A LOG RESOLUTION FOR THE THETA DIVISOR OF A HYPERELLIPTIC CURVE

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ABSTRACT. In this paper, we prove that the theta divisor of a smooth hyperelliptic curve has a natural and explicit embedded resolution of singularities using iterated blowups of Brill-Noether subvarieties. We also show that the Brill-Noether strification of the hyperelliptic Jacobian is a Whitney strification.

INTRODUCTION

Let $C$ be a smooth projective curve of genus $g \geq 1$. Let $\text{Jac}(C)$ be the Jacobian of $C$, and let $\Theta \subseteq \text{Jac}(C)$ be the theta divisor. The purpose of this paper is to give a natural and explicit log resolution of the pair $(\text{Jac}(C), \Theta)$ when $C$ is a hyperelliptic curve.

Recall that the Brill-Noether variety $W_{g-1}^r(C)$ parametrizes line bundles $L \in \text{Pic}^{g-1}(C)$ of degree $g - 1$ with $h^0(L) \geq r + 1$. According to a theorem by Riemann, we can choose an isomorphism $\text{Jac}(C) \cong \text{Pic}^{g-1}(C)$ so that the theta divisor $\Theta$ becomes identified with $W_{g-1}(C) := W_{g-1}^0(C)$. The Abel-Jacobi map from the symmetric product $C_{g-1} := \text{Sym}^{g-1}(C)$ gives a resolution of singularities of $\Theta$, which is useful for answering many geometric questions about Jacobian varieties. However, if one wants to investigate the geometry of the embedding $\Theta \subseteq \text{Jac}(C)$, one needs instead a log resolution of the pair $(\text{Jac}(C), \Theta)$. Inspired by a global study of the vanishing cycle functor for divisors, we are lead to the question of finding an explicit log resolution in the case of hyperelliptic theta divisors. Since the log resolution is of a purely geometric nature, we leave the actual computation of vanishing cycles to another paper (see the application of this log resolution to the computation of minimal exponents and related invariants in [10, §9.1]).

When $C$ is a hyperelliptic curve of genus $g \geq 1$, we have a lot of very precise information about the chain of subvarieties

\begin{equation}
\Theta = W_{g-1}^r(C) \supseteq W_{g-1}^{r+1}(C) \supseteq \cdots \supseteq W_{g-1}^{n}(C),
\end{equation}

where $n = \left\lfloor \frac{g+1}{2} \right\rfloor$ is the maximal integer such that $W_{g-1}^n(C) \neq \emptyset$. First, the dimension of $W_{g-1}^r(C)$ is equal to $g - 1 - 2r$ and $W_{g-1}^{r+1}(C)$ is reduced (see Proposition [A.1]). Second, the singular locus of $W_{g-1}^{r+1}(C)$ is exactly $W_{g-1}^{r+1}(C)$. Third, the multiplicity of the theta divisor at a point $L \in \text{Pic}^{g-1}(C)$ is equal to $h^0(L)$ by the Riemann singularity theorem, and so $W_{g-1}^{r+1}(C)$ is exactly the set of points of multiplicity $\geq r + 1$ on $\Theta$ (see [I] Chapter IV, §4 for details).

These facts immediately suggest that one might be able to get a log resolution of the pair $(\text{Jac}(C), \Theta)$ by successively blowing up the Brill-Noether subvarieties $W_{g-1}^r(C)$ in the order from smallest to largest. This guess turns out to be correct, but it requires quite a bit of work to prove rigorously that it works.

More precisely, we use the following iterative procedure, consisting of $n$ steps. In the first step, we blow up $\text{Jac}(C)$ along the smallest subvariety $W_{g-1}^n(C)$, and denote the blowup by $\pi_1: \text{bl}_1(\text{Jac}(C)) \to \text{Jac}(C)$. In the second step, we blow up $\text{bl}_1(\text{Jac}(C))$ along the strict transform of $W_{g-1}^{n-1}(C)$, and denote the new blowup by $\pi_2: \text{bl}_2(\text{Jac}(C)) \to \text{Jac}(C)$. In the $i$-th step, we blow up $\text{bl}_{i-1}(\text{Jac}(C))$ along the strict transform of $W_{g-1}^{n-1}(C)$,
and denote the new blowup by $\pi_i: \text{bl}_i(\text{Jac}(C)) \to \text{Jac}(C)$. This process stops after the $n$-th step. The strict transforms of the exceptional divisor give us a sequence of divisors $Z_0, Z_1, \ldots, Z_{n-1}$, with $Z_i$ sitting over the locus $W_{g-1}^{n-i}(C)$ of points of multiplicity $\geq n + 1 - i$. Let $\tilde{\Theta}$ denote the strict transform of the theta divisor. With this notation, our main result is the following.

**Theorem A.** If $C$ is a smooth hyperelliptic curve, then in the sequence of blowups described above, $\pi_n: \text{bl}_n(\text{Jac}(C)) \to \text{Jac}(C)$ is a log resolution of $(\text{Jac}(C), \Theta)$, where

$$\pi_n^*(\Theta) = \tilde{\Theta} + \sum_{i=0}^{n-1}(n + 1 - i)Z_i$$

is a divisor with simple normal crossing support. Moreover, at the $i$-th stage of the construction, the strict transform of $W_{g-1}^{n-i}(C)$ becomes smooth, and so each blowup in the sequence is a blowup along a smooth center.

We can also describe the generic structure of the exceptional divisors.

**Corollary B.** For $r = 1, \ldots, n$, every fiber of the projection

$$Z_{n-r} \setminus (Z_0 \cup \cdots \cup Z_{n-r-1} \cup Z_{n-r+1} \cup \cdots \cup Z_{n-1} \cup \tilde{\Theta}) \to W_g^r(C) \setminus W_g^{r+1}(C)$$

is isomorphic to the complement of a hypersurface of degree $r + 1$ in $\mathbb{P}^{2r}$; the hypersurface is the $(r-1)^{th}$ secant variety of a rational normal curve of degree $2r$ in $\mathbb{P}^{2r}$.

There are a few other examples in the literature where this simple-minded procedure of successive blowups along singular loci produces a log resolution:

1. Let $X$ be the affine space of $n$-by-$n$ matrices and let $D$ be the hypersurface defined by the vanishing of the determinant. Let $D_1 \subseteq D$ be the set of matrices of rank $\leq n - i$. According to [1, p. 69], one has $(D_i)_{\text{sing}} = D_{i+1}$, and $D_i$ is exactly the set of points of multiplicity $\geq i$ on $D$. It is proved in [8, Chapter 4] and in [11] (using complete collineations) that one can construct a log resolution of the pair $(X, D)$ by successively blowing up $D_n, D_{n-1}, \ldots, D_2$.

2. Let $X = \mathbb{P}H^0(C, M)$ and let $D = \text{Sec}^n(C)$ be the $n$-th secant variety of a smooth projective curve $C$, embedded by a line bundle $M$ with $h^0(M) = 2n + 3$ that separates $2n + 2$ points. Setting $D_i = \text{Sec}^{n+i-1}(C)$, Bertram [2, Page 440] proved that $(D_i)_{\text{sing}} = D_{i+1}$ and that $D_i$ is again the set of points of multiplicity $\geq i$ on $D$. He also showed [2, Corollary 2.4] that successively blowing up $D_n, D_{n-1}, \ldots, D_2$ produces a log resolution of the pair $(X, D)$.

Another common feature of these examples is that the divisor in question is a determinantal variety: this is easy to see for the space of matrices or for the secant variety of a rational normal curve, and is also true for the theta divisor of a hyperelliptic curve (which is the determinantal variety associated to a morphism of vector bundles over $\text{Pic}^{g-1}(C)$).

The subvarieties in (1) induces a stratification

$$\text{Jac}(C) = (\text{Jac}(C) - \Theta) \sqcup \bigcup_{0 \leq r \leq n} (W_{g-1}^r(C) - W_{g-1}^{r+1}(C)),$$

which is called the Brill-Noether stratification. For the computation of global nearby and vanishing cycles along $\Theta$, it is useful to have the following

**Proposition C.** If $C$ is a smooth hyperelliptic curve, then the Brill-Noether stratification of $\text{Jac}(C)$ defined above is a Whitney stratification.
Ideas of the proof. Let us discuss the method we use to prove Theorem A. Our main tool is Bertram’s blowup construction for a chain of maps \(\mathbb{P}^1\) (for a detailed and complete review of this construction, see §1).

One inconvenient point in the process above is that the Brill-Noether varieties \(W^r_{g-1}(C)\) are not smooth, which makes it hard to keep track of conormal bundles and exceptional divisors in the various blowups. Fortunately, on a hyperelliptic curve, each \(W^r_{g-1}(C)\) has a natural resolution of singularities by \(C_{g-1-2r}\), the \((g-1-2r)\)-th symmetric product of the curve, viewed as the space of effective divisors of degree \(g-1-2r\) on \(C\). Let \(g_1\) be the line bundle corresponding to the hyperelliptic map \(h: C \to \mathbb{P}^1\). For \(0 \leq r \leq n\), the resolution of singularities is the Abel-Jacobi mapping

\[
\delta_{n-r}: C_{g-1-2r} \to W^r_{g-1}(C), \quad D \mapsto r g_1^1 \otimes O_C(D),
\]

where \(n = \lfloor \frac{g-1}{2} \rfloor\). Since it is easier to blow up smooth varieties, we therefore modify the construction from above, and instead of the subvarieties \(W^r_{g-1}(C)\), we work with the chain of maps \(\{\delta_i\}_{i=0}^n\). The advantage is that we do not need to analyze the singularities of the proper transforms of \(W^r_{g-1}(C)\) and how they intersect with exceptional divisors; instead, we transform the problem into checking that certain maps are embeddings (see Lemma 1.20), which eventually reduces to the calculation of certain conormal bundles. The projectivized conormal bundles that show up as exceptional divisors are closely related to secant bundles over symmetric products of \(\mathbb{P}^1\); for that reason, Bertram’s result about these secant bundles is another important tool that we use.

Outline of the paper. In §1 we recall Bertram’s blowup construction in detail. In §2 we set up notations for the Abel-Jacobi maps and reduce the proof of Theorem A to two propositions (Proposition 2.1 and Proposition 2.2), which deal with the properties of two specific chains of maps between symmetric products and Jacobians. In §3 we review the construction of secant bundles and describe Bertram’s results. In §4 we study some basic properties of Abel-Jacobi maps and the addition maps among symmetric products. In §5-§6 we prove Proposition 2.2 and Proposition 2.1 and thereby complete the proof of Theorem A for hyperelliptic curves of odd genus. The proof of Corollary B can be found at the end of §5. Finally, in §7 we outline a proof for hyperelliptic curves of even genus, which goes along the same line but requires a few changes in the notation. In §8 we prove Proposition C, which is a technical result needed for the computation of vanishing cycles along theta divisors. In §9 we propose some questions in the direction of this paper.

Notation.

1. If \(V\) is a vector space, \(\mathbb{P}(V)\) stands for the projective space of one-dimensional quotients of \(V\). We use the same notation for vector bundles.

2. Let \(f: X \to Y\) be a morphism between smooth projective varieties. Let \(Y_1 \subseteq Y\) be a subvariety. We use the notation

\[
f^{-1}(Y_1) := Y_1 \times_Y X,
\]

exclusively for the scheme-theoretic preimage, which is the fiber product of the two morphisms \(X \to Y\) and \(Y_1 \to Y\).

3. Let \(f: X \to Y\) be a morphism between smooth varieties. We denote by

\[
df: f^*T^*_Y \to T^*_X
\]

the induced morphism between cotangent bundles, and by

\[
N^*_f = \text{Ker}(df: f^*T^*_Y \to T^*_X)
\]
the conormal bundle of the morphism. In the case of a closed embedding $X \subseteq Y$, we also use the notation $N^*_X|_Y$.

(4) We use the hooked arrow $\hookrightarrow$ for embeddings of algebraic varieties.

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1. Bertram’s blowup construction

In this section, we review Bertram’s construction from [2, §2] and introduce the notion of NCD chains. We consider sequences of blowups whose centers are determined by a chain of morphisms; the main result of this section is an inductive criterion for checking that such a sequence of blowups is a log resolution (see Lemma 1.20).

1.1. Chains and maps of chains. Let $X$ be a projective variety, not necessarily smooth.

Definition 1.1. A proper chain is a sequence of morphisms $\{f_i : X_i \rightarrow X\}_{i=0}^n$ from projective varieties $X_i$ with the property that for each $0 \leq i < j \leq n$, there exists a commutative diagram

$$
\begin{array}{ccc}
X_{i,j} & \xrightarrow{g_{i,j}} & X_i \\
\downarrow{h_{i,j}} & & \downarrow{f_i} \\
X_j & \xrightarrow{f_j} & X \\
\end{array}
$$

so that $g_{i,j}$ is surjective and there is a proper inclusion $f_i(X_i) \subseteq f_j(X_j)$.

Remark 1.2. Note that in Definition 1.1 it is sufficient to take $X_{i,j}$ to be the fiber product of $f_i$ and $f_j$. However, in practice $X_{i,j}$ is usually not the fiber product. To explain this point, let us borrow notations from §2 and look at the diagram from (9)

$$
\begin{array}{ccc}
C_{2i} \times P^{j-i} & \xrightarrow{p_i} & C_{2i} \\
\downarrow{\gamma_{i,j}} & & \downarrow{\delta_i} \\
C_{2j} & \xrightarrow{\delta_j} & \text{Jac}(C), \\
\end{array}
$$

which is used to show the chain of Abel-Jacobi maps $\{\delta_i\}$ is a proper chain. But the diagram above is not Cartesian (only Cartesian over suitable open subsets, see Lemma 4.2(a)). On the other hand, this diagram becomes Cartesian after sufficiently many blow-ups: in Proposition 2.1 we show that the $i$-th blow-up of the diagram above looks like

$$
\begin{array}{ccc}
\text{bl}_i(C_{2i}) \times P^{j-i} & \xrightarrow{p_i} & \text{bl}_i(C_{2i}) \\
\downarrow{\text{bl}_i(\gamma_{i,j})} & & \downarrow{\text{bl}_i(\delta_i)} \\
\text{bl}_i(C_{2j}) & \xrightarrow{\text{bl}_i(\delta_j)} & \text{bl}_i(\text{Jac}(C)) \\
\end{array}
$$

where is Cartesian. In other words, $X_{i,j}$ becomes a fiber product after sufficiently many blow-ups.

We now define the associated sequence of blowups for a proper chain, which exists under the assumption that certain morphisms are closed embeddings.
Definition 1.3. Let \( \{f_j : X_j \to X\} \) be a proper chain. Assume \( f_0 \) is an embedding, we identify \( X_0 \) with its image and define:

\[
\begin{align*}
\text{bl}_1(X) & := \text{the blowup of } X \text{ along } X_0. \\
\text{bl}_1(X_j) & := \text{the blowup of } X_j \text{ along } f_j^{-1}(X_0). \\
\text{bl}_1(f_j) & := \text{the unique lift of } f_j \text{ to a map } \text{bl}_1(X_j) \to \text{bl}_1(X),
\end{align*}
\]

where the last morphism exists by the universal property of blowing up. Assume for some \( 1 \leq i \leq n-1 \) that \( \text{bl}_i(X) \), \( \text{bl}_i(X_j) \) and \( \text{bl}_i(f_j) \) have already been defined for all \( j \geq i \), and that the map

\[
\text{bl}_i(f_i) : \text{bl}_i(X_i) \to \text{bl}_i(X)
\]
is an embedding. Under these assumptions, we can identify \( \text{bl}_i(X_i) \) with its image and set

\[
\begin{align*}
\text{bl}_{i+1}(X) & := \text{the blowup of } \text{bl}_i(X) \text{ along } \text{bl}_i(X_i). \\
\text{bl}_{i+1}(X_j) & := \text{the blowup of } \text{bl}_i(X_j) \text{ along } \text{bl}_i(f_j)^{-1}(\text{bl}_i(X_i)). \\
\text{bl}_{i+1}(f_j) & := \text{the unique lift of } \text{bl}_i(f_j) \text{ to a map } \text{bl}_{i+1}(X_j) \to \text{bl}_{i+1}(X).
\end{align*}
\]

Notation 1.4. To have a uniform notation, we set

\[
\begin{align*}
\text{bl}_0(f_i) & := f_i, \\
\text{bl}_0(X_i) & := X_i, \\
\text{bl}_0(X) & := X.
\end{align*}
\]

This notation is going to be useful in the inductive proofs later.

Definition 1.5. Let \( \{f_i : X_i \to X\}_{i=0}^{n} \) be a proper chain. If \( \text{bl}_{n+1}(X) \) is defined in Definition 1.3, we say that \( \{f_i\} \) is a chain of centers. Concretely, this amounts to the (recursive) condition that the \( n+1 \) morphisms \( f_0, \text{bl}_1(f_1), \ldots, \text{bl}_n(f_n) \) should all be closed embeddings. If \( X, X_0, \text{bl}_1(X_1), \ldots, \text{bl}_n(X_n) \) are all smooth, then we say that \( \{f_i\} \) is a chain of smooth centers.

We formulate an additional definition which ensures that the exceptional divisors in the final blowup \( \text{bl}_{n+1}(X) \) form a simple normal crossing divisor. We are going to refer to these conditions as the NCD conditions.

Definition 1.6. Let \( \{f_i : X_i \to X\}_{i=0}^{n} \) be a chain of smooth centers. We define divisors

\[
E_{i,j} \subseteq \text{bl}_j(X), \quad \text{for } j \geq i + 1,
\]
as follows. First \( E_{i,i+1} \subseteq \text{bl}_{i+1}(X) \) is the exceptional divisor for the blowing-up of \( \text{bl}_i(X) \) along \( \text{bl}_i(X_i) \). For each \( j \geq i+1 \), let \( E_{i,j} \subseteq \text{bl}_j(X) \) be the scheme-theoretic inverse image of \( E_{i,i+1} \) under the later blowups, as in the following Cartesian diagrams:

\[
\begin{array}{ccc}
\text{bl}_j(X) & \longrightarrow & \text{bl}_{i+1}(X) \\
\uparrow & & \uparrow \\
E_{i,j} & \longrightarrow & E_{i,i+1} \\
\text{bl}_j(X_i) & \longrightarrow & \text{bl}_i(X_i)
\end{array}
\]

Moreover, we set

\[
E_i := E_{i,n+1} \subseteq \text{bl}_{n+1}(X)
\]
for the divisors in the final blowup space and we say that \( \{E_i\}_{i=0}^{n} \) is the set of exceptional divisors of the chain \( \{f_i\}_{i=0}^{n} \).

We say that a chain of smooth centers \( \{f_i\}_{i=0}^{n} \) is an NCD chain if, for each \( j \leq n \), the intersection

\[
\text{bl}_j(X_j) \cap E_{i,j} \subseteq \text{bl}_j(X)
\]
is transverse for all $i < j$ and the divisor $E_{0,j} + \cdots + E_{j-1,j} \subseteq \mathrm{bl}_j(X)$ has simple normal crossings. It follows that for any NCD chain, the divisor

$$E_0 + E_1 + \cdots + E_n \subseteq \mathrm{bl}_{n+1}(X)$$

is a simple normal crossing divisor.

**Remark 1.7.** Let $\{f_i : X_i \to X\}_{i=0}^n$ be a chain of smooth centers. Then there is a natural embedding of $X - f_n(X_n)$ into $\mathrm{bl}_{n+1}(X)$ such that

$$\mathrm{bl}_{n+1}(X) - \bigcup_{0 \leq i \leq n} E_i = X - f_n(X_n).$$

This follows from the construction of the iterated blowups, as the set on the right is exactly the complement of all the centers in the sequence of blowups.

In Proposition 2.2, the map $X_{j,k} \xrightarrow{h_{j,k}} X_k$ for $j < k$ in Definition 1.4 is going to take the form

$$h_{j,k} : X_{j,k} = X_j \times S_{j,k} \to X_k$$

where $S_{j,k}$ is a smooth variety so that $\{h_{j,k}\}_{j=0}^{n+1}$ is also a proper chain. For induction purposes, one usually fixes $j,k$ and produces another proper chain

$$\{h_{i,j} \times \mathrm{id} : (X_i \times S_{ij}) \times S_{jk} \to X_j \times S_{jk}\}_{i=0}^{n-1}.$$ 

Therefore, the following notation becomes convenient.

**Notation 1.8.** Let $S$ be a smooth variety and let $\{f_i : X_i \to X\}_{i=0}^n$ be a proper chain. It induces a new proper chain $\{f_i \times \mathrm{id} : X_i \times S \to X \times S\}_{i=0}^n$, i.e. the collection of maps that are $f_i$ on the first factor and the identity on $S$.

**Lemma 1.9.** Let $\{f_i : X_i \to X\}_{i=0}^n$ be an NCD chain. For each $0 \leq i < j \leq n + 1$, let $E_{i,j} \subseteq \mathrm{bl}_j(X)$ and $F_{i,j} \subseteq \mathrm{bl}_j(X \times S)$ be the exceptional divisors of the chains $\{f_i\}_{i=0}^n$ and $\{f_i \times \mathrm{id}\}_{i=0}^n$ as in Definition 1.4. Then there are natural isomorphisms

$$\mathrm{bl}_i(X \times S) = \mathrm{bl}_i(X) \times S, \quad \mathrm{bl}_i(X_j \times S) = \mathrm{bl}_i(X_j) \times S,$$

$$\mathrm{bl}_i(f_j \times \mathrm{id}) = \mathrm{bl}_i(f_j) \times \mathrm{id}, \quad F_{i,j} = E_{i,j} \times S.$$ 

Moreover, $\{f_i \times \mathrm{id}\}_{i=0}^n$ is again an NCD chain.

