ON A DEGENERATE PARABOLIC EQUATION ARISING IN PRICING OF ASIAN OPTIONS

SEICK KIM

ABSTRACT. We study a certain one dimensional, degenerate parabolic partial differential equation with a boundary condition which arises in pricing of Asian options. Due to degeneracy of the partial differential operator and the non-smooth boundary condition, regularity of the generalized solution of such a problem remained unclear. We prove that the generalized solution of the problem is indeed a classical solution.

1. Introduction and Main result

In [5], Veˇ ceˇ r proposed a unified method for pricing Asian options, which lead to a simple one-dimensional partial differential equation

\[ u_t + \frac{1}{2} \left( x - e^{-\int_0^t q(s) \, ds} \right)^2 \sigma^2 u_{xx} = 0 \]

with the boundary condition

\[ u(T, x) = (x - K_1)_+ := \max(x - K_1, 0). \]

Here, \( \nu(t) \) is the measure representing the dividend yield, \( \sigma \) is the volatility of the underlying asset, \( q(t) \) is the trading strategy given by

\[ q(t) = \exp \left\{ -\int_t^T \, d\nu(s) \right\} \cdot \int_t^T \exp \left\{ -r(T - s) + \int_s^T \, d\nu(\tau) \right\} \, d\mu(s), \]

where \( r \) is the interest rate and \( \mu(t) \) represents a general weighting factor. In the fixed strike Asian call option, we have \( K_1 = 0 \) in the boundary condition (1.2); see [4][5] for details. If we assume that \( d\mu(t) = \rho(t) \, dt \) for some \( \rho \in L^\infty([0, T]) \) satisfying \( 0 < \rho_0 \leq \rho(t) \), then it is readily seen that

\[ e^{-\int_t^T \, d\nu(s)} q(t) = c \int_t^T \exp \left\{ -r(T - s) + \int_s^T \, d\nu(\tau) \right\} \, d\mu(s) \quad (c = e^{-\int_0^T \, d\nu(t)} > 0) \]

is a monotone decreasing Lipschitz continuous function. We are thus lead to consider the following one-dimensional parabolic PDE

\[ u_t + \frac{1}{2} (b(t) - x)^2 u_{xx} = 0 \]

in \( H_T := (0, T) \times \mathbb{R} \) with the boundary condition

\[ u(T, x) = x_+, \]

where \( b(t) \) is a Lipschitz continuous function defined on \([0, T]\) such that \( b(T) = 0 \) and

\[ m_1 \leq -b'(t) \leq m_2, \quad \text{for a.e. } t \in (0, T) \quad \text{for some } m_1, m_2 > 0. \]
In this article we are mainly concerned with regularity of the (generalized) solution $u(t, x)$ of the problem (1.3), (1.4). It is a rather nontrivial task to show that the problem (1.3), (1.4) has a solution in the classical sense. First of all, it should be noted that even though the coefficient which appears in (1.3) is Lipschitz continuous, the classical approach based on Schauder theory is not applicable here, for the operator in (1.3) becomes degenerate along the curve $x = b(t)$. Nevertheless, it is possible to show that the problem (1.3), (1.4) admits the “probabilistic” solution: Let
\begin{equation}
 u(t, x) := \mathbb{E} f(X_T(t, x)),
\end{equation}
where $f(x) := x_+$ and $X_t$ is the stochastic process which satisfies, for $t \in [0, T]$ and $x \in \mathbb{R}$,
\begin{equation}
 \begin{cases}
 dX_t(x, t) = (b_s - X_s(t, x)) \, dw_s, & s \geq t, \quad (b_s = b(s)) \\
 X_0(t, x) = x.
\end{cases}
\end{equation}
It is known that such a process $X_t$ exists and that if $f$ is twice continuously differentiable, then $u(t, x)$ given by (1.6) is a classical solution of (1.3) in $H_T$ (i.e., $u(t, x)$ is continuously differentiable with respect to $t$ and twice continuously differentiable with respect to $x$ in $H_T$ and satisfies (1.3) there) with the boundary condition $u(T, x) = f(x)$; see e.g. [2]. Unfortunately, $f(x) = x_+$ is not twice continuously differentiable and the above method is not directly applicable here. On the other hand, it should be also noted that if $b(t)$ is smooth enough and $b'(t) \neq 0$ everywhere, then the differential operator in (1.3) satisfies Hörmander’s conditions for hypoellipticity (see [1]). Therefore, in this case, it is not hard to see that $u(t, x)$ given by (1.6) becomes a classical solution of the problem (1.3). (1.4).

