Finite Temperature Scalar Potential
from a $1/N$ Expansion

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Abstract

We compute the leading and next-to-leading corrections to the finite temperature scalar potential for a 3+1 dimensional $\phi^4$ theory using a systematic $1/N$ expansion. Our approach automatically avoids problems associated with infrared divergences in ordinary perturbation theory in $\hbar$. The leading order result does not admit a first order phase transition. The subleading result shows that the exact theory can admit at best only a very weak first order phase transition. For $N = 4$ and weak scalar coupling we find that $T_1$, the temperature at which tunneling from the origin may begin in the case of a first order transition, must be less than about 0.5 percent larger than $T_2$, the temperature at which the origin changes from being a local minimum to being a local maximum. We compare our results to the effective potential found from a sum of daisy graphs.
1. Introduction.

There has been much recent interest in the nature of the electroweak phase transition motivated by the possibility of baryogenesis within the standard model itself. As noted by Kirzhnits and Linde [1], Weinberg [2] and Dolan and Jackiw [3] over 18 years ago, a spontaneously broken field theory may have its symmetry restored at high enough temperature. For example, in a spontaneously broken $\phi^4$ theory, temperature corrections give a positive mass–squared contribution at the origin ($\phi = 0$), and at high enough temperature this correction results in a global symmetry–unbroken minimum at the origin.

In such a model there are two important temperatures as we lower the temperature from a very high value. The first, $T_1$, is the temperature at which a second possible minimum appears degenerate in energy with the minimum at the origin. The second, $T_2$, is the temperature at which the effective mass at the origin vanishes, i.e. when the origin changes from being a local minimum to a local maximum.

The phase transition from the symmetric to nonsymmetric phase as we cool a system described by such a model may therefore proceed in two ways, by tunneling when the temperature is between $T_1$ and $T_2$, or by a rollover when the temperature drops below $T_2$. At $T = T_1$, isolated bubbles of the symmetry broken phase will start to be created by tunneling. If the phase transition completes by bubble nucleation we will call the transition first order, otherwise if it completes mostly by a rollover after $T < T_2$ then we will call the transition second order. If the finite temperature scalar potential possesses only the local minimum at $\phi^2 = 0$ until $T = T_2$ then there is no $T_1$ and the phase transition is necessarily second order. Even if there is a $T_1$ we will say only that the system admits a first order transition since if $T_1$ is close enough to $T_2$ we expect the phase transition to complete by a rollover.

In the context of the standard model, it has been suggested that sufficient baryogenesis may occur [4,5,6] if the phase transition is first order. Unfortunately, an accurate determination of the nature of the phase transition has proven difficult in this case. One needs a reliable determination of the temperature dependent effective potential of the scalars near the origin for temperatures between $T_1$ and $T_2$. One–loop calculations suggest that baryogenesis can only occur in the minimal standard model for Higgs mass $M_H < 45 \text{ GeV}$ [7,8], a possibility that appears ruled out by experiment [9]. However, as noted by both Weinberg [2] and Dolan and Jackiw [3], naive finite temperature perturbation theory for even the four dimensional $\phi^4$ model suffers from infrared divergences. These divergences invalidate the one–loop approximation. In particular, at temperatures close to $T_2$ perturbation theory in $\hbar$ begins to break down. The reason for this can be easily understood: even the one–loop correction drastically modifies the tree level potential; it is clearly not a small perturbation. Due to such problems, the one–loop result in pure $\phi^4$ theory gives a complex potential for small values of $\phi^2$ and cannot be used to study the nature of the phase transition.

In the standard model, one may argue that if the phase transition completes “well before” $T_2$ then the one–loop calculation can be used. However, in the case of the standard
model the phase transition is expected to be weakly first order, if it is first order at all. In addition, it is not \textit{a priori} clear that resumming some infinite class of diagrams does not rule out a first order transition. The task then is to extract the leading corrections near $T = T_2$ to all orders in perturbation theory. For example, Dolan and Jackiw summed a class of diagrams, the “super–daisies” to circumvent the infrared divergence problem in pure $\phi^4$ theory (with $N$ scalars) to get a reliable estimate of $T_2$. Such diagrams correspond to iterating the daisy graphs

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\end{align*}
$$

so that each petal has an arbitrary number of petals and so on. Such a summation corresponds to the leading term in a $1/N$ expansion and greatly modifies the $1$–loop estimate.

In this article we will use the systematic $1/N$ expansion method to calculate the leading and next–to–leading corrections to the finite temperature effective scalar potential near the origin in a weakly coupled $\phi^4$ theory in four dimensions. In the case of the standard model, one also needs to incorporate corrections to the scalar potential from gauge loops. We do not address this problem here, but will later comment on the applicability of our methodology to a gauged $\phi^4$ theory. In section two we review the $1/N$ expansion for a $\phi^4$ theory and present our computations. In section three we discuss our results. Qualitatively, we find that the leading order in $N$ correction does not admit a first order transition. The next–to–leading may admit a first order transition, but an extremely weak one. We stress that this approach automatically avoids the infrared divergence problem of ordinary perturbation theory. In particular, for the range of $\phi^2$ and $T$ we are interested in the potential does not become complex.

In the context of the standard model there have been several recent papers on improving the $1$–loop estimate [10,11,12]. These attempts mainly involve summing the daisy diagrams. The simplest procedure [13] has been to use the $1$–loop generated temperature dependent mass to calculate $1$–loop corrections. In pure $\phi^4$ theory, if properly done, this can be interpreted as summing the daisy diagrams. This procedure has its roots in the work of Weinberg [2]. It is useful to briefly discuss this procedure, and at the same time introduce some necessary formulas, before presenting our results.

