A PRIORI ESTIMATES FOR SEMISTABLE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider positive semistable solutions $u$ of $Lu + f(u) = 0$ with zero Dirichlet boundary condition, where $L$ is a uniformly elliptic operator and $f \in C^2$ is a positive, nondecreasing, and convex nonlinearity which is superlinear at infinity. Under these assumptions, the boundedness of all semistable solutions is expected up to dimension $n \leq 9$, but only established for $n \leq 4$.

In this paper we prove the $L^\infty$ bound up to dimension $n = 5$ under the following further assumption on $f$: for every $\varepsilon > 0$, there exist $T = T(\varepsilon)$ and $C = C(\varepsilon)$ such that $f'(t) \leq Cf(t)^{1+\varepsilon}$ for all $t > T$. This bound will follow from a $L^p$-estimate for $f'(u)$ for every $p < 3$ (and for all $n \geq 2$). Under a similar but more restrictive assumption on $f$, we also prove the $L^\infty$ estimate when $n = 6$. We remark that our results do not assume any lower bound on $f'$.

1. Introduction. In this note we consider semistable solutions of the boundary value problem

$$\begin{cases}
Lu + f(u) &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain with $n \geq 2$, $f \in C^2$, and $Lu := \partial_i(a^{ij}(x)u_j)$ is uniformly elliptic. More precisely, we assume that $(a^{ij}(x))$ is a symmetric $n \times n$ matrix with bounded measurable coefficients, i.e., $a^{ij} = a^{ji} \in L^\infty(\Omega)$, for which there exist positive constants $c_0$ and $C_0$ satisfying

$$c_0|\xi|^2 \leq a^{ij}(x)\xi^i\xi^j \leq C_0|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \ x \in \Omega. \quad (1.2)$$

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By semistability of the solution \( u \), we mean that the lowest Dirichlet eigenvalue of the linearized operator at \( u \) is nonnegative. That is, we have the semistability inequality

\[
\int_{\Omega} f'(u)\eta^2 \, dx \leq \int_{\Omega} a^{ij}(x) \eta_i \eta_j \, dx \quad \text{for all } \eta \in H^1_0(\Omega).
\]

(1.3)

There is a large literature on a priori estimates for semistable solutions, beginning with the seminal paper of Crandall and Rabinowitz [5]. In [5] and subsequent works, a basic and standard assumption is that \( u \) is positive in \( \Omega \) and \( f \in C^2 \) is positive, nondecreasing, and superlinear at infinity:

\[
f(0) > 0, \quad f' \geq 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{f(t)}{t} = \infty.
\]

(1.4)

Note that, under these assumptions and with \( f(u) \) replaced by \( \lambda f(u) \) with \( \lambda \geq 0 \), semistable solutions do exist for an interval of parameters \( \lambda \in (0, \lambda^*) \); see [5].

In recent years there have been strong efforts to obtain a priori bounds under minimal assumptions on \( f \) (essentially (1.4)), mainly after Brezis and Vázquez [2] raised several open questions (see also the open problems raised by Brezis in [1]).

The following are the main results in this direction. The important paper of Nedev [6] obtains the \( L^\infty \) bound for \( n = 2 \) and \( 3 \) if \( f \) satisfies (1.4) and in addition \( f \) is convex. Nedev states his result for \( L = \Delta \) but it is equally valid for general \( L \). When \( 2 \leq n \leq 4 \) and \( L = \Delta \), Cabré [3] established that the \( L^\infty \) bound holds for arbitrary \( f \) if in addition \( \Omega \) is convex. Villegas [10] replaced the condition that \( \Omega \) is convex in Cabré’s result assuming instead that \( f \) is convex. For the radial case, Cabré and Capella [4] proved the \( L^\infty \) bound for arbitrary \( f \) when \( n \leq 9 \). On the other hand, it is well known that there exist unbounded semistable solutions when \( n \geq 10 \) (for instance, for the exponential nonlinearity \( e^u \)).

