LANG’S CONJECTURE FOR $y^2 = x^3 + b$

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Abstract. For $E_b : y^2 = x^3 + b$, we establish Lang’s conjecture on a lower bound for the canonical height of non-torsion points with good constants. In most cases, our results are actually best-possible.

1. Introduction

The canonical height, $\hat{h}$, on an elliptic curve $E$ defined over a number field $\mathbb{K}$ is a measure of the arithmetic complexity of points on the curve. It has many desirable properties. For example, $\hat{h}(P) = 0$ if and only if $P$ is a torsion point and it is a positive definite quadratic form on the lattice $E(\mathbb{K})/(\text{torsion})$. See [11, Chapter VIII] and [1, Chapter 9] for more information on this height.

Since the canonical height is positive for non-torsion points, a natural question that arises is how small the height can be for such points. Lang’s Conjecture states how the lower bound varies with the curve.

Conjecture 1.1 (Lang’s Conjecture). Let $E/\mathbb{K}$ be an elliptic curve with minimal discriminant $D_{E/\mathbb{K}}$. There exist constants $C_1 > 0$ and $C_2$, depending only on $[\mathbb{K} : \mathbb{Q}]$, such that for all non-torsion points $P \in E(\mathbb{K})$ we have

$$\hat{h}(P) > C_1 \log \left( N_{\mathbb{K}/\mathbb{Q}} (D_{E/\mathbb{K}}) \right) + C_2.$$

See page 92 of [8] along with the strengthened version in Conjecture VIII.9.9 of [11].

Such lower bounds have applications to counting the number of integral points on elliptic curves (see [6]), problems involving elliptic divisibility sequences [4, 5, 15], . . .

Silverman [10] showed that Lang’s conjecture holds for any elliptic curve with integral $j$-invariant over any number field (note that this includes our
curves, \( E_b \), since their \( j \)-invariant is 0). Hindry and Silverman \[6\] later proved an explicit version of Lang’s conjecture whenever Szpiro’s ratio, \( \sigma_{E/K} \), of \( E/K \) is known. Hence Lang’s conjecture follows from Szpiro’s conjecture (or the \( ABC \) conjecture).

Here we consider elliptic curves, \( E_b/\mathbb{Q} \), given by the Weierstrass equation \( y^2 = x^3 + b \) with \( b \in \mathbb{Z} \). It can be shown that \( \sigma_{E_b/\mathbb{Q}} < 5 \), hence from Theorem 0.3 of \[6\],

\[ \hat{h}(P) > 2 \cdot 10^{-28} \log |\Delta (E_b)|. \]

Subsequently, David \[3\] and Petsche \[9\] improved Hindry and Silverman’s result. From Petsche’s Theorem 2, for example, it follows that \( 2 \cdot 10^{-28} \) above can be replaced by \( 2 \cdot 10^{-22} \).

For \( E_b/\mathbb{Q} \) in the special case of \( b = -432m^2 \) for a cube-free integer \( m \), Jedrzejak \[7\] proved a much sharper result, which was improved further by Everest, Ingram and Stevens \[4\] in their Lemma 4.3:

\[ \hat{h}(P) \geq \frac{1}{27} \log (m) - \frac{1}{27} \log (2) - \frac{1}{12} \log (3) = \frac{1}{54} \log |b| - \frac{1}{9} \log (2) - \frac{5}{36} \log (3). \]

We provide a version for \( E_b/\mathbb{Q} \) here for all \( b \in \mathbb{Z} \) which are neither divisible by a sixth-power nor congruent to 16 mod 64 (i.e., global minimal Weierstrass equations for all \( E_b/\mathbb{Q} \) – see Lemma 4.7).

1.1. Results.

**Theorem 1.2.** Let \( b \) be an integer which is sixth-power-free and not congruent to 16 mod 64. Let \( P \in E_b(\mathbb{Q}) \) be a nontorsion point.

(a) If \( b < 0 \), then

\[ \hat{h}(P) \geq \frac{1}{216} \log |b| + \frac{1}{144} \log (3). \]

(b) If \( b > 0 \), then

\[ \hat{h}(P) > \frac{1}{216} \log |b| + \frac{1}{108} \log (2). \]

**Remark.** This result has a better constant term than Jedrzejak’s result \[7\] in his Corollary 1. In addition, our result applies to all curves of the form \( E_b \).

**Theorem 1.3.** Let \( b \) be an integer which is sixth-power-free, not congruent to 16 mod 64 and suppose that \( \text{ord}_p(b) \neq 3 \) for all primes, \( p > 3 \). Let \( P \in E_b(\mathbb{Q}) \) be a nontorsion point.
(a) If $b < 0$, then

$$\hat{h}(P) \geq \frac{1}{54} \log |b| - \frac{1}{18} \log(2) - \frac{1}{18} \log(3).$$

(b) If $b > 0$, then

$$\hat{h}(P) > \frac{1}{54} \log |b| - \frac{1}{54} \log(2) - \frac{1}{12} \log(3).$$

**Remark.** This result is somewhat stronger than the result of Everest, Ingram and Stevens [4] in their Lemma 4.3 (as with Theorem 1.2 we have improved the constant). Once again, our result applies to a much more general class of curves of the form $E_b$.

**Remark.** The conditions here are those that apply to a subset of all values of $b$ such that $c_p = 1$ or 3 for all primes, $p > 3$, where $c_p$ is the Tamagawa index of the curve at $p$. See Lemma 4.1 for the precise conditions required to have $c_p = 1$ or 3.

Also see Conjecture 1.7 for what is likely the attained lower bound in this case.

In [16], we were able to show that our results were best possible. In many cases, we are able to do that for $E_b(\mathbb{Q})$ too. Theorems 1.4 and 1.5 cover over 98.3% of values of $b$ giving rise to global minimal Weierstrass equations.

**Theorem 1.4.** Let $b$ be an integer which is sixth-power-free, not congruent to 16 mod 64. Furthermore, if $p^e || b$ for any $p > 3$ with $e = 2, 3$ or 4, then assume that $(-1)^e b/p^e$ is not a second, third or second power residue modulo $p$, respectively.

(a) If $b < 0$, then

$$\hat{h}(P) \geq \frac{1}{6} \log |b| - \frac{2}{3} \log(2) - \frac{1}{2} \log(3).$$

(b) If $b > 0$, then

$$\hat{h}(P) > \frac{1}{6} \log |b| - \frac{1}{3} \log(2) - \frac{3}{4} \log(3).$$

**Remark.** Note that this includes square-free $b$. The conditions here are those such that $c_p = 1$ for all primes, $p > 3$.

Over 93.7% of values of $b$ giving rise to global minimal Weierstrass equations are covered by the conditions in this Theorem.
See Section 6 for examples showing that Theorem 1.4 is best-possible.

**Theorem 1.5.** Let \( b \) be an integer which is sixth-power-free, not congruent to \( 16 \mod 64 \), and if \( p^e \mid b \) for any \( p > 3 \) with \( e = 2 \) or \( e = 4 \), then \( (b/p^e) \) is not a quadratic residue modulo \( p \).

(a) If \( b < 0 \), then

\[
\hat{h}(P) \geq \frac{1}{24} \log |b| - \frac{1}{6} \log(2) - \frac{5}{48} \log(3).
\]

(b) If \( b > 0 \), then

\[
\hat{h}(P) > \frac{1}{24} \log |b| - \frac{1}{12} \log(2) - \frac{1}{6} \log(3).
\]

**Remark.** The conditions here are those such that \( c_p \neq 3 \) for all primes, \( p > 3 \). In fact, under the conditions here, every element of the component group for each prime, \( p > 3 \), is of order at most 2.

**Remark.** Note that if \( b = -432m^2 \) and all prime divisors, \( p > 3 \), of \( m \) are congruent to \( 5 \mod 6 \), then \(-3\) is not a quadratic residue modulo \( p \) and hence this theorem applies. So we get a much stronger version of the result of Everest, Ingram and Stevens [14] in equation (13) of their Lemma 4.3.

In addition, Theorem 1.5 is more general too.

See Section 6 for examples showing that Theorem 1.5 is best-possible.

Lastly, note that while the formulation of our results is not in terms of \( \Delta(E_b) \), it is equivalent to such a formulation since \( \Delta(E_b) = -432b^2 \).

1.2. **Conjectures.** Theorems 1.2 and 1.3 are not best possible. However, through computations based on the lemmas used to prove our theorems, we have strong evidence that the following conjectures are best possible. Furthermore, we have been able to find families of points and curves showing that these conjectures cannot be improved (see the relevant subsections of Section 6).

The following appears to be the sharp version of Theorem 1.2 with the same lower bound holding for both \( b > 0 \) and \( b < 0 \).

**Conjecture 1.6.** Let \( b \) be an integer which is sixth-power-free, not congruent to \( 16 \mod 64 \). If \( P \in E_b(\mathbb{Q}) \) be a nontorsion point, then

\[
\hat{h}(P) > \frac{1}{30} \log |b| - \frac{\log(6)}{6}.
\]
The following appears to be the sharp version of Theorem 1.3.

**Conjecture 1.7.** Let \( b \) be an integer which is sixth-power-free, not congruent to 16 mod 64 and suppose that \( c_p \mid 3 \) for all primes, \( p > 3 \) and that \( c_p = 3 \) for at least one prime, \( p > 3 \). Let \( P \in E_b(\mathbb{Q}) \) be a nontorsion point.

(a) If \( b < 0 \), then

\[
\hat{h}(P) > \frac{1}{18} \log |b| - \frac{\log(2)}{18} - \frac{\log(3)}{4}.
\]

(b) If \( b > 0 \), then

\[
\hat{h}(P) > \frac{1}{18} \log |b| - \frac{\log(6)}{6}.
\]

As in [16], our proof is based on the decomposition of the canonical height as the sum of local height functions. However, there are differences in the behaviour of the curves in each family. One of particular interest to us, and one that caused additional complications here, is that the local archimedean height function here has an error term near its critical point that is \( O(\epsilon^2) \), whereas in [15], the analogous error term is \( O(\epsilon) \).

In general, it appears that the archimedean height function for all elliptic curves behaves in one of these two ways. Our work to understand this function better is ongoing, but it appears that the error term changes from \( O(\epsilon) \) to \( O(\epsilon^2) \) for the curve defined by \( y^2 = x^3 + ax + b \) once \( b \geq 0.9 \ldots |a|^{3/2} \), roughly.

To obtain our results here, we require precise bounds on the archimedean height on \( E_b \) (Section 3, in particular, Lemmas 3.1 and 3.4), along with a complete analysis of the \( p \)-adic reduction of \( E_b \) (Section 4). In Section 5, we prove our theorems and Section 6 contains infinite families of examples pertaining to our theorems and conjectures.

2. Notation

For what follows in the remainder of this paper, we will require some standard notation (see [11, Chapter 3], for example).

Let \( \mathbb{K} \) be a number field and let \( E/\mathbb{K} \) be an elliptic curve given by the Weierstrass equation

\[
E: y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,
\]

with \( a_1, \ldots, a_6 \in \mathbb{K} \).
Put
\[ b_2 = a_1^2 + 4a_2, \]
\[ b_4 = 2a_4 + a_1a_3, \]
\[ b_6 = a_3^2 + 4a_6, \]
\[ b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2, \]
then \( E/K \) is also given by \( y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6 \).

For a point \( P \in E(\mathbb{K}) \), we define the canonical height of \( P \) by
\[ \hat{h}(P) = \frac{1}{2} \lim_{n \to \infty} \frac{h(2^n P)}{4^n}, \]
with \( h(P) = h(x(P)) \), where \( h(P) \) and \( h(x(P)) \) are the absolute logarithmic heights of \( P \) and \( x(P) \), respectively (see Sections VIII.6,7 and 9 of [11]). Also recall that for \( \mathbb{Q} \), \( h(s/t) = \log \max\{|s|, |t|\} \) with \( s/t \) in lowest terms is the absolute logarithmic height of \( s/t \).

Let \( M_\mathbb{K} \) be the set of valuations of \( \mathbb{K} \) and for each \( v \in M_\mathbb{K} \), let \( n_v \) be the local degree and let \( \hat{\lambda}_v(P) : E(\mathbb{K}_v) \setminus \{O\} \to \mathbb{R} \) be the local height function, where \( \mathbb{K}_v \) is the completion of \( \mathbb{K} \) at \( v \). From Theorem VI.2.1 of [13], we have the following decomposition of the canonical height into local height functions
\[ \hat{h}(P) = \sum_{v \in M_\mathbb{K}} n_v \hat{\lambda}_v(P). \]

For \( \mathbb{K} = \mathbb{Q} \), the non-archimedean valuations on \( \mathbb{K} \) can be identified with the set of rational primes. For a non-archimedean valuation, \( v \), we let \( q_v \) be the associated prime and
\[ v(x) = -\log |x|_v = \text{ord}_{q_v}(x) \log (q_v). \]

Remark. We refer the reader to [2, Section 4] and [11, Remark VIII.9.2] for notes about the various normalisations of both the canonical and local height functions. In what follows, our local height functions, \( \hat{\lambda}_v(P) \), are those that [2] denotes as \( \lambda_v^{\text{SilB}}(P) \), that is as defined in Silverman’s book [13, Chapter VI]. So as stated in (11) of [2], their \( \lambda_v(P) \) equals \( 2\hat{\lambda}_v(P) + (1/6) \log |\Delta(E)|_v \) here.

