Anisotropic variable Campanato-type spaces and their Carleson measure characterizations

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Abstract
Let \( p(\cdot) : \mathbb{R}^n \to (0, \infty) \) be a variable exponent function satisfying the globally log-Hölder continuous condition and \( A \) a general expansive matrix on \( \mathbb{R}^n \). In this article, the authors introduce the anisotropic variable Campanato-type spaces and give some applications. Especially, using the known atomic and finite atomic characterizations of anisotropic variable Hardy space \( H_A^{p(\cdot)}(\mathbb{R}^n) \), the authors prove that this Campanato-type space is the appropriate dual space of \( H_A^{p(\cdot)}(\mathbb{R}^n) \) with full range \( p(\cdot) \). As applications, the authors first deduce several equivalent characterizations of these Campanato-type spaces. Furthermore, the authors also introduce the anisotropic variable tent spaces and show their atomic decomposition. Combining this and the obtained dual theorem, the Carleson measure characterizations of these anisotropic variable Campanato-type spaces are established.

Keywords (variable) Campanato-type space · (variable) Hardy space · Tent space · Duality · Carleson measure · Fractional integral operators

Mathematics Subject Classification 42B30 · 46E30 · 28C20 · 26A33

1 Introduction
Recall that the classical Campanato space was introduced by Campanato [2] in 1964, which includes the bounded mean oscillation function space \( \text{BMO}(\mathbb{R}^n) \) of John and
Nirenberg [20]. Later, Taibleson and Weiss [39] showed that the Campanato space is the dual space of the well-known Hardy space \( H^p(\mathbb{R}^n) \) for \( p \in (0, 1) \) in 1980, which generalized the celebrated dual theorem of Fefferman and Stein [11]. Namely, the bounded mean oscillation function space BMO(\mathbb{R}^n) is the dual space of the Hardy space \( H^1(\mathbb{R}^n) \). Nowadays, the Campanato space plays an important role in harmonic analysis and partial differential equations, and has been systematically studied and developed so far. For instance, Cianchi and Pick [3] studied the Sobolev embedding into Campanato spaces; Nakai [25] extended the Campanato spaces into the spaces of homogeneous type; Nakai [26] investigated fractional integral operators and singular integral operators on Campanato spaces or their predual spaces; Nakai and Yoneda [28] gave some applications of Campanato spaces with variable growth condition to the Navier–Stokes equation; Mizuta et al. [24] recently considered Campanato–Morrey spaces for the double phase functionals; Ho [14] investigated integral operators on BMO and Campanato spaces. For more developments on Campanato spaces, we refer the reader to [1, 15, 17, 27, 32, 35, 40].

In addition, based on variable Lebesgue spaces, several variable function spaces have rapidly been developed in the past two decades; see Ho [13] for fractional integrals on Morrey spaces with variable exponent, [12, 19] and Weisz [41] for variable martingale Hardy spaces and Xu [42] for variable Besov and Triebel-Lizorkin spaces as well as the book [38] for various function spaces. Note that, the study of variable Lebesgue spaces originates from Orlicz [29]. However, they have been the subject of more intensive study since the early 1990s due to their intrinsic interest for applications into fractional calculus, harmonic analysis, partial differential equations and variational integrals with nonstandard growth conditions (see books [5] and [8]). Recall that, let a measurable exponent function \( p(\cdot) : \mathbb{R}^n \to (0, \infty) \) satisfy the so-called globally log-Hölder continuous condition [see (2.4) and (2.5) below], \( p_- := \text{ess inf}_{x \in \mathbb{R}^n} p(x) \), and \( p_+ := \text{ess sup}_{x \in \mathbb{R}^n} p(x) \). In 2012, Nakai and Sawano [27] introduced the variable Hardy space \( H^{p(\cdot)}(\mathbb{R}^n) \) and generalized the Campanato spaces into the variable exponents setting, in which they also showed that the variable Campanato space is the dual of \( H^{p(\cdot)}(\mathbb{R}^n) \) when \( 0 < p_- \leq p_+ < 1 \); see [27, Theorem 7.5]. After that, Sawano [37] improved the corresponding result in [32] via extending the atomic characterization of \( H^{p(\cdot)}(\mathbb{R}^n) \). Moreover, Zhuo et al. [45] established equivalent characterizations of \( H^{p(\cdot)}(\mathbb{R}^n) \) via intrinsic square functions. In particular, Cruz-Uribe and Wang [6] also independently investigated the variable Hardy space \( H^{p(\cdot)}(\mathbb{R}^n) \) with \( p(\cdot) \) satisfying some weaker hypothesis than those used in [27]. Very recently, Cruz-Uribe et al. [7] further developed a new approach to show norm inequalities for Calderón–Zygmund singular operators and fractional integral operators on variable Hardy spaces with Muckenhoupt weight.

On the other hand, under the framework of Hardy-type spaces associated with ball quasi-Banach function spaces, Zhang et al. [43] recently obtained the dual result of \( H^{p(\cdot)}(\mathbb{R}^n) \) on the full range \( 0 < p_- \leq p_+ < \infty \), which extended the dual theorem of Nakai and Sawano [27]. Moreover, let \( A \) be a general expansive matrix on \( \mathbb{R}^n \). Recall that the variable Hardy space \( H^{p(\cdot)}_A(\mathbb{R}^n) \) associated with \( A \) was first introduced and studied by Liu et al. [21], in which they characterized \( H^{p(\cdot)}_A(\mathbb{R}^n) \) in terms of maximal functions, atoms, finite atoms, and Littlewood–Paley functions. Furthermore,
the anisotropic variable Campanato space associated with $A$ was introduced by Wang [40] very recently. In [40], the author proved that the anisotropic variable Campanato space is the dual space of the anisotropic variable Hardy space $H^{p(\cdot)}_A(\mathbb{R}^n)$ with $0 < p_- \leq p_+ \leq 1$. In addition, note that if $p_- \in (1, \infty)$, then

$$H^{p(\cdot)}_A(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$$

with equivalent quasi-norms (see [44, Corollary 4.20]). Thus, in this case, it follows from Cruz-Uribe and Fiorenza [5, Theorem 2.80] that the dual space of $H^{p(\cdot)}_A(\mathbb{R}^n)$ is just the variable Lebesgue space $L^{p'(\cdot)}(\mathbb{R}^n)$, where conjugate exponent function $p'(\cdot)$ is defined by setting $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. Obviously, there is a gap between two ranges $0 < p_- \leq p_+ \leq 1$ and $p_- \in (1, \infty)$. This means that when $0 < p_- \leq 1 < p_+ < \infty$, the dual space of $H^{p(\cdot)}_A(\mathbb{R}^n)$ is still missing until now. Indeed, the main difficulty in this case is that the considered Hardy space does not have a concave quasi-norm.

To solve this problem and also to enrich the theory of anisotropic variable Campanato spaces, in this article, by viewing the finite linear combinations of atoms as a whole, we introduce the anisotropic variable Campanato-type spaces and give some applications. Especially, we show that these anisotropic variable Campanato-type spaces include the anisotropic variable Campanato space in [40]. Moreover, inspired by [43] and via the known atomic and finite atomic characterizations of the space $H^{p(\cdot)}_A(\mathbb{R}^n)$, we prove that this Campanato-type space is the appropriate dual space of $H^{p(\cdot)}_A(\mathbb{R}^n)$ with $0 < p_- \leq p_+ < \infty$, which extends the known dual result of $H^{p(\cdot)}_A(\mathbb{R}^n)$. We should point out that, even in the isotropic case, the obtained dual theorem gives a complete answer to the open question proposed by Izuki et al. in [18, Section 9.3]. As applications, we deduce several equivalent characterizations of these Campanato-type spaces. Furthermore, we also introduce the anisotropic variable tent spaces and show their atomic decomposition. Combining this and the dual theorem, we finally establish the Carleson measure characterizations of the anisotropic variable Campanato-type spaces.

Precisely, this article is organized as follows.

In Section 2, we recall some notions on dilations and variable Lebesgue spaces. Then we introduce the anisotropic variable Campanato-type space and show some basic properties. Section 3 is devoted to proving that the anisotropic variable Campanato-type space is the dual space of $H^{p(\cdot)}_A(\mathbb{R}^n)$ for full range $0 < p_- \leq p_+ < \infty$ (see Theorem 1 below). To this end, we first recall the atomic and the finite atomic characterizations of the anisotropic variable Hardy space $H^{p(\cdot)}_A(\mathbb{R}^n)$ established in [21, Theorems 4.8 and 5.4]. Combining these, the special structure of the anisotropic variable Campanato-type space, and some basic tools from functional analysis, we identify the Campanato-type space $L^{p(\cdot), q(\cdot), s, p(\cdot)}_A(\mathbb{R}^n)$ with the dual space of the Hardy space $H^{p(\cdot)}_A(\mathbb{R}^n)$ for the full range $0 < p_- \leq p_+ < \infty$. We point out that, as a special case, the dual theorem of $H^{p(\cdot)}_A(\mathbb{R}^n)$ with $0 < p_- \leq p_+ \leq 1$ is obviously obtained, which covers the result of [40, Theorem 4.4] (see Remark 6 below).
In Section 4, we first establish several equivalent characterizations of the anisotropic variable Campanato-type space (see Theorem 2 and Corollary 3 below). Then, applying this and the obtained dual result in Theorem 1, we further establish the Carleson measure characterization of the anisotropic variable Campanato-type space $\mathcal{L}^p_A(\mathbb{R}^n)$ (see Theorem 3 below). To show this Carleson measure characterization, we introduce the anisotropic variable tent space (see Definition 9 below) and give their atomic decomposition (see Lemma 5 below), which plays a key role in the proof of Theorem 3. Indeed, applying this atomic decomposition, the Lusin area function characterization of $\mathcal{H}^p(\mathbb{R}^n)$ proved in [21, Theorem 6.1] (see also [22, Theorem 4.4(i)]), and the obtained dual theorem, we further conclude Theorem 3.

Finally, we make some conventions on notation. Let

\[ N = \{1, 2, \ldots\}, \quad Z_+ = \{0\} \cup N, \quad \text{and} \quad Z^n_+ = (Z_+)^n, \]

and use 0 to denote the origin of $\mathbb{R}^n$. For any multi-index $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ and $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, let

\[ |\alpha| := \alpha_1 + \cdots + \alpha_n, \quad x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \text{and} \quad \partial^\alpha := \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}. \]

We always denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. The notation $f \lesssim g$ means $f \leq Cg$ and, if $f \lesssim g \lesssim f$, then we write $f \sim g$. If $f \lesssim Cg$ and $g = h$ or $g \lesssim h$, then we write $f \lesssim g \sim h$, rather than $f \lesssim g = h$ or $f \lesssim g \leq h$. For any $q \in [1, \infty]$, we denote by $q'$ its conjugate exponent, namely, $1/q + 1/q' = 1$. The symbol $[s]$ for any $s \in \mathbb{R}$ denotes the largest integer not greater than $s$. In addition, for any set $E \subset \mathbb{R}^n$, we denote the set $\mathbb{R}^n \setminus E$ by $E^c$, its characteristic function by $1_E$, and its $n$-dimensional Lebesgue measure by $|E|$. Throughout this article, the symbol $C^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^n$.

## 2 Anisotropic variable Campanato-type spaces

In this section, we first recall some notions on dilations and variable Lebesgue spaces. Then we introduce the anisotropic variable Campanato-type space and give some basic properties. To begin with, we present the definition of dilations from [1, p. 5, Definition 2.1].

**Definition 1** A real $n \times n$ matrix $A$ is called an expansive matrix, shortly, a dilation if

\[ \min_{\lambda \in \sigma(A)} |\lambda| > 1, \]

here and thereafter, $\sigma(A)$ denotes the set of all eigenvalues of $A$.

Let $b := |\det A|$. Then, from [1, p. 6, (2.7)], it follows that $b \in (1, \infty)$. By the fact that there exist an open ellipsoid $\Delta$, with $|\Delta| = 1$, and $r \in (1, \infty)$ such that $\Delta \subset r \Delta \subset A \Delta$ (see [1, p. 5, Lemma 2.2]), we find that, for any $k \in \mathbb{Z}$, $B_k := A^k \Delta$ is open, $B_k \subset r B_k \subset B_{k+1}$ and $|B_k| = b^k$. For any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, an ellipsoid
$x + B_k$ is called a \textit{dilated ball}. In what follows, we always let $\mathcal{B}$ be the set of all such dilated balls, namely,

$$\mathcal{B} := \{x + B_k : x \in \mathbb{R}^n \text{ and } k \in \mathbb{Z}\}$$  \hspace{1cm} (2.1)

and

$$\omega := \inf \left\{ \ell \in \mathbb{Z} : r^\ell \geq 2 \right\}. \hspace{1cm} (2.2)$$

The following notion of homogeneous quasi-norms is just \cite[p. 6, Definition 2.3]{1}.

\begin{definition}
A \textit{homogeneous quasi-norm}, associated with a dilation $A$, is a measurable mapping $\rho : \mathbb{R}^n \to [0, \infty)$ satisfying

(i) if $x \neq 0$, then $\rho(x) \in (0, \infty)$;

(ii) for any $x \in \mathbb{R}^n$, $\rho(Ax) = b \rho(x)$;

(iii) there exists an $H \in [1, \infty)$ such that, for any $x, y \in \mathbb{R}^n$,

$$\rho(x + y) \leq H[\rho(x) + \rho(y)].$$

In what follows, for a given dilation $A$, by \cite[p. 6, Lemma 2.4]{1}, we may use, for both simplicity and convenience, the \textit{step homogeneous quasi-norm} $\rho$ defined by setting, for any $x \in \mathbb{R}^n$,

$$\rho(x) := \sum_{k \in \mathbb{Z}} b^k 1_{B_{k+1}\setminus B_k}(x) \quad \text{when } x \neq 0, \text{ or else } \rho(0) := 0.$$

A measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ is called a \textit{variable exponent}. Denote by $\mathcal{P}(\mathbb{R}^n)$ the \textit{set of all variable exponents} $p(\cdot)$ satisfying

$$0 < p_- := \operatorname{ess inf}_{x \in \mathbb{R}^n} p(x) \leq \operatorname{ess sup}_{x \in \mathbb{R}^n} p(x) =: p_+ < \infty.$$  \hspace{1cm} (2.3)

For any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and a measurable function $f$, the \textit{modular functional} $\varrho_{p(\cdot)}(f)$ is defined by setting

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx$$

and the \textit{Luxemburg} (also called \textit{Luxemburg-Nakano}) \textit{quasi-norm} $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ of $f$ is given by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \varrho_{p(\cdot)}(f/\lambda) \leq 1 \right\}.$$

Then the \textit{variable Lebesgue space} $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ such that $\varrho_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. 

