A Note on the Inapproximability of Induced Disjoint Paths

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Abstract

We study the inapproximability of the induced disjoint paths problem on an arbitrary
\( n \)-node \( m \)-edge undirected graph, which is to connect the maximum number of the \( k \) source-
sink pairs given in the graph via induced disjoint paths. It is known that the problem is
NP-hard to approximate within \( m^{1/2-\varepsilon} \) for a general \( k \) and any \( \varepsilon > 0 \). In this paper, we prove
that the problem is NP-hard to approximate within \( n^{1-\varepsilon} \) for a general \( k \) and any \( \varepsilon > 0 \) by
giving a simple reduction from the independent set problem.

Keywords: Approximation algorithm, Inapproximability, Hardness of approximation,
Induced disjoint paths

1. Problem and Hardness of Approximation

The concept of induced disjoint paths is recently introduced to characterize some non-
interfering situation during data transmission in wireless networks [1, 2]. A set of paths in
an undirected graph (graph, for short) are called induced disjoint paths if each one of them
has no chords (i.e., is an induced path) and any two of them have neither common nodes
nor adjacent nodes. Given a graph \( G \) and a collection of \( k \) source-sink pairs in \( G \) (\( k \) as part of
the input of the problem), the induced disjoint paths problem (IDPP) is to connect the
maximum number of these source-sink pairs via induced disjoint paths. Zhang et al. [2]
have shown that for any \( \varepsilon > 0 \), it is NP-hard to approximate IDPP to within \( m^{1/2-\varepsilon} \) on an
arbitrary \( n \)-node \( m \)-edge graph, and there is a greedy algorithm with approximation ratio
\( \sqrt{m} \), matching the lower bound \( \sqrt{m} \). In this paper, we will prove that for any \( \varepsilon > 0 \), it is
NP-hard to approximate IDPP to within \( n^{1-\varepsilon} \) on an arbitrary \( n \)-node graph. Our method
is based on a simple approximation-preserving reduction from the independent set problem
to IDPP, and an inapproximability result for finding maximum independent sets in general
graphs.

IDPP is an extension of the node-disjoint paths problem (DPP), one simple version of
the disjoint paths problem that is one of the classic NP-hard combinatorial optimization
problems. In DPP, we are given a graph \( G \) and a set of \( k \) source-sink pairs in \( G \) (\( k \) as part of

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the input of the problem), and the objective is to find a largest subset of the source-sink pairs that can be simultaneously connected in an node-disjoint manner. In fact, any instance of DPP can be reduced to an instance of IDPP by subdividing every edge into two edges. Thus, IDPP is harder than DPP. As a comparison, the best known approximation guarantee for DPP on an arbitrary \( n \)-node graph is \( O(\sqrt{n}) \) \cite{3,4}. Recently Andrews et al. \cite{5} have shown that for any \( \varepsilon > 0 \), DPP is hard to approximate within \( \log^{1-\varepsilon} n \) on an arbitrary \( n \)-node graph, unless \( \text{NP} \subseteq \text{ZPTIME}(n^{\text{poly}(\log n)}) \). This might be the best known inapproximability result for DPP on general graphs. An interesting open question is whether there exists an \( \varepsilon > 0 \) so that there is no polynomial time \( O(n^\varepsilon) \)-approximation algorithm for DPP on an arbitrary \( n \)-node graph, unless \( \text{P} = \text{NP} \) \cite{6}.

2. New Result on Hardness of Approximation

This section is devoted to the proof of a new approximation hardness result for IDPP on general graphs. The main theorem is as follows.

**Theorem 1.** On an arbitrary \( n \)-node graph, there can be no polynomial time algorithm that approximates IDPP to within \( n^{1-\varepsilon} \) for any \( \varepsilon > 0 \), unless \( \text{P} = \text{NP} \).

For the purpose of clarification, we prove below two lemmas that together immediately lead to Theorem 1.

**Lemma 1.** On an arbitrary \( n \)-node graph, if there exists a polynomial time algorithm for IDPP with approximation ratio \( n^{1-\varepsilon} \) for a certain \( \varepsilon > 0 \), then there exists a polynomial time algorithm for IDPP with approximation ratio \( (\frac{n}{3})^{1-\varepsilon'} \) for some \( \varepsilon' > 0 \).

**Proof.** Assume that we have a polynomial time algorithm app-IDPP with approximation ratio \( n^{1-\varepsilon} \) for a certain \( \varepsilon > 0 \). We now describe a polynomial time algorithm and claim that the algorithm has approximation ratio \( (\frac{n}{3})^{1-\varepsilon'} \) for some \( \varepsilon' > 0 \). The algorithm uses as input an instance of IDPP, and calls a procedure to solve the instance according to whether or not \( n \) is smaller than \( 3^{1+\frac{1}{\varepsilon}} \).

**Case 1** (\( n < 3^{1+\frac{1}{\varepsilon}} \)): It calls a brute-force procedure to solve the instance. This can be done in a constant time that depends on \( \frac{1}{\varepsilon} \).

**Case 2** (\( n \geq 3^{1+\frac{1}{\varepsilon}} \)): It calls app-IDPP to solve the instance. This can be done in polynomial time according to the above assumption.