Our canonical height also follows Silverman and is one-half that found in [2] as well as one-half that returned from the height function, ellheight, in PARI.
3. archimedean Estimates

3.1. $b < 0$.

Lemma 3.1. Suppose $b \in \mathbb{R}$ is negative and let $P = (x(P), y(P)) \in E_b(\mathbb{R})$ be a point of infinite order. Then

\begin{equation}
\hat{\lambda}_\infty(P) > \frac{1}{6} \log |b| + \frac{1}{4} \log(3) - \frac{1}{12} \log |\Delta(E_b)|.
\end{equation}

Remark. The lower bound in (3.1) is approached as $x(P) \to |b|^{1/3}$.

Proof. We will estimate the archimedean contribution to the canonical height by using Tate’s series (see [14] as well as the presentation in [12]). Let

\begin{align*}
t(P) &= 1/x(P) \quad \text{and} \quad z(P) = 1 - b_4t(P)^2 - 2b_6t(P)^3 - b_8t(P)^4,
\end{align*}

for a point $P = (x(P), y(P)) \in E(\mathbb{R})$. Then the archimedean local height of $P \in E(\mathbb{R})$ is given by the series

\begin{equation}
\hat{\lambda}_\infty(P) = \frac{1}{2} \log |x(P)| + \frac{1}{8} \sum_{k=0}^{\infty} 4^{-k} \log |z(2^kP)| - \frac{1}{12} \log |\Delta(E)|.
\end{equation}

Here we have $b_2 = b_4 = b_8 = 0$ and $b_6 = 4b$, so $t(P) = 1/x(P)$ and $z(P) = 1 - 8bt(P)^3$.

Since $b < 0$, for any $Q \in E_b(\mathbb{R})$, $x(Q) \geq |b|^{1/3}$. Hence $1 \leq z(Q) \leq 9$. In particular, $1 \leq z(2^kP) \leq 9$.

Applying this inequality for $k \geq 1$ and the definition of $z(P)$ to (3.2), we obtain

\begin{align*}
0 &\leq \hat{\lambda}_\infty(P) - \left( \frac{1}{8} \log (x(P)^4 - 8bx(P)) - \frac{1}{12} \log |\Delta(E_b)| \right) \\
&\leq \frac{1}{12} \log(3).
\end{align*}

Since $x(P) \geq |b|^{1/3}$, $x(P)^4 - 8bx(P) \geq 9|b|^{4/3}$. Hence

\begin{align*}
\hat{\lambda}_\infty(P) &\geq \frac{1}{8} \log (x(P)^4 - 8bx(P)) - \frac{1}{12} \log |\Delta(E_b)| \\
&\geq \frac{1}{6} \log |b| + \frac{1}{4} \log(3) - \frac{1}{12} \log |\Delta(E_b)|.
\end{align*}

\hfill \square

3.2. $b > 0$. We use Tate’s series here to estimate the archimedean contribution to the canonical height. However for $b > 0$, $E_b(\mathbb{R})$ includes the point $(0, b^{1/2})$, which causes a problem since we require $x(2^kP)$ to be bounded away from 0 to ensure that Tate’s series converges. To get around this, we use an idea of Silverman’s (see page 340 of [12]) and translate the curve to
the right using \( x' = x + 2b^{1/3} \), noting that \( \hat{\lambda}_\infty \) is fixed under such translations. In this way, we obtain the elliptic curve

\[
E_b': y^2 = x^3 - 6b^{1/3}x^2 + 12b^{2/3}x - 7b
\]

and every point, \((x, y)\), in \( E_b'(\mathbb{R}) \) satisfies \( x \geq b^{1/3} \).

For \( E_b' \), we have \( b_2 = -24b^{1/3}, b_4 = 24b^{2/3}, b_6 = -28b \) and \( b_8 = 24b^{4/3} \). Hence

\[
t(P') = 1/x(P') \quad \text{and} \quad z(P') = 1 - 24b^{2/3}t(P')^2 + 56bt(P')^3 - 24b^{4/3}t(P')^4
\]

for any \( P' \in E_b'(\mathbb{R}) \).

The approach that we take is similar to that taken for the proof of Lemma 3.4 of [16]. However there is one significant complication. Whereas there for \( x(P') = (1 + \epsilon)\sqrt{a} \),

\[
\hat{\lambda}_\infty(P') - \{(1/4) \log(a) - (1/12) \log|\Delta(E_a)|\} = O(\epsilon),
\]

here we find that for \( x(P') = 2(1 + \epsilon)b^{1/3} \),

\[
\hat{\lambda}_\infty(P') - \{(1/6) \log(b) + (1/3) \log(2) - (1/12) \log|\Delta(E_b)|\} = O(\epsilon^2),
\]

so we need to proceed more carefully. This is apparent with the quintic lower bounds for \( \log(x(2^kP')) \) in Lemma 3.2 as well as the estimates in Lemma 3.3 which will be required for some large values of \( k \).

**Lemma 3.2.** Suppose \( b \in \mathbb{R} \) is positive and \( P' \in E_b'(\mathbb{R}) \) where \( x(P') = 2(1 + \epsilon)b^{1/3} \).

(a) Then

\[
x(P')^4z(P') = 8b^{4/3}(1 - 2\epsilon + 8\epsilon^3 + 2\epsilon^4).
\]

Also,

\[
(3.3) \quad \log|x(P')^4z(P')| \geq \log(8b^{4/3}) - 2\epsilon - 2\epsilon^2 + \frac{16}{3}\epsilon^3 + 9.7\epsilon^4,
\]

for \(-0.1745 \leq \epsilon \leq 0.6 \).

(b) Suppose \( k \) is a positive integer and that \(-0.379 \leq (-2)^{k-1}\epsilon \leq 1.044 \). Then

\[
(3.4) \quad \left| \frac{x(2^kP')}{2b^{1/3}} - \{1 + (-2)^k\epsilon + ((-2)^{4k} - (-2)^k)\epsilon^4\} \right| \leq 2 \cdot 2^{7k} |\epsilon|^7.
\]

(c) For \(-0.5 \leq \epsilon \leq 8.6 \),

\[
(3.5) \quad \log(z(2P')) \geq -\log(2) + 12\epsilon - 32\epsilon^3 + 3.7\epsilon^4.
\]
For $-0.18 \leq \epsilon \leq 0.6$,

\begin{equation}
(3.6) \log (z(2P')) \geq -\log(2) + 12\epsilon - 32\epsilon^3 + 132\epsilon^4 - \frac{4608}{5}\epsilon^5 + 1200\epsilon^6.
\end{equation}

(d) Suppose $k$ is a positive integer and that $-0.78 \leq (-2)^k\epsilon \leq 0.79$. Then

\begin{equation}
(3.7) \log (z(2^kP')) \geq -\log(2) - 6(-2)^k\epsilon + 4(-2)^3\epsilon^3.
\end{equation}

If $k$ is a positive integer and $-1.0 \leq (-2)^k\epsilon \leq 0.36$, then

\begin{equation}
(3.8) \log (z(2^kP')) \geq -\log(2) - 6(-2)^k\epsilon + 4(-2)^3\epsilon^3 + \left(9(-2)^4k + 6(-2)^k\right)\epsilon^4 + (144/5)(-2)^5k\epsilon^5.
\end{equation}

Remark. The ranges for most of these inequalities are nearly sharp.

This is true for the inequalities in parts (a) and (c).

Also in part (d), (3.8) holds for $\epsilon \leq 0.102$ for $k = 2$ (we find $\epsilon \leq 0.09$ above), for $-0.051 \leq \epsilon$ for $k = 3$ (we find $-0.045 \leq \epsilon$ above) and for $\epsilon \leq 0.025$ for $k = 4$ (we find $-0.022 \leq \epsilon$ above).

It appears that the lower bound for $(-2)^k\epsilon$ in order for (3.8) to hold is not required. This is certainly true for $k = 1, 2$ and 3.

For (3.7), our ranges are not as close, but are still within a factor of less than two of the correct ranges. It holds for $\epsilon \leq 0.319$ for $k = 2$ (we find $\epsilon \leq 0.197$ above), for $-0.159 \leq \epsilon$ for $k = 3$ (we find $-0.098 \leq \epsilon$ above) and for $\epsilon \leq 0.079$ for $k = 4$ (we find $-0.049 \leq \epsilon$ above).

Again, it appears that the lower bound for $(-2)^k\epsilon$ is not required.

Remark. As there are multiple inequalities here for the same quantities, let us explain how and where they will be used.

Part (a) is used to prove Lemma 3.4 for “large” $\epsilon > 0$ when $x(P') = 2(1 + \epsilon)b^{1/3}$ as well as for $\epsilon$ near 0.

Part (b) is used to prove part (d) of this lemma.

From part (c), (3.5) will be used to prove Lemma 3.4 for “large” $\epsilon > 0$ when $x(P') = 2(1 + \epsilon)b^{1/3}$, as well as in the proof of (3.7) in part (d); while (3.6) will be used to prove (3.7) in part (d) (note that we need the $\epsilon^6$ term here to eliminate $\epsilon^6$ terms in part (d) and this inequality provides no significant benefit for “large” $\epsilon > 0$).

Lastly, part (d) will be used to prove Lemma 3.4 for $\epsilon$ near 0 when $x(P') = 2(1 + \epsilon)b^{1/3}$. We will use (3.8) mostly, but will need (3.7) in one instance.
The inequalities in part (d) are also correct for the terms appearing in them. That is, the error in (3.7) is $O(\epsilon^4)$, while the error in (3.8) is $O(\epsilon^6)$.

Proof. (a) We can write

$$x(P')^4 z(P') = x(P')^4 - 24b^{2/3} x(P')^2 + 56bx(P') - 24b^{4/3}. \tag{3.9}$$

The equality in part (a) follows immediately by substitution.

$$x(P')^4 z(P') = b^{4/3} \left((2(1 + \epsilon))^4 - 24(2(1 + \epsilon))^2 + 56(2(1 + \epsilon)) - 24\right) = 8b^{4/3} \left(1 - 2\epsilon + 8\epsilon^3 + 2\epsilon^4\right).$$

To prove that (3.3) holds over the stated range, we use the critical points of the function formed by subtracting the lower bound from $\log \left(1 - 2\epsilon + 8\epsilon^3 + 2\epsilon^4\right)$.

Consider

$$\log \left(1 - 2\epsilon + 8\epsilon^3 + 2\epsilon^4\right) - \left(-2\epsilon - 2\epsilon^2 + \frac{16}{3}\epsilon^3 + 9.7\epsilon^4\right).$$

Its derivative is a rational function whose numerator is of degree 7 and whose denominator is of degree 4. The denominator has two real roots, but both are less than $-0.7188$. The numerator has seven real roots: at $\epsilon = -3.9273\ldots, -0.8336\ldots, -0.1388\ldots, 0$ (a triple root), and $0.4875\ldots$. At the roots of the numerator larger than $-0.5$, our lower bound holds. It also holds for $\epsilon = -0.1745$ and $\epsilon = 0.6$, but it fails at $\epsilon = -0.175$ and $0.617$.

(b) From the duplication formula,

$$x(2P') = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6} = \frac{x^4 - 24b^{2/3}x^2 + 56bx - 24b^{4/3}}{4x^3 - 24b^{1/3}x^2 + 48b^{2/3}x - 28b}.$$

We have

$$\left(2\epsilon^4 + 8\epsilon^3 - 2\epsilon + 1\right) - \left(1 - 2\epsilon + 18\epsilon^4\right) \left(8\epsilon^3 + 1\right) = -144\epsilon^7.$$  

Hence, for $\epsilon \geq 0$, the desired lower bound holds since $1 + 8\epsilon^3 \geq 1$ for such $\epsilon$. Similarly,

$$\frac{x(2P')}{2b^{1/3}} - (1 - 2\epsilon + 18\epsilon^4) \leq 0.$$

For $\epsilon < 0$, the desired upper bound holds provided $144/ \left(1 + 8\epsilon^3\right) \leq 256$, this is true if $-0.379 \leq \epsilon < 0$. Similar to the case of $\epsilon \geq 0$,

$$0 \leq \frac{x(2P')}{2b^{1/3}} - (1 - 2\epsilon + 18\epsilon^4),$$
provided $8\varepsilon^3 + 1 > 0$ (i.e., $\varepsilon > -0.5$). Hence the desired inequalities hold for $k = 1$ if $\varepsilon \geq -0.379$.

