk^*(E)$ project to the upper half plane so that the $J$-polygons that compute the operations $\mu^k$ of $\mathcal{F}uk^*(\hat{E})$ (for objects that are in $\mathcal{F}uk^*(E)$) project to curves $v$ in $\mathbb{C}$ with boundary inside the upper half plane. Our choice of almost complex structures imply that such a curve $v$ can be assumed to be holomorphic outside (possibly a slightly bigger set containing) $U$ and, by the open mapping theorem, we
deduce that \( v \) can not extend outside of the region where \( E \) and \( \hat{E} \) coincide. Thus, for objects picked in \( \mathcal{F}_{uk}^*(E) \), the operations \( \mu_k \) are the same in \( \mathcal{F}_{uk}^*(\hat{E}) \) and in \( \mathcal{F}_{uk}^*(E) \).

Let \( \mathcal{Y}(W) \) be the Yoneda module associated to an object \( W \in Ob(\mathcal{F}_{uk}^0(\hat{E})) \). We denote by \( W_E \) the pull-back module:

\[
W_E = (J^{E,\hat{E}})^*(\mathcal{Y}(W))
\]

In case \( W \) is remote with respect to \( E \) we say that the module \( W_E \) is a remote \( \mathcal{F}_{uk}^*(E) \)-module.

**Proposition 4.3.1.** With the terminology above, assume that \( W \in Ob(\mathcal{F}_{uk}^*(\hat{E})) \) is remote relative to \( E \), \( W : \emptyset \sim (L_1, \ldots, L_s) \), then \( W_E \in Ob(D\mathcal{F}_{uk}^*(E)) \) and admits a decomposition in \( D\mathcal{F}_{uk}^*(E) \):

\[
W_E = (\gamma_s \times L_s \to \gamma_{s-1} \times L_{s-1} \to \ldots \to \gamma_2 \times L_2)
\]

To unwrap a bit the meaning of this Proposition consider a cobordism \( W \) in \( E \). If there is a horizontal hamiltonian isotopy \( \phi : \hat{E} \to \hat{E} \) that pushes \( W \) away from the singularities of \( \pi \), in the sense that \( \pi(\phi(W)) \cap U = \emptyset \), then the Proposition implies that \( W \) admits a decomposition as claimed in Theorem 4.2.1 but with all the modules \( E_i = 0 \). As a particular case that is already of interest, if \( \pi \) has no singularities \( E = \mathbb{C} \times M \) (\( U = \emptyset \) and \( m = 0 \)), then Proposition 4.3.1 applies to any cobordism \( W \subset E = \mathbb{C} \times M \). Thus, for \( E = \mathbb{C} \times M \), Proposition 4.3.1 implies Theorem 4.2.1.

**Proof of Proposition 4.3.1.** By possibly applying a horizontal isotopy to \( W \) we may assume that not only \( W \subset \hat{\pi}^{-1}(\mathbb{R} \times (-\infty, 0] \cup Q_U^-) \) as in the definition of remote cobordisms but that,
moreover, the intersection
\[ W^- = W \cap Q^-_U \]
coincides with a disjoint union of cylindrical ends of \( W \). In other terms
\[ W^- = \bigcup_{i=1}^n \alpha_i \times L_i \]
where \( \alpha_i \) are curves in \( C \) as in Figure 11. In particular, for any object \( X \in \text{Ob}(\mathcal{F}uk^*(E)) \), the intersection \( W \cap X \) consists of a union of intersections of the ends of \( W \) with the ends of \( X \) and is included in the quadrant \( Q^-_U \).

The main part of the proof makes essential use of constructions that appear in [BC3]. It consists of three main steps.

**Step 1: Repositioning \( W \).** Here we replace the module \( W_E \) with a quasi-isomorphic module corresponding to a cylindrical Lagrangian that can be handled easier geometrically. For this purpose we include the two \( A_\infty \)-categories \( \mathcal{F}uk^*(E) \) and \( \mathcal{F}uk^*(\hat{E}) \) in two other \( A_\infty \)-categories, respectively, \( \mathcal{F}uk^+_1(E) \) and \( \mathcal{F}uk^+_2(\hat{E}) \). These two categories have objects that are again cobordisms as before with the difference that their ends have heights \( \in \frac{1}{2} \mathbb{Z} \subset \mathbb{Q} \). In other words, compared with Definition 2.2.3, the difference is that \( V \cap \pi^{-1}(Q^-_U) = \bigcup_{i \in \mathbb{N}^*}((-\infty, -a_U] \times \{\frac{1}{2}\}) \times L_i \). The inclusion \( \mathcal{F}uk^*(E) \to \mathcal{F}uk^+_1(E) \) is obvious and is clearly full and faithful and similarly for the two categories associated to \( \hat{E} \). We now perturb \( W \) by a (non-horizontal) Hamiltonian isotopy so as to obtain an object \( W' \) of \( \mathcal{F}uk^+_1(\hat{E}) \) that differs from \( W \) only inside \((-\infty, -a_U-2] \times [\frac{1}{2}, +\infty) \) and is so that the ends of \( W' \) restricted to \((-\infty, -a_U-4-s] \times [\frac{1}{2}, +\infty) \) are of the form \((-\infty, -a_U-4-s] \times \{i-\frac{1}{2}\} \times L_i \) (for all the definitions involved to be coherent we might need to enlarge here the set \( U \)). In other words, the ends of \( W' \) are shifted down by \( \frac{1}{2} \) compared to the ends of \( W \). Let \( W'_E \) be the \( \mathcal{F}uk^*(E) \)-module obtained as pull-back over the inclusions
\[ \mathcal{F}uk^*(E) \to \mathcal{F}uk^*(\hat{E}) \to \mathcal{F}uk^+_2(\hat{E}) \]
from the \( \mathcal{F}uk^+_1(\hat{E}) \)-module \( \mathcal{Y}(W') \). It is easy to see that \( W_E \) and \( W'_E \) are quasi-isomorphic. This is a direct consequence of the fact that our definition of \( \text{Mor}_{\mathcal{F}uk^*(\hat{E})}(X, W) = CF(X, W) \) involved a perturbation of \( W \) in which its the negative ends are “moved” down compared to those of \( X \). More precisely, recall from [BC3] that \( CF(X, W) \) is defined by using a specific profile function \( h \) and an associated Hamiltonian \( \tilde{H}_X,W \). With these choices \( CF(X, W) \) is identified with \( CF(X, (\phi_{1}^{R_X,W})^{-1}(W)) \) (under the assumption that \( X \) and \( (\phi_{1}^{R_X,W})^{-1}(W) \) intersect transversely). The projection of \( (\phi_{1}^{R_X,W})^{-1}(W) \) to \( \mathbb{C} \) is as in Figure 12.

To summarize this first step, we have replaced in our argument the cobordism \( W \) by the cobordism \( W' \). Moreover, by a horizontal Hamiltonian isotopy, we may assume that \( W' \) has a projection as in Figure 13. More precisely, we assume that \( (W')^- = W' \cap Q^-_U \) is a disjoint
union of components $\alpha_i \times L_i$ so that $\alpha_i$ is obtained by rounding the corner of the union of two intervals $(-\infty, -a_U - 4 - s + i] \times \{i - \frac{1}{2}\} \cup \{-a_U - 4 - s + i \times [0, i - \frac{1}{2}\}$. In particular, the intersections of $X$ and $W'$ project onto $\mathbb{C}$ to the points $b_{ij} = \{-a_U - 4 - s + i \times \{j\}$ with $i > j$, $i, j \in \mathbb{N}^*$, $i = 1, 2, \ldots, s$; $b_{ij}$ is precisely the projection of the intersection of the $i$-th end of $W'$ with the $j$-th end of $X$.

We may also assume, by a slight additional horizontal isotopy, that $W' \cap \pi^{-1}(\mathbb{R} \times [-\frac{1}{2}, \infty))$ is a union of cylindrical ends.

**Step 2:** "Snaky" perturbation data. This step of the proof consists in choosing the perturbation data used in the definition of $\mathcal{F}uk^*(E)$ and $\mathcal{F}uk^*(\hat{E})$ in a convenient way. Recall that $W'$ is already fixed as discussed at step 1. The perturbation data in question are chosen as described in §3.2 except that the profile function $h$ as well as the almost complex structure $J$ will be picked with some additional properties described below.

We start with the choice of the profile function $h$. As can be seen from §3.2 the fundamental ingredient in the definition of $h$ are the functions $h_{\pm}$. We start with $h_{+}$: the only requirement in this case is that $h_{+} : [a_U + \frac{3}{2}, \infty) \rightarrow \mathbb{R}$ has its single critical point (the bottleneck) at $a_U + 2$. In other words the difference with respect to the construction at §3.2.2 is that the value $\frac{1}{2}$ is replaced with $a_U$. In fact, as we only consider cobordisms without positive ends the choice of $h_{+}$ is not particularly important as long as the bottlenecks are away from $U$. We
now discuss the function $h_-$. This is a smooth function $h_- : (-\infty, -a_U - 1] \to \mathbb{R}$ with the following additional properties - see Figure 14:

$a'$. The function $h_-$ has critical points $o_i = -a_U - 3 - i$, $i = 0, 1, \ldots, s$ that are non-degenerate local maxima.

$a''$. The function $h_-$ has critical points $o'_i = -a_U - \frac{7}{2} - i$, $i = 0, 1, \ldots, s - 1$ that are non-degenerate local minima.

$a'''$. $h_-$ has no other critical points than those at $a'$, $a''$ above and for all $x \in (-\infty, a_U - 4 - s]$ we have $h_-(x) = \alpha^- x + \beta^-$ for some constants $\alpha^-, \beta^-, \alpha^- > 0$.

Figure 13. The remote cobordism $W' \subset \dot{E}$, the object $X \in \text{Ob}(\text{Fuk}^*(E))$ and the curves $\alpha_i$. The height of the $i$-th end of $W'$ is $i - \frac{1}{2}$ while the $i$-th end of $X$ has height $i$.

Figure 14. The graph of $(\phi^i)^{-1}(\mathbb{R})$ for $s = 4$. 
Beyond this, the properties of the function \( h \) are obtained by direct analogy with those given at the points i, ii, iii, iv in §3.2.2 but with the point \( a \) replaced by the three conditions \( a', a'', a''' \) above. In particular, the set \( W_i^- \) now becomes \( W_i^- = (-\infty, -a_U - 1] \times [i - \epsilon, i + \epsilon] \) and \( T_i^- = (-\infty, -a_U - 1] \times [i - \epsilon/2, i + \epsilon/2] \). From this point on, the construction continues along the same approach as in §3.2. In particular, the properties of the family \( \Theta \) and those of \( J \) are just the same as properties \( a, b, c \) in §3.2.3 but they are relative to a set \( K_{V_1, \ldots, V_{k+1}} \) that satisfies different requirements compared to those in §3.2.3.

We now discuss the two properties required of \( K_{V_1, \ldots, V_{k+1}} \). We start by underlining that while \( V_1, \ldots, V_k \) are elements of \( \mathcal{L}^*(E) \), \( V_{k+1} \) is either an element of \( \mathcal{L}^*(E) \) or \( V_{k+1} = W' \). Further, we fix small disks \( D_{ij} \subset \mathbb{C} \) of radius smaller than \( \frac{1}{8} \) that are respectively centered at the points \( (\delta, j), i = 0, \ldots, s - 1, j \in \{1, \ldots, s_{V_1, \ldots, V_{k+1}} \} \). We denote by \( D'_{ij} \subset D_{ij} \) the disk with the same center but with radius half of that of \( D_{ij} \). Recall, that \( s_{V_1, \ldots, V_{k+1}} \) is the smallest \( l \in \mathbb{N} \) so that \( \pi(V_1 \cup V_2 \cup \ldots \cup V_{k+1}) \subset [\frac{1}{2}, l] \). We also pick a compact set \( Z \subset \mathbb{R} \times (-\infty, -\frac{1}{2}] \) which contains in its interior \( \pi(W') \cap \mathbb{R} \times (-\infty, -\frac{1}{2}] \) (recall that \( W' \) is cylindrical outside \( \pi^{-1}(\mathbb{R} \times (-\infty, -\frac{1}{2}]) \) as well as a slightly bigger set \( Z' \subset \mathbb{R} \times (-\infty, -\frac{1}{4}] \). We require:

\[
K_{V_1, \ldots, V_{k+1}} \supset \bigcup_{i,j} D'_{ij} \cup [-a_U - \frac{13}{4}, a_U + 2] \times [\frac{1}{8}, +\infty) \cup Z'.
\]

and

\[
K_{V_1, \ldots, V_{k+1}} \subset \bigcup_{i,j} D_{ij} \cup [-a_U - \frac{13}{4}, a_U + 2] \times [\frac{1}{8}, +\infty) \cup Z'.
\]

We now will see that this class of perturbation data is sufficient to insure the regularity and the compactness of the moduli spaces appearing in the definition of the category \( \mathcal{F}uk^*(E) \) and of the \( \mathcal{F}uk^*(E) \)-module \( W'_E \). In the next section we will use these specific perturbations to extract the exact triangles claimed in the statement.

Let \( u : S_r \to E \) be a solution of (18) that satisfies the boundary and asymptotic conditions required to define the multiplications \( \mu_k \) for \( \mathcal{F}uk^*(E) \) or for the definition of the module \( W_E \). In the first case the boundary conditions are along cobordisms \( V_1, \ldots, V_{k+1} \) \( (V_i \in \mathcal{L}^*(E), \) in particular, \( V_i \) projects on the upper half plane). In the second case, the curve is defined on a punctured polygon so that the component \( C_i \) of the boundary of the polygon is mapped to \( V_i \) for \( 1 \leq i \leq k \) and the \( k+1 \)-th component \( C_{k+1} \) is mapped to \( W' \).

By the change of variables in §3.2.4, (and by taking \( h \) sufficiently small) we deduce that there exists some small \( \delta > 0 \) so that if \( u : S_r \to E \) satisfies (18) with the choice of perturbation data as just above and if \( v : S_r \to E \) is defined by \( u(z) = \phi_{a_v(z)}^h(v(z)) \), then \( v' = \pi \circ v \) is holomorphic outside of the set

\[
\tilde{K} = \bigcup_{i,j} D''_{ij} \cup [-a_U - \frac{13}{4} - \delta, a_U + 2 + \delta] \times [\frac{1}{8} - \delta, +\infty) \cup Z''.
\]
where $D''_{ij}$ is a disk with the same center as $D_{ij}$ but slightly bigger and, similarly, $Z''$ is a set slightly bigger than $Z'$ - see Figure 15. In view of this transformation, compactness for the

Figure 15. The set $\tilde{K}$ outside which $\nu'$ is holomorphic is the union of all the regions in pink: the disks $D''_{ij}$, the box

$$B = [-a_U - \frac{13}{4} - \delta, a_U + 2 + \delta] \times [\frac{1}{8} - \delta, +\infty)$$

and the neighborhood $Z''$ of the non-cylindrical part of $\pi(W')$. Are also pictured the points $b_{ij}$. Here $s = 3$. The non cylindrical part of the cobordisms $X \in \mathcal{L}^*(E)$ projects inside $B$.

relevant moduli spaces follows without difficulty by the usual bottleneck argument. Thus, the only issue that requires some attention is regularity. Denote

$$K' = \bigcup_{i,j} D'_{ij} \cup [-a_U - \frac{11}{4}, a_U + \frac{7}{4}] \times [\frac{1}{4}, s_{V_1,\ldots,V_{k+1}} + 1] \cup Z .$$

Given that $K' \subset K_{V_1,\ldots,V_k}$, the perturbation data can be chosen essentially freely over $K'$ and thus, for all moduli spaces consisting of curves whose image intersects $\pi^{-1}(K')$ regularity can be handled in the standard fashion as in [Sei3]. Therefore, we are left to analyze the curves $u : S_r \to E$ so that $\pi(u)$ has an image disjoint from $K'$. Assume first that $u$ appears in the definition of the higher structures of $\mathcal{F}uk^*(E)$. In this case, the condition $\pi^{-1}(K') \cap \text{Image}(u) = \emptyset$
implies that all the boundary of \( u \) projects onto \( \mathbb{C} \) along a single line \((-\infty, -a_\ell - 2] \times \{j\}\). Given that \((o^i, j) \in K'\), it follows that the image of \( \pi(u) \) can not cross any of the points \((o^i, j)\), nor can it have one of these points as asymptotic limit. As a consequence, the asymptotic limits of \( u \) have to project to just one of the points \((o_i, j)\). But by now taking a look to \( u' \) which is holomorphic around \((o_i, j)\) one sees immediately that \( u' \) and thus \( \pi(u) \) has to be constant (indeed, \((o_i, j)\) can not be the exit point of \( u' \) by an application of the open mapping theorem). The second possibility to consider is if \( u \) appears in the definition of the module structure of \( W'_E \). It is immediate, in this case too that \( \pi^{-1}(K') \cap \text{Image}(u) = \emptyset \) implies that all asymptotic limits of \( u \) coincide with a single point \( b_{ij} \) (which is, of course, also of the form \((o_i, j)\)). It is easy to see by an application of the open mapping theorem that in this case \( \pi(u) \) has again to be constant. To conclude this argument, the only moduli spaces for which regularity is in question consist of curves \( u \) so that \( \pi(u) \) is constant equal to one of the point \((o_i, j)\). That means that these curves take values in the fiber over \((o_i, j)\) and, because \( o_i \) is a local maximum of \( h_- \), one can see, as in §4.2 [BC3] that by picking regular data in the fiber these moduli spaces are regular too.

Thus the regularity of all the moduli spaces involved can be achieved by generic choices of data. We work from now on with such data associated to the “snaky” perturbations constructed at this step.

**Step 3: The proof of (26).** We will show now that there is a sequence of \( \mathcal{F}uk^*(E) \)-modules \( \tilde{L}_i, W'_{E,i}, i = 1, \ldots, s \), with \( W'_{E,i} \) being submodules of \( W'_E \), so that:

i. \( W'_{E,1} = 0, W'_{E,s} = W'_E \) and for \( i \geq 2 \) there exist exact sequences of \( \mathcal{F}uk^*(E) \)-modules

\[
0 \to W'_{E,i-1} \to W'_{E,i} \to \tilde{L}_i \to 0
\]

ii. there exists a quasi-isomorphism of \( \mathcal{F}uk^*(E) \)-modules

\[
\tilde{L}_i \cong \mathcal{Y}(\gamma_i \times L_i),
\]

where \( \mathcal{Y} \) is the Yoneda embedding for \( \mathcal{F}uk^*(E) \).

These points immediately imply the statement of Proposition 4.3.1. We now proceed to the construction of \( W'_{E,i} \) and to prove the points i, ii above.

Let \( X \in \mathcal{L}^*(E) \) and let \( W' \) be the remote cobordism as discussed at the first step. We now assume “snaky” perturbations picked as described at the second step. In particular, the complex \( CF(X, W') \) is well defined. The generators of this complex are identified with the intersection \( X \cap (\phi^h_1)^{-1}(W') \). Notice that due to the choice of snaky perturbations \( \pi(X \cap (\phi^h_1)^{-1}(W')) \) and \( \pi(X \cap (\phi^h_1)^{-1}(W')) = \{b_{rs}\}_{r,s} \) see Figure 16. We now put

\[
P_{rs}(X) = X \cap (\phi^h_1)^{-1}(W') \cap \pi^{-1}(b_{rs})
\]
and we define

\[ W'_{E,i}(X) = A\langle \cup_{1 \leq r \leq i; s < r} P_{rs} \rangle \subset CF(X, W') . \]

In other words, the generators of \( W'_{E,i}(X) \) are the intersection points of \( X \) with the first \( i \) branches of the \( W' \). It is clear from the construction that \( W'_{E,1} = 0 \) and that \( W'_{E,s} = W'_E \). We will show now that, for each \( 1 \leq i \leq s \), the structural maps \( \mu_k \) of \( W'_E \) when restricted to \( W'_{E,i} \) have values into \( W'_{E,i} \). In other words

\[ (30) \quad \mu_k|_{W'_{E,i}} : CF(V_1, V_2) \otimes \ldots \otimes CF(V_{k-1}, V_k) \otimes W'_{E,i}(V_k) \to W'_{E,i}(V_1) . \]

This property immediately implies that the \( W'_{E,i} \) are indeed \( A_\infty \)-modules and moreover that the inclusions of vector spaces \( W'_{E,i-1}(-) \subset W'_{E,i}(-) \) are actually inclusions of \( \mathcal{F}_{uk}^*(E) \)-modules. The modules \( \tilde{L}_i \) defined as the respective quotients. With these definition for \( W'_{E,i} \) and assuming (30) it is a simple exercise to show point ii. In summary, to conclude the proof of the proposition it remains to show (30).

Our argument is based on properties of the curve \( v' = \pi(v) \) where \( v \) is related to a curve \( u : S_r \to E \) by equation (19) and \( u \) is a solution of (18) contributing to the module structural map \( \mu_k \). Here \( S_r \) is the disk with \( k + 1 \) boundary punctures, of which \( k \) are the entries and the last one is an exit puncture. The last entry, denoted \( m \), is the “module” entry and is asymptotic to a generator of \( CF(V_{k-1}, W'_{E,i}) \). The exit, denoted \( e \), is asymptotic to a generator of \( CF(V_1, W'_{E,i}) \).
We will make the following simplifying assumption: we assume that the transition functions used in the definition of moduli spaces associated to the module multiplication are so that:

\[(31) a_r(z) = 1 \quad \forall z \in C_{k+1},\]

where \(C_{k+1}\) is the component of the boundary of the punctured disk \(S_r\) that joins \(m\) to \(e\). (See Figure 7 for an illustration of the case \(k = 3\), where \(C_4\) bounds both \(\epsilon_3\) and \(\epsilon_4\).) In other words we use transition functions as in §3.2.1 except that we add (31) and we modify conditions \(i. c\) and \(ii. c\)' in §3.2.1 such as to no longer require \(a_r \circ \epsilon(s,t) = 0\) for \((s,t) \in \{0\} \times [0,1]\) for \(\epsilon\) for the strip like ends associated to \(m\) and to \(e\). By imposing (31) just to the moduli spaces appearing in the definition of modules over \(\mathcal{F}uk^*(E)\) (and not to those defining the multiplication in \(\mathcal{F}uk^*(E)\) itself) we easily see that, on one hand, condition (31) is compatible with gluing and splitting and, moreover, it does not contradict the definition of the multiplication in \(\mathcal{F}uk^*(E)\) itself. At the same time, this means that we get two presumtive definitions for the Yoneda modules of objects in \(\mathcal{F}uk^*(E)\): one using the conditions in §3.2.1 and the other making use of (31). However, it is easy to see that the two resulting modules are quasi-isomorphic and thus our simplifying condition does not affect any further arguments.

The geometric advantage of this simplifying assumption on \(a_r\) is that \(v\) no longer satisfies a moving boundary condition along \(C_{k+1}\), rather \(v\) maps all of \(C_{k+1}\) to \(W'' = (\phi^h_1)^{-1}(W')\). We also remark that, by the definition of \(h\), and the position of \(\pi(W')\) relative to the ends of cobordisms \(\in \mathcal{L}^*(E)\) - as in Figure 16 - we have that \(W''\) is just a close perturbation of \(W'\) and \(\pi(W'')\) intersects the horizontal lines of positive, integral imaginary coordinates transversely and in the same points as \(\pi(W')\).

Our claim (30) reduces to showing that if \(v'(m) = b_{\alpha\beta}\) and \(v'(e) = b_{rs}\), then \(r \leq \alpha\).

We first fix some notation relative to certain regions in \(Q_{-U}^-\). First we denote by \(F\) the region given as

\[F = \bigcup_{0 \leq t \leq 1, j \in \mathbb{Z}} \phi^{-1}_{-t}((-\infty, -a_U) \times \{j\}) \cup W''.\]

In short, \(F\) is the set swiped by all the potential boundary conditions of the curves \(v'\). Further, we denote \(\widehat{F} = F \cup \hat{K}\) (see (29)) and we put \(G = \mathbb{C} \setminus \widehat{F}\) - see Figure 17.

From step 2 we know that \(v'\) is holomorphic over \(G\) and clearly, the boundary of \(S_r\) is so that \(v'(\partial S_r) \cap G = \emptyset\). It is an elementary fact (see for instance Proposition 3.3.1 in [BC3]) that as soon as \(\text{Image}(v')\) intersects a connected component of \(G\), the full component has to be contained in \(\text{Image}(v')\). In particular, this means that \(\text{Image}(v')\) can not intersect an unbounded component of \(G\).

Each point \(b_{ij}\) is in the closure of four components of \(G\) that meet, basically, as four quadrants at \(b_{ij}\). Our argument will make use of the following:
Lemma 4.3.2. Suppose that $b_{ij}$ is different from both $v'(e)$ and $v'(m)$ and that the component corresponding to the fourth quadrant at $b_{ij}$ is in the image of $v'$, then at least one among the first or third quadrants are also in the image of $v'$.

For an illustration of the statement of the Lemma take a look to Figure 18 and the point $b_{42}$ there. The claim of the Lemma is that if the green region having $b_{42}$ in its boundary is included in $\text{Image}(v')$, then one of the yellow regions next to $b_{42}$ is also contained in this image.

Proof of Lemma 4.3.2. Consider a small segment $I \subset \pi(W'')$ that ends up at $b_{ij}$ and is included in the closure of the fourth quadrant. We have $I \subset \text{Image}(v')$. Let $x \in I$. If $x$ is the image of a point $z \in \text{Int}(S_r)$, then, by the open mapping theorem, the image of $v'$ also intersects the third quadrant which implies our claim. Thus it is sufficient to consider the case when all the points of $I$ are in the image of boundary points of $S_r$. The only boundary component that is mapped to $W''$ is $C_{k+1}$ so that $I \subset v'(C_{k+1})$. Moreover, as $b_{ij}$ is not the asymptotic image of the ends of $C_{k+1}$, it follows that $b_{ij} \in v'(C_{k+1})$. Let $z \in C_{k+1}$ so that $v'(z) = b_{ij}$. As shown at step 2, $v'$ is holomorphic outside of $\hat{K}$ and thus, in particular, around $b_{ij}$. Given that (around $b_{ij}$) $v'(C_{k+1})$ is contained in the vertical line through $b_{ij}$ and, due to the

Figure 17. The region $\hat{F}$ is the union of $\hat{K}$ (the union of all the pink regions) and $F$ (the region in red).
bottleneck structure around $b_{ij}$, the open mapping theorem implies that $\text{Image}(v')$ intersects the region of $G$ corresponding to the first quadrant and ends the proof of the lemma. □

Figure 18. We take here $s \geq 5$ and in blue are the projections of the ends of $W^m$. Assume $v'(m) = b_{41}$ and suppose $v'(e) = b_{rs}$ with $r \geq 4$; $v'$ exits $b_{41}$ through one of the green regions which is therefore included in $\text{Image}(v')$; Lemma 4.3.2 applied to $b_{42}$ and $b_{41}$ shows that one of the yellow regions $\subset \text{Image}(v')$; by applying again Lemma 4.3.2 to one of the upper left corners of the yellow regions - in light gray - we get that an unbounded region of $G$ is contained in $\text{Image}(v')$. Thus, we reach a contradiction in three steps.

