Entanglement entropy in the Hamming networks

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Abstract

We investigate the Hamming networks that their nodes are considered as quantum harmonic oscillators. The entanglement of the ground state can be used to quantify the amount of information each part of a network shares with the rest of the system via quantum fluctuations. Therefore, the Schmidt numbers and entanglement entropy between two special parts of Hamming network, can be calculated. To this aim, first we use the stratification method to rewrite the adjacency matrix of the network in the stratification basis. Then the entanglement entropy and Schmidt number for special partitions are calculated analytically by using the generalized Schur complement method. Also, we calculate the entanglement entropy between two arbitrary subsets (two equal subsets have the same number of vertices) in $H(2,3)$ and $H(2,4)$ numerically, and we

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give the minimum and maximum values of entanglement entropy in these two Hamming network.
1 Introduction

Entanglement plays a crucial role in quantum information processing, including quantum communication [1,2] and quantum computation [3-5]. It is one of the remarkable features that distinguishes quantum mechanics from classical mechanics.

For decades, entanglement has been the focus of much work in the foundations of quantum mechanics, being associated particularly with quantum nonseparability and the violation of Bells inequalities [6]. Since entanglement has become regarded as such an important resource, there is a need for a means of quantifying it. For the case of bipartite entanglement, a recent exhaustive review was written by the Horodecki family [7] and entanglement measures have been reviewed in detail by Virmani and Plenio [8]. One of the operational entanglement criteria is the Schmidt decomposition [9-11]. The Schmidt decomposition is a very good tool to study entanglement of bipartite pure states. The Schmidt number provides an important variable to classify entanglement. The entanglement of a partly entangled pure state can be naturally parametrized by its entropy of entanglement, defined as the von Neumann entropy, or equivalently as the Shannon entropy of the squares of the Schmidt coefficients [9-11]. The situation simplifies if only so called Gaussian states of the harmonic oscillator modes are considered [12-17]. The importance of gaussian states is two-fold; firstly, its structural mathematical description makes them much more amenable than any other continuous variable system (Continuous variable systems are those described by canonical conjugated coordinates x and p endowed with infinite dimensional Hilbert spaces). Secondly, its production, manipulation and detection with current optical technology can be done with a very high degree of accuracy and control. In [18] the authors quantified the amount of information that a single element of a quantum network shares with the rest of the system. They considered a network of quantum harmonic oscillators and analyzed its ground state to compute the entropy of entanglement that vacuum fluctuations creates between single nodes and the rest of the network by using
the Von Neumann entropy. In [19], jafarizadeh et al, quantify the entanglement entropy between two parts of the network. To this aim, they compute the vacuum state of bosonic modes harmonically coupled through the specific adjacency matrix of a given network.

In this paper, we want to calculate the Schmidt numbers and entanglement entropy between two special parts of Hamming network by generalized Schur complement method.

In section II, we give some preliminaries such as definitions related to association schemes, corresponding stratification and The Terwilliger algebra.

In section III, the generalized Schur complement method is used for calculating the Schmidt numbers and entanglement entropy. In this method, we will apply the generalized Schur complement method to the potential matrix in the stratification basis several times to calculate the entanglement entropy between two arbitrary parts in Hamming network. Then, by using this method, the entanglement entropy and Schmidt number for special partitions are calculated analytically. Also, we calculate the entanglement entropy between two arbitrary subsets (two equal subsets have the same number of vertices) in $H(2, 3)$ and $H(2, 4)$ numerically, and we give the minimum and maximum values of entanglement entropy in these two Hamming network.

2 Preliminaries

In this section we give some preliminaries such as definitions related to association schemes, corresponding stratification and The Terwilliger algebra.

2.1 The model and Hamiltonian

We consider nodes as identical quantum oscillators, interacting as dictated by the network topology encoded in the Laplacian $L$. The Laplacian of a network is defined from the Adjacency matrix as $L_{ij} = k_i \delta_{ij} - A_{ij}$, where $k_i = \sum_j A_{ij}$ is the connectivity of node $i$, i.e., the number
of nodes connected to $i$. The Hamiltonian of the quantum network thus reads:

$$H = \frac{1}{2}(p^T P + x^T (I + 2gL)X)$$  \hspace{1cm} (2-1)$$

here $I$ is the $N \times N$ identity matrix, $g$ is the coupling strength between connected oscillators while $p^T = (p_1, p_2, ..., p_N)$ and $x^T = (x_1, x_2, ..., x_N)$ are the operators corresponding to the momenta and positions of nodes respectively, satisfying the usual commutation relations: $[x, p^T] = i\hbar I$ (we set $\hbar = 1$ in the following) and the matrix $V = I + 2gL$ is the potential matrix. Then the ground state of this Hamiltonian is:

$$\psi(X) = (\det(I + 2gL))^{1/4} \frac{\pi^{N/4}}{A_g} \exp(-\frac{1}{2}(X^T (I + 2gL)X))$$  \hspace{1cm} (2-2)$$

Where the $A_g = (\det(I + 2gL))^{1/4} \frac{\pi^{N/4}}{\pi^{N/4}}$ is the normalization factor for wave function. The elements of the potential matrix in terms of entries of adjacency matrix is

$$V_{ij} = (1 + 2g\kappa_i)\delta_{ij} - 2gA_{ij}$$

### 2.2 Schmidt decomposition and entanglement entropy

Any bipartite pure state $|\psi\rangle_{AB} \in H = H_A \otimes H_B$ can be decomposed, by choosing an appropriate basis, as

$$|\psi\rangle_{AB} = \sum_{i=1}^{m} \alpha_i |a_i\rangle \otimes |b_i\rangle$$  \hspace{1cm} (2-3)$$

where $1 \leq m \leq \min\{\dim(H_A); \dim(H_B)\}$, and $\alpha_i > 0$ with $\sum_{i=1}^{m} \alpha_i^2 = 1$. Here $|a_i\rangle$ ($|b_i\rangle$) form a part of an orthonormal basis in $H_A$ ($H_B$). The positive numbers $\alpha_i$ are called the Schmidt coefficients of $|\psi\rangle_{AB}$ and the number $m$ is called the Schmidt rank of $|\psi\rangle_{AB}$.

Entropy of entanglement is defined as the von Neumann entropy of either $\rho_A$ or $\rho_B$:

$$E = -Tr\rho_A \log_2 \rho_A = Tr\rho_B \log_2 \rho_B = -\sum_{i} \alpha_i^2 \log_2 \alpha_i^2$$  \hspace{1cm} (2-4)$$
2.3 Association scheme

First we recall the definition of association schemes. The reader is referred to Ref.[?], for further information on association schemes.

Definition 2.1 (Symmetric association schemes). Let $V$ be a set of vertices, and let $R_i (i = 0, 1, \ldots, d)$ be nonempty relations on $V$ (i.e., subset of $V \times V$). Let the following conditions (1), (2), (3) and (4) be satisfied. Then, the relations $\{R_i\}_{0 \leq i \leq d}$ on $V \times V$ satisfying the following conditions

1. $\{R_i\}_{0 \leq i \leq d}$ is a partition of $V \times V$
2. $R_0 = (\alpha, \alpha) : \alpha \in V$
3. $R_i = R_i^t$ for $0 \leq i \leq d$, where $R_i^t = (\beta, \alpha) : (\alpha, \beta) \in R_i$
4. For $(\alpha, \beta) \in R_k$, the number $p_{ij}^k = |\gamma \in V : (\alpha, \gamma) \in R_i$ and $(\gamma, \beta) \in R_j|$ does not depend on $(\alpha, \beta)$ but only on $i, j$ and $k$, define a symmetric association scheme of class $d$ on $V$ which is denoted by $Y = (V, \{R_i\}_{0 \leq i \leq d})$. Furthermore, if we have $p_{ij}^k = p_{ji}^k$ for all $i, j, k = 0, 2, \ldots, d$, then $Y$ is called commutative.