Now we assume that (b) holds for some positive $k$ and proceed to show that it is also true for $k + 1$.

Supposing that $x(Q') = 2(1+\eta)b^{1/3}$, consider the function $x(2Q')/ (2b^{1/3})$:

$$\frac{2\eta^4 + 8\eta^3 - 2\eta + 1}{8\eta^3 + 1}. \tag{3.10}$$

We consider $(-2)^k \varepsilon \geq 0$ and $(-2)^k \varepsilon < 0$ separately and start with $(-2)^k \varepsilon \geq 0$.

For the upper bound, we substitute $(-2)^k \varepsilon + ((-2)^{4k} - (-2)^k) \varepsilon^4 + c(-2)^k \varepsilon^7$ for $\eta$ in (3.10) and show that the numerator of the resulting expression minus $1 + (-2)^{k+1} \varepsilon + ((-2)^{4(k+1)} - (-2)^{k+1}) \varepsilon^4 + 256(-2)^{7k}\varepsilon^7$ times its denominator is not positive for $-2 \leq c \leq 2$ and $0 \leq (-2)^k \varepsilon \leq 1.044$. The polynomial that we obtain in this way is of degree 28 in $\varepsilon$ and only contains monomials in $\varepsilon$ whose degree is congruent to 1 mod 3. Its leading coefficient is $(2c^4 - 2048c^3)(-2)^{28k}$ and its monomial of least degree is $((-2c - 328)(-2)^{7k} - 72(-2)^{4k}) \varepsilon^7$. We denote this polynomial by $f_{UB,+}((-2)^k, c, \varepsilon)$.

We now proceed as follows. We describe it in detail in this case as we will proceed in the same way for the other cases too.

(1) $1 \leq k \geq 10$: here we create the polynomial, $g_{UB,+},k(c, x) = f_{UB,+}((-2)^k, c, x/(-2)^k)$.

We find the largest value of $x_{\text{max}}$ such that $g_{UB,+},k(c, x) \leq 0$ for all $-2 \leq c \leq 2$ and all $0 \leq x \leq x_{\text{max}}$.

For a given value of $x_{\text{max}}$, we find the maximum value of $g_{UB,+},k(c, x)$ for all $-2 \leq c \leq 2$ and all $0 \leq x \leq x_{\text{max}}$ as follows.

We take the resultant of $(\partial/\partial x)g_{UB,+},k(c, x)$ and $(\partial/\partial c)g_{UB,+},k(c, x)$ with respect to $x$ (first removing any common factors of $x$ from the two derivatives). This gives us a polynomial in $c$ and any critical points of $g_{UB,+},k(c, x)$ must have one of these roots as their $c$-coordinate. We then find the maximum value of $g_{UB,+},k(c, x)$ for each such value of $c$, as well as for $c = \pm 2$, with $0 \leq x \leq x_{\text{max}}$.

We do this for $x_{\text{max}} = 0$ and $x_{\text{max}} = 10$ and then use bisection to find the largest $x_{\text{max}}$ with $g_{UB,+},k(c, x)$ for all $-2 \leq c \leq 2$ and all $0 \leq x \leq x_{\text{max}}$.

In this way, we find that for $1 \leq k \leq 10$, we can take $x_{\text{max}} = 1.04473 \ldots$, with the smallest value of $x_{\text{max}}$ occurring for $k = 2$. 
(2) \( k \geq 11 \): the benefit of step (1) is that the coefficient of \( f_{UB,+}((-2)^k, c, \epsilon) \) for \( \epsilon^j \) is a polynomial in \((-2)^k\) of degree \( j \) (i.e., it is of the form \( c_j(-2)^{jk} \)) lower-order terms, where \( c_j \) is a polynomial in \( c \). For \( k \) large, these lower-order terms divided by \((-2)^{jk}\) are very small and hence make a small bound contribution. Therefore, we can bound \( f_{UB,+}((-2)^k, c, \epsilon) \) from above by a polynomial in two variables \( c \) and \( x(= (-2)^k \epsilon) \).

From the highest-order terms, we get the polynomial

\[
(2c^4 - 2048c^3) x^{28} - (120c^3 + 6144c^2) x^{25} + (24c^3 - 6516c^2 - 6144c) x^{22} \\
- (312c^2 + 12664c + 2048) x^{19} + (60c^2 - 6840c - 6270) x^{16} \\
- (264c + 6504) x^{13} + (56c - 2372) x^{10} - (2c + 328)x^7.
\]

From the lower-order terms, we get the polynomial

\[
3 \cdot 10^{-6}x^{25} + 4 \cdot 10^{-6}x^{22} + 4 \cdot 10^{-6}x^{19} + 2 \cdot 10^{-6}x^{16} + 9 \cdot 10^{-7}x^{15} + 3 \cdot 10^{-8}x^{10} + 9 \cdot 10^{-9}x^{7}.
\]

Our method of obtaining these coefficients is simple use of the triangle inequality along with the conditions that \( |c| \leq 2 \) and \( |(-2)^k| \geq 2^{11} \). But, as is apparent, the resulting coefficients are so small that this suffices.

Adding these two polynomials, we get the desired polynomial in two variables, \( c \) and \( x \), which is larger than \( f_{UB,+}((-2)^k, c, \epsilon) \) for all \( k \geq 11 \), all \(-2 \leq c \leq 2 \) and all \( 0 \leq x(= (-2)^k \epsilon) \).

We now apply the same process to this polynomial as we did for each \( g_{UB,+}^{(k)}(c, x) \) to find \( x_{max} \). In this way, we find that for \( k \geq 11 \), our desired bound holds for all \( 0 \leq (-2)^k \epsilon \leq 1.0462 \ldots \).

We proceed in a similar way for the lower bound for \((-2)^k \epsilon \geq 0\), again substituting \((-2)^k \epsilon + ((-2)^{4k} - (-2)^k) \epsilon^4 + c(-2)^{7k} \epsilon^7 \) for \( \eta \) in (3.10) and showing that the numerator of the resulting expression minus \( 1 + (-2)^{k+1} \epsilon + ((-2)^{4(k+1)} - (-2)^{k+1}) \epsilon^4 - 256(-2)^{7k} \epsilon^7 \) times its denominator is not negative. Again, the polynomial that we obtain in this way, \( f_{LB,+}((-2)^k, c, \epsilon) \), is of degree 28 in \( \epsilon \), and only contains monomials in \( \epsilon \) whose degree is congruent to 1 mod 3. Its leading coefficient is \((2c^4 + 2048c^3)(-2)^{28k}\) and its monomial of least degree is \(((-2c + 184)(-2)^{7k} - 72(-2)^k) \epsilon^7\).

For each \( 1 \leq k \leq 10 \), we define \( g_{LB,+}^{(k)}(c, x) \) and \( x_{max} \) in an analogous way to above and find that for such \( k \), we can take \( x_{max} = 1.04482 \ldots \), with the smallest value of \( x_{max} \) occurring for \( k = 2 \).
From the highest-order terms, we get the polynomial
\[
(2c^4 + 2048c^3) x^{28} - (120c^3 - 6144c^2) x^{25} + (24c^3 + 5772c^2 + 6144c) x^{22}
+ (-312c^2 + 11912c + 2048) x^{19} + (60c^2 + 5448c + 6018) x^{16}
+ (-264c + 5784) x^{13} + (56c + 1724) x^{10} + (-2c + 184)x^7.
\]

From the lower-order terms, we get the polynomial
\[
3\cdot10^{-6}x^{25}+4\cdot10^{-6}x^{22}+4\cdot10^{-6}x^{19}+2\cdot10^{-6}x^{16}+7\cdot10^{-7}x^{13}+3\cdot10^{-8}x^{10}+9\cdot10^{-9}x^7.
\]

We find that for \( k \geq 11 \), our desired bound holds for all \( 0 \leq (-2)^k \epsilon \leq 1.0462 \ldots \). Note that here we subtract the “lower-terms” polynomial from the “higher-terms” one because we are trying to bound \( f_{LB, +} \) from below here.

Hence our bounds hold for \((-2)^k \epsilon \geq 0\).

We now consider \((-2)^k \epsilon < 0\).

For the upper bound, we again proceed as above and substitute \((-2)^k \epsilon + ((-2)^k - (-2)^k) \epsilon^4 + c(-2)^7\epsilon^7\) for \( \eta \) in (3.10) and show that the numerator of the resulting expression minus \(1 + (-2)^k \epsilon + ((-2)^{4(k+1)} - (-2)^{k+1}) \epsilon^4 - 256(-2)^7\epsilon^7\) times its denominator is not positive.

For \(1 \leq k \leq 10\), we find that the desired upper bound holds for \((-2)^k \epsilon \geq -0.50153 \ldots\), which is attained for \( k = 2 \).

From the highest-order terms, we get the polynomial
\[
(2c^4 + 2048c^3) x^{28} - (120c^3 + 6144c^2) x^{25} + (24c^3 + 5772c^2 + 6144c) x^{22}
+ (-312c^2 + 11912c + 2048) x^{19} + (60c^2 + 5448c + 6018) x^{16}
+ (-264c + 5784) x^{13} + (56c + 1724) x^{10} + (-2c + 184)x^7.
\]

From the lower-order terms, we get the polynomial
\[
-3\cdot10^{-6}x^{25}+4\cdot10^{-6}x^{22}+4\cdot10^{-6}x^{19}+2\cdot10^{-6}x^{16}+7\cdot10^{-7}x^{13}+3\cdot10^{-8}x^{10}+9\cdot10^{-9}x^7,
\]

note that we want the coefficients to be negative for the odd powers of \( x \) and positive for the even powers since we are considering \( x = (-2)^k \epsilon < 0 \) here.

We find that for \( k \geq 11 \), our desired bound holds for all \(-0.50465 \ldots \leq (-2)^k \epsilon \leq 0\).

Lastly, for the lower bound, we substitute \((-2)^k \epsilon + ((-2)^{4k} - (-2)^k) \epsilon^4 + c(-2)^{7k}\epsilon^7\) for \( \eta \) in (3.10) and show that the numerator of the resulting
expression minus 1 + \((-2)^{k+1}\epsilon + \(((-2)^{4(k+1)} - (-2)^{k+1}\) \epsilon^4 + 256(-2)^{7k}\epsilon^7\times its denominator is not negative.

For 1 \leq k \leq 10, we find that the desired lower bound holds for \((-2)^{k}\epsilon \geq -0.61349\ldots\), which is attained for \(k = 2\).

From the highest-order terms, we get the polynomial
\[
(2c^4 - 2048c^3)x^{28} - (120c^3 + 6144c^2)x^{25} + (24c^3 - 6516c^2 + 6144c)x^{22}
- (312c^2 + 12664c + 2048)x^{19} + (60c^2 - 6840c - 6270)x^{16}
- (264c + 6504)x^{13} + (56c - 2372)x^{10} - (2c + 328)x^7.
\]

From the lower-order terms, we get the polynomial
\[
-3 \times 10^{-6}x^{25} + 4 \times 10^{-6}x^{22} - 4 \times 10^{-6}x^{19} + 2 \times 10^{-6}x^{16} - 9 \times 10^{-7}x^{13} + 3 \times 10^{-8}x^{10} - 9 \times 10^{-9}x^7.
\]

We find that for \(k \geq 11\), our desired bound holds for all \(-0.61498\ldots \leq (-2)^k\epsilon \leq 0\).

Hence, the desired inequalities also hold for \((-2)^k\epsilon < 0\).

(c) We proceed as in part (a), considering critical points. We can write \((d/\epsilon)\log (2z(2P'))\) as a rational function of \(\epsilon\) whose numerator is of degree 17 with 12288 as its leading coefficient and whose denominator is of degree 20 with 64 as its leading coefficient. Moreover, the largest root of the denominator is at \(\epsilon = -0.5493\ldots\), so the denominator is positive for \(\epsilon \geq -0.5\). Denoting these polynomials by \(z_a(\epsilon)\) and \(z_d(\epsilon)\) respectively, we consider
\[
(3.11)
\]
\[
z_a(\epsilon) - z_d(\epsilon)(d/\epsilon) \left(12 \epsilon - 32 \epsilon^3 + 3.7 \epsilon^4\right) = z_a(\epsilon) - z_d(\epsilon) \left(12 - 96 \epsilon^2 + 14.8 \epsilon^3\right).
\]

This is a polynomial of degree 23 with leading coefficient \(-947.2\). It has 11 real roots, including one at \(-0.8485\ldots\) (and three roots less than this), its next largest root is at \(-0.4409\ldots\), followed by a triple root at 0, and positive roots at 0.2290\ldots, 0.3580\ldots, 6.4673\ldots.