**Proof.** This follows from the fact that blowup maps commute with taking Cartesian product with the smooth variety $S$, and that transversality is also preserved under product with $S$. $\square$

To prove that a chain of centers is an NCD chain, it is useful to have the following notion (see [2, Page 442]).

**Definition 1.10.** Suppose that $\{f_i : X_i \to X\}_{i=0}^n$ and $\{g_i : Y_i \to Y\}_{i=0}^n$ are two chains of centers. We say that a map $\phi : X \to Y$ is a map of chains of centers if it satisfies the following conditions.

- First $\phi^{-1}(Y_0) = X_0$, so $\mathrm{bl}_i(X) \to \mathrm{bl}_i(Y)$ is defined, and
- inductively, assume that for some $0 < i \leq n$ the map $\mathrm{bl}_i(\phi) : \mathrm{bl}_i(Y) \to \mathrm{bl}_i(X)$ exists, then one has

$$\mathrm{bl}_i(\phi)^{-1}(\mathrm{bl}_i(Y_i)) = \mathrm{bl}_i(X_i).$$

Consequently, one can define

$$\mathrm{bl}_{i+1}(\phi) : \mathrm{bl}_{i+1}(X) \to \mathrm{bl}_{i+1}(Y)$$

to be the unique lift of $\mathrm{bl}_i(\phi)$, which exists by the universal property of blowing up.
We say that the map $\phi$ is an injective map of chains of centers if in addition, $\text{bl}_{n+1}(\phi)$ is injective.

The following lemma gives an inductive criterion for a chain to be NCD.

**Lemma 1.11.** Let $\{\phi_j : X_j \to X\}_{j=0}^n$ be a proper chain, with diagrams (as in Definition [LJ]) for $0 \leq i < j \leq n$:

\[
\begin{array}{ccc}
X_{i,j} & \longrightarrow & X_i \\
\downarrow f_{i,j} & & \downarrow \phi_i \\
X_j & \longrightarrow & X \\
\phi_j & & \\
\end{array}
\]

Suppose that $X$, $X_i$, $X_{i,j}$ are all smooth projective varieties and assume the following conditions are satisfied:

1. For each $j \leq n$, $\{f_{i,j} : X_{i,j} \to X_j\}_{i=0}^{j-1}$ is an NCD chain.
2. $\{\phi_j\}_{j=0}^n$ is a chain of smooth centers.
3. Each $\phi_i$ is a map of chains of centers.

Then the chain $\{\phi_j\}_{j=0}^n$ is an NCD chain.

**Proof.** The two assumptions – that $\phi_j$ is a map of chains of centers and that $\{\phi_j\}_{j=0}^n$ is a chain of smooth centers – imply that $\phi_j$ is an injective map of chains of centers. By Remark [1.7], the NCD conditions on $\{f_{i,j}\}_{i=0}^{j-1}$ guarantee that the complement of $X_j - f_{j-1,j}(X_{j-1,j})$ in $\text{bl}_j(X_j)$ is a simple normal crossing divisor with $j$ components. Therefore we can apply [2, Lemma 2.1]. Moreover, it follows from the proof of [2, Lemma 2.1] that for each $j \leq n$, $\text{bl}_j(X_j)$ intersects each exceptional divisor in $\text{bl}_j(X)$ transversely, and that the new exceptional divisors in $\text{bl}_{j+1}(X)$ also intersect transversely. This implies that $\{\phi_j\}_{j=0}^n$ is an NCD chain. \[\square\]

In the rest of this section, we discuss how one can replace the conditions (2) and (3) in Lemma [1.11] by certain conditions on exceptional divisors and their complements, in the presence of transversality condition from $\{f_{i,j}\}$ being NCD chains, see Lemma [1.20]. These conditions are embedded in the proof of [2, Proposition 2.2]. By extracting them, we hope it will make our proof of Theorem [A] more transparent.

### 1.2. Criteria for maps of chains of centers.

For a map $\phi : X \to Y$ between chains $\{f_i : X_i \to X\}$ and $\{g_i : Y_i \to Y\}$, to show it is a map of chains of centers, one needs to check $\text{bl}_i(\phi)^{-1}(\text{bl}_i(Y_i)) = \text{bl}_i(X_i)$ for each $i$. In the presence of certain transversality conditions, we can apply the following lemma. First let us fix some notation.

**Notation 1.12.** For a closed embedding of smooth varieties $A \subset B$, we use $N_{A|B}^*$ to denote the conormal bundle of $A$ inside $B$, i.e.

$N_{A|B}^* = \text{Ker}\{d\iota : \iota^*T_B \to T_A^*\}$.

**Notation 1.13.** For a sequence of divisors $\{E_i\}_{i=0}^n$, we denote by

$E_i^O := E_i - (E_0 \cup E_1 \cup \cdots \cup E_{i-1})$.

Note that we are removing only the intersections with the previous divisors.

**Lemma 1.14.** Let $\phi : X \to Y$ be a morphism between smooth projective varieties. Let $\{E_i\}_{i=0}^{j-1}$ and $\{F_i\}_{i=0}^{j-1}$ be two sequences of smooth divisors in $X$ and $Y$ such that

$\phi^{-1}(F_i) = E_i, \quad \forall 0 \leq i \leq j - 1$.

Let $X_1 \subset X$ and $Y_1 \subset Y$ be two smooth subvarieties. Assume that for each $0 \leq i \leq j - 1$,

(a) the intersections $X_1 \cap E_i$ and $Y_1 \cap F_i$ are transverse,
Lemma shows that this is equivalent to the surjectivity of $\phi$. Then $\phi^{-1}(Y_1) = X_1$.

Proof. From the assumption, we know that the set-theoretic preimage of $Y_1$ under $\phi$ is $X_1$. In order to show that this also holds scheme-theoretically, we need to know that $\phi^*\mathcal{I}_{Y_1} \to \mathcal{I}_{X_1}$ is surjective. Since $N_{X_1|X}^* = \mathcal{I}_{X_1}/\mathcal{I}_{X_1}^2$ and $N_{Y_1|Y}^* = \mathcal{I}_{Y_1}/\mathcal{I}_{Y_1}^2$, Nakayama’s lemma shows that this is equivalent to the surjectivity of $d\phi : \phi^*N_{Y_1|Y}^* \to N_{X_1|X}^*$.

This can be checked over $X_1 - \cup_i E_i = X_1 - \cup_i E_i^o$ and $X_1 \cap E_i^o$ separately. Condition (c) implies that $d\phi$ is surjective over $X_1 - \cup_i E_i$. On the other hand, using condition (a), we have the following commutative diagram

$$
\begin{array}{ccc}
\phi^*N_{Y_1|Y}\big|_{Y_1\cap F_i^o} & \xrightarrow{d\phi|_{Y_1\cap F_i^o}} & N_{X_1|X}\big|_{X_1\cap E_i^o} \\
\cong & & \cong \\
\phi^*N_{Y_1\cap F_i^o|F_i^o} & \xrightarrow{d(\phi|_{F_i^o})} & N_{X_1\cap E_i^o|E_i^o}.
\end{array}
$$

The bottom map is induced by $\phi|_{E_i^o} : E_i^o \to F_i^o$. The vertical maps are isomorphisms because of the transversality condition (which implies that the intersections $X_1 \cap E_i^o$ and $Y_1 \cap F_i^o$ are transverse). Therefore condition (b) implies that $d\phi$ is also surjective over $X_1 \cap E_i^o$ for each $i$. □

1.3. Criterion for a proper chain to have smooth centers. Let $\{f_i : X_i \to X\}$ be a proper chain where $X_i$ and $X$ are all smooth. Assuming that the map $\bl_i(f_i) : \bl_i(X_i) \to \bl_i(X)$ exists, we discuss how to check it is an embedding under some transversality conditions. Notation [4,12] is used throughout.

Lemma 1.15. Let $f : X \to Y$ be a morphism between two smooth projective varieties. Let $F \subseteq Y$ be a smooth divisor. If the (scheme-theoretic) preimage $E := f^{-1}(F)$ is also a smooth divisor, then

$$f^*N^*_{F|Y} = N^*_{E|X},$$

where $N^*_{E|X}$ is the conormal bundle of $E$ in $X$, as in Notation [4,12].

Proof. Since $F$ is a smooth divisor, we have $N_{F|Y} = \mathcal{O}_Y(-F)|_F$ by the conormal sequence. The same holds for the scheme-theoretic preimage $E = f^{-1}(F)$. Therefore

$$f^*N^*_{F|Y} = f^*\mathcal{O}_Y(-F)|_F = \mathcal{O}_X(-E)|_E = N^*_{E|X}.$$ □

Lemma 1.16. Let $f : X \to Y$ be a morphism between two smooth projective varieties. Let $Z \subseteq Y$ be a smooth subvariety such that $W := f^{-1}(Z)$ is smooth and properly contained in $X$, and denote by $\tilde{f} : \tilde{X} \to Y$ the induced morphism between the two blowups $\tilde{Y} = \bl_Z Y$ and $\tilde{X} = \bl_W X$, as in the following diagram

$$
\begin{array}{ccccc}
E & \xleftarrow{f} & \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} & \xleftarrow{f} & F \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
W & \xleftarrow{f} & X & \xrightarrow{f} & Y & \xleftarrow{f} & Z
\end{array}
$$

where $F$ and $E$ are the exceptional divisors. Then $\tilde{f}^{-1}(F) = E$ and

$$\tilde{f}^*N^*_{F|\tilde{Y}} = N^*_{E|\tilde{X}}.$$ (3)
Proof. It is proved in [2] Page 442, Fact A] that \( \tilde{f}^{-1}(F) = E \). By assumption, \( Z \) and \( W \) are smooth, therefore \( E \) and \( F \) are smooth divisors and we can apply Lemma 1.15 to get (3). □

Lemma 1.17. Let us keep the assumptions from Lemma 1.16 and further assume that

(a) the map \( \tilde{f}\big|_E : E \to F \) is an embedding,

(b) the map \( f : X - W \to Y - Z \) is an embedding.

Then \( \tilde{f} \) is an embedding.

Proof. By condition (b), \( \tilde{f} \) is an embedding away from \( E \). Condition (a) implies that \( \tilde{f} \) is set-theoretically injective over \( E \). Therefore, it suffices to show that

\[
d\tilde{f} : \tilde{f}^*T_Y^* \to T_X^*
\]

is surjective over \( E \). Consider the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{f}^*N_{F|Y}^* & \longrightarrow & \tilde{f}^*T_Y^*|_E & \longrightarrow & \tilde{f}^*T_F^* & \longrightarrow & 0 \\
& & \downarrow{\cong} & & \downarrow{df} & & \downarrow{d(f|_E)} & & \\
0 & \longrightarrow & N_{E|X}^* & \longrightarrow & T_X^*|_E & \longrightarrow & T_E^* & \longrightarrow & 0 \\
\end{array}
\]

By (3), the arrow on the left is an isomorphism. Since \( \tilde{f}\big|_E \) is an embedding, the arrow on the right is surjective. By the snake lemma, we conclude that the arrow in the middle is also surjective, and conclude that \( \tilde{f} \) is an embedding. □

Consider a proper chain \( \{f_j : X_j \to X\} \) and assume the map

\[
\text{bl}_j(f_j) : \text{bl}_j(X_j) \to \text{bl}_j(X)
\]

exists for some \( j \). To prove that \( \text{bl}_j(f_j) \) is an embedding, as in Lemma 1.11 one usually first shows that \( \{f_{i,j} : X_{i,j} \to X_j\}_{i=0}^{j-1} \) is a NCD chain and \( f_j \) is a map of chains of centers. Therefore we need a generalization of Lemma 1.17 to such situation.

Let \( \{f_i : X_i \to X\}_{i=0}^{j-1} \) and \( \{g_i : Y_i \to Y\}_{i=0}^{j-1} \) be two chains of centers, and let \( \phi : X \to Y \) be a map of chains of centers, meaning that the map \( \text{bl}_i(\phi) : \text{bl}_i(X) \to \text{bl}_i(Y) \) exists for each \( i \leq j \) and

\[
\text{bl}_i(\phi)^{-1}(\text{bl}_i(X_i)) = \text{bl}_i(Y_i), \quad \forall 0 \leq i \leq j - 1.
\]

We want conditions to make sure the final map

\[
\text{bl}_j(\phi) : \text{bl}_j(X) \to \text{bl}_j(Y)
\]

is an embedding. For each \( i \), consider the following diagram:

\[
\begin{array}{cccccc}
E_i & \hookrightarrow & \text{bl}_j(X) & \xrightarrow{\text{bl}_i(\phi)} & \text{bl}_j(Y) & \hookleftarrow & F_i \\
& & \downarrow & & \downarrow & & \\
& & \text{bl}_i(X_i) & \xrightarrow{\text{bl}_i(\phi)} & \text{bl}_i(Y_i) & & \\
\end{array}
\]

Here \( E_i \) is the exceptional divisor over \( \text{bl}_i(X_i) \) and \( F_i \) is defined similarly. By Lemma 1.16 and (6), one can deduce that

\[
\text{bl}_j(\phi)^{-1}(F_i) = E_i, \quad \forall 0 \leq i \leq j - 1.
\]

Lemma 1.18. Using the notation above. Further assume that \( X, Y \) are smooth projective varieties and

(a) the two chains \( \{f_i\}_{i=0}^{j-1}, \{g_i\}_{i=0}^{j-1} \) satisfy the NCD conditions,
(b) for every $0 \leq i \leq j - 1$, the induced map 
\[ \text{bl}_j(\phi) : E_i^\circ \to F_i^\circ \]

is an embedding.
(c) the map $\phi : X - f_{j-1}(X_{j-1}) \to Y$ is an embedding.

Then $\text{bl}_j(\phi)$ is an embedding.

Proof. The transversality condition (a) guarantees that $F_i$ and $E_i$ are smooth divisors in smooth projective varieties. Therefore, by (6) and Lemma 1.15, we have 
\[ \text{bl}_j(\phi)^*N_{F_i|\text{bl}_j(Y)}^* = N_{E_i|\text{bl}_j(X)}^* . \]

By restriction to $E_i^\circ$ and $F_i^\circ$, we get
\[ (7) \quad \text{bl}_j(\phi)^*N_{F_i^\circ|\text{bl}_j(Y)}^* = N_{E_i^\circ|\text{bl}_j(X)}^*. \]

Consider the following commutative diagram:
\[
\begin{array}{ccc}
0 & \rightarrow & \text{bl}_j(\phi)^*N_{F_i^\circ|\text{bl}_j(Y)}^* \\
\downarrow_{\cong} & & \downarrow_{d\text{bl}_j(\phi)} \\
0 & \rightarrow & T_{E_i^\circ|\text{bl}_j(X)}^* \\
\end{array}
\]

Using (7), the condition (b) and the snake lemma, we see that
\[ d\text{bl}_j(\phi) : \text{bl}_j(\phi)^*T_{E_i^\circ|\text{bl}_j(Y)}^* \rightarrow T_{E_i^\circ|\text{bl}_j(X)}^* \]
is surjective over each $E_i^\circ$. By Remark 1.19, the set $X - f_n(X_n)$ naturally embeds into $\text{bl}_j(X)$ with complement $\cup_i E_i$, and condition (c) says that $d\text{bl}_j(\phi)$ is surjective away from $\cup_i E_i$. Since $\bigcup_i E_i = \bigcup_i E_i^\circ$, we conclude that $\text{bl}_j(\phi)$ is an embedding. \hfill \square

Remark 1.19. In condition (b), we do not ask for $\text{bl}_j(\phi) : E_i \to F_i$ to be an embedding; the reason is that it is much easier to check this condition on open subsets of exceptional divisors in practice.

1.4. Criterion for a proper chain to be NCD. Putting everything together, we can prove the following inductive lemma for verifying the conditions (2) and (3) in Lemma 1.11. We will see that the NCD conditions play an important role.

Lemma 1.20. Let $\{\phi_j : X_j \to X\}_{j=0}^n$ be a proper chain with diagrams ($j < k$)

\[
\begin{array}{ccc}
X_{j,k} & \rightarrow & X_j \\
\downarrow_{f_{j,k}} & & \downarrow_{\phi_j} \\
X_k & \rightarrow & X \\
\phi_k
\end{array}
\]
as in Definition 1.14. Let $k \leq n$ be an integer, and assume the following conditions are satisfied.

(I) The chain $\{f_{j,k} : X_{j,k} \to X_k\}_{j=0}^{k-1}$ is NCD.

(II) The map $\phi_k : X_k - f_{k-1,k}(X_{k-1,k}) \to X$ is an embedding.

(III) The chain $\{\phi_j\}_{j=0}^{k-1}$ is NCD, so that the following map exist
\[ \text{bl}_i(\phi_k) : \text{bl}_i(X_k) \to \text{bl}_i(X), \quad \forall i \leq k, \]
and $\text{bl}_i(\phi_i) : \text{bl}_i(X_i) \to \text{bl}_i(X)$ are embeddings for $i < k$. 

For every $j \leq k$, the following holds. Assume one has
\[ \text{bl}_i(\phi_k)^{-1}(\text{bl}_i(X_j)) = \text{bl}_i(X_{i,k}), \quad \forall i \leq j - 1, \]
so that $\text{bl}_i(X_k)$ coincides with the blowup space associated to $\{f_{i,k} : X_{i,k} \rightarrow X_k\}_{i=0}^{j-1}$ and the following diagram exists for all $0 \leq i \leq j - 1$:
\[
\begin{array}{ccc}
E_i & \longrightarrow & \text{bl}_j(X_k) \\
\downarrow & & \downarrow \\
\text{bl}_i(X_{i,k}) & \longrightarrow & \text{bl}_i(X_k) \\
\end{array}
\]
\[ \text{bl}_i(X_{i,k}) \longrightarrow \text{bl}_i(X_k) \longrightarrow F_i \rightarrow \text{bl}_i(X_i) \]

where $E_i,F_i$ are exceptional divisors over $\text{bl}_i(X_{i,k})$ and $\text{bl}_i(X_i)$, respectively. Then the following holds:
\[
(*) \quad \text{bl}_i(\phi_k)^{-1}(\text{bl}_j(X_j) - \bigcup_{i<j}E_i) = \text{bl}_j(X_{j,k}) - \bigcup_{i<j}E_i, \quad \text{if } j < k,
\]
\[(**) \quad \text{bl}_j(\phi_k)^{-1}(\text{bl}_j(X_j) \cap F_i^0) = \text{bl}_j(X_{j,k}) \cap E_i^0 \quad \forall i < j, \quad \text{if } j < k,
\]

where $E_i^0 = E_i - \bigcup_{h<i}E_h$ as in Notation I.13.

Then for each $j \leq k - 1$, we have
\[ \text{bl}_j(\phi_k)^{-1}(\text{bl}_j(X_j)) = \text{bl}_j(X_{j,k}). \]

In particular, $\phi_k$ is a map of chains of centers. Assume the above diagram exists for $j = k$, then
\[ (***) \quad \text{bl}_k(\phi_k) : E_i^0 \rightarrow F_i^0 \text{ is an embedding whenever } i < k. \]

Moreover, the chain $\{\phi_j\}_{j=0}^k$ is a chain of smooth centers.

**Proof.** By definition, it suffices to prove the following statements
(i) We have $\text{bl}_j(\phi_k)^{-1}(\text{bl}_j(X_j)) = \text{bl}_j(X_{j,k})$ for all $j < k$.
(ii) $\text{bl}_k(X_k)$ is smooth.
(iii) We have an embedding $\text{bl}_k(\phi_k) : \text{bl}_k(X_k) \hookrightarrow \text{bl}_k(X)$.

Let $j < k$, we first prove statement (i). By assumption, we have
\[ \text{bl}_i(\phi_k)^{-1}(\text{bl}_i(X_i)) = \text{bl}_i(X_{i,k}), \quad \forall i < j. \]

It follows that
\[ \text{bl}_j(\phi_k)^{-1}(F_i) = E_i, \quad \forall i < j. \]

Let us apply Lemma 1.14 to the map $\text{bl}_j(\phi_k) : \text{bl}_j(X_k) \rightarrow \text{bl}_j(X)$, the subvarieties $\text{bl}_j(X_j), \text{bl}_j(X_{j,k})$, the collection of divisors $\{E_i^0\}_{i=0}^{j-1}, \{F_i^0\}_{i=0}^{j-1}$ and the chains $\{f_{j,k}\}_{j=0}^{k-1}, \{\phi_j\}_{j=0}^{k-1}$. As $j < k$ and $\{f_{j,k}\}_{j=0}^{k-1}, \{\phi_j\}_{j=0}^{k-1}$ are NCD chains by assumptions (I),(III), we know $\text{bl}_j(X_j), \text{bl}_j(X_{j,k})$ are smooth and thus the assumption (a) in Lemma 1.14 is satisfied. Together with assumptions $(*)$, $(**)$, we can use Lemma 1.14 to conclude that the statement (i) holds for $j$.

For statement (ii), statement (i) implies that
\[ \text{bl}_j(\phi_k)^{-1}(\text{bl}_j(X_j)) = \text{bl}_j(X_{j-1,k}), \quad \forall j \leq k - 1. \]

Since $\{f_{j,k}\}_{j=0}^{k-1}$ is an NCD chain by assumption (I), we have
\[ \text{bl}_j(X_{j,k}) \hookrightarrow \text{bl}_j(X_k), \]
Lemma 1.22. Let $\phi : X \to Y$ be a $B$-morphism of smooth algebraic varieties over a smooth variety $B$, such that $f$ and $g$ are smooth morphisms.

\[
\begin{array}{c}
  X \\ \downarrow f \quad \downarrow \phi \\
  Y \\
\end{array}
\]

Denote the induced map over a closed point $t \in B$ by the symbol $\phi_t : X_t \to Y_t$. Then the following hold.

