Stable limit laws for the parabolic Anderson model between quenched and annealed behaviour\(^1\)

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Abstract. We consider the solution to the parabolic Anderson model with homogeneous initial condition in large time-dependent boxes. We derive stable limit theorems, ranging over all possible scaling parameters, for the rescaled sum over the solution depending on the growth rate of the boxes. Furthermore, we give sufficient conditions for a strong law of large numbers.

Résumé. Nous considérons la solution du modèle parabolique d’Anderson avec condition initiale homogène sur de grandes boîtes dépendantes du temps. Nous dérivons des théorèmes limites stables, pour toutes les valeurs possibles des paramètres d’échelle, pour la somme de la solution changée d’échelle en fonction du taux de croissance des boîtes. De plus, nous donnons des conditions suffisantes pour une loi des grands nombres.

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1. Introduction

1.1. The problem

The parabolic Anderson model (PAM) is the heat equation on the lattice with a random potential, given by

$$\begin{align*}
\frac{\partial}{\partial t} u(t,x) &= \kappa \Delta u(t,x) + \xi(x)u(t,x), \quad (t,x) \in (0, \infty) \times \mathbb{Z}^d, \\
u(0,x) &= u_0(x), \quad x \in \mathbb{Z}^d,
\end{align*}$$

(1)

where \(\kappa > 0\) denotes a diffusion constant, \(u_0\) a nonnegative function, and \(\Delta\) the discrete Laplacian, defined by

$$\Delta f(x) := \sum_{y \in \mathbb{Z}^d : |x-y| = 1} \left[ f(y) - f(x) \right], \quad x \in \mathbb{Z}^d, \quad f : \mathbb{Z}^d \to \mathbb{R}.$$  

Furthermore, \(\xi := \{\xi(x), x \in \mathbb{Z}^d\}\) is an i.i.d. random potential. We will stick in this paper to the homogeneous initial condition \(u_0 \equiv 1\).

The solution \(u\) depends on two effects. The Laplacian tends to make it flat whereas the potential tends to make it uneven. In combination this causes the occurrence of small regions where almost all mass of the system is located. This

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effect is called intermittency and it is present for all potentials that are not almost surely constant, see [9], Theorem 3.2. It turns out that the intermittency effect becomes the more pronounced the more heavy tailed the potential is.

Basically, there are two ways of looking at the solution. On the one hand one can pick one realisation of the potential field and consider the almost sure behaviour of $u$. This is the so-called quenched setting. On the other hand one can take expectation with respect to the potential and consider the averaged behaviour of $u$. This is the so-called annealed setting. Expectation with respect to $\xi$ will be denoted by $\langle \cdot \rangle$, and the corresponding probability measure will be denoted by $\mathbb{P}$. Those sites $x \in \mathbb{Z}^d$ whose peaks $\xi(x)$ govern the quenched behaviour of $u$ differ heavily from those that govern the annealed behaviour, see [9]. Therefore, it is interesting to understand the transition mechanism from quenched to annealed behaviour.

To this end we are interested in expressions such as $\frac{1}{|Q|} \sum_{x \in Q} u(t, x)$ where $Q$ is a large centred box. If $Q$ has a fixed size then $\frac{1}{|Q|} \sum_{x \in Q} u(t, x)$ follows quenched behaviour as $t$ tends to infinity, i.e., $\frac{1}{|Q|} \sum_{x \in Q} u(t, x)/u(t, 0) \to 1$ a.s. This can be deduced from the Feynman–Kac representation of $u$ given by

$$u(t, x) = \mathbb{E}_x \exp \left\{ \int_0^t \xi(X_s) \, ds \right\} u_0(X_t), \quad (t, x) \in [0, \infty) \times \mathbb{Z}^d,$$

where $X$ is a simple, symmetric, continuous time random walk with generator $\kappa \Delta$ and $\mathbb{P}_x (\mathbb{E}_x)$ denotes the corresponding probability measure (expectation) if $X_0 = x$ a.s.

On the other hand, if we fix $t$ and let the size of $Q$ tend to infinity then (due to the homogeneous initial condition) by Birkhoff’s ergodic theorem $\frac{1}{|Q|} \sum_{x \in Q} u(t, x)$ displays annealed behaviour almost surely, i.e., $\frac{1}{|Q|} \sum_{x \in Q} u(t, x)/\langle u(t, 0) \rangle \to 1$ a.s. Therefore, a natural question is what happens if the box $Q$ is time dependent.

More precisely, we want to find for all $\alpha \in (0, 2)$ a large box $Q_{L_\alpha(t)}$, with $Q_{r(t)} = [-r(t), r(t)]^d \cap \mathbb{Z}^d$, for any $r(t) > 0$, and numbers $A(t)$, $B_\alpha(t)$ such that

$$\sum_{x \in Q_{L_\alpha(t)}} \frac{u(t, x) - A(t)}{B_\alpha(t)} \overset{\text{a.s.}}{\to} F_\alpha, \quad t \to \infty$$

with $F_\alpha$ a suitable stable distribution.

In the case $\kappa = 0$, i.e., if the solutions at different sites are independent, the problem has been addressed in [1] under the assumption that the logarithmic tail of the distribution is normalized regularly varying. A wider class of distributions was considered in [5]. In [13] a conceptual treatment of several classes of time-dependent sums is offered, in particular explaining the universality of the limit laws in different cases. In [1] the authors also give sufficient and necessary conditions on the growth rate of $Q$ for a weak law of large numbers (WLLN) and for a central limit theorem (CLT) to hold. Corresponding results for a WLLN and a CLT for the PAM, i.e., $\kappa \neq 0$, were derived in [2] and in [3]. They state that, under appropriate regularity assumptions, there exist $J(t)$ and $\gamma_1 < \gamma_2$, all depending on the tails of $\xi$, such that:

(i) $\frac{1}{|Q_{J(t)}|} \sum_{x \in Q_{J(t)}} u(t, x) \sim \langle u(t, 0) \rangle$, as $t \to \infty$ if $\gamma > \gamma_1$, in probability, $\frac{1}{|Q_{J(t)}|} \sum_{x \in Q_{J(t)}} u(t, x) = o(\langle u(t, 0) \rangle)$, as $t \to \infty$ if $\gamma < \gamma_1$, in probability.