Let \( C^{\log}(\mathbb{R}^n) \) be the set of all functions \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) satisfying the globally log-Hölder continuous condition, namely, there exist \( C_\log(p), C_\infty \in (0, \infty) \), and \( p_\infty \in \mathbb{R} \) such that, for any \( x, y \in \mathbb{R}^n \),

\[
|p(x) - p(y)| \leq \frac{C_\log(p)}{\log(e + 1/\rho(x - y))}
\]

and

\[
|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + \rho(x))}.
\]

In what follows, for any \( s \in \mathbb{Z}_+ \), \( \mathbb{P}_s(\mathbb{R}^n) \) denotes the set of all polynomials on \( \mathbb{R}^n \) with degree not greater than \( s \); for any ball \( B \in \mathcal{B} \) with \( \mathcal{B} \) as in (2.1) and any locally integrable function \( g \) on \( \mathbb{R}^n \), we use \( P_s^B g \) to denote the minimizing polynomial of \( g \) with degree not greater than \( s \), which means that \( P_s^B g \) is the unique polynomial \( f \in \mathbb{P}_s(\mathbb{R}^n) \) such that, for any \( h \in \mathbb{P}_s(\mathbb{R}^n) \),

\[
\int_B [g(x) - f(x)] h(x) \, dx = 0.
\]

We next introduce the anisotropic variable Campanato-type spaces. For any given \( q \in [1, \infty) \), we use \( L^q_{\text{loc}}(\mathbb{R}^n) \) to denote the set of all \( q \)-order locally integrable functions on \( \mathbb{R}^n \).

**Definition 3** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), \( q \in [1, \infty) \), \( \eta \in (0, \infty) \), and \( s \in \mathbb{Z}_+ \). Then the anisotropic variable Campanato-type space \( L^A_{p(\cdot), q, s, \eta}(\mathbb{R}^n) \) is defined to be the set of all \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \) such that

\[
\left\| f \right\|_{L^A_{p(\cdot), q, s, \eta}(\mathbb{R}^n)} \quad := \quad \sup \left\{ \sum_{i=1}^m \left[ \frac{\lambda_i}{\|1_B^{(i)}\|_{L^p(\mathbb{R}^n)}} \right]^\eta \right\}^{\frac{1}{\eta}} \times \left\{ \sum_{j=1}^m \left[ \frac{\lambda_j |B^{(j)}|}{\|1_B^{(j)}\|_{L^p(\mathbb{R}^n)}} \right]^{\frac{1}{q}} \left[ \int_{B^{(j)}} \left| f(x) - P_{B^{(j)}}^g f(x) \right|^q \, dx \right]^{\frac{1}{q}} \right\}^{\frac{1}{q}}
\]

is finite, where the supremum is taken over all \( m \in \mathbb{N}, \{B^{(j)}\}_{j=1}^m \subset \mathcal{B} \), and \( \{\lambda_j\}_{j=1}^m \subset [0, \infty) \) with \( \sum_{j=1}^m \lambda_j \neq 0 \).

**Remark 1**

(i) We point out that \( \mathbb{P}_s(\mathbb{R}^n) \subset L^A_{p(\cdot), q, s, \eta}(\mathbb{R}^n) \). Indeed,

\[
\left\| f \right\|_{L^A_{p(\cdot), q, s, \eta}(\mathbb{R}^n)} = 0
\]
if and only if \( f \in \mathbb{P}_s(\mathbb{R}^n) \). Throughout this article, we always identify \( f \in L^A_{p(\cdot),q,s,\eta}(\mathbb{R}^n) \) with \( \{ f + P : P \in \mathbb{P}_s(\mathbb{R}^n) \} \).

(ii) For any \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \), define

\[
\| f \|_{L^A_{p(\cdot),q,s,\eta}(\mathbb{R}^n)} := \sup \inf \left\{ \left\| \sum_{i=1}^{m} \left[ \frac{\lambda_i}{\| B(i) \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^\eta \mathbf{1}_{B(i)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}^{-1} \times \sum_{j=1}^{m} \left\{ \frac{\lambda_j |B(j)|}{\| B(j) \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|B(j)|} \int_{B(j)} |f(x) - P_s B(j) f(x)|^q \, dx \right]^{\frac{1}{q}} \right\},
\]

where the supremum is the same as in Definition 3 and the infimum is taken over all \( P \in \mathbb{P}_s(\mathbb{R}^n) \). Then we can easily show that \( \| f \|_{L^A_{p(\cdot),q,s,\eta}(\mathbb{R}^n)} \) is an equivalent quasi-norm of the Campanato-type space \( L^A_{p(\cdot),q,s,\eta}(\mathbb{R}^n) \) and we omit the details here.

(iii) In 2010, Nakai [26] studied the fractional integrals on Campanato spaces with variable growth conditions and recently Rafeiro and Samko [31, 33, 34] as well as Samko [36] have investigated properties of fractional integrals. We should point out that it is also meaningful to consider the boundedness of fractional integrals and their commutators on the anisotropic variable Campanato-type space as in Definition 3.

For the space \( L^A_{p(\cdot),q,s,\eta}(\mathbb{R}^n) \), we have the following equivalent quasi-norm characterization.

**Proposition 1** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), \( q \in [1, \infty) \), \( \eta \in (0, \infty) \), and \( s \in \mathbb{Z}_+ \). For any \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \), define

\[
\| f \|_{L^A_{p(\cdot),q,s,\eta}(\mathbb{R}^n)} := \sup \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_j |B(j)|}{\| B(j) \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|B(j)|} \int_{B(j)} |f(x) - P_s B(j) f(x)|^q \, dx \right]^{\frac{1}{q}} \right] \right\},
\]

where the supremum is taken over all \( \{ B(j) \}_{j \in \mathbb{N}} \subset \mathfrak{B} \) and \( \{ \lambda_j \}_{j \in \mathbb{N}} \subset [0, \infty) \) satisfying

\[
\left\| \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_j}{\| B(j) \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^\eta \mathbf{1}_{B(j)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \in (0, \infty).
\]
Then, for any \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \),
\[
\|f\|_{L^q_{\rho(\cdot),q,s,\eta}(\mathbb{R}^n)} = \|f\|_{L^q_{\rho(\cdot),q,s,\eta}(\mathbb{R}^n)}.
\]

**Proof** Let \( p(\cdot), q, s, \) and \( \eta \) be as in the present proposition and \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \). Obviously,
\[
\|f\|_{L^q_{\rho(\cdot),q,s,\eta}(\mathbb{R}^n)} \leq \|f\|_{L^q_{\rho(\cdot),q,s,\eta}(\mathbb{R}^n)}.
\]
Conversely, let \( \{B(j)\}_{j \in \mathbb{N}} \subset \mathcal{B} \) and \( \{\lambda_j\}_{j \in \mathbb{N}} \subset [0, \infty) \) satisfy (2.6). Observe that

\[
\lim_{m \to \infty} \left\{ \sum_{i=1}^{m} \left[ \frac{\lambda_i}{\|1_{B(i)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\eta} 1_{B(i)} \right\}^{-\frac{1}{\eta}} L^{p(\cdot)}(\mathbb{R}^n)
\]

\[
\times \sum_{j=1}^{m} \left[ \frac{\lambda_j |B(j)|}{\|1_{B(j)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right] \left[ \frac{1}{|B(j)|} \int_{B(j)} |f(x) - P_{B(j)}^s f(x)|^q \, dx \right]^{\frac{1}{q}}.
\]

Therefore, for any given \( \varepsilon \in (0, \infty) \), there exists an \( m_0 \in \mathbb{N} \) such that \( \sum_{j=1}^{m_0} \lambda_j \neq 0 \) and

\[
\left\{ \sum_{i=1}^{m_0} \left[ \frac{\lambda_i}{\|1_{B(i)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\eta} 1_{B(i)} \right\}^{-\frac{1}{\eta}} L^{p(\cdot)}(\mathbb{R}^n)
\]

\[
\times \sum_{j=1}^{m_0} \left[ \frac{\lambda_j |B(j)|}{\|1_{B(j)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right] \left[ \frac{1}{|B(j)|} \int_{B(j)} |f(x) - P_{B(j)}^s f(x)|^q \, dx \right]^{\frac{1}{q}} + \varepsilon
\]

\[
\leq \|f\|_{L^q_{\rho(\cdot),q,s,\eta}(\mathbb{R}^n)} + \varepsilon.
\]
which, combined with the arbitrariness of \( \{ B^{(j)} \}_{j \in \mathbb{N}} \subset \mathcal{B} \) and \( \{ \lambda_j \}_{j \in \mathbb{N}} \subset [0, \infty) \) satisfying (2.6), and \( \varepsilon \in (0, \infty) \), further implies that

\[
\| \bar{f} \|_{L^A_{p(\cdot), q, s, \eta}(\mathbb{R}^n)} \leq \| f \|_{L^A_{p(\cdot), q, s, \eta}(\mathbb{R}^n)}.
\]

This finishes the proof of Proposition 1. \( \square \)

Recall that the following anisotropic variable Campanato space \( L^A_{p(\cdot), q, s}(\mathbb{R}^n) \) was introduced in [40, Definition 4.1].

**Definition 4** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), \( q \in [1, \infty) \), and \( s \in \mathbb{Z}_+ \). The anisotropic variable Campanato space \( L^A_{p(\cdot), q, s}(\mathbb{R}^n) \) is defined to be the set of all \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \) such that

\[
\| f \|_{L^A_{p(\cdot), q, s}(\mathbb{R}^n)} := \sup_{B \in \mathcal{B}} \inf_{P \in \mathcal{P}(\mathbb{R}^n)} \frac{|B|}{|B|} \left[ \frac{1}{|B|} \int_B |f(x) - P f(x)|^q \, dx \right]^{\frac{1}{q}}
\]

is finite.

**Remark 2**

(i) For any \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \), define

\[
\| f \|_{L^A_{p(\cdot), q, s}(\mathbb{R}^n)} := \sup_{B \in \mathcal{B}} \inf_{P \in \mathcal{P}(\mathbb{R}^n)} \frac{|B|}{|B|} \left[ \frac{1}{|B|} \int_B |f(x) - P f(x)|^q \, dx \right]^{\frac{1}{q}}.
\]

It is easy to see that \( \| \cdot \|_{L^A_{p(\cdot), q, s}(\mathbb{R}^n)} \) is an equivalent quasi-norm of the Campanato space \( L^A_{p(\cdot), q, s}(\mathbb{R}^n) \).

(ii) From Remark 1(ii) and Definition 4, it immediately follows that

\[
L^A_{p(\cdot), q, s, \eta}(\mathbb{R}^n) = L^A_{p(\cdot), q, s}(\mathbb{R}^n)
\]

for all indices as in Definition 3, and this inclusion is continuous.

We now investigate the further relation between two spaces \( L^A_{p(\cdot), q, s, \eta}(\mathbb{R}^n) \) and \( L^A_{p(\cdot), q, s}(\mathbb{R}^n) \). Indeed, the next proposition shows that the inverse inclusion of Remark 2(ii) also holds true for certain ranges of indices.

**Proposition 2** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), \( p_+ \in (0, 1] \) with \( p_+ \) as in (2.3), \( q \in [1, \infty) \), \( \eta \in (0, 1] \), and \( s \in \mathbb{Z}_+ \). Then

\[
L^A_{p(\cdot), q, s, \eta}(\mathbb{R}^n) = L^A_{p(\cdot), q, s}(\mathbb{R}^n)
\]

with equivalent quasi-norms.
**Proof** Let \( p(\cdot), p_+, q, \eta, \) and \( s \) be as in the present proposition. To prove this proposition, by Remark 2(ii), it suffices to show that

\[
\mathcal{L}^A_{p(\cdot),q,s}(\mathbb{R}^n) \subset \mathcal{L}^A_{p(\cdot),q,s,\eta}(\mathbb{R}^n)
\]

and the inclusion is continuous. To do this, let \( f \in \mathcal{L}^A_{p(\cdot),q,s}(\mathbb{R}^n) \). Then, from the assumptions that \( p_+ \in (0, 1] \) and \( \eta \in (0, 1] \), together with the monotonicity of \( \ell^\eta \) and Remark 2(i), we deduce that

\[
\|f\|_{\mathcal{L}^A_{p(\cdot),q,s,\eta}(\mathbb{R}^n)} \lesssim \sup \left( \sum_{i=1}^m \lambda_i \right)^{-1} \sum_{j=1}^m \left\{ \frac{\lambda_j |B^{(j)}|}{\|1_{B^{(j)}}\|_{L^p(\mathbb{R}^n)}} \left[ \frac{1}{|B^{(j)}|} \int_{B^{(j)}} |f(x) - P_{B^{(j)}}^s f(x)|^q \, dx \right]^{\frac{1}{q}} \right\}
\]

\[
\lesssim \sup \left( \sum_{i=1}^m \lambda_i \right)^{-1} \sum_{j=1}^m \lambda_j \|f\|_{\mathcal{L}^A_{p(\cdot),q,s}(\mathbb{R}^n)} \sim \|f\|_{\mathcal{L}^A_{p(\cdot),q,s}(\mathbb{R}^n)},
\]

where the supremum is taken over all \( m \in \mathbb{N} \), \( \{B^{(j)}\}_{j=1}^m \subset \mathcal{B} \), and \( \{\lambda_j\}_{j=1}^m \subset [0, \infty) \) with \( \sum_{j=1}^m \lambda_j \neq 0 \). This finishes the proof of Proposition 2.