Evaluating \log (2z(2P')) - (12 \epsilon - 32 \epsilon^3 + 3.7 \epsilon^4) at each of these roots and at \(\epsilon = -0.5\) and \(\epsilon = 8.6\), we find that it is always positive and hence \((3.5)\) holds.

We proceed in the same way to prove \((3.6)\).

(d) We proceed as in the proof of part (b) using the bounds in part (b) and \((3.6)\) in part (c) to obtain \((3.8)\).
Substituting \((-2)^k \epsilon + ((-2)^{4k} - (-2)^k) \epsilon^4 + c(-2)^7k \epsilon^7\) for \(\epsilon\) in (3.6) in part (c), we obtain a polynomial in \(\epsilon\) of degree 42 whose coefficients are polynomials in \((-2)^k\) and \(c\) and we subtract \(- \log(2) - 6(-2)^{k+1} \epsilon + 4(-2)^3(k+1) \epsilon^3 + (9(-2)^4(k+1) + 6(-2)^k) \epsilon^4 + (144/5)(-2)^5(k+1) \epsilon^5\) from this polynomial. As in part (b), we consider \((-2)^k \epsilon \geq 0\) and \((-2)^k \epsilon < 0\) separately.

For \((-2)^k \epsilon \geq 0\) and \(1 \leq k \leq 10\), we find that the desired inequality holds for \((-2)^k \epsilon \leq 10\) (at least).

For \((-2)^k \epsilon \geq 0\) and \(k \geq 11\), we find that the desired inequality holds for \((-2)^k \epsilon \leq 1.62\).

For \((-2)^k \epsilon < 0\) and \(1 \leq k \leq 10\), we find that the desired inequality holds for \(-0.3416 \leq (-2)^k \epsilon\), which occurs for \(k = 1\). For \(k \geq 11\), \(-0.3605 \leq (-2)^k \epsilon\).

We need to apply the constraints from part (b) and part (c) to these four bounds on \((-2)^k \epsilon\).

We can only substitute \((-2)^k \epsilon + ((-2)^{4k} - (-2)^k) \epsilon^4 + c(-2)^7k \epsilon^7\) for \(\epsilon\) with \(|c| \leq 2\) if \(-2.088 \leq (-2)^k \epsilon \leq 0.758\).

Second, we need \(-0.18 \leq (-2)^k \epsilon + ((-2)^{4k} - (-2)^k) \epsilon^4 + c(-2)^7k \epsilon^7 \leq 0.6\).

The lower bound holds for \(-0.18 \leq (-2)^k \epsilon \leq 1.01\). The upper bound holds for \(-0.87 \leq (-2)^k \epsilon \leq 0.5\).

Hence the desired bounds hold for \(-0.18 \leq (-2)^k \epsilon \leq 0.5\).

We prove (3.7) in a similar way, using the bounds in part (b) and (3.5) in part (c) to obtain (3.7).

For \((-2)^k \epsilon \geq 0\) and \(1 \leq k \leq 10\), we find that the desired inequality holds for \((-2)^k \epsilon \leq 0.392\).

For \((-2)^k \epsilon \geq 0\) and \(k \geq 11\), we find that the desired inequality holds for \((-2)^k \epsilon \leq 0.398\).

For \((-2)^k \epsilon < 0\) and \(1 \leq k \leq 10\), we find that the desired inequality holds for \(-0.399 \leq (-2)^k \epsilon\). For \(k \geq 11\), \(-0.402 \leq (-2)^k \epsilon\).

Again, we need to apply the constraints from part (b) and part (c) to these four bounds on \((-2)^k \epsilon\).

We can only substitute \((-2)^k \epsilon + ((-2)^{4k} - (-2)^k) \epsilon^4 + c(-2)^7k \epsilon^7\) for \(\epsilon\) with \(|c| \leq 2\) if \(-2.088 \leq (-2)^k \epsilon \leq 0.758\).

Second, we need \(-0.5 \leq (-2)^k \epsilon + ((-2)^{4k} - (-2)^k) \epsilon^4 + c(-2)^7k \epsilon^7 \leq 8.6\).

The lower bound holds for \(-0.56 \leq (-2)^k \epsilon \leq 1.04\). The upper bound holds for \(-1.2 \leq (-2)^k \epsilon \leq 1.15\).
Hence the desired bounds hold for $-0.399 \leq (-2)^k\epsilon \leq 0.392$.

All the required calculations, polynomial manipulations and root-finding were done using MAPLE.

Lemma 3.3. Suppose $b \in \mathbb{R}$ is positive and $Q' \in E'_b(\mathbb{R})$.

(a) If $x(Q') = 2(1 + \epsilon) b^{1/3}$ with $-0.5 \leq \epsilon \leq -0.17$, then

$$\log (z(2Q')) > 96.79\epsilon^3 + 102.8\epsilon^2 + 26.56\epsilon - 0.3231.$$  

Furthermore, the lower bound is a decreasing function of $\epsilon$ in this range.

(b) If $x(Q') = 2(1 + \epsilon) b^{1/3}$ with $-0.4 \leq \epsilon \leq -0.17$, then

$$\log (z(4Q')) > -272.3\epsilon^3 - 281.5\epsilon^2 - 70.33\epsilon - 2.964.$$  

Furthermore, the lower bound is an increasing function of $\epsilon$ in this range and for $-0.627 \leq \epsilon \leq -0.4$, the lower bound is less than $-\log(11)$.

(c) If $x(Q') = 2(1 + \epsilon) b^{1/3}$ with $-0.5 \leq \epsilon \leq 0$, then

$$\log (x(Q')^4 z(Q')) > \frac{4}{3} \log (b) - 4.346\epsilon^2 - 2.469\epsilon + 2.049.$$  

Remark. Parts (a) and (b) are used in the proof of Lemma 3.4 for both large and small $|\epsilon|$.

Part (c) is only used in the proof of Lemma 3.4 for $\epsilon \leq -0.1743$.

Remark. In part (a), the maximum gap in the inequality is less than 0.048, which occurs at $\epsilon = -0.17$.

In part (b), the maximum gap in the inequality is less than 0.16, which occurs near $\epsilon = -0.4$ and $-0.26$.

In part (c), the maximum gap in the inequality is less than 0.031, which occurs near $\epsilon = -0.4$ and 0.

Proof. (a) With our expression for $x(Q')$, $z(2Q')$ is a rational function of $\epsilon$, we take the derivative with respect to $\epsilon$ of

$$f(\epsilon) = \log (z(2Q')) - (96.79\epsilon^3 + 102.8\epsilon^2 + 26.56\epsilon - 0.3231).$$

Using MAPLE, we find that this results in a rational function of degree 22 in the numerator and degree 20 in the denominator. The six real roots of the denominator are all at most 0.5493..., while the real roots of the numerator which are between $-0.5$ and $-0.17$ are $-0.4683\ldots$, $-0.4102\ldots$, $-0.3215\ldots$ and $-0.2157\ldots$. Since $f(\epsilon)$ takes the values $0.0421\ldots$, $0.0305\ldots$, $0.0476\ldots$ and $0.000029\ldots$ at these roots of the numerator respectively and since
LANG’S CONJECTURE FOR $y^2 = x^3 + b$

$f(-0.5) = 0.00185$ and $f(-0.17) = 0.0478\ldots$, it follows that $f(\epsilon) > 0$ in the required range.

The statement that the polynomial lower bound is decreasing in this range follows by taking its derivative and solving for where it is zero (only at $\epsilon = -0.5381\ldots$ and $-0.1699\ldots$, both outside our range).

(b-c) We proceed in the same way here as in the proof of part (a).

The expressions in these results were found using the \texttt{minimax} function in MAPLE, which is based on the Remez algorithm. In each case, we applied this MAPLE function, obtained an approximation of the desired form, and then subtracted from the approximation the \texttt{maxerror} variable obtained from the function to ensure that the approximation was a lower bound. Lastly, we rounded the coefficients in the approximation appropriately. □

\textbf{Lemma 3.4.} Let $b \in \mathbb{R}$ be a positive real number and let $P \in E_b(\mathbb{R})$ be a point of infinite order. Then

\begin{equation}
\hat{\lambda}_\infty(P) > \frac{1}{6} \log |b| + \frac{1}{3} \log(2) - \frac{1}{12} \log |\Delta(E_b)|.
\end{equation}

\begin{description}
\item[Remark.] This lower bound is approached as $x(P)/|b|^{1/3} \to 0$.
\end{description}

\textbf{Proof.} We start by obtaining bounds for $z(P')$ that hold for all $P' \in E_b'(\mathbb{R})$.

With $x(P') = c(P')b^{1/3}$, from (3.9), we have

\begin{equation}
x(P')^4 z(P') = b^{4/3} \left( c(P')^4 - 24c(P')^2 + 56c(P') - 24 \right)
\end{equation}

and

\begin{equation}
\frac{d}{dc(P')} \left( c(P')^4 - 24c(P')^2 + 56c(P') - 24 \right) = 4c(P')^3 - 48c(P') + 56.
\end{equation}

The derivative has roots at $-3.9433\ldots$, $1.3909\ldots$ and $2.5524\ldots$. Hence $x(P')^4 z(P') > 5b^{4/3}$ for $x(P') \geq b^{1/3}$ (with the smallest constant occurring for $c(P') = 2.5524\ldots$).

Also,

\begin{equation}
\frac{d}{dx(P')} z(P') = \frac{24b^{2/3} \left( 2x^2 - 7b^{1/3}x + 4b^{2/3} \right)}{x^5},
\end{equation}

which has zeroes at $(7 \pm \sqrt{17})b^{1/3}/4$. Since $(7 - \sqrt{17})/4 < 1$, we need only consider the root at $(7 + \sqrt{17})b^{1/3}/4$. 
So \( z(P') = 9 \) for \( x(P') = b^{1/3} \), it decreases to a minimum at \( x(P') = (7 + \sqrt{17})b^{1/3}/4 \) where \( z(P') = (497 - 119\sqrt{17})/64 = 0.09922 \ldots \) and asymptotically approaches 1 from below for larger \( x(P') \). Hence

\[
\frac{1}{11} < \frac{497 - 119\sqrt{17}}{64} \leq z(P') \leq 9,
\]

for all \( P' \in E_b(\mathbb{R}) \).

Write \( x(P') = 2(1 + \epsilon)b^{1/3} \) where \( \epsilon \geq -0.5 \) (i.e., we use the point where \( \hat{\lambda}_\infty(P') \) takes its minimum value as the centre).

Let \( -\alpha \) be the root of \(-1 - 4X + 8X^2 - 16X^3 - 28X^4 - 16X^5 - 64X^6 + 32X^7 + 8X^8 \) that is approximately \(-0.1743319 \ldots \). This is the polynomial whose roots, \( \epsilon \), are such that if \( x(P') = 2(1 + \epsilon)b^{1/3} \), then \( x(4P') = b^{1/3} \).

- \( \epsilon \geq 2\alpha = (0.348 \ldots) \).

Using elementary calculus, we find that \( 8 - 16\epsilon + 64\epsilon^3 + 16\epsilon^4 = x(P')^4z(P')/b^{1/3} \) is increasing for \( \epsilon > 0.276 \ldots \) and so it is larger than 14.2 for \( \epsilon > 0.6 \). Hence

\[
\hat{\lambda}_\infty(P') = \frac{1}{8} \log |x(P')^4z(P')| + \frac{1}{8} \sum_{k=1}^{\infty} 4^{-k} \log |z(2^k P')| - \frac{1}{12} \log |\Delta(E_b)|
\]

\[
> \frac{1}{6} \log |b| + \frac{1}{8} \log(14.2) - \frac{\log(11)}{24} - \frac{1}{12} \log |\Delta(E_b)|
\]

\[
> \frac{1}{6} \log |b| + \frac{1}{3} \log(2) - \frac{1}{12} \log |\Delta(E_b)|,
\]

for all \( P' \) with \( x(P') > 3.2b^{1/3} \).

Now we estimate \( \hat{\lambda}_\infty(P') \) from below for \( \epsilon \leq 0.6 \), using our estimates in Lemma 3.2(a) (in particular, (3.3)) and (c) (3.5) applied to the \( k = 1 \) term in the summation).

