Planar Turán Number of Double Stars

Debarun Ghosh\textsuperscript{1,2}  Ervin Győri\textsuperscript{1,2}  Addisu Paulos\textsuperscript{1,2}  Chuanqi Xiao\textsuperscript{1,2}

\textsuperscript{1}Central European University, Budapest
ghosh.debarun@phd.ceu.edu, xiao.chuanqi@outlook.com, addisu.2004@yahoo.com

\textsuperscript{2}Alfréd Rényi Institute of Mathematics, Budapest
gyori.ervin@renyi.hu

Abstract

Given a graph $F$, the planar Turán number of $F$, denoted by $\text{ex}_P(n, F)$, is the maximum number of edges in an $n$-vertex $F$-free planar graph. Such an extremal graph problem was initiated by Dowden while determining the sharp upper bound for $\text{ex}_P(n, C_4)$ and $\text{ex}_P(n, C_5)$, where $C_4$ and $C_5$ are cycles of length four and five, respectively. In this paper, we determine the upper bounds for $\text{ex}_P(n, S_{2,2})$, $\text{ex}_P(n, S_{2,3})$, $\text{ex}_P(n, S_{2,4})$, $\text{ex}_P(n, S_{2,5})$, $\text{ex}_P(n, S_{3,3})$ and $\text{ex}_P(n, S_{3,4})$, where $S_{m,n}$ is a double star with $m$ and $n$ leaves. Moreover, the bounds for $\text{ex}_P(n, S_{2,2})$ and $\text{ex}_P(n, S_{2,3})$ are sharp.

1 Introduction

One of the famous problems in extremal graph theory is determining the number of edges in an $n$-vertex graph to force a particular graph structure. The well-known result of Turán’s \cite{9} gives the maximum number of edges in an $n$-vertex graph containing no complete graph of a given order. The Turán number of a graph $H$, denoted by $\text{ex}(n, H)$, is the maximum number of edges in an $n$-vertex graph that does not contain $H$ as a subgraph. Let $\text{EX}(n, H)$ denote the set of extremal graphs, that is the set of all $n$-vertex, $H$-free graph $G$ such that $e(G) = \text{ex}(n, H)$. Erdős, Stone and Simonovits \cite{3, 2} gave a more generalized result where they determined the asymptotics of $\text{ex}(n, F)$ for all non-bipartite graphs $F$. They proved that $\text{ex}(n, F) = (1 - \frac{1}{\chi(F)-1}) \binom{n}{2} + o(n^2)$, where $\chi(F)$ denotes the chromatic number of $F$.

Over the last decade, a considerable amount of research work has been carried out on Turán type problems. For example, when the host graphs are $K_n$, $k$-uniform hypergraphs, and $k$-partite graphs, see \cite{2, 10}. In 2015, Dowden \cite{1} initiated the study of Turán-type problems when the host graph is planar graph, that is, how many edges a planar graph on $n$ vertices can have without
containing a given smaller graph? The planar Turán number of a graph \( F \), denoted by \( \text{ex}_p(n, F) \), is the maximum number of edges in a planar graph on \( n \) vertices containing no \( F \) as a subgraph. Dowden [1] determined a sharp upper bound for \( \text{ex}_p(n, C_4) \) and \( \text{ex}_p(n, C_5) \), where \( C_4 \) and \( C_5 \) are a cycle of length four and five respectively. Very recently, Ghosh et al. [4] obtained a sharp upper bound for \( \text{ex}_p(n, C_6) \), where \( C_6 \) is a cycle of length six.

The analog to Turán’s theorem in the case of planar graphs is fairly trivial. Since \( K_5 \) is not planar, there are only two meaningful cases. For the \( K_3 \), the extremal number of edges is \( 2n - 4 \), and the extremal graph is \( K_{2,n-2} \) (since all faces have size four when drawn in the plane). Note that there exist planar triangulations not containing \( K_4 \) (e.g., take a cycle of length \( n - 2 \) and then add two new vertices which are adjacent to all those in the cycle). Thus, the extremal number in the case of \( K_4 \) is \( 3n - 6 \). The planar Turán number when the forbidden subgraph is a star is also fairly trivial. The authors in [4] proved that \( \text{ex}_p(n, H) = 3n - 6 \) for all \( H \) with \( n > |H| + 2 \) and either \( \chi(H) = 4 \) or \( \chi(H) = 3 \) and \( \Delta(H) > 7 \). They also completely determined \( \text{ex}_p(n, H) \) when \( H \) is a wheel or a star, and the case when \( H \) is a \((t, r)\)-fan, that is, \( H \) is isomorphic to \( K_1 + tK_{r-1} \), where \( t > 2 \) and \( r > 3 \) are integers. The next most natural type of graph to investigate is perhaps a path. For extremal planar Turán number for paths of length \{6, 7, 8, 9, 10, 11\}, we refer the reader to [6] and [7]. Planar Turán number of other graphs, for instance, wheels and Theta graphs also be considered. We refer the reader to [5, 6, 8] for detailed results. The next natural extension of the topic is considering double stars as the forbidden graph.

**Definition 1.** An \((m,n)\)-double star, denoted by \( S_{m,n} \), is the graph obtained by taking an edge, say \( xy \), and joining one of its end vertices, say \( x \), with \( m \) vertices and the other end vertex, \( y \), with \( n \) vertices which are different from the \( m \) vertices. The edge \( xy \) is called the backbone of the double star. The vertices adjacent to an end vertex of the backbone are called the leaf-sets of the double star. Figure 1 shows an \( m,n \) double star such that the backbone is \( xy \) and the leaf-sets are \( \{x_1, x_2, \ldots, x_m\} \) and \( \{y_1, y_2, \ldots, y_n\} \), respectively.

In this paper, we address the upper bounds of planar Turán number of the following double stars: \( S_{2,2}, S_{2,3}, S_{2,4}, S_{2,5}, S_{3,3} \) and \( S_{3,4} \). Moreover, the bound for \( \text{ex}_p(n, S_{2,2}) \) and \( \text{ex}_p(n, S_{2,3}) \) is sharp. The bound for \( \text{ex}_p(n, S_{3,3}) \) is sharp up-to a linear term. Before proceeding to our results, we mention notations and terminologies to be used in next.

All the graphs we consider in this paper are simple and finite. Let \( G \) be a graph. We denote the vertex and edge set of \( G \) by \( V(G) \) and \( E(G) \), respectively. Let \( e(G) \) and \( v(G) \) denote the number of
edges and vertices, respectively. We denote the degree of a vertex \( v \) by \( d(v) \), the minimum degree in graph \( G \) by \( \delta(G) \) and the maximum degree in graph \( G \) by \( \Delta(G) \). The subgraph induced by \( S \subseteq V(G) \), is denoted by \( G[S] \). Moreover, \( N(v) \) denotes the set of vertices in \( G \) adjacent to \( v \). Let \( A \) and \( B \) be disjoint subsets of \( V(G) \). Let \( e(A, B) \) denote the number of edges in \( G \), that joins a vertex in \( A \) and a vertex in \( B \). An \( m\)-\( n \) edge is an edge such that the end vertices of the edge are with degree \( m \) and \( n \). The join \( G = G_1 + G_2 \) of graphs \( G_1 \) and \( G_2 \) with disjoint vertex sets \( V_1 \) and \( V_2 \) and edge sets \( X_1 \) and \( X_2 \) is the graph union \( G_1 \cup G_2 \) together with all the edges joining \( V_1 \) and \( V_2 \).

The following theorem summarizes the main results:

**Theorem 1.** Estimates on the planar Turán number of double stars \( S_{m,n} \) for given values of \( \{m, n\} \) are as follows:

(i) For any \( n \geq 16 \), \( \exp(n, S_{2,2}) = 2n - 4 \).

(ii) For any \( n \geq 1 \), \( \exp(n, S_{2,3}) = 2n \).

(iii) For any \( n \geq 1 \), \( \frac{15}{7}n \leq \exp(n, S_{2,4}) \leq \frac{8}{3}n \).

(iv) For \( n \geq 1 \), \( \frac{5}{2}n \leq \exp(n, S_{2,5}) \leq \frac{20}{7}n \).

(v) For \( n \geq 3 \), \( \frac{5}{2}n - 5 \leq \exp(n, S_{3,3}) \leq \frac{5}{2}n - 2 \).

(vi) For \( n \geq 1 \), \( \frac{9}{4}n \leq \exp(n, S_{3,4}) \leq \frac{20}{7}n \).

**2 Planar Turán number of \( S_{2,2} \)**

We start by proving the following weaker bounds:

**Lemma 1.** Let \( G \) be an \( S_{2,2} \)-free plane graph on \( n \) \((n \neq 5)\) vertices, then \( e(G) \leq 2n - 2 \).
Proof. Suppose that $G$ contains 6 vertices. There are only two 6-vertex maximal planar graphs $M_1$ and $M_2$ as shown in Figure 2. It can be checked that $M_1^-$ and $M_2^-$ both contain an $S_{2,2}$. Thus, $e(G) \leq 10 = 2n - 2$, when $n = 6$. Now let $G$ be an $S_{2,2}$-free planar graph on 7 vertices. If $G$ contains a vertex of degree at most 2, we are done by induction. Moreover, there is no vertex of degree at least 5 in $G$. Suppose there is a vertex $x \in V(G)$ such that $d(x) = k$, where $k \geq 5$. In this case, each vertex in $N(x)$ must be of degree at most 2. Otherwise, it is easy to find an $S_{2,2}$ in $G$, which is a contradiction.

Assume that $\Delta(G) \leq 4$. Let the number of vertices in $G$ with degree at most 3 be $k$. Hence, $G$ contains at least $n - k$ vertices of degree 4, which implies

$$e(G) \leq \frac{4(n - k) + 3k}{2} = 2n - \frac{k}{2}.$$ 

If there are at least 4 vertices of degree at most 3, then $e(G) \leq 2n - \frac{4}{2} = 2n - 2$. Let $v$ be a degree 4 vertex in $G$. If each vertex in $N(v)$ is of degree at most 3, then $e(G) \leq 2n - 2$. So, there is a vertex $u \in N(v)$, such that $uv$ is a 4-4 edge in $G$. Since $G$ is an $S_{2,2}$-free plane graph, $uv$ must be contained in 3 triangles. Let $N(u) \cap N(v) = \{x_1, x_2, x_3\}$ and $S = V(G)\backslash\{u, v, x_1, x_2, x_3\}$. Observe that no vertex in $S$ is adjacent to a vertex in $\{x_1, x_2, x_3\}$. Deleting the vertices $\{u, v, x_1, x_2, x_3\}$. We deleted at most $3 \cdot 5 - 6 = 9$ edges. Therefore, $e(G) = e(G - \{u, v, x_1, x_2, x_3\}) + 9 \leq 2(n - 5) - 2 + 9 \leq 2n - 2$. Hence, we are done by induction.

Lemma 2. Let $G$ be an $S_{2,2}$-free plane graph on $n$ ($n \geq 8$) vertices. If there is a vertex with degree at least 5, then $e(G) \leq 2n - 4$.