**Remark 3** By Proposition 2 with \( p(\cdot) \equiv p \in (0, 1] \) and [16, Remark 11(iii)], we find that the new introduced Campanato-type space as in Definition 3 includes the classical Campanato space \( L_{\frac{1}{p}-1,q,s}(\mathbb{R}^n) \), introduced by Campanato [2], and the space \( \text{BMO}(\mathbb{R}^n) \), introduced by John and Nirenberg [20], as special cases.

**Remark 4** Rafeiro and Samko [30, Theorem 3.15] proved the coincidence of variable exponent Campanato spaces with variable exponent Hölder spaces in some cases. However, by the definition of variable exponent Campanato spaces in [30], we find that these Campanato spaces are different from the Campanato spaces introduced in this article. Moreover, we should point out that, from Remark 3 above and [2] (see also [25]), we infer that the new introduced Campanato-type space as in Definition 3 also includes the Hölder space \( \Lambda_\alpha \) with \( \alpha \in (0, 1] \) as special cases.

### 3 Duality between \( H^p_{A}(\mathbb{R}^n) \) and \( L^A_{p(\cdot),q',s,p}(\mathbb{R}^n) \)

In this section, we establish the relation between the anisotropic variable Campanato-type space and the anisotropic variable Hardy space via duality. To be precise, based on the structure of the anisotropic variable Campanato-type space introduced in Definition 3, and by the atomic and the finite atomic characterizations of the anisotropic variable Hardy space \( H^p_{A}(\mathbb{R}^n) \) established in [21] as well as some basic tools.
from functional analysis, we show that the new introduced Campanato-type space $L^A_{p(\cdot), q(\cdot), r(\cdot), p(\cdot), R^n}$ is the dual space of the Hardy space $H^{p(\cdot)}_A(R^n)$ for the full range $p(\cdot) \in C^{\log}(R^n)$, which extends the known dual result of [40].

Recall that a $C^\infty(R^n)$ function $\varphi$ is called a Schwartz function if, for any $m \in \mathbb{Z}_+$ and multi-index $\alpha \in \mathbb{Z}^n_+$,

$$\| \varphi \|_{\alpha, m} := \sup_{x \in \mathbb{R}^n} [\rho(x)]^m |\partial^\alpha \varphi(x)| < \infty.$$  

Denote by $S(R^n)$ the set of all Schwartz functions, equipped with the topology determined by $\{\| \cdot \|_{\alpha, m}\}_{\alpha \in \mathbb{Z}^n_+, m \in \mathbb{Z}_+}$, and $S'(R^n)$ the dual space of $S(R^n)$, equipped with the weak-$*$ topology. For any $N \in \mathbb{Z}_+$, let

$$S_N(R^n) := \{ \varphi \in S(R^n) : \| \varphi \|_{\alpha, \ell} \leq 1, |\alpha| \leq N, \ell \leq N \},$$
equivalently,

$$\varphi \in S_N(R^n) \iff \| \varphi \|_{S_N(R^n)} := \sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} \left[ |\partial^\alpha \varphi(x)| \max \left\{ 1, [\rho(x)]^N \right\} \right] \leq 1.$$  

Let $\lambda_-, \lambda_+ \in (1, \infty)$ be two numbers such that

$$\lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$  

We should point out that, if $A$ is diagonalizable over $\mathbb{C}$, then we may let $\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\}$ and $\lambda_+ := \max\{|\lambda| : \lambda \in \sigma(A)\}$. Otherwise, we may choose them sufficiently close to these equalities in accordance with what we need in our arguments.

Applying [21, Definition 2.4 and Theorem 3.10], we now recall the following equivalent definition of the anisotropic variable Hardy space $H^{p(\cdot)}_A(R^n)$ via the radial maximal function.

**Definition 5** Let $p(\cdot) \in C^{\log}(R^n)$ and $\varphi \in S(R^n)$ satisfy $\int_{\mathbb{R}^n} \varphi(x) \, dx \neq 0$. The anisotropic variable Hardy space $H^{p(\cdot)}_A(R^n)$ is defined to be the set of all $f \in S'(R^n)$ such that

$$\| f \|_{H^{p(\cdot)}_A(R^n)} := \left\| M_\varphi^0(f) \right\|_{L^{p(\cdot)}(R^n)} < \infty,$$

where $M_\varphi^0(f)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\varphi^0(f)(x) := \sup_{k \in \mathbb{Z}} |f \ast \varphi_k(x)|,$$
equivalently,

here and thereafter, for any $k \in \mathbb{Z}$, $\varphi_k(\cdot) := b^k \varphi(A^k \cdot)$.
We also need the following notions of anisotropic variable \((p(\cdot), r, s)\)-atoms and anisotropic variable finite atomic Hardy spaces from [21, Definitions 4.1 and 5.1].

**Definition 6** Let \(p(\cdot) \in \mathcal{P}(\mathbb{R}^n), r \in (1, \infty], \) and \(s \) be as in (3.1). A measurable function \(a \) on \(\mathbb{R}^n \) is called an anisotropic variable \((p(\cdot), r, s)\)-atom if

\[
\begin{align*}
&\text{(i) } \text{supp } a := \{ x \in \mathbb{R}^n : a(x) \neq 0 \} \subset B \in \mathcal{B}; \\
&\text{(ii) } \| a \|_{L^r(\mathbb{R}^n)} \leq \| B \|_{\mathcal{P}(\mathbb{R}^n)}^{1/r}; \\
&\text{(iii) for any } \gamma \in \mathbb{Z}^n_+ \text{ with } |\gamma| \leq s, \int_{\mathbb{R}^n} a(x) x^\gamma \, dx = 0.
\end{align*}
\]

**Definition 7** Let \(p(\cdot) \in C^{\log}(\mathbb{R}^n), r \in (1, \infty], \) and \(s \) be as in (3.1). The anisotropic variable finite atomic Hardy space \(H_{A, \text{fin}}^{p(\cdot), r,s}(\mathbb{R}^n) \) is defined to be the set of all \( f \) in \( S'(\mathbb{R}^n) \) satisfying that there exist an \( I \in \mathbb{N}, \{ \lambda_i \}_{i=1}^I \subset [0, \infty), \) and a finite sequence \(\{ a_i \}_{i=1}^I\) of \((p(\cdot), r, s)\)-atoms supported, respectively, in \( \{ B_i \}_{i=1}^I \subset \mathcal{B} \) such that \( f = \sum_{i=1}^I \lambda_i a_i \) in \( S'(\mathbb{R}^n) \). Moreover, for any \( f \in H_{A, \text{fin}}^{p(\cdot), r,s}(\mathbb{R}^n) \), let

\[
\| f \|_{H_{A, \text{fin}}^{p(\cdot), r,s}(\mathbb{R}^n)} := \inf \left\{ \left. \sum_{i=1}^I \left[ \frac{\lambda_i}{\| B_i \|_{L^p(\mathbb{R}^n)}} \right]^{p} \right\}^{1/p} \right\}_{L^p(\mathbb{R}^n)},
\]

where the infimum is taken over all decompositions of \( f \) as above and, here and thereafter, \( p_- := \min\{ p_-, 1 \} \) with \( p_- \) as in (2.3).

Next, we state the dual result between \( H_{A}^{p(\cdot)}(\mathbb{R}^n) \) and \( \mathcal{L}^A_{p(\cdot), q', s, p_-}(\mathbb{R}^n) \) as follows.

**Theorem 1** Let \( p(\cdot) \in C^{\log}(\mathbb{R}^n), q \in (\max\{1, p_+\}, \infty] \) with \( p_+ \) as in (2.3), and

\[
s \in \left[ \left( \frac{1}{p_-} - 1 \right) \frac{\ln b}{\ln \lambda_-}, \infty \right) \cap \mathbb{Z}_+,
\]

where \( \lambda_- \) and \( p_- \) are as in (2.5) and (2.3), respectively. Then the dual space of \( H_{A}^{p(\cdot)}(\mathbb{R}^n) \), denoted by \( (H_{A}^{p(\cdot)}(\mathbb{R}^n))^* \), is \( \mathcal{L}^A_{p(\cdot), q', s, p_-}(\mathbb{R}^n) \) in the following sense:

(i) Let \( g \in \mathcal{L}^A_{p(\cdot), q', s, p_-}(\mathbb{R}^n) \). Then the linear functional

\[
\Lambda_g : f \to \Lambda_g(f) := \int_{\mathbb{R}^n} f(x) g(x) \, dx,
\]

initially defined for any \( f \in H_{A, \text{fin}}^{p(\cdot), q,s}(\mathbb{R}^n) \), has a bounded extension to \( H_{A}^{p(\cdot)}(\mathbb{R}^n) \).

(ii) Conversely, any continuous linear functional on \( H_{A}^{p(\cdot)}(\mathbb{R}^n) \) arises as in (3.2) with a unique \( g \in \mathcal{L}^A_{p(\cdot), q', s, p_-}(\mathbb{R}^n) \).

Moreover, \( \| g \|_{\mathcal{L}^A_{p(\cdot), q', s, p_-}(\mathbb{R}^n)} \sim \| \Lambda_g \|_{(H_{A}^{p(\cdot)}(\mathbb{R}^n))^*} \), where the positive equivalence constants are independent of \( g \).
To prove Theorem 1, we need the following atomic and the finite atomic characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$, which were established in [21, Theorems 4.8 and 5.4].

**Lemma 1** Let $p(\cdot), q, s$ be as in Theorem 1, $\{a_j\}_{j \in \mathbb{N}}$ a sequence of $(p(\cdot), q, s)$-atoms supported, respectively, in $\{B^{(j)}\}_{j \in \mathbb{N}} \subset \mathcal{B}$, and $\{\lambda_j\}_{j \in \mathbb{N}} \subset [0, \infty)$ such that

$$
\left\| \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_j}{\|1_{B^{(j)}}\|_{L^p(\mathbb{R}^n)}} \right]^p 1_{B^{(j)}} \right\|_{L^p(\mathbb{R}^n)} < \infty.
$$

Then the series $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ converges in $H_A^{p(\cdot)}(\mathbb{R}^n)$, $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$, and there exists a positive constant $C$, independent of $f$, such that

$$
\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_j}{\|1_{B^{(j)}}\|_{L^p(\mathbb{R}^n)}} \right]^p 1_{B^{(j)}} \right\|_{L^p(\mathbb{R}^n)}.
$$

**Lemma 2** Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $s$ be as in (3.1).

(i) If $r \in (\max\{1, p_+\}, \infty)$ with $p_+$ as in (2.3), then $\|\cdot\|_{H_A^{p(\cdot), r, s}(\mathbb{R}^n)}$ and $\|\cdot\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}$ are equivalent quasi-norms on $H_A^{p(\cdot), r, s}(\mathbb{R}^n)$;

(ii) $\|\cdot\|_{H_A^{p(\cdot), \infty, s}(\mathbb{R}^n)}$ and $\|\cdot\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}$ are equivalent quasi-norms on $H_A^{p(\cdot), \infty, s}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$,

where $C(\mathbb{R}^n)$ denotes the set of all continuous functions on $\mathbb{R}^n$.

The following lemma is needed for establishing the dual theorem, whose proof is similar to that of [43, Proposition 3.13]; we omit the details.

**Lemma 3** Let $p(\cdot)$ and $s$ be as in Lemma 2. Then $H_A^{p(\cdot), \infty, s}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $H_A^{p(\cdot)}(\mathbb{R}^n)$.

We now show Theorem 1.

**Proof** Let all the notation be as in the present theorem. We first prove (i) by considering the range of $q$ in the following two cases.

**Case 1.** $q \in (\max\{1, p_+\}, \infty)$. In this case, let $g \in L_{p(\cdot), q, s}(\mathbb{R}^n)$. For any $f \in H_A^{p(\cdot), q, s}(\mathbb{R}^n)$, from Definition 7, it follows that there exist $\{\lambda_j\}_{j=1}^m \subset [0, \infty)$ and $\{a_j\}_{j=1}^m$ of $(p(\cdot), q, s)$-atoms supported, respectively, in the balls $\{B^{(j)}\}_{j=1}^m \subset \mathcal{B}$ such that $f = \sum_{j=1}^m \lambda_j a_j$ in $S'(\mathbb{R}^n)$ and

$$
\left\| \sum_{j=1}^m \left[ \frac{\lambda_j}{\|1_{B^{(j)}}\|_{L^p(\mathbb{R}^n)}} \right]^p 1_{B^{(j)}} \right\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{H_A^{p(\cdot), q, s}(\mathbb{R}^n)}.
$$
By this, the vanishing moments of \( \alpha_j \), the Hölder inequality, the size condition of \( \alpha_j \), Remark 1(ii), and Lemma 2(i), we conclude that

\[
|A_g(f)| \leq \sum_{j=1}^{m} \lambda_j \left| \int_{\mathbb{R}^n} a_j(x) g(x) \, dx \right|
\]

\[
= \sum_{j=1}^{m} \lambda_j \inf_{P \in \mathcal{P}_x(\mathbb{R}^n)} \left| \int_{B(j)} a_j(x) [g(x) - P(x)] \, dx \right|
\]

\[
\leq \sum_{j=1}^{m} \frac{\lambda_j |B(j)|}{\|1_{B(j)}\|_{L^p(\mathbb{R}^n)}} \inf_{P \in \mathcal{P}_x(\mathbb{R}^n)} \left[ \frac{1}{|B(j)|} \int_{B(j)} |g(x) - P(x)|^{q'} \, dx \right]^{\frac{1}{q'}}
\]

\[
\sim \|g\|_{L^A_{p},q,s}(\mathbb{R}^n) \left\{ \left\| \sum_{i=1}^{m} \frac{\lambda_i}{\|1_{B(i)}\|_{L^p(\mathbb{R}^n)}} \right\|_{L^p(\mathbb{R}^n)} \right\}^{\frac{1}{p}}
\]

which, together with the fact that \( H_{A,\text{fin}}^{p(\cdot),q,s}(\mathbb{R}^n) \) is dense in \( H_A^{p(\cdot)}(\mathbb{R}^n) \), further implies that (i) holds true in this case.

