SYMMETRIES, NEWTONOID VECTOR FIELDS AND CONSERVATION LAWS IN THE LAGRANGIAN $k$-SYMPLECTIC FORMALISM

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Abstract. In this paper we study symmetries, Newtonoid vector fields, conservation laws, Noether’s Theorem and its converse, in the framework of the $k$-symplectic formalism, using the Frölicher-Nijenhuis formalism on the space of $k^1$-velocities of the configuration manifold.

For the case $k = 1$, it is well known that Cartan symmetries induce and are induced by constants of motions, and these results are known as Noether’s Theorem and its converse. For the case $k > 1$, we provide a new proof for Noether’s Theorem, which shows that, in the $k$-symplectic formalism, each Cartan symmetry induces a conservation law. We prove that, under some assumptions, the converse of Noether’s Theorem is also true and we provide examples when this is not the case. We also study the relations between dynamical symmetries, Newtonoid vector fields, Cartan symmetries and conservation laws, showing when one of them will imply the others. We use several examples of partial differential equations to illustrate when these concepts are related and when they are not.

1. Introduction

The $k$-symplectic formalism [39] is the generalization to field theories of the standard symplectic formalism in Mechanics [1, 2], which is the geometric framework for describing autonomous dynamical systems. A natural extension of this formalism is the so-called $k$-cosymplectic formalism, [33, 34], which is a generalization to field theories of the cosymplectic formalism describing non-autonomous mechanical systems. One of the advantages of using these formalisms is that only the tangent and cotangent bundles of the configuration manifold are required to develop them. Others papers related with the $k$-symplectic and $k$-cosymplectic formalism are [20, 27, 28, 37, 40, 46, 47].

The polysymplectic formalism developed by Giachetta, Mangiarotti and Sardanashvily in [14], which is based on a vector-valued form defined on some associated fiber bundle, is a different description of classical field theories of first order than the $k$-symplectic formalism. See also [21], for other considerations regarding this aspect. The soldering form on linear frames bundle is a polysymplectic form, and its study and applications to field theory, constitute the $n$-symplectic geometry developed by Norris in [41, 42, 43, 44].

Alternatively, one can derive the field equations by use of the so-called multisymplectic formalism, which was developed by Tulczyjew’s school, [22, 23, 24, 48, 49], and independently by García and Pérez-Rendón [12, 13] and Goldschmidt and Sternberg [15]. This approach was revised by Gotay et al. [16, 17, 18, 19] and more recently by Castrillón et al. [8, 9]. The relationship of the $k$-symplectic formalism with the multisymplectic formalism is studied in [46].

The aim of this paper is to study Noether’s Theorem for first-order classical field theories, using the Lagrangian $k$-symplectic formalism. This study was initialized in [47] where large part of the discussion is a generalization of the results obtained for non-autonomous mechanical systems. See,
in particular [29] and references quoted therein. We introduce the set of Newtonoid vector fields and prove that any Cartan symmetry is a Newtonoid vector field. Furthermore, we show that under some assumptions, Newtonoid vector fields are Cartan symmetries and they induce conservation laws. This result extends the work developed by Marmo and Mukunda in [36]. The study of symmetries in field theory, using various geometric frameworks, has been done in [15, 11, 19, 30, 37].

The structure of the paper is as follows. In Section 2 we review the \( k \)-symplectic Lagrangian formalism, and hence the field theoretic state space of velocities is introduced in Section 2.1 as the Whitney sum \( T_k^1 Q \) of \( k \)-copies of the tangent bundle \( TQ \) of a manifold \( Q \). This manifold has a canonical \( k \)-tangent structure defined by \( k \) tensor fields \( (J^1, \ldots, J^k) \) of type \((1,1)\), see [31, 32]. In the case \( k = 1 \), \( J^1 \) is the canonical tangent structure of the tangent bundle \( TQ \). The canonical \( k \)-tangent structure of \( T_k^1 Q \) is used to construct the Poincaré-Cartan forms.

A particular type of second order partial differential equations, which we call SOPDE, are introduced in Section 2.2. They are a generalization of SODE’s (semisprays) found in Geometric Mechanics. The Lagrangian formalism is developed in Section 2.3.

In Section 3 we discuss symmetries and conservation laws for Lagrangian functions on \( T_k^1 Q \). We prove Noether’s Theorem [39] which shows that each Cartan symmetry induces a conservation law. We provide in Proposition 3.11 some conditions under which the converse of Noether’s Theorem is true. Noether’s Theorem [39] was proved previously in [17] using local coordinates. Here we present a direct global proof using the Frölicher-Nijenhuis formalism. For a modern description of the Frölicher-Nijenhuis formalism see [25, §8]. In Section 3.2 we introduce the set of Newtonoid vector fields in the framework of \( k \)-symplectic formalism, extending the work of Marmo and Mukunda [36] for the case \( k = 1 \). In Proposition 3.8 we prove that Cartan symmetries are always Newtonoid vector fields. In Theorem 3.13 we show that, under some assumptions, Newtonoid vector fields are Cartan symmetries and hence they provide conservation laws.

2. Review of Lagrangian \( k \)-symplectic formalism

In this section we briefly recall the Lagrangian \( k \)-symplectic formalism. We refer the reader to [39, 47] for more details about this formalism. We present first the geometric framework for this formalism, which is given by the tangent bundle of \( k \)-velocities \( T_k^1 Q \) of the configuration manifold \( Q \), together with the canonical structures. For a Lagrangian on \( T_k^1 Q \), the geometric informations we need for the Lagrangian \( k \)-symplectic formalism are encoded in the Poincaré-Cartan forms. We discuss further systems of second order partial differential equations (SOPDE) as well as their relations with Euler-Lagrange equations.

2.1. Geometric framework.

The tangent bundle of \( k^1 \)-velocities of a manifold. Canonical structures. In this work we consider \( Q \) a real, \( n \)-dimensional and \( C^\infty \)-smooth manifold. Throughout the paper, we assume that all objects are \( C^\infty \)-smooth where defined. Consider \((TQ, \tau, Q)\) the tangent bundle of the manifold \( Q \). We denote by \( C^\infty (Q) \) the ring of smooth functions on \( Q \), and by \( \mathfrak{X}(Q) \) the \( C^\infty (Q) \)-module of vector fields on \( Q \).

Let us denote by \( T_k^1 Q \) the Whitney sum \( TQ \oplus \mathbb{R}^k \). \( \oplus TQ \) of \( k \) copies of \( TQ \), with projection \( \tau_Q : T_k^1 Q \to Q \). \( T_k^1 Q \) can be identified with the manifold \( J_k^1 (\mathbb{R}^k, Q) \) of the \( k^1 \)-velocities of \( Q \); that is, \( 1 \)-jets of maps \( \sigma : \mathbb{R}^k \to Q \), with source at 0 \( \in \mathbb{R}^k \). For this reason the manifold \( T_k^1 Q \) is called the tangent bundle of \( k^1 \)-velocities of \( Q \), see [25]. If \((q^i)\) are local coordinates on \( U \subseteq Q \), then the induced local coordinates in \( T_k^1 U = \tau_Q^{-1} (U) \) are denoted by \((q^i, v^\alpha)\), \( 1 \leq i \leq n, 1 \leq \alpha \leq k \). Throughout this work we implicitly assume summation over repeated covariant and contravariant latin indices \( i, j, l, \ldots \in \{1, \ldots, n\} \), as well as summation over repeated greek indices \( \alpha, \beta, \ldots \in \{1, \ldots, k\} \).
The canonical $k$-tangent structure on $T^1_kQ$, see [39], is the family $J = (J^1, \ldots, J^k)$ of $k$ tensor fields of type $(1,1)$, which are locally given by

$$ J^α = \frac{∂}{∂v^α_i} \otimes dq^i, \quad α \in \{1, \ldots, k\}. \quad (2.1) $$

In the case $k = 1$, $J^1$ is the well-known canonical tangent structure of the tangent bundle.

The Liouville vector field, $C \in \mathfrak{X}(T^1_kQ)$, is the infinitesimal generator of the following flow

$$ ψ: \mathbb{R} \times T^1_kQ \rightarrow T^1_kQ, \quad ψ(s, (q,v_1,\ldots,v_k)) = (q, e^s v_1, \ldots, e^s v_k). $$

In local coordinates, the Liouville vector field has the form

$$ C = v^i_α \frac{∂}{∂v^i_α}. \quad (2.2) $$

The vertical distribution is the $kn$-dimensional distribution on $T^1_kQ$ given by $V: u \in T^1_kQ \rightarrow V(u) = \text{Ker} d_u J_ρ = \text{Ker} J_ρ \subset T_uT^1_kQ$. The vertical distribution $V$ splits into $k$ subdistributions $\tilde{V}^α(u), α \in \{1, \ldots, k\}$. Each of these vertical subdistributions is $n$-dimensional and integrable since $\tilde{V}^α(u) = \text{span} \{∂/∂v^i_α, 1 ≤ i ≤ n\}$.

**Poincaré-Cartan forms on $T^1_kQ$.** The Lagrangian $k$-symplectic formalism for a Lagrangian function $L$ on $T^1_kQ$ can be developed from the corresponding Poincaré-Cartan forms.