We return to the proof of the proposition and we recall $v'(m) = b_{\alpha \beta}$, $v'(e) = b_{rs}$. Assume that $r > \alpha$. As $m$ is an entry point, by orientation reasons, $\text{Image}(v')$ has to contain at least one of the first or third quadrants at $b_{\alpha \beta}$. In both cases, the upper left corner of the respective quadrant, that we denote by $b_{i_1j_1}$, is so that $i_1 \leq \alpha$. Thus Lemma 4.3.2 can be applied to $b_{i_1j_1}$ and it implies that the first or third quadrant at $b_{i_1j_1}$ is contained in $\text{Image}(v')$. Let $b_{i_2j_2}$ be the upper left corner of the respective quadrant. We have $i_2 \leq i_1$. This process can be pursued recursively, thus getting a sequence of points $b_{i_1j_1}, b_{i_2j_2}, \ldots$ and associated quadrants $\subset \text{Image}(v')$ by picking at each step the upper left corner of a quadrant obtained from Lemma 4.3.2 applied to the previous point in the sequence. This process continues till one the quadrants in question is an unbounded region. But this contradicts the fact that
the image of $v'$ can not intersect such a region. See Figure 18 for an illustration of this argument. □

4.4. Disjunction via Dehn twists. This subsection is purely geometric in nature and is of independent interest. Monotonicity assumptions are not required in this part. The main purpose here is to show that certain Dehn twists of a cobordism are Hamiltonian isotopic to remote cobordisms and therefore can be decomposed by means of Proposition 4.3.1. The idea is the following. Given a cobordisms $V \subset E$, we first add specific singularities to $E$ (with critical values in the lower half plane) so that we can join each initial singularity $x_i$ of $E$ to one of the “new” ones, $x'_i$, by a matching cycle $S_i$. We then show that, with appropriate choices for the matching cycles and the other elements of the construction, the iterated Dehn twist $\tau_{S_m} \circ \ldots \circ \tau_{S_i} \circ \ldots \circ \tau_{S_1}$ transforms $V$ into a remote cobordism $V'$.

4.4.1. The case of a single singularity. We start with the core of the geometric argument. This appears in the case of a fibration with a single singularity.

Fix $S \subset M$, a framed - or parametrized - Lagrangian sphere. We use Seidel’s terminology here [Sei2] (see also [Sei3]) so that this means $S$ is Lagrangian and that we fix a specific parametrization $e : S^n \to S$. Consider a tame Lefschetz fibration $\pi : E \to \mathbb{C}$ with a single singularity $x_1$ with a vanishing cycle that coincides with $S$ and which is tame outside $U \subset \mathbb{R} \times \left[\frac{1}{2}, +\infty\right) \subset \mathbb{C}$. We will assume that the singularity has critical value $v_1 = (1, \frac{3}{2})$. Fix also a cobordism $V : \emptyset \to (L_1, L_2, \ldots, L_s), V \subset E$.

We will make use of an auxiliary Lefschetz fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ that coincides with $E$ over the upper half plane and that has an additional critical point $x'_1$ with corresponding critical value $v'_1 = (-1, -\frac{3}{2})$ and with a vanishing cycle $S' \subset M$ that is related to the fibration $E$ in the following way. Fix a small disk $D$ around $v'_1$ (that is included in the lower half plane). Fix also a path $\gamma$ that joins $v_1$ to a point $v_0$ on the boundary of $D$. We let $T_\gamma$ be the thimble originating at $x_1$ and whose planar projection is $\gamma$. Let $S_0 \subset \pi^{-1}(v_0)$ be the boundary of $T_\gamma$. Finally, the fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ is so that it admits the sphere $S_0$ as vanishing cycle and it is tame outside a set $\hat{U}$ so that (as pictured in Figure 11) $\hat{U} \subset (-\infty, a_\hat{U}] \times [-K, +\infty)$ and, additionally, $v_0 \not\in \hat{U}$. Given that $E$ is trivial over the lower half-plane, the construction of $\hat{E}$ follows directly from the constructions in [Sei3]. In particular, if we extend the curve $\gamma$ to a curve (that we shall continue to denote by $\gamma$) that joins $v_1$ to $v'_1$ this is covered by a matching cycle $\hat{S}_\gamma \subset \hat{E}$.

For further use, we now fix another thimble $T$ originating at $x_1$ and whose projection is the vertical half-line $\{1\} \times \left[\frac{3}{2}, \infty\right)$. 

Proposition 4.4.1. There exists a curve $\gamma$, depending on $V$, and a Lagrangian sphere $S'$ in $\hat{E}$, hamiltonian isotopic to the matching sphere $\hat{S}_\gamma$ so that the Lagrangian $V' = \tau_{S'} V$ is disjoint from $T$ and the intersection $V' \cap S'$ is contained in $D$. 

Proof. We start the proof by recalling the definition of the Dehn twist following [Sei2]. We begin with the model Dehn twist. Let $g$ be the standard round metric on $S^n$ and for $0 < \lambda$ denote by $D_{\lambda}^*S^n \subset T^*S^n$ the disk bundle consisting of cotangent vectors of norm $\leq \lambda$. We have identified here $T^*S^n$ with $TS^n$ via the metric $g$. Our conventions are such that the symplectic form on the cotangent bundle $T^*S^n$ is $dp \wedge dq$ where $q$ is the “base” coordinate and $p$ is the coordinate along the fibre.

Denote by $\psi_t : D_{\lambda}^*S^n \setminus 0_{S^n} \rightarrow D_{\lambda}^*S^n \setminus 0_{S^n}$ the normalized geodesic flow corresponding to $g$, defined on the complement of the zero-section. With our conventions this flow is the Hamiltonian flow of the function $H(p,q) = |p|^2$.

Denote by $\sigma : S^n \rightarrow S^n$ the antipodal map. Note that $\psi_{\pi}$ extends to the zero-section by $\sigma$.

Given $0 < \lambda$, pick a smooth function $\rho_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

1. $\rho(t) + \rho(-t) = 1$ for every $|t| \leq \delta$ for some $0 < \delta < \lambda$.
2. $\text{supp}(\rho) \subset (-\lambda, \lambda)$; $\rho(t) \geq 0$, $\forall t > 0$.

Note that we have $\rho(0) = \frac{1}{2}$.

With the above at hand we define the model Dehn twist $\tau_{S^n} : D_{\lambda}^*S^n \rightarrow D_{\lambda}^*S^n$ by the formula

\[
\tau_{S^n}(x) = \begin{cases} 
\psi_{2\pi \rho(||x||)}^g(x), & x \in T_{\leq \lambda}^*S^n \setminus 0_{S^n}; \\
\sigma(x), & x \in 0_{S^n}.
\end{cases}
\]

Note that $\tau_{S^n}$ is the identity near the boundary of $D_{\lambda}^*S^n$.

Now let $N$ be a symplectic manifold and $f : S^n \rightarrow N$ a Lagrangian embedding of the $n$-sphere. Denote by $S = f(S^n) \subset N$ its image. By the Darboux-Weinstein theorem there exists a neighborhood $U(S) \subset N$ of $S$, a $0 < \lambda$, and a symplectic diffeomorphism $i : D_{\lambda}^*S^n \rightarrow U(S)$ that maps $0_{S^n}$ to $S$ via the map $f$. Define now the Dehn-twist along $S$, $\tau_S : N \rightarrow N$, by setting $\tau_S = i \circ \tau_{S^n} \circ i^{-1}$ on the image of $i$ and extend it as the identity to the rest of $N$.

By the results of [Sei2] the diffeomorphism $\tau_S$ is symplectic and moreover, its isotopy class is independent of the choices of $\rho$ and $\lambda$, but possibly not of the class of parametrization of the Lagrangian sphere $f : S^n \rightarrow S$. We call the symplectomorphism $\tau_S$ the Dehn twist along $S$ (despite the fact that it does not depend solely on $S$).

In the rest of the proof the place of $N$ will be taken by $\hat{E}$ and the role of $S$ by the matching cycle $\hat{S}_\gamma$.

To start the actual proof we first assume that, after a possible Hamiltonian isotopy of $V$, $T$ intersects $V$ transversely in the points $p_1, \ldots, p_k \in T$.

All along the argument it is useful to refer to Figure 19.
The Lefschetz fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ coincides with $E$ over the upper semi-plane; $\hat{\pi}$ has two singularities of critical values $v_1$ and $v'_1$ and is symplectically trivial outside of $\hat{U}$. Are pictured (in projection on $\mathbb{C}$): the "straight" vertical thimble $T$ and its deformation $\hat{T}$; the matching cycle $S$ that coincides with $\hat{T}$ from $v_1$ to $e_0$; the disk $D$; $S \cap V = \{p_1, p_2, p_3\}$; $q_i = \sigma(p_i)$ (where $\sigma$ is the antipodal map); the neighborhood $U(S)$ where is supported $\tau_S$; the portion $\hat{T}'$ of $\hat{T}$ that differs from $S$ and is included in $U(S)$; the projections $I_1, I'_1$ of two disks $K_1, K'_1$ in $S$ around the two singularities of $\hat{\pi}$ so that $S_0 = S \setminus (K_1 \cup K'_1)$ lies inside a trivial symplectic fibration. Notice that the domain $\hat{U}$ is generally unbounded along some additional directions compared to the domain outside which $E$ is tame. This is required so that the fibration $\hat{E}$, that agrees with $E$ over the upper half plane, has additional singularities compared to $E$. Our choice is for this unbounded direction to be in the lower left corner, as in the picture.
**Step 1:** Choice of the curve \( \gamma \). Recall that the fibration \( \pi : E \to \mathbb{C} \) is tame outside the set \( U \subset \mathbb{C} \) and the fibration \( \hat{\pi} : \hat{E} \to \mathbb{C} \) is tame outside the larger set \( \hat{U} \). We fix two neighborhoods \( U(V) \subset U'(V) \) of \( V \). We consider an auxiliary thimble \( \bar{T} \) whose projection on \( \mathbb{C} \) is as in Figure 19. In particular, \( \bar{T} \) coincides with \( T \) inside \( U(V) \) as well as outside of \( U'(V) \) and \( \pi^{-1}(\mathbb{C} \setminus \hat{U}) \cap \bar{T} \neq \emptyset \) but \( \pi^{-1}(\mathbb{C} \setminus \hat{U}) \cap \bar{T} \cap U(V) = \emptyset \). We notice that \( \bar{T} \) is hamiltonian isotopic to \( T \) by an isotopy supported away from \( U(V) \cup \pi^{-1}(\mathbb{R} \times (-\infty, 0]) \) (\( T \) and \( T \) are Lagrangian isotopic and it is easy to check that this isotopy is exact).

Denote by \( \bar{\eta} = \pi(\bar{T}) \). We assume that - as in Figure 19, \( \bar{\eta} \) can be written as the union of three closed connected sub-segments \( \bar{\eta} = \bar{\eta}' \cup \bar{\eta}'' \cup \bar{\eta}''' \) so that \( \bar{\eta}' \cup \bar{\eta}''' \) is the closure of \( \hat{U} \cap \bar{\eta} \). Thus, the interior of \( \bar{\eta}'' \) is disjoint from \( \hat{U} \). We also assume to fix that \( \bar{\eta}'' \subset [1, \infty) \times [1, \infty) \). Consider a point \( e_0 \) inside the segment \( \bar{\eta}'' \) so that \( \bar{\eta}'' = \bar{\eta}''_1 \cup \bar{\eta}''_2 \) with \( \bar{\eta}''_1 \) and \( \bar{\eta}''_2 \) the closures of the two sub-segments given by \( \bar{\eta}'' \setminus \{e_0\} \) with \( e_0 \) being the end-point of \( \bar{\eta}''_1 \) and the starting point of \( \bar{\eta}''_2 \). We now pick the curve \( \gamma \subset \mathbb{C} \) that joins \( v_1 \) to \( v'_1 \) so that \( \gamma \) can be written as a union of two connected, closed parts \( \gamma = \gamma_1 \cup \gamma_2 \) so that \( \gamma_1 \) originates in \( v_1 \) and coincides with \( \bar{\eta}' \cup \bar{\eta}''_1 \), \( \gamma_2 \) is disjoint from \( U(V) \), it intersects \( \bar{\eta} \) only in \( e_0 \), it ends in \( v'_1 \) and \( \gamma_2 \setminus D \subset \mathbb{C} \setminus \hat{U} \). Clearly, \( e_0 \) is a point where \( \bar{\eta} \) and \( \gamma \) are tangent and after this point \( \gamma \) is to the “right” of \( \bar{\eta} \) and is included in \( \mathbb{C} \setminus \hat{U} \) till (and including) the moment it reaches \( D \).

Notice that if we show that:

\[
(33) \quad \tau_{S_{\gamma}} V \cap \bar{T} = \emptyset \quad \text{and} \quad \tau_{\hat{S}_{\gamma}} V \cap \hat{S}_{\gamma} \subset D
\]

then by using the Hamiltonian isotopy \( \psi \) that carries \( \bar{T} \) to \( T \) and such that \( \psi(V) = V \), we deduce that there is a Lagrangian sphere \( S' = \psi(\hat{S}_{\gamma}) \) so that \( \tau_{S'} V \) is disjoint from \( T \) and \( \tau_{\hat{S}} V \cup S' \subset D \). For this argument, \( \tau_{S'} \) is defined by using the choice of framing so that \( \tau_{S'}^{-1} = \psi \circ \tau_{\hat{S}_{\gamma}}^{-1} \circ \psi^{-1} \) (hence \( \tau_{S'}^{-1}(V) = \psi \circ \tau_{\hat{S}_{\gamma}}^{-1}(V) \)). In short, it remains to show (33).

**Step 2:** Other choices involved in the definition of the twist. From now on, to simplify notation, we put \( S = \hat{S}_{\gamma} \). We first choose a small Weinstein neighborhood \( U(S) \) of \( S \). The Dehn twist \( \tau_S \) will be supported inside this neighborhood. We notice, by construction, that \( \{p_1, \ldots, p_k\} = T \cap V = \bar{T} \cap V = S \cap V \). We may assume that \( V \cap U(S) \) is a union of small disks \( D \subset V \) centered at \( p_i \), for convenience, we may assume are included in the fiber of \( T^*S \) through \( p_i \) under the identification of \( U(S) \) with a disk bundle of \( T^*S \). Further, we denote by \( \bar{T}' \) the closure of \( (\bar{T} \setminus S) \cap U(S) \). We now consider a disk \( K_1 \subset S \) centered at \( x_1 \) so that \( U(V) \cap S \subset K_1 \). Similarly we also consider a disk \( K'_1 \subset S \) centered at \( x'_1 \). We assume that both \( K_1 \) and \( K'_1 \) are preimages of segments \( I_1 \) and \( I'_1 \) contained in \( \gamma \) and we suppose that the two disks are so that \( \gamma_0 = \gamma \setminus (I_1 \cup I'_1) \subset \mathbb{C} \setminus \hat{U} \), \( e_0 \in \gamma_0 \) and \( I'_1 \subset D \). We further pick \( U(S) \), \( K_1 \) and \( K'_1 \) so that \( \bar{T}' \) is disjoint from both \( K_1 \) and \( K'_1 \). We consider the curve oriented so that it starts at \( v_1 \) and ends at \( v'_1 \).
The boundary of $K_1$ is a Lagrangian sphere $A \subset (M, \omega)$ and the boundary of $K'_1$ is the same sphere transported to the end of $\gamma_0$ (parallel transport is trivial along $\gamma_0$ because $\hat{\pi}$ is symplectically trivial outside $\hat{U}$). We denote the sphere that appears as boundary of $K'_1$ by $A'$. The region $S_0 = S \setminus \text{Int}(K_1 \cup K'_1)$ is diffeomorphic to a cylinder $C = [-a, a] \times A$. We think about this cylinder so that $\{-a\} \times A$ corresponds to the boundary of $K_1$ and $\{a\} \times A$ corresponds to the boundary of $K'_1$.

Denote by $U(S_0)$ the restriction of the neighborhood $U(S)$ (identified with a disk bundle in $T^*S$) to $S_0$. We assume $U(S)$ small enough so that $\pi(U(S_0)) \subset \mathbb{C} \setminus \hat{U}$. As $\hat{\pi}$ is trivial over $U(S_0)$, by possibly reducing $U(S)$ further, we obtain the existence of a symplectomorphism:

$$k : D_rT^*[-a, a] \times D_rT^*A \to U(S_0) \approx D_sT^*S_0 \subset \hat{E}$$

(here $D_r(-)$ are the respective radius-$r$ disk bundles). After picking $a$ appropriately, this symplectomorphism can be made also compatible with the almost complex structures involved so that $\pi' = \hat{\pi} \circ k$ is holomorphic with respect to the split standard complex structure in the domain and the standard complex structure in $\mathbb{C}$.

**Step 3: The parametrization of $S$.** This step consists in picking a particular framing of $S$ so that the associated Dehn twist $\tau_S$ can be tracked explicitly. To simplify slightly notation we assume $a = 1 - \delta$ with $\delta$ very small.

We fix a diffeomorphism $\varphi : S^n \to A$. Let $h : S^{n+1} \to \mathbb{R}$ be the height function defined on the standard round sphere in $\mathbb{R}^{n+2}$ and let $S_\delta = h^{-1}([-a, a])$. We now pick a parametrization of $\alpha : S^{n+1} \to S$ so that the restriction of this parametrization to $S_\delta$ is a diffeomorphism $\alpha_0 = \alpha|_{S_\delta} : S_\delta \to C$ with the property that for each $t \in [-a, a]$, $\alpha|_{h^{-1}(t)} : h^{-1}(t) \to \{t\} \times A \subset C$ is a rescaling of $\varphi$, and so that $h(\alpha^{-1}(x)) = -1$, $h(\alpha^{-1}(x')) = 1$ (recall that $x, x' \in \hat{E}$ are the critical points of $\pi$ lying over $v_1, v'_1$ respectively). Clearly, $\alpha_0$ extends to a symplectic diffeomorphism $\tilde{\alpha}_0 : T^*S_\delta \to T^*C$ so that $T^*h^{-1}(t)$ is mapped by a symplectomorphism to $\{t\} \times T^*A$. Basically, we are parametrizing here the “flat” cylinder $C$ (which is identified with $S_0$) by the “round” cylinder $S_\delta$ and we then extend this parametrization as symplectomorphisms at the level of the cotangent bundles. All the parametrizations involved identify level sets of the height function on $S_\delta$ to slices of the cylinder $C$.

We denote by $\sigma : S \to S$ the antipodal map defined using this parametrization. This means, in particular, that the points $q_i = \sigma(p_i)$ are contained in $D$ (the disk appearing in the statement of the proposition). As showed in [Sei2], with an appropriate choice of function $\rho$ in the definition of the Dehn twist (which we have assumed here) the intersection $\tau_S V \cap S$ is transverse and consists precisely of the antipodal of the intersection $S \cap V$. Thus, $\tau_S V \cap S = \{q_1, \ldots, q_k\} \subset D$ as claimed in the second part of (33). It remains to show the main part of the claim: $\tau_S V \cap \bar{T} = \emptyset$. As $\tau_S V \cap S = \{q_1, \ldots, q_k\}$, the Dehn twist $\tau_S$ is supported inside $U(S)$ and given that $\bar{T}$ and $S$ coincide along the segment of $\gamma$ that starts at $v_1$ and
ends at $e_0$ it follows that
\begin{equation}
\tau_S V \cap \bar{T} = \tau_S V \cap \bar{T}' = \tau_S (V \cap \tau_S^{-1}(\bar{T}'))
\end{equation}
Thus, to conclude the proof, it is enough to show $\tau_S^{-1}(\bar{T}') \cap V = \emptyset$.

**Step 4:** Showing $\tau_S^{-1}(\bar{T}') \cap V = \emptyset$. By possibly adjusting the neighborhood $U(S)$ we may assume that $U$ can be written as $U(S) = (k \circ \tilde{a}_0)(U(S^{n+1}))$ for some neighborhood $U(S^{n+1})$ of the zero section inside $T^*S_{\delta}$. Let $\bar{T}' = (k \circ \tilde{a}_0)^{-1}(T')$. We denote by $U(S_{\delta})$ the corresponding neighborhood of $S_{\delta}$ (so that $U(S_{\delta})$ is the preimage of $U(S_0)$) and we let $\bar{K}_1$ be the cap $\bar{K}_1 = h^{-1}(-1, -1 + \delta] = (k \circ \tilde{a}_0)^{-1}(K_1)$. Further, we let $U(\bar{K}_1)$ be the restriction of $U(S^{n+1})$ over $\bar{K}_1$. Clearly $\bar{T}' \subset U(S_{\delta})$, and to show the claim it is enough to notice that $\tau_S^{-1}(\bar{T}') \cap U(\bar{K}_1) = \emptyset$ where now $\tau_S$ is the standard model for the Dehn twist.

Let $(x, v) \in \bar{T}' \subset T^*S_{\delta}$ with $v \in T_x^*S^{n+1}$, $v \neq 0$. We now notice that the condition that $\bar{T}'$ is to the “left” of $S$ in Figure 19 translates to the fact that
\begin{equation}
\langle v, J\nabla h(x) \rangle > 0 .
\end{equation}
Here $J$ is an almost complex structure on $T^*S_{\delta}$ with respect to which, as at Step 2, the map $\pi' = \pi \circ k$ is holomorphic. This follows from the same inequality that is valid for the planar projection of $\bar{T}'$ relative to $\gamma_0$. Equation (35) implies that the geodesic flow with origin $(x, v)$ has its vertical component pointing in the direction of $-\nabla h$ (because if $\langle v, w \rangle > 0$, then the geodesic associated to $v$ points in the direction of $Jw$). Thus, the inverse of the geodesic flow points in the direction of $\nabla h$ and therefore away from $\bar{K}_1$. As a consequence, it is easy to see that the orbit $\phi^t_\pi(x, v)$ for $-\pi \leq t \leq 0$ does not intersect $U(\bar{K}_1)$ and, as a consequence, $\tau_S^{-1}(\bar{T}') \cap V = \emptyset$ - see also Figure 20. □

**Corollary 4.4.2.** With the notation in Proposition 4.4.1 the cobordism $\tau_S V$ is hamiltonian isotopic - via an isotopy with compact support - to a cobordism that is remote relative to $E$.

**Proof.** We already know from Proposition 4.4.1 that $V' = \tau_S V$ is disjoint from $T$. Consider an $\Omega$-compatible almost complex structure $J$ on $E$ with the additional property that $\pi : E \to \mathbb{C}$ is $J$-holomorphic. It is well known that the function $\text{Im}(\pi) : E \to \mathbb{R}$ defines a Morse function on $E$ whose negative gradient flow $\xi$ (with respect to the metric induced by $(\Omega, J)$) is also Hamiltonian. Moreover $\xi$ has the thimble $T$ as a stable manifold. Write $\xi = X^H$ with $H : E \to \mathbb{R}$. Now consider a smooth function $\eta : \mathbb{C} \to \mathbb{R}$ so that $\eta(z) = 1$ if $z \in [-a_U - 1, a_U + 1] \times [-\frac{1}{2}, +\infty)$ and $\eta(z) = 0$ if $z \in ((-\infty, -a_U - 2] \times \mathbb{R}) \cup ([-a_U - 2, a_U + 2] \times (-\infty, -\frac{3}{2}]) \cup ([a_U + 2, \infty) \times \mathbb{R})$. Let $\xi'$ be the Hamiltonian flow of the function $(\eta \circ \pi)H$ defined on $E$. It is easy to see that, after sufficient time, the flow $\xi'$ isotopes $V'$ to a new cobordism $V''$ that is included in $\tilde{h}^{-1}(\mathbb{R} \setminus (-\infty, 0] \times \mathbb{R} \cup Q_U^-)$. Therefore, $V''$ is remote relative
to $E$. Moreover, as the ends of $V'$ are not moved by this isotopy, it is easy to see that, by a further truncation of $\xi'$, $V''$ is hamiltonian isotopic to $V'$ through a compactly supported isotopy.

\[ \square \]

4.4.2. Multiple singularities. Consider a Lefschetz fibration $\pi : E \to \mathbb{C}$ as in §4.1, thus possibly with more than one singularity.

We fix $V \in \mathcal{O}(\mathcal{F}uk^*(E), V : \emptyset \sim (L_1, \ldots, L_s))$. The purpose of this subsection is to describe an extension of Proposition 4.4.1 and Corollary 4.4.2 to the case of multiple singularities.

We will consider a fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ that extends $E$ and has one more singularity $x'_i$ for each singular point $x_i$, $1 \leq i \leq m$, of $\pi$ so that the vanishing cycles of $x_i$ and $x'_i$ can be related by matching cycles $\hat{S}_i$ that are the analogues of the matching cycle $\hat{S}_\gamma$ from Proposition 4.4.1. We then obtain Lagrangian spheres, $S'_i$ that are hamiltonian isotopic to $\hat{S}_i$, as in Figure 21, and we then consider the image of $V$ under the iterated Dehn twist

\[ V' = \tau_{S'_m} \circ \tau_{S'_{m-1}} \circ \cdots \circ \tau_{S'_1}(V) \]

inside $\hat{E}$ as well as the following Hamiltonian isotopic copy of it $V'' = \tau_{S''_m} \circ \tau_{S''_{m-1}} \circ \cdots \circ \tau_{S''_1}(V)$ obtained by applying an iterated Dehn twist along the Lagrangian spheres $S''_j$ which are Hamiltonian isotopic to the $S'_j$'s.

Let $\mathcal{T}_i$ be the vertical thimble with origin the critical point $x_i$ and projecting to the vertical half-line $\{i\} \times [\frac{3}{2}, \infty)$. The thimbles $\mathcal{T}_i$ generalize the thimble $T$ considered earlier (just before Proposition 4.4.1) in the context of one singularity to the case of multiple singularities. We
denote them by $\mathcal{T}$ (this avoids confusion with the thimbles $T_i$ that are horizontal at infinity and are associated to the curves $t_i$, see Figure 10).