The number $v$ of the vertices, $|V|$, is called the order of the association scheme and $R_i$ is called $i$-th relation.

The intersection number $p_{ij}^k$ can be interpreted as the number of vertices which have relation $i$ and $j$ with vertices $\alpha$ and $\beta$, respectively provided that $(\alpha, \beta) \in R_k$, and it is the same for all elements of relation $R_k$. For all integers $i (0 \leq i \leq d)$, set $\kappa_i = p_{ii}^0$ and note that $\kappa_i \neq 0$, since $R_i$ is non-empty. We refer to $\kappa_i$ as the $i$-th valency of $Y$.

Let $Y = (X, \{R_i\}_{0 \leq i \leq d})$ be a commutative symmetric association scheme of class $d$, then the matrices $A_0, A_1, \ldots, A_d$ defined by

$$ (A_i)_{\alpha, \beta} = \begin{cases} 1 & \text{if } (\alpha, \beta) \in R_i, \\ 0 & \text{otherwise } (\alpha, \beta) \in V \end{cases} $$

(2-5)
are adjacency matrices of $Y$ such that

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k.$$  

(2-6)

From (2-6), it is seen that the adjacency matrices $A_0, A_1, ..., A_d$ form a basis for a commutative algebra $A$ known as the Bose-Mesner algebra of $Y$. This algebra has a second basis $E_0, ..., E_d$ (known as primitive idempotents of $Y$) so that

$$E_0 = \frac{1}{n} J, \quad E_i E_j = \delta_{ij} E_i, \quad \sum_{i=0}^{d} E_i = I.$$  

(2-7)

where, $J$ is the all-one matrix in $A$. Let $P$ and $Q$ be the matrices relating the two bases for $A$:

$$A_i = \sum_{j=0}^{d} P_{ij} E_j, \quad 0 \leq j \leq d,$$

$$E_i = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_j, \quad 0 \leq j \leq d.$$  

(2-8)

Then clearly

$$PQ = QP = nI.$$  

(2-9)

It also follows that

$$A_i E_j = P_{ij} E_j,$$  

(2-10)

which shows that the $P_{ij}$ (resp. $Q_{ij}$) is the $j$-th eigenvalue (resp. the $j$-th dual eigenvalue) of $A_i$ (resp. $E_i$) and that the columns of $E_j$ are the corresponding eigenvectors. Thus, $m_i = \text{rank} (E_i)$ is the multiplicity of the eigenvalue $P_{ij}$ of $A_i$ (provided that $P_{ij} \neq P_{kj}$ for $k \neq i$). We see that $m_0 = 1, \sum_i m_i = n$, and $m_i = \text{trace} E_i = n (E_i)_{jj}$ (indeed, $E_i$ has only eigenvalues 0 and 1, so $\text{rank} (E_k)$ equals the sum of the eigenvalues).

Clearly, each non-diagonal (symmetric) relation $R_i$ of an association scheme $Y = (V, \{R_i\}_{0 \leq i \leq d})$ can be thought of as the network $(V, R_i)$ on $V$, where we will call it the underlying network of association scheme $Y$. In other words, the underlying network $\Gamma = (V, R_1)$ of an association scheme is an undirected connected network, where the set $V$ and $R_1$ consist of its vertices and edges, respectively. Obviously replacing $R_1$ with one of the other relations such as $R_i$,
for \( i \neq 0, 1 \) will also give us an underlying network \( \Gamma = (V, R_i) \) (not necessarily a connected network) with the same set of vertices but a new set of edges \( R_i \).

### 2.4 Stratification

For an underlying network \( \Gamma \), let \( W = \mathbb{C}^n \) (with \( n = |V| \)) be the vector space over \( \mathbb{C} \) consisting of column vectors whose coordinates are indexed by vertex set \( V \) of \( \Gamma \), and whose entries are in \( \mathbb{C} \). For all \( \beta \in V \), let \( |\beta\rangle \) denotes the element of \( W \) with a 1 in the \( \beta \) coordinate and 0 in all other coordinates. We observe \( \{|\beta\rangle | \beta \in V \} \) is an orthonormal basis for \( W \), but in this basis, \( W \) is reducible and can be reduced to irreducible subspaces \( W_i, i = 0, 1, ..., d \), i.e.,

\[
W = W_0 \oplus W_1 \oplus ... \oplus W_d, \tag{2-11}
\]

where, \( d \) is diameter of the corresponding association scheme. If we define \( \Gamma_i(o) = \{ \beta \in V : (o, \beta) \in R_i \} \) for an arbitrary chosen vertex \( o \in V \) (called reference vertex), then, the vertex set \( V \) can be written as disjoint union of \( \Gamma_i(o) \), i.e.,

\[
V = \bigcup_{i=0}^{d} \Gamma_i(o). \tag{2-12}
\]

In fact, the relation (2-12) stratifies the network into a disjoint union of strata (associate classes) \( \Gamma_i(o) \). With each stratum \( \Gamma_i(o) \) one can associate a unit vector \( |\phi_i\rangle \) in \( W \) (called unit vector of \( i \)-th stratum) defined by

\[
|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in \Gamma_i(o)} |\alpha\rangle, \tag{2-13}
\]

where, \( |\alpha\rangle \) denotes the eigenket of \( \alpha \)-th vertex at the associate class \( \Gamma_i(o) \) and \( \kappa_i = |\Gamma_i(o)| \) is called the \( i \)-th valency of the network (\( \kappa_i := p_{ii}^0 = |\{ \gamma : (o, \gamma) \in R_i \}| = |\Gamma_i(o)| \)). For \( 0 \leq i \leq d \), the unit vectors \( |\phi_i\rangle \) of Eq.(2-13) form a basis for irreducible submodule of \( W \) with maximal dimension denoted by \( W_0 \). Since \( \{|\phi_i\rangle\}_{i=0}^{d} \) becomes a complete orthonormal basis of \( W_0 \), we often write

\[
W_0 = \bigoplus_{i=0}^{d} \mathbb{C}|\phi_i\rangle. \tag{2-14}
\]
Let $A_i$ be the adjacency matrix of the underlying network $\Gamma$. From the action of $A_i$ on reference state $|\phi_0\rangle$ ($|\phi_0\rangle = |o\rangle$, with $o \in V$ as reference vertex), we have

$$A_i|\phi_0\rangle = \sum_{\beta \in V_i(o)} |\beta\rangle.$$  

(2-15)

Then by using (2-13) and (2-15), we obtain

$$A_i|\phi_0\rangle = \sqrt{\kappa_i}|\phi_i\rangle.$$  

(2-16)

### 3 Entanglement entropy between two parts of a network

In order to calculate the entanglement entropy between two equal parts in the graph (half first strata are in one subset and the other strata are in the second subset), we introduce the following process:

First we want to generalize Schur complement method [19]: Suppose we have the following matrix which is composed of block matrices.