**Definition 2.1.** A Lagrangian is a smooth function $L$ on $T^1_kQ$. A Lagrangian $L \in C^∞(T^1_kQ)$ is called regular if the Hessian matrix of $L$ with respect to the fibre coordinates, has maximal rank $kn$ on $T^1_kQ$.

For a Lagrangian $L$, the energy function is $E_L = C(L) - L \in C^∞(T^1_kQ)$, with local expression

$$ E_L = v^i_α \frac{∂L}{∂v^i_α} - L. \quad (2.4) $$

For each Lagrangian $L \in C^∞(T^1_kQ)$ we consider the family of Poincaré-Cartan 1-forms $θ_L^α = d_J L = dL \circ J^α$, as well as the family of Poincaré-Cartan 2-forms on $T^1_kQ, \omega^α_L = -dθ^α_L$.

In induced local coordinates on $T^1_kQ$, the Poincaré-Cartan forms are given by

$$ \theta^α_L = \frac{∂L}{∂v^i_α} dq^i, \quad (2.5) $$

$$ \omega^α_L = \frac{1}{2} \left( \frac{∂^2 L}{∂q^i∂v^α_j} - \frac{∂^2 L}{∂q^j∂v^α_i} \right) dq^i ∧ dq^j + \frac{∂^2 L}{∂v^i_α∂v^j_β} dq^i ∧ dv^j_β. $$

We recall now the definition of a $k$-symplectic structure, see [31, 4].

**Definition 2.2.** A $k$-symplectic structure on a $k+nk$-dimensional manifold $M$ is given by a family of $k$ closed 2-forms $(ω^1, \ldots, ω^k)$ and an integrable $kn$-dimensional distribution $V$ on $M$ such that

i) $\bigcap_{α=1}^k \text{Ker}(ω^α) = \{0\}$, ii) $ω^α|_v = 0, \quad α \in \{1, \ldots, k\}. $$

Using formulae (2.3) and (2.5) one obtains that a Lagrangian $L$ is regular if and only if the Poincaré-Cartan 2-forms and the vertical distribution, $(ω^1_L, \ldots, ω^k_L, V = \text{ker} \tau_Q)$, define a $k$-symplectic structure on $T^1_kQ$, see [39].

3
Complete lifts of vector fields. The lifting process of some geometric structures from a base manifold to the total space of some fibre bundle has proven its usefulness for studying the corresponding geometric structures [38, 50]. For example, the complete lift of a system of second order ordinary differential equations contains informations about its symmetries and first order variations, [7].

**Definition 2.3.** Let $\phi: Q \to Q$ be a differentiable map, then the first order prolongation of $\phi$ to $T^1_kQ$ is the map $T^1_k\phi: T^1_kQ \to T^1_kQ$, defined by $T^1_k\phi(j^1_k\sigma) = j^1_k(\phi \circ \sigma)$. This means that for $(q, v_1, \ldots, v_k) \in T_qQ$, $q \in Q$, we have

$$T^1_k\phi(q, v_1, \ldots, v_k) = (d_q\phi(v_1), \ldots, d_q\phi(v_k)).$$

If $Z$ is a vector field on $Q$, with local 1-parametric group of transformations $h_s: Q \to Q$, then the local 1-parametric group of transformations $T^1_kh_s: T^1_kQ \to T^1_kQ$ generates a vector field $Z^C$ on $T^1_kQ$, which is called the complete lift of $Z$ to $T^1_kQ$. If locally $Z = Z'\partial/\partial q^i$, then the complete lift is given by

$$Z^C = Z'\frac{\partial}{\partial q^i} + v_j^1\frac{\partial Z^j}{\partial q^i} + v_{ij}^1\frac{\partial}{\partial v_{ij}^1}.$$  

If we consider also the vertical lifts $Z^{V_s} = J^sZ^C$, then the following properties are well known, see [38],

$$[X^C, Y^C] = [X, Y]^C, \quad [X^C, Y^{V_s}] = [X, Y]^{V_s}, \quad [X^{V_s}, Y^{V_s}] = 0.$$  

These formulae extend the well known properties of Lie brackets for vertical and complete lifts of vector fields to $TQ$, [50].

2.2. Systems of first and second-order partial differential equations. A vector field on a manifold $M$ defines a system of first-order ordinary differential equations. Accordingly, a $k$-vector field on $M$, for some $k > 1$, defines a system of first-order partial differential equations. Furthermore, some special $k$-vector field on the manifold $M = T^1_kQ$ defines a system of second-order partial differential equations.

**First-order partial differential equations on a manifold.** In this subsection, we briefly show how $k$-vector fields determine systems of first-order partial differential equations.

**Definition 2.4.** A $k$-vector field on an arbitrary manifold $M$ is a section $X: M \longrightarrow T^1_kM$ of the canonical projection $\tau_M: T^1_kM \longrightarrow M$.

Since $T^1_kM$ is the Whitney sum $TM \oplus k \cdot \oplus TM$ of $k$ copies of $TM$, we deduce that a $k$-vector field $X$ defines a family of $k$ vector fields $X_1, \ldots, X_k \in \mathfrak{X}(M)$ by projecting $X$ onto every factor; that is, $X_\alpha = \tau_\alpha \circ X$, where $\tau_\alpha: T^1_kM \longrightarrow TM$ is the canonical projection on the $\alpha^{th}$-copy $TM$ of $T^1_kM$.

**Definition 2.5.** An integral section of the $k$-vector field $X = (X_1, \ldots, X_k)$, passing through a point $x \in M$, is a map $\psi: U \subset \mathbb{R}^k \longrightarrow M$, defined on some open neighborhood $U$ of $0 \in \mathbb{R}^k$, such that

$$\psi(0) = x, \quad d_t\psi\left(\frac{\partial}{\partial x^\alpha} \bigg|_t\right) = X_\alpha(\psi(t)) \in T_{\psi(t)}M, \quad \text{for every } t \in U, \ 1 \leq \alpha \leq k.$$  

Equivalently, $\psi$ satisfies $X \circ \psi = \psi^{(1)}$, where $\psi^{(1)}: U \subset \mathbb{R}^k \longrightarrow T^1_kM$ is the first-order prolongation of $\psi$ to $T^1_kM$ defined by

$$\psi^{(1)}: U \subset \mathbb{R}^k \longrightarrow T^1_kM \quad t \longrightarrow \psi^{(1)}(t) = \left(d_t\psi\left(\frac{\partial}{\partial x^\alpha} \bigg|_t\right), \ldots, d_t\psi\left(\frac{\partial}{\partial x^\alpha} \bigg|_t\right)\right).$$
In local coordinates, if \( \psi(t) = (\psi^i(t)) \), then we have

\[
(2.7) \quad \psi^{(1)}(t) = \left( \psi^i(t), \frac{\partial \psi^i}{\partial x^\alpha}(t) \right).
\]

A k-vector field \( X = (X_1, \ldots, X_k) \) on \( M \) is **integrable** if there is an integral section passing through every point of \( M \).

Consider \( X = (X_1, \ldots, X_k) \) a k-vector field, where \( X_\alpha = X^i_\alpha \partial / \partial x^i \) in a coordinate system \((U, x^i)\) on \( M \). The k-vector field \( X \) induces a system of first-order partial differential equations on \( M \), which is given by

\[
X^i_\alpha(x^i(t)) = \frac{\partial x^i}{\partial t^\alpha} \bigg|_t, \quad \alpha \in \{1, \ldots, k\}, \quad i \in \{1, \ldots, \dim M\}.
\]

From Definition 2.5 we deduce that \( \psi \) is an integral section of \( X \) if \( \psi \) is a solution to the above system of first-order partial differential equations, which means that it satisfies

\[
X^i_\alpha(\psi(t)) = \frac{\partial \psi^i}{\partial x^\alpha}(t), \quad \alpha \in \{1, \ldots, k\}, \quad i \in \{1, \ldots, \dim M\}.
\]

**Systems of second-order partial differential equations.** In this part we characterize those integrable k-vector fields on \( M = T^1_kQ \) that have as integral sections first order prolongations \( \phi^{(1)} \) of maps \( \phi : U \subset \mathbb{R}^k \to Q \). Such k-vector fields define integrable systems of second-order partial differential equations on the base manifold \( Q \).

As we recalled, a k-vector field in \( T^1_kQ \) is a section \( \xi : T^1_kQ \to T^1_k(T^1_kQ) \) of the canonical projection \( \pi^{(1)}_Q : T^1_k(T^1_kQ) \to T^1_kQ \). We note that there are systems of partial differential equations that are not induced by k-vector fields. However, in this work we are interested only in those systems of PDE that are induced by such k-vector fields. In view of these considerations, we consider the following definition.

**Definition 2.6.** A system of second-order partial differential equations (SOPDE) on \( Q \) is a k-vector field \( \xi = (\xi_1, \ldots, \xi_k) \) on \( T^1_kQ \), which is a section of the projection \( T^1_kT^1_kQ \to T^1_kQ \), namely \( T^1_k\pi_Q \circ \xi = \mathrm{Id}_{T^1_kQ} \), and this is equivalent to

\[
(2.8) \quad d\tau_Q \circ \xi_\alpha = \tau_\alpha : T^1_kQ \to TQ, \quad \alpha \in \{1, 2, \ldots, k\}.
\]

Equivalently, above equations can be written as follows

\[
d(q,v)\tau_Q(\xi_\alpha(q,v)) = (q, v_\alpha), \quad \text{for} \ (q, v_1, \ldots, v_k) \in T^1_kQ, \quad \alpha \in \{1, \ldots, k\}.
\]

In the case \( k = 1 \), Definition 2.6 reduces to the definition of a system of second-order ordinary differential equations (SOODE).