To understand the meaning of such a procedure, it is necessary to recall an important formula in the background field method. To be precise we consider a simple $O(N)$ scalar field theory with action ($i = 1 \ldots N$)

$$
S[\phi] = \frac{1}{2} \int \delta^{ij} \partial_\mu \phi^i \partial^\mu \phi^j - \frac{\lambda}{4!} \int (\phi^2 - v^2)^2.
$$

(2)
One calculates [14]

\[
\Gamma[\phi] = -i\hbar \ln \int [D\hat{\phi}] \exp \left( \frac{i}{\hbar} \left( S[\phi + \hat{\phi}] - \frac{\delta S[\phi]}{\delta \phi^i} \hat{\phi}^i \right) \right) \\
= S[\phi] - i\hbar \ln \int [D\hat{\phi}] \exp \left( \frac{i}{\hbar} \int \left( -\frac{1}{2} \hat{\phi}^i \Delta_{ij}^{-1} \hat{\phi}^j - \frac{\lambda}{3!} \phi_k \hat{\phi}^k \hat{\phi}^2 - \frac{\lambda}{4!} (\hat{\phi}^2)^2 \right) \right). \tag{3}
\]

The first term on the RHS of the last line is the classical action. The inverse scalar propagator is

\[
\Delta_{ij}^{-1} = [\delta_{ij} \partial^2 + M_{ij}^2(\phi)]. \tag{4}
\]

\(M_{ij}^2\) is the second derivative of the classical potential w.r.t. the fields \(\phi^i, \ M_{ij}^2 = \partial_i \partial_j V = (\lambda/6)(\delta_{ij}(\phi^2 - v^2) + 2\phi_i \phi_j)\).

Equation (3) is an effective action in that it incorporates the effects of all one–particle–irreducible (1PI) diagrams and can be given to any order in perturbation theory. It may be evaluated in perturbation theory as

\[
\Gamma[\phi] = S[\phi] + \frac{1}{2} i\hbar \text{Tr} \ln (\delta^k \Delta_{ij}^{-1}) + \bar{\Gamma}[\phi], \tag{5}
\]

The second term is the familiar one–loop contribution and the rest contains all higher loop corrections from 1PI graphs. The one-loop corrected potential is

\[
\tilde{V}(\phi) = V(\phi) - \frac{1}{2} i \int \text{Tr} \ln \left[ p^2 \delta_{ij} - M_{ij}^2 \right], \tag{6}
\]

where the trace is over internal indices and for zero temperature the integration measure over momentum is \(d^4p/(2\pi)^4\). For finite \(T\), in the imaginary time formalism, the integral over four momentum goes over to an integral over three momentum and a discrete sum in the manner given in [3]. We will assume the reader is familiar with finite temperature field theory.

We now return to problem of understanding what it means to use the one–loop corrected mass to calculate one–loop corrections. If we iterate the one–loop result once we obtain

\[
\tilde{V}(\phi) = V(\phi) - \frac{1}{2} i \int \text{Tr} \ln \left[ p^2 \delta_{ij} - M_{ij}^2 + \frac{1}{2} i \int \partial_i \partial_j \text{Tr} \ln \left[ k^2 - M_{kl}^2 \right] \right]. \tag{7}
\]

If we keep only the contributions from (7) when the \(\partial_i \partial_j\) both act on the same mass–squared matrix we obtain, by carrying out the differentiation and expanding the log,

\[
\tilde{V} = \tilde{V} + i \sum_{n=1}^{\infty} \text{Tr} \int \frac{1}{n} \left[ i[p^2 - M^2]_{ij}^{-1} \int \frac{\lambda}{6} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) [k^2 - M^2]_{kl}^{-1} \right]^n. \tag{8}
\]

Here the term inside the square brackets is understood as a matrix with two indices; the trace is over powers of this matrix. It may be verified that the RHS is the tree–level plus
contributions from the daisy graphs of (1), with the correct combinatorics except for the two loop graph \((n=1)\) which is over counted. The correct factor for the two–loop graph is half the one appearing from this procedure \([14]\). In the large \(N\) limit if we keep only the leading order in \(N\) and account for the two loop difference, this will reproduce the result of Dolan and Jackiw \([3]\) for the finite temperature daisy sum correction to the effective mass at the origin:

\[
\frac{\partial\tilde{V}}{\partial\phi^2} = \frac{\partial\tilde{V}}{\partial\phi^2} + \frac{iN}{2} \sum_{n=1}^{\infty} \int_{p} \frac{1}{n} \left( \int_{k} [k^2 - \frac{\lambda}{6} (\phi^2 - v^2)]^{-1} \right)^n \frac{\partial}{\partial\phi^2} \left( \frac{iN\lambda}{6} [p^2 - \frac{\lambda}{6} (\phi^2 - v^2)]^{-1} \right)^n .
\]

(9)

Following \([3]\), one keeps only the most infrared divergent contributions. This amounts to letting the \(\partial/\partial\phi^2\) act only on \([p^2 - \frac{\lambda}{6} (\phi^2 - v^2)]^{-n}\) for \(n \geq 2\). For the two loop case it may act on either \([p^2 - \frac{\lambda}{6} (\phi^2 - v^2)]^{-1}\) or \([k^2 - \frac{\lambda}{6} (\phi^2 - v^2)]^{-1}\) which again explains why this case is different. As shown in \([3]\), however, the daisy graphs do not exhaust all dominant \(N\) contributions. For this one needs to sum the superdaisy graphs. Therefore, even for large \(N\), this approximation scheme is not complete. Furthermore, in (7) there are additional corrections when the \(\partial_i\) and \(\partial_j\) both act on different mass–squared matrices. These corrections cannot be written as a sum of 1PI diagrams. Therefore a sensible approximation scheme can only be achieved by ignoring these corrections (which should in any case be subleading in \(N\)). In the last section we will compare our \(1/N\) expansion result with the leading \(O(N)\) correction from using an effective temperature dependent mass.

2. \(\phi^4\) theory to subleading order.

The systematic \(1/N\) expansion allows us to calculate (3) as a perturbation in \(1/N\) near \(\phi = 0\) \([15,16]\). Root \([16]\) has evaluated the leading and next–to–leading corrections to the zero temperature scalar potential in 4,3,2 and 1 dimensions. The procedure at finite temperature is very similar.

At very high temperature, our results for four dimensions should be essentially those of a three dimensional euclidean field theory with a dimensionful \(\phi^4\) coupling. In three dimensions and zero temperature, the leading \(O(N)\) potential has long been known \([15]\). It has exactly the same form as the sum of finite temperature superdaisy graphs that were computed by Dolan and Jackiw \([3]\) for a four dimensional \(\phi^4\) theory. An important point in our approach is that the effect of introducing an auxiliary field \(\sigma\) is to shift the \(\phi\) mass term and as a result there are no infrared divergences in this formalism.

To proceed, we first set \(\lambda = f/N\). The \(1/N\) expansion assumes \(f\) is fixed as \(N\) increases, not \(\lambda\). Then by introducing a dimension two auxiliary field \(\sigma\) we rewrite the
The auxiliary field has eliminated the $\phi^4$ term; the original form of (3) is easily recovered by use of the equation of motion for $\sigma$.