**Case 2.** \( q = \infty \). In this case, using Lemma 3 and repeating the proof of Case 1, we then find that any \( g \in L^A_{p(\cdot),1,s}(\mathbb{R}^n) \) induces a bounded linear functional on \( H_A^{p(\cdot)}(\mathbb{R}^n) \), which is initially defined on \( H_{A,\text{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \) by setting, for any \( f \in H_{A,\text{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \),

\[
A_g : f \mapsto A_g(f) := \int_{\mathbb{R}^n} f(x) g(x) \, dx,
\]

and then has a bounded extension to \( H_A^{p(\cdot)}(\mathbb{R}^n) \). Thus, to show (i) in this case, it remains to prove that, for any \( f \in H_{A,\text{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n) \),

\[
A_g(f) = \int_{\mathbb{R}^n} f(x) g(x) \, dx.
\]

For this purpose, assume that \( f \in H_{A,\text{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n) \) and \( \text{supp} \ (f) \subset B(0,L) \) with some \( L \in (0,\infty) \). Let \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) satisfy \( \text{supp} \ \varphi \subset B(0,1) \) and \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 1 \). Thus, for any \( t \in (0,1) \), \( \varphi_t * f \in H_{A,\text{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \), here and thereafter, for any \( t \in (0,\infty) \), \( \varphi_t(\cdot) := t^{-n} \varphi(t^{-1}) \). Letting \( r \in (\max\{1,p_+\},\infty) \), then \( f \in L^r(\mathbb{R}^n) \) and hence, from [9, Theorem 2.1], it follows that

\[
\lim_{t \in (0,\infty),t \to 0} \|f - \varphi_t * f\|_{L^r(\mathbb{R}^n)} = 0.
\]
By this and the Riesz lemma, we know that there exists a sequence \( \{ t_k \} \subset (0, 1) \) such that \( \lim_{k \to \infty} t_k = 0 \) and, for almost every \( x \in \mathbb{R}^n \), \( \lim_{k \to \infty} \varphi_{t_k} \ast f(x) = f(x) \). Therefore, we have

\[
\lim_{k \to \infty} \| f - \varphi_{t_k} \ast f \|_{H_A^{p(\cdot)}(\mathbb{R}^n)} = 0. \tag{3.5}
\]

Indeed, we only need to show that, for any given \((p(\cdot), \infty, s)\)-atom, (3.5) holds true. To this end, let \( a \) be a \((p(\cdot), \infty, s)\)-atom supported in the ball \( B(x_B, \ell(B)) \). Then, applying [9, Theorem 2.1], we have

\[
\lim_{t \in (0, 1), t \to 0} \| a - \varphi_t \ast a \|_{L'(\mathbb{R}^n)} = 0, \tag{3.6}
\]

Note that, for any \( t \in (0, 1) \), \( \frac{|B(x_B, \ell(B) + 2)|^\tau}{\|B(x_B, \ell(B) + 2)\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \| a - \varphi_t \ast a \|_{L'(\mathbb{R}^n)} \) is an \((p(\cdot), r, s)\)-atom supported in the ball \( B(x_B, \ell(B) + 2) \). Combining this, Lemma 1, and (3.6), we further conclude that

\[
\| a - \varphi_t \ast a \|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \lesssim \frac{\|B(x_B, \ell(B) + 2)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \| a - \varphi_t \ast a \|_{L'(\mathbb{R}^n)}}{|B(x_B, \ell(B) + 2)|^\tau} \lesssim \| a - \varphi_t \ast a \|_{L'(\mathbb{R}^n)} \to 0
\]

as \( t \to 0 \). This implies that (3.5) holds true.

Furthermore, from (3.5), (3.3), the fact that

\[
\left| (\varphi_{t_k} \ast f) g \right| \leq \| f \|_{L^\infty(\mathbb{R}^n)} \int_{B(0, L+1)} |g| \in L^1(\mathbb{R}^n),
\]

and the Lebesgue dominated convergence theorem, we deduce that

\[
A_g(f) = \lim_{k \to \infty} A_g(\varphi_{t_k} \ast f) = \lim_{k \to \infty} \int_{\mathbb{R}^n} \varphi_{t_k} \ast f(x) g(x) \, dx = \int_{\mathbb{R}^n} f(x) g(x) \, dx,
\]

which completes the proof of (3.4) and hence of (i).

We now show (ii). For any \( \Lambda \in (H_A^{p(\cdot)}(\mathbb{R}^n))^* \), applying an argument similar to that used in the proof of [16, Theorem 7.4], we conclude that there exists a unique \( g \in \mathcal{L}_p^A(\mathbb{R}^n) \) such that, for any \( f \in H_A^{p(\cdot), q, s}(\mathbb{R}^n) \),

\[
\Lambda(f) = \int_{\mathbb{R}^n} f(x) g(x) \, dx.
\]

To finish the proof of (ii), we then only need to show that \( g \in \mathcal{L}_p^{A(\mathbb{J})}(\mathbb{R}^n) \). Indeed, for any \( m \in \mathbb{N} \), \( \{ B^{(j)} \}_{j=1}^m \subset \mathcal{B} \), and \( \{ \lambda_j \}_{j=1}^m \subset [0, \infty) \) with \( \sum_{j=1}^m \lambda_j \neq 0 \), let \( \eta_j \in L^q(B^{(j)}) \) with \( \| \eta_j \|_{L^q(\mathcal{B}^{(j)})} = 1 \) satisfy that

\[ \small{\mathcal{S}} \]
\[
\left[ \int_{B(j)} \left| g(x) - P_{B(j)}^s g(x) \right|^{q'} \, dx \right]^{\frac{1}{q'}} = \int_{B(j)} \left[ g(x) - P_{B(j)}^s g(x) \right] \eta_j(x) \, dx
\] (3.7)

and, for any \( x \in \mathbb{R}^n \), define
\[
a_j(x) := \frac{|B(j)|^{\frac{1}{q'}} \left[ \eta_j(x) - P_{B(j)}^s \eta_j(x) \right] \mathbb{1}_{B(j)}}{\| \mathbb{1}_{B(j)} \|_{L^p(\mathbb{R}^n)} \| \eta_j - P_{B(j)}^s \eta_j \|_{L^q(B(j))}}.
\]

Thus, for any \( j \in \{1, \ldots, m\} \), \( a_j \) is an \((p(\cdot), q, s)\)-atom. By this and Lemma 1, we know that \( \sum_{j=1}^m \lambda_j a_j \in H_A^{p(s)}(\mathbb{R}^n) \). This, together with (3.7) and the facts that \( \Lambda \in (H_A^{p(s)}(\mathbb{R}^n))^* \) and
\[
\| \eta_j - P_{B(j)}^s \eta_j \|_{L^q(B(j))} \lesssim 1,
\]

further implies that
\[
\sum_{j=1}^m \lambda_j |B(j)| \left[ \int_{B(j)} \left| g(x) - P_{B(j)}^s g(x) \right|^{q'} \, dx \right]^{\frac{1}{q'}} = \sum_{j=1}^m \lambda_j |B(j)| \frac{1}{|B(j)|} \int_{B(j)} g(x) - P_{B(j)}^s g(x) \eta_j(x) \, dx
\]
\[
= \sum_{j=1}^m \lambda_j |B(j)|^{\frac{1}{q'}} \int_{B(j)} \left[ \eta_j(x) - P_{B(j)}^s \eta_j(x) \right] g(x) \mathbb{1}_{B(j)}(x) \, dx
\]
\[
\lesssim \sum_{j=1}^m \lambda_j \int_{B(j)} a_j(x) g(x) \, dx \sim \sum_{j=1}^m \lambda_j A(a_j) \sim A \left( \sum_{j=1}^m \lambda_j a_j \right)
\]
\[
\lesssim \left\| \sum_{j=1}^m \lambda_j a_j \right\|_{H_A^{p(s)}(\mathbb{R}^n)} \lesssim \left\{ \sum_{j=1}^m \left[ \frac{\lambda_j}{\| \mathbb{1}_{B(j)} \|_{L^p(\mathbb{R}^n)}} \right]^p \mathbb{1}_{B(j)} \right\}^{\frac{1}{p}}.
\]

From this, it follows that \( g \in L^A_{p(\cdot), q', s', p}(\mathbb{R}^n) \), which completes the proof of (ii) and hence of Theorem 1. \(\square\)

**Remark 5** Let \( p(\cdot), q, s, \) and \( p \) be as in Theorem 1. Recall that Liu et al. [21, Theorem 4.8] showed \( H_A^{p(s)}(\mathbb{R}^n) = \overline{H_A^{p(s)}(\mathbb{R}^n)} \) (anisotropic variable atomic Hardy spaces) with equivalent quasi-norms. Therefore, combining this with Theorem 1, we conclude that
\[
\left( H_A^{p(s)}(\mathbb{R}^n) \right)^* = L^A_{p(\cdot), q', s', p}(\mathbb{R}^n),
\]

which reflects the result obtained in Theorem 1 more intuitively.
Remark 6 We point out that the dual of the anisotropic variable Hardy space \( H^p_A(\mathbb{R}^n) \) as \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \) with \( p_+ \in (0, 1) \) was obtained by Wang [40, Theorem 4.4]. However, using Proposition 2, we find that [40, Theorem 4.4] is a special case of Theorem 1. Indeed, note that, in the case \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \) with \( 0 < p_- \leq 1 < p_+ < \infty \), the dual space of \( H^p_A(\mathbb{R}^n) \) can not be deduced from [40, Theorem 4.4] anymore. But, even in this case, the dual of \( H^p_A(\mathbb{R}^n) \) is also contained in Theorem 1. This is the main contribution of Theorem 1. Namely, this theorem identifies the new introduced Campanato-type space \( L^{A}_{p(\cdot), q', s, p}(\mathbb{R}^n) \) with the dual space of the Hardy space \( H^p_A(\mathbb{R}^n) \) for all \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \).

Remark 7 Let \( I_{n \times n} \) denote the \( n \times n \) unit matrix. When \( A := dI_{n \times n} \) for some \( d \in \mathbb{R} \) with \( |d| \in (1, \infty) \), then the space \( H^p_A(\mathbb{R}^n) \) goes back to the variable Hardy space studied in [6, 27]. In this case, Theorem 1 shows the dual theorem of \( H^p_A(\mathbb{R}^n) \) with \( 0 < p_- \leq p_+ < \infty \), which gives a complete answer to the question proposed by Izuki et al. in [18, Section 9.3].

By Theorem 1, we easily obtain the following equivalence of the Campanato-type space; we omit the details.

Corollary 1 Let \( p(\cdot), s, \) and \( p \) be as in Theorem 1, \( q \in [1, \infty) \) when \( p_+ \in (0, 1) \), or \( q \in [1, p'_+) \) when \( p_+ \in [1, \infty) \). Then

\[
L^A_{p(\cdot), q', s, p}(\mathbb{R}^n) = L^A_{p(\cdot), 1, s_0, p_0}(\mathbb{R}^n)
\]

with equivalent quasi-norms, where \( s_0 := \lceil (1/p_- - 1) \ln b/\ln \lambda \rceil \) and \( p_0 := \min\{1, p_-\} \) with \( p_- \) as in (2.3).

Let \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \) with \( p_- \in (1, \infty) \). Combining Theorem 1, the fact that \( (L^p_A(\mathbb{R}^n))^* = L^{p'}_{p(\cdot)}(\mathbb{R}^n) \) (see [5, Theorem 2.80]), [44, Corollary 4.20], and Corollary 1, we conclude the following conclusion; we omit the details.

Corollary 2 Let \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \) with \( p_- \in (1, \infty) \), and \( s_0 \) and \( p_0 \) be as in Corollary 1. Then

\[
L^{p(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = L^A_{p(\cdot), 1, s_0, p_0}(\mathbb{R}^n)
\]

with equivalent quasi-norms.