We have

\[
\hat{\lambda}_\infty(P') = \frac{1}{8} \log |x(P')^4z(P')| + \frac{1}{8} \sum_{k=1}^{\infty} 4^{-k} \log |z(2^k P')| - \frac{1}{12} \log |\Delta(E_b)|
\]

\[
> \frac{1}{6} \log |b| + \frac{11 \log(2)}{32} + \frac{\epsilon}{8} - \frac{\epsilon^2}{4} - \frac{\epsilon^3}{3} + 1.328 \epsilon^4 - \frac{\log(11)}{96} - \frac{1}{12} \log |\Delta(E_b)|.
\]

Since the polynomial

\[
\frac{11 \log(2)}{32} - \frac{\log(11)}{96} - \frac{\log(2)}{3} + \frac{\epsilon}{8} - \frac{\epsilon^2}{4} - \frac{\epsilon^3}{3} + 1.328 \epsilon^4
\]

is positive for all \( \epsilon > 0.329 \ldots \), our desired lower bound for \( \hat{\lambda}_\infty(P') \) holds for all \( P' \) with \( x(P') \geq 2.66b^{1/3} \) (i.e., \( \epsilon \geq 0.33 \)).

- \(-0.5 \leq \epsilon \leq -\alpha(= -0.1743319 \ldots) \).
Using Lemma 3.3 parts (a) and (c) along with (3.16), we have

\[ \tilde{\lambda}_\infty(P') > \frac{1}{6} \log(b) + \frac{-4.346\epsilon^2 - 2.469\epsilon + 2.049}{8} + \frac{96.79\epsilon^3 + 102.8\epsilon^2 + 26.56\epsilon - 0.3231}{32} - \frac{\log(11)}{96} - \frac{1}{12} \log|\Delta(E_b)|. \]

We require

\[ 3.025\epsilon^3 + 2.669\epsilon^2 + 0.522\epsilon + 0.221 > \frac{\log(2)}{3}, \]

which holds for \(-0.5 \leq \epsilon \leq -0.3315\).

Using Lemma 3.3(b) too, we require

\[ \frac{-4.346\epsilon^2 - 2.469\epsilon + 2.049}{8} + \frac{96.79\epsilon^3 + 102.8\epsilon^2 + 26.56\epsilon - 0.3231}{32} + \frac{-272.3\epsilon^3 - 281.5\epsilon^2 - 70.33\epsilon - 2.964}{128} - \frac{\log(11)}{384} > \frac{\log(2)}{3}. \]

This holds for \(-0.34 \leq \epsilon \leq -0.1743\), establishing the lemma in the desired interval.

\( \bullet \) \(-\alpha < \epsilon < 2\alpha \).

If \( \epsilon < 0 \), then let \( N \geq 3 \) be the largest odd integer such that \(-\alpha < (-2)^{N-3}\epsilon \).

If \( \epsilon > 0 \), then let \( N \geq 2 \) be the largest even integer such that \(-\alpha < (-2)^{N-3}\epsilon \).

The reason for this choice is that with it, \( N-1 \) is the largest integer such that \( b^{1/3} < x (2^{N-1}P') < x (2^{N-3}P') < \ldots \). Such \( x (2^k P') \) are well-behaved (see Lemma 3.2(b)), leading to good bounds for \( z (2^k P') \) for \( k = 1, \ldots, N-1 \) (seeLemma 3.2(d)).

Applying (3.8) for \( k = 1, \ldots, N-2 \) (note that \(-1.0 < -\alpha < (-2)^{N-3}\epsilon \leq -\alpha/4 \) and \( \alpha/2 \leq (-2)^{N-2}\epsilon < 2\alpha < 0.36 \), so this inequality is applicable for such \( k \)), (3.7) for \( k = N-1 \) (note that \(-0.78 < -4\alpha < (-2)^{N-1}\epsilon \leq -\alpha \), so this inequality is applicable for such \( k \), as well as (3.3) and the lower
bound in \((3.16)\) for the remainder terms, we obtain

\[
\hat{\lambda}_\infty(P') \geq \frac{1}{6} \log |b| + \frac{3}{8} \log(2) - \frac{\epsilon}{4} - \frac{\epsilon^2}{4} + \frac{2\epsilon^3}{3} + \frac{9.7\epsilon^4}{8} - \frac{1}{12} \log |\Delta(E_b)|
\]

\[
- \sum_{k=1}^{N} \log(2) - \sum_{k=1}^{N} \frac{6 \cdot (-2)^k \epsilon}{8 \cdot 4^k} + \sum_{k=1}^{N} \frac{4 \cdot (-2)^{3k} \epsilon^3}{8 \cdot 4^k}
\]

\[
+ \sum_{k=1}^{N-2} \frac{(9 \cdot (-2)^{4k} + 6 \cdot (-2)^k) \epsilon^4}{8 \cdot 4^k} + \sum_{k=1}^{N-2} \frac{144 \cdot (-2)^{5k} \epsilon^5}{5 \cdot 8 \cdot 4^k}
\]

\[
+ \frac{1}{8 \cdot 4^N} \log \left( z \left( 2^N P' \right) \right) + \frac{1}{8 \cdot 4^{N+1}} \log \left( z \left( 2^{N+1} P' \right) \right) - \frac{\log(11)}{96 \cdot 4^N}.
\]

Note that there is some overlap with the “large \(\epsilon\)” work above. E.g., here with \(N = 2\), we consider \(0.087 \ldots = \alpha/2 \leq \epsilon < 2\alpha = 0.34 \ldots\) and our sums have at most one term. Above, we also used such one-term sums when considering \(-0.33 \leq \epsilon \leq 0.6\). The difference here is that we will use sharper estimates for some of the remaining terms \((k \geq N)\). In particular, \(k = N\) and sometimes \(k = N + 1\) too.

By evaluating the sums in this inequality, we find that \(\hat{\lambda}_\infty(P')\) is greater than or equal to

\[
(3.17)
\]

\[
\frac{1}{6} \log |b| + \frac{\log(2)}{3} - \frac{\epsilon^2}{4} + \frac{\epsilon^3}{3} + (-2)^{-N} \epsilon^4 - \frac{43\epsilon^4}{80} - \frac{16\epsilon^5}{5} - \frac{1}{12} \log |\Delta(E_b)|
\]

\[
+ 4^{-N} \left\{ \frac{\log(2^{16/11})}{96} + \frac{(-2)^N}{2} \epsilon - \frac{(-2)^{3N}}{6} \epsilon^3 + \frac{3(-2)^{4N}}{32} \epsilon^4 + \frac{(-2)^{5N}}{20} \epsilon^5 \right\}
\]

\[
+ \frac{1}{8 \cdot 4^N} \log \left( z \left( 2^N P' \right) \right) + \frac{1}{8 \cdot 4^{N+1}} \log \left( z \left( 2^{N+1} P' \right) \right).
\]

Next we estimate \(\log \left( z \left( 2^N P' \right) \right)\) from below. Note that \((3.16)\) is too weak to allow us to prove our lemma.

Write \(Q' = 2^{N-1} P'\) and put \(x(Q') = 2 \left( 1 + \epsilon_1 \right) b^{1/3}\). Since \(\alpha/2 \leq (-2)^{N-2} \epsilon \leq 2\alpha\) by the definition of \(N\), we can apply Lemma \([3.2](b)\), finding that

\(-0.5 \leq \epsilon_1 \leq (-2)^{N-1} \epsilon + \left( (-2)^{4(N-1)} - (-2)^{N-1} \right) \epsilon^4 + 2 \cdot 2^{7(N-1)} |\epsilon|^7 \leq -0.17\). Furthermore, since

\((-2)^{4(N-1)} - (-2)^{N-1} \leq (9/8)(-2)^{4(N-1)}\) for \(N \geq 2\) and

\(2 \cdot 2^{7(N-1)} |\epsilon|^7 = 16 \left( 2^{N-2} |\epsilon| \right)^3 2^{4(N-1)} \epsilon^4 < 0.68 \cdot 2^{4(N-1)} \epsilon^4\),

we find that

\(-0.5 \leq \epsilon_1 < (-2)^{N-1} \epsilon + 1.81(-2)^{4(N-1)} \epsilon^4 < -0.17\).
We now apply Lemma 3.3(a) and find that $\log \left( (2^N P') \right)$ is at least

\begin{equation}
(3.18) \quad 0.14 \left( (-2)^N \epsilon \right)^{12} - 1.858 \left( (-2)^N \epsilon \right)^9 + 1.315 \left( (-2)^N \epsilon \right)^8 + 8.212 \left( (-2)^N \epsilon \right)^6 \\
-11.63 \left( (-2)^N \epsilon \right)^5 + 3.004 \left( (-2)^N \epsilon \right)^4 - 12.1 \left( (-2)^N \epsilon \right)^3 \\
+ 25.7 \left( (-2)^N \epsilon \right)^2 - 13.28 \left( (-2)^N \epsilon \right) - 0.3231.
\end{equation}

Now we estimate $\log \left( (2^{N+1} P') \right)$ from below. Here (3.16) does allow us to prove our lemma for $-\alpha \leq (-2)^{-N-3} \epsilon \leq -0.085$, but not all the way to $-\alpha/4$.

Since $(-2)^4(N-1) - (-2)^N \geq (63/64)(-2)^4(N-1)$ for $N \geq 2$, as above, we find that

$$-0.627 < (-2)^{N-1} \epsilon + 0.3(-2)^{4(N-1)} \epsilon^4 < \epsilon_1 < -0.17.$$  

Hence we can apply Lemma 3.3(b) with $\epsilon$ there set to $(-2)^{N-1} \epsilon + 0.3 \left( (-2)^{N-1} \epsilon \right)^4$ and find that $\log \left( (2^{N+1} P') \right)$ is greater than

\begin{equation}
(3.19) \quad -0.0018 \left( (-2)^N \epsilon \right)^{12} + 0.1435 \left( (-2)^N \epsilon \right)^9 - 0.099 \left( (-2)^N \epsilon \right)^8 \\
-3.83 \left( (-2)^N \epsilon \right)^6 + 5.278 \left( (-2)^N \epsilon \right)^5 - 1.319 \left( (-2)^N \epsilon \right)^4 \\
+ 34.03 \left( (-2)^N \epsilon \right)^3 - 70.38 \left( (-2)^N \epsilon \right)^2 + 35.16(-2)^N \epsilon - 2.964.
\end{equation}

Recall that the lower bound in Lemma 3.3(b) is less than $-\log(11)$ for $\epsilon$ there between $-0.627$ and $-0.4$, so this lower bound for $\log \left( (2^{N+1} P') \right)$ also holds for $\epsilon$ here satisfying $-0.627 < (-2)^{N-1} \epsilon + 0.3(-2)^{4(N-1)} \epsilon^4$.

Substituting $c/(-2)^N$ for $\epsilon$, we find that

$$-1/4 + \epsilon/3 + (-2)^{-N} \epsilon^2 - (43/80) \epsilon^2 - (16/5) \epsilon^3 = -\frac{1}{4} + \frac{c}{3 \cdot (-2)^N} + \frac{c^2}{(-2)^{3N}} - \frac{43c^2}{80(-2)^{2N}} - \frac{16c^3}{5(-2)^{3N}}.$$  

For $0 \leq c \leq 1.424$, we find that the smallest of these polynomials for $N \geq 2$ is the one with $N = 3$ (this is easily done by examining the coefficients and also comparing the polynomials for $N = 2$ and $N = 3$, which is how we obtain the upper bound on $c$ here). That is,

$$-1/4 + \epsilon/3 + (-2)^{-N} \epsilon^2 - (43/80) \epsilon^2 - (16/5) \epsilon^3 \geq -1/4 - (1/24)c - (53/5120)c^2 + (1/160)c^3.$$
Combining (3.17), (3.18) and (3.19) with this estimate and writing \( \varepsilon = (-2)^{-N}c \), it follows that \( \hat{\lambda}_\infty(P') \) is greater than
\[
\frac{1}{6} \log |b| + \frac{\log(2)}{3} + \left(-1/4 - (1/24)c - (53/5120)c^2 + (1/160)c^3\right)\varepsilon^2 - \frac{1}{12} \log |\Delta (E_b)|
+ \frac{\varepsilon^2}{c^2} \left(0.0174c^{12} - 0.2278c^9 + 0.1612c^8 + 0.9068c^6 - 1.2388c^5 + 0.428c^4
- 0.6158c^3 + 1.0131c^2 - 0.0613c - 0.0425\right).
\]

Hence, we need to show that
\[
0.0174c^{12} - 0.2278c^9 + 0.1612c^8 + 0.9068c^6 - 1.2326c^5 + 0.4176c^4
- 0.6575c^3 + 0.7631c^2 - 0.0613c - 0.0425 \geq 0.
\]

This polynomial has one positive roots at 0.3382... and has a positive leading coefficient. Hence the polynomial is in the desired range, completing the proof of the lemma.  \( \square \)

4. Non-archimedean Estimates

4.1. Non-archimedean Estimates for \( q_v > 3 \).

**Lemma 4.1.** Let \( v \) be a non-archimedean valuation on \( \mathbb{Q} \) associated with an odd prime number, \( q_v > 3 \), and let \( b \) be an integer such that \( q_v^6 \nmid b \). The Kodaira types and Tamagawa indices of \( E_b \) at \( v \) are as in Table 1.