For convex nonlinearities \( f \) and under extra assumptions involving the two numbers

\[
\tau_- := \liminf_{t \to \infty} \frac{f(t)f''(t)}{f'(t)^2} \leq \tau_+ := \limsup_{t \to \infty} \frac{f(t)f''(t)}{f'(t)^2}
\]

(1.5)

much more is known (see more detailed comments after Corollary 1.3). For instance, Sanchón [7] proved that \( u \in L^\infty(\Omega) \) whenever \( \tau_- = \tau_+ \geq 0 \) and \( n \leq 9 \). This hypothesis is satisfied by \( f(u) = e^u \), as well as by \( f(u) = (1 + u)^m \), \( m > 1 \).

It is still an open problem to establish an \( L^\infty \) estimate in general domains \( \Omega \) when \( 5 \leq n \leq 9 \) under (1.4) and the convexity of \( f \) as the only assumptions on the nonlinearity.

Our purpose here is to prove the following results:

**Theorem 1.1.** Let \( f \in C^2 \) be convex and satisfy (1.4). Assume in addition that for every \( \varepsilon > 0 \), there exist \( T = T(\varepsilon) \) and \( C = C(\varepsilon) \) such that

\[
f'(t) \leq Cf(t)^{1+\varepsilon} \quad \text{for all } t > T.
\]

(1.6)

If \( u \) is a positive semistable solution of (1.1), then we have \( f'(u) \in L^p(\Omega) \) for all \( p < 3 \) and \( n \geq 2 \), while \( f(u) \in L^p(\Omega) \) for all \( p < \frac{n}{n-3} \) and \( n \geq 6 \).

As a consequence, we deduce respectively:

(a) If \( n \leq 5 \), then \( u \in L^\infty(\Omega) \).

(b) If \( n \geq 6 \), then \( u \in W^{1,p}_0(\Omega) \) for all \( p < \frac{n}{n-6} \) and \( u \in L^p(\Omega) \) for all \( p < \frac{n}{n-6} \). In particular, if \( n \leq 9 \) then \( u \in H^1_0(\Omega) \).
Theorem 1.1 establishes the $L^\infty$ bound up to dimension 5 when \((1.6)\) holds for every $\varepsilon > 0$ (with $C = C(\varepsilon)$ and $T = T(\varepsilon)$). If we assume more about $f$ we can obtain an $L^\infty$ bound up to dimension $n = 6$.

**Theorem 1.2.** Let $f \in C^2$ be convex and satisfy \((1.4)\). Assume in addition that there exist positive constants $\varepsilon \in (0, 1)$, $T$, and $C$ such that

$$f'(t) \leq Cf(t)^{1-\varepsilon} \quad \text{for all } t > T. \quad (1.7)$$

If $u$ is a positive semistable solution of \((1.1)\), then we have $f'(u) \in L^{\frac{3-\varepsilon}{1-\varepsilon}}(\Omega)$ for all $n \geq 2$, while $f(u) \in L^p(\Omega)$ for all $p < \frac{(1-\varepsilon)n}{(1-\varepsilon)n - 4 + 2\varepsilon}$ and $n \geq 6 + \frac{4\varepsilon}{1-\varepsilon}$.

As a consequence, we deduce respectively:

(a) If $n < 6 + \frac{4\varepsilon}{1-\varepsilon}$, then $u \in L^\infty(\Omega)$.
(b) If $n \geq 6 + \frac{4\varepsilon}{1-\varepsilon}$, then $u \in W^{1,p}_0(\Omega)$ for all $p < \frac{(1-\varepsilon)n}{(1-\varepsilon)n - 4 + 2\varepsilon}$ and $u \in L^p(\Omega)$ for all $p < \frac{(1-\varepsilon)n}{(1-\varepsilon)n - 6 + 4\varepsilon}$. In particular, if $n < 10 + \frac{4\varepsilon}{1-\varepsilon}$ then $u \in H^1_0(\Omega)$.

The main novelty of our results is twofold. On the one hand, we do not assume any lower bound on $f'$ to obtain our estimates, nor any bound on $f''$ as in [5] or [7] (as commented next). On the other hand, we obtain $L^p$ estimates for $f'(u)$. To our knowledge such estimates do not exist in the literature. In fact, using the $L^p$ estimate for $f(u)$ established in Theorem 1.2 and standard regularity results for uniformly elliptic equations, it follows that $u$ is bounded in $L^\infty(\Omega)$ whenever $n < 6 + \frac{4\varepsilon}{1-\varepsilon}$. Note that the range of dimensions obtained in Theorem 1.2 (a), $n < 6 + \frac{4\varepsilon}{1-\varepsilon}$, is bigger than this one. This will follow from the $L^p$ estimate on $f'(u)$.

