Bloch bound states in the radiation continuum in a periodic array of dielectric rods

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We consider an infinite periodic array of dielectric rods in vacuum with the aim to demonstrate three types of a Bloch bound states in the continuum (BSC), symmetry protected with a zero Bloch vector, embedded into one diffraction channel with nonzero Bloch vector, and embedded into two and three diffraction channels. The first and second types of the BSC exist in a wide range of material parameters of the rods, while the third occurs only at a specific value of the radius of the rods. We show that the second type supports the power flux along the array. In order to find BSC we put forward an approach based on the expansion over the Hankel functions. We show how the BSC reveals itself in the scattering function when the singular BSC point is approached along a specific path in the parametric space.

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I. INTRODUCTION

In 1929, von Neumann and Wigner [1] predicted the existence of discrete solutions of the single-particle Schrödinger equation embedded in the continuum of positive energy states, bound states in the continuum (BSC). Their analysis examined by Stillinger and Herrick [2] for a long time was regarded as mathematical curiosity because of certain spatially oscillating central symmetric potentials. That situation cardinaly changed when Friedrich and Wintgen [3] formulated a generic two-level Fano-Anderson model and derived a condition for BSC as a resonant state whose width tends to zero as at least one physical parameter varies continuously (see, also [4–6]). Nowadays one can see a rapid growth in the amount of publications, both theoretical and experimental devoted to the BSCs in different physical systems [7–23].

The BSC can be classified by the mechanism responsible for the localization of waves [15, 16]. The most obvious mechanism is related to the symmetry [7, 11] when the continuum states and the BSCs have different symmetries. The second mechanism is a Fabry-Perot resonator with the wave trapped between the mirrors as the distance between the mirrors is tuned. That mechanism was used in the system of quantum dots coupled by a wire [24, 25] as well as in photonics [9, 10, 13, 26, 27]. The above mentioned types of the BSC were extensively studied in different photonics structures and experimentally observed [14, 17–20]. The third mechanism is full destructive interference of two resonances which was originally put forward by Friedrich and Wintgen [3] and later developed in Refs. [6, 21]. It should be noted however that experimental observation of this type of BSC is difficult because it is necessary to vary material parameters [20]. Fourth, at some value of parameters the coupling of a bound state of the closed system with the continuum channel can turn to zero accidentally to give rise to the accidental BSC [16]. We speculate that this mechanism underlies the robust BSC observed in photonic crystal slab [22, 23]. In the present paper we consider a one-dimensional periodic array of GaAs cylindrical rods. We show that the array is capable to support multiple symmetry protected Bloch BSC. The Bloch BSC with nonzero wave vector supports power flux. We also show that there are Bloch BSCs embedded into two and three continua.

II. BASIC EQUATIONS FOR EM WAVE SCATTERING BY PERIODIC ARRAY OF RODS

The system under consideration is infinitely long array of GaAs rods. The rods are infinitely long in the z direction, parallel to each other, and periodically spaced with the distance $h$ along the x-axis on the x-y plane as shown in Fig. 1. In what follows we take $h = 1$. We consider the scattering of TM electromagnetic waves by this array. The scattering of plane waves by cylinders has been the subject of many investigations long time ago. Most of difficulties in this connection is caused by multiple scattering by cylinders [28]. For the case of two cylinders [29] and infinite periodic row of cylinders [30, 31] the problem was considered to bring it to a simple and tractable formulation. In this section we present the basic equations of that theory for the reader’s convenience based on Ref. [31].

Assume that the periodic array of rods is illuminated by a plane wave

$$\psi_{inc}(x, y) = \sqrt{\frac{2}{k_y}} e^{i(k_x x + k_y y)}$$
where $\psi$ is the electric field directed along the rods, $k_x = -k_0 \cos \varphi_i, k_y = -k_0 \sin \varphi_i, k_0 = \omega/c$ and $\varphi_i$ defines the angle of incidence. The plane wave can be expanded over the Bessel functions

$$
\psi_{\text{inc}}(r, \varphi) = \sqrt{\frac{2}{|k_y|}} \sum_m (-i)^m e^{im(\varphi - \varphi_i)} J_m(k_0 r) = \sum_m \psi_{\text{inc},m} e^{im\varphi} J_m(k_0 r),
$$

where

$$
\psi_{\text{inc},m} = \sqrt{\frac{2}{|k_y|}} (-i)^m e^{-im\varphi_i} = \sqrt{\frac{2}{|k_y|}} \left( \frac{i k_x + k_y}{k_0} \right)^m.
$$