$$V = \begin{pmatrix}
V_{11} & V_{12} & 0 \\
V_{21} & V_{22} & V_{23} \\
0 & V_{32} & V_{33}
\end{pmatrix}$$  

(3-17)

Then we do the generalized Schur complement transformation. This transformation is:

$$\begin{pmatrix}
V_{11} & V_{12} & 0 \\
V_{21} & V_{22} & V_{23} \\
0 & V_{32} & V_{33}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & V_{23}V_{33}^{-1} \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
V_{11} & V_{12} & 0 \\
V_{12}^T & V_{22} - V_{23}V_{33}^{-1}V_{32} & 0 \\
0 & 0 & V_{33}
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & V_{33}^{-1}V_{32} & 1
\end{pmatrix}$$  

(3-18)

In our work, we will apply the generalized Schur complement method to the potential matrix in the stratification basis several times. So in the transformed matrix all of the blocks are scalar. Therefore the potential matrix is transformed to a $2 \times 2$ matrix finally.

$$V = \begin{pmatrix}
a_{11} & a_{12} \\
a_{12}^T & a_{22}
\end{pmatrix}$$  

(3-19)
The wave function in this stage is

$$\psi(x, y) = A_g e^{\frac{1}{2} \left(-\frac{1}{2} (x^2 + y^2)\right)} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(3-20)

by rescaling the variables $x$ and $y$:

$$\tilde{x} = a_{11}^{1/2} x$$

$$\tilde{y} = a_{22}^{1/2} y$$

the ground state wave function is transformed to

$$\psi(\tilde{x}, \tilde{y}) = A_g e^{\frac{1}{2} \left(-\frac{1}{2} (\tilde{x}^2 + \tilde{y}^2)\right)} \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

(3-21)

where

$$\gamma = a_{11}^{-1/2} a_{12} a_{22}^{-1/2}$$

(3-22)

So the ground state wave function is

$$\psi(\tilde{x}, \tilde{y}) = A_g e^{-\tilde{x}^2/2 - \tilde{y}^2/2 - \gamma \tilde{x} \tilde{y}}$$

(3-23)

Then we can use following identity to calculate the schmidt number of this wave function,

$$\frac{1}{\pi^{1/2}} e^{-\frac{1}{2} \left((x^2 + y^2)\right) + \frac{2t}{1 - t^2} xy} = (1 - t^2)^{1/2} \sum_n t^n \psi_n(x) \psi_n(y)$$

(3-24)

In order to calculating the entropy, we apply a change of variable as

$$1 - t^2 = \frac{2}{\nu + 1}$$

$$t^2 = \frac{\nu - 1}{\nu + 1}$$

So the above identity becomes

$$\frac{1}{\pi^{1/2}} e^{-\frac{\nu}{2} \left((x^2 + y^2)\right) + (\nu^2 - 1)^{1/2} xy} = \left(\frac{2}{\nu + 1}\right)^{1/2} \sum_n \left(\frac{\nu - 1}{\nu + 1}\right)^{n/2} \psi_n(x) \psi_n(y)$$

(3-25)
and the reduced density matrix is

$$\rho = \frac{2}{\nu + 1} \sum_n (\frac{\nu - 1}{\nu + 1})^n |n\rangle \langle n|$$

(3-26)

the entropy is

$$S(\rho) = -\sum_n p_n \log(p_n)$$

(3-27)

where

$$p_n = \frac{2}{\nu + 1} (\frac{\nu - 1}{\nu + 1})^n$$

$$\sum_n p_n \log(p_n) = \log(\frac{2}{\nu + 1}) + \langle n \rangle \log(\frac{\nu - 1}{\nu + 1})$$

(3-28)

and

$$\langle n \rangle = \frac{\nu - 1}{2}$$

$$S(\rho) = \frac{\nu + 1}{2} \log(\frac{\nu + 1}{2}) - \nu \log(\nu - 1)$$

(3-29)

By comparing the wave function (3-24) and the identity (3-25) and define the scale $\mu^2$, we conclude that

$$\nu = 1 \times \mu^2$$

$$(\nu^2 - 1)^{1/2} = -\gamma \times \mu^2$$

After some straightforward calculation we obtain

$$\nu = (\frac{1}{1 - \gamma^2})^{1/2}$$

(3-30)

By above discussion we conclude that

$$e^{-\langle x^2 \rangle/2 - \langle y^2 \rangle/2 - \gamma xy} = \sum_n \lambda_n \psi_n(x) \psi_n(y)$$

where

$$\lambda_n = (\frac{2}{\nu + 1})^{1/2} (\frac{\nu - 1}{\nu + 1})^n$$

and the entropy is

$$S(\rho) = \frac{\nu + 1}{2} \log(\frac{\nu + 1}{2}) - \nu \log(\nu - 1)$$

(3-31)
3.1 Hamming network

The symmetric product of $d$-tuples of trivial scheme $K_n$ with adjacency matrices of $I_n$, $J_n - I_n$ is association scheme with the following adjacency matrices (generators of its Bose-Mesner algebra)

$$A_0 = I_n \otimes I_n \otimes \ldots \otimes I_n$$

$$A_1 = \sum_{\text{permutation}} (J_n - I_n) \otimes I_n \otimes \ldots \otimes I_n$$

$$\vdots$$

$$A_i = \sum_{\text{permutation}} (J_n - I_n) \otimes (J_n - I_n) \otimes \ldots \otimes (J_n - I_n) \otimes I_n \otimes \ldots \otimes I_n$$  \hspace{1cm} (3-32)

Where $J_n$ is $n \times n$ matrix with all matrix elements equal to one. This scheme is the well known Hamming scheme with intersection number

$$a_i = \frac{(n - 1)^i d(d - 1) \ldots (d - i + 1)}{i!}, \hspace{1cm} 1 \leq i \leq d$$

$$b_i = i, \hspace{1cm} 1 \leq i \leq d,$$

$$c_i = (n - 1)(d - i), \hspace{1cm} 0 \leq i \leq d - 1,$$  \hspace{1cm} (3-33)

where its underlying graph is the cartesian product of $d$-tuples of cyclic group $Z_n$. We can rewrite the matrix $J_n - I_n$ as the following and then calculate the singular value decomposition of matrix $B$

$$\begin{pmatrix}
0 & B \\
B^T & J_{n-1} - I_{n-1}
\end{pmatrix} \equiv \begin{pmatrix}
0 & 1 & 1 & \ldots & 1 \\
1 & & & & \\
1 & & & & \\
\vdots & & & & \\
1
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 1 & \ldots & 1 \\
1 \\
1 \\
\vdots \\
1
\end{pmatrix}$$
\[
= \begin{pmatrix}
1 & 0 \\
0 & O_{n-1}
\end{pmatrix}
\begin{pmatrix}
0 & \sqrt{n-1} & 0 & \ldots & 0 \\
\sqrt{n-1} & n-2 & 0 & \ldots & 0 \\
0 & 0 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & O_{n-1}
\end{pmatrix}
\] (3-34)

Now we want to introduce new basis proportional to this transformation

\[
|\bar{0}\rangle = |0\rangle
\]

\[
|\bar{1}\rangle = \frac{1}{\sqrt{n-1}}(|1\rangle + |2\rangle + \ldots + |n-1\rangle)
\]

\[
|\bar{k}\rangle = \frac{1}{\sqrt{n-1}} \left( \sum_{j=1}^{n-1} w^{(k-1)(j-1)} |j\rangle \right) \quad k = 2, 3, \ldots, n-1
\] (3-35)