Of course, in both results (and also in the rest of the paper), $u \in L^p$ or $u \in W^{1,p}$ means that $u$ is bounded in $L^p$ or in $W^{1,p}$ by a constant independent of $u$.

Our assumptions \((1.6)\) and \((1.7)\) in Theorems 1.1 and 1.2 are related to the hypothesis $\tau_+ \leq 1$ (recall \((1.5)\)). Indeed, by the definition of $\tau_+$, for every $\delta > 0$ there exists $T = T(\delta)$ such that $f(t)f''(t) \leq (\tau_+ + \delta)f'(t)^2$ for all $t > T$, or equivalently, $\frac{d}{dt} \frac{f'(t)}{(\tau_+ + \delta)} \leq 0$ for all $t > T$. Thus,

$$\frac{f'(t)}{(\tau_+ + \delta)} \leq \frac{f'(T)}{(\tau_+ + \delta)} = C \quad \text{for all } t > T.$$

From this, it is clear that if $\tau_+ \leq 1$, then assumption \((1.6)\) holds choosing $\delta = \varepsilon$. This (and also the following statement) is shown using the previous argument, that is, establishing an inequality for $\frac{d}{dt} \frac{f'(t)}{f(t)}$. Note that $0 \leq \tau_- \leq 1$ always holds since $f$ is a continuous function defined in $[0, +\infty)$. If instead $\tau_+ < 1$, then \((1.7)\) is satisfied with $\varepsilon = 1 - \tau_+ - \delta$, where $\delta > 0$ is arbitrarily small. Therefore as an immediate consequence of part (i) of Theorems 1.1 and 1.2 we obtain the following.

**Corollary 1.3.** Let $f \in C^2$ be convex and satisfy \((1.4)\). Let $u$ be a positive semistable solution of \((1.1)\). The following assertions hold:

(a) If $\tau_+ = 1$ and $n < 6$, then $u \in L^\infty(\Omega)$.
(b) If $\tau_+ < 1$ and $n < 2 + \frac{4}{\tau_+}$, then $u \in L^\infty(\Omega)$.

If $\tau_+ < 1$, then for every $\varepsilon \in (0, 1 - \tau_+)$, there exists a positive constant $C$ such that $f(t) \leq C(1 + t)^{-\frac{\varepsilon}{\tau_+ - \varepsilon}}$ for all $t \geq 0$ (this can be easily seen integrating twice in the definition of $\tau_+$). Thus under this hypothesis, $f$ has at most polynomial growth.

All the results in the literature considering $\tau_-$ and $\tau_+$ (defined in \((1.5)\)) assume $\tau_- > 0$. Instead in Corollary 1.3, no assumption is made on $\tau_-$. 
Crandall and Rabinowitz [5] proved an a priori $L^\infty$ bound for semistable solutions when $0 < \tau_- < \tau_+ < 2 + \tau_- + 2\sqrt{\tau_-}$ and $n < 4 + 2\tau_+ + 4\sqrt{\tau_-}$. Note that for nonlinearities $f$ such that $\tau_- = 1$ and $\tau_+ < 5$ one obtains the $L^\infty$ bound if $n \leq 9$ (a dimension which is optimal). This is the case for many exponential type nonlinearities, as for instance $f(u) = e^{au}$ for any $a \in \mathbb{R}^+$. The results in [5] were improved by Sanchón in [7] establishing that $u \in L^\infty(\Omega)$ whenever $\tau_- > 0$ and $n < 6 + 4\sqrt{\tau_-}$ (remember that $\tau_- \leq 1$). Moreover, if $0 < \tau_- \leq \tau_+ < 1$, then using an iteration argument in [5], one has that $u \in L^\infty(\Omega)$ whenever $n < 2 + \frac{4}{\tau_+} \left(1 + \sqrt{\tau_-}\right)$. Note that Corollary 1.3 coincides with these results in the case where $\tau_- = 0$.

Let us make some further comments on conditions (1.6) and (1.7) in Theorems 1.1 and 1.2, respectively.