4 Applications

As applications, this section aims to establish equivalent characterizations of the new anisotropic variable Campanato-type spaces and the Carleson measure characterizations of \( L^A_{p(\cdot), 1, s, p}(\mathbb{R}^n) \) via the dual result obtained in Theorem 1. To begin with, we show the following theorem.
Theorem 2 Let $p(\cdot)$, $q$, $s$, and $p$ be as in Corollary 1 and $\varepsilon \in ([2/r - 1] \ln b / \ln \lambda, \infty)$ for some $r \in (0, 1)$. Then the following statements are mutually equivalent:

(i) $f \in \mathcal{L}_{p(\cdot), q, s, p}^A(\mathbb{R}^n)$;

(ii) $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ and

\[
\|f\|_{\mathcal{L}_{p(\cdot), q, s, p}^A(\mathbb{R}^n)} := \sup_{m} \left\{ \sum_{i=1}^{m} \left[ \frac{\lambda_i}{\|1_{x_i + B_{l_i}}\|_{L^p(\mathbb{R}^n)}} \right]^p 1_{x_i + B_{l_i}} \right\}^{\frac{1}{p}} \times \prod_{j=1}^{m} \left[ \frac{\lambda_j}{\|1_{x_j + B_{l_j}}\|_{L^p(\mathbb{R}^n)}} \right] \int_{x_j + B_{l_j}} \left| f(x) - P_{x_j + B_{l_j}} f(x) \right| dx \right\}^{\frac{1}{q}} < \infty,
\]

where the supremum is taken over all $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^{m} \subset \mathcal{B}$ with $\{x_j\}_{j=1}^{m} \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^{m} \subset \mathbb{Z}$, and $\{\lambda_j\}_{j=1}^{m} \subset [0, \infty)$ with $\sum_{j=1}^{m} \lambda_j \neq 0$;

(iii) $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ and

\[
\|f\|_{\mathcal{L}_{p(\cdot), q, s, p}^A(\mathbb{R}^n)} := \sup_{m} \left\{ \sum_{i=1}^{m} \left[ \frac{\lambda_i}{\|1_{x_i + B_{l_i}}\|_{L^p(\mathbb{R}^n)}} \right]^p 1_{x_i + B_{l_i}} \right\}^{\frac{1}{p}} \times \prod_{j=1}^{m} \left[ \frac{\lambda_j}{\|1_{x_j + B_{l_j}}\|_{L^p(\mathbb{R}^n)}} \right] \int_{x_j + B_{l_j}} \left| f(x) - P_{x_j + B_{l_j}} f(x) \right| dx \right\}^{\frac{1}{q}} < \infty,
\]

where the supremum is the same as in (ii) and the infimum is taken over all $P \in \mathfrak{P}_s(\mathbb{R}^n)$;

(iv) $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ and
\[ \| f \|_{A^{p,1}_{\bar{p}(\cdot),1,x,\rho}(\mathbb{R}^n)} := \sup \left\{ \sum_{i=1}^{m} \left( \frac{\lambda_i}{\| 1_{x_i+B_i} \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right)^{\frac{1}{p}} \left( 1_{x_i+B_i} \right) \right\}^{\frac{1}{p}} \times \sum_{j=1}^{m} \left( \frac{\lambda_j |x_j+B_{lj}|}{\| 1_{x_j+B_{lj}} \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \int_{\mathbb{R}^n} b_j^{\mathfrak{f}(1+\varepsilon \ln \frac{\lambda_j}{b_j})} |f(x) - P_{x_j+B_{lj}} f(x)| \, dx \right) \right\}^{\frac{1}{p}} < \infty, \]

where the supremum is the same as in (ii).

Moreover, for any \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \),

\[ \| f \|_{A^{p,1}_{\bar{p}(\cdot),1,x,\rho}(\mathbb{R}^n)} \sim \| f \|_{A^{p,1}_{\bar{p}(\cdot),1,x,\rho}(\mathbb{R}^n)} \sim \| f \|_{A^{p,1}_{\bar{p}(\cdot),1,x,\rho}(\mathbb{R}^n)} \sim \| f \|_{A^{p,1}_{\bar{p}(\cdot),1,x,\rho}(\mathbb{R}^n)} \]

with the positive equivalence constants independent of \( f \).

To prove Theorem 2, we need the following inequality, which is just a consequence of [21, Lemma 4.4].

**Lemma 4** Let \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \) and \( r \in (0, \min\{1, p_\rightarrow\}) \) with \( p_\rightarrow \) as in (2.3). Then there exists a positive constant \( C \) such that, for any \( \{x_j + B_{lj}\}_{j=1}^{m} \subset \mathfrak{B} \) with \( \{x_j\}_{j=1}^{m} \subset \mathbb{R}^n \) and \( \{l_j\}_{j=1}^{m} \subset \mathbb{Z} \), and \( k \in \mathbb{N} \),

\[ \left\| \sum_{j \in \mathbb{N}} 1_{x_j+B_{lj}+k} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C b^{k/r} \left\| \sum_{j \in \mathbb{N}} 1_{x_j+B_{lj}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \]

Now we can prove Theorem 2.

**Proof** From Corollary 1, we deduce the equivalence of (i) and (ii). Moreover, (i) obviously implies (iii) and conversely, by an argument similar to the proof of [23, Proposition 4.1], we know that (iii) implies (i). Therefore, to prove this theorem, it suffices to show that (ii) is equivalent to (iv).

Assume that (iv) holds true. We now show (ii). Indeed, for any \( m \in \mathbb{N} \), let \( \{x_j + B_{lj}\}_{j=1}^{m} \subset \mathfrak{B} \) with \( \{x_j\}_{j=1}^{m} \subset \mathbb{R}^n \) and \( \{l_j\}_{j=1}^{m} \subset \mathbb{Z} \), and \( \{\lambda_j\}_{j=1}^{m} \subset [0, \infty) \) with \( \sum_{j=1}^{m} \lambda_j \neq 0 \). Then, for any \( \varepsilon \in (0, \infty) \) and \( j \in \{1, \ldots, m\} \), we have
\[
\int_{\mathbb{R}^n} \frac{b^{\varepsilon l_j} \ln \lambda_{x_j}}{\ln b} \frac{\left| f(x) - P_{x_j + B_{l_j}} f(x) \right|}{b_{l_j}^{(1 + \varepsilon) \ln \lambda_{x_j}} + [\rho(x - x_j)]^{1 + \varepsilon} \ln \lambda_{x_j}} \, dx
\]

\[
\lesssim \int_{x_j + B_{l_j}} \frac{b^{\varepsilon l_j} \ln \lambda_{x_j}}{\ln b} \frac{\left| f(x) - P_{x_j + B_{l_j}} f(x) \right|}{b_{l_j}^{(1 + \varepsilon) \ln \lambda_{x_j}}} \, dx
\]

\[
\sim \frac{1}{|x_j + B_{l_j}|} \int_{x_j + B_{l_j}} \left| f(x) - P_{x_j + B_{l_j}} f(x) \right| \, dx,
\]

which, together with the assumption \( \|f\|_{L^p(\mathbb{R}^n)} < \infty \), implies that (ii) holds true. Thus, (iv) implies (ii).

Next assume that (ii) holds true. We prove (iv). Note that, for any \( m \in \mathbb{N} \), \( \{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B} \) with \( \{x_j\}_{j=1}^m \subset \mathbb{R}^n \) and \( \{l_j\}_{j=1}^m \subset \mathbb{Z} \), and \( \{\lambda_j\}_{j=1}^m \subset [0, \infty) \) with \( \sum_{j=1}^m \lambda_j \neq 0 \), we have

\[
\sum_{j=1}^m \left\{ \frac{\lambda_j |x_j + B_{l_j}|}{\|1_{x_j + B_{l_j}}\|_{L^p(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \frac{b^{\varepsilon l_j} \ln \lambda_{x_j}}{\ln b} \frac{\left| f(x) - P_{x_j + B_{l_j}} f(x) \right|}{b_{l_j}^{(1 + \varepsilon) \ln \lambda_{x_j}}} + [\rho(x - x_j)]^{1 + \varepsilon} \ln \lambda_{x_j} \, dx \right\}
\]

\[
= \sum_{j=1}^m \left\{ \frac{\lambda_j |x_j + B_{l_j}|}{\|1_{x_j + B_{l_j}}\|_{L^p(\mathbb{R}^n)}} \left[ \int_{x_j + B_{l_j}} + \sum_{k=0}^{\infty} \int_{b_{l_j}^{l_j+k}}^{b_{l_j}^{l_j+k+1}} \frac{b^{\varepsilon l_j} \ln \lambda_{x_j}}{\ln b} \frac{\left| f(x) - P_{x_j + B_{l_j}} f(x) \right|}{b_{l_j}^{(1 + \varepsilon) \ln \lambda_{x_j}}} \, dx \right] \right\}
\]

\[
\leq \sum_{j=1}^m \left\{ \frac{\lambda_j}{\|1_{x_j + B_{l_j}}\|_{L^p(\mathbb{R}^n)}} \int_{x_j + B_{l_j}} \left| f(x) - P_{x_j + B_{l_j}} f(x) \right| \, dx \right\}
\]

\[
+ \sum_{j=1}^m \left\{ \frac{\lambda_j}{\|1_{x_j + B_{l_j}}\|_{L^p(\mathbb{R}^n)}} \times \int_{b_{l_j}^{l_j+k}}^{b_{l_j}^{l_j+k+1}} \left| f(x) - P_{x_j + B_{l_j}} f(x) \right| \, dx \right\}.
\]

Therefore,
\[
\left\| \sum_{i=1}^{m} \left[ \frac{\lambda_i}{\|1_{x_i+B_t} \|_{L^{p}(\mathbb{R}^n)}} \right] \right\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)}^{-1} \times \sum_{j=1}^{m} \left[ \frac{\lambda_j}{\|1_{x_j+B_t} \|_{L^{p}(\mathbb{R}^n)}} \right] \left\| f(x) - P_{x_j+B_t} f(x) \right\|_{L^{p}(\mathbb{R}^n)} \right\} \text{dx} \\
\lesssim \| f \|_{L^{p(\cdot)}_{1,*}, s}^{A}(\mathbb{R}^n) + \Theta,
\]

where

\[
\Theta := \left\| \sum_{i=1}^{m} \left[ \frac{\lambda_i}{\|1_{x_i+B_t} \|_{L^{p}(\mathbb{R}^n)}} \right] \right\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)}^{-1} \times \sum_{j=1}^{m} \left[ \frac{\lambda_j}{\|1_{x_j+B_t} \|_{L^{p}(\mathbb{R}^n)}} \right] \left\| f(x) - P_{x_j+B_t} f(x) \right\|_{L^{p}(\mathbb{R}^n)} \text{dx} \\
\times \sum_{k=0}^{\infty} b^{-k(1+\varepsilon \ln \frac{\alpha}{\ln b})} \left\| f(x) - P_{x_j+B_t} f(x) \right\|_{L^{p}(\mathbb{R}^n)} \text{dx} \\
\times \sum_{k=0}^{\infty} b^{-k(1+\varepsilon \ln \frac{\alpha}{\ln b})} \left\| f(x) - P_{x_j+B_t} f(x) \right\|_{L^{p}(\mathbb{R}^n)} \text{dx} \\
+ \left\| \sum_{i=1}^{m} \left[ \frac{\lambda_i}{\|1_{x_i+B_t} \|_{L^{p}(\mathbb{R}^n)}} \right] \right\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)}^{-1} \times \sum_{j=1}^{m} \left[ \frac{\lambda_j}{\|1_{x_j+B_t} \|_{L^{p}(\mathbb{R}^n)}} \right] \left\| f(x) - P_{x_j+B_t} f(x) \right\|_{L^{p}(\mathbb{R}^n)} \text{dx} \\
\times \sum_{k=0}^{\infty} b^{-k(1+\varepsilon \ln \frac{\alpha}{\ln b})} \left\| f(x) - P_{x_j+B_t} f(x) \right\|_{L^{p}(\mathbb{R}^n)} \text{dx} \right). \]

This, together with the Tonelli theorem, \( r \in (0, p) \), and Lemma 4, implies that

\[
\Theta \lesssim \| f \|_{L^{p(\cdot)}_{1,*}, s}^{A}(\mathbb{R}^n) \sum_{k=0}^{\infty} b^{-k(1-\frac{2}{p} + \varepsilon \ln \frac{\alpha}{\ln b})} \]
\[
+ \left\| \sum_{i=1}^{m} \left\{ \frac{\lambda_i}{\| 1_{x_i+B_i} \|_{L^p(\mathbb{R}^n)}} \right\}_i \right\|_{L^p(\mathbb{R}^n)}^{-1} \times \sum_{j=1}^{m} \left\{ \frac{\lambda_j}{\| 1_{x_j+B_j} \|_{L^p(\mathbb{R}^n)}} \right\} \times \int_{x_j+B_j+k} \left| P_{x_j+B_j+k}^s f(x) - P_{x_j+B_j}^s f(x) \right| \, dx \right\} \]. (4.2)

On another hand, from [1, (8.9)], we deduce that, for any \( x \in x_j + B_{j+k} \),

\[
\left| P_{x_j+B_j+k}^s f(x) - P_{x_j+B_j}^s f(x) \right| \leq \sum_{v=1}^{k} \left| P_{x_j+B_{j+v}}^s f(x) - P_{x_j+B_{j+v-1}}^s f(x) \right|
\]

\[
= \sum_{v=1}^{k} \left| P_{x_j+B_{j+v-1}}^s \left( f - P_{x_j+B_{j+v}}^s f \right)(x) \right|
\]

\[
\leq \sum_{v=1}^{k} \frac{1}{|x_j+B_{j+v-1}|} \int_{x_j+B_{j+v}} \left| f(x) - P_{x_j+B_{j+v}}^s f(x) \right| \, dx,
\]

which, combined with (4.2), the Tonelli theorem, Lemma 4, and the fact that \( \varepsilon \in ([2/r - 1] \ln b/\ln \lambda_{\gamma}, \infty) \), further implies that

\[
\Theta \lesssim \| f \|_{L^A_p(\mathbb{R}^n)} \sum_{k \in \mathbb{N}} b^{-k\left(1 - \frac{2}{r} + \varepsilon \frac{\ln \lambda_{\gamma}}{\ln b}\right)}
\]

\[
+ \left\| \sum_{i=1}^{m} \left\{ \frac{\lambda_i}{\| 1_{x_i+B_i} \|_{L^p(\mathbb{R}^n)}} \right\}_i \right\|_{L^p(\mathbb{R}^n)}^{-1} \times \sum_{j=1}^{m} \left\{ \frac{\lambda_j}{\| 1_{x_j+B_j} \|_{L^p(\mathbb{R}^n)}} \right\} \times \int_{x_j+B_j+k} \left| f(x) - P_{x_j+B_j+k}^s f(x) \right| \, dx \right\} \lesssim \| f \|_{L^A_p(\mathbb{R}^n)} \sum_{k \in \mathbb{N}} b^{-k\left(1 - \frac{2}{r} + \varepsilon \frac{\ln \lambda_{\gamma}}{\ln b}\right)}
\]

\[
+ \| f \|_{L^A_p(\mathbb{R}^n)} \sum_{k \in \mathbb{N}} b^{-k\varepsilon \frac{\ln \lambda_{\gamma}}{\ln b}} \sum_{v=1}^{k} b^{v(2/r-1)}
\]
\[ \lesssim \| f \|_{L^p_{\ell^0,1,s}}(\mathbb{R}^n) \sum_{k \in \mathbb{N}} b^{-k(1-\frac{2}{p}+\frac{2 \ln \lambda}{\ln b})} \lesssim \| f \|_{L^p_{\ell^0,1,s}}(\mathbb{R}^n). \]

Combining this and (4.1), we finally conclude that (iv) holds true and hence finish the proof of Theorem 2. \hfill \Box

From Proposition 1 and Theorem 2, we immediately deduce the following equivalent characterizations of \( L^p_{\ell^0,1,s} (\mathbb{R}^n) \); we omit the details.

