Klein factors and Fermi-Bose Equivalence

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Generalizing the kink operator of the Heisenberg spin 1/2 model, we construct a set of Klein factors explicitly such that (1 + 1) dimensional fermion theories with arbitrary number of species are mapped onto the corresponding boson theories with the same number of species and vice versa. The actions for the resultant theories do not possess any nontrivial Klein factor. With this set of Klein factors, we are also able to map the simple boundary states such as the Neumann and the Dirichlet boundary states, of the fermion (boson) theory onto those of the boson (fermion) theory. Applications of the Fermi-Bose equivalence with the constructed Klein factors to well-known (1 + 1) dimensional theories have been discussed.

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I. INTRODUCTION

One of the most powerful tools to analyze (1 + 1) dimensional interacting theories is the Fermi-Bose equivalence [1, 2], which is also called as fermionization or bosonization in the literature [4–12]. In (1 + 1) dimensions a fermion theory is mapped onto a boson theory completely by the fermionization procedure and a boson theory is mapped onto its counterpart fermion by the bosonization. The most useful advantage of the fermionization or the bosonization among others is that the strong coupling regimes of the theory can be mapped onto the weak coupling regimes of its counterpart theory. Often the counterpart theory turns out to be exactly solvable. For this reason the Fermi-Bose equivalence has been employed to study various subjects in both condensed matter physics and string theory. The applications of the Fermi-Bose equivalence range from various important subjects in condensed matter physics, such as the Hubbard model [13–16], the anisotropic Heisenberg spin-1/2 model [17, 18], the Kondo problem [20, 21], the Tomonaga-Luttinger liquid [12, 24–28], junctions of quantum wires [29, 30] and to those in string theory, which include the string partition functions on Riemann surfaces [31–33], the string field theory [34], and the rolling tachyon [35–40].

For the theory with a fermion (or boson) of a single species the equivalence is well established by the Mattis-Mandelstam formula [1, 2]. By the formula fermion field operators are expressed in terms of the boson field operators in such a way that the excited states in the fermion theory can be represented as coherent states created by boson field operators. The Fermi-Bose equivalence has been the main tool for studying the non-perturbative properties of (1+1) dimensional theories since the seminal paper by Coleman [3], where the fermion theory of the massive Thirring model is shown to be equivalent to the boson theory of the sine-Gordon model. The novel feature of the Mattis-Mandelstam formula is that the fermion field operators, represented by the boson field operators only, satisfy the Fermi statistics correctly. However, if the theory contains fermion fields of more than two species, one immediately finds that the Mattis-Mandelstam does not work if extended naively. Since the boson field operators of different species are independent of each other, they commute, hence the fermion field operators constructed by the corresponding boson field operators. The fermi statistics between different species is not reproduced by the fermi field operators constructed in terms of the boson field operators by the Mattis-Mandelstam formula. In order to remedy this drawback we may modify the Mattis-Mandelstam formula by introducing additional operators in front of the fermion operators, which are termed as Klein factors. In order to ensure the anti-commutation relations between the fermion operators correctly, they have to satisfy certain conditions. In the literature one may find various useful discussions on the construction of the Klein factors and their applications. It may not be difficult to make the fermion operators of different species anticommute each other by introducing the Klein factors which have some simple structures. For an example, to represent them explicitly, one may choose the Dirac gamma matrices, which satisfy the Clifford algebra.

But if the theory contains some non-trivial interactions, the bosonized or fermionized action still contains Klein factors. Although these Klein factors may not appear explicitly in evaluation of some physical quantities, their appearances in the resultant action could be sources of controversy in some cases. We may encounter a similar difficulty when we try to apply the boundary state formulation [41, 42] to the (1+1) dimensional theories in condensed matter physics and string theory. The boundary state formulation is one of the most efficient framework to calculate the
correlation functions of physical operators in the field theory defined on the one dimensional space with a boundary. In condensed matter physics many of the aforementioned \((1 + 1)\) dimensional models are defined on the one dimensional space with a boundary. In string theory, the open strings attached on D-branes are described by \((1 + 1)\) dimensional worldsheets action defined on the one dimensional space with boundaries. The simple boundary states such as the Neumann state and the Dirichlet state are of course well defined both in the boson theory and in the fermion theory. However, the boundary states of the boson theory and those of its corresponding fermion theory are not mapped precisely onto each other by the Fermi-Bose equivalence. The Klein factors enter. Suppose that we define those simple boundary states in a boson theory and apply the fermionization with the conventional representations of the Klein factors to the boundary states. Since the boundary conditions of the boson theory are linear in terms of the boson fields, the Klein factors explicitly enter the boundary conditions.

In recent papers, in order to resolve those difficulties, associated with the conventional representations of the Klein factors, a new representation of the Klein factors has been proposed and applied to the critical boundary sine-Gordon model \([38]\), the rolling tachyon \([37, 39, 40]\), and the quantum Brownian motion on a triangular lattice \([43]\). Since the new representation of Klein factors only utilize the zero modes of the boson field operators and reduce to \(c\)-numbers upon the simple boundary conditions, we can map the boundary states of the boson theory onto those of the fermion theory precisely and vice versa. In this paper we extend this construction to the most general cases and present an explicit expression of Klein factors. Then we apply them to some well-known \((1 + 1)\) dimensional field theories, which include \(SU(2)\) Thirring model, the Gross-Neveu models with \(N = 2\) and the chiral Gross-Neveu model with \(N = 2\) and \(N = 3\). Some of the results are new.

## II. KLEIN FACTORS AND BOUNDARY STATES

We begin with the Mattis-Madelstam formula for the fermion field of a single species. Let us consider a left moving free boson field in the \((1 + 1)\) dimensional Euclidean space-time, \(\phi_L(\tau, \sigma) = \phi_L(\tau + i\sigma)\). With a periodic boundary condition \(\phi_L(\tau, \sigma + 2\pi) = \phi_L(\tau, \sigma)\), it may be written in terms of the oscillator operators as follows

\[
\phi_L(\tau + i\sigma) = \frac{1}{\sqrt{2}} x_L - \frac{i}{\sqrt{2}} p_L(\tau + i\sigma) + \sum_{n=1}^{\infty} \frac{i}{2n} (a_n e^{-n(\tau + i\sigma)} - a_n^\dagger e^{n(\tau + i\sigma)}),
\]

where the fundamental boson commutation relations are

\[
[x_L, p_L] = i, \quad [a_m, a_n^\dagger] = \delta_{mn}.
\]

The Mattis-Madelstam formula states that the free fermion field operator \(\psi_L\) and \(\psi_L^\dagger\), satisfying anticommutation relations

\[
\{\psi_L(\sigma), \psi_L(\sigma')\} = 0, \quad \{\psi_L^\dagger(\sigma), \psi_L(\sigma')\} = 2\pi \delta(\sigma - \sigma'),
\]

in the \((1 + 1)\) dimensions can be represented in terms of the boson field operators as follows

\[
\psi_L = e^{-i\sqrt{2} \phi_L^\dagger}, \quad \psi_L^\dagger = e^{i\sqrt{2} \phi_L}.
\]

Now let us introduce a right moving free boson field, \(\phi_R(\tau, \sigma) = \phi_R(\tau - i\sigma)\)

\[
\phi_R(\tau - i\sigma) = \frac{1}{\sqrt{2}} x_R - \frac{i}{\sqrt{2}} p_R(\tau - i\sigma) + \sum_{n=1}^{\infty} \frac{i}{2n} (\tilde{a}_n e^{-n(\tau - i\sigma)} - \tilde{a}_n^\dagger e^{n(\tau - i\sigma)}),
\]

with the fundamental boson commutation relations

\[
[x_R, p_R] = i, \quad [\tilde{a}_m, \tilde{a}_n^\dagger] = \delta_{mn}.
\]

We may define the right moving fermion field in terms of the right moving boson as in the case of the left moving fermion field

\[
\psi_R = e^{-i\sqrt{2} \phi_R^\dagger}, \quad \psi_R^\dagger = e^{i\sqrt{2} \phi_R}.
\]

They satisfy the anticommutation relations of the fermion fields

\[
\{\psi_R(\sigma), \psi_R(\sigma')\} = 0, \quad \{\psi_R^\dagger(\sigma), \psi_R(\sigma')\} = 2\pi \delta(\sigma - \sigma').
\]
At this point we already encounter the difficulty associated with the bosonization: Since the left moving boson field operator and the right moving boson field operators are completely independent of each other, they commute
\[ [\phi_L(\sigma), \phi_R(\sigma')] = 0. \] (9)

It follows from this that the left moving fermion field operator and the right moving fermion field operator defined by Eqs. (11) satisfy the commutation relations instead of the anticommutation relations. We need to introduce additional factors \( \eta_L, \eta_R \) in front of the fermion operators
\[ \psi_L = \eta_L : e^{-i\sqrt{2} \phi_L} :, \quad \psi_R = \eta_R : e^{i\sqrt{2} \phi_R} :, \] (10)
so that the anti-commutation relations between fermion operators are reproduced
\[ \{\psi_L(\sigma), \psi_R(\sigma')\} = 0, \quad \{\psi_L^\dagger(\sigma), \psi_R(\sigma')\} = 0, \quad \{\psi_L(\sigma), \psi_L^\dagger(\sigma')\} = 0, \quad \{\psi_L^\dagger(\sigma), \psi_R^\dagger(\sigma')\} = 0. \] (11)

Note that if the Klein factors \( \eta_L, \eta_R \) commute with the boson operators \( \phi_L, \phi_R \) and satisfy the Clifford algebra
\[ \{\eta_i, \eta_j\} = 2\delta_{ij}, \quad i, j = L, R, \] (12)
the constructed fermion operators Eqs. (10) satisfy the correct anticommutation relations among the fermion operators Eqs. (3, 8, 11). One may represent them by the Pauli matrices. To be explicit, we choose
\[ \eta_L = \sigma_1, \quad \eta_R = \sigma_2. \] (13)

For the theory with fermion fields of more than two species, in general one may choose the Dirac matrices to represent the Klein factors. This is one of the conventional methods to construct the Klein factors. With this representation of the Klein factors, the anti-commutation relations between the fermion operators are easily reproduced, but the Klein factors make their explicit appearances in the bosonized action. For an example, consider the Dirac mass term in the fermion theory \( \bar{\psi}\psi \), which may be written in terms of the boson field operators as
\[ \bar{\psi}\psi = \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L = \eta_L^\dagger \eta_R : e^{\sqrt{2} \phi} : + \eta_R^\dagger \eta_L : e^{-\sqrt{2} \phi} := i\sigma_3 \left( e^{\sqrt{2} \phi} : - e^{-\sqrt{2} i \phi} : \right). \] (14)

It is not clear how to deal with the Klein factors in the bosonized action. Often the Klein factors in the actions are ignored. Or one of the eigenstates of the Klein factors is chosen, while others are projected out. It is certainly ambiguous. For the example at hand, among
\[ \pm i \left( e^{\sqrt{2} i \phi} : - e^{-\sqrt{2} i \phi} : \right) \] (15)
which boson operators we would choose to represent the Dirac mass term?