Proof. Let $x \in V(G)$ such that $d(x) = k \geq 5$. Let $N(x) = \{x_1, x_2, x_3, \ldots, x_k\}$, and $S$ be the set of vertices in $V(G) \backslash N(x)$. Each vertex in $N(x)$ is adjacent to at most one other vertex in $N(x)$. Otherwise, it is easy to show that $G$ contains an $S_{2,2}$. Similarly, for an edge $x_i x_j$, where $x_i, x_j \in N(x)$, there is no vertex in $S$ which is adjacent to either $x_i$ or $x_j$. Thus, the number

\[ e(G) \leq 2n - 2. \]
of edges joining a vertex in \( N(x) \) and a vertex in \( S \) is at most \( k \). If \(|S|\neq 5\), using Lemma 1, \( e(G[S]) \leq 2(n-k-1)-2 \). Therefore, \( e(G) \leq 2(n-k-1)-2+k+k = 2n-4 \), and we are done. Let \(|S|=5\). Let the graph induced by \( S \) be a \( K_5^- \). Since \( G \) is an \( S_{2,2} \)-free plane graph, no vertex in \( N(x) \) is adjacent to any vertex in \( S \). This implies that \( e(G[S]) = 9 \) and \( e(G[N(x)]) \leq \frac{k}{2} \). Hence, \( e(G) \leq k + \frac{k}{2} + 9 \). Since \( n = k + 6 \), we have \( e(G) \leq (n-6) + \frac{n-6}{2} + 9 = \frac{3n}{2} \leq 2n-4 \) for \( n \geq 8 \). \( \square \)

**Proof of Theorem 4(i).** The lower bound is attained by considering the graph \( K_{2,n-2} \), which is \( S_{2,2} \)-free and contains \( 2n-4 \) edges. From Lemma 2 we may assume that \( \Delta(G) \leq 4 \). Let \( k \) be the number of vertices in \( G \) whose degree is at most 3. Then \( e(G) \leq \frac{4(n-k)+3k}{2} = 2n - \frac{k}{2} \). If \( k \geq 8 \), then we are done. We may assume that the number of degree 4 vertices in \( G \) is at least 9, since \( n \geq 16 \). We start by proving the following claims:

**Claim 1.** There is no degree 4 vertex in \( G \) such that all its neighbors are of degree at most 3.

**Proof.** Suppose not. Let \( x \) be a degree 4 vertex in \( G \), such that \( d(y) \leq 3 \) for all \( y \in N(x) \). It is easy to check that for each \( y \in N(x) \), if \( y \) is adjacent to any vertex in \( V(G)\setminus \{x\} \cup N(x) \), then \( d(y) = 2 \). Otherwise, \( G \) contains an \( S_{2,2} \). Therefore, using Lemma 1, \( e(G) = e(G[S]) + 4 + 4 \leq 2(n-5) - 2 + 8 = 2n-4 \). Assume that for each \( y \in N(x) \), \( y \) is not adjacent to any vertex in \( V(G)\setminus \{x\} \cup N(x) \). Then \( e(G) \leq 8 \). Therefore, using Lemma 1, \( e(G) = e(G[S]) + 8 \leq 2(n-5) - 2 + 8 = 2n-4 \). \( \square \)

**Claim 2.** The number of 4-4 edges in a matching in \( G \) is at least 3.

**Proof.** Suppose not. Let the number of 4-4 edges in a matching in \( G \) be 2. Denote the 4-4 edges in the matching by \( uv \) and \( xy \). Each of the edges is contained in 3 triangles. Let \( N(u) \cap N(v) = \{u_1, u_2, u_3\} \) and \( N(x) \cap N(y) = \{x_1, x_2, x_3\} \). Since \( G \) is an \( S_{2,2} \)-free plane graph, no vertex in \( \{u_1, u_2, u_3\} \) is adjacent to a vertex in \( V(G)\setminus \{x, y, x_1, x_2, x_3\} \) and no vertex in \( \{x_1, x_2, x_3\} \) is adjacent to a vertex in \( V(G)\setminus \{u, v, u_1, u_2, u_3\} \). Moreover, at least two vertices in \( \{x_1, x_2, x_3\} \) and in \( \{u_1, u_2, u_3\} \) are of degree at most three. The number of degree 4 vertices in \( G \) is at least 9. Thus, there is a degree 4 vertex, say \( z \), in \( V(G)\setminus \{x, y, u, v, x_1, x_2, x_3, u_1, u_2, u_3\} \) such that \( d(t) \leq 3 \) for each \( t \in N(z) \). This contradicts Claim 1. A similar argument can be given, if we assume that the number of 4-4 edges in a matching in \( G \) is 1. \( \square \)

From now on, suppose that the number of 4-4 edges in a matching in \( G \) is at least 3. We distinguish the following two cases:
Case 1: The number of 4-4 edges in a matching in $G$ is at least 4. Denote the 4-4 edges in the matching in $G$ by $x_1x_2$, $x_3x_4$, $x_5x_6$ and $x_7x_8$, respectively. Recall that each edge is contained in 3 triangles. Moreover, at least two vertices in $N(x_i) \cap N(x_{i+1})$ are of degree at most 3, for each $i \in \{1, 3, 5, 7\}$. Thus, we have at least 8 vertices in $G$ whose degree is at most 3. Hence, we are done in this case.

Case 2: The number of 4-4 edges in a matching in $G$ is 3. Denote the 4-4 edges in the matching in $G$ by $x_1x_2$, $x_3x_4$ and $x_5x_6$, respectively. At least two vertices in $N(x_i) \cap N(x_{i+1})$ are of degree at most 3, for $i \in \{1, 3, 5\}$. This implies that $G$ contains at least 6 vertices whose degree is at most 3. Moreover, if a vertex in $N(x_i) \cap N(x_{i+1})$ is of degree at most 2, for some $i \in \{1, 3, 5\}$, then the remaining two vertices are of degree at most 3. Observe that in this case, $e(G) \leq \frac{4(n-7)+3\cdot 6+2}{2} = 2n - 4$ and we are done.

So, we assume exactly two vertices in $N(x_i) \cap N(x_{i+1})$ are of degree 3, for each $i \in \{1, 3, 5\}$. In this case, the remaining vertex in $N(x_i) \cap N(x_{i+1})$ is of degree 4. Thus, the vertices $\{x_i, x_{i+1}\} \cup \left(N(x_i) \cap N(x_{i+1})\right)$, for each $i \in \{1, 3, 5\}$, induce a $K_5^-$, and it is a component in $G$. If $n = 16$, then there is an isolated vertex. In this case, $e(G) = 27 < 2n - 4$. If $n > 17$, it is easy to find 2 more vertices of degree at most 3 since the number of 4-4 edges in a matching in $G$ is 3. Thus, there are at least 8 degree 3 vertices in $G$. This completes the proof of Theorem 1(ii).

3 Planar Turán number of $S_{2,3}$

Proof of theorem 1(ii). Let $G$ be an $S_{2,3}$-free plane graph on $n$ vertices. Since the graph $S_{2,3}$ contains 7 vertices, a maximal planar graph with $n \leq 6$ vertices does not contain an $S_{2,4}$. Thus, the lower bound is attained by considering disjoint copies of the maximum planar graphs on 6 vertices, i.e., $M_1$ or $M_2$ (see Figure 2). If the maximum degree in $G$ is at most 4, then $e(G) \leq 2n$. Now we separate the rest of the proof into 2 cases:

Case 1: There exists a vertex $v \in V(G)$, such that $d(v) \geq 6$. It is easy to check that for each $u \in N(v)$, $d(u) \leq 2$, otherwise, we find a copy of $S_{2,3}$ in $G$. Delete a vertex $u \in N(v)$, then the number of deleted edges is at most 2. By the induction hypothesis, we get $e(G - \{u\}) \leq 2(n - 1)$. Hence, $e(G) = e(G - \{u\}) + d(u) \leq 2(n - 1) + 2 = 2n$.

Case 2: There exists a vertex $v \in V(G)$, such that $d(v) = 5$. It is easy to check that for each $u \in N(v)$, if $u$ is adjacent to any vertex in $V(G) \setminus (\{v\} \cup N(v))$, then $d(u) = 2$. Otherwise, $G$ contains an $S_{2,3}$. As in Case 1, we are done by induction. Assume that for each $u \in N(v)$, $u$ is not
adjacent to any vertex in $V(G) \setminus \{v \cup N(v)\}$. Then $e(G[v \cup N(v)]) \leq 3 \cdot 6 - 6 = 12$. By the induction hypothesis, $e(G - \{v \cup N(v)\}) \leq 2(n - 6)$. Therefore, $e(G) = e(G - \{v \cup N(v)\}) + e(G[v \cup N(v)]) \leq 2n$. 

4 Planar Turán number of $S_{2,4}$

Let $G$ be an $S_{2,4}$-free plane graph on $n$ vertices. Since $S_{2,4}$ contains 8 vertices, a maximal planar graph with $n \leq 7$ vertices, does not contain an $S_{2,4}$. Let $7 \mid n$. Consider the plane graph consisting of $\frac{n}{7}$ disjoint copies of maximal planar graphs on 7 vertices. This graph does not contain an $S_{2,4}$. Hence, $\text{ex}_P(n, S_{2,4}) \geq \frac{15}{7}n$.

Claim 3. Let $G$ be an $S_{2,4}$ on $(1 \leq n \leq 18)$ vertices. The number of edges in $G$ is at most $\frac{8}{3}n$.

Proof. Recall that, an $n$-vertex maximal planar graph contains $3n - 6$ edges. Since $3n - 6 \leq \frac{8}{3}n$ for $n \leq 18$, $e(G) \leq \frac{8}{3}n$ holds for all $n$, $1 \leq n \leq 18$.

Lemma 3. If $G$ contains a vertex of degree greater than or equal to 7, then $e(G) \leq \frac{8}{3}n$.

Proof. Let $v \in V(G)$, such that $d(v) = 7$. It is easy to check that for each $u \in N(v)$, $d(u) \leq 2$, otherwise, we find a copy of $S_{2,4}$ in $G$. Delete a vertex $u \in N(v)$, then the number of deleted edges is at most 2. By the induction hypothesis, we get $e(G - \{u\}) \leq \frac{8}{3}(n - 1)$. Hence, $e(G) = e(G - \{u\}) + d(u) \leq \frac{8}{3}(n - 1) + 2 \leq \frac{8}{3}n$.

Lemma 4. If $G$ contains a vertex of degree 6, then $e(G) \leq \frac{8}{3}n$.

Proof. Let $v \in V(G)$, such that $d(v) = 6$ and let $H = N(v) \cup \{v\}$. It is easy to check that for each $u \in N(v)$, if $u$ is adjacent to any vertex in $V(G) \setminus H$, then $d(u) = 2$. Otherwise, $G$ contains an $S_{2,4}$. We are done by induction in this case. Assume that for each $u \in N(v)$, $u$ is not adjacent to any vertex in $V(G) \setminus H$. Then $e(G[H]) \leq 3 \cdot 7 - 6 = 15$. Hence, $e(G) = e(G - H) + e(G[H]) \leq \frac{8}{3}(n - 6) + 15 \leq \frac{8}{3}n$. We are done by induction.

Proof of Theorem 7(iii). If $G$ contains a vertex of degree at least 6, we are done by Lemmas 3 and 4. Hence, we can assume that $\Delta(G) \leq 5$.

Claim 4. If $G$ contains a 5 - 5 edge, then $e(G) \leq \frac{8}{3}n$. 

7
Proof. Let $xy \in E(G)$ be a 5−5 edge. There are at least 3 triangles sitting on the edge $xy$, otherwise $G$ contains an $S_{2,4}$. We subdivide the cases based on the number of triangles sitting on the edge $xy$.

1. **There are 4 triangles sitting on the edge $xy$.** Let $a, b, c$ and $d$ be the vertices in $G$ which are adjacent to both $x$ and $y$ (see Figure 3). Let $S_1 = \{a, b, c, d\}$ and $H = \{x, y, a, b, c, d\}$. Delete the vertices in $H$. The vertices in $S_1$ can have at most one neighbor in $V(G)\setminus H$ each and can form a path of length 3 in $S_1$. Hence, the number of edges be deleted is at most $9 + 4 + 3 = 16$. Using the induction hypothesis, $e(G) \leq e(G - H) + 16 \leq \frac{8}{3}(n - 6) + 16 = \frac{8n}{3}$.