The scattered wave outside the rod is given by the Hankel functions

$$
\Psi_s(r, \varphi) = \sqrt{\frac{2}{|k_y|}} \sum_m a_m \exp (im\varphi) H^{(1)}_m(k_0 r)
$$

The relation between incident and scattered waves is given by

$$
a_m = \sum_n T_{mn} \psi_{\text{inc},n},
$$

where the transition matrix has the following form

$$
T_{mn} = \delta_{mn} \frac{\sqrt{\epsilon J'_m(qR) J_m(k_0 R) - J''_m(k_0 R) J_m(qR)} - \sqrt{\epsilon J'_m(qR) J_m(k_0 R) - J''_m(k_0 R) H^{(1)}_m(k_0 R)}}{H^{(1)}_m(k_0 R) - \sqrt{\epsilon J'_m(qR) J_m(k_0 R) - J''_m(k_0 R) H^{(1)}_m(k_0 R)}}.
$$

$q = \sqrt{\epsilon} k_0$, and $\epsilon$ is the permittivity of the rod of radius $R$. The scattering function is a sum of waves scattered from each rod:

$$
\Psi_s(x, y) = \sqrt{\frac{2}{|k_y|}} \sum_j \sum_m a_{m,j} \exp (ikx_j) \exp (im\varphi_j) H^{(1)}_m(k_0 r_j)
$$

where $(r_j, \varphi_j)$ is the local polar coordinate system whose origin is located in the center of the $j$-th rod and $a_{m,j}$ denotes the amplitude of the field scattered from the $j$-th rod. The periodicity of the structure requires that scattered field satisfies the Bloch theorem:

$$
\Psi_s(x + 1, y) = \exp (ikx) \Psi_s(x, y)
$$

with $k_x$ as the Bloch vector. Substituting Eq. (7) into Eq. (6) we have

$$
\Psi_s(x, y) = \sqrt{\frac{2}{|k_y|}} \sum_j \sum_m \exp (ikx_j) a_m \exp (im\varphi_j) H^{(1)}_m(k_0 r_j),
$$
where \( a_m = a_{m,0} \). The total wave function therefore is \( \Psi = \Psi_{inc} + \Psi_s \). Consequently, Eq. \((7)\) defines the wave vectors of diffraction channels \([27, 31]\)

\[
k_{y,l} = \sqrt{k_0^2 - k_{x,l}^2}, \quad k_{x,l} = k_x + 2\pi l, l = 0, \pm 1, \pm 2, \ldots
\]

(9)

The infinite periodic array of rods scatters the wave only into a number of directions defined by wave vectors \( k_{x,l}, k_{y,l} \). In the other words, the system under consideration supports only finite number of continua or diffraction channels notified by integer \( l \) in Eq. \((9)\) for which \( k_{y,l} \) is real. The diffraction channel waves are given by

\[
\psi_{r,t}^l = \sqrt{\frac{2}{k_{y,l}}} e^{ik_{y,l}x \pm ik_{y,l}y}.
\]

(10)

Here indexes \( t, r \) imply the transmitted and reflected plane waves which are schematically shown in Fig. \(1\) by blue thin arrows.

Following Ref. \([31]\) we introduce the aggregate matrix \( \hat{A} \) which relates the amplitudes \( \Psi = \{\psi_{inc,m}\} \) in the expansion of the incident plane wave \((1)\) to the amplitudes \( \mathbf{a} = \{a_m\} \) of scattered wave \((8)\)

\[
\mathbf{a} = \hat{A} \Psi_{inc}.
\]

(11)

That matrix could be considered as an analogue of the scattering matrix however there is important difference. The last connects the scattering channels notified by \( l \) with the incident wave while the former connects the amplitudes of the Hankel functions in the scattered wave with the incident wave.