**Remark 1.4.** (i) Condition (1.6) is equivalent to

$$\limsup_{t \to +\infty} \frac{\log f'(t)}{\log f(t)} \leq 1,$$

since (1.6) holds if and only if

$$\frac{\log f'(t)}{\log f(t)} \leq (1 + \varepsilon) + \frac{C}{\log f(t)}$$

for all $t > T$; note that $f(t) \to +\infty$ as $t \to +\infty$ by (1.4). Many nonlinearities $f$ satisfy this condition (like exponential or power type nonlinearities).

(ii) Setting $s = f(t)$ and $t = \gamma(s)$, (1.6) is equivalent to the condition $\gamma'(s) \geq \theta s^{-1-\varepsilon}$ for some $\theta > 0$ and for all $s$ sufficiently large. This clearly shows that (1.6) does not follow from the convexity of $f$ alone (which is equivalent to $\gamma'$ being nonincreasing).

Instead, condition (1.7) is equivalent to $\gamma'(s) \geq \theta s^{-1+\varepsilon}$ for some $\theta > 0$ and for all $s$ sufficiently large. In particular, $\gamma(s) \geq \theta s^\varepsilon$ for $s$ large enough, or equivalently, $f(t) \leq C(1 + t)^{1/2}$ for some constant $C > 0$ and for all $t$.

On the other hand, $f(t)f''(t)/f'(t)^2 = -s\gamma''(s)/\gamma'(s)$, a second derivative condition on $\gamma$, in contrast with the first derivative conditions of (1.6) and (1.7). Therefore, for most nonlinearities satisfying (1.6) (or (1.7)), the limit $f_f''(f')^2$ at infinity does not exist (i.e., $\tau_- < \tau_+$) and in addition, it may happen that $\tau_- = 0$.

(iii) Note that by convexity, $\varepsilon f'(t) \leq f(t+\varepsilon) - f(t) \leq f(t+\varepsilon)$ for all $t$. Therefore, (1.6) holds if $f(t+\varepsilon) \leq C f(t)^{1+\varepsilon}$ for all $t$ sufficiently large.

2. Preliminary estimates. We start by recalling the following standard regularity result for uniformly elliptic equations.

**Proposition 2.1.** Let $a^{ij} = a^{ji}$, $1 \leq i, j \leq n$, be measurable functions on a bounded domain $\Omega$. Assume that there exist positive constants $c_0$ and $C_0$ such that (1.2) holds. Let $u \in H^2_0(\Omega)$ be a weak solution of

$$\begin{cases}
Lu + c(x)u = g(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

with $c, g \in L^p(\Omega)$ for some $p \geq 1$.

Then, there exists a positive constant $C$ independent of $u$ such that the following assertions hold:

(i) If $p > n/2$, then $\|u\|_{L^\infty(\Omega)} \leq C(\|u\|_{L^1(\Omega)} + \|g\|_{L^p(\Omega)})$.

(ii) Assume $c \equiv 0$. If $1 \leq p < n/2$, then $\|u\|_{L^r(\Omega)} \leq C\|g\|_{L^r(\Omega)}$ for every $1 \leq r < np/(n-2p)$. Moreover, $\|u\|_{W^{1,\gamma}_c(\Omega)} \leq C$ for every $1 \leq r < np/(n-p)$.
Part (i) of Proposition 2.1 is established in Theorem 3 of [8] with the $L^2$-norm of $u$ instead of the $L^1$-norm. However, an immediate interpolation argument shows that the result also holds with $\|u\|_{L^1(\Omega)}$. Note also that in the right hand side of this estimate, $\|u\|_{L^1(\Omega)} \leq C(\|u\|_{L^1(\Omega)} + \|g\|_{L^p(\Omega)})$, the term $\|u\|_{L^1(\Omega)}$ cannot be removed (think on the equation with $g \equiv 0$ satisfied by the eigenfunctions of the Laplacian). For part (ii) we refer to Theorems 4.1 and 4.3 of [9].

As an easy consequence of Proposition 2.1 (i) we obtain the following:

**Corollary 2.2.** Let $u \in H^1_0(\Omega)$ be a nonnegative weak solution of (1.1) with $f$ nondecreasing and convex. Assume $p > n/2$. If there exists a positive constant $C_1$ such that $\|u\|_{L^1(\Omega)} \leq C_1$ and $\|f'(u)\|_{L^p(\Omega)} \leq C_1$, then $\|u\|_{L^\infty(\Omega)} \leq C$ for some positive constant $C$ independent of $u$ (but depending on $C_1$).