2. **There are 3 triangles sitting on the edge $xy$.** Let $a, b$ and $c$ be the vertices in $G$ which are adjacent to both $x$ and $y$. Let $d$ be the vertex adjacent to $x$ but not adjacent to $y$, and $e$ be the vertex adjacent to $y$ but not adjacent to $x$. Let $S_1 = \{a, b, c\}$ and $H = \{x, y, d, e\} \cup S_1$. Delete the vertices in $H$. The vertices $d$ and $e$ can have at most one neighbor in $V(G)\setminus H$ each. We distinguish the cases as follows:

   (a) **The vertices $d$ and $e$ have no neighbors in $S_1$, see Figure 4(i).** The vertices in $S_1$ can have at most one neighbor in $V(G)\setminus H$ each and can form a path of length 2 in $S_1$. If the vertices $d$ and $e$ are adjacent, they cannot have a neighbor in $V(G)\setminus H$. Otherwise, we have an $S_{2,4}$ with $dx$ (or $ey$) as the backbone. Thus, the number of edges deleted is at most $9 + 2 + 3 + 2 = 16$.

   (b) **One of the vertices $d$ or $e$ has one neighbor in $S_1$, and the other has none.** Without loss of generality, suppose $a$ is the neighbor of $d$, see Figure 4(ii). Note that $a$ cannot have a neighbor in $V(G)\setminus H$. Otherwise, we have an $S_{2,4}$ with $ay$ as the backbone. If the vertices $d$ and $e$ are adjacent, they cannot have a neighbor in $V(G)\setminus H$. Similarly,
Figure 4: The graph $G$ has a $5 - 5$ edge $xy$, with 3 triangles sitting on the edge $xy$.

(i) The vertices $d$ and $e$ have no neighbors in $S_1$.

(ii) The vertex $d$ has one neighbor in $S_1$, and $e$ has none.

(iii) The vertices $d$ and $e$ have one common neighbor in $S_1$.

(iv) The vertices $d$ and $e$ have one distinct neighbor in $S_1$.

(v) The vertex $d$ has two neighbors in $S_1$, while $e$ has none.

(vi) The vertex $d$ is the neighbor of $a$ and $c$, and while $e$ is the neighbor of $c$.

(vii) The vertex $d$ is the neighbor of $a$ and $c$, while $e$ is the neighbor of $b$.

(viii) The vertices $d$ and $e$ are neighbors of $a$ and $c$ both.

(ix) The vertex $d$ is the neighbor of $a$ and $c$, while $e$ is the neighbor of $a$ and $b$. 
as before, the vertices $b$ and $c$ can have at most one neighbor in $V(G) \setminus H$ each and the vertices $\{a, b, c\}$ can form a path of length 2 in $S_1$. Thus, the number of edges deleted is at most $10 + 2 + 2 + 2 = 16$.

(c) **The vertices $d$ and $e$ have one neighbor in $S_1$.** There are two possibilities. In the first case, without loss of generality, suppose $a$ is the common neighbor of $d$ and $e$, see Figure 4(iii). Similarly, as before, $a$ cannot have a neighbor in $V(G) \setminus H$. If the vertices $d$ and $e$ are adjacent, they cannot have a neighbor in $V(G) \setminus H$. The vertices $b$ and $c$ can have at most one neighbor in $V(G) \setminus H$ each, and the vertices $\{a, b, c\}$ can form a path of length 2 in $S_1$. Thus, the number of edges deleted is at most $11 + 2 + 2 + 2 = 17$.

Without loss of generality, suppose $a$ is the neighbor of $d$ while $c$ is the neighbor of $e$, see Figure 4(iv). Similarly, as before, the vertices $a$ and $c$ cannot have a neighbor in $V(G) \setminus H$. The vertex $b$ can have at most one neighbor in $V(G) \setminus H$ and the vertices $\{a, b, c\}$ can form a path of length 2 in $S_1$. Thus, the number of edges deleted is at most $11 + 2 + 1 + 2 = 16$. (If the vertices $d$ and $e$ are adjacent, there can only be a path of length 1 inside $S_1$. Again, this precision is unnecessary. We skip this in the following cases also.)

(d) **One of the vertices $d$ or $e$ has two neighbors in $S_1$, while the other has none.** Without loss of generality, suppose $d$ is the neighbor of $a$ and $c$, see Figure 4(v). Similarly, as before, the vertices $a$ and $c$ cannot have a neighbor in $V(G) \setminus H$. The vertex $b$ can have at most one neighbor in $V(G) \setminus H$ and the vertices $\{a, b, c\}$ can form a path of length 2 in $S_1$. If the vertices $d$ and $e$ are adjacent, they cannot have a neighbor in $V(G) \setminus H$. Thus, the number of edges deleted is at most $11 + 2 + 1 + 2 = 16$.

(e) **One of the vertices $d$ or $e$ has two neighbors in $S_1$, while the other has one neighbor.** There are two possibilities. In the first case, without loss of generality, suppose $d$ is the neighbor of $a$ and $c$, and $e$ is the neighbor of $c$ (see Figure 4(vi)). Similarly, as before, the vertices $a$ and $c$ cannot have a neighbor in $V(G) \setminus H$. The vertex $b$ can have at most one neighbor in $V(G) \setminus H$ and the vertices $\{a, b, c\}$ can form a path of length 2 in $S_1$. If the vertices $d$ and $e$ are adjacent, they cannot have a neighbor in $V(G) \setminus H$. Thus, the number of edges deleted is at most $12 + 2 + 1 + 2 = 17$.

In the other case, without loss of generality, assume that $d$ is the neighbor of $a$ and $c$, and $e$ is the neighbor of $b$ (see Figure 4(vii)). The vertices $a, b$ and $c$ cannot have a
neighbor in \(V(G)\setminus H\), but they can form a path of length 2 in \(S_1\). Thus, the number of edges deleted is at most \(12 + 2 + 2 = 16\).

(f) **Both the vertices \(d\) and \(e\) have two neighbors in \(S_1\).** There are two possibilities. In the first case, without loss of generality, suppose \(d\) and \(e\) are the neighbors of \(a\) and \(c\) both (see Figure 4(viii)). Similarly, as before, the vertices \(a\) and \(c\) cannot have a neighbor in \(V(G)\setminus H\). The vertex \(b\) can have one neighbor in \(V(G)\setminus H\). The vertices \(\{a, b, c\}\) can form a path of length 2 in \(S_1\). If the vertices \(d\) and \(e\) are adjacent, they cannot have a neighbor in \(V(G)\setminus H\). Thus, the number of edges deleted is at most \(13 + 1 + 2 + 2 = 18\).

On the other hand, without loss of generality, assume \(d\) is the neighbor of \(a\) and \(c\), while \(e\) is the neighbor of \(a\) and \(b\) (see Figure 4(ix)). The vertices \(a, b\) and \(c\) cannot have a neighbor in \(V(G)\setminus H\), but they can form a path of length 2 in \(S_1\). Thus, the number of edges deleted is at most \(13 + 2 + 2 = 17\).

Using the induction hypothesis,

\[
e(G) \leq e(G - H) + 18 \leq \frac{8}{3}(n - 7) + 18 = \frac{8}{3}n.
\]

This completes the proof. \(\square\)

Take \(x, y \in V(G)\). By the previous claims, if \(d(x) + d(y) \geq 10\), we are done by induction. Assume that \(d(x) + d(y) \leq 9\). Summing it over all the edge pairs in \(G\), we have \(9e \geq \sum_{xy \in E(G)} (d(x) + d(y)) = \sum_{x \in V(G)} (d(x))^2 \geq nd^2 = n(\frac{2e}{n})^2\), where \(d\) is the average degree in \(G\). This gives us \(e \leq \frac{9}{4}n \leq \frac{8}{7}n\) for \(n \geq 1\). \(\square\)

### 5 Planar Turán number of \(S_{2,5}\)

Let \(G\) be an \(S_{2,5}\)-free plane graph on \(n\) vertices. Let \(12|n\). Consider the plane graph consisting of \(\frac{n}{12}\) disjoint copies of 5-regular maximal planar graphs with \(12\) vertices, see Figure 5. This graph does not contain an \(S_{2,5}\), since it is a 5–regular graph. Hence, \(\text{ex}_P(n, S_{2,4}) \geq \frac{5}{2}n\).

**Claim 5.** Let \(G\) be an \(S_{2,5}\) on \(n\) \((1 \leq n \leq 42)\) vertices. The number of edges in \(G\) is at most \(\frac{20}{7}n\).

**Proof.** Recall that, an \(n\)-vertex maximal planar graph contains \(3n - 6\) edges. Since \(3n - 6 \leq \frac{20}{7}n\) for \(n \leq 42\), we get \(e(G) \leq \frac{20}{7}n\) holds for all \(n\), when \(1 \leq n \leq 42\). \(\square\)
Figure 5: \((n/12)\)-disjoint copies of 5-regular maximal planar graphs with 12 vertices.

**Lemma 5.** If \(G\) contains a vertex \(v\) of degree greater than or equal to 8, then \(e(G) \leq \frac{20}{7}n\).

*Proof.* Let \(v \in V(G)\), such that \(d(v) \geq 8\). It is easy to check that for each \(u \in N(v)\), \(d(u) \leq 2\). Otherwise, we find a copy of \(S_{2,5}\) in \(G\). Delete a vertex \(u \in N(v)\), then the number of deleted edges is at most 2. By the induction hypothesis, we get \(e(G - \{u\}) \leq \frac{20}{7}(n - 1)\). Hence, \(e(G) = e(G - \{u\}) + d(u) \leq \frac{20}{7}(n - 1) + 2 \leq \frac{20}{7}n\). \(\blacksquare\)

**Lemma 6.** If \(G\) contains a vertex \(v\) of degree equal to 7, then \(e(G) \leq \frac{20}{7}n\).

*Proof.* Let \(v \in V(G)\), such that \(d(v) = 7\) and let \(H = N(v) \cup \{v\}\). It is easy to check that for each \(u \in N(v)\), if \(u\) is adjacent to any vertex in \(V(G) \setminus H\), then \(d(u) = 2\). Otherwise, \(G\) contains an \(S_{2,5}\). We are done by induction in this case. In the other case, assume that for each \(u \in N(v)\), \(u\) is not adjacent to any vertex in \(V(G) \setminus H\). Then \(e(G[H]) \leq 3 \cdot 8 - 6 = 18\). Hence, \(e(G) = e(G - H) + e(G[H]) \leq \frac{20}{7}(n - 8) + 18 = \frac{20}{7}n\). We are done by induction. \(\blacksquare\)

*Proof of Theorem 1(iv).* If \(G\) contains a vertex of degree at least 7, we are done by Lemmas 5 and 6. Hence, we can assume \(\Delta(G) \leq 6\).

**Claim 6.** If \(G\) contains a 6−6 edge, then \(e(G) \leq \frac{20}{7}n\).

*Proof.* Let \(xy \in E(G)\) be a 6-6 edge. There are at least 4 triangles sitting on the edge \(xy\), otherwise \(G\) contains an \(S_{2,5}\). We subdivide the cases based on the number of triangles sitting on the edge \(xy\).

