A Linear Exponential Comonad in s-finite Transition Kernels and
Probabilistic Coherent Spaces

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Abstract

This paper concerns a stochastic construction of probabilistic coherent spaces by employing novel ingredients
(i) linear exponential comonad arising properly in the measure-theory (ii) continuous orthogonality between
measures and measurable functions.

A linear exponential comonad is constructed over a symmetric monoidal category of transition kernels,
relaxing Markov kernels of Panangaden’s stochastic relations into s-finite kernels. The model supports
an orthogonality in terms of an integral between measures and measurable functions, which can be seen
as a continuous extension of Girard-Danos-Ehrhard’s linear duality for probabilistic coherent spaces. The
orthogonality is formulated by a Hyland-Schalk double glueing construction, into which our measure theoretic
monoidal comonad structure is accommodated. As an application to countable measurable spaces, a dagger
compact closed category is obtained, whose double glueing gives rise to the familiar category of probabilistic
coherent spaces.

Keywords: Stochastic Relations, Transition Kernels, Linear Exponential Comonad, Linear Logic,
Orthogonality, Measure Theory, s-finite, Exponential Measurable Space, Double Glueing, Tight
Orthogonality Category, Categorical Model, Probabilistic Denotational Semantics

Introduction

Coherent spaces [20], the original model in which Girard discovered linear logic, provide a denotational
semantics of functional programming languages as well as logical systems. Each space is a set endowed with
a graph structure, called a web, in which a proof (hence a program) is interpreted by a certain subset, called
a clique. The distinctive feature of this model is the linear duality, stating that a clique \( x \subseteq X \) and an
anti-clique \( x' \subseteq X' \) intersect in at most a singleton \( \#(x \cap x') \leq 1 \). The linear duality arising intrinsically to
the coherent spaces goes along with constructive modelling of logical connectives. There arises a dual pair
of multiplicative connectives and of additive ones, together with linear implication for multiplicative closed
structure (i.e., *-autonomy of denotational semantics).

Category theoretically (freely from the web-based method), the coherent spaces are realised by Hyland-
Schalk’s double glueing construction [28] \( G(\text{Rel}) \) over the category of relations \( \text{Rel} \), which is the most primary
self dual denotational semantics with the tensor (the cartesian product of sets) and the biproduct (the
disjoint union of sets). The double glueing lifts the degenerate duality of \( \text{Rel} \) into a nondegenarate one,
called orthogonality, which in turn gives rise to the linear duality so that the coherent spaces reside as an
orthogonal subcategory.

Developing the web method, Ehrhard investigates the linear duality in the mathematically richer structures
of Köthe spaces [11] and finiteness spaces [12]. His investigation of duality leads Danos-Ehrhard [8] to
formulate a probabilistic (fuzzy) version of duality in their probabilistic coherent spaces \( \text{Pcoh} \). Their
construction starts with giving non-negative real valued functions on a web $I$, reminiscent of probabilistic distributions (but not necessarily to the interval $[0, 1]$) on the web, so that a clique becomes a subset of $\mathbb{R}_+^I$. Then, in their probabilistic setting, the linear duality becomes formulated $\langle x, x' \rangle = \sum_{i \in I} x_i x'_i \leq 1$ for $x, x' \in \mathbb{R}_+^I$. The precursor of the formulation is addressed earlier in Girard [23]. The probabilistic linear duality is accommodated into the linear exponential $!$ by generalising the original finite multiset functor construction of Girard [20] with careful analysis of permutations and combinations on enumerating members of multisets. The canonicity of their exponential construction is ensured in [7].

The recent trend of probabilistic semantics is more widely applied to transition systems with continuous state spaces for concurrent systems such as stochastic process calculi. The stochastic relation $\text{SRel}$, explored by Panangaden [34, 33], provides a fundamental categorical ingredient to the study, analogous to how the category $\text{Rel}$ of the relations has been to deterministic discrete systems. Recalling that $\text{Rel}$ is the Kleisli category of the powerset monad, $\text{SRel}$ is a probabilistic analog of the Giry monad [24], whereby powerset is replaced by a probability measure on a set, giving random choice of points, hence collections of fuzzy subsets are obtained. $\text{SRel}$ also provides coalggebraic reasoning for continuous time branching logics [10]. Despite the lack of cartesian closed structure, Markov kernels provide a measure theoretic foundation of recent development of various denotational semantics for higher-order probabilistic computations [17, 40], theoretically with adequacy and practically with continuous distributions for Monte Carlo simulation. We also remark an intermediate approach on the weighted relational model [30] confining the discrete probability but acquiring $*$-autonomy and exponential structure for Linear Logic.

This paper intends to present a general machinery inspired by $\text{SRel}$, amalgamating category and measure theories. This integration leads to the development of two fundamental constructions: (i) linear exponential comonad tailored for stochastic processes and (ii) linear duality and probabilistic orthogonality in continuous spaces. The two parts involve comonad, widely used in computer science, but exploration in its continuous stochastic aspect is initiated just recently by [14, 18, 35] in the higher order probabilistic programming.

Our results for each can be summarized as follows: (i) The counting process [5] in the realm of stochastic processes introduces a novel categorical representation of linear exponential comonad, capturing the exponential modality of linear logic. Specifically, for countable measurable spaces, this approach simplifies the understanding of the exponential structure within $\text{Pcoh}$ by representing it as a discrete collapse of measure-based probability. Furthermore, our linear exponential comonad based on transition kernels can be viewed as a continuous version of the weighted relational model outlined in [30] utilising $\mathbb{R}_+$-weighted $\text{Rel}$ for the analytic exponential. (ii) Within a broader continuous framework, our study of transition kernels offers a new perspective on Hyland-Schalk orthogonality. This perspective provides insight into the continuous extension of linear duality in terms of measures and measurable functions.

It is important to note that both (i) and (ii) do not incorporate any closed structure for monoidal products in the continuous framework, making them inconclusive as a complete model of linear logic.

The paper commences by introducing a stochastic framework involving transition kernels [1] that establishes a category $\text{Tker}$ with biproduct. This framework elucidates the necessity for kernels to encompass the infinite real values $\infty$, especially in the context of transition kernels between measurable spaces and measurable functions, which serve as a relaxation of sub-Markov kernels within Panangaden’s $\text{SRel}$ of subprobability measures.

The monoidal product is straightforwardly derivable through the measure theoretic direct product, similar to $\text{SRel}$. However within the relaxation in our framework, careful examination of the functoriality of the product becomes imperative. In measure theory, the functorial monoidal product is guaranteed by the fundamental Fubini-Tonelli Theorem, where $\sigma$-finiteness [34], including finiteness and subMarkov properties, plays a crucial role. However, category theory presents challenges as $\sigma$-finiteness is not preserved under categorical composition. Although a smaller class of finiteness [4] maintains both categorical composition and functorial monoidalness, it proves insufficient for accommodating the exponential modality as required in this paper. The concept of $s$-finiteness, recently explored in Staton’s work [39], extends the traditional $\sigma$-finiteness to preserve composition while upholding Fubini-Tonelli for functorial monoidal product. We demonstrate that $s$-finiteness also facilitates an exponential construction, both in terms of measure theory and category theory. The exponential construction is characterised through “counting measures” [5] in the
realm of stochastic processes, where the counting function for multisets becomes measurable. In our study, we establish an exponential endofunctor within the monoidal category $\text{TsKer}$, consisting of $s$-finite transition kernels. Furthermore, we devise a linear exponential comonad in $\text{TsKer}^*$, serving as a model for the exponential modality in linear logic from a category-theoretical perspective [2, 28, 31].

Secondly, we utilise Hyland-Schalk general categorical construction of double glueing [28] to $\text{TsKer}^\text{op}$. In the double glueing $G(\text{TsKer}^\omega)$, the coproduct and product operations, as well as the monoidal tensor and cotensor, exhibit distinctions. Our primary focus is how to lift the linear exponential comonad in $\text{TsKer}^\text{op}$ to the double glueing. By applying the general methodology [28] to our specific exponential kernels, distinct linear comonad structures are established within $G(\text{TsKer}^\omega)$. Subsequently, by observing a contravariant equivalence between $\text{TKer}$ and $M_E$ (representing measurable functions and linear positive maps preserving monotone convergence), we formulate an orthogonality between a measurable map $f$ and a measure $\mu$ so that $f \in \text{TKer}(X, I)$ and $\mu \in \text{TKer}(I, X)$ are orthogonal if $\int f d\mu \leq 1$. This orthogonality serves as a continuous version of Danos-Ehrhard’s linear duality for $\text{Pcoh}$. Our orthogonality includes an adjunction between operators $\kappa^*$ and $\kappa_*$ associated with a kernel $\kappa$ respectively on measurable functions and on measures. This adjunction enforces a coherence condition for the orthogonality concerning the exponential comonad. It notably simplifies the general construction by Hyland-Schalk. The introduced orthogonality concept enables the construction of certain double glueing subcategories, including tight orthogonality one, on which our primary focus lies.

Finally, we delve into the full subcategory $\text{TsKer}_\omega$ of countable measurable spaces, where morphisms of transition kernels collapse into transition matrices. Within $\text{TsKer}_\omega$, there exists a dagger functor internalising the contravariant equivalence in the subcategory restriction. This results in a monoidal closed structure within $\text{TsKer}_\omega$, rendering the category dagger compact closed. Consequently the double glueing $G(\text{TsKer}_\omega)$ becomes $^\ast$-autonomous. Our goal is to establish an equivalence of the tight orthogonal subcategory $T(\text{TsKer}_\omega^\omega)$ to the category $\text{Pcoh}$ of probabilistic coherent spaces. Notably, this equivalence marks the first precise formulation of the folklore among the linear logic community (cf. [32]).

The paper is organised as follows: Section 2 presents various categories of transition kernels and measurable spaces. Section 3 starts with a measure theoretic study on exponential measurable spaces and establishes exponential transition kernels. Section 3 constructs a linear exponential comonad over a monoidal category $\text{TsKer}^\text{op}$. Section 4 is an application of Hyland-Schalk double glueing to our measure theoretic construction. Section 5 restricts to the countable measurable spaces in particular for obtaining $\text{Pcoh}$ as a double glueing.
1. Preliminaries from Measure Theory

This section recalls some basic definitions and a theorem from measure theory, necessary in this paper.

**Terminology** \( \mathbb{N} \) denotes the set of non negative integers. \( \mathbb{R}_+ \) denotes the set of non negative reals. \( \mathbb{R}^+ \) denotes \( \mathbb{R}_+ \cup \{\infty\} \). \( \mathfrak{S}_n \) denotes the symmetric group over \( \{1, \ldots, n\} \). \( \delta_{x,y} \) is the Kronecker delta. For a subset \( A \), \( \chi_A \) denotes the characteristic function of \( A \). The Dirac delta \( \delta_x(A) \) is \( \chi_A(x) \). \( \psi \) denotes the disjoint union of sets.

**Definition 1.1** \((\sigma\text{-field } \mathcal{X} \text{ and measurable space } (X, \mathcal{X}))\).
A \( \sigma \)-field over a set \( X \) is a family \( \mathcal{X} \) of subsets of \( X \) containing \( \emptyset \), closed under the complement and countable union. A pair \( (X, \mathcal{X}) \) is called a measurable space. The members of \( \mathcal{X} \) are called measurable sets.

The measurable space is often written simply by \( X \), as \( X \) is the largest element in \( \mathcal{X} \). For a measurable set \( Y \in \mathcal{X} \), the measurable subspace \( \mathcal{X} \cap Y \), called the restriction on \( Y \), is defined by \( \mathcal{X} \cap Y := \{A \cap Y \mid A \in \mathcal{X}\} \).

**Definition 1.2** \((\sigma(\mathcal{F}) \text{ and Borel } \sigma\text{-field } \mathcal{B}_+))\).
For a family \( \mathcal{F} \) of subsets of \( X \), \( \sigma(\mathcal{F}) \) denotes the \( \sigma \)-field generated by \( \mathcal{F} \), i.e., the smallest \( \sigma \)-field containing \( \mathcal{F} \). When \( X \in \mathbb{R}_+ \) and \( \mathcal{F} \) is the family \( \mathcal{O}_{\mathbb{R}_+} \) of the open sets in \( \mathbb{R}_+ \) (with the topology whose basis consists of the open intervals in \( \mathbb{R}_+ \) together with \( (a, \infty) := \{x \mid a < x\} \) for all \( a \in \mathbb{R}_+ \)), the \( \sigma \)-field is denoted by \( \mathcal{B}_+ \), whose members are called Borel sets over \( \mathbb{R}_+ \).

**Definition 1.3** \((\text{measurable function})\).
For measurable spaces \((X, \mathcal{X})\) and \((Y, \mathcal{Y})\), a function \( f : X \to Y \) is \((\mathcal{X}, \mathcal{Y})\)-measurable (often just measurable) if \( f^{-1}(B) \in \mathcal{X} \) whenever \( B \in \mathcal{Y} \). In this paper, a measurable function unless otherwise mentioned is to the Borel set \( \mathcal{B}_+ \) over \( \mathbb{R}_+ \) from some measurable space \((X, \mathcal{X})\).

**Definition 1.4** \((\text{measure})\).
A measure \( \mu \) on a measurable space \((X, \mathcal{X})\) is a function from \( \mathcal{X} \) to \( \mathbb{R}_+ \) satisfying (\( \sigma \)-additivity): If \( \{A_i \mid i \in I\} \) is a countable family of pairwise disjoint sets, then \( \mu(\bigcup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i) \).

**Definition 1.5** \((\text{integration})\).
For a measure \( \mu \) on \((X, \mathcal{X})\), and a \((\mathcal{X}, \mathcal{B}_+)\)-measurable function \( f \), the integral of \( f \) over \( X \) wrt the measure \( \mu \) is defined by \( \int_X f(x) \mu(dx) \), which is simply written \( \int_X f \, d\mu \). It is also written \( \int_X f \, d\mu \).

**Theorem 1.6** \((\text{monotone convergence})\). Let \( \mu \) be a measure on a measurable space \((X, \mathcal{X})\). For an monotonic sequence \( \{f_n\} \) of \((\mathcal{X}, \mathcal{B}_+)\)-measurable functions, if \( f = \sup_n f_n \), then \( f \) is measurable and \( \sup \int_X f_n \, d\mu = \int_X f \, d\mu \).

**Definition 1.7** \((\text{push forward measure } \mu \circ F^{-1} \text{ along a measurable function } F)\).
For a measure \( \mu \) on \((Y, \mathcal{Y})\) and a measurable function \( F \) from \((Y, \mathcal{Y})\) to \((Y', \mathcal{Y}')\), \( \mu' = \mu(F^{-1}(\{B\})) \) becomes a measure on \((Y', \mathcal{Y}')\), called push forward measure of \( \mu \) along \( F \). The push forward measure \( \mu' \) has the following property, called “variable change of integral along push forward \( F \)”: \( \int_{Y'} g \, d\mu' = \int_Y (g \circ F) \, d\mu \).

That is, \( \int_{Y'} g(y') \, d\mu' = \int_Y g(F(y)) \, d\mu \) for all \( g \in \mathcal{Y}' \). The push forward measure \( \mu' \) is often denoted by \( \mu \circ F^{-1} \) by abuse of notation.

2. **Category TKer, its Dual \( M_\varepsilon \) and Monoidal Subcategory \( TsKer \) of \( s \)-finite Transition Kernels**

This section starts with introducing a category \( TKer \) of transition kernels with convolution (i.e., an integral transform on the product) as categorical composition. Measures and measurable function on a measurable space both arise as certain morphisms in the category. A contravariant equivalence is shown to a category \( M_\varepsilon \) of measurable functions. When imposing \( s \)-finiteness to kernels, a monoidal subcategory \( TsKer \) with (countable) biproducts is obtained.
2.1. Transition kernels and Contravariant Equivalence

**Definition 2.1** (transition kernel). For measurable spaces \((X, \mathcal{X})\) and \((Y, \mathcal{Y})\), a transition kernel from \((X, \mathcal{X})\) to \((Y, \mathcal{Y})\) is a function

\[
\kappa : X \times \mathcal{Y} \rightarrow \mathbb{R}_+ 
\]

such that

(i) For each \(x \in X\), the function \(\kappa(x, -) : \mathcal{Y} \rightarrow \mathbb{R}_+\) is a measure on \((Y, \mathcal{Y})\).

(ii) For each \(B \in \mathcal{Y}\), the function \(\kappa(-, B) : X \rightarrow \mathbb{R}_+\) is measurable on \((X, \mathcal{X})\).

**Definition 2.2** (Operations \(\kappa_*\) and \(\kappa^*\) of a kernel \(\kappa\) on measures and measurable functions). Let \(\kappa : (X, \mathcal{X}) \rightarrow (Y, \mathcal{Y})\) be a transition kernel.

- For a measure \(\mu\) on \(\mathcal{X}\),

  \[
  (\kappa_* \mu)(B) := \int_X \kappa(x, B) \mu(dx)
  \]

  is a measure on \(\mathcal{Y}\), where \(B \in \mathcal{Y}\).

  In particular, for a Dirac measure \(\delta_a\) with any \(a \in X\),

  \[
  (\kappa_* \delta_a)(B) = \int_X \kappa(x, B) \delta_a(dx) = \kappa(a, B)
  \]

- For a measurable function \(f\) on \(\mathcal{Y}\),

  \[
  (\kappa^* f)(x) := \int_Y f(y) \kappa(x, dy)
  \]

  is measurable on \(\mathcal{X}\), where \(x \in X\).

  In particular, for a characteristic function \(\chi_B\) for any \(B \in \mathcal{Y}\),

  \[
  (\kappa^* \chi_B)(x) := \kappa(x, B) \quad (1)
  \]

It is direct to check, by the monotone convergence theorem 1.6, that \(\kappa^* f\) is measurable.

A characterization is known in (1) for which general mappings \(\alpha\) in place of \(\kappa^*\) in turn define transition kernels as follows:

**Proposition 2.3** (Lemma 36.2 [1]). Let \(\mathcal{E}(\mathcal{X})\) denote the set of all \(\mathbb{R}_+\)-valued measurable functions on a measurable space \(\mathcal{X}\). If a function \(\alpha : \mathcal{E}(\mathcal{Y}) \rightarrow \mathcal{E}(\mathcal{X})\) is linear (that is, \(\alpha(0) = 0\) and \(\alpha(r f + s g) = r \alpha(f) + s \alpha(g)\) for \(r, s \in \mathbb{R}_+\)) and preserves monotone convergence (that is, \(\sup \alpha(f_n) = \alpha(\sup f_n)\) for any monotone sequence \(\{f_n\}\) in \(\mathcal{E}(\mathcal{Y})\)), then

\[
\alpha(\chi_B)(x) := \kappa(x, B)
\]

becomes a transition kernel from \((X, \mathcal{X})\) to \((Y, \mathcal{Y})\). Moreover \(\kappa\) is the unique transition kernel satisfying \(\kappa^* f = \alpha(f)\) for all \(f \in \mathcal{E}(\mathcal{X})\).

**Definition 2.4** (categories \(\text{TKer}\) and \(\text{M}_\mathcal{E}\)).

- \(\text{TKer}\) denotes the category where each object is a measurable space \((X, \mathcal{X})\) and a morphism is a transition kernel \(\kappa\) from \((X, \mathcal{X})\) to \((Y, \mathcal{Y})\). The composition is the convolution of two kernels \(\kappa : (X, \mathcal{X}) \rightarrow (Y, \mathcal{Y})\) and \(\iota : (Y, \mathcal{Y}) \rightarrow (Z, \mathcal{Z})\):

  \[
  \iota \circ \kappa(x, C) = \int_Y \kappa(x, dy) \iota(y, C) \quad (2)
  \]

\(\text{id}_{(X, \mathcal{X})}\) is the unit kernel \(\delta : (X, \mathcal{X}) \rightarrow (X, \mathcal{X})\), defined for \(x \in X\) and \(A \in \mathcal{X}\) by;

- if \(x \in A\) then \(\delta(x, A) = 1\), else \(\delta(x, A) = 0\).

That is, for each \(x \in X\), \(\delta(x, -)\) is the Dirac measure on \((X, \mathcal{X})\).
- \( \mathcal{M}_E \) denotes the category whose objects are measurable spaces, same as \( \text{T Ker} \), but whose morphisms \( \mathcal{M}_E(\mathcal{X}, \mathcal{Y}) \) consists of any linear positive map \( \alpha : \mathcal{E}(\mathcal{X}) \to \mathcal{E}(\mathcal{Y}) \) preserving monotone convergence. The composition is simply that of the functions.

It is now well known that the composition (2) for sub-Markov kernels (cf. below Remark 2.6) comes from Giry’s probabilistic monad, resembling the power set monad of the relational composition (cf. [24, 34]). Instead of using the monad applied to our general setting for the transition kernels, we give a simpler intuition how the composition (2) arises via the simpler composition of \( \mathcal{M}_E \) by assuming the expected functoriality \((\iota \circ \kappa)^* f = (\iota^* \circ \kappa^*)(f)\). That is, for any \( x \in X \), we see

\[
(i \circ \kappa)^* f(x) = ((\kappa^* \circ \iota^*)(f))(x) = (\kappa^*(\iota^* f))(x) = \int_Y \iota^* f(y) \kappa(x, dy) = \int_Y (\int_Z f(z) \iota(y, dz)) \kappa(x, dy)
\]

In particular, taking \( f = \chi_C \) yields

\[
(i \circ \kappa)^* \chi_C(x) = \int_Y \iota(y, C) \kappa(x, dy),
\]

which by (1) imposes the definition of the composition of the two kernels.