**Proof.** Rewrite equation (1.1) as $Lu + c(x)u = -f(0)$ where $c(x) = (f(u) - f(0))/u$. Then by convexity, $0 \leq c(x) \leq f'(u)$ and the result follows by Proposition 2.1 (i). \qed

The following estimates involving

$$\tilde{f}(u) := f(u) - f(0)$$

are due to Nedev [6] when $L = \Delta$. We give here a new proof of the estimates consistent with our own approach. Note that assumptions (1.6) and (1.7) in Theorems 1.1 and 1.2, respectively, also hold replacing $f$ by $\tilde{f}$ on their right hand side, since $f(t) \leq 2(f(t) - f(0)) = 2\tilde{f}(t)$ for $t$ large enough. We will use this fact in the proof of both results.

**Lemma 2.3.** Let $f \in C^2$ be convex and satisfy (1.4). If $u$ is a positive semistable solution of (1.1), then there exists a positive constant $C$ independent of $u$ such that

$$\int_{\Omega} \tilde{f}(u)f'(u)\,dx \leq C \quad \text{and} \quad \int_{\Omega} \tilde{f}(u)f''(u)a^{ij}(x)u_iu_j\,dx \leq C. \quad (2.1)$$

**Proof.** Let $u$ be a semistable solution of (1.1) and $\tilde{f}(u) = f(u) - f(0)$. Note that $\tilde{f}(u)$ satisfies

$$L(\tilde{f}(u)) + f'(u)\tilde{f}(u) = -f(0)f'(u) + f''(u)a^{ij}(x)u_iu_j.$$ 

Multiplying the previous identity by $\tilde{f}(u)$ and using the semistability condition (1.3), we obtain

$$0 \leq \int_{\Omega} \left( a^{ij}(x)\tilde{f}(u)i(\tilde{f}(u))j - f'(u)\tilde{f}(u)^2 \right)\,dx$$

$$= f(0)\int_{\Omega} f'(u)\tilde{f}(u)\,dx - \int_{\Omega} \tilde{f}(u)f''(u)a^{ij}(x)u_iu_j\,dx,$$

or equivalently,

$$\int_{\Omega} \tilde{f}(u)f''(u)a^{ij}(x)u_iu_j\,dx \leq f(0)\int_{\Omega} \tilde{f}(u)f'(u)\,dx. \quad (2.2)$$

As a consequence, the second estimate in (2.1) follows by the first one.

Multiplying the equation (1.1) by the test functions $\zeta = f'(u) - f'(0)$ and

$$\zeta = \begin{cases} 
0 & \text{if } u \leq M \\
 f'(u) - f'(M) & \text{if } u > M,
\end{cases}$$