To calculate the effective potential $V(\phi)$ one proceeds as follows. First, using the background field method one computes the effective potential as a function of backgrounds of $\phi$ and $\sigma$. Then, the background of $\sigma$ is eliminated by its equation of motion.

The systematic $1/N$ expansion is performed by expanding the action $S[\phi, \sigma]$ about backgrounds $\phi$ and $\chi$ thus:

$$\phi^i \to \sqrt{N}\phi^i + \hat{\phi}^i, \quad \sigma \to \chi + \frac{\hat{\sigma}}{\sqrt{N}}. \quad (11)$$

The factors of $\sqrt{N}$ have been inserted with hindsight. In particular, the rescaling of the background $\phi$ is necessary in order to ensure that the effective potential is renormalizable to each order in the $1/N$ expansion. This will be seen later in the explicit calculations.

One now writes a formula similar to (3), with an integral over $\hat{\phi}$ and $\hat{\sigma}$:

$$\Gamma[\phi, \chi] = S[\sqrt{N}\phi, \chi] - i\hbar \ln \int [D\hat{\phi}] [D\hat{\sigma}] \exp \left( \frac{i}{\hbar} \int \left( -\frac{1}{2} \hat{\phi}^i \Delta^{-1}_{ij} \hat{\phi}^j - \frac{\hat{\phi}^2 \hat{\sigma}}{2\sqrt{N}} - \phi^i \phi^i \sigma + \frac{3\hat{\sigma}^2}{2f} \right) \right). \quad (12)$$

The factor of $\hbar$ is useful in order to compare results in the $1/N$ expansion to those from the usual perturbation theory in $\hbar$; for practical purposes we will take $\hbar = 1$. All terms linear in the quantum fields have been discarded, and we have defined

$$\Delta^{-1}_{ij} = \delta_{ij}(\partial^2 + \chi). \quad (13)$$

The integral over $\hat{\phi}$ is gaussian and can (formally) be performed exactly to give the leading $O(N)$ contribution to the effective potential. The integral over over $\hat{\sigma}$ also produces $\hat{\sigma}$ terms which may be expanded in a power series. It is known that the $\hat{\sigma}^2$ terms give rise to the next–to–leading corrections in the $1/N$ expansion, which is all we are interested in. Furthermore, to calculate the effective potential we can take $\phi$ and $\chi$ as space–time constants, but not $\hat{\sigma}$. The integral over $\hat{\phi}$ produces a kinetic term for $\hat{\sigma}$ which must be carefully determined in order to evaluate the next–to–leading corrections.

The $\hat{\phi}\hat{\sigma}$ term in (12) can be rewritten as a term purely in $\hat{\sigma}$ by shifting the variable of integration $\hat{\phi}$. This can easily be achieved to all orders in the $1/N$ expansion, but since
we are only interested in the leading and next-to-leading terms it suffices to write the argument of the exponential in (12) as

\[
\frac{i}{\hbar} \int \left( -\frac{1}{2} \hat{\phi}^i \Delta_{ij}^{-1} \hat{\phi}^j - \frac{1}{2\sqrt{N}} \hat{\phi}^2 + \frac{1}{2} \hat{\phi}^i \Delta_{ij} \hat{\phi}^j + \frac{3}{2} \hat{\phi}^2 \right) .
\] (14)

Using \( \hat{\phi}(x) = -i \hbar \frac{\delta}{\delta J_i(x)} \exp \left( \frac{i}{\hbar} \int J_k \hat{\phi}^k \right) \) evaluated at \( J_i = 0 \), and then rescaling the source and field \( \hat{\phi} \) by \( \frac{1}{\sqrt{\hbar}} \) we obtain for the integral over \( \hat{\phi} \) in (12),

\[
\Gamma[\phi, \chi, \hat{\sigma}] = -i \hbar \ln \int [D\hat{\phi}] \exp \left( \frac{i}{\hbar} \int \left( -\frac{1}{2} \hat{\phi}^i \Delta_{ij}^{-1} \hat{\phi}^j - \frac{1}{2\sqrt{N}} \hat{\phi}^2 \right) \right)
\] = \( \frac{1}{2} i \hbar \text{Tr} \ln \Delta_{ij}^{-1} - i \hbar \ln \left[ \exp \left( \frac{i}{\hbar} S_I \right) \right] \exp \left( -\frac{i}{2} \int J_i \Delta_{ij} \Delta_{ij} \right) \), (15)

evaluated at \( J = 0 \), where

\[
S_I[\hat{\phi}] = -\frac{1}{2\sqrt{N}} \int \hat{\sigma} \hat{\phi}^2 .
\] (16)

Of course, eq. (15) does nothing more than define a perturbative expansion with a scalar propagator \( \Delta_{ij} \). The first term in (13) is a one-loop result. It has been evaluated many times before and is given by [3]

\[
0 \equiv \frac{i}{2} \hbar \int_{x,p} \text{Tr} \ln \left[ p^2 - \delta_{ij} \chi \right] = -\int_x V_0 + V_T .
\] (17)

\( V_0 \) is the zero temperature result which is divergent and must be regulated. With a sharp momentum cutoff \( \Lambda \),

\[
V_0 = \frac{N}{32\pi^2} \left[ \frac{1}{2} \chi^2 \ln[\chi/\Lambda^2] - \frac{1}{4} \chi^2 + \chi \Lambda^2 \right] .
\] (18)

\( V_T \) is the finite, temperature dependent, result:

\[
V_T = N \left[ -\frac{\pi^2}{90\beta^4} + \frac{\chi}{24\beta^2} - \frac{\chi^2}{12\pi^2} - \frac{1}{64\pi^2} \chi^2 \ln[\chi/\beta^2] + \frac{c}{64\pi^2} \chi^2 \right] .
\] (19)

We have dropped \( O(\beta^2) \) terms and higher, and \( c \approx 5.41 \). This result is valid at high enough temperatures. In the electroweak model, it is known [8] that the high temperature expansion is well justified at temperatures relevant for the study of the phase transition.