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure21.png}
\caption{The cobordism $V : \emptyset \to (L_1, L_2, L_3, L_4)$, the Lagrangian spheres $S'_1, S'_2, S'_3$ together with the vertical thimbles $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ so that $V'' = \tau_{S'_m} \circ \tau_{S'_{m-1}} \circ \cdots \circ \tau_{S'_1}(V)$ is disjoint from the $\mathcal{T}_i$'s.}
\end{figure}

**Corollary 4.4.3.** It is possible to construct $\hat{E}$ and the Lagrangian spheres $S'_i$ so that the cobordism $V''$ is disjoint from all the thimbles $\mathcal{T}_i$. As a consequence, there exists a horizontal Hamiltonian isotopy $\phi$ so that the cobordism $\phi(V'') \subset \hat{E}$ is remote relative to $E$. In particular, in $D\text{Fuk}^*(E)$, there exists a cone decomposition:

$$V' \cong (\gamma_s \times L_s \to \gamma_{s-1} \times L_{s-1} \to \cdots \to \gamma_2 \times L_2).$$

**Proof.** The first part of the proof is to construct iteratively fibrations $\hat{\pi}_i : \hat{E}_i \to C$ with $\hat{E}_0 = E$ and with the final fibration $\hat{E} = \hat{E}_m$ so that $\hat{E}_{i+1}$ extends $\hat{E}_i$ and has one more singularity, $x'_{i+1}$, compared to $\hat{E}_i$. At each step we also construct the matching cycles $\hat{S}_i$ joining $x_i$ to $x'_i$ and their Hamiltonian isotopic images $S'_i$ so that the relevant properties are satisfied. Here are more details on the induction step. Assume that $\hat{E}_k$ has already been constructed together with the matching cycles $\hat{S}_i$ and their hamiltonian isotopic copies $S'_i$, $1 \leq i \leq k$ so that $V''_k = \tau_{S'_k} \circ \tau_{S'_{k-1}} \circ \cdots \circ \tau_{S'_1}(V)$ is disjoint from $\mathcal{T}_i$, $1 \leq i \leq k$. We now consider the cobordism $V''_k$ and the vertical thimble $\mathcal{T}_{k+1}$ and we apply to them the construction described in the proof of Proposition 4.4.1. This produces first a new fibration $\hat{E}_{k+1}$ that has an additional singularity denoted now by $x'_{k+1}$. Here, the only difference with respect to the construction of $\hat{E}$ in Proposition 4.4.1 is that the coordinates of the critical value $v'_{k+1}$ associated to $x'_{k+1}$ is
now \((-1, -k - \frac{3}{2})\) and the set \(\hat{U}\), outside which \(\hat{E}_{k+1}\) is tame, is extended appropriately inside the third-quadrant. Further, just as in the proof of Proposition 4.4.1 we can construct the deformed thimble \(\mathcal{T}_{k+1}\) as well as the matching cycle \(\hat{S}_\gamma\) so that \(\hat{S}_\gamma\) coincides with \(\mathcal{T}_{k+1}\) over a certain sub-segment of \(\gamma\). Two important points should be made here: first, the place of \(V\) in the proof of Proposition 4.4.1 is taken here by \(V''_{k+1}\); second \(\mathcal{T}_{k+1}\) as well as \(\mathcal{T}_{k+1}\) and \(\hat{S}_\gamma\) are all disjoint from \(\mathcal{T}_i\) for \(i \leq k\). Now, again as in the proof of Proposition 4.4.1, we obtain that there exists a Hamiltonian isotopy \(\psi_{k+1}\) supported outside a neighborhood of \(V''_{k+1}\) so that \(S'_{k+1} = \psi_{k+1}(\hat{S}_\gamma)\) has the property that \(V''_{k+1} = \tau_{S'_{k+1}} V''_{k}\) is disjoint from \(\mathcal{T}_{k+1}\). One additional point appears here: it is easy to see that the isotopy \(\psi_{k+1}\) can be assumed to leave fixed \(\mathcal{T}_i\) for \(i \leq k\). By defining \(V''_{k+1}\) by using a sufficiently small neighborhood \(U(S'_{k+1})\) of \(S'_{k+1}\) so that \(U(S'_{k+1}) \cap \mathcal{T}_i = \emptyset\) for all \(i \leq k\), we also deduce \(V''_{k+1} \cap \mathcal{T}_i = \emptyset\) \(1 \leq i \leq k\) and the induction step is completed.

We now put \(V'' = V''_m\) and we know that \(V''\) is disjoint from all the thimbles \(\mathcal{T}_i\). Constructing the horizontal isotopy that transforms \(V''\) into a cobordism \(V''\) remote relative to \(E\) is a simple exercise by, possibly, iterating the construction in Corollary 4.4.2.

Finally, the cone-decomposition in the statement follows by applying to \(V'''\) Proposition 4.3.1. 

4.4.3. **Dehn twist as multiple surgery.** Here we give an interpretation of the action of a Dehn twist on Lagrangian submanifolds in terms of surgery. Fix \(S^n \rightarrow S \subset M\), a parametrized Lagrangian sphere and let \(L\) be another Lagrangian submanifold of \((M, \omega)\). It is known that if \(L\) and \(S\) intersect transversely and in a single point, then Lagrangian surgery at this point produces a Lagrangian \(S\# L\) that is Hamiltonian isotopic to the Dehn twist \(\tau_S L\) of \(L\) along \(S\) (see e.g. [Sei1, Tho]). See [Pol] as well as [LS] for the definition of Lagrangian surgery). Assume now that \(L\) is still transverse to \(S\) but that the number of intersection points \(L \cap S\) is more than one. In this case too, one can express the Dehn twist \(\tau_S(L)\) as a certain type of surgery. The construction goes as follows. Assume that \(L \cap S = \{p_1, \ldots, p_r\}\). Fix an additional point \(p_0 \in S\) and a small neighborhood of it \(V \subset S\).

i. Consider \(r\) Hamiltonian diffeomorphisms \(\phi^j, 1 \leq j \leq r\) supported in a small Weinstein neighborhood of \(S\), so that \(S_j = \phi^j(S)\) is transverse to \(S\) and \(S_j \cap S = \{p_j, p'_j\}\) for some additional point \(p'_j \in V\).

ii. Pick small disks \(D^L_j \subset L\) centered at \(p_j\) and disks \(D^{S_j} \subset S_j\) also centered at \(p_j\) as well as Lagrangian handles \(H_j \subset M\) defined in a small neighborhood of \(p_j\) that join \(S_j\) to \(L\) so that \((L \setminus D^L_j) \cup (S_j \setminus D^{S_j}) \cup H_j\) is the usual Lagrangian surgery \(L \# S_j\) between \(L\) and \(S_j\) at the point \(p_j\) (this is, in general, an immersed Lagrangian). Notice that there are two choices for Lagrangian surgery at each intersection point. The choice used here is the same at each point and is the one defined as follows (this is the same convention
as in [BC2]). The sphere $S$ is oriented hence so are the $S_j$’s. This induces a local orientation on $L$ (even if $L$ is not orientable) near each intersection point $p_j$ in such a way that $T_{p_j}S_j \oplus T_{p_j}L$ gives the orientation of $T_{p_j}M$. We then symplectically identify a neighborhood of $p_j \in M$ with a neighborhood of 0 in $\mathbb{R}^{2n}$ in such a way that $D^S_j$ is identified with a small disk around 0 in $\mathbb{R}^n \times \{0\}$ and $D^L_j$ with a small disk around 0 in $\{0\} \times \mathbb{R}^n$, with the last two identifications being orientation preserving. The model Lagrangian handle is then defined to be

$$H_j = \bigcup_{t \in [-1,1]} \gamma(t) S_{n-1} \in \mathbb{C}^n \cong \mathbb{R}^{2n},$$

where $\gamma(t) : [-1, 1] \rightarrow \mathbb{C}$ is an appropriately chosen curve whose image is in the 2’nd quadrant and such that $\gamma(t) \in \mathbb{R}_{<0}$ for $t$ close to $-1$ and $\gamma(t) \in i\mathbb{R}_{>0}$ for $t$ close 1.

iii. Define $S \#_r L$ by

$$S \#_r L = (\bigcup_j S_j \setminus D^S_j) \cup (L \setminus \bigcup_j D^L_j) \cup (\bigcup_j H_j).$$

In other words $S \#_r L$ is obtained by performing simultaneously, for all $1 \leq j \leq r$, the one point surgery at $p_j$ between $S_j$ and $L$.

Figure 22. Dehn twist as multiple surgery for $n = 1$ assuming two intersection points $p_1, p_3$ between $L$ and $S$.

Either by a direct argument - this is instructive to draw in dimension two as in Figure 22 - or by comparing this multiple surgery construction with the definition of $\tau_{SL}$, we see that there exist choices of $\phi^j, D^L_j, D^S_j, H_j$ so that:

i. $S \#_r L$ is embedded and is Hamiltonian isotopic to $\tau_{SL}$.

ii. $S \#_r L$ is transverse to $S$ and it intersects $S$ in the r points $p_j \in V, 1 \leq j \leq r$.

iii. If both $L$ and $S$ are monotone of monotonicity constant $\rho$, then so is $S \#_r L$. 

As explained above, the local model for surgery at a point requires an order among the two Lagrangians involved. By reversing the order for all the one-point surgeries, we obtain again a Lagrangian denoted now \( L \#_r S \). This has properties similar to i, ii, iii above except that it is hamiltonian isotopic to \( \tau_S^{-1} L \). From this perspective, Proposition 4.4.1 claims that, with appropriate choices of handles, we have \( (S' \#_r V) \cap T = \emptyset \).

**Remark 4.4.4.** The “doubling” of singularities used in Proposition 4.4.1 first appeared in a somewhat different form and with a different purpose in the work of Seidel [Sei3]. From the perspective of our paper, the initial approach to the setting of Proposition 4.4.1 was to consider a thimble \( T' \) (inside \( E \)) that projects over the curve \( \gamma \) in Figure 19 and continues horizontally to \( -\infty \). The idea was to disjoin \( V \) from \( T \) by a process of multiple surgery with multiple copies of \( T' \), in other words to define \( V' = T' \#_r V \) so that \( V' \cap T = \emptyset \). Purely geometrically, this operation is possible. However, the problem in drawing algebraic conclusions from it is that the condition \( V' \cap T = \emptyset \) turns out to force that the copies of \( T' \) used in the surgery are not cylindrical at infinity. As a consequence, \( V' \) is not cylindrical at infinity either and all the machinery involving \( J \)-holomorphic curves can not be applied directly to it. On the other hand, by compactifying \( T' \) to the sphere \( S' \) - as described in the paper - this issue is no longer present. The price to pay is that we need to add singularities to the initial fibration \( E \).

### 4.5. A cobordism viewpoint on Seidel’s exact triangle.

In this section we present a new proof of Seidel’s exact triangle [Sei2, Sei3]. This is the last essential ingredient for the proof of Theorem 4.2.1. Our proof is based on cobordism considerations and is valid in the monotone setting. We give full details not only for the sake of self-containedness but also in order to emphasize the reason why the Novikov ring \( \mathcal{A} \) is required in the proof of Theorem 4.2.1: this is precisely in establishing Seidel’s exact triangle. Additionally, in the proof of Theorem 4.2.1 we need a variant of the exact triangle that applies to the case when the Lagrangian to which is applied the Dehn twist is itself a cobordism in the total spaces of a Lefschetz fibration and the proof is robust enough to cover this case with minimal adjustment.

Seidel’s proof [Sei3] assumes an exact setting but his argument adapts to the monotone case too and also admits further generalizations as in [WW].

#### 4.5.1. The exact triangle.

We work, as in the rest of the paper, with coefficients in the universal Novikov ring \( \mathcal{A} \) over \( \mathbb{Z}_2 \) and with monotone Lagrangians assumed to be of class \( * \). Floer complexes and Fukaya categories are ungraded.

Below we will have two versions of the Seidel’s exact triangle. The first is for symplectic manifolds \( X \) (which are either closed or symplectically convex at infinity) and their compact Fukaya categories (i.e. the Fukaya categories whose objects are *closed* Lagrangian submanifolds). The second version is specially tailored to the situation when \( X \) is itself the total space of a Lefschetz fibration and the Fukaya category considered in \( X \) is that of negatively ended
cobordisms in $X$. It is the second version that will be used in the proof of Theorem 4.2.1. We will later exhibit $X$ as a fiber in a Lefschetz fibration denoted by $\mathcal{E}$. The choice of notation ($\mathcal{E}$ and $X$) is intentional, in order to avoid confusion with the Lefschetz fibrations $E \to C$ and their fibers $M$ that appear in the rest of the paper.

We begin with the version for the compact Fukaya category. Let $(X^{2n}, \omega)$ be a symplectic manifold which is either closed or symplectically convex at infinity. Let $S$ a parametrized Lagrangian sphere in $X$, i.e. a Lagrangian submanifold $S \subset X$ together with a diffeomorphism $i_S : S^n \to S$. Recall that we denote by $\tau_S : X \to X$ the Dehn twist associated to $S$. Assume further that $S \subset X$ is monotone and denote by $*$ its monotonicity class. Following the conventions of the paper, we write $\mathcal{F}uk^*(X)$ for the Fukaya category of monotone closed Lagrangian submanifolds of $X$ of monotonicity class $*$.

The following important result was proved by Seidel [Sei2] in the exact case. As mentioned above, we extend the result to the monotone case and provide an independent proof.

**Proposition 4.5.1.** Let $X$, $S$ be as above and let $Q \subset X$ be another monotone closed Lagrangian submanifold of monotonicity class $*$. In $D\mathcal{F}uk^*(X)$ there is an exact triangle of the form:

\begin{equation}
\begin{array}{ccc}
\tau_S(Q) & \rightarrow & Q \\
\downarrow & & \downarrow \\
S \otimes \mathcal{H}F(S,Q) & \leftarrow & \\
\end{array}
\end{equation}

The proof of this result will occupy most of §4.5.3 below.

**Remark 4.5.2.** If one restricts the objects in the Fukaya category of $X$ to orientable Lagrangians, our proof should hold also with a $\mathbb{Z}_2$-grading. Similarly, under more assumptions on the Lagrangians (and additional structures) the proof is expected to carry over with a $\mathbb{Z}$-grading as well as, if one assumes all Lagrangians to be endowed with spin structures, with coefficients in $\mathbb{Z}$.

4.5.2. Second version of the exact triangle: the case when $X$ is a Lefschetz fibration. Here we assume that $X$ is the total space of a tame Lefschetz fibration $\pi_X : X \to C$, as defined in §2. (The assumption that $X$ is symplectically convex at infinity is now dropped.) We denote by $\mathcal{F}uk^*(X)$ the Fukaya category of $X$ whose objects are negatively ended Lagrangian cobordisms in $X$ of monotonicity class $*$ as defined in §3.2.

**Proposition 4.5.3.** For $X$ as above, let $S \subset X$ be a monotone Lagrangian sphere of class $*$ and let $Q \subset X$ be a monotone Lagrangian cobordism (possibly without ends) of the same monotonicity class. Then in $D\mathcal{F}uk^*(X)$ there is an exact triangle as in (37).
The proof is very similar to the proof of Proposition 4.5.1 (which is given in §4.5.3 below),
the only difference being that now \( Q \) is allowed to be a cobordism rather than just a closed
Lagrangian (and similarly for the objects of \( \mathcal{F}uk^*(X) \)). We explain the necessary modifications
to the proof with respect to compact case in §4.5.7 below.

4.5.3. Outline of the proof of Proposition 4.5.1. The proof is simple and we summarize it
here (the precise details are given in §4.5.4 below). By the geometric interpretation of the
monodromy around an isolated Lefschetz singularity - as described in [Sei2] - there exists a
Lefschetz fibration \( \pi : E \to \mathbb{C} \) with a single singularity - chosen at the origin - and a cobordism
\( V \subset E \) as in Figure 24, that projects to the curve \( \gamma'' \) there, and has ends \( Q \) and \( \tau S Q \).
Consider a second cobordism \( W \), as in the same picture, obtained as the trail of \( N \) along
the curve \( \gamma' \), where \( N \) is any Lagrangian in \( L^*(X) \). The cobordism machinery produces an associated
chain morphism \( CF(N,\tau S Q) \to CF(N,Q) \) given by counting the Floer strips going from
the intersections of \( W \) and \( V \) that project to \( w_1 \) to the intersections that project to \( w_0 \) and
the cone - in the sense of chain complexes - over this morphism is \( CF(W,V) \). The proof
reduces to finding a quasi-isomorphism \( CF(N,S) \otimes CF(S,Q) \to CF(W,V) \). The next step
is again geometric and is based on the well-known fact that the function \( \text{Re}(\pi) \) is Morse with
a single singularity at the origin and that its gradient with respect to the standard metric is
Hamiltonian. Moreover, the positive horizontal thimble originating at 0 is the stable manifold
of \( \text{Re}(\pi) \) and the negative horizontal thimble is the unstable manifold of \( \text{Re}(\pi) \). To start this
stage in the proof, we use the flow of \( \nabla \text{Re}(\pi) \) to push \( W \) to the right in picture Figure 24
thus getting \( \tilde{W} \); similarly, we use the gradient of \( -\text{Re}(\pi) \) to push \( V \) to the left in the same
picture thus getting \( \tilde{V} \) - see Figure 25. We notice that \( CF(\tilde{W},\tilde{V}) \cong CF(W,V) \) and analyze
the complex \( CF(\tilde{W},\tilde{V}) \). Assuming all relevant intersections are generic, by standard Morse
theory, if \( W \) is pushed enough to the right, \( \tilde{W} \) intersects a neighborhood around the singularity
in a number \( n_1 \) of copies of the stable manifold of \( \text{Re}(\pi) \). Moreover, \( n_1 \) is equal to the number
of intersections of \( W \) with the unstable manifold of \( \text{Re}(\pi) \). Similarly, \( \tilde{V} \) intersects the same
neighborhood in \( n_2 \) copies of the unstable manifold of \( \text{Re}(\pi) \) and \( n_2 \) is equal to the number
of intersections of \( V \) with the stable manifold of \( \text{Re}(\pi) \). The interpretation of the stable and
unstable manifolds as thimbles (and our transversality assumptions) immediately imply that
\( n_1 \) equals the number of intersection points \( N \cap S \) and \( n_2 \) is the number of intersections \( S \cap Q \).
Moreover, each copy of the stable manifold that is associated to \( W' \) intersects precisely once
each copy of the unstable manifold that is contained in \( V' \). In short, it follows that there is a
bijection \( \Xi \) between the following two sets \( (N \cap S) \times (S \cap Q) \equiv (\tilde{W} \cap \tilde{V}) \). The last step of
the proof is more technical and shows that \( \Xi \) extends to a quasi-isomorphism of chain complexes.
The basic idea here is to compare the \( \Xi \) with the product \( \mu_2 : CF(\tilde{W},T_{\Delta}) \otimes CF(T_{\Delta},\tilde{V}) \to CF(\tilde{W},\tilde{V}) \) where \( T_{\Delta} \) is a thimble as in Figure 24. The key part of the argument is to notice
that the $J$-holomorphic triangles giving this product decompose in two classes: “short” ones, of small area, and “long” ones, of big area, and that the short component of $\mu_2$ is a bijection identified to $\Xi$. Because we work over $\mathcal{A}$ this means that the product $\mu_2$ is a quasi isomorphism and the wanted statement easily follows.

4.5.4. Proof of Proposition 4.5.1. The actual proof consists of seven steps that follow below. Two auxiliary Lemmas that are used along the way are proved in §4.5.5 and §4.5.6.

To fix ideas, we first carry out the proof under the assumption that $X$ is closed. We discuss the non-compact case at the end.

**Step 1: Constructing an appropriate Lefschetz fibration.**

We first claim that there exists a Lefschetz fibration $\pi : \mathcal{E} \rightarrow \mathbb{C}$ with symplectic structure $\Omega$ so that $\mathcal{E}$ is tame over a subset $\mathcal{W} \subset \mathbb{C}$ as in Figure 23, and there is a symplectic trivialization $\psi$ over $\mathcal{W}$ (see Definition 2.2.2), such that $\mathcal{E}$, $\Omega$ and $\psi$ have the following properties:

1. The fibration has only one critical point $p \in \mathcal{E}$ lying over $0 \in \mathbb{C}$.
2. The fiber $(\mathcal{E}_{z_0}, \Omega|_{\mathcal{E}_{z_0}})$ over $z_0 = -10 \in \mathbb{C}$ is symplectomorphic via the trivialization $\psi$ to $(X, \omega)$. (Henceforth we make this identification.)
3. The vanishing cycle in $\mathcal{E}_{z_0}$ associated to the path going from $z_0$ to 0 along the $x$-axis is $S$.
4. The monodromy associated to a loop $\lambda$ based at $z_0$ that goes around 0 counterclockwise is Hamiltonian isotopic to $\tau_S$.

![Figure 23. Constructing the fibration $\mathcal{E}$.](image-url)
To prove this we first construct a Lefschetz fibration $\mathcal{E} \to \mathbb{C}$ (not necessarily tame) whose total space is endowed with a symplectic structure $\Omega^*$ and with the following properties:

1. The fibration has only one critical point $p \in \mathcal{E}$ lying over $0 \in \mathbb{C}$.
2. The fiber over $z_0^* = -1 \in \mathbb{C}$ is $(\mathcal{E}_{z_0^*}, \Omega|_{\mathcal{E}_{z_0^*}}) = (X, \omega)$.
3. The vanishing cycle in $\mathcal{E}_{z_0^*}$ associated to the path going from $z_0^*$ to 0 along the $x$-axis is Hamiltonian isotopic to $S$.
4. The monodromy around a loop $\lambda^*$ based at $z_0^*$ which goes counterclockwise around the critical value 0 is Hamiltonian isotopic to the Dehn twist $\tau_S$.

The proof that such a Lefschetz fibration exists follows from [Sei2] (see also Chapter 16e in [Sei3]), where it is proved for exact Lagrangian spheres. However, that proof extends to the case when $X$ is possibly not exact.

Given the fibration $\mathcal{E} \to \mathbb{C}$ and $\Omega^*$ we apply Proposition 2.3.1 with appropriate subsets $\mathcal{N}$ and $\mathcal{W}$ as in Figure 23 and base point $z_0 = -10$. We obtain a new symplectic structure $\Omega'$ on $\mathcal{E}$ with respect to which the fibration is tame over $\mathcal{W}$ such that $\Omega'$ coincides with $\Omega^*$ over $\mathcal{N}$. We thus obtain a trivialization $\psi' : (\mathcal{W} \times X', c\omega_C \oplus \omega') \to (\mathcal{E}|_W, \Omega')$, where $(X', \omega') = (\mathcal{E}|_{z_0}, \Omega'|_{\mathcal{E}_{z_0}})$ and $c > 0$.

Consider the loop $\lambda$ which starts at $z_0$, goes to $z_0^*$ along the $x$-axis, then goes along $\lambda^*$ and finally comes back to $z_0$ along the $x$-axis. Parallel transport along the straight segment connecting $z_0$ to $z_0^*$ and with respect to the connection $\Gamma' = \Gamma(\Omega')$ gives a symplectomorphism $\varphi : (X', \omega') \to (X, \omega)$. Put $S' = \varphi^{-1}(S)$. Clearly the monodromy (with respect to $\Gamma'$) along $\lambda$ is $\varphi^{-1} \circ \tau_S \varphi = \tau_{S'}$.

Finally, the desired symplectic structure on $\mathcal{E}$ and the trivialization are obtained by taking $\Omega = \Omega'$ and $\psi = \psi' \circ (\text{id} \times \varphi^{-1})$.

From now on the trivialization $\psi$ will be implicitly assumed and will just make the following identification

$$(\mathcal{E}|_W, \Omega|_{\pi^{-1}(W)}) = (\mathcal{W} \times X, c\omega_C \oplus \omega).$$

**Step 2:** Translating the problem to cobordisms.

Let $\gamma' \subset \mathbb{C}$ be the curve depicted in Figure 24. In a similar way to [BC3] $\gamma'$ gives rise to an inclusion functor

$$\mathcal{I}_{\gamma'} : \mathcal{F}uk^*(X) \to \mathcal{F}uk^*(\mathcal{E})$$

whose action on objects is $\mathcal{I}_{\gamma'}(N) = \gamma'N$, where $\gamma'N \subset \mathcal{E}$ stands for the trail of $N$ along the curve $\gamma'$ (see §2.1.1). Here, by $\mathcal{F}uk^*(\mathcal{E})$ we mean the Fukaya category of cobordisms in $\mathcal{E}$ of monotonicity class $*$ but we do not require the cobordisms to be only negatively ended. This category is defined, following the recipe in [BC3] as described in §3.2, but by also using perturbations and bottlenecks associated to the positive ends. For the purpose of the proof
below, it is actually enough to restrict to a subcategory whose objects are just cylindrical 
cobordisms in $\mathcal{E}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure24.png}
\caption{The cobordisms $V$, $W$ and $T_{\Delta}$.}
\end{figure}

Next, consider the curve $\gamma'' \subset \mathbb{C}$ as depicted in Figure 24 and fix a base point $w_0 \in \gamma'' \cap \mathcal{W}$. Define $V \subset (\mathcal{E}, \Omega)$ to be the Lagrangian submanifold obtained as the trail of $Q \subset \mathcal{E}_{w_0} = X$ along $\gamma''$.

Note that since the fibration $(\mathcal{E}, \Omega)$ is symplectically trivial over $\mathcal{W}$ the lower end of $V$ is identified with $Q$ and due to the homotopy class of $\gamma''$ (in $(\mathbb{C} \setminus \{0\}, \text{rel } \infty)$) the upper end of $V$ is a Lagrangian submanifold of $X$ which is Hamiltonian isotopic to $\tau_S(Q)$. Similarly, the lower end of $W$ is cylindrical over $\mathcal{N}$ and the upper end is cylindrical over $\tau_S^{-1}(\mathcal{N})$.

Below we will work with the Fukaya categories of both $X$ and $\mathcal{E}$. Our choices of auxiliary parameters (Floer and perturbation data, etc.) for these categories will be as described in §3. We therefore omit them from the notation in Floer complexes and the other $A_\infty$-structures. There are a few modifications compared to the conventions used in §3: we assume that the ends of the curve $\gamma'$ are at height $-2$ and $2$ and the ends of $\gamma''$ are at $-1$ and $1$. In other words, to fit precisely the setting in §3 we need to translate the whole picture by $+3i$. Clearly, this adjustment is completely formal, and it has no impact for the argument.

Denote by $\mathcal{Y}_X: \text{Fuk}^*(X) \to \text{mod}(\text{Fuk}^*(X))$ and $\mathcal{Y}: \text{Fuk}^*(\mathcal{E}) \to \text{mod}(\text{Fuk}^*(\mathcal{E}))$ the Yoneda embeddings associated to the Fukaya categories of $X$ and $\mathcal{E}$ respectively. When no confusion may arise we will simplify the notation and denote the module $\mathcal{Y}_X(L)$ associated to a Lagrangian $L \subset X$ simply by $L$ and similarly for $\mathcal{E}$. 
We now analyze the pullback module $\mathcal{I}_\gamma^* V \in \text{mod}(\text{Fuk}^*(X))$. Similar arguments to [BC3] (see also §4.3 in this paper) show that we have a quasi-isomorphism:

$$\mathcal{I}_\gamma^* V \cong \text{cone}(\tau_S(Q) \xrightarrow{\varphi} Q),$$

for some morphism of $A_\infty$-modules $\varphi$ that is induced by counting holomorphic strips (and polygons) going from the intersection of $V$ with $W$ at the $\tau_S(Q)$ end to the intersection of $V$ and $W$ at the $Q$ end - see Figure 24.

Let $T_\Delta \subset E$ be the thimble corresponding to the “diagonal” curve $\Delta$ depicted in Figure 24. We view $T_\Delta$ as an object of $\text{Fuk}^*(E)$.

Consider now the $\text{Fuk}^*(E)$-module

$$M = T_\Delta \otimes CF(T_\Delta, V),$$

where the second factor in the tensor product is regarded as a chain complex (see Chapter 3c in [Sei3] for the definition of the tensor product of an $A_\infty$-module and a chain complex).