1. If $\phi_t$ is an embedding for each $t$, then $\phi$ is an embedding.
2. Let $X_1 \subset X$ and $Y_1 \subset Y$ be smooth subvarieties. If $Y_t \cap Y_1, X_t \cap X_1$ are smooth and $\phi_t^{-1}(Y_t \cap Y_1) = X_t \cap X_1$ for each $t$, then $\phi^{-1}(Y_1) = X_1$.
3. Suppose $\phi$ is an embedding, then $\text{bl}_X(Y)$ is a $B$-variety and the fiber over $t \in B$ is $\text{bl}_X(Y_t)$.

and $\beta_j(X_{j,k})$ are all smooth for each $j \leq k - 1$. Thus $\beta_k(X_k)$ is also the blow up space obtained using the chain $\{f_{j,k}\}_{j=0}^{k-1}$, i.e. it is the iterated blow up of the smooth variety $X_k$ along smooth subvarieties $\beta_j(X_{j,k})$ for $j \leq k - 1$. Hence $\beta_k(X_k)$ must be smooth. This proves statement (ii).

Lastly we prove statement (iii). Statement (i) implies that $\phi_k : X_k \to X$ is a map of chains of centers with respect to the chains $\{f_{i,k}\}_{i=0}^{k-1}$ and $\{\phi_i\}_{i=0}^{k-1}$, which are NCD chains by assumption (I) and (III), respectively. Assumptions (II) and (***) verify the assumptions (b),(c) in Lemma 1.18. Therefore, we can apply Lemma 1.18 to $\phi_k$ to obtain that $\beta_k(\phi_k) : \beta_k(X_k) \to \beta_k(X)$ is an embedding.

Remark 1.21. According to this lemma, to prove a proper chain $\{\phi_k : X_k \to X\}_{k=0}^n$ is NCD, it suffices to show that auxiliary chains $\{f_{i,k} : X_{i,k} \to X_k\}_{i=0}^{k-1}$ satisfy various conditions on pull-backs and exceptional divisors. We generally verify Assumptions (II) and (*) directly, whereas assumption (**)(***) rely on the computation of conormal bundles. However, assumption (I) is the most difficult to check and requires a further induction as below. For our application, the chain $\{\phi_j\}$ takes the form $(j < k)$

\[
\begin{array}{c}
X_{j,k} = X_j \times S_{j,k} \\
\downarrow f_{j,k} \\
X_k \\
\downarrow \phi_k \\
X \\
\end{array}
\]

where $p_1$ is the first projection and $S_{j,k}$ are smooth projective. Moreover, if we fix $k$, then $\{f_{j,k}\}_{j=0}^{k-1}$ is a proper chain via the diagram $(i < j < k)$

\[
\begin{array}{c}
X_i \times S_{i,j} \times S_{j,k} \\
\downarrow f_{i,j} \times \text{id} \\
X_j \times S_{j,k} \\
\downarrow f_{j,k} \\
X_k \\
\end{array}
\]

where $r : S_{i,j} \times S_{j,k} \to S_{i,k}$ is a surjective map depends on the specific property of $S_{j,k}$. This makes induction possible: the study of the chain $\{f_{j,k}\}_{j=0}^{k-1}$ can be further reduced to the proper chain $\{f_{i,j} \times \text{id}\}_{j=0}^{j-1}$, again using Lemma 1.20. Luckily, in our case, the process stops at this point and one does not need any further auxiliary chains.

Since the exceptional divisors in smooth blowups are projective bundles, to verify the assumptions (**) and (***) in Lemma 1.20 in practice, we need a relative version of some of the lemmas above.
Proof. For (1), it suffices to check that the differential $d\phi : \phi^*T_Y^* \to T_X^*$ is surjective, which can be checked by restriction to each $X_i$; the proof is similar to that of Lemma 1.17 by using the isomorphism

$$\phi^*N^*_i|_Y \cong N^*_X|_X.$$  

For (2), it suffices to check the surjectivity of the induced map

$$d\phi : \phi^*N^*_i|_Y \to N^*_X|_X.$$  

The proof is similar to that of Lemma 1.14 and uses the isomorphism

$$N^*_i|_{Y_i \cap Y_i} \cong N^*_i|_{Y_i}$$  

because $Y_i \cap Y_i$ is smooth; likewise for $N^*_X|_X$. (3) follows from local computations.  

2. Abel-Jacobi maps and addition maps on symmetric products

In this section, we reduce the proof of Theorem A to two somewhat technical propositions about certain natural proper chains being NCD chains, using the general framework in §5. Let $C$ be a hyperelliptic curve of odd genus $g = 2n + 1$. The even genus case is similar, and will be treated separately in §7. Let $g^1_2$ be the line bundle corresponding to the hyperelliptic map $h : C \to \mathbb{P}^1$ and denote by $C_j := \text{Sym}^j(C)$ the $j$-th symmetric product of $C$; we view the closed points of $C_j$ as effective divisors of degree $j$ on the curve $C$, and let $C_0$ be the one-point set consisting of the trivial divisor. Consider the following sequence of morphisms $\{\delta_j\}_{j=0}^n$ to $\text{Jac}(C)$, where

$$\delta_j : C_{2j} \to \text{Jac}(C) = \text{Pic}^{g-1}(C), \quad 0 \leq j \leq n,$$

$$D \mapsto (n-j)g^1_2 \otimes \mathcal{O}_C(D).$$

By the Abel-Jacobi theorem, we have $\delta_j(C_{2j}) = W^j_{g-1}$ and there is a natural embedding $\mathbb{P}^j = \delta_j^{-1}(ng^1_2) \hookrightarrow C_{2j}$.

It is easy to see that $\{\delta_j\}_{j=0}^n$ is a proper chain in the sense of Definition 1.1. Indeed, for $i < j$, we have a commutative diagram

$$\begin{array}{ccc}
C_{2i} \times \mathbb{P}^{j-i} & \xrightarrow{p_1} & C_{2i} \\
\downarrow_{\gamma_{i,j}} & & \downarrow_{\delta_i} \\
C_{2j} & \xrightarrow{\delta_j} & \text{Jac}(C),
\end{array}$$

where $p_1$ denotes the projection to the first factor and $\gamma_{i,j}$ is defined by

$$\gamma_{i,j} : C_{2i} \times \mathbb{P}^{j-i} \hookrightarrow C_{2i} \times C_{2j-2i} \to C_{2j}, \quad 0 \leq i \leq j - 1.$$  

The first map is induced by the natural embedding $\mathbb{P}^{j-i} \hookrightarrow C_{2j-2i}$ and the second map is the addition map on symmetric products. Since $p_1$ is surjective and

$$\delta_i(C_{2i}) = W^j_{g-1} \subsetneq W^{j-1}_{g-1} = \delta_j(C_{2j}),$$

the chain $\{\delta_j\}_{j=0}^n$ is a proper chain, as claimed. Note that the diagram (9) is not Cartesian. It is also not hard to check that $\{\gamma_{i,j}\}_{i=0}^{j-1}$ is a proper chain, see the discussion below.

The proof of Theorem A for hyperelliptic curves of odd genus can be reduced to the following proposition, whose proof we postpone until §5. It describes the properties of the chain $\{\delta_j\}_{j=0}^n$, using the language from §4.

**Proposition 2.1.** The proper chain $\{\delta_j : C_{2j} \to \text{Jac}(C)\}_{j=0}^n$ is an NCD chain. Concretely, this means the following things:
(a) For $0 \leq j \leq n$, the maps $\bl_j(\delta_k)$ associated to the chain $\{\delta_j\}_{j=0}^n$ exist and there are embeddings of smooth projective varieties

$$\bl_j(\delta_j) : \bl_j(C_{2j}) \hookrightarrow \bl_j(Jac(C)),$$

whose images intersect the union of all the exceptional divisors in $\bl_j(Jac(C))$ transversely.

(b) There is a natural embedding $Jac(C) - \delta_n(C_{2n}) \hookrightarrow \bl_{n+1}(Jac(C))$, whose complement has $n + 1$ smooth components with normal crossings.

Moreover, the map $\delta_k : C_{2k} \to Jac(C)$ is a map of chains of centers between the chains $\{\gamma_{j,k}\}_{j=0}^{k-1}$ and $\{\delta_j\}_{j=0}^{n}$, so that $\bl_j(C_{2k})$ coincides with the space associated to the chain $\{\gamma_{j,k} : C_{2j} \times \mathbb{P}^{k-j} \to C_{2k}\}_{j=0}^{k-1}$. Finally, there is a natural identification

$$\bl_j(C_{2j} \times \mathbb{P}^{k-j}) = \bl_j(C_{2j}) \times \mathbb{P}^{k-j},$$

which induces a Cartesian diagram

$$\begin{array}{ccc}
\bl_j(C_{2j}) \times \mathbb{P}^{k-j} & \xrightarrow{p_1} & \bl_j(C_{2j}) \\
\downarrow{\bl_j(\gamma_{j,k})} & & \downarrow{\bl_j(\delta_k)} \\
\bl_j(C_{2k}) & \xrightarrow{\bl_j(\delta_k)} & \bl_j(Jac(C))
\end{array}$$

Assuming this proposition, we can easily deduce the main theorem for hyperelliptic curves of odd genus $g = 2n + 1$ (the even genus case will follow in the same way from Proposition 7.1).

**Proof of Theorem A.** For the sake of clarity, let us denote by $\{\bl'_i(Jac(C))\}_{i=1}^n$, the sequence of blowups described in the introduction, where at the $i$-th stage to get $\bl'_i(Jac(C))$, we blow up the strict transform of the Brill-Noether variety $W_{g-1-i}^{n+1-i}(C)$. We are going to argue that, in fact,

$$\bl'_i(Jac(C)) = \bl_i(Jac(C)).$$

First, since $g = 2n + 1$, the image of $\bl_n(C_{2n})$ in $\bl_n(Jac(C))$ is a divisor, and so $\bl_{n+1}(Jac(C)) = \bl_n(Jac(C))$; therefore both sequences really have only $n$ steps. Note that $\delta_i(C_{2i})$ is equal to the subset $W_{g-1-i}^{n+1-i}(C)$ of $Jac(C) = Pic^{g-1}(C)$. By Proposition 2.1, one has an embedding

$$\bl_i(C_{2i}) \hookrightarrow \bl_i(Jac(C)).$$

By induction on $1 \leq i \leq n$, it then follows easily that $\bl'_i(Jac(C)) = \bl_i(Jac(C))$, and that the proper transform of $W_{g-1-i}^{n-i}(C)$ under the birational morphism $\pi_i : \bl_i(Jac(C)) \to Jac(C)$ is equal to the image of $\bl_i(C_{2i}) \hookrightarrow \bl_i(Jac(C))$, hence smooth.

The conclusion is that $\bl'_n(Jac(C)) = \bl_n(Jac(C))$ is smooth, and that the strict transform $\tilde{\Theta}$ is the image of $\bl_n(C_{2n}) \hookrightarrow \bl_n(Jac(C))$, hence also smooth. Since $\{\delta_j\}_{j=0}^n$ is an NCD chain, the pullback $\pi^*\tilde{\Theta}$ is a divisor with simple normal crossings. The multiplicity of the exceptional divisor $Z_i$ equals the multiplicity of $\Theta$ at a point in $W_{g-1}^{n-i} - W_{g-1}^{n+1-i}$, which is $n + 1 - i$ by the Riemann Singularity Theorem. 

The proof of Proposition 2.1 is by a rather tricky inductive argument. Along the way, we need several auxiliary chains that we now describe. The first such chain lives over $C_{2j}$, and its shape is suggested by the commutative diagram in (9).

For each $1 \leq k \leq n$, we have the chain of maps

$$\gamma_{i,k} : C_{2i} \times \mathbb{P}^{k-i} \to C_{2k},$$

for which

$$\{\gamma_{i,k} : C_{2i} \times \mathbb{P}^{k-i} \to C_{2k}\}_{i=0}^{k-1}$$
follows from the proof of Lemma 4.2 that \( D \) to \( C \) further chains are needed. because of the product structure of this chain, the inductive process stops here and no

\[
\begin{array}{ccc}
C_{2i} \times P^{j-i} \times P^{k-j} & \xrightarrow{id \times r} & C_{2i} \times P^{k-i} \\
\downarrow \gamma_{i,j} \times id & & \downarrow \gamma_{i,k} \\
C_{2j} \times P^{k-j} & \xrightarrow{\gamma_{j,k}} & C_{2k}
\end{array}
\]

Here \( r \) is the restriction of the addition map \( C_{2(j-i)} \times C_{2(k-j)} \to C_{2(k-i)} \), which can also be viewed as the addition map for symmetric products of \( P^1 \) if we think of \( P^i \) as Sym\(^i\)\( P^1 \). It follows from the proof of Lemma 4.2 that \( \gamma_{i,k}(C_{2i} \times P^{k-i}) \) parametrizes effective divisors \( D \) of degree \( 2k \) such that \( h^0(O_C(D)) \geq k - i + 1 \). Therefore for \( i < j \), one has

\[
\gamma_{i,k}(C_{2i} \times P^{k-j}) \subset \gamma_{j,k}(C_{2j} \times P^{k-j}),
\]

and so \( \{\gamma_{i,k}\}_{i=0}^{k-1} \) is also a proper chain, as claimed.

The following commutative diagram:

\[
\begin{array}{ccc}
\{\gamma_{i,j} \times id : (C_{2i} \times P^{j-i}) \times P^{k-j} \to C_{2j} \times P^{k-j}\}_{i=0}^{j-1} & & \\
\end{array}
\]

which is induced by taking the product of the chain \( \{\gamma_{i,j}\}_{i=0}^{j-1} \) with \( P^{k-j} \). Fortunately, because of the product structure of this chain, the inductive process stops here and no further chains are needed.

The key step in the proof of Proposition 2.2 is the following result. Denote by \( bl_i(C_{2j} \times P^{j-k}) \) and \( bl_i(C_{2k}) \) the spaces associated to the chain \( \{\gamma_{j,k}\}_{j=0}^{k-1} \). Denote by \( bl_i(C_{2i} \times P^{j-i} \times P^{k-j}) \) the spaces associated to the chain \( \{\gamma_{i,j} \times id\}_{i=0}^{j-1} \).

**Proposition 2.2.** Let \( k \) be an integer such that \( 1 \leq k \leq n \). Then the proper chain

\[
\{\gamma_{j,k} : C_{2j} \times P^{k-j} \to C_{2k}\}_{j=0}^{k-1}
\]

is an NCD chain and for each \( j < k \), the map

\[
\gamma_{j,k} : C_{2j} \times P^{k-j} \to C_{2k}
\]

is a map of chains of centers from \( \{\gamma_{i,j} \times id\}_{i=0}^{j-1} \) to \( \{\gamma_{i,k}\}_{i=0}^{j-1} \). Concretely, this means the following things:

(a) For \( 0 \leq i < k \), there is a closed embedding

\[
bl_i(\gamma_{i,k}) : bl_i(C_{2i} \times P^{k-i}) \hookrightarrow bl_i(C_{2k}),
\]

whose image intersects the union of all the exceptional divisors in \( bl_i(C_{2k}) \) transversely.

(b) There is a natural embedding \( C_{2k} - \gamma_{k-1,k}(C_{2k-2} \times P^1) \hookrightarrow bl_k(C_{2k}) \), whose complement has \( k \) smooth components with normal crossings.

(c) For each \( i < j < k \), one has a Cartesian diagram

\[
\begin{array}{ccc}
\text{bl}_i(C_{2i} \times P^{j-i} \times P^{k-j}) & \xrightarrow{\text{bl}_i(\gamma_{i,j} \times id)} & \text{bl}_i(C_{2i} \times P^{k-i}) \\
\downarrow \text{bl}_i(\gamma_{i,j}) & & \downarrow \text{bl}_i(\gamma_{i,k}) \\
\text{bl}_i(C_{2j} \times P^{k-j}) & \xrightarrow{\text{bl}_i(\gamma_{j,k})} & \text{bl}_i(C_{2k})
\end{array}
\]
3. Secant bundles and maps between them

The proofs of Proposition 2.2 and Proposition 2.1 rely on certain results about secant bundles over symmetric products of curves. In this section, we review the necessary definitions and results, following the notation in Bertram’s work [2].

Let \( C \) be a smooth projective curve of genus \( g \geq 0 \), let \( M \) be a line bundle on \( C \), and let \( j \geq 0 \) be an integer. We denote by \( C_j = \text{Sym}^j C \) the \( j \)-th symmetric product of the curve. Consider the following diagram:

\[
\begin{array}{cccc}
\mathcal{D}_{j+1} & \hookrightarrow & C \times C_{j+1} \\
\downarrow & & \downarrow p_1 & \downarrow p_2 \\
C & & C_j & \to C_{j+1}
\end{array}
\]

Here \( \mathcal{D}_{j+1} = C \times C_j \) is the universal divisor of degree \( j + 1 \) over \( C_{j+1} \), embedded via \((p, D) \mapsto (p, p + D)\). We say that \( M \) separates \( d \) points if

\[
h^0(C, M) = h^0(C, M(-D)) + d, \quad \forall D \in C_d.
\]

If \( M \) separates \( j + 1 \) points, then the sequence of sheaves

\[
0 \to p_1^* M \otimes \mathcal{O}(-\mathcal{D}_{j+1}) \to p_1^* M \to p_1^* M \otimes \mathcal{O}_{\mathcal{D}_{j+1}} \to 0
\]

on \( C \times C_{j+1} \) remains exact when pushed down to \( C_{j+1} \).

**Definition 3.1.** The secant bundle (with respect to \( M \)) of \( j \)-planes over \( C_{j+1} \) is

\[
B_j^j(M) := \mathbb{P}(p_2)_*(p_1^* M \otimes \mathcal{O}_{\mathcal{D}_{j+1}}).
\]

This is a \( \mathbb{P}^j \)-bundle over the symmetric product \( C_{j+1} \); for \( j = 0 \), we have \( B^0(M) = C \).

If \( M \) separates \( j + 1 \) points, the natural map to \( \mathbb{P}H^0(C, M) \) is

\[
\beta_j : B_j^j(M) \to \mathbb{P}(p_2)_*(p_1^* M) = \mathbb{P}H^0(C, M) \times C_{j+1} \to \mathbb{P}H^0(C, M),
\]

where the last map is the projection to \( \mathbb{P}H^0(C, M) \).

Assuming that \( M \) separates \( m + 1 \) points, we get a proper chain

\[
\{\beta_j : B_j^j(M) \to \mathbb{P}H^0(C, M)\}_{j=0}^m,
\]

using the following diagram

\[
\begin{array}{ccc}
B^i(M) \times C_{j-i} & \xrightarrow{p_1} & B^i(M) \\
\downarrow \alpha_{i,j} & & \downarrow \beta_i \\
B^j(M) & \xrightarrow{\beta_j} & \mathbb{P}H^0(C, M)
\end{array}
\]

Here \( p_1 \) is again the projection to the first coordinate. For \( i < j \), the map

\[
\alpha_{i,j} : B^i(M) \times C_{j-i} \to B^j(M)
\]

is induced by the addition map \( r : C_{i+1} \times C_{j-i} \to C_{j+1} \) (see the second definition in [2, Page 432]). This chain is proper because the image \( \beta_j(B^j(M)) \) is exactly the usual secant variety \( \text{Sec}^j(C) \) of \( j \)-planes through \( j + 1 \) points of \( C \) inside \( \mathbb{P}H^0(C, M) \) (see also [2, Page 432]).

In order to study this chain, Bertram introduced certain auxiliary chains, just as in the previous section. Fix \( k < m \), one can show that the chain \( \{\alpha_{i,k} : B^i(M) \times C_{k-i} \to \)
Remark 3.3. Using Definition 1.6, Bertram’s proof actually shows that \( \{B^i(M) \times C_j = 0 \}_{i=0}^{k-1} \) is proper because the image \( \alpha_{i, j} \) can be thought as the relative secant variety of \( \langle \rangle \)-planes in \( B^k(M) \), see [2] Page 433.

Lastly, using the construction in Notation 1.8, we have for each pair \((j, k)\) with \( j < k \) a proper chain
\[
\{ \alpha_{i, j} \times \text{id} : (B^i(M) \times C_{j-i}) \times C_{k-j} \rightarrow B^j(M) \times C_{k-j} \}_{i=0}^{j-1}.
\]

In [2] Proposition 2.2, Proposition 2.3, Bertram proved the following result.

**Proposition 3.2** (Bertram). Let \( M \) be a line bundle on \( C \).

(a) If \( M \) separates \( m + 1 \) points, then for \( j < k < m \), both \( \{\alpha_{i, k}\}_{i=0}^{k-1} \) and \( \{\alpha_{i, j} \times \text{id}\}_{i=0}^{j-1} \) are chains of smooth centers, and the map
\[
\alpha_{j, k} : B^j(M) \times C_{k-j} \rightarrow B^k(M)
\]
is an injective map of chains from \( \{\alpha_{i, j} \times \text{id}\}_{i=0}^{j-1} \) to \( \{\alpha_{i, k}\}_{i=0}^{k-1} \).

(b) If \( M \) separates \( 2k + 2 \) points, then \( \{\beta_j\}_{j=0}^{k} \) is a chain of smooth centers, and
\[
\beta_k : B^k(M) \rightarrow \mathcal{P}H^0(C, M)
\]
is an injective map of chains from \( \{\alpha_{j, k}\}_{j=0}^{k-1} \) to \( \{\beta_j\}_{j=0}^{k-1} \).