Hörmander’s theorem is not available under a mere assumption that $b(t)$ is a Lipschitz continuous function satisfying (1.5).

The main goal of this article is to present a technique to prove that the generalized solution $u(t, x)$ of the problem (1.3), (1.4) is indeed a classical solution. Let us now state our main theorem.

**Theorem 1.8.** For $t \in [0, T]$ and $x \in \mathbb{R}$, let $X_t = X_t(t, x)$ be the stochastic process which satisfies (1.7) and let $u(t, x)$ be defined as in (1.6) with $f(x) := x_+$. Then $u(t, x)$ is a classical solution of the equation (1.3) in $H_T = (0, T) \times \mathbb{R}$ satisfying the boundary condition (1.4).

The organization of this paper is as follows. In Sec. 2 we introduce some notations and present a preliminary lemma which will be used in the proof of the main result. In Sec. 3 we give the proof of our main result, Theorem 1.8. An outline of the proof is as follows. We first split $u = u_1 + u_2$, where $u_i$ are the probabilistic solutions of (1.3) satisfying $u_i(T, x) = f_i(x)$ with $f_1(x) = x$ and $f_2(x) = (-x)_+$. It can be readily seen that $u_1$ is a classical solution of (1.3) in $H_T$. Next, we show that $u_2 \equiv 0$ in the set $\{(t, x) \in H_T : x \geq b(t)\}$. Then, by using a suitable rescaling and the lemma in Sec. 2, we show that $u_2$ decays very rapidly to zero near the curve $x = b(t)$. This is the key point of the proof. Then, we apply the interior Schauder estimates to $u_2$ to conclude that $\partial u_2, \partial_x u_2$, and $\partial_{xx} u_2$ all decay rapidly to zero near the curve $x = b(t)$, from which we will be able to complete the proof. Finally, in Sec. 4 we reformulate the key lemma of the proof in more general settings, in the hope that this technique might be useful to some other problems as well.

2. Notations and preliminaries

2.1. Some notations. We introduce some notations which will be used in the proof. We define the parabolic distance between the points $z_1 = (t_1, x_1)$ and $z_2 = (t_2, x_2)$ as
\[ |z_1 - z_2|_p := \max(|t_1 - t_2|, |x_1 - x_2|). \]
Let \( \alpha \in (0, 1) \) be a fixed constant. If \( u \) is a function in a domain \( Q \subset \mathbb{R}^2 \), we denote
\[
[u]_{\alpha/2, \sigma_0} = \sup_{z_1, z_2 \in Q} \frac{|u(z_1) - u(z_2)|}{|z_1 - z_2|^{\alpha/2}}, \quad [u]_{0, Q} = \sup_Q |u|,
\]
\[
[u]_{\alpha/2, \sigma_0, Q} = [u]_{\alpha/2, \sigma_0} + [u]_{0, Q}.
\]
By \( C^{\alpha/2, \sigma_0}(Q) \) we denote the space of all functions for which \( [u]_{\alpha/2, \sigma_0, Q} < \infty \). We also introduce the space \( C^{1+\alpha/2, 2+\alpha}(Q) \) as the set of all functions \( u \) defined in \( Q \) for which both
\[
[u]_{1+\alpha/2, 2+\alpha, Q} := [u]_{\alpha/2, \sigma_0, Q} + [u_{xx}]_{\alpha/2, \sigma_0, Q} < \infty \quad \text{and}
\]
\[
[u]_{1+\alpha/2, 2+\alpha, Q} := [u]_{0, Q} + [u_{x0}]_{0, Q} + [u_{x0x0}]_{0, Q} + [u_{xx}]_{1+\alpha, 2+\alpha, Q} < \infty.
\]
The function space \( C^{1,2}(Q) \) denotes the set of all functions defined in \( Q \) for which
\[
[u]_{0, Q} + [u_{x0}]_{0, Q} + [u_{x0x0}]_{0, Q} < \infty.
\]
We say \( u \in C^{1+\alpha/2, 2+\alpha}_{loc}(Q) \) if \( u \in C^{1+\alpha/2, 2+\alpha}(Q') \) for all compact set \( Q' \subset Q \) and similarly, \( u \in C^{1,2}_{loc}(Q) \) if \( u \in C^{1,2}(Q') \) for all compact set \( Q' \subset Q \).