**Remark 2.5** (measures and measurable functions as \( \text{T Ker} \) morphisms.). Measures and measurable functions both reside as morphisms in \( \text{T Ker} \): Let \((I, \mathcal{I})\) be the singleton measurable space with \( I = \{\ast\} \), hence \( \mathcal{I} = \{\emptyset, \{\ast\}\} \), then

\[
\text{T Ker}(\mathcal{I}, \mathcal{X}) = \{\kappa(\ast, -) : \mathcal{X} \to \mathbb{R}_+ \mid \text{kernel } \kappa \text{ with domain } \mathcal{I}\} = \{ \text{the measures } \mu \text{ on } (X, \mathcal{X}) \}
\]

\[
\text{T Ker}(\mathcal{X}, \mathcal{I}) = \{\kappa(-, \{\ast\}) : X \to \mathbb{R}_+ \mid \text{kernel } \kappa \text{ with codomain } \mathcal{I}\} \cup \{\kappa(-, \emptyset) = 0\} = \{ \text{the measurable functions } f \text{ on } (X, \mathcal{X}) \text{ to } \mathbb{R}_+ \}
\]

The operations \( \kappa_* \) and \( \kappa^* \) of Definition 2.2 are respectively categorical precomposition and composition with \( \kappa \) in \( \text{T Ker} \) so that \( \kappa_* \mu = \kappa \circ \mu \) and \( \kappa^* f = f \circ \kappa \).

**Remark 2.6** (\( \text{S Rel} \) [34, 33]). The category \( \text{S Rel} \) of stochastic relations is a wide subcategory of \( \text{T Ker} \) strengthening the conditions of Definition 2.1 into (i) \( \kappa(x, -) \) is a sub-probability measure (i.e., a measure from \( \mathcal{Y} \) to \([0, 1])\) and (ii) \( \kappa(-, B) \) is a bounded measurable function.

The morphisms of \( \text{S Rel} \) are called sub-Markov kernels. They are called Markov kernels when \( \kappa(x, B) = 1 \) for any \( x \in X \).

Note: The bounded condition of (ii) is derivable from (i), thus the condition (ii) is redundant when defining \( \text{S Rel} \) as a subcategory of \( \text{T Ker} \).

We also remark here a crucial reason seen immediately in the next Subsection 2.2 why \( \text{S Rel} \) needs to be extended to \( \text{T Ker} \) in this paper: The coproduct of \( \text{S Rel} \) given in [34, 33] is not a biproduct in \( \text{S Rel} \), but it is so in \( \text{T Ker} \) (cf. Proposition 2.9). The biproduct existing in \( \text{T Ker} \) will be crucial to the main purpose of the paper in order to construct an exponential structure in the s-finite subcategory introduced in Section 2.3 below.

Proposition 2.3 says category theoretically;

**Proposition 2.7.** \( \text{T Ker} \) and \( \mathcal{M}_E \) are contravariantly equivalent. The equivalence is given by the contravariant functor

\[
(\ )^* : \text{T Ker} \cong (\mathcal{M}_E)^{\text{op}}
\]

On the objects, \((\ )^* \) acts as the identity. On the morphisms, the functoriality \((i \circ \kappa)^* = \kappa^* \circ \iota^* \) is checked above.

The contravariant equivalence \((\ )^* \) in particular gives a direct account on the measurable functions as the homset in Remark 2.5 by \( \text{T Ker}(\mathcal{X}, \mathcal{I}) \cong \mathcal{M}_E(\mathcal{I}, \mathcal{X}) \).
Remark 2.8. The contravariant functor \((\cdot)^*\) when restricted to the Markov kernels \(\mathcal{S}\mathcal{R}\mathcal{E}\) gives a contravariant equivalence to \(\mathbf{Vec}_{\mathcal{E}}\), where each object is the subspace \(\mathcal{E}(\mathcal{X})\) of bounded measurable functions, which forms a vector space. The boundedness makes the space not only a vector space but moreover a Banach space with the uniform norm \(\|f\| = \sup_{x \in \mathcal{X}} |f(x)|\). The opposite category is studied in [34] as the category of the predicate transformers, stemming from Kozen’s precursory work on probabilistic programming. Taking measurable functions as predicates and measures as states, the ordinary satisfaction relation, say \(\mu \models f\), is generalised into integrals, say \(\int fd\mu\) giving a value in the interval \([0, 1]\). In the present paper in Section 4.2, this satisfaction relation will be explored in terms of the orthogonality relation.

2.2. Countable Biproducts in \(\mathbf{T}\mathbf{K}\mathbf{e}\mathbf{r}\)

The transition kernels have an intrinsic category theoretical property.

Proposition 2.9 (biproduct \(\biguplus\)). \(\mathbf{T}\mathbf{K}\mathbf{e}\mathbf{r}\) has countable biproducts.

Proof. Given a countable family \(\{(X_i, \mathcal{X}_i)\}_i\) of measurable spaces, we define

\[
\biguplus_i (X_i, \mathcal{X}_i) := \left(\bigcup_i \{i\} \times X_i, \bigcup_i \mathcal{X}_i\right),
\]

where \(\bigcup_i \mathcal{X}_i := \{\bigcup_i \{i\} \times A_i \mid A_i \in \mathcal{X}_i\}\) is the \(\sigma\)-field generated by the measurable sets of each summands.

(Coproduct): (3) defines a coproduct for \(\mathbf{T}\mathbf{K}\mathbf{e}\mathbf{r}\). The injection \(i_j : (X_j, \mathcal{X}_j) \rightarrow (\bigcup_i \{i\} \times X_i, \bigcup_i \mathcal{X}_i)\) is defined by \(i_j(x_j, \sum_i A_i) := \delta(x_j, A_i)\). The mediating morphism \(\oplus_{i \in I} f_i : \left(\bigcup_i \{i\} \times X_i, \bigcup_i \mathcal{X}_i\right) \rightarrow (Y, \mathcal{Y})\) for given morphisms \(f_i : (X_i, \mathcal{X}_i) \rightarrow (Y, \mathcal{Y})\) is defined by \((\oplus_{i \in I} f_i)((i, x), B) = f_i(x, B)\). Note (3) is the same instance as the known coproduct in \(\mathcal{S}\mathcal{R}\mathcal{E}\). However, in the relaxed structure of \(\mathbf{T}\mathbf{K}\mathbf{e}\mathbf{r}\), we have moreover;

(Product): (3) becomes a product for \(\mathbf{T}\mathbf{K}\mathbf{e}\mathbf{r}\). The projection \(\pi_j : \biguplus_i (X_i, \mathcal{X}_i) \rightarrow (X_i, \mathcal{X}_i)\) is given by \(\pi_j((j, x), A_i) := \delta_{i,j} \cdot \delta(x, A_i)\). The mediating morphism \(\&_j g_i\) for given morphisms \(g_i : (Y, \mathcal{Y}) \rightarrow (X_i, \mathcal{X}_i)\) is defined to be \((\&_j g_i)(b, \bigcup_i \{i\} \times A_i) := \sum_j g_i(b, A_i)\). Note the construction for the meditating morphism is not closed in Markov kernels, but is so in transition kernels. This construction shows how values of measurable functions include the infinite real \(\infty\) when \(I\) becomes infinite. We check the uniqueness of the mediating morphism, say \(m\):

\[
(\pi_j \circ m)(b, \{j\} \times A_j) = \int_{\mathcal{X}_j} m(b, d(i, x))\pi_j((i, x), \{j\} \times A_j) = \int_{\mathcal{X}_j} m(b, d(i, x)) \delta_{j,i} \cdot \delta((i, x), \{j\} \times A_j) = \int_{X_j} m(b, d(j, x)) \delta((j, x), \{j\} \times A_j) = m(b, \{j\} \times A_j)
\]

(4)

The required commutativity for \(m\) is \(\pi_j \circ m = g_j\), which holds by (4) if and only if \(m(b, A_j) = g_j(b, A_j)\) for all \(j\). Since \(m(b, -)\) is a measure and \(\{A_j\}_{j \in J}\) are disjoint, this yields the definition \(\&_j g_i\) of the mediating morphism.

The unit of the biproduct is the null measurable space \(\mathcal{T} = (\emptyset, \{\emptyset\})\).

This subsection ends with the following remark, which though is not required to comprehend the paper.

Remark 2.10 (\(\mathbf{T}\mathbf{K}\mathbf{e}\mathbf{r}\) is traced wrt the biproduct). \(\mathbf{T}\mathbf{K}\mathbf{e}\mathbf{r}\) is a unique decomposition category [25, 26], which is a generalisation of Arbib-Manes partially additive category studied in [34] for \(\mathcal{S}\mathcal{R}\mathcal{E}\). A countable family of \(\mathbf{T}\mathbf{K}\mathbf{e}\mathbf{r}\) morphisms \(\kappa_i : (X, \mathcal{X}) \rightarrow (Y, \mathcal{Y})\) is summandable so that \(\sum_{i \in I} \kappa_i(x, B)\) is a transition kernel. Then \((\mathbf{T}\mathbf{K}\mathbf{e}\mathbf{r}, \biguplus)\) is traced so that any \(\kappa : (X, \mathcal{X}) \biguplus (Z, \mathcal{Z}) \rightarrow (Y, \mathcal{Y}) \biguplus (Z, \mathcal{Z})\) yields \(\text{Tr}_{\mathcal{X}, \mathcal{Z}}(\kappa) : (X, \mathcal{X}) \rightarrow (Y, \mathcal{Y})\) which is the standard trace formula corresponding to Girard’s execution formula for Geometry of Interaction [21] and is defined further in [27] using the execution concept. The trace operator enables modelling of both feed back and iteration on a given morphism. Notably the int construction by Joyal-Street-Verity [29] results in a compact closed completion of \(\mathbf{T}\mathbf{K}\mathbf{e}\mathbf{r}\) with \(\biguplus\) serving as tensor. It is important to note that the monoidal product discussed in this paper is distinct from this one, but the measure theoretic direct product as introduced in Definitions 2.11 and 2.19 below.
2.3. Monoidal Product and Countable Biproducts in **TsKer**

This subsection introduces a subcategory **TsKer** of s-finite transition kernels. The s-finiteness is a relaxation of a standard measure theoretic class of the σ-finiteness so that the σ-finiteness resides intermediately between finiteness and s-finiteness. The relaxed class of the s-finite kernels is closed under composing kernels, which is not the case in the class of σ-finite kernels. Inside the subcategory **TsKer**, the monoidal product of morphisms is functorially defined to accommodate Fubini-Tonelli Theorem for the unique integration over product measures. **TsKer** is also shown to retain the countable biproducts in **TKer** of the previous subsection.

**Definition 2.11** (product of measurable spaces). The product of measurable spaces \((X_1, \mathcal{X}_1)\) and \((X_2, \mathcal{X}_2)\) is the measurable space \((X_1 \times X_2, \mathcal{X}_1 \otimes \mathcal{X}_2)\), where \(\mathcal{X}_1 \otimes \mathcal{X}_2\) denotes the σ-field over the cartesian product \(X_1 \times X_2\) generated by measurable rectangles \(A_1 \times A_2\)’s such that \(A_i \in \mathcal{X}_i\).

In order to accommodate measures into the product of measurable spaces, each measures \(\mu_i\) on \((X_i, \mathcal{X}_i)\) need to be extended uniquely to that on the product. The condition of σ-finiteness ensures this, yielding the unique product measure over the product measurable space:

**Definition 2.12** (σ-finiteness). A measure \(\mu\) on \((X, \mathcal{X})\) is σ-finite when the set \(X\) is written as a countable union of sets of finite measures. That is, \(\exists A_1, A_2, \ldots \in \mathcal{X}\) such that \(\mu(A_i) < \infty\) and \(X = \bigcup_{i=1}^{\infty} A_i\).

**Definition 2.13** (product measure). For a σ-finite measures \(\mu_i\) on \((X_i, \mathcal{X}_i)\) with \(i = 1, 2\), there exists a unique measure \(\mu\) on \((X_1 \times X_2, \mathcal{X}_1 \otimes \mathcal{X}_2)\) such that \(\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)\). \(\mu\) is written \(\mu_1 \otimes \mu_2\) and called the product measure of \(\mu_1\) and \(\mu_2\).

The product measure derived from σ-finite measures guarantees a basic theorem in measure theory, stating double integration is treated as iterated integration.

**Theorem 2.14** (Fubini-Tonelli). For σ-finite measures \(\mu_i\) on \((X_i, \mathcal{X}_i)\) with \(i = 1, 2\) and a \((\mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{B}_+)-\)measurable function \(f\),

\[
\int_{X_1 \times X_2} f \, d(\mu_1 \otimes \mu_2) = \int_{X_2} d\mu_2 \left( \int_{X_1} f \, d\mu_1 \right) = \int_{X_1} d\mu_1 \left( \int_{X_2} f \, d\mu_2 \right)
\]

The measure theoretical basic Fubini-Tonelli Theorem will become crucial also to the categorical study of the present paper, not only dealing with functoriality of morphisms on the product measurable spaces (cf. Proposition 3.11 below), but also giving a new instance of the orthogonality using the measure theory in Section 4.

Although one can impose σ-finiteness for the transition kernels \(\kappa(x, -)\) (uniformly or non-uniformly in \(x\)), this class of kernel is not closed in general under the composition in the category **TKer**. For the sake of category theory, one remedy for ensuring the compositionality is to tighten the class into the finite kernels. This class confined to the measures is used in finite measure transformer semantics [4] for probabilistic programs. However the class of the finite kernels is not closed under our exponential construction (Definition 3.9) later seen in Section 3.2. Thus, we need another remedy to loosen the condition contrarily, which is how s-finiteness arises below. While its notion was earlier established in [38, 19], the s-finiteness is recently studied by Staton [39] in modelling programming semantics. In addition to the compositionality in our categorical setting, the relaxed class of the s-finite kernels is shown to retain the Fubini-Tonelli Theorem (Proposition 2.18) working with the uniquely defined product measure.

**Definition 2.15** (s-finite kernels [38, 39]). Let \(\kappa\) be a transition kernel from \((X, \mathcal{X})\) to \((Y, \mathcal{Y})\).

- \(\kappa\) is called finite when \(\sup_{x \in X} \kappa(x, Y) < \infty\); i.e., the condition says that up to the scalar \(0 < a < \infty\) factor determined by the sup, \(\kappa\) is Markovian.

- \(\kappa\) is called s-finite when \(\kappa = \sum_{i \in \mathbb{N}} \kappa_i\) where each \(\kappa_i\) is a finite kernel from \((X, \mathcal{X})\) to \((Y, \mathcal{Y})\) and the sum is defined by \(\sum_{i \in \mathbb{N}} \kappa_i(x, B) := \sum_{i \in \mathbb{N}} \kappa_i(x, B)\). This is well-defined because any countable sum of kernels from \((X, \mathcal{X})\) to \((Y, \mathcal{Y})\) becomes a kernel of the same type.
In the definition of s-finiteness, note that \((\sum_{i\in\mathbb{N}} \kappa_i)^* = \sum_{i\in\mathbb{N}} \kappa_i^*\) and \((\sum_{i\in\mathbb{N}} \kappa_i)_* = \sum_{i\in\mathbb{N}} (\kappa_i)_*\) for the operations of Definition 2.2: That is, the preservation of the operation \((\ )^*\) (resp. \((\ )_*\)) means the commutativity of integral over countable sum of measures (resp. of measurable functions).

**Remark 2.16.** Both classes of the finite kernels and of the s-finite kernels are closed under the categorical composition of \(\text{Tker}\). This is directly calculated for the finite kernels, to which the s-finite ones are reduced by the note in the above paragraph. We refer to the proof of Lemma 3 of [39] for the calculation. In particular, the class of s-finite kernels is closed under push forwards along measurable functions. The both classes form wide subcategories of \(\text{TsKer}\) introduced below Definition 2.19.

The definition subsumes that of \emph{s-finite measures} when \((X,\mathcal{X})\) is in particular the singleton measurable space \((I,\mathcal{I})\). Note that every \(\sigma\)-finite measure is s-finite, but not vice versa: E.g., the infinite measure \(\infty\cdot \delta_a\) for the Dirac \(\delta_a\) with \(a \in X\) is not \(\sigma\)-finite, but s-finite.

A characterization of s-finite kernels is directly derived:

**Proposition 2.17** (Proposition 7 of [39]). A kernel is s-finite if and only if it is a push forward of a \(\sigma\)-finite kernel.

**Proof.** We prove "only if" part as "if part" is direct because of the inclusion of \(\sigma\)-finiteness into s-finiteness and of the closedness of s-finiteness under push forward.

Given a s-finite kernel \(\kappa = \sum_{i\in\mathbb{N}} \kappa_i\) with finite kernels \(\kappa_i\)'s from \((X,\mathcal{X})\) to \((Y,\mathcal{Y})\), a \(\sigma\)-finite kernel \(\nu\) from \((X,\mathcal{X})\) to \((\mathbb{N},2^{\mathbb{N}}) \times (Y,\mathcal{Y})\) is defined by \(\nu(x,V) := \sum_{i\in\mathbb{N}} \kappa_i(x,\{y \mid (i,y) \in V\})\). Then \(\kappa\) is the push forward of \(\nu\) along the projection \(\mathbb{N} \times Y \rightarrow Y\).

The original Fubini-Tonelli (Theorem 2.14) for the \(\sigma\)-finite measures extends to the s-finite measures:

**Proposition 2.18** (Fubini-Tonelli extending for s-finite measures (cf. Proposition 5 of Staton [39])). For the same \(f\) as Theorem 2.14 but \(\mu_1\) and \(\mu_2\) are s-finite measures, it holds;

\[
\int_{X_2} d\mu_2 \int_{X_1} f \, d\mu_1 = \int_{X_1} d\mu_1 \int_{X_2} f \, d\mu_2
\]

**Proof.** Write \(\mu_1 = \sum_{i\in\mathbb{N}} \mu_1^i\) and \(\mu_2 = \sum_{j\in\mathbb{N}} \mu_2^j\) with finite kernels \(\mu_1^i\)'s and \(\mu_2^j\)'s, then the following is from (LHS) to (RHS):

\[
\int_{X_2} \sum_{j\in\mathbb{N}} \mu_2^j(dx_2) \int_{X_1} f \sum_{i\in\mathbb{N}} \mu_1^i(dx_1) = \sum_{j\in\mathbb{N}} \int_{X_2} \mu_2^j(dx_2) \sum_{i\in\mathbb{N}} \int_{X_1} \mu_1^i(dx_1)
\]

\[
= \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \int_{X_2} \mu_2^j(dx_2) \int_{X_1} \mu_1^i(dx_1) = \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \int_{X_2} \mu_1^i(dx_1) \int_{X_1} \mu_2^j(dx_2)
\]

The first and the last (resp. the second and the second last) equations are by the commutativity of integral over countable sum of measurable functions (resp. of measures) (cf. the note in Definition 2.15). The middle equation is the original Fubini-Tonelli for the \(\sigma\)-finite measures, hence here in particular for the finite ones.

Finally it is derived that the s-finite transition kernels form a monoidal category.

**Definition 2.19** (monoidal subcategories \(\text{TsKer}\) of s-finite kernels and \(\text{Tker}\) of finite ones). \(\text{TsKer}\) is a wide subcategory of \(\text{Tker}\), whose morphisms are the \emph{s-finite} transition kernels. The \(s\)-finiteness of kernels is preserved under the composition of \(\text{Tker}\). \(\text{TsKer}\) has a symmetric monoidal product \(\otimes\): On objects is by Definition 2.13. Given morphisms \(\kappa_1 : (X_1,\mathcal{X}_1) \rightarrow (Y_1,\mathcal{Y}_1)\) and \(\kappa_2 : (X_2,\mathcal{X}_2) \rightarrow (Y_2,\mathcal{Y}_2)\), their product is defined explicitly:

\[
(\kappa_1 \otimes \kappa_2)((x_1, x_2), C) := \int_{Y_1} \kappa_1(x, dy_1) \int_{Y_2} \kappa_2(x, dy_2) \chi_C((y_1, y_2))
\]
Alternatively, thanks to Fubini-Tonelli (Proposition 2.18), the monoidal product is implicitly defined as the unique transition kernel \( \kappa_2 \otimes \kappa_2 : (X_1 \times X_1, X_1 \otimes X_1) \rightarrow (Y_1 \times Y_1, Y_1 \otimes Y_1) \) satisfying the following for any rectangle \( B_1 \times B_2 \) with \( B_i \in X_i \) for \( i = 1, 2 \):

\[
(\kappa_1 \otimes \kappa_2)((x_1, x_2), B_1 \otimes B_2) = \kappa_1(x_1, B_1) \kappa_2(x_2, B_2)
\]

The unit of the monoidal product is the singleton measurable space \((I, I)\).

\( \text{T} \ker \) is a monoidal wide subcategory of \( \text{TsKer} \) whose morphisms are finite transition kernels.

Proposition 2.20 (The subcategory \( \text{TsKer} \) retains the countable biproducts of \( \text{T} \ker \)).

\( \text{TsKer} \) has countable biproducts which are those in \( \text{T} \ker \) residing inside the subcategory.