A similar difficulty is encountered when we construct the boundary states. The Neumann boundary condition and the Dirichlet boundary condition are simply given in terms of boson field operators as
\[ \phi_L|N\rangle = \phi_R|N\rangle, \quad \phi_L|D\rangle = -\phi_R|D\rangle. \] (16a, 16b)

But if we apply the fermion representations Eq. (10) to express the boundary condition in terms of fermion operators, we realize that it cannot be written in terms of the fermion fields only
\[ \psi_L|N\rangle = \eta_L \eta_R \psi_L^\dagger|N\rangle = i\sigma_3 \psi_R^\dagger|N\rangle, \quad \psi_L^\dagger|N\rangle = \eta_L^\dagger \eta_R \psi_R|N\rangle = i\sigma_3 \psi_R|N\rangle, \] (17a)
\[ \psi_L|D\rangle = \eta_L \eta_R \psi_R|D\rangle = i\sigma_3 \psi_R|D\rangle, \quad \psi_L^\dagger|D\rangle = \eta_L^\dagger \eta_R \psi_R^\dagger|D\rangle = i\sigma_3 \psi_R^\dagger|D\rangle. \] (17b)

The Klein factors enter the boundary conditions explicitly. The free boson field theory defined on a space-time with a boundary is not mapped precisely onto a free fermion field theory by the fermi-bose equivalence with the Klein factors represented by the Pauli matrices.

Other conventional representations of Klein factors also lead us to the same conclusion. An alternative representation of the Klein factors is
\[ \eta_i = \exp \left( i\pi \sum_{i<j} p_{ij} \right), \quad i, j = 1, 2, \ldots, N. \] (18)
For the single boson theory, we may write

$$\psi_L = e^{-i\sqrt{2}\phi_L} : \psi_R = e^{i\pi p_L} e^{i\sqrt{2}\phi_R} : .$$

(19)

The Klein factors in this representation do not commute with the Mattis-Mandelstam operators, but commute with each other, in contrast to the Dirac matrix representation of the Klein factors. In fact, the anti-commutation relations between the fermion field operators are ensured by the non-commutative algebra between the Klein factors and the Mattis-Mandelstam operators,

$$\{ \psi_L(\sigma), \psi_R(\sigma') \} = \{ e^{-i\sqrt{2}\phi_L}, e^{i\pi p_L} \} : e^{i\sqrt{2}\phi_R} := 0.$$  

(20)

Here we make use of the Baker-Campbell-Hausdor formula

$$e^A e^B = e^B e^A [A, B], \quad \text{if } [A, B] = c\text{-number}. \quad (21)$$

Although the anti-commutation relations between the fermion field operators are satisfied, the difficulties associated with the Dirac matrix representation of the Klein factors still remain unresolved. If we bosonize the Dirac fermion mass term by using this representation, we have

$$\bar{\psi} \psi = \psi_R \psi_L + \psi_L \psi_R = -ie^{i\pi p_L} : e^{i\sqrt{2}\phi} : + e^{-i\pi p_L} : e^{-i\sqrt{2}\phi} : .$$

(22)

The Klein factors $e^{\pm i\pi p_L}$ enter explicitly. The same problem occurs when we map the simple boundary states. The Neumann boundary condition in the boson theory Eq. (16a) is now expressed in terms of the fermion field operators as

$$\psi_L|N\rangle = e^{i\pi p_L} \psi_R^\dagger|N\rangle, \quad \psi_R^\dagger|N\rangle = e^{-i\pi p_L} \psi_L|N\rangle.$$  

(23)

Note that the Klein factors enter the boundary condition as before. Thus, this alternative representation of the Klein factors also suffers the same problems.

In recent works, a new representation of the Klein factors has been proposed. The new representation only utilizes the zero modes of the momentum operators $p_L$ and $p_R$. In the case of the theory with a single boson the most general forms of the Klein factors may be written as

$$\psi_L = e^{-\frac{\pi}{2}(\alpha L p_L + \beta R p_R)} e^{-\sqrt{2}X_L}, \quad \psi_R = e^{\frac{\pi}{2}(\beta L p_R + \alpha R p_R)} e^{\sqrt{2}X_R}.$$  

(24)

where $\alpha^L, \alpha^R, \beta^L, \beta^R$ are constant parameters, to be fixed by suitable conditions. We may derive the conditions for the Klein factors to satisfy, requiring followings:

- The fundamental anti-commutation relations between fermion operators are properly reproduced.
- The boundary conditions for the simple boundary states of the boson theory are mapped onto those of the corresponding fermion theory by the Fermi-Bose equivalence.
- Relevant interaction terms of the fermion theory, including the Dirac mass term, should be mapped onto the corresponding terms in the boson theory, which do not contain any Klein factor.

It is not difficult to find a solution satisfying all those conditions for the theory with a single boson:

$$\psi_L = e^{-\frac{\pi}{4}(p_L + p_R)} e^{-\sqrt{2}\phi_L}, \quad \psi_R = e^{-\frac{\pi}{4}(p_L + p_R)} e^{\sqrt{2}\phi_R}.$$  

(25)

One can easily check that the anti-commutation relations Eqs. (3, 8, 11) between the fermion field operators are satisfied. The Dirac mass operator in the fermion theory is now unambiguously mapped onto the periodic potential of the boson theory

$$\bar{\psi} \psi = e^{i\sqrt{2}\phi} : + e^{-i\sqrt{2}\phi} : = 2 : \cos(\sqrt{2}\phi) : .$$  

(26)

The Neumann boundary condition of the boson theory Eq. (16a) is mapped onto the following linear condition in terms of the fermion field operators

$$\psi_L|N\rangle = i\psi_R^\dagger|N\rangle, \quad \psi_R^\dagger|N\rangle = i\psi_L|N\rangle.$$  

(27)

Thus, at least for the theory of a single boson field, we have a proper representation of the Klein factors, which is free from the associated problems. The Klein factors for the case of the model with two boson fields have been constructed.
and applied to the rolling tachyon in string theory. The new representation of the Klein factors also has been applied to the quantum Brownian motion on a triangular lattice, which requires three boson fields.

We may recall the kink or soliton operator in the anisotropic Heisenberg (XXZ) model, which changes the statistics of the spin-1/2 operator. By the Jordan-Wigner transformation the spin-1/2 operators can be mapped to spinless fermions. Bu the fermion operators on different sites anticommute while the spin-1/2 operators on different sites commute. In order to give the correct statistics to the fermion operators, one may introduce the kink operator, which do not commute with the boson fields. The new representation of the Klein factors, which we are about to construct, may be considered as generalized kink operators, which have a more complex structure.

III. CONDITIONS FOR KLEIN FACTORS

We may write the most general form of Klein factors for the fermion fields which correspond to the N boson fields as

\[ \psi_L^a = e^{-\frac{i}{\hbar} \int \left( \sum_{n=1}^{N} \left( \alpha_{ab} p_L^a + \beta_{ab} p_R^b \right) \right) \phi^a_{L} d\sigma}, \quad \psi_R^a = e^{-\frac{i}{\hbar} \int \left( \sum_{n=1}^{N} \left( \alpha_{ab} p_L^a + \beta_{ab} p_R^b \right) \right) \phi^a_{R} d\sigma}, \]

(29a)

\[ \psi_L^{\dagger a} = e^{\frac{i}{\hbar} \int \left( \sum_{n=1}^{N} \left( \alpha_{ab} p_L^a + \beta_{ab} p_R^b \right) \right) \phi^{\dagger a}_{L} d\sigma}, \quad \psi_R^{\dagger a} = e^{\frac{i}{\hbar} \int \left( \sum_{n=1}^{N} \left( \alpha_{ab} p_L^a + \beta_{ab} p_R^b \right) \right) \phi^{\dagger a}_{R} d\sigma}. \]

(29b)

where \( a, b = 1, 2, \ldots, N \) and \( \alpha_{ab}, \beta_{ab} \) are constant parameters to be fixed by the conditions, imposed on the Klein factors. The left moving boson field operators and the right moving ones may be expanded in terms of the oscillator modes as follows:

\[ \phi_L^a(\tau + i\sigma) = \frac{1}{\sqrt{2}} \psi_L^a - \frac{i}{\sqrt{2}} \psi_L^{\dagger a} (\tau + i\sigma) + \sum_{n=1}^{N} \frac{i}{\sqrt{2N}} \left( a_n e^{-n(\tau + i\sigma)} - a_n^{\dagger} e^{n(\tau + i\sigma)} \right) \]

(30a)

\[ \phi_R^a(\tau - i\sigma) = \frac{1}{\sqrt{2}} \psi_R^a - \frac{i}{\sqrt{2}} \psi_R^{\dagger a} (\tau - i\sigma) + \sum_{n=1}^{N} \frac{i}{\sqrt{2N}} \left( a_n e^{-n(\tau - i\sigma)} - a_n^{\dagger} e^{n(\tau - i\sigma)} \right) \]

(30b)

with the non-vanishing fundamental commutators

\[ [x_L^a, x_L^b] = i\delta^{ab}, \quad [x_R^a, x_R^b] = i\delta^{ab}, \]

(31a)

\[ [a_m^a, a_n^b] = \delta^{ab}\delta_{mn}, \quad [\tilde{a}_m^a, \tilde{a}_n^a] = \delta^{ab}\delta_{mn}. \]

(31b)

A. Anti-Commutation Relations between the Fermion Operators

The first conditions to be imposed on the Klein factors are obtained by requiring the anti-commutation relations between fermion operators

\[ \{ \psi_L^a(\sigma), \psi_L^b(\sigma') \} = 0, \quad \{ \psi_R^a(\sigma), \psi_R^b(\sigma') \} = 0, \]

(32a)

\[ \{ \psi_R^a(\sigma), \psi_L^b(\sigma') \} = 0, \quad \{ \psi_L^a(\sigma), \psi_R^b(\sigma') \} = 0, \]

(32b)

\[ \{ \psi_L^{\dagger a}(\sigma), \psi_L^b(\sigma') \} = \delta^{ab}2\pi \delta(\sigma - \sigma'), \]

(32c)

\[ \{ \psi_R^{\dagger a}(\sigma), \psi_R^b(\sigma') \} = \delta^{ab}2\pi \delta(\sigma - \sigma'). \]

(32d)

By some algebra we find that the anti-commutation relations between the fermion field operators \( \psi_{aL}, \psi_{aL}^\dagger, \psi_{aR}, \psi_{aR}^\dagger \), are ensured if the following conditions are satisfied

\[ e^{\frac{2\pi i}{\hbar}(\alpha_{ab} - \alpha_{ab}^*)} = -1 \quad \text{for} \ a \neq b, \]

(33a)

\[ e^{\frac{2\pi i}{\hbar}(\beta_{ab} - \beta_{ab}^*)} = -1 \quad \text{for} \ a \neq b, \]

(33b)

\[ e^{\frac{2\pi i}{\hbar}(\alpha_{ab} - \alpha_{ab})} = -1. \]

(33c)
We note that the anti-commutation relations between the fermion operators alone cannot fix the Klein factors. We only have \((2N^2 - N)\) conditions to fix \(4N^2\) parameters. Additional conditions would be obtained by requiring that the simple boundary states are linearly represented in terms of the fermion fields and the interaction terms are also uniquely represented in terms of the fermion fields only in the fermion theory.