Locally, a SOPDE \( \xi = (\xi_1, \ldots, \xi_k) \) is given by

\[
(2.9) \quad \xi_\alpha = v^i_\alpha \frac{\partial}{\partial q^i} + \xi^i_{\alpha\beta} \frac{\partial}{\partial v^\beta}, \quad \alpha \in \{1, 2, \ldots, k\},
\]

where \( \xi^i_{\alpha\beta} \) are smooth functions defined on domains of induced charts on \( T^1_kQ \).

All these considerations allow us to reformulate the definition for a SOPDE, using the k-tangent structure and the Liouville vector field \( C \) (see formulae (2.1) and (2.2)), as follows.

**Proposition 2.7.** A k-vector field \( \xi = (\xi_1, \ldots, \xi_k) \) on \( T^1_kQ \) is a SOPDE if and only if \( J^\alpha(\xi_\alpha) = C \).

If \( \psi : U \subset \mathbb{R}^k \to T^1_kQ \), locally given by \( \psi(t) = (\psi^i(t), \psi^i_\alpha(t)) \), is an integral section of a SOPDE \( \xi = (\xi_1, \ldots, \xi_k) \) then from Definition 2.5 and formula 2.9 it follows

\[
(2.10) \quad \frac{\partial \psi^i}{\partial t^\alpha} \bigg|_t = \psi^i_\alpha(t), \quad \frac{\partial \psi^i_\alpha}{\partial v^\beta} \bigg|_t = \xi^i_{\alpha\beta}(\psi(t)).
\]
Using formulae (2.7) and (2.10) we obtain the following characterization for the integral maps of a SOPDE.

**Proposition 2.8.** Let $\xi = (\xi_1, \ldots, \xi_k)$ be an integrable SOPDE. If $\psi$ is an integral section of $\xi$, then $\psi = \phi^{(1)}$, where $\phi^{(1)}$ is the first-order prolongation of the map

$$\phi = \tau_Q \circ \psi : U \subset \mathbb{R}^k \rightarrow T^1_k Q \rightarrow Q,$$

and $\phi$ is a solution to the system of second-order partial differential equations

$$(2.11) \quad \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta}(t) = \xi^i_{\alpha\beta}(\phi^j(t), \frac{\partial \phi^j}{\partial v^\gamma}(t)).$$

Conversely, if $\phi : U \subset \mathbb{R}^k \rightarrow Q$ is any map satisfying the system (2.11), then $\phi^{(1)}$ is an integral section of $\xi = (\xi_1, \ldots, \xi_k)$.

**Definition 2.9.** If $\phi^{(1)}$ is an integral section of a SOPDE $\xi = (\xi_1, \ldots, \xi_k)$, then map $\phi$ will be called a solution to $\xi$.

From equations (2.11) we deduce that if $\xi$ is an integrable SOPDE then necessarily we have the symmetry $\xi^i_{\alpha\beta} = \xi^i_{\beta\alpha}$ for all $\alpha, \beta = 1, \ldots, k$. Therefore, for a SOPDE $\xi$, locally given by formula (2.9), we require the following integrability conditions [26]:

$$\xi^i_{\alpha\beta} = \xi^i_{\beta\alpha}, \quad \xi^i_{\alpha} = \xi^i_{\beta}(\xi^j_{\gamma}), \quad \forall \alpha, \beta, \gamma \in \{1, \ldots, k\}. \tag{2.12}$$

Integrability conditions (2.12) are equivalent to the fact that $[\xi_\alpha, \xi_\beta] = 0$, $\forall \alpha, \beta \in \{1, \ldots, k\}$. The integrability conditions (2.12) have been also proved in [35]. Due to the first symmetry condition (2.12) we have that the system (2.11) is a system of $nk(k + 1)/2$ second-order partial differential equations.

2.3. **Euler-Lagrange equations.** An important class of SOPDE on a manifold $Q$ contains those whose solutions are among the solutions of the Euler-Lagrange equations for some Lagrangian function on $T^1_k Q$. In Proposition 2.11 we characterize this class, while in Proposition 2.12 we discuss the relation between the solutions of a SOPDE in this class and the solutions of the corresponding Euler-Lagrange equations.

The variational problem for a Lagrangian $L$ on $T^1_k Q$ leads to the following system of Euler-Lagrange equations

$$(2.13) \quad \frac{\partial}{\partial t^\alpha} \left( \frac{\partial L}{\partial v^i_{\alpha}} \right) - \frac{\partial L}{\partial q^i} = 0.$$

Euler-Lagrange equations (2.13) can be written as

$$(2.14) \quad \eta^i_{\alpha} \frac{\partial^2 q^j}{\partial t^\alpha \partial t^\beta} + \frac{\partial^2 L}{\partial q^i \partial v^i_{\alpha}} \frac{\partial v^i_{\alpha}}{\partial q^j} - \frac{\partial L}{\partial q^j} = 0,$$

which represents a system of $n$ second-order partial differential equations on $Q$.

Denote by $\mathcal{X}_L(T^1_k Q)$ the set of $k$-vector fields $\xi = (\xi_1, \ldots, \xi_k)$ on $T^1_k Q$, which are solutions to the equation

$$i_{\xi_\alpha} \omega^L = dE_L. \tag{2.15}$$

If each $\xi_\alpha$ is locally given by

$$\xi_\alpha = \xi^i_{\alpha} \frac{\partial}{\partial q^i} + \xi^i_{\alpha\beta} \frac{\partial}{\partial v^i_{\beta}}, \quad \alpha \in \{1, \ldots, k\}, \tag{2.16}$$
then \((\xi_1, \ldots, \xi_k)\) is a solution to (2.19) if and only if the functions \(\xi_i^1\) and \(\xi_i^{1,\beta}\) satisfy the following system of equations

\[
\begin{align*}
\frac{\partial^2 L}{\partial q^i \partial v^a} \xi_i^1 - \frac{\partial^2 L}{\partial q^i \partial v^a} \xi_i^{1,\beta} &= v_i^a \frac{\partial^2 L}{\partial q^i \partial v^a} - \frac{\partial L}{\partial q^i}, \\
\frac{\partial^2 L}{\partial v_i^a \partial v^a} \xi_i^1 &= \frac{\partial^2 L}{\partial v_i^a \partial v^a} v_i^a.
\end{align*}
\]

If \(L\) is a regular Lagrangian, the above equations are equivalent to the following equations

\[
\frac{\partial^2 L}{\partial v_i^a \partial v^a} \xi_i^1 + \frac{\partial^2 L}{\partial q^i \partial v^a} v_i^a - \frac{\partial L}{\partial v^i} = 0, \quad \xi_i^1 = v_i^a.
\]

Using equations (2.18) we deduce the following lemma.

**Lemma 2.10.** Consider \(L \in C^\infty(T_k^1 Q)\) a Lagrangian.

1) If \(L\) is regular, then any \(k\)-vector field \(\xi \in \mathfrak{X}_L^k(T_k^1 Q)\) is a SOPDE, it is locally given by formula (2.17) and satisfies equations (2.18).

2) If \(\xi \in \mathfrak{X}_L^k(T_k^1 Q)\) and \(\xi\) is a SOPDE, then it is locally given by formula (2.17) and satisfies equations (2.18).

Next proposition characterizes the set of SOPDEs that are in \(\mathfrak{X}_L^k(T_k^1 Q)\).

**Proposition 2.11.** Let \(L \in C^\infty(T_k^1 Q)\) be a Lagrangian and let \(\xi = (\xi_1, \ldots, \xi_k)\) be a SOPDE on \(T_k^1 Q\). Then \(\xi \in \mathfrak{X}_L^k(T_k^1 Q)\) if and only if it satisfies the following condition:

\[
\mathcal{L}_{\xi, \theta_L^a} = dL, \quad \text{or locally} \quad \xi_a \left( \frac{\partial L}{\partial v^a} \right) = \frac{\partial L}{\partial q^i}.
\]

**Proof.** We will start by proving that the first equation in (2.19) is equivalent to equation (2.15). Since \(\xi\) is a SOPDE we have that \(J^a \xi_a = \mathbb{C}\) and hence it follows that \(i_{\xi, \theta_L^a} = \theta_L^a (\xi_a) = (dL \circ J^a)(\xi_a) = CL\). Using the fact that \(\omega_L^1 = -d\theta_L^1\) we obtain

\[
\mathcal{L}_{\xi, \theta_L^a} \xi_a = di_{\xi, \theta_L^a} + i_{\xi, \theta_L^a} \omega_L^1 = d(CL) - i_{\xi, \omega_L^1} = dL + (dE_L - i_{\xi, \omega_L^1}).
\]

It follows that the first equation in (2.19) is equivalent to equation (2.15).