Proof. The coproduct construction of Proposition 2.9 all works under the additional constraint of the \( s \)-finiteness of kernels. For the product construction, the only construction necessary to be checked is that of the mediating morphism \( \&_{i,j} g_i \), employing the sum over \( i \in I \) for a countable infinite \( I \): If given \( g_i \)'s of the product construction in Proposition 2.9 are \( s \)-finite, then each is written \( g_i = \sum_{j \in N} g_{i(j)} \), where each \( g_{i(j)} \) is a finite kernel from \((Y, Y)\) to \((X_i, X_i)\). In what follows, the index set \( I \) is identified with \( N \). A transition kernel \( h_n \) is defined for each \( n \in N \):

\[
h_n := \sum_{i+j=n} \iota_i \circ g_{i(j)} : (Y, Y) \rightarrow \prod_{j \in I} (X_j, X_j)
\]

where \( \iota_i : (X_i, X_i) \rightarrow \prod_{j \in I} (X_j, X_j) \) is the coproduct injection. Note \( h_n \) is a finite kernel, as the sum specified by the subscript \( i + j = n \) is finite. Then in terms of the finite kernels, the mediating morphism \( \&_{i} g_i \), constructed in Proposition 2.9 is represented as follows to be \( s \)-finite:

\[
\bigwedge_{i \in N} g_i = \sum_{n \in N} h_n
\]

Remark 2.21 (the infinite biproduct as colimit in \( \text{TsKer} \)). The countable infinite biproduct in \( \text{TsKer} \) is characterised by the colimit inside the subcategory: Given the direct system \(< \prod_{n=0}^{k} (X_n, X_n), \iota_n^k \rangle_{k \in N, \ell \geq k} \) in \( \text{TsKer} \), the colimit \( \lim_k \prod_{n=0}^{k} (X_n, X_n) \) coincides with the infinite biproduct \( \prod_{n=0}^{\infty} (X_n, X_n) \) in \( \text{TsKer} \). Hence, the colimit is closed in the subcategory \( \text{TsKer} \), but not necessarily in \( \text{T} \ker \).

3. A Linear Exponential Comonad over \( \text{TsKer}^a \)

3.1. Exponential Measurable Space \((X_e, X_e)\)

This subsection concerns a measure theoretic study on exponential measurable spaces. [5] is a good reference for the subsection.

Definition 3.1 (exponential monoid \( X_e \)). \( X_e \) denotes the free abelian monoid (the free semi group with identity) generated by a set \( X \): The members of \( X_e \) are the formal products \( x_1 x_2 \cdots x_n \) where \( x_i \in X \) and \( n \in N \) so that order of factor is irrelevant. The monoid operation for members of \( X_e \) is obviously the free product. When \( n = 0 \), under the convention \( x_1 x_2 \cdots x_n = 0 \), this is the monoid identity (in spite of the multiplicative notation), which is equated with the empty sequence. The monoid operator is written by a product \((x, y) \rightarrow xy\). Each member \( \pi \in X_e \) is identified with a finite multiset of elements in \( X \) and vice versa. Hence \( \pi \) is seen as an integer valued function on \( X \), which vanishes to zero outside the finite sets;

\[
\pi(t) = \text{multiplicity of } t \in X \text{ as a factor of } \pi.
\]

That is, \( \pi \) represents the unique multiset of elements \( X \), and vice versa.
For $A \subseteq X$, we define

$$\pi(A) := \sum_{t \in A} \pi(t)$$

Then $\pi(A)$ represents the number of elements in $A$.

The counting function $n_A$ on $X_e$ is defined for each $A \subseteq X$,

$$n_A(x) = \pi(A)$$

Note if $x$ is (i.e., the singleton sequence of $x \in X$), then $\pi(A) = n_A(x) = \delta(x, A)$ for any subset $A$ of $X$.

The members of $X_e$ can be seen as equivalence classes of ordered sequences in $X_e^\bullet$ defined below under rearrangement (permutations of factors):

**Definition 3.2** (non-abelian monoid $X_e^\bullet$). Using $\cdot$ for ordered sequences, $X_e^\bullet$ denotes the nonabelian monoid generated by $X$, consisting of ordered sequences $x_1 \cdot x_2 \cdots \cdot x_n$ where $x_i \in X$ and $n \geq 0$. The monoid operation for members of $X_e^\bullet$ is obviously the operation $\cdot$ joining sequences in order. Then the abelian monoid $X_e$ is the image of the nonabelian homomorphism $F$ forgetting the order of the factors:

$$F : X_e^\bullet \rightarrow X_e \quad x_1 \cdot \cdots \cdot x_n \mapsto x_1 \cdots x_n.$$ 

Obviously $F^{-1}(F(A))$ is the smallest symmetric set containing $A \subset X_e^\bullet$. The set $X_e^\bullet$ is a disjoint union

$$X_e^\bullet = \bigcup_{n \geq 0} X_e^\bullet_n$$

where the set $X_e^\bullet_n$ denotes $\{x_1 \cdots x_n \mid x_i \in X\}$, which is isomorphic to the $n$-ary cartesian product $X^n$ of $X$.

**(Notation)** For any family $\mathcal{F}$ of subsets of $X$, $\mathcal{P}^\bullet(\mathcal{F})$ denotes the class of all finite ordered sequences $A_1 \cdots A_n := \{a_1, \cdots, a_n \mid a_i \in A_i\}$ with $A_i \in \mathcal{F}$ and $n \in \mathbb{N}$. Similar notation for $\mathcal{P}(\mathcal{F})$ denoting the class of all symmetric formal product $A_1 \cdots A_n := \{a_1 \cdots a_n \mid a_i \in A_i\}$ so that the order of factors is irrelevant.

**Definition 3.3** (measurable space $(X_e^\bullet, \mathcal{X}_e^\bullet)$ induced by $(X, \mathcal{X})$). Every measurable space $\mathcal{X}$ on $X$ induces a corresponding measurable space on the set $X_e^\bullet$ defined by:

$$(X_e^\bullet, \mathcal{X}_e^\bullet) = \left( \bigcup_{n \geq 0} X_e^\bullet_n, \bigcup_{n \geq 0} \mathcal{X}_e^\bullet_n \right),$$

whose $\sigma$-field $\mathcal{X}_e^\bullet$ is the disjoint union of the measure theoretic $n$-ary direct product of $\mathcal{X}$, on the set $X_e^\bullet_n$. That is, $\bigcup_{n \geq 0} X_e^\bullet_n = \{\bigcup_{n \geq 0} A_n \mid A_n \in \mathcal{X}_e^\bullet_n\}$. Note by this definition, $\mathcal{X}_e^\bullet$ is the $\sigma$-field generated by $\mathcal{P}^\bullet(\mathcal{X})$ and the subspace of $\mathcal{X}_e^\bullet$ restricted to $X_e^\bullet_n$ coincides with the $n$-ary direct product of the measurable space $\mathcal{X}$: i.e.,

$$\mathcal{X}_e^\bullet = \sigma(\mathcal{P}^\bullet(\mathcal{X})) \quad \text{and} \quad \mathcal{X}_e^\bullet \cap X_e^\bullet_n = \mathcal{X}_e^\bullet_n \quad \text{for any } n \geq 0.$$ 

In particular when $n = 0$, $\mathcal{X}_e^\bullet_0$ is the only $\sigma$-field $\{\emptyset, \{\emptyset\}\}$ over $X_e^\bullet_0 = \emptyset$.

In terms of category theory, Definition 3.3 says

**Proposition 3.4.** In $\text{Tker}$, the measurable space $(X_e^\bullet, \mathcal{X}_e^\bullet)$ of Definition 3.3 is the countable infinite coproduct $\bigcap_{n=0}^{\infty} (X_e^\bullet_n, \mathcal{X}_e^\bullet_n)$, whose inclusion in $\bigcap_{n=0}^{\infty} (X_e^\bullet_n, \mathcal{X}_e^\bullet_n)$ from the $m$-th component is given:

$$(X_e^\bullet, \mathcal{X}_e^\bullet) \cong \bigcap_{n=0}^{\infty} (X_e^\bullet_n, \mathcal{X}_e^\bullet_n) \cong_{in_m} (X_e^m, \mathcal{X}_e^m),$$

where for $x_1 \cdots x_m \in X_e^m$ and $A \in \mathcal{X}_e^\bullet$
\[ \inf_m(x_1, \ldots, x_m, \mathcal{A}) = \delta(x_1, \ldots, x_m, \mathcal{A} \cap X^m) = \delta(x_1, \ldots, x_m, \mathcal{A}) \]

Note that the injection factors through the colimit inclusion \( \inf_l \) of the direct system of Remark 2.21 such that \( \inf_m = \inf_l \circ \inf_m^l \) for any \( l \geq m \).

The infinite coproduct simultaneously becomes infinite biproducts, whose \( m \)-th projection \( \text{pr}_m^\infty(x_1, \ldots, x_k, \mathcal{B}) \) is

\[ \text{pr}_m^\infty(x_1, \ldots, x_k, \mathcal{B}) = \delta_k, m \delta(x_1, \ldots, x_k, \mathcal{B}) \quad \text{for} \quad x_1, \ldots, x_k \in X_e^c \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad \mathcal{B} \in \mathcal{X}^m. \]

Because of Remark 2.21, the construction of Proposition 3.4 is closed inside the subcategory \( \text{Tker} \) of \( s \)-finite kernels (but not in \( \text{Tker} \) of finite kernels).

Finally, the exponential measurable space \((X_e, \mathcal{X}_e)\) is obtained by the following equivalent characterisations of a \( \sigma \)-field \( \mathcal{X}_e \).

**Proposition 3.5** (\( \sigma \)-field \( \mathcal{X}_e \) over \( X_e \) (cf. Theorem 4.1 [5])). For a measurable space \((X, \mathcal{X})\), the following families of subsets of \( X_e \) all coincide with the \( \sigma \)-field \( \sigma(\mathcal{P}(X)) \), which is denoted by \( \mathcal{X}_e \).

(i) The quotient \( X_e^* \) wrt rearrangement \( \{ A \subseteq X_e \mid F^{-1}(A) \in \mathcal{X}_e^* \} \).

(ii) The projection of \( X_e^* \) by \( F \): i.e., the \( \sigma \)-field \( \{ F(A) \mid A \in \mathcal{X}_e^* \} \).

(iii) The smallest \( \sigma \)-field wrt which the counting functions \( n_A \) are measurable for all \( A \in \mathcal{X} \).

(iv) The largest \( \sigma \)-field \( \mathcal{Y} \) for \( X_e \) having \( \mathcal{X} \) as a subspace and such that the monoid product is measurable from \( \mathcal{Y} \times \mathcal{Y} \) to \( \mathcal{Y} \).

(v) The smallest \( \sigma \)-field for \( X_e \) containing \( \mathcal{X} \) and for which the monoid product preserves measurability.

(vi) The smallest \( \sigma \)-field for \( X_e \) containing \( \mathcal{X} \) and closed under the symmetric product.

**Definition 3.6** (exponential measurable space \((X_e, \mathcal{X}_e)\)). The measurable space \((X_e, \mathcal{X}_e)\), whose \( \mathcal{X}_e \) is defined by Proposition 3.5 such that \( \mathcal{X}_e = \sigma(\mathcal{P}(X)) \), is called the exponential measurable space of \((X, \mathcal{X})\).

This section ends with a measure theoretic proposition on isomorphisms relating the biproduct and the tensor via the exponential:

**Proposition 3.7.** The following holds for any measurable spaces \((X, \mathcal{X})\) and \((Y, \mathcal{Y})\):

(i) \( (X, \mathcal{X}) \coprod (Y, \mathcal{Y}) \)

(ii) \( T_e = (\emptyset, \{ \emptyset, \{ \emptyset \} \}) \), which is isomorphic to the monoid unit \((I, I)\).

**Proof.** We prove (i) since (ii) is direct, as the monoid identity of the exponential monoid of Definition 3.1 is given by the empty sequence.

First, the monoid isomorphism \((X \cup Y)_e \cong X_e \times Y_e \) between the largest measurable sets of each side is given as follows: For any \( z = z_1 \cdots z_n \in (X \cup Y)_e \), there exist a rearrangement \( \sigma \) and \( 0 \leq \ell \leq n \) such that \( z' := z_{\sigma(1)} \cdots z_{\sigma(\ell)} \in X_e \) and \( z'' := z_{\sigma(\ell+1)} \cdots z_{\sigma(n)} \in Y_e \). Note \( z' \) and \( z'' \) are uniquely independent of the choice of the rearrangement, thus mapping \( z \) to \((z', z'')\) gives a monoid isomorphism.

Second, the monoid iso is shown to induce the set theoretical isomorphism of the \( \sigma \)-fields of both sides \((X \cup Y)_e \cong X_e \times Y_e \).

By the definition of the product of two measurable spaces and Proposition 3.5 (iii), \( X_e \otimes Y_e \) is the smallest \( \sigma \)-field in which the product of counting functions \( n_A n_B : (x, y) \mapsto n_A(x) n_B(y) \) becomes measurable for all \( A \in \mathcal{X} \) and \( B \in \mathcal{Y} \). As the exponential function is one to one, Proposition 3.5 (iii) holds with \( n_A \) replaced by \( 2^{n_A} \). Since \( n_{A \otimes B}(x y) = n_A(x) + n_B(y) \), the following commutes so that the isomorphism becomes that between the two \( \sigma \)-fields.

\[
\begin{array}{ccc}
2^{n_{A \otimes B}} & \cong & 2^{n_A} 2^{n_B} \\
\xrightarrow[n_{A \otimes B}(x y)]{} & & \xrightarrow{} \\
(X \coprod Y)_e & \cong & X_e \otimes Y_e
\end{array}
\]

That is, \((X \coprod Y)_e\) and \(X_e \otimes Y_e\) are the smallest \( \sigma \)-fields making the respective functions \( 2^{n_{A \otimes B}} \) and \( 2^{n_A} 2^{n_B} \) measurable for all \( A \in \mathcal{X} \) and \( B \in \mathcal{Y} \). \(\square\)
Remark 3.8 (Seely isomorphism). The isomorphism (i) of Proposition 3.7 is a Seely isomorphism [36] in an appropriate category theoretical model of linear logic, as our binary biproduct models the logical connective &. The Seely isomorphism is known derivable [2, 31] using category theoretic abstraction from any linear exponential comonad structure with product, which structure will be obtained for a certain class of transition kernels in the next Section 3.3 (cf. Theorem 3.25).

3.2. Exponential Kernel $\kappa_e$ in s-finiteness

This subsection concerns a categorical investigation in $\text{TsKer}$ on the exponential measurable spaces of Section 3.1. This section starts with seeing the exponential acts not only on objects as defined in Section 3.1 but on the morphisms on $\text{TsKer}$, hence becomes an endofunctor.

(Notation) For a measurable space $(X, \mathcal{X})$ and $m \in \mathbb{N}$,

$$X^{(m)} := \{ x \in X_e \mid n_X(x) = m \} \subset X_e$$

This divides the set $X_e$ into the following disjoint union:

$$X_e = \bigcup_{n \geq 0} X^{(n)} \quad (5)$$

For any $A \in \mathcal{X}$, $A^{(m)}$ is defined same for the subspace $\mathcal{X} \cap A$.

Definition 3.9 (exponential kernel $\kappa_e$). In $\text{TsKer}$, every transition kernel $\kappa : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ induces a corresponding exponential kernel $\kappa_e : (X_e, \mathcal{X}_e) \to (Y_e, \mathcal{Y}_e)$, which we shall define in (11).

In what follows, $\kappa^n$ denotes the $n$-ary cartesian product $\kappa^n : (X, \mathcal{X})^n \to (Y, \mathcal{Y})^n$, for which the $n$-ary cartesian product of an object is given by $(X, \mathcal{X})^n = (X^\bullet \otimes X^\bullet)^n$.

The characterisation of Proposition 3.4 ensures the unique morphism $\kappa_e$ from $(X_e^\bullet, \mathcal{X}_e^\bullet)$ to $(Y_e^\bullet, \mathcal{Y}_e^\bullet)$ in $\text{TsKer}$;

$$\kappa_e := \lim_{n \to \infty} \prod_{n=0}^l (\text{in}^m \circ \kappa^n) \simeq \prod_{n=0}^\infty (\text{in}^m \circ \kappa^n) \quad (6)$$

See the following commutative diagram for the definition (6):

Explicitly, $\kappa_e$ is defined for any $(x_1, \ldots, x_n) \in X_e \cap X^{(n)}$ with any $n$ and $B \in \mathcal{Y}_e$,

$$\kappa_e((x_1, \ldots, x_n), B) = \kappa_e((x_1, \ldots, x_n), B \cap Y^\bullet \otimes \cdots \otimes Y^\bullet) = \kappa^n((x_1, \ldots, x_n), B \cap Y^\bullet \otimes \cdots \otimes Y^\bullet) \quad (7)$$

This is by virtue that $B = \sum_{i=0}^\infty (B \cap Y^\bullet \otimes \cdots \otimes Y^\bullet)$ and $\text{in}^m_{n}(y_1, \ldots, y_n), B \cap Y^\bullet \otimes \cdots \otimes Y^\bullet) = 0$ unless $m = n$. 

13
Since \( \kappa_e^* : X_e^* \times Y_e^* \to \mathbb{R}_+ \) is a transition kernel and the forgetful \( F : (Y_e^*, \mathcal{Y}_e^*) \to (Y_e, \mathcal{Y}_e) \) is \((Y_e^*, \mathcal{Y}_e^*)\)-measurable, the pushforward measure on \( Y_e \) along \( F \) is defined for each fixed \((x_1, \ldots, x_m) \in \mathcal{X}_e^*\), which we denote (under the convention of Definition 1.7) by

\[
\kappa_e^*((x_1, \ldots, x_m), -) \circ F^{-1} : Y_e \to \mathbb{R}_+
\]

This determines the following transition kernel, denoted by \( \kappa_e^* \circ F^{-1} \), from \((X_e^*, \mathcal{X}_e^*)\) to \((Y_e, \mathcal{Y}_e)\):

\[
\kappa_e^* \circ F^{-1} : X_e^* \times Y_e \to \mathbb{R}_+ \quad ((x_1, \ldots, x_m), \mathcal{B}) \mapsto \kappa_e^*((x_1, \ldots, x_m), F^{-1}(\mathcal{B}))
\]

for any \((x_1, \ldots, x_m) \in X_e^*\) and any \( \mathcal{B} \in \mathcal{Y}_e \cap \mathcal{Y}^{(m)} \) with any \( m \in \mathbb{N} \).

Directly from the definition, for any permutation \( \sigma \in \mathfrak{S}_m \),

\[
\kappa_e^*((x_1, \ldots, x_m), F^{-1}(\mathcal{B})) = \kappa_e^*((x_{\sigma(1)}, \ldots, x_{\sigma(m)}), F^{-1}(\mathcal{B})).
\]

(9) is implied using (7) by the following (10) for any \( D \in \mathcal{Y}_e^{(m)} \) and any permutation \( \sigma \in \mathfrak{S}_m \),

\[
k^m((x_1, \ldots, x_m), \sigma(D)) = k^m((x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(m)}), D)
\]

It is sufficient to check (10) for any rectangle \( D := C_1 \times \cdots \times C_m \) such that \( 1 \leq i \leq m \ C_i \in \mathcal{Y} \), but for which (10) is \( \kappa(x_{\sigma(1)}), \ldots, \kappa(x_{\sigma(m)}) = \kappa(x_1, C_{\sigma^{-1}(1)}), \ldots, \kappa(x_m, C_{\sigma^{-1}(m)}) \) by the definition of the product measurable space.

Observing

\[
F^{-1}(\mathcal{B} \cap \mathcal{Y}^{(m)}) = F^{-1}(\mathcal{B}) \cap \mathcal{Y}^{(m)}
\]

We thus finally define \( \kappa_e : X_e \times \mathcal{Y}_e \to \mathbb{R}_+ \) for any \( x_1 \cdots x_m \in X_e \) and any \( \mathcal{B} \in \mathcal{Y}_e \cap \mathcal{Y}^{(m)} \)

\[
\kappa_e(x_1 \cdots x_m, \mathcal{B}) := (\kappa_e^* \circ F^{-1})((x_1, \ldots, x_m), \mathcal{B})
= \kappa_e^*((x_1, \ldots, x_m), F^{-1}(\mathcal{B})),
\]

which definition does not depend on the ordering of \( x_1 \cdots x_m \).

We need to check \( \kappa_e \) defined above is a transition kernel: The second argument of \( \kappa_e \) giving a measure over \( \mathcal{Y}_e \) is direct by the definition (11) because so does the second argument of \( \kappa_e^* \).

For measurability in \( \mathcal{X}_e \) for the first argument of \( \kappa_e \), by virtue of Proposition 3.5 (ii), it suffices to show that \((\kappa_e)^{-1}(F^{-1}(-), \mathcal{B}) \) is measurable in \( \mathcal{X}_e^* \). But this is derived from the measurability of the first argument \( \kappa_e^* \) in \( \mathcal{X}_e^* \) because by the commutative diagram below, yielding

\[
(k_e)^{-1}(F^{-1}(-), \mathcal{B}) = (F \times \text{Id})^{-1} \circ (k_e)^{-1}(-, \mathcal{B}) = (k_e \circ (F \times \text{Id}))^{-1}(-, \mathcal{B}) = (k_e^* \circ F^{-1})^{-1}(-, \mathcal{B}).
\]

The so constructed \( \kappa_e \) is s-finite, as \( \kappa_e^* \) resides in \( \text{TsKer} \) (cf. Proposition 3.4) and s-finiteness is closed under the push forward along \( F \) (cf. Remark 2.16).

In order to show the functoriality of the exponential over kernels of Definition 3.9, we prepare the following lemma on \( \pi \)-system and Dynkin system.

**Lemma 3.10** (for Proposition 3.11: a \( \pi \)-system for \( \mathcal{X}_e^* \)). Let \((X, \mathcal{X})\) be a measurable space.
(a) For any \( n \geq 0 \), the following family consisting of subsets of \( X^n \)

\[
D_n := \left\{ \bigcup_{k=0}^{\infty} C_{k,1} \times \cdots \times C_{k,n} \mid C_{k,j} \in \mathcal{X} \right\}
\]  

(12)

is both (i) a \( \pi \)-system and (ii) a Dynkin system.