B. The Simple Boundary States and Klein Factors

Often the condensed matter systems and the string theory are defined on one dimensional space with a boundary. The boundary conditions for simple boundary states such as \(|N\rangle = |N, \ldots, N\rangle\) and \(|D\rangle = |D, \ldots, D\rangle\) should be realized linearly in terms of the fermion operators without the Klein factors. In condensed matter system interactions between physical fields defined on the bulk space and the impurities on the boundary are the main subjects of study. In string theory the open strings attached on D-branes are described by the \((1 + 1)\) dimensional field theory defined on one dimensional space with a boundary. Thus, when we apply the bosonization or the fermionization to these theories we encounter the difficulty, similar to those discussed in the previous section if the Klein factors enter the boundary condition explicitly. This requirement will impose some conditions for the Klein factors and yield the relations between \(\alpha^{L}_{ab}, \alpha^{R}_{ab}, \beta^{L}_{ab}, \beta^{R}_{ab}\). We begin with the boundary state \(|N\rangle\).

1. The Neumann Boundary State

If the theory does not have any non-trivial interaction both in the bulk and the boundary, the fields should be subject to the Neumann condition. The boundary condition for the state \(|N\rangle\) is given linearly in terms of the bosonic operator as

\[
\phi^{a}_{L}|N\rangle = \phi^{a}_{R}|N\rangle, \quad a = 1, 2, \ldots, N,
\]

which can be read in terms of normal modes as

\[
x^{a}_{L}|N\rangle = x^{a}_{R}|N\rangle, \quad p^{a}_{L}|N\rangle = -p^{a}_{R}|N\rangle, \quad a^{a}_{n}|N\rangle = -\tilde{a}^{a\dagger}_{n}|N\rangle.
\]

Applying the fermion field operator \(\psi^{a}_{L}\) on the Neumann boundary state

\[
\psi^{a}_{L}|N\rangle = e^{i\beta_{ab}^{L}a^{\dagger}_{aR}}\psi^{a}_{aR}|N\rangle + e^{i\sum_{b}(\alpha^{R}_{ab} - \beta^{R}_{ab} - \alpha^{L}_{ab} + \beta^{L}_{ab})}\psi^{a}_{R}|N\rangle,
\]

we find that the Neumann boundary condition can be expressed linearly in terms of the fermion operators if the following conditions are satisfied

\[
\alpha^{L}_{ab} - \alpha^{R}_{ab} - \beta^{L}_{ab} + \beta^{R}_{ab} = 0. \quad (36)
\]

Under the condition Eq. (36) we write the Neumann boundary condition in the fermion theory as

\[
\psi^{a}_{L}\langle N\rangle = e^{i\beta_{ab}^{L}a^{\dagger}_{aR}}\psi^{a}_{aR}\langle N\rangle, \quad \psi^{\dagger a}_{L}\langle N\rangle = e^{-i(\alpha^{R}_{ab} - \beta^{R}_{ab} + \beta^{L}_{ab})}\psi^{\dagger a}_{R}\langle N\rangle.
\]

These fermion boundary conditions should be also consistent with the fundamental anti-commutation relations between the fermion field operators Eqs. (39a), (39b), (39c), (39d),

\[
\{\psi^{a}_{L}(\sigma), \psi^{b}_{L}(\sigma')\}|N\rangle = -e^{i\beta_{ab}^{L}}e^{-i(\alpha^{R}_{ab} - \beta^{R}_{ab} + \beta^{L}_{ab})}\{\psi^{b}_{R}(\sigma'), \psi^{a}_{R}(\sigma)\}|N\rangle = \delta^{ab}2\pi\delta(\sigma - \sigma')|N\rangle.
\]

If \(a \neq b\), LHS=RHS=0. So it does not requires any additional condition. But if \(a = b\), the phase factor on RHS must be 1. It yields the following additional conditions

\[
e^{-i(\alpha^{R}_{aa} - 2\beta^{R}_{aa} + \beta^{L}_{aa})} = -1, \quad a = 1, 2, \ldots, N. \quad (39)
\]
2. The Dirichlet Boundary State

In string theory the end points of the string coordinate fields in the directions, orthogonal to the worldsheet of the D-branes satisfy the Dirichlet condition. In the condensed matter systems, if the interactions between the impurities on the boundary and the physical fields on the bulk space become strong, the fixed point of the RG (renormalization group) flow is the Dirichlet state. We require that the Dirichlet boundary condition in the boson theory must be also mapped onto that in the fermion theory by the fermionization as in the case of the Neumann boundary condition. The boundary condition for $|D\rangle$ is given in terms of the bosonic operator as

$$\phi^a_D|D\rangle = -\phi^a_R|D\rangle, \quad a = 1, 2, \ldots, N. \quad (40)$$

If it is written in terms of normal modes,

$$x^a_D|D\rangle = -a^a_R|D\rangle, \quad p^a_L|D\rangle = p^a_R|D\rangle, \quad a^a_n|D\rangle = \bar{a}^a_n|D\rangle. \quad (41)$$

Since it may be written in terms of fermion operator as

$$\psi^a_D|D\rangle = e^{-\frac{i}{\hbar}((\alpha^a_u + \beta^a_u)\psi^a_R e^{-\frac{i}{\hbar} \sum_k (\alpha^a_k + \beta^a_k + \beta^a_k p^k_L)|D\rangle}}, \quad (41)$$

we should impose the following condition

$$\alpha^a_{ab} + \alpha^a_R + \beta^a_{ab} + \beta^a_R = 0. \quad (42)$$

Then the Dirichlet boundary condition is linearly represented by the fermion field operators without the Klein factors

$$\psi^a_D|D\rangle = e^{-\frac{i}{\hbar}((\beta^a_u + \beta^a_u)\psi^a_R|D\rangle}, \quad (43)$$

Under the condition Eq. (42), we also find

$$\psi^a_D|D\rangle = e^{-\frac{i}{\hbar}((\alpha^a_u + \beta^a_u)\psi^a_R|D\rangle}. \quad (44)$$

These two boundary conditions Eq. (43) and Eq. (44) should be compatible with the fundamental fermion anticommutation relations. It follows from this requirement that

$$e^{-\frac{i}{\hbar}((\alpha^a_u + \beta^a_u + \beta^a_R) = -1, \quad a = 1, 2, \ldots, N. \quad (45)$$

Other boundary states and Klein factors: We may repeat the same procedure for other mixed boundary states such as $|D, N, \ldots, N\rangle$ or $|N, \ldots, D\rangle$. However, it does not produce any additional condition for the Klein factors, since they are all related by $SU(N)$ global transformations.

C. The Periodic Potential and Fermion Mass Term

The boson theory in $(1 + 1)$ dimensions may contain the periodic potential terms,

$$U_a + U_{a}^{\dagger} = e^{i\sqrt{2}\phi^a} + e^{-i\sqrt{2}\phi^a}, \quad a = 1, 2, \ldots, N, \quad (46)$$

which arise in the bulk or on the boundary. In a condensed matter theory these terms correspond to the bosonized form of the Umklapp process [7]. When fermionizing the periodic potential terms, we may rewrite them as bilinear terms in fermion operators

$$e^{i\sqrt{2}\phi^a} = :e^{i\sqrt{2}(\phi^a_+ + \phi^a_-)}: = \psi^a_L e^{-\frac{i}{\hbar} \sum_k (\alpha^a_k + \beta^a_k p^k_L)} e^{-\frac{i}{\hbar} \sum_k (\alpha^a_k + \beta^a_k p^k_R)} \bar{\psi}^a_R. \quad (47)$$

We do not want the Klein factors to enter explicitly in the fermion bilinear operators. It follows from this constraint that

$$\alpha^a_{ab} + \alpha^a_R = 0, \quad \beta^a_{ab} + \beta^a_R = 0, \quad a, b = 1, 2, \ldots, N. \quad (48)$$

Under these conditions they could be interpreted as the Dirac mass terms

$$U_a + U_{a}^{\dagger} = \psi^a_L \psi^a_R + \bar{\psi}^a_R \psi^a_L = \bar{\psi}^a \psi^a, \quad (49)$$

of the relativistic Dirac fermion theory.
D. Interactions and The Klein Factors

When we apply the fermionization or the bosonization to $(1+1)$ dimensional theories, we often find that non-trivial Klein factors also appear in the interaction terms of the counterpart theories. These Klein factors are sources of controversy. Since there are still some rooms to impose additional conditions for the Klein factors, we may impose further conditions consistently to get rid of those Klein factors. Let us consider following interaction terms between boson fields

\[ V_a = V_{a,a+1} + V_{a,a+1}^\dagger = e^{-\frac{i\pi}{2}(\phi^a - \phi^{a+1})} + e^{-\frac{i\pi}{2}(\phi^a - \phi^{a+1})}, \]

where $a = 1, 2, \ldots, N$, $\phi^{N+1} = \phi^1$. The interaction terms of this type arise in the theory of junctions of quantum wires as boundary interactions. In $(1+1)$ dimensional quantum field theory interaction terms of this type also appear in the bosonized bulk action of the chiral Gross-Neveu model. If these terms arise in the bulk, we may scale the boson fields as $\phi^a \rightarrow 2\phi^a$

\[ V_a = V_{a,a+1} + V_{a,a+1}^\dagger = e^{i\pi}(\phi^a - \phi^{a+1}) + e^{-i\pi}(\phi^a - \phi^{a+1}) \]

Then by using the condition Eq.$(48)$, we may rewrite them as four fermi terms in the fermion theory without nontrivial Klein factors

\[ V_a = U^{a+1}U^a + U^{a+1}U^a = \left( \psi_R^{a+1}\psi_L^a \right) \left( \psi_L^a\psi_R^a \right) + \left( \psi_R^{a+1}\psi_L^a \right) \left( \psi_L^a\psi_R^a \right) \]

We do not need to impose additional conditions to fermionize the bulk interaction terms of this type.