1. **There are 5 triangles sitting on the edge** \(xy\). Let \(a, b, c, d\) and \(e\) be the vertices in \(G\) which are adjacent to both \(x\) and \(y\) (see Figure 6). Let \(S_1 = \{a, b, c, d, e\}\) and \(H = \{x, y, a, b, c, d, e\}\). Delete the vertices in \(H\). The vertices in \(S_1\) can have at most one neighbor in \(V(G) \setminus H\) each and can form a path of length 4 in \(S_1\). Hence, the number of edges deleted is at most \(11 + 5 + 4 = 20\). Using the induction hypothesis, \(e(G) \leq e(G - H) + 20 \leq \frac{20}{7}(n - 7) + 20 = \frac{20}{7}n\).
2. **There are 4 triangles sitting on the edge** $xy$. Let $a, b, c$ and $d$ be the vertices in $G$ which are adjacent to both $x$ and $y$. Let $e$ be the vertex adjacent to $x$ but not adjacent to $y$, and $f$ be the vertex adjacent to $y$ but not adjacent to $x$. Let $S_1 = \{a, b, c, d\}$ and $H = \{x, y\} \cup S_1 \cup \{e, f\}$. Delete the vertices in $H$. The vertices $e$ and $f$ can have at most one neighbor in $V(G)\setminus H$ each. We distinguish the cases based on the neighbors of $e$ and $f$ as follows:

(a) **The vertices $e$ and $f$ have no neighbors in** $S_1$, see Figure 7(i). The vertices in $S_1$ can have at most one neighbor in $V(G)\setminus H$ each and can form a path of length 3 in $S_1$. If the vertices $e$ and $f$ are adjacent, then they cannot have a neighbor in $V(G)\setminus H$. Otherwise, we have an $S_{2,5}$ with $ex$ (or $fy$) as the backbone. Thus, the number of edges deleted is at most $11 + 3 + 4 + 2 = 20$.

(b) **One of the vertices $e$ or $f$ has one neighbor in** $S_1$, and the other has none. Without loss of generality, suppose $a$ and $e$ are adjacent (see Figure 7(ii)), then $a$ cannot have a neighbor in $V(G)\setminus H$. Otherwise, we have an $S_{2,5}$ with $ay$ as the backbone. If the vertices $e$ and $f$ are adjacent, then they cannot have a neighbor in $V(G)\setminus H$. Similarly, as before, the vertices $b, c$ and $d$ can have at most one neighbor in $V(G)\setminus H$ each and the vertices $\{a, b, c, d\}$ can form a path of length 3 in $S_1$. Thus, the number of edges deleted is at most $12 + 3 + 3 + 2 = 20$.

(c) **The vertices $e$ and $f$ have one neighbor in** $S_1$. There are two possibilities. In the first case, without loss of generality, suppose $a$ is the common neighbor of $e$ and $f$, see
Figure 7: The graph $G$ has a $6 - 6$ edge $xy$, with 4 triangles sitting on the edge $xy$.

(i) the vertices $e$ and $f$ have no neighbors in $S_1$.

(ii) The vertex $e$ has one neighbor in $S_1$ and $f$ has none.

(iii) The vertices $e$ and $f$ have one common neighbor in $S_1$.

(iv) The vertices $e$ and $f$ have one distinct neighbor in $S_1$.

(v) The vertex $e$ has two neighbors in $S_1$ and $f$ has none.

(vi) The vertex $e$ is the neighbor of $a$ and $d$, and $f$ is the neighbor of $c$.

(vii) The vertex $e$ is the neighbor of $a$ and $d$, and $f$ is the neighbor of $b$.

(viii) The vertices $e$ and $f$ are neighbors of both $a$ and $d$.

(ix) The vertex $e$ is the neighbor of $a$ and $d$, while $f$ is the neighbor of $a$ and $b$.

(x) The vertex $e$ is the neighbor of $a$ and $d$, while $f$ is the neighbor of $b$ and $c$.

Figure 7(iii). Similarly, as before, $a$ cannot have a neighbor in $V(G) \setminus H$. If the vertices $e$ and $f$ are adjacent, they cannot have a neighbor in $V(G) \setminus H$. The vertices $b, c$ and $d$ can have at most one neighbor in $V(G) \setminus H$ each, and the vertices $\{a, b, c, d\}$ can form a path of length 3 in $S_1$. Thus, the number of edges deleted is at most $13 + 3 + 3 + 2 = 21$.

Without loss of generality, suppose $a$ is the neighbor of $e$ while $d$ is the neighbor of $f$, see Figure 7(iv). Similarly, as before, the vertices $a$ and $d$ cannot have a neighbor in $V(G) \setminus H$. The vertex $b$ and $c$ can have at most one neighbor in $V(G) \setminus H$ each, and the vertices $\{a, b, c, d\}$ can form a path of length 3 in $S_1$. If the vertices $e$ and $f$ are adjacent, then they cannot have a neighbor in $V(G) \setminus H$. Thus, the number of edges deleted is at most $13 + 3 + 2 + 2 = 20$. (In fact, it can be shown that if the vertices $w$ and $f$ are adjacent, there can only be a 2-path inside $S_1$. However, this precision is unnecessary. We skip this in the following cases also.)

(d) **One of the vertices $e$ or $f$ has two neighbors in $S_1$, while the other has none.**

Without loss of generality, suppose $e$ is the neighbor of $a$ and $d$, see Figure 7(v). Similarly, as before, the vertices $a$ and $d$ cannot have a neighbor in $V(G) \setminus H$. The vertices $b$ and $c$ can have at most one neighbor in $V(G) \setminus H$ each, and the vertices $\{a, b, c, d\}$ can form a path of length 3 in $S_1$. If the vertices $e$ and $f$ are adjacent, then they cannot have a
neighbor in \( V(G) \setminus H \). Thus, the number of edges deleted is at most \( 13 + 3 + 2 + 2 = 20 \).

(e) **One of the vertices \( e \) or \( f \) has two neighbors in \( S_1 \), while the other has one neighbor.** There are two possibilities. In the first case, without loss of generality, suppose \( e \) is the neighbor of \( a \) and \( d \), and \( f \) is the neighbor of \( d \) (see Figure 7(vi)). Similarly, as before, the vertices \( a \) and \( d \) cannot have a neighbor in \( V(G) \setminus H \). The vertices \( b \) and \( c \) can have at most one neighbor in \( V(G) \setminus H \) each, and the vertices \( \{a, b, c, d\} \) can form a path of length 3 in \( S_1 \). If the vertices \( e \) and \( f \) are adjacent, then they cannot have a neighbor in \( V(G) \setminus H \). Thus, the number of edges deleted is at most \( 14 + 3 + 2 + 2 = 21 \).

In the other case, without loss of generality, assume that \( e \) is the neighbor of \( a \) and \( d \), and \( f \) is the neighbor of \( b \) (see Figure 7(vii)). The vertices \( a, b \) and \( d \) cannot have a neighbor in \( V(G) \setminus H \), but they along with \( c \) can form a path of length 3 in \( S_1 \). The vertex \( c \) can have at most one neighbor in \( V(G) \setminus H \). Thus, the number of edges deleted is at most \( 14 + 3 + 1 + 2 = 20 \).

(f) **Both the vertices \( e \) and \( f \) have two neighbors in \( S_1 \).** There are three possibilities. In the first case, without loss of generality, suppose \( e \) and \( f \) are neighbors of both \( a \) and \( d \), see Figure 7(viii). Similarly, as before, the vertices \( a \) and \( d \) cannot have a neighbor in \( V(G) \setminus H \). The vertices \( b \) and \( c \) can have at most one neighbor in \( V(G) \setminus H \) each. The vertices \( \{a, b, c, d\} \) can form a path of length 3 in \( S_1 \). If the vertices \( e \) and \( f \) are adjacent, then they cannot have a neighbor in \( V(G) \setminus H \). Thus, the number of edges deleted is at most \( 15 + 3 + 2 + 2 = 22 \).

On the other hand, without loss of generality, assume \( e \) is the neighbor of both \( a \) and \( d \), while \( f \) is the neighbor of \( a \) and \( b \) (see Figure 7(ix)). The vertices \( a, b \) and \( d \) cannot have a neighbor in \( V(G) \setminus H \), but they along with \( c \) can form a path of length 3 in \( S_1 \). The vertex \( c \) can have at most one neighbor in \( V(G) \setminus H \) each. Thus, the number of edges deleted is at most \( 15 + 3 + 1 + 2 = 21 \).

In the last case, without loss of generality, assume \( e \) is the neighbor of \( a \) and \( d \) both, while \( f \) is the neighbor of \( b \) and \( c \) (see Figure 7(x)). The vertices \( a, b, c \) and \( d \) cannot have a neighbor in \( V(G) \setminus H \), but they can form a path of length 3 in \( S_1 \). Thus, the number of edges deleted is at most \( 15 + 3 + 2 = 20 \).
Using the induction hypothesis,
\[ e(G) \leq e(G - H) + 22 \leq \frac{20}{7}(n - 8) + 22 = \frac{20}{7}n - \frac{6}{7}. \]

This completes the proof of the claim. \[\square\]

Consider \(x, y \in V(G)\). By the previous claims, if \(d(x) + d(y) \geq 12\), we are done by induction. Assume that \(d(x) + d(y) \leq 11\). Summing it over all the edge pairs in \(G\), we have
\[ 11e(G) \geq \sum_{xy \in E(G)} (d(x) + d(y)) = \sum_{x \in V(G)} (d(x))^2 \geq n\overline{d}^2 = n\left(\frac{2e(G)}{n}\right)^2, \]
where \(\overline{d}\) is the average degree in \(G\). This gives us \(e(G) \leq \frac{11}{4}n \leq \frac{20}{7}n\), for \(n \geq 1\). \[\square\]

6 Planar Turán number of \(S_{3,3}\)

We show that for infinitely many integer values of \(n\), we can construct an \(n\)-vertex \(S_{3,3}\)-free plane graph \(G_n\) with \(\frac{5}{2}n - 5\) edges. This is to verify that the bound we have is best up to the linear term. Consider a plane graph \(G_n\) which is obtained by joining every vertex of the maximal matching on \(n - 2\) vertices with two vertices. Constructions of \(G_n\) when \(n\) is even or odd is shown in Figure 8. Each edge in \(G_n\) has a degree 3 end vertex. Thus, \(G_n\) is an \(S_{3,3}\)-free planar graph. Moreover, \(e(G_n) = \left\lfloor \frac{5}{2}n \right\rfloor - 5\).

![Figure 8: Extremal Constructions for the lower bound of planar Turán number of \(S_{3,3}\).](image)

Claim 7. Let \(G\) be an \(S_{3,3}\) on \(n(1 \leq n \leq 8)\) vertices. The number of edges in \(G\) is at most \(\frac{5}{2}n - 2\).

Proof. Recall that, an \(n\)-vertex maximal planar graph contains \(3n - 6\) edges. Since \(3n - 6 \leq \frac{5}{2}n - 2\) for \(n \leq 8\), \(e(G) \leq \frac{5}{2}n - 2\) holds for all \(n, 1 \leq n \leq 8\). \[\square\]
Let $u$ be a vertex in $G$ with degree at most 2. By the induction hypothesis, we get $e(G - \{u\}) \leq \frac{20}{7}(n - 1)$. Hence, $e(G) = e(G - \{u\}) + d(u) \leq \frac{5}{7}(n - 1) - 2 + \frac{5}{7}n - 2$. Similarly, if we have a 3-3 edge in $G$, we can finish the proof by induction. From now on, we may assume that $G$ contains no vertex of degree at most 2 and no 3-3 edge. The following claims deal with the different cases of degree pairs in $G$:

**Claim 8.** No vertex in $G$ with degree at least 7 is adjacent to a vertex of degree at least 4.

**Proof.** Suppose not. Let $xy$ be an edge in $G$ such that $d(x) \geq 7$ and $d(y) \geq 4$. Obviously, there are three vertices in $V(G) \setminus \{x\}$, say $y_1, y_2$, and $y_3$, which are adjacent to $y$. Since $|N(x) \setminus \{y\}| \geq 6$, there are three vertices $x_1, x_2$, and $x_3$, not in $\{y, y_1, y_2, y_3\}$ which are adjacent to $x$. This implies that we got an $S_{3,3}$ in $G$ with backbone $xy$ and leaf-sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$, respectively, which is a contradiction. This completes the proof of Claim 8.