Recall that a nonempty family of subsets of a universal set is a \( \pi \)-system if the family is closed under finite intersections. It is a Dynkin system if the family contains \( \emptyset \) and is closed both under complements and under countable disjoint unions.

(b) \( D_n \) becomes a \( \sigma \)-field, hence coincides with the measurable space \( \mathcal{X}^n \) for any \( n \geq 0 \). Thus we characterise

\[ \mathcal{X}^n = \bigcup_{n \geq 0} D_n \]

Proof. As (b) is a consequence of (a) by Dynkin Theorem (cf. [1] for the theorem) stating that any Dynkin system which is also a \( \pi \)-system is a \( \sigma \)-field, we prove (a):

(a) (i) Direct by (A) \( (A_1 \times \cdots \times A_n) \cap (B_1 \times \cdots \times B_n) = (A_1 \cap B_1) \times \cdots \times (A_n \cap B_n) \).
(ii) As the empty set is contained in \( (12) \), we check the other two conditions:

(Closedness under countable disjoint unions) Immediate from the definition (12), by observing

\[
\bigcup_{i=1}^{\infty} \bigcup_{k=0}^{\infty} C_{k,1} \times \cdots \times C_{k,n} = \bigcup_{m=0}^{\infty} \bigcup_{k+i+m}^{\infty} C_{k,1} \times \cdots \times C_{k,n}.
\]

(Closedness under the complement) First we observe \( (A_1 \times \cdots \times A_n)^c = \bigcup_{\varepsilon \in \{1,\varepsilon\}^n} A^{(1)}_1 \times \cdots \times A^{(n)}_n \), where \( [n] := \{1,\ldots,n\} \) and \( X^1 = X \). Then using De Morgan and distribution of intersection over union:

\[
\bigcup_{k=0}^{\infty} C_{k,1} \times \cdots \times C_{k,n}^c = \bigcap_{k=0}^{\infty} (C_{k,1} \times \cdots \times C_{k,n})^c = \bigcap_{k=0}^{\infty} \bigcup_{\varepsilon \in \{1,\varepsilon\}^n} C_{k,1}^{(1)} \times \cdots \times C_{k,n}^{(n)} = \bigcup_{\varepsilon \in \{1,\varepsilon\}^n} \bigcap_{k=0}^{\infty} C_{k,1}^{(1)} \times \cdots \times \bigcap_{k=0}^{\infty} C_{k,n}^{(n)}
\]

which belongs to (12) as \( \forall j \in \{1,\ldots,n\} \bigcap_{k=0}^{\infty} C_{k,j} = 0 \in \mathcal{X} \).

Proposition 3.11 (functoriality of \((\cdot)_e\)).

\((\cdot)_e\) of Definition 3.9 becomes an endofunctor on the category \text{T}\text{S}\text{Ker}.

Proof. The condition \( (\iota \circ \kappa)_e = \iota_e \circ \kappa_e \) for \( \kappa : (X,\mathcal{X}) \to (Y,\mathcal{Y}) \) and \( \iota : (Y,\mathcal{Y}) \to (Z,\mathcal{Z}) \) is proved. For this, the following variable change (cf. Definition 1.7) plays a crucial role:

(Variable change of integral along \( F : (Y^e,\mathcal{Y}^e) \to (Y^e,\mathcal{Y}^e) \))

\[
\int_{Y^e} \kappa_e^*(-,F^{-1}d\bar{y}) \iota_e(y,\sim) = \int_{Y^e} \kappa_e^*(-,d\bar{y}) \iota_e(F(\bar{y}),\sim)
\]

(13)

where \( \bar{y} = (y_1,\ldots,y_m) \in \mathcal{Y}^e \) so that \( F(\bar{y}) = y_1 \cdots y_m = y \in \mathcal{Y}^e \) for any \( m \).

The equation (13) is that of Definition 1.7 when the push forward measure \( \mu' = \mu \circ F^{-1} \) is defined for \( \mu(B) := \kappa_e^*(-,B) \) with any fixed \(-\) (cf. (8)), and the measurable function \( g \) on \( \mathcal{Y}^e \) is given by \( \iota_e(y,\sim) \) with any fixed \( \sim \).

For any \( x_1 \cdots x_n \in X_e \) and any \( C \in Z_e \cap Z^{(n)} \) such that, by Lemma 3.10 (b),

\[
F^{-1}(C) = \bigcup_{k=0}^{\infty} C_{k,1} \times \cdots \times C_{k,n} \in Z_e^e \cap Z^{*n},
\]

\[
(\iota_e \circ \kappa_e)(x_1 \cdots x_n, C)
= \int_{Y^e} \kappa_e^*(x_1 \cdots x_n, d\bar{y}) \iota_e(y, C)
= \int_{Y^e} \kappa_e^*(x_1,\ldots,x_n,F^{-1}(d\bar{y})) \iota_e(y, C)
= \int_{Y^e} \kappa_e^*(x_1,\ldots,x_n, d\bar{y}) \iota_e(F(\bar{y}), C)
\]

by the def of \( \kappa_e \) in (11)

by (13) of variable change
\[= \int_{Y^*} \kappa_e^*((x_1, \ldots, x_n), d(y_1, \ldots, y_n)) \omega_e(F((y_1, \ldots, y_n), \mathbb{C})) \quad \text{by } \omega_e = (y_1, \ldots, y_n) \in Y^* \cap Y^{*n}\]

\[= \int_{Y^*} \kappa_e^*((x_1, \ldots, x_n), d(y_1, \ldots, y_n)) *_e((y_1 \ldots y_n), F^{-1}(\mathbb{C})) \quad \text{by the def of } \omega_e\]

\[= \int_{Y^*} \kappa^e((x_1, \ldots, x_n), d(y_1, \ldots, y_n)) \omega_e^n((y_1, \ldots, y_n), \bigcup_{k=0}^{\infty} C_{k,1} \times \cdots \times C_{k,n}) \quad \text{by } \sigma\text{-additivity}\]

\[= \sum_{k=0}^{\infty} \int_{Y^*} \kappa^e((x_1, \ldots, x_n), d(y_1, \ldots, y_n)) \omega_e^n((y_1, \ldots, y_n), C_{k,1} \times \cdots \times C_{k,n}) \quad \text{by the product measure}\]

\[= (\omega \circ \kappa)^e((x_1, \ldots, x_n), F^{-1}(\mathbb{C})) \quad \text{by the def of } (\omega \circ \kappa)_e^e\]

\[= (\omega \circ \kappa)_e(x_1 \cdots x_n, \mathbb{C}) \quad \text{by the def of } (\omega \circ \kappa)_e\]

\[\square\]

**Remark 3.12** (The exponential construction \((-)\) preserves s-finiteness, but not finiteness.). In addition that the class retains Fubini-Tonelli for the functorial monoidal product in Section 2.3, the class of s-finiteness is employed in this paper because it makes \((-)\) an endofunctor as shown above. E.g., its restriction on \(\mathsf{Tker}\) of the finite kernels is no more an endofunctor but from \(\mathsf{Tker}\) to \(\mathsf{TsKer}\).

**3.3. A Linear Exponential Comonad over \(\mathsf{TsKer}^p\)**

The exponential presented in Section 3.1 and Section 3.2 is shown to provide a linear exponential comonad over the monoidal category \(\mathsf{TsKer}^p\) with countable biproducts, hence a categorical model of the exponential modality of linear logic [2, 28, 31].

Due to the asymmetry between the first (measures) and the second (measurable functions) arguments of transition kernels in continuous measurable spaces, the exponential comonad considered in Subsection 3.3 is for the opposite category \(\mathsf{TsKer}^{op}\) 2.

**Notation for morphisms in the opposite \(\mathsf{TsKer}^{op}\):** The category considered in this section is the opposite category \(\mathsf{TsKer}^{op}\) so that the composition is converse to \(\mathsf{TsKer}\). In \(\mathsf{TsKer}^{op}\), a morphism \(\kappa : (X, A') \rightarrow (Y, Y')\) is a transition kernel from \((Y, Y')\) to \((X, A')\). Accordingly, a morphism \(\kappa\) is denoted by \(\kappa(A, y)\) meaning that its left (resp. right) argument determines a measure (resp. a measurable function). In particular, the Dirac delta measure which is the identity morphism on \((X, A')\) is written by \(\delta(A, x)\). Accordingly, the composition of two morphisms \(\kappa(A, y) : (X, A') \rightarrow (Y, Y')\) and \(\iota(B, z) : (Y, Y') \rightarrow (Z, Z')\) in \(\mathsf{TsKer}^{op}\) is

\[\iota \circ \kappa(A, z) = \int_Y \kappa(A, y) \iota(dy, z)\]

\[\text{By the monotone convergence theorem together with the commuting integral with finite sum}\]

\[\text{Our choice of opposite later in Section 5 turns out to coincide with Danos-Ehrhard’s (left) choice of permutation of their formulation of exponential in \(\mathsf{Pcoh}\). See Remark 5.21.}\]
**Typographic Convention:** In what follows, the following typography is used to discriminate levels of the exponential measurable spaces: \(x, y, z, \ldots \in X\) and \(A, B, C, \ldots \in \mathcal{X}\) for \((X, \mathcal{X})\). \(x, y, z, \ldots \in X_e\) and \(A, B, C, \ldots \in \mathcal{X}_e\) for \((X_e, \mathcal{X}_e)\). \(r, i, j, \ldots \in X_{ee}\) and \(A, B, C, \ldots \in \mathcal{X}_{ee}\) for \((X_{ee}, \mathcal{X}_{ee})\).

We recall the definition of linear exponential comonad.

**Definition 3.13** (linear exponential comonad [28, 31]). Let \((\mathcal{C}, \otimes, I)\) be a symmetric monoidal category. A linear exponential comonad on \(\mathcal{C}\) is a monoidal comonad

\[
(! : C \rightarrow C, s : ! \rightarrow !, d : ! \rightarrow \text{Id}_C, m_{X,Y} : !X \otimes !Y \rightarrow !(X \otimes Y), m_I : I \rightarrow !I)
\]

equipped with two monoidal natural transformations \(c : ! \rightarrow \Delta \circ !\) (with \(\Delta\) denoting the diagonal functor for the tensor) and \(w : ! \rightarrow I\) such that the following holds for each \(X\):

- \((!X, c_X, w_X)\) forms a commutative comonoid.
- \(c_X\) is a coalgebra morphism from \((!X, s_X)\) to \((!X \otimes X, m_{X,X} \circ (s_X \otimes s_X))\).
- \(w_X\) is a coalgebra morphism from \((!X, s_X)\) to \((I, m_I)\).
- \(s_X\) is a comonoid morphism from \((!X, c_X, w_X)\) to \(((!X, c_X, w_X))\).

We start to construct the structure maps in \(TsKer^{op}\) for the linear exponential comonad.

**Proposition 3.14** (Dereeliction). \(d_X : (X_e, \mathcal{X}_e) \rightarrow (X, \mathcal{X})\) is defined for \(A \in \mathcal{X}_e\) and \(x \in X\)

\[
d_X(A, x) := \delta(A \cap X^{(1)}, x)
\]

Recall that \(A \cap X^{(1)} \subset X\). Then, this gives a natural transformation \(d : (\_)_e \rightarrow \text{Id}_{TsKer^{op}}\).

**Proof.**

Let \(A \in \mathcal{X}_e\) and \(y \in Y\). Given \(\kappa : \mathcal{X} \rightarrow \mathcal{Y}\), \(d_Y \circ \kappa_e(A, y) = \int_{X_e} \kappa_e(A, x) d_Y(dx, y) = \int_{Y_e \cap Y^{(1)}} \kappa_e(A, x) \delta(dx \cap Y^{(1)}, y) = \kappa_e(A, y)\). While, \(\kappa \circ d_X(A, y) = \int_X d_X(A, x) \kappa(dx, y) = \int_X \delta(A \cap X^{(1)}, x) \kappa(dx, y) = \kappa(A \cap X^{(1)}, y)\). The both HSs coincide because \(\kappa_e(A, y) = \kappa(A \cap X^{(1)}, y)\) for \(y \in Y_e \cap Y^{(1)}\).

In order to introduce the storage in Proposition 3.17, we prepare;

**Definition 3.15** (\(| : X_{ee} \rightarrow X_e\)). For a set \(X\), the mapping | is defined by

\[
| : X_{ee} \rightarrow X_e \quad a_1 \cdots a_n \mapsto |a_1 \cdots a_n| := a_1 \cdots a_{1k_1} \cdots a_{nk_n}
\]

where every \(a_i \in X\) is \(a_{1i} \cdots a_{ki}\) with each \(a_{ij} \in X\).

Note:

- \(| : X_{ee} \cap (X_e)^{(1)}|\) is the identity. That is, when \(n = 1\) so that \(a_1 \in X_{ee}\), it holds \(|a_1| = a_1\).
- \(| : X_{ee} \cap (X_e \cap X^{(1)})^{(n)}|\) is the identity. That is, when \(k_i = 1\) for all \(i = 1, \ldots, n\) so that \(a_{11} \cdots a_{n1} \in X_{ee}\), it holds \(|a_{11} \cdots a_{n1}| = a_{11} \cdots a_{n1}\).

For any \(A \in \mathcal{X}_e\), its inverse image along | is defined by

\[
|A|^{-1} = \{ \eta \in X_{ee} \mid \text{ such that } \mid \eta \mid \in A \}
\]

The following lemma 3.16 ensures that the inverse image \(|A|^{-1}\) belongs to \(\mathcal{X}_{ee}\).
Lemma 3.16. The function $| \cdot |$ of Definition 3.15 is $(\mathcal{X}_e, \mathcal{X}_e)$-measurable.

Proof. For any $A \in \mathcal{X}_e \cap X^{(n)}$ with an arbitrary $n$, we show that $|A|^{-1}$ (i.e., the the inverse image of $A$ along $| \cdot |$) belongs to $\mathcal{X}_e$. But this is equivalent to show that the inverse image of $A$ along the composition $| \cdot | \circ F : (X_e)_e \to (X_e)_e \to X_e$, 

$$\left( | \cdot | \circ F \right)^{-1}(A) = F^{-1}(|A|^{-1}) \quad (14)$$

belongs to $(X_e)_e^*$, for which $F$ denotes the forgetful map in Definition 3.2 for the adequate type. Observe that (14) becomes a subset of $\bigcup X^{(n_1)} \times \cdots \times X^{(n_k)}$, whose union ranges over $(n_1, \ldots, n_k)$ such that $\sum_{i=1}^k n_i = n$.

In what follows in the proof all the $U$ is the same as this.

Consider the $k$-folding cartesian product $F^k$ of the forgetful $F : X_e^* \to X_e$, whose restriction on the domain yields $X^{n_1} \times \cdots \times X^{n_k} \to X^{(n_1)} \times \cdots \times X^{(n_k)}$. The inverse image of (14) along the union $\bigcup$ of the foldings coincides with $F^{-1}(A)$. That is,

$$\left( \bigcup (F^k)^{-1} \right)(14) := \bigcup (F^k)^{-1}(F^{-1}(|A|^{-1})) = F^{-1}(A), \quad (15)$$

whose RHS belongs to $X_e^* \times X^{n_1}$ by the choice $A$. This means (14) belongs to

$$\bigcup (X_e \cap X^{(n_1)}) \times \cdots \times (X_e \cap X^{(n_k)}) \quad (16)$$

as $X_e^* \times X^{n_1} = \bigcup (F^k)^{-1}(X_e \cap X^{(n_1)}) \times \cdots \times (X_e \cap X^{(n_k)}))$. The assertion has been proven as (16) $\subseteq |\bigcup (X_e)_e^*| \subseteq (X_e)_e^*$.

Proposition 3.17 (Storage). Storage, also called digging, $s_{\mathcal{X}} : (X_e, \mathcal{X}_e) \to (X_{ee}, \mathcal{X}_{ee})$ is defined for $A \in \mathcal{X}_e$ and $\eta \in X_{ee}$

$$s_{\mathcal{X}}(A, \eta) := \delta(A, |\eta|) = \delta(|A|^{-1}, \eta)$$

Then, this gives a natural transformation $s : (\_)_e \to (\_)_{ee}$.

Proof. We show that $\kappa_{ee} \circ s_{\mathcal{X}} = s_{\mathcal{Y}} \circ \kappa_e$ for any $\kappa : \mathcal{X} \to \mathcal{Y}$. Let $A \in \mathcal{X}_e$ and $y_1 \cdots y_k \in Y_{ee}$.

$$\begin{align*}
LHS(A, y_1 \cdots y_k) &= \int_{X_{ee}} s_{\mathcal{X}}(A, z_1 \cdots z_k) \kappa_{ee}(dz_1 \cdots z_k, y_1 \cdots y_k) \\
&= \int_{X_{ee}} \delta(|A|^{-1}, z_1 \cdots z_k) \kappa_{ee}(dz_1 \cdots z_k, y_1 \cdots y_k) = \kappa_{ee}(|A|^{-1}, y_1 \cdots y_k) \\
RHS(A, y_1 \cdots y_k) &= \int_{Y_{ee}} \kappa_e(A, z) s_{\mathcal{Y}}(dz, y_1 \cdots y_k) \\
&= \int_{Y_{ee}} \kappa_e(A, z) \delta(dz, |y_1 \cdots y_k|) = \kappa_e(A, |y_1 \cdots y_k|)
\end{align*}$$

The both HSs coincide by the following Lemma 3.18. \qed

Lemma 3.18. For any $\kappa : \mathcal{X} \to \mathcal{Y}$, the following holds for any $A \in \mathcal{X}_e$ and $\eta \in X_{ee}$:

$$\kappa_{ee}(|A|^{-1}, \eta) = \kappa_e(A, |\eta|)$$

Proof. Let $\eta$ be $y_1 \cdots y_r \in X_{ee}$ so that $y_i = y_{i1} \cdots y_{in_i} \in X_e$ with $i = 1, \ldots, r$. Then it suffices to consider $A \in \mathcal{X}_e \cap X^{(n)}$ with $n = \sum_{i=1}^r n_i$.

In the following $F$ and $F$ are the same as in the proof of Lemma 3.16.

$$\begin{align*}
RHS(A, |y_1 \cdots y_r|) &= \kappa_e(A, y_{11} \cdots y_{1n_1} \cdots y_{r1} \cdots y_{rn_r}) \\
&= \kappa_e^{(F^{-1}(A), (y_{11}, \ldots, y_{1n_1}, \ldots, y_{r1}, \ldots, y_{rn_r}))} \\
&= \kappa_{ee}^{(F^{-1}(A) \cap X^{n}((y_{11}, \ldots, y_{1n_1}, \ldots, y_{r1}, \ldots, y_{rn_r})) \text{ by the choice } A \text{ with } n = \sum_{i=1}^r n_i,}
\end{align*}$$
Lemma 3.16.

Proposition 3.21.

\[ \text{The both HSs coincide by (15) of Lemma 3.16. The coincidence presumes the associativity of of the cartesian product so that} \]
\[ (\mathcal{Y}_1, \ldots, \mathcal{Y}_r) = (y_{11}, \ldots, y_{1n_1}, \ldots, y_{r1}, \ldots, y_{rn_r}). \]

\[ \square \]

Definition 3.19 (Monoidalness). In the definition, \( *^n \) denotes \( \ast \cdots \ast \in I_e \) for \( n \in \mathbb{N} \).

- \( m_I : (I, I) \rightarrow (I_e, I_e) \) is defined:

\[ m_I(I, *^n) := \begin{cases} 1 & \text{for } n \geq 1 \\ 0 & \text{for } n = 0 \end{cases} \]

To be short, \( m_I(I, *^n) = \min(n, 1) \).

- \( m_{X, Y} : (X_e, \mathcal{X}_e) \otimes (Y_e, \mathcal{Y}_e) \rightarrow ((X \times Y)_e, (X \otimes Y)_e) \) is defined for every rectangle \( A \times B \) with \( A \in \mathcal{X}_e, B \in \mathcal{Y}_e \) and every \( (x_1, y_1) \cdots (x_n, y_n) \in (X \times Y)_e \) for any \( n \in \mathbb{N} \).

\[ m_{X, Y}(A \times B, (x_1, y_1) \cdots (x_n, y_n)) := \delta(A, x_1 \cdots x_n) \delta(B, y_1 \cdots y_n) \]

Note for the definition, the finite rectangles with same dimensions suffice as the following holds

\[ m_{X, Y}(A \times B, (x_1, y_1) \cdots (x_n, y_n)) = m_{X, Y}((A \times B) \cap (X^{(n)} \times Y^{(n)}), (x_1, y_1) \cdots (x_n, y_n)) \]

Proposition 3.20. The dereliction \( d \) is a monoidal natural transformation with respect to the monoidalness \( m \) of Definition 3.19.

Proof. The two conditions (i) and (ii) are checked:

(i) The composition \( I \xrightarrow{m_I} I_e \xrightarrow{d_I} I \) is the identity.

\[ d_I \circ m_I(-, *) = \int_{I_e} m_I(-, z) d_I(dz, *) = \int_{I_e} m_I(-, z) \delta(dz \cap I^{(1)}, *) = \int_{I_e \cap I^{(1)}} m_I(-, z) \delta(dz, *) = m_I(-, *) \]

(ii) \( d_{X \otimes Y} \circ m_{X, Y} = d_X \otimes d_Y \)

\[ \text{LHS}(A \times B, (x, y)) = \int_{(X \times Y)_e} m_{X, Y}(A \times B, z) d_{X \otimes Y}(dz, (x, y)) \]

\[ = \int_{(X \times Y)_e} m_{X, Y}(A \times B, z) \delta(dz \cap (X \times Y)^{(1)}, (x, y)) \]

\[ = \int_{(X \times Y)_e \cap (X \times Y)^{(1)}} m_{X, Y}(A \times B, z) \delta(dz, (x, y)) \]

\[ = m_{X, Y}(A \times B, (x, y)) = \delta(A, x) \delta(B, y) = \delta(A \cap X^{(1)}, x) \delta(B \cap Y^{(1)}, y) \]

\[ = d_X(A, x) d_Y(B, y) = RHS(A \times B, (x, y)) \]

\[ \square \]

Proposition 3.21. \( (( )_e, d_X, s_X) \) is a comonad on \( \text{TsKer}^* \).
Proof. The two conditions (i) and (ii) are checked. In the proof $\varepsilon = x_1 \cdots x_n \in X_e \cap X^{(n)}$ for any $n$ and $A \in \mathcal{X}_e$.