2.2. A lemma on Gaussian estimates. Let \( R > 0 \) be fixed and \( g(x) \) be a continuous function defined on \([-R, R]\) satisfying \( 1/2 \leq g(x) \leq 3/2 \) for \( x \in [-R, R] \). We denote
\[
Q := \{(t, x) \in \mathbb{R}^2 : 0 < t < 2, |x| < R\},
\]
\[
\Omega := \{(t, x) \in Q : t > g(x)\}, \quad \Sigma := \{(t, x) \in Q : t = g(x)\}.
\]

**Lemma 2.1.** Let \( \Omega \) and \( \Sigma \) be defined as above and let \( a(t, x) \) be a function satisfying
\[
(2.2) \qquad 0 \leq a(t, x) \leq 1, \quad \forall (t, x) \in \Omega.
\]
Assume that \( u \in C^{1,2}_{loc}(\Omega) \cap C(\overline{\Omega}) \) and satisfies
\[
\begin{align*}
Lu := & \quad u_t - a(t, x)u_{xx} = 0 \quad \text{in} \quad \Omega \\
& \quad u = 0 \quad \text{on} \quad \Sigma.
\end{align*}
\]
Then, we have the following estimate:
\[
(2.3) \quad [u]_{0, \Omega} \leq (16/\sqrt{\pi}) R^{-1} e^{-R^2/32} [u]_{0, \Omega}, \quad \text{where} \quad \Omega' := \{(t, x) \in \Omega : |x| < R/2\}.
\]

**Proof.** By changing \( u \to u / [u]_{0, \Omega} \), we may assume \( [u]_{0, \Omega} = 1 \). Let \( \Phi(t, x) \) be the fundamental solution of the heat equation in \((0, \infty) \times \mathbb{R} \); i.e.,
\[
\Phi(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.
\]
Let \( v(t, x) \) be a function on \((0, \infty) \times \mathbb{R} \) defined by
\[
(2.4) \quad v(t, x) = 2 \int_E \Phi(t, x - y) dy, \quad \text{where} \quad E := \bigcup_{j \in \mathbb{Z}} ((4j + 1)R, (4j + 3)R).
\]
Denote \( D := \{(t, x) \in \mathbb{R}^2 : t > 0, |x| < R\} \). From \((2.4)\), it follows that \( v \geq 0 \) and satisfies
\[
\begin{align*}
\left\{ \begin{array}{l}
\quad v_t - v_{xx} = 0 \quad \text{in} \quad D, \\
\quad v = 0 \quad \text{on} \quad \partial_x D := \{(t, x) \in \mathbb{R}^2 : t = 0, |x| < R\}, \\
\quad v = 1 \quad \text{on} \quad \partial_x D := \{(t, x) \in \mathbb{R}^2 : t > 0, |x| = R\}.
\end{array} \right.
\end{align*}
\]
Moreover, by the comparison principle, we see that \( v(t, x) \leq v(t + h, x) \) in \( D \) for any \( h > 0 \), and thus it follows that
\[
(2.6) \quad v_{xx} = v_t \geq 0 \quad \text{in} \quad D.
\]
On the other hand, for \( \alpha \), therefore by the maximum principle and (2.6), we have

\[
\n\]

\( \varepsilon \) satisfies the boundary condition (1.4).