**Claim 9.** If there is a 6-6 edge in $G$, then $e(G) \leq \frac{5}{7}n - 2$.

**Proof.** Let $xy \in E(G)$ be a 6-6 edge. Since $G$ is an $S_{3,3}$-free plane graph, $xy$ must be contained in 5 triangles, see Figure 9. Let $a, b, c, d$, and $e$ be the vertices in $G$ which are adjacent to both $x$ and $y$. Let $S_1 = \{a, b, c, d, e\}$ and $H = \{x, y, a, b, c, d, e\}$. Delete the vertices in $H$. Suppose $a$ has two neighbors in $V(G) \setminus H$. We immediately get an $S_{3,3}$ with $xa$ or $ya$ as the backbone. Thus, any vertex in the set $S_1$ can have at most 1 neighbor in $V(G) \setminus H$. If there are no edges between the vertices in $S_1$, we deleted at most $11 + 5 = 16$ edges. Assume that there is an edge between the vertices in $S_1$, say $ab$. If $a$ (or $b$) has a neighbor in $V(G) \setminus H$, $xa$ (or $xb$) is the backbone of an $S_{3,3}$.
Figure 10: The graph $G$ has a 5–6 edge $xy$.

(i) The vertex $e$ has no neighbors in $S_1$.

(ii) The vertex $e$ has one neighbor in $S_1$.

(iii) The vertex $e$ has two neighbors in $S_1$.

Similarly, for the other edges in $S_1$, both vertices cannot have a neighbor in $V(G) \setminus H$. Thus, if there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $11 + 4 = 15$. Using the induction hypothesis,

$$e(G) \leq e(G - H) + 16 \leq \frac{5}{2}(n - 7) - 2 + 16 = \frac{5}{2}n - 3.5 < \frac{5}{2}n - 2.$$ 

This completes the proof. \qed

**Claim 10.** If there is a 5-6 edge in $G$, then $e(G) \leq \frac{5}{2}n - 2$.

**Proof.** Let $xy$ be a 5-6 edge in $G$. Since $G$ is an $S_{3,3}$-free plane graph, $xy$ must be contained in 4 triangles, see Figure 10. Let $a, b, c$ and $d$ be the vertices in $G$ which are adjacent to both $x$ and $y$. Let $e$ be the vertex adjacent to $y$ but not adjacent to $x$. Let $S_1 = \{a, b, c, d\}$ and $H = \{x, y, a, b, c, d, e\}$. Delete the vertices in $H$. Any vertex in $S_1$ can have at most 1 neighbor in $V(G) \setminus H$. If there is an edge joining any two vertices in $S_1$, say $ab$. Similarly, as before, the vertices $a$ and $b$ cannot have a neighbor in $V(G) \setminus H$. We further distinguish the cases based on the number of edges from $e$ as follows:

1. **The vertex $e$ has no neighbors in $S_1$, see Figure 10(i).** Clearly, the vertex $e$ can have at most 2 neighbors in $V(G) \setminus H$. If there are no edges between the vertices in $S_1$, we deleted at
most $10 + 2 + 4 = 16$ edges. If there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $10 + 2 + 3 = 15$.

2. **The vertex $e$ has one neighbor in $S_1$.** Without loss of generality, suppose $a$ is the neighbor of $e$, see Figure 10(ii). The vertex $e$ can have at most one neighbor in $V(G) \setminus H$. If the vertex $a$ has a neighbor in $V(G) \setminus H$, we have an $S_{3,3}$ with $xa$ as the backbone. If there are no edges between the vertices in $S_1$, we have deleted at most $11 + 3 + 1 = 15$ edges. If there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $11 + 3 + 1 = 15$.

3. **The vertex $e$ has two neighbors in $S_1$.** Without loss of generality, suppose $e$ is the neighbor of $a$ and $d$, see Figure 10(iii). The vertex $e$ cannot have a neighbor in $V(G) \setminus H$, otherwise we have an $S_{3,3}$ with $ye$ as the backbone. If either $a$ or $d$ has a neighbor in $V(G) \setminus H$, we have an $S_{3,3}$ with $xa$ or $xd$ as the backbone, respectively. Suppose there is no edge between the vertices in $S_1$. The total number of edges deleted is at most $12 + 2 = 14$. Suppose $a$ and $b$ are adjacent, then $b$ cannot have a neighbor in $V(G) \setminus H$. Similarly, for the other edges in $S_1$ except $ad$, which is still possible without any extra restrictions. If there is an edge joining any two vertices in $S_1$, the total number of edges deleted is at most $12 + 3 = 15$.

Thus, by induction

$$e(G) = e(G - H) + 16 \leq \frac{5}{2}(n - 7) - 2 + 16 < \frac{5}{2}n - 2,$$

and we are done. \qed

**Claim 11.** If there is a 4-6 edge in $G$, then $e(G) \leq \frac{5}{2}n - 2$.

**Proof.** Let $xy$ be a 4 – 6 edge in $G$. Since $G$ is an $S_{3,3}$-free plane graph, $xy$ must be contained in 3 triangles, see Figure 11. Let $a, b$ and $c$ be the vertices in $G$ which are adjacent to both $x$ and $y$. Let $d$ and $e$ be the vertices adjacent to $y$ but not to $x$. Let $S_1 = \{a, b, c\}$ and $H = \{x, y, a, b, c, d, e\}$. Delete the vertices in $H$. The vertices $d$ and $e$ can have at most two neighbors in $V(G) \setminus H$ each. The vertices in $S_1$ can have at most one neighbor in $V(G) \setminus H$ each. If there is an edge joining any two vertices in $S_1$, say $ab$. Similarly, as before, the vertices $a$ and $b$ cannot have a neighbor in $V(G) \setminus H$. We distinguish the cases based on the neighbors of $d$ and $e$ as follows:

1. **The vertices $d$ and $e$ have no neighbors in $S_1$, see Figure 11(i).** If the vertices $d$ and $e$ are adjacent, they can have at most one neighbor in $V(G) \setminus H$ each. Otherwise, we have an
Figure 11: The graph $G$ has a 4−6 edge $xy$.

(i) The vertices $d$ and $e$ have no neighbors in $S_1$.
(ii) The vertex $d$ has one neighbor in $S_1$, and $f$ has none.
(iii) Both the vertices $d$ and $e$ have one common neighbor in $S_1$.
(iv) The vertices $d$ and $e$ have one distinct neighbor in $S_1$.
(v) The vertex $d$ has two neighbors in $S_1$, while $e$ has none.
(vi) The vertex $d$ is the neighbor of $a$ and $c$, while $e$ is the neighbor of $c$.
(vii) The vertex $d$ is the neighbor of $a$ and $c$, while $e$ is the neighbor of $b$.
(viii) Both the vertices $d$ and $e$ have two neighbors in $S_1$. 
$S_{3,3}$ with $dy$ (or $ey$) as the backbone. If there are no edges between the vertices in $S_1$, the number of edges deleted is at most $9 + 3 + 4 = 16$. If there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $9 + 2 + 4 = 15$.

2. **One of the vertices $d$ or $e$ has one neighbor in $S_1$, while the other has none.** Without loss of generality, suppose the vertices $a$ and $d$ are adjacent (see Figure III(ii)). The vertex $a$ cannot have a neighbor in $V(G)\setminus H$, and $d$ can have at most one neighbor in $V(G)\setminus H$. If the vertices $d$ and $e$ are adjacent, then $d$ cannot have a neighbor in $V(G)\setminus H$ and $e$ can have at most one neighbor in $V(G)\setminus H$. Otherwise, we have an $S_{3,3}$ with $dy$ (or $ey$) as the backbone. If there are no edges between the vertices in $S_1$, the number of edges deleted is at most $10 + 2 + 3 = 15$. If there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $10 + 2 + 3 = 15$.

3. **Both the vertices $d$ and $e$ have one neighbor in $S_1$.** There are two possibilities. In the first case, without loss of generality, suppose $a$ is the common neighbor of $d$ and $e$ (see Figure III(iii)). Similarly, as before, $a$ cannot have a neighbor in $V(G)\setminus H$, and $d$ and $e$ can have at most one neighbor in $V(G)\setminus H$ each. If the vertices $d$ and $e$ are adjacent, they cannot have a neighbor in $V(G)\setminus H$. If there are no edges between the vertices in $S_1$, the number of edges deleted is at most $11 + 2 + 2 = 15$. If there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $11 + 2 + 2 = 15$.

4. **One of the vertices $d$ or $e$ has two neighbors in $S_1$, while the other has none.** Without loss of generality, suppose $d$ is the neighbor of $a$ while $c$ is the neighbor of $e$ (see Figure III(iv)). Similarly, as before, the vertices $a$ and $c$ cannot have a neighbor in $V(G)\setminus H$, and $d$ and $e$ can have at most one neighbor in $V(G)\setminus H$ each. If the vertices $d$ and $e$ are adjacent, they cannot have a neighbor in $V(G)\setminus H$. If there are no edges between the vertices in $S_1$, the number of edges deleted is at most $11 + 1 + 2 = 14$. If $a$ and $b$ are adjacent, then $b$ does not have a neighbor in $V(G)\setminus H$. Similarly, for the edge $bc$. The vertices $a$ and $c$ can be adjacent without any constraints. Thus, if there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $11 + 2 + 2 = 15$. 

4. **One of the vertices $d$ or $e$ has two neighbors in $S_1$, while the other has none.** Without loss of generality, suppose $d$ is the neighbor of $a$ and $c$, while $e$ has no neighbors in $S_1$ (see Figure III(v)). The vertex $d$ cannot have a neighbor in $V(G)\setminus H$. Similarly, as before, the vertices $a$ and $c$ cannot have a neighbor in $V(G)\setminus H$, whereas $e$ can have at most
two neighbors in $V(G) \setminus H$. If $d$ and $e$ are adjacent, then $e$ can have at most one neighbor in $V(G) \setminus H$. If there are no edges between the vertices in $S_1$, the number of edges deleted is at most $11 + 1 + 2 = 14$. If $a$ and $b$ are adjacent, then $b$ does not have a neighbor in $V(G) \setminus H$. Similarly, for the edge $bc$. The vertices $a$ and $c$ can be adjacent without any constraints. Thus, if there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $11 + 2 + 2 = 15$.

5. **One of the vertices $d$ or $e$ has two neighbors in $S_1$, while the other has one.** There are two possibilities. In the first case, without loss of generality, suppose $d$ is the neighbor of $a$ and $c$, while $e$ is the neighbor of $c$ (see Figure 11(vi)). Similarly, as before, the vertices $a, c$ and $d$ cannot have a neighbor in $V(G) \setminus H$, whereas $e$ can have at most one neighbor in $V(G) \setminus H$. If $d$ and $e$ are adjacent, then $e$ cannot have a neighbor in $V(G) \setminus H$. If there are no edges between the vertices in $S_1$, the number of edges deleted is at most $12 + 1 + 1 = 14$. In the other case, the vertices $a$ and $c$ can be adjacent without any constraints. Thus, if there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $12 + 2 + 1 = 15$. Without loss of generality, suppose $d$ is the neighbor of $a$ and $c$, while $e$ is the neighbor of $b$ (see Figure 11(vii)). Similarly, as before, the vertices $a, b, c$ and $d$ cannot have a neighbor in $V(G) \setminus H$, whereas $e$ can have at most one neighbor in $V(G) \setminus H$. If $d$ and $e$ are adjacent, then $e$ cannot have a neighbor in $V(G) \setminus H$. If there are no edges between the vertices in $S_1$, the number of edges deleted is at most $12 + 1 = 13$. In the other case, the vertices $a, b$ and $c$ can be adjacent without any constraints. Hence, the number of edges deleted is at most $12 + 2 + 1 = 15$.

6. **Both the vertices $d$ and $e$ have two neighbors in $S_1$.** Without loss of generality, suppose $d$ is the neighbor of $a$ and $c$, while $e$ is the neighbor of $b$ and $c$ (see Figure 11(viii)). Similarly, as before, the vertices $a, b, c, d$ and $e$ cannot have a neighbor in $V(G) \setminus H$. The vertex $c$ can be adjacent to $a$ and $b$ without any constraints. Hence, the number of edges deleted is at most $13 + 2 = 15$.