(i) $d_{\mathcal{X}_e} \circ s_{\mathcal{X}} = \mathrm{Id}_{\mathcal{X}_e} = (d_{\mathcal{X}})_e \circ s_{\mathcal{X}}$

$LHS(A, \varepsilon) = \int_{X^{(n)}} s_{\mathcal{X}}(A, \eta) d_{\mathcal{X}}(d\eta, \varepsilon) = \int_{X^{(n)} \cap (X_e^{(1)})} s_{\mathcal{X}}(A, \eta) \delta(d\eta, \varepsilon) = s_{\mathcal{X}}(A, \varepsilon) = \delta(|A|^{-1}, \varepsilon)$

$RHS(A, \varepsilon) = \int_{X^{(n)}} s_{\mathcal{X}}(A, \eta) (d_{\mathcal{X}})_e(d\eta, \varepsilon) = \int_{X^{(n)}} \delta(|A|^{-1}, \eta) (d_{\mathcal{X}})_e(d\eta, \varepsilon) = (d_{\mathcal{X}})_e(|A|^{-1}, \varepsilon)$

$= (d_{\mathcal{X}})^\bullet_n(F^{-1}(|A|^{-1}) \cap (X_e^{(n)}, (x_1, \ldots, x_n))) = \delta^\bullet_n(F^{-1}(|A|^{-1}) \cap (X_e \cap X^{(1)})^{*n}, (x_1, \ldots, x_n))$

$= \delta_e(|A|^{-1}, x_1 \cdots x_n) = \delta(|A|^{-1}, x_1 \cdots x_n)$

The both HSs coincide with $\mathrm{Id}_{\mathcal{X}_e}$ as $\delta(|A|^{-1}, x_1 \cdots x_n) = \delta(A, |x_1 \cdots x_n|)$ and $|x_1 \cdots x_n| = x_1 \cdots x_n$.

(ii) $s_{X_e} \circ s_{\mathcal{X}} = (s_{\mathcal{X}})_e \circ s_{\mathcal{X}}$

$LHS(A, -) = \int_{X^{(n)}} s_{\mathcal{X}}(A, \eta) s_{\mathcal{X}}(d\eta, -) = \int_{X^{(n)}} \delta(|A|^{-1}, \eta) s_{\mathcal{X}}(d\eta, -)$

$= s_{\mathcal{X}}(|A|^{-1}, -) = \delta(|A|^{-1}, -) = \delta(A, |\cdot|(|\cdot|))$

For RHS, let $- \in \mathcal{X}_e$ instantiated with $\eta_1 \cdots \eta_n \in X^{(n)}$ such that $\eta_i \in X_{ee}$ with $i = 1, \ldots, n$.

$RHS(A, \eta_1 \cdots \eta_n) = \int_{X^{(n)}} s_{\mathcal{X}}(A, \eta_1) (s_{\mathcal{X}})_e(d\eta_1, \eta_2 \cdots \eta_n) = \int_{X^{(n)}} \delta(|A|^{-1}, \eta_1) (s_{\mathcal{X}})_e(d\eta_1, \eta_2 \cdots \eta_n)$

$= (s_{\mathcal{X}})_e(|A|^{-1}, \eta_1 \cdots \eta_n) = (s_{\mathcal{X}})^\bullet_n(F^{-1}(|A|^{-1}) \cap X^{*n}, (\eta_1, \ldots, \eta_n)) = \delta^\bullet_n(F^{-1}(|A|^{-1}) \cap X^{*n}, (|\eta_1|, \ldots, |\eta_n|))$

$= \delta_n(F^{-1}(|A|^{-1}), (|\eta_1|, \ldots, |\eta_n|)) = \delta_e(|A|^{-1}, |\eta_1| \cdots |\eta_n|) = \delta(|A|^{-1}, |\eta_1| \cdots |\eta_n|)$

The both HSs coincide because of the following equality in $X_e$:

$$||\eta_1 \cdots \eta_n|| = ||\eta_1|| \cdots ||\eta_n||$$

In terms of the monoidalness, Proposition 3.17 is strengthened into

**Proposition 3.22 (Monoidality of $s$).** The natural transformation storage $s$ is monoidal. That is

$$s_{X \otimes Y} \circ m_{X, Y} = (m_{X, Y})_e \circ m_{X_e, Y_e} \circ (s_{X} \otimes s_{Y})$$

Note that the monoidality on the functor $(\cdot)_{ee}$ is given by $(m_{X, Y})_e \circ m_{X_e, Y_e}$.

Proof. In the proof, it is sufficient to consider an instantiation at any rectangle $A \times B \in \mathcal{X}_e \otimes \mathcal{Y}_e$ such that $A \in \mathcal{X}_e \cap X^{(n)}$ and $B \in \mathcal{Y}_e \cap Y^{(n')}$ for any $n, n' \geq 0$.

For LHS, by virtue of the note on $m_{X, Y}$ below Definition 3.19, we calculate the case $n = n'$, as the other case $n \neq n'$ directly makes LHS zero.

$LHS(A \times B, -) = \int_{(X \times Y)} m_{X, Y}(A \times B, z) s_{Y \otimes Y}(dz, -)$

$= \int_{(X \times Y)(n)} m_{X, Y}(A \times B, (x_1, y_1) \cdots (x_n, y_n)) s_{Y \otimes Y}(d(x_1, y_1) \cdots (x_n, y_n), -)$

$= \int_{(X \times Y)(n)} \delta(A, x_1 \cdots x_n) \delta(B, y_1 \cdots y_n) s_{Y \otimes Y}(d(x_1, y_1) \cdots (x_n, y_n), -)$

$= s_{Y \otimes Y}([A, B], -),$ in which $[A, B] := \{(x_1, y_1) \cdots (x_n, y_n) \mid x_1 \cdots x_n \in A \ y_1 \cdots y_n \in B, n \in \mathbb{N}\}$.

Note $(x_1, y_{\sigma(1)}) \cdots (x_n, y_{\sigma(n)}) \in [A, B]$ for any $\sigma \in S_n$ whenever $(x_1, y_1) \cdots (x_n, y_n) \in [A, B]$. (Symmetrically $(x_{\sigma(1)}, y_1) \cdots (x_{\sigma(n)}, y_n) \in [A, B]$ under the same condition.)
Let $-$ be instantiated with any element $e_1 \cdots e_m \in (X \otimes Y)_{ee} \cap ((X \times Y)_e)^{(m)}$ for any $m$ so that each $e_i = (x_{i1}, y_{i1}) \cdots (x_{im}, y_{im}) \in (X \times Y)_e$. Then
\[
LHS(A \times B, e_1 \cdots e_m) = s_X \otimes (s_B)(A, B), e_1 \cdots e_m) = \delta([A, B], [e_1 \cdots e_m]) \\
= \delta(A, x_{11} \cdots x_{i1} \cdots x_{im} \cdots x_{m1} \cdots x_{mn1} \cdots x_{mnm}) \delta(B, y_{11} \cdots y_{11} \cdots y_{1m} \cdots y_{mn1} \cdots y_{mnm})
\]
For (RHS), first observe,
\[
m_{x_1, y_1} \circ (s_X \otimes s_Y)(A \times B, -) = \int_{X_{ee}} \int_{Y_{ee}} (s_X \otimes s_Y)(A \times B, (x_1, y_2)) m_{x_1, y_1}(x_1, y_1) \delta(x_1, x_1) \delta(y_1, y_2) \delta(-, -) \\
= \int_{X_{ee}} \int_{Y_{ee}} \delta(|A|^{-1}, x_1) \delta(|B|^{-1}, y_1) m_{x_1, y_1}(x_1, y_1) \delta(x_1, x_1) \delta(y_1, y_1) \delta(-, -) = m_{x_1, y_1}(|A|^{-1} \times |B|^{-1}, -)
\]
Using the observation, the following is calculated in which $-$ denotes an arbitrary instantiation $e_1 \cdots e_m \in (X \times Y)_{ee} \cap ((X \times Y)_e)^{(m)}$ with any $m$ and $e_i \in (X \times Y)_e$.
\[
RHSA \times B, - \\
= \sum_{k=0}^{\infty} \int_{X_{ee}} \int_{Y_{ee}} m_{x_1, y_1} \circ (s_X \otimes s_Y)(A \times B, z) m_{x_1, y_1}(e_1 \cdots e_m)(dz, -) \\
= \sum_{k=0}^{\infty} \int_{X_{ee}} \int_{Y_{ee}} m_{x_1, y_1}(|A|^{-1} \times |B|^{-1}, (x_1, y_1) \cdots (x_k, y_k)(x_k, y_k)) (m_{x_1, y_1}(e_1 \cdots e_m)(d(x_1, y_1) \cdots (x_k, y_k), -)
\]
by (5) and commuting integral over countable sum
\[
= \sum_{k=0}^{\infty} \int_{X_{ee}} \int_{Y_{ee}} m_{x_1, y_1}(|A|^{-1} \times |B|^{-1}, (x_1, y_1) \cdots (x_k, y_k), -) m_{x_1, y_1}(e_1 \cdots e_m)(d(x_1, y_1) \cdots (x_k, y_k)) \\
= \int_{X_{ee}} \int_{Y_{ee}} \delta(|A|^{-1}, x_1) \cdots \delta(|B|^{-1}, y_1) \cdots \delta(|A|^{-1}, x_m) \cdots \delta(|B|^{-1}, y_m) m_{x_1, y_1}(e_1 \cdots e_m)(d(x_1, y_1) \cdots (x_m, y_m), -)
\]
as the sum solely contributes when $k = m$ (i.e., zero if $k \neq m$) by (7)
\[
= \int_{X_{ee}} \int_{Y_{ee}} \left[ \delta(|A|^{-1}, x_1 \cdots x_m) \delta(|B|^{-1}, y_1 \cdots y_m) \right] \left[ \delta(|A|^{-1}, x_1 \cdots x_m) \delta(|B|^{-1}, y_1 \cdots y_m) \right] \\
= \int_{X_{ee}} \int_{Y_{ee}} \left[ \delta(|A|^{-1}, x_1 \cdots x_m) \delta(|B|^{-1}, y_1 \cdots y_m) \right] \left[ \delta(|A|^{-1}, x_1 \cdots x_m) \delta(|B|^{-1}, y_1 \cdots y_m) \right] \\
by \text{the def of } (m_{x_1, y_1})_e
\]
by the variable change (13) along $F : (X_e \times Y_e)_e \rightarrow (X_e \times Y_e)_e$
\[
= \int_{X_{ee}} \int_{Y_{ee}} \delta(F^{-1}(|A|^{-1}), (x_1, \ldots, x_m)) \delta(F^{-1}(|B|^{-1}), (y_1, \ldots, y_m)) \prod_{i=1}^{m} m_{x_1, y_1}(d(x_i, y_i), e_i) \\
by \text{the def of } (m_{x_1, y_1})_e \text{ using the product measure}
\]
by putting explicitly $e_i = (x_{i1}, y_{i1}) \cdots (x_{im}, y_{im}) \in (X \times Y)_e$ with $i = 1, \ldots, m$
\[
= \delta(F^{-1}(|A|^{-1}), (x_{i1} \cdots x_{im}), \ldots, (x_{i1} \cdots x_{im})) \delta(F^{-1}(|B|^{-1}), (y_{i1} \cdots y_{im}), \ldots, (y_{i1} \cdots y_{im})) \\
= \delta(|A|^{-1}, x_1 \cdots x_m) \delta(|B|^{-1}, y_1 \cdots y_m) \\
\text{with } x_i = x_{i1} \cdots x_{im} \text{ and } y_i = y_{i1} \cdots y_{im}, (1 \leq i \leq m). \\
= \delta(A, |A|^{-1}, x_1 \cdots x_m) \delta(B, |B|^{-1}, y_1 \cdots y_m) \\
= \delta(A, x_{11} \cdots x_{i1} \cdots x_{im} \cdots x_{m1} \cdots x_{mn1} \cdots x_{mnm}) \delta(B, y_{11} \cdots y_{11} \cdots y_{1m} \cdots y_{mn1} \cdots y_{mnm})
\]
Both HSs coincide. \qed

**Proposition 3.23** (weakening and contraction).
Monoidal natural transformations $w_X$ and $c_X$ are defined:
(b) The commutative comonoid conditions are the following (a), (b) and (c): 

\[
\text{LHS}(X) = \text{RHS}(X) = 0
\]

The conditions for the coalgebra morphisms are the following (i) and (ii) respectively for the weakening and for the contraction:

(i) \( \text{Id} \) is the identity diagram.

(ii) \( \delta \) is the coproduct diagram.

Then \( (X \ltimes \text{Id} X \ltimes X) \) forms a commutative comonoid. Moreover \( \text{w}_X \) is a coalgebra morphism from \( (X \ltimes \text{Id} X \ltimes X) \) to \( (I, m_I) \) and \( \text{c}_X \) is a coalgebra morphism from \( (X \ltimes \text{Id} X \ltimes X) \) to \( (X \ltimes \text{Id} X \ltimes X) \) for the monoidal unit. The condition is checked as follows:

\[
\text{LHS}(A, \ast^n) = \int_{X} \text{w}_X(A, x) m_X(dx, \ast^n) = \text{w}_X(A, \ast) \triangleleft \ast^n = \delta(A, 0) \times \text{min}(n, 1)
\]

\[
\text{RHS}(A, \ast^n) = \int_{X} \text{c}_X(A, \eta) (\text{w}_X)_{c}(dn, \ast^n) = \int_{X} \delta(|A|^{-1}, \eta) (\text{w}_X)_{c}(dn, \ast^n)
\]

Thus, using this at the following final line,

\[
\text{RHS}(A, (z_1, y_1) \cdots (z_n, y_n)) = \int_{X} \text{c}_X(A, (\eta_1, \eta_2)) m_{X \ltimes X} (\eta_1 \times \eta_2, (z_1, y_1) \cdots (z_n, y_n))
\]
\[ \delta(A, || x_1 \cdots x_n ||, \ldots || y_1 \cdots y_n ||) \]

On the other hand,

\[ LHS(A, (x_1, y_1) \cdots (x_n, y_n)) = \int_{X_{x}} s_X(A, \eta)(c_X)_e(d\eta, (x_1, y_1) \cdots (x_n, y_n)) \]

\[ = \int_{X_{x \circ c_X}} \delta(|A|^{-1}, \eta)(c_X)_e(d\eta, (x_1, y_1) \cdots (x_n, y_n)) = (c_X)_e(|A|^{-1}, (x_1, y_1) \cdots (x_n, y_n)) \]

\[ = (c_X)_e(F^{-1}(|A|^{-1}), ((x_1, y_1), \ldots, (x_n, y_n))) = (c_X)_e(F^{-1}(|A|^{-1}) \cap X^\bullet, ((x_1, y_1), \ldots, (x_n, y_n))) \]

\[ = \delta_e(|A|^{-1}, |x_1 y_1| \cdots |x_n y_n|) = \delta(|A|^{-1}, |x_1 y_1| \cdots |x_n y_n|) = \delta(A, || x_1 y_1 | \cdots | x_n y_n ||) \]

The both HSs coincide because of the following equality in \( X_e \).

\[ || x_1 \cdots x_n ||, || y_1 \cdots y_n || = || x_1 y_1 | \cdots | x_n y_n || \]

We end this section with Theorem 3.25 summarising this section after the lemma below:

**Lemma 3.24** (comonoidality of \( s \)). \( s_X \) is a comonoid morphism from \( (X_e, c_X, w_X) \) to \( (X_e, c_X, w_X) \).

**Proof.** The two conditions need to be checked:

- \((s_X \otimes s_X) \circ c_X = c_{X_e} \circ s_X\)
  \[ LHS(A, (\eta_1, \eta_2)) = \int_{X_{x \circ c_X}} s_X(A, \tau) c_{X_e}(d\tau, (\eta_1, \eta_2)) = \int_{X_{x \circ c_X}} s_X(A, \tau) \delta(d\tau, |\eta_1, \eta_2|) = s_X(A, |\eta_1, \eta_2|) = (c_X)_e(|A|, |\eta_1, \eta_2|) \]

- \(w_{X_e} \circ s_X = w_X\)
  \[ LHS(A, *) = \int_{X_{x \circ c_X}} s_X(A, \eta) w_{X_e}(d\eta, *) = \int_{X_{x \circ c_X}} s_X(A, \eta) \delta(d\eta, 0) = s_X(A, 0) = \delta(A, |0|) \]

4. **Double Glueing and Orthogonality over \( \text{TsKer}^* \)**

This section constructs the double glueing over \( \text{TsKer}^* \) in accordance with Hyland-Schalk’s general categorical framework [28] for constructing the structure of linear logic, but without the assumption of any closed structure of the base category. In Section 4.1, a crude but non degenerate opposite pair is obtained between product and coproduct as well as between tensor and cotensor, lifting those but collapsed in the monoidal category \( \text{TsKer}^* \). Furthermore an exponential comonad is constructed for the glueing \( G(\text{TsKer}^*) \) over \( \text{TsKer}^* \). In Section 4.2, a new instance of Hyland-Schalk orthogonality is given in terms of Lebesgue integral between measures and measurable functions, owing to the measure theoretic study in the preceding sections. The instance in \( \text{TsKer} \) has an adjunction property, called reciprocal, in terms of an inner product using the integral. The reciprocal orthogonality enables us to retain the exponential comonad to the slack subcategory \( \mathbf{S}(\text{TsKer}^*) \). Following the framework [28], the double glueing considered in this paper is along hom-functors to the category of sets.
4.1. Double Glueing \( G(TKer^w) \) with Exponential Comonad

**Definition 4.1** (The category \( G(TKer^w) \)).
An object is a tuple \( (X, U, R) \) such that \( X \) is an object of \( TKer^w \), and \( U \) and \( R \) are sets \( U \subseteq TKer^w(I, X) \) and \( R \subseteq TKer^w(X, I) \). That is, \( U \) and \( R \) comprise specific classes of measurable functions and of measures respectively.

Each map from \( X = (X, U, R) \) to \( Y = (Y, V, S) \) is any \( TKer^w \) map \( \kappa : X \longrightarrow Y \) satisfying:
- \( \forall g : I \longrightarrow X \) in \( U \), the composition \( \kappa \circ g : I \longrightarrow X \longrightarrow Y \) belongs to \( V \).
- \( \forall \mu : Y \longrightarrow I \) in \( S \), the composition \( \kappa \circ \mu : X \longrightarrow Y \longrightarrow I \) belongs to \( R \).

The forgetful functor exists \( G(TKer^w) \longrightarrow TKer^w \) forgetting the second and the third components of the objects.

The double glueing category \( G(TKer^w) \) is defined the same over \( TKer^w \) and becomes a subcategory of \( G(TKer^w) \). The general result of Hyland-Schalk [28] applies to the subcategory.

**Proposition 4.2.** \( G(TKer^w) \) is a monoidal category with product and coproduct, which is collapsed to the corresponding structures of \( TsKer^w \) by the forgetful functor.

Given objects \( X = (X, U, R) \) and \( Y = (Y, V, S) \) of \( G(TKer^w) \),

- **Tensor product** \( X \otimes Y = (X \otimes Y, U \otimes V, T) \), where
  
  \[ U \otimes V = \{ f \otimes g : I \cong I \otimes I \longrightarrow X \otimes Y | f \in U, g \in V \} \]

  \[ T = \{ \nu : X \otimes Y \longrightarrow I | T_{f \otimes g}^{\nu} \in X, \nu^*(f \otimes \delta Y) \in S \text{ and } I_{\nu g}^{\nu Y} \in Y, \nu^*(\delta X \otimes g) \in R \} \]

  Note \( \nu^*(f \otimes \delta Y) : Y \cong I \otimes Y \longrightarrow X \otimes Y \nu \longrightarrow I \) and \( \nu^*(\delta X \otimes g) : X \cong X \otimes I \longrightarrow X \otimes Y \nu \longrightarrow I \).

  The tensor unit is given \( I = (I, \{Id_I\}, TsKer^w(I, I)) \).

  For a subset \( U \) of a homset and a morphism \( f \) of appropriate type, \( U \circ f \) and \( f \circ U \) denote the respective subsets composed and precomposed with \( f \) element-wisely to \( U \).

  **Product** \( X \& Y = (\chi \mathop{\Pi} Y, U \& V, (R \circ pr_X) \cup (S \circ pr_X)) \), where

  \[ U \& V := \{ u \& v : I \longrightarrow \chi \mathop{\Pi} Y | u \in U, v \in V \} \]

  Note \( u \& v \) denotes the mediating morphism for \( I \) as the product in \( TsKer^w \).

- **Coproduct** \( X \oplus Y = (\chi \mathop{\Pi} Y, (in_X \circ U) \cup (in_Y \circ V), R \oplus S) \), where

  \[ R \oplus S := \{ r \oplus s : \chi \mathop{\Pi} Y \longrightarrow I | r \in R, s \in S \} \]

  Note \( r \oplus s \) denotes the mediating morphism for \( I \) as the coproduct in \( TsKer^w \).

  The unit for the coproduct is \( (T, \emptyset, \{\emptyset\}) \).