For the infinite periodic array of rods multiple scattering events can be summated as follows \([31]\)

\[
\hat{A} = \left(1 - \hat{T} \hat{L}\right)^{-1} \hat{T}
\]

(12)

where \( \hat{T} \)-matrix is given by Eq. \((5)\) and

\[
L_{n-m} = \sum_{l=1}^{\infty} H_n^{(1)}(lk_0 h)[e^{ik_{x,l} x} + (-1)^{n-m} e^{-ik_{x,l} x}].
\]

(13)

Let us introduce components of waves transmitted and reflected in the \( l \)-th diffraction channel

\[
\mathbf{v}_{l}^{(t,r)} = \sqrt{\frac{2}{k_{y,l}}} \left\{ \left( \frac{ik_{x,l} + k_{y,l}}{k_0} \right)^m \right\}
\]

(14)

in terms of which we rewrite Eq. \((11)\) in accordance to Eq. \((2)\)

\[
\mathbf{a} = \hat{A} \mathbf{v}_0^{(t)}.
\]

(15)

Then we have \([31]\)

\[
r_l = [\mathbf{v}_{l}^{(r)}]^+ \mathbf{a}, \quad t_l = \delta_{l,0} + [\mathbf{v}_{l}^{(t)}]^+ \mathbf{a},
\]

(16)

which are the reflection and transmission amplitudes respectively for open channels. The scattering function \((6)\) can be written as follows \([31]\)

\[
\Psi_s(x, y) = \begin{cases} 
\sum_t r_t \psi_{t}^{(r)}(x, y) & \text{if } y > 0 \\
\sum_t t_t \psi_{t}^{(t)}(x, y) & \text{if } y < 0
\end{cases}
\]

(17)

where both open and closed channels are summated.

III. CALCULATION OF BOUND STATES IN THE CONTINUUM

Typically, a plane wave is scattered by the array of rods into diffraction channels according to Eq. \((15)\). However there could be a unique case when the matrix \( \hat{A} \) in Eq. \((15)\) is singular and the solution for the scattering wave exists
irrespective to the incident wave amplitude $v_0(t)$. As will be shown below this unique solution is decoupled from the diffraction channels and localized nearby the array. This solution is the bound state with frequency embedded in the diffraction continua, i.e., a BSC. In this section we adapt the approach of the effective non-Hermitian Hamiltonian for calculation of BSC in the periodic array of rods.

As it could be seen from Eq. (15) the aggregate matrix $\hat{A}$ must be singular for the BSC to exist. This matrix is not hermitian, and similar to the approach of the effective Hamiltonian it is fruitful to introduce the biorthogonal basis of the eigenvectors of the matrix

$$\hat{\Lambda} = \hat{A}^{-1} = \hat{T}^{-1} - \hat{L}$$

as follows

$$\hat{\Lambda} x_f = \lambda_f x_f, \quad \hat{\Lambda}^+ y_f = \lambda_f^+ y_f, \quad y_f^+ x_g = \delta_{fg}. \quad (19)$$

One can show that

$$\sum_f x_f y_f^+ = 1, \quad \hat{\Lambda}^{-1} = \sum_f \frac{x_f y_f^+}{\lambda_f}. \quad (20)$$

Then by the use of Eqs. (19) and (20) the scattering state in Eq. (15) can be expanded over the biorthogonal basis as follows

$$a = \sum_f \frac{(y_f^+ v_0^{(l)}) x_f}{\lambda_f}. \quad (21)$$

By substituting Eq. (21) into (16) we obtain the scattering amplitudes in the following form

$$r_l = \sum_f W_{f,l}^{(r)} \overline{W}_{f,0}^{(r)} \frac{t_l}{\lambda_f}, \quad t_l = \delta_{l,0} + \sum_f \frac{W_{f,l}^{(r)} \overline{W}_{f,0}^{(r)} \lambda_f}{\lambda_f}. \quad (22)$$

where we introduced the coupling constants between the $f$-th eigenvector of the matrix $\Lambda$ and the $l$-th diffraction channel vector $v_0^{(l)}$

$$W_{f,l}^{(r,t)} = [v_t^{(r,t)}]^+ x_f, \quad \overline{W}_{f,0}^{(r)} = y_f^+ v_0^{(l)}. \quad (23)$$

Eqs. (22) are similar to those derived in Ref. [34].