Thus, by induction

$$e(G) = e(G - H) + 16 \leq \frac{5}{2}(n - 7) - 2 + 16 \leq \frac{5}{2}n - 2,$$

and we are done.
The following lemma completes the proof of the Theorem I(v):

**Lemma 7.** Let $G$ be an $S_{3,3}$-free plane graph on $n$ vertices, then $e(G) \leq \frac{5}{2}n - 2$.

**Proof.** Let $A = \{ x \in V(G) \mid d(x) = 3 \}$, $B = \{ x \in V(G) \mid d(x) = 4 \text{ or } d(y) = 5 \}$ and $C = \{ x \in V(G) \mid d(x) \geq 6 \}$.

Since there is no 3-3 edge, the vertices in $A$ are independent. From Claims II and III there is no edge between the sets $B$ and $C$. Moreover, $C$ is independent by Claim II. The distribution of the edges in $G$ is shown in Figure 12.

![Figure 12: A graph showing the distribution of edges in $G$.](image)

Let $|A| = a$, $|B| = b$ and $|C| = c$. Let $x$ be the number of edges between the sets $A$ and $B$, i.e., $e(A, B) = x$. Since the maximum degree in $B$ is 5, we have $2e(G[B]) = \sum_{v \in B} d(v) - x$ which implies $e(G) \leq \frac{5b-x}{2}$. Each vertex in $A$ has degree 3 and the vertices in $A$ are independent, hence $e(A, C) = 3a - x$. Thus, the number of edges in $G$ is

$$e(G) = e(G[B]) + e(A, B) + e(A, C) \leq \frac{5b-x}{2} + x + 3a - x = \frac{5a+5b+a-x}{2}.$$  

On the other hand, since $G$ is a plane graph and the graph induced by the vertices in $A$ and the vertices in $C$ is bipartite, $e(A, C) = 3a - x \leq 2(a+c) - 4$. This implies that $\frac{a-x}{2} \leq c-2 = \frac{5c-3}{2} - 2$ for all $c \geq 0$. Therefore, using the inequality in (I), we get

$$e(G) \leq \frac{5}{2}a + \frac{5}{2}b + \frac{5}{2}c - \left( \frac{3}{2}c + 2 \right) = \frac{5}{2}(a+b+c) - \left( \frac{3}{2}c + 2 \right) \leq \frac{5}{2}n - 2. \quad (1)$$

24
The last inequality in (11) holds (and hence Lemma 7) if $a \neq 0$ and $c \neq 0$. To finish the proof, we distinguish the following two cases:

**Case 1:** $a \neq 0$ and $c = 0$. Observe that $e(G) \leq \frac{5(n-a)+3a}{2} = \frac{5n}{2} - a$. If $a \geq 2$, then we are done. Thus, $a = 1$. Let the number of degree 4 vertices in $G$ be $k$. Hence, $e(G) = \frac{5(n-k-1)+4k+3}{2} = \frac{5n}{2} - k - 1$. If $k \geq 2$, then we are done.

Let the number of degree 4 vertices in $G$ be at most 1. Let $A = \{x\}$ and $N(x) = \{x_1, x_2, x_3\}$. Let $d(x_i) = 5$, for every $i \in \{1, 2, 3\}$. Considering that there is at most one degree 4 vertex in $G$, it is easy to find a degree 5 vertex in $\bigcup_{i=1}^{3} N(x_i)$, such that all its 5 neighboring vertices are of degree 5. Moreover, the same property holds if one vertex in $N(x)$ is of degree 4. Let $v$ be a degree 5 vertex in $G$, such that all its 5 neighboring vertices are of degree 5. Let $N(v) = \{x_1, x_2, x_3, x_4, x_5\}$, such that a plane drawing of $G$ results in a clockwise alignment of the vertices $x_1, x_2, x_3, x_4, x_5$ around $v$. Since $G$ is an $S_{3,3}$-free plane graph, every 5-5 edge in $G$ must be contained in at least 3 triangles. Thus, an edge $x_1v$ must be contained in at least 3 triangles. This implies, $x_1$ must be adjacent to at least one vertex in $\{x_3, x_4\}$. Without loss of generality, assume $x_1$ and $x_3$ are adjacent. Then the 5-5 edge $x_2v$ is contained in at most 2 triangles, which results in an $S_{3,3}$ in $G$ with $x_2v$ as the backbone.

**Case 2:** $a = 0$. Let the number of degree 4 vertices in $G$ be $k$. Thus, $e(G) = \frac{5(n-k)+4k}{2} = \frac{5n}{2} - k$. If the number of degree 4 vertices is at least 4, then $e(G) \leq \frac{5n}{2} - 2$ and we are done. Now assume that the number of degree 4 vertices in $G$ is at most 3. Notice that, $v(G) \geq 8$. Otherwise, taking any maximal planar graph on $n$ vertices, it can be checked that $3n - 6 < \frac{5}{2}n - 2$.

Let $v$ be a degree 5 vertex in $G$, and $N(v) = \{x_1, x_2, x_3, x_4, x_5\}$. At least two vertices in $N(v)$ must be of degree 5. Otherwise, the number of degree 4 vertices is at least 4 and we are done. Let the plane drawing of $G$ result in a clockwise alignment of the vertices $x_1, x_2, x_3, x_4, x_5$ around $v$. There are exactly 2 vertices in $N(v)$, which are of degree 5. Indeed, suppose that the number of degree 5 vertices is at least 3. We can assume that for some $i \in [5]$, $d(x_i) = d(x_{i+1}) = 5$. Without loss of generality, assume that these vertices are $x_1$ and $x_2$. Since $x_1v$ and $x_2v$ are 5-5 edges, they must be contained in at least 3 triangles. Thus, both $x_1$ and $x_2$ must be adjacent to $x_4$. On the other hand, it is easy to see that $x_3v$ and $x_5v$ are 4-5 edges. Thus, the vertex $x_3$ must be adjacent to $x_2$ and $x_4$. Similarly, the vertex $x_5$ must be adjacent to $x_1$ and $x_4$. Since $d(x_3) = 4$, there must be a vertex $x_6$, such that $x_3x_6 \in E(G)$. If $d(x_6) = 5$, then $x_6$ must be adjacent to $x_4$. This is impossible, as $d(x_4) = 5$. Hence, $d(x_6) = 4$. Similarly, we have another vertex $x_7$ adjacent to $x_5$. 

25
and \(d(x_7) = 4\). This is a contradiction, as we found 4 vertices of degree 4, namely \(x_3, x_5, x_6\) and \(x_7\).

Thus, we can assume that only two vertices in \(N(v)\) are of degree 5. Moreover, the vertices are not consecutive with respect to the alignment in the clockwise direction. Without loss of generality, assume that the vertices are \(x_2\) and \(x_4\). It can be checked that the vertices \(x_2\) and \(x_4\) are adjacent.

Since \(x_1v\) and \(x_5v\) are 4-5 edges, then the edges \(x_1x_2, x_1x_5\) and \(x_5x_4\) are in \(G\). Since \(d(x_3) = 4\), there must exist a vertex \(x_6\) adjacent to the vertex \(x_3\). If this vertex is of degree 4, then it is a contradiction as we found 4 vertices of degree 4, namely \(x_1, x_3, x_5\) and \(x_6\). Hence, \(d(x_6) = 5\) and the edges \(x_2x_6\) and \(x_4x_6\) are in \(G\). Since \(d(x_1) = 4\), there must exist a vertex \(x_7\) such that \(x_1x_7 \in E(G)\).

If \(d(x_7) = 5\), then \(x_7\) is adjacent to \(x_2\) and \(d(x_2) \geq 6\), which is a contradiction. Hence, \(x_7\) must be a vertex of degree 4. This is a contradiction, as we found 4 vertices of degree 4, namely \(x_1, x_3, x_5\) and \(x_7\). This completes the proof of Claim 7 and subsequently the proof of Theorem 1(vi). \(\square\)

7 Planar Turán number of \(S_{3,4}\)

**Proof of the Theorem 1(vi).** Let \(G\) be an \(n\)-vertex \(S_{3,4}\)-free plane graph. Since \(S_{3,4}\) contains 9 vertices, a maximal planar graph with \(n \leq 8\) vertices, does not contain an \(S_{3,4}\). Let \(8 | n\). Consider the plane graph consisting of \(\frac{n}{8}\) disjoint copies of maximal planar graphs on 8 vertices. This graph does not contain an \(S_{3,4}\). Hence, \(\exp(n, S_{3,4}) \geq \frac{9}{7}n\).

**Claim 12.** Let \(G\) be an \(S_{3,4}\) on \(n\) \((1 \leq n \leq 42\) vertices. The number of edges in \(G\) is at most \(\frac{20}{7}n\).

**Proof.** Recall that, an \(n\)-vertex maximal planar graph contains \(3n - 6\) edges. Since \(3n - 6 \leq \frac{20}{7}n\) for \(n \leq 42\), \(e(G) \leq \frac{20}{7}n\) holds for all \(n\), \(1 \leq n \leq 42\). \(\square\)

The following claims deal with the different cases of degree pairs in \(G\):

**Claim 13.** No vertex in \(G\) with a degree at least 8 is adjacent to a vertex of degree at least 4.

**Proof.** Suppose not. Let \(xy\) be an edge in \(G\) such that \(d(x) \geq 8\) and \(d(y) \geq 4\). Obviously, there are three vertices in \(V(G) \setminus \{x\}\), say \(y_1, y_2\) and \(y_3\), which are adjacent to \(y\). Since \(|N(x) \setminus \{y\}| \geq 7\), there are four vertices \(x_1, x_2, x_3\) and \(x_4\), not in \(\{y, y_1, y_2, y_3\}\) which are adjacent to \(x\). This implies we got an \(S_{3,4}\) in \(G\) with backbone \(xy\) and leaf-sets \(\{x_1, x_2, x_3, x_4\}\) and \(\{y_1, y_2, y_3\}\), respectively, which is a contradiction. This completes the proof. \(\square\)

**Claim 14.** If there is a 7-7, 6-7, 5-7 and 6-6 edge in \(G\), then \(e(G) \leq \frac{20}{7}n\).
Proof. The proofs are similar to the Claims 9, 10, 11, and 6 respectively. We skip it here for conciseness, but it is provided in the Appendix A.

Take $x, y \in V(G)$. By the previous claims, if $d(x) + d(y) \geq 12$, we are done by induction. Assume that $d(x) + d(y) \leq 11$. Summing it over all the edge pairs in $G$, we have $11e \geq \sum_{xy \in E(G)}(d(x) + d(y)) \geq nd^2 = n(\frac{2e}{n})^2$, where $d$ is the average degree in $G$. This gives us $e \leq \frac{11}{4}n \leq \frac{20}{7}n$ for $n \geq 1$.

8 Concluding remarks and Conclusions

Concerning the exact value of $\text{exp}_P(n, S_{3,3})$, we conjecture the following:

**Conjecture 1.**

$$\text{exp}_P(n, S_{3,3}) = \begin{cases} 
3n - 6, & \text{if } 3 \leq n \leq 7, \\
16, & \text{if } n = 8, \\
18, & \text{if } n = 9, \\
\left\lfloor \frac{5}{2}n \right\rfloor - 5, & \text{otherwise}.
\end{cases}$$

9 Acknowledgements

Györi’s research was partially supported by the National Research, Development, and Innovation Office NKFIH, grants K132696, SNN135643 and K126853.

References

[1] Chris Dowden. “Extremal $C_4$-Free/$C_5$-Free Planar Graphs”. *J. Graph Theory* 83.3 (2016), pp. 213–230. DOI: [10.1002/jgt.21991](https://doi.org/10.1002/jgt.21991).