(ii) $\frac{1}{|Q_{J(t)}|} \sum_{x \in Q_{J(t)}} \frac{u(t, x) - \langle u(t, 0) \rangle}{\sqrt{\langle u(t, 0) \rangle^2}} \overset{\text{a.s.}}{\to} \mathcal{N}(0, 1)$, as $t \to \infty$ if $\gamma > \gamma_2$, $\frac{1}{|Q_{J(t)}|} \sum_{x \in Q_{J(t)}} \frac{u(t, x) - \langle u(t, 0) \rangle}{\sqrt{\langle u(t, 0) \rangle^2}} = o(1)$, as $t \to \infty$ if $\gamma < \gamma_2$, in probability.

Here, $\mathcal{N}(0, 1)$ denotes the law of the standard normal distribution with variance 1.

However, $\alpha$-stable limits for the PAM have not been investigated so far. Furthermore, we give sufficient conditions on the growth rate of $Q$ for a strong law of large numbers to hold. So far this has been done neither for the PAM nor for the $\kappa = 0$ case.

For a general overview of the parabolic Anderson model see, for instance [15] and [8]. A WLLN and a CLT for the PAM with time-dependent white noise potential using rather different techniques can be found in [7]. The critical growth rates in that model are of the same order as for the double-exponential case in our model. Similar questions concerning a version of the random energy model were investigated in [6].
1.2. Main results

To state the main results we need to introduce some notation. Let

\[ \varphi(h) := -\log P(\xi(0) > h) \]

and \( h_t \) a solution to

\[ \sup_{h \in (0, \infty)} (th - \varphi(h)) = th_t - \varphi(h_t). \]

If \( \varphi \) is ultimately convex then \( h_t \) is unique for any large \( t \). Throughout this paper we will assume that \( \xi(0) \) is unbounded from above and has finite exponential moments of all orders. Under these circumstances the left-continuous inverse of \( \varphi \),

\[ \psi(s) := \min\{r: \varphi(r) \geq s\}, \quad s > 0, \]

is well defined. Furthermore, this implies that the cumulant generating function

\[ H(t) := \log \langle \exp\{t\xi(0)\}\rangle, \quad t \geq 0, \]

is well-defined and that \( H(t) < \infty \) for all \( t \) with \( \lim_{t \to \infty} H(t)/t = \infty \). If \( \varphi \in C^2 \) is ultimately convex and satisfies some mild regularity assumptions then the Laplace method yields that \( H(t) = th_t - \varphi(h_t) + o(t) \).

In [10], Theorems 1.2 and 2.2, the authors prove that there exists \( \chi = \chi(\rho) \in [0, 2d\kappa] \) such that

\[ \frac{\log u(t, 0)}{t} = \xi^{(1)}_{Q_t} - \chi + o(1), \quad \text{a.s.,} \quad (2) \]

with \( \xi^{(1)}_A = \sup\{\xi(x): x \in A\} \), if Assumption F is satisfied, and

\[ \frac{\log \langle u(t, 0)^p \rangle}{t} = \frac{H(pt)}{t} - p\chi + o(1), \quad p \in \mathbb{N}, \quad (3) \]

if Assumption H is satisfied. Notice that Assumption F implies Assumption H. Furthermore, it turns out that \( \chi = \chi(\rho) \) is strictly increasing in \( \rho \) with \( \chi(0) = 0 \) and \( \chi(\infty) = 2d\kappa \). For details see [10].

Prominent examples satisfying Assumption F are the double exponential distribution, i.e., \( P(X > x) = \exp\{-\exp\{x/\rho\}\}, x > 0, \) for \( \rho \in (0, \infty) \) and the Weibull distribution, i.e., \( P(X > x) = \exp\{-x^\gamma\}, x > 0 \) with \( \gamma > 1 \) for \( \rho = \infty \).

For \( \alpha \in (0, 2) \) let \( \mathcal{F}_{\alpha} \) be the \( \alpha \)-stable distribution with characteristic function

\[ \phi_{\alpha}(u) = \begin{cases} \exp\{-\Gamma(1 - \alpha)|u|^\alpha \exp\{-\frac{\pi u}{2} \text{sign } u\}\}, & \alpha \neq 1, \\ \exp\{iu(1 - \gamma) - \frac{\pi}{2}u(1 + 2\pi i \log |u| \text{sign } u)\}, & \alpha = 1. \end{cases} \]
Moreover, let

\[ L_\alpha(t) := \exp \{ \varphi(h_\alpha t) \} \quad \text{and} \quad B_\alpha(t) := \exp \{ t \cdot (h_\alpha - \chi + o(1)) \} , \]

where the error term of \( B_\alpha(t) \) is chosen in a suitable way. Then we find our main result:

**Theorem 1 (Stable limit laws).** Let \( \varphi \in C^2 \) be ultimately convex and Assumption F be satisfied. Then for \( \alpha \in (0, 2) \),

\[
\sum_{x \in Q_{L_\alpha(t)}} \frac{u(t,x) - A(t)}{B_\alpha(t)} \xrightarrow{t \to \infty} F_\alpha ,
\]

with

\[
A(t) = \begin{cases} 
0, & \text{if } \alpha \in (0, 1), \\
\langle u(t,0) \rangle, & \text{if } \alpha \in (1, 2), \\
\langle u(t,0)\rangle \mathbb{1}_{u(t,0) \leq B_\alpha(t)}, & \text{if } \alpha = 1.
\end{cases}
\]