The BSC occurs when, at least, one of complex eigenvalues $\lambda$ equals zero

$$\hat{\Lambda} \Phi_{BSC} = 0 \quad (24)$$

Therefore, the BSC is in fact a null eigenvector. Next, Eq. (22) shows that for the scattering amplitudes to remain finite we have to imply the condition

$$[v_t^{(r,t)}]^+ \Phi_{BSC} = 0. \quad (25)$$

This equation shows that the BSC is decoupled from all open diffraction channels and therefore does not leak into the radiation continuum.

**IV. SYMMETRY PROTECTED BSC**

In what follows we take the dielectric constant of the rods $\epsilon = 12$ (GaAs rods). We briefly describe numerical procedure of calculation of the BSC function. At the first step the eigenvalue problem for the matrix $\Lambda$ defined by Eqs. (13) and (18) is solving. Numerically we truncate the rank of the matrix $\Lambda$ by $|m| \leq 10$ to ensure sufficient accuracy. Then the eigenvector components $a_m$ are substituted into Eq. (15) and after into Eq. (17) to find the BSC wave function outside the rods. A summation over $l$ includes only closed channels, i.e. evanescent modes. The BSC function inside the rods is to be expanded over the Bessel functions where the coefficients are calculated by means of the matching of the outside and inside functions on the boundary of rods.
Let us assume that only diffraction channel $l = 0$ is open. Examples of the Bloch states embedded into this channel are shown in Fig. 2 for $k_x = 0$ where the pattern of the BSC in Fig. 2 (a) was first presented by Shipman and Venakides [7]. We show that these Bloch BSCs are symmetry protected. For $k_x = 0, \pi$ the matrix $\hat{\Lambda}$ has the property

$$\Lambda_{m,n} = \Lambda_{|m-n|}. \quad (26)$$

Let us introduce the operator which inverts the indexes of $a_m$

$$\hat{P}a_m = a_{-m}. \quad (27)$$

Then it follows from Eq. (26) that operator $\hat{P}$ commutes with the matrix $\tilde{\Lambda}$. Therefore, the eigenvectors of the matrix $\tilde{\Lambda}$ are only symmetric or antisymmetric relative to $m \to -m$, in particular the same holds true for the BSC

$$\Psi_{m,BSC}^{s,a} = \pm \Psi_{-m,BSC}^{s,a} \quad (28)$$

where $s$ and $a$ stand for symmetric and antisymmetric states respectively. For $k_x = 0, \pi$ an additional property arises as it could be seen from Eq. (13)

$$\Lambda_{|2m+1|} = 0. \quad (29)$$

That allows to split the BSCs into the following types. The BSCs of the first type have only odd components $a_{2m} = 0$, while the BSC of the second type have only even components $a_{2m+1} = 0$. Thus, the BSCs for $k_x = 0, \pi$ can be split into four types: $(s,o), (s,e), (a,o), (a,e)$.

For diffraction channel $l = 0$ with $k_x = 0$ we have from Eq. (14)

$$v_{l=0,m}^{(t)} = \sqrt{\frac{2}{k_{y,0}}} (-1)^m, \quad v_{l=0,m}^{(r)} = \sqrt{\frac{2}{k_{y,0}}}. \quad (30)$$
Substituting Eq. (30) into Eq. (25) we find that only two types of the BSC \((a, o)\) and \((a, e)\) are symmetry protected. Let us consider first the Bloch BSC which belongs to the type \((a, o)\). Matching the solutions Eq. (8) on the boundary of the rod we find the BSC wave function interior of the \(j = 0\) rod as follows

\[
\Phi(r, \varphi)^{(a,o)} = \sum_{m=1}^{\infty} A_{2m-1}(r) \cos(2m-1)\varphi, \quad (31)
\]

with

\[
A_m(r) = \frac{J_m(k_0 R)}{T_{mm}} + \frac{H_1^{(1)}(k_0 R)}{J_m(q R)} a_m J_m(q r)
\]