[2] Paul Erdős. “On the number of complete subgraphs contained in certain graphs”. *Publ. Math. Inst. Hung. Acad. Sci.* 7 (1962), pp. 459–464.

[3] Paul Erdős. “On the structure of linear graphs”. *Israel Journal of Mathematics* 1 (1963), pp. 156–160. DOI: [10.1007/BF02759702](https://doi.org/10.1007/BF02759702).

[4] Debarun Ghosh, Ervin Györi, Ryan Martin, Addisu Paulos, and Chuanqi Xiao. “Planar Turán number of the 6-cycle”. *arXiv:2004.14094* (2020).

[5] Debarun Ghosh, Ervin Györi, Addisu Paulos, Chuanqi Xiao, and Oscar Zamora. “Planar Turán Number of the $\Theta_6$”. *arXiv:2006.00994* (2020).

[6] Yongxin Lan and Yongtang Shi. “Planar Turán Numbers of Short Paths”. *Graphs Comb.* 35.5 (2019), pp. 1035–1049. DOI: [10.1007/s00373-019-02055-v](https://doi.org/10.1007/s00373-019-02055-v).

[7] Yongxin Lan, Yongtang Shi, and Zi-Xia Song. “Extremal H-Free Planar Graphs”. *Electron. J. Comb.* 26.2 (2019), P2.11.
Yongxin Lan, Yongtang Shi, and Zi-Xia Song. “Extremal Theta-free planar graphs”. *Discret. Math.* 342.12 (2019). doi: [10.1016/j.disc.2019.111610](https://doi.org/10.1016/j.disc.2019.111610)

Pál Turán. “On an extremal problem in Graph Theory”. *Matematikai és Fizikai Lapok* 48 (Nov. 1940), pp. 436–452.

Alexander Aleksandrovich Zykov. “On some properties of linear complexes”. *Matematiceskii Sbornik* 66.2 (1949), pp. 163–188.
A Proof of Lemma 14

Figure 13: $G$ has a 7−7 edge $xy$.

Lemma 8. If there is a 7-7 edge in $G$, then $e(G) \leq \frac{20}{7}n$.

Proof. Let $xy \in E(G)$ be a 7-7 edge. Since $G$ is an $S_{3,4}$-free plane graph, $xy$ must be contained in 6 triangles. Let $a, b, c, d, e$ and $f$ be the vertices in $G$ which are adjacent to both $x$ and $y$, see Figure 13(i). Let $S_1 = \{a, b, c, d, e, f\}$ and $H = \{x, y, a, b, c, d, e, f\}$. Delete the vertices in $H$. Assume $a$ has two neighbors in $V(G) \setminus H$. We immediately get an $S_{3,4}$ with $xa$ or $ya$ as the backbone. Thus, any vertex in the set $S_1$ can have at most 1 neighbor in $V(G) \setminus H$. If there are no edges between the vertices in $S_1$, we deleted at most $13 + 6 = 19$ edges. Assume that there is an edge between the vertices in $S_1$, say $ab$. If $a$ (or $b$) has a neighbor in $V(G) \setminus H$, then $xa$ (or $xb$) is the backbone of an $S_{3,3}$. Similarly, for the other edges in $S_1$, both the vertices cannot have a neighbor in $V(G) \setminus H$. Thus, if there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $13 + 5 = 18$. By the induction hypothesis, we get $e(G - H) \leq \frac{20}{7}(n - 8)$. Hence, $e(G) = e(G - H) + 19 \leq \frac{20}{7}(n - 8) + 19 \leq \frac{20}{7}n$. \hfill \qed

Lemma 9. If there is a 6-7 edge in $G$, then $e(G) \leq \frac{20}{7}n$.

Proof. Let $xy \in E(G)$ be a 6-7 edge. Since $G$ is an $S_{3,4}$-free plane graph, $xy$ must be contained in 5 triangles. Let $a, b, c, d$ and $e$ be the vertices in $G$ which are adjacent to both $x$ and $y$. Let $f$ be the vertex adjacent to $y$ but not to $x$, see Figure 14. Let $S_1 = \{a, b, c, d, e\}$ and $H = \{x, y, a, b, c, d, e, f\}$. 

29
Figure 14: $G$ has a $6 - 7$ edge $xy$.

(i) The vertex $f$ has no neighbors in $S_1$.
(ii) The vertex $f$ has one neighbor in $S_1$.
(iii) The vertex $f$ has two neighbors in $S_1$.

Delete the vertices in $H$. Any vertex in $S_1$ can have at most 1 neighbor in $V(G) \setminus H$. If there is an edge joining any two vertices in $S_1$, say $ab$. Similarly, as before, the vertices $a$ and $b$ cannot have a neighbor in $V(G) \setminus H$. We further distinguish the cases based on the number of edges from $f$ as follows:

1. **The vertex $f$ has no neighbors in $S_1$,** see Figure 14(i). Clearly, the vertex $f$ can have at most 2 neighbors in $V(G) \setminus H$. If there are no edges between the vertices in $S_1$, we deleted at most $12 + 2 + 5 = 19$ edges. If there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $12 + 2 + 4 = 18$.

2. **The vertex $f$ has one neighbor in $S_1$.** Without loss of generality, suppose $a$ is the neighbor of $f$, see Figure 14(ii). The vertex $f$ can have at most one neighbor in $V(G) \setminus H$. If the vertex $a$ has a neighbor in $V(G) \setminus H$, we have an $S_{3,4}$ with $xa$ as the backbone. If there are no edges between the vertices in $S_1$, we have deleted at most $13 + 4 + 1 = 18$ edges. If there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $13 + 4 + 1 = 18$.

3. **The vertex $f$ has two neighbors in $S_1$.** Without loss of generality, suppose $f$ is the neighbor of $a$ and $e$, see Figure 14(iii). The vertex $f$ cannot have a neighbor in $V(G) \setminus H$,
otherwise we have an $S_{3,4}$ with $yf$ as the backbone. If either $a$ or $e$ has a neighbor in $V(G)\setminus H$, we have an $S_{3,4}$ with $xa$ or $xe$ as the backbone, respectively. Suppose there is no edge between the vertices in $S_1$. The total number of edges deleted is at most $14 + 3 = 17$. Suppose $a$ and $b$ are adjacent, then $b$ cannot have a neighbor in $V(G)\setminus H$. Similarly, for the other edges in $S_1$ except $ad$, which is still possible without any extra restrictions. If there is an edge joining any two vertices in $S_1$, the total number of edges deleted is at most $14 + 4 = 18$.

Thus, by induction
\[ e(G) = e(G - H) + 19 \leq \frac{20}{7}(n - 8) + 19 < \frac{20}{7}n, \]
and we are done.

Lemma 10. If there is a 5-7 edge in $G$, then $e(G) \leq \frac{20}{7}n$.

Proof. Let $xy$ be a 5–7 edge in $G$. Since $G$ is an $S_{3,4}$-free plane graph, $xy$ must be contained in 4 triangles. Let $a, b, c$ and $d$ be the vertices in $G$ which are adjacent to both $x$ and $y$. Let $e$ and $f$ be the vertices adjacent to $y$ but not to $x$, see Figure 15. Let $S_1 = \{a, b, c, d\}$ and $H = \{x, y, a, b, c, d, e, f\}$. Delete the vertices in $H$.

The vertices $e$ and $f$ can have at most two neighbors in $V(G)\setminus H$ each. The vertices in $S_1$ can have at most one neighbor in $V(G)\setminus H$ each. If there is an edge joining any two vertices in $S_1$, say $ab$. Similarly, as before, the vertices $a$ and $b$ cannot have a neighbor in $V(G)\setminus H$. We distinguish the cases based on the neighbors of $e$ and $f$ as follows:

1. The vertices $e$ and $f$ have no neighbors in $S_1$, see Figure 15(i)). If the vertices $e$ and $f$ are adjacent, they can have at most one neighbor in $V(G)\setminus H$ each. Otherwise, we have an $S_{3,4}$ with $ey$ (or $fy$) as the backbone. If there are no edges between the vertices in $S_1$, the number of edges deleted is at most $11 + 4 + 4 = 19$. If there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $11 + 3 + 4 = 18$.

2. One of the vertices $e$ or $f$ has one neighbor in $S_1$, while the other has none. Without loss of generality, suppose $e$ and $a$ are adjacent (see Figure 15(ii)). The vertex $a$ cannot have a neighbor in $V(G)\setminus H$, and $e$ can have at most one neighbor in $V(G)\setminus H$. If the vertices $e$ and $f$ are adjacent, then $e$ cannot have a neighbor in $V(G)\setminus H$ and $f$ can have at most one neighbor in $V(G)\setminus H$. If there are no edges between the vertices in $S_1$, the number
Figure 15: $G$ has a $5−7$ edge $xy$.

(i) The vertices $e$ and $f$ have no neighbors in $S_1$.

(ii) The vertex $e$ has one neighbor in $S_1$, and $f$ has none.

(iii) Both the vertices $e$ and $f$ have one common neighbor in $S_1$.

(iv) The vertices $e$ and $f$ have one neighbor in $S_1$ and they are distinct.

(v) One of the vertices $e$ or $f$ has two neighbors in $S_1$.

(vi) Suppose $e$ is neighbor of $a$ and $d$, while $f$ is neighbor of $d$.

(vii) Suppose $e$ is neighbor of $a$ and $d$, while $f$ is neighbor of $c$.

(viii) Suppose $e$ is neighbor of $a$ and $d$, while $f$ is neighbor of $c$ and $d$.

(ix) Suppose $e$ is neighbor of $a$ and $d$, while $f$ is neighbor of $b$ and $c$.

of edges deleted is at most $12 + 3 + 3 = 18$. If there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $12 + 3 + 3 = 18$.

3. **Both the vertices $e$ and $f$ have one neighbor in $S_1$**. There are two possibilities. In the first case, without loss of generality, suppose $a$ is the common neighbor of $e$ and $f$ (see Figure 15(iii)). Similarly, as before, the vertex $a$ cannot have a neighbor in $V(G)\setminus H$, and $e$ and $f$ can have at most one neighbor in $V(G)\setminus H$ each. If the vertices $e$ and $f$ are adjacent, $e$ and $f$ have no neighbors in $V(G)\setminus H$. If there are no edges between the vertices in $S_1$, the number of edges deleted is at most $13 + 3 + 2 = 18$. If there is an edge joining any two vertices in $S_1$, the number of edges deleted is at most $13 + 3 + 2 = 18$.

Without loss of generality, suppose $a$ is the neighbor of $e$ and $d$ is the neighbor of $f$ (see Figure 15(iv)). Similarly, as before, the vertices $a$ and $d$ cannot have a neighbor in $V(G)\setminus H$, and $e$ and $f$ can have at most one neighbor in $V(G)\setminus H$ each. If the vertices $e$ and $f$ are adjacent, then they have no neighbor in $V(G)\setminus H$. If there are no edges between the vertices in $S_1$, the number of edges deleted is at most $13 + 2 + 2 = 17$. If $a$ and $b$ are adjacent, then $b$ does not have a neighbor in $V(G)\setminus H$. Similarly, for the other edges in $S_1$, except $ad$. The vertices $a$ and $d$ can be adjacent without any constraints. Thus, the number of edges deleted is at most $13 + 3 + 2 = 18$. 

33
4. **One of the vertices e or f has two neighbors in S₁, while the other has none.**

Without loss of generality, suppose e is the neighbor of a and d, while f has no neighbors in S₁ (see Figure 15(v)). The vertex e cannot have a neighbor in V(G)\H. Similarly, as before, the vertices a and d cannot have a neighbor in V(G)\H, whereas f can have at most two neighbors in V(G)\H. If e and f are adjacent, then f can have at most one neighbor in V(G)\H. If there are no edges between the vertices in S₁, the number of edges deleted is at most 13 + 2 + 2 = 17. If a and b are adjacent, then b does not have a neighbor in V(G)\H. Similarly, for the other edges in S₁, except ad. The vertices a and d can be adjacent without any constraints. Hence, the number of edges deleted is at most 13 + 3 + 2 = 18.