**Remark 3.3.** Using Definition 1.6, Bertram’s proof actually shows that \( \{\alpha_{j, k}\}_{j=0}^{k-1} \) and \( \{\beta_j\}_{j=0}^{k} \) are NCD chains. But these facts will not be used later.

Let us spell out in detail what Bertram’s theorem says in the case of \( \mathbb{P}^1 \), where the images of the secant bundles for \( \mathcal{O}_{\mathbb{P}^1}(d) \) are the secant varieties to the rational normal curve of degree \( d \) in \( \mathbb{P}^d \).

**Corollary 3.4.** Let \( d \geq 2k + 1 \) and consider the line bundle \( M = \mathcal{O}_{\mathbb{P}^1}(d) \) on \( \mathbb{P}^1 \). Then \( \{\beta_j\}_{j=0}^{k} \) is a chain of smooth centers. Fix \( 0 \leq i < j < k \), then

(a) The diagram
\[
\begin{array}{ccc}
\text{bl}_i(B^i(M) \times \mathbb{P}^{j-i}) \times \mathbb{P}^{k-j} & \longrightarrow & \text{bl}_i(B^i(M) \times \mathbb{P}^{k-j}) \\
\downarrow \text{bl}_i(\alpha_{i, j}) \times \text{id} & & \downarrow \text{bl}_i(\alpha_{i, k}) \\
\text{bl}_i(B^j(M) \times \mathbb{P}^{k-j}) & \longrightarrow & \text{bl}_i(B^k(M))
\end{array}
\]
is Cartesian and the two vertical arrows are embeddings. In particular, \( \alpha_{j, k} \) is a map of chains of centers.

(b) The diagram
\[
\begin{array}{ccc}
\text{bl}_i(B^i(M) \times \mathbb{P}^{j-i}) & \longrightarrow & \text{bl}_i(B^j(M)) \\
\downarrow \text{bl}_i(\alpha_{i, j}) & & \downarrow \text{bl}_i(\beta_j) \\
\text{bl}_i(B^j(M)) & \longrightarrow & \text{bl}_i(\mathbb{P}^d)
\end{array}
\]
is Cartesian and the two vertical arrows are embeddings. In particular, \( \beta_j \) is a map of chains of centers.
(c) There is a natural isomorphism
\[ \text{bl}_i(B^i(M) \times P^{j-i}) \cong \text{bl}_i B^i(M) \times P^{j-i}. \]

Proof. Since \( M = \mathcal{O}_{P^1}(d) \) separates \( d + 1 \) points on \( P^1 \), we can apply Proposition 3.2 and use the isomorphisms \( P^k \cong \text{Sym}^k P^1 \) and \( P^d \cong PH^0(P^1, M) \). The last statement follows from Lemma 1.9.

\[ \square \]

4. Properties of Abel-Jacobi maps and addition maps

In this section, as a preparation for the proofs of Proposition 2.2 and Proposition 2.1, we establish some basic properties of the map \( \gamma_{i,j} : C_{2i} \times P^{j-i} \to C_{2j} \) from (10) and of the Abel-Jacobi map \( \delta_j : C_{2j} \to \text{Jac}(C) \). In particular, their conormal bundles are calculated in terms of the secant bundles over symmetric products of \( P^1 \). In fact, it is known that the conormal bundle of the Abel-Jacobi map can be described in terms of Steiner bundles (see [4, Theorem 1.1]). For our purpose, it is more natural to use secant bundles.

**Notation 4.1.** For each \( j \), we define \( U_{2j} := C_{2j} - \gamma_{j-1,j}(C_{2j-2} \times P^1) \). In other words, \( U_{2i} \) consists of divisors \( D \) where none of the degree 2 subdivisors of \( D \) form a hyperelliptic pair and \( h^0(O_C(D)) = 1 \). By Remark 1.7, this is exactly the open subset of \( C_{2j} \) which is the complement of exceptional divisors in \( \text{bl}_i(C_{2j}) \) associated to the chain \( \{\gamma_{i,j}\}_{i=0}^{j-1} \), once we prove this chain is a chain of smooth centers.

**Lemma 4.2.** Let \( C \) be a hyperelliptic curve of odd genus \( g = 2n + 1 \). Fix \( 0 \leq i < j \leq n \).

(a) The maps \( \delta_j : U_{2j} \to \text{Jac}(C), \quad \gamma_{i,j} : U_{2i} \times P^{j-i} \to C_{2j} \)

are embeddings and the restriction of the diagram (11) is Cartesian:

\[
\begin{array}{ccc}
U_{2i} \times P^{j-i} & \xrightarrow{id \times r} & U_{2i} \\
\gamma_{i,j} \downarrow & & \delta_j \downarrow \\
C_{2j} & \xrightarrow{\delta_i} & \text{Jac}(C)
\end{array}
\]

Equivalently, we have \( \delta_i^{-1}(U_{2i}) = U_{2i} \times P^{j-i} \).

(b) For \( 0 \leq i < j < k \leq n \), the restriction of the diagram (12) is Cartesian:

\[
\begin{array}{ccc}
(U_{2i} \times P^{j-i}) \times P^{k-j} & \xrightarrow{id \times r} & U_{2i} \times P^{k-j} \\
\gamma_{i,j} \times id \downarrow & & \gamma_{j,k} \downarrow \\
C_{2j} \times P^{k-j} & \xrightarrow{\gamma_{j,k}} & C_{2k}
\end{array}
\]

Equivalently, we have \( \gamma_{j,k}^{-1}(U_{2i} \times P^{k-j}) = U_{2i} \times P^{j-i} \times P^{k-j} \).

(c) In particular, for \( j < k \leq n \) and \( i = 0 \), we have \( \gamma_{j,k}^{-1}(C_0 \times P^k) = C_0 \times P^j \times P^{k-j} \).

Proof. Since \( C \) is a hyperelliptic curve, for each divisor \( D \) with \( h^0(\mathcal{O}_C(D)) = r + 1 \) and degree \( d \leq g(C) = 2n + 1 \), there is a unique decomposition

\[ D = E + \sum_{\ell=1}^{r} (p_\ell + q_\ell), \]
such that $p_{\ell} + q_{\ell}$ are hyperelliptic pairs and $E$ is a degree $d - 2r$ divisor with $h^0(\mathcal{O}_C(E)) = 1$. Similarly, we also have for any $L \in \text{Jac}(C)$ with $h^0(L) = r + 1$, there is a unique decomposition

$$L = rg_2^1 \otimes L'$$

such that $h^0(L') = 1$, see [I] Page 13.

As $\delta_i(D) = (n - i)g_2^1 \otimes \mathcal{O}_C(D)$ by (8), it follows immediately from the uniqueness of the decomposition (17) that $\delta_i : U_{2i} \to \text{Jac}(C)$ is injective, and that the image $\delta_i(U_{2i})$ consists of line bundles $L \in \text{Jac}(C)$ such that $h^0(L) = n - i + 1$. Similarly, one can show that the map $\gamma_{i,j} : U_{2i} \times C_{j-i} \to C_{2j}$ is injective, and any divisor in its image can be written as

$$D = E + \sum_{\ell=1}^{j-i}(p_{\ell} + q_{\ell})$$

where $E$ is a degree 2i divisor with $h^0(\mathcal{O}_C(E)) = 1$ and $p_{\ell} + q_{\ell}$ are hyperelliptic pairs. Therefore the image $\gamma_{i,j}(U_{2i} \times C_{j-i})$ consists of divisors $D$ of degree 2j such that $h^0(\mathcal{O}_C(D)) = j - i + 1$. To show the restricted maps $\delta_j$ and $\gamma_{i,j}$ are embeddings, one needs the surjectivity of $d\delta_j$ and $d\gamma_{i,j}$, which follows from Lemma [1.3](a) and Lemma [4.7](a) below (whose proofs are independent of this lemma).

Now we want to argue the diagrams in (a),(b) are Cartesian. We will identify $U_{2i}$ and $U_{2i} \times \mathbb{P}^{j-i}$ with their images in $\text{Jac}(C)$ and $C_{2j}$. First, let us show that

$$\delta_j^{-1}(U_{2i}) = U_{2i} \times \mathbb{P}^{j-i}.$$ 

Let us first prove this set-theoretically. Suppose $D \in \delta_j^{-1}(U_{2i})$, this means that

$$h^0(\delta_j(D)) = h^0((n - j)g_2^1 \otimes \mathcal{O}_C(D)) = n - i + 1$$

by the characterization of the image of $U_{2i}$ in $\text{Jac}(C)$. Using (17), we must have

$$h^0(\mathcal{O}_C(D)) = (n - i + 1) - (n - j) = j - i + 1$$

and conclude that $D \in U_{2i} \times \mathbb{P}^{j-i}$, by the characterization of its image in $C_{2j}$. The argument for the set-theoretic part of (b),(c) is similar. To finish the proof of (a), we need to show the surjectivity of

$$d\delta_j : \delta_j^* N_{U_{2i}/\text{Jac}(C)} \to N_{U_{2i} \times \mathbb{P}^{j-i}/C_{2j}}^*,$$

which follows from Corollary 4.8. Similarly, the statement that the diagram in (b) is Cartesian follows from Corollary 4.6. \qed

**Notation 4.3.** As above, the open subset $U_{2j} \subseteq C_{2j}$ consists of divisors $D$ of degree 2j such that there is no hyperelliptic pair contained as an effective subdivisor of $D$. In particular, any $D \in U_{2j}$ gives a degree 2j divisor on $\mathbb{P}^1$ by pushforward along the hyperelliptic map $h : C \to \mathbb{P}^1$. We denote this divisor on $\mathbb{P}^1$ by the symbol $h_*D$ and define

$$\mathcal{O}_{\mathbb{P}^1}(g - 1 - h_* D) := \mathcal{O}_{\mathbb{P}^1}(g - 1) \otimes \mathcal{O}_{\mathbb{P}^1}(-h_* D),$$

which is a line bundle of degree $g - 1 - 2j$ on $\mathbb{P}^1$. Note that we have $h_* \mathcal{O}_D \cong \mathcal{O}_{h_* D}$.

The divisor $h_* D$ shows up in the following way. Since the curve $C$ is hyperelliptic, we have $\omega_C \cong h^* \mathcal{O}_{\mathbb{P}^1}(g - 1)$, and therefore

$$h_*\omega_C \cong \omega_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(g - 1).$$

If we twist the canonical bundle by an effective divisor $D \in U_{2j}$, we instead get

$$h_*\omega_C(-D) \cong \omega_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(g - 1 - h_* D).$$

(18)
This follows from pushing the short exact sequence \(0 \rightarrow \omega_C(-D) \rightarrow \omega_C \rightarrow \omega_C \otimes O_D \rightarrow 0\) forward along \(h: C \rightarrow \mathbb{P}^1\), and using \(h_*(\omega_C \otimes O_D) \cong O_{\mathbb{P}^1}(g-1) \otimes O_{h_*D}\) via the projection formula and \(h_*O_D \cong O_{h_*D}\).

**Lemma 4.4.** Let \(C\) be a hyperelliptic curve of genus \(g = 2n + 1\). For \(0 \leq i < j \leq n\) and \(\gamma_{i,j}\) from \([10]\), we have

\(\text{(a)}\) \(d\gamma_{i,j}: \gamma_{i,j}^* T_{C_{2j}}^* \rightarrow T_{C_{2i} \times X^{j-i}}^*\) is surjective when restricted to \(U_{2i} \times \mathbb{P}^{j-i}\).

\(\text{(b)}\) For \(i = 0\), we have an isomorphism

\[\text{PN}_{\mathbb{P}^j|C_{2j}} \cong B^{j-1}(O_{\mathbb{P}^1}(g - 1)),\]

the latter is the secant bundle over \(\text{Sym}^{j} \mathbb{P}^1 = \mathbb{P}^j\) with respect to \(O_{\mathbb{P}^1}(g - 1)\).

\(\text{(c)}\) For \(i \geq 1\), the space \(\text{PN}_{n_{i,j}|D} \times \mathbb{P}^{j-i}\) is smooth over \(U_{2i}\), such that over \(D \in U_{2i}\) we have an isomorphism:

\[\text{PN}_{n_{i,j}|D} \times \mathbb{P}^{j-i} \cong B^{j-i-1}(O_{\mathbb{P}^1}(g - 1 - h_*D)),\]

the secant bundle over \(\mathbb{P}^{j-i}\) with respect to \(O_{\mathbb{P}^1}(g - 1 - h_*D)\).

Here for a map \(f: X \rightarrow Y\), we denote by \(N_f^* = \text{Ker}(df: f^* T^* Y \rightarrow T^* X)\).

**Proof.** As a warm-up, let us calculate the conormal bundle of \(\mathbb{P}^j\) inside \(C_{2j}\). Recall that for any divisor \(D \in C_{2j}\), there is a canonical identification \([1]\) Page 160

\[T_{C_{2j}}^*|D \cong H^0(C, \omega_C \otimes O_D).\]

Using the isomorphism \(\mathbb{P}^j \cong \text{Sym}^j \mathbb{P}^1\), the morphism \(\mathbb{P}^j \rightarrow C_{2j}\) associates an effective divisor \(E\) of degree \(j\) on \(\mathbb{P}^1\) to an effective divisor \(h^* E\) of degree \(2j\) on \(C\). Similarly, we have

\[T_{\mathbb{P}^j|E}^* \cong H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1} \otimes O_E)\]

for the cotangent space of \(\mathbb{P}^j\) at the point \(E\). Moreover, \(T_{C_{2j}}^*|_{h^* E}\) and \(T_{\mathbb{P}^j|E}^*\) can be related in the following way. One has

\[T_{C_{2j}}^*|_{h^* E} \cong H^0(C, \omega_C \otimes h^* O_E) \cong H^0(\mathbb{P}^1, h_* \omega_C \otimes O_E) \]

\[\cong H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1} \otimes O_E) \oplus H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(g - 1) \otimes O_E)\]

\[= T_{\mathbb{P}^j|E}^* \oplus H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(g - 1) \otimes O_E).\]

so that the morphism between the two cotangent spaces is the projection to the first summand. It follows that the map \(T_{C_{2j}}^*|_{h^* E} \rightarrow T_{\mathbb{P}^j|E}^*\) induced by \(\mathbb{P}^j \hookrightarrow C_{2j}\) is surjective (which means that \(\mathbb{P}^j \hookrightarrow C_{2j}\) is a closed embedding), with kernel

\[N_{\mathbb{P}^j|C_{2j}}|_{E} \cong H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(g - 1) \otimes O_E).\]

Varying this isomorphism over \(E \in \mathbb{P}^j\) gives

\[N_{\mathbb{P}^j|C_{2j}} \cong (p_2)_* (p_1^* O_{\mathbb{P}^1}(g - 1) \otimes O_{\varepsilon_j}).\]

where \(\varepsilon_j\) denotes the universal divisor over \(\mathbb{P}^j \cong \text{Sym}^j \mathbb{P}^1\), as in the following diagram:

\[\begin{array}{ccc}
\varepsilon_j & \hookrightarrow & \mathbb{P}^1 \times \text{Sym}^j \mathbb{P}^1 \\
p_1 & \downarrow & \downarrow p_2 \\
\mathbb{P}^1 & \rightarrow & \text{Sym}^j \mathbb{P}^1
\end{array}\]

In particular, the right hand side is the secant bundle \(B^{j-1}(O_{\mathbb{P}^1}(g - 1))\), and so we have proved (b).
For (a) and (c), consider \( D \in U_{2i} \) and \( E \in \mathbb{P}^{j-i} \). The morphism \( \gamma_{i,j} : U_{2i} \times \mathbb{P}^{j-i} \to C_{2j} \) takes the pair \((D, E)\) to the divisor \( D + h^* E \) of degree \( 2j \) on \( C \). Because \( D \in U_{2i} \), we have \( H^0(C, \mathcal{O}_C(D + h^* E)) = H^0(C, \mathcal{O}_C(h^* E)) \) by \([17]\). After a little bit of diagram chasing, this gives us a short exact sequence

\[
0 \to H^0(C, \omega_C(-D) \otimes \mathcal{O}_{h^* E}) \to H^0(C, \omega_C \otimes \mathcal{O}_{D+h^* E}) \to H^0(C, \omega_C \otimes \mathcal{O}_D) \to 0
\]

and the morphism between the cotangent spaces of \( C_{2j} \) and \( U_{2i} \) is the morphism in this short exact sequence. Consequently using \([18]\), one has

\[
\ker \left( T_{C_{2j}}|_{D+h^* E} \to T_{U_{2i}}|_D \right) \cong H^0(C, \omega_C(-D) \otimes h^* \mathcal{O}_E)
\]

\[
\cong H^0(\mathbb{P}^1, h_* \omega_C(-D) \otimes \mathcal{O}_E)
\]

\[
\cong H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1} \otimes \mathcal{O}_E) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1-h_* D) \otimes \mathcal{O}_E)
\]

\[
= T_{\mathbb{P}^{j-i}}|_E \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1-h_* D) \otimes \mathcal{O}_E),
\]

and the morphism to the cotangent space of \( \mathbb{P}^{j-i} \) is the projection to the first summand.

Hence, we deduce that

\[
d\gamma_{i,j} : T_{C_{2j}}|_{D+h^* E} \to T_{U_{2i}}|_D \oplus T_{\mathbb{P}^{j-i}}|_E
\]

is surjective, proving (a); and that its kernel can be identified with

\[
N_{\gamma_{i,j}|(D,E)}^* \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1-h_* D) \otimes \mathcal{O}_E).
\]

This isomorphism is natural in \( E \in \mathbb{P}^{j-i} \), and therefore

\[
N_{\gamma_{i,j}|(D) \times \mathbb{P}^{j-i}}^* \cong (p_2)_* \left( p_1^* \mathcal{O}_{\mathbb{P}^1}(g-1-h_* D) \otimes \mathcal{O}_{\mathbb{P}^{j-i}} \right).
\]

The latter is a vector bundle on \( \mathbb{P}^{j-i} \) because the line bundle \( \mathcal{O}_{\mathbb{P}^1}(g-1-h_* D) \) separates \( j-i \) points (on account of the inequality \( g-1-2i = 2n-2i > j-i \)). Varying \( D \in U_{2i} \), we see that the projectivized conormal bundle \( N_{\gamma_{i,j}}^*|_{U_{2i} \times \mathbb{P}^{j-i}} \) is a projective bundle over \( U_{2i} \), hence is smooth over \( U_{2i} \). Moreover, its fiber over \( D \in U_{2i} \) is the secant bundle \( B^{j-i-1}(\mathcal{O}_{\mathbb{P}^1}(g-1-h_* D)) \), which proves (c). \( \square \)

**Remark 4.5.** This lemma is parallel to \([2, \text{Lemma 1.3}]\), with the difference that the relevant divisor is \( h_* D \) (not \( 2h_* D \) as in Bertram’s case).

From the proof of Lemma \([14]\) we can deduce one additional useful fact. For \( 0 \leq i < j < k \leq n \), consider the commutative diagram induced by \([12]\) via restriction:

\[
(U_{2i} \times \mathbb{P}^{j-i}) \times \mathbb{P}^{k-j} \xrightarrow{id \times r} U_{2i} \times \mathbb{P}^{k-i}
\]

\[
\gamma_{i,j} \times id \downarrow \quad \gamma_{i,k} \downarrow
\]

\[
C_{2j} \times \mathbb{P}^{k-j} \xrightarrow{\gamma_{j,k}} C_{2k}
\]

**Corollary 4.6.** For \( D \in U_{2i} \), the induced map of conormal bundles

\[
e : (id \times r)^* N_{\gamma_{i,k}|(D) \times \mathbb{P}^{j-i}}^* \to N_{\gamma_{i,j} \times id|(D) \times \mathbb{P}^{j-i} \times \mathbb{P}^{k-j}}^*
\]
on \( \{D\} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j} \) is surjective, and the diagram

\[
\begin{array}{ccc}
\mathbf{P} \mathcal{N}_{\mathcal{H},j}^* \times \text{id} |_{\{D\} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j}} & \overset{\alpha}{\longrightarrow} & \mathbf{P} \mathcal{N}_{\mathcal{H},k}^* |_{\{D\} \times \mathbf{P}^{k-i}} \\
\downarrow \text{id}_{\mathbf{P}^{k-j}} & & \downarrow \text{id} \\
B^{j-i-1}(M) \times \mathbf{P}^{k-j} & \overset{\alpha_{j-1,k,i-1}}{\longrightarrow} & B^{k-i-1}(M)
\end{array}
\]

commutes. Here \( \alpha \) is induced by \( \epsilon \) and the projection to \( \mathbf{P} \mathcal{N}_{\mathcal{H},k}^* |_{\{D\} \times \mathbf{P}^{k-i}} \), both vertical arrows are isomorphisms, and \( \alpha_{j-1,k,i-1} \) is the map in (16) for the curve \( \mathbf{P}^1 \) and the line bundle \( M = \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_sD) \).

Proof. To simplify the notation, fix a point \( D \in U_{2i} \) and define \( M = \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_sD) \). For \( E_1 \in \mathbf{P}^{j-i} \) and \( E_2 \in \mathbf{P}^{k-j} \), according to (21) from the proof of Lemma 4.4 the map

\[
\epsilon |_{\{(D,E_1,E_2)\}} : \mathcal{N}_{\mathcal{H},j}^* |_{\{D,E_1+E_2\}} \to \mathcal{N}_{\mathcal{H},j}^* |_{\{D,E_1\}}
\]

between the fibers of the two conormal bundles is identified with the map

\[
H^0(\mathbf{P}^1, M \otimes \mathcal{O}_{E_1+E_2}) \to H^0(\mathbf{P}^1, M \otimes \mathcal{O}_{E_1}),
\]

which is obviously surjective. The remaining assertion is clear from (16). \( \square \)

Next, we prove analogous results for Abel-Jacobi maps.