But if the interaction terms of this type arise on the boundary, we should treat them with some care. On the boundary they cannot be written as four fermi terms. If we turn off the interaction, the boundary state should reduce to the Neumann state. And the boundary state $|B\rangle$ in the presence of the interaction is obtained by applying the boundary interaction terms on the simple boundary state $|N\rangle$

\[ |B\rangle = V_a|N\rangle = \left( U^{a+1}U^a + U^{a+1}U^a \right)|N\rangle = \left( \psi_R^{a+1}\psi_L^a \right) \left( \psi_L^a\psi_R^a \right) + \left( \psi_R^{a+1}\psi_L^a \right) \left( \psi_L^a\psi_R^a \right)|N\rangle. \]

If we make use of the Neumann condition Eq.$(37)$, we find that $U^a|N\rangle$ is vanishing

\[ U^a|N\rangle = \left( \psi_L^a\psi_R^a \right)|N\rangle = e^{i\pi(\alpha^L_{aa} - \beta^L_{ab} + \beta^R_{ab})}\left( \psi_L^{a+1}\psi_R^a \right)|N\rangle = 0, \]

thanks to the fermi statistics. Thus, if we rewrite the interaction term $V_a$ as a four fermi term on the boundary, it becomes a null operator.

On the boundary the interaction term $V_a$ may be represented as a bilinear operator of fermi fields instead. The boundary interaction term $V_{a,a+1}$ acting on the Neumann boundary state may be written in terms of the left moving fermion field operators as

\[ V_{a,a+1}|N\rangle = e^{-\frac{i\pi}{2}(\alpha^L_{aa} - \alpha^L_{a+1,a})}\psi_R^a|N\rangle e^{-\frac{i\pi}{2}\sum_b(\alpha^L_{a+1,b} - \alpha^L_{a,b} - \beta^L_{a+1,b} + \beta^R_{a,b})}\psi_L^b|N\rangle. \]

(55)

It follows that in order to remove non-trivial Klein factors in the fermion form of the interaction term we should impose the following condition

\[ \alpha^L_{a+1,b} - \alpha^L_{ab} - \beta^L_{a+1,b} + \beta^R_{ab} = 0, \quad a, b = 1, 2, \ldots, N. \]

(56)

Once this condition is imposed, the boundary interaction term $V_a$ may be written as

\[ V_{a,a+1}|N\rangle = e^{-\frac{i\pi}{2}(\alpha^L_{aa} - \alpha^L_{a+1,a})}\psi_R^a|N\rangle. \]

(57)

We note that the boundary term $V_{a,a+1}$ can be also equally written in terms of the right moving chiral fermion field operators as

\[ V_{a,a+1}|N\rangle = e^{-\frac{i\pi}{2}(\beta^R_{a+1,a+1} - \beta^R_{aa+1})}\psi_R^a|N\rangle e^{\frac{i\pi}{2}\sum_b(\alpha^R_{a,b} - \beta^R_{a,b})}\psi_L^b|N\rangle. \]

(58)
Thus, if the following condition is satisfied
\[ \alpha_{ab}^{R} - \alpha_{a+1,b}^{R} - \beta_{ab}^{R} + \beta_{a+1,b}^{R} = 0, \] (58)
the boundary term can be written as a bilinear of the right moving fermion field operators
\[ V_{a,a+1} |N\rangle = e^{\frac{\pi}{2} i (\alpha_{a+1,a+1}^{L} - \beta_{a+1,a+1}^{R})} \psi_{a+1,R}^\dagger \psi_{a+1,L} |N\rangle. \] (59)

As we repeat the same procedure for the boundary interaction term \( V_{a,a+1}^\dagger \), we obtain the fermion form of the boundary interaction term as
\[ V_{a,a+1}^\dagger |N\rangle = e^{\frac{\pi}{2} i (\alpha_{a+1,a+1}^{L} - \beta_{a+1,a+1}^{R})} \psi_{a+1,R}^\dagger \psi_{a+1,L} |N\rangle. \] (60)

A non-trivial Klein factor does not arise if the condition Eq. (56) is satisfied. Rewriting the boundary interaction term \( V_{a}^\dagger \) in terms of the right moving fermion operators, we have
\[ V_{a}^\dagger |N\rangle = e^{\frac{\pi}{2} i (\beta_{a+1,a+1}^{R} - \beta_{a,a}^{R})} \psi_{a+1,R}^\dagger \psi_{a,R} |N\rangle \] (61)
under the condition Eq. (58). Therefore, we may write the boundary interaction term \( V_{a} = V_{a,a+1} + V_{a,a+1}^\dagger \) in terms of the left moving fermion field operators as
\[ V_{a} |N\rangle = \left( e^{\frac{\pi}{2} i (\alpha_{a,a}^{L} - \alpha_{a+1,a+1}^{L})} \psi_{a+1,L}^\dagger \psi_{aL} + e^{\frac{\pi}{2} i (\alpha_{a+1,a+1}^{L} - \alpha_{a,a}^{L})} \psi_{a+1,L}^\dagger \psi_{aL} + e^{\frac{\pi}{2} i (\beta_{a+1,a+1}^{R} - \beta_{a,a}^{R})} \psi_{a+1,R}^\dagger \psi_{a,R} \right) |N\rangle, \] (62)
or in terms of the right moving fermion field operators as
\[ V_{a} |N\rangle = \left( e^{\frac{\pi}{2} i (\alpha_{a,a}^{L} - \alpha_{a+1,a+1}^{L})} \psi_{a+1,R}^\dagger \psi_{a+1,L}^\dagger + e^{\frac{\pi}{2} i (\alpha_{a+1,a+1}^{L} - \alpha_{a,a}^{L})} \psi_{a+1,R}^\dagger \psi_{a+1,L} + e^{\frac{\pi}{2} i (\beta_{a+1,a+1}^{R} - \beta_{a,a}^{R})} \psi_{a+1,R}^\dagger \psi_{a,R} \right) |N\rangle. \] (63)

Note that, however, the fermion form of \( V_{a} \) is not manifestly Hermitian unless appropriate conditions for the Klein factors are imposed. It can be achieved by introducing the following conditions
\[ e^{\frac{\pi}{2} i (\alpha_{a+1,a+1}^{L} - \beta_{a+1,a+1}^{R})} = 1, \] (64a)
\[ e^{\frac{\pi}{2} i (\alpha_{a+1,a+1}^{L} - \beta_{a+1,a+1}^{R})} = 1. \] (64b)

**IV. SOLUTIONS**

In the previous section we exhaust the necessary conditions for the Klein factors to satisfy. If there exists a set of Klein factors, which satisfy all those conditions consistently, we can map the boson theories onto the corresponding fermion theories which do not contain any non-trivial Klein factors in their actions and vice versa. In this section we show that those conditions are consistent and there are many sets of solutions parameterized by some integers. At the end we present an explicit solution, which may be the simplest one.

Making use of Eqs. (60, 62), we may replace all \( \beta_{ab}^{L} \) and \( \beta_{ab}^{R} \) by \( \alpha_{ab}^{L} \) and \( \alpha_{ab}^{R} \) in the conditions
\[ \beta_{ab}^{L} = -\alpha_{ab}^{R}, \quad \beta_{ab}^{R} = -\alpha_{ab}^{L}, \quad a, b = 1, 2, \ldots, N. \] (65)

Then the equations Eq. (59) and Eq. (58) reduce to the following conditions
\[ \alpha_{a+1,b}^{L} - \alpha_{a,b}^{L} + \alpha_{a+1,b}^{R} - \alpha_{a,b}^{R} = 0, \quad a, b = 1, 2, \ldots, N. \] (66)

These conditions become redundant if the conditions Eq. (58) are chosen. Under the conditions Eq. (65), Eqs. (39, 45) may be written as \( e^{-\pi i \alpha_{aa}^{R}} = -1, \ i.e., \)
\[ \alpha_{aa}^{R} = 2n_{aa}^{R} + 1, \quad n_{aa}^{R} \in \mathbb{Z}, \quad a = 1, 2, \ldots, N. \] (67)

The conditions Eqs. (38, 39, 45) which ensure the anticommutation relations between fermion operators, \( \psi_{L}^{a}, \psi_{R}^{a} \) reduce to
\[ e^{\frac{\pi}{2} i (\alpha_{ab}^{L} - \alpha_{ab}^{R})} = -1, \quad e^{\frac{\pi}{2} i (\alpha_{ab}^{R} + \alpha_{ab}^{L})} = -1. \] (68)
These conditions are satisfied by rewriting the antisymmetric part of \((\alpha^L)\) and the symmetric part of \((\alpha^R)\) by integers \(m_{ab}^L\) and \(n_{ab}^R\)

\[
\alpha_{ab}^L - \alpha_{ba}^L = 2(2m_{ab}^L + 1), \quad \alpha_{ab}^R + \alpha_{ba}^R = 2(2n_{ab}^R + 1), \quad a < b, \quad m_{ab}^L, n_{ab}^R \in \mathbb{Z}.
\] (69)

With the conditions Eq. (43), Eq. (69) is rewritten in terms of \(\alpha_{ab}^L\)

\[
\alpha_{ab}^L - \alpha_{ba}^L = 2(2m_{ab}^L + 1), \quad \alpha_{ab}^L + \alpha_{ba}^L = -2(2n_{ab}^R + 1).
\] (70)

If a set of solution for the components of \((\alpha^L)\) is found, the components of \((\alpha^R), (\beta^L), (\beta^R)\) are determined by

\[
\alpha_{ab}^L = -\alpha_{ab}^R = \beta_{ab}^L = -\beta_{ab}^R.
\] (71)

The diagonal components may be solved by some integers \(n_a\)

\[
\alpha_{aa}^L = -\alpha_{aa}^R = \beta_{aa}^L = -\beta_{aa}^R = 2n_a + 1, \quad n_a \in \mathbb{Z},
\] (72)

where \(n_a = -n_{aa}^R - 1\). The Hermitian conditions Eq. (111) set a condition to be satisfied by the integers \(n_a\)

\[
e^{\pi i(n_{a+1}+n_a)} = 1.
\] (73)