5. **One of the vertices e or f has two neighbors in S₁, while the other has one.** There are two possibilities. In the first case, without loss of generality, suppose e is the neighbor of a and d, while f is the neighbor of d (see Figure 15(vi)). Similarly, as before, the vertices a, d and e cannot have a neighbor in V(G)\H, whereas f can have at most one neighbor in V(G)\H. If e and f are adjacent, then f cannot have a neighbor in V(G)\H. If there are no edges between the vertices in S₁, the number of edges deleted is at most 14 + 2 + 1 = 17. In the other case, the vertices a and d can be adjacent without any constraints. Hence, the number of edges deleted is at most 14 + 3 + 1 = 18.

Without loss of generality, suppose e is the neighbor of a and d, while f is the neighbor of c (see Figure 15(vii)). Similarly, as before, the vertices a, c, d and e cannot have a neighbor in V(G)\H, whereas f can have at most one neighbor in V(G)\H. If there are no edges between the vertices in S₁, the number of edges deleted is at most 14 + 1 + 1 = 16. In the other case, the vertices a, c and d can be adjacent without any constraints. Hence, the number of edges deleted is at most 14 + 3 + 1 = 18.

6. **Both the vertices e and f have two neighbors in S₁.** There are two possibilities. In the first case, without loss of generality, suppose e is the neighbor of a and d, while f is the neighbor of c and d (see Figure 15(viii)). Similarly, as before, the vertices a, c, d, e and f cannot have a neighbor in V(G)\H. If there are no edges between the vertices in S₁, the number of edges deleted is at most 15 + 1 = 16. In the other case, the vertices a, c and d can be adjacent without any constraints. Thus, the number of edges deleted is 15 + 3 = 18.

Without loss of generality, suppose e is the neighbor of a and d, while f is the neighbor of
b and c (see Figure 15(ix)). Similarly, as before, the vertices a, b, c, d, e and f cannot have a neighbor in $V(G) \setminus H$. The vertices a, b, c and d can be a path of length 3 without any constraints. In this case, the number of edges deleted is $15 + 3 = 18$.

Thus, by induction

$$e(G) = e(G - H) + 19 \leq \frac{20}{7}(n - 8) + 19 < \frac{20}{7}n,$$

and we are done. \hfill \Box

Lemma 11. If there is a 6-6 edge in G, then $e(G) \leq \frac{20}{7}n$.

Proof. Let $xy \in E(G)$ be a 6-6 edge. There are at least 4 triangles sitting on the edge $xy$, otherwise $G$ contains an $S_{3,4}$. We subdivide the cases based on the number of triangles sitting on the edge $xy$.

1. **There are 5 triangles sitting on the edge $xy$.** Let $a, b, c, d$ and $e$ be the vertices in $G$ which are adjacent to both $x$ and $y$, see Figure 8. Let $S_1 = \{a, b, c, d, e\}$, and $H = S_1 \cup \{x, y\}$. Delete the vertices in $H$. The vertices in $S_1$ can have at most one neighbor in $V(G) \setminus H$ each and can form a path of length 4 within $S_1$. Hence, the number of edges deleted is $11 + 5 + 4 = 20$. By the induction hypothesis, we get $e(G - H) \leq \frac{20}{7}(n - 7)$. Hence, $e(G) = e(G - H) + 20 \leq \frac{20}{7}(n - 7) + 20 \leq \frac{20}{7}n$.

2. **There are 4 triangles sitting on the edge $xy$.** Let $a, b, c$ and $d$ be the vertices in $G$ which are adjacent to both $x$ and $y$. Let $e$ be the vertex adjacent to $x$ but not adjacent to $y$, and $f$ be adjacent to $y$ but not adjacent to $x$. Let $S_1 = \{a, b, c, d\}$ and $H = \{x, y\} \cup S_1 \cup \{e, f\}$. Delete the vertices in $H$. The vertices $e$ and $f$ can have at most two neighbors in $V(G) \setminus H$ each. We distinguish the cases based on the neighbors of $e$ and $f$ as follows:

   (a) **The vertices $e$ and $f$ have no neighbors in $S_1$, see Figure 7(i)).** The vertices in $S_1$ can have at most one neighbor in $V(G) \setminus H$ each and can form a path of length 3 in $S_1$. If the vertices $e$ and $f$ are adjacent, then both can have at most one neighbor in $V(G) \setminus H$. Otherwise, we have an $S_{3,4}$ with $ex$ (or $fy$) as the backbone. Thus, the number of edges deleted is at most $11 + 3 + 4 + 4 = 22$.

   (b) **One of the vertices $e$ or $f$ has one neighbor in $S_1$, while the other has none.** Without loss of generality, assume that $e$ and $a$ are adjacent, see Figure 7(ii). The vertex
a cannot have a neighbor in $V(G) \setminus H$, and $e$ can have at most one neighbor in $V(G) \setminus H$. Otherwise, if $a$ has a neighbor in $V(G) \setminus H$, $ya$ is the backbone of an $S_{3,4}$. If the vertices $e$ and $f$ are adjacent, then the vertex $e$ cannot have a neighbor in $V(G) \setminus H$, and $f$ can have at most one neighbor in $V(G) \setminus H$. Otherwise, we have an $S_{3,4}$ with $ex$ (or $fy$) as the backbone. Similarly, as before, the vertices $b, c$ and $d$ can have at most one neighbor in $V(G) \setminus H$ each and the vertices \{a, b, c, d\} can form a path of length 3 in $S_1$. Thus, the number of edges deleted is at most $12 + 3 + 3 + 3 = 21$.

(c) **The vertices $e$ and $f$ have one neighbor in $S_1$.** There are two possibilities. In the first case, without loss of generality, suppose $a$ is the common neighbor of $e$ and $f$, see Figure 7(iii). Similarly, as before, $a$ cannot have a neighbor in $V(G) \setminus H$, and $e$ and $f$ can have at most one neighbor in $V(G) \setminus H$ each. The vertices $b, c$ and $d$ can have at most one neighbor in $V(G) \setminus H$ each, and the vertices \{a, b, c, d\} can form a path of length 3 in $S_1$. If the vertices $e$ and $f$ are adjacent, then they cannot have a neighbor in $V(G) \setminus H$. Thus, the number of edges deleted is at most $13 + 3 + 3 + 2 = 21$.

Without loss of generality, suppose $a$ is the neighbor of $e$, and $d$ is the neighbor of $f$, see Figure 7(iv). Similarly, as before, the vertices $a$ and $d$ cannot have a neighbor in $V(G) \setminus H$, and $e$ and $f$ can have at most one neighbor in $V(G) \setminus H$ each. The vertices $b$ and $c$ can have at most one neighbor in $V(G) \setminus H$ each, and the vertices \{a, b, c, d\} can form a path of length 3 in $S_1$. If the vertices $e$ and $f$ are adjacent, then they cannot have a neighbor in $V(G) \setminus H$. Thus, the number of edges deleted is at most $13 + 3 + 2 + 2 = 20$.

(In fact, it can be shown that if the vertices $e$ and $f$ are adjacent, there can only be a $2$-path inside $S_1$. However, this precision is unnecessary. We skip this in the following cases also.)

(d) **One of the vertices $e$ or $f$ has two neighbors in $S_1$, while the other has none.** Without loss of generality, suppose $e$ is the neighbor of $a$ and $d$, see Figure 7(v).

Similarly, as before, $a$ and $d$ cannot have a neighbor in $V(G) \setminus H$, and $e$ at most one neighbor in $V(G) \setminus H$. The vertices $b$ and $c$ can have at most one neighbor in $V(G) \setminus H$ each, and the vertices \{a, b, c, d\} can form a path of length 3 in $S_1$. If the vertices $e$ and $f$ are adjacent, then $e$ cannot have a neighbor in $V(G) \setminus H$ and $f$ can have at most one neighbor in $V(G) \setminus H$. Thus, the number of edges deleted is at most $13 + 3 + 2 + 3 = 21$.

(e) **One of the vertices $e$ or $f$ has two neighbors in $S_1$, while the other has one**
neighbor. There are two possibilities. In the first case, without loss of generality, suppose \( e \) is the neighbor of \( a \) and \( d \), and \( f \) is the neighbor of \( d \) (see Figure 7(vi)). Similarly, as before, the vertices \( a \) and \( d \) cannot have a neighbor in \( V(G) \setminus H \). The vertices \( e \) and \( f \) can have at most one neighbor in \( V(G) \setminus H \) each. On the other hand, the vertices \( b \) and \( c \) can have at most one neighbor in \( V(G) \setminus H \) each and the vertices \( \{a,b,c,d\} \) can form a path of length 3 in \( S_1 \). If the vertices \( e \) and \( f \) are adjacent, \( e \) and \( f \) cannot have a neighbor in \( V(G) \setminus H \). Thus, the number of edges deleted is at most \( 14 + 3 + 2 + 2 = 21 \).

Without loss of generality, assume that \( e \) is the neighbor of \( a \) and \( d \), and \( f \) is the neighbor of \( b \) (see Figure 7(vii)). The vertices \( a, b \) and \( d \) cannot have a neighbor in \( V(G) \setminus H \), but they along with \( c \) can form a path of length 3 in \( S_1 \). The vertices \( c, e \) and \( f \) can have at most one neighbor in \( V(G) \setminus H \) each. Thus, the number of edges deleted is at most \( 14 + 3 + 1 + 2 = 20 \).

(f) **Both the vertices \( e \) and \( f \) have two neighbors in \( S_1 \).** There are three possibilities.

In the first case, without loss of generality, suppose \( e \) and \( f \) are neighbors of \( a \) and \( d \) both, see Figure 7(viii). Similarly, as before, the vertices \( a \) and \( d \) cannot have a neighbor in \( V(G) \setminus H \). The vertices \( e \) and \( f \) can have at most one neighbor in \( V(G) \setminus H \) each. The vertices \( b \) and \( c \) can have at most one neighbor in \( V(G) \setminus H \) each. The vertices \( \{a,b,c,d\} \) can form a path of length 3 in \( S_1 \). If the vertices \( e \) and \( f \) are adjacent, \( e \) and \( f \) cannot have a neighbor in \( V(G) \setminus H \). Thus, the number of edges deleted is at most \( 15 + 3 + 2 + 2 = 22 \).

On the other hand, without loss of generality, assume \( e \) is the neighbor of \( a \) and \( d \) both, while \( f \) is the neighbor of \( a \) and \( b \) (see Figure 7(ix)). The vertices \( a, b \) and \( d \) cannot have a neighbor in \( V(G) \setminus H \), but they along with \( c \) can form a path of length 3 in \( S_1 \). The vertices \( c, e \) and \( f \) can have at most one neighbor in \( V(G) \setminus H \) each. Thus, the number of edges deleted is at most \( 15 + 3 + 1 + 2 = 21 \).

In the last case, without loss of generality, assume \( e \) is the neighbor of \( a \) and \( d \) both, while \( f \) is the neighbor of \( b \) and \( c \) (see Figure 7(x)). The vertices \( a, b, c \) and \( d \) cannot have a neighbor in \( V(G) \setminus H \), but they can form a path of length 3 in \( S_1 \). The vertices \( e \) and \( f \) can have at most one neighbor in \( V(G) \setminus H \) each. Thus, the number of edges deleted is at most \( 15 + 3 + 2 = 20 \).
By the induction hypothesis, we get \( e(G - H) \leq \frac{20}{7}(n - 8) \). Hence, \( e(G) = e(G - H) + 22 \leq \frac{20}{7}(n - 8) + 22 \leq \frac{20}{7}n \). \qed
This figure "S35L.png" is available in "png" format from:

http://arxiv.org/ps/2110.10515v3