The effect of matrix \(J_n - I_n\) on the above basis from equation (3-35) will be

\[
(J_n - I_n)|\bar{0}\rangle = \sqrt{n-1}|\bar{1}\rangle
\]

\[
(J_n - I_n)|\bar{1}\rangle = (n-2)|\bar{1}\rangle + \sqrt{n-1}|\bar{0}\rangle
\]

\[
(J_n - I_n)|\bar{k}\rangle = -|\bar{k}\rangle, \quad k = 2, 3, \ldots, n-1
\] (3-36)

So the matrices \((J_n - I_n)_-\), \((J_n - I_n)_0\) and \((J_n - I_n)_+\) are

\[
(J_n - I_n)_- = \sqrt{n-1}|\bar{1}\rangle\langle\bar{0}|
\]

\[
(J_n - I_n)_0 = (n-2)|\bar{1}\rangle\langle\bar{1}| - \sum_{k=2}^{n-1} |\bar{k}\rangle\langle\bar{k}|
\]

\[
(J_n - I_n)_+ = \sqrt{n-1}|\bar{0}\rangle\langle\bar{1}|
\] (3-37)

Then we introduce the matrices \(A_+\), \(A_-\) and \(A_0\) as

\[
A_+ = \sum I_n \otimes \ldots \otimes I_n \otimes (J_n - I_n)_+ \otimes I_n \otimes \ldots \otimes I_n
\]

\[
A_- = \sum I_n \otimes \ldots \otimes I_n \otimes (J_n - I_n)_- \otimes I_n \otimes \ldots \otimes I_n
\]
In this stage we want to construct the first stratum of Hamming network. We begin with

$$|\phi_0\rangle_1 = |00\ldots0\rangle = |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle$$

By using the equation (3-38), we conclude that

$$A_+|\phi_0\rangle_1 = 0$$

$$A_-|\phi_0\rangle_1 = \sqrt{(n-1)d}|\phi_1\rangle_1$$

So we have

$$|\phi_1\rangle_1 = \frac{1}{\sqrt{d}}(|\bar{0}\ldots\bar{0}\rangle + |\bar{0}\ldots0\rangle + \ldots + |00\ldots\bar{1}\rangle) = \frac{1}{\sqrt{d}} \sum_i |\bar{0}\ldots\bar{i}\ldots\bar{0}\rangle$$

$$|\phi_2\rangle_1 = \frac{1}{\sqrt{C_2^d}} \sum_{i,j=1}^d |\bar{0}\ldots\bar{i}\ldots\bar{0}\ldots\bar{1}\ldots\bar{i}\ldots\bar{0}\rangle$$

$$|\phi_j\rangle_1 = \frac{1}{\sqrt{C_j^d}} \sum_{i_1<i_2<\ldots<i_d=1}^d |\bar{0}\ldots\bar{0}\ldots\bar{0}\ldots\bar{1}\ldots\bar{0}\ldots\bar{1}\ldots\bar{0}\rangle$$

We are interesting in the effect of $A_-$ on the one stratum of first strata $|\phi_j\rangle_1$,

$$A_-|\phi_j\rangle_1 = \sqrt{n-1}\sqrt{(j+1)(d-j)}|\phi_{j+1}\rangle_1 \tag{3-40}$$

from the above equation we conclude that, without the coefficient $\sqrt{n-1}$, this graph is the spin 1/2 representation of the angular momentum operator.

Also we have

$$A_0|\phi_j\rangle_1 = j(n-2)|\phi_j\rangle_1$$

$$A_+|\phi_j\rangle_1 = \sqrt{n-1}\sqrt{(j+1)(d-j)}|\phi_{j-1}\rangle_1 \tag{3-41}$$

The above basis have only the elements 0, 1. For $\alpha_i$, $i = 2, 3, \ldots, n-1$. We know that, the number of all states are $n^d$. If all elements are consisted from 0,1, we have, $2^d$ states. Now, imagine one of the elements of state become $\alpha_i$. So, $\alpha_i$ can take $n-2$ value and can be in
\[ C_d^1 \] place, therefore the number of states with one parameter \( \alpha_i \), is \( (n - 2)C_d^1 \). Therefore, we have \( 2^{d-1} \) states of this kind and each state, is repeated by \( (n - 2)C_d^1 \). So, the number of all states with one \( \alpha_i \) is \( 2^{d-1}(n - 2)C_d^1 \). By the same way, let we have two different \( \alpha_i \) and \( \alpha_j \), then the number of states will be \( 2^{d-2}(n - 2)^2C_d^2 \). Similarity, if we have \( d \) different \( \alpha_i \), then we have \( (n - 2)^d \) states, that all of them are singlet. We can extend the term \( n^d \), as following:

\[
\begin{align*}
n^d &= 2^d + 2^{d-1}(n-2)C^1_d + 2^{d-2}(n-2)^2C^2_d + \ldots + 2^{d-m}(n-2)^mC^m_d + \ldots + 2(n-2)^{d-1}C_{d-1}^d + (n-2)^d  \\
&= \sum_{m=0}^{d} 2^{d-m}(n-2)^mC^m_d
\end{align*}
\]

The first term in the above is the number of all states contained only 0 and 1, i.e. there is not the parameter \( \alpha_i \) in them. The second term in the extension is the number of all states which have only one \( \alpha_i \). In fact the \( m \)th term in the above, is the number of all states that these states are contained \( m \) different \( \alpha_i \)s.

The adjacency matrix for these states with \( m \) different \( \alpha_i \) is similar to the adjacency matrix for the \( |\phi_i\rangle \)s, but it will be for \( d' = d - m \), i.e., the effect of operators \( A_+ \) and \( A_- \) on the states \( (|\varphi_j\rangle) \) is similar to the binary states \( (|\phi_j\rangle) \). The diagonal elements can be extract from following equation:

\[
A_0|\varphi_j\rangle_{\alpha_1,\alpha_2,\ldots,\alpha_m} = (j(n-2) - m)|\varphi_j\rangle_{\alpha_1,\alpha_2,\ldots,\alpha_m} \quad j = 0, 1, \ldots, d - m
\]

For other strata, we begin with

\[
|\phi_1\rangle_k = \frac{1}{\sqrt{d}} \sum (w_d)^{kj} |0\ldots0\underbrace{1}_j0\ldots0\rangle
\]

then apply the operator \( A_- \) to this state and obtain the state \( |\phi_2\rangle_k \), So

\[
|\phi_2\rangle_k = \frac{1}{\sqrt{d(d-2)}} \sum_{i_1 < i_2}^d (w^{k_{i_1} + w^{k_{i_2}}}) |0\ldots0\underbrace{1}_{i_1}0\ldots0\underbrace{1}_{i_2}0\ldots0\rangle
\]

\[
|\phi_3\rangle_k = \frac{1}{\sqrt{d(d-2)(d-3)/2}} \sum_{i_1 < i_2 < i_3}^d (w^{k_{i_1} + w^{k_{i_2}} + w^{k_{i_3}}}) |0\ldots0\underbrace{1}_{i_1}0\ldots0\underbrace{1}_{i_2}0\ldots0\underbrace{1}_{i_3}0\ldots0\rangle
\]

\[
\vdots
\]
Entanglement entropy

\[ |\phi_m\rangle_k = \frac{1}{\sqrt{\frac{(d-2)\cdots(d-m)}{(m-1)!}} \sum_{i_1 < i_2 < \cdots < i_m} (w^{k_{i_1} + w^{k_{i_2}} + \cdots + w^{k_{i_m}}})|0 \cdots 0 \underbrace{1}_{i_1} 0 \cdots 0 \underbrace{1}_{i_2} 0 \cdots 0 \underbrace{1}_{i_m} 0 \cdots 0\rangle \]  