**Proof.** We use Tate’s algorithm with \( K = \mathbb{Q}_v \) (using the steps and notation in Silverman’s presentation of Tate’s algorithm in Section IV.9 of [13]).

- **Step 1.** This step applies when \( \text{ord}_{q_v}(\Delta(E_b)) = 0 \). Since \( \Delta(E_b) = -432b^2 \) and \( 432 = 2^43^3 \), the reduction type is \( I_0 \) at \( v \) when \( \text{ord}_{q_v}(b) = 0 \).
- **Step 2.** We have \( \text{ord}_{q_v}(\Delta(E_b)) > 0 \). The singular point, \( P = (x(P), y(P)) \), is already at \((0,0)\) since \( \text{ord}_{q_v}(2y(P)), \text{ord}_{q_v}(3x(P)) > 0 \) implies that \( \text{ord}_{q_v}(x(P)) > 0 \) too, so no change of variables is needed. Therefore, \( b_2 = 0 \) and hence \( \text{ord}_{q_v}(b_2) > 0 \). Thus Step 2 does not apply.
- **Step 3.** Since \( a_6 = b \). If \( \text{ord}_{q_v}(b) = 1 \), then the reduction type is \( II \).
- **Step 4.** We may now assume that \( \text{ord}_{q_v}(b) \geq 2 \). Note that \( b_6 = 4b \) and \( b_8 = 0 \). Hence \( \text{ord}_{q_v}(b_8) \geq 3 \) and so step 4 cannot apply.
- **Step 5.** If \( \text{ord}_{q_v}(b) = 2 \), then the reduction type is \( IV \). If \( b/q_v^2 \) is a quadratic residue modulo \( q_v \), then \( c_v = 3 \). Otherwise, \( c_v = 1 \).
Step 6. We write $P(T) = T^3 + b/q_v^3$, since $a_2 = a_4 = 0$. Its discriminant is $-27b^2/q_v^6$. If $\text{ord}_{q_v}(b) = 3$, then the discriminant is not zero modulo $q_v$ and the reduction type is $I_0^*$. If $-b/q_v^3$ is a cubic residue modulo $q_v$, then $P(T)$ has at least one root in $k$. If $-3$ is a quadratic residue modulo $q_v$ (that is $q_v \equiv 1 \mod 6$), then $P(T)$ has three roots in $k$ and $c_v = 4$, otherwise (that is, $q_v \equiv 5 \mod 6$) it only has 1 root in $k$ and $c_v = 2$.

If $-b/q_v^3$ is not a cubic residue modulo $q_v$, then $c_v = 1$.

Step 7. Here we assume that $P(T)$ has one simple root and one double root. But the third roots of unity are distinct, since $q_v > 3$, so this is not possible.

Step 8. Again, since the third roots of unity are distinct, this can only occur if the triple root of $P(T)$ is zero. That is, $\text{ord}_{q_v}(b) > 3$. So we consider the polynomial $Y^2 - b/q_v^4$. It has distinct roots if and only if $\text{ord}_{q_v}(b) = 4$.

If $\text{ord}_{q_v}(b) = 4$ and $b/q_v^4$ is a quadratic residue modulo $q_v$, then the reduction type is $IV^*$ and $c_v = 3$. If $\text{ord}_{q_v}(b) = 4$ and $b/q_v^4$ is a non-quadratic residue modulo $q_v$, then the reduction type is $IV^*$ and $c_v = 1$.

Step 9. Since $a_4 = 0$, this step does not apply.
• Step 10. This is the last remaining case if \( b \) is sixth-power-free. Here the reduction type is \( II^* \).

This completes the proof. \( \Box \)

**Lemma 4.2.** Let \( v \) be a non-archimedean valuation on \( \mathbb{Q} \) associated with an odd prime number, \( q_v > 3 \), and let \( b \) be an integer such that \( q_v^6 \nmid b \).

(a) \( P \in E_b(\mathbb{Q}_v) \) is singular if and only if \( \text{ord}_{q_v}(x(P)), \text{ord}_{q_v}(y(P)) > 0 \).

(b) For any \( P \in E_b(\mathbb{Q}_v) \), \( 6P \) is always non-singular. Furthermore, if \( P \neq O \) is non-singular, then

\[
\widehat{\lambda}_v(P) = \frac{1}{2} \log \max\{1, |x(P)|_v\} - \frac{\log |\Delta(E_b)|_v}{12}.
\]

(c) For any \( P \in E_b(\mathbb{Q}_v) \setminus \{O\} \),

\[
\widehat{\lambda}_v(P) = \frac{1}{2} \log \max\{1, |x(P)|_v\} - \frac{\log |\Delta(E_b)|_v}{12}
\]

\[
\begin{cases} 
(1/3) \log(q_v) & \text{if } \text{ord}_{q_v}(x(P)) > 0, \text{ord}_{q_v}(b) = 2, \\
(1/2) \log(q_v) & \text{if } \text{ord}_{q_v}(x(P)) > 0, \text{ord}_{q_v}(b) = 3, \\
(2/3) \log(q_v) & \text{if } \text{ord}_{q_v}(x(P)) > 0, \text{ord}_{q_v}(b) = 4, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** (a) We require \( \text{ord}_{q_v}(3x(P)^2) = 2 \text{ord}_{q_v}(x(P)) > 0 \) and \( \text{ord}_{q_v}(2y(P)) = \text{ord}_{q_v}(y(P)) > 0 \).

(b) The first statement follows from the fact that 6 is the least common multiple of the maximum orders of elements in the possible component groups, \( E_b(K)/E_{b,0}(K) \), in Table 1. Note that although \( c_v = 4 \) when \( \text{ord}_{q_v}(b) = 3, q_v \equiv 1 \mod 6, \) and \( -b/q_v^3 \) is a cubic residue modulo \( q_v \), the component group is \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) here (see Table 4.1 in Section IV.9 of [13]) and hence \( 2P \) is always non-singular modulo \( q_v \) in this case.

The expression for the local height is equation (26) in [12].

(c) This follows from our results in Lemma 4.1 along with Proposition 6 and the accompanying Table 2, as well as equation (11), of [2]. \( \Box \)

4.2. Non-archimedean Estimates for \( q_v = 3 \).

**Lemma 4.3.** Let \( b \) be an integer and suppose that \( 3^6 \nmid b \). The Kodaira types and Tamagawa indices of \( E_b \) at 3 are as in Table 2.
Proof. As in the proof of the previous lemma, we use Tate’s algorithm here.

• Step 1. Since $\Delta (E_b) = -432b^2$, 3 is always a divisor of $\Delta (E_b)$ and so Step 1 never applies.

• Step 2. If $3 | b$, then the singular point is at $(0,0)$ so no translation is required in Step 2. In this case, $b_2 = a_1^2 + 4a_2 = 0$, so Step 2 does not apply.

  If $b \equiv 1 \mod 3$, then the singular point is at $(2,0)$ so we must use the change of variables $x = x' + 2$ and we have $a_1 = a_3 = 0$, $a_2 = 6$, $a_4 = 12$ and $a_6 = b + 8$. In this case, $b_2 = a_1^2 + 4a_2 = 24 \equiv 0 \mod 3$ and again Step 2 does not apply.

  If $b \equiv 2 \mod 3$, then the singular point is at $(1,0)$ so we must use the change of variables $x = x' + 1$ and we have $a_1 = a_3 = 0$, $a_2 = 3$, $a_4 = 3$ and $a_6 = b + 1$. In this case, $b_2 = a_1^2 + 4a_2 = 12 \equiv 0 \mod 3$ and again Step 2 does not apply.

• Step 3. If $3 | b$, then Step 3 applies if $\text{ord}_3(b) = 1$. In this case, the reduction type is $II$.

  If $b \equiv 1 \mod 3$, then Step 3 applies if $\text{ord}_3(b + 8) = 1$. In this case, the reduction type is $II$.

  If $b \equiv 2 \mod 3$, then Step 3 applies if $\text{ord}_3(b + 1) = 1$. In this case, the reduction type is $II$.

• Step 4. If $3 | b$, then $9 | b$ (since we have $9 | a_6$ here). However, $b_8 = 0$ and so Step 4 does not apply.

  If $b \equiv 1 \mod 3$, then $b_8 = 24b + 48$ and since we assume in this step that $\text{ord}_3(b + 8) \geq 2$, it follows that $b_8 \equiv 72 \mod 216$.

  Here the reduction type is $III$. 

| $b$ | Kodaira type | $c_3$ |
|-----|-------------|------|
| $b \equiv 2, 3, 4, 5, 6, 7 \mod 9$ | $II$ | 1 |
| $b \equiv 1, 8 \mod 9$ | $III$ | 2 |
| $b \equiv 9 \mod 27$ | $IV$ | 3 |
| $b \equiv 18 \mod 27$ | $IV$ | 1 |
| $b \equiv 54, 81, 108 \mod 243$ | $IV^*$ | 3 |
| $b \equiv 135, 162, 189 \mod 243$ | $IV^*$ | 1 |
| $b \equiv 27, 216 \mod 243$ | $III^*$ | 2 |
| $b \equiv 0 \mod 243$ | $II^*$ | 1 |

Table 2. $E_b$ reduction information for $q_v = 3$
If $b \equiv 2 \mod 3$, then $b_8 = 3b + 12$ and since we assume in this step that $\text{ord}_3(b + 1) \geq 2$, it follows that $b_8 \equiv 99 \mod 108$.

Here the reduction type is $III$.

The only remaining case is $b \equiv 0 \mod 9$.

- Step 5. Since $3 | b$, we have $b_6 = 4b$. If $\text{ord}_3(b) = 2$, then the reduction type is $IV$.

If $b/9$ is a quadratic residue modulo 3, then $c_3 = 3$. Otherwise, $c_3 = 1$.

- Step 6. Since $3 | b$, there is no need to change coordinates. $P(T) = T^3 + b/27$ and $\text{disc}(P) = -b^2/27$.

We know that $b \equiv 0 \mod 27$ and hence $\text{disc}(P) \equiv 0 \mod 3$. Therefore, Step 6 does not apply.

- Step 7. There is just one third root of unity (with multiplicity 3) here. Hence $P(T)$ can never have a simple root and a double root, so Step 7 does not apply.

- Step 8. $P(T) = T^3 + b/27$ has a triple root.

We have three cases to consider.

(1) If $\text{ord}_3(b) \geq 4$, then this triple root is at $T = 0$ and we consider the polynomial $Y^2 - b/81$.

If $\text{ord}_3(b) = 4$, then this polynomial has simple roots, so the reduction type is $IV^*$. If $b/81$ is a quadratic residue modulo 3 (i.e., if $b/81 \equiv 1 \mod 3$), then $c_3 = 3$. If $b/81 \equiv 2 \mod 3$, then $c_3 = 1$.

If $\text{ord}_3(b) > 4$, then Step 8 does not apply.

(2) If $b/27 \equiv 1 \mod 3$, then we replace $x$ by $x + 6$ and obtain $y^2 = x^3 + 18x^2 + 108x + b + 216$. Observe that $b + 216 \equiv 0 \mod 81$, since $b \equiv 27 \mod 81$. Here $a_3 = 0$ and $a_6 = b + 216$, so $Y^2 + a_{3,2}Y - a_{6,4} = Y^2 - (b + 216)/81$.

If $\text{ord}_3(b + 216) = 4$, then this polynomial has simple roots, so the reduction type is $IV^*$. If $(b + 216)/81$ is a quadratic residue modulo 3 (i.e., if $(b + 216)/81 \equiv 1 \mod 3$), then $c_3 = 3$. If $(b + 216)/81 \equiv 2 \mod 3$, then $c_3 = 1$.

If $\text{ord}_3(b + 216) > 4$, then Step 8 does not apply.

(3) If $b/27 \equiv 2 \mod 3$, then we replace $x$ by $x + 3$ and obtain $y^2 = x^3 + 9x^2 + 27x + b + 27$. Observe that $b + 27 \equiv 0 \mod 81$, since $b \equiv 54 \mod 81$. Here $a_3 = 0$ and $a_6 = b + 27$, so $Y^2 + a_{3,2}Y - a_{6,4} = Y^2 - (b + 27)/81$. 

If \( \text{ord}_3(b + 27) = 4 \), then this polynomial has simple roots, so the reduction type is \( IV^* \). If \((b + 27)/81\) is a quadratic residue modulo 3 (i.e., if \((b + 27)/81 \equiv 1 \mod 3\)), then \( c_3 = 3 \). If \((b + 27)/81 \equiv 2 \mod 3\), then \( c_3 = 1 \).

If \( \text{ord}_3(b + 27) > 4 \), then Step 8 does not apply.

- **Step 9.** If \( \text{ord}_3(b) \geq 5 \), then \( a_4 = 0 \), so this step does not apply.