**Remark 4.3** (product/coproduct and tensor/cotensor). The product and the coproduct of \( G(TKer^w) \) do not coincide, despite that the forgetful functor makes them collapse into the biproduct in \( TsKer^w \). Similarly, another tensor product is defined, say the cotensor \( \check{Y} \), owing to the nonsymmetricity of the second and the third components for the tensor object:

- **Cotensor product** \( X \check{\otimes} Y = (X \otimes Y, W, R \otimes S) \), where

  \[ R \otimes S = \{ \kappa \otimes \tau : X \otimes Y \longrightarrow I \otimes I \cong I | \kappa \in R, \tau \in S \} \]

  \[ W = \{ h : I \longrightarrow X \otimes Y | X^{\nu \kappa \in R} \longrightarrow I, (\kappa \otimes \delta Y)^* h \in V \text{ and } X^{\nu \tau \in S} \longrightarrow I, (\delta_X \otimes \tau)^* h \in U \} \]

  Note \( (\kappa \otimes \delta Y)^* h : I \longrightarrow X \otimes Y \longrightarrow I \otimes Y \cong Y \) and \( (\delta_X \otimes \tau)^* h : I \longrightarrow X \otimes Y \longrightarrow X \otimes I \cong X \).
Lemma 4.6. A natural transformation \( k : \text{Tsker}^\ast(I, -) \to \text{Tsker}^\ast(I, (-)_c) \) is defined by the following instance \( k_X(u) : I \to X \) for every \( u : I \to X \) in \( \text{Tsker}^\ast \):

\[
k_X(u)(I, x_1 \cdots x_n) := u_e(I^{(n)}, x_1 \cdots x_n)
\]

\[
= u^\ast(F^{-1}(I^{(n)}), (x_1, \ldots, x_n))
\]

\[
= u^n(I^n, (x_1, \ldots, x_n))
\]

\[
= \prod_{i=1}^n u(I, x_i)
\]

Definition 4.4 is well defined so that the naturality of \( k \):

\[
t_e \circ k_X(u) = k_Y(e \circ u) : I \to Y_c\quad \text{for any } e : X \to Y\text{ and } u : I \to X\text{ in } \text{Tsker}^\ast
\]

is shown as follows with any \( y_1, \ldots, y_n \in Y_c; \)

\[
RHS(I, y_1 \cdots y_n) = \prod_{i=1}^n (e \circ u)(I, y_i) = \prod_{i=1}^n \int_X u(I, x) e(dx, y_i)
\]

\[
LHS(I, y_1 \cdots y_n) = \int_X k_X(I, x) e(dx, y_1 \cdots y_n)
\]

\[
= \int_X k_X(I, x) e^\ast(F^{-1}(dx), (y_1, \ldots, y_n))
\]

\[
= \int_X^n k_X(I, x_1 \cdots x_n) e^\ast(dx(x_1, \ldots, x_n), (y_1, \ldots, y_n))
\]

\[
= \int_X^n \prod_{i=1}^n u(I, x_i) e(dx_i, y_i)
\]

\[
= \prod_{i=1}^n \int_X u(I, x) e(dx, y_i)
\]

Lemma 4.5. For any \( u : I \to X \),

\[
(d_X)^\ast(k_X(u)) = u
\]

That is, the natural transformation \( \text{Tsker}^\ast(I, d) : \text{Tsker}^\ast(I, -) \to \text{Tsker}^\ast(I, (-)_c) \) induced by \( d \) of Definition 3.14 is a left inverse of \( k \) so that \( \text{Tsker}^\ast(I, d) \circ k = \text{Id}_{\text{Tsker}^\ast(I, -)} \).

Proof.

\[
(d_X \circ k_X(u))(I, x) = \int_{X_c} k_X(u)(I, y) d_X(dy, x) = \int_{X_c \cap X^{(1)}} k_X(u)(I, y) d_X(dy, x) = k_X(u)(I, x) = u(I, x)
\]

The third equation is because \( d_X(dy, x) = \delta(dy, a) \) as \( y \in X_c \cap X^{(1)} \). \( \square \)

In order to define certain exponential comonad in \( G(\text{Tsker}^\ast) \), the following linear distributivity is crucial, guaranteeing to the comonad structure of \( \text{Tsker}^\ast \).

Lemma 4.6 (linear distributivity of \( k_X(-) \)). The natural transformation \( k_X(-) : \text{Tsker}^\ast(I, -) \to \text{Tsker}^\ast(I, (-)_c) \) meets the following criteria (i), (ii) and (iii) of Hyland-Schalk (cf. pg.209 [28]) in order to make \( \text{Tsker}^\ast(I, -) \) linear distributive:
(Remind the notation below that $C(I, -)$ is a functor so instantiated both by object and by morphism.)

(i) well-behavior wrt the comonad structure

\[
\begin{array}{ccc}
\text{Id}_{C(I, X)} & \xrightarrow{k_X} & C(I, X) \\
\downarrow & & \downarrow k_X \\
C(I, X) & \xrightarrow{k_X} & C(I, !X)
\end{array}
\]

(ii) respecting the comonoid structure

\[
\begin{array}{ccc}
\theta & \xleftarrow{\Delta_{C(I, X)}} & C(I, X) \otimes C(I, X) \\
\downarrow & & \downarrow k_X \times k_X \\
C(I, I) & \xleftarrow{C(I, w_X)} & C(I, !X) \\
\downarrow c_X & & \downarrow c_{I, !X} \\
C(I, I) & \xrightarrow{C(I, e_X)} & C(I, !X \otimes !X)
\end{array}
\]

(iii) monoidal

\[
\begin{array}{ccc}
C(I, I) & \xrightarrow{k_I} & C(I, !I) \\
\downarrow k_X & & \downarrow k_X \times k_Y \\
C(I, X) \times C(I, Y) & \xrightarrow{\otimes} & C(I, X \otimes Y) \\
\downarrow k_{X \otimes Y} & & \downarrow k_{X \otimes Y} \\
C(I, I) & \xrightarrow{C(I, m_g)} & C(I, !I)
\end{array}
\]

Proof. (ii) and (iii) are direct as so are the following equations stipulating the commutativity diagrams:

(ii) $k_X(u) \otimes k_X(u) = c_X \circ k_X(u)$ and $d_X \circ k_X(u) = \varphi \circ \exists_0$ where $\exists_0$ is the unique morphism to the empty measurable space (empty product as terminal object) and $\varphi : 0 \to \text{TsKer}^n(I, I)$.

(iii) $k_{X \otimes Y}(u \otimes v) = m_{X, Y}(k_X(u) \otimes k_Y(v))$ and $k_X(u) = m_{X} \circ u$

Hence, we need to prove (i) having two equalities:

(i-a) $k_X(k_X(u)) = s_X \circ k_X(u)$.

Let $x_1 \cdots x_m \in X_{ce}$ so that $x_i = x_{i1} \cdots x_{in_i}$ for certain $n_i$ with $i = 1, \ldots, m$.

Recall Definition 3.15 that $|x_1 \cdots x_m| = x_{11} \cdots x_{1n_1} \cdots x_{m1} \cdots x_{mn_m}$.

RHS$(I, x_1 \cdots x_m) = \int_{X_n} k_X(u)(I, y) s_X(dy, x_1 \cdots x_m) = \int_{X_n} k_X(u)(I, y) \delta(dy, |x_1 \cdots x_m|)$

\[
= k_X(u)(I, |x_1 \cdots x_m|) = u(I, x_{11}) \cdots u(I, x_{1n_1}) \cdots u(I, x_{m1}) \cdots u(I, x_{mn_m})
\]

LHS$(I, x_1 \cdots x_m) = \prod_{i=1}^{m} k_X(u)(I, x_i) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} u(I, x_{ij})$

(i-b) $d_X \circ k_X(u) = u$

This is by Lemma 4.5.

With Lemma 4.6, the following is an instance of $C = \text{TsKer}^n$ of Hyland-Schalk’s Proposition 36 of [28] on $G(C)$.

**Proposition 4.7** (Hyland-Schalk exponential comonad on glueing [28],). There are two kinds (I) and (II) of linear exponential comonad on $G(\text{TsKer}^n)$ as follows so that the forgetful functor to $\text{TsKer}^n$ preserves the structure. For an object $\mathcal{X} = (\mathcal{X}, U, R)$ in $G(\text{TsKer}^n)$,
(I) \( \mathcal{X}_c = (\mathcal{X}_c, k_\mathcal{X}(U), \text{TsKer}^\ast(\mathcal{X}_c, \mathcal{I})) \), where \( k_\mathcal{X}(U) = \{ k_\mathcal{X}(u) : \mathcal{I} \to \mathcal{X}_c \mid u \in U \} \)

(II) \( \mathcal{X}_c = (\mathcal{X}_c, k_\mathcal{X}(U), ?R) \), where \( k_\mathcal{X}(U) \) as above, but \( ?R \) is the smallest subset of \( \text{TsKer}^\ast(\mathcal{X}_c, \mathcal{I}) \)

(a) Containing \( \{(d_\mathcal{X})_* \mu : \mathcal{X}_c \xrightarrow{d_\mathcal{X}} \mathcal{X} \xrightarrow{\mu} \mathcal{I} \mid \mu \in R\} \)

(b) Containing the weakening \( w_\mathcal{X} : \mathcal{X}_c \to \mathcal{I} \)

(c) Closed under the following for the contraction \( c_\mathcal{X} : \)

For any \( h : \mathcal{X}_c \otimes \mathcal{X}_c \to \mathcal{I} \),

\[
\forall u \in U \ [ (k_\mathcal{X}(u) \otimes \text{Id}_{\mathcal{X}_c})_* h \in ?R \land (\text{Id}_{\mathcal{X}_c} \otimes k_\mathcal{X}(u))_* h \in ?R ] \implies (c_\mathcal{X})_* h \in ?R.
\]

### 4.2. Orthogonality as Relation between Measures and Measurable Functions

The Hyland-Schalk orthogonality relation [28], when applied concretely to the measure theoretic framework in the present paper, becomes a relation between measures and measurable functions over a measurable space. The relation is shown to satisfy a property “reciprocity”, which is derivable from the adjunction of the inner product in terms of Lebesgue integral of a measurable function over a measure. Our reciprocal orthogonality is strong enough to guarantee a certain relevant structure maps Hyland-Schalk employed in [28] to obtain the product and exponential structures for the slack orthogonality category \( \mathbf{S}(\mathcal{C}) \), which forms a subcategory of the double glueing \( \mathbf{G}(\mathcal{C}) \). Note in this subsection, we do not assume any closed structure (i.e., the linear implication). The category \( \mathcal{C} \) considered in this subsection is either \( \text{TsKer}^\ast \) or \( \text{TKer}^\ast \).

**Definition 4.8** (orthogonality on a monoidal category \( \mathcal{C} \)). An orthogonality on a monoidal category \( \mathcal{C} \) is a family of relation \( \perp_R \) between maps \( I \to R \) and those \( R \to I \) satisfying the following conditions on isomorphism, identity and on tensor for a monoidal category by Hyland-Schalk (cf. Definition 45 [28]).

Note: Although the original orthogonality is for a monoidal closed category, we consider a general monoidal one without the implication.

**Definition 4.8** (orthogonality on a monoidal category \( \mathcal{C} \)). An orthogonality on a monoidal category \( \mathcal{C} \) is a family of relation \( \perp_R \) between maps \( I \to R \) and those \( R \to I \) satisfying the following conditions on isomorphism, identity and on tensor for a monoidal category by Hyland-Schalk (cf. Definition 45 [28]).

\( \text{(isomorphism)} \)

If \( \iota : R \to S \) is an isomorphism then for any \( u : I \to R \) and \( x : R \to I \),

\[
u \perp_R x \iff \iota \circ u \perp_S x \circ \iota^{-1}
\]

\( \text{(identity)} \)

For all \( u : I \to R \) and \( x : R \to I \),

\[
u \perp_R x \quad \text{implies} \quad \text{Id}_I \perp_I x \circ u
\]

\( \text{(tensor)} \)

Given \( u : I \to R \), \( v : I \to S \) and \( h : R \otimes S \to I \), \( u \perp_R h \circ (\text{Id}_R \otimes v) \) and \( v \perp_S h \circ (u \otimes \text{Id}_S) \) imply \( u \otimes v \perp_{R\otimes S} h \).

For \( U \subseteq \mathcal{C}(I, R) \), its orthogonal \( U^\circ \subseteq \mathcal{C}(R, J) \) is defined by

\[
U^\circ := \{ x : R \to J \mid \forall u \in U \ u \perp_R x \}
\]

This gives a Galois connection so that \( U^{\circ \circ} = U^\circ \). The operator \( (\ )^\circ \) is called the closure operator in the sequel.

**Lemma 4.9** (reciprocal orthogonality). If a family of relation satisfies the following for every \( u : I \to R \), \( x : S \to J \), and \( f : R \to S \),

\[
u \perp_R x \circ f \quad \text{if and only if} \quad f \circ u \perp_S x,
\]

then this becomes an orthogonality relation. An orthogonality satisfying (17) is called reciprocal.
Proof. We show the condition (17) entails the three conditions (isomorphism), (identity) and (tensor) of Definition 4.8. Moreover, the (tensor condition) is strengthened into the (precise tensor) obtained by replacing “imply” with “iff”. The derivation is direct for the isomorphism and the identity conditions. For the precise tensor condition, observing 

\[ u \otimes v = (\text{Id}_R \otimes v) \circ u = (u \otimes \text{Id}_R) \circ v, \]

one antecedent implies the descendant by reciprocity either on \( \text{Id}_R \otimes v \) or on \( u \otimes \text{Id}_S \), and vice versa. 

In order to construct an orthogonality on \( \text{TsKer}^\circ \), we define an inner product of \( \text{TsKer}^\circ \), which has the adjunction property:

**Definition 4.10** (inner product). For a measure \( \mu \in \text{TKer}^\circ (\mathcal{X}, \mathcal{I}) \) and a measurable function \( f \in \text{TKer}^\circ (\mathcal{I}, \mathcal{X}) \), we define

\[ \langle f \mid \mu \rangle_{\mathcal{X}} := \int_{\mathcal{X}} f d\mu \]

Then the two operators in Definition 2.2 become characterised as follows:

**Lemma 4.11** (adjunction between \( \kappa^* \) and \( \kappa_* \)). In \( \text{TsKer}^\circ \), for any measure \( \mu : \mathcal{X} \to \mathcal{I} \), any measurable function \( f : \mathcal{I} \to \mathcal{Y} \) and any transition kernel \( \kappa : \mathcal{Y} \to \mathcal{X} \),

\[ \langle f \mid \kappa_* \mu \rangle_{\mathcal{Y}} = \langle \kappa^* f \mid \mu \rangle_{\mathcal{X}} \]

Remind that \( \kappa_* \mu = \mu \circ \kappa \) and \( \kappa^* f = \kappa \circ f \).

**Proof.** The following starts from LHS and ends with RHS of the assertion, using Fubini-Tonelli:

\[ \int_{\mathcal{Y}} f(y)(\kappa_* \mu)(dy) = \int_{\mathcal{Y}} f(y) \int_{\mathcal{X}} \kappa(dy, x) \mu(dx) = \int_{\mathcal{X}} \mu(dx) \int_{\mathcal{Y}} f(y) \kappa(dy, x) = \int_{\mathcal{X}} (\kappa^* f)(x) \mu(dx) \]

Using the inner product, an orthogonality relation on \( \text{TsKer}^\circ \) is defined.

**Definition 4.12** (orthogonality in terms of integral). For a measurable function \( f \in \text{TsKer}^\circ (\mathcal{I}, \mathcal{X}) \) and a measure \( \mu \in \text{TsKer}^\circ (\mathcal{X}, \mathcal{I}) \), the relation \( \perp_{\mathcal{X}} \subset \text{TsKer}^\circ (\mathcal{I}, \mathcal{X}) \times \text{TsKer}^\circ (\mathcal{X}, \mathcal{I}) \) is defined

\[ f \perp_{\mathcal{X}} \mu \iff \langle f \mid \mu \rangle_{\mathcal{X}} \leq 1 \]

**Proposition 4.13** (\( \perp_{\mathcal{X}} \) is a reciprocal orthogonality in \( \text{TsKer}^\circ \)). The relation \( \perp_{\mathcal{X}} \) defined in Definition 4.12 is an orthogonality in \( \text{TsKer}^\circ \), and moreover is reciprocal.

**Proof.** By Lemma 4.9, it suffices to show that the relation is reciprocal to satisfy the condition (17), which is derived by Lemma 4.11.

The orthogonality of Definition 4.12 gives rise to the following full subcategory of \( G(\text{TsKer}^\circ) \).

**Definition 4.14** (slack category \( S(\text{TsKer}^\circ) \) (cf. [28] for the general definition)). The slack orthogonality category \( S(\text{TsKer}^\circ) \) is the full subcategory of \( G(\text{TsKer}^\circ) \) on those objects \( (\mathcal{X}, U, R) \) such that \( U \subseteq R^\circ \) and \( R \subseteq U^\circ \).
Example 4.15 (objects of $\mathbf{S}(\mathbf{TsKer}^w)$). 3
The following independent examples (i) and (ii) guarantee that the slack category of Definition 4.14 is not
degenerate so that $U$ and $R$ become in general continuous.

(i) When $(X, \mathcal{X})$ is the Borel-field $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let $\mu$ be the Lebesgue measure over $\mathcal{B}(\mathbb{R})$ and $f$ be a density function of a probability distribution on $X$. That is, $f$ is Lebesgue integrable such that $\int_{-\infty}^{a} f(x)dx = \mu((-\infty, a])$ for a certain probability measure $\mu$ on $X$ with any $a$. A particular example is well known for the exponential distribution with rate $r$, for which $f$ is given by $f(x) = re^{-rx}$ for $x \geq 0$, where it equals zero otherwise.

Define $U_f := \{\chi_{(a,b)}f \mid a < b \in \mathbb{R}\}$ and $R_\mu := \{\lambda A \in \mathcal{X}, \mu(A \cap (a,b)) \mid a < b \in \mathbb{R}\}$,
in which $\chi_{(a,b)}$ is the characteristic function of an interval $(a, b)$.
Then $(X, U_f, R_\mu)$ is an object of the slack category.

\[
\int_{-\infty}^{\infty}(\chi_{(a,b)}f)(x)dx = \int_{a}^{b}f(x)dx = \int_{(a,b)} f(x)dx \leq \int_{-\infty}^{\infty} f(x)dx = 1
\]

The last equality and inequality are the properties of density functions.

The density function for a probabilistic distribution is widely recognised for providing the importance sampling procedure in probabilistic programming. This procedure involves drawing a sample from a distribution based on its density function, which describes the likelihood of a scored point. For syntax and semantics on sampling concerning the score, refer to [3, 39, 40]. In this context, our second component $U_f$ may be seen to comprise sampling procedures whose $\chi_{(a,b)}$ function serves to confine the sampling space within the interval in $\mathbb{R}$. Each sampling choice is implemented through a double-glueing morphism $\kappa$ from the slack tensor unit $(I, \{\text{Id}_I\}, \{\text{Id}_I\}^\circ)$ (cf. Lemma 4.16 below) to the object as dictated by the first morphism condition $\kappa \circ \text{Id}_I \in U_f$ in Definition 4.1, while the second condition, becoming the orthogonality $\mu' \circ \kappa = \int_{a}^{b} \kappa(x)dx \leq 1$ for $\mu' \in R_\mu$, is automatic by the slack object. Note that a kernel $\kappa$ of this type in $\mathbf{TsKer}^w$ is identified with a measurable function on $X$.

(ii) Given any measurable space $(X, \mathcal{X})$ and any $s$-finite measure $\mu$ (i.e., an element $\mathbf{TsKer}^w((X, \mathcal{X}))$). $L_1^+(X, \mu)$ denotes the subclass of non-negative measurable $h$ whose $L_1$-norm is finite; i.e., $\|h\| := \int_X h d\mu < \infty$. Note that $L_1^+(X, \mu)$ is contained in $\mathcal{E}(X)$ defined in Proposition 2.3. Recall in the proposition that a transition kernel $\kappa : X \times \mathcal{X} \to \mathbb{R}_+$ induces the endomap $\kappa^*$ on $\mathcal{E}(X)$, hence on $L_1^+(X, \mu)$. In this example, $\kappa$ is called a $\mu$-contraction when $\|\kappa^*x\| \leq \|\mu\|$ for any $x \in L_1^+(X, \mu)$. That is, $\kappa^*$ on $L_1^+(X, \mu)$ contracts the norm.

Fix an arbitrary $f \in \mathbf{TsKer}^w((I, \mathcal{X}))$ such that $\|f\| \leq 1$. Define

\[
U_f := \{\kappa^*f \mid \kappa : \mathcal{X} \to \mathcal{X} \text{ a contraction } s\text{-finite kernel}\} \\
R_\mu := \{\kappa_\mu \mid \kappa : \mathcal{X} \to \mathcal{X} \text{ is a contraction } s\text{-finite kernel}\}
\]

Note that $f \in U_f$ and $\mu \in R_\mu$ as we may take $\kappa$ to be the unit kernel which is a $\mu$-contraction.

$(X, U_f, R_\mu)$ is an object of the slack category as the two inclusions of Definition 4.14 hold by the following
for any $\mu$-contraction kernels $\kappa$ and $\tau$ on $\mathcal{X}$:

\[
\langle \kappa^*f | \tau_\mu \rangle_X = \langle \tau^*\kappa^*f | \mu \rangle_X = \langle (\kappa \circ \tau)^*f | \mu \rangle_X \leq \langle f | \mu \rangle_X \leq 1
\]

The first equality is by reciprocity and third inequality is by preservation of contraction under kernel com-
position.