Furthermore, we find:

**Theorem 2 (Strong law of large numbers).** Let Assumption H be satisfied, and \( r(t) \) be so large that \( \lim_{t \to \infty} \frac{1}{t} \times (\log |Q_{r(t)}| - H(2t) + 2H(t)) > 0 \) then for every sequence \( (t_n)_{n \in \mathbb{N}} \) satisfying \( \sum_n \exp \{-t_n\} < \infty \),

\[
\frac{1}{|Q_{r(t_n)}|} \sum_{x \in Q_{r(t_n)}} \frac{u(t_n,x)}{\langle u(t_n,0) \rangle} \xrightarrow{t_n \to \infty} 1 \quad \text{a.s.}
\]

Notice that the necessary growth rate of \( Q \) for a WLLN to hold is the same as in Theorem 1 for \( \alpha = 1 \) and that the necessary growth rate of \( Q \) for a CLT to hold corresponds to \( \alpha = 2 \), see [3]. The growth rate in Theorem 2 is of the same order as in the CLT case. Notice, that Theorem 1 is closer to the i.i.d. case than to the case of a random walk among random obstacles as considered in [2], Theorem 3, where the limiting distributions are not stable laws, but infinite divisible distributions with Levy spectral functions that are not continuous. It seems as if the discrete character of the random walk mentioned in the comment on [2], Theorem 3, is more decisive for that model than for ours which can be reduced to the i.i.d. case by virtue of an appropriate coarse-graining.

To get a feeling for the numbers involved we give them for the two examples mentioned above in Table 1 below.

Notice that in the Weibull case we have

\[
\log B_\alpha(t) = \frac{1}{\alpha} \left( \log |u(t,0)^\alpha| + \log |Q_{L_\alpha(t)}| \right) + o(t),
\]

see [11], i.e., the asymptotics of \( B_\alpha(t) \), \( Q_{L_\alpha(t)} \) and the \( \alpha \)th moment of \( u \) are closely linked. Because of (3) and our considerations in Section 2 this relationship seems to be true in the double exponential case, as well.

### 2. Stable limit laws

Let us explain our strategy of the proof of Theorem 1. We decompose the large box \( Q_{L_\alpha(t)} \) into boxes \( Q_{l(t)}^{(i)} \), \( i = 1, \ldots, \lfloor |Q_{L_\alpha(t)}|/|Q_{l(t)}| \rfloor \) of much smaller size. In each subbox we approximate \( u \) by \( u^{(i)} \), the solution with Dirichlet boundary conditions in \( Q_{l(t)}^{(i)} \). In this way, we reduce the problem to the case of i.i.d. random variables. A spectral

| Distribution | \( \varphi(x) \) | \( \log L_\alpha(t) \) | \( \log B_\alpha(t) \) |
|--------------|-----------------|-----------------|-----------------|
| Weibull      | \( x^\gamma, \gamma > 1 \) | \( \left( \frac{\alpha t}{\Gamma} \right)^{\gamma/(\gamma-1)} \) | \( t^\left( \frac{\alpha t}{\Gamma} \right)^{1/(\gamma-1)} - 2dxat + o(t) \) |
| Double-exponential | \( \exp[x/\rho] \) | \( \rho \alpha t \) | \( t \rho \log \rho at - \chi(\rho) at + o(t) \) |
representation shows that \( \sum_{x \in Q_{l(t)}} u^{(i)}(t, x) \) can be approximated by \( e^{t \lambda_1^{(i)}} \), where \( \lambda_1^{(i)} \) is the principal Dirichlet eigenvalue of \( \Delta + \xi \) in \( Q_{l(t)}^{(i)} \). Then a classical result on stable limits for sums of \( t \)-dependent i.i.d. random variables yields the result.

Note that we cannot apply the results of [1] since they require the function \( \varphi \) to be normalized regularly varying, which is for instance not true in the important case of double-exponential tails. An alternative approach could be to adopt the techniques from [5].

Let us turn to the details. We first work on the \( u^{(i)} \) and show in the end how to approximate \( u^{(i)} \) by \( u^{(i)} \). We assume that \( Q_{l(t)}^{(i)} \) are translated copies of \( Q_{l(t)} \). We consider the solution \( u^{(i)} \) to the PAM in \( Q_{l(t)}^{(i)} \) with Dirichlet boundary conditions, i.e., \( \xi(x) = -\infty \) for all \( x \notin Q_{l(t)}^{(i)} \), where \( l(t) = \max \{ t^2 \log^2 t, H(4t) \} \). The corresponding Laplacian will be denoted \( \Delta_{Q_{l(t)}^{(i)}}^{0} \).

By \( \tau_U := \inf \{ t > 0 : X_t \in U \} \) we denote the first hitting time of a set \( U \) by a random walk \( X \). The Feynman–Kac representation of \( u^{(i)} \) reads

\[
\sum_{x \in Q_{l(t)}^{(i)}} u^{(i)}(t, x) = \sum_{x, y \in Q_{l(t)}^{(i)}} |Q_{l(t)}^{(i)}| \sum_{k=1}^{\lfloor Q_{l(t)}^{(i)} \rfloor} e^{x_k^{(i)}} e_k^{(i)}(x) e_k^{(i)}(y), \quad t \in [0, \infty).
\]

For simplicity we have suppressed the time dependence of the eigenvalues and eigenvectors that arises because the boxes are time dependent. From Parseval’s inequality, the fact that \( l(t) \) is of subexponential order and the proof of Theorem 2.2 in [10] it follows that there exists \( \bar{e}^{(i)}(t) \) such that

\[
\sum_{x \in Q_{l(t)}^{(i)}} u^{(i)}(t, x) = e^{\mu_t^{(i)}}, \quad \text{where} \quad \mu_t^{(i)}(\xi) = \lambda_1^{(i)} + \bar{e}^{(i)}(t).
\]

Sometimes we will write \( \mu_t \) instead of \( \mu_t^{(i)} \), \( \lambda_1^{(i)} \) and \( \bar{e}(t) \) for \( \lambda_1^{(i)} \) and \( \bar{e}^{(i)}(t) \).