The \( O(\hat{\sigma}^2) \) correction is a 1–loop contribution as well. Extracting the 1PI part from (15) we obtain to leading order

\[
0 \equiv -\frac{i}{4} \hbar \int_{x,p} \frac{1}{p^2 - \chi} \hat{\sigma} \frac{1}{(p+i\partial)^2} \hat{\sigma} .
\] (20)
In deriving this result we took the functional derivatives in (13), Fourier transformed the resulting $\delta$–functions and dropped all total divergences. We further used properties of the translation operators: $\exp(-ip \cdot x) H(-i\partial_p) \exp(ip \cdot x) = H(x - i\partial_p) = \exp(-i\partial_x \cdot \partial_p) H(x) \exp(i\partial_x \cdot \partial_p)$. Dropping total divergences assumes that the integral is well regulated (in three dimensions it is finite). Eq. (20) may be evaluated using Feynman parameters; the result is

$$
\Gamma[\phi, \chi] = S[\sqrt{N} \phi, \chi] + 1/2 i\hbar \text{Tr} \ln \Delta^{-1}_{ij} + \Gamma_1[\phi, \chi],
$$

where the next–to–leading contribution from the gaussian integral over $\hat{\sigma}$ is

$$
\Gamma_1[\phi, \chi] = 1/2 i\hbar \text{Tr} \ln \left[ 1 + \frac{hf}{24\pi \beta \sqrt{\partial^2}} \sin^{-1} \left( \frac{1}{\sqrt{1 + 4\chi / \partial^2}} \right) + \frac{f}{3} \phi_i \Delta^{ij} \phi_j \right].
$$

We now have all the formalism behind us. Before presenting explicit calculations it is useful to rederive the leading order “mass–gap” equation of [3] to see how the infrared problem is automatically avoided.

To leading order in $N$ one drops the $\hat{\sigma}$ terms and eq. (12) reduces to

$$
\bar{\Gamma}[\phi, \chi] = \int_x N \left( (\partial \phi)^2 + \frac{3N}{2f} \chi^2 - \frac{N}{2} \chi (\phi^2 - v^2) + \frac{1}{2} i\hbar \text{Tr} \ln \Delta^{-1}_{ij} \right)
$$

$$
= N \int_x \left[ \frac{3N}{2f} \chi^2 - \frac{1}{2} \chi (\phi^2 - v^2) + \frac{1}{2} i\hbar \text{Tr} \ln \left[ p^2 - \chi \right] \right].
$$

For convenience, we have rescaled the vev, $v^2 \to Nv^2$ with respect to the last section. The equation of motion for $\chi$,

$$
\frac{\partial \Gamma[\phi, \chi]}{\partial \chi} = 0
$$

(25)
gives the following equation

\[
\chi = \frac{f}{6} (\phi^2 - v^2) - i \hbar \frac{f}{6} \int \frac{1}{p^2 - \chi}. \tag{26}
\]

But \(\chi\) is also just \(-2\partial \tilde{\Gamma}/\partial \phi^2\), which is the radiatively corrected “effective mass” to leading order and is the reason why (26) is called the “mass-gap” equation. It is exactly the result derived in [3] by summing superdaisy graphs, each of which is infrared divergent. The finite temperature, \(T = 1/\beta\), result in four dimensions is [3,15,16]

\[
\chi = \frac{f}{6} (\phi^2 - v^2) + \frac{\hbar f}{6} \left( \frac{1}{12\beta^2} - \frac{\sqrt{\chi}}{4\pi^2} \right) + \ldots, \tag{27}
\]

where the “…” refer to

\[
\left( c - \frac{1}{2} - \ln[\Lambda^2 \beta^2] \right) \frac{\hbar f}{96\pi^2} \chi + \frac{\hbar f \Lambda^2}{96\pi^2}. \tag{28}
\]

This result is the sum of the \(T\) dependent finite part and the \(T = 0\) divergent part. The divergences may be absorbed by renormalizing \(v^2\) and \(f\) and by introducing an arbitrary mass scale \(M\) [15]. Then for \(f \sim O(1)\), and for all reasonable values of \(M\beta\) we can forget the extra terms at high enough temperatures.

To compare, the zero temperature result for the action (2) in three dimensions is [15]

\[
\chi = \frac{f}{6} (\phi^2 - v^2) - \frac{\hbar f}{6} \frac{\sqrt{\chi}}{4\pi}. \tag{29}
\]

Eq. (27) determines \(T_2\), the temperature when the effective mass vanishes at the origin, to leading order in \(N\). Putting \(\chi = 0\) we get \((\hbar = 1)\)

\[
\frac{1}{\beta_2^2} = 12v^2. \tag{30}
\]

Furthermore, at the origin, \(\sqrt{\chi}\) has a remarkably simple solution for \(T\) just above \(T_2\) [3]

\[
\sqrt{\chi} = \frac{2\pi}{3} \left( \frac{1}{\beta} - \frac{1}{\beta_2} \right). \tag{31}
\]

In general, we find for \(\sqrt{\chi}\) the leading result

\[
\sqrt{\chi} = \frac{f}{48\pi \beta} \left[ \sqrt{1 + \frac{32\pi^2}{f} (12\phi^2 \beta^2 - 12v^2 \beta^2 + 1)} - 1 \right]. \tag{32}
\]

We have chosen the sign in the solution (32) of (27) so that \(\sqrt{\chi}\) is positive for \(T\) bigger than \(T_2\). Eq. (32) simplifies in various limits. At \(\phi^2 = 0\), for \(T\) just above \(T_2\) we obtain (31). At \(384\pi^2 \phi^2 \beta_2^2 \gg f\) and \(T\) just above \(T_2\),

\[
\sqrt{\chi} \approx \sqrt{\frac{f \phi^2}{6}}. \tag{33}
\]
Finally, when $T/T_2 - 1 \gg f/(64\pi^2)$ (i.e. $32\pi^2(1 - 12v^2\beta^2)/f \gg 1$) then for all $\phi$ we have the simplification
\[
\sqrt{\phi} \approx \sqrt{\frac{f}{6\pi}} \left[ \frac{1}{12} ((T - T_2)^2 + 2(T - T_2)T_2) + \phi^2. \right]
\] (34)

We now return to the computation of $\Gamma_1$ in the high temperature limit. Some care must be taken to ensure that the final answer contains no temperature dependent divergences [17]. Also, to the order we are working in, it is sufficient to use the solution of (27) for $\chi$ [16]. We will drop all field independent divergences.