Contraction kernels $\kappa$’s in (18) are exemplified in terms of conditional expectations for a probability measure $\mu$: Given a probability space $(X, \mathcal{X}, \mu)$ and a sub $\sigma$-field $\mathcal{G}$ of $X$, the conditional expectation $E[g | \mathcal{G}]$ of

3The example is not a prerequisite for the rest.
\( g \in L^+_1((X, \mathcal{X}), \mu) \) is the \( \mathcal{G} \)-measurable function such that \( \int_A E[g \mid \mathcal{G}] d\mu = \int_A g d\mu \) for all measurable \( A \in \mathcal{G} \). Then the following inequality is directly derivable

\[
\|E[g \mid \mathcal{G}]\| \leq \|g\|
\]

from the well know property \( E[E[g \mid \mathcal{G}] ] = E[g] \) together with Jensen’s inequality \( |E[g \mid \mathcal{G}]| \leq E[|g| \mid \mathcal{G}] \) (cf. respectively (15.6) and (15.12) of \cite{1}).

The inequality means the conditional expectation contracts the \( L_1 \)-norm:

\[
E[\sim \mid \mathcal{G}] : L^+_1((X, \mathcal{X}), \mu) \longrightarrow L^+_1((X, \mathcal{G}), \mu) \quad g \mapsto E[g \mid \mathcal{G}]
\]  

(19)

The conditional expectation map (19) is known to be linear, positive and preserving monotone convergence (cf. Section 15 of \cite{1} in particular (15.13) for the conditional monotone convergence). These conditions happen to be the same as those for morphisms in \( \mathcal{M}_E \) used in Proposition 2.3. Hence the same argument applies to conclude that the map (19) coincides with \( \kappa^* \) on \( L^+_1((X, \mathcal{X}), \mu) \) for certain kernel \( \kappa \).

Consider when \( \mathcal{X} = \mathcal{B}(\mathbb{R}) \) and \( \mathcal{G} \) is generated by the Borel subsets of an interval \( (a, b) \). Then the conditional expectation of (19) is

\[
E[g \mid \mathcal{G}] = \frac{1}{b-a} \chi_{(a,b)} \int_a^b g(t) dt
\]

This determines the action \( \kappa^* \) on functions, and correspondingly the action \( \kappa_* \) on measures by means of \( \kappa_* \mu(B) = \int_X \kappa^*_B(x) \mu(dx) \) (cf. (1)). In particular, \( U_f \) and \( R_\mu \) of (18) contain the following respective uncountable subsets:

\[
\{ \frac{1}{b-a} \chi_{(a,b)} \int_a^b f(t) dt \mid a < b \in \mathbb{R} \} \quad \text{and} \quad \{ \lambda A \in \mathcal{X}, \mu(A \cap (a,b)) \mid a < b \in \mathbb{R} \}
\]

The above construction of the conditional expectation is generalised when a given measure \( \mu \) is s-finite. We may write \( \mu = \sum_i \mu_i \) with each \( \mu_i \) being a probability measure. For each probability space \( ((X, \mathcal{X}), \mu_i) \), the conditional expectation \( E_i[\sim \mid \mathcal{G}] \) yields an endomap \( (\kappa_i)^* \) on \( \mathcal{E}(\mathcal{X}) \) for certain contraction kernel \( \kappa_i \). Then s-finite \( \kappa := \sum_i \kappa_i \) becomes a contraction s-finite kernel by virtue of the property on the mixture of measures \( \int_X g d(\sum_i \mu_i) = \sum_i (\int_X g d\mu_i) \).

**Lemma 4.16** (monoidal product in \( \mathbf{S}(\mathbf{TsKer}^w) \)). \( \mathbf{S}(\mathbf{TsKer}^w) \) is a monoidal subcategory of \( \mathbf{G}(\mathbf{TsKer}^w) \) with the tensor unit \( (\mathcal{I}, \{\text{Id}_\mathcal{I}\}, \{\text{Id}_\mathcal{I}\}^\circ) \).

**Proof.** By virtue of the tensor condition for orthogonality, the monoidal product of Proposition 4.2 is shown closed in the subcategory \( \mathbf{S}(\mathbf{TsKer}^w) \): It suffices to show that the third component of \( (\mathcal{X}, U, R) \otimes (\mathcal{Y}, V, S) \) is perpendicular to \( U \otimes V \). Take \( \nu \) from the third component, then \( \forall f \in U \nu^*(f \otimes \delta_y) \in S \subseteq U^w \) and \( \forall g \in V \nu^*(\delta_X \otimes g) \in R \subseteq U^w \), but which implies \( \nu \bot_{\mathcal{X} \otimes \mathcal{Y}} f \otimes g \) by the tensor condition of the orthogonality of Definition 4.8.

In [28], to obtain an exponential structure as well as an additive one for the slack category, Hyland-Schalk employ certain relevant structure maps for a general category \( \mathcal{C} \). We remark that \( \mathcal{C} = \mathbf{TsKer}^w \) in this paper, automatically validates their structure maps:

**Remark 4.17** (Hyland-Schalk’s positive and negative maps are implicated by reciprocity). Hyland-Schalk (in Definition 51 of [28]) call a map \( f : R \rightarrow S \) is positive (resp. negative) with respect to \( U \subseteq \mathcal{C}(\mathcal{I}, R) \) and \( Y \subseteq \mathcal{C}(S, I) \) when \( f \circ u \perp_S y \) implies (resp. is implied by) \( u \perp_R y \circ f \) for all \( u \in U, y \in Y \). When \( U \) and \( Y \) are the whole homsets, they say positive and negative outright. When a map is both positive and negative, it is called focused. We remark that our reciprocity (17) in \( \mathbf{TsKer}^w \) ensures these property on maps: That is, if an orthogonality is reciprocal, then any map is automatically both positive and negative hence focused with respect to any \( U \) and \( Y \).
By the remark, the slack category $S(TsKer^*)$ over $TsKer^*$, has the product and coproduct, and moreover the exponential comonad as follows.

**Proposition 4.18 (product and coproduct in $S(TsKer^*)$).** The slack category $S(TsKer^*)$ is closed under the product and the coproduct in $G(TsKer^*)$ of Proposition 4.2.

**Proof.** By Hyland-Schalk’s Propositions 52 in [28] because their presupposition positivity (res. negativity) of the projection (resp. injection) of product (resp. coproduct) is entailed from our reciprocal orthogonality condition by Remark 4.17. □

**Proposition 4.19 (exponential comonad on the slack category).** $S(TsKer^*)$ has the following exponential comonad:

$$!(\mathcal{X}, U, R) = (!\mathcal{X}, k_{\mathcal{X}}(U), ?R)$$

where $?R$ is defined as in Proposition 4.7, but the clause (b) is replaced by:

$$\{\chi \cdot w_\mathcal{X} | \text{Id}_I \perp \chi\} \subseteq ?R$$

**Proof.** By Proposition 53 of [28], which supposition on the positivity of the three structure maps $d$, $w$ and $c$ (for $k$) is a direct consequence in $TsKer^*$ by Remark 4.17 □

5. **Discretisation $TsKer_\omega$ and Probabilistic Coherent Spaces**

This section starts with considering a discrete (i.e., countable) restriction of the transition kernels within the transition matrices. The restriction makes the integral for the categorical composition into simpler algebraic sum, and turns out to give an involution in the full subcategory $T\text{Ker}_\omega$ of the countable measurable spaces. The involution is directly shown to imply dagger compact closedness of the subcategory $TsKer_\omega$.

Second, the double glueing is constructed over the dagger compact closed category, so that a $*$-autonomous structure is obtained. Finally, the orthogonality of the previous section is extended over the involution, and the tight orthogonality subcategory $T(TsKer_\omega)$ of the double glueing is shown to coincide with Danos-Ehrhard’s category of probabilistic coherent spaces [8].

### 5.1. Involution in $T\text{Ker}_\omega$ and Dager Compact Closed $TsKer_\omega$

When the set $Y$ of $(Y, \mathcal{Y})$ is countable, the integral of the composition (2) is replaced by the cruder sum:

$$\iota \circ \kappa(x, C) = \sum_{y \in Y} \kappa(x, \{y\}) \iota(y, C)$$  \hspace{1cm} (20)

In the countable case, $\kappa(x, \{y\})$ is written simply by $\kappa(x, y)$, and the collection $(\kappa(x, y))_{x \in X, y \in Y}$ is called a transition matrix, as the composition (20) becomes the matrix multiplication, under the same countable condition making $\iota(y, C)$ into $\iota(y, \{c\})$. This yields the full subcategory $T\text{Ker}_\omega$ consisting of the countable measurable spaces in $T\text{Ker}$.

**Definition 5.1 ($T\text{Ker}_\omega$).** A measurable space $(X, \mathcal{X})$ is **countable** when the set $X$ is countable. $T\text{Ker}_\omega$ is the full subcategory whose objects are the countable measurable spaces in $T\text{Ker}$. Then the morphisms of $T\text{Ker}_\omega$ are characterised as the transition matrices between two countable measurable spaces.

**Proposition 5.2 (involution $(\ )^*$).**

$T\text{Ker}_\omega$ is a dagger category [37] with the following self involutive functor $(\ )^*$, which is contravariant and the identity on the objects.

(On morphisms) For a transition matrix $\kappa : (X, \mathcal{X}) \rightarrow (Y, \mathcal{Y})$, $\kappa^* : (Y, \mathcal{Y}) \rightarrow (X, \mathcal{X})$ is given by the transpose of the matrix:

$$(\kappa^*(y, x))_{y \in Y, x \in X} := (\kappa(x, y))_{x \in X, y \in Y}$$
Remark 5.3. The involution \((\cdot)^*\) is an internalisation of the contravariant equivalence of Proposition 2.7 restricting the subcategory \(\mathsf{TKer}_\omega\).

The involution of Proposition 5.2 with the monoidal product directly yields the compact closed structure of the subcategory \(\mathsf{TsKer}_\omega\) of \(\mathsf{TKer}_\omega\).

**Proposition 5.4** (dagger compact closed category \(\mathsf{TsKer}_\omega\)). Let \(\mathsf{TsKer}_\omega\) be the full subcategory of \(\mathsf{TsKer}\) consisting of the countable measurable spaces. Then the dagger \(\mathsf{TsKer}_\omega\) of Proposition 5.2 becomes compact closed whose dual object (and its extension to the contravariant functor) is given by the involution \((\cdot)^*\). In particular, the monoidal closed structure is by one to one correspondence between the transition matrices \(\kappa((x,y), z)\) and \(\kappa(x, (y, z))\):

\[\mathsf{TsKer}_\omega(\mathcal{X} \otimes \mathcal{Y}, Z) \cong \mathsf{TsKer}_\omega(\mathcal{X}, \mathcal{Y}^* \otimes Z)\]

**Proof.** The unit \(\phi_\mathcal{X} : I \to \mathcal{X}^* \otimes \mathcal{X}\) for the compact closedness is given by the matrix whose element for each \((*, (x_1, x_2)) \in I \times (X \times X)\) is \(\phi_\mathcal{X}(*, (x_1, x_2)) = \delta_{x_1, x_2}\). The counit \(\psi_\mathcal{X} : \mathcal{X} \otimes \mathcal{X}^* \to I\) is the transpose matrix of the unit. This directly yields the dagger compact closedness \(\phi_\mathcal{X} = \sigma_{\mathcal{X}, \mathcal{X}^*} \circ (\psi_\mathcal{X})^*\).

**Remark 5.5** (\(\mathsf{TsKer}_\omega = \mathsf{TKer}_\omega\)). \(\mathsf{TsKer}_\omega\) coincides with \(\mathsf{TKer}_\omega\). As the former is a wide subcategory of the latter, we need to check the fullness for the subcategory: Every morphism in \(\mathsf{TKer}_\omega\) is a transition matrix \((\kappa(x,y))_{x \in X, y \in Y}\), whose each element \(\kappa(x,y)\) is approximated as a countable sum \(\sum_{i \in N} \kappa_i(x,y)\) with finite \(\kappa_i(x,y)\)’s. Thus \(\kappa = \sum_{i \in N} \sum_{(x,y) \in X \times Y} \kappa_i(x,y)\), where each \(\kappa_i(x,y)\) determines the transition matrix \((\delta_{x_i, x_j} \kappa_i(x,y))_{x \in X, y \in Y}\). The coincidence of the two categories means that the discretisation makes the s-finiteness redundant so that the well behaved monoidal composition is freely obtained in \(\mathsf{TKer}_\omega\) with respect to Fubini-Tonelli. The well behavior is a direct consequence that the monoidal product and composition become algebraic when the morphisms of continuous kernels collapse into transition matrices in \(\mathsf{TKer}_\omega\).

In spite of the remark, in what follows in Subsection 5.2, we continue to use \(\mathsf{TsKer}_\omega\), whereby the connection to the continuous case studied in the previous sections is seen direct.

### 5.2. Double Glueing and Probabilistic Coherent Spaces

#### 5.2.1. *-autonomy of \(\mathbb{G}(\mathsf{TsKer}_\omega^\omega)\)

The base category considered in this subsection is \((\mathsf{TsKer}_\omega^\omega)\) denoting the full subcategory of \(\mathsf{TsKer}_\omega^\omega\) consisting of the countable measurable spaces, which is equal to \((\mathsf{TsKer}_\omega^\omega)^\omega\). Hence the category is denoted simply by \(\mathsf{TsKer}_\omega^\omega\). When Proposition 4.2 takes \(\mathsf{TsKer}_\omega^\omega\) as the base category, the double glueing category \(\mathbb{G}(\mathsf{TsKer}_\omega^\omega)\) has a stronger form, inheriting the dagger compact closed structure of \(\mathsf{TsKer}_\omega^\omega\) of Proposition 5.4.

**Proposition 5.6** (*-autonomy of \(\mathbb{G}(\mathsf{TsKer}_\omega^\omega)\)). \(\mathbb{G}(\mathsf{TsKer}_\omega^\omega)\) is self involutive.

\[(\mathcal{X}, U, R)^\perp = (\mathcal{X}^*, R, U)\]

modulo the equivalence \((\cdot)^*: \mathsf{TsKer}_\omega^\omega(\mathcal{I}, \mathcal{X}) \cong \mathsf{TsKer}_\omega^\omega(\mathcal{X}, \mathcal{I})\). Moreover, \(\mathbb{G}(\mathsf{TsKer}_\omega^\omega)\) becomes *-autonomous, whose monoidal closedness is given by the following implication in terms of the involution and the cotensor \(\gamma\):

\[(implication)\]

\[\mathcal{X} \rightarrow \mathcal{Y} := \mathcal{X}^\perp \mathcal{Y} = (\mathcal{X}^* \otimes \mathcal{Y}, W, U^* \otimes S),\]

where

\[W = \{v : \mathcal{I} \rightarrow \mathcal{X}^* \otimes \mathcal{Y} | \mathcal{I} \xrightarrow{\nu \otimes U} \mathcal{X}, \; \nu^*(\kappa^* \otimes \delta_\mathcal{Y}) \in V \; \text{and} \; \mathcal{Y} \xrightarrow{\sigma_\mathcal{Y} \otimes S} \mathcal{I}, \; \nu^*(\delta_\mathcal{X} \otimes g) \in R\}\]

\[U^* \otimes S = \{f^* \otimes g : \mathcal{X}^* \otimes \mathcal{Y} \rightarrow \mathcal{I} \otimes \mathcal{I} \cong \mathcal{I} | f \in U \; g \in S\}\]

Note that \(W\) represents the homset \(\mathbb{G}(\mathsf{TsKer}_\omega^\omega)(\mathcal{X}, \mathcal{Y})\).

A direct corollary is obtained when the slack category studied in Section 4.2 is restricted to the discrete measurable spaces. The corollary states that the slack subcategory is a model of classical linear logic.
Corollary 5.7. The slack orthogonality subcategory \( S(TsKer^a) \) is defined to be the full subcategory of \( S(TsKer^a) \) whose \( X \) of Definition 4.14 is an object from \( TsKer^a \). Then \( S(TsKer^a) \) is \(*\)-autonomous with product (hence coproduct), and equipped with linear exponential comonad.

Proof. It suffices to show the following (i), (ii) and (iii), but whose latter two are direct and (i) is by Theorem 54 of [28], whose supposition on the positive (resp. negative) projections (resp. injections) and the three positive structure maps \( d, w \) and \( c \) (for \( k \)) is by Remark 4.17: (i) The \(*\)-autonomy of the slack category \( S(TsKer^a) \) is inherited from that of \( G(TsKer^a) \) of Proposition 5.6 by Lemma 4.16. (ii) The product of Proposition 4.18 is closed in the discrete subcategory. (iii) The linear exponential comonad in Proposition 4.19 is closed in the discrete subcategory. \qed

5.2.2. Exponential Comonad for Tight Category \( T(TsKer^a) \)

This part starts with introducing the tight orthogonality subcategory \( T(TsKer^a) \) of \( G(TsKer^a) \), and shows that the subcategory is also a categorical model of linear logic.

Definition 5.8 (tight category \( T(TsKer^a) \) (cf. Definition 47 [28] for the general definition)). The tight orthogonality category \( T(TsKer^a) \) is the full subcategory of \( G(TsKer^a) \) on those objects \((X, U, R)\) such that \( U = R^0 \) and \( R = U^0 \).

Example 5.9 (objects of \( T(TsKer^a) \)). Let \( C = TsKer^a \). For any subset \( U \) of \( C(I, X) \), \((X, U^0, U^0)\) becomes an object of the tight \( T(C) \). Dually for any \( R \subset C(X, I) \), \((X, R^0, R^0)\) becomes an object of the tight \( T(C) \).

A general lemma is prepared on the orthogonality broadly for the continuous \( TsKer^a \).

Lemma 5.10 (stable tensor). Any reciprocal orthogonality on a monoidal category \( C \) stabilises the monoidal product: That is, For all \( U \subseteq C(I, R) \) and \( V \subseteq C(I, S) \), (stable tensor)

\[
(U^0 \otimes V^0)^0 = (U^0 \otimes V)^0 = (U \otimes V^0)^0
\]

Hence in particular, the orthogonality in \( TsKer^a \) stabilises the monoidal product.

Proof. We prove \((\supseteq)\) of the stable tensor condition as the converse is tautological. Take any \( \nu \in RHS \), which means \( \forall f \in U^0 \forall g \in V \ f \otimes g = (f \otimes S) \circ (I \otimes g) \downarrow_{R \otimes S} R \otimes S \quad \nu \downarrow_{J} J \) if by reciprocity \( g \downarrow_S S \cong I \otimes S \quad \downarrow_{R \otimes S} R \otimes S \downarrow_{J} J \). But this means \( \forall h \in V^0 \quad \h \downarrow_S \nu \circ (f \otimes S) \) if by reciprocity \( f \otimes h = (f \otimes S) \circ (I \otimes h) \downarrow_{R \otimes S} \nu \), which means \( \nu \in LHS \). \qed

In the presence of the involution in \( TsKer^a \), it is necessary to enhance the coherence of the orthogonality relation in Definition 4.8 to account for the involution \((\cdot)^*\). This augmented condition is called as "symmetry orthogonality" within the general context in [28].

(symmetry) Given \( u : I \rightarrow X \) and \( v : X \rightarrow I \),

\[
u \perp_X u \iff v^* \perp_X u^*
\]

Then this additional condition is satisfied in \( TsKer^a \), to yield the following proposition, corresponding to Proposition 4.13.

Proposition 5.11. \( \perp \) is a symmetric orthogonal relation in \( TsKer^a \).

Proof. The additional condition of symmetry is checked:

\[
(f \circ \mu)_X = f \circ (f \circ \mu)^* = \mu^* \circ f^* = (\mu^* f^*)_X
\]

The second equality is because the transpose of the scalars (i.e., of the homset \( TsKer^a(I, I) \)) is the identity. \qed
Moreover, \( \text{TsKer}_w^w \) has a monoidal closed structure, so the the coherence of the orthogonality on implication of \([28]\) needs to be augmented in Definition 4.8:

(implication)

Given \( u: I \rightarrow R \), \( y: S \rightarrow I \) and \( f: R \rightarrow S \),

\[
u \perp_R y \circ f \quad \text{and} \quad f \circ u \perp_S y \quad \text{imply} \quad \hat{f} \perp_{R \rightarrow S} u \rightarrow y
\]

(21)

where \( \hat{f} \) is the transpose of \( f \).

The strengthened condition is called \textit{precise implication} when “imply” is replaced by “iff” (same as the tensor condition was called).

\begin{lemma}
The implication orthogonality is derivable from the symmetry orthogonality together with the tensor one. Moreover the precise implication is derivable when the tensor condition is precise. Hence, in particular \( \text{TsKer}_w^w \) validates the precise implication condition.
\end{lemma}

\begin{proof}
Apply the symmetry condition to the descendant of (21), then \( y^* \otimes u \perp_{S^* \otimes R} (\hat{f})^* \). This happens to be a descendant of (tensor) whose two antecedents are \( y^* \perp_{S^*} (\hat{f})^* \circ (S^* \otimes u) \) and \( u \perp_R (\hat{f})^* \circ (y^* \otimes R) \). But (RHS)* of the first orthogonality is \((u \rightarrow S) \circ \hat{f} = (f \circ u) = f \circ u \) and (RHS)* of the second orthogonality is \(((R \rightarrow y) \circ \hat{y})^* = (y \circ \hat{f})^* = y \circ f \). Hence the two orthogonality are the antecedent of the tensor condition.