**Corollary 3** If \( p(\cdot), q, s, p, \) and \( \varepsilon \) are as in Theorem 2, then all the results of Theorem 2 still hold true with \( m \) replaced by \( \infty \), and the supremum therein taken over all \( \{ x_j + B_{l_j} \}_{j \in \mathbb{N}} \subseteq \mathcal{B} \) with \( \{ x_j \}_{j \in \mathbb{N}} \subseteq \mathbb{R}^n \) and \( \{ l_j \}_{j \in \mathbb{N}} \subseteq \mathbb{Z} \), and \( \{ \lambda_j \}_{j \in \mathbb{N}} \subseteq [0, \infty) \) satisfying

\[ \left\| \left\{ \sum_{j \in \mathbb{N}} \left( \frac{\lambda_j}{1_{x_j + B_{l_j}}} \right)^p 1_{x_j + B_{l_j}} \right\}_{x_j + B_{l_j} \subseteq \mathbb{R}^n} \right\|_{L^p(\mathbb{R}^n)} \in (0, \infty). \]

Via these obtained equivalent characterizations of the anisotropic variable Campanato-type space, we establish the Carleson measure characterization of \( L^p_{\ell^0,1,s} (\mathbb{R}^n) \). To begin with, we introduce the \( p(\cdot) \)-Carleson measure as follows.

**Definition 8** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \). A Borel measure \( d\mu \) on \( \mathbb{R}^n \times \mathbb{Z} \) is called a \( p(\cdot) \)-Carleson measure if

\[ \| d\mu \|_{C^p(\mathbb{R}^n)} := \sup \left\{ \left\| \sum_{i=1}^m \left( \frac{\lambda_i}{1_{B^{(i)}}} \right)^\eta \right\|_{L^p(\mathbb{R}^n)}^{1/\eta} \right\} \times \sum_{j=1}^m \frac{\lambda_j |B^{(j)}|^{1/2}}{1_{B^{(j)}} \| 1_{B^{(j)}} \|_{L^p(\mathbb{R}^n)}} \left[ \int_{B^{(j)}} |d\mu(x, k)| \right]^{1/2} \]

is finite, where \( \eta \in (0, \infty) \) and the supremum is taken over all \( m \in \mathbb{N}, \{ B^{(j)} \}_{j=1}^m \subseteq \mathcal{B} \), and \( \{ \lambda_j \}_{j=1}^m \subseteq [0, \infty) \) with \( \sum_{j=1}^m \lambda_j \neq 0 \), and, for any \( j \in \{ 1, \ldots, m \} \), \( B^{(j)} \) denotes the tent over \( B^{(j)} \), namely,

\[ B^{(j)} := \{ (y, k) \in \mathbb{R}^n \times \mathbb{Z} : y + B_k \subseteq B^{(j)} \}. \]

**Remark 8** Let \( p(\cdot), \eta, \) and \( d\mu \) be as in Definition 8 and

\[ \| d\mu \|_{C^p(\mathbb{R}^n)} := \sup \left\{ \left\| \sum_{i=1}^m \left( \frac{\lambda_i}{1_{B^{(i)}}} \right)^\eta \right\|_{L^p(\mathbb{R}^n)}^{1/\eta} \right\} \times \sum_{j \in \mathbb{N}} \frac{\lambda_j |B^{(j)}|^{1/2}}{1_{B^{(j)}} \| 1_{B^{(j)}} \|_{L^p(\mathbb{R}^n)}} \left[ \int_{B^{(j)}} |d\mu(x, k)| \right]^{1/2}, \]

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where the supremum is taken over all \( \{ B^{(j)} \}_{j \in \mathbb{N}} \subset \mathcal{B} \) and \( \{ \lambda_j \}_{j \in \mathbb{N}} \subset [0, \infty) \) satisfying

\[
\left\| \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_j}{\| 1_{B^{(j)}} \|_{L^p(\mathbb{R}^n)}} \right]^\eta \right\}^{1/\eta} \right\|_{L^p(\mathbb{R}^n)} \in (0, \infty).
\]

Then \( \| d\mu \|_{C_p(A)} = \| d\mu \|_{C_p(\mathbb{R}^n)} \).

In what follows, for any given \( k \in \mathbb{Z} \), define

\[
\delta_k(j) := \begin{cases} 1 & \text{when } j = k, \\ 0 & \text{when } j \neq k. \end{cases}
\]

Let \( C_c^\infty(\mathbb{R}^n) \) denote the collection of all infinitely differentiable functions with compact support on \( \mathbb{R}^n \) and, for any \( \varphi \in \mathcal{S}(\mathbb{R}^n) \), \( \hat{\varphi} \) denote its Fourier transform, namely, for any \( \xi \in \mathbb{R}^n \),

\[
\hat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x)e^{-2\pi i x \cdot \xi} \, dx,
\]

where \( i := \sqrt{-1} \) and \( x \cdot \xi := \sum_{i=1}^n x_i \xi_i \) for any \( x := (x_1, \ldots, x_n) \), \( \xi := (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \). Let \( s \in \mathbb{Z}_+ \) and \( \phi \in C_c^\infty(\mathbb{R}^n) \) satisfy

\[
supp \phi \subset B_0, \quad \int_{\mathbb{R}^n} x^\gamma \phi(x) \, dx = 0, \quad \forall \gamma \in \mathbb{Z}^n_+ \text{ with } |\gamma| \leq s, \quad (4.3)
\]

and there exists a positive constant \( C \) such that

\[
|\hat{\phi}(\xi)| \geq C \quad \text{when } \xi \in \left\{ x \in \mathbb{R}^n : (2\|A\|)^{-1} \leq \rho(x) \leq 1 \right\}, \quad (4.4)
\]

where the dilation \( A := (a_{i,j})_{1 \leq i, j \leq n} \) and \( \| A \| := (\sum_{i,j=1}^n |a_{i,j}|^2)^{1/2} \).

We now show the Carleson measure characterization of \( \mathcal{L}_p^A(\mathbb{R}^n) \) as follows.

**Theorem 3** Let \( p(\cdot), s, \) and \( p \) be as in Theorem 1, \( p_+ \in (0, 2) \), and \( \phi \in \mathcal{S}(\mathbb{R}^n) \) be a radial real-valued function satisfying (4.3) and (4.4).

(i) If \( b \in \mathcal{L}_p^A(\mathbb{R}^n) \), then, for any \( (x, k) \in \mathbb{R}^n \times \mathbb{Z} \), \( d\mu(x, k) := \sum_{\ell \in \mathbb{Z}} \phi_{-\ell} * b(x) \, dx \delta_\ell(k) \) is a \( p(\cdot) \)-Carleson measure on \( \mathbb{R}^n \times \mathbb{Z} \); moreover, there exists a positive constant \( C \), independent of \( b \), such that

\[
\| d\mu \|_{C_p(\mathbb{R}^n)} \leq C \| b \|_{\mathcal{L}_p^A(\mathbb{R}^n)}.
\]
(ii) If \( b \in L^2_{\text{loc}}(\mathbb{R}^n) \) and, for any \((x, k) \in \mathbb{R}^n \times \mathbb{Z}, \) \( d\mu(x, k) := \sum_{\ell \in \mathbb{Z}} |\phi_{-\ell} * b(x)|^2 \, dx \, \delta_{x}(k) \) is a \( p(\cdot) \)-Carleson measure on \( \mathbb{R}^n \times \mathbb{Z}, \) then \( b \in \mathcal{L}^n_{\mu(\cdot)}(\mathbb{R}^n) \) and, moreover, there exists a positive constant \( C, \) independent of \( b, \) such that

\[
\|b\|_{\mathcal{L}^n_{\mu(\cdot)}(\mathbb{R}^n)} \leq C \|d\mu\|_{\mathcal{E}^{p(\cdot),\mu}(\mathbb{R}^n)}.
\]

To prove Theorem 3, we need the anisotropic variable tent space and its atomic decomposition. However, as far as we know, the theory on anisotropic variable tent spaces is blank in the literature. Thus, we first introduce the anisotropic variable tent space and then establish its atomic decomposition. For any \( x \in \mathbb{R}^n, \) let

\[
\Gamma(x) := \{(y, k) \in \mathbb{R}^n \times \mathbb{Z} : y \in x + B_k\}
\]

be the cone of aperture 1 with vertex \( x \in \mathbb{R}^n. \) For any measurable function \( G \) on \( \mathbb{R}^n \times \mathbb{Z}, \) the anisotropic discrete Lusin area function \( A(G) \) is defined by setting, for any \( x \in \mathbb{R}^n, \)

\[
A(G)(x) := \left( \sum_{\ell \in \mathbb{Z}} b^{-\ell} \int_{\{y \in \mathbb{R}^n : (y, \ell) \in \Gamma(x)\}} |G(y, \ell)|^2 \, dy \right)^{1/2}.
\]

Via this anisotropic discrete Lusin area function, we introduce the following anisotropic variable tent space.

**Definition 9** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n). \) The anisotropic variable tent space \( T^{p(\cdot)}_\Lambda(\mathbb{R}^n \times \mathbb{Z}) \) is defined to be the set of all measurable functions \( G \) on \( \mathbb{R}^n \times \mathbb{Z} \) such that \( A(G) \in L^{p(\cdot)}(\mathbb{R}^n) \) and equipped with the quasi-norm \( \|G\|_{T^{p(\cdot)}_\Lambda(\mathbb{R}^n \times \mathbb{Z})} := \|A(G)\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \)

We next give the definition of anisotropic \((p(\cdot), \infty)\)-atoms.

**Definition 10** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( q \in (1, \infty). \) A measurable function \( a \) on \( \mathbb{R}^n \times \mathbb{Z} \) is called an anisotropic \((p(\cdot), q)\)-atom if there exists a ball \( B \in \mathcal{B} \) such that

(i) \( \text{supp } a := \{(x, k) \in \mathbb{R}^n \times \mathbb{Z} : a(x, k) \neq 0\} \subset \overline{B}, \)

(ii) \( \|a\|_{T^q_\Lambda(\mathbb{R}^n \times \mathbb{Z})} := \|A(a)\|_{L^q(\mathbb{R}^n)} \leq \frac{|B|^{1/q}}{\|B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}. \)

Moreover, if \( a \) is an anisotropic \((p(\cdot), q)\)-atom for any \( q \in (1, \infty), \) then \( a \) is called an anisotropic \((p(\cdot), \infty)\)-atom.

For functions in the anisotropic variable tent space \( T^{p(\cdot)}_\Lambda(\mathbb{R}^n \times \mathbb{Z}), \) we have the following atomic decomposition.

**Lemma 5** Let \( p(\cdot) \in C^{\log}(\mathbb{R}^n). \) Then, for any \( G \in T^{p(\cdot)}_\Lambda(\mathbb{R}^n \times \mathbb{Z}), \) there exist \( \{\lambda_j\}_{j \in \mathbb{N}} \subset [0, \infty), \) \( \{B^{(j)}\}_{j \in \mathbb{N}} \subset \mathcal{B}, \) and a sequence \( \{A_j\}_{j \in \mathbb{N}} \) of anisotropic...
(\(p(\cdot), \infty\))-atoms supported, respectively, in \(\{B^{(j)}\}_{j \in \mathbb{N}}\) such that, for almost every \((x, k) \in \mathbb{R}^n \times \mathbb{Z}\),

\[ G(x, k) = \sum_{j \in \mathbb{N}} \lambda_j A_j(x, k) \quad \text{and} \quad |G(x, k)| = \sum_{j \in \mathbb{N}} \lambda_j |A_j(x, k)| \]

pointwisely, and

\[
\left\| \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_j}{\|1_{B^{(j)}}\|_{L^p(\mathbb{R}^n)}} \right]^p 1_{B^{(j)}} \right\}^{1/p} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|G\|_{T^{p(\cdot)}(\mathbb{R}^n \times \mathbb{Z})}, \tag{4.5}
\]

where the implicit positive constant is independent of \(G\).