**Remark.** The above already implies that for \( \log r(t) = o(H(t)) \) the quenched setting is prominent in the following sense,

\[
\lim_{t \to \infty} \frac{\log u(t, 0)}{\log \sum_{x \in Q_{l(t)}} u(t, x)} = 1, \quad \text{a.s.}
\]

In the next lemma we show how the distributions of \( \mu_t \) and \( \xi(0) \) are linked.

**Lemma 3.** Let Assumption F be satisfied. Then for all functions \( h \) with \( \lim_{t \to \infty} |Q_{l(t)}| P(\xi(0) > h(t)) = 0 \) there exists \( \varepsilon(t) = \varepsilon(\xi, t) = o(1) \) such that

\[
P(\mu_t > h(t)) \sim |Q_{l(t)}| P(\xi(0) > h(t) + \chi - \varepsilon(t)), \quad t \to \infty.
\]

**Proof.** In [10], Proof of Theorem 2.16, the authors show that the first eigenvalue of \( \Delta_{Q_{l(t)}^{(i)}}^{0} + \xi \) satisfies

\[
\lambda_1 = \xi^{(i)}_{Q_{l(t)}^{(i)}} - \chi + \bar{e}(t),
\]
with $\bar{\varepsilon}(t) = \bar{\varepsilon}(\xi, t) = o(1)$. Let

$$\varepsilon(t) := \bar{\varepsilon}(t) + \bar{\varepsilon}(t).$$

Then

$$P(\mu_t > h(t)) = P(\xi^{(1)}_{\mathcal{Q}(t)} > h(t) + \chi - \varepsilon(t))$$

$$= 1 - \left(1 - P(\xi(0) > h(t) + \chi - \varepsilon(t))\right)|\mathcal{Q}(t)|$$

$$\sim 1 - \exp\left\{-|\mathcal{Q}(t)|P(\xi(0) > h(t) + \chi - \varepsilon(t))\right\}$$

$$\sim |\mathcal{Q}(t)|P(\xi(0) > h(t) + \chi - \varepsilon(t)), \quad t \to \infty.$$ 

In the third line we use L'Hopital's rule. □

Let

$$\tilde{\varphi}_t(x) = -\log P(\mu_1 > x),$$

and $\tilde{h}_t$ a solution to

$$\sup_{h \in (0, \infty)} (th - \tilde{\varphi}_t(h)) = \tilde{h}_t - \tilde{\varphi}_t(\tilde{h}_t).$$

(5)

It follows that $\tilde{h}_t = h_t + \chi + o(1)$ and if $\tilde{\varphi}_t$ is ultimately convex then $\tilde{h}_t$ is the unique solution to (5). Then, an application of the Laplace method yields

$$\langle u(t, 0) \rangle \sim \langle u(i)(t, 0) \rangle \sim t \int_0^\infty \exp\{th - \tilde{\varphi}(h)\} \, dh = \exp\left\{[t\tilde{h}_t - \tilde{\varphi}(\tilde{h}_t)](1 + o(1))\right\}.$$ 

The first asymptotics follow from [11], Proposition 7. Hence, we obtain together with Lemma 3 that

$$\log B_{\alpha}(t) = t\tilde{h}_{\alpha t} = \frac{1}{\alpha} \left(\log \langle u(t, 0)^\alpha \rangle + \log |\mathcal{Q}_L(t)|\right)(1 + o(1)).$$

To prove convergence of $\sum_{i:\mathcal{Q}(t) \subset \mathcal{Q}_L(t)} (e^{R(t)} - \tilde{A}(t))/B_{\alpha}(t)$, as $t \to \infty$, to an infinitely divisible distribution with characteristic function equal to

$$\phi(u) = \exp \left\{ iau - \frac{\sigma^2 u^2}{2} + \int_{|x| > 0} \left( e^{iu/t} - 1 - \frac{iu}{1 + x^2} \right) d\tilde{L}(x) \right\},$$

we have to verify the following condition (see [16], Chapter IV).

**Condition P.**

(i) Condition of infinite smallness:

$$\lim_{t \to \infty} \max_{i:\mathcal{Q}(t) \subset \mathcal{Q}_L(t)} P \left( \frac{e^{R(t)}}{B_{\alpha}(t)} \geq \varepsilon \right) = 0, \quad \varepsilon > 0.$$

(ii) In all points $x$ of continuity, the function $\tilde{L}$ satisfies:

$$\tilde{L}(x) = -\lim_{t \to \infty} \frac{\mathcal{Q}_L(t)}{|\mathcal{Q}(t)|} P \left( \frac{e^{R(t)}}{B_{\alpha}(t)} > x \right).$$
(iii) The constant $\sigma^2$ satisfies:

$$
\sigma^2 = \lim_{\tau \to 0} \lim_{t \to \infty} \frac{|Q_{L_a(t)}|}{|Q(t)|} \text{Var} \left( \frac{e^{t\mu t}}{B_a(t)} \mathbb{1}_{(e^{t\mu t} / B_a(t)) \leq \tau} \right) \\
= \lim_{\tau \to 0} \lim_{t \to \infty} \frac{|Q_{L_a(t)}|}{|Q(t)|} \text{Var} \left( \frac{e^{t\mu t}}{B_a(t)} \mathbb{1}_{(e^{t\mu t} / B_a(t)) \leq \tau} \right).
$$

(iv) For every $\tau > 0$ the constant $a$ satisfies:

$$
\lim_{t \to \infty} \left\{ |Q_{L_a(t)}| \left( \frac{e^{t\mu t}}{B_a(t)} \mathbb{1}_{(e^{t\mu t} / B_a(t)) \leq \tau} \right) - \tilde{A}(t) \right\} = a + \int_0^\tau x^3 dL(x) - \int_\tau^\infty \frac{x}{1 + x^2} dL(x).
$$

Items (i) and (ii) will follow from the next lemma, and (iii) and (iv) from the next proposition.