It is evident that the algorithm has a polynomial time complexity, and the solution yielded in Case 1 is optimal. Therefore, the approximation ratio of the algorithm relies on the solution yielded by app-IDPP in Case 2. Since app-IDPP is an \( n^{1-\varepsilon} \)-approximation algorithm, so is the algorithm. In order to prove that the algorithm has the claimed approximation ratio, \( (\frac{n}{3})^{1-\varepsilon'} \), it suffices to show that there exists \( \varepsilon' > 0 \) such that \( (\frac{n}{3})^{1-\varepsilon'} \geq n^{1-\varepsilon} \). For this purpose, letting \( \varepsilon' = \varepsilon^2 \), we readily have from \( n \geq 3^{1+\frac{1}{\varepsilon}} \):

\[
\begin{align*}
    n^{(1-\varepsilon)\varepsilon} &\geq 3^{(1-\varepsilon)(1+\varepsilon)}, \\
    n^{1-\varepsilon^2} &\geq 3^{1-\varepsilon^2}n^{1-\varepsilon},
\end{align*}
\]
\[(\frac{n}{3})^{1-\varepsilon'} \geq n^{1-\varepsilon}.\]

This completes the proof. \qed

The complexity of approximating IDPP can be related to that of finding maximum independent sets in graphs by a reduction from the independent set problem to IDPP in the proof of Lemma 2 below. Recall that an independent set of a graph is a set of nodes no two of which are connected by an edge, and the independent set problem is to find a maximum independent set in the graph. The independent set problem and the clique problem are complementary: a clique in \(G\) is an independent set in the complement graph of \(G\) and vice versa. It is known that on an arbitrary \(n\)-node graph, it is NP-hard to approximate the maximum clique problem to within \(n^{1-\varepsilon}\) for any \(\varepsilon > 0\) \[^7\]. This inapproximability result for the clique problem is applied equally well to the independent set problem. Now we prove the following lemma.

**Lemma 2.** On an arbitrary \(n\)-node graph, there can be no polynomial time algorithm that approximates IDPP to within \(\left(\frac{n}{3}\right)^{1-\varepsilon}\) for any \(\varepsilon > 0\), unless \(P=NP\).

**Proof.** We consider a reduction from the independent set problem to IDPP. Given an instance of the independent set problem, a graph \(G'\) with \(n'\) nodes and \(m'\) edges, we below construct an instance of IDPP, which contains a graph \(G\) and \(k\) pairs of nodes in \(G\).

- From the graph \(G'\), we form the graph \(G\) by adding \(n'\) pairs of nodes to \(G'\), one pair for each node in \(G'\), and adding \(2n'\) edges to \(G'\) to join each pair of added nodes to the corresponding node in \(G'\). The resulting graph \(G\) has \(3n'\) nodes and \(m' + 2n'\) edges.
- Letting \(k = n'\), we form \(k\) pairs of nodes in \(G\) by selecting the \(n'\) pairs of added nodes.

The above reduction can be done in polynomial time. Moreover, it is obvious that the given instance has an independent set of size \(t\) if and only if the constructed instance of IDPP has \(t\) induced disjoint paths. This implies that the reduction is an approximability-preserving one. By the above inapproximability result for the independent set problem on a general graph, unless \(P=NP\), there does not exist a polynomial time algorithm for approximating IDPP to within a factor of \(\left(\frac{n}{3}\right)^{1-\varepsilon}\) of the optimal solution since \(G\) has \(3n'\) nodes. The proof is complete. \qed

### 3. Conclusion

A new result on hardness of approximation for IDPP on general graphs has been proved in this paper. That is, on an arbitrary \(n\)-node \(m\)-edge graph, for all \(\varepsilon > 0\), approximating IDPP to within \(n^{1-\varepsilon}\) is NP-hard. The new result is distinct from the known one in the literature that states that for all \(\varepsilon > 0\), approximating IDPP to within \(m^{\frac{1}{2} - \varepsilon}\) is NP-hard. Since \(m^{1/2} < n\) always holds for any \(n\)-node \(m\)-edge graph, the new result yields a stronger lower bound on the best possible approximation guarantee for IDPP than the known result can yield, namely, \(m^{\frac{1}{2} - \varepsilon} < n^{1-\varepsilon}\) for any \(\varepsilon > 0\). On those graphs with \(m = O(n^\alpha)\) for \(\alpha < 2\) (for example, sparse graphs), the new result obviously can give such a tighter lower bound.
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References

[1] K. Kawarabayashi and Y. Kobayashi, The induced disjoint paths problem, LNCS 5035 (2008) 47-61.

[2] K. Zhang, H. Gao, and J. Li, Finding multiple induced disjoint paths in general graphs, Information Processing Letters 111 (2011) 1022-1026.

[3] C. Chekuri, S. Khanna, F.B. Shepherd, An $O(\sqrt{n})$ approximation and integrality gap for disjoint paths and unsplittable flow, Theory of Computing 2 (2006) 137-146.

[4] T. Nguyen, On the disjoint paths problem, Operations Research Letters, 35 (2007) 10-16.

[5] M. Andrews, J. Chuzhoy, V. Guruswami, S. Khanna, K. Talwar, L. Zhang, Inapproximability of edge-disjoint paths and low congestion routing on undirected graphs, Combinatorica 30 (5) (2010) 485-520.

[6] J.M. Kleinberg, Approximation algorithms for disjoint paths problems, Ph.D.Thesis, MIT, Cambridge, MA, 1996.

[7] D. Zuckerman, Linear degree extractors and the inapproximability of max clique and chromatic number, Theory of Computing 3 (2007) 103-128.