The $A_\infty$-operations $\mu_k$, $k \geq 2$, induce a homomorphism of modules $M \rightarrow V$. Pulling back by $\mathcal{I}_\gamma^*$, this homomorphism induces a homomorphism of $\text{Fuk}^*(X)$-modules:

$$\nu : \mathcal{I}_\gamma^* M \rightarrow \mathcal{I}_\gamma^* V.$$

We claim that Proposition 4.5.1 reduces to the next statement:

**Proposition 4.5.4.** The homomorphism $\nu$ is a quasi-isomorphism.

This is so due to the following quasi-isomorphisms:

$$\mathcal{I}_\gamma^* M = \mathcal{I}_\gamma^* T_\Delta \otimes CF(T_\Delta, V) \cong (\tau_S^{-1})^* S \otimes CF(S, Q) \cong S \otimes CF(S, Q).$$

Here $(\tau_S^{-1})^* S$ stands for the pull-back of the module $S$ (i.e. the image of $S$ under the Yoneda embedding) via the $A_\infty$-functor induced by $\tau_S^{-1}$. Note also that we are not taking orientations into account so $\tau_S(S) = S$, hence the last quasi-isomorphisms.

In turn, by the general theory of $A_\infty$-categories, in order to prove Proposition 4.5.4 it is enough to show that for every Lagrangian $N \in \text{Ob}(\text{Fuk}^*(X))$ the map

$$\mu_2 : CF(\gamma'/N, T_\Delta) \otimes CF(T_\Delta, V) \rightarrow CF(\gamma'/N, V)$$

is a quasi-isomorphism. (Recall that $\gamma'/N$ stands for the trail of $N$ along $\gamma'$.)

**Remark 4.5.5.** We have not indicated at this moment the choices of Floer and perturbation data in (42) for two reasons. This is because, whether or not the map in (42) is a quasi-isomorphism does not depend on these specific choices (the induced product in homology is canonical). Moreover, later on in the proof we will actually make use of a very specific choice of parameters (which is different than the one used in §3 when setting up the entire
Fukaya category of $\mathcal{E}$) for which it will be convenient to prove that the map in (42) is a quasi-isomorphism.

The rest of this section will be devoted to proving that (42) is a quasi-isomorphism. For brevity we denote from now on $W = \gamma'N \subset \mathcal{E}$ (see Figure 24).

**Step 3: Stretching the cobordisms.**

Write the projection $\pi : \mathcal{E} \rightarrow \mathbb{C}$ as

$$\pi = \text{Re}(\pi) + \text{Im}(\pi)i.$$ 

Denote by $Z = -\nabla\text{Re}(\pi)$ the negative gradient vector field of the real part of $\pi$ with respect to the Riemannian metric induced on $\mathcal{E}$ by $(\Omega, J_\mathcal{E})$. Since the functions $\text{Re}(\pi)$ and $\text{Im}(\pi)$ are harmonic conjugate (recall that $\pi$ is holomorphic), it follows that $Z$ is also the Hamiltonian vector field associated to the function $\text{Im}(\pi)$.

The flow of the vector field $Z$ will be used extensively throughout the rest of the proof. However, due to the non-compactness of $\mathcal{E}$, it might lack to be defined for all times. To overcome this difficulty we proceed as follows. Write $y_1 + iy_2 \in \mathbb{C}$ for the standard coordinates on $\mathbb{C}$. Denote by $R^\Omega$ the curvature of the connection $\Gamma(\Omega)$. (Recall that this is a 2-form on $\mathbb{C}$ with values in the space of Hamiltonian functions of the fibers of $\mathcal{E}$.) A straightforward calculation shows that for every $z \in \mathbb{C}, p \in \mathcal{E}_z$ we have:

$$(43) \quad Z_{(z,p)} = \frac{-1}{C(z) - R^\Gamma_z(\partial_{y_1}, \partial_{y_2})(p)(\partial_{y_1})^{\text{hor}}},$$

where $C : \mathbb{C} \rightarrow \mathbb{R}$ is a function and $(\partial_{y_1})^{\text{hor}}$ stands for the horizontal lift of $\partial_{y_1}$. Since $Z = -\nabla\text{Re}(\pi)$ it follows that the denominator on the right-hand side of (43) is always positive. Fix a positive real number $a > 0$ and define

$$\Omega' = \Omega + a\pi^*dy_1 \wedge dy_2.$$ 

Note that $J_\mathcal{E}$ continues to be compatible with $\Omega'$. Denote by $Z'$ the negative gradient of the same function, $\text{Re}(\pi)$, but now defined via the metric associated to $(\Omega', J_\mathcal{E})$. A simple calculation shows that:

$$(44) \quad Z'_{(z,p)} = \frac{-1}{a + C(z) - R^\Gamma_z(\partial_{y_1}, \partial_{y_2})(p)(\partial_{y_1})^{\text{hor}}}.$$ 

Clearly the coefficient standing before $(\partial_{y_1})^{\text{hor}}$ on the right-hand side of (44) is bounded from above by $1/a$. It now easily follows that the flow of $Z'$ exists for all times (recall that we are under the assumption that the fiber $X$ is compact). Finally, note that the connections of $\Omega$ and $\Omega'$ are the same and moreover, $V$ and $W$ continue both to be Lagrangian cobordisms with respect to the new form $\Omega'$. 
Summarizing the preceding procedure, by replacing $\Omega$ by $\Omega'$ we may assume that the negative gradient flow of $\text{Re}(\pi)$ exists for all times. For simplicity we continue to denote the symplectic structure of $E$ by $\Omega$.

Denote by $\phi_t, t \in \mathbb{R}$, the flow of $Z$. Note that the function $\text{Re}(\pi)$ is a Morse function with exactly one critical point $p \in E$ lying over $0 \in \mathbb{C}$. The Morse index of $\text{Re}(\pi)$ at $p$ is precisely $m = \text{dim}_\mathbb{C} E$. Denote by $\phi_t, t \in \mathbb{R}$, the flow of $Z$. The stable submanifold of $Z$ is the thimble $T'$ lying over the positive $x$-axis and the unstable submanifold of $Z$ is the thimble $T''$ lying over the negative $x$-axis. Note that we have $J_\xi T_p(T') = T_p(T'')$.

**Figure 25.** The cobordisms $V, W$ after the flows $\phi_t$ and $\phi_t^{-1}$ are applied to them for large time $t$.

From now on we set:

$$m = \frac{1}{2} \dim E = n + 1, \quad \text{where} \quad n = \frac{1}{2} \dim X.$$ 

Denote by $B'(r) = B''(r) = B^m(r) \subset \mathbb{R}^m$ two copies of the $m$-dimensional closed Euclidean ball of radius $r$ around $0 \in \mathbb{R}^m$. (Since each of these two balls corresponds to a different factor of $\mathbb{R}^m \times \mathbb{R}^m$ we have chosen to denote them by different symbols.)

Fix a little neighborhood $Q_p \subset E$ of $p$ which is symplectomorphic to a product $B'(r_0) \times B''(r_0) \subset (\mathbb{R}^m \times \mathbb{R}^m, \omega_{\text{can}} = dp_1 \wedge dq_1 + \cdots dp_m \wedge dq_m)$ for some small $r_0$. Below we will abbreviate $B' = B'(r_0)$, $B'' = B''(r_0)$.

We may assume that the symplectic identification $Q_p \approx B' \times B''$ sends $T' \cap Q_p$ to $B' \times \{0\}$ and $T'' \cap Q_p$ to $\{0\} \times B''$ and $T_\Delta$ to the diagonal $\{(x, y) \in B' \times B'' \mid x = y\}$. From now on we assume the identification $Q_p \approx B' \times B''$ explicit and when convenient view $Q_p$ as a subset of $\mathbb{R}^{2m}$.
We now apply the flow $\phi_t$ to $V$ and $\phi_t^{-1}$ to $W$ (see Figures 25, 26). Standard arguments in Morse theory imply that for $t_0 \gg 1$ we have

$$\phi_{t_0}^{-1}(W) \cap Q_p = \bigcup_{i=1}^{s''} D'_i, \quad \phi_{t_0}(V) \cap Q_p = \bigcup_{j=1}^{s'} D''_j,$$

where $D'_i \subset Q_p$ are graphs of exact 1-forms on $B'$ and $D''_j \subset Q_p$ are graphs of exact 1-forms on $B''$. Here $s'' = \#(W \cap T'')$ and $s' = |V \cap T'|$ are the number of intersection points (counted without signs) of $W \cap T''$ and $V \cap T'$ respectively. Note also that by our construction of $E$ we have $s'' = \#(N \cap S)$ and $s' = \#(Q \cap S')$, where $S'$ is the vanishing sphere $T' \cap E_x$ with $0 < x$ large enough so that $x \in W$. Note that $S'$, when viewed as a Lagrangian in $(X, \omega)$ is Hamiltonian isotopic to $S$.

Fix $0 < \delta_0 \ll 1/3$. By taking $t_0$ large enough we may assume that

$$\phi_{t_0}^{-1}(W) \cap Q_p \subset B' \times B''(\delta_0 r_0), \quad \phi_{t_0}(V) \cap Q_p \subset B'(\delta_0 r_0) \times B''$$

and moreover that each of the $D'_i$ (resp. $D''_j$’s) is $C^1$-close to a constant section of $B' \times B''$ (resp. $B' \times B''$). See Figure 26.

![Figure 26. The parts of $\phi_t(V)$ and $\phi_t^{-1}(W)$ that lie in $Q_p$.](image)

Thus by applying a suitable Hamiltonian diffeomorphism of $Q_p$ (which extends to the rest of $E$) we may assume that

$$\phi_{t_0}^{-1}(W) \cap Q_p = \bigsqcup_{i=1}^{s''} B' \times \{a''_i(t_0)\}, \quad \phi_{t_0}(V) \cap Q_p = \bigsqcup_{j=1}^{s'} \{a'_j(t_0)\} \times B''.$$
where $|a'_i(t_0)|, |a''_j(t_0)| < \delta_0 r_0$. See Figure 27.

Fix now $t_0$ large enough as above and set

$$\tilde{V} := \phi_{t_0}(V), \quad \tilde{W} = \phi_t^{-1}(t_0)(W).$$

For $r', r'' < r_0$ we abbreviate $B(r', r'') := B'(r') \times B''(r'')$ and also $B = B(r_0, r_0) = B' \times B''$.

**Step 4: A further isotopy of $\tilde{V}$ and $\tilde{W}$**.

We claim there exist two Hamiltonian isotopies $\psi'_t, \psi''_t$, $0 \leq t < 1$, with $\psi'_0 = \psi''_0 = \text{id}$ and with the following properties for every $0 \leq t < 1$:

1. $\psi'_t$, $\psi''_t$ are both supported in $\text{Int}(B)$.
2. $\psi'_t(\tilde{W}) \cap B(r_0/3, r_0/3) = \bigcup_{i=1}^{s''} B'(r_0/3) \times \{b'_i(t)\}$ with $|b'_i(t)| \leq (1-t)\delta_0 r_0$ for every $i$.
3. $\psi''_t(\tilde{W}) \cap B(r_0/3, r_0/3) = \bigcup_{j=1}^{s''} \{b''_j(t)\} \times B''(r_0/3)$ with $|b''_j(t)| \leq (1-t)\delta_0 r_0$ for every $j$.
4. $\psi'_t(\tilde{W}) \cap \left((B'(r_0) \setminus B'(2r_0/3)) \times B''(r_0)\right) = \tilde{W} \cap \left((B'(r_0) \setminus B'(2r_0/3)) \times B''(r_0)\right)$.
5. $\psi''_t(\tilde{W}) \cap \left(B'(r_0) \times (B''(r_0) \setminus B''(2r_0/3))\right) = \tilde{V} \cap \left(B'(r_0) \times (B''(r_0) \setminus B''(2r_0/3))\right)$.
6. $\psi'_t(\tilde{W})$ and $\psi''_t(\tilde{V})$ intersect only inside $B(\delta_0 r_0, \delta_0 r_0)$. Moreover, their intersection is:

$$\psi'_t(\tilde{W}) \cap \psi''_t(\tilde{V}) = \{(b'_i(t), b''_j(t)) \mid 1 \leq i \leq s'', 1 \leq j \leq s\}.$$
Figure 28. The isotopies $\psi'_t(\tilde{V}), \psi''_t(\tilde{W})$

See Figure 28. The construction of the isotopies $\psi'_t, \psi''_t$ is elementary and can be done quite explicitly. For point (7) one might need to reduce the size of the parameter $\delta_0$ from (45), which can be done in advance.

To keep the notation short we now set:

$$\tilde{V}_t = \psi''_t(\tilde{V}), \quad \tilde{W}_t = \psi'_t(\tilde{W}).$$

Note that $\tilde{W}_t \cap B(r_0, r_0)$ is disconnected and has precisely $s''$ connected components, each of which looks like a copy of $B' \times \{0\}$ which is (non-linearly) translated along the $B''$-axis. These components lie in “parallel” position one with respect to the other (see Figure 28). We will refer to these components as the sheets of $\tilde{W}$ inside $B(r_0, r_0)$ and denote them by $\mathcal{S}^W_i(t), \; i = 1, \ldots, s''$. The indexing here is so that $\mathcal{S}^W_i(t)$ coincides with $B'(r_0/3) \times \{b''_i(t)\}$ inside $B(r_0/3, r_0/3)$. Similarly, $\tilde{V}_t \cap B(r_0, r_0)$ is disconnected and consists of $s'$ “parallel” sheets which are all “translates” of $\{0\} \times B''$. We denote them by $\mathcal{S}^V_j(t), \; j = 1, \ldots, s'$, where the indexing is done so that $\mathcal{S}^V_j(t)$ coincides with $\{b_j(t)\} \times B''(r_0/3)$ inside $B'(r_0/3, r_0/3)$. See Figure 28. Clearly we have

\begin{align*}
\mathcal{S}^W_i(t) \cap \mathcal{S}^V_j(t) &= \{(b'_j(t), b''_i(t))\}, \\
\mathcal{S}^W_i(t) \cap T_\Delta &= \{(b''_i(t), b''_i(t))\}, \quad T_\Delta \cap \mathcal{S}^V_j(t) = \{(b'_j(t), b'_j(t))\}.
\end{align*}
**Step 5:** Area estimates for large holomorphic triangles. Let \( D' = D \setminus \{z_1, z_2, z_3\} \) be the unit disk punctured at three boundary points \( z_1, z_2, z_3 \) ordered clock-wise along \( \partial D \). Fix strip-like ends around the punctures (see §3), and denote by \( \partial_i D' \), the arc on \( \partial D' \) connecting \( z_i \) with \( z_j \).

We will now consider a special almost complex structure \( J^0_B \) on \( B = B' \times B'' \). We identify \( \mathbb{R}^m \times \mathbb{R}^m \) with \( \mathbb{C}^m \) in the obvious way via \((x_1, \ldots, x_m, y_1, \ldots, y_m) \mapsto (x_1 + iy_1, \ldots, x_m + iy_m)\). This induces a complex structure \( J_{\text{std}} \) on \( \mathbb{R}^m \times \mathbb{R}^m \). We define \( J^0_B \) to be the restriction of \( J_{\text{std}} \) to \( B \subset \mathbb{R}^m \times \mathbb{R}^m \). Define now \( J_0 \) to be the space of \( \Omega \)-compatible domain-dependent almost complex structures \( J = \{ J_z \}_{z \in D'} \) which coincide with \( J^0_B \) on \( B \). For elements \( J \in J_0 \), and \( z \in D', p \in \mathcal{E} \) we will also write \( J(z, p) \) for the restriction of \( J_z \) to \( T_p \mathcal{E} \).

Consider now finite energy solutions to the Floer equation with boundary conditions on the Lagrangians \( \tilde{W}_t, T_\Delta, \tilde{V}_t \):

\[
\begin{align*}
  u : D' &\to \mathcal{E}, \quad E(u) < \infty, \\
  Du + J(z, u) \circ Du \circ j &\equiv 0, \\
  u(\partial_{3,1} D') &\subset \tilde{W}_t, \quad u(\partial_{1,2} D') \subset T_\Delta, \quad u(\partial_{2,3} D') \subset \tilde{V}_t
\end{align*}
\]

(47)

Together with the requirement that \( u \) converges along each strip-like end of \( D' \) to an intersection point between the corresponding pair of Lagrangians (associated to the two arcs of \( \partial D' \) that neighbor a given puncture). Thus \( u \) extends continuously to a map \( u : D \to \mathcal{E} \) with

\[
\begin{align*}
  u(z_1) &\in \tilde{W}_t \cap T_\Delta, \quad u(z_2) \in T_\Delta \cap \tilde{V}_t, \quad u(z_3) \in \tilde{W}_t \cap \tilde{V}_t.
\end{align*}
\]

In what follows we denote for a (finite energy) map \( u : D \to \mathcal{E} \) by \( A_\Omega(u) = \int_{D'} u^* \Omega \) its symplectic area.

We now fix once and for all \( r_1 \) with \( 2r_0/3 < r_1 < r_0 \).

**Lemma 4.5.6.** There exists a constant \( C = C(r_1, \tilde{W}, \tilde{V}) > 0 \) (that depends only on \( r_1 \) and \( \tilde{W} = \tilde{W}_0, \tilde{V} = \tilde{V}_0 \)) with the following property. Let \( 0 \leq t < 1 \) and \( J \in J_0 \). Then every solution \( u : D' \to \mathcal{E} \) of (47) with \( u(D') \not\subset B(r_1, r_1) \) must satisfy \( A_\Omega(u) \geq C \).

The proof of the lemma is given in §4.5.5 below.

Next consider the intersections between any of \( \tilde{W}_t, \tilde{V}_t \) and \( T_\Delta \). Recall from (46) the intersections between \( S^W_i(t), S^V_j(t) \) and \( T_\Delta \). For simplicity we set

\[
\begin{align*}
  w_i(t) = (b''_i(t), b'_i(t)), \quad v_j(t) = (b'_j(t), b_j(t)), \quad x_{i,j}(t) = (b'_j(t), b''_i(t)).
\end{align*}
\]

With this notation we have:

\[
\begin{align*}
  \tilde{W}_t \cap T_\Delta &\{ w_i(t) \mid 1 \leq i \leq s'' \}, \quad T_\Delta \cap \tilde{V}_t = \{ v_j(t) \mid 1 \leq j \leq s' \}, \\
  \tilde{W}_t \cap \tilde{V}_t &\{ x_{i,j}(t) \mid 1 \leq i \leq s'', 1 \leq j \leq s' \}.
\end{align*}
\]

(48)

As a consequence from Lemma 4.5.6 we have:
Corollary 4.5.7. Let $0 \leq t < 1$, $J \in \mathcal{J}_0$ and $u : D' \to \mathcal{E}$ a solution of (47). If

$$u(z_1) = w_i(t), \quad u(z_2) = v_j(t), \quad u(z_3) \neq x_{i,j}(t),$$

then $A_\Omega(u) \geq C$, where $C$ is the constant from Lemma 4.5.6.

Proof of Corollary 4.5.7. Let $u : D' \to \mathcal{E}$ be as in the statement of the corollary. We claim that $u(\partial D') \not\subset B(r_1, r_1)$.

To prove this, assume the contrary were the case, i.e. that $u(\partial D') \subset B(r_1, r_1)$. Since $u(z_1) = w_i(t)$ it follows that $u(\partial_1 D') \subset S_i^W(t)$. Similarly, from $u(z_2) = v_j(t)$ we conclude that $u(\partial_1 D') \subset S_j^Y(t)$. It now follows that $u(z_3) \in S_i^W(t) \cap S_j^Y(t) = \{x_{i,j}(t)\}$, which is a contradiction. This proves that $u(\partial D') \not\subset B(r_1, r_1)$. By Lemma 4.5.6 we have $A_\Omega(u) \geq C$. □

Step 6: Estimating the small holomorphic triangles.

Lemma 4.5.8. There exists $\epsilon > 0$ and a constant $C' > 0$ such that the following holds. Let $1 - \epsilon \leq t < 1$ and $1 \leq i \leq s''$, $1 \leq j \leq s'$ and $J \in \mathcal{J}_0$. Then among the solutions of equation (47) there exists a unique one $u$ with the following two properties:

1. $u(z_1) = w_i(t), \quad u(z_2) = v_j(t), \quad u(z_3) = x_{i,j}(t)$.
2. $A_\Omega(u) < C'$.

Moreover, this solution $u$ satisfies $u(D') \subset B(r_0/3, r_0/3)$ and $A_\Omega(u) \leq \sigma(t)$, where $\sigma(t) \xrightarrow{t \to 1^{-}} 0$. Furthermore $J$ is regular for the solution $u$ in the sense that the linearized $\overline{\partial}$ operator is surjective at $u$.

The proof is given in §4.5.6 below.

Step 7: End of the proof. We are now ready to prove that the map in (42) is a quasi-isomorphism, thus proving Proposition 4.5.4.

Following Steps 1-6 above it is enough to show that the map

$$\mu_2 : CF(\tilde{W}_t, T_\Lambda) \otimes CF(T_\Lambda, \tilde{V}_t) \to CF(\tilde{W}_t, \tilde{V}_t) \quad (49)$$

is a quasi-isomorphism for some $0 \leq t < 1$.

Next, note that the whether or not (49) (or (42)) is a quasi-isomorphism is independent of the Floer and perturbation data used for the respective Floer complexes and for the operation $\mu_2$. Therefore for the sake of our proof any choice of such data would do as long as it is regular and amenable to the situation of cobordisms. (In contrast, consistency with respect to the perturbation data used for the higher $\mu_k$’s is irrelevant for our present purposes.) We therefore choose for (49) Floer data for which the Hamiltonian perturbations are 0 and $J \in \mathcal{J}_0$.

By construction, $CF(\tilde{W}_t, T_\Lambda)$ has the elements $w_1(t), \ldots, w_{s''}(t)$ as a basis. Similarly $CF(T_\Lambda, \tilde{V}_t)$ has a basis consisting of $v_1(t), \ldots, v_{s'}(t)$ and $CF(\tilde{W}_t, \tilde{V}_t)$ can be endowed with
the basis \( \{x_{i,j}(t)\}_{1 \leq i \leq s'', 1 \leq j \leq s'} \). Thus we have a 1-1 correspondence between the associated basis of \( CF(\widetilde{W}_t, T_\Delta) \otimes CF(T_\Delta, \widetilde{V}_t) \) and the basis of \( CF(\widetilde{W}_t, \widetilde{V}_t) \), given by
\[
w_i(t) \otimes v_j(t) \leftrightarrow x_{ij}(t).
\]

We will now show that for \( t < 1 \) close enough to 1 and appropriate \( J \), the matrix of \( \mu_2 \) with respect to these bases is invertible. This will prove that for such a choice of \( t \) and \( J \), \( \mu_2 \) is in fact a chain isomorphism (hence a quasi-isomorphism for any other choice). Below we will denote the matrix of \( \mu_2 \) with respect to these bases by \( M \).

Fix a generic \( J \in J_0 \) and \( t_0 \) with \( 1 \leq t_0 < 1 - \epsilon \) such that \( \sigma(t_0) \ll C' \), where \( \epsilon, C' \) and \( \sigma \) are as in Proposition 4.5.8. By Proposition 4.5.8 the entries in the diagonal of \( M \) have the form
\[
M_{k,k}(T) = T^{\alpha_k} + O(T^{C'}),
\]
with \( 0 \leq \alpha_k \leq \sigma(t_0) \). Here \( o(T^{C'}) \) stands for an element of the Novikov ring in which every monomial is of the form \( c_l T^{\lambda_l} \) with \( c_l \in \mathbb{Z}_2 \) and \( \lambda_l \geq C' \).

Similarly, by Corollary 4.5.7, the elements of \( M \) that are off the diagonal are all of the form
\[
M_{k,l} = O(T^C), \quad \forall k \neq l,
\]
where \( C \) is the constant from Corollary 4.5.7 and Lemma 4.5.6.

By choosing \( t_0 \) close enough to 1 we obtain \( \alpha_k \) as close as we want to 0. It easily follows that for such a choice of \( t_0 \) the matrix \( M \) can be transformed via elementary row operations to an upper triangular matrix with non-zero elements in the diagonal. It follows that \( M \) is invertible. \( \square \)

**Remark 4.5.9.** It is easy to see that the map \( \varphi \) from (38) is chain-homotopic to the corresponding map constructed by Seidel (in the exact case) in his construction of the exact triangle associated to a Dehn twist. As a consequence, the exact triangle constructed above coincides with his.

**4.5.5. Proof of Lemma 4.5.6.** Throughout the proof we will denote by \( \text{Ball}_x(r) \subset \mathbb{R}^m \times \mathbb{R}^m \) the open Euclidean ball of radius \( r \) centered at \( x \).

Fix \( r_2 \) with \( 2r_0/3 < r_2 < r_1 \) and let \( \rho_2 > 0 \) small enough so that:

1. For \( i \) and every \( x \in S^W_i(t) \cap (\partial B'(r_2) \times B'') \) the closed ball \( \text{Ball}_x(\rho_1) \) is disjoint from all \( S^W_k(t) \) for every \( k \neq i \) as well as from \( \widetilde{W}_t \) and from \( T_\Delta \).
2. For \( j \) and every \( x \in S^V_j(t) \cap (B' \times \partial B''(r_2)) \) the closed ball \( \text{Ball}_x(\rho_1) \) is disjoint from all \( S^V_k(t) \) for every \( k \neq j \) as well as from \( \widetilde{V}_t \) and from \( T_\Delta \).
3. For every \( x \in T_\Delta \cap (\partial B'(r_2) \times \partial B''(r_2)) \) the closed ball \( \text{Ball}_x(\rho_1) \) is disjoint from \( \widetilde{W}_t \) and \( \widetilde{V}_t \).
Estimating the area of holomorphic curves that go out of $B(2r_0/3, 2r_0/3)$.

By construction, such a $\rho_1$ exists and can be chosen to be independent of $0 \leq t < 1$. (Recall that $\tilde{W}_t \cap (B(r_0, r_0) \setminus B(2r_0/3, r_0))$ is independent of $t$.) See Figure 29.

Similarly, choose $\rho_2 > 0$ such that for every $x \in \partial B(r_1, r_1)$ the closed ball $\overline{\text{Ball}}_x(\rho_2)$ is disjoint from $B(r_2, r_2)$ and is also contained inside $B = B(r_0, r_0)$.

Set $C := \min \{ \frac{\pi}{2} \rho_2^2, \pi \rho_1^2 \}$.

Now let $u : D' \rightarrow E$ be a solution of (47) and assume first that $u$ satisfies the following special assumption: $u(\partial D') \not\subset B(r_2, r_2)$. We will prove that $A_\Omega(u) \geq C$.

Since $u(z_i) \in B(2r_0/3, 2r_0/3)$ (recall $z_i$ are the punctures of $D'$) it follows that there exists $z_\ast \in \partial D'$ such that $u(z_\ast)$ lies in one of the following three:

1. $S_i^W(t) \cap (\partial B'(r_2) \times B'')$ for some $i$; or
2. $S_j^V(t) \cap (B' \times \partial B''(r_2))$ for some $j$; or
3. $T_\Delta \cap (\partial B'(r_2) \times \partial B''(r_2))$.

Consider now the intersection $u(D') \cap \overline{\text{Ball}}_{u(z_\ast)}(\rho_2)$. By the Lelong inequality (applied after a reflection in the ball with respect to the corresponding Lagrangian) it follows that

$$A_\Omega(u) \geq \frac{\pi}{2} \rho_2^2 \geq C.$$
We are now ready to prove the general case. Assume that \( u(D') \not\subset B(r_1, r_1) \). There are two cases (mutually not exclusive): either \( u(\partial D') \not\subset B(r_1, r_1) \), or \( u(\text{Int} \, D') \not\subset B(r_1, r_1) \).