We make the replacement $\partial^2 \rightarrow -p^2$ and the trace becomes an integral over space–time and momentum. At finite $T$, $i \int_p \rightarrow -\beta^{-1} \int d^3p^0/(2\pi)^3 \sum_n = -[4\pi^2\beta]^{-1} \int_0^\infty (d\bar{p}^2) \sqrt{\bar{p}^2} \sum_n$ with $-p^2 \rightarrow \bar{p}^2 + 4\pi^2n^2T^2$ in the integrand. The discrete sum over $n$ runs over all integers. One then obtains,
\[
\Gamma_1 = -\frac{h}{2\beta} \int \frac{d^3p^0}{(2\pi)^3} \sum_{n=-\infty}^{+\infty} \ln \left[ 1 + \frac{hf/(24\pi \beta)}{\sqrt{\bar{p}^2 + 4\pi^2n^2T^2}} \sin^{-1}\left( \frac{1}{\sqrt{1 + 4\chi/(\bar{p}^2 + 4\pi^2n^2T^2)}} \right) + \frac{f\phi^2/3}{\bar{p}^2 + 4\pi^2n^2T^2 + \chi} \right].
\] (35)

The integral is not infrared divergent, but is ultraviolet divergent. The ultraviolet divergences should be the same as those appearing if we did the momentum integral over four momenta. [This not quite the same as taking $\beta \rightarrow \infty$ here, because we use the high temperature result for the $\bar{\sigma}$ propagator.] We must isolate these divergences before calculating the temperature dependent part in the high temperature limit. Furthermore, even for say the $n = 0$ contribution we were not able to find an analytic expression for the integral. We can however evaluate it in some limits. [For the rest of this section, we delete overall integrals over space–time.]

Consider the $n = 0$ contribution. We then change variables: $t = \sqrt{4\chi/\bar{p}^2}$. Equation (35) becomes
\[
\Gamma_1(n = 0) = -\frac{(4\chi)^{3/2}}{(2\pi)^2\beta} \frac{h}{t^4} \int_0^\infty \frac{dt}{t} \ln \left[ 1 + \frac{\alpha t \tan^{-1}\left( \frac{1}{t} \right)}{3\chi t^2 + 4} \right],
\] (36)
where we have defined $\alpha = hf/(24\pi \beta \sqrt{4\chi})$ and used the identity $\sin^{-1}(1 + t^2)^{-1/2} = \tan^{-1}t^{-1}$. At $\phi^2=0$, $\alpha$ is small to temperatures just above $T_2$ because from (31) we have $\alpha < 1$ as long as $1 - \beta/\beta_2 > f/(32\pi^2)$ at the origin. In what follows, we will assume small $\alpha$ and small enough $\phi^2/\chi$. In (30), as $t$ ranges from 0 to $\infty$, $t \tan^{-1}t^{-1}$ ranges from 0 to 1. At $t = 1$, $t \tan^{-1}t^{-1} = \pi/4$. Therefore for small $\alpha$, $\alpha t \tan^{-1}t^{-1}$ remains small for all values of $t$ in the integral. Hence, for small $\alpha$ and small $\phi^2$ we can use the simplification $\ln[1 + r] \approx r$ to simplify (30) or (35).
For the moment let us ignore the explicit $\phi^2$ term in $\Gamma_1$. To calculate (35) for small $\alpha$ we note the following observation for the $n \neq 0$ contributions. If $\sqrt{\chi} < \pi T$ then the arcsine possesses an expansion in inverse powers of the momentum. We have,

$$\sin^{-1}\left(\frac{1}{\sqrt{1+4\chi/\bar{p}^2}}\right) = \frac{\pi/2}{\sqrt{\bar{p}^2 + 4\pi^2 n^2 T^2}} - \sum_{l=0}^{\infty} (-1)^l (4\chi)^{l+\frac{1}{2}} \left(2l + 1\right) \left(\bar{p}^2 + 4\pi^2 n^2 T^2\right)^{l+1}. \quad (37)$$

The $n = 0$ contribution unfortunately does not possess such a straightforward expansion for all $\bar{p}^2$. If $\bar{p}^2 > 4\chi$ then the expansion is as above, otherwise it is

$$\sin^{-1}\left(\frac{1}{\sqrt{1+4\chi/\bar{p}^2}}\right) = \frac{1}{\sqrt{\chi}} \sum_{l=0}^{\infty} (-1)^l (\bar{p}^2)^l. \quad (38)$$

We then split up $\Gamma_1$ into four parts: $A$, the contribution from $n = 0$ for $\bar{p}^2 > 4\chi$; $B$, the contribution from $n = 0$ for $\bar{p}^2 < 4\chi$; $C$, the contribution from $n \neq 0$ for $\bar{p}^2 > 4\chi$; $D$, the contribution from $n \neq 0$ for $\bar{p}^2 < 4\chi$. $B$ is found using the expansion (38), whereas all the rest involve the expansion (37). We denote the contributions to $A$, $C$ and $D$ from the first term in (37) by $a$, $c$ and $d$, respectively. We further define $b$ to be a contribution similar to $a$ but with the momentum integral up to $\bar{p}^2 < 4\chi$. Then it can be seen $a + b + c + d$ is field independent and can be dropped, i.e. we evaluate $\Gamma_1 = (A - a) + B + (C - c) + (D - d) - b$. For $B$ we get

$$B = -\frac{(4\chi)^{\frac{3}{2}} \hbar}{(2\pi)^2 \beta} \int_1^\infty \frac{dt}{t^3} \alpha \tan^{-1} \left(\frac{1}{t}\right) = \frac{f\hbar^2 \chi}{84\pi^3 \beta^2}. \quad (39)$$

For $b$ we get

$$b = -\frac{\hbar^2 f}{48\pi \beta^2} \int_0^{\bar{p}^2 = 4\chi} \frac{d^3 \bar{p}}{(2\pi)^3 2\sqrt{\bar{p}^2}} \frac{\pi}{2} = -\frac{\hbar^2 f \chi}{96\pi^2 \beta^2}. \quad (40)$$

We also get

$$(A - a) + (C - c) = \frac{\hbar^2 f}{48\pi \beta} \sum_{l=0}^{\infty} I_{l+1} (-1)^l (4\chi)^{l+\frac{1}{2}} \left(2l + 1\right) \approx \frac{\hbar^2 f \chi}{24\pi \beta} \left(\frac{A^2}{16\pi^2} + \frac{1}{12\beta^2} - \frac{3h^2 f \chi}{56\pi^3 \beta^2}\right) \quad (41)$$

where $I_l$ is given in the appendix (for $\epsilon = \sqrt{4\chi}, \chi = 0$). Furthermore it is straightforward to show that $D - d$ is subleading in temperature and may be dropped.