**Proof** Let \(p(\cdot) \in C^{1, \log}(\mathbb{R}^n)\) and \(G \in T^{p(\cdot)}_A(\mathbb{R}^n \times \mathbb{Z})\). For any \(j \in \mathbb{Z}\), let \(O_j := \{x \in \mathbb{R}^n : A(G)(x) > 2^j\}, G_j := (O_j)^6\), and, for any given \(\gamma \in (0, 1)\),

\[ (O_j)^*_\gamma := \{x \in \mathbb{R}^n : M_{HL}(1_{O_j})(x) > 1 - \gamma\}, \]

here and thereafter, \(M_{HL}\) denotes the anisotropic Hardy–Littlewood maximal operator, namely, for any \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\) and \(x \in \mathbb{R}^n\),

\[ M_{HL}(f)(x) := \sup_{k \in \mathbb{Z}} \sup_{y + B_k \ni x} \frac{1}{|B_k|} \int_{y + B_k} |f(z)| \, dz. \]

Then, by the proof of [10, (1.14)], we find that

\[ \text{supp } G \subset \left[ \bigcup_{j \in \mathbb{Z}} (O_j)^*_\gamma \cup E \right], \tag{4.6} \]

where \(E \subset \mathbb{R}^n \times \mathbb{Z}\) satisfies that \(\sum_{\ell \in \mathbb{Z}} \int_{\{y \in \mathbb{R}_p : (y, \ell) \in E\}} dy = 0\). Moreover, applying [10, (1.15)], we know that, for any \(j \in \mathbb{Z}\), there exist an integer \(N_j \in \mathbb{N} \cup \{\infty\}\), \(\{x^j_k\}_{k=1}^{N_j} \subset (O_j)^*_\gamma\), and \(\{l^j_k\}_{k=1}^{N_j} \subset \mathbb{Z}\) such that \(\{x^j_k + B^j_{l_k}\}_{k=1}^{N_j}\) has finite intersection property and

\[ \square \]
\[(O_j)_{\gamma}^* = \bigcup_{k=1}^{N_j} \left( x_k^j + B_{l_k}^j \right) \]
\[= \left( x_1^j + B_{l_1}^j \right) \cup \left( \left( x_2^j + B_{l_2}^j \right) \setminus \left( x_1^j + B_{l_1}^j \right) \right) \cup \cdots \]
\[\cup \left\{ \left( x_{N_j}^j + B_{l_{N_j}}^j \right) \setminus \bigcup_{i=1}^{N_j-1} \left( x_i^j + B_{l_i}^j \right) \right\} \]
\[= \bigcup_{k=1}^{N_j} B_{j,k}. \tag{4.7} \]
\end{equation}

Note that, for any \( j \in \mathbb{Z} \), \{\( B_{j,k} \)\}_{k=1}^{N_j} are mutually disjoint. Thus, \( (O_j)_{\gamma}^* = \bigcup_{k=1}^{N_j} \widehat{B_{j,k}}. \)

For any \( j \in \mathbb{Z} \) and \( k \in \{1, \ldots, N_j\} \), let

\[C_{j,k} := \widehat{B_{j,k}} \cap \left[ (O_j)_{\gamma}^* \setminus (O_{j+1})_{\gamma}^* \right],\]

\[A_{j,k} := 2^{-j} \left\| 1_{x_k^j + B_{l_k}^j} \right\|_{L^p(\mathbb{R}^n)}^{-1} G 1_{C_{j,k}}, \tag{4.9} \]

and \( \lambda_{j,k} := 2^{j} \left\| 1_{x_k^j + B_{l_k}^j} \right\|_{L^p(\mathbb{R}^n)}. \) Therefore, from (4.6), it follows that

\[G = \sum_{j \in \mathbb{Z}} \sum_{k=1}^{N_j} \lambda_{j,k} A_{j,k} \quad \text{and} \quad |G| = \sum_{j \in \mathbb{Z}} \sum_{k=1}^{N_j} \lambda_{j,k} |A_{j,k}|\]

almost everywhere on \( \mathbb{R}^n \times \mathbb{Z} \). We now show that, for any \( j \in \mathbb{Z} \) and \( k \in \{1, \ldots, N_j\} \), \( A_{j,k} \) is an anisotropic \((p(\cdot), \infty)\)-atom supported in \( x_k^j + B_{l_k}^j \). Obviously, \( \text{supp} A_{j,k} \subset C_{j,k} \subset \widehat{B_{j,k}} \subset x_k^j + B_{l_k}^j \). In addition, let \( q \in (1, \infty) \) and \( h \in T_2^{q'}(\mathbb{R}^n \times \mathbb{Z}) \) satisfy \( \|h\|_{T_2^{q'}(\mathbb{R}^n \times \mathbb{Z})} \leq 1 \). Note that \( C_{j,k} \subset (O_{j+1})_{\gamma}^* = \bigcup_{x \in (G_{j+1})_{\gamma}} \Gamma(x) \). Applying this, [10, Lemma 1.3], the Hölder inequality, and (4.9), we find that

\[\left| \langle A_{j,k}, h \rangle \right| \leq \sum_{\ell \in \mathbb{Z}} \int_{C_{j,k}} A_{j,k}(y, \ell) h(y, \ell) \, dy \]
\[\lesssim \int_{G_{j+1}} \sum_{\ell \in \mathbb{Z}} \int_{\{y \in \mathbb{R}^n: (y, \ell) \in \Gamma(x)\}} b^{-\ell} \left| A_{j,k}(y, \ell) h(y, \ell) \right| \, dy \, dx \]
\[\lesssim \int_{(O_{j+1})_{\gamma}} \mathcal{A}(A_{j,k})(x) \mathcal{A}(h)(x) \, dx \]

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which, combined with \((T_2^q(\mathbb{R}^n \times \mathbb{Z}))^* = T_2^q(\mathbb{R}^n \times \mathbb{Z})\) (see [4, 10]), further implies that \(\|A_{j,k}\|_{T_2^q(\mathbb{R}^n \times \mathbb{Z})} \lesssim \|1_{x_k^j+B_k^j} 1^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\). This implies that, for any \(j \in \mathbb{Z}\) and \(k \in \{1, \ldots, N_j\}\), \(A_{j,k}\) is an anisotropic \((p(\cdot), q)\)-atom up to a harmless constant multiple for all \(q \in (1, \infty)\). Thus, for any \(j \in \mathbb{Z}\) and \(k \in \{1, \ldots, N_j\}\), \(A_{j,k}\) is an anisotropic \((p(\cdot), \infty)\)-atom up to a harmless constant multiple.

We next prove (4.5). To achieve this, from (4.7), the finite intersection property of \(\{x_k^j+B_k^j\}_{k=1}^{N_j}\), the fact that \(1_{(O_j)^\gamma_j} \lesssim [M_{HL}(1_{O_j})]^{1/\tau}\) with \(\tau \in (0, p)\), and [21, Lemma 4.4], we deduce that

\[
\lambda \left\{ \sum_{j \in \mathbb{Z}} \sum_{k=1}^{N_j} \left[ \frac{\lambda_{j,k}}{\|1_{x_k^j+B_k^j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[ \frac{2^j 1_{O_j}^\gamma_j}{} \right]^p \right\}^{1/p} \lesssim \|A(G)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|G\|_{T_A^{p(\cdot)}(\mathbb{R}^n \times \mathbb{Z})}.
\]

This implies that (4.5) holds true and hence finishes the proof of Lemma 5. \(\square\)

With the help of Theorems 1 and 2 and Lemma 5, we can now prove Theorem 3.

**Proof** Let \(p(\cdot)\), \(s\), \(p_\cdot\), and \(\phi\) be as in the present theorem. We first prove (i). To this end, let \(b \in L^{p(\cdot),1,s}_{p_\cdot,\phi}(\mathbb{R}^n)\) and \((x_j + B_j)_{j=1}^m \subset \mathcal{B}\), where \(m \in \mathbb{N}\), \((x_j)_{j=1}^m \subset \mathbb{R}^n\),
and \( \{l_j\}_{j=1}^m \subset \mathbb{Z} \). Then, for any \( j \in \{1, \ldots, m\} \), we have

\[
b = P^s_{x_j + B_l} b + \left( b - P^s_{x_j + B_l} b \right) 1_{x_j + B_l + \omega} + \left( b - P^s_{x_j + B_l} b \right) 1_{(x_j + B_l + \omega)^c} \equalcolon b_j^{(1)} + b_j^{(2)} + b_j^{(3)}
\]

(4.10)

with \( \omega \) as in (2.2). Note that, for any \( \alpha \in \mathbb{Z}_n^+ \) with \( |\alpha| \leq s \), \( \int_{\mathbb{R}^n} \phi(x) x^\alpha \, dx = 0 \). Therefore, for any \( k \in \mathbb{Z} \) and \( j \in \{1, \ldots, m\} \), \( \phi_k \ast b_j^{(1)} \equiv 0 \) and hence, for any \( j \in \{1, \ldots, m\} \), we have

\[
\sum_{\ell \in \mathbb{Z}} \int_{\{x \in \mathbb{R}^n: (x, \ell) \in x_j + B_l\}} \left| \phi_{-\ell} \ast b_j^{(1)}(x) \right|^2 \, dx = 0. \tag{4.11}
\]

In addition, from the Tonelli theorem and the boundedness of the \( g \)-function (see, for instance, [16, Theorem 6.3]), we deduce that, for any \( j \in \{1, \ldots, m\} \),

\[
\sum_{\ell \in \mathbb{Z}} \int_{\{x \in \mathbb{R}^n: (x, \ell) \in x_j + B_l\}} \left| \phi_{-\ell} \ast b_j^{(2)}(x) \right|^2 \, dx
\leq \int_{\mathbb{R}^n} \sum_{\ell \in \mathbb{Z}} \left| \phi_{-\ell} \ast b_j^{(2)}(x) \right|^2 \, dx \lesssim \left\| b_j^{(2)} \right\|_{L^2(\mathbb{R}^n)}
\sim \int_{x_j + B_l + \omega} \left| b(x) - P^s_{x_j + B_l} b(x) \right|^2 \, dx
\lesssim \int_{x_j + B_l + \omega} \left| b(x) - P^s_{x_j + B_l + \omega} b(x) \right|^2 \, dx
+ \int_{x_j + B_l + \omega} \left| P^s_{x_j + B_l + \omega} b(x) - P^s_{x_j + B_l + \omega} b(x) \right|^2 \, dx. \tag{4.12}
\]

Moreover, by [1, (8.9)] (see also [23, Lemma 4.1]), we know that, for any \( x \in x_j + B_l + \omega \),

\[
\left| P^s_{x_j + B_l + \omega} b(x) - P^s_{x_j + B_l} b(x) \right|
\geq \frac{1}{|x_j + B_l|} \int_{x_j + B_l + \omega} \left| b(y) - P^s_{x_j + B_l + \omega} b(y) \right| \, dy,
\]

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which, together with (4.12) and Lemma 4, further implies that, for any \( m \in \mathbb{N} \), \( \{x_j + B_{I_j}\}_{j=1}^m \subset \mathfrak{B} \), and \( \{\lambda_j\}_{j=1}^m \subset [0, \infty) \) with \( \sum_{j=1}^m \lambda_j \neq 0 \),

\[
\left\| \sum_{i=1}^m \left[ \frac{\lambda_i}{\|1_{x_i} + B_{I_i}\|_{L^p(\mathbb{R}^n)}} \right]_1^{1/p} \left( 1_{x_i} + B_{I_i} \right)^{1/p} \right\|_{L^p(\mathbb{R}^n)}^{-1} \sum_{j=1}^m \frac{\lambda_j |x_j + B_{I_j}|^{1/2}}{\|1_{x_j} + B_{I_j}\|_{L^p(\mathbb{R}^n)}} \right\|
\times \left[ \sum_{j=1}^m \sum_{\ell \in \mathbb{Z}^n} \left| \phi_{-\ell} * B_j^{(2)}(x) \right|^2 dx \right]^{1/2} \leq \sum_{j=1}^m \frac{\lambda_j |x_j + B_{I_j}|^{1/2}}{\|1_{x_j} + B_{I_j}\|_{L^p(\mathbb{R}^n)}} \right\|_{L^p(\mathbb{R}^n)}
\leq \|b\|_{L^{\lambda, 1, 1}(\mathbb{R}^n)}. \tag{4.13}
\]

We now estimate \( b_j^{(3)} \). For this purpose, let \( r \in (0, p) \) and \( \varepsilon \in ([2/r - 1] \ln b/\ln \lambda_-, \infty) \). Then, for any \( j \in \{1, \ldots, m\} \) and \( (x, \ell) \in x_j + B_{I_j} \), we have

\[
\left| \phi_{-\ell} * b_j^{(3)}(x) \right| \lesssim \int_{(x_j + B_{I_j} + \omega)^c} \frac{b_{\varepsilon \ell \ln \lambda_+ / \ln b}}{(b_{\varepsilon \ell} + \rho(x - y))^{1+\varepsilon \ln \lambda_+ / \ln b}} |b(y) - b_{x_j + B_{I_j}} b(y)| dy
\lesssim \frac{b_{\varepsilon \ell} \ln \lambda_+ / \ln b}{b_{\varepsilon \ell}} \int_{(x_j + B_{I_j} + \omega)^c} \frac{b_{\varepsilon \ell \ln \lambda_+ / \ln b}}{b_j^{(1+\varepsilon \ln \lambda_+ / \ln b)}} dy.
\]
By this, Theorem 2, and the Tonelli theorem, we find that, for any $m \in \mathbb{N}$, \((x_j + B_{l_j})_{j=1}^m \subset \mathcal{B}\), and \((\lambda_j)_{j=1}^m \subset (0, \infty)\) with \(\sum_{j=1}^m \lambda_j \neq 0\),

\[
\left\| \sum_{i=1}^m \left[ \frac{\lambda_i}{\| 1_{x_i + B_{l_i}} \|_{L^p(\mathbb{R}^n)}} \right]^{1/p} 1_{x_i + B_{l_i}} \right\|_{L^{p'}(\mathbb{R}^n)}^{-1} \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j}|^{1/2}}{\| 1_{x_j + B_{l_j}} \|_{L^p(\mathbb{R}^n)}}
\times \left\| \sum_{\ell \in \mathbb{Z}} \int_{\{x \in \mathbb{R}^n : (x, \ell) \in x_{j} + B_{l_j} \}} \left| \phi_{-\ell} * b_j^{(3)}(x) \right|^2 dx \right\|^{1/2}
\geq \|b\|_{L^{A,X}_{p(\cdot),1,x,1_p}(\mathbb{R}^n)} \sim \|b\|_{L^{A}_{p(\cdot),1,x,1_p}(\mathbb{R}^n)}.
\]

Combining this, (4.10), (4.11), and (4.13), we conclude that

\[
\left\| \sum_{i=1}^m \left[ \frac{\lambda_i}{\| 1_{x_i + B_{l_i}} \|_{L^p(\mathbb{R}^n)}} \right]^{1/p} 1_{x_i + B_{l_i}} \right\|_{L^{p'}(\mathbb{R}^n)}^{-1} \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j}|^{1/2}}{\| 1_{x_j + B_{l_j}} \|_{L^p(\mathbb{R}^n)}}
\times \left\| \sum_{\ell \in \mathbb{Z}} \int_{\{x \in \mathbb{R}^n : (x, \ell) \in x_{j} + B_{l_j} \}} \left| \phi_{-\ell} * b(x) \right|^2 dx \right\|^{1/2}
\leq \|b\|_{L^{A}_{p(\cdot),1,x,1_p}(\mathbb{R}^n)},
\]

which implies that, for any \((x, k) \in \mathbb{R}^n \times \mathbb{Z},\)

\[
d\mu(x, k) := \sum_{\ell \in \mathbb{Z}} \left| \phi_{-\ell} * b(x) \right|^2 dx \delta_k(k)
\]

is a \(p(\cdot)\)-Carleson measure on \(\mathbb{R}^n \times \mathbb{Z}\) and \(\|d\mu\|_{c_{p(\cdot),A}} \lesssim \|b\|_{L^{A}_{p(\cdot),1,x,1_p}(\mathbb{R}^n)}\). This finishes the proof of (i).