If the first case occurs then clearly \( u(\partial D') \not\subset B(r_2, r_2) \) and we are done. Therefore we may assume that \( u(\partial D') \subset B(r_2, r_2) \) and that the second case occurs, namely \( u(\text{Int} \, D') \not\subset B(r_1, r_1) \). It follows that there is \( z_* \in \text{Int} \, D' \) with \( u(z_*) \in \partial B(r_1, r_1) \). Applying the Lelong inequality for \( u(D') \cap \overline{\text{Ball}_{u(z_*)}(\rho_1)} \) we obtain

\[
A_{g_1}(u) \geq \pi \rho_1^2 \geq C. 
\]

\[\square\]

4.5.6. \textit{Proof of Lemma 4.5.8.} Before defining the constant \( C'' \), we first consider solutions \( u \) of (47) that satisfy property (1) of our proposition as well as property (2) with the constant \( C'' \) replaced by the constant \( C \) from Lemma 4.5.6. (The constant \( C'' \), defined below, will have the property that \( 0 < C'' \leq C \).) By Lemma 4.5.6 we have \( u(D') \subset B(r_1, r_1) \). Since

\[
\widetilde{W}_t \cap B(r_1, r_2) = \bigcap_{k=1}^{s''} S^W_k(t), \quad \widetilde{V}_t \cap B(r_1, r_2) = \bigcap_{k=1}^{s'} S^V_k(t)
\]

it follows that

\[
u(\partial_{3,1} D') \subset S^W_i(t), \quad u(\partial_{1,2} D') \subset T_\Delta, \quad u(\partial_{2,3} D') \subset S^V_j(t).
\]

Thus we are considering here finite energy solutions \( u : D' \rightarrow B' \times B'' \) of (47) subject to the boundary condition (50) and the asymptotics (see Figure 30)

\[
u(z_1) = w_i(t), \quad u(z_2) = v_j(t), \quad u(z_3) = x_{i,j}(t).
\]

Recall also that our almost complex structure \( J \) is in \( J_0 \), hence by definition \( J \equiv J^0_B \) on \( B = B' \times B'' \).

We now claim that there is a constant \( 0 < C'' \leq C \) such that all solutions \( u \) of (50) with asymptotics (51) and with \( A_{g_1}(u) \leq C'' \) must satisfy \( u(D') \subset B(r)/3, r_0/3) \). The proof of this claim is very similar to that of Lemma 4.5.6 and in fact even simpler since we are considering here boundary conditions only on one pair of sheets \( (S^W_i(t), S^V_j(t)) \) and \( T_\Delta \), and the distance between each pair of these three Lagrangians outside of \( B(r_0/3, r_0/3) \) is uniformly bounded below.

This proves that all solutions \( u : D' \rightarrow E \) that satisfy assumptions (1) and (2) of our proposition have their images inside \( B(r_0/3, r_0/3) \).
Figure 30. Holomorphic triangles going from $w_i(t), v_j(t)$ to $x_{i,j}(t)$.

It remains to show the existence and uniqueness of such solutions, the area estimate and the regularity. To this end, set:

$$S_W(t) := \mathbb{R}^m \times \{ b''_i(t) \}, \quad S_V(t) := \{ b'_j(t) \} \times \mathbb{R}^m, \quad T_\Delta = \{ (x, x) \mid x \in \mathbb{R}^m \}.$$

Clearly $S_W(t)$ coincides with $S_W(t)$ inside $B(r_0/3, r_0/3)$ and similarly for $S_V(t)$ as well as for $T_\Delta$ and $T_\Delta$. Thus for our purposes we can consider now the equation (47) for maps $u : D' \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ with $J = J_{\text{std}}$ and with the following boundary condition and asymptotics:

$$u(\partial_{3,1}D') \subset S_W(t), \quad u(\partial_{1,2}D) \subset T_\Delta(t), \quad u(\partial_{2,3}D') \subset S_V(t),$$

$$u(z_1) = w_i(t), \quad u(z_2) = v_j(t), \quad u(z_3) = x_{i,j}(t).$$

Note that this problem splits. If we rearrange the coordinates by identifying of $\mathbb{R}^m \times \mathbb{R}^m \simeq (\mathbb{R}^2)^m$ via the symplectic isomorphism $(p_1, \ldots, p_m, q_1, \ldots, q_m) \mapsto (p_1, q_1, \ldots, p_m)$ then $J_{\text{std}}$ is sent to the standard split complex structure (which we continue to denote $J_{\text{std}}$), and $S_W(t)$ becomes $(\mathbb{R} \times q_1(t)) \times \cdots \times (\mathbb{R} \times q_m(t))$, where $b''_i(t) = (q_1(t), \ldots, q_m(t))$. Similarly $S_V(t)$ becomes $(p_1(t) \times \mathbb{R}) \times \cdots \times (p_m(t) \times \mathbb{R})$, where $b'_j(t) = (p_1(t), \ldots, p_m(t))$. Finally, $T_\Delta$ becomes $\Delta_1 \times \cdots \Delta_m$ where $\Delta_i$ is the diagonal in each of the $\mathbb{R}^2$ factors. We continue to denote the corresponding three Lagrangians by $S_W(t), S_V(t)$ and $T_\Delta$.

We will now write maps $u : D' \rightarrow (\mathbb{R}^2)^m$ as: $u(z) = (u_1(z), \ldots, u_m(z))$ with $u_k(z) \in \mathbb{R}^2$. Clearly each of the maps $u_k : D' \rightarrow \mathbb{R}^2 \simeq \mathbb{C}$ is holomorphic (in the usual sense) and satisfies
the boundary conditions and asymptotics (see Figure 31):

\[ u(\partial_{3,1}D') \subset \mathbb{R} \times q_k(t), \quad u(\partial_{1,2}D') \subset \Delta_k, \quad u(\partial_{2,3}D') \subset p_k(t) \times \mathbb{R}, \]

\[ u(z_1) = (q_k(t), q_k(t)), \quad u(z_2) = (p_k(t), p_k(t)), \quad u(z_3) = (p_k(t), q_k(t)). \]

Figure 31. Holomorphic triangles in \( \mathbb{R}^2 \) corresponding to the projection on the \( k' \)th factor of \( u \).

Standard 1-dimensional complex analysis show that there is a unique holomorphic map \( u^0_k : D' \rightarrow \mathbb{C} \cong \mathbb{R}^2 \) with the boundary conditions (53), the image of which is precisely the triangle consisting of the convex hull of the three points \((q_k(t), q_k(t)), (p_k(t), p_k(t)), (p_k(t), q_k(t))\). Moreover, a straightforward calculation (using e.g. the methods from Chapter 13 of [Sei3]) shows that the Maslov index of \( u^0_k \) is 0 and that the standard complex structure of \( \mathbb{C} \) is regular for this solution.

Note that the mutual position of the three Lagrangians from (53) plays a crucial role here. If for example, one would replace \( \Delta_k \) by the anti-diagonal line \( \{(x, -x) : x \in \mathbb{R}\} \) then there would be no solutions with the boundary conditions (53), the reason being that the order of the punctures \( z_1, z_2, z_3 \) on \( \partial D \) is “wrong”.

It follows that \( u^0(z) = (u^0_1(z), \ldots, u^0_m(z)) \) is the unique holomorphic map \( u : D' \rightarrow (\mathbb{R}^2)^m \) satisfying (52). Since the \( \overline{\partial} \)-operator splits in a compatible way with the splitting \( (\mathbb{R}^2)^m \) it follows that the index of \( u^0 \) is 0 and that \( J_{\text{std}} \) is regular.

Finally, it is clear that the symplectic area \( A_\Omega(u^0) \) of \( u^0 \) is the sum of the areas of the triangles \( u^0_k, k = 1, \ldots, m \). Since \( p_k(t), q_k(t) \xrightarrow{t \to 1^-} 0 \) it follows that \( A_\Omega(u^0) \xrightarrow{t \to 1^-} 0 \).

This concludes the proof of the proposition.
Remark. An alternative calculation of the index and regularity can be done by degenerating
the problem to $t = 1$. Then the three Lagrangians forming the boundary conditions in (53)
become $\mathbb{R} \times \{0\}$, $\Delta_k$ and $\{0\} \times \mathbb{R}$. The asymptotics at the punctures become $u_k(z_1) = u_k(z_2) = u_k(z_3) = (0,0)$. It is easy to see that the only solution now is the constant solution at $(0,0)$.

The fact that its index is 0 and that $J$ is regular follow e.g. from [BC3] (section 4.3). By
a standard implicit function theorem it follows that the same holds for $t$’s close enough to 1. Note that also here, if one would replace $\Delta_k$ by a line going through the 2’nd and 4’th
quadrants, e.g. $\{(x, -x) : x \in \mathbb{R}\}$, things would go wrong. The constant map at 0 would still
be a solution but its index would be negative and $J$ would not be regular with respect to it.

It remains to discuss the case when $X$ is non-compact but symplectically convex at $\infty$. The
proof is very similar to the one for the case when $X$ is closed. Recall that although now $X$
is not compact the objects of $\mathcal{F}uk^*(X)$ (i.e. the Lagrangians in $X$) are still assumed to be
compact.

The results of Seidel (see Chapter 16e of [Sei3] and [Sei2]) can be used to produce a fibration
$\mathcal{E}$ of generic fibre $X$, in the sense of the definitions in §2.1, in particular this fibration verifies
assumption $T_\infty$. As in the compact fibre case, we then use the Proposition 2.3.1 to transform
the fibration into a tame one that continues to satisfy $T_\infty$. The proof then pursues just as
in the compact case. Indeed, notice that Assumption $T_\infty$ implies that the monodromy is
well defined over any path in $\mathbb{C} \setminus \text{Critv}(\pi)$ (and in fact over any path in $\mathbb{C}$ if we restrict the
monodromy to “infinity in the fibers”). Similarly, the procedure from page 66 that ensures
that the negative gradient flow of $\text{Re}(\pi)$ is defined for all times continues to work in the present
setting. Indeed, the fact that the fibers of $\mathcal{E}$ are not compact does not pose any problems
because (in the notation of Assumption $T_\infty$) on $\mathcal{E}^\infty \approx \mathcal{E}_{w_0}^\infty \times \mathbb{C}$ this flow is just a translation
in the $\mathbb{C}$-direction. Finally, in what concerns the Floer and perturbation data we use as in
§3.2.5 almost complex structures that are split at $\infty$ as $i \oplus J_0$ with $J_0$ compatible with the
symplectic convexity of (the end) of $X$.

4.5.7. Proof of Proposition 4.5.3. We now explain how to modify the proof of Proposition 4.5.1
under the assumptions of Proposition 4.5.3, namely that $X$ is itself the total space of a tame
Lefschetz fibration $\pi_X : X \longrightarrow \mathbb{C}$ as described in §4.5.2. Denote by $(N, \omega)$ the generic fibre of
$\pi_X$ which is compact or symplectically convex at infinity.

As in the the proof of Proposition 4.5.1, we again construct a Lefschetz fibration $\pi_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathbb{C}$ with fiber over $w_0$ being $X$. As before, the fibration $\mathcal{E}$ can be assumed to satisfy
Assumption $T_\infty$ as well as the other assumptions in §2.1. By applying to this fibration the
same procedure as in the proof of Proposition 2.3.1 we may further assume that this fibration
is also tame.
In what concerns the Fukaya category $\mathcal{F}uk^*(\mathcal{E})$ of $\mathcal{E}$, by inspecting the proof of Proposition 4.5.1, we see that we can actually use here only a smaller category whose objects are cylindrical cobordisms $V \subset \mathcal{E}$ (not necessarily negatively ended) obtained by taking the trail of a given cobordism $Q \subset \mathcal{E}_{v_0} = X$ along a curve $\gamma \subset \mathbb{C} \setminus \text{Crit}v(\pi_X)$. To avoid confusion denote the Fukaya category involved here by $\mathcal{F}uk^*_{r}(\mathcal{E})$ (where $r$ indicates that our objects are restricted as above). Notice that later in the proof we apply certain isotopies (e.g. the negative gradient flow of $\text{Re}(\pi_X)$) to these cobordisms that might not keep them everywhere cylindrical. However, as we shall see below, this is not a problem since for that stage of the proof we do not need the entire Fukaya category anymore but only Floer homology calculations.

Using the notation from Assumption $T_\infty$, put $X^\infty = \mathcal{E}_{v_0}^{\infty}$ and fix a symplectic identification $\approx$

\begin{equation}
\mathcal{E}^\infty \approx \mathbb{C}_\mathcal{E} \times X^\infty.
\end{equation}

Here $\mathbb{C}_\mathcal{E}$ stands for the base of the fibration $\mathcal{E}$, which is just a copy of $\mathbb{C}$. The subscript $\mathcal{E}$ is there only in order to emphasize the relation to $\mathcal{E}$. Denote by $\widehat{\pi}_X : \mathcal{E}^\infty \rightarrow \mathbb{C}_X$ the projection (on the other copy of $\mathbb{C}$) induced via (54) by $\pi_X : X^\infty \rightarrow \mathbb{C}_X$.

Notice that due to the $T_\infty$ assumption a cobordism $V \in \mathcal{O}b(\mathcal{F}uk^*_r(\mathcal{E}))$ has the property that $V \cap \mathcal{E}^\infty$ is a union of finitely many components of the form $\gamma \times l_i \times L_i$ where $\gamma \in \mathbb{C}_\mathcal{E}$ is the projection of $V$ onto $\mathbb{C}_\mathcal{E}$, $l_i$ is a negative ray in $\mathbb{C}_X$ (of imaginary coordinate $i$ in) and $L_i \subset N$ is a Lagrangian in $N$. To fix ideas we will call these Lagrangians $L_i$, the ends of $V$ in the direction of $\mathbb{C}_X$. The important fact to keep in mind is that these ends remain constant along $\gamma$. Obviously, there are also the “usual” ends of $V$ that are of the form $l_i \times C_i$ where $l_i$ is a ray (negative or positive) in $\mathbb{C}_\mathcal{E}$ and $C_i \subset X$ is a negative-ended cobordism in $X$. We will refer to these cobordisms $C_i$ as the ends of $V$ in the direction of $\mathbb{C}_\mathcal{E}$. For each $V$ there are obviously at most two such ends. Notice also that the ends of $C_i$, itself viewed as cobordism, are Lagrangians in $N$ that coincide with the ends of $V$ in the direction of $\mathbb{C}_X$.

We now pass to explaining the choices of Floer and perturbation data required to define the category $\mathcal{F}uk^*_r(\mathcal{E})$. We first pick a profile function $h_X : C_x \rightarrow \mathbb{R}$ such as in §3.2.2 but with the property that the bottlenecks are inside $\pi_X(X^\infty)$.

Consider $V_1, \ldots, V_{k+1} \in \mathcal{O}b(\mathcal{F}uk^*_r(\mathcal{E}))$. Let $C^1, \ldots, C^s \in \mathcal{O}b(\mathcal{F}uk^*(X))$ be the collection of all the ends in the direction of $\mathbb{C}_X$ of the objects $V_1, \ldots, V_{k+1}$. We use the function $h_X$ and the method in §3.2.3 to construct the Floer and perturbation data, associated to $C^1, \ldots, C^s$ as objects of the category $\mathcal{F}uk^*(X)$ associated to the tame Lefschetz fibration $\pi_X : X \rightarrow \mathbb{C}$. We denote all this data by $\mathcal{D}^X_{V_1, \ldots, V_{k+1}}$. As described in §3.2, this data consists of particular choices of Hamiltonians on $X$, that are grouped here in $\mathcal{H}^X_{V_1, \ldots, V_{k+1}}$, and almost complex structures on $X$, grouped in $\mathcal{J}^X_{V_1, \ldots, V_{k+1}}$ so that $\mathcal{D}^X_{V_1, \ldots, V_{k+1}} = (\mathcal{H}^X_{V_1, \ldots, V_{k+1}}, \mathcal{J}^X_{V_1, \ldots, V_{k+1}})$.

Pick a profile function $h_{\mathcal{E}} : \mathbb{C}_\mathcal{E} \rightarrow \mathbb{R}$ again as described in §3.2.2. Let $\gamma_l$ be the projection of $V_l$ onto $\mathbb{C}_\mathcal{E}$. Now modify $h_{\mathcal{E}}$, away from the region of the bottlenecks, in such a way that the new
function \( h_{V_1,\ldots,V_{k+1}} \) conserves the same bottlenecks as \( h_E \) and, additionally, \((\phi^{h_{V_1,\ldots,V_{k+1}}}_i)^{-1}(\gamma_i)\) is transverse to \( \gamma_j \) for all \( i,j \). Now define a new set of Hamiltonians, this time defined on \( \mathbb{C}_E \times X \) as follows: \( \mathcal{H}_{V_1,\ldots,V_{k+1}}' = \{ h_{V_1,\ldots,V_{k+1}} + H : H \in \mathcal{H}_{V_1,\ldots,V_{k+1}}^X \} \).

With these choices, we can describe the constraints on the class of Hamiltonians \( \mathcal{H}_{V_1,\ldots,V_{k+1}}^E \) defined on \( E \) that are part of the perturbation data \( \mathcal{D}_{V_1,\ldots,V_{k+1}} = (\mathcal{H}_{V_1,\ldots,V_{k+1}}^E, \mathcal{J}_{V_1,\ldots,V_{k+1}}^E) \) that we associate to the family \( V_1,\ldots,V_{k+1} \), as required to define \( \mathcal{F}_{uk}^*(E) \). There is a compact set \( K_{V_1,\ldots,V_{k+1}} \subset \mathbb{C}_E \) away from the bottlenecks of \( h_E \) and a compact set \( K_{V_1,\ldots,V_{k+1}}' \subset E^\infty \) away from the bottlenecks of \( h_X \) so that the hamiltonians in \( \mathcal{H}_{V_1,\ldots,V_{k+1}}^E \) coincide with corresponding Hamiltonians in \( \mathcal{H}_{V_1,\ldots,V_{k+1}}' \) on the set

\[
\mathcal{S}_{V_1,\ldots,V_{k+1}} = (E^\infty \setminus K_{V_1,\ldots,V_{k+1}}') \cup \pi_1^{-1}(\mathbb{C}_E \setminus K_{V_1,\ldots,V_{k+1}}).
\]

It is useful to notice at this point that, because the ends of \( V_i \) in the direction of \( \mathbb{C}_X \) do not change along \( \gamma_i \) this choice of Hamiltonian perturbations insures the required transversality at \( \infty \) both in the \( \mathbb{C}_E \) direction as well as in the \( \mathbb{C}_X \) direction. As the Hamiltonians in \( \mathcal{H}_{V_1,\ldots,V_{k+1}}^E \) are basically arbitrary perturbations of the Hamiltonians in \( \mathcal{H}_{V_1,\ldots,V_{k+1}}' \) outside of \( \mathcal{S}_{V_1,\ldots,V_{k+1}} \) this (together with the choice of almost complex structures as detailed below) is also sufficient to achieve the regularity of the relevant moduli spaces.

The family of almost complex structures \( \mathcal{J}_{V_1,\ldots,V_{k+1}}^E \) associated to \( V_1,\ldots,V_{k+1} \) satisfies similar constraints. Namely, over \( \mathcal{S}_{V_1,\ldots,V_{k+1}} \) they are of the form \( i_E \oplus J \) with \( J \in \mathcal{J}_{V_1,\ldots,V_{k+1}}^X \) but can be perturbed freely, so as to insure regularity, outside of \( \mathcal{S}_{V_1,\ldots,V_{k+1}} \).

With these choices the compactness results required to define the category \( \mathcal{F}_{uk}^*(E) \) are valid. More specifically, all solutions \( u \) of the relevant perturbed Cauchy-Riemann equation lie in a prescribed compact subset. The argument is very similar to the one in [BC3]. We consider a hamiltonian \( \bar{h} : E \to \mathbb{R} \) so that away from \( \mathcal{S}_{V_1,\ldots,V_{k+1}} \), \( \bar{h} \) coincides with \( h_E \oplus h_X \). We then use the naturality transformation involving \( \bar{h} \), as summarized in §3.2.4, to turn the solutions \( u \) into curves \( v \) that are (non-perturbed) \( J \)-holomorphic away from \( \mathcal{S}_{V_1,\ldots,V_{k+1}} \). We then apply the open mapping theorem to the projections \( \pi_X \circ v \) and \( \pi_E \circ v \). To summarize, the arguments for both regularity and compactness of the relevant moduli spaces follow closely the corresponding arguments in [BC3] that are used to set up the Fukaya category of cobordisms in \( \mathbb{C} \times M \).

Beyond the definition of \( \mathcal{F}_{uk}^*(E) \) an additional remark is in order. A key part of the proof in §4.5.4 uses the Floer homology for the pairs \((W,V),(W,T_\Delta)\) and \((T_\Delta,V)\). In the course of the proof we apply to \( W \) and \( V \) the negative and positive gradient flows of \( \text{Re}(\pi_E) \). While \( V \) and \( W \) are cylindrical, these flows do not preserve cylindricity. Nevertheless, cylindricity is preserved at infinity in the fiber-direction due to Assumption \( T_\infty \) on \( E \). Therefore the Floer data can easily be adjusted in this case too by using possibly another compactly supported perturbation to insure transversality.
With this remark taken into account and with the definition of $\mathcal{F}uk^*_r(E)$ as above the remainder of the proof proceeds just as in the proof of Proposition 4.5.1.

4.6. The decomposition in Theorem A. To construct this decomposition we start with the proof of Theorem 4.2.1.

4.6.1. Proof of Theorem 4.2.1. We assume for the moment that we are in the setting of §4.2. In particular, $\pi : E \to \mathbb{C}$ is a tame Lefschetz fibration with the properties listed there.

Let $V : \emptyset \sim (L_1, \ldots, L_s)$ and consider the Lefschetz fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ obtained from $E$ by adding singularities as described in §4.4.2. Consider also the cobordism $V' = \tau_{\hat{S}_m} \circ \tau_{\hat{S}_{m-1}} \circ \cdots \circ \tau_{\hat{S}_1}(V) \subset \hat{E}$.

Given $W \in L^*(E)$ we rewrite the exact sequence in Proposition 4.5.3 as

$$W = \text{cone}(S \otimes HF(S, W) \to \tau S W)$$

and deduce that in $DFuk^*(\hat{E})$ we have the following decomposition of $V$:

$$V \cong (\hat{S}_1 \otimes E_1 \to \hat{S}_2 \otimes E_2 \to \cdots \to \hat{S}_m \otimes E_m \to V')$$

where

$$E_i = HF(\hat{S}_i, \tau_{\hat{S}_{i-1}} \circ \cdots \circ \tau_{\hat{S}_1}(V)) .$$

Notice that in $DFuk^*(E)$ we have $T_i \cong (J^{E,\hat{E}})_{\hat{S}_i}^{*}(\hat{S}_i)$ where $J^{E,\hat{E}}$ is the inclusion (24) and $T_i$ are the thimbles in the statement of Theorem 4.2.1. Thus, in $DFuk^*(E)$ we have the decomposition:

$$V \cong (T_1 \otimes E_1 \to T_2 \otimes E_2 \to \cdots \to T_m \otimes E_m \to V')$$

By Corollary 4.4.3 we know that inside $DFuk^*(E)$ we have:

$$V' \cong (\gamma_s \times L_s \to \gamma_{s-1} \times L_{s-1} \to \cdots \to \gamma_2 \times L_2)$$

Splicing together (56) and (57) we obtain:

$$V \cong (T_1 \otimes E_1 \to \cdots \to T_m \otimes E_m \to \gamma_s \times L_s \to \cdots \to \gamma_2 \times L_2)$$

which concludes the proof of Theorem 4.2.1. \qed
4.6.2. The decomposition in Theorem A. We assume the setting from Theorem 4.1.1 (which we recall is just a more precise reformulation of Theorem A) and recall a bit of the necessary background. The fibration \( \pi : E \to C \) is no longer assumed to be tame. All the singularities of \( \pi \) are included in \( \pi^{-1}(S_{x,y}) \) and there is a tame fibration \( \pi : E_\tau \to C \) that coincides with \( E \) over \([x-3, y+3] \times [-\frac{1}{2}, \infty)\) and is tame outside of a set \( U \) that contains \((x-4, y+4) \times (-1, \infty)\). Recall from §3.3 that the objects of the category \( \mathcal{F}uk^*(E; \tau) \) are uniformly monotone cobordisms \( V \subset E \) that are cylindrical outside \( S_{x,y} \) and the operations \( \mu_k \) of \( \mathcal{F}uk^*(E; \tau) \) are defined by means of the corresponding operations in the category \( \mathcal{F}uk^*(E_\tau) \) associated to the tame fibration \( E_\tau \).

The decomposition in Theorem 4.1.1 (and thus that in Theorem A) follows rapidly from that in Theorem 4.2.1. Indeed, recall from §3.3 that we have an inclusion:

\[
\mathcal{F}uk^*(E; \tau) \to \mathcal{F}uk^*(E_\tau)
\]

that is a quasi-equivalence and which, on objects, is defined by \( V \to \overline{V} \) where \( \overline{V} \) is obtained by cutting off the the ends of \( V \) along the line \( \{x - 3\} \times \mathbb{R} \) and extending them horizontally by parallel transport in the fibration \( E_\tau \). As \( E_\tau \) is a tame fibration, Theorem 4.2.1 can be applied to it. We deduce decompositions involving two types of curves in the plane, the \( t_k \)'s and \( \gamma_i \)'s as in Figure 10. The curves \( \gamma_i \) appearing here are included in the negative quadrant \( Q_U = (-\infty, -a_U] \times [0, \infty) \) and they are away from \( U \). For reasons that will become clear in a moment, it is convenient to refine the notation for these curves such as to explicitly indicate their dependence on \( U \). Thus we will further denote them by \( \gamma_i^U \).

The decomposition result that we want to show here - in the statement of Theorem 4.1.1 - applies to \( \mathcal{F}uk^*(E; \tau) \). It again involves the same thimbles \( T_k \) associated to the curves \( t_k \) as before as well certain “trails” denoted in Theorem 4.1.1 by \( \gamma_i L_i \). It is important to notice at this point that as \( a_U < x - 4 \), the curves \( \gamma_i \) appearing in the statement of Theorem 4.1.1 do not coincide with the \( \gamma_i^U \)'s above - see also Figure 32. Nonetheless, for \( L \in L^*(M) \) and any curve \( \gamma_i \) consider the cobordism \( \overline{\gamma_i L} \) as an object of \( \mathcal{F}uk^*(E_\tau) \). This object is quasi-isomorphic to \( \gamma_i^U \times L \) (this can proved directly, but it also follows immediately from Theorem 4.2.1 itself). As a consequence, we may replace in the decomposition given by Theorem 4.2.1 the objects \( \gamma_i^U \times L_i \) by the objects \( \overline{\gamma_i L_i} \) and by pulling back the resulting decomposition form \( \mathcal{F}uk^*(E_\tau) \) to \( \mathcal{F}uk^*(E; \tau) \) via the inclusion (58) we obtain the decomposition claimed in Theorem 4.1.1. □

5. Main consequences

5.1. From the total space to the fiber and back. We will work in this subsection only with tame Lefschetz fibrations - see Definition 2.2.2. In view of §2.3 this is not restrictive.
Figure 32. The Lagrangian $\gamma^U_3 \times L$ is an object in $\mathcal{F}uk^*(E'; a_U, a_U)$ but is not cylindrical outside of $[x - 4, y + 4] \times \mathbb{R}$ and thus it not an object in $\mathcal{F}uk^*(E'; x - 4, y + 4)$.