- If \( \text{ord}_3(b + 216) \geq 5 \), then \( a_4 = 108 \) (as we found in Step 8). So the reduction type is \( III^* \).

- If \( \text{ord}_3(b + 27) \geq 5 \), then \( a_4 = 27 \) (as we found in Step 8). So the reduction type is \( III^* \).

- **Step 10.** If \( \text{ord}_3(b) = 5 \), then the reduction type is \( III^* \).

This concludes the proof of the lemma. \( \square \)

**Lemma 4.4.** Let \( b \) be an integer and suppose that \( 3^6 \nmid b \).

(a) \( P \in E_b(\mathbb{Q}_3) \) is singular if and only if \( \text{ord}_3(x(P) + b) > 0 \).

(b) For any \( P \in E_b(\mathbb{Q}_3) \), \( 6P \) is always non-singular. Furthermore, if \( P \neq O \) is non-singular, then

\[
\hat{\lambda}_3(P) = \frac{1}{2} \log \max \{1, |x(P)|_3\} - \frac{|\Delta(E_b)|_3}{12}.
\]

(c) For any \( P \in E_b(\mathbb{Q}_3) \setminus \{O\} \),

\[
\hat{\lambda}_3(P) = \frac{1}{2} \log \max \{1, |x(P)|_3\} - \frac{|\Delta(E_b)|_3}{12}
\]

\[
= \begin{cases} 
(1/4) \log(3) & \text{if } b \equiv 1, 8 \mod 9 \text{ and } \text{ord}_3(x(P) + b) > 0 \\
(1/3) \log(3) & \text{if } b \equiv 9 \mod 27 \text{ and } \text{ord}_3(x(P)) > 0 \\
(2/3) \log(3) & \text{if } b \equiv 54, 81, 108 \mod 243 \text{ and } \text{ord}_3(x(P)) > 0 \\
(3/4) \log(3) & \text{if } b \equiv 27, 216 \mod 243 \text{ and } \text{ord}_3(x(P)) > 0 \\
0 & \text{otherwise}.
\end{cases}
\]  

(4.2)

**Proof.** (a) We require \( \text{ord}_3(3x(P)^2) > 0 \) and \( \text{ord}_3(2y(P)) = \text{ord}_3(y(P)) > 0 \). If \( \text{ord}_3(x(P)) > 0 \), then we must have \( \text{ord}_3(b) > 0 \) too. If \( \text{ord}_3(x(P)) = 0 \), then we must have \( \text{ord}_3(x(P)^3 + b) > 0 \) too. Writing \( x(P) = x_n/x_d \), we have \( x(P)^3 + b = (x_n^3 + bx_d^3)/x_d^3 \). Since \( x^2 \equiv 1 \mod 3 \) for all \( x \neq 0 \), we see that \( x_n^3 + bx_d^3 \equiv x_n + bx_d \mod 3 \) and so \( \text{ord}_3(x(P)^3 + b) = \text{ord}_3(x(P) + b) > 0 \).

This last condition holds in all cases when \( P \) is singular. Furthermore, this condition implies that \( \text{ord}_3(y(P)) > 0 \), so \( \text{ord}_3(x(P) + b) > 0 \) is a necessary and sufficient condition for \( P \) to be singular.
(b) As in the proof of Lemma 4.2(b), 6 is the least common multiple of the maximum orders of elements in the possible component groups, $E_b(K)/E_{b,0}(K)$.

The argument is the same as for Lemma 4.2(b).

(c) This follows from Lemma 4.3 along with Proposition 6 and equation (11) of [2].

4.3. Non-archimedean Estimates for $q_v = 2$.

Lemma 4.5. Let $b$ be an integer and suppose that $b \not\equiv 0, 16 \text{ mod } 64$. The Kodaira types and Tamagawa indices of $E_b$ at 2 are as in Table 3.

| $b$            | Kodaira type | $c_2$ |
|----------------|--------------|-------|
| $b \equiv 16 \text{ mod } 64$ | $I_0$        | 1     |
| $b \equiv 2, 3 \text{ mod } 4$  | $II$         | 1     |
| $b \equiv 5 \text{ mod } 8$    | $IV$         | 1     |
| $b \equiv 1 \text{ mod } 8$    | $IV$         | 3     |
| $b \equiv 8, 12 \text{ mod } 16$ | $I_0^*$      | 2     |
| $b \equiv 4 \text{ mod } 32$   | $IV^*$       | 3     |
| $b \equiv 20 \text{ mod } 32$  | $IV^*$       | 1     |
| $b \equiv 32, 48 \text{ mod } 64$ | $II^*$      | 1     |

Table 3. $E_b$ reduction information for $q_v = 2$

Proof. As above, we apply Tate’s algorithm.

- Step 1. Since $\Delta(E_b) = -432b^2$, 2 is always a divisor of $\Delta(E_b)$ and so Step 1 never applies.

- Step 2. If $b$ is even, then the singular point is at $(0, 0)$ so no translation is required in Step 2. In this case, $b_2 = a_1^2 + 4a_2 = 0$, so Step 2 does not apply.

If $b$ is odd, then the singular point is at $(0, 1)$, since if $P$ is a singular point then $\text{ord}_2(3x(P)) > 0$. So use the change of variables $y = y' + 1$ and we have $a_1 = a_2 = 0$, $a_3 = 2$, $a_4 = 0$ and $a_6 = b - 1$.

With this change of variables, the conditions $2|a_3$, $2|a_4$ and $2|a_6$ are satisfied. Here $b_2 = a_1^2 + 4a_2 = 0 \equiv 0 \text{ mod } 2$ and again Step 2 does not apply.

- Step 3. If $b$ is even, then Step 3 applies if $\text{ord}_2(a_6) = \text{ord}_2(b) = 1$.

If $b$ is odd, then Step 3 only applies if $b \equiv 3 \text{ mod } 4$, since $a_6 = b - 1$. 

So the Kodaira type is $II$ at 2 when $b \equiv 2, 3 \mod 4$.

- Step 4. If $b$ is even, then $b_8 = 0$ and hence Step 4 does not apply.
  If $b$ is odd, then $b_8 = 0$ and again Step 4 does not apply.

- Step 5. If $b$ is even, then $b_6 = 4b$ and this step does not apply.
  If $b$ is odd (and recall that we need only consider $b \equiv 1 \mod 4$ here), then $b_6 = 4b$ and $8 \nmid b_6$. Therefore, the Kodaira type is $IV$.

To determine the Tamagawa index, we consider the polynomial $T^2 + a_{3,1}T - a_{6,2} = T^2 + T - (b - 1)/4$. This will be irreducible if $(b - 1)/4 \not\equiv 0 \mod 2$, that is $b \equiv 5 \mod 8$. Hence $c_2 = 3$ if $b \equiv 1 \mod 8$ and $c_2 = 1$ if $b \equiv 5 \mod 8$.

The only remaining case is $b \equiv 0 \mod 4$.

- Step 6. If $\text{ord}_2(b) = 2$, then we apply the change of variables so that $8|a_6$.
  Thus $a_1 = a_2 = a_4 = 0$, $a_3 = 4$, $a_6 = b - 4$. Now we consider the polynomial $P(T) = T^3 + (b - 4)/8$. If $b \equiv 4 \mod 16$, then $P(T)$ has a triple root and so Step 6 (and 7) does not apply. If $b \equiv 12 \mod 16$, then $P(T)$ has distinct roots and the reduction type is $I_0^*$. $P(T)$ factors as $(T + 1)(T^2 + T + 1)$, so $c_2 = 2$.

  If $8|b$, then we consider the polynomial $P(T) = T^3 + a_{2,1}T^2 + a_{4,2}T + a_{6,3} = T^3 + b/8$.

  If $16|b$, then $P(T)$ has a triple root and so Step 6 (and 7) does not apply.

  If $\text{ord}_2(b) = 3$, then $P(T)$ has distinct roots and the reduction type is $I_0^*$. $P(T)$ factors as $(T + 1)(T^2 + T + 1)$, so $c_2 = 2$.

- Step 7. This step does not apply as we saw in Step 6.

- Step 8. We have two cases to consider here.

  1. $b \equiv 4 \mod 16$. Here we consider $Y^2 + a_{3,2}Y - a_{6,4} = Y^2 + Y - (b-4)/16$. This polynomial has distinct roots and hence the reduction type is $IV^*$. If $b \equiv 4 \mod 32$, then $c_2 = 3$. Otherwise, that is if $b \equiv 20 \mod 32$, then $c_2 = 1$.

  2. $b \equiv 0 \mod 16$. Here we consider $Y^2 + a_{3,2}Y - a_{6,4} = Y^2 - b/16$. This polynomial does not have distinct roots and hence Step 8 does not apply.

- Step 9. If $b \equiv 0 \mod 32$, then the double root of $Y^2 + a_{3,2}Y - a_{6,4} = Y^2$ is at $Y = 0$. However, $a_4 = 0$, so Step 9 does not apply.

  If $b \equiv 16 \mod 32$, then the double root of $Y^2 + a_{3,2}Y - a_{6,4} = Y^2 + 1$ is at $Y = 1$, so we must apply a translation. Using the translation $y = y' + 4$, we obtain the curve $y^2 + 8y = x^3 + b - 16$, so $a_1 = a_2 = a_4 = 0$, $a_3 = 8$ and $a_6 =$
$b - 16$. Since $b \equiv 16 \mod 32$, $b - 16 \equiv 0 \mod 32$ and $Y^2 + a_{3,2}Y - a_{6,4} = Y^2$. But again, $a_4 = 0$, so Step 9 does not apply.

- **Step 10.** Here we have two cases.
  1. $b \equiv 0 \mod 32$ with $16 \mid a_4$ and $64 \nmid a_6$ since we are assuming that $b$ is sixth-power-free. Hence Step 10 applies and the reduction type is $II^*$. If $64 \nmid b - 16$, that is $b \not\equiv 16 \mod 64$, then Step 10 applies and the reduction type is $II^*$.

  If $b \equiv 16 \mod 64$, then Step 10 does not apply.

- **Step 11.** Here we have $b \equiv 16 \mod 64$, our Weierstrass equation is not minimal and we obtain a new Weierstrass equation

  \[ y'^2 + y' = x'^3 + (b - 16)/64. \]

  The discriminant of this Weierstrass equation is $-27b^2/2^8$. Since this is odd, the reduction type is $I_0$. \qed

**Lemma 4.6.** Let $b$ be an integer and suppose that $b \not\equiv 0, 16 \mod 64$.

(a) $P \in E_b(\mathbb{Q}_2)$ is singular if and only if $\text{ord}_2(x(P)) > 0$.

(b) For any $P \in E_b(\mathbb{Q}_2)$, $6P$ is always non-singular. Furthermore, if $P \not\equiv O$ is non-singular, then

\[
\hat{\lambda}_2(P) = \frac{1}{2} \log \max \{1, |x(P)|_2\} - \frac{\Delta(E_b)|_2}{12}.
\]

(c) For any $P \in E_b(\mathbb{Q}_2) \setminus \{O\}$,

\[
\hat{\lambda}_2(P) = \frac{1}{2} \log \max \{1, |x(P)|_2\} - \frac{\Delta(E_b)|_2}{12} - \begin{cases} 
(1/3) \log(2) & \text{if } b \equiv 1 \mod 8 \text{ and } \text{ord}_2(x(P)) > 0 \\
(1/2) \log(2) & \text{if } b \equiv 8, 12 \mod 16 \text{ and } \text{ord}_2(x(P)) > 0 \\
(2/3) \log(2) & \text{if } b \equiv 4 \mod 32 \text{ and } \text{ord}_2(x(P)) > 0 \\
0 & \text{otherwise.}
\end{cases}
\] (4.3)

**Proof.** (a) We require $\text{ord}_2(3x(P)^2) = 2\text{ord}_2(x(P)) > 0$ and $\text{ord}_2(2y(P)) > 0$. Since $b \in \mathbb{Z}$ and $\text{ord}_2(x(P)) > 0$, $\text{ord}_2(2y(P)) > 0$ always holds. Hence $\text{ord}_2(x(P)) > 0$ is a necessary and sufficient condition.

(b) and (c) Again, 6 is the least common multiple of the maximum orders of elements in the possible component groups, $E_b(K)/E_{b,0}(K)$ and the proofs are identical to those for parts (b) and (c) of Lemmas 4.2 and 4.4. \qed
4.4. Global Minimal Weierstrass Equation for $E_b/\mathbb{Q}$. Putting together the information we obtained from Tate’s algorithm in the above three subsections we obtain the following result.