**Lemma 4.** Let Assumption F be satisfied and $\varphi \in C^2$ be ultimately convex. Then

$$
\lim_{t \to \infty} \frac{|Q_{L_a(t)}|}{|Q(t)|} P \left( \mu_t > \log \frac{B_a(t)}{t} + \log \frac{x}{t} \right) = x^{-a}.
$$

**Proof.** Lemma 3 and a first order Taylor expansion yield

$$
P \left( \mu_t > \log \frac{B_a(t)}{t} + \log \frac{x}{t} \right) \\
\sim \frac{|Q(t)|}{|Q(t)|} P \left( \xi(0) > \log \frac{B_a(t)}{t} + \log \frac{x}{t} + \chi + \varepsilon(t) \right) \\
= \exp \left\{ \log |Q(t)| - \varphi \left( \frac{\log B_a(t)}{t} + \chi + o(1) \right) - \varphi' \left( \frac{\log B_a(t)}{t} + \chi + o(1) \right) \log \frac{x}{t} + o(1) \right\}.
$$

Since $\varphi$ is ultimately convex and $\xi$ is unbounded from above we find that $\varphi''(h_{at}) = 1 / h_{at}' = o(t^2)$. From this we can conclude that the error term in the Taylor expansion above vanishes asymptotically. Moreover, by our choice of $l(t)$ and $B_a(t)$ it follows that

$$
\log |Q_{L_a(t)}| = \varphi \left( \frac{\log B_a(t)}{t} + \chi + o(1) \right) \quad \text{and} \quad \lim_{t \to \infty} \frac{\varphi'((\log B_a(t)/t) + \chi + o(1))}{t} = \alpha.
$$

**Proposition 5.** Let Assumption F be satisfied and $\varphi \in C^2$ be ultimately convex. Then, for any $\tau > 0$,

(i) If $p > \alpha$ then

$$
\lim_{t \to \infty} \frac{|Q_{L_a(t)}|}{|Q(t)|} \left( \frac{e^{pt\mu t}}{B_a(t)^p} \mathbb{1}_{(e^{pt\mu t} / B_a(t)) \leq \tau} \right) = \frac{\alpha}{p - \alpha} \tau^{p-\alpha};
$$

(ii) If $p < \alpha$ then

$$
\lim_{t \to \infty} \frac{|Q_{L_a(t)}|}{|Q(t)|} \left( \frac{e^{pt\mu t}}{B_a(t)^p} \mathbb{1}_{(e^{pt\mu t} / B_a(t)) > \tau} \right) = \frac{\alpha}{\alpha - p} \tau^{p-\alpha};
$$

(iii) If $p = \alpha$ then

$$
\lim_{t \to \infty} \frac{|Q_{L_a(t)}|}{|Q(t)|} \left( \frac{e^{pt\mu t}}{B_a(t)^p} (\mathbb{1}_{(e^{pt\mu t} / B_a(t)) \leq \tau} - \mathbb{1}_{(e^{pt\mu t} / B_a(t)) \leq 1}) \right) = \alpha \log \tau.
$$
Proof. (i) Integration by parts yields
\[
\left\{ e^{pt\mu_t} \mathbf{1}_{\{ e^{pt\nu_t}/B_{a(t)} \leq t \} } \right\} = \int_0^{h_{at} + \log \tau/t} e^{ptx} d(1 - \tilde{F}_{\mu_t}(x)) = \left. \left[ e^{ptx - \varphi(x)} \right]_{x=0}^{x=h_{at} + \log \tau/t} + pt \int_0^{h_{at} + \log \tau/t} e^{ptx - \varphi(x)} dx. \right.
\]
Here \( \tilde{F}_{\mu_t} \) denotes the tail distribution function of \( \mu_t \). Similarly as in the proof of Lemma 4 we find with the help of a first order Taylor expansion of \( \varphi \) that uniformly in \( \tau \),
\[
\tilde{\varphi}(\ln h_{at} + \log \tau/t) \sim \tilde{\varphi}(h_{at}) - \alpha \log \tau, \quad t \to \infty.
\]
Substituting \( x = \ln h_{at} + \log \tau/t u \) for \( \tau \neq 1 \) we find that
\[
pt \int_0^{h_{at} + \log \tau/t} e^{ptx - \varphi(x)} dx \sim e^{pth_{at} - \tilde{\varphi}(h_{at})} p \log \tau \int_{-\infty}^{1} e^{u(p-\alpha) \log \tau} du \sim \frac{p}{p-\alpha} e^{pth_{at} - \tilde{\varphi}(h_{at}) + (p-\alpha) \log \tau}, \quad t \to \infty.
\]
Altogether this proves the claim.

(ii) and (iii) follow similarly. \( \square \)

Overall we find:

Theorem 6. Let Assumption F be satisfied and \( \varphi \in C^2 \) be ultimately convex. Then for \( \alpha \in (0, 2) \),
\[
\sum_{i: Q(i) \subset Q_{\mu_t}} e^{\mu_t(i)} - \tilde{A}(t) \to \infty \Rightarrow \mathbb{F}_\alpha,
\]
with
\[
\tilde{A}(t) = \begin{cases} 
0, & \text{if } \alpha \in (0, 1), \\
\left\{ e^{\mu_t(i)} \right\}, & \text{if } \alpha \in (1, 2), \\
\left\{ e^{\mu_t(i) \mathbf{1}_{\mu_t(i) \leq 1}} \right\}, & \text{if } \alpha = 1.
\end{cases}
\]