**Lemma 4.7.** Let \( C \) be a hyperelliptic curve of odd genus \( g = 2n + 1 \). For \( 0 \leq j \leq n \), consider the map \( \delta_j \) from (5). Then the following holds.

(a) \( d\delta_j : \delta_j^* T^*_j \text{Jac}(C) \to T^*_j C \) is surjective when restricted to \( U_{2j} \).

(b) The fiber of \( \mathcal{N}^*_j \) over \( D \in U_{2j} \) is \( H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_sD)) \).

Proof. We only sketch the proof, as it is similar to that of Lemma 4.4. The cotangent bundle \( T^*_j \text{Jac}(C) \) is a trivial bundle with fibers isomorphic to \( H^0(C, \omega_C) \), and the map \( d\delta_j \) over \( D \in C_{2j} \) can be identified with

\[
H^0(C, \omega_C) \to H^0(C, \omega_C \otimes \mathcal{O}_D).
\]

The cokernel of this map is the kernel of \( H^1(\omega_C(-D)) \to H^1(\omega_C) \); dually one has the natural map \( H^0(\mathcal{O}_C) \to H^0(\mathcal{O}_C(D)) \). If \( D \in U_{2j} \), then \( h^0(\mathcal{O}_C(D)) = 1 \) and the last map is an isomorphism. Therefore the map (22) is surjective and we have

\[
\mathcal{N}^*_j |_D \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_sD)),
\]

by (18), as \( h^0(\mathbf{P}^1, \omega_{\mathbf{P}^1}) = 0 \). \( \square \)

Let us record one additional useful fact. Consider the commutative diagram induced by (9) via restriction \( (i < j) \):

\[
\begin{array}{ccc}
U_{2i} \times \mathbf{P}^{j-i} & \overset{p_1}{\longrightarrow} & U_{2i} \\
\downarrow \gamma_{i,j} & & \downarrow \delta_j \\
C_{2j} & \overset{\delta_j}{\longrightarrow} & \text{Jac}(C)
\end{array}
\]

\textbf{Corollary 4.8.} Fix \( 0 \leq i < j \leq n \). For \( D \in U_{2i} \), the induced map of conormal bundles

\[
\epsilon : p^*_1 \mathcal{N}_{\mathcal{H},i}^*|_D \to \mathcal{N}_{\mathcal{H},j}^* |_{\{D\} \times \mathbf{P}^{j-i}}
\]
over \(\{D\} \times P^{j-i}\) is surjective, and the diagram

\[
\begin{array}{ccc}
P_{\gamma_{i,j}}^*|_{\{D\} \times P^{j-i}} & \xrightarrow{\beta} & \begin{array}{c}P_{\delta_i}^*|_D
\end{array} \\
\downarrow & & \downarrow \\
B^{j-i-1}(M) & \xrightarrow{\beta_j} & PH^0(P^1, M)
\end{array}
\]

commutes. Here the first vertical map is an isomorphism by Lemma 4.4(c), the second isomorphism comes from Lemma 4.7, and the map \(\beta_j\) is the map (14) for the curve \(P_1\) and the line bundle \(M = O_{P_1}(g - 1 - h_sD)\), and \(\beta\) is induced by \(\epsilon\) and the projection to \(P_{\gamma_{i,j}}^*|_D\).

Proof. Fix a point \(D \in U_{2i}\) and define \(M = O_{P_1}(g - 1 - h_sD)\). For \(E \in P^{j-i}\), by (21) and (23), the map \(\epsilon|_{(D,E)} : N_{\delta_i}^*|_D \to N_{\gamma_{i,j}}^*|_{(D,E)}\) between the fibers of the two conormal bundles is identified with the map

\[
H^0(P^1, M) \to H^0(P^1, M \otimes O_E),
\]

which is surjective by degree reasons (\(\deg M = g - 1 - 2i > j - i = \deg E\)). The remaining assertion is clear from (14). \(\square\)

5. The proof of Proposition 2.1 and Corollary B

In this section, we prove Proposition 2.1 under the following assumption.

Assumption. Proposition 2.2 and Claim 6.1-Claim 6.5 hold.

The reason is that the proofs of Proposition 2.1 and Proposition 2.2 follow the same lines, but the notation for Proposition 2.2 is more complicated, so we postpone its proof (as well as the proofs of these claims) to §6. We also prove Corollary B at the end of this section.

Let \(C\) be a smooth hyperelliptic curve of odd genus \(g = 2n + 1\). Consider the proper chain \(\{\delta_k : C_{2k} \to Jac(C)\}_{k=0}^n\) from (8) with the diagram from (9) for \(j < k\)

\[
C_{2j} \times P^{k-j} \xrightarrow{\pi_1} C_{2j} \\
\downarrow \gamma_{j,k} \quad \quad \downarrow \delta_j \\
C_{2k} \xrightarrow{\delta_k} Jac(C)
\]

Let \(k\) be an integer such that \(0 \leq k \leq n\). Denote by

\[bl_j(C_{2k}), \quad bl_j(Jac(C))\]

the spaces associated to the chain \(\{\delta_k\}_{k=0}^n\). Denote by

\[bl_i(C_{2j} \times P^{k-j})\]

the space associated to the chain \(\{\gamma_{j,k}\}_{j=0}^{k-1}\). We break the proof of Proposition 2.1 into following statements, together with some auxiliary statements.

Claim 5.1. \(\{\delta_j\}_{j=0}^k\) is a NCD chain.

Claim 5.2. Assume Claim 5.1 holds for \(k - 1\), so that \(bl_j(\delta_k)\) exists for all \(j \leq k\) and there is an embedding \(bl_j(C_{2j}) \xrightarrow{bl_j(\delta_j)} bl_j(Jac(C))\) for all \(j < k\), then

\[bl_j(\delta_k)^{-1}(bl_j(C_{2j})) = bl_j(C_{2j} \times P^{k-j}), \quad \forall j < k.\]

As a consequence, the following hold.
• $\phi_k$ is a map of chains of centers.

• For $j \leq k$, the space $\text{bl}_j(C_{2k})$ coincides with the space associated to the chain $\{\gamma_{j,k} : C_{2j} \times \mathbb{P}^{k-j} \to C_{2k}\}_{j=0}^{k-1}$, by Proposition 5.4. Therefore there is a natural isomorphism

$$(24) \quad \text{bl}_j(C_{2k} \times \mathbb{P}^{t-k}) = \text{bl}_j(C_{2k}) \times \mathbb{P}^{t-k}, \quad \forall j \leq k \leq t \leq n,$$

by (23) in Claim 5.2 (note that there $\text{bl}_j(C_{2k})$ is defined to be the space associated to the chain $\{\gamma_{j,k}\}_{j=0}^{k-1}$).

• There exists a Cartesian diagram

$$\begin{array}{ccc}
\text{bl}_j(C_{2j}) \times \mathbb{P}^{k-j} \\ \downarrow \quad \text{bl}_j(\gamma_{j,k}) \quad \downarrow \quad \text{bl}_j(\delta_k) \\ \text{bl}_j(C_{2k}) \quad \text{bl}_j(\text{Jac}(C)) \\
\end{array}$$

Claim 5.3. Assume Claim 5.1 holds for $k = j - 1$ and Claim 5.2 holds for $k' < k$ and $(i,k)$ with $i \leq j - 1$, so that the following diagram exists for $i < j \leq k$:

$$\begin{array}{cccc}
\text{bl}_j(C_{2j}) & \text{bl}_j(C_{2k}) & \text{bl}_j(\text{Jac}(C)) & \text{bl}_j(C_{2i}) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
E_{i,j} & \text{bl}_i(C_{2k}) & \text{bl}_i(\text{Jac}(C)) & \text{bl}_i(C_{2i}) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
E_{i,i+1} & \text{bl}_{i+1}(C_{2k}) & \text{bl}_{i+1}(\text{Jac}(C)) & \text{bl}_{i+1}(C_{2i}) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{bl}_i(C_{2i} \times \mathbb{P}^{k-i}) & \text{bl}_i(C_{2k}) & \text{bl}_i(\text{Jac}(C)) & \text{bl}_i(C_{2i}) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{bl}_i(C_{2i}) \times \mathbb{P}^{k-i} & \text{bl}_i(C_{2k}) & \text{bl}_i(\text{Jac}(C)) & \text{bl}_i(C_{2i}) \\
\end{array}$$

where $F_{i,j}$, $E_{i,j}$ are the exceptional divisors over $\text{bl}_i(C_{2i})$ and $\text{bl}_i(C_{2i} \times \mathbb{P}^{k-i})$ respectively, and $\text{bl}_i(C_{2i} \times \mathbb{P}^{k-i}) = \text{bl}_i(C_{2i} \times \mathbb{P}^{k-i})$ by the inductive Claim 5.2, then

$$(25) \quad \text{bl}_j(\delta_k)^{-1}(F_{i,j}) = E_{i,j}, \quad \forall i < j \leq k.$$

$$(26) \quad \text{bl}_j(C_{2j}) - \bigcup_{i<j} F_{i,j} = U_{2j}, \quad \forall j < k,$$

where $U_{2j} = C_{2j} - \gamma_{j,j-1}(C_{2j-2} \times \mathbb{P}^1)$.

Claim 5.4. With the same assumption in Claim 5.3. For $0 \leq i < j \leq k$, denote by

$$E_{i,j}^o = E_{i,j} - \bigcup_{h<i} E_{h,j}, \quad F_{i,j}^o = F_{i,j} - \bigcup_{h<i} F_{h,j}.$$

Then (25) induces a morphism of $U_{2i}$-varieties

$$\text{bl}_j(\delta_k) : E_{i,j}^o \to F_{i,j}^o,$$

such that over $D \in U_{2i}$ the morphism is

$$\text{bl}_{j-1}(\beta_{k-1}) : \text{bl}_{j-1} B^{k-i-1}(M) \to \text{bl}_{j-1} \mathbb{P}^H(M),$$

the map associated to the chain $\{\beta_t\}_{t=0}^{k-i-1}$ in (13) for the line bundle $M = \mathcal{O}_{\mathbb{P}^1}(g-1-hsD)$ (c.f. Notation 4.3) over $\mathbb{P}^1$. 

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In particular, assume Claim 5.1 holds for \( k \), so that there is an embedding \( \bl_k(\delta_k) : \bl_k(C_{2k}) \hookrightarrow \bl_k(\Jac(C)) \),  then  \[
(27) \quad \bl_k(C_{2k}) \cap F_{i,k}^0 = \bl_k(\delta_k)^{-1}(F_{i,k}^0) \text{ is a } U_{2i}-\text{variety,}
\]
with fiber over \( D \) being \( \bl_{k-i-1}B^{k-i-1}(M) \).

**Claim 5.5.** With the same assumption and notation in Claim 5.4, then

\[
(*) \quad \bl_j(\delta_k)^{-1}(\bl_j(C_{2j}) - \bigcup_{i<j}F_{i,j}) = \bl_j(C_{2j} \times \mathbb{P}^{k-j}) - \bigcup_{i<j}E_{i,j}, \quad \forall j < k;
\]

\[
(**) \quad \bl_j(\delta_k)^{-1}(\bl_j(C_{2j}) \cap F_{i,j}^0) = \bl_j(C_{2j} \times \mathbb{P}^{k-j}) \cap E_{i,j}^0, \quad \forall i < j < k;
\]

\[
(***) \quad \bl_k(\delta_k) : E_{i,k}^0 \rightarrow F_{i,k}^0 \text{ is an embedding, } \forall i < k.
\]

We prove these claims by induction on \( k \). The base case is \( k = 0 \). Claim 5.1 holds as \( \delta_0 : C_0 \hookrightarrow \Jac(C) \) is an embedding. There is nothing to check for other claims.

To make the inductive proof more readable, here is a summary:

**Step 1:** Inductive Claim 5.1, 5.2, Claim 5.2 for \((i, k), i \leq j - 1 \implies \text{Claim 5.3 for } k\).

**Step 2:** Inductive Claim 5.1, 5.4 + Claim 5.3 for \( k \implies \text{Claim 5.4 for } k\).

**Step 3:** Claim 5.4 for \( k \implies \text{Claim 5.5 for } k\).

**Step 4:** Claim 5.3-5.5 for \( k + \text{Claim 5.2 for } (i, k), i \leq j - 1 \implies \text{Claim 5.2 for } (j, k)\).

**Step 5:** Inductive Claim 5.1 + Claim 5.2 for \( k \implies \text{Claim 5.1 for } k\).

Assume Claim 5.1, Claim 5.5 hold for all \( k' < k \). The main part of the proof is Claim 5.2 for \( k \). We do this by induction on \( j \) and fixing \( k \). The base case \( j = 0 \) is clear, as Abel-Jacobi theorem gives

\[
\bl_0(\delta_k^{-1})(C_0) = \delta_k^{-1}(C_0) = C_0 \times \mathbb{P}^k.
\]

Assume Claim 5.2 holds for \((i, k)\) with \( i \leq j - 1 \) for some \( j \), we want to show Claim 5.2 holds for \((j, k)\). To do this, we prove Claim 5.3, 5.5 for \( k \).

**Step 1** Claim 5.3 for \( k \). By the inductive Claim 5.1 and Claim 5.2, the assumptions in Claim 5.3 are satisfied, so one has

\[
\bl_i(\delta_k)^{-1}(\bl_i(C_{2i})) = \bl_i(C_{2i} \times \mathbb{P}^{k-i}), \quad \forall i \leq j - 1.
\]

Moreover, we know the varieties \( \bl_i(C_{2i}), \bl_i(C_{2i} \times \mathbb{P}^{k-i}) \) are smooth for \( i < j \), as \( \{\delta_i\}_{i=0}^{k-1}, \{\gamma_{i,k}\}_{i=0}^{k-1} \) are NCD chains (the smoothness of \( \bl_i(C_{2i} \times \mathbb{P}^{k-i}) \) also follows from the inductive \( (22) \)). Now we can apply Lemma 1.16 to conclude that

\[
\bl_{i+1}(\delta_k)^{-1}(F_{i,i+1}) = E_{i,i+1}, \quad \forall i \leq j - 1.
\]

Then (25) hold for \( k \) by doing an induction on \( j \), the number of blowups. To prove (26) for \( k \), we can understand the intersection \( \bl_j(C_{2j}) \cap F_{i,j} \) via the following diagram:

\[
\begin{array}{ccc}
E'_{i,j} & \rightarrow & \bl_j(C_{2j}) \rightarrow \bl_j(\Jac(C)) \leftarrow F_{i,j} \\
\downarrow & & \downarrow & & \downarrow \\
\bl_i(C_{2i} \times \mathbb{P}^{j-i}) & \rightarrow & \bl_i(C_{2i}) \rightarrow \bl_i(\Jac(C)) & \rightarrow & \bl_i(C_{2i}),
\end{array}
\]

where \( E'_{i,j} \) is the exceptional divisor over \( \bl_i(C_{2i} \times \mathbb{P}^{j-i}) \); this diagram exists because \( \{\delta_i\}_{i=0}^{j-1} \) and \( \{\gamma_{i,j}\}_{i=0}^{j-1} \) are NCD chains by the inductive Claim 5.1 and Proposition 2.2 respectively, and the inductive Claim 5.2 gives

\[
\bl_i(\delta_j)^{-1}(\bl_i(C_{2i})) = \bl_i(C_{2i} \times \mathbb{P}^{j-i}), \quad \forall i < j,
\]
which implies that \( \text{bl}_j(C_{2j}) \) coincides with the blow-up space associated to the chain \( \{\gamma_{i,j}\}_{i=0}^{j-1} \). Using the smoothness of \( \text{bl}_i(C_{2i}) \) and \( \text{bl}_i(C_{2i} \times \mathbb{P}^{j-i}) \) for all \( i < j \), it follows again from Lemma 1.16 that

\[
\text{bl}_j(C_{2j}) \cap F_{i,j} = \text{bl}_j((\delta_j)^{-1}(F_{i,j})) = E'_{i,j}.
\]

By construction, \( \{E'_{i,j}\}_{i=0}^{j-1} \) is the set of exceptional divisors associated to the NCD chain \( \{\gamma_{i,j}\}_{i=0}^{j-1} \), thus it follows from Remark 1.7 that

\[
\text{bl}_j(C_{2j}) - \bigcup_{i < j} F_{i,j} = \text{bl}_j(C_{2j}) - \bigcup_{i < j} E'_{i,j} = C_{2j} - \gamma_{j-1,j}(C_{2j} - \times \mathbb{P}^1) = U_{2j}.
\]

This finishes the proof of Claim 5.3 for \( F \).

**Step 2:** let us turn to Claim 5.4, which is the most crucial one. Fix \( j \) and \( k \), we prove Claim 5.4 for \( j \) by induction on \( j \) (with \( i + 1 \leq j < k \)). The base case is \( j = i + 1 \).

By construction, \( F_{i,i+1} \) and \( F_{h,i+1} \) (\( h < i \)) are the exceptional divisors over \( \text{bl}_i(C_{2i}) \) and \( F_{h,i} \), respectively, associated to the blow up of \( \text{bl}_i(\text{Jac}(C)) \) along \( \text{bl}_i(C_{2i}) \). To understand \( F_{i,i+1}^\circ = F_{i,i+1} - \bigcup_{h < i} F_{h,i+1} \), we want to know how \( F_{h,i} \) intersects \( \text{bl}_i(C_{2i}) \). By the inductive Claim 5.1 the chain \( \{\delta_h\}_{h=0}^{k-1} \) is NCD, and hence we have transverse intersections (as \( i < k \))

\[
\text{bl}_i(C_{2i}) \cap F_{h,i} \subseteq \text{bl}_i(\text{Jac}(C)), \quad \forall h < i.
\]

Therefore we conclude that

\[
F_{i,i+1}^\circ = F_{i,i+1} - \bigcup_{h < i} F_{h,i+1}
\]

is the exceptional divisor for the blow up of \( \text{bl}_i(\text{Jac}(C)) \) along

\[
\text{bl}_i(C_{2i}) - \bigcup_{h < i} F_{h,i} = U_{2i},
\]

where the equality follows from (26) for \( k \). Since \( U_{2i} \) is away from all the exceptional divisors \( F_{h,i} \) associated to the previous blowups of \( \text{Jac}(C) \), \( F_{i,i+1}^\circ \) can also be identified with the exceptional divisor for the blow up of \( \text{Jac}(C) \) along \( U_{2i} \) (identified with the image \( \delta_i(U_{2i}) \)). Let \( D \in U_{2i} \) and denote by \( M := \mathcal{O}_{\mathbb{P}^2}(g - 1 - h_s D) \). It follows from Lemma 1.7(b) that \( F_{i,i+1}^\circ \) is a \( U_{2r} \)-variety with fiber over \( D \) being \( \mathbb{P}^H(\mathbb{H}(M)) \). On the other hand, Claim 6.4 says that \( E_{i,i+1}^\circ \) is a \( U_{2r} \)-variety with fiber over \( D \) being \( \beta_{k-i-1} : B^{k-i-1}(M) \rightarrow \mathbb{P}^H(\mathbb{H}(M)) \).

We conclude the base case \( j = i + 1 \).

Assume Claim 5.4 holds for all \( j' < j \). Let us look at the diagram

\[
\begin{array}{cccccc}
E_{i,j}^\circ & \longrightarrow & \text{bl}_j(C_{2j}) & \text{bl}_j(\text{Jac}(C)) & \text{bl}_j(C_{2j}) & \longrightarrow & F_{i,j}^\circ \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_{i,j-1}^\circ & \longrightarrow & \text{bl}_{j-1}(C_{2j}) & \text{bl}_{j-1}(\text{Jac}(C)) & \text{bl}_{j-1}(C_{2j}) & \longrightarrow & F_{i,j-1}^\circ \\
\text{bl}_{j-1}(C_{2j-2} \times \mathbb{P}^{j-1}) & \leftrightarrow & \text{bl}_{j-1}(\text{Jac}(C)) & \leftrightarrow & \text{bl}_{j-1}(C_{2j-2}) & & \\
\end{array}
\]

As \( j - 1 \leq k - 1 \), the inductive Claim 5.4 implies that the chain \( \{\delta_i\}_{i=0}^{j-1} \) is NCD. This means that there is an embedding

\[
\text{bl}_{j-1}(\delta_{j-1}) : \text{bl}_{j-1}(C_{2j-2}) \hookrightarrow \text{bl}_{j-1}(\text{Jac}(C)),
\]
and the intersection \( \text{bl}_{j-1}(C_{2j-2}) \cap F_{i,j-1}^0 \) is transverse. Consequently, \( F_{i,j}^0 \) is the blowup of \( F_{i,j-1}^0 \) along \( \text{bl}_{j-1}(C_{2j-2}) \cap F_{i,j-1}^0 \), which can be analyzed as a \( U_{2i} \)-variety of blowups. Let \( D \in U_{2i} \) and set \( M = \mathcal{O}_{\mathbb{P}^1}(g-1-h_D) \), which has degree
\[
g - 1 - 2i = 2n - 2i \geq 2(k - i - 1) + 1
\]
(recall that \( g(C) = 2n + 1 \) and \( k \leq n \)). This allows us to apply Corollary 3.4 to the chain \( \{ \beta_j \}_{j=0}^{k-1} \) in the following argument. We know the following hold:

- \( F_{i,j}^0 \) is a \( U_{2i} \)-variety with fiber over \( D \) being \( \text{bl}_{j-1-2} \mathbb{P} H^0(M) \), by Claim 5.4 for \( j' < j \).
- \( \text{bl}_{j-1}(C_{2j-2}) \cap F_{i,j-1}^0 \) is a \( U_{2i} \)-variety with fiber over \( D \) being \( \text{bl}_{j-1}(\beta_{i-1}) B^{j-1-i-1}(M) \), by (27) in the inductive Claim 5.4.