we obtain a similar estimate to (2.1) as a consequence of Proposition 2.1 (ii) when $u$ satisfies (1.6) and (1.7) with $f(0) = 0$.
we find
\[ \int_{\Omega} f''(u)a^{ij}(x)u_iu_j \, dx = \int_{\Omega} f(u)(f'(u) - f'(0)) \, dx \quad (2.3) \]
and
\[ \int_{\{u > M\}} f''(u)a^{ij}(x)u_iu_j \, dx = \int_{\{u > M\}} f(u)(f'(u) - f'(M)) \, dx, \quad (2.4) \]
respectively.
Combining (2.2) and (2.3), we obtain
\[ \int_{\Omega} (f(u) - 2f(0))f''(u)a^{ij}(x)u_iu_j \, dx \leq f(0)f'(0) \int_{\Omega} f(u) \, dx - f(0)^2 \int_{\Omega} f'(u) \, dx. \quad (2.5) \]
Choose \( M \) (depending on \( f \)) such that \( f(t) > 2f(0) + 2 \) for all \( t \geq M \). On the one hand, using (2.4), the convexity of \( f \), and that \((a^{ij})\) is a positive definite matrix, we obtain
\[ 2 \int_{\{u > M\}} f(u)(f'(u) - f'(M)) \, dx = 2 \int_{\{u > M\}} f''(u)a^{ij}(x)u_iu_j \, dx \leq \int_{\{u > M\}} (f(u) - 2f(0))f''(u)a^{ij}(x)u_iu_j \, dx. \]
On the other hand, for some constant \( C \) depending only on \( f \) (and \( M \)), there holds
\[ - \int_{\{u \leq M\}} (f(u) - 2f(0))f''(u)a^{ij}(x)u_iu_j \, dx \leq C \int_{\{u \leq M\}} a^{ij}(x)u_iu_j \, dx \leq CM \int_{\Omega} f(u) \, dx, \]
where the last inequality follows from multiplying equation (1.1) by \( \min\{u, M\} \).
Combining the previous bounds with (2.5), it follows that
\[ 2 \int_{\{u > M\}} f(u)(f'(u) - f'(M)) \, dx \leq f(0)f'(0) \int_{\Omega} f(u) \, dx + CM \int_{\Omega} f(u) \, dx. \quad (2.6) \]
Finally, choose \( \overline{M} > M \) (depending only on \( f \)) such that \( f'(M) < \frac{f'(t)}{t} \) if \( t > \overline{M} \) (see Remark 2.4 below). Then (2.6) implies
\[ \int_{\{u > \overline{M}\}} f(u)f'(u) \, dx \leq C \int_{\Omega} f(u) \, dx, \]
and using that \( f'(t) \to +\infty \) at infinity (again by Remark 2.4), we conclude
\[ \int_{\Omega} f(u)f'(u) \, dx \leq C, \]
where \( C \) is independent of \( u \). \( \square \)

**Remark 2.4.** Note that \( \bar{f}(t)/t \leq f'(t) \) for all \( t \geq 0 \) since \( f \) is convex. In particular, by condition (1.4), we obtain \( \lim_{t \to \infty} f'(t) = \infty \). Therefore, as a consequence of estimate (2.1) we obtain
\[ \int_{\Omega} f(u) \, dx \leq C, \quad (2.7) \]
where \( C \) is a constant independent of \( u \). As in [6], from this and Proposition 2.1 (ii), one deduces
\[ u \text{ in } L^q(\Omega) \text{ for all } q < n/(n-2). \quad (2.8) \]
Our results improve this estimate under the additional assumptions on \( f \) of Theorems 1.1 and 1.2.
Therefore, by the arbitrariness of $\varepsilon$, a consequence, by Corollary 2.2 and since $u$ in particular, by Lemma 2.3 and the bound (2.7), we obtain by convexity of $\varepsilon$ independent of $u$, all $p < n / 2$.

In the following, the constants $C$ assume that (1.6) holds replacing $f$ by $\tilde{f}$: for every $\varepsilon > 0$, there exist $T = T(\varepsilon)$ and $C = C(\varepsilon)$ such that

$$f'(t) \leq C\tilde{f}(t)^{1+\varepsilon} \quad \text{for all } t > T. \quad (3.1)$$

In the following, the constants $C$ may depend on $\varepsilon$ and $T$ but are independent of $u$.

We start by proving that $f'(u) \in L^p(\Omega)$ for all $p < 3$ and as a consequence the statement in part (a). Let $\alpha = \frac{3+\varepsilon}{1+\varepsilon}$ (with $\varepsilon$ as in (3.1)). Multiplying (1.1) by $(f'(u) - f'(0))^\alpha$ and integrating by parts we obtain

$$\int_{\Omega} \frac{f'(u)}{1 + f'(u)}(f'(u) - f'(0))^\alpha dx + \int_{\Omega} \frac{f''(u)}{(1 + f'(u))^2} (f'(u) - f'(0))^\alpha a^{ij}(x)u_iu_j dx$$

$$= \alpha \int_{\Omega} \frac{f''(u)}{1 + f'(u)}(f'(u) - f'(0))^\alpha - 1 a^{ij}(x)u_iu_j dx$$

$$\leq \alpha \int_{\{u \leq T\}} \frac{f'(u)^{\alpha-1}f''(u)}{1 + f'(u)} a^{ij}(x)u_iu_j dx$$

$$+ C \int_{\{u > T\}} (f(u)^{(1+\varepsilon)(\alpha-1)-1}f''(u)a^{ij}(x)u_iu_j dx$$

$$\leq C \left\{ \int_{\{u < T\}} a^{ij}(x)u_iu_j dx + \int_{\Omega} \tilde{f}(u)f''(u)a^{ij}(x)u_iu_j dx \right\}$$

$$\leq C \left\{ T \int_{\Omega} f'(u) dx + \int_{\Omega} \tilde{f}(u)f''(u)a^{ij}(x)u_iu_j dx \right\}. \quad (3.2)$$