Since \(\xi\) is a SOPDE it follows that \(\xi_a(q^i) = v_i^a\). Therefore, we have

\[
\mathcal{L}_{\xi, \theta_L^a} - dL = \mathcal{L}_{\xi_a} \left( \frac{\partial L}{\partial v^a} dq^i \right) - dL = \xi_a \left( \frac{\partial L}{\partial v^a} \right) dq^i + \left( \frac{\partial L}{\partial v^a} \right) dv^a - dL = \left[ \xi_a \left( \frac{\partial L}{\partial v^a} \right) - \frac{\partial L}{\partial q^i} \right] dq^i,
\]

and hence the two equations in (2.19) are equivalent. \(\square\)

We will discuss now the relation between solutions of the Euler-Lagrange equations (2.13) or (2.14) and integral sections of \(k\)-vector fields in \(\mathfrak{X}_L^k(T_k^1 Q)\).

**Proposition 2.12.** Consider a Lagrangian \(L\) on \(T_k^1 Q\) and a \(k\)-vector field \(\xi \in \mathfrak{X}_L^k(T_k^1 Q)\).

1) If \(\xi\) is a SOPDE, then a map \(\phi : U \subset \mathbb{R}^k \to Q\) is a solution to the Euler-Lagrange equations (2.13) if and only if

\[
\theta_{ij}^{\alpha \beta} \circ \phi(1) \left( \xi_j^{1,\alpha \beta} \circ \phi(1) - \frac{\partial^2 \phi_j}{\partial v^a \partial q^i} \right) = 0.
\]

2) If the Lagrangian \(L\) is regular, then \(\xi\) is a SOPDE, and if \(\phi : U \subset \mathbb{R}^k \rightarrow Q\) is a solution to \(\xi\), then \(\phi\) is a solution to the Euler-Lagrange equations (2.13).
3) If $\xi$ is integrable, and $\phi^{(1)}: U \subset \mathbb{R}^k \to T^1_k Q$ is an integral section, then $\phi: U \subset \mathbb{R}^k \to Q$ is a solution to the Euler-Lagrange equations (2.13).

**Proof.**

1) Consider a map $\phi: U \subset \mathbb{R}^k \to Q$. If $\phi$ is a solution to the Euler-Lagrange equations (2.14), then we have

$$ \frac{\partial^2 L}{\partial q^i \partial v^j} \circ \phi^{(1)} + \frac{\partial^2 L}{\partial \dot{q} \partial v^j} \circ \phi^{(1)} \frac{\partial \phi}{\partial t^a} - \frac{\partial L}{\partial \dot{q}} \circ \phi^{(1)} = 0. $$

If the $k$-vector field $\xi$ is a SOPDE, then $\xi \in \mathcal{X}_L^k(T^1_k Q)$ if and only if it satisfies the equations

$$ \frac{\partial^2 L}{\partial q^i \partial v^j} v^j_{\alpha} + \frac{\partial^2 L}{\partial v^j \partial v^j} \xi_{j} \epsilon_{\alpha \beta} = \frac{\partial L}{\partial \dot{q}^i} \circ \phi^{(1)}. $$

If we restrict equation (2.22) to the image of $\phi^{(1)}$ we obtain

$$ \frac{\partial^2 L}{\partial q^i \partial v^j} \circ \phi^{(1)} \xi_{j} \epsilon_{\alpha \beta} \circ \phi^{(1)} + \frac{\partial^2 L}{\partial \dot{q} \partial v^j} \circ \phi^{(1)} \frac{\partial \phi}{\partial t^a} - \frac{\partial L}{\partial \dot{q}} \circ \phi^{(1)} = 0. $$

Using equations (2.23) it follows that $\phi$ satisfies (2.20) if and only if it satisfies (2.21) that are equivalent to Euler-Lagrange equations (2.14).

2) If $\phi: U \subset \mathbb{R}^k \to Q$ is a solution to $\xi$ then it satisfies equations (2.21). Therefore, equations (2.20) are automatically satisfied and hence $\phi$ is a solution of the Euler-Lagrange equations (2.14).

3) Since $\xi \in \mathcal{X}_L^k(T^1_k Q)$ it follows that $\xi$ satisfies first equation (2.17). If we restrict this equation to $\phi^{(1)}: U \subset \mathbb{R}^k \to T^1_k Q$, which is an integral map of $\xi$, we obtain that $\phi$ satisfies the Euler-Lagrange equations (2.13).

**Remark 2.13.** The results of Lemma 2.10 and results 2) and 3) of Proposition 2.12 are the fundamentals of Lagrangian $k$-symplectic formalism and equation (2.14) can be seen as a geometric version of the Euler-Lagrange field equations.

**Remark 2.14.** Formula (2.20) does not require any relationship between the $k$-vector field $\xi \in \mathcal{X}_L^k(T^1_k Q)$ and the solution $\phi$ to the Euler-Lagrange equations (2.13). In other words, we might have $\phi$ a solution to the Euler-Lagrange equations (2.13) which may not be a solution for any $\xi \in \mathcal{X}_L^k(T^1_k Q)$.

**Example 2.15.** In this example we consider the theory of a vibrating string. Coordinates $(t^1, t^2)$ are interpreted as the time and the distance along the string, respectively. If $\phi: (t^1, t^2) \in \mathbb{R}^2 \to \phi(t^1, t^2) \in \mathbb{R}$ denotes the displacement of each point of the string as function of the time $t^1$ and the position $t^2$, the motion equation is

$$ \sigma \frac{\partial^2 \phi}{\partial (t^1)^2} - \tau \frac{\partial^2 \phi}{\partial (t^2)^2} = 0, $$

where $\sigma$ and $\tau$ are certain constants of the mechanical system.

Equation (2.24) is the Euler-Lagrange equation for the regular Lagrangian

$$ L: T^2 \mathbb{R} \to \mathbb{R}, \quad L(q, v_1, v_2) = \frac{1}{2} (\sigma v_1^2 - \tau v_2^2). $$

From formulae (2.25) and (2.25) we deduce that

$$ \omega^1 = \sigma dq \wedge dv_1, \quad \omega^2 = -\tau dq \wedge dv_2, \quad dE_L = \sigma v_1 dv_1 - \tau v_2 dv_2. $$

Therefore, a SOPDE $(\xi_1, \xi_2) \in \mathcal{X}(T^2 \mathbb{R})$

$$ \xi_1 = v_1 \frac{\partial}{\partial q} + \xi_{11} \frac{\partial}{\partial v_1} + \xi_{12} \frac{\partial}{\partial v_2}, \quad \xi_2 = v_2 \frac{\partial}{\partial q} + \xi_{12} \frac{\partial}{\partial v_1} + \xi_{22} \frac{\partial}{\partial v_2}. $$
is a solution to equation (2.15) if and only if it satisfies

\[ \sigma \xi_{11} - \tau \xi_{22} = 0. \]

The integrability conditions (2.12) are in this case

\[ \frac{\partial \xi_{11}}{\partial v_1} = \frac{\partial \xi_{12}}{\partial v_2}, \quad \sigma \frac{\partial \xi_{11}}{\partial v_1} = \tau \frac{\partial \xi_{12}}{\partial v_1}. \]

An example of an integrable SOPDE, which is a solution to (2.27) is given by

\[ \xi_1 = v_1 \frac{\partial}{\partial q} + \tau \left( \sigma (v_1)^2 + \tau (v_2)^2 \right) \frac{\partial}{\partial v_1} + 2 \sigma \tau v_1 v_2 \frac{\partial}{\partial v_1}, \]

\[ \xi_2 = v_2 \frac{\partial}{\partial q} + 2 \sigma \tau v_1 v_2 \frac{\partial}{\partial v_1} + \sigma \left( \sigma (v_1)^2 + \tau (v_2)^2 \right) \frac{\partial}{\partial v_2}. \]

Thus any solution \( \phi \) of the integrable SOPDE \( (\xi_1, \xi_2) \) in the formulae above is a solution of the vibrating string equation (2.24).

3. Noether’s theorem

In this section we discuss symmetries and conservation laws for Lagrangian functions on \( T^1_k Q \). We introduce the Newtonoid vector fields in this framework, extending the work of Marmo and Mukunda [36] for the case \( k = 1 \). We provide a new proof for Noether’s Theorem 3.9 as well as some conditions under which its converse is true. Noether’s Theorem 3.9 was proved previously in [47] using local coordinates. Here we present a direct global proof using the Frölicher-Nijenhuis formalism.

3.1. Conservation laws and Cartan symmetries. For a regular Lagrangian on \( TQ \), the corresponding Euler-Lagrange equations are equivalent to a SODE. This implies that its dynamical symmetries are equivalent to Cartan symmetries, which (locally) determine and are determined by constants of motions [10, §13.8]. For \( k > 1 \) and a Lagrangian \( L \) on \( T^1_k Q \) none of the above equivalences are true anymore in the very general context. However, some relations remain true. In this subsection we discuss such relations between Cartan symmetries and conservation laws.