(32)

where the property of the Bessel functions

\[ J_{-m}(x) = (-1)^m J_m(x) \]

was taken into account. One can see that the BSC function has the nodal line at \(\varphi = \pi/2\), i.e., at \(x = 0\). However because the BSC is a Bloch wave we obtain a periodical set of nodal lines at \(x = 0, \pm 1, \pm 2, \ldots\). Note that the BSC exists not only within the rods. Although there is no contribution of \(l = 0\) diffraction channel into the scattering function (17) because \(r_{l=0} = 0, t_{l=0} = 0\) there are contributions of closed evanescent diffraction channels with \(l \neq 0\). According to Eq. (9) the dominant contribution \(l = 1\) gives us the localization scale in the \(y\)-direction as

\[ y_{BSC} \sim \frac{1}{\sqrt{4\pi^2 - k_0^2}}. \]

(33)

The patterns of the antisymmetric odd Bloch BSC are shown in Fig. 2 (a), (c) and (d).

For the antisymmetric even BSC \((a, e)\) we have the following solution within the rods

\[
\Phi(r, \varphi)^{(a,e)} = \sum_{m=1}^{\infty} A_{2m}(r) \sin 2m\varphi, \quad (34)
\]

which has the nodal lines at \(\varphi = 0, \pi/2\). The numerically calculated pattern of this BSC is shown in Fig. 2(b). Thus, the antisymmetric Bloch BSC exists provided that the BSC frequency \(k_0\) does not exceed the threshold of the next diffraction channels with \(l \neq 0\). Fig. 3 shows that all symmetry protected BSCs from Fig. 2 coexist in a wide range of the radius of the rods.

![FIG. 3: BSC frequencies \(k_0 = \omega/c\) vs radius \(R\) for the BSCs shown in Fig. 2](image)

Next we consider symmetric odd BSC \((s, o)\) in the diffraction channel \(k_x = 0, l = 0\). Substituting Eqs. (30) and (28) into Eq. (25) we obtain

\[
[v^r,t]_l^{s,o} \Phi^{(s,o)} = 2 \sqrt{\frac{2}{k_y}} \sum_{m=1}^{\infty} \Phi_{2m-1}^{(s,o)}. \quad (35)
\]

In contrast to the antisymmetric wave the right-hand part of this equation can be equal to zero only accidentally as one varies the radius of the rods. The corresponding BSC function

\[
\Phi(r, \varphi)^{(s,o)} = \sum_{m=1}^{\infty} A_{2m-1}(r) \sin(2m - 1)\varphi
\]

(36)
is presented in Fig. 4. This BSC has nodal lines at \( y = 0 \). The symmetric even BSC \((s, e)\) wave function has the following form for \( r < R \)

\[
\Phi(r, \varphi)^{(s, e)} = \sum_{m=0}^{\infty} A_{2m}(r) \cos 2m \varphi.
\]  

(37)

However for given parameter \( \epsilon = 12 \) our computation revealed only the symmetric odd BSC. Thus, the symmetric

BSC both odd and even can not be classified as symmetry protected because of the necessity to adjust the right-hand part of Eq. (35) to zero.

V. THE BOUND STATES EMBEDDED INTO TWO AND THREE DIFFRACTION CHANNELS

Now we show that the BSC can exist even when more than one diffraction channel is open. That phenomenon was considered by Ndangali and Shabanov in the double arrays of dielectric rods [27]. They showed that tuning for BSC requires higher dimensionality of the parametric space. In Ref. [16] we demonstrated that the BSC can be robust relative to a few open channels owe to the symmetry of the resonator. We show in this section that the present system can also support BSC embedded into a few diffraction channels. First, we assume that only two diffraction channels \( l = 0, k_{x,0} = k_x = \pi \) and \( l = -1, k_{x,-1} = -\pi \) are open. The components of ingoing and outgoing waves (14) satisfy the following relationships

\[
v^{(r,t)}_{l=0,m} = [v^{(r,t)}_{l=-1,m}]^* = v^{(r,t)}_{l=-1,-m}.
\]  

(38)

In this case all symmetric properties established in Eqs. (28) and (29) still hold true. We start with the symmetric odd state \((s, o)\) and consider its coupling with the channels (38) to establish the following equalities

\[
W^{(r,t)}_{l=0} = W^{(r,t)}_{l=-1}, \quad W^{(r)}_{l=0} = -W^{(t)}_{l=0}.
\]  

(39)

Here, for brevity we omitted the superscript \((s, o)\). One can see that all coupling constants equal zero if \( W^{(t)}_{l=0} = 0 \) which can be fulfilled through variation of the radius of the rod. Such Bloch BSC function is shown in Fig. 5. For \( \epsilon = 12 \) we did not find the BSC of other symmetry types: \((s, e), (a, o), (a, e)\). Although that does not mean that they are impossible for different \( \epsilon \).