The $\phi^2$ term from the expansion of the log is more easily evaluated and contributes

$$\Gamma_1 \ni -\frac{\hbar f}{6} \phi^2 I_1 = -\frac{\hbar f}{6} \phi^2 \left(\frac{A^2}{16\pi^2} + \frac{1}{12\beta^2} - \frac{\sqrt{\chi}}{4\pi \beta}\right), \quad (42)$$

where $I_1$ is taken from the appendix ($\epsilon = 0, \chi \neq 0$). If we keep all powers of $\phi^2$, but ignore the arcsine term, we must also include

$$\Gamma_1 \ni \frac{\hbar \chi^{\frac{3}{2}}}{4\pi \beta} \sum_{l=2}^{\infty} \frac{(-f \phi^2 / 6\chi)^l (2l - 5)!!}{l!} \quad (43)$$

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This series converges for all values of $\phi^2/\chi$ we will be interested in.

Each of the $O(\Lambda^2)$ divergent terms in the above equations is not separately renormalizable, however their sum is. This is because for the next–to–leading terms we can use eq. (27) to express $\phi^2 - \hbar \sqrt{\chi}/4\pi\beta = 6\chi/f + v^2 - h(12\beta^2)^{-1} + O(1/N)$. Since the tree-level potential contains a term proportional to $v^2\chi$, we see that the above divergences in $\Gamma_1$ can be absorbed by renormalizing $v^2$ (throughout we have neglected field independent constants). The dominant contributions to $\Gamma_1$ are found by summing the above results; we obtain

$$\Gamma_1 \approx -\hbar \chi \left( \frac{\Lambda^2}{16\pi^2} + \frac{1}{12\beta^2} \right) + \frac{\hbar f \phi^2 \sqrt{\chi}}{24\pi\beta} + \frac{\hbar \chi^\frac{3}{2}}{4\pi\beta} \sum_{l=2}^{\infty} (-f \phi^2/6\chi)^l (2l - 5)!/l!,$$

$$\approx -\hbar \chi \left( \frac{\Lambda^2}{16\pi^2} + \frac{1}{12\beta^2} \right) + \frac{\hbar \chi^\frac{3}{2}}{12\pi\beta} \left[ (1 + f \phi^2/3\chi)^\frac{3}{2} - 1 \right]. \tag{44}$$

Altogether, summing the various partial results we find for the dominant high temperature effective scalar potential from (12), $V(\phi) = -\Gamma[\phi, \chi(\phi)]$, the following renormalized result up to a constant ($\hbar = 1$):

$$V(\phi) = N \left[ \frac{1}{2} \chi(\phi^2 - v^2) - \frac{3}{2f} \chi^2 \right] + (N + 2) \frac{\chi^2}{24\beta^2} - \frac{\chi^\frac{3}{2}}{12\pi\beta} \left[ (1 + f \phi^2/3\chi)^\frac{3}{2} + N - 1 \right]. \tag{45}$$

where $\chi$ is a solution of $\partial V/\partial \chi = 0$. We do not give details of the renormalization which for zero temperature and in 3 and 4 dimensions may be found in [15,16].

We will use these results in the last section of this paper, where we also discuss the range of $\phi^2$ and $T$ for which these results are the most dominant.

3. Discussion.

Our main result is the leading order in $N$ and next–to–leading order expression for the high temperature scalar potential, eq. (13). The tree level potential, with the normalizations of the last section is

$$V(\phi) = \frac{Nf}{4!} (\phi^2 - v^2)^2. \tag{46}$$

Eq. (45) includes loop corrections to this. We assumed $\beta \sqrt{\chi} \ll 1$ and $\beta \sqrt{\chi} \gg f/48\pi \approx f/150$ in obtaining the next–to–leading result. At $T = T_2$ for the leading order result, this gives approximately $100v^2/f \gg \phi^2 \gg fv^2/100$. At $\phi^2 = 0$ we require $T - T_2 \gg fT_2/300$.

To study the nature of the phase transition we may look for zeros of $dV/d\phi^i$. There is always one zero at the origin, $\phi^i = 0$ for all $i$. Away from the origin we can look for zeros of $dV/d\phi^2 = (\partial V/\partial \chi)(\partial \chi/\partial \phi^2) + \partial V/\partial \phi^2 = \partial V/\partial \phi^2$. Since to $O(N)$, $dV/d\phi^2 \propto \chi$,
the leading order result has one such zero away from the origin when \( T < T_2 \) (which is the local minimum at temperatures below \( T_2 \)). Thus, the leading order result does not admit a first order phase transition, for which we need zeros away from the origin for temperatures \( T \geq T_2 \).

At next-to-leading order, the critical temperature \( T_2 \) is modified from the leading order result. \( \partial V/\partial \phi^2 = 0 \) for \( V \) given by (45) still has a solution when \( \chi = 0 \). Writing out \( \partial V/\partial \chi \) at \( \phi^2 = 0 \) and setting \( \chi = 0 \) immediately gives for the temperature \( T_2 \),

\[
\left( 1 + \frac{2}{N} \right) T_2^2 = 12v^2.
\]

(47)

\( T_2 \) has been reduced from its leading order value. \( \partial V/\partial \phi^2 = 0 \) has one further solution at the origin. \( \partial V/\partial \phi^2 = 0 \) gives

\[
\frac{N}{2} \sqrt{\chi} = \frac{f}{24\pi \beta} (1 + f\phi^2/3\chi)^{3/2},
\]

(48)

when \( \sqrt{\chi} \neq 0 \). At \( \phi^2 = 0 \) this has the solution \( \sqrt{\chi} = f/(12N\pi \beta) \). Writing out \( \partial V/\partial \chi = 0 \) and inserting the above expression for \( \sqrt{\chi} \) gives a second (slightly higher) temperature when the effective mass at the origin vanishes. To \( O(1/N) \) it is given by

\[
\left( 1 + \frac{2 - f/4\pi^2}{N} \right) T^2 = 12v^2.
\]

(49)

For \( N = 4, f = 1 \) this is within 0.5 percent of \( T_2 \). Between this temperature and \( T_2 \) the effective mass at the origin is negative! However, since our approximations break down at \( \beta \sqrt{\chi} \) less than about \( f/48\pi \) such a phenomenon is not necessarily physical; here it is an artifact of our approximations.