**Definition 3.1.** A map \( f = (f^1, \ldots, f^k): T^1_k Q \to \mathbb{R}^k \) is called a conservation law (or a conserved quantity) for the Euler-Lagrange equations (2.13) if the divergence of

\[ f \circ \phi^{(1)} = (f^1 \circ \phi^{(1)}, \ldots, f^k \circ \phi^{(1)}): U \subset \mathbb{R}^k \to \mathbb{R}^k \]

is zero, for every \( \phi: U \subset R^k \to M \) solution to the Euler-Lagrange equations (2.13), that is

\[ 0 = \frac{\partial (f^\alpha \circ \phi^{(1)})}{\partial \alpha} \bigg| _t = \frac{\partial f^\alpha}{\partial q^i} \bigg|_{\phi^{(1)}(t)} \frac{\partial \phi^i}{\partial \alpha} \bigg| _t + \frac{\partial f^\alpha}{\partial v^\beta_{j}(\alpha)} \bigg|_{\phi^{(1)}(t)} \frac{\partial \alpha^j}{\partial \beta \partial \alpha^i} \bigg| _t. \]

Now, we present a simple example of conservation law.

**Example 3.2.** The following two functions \( f^\alpha: T^1_2 \mathbb{R} \to \mathbb{R}, \alpha \in \{1, 2\} \), where

\[ f^1(v_1, v_2) = -2 \sigma v_1 v_2, \quad f^2(v_1, v_2) = \sigma (v_1)^2 + \tau (v_2)^2, \]

give a conservation law for the Euler-Lagrange equation (2.24). In fact, if \( \phi \) is a solution to the Euler-Lagrange equations (2.24), using (3.2) we deduce that

\[ \frac{\partial (f^1 \circ \phi^{(1)})}{\partial t^1} + \frac{\partial (f^2 \circ \phi^{(1)})}{\partial t^2} = 0. \]

Hence the functions (3.2) give a conservation law for the Lagrangian (2.25).
Lemma 3.3. Let \( f = (f^1, \ldots, f^k) : T_k^1Q \to \mathbb{R}^k \) be a conservation law. Let \( \xi = (\xi_1, \ldots, \xi_k) \) be an integrable sopde in \( \mathfrak{X}_k^l(T_k^1Q) \), then
\[
(3.3) \quad \xi_\alpha(f^\alpha) = 0.
\]

Proof. Since \( \xi \) is integrable we know that for every point \( x \in T_k^1Q \) there exists an integral section \( \phi^{(1)} : U \subset \mathbb{R}^k \to T_k^1Q \) such that

1) \( \phi \) is a solution to the Euler-Lagrange equations, because \( \xi \in \mathfrak{X}_k^l(T_k^1Q) \),
2) \( \phi \) satisfies
\[
(3.4) \quad v_\alpha^j(\phi^{(1)}(t)) = \frac{\partial \phi^j}{\partial t} \bigg|_{t=0}, \quad \xi_\alpha^j(\phi^{(1)}(t)) = \frac{\partial^2 \phi^j}{\partial t^\alpha \partial t^\beta} \bigg|_{t=0},
\]

Since \( f = (f^1, \ldots, f^k) \) is a conservation law, using formula (3.11) at \( t = 0 \), and using formulae (3.4), we have
\[
0 = \frac{\partial (f^\alpha \circ \phi^{(1)})}{\partial t^\alpha} \bigg|_0 = \frac{\partial f^\alpha}{\partial q^i} \bigg|_{\phi^{(1)}(0)} \frac{\partial \phi^i}{\partial t^\alpha} \bigg|_0 + \frac{\partial f^\alpha}{\partial v_\beta} \bigg|_{\phi^{(1)}(0)} \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta} \bigg|_0
= \frac{\partial f^\alpha}{\partial q^i} \bigg|_x \frac{\partial \phi^i}{\partial t^\alpha} \bigg|_0 + \frac{\partial f^\alpha}{\partial v_\beta} \bigg|_x \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta} \bigg|_0 = \frac{\partial f^\alpha}{\partial q^i} \bigg|_x v_\alpha^i(x) + \frac{\partial f^\alpha}{\partial v_\beta} \bigg|_x \xi_\alpha^j(x) = \xi_\alpha(x)f^\alpha.
\]

\( \square \)

The converse of Lemma 3.3 may not be true, and the reason is that, as we can see from formula (2.26), we might have solutions \( \phi \) of the Euler-Lagrange equations (2.14) that are not solutions to some \( \xi \in \mathfrak{X}_k^l(T_k^1Q) \).

However, we will see in the following Lemma that, under some assumption on functions \( f^\alpha \), this converse is true.

Lemma 3.4. Let \( L \in C^\infty(T_k^1Q) \) be a Lagrangian and assume that there exists a vector field \( \chi \in \mathfrak{X}(T_k^1Q) \) such that
\[
(3.5) \quad i_\chi \omega^\alpha_L = df^\alpha, \quad \forall \alpha \in \{1, \ldots, k\},
\]

for some functions \( f^\alpha : T_k^1Q \to \mathbb{R} \).

Then, \( f^\alpha \) is a conservation law for the Euler-Lagrange equations (2.13) if and only if \( \xi_\alpha(f^\alpha) = 0 \), for all integrable sopde \( \xi \in \mathfrak{X}_k^l(T_k^1Q) \).

Proof. The direct implication is given by Lemma 5.3. Note that for this implication we do not need the assumption on the existence of the vector field \( \chi \) that satisfies (3.5).

For the converse implication consider \( X = X^i \partial / \partial q^i + X_\alpha \partial / \partial v_\alpha \) a vector field on \( T_k^1Q \) that satisfies (3.5). In view of the second formula (2.26) we can write equation (3.5) as follows
\[
\left( \frac{\partial^2 L}{\partial q^i \partial v_\alpha} - \frac{\partial^2 L}{\partial q^j \partial v_\beta} \right) X^j - g^{\alpha \beta} X^j \partial g^{\alpha \beta} / \partial q^i \partial v_\alpha = \frac{\partial f^\alpha}{\partial q^i} dq^i + \frac{\partial f^\alpha}{\partial v_\beta} dv_\beta,
\]

and necessarily we have
\[
(3.6) \quad \frac{\partial f^\alpha}{\partial v_\beta} = g^{\alpha \beta} X^i.
\]
Consider now \( \phi \) any solution to the Euler-Lagrange equations (2.14) (which may not be a solution of any \( \xi \)). It follows that \( \phi \) satisfies equations (2.20), since \( \xi \) is assumed to be an integrable SOPDE.

If we contract equations (2.20) by \( X^i \circ \phi^{(1)} \), we obtain

\[(3.7) \quad (X^i \circ \phi^{(1)})(g^{\alpha \beta}_i \circ \phi^{(1)}) \left( \frac{\partial^2 \phi^j}{\partial t^\alpha \partial t^\beta} - \xi^{j \alpha \beta}_i \circ \phi^{(1)} \right) = 0.\]

If we replace formula (3.3) in equation (3.7) we obtain

\[0 = \frac{\partial f^\alpha}{\partial v^j} \circ \phi^{(1)} \left( \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta} - \xi^{j \alpha \beta}_i \circ \phi^{(1)} \right) = \frac{\partial (f^\alpha \circ \phi^{(1)})}{\partial t^\alpha} - \xi^\alpha_\alpha (f^\alpha) \circ \phi^{(1)},\]

and this formula proves the result. \( \square \)

Let us recall the definition of Cartan symmetry for a Lagrangian \( L \), see [47].

**Definition 3.5.** A vector field \( X \in \mathfrak{X}(T^*_k Q) \) is called a Cartan symmetry of the Lagrangian \( L \), if \( \mathcal{L}_X \omega^\alpha_L = 0 \) for all \( \alpha \in \{1, \ldots, k\} \) and \( \mathcal{L}_X E_L = 0 \).

In this case the flow \( \phi_t \) of \( X \) transforms solutions to the Euler-Lagrange equations on solutions to the Euler-Lagrange equations, that is, each \( \phi_t \) is a symmetry of the Euler-Lagrange equations, see [47].

From condition \( \mathcal{L}_X \omega^\alpha_L = 0 \) one obtains that there exists locally defined functions \( f^\alpha \) such that \( i_X \omega^\alpha_L = df^\alpha \). Thus if \( X \) is a Cartan symmetry Lemma 3.4 holds locally.

### 3.2. Newtonoid vector fields

In this subsection we study some properties of the set of Newtonoid vector fields associated to a SOPDE, generalizing the \( k = 1 \) case, introduced by Marmo and Mukunda in [30]. Properties of the Newtonoid vector fields associated to a SODE and their relations to symmetries and first order variation of geodesics were studied [10]. We extend some of these properties to the case \( k > 1 \). In Proposition 3.8 we prove that Cartan symmetries of a regular Lagrangian \( L \) are Newtonoid vector fields for all corresponding SOPDE \( \xi \in \mathfrak{X}(T^*_k Q) \).