In particular, by Lemma 2.3 and the bound (2.7), we obtain

$$\int_{\Omega} f'(u)^\alpha dx \leq C \quad \text{where } \alpha = \frac{3 + \varepsilon}{1 + \varepsilon}.$$ 

Therefore, by the arbitrariness of $\varepsilon > 0$, we obtain $f'(u) \in L^p(\Omega)$ for all $p < 3$. As a consequence, by Corollary 2.2 and since $u \in L^1(\Omega)$ (see Remark 2.4), we obtain the $L^\infty$ estimate established in part (a), i.e., if $n < 6$ then $\|u\|_{L^\infty(\Omega)} \leq C$.

In the following, we may assume $n \geq 6$. Let us prove now that $f'(u) \in L^p(\Omega)$ for all $p < n/(n - 4)$, and as a consequence, the statement in part (b). Now we take
\(\alpha = 1 + \frac{1}{1+\varepsilon}\). Multiplying (1.1) by \((f'(u) - f'(0))^\alpha\) and using (3.1) and Lemma 2.3, we obtain
\[
\int_{\Omega} f(u)(f'(u) - f'(0))^\alpha \, dx = \alpha \int_{\Omega} (f'(u) - f'(0))^{\alpha-1} f''(u)a^{ij}(x)u_i u_j \, dx \leq C \int_{\Omega} f''(u)a^{ij}(x)u_i u_j \, dx \leq C.
\]
Hence, using the convexity of \(f\) and that \(f'(0) \leq f'(t)/2\) for \(t\) large, we obtain
\[
\int_{\Omega} \tilde{f}(u)^{\alpha+1} \, dx \leq C \quad \text{for all} \ \alpha \in (1,2).
\]

We now repeat the iteration argument of Nédélec [6]. Assume that \(u \in L^p(\Omega)\) for all \(1 \leq p < p_0\). Given any positive number \(\beta\), set
\[
\Omega_1 := \{x \in \Omega : \frac{\tilde{f}(u)^{\alpha+1}}{u^\alpha} > \tilde{f}(u)^{\alpha+1-\beta}\}, \quad \Omega_2 := \Omega \setminus \Omega_1 = \{x \in \Omega : \tilde{f}(u) \leq u^\beta\}.
\]
By (3.4), we have
\[
\int_{\Omega_1} \tilde{f}(u)^{\alpha+1-\beta} \, dx \leq C.
\]
Moreover,
\[
\int_{\Omega_2} \tilde{f}(u)^p \, dx \leq \int_{\Omega_2} u^\beta dx \leq C \quad \text{for all} \ \beta < \frac{\beta}{\alpha}p_0.
\]
Choose \(\beta\) such that \(\alpha + 1 - \beta = \frac{\beta}{\alpha}p_0\), i.e., \(\beta = (\alpha + 1)/(1 + \frac{p_0}{\alpha})\). Then, by (3.5), (3.6), and letting \(\alpha \uparrow 2\), we obtain \(\tilde{f}(u) \in L^p(\Omega)\) for all \(1 < p < \frac{3p_0}{2+p_0}\). Hence, by elliptic regularity theory (see Proposition 2.1 (ii)),
\[
u \in L^p(\Omega) \quad \text{for all} \ 1 < p < p_1 := \frac{n}{2+p_0} = \frac{3p_0}{2n+n-6}.
\]
By (2.8) we can start the iteration process with \(p_0 = n/(n-2)\). Set \(p_{k+1} := \frac{3p_k}{2n+(n-6)p_k}\) for \(k \geq 1\). Note that \(p_k \leq n/(n-6)\) for all \(k \geq 1\) (by induction and since \(p_0 = n/(n-2) \leq n/(n-6)\)). Moreover \(p_{k+1} > p_k\) (in fact, this is equivalent to \(p_k < n/(n-6)\)), and hence, \(\lim_{k \to \infty} p_k = n/(n-6) =: p_\infty\).