To ascertain the nature of the phase transition we have to study the shape of the potential near \( T = T_2 \). If a second minimum degenerate in energy with the one at the origin does not appear by temperature \( T_1 > (1 + 5f \times 10^{-3})T_2 \) then we cannot reliably determine if the theory admits a first order phase transition. However, in this case, if the exact theory does admit a first order phase transition then it will be only a very weak one.

The general solution of \( \chi(\phi) \) is complicated. As noted by Root [16], one can use the leading order result for \( \chi(\phi) \) in the next-to-leading potential (44) to find \( V(\phi) \) to \( O(1) \) in the \( 1/N \) expansion. However, (52) can develop an imaginary part for temperatures below the leading order value for \( T_2 \). Therefore, it is preferable to use the next-to-leading order solution for \( \chi(\phi) \). Our results in what follows are correct to \( O(1/N) \), barring the other approximations we made. At \( T \) near \( T_2 \) given by (17) we once again have (53) when \( \phi^2 \gg v^2/300 \). To see this, we write out the equation for \( \chi(\phi) \),

\[
\frac{6\chi}{f} = \phi^2 - v^2 + \frac{1 + 2/N}{12\beta^2} - \frac{\sqrt{\chi}}{4\pi \beta} \\
- \frac{\sqrt{\chi}}{12\pi \beta N} \left[ (1 + f\phi^2/3\chi)^{3/2} - 1 \right] + \frac{f\phi^2}{12\pi \beta \sqrt{\chi}N} (1 + f\phi^2/3\chi)^{3/2}.
\]

(50)
In this put $\beta = \beta_2$ and $\sqrt{\chi} = \sqrt{f \phi^2/6 + \delta}$. Assuming that $\delta$ is small one obtains $\delta \approx -f/48\pi\beta[1 + (\sqrt{3} - 1)/N]$ which can be ignored when $\delta \ll \sqrt{f \phi^2/6}$. We can improve on this estimate for $\sqrt{\chi}$ by noting that when $\sqrt{\chi} \approx \sqrt{f \phi^2/6}$ the last two terms in (50) add to an amount which for large enough $N$ (say $N > 3$) are much smaller than the third last term. Hence we have the approximate solution

$$\sqrt{\chi} = \frac{f}{48\pi\beta} \left[ 1 + \frac{32\pi^2}{f} \left( \frac{12\phi^2\beta^2 - 12v^2\beta^2 + 1 + 2/N}{6} \right) - 1 \right].$$

(51)

This should be valid at temperatures near $T_2$ and $\phi^2$ sufficiently far from the origin.

To see if a first order phase transition is possible we must find the zeros of $dV/d\phi^2 = \partial V/\partial \phi^2$. There is one solution at $\sqrt{\chi} = 0$ and another given by solving (48), which at $T = T_2$ and $\chi \approx f \phi^2/6$ has a solution at $\sqrt{\phi^2} \approx v/\sqrt{3f/[2N(N + 2)]}$. This result holds if $N$ is not too large ($\sqrt{\phi^2}$ is not too small). At $N = 4$ we have $\sqrt{\phi^2} \approx \sqrt{fv}/4$. We can do better numerically. We studied the case $N = 4$, $f = 1$. The approximate solution (51) suggests $T_1 = T_2(1 + 3.5 \times 10^{-3})$, with $\sqrt{\phi^2} \approx 0.08v$ at the second minimum at $T = T_1$. The height of the barrier separating the two minima is at most of $O(2 \times 10^{-7}v^4)$. At $T = T_2$ the minimum is at $\sqrt{\phi^2} \approx 0.15v$, which is slightly smaller than our cruder estimate above.

All the numbers above occur as our approximations break down so we cannot determine the exact nature of the phase transition. However, as argued above we may deduce from them that the exact model can admit only a very weak first order phase transition. To get a more accurate picture one should evaluate (35) in the limit $\beta \sqrt{\chi} \ll f/48\pi$.

To compare these results with the effective mass insertion procedure discussed in the introduction, we note that the leading $O(N)$ one–loop corrections to (46) give the (renormalized) potential [3]

$$V = N \left[ \frac{f}{4!} (\phi^2 - v^2)^2 + \frac{f(\phi^2 - v^2)/6}{24\beta^2} - \frac{(f(\phi^2 - v^2)/6)\beta}{12\pi\beta} \right].$$

(52)

As indicated in the introduction, this is complex for $\phi^2 < v^2$. The leading $O(N)$ one–loop corrected mass is

$$\frac{Nf}{6} \left( \phi^2 - v^2 + \frac{1}{12\beta^2} \right).$$

(53)

Using this to perform the one–loop corrections, one gets

$$V = N \left[ \frac{f}{4!} (\phi^2 - v^2)^2 + \frac{f(\phi^2 - v^2 + 1/12\beta^2)/6}{24\beta^2} - \frac{(f(\phi^2 - v^2 + 1/12\beta^2)/6)\beta}{12\pi\beta} \right].$$

(54)

The effective mass at the origin vanishes at $T_2^2 = 12v^2$, as for the leading $O(N)$ result from the $1/N$ expansion. However, there is a higher temperature at which this also occurs.
Namely, if we put $T = T_2 + t$ then for small $t$ the appropriate equation is easy to solve. The result is that the effective mass at the origin also vanishes at $T \approx (1 + f/144)T_2$. Between this temperature and $T_2$ the effective mass at the origin is negative. Thus compared to our results from the $1/N$ expansion, we see that the effective mass insertion technique is not reliable at temperatures less than about 1 percent above $T_2$ even in the large $N$ limit.

In conclusion, for large enough $N$ the model can admit only a very weak first order phase transition. For $N = 4$ it would be interesting to see how $O(1/N)$ contributions modify the results above. At $N = 4$ the next–to–leading corrections are certainly smaller than the $O(N)$ corrections and we do not expect $O(1/N)$ corrections to compete with the $O(N)$ or $O(1)$ contributions. However, it would be necessary to perform an explicit calculation to check this belief. Finally, in the context of the standard model, one needs to add gauge fields $A$ to this model. In fact, the contributions from gauge loops are expected to dominate over those from scalar loops in the weak scalar self–coupling limit. The main problem in the case of a gauged model is the gauge fixing dependence of the computations. Otherwise, in principle, is should be possible to determine the dominant contributions to the finite temperature scalar potential. For example, for $SU(2)$, the gauge kinetic term has two types of $A^4$ interaction ($[A_{\mu}^i A_{\mu}^i]^2$ and $[A_{\nu}^i A_{\nu}^i]^2$, where $i = 1, 2, 3$) as well as an $A^3$ interaction. Loops from the second $A^4$ interaction and loops from the $A^3$ interaction are subleading to those from the first $A^4$ interaction. If we drop these last two interactions then what remains can be treated very similarly to the pure $\phi^4$ theory. One must then determine what gauge fixing invariant quantities can be extracted from such a computation. It is expected that, if carefully done, physically measurable quantities should be gauge fixing independent. Work along these lines is in progress.