Proof. Since the \( u^{(i)} \) are i.i.d., the \( \mu_t^{(i)} \) are as well. Hence, we have to check the four points of Condition P. Items (i) and (ii) follow from Lemma 4. We find that \( \tilde{L}(x) = x^{-\alpha} \). It follows from Proposition 5 that \( \sigma^2 = 0 \). Furthermore, Proposition 5 together with [1], Proposition 6.4, yields the constant \( \alpha \) from which we can deduce \( \phi \). The stability of the limit law follows from [16], Theorem IV.12, since \( \sigma^2 = 0 \) and \( \tilde{L}(x) = x^{-\alpha} \). \( \square \)

Remark. An infinitely divisible law with characteristic function as in (6) is stable if and only if either \( \tilde{L} \equiv 0 \) or \( \sigma^2 = 0 \) and \( \tilde{L}(x) = cx^{-\alpha}, c > 0, \alpha \in (0, 2) \), see [16], Theorem IV.12.

We extend the functions \( u^{(i)} \) to a function \( \tilde{u}: Q_{\mu_t(t)} \to [0, \infty) \) by putting \( \tilde{u}(t, x) = u^{(i)}(t, x) \) for \( x \in Q^{(i)}_{\mu_t(t)} \). Now it remains to show that
\[
\sum_{x \in Q_{\mu_t(t)}} \frac{u(t, x) - A(t)}{B_{a(t)}} \quad \text{and} \quad \sum_{x \in Q_{\mu_t(t)}} \frac{\tilde{u}(t, x) - \tilde{A}(t)/Q_{\mu_t(t)}}{B_{a(t)}} = \sum_{i: Q^{(i)}_{\mu_t(t)} \subset Q_{\mu_t(t)}} \frac{\exp\{t\mu_t^{(i)}(t) - \tilde{A}(t)\}}{B_{a(t)}}
\]
Fig. 1. Coarse-graining.

have the same $\alpha$-stable limit distribution. To this end let $\mathcal{T}_t^c = \bigcup_{i: Q_l^{(i)} \subset Q_{t;\alpha}(t)} Q_l^{(i)} \setminus Q_l^{(i)}(1 - 1/t)$ and $\mathcal{T}_t = Q_{L_{\alpha}(t)} \setminus \mathcal{T}_t^c$ (see Fig. 1).

Notice that

$$u(t, x) - \tilde{u}(t, x) = \mathbb{E}_x \exp\left\{ \int_0^\infty \xi(X_s) \, ds \right\} \mathbb{I}_{\tau(Q_l)^c \leq t}, \quad (t, x) \in [0, \infty) \times Q_l^{(i)}.$$

In the next lemma we show that those paths of the random walk in the Feynman–Kac formula that start in $\mathcal{T}_t$ and leave $Q_l(t)$ before time $t$ are asymptotically negligible.

**Lemma 7.** Almost surely,

$$\lim_{t \to \infty} \sup_{x \in Q_l(t)(1 - 1/t)} \mathbb{E}_x \exp\left\{ \int_0^\infty \xi(X_s) \, ds \right\} \mathbb{I}_{\tau(Q_l)^c \leq t} = 0.$$

**Proof.** We find that

$$\sup_{x \in Q_l(t)(1 - 1/t)} \mathbb{E}_x \exp\left\{ \int_0^\infty \xi(X_s) \, ds \right\} \mathbb{I}_{\tau(Q_l)^c \leq t} \leq \exp\left\{ t \sup_{x \in Q_l(t)} \xi(x) \right\} \mathbb{P}_0(\tau(Q_l)^c \leq t) \leq 2d+1 \exp\left\{ t \cdot o(\log(|Q_l(t)|)) - |Q_l(t)/t| \log\left( \frac{|Q_l(t)|}{dt} \right) \right\} \to 0.$$

In the last inequality we use [10], Lemma 2.5 and Corollary 2.7.

**Lemma 8.** For all $\varepsilon > 0$,

(i) if $\alpha \in (0, 1]$ then

$$\lim_{t \to \infty} \mathbb{P} \left( \frac{1}{B_{\alpha}(t)} \sum_{x \in Q_{L_{\alpha}(t)}} \left[ u(t, x) - \tilde{u}(t, x) \right] > \varepsilon \right) = 0;$$

(ii) if $\alpha \in [1, 2)$ then

$$\lim_{t \to \infty} \mathbb{P} \left( \frac{1}{B_{\alpha}(t)} \sum_{x \in Q_{L_{\alpha}(t)}} \left[ u(t, x) - \langle u(t, x) \rangle - \tilde{u}(t, x) + \langle \tilde{u}(t, x) \rangle \right] > \varepsilon \right) = 0;$$
(iii) if $\alpha = 1$ then
\[ \lim_{t \to \infty} P\left( \sum_{x \in Q_{L\alpha}(t)} \frac{u(t, x) - \langle u(t, x) \rangle_{x \in B_\alpha(t)}}{B_\alpha(t)} > \varepsilon \right) = 0. \]

**Proof.** (i) From Lemma 7 and the fact that $|I_t| < B_\alpha(t)$ for all $t$ it follows that for $t \to \infty$,
\[ P\left( \frac{1}{B_\alpha(t)} \sum_{x \in Q_{L\alpha}(t)} u(t, x) - \tilde{u}(t, x) > \varepsilon \right) \sim P\left( \frac{1}{B_\alpha(t)} \sum_{x \in I_t^c} u(t, x) - \tilde{u}(t, x) > \varepsilon \right). \]

By the definition of $B_\alpha(t)$ and by Markov’s inequality it follows that
\[ P\left( \frac{1}{B_\alpha(t)} \sum_{x \in I_t^c} u(t, x) - \tilde{u}(t, x) > \varepsilon \right) \leq \frac{1}{\varepsilon} \frac{\langle \sum_{x \in I_t^c} u(t, x) \rangle^\alpha}{|Q_{L\alpha}(t)| \langle u(t, 0) \rangle^\alpha \langle u(t, 0) \rangle^\alpha} \]
\[ \to 0 \text{ as } t \to \infty. \]