Then it follows from Lemma 1.22(3) that \( F_{i,j}^0 \) is a \( U_{2i} \)-variety with fiber over \( D \) being the blow up of \( \text{bl}_{j-1-2} \mathbb{P} H^0(M) \) along \( \text{bl}_{j-1-2} B^{j-1-i-2}(M) \), which by definition is \( \text{bl}_{j-1-1} \mathbb{P} H^0(M) \).

A similar argument using Claim 5.4 says that \( E_{i,j}^0 \subseteq \text{bl}_j(C_{2j}) \) is a \( U_{2i} \)-variety with fiber over \( D \) being \( \text{bl}_{j-1-2} B^{j-i-1}(M) \), the blow up of \( \text{bl}_{j-1-2} B^{j-i-1}(M) \) along \( \text{bl}_{j-1-2} B^{i-1-2}(M) \times \mathbb{P}^{k-j+1} \). Since \( \text{bl}_{j-1}(\beta_k) \) over \( D \in U_{2i} \) is \( \text{bl}_{j-1}(\beta_{k-i-1}) \) (by inductive assumption) and by Corollary 3.4(b), \( \beta_{k-i-1} \) is a map of chains of centers, i.e.
\[
\text{bl}_{j-2}(\beta_{k-i-1})^{-1}(\text{bl}_{j-2} B^{j-i-2}(M)) = \text{bl}_{j-2}(B^{j-i-2}(M) \times \mathbb{P}^{k-j+1}).
\]

We conclude from Lemma 1.22(2) that \( \text{bl}_j(\delta_k) \) is a \( U_{2i} \)-morphism and the fiber over \( D \) is
\[
\text{bl}_{j-1-1}(\beta_{k-i-1}) : \text{bl}_{j-1} B^{j-i-1}(M) \to \text{bl}_{j-1} \mathbb{P} H^0(M).
\]

This finishes the inductive proof on \( j \) of Claim 5.4 for \( k \).

**Step 3:** Claim 5.5 for \( k \). For \((*)\), we know from the discussion in Step 2 that \( U_{2j} \) is not touched by all the blowups associated to \( \text{bl}_j(\delta_k) \), thus
\[
\text{bl}_j(\delta_k)^{-1}(U_{2j}) = \delta_k^{-1}(U_{2j}) = U_{2j} \times \mathbb{P}^{k-j},
\]
where the last equality comes from Lemma 1.2(a). The (32) in Claim 6.3 implies that
\[
\text{bl}_j(C_{2j} \times \mathbb{P}^{k-j}) \cap \bigcup_{i<j} E_{i,j}^0 = U_{2j} \times \mathbb{P}^{k-j}.
\]

Putting these two equations together with (26) for \( k \), we obtain \((*)\) for \( k \).

Let us turn to \((***)\). By Claim 5.3 for \( k' \geq k \) and Claim 6.4 the diagram
\[
\begin{array}{ccc}
\text{bl}_j(C_{2j} \times \mathbb{P}^{k-j}) \cap E_{i,j}^0 & \longrightarrow & \text{bl}_j(C_{2j}) \cap F_{i,j}^0 \\
\downarrow & & \downarrow \\
E_{i,j}^0 & \longrightarrow & F_{i,j}^0
\end{array}
\]
is a diagram of \( U_{2i} \)-varieties, whose fiber over \( D \in U_{2i} \) is
\[
\begin{align*}
\text{bl}_{j-i-1}(B^{j-i-1}(M) \times \mathbb{P}^{k-j}) & \longrightarrow \text{bl}_{j-i-1} B^{j-i-1}(M) \\
\text{bl}_{j-i-1}(\beta_{j-i-1}) & \longrightarrow \text{bl}_{j-i-1}(\beta_{j-i-1}) \\
\text{bl}_{j-i-1} B^{k-i-1}(M) & \longrightarrow \text{bl}_{j-i-1} \mathbb{P} H^0(M)
\end{align*}
\]
Here \( M \) denotes the line bundle \( \mathcal{O}_{\mathbb{P}^1}(g-1-h_D) \). By Corollary 3.4(b), the above diagram is Cartesian, i.e.
\[
\text{bl}_{j-i-1}(\beta_{j-i-1})^{-1}(\text{bl}_{j-i-1} B^{j-i-1}(M)) = \text{bl}_{j-i-1}(B^{j-i-1}(M) \times \mathbb{P}^{k-j}).
\]
As (bl_{\gamma_j}(C_{2j}) \times \mathbb{P}^{k-j}) \cap E_{i,j}^0 and bl_{\gamma_j}(C_{2j}) \cap F_{i,j}^0 are both smooth, we can apply Lemma [1.22(2)] to the U_{2i}-morphism $\text{bl}_{\gamma_j}(\delta_k) : E_{i,j}^0 \to F_{i,j}^0$ to conclude that the previous diagram is also Cartesian, i.e. (***) hold for $k$.

Last, for (***) Claim 5.4 for $k$ implies that the fiber of the map

$$\text{bl}_k(\delta_k) : E_{i,k}^0 \to F_{i,k}^0$$

over $D \in U_{2i}$ is $\text{bl}_{k-i-1}(\beta_{k-i-1})$, which is an embedding by Corollary [3.3(b)]. As $E_{i,k}^0, F_{i,k}^0$ are both smooth, we conclude from Lemma [1.22(1)] that (***) holds for $k$.

**Step 4:** Claim [5.2] for $(j, k)$. We apply Lemma [1.20] to the chain $\{\delta_j^k\}_{j=0}^k$ with

$$X = \text{Jac}(C), \quad X_k = C_{2k}, \quad X_{j,k} = C_{2j} \times \mathbb{P}^{k-j}, \quad f_{j,k} = \gamma_{j,k}, \quad \phi_k = \delta_k.$$

The assumptions in Lemma [1.20] can be checked as follows:

(I) The chain $\{\gamma_{j,k}\}_{j=0}^{k-1}$ is NCD, as we assume Proposition [2.2] holds.

(II) The map $\delta_k : C_{2k} - \gamma_{k-1,k}(C_{2k-2} \times \mathbb{P}^1) \to \text{Jac}(C)$ is an embedding, by Lemma [4.2(a)] and Notation [4.1].

(III) The chain $\{\delta_j^k\}_{j=0}^k$ is NCD, by the inductive Claim 5.1.

(IV) We have $\text{bl}_j(\delta_k) = \text{bl}_j(C_{2l} \times \mathbb{P}^{k-i})$ for all $i \leq j - 1$, because we assume Claim [5.2] holds for $(i, k), i \leq j - 1$. Moreover, the conditions (**), (**), (***) are satisfied because Claim [5.3] holds for $k$.

As a consequence, we conclude that

$$\text{bl}_j(\delta_k) = \text{bl}_j(C_{2j}),$$

and $\{\delta_j^k\}_{j=0}^{k-1}$ is a chain of smooth centers. Therefore, Claim [5.2] holds for $(j, k)$ and this finishes the inductive proof on $j$ of Claim [5.2] for $k$.

**Step 5:** Claim [5.1] for $k$. We apply Lemma [1.11] to the chain $\{\delta_j^k\}_{j=0}^k$ and the chains $\{\gamma_{i,j}\}_{i=0}^{j-1}$ for $j \leq k$. At this point, the following hold:

1. For each $j \leq k$, the chain $\{\gamma_{i,j}\}_{i=0}^{j-1}$ is NCD, by Proposition [2.2].
2. The proper chain $\{\delta_{j,k}\}_{j=0}^{k}$ is a chain of smooth centers, by Step 4.
3. For each $j \leq k$, $\phi_j$ is a map of chains of centers, by Step 4 and inductive Claim [5.2] for $k' < k$.

Moreover, $\text{Jac}(C), C_{2k}, C_{2i} \times \mathbb{P}^{j-i}$ for all $0 \leq i < j \leq k$ are evidently smooth projective varieties. Consequently, $\{\delta_j^k\}_{j=0}^k$ is an NCD chain. This proves Claim [5.1] for $k$.

Therefore, we finish the inductive proof for all the claims above and also the proof of Proposition [2.1].

**Proof of Corollary [2]** For $0 \leq i \leq n - 1$, the exceptional divisor $Z_i \subseteq \text{bl}_{n+1}(\text{Jac}(C))$ defined in the introduction is the exceptional divisor $F_{i,n+1} \subseteq \text{bl}_{n+1}(\text{Jac}(C))$, defined as the preimage of $F_{i,n} \subseteq \text{bl}_n(\text{Jac}(C))$, in the notation of this section. The proper transform $\tilde{\Theta}$ is $F_{n,n+1}$, the exceptional divisor for the blow up of $\text{bl}_n(\text{Jac}(C))$ along $\text{bl}_n(C_{2n})$. Note that there is a natural identification

$$W_{g-1}^{n-i}(C) - W_{g-1}^{n-i+1}(C) \cong U_{2i}$$

induced by $\delta_i : C_{2i} \to W_{g-1}^{n-i}(C) \subseteq \text{Jac}(C)$ (c.f. the discussion after (8)), because $\delta_i$ is an isomorphism over $U_{2i}$ and $\delta_i(U_{2i})$ consists of line bundles $L$ with $\deg L = g - 1$ and $h^0(C, L) = n - i + 1$ (which follows from the hyperelliptic property of $C$, see the proof of Lemma [4.2]).
Let $D \in U_2$ and set $M := \mathcal{O}_{D}(g - 1 - h_{s}D)$ (c.f. Notation 4.3). Claim 5.4 implies that the fiber of the projection

$$F_{i,n}^{\circ} = F_{i,n} - \bigcup_{h < i} F_{h,n} \to W_{g - 1}^{n - i}(C) - W_{g - 1}^{n - i + 1}(C) = U_2$$

over $D$ is $\text{bl}_{n-i} \mathbb{P}H^{0}(\mathbb{P}^{1}, M)$, the blowup space associated to the chain

$$\{ \beta_{k-i-1} : B^{k-i-1}(M) \to \mathbb{P}H^{0}(M) \}_{k+i+1}.$$

Blowing up one more time, we can argue as in the proof of Claim 5.4 (Step 2 of the proof above) that the fiber of $F_{i,n+1}^{\circ} = F_{i,n+1} - \bigcup_{h < i} F_{h,n+1}$ over $D$ is $\text{bl}_{n-i} \mathbb{P}H^{0}(\mathbb{P}^{1}, M)$. Let $k$ be an integer such $i + 1 \leq k \leq n$. \text{Claim 5.4} says that

$$F_{i,k}^{\circ} \cap \text{bl}_{k}(C_{i,n}) \subseteq \text{bl}_{k}(\text{Jac}(C)).$$

is a $U_{2r}$-variety with fiber over $D$ is $\text{bl}_{k-i-1} B^{k-i-1}(M)$. Then in the final blow up space we have

$$F_{i,n+1}^{\circ} \cap F_{k,n+1} \subseteq \text{bl}_{n+1}(\text{Jac}(C))$$

is a $U_{2r}$-variety with fiber over $D$ is the exceptional divisor

$$H_{k-i-1} \subseteq \text{bl}_{n-i} \mathbb{P}H^{0}(\mathbb{P}^{1}, M)$$

over $\text{bl}_{k-i-1} B^{k-i-1}(M)$, associated to the chain $\{ \beta_{k-i-1} \}_{k+i+1}^{n}$. As $\{ \beta_{k-i-1} \}_{k+i+1}^{n}$ is a smooth chain by Corollary 3.4(b), Remark 1.7 implies that the fiber of

$$Z_{i} = \bigcup_{0 \leq j \leq n-1, j \neq i} Z_{j} \to \hat{\Theta} = F_{i,n+1}^{\circ} - \bigcup_{i+1 \leq k \leq n} F_{k,n+1}$$

over $D$ is

$$\text{bl}_{n-i} \mathbb{P}H^{0}(\mathbb{P}^{1}, M) - \bigcup_{i+1 \leq k \leq n} H_{k-i-1} \cong \mathbb{P}H^{0}(\mathbb{P}^{1}, M) - \beta_{n-i-1}(B^{n-i-1}(M)).$$

By definition, the image $\beta_{n-i-1}(B^{n-i-1}(M))$ is the $(n - i - 1)$-th secant variety

$$\text{Sec}^{n-i-1}(\mathbb{P}^{1}) \subseteq \mathbb{P}H^{0}(\mathbb{P}^{1}, M) = \mathbb{P}^{2(n-i)},$$

where $\mathbb{P}^{1}$ is the rational normal curve $\beta_0(B^{0}(M))$. Note that the secant variety $\text{Sec}^{n-i-1}(\mathbb{P}^{1})$ has degree $(n - i) + 1$ and the rational curve has degree $2(n - i)$. Applying $i = n - r$, we obtain Corollary 5.7. \hfill \Box

6. The proof of Proposition 2.2

In this section, we prove Proposition 2.2 whose proof follows the same line as Proposition 2.1. Therefore, some details will be omitted. The essential difference is, besides the already existing induction, we need to use an extra induction on $t$. The inductive hypothesis on $t$ plays the role of Proposition 2.2 in the proof of Proposition 2.1.

Let $C$ be a smooth hyperelliptic curve of genus $g = 2n + 1$. Let $t$ be an integer with $t \leq n$, consider the proper chain $\{ \gamma_{k,t} : C_{2k} \times \mathbb{P}^{t-k} \to C_{2t} \}_{k=0}^{\lceil \frac{t}{2} \rceil}$ from (1) with the diagram (j < k < t) from (12)

$$C_{2j} \times \mathbb{P}^{k-j} \times \mathbb{P}^{t-k} \xrightarrow{id \times \gamma} C_{2j} \times \mathbb{P}^{t-j}$$

Denote by

$$\text{bl}_{j}(C_{2k} \times \mathbb{P}^{t-k}), \quad \text{bl}_{j}(C_{2t})$$

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the spaces associated to the chain \( \{ \gamma_{k,t} \}_{k=0}^{t-1} \) and denote by
\[
\text{bl}_i(C_2) \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}
\]
the space associated to the chain \( \{ \gamma_{j,k} \times \text{id} \}_{j=0}^{k-1} \).

Let \( k \) be an integer with \( 0 \leq k < t \). We break the proof of Proposition 2.2 into the following claims.

**Claim 6.1.** The chain \( \{ \gamma_{j,k} \}_{j=0}^{k} \) is a NCD chain.

**Claim 6.2.** Assume Claim 6.1 holds for \( k = 1 \), then
\[
\text{bl}_j(\gamma_{k,t})^{-1}(\text{bl}_j(C_2) \times \mathbf{P}^{t-j})) = \text{bl}_j(C_2) \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}) , \quad \forall j < k < t .
\]

As a consequence, for \( j \leq k \), \( \text{bl}_j(C_{2k} \times \mathbf{P}^{t-k}) \) coincides with the space associated to the chain \( \{ \gamma_{j,k} \times \text{id} \}_{j=0}^{k-1} \), \( \gamma_{j,k} \) is a map of chains of centers, and by Lemma 1.9 there are natural isomorphisms
\[
\text{bl}_j(C_{2k} \times \mathbf{P}^{t-k}) = \text{bl}_j(C_{2k}) \times \mathbf{P}^{t-k}, \quad j < k ,
\]
\[
\text{bl}_j(C_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}) = \text{bl}_j(C_{2j}) \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}, \quad i < j < k .
\]

**Claim 6.3.** Assume the following diagram exists for \( i < j < k < t \)
\[
\begin{array}{ccc}
\text{bl}_j(C_{2j}) \times \mathbf{P}^{t-j} & \xrightarrow{\text{bl}_j(\gamma_{k,t})} & \text{bl}_j(C_2) \\
\downarrow & & \downarrow \\
\text{bl}_j(C_{2k} \times \mathbf{P}^{k-i} \times \mathbf{P}^{t-k}) & \xrightarrow{\text{bl}_j(\gamma_{k,t})} & \text{bl}_i(C_2) \\
\end{array}
\]

where \( G_{i,j}, E_{i,j} \) are the exceptional divisors over \( \text{bl}_j(C_{2j} \times \mathbf{P}^{k-i} \times \mathbf{P}^{t-k}) \) and \( \text{bl}_i(C_{2i} \times \mathbf{P}^{t-i}) \), respectively, then
\[
\text{bl}_j(\gamma_{k,t})^{-1}(E_{i,j}) = G_{i,j}, \quad \forall j \leq k ,
\]
\[
\text{bl}_j(C_{2j} \times \mathbf{P}^{t-j}) = \bigcup_{i < j} E_{i,j} = U_{2j} \times \mathbf{P}^{t-j}, \quad \forall j < k .
\]

**Claim 6.4.** With the assumption in Claim 6.3 Denote by \( E_{i,j}^o = E_{i,j} - \cup_{h < i} E_{h,j} \); the same for \( G_{i,j}^o \). Then the map induced from (31)
\[
\text{bl}_j(\gamma_{k,t}) : G_{i,j}^o \to E_{i,j}^o
\]
is a morphism of \( U_{2i} \)-varieties, whose fiber over \( D \in U_{2i} \) is \( \text{bl}_{j-i-1} (\alpha_{k-i-1,t-i-1}) \), where
\[
\alpha_{k-i-1,t-i-1} : B^{k-i-1} \times \mathbf{P}^{t-k} \to B^{t-i-1}(M)
\]
is the map from (10) for the line bundle \( M = \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_s D) \) (see Notation 4.3) and \( \text{bl}_{j-i-1}(\alpha_{k-i-1,t-i-1}) \) is the map associated to the chain \( \{ \alpha_{t-i-1} \}_{t=0}^{k-i-1} \).

As a consequence, assume Claim 6.7 holds for \( k \), so \( \text{bl}_k(\gamma_{k,t}) \) is an embedding, then
\[
\text{bl}_k(C_{2k} \times \mathbf{P}^{t-k}) \cap E_{i,k}^o = \text{bl}_k(\gamma_{k,t})^{-1}(E_{i,k}^o)
\]
is a variety over \( U_{2i} \) with fiber over \( D \) being \( \text{bl}_{k-i-1}(B^{k-i-1}(M) \times \mathbf{P}^{t-k}) \).
Claim 6.5. With the same assumption and notation in Claim 6.4, then
\begin{align*}
(*) & \quad \text{bl}_j(\gamma_{k,t}^{-1})(\text{bl}_j(C_2 j \times \mathbb{P}^{t-j}) - \bigcup_{i < j} E_{i,j}) = \text{bl}_j(C_2 j \times \mathbb{P}^{k-j} \times \mathbb{P}^{t-k}) - \bigcup_{i < j} G_{i,j}, \quad \forall j < k, \\
(**) & \quad \text{bl}_j(\gamma_{k,t}^{-1})(\text{bl}_j(C_2 j \times \mathbb{P}^{t-j}) \cap E_{i,j}^o) = \text{bl}_j(C_2 j \times \mathbb{P}^{k-j} \times \mathbb{P}^{t-k}) \cap G_{i,j}^o, \quad \forall i < j < k, \\
(***) & \quad \text{bl}_k(\gamma_{k,t}) : G_{i,k}^o \to E_{i,k}^o \text{ is an embedding for all } i < k.
\end{align*}

We prove the claims above by induction on $t$. The base case $t = 1$ for Claim 6.1 follows from the fact that $\gamma_{0,1} : \mathbb{P}^1 \hookrightarrow C_2$ is an embedding of smooth varieties. There is nothing to check for the other claims. Then we work with the following

Assumption. Claim 6.7 - Claim 6.3 hold for all $t' < t$, for some $t$.

To prove these claims for $t$, we fix $t$ and do an extra induction on $k$ with $k < t$. The base case $k = 0$ for Claim 6.1 follows from the fact that $\gamma_{0,t} : C_0 \times \mathbb{P}^t \hookrightarrow C_2 t$ is an embedding of smooth varieties. Nothing needs to be done about the other claims. Assume Claim 6.1 - Claim 6.5 hold for all $k'$ with $k' < k < t$ for some $k$.

The proof for $k$ is divided into 5 steps. The key point is to prove Claim 6.2 for $k$, which we prove by doing an additional induction on $j$ with $j < k$. The base case $j = 0$ follows from Lemma 4.2(c), which says that

$$
\gamma_{k,t}^{-1}(C_0 \times \mathbb{P}^t) = C_0 \times \mathbb{P}^k \times \mathbb{P}^{t-k}.
$$

Assume Claim 6.2 holds for all $(j', k)$, with $j' < j < k$, for some $j$. Our goal is to prove Claim 6.3 for $(j, k)$. To achieve this, we prove Claim 6.3 for $k$, and Claim 6.4 Claim 6.5 for $(j, k)$.

Step 1: Claim 6.3 for $k$. By inductive Claim 6.1 and Claim 6.2, the assumption of Claim 6.3 is satisfied and $\text{bl}_i(C_2 i \times \mathbb{P}^{t-i}), \text{bl}_i(C_2 i \times \mathbb{P}^{k-i} \times \mathbb{P}^{t-k})$ are all smooth. Using the inductive (28) in Claim 6.2, we can apply Lemma 1.16 to get (31) for $k$.