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Appendix.

The finite temperature quantities we need to evaluate in this paper are all of the form

\[ I_l = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int_{p^2=\epsilon^2}^{+\infty} \frac{d^3p}{(2\pi)^3} [p^2 + 4\pi^2 n^2 T^2 + \chi]^{-l} \]

for \( l \) a positive integer. The discrete sum runs over all integer \( n \). We need only the cases \( \epsilon = 0 \) and \( \chi \neq 0 \) or \( \epsilon \neq 0 \) and \( \chi = 0 \). We assume \( \beta \sqrt{\chi} \ll 1 \), but \( \beta \sqrt{\chi} \gg f/(48\pi) \approx f/150 \).

For \( l > 1 \) we can find the answer by differentiating the \( l = 1 \) result \( l - 1 \) times w.r.t. \( \chi \). To evaluate the expressions we follow [3]. The sum over \( n \) is explicitly performed to give

\[ I_1 = \int_{p^2=\epsilon^2}^{+\infty} \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{2\sqrt{p^2 + \chi}} + \frac{1}{\sqrt{p^2 + \chi} \left( \exp(\beta \sqrt{p^2 + \chi}) - 1 \right)} \right]. \]

The first term in this expression is the \( T = 0 \) piece and contains all the ultraviolet divergences; the second term is the \( T \) dependent piece and is finite. Changing variables, the \( T \) dependent piece is

\[ I_1(T \neq 0) = \frac{1}{2\pi^2 \beta^2} \int_{\beta \epsilon}^{+\infty} \frac{z^2 dz}{\sqrt{z^2 + \beta^2 \chi}} \left( \frac{1}{\exp(\sqrt{z^2 + \beta^2 \chi}) - 1} \right). \]

In the high temperature limit, and for \( \epsilon = 0 \), we have the leading result of [3],

\[ I_1(\epsilon = 0) = \frac{\Lambda^2}{16\pi^2} + \frac{\chi}{8\pi^2} \ln \left[ \frac{\sqrt{\chi}}{\Lambda} \right] + \frac{1}{12\beta^2} - \frac{\sqrt{\chi}}{4\pi \beta} + \frac{\chi}{16\pi^2} \left[ c - \frac{1}{2} - 2 \ln[\beta \sqrt{\chi}] \right]. \]

The log pieces combine to give a single term proportional to \( \chi \ln[\beta \Lambda] \). Such log–divergent terms in the final answer must be renormalizable. The coefficient of this term is of \( O(\beta \sqrt{\chi}/(2\pi)) \) down from the preceeding \( O(T) \) term, and in our approximation all \( O(\chi) \) terms can be neglected in comparison to \( O(\sqrt{\chi}/\beta) \) terms. Furthermore, to properly renormalize such terms in the effective potential we should keep subleading corrections to the \( \sigma \) propagator, eq. (21), which we discarded. Therefore it is consistent to drop all the terms proportional to \( \chi \). This is what we do in the following.

For \( \epsilon \neq 0 \) we subtract from the above the temperature dependent piece

\[ \frac{1}{2\pi^2 \beta^2} \int_{0}^{\beta \epsilon} \frac{z^2 dz}{z^2 + \beta^2 \chi} \cdot \frac{\sqrt{z^2 + \beta^2 \chi}}{\exp(\sqrt{z^2 + \beta^2 \chi}) - 1} \approx \frac{1}{2\pi^2 \beta^2} \int_{0}^{\beta \epsilon} \frac{z^2 dz}{z^2 + \beta^2 \chi} \left( 1 - \frac{1}{2} \frac{1}{\sqrt{z^2 + \beta^2 \chi}} \right) \]

\[ \approx \frac{\epsilon}{2\pi^2 \beta} \tan^{-1} \left[ \frac{\epsilon}{\sqrt{\chi}} \right] - \frac{\sqrt{\epsilon^2 + \chi}}{8\pi^2} + \frac{\chi}{8\pi^2} \ln \left[ \frac{\sqrt{\epsilon^2 + \chi + \epsilon}}{\sqrt{\chi}} \right], \]

and the zero temperature piece,

\[ \frac{1}{4\pi^2} \int_{0}^{\epsilon} \frac{z^2 dz}{\sqrt{z^2 + \chi}} = \frac{\epsilon \sqrt{\epsilon^2 + \chi}}{8\pi^2} - \frac{\chi}{8\pi^2} \ln \left[ \frac{\sqrt{\epsilon^2 + \chi + \epsilon}}{\sqrt{\chi}} \right], \]
to arrive at

\[
I_1(\epsilon \neq 0) = \frac{\Lambda^2}{16\pi^2} + \frac{1}{12\beta^2} \frac{\epsilon}{2\pi^2\beta} + \frac{\sqrt{\chi}}{2\pi^2\beta} \left( \tan^{-1} \left( \frac{\epsilon}{2\sqrt{\chi}} \right) - \frac{\pi}{2} \right).
\]  

(61)

In the limit \( \chi \to 0 \) this expression has an expansion in positive integer powers of \( \chi \). Therefore, its \( l \)th derivative w.r.t. \( \chi \) evaluated at \( \chi = 0 \) is well defined. By expanding the arctan in this limit we obtain for \( l \) a positive integer and \( \chi = 0 \),

\[
I_l(\epsilon \neq 0) = \left[ \frac{\Lambda^2}{16\pi^2} + \frac{1}{12\beta^2} \right] \delta_{l-1} + \frac{\epsilon^{3-2l}}{2\pi^2\beta(2l-3)}.
\]  

(62)

\( I_l \) for \( \epsilon = 0 \) and \( \chi \neq 0 \) is simply found by differentiating (61) w.r.t. \( \chi \) the appropriate number of times. The answer is, for \( l > 1 \),

\[
I_l(\chi \neq 0) = \frac{\chi^{\frac{3}{2}-l}}{4\pi\beta} \frac{(2l-5)!!}{2^{l-1}(l-1)!}.
\]  

(63)
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