Next, we consider the case of three open diffraction channels \( l = 0, k_{x,0} = k_x = \pi, \) \( l = 1, k_{x,1} = \pi, \) and \( l = -1, k_{x,-1} = -\pi, \) the utmost number of open channels with an embedded bound state. Our simulations show that the only antisymmetric odd BSC \((a, o)\) can exist for which the symmetry establishes five equalities for six coupling constants

\[
W^{(r,t)}_{l=-1} = -W^{(r,t)}_{l=1}, \quad W^{(r)}_{l=-1} = W^{(t)}_{l=1}, \quad W^{(r)}_{l=0} = W^{(t)}_{l=0} = 0.
\]  

(40)

Similar to the previous case in order to cancel all couplings it is enough to fulfill \( W^{(t)}_{l=0} = 0 \). The last equality can be achieved by tuning the radius of the rods. The wave function within the rods has the following form

\[
\Phi(r, \varphi)^{(a, o)} = \sum_{m=1}^{\infty} A_{2m-1}(r) \cos(2m - 1) \varphi.
\]  

(41)
FIG. 5: (Color online) Real part of the \((s, o)\) BSC wave function embedded into the open \(l = 0, l = -1\) diffraction channels for \(R = 0.2107, k_0 = 6.87, k_x = \pi\).

The Bloch wave function is shown in Fig. 5. One can see that nodal lines at \(x = 0, \pm 1/2, \pm 1, \ldots\) are similar to the

FIG. 6: (Color online) Real part of the \((a, o)\) BSC wave function embedded into the three open diffractions channels: \(l = 0, l = \pm 1\) for \(R = 0.3193, k_0 = 9.876, k_x = 0\).

antisymmetric odd Bloch BSC embedded in the first diffraction channel shown in Fig. 2.

VI. BLOCH BSC WITH POYNTING VECTOR

Could the Bloch BSC occur at \(\pi > k_x > 0\) in the continuum of free-space modes? This question was first answered positively by Porter and Evans [8] who considered acoustic trapping in the array of rods of rectangular cross-section. Later Marinica et al [10] demonstrated the existence of the Bloch BSC at \(k_x > 0\) in two parallel dielectric gratings and Ndangali and Shabanov [27] in two parallel arrays of dielectric rods. Each array has a transmission zero for definite incident angle and frequency to form the Fabry-Perot resonator that support BSCs trapped between the "mirrors" [10]. In a single array of rods positioned on the surface of bulk 2d photonic crystal multiple BSCs with \(k_x \geq 0\) were considered by Wei et al [20]. That system can be also seen as a Fabry-Perot resonator with the photonic crystal playing the role of bottom mirror provided that the frequency is in the band gap. The periodic array of rods plays the role of top "mirror" at certain frequency and the radius of the rods. Such BSCs are one-dimensional Bloch surface states which do not leak into the radiation continuum [19]. The next important step was undertaken by Zhen et al [36] who presented the Bloch BSC in a PhC slab with one-dimensional periodicity in \(x\) where the Bloch BSC evolves with \(k_x \neq 0, k_z = 0\) into the BSC with \(k_x = 0, k_z \neq 0\) for decreasing of the slab thickness.

In the present paper we show that the electromagnetic Bloch BSC with \(\pi > k_x > 0\) occurs in a periodic array of rods of the circular cross-section as the result of zero coupling of the Bloch state with diffraction channel at specific
values of $k_x$ and $k_0$. We stress that similar to the systems described above [20, 22, 23] there is no necessity to tune the material parameters of the rods. The only condition is that the dielectric constant and the radius of the rods are large enough for only $l = 0$ diffraction channel to be open as seen from Fig. 7. These Bloch BSCs are degenerate due to $k_x \to -k_x$ symmetry. Fig. 8 shows the pattern of the BSC (real part) with the Poynting vector field that indicates a power flux along the array. The diffraction channel parameters can be changed as one changes the radius of the rods as shown in Fig. 7.