Let us write the upper triangle components of \((\alpha^L)\) as

\[
\alpha_{ab}^L = 2n_{ab} = 2(m_{ab}^L - n_{ab}^R), \quad \text{for} \quad a < b
\] (74)

where \(n_{ab}\) is an integer. Then the lower triangle components of \((\alpha^L)\) is written by

\[
\alpha_{ba}^L = \alpha_{ab}^L - 4m_{ab}^L - 2, \quad \text{for} \quad a < b.
\] (75)

In summary, we have solutions parameterized by some integers \(n_{ab}, m_{ab}, n_a, a < b,\)

\[
\begin{align*}
\alpha_{ab}^L &= 2n_{ab}, & \alpha_{ba}^L &= 2n_{ab} - 2(m_{ab}^L + 1), & \text{for} \quad a < b, \\
\alpha_{aa}^L &= 2n_a + 1, & e^{\pi i(n_{a+1}+n_a)} &= 1,
\end{align*}
\] (76a, 76b)

\[
(\alpha^L) = - (\alpha^R) = (\beta^L) = - (\beta^R).
\] (76c)

**Some Simple Solutions:** The simplest choice may be \(n_{ab} = 0\) for \(a < b\) and \(n_a = 0\). The upper triangle of \((\alpha^L)\) is null and the diagonal components are 1. By choosing \(m_{ab} = -1\), we can make the off-diagonal components of \((\alpha^L)\) have the same value of 2,

\[
(\alpha^L) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
2 & 2 & \cdots & 1
\end{pmatrix} = - (\alpha^R) = (\beta^L) = - (\beta^R).
\] (77)

We can choose alternatively, \(\alpha_{ab}^L = 0, m_{ab}^L = -1\) for \(a > b\) and \(n_a = 0\). This yields a solution for \((\alpha^L)\), of which lower triangle is null

\[
(\alpha^L) = \begin{pmatrix}
1 & 2 & \cdots & 2 \\
0 & 1 & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix} = - (\alpha^R) = (\beta^L) = - (\beta^R).
\] (78)

In the previous work [43] on the Brownian motion on a triangular lattice, we apply the \(N = 3\) solution, of which explicit expression is given by

\[
(\alpha^L) = \begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 2 \\
2 & 0 & 1
\end{pmatrix} = - (\alpha^R) = (\beta^L) = - (\beta^R).
\] (79)

It corresponds to the solution Eqs. (76a, 76b) with

\[
n_1 = n_2 = n_3 = 0, \quad n_{12} = 1, \quad n_{13} = 0, \quad n_{23} = 1, \quad m_{12} = 0, \quad m_{23} = 0, \quad m_{13} = -1.
\] (80)
V. APPLICATIONS OF BOSE-FERMI EQUIVALENCE AND KLEIN FACTORS

Here we apply the set of Klein factors, which constructed in the last section, to some well-known models in quantum field theory and condensed matter physics. We begin with the model with a single boson field. As we discussed in the introduction we have some difficulties with the conventional representations of the Klein factor even for the case of theories with a single boson field. Since in $(1+1)$ dimensions the left moving boson and the right moving boson are completely independent of each other, they must be treated as two independent species.

A. Theory with a Single Boson Field: N=1 System

If the theory has only a single boson, the matrices $(\alpha^{L/R})$ and $(\beta^{L/R})$ reduce to numbers. The simple solution Eq.(77) is written as

$$\alpha^L = 1, \quad \alpha^R = -1, \quad \beta^L = 1, \quad \beta^R = -1. \quad (81)$$

The fermion fields $\psi^{L/R}$ are represented in terms of chiral boson fields $\phi^{L/R}$ as

$$\psi_L = e^{-\frac{\pi}{2}(\alpha^Lp^L + \beta^L\gamma^5)}e^{-\sqrt{2}\phi_L}, \quad \psi_R = e^{\frac{\pi}{2}(\alpha^Rp^L + \beta^R\gamma^5)}e^{\sqrt{2}\phi_R}. \quad (82a)$$

$$\psi_R = e^{\frac{\pi}{2}(\alpha^Rp^L + \beta^R\gamma^5)}e^{\sqrt{2}\phi_R}. \quad (82b)$$

It is easy to confirm that the periodic potentials of the boson theory are mapped onto the Dirac mass term and the chiral mass term of the fermion theory

$$e^{\sqrt{2}\phi} + e^{-\sqrt{2}\phi} = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L = \bar{\psi}\psi, \quad (83a)$$

$$e^{\sqrt{2}\phi} - e^{-\sqrt{2}\phi} = \bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L = \bar{\psi}\gamma^5\psi. \quad (83b)$$

Here the Dirac gamma matrices are

$$\begin{align*}
\gamma^0 &= \sigma_1, \quad \gamma^1 = \sigma_2, \quad \gamma^5 = \sigma_3 = -i\gamma^0\gamma^1, \\
\sigma_1 &= \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).
\end{align*} \quad (84a)$$

$$\begin{align*}
\sigma_1 &= \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).
\end{align*} \quad (84b)$$

We also easily derive the well-known bosonization rules

$$\partial_\tau\phi_L = -\frac{i}{\sqrt{2}}\psi_L^\dagger\psi_L, \quad \partial_\sigma\phi_L = \frac{1}{\sqrt{2}}\psi_L^\dagger\psi_L, \quad (85a)$$

$$\partial_\tau\phi_R = \frac{i}{\sqrt{2}}\psi_R^\dagger\psi_R, \quad \partial_\sigma\phi_R = \frac{1}{\sqrt{2}}\psi_R^\dagger\psi_R. \quad (85b)$$

1. Thirring Model

By the work of Coleman \[3\] it is well known that the massless Thirring model \[44\] is equivalent to the free boson model. It follows from the above bosonization rules.

$$L = \frac{1}{2\pi} \left( \bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{g}{4\pi} j^\mu j_\mu \right)$$
$$= \frac{1}{4\pi}(\partial\phi)^2 + \frac{g}{4\pi^2}(\partial\phi)^2 \quad (86)$$

$$= \frac{1}{4\beta^2}(\partial\phi)^2,$$

where

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \quad \beta^2 = \frac{1}{1 + g/\pi}. \quad (87)$$
This equivalence can be extended to a more general case of equivalence between the massive Thirring model $L_{Th}$ and the sine-Gordon model $L_{SG}$,

\begin{align*}
L_{Th} & = \frac{1}{2\pi} \left( \bar{\psi} \gamma \cdot \partial \psi + \frac{g}{4\pi} j^\mu j_\mu \right) + \frac{m}{2} \bar{\psi} \gamma^\mu \gamma_\mu \psi, \\
L_{SG} & = \frac{1}{4\pi^2} \partial \phi \partial \phi + \frac{m}{2} \left( e^{\sqrt{2} \bar{\psi} \phi} + e^{-\sqrt{2} \bar{\psi} \phi} \right) + \frac{mc}{2i} \left( e^{\sqrt{2} \bar{\psi} \phi} - e^{-\sqrt{2} \bar{\psi} \phi} \right).
\end{align*} \tag{88a,b}

\[ \square \]

2. The Simple Boundary States

The boundary state is a closed string state which is annihilated by the boundary condition that would be imposed on an open string embedding function when the open string world-sheet ends on the D-brane world-volume. Depending on whether the open string embedding function is longitudinal or transverse to the brane, the boundary condition is Neumann or Dirichlet, respectively. The Neumann and Dirichlet boundary states for the bosonic string obey

\begin{align*}
\phi_L |N\rangle &= \phi_R |N\rangle, \tag{89a} \\
\phi_L |D\rangle &= -\phi_R |D\rangle. \tag{89b}
\end{align*}

The conditions (89a) and (89b) are solved by

\begin{align*}
|N\rangle &= \sum_{p_L} \prod_{n=1}^\infty \exp \left( -a_n^L \bar{a}_n^L \right) |p_L, -p_L\rangle, \tag{90a} \\
|D\rangle &= \sum_{p_L} \prod_{n=1}^\infty \exp \left( a_n^L \bar{a}_n^L \right) |p_L, p_L\rangle. \tag{90b}
\end{align*}

In the fermion representation with Eqs. (82a) and (82b), the boundary state conditions of boson theory Eqs. (89a) and (89b) are mapped onto the following boundary state conditions of fermion theory

\begin{align*}
\psi_L |N\rangle &= i \psi_R^\dagger |N\rangle, \quad \psi_L^\dagger |N\rangle = i \psi_R |N\rangle, \tag{91a} \\
\psi_L |D\rangle &= \psi_R |D\rangle, \quad \psi_L^\dagger |D\rangle = -\psi_R^\dagger |D\rangle. \tag{91b}
\end{align*}

With the simplest solution for the Klein factors Eq. (77), we can also transcribe the Neumann boundary condition and the Dirichlet boundary condition in the fermion theory of $N$ species

\begin{align*}
\phi_L^a |N\rangle &= \phi_R^a |N\rangle, \tag{92a} \\
\phi_L^a |D\rangle &= -\phi_R^a |D\rangle, \quad a = 1, 2, \ldots, N. \tag{92b}
\end{align*}

into the Neumann boundary condition and the Dirichlet boundary condition in the fermion theory of $N$ species as follows

\begin{align*}
\psi_L^a |N\rangle &= i \psi_R^a |N\rangle, \quad \psi_L^{a\dagger} |N\rangle = i \psi_R^a |N\rangle, \tag{93a} \\
\psi_L^a |D\rangle &= \psi_R^a |D\rangle, \quad \psi_L^{a\dagger} |D\rangle = -\psi_R^a |D\rangle, \quad a = 1, 2, \ldots, N. \tag{93b}
\end{align*}

At a glance the boundary conditions for the Neumann and the Dirichlet states Eqs. (93a, 93b) appear to be asymmetric. By a constant phase shift,

\begin{align*}
\psi_L^a &\rightarrow e^{-i\pi/4} \psi_L^a, \quad \psi_L^{a\dagger} \rightarrow e^{i\pi/4} \psi_L^{a\dagger}, \tag{94a} \\
\psi_R^a &\rightarrow e^{i\pi/4} \psi_R^a, \quad \psi_R^{a\dagger} \rightarrow e^{-i\pi/4} \psi_R^{a\dagger}. \tag{94b}
\end{align*}

they can be made resemble the boundary state conditions of the boson theory

\begin{align*}
\psi_L^a |N\rangle &= i \psi_R^a |N\rangle, \quad \psi_L^{a\dagger} |N\rangle = i \psi_R^a |N\rangle, \tag{95a} \\
\psi_L^a |D\rangle &= i \psi_R^a |D\rangle, \quad \psi_L^{a\dagger} |D\rangle = -i \psi_R^a |D\rangle, \quad a = 1, 2, \ldots, N. \tag{95b}
\end{align*}