We fix a SOPDE \( \xi \) and consider the following set of vector fields on \( T^*_k Q \)

\[(3.8) \quad \mathfrak{X}_\xi = \text{Ker} \left( J^\alpha \circ \mathcal{L}_{\xi_\alpha} \right) \subset \mathfrak{X}(T^*_k Q).\]

The set \( \mathfrak{X}_\xi \) can be expressed locally as follows

\[(3.9) \quad \mathfrak{X}_\xi = \left\{ X \in \mathfrak{X}(T^*_k Q), \ X = X^i \frac{\partial}{\partial q^i} + \xi^i_\alpha (X^i) \frac{\partial}{\partial v^\alpha} \right\}.\]

Indeed, for a vector field \( X \in \mathfrak{X}(T^*_k Q) \), we have

\[(3.10) \quad [\xi_\alpha, X] = \left[ v^i_\alpha \frac{\partial}{\partial q^i} + \xi^i_\alpha (X^i) \frac{\partial}{\partial v^\alpha}, X^i \frac{\partial}{\partial q^i} + X^i \frac{\partial}{\partial v^i} \right] = \left( \xi_\alpha (X^i) - X^i \right) \frac{\partial}{\partial q^i} + \left( \xi^j_\alpha (X^j) - X^j \right) \frac{\partial}{\partial v^\alpha},\]

and therefore \( J^\alpha \circ \mathcal{L}_{\xi_\alpha} (X) = 0 \) if and only if \( \xi_\alpha (X^i) = X^i \).

**Definition 3.6.** Consider \( \xi \) a SOPDE.

1. A vector field \( X \in \mathfrak{X}_\xi \) is called a Newtonoid for \( \xi \).
2. A vector field \( X \in \mathfrak{X}(T^*_k Q) \) is called a dynamical symmetry of \( \xi \) if \( \left[ \xi_\alpha, X \right] = 0 \), for all \( \alpha \in \{1, \ldots, k\} \).

From formula (3.10) it follows that any dynamical symmetry for a SOPDE \( \xi \) is a Newtonoid for \( \xi \). For \( k = 1 \), the set \( \mathfrak{X}_\xi \) was introduced in [30] and it was called the set of Newtonoid vector fields. In the next lemma we provide some properties of the set of Newtonoid vector fields.
Lemma 3.7. For a SOPDE $\xi$ consider the map $\pi_\xi : \mathfrak{X}(T^1_kQ) \rightarrow \mathfrak{X}(T^1_kQ)$, given by $\pi_\xi = \text{Id} + J^\alpha \circ L_{\xi^\alpha}$.

1) The map $\pi_\xi$ satisfies $\pi_\xi \circ \pi_\xi = \pi_\xi$, for any two SOPDEs $\xi$ and $\xi'$. In particular we have $\pi_\xi^2 = \pi_\xi$ and hence $\pi_\xi$ is a projector;

2) $\text{Im } \pi_\xi = \mathfrak{X}_\xi$, Ker $\pi_\xi = \mathfrak{X}'(T^1_kQ)$, and hence the following sequence is exact

$$0 \rightarrow \mathfrak{X}'(T^1_kQ) \xrightarrow{\iota} \mathfrak{X}(T^1_kQ) \xrightarrow{\pi_\xi} 0.$$ 

3) For $f \in C^\infty(T^1_kQ)$ and $X \in \mathfrak{X}_\xi$, we define the product

$$f * X = \pi_\xi(fX) = fX + \xi_\alpha(f)J^\alpha X \in \mathfrak{X}_\xi.$$ 

The set $\mathfrak{X}_\xi$ is a $C^\infty(T^1_kQ)$-module with respect to the $*$ product.

4) A vector field $X$ on $T^1_kQ$ is a Newtonoid for $\xi$ if and only if it has the local expression

$$X = X^i(q,v) * \frac{\partial}{\partial q^i}.$$ 

Proof. Using the definition of the map $\pi_\xi$ it follows that $\pi_\xi \circ \pi_\xi' = \text{Id} + J^\alpha \circ L_{\xi^\alpha} + J^\alpha \circ L_{\xi'_\alpha} + J^\alpha \circ L_{\xi^\alpha} \circ J^\beta \circ L_{\xi'_\beta}$. Now using the formula $J^\alpha \circ L_{\xi^\alpha} \circ J^\beta = -J^\beta$ it follows that $\pi_\xi \circ \pi_\xi' = \pi_\xi$, which shows that first part of the lemma is true.

Using formula (3.10), we have

$$\pi_\xi \left( X^i \frac{\partial}{\partial q^i} + X^i_{\alpha} \frac{\partial}{\partial v_{\alpha}} \right) = X^i \frac{\partial}{\partial q^i} + \xi_\alpha(X^i) \frac{\partial}{\partial v_{\alpha}},$$

which shows that $\text{Im } \pi_\xi = \mathfrak{X}_\xi$ and Ker $\pi_\xi = \mathfrak{X}'(T^1_kQ)$.

With the $*$ product defined in formula (3.11), the map $\pi_\xi$ transfers the $C^\infty(T^1_kQ)$-module structure of $(\mathfrak{X}(T^1_kQ), \cdot)$ to $(\mathfrak{X}_\xi, *)$.

We have that $\text{Im } \pi_\xi = \mathfrak{X}_\xi$. Therefore $X \in \mathfrak{X}_\xi$ if and only if $X = \pi_\xi(X)$. Using the above properties of map $\pi_\xi$, a vector field $X$ on $T^1_kQ$ is locally given as in the last part of formula (3.12) if and only if it is given by formula (3.12).

From formula (3.10) it follows that the complete lift $Z^\xi \in \mathfrak{X}(T^1_kQ)$ of a vector field $Z \in \mathfrak{X}(Q)$ is a Newtonoid vector field for an arbitrary SOPDE $\xi$. In the next proposition we will see that the set of Newtonoid vector fields contains also Cartan symmetries.

Proposition 3.8. Consider $L$ a regular Lagrangian on $T^1_kQ$ and $X \in \mathfrak{X}(T^1_kQ)$ a Cartan symmetry of $L$. Then $X$ is a Newtonoid vector field for every $\xi \in \mathfrak{X}_L(T^1_kQ)$.

Proof. Consider $X \in \mathfrak{X}(T^1_kQ)$ a Cartan symmetry of $L$ and $\xi \in \mathfrak{X}_L(T^1_kQ)$. Since $L$ is regular it follows that $\xi$ is a SOPDE. Moreover, $\xi$ is a solution of the equation $i_{\xi^\alpha} \omega_L^\alpha = dE_L$. If we apply $L_X$ to both sides of this equation and use the commutation rules we obtain

$$i_{[\xi_\alpha,X]} \omega_L^\alpha - i_{\xi_\alpha,X} \omega_L^\alpha = dL_X E_L.$$ 

Using now the fact $L_X \omega_L^\alpha = 0$ and $L_X E_L = 0$ it follows that

$$i_{[\xi_\alpha,X]} \omega_L^\alpha = 0.$$ 

We will prove now that equation (3.13) implies that $J^\alpha [\xi_\alpha, X] = 0$ and hence $X$ is a Newtonoid vector field for $\xi$. Using formula (3.10), we have

$$[\xi_\alpha, X] = V^i_\alpha \frac{\partial}{\partial q^i} + V^i_\alpha \frac{\partial}{\partial v^i},$$

where $V^i_\alpha = \xi_\alpha(X^i) - X^i_\alpha$ and $V^i_\alpha = \xi_\alpha(X^i) - X^i_\alpha$. Using formula (3.10), it follows that the $k$-symplectic 2-forms $\omega_L^\alpha$ can be written as follows

$$\omega_L^\alpha = a^\alpha_{ij} dq^i \wedge dq^j + g^\alpha_{ij} dq^i \wedge dv^j,$$
where
\[ a_{ij}^j = \frac{1}{2} \left( \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} - \frac{\partial^2 L}{\partial q^i \partial \dot{q}^i} \right), \quad g_{ij}^\alpha = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}. \]

If we replace now formulae (3.14) and (3.15) in equation (3.13) we obtain
\[ \left( 2a_{ij}^\alpha V_i^j - g_{ij}^{\alpha \beta} V_i^j \right) dq^i + g_{ij}^{\alpha \beta} V_i^\alpha d\dot{q}^j = 0, \]
which implies that \( g_{ij}^{\alpha \beta} V_i^\alpha = 0 \). Using the fact that the Lagrangian \( L \) is regular it follows that \( g_{ij}^{\alpha \beta} \) has maximal rank and hence \( V_i^\alpha = \xi_\alpha (X^i) - X_i^\alpha = 0 \), which shows that \( X \) is a Newtonoid vector field for \( \xi \).

3.3. **Noether’s Theorem.** For the \( k = 1 \) case it is well known that Cartan symmetries induce and are induced by constants of motion, and these results are known as Noether’s Theorem and its converse. For \( k > 1 \), Noether’s Theorem is also true, each Cartan symmetry induces a conservation law, see Theorem 3.9. However, its converse may not be true. In Proposition 3.11 we discuss when this is the case.

The following theorem is proved in [47, Thm 3.13] using local coordinates. Here we give a direct proof of Noether’s Theorem using the Frölicher-Nijenhuis formalism on \( T_k^1 Q \). This proof will allow us to discuss also when the converse of Noether’s Theorem is true, for \( k > 1 \). To show that there are cases where the converse of Noether’s Theorem is not true, we provide examples of conservation laws that are not induced by Cartan symmetries.

**Theorem 3.9. (Noether’s Theorem)** Consider \( L \) a Lagrangian on \( T_k^1 Q \) and \( X \in \mathfrak{X}(T_k^1 Q) \) a Cartan symmetry for \( L \). Then, there exists (locally defined) functions \( g^\alpha \) on \( T_k^1 Q \) such that
\[ L_X \theta^\alpha_L = dg^\alpha \]
and the following functions
\[ f^\alpha = \theta^\alpha_L(X) - g^\alpha \]
give a conservation law for the Euler-Lagrange equations.