(ii) Similarly as in case (i) we find that asymptotically
\[ P\left( \frac{1}{B_\alpha(t)} \sum_{x \in Q_{L\alpha}(t)} \left[ u(t, x) - \langle u(t, x) \rangle - \tilde{u}(t, x) + \langle \tilde{u}(t, x) \rangle \right] > \varepsilon \right) \leq \frac{1}{\varepsilon} \frac{\langle \sum_{x \in I_t^c} u(t, x) \rangle^\alpha}{|Q_{L\alpha}(t)| \langle u(t, 0) \rangle^\alpha} + o(1). \]

Furthermore, we have
\[ \left\langle \left( \sum_{x \in I_t^c} u(t, x) \right)^\alpha \right\rangle \leq \left\langle \left( \sum_{x \in I_t^c} u(t, x)^2 + \sum_{y \in I_t^c: |x-y| \leq l(t)/t} u(t,x)u(t,y) + \sum_{y \in I_t^c: |x-y| > l(t)/t} u(t,x)u(t,y) \right)^{\alpha/2} \right\rangle \]
\[ \leq \sum_{x \in I_t^c} \left\langle t(t)(1 - 1/t) \left| u(t, x) \right|^\alpha \right\rangle + \sum_{x,y \in I_t^c: |x-y| > l(t)/t} \left\langle \left( u(t, x)u(t, y) \right)^{\alpha/2} \right\rangle. \]

The first summand can be treated as in case (i) whereas the second summand can be treated similarly as in the proof of Lemma 9.

(iii) follows analogously. $\square$

Now we are able to prove Theorem 1.

**Proof of Theorem 1.** We only consider the case $\alpha \in (0, 1)$. The other cases follow similarly. It follows from Lemma 8 that for every $\varepsilon > 0$,
\[ \lim_{t \to \infty} P\left( \sum_{i: Q(i) \in Q_{L\alpha}(t)} \frac{\sum_{x \in Q(i)} u(t, x) - \exp\{t \mu(i)\}}{B_\alpha(t)} > \varepsilon \right) = 0. \]
while Theorem 6 states that under the same conditions as in Theorem 1,

\[
\sum_{i : Q_{l(t)}(t) \subseteq Q_{l(t)}} \frac{e^{\mu_i(t)}}{B_{a}(t)} \xrightarrow{t \to \infty} F_a.
\]

Therefore, the claim follows from Slutzky’s theorem. \(\square\)

**Remark.** We expect that a similar result as Theorem 1 with the same stable limit distribution also holds for potential tails that are bounded from above as considered in [4] and [12]. However, since in that case we do not have such a close link between \(\mu_t\) and \(\xi_{i(t)}^{(1)}\) we cannot determine the distribution of \(\mu_t\) and therefore \(L_a(t) = -\log P(\mu_t > \tilde{h}_t)\) cannot be made as explicit as under Assumption F.

For more heavy tailed potentials than considered in this paper the cumulant generating function \(H\) is not finite any more and annealed asymptotics do not exist. Therefore, our approach is not feasible in that situation. However, there are some recent papers on the PAM with localised initial condition \(\delta_0\) that derive scaling limit theorems for exponential tails, see [14] or Weibull tails with parameter \(\gamma < 1\), see [17].

### 3. Strong law of large numbers

Recall that \(l(t) = \max\{t^2 \log^2 t, H(4t)\}\) and that \(x + Q_{l(t)}(t)\) is the lattice box with centre \(x\) and sidelength \(l(t)\).

**Lemma 9.** Let Assumption H be satisfied and \(r(t)\) be chosen as in Theorem 2, then

\[
\lim_{t \to \infty} \frac{1}{|Q_{r(t)}(t)|^2} \sum_{x, y \in Q_{r(t)}(t); |x - y| > 2l(t)} \left( \frac{\langle u(t, x) u(t, y) \rangle}{\langle u(t, 0) \rangle^2} - 1 \right) = 0.
\]

**Proof.** For \(t > 0\) and \(x \in Q_{r(t)}(t)\) let

\[
u^{(1)}(t, x) = \mathbb{E}_x \exp \left\{ \int_0^\infty \xi(X_s) \, ds \right\} \mathbb{1}_{\tau (x + Q_{l(t)}) < t}
\]

and

\[
u(t, x, y) = \mathbb{E}_{x, y} \exp \left\{ \int_0^\infty \xi(X_s) \, ds \right\} \exp \left\{ \int_0^\infty \xi(Y_s) \, ds \right\} \mathbb{1}_{\tau X (x + Q_{l(t)}) < t \text{ or } \tau Y (y + Q_{l(t)}) < t},
\]

where \(X\) and \(Y\) are two independent random walks starting in \(x\), \(y\), respectively, \(\mathbb{E}_{x, y}\) is their joint expectation, and \(\tau X\), \(\tau Y\) are their exit times from a set \(A \subset \mathbb{Z}^d\), respectively. If \(|x - y| > 2l(t)\) then \(\nu^{(1)}(t, x)\) and \(\nu^{(1)}(t, y)\) are independent, and hence

\[
\sum_{x, y \in Q_{r(t)}(t); |x - y| > 2l(t)} \left( \frac{\langle u(t, x) u(t, y) \rangle}{\langle u(t, 0) \rangle^2} - 1 \right) = \sum_{x, y \in Q_{r(t)}(t); |x - y| > 2l(t)} \frac{\langle u(t, x, y) \rangle}{\langle u(t, 0) \rangle^2}.
\]