To prove (32), consider the following diagram for any $(i, j)$ with $i < j < k$

\[
\begin{array}{ccc}
G_{i,j}^t & \hookrightarrow & \text{bl}_j(C_2 j \times \mathbb{P}^{t-j}) \\
\downarrow & & \downarrow \\
\text{bl}_i(C_2 i \times \mathbb{P}^{j-i} \times \mathbb{P}^{t-j}) & \hookrightarrow & \text{bl}_i(C_2 i \times \mathbb{P}^{t-i}) \\
\end{array}
\]

Here $G_{i,j}^t$ is the exceptional divisor over $\text{bl}_i(C_2 i \times \mathbb{P}^{j-i} \times \mathbb{P}^{t-j})$. This diagram exists because $\{\gamma_{i,t}\}_{i=0}^{j-1}$ and $\{\gamma_{i,j} \times \text{id}\}_{i=0}^{j-1}$ are NCD chains by the inductive Claim 6.1 and Lemma 1.9 also the inductive hypothesis that $\text{bl}_i(C_2 i \times \mathbb{P}^{t-j})$ coincides with the space associated to the chain $\{\gamma_{i,j} \times \text{id}\}_{i=0}^{j-1}$ for all $i \leq j$. Using the inductive (28), Lemma 1.16 implies that

$$
\text{bl}_i(C_2 j \times \mathbb{P}^{t-j}) \cap E_{i,j} = \text{bl}_i(\gamma_{j,t}^{-1})(E_{i,j}) = G_{i,j}^t.
$$

Since $\{G_{i,j}^t\}_{i=0}^{j-1}$ is the set of exceptional divisors associated to the NCD chain $\{\gamma_{i,j} \times \text{id}\}_{i=0}^{j-1}$, using Remark 4.7 one has

\[
\text{bl}_j(C_2 j \times \mathbb{P}^{t-j}) - \bigcup_{i < j} E_{i,j} = \text{bl}_j(C_2 j \times \mathbb{P}^{t-j}) - \bigcup_{i < j} G_{i,j}^t &= C_2 j \times \mathbb{P}^{t-j} - (\gamma_{j-1,j} \times \text{id})(C_2 j \times \mathbb{P}^{t-j}) \\
&= U_{2j} \times \mathbb{P}^{t-j}.
\]

Therefore (32) holds for $k$, and we have completed the proof of Claim 6.3 for $k$.

Step 2: Claim 6.4 for $(j, k)$. We fix $i$ and induct on $\ell$ with $i + 1 \leq \ell \leq j$. The reason is that, in this range, $G_{i,j}$ can be understood using the inductive hypothesis. Consider the
following diagram associated to the chain \( \{ \gamma_{i,k} \}_{i=0}^{k-1} \) (which exists by the inductive Claim 6.4):

\[
E^k_{i,j} \xrightarrow\bl_i(C_{2i}) \xrightarrow{\bl_i(C_{2i} \times \mathbb{P}^{k-i})} \xrightarrow{bl_i(\gamma_{i,k})} \bl_i(C_{2k})
\]

Here \( E^k_{i,j} \) is the exceptional divisor over \( \bl_i(C_{2i} \times \mathbb{P}^{k-i}) \). Since Claim 6.2 holds for \((i, k)\) with \(i \leq j - 1\), we know that \( \bl_i(C_{2i} \times \mathbb{P}^{t-k}) \) coincides with the space associated to the chain \( \{ \gamma_{j,k} \times \text{id} \}_{j=0}^{k-1} \) for \(i \leq j\), thus Lemma 1.9 imply

\[
\bl_i(C_{2k} \times \mathbb{P}^{t-k}) = \bl_i(C_{2k}) \times \mathbb{P}^{t-k}, \quad \forall i \leq j.
\]

As \( G_{i,j} \subseteq \bl_i(C_{2k} \times \mathbb{P}^{t-k}) \) are exceptional divisors associated to the chain \( \{ \gamma_{j,k} \times \text{id} \}_{j=0}^{k-1} \), using Lemma 1.9 again, we have

(33) \[
G_{i,j} = E^k_{i,j} \times \mathbb{P}^{t-k}, \quad \forall i \leq j.
\]

Now we start the inductive proof of Claim 6.4 for \((j, k)\). For \(D \in U_{2i}\), we denote by

\[
M := \mathcal{O}_{\mathbb{P}^1}(g - 1 - h_s D).
\]

The base case is \(j = i + 1\). As in the proof of Claim 5.4 because the chain \( \{ \gamma_{\ell,\ell} \}_{\ell=0}^{k-1} \) is NCD and \(i \leq k - 1\), one can show that \( E^0_{i,i+1} \) is the exceptional divisor over

\[
\bl_i(C_{2i} \times \mathbb{P}^{t-i}) - \bigcup_{\ell < i} E^i_{\ell,\ell} \cup U_{2i} \times \mathbb{P}^{t-i} \subseteq \bl_i(C_{2i}),
\]

thus it is also the exceptional divisor for blowing up \( C_{2i} \) along \( U_{2i} \times \mathbb{P}^{t-i} \). By Lemma 4.4(c), we conclude that \( E^0_{i,i+1} \) is a \( U_{2i}\)-variety with fiber over \( D \) being \( B^{t-i-1}(M) \).

For \( G^0_{i,i+1} \), it follows from the inductive Claim 6.3 and (33) that \( G^0_{i,i+1} \) is a \( U_{2i}\)-variety with fiber over \( D \) being \( B^{k-i-1}(M) \times \mathbb{P}^{t-k} \). Moreover, by Corollary 4.6 \( \bl_{i+1}(\gamma_{k,\ell}) : G^0_{i,i+1} \to E^0_{i,i+1} \) is a \( U_{2i}\)-morphism and the fiber over \( D \) is

\[
\alpha_{k-i-1,\ell-i-1} : B^{k-i-1}(M) \times \mathbb{P}^{t-k} \to B^{t-i-1}(M).
\]

This concludes the base case \(\ell = i + 1\).

Assume Claim 6.4 is true for all \(\ell' < \ell\), with \(\ell \leq j\). Consider the following diagram

\[
\begin{array}{ccc}
G^0_{i,\ell} & \xrightarrow{\bl_{i}(C_{2\ell})} & \xrightarrow{bl_{i}(\gamma_{\ell,i})} \bl_{i}(C_{2i}) \xleftarrow{\bl_{i}(C_{2i} \times \mathbb{P}^{t-k})} E^0_{i,\ell} \\
\downarrow & & \downarrow \\
G^0_{i,\ell-1} & \xrightarrow{bl_{i-1}(C_{2\ell-2} \times \mathbb{P}^{k-1}(\ell-1))} \xrightarrow{bl_{i-1}(\gamma_{\ell-1,i})} \bl_{i-1}(C_{2i-1}) & \xleftarrow{\bl_{i-1}(C_{2i-1} \times \mathbb{P}^{t-k})} \xrightarrow{\bl_{i-1}(\gamma_{\ell-1,i})} E^0_{i,\ell-1} \\
& \xleftarrow{bl_{i-1}(\gamma_{\ell-1,i,k})} & \\
& \bl_{i-1}(C_{2\ell-2}) & \xrightarrow{bl_{i-1}(\gamma_{\ell-1,i,k})} \\
\end{array}
\]

By the inductive Claim 6.4 we know the chain \( \{ \gamma_{i,k} \}_{i=0}^{k-1} \) is NCD and hence \( E^0_{i,\ell} \) is the blow up of \( E^0_{i,\ell-1} \) along \( E^0_{i,\ell-1} \cap \bl_{i-1}(C_{2\ell-2} \times \mathbb{P}^{(\ell-1)}) \). By inductive Claim 6.4 and Lemma 1.22(3), we deduce that \( E^0_{i,\ell} \) is a \( U_{2i}\)-variety and the fiber over \( D \) is \( \bl_{i-1} B^{t-i-1}(M) \).

Similarly, using (33), the inductive Claim 6.4 and Lemma 1.9 one can deduce that \( G^0_{i,\ell} \) as a \( U_{2i}\)-variety with fiber \( \bl_{i-1}(B^{k-i-1}(M) \times \mathbb{P}^{t-k}) \). Moreover, by Corollary 5.4(a), the map \( \bl_{i}(\gamma_{k,\ell}) : G^0_{i,\ell} \to E^0_{i,\ell} \) is a \( U_{2i}\)-morphism with fiber over \( D \) being

\[
\bl_{i-1}(\alpha_{k-i-1,\ell-i-1}) : \bl_{i-1} B^{k-i-1}(M) \times \mathbb{P}^{t-k} \to \bl_{i-1} B^{t-i-1}(M).
\]
This finishes the inductive proof of Claim 6.4 for $(j, k)$.

**Step 3**: Claim 6.5 for $(j, k)$ (without (**)). For (**), using [33] and the inductive [32], we can show that

$$\text{bl}_j(C_{2j} \times \mathbb{P}^{k-j} \times \mathbb{P}^{t-k}) - \bigcup_{i<j} G_{i,j} = (U_{2j} \times \mathbb{P}^{k-j}) \times \mathbb{P}^{t-k}.$$ 

On the other hand, Lemma 1.22(b) implies that

$$\text{bl}_j(\gamma_{k,t})^{-1}(U_{2j} \times \mathbb{P}^{t-j}) = \gamma_{k,t}^{-1}(U_{2j} \times \mathbb{P}^{t-j}) = (U_{2j} \times \mathbb{P}^{k-j}) \times \mathbb{P}^{t-k}.$$ 

Together with [32], we obtain (*) for $(j, k)$.

As (**) for $(j, k)$, the fiberwise equality over $D \in U_{2j}$ follows from Claim 6.4 for $(j, k)$ and Corollary 3.4(a). Then we can apply Lemma 1.22(2) to obtain (**) for $(j, k)$.

**Step 4**: Claim 6.2 for $(j, k)$. We apply Lemma 1.20 to the chain $\{\gamma_{j,t}\}_{j=0}^n$ with

$$X = C_2, \quad X_k = C_{2k} \times \mathbb{P}^{t-k}, \quad X_{j,k} = C_{2j} \times \mathbb{P}^{k-j} \times \mathbb{P}^{t-k}, \quad f_{j,k} = \gamma_{j,k} \times \text{id}, \quad \phi_k = \gamma_{k,t}.$$ 

For each $k$, the assumptions in Lemma 1.20 are satisfied:

(I) The chain $\{\gamma_{j,k} \times \text{id}\}_{j=0}^{k-1}$ is a NCD chain, by the inductive Claim 6.1 and Lemma 1.9

(II) The map

$$\gamma_{k,t} : C_{2k} \times \mathbb{P}^{t-k} \to (C_{2k-2} \times \mathbb{P}^1 \times \mathbb{P}^{t-k}) = U_{2k} \times \mathbb{P}^{t-k} \to C_{2t}$$

is an embedding, by Lemma 4.2 Notation 4.1

(III) The chain $\{\gamma_{j,t}\}_{j=0}^{k-1}$ is NCD by the inductive Claim 6.1

(IV) By Claim 6.2 for $(i, k)$ with $i < j - 1$, we have

$$\text{bl}_i(\gamma_{k,t})^{-1}(\text{bl}_i(C_{2i} \times \mathbb{P}^{t-i})) = \text{bl}_i(C_{2i} \times \mathbb{P}^{k-i} \times \mathbb{P}^{t-k}), \quad \forall i \leq j - 1.$$ 

Moreover, the conditions (*) (**) are satisfied for $i < j$ by Step 3.

Consequently, we conclude that Claim 6.2 holds for $(j, k)$. This finishes the proof of Claim 6.2 for $k$.

In particular, running Step 2 for $j = k - 1$ shows that Claim 6.4 holds for $k$ (including the case $j = k$). Together with Corollary 3.4(a) Lemma 1.22(1), we obtain (***) for $k$ in Claim 6.5, so Claim 6.5 also holds for $k$. Applying Lemma 1.20 again with (***) we know that $\{\gamma_{j,t}\}_{j=0}^k$ is a chain of smooth centers.

**Step 5**: Claim 6.1 for $k$. This follows from applying Lemma 1.11 to the chain $\{\gamma_{j,t}\}_{j=0}^k$ and using that Claim 6.1 holds for $k - 1$, Claim 6.2 holds for $k' \leq k$, and $\{\gamma_{j,t}\}_{j=0}^k$ is a chain of smooth centers (from Step 4).

Therefore, we finish the inductive proof on $k$ and $t$ for Claim 6.1 - Claim 6.5. As a consequence, we obtain Proposition 2.2.

### 7. Even Genus Case

In this section, let $C$ be a smooth hyperelliptic curve of even genus $g = 2n + 2$. We sketch a proof of Theorem 3.1 for $C$. The ideas are essentially the same by reducing to the calculation of conormal bundles, but for parity reasons, the corresponding maps need some modification. First, we have a chain of maps $\{\delta_j\}_{j=0}^n$ to $\text{Jac}(C)$, where

$$\delta_j : C_{2j+1} \to \text{Pic}^{g-1}(C) = \text{Jac}(C), \quad 0 \leq j \leq n$$

$$D \mapsto (n - j)g_2^1 \otimes O_C(D).$$
The image $\delta_j(C_{2j+1})$ is $W_{g-1}^{-j}$, hence this is a proper chain. By the Abel-Jacobi theorem, for each $\ell \geq 1$, we have $P_{\ell} \subseteq C_{2\ell}$, then for each $j \geq 1$, there is a proper chain of maps $(\gamma_{i,j})_{i=0}^{j-1}$ induced by the addition maps:

$$\gamma_{i,j} : C_{2i+1} \times P^{j-i} \hookrightarrow C_{2i+1} \times C_{2j-2i} \to C_{2j+1}, \quad 0 \leq i < j.$$ 

The even genus case of Theorem A is reduced to the following analogue of Proposition 2.2 and Proposition 2.1.

**Proposition 7.1.**

1. For each $1 \leq j \leq n$, the chain

$$\{\gamma_{i,j} : C_{2i+1} \times P^{j-i} \to C_{2j+1}\}_{i=0}^{j-1}$$

is a NCD chain and for each $1 \leq j \leq n$, the map $\gamma_{i,j}$ is a map of chains of centers.

2. The chain $\{\delta_j : C_{2j+1} \to \text{Jac}(C)\}_{j=0}^{n}$ is a NCD chain and for each $1 \leq j \leq n$, the map $\delta_j : C_{2j+1} \to \text{Jac}(C)$ is a map of chains of centers.

As in the proof of Proposition 2.2 and Proposition 2.1, the proof of Proposition 7.1 relies on parallel statements for Abel-Jacobi maps and conormal bundles as in Lemma 4.3. We will only give the statements, and leave the proofs to the interested reader.

For each $j \geq 0$, denote

$$U_{2j+1} := C_{2j+1} - \gamma_{j-1,j}(C_{2j-1} \times P^1).$$

Note that for $j = 0$, we have $U_1 = C$. Any divisor $D \in U_{2j+1}$ gives a degree $2j+1$ divisor on $P^1$ via the hyperelliptic map $C \to P^1$, which is denoted to be $h_s D$ and the associated line bundle is

$$\mathcal{O}_{P^1}(g - 1 - h_s D) := \mathcal{O}_{P^1}(g - 1) \otimes \mathcal{O}_{P^1}(-h_s D).$$

**Lemma 7.2.** Let $C$ be a hyperelliptic curve of genus $g = 2n + 2$. Then:

(a) for any $0 \leq i < j$, the map $\gamma_{i,j} : C_{2i+1} \times P^{j-i} \to C_{2j+1}$ is an embedding when restricting to $U_{2i+1} \times P^{j-i}$ and for $0 \leq \ell < i < j$ we have

$$\gamma_{i,j}^{-1}(U_{2i+1} \times P^{j-\ell}) = U_{2\ell+1} \times P^{j-\ell} \times P^{j-i}.$$

(b) for $0 \leq j \leq n$, the map $\delta_j : C_{2j+1} \to \text{Jac}(C)$ is an embedding when restricting to $U_{2j+1}$, and for $0 \leq i < j$, we have

$$\delta_j^{-1}(U_{2i+1}) = U_{2i+1} \times P^{j-i}.$$

In particular, since $U_1 = C$ we have

$$\delta_j^{-1}(C) = C \times P^j \subseteq C_{2j+1}.$$

**Lemma 7.3.** For $0 \leq i < j$, consider the map $\gamma_{i,j} : C_{2i+1} \times P^{j-i} \to C_{2j+1}$ and let $D \in U_{2i+1}$. Then

(a) $d\gamma_{i,j} : \gamma_{i,j}^* T_{C_{2j+1}} \to T_{C_{2i+1} \times P^{j-i}}$ is surjective when restricted to $U_{2i+1} \times P^{j-i}$.

(b) $P N_{\gamma_{i,j}}^*:|U_{2i+1} \times P^{j-i}|$ is a smooth variety over $U_{2i+1}$ such that over $D \in U_{2i+1}$ we have an isomorphism

$$P N_{\gamma_{i,j}}^*:|D| \times P^{j-i} \cong B_{i-j-1}(\mathcal{O}_{P^1}(g - 1 - h_s D)).$$

Furthermore, for $\ell < i$, consider the commutative diagram

$$\begin{array}{ccc}
(C_{2\ell+1} \times P^{i-\ell}) \times P^{j-i} & \xrightarrow{id \times r} & C_{2\ell+1} \times P^{j-\ell} \\
\downarrow \gamma_{i,i} \times \text{id} & & \downarrow \gamma_{i,j}
\end{array}$$

$$C_{2i+1} \times P^{j-i} \xrightarrow{\gamma_{i,j}} C_{2j+1}.$$
Here $r$ is the addition map. For any $D \in U_{2t+1}$, there is an induced map of conormal bundles on $\{D\} \times P^{i-\ell} \times P^{j-i}$:

$$\epsilon : (id \times r)^* N^*_\gamma \mid_{\{D\} \times P^{i-\ell} \times P^{j-i}} \rightarrow N^*_m \mid_{\{D\} \times P^{i-\ell} \times P^{j-i}}.$$ 

Then:

(c) $\epsilon$ is surjective.

(d) The following diagram commutes:

$$\begin{array}{ccc}
P N^*_m \mid_{\{D\} \times P^{i-\ell} \times P^{j-i}} & \xrightarrow{\alpha} & P N^*_m \mid_{\{D\} \times P^{i-\ell}} \\
\downarrow \approx & & \downarrow \approx \\
B^{i-\ell-1}(M) \times P^{j-i} & \xrightarrow{\alpha^* \otimes 1, j, \cdot} & B^{i-\ell-1}(M)
\end{array}$$

Here $\alpha^* \otimes 1, j, \cdot$ is the map $[16]$ for the curve $P^1$ and the line bundle $M = \mathcal{O}_{P^1}(g - 1 - h, D)$; and $\alpha$ is the map induced by $\epsilon$ composed with a projection to $P N^*_m (D \times P^{i-\ell})$.

**Lemma 7.4.** With the notation in Lemma 7.3. For $0 \leq j \leq n$, consider the map $\delta_j : C_{2j+1} \rightarrow \text{Jac}(C)$. Then:

(a) $d \delta_j : \delta_j^* T^*_{\text{Jac}(C)} \rightarrow T^*_{C_{2j+1}}$ is surjective when restricted to $U_{2j+1}$.

(b) the fiber of $N^*_m \mid_{\delta}$ over $D \in U_{2j+1}$ is $H^0(P^1, \mathcal{O}_{P^1}(g - 1 - h, D))$.

Furthermore, consider the diagram for $i < j$,

$$\begin{array}{ccc}
C_{2i+1} \times P^{j-i} & \xrightarrow{p_1} & C_{2i+1} \\
\downarrow \gamma_{i,j} & & \downarrow \delta_i \\
C_{2j+1} & \xrightarrow{\delta_j} & \text{Jac}(C)
\end{array}$$

Then for $D \in U_{2i+1}$, we get the induced map of conormal bundles over $\{D\} \times P^{j-i}$:

$$\epsilon : p_1^* N^*_m \mid_{D} \rightarrow N^*_m \mid_{\{D\} \times P^{j-i}}$$

and we have

(c) $\epsilon$ is surjective.

(d) The following diagram commutes:

$$\begin{array}{ccc}
P N^*_m \mid_{\{D\} \times P^{j-i}} & \xrightarrow{\beta} & P N^*_m \mid_{D} \\
\downarrow \approx & & \downarrow \approx \\
B^{j-i-1}(M) & \xrightarrow{\beta^* \otimes 1, j, \cdot} & PH^0(P^1, M)
\end{array}$$

The first vertical isomorphism comes from (b) of Lemma 7.3 and $\beta^* \otimes 1, j, \cdot$ is the map $[14]$ for the curve $P^1$ and the line bundle $M = \mathcal{O}_{P^1}(g - 1 - h, D)$. The map $\beta$ is induced from $\epsilon$ composed with a projection to $P N^*_m \mid_{D}$.

**8. Brill-Noether Stratifications are Whitney**

In this section, let $C$ be a smooth projective hyperelliptic curve of genus $g$. We show that the Brill-Noether stratification of $\text{Jac}(C)$ determined by

$$\text{Jac}(C) \supset \Theta = W_{g-1}(C) \supset W_{g-1}^1(C) \supset \cdots \supset W_{g-1}^n(C),$$

is a Whitney stratification, where $n = \left\lfloor \frac{g-1}{2} \right\rfloor$. We will assume $g = 2n + 1$; the even genus case is similar.
8.1. Whitney stratifications. Let $Z$ be a smooth real manifold and let $X, Y \subseteq Z$ be two embedded smooth real sub-manifolds. Suppose $Y \subseteq \overline{X}$, where the closure is taken inside $Z$ with respect to the Euclidean topology.

Definition 8.1. We say that the pair $(X, Y)$ satisfies the Whitney conditions if for any point $y \in Y$ the following two conditions hold:

(A) If $\{x_i\} \subseteq X$ is a sequence of points converging to $y$, and if the sequence of tangent spaces $T_{x_i}X$ converges to a linear space $T$ of the same dimension, then $T_yY \subseteq T$.