These representations of the simple boundary states may be also useful to discuss the resonant multilead point-contact tunneling 45 and the multi-channel Kondo problem 46, 47 in the framework of the boundary state formulation.
B. Theory with Two Boson Fields: N=2 System

For the theory with two boson fields or fermion fields with two flavors, the matrices \((\alpha^{L/R})\) and \((\beta^{L/R})\) are given by \(2 \times 2\) matrices. The simple solution for this case is

\[
(\alpha^L) = -(\alpha^R) = (\beta^L) = -(\beta^R) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.
\]

We can spell out the explicit expressions of the fermion field operators as follows:

\[
\begin{align*}
\psi^1_L &= e^{-\frac{i}{2}(\pi \gamma^1 + p \gamma^5 + p \gamma^5 + 2 \pi \gamma^1)} e^{-\sqrt{2} i \phi_1}, \\
\psi^2_L &= e^{-\frac{i}{2}(\pi \gamma^1 + p \gamma^5 + p \gamma^5 + 2 \pi \gamma^1)} e^{-\sqrt{2} i \phi_2}, \\
\psi^1_R &= e^{-\frac{i}{2}(\pi \gamma^1 + p \gamma^5 + p \gamma^5 + 2 \pi \gamma^1)} e^{-\sqrt{2} i \phi_1}, \\
\psi^2_R &= e^{-\frac{i}{2}(\pi \gamma^1 + p \gamma^5 + p \gamma^5 + 2 \pi \gamma^1)} e^{-\sqrt{2} i \phi_2}.
\end{align*}
\]

This is the same representation of the Klein factors we adopted for the rolling tachyon in the previous work \[40\]. In the following we discuss applications of the bosonization to the Gross-Neveu model \[48, 49\], chiral Gross-Neveu model and the Thirring model with two fermion fields. Comparing the bosonized actions of the models, we observe some dissimilarities and similarities between the models. The Gross-Neveu model and the chiral Gross-Neveu model have been proposed to study the dynamical breaking of the chiral symmetry. Because the models share many important features with the quantum chromodynamics in \((3 + 1)\) dimensions, the asymptotic freedom for an example, they are extensively studied in literature. But the studies on the models are mainly focused on the perturbative analysis based on the \(1/N\) expansion. The Fermi-Bose equivalence may enlarge the scope of the analysis, as it enables us to explore non-perturbative domains of the models.

1. Gross-Neveu model with \(N = 2\)

The Gross-Neveu model with \(N\) fermion fields is described by the following Lagrangian \[48\]

\[
L = \frac{1}{2\pi} \left[ \sum_{a=1}^{N} \psi^a \gamma^\mu \partial_\mu \psi^a + \frac{g}{4\pi} \left( \sum_{a=1}^{N} \bar{\psi}^a \psi^a \right)^2 \right].
\]

(98)

Making use of the bosonization rule Eqs.\[97a, 97b, 97c, 97d\] and the following identity which can be proved by the boson field representations of the fermion field operators Eqs.\[97a, 97b, 97c, 97d\]

\[
:\left( \bar{\psi}^a \psi^a \right)^2 : = -2 \psi^a_L \bar{\psi}^a_L \psi^a_R \bar{\psi}^a_R = - \left( \partial \phi^a \right)^2,
\]

(99)

we can map the \(N = 2\) Gross-Neveu model onto the following Lagrangian for two boson fields \[49\]

\[
\begin{align*}
L &= \frac{1}{4\pi} \left( 1 - \frac{g}{2\pi} \right)^2 \sum_{a=1}^{2} \partial \phi^a \partial \phi^a \\
&\quad + \frac{g}{4\pi^2} \left( e^{\sqrt{2}(\phi^1 + \phi^2)} + e^{-\sqrt{2}(\phi^1 + \phi^2)} + e^{\sqrt{2}(\phi^1 - \phi^2)} + e^{-\sqrt{2}(\phi^1 - \phi^2)} \right).
\end{align*}
\]

(100)

Defining two boson fields \(\phi^a, a = 1, 2\), which are related to two boson fields \(\phi^a, a = 1, 2\) by a \(SO(2)\) transformation

\[
\phi^1 = \frac{1}{\sqrt{2}} \left( \phi^1 + \phi^2 \right), \quad \phi^2 = \frac{1}{\sqrt{2}} \left( \phi^1 - \phi^2 \right),
\]

(101)

we rewrite the Lagrangian as

\[
L = \frac{1}{4\pi} \left( 1 - \frac{g}{2\pi} \right)^2 \sum_{a=1}^{2} \partial \phi^a \partial \phi^a + \frac{g}{4\pi^2} \left( e^{2i\phi^1} + e^{-2i\phi^1} + e^{2i\phi^2} + e^{-2i\phi^2} \right).
\]

(102)
Scaling $\varphi^a \to \varphi^a/\sqrt{2}$, we find that the Gross-Neveu model with $N = 2$ is equivalent to a direct sum of two sine-Gordon models

$$L = \sum_{a=1}^{N} \sum_{\alpha=1}^{2} \left\{ \frac{1}{8\pi} \left( 1 - \frac{g}{2\pi} \right) \partial \varphi^\alpha \partial \varphi^\alpha + \frac{g}{4\pi^2} \left( e^{i\sqrt{2} \varphi^\alpha} + e^{-i\sqrt{2} \varphi^\alpha} \right) \right\}. \quad (103)$$

This model is critical at $g = -2\pi$, where it can be shown to be equivalent to a free massive fermion theory with two flavors by refermionization

$$L = \frac{1}{2\pi} \sum_{a=1}^{2} \bar{\psi}^a \left( \gamma^\mu \partial_\mu - m \right) \psi^a, \quad m = 1. \quad (104)$$

2. *Chiral Gross-Neveu model with $N = 2$*

The chiral Gross-Neveu model with fermion fields of $N$ flavors is described by

$$L = \frac{1}{2\pi} \sum_{a=1}^{N} \bar{\psi}^a \gamma^\mu \partial_\mu \psi^a + \frac{g}{4\pi^2} \left( \sum_{a=1}^{N} \bar{\psi}^a \psi^a \right)^2 \left( \sum_{a=1}^{N} \bar{\psi}^a \gamma^5 \psi^a \right)^2. \quad (105)$$

Making use of the identities of the bosonization, which follow from the boson representation of the fermion field operators,

$$: \bar{\psi}^a \psi^a : = : e^{\sqrt{2} i \varphi^a} + : e^{-\sqrt{2} i \varphi^a} := 2 : \cos \sqrt{2} \varphi^a :,$$
$$: \bar{\psi}^a \gamma^5 \psi^a : = : e^{\sqrt{2} i \varphi^a} - : e^{-\sqrt{2} i \varphi^a} := 2 i : \sin \sqrt{2} \varphi^a :,$$
$$: \left( \bar{\psi}^a \gamma^5 \psi^a \right)^2 : = : \left( \bar{\psi}_L^a \psi_R^a - \bar{\psi}_R^a \psi_L^a \right)^2 : = : \left( \partial \varphi^a \right)^2 :,$$
and Eq. (104), we can map the $N = 2$ chiral Gross-Neveu model onto the following boson model

$$L = \frac{1}{4\pi} \left( 1 - \frac{g}{\pi} \right) \sum_{a=1}^{2} \partial \varphi^a \partial \varphi^a + \frac{g}{2\pi^2} \left( e^{2i\varphi^1} + e^{-2i\varphi^1} \right) \left( e^{2i\varphi^2} + e^{-2i\varphi^2} \right). \quad (107)$$

If we take the $SO(2)$ transformation Eq. (101), as in the case of the Gross-Neveu model, we find

$$L = \frac{1}{4\pi} \left( 1 - \frac{g}{\pi} \right) \sum_{a=1}^{2} \partial \varphi^a \partial \varphi^a + \frac{g}{2\pi^2} \left( e^{2i\varphi^2} + e^{-2i\varphi^2} \right). \quad (108)$$

that the Chiral Gross-Neveu $N = 2$ model is equivalent to a direct sum of a sine-Gordon model and a free boson theory. Scaling

$$\varphi^1 \to \frac{\varphi^1}{\sqrt{1 - \frac{g}{2\pi}}}, \quad \varphi^2 \to \frac{\varphi^2}{\sqrt{2}}, \quad (109)$$

we have

$$L = \frac{1}{4\pi} \partial \varphi^1 \partial \varphi^1 + \frac{1}{8\pi} \left( 1 - \frac{g}{\pi} \right) \partial \varphi^2 \partial \varphi^2 + \frac{g}{2\pi^2} \left( e^{i\varphi^2} + e^{-i\varphi^2} \right). \quad (110)$$

This model is critical at $g = -\pi$, where it can be mapped onto a free fermion model by refermionization

$$L = \frac{1}{2\pi} \bar{\psi}^1 \gamma \cdot \partial \psi^1 + \frac{1}{2\pi} \bar{\psi}^2 \left( \gamma \cdot \partial - m \right) \psi^2, \quad m = 1. \quad (111)$$

Here we can see the difference between the Gross-Neveu model and the chiral Gross-Neveu model in their spectrums.
3. SU(2) Thirring Model

The bosonization of the $U(1)$ Thirring model, discussed as an example model with $N = 1$, can be generalized to the bosonization of $SU(N) \times U(1)$ Thirring model, of which Lagrangian is

$$L = \frac{1}{2\pi} \left\{ \sum_{a=1}^{N} \bar{\psi}^a \gamma^5 \cdot \partial \psi^a + \frac{g_{SU(1)}}{4\pi} J^\mu J_\mu + \frac{g_{SU(N)}}{4\pi} \sum_i J^{\mu (i)} J^{(i) \mu} \right\},$$

(112)

where $J^\mu$ and $J^{\mu (i)}$ are $U(1)$ current and $SU(N)$ current respectively,

$$J^\mu = \sum_{a=1}^{N} \bar{\psi}^a \gamma^\mu \psi^a, \quad J^{\mu (i)} = \sum_{a,b=1}^{N} \bar{\psi}^a \gamma^\mu \lambda^{(i)}_{ab} \psi^b, \quad i = 1, 2, \ldots, N,$$

(113)

and $\lambda^{(i)}_{ab}$, $i = 1, 2, \ldots, N$, are generators of $SU(N)$ group. The case of $N = 2$, i.e., $SU(2)$ Thirring model has been discussed in the literature, in the context of bosonization. But an explicit representation of the Klein factors has not been given and the bosonized action contains the products of Klein factors. Here we discuss bosonization of the $SU(2)$ Thirring model explicitly, using the Klein factors given by Eqs.(97a, 97b, 97c, 97d). We shall see that the bosonized action of the model does not contain any non-trivial Klein factor.