(3-44)

It can be shown that for these states

\[ A_- |\phi_m\rangle_k = \sqrt{n - 1} \sqrt{m(d - m - 1)} |\phi_{m+1}\rangle_k \]  

(3-45)

These states are equivalent to the angular momentum \(2J = d - 2\). For the third stratum we have

\[ |\phi_{1,1,2}\rangle_{k1,k2} = \frac{1}{2d} \sum_{i_1 \neq i_2}^d (w^{k_{i_1} + k_{i_2}} - w^{k_{i_2} + k_{i_1}})|0 \cdots 0 \underbrace{1}_{i_1} 0 \cdots 0 \underbrace{1}_{i_2} 0 \cdots 0\rangle \]  

(3-46)

These states are equivalent to the representation of \(2J = d - 4\) angular momentum. The highest weight of these kind of basis, is

\[ |\phi_{1,1,\ldots,k_m}\rangle = \frac{1}{m! \sqrt{d^m}} \sum_{i_1 < i_2 < \cdots < i_m}^d \sum_{\alpha_1,\alpha_2,\ldots,\alpha_m} (w^{k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_m}})|0 \cdots 0 \underbrace{1}_{i_1} 0 \cdots 0 \underbrace{1}_{i_2} 0 \cdots 0 \underbrace{1}_{i_m} 0 \cdots 0\rangle \]  

(3-47)

For the network with an arbitrary \(d\), the number of strata is \([\frac{d}{2}] + 1\).

For calculating the entanglement entropy, we use the following equations

\[ A_- |\phi_j\rangle_1 = \sqrt{n - 1} \sqrt{(j + 1)(d - j)} |\phi_{j+1}\rangle_1 \]

\[ A_0 |\phi_j\rangle_1 = j(n - 2) |\phi_j\rangle_1 \]

\[ A_+ |\phi_j\rangle_1 = \sqrt{n - 1} \sqrt{(j + 1)(d - j)} |\phi_{j-1}\rangle_1 \]  

(3-48)
So the Adjacency matrix of each strata is

\[
A = \begin{pmatrix}
0 & c\sqrt{d'} & 0 & 0 & \ldots & 0 \\
c\sqrt{d'} & n-2 & c\sqrt{2(d'-1)} & 0 & \ldots & 0 \\
0 & c\sqrt{2(d'-1)} & 2(n-2) & c\sqrt{3(d'-2)} & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & c\sqrt{2(d'-1)} & (d'-1)(n-2) & c\sqrt{d'} \\
0 & 0 & \ldots & 0 & c\sqrt{d'} & d'(n-2)
\end{pmatrix}
\] (3-49)

Where \( c = \sqrt{n-1} \) and \( d' \) is a parameter which is related to the number of strata, for the first term in equation (3-48) in the binary Hamming, the \( d' \) is equal to \( d \) and for the second term in binary Hamming the parameter \( d' = d - 2 \), therefore in each stage this parameter decreases consecutively.

### 3.1.1 Entanglement entropy in Hamming network between two equal parts: First half and second half strata

In this section, we want to calculate bipartite entanglement between two equal parts of strata in Hamming network. After applying the generalized Schur complement method to the potential matrices of each strata in Hamming networks, we have a \( 2 \times 2 \) matrix finally, which it’s entries are

\[
a_{12} = -2gc\sqrt{\left(\frac{d'+1}{2}\right)\left(\frac{d'+1}{2}\right)} = -gc(d'+1) = -g\sqrt{n-1}(d'+1)
\]

\[
a_{11} = x - \alpha_k - \frac{\omega_{k-1}}{x - \alpha_{k-1} - \frac{\omega_{k-2}}{x - \alpha_{k-2} - \frac{\omega_{k-3}}{x - \alpha_{k-3} - \frac{\omega_2}{x - \alpha_2 - \frac{1}{\alpha_1}}}}}
\]

Where \( x = 1 + 2gd(n-1) \),

\[
\alpha_i = -2g(i-1)(n-2), \quad i = 1, 2, \ldots, \frac{d'+1}{2}
\]

\[
\omega_i = 4g^2c^2i(d'-i+1), \quad i = 1, 2, \ldots, \frac{d'-1}{2}
\] (3-50)
And

\[ a_{22} = x - \alpha'_k - \frac{\omega'_{k-1}}{x - \alpha'_{k-1}} \frac{\omega'_{k-2}}{x - \alpha'_{k-2}} \ldots \]

\[ \alpha'_i = -2g(d' - i + 1)(n - 2), \quad i = 1, 2, \ldots, \frac{d' + 1}{2} \]

\[ \omega'_i = 4g^2c^2d'(d' - i + 1), \quad i = 1, 2, \ldots, \frac{d' - 1}{2} \] (3-51)

Then the parameter \( \gamma_i \) (\( \gamma_i \) for \( i \)th stratum) for each strata is obtained from (3-22):

\[ \gamma = \frac{a_{12}}{\sqrt{a_{11}a_{22}}} \] (3-52)

### 3.1.2 Entanglement entropy in Hamming network between two equal parts: even strata are in first subset and odd strata are in second subset

The entanglement entropy and Schmidt number are considered between two equal parts of hamming network, that even strata are in first subset and odd strata are in second subset. In this kind of partitioning we can consider for two case. First case for \( d = odd \) and second case for \( d = even \).

**Case I (\( d \) is odd):** In this case the potential matrix \( V_{11} \) can be written as following:

\[
\begin{pmatrix}
1 + 2gdc^2 & 0 & \ldots & 0 \\
0 & 1 + 2g(d^2c^2 - 2(c^2 - 1)) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 + 2g(d^2c^2 - (d - 1)(c^2 - 1))
\end{pmatrix}_{\frac{d+1}{2} \times \frac{d+1}{2}}
\] (3-53)
And $V_{22}$ is
\[
\begin{pmatrix}
1 + 2g(d c^2 - (c^2 - 1)) & 0 & \ldots & 0 \\
0 & 1 + 2g(d c^2 - 3(c^2 - 1)) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 + 2g(d c^2 - d(c^2 - 1))
\end{pmatrix}_{\frac{d+1}{2} \times \frac{d+1}{2}}
\]
(3-54)

And the connection matrix of potential matrix $V_{12}$ is:
\[
\begin{pmatrix}
-2gc\sqrt{d} & 0 & \ldots & 0 \\
-2gc\sqrt{2(d-1)} & -2gc\sqrt{3(d-2)} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & -2gc\sqrt{2(d-1)} & -2gc\sqrt{d}
\end{pmatrix}_{\frac{d+1}{2} \times \frac{d+1}{2}}
\]  
(3-55)

Therefore, we have $\frac{d+1}{2}$ parameters $\gamma$

\[
\gamma_1 = \frac{2g\sqrt{n-1}}{\sqrt{1 + 2g(n(d-1))}\sqrt{1 + 2g(n(d-1) - (n-2))}}
\]

\[
\gamma_2 = \frac{(2 + 4)g\sqrt{n-1}}{\sqrt{1 + 2g(n(d-1) - 2(n-2))}\sqrt{1 + 2g(n(d-1) - 3(n-2))}}
\]

\[
\vdots
\]

\[
\gamma_{\frac{d+1}{2}} = \frac{(2 + 4\frac{d+1}{2})g\sqrt{n-1}}{\sqrt{1 + 2g(n(d-1) - (d-1)(n-2))}\sqrt{1 + 2g(n(d-1) - d(n-2))}}
\]  
(3-56)

Where $c$ is $\sqrt{n-1}$.