**Lemma 4.7.** Let $b_1$ be the sixth-power-free part of $b$.

If $b_1 \equiv 16 \mod 64$, then a global minimal Weierstrass equation for $E_b/\mathbb{Q}$ is

$$y^2 + y = x^3 + (b_1 - 16)/64.$$  

Otherwise, a global minimal Weierstrass equation for $E_b/\mathbb{Q}$ is

$$y^2 + y = x^3 + b_1.$$

5. Proof of Results

5.1. **Proof of Theorem 1.2.** We compute the canonical height by summing local heights.

Taking the sum of Lemmas 4.2, 4.4 and 4.6 over all primes gives the inequality

$$\sum_{v \neq \infty} \lambda_v(6P) \geq \frac{1}{12} \log |\Delta (E_b)|.$$  

• $b < 0$

Adding (5.1) to the lower bound (3.1) for $\lambda_\infty(6P)$, we obtain

$$\hat{h}(6P) \geq \frac{1}{6} \log |b| + \frac{1}{4} \log(3).$$

Since $\hat{h}(6P) = 36\hat{h}(P)$, this proves Theorem 1.2(a).

• $b > 0$

Adding (5.1) to the lower bound (3.15) for $\lambda_\infty(6P)$, we obtain

$$\hat{h}(6P) > \frac{1}{6} \log |b| + \frac{1}{3} \log(2).$$

Once again, since $\hat{h}(6P) = 36\hat{h}(P)$, Theorem 1.2(b) immediately follows.

5.2. **Proof of Theorem 1.3.** Here too, we compute the canonical height by summing local heights.

From Lemma 4.1 and our hypotheses here, $3P$ is non-singular for all $P \in E_b(\mathbb{Q}_v)$ and all primes $q_v > 3$, so we can apply Lemma 4.2(b) for these local heights. Combining this with Lemmas 4.3(c) and 4.6(c) (note...
the worst case for \( q_v = 2 \) is \( b \equiv 8, 12 \mod 16 \), since for \( b \equiv 4 \mod 32 \), we have \( c_2 = 3 \) and hence \( 3P \) is non-singular) gives the inequality

\[
\sum_{v \neq \infty} \hat{\lambda}_v(3P) \geq -(1/2) \log(2) - (3/4) \log(3) + \frac{1}{12} \log |\Delta(E_b)|.
\]

\[ \text{• } b < 0 \]

Adding (5.2) to the lower bound obtained from (3.1) for \( \hat{\lambda}_\infty(3P) \), we have

\[
\hat{h}(3P) \geq \frac{1}{6} \log |b| - (1/2) \log(2) - (1/2) \log(3).
\]

Since \( \hat{h}(3P) = 9\hat{h}(P) \), this proves Theorem 1.3(a).

\[ \text{• } b > 0 \]

Adding (5.2) to the lower bound from (3.15) for \( \hat{\lambda}_\infty(3P) \), we obtain

\[
\hat{h}(3P) > \frac{1}{6} \log |b| - \frac{1}{6} \log(2) - \frac{3}{4} \log(3).
\]

Once again, since \( \hat{h}(3P) = 9\hat{h}(P) \), Theorem 1.3(b) immediately follows.

5.3. **Proof of Theorem 1.4.** Once again, we compute the canonical height by summing local heights.

From Lemma 4.1 and our hypotheses, \( P \) is non-singular for all \( P \in E_b(\mathbb{Q}_v) \) and all primes \( q_v > 3 \). Hence we can apply Lemma 4.2(b) for these primes. Combining this with Lemmas 4.4(c) and 4.6(c) gives the inequality

\[
\sum_{v \neq \infty} \hat{\lambda}_v(P) \geq -(2/3) \log(2) - (3/4) \log(3) + \frac{1}{12} \log |\Delta(E_b)|.
\]

\[ \text{• } b < 0 \]

Adding (5.3) to the lower bound obtained from (3.1) for \( \hat{\lambda}_\infty(P) \), we have

\[
\hat{h}(P) \geq \frac{1}{6} \log |b| - \frac{2}{3} \log(2) - \frac{1}{2} \log(3).
\]

\[ \text{• } b > 0 \]

Adding (5.3) to the lower bound from (3.15) for \( \hat{\lambda}_\infty(P) \), we obtain

\[
\hat{h}(P) > \frac{1}{6} \log |b| - \frac{1}{3} \log(2) - \frac{3}{4} \log(3).
\]
5.4. **Proof of Theorem 1.5.** Once more, we compute the canonical height by summing local heights.

From Lemma 4.1 and our hypotheses, $2P$ is non-singular for all $P \in E_b(\mathbb{Q}_v)$ and all primes $q_v > 3$. Hence we can apply Lemma 4.2(b) for these primes. Combining this with Lemmas 4.4(c) and 4.6(c) gives the inequality

$$\sum_{v \neq \infty} \lambda_v(2P) \geq -(2/3)\log(2) - (2/3)\log(3) + \frac{1}{12} \log |\Delta(E_b)|.$$  

(5.4)

Note the worst cases occur when $b \equiv 54, 81, 108 \mod 243$ (the case of $b \equiv 27, 216 \mod 243$ does not apply to $2P$ since $c_3 = 2$) and when $b \equiv 4 \mod 32$.

- $b < 0$

Adding (5.4) to the lower bound obtained from (3.1) for $\hat{h}(2P)$, we have

$$\hat{h}(2P) \geq \frac{1}{6} \log |b| - (2/3)\log(2) - (5/12)\log(3).$$

Since $\hat{h}(2P) = 4\hat{h}(P)$, Theorem 1.5(a) immediately follows.

- $b > 0$

Adding (5.3) to the lower bound from (3.1) for $\lambda_\infty(2P)$, we obtain

$$\hat{h}(2P) > \frac{1}{6} \log |b| - (1/3)\log(2) - (2/3)\log(3).$$

Since $\hat{h}(2P) = 4\hat{h}(P)$, Theorem 1.5(b) immediately follows.

6. **Sharpness of Results**

6.1. **Theorem 1.4.** That this result is best-possible for such $b$ can be seen from the following infinite families of pairs of curves and points.

- $b > 0$

For $b > 0$, take $b = 11664b_1^2 - 11664b_1 + 4644$ and $P = (-12, 54(2b_1 - 1))$, where $b_1$ is an integer and we let it approach $+\infty$.

For such pairs of curves and points, we find that $x(P)/|b|^{1/3} \to 0$ as $b_1 \to +\infty$ and hence the archimedean height approaches the lower bound in Lemma 3.4. We see that $b \equiv 4 \mod 32$ and $b \equiv 27 \mod 243$, so such values of $b$ have the smallest non-archimedean height functions at both 2 and 3. Furthermore, by our conditions on $b$ in the Theorem 1.4, our points $P$ are non-singular for the other primes.

- $b < 0$
For $b < 0$, we put $b = -46656b_1^3 - 93312b_1^2 - 62208b_1 - 10908$ and $P = (36b_1 + 24, 54)$ where $b_1$ is an integer and we let it approach $+\infty$. Again, we find that $x(P) \to |b|^{1/3}$ and that the required conditions at each of the primes are satisfied too.

6.2. Theorem 1.5. As in the previous subsection, here we produce infinite families of pairs of curves and points demonstrating that Theorem 1.5 is best-possible.

First, we note that our non-archimedean results are exact, so any gap between the actual height of points and our lower bound in Theorem 1.5 must arise from the archimedean local height hence we construct points on our curves such that the archimedean height of the point approaches our lower bound for it.

- **$b > 0$**
  
  For $b > 0$, we let $b_1$ be an integer, put $P = \left(72b_1^4 - 12b_1, 18 (6b_1^3 - 1)^2\right)$ and $b = 46656b_1^{12} - 93312b_1^9 + 38880b_1^6 - 6048b_1^3 + 324 = (2b_1^3 - 3) (6b_1^3 - 1)^3$.

  For such pairs, $x(2P) = 24b_1$, so $x(2P)/|b|^{1/3} \to 0$ as $b \to +\infty$ and the lower bound for the archimedean height in Lemma 3.4 is sharp as $b_1 \to +\infty$. As above, the desired conditions on all the primes are satisfied too (note $b \equiv 27b_1^3 + 81 \mod 243$).

- **$b < 0$**
  
  For $b < 0$, let $b_1$ be an odd positive integer and put $b_2 = \left[6b_1^2 / (3 + 2\sqrt{3})\right]$ where $[z]$ is the nearest integer to $z$.

  Put $P = \left(12b_1 (6b_1^2 - b_2), 18 (6b_1^2 - b_2)^2\right)$ and $b = 108 (b_2 - 6b_1^2)^3 (3b_2 - 2b_1^2)$.

  For such pairs, $x(2P) = 24b_1b_2$ and $x(2P)^3 = 2985984b_1^{12}/(3 + 2\sqrt{3})^3$. We also find that $b = -2985984b_1^{12}/(3 + 2\sqrt{3})^3 + O(b_1^9)$. Therefore, as $b_1 \to +\infty$, $x(2P) \to |b|^{1/3}$ and the lower bound for the archimedean height is sharp.

6.3. Conjecture 1.6.

- **$b > 0$**
  
  Let $b_1$ be a positive integer not divisible by 3 and congruent to 3 mod 4, put $b = -27b_1^3 (3b_1 + 1)^2$ and $P = (3b_1 (3b_1 + 1), 9b_1^2 (3b_1 + 1))$. Note $x(P) \approx 9b_1^2$ and $b \approx -243b_1^5$.

  $z(P) = 1 - 8b/x(P)^3 \approx 1 + 8/(3b_1) \to 1$ and $x^4(P)z(P) \approx 3^8b_1^8 = |b|^{8/5}$. Furthermore, $x(2^n P) \approx x(P)/4^n \approx 9b_1^2/4^n$ so $z(2^n P) \approx 1$. Hence $(1/8) \sum_{n=1}^\infty 4^{-n} \log (z(2^n P)) \to 0$. Therefore $\lambda_\infty(P') \to (1/5) \log |b|$.
If we put $q_1 = b_1$ and $q_2 = 3b_1 + 1$, then $q_1 \approx |b|^{1/5}/3$ and $q_2/2 \approx |b|^{1/5}$ (note that since $b_1 \equiv 3 \mod 4$, $q_2 \equiv 2 \mod 4$). In the worst case for the heights (i.e., $q_1 q_2$ has distinct prime factors and we get quadratic and cubic residues from all the prime factors of $q_1$ and $q_2$, respectively), then the contribution of the non-archimedean heights for primes larger than 3 is $-(1/2) \log (q_1) - (1/3) \log (q_2/2) = -(1/10) \log(b) + (1/2) \log(3) - (1/15) \log(b) + (1/3) \log(2)$. Lastly, we have $b \equiv 54, 108 \mod 243$ (if $b_1$ is odd or even, respectively) and $b \equiv 8, 12 \mod 16$ (if $b_1$ is congruent to 2 or 3 mod 4, respectively) so we must subtract $(2/3) \log(3)$ and $(1/2) \log(2)$.

Therefore $\hat{h}(P) \to (1/30) \log |b| - (1/6) \log(6)$ as $b_1 \to +\infty$.

• $b > 0$

Let $b_1$ be a positive integer not divisible by 3 and congruent to 1 mod 4 (otherwise, we get a non-minimal Weierstrass equation), put $b = 27b_1^2 (3b_1 - 1)^2$ and $P = (3b_1 (3b_1 - 1), 9b_1^2 (3b_1 - 1))$. Note $x(P) \approx 9b_1^2$ and $b \approx 243b_1^5$.

Arguing as in the case of $b < 0$, we find that $\hat{h}(P) \to (1/30) \log |b| - (1/6) \log(6)$ as $b_1 \to +\infty$.

Other families of examples were also found for $b > 0$. E.g., $b = 3^3 \cdot 5 (6b_1 + 5)^2 (16b_1 + 15)^3$ with $x(P) = 6 (6b_1 + 5) (16b_1 + 15)$ and $b = 3^3 \cdot 7 (6b_1 + 1)^2 (16b_1 + 5)^3$ with $x(P) = 6 (6b_1 + 1) (16b_1 + 5)$.

6.4. Conjecture 1.7

• $b < 0$

Let $b_1$ be a positive integer, put $b = -108 (9b_1^2 - 2) (18b_1^2 - 1)^2$ and $P = (6 (18b_1^2 - 1), 54b_1 (18b_1^2 - 1))$. Here (again using the same argument as for Conjecture 1.6 with $b < 0$), we find that $\hat{h}(P) \to (1/18) \log |b| - (1/18) \log(2) - (1/4) \log(3)$ as $b_1 \to +\infty$. From our examples, it approaches this value from above.

• $b > 0$

Let $b_1$ be a positive integer, put $b = 108 (3b_1^2 + 2) (12b_1^2 - 1)^2$ and $P = (6 (12b_1^2 - 1), 54b_1 (12b_1^2 - 1))$. In this case, $\hat{h}(P) \to (1/18) \log |b| - (1/6) \log(6)$ as $b_1 \to +\infty$.

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