Thus we assume that $\pi : E \to \mathbb{C}$ is a Lefschetz fibration which is tame outside of $U \subset \mathbb{C}$ and $(M, \omega)$ is the generic fibre. The fibration $E$ has singularities $x_1, \ldots, x_m$ of respective critical values $v_1, \ldots, v_m$ (assumed to be, for simplicity, $v_k = (k, \frac{3}{2})$). Denote by $O \in \mathbb{C}$ the origin and recall that the fibration $E$ is assumed to be tame over a region that contains $O$. Connect each critical value $v_k$ to $O$ by a straight segment, and denote by $S_k \in \pi^{-1}(O) = M$ the vanishing cycle associated to that path.

We use the rest of the set-up and notation from §4.2. The results described below are all consequences of Theorem 4.2.1.

5.1.1. **Descent: from decompositions in $D\mathcal{F}uk^*(E)$ to decompositions in $D\mathcal{F}uk^*(M)$**.

**Corollary 5.1.1.** As in Theorem 4.2.1, let $V \in \mathcal{L}^*(E)$, $V : \emptyset \to (L_1, \ldots, L_s)$. Then there exists an iterated cone decomposition that depends on $V$ and takes place in $D\mathcal{F}uk^*(M)$:

$$L_1 \cong (\tilde{\tau}_{2, \ldots, m}^{-1} S_1 \otimes E_1 \to \tilde{\tau}_{3, \ldots, m}^{-1} S_2 \otimes E_2 \to \cdots \to \tilde{\tau}_{i+1, \ldots, m}^{-1} S_i \otimes E_i \to \cdots \to S_m \otimes E_m \to L_s \to L_{s-1} \to \cdots \to L_2),$$

where $\tilde{\tau}_{i, \ldots, m}$ stands for the composition:

$$\tilde{\tau}_{i, \ldots, m} = \tau_{S_i} \circ \tau_{S_{i+1}} \circ \cdots \circ \tau_{S_m}.$$

**Proof.** In this proof it is convenient to consider again the category $D\mathcal{F}uk^*_\mathbb{H}(E)$ from §4.3. Recall that the difference between this category and $D\mathcal{F}uk^*(E)$ is that the objects $V$ of the underlying category $\mathcal{F}uk^*_\mathbb{H}(E)$ are more general cobordisms than those given in Definition 2.2.3 in that the imaginary coordinates of the ends of $V$ are allowed to also be positive half-integers.
In other words, $V$ has only negative ends and

$$V \cap \pi^{-1}(Q^-_U) = \bigcup_i ((-\infty, -a_U] \times \frac{i}{2}) \times L_i.$$ 

We now consider curves $\eta_i$ as in Figure 33.

![Figure 33. The auxiliary curves $\eta_i$ together with the cobordism $V \in \mathcal{L}^*(E)$.](image)

These curves satisfy

$$\eta_i((\infty, -1]) = (-\infty, -a_U - 2] \times \frac{2i - 1}{2}, \quad \eta_i([1, +\infty)) = (-\infty, -a_U - 2] \times \frac{2i + 1}{2}$$

and $\eta_i(\mathbb{R}) \subset Q^-_U$.

As shown in [BC3] there exists an $A_\infty$-functor:

$$i^n_\eta : \text{Fuk}^*(M) \to \text{Fuk}_{\frac{1}{2}}^*(E)$$

which acts on objects by $L \mapsto \eta_j \times L$. Consider now the pull-back functor:

$$(i^n_\eta)^* : \text{mod}(\text{Fuk}_{\frac{1}{2}}^*(E)) \to \text{mod}(\text{Fuk}^*(M)).$$

Notice that there is a full and faithful embedding $e : \text{Fuk}^*(E) \to \text{Fuk}_{\frac{1}{2}}^*(E)$. Consider the Yoneda embeddings $\mathcal{Y} : \text{Fuk}^*(E) \to \text{mod}(\text{Fuk}^*(E))$ and $\mathcal{Y}_{\frac{1}{2}} : \text{Fuk}_{\frac{1}{2}}^*(E) \to \text{mod}(\text{Fuk}_{\frac{1}{2}}^*(E))$.

Let $\mathcal{Y}' : \text{Fuk}^*(E) \to \text{mod}(\text{Fuk}_{\frac{1}{2}}^*(E))$ be $\mathcal{Y}' = \mathcal{Y}_{\frac{1}{2}} \circ e$. The homology category associated to the triangular completion $(\text{Image}(\mathcal{Y}'))^\wedge$ of the image of $\mathcal{Y}'$ inside $\text{mod}(\text{Fuk}_{\frac{1}{2}}^*(E))$ is easily seen to be quasi-equivalent to $D\text{Fuk}^*(E)$ (see also §3.1).

For an object $V \in \text{Fuk}^*(E)$ let $\mathcal{M}_V' = \mathcal{Y}'(V)$. Notice that $(i^n_{\eta})^*(\mathcal{M}_V')$ is precisely the Yoneda module associated to the $j$-end of $V$. Thus $i^n_\eta$ takes Yoneda modules to Yoneda
modules and given that \( H_0(Image(\mathcal{Y}')) = D\mathcal{F}uk^*(E) \) we deduce that the functor \((i^!j)^*\) induces a functor of triangulated categories

\[
\mathcal{R}_j : D\mathcal{F}uk^*(E) \rightarrow D\mathcal{F}uk^*(M)
\]

that we will refer to as the restriction to the \( j \)-th end.

The decomposition in the statement is obtained by applying \( \mathcal{R}_1 \) to the decomposition in Theorem 4.2.1. It is easy to see that the end of the thimble \( T \) strictly bigger than any element of \( N \) strictly in a way that respects the order of \( N \) over a given ring \( A \) that, on objects, associates to each cobordism \( V \) that we will refer to as the restriction to the \( j \)-th end. At the derived level we also have \( \mathcal{R}_j \circ i^!j = \text{id} \). Notice also that the pull-back functor

\[
\hat{\mathcal{R}}^*_j : \text{mod}(\mathcal{F}uk^*(M)) \rightarrow \text{mod}(\mathcal{F}uk^*(E))
\]

takes the Yoneda module \( \mathcal{Y}(L) \) to the Yoneda module \( \mathcal{Y}(\eta_j \times L) = i^!j(L) \).

5.1.2. Ascent: from \( D\mathcal{F}uk^*(M) \) to the category \( D\mathcal{F}uk^*(E) \). We assume the same setting as fixed at the beginning of §5.1 and start with some algebraic notation. Let \( \mathcal{B} \) be an \( A_\infty \)-category (over a given ring \( A \), e.g. the Novikov ring) and \( R_1, \ldots R_m \) a collection of \( m \) objects of \( \mathcal{B} \). The following construction is a straightforward extension of the notion of directed \( A_\infty \) category as it appears in [Sei3] (see, in particular, (5m) there).

Consider the ordered set \( I_m = \{1, \ldots, m\} \) and let \( \mathbb{N}_{+m} \) be the disjoint union \( \mathbb{N} \cup I_m \) ordered strictly in a way that respects the order of \( \mathbb{N} \) and \( I_m \) and so that each element in \( I_m \) is strictly bigger than any element of \( \mathbb{N} \). We still denote the resulting order relation by \( \geq \). For any two \( i, j \in \mathbb{N}_{+m} \) we put \( \xi^{i,j} = 1 \) if \( i \geq j \) and \( \xi^{i,j} = 0 \) if \( i < j \) and we let \( \xi^{i_1,i_2,\ldots,i_{k+1}} = \xi^{i_1,i_2} \xi^{i_3,i_4} \cdots \xi^{i_{k+1}} \).

We denote by \( \mathbb{N}_{+m} \otimes \mathcal{B} \) the unique \( A_\infty \)-category with the properties:

i. The objects of \( \mathbb{N}_{+m} \otimes \mathcal{B} \) are couples \((i, L)\) with \( i \in \mathbb{N}_{+m} \) and \( L \) an object of \( \mathcal{B} \) with the constraint that if \( i \in I_m \), then \( L = R_i \). We will write the couples \((i, L)\) as \( i \times L \).

ii. The morphisms of \( \mathbb{N}_{+m} \otimes \mathcal{B} \) are defined by:

\[
\text{Mor}(i \times L, j \times L') = \xi^{i,j} \text{Mor}(L, L')
\]

except if \( i = j \in I_m \). In this case \( \text{Mor}(i \times R_i, i \times R_i) = \mathcal{A} \epsilon_{R_i} \). Here \( \epsilon_{R_i} \) is, by definition, a strict unit in the category \( \mathbb{N}_{+m} \otimes \mathcal{B} \).
iii. We denote by
\[ \mu_k : \text{Mor}(L_1, L_2) \otimes \text{Mor}(L_2, L_3) \otimes \ldots \otimes \text{Mor}(L_k, L_{k+1}) \to \text{Mor}(L_1, L_{k+1}) \]
the multiplications in \( \mathcal{B} \). Consider successive indexes \((i_1, i_2, \ldots, i_{k+1})\) so that no two successive indexes \(i_r, i_{r+1}\) satisfy \(i_r = i_{r+1} \in I_m\). Then the multiplications in \( \mathbb{N}^+_m \otimes \mathcal{B} \) are given by:
\[ \mu'_k : \text{Mor}(i_1 \times L_1, i_2 \times L_2) \otimes \text{Mor}(i_2 \times L_2, i_3 \times L_3) \otimes \ldots \otimes \text{Mor}(i_k \times L_k, i_{k+1} \times L_{k+1}) \to \text{Mor}(i_1 \times L_1, i_{k+1} \times L_{k+1}) \]
(61)
\[ \mu'_k = \xi_{i_1 \ldots i_{k+1}} \mu_k. \]

In case for some index \( r \) we have \( i_r = i_{r+1} \in I_m \), then \( \mu'_k \) is completely described by the requirement that \( e_{R_i} \) be a strict unit: \( \mu'_k \) vanishes if \( k \neq 2 \) and \( \mu'_2(a, e_{R_i}) = a, \mu'_2(e_{R_i}, b) = b. \)

The notation \( \mathbb{N}^+_m \otimes \mathcal{B} \) is slightly imprecise as this category actually depends on the choice of objects \( R_1, \ldots, R_m \). In case the \( A_\infty \) category \( \mathcal{B} \) is such that the objects \( R_i \) have strict units \( e'_{R_i} \in \text{Mor}_B(R_i, R_i) \), then by taking \( e_{R_i} = e'_{R_i} \), equation (61) applies without treating separately the case \( i_r = i_{r+1} \in I_m \). In general, when the \( R_i \)'s do not have strict units, we treat the \( e_{R_i} \)'s as formal elements, part of the construction of \( \mathbb{N}^+_m \otimes \mathcal{B} \).

**Corollary 5.1.3.** There exists a choice of Lagrangians spheres \( R_1, \ldots, R_m \in \mathcal{L}^*(M) \) and an equivalence of categories:
\[ \mathcal{I} : D(\mathbb{N}_m \otimes \mathcal{F}uk^*(M)) \to D\mathcal{F}uk^*(E) . \]

**Proof.** Consider the full and faithful subcategory \( \mathcal{F}(E) \) of \( \mathcal{F}uk^*(E) \) whose objects consist of the following two collections:

i. \( \gamma_{i+2} \times L \) with \( i \in \mathbb{N} \) and \( L \in \mathcal{L}^*(M) \). Here \( \gamma_k, k \geq 2, \) are the plane curves defined in §4.1.1 (see also Figure 10).

ii. the thimbles \( T_j, j \in I_m \).

The generation Theorem 4.2.1 combined with the algebraic Lemma 3.34 in [Sei3] implies that there is an equivalence of categories
\[ D\mathcal{F}(E) \to D\mathcal{F}uk^*(E) \]
induced by the inclusion
\[ \mathcal{F}(E) \to \mathcal{F}uk^*(E) . \]

We now intend to show the existence of a quasi-equivalence of \( A_\infty \)-categories:
\[ \Xi : \mathbb{N}_m \otimes \mathcal{F}uk^*(M) \to \mathcal{F}(E) . \]
To this end we first pick a specific family of objects $R_1, \ldots, R_m$ in $\text{Fuk}^*(M)$. By definition, these objects are the following Lagrangian spheres:

$$R_{m+1-i} := \tilde{\tau}_{i+1, \ldots, m}(S_i), \ i = 1, \ldots, m$$

- see Corollary 5.1.1 for the notation. For $i \in \mathbb{N}$, and $L \in \mathcal{L}^*(M)$, we define $\Xi'(i \times L) = \gamma_{i+2} \times L$.

For $i \in I_m$ we define $\Xi'(i \times R_i) = T_{m+1-i}$. It is not difficult to see - as in the construction of the inclusion functor $\mathcal{I}_{\gamma,h}$ in [BC3], in particular Proposition 4.2.3 there - that by using appropriate choices for the curves $\gamma_i$ as well as almost complex structures and perturbation data, we can describe the morphisms and higher products in $\mathcal{F}(E)$ by the formulas corresponding to $\mathbb{N}^+ \otimes \text{Fuk}^*(M)$. There is however one exception concerning this correspondence and due to it the map $\Xi'$ can not be assumed directly to be a morphism of $A_\infty$ categories: the difficulty comes from the fact that the objects $T_j$ of $\mathcal{F}(E)$ do not, in general, have strict units. However, there is an algebraic argument - Lemma 5.20 in §(5n) in [Sei3] - that applies also to our case with minor modifications and implies that we can replace $\Xi'$ by a true $A_\infty$ functor: $\Xi: \mathbb{N}^+ \otimes \text{Fuk}^*(M) \to \mathcal{F}(E)$ that acts on objects in the same way as $\Xi'$ and so that $\Xi$ is a quasi-equivalence. Clearly, this implies the equivalence of the associated derived categories and the existence of $\mathcal{I}$. □

Remark 5.1.4. a. Corollary 5.1.3 extends a result of Seidel in §18 of [Sei3] (see also [Sei4]) which provides a similar description for the subcategory of $D\text{Fuk}^*(E)$ that is generated by the thimbles $T_i$.

b. It is easy to see by direct calculation that there are inclusions $J_s: D\text{Fuk}^*(M) \to D(\mathbb{N}^+ \otimes \text{Fuk}^*(M))$ induced by $L \to (s, L)$ for all $s \in \mathbb{N}$. The compositions $J'_s = \mathcal{I} \circ J_s$ have a simple geometric interpretation. Consider the inclusion $i_{s+2}: \text{Fuk}^*(M) \to \text{Fuk}^*(E)$ which acts on objects as $L \to \gamma_{s+2} \times L$. This induces a functor $i_{s+2}: D\text{Fuk}^*(M) \to D\text{Fuk}^*(E)$ that coincides with $J'_s$.

c. An obvious by-product of this Corollary is that the derived categories $D\text{Fuk}^*(E; \tau)$ from the statement of Theorem 4.1.1 are independent of the choice of tame fibration $E_\tau$ up to equivalence. Together with §4.6.2 this concludes the proof of Theorem 4.1.1.

5.2. The Grothendieck group. The purpose of this section is to discuss a variety of consequences of Theorem 4.2.1 in what concerns the morphism $\Theta$ from (1) as well as the Grothendieck group itself.

5.2.1. Cobordism groups and the Grothendieck group. We start by defining the appropriate cobordism groups that will be of interest to us here. We will restrict here too the discussion to tame Lefschetz fibrations. Fix such a fibration $\pi: E \to \mathbb{C}$ that is tame outside $U \subset \mathbb{C}$. Let $(M, \omega)$ be the fibre of $\pi$ at a point $z_0 \in \mathbb{C} \setminus U$. Let $\Omega^*_{\text{ab}}(M; E)$ be the abelian group defined as the quotient of the free abelian group generated by the Lagrangians $L \in \mathcal{L}^*(M)$...
modulo the relations $\mathcal{R}_{\text{cob}}^E$ generated by the cobordisms $V : \emptyset \sim (L_1, \ldots, L_s)$, $V \in \mathcal{L}^*(E)$ in the sense that to each such $V$ we associate the relation $L_1 + \ldots + L_s \in \mathcal{R}_{\text{cob}}^E$. Basically, the point of view here is that cobordisms are relators among their ends. As we do not take into account orientations this group is obviously 2-torsion. Notice that all vanishing spheres $S \subset M$ (associated to any path between a critical value of $\pi$ and $z_0$) belong to $\mathcal{R}_{\text{cob}}^E$, hence there cobordism class is $0 \in \Omega^*_{\text{Lag}}(M; E)$. This follows from the fact that a vanishing sphere is the single end of a cobordism which is a thimble of some path going from one critical value of $\pi$ to $z_0$.

In case $\pi : E \rightarrow \mathbb{C}$ is the trivial fibration (i.e. $E$ splits symplectically as $E = \mathbb{C} \times M$ and $\pi = \text{pr}_{\mathbb{C}}$) we will abbreviate $\Omega^*_{\text{Lag}}(M; E)$ for $\Omega^*_{\text{Lag}}(M)$. 

**Remark 5.2.1.**

a. While we will not explore this issue here, notice that the group $\Omega^*_{\text{Lag}}(M; E)$ is the abelianization of a group $\mathcal{G}^*_{\text{Lag}}(M; E)$ that is defined as the free non-abelian group generated by the $L \in \mathcal{L}^*(M)$ modulo relations $L_1 \cdot L_2 \cdot \ldots \cdot L_s$ associated as before to cobordisms $V : \emptyset \sim (L_1, \ldots, L_s)$. In other words, in this case we take into account the geometric order of the ends of $V$.

b. It is easy to adjust the definition of the groups $\Omega^*_{\text{Lag}}(-)$ to the case of non-tame fibrations. However, in view of §2.3, all interesting phenomena concerning these cobordism groups are already present in the case of tame fibrations.

Recall the Grothendieck group $K_0(D\mathcal{F}uk^*(M))$ that is associated to the triangulated category $D\mathcal{F}uk^*(M)$ as in §3.1. Notice that this group too is 2-torsion because we work in an ungraded setting. We are interested in a quotient of this Grothendieck group that is associated to our tame fibration $\pi : E \rightarrow \mathbb{C}$. To construct it assume $x_1, \ldots, x_m$ are the critical points of $\pi$ and let the corresponding critical values be $v_1, \ldots, v_m$. Then for each $i$ pick a path in $\mathbb{C}$ from $v_i$ to $z_0$ (such as, for instance, the paths $t_i$ in Figure 10). There is an associated thimble to each such path and let $\Sigma_i$ be the vanishing sphere in $M = \pi^{-1}(z_0)$ that is the end of the thimble from $x_i$ to $M$. Denote by $\mathcal{S}_E$ the subgroup in $K_0(D\mathcal{F}uk^*(M))$ that is generated by the spheres $\Sigma_i$. Finally, define the quotient:

$$K_0(D\mathcal{F}uk^*(M); E) = K_0(D\mathcal{F}uk^*(M))/\mathcal{S}_E.$$ 

**Corollary 5.2.2.** The group $K_0(D\mathcal{F}uk^*(M); E)$ does not depend on the choices made in its construction and there exists a morphism of groups:

$$\Theta^E : \Omega^*_{\text{Lag}}(M; E) \rightarrow K_0(D\mathcal{F}uk^*(M); E)$$

that is induced by $L \rightarrow L$.

This morphism extends the Lagrangian Thom morphism initially constructed in [BC3] and already mentioned at (1)

$$\Theta : \Omega^*_{\text{Lag}}(M) \rightarrow K_0(D\mathcal{F}uk^*(M))$$
Proof. We first discuss the independence of $K_0(D\mathcal{F}uk^*(M); E)$ of the choices of the vanishing spheres $\Sigma_i$. Assume for instance that one of these spheres, say $\Sigma_1$ - that is the end of a thimble $K_1$ that projects to a path $k_1$ from $v_1$ to $z_0$ - is replaced with a sphere $\Sigma'_1$ which is the end of a thimble $K'_1$, associated to a different path, $k'_1$. By the results of Seidel [Sei3], the difference between $\Sigma_1$ and $\Sigma'_1$ (up to hamiltonian isotopy) can be described as follows: one sphere is obtained from the other by an applying a symplectic diffeomorphism $\phi$ which can be written as word in the elements $\tau_{\Sigma_2}, \ldots, \tau_{\Sigma_m}$ (i.e. $\phi$ is a composition of Dehn twists and their inverses along spheres from the collection $\Sigma_2, \ldots, \Sigma_m$). From Seidel’s exact triangle as given in Proposition 4.5.1 we see that the subgroups generated, respectively, by $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ and $\Sigma'_1, \Sigma_2, \ldots, \Sigma_m$ are the same.

The existence of the morphism $\Theta^E$ is now an immediate consequence of the decomposition in Corollary 5.1.1. □

5.2.2. The Grothendieck group as an algebraic cobordism group. We now focus our attention on the category $\mathcal{F}uk^*(E)$. For each module $\mathcal{M} \in \text{Ob}(D\mathcal{F}uk^*(E))$, define $[\mathcal{M}]_j \in \text{Ob}(D\mathcal{F}uk^*(M))$ by

$$[\mathcal{M}]_j = R_j(\mathcal{M})$$

where $R_j$ are the restriction functors defined in the proof of Corollary 5.1.1 (see also Remark 5.1.2). Basically, this extends to all objects in $D\mathcal{F}uk^*(E)$ the operation that associates to a cobordism $V$ its $j$-th end. It is easy to see that for all objects $\mathcal{M}$ of $D\mathcal{F}uk^*(E)$ there are only finitely many non-vanishing $[\mathcal{M}]_j$’s.

We now define another group $\Omega^*_{Alg}(M; E)$, which we call the algebraic cobordism group, as the free abelian group generated by all the isomorphisms types of objects $\in \text{Ob}(D\mathcal{F}uk^*(M))$ modulo the relations

$$[\mathcal{M}]_1 + [\mathcal{M}]_2 + [\mathcal{M}]_3 + \ldots = 0$$

for each $\mathcal{M} \in \text{Ob}(D\mathcal{F}uk^*(E))$.

The group $\Omega^*_{Alg}(M; E)$ can be viewed as an algebraic cobordism group in the following sense. The generators of this group are the (isomorphism type of) objects of $D\mathcal{F}uk^*(M)$, thus they are obtained by completing algebraically the objects of $\mathcal{F}uk^*(M)$ as in the construction of the derived Fukaya category. Similarly, the relations defining the group are again an algebraic completion - in a similar sense but now involving the categories $\mathcal{F}uk^*(E)$ and $D\mathcal{F}uk^*(E)$ - of the relations providing $\Omega^*_{Lag}(M; E)$. By definition, there is an obvious group morphism:

$$q : \Omega^*_{Lag}(M; E) \to \Omega^*_{Alg}(M; E).$$

Corollary 5.2.3. There is a group isomorphism

$$\Theta^E_{Alg} : \Omega^*_{Alg}(M; E) \to K_0(D\mathcal{F}uk^*(M); E).$$
so that \( \Theta^E = \Theta_{\text{Alg}}^E \circ q \).

Proof. Throughout the proof we abbreviate \( K_0 = K_0(\mathcal{D\mathcal{F}uk^*}(M); E) \).

At the level of generators we define \( \Theta_{\text{Alg}}^E \) to be the identity. The surjectivity of \( \Theta_{\text{Alg}}^E \) is clear as well as the relation \( \Theta^E = \Theta_{\text{Alg}}^E \circ q \). The only two things to check are that this map is well-defined and injective.

To show that \( \Theta_{\text{Alg}}^E \) is well-defined we need to prove that if \( M \) is an object of \( \mathcal{D\mathcal{F}uk^*}(E) \), then \( \sum_i [\mathcal{M} ]_i = 0 \) in \( K_0(\mathcal{D\mathcal{F}uk^*}(M); E) \). To see this recall that, by the definition of \( \mathcal{D\mathcal{F}uk^*}(E) \), there are \( V_j \in \mathcal{L}^*(E) \) so that:

\[
\mathcal{M} \cong (V_m \to V_{m-1} \to \ldots \to V_2 \to V_1).
\]

By Theorem 4.2.1, in \( K_0 \) we have:

\[
\sum_i [V_j]_i = 0 \quad \forall j.
\]

Moreover, \( \forall i \), we have the following cone decomposition of \( [\mathcal{M}]_i \) in \( \mathcal{D\mathcal{F}uk^*}(M) \):

\[
[\mathcal{M}]_i \cong ([V_m]_i \to [V_{m-1}]_i \to \ldots \to [V_2]_i \to [V_1]_i)
\]

because the functor \( \mathcal{R}_i \) is triangulated. This means that in \( K_0 \):

\[
\sum_i [\mathcal{M}]_i = \sum_{i,j} [V_j]_i = 0.
\]

This concludes the proof of the well-definedness of the map \( \Theta_{\text{Alg}}^E \).

It remains to show that \( \Theta_{\text{Alg}}^E \) is injective. We start by proving the injectivity in the case when \( \pi \) is trivial and so \( E = \mathbb{C} \times M \). We omit \( E \) from the notation of \( \Theta_{\text{Alg}} \) in this case and, similarly, we put \( \Omega_{\text{Alg}}(M) = \Omega_{\text{Alg}}(M; \mathbb{C} \times M) \). Assume that

\[
\mathcal{M} \to \mathcal{M}' \to \mathcal{M}''
\]

is an exact triangle of \( \mathcal{F}uk^*(M) \)-modules. The injectivity of \( \Theta_{\text{Alg}} \) follows by constructing for each such triangle an object \( T \) in \( \mathcal{D\mathcal{F}uk^*}(\mathbb{C} \times M) \) so that \( [T]_1 = \mathcal{M}'' \), \( [T]_2 = \mathcal{M}' \) and \( [T]_3 = \mathcal{M} \). Indeed, this implies that all the relations that are used in the definition of \( K_0 \) also appear among the relations that define \( \Omega_{\text{Alg}}(M) \) which means that \( \Theta_{\text{Alg}} \) is invertible.

To construct this object \( T \) we proceed as follows. We first recall that, by definition, \( \mathcal{M}'' \) is - up to isomorphism - the cone over a module map \( f : \mathcal{M} \to \mathcal{M}' \).

Now recall the \( A_\infty \)-category \( \mathbb{N} \otimes \mathcal{F}uk^*(M) \) as in §5.1.2 (notice that now \( m = 0 \)). We first construct an object \( \tilde{T} \) of \( \mathbb{N} \otimes \mathcal{F}uk^*(M) \). This consists of two steps. First, for each \( \mathcal{F}uk^*(M) \)-module \( \mathcal{N} \) and each curve \( \gamma_i \) we define a \( \mathbb{N} \otimes \mathcal{F}uk^*(M) \)-module denoted by \( \gamma_i \times \mathcal{N} \). On objects \( \gamma_j \times L \) we put \( (\gamma_i \times \mathcal{N})(\gamma_j \times L) = \xi^{j,i} \mathcal{N}(L) \). The \( A_\infty \)-module operations are defined by a direct
adaptation of the formulas giving the operations in $\mathbb{N} \otimes \mathcal{F}uk^*(M)$. The second step is to define a morphism

$$\tilde{f} : \gamma_3 \times \mathcal{M} \to \gamma_2 \times \mathcal{M}'$$.