Therefore, we obtain \(f(u) \in L^p(\Omega)\) for all \(1 < p < \frac{3p_0}{2+p_0} = \frac{n}{n-\varepsilon}\). The remainder of the statements of Theorem 1.1 follow from standard elliptic regularity theory (see Proposition 2.1 (ii)).

\[\square\]

**Proof of Theorem 1.2.** The proof of Theorem 1.2 is essentially the same. Using assumption (1.7) in (3.2), the first part of the proof gives \(f'(u) \in L^\alpha(\Omega)\) with \(\alpha = 1 + \frac{1}{1-\varepsilon}\). Therefore, by Corollary 2.2 and since \(u \in L^1(\Omega)\) (see Remark 2.4), \(u \in L^\infty(\Omega)\) when \(\varepsilon > \frac{1}{2}\), or equivalently, when \(n < 6 + \frac{4}{1-\varepsilon}\).

Assume \(n \geq 6 + \frac{4}{1-\varepsilon}\). To obtain the estimate on \(f(u)\), we deduce (3.3) with \(\alpha = 1 + \frac{1}{1-\varepsilon}\) using now (1.7) instead of (1.6). In particular,
\[
\int_{\Omega} \tilde{f}(u)^{\alpha+1} \, dx \leq C \quad \text{for} \ \alpha = 1 + \frac{1}{1-\varepsilon}.
\]
At this point, we repeat the previous iteration argument to obtain the increasing sequence

\[ p_0 = \frac{n}{n - 2}, \quad p_{k+1} = \frac{(3 - 2\varepsilon)p_k}{(2 - \varepsilon + (1 - \varepsilon)p_k)n - 2(3 - 2\varepsilon)p_k}, \quad \text{for all } k \geq 0, \]

with limit \( p_\infty = \frac{(1-\varepsilon)n}{(1-\varepsilon)n-6n+6}. \) As a consequence, \( f(u) \in L^p(\Omega) \) for all \( p < \frac{\beta}{\alpha}p_\infty \) where \( \beta = (\alpha + 1)/(1 + \frac{p_\infty}{\alpha}) \), i.e., \( f(u) \in L^p(\Omega) \) for all \( p < \frac{(1-\varepsilon)n}{(1-\varepsilon)n-4+2\varepsilon} \). The remainder of the statements of Theorem 1.2 follow from standard elliptic regularity theory (see Proposition 2.1 (ii)).

**REFERENCES**

[1] H. Brezis, Is there failure of the Inverse Function Theorem?, *Morse Theory, Minimax Theory and Their Applications to Nonlinear Differential Equations* Int. Press, Somerville, 1 (2003), 23–33.

[2] H. Brezis and J. L. Vázquez, Blow-up solutions of some nonlinear elliptic problems, *Rev. Mat. Univ. Complut. Madrid*, 10 (1997), 443–469.

[3] X. Cabré, Regularity of minimizers of semilinear elliptic problems up to dimension 4, *Comm. Pure Appl. Math.*, 63 (2010), 1362–1380.

[4] X. Cabré and A. Capella, Regularity of radial minimizers and extremal solutions of semilinear elliptic equations, *J. Funct. Anal.*, 238 (2006), 709–733.

[5] M. G. Crandall and P. H. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, *Arch. Ration. Mech. Anal.*, 58 (1975), 207–218.

[6] G. Nedev, Regularity of the extremal solution of semilinear elliptic equations, *C. R. Acad. Sci. Paris*, 330 (2000), 997–1002.

[7] M. Sanchón, Boundedness of the extremal solution of some \( p \)-Laplacian problems, *Nonlinear Analysis*, 67 (2007), 281–294.

[8] J. Serrin, Local behavior of solutions of quasilinear elliptic equations, *Acta Math.*, 111 (1964), 247–302.

[9] N. S. Trudinger, Linear elliptic operators with measurable coefficients, *Ann. Scuola Norm. Sup. Pisa. (3)*, 27 (1973), 265–308.

[10] S. Villegas, Boundedness of extremal solutions in dimension 4, *Adv. Math.*, 235 (2013), 126–133.

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