**Proof.** Since \( X \) is a Cartan symmetry for \( L \) it follows that \( L_X \omega^\alpha_L = 0 \) and hence the 1-forms \( L_X \theta^\alpha_L \) are closed. Locally, on \( T_k^1 Q \), one can find function \( g^\alpha \) such that \( L_X \theta^\alpha_L = dg^\alpha \), thus
\[ i_X d\theta^\alpha_L + d_i \theta^\alpha_L = dg^\alpha, \]
or equivalently
\[ i_X \omega^\alpha_L = d(\theta^\alpha_L(X) - g^\alpha). \]

We will show now that functions \( f^\alpha = \theta^\alpha_L(X) - g^\alpha \), in formula (3.17), give a conservation law. We will compute first \( \xi_\alpha(f^\alpha) \).

Using formula (3.17) we have
\[
\xi_\alpha(f^\alpha) = \mathcal{L}_{\xi_\alpha} i_X \theta^\alpha_L - \xi_\alpha(g^\alpha) = i_X L_{\xi_\alpha} d_J \alpha + i_{[\xi_\alpha, X]} d_J \alpha - L_{\xi_\alpha}(g^\alpha)
\]
\[ = i_X dL + i_{[\xi_\alpha, X]} d_J \alpha - L_{\xi_\alpha}(g^\alpha). \]

We apply now \( i_{\xi_\alpha} \) to both terms in formula \( L_X d_J \alpha = dg^\alpha \), sum over \( \alpha \), and obtain
\[
\xi_\alpha(g^\alpha) = i_{\xi_\alpha} dg^\alpha = i_{\xi_\alpha} L_X d_J \alpha = L_X i_{\xi_\alpha} d_J \alpha L + i_{[\xi_\alpha, X]} d_J \alpha L
\]
\[ = L_X C(L) + i_{[\xi_\alpha, X]} d_J \alpha L. \]

If we replace now, \( \xi_\alpha(g^\alpha) \), from the above formula in formula (3.18) we obtain \( \xi_\alpha(f^\alpha) = -L_X (E_L) = 0 \). Therefore we have:
\[ \xi_\alpha(f^\alpha) = 0, \quad i_X \omega^\alpha_L = df^\alpha. \]

Now using Lemma 3.4 it follows that \( f^\alpha \) is a conservation law for \( L \). □
We have seen that if \( X \) is a Cartan symmetry for a Lagrangian \( L \) on \( T^1_k \mathbb{R} \) then the functions \( f^\alpha \in \mathcal{C}^\infty(T^1_k \mathbb{Q}) \), which satisfy the equation \( i_X \omega^\alpha = df^\alpha \), give a conservation law for \( L \). We say that this conservation law \( f^\alpha \) is induced by the Cartan symmetry \( X \). For \( k > 1 \) there are conservation laws that are not induced by Cartan symmetries. Next we provide such an example.

**Example 3.10.**

\( a \) We have seen in Example 3.2 that the functions \( f^\alpha : T^1 \mathbb{R} \to \mathbb{R} \), given by formula (3.2), give a conservation law for the Euler-Lagrange equations (2.24). We will prove now that this conservation law is not induced by a Cartan symmetry, and hence it will show that the converse of Noether’s Theorem 3.9 is not true, unless the assumptions (3.5) are satisfied. Consider \( X \in \mathfrak{X}(T^1_2 \mathbb{R}) \), locally given by

\[
X = Z \frac{\partial}{\partial q} + Z_1 \frac{\partial}{\partial v_1} + Z_2 \frac{\partial}{\partial v_2}
\]

Using formulae (2.20), first equation (3.20), for \( \alpha = 1 \), can be written as follows

\[
i_X \omega^1 = \sigma(Z dv_1 - Z_1 dq) = df^1 = \frac{\partial f^1}{\partial q} dq + \frac{\partial f^1}{\partial v_1} dv_1 + \frac{\partial f^1}{\partial v_2} dv_2.
\]

This implies that \( \frac{\partial f^1}{\partial v_2} = 0 \), which is not true, since in our case \( \frac{\partial f^1}{\partial v_2} = -2 \sigma v_1 \).

\( b \) Consider the homogeneous isotropic 2-dimensional wave equation

(3.19)

\[ u_{tt} - c u_{xx} - c u_{yy} = 0.\]

Let us make the following notations \( t^1 = t, t^2 = x, t^3 = y \) and \( q = u \). The regular Lagrangian function \( L \in \mathcal{C}^\infty(T^1_3 \mathbb{R}) \) for the wave equation (3.19) is

(3.20)

\[ L = \frac{1}{2} ((v_1)^2 - c (v_2)^2 - c (v_3)^2).\]

Each of the following three sets of functions on \( T^1_3 \mathbb{R} \) will give a conservation law for the Lagrangian \( L \) in formula (3.20):

\[
\begin{align*}
f^1(v_1, v_2, v_3) &= (v_1)^2 + c (v_2)^2 + c (v_3)^2, \\
f^2(v_1, v_2, v_3) &= -2 c v_1 v_2, \\
f^3(v_1, v_2, v_3) &= -2 c v_1 v_3.
\end{align*}
\]

\[
\begin{align*}
f^1(v_1, v_2, v_3) &= 2 v_1 v_2, \\
f^2(v_1, v_2, v_3) &= -(v_1)^2 - c (v_2)^2 + c (v_3)^2, \\
f^3(v_1, v_2, v_3) &= -2 c v_2 v_3.
\end{align*}
\]

\[
\begin{align*}
f^1(v_1, v_2, v_3) &= 2 v_1 v_3, \\
f^2(v_1, v_2, v_3) &= -2 c v_2 v_3, \\
f^3(v_1, v_2, v_3) &= -(v_1)^2 + c (v_2)^2 - c (v_3)^2.
\end{align*}
\]

None of these conservation laws are induced by Cartan symmetries.

Theorem 3.9 shows that any Cartan symmetry of a Lagrangian \( L \) induces (locally defined) conservation laws, for \( k \geq 1 \).

For the case \( k = 1 \) the converse of this theorem is also true: any conservation law of a Lagrangian is induced by a Cartan symmetry.

In the case \( k > 1 \) such result is not true anymore, unless we require some extra assumptions. As we have already seen in Example 3.10 there are examples of conservation laws for some Lagrangians, that are not induced by any Cartan symmetries.

Part of the next proposition will show when conservation laws for a Lagrangian are induced by Cartan symmetries.

**Proposition 3.11.** Consider \( L \in \mathcal{C}^\infty(T^1_k \mathbb{Q}) \) a Lagrangian, functions \( f^1, \ldots, f^k \in \mathcal{C}^\infty(T^1_k \mathbb{Q}) \), and a vector field \( X \in \mathfrak{X}(T^1_k \mathbb{Q}) \) such that equations (3.20) are satisfied. Then \( f^\alpha \) is a conservation law for \( L \) if and only if \( X \) is a Cartan symmetry.

**Proof.** In view of Lemma 3.4 we will have to prove that \( X \) is a Cartan symmetry if and only if \( \xi_\alpha(f^\alpha) = 0 \) for all integrable SOPDE \( \xi \in \mathfrak{X}^1_k(T^1_k \mathbb{Q}) \).
Using formula \( (3.25) \), and the fact that \( \xi \in \mathfrak{X}_k^1(T^1_kQ) \), we have
\[
(3.21) \quad X(E_L) = i_XdE_L = -i_Xi_{\xi^a}\omega^a = i_{\xi,}\omega^a = i_{\xi,}df^a = \xi_{\alpha}(f^a).
\]
From formula \( (3.25) \) it follows that \( \mathcal{L}X\omega^a = 0 \). Therefore, \( X \) is a Cartan symmetry if and only if \( X(E_L) = 0 \) and, in view of formula \( (3.21) \), this is equivalent to \( \xi_{\alpha}(f^a) = 0 \). \( \square \)

For the case \( k = 1 \), the regularity condition of the Lagrangian implies that the Poincaré-Cartan 2-form \( \omega \) is a symplectic form and hence equation \( (3.5) \) always has a unique solution. In the case \( k > 1 \), for some given functions \( f^\alpha \in C^\infty(T^1_kQ) \), the system \( (3.5) \) is overdetermined and it may not have solutions \( X \in \mathfrak{X}(T^1_kQ) \). We will provide examples when this is the case.

**Example 3.12.**

1) Let us consider the following Lagrangians \( L : T^1_2\mathbb{R} \to \mathbb{R} \):
\[
(a) \quad L(q, v_1, v_2) = \frac{1}{2}(\sigma v_1^2 - \tau v_2^2), \quad (b) \quad L(q, v_1, v_2) = \sqrt{1 + (v_1)^2 + (v_2)^2}.
\]

The vector field \( X = \partial/\partial q \) on \( T^1_2Q \) is a Cartan symmetry for both Lagrangians and the corresponding conservation laws are
\[
(a) \quad f^1 = \sigma v_1, \quad f^2 = -\tau v_2, \quad (b) \quad f^1 = \frac{v_1}{\sqrt{1 + (v_1)^2 + (v_2)^2}}, \quad f^2 = \frac{v_2}{\sqrt{1 + (v_1)^2 + (v_2)^2}}.
\]

The above Lagrangians correspond to the vibrating string equations and the equation of minimal surfaces, respectively, see [40, 45].