We then define $\tilde{T}$ by $\tilde{T} = cone(\tilde{f})$. The morphism $\tilde{f}$ is induced by $f$ and is given by a formula again perfectly similar to the formula of the multiplication in $\mathbb{N} \otimes \mathcal{F}uk^*(M)$, but using $f$ instead of $\mu_k$ and replacing $\text{Mor}(i_k \times L_k, i_{k+1} \times L_{k+1})$ by $(\gamma_3 \times \mathcal{M})(\gamma_{i_{k-2}} \times L_{k+1})$ and $\text{Mor}(i_1 \times L_1, i_{k+1} \times L_{k+1})$ by $(\gamma_2 \times \mathcal{M})(\gamma_{i_1-2} \times L_1)$. We now consider the sequence of functors, the first two being equivalences and the last a full and faithful embedding:

(62) \[ D(\mathbb{N} \otimes \mathcal{F}uk^*(M)) \to D\mathcal{F}(\mathbb{C} \times M) \to D\mathcal{F}uk^*(\mathbb{C} \times M) \to D\mathcal{F}uk^*_2(\mathbb{C} \times M). \]

Here, the $A_\infty$-category $D\mathcal{F}(\mathbb{C} \times M)$ is defined as in the proof of Corollary 5.1.3. We now use the composition of the functors in (62) to define $[\mathcal{H}]_j = (i^n)^*(\mathcal{H})$ for each module $\mathcal{H}$ in $D(\mathbb{N} \otimes \mathcal{F}uk^*(M))$ - see the proof of Corollary 5.1.1 for the definition of $i^n$. We take $T$ to be the image of $\tilde{T}$ by the first two equivalences in (62) and we claim that:

1. for each object $\mathcal{N}$ in $D\mathcal{F}uk^*(M)$ we have that $[(\gamma_i \times \mathcal{N})]_j \cong \mathcal{N}$ if $i = j$ or $j = 1$ and is 0 otherwise. Moreover, $(i^n)^*(\tilde{f}) \cong f$.
2. $[T]_1 = \mathcal{M}'$, $[T]_2 = \mathcal{M}'$, $[T]_3 = \mathcal{M}$ and $[T]_i = 0$ whenever $i \geq 4$.

Notice that point b concludes the proof for $E = \mathbb{C} \times M$. Given that the equivalences in (62) are triangulated, point b follows directly from a. Thus, it remains to check a. For this we notice that pull-back respects triangles and as each object $\mathcal{N}$ is isomorphic to an iterated cone of objects $L \in \mathcal{F}uk^*(M)$ it is enough to verify the statement for the Yoneda modules $\gamma_i \times L$, $L \in \mathcal{L}^*(M)$. But for these modules the statement is obvious. The statement for $\tilde{f}$ follows in a similar fashion.

We are left to show the more general statement for a Lefschetz fibration $\pi : E \to \mathbb{C}$ that is not trivial. For this we recall that, for each thimble $T_i$ we have $(i^n)^*(T_i) = \tau_{i+1,...,m}^{-1}S_i$. (The definition of the spheres $S_i$ appears in §5.) Thus, by the definition of the groups involved, we have a quotient map

(63) \[ \Omega^*_{Alg}(M)/S'_E \to \Omega^*_{Alg}(M; E) \xrightarrow{\Theta^e_{Alg}} K_0(D\mathcal{F}uk^*(M); E), \]

where $S'_E$ is the subgroup generated by the vanishing spheres of $\pi$. To conclude the proof of the theorem it is enough to show that the composition of maps in (63) is an isomorphism. Recall that

$$K_0(D\mathcal{F}uk^*(M); E) = K_0(D\mathcal{F}uk^*(M))/S_E$$

and notice that the isomorphism $\Theta_{Alg}$ - associated to the trivial fibration $\mathbb{C} \times M$ - has the property that $\Theta_{Alg}(S'_E) = S_E$. Therefore the composition of maps in (63) is an isomorphism and this concludes the proof. \qed
5.2.3. **Comparison with ambient quantum homology.** There is an obvious morphism:

\[ i : \Omega^\ast_{\text{Lag}}(M) \rightarrow QH_\ast(M) \]

that associates to each Lagrangian \( L \) its homology class \([L] \in H_n(M; \mathbb{Z}_2) \subset QH_\ast(M)\). From the point of view of Corollary 5.2.3 it is natural to expect that \( i \) factors through a morphism:

\[ i' : \Omega^\ast_{\text{Alg}}(M) \rightarrow QH_\ast(M) \]

This is indeed true as we will see below.

**Corollary 5.2.4.** Consider a module \( \mathcal{M} \in \text{Ob}(DFuk^\ast(M)) \). Such a module admits a cone-decomposition (up to quasi-isomorphism)

\[ \mathcal{M} \cong (L_s \rightarrow L_{s-1} \rightarrow \ldots \rightarrow L_1) . \]

With this notation, the equation

\[ (64) \quad i'(\mathcal{M}) = \sum_j [L_j] \in QH_\ast(M) \]

provides a well-defined group morphism

\[ i' : \Omega^\ast_{\text{Alg}}(M) \rightarrow QH_\ast(M) \]

so that \( i = i' \circ q \).

**Proof.** While this definition of \( i' \) seems very simple the fact that \( i' \) is a well-defined morphism of groups is somewhat surprising. We only know a proof of this fact which follows from the indirect construction that we give below.

We will write \( i' \) as a composition of two morphisms \( i' = \tilde{i}' \circ \Theta_{\text{Alg}} \) where \( \Theta_{\text{Alg}} : \Omega^\ast_{\text{alg}}(M) \rightarrow K_0(DFuk^\ast(M)) \) is the isomorphism in Corollary 5.2.3 and

\[ \tilde{i}' : K_0(DFuk^\ast(M)) \rightarrow QH_\ast(M) \]

is a morphism that is known to experts, see for instance §5 in [Sei5]. The definition of \( \tilde{i}' \) is somewhat subtle so we review it here.

The morphism \( \tilde{i}' \) is a composition of morphisms:

\[ K_0(DFuk^\ast(M)) \xrightarrow{f_1} K_0(\mathcal{Y}(Fuk^\ast(M))^\wedge) \xrightarrow{f_2} H_\ast(\mathcal{Y}(Fuk^\ast(M))^\wedge) \xrightarrow{f_3} HH_\ast(Fuk^\ast(M)) \xrightarrow{f_4} QH(M) . \]

Here, the category \( \mathcal{Y}(Fuk^\ast(M)) \) is the Yoneda image of \( Fuk^\ast(M) \); \( \mathcal{Y}(Fuk^\ast(M))^\wedge \) is its triangular completion (as \( A_\infty \)-category); \( HH_\ast(B) \) is the Hochschild homology of the \( A_\infty \)-category \( B \) with values in itself (generally denoted by \( HH_\ast(B, B) \)). The morphisms involved are as follows: \( f_1 \) is an obvious isomorphism that reflects the definition of the triangular structure of \( DFuk^\ast(M) \), the morphism \( f_2 \) sends each module in \( \mathcal{M} \in \mathcal{Y}(Fuk^\ast)^\wedge \) to the
Hochschild homology class of its unit endomorphism $e_M \in \text{hom}(M, M)$. The latter descends to $K_0$ because, as it follows from Proposition 3.8 in [Sei3], if $M' \to M \to M''$ is an exact triangle in a triangulated $A\infty$-category $\mathcal{A}$, then $e_M = e_{M'} + e_{M''}$ in $HH_\ast(\mathcal{A})$. The morphism $f_3$ comes from the fact that the natural inclusion $\mathcal{F}uk\ast(M) \to \mathcal{Y}(\mathcal{F}uk\ast(M))$ induces an isomorphism in Hochschild homology (this is sometimes referred to as a form of Morita invariance. See [Toe] for the analogous though different context of dg-categories); $f_3$ is the inverse of this isomorphism. Finally, $f_4$ is the open-closed map (see, for instance, [Sei5]).

Remark 5.2.5. Assume that $M'$ is another module in $D\mathcal{F}uk\ast(M)$ as in the statement of the corollary such that $M' \cong M$ and $M' = (L'_r \to L'_{r-1} \to \ldots \to L'_1)$. The existence of $i'$ then implies that $\sum_j [L'_j] = \sum_k [L_k]$. It is interesting to note that the only way we know to show this fact is through the indirect method contained in the proof of the Corollary.

5.2.4. The periodicity isomorphism (2). In view of Corollary 5.1.3 it is natural to expect that $K_0(D\mathcal{F}uk\ast(E))$ can be calculated in terms of $K_0(D\mathcal{F}uk\ast(M))$. We will give here such a calculation but only in the case when $E$ is the trivial fibration $E = \mathbb{C} \times M$. An analogous statement for non-trivial fibrations is expected to also hold, but would require further algebraic elaboration.

Corollary 5.2.6. There exists a canonical isomorphism $$K_0(D\mathcal{F}uk\ast(\mathbb{C} \times M)) \cong \mathbb{Z}_2[t] \otimes K_0(D\mathcal{F}uk\ast(M))$$ induced by the map that sends $M \in \mathcal{O}b(D\mathcal{F}uk\ast(\mathbb{C} \times M))$ to $\sum_{i \geq 2} t^{i-2} \otimes R_i(M)$, where $R_i$ is the restriction functor from (60).

Proof. From Corollary 5.1.3 it is enough to show that $$K_0(D(\mathbb{N} \otimes \mathcal{F}uk\ast(M))) \cong \mathbb{Z}_2[t] \otimes K_0(D\mathcal{F}uk\ast(M)).$$ To simplify notation we denote $G_1 = K_0(D(\mathbb{N} \otimes \mathcal{F}uk\ast(M)))$ and $G_2 = \mathbb{Z}_2[t] \otimes K_0(D\mathcal{F}uk\ast(M))$. Given a module $M$ which is an object of $D(\mathbb{N} \otimes \mathcal{F}uk\ast(M))$ we use the composition in (62) to define the restriction modules $[M]_i$ that are objects of $D\mathcal{F}uk\ast(M)$ and define the sum $\phi(M) = \sum_{i \geq 2} t^{i-2} \otimes [M]_i \in G_2$. Because the restriction functors $R_j$ are triangulated it is easy to see that this map descends to a morphism $\phi : G_1 \to G_2$. The construction of the modules $\gamma_i \times \mathcal{N}$ in the proof of Corollary 5.2.3, in particular point (a) in the course of
that proof, shows that $\phi$ is surjective. To show that $\phi$ is injective we construct an inverse $\psi : G_2 \to G_1$. We define $\psi(t^i \otimes \mathcal{N}) = \gamma_{i+2} \times \mathcal{N}$ for each object in $\mathcal{N} \in D\mathit{Fuk}^*(M)$, where we have used here the notation from the proof of Corollary 5.1.3. Once we show that $\psi$ is well defined (in other words, that it respects the relations giving $K_0$) it immediately follows that it is an inverse of $\phi$ by the point (a) in the proof of Corollary 5.2.3. But again as in the proof of Corollary 5.2.3, namely the construction of $\tilde{T}$, it is easy to see that the map $\mathcal{N} \mapsto \gamma_i \times \mathcal{N}$ respects triangles. As a consequence, $\psi$ is well defined and this concludes the proof. □

6. Examples

The purpose of this section is to exemplify various aspects of the machinery in the paper. We start by making more explicit the structure contained in the writing of the cone-decompositions in Theorem A and exemplify this in the simplest possible setting consisting of cobordisms in $\mathbb{C}$. We then indicate how the cone-decompositions associated to cobordisms in our previous paper [BC3] are a consequence of the results here. We pursue with some cobordism examples in non-trivial Lefschetz fibrations. We first consider a simple horse-shoe like curve in a Lefschetz fibration with just one critical value and make explicit how Seidel’s exact sequence follows by applying our machinery to this case. Finally, and this is the novel and longest part of the section, we discuss real Lefschetz fibrations and their relation to Lagrangian cobordism.

6.1. Unwrapping cone-decompositions. The decompositions provided by Theorem A contain more structure than it appears superficially in the writing:

$$V \cong (T_1 \otimes E_1 \to T_2 \otimes E_2 \to \ldots \to T_m \otimes E_m \to \gamma_s L_s \to \gamma_{s-1} L_{s-1} \to \ldots \to \gamma_2 L_2).$$

Namely, see also §3.1.1, writing

$$V \cong (C_3 \to C_2 \to C_1)$$

actually means

$$V \cong \text{cone}(C_3 \xrightarrow{f_3} \text{cone}(C_2 \xrightarrow{f_2} C_1))$$

and the attaching maps $f_i$ as well as the intermediate cones are, of course, crucial in determining the result of the iterated cone.

This point is already in evidence in the simplest setting to which can be applied the machinery of the paper: cobordisms in $\mathbb{C}$ without any positive ends (and with the negative ends having integral imaginary coordinates). Obviously, these cobordisms are simply disjoint unions of circles and arcs diffeomorphic to $\mathbb{R}$ with horizontal ends pointing in the negative direction. Notice that due to the uniform monotonicity condition all circles have to enclose the same area. At the same time, circles do not play a significant role here since they have vanishing quantum homology and thus they are not seen by Floer and Fukaya category machinery.

Consider two Lagrangians $V$ and $V'$ as in Figure 34 below.
Namely, $V$ consists of two connected components: $V_0$ and $V_1$ with $V_0$ an arc with ends at height 2 and 6 and $V_1$ an arc with ends at height 3 and 5'; $V'$ has also two components $V'_0$ an arc with ends at height 2 and 3 and $V'_1$ again an arc with ends at height 5 and 6. It is easy to see that $V$ and $V'$ are the results of the two types of surgery on the Lagrangians $W$ and $W'$ in the middle part of Figure 34. This means, in particular, as seen in [BC2] that $V$ and $V'$ are themselves Lagrangian cobordant.

![Figure 34](image.png)

**Figure 34.** The planar cobordisms $V = V_0 \cup V_1$ and $V' = V'_0 \cup V'_1$. They are obtained through the two types of surgery on $W$ and $W'$. We have $HF(\gamma_4, V) \neq HF(\gamma_4, V')$.

Theorem A applied to $V$ and $V'$ produces decompositions that, formally, in the writing of the statement of that Theorem both look as:

$$(\gamma_6 \to \gamma_5 \to \gamma_3 \to \gamma_2) .$$

However, it is easy to see that $V$ and $V'$ are not isomorphic objects in $\mathcal{DFuk}^*(\mathbb{C})$. Indeed, $HF(\gamma_4, V) \neq 0$ but $HF(\gamma_4, V') = 0$ and it is an easy exercise to see that the actual two cone decompositions associated to $V$ and $V'$ by Theorem A are different: the intermediate cones and the relevant attaching maps are not the same.

Other examples relevant in this context are associated to elementary Lagrangian cobordisms $W : Q \sim Q$, $W \subset \mathbb{C} \times M$ (here $(M, \omega)$ is our fixed symplectic manifold). Examples of such cobordisms are provided by Lagrangian suspension (see [CC] for definitions and an in-depth description of the relation between Lagrangian suspension and cobordism). To such a $W$ we easily can associate a cobordism $V : \emptyset \sim (\emptyset, Q, Q)$. This can be done by first translating $W$
by using \((z, x) \to (z + i, x)\) and then bending the positive end to the right and extending it to \(-\infty\) so that it has height 3. The ends of \(V\) have height 2 and 3 - as in Figure 35. Of course,

\[
\begin{tikzpicture}
  \node (Q) at (0,0) {Q};
  \node (W) at (2,0) {W};
  \node (Q) at (4,0) {Q};
  \draw[->] (Q) -- (W);
  \draw[->] (W) -- (Q);
\end{tikzpicture}
\]

**Figure 35.** The cobordism \(V\) is obtained by bending the positive end of the elementary cobordism \(W : Q \leadsto Q\).

the simplest such example, \(V_0\), is associated to the trivial cobordism \(W_0 = \mathbb{R} \times \{0\} \times Q\).

The first remark for this class of examples is that all such \(V\)'s are isomorphic in \(DFuk^*(\mathbb{C} \times M)\) to \(V_0\). The reason is that from Theorem A we have a decomposition:

\[
V \cong \text{cone}(\gamma_3 \times Q \xrightarrow{\varphi_V} \gamma_2 \times Q) .
\]

The morphism \(\varphi_V\) can be identified with a class \(\varphi_V \in HF(Q, Q)\) which is given by the image of the fundamental class \([Q] \in HF(Q, Q)\) under the morphism \(\varphi\) defined as in Equation (38) - see also Figure 24 (of course, in our discussion here the fibration is trivial so that both ends of \(V\) in Figure 24 are equal to \(Q\)). Moreover, \(\varphi_V\) is an invertible element (see also [BC2]). As a consequence, the cone over \(\varphi_V\) is easily identified with the cone over \(\varphi_{V_0}\), where \(\varphi_{V_0} = [Q]\).

In short, the two decompositions are isomorphic as in the diagram below

\[
\begin{array}{ccc}
\gamma_3 \times Q & \xrightarrow{\varphi_V} & \gamma_2 \times Q \\
\downarrow{id} & & \downarrow{\varphi_V^{-1}} \\
\gamma_3 \times Q & \xrightarrow{\varphi_{V_0}} & \gamma_2 \times Q
\end{array}
\]

but they are not identical.

6.2. **Decompositions in \(DFuk^*(M)\) induced from cobordisms in \(\mathbb{C} \times M\).** Let \(V'\) be a cobordism \(V' : \emptyset \leadsto (L_1, \ldots, L_k)\), \(V' \subset (\mathbb{C} \times M, \omega_0 \oplus \omega)\). Theorem A and its Corollary 5.1.1 associate to \(V'\) a cone decomposition

\[
L_1 \cong (L_k \to L_{k-1} \to \ldots \to L_2)
\]
At the same time, the machinery in [BC3] applies to cobordisms $V'' : L \sim (L_1, \ldots, L_k)$ and associates to such a $V''$ another cone decomposition:

$$L \cong (L_k \to L_{k-1} \to \ldots L_1)$$

We want to briefly remark here that the decomposition (67) is a consequence of (66). By elementary manipulations, to see this it is sufficient to consider a cobordism $V : \emptyset \sim (L_2, L_3, \ldots, L_k)$ without positive ends and with the first negative end, $L_1$, also empty and show that the cone decompositions (67) and (66), both associated to $V$, coincide.

For this, notice that, by following the proofs of Theorem A and Corollary 5.1.1, the cone decomposition (66) is deduced from the following exact sequences of $\mathcal{F}_{uk}^*(M)$ modules:

$$W_{E,i}^\prime (r \times Y) \to \mathcal{M}_{V,i} \to \mathcal{Y}(L_i).$$

Here $W_{E,i}^\prime$ are the $\mathcal{F}_{uk}^*(\mathbb{C} \times M)$ modules that are introduced at the Step 3 of the proof of Proposition 4.3.1, $r$ is the horizontal line $r = \mathbb{R} \times \{1\}$ and $Y$ stands for a variable $Y \in \text{Ob}(\mathcal{F}_{uk}^*(M))$. The first map in (68) is an inclusion and the second a quotient. There is a slight abuse here as cobordisms of type $r \times Y$ have obviously a positive end by contrast to the objects considered in most of this paper, still the modules $W_{E,i}^\prime (r \times Y)$ are well defined. Indeed, as explained at the Step 3 of the proof of Proposition 4.3.1, $W_{E,i}^\prime (r \times Y)$ is generated by the intersection points of $r \times Y$ with the first $i$ branches of $W'$ where $W'$ is, in our case, obtained from $V$ by a Hamiltonian isotopy that keeps its ends fixed and moves the non-cylindrical part of $V$ in the lower half-plane - see, for instance, Figure 15. By inspecting [BC3], we see that the cone decomposition (67) follows from exact sequences of $\mathcal{F}_{uk}^*(M)$ modules:

$$\mathcal{M}_{V,i} \to \mathcal{M}_{V,i-1} \to \mathcal{Y}(L_i).$$

For the description of these modules see Figure 4 and Equation (4) in [BC3]. It immediately, follows that $\mathcal{M}_{V,i} = W_{E,i}^\prime (r \times Y)$ and thus (66) and (67) are identified.

6.3. A simple cobordism in a Lefschetz fibration with a single critical point. Consider a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ of fibre $(M, \omega)$ and with a single singularity $x_1$ of critical value $v_1$. We assume that the fibration is tame outside a set $U \subset \mathbb{C}$ as in Figure 36 and we consider a cobordism $V \subset E$ that projects to the curve $\gamma \subset \mathbb{C}$. As in the picture this curve turns once around $v_1$. Are also pictured there the curves $\gamma_2$ and $t_1$ that appear in the statement of Theorem A as well as the “mirror” singularity $x_1'$ and the matching sphere $\hat{S}_1$ that appear in the proof of this theorem (see §4.6).

By the relation between the Dehn twist and the monodromy of Lefschetz fibrations, the ends of $V$ are so that if the first end of $V$ is the Lagrangian $L \subset M$, then the second end is $\tau_S L$ for $S$ an appropriate vanishing sphere associated to $x_1$, this can be taken to be the sphere over the end of the curve $t_1$. 

The curves $\gamma$, $\gamma_2$, $t_1$, the region $U$ outside which the fibration $\pi : E \to C$ is tame and the matching sphere $\hat{S}_1$ that is included in the enriched fibration $\hat{\pi} : \hat{E} \to C$.

Theorem A applied to $V$ shows that:

\begin{equation}
V \cong \text{cone}(T_1 \otimes E_1 \to \gamma_2 \times \tau_{\hat{S}}L)
\end{equation}

where, as in (55), $E_1 = HF(\hat{S}_1, V)$. In this case we easily see that $HF(\hat{S}_1, V) \cong HF(S, L)$. By applying the restriction functor $R_1$ to the equation (69) we obtain

\[ L \cong \text{cone}(S \otimes HF(S, L) \to \tau_{\hat{S}}L) \]

which is just another way to express Seidel’s exact triangle.

6.4. Changes of generators. The generators appearing in Theorem A, in particular, the $T_i$’s are not always the most convenient for calculations even if they appear naturally in our proof. It is however easy to change generators in case a different choice is preferable. We exemplify this in the case of one Lefschetz fibration which we assume to fit the setting of Theorem 4.2.1 and with only three critical points, thus $m = 3$.

We consider two families of thimbles $T_1, T_2, T_3$ that are like in the statement of Theorem A and they cover curves $t_i$ and $T'_1, T'_2, T'_3$ that cover curves $t'_i$. Both types of curves are drawn in Figure 37.

It is easy to see that by applying Theorem A to the thimbles $T'_i$ we obtain first $T'_3 \cong T_3$. Further, $T'_2 \cong \text{cone}(T_2 \to T_3 \otimes E_3^2)$, with $E_3^2 = HF(\hat{S}_3, \tau_{\hat{S}_2} \tau_{\hat{S}_1} T'_2)$. Notice also $\tau_{\hat{S}_1} T'_2 = T'_2$ and $\tau_{\hat{S}_2} T'_2$ is just the one point surgery between $S_2$ and $T'_2$. It follows $E_3^2 \cong HF(S_3, S_2)$ where $S_i$ are vanishing spheres associated to the singularity $x_i$ (inside a fixed fibre $(M, \omega) = \pi^{-1}(\bar{z}_0)$).
Thus $T_2 \cong (T'_3 \otimes HF(S_3, S_2) \rightarrow T'_2)$.

Similarly, $T'_1 \cong \text{cone}(T_1 \rightarrow T_2 \otimes E_2 \rightarrow T_3 \otimes E_3)$ and we can again estimate: $E_2^1 = HF(\hat{S}_2, \tau_{\hat{S}_1} T'_1) \cong HF(S_2, S_1)$, $E_3^1 = HF(\hat{S}_3, \tau_{\hat{S}_2} \tau_{\hat{S}_1} T'_1)$. Thus we get:

$$T_1 \cong (T'_3 \otimes HF(S_3, S_2) \otimes HF(S_2, S_1) \rightarrow T'_2 \otimes HF(S_2, S_1) \rightarrow T'_3 \otimes E_3^1 \rightarrow T'_1).$$

This expression can be further simplified. For instance, the second and third terms can be switched because $\text{Mor}(T'_2, T'_3) = 0$. In conclusion, we can write

$$T_1 \cong (T'_3 \otimes E_3 \rightarrow T'_2 \otimes E_2 \rightarrow T'_1)$$

for appropriate $\mathcal{A}$-modules $E_3$, $E_2$. Using these arguments the decompositions given by Theorem A can be re-written in the generators $T'_i$: the sequence $(T_1 \otimes E_1 \rightarrow \ldots T_3 \otimes E_3)$ inside the cone-decomposition provided by that theorem will be replaced by $(T'_3 \otimes G_3 \rightarrow T'_2 \otimes G_2 \rightarrow T'_1 \otimes G_1)$ for appropriate modules $G_i$.

The manipulations above can be extended to fibrations with more than three singularities in an obvious way. The main difficulty in making these changes of generators explicit is in determining the modules $G_i$. In this respect, it is useful to note that there exists an alternative proof of the decompositions in Theorem A that avoids the geometric disjunction step contained in §4.4 and implements iteratively the stretching argument in §4.5 to the case of more singularities. While this method becomes quite involved for more than a few singularities, it offers sometimes a more direct way to estimate the relevant modules for specific generating families of thimbles.

6.5. **Real Lefschetz fibrations.** Real Lefschetz fibrations have recently been studied from the topological and real algebraic geometry viewpoints (see e.g. [DS, Sal1, Sal2, Sal3]). Lagrangian cobordism is naturally related to this notion and we describe this relationship in the
first subsection below. We then pursue with a construction of such fibrations and, in the last subsection, with a concrete example.

6.5.1. Lagrangian cobordism and real Lefschetz fibrations. Let $\pi : E \to \mathbb{C}$ be a Lefschetz fibration endowed with a symplectic structure $\Omega$, as in Definition 2.1.1. Denote by $(M,\omega)$ the general fiber of $(E,\Omega)$. Let $c_E : E \to E$ be an anti-symplectic involution, i.e. $c_E^* \Omega = -\Omega$ and $c_E \circ c_E = \text{id}$. Assume further that $c_E$ covers the standard complex conjugation $c_C : \mathbb{C} \to \mathbb{C}$, namely $\pi \circ c_E = c_C \circ \pi$. Denote by $V = \text{Fix}(c_E)$ the fixed point locus of $c_E$. Note that the projection $\pi(V)$ of $V$ to $\mathbb{C}$ is a subset of $\mathbb{R}$. The following proposition shows that $V$ is a Lagrangian cobordism and also gives a criterion for its monotonicity.

**Proposition 6.5.1.** Under the above assumptions $V$ is a Lagrangian cobordism with at most one positive end and at most one negative one (but possibly without any ends at all). Its projection $\pi(V) \subset \mathbb{R}$ is of the form $\bigcup_{j \in S} \bar{I}_j$, where $S$ is a subset of the set of connected components of $\mathbb{R} \setminus \text{Crit}(\pi)$, $I_j$ stands for the path connected component corresponding to $j$ and $\bar{I}_j$ is the closure of $I_j$. Thus $\partial \pi(V)$ is a subset of $\text{Crit}(\pi) \cap \mathbb{R}$.