The bosonized form of $U(1)$ current is found as

$$J^0 = \sqrt{2} \sum_{a=1}^{2} \partial_\sigma \phi^a, \quad J^1 = - \sqrt{2} \sum_{a=1}^{2} \partial_\tau \phi^a.$$

(114)

Thus, we may rewrite the $U(1)$ current term as

$$J^\mu J_\mu = (J^0)^2 + (J^1)^2 = 2 \left( \sum_{a=1}^{2} \partial_\sigma \phi^a \right)^2 + 2 \left( \sum_{a=1}^{2} \partial_\tau \phi^a \right)^2 = 2 \sum_{a,b=1}^{2} \partial \phi^a \partial \phi^b.$$

(115)

In order to find the bosonized form of the $SU(2)$ Thirring interaction term, we need to do some algebra. Choosing $\lambda^{(i)}_{ab} = \sigma^{i}_{ab}$, $i = 1, 2, 3$ for $SU(2)$, we may write the $SU(2)$ currents as

$$J^{\mu (1)} = \frac{1}{2} \bar{\psi}^a \gamma^\mu (\sigma^1)_{ab} \psi^b = \frac{1}{2} \left( \bar{\psi}^1 \gamma^\mu \psi^1 + \bar{\psi}^2 \gamma^\mu \psi^2 \right),$$

(116a)

$$J^{\mu (2)} = \frac{1}{2} \bar{\psi}^a \gamma^\mu (\sigma^2)_{ab} \psi^b = \frac{i}{2} \left( \bar{\psi}^1 \gamma^\mu \psi^1 - \bar{\psi}^2 \gamma^\mu \psi^2 \right),$$

(116b)

$$J^{\mu (3)} = \frac{1}{2} \bar{\psi}^a \gamma^\mu (\sigma^3)_{ab} \psi^b = \frac{1}{2} \left( \bar{\psi}^1 \gamma^\mu \psi^1 - \bar{\psi}^2 \gamma^\mu \psi^2 \right).$$

(116c)

Although the $SU(2)$ currents may not be expressed entirely in terms of the boson fields, the $SU(2)$ Thirring interaction term may be bosonized to be entirely rewritten in terms of boson fields only. The $SU(2)$ Thirring interaction term may be written as

$$\sum_{\mu, i} J^{\mu (i)} J^{\mu (i)} = \frac{1}{4} \sum_{\mu, i, a,b,c,d} (\bar{\psi}^\gamma \mu)_{a \sigma^{i}_{ab}} \psi_{b} (\bar{\psi}^\gamma \mu)_{c \sigma^{i}_{cd}} \psi_{d}$$

$$= \frac{1}{4} \sum_{\mu, a,b} \left[ 2 (\bar{\psi}^\gamma \mu)_{a} \psi_{b} (\bar{\psi}^\gamma \mu)_{b} \psi_{a} - (\bar{\psi}^\gamma \mu)_{a} \psi_{b} (\bar{\psi}^\gamma \mu)_{b} \psi_{a} \right],$$

(117)

where the completeness relation of the Pauli matrices is used

$$\sum_{i=1}^{3} \sigma^{i}_{ab} \sigma^{i}_{cd} = 2 \delta_{ad} \delta_{bc} - \delta_{ab} \delta_{cd}.$$

(118)
Rewriting the $SU(2)$ Thirring interaction term explicitly in terms of the $SU(2)$ components of the fermi fields, we have

\[ \sum_{\mu,i} J_{\mu}^{(i)} \mu_{(i)} = \sum_{a,b} \left( \psi_L^{a\dagger} \psi_R^{a} + \psi_R^{a\dagger} \psi_L^{a} \right) - \frac{1}{4} \left( \sum_{a} \psi^a \gamma^\mu \psi^a \right)^2 \]

\[ = 2 \sum_{a} \psi_L^{a\dagger} \psi_R^{a} - \sum_{a \neq b} \left( U_a U_b^\dagger + U_b U_a^\dagger \right) - \frac{1}{4} J^\mu J_\mu \]

\[ = \sum_a (\partial \phi^a)^2 - \sum_{a \neq b} \left[ e^{i \sqrt{2} (\phi^a - \phi^b)} + e^{-i \sqrt{2} (\phi^a - \phi^b)} \right] - \frac{1}{4} J^\mu J_\mu \quad (119) \]

Here it should be noted that with the representation the fermion field operators Eqs. (97a, 97b, 97c, 97d), the Klein factors do not explicitly enter the bosonized $SU(2)$ interaction term.

Making use of the bosonized form of the $U(1)$ and $SU(2)$ Thirring interaction terms, we are able to rewrite the Lagrangian of $SU(2) \times U(1)$ Thirring model entirely in terms of the boson fields

\[ L = \frac{1}{4\pi} \left( 1 + \frac{g_{SU(2)}}{2\pi} \right) \sum_{a=1}^2 (\partial \phi^a)^2 + \frac{1}{4\pi^2} g_{U(1)} \sum_{a,b=1}^2 \partial \phi^a \partial \phi^b \]

\[ - \frac{g_{SU(2)}}{4\pi^2} \left( e^{i \sqrt{2} (\phi^1 - \phi^2)} + e^{-i \sqrt{2} (\phi^1 - \phi^2)} \right) \quad (120) \]

The boson Lagrangian may be simplified if rewritten by the two boson fields $\phi^a \ a = 1, 2$, defined by Eq. (101) as in the case of the $N = 2$ Gross-Neveu model

\[ L = \frac{1}{4\pi} \left( 1 + \frac{2g_{U(1)}}{\pi} \right) \partial \phi^1 \partial \phi^1 + \frac{1}{4\pi} \left( 1 + \frac{g_{SU(2)}}{2\pi} \right) \partial \phi^2 \partial \phi^2 - \frac{g_{SU(2)}}{4\pi^2} \left( e^{2i\phi^2} + e^{-2i\phi^2} \right) \quad (121) \]

Scaling two boson fields $\phi^1$ and $\phi^2$ as

\[ \phi^1 \rightarrow \sqrt{\frac{1 + \frac{2g_{U(1)}}{\pi}}{\frac{2g_{U(1)}}{\pi}}}, \quad \phi^2 \rightarrow \sqrt{2} \quad (122) \]

we find that the $SU(2) \times U(1)$ Thirring model is equivalent to a direct sum of a free boson model and a sine-Gordon model

\[ L = \frac{1}{4\pi} \partial \phi^1 \partial \phi^1 + \frac{1}{8\pi} \left( 1 + \frac{g_{SU(2)}}{2\pi} \right) \partial \phi^2 \partial \phi^2 - \frac{g_{SU(2)}}{4\pi^2} \left( e^{i \sqrt{2} \phi^2} + e^{-i \sqrt{2} \phi^2} \right) \quad (123) \]

This model is critical at the point where $g_{SU(2)} = 2\pi$. Refermionizing the model at this point brings us to a free fermion model which contains a massless fermion field and a massive fermion field

\[ L = \frac{1}{2\pi} \bar{\psi} \gamma \cdot \partial \psi^1 + \frac{1}{2\pi} \bar{\psi}^2 (\gamma \cdot \partial - m) \psi^2, \quad m = 1 \quad (124) \]

It is interesting to note that the $SU(2)$ Thirring model is equivalent to the $N = 2$ chiral Gross-Neveu model. The spectra of both models at the critical point are in exact agreement.

C. Theory with Three Boson Fields: $N = 3$ System

For the theory with three boson fields or fermion fields with three flavors, the matrices $(\alpha^L/R)$ and $(\beta^L/R)$ are given by $3 \times 3$ matrices. The simple solution for this case is

\[ (\alpha^L) = (\alpha^R) = (\beta^L) = (\beta^R) = \begin{pmatrix} 1 & 2 & 2 \\
0 & 1 & 2 \\
0 & 0 & 1 \end{pmatrix} \quad (125) \]
The explicit expressions of the fermion field operators are written as follows

\[ \psi_L^1 = e^{-\frac{i}{\hbar}(p_L^1 + 2p_L^2 + 2p_L^3 + p_R^1 + 2p_R^2 + 2p_R^3)} e^{-\sqrt{2}i\phi_L^1}, \]
\[ \psi_L^2 = e^{-\frac{i}{\hbar}(p_L^2 + 2p_L^3 + p_R^1 + 2p_R^2 + 2p_R^3)} e^{-\sqrt{2}i\phi_L^2}, \]
\[ \psi_L^3 = e^{-\frac{i}{\hbar}(p_L^3 + p_R^1 + 2p_R^2 + 2p_R^3)} e^{-\sqrt{2}i\phi_L^3}, \]
\[ \psi_R^1 = e^{-\frac{i}{\hbar}(p_L^1 + 2p_L^2 + 2p_L^3 + p_R^1 + 2p_R^2 + 2p_R^3)} e^{\sqrt{2}i\phi_R^1}, \]
\[ \psi_R^2 = e^{-\frac{i}{\hbar}(p_L^2 + 2p_L^3 + p_R^1 + 2p_R^2 + 2p_R^3)} e^{\sqrt{2}i\phi_R^2}, \]
\[ \psi_R^3 = e^{-\frac{i}{\hbar}(p_L^3 + p_R^1 + 2p_R^2 + 2p_R^3)} e^{\sqrt{2}i\phi_R^3}. \]

As example models for the theory with three boson fields, we choose to discuss the quantum Brownian motion on a triangular lattice and the chiral Gross-Neveu model with \( N = 3 \).