2) For the Lagrangian \( L : T^1_3\mathbb{R} \to \mathbb{R} \) defined by
\[
L(q, v_1, v_2, v_3) = \frac{1}{2}((v_1)^2 + (v_2)^2 + (v_3)^2),
\]
the vector field \( X = \partial/\partial q \) is a Cartan symmetry, and the induced conservation law is
\[
f^1(v_1) = v_1, \quad f^2(v_2) = v_2, \quad f^3(v_3) = v_3.
\]
The Euler-Lagrange equations corresponding to \( L \) are the Laplace equations.

3) For the Lagrangian \( L : T^2_2\mathbb{R}^2 \to \mathbb{R} \) defined by
\[
L(q^1, q^2, v_1^1, v_1^2, v_2^1, v_2^2) = \left( \frac{1}{2} \lambda + \nu \right) \left[ (v_1^1)^2 + (v_2^1)^2 \right] + \frac{1}{2} \nu \left[ (v_1^2)^2 + (v_2^2)^2 \right] + (\lambda + \nu)v_1^1 v_2^2,
\]
the vector field \( X = \partial/\partial q^1 + \partial/\partial q^2 \) is again a Cartan symmetry. The induced conservation law is
\[
f^1 = (\lambda + 2\nu)v_1^1 + \nu v_1^2 + (\lambda + \nu)v_2^2, \quad f^2 = (\lambda + \nu)v_1^1 + \nu v_1^2 + (\lambda + 2\nu)v_2^2.
\]
The Euler-Lagrange equations corresponding to \( L \) are the Navier equations, see [40, 45].

In Proposition 3.8 we have seen that Cartan symmetries are Newtonoid vector fields. Next theorem shows that under some assumptions Newtonoid vector fields provide Cartan symmetries and hence conservation laws. This theorem generalizes the result obtained in the case \( k = 1 \) by Marmo and Mukunda [36] for regular Lagrangians.

**Theorem 3.13.** Consider \( L \) a regular Lagrangian on \( T^1_kQ \). We assume that there exists \( X \in \mathfrak{X}(T^1_kQ) \) and \( g^\alpha \in C^\infty(T^1_kQ) \) such that
\[
(3.22) \quad \pi_\xi(X)(L) = \xi_\alpha(g^\alpha), \forall \text{ SOPDE} \ \xi_\alpha.
\]
Then, it follows:

1) If \( \xi \in \mathfrak{X}_k^1(T^1_kQ) \) we have that \( \pi_\xi(X) \) is a Cartan symmetry for \( L \).

2) The functions \( f^\alpha = \theta^\alpha_k(X) - g^\alpha \) give a conservation law for \( L \).
Proof. 1) We have to prove that the following two conditions are satisfied

\[ \mathcal{L}_{\pi_\xi(X)} \omega^L_\alpha = 0, \quad (b) \quad \mathcal{L}_{\pi_\xi(X)} E_L = 0. \]

(a) For each \( \alpha \in \{1, ..., k\} \) we denote the 1-forms

\[ \eta^\alpha = \mathcal{L}_{\pi_\xi(X)} \theta^L_\alpha - dg^\alpha. \]

First we show that

\[ i_{V_\alpha} \eta^\alpha = 0, \quad L_{\xi_\alpha} \eta^\alpha (V) = 0 \]

for arbitrary vertical vector fields \( V, V_1, V_2, ..., V_k \). First condition above is equivalent to the fact that \( \eta^\alpha = \eta^\alpha dq^i \) are semi-basic 1-forms. Moreover, using the fact that \( \mathcal{L}_{\xi_\alpha} \eta^\alpha = \xi_\alpha (\eta^\alpha) dq^i + \eta^\alpha dv_\alpha^i \), it follows that the second condition will imply \( \eta^\alpha = 0 \).

Let \( \xi_\alpha \in \mathfrak{X}(T^1_kQ) \) be a sopde and \( \nu_\alpha \) be vertical vector fields. It follows that \( \xi'_\alpha = \xi_\alpha - \nu_\alpha \) are also sopdes. Using the fact that \( \theta^L_\alpha \) are semi-basic 1-forms and \( \nu_\alpha \) are vertical vector fields, we have that \( i_{\nu_\alpha} \theta^L_\alpha = 0 \). Therefore, making use of the corresponding commutation rules, we have

\[ i_{V_\alpha} \eta^\alpha = i_{V_\alpha} \mathcal{L}_{\pi_\xi(X)} \theta^L_\alpha - i_{V_\alpha} dg^\alpha = i_{V_\alpha} \mathcal{L}_{\pi_\xi(X)} \theta^L_\alpha - L_{\pi_\xi(X)} i_{V_\alpha} \theta^L_\alpha - i_{V_\alpha} dg^\alpha = i_{[V_\alpha, \pi_\xi(X)]} \theta^L_\alpha - i_{[V_\alpha, \pi_\xi(X)]} dL - i_{V_\alpha} dg^\alpha = \mathcal{L}_{\xi_\alpha} \theta^L_\alpha - \mathcal{L}_{\xi_\alpha} dg^\alpha = \xi_\alpha (g^\alpha) - \xi'_\alpha (g^\alpha) - \nu_\alpha (g^\alpha) = 0. \]

In the above calculations we used the fact that \( \pi_\xi(X) - \pi_\xi(X) = J^\alpha [\nu_\alpha, \pi_\xi(X)] \) and the fact that the sopdes \( \xi \) and \( \xi' \) satisfy the hypothesis \( 3.22 \).

We fix now \( \xi \in X^1(T^1_kQ) \), and since \( L \) is regular this means that \( \mathcal{L}_{\xi_\alpha} \theta^L_\alpha = dL \). Using the notation \( 3.22 \) we have

\[ \mathcal{L}_{\xi_\alpha} \eta^\alpha = \mathcal{L}_{\xi_\alpha} \mathcal{L}_{\pi_\xi(X)} \theta^L_\alpha - \mathcal{L}_{\xi_\alpha} dg^\alpha = \mathcal{L}_{\xi_\alpha} \pi_\xi(X) \theta^L_\alpha + \mathcal{L}_{\xi_\alpha} \mathcal{L}_{\pi_\xi(X)} \xi_\alpha \theta^L_\alpha - \mathcal{L}_{\xi_\alpha} dg^\alpha = i_{[\xi_\alpha, \pi_\xi(X)]} \omega^L_\alpha + \mathcal{L}_{\pi_\xi(X)} dL - d\xi_\alpha (g^\alpha) = i_{[\xi_\alpha, \pi_\xi(X)]} \omega^L_\alpha. \]

Using the fact that \( [\xi_\alpha, \pi_\xi(X)] \) is a vertical vector fields, and the \( k \)-symplectic structure in formula \( 2.5 \) vanishes on pairs of vertical vector fields, it follows that for an arbitrary vertical vector field \( V \) we have

\[ \mathcal{L}_{\xi_\alpha} \eta^\alpha (V) = \omega^L_\alpha ([\xi_\alpha, \pi_\xi(X)] , V) = 0. \]

Hence, we proved that \( \eta^\alpha = 0 \), which means that

\[ \mathcal{L}_{\pi_\xi(X)} \theta^L_\alpha = dg^\alpha. \]

If we take the exterior derivative in the above formula it follows that \( \mathcal{L}_{\pi_\xi(X)} \omega^L_\alpha = 0 \).

(b) In order to prove that \( \pi_\xi(X) \) is a Cartan symmetry it remains to show that \( \pi_\xi(X)(E_L) = 0 \). For this we use the fact that \( C = J^\alpha (\xi_\alpha) \) and hence \( C(L) = i_C dL = i_{\xi_\alpha} \theta^L_\alpha \). Therefore,

\[ \pi_\xi(X)(E_L) = \pi_\xi(X)(C(L)) - \pi_\xi(X)(L) = \mathcal{L}_{\pi_\xi(X)} i_{\xi_\alpha} \theta^L_\alpha - \mathcal{L}_{\pi_\xi(X)} L = i_{\xi_\alpha} (\mathcal{L}_{\pi_\xi(X)} \theta^L_\alpha - dg^\alpha) = 0. \]

2) So far we have proved that \( \pi_\xi(X) \) is a Cartan symmetry and it satisfies formula \( 3.24 \). Using the fact \( J^\alpha \circ \pi_\xi = J^\alpha \) and Noether’s theorem \( 3.39 \) it follows that the functions \( f^\alpha = \theta^L_\alpha (\pi_\xi(X)) - g^\alpha = \theta^L_\alpha (X) - g^\alpha \) give a conservation law for \( L \).

Theorem 3.13 extends the results in Corollary 3.15 from [17]. Indeed if \( X = Z^C \) for some \( Z \in \mathfrak{X}(M) \) and \( g^\alpha \in C^\infty (M) \) the condition \( 3.22 \) becomes \( Z^C(L) = v_0^i \partial g^\alpha / \partial q^i \). It follows that \( Z^C \) is a Cartan symmetry and the functions \( f^\alpha = Z^C (L) - g^\alpha \) define a conservation law.
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