Moreover, for every $z \in \mathbb{R} \setminus \text{Crit}(\pi)$ the part of $V$ lying over $z$, $V_z := E_z \cap V$, coincides with the fixed point locus of the anti-symplectic involution $c_E|_{E_z}$ hence is either empty or a smooth Lagrangian submanifold of $E_z$ (possibly disconnected). In particular, the Lagrangians corresponding to the ends of $V$ (if they exist) are real with respect to restriction of $c_E$ to the regular fibers over the real axis at $\pm \infty$.

If $(E,\Omega)$ is a spherically monotone symplectic manifold then $V$ is a monotone Lagrangian submanifold of $E$. Its minimal Maslov number satisfies $N_V = C_E$ where $C_E$ is the minimal Chern number on spherical classes in $E$.

If $\dim_{\mathbb{C}} M \geq 2$ and $(M,\omega)$ is spherically monotone then $(E,\Omega)$ is spherically monotone too, hence $V$ is a monotone Lagrangian cobordism. Its minimal Maslov number is $N_V = C_M$.

**Proof.** That $V$ is a (smooth) Lagrangian submanifold follows from it being the fixed point locus of an anti-symplectic involution.

We now show that $V$ is a cobordism and prove the other statements about the projection $\pi(V)$. Since $V$ is Lagrangian, $D\pi_x|_{T_x V} \to \mathbb{R}$ vanishes iff $x \in \text{Crit}(\pi)$ (see e.g. Chapter 16 of [Sei3]). It follows that $\pi(V) \setminus \text{Crit}(\pi)$ is an open subset of $\mathbb{R}$ and all the points in this subset are regular values of the projection $\pi|_V : V \to \mathbb{R}$. By construction $V \subset E$ is a closed subset. Therefore if $I \subset \mathbb{R} \setminus \text{Crit}(\pi)$ is a connected component and $\pi(V) \cap I \neq \emptyset$ then $I \subset \pi(V)$. Next, notice that since $V$ is Lagrangian it is invariant with respect to parallel transport along any intervals $I \subset \pi(V) \setminus \text{Crit}(\pi)$.

The statements about $V_z = \text{Fix}(c_E|_{E_z})$ follow directly from the definitions.
We now address the monotonicity of $V$. This follows from spherical monotonicity of $(E, \Omega)$, by a standard reflection argument based on the existence of the anti-symplectic involution $c_E$ and the fact that $V = \text{Fix}(c_E)$.

Finally, it remains to prove the statement relating the spherical monotonicity of $(M, \omega)$ with that of $E$. Let $E_{z_0} \subset E$ be a smooth fiber endowed with the symplectic structure induced by $\Omega$ (so that $(M, \omega)$ is symplectomorphic to $E_{z_0}$). Assume that $\dim_{\mathbb{C}} E_{z_0} \geq 2$ and that $E_{z_0}$ is spherically monotone. It is easy to see that the inclusion, $\pi_2(E_{z_0}) \to \pi_2(W)$ is surjective and this implies the monotonicity statement. \hfill \Box

In the next subsection we will show how to construct real Lefschetz fibrations out of Lefschetz pencils arising in real algebraic geometry.

6.5.2. Constructing real Lefschetz fibrations. Let $X$ be a smooth complex projective variety endowed with a real structure, namely an anti-holomorphic involution $c_X : X \to X$. Let $\mathcal{L}$ be a very ample line bundle on $X$ and assume further that it is endowed with a real structure compatible with $c_X$. By this we mean an anti-holomorphic involution $c_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ covering $c_X$, i.e. $\text{pr} \circ c_{\mathcal{L}} = c_X \circ \text{pr}$, where $\text{pr} : \mathcal{L} \to X$ is the bundle projection.

Denote by $H^0(\mathcal{L})$ the space of holomorphic sections of $\mathcal{L}$ and by $P := \mathbb{P}(H^0(\mathcal{L}))^*$ the projectivization of its dual (which can also be thought of as the space of hyperplanes in $H^0(\mathcal{L})$). We denote by $P^* := \mathbb{P}H^0(\mathcal{L})$ the projectivization of the space of sections itself. Note that $P^*$ is the dual projective space of $P$, hence the notation.

The real structure of $\mathcal{L}$ induces a real structure $c_H$ on $H^0(\mathcal{L})$ defined by $c_H(s) = c_{\mathcal{L}} \circ s \circ c_X$. Denote by $H^0_R(\mathcal{L}) \subset H^0(\mathcal{L})$ the space of real sections of $\mathcal{L}$ (i.e. sections $s$ with $c_H(s) = s$). The real structure $c_H$ descends to real structures on $P^*$ and $P$ which, by abuse of notation, we continue to denote both by $c_H$. The fixed point locus of $c_H$ on $P$ will be denoted by $P^*_R$ and that on $P^*$ by $P^*_R$.

Consider now the projective embedding defined using the sections of $\mathcal{L}$, $X \hookrightarrow P$. This embedding is real in the sense that it commutes with $(c_X, c_H)$. Furthermore, there is an isomorphism between $P$ and $\mathbb{C}P^n$ which sends $c_H$ to the standard real structure $c_{\mathbb{C}P^n}$ of $\mathbb{C}P^n$ (hence $P^*_R$ is sent under this isomorphism to $\mathbb{R}P^n$). We fix once and for all such an isomorphism. Denote by $\omega_{\mathbb{C}P^n}$ the standard symplectic structure of $\mathbb{C}P^n$ normalized so that the area of $\mathbb{C}P^1$ is $1$. Since $c_{\mathbb{C}P^n}$ is anti-symplectic with respect to $\omega_{\mathbb{C}P^n}$ the previously mentioned isomorphism yields a Kähler form $\omega_P$ on $P$ and therefore also a Kähler form $\omega_X$ on $X$ so that $c_X$ is anti-symplectic with respect to $\omega_X$.

Let $\Delta(\mathcal{L}) \subset P^*$ be the discriminant locus (a.k.a. the dual variety of $X$), which by definition is the variety consisting of all section $[s] \in P^*$ (up to a constant factor) which are somewhere non-transverse to the zero-section. Denote by $\Delta_R(\mathcal{L}) = \Delta(\mathcal{L}) \cap P^*_R$ its real part.
Let $\ell \subset P^*$ be a line which is invariant under $c_H$ and intersects $\Delta(\mathcal{L})$ only along its smooth strata and transversely. Fix an isomorphism $\ell \approx \mathbb{C}P^1$ and endow $\ell$ with a standard Kähler structure $\omega_\ell$ normalized so that its total area is 1. Consider the symplectic manifold $\ell \times X$ endowed with the symplectic structure $\omega_\ell \oplus \omega_X$. For every $\lambda \in P^*$ denote by $\Sigma^{(\lambda)} = s^{-1}(0) \subset X$ the zero locus corresponding to a section $s$ representing $\lambda$. (The varieties $\Sigma^{(\lambda)}$ are sometimes called hyperplane sections since they can also be viewed as the intersection of the image of $X$ in $P$ with linear hyperplanes.) Note that for all $\lambda \notin \Delta(\mathcal{L})$, the variety $\Sigma^{(\lambda)}$ is smooth. We endow these varieties with the symplectic structure induced from $\omega_X$. The complement of the discriminant, $P^* \setminus \Delta(\mathcal{L})$, is path connected (since $\Delta(\mathcal{L})$, being a proper complex subvariety of $P^*$, has real codimension $\geq 2$). Therefore all the symplectic manifolds $\Sigma^{(\lambda)}$, $\lambda \in P^* \setminus \Delta(\mathcal{L})$, are mutually symplectomorphic.

For every $\lambda \in P^*_\mathbb{R} \setminus \Delta_\mathbb{R}(\mathcal{L})$ the manifold $\Sigma^{(\lambda)}$ has a real structure induced by $c_X$. Denote its real part by $\Sigma^{(\lambda)}_\mathbb{R}$. We stress that in contrast to $P^* \setminus \Delta(\mathcal{L})$, its real part $P^*_\mathbb{R} \setminus \Delta_\mathbb{R}(\mathcal{L})$ is in general disconnected and the topology of $\Sigma^{(\lambda)}_\mathbb{R}$ depends on the connected component $\lambda$ belongs to. Define now

$$\hat{E} = \{ (\lambda, x) \mid \lambda \in \ell, \ x \in \Sigma^{(\lambda)} \} \subset \ell \times X.$$ 

Due to the transversality assumptions between $\ell$ and $\Delta(\mathcal{L})$ the variety $\hat{E}$ is smooth. We endow it with the symplectic structure $\Omega$ induced by $\omega_\ell \oplus \omega_X$.

The space $\hat{E}$ comes with two “projections”, $\pi : \hat{E} \rightarrow \ell$ and $p_X : \hat{E} \rightarrow X$, induced by the two projections from $\ell \times X$ to its factors. The first one is a Lefschetz fibration (whose base is $\ell \approx \mathbb{C}P^1$). The fact that the critical points of $\pi$ are non-degenerate follows from the transversality assumptions on the intersection of $\ell$ and $\Delta(\mathcal{L})$. The second projection (which will not be used here) realizes $\hat{E}$ as the blow-up $\text{Bl}_B(X) \rightarrow X$ of $X$ along the base locus $B$ of the pencil $\ell$ (i.e. $B = \{ x \in X \mid x \in \Sigma^{(\lambda)} \forall \lambda \in \ell \}$). The involutions $c_H$ and $c_X$ induce an anti-holomorphic involution on $\hat{E}$ which is also anti-symplectic with respect to $\hat{\Omega}$.

Let $D \subset \ell$ be a closed disk which is invariant under $c_H$. Identify $\ell \setminus D$ with $\mathbb{C}$ via an orientation preserving diffeomorphism which commutes with $(c_H, c_C)$, where $c_C$ is the standard conjugation on $\mathbb{C}$. The real part $\ell_\mathbb{R} \setminus D$ of $\ell \setminus D$ is sent by this diffeomorphism to $\mathbb{R}$.

By restricting $\pi$ to the complement of $D$ we obtain a Lefschetz fibration $E = \pi^{-1}(\ell \setminus D)$ over $\ell \setminus D \cong \mathbb{C}$. We endow $E$ with the symplectic structure $\Omega$ coming from $\hat{\Omega}$ and by a slight abuse of notation denote its projection by $\pi : E \rightarrow \mathbb{C}$. Restricting the preceding anti-symplectic involution of $\hat{E}$ to $E$ we obtain an anti-symplectic involution $c_E$ on $E$ which covers the standard conjugation $c_C$ as in §6.5. The critical values of $\pi$ are precisely $(\ell \setminus D) \cap \Delta(\mathcal{L})$. Some of them lie on $\ell_\mathbb{R}$ (i.e. the real axis) and the others come in pairs of conjugate points.

Note that $\ell_\mathbb{R} \setminus \Delta(\mathcal{L})$ might have several connected components. If $\lambda', \lambda'' \in \ell_\mathbb{R} \setminus \Delta(\mathcal{L})$ are in the same component then $\Sigma^{(\lambda')}_{\mathbb{R}}$ and $\Sigma^{(\lambda'')}_{\mathbb{R}}$ are diffeomorphic, but otherwise not necessarily.
Consider now the fixed point locus \( V = \text{Fix}(c_E) \subset E \). By Proposition 6.5.1, \( V \) is a Lagrangian cobordism. Its ends correspond to \( \Sigma^{(\lambda)}_\pm \) and \( \Sigma^{(\lambda \pm)}_\pm \), where \( \lambda_-, \lambda_+ \in \ell_R \setminus D \) are close enough to the two boundary points of \( \ell_R \cap D \). As hinted above, any of the \( \Sigma^{(\lambda \pm)}_\pm \) might be disconnected. At the other extremity any of these ends might also be void.

Finally we address the issue of monotonicity. Assume that \( \dim \mathbb{C} \mathcal{X} \geq 3 \) and that the symplectic manifold \( (\Sigma^{(\lambda)}, \omega_X|_{\Sigma^{(\lambda)}}), \lambda \notin \Delta(\mathcal{L}) \), is spherically monotone. By Proposition 6.5.1 the Lagrangian cobordism \( V \) is monotone.

Turning to more algebraic-geometric terms, here is a criterion that assures monotonicity of the \( \Sigma^{(\lambda)}_\pm \)'s. For an algebraic variety we denote by \( -K_X \) its canonical class. The following follows easily from adjunction.

**Proposition 6.5.2.** Let \( X \) be a Fano manifold with \( \dim \mathbb{C} \mathcal{X} \geq 3 \) and write \( -K_X = rD \), with \( r \in \mathbb{N} \) and \( D \) a divisor class. Further, suppose that \( \mathcal{L} = qD \) with \( 0 < q \in \mathbb{Q} \) and \( q < r \). Then the symplectic manifolds \( (\Sigma^{(\lambda)}, \omega_X|_{\Sigma^{(\lambda)}}), \lambda \notin \Delta(\mathcal{L}) \), are monotone. In particular \( V \) is a monotone Lagrangian cobordism.

### 6.5.3. A concrete example - real quadric surfaces.

We present here a concrete example of a real Lefschetz fibration associated to a pencil of complex quadric surfaces in \( \mathbb{C} \mathbb{P}^3 \). The example can be easily generalized to higher dimensions.

Let \( X = \mathbb{C} \mathbb{P}^3 \) and \( \mathcal{L} = \mathcal{O}_{\mathbb{C} \mathbb{P}^3}(2) \), both endowed with their standard real structures (induced by complex conjugation). Clearly \( \mathcal{L} \) is very ample and gives rise to the so called degree-2 Veronese embedding which we describe shortly.

Using coordinates \([X_0 : X_1 : X_2 : X_3]\) on \( \mathbb{C} \mathbb{P}^3 \) we identify the space \( H^0(\mathcal{L}) \) of sections of \( \mathcal{L} \) with the space of quadratic homogeneous polynomials \( \lambda(X) \) in the variables \( X = (X_0, X_1, X_2, X_3) \):

\[
\lambda(X) = \sum_{0 \leq i \leq j \leq 3} a_{i,j} X_i X_j.
\]

(70)

Taking \( X_i X_j, 0 \leq i \leq j \leq 3 \), as a basis for this space we obtain an identifications \( \mathbb{P} \cong \mathbb{C} \mathbb{P}^9 \) under which the projective embedding \( X \hookrightarrow \mathbb{C} \mathbb{P}^9 \) is given by:

\[
[z_0 : z_1 : z_2] \longmapsto [z_0^2 : z_0 z_1 : \cdots : z_1 z_j : \cdots : z_2 z_3 : z_3^2],
\]

where the coordinates on the right-hand side go over all \((i, j)\) with \( 0 \leq i \leq j \leq 3 \).

The hyperplane section corresponding to the polynomial \( \lambda \) is a quadric surface

\[
\Sigma^{(\lambda)} = \{ [z_0 : z_1 : z_2 : z_3] \mid \lambda(z_0, z_1, z_3) = 0 \} \subset \mathbb{C} \mathbb{P}^3.
\]
A straightforward calculation shows that \( \lambda \in \Delta(\mathcal{L}) \) if and only if
\[
\text{det} \begin{pmatrix} 2a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & 2a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & 2a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & 2a_{33} \end{pmatrix} = 0.
\]

This shows that the discriminant \( \Delta(\mathcal{L}) \) is a variety of degree 4 in \( \mathbb{P}^* \cong \mathbb{C}P^9 \). The smooth stratum of \( \Delta(\mathcal{L}) \) consists of those \( \lambda \)'s where the matrix in (71) has rank 3.

The real part \( \Delta_R(\mathcal{L}) \) of the discriminant consists of those polynomials \( \lambda \) which in addition to (71) have real coefficients (i.e. \( a_{i,j} \in \mathbb{R} \) for every \( i, j \)).

It is well known that for \( \lambda \not\in \Delta(\mathcal{L}) \) the variety \( \Sigma^{(\lambda)} \) is isomorphic to \( \mathbb{C}P^1 \times \mathbb{C}P^1 \), and moreover when viewed as a symplectic manifold (endowed with the structure induced from the projective embedding) it is symplectomorphic to \( (\mathbb{C}P^1 \times \mathbb{C}P^1, 2\omega_{\mathbb{C}P^1} + 2\omega_{\mathbb{C}P^1}) \), where \( \omega_{\mathbb{C}P^1} \) is normalized so that the area of \( \mathbb{C}P^1 \) is 1.

Consider now the following two sections
\[
\lambda_0(X) = X_0^2 + X_1^2 + X_2^2 - X_3^2, \quad \lambda_1(X) = X_0X_3 - X_1X_2.
\]

A simple calculation shows that \( \lambda_0, \lambda_1 \not\in \Delta(\mathcal{L}) \). Denote the real part of \( \Sigma^{(\lambda)} \) by \( L^{(\lambda)} \), \( i = 0, 1 \). It is easy to see that \( L^{(\lambda)} \) is a Lagrangian tours and moreover we can find a symplectomorphism \( \phi^{(\lambda)} : \Sigma^{(\lambda)} \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1 \) so that \( \phi^{(\lambda)}(L^{(\lambda)}) \) is the split torus \( T = \mathbb{R}P^n \times \mathbb{R}P^1 \). We fix such a diffeomorphism \( \phi^{(\lambda)} \). Similarly, there is a symplectomorphisms \( \phi^{(\lambda_0)} : \Sigma^{(\lambda_0)} \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1 \) that sends \( L^{(\lambda_0)} \) to the Lagrangian sphere \( S = \{(z, \bar{z}) \mid z \in \mathbb{C}P^1\} \subset \mathbb{C}P^1 \times \mathbb{C}P^1 \) which is so-called the anti-diagonal.

We now consider the pencil \( \ell \subset \mathbb{P}^* \) that passes through the two points \( \lambda_0 \) and \( \lambda_1 \). Clearly \( \ell \) is invariant under the anti-holomorphic involution \( c_H \). We can parametrize \( \ell \) by
\[
\mathbb{C}P^1 \ni [t_0 : t_1] \longmapsto \lambda_{[t_0, t_1]} := t_0\lambda_0 + t_1\lambda_1.
\]
A simple calculation shows that the intersection points of \( \ell \) with \( \Delta(\mathcal{L}) \) occur for the following values of \( [t_0 : t_1] \):
\[
[t_0 : t_1] \in \{[1 : 2], [1 : -2], [1 : 2i], [1 : -2i]\},
\]
and that \( \ell \) intersects \( \Delta(\mathcal{L}) \) only along the regular stratum. Moreover this intersection is transverse. See the left part of Figure 38.

We now appeal to the construction in §6.5.2. Below we will often identify \( \mathbb{C} \cong \mathbb{R}^2 \) in the obvious way. Choose a disk \( D \subset \ell \) which is invariant under \( c_H \) and contains the point \( [1 : 2], [1 : 2i], [1 : -2i] \) but not the point \( [1 : -2] \). Fix an orientation preserving diffeomorphism
\[ \beta: \ell \setminus D \to \mathbb{C} \cong \mathbb{R}^2 \] such that:
\[ \beta(\lambda_1) = (-1, 0), \quad \beta(\lambda_0) = (1, 0), \quad \beta([1 : -2]) = (0, 0). \]

See the right part of Figure 38. From now on we use the identification \( \beta \) implicitly and simply write \( \lambda_1 = (-1, 0), \quad \lambda_0 = (1, 0). \)

Restricting \( \tilde{E} \) to \( \ell \setminus D \) and applying a base change via \( \beta \) we obtain a Lefschetz fibration \( \pi: E \to \mathbb{C} \) with general fiber \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) and with a real structure. The projection \( \pi \) has exactly one critical value at \( 0 \in \mathbb{C} \) (corresponding to \([1 : -2] \in \ell\)). The real part \( V \) of \( E \) is a cobordism with one negative end associated to \( L^- = L^{(\lambda_1)} \) which is a Lagrangian torus, and one positive end associated to \( L^+ = L^{(\lambda_0)} \) which is a Lagrangian sphere. Finally, by Proposition 6.5.2, \( V \) is monotone with minimal Maslov number \( N_V = 2 \). (It might be interesting to notice that \( N_{L^-} = 2 \) while \( N_{L^+} = 4 \). Note also that \( d_{L^-} = d_{L^+} = 0 \).)

**Transforming \( V \) to a negative ended cobordism.** In order to obtain a cobordism with only negative ends (as considered in the rest of the paper) we proceed as follows. Take the Lefschetz fibration \( \pi: E \to \mathbb{C} \) and \( V \subset E \) as constructed above. Recall that \( 0 \in \mathbb{C} \) was the (single) critical value of \( \pi \). Consider a smooth embedding \( \alpha': [0, \infty) \to \mathbb{R}^2 \) so that:

1. \( \alpha'(t) = (t, 0) \) for every \( 0 \leq t \leq 1 \).
2. For \( 1 < t \), \( \alpha' \) lies in the lower half plane and \( \alpha'(2) = (0, -1) \).
3. For every \( 2 \leq t \), \( \alpha'(t) = (2 - t, -1) \).

Now take the part of the cobordism \( V \) that lies over \((-\infty, 1] \times \mathbb{R} \subset \mathbb{R}^2\) and glue to its right hand side the trail of the Lagrangian sphere \( L^{(\lambda_0)} = V|_{(1,0)} \) along the curve \( \alpha'|_{(1,\infty)} \). Denote the result by \( W \). It is easy to see that \( W \) is a smooth Lagrangian cobordism with two negative
ends. The lower end is a Lagrangian sphere and the upper end is a Lagrangian torus, both living inside symplectic manifolds that are symplectomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. See Figure 39.

![Figure 39. The cobordism $W$ with two negative ends, and the parallel transport of the sphere $L^{(\lambda_0)}$ to the fiber over $\lambda_1$.](image)

Note that the Lefschetz fibration $E$ is not tame. Therefore in order to apply the cone decomposition from Corollary 5.1.1 we need to identify fibers over different ends. To this end, denote by $\alpha''$ the straight segment connecting $\alpha'(3) = (-1, -1)$ to $\lambda_1 = (-1, 0)$. Denote by $\alpha = \alpha'|_{[1,3]} \ast \alpha''$ the concatenation of $\alpha'|_{[1,3]}$ with $\alpha''$. Denote by $\Pi_\alpha : E_{\lambda_0} \rightarrow E_{\lambda_1}$ the parallel transport along $\alpha$. Let $S^{(\lambda_1)} = \Pi_\alpha(L^{(\lambda_0)})$ be the parallel transport of the Lagrangian sphere $L^{(\lambda_0)}$ to the fiber $\Sigma^{(\lambda_1)} = E_{\lambda_1}$ of $E$ over $\lambda_1$. See Figure 39. By Corollary 5.1.1 we have in $D\mathcal{F}uk^\ast(\Sigma^{(\lambda_1)})$ an isomorphism:

$$S^{(\lambda_1)} \cong \text{cone}(S_1 \otimes E \rightarrow L^{(\lambda_1)}),$$

where $S_1 \subset \Sigma^{(\lambda_1)}$ is the vanishing cycle associated to the critical point of $\pi$ over 0 and the path $\alpha'|_{[0,3]} \ast \alpha''$. According to (55), the space $E$ is $HF(\hat{S}_1, W)$, where $\hat{S}_1$ is the matching cycle emanating from $z_1$, which lies in a suitable extension of the fibration $E$ (see §4.4.2).

In our case, it is not hard to see that $\hat{S}_1$ intersects $W$ at a single point and the intersection is transverse. Therefore $E$ is a 1-dimensional space. Applying $\phi^{(\lambda_1)}$ to (73) we now obtain the following isomorphism in $D\mathcal{F}uk^\ast(\mathbb{C}P^1 \times \mathbb{C}P^1)$:

$$\phi^{(\lambda_1)}(S^{(\lambda_1)}) \cong \text{cone}(\phi^{(\lambda_1)}(S_1) \rightarrow T).$$

By a result of Hind [Hin] all Lagrangian spheres in $\mathbb{C}P^1 \times \mathbb{C}P^1$ are Hamiltonian isotopic. In particular $\phi^{(\lambda_1)}(S^{(\lambda_1)})$ and $\phi^{(\lambda_1)}(S_1)$ are both Hamiltonian isotopic to the anti-diagonal $S$. It follows that:

$$S \cong \text{cone}(S \rightarrow T).$$

By rotating the exact triangle corresponding to (74) we obtain the following result:
Corollary 6.5.3. Let \( M = \mathbb{C}P^1 \times \mathbb{C}P^1 \), endowed with the symplectic structure \( \omega_{\mathbb{C}P^1} \oplus \omega_{\mathbb{C}P^1} \). Denote by \( S = \{(z, \bar{z}) \mid z \in \mathbb{C}P^1\} \subset M \) the anti-diagonal and by \( T = \mathbb{R}P^1 \times \mathbb{R}P^1 \subset M \) the split torus. Then in \( DFuk^*(M) \) there is an isomorphism
\begin{equation}
T \cong \text{cone}(S \rightarrow S).
\end{equation}

Remarks. a. The existence of an isomorphism of the type \((75)\) could probably be derived also by the following construction whose details need to be precisely worked out. Consider a Hamiltonian isotopic copy \( S' \) of \( S \) so that \( S' \) intersects \( S \) transversely at exactly two points. By performing Lagrangian surgery of \( S' \) and \( S \) at the intersection points (with appropriate choices of handles) one obtains a Lagrangian torus \( T' \subset M \). Moreover, for a suitable choice of \( S' \) and choices of handles the torus \( T' \) should be Hamiltonian isotopic to the split torus \( T \). Applying the “figure-Y” surgery construction from [BC2] we obtain a cobordism \( V \) in \( \mathbb{R}^2 \times M \) with two negative ends \( S, S' \) and one positive end \( T' \). The cobordism \( V \) should also be monotone for suitable choices of handles in the figure-Y surgery. The cone decomposition in \((75)\) would now follow from the main results of [BC3].

b. Our work does not provide much information about the precise morphism \( S \rightarrow S \) from \((75)\). It would be interesting to determine the precise map and also to figure out how \((75)\) behaves with respect to grading (in this case a \( \mathbb{Z}_2 \)-grading).

A few variations on the same example. One can alter the construction of \( E \) and \( V \) to obtain a Lefschetz fibrations \( \pi : E' \rightarrow \mathbb{C} \) with more critical values. This can be done for example by choosing the disk \( D \) to contain the point \([1 : -2]\) and none of the other points from \((72)\). The result will then be a fibration with three critical values - one lying on the \( x \)-axis and another pair of critical points conjugate one to the other. The cobordism \( V \) in this case would still be between a Lagrangian sphere and a torus.

If one chooses the disk \( D \) not to contain any of the points in \((72)\) and its center to lie somewhere along the interval \([1 : x], x \in [-2, 2]\), then the fibration will have four critical values, two real ones and to conjugate ones. The cobordism \( V \) will have a Lagrangian \( S^2 \) on its both ends, and the topology of \( V \) will still be non-trivial (i.e. \( V \) will not be diffeomorphic to \( \mathbb{R} \times S^2 \)). A similar example with Lagrangian \( T^2 \)'s on both ends can be constructed by taking the disk to have its center somewhere along \([1 : x], x > 2\).

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