FIG. 7: $k_x$ (solid line) and $k_0$ (dash line) vs radius of rods $R$.

FIG. 8: (Color online) Real part of the propagating BSC for $R = 0.3, k_0 = 3.5577, k_x = 1.2556$. White arrows show the Poynting vector carrying power current in the Bloch BSC.

VII. EMERGENCE OF THE BSC IN SCATTERING FUNCTION

It is clear that probing the BSC by a wave injected into a diffraction channel is impossible because the BSC is orthogonal to the channel. Nevertheless one can show there is a path in the parametric space leading to the BSC point. Approaching the BSC point along this path reveals the BSC as dominant contribution in the scattering wave function (21). Assume that among all complex eigenvalues $\lambda_f$ only one tends to zero $\lambda_s \to 0$ while the others remain finite in the vicinity of the BSC point. In what follows we suppose that only one diffraction channel $l = 0$ is open.
Then we can write the singular part in the scattering wave (21) as follows

$$a_s = \frac{(y^+ v_0^{(t,r)})x_s}{\lambda_s}$$

(42)

where $x_s$ is the eigenvector which tends to the null vector. Respectively, we can present the singular parts in the reflection and transmission amplitudes (16), i.e., the scattering matrix as follows

$$r_{sl} = \frac{W_s^{(r)} W_s^{(t)}}{\lambda_s}, \quad t_{sl} = \frac{W_s^{(t)} W_s^{(t)}}{\lambda_s}.$$  

(43)

Fig. 9 shows the transmittance vs $k_x$ and $k_0$. One can see a singular point where unit transmittance touches zero transmittance which corresponds to the collapse of the Fano resonance [24]. This singular point $k_{xs}, k_{0s}$ corresponds to the Bloch BSC shown in Fig. 8. To find the path leading to the BSC like scattering wave we following Ref. [32]

In Fig. 10 we plot the singular part in the scattering state (42) $|a_s|$ and reflection coefficient $|r_{sl}|$ as a function of the angle variable $\alpha$ defined via $k_x - k_{xs} = \rho \cos \alpha$, $k_0 - k_{0s} = \rho \sin \alpha$ with $\rho = 0.1$. The angular behavior demonstrates sharp peaks. The smaller is the radius $\rho$ the sharper are the peaks associated with the BSC. Therefore it follows that there is a path leading to the BSC-like scattering function. This path is shown by white solid line in Fig. 9 which corresponds to the unit transmittance in the collapsing Fano resonance.

**VIII. SUMMARY**

In the present paper we used the approach presented in Ref. [31] for cylindrical rods. The approach is based on expansion of incident and scattered waves over the Hankel or Bessel functions and has an advantage that the number of functions could be taken rather small. The symmetry of the system waves implies the symmetry selection rules for coefficients $a_m$ in the expansion of the Bloch BSC. That results in four types for the Bloch BSC with wave vector $k_x = 0, \pi$ classified in Section IV. In turn the symmetry restrictions for $a_m$ are reflected in the space symmetry as demonstrated in Fig. 2. It is important the symmetry protected Bloch BSC exist in a wide range of the material
parameters of the rods. This symmetry property allows the existence of the Bloch BSC embedded into two and three diffraction channels although at the price of tuning the radius of the rods. Such BSCs are presented in Figs. 4-6. The next interesting class of the Bloch BSC is the BSC with nonzero Bloch wave vector shown in Fig. 8. This BSC carries a power flux along the array.

The BSC exist in a selected point in the parametric space. So it might be thought that the BSC are not important beyond this point. Following Ref. 32, we show that the scattering wave function becomes BSC-like if one approaches to the BSC point along the path where the transmission coefficient equals unit (see Figs. 9 and 10). This provides a cue to experimental observation of the BSC.

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\[ |a_s| (a) \text{ and reflection coefficient (b) vs angular variable } \alpha \text{ for encircling of the BSC point shown in Fig. 9.} \]

FIG. 10: (Color online) Behavior of $|a_s|$ (a) and reflection coefficient (b) vs angular variable $\alpha$ for encircling of the BSC point shown in Fig. 9.

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