(B) If $\{x_i\} \subseteq X$ and $\{y_i\} \subseteq Y$ are two sequences of points that both converge to $y$, if the sequence of real secant lines between $x_i$ and $y_i$ converges to a real line $L$, and if the sequence of tangent spaces $T_{x_i}X$ converges to a linear subspace $T$ of the same dimension, then $L \subseteq T$.

The Whitney condition (B) involves real secant lines (in local coordinates), and is therefore not so easy to verify in practice. Instead, in the case of complex algebraic varieties, there is a condition (W) introduced by Kuo [9] and Verdier [12], which implies the Whitney conditions and is easier to work with in our situation. It is proved by Teissier that, for complex analytic stratifications, the Whitney conditions are equivalent to condition (W), but we will not need this fact.

Definition 8.2 (Distance). Let $V$ be a complex vector space and let $A, B \subseteq V$ be two linear subspaces. Fix an inner product $(\cdot, \cdot)$ on $V$. The distance from $A$ to $B$ is defined to be

$$d(A, B) := \sup_{a \in A, \neq 0} \inf_{b \in B, \neq 0} \sin \theta(a, b).$$

Here $\theta(a, b)$ is the angle between two vectors $a, b$ determined by the inner product $(\cdot, \cdot)$.

Here are some basic properties of $d(A, B)$. Note that it is not symmetric in $A$ and $B$ (which is why we don’t call it the distance “between” $A$ and $B$).

Fact 8.3. -

1. $d(A, B) = 0$ if and only if $A \subseteq B$.
2. Let $A \subseteq A'$ be two subspaces, then $d(A, B) \leq d(A', B)$.
3. Identify $V$ with the conjugate dual space $V^*$ via the inner product $(\cdot, \cdot)$ so that $\ker(V^* \to B^*)$ is identified with the orthogonal complement $B^\perp$. Then

$$d(\ker(V^* \to B^*), \ker(V^* \to A^*)) = d(B^\perp, A^\perp) = d(A, B).$$

After choosing an orthonormal basis, this comes down to the fact that a linear operator and its adjoint (between two finite dimensional Hilbert spaces) have the same operator norm.

From now on, let $Z$ be a complex manifold. Let $X, Y$ be two embedded smooth complex submanifolds of $Z$ such that $Y \subseteq \overline{X}$.

Definition 8.4. We say that the pair $(X, Y)$ satisfies Condition (W) if for any point $y \in Y$, and for any sequence of points $\{x_i\} \subseteq X$ converging to $y$, there exists a constant $C > 0$ such that for $i \gg 0$, we have

$$d(T_yY, T_{x_i}X) \leq C \cdot d(y, x_i)$$

where we view $T_{x_i}X$ as a subspace of $T_yZ$ using a local trivialization of the tangent bundle $T_Z$, and $d(y, x_i)$ is the Euclidean distance between $y$ and $x_i$ in a local coordinate chart.

Kuo [9] (see also Verdier [12, Théorème 1.5]) proved the following.
Theorem 8.5. Let $Z$ be a complex manifold. Let $X, Y$ be two embedded smooth complex submanifolds of $Z$ such that $Y$ is contained in the closure of $X$. If the pair $(X, Y)$ satisfies Condition $(W)$, then the pair $(X, Y)$ satisfies the Whitney conditions $(A), (B)$.

We are going to use this result in the following form.

Lemma 8.6. Same assumptions as above. Assume the pair $(X, Y)$ satisfies the Whitney condition $(A)$. Then the pair $(X, Y)$ satisfies the Whitney condition $(B)$ if the following condition holds: Let $y \in Y$ be any point, and let $\{x_i\} \subseteq X$ be a sequence of points converging to $y$ such that $T = \lim_{i \to \infty} T_{x_i}X$ exists. Then there is a constant $C$ such that

$$d(T, T_{x_i}X) \leq C \cdot d(y, x_i).$$

Equivalently, there exists a constant $C$ such that

$$d((N^*_X|Z)_{x_i}, \lim_{i \to \infty} (N^*_X|Z)_{x_i}) \leq C \cdot d(y, x_i),$$

where $N^*_X|Z$ denotes the conormal bundle, see Notation 1.12.

Proof. By Whitney condition $(A)$, we know that $T_yY \subseteq T$. By the property (2) of the distance function in Fact 8.3 we conclude that

$$d(T_yY, T_{x_i}X) \leq d(T, T_{x_i}X) \leq C \cdot d(y, x_i).$$

This verifies the Condition $(W)$ and thus gives the Whitney condition $(B)$ by Theorem 8.5. The last statement uses the property (3) of the distance function in Fact 8.3. □

Definition 8.7. Let $X$ be a complex algebraic variety and suppose there is a finite algebraic stratification

$$X = \bigsqcup S_i$$

by connected algebraic varieties whose irreducible components are smooth. We say this is a Whitney stratification if for any $S_j \subseteq \overline{S_i}$, the pair $(S_i, S_j)$ satisfies the Whitney conditions $(A)$ and $(B)$.

8.2. Brill-Noether stratifications are Whitney. Recall that $C$ is a genus $2n + 1$ smooth hyperelliptic curve. For each $0 \leq r \leq n$, denote

$$W^r_{g-1}(C)^\circ := W^r_{g-1}(C) - W^{r+1}_{g-1}(C),$$

which is a connected smooth algebraic variety, and parametrizes degree $g - 1$ line bundles with exactly $r + 1$ independent sections. The subvariety $\text{Jac}(C) - \Theta$ is also smooth and parametrizes degree $g - 1$ lined bundles with no sections. The Brill-Noether stratification of $\text{Jac}(C)$ is defined to be

$$\text{Jac}(C) = (\text{Jac}(C) - \Theta) \sqcup \bigsqcup_{0 \leq r \leq n} W^r_{g-1}(C)^\circ.$$

Proposition 8.8. The Brill-Noether stratification of $\text{Jac}(C)$ is a Whitney stratification.

Proof. Note that $\overline{W^i_{g-1}(C)^\circ} = W^i_{g-1}(C)$ and for $i < j$, we have

$$W^j_{g-1}(C)^\circ \subseteq W^j_{g-1}(C) \subseteq W^i_{g-1}(C).$$

We also have $\text{Jac}(C) - \Theta = \text{Jac}(C)$. By Definition 8.7 it suffices to show that for each $i < j$, the pair $(W^i_{g-1}(C)^\circ, W^j_{g-1}(C)^\circ)$ satisfies the Whitney conditions, and the same holds for the pair $(\text{Jac}(C) - \Theta, W^i_{g-1}(C)^\circ)$. To apply Lemma 8.6 we need to understand the conormal bundles of the Brill-Noether strata. Recall that for each $0 \leq r \leq n$, the Abel-Jacobi map $\delta_{(g-1-2r)/2} = \delta_{n-r}$ induces an isomorphism

$$\delta_{(g-1-2r)/2} : U_{g-1-2r} \xrightarrow{\sim} W^r_{g-1}(C)^\circ, \quad D \mapsto \mathcal{O}_C(D) \otimes \tau g_2^1,$$
where $U_{g-1-2r}$ is defined in Notation 4.14 and is the open subset of $C_{g-1-2r}$ consisting of divisors $D$ such that $h^0(C, \mathcal{O}_C(D)) = 1$. By Lemma 4.17, for any $D \in U_{g-1-2r}$ and $L := \mathcal{O}_C(D) \otimes r g_2^1$, we have

$$(N^*_W ^r g_{-1-2r} \mathcal{O}_C / \text{Jac}(C))^L = (N^*_W ^r g_{-1-2r}/2)^D = H^0(C, \omega_C(-D)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1 - h_\delta D))$$

where the last isomorphism is induced by $h : C \to \mathbb{P}^1$, the hyperelliptic map determined by the unique $g_2^1$ and $h_\delta D$ is the degree $g - 1 - 2r$ divisor defined in Notation 4.13.

For each $i < j$, let $\{L_k\} \subseteq W_{g-1}^r \mathcal{O}_C$ be a sequence of line bundles converging to $L \in W_{g-1}^r \mathcal{O}_C$. Using the isomorphism $[38]$, we can write

$$L_k = \mathcal{O}_C(D_k) \otimes i g_2^1, \quad L = \mathcal{O}_C(D) \otimes j g_2^1$$

such that $D_k \in U_{g-1-2i}$ and $D \in U_{g-1-2j}$. From the discussion above, we know that

$$(N^*_W ^r g_{-1-2i} \mathcal{O}_C / \text{Jac}(C))^L_k = H^0(C, \omega_C(-D_k)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1 - h_\delta D_k))$$

$$(N^*_W ^r g_{-1-2j} \mathcal{O}_C / \text{Jac}(C))^L = H^0(C, \omega_C(-D)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1 - h_\delta D))$$

If we denote $\overline{D} := \lim_{k \to \infty} D_k \in C_{g-1-2i}$ to be the limit divisor, since $\lim_{k \to \infty} L_k = L$, we see that $D$ is an effective subdivisor of $\overline{D}$. Therefore,

$$\lim_{k \to \infty} (N^*_W ^r g_{-1-2i} \mathcal{O}_C / \text{Jac}(C))^L_k = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1 - h_\delta \overline{D}))$$

$$\subseteq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1 - h_\delta D)) = (N^*_W ^r g_{-1-2j} \mathcal{O}_C / \text{Jac}(C))^L,$$

where the first equality uses the fact that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) = 0$ for any $k \geq 0$ and hence we can take limits. This verifies the Whitney condition (A) for the pair $(W_{g-1}^r \mathcal{O}_C, W_{g-1-2r}^r \mathcal{O}_C)$, by going to the dual spaces. Now by Lemma 8.3, in order to prove the Whitney condition (B), we just need to show that there exists a constant $A$ such that

$$d(H^0(C, \omega_C(-\overline{D})), H^0(C, \omega_C(-D_k))) \leq A \cdot d(L, L_k) = d(\overline{D}, D_k),$$

where the distance function on the left is induced by an inner product on the vector space $H^0(C, \omega_C)$ and the distance function on the right is induced by the Euclidean norm on a neighborhood of $\overline{D}$ in $C_{g-1-2i}$. Since the hyperelliptic map $h : C \to \mathbb{P}^1$ is either a local isomorphism (off the branch locus) or locally of the form $t \mapsto t^2$ (on the branch locus), we can push everything down to $\mathbb{P}^1$; there, it suffices to prove that

$$d(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1 - h_\delta \overline{D})), H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1 - h_\delta D_k))) \leq A \cdot d(h_\delta \overline{D}, h_\delta D_k),$$

which follows from the interpretation of the space $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1 - h_\delta D))$ as the space of degree $g - 1$ homogeneous polynomials vanishing along the divisor $h_\delta D$ and explicit computations.

For the pair $(\text{Jac}(C) - \Theta, W_{g-1}^r \mathcal{O}_C)$, Condition (W) is vacuous because Jac$(C)$ is a complex manifold (using the property (1) of the distance function in Fact 8.3). \(\square\)

9. Questions and open problems

This section is devoted to some questions and open problems.

The log resolution of the hyperelliptic theta divisor is rather intricate. To have a better understanding of it, we ask

**Question 9.1.** Is there a modular interpretation of the log resolution in Theorem A?\(^4\)

\(^4\)Botong Wang pointed out that one can view this as a Lipschitz property of the map between compact manifolds $\text{Sym}^{g-1-2r} \mathbb{P}^1 \to \text{Grass}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1)), 2i)$ which sends $E$ to $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g - 1 - E))$. \(\square\)
Let $C$ be a Brill-Noether general curve. The Brill-Noether varieties $W^r_{g-1}(C)$ behave like generic determinantal varieties. It is natural to ask for an extension of our results:

**Problem 9.2.** Prove that Theorem [A] and Proposition [C] hold for such a curve $C$.

### Appendix A. Reducedness of $W^r_d(C)$

In this appendix, we provide the proof of the following result, due to the lack of suitable references. We thank Nero Budur for bringing this matter to our attention.

**Proposition A.1.** Let $C$ be a smooth hyperelliptic curve of genus $g$. Let $d, r \in \mathbb{N}$ be integers such that $0 \leq r \leq d \leq g$. Then the Brill-Noether variety $W^r_d(C)$ is reduced.

We recall the following result saying that reducedness can be checked on the level of tangent cones.

**Lemma A.2.** Let $Z \subseteq X$ be a closed subscheme of a smooth variety $X$ and $x \in Z$ be a closed point. If the tangent cone $TC_xZ \subseteq T_xX$ is reduced, then $Z$ is reduced in an open neighborhood of $x$.

**Proof.** Equip the tangent cone $TC_xZ$ with its natural scheme structure, then there is a flat specialization of (a neighborhood of $x$ in) $Z$ to $TC_xZ$. The desired result follows from the fact that reducedness is an open condition in flat families, c.f. [7] Theorem 12.1.1 (vii).

By Lemma [A.2] Proposition [A.1] is reduced to the following

**Lemma A.3.** The tangent cone $TC_LW^r_d(C)$ is reduced for any $L \in W^r_d(C)$.

**Proof.** To simplify the notation, we denote $W^r_d = W^r_d(C)$. Fix $L$ a Poincaré line bundle on $C \times \text{Pic}^d(C)$ and let $pr_2 : C \times \text{Pic}^d(C) \to \text{Pic}^d(C)$ be the second projection. Let $L \in W^r_d$ be a line bundle of degree $d$ and assume $h^0(L) = s + 1$ with $s \geq r$. In a neighborhood of $L$ in $\text{Pic}^d(C)$, we can produce a minimal complex computing $W^r_d$, by a variant of the method in [11] Chapter IV, §3]. Note that we can always pick a point $p \in C$ such that $h^1(L(p)) = h^1(L) - 1$ and $h^0(L) = h^0(L(p))$. Iterating this, we can pick an effective divisor $D$ of degree $h^1(L) = g - d + s$ with the property that $H^1(L(D)) = 0$ and in the short exact sequence

$$0 \to L \to L(D) \to L(D) \otimes \mathcal{O}_D \to 0,$$

the induced connecting map $H^0(D, L(D) \otimes \mathcal{O}_D) \to H^1(C, L)$ is an isomorphism (equivalently, $H^0(C, L) \to H^0(C, L(D))$ is an isomorphism). Denote by $\mathcal{D} = pr_2^*D$ the effective divisor on $C \times \text{Pic}^d(C)$. Then on some neighborhood of the point $L$, we have a short exact sequence

$$0 \to pr_{2,*}\mathcal{L}(\mathcal{D}) \to pr_{2,*}(\mathcal{L}(\mathcal{D}) \otimes \mathcal{O}_D) \to R^1pr_{2,*}\mathcal{L} \to 0,$$

where $R^1pr_{2,*}\mathcal{L}(\mathcal{D})$ vanishes on the neighborhood in question. Here $pr_{2,*}\mathcal{L} = 0$ because it is torsion-free and vanishes at a general point in the neighborhood of $L$. This gives us a presentation

$$0 \to E^0 \xrightarrow{A} E^1 \to R^1pr_{2,*}\mathcal{L} \to 0$$

where $E^0$ and $E^1$ are vector bundles of rank $h^0(L) = s + 1$, respectively $h^1(L) = g - d + s$. Moreover, the differential, viewed as a matrix $A$, vanishes at the point $L$. Let $A_1$ be the linear part of $A$; that has entries in $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal at $L$. 

Now $W^r_d$ is defined, in a neighborhood of the point $L$, by the vanishing of all the $(s - r + 1) \times (s - r + 1)$ minors of $A$. Because for $L' \in W^r_d$, the condition is
\[ h^0(L') \geq r + 1 \iff h^1(L') \geq g - d + r \]
\[ \iff \text{rank}(A)_{L'} \leq (g - d + s) - (g - d + r) = (s - r). \]
It follows from the tangent cone theorem in generic vanishing theory (c.f. [6, Theorem 4]) that one has the following containments:
\[ \mathcal{I}_1 \subseteq \mathcal{I}_{TC_L} W^r_d \subseteq \sqrt{\mathcal{I}_{TC_L} W^r_d}, \]
where the first ideal is generated by all the $(s - r + 1) \times (s - r + 1)$ minors of $A_1$. If one knows that the first ideal $\mathcal{I}_1$ is a radical ideal, and that both $\mathcal{I}_1$ and $\sqrt{\mathcal{I}_{TC_L} W^r_d}$ define the same conical subset in $T_L \text{Pic}^d(C)$ (forgetting about the scheme structure), then the tangent cone $TC_L W^r_d$ is reduced.

Since $W^r_d \cong W_{d-2r}$ as sets, one has
\[ \dim TC_L W^r_d = \dim W^r_d = d - 2r. \]
By the discussion above, it suffices to show that the $(s - r + 1) \times (s - r + 1)$ minors of the matrix $A_1$ defines a reduced, irreducible subscheme of $T_L \text{Pic}^d(C)$ of dimension $d - 2r$. This boils down to the following two claims.

**Claim 1:** The matrix $A_1$ is a Hankel/Catalecticant matrix, i.e.
\[ A_1 = \begin{pmatrix} x_1 & x_2 & \cdots & x_{g-d+s} \\ x_2 & x_3 & \cdots & x_{g-d+s+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{s+1} & \cdots & \cdots & x_{g-d+2s} \end{pmatrix} \]
up to a change of local coordinates.

**Proof of Claim 1:** We learned this argument from Nero Budur, see [3, Proposition 8.17]. By [1], the matrix $A_1$ is the one given by the map
\[ H^0(L) \rightarrow H^1(L) \otimes H^0(\omega_C), \]
which is equivalent to the Petri map
\[ \pi_L : H^0(\omega_C) \otimes L^{-1} \rightarrow H^0(\omega_C). \]
Since $C$ is hyperelliptic and $h^0(L) = s + 1$, we can write
\[ L = s \eta_2 + p_1 + \cdots + p_{d-2s}, \]
\[ \omega_C \otimes L^{-1} = (g - 1 - s - (d - 2s)) \eta_2 + q_1 + \cdots + q_{d-2s}, \]
where $p_i + q_i$ is a hyperelliptic pair for each $1 \leq i \leq d - 2s$ and no two $p_i$ lie in the same fiber of the hyperelliptic involution $C \rightarrow \mathbb{P}^1$. Then the Petri map corresponds to
\[ H^0(\mathbb{P}^1, \mathcal{O}(s)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(g - 1 - d + s)) \rightarrow H^0(\mathbb{P}^1, \mathcal{O}(g - 1 - d + 2s)) \rightarrow H^0(\mathbb{P}^1, \mathcal{O}(g - 1)) \]
The last map is the tensor product with the section $\eta \in H^0(\mathbb{P}^1, \mathcal{O}(d - 2s))$, where $\eta$ is the product of all linear forms defining the image of $p_i$ in $\mathbb{P}^1$ for $1 \leq i \leq d - 2s$. Write $V = H^0(\mathbb{P}^1, \mathcal{O}(1))$, then the Petri map is the natural multiplication map
\[ \text{Sym}^* V \otimes \text{Sym}^{g-1-d+s} V \rightarrow \text{Sym}^{g-1-d+2s} V, \]
which clearly gives a Catalecticant matrix since $\dim V = 2$. 
Claim 2: Let $C_{v,w}$ be a $v \times w$ Catalecticant matrix with $v \geq w$, i.e.

$$
C_{v,w} = \begin{pmatrix}
x_1 & x_2 & \cdots & x_w \\
x_2 & x_3 & \cdots & x_{w+1} \\
\vdots & \vdots & \ddots & \vdots \\
x_v & \cdots & \cdots & x_{v+w-1}
\end{pmatrix}
$$

Then for $k < w$, the ideals of $(k+1) \times (k+1)$ minors of $C_{v,w}$ defines a reduced irreducible subscheme $Z$ of dimension $2k$ in $\mathbb{C}^{v+w-1}$.

Proof of Claim 2: We use notations in [5]. Let $M = \text{Cat}(v,w) \subseteq \mathbb{P} \mathbb{C}^{v+w}$ be the Catalecticant space, which is of dimension $v + w - 2$ (c.f. [5, Page 561]). Let $M_k$ be the subscheme of matrices of rank $\leq k$ in $M$, the linear space corresponds to all the minors of $C_{v,w}$ of size $k + 1$. By [5, Proposition 4.3], one has

$$
\text{codim}_M M_k = v + w - 1 - 2k,
$$

and $M_k$ is the $k$-secant variety of a rational normal curve. Thus

$$
\text{dim } M_k = \text{dim } M - (v + w - 1 - 2k) = (v + w - 2) - (v + w - 1 - 2k) = 2k - 1
$$

and $M_k$ is irreducible. Moreover, Eisenbud [5, Proposition 4.3 and after] observes that $M_k$ is reduced. Therefore the corresponding space $Z \subseteq \mathbb{C}^{v+w-1}$ is reduced, irreducible and has dimension $2k$.

Now we can finish the proof of this lemma. If $d \geq g - 1$, then $s - r < s + 1 \leq g - d + s$; if $d = g$, then we can assume $r \geq 1$ ($W^0_g(C)$ is reduced by a theorem of Kempf) and still get $s - r < s = g - d + s \leq s + 1$. Therefore we can apply Claim 1 and 2 to obtain that the $(s - r + 1) \times (s - r + 1)$ minors of the matrix $A_1$ defines a reduced, irreducible subscheme in $T^1_{\text{Pic}} \mathbb{P}^d(C) = \mathbb{C}^q$ of dimension $2(s - r) + (d - 2s)$ (because only the variables $x_1, \ldots, x_{g-d+2s}$ show up in the matrix $A_1$ and the other variables provide an additional $d - 2s$ dimensions). This gives what we want and therefore we finish the proof that $TC^1_{\text{r}}W^d_g(C)$ is reduced.

As a consequence, $W^d_g(C)$ is reduced for any $0 \leq r \leq d \leq g$. \hfill \Box

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