**Case II (d is even):** In this case the potential matrix $V_{11}$ is
\[
\begin{pmatrix}
1 + 2g d c^2 & 0 & \ldots & 0 \\
0 & 1 + 2g(d c^2 - 2(c^2 - 1)) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 + 2g(d c^2 - d(c^2 - 1))
\end{pmatrix}_{\frac{d+1}{2} \times \frac{d+1}{2}}
\]  
(3-57)
And $V_{22}$ is
\[
\begin{pmatrix}
1 + 2g(dc^2 - (c^2 - 1)) & 0 & \ldots & 0 \\
0 & 1 + 2g(dc^2 - 3(c^2 - 1)) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 + 2g(dc^2 - (d-1)(c^2 - 1))
\end{pmatrix}
\]
\[\frac{4}{4} \times \frac{4}{4} \tag{3-58}\]

And the connection matrix of potential matrix $V_{12}$ is:
\[
\begin{pmatrix}
-2gc\sqrt{d} & 0 & \ldots & 0 \\
-2gc\sqrt{2(d-1)} & -2gc\sqrt{3(d-2)} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & -2gc\sqrt{2(d-1)} & -2gc\sqrt{d}
\end{pmatrix}
\]
\[\frac{4}{4+1} \times \frac{4}{4} \tag{3-59}\]

Therefore, we have $\frac{d}{2}$ parameters $\gamma$
\[
\gamma_1 = \frac{4g\sqrt{n-1}}{\sqrt{1 + 2g(n(d-1))\sqrt{1 + 2g(n(d-1) - (n-2))}}}
\]
\[
\gamma_2 = \frac{(4 + 4)g\sqrt{n-1}}{\sqrt{1 + 2g(n(d-1) - 2(n-2))\sqrt{1 + 2g(n(d-1) - 3(n-2))}}}
\]
\[
\vdots
\]
\[
\gamma_{\frac{d}{2} + 1} = \frac{(2 + 4(\frac{d}{2} - 1))g\sqrt{n-1}}{\sqrt{1 + 2g(n(d-1) - (d-2)(n-2))\sqrt{1 + 2g(n(d-1) - (d-1)(n-2))}}}
\]
\[\tag{3-60}\]

Finally the entropy of entanglement of each strata, i.e. $S(\rho)$, is obtained from Eq.(3-29). So the total entropy is $S(\rho) = \Sigma_i S(\rho_i)$.

### 3.1.3 Entanglement entropy in Hamming network between two equal parts of adjacency matrix

In this part, we want to calculate the entanglement entropy in Hamming network between two equal parts of adjacency matrix.
Case I (n is even): we can define the blocks of adjacency matrix as following:

\[
A_{11} = A_{22} = (J_n - I_n) \otimes I_n \otimes \cdots \otimes I_n + I_n \otimes (J_n - I_n) \otimes \cdots \otimes I_n + \cdots + I_n \otimes I_n \otimes \cdots I_n \otimes (J_n - I_n) \quad (3-61)
\]

And, the connection matrix is

\[
A_{12} = J_n \otimes I_n \otimes \cdots \otimes I_n \quad (3-62)
\]

So, we have the \(d\) types of parameter \(\gamma\)

\[
\gamma_1 = \frac{ng}{1 + ng}
\]

\[
\gamma_2 = \frac{ng}{1 + 3ng}
\]

\[
\gamma_3 = \frac{ng}{1 + 5ng}
\]

\[
\vdots
\]

\[
\gamma_d = \frac{ng}{1 + (2d - 1)ng}
\]

(3-63)

For \(d = 2\), the degeneracy number of \(\gamma_1\) is 1 and the degeneracy number of \(\gamma_2\) is \(n^{d-1} - 1\).

For \(d \geq 3\), the degeneracy numbers of \(\gamma_i\) respectively are

\[1, (d - 1)(n - 1), (d - 2)(n - 1)^2, (d - 3)(n - 1)^3, \ldots, (d - (d - 1))(n - 1)^{d-1}\]

Case II (n is odd): we can define the blocks of adjacency matrix as following:

\[
A_{11} = (J_{n+1} - I_{n+1}) \otimes I_n \otimes \cdots \otimes I_n + I_{n+1} \otimes (J_n - I_n) \otimes \cdots \otimes I_n + \cdots + I_{n+1} \otimes I_n \otimes \cdots \otimes I_n \otimes (J_n - I_n) \quad (3-64)
\]

\[
A_{22} = (J_{\frac{n+1}{2}} - I_{\frac{n+1}{2}}) \otimes I_n \otimes \cdots \otimes I_n + I_{\frac{n+1}{2}} \otimes (J_n - I_n) \otimes \cdots \otimes I_n + \cdots + I_{\frac{n+1}{2}} \otimes I_n \otimes \cdots \otimes I_n \otimes (J_n - I_n) \quad (3-65)
\]

And, the connection matrix is

\[
A_{12} = J_{\frac{n+1}{2} \times \frac{n+1}{2}} \otimes I_n \otimes \cdots \otimes I_n \quad (3-66)
\]

Where \(J_{\frac{n+1}{2} \times \frac{n+1}{2}}\) is the matrix with \(\frac{n+1}{2}\) rows and \(\frac{n+1}{2}\) columns, with all elements are 1.

So, we have the \(d\) types of parameter \(\gamma\)

\[
\gamma_1 = \frac{\sqrt{(n + 1)(n - 1)g}}{\sqrt{1 + (n + 1)g} \sqrt{1 + (n - 1)g}}
\]
\[ \gamma_2 = \frac{\sqrt{(n+1)(n-1)g}}{\sqrt{1 + (3n+1)g}\sqrt{1 + (3n-1)g}} \]
\[ \gamma_3 = \frac{\sqrt{(n+1)(n-1)g}}{\sqrt{1 + (5n+1)g}\sqrt{1 + (5n-1)g}} \]
\[ \vdots \]
\[ \gamma_d = \frac{\sqrt{(n+1)(n-1)g}}{\sqrt{1 + ((2d-1)n+1)g}\sqrt{1 + ((2d-1)n-1)g}} \] (3-67)

For \( d = 2 \), the degeneracy number of \( \gamma_1 \) is 1 and the degeneracy number of \( \gamma_2 \) is \( n^{d-1} - 1 \).

For \( d \geq 3 \), the degeneracy numbers of \( \gamma_i \) respectively are

1, \((d - 1)(n - 1)\), \((d - 2)(n - 1)^2\), \((d - 3)(n - 1)^3\), ..., \((d - (d - 1))(n - 1)^{d-1}\). Finally the entropy of entanglement of each strata, i.e. \( S(\rho) \), is obtained from Eq.(3-29). So the total entropy is \( S(\rho) = \Sigma_i S(\rho_i) \).