1. Quantum Brownian Motion on a Triangular Lattice

The application of the Fermi-Bose equivalence has been studied in ref.\[43\] in some detail. Here we review the subject briefly for the purpose of the purpose of comparison. The Euclidean action for the quantum Brownian motion (QBM) is given as follows

\[ S_{QBM} = \frac{\eta}{4\pi} \int_0^\beta dt dt' \frac{(X(t) - X(t'))^2}{(t - t')^2} + \frac{M}{2} \int_0^\beta dt \dot{X}^2 + V_0 \int_0^\beta dt \sum_{i=1}^3 \cos(2\pi k_i \cdot X) \]  

(127)

where \( \beta = 1/T \) and

\[ k_1 = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad k_2 = \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \quad k_3 = (-1, 0). \]  

(128)

The first non-local action depicts the frictional force due to the coupling of the particles to a bath or an environment which consists of an infinite set of Harmonic oscillators \[51, 52\] and the third term is the periodic potential on the triangular lattice. Since the triangular lattice spans a two dimensional plane, the model contains only two boson fields initially. But in order to fermionize the model, it is necessary to introduce an auxiliary boson field \( X^3 \). Trading the non-local term on one dimension with the Polyakov local action of string theory on two dimensions, we may rewrite the QBM action as

\[ S = \frac{\alpha}{4\pi} \int dt d\sigma \partial_\sigma \phi^a \partial^a \phi^a + \frac{V}{2} \int d\sigma \sum_{a=1}^3 \left( e^{-\int_{\sigma} \phi^a - \phi^{a+1}} + e^{-\int_{\sigma} \phi^a - \phi^{a+1}} \right) \]  

(129)

where \( \phi^{a+3} = \phi^a \). The three boson fields \( \phi^a \), \( a = 1, 2, 3 \) are related to \( X^a \), \( a = 1, 2, 3 \) by an \( O(3) \) rotation

\[ \phi^1 = \frac{1}{\sqrt{2}} X^1 + \frac{1}{\sqrt{6}} X^2 + \frac{1}{\sqrt{3}} X^3, \]  

(130a)

\[ \phi^2 = -\frac{1}{\sqrt{2}} X^1 + \frac{1}{\sqrt{6}} X^2 + \frac{1}{\sqrt{3}} X^3, \]  

(130b)

\[ \phi^3 = -\frac{\sqrt{2}}{6} X^2 + \frac{1}{\sqrt{3}} X^3. \]  

(130c)

If the periodic boundary interaction is absent, the boundary conditions for \( \phi^a \), \( a = 1, 2, 3 \), would be Neumann: \( (\phi_L^a - \phi_R^a) |_{\sigma=0} = 0 \). Thus, the boundary state for QBM may be written formally as

\[ |B_{QBM}\rangle = \exp \left[ -\frac{V}{2} \int d\sigma \sum_{a=1}^3 \left( e^{-\int_{\sigma} \phi^a - \phi^{a+1}} + e^{-\int_{\sigma} \phi^a - \phi^{a+1}} \right) \right] |N\rangle \]  

(131)

where \( (\phi_L^a - \phi_R^a) |N\rangle = 0, a = 1, 2, 3. \)

\[ \]
With the Neumann condition, we may write the boundary state for QBM at the critical point in fermion theory as

\[ |B_{QBM}\rangle = \exp \left[ \frac{V}{2} \int d\sigma \sum_{a=1}^{3} \left( \psi_{L}^{a+1} \psi_{L}^{a} - \psi_{R}^{a+1} \psi_{R}^{a} \right) \right] : |N\rangle, \]

\[ = \exp \left[ \frac{V}{2} \int d\sigma \sum_{a=1}^{3} \left( \psi_{R}^{a} \psi_{L}^{a} - \psi_{R}^{a+1} \psi_{R}^{a} \right) \right] : |N\rangle, \]  

(132)

where \( \psi_{L/R}^{a+3} = \psi_{L/R}^{a} \). From the boundary state Eq. (132), rewritten in terms of the fermion fields, we may deduce the fermionized action for the QBM as follows

\[ S_{QBM} = \frac{1}{2\pi} \int d\sigma \sum_{a=1}^{3} \left( \bar{\psi}^{a} \gamma^{\mu} \partial_{\mu} \psi^{a} + \frac{g}{4\pi} j^{a \mu} \right) + \frac{V}{4} \int d\sigma \sum_{a=1}^{3} \left( \bar{\psi}^{a-1} \psi^{a+1} - \bar{\psi}^{a+1} \gamma^{1} \psi^{a} \right) \]  

(133)

where \( g = \pi(\alpha - 1) \) and \( j^{a \mu} = \bar{\psi}^{a} \gamma^{\mu} \psi^{a} \). Thus, the QBM model is equivalent to a generalized Thirring model with boundary terms, which are quadratic in fermion fields.

2. Chiral Gross-Neveu Model with \( N = 3 \)

Applying the Fermi-Bose equivalence with the constructed Klein factors to the chiral Gross-Neveu model with \( N = 3 \),

\[ L = \frac{1}{2\pi} (1 - \frac{g}{\pi}) \sum_{a=1}^{3} \partial \phi^{a} \partial \phi^{a} + \frac{g}{2\pi^{2}} \sum_{a=1}^{3} \left( e^{\sqrt{2}(\phi^{a} - \phi^{a+1})} + e^{-\sqrt{2}(\phi^{a} - \phi^{a+1})} \right) \]  

(134)

we find that its bosonized Lagrangian is given as

\[ L = \frac{1}{4\pi} \left( 1 - \frac{g}{\pi} \right) \sum_{a=1}^{3} \partial \varphi^{a} \partial \varphi^{a} + \frac{g}{2\pi^{2}} \sum_{a=1}^{3} \left( e^{\sqrt{2}(\varphi^{a} - \varphi^{a+1})} + e^{-\sqrt{2}(\varphi^{a} - \varphi^{a+1})} \right) \]  

(135)

where \( \varphi^{a+1} = \varphi^{a} \). Introducing three boson fields \( \varphi^{a}, a = 1, 2, 3 \), which are related to the three boson fields \( \phi^{a}, a = 1, 2, 3 \) by scaling and SO(3) rotation

\[ \varphi^{1} = \sqrt{2}\phi^{1} - \sqrt{2}\phi^{2}; \]  

(136a)

\[ \varphi^{2} = \sqrt{\frac{7}{3}}\phi^{1} + \sqrt{\frac{7}{3}}\phi^{2} - 2\sqrt{\frac{7}{3}}\phi^{3}; \]  

(136b)

\[ \varphi^{3} = \frac{2}{\sqrt{3}}\phi^{1} + \frac{2}{\sqrt{3}}\phi^{2} + \frac{2}{\sqrt{3}}\phi^{3}; \]  

(136c)

we may rewrite the bosonized Lagrangian of the chiral Gross-Neveu model with \( N = 3 \) as

\[ L = \frac{1}{16\pi} \left( 1 - \frac{g}{\pi} \right) \sum_{a=1}^{3} \partial \varphi^{a} \partial \varphi^{a} + \frac{g}{2\pi^{2}} \sum_{i=1}^{3} \cos (k_{i} \cdot \varphi). \]  

(137)

Here \( k_{i}, i = 1, 2, 3 \), are the three vectors Eq. (125), which spans the triangular lattice and \( \varphi \) in the periodic potential is a two dimensional vector \( (\varphi^{1}, \varphi^{2}) \). Since the boson field \( \varphi^{3} \) does not enter the periodic potential, it may trivially integrated out. Thus, the chiral Gross-Neveu model with \( N = 3 \) is equivalent to the sine-Gordon model on a two dimensional triangular lattice.

VI. CONCLUSIONS

The Klein factors are important ingredients of the Fermi-Bose equivalence, which has been an essential tool to analyze a wide variety of \((1 + 1)\) dimensional models in condensed matter physics, quantum field theory and string
theory. However, the conventional representations of the Klein factors have room for improvement. They may enter the action and the boundary state conditions explicitly if we transcribe a boson theory into its corresponding fermion theory and vice versa, using the Fermi-Bose equivalence. It is certainly an undesirable feature of the conventional representations of the Klein factors. In this paper we completed the construction of a new representation of the Klein factors, which has been initiated in recent works [37–40, 43]. The new representation of the Klein factors is a generalization of the kink or soliton operator in the anisotropic Heisenberg model, of which role is to change the statistics of the spin-1/2 operator. The new representation of the Klein factors resolves the problems commonly shared by the conventional representations as they do not make an explicit appearance in the action and the boundary state conditions.

We wrote down the most general form of the Klein factors for the theory with $N$ boson fields, which has $4N^2$ parameters. Even if we impose the condition that the fermion field operators satisfy the anti-commutation relations, $(2N^2 + 1)$ parameters remain unfixed. So we are able to impose further constraints such that the Klein factors do not appear in the actions and the boundary states when we map the $(1 + 1)$ dimensional theories by applying the Fermi-Bose equivalence. By an explicit construction we have shown that these conditions are consistent and there exist solutions satisfying all these conditions. A simple solution has been presented as an explicit example and applied to some well-known $(1 + 1)$ dimensional theories.

We applied the Fermi-Bose equivalence to the simple boundary states of the theory with arbitrary number of boson fields. Then we chose Gross-Neveu model with $N = 2$, chiral Gross-Neveu models with $N = 2$ and 3, and $SU(2)$ Thirring model to apply the Fermi-Bose equivalence, using the newly constructed representation of Klein factors. The Gross-Neveu model with $N = 2$ was shown to be equivalent to a direct sum of two sine-Gordon models while the chiral Gross-Neveu model with $N = 2$ is equivalent to a direct sum of a sine-Gordon model and a free boson model. An interesting observation is that the $N = 2$ chiral Gross-Neveu model is equivalent to the $SU(2)$ Thirring model. As example models for the theory with three boson fields, we discussed a model for the quantum Brownian motion (QBM) on a triangular lattice and the $N = 3$ chiral Gross-Neveu model. The model for QBM is defined initially with two boson fields. But to fermionize the model an auxiliary boson field is introduced additionally. Thus, in order to apply the Fermi-Bose equivalence, we need to redefine the model with three boson fields. Applying the Fermi-Bose equivalence, we obtained a Thirring model type action with a fermion bilinear interaction on the boundary. Since a similar action arises in the single-channel spinless Tomonaga-Luttinger liquid model for the junction of three quantum wires [30], the new representation of the Klein factors may be useful to study the multi-junctions of quantum wires.

A more interesting example model may be the chiral Gross-Neveu model with $N = 3$. If the model is bosonized, the phase interaction, which appears on the spatial boundary in the case of QBM model, emerges in the bulk action of the model. Then by taking a $SO(3)$ rotation of the three boson fields, the bosonized action may be rewritten as a sine-Gordon model on a two dimensional triangular lattice. This action can be identified as the effective field theory action for the long wavelength fluctuations of the fully packed loop (FPL) model in statistical physics [33, 55]. The chiral Gross-Neveu model with $N = 3$ corresponds to the FPL model with loop fugacity $n = 2$ model, which undergoes a Kosterlitz-Thouless type transition [33, 56] into a long-range ordered state. Applications of the Fermi-Bose equivalence to the FPL models and the reinterpretation of the phase transitions of the FPL model in the context of the chiral Gross-Neveu model could be excellent subjects, which deserve further study.

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