On Weakly Regular Rings and SSF-rings

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1- Introduction

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary ring R-modules. For any non-empty subset X of ring R, r(X) and L(X) denote the right annihilator of X and the left annihilator of X, respectively. Z(R), J(R) will denote respectively the center of R and Jacobson radical of R. Recall that:

1- R is called reduced if R has no non-zero nilpotent elements.
2- R is said to be von Neumann regular (or just regular) if, for any ideal I of R, R/I is flat if and only if for each a ∈ I, there exists b ∈ I such that a = ba.
3- A ring R is said to be right (left) quasi duo ring [3], if every maximal right (left) ideal is a two-sided ideal of R.
4- Following [6], for any ideal of R, R/I is flat if and only if for each a ∈ I, there exists b ∈ I such that a = ba.
5- A ring R is called weakly right duo [4] if for any a ∈ R, there exists a positive integer n such that a^nR is a two-sided ideal of R.
2- Rings Whose Simple Singular Modules are Flat

**Definition 2-1**:--
A ring $R$ is called a right (left) SSF-ring, if every simple singular right (left) $R$-module is flat.

The following lemma is well-known, so we omit its proof.

**Lemma 2-2**:--
For any $a \in Z(R)$, if $ara=a$ for some $r \in R$, then there exists $b \in Z(R)$ such that $a=aba$.

**Proof**:-- see[7]

We consider the condition (*) : $R$ satisfies $L(a) \subseteq r(a)$ for any $a \in R$.

**Proposition 2-3**:--
If $R$ satisfies (*), SSF-ring, then the center $Z(R)$ of $R$ is a Von Neumann regular ring.

**Proof**:-- First we will show that $aR+r(a)=R$ for any $a \in Z(R)$.

If not, there exists a maximal right ideal $M$ of $R$ such that $aR+r(a) \subseteq M$. Since $a \in Z(R)$, $aR+r(a)$ is an essential right ideal and so $M$ must be an essential right ideal of $R$. Since $R/M$ is flat and $a \in M$, then there exists $b \in M$ such that $a=ba$ and hence $(1-b) \in L(a) \subseteq r(a) \subseteq M$ implies $1 \in M$, which is a contradiction. Therefore $aR+r(a)=R$ for any $a \in Z(R)$ and so we have $a=ara$ for some $r \in R$. Applying Lemma (2-2) $Z(R)$ is a Von Neumann regular ring.

Recall that a ring $R$ is right (left) weakly regular if $I^2=I$ for each right (left) ideal $I$ of $R$; equivalently, $a \in aRa$ ($a \in RaR$) for every $a \in R$. $R$ is weakly regular if it is both right and left weakly regular[5].

**Lemma 2-4**:--
If $R$ satisfies (*), then $RaR+r(a)$ is an essential right ideal.

**Proof**:-- see[7]

**Theorem 2-5**:--
If $R$ satisfies (*), and SSF-ring, then $R$ is a reduced weakly regular ring.

**Proof**:-- Let $a^2=0$. Suppose that $a \neq 0$. By Lemma (2-4), $r(a)$ is essential right ideal of $R$. Since $a \neq 0$, $r(a) \neq R$. Thus there exists a maximal essential right ideal $M$ of $R$ containing $r(a)$. Since $R/M$ is flat and $a \in M$ there exists $b \in M$ such that $a=ba$ and hence $(1-b) \in L(a) \subseteq r(a) \subseteq M$ and so $1 \in M$, which is a contradiction. Hence $a=0$ and so $R$ is reduced.

Now, we will show that $RaR+r(a)=R$ for any $a \in R$. Suppose that there exists $b \in R$ such that $RbR+r(b) \neq R$. Then there exists a maximal right ideal $M$ of $R$ containing $RbR+r(b)$. By Lemma (2-4), $M$ must be essential in
R. Therefore \( R/M \) is flat. Then there exists \( c \in M \) such that \( b = cb \) and hence \((1-c)e \in L(b) \subseteq r(b) \subseteq M \) and so \( 1 \in M \), which is a contradiction. Therefore \( RaR + r(a) = R \) for any \( a \in R \). Hence \( R \) is a right weakly regular ring. Since \( R \) is reduced, it also can be easily verified that \( R \) is a weakly regular ring.

**Corollary 2-6:**

If \( R \) is a reduced and SSF-ring, then \( R \) is a weakly regular ring.

**Lemma 2-7:**

Let \( R \) be a right quasi duo ring. Then \( R/J(R) \) is a reduced ring.