### 3.2 Entanglement entropy between all kinds of two parts in Hamming H(2,3) network

In this section we separated Hamming H(2,3) network into two parts, the first part contains 5 nodes and the second part contains 4 nodes, then calculated the entanglement entropy between all kinds of bisection numerically. In this case, there are 126 kinds of partitioning in a way that two parts. Numerical calculations show that some of these 126 sets have the same entropy, such that there are 5 kinds of different values for entanglement entropies.
The maximum entanglement entropy in this section is for subset: 1 of table 1. In this case there are the maximum edges between two parts. The minimum entanglement entropy in this section are for subset:5 of table 1. In these cases there are the minimum number of edges between two parts.

3.3 Entanglement entropy between all kinds of two parts in Hamming $H(2,4)$ network

In this section we separated Hamming $H(2,4)$ network into to equal parts, then calculated the entanglement entropy between all kinds of bisection numerically. The order of name of vertices are based on $A = I_4 \otimes (J_4 - I_4) + (J_4 - I_4) \otimes I_4$.

In this case, there are 6435 kinds of partitioning in a way that two parts. Numerical calculations show that some of these 6435 sets have the same entropy, such that there are 22 kinds of different values for entanglement entropies. The maximum entropy is in the partition that the vertices $(1, 2, 5, 7, 10, 12, 15, 16)$ in the first part. And, the minimum entropy is in the partition that the vertices $(1, 2, 3, 4, 5, 6, 7, 8)$ in the first part and other vertices is in the second part. In the table 2, the assortment of entropy from maximum to minimum is given respect to abundance and agent of each group.
| set | partitions |
|-----|------------|
| 1   | (2, 3, 4, 5, 6), (2, 3, 4, 5, 7), (2, 3, 4, 5, 8), (2, 3, 4, 5, 9), (2, 3, 4, 6, 9), (2, 3, 4, 7, 8), (2, 3, 5, 6, 9), (2, 3, 5, 7, 8), (2, 4, 5, 6, 9), (2, 4, 5, 7, 8), (3, 4, 5, 6, 9), (3, 4, 5, 7, 8), (3, 5, 6, 7, 8), (2, 4, 7, 8, 9), (3, 4, 6, 8, 9), (2, 5, 6, 7, 9), (1, 5, 6, 7, 9), (1, 5, 6, 7, 8), (1, 4, 7, 8, 9), (1, 4, 6, 8, 9), (1, 3, 6, 7, 8), (1, 3, 6, 8, 9), (1, 2, 7, 8, 9), (1, 2, 6, 7, 9), (1, 2, 4, 7, 8), (1, 3, 5, 7, 8), (1, 3, 5, 6, 9), (1, 3, 5, 6, 7), (1, 3, 4, 8, 9), (1, 3, 4, 6, 9), (1, 3, 4, 6, 8), (1, 2, 5, 6, 7), (1, 2, 5, 6, 9), (1, 2, 5, 7, 9), (1, 2, 4, 7, 9), (1, 2, 4, 8, 9), |
| 2   | (2, 3, 4, 8, 9), (2, 3, 4, 5, 7), (2, 4, 5, 7, 9), (3, 4, 5, 6, 8), (1, 6, 7, 8, 9), (1, 3, 4, 7, 8), (1, 3, 5, 6, 9), (2, 3, 4, 8, 9), (1, 2, 4, 6, 9), |
| 3   | (3, 4, 5, 8, 9), (3, 4, 5, 6, 7), (2, 4, 5, 8, 9), (2, 4, 5, 6, 7), (2, 3, 5, 6, 8), (2, 3, 5, 7, 9), (2, 3, 4, 7, 9), (2, 3, 4, 6, 8), (2, 4, 6, 7, 9), (2, 4, 6, 8, 9), (2, 5, 6, 7, 8), (2, 5, 7, 8, 9), (3, 4, 6, 7, 8), (3, 4, 7, 8, 9), (3, 5, 6, 7, 9), (3, 5, 6, 8, 9), (1, 5, 7, 8, 9), (1, 5, 6, 8, 9), (1, 4, 6, 7, 9), (1, 4, 6, 7, 8), (1, 3, 7, 8, 9), (1, 3, 6, 7, 9), (1, 2, 6, 7, 8), (1, 2, 6, 8, 9), (1, 2, 3, 7, 8), (1, 4, 5, 6, 9), (1, 4, 5, 7, 8), (1, 2, 3, 6, 9), (1, 3, 4, 5, 6), (1, 3, 4, 5, 8), (1, 2, 4, 5, 7), (1, 2, 4, 5, 9), (1, 2, 3, 5, 7), (1, 2, 3, 5, 6), (1, 2, 3, 4, 9), (1, 2, 3, 4, 8) |
| 4   | (3, 4, 5, 7, 9), (2, 4, 5, 6, 8), (2, 3, 5, 8, 9), (2, 3, 4, 6, 7), (4, 5, 7, 8, 9), (4, 5, 6, 8, 9), (4, 5, 6, 7, 9), (4, 5, 6, 7, 8), (2, 3, 7, 8, 9), (2, 3, 6, 8, 9), (2, 3, 6, 7, 9), (2, 3, 6, 7, 8), (5, 6, 7, 8, 9), (4, 6, 7, 8, 9), (3, 6, 7, 8, 9), (2, 6, 7, 8, 9), (1, 2, 4, 6, 8), (1, 4, 5, 6, 8), (1, 4, 5, 7, 9), (1, 3, 5, 7, 9), (1, 3, 5, 8, 9), (1, 3, 4, 7, 9), (1, 3, 4, 6, 7), (1, 2, 5, 6, 8), (1, 2, 5, 8, 9), (1, 2, 3, 6, 7), (1, 2, 3, 8, 9), (1, 2, 4, 6, 7), (1, 3, 4, 5, 7), (1, 3, 4, 5, 9), (1, 2, 4, 5, 6), (1, 2, 4, 5, 8), (1, 2, 3, 5, 8), (1, 2, 3, 5, 9), (1, 2, 3, 4, 7), (1, 2, 3, 4, 6) |
| 5   | (2, 4, 6, 7, 8), (2, 5, 6, 8, 9), (3, 4, 6, 7, 9), (3, 4, 6, 7, 8), (1, 4, 5, 8, 9), (1, 4, 5, 6, 7), (1, 2, 3, 6, 8), (1, 2, 3, 7, 9), (1, 2, 3, 4, 5) |

Table 1: All equal subsets in $H(2, 3)$. The entanglement entropy for these subsets are $S_1 > S_2 > S_3 > S_4 > S_5$. 

Table 2: agent of equal subsets in $H(2, 4)$

| set | abundance | agent of partitions |
|-----|-----------|---------------------|
| 1   | 36        | (1, 2, 6, 7, 10, 12, 15, 16) |
| 2   | 9         | (1, 2, 5, 6, 11, 12, 15, 16) |
| 3   | 288       | (1, 2, 3, 5, 8, 10, 12, 15) |
| 4   | 288       | (1, 2, 3, 5, 6, 11, 12, 16) |
| 5   | 576       | (1, 2, 3, 5, 6, 9, 12, 15) |
| 6   | 576       | (1, 2, 3, 5, 6, 9, 12, 16) |
| 7   | 432       | (1, 2, 3, 5, 6, 8, 11, 16) |
| 8   | 288       | (1, 2, 3, 5, 6, 9, 11, 16) |
| 9   | 24        | (1, 2, 3, 5, 6, 7, 12, 16) |
| 10  | 576       | (1, 2, 3, 5, 6, 8, 9, 15) |
| 11  | 288       | (1, 2, 3, 5, 6, 7, 9, 16) |
| 12  | 288       | (1, 2, 3, 4, 5, 6, 11, 16) |
| 13  | 72        | (1, 2, 3, 5, 6, 8, 9, 14) |
| 14  | 1152      | (1, 2, 3, 4, 5, 6, 9, 15) |
| 15  | 288       | (1, 2, 3, 4, 5, 6, 11, 15) |
| 16  | 72        | (1, 2, 3, 4, 5, 6, 11, 12) |
| 17  | 288       | (1, 2, 3, 4, 5, 6, 9, 14) |
| 18  | 288       | (1, 2, 3, 4, 5, 6, 9, 11) |
| 19  | 96        | (1, 2, 3, 4, 5, 6, 7, 12) |
| 20  | 360       | (1, 2, 3, 4, 5, 6, 7, 9) |
| 21  | 144       | (1, 2, 3, 4, 5, 6, 9, 13) |
| 22  | 6         | (1, 2, 3, 4, 5, 6, 7, 8) |
4 Conclusion