We now prove (ii). Indeed, let \(f \in H_{A, \text{fin}}^{p(\cdot), \infty, \text{loc}}(\mathbb{R}^n)\) with the norm greater than zero. Then \(f \in L^\infty(\mathbb{R}^n)\) with compact support. Therefore, by the fact that \(b \in L^{2}_{\text{loc}}(\mathbb{R}^n)\) and \([10, (2.10)]\), we know that

\[
\int_{\mathbb{R}^n} f(x)\overline{b(x)} \, dx \sim \int_{\mathbb{R}^n} \phi_{-\ell} * f(x) \overline{b(x)} \, dx,
\]

(4.14)
where $\varphi \in S(\mathbb{R}^n)$ satisfies that $\text{supp} \widehat{\varphi}$ is compact and away from the origin and, for any $\xi \in \mathbb{R}^n \setminus \{0\}$, $\sum_{k \in \mathbb{Z}} \widehat{\varphi}((A^*)^k \xi) \hat{\varphi}((A^*)^k \xi) = 1$ with $A^*$ as the adjoint matrix of $A$. Furthermore, from the Lusin area function characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$ (see [22, Theorem 4.4(i)]) and the fact that $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$, it follows that

$$
\| \varphi_{-\ell} \ast f \|_{T_A^{p(\cdot)}(\mathbb{R}^n \times \mathbb{Z})} \lesssim \| f \|_{H_A^{p(\cdot)}(\mathbb{R}^n)} < \infty.
$$

This, together with Lemma 5, further implies that there exist $\{\lambda_j\}_{j \in \mathbb{N}} \subset [0, \infty)$ and a sequence $\{A_j\}_{j \in \mathbb{N}}$ of anisotropic $(p(\cdot), \infty)$-atoms supported, respectively, in $\{B(j)\}_{j \in \mathbb{N}}$ with $\{B(j)\}_{j \in \mathbb{N}} \subset \mathcal{B}$ such that, for almost every $(x, \ell) \in \mathbb{R}^n \times \mathbb{Z},$

$$
\varphi_{-\ell} \ast f(x) = \sum_{j \in \mathbb{N}} \lambda_j A_j(x, \ell)
$$

and

$$
0 < \left\| \left\{ \frac{\lambda_j}{\|1_{B(j)}\|_{L^p(\mathbb{R}^n)}} \right\|^{1/p} \right\|_{L^q(\mathbb{R}^n)} \lesssim \| f \|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.
$$

Combining this, (4.14), the Hölder inequality, the Tonelli theorem, and the size condition of $A_j$, we conclude that, for any $f \in H_A^{p(\cdot), \infty, s}(\mathbb{R}^n),$

$$
\left| \int_{\mathbb{R}^n} f(x) b(x) \, dx \right| \\
\lesssim \sum_{j \in \mathbb{N}} \lambda_j \int_{\mathbb{R}^n} \left| A_j(x, \ell) \right| \left| \varphi_{-\ell} \ast b(x) \right| \, dx \\
\lesssim \sum_{j \in \mathbb{N}} \lambda_j \left[ \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^n} \left| A_j(x, \ell) \right|^2 \, dx \right]^{1/2} \\
\times \left[ \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^n} \left| \varphi_{-\ell} \ast b(x) \right|^2 \, dx \right]^{1/2} \\
\lesssim \sum_{j \in \mathbb{N}} \lambda_j \left\| A_j \right\|_{T_2^2(\mathbb{R}^n \times \mathbb{Z})} \left[ \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^n} \left| \varphi_{-\ell} \ast b(x) \right|^2 \, dx \right]^{1/2} \\
\lesssim \sum_{j \in \mathbb{N}} \frac{\lambda_j}{\|1_{B(j)}\|_{L^p(\mathbb{R}^n)}} \left[ \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^n} \left| \varphi_{-\ell} \ast b(x) \right|^2 \, dx \right]^{1/2} \\
\lesssim \| f \|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \|d\mu\|_{C_{p(\cdot), s}}.
$$
From this, Theorem 1, $p_+ \in (0, 2)$, Corollary 1, and Remark 8, we further conclude that

$$\| b \|_{L^A_{p(+),1,s} (\mathbb{R}^n)} \lesssim \| d \mu \|_{C_{p(+),A}},$$

which completes the proof of (ii) and hence of Theorem 3.

\[ \Box \]

**Remark 9** Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0, 1]$, and $s$ be as in (3.1). Then, using Proposition 2, we know that Theorem 3 also gives the Carleson measure characterization of the anisotropic variable Campanato space $L^A_{p(\cdot),1,s} (\mathbb{R}^n)$. We should point out that, even in this case, Theorem 3 is also new.

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**References**

1. Bownik, M.: Anisotropic Hardy Spaces and Wavelets. Vol. 164(781), American Mathematical Society, Providence, RI (2003)
2. Campanato, S.: Proprietà di una famiglia di spazi funzionali. Ann. Scuola Norm. Sup. Pisa 3(18), 137–160 (1964)
3. Cianchi, A., Pick, L.: Sobolev embeddings into spaces of Campanato, Morrey, and Hölder type. J. Math. Anal. Appl. 282(1), 128–150 (2003)
4. Coifman, R.R., Meyer, Y., Stein, E.M.: Some new function spaces and their applications to harmonic analysis. J. Funct. Anal. 62(2), 304–335 (1985)
5. Cruz-Uribe, D.V., Fiorenza, A.: Variable Lebesgue Space. Foundations and Harmonic Analysis. Appl. Number. Harmon Aanl, Birkhäuser/Springer, Heidelberg (2013)
6. Cruz-Uribe, D.V., Wang, L.-A.D.: Variable Hardy spaces. Indiana Univ. Math. J. 63(2), 447–493 (2014)
7. Cruz-Uribe, D.V., Moen, K., Nguyen, H.V.: A new approach to norm inequalities on weighted and variable Hardy spaces. Ann. Acad. Sci. Fenn. Math. 45(1), 175–198 (2020)
8. Diening, L., Harjulehto, P., Hästö, P., Růžička, M.: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics 2017, Springer, Heidelberg (2011)
9. Duoandikoetxea, J.: Fourier Analysis. Graduate Studies in Mathematics 29, American Mathematical Society, Providence, RI (2001)
10. Fan, X., Li, B.: Anisotropic tent spaces of Musielak-Orlicz type and their applications. Adv. Math. (China) 45(2), 233–251 (2016)
11. Fefferman, C., Stein, E.M.: $H^p$ spaces of several variables. Acta Math. 129(3–4), 137–193 (1972)
12. Hao, Z., Jiao, Y.: Fractional integral on martingale Hardy spaces with variable exponents. Fract. Calc. Appl. Anal. 18(5), 1128–1145 (2015). https://doi.org/10.1515/fca-2015-0065
13. Ho, K.-P.: Fractional integral operators with homogeneous kernels on Morrey spaces with variable exponents. J. Math. Soc. Japan 69(3), 1059–1077 (2017)
14. Ho, K.-P.: Integral operators on BMO and Campanato spaces. Indag. Math. (N.S.) 30(6), 1023–1035 (2019)
15. Huang, L., Liu, J., Yang, D., Yuan, W.: Dual spaces of anisotropic mixed-norm Hardy spaces. Proc. Amer. Math. Soc. 147(3), 1201–1215 (2019)
16. Huang, L., Liu, J., Yang, D., Yuan, W.: Real-variable characterizations of new anisotropic mixed-norm Hardy spaces. Comm. Pure Appl. Anal. 19(6), 3033–3082 (2020)
17. Huang, L., Yang, D., Yuan, W.: Anisotropic mixed-norm Campanato-type spaces with applications to duals of anisotropic mixed-norm Hardy spaces. Banach J. Math. Anal. 15(4), Art. 62, 36 pp (2021)
18. Izuki, M., Nakai, E., Sawano, Y.: Hardy spaces with variable exponent. In: Harmonic Analysis and Nonlinear Partial Differential Equations. RIMS Kôkyûroku Bessatsu, B42, Res. Inst. Math. Sci., Kyoto, 109–136 (2013)
19. Jiao, Y., Zhou, D., Weisz, F., Hao, Z.: Corrigendum: fractional integral on martingale Hardy spaces with variable exponents. Fract. Calc. Appl. Anal. 20(4), 1051–1052 (2017). https://doi.org/10.1515/fca-2017-0055
20. John, F., Nirenberg, L.: On functions of bounded mean oscillation. Commun. Pure Appl. Math. 14, 415–426 (1961)
21. Liu, J., Weisz, F., Yang, D., Yuan, W.: Anisotropic Hardy spaces and their applications. Taiwanese J. Math. 22(5), 1173–1216 (2018)
22. Liu, J., Haroske, D.D., Yang, D.: A survey on some anisotropic Hardy-type function spaces. Anal. Theory Appl. 36(4), 373–456 (2020)
23. Lu, S.-Z.: Four Lectures on Real $H^p$ Spaces. World Scientific Publishing Co., Inc, River Edge, NJ (1995)
24. Mizuta, Y., Nakai, E., Ohno, T., Shimomura, T.: Campanato–Morrey spaces for the double phase functionals with variable exponents. Nonlinear Anal. 197, Art. 111827, 19 pp (2020)
25. Nakai, E.: The Campanato, Morrey and Hölder spaces on spaces of homogeneous type. Studia Math. 176(1), 1–19 (2006)
26. Nakai, E.: Singular and fractional integral operators on Campanato spaces with variable growth conditions. Rev. Mat. Complut. 23(2), 355–381 (2010)
27. Nakai, E., Sawano, Y.: Hardy spaces with variable exponents and generalized Campanato spaces. J. Funct. Anal. 262(9), 3665–3748 (2012)
28. Nakai, E., Yoneda, T.: Applications of Campanato spaces with variable growth condition to the Navier-Stokes equation. Hokkaido Math. J. 48(1), 99–140 (2019)
29. Orlicz, W.: Über konjugierte Exponentenfolgen. Studia Math. 3, 200–211 (1931)
30. Rafeiro, H., Samko, S.: Variable exponent Campanato spaces. J. Math. Sci. (N.Y.) 172(1), 143–164 (2011)
31. Rafeiro, H., Samko, S.: Fractional integrals and derivatives: mapping properties. Fract. Calc. Appl. Anal. 19(3), 580–607 (2016). https://doi.org/10.1515/fca-2016-0032
32. Rafeiro, H., Samko, S.: On the Riesz potential operator of variable order from variable exponent Morrey space to variable exponent Campanato space. Math. Methods Appl. Sci. 43(16), 9337–9344 (2020)
33. Rafeiro, H., Samko, S.: A note on vanishing Morrey $\rightarrow$ VMO result for fractional integrals of variable order. Fract. Calc. Appl. Anal. 23(1), 298–302 (2020). https://doi.org/10.1515/fca-2020-0013
34. Rafeiro, H., Samko, S.: Fractional operators of variable order from variable exponent Morrey spaces to variable exponent Campanato spaces on quasi-metric measure spaces with growth condition. Ricerche Math. (2021). https://doi.org/10.1007/s11587-021-00639-4
35. Rafeiro, H., Samko, S.: Addendum to “On the Riesz potential operator of variable order from variable exponent Morrey space to variable exponent Campanato space.” Math. Methods Appl. Sci. 45(1), 557–560 (2022)
36. Samko, S.: A note on Riesz fractional integrals in the limiting case $\alpha(x)p(x) \equiv n$. Fract. Calc. Appl. Anal. 16(2), 370–377 (2013). https://doi.org/10.2478/s13540-013-0023-x
37. Sawano, Y.: Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operator. Integral Equ. Oper. Theory 77(1), 123–148 (2013)
38. Sawano, Y.: Theory of Besov Spaces. Spinger, Singapore (2018)
39. Taibleson, M.H., Weiss, G.: The molecular characterization of certain Hardy spaces. In: Coifman, R.R., Rochberg, R., Taibleson, M.H., Weiss, G. (eds.) Representation Theorems for Hardy Spaces, pp. 67–149. Astérisque, vol. 77. Soc. Math. France, Paris (1980)
40. Wang, W.: Dualities of variable anisotropic Hardy spaces and boundedness of singular integral operators. Bull. Korean Math. Soc. 58(2), 365–384 (2021)
41. Weisz, F.: Characterizations of variable martingale Hardy spaces via maximal functions. Fract. Calc. Appl. Anal. 24(2), 393–420 (2021). https://doi.org/10.1515/fca-2021-0018
42. Xu, J., Yang, X.: The $B_{p,q}^\alpha$ type Morrey–Triebel–Lizorkin spaces with variable smoothness and integrability. Nonlinear Anal. 202, Art. 112098, 19 pp (2021)
43. Zhang, Y., Huang, L., Yang, D., Yuan, W.: New ball Campanato-type function spaces and their applications. J. Geom. Anal. 32(2), Art. 99, 42 pp (2022)
44. Zhuo, C., Sawano, Y., Yang, D.: Hardy spaces with variable exponents on RD-spaces and applications. Dissertationes Math. (Rozprawy Mat.) 520, 1–74 (2016)
45. Zhuo, C., Yang, D., Liang, Y.: Intrinsic square function characterizations of Hardy spaces with variable exponents. Bull. Malays. Math. Sci. Soc. 39(4), 1541–1577 (2016)

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