**Proof:**

see [6]

**Proposition 2-8:**

Let \( R \) be a right quasi duo ring. The following statements are equivalent.

a) \( R \) is a right weakly regular ring.

b) \( R \) is a strongly regular ring.

**Proof:**

see [6]

**Proposition 2-9:**

Every weakly right (left) duo ring is right (left) quasi-duo.

**Proof:**

see [1]

**Proposition 2-10:**

Let \( R \) be a right (left) quasi duo ring. If every simple singular right \( R \)-module is flat, then \( R/J(R) \) is strongly regular.

**Proof:**

Let \( 0 \neq \overline{a} \in \overline{R} = R/J(R) \). We will show that \( \overline{RaR} + r(\overline{a}) = \overline{R} \).

Suppose not. Then there exists a maximal right ideal \( M \) of \( R \) such that \( \overline{RaR} + r(\overline{a}) \subseteq M/J(R) \). From Lemma (2-7), \( \overline{R} \) is reduced we have \( r(\overline{a}) = L(\overline{a}) \) for any \( \overline{a} \in \overline{R} \). Then by Lemma (2-4) \( \overline{RaR} + r(\overline{a}) \) is an essential right ideal of \( \overline{R} \). Thus \( R/J(R) \) must be right essential in \( \overline{R} \). Therefore \( R/M \) is a simple singular right \( R \)-module and so \( R/M \) is flat, then there exists \( c \in M \) such that \( a = ca \) and hence \((1-c)e \in L(a) \subseteq r(a) \subseteq M \) and so \( 1 \in M \), which is a contradiction. Therefore \( R/J(R) \) is right weakly regular since \( R/J(R) \) is reduced it also can be easily verified that \( R/J(R) \) is a weakly regular ring. By proposition (2-8), \( R \) is a strongly regular ring.

**Corollary 2-11:**

Let \( R \) be a weakly right duo, SSF-ring. Then \( R/J(R) \) is a right weakly regular ring.
Proof: By Proposition (2-9) R is a right quasi duo ring. Also by Proposition (2-10) \(R/J(R)\) is a right weakly regular ring.

A ring \(R\) is called strongly right bounded (briefly SRB) [2] if every non-zero right ideal contains a non-zero two-sided ideal of \(R\).

Lemma 2-12: -
If \(R\) is a semi prime SRB ring, then \(R\) is reduced.
Proof: - see[2]

Theorem 2-13: -
Let \(R\) be a SRB and SSF-ring. Then \(R\) is a reduced weakly regular.

Proof: - By Corollary(2-6) and Lemma (2-12), it is enough to show that \(R\) is semi prime. Suppose that there exists a non-zero right ideal \(A\) of \(R\) such that \(A^2=0\). Then there exists \(0\neq a\in A\) such that \(a^2=0\). First observe that \(r(a)\) is an essential right ideal of \(R\). If not there exists a non-zero right ideal \(K\) such that \(r(a)\oplus K\) is right essential in \(R\). Since \(R\) is SRB, there is a non-zero ideal \(I\) of \(R\) such that \(I\subseteq K\).

Now \(aI\subseteq aR\cap I\subseteq r(a)\cap K=0\). Hence \(I\cap r(a)\cap K=0\) (\(aI\subseteq I\)).

This is a contradiction. Thus \(r(a)\) must be a proper essential right ideal of \(R\). Hence there exists a maximal right ideal \(M\) of \(R\) containing \(r(a)\).

Clearly \(M\) is an essential right ideal of \(R\), \(R/M\) is flat, then there exists \(c\in M\) such that \(a=ca\). Now \(aca\in aRa\subseteq A^2=0\) and so \(1\in M\), which is a contradiction.

Therefore \(R\) must be semi-prime, hence \(R\) is a reduced weakly regular.
REFERENCES

[1] Hua-ping (1995) On quai-duo rings, Glas. Math. J. 37.

[2] Kim, N.K.; Nam, S.B. and Kim J.Y.(1999) On Simple singular Gp-injective modules, Comm. in Alg. 27(5), 2087-2096.

[3] Ming, R.Y.C.(1974) On Von Neumann regular rings, proc. Edin. Math. Soc. 19, 89-91.

[4] Ming, R.Y.C.(1992) A note on injective rings, Hokk. Math. J. Vol.21, 231-238.

[5] Ramamurthi, V.S.(1973) Weakly regular ring, Cand Math. Bull. 16, 317-321.

[6] Rege, M.B.,(1986) On Von Neumann regular ring and SF-ring Math. Japonica 31, 927-936.

[7] Sang, B.N.(1999) Anote on simple singular GP-injective modules, Kang. Kyu. Math. J. 7 no.2, 215-218.