The entanglement entropy is obtained between two parts in the Hamming networks that their nodes are considered as quantum harmonic oscillators. The generalized Schur complement method is used to calculate the Schmidt numbers and entanglement entropy between two parts of Hamming graph. Analytically, entanglement entropy in two special partitions are calculated and the maximum entropy in different partitioning is given. Numerical results show that, the maximum and minimum entanglement entropy have a relation with the edges between two parts.

One expects that the entanglement entropy and Schmidt numbers can be calculated in Johnson networks.

5 Appendix

5.1 Other expression for stratification of Hamming graph

In this section, we want obtain the matrix $A_{ij}$ by straight way. We know the basis of different strata is

$$|00...0), |10...0), |01...0), ..., |00...1), ..., |11...1)$$

(5-68)

So, we know basis of each strata from number of 1. in $m$’th strata, the basis are: $e_{i_1,i_2,...,i_m}$, where $i_1 < i_2 < ... < i_m$. Therefore, the adjacency matrix is defined as following

$$A_{i_1,i_2,...,i_m,j_1,j_2,...,j_{m+1}} = \delta_{i_1j_1}\delta_{i_2j_2}...\delta_{i_mj_m} \prod_{k=1}^{m} (1 - \delta_{i_kj_{m+1}}) + \delta_{i_1j_2}\delta_{i_2j_3}...\delta_{i_mj_{m+1}} \prod_{k=1}^{m} (1 - \delta_{i_kj_k}) + ...$$

(5-69)

From the equation (3 – 32), The adjacency matrix in Hamming graph commutate with permutation. If $\pi$ is permutation, we have

$$[A, \pi] = 0$$

(5-70)
It means that the adjacency matrix under displacement of particles is invariant.

\[ \pi_m A = A \pi_{m+1} \tag{5-71} \]

Let \(|\psi\rangle\) is the state, that is invariant under permutation of \(\pi_{m+1}\)

\[ \pi_{m+1} |\psi\rangle = \theta |\psi\rangle \tag{5-72} \]

That \(\theta\) is the phase. So, the acting of \(A\) on the \(|\psi\rangle\) is the eigenvector of \(\pi_m\), so, that is enough finding the eigenvectors of permutation group. And, we know that the eigenvectors of permutation group is discrete fourier transform. So, we have

\[
|i_1, i_2, \ldots, i_m\rangle = \frac{1}{\sqrt{m!d^m}} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m} \varepsilon_{\alpha_1, \alpha_2, \ldots, \alpha_m} \omega^{(i_{\alpha_1}-1)(j_1-1)+(i_{\alpha_2}-1)(j_2-1)+\ldots+(i_{\alpha_m}-1)(j_m-1)} |0 \underbrace{1 \ 0 \ldots \ 1} \ 0 \ldots \ 0\rangle \tag{5-73} \]

Where \(\varepsilon\) is Levi-Civita symbol

### 5.2 Finding the normalization factor for strata of Hamming network

Let

\[ |\tilde{\phi}_2\rangle_k = \sum_{i_1 < i_2} (\omega^{k_{i_1}} + \omega^{k_{i_2}})|0\ldots0 \underbrace{1 \ 0 \ldots \ 1} \ 0 \ldots \ 0\rangle \tag{5-74} \]

Now, we must find the normalization multiplier of \(|\tilde{\phi}_2\rangle_k\). Therefore, we define \(|\tilde{\phi}_2\rangle_{k'}\) orthogonal to \(|\tilde{\phi}_2\rangle_k\).

\[ |\tilde{\phi}_2\rangle_{k'} = \sum_{i_1 < i_2} (\omega^{k'_{i_1}} + \omega^{k'_{i_2}})|0\ldots0 \underbrace{1 \ 0 \ldots \ 1} \ 0 \ldots \ 0\rangle \tag{5-75} \]

Therefore

\[
k \langle \tilde{\phi}_2 | \tilde{\phi}_2 \rangle_{k'} = \frac{1}{d(d-2)} \sum_{i_1 \neq i_2} (\omega^{k_{i_1}} + \omega^{k_{i_2}})(\omega^{k'_{i_1}} + \omega^{k'_{i_2}}) = \]

\[
= \frac{1}{2}((d-2)\omega^{(k-k')i_1} + (d-2)\omega^{(k-k')i_2}) = d(d-2) \tag{5-76} \]
So, the $|\tilde{\phi}_2\rangle_k$ is
\begin{equation}
|\tilde{\phi}_2\rangle_k = \frac{1}{\sqrt{d(d-2)}} \sum_{i_1 < i_2} (\omega_d^{k_{i_1}} + \omega_d^{k_{i_2}})|0...0\ 1\ 0...0\ 1\ 0...0\rangle
\end{equation}
(5-77)

finally, by the same way, we have
\begin{equation}
|\tilde{\phi}_m\rangle_k = \sum_{i_1 < i_2 < ... < i_m} (\omega_d^{k_{i_1}} + \omega_d^{k_{i_2}} + ... + \omega_d^{k_{i_m}})|0...0\ 1\ 0...0\ 1\ 0...0\rangle
\end{equation}
(5-78)

and
\begin{equation}
|\tilde{\phi}_m\rangle_{k'} = \sum_{i_1 < i_2 < ... < i_m} (\omega_d'^{k_{i_1}} + \omega_d'^{k_{i_2}} + ... + \omega_d'^{k_{i_m}})|0...0\ 1\ 0...0\ 1\ 0...0\rangle
\end{equation}
(5-79)

Therefore
\begin{equation}
k(\tilde{\phi}_m|\tilde{\phi}_m\rangle_{k'}) = \frac{1}{m!} \sum_{i_1 \neq i_2 \neq ... \neq i_m} (\omega_d^{k_{i_1}} (\omega_d'^{k_{i_2}} + \omega_d'^{k_{i_2}} + ... + \omega_d'^{k_{i_m}}) = \frac{d(d-2)(d-3)...(d-m-1)(d-m)}{(m-1)!}
\end{equation}
(5-80)

So, the $|\tilde{\phi}_m\rangle_k$ is
\begin{equation}
|\tilde{\phi}_m\rangle_k = \sqrt{\frac{(m-1)!}{d(d-2)...(d-m)}} \sum_{i_1 < i_2 < ... < i_m} (\omega_d^{k_{i_1}} + \omega_d^{k_{i_2}} + ... + \omega_d^{k_{i_m}})|0...0\ 1\ 0...0\ 1\ 0...0\rangle
\end{equation}
(5-81)

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