The second assertion is by the reversibility of the above argument using precise tensor. The third assertion is by Lemma 5.10.
\end{proof}

\begin{lemma}
If an orthogonality is symmetric, the stable tensor implies the stable implication; That is, for all \( U \subseteq \mathcal{C}(I, R) \) and \( Y \subseteq \mathcal{C}(S, I) \),

(implication)

\[
(U^\circ \rightarrow Y^\circ)^\circ = (U \rightarrow Y^\circ)^\circ = (U^\circ \rightarrow Y)^\circ
\]

Hence the orthogonality in \( \text{TsKer}_w^w \) stabilises the implication.
\end{lemma}

\begin{proof}
The *-autonomy \( X \rightarrow Z = (X \otimes Z^*)^* \) of \( \text{TsKer}_w^w \) makes the stable implication into a stable tensor via \((U^\circ)^* = (U^*)^\circ \), which equality is obtained directly by the symmetry orthogonality. The second assertion is by Lemma 5.10 and Proposition 5.11.
\end{proof}

An orthogonality is called \textit{stable} when it satisfies both stable tensor and stable implication.

The rest of this part is devoted to observing that the stable orthogonality of \( \text{TsKer}_w^w \) ensures, accordingly to Hyland-Schalk \([28]\), an exponential comonad on \( \mathbf{T}(\text{TsKer}_w^w) \) as well as both monoidal product and product and coproduct. That is,

\begin{theorem}
The tight category \( \mathbf{T}(\text{TsKer}_w^w) \) is a model of classical linear logic.
\end{theorem}

\begin{proof}
By the following three Propositions 5.15, 5.16, and 5.17.
\end{proof}

\begin{proposition}[monoidal product in the tight orthogonality category]
The tight category \( \mathbf{T}(\text{TsKer}_w^w) \) has the following monoidal product so that the forgetful to \( \text{TsKer}_w^w \) preserves the structure:

\[
(\mathcal{X}, U, R) \otimes (\mathcal{Y}, V, S) = (\mathcal{X} \otimes \mathcal{Y}, (U \otimes V)^\circ, (U \otimes V)^\circ)
\]

with the tensor unit \((\mathcal{I}, \{\text{Id}_x\}^\circ, \{\text{Id}_x\}^\circ)\). Moreover, the monoidal structure is closed with the following linear implication space so that the forgetful to \( \text{TsKer}_w^w \) preserves the structure:

\[
(\mathcal{X}, U, R) \rightarrow (\mathcal{Y}, V, S) = (\mathcal{X} \rightarrow \mathcal{Y}, (U \rightarrow S)^\circ, (U \rightarrow S)^\circ)
\]

\end{proposition}
Proof. By Proposition 61 of Hyland-Schalk [28] for a general $C$ with stable orthogonality, which proposition is applicable to $C = \text{TsKer}_w^a$ thanks to Lemma 5.12.

Proposition 5.16 (product and coproduct in $\mathbf{T}(\text{TsKer}_w^a)$). The tight category $\mathbf{T}(\text{TsKer}_w^a)$ has the following product and coproduct so that the forgetful functor to $\text{TsKer}_w^a$ preserves the structures:

$$(x, U, R) \land (y, V, S) = (x \uplus y, U \& V, (U \& V)^\circ)$$

$$(x, U, R) \lor (y, V, S) = (x \uplus y, (R \lor S)^\circ, R \lor S)$$

Note by the self involution of the gluing (acting *, hence identity on the first component, and flipping second and third components), $\land$ and $\lor$ are mutually definable each other.

Proof. By Hyland-Schalk’s Proposition 63 in [28] for a general monoidal category with a stable orthogonality because their presupposition positivity (resp. negativity) of the projection (resp. injection) of product (resp. coproduct) is entailed from our reciprocal orthogonality condition by Remark 4.17.

Finally, exponential structure for the tight category $\mathbf{T}(\text{TsKer}_w^a)$ is obtained as an instance of Hyland-Schalk general construction for $\mathbf{T}(C)$ in [28].

Proposition 5.17 (exponential comonad on $\mathbf{T}(\text{TsKer}_w^a)$). $\mathbf{T}(\text{TsKer}_w^a)$ has the following exponential comonad so that the forgetful to $\text{TsKer}_w^a$ preserves the structure:

$$!(x, U, R) = (|x|, k_X(U)^\circ, k_X(U)^\circ)$$

Proof. By Theorem 65 of [28] for a general monoidal category with a stable orthogonality because their presupposition for the theorem (described below) is automatically derived by Remark 4.17: All the structure maps $d, s, w, c$ for linear exponential comonad and all maps of the form $?$ are positive for $k$ (i.e., with respect to $\forall u k_X(U)$ and $C(I, I)$) and the product projections are focused.

5.2.3. $\text{Pcoh}$ and its Equivalence to $\mathbf{T}(\text{TsKer}_w^a)$

Definition 5.18 ($\text{Pcoh}$ [8, 7]). The definition of the Danos-Ehrhard’s category $\text{Pcoh}$ of probabilistic coherent spaces starts with the inner product and the polar:

(inner product) $\langle x, x \rangle := \sum_{a \in A} x_a x_a^*$ for $x, x' \subseteq \mathbb{R}_+^A$ with a countable set $A$.

(polar) $P^+ := \{ x' \in \mathbb{R}_+^A \mid \\forall x \in P \langle x, x' \rangle \leq 1 \}$ for $P \subseteq \mathbb{R}_+^A$.

Then $\text{Pcoh}$ is defined as follows:

(object) $X = (|X|, PX)$, where $|X|$ is a countable set, $PX \subseteq \mathbb{R}_+^{|X|}$ such that $PX^\perp \subseteq PX$, and $0 < \sup \{ x_a \mid x \in PX \} < \infty$ for all $a \in |X|$.

(morphism) A morphism from $X$ to $Y$ is an element $u \in P(X \times Y^\perp)$, which can be seen as a matrix $(u)_{a \in |X|, b \in |Y|}$ of columns from $|X|$ and of rows from $|Y|$. Composition is the product of two matrices such that $(uv)_{a \in |X|, b \in |Y|} = \sum_{c \in |Y|} u_{a, c} v_{c, b}$ for $u : X \rightarrow Y$ and $v : Y \rightarrow Z$.

(dual) $X^\perp = (|X|, PX^\perp)$ and $u^\perp \in \text{Pcoh}(Y^\perp, X^\perp)$ is the transpose of a matrix $u \in \text{Pcoh}(X, Y)$.

(tensor $\otimes$) $X \otimes Y = (|X| \times |Y|, \{ x \otimes y \mid x \in PX \ y \in PY \}^\perp)$.

For $u \in \text{Pcoh}(X_1, Y_1)$ and $v \in \text{Pcoh}(X_2, Y_2)$, $u \otimes v \in \text{Pcoh}(X_1 \otimes X_2, Y_1 \otimes Y_2)$ is $(u \otimes v)_{(a_1, a_2), (b_1, b_2)} = u_{a_1, b_1} v_{a_2, b_2}$.

(product $\&$) $X_1 \& X_2 = (|X_1| \uplus |X_2|, \{ x \in \mathbb{R}_+^{\uplus |X_1|} \mid \forall i, \pi_i(x) \in P(X_i) \})$, where $\pi_i(x)_a$ is $x_{(i, a)}$. $P(X_1 \& X_2)$ becomes automatically closed under the bipolar.

(exponential) The original definition using finite multisets is rewritten by the exponential monoid and the counting function in Section 3.1 of this paper.

$$X = (|X|, \{ x^\perp \mid x \in PX \}^\perp),$$

where

$$x^\perp(a) := \prod_{a \in |X|} x_a^a$$  (22)
This is well defined as the a's support set \{a \in \mathbb{X} \mid a(a) \neq 0\} is finite. When a is explicitly written by \(a = a_1 \cdots a_k\) with \(a_i \in \mathbb{X}\), \(x(\alpha) = \prod_{i=1}^{k} x_{a_i}\), since each \(a_i\) has \(a(a_i)\)-times multiplicity in \(a\).

For \(t \in \mathbb{Pcoh}(\mathbb{X}, \mathbb{Y})\), \(! t \in \mathbb{Pcoh}(!\mathbb{X}!\mathbb{Y})\) is defined by

\[
(!!)_{a,b} := \sum_{e \in L((a,b))} \frac{b!}{e!} t_{\sigma(e),b}^a
\]

in which for \(a \in \mathbb{X}|_e\) and \(b \in \mathbb{Y}|_e\):

\[
L(a,b) := \left\{ e \in (|\mathbb{X}| \times |\mathbb{Y}|)_e \mid \forall a \in |\mathbb{X}| \sum b_{(e)} c((a,b)) = a(a) \right\}
\]

\(b! := \prod_{b \in |\mathbb{Y}|} b(b)!\) and \(e! := \prod_{(a,b) \in |\mathbb{X}| \times |\mathbb{Y}|} e((a,b))\)!

To be explicit, when \(a\) and \(b\) are given explicitly by \(a = a_1 \cdots a_n\) and \(b = b_1 \cdots b_n\), (23) is written

\[
(!!)_{a,b} = \sum_{\sigma \in \mathbb{S}_n/S_{a}} \prod_{i=1}^{n} t_{\sigma(a_i),b_i}
\]

in which \(S_{a}\) denotes the stabiliser subgroup of \(\mathbb{S}_n\) at \(\tilde{a} := (a_1, \ldots, a_n)\) defined by

\[
S_{a} := \{ \sigma \in \mathbb{S}_n \mid a_i = a_{\sigma(i)} \forall i = 1, \ldots, n \}
\]

Note that the definition (24) does not depend on the ordering \(\bar{a}\) of \(a\) for the stabiliser subgroup \(S_{a}\) as \(\mathbb{S}_n/S_{\bar{a}} = \mathbb{S}_n/S_{\tilde{a}}\) for any permutation \(\sigma \in \mathbb{S}_n\), hence its action on \(a\) is well defined.

The category \(\mathbb{Pcoh}\) turns out to be equivalent to the \(*\)-autonomous \(\mathbb{T}(\mathbb{Tk}^\mathbb{op}/\mathbb{X})\) with the linear exponential comonad. The equivalence is by the following Theorem 5.19 and Proposition 5.20.

**Theorem 5.19 (equivalence of \(\mathbb{Pcoh}\)).** The tight orthogonality category \(\mathbb{T}(\mathbb{Tk}^\mathbb{op}/\mathbb{X})\) is equivalent to the category \(\mathbb{Pcoh}\).

**Proof.** The key property for the equivalence is that the measures (i.e., the homset \(\mathbb{Tk}^\mathbb{op}/\mathbb{X}\)), and the measurable functions (i.e., the homset \(\mathbb{Tk}^\mathbb{op}/\mathbb{X}\)) become isomorphic in \(\mathbb{Tk}^\mathbb{op}\) by virtue of the involution \((\cdot)^*\), and furthermore in \(\mathbb{Tk}^\mathbb{op}\) they both collapse to bounded functions from \(\mathbb{X}\) to \(\mathbb{R}_+\), hence residing in \(\mathbb{R}_+^{\mathbb{X}}\) in \(\mathbb{Pcoh}\).

The orthogonality \(\langle, \rangle\) in \(\mathbb{Pcoh}\) coincides with \(\langle | \rangle\) in \(\mathbb{Tk}^\mathbb{op}\), as the integral of Definition 4.12 collapses to the sum in the subcategory of discrete measurable spaces.

An object \(\mathbb{X} = (|\mathbb{X}|, \mathbb{PX})\) in \(\mathbb{Pcoh}\) corresponds one to one to the object \(\mathbb{X} = (|\mathbb{X}|, \mathbb{PX}, (\mathbb{PX})^\mathbb{op})\) in \(\mathbb{T}(\mathbb{Tk}^\mathbb{op}/\mathbb{X})\), preserving the involution \((\cdot)^*\). Every morphism from \(\mathbb{X}\) to \(\mathbb{Y}\) in \(\mathbb{Pcoh}\) is by definition an element \(\mathbb{P}(\mathbb{X}^\mathbb{op} \times \mathbb{Y})\), which is the second component of \(\mathbb{X} \rightarrow \mathbb{Y}\) in \(\mathbb{T}(\mathbb{Tk}^\mathbb{op}/\mathbb{X})\). Composition of \(\mathbb{T}(\mathbb{TKer}^\mathbb{op}_\omega)\) is the product of matrices, same as \(\mathbb{Tk}^\mathbb{op}\). E.g., in particular their map \(fun(u) : \mathbb{PX} \rightarrow \mathbb{PY}\) for \(u \in \mathbb{Pcoh}(\mathbb{X}, \mathbb{Y})\) (cf. Section 1.2.2 [8]) is written in \(\mathbb{Tk}^\mathbb{op}\) simply by \(fun(x) = u^*x \in \mathbb{PY}\). Since the tensor and the additive structures are direct, only the exponential structure is checked on (i) objects and on (ii) morphisms. In the both levels, Danos-Ehrhard's exponential construction in \(\mathbb{Pcoh}\) turns out to coincide with that of Hyland-Schalk for double glueing applied to our \(\mathbb{Tk}^\mathbb{op}_\omega\).

(i) It is shown that \((\cdot)^{\mathbb{op}} : \mathbb{R}^{\mathbb{X}} \rightarrow \mathbb{R}^{\mathbb{X}}\) of (22) in \(\mathbb{Pcoh}\) defines the same map as \(k_{\mathbb{X}}(\cdot)^{\mathbb{op}} : \mathbb{Tk}^\mathbb{op}_\omega(\mathbb{I}, \mathbb{X}) \rightarrow \mathbb{Tk}^\mathbb{op}_\omega(\mathbb{I}, \mathbb{X})\) of Definition 4.4: For \(f \in \mathbb{R}^{\mathbb{X}}\), \(f^\mathbb{op}(a) := \prod_{a \in |\mathbb{X}|} f(a)_a\). On the other hand in \(\mathbb{Tk}^\mathbb{op}_\omega\), for \(a = a_1 \cdots a_k \in X_e\), \(k_{\mathbb{X}}(f)((*, a)) = \prod_{a \in |\mathbb{X}|} f(a)_a\). Thus \(f^\mathbb{op} = k_{\mathbb{X}}(f)^{\mathbb{op}}\), for which the right \(f : \mathbb{I} \rightarrow \mathbb{X}\) is identified with a measurable function on \(\mathbb{X}\).

(ii) First note \(\forall e \in L((a,b)) |\mathbb{X}|(e) = |\mathbb{X}|(a) = |\mathbb{Y}|(b)\), whose number is denoted by \(r\) such that \(a = a_1 \cdots a_r\) and \(b = b_1 \cdots b_r\). Then in \(\mathbb{Tk}^\mathbb{op}_\omega\),

\[
t_u(a,b) = t_u(a_1 \cdots a_r, b_1 \cdots b_r) = t_u(F^{-1}(a_1 \cdots a_r), (b_1, \ldots, b_r))
\]

by \((\cdot)_e^{\mathbb{op}}\) in \(\mathbb{Tk}^\mathbb{op}_\omega\)

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= t^r(S \sigma \in S_r \{(a_{\sigma(1)}, \ldots, a_{\sigma(r)})\}, (b_1, \ldots, b_r)) \text{ by the def of } F \\
= t^r(S \sigma \in S_r/S_{\vec{a}} \{(a_{\sigma(1)}, \ldots, a_{\sigma(r)})\}, (b_1, \ldots, b_r)) \text{ thanks to the quotient by } S_{\vec{a}} \\
= \sum_{\sigma \in S_r/S_{\vec{a}}} t^r((a_{\sigma(1)}, \ldots, a_{\sigma(r)}), (b_1, \ldots, b_r)) \text{ by } \sigma\text{-additivity} \\
= \sum_{\sigma \in S_r/S_{\vec{a}}} \prod_{i=1}^r t(a_{\sigma(i)}, b_i) \text{ by the def of } t^r

This has shown that \( t_e(a, b) = (t)_{a,b} \).

The comonad structure of the exponential of \( \mathbb{P}_{coh} \) is given in [8] as follows:

(dereliction) \( (d_X)_{a,a} := \delta_a, [a] \) where \( d_X \in \mathbb{P}_{coh}(!X, X) \)

(storage) \( (s_X)_{a,M} := \delta_a, \sum M \) where \( s_X \in \mathbb{P}_{coh}(!X, !!X) \)

**Proposition 5.20.** The comonad structure of \( \mathbb{P}_{coh} \) is the discretisation of that of \( T(TsKer^{\omega}_m) \) under the equivalence of Theorem 5.19.

**Proof.** The dereliction and the storage become directly the discretisation of the corresponding maps in \( TsKer^{\omega}_m \) (hence of \( T(TsKer^{\omega}_m) \)), defined respectively in Propositions 3.14 and 3.17.

**Remark 5.21** (The opposite \( TsKer^{\omega}_m \) coincides with \( \mathbb{P}_{coh} \)'s left enumeration). Our choice taking the opposite of \( TsKer^{\omega}_m \) starting from Section 3.3 turns out to yield Danos-Ehrhard’s choice of left enumeration in formalising the exponential in (24). When the right enumeration is chosen oppositely, there arises another exponential, say \( \nabla \),

\[
(\nabla t)_{a,b} := \sum_{c \in L(a,b)} \frac{n!}{c_1! \cdots c_n!} t^c = \sum_{\sigma \in S_{\vec{a}}/S_{\vec{a}}} \prod_{i=1}^n t(a_{\sigma(i)}, b_{\sigma(i)})
\]

Compare the first and second formulas respectively with (23) and (24) to see the opposite enumeration. It holds \( (\nabla t)_{a,b} = \frac{n!}{\vec{a}!} (t)_{a,b} \). The two exponentials become isomorphic [13], by the following natural isomorphism in terms of the multinomial coefficient \( m : ! \rightarrow \nabla \), defined by

\[
(m_X)_{a,b} = m(a) \delta_{a, b}.
\]

Remind the multinomial coefficient of \( a \in X_e \cap X^{(n)} \) is defined by \( m(a) := \frac{n!}{\prod_{a \in X^{(n)} \setminus \{a\}} a(a)!} \), which number is equal to the cardinality \( |S_{\vec{a}}/S_{\vec{a}}| \) of the quotient independently of the ordering \( \vec{a} \) of \( a \).

6. Conclusion

This paper offers four main contributions:

(i) Presenting a monoidal category \( TsKer \) of s-finite transition kernels between measurable spaces after Staton [39], with countable biproducts. Showing a construction of exponential kernels in \( TsKer \) by accommodating the exponential measurable spaces for counting process into the category.

(ii) Constructing a linear exponential comonad over \( TsKer^{\omega}_m \), modelling the exponential modality in linear logic. Although this initiates a continuous linear exponential comonad employing a general measure theory, though we leave it a future work on any monoidal closed structure inside \( TsKer \) required for modelling the multiplicative fragment of the logic.
(iii) Giving a measure theoretic instance of Hyland-Schalk orthogonality in terms of an integral between measures and measurable functions. The instance is inspired by the contravariant equivalence between $\text{TKer}$ of the transition kernels and $M_E$ of measurable functions, and realised by adjunction of a kernel acting on both sides. We examine a monoidal comonad in the double glueing $G(\text{TsKer}^\omega)$ inheriting from $\text{TsKer}^\omega$ of (i) and also that in the slack orthogonality subcategory $S(\text{TsKer}^\omega)$.

(iv) (Discretisation of (i), (ii) and (iii)):
Obtaining a dagger compact closed category $\text{TsKer}_\omega$ when restricting $\text{TsKer}$ of (i) to the countable measurable spaces. We show an equivalence of the tight orthogonality category $T(\text{TsKer}^\omega)$ to $\text{Pcoh}$ of probabilistic coherent spaces by virtue of the discrete collapse of the orthogonality of (ii) into the linear duality of $\text{Pcoh}$.

We now discuss some future directions. Our categories $\text{TKer}$ with countable biproducts and $\text{TsKer}$ with tensor are inspired from the standard measure-theoretic formalisation of probability theory, and similarly the linear exponential comonad over $\text{TsKer}^\omega$ from the counting process for exponential measurable spaces. We believe our semantics of transition kernels will provide a general tool for semantics of higher order probabilistic programming languages such as probabilistic PCF [8, 16], of which $\text{Pcoh}$ is a denotational semantics. We need to examine a concrete example making continuous Markov kernels indispensable (rather than discrete Markov matrices) for interpreting probabilistic computational reductions as a stochastic process. For this, any monoidal closed structure fundamental to denotational semantics needs to be explored in continuous measure spaces. Recent development [16, 6] on CCC extension induced by $\text{Pcoh}$ for continuous probabilities may be seen as a mutual construction of our construction because ours starts with the continuity to obtain $\text{Pcoh}$ as a discretisation.

An important future work is to connection to Staton’s denotational semantics [39] for commutativity of first-order probabilistic functional programming, in which s-finiteness of kernels characterises commutativity of programming languages. We are interested in how our trace structure for feedback and probabilistic iteration (cf. Remark 2.10) may play any role in his probabilistic data flow analysis using categorical arrows. That is, a direction towards a probabilistic Geometry of Interaction employing the continuous categories of the present paper.

Another promising further study is association with point process monad in [9] using distribution between Giry monad and multisets. This may provide a general categorical understanding how our measure theoretic commutative monoid, seen as counting process, yields the exponential comonad for linear logic.

Relating our model to Girard’s coherent Banach spaces [22] on one hand involves analysing the contravariant equivalence of Proposition 2.7 under certain constraints required from logical and type systems. On the other hand, the double glueing in Section 4 will give a direct bridge to the duality of coherent Banach spaces, employing a (variant of) a Chu construction, which is known as another instance of Hyland-Schalk orthogonality.

After the submission of the earlier version of the paper, series of pioneering works are published [14, 18, 35] on continuous exponentials for higher ordered programming. Their approach to measure-theoretic continuity would warrant our future work mentioned above. Especially, Ehrhard’s measurable exponential [14] in cones, giving a continuous extension of discrete probability, could be quite beneficial.

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