Hölder’s inequality and [9], Lemma 2.4 and Theorem 3.1, yield for all \(x, y \in Q_{r(t)}(t) \setminus Q_{l(t)}\),

\[
\langle u(t, x, y) \rangle \leq \sqrt{\langle u(t, 0) \rangle^4 2 \mathbb{P}_x (\tau (x + Q_{l(t)}) < t)}
\]

\[\leq \exp \left\{ \frac{1}{2} \left( l(t) - l(t) \log l(t) + o(t) \right) \right\} \xrightarrow{t \to \infty} 0.\]
Proof of Theorem 2. By Chebyshev’s inequality we find that for every $s > 0$,
\[
\mathbb{P}\left( \sup_{t_n > s} \frac{1}{|Q_{r(t_n)}|} \sum_{x \in Q_{r(t_n)}} \left( \frac{u(t_n, x)}{\langle u(t_n, 0) \rangle} - 1 \right) > \varepsilon \right) \leq \sum_{t_n > s} \frac{1}{\varepsilon^2 \text{Var}\left( \frac{1}{|Q_{r(t_n)}|} \sum_{x \in Q_{r(t_n)}} \left( \frac{u(t_n, x)}{\langle u(t_n, 0) \rangle} - 1 \right) \right)}.
\]
As $t$ tends to infinity it follows with Lemma 9 that
\[
\text{Var}\left( \frac{1}{|Q_{r(t)}|} \sum_{x \in Q_{r(t)}} \left( \frac{u(t, x)}{\langle u(t, 0) \rangle} - 1 \right) \right) \sim \frac{1}{|Q_{r(t)}|} \sum_{x, y \in Q_{r(t)}} \left( \frac{\langle u(t, 0) u(t, y) \rangle}{\langle u(t, 0) \rangle^2} - 1 \right)
\sim \frac{1}{|Q_{r(t)}|} \sum_{x \in Q_{r(t)}} \left( \frac{\langle u(t, 0) u(t, x) \rangle}{\langle u(t, 0) \rangle^2} - 1 \right)
\leq \frac{|Q_{r(t)}|}{|Q_{r(t)}|} \left( \frac{\langle u(t, 0) \rangle^2}{\langle u(t, 0) \rangle^2} - 1 \right)
= \exp\left\{ - \log |Q_{r(t)}| + H(2t) - 2H(t) + o(t) \right\}.
\]
The last asymptotics are due to (3). Now the claim follows because for our choice of $r(t)$,
\[
\lim_{t_n \to \infty} \sum_{t_n > s} \exp\left\{ - \log |Q_{r(t_n)}| + H(2t_n) - 2H(t_n) + o(t_n) \right\} = 0.
\]
\[\square\]

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References

[1] G. Ben Arous, L. V. Bogachev and S. A. Molchanov. Limit theorems for sums of random exponentials. Probab. Theory Related Fields 132 (2005) 579–612. MR2198202
[2] G. Ben Arous, S. Molchanov and A. Ramirez. Transition from the annealed to the quenched asymptotics for a random walk on random obstacles. Ann. Probab. 33 (2005) 2149–2187. MR2184094
[3] G. Ben Arous, S. Molchanov and A. Ramirez. Transition asymptotics for reaction–diffusion in random media. In Probability and Mathematical Physics: A Volume in Honor of Stanislav Molchanov. CRM Proc. Lecture Notes 42 1–40. Amer. Math. Soc., Providence, RI, 2007. MR2352279
[4] M. Biskup and W. König. Long-time tails for the parabolic Anderson model with bounded potential. Ann. Probab. 29 (2001) 636–682. MR1849173
[5] L. Bogachev. Limit laws for norms of IID samples with Weibull tails. J. Theoret. Probab. 19 (2006) 849–873. MR2279606
[6] A. Bovier, I. Kurkova and M. Löwe. Fluctuations of the free energy in the REM and the $p$-spin SK models. Ann. Probab. 30 (2002) 605–651. MR1905853
[7] M. Cranston and S. A. Molchanov. Quenched to annealed transition in the parabolic Anderson problem. Probab. Theory Related Fields 138 (2007) 177–193. MR2288068
[8] J. Gärtner and W. König. The parabolic Anderson model. In Interacting Stochastic Systems 153–179. J.-D. Deuschel and A. Greven (Eds). Springer, Berlin, 2005. MR2118574
[9] J. Gärtner and S. A. Molchanov. Parabolic problems for the Anderson model. I. Intermittency and related topics. Comm. Math. Phys. 132 (1990) 613–655. MR1069840
[10] J. Gärtner and S. A. Molchanov. Parabolic problems for the Anderson model. II. Second-order asymptotics and structure of high peaks. Probab. Theory Related Fields 111 (1998) 17–55. MR1626766
[11] J. Gärtner and A. Schnitzler. Time correlations for the parabolic Anderson model. Electron. J. Probab. 16 (2011) 1519–1548. MR2827469
[12] R. van der Hofstad, W. König and P. Mörters. The universality classes in the parabolic Anderson model. Comm. Math. Phys. 267 (2006) 307–353. MR2249772
[13] A. Janssen. Limit laws for power sums and norms of i.i.d. samples. Probab. Theory Related Fields 146 (2010) 515–533. MR2574737
[14] H. Lacoin and P. Mörters. A scaling limit theorem for the parabolic Anderson model with exponential potential. In Probability in Complex Physical Systems. In Honour of J. Gärtner and E. Bolthausen 247–271. Springer Proceedings in Mathematics 11. Springer, Berlin, 2012. DOI:10.1007/978-3-642-23811-6_10.
[15] S. A. Molchanov. Lectures on random media. Lecture Notes in Math. 1581 242–411. Springer, Berlin, 1994. MR1307415
[16] V. V. Petrov. Sums of Independent Random Variables. Springer, New York, 1975. MR0388499
[17] N. Sidorova and A. Twarowski. Localisation and ageing in the parabolic Anderson model with Weibull potential. Ann. Probab. To appear. Available at arXiv:1204.1233v2, 2012.