Denote by \( \partial_p \Omega \) the parabolic boundary of \( \Omega \) (see e.g., [3] for its definition) and observe that \( \Sigma' := \partial_p \Omega \setminus \Sigma \subset \partial_s D \). Then, by (2.5), we find (recall that we assume \( |\partial_0 \Omega| = 1 \))

\[
\n\]

\( v \geq 0 \) on \( \partial_p \Omega \).

Therefore, by the maximum principle and (2.6), we have

\[
\n\]

\( |u(t, x)| \leq \varepsilon (t, x) \leq 2 \varepsilon (x), \quad \forall (t, x) \in \Omega. \)

On the other hand, for \( |x| < R/2 \), we estimate \( \varepsilon (2, x) \) by

\[
\varepsilon (2, x) = 2 \int_E \Phi (2, x - y) \, dy \leq 4 \int_{R^{-1} |x|}^{\infty} \Phi (2, y) \, dy \leq 4 \int_{R/2}^{\infty} \Phi (2, y) \, dy
\leq \frac{8}{\sqrt{8\pi R}} \int_{R/2}^{\infty} ye^{-y^2/8} \, dy = \frac{16}{\sqrt{2\pi}} R^{-1} e^{-R^2/32}.
\]

The lemma is proved. \( \Box \)

3. Proof of Theorem 1.8

For \( t \in [0, T] \) and \( x \in \mathbb{R} \), let \( X_t = X_s (t, x) \) be the stochastic process which satisfies (1.7). It is well known that such a process \( X_t \) exists; see e.g., [2] Theorem V.1.1. Denote

\[
(3.1) \quad u_1 (t, x) = \mathbb{E} f_1 (X_T(t, x)), \quad u_2 (t, x) = \mathbb{E} f_2 (X_T(t, x)),
\]

where \( f_1 (x) = x \) and \( f_2 (x) = (-x) \), so that \( f (x) = f_1 (x) + f_2 (x) \). By [2] Theorem V.7.4], the function \( u_1 \) and its derivatives \( \partial_x u_1 \) are continuous in \( H_T \) and \( u_1 \) satisfies the equation (1.3) there. In other words, the function \( u_1 \) is a classical solution of (1.3) in \( H_T \). Also, it is readily seen that \( u_i \in C (\overline{H_T}) \) \((i = 1, 2) \). Therefore, it is clear that \( u = u_1 + u_2 \) satisfies the boundary condition (1.4).

Let us further analyze the function \( u_2 \). Once we prove that \( u_2 \) is also a classical solution of (1.3) in \( H_T \), then we are done. Let \( \{g_k\}_{k=1}^{\infty} \) be smooth approximations of \( f_2 \), say obtained by using mollifiers, such that \( g_k \to f_2 \) uniformly. Denote

\[
\varepsilon_k (t, x) = \mathbb{E} g_k (X_T(t, x)).
\]

Then by the same reasoning as above, the functions \( \{\varepsilon_k\}_{k=1}^{\infty} \) are classical solution of (1.3) in \( H_T \). Note that by interior Schauder estimates, \( C^{1+a/2,2+a/2} \)-norm of \( \varepsilon_k \) in any compact set belonging to \( H_T \setminus \{(t, x) : x = b(t)\} \) is estimated through its supremum over a bounded domain containing the set. Since \( g_k \to f_2 \) uniformly, we also have \( \varepsilon_k \to \varepsilon_2 \) uniformly, and thus we get

\[
\varepsilon_2 \in C^{1+a/2,2+a} (\Omega), \quad \Omega := H_T \setminus \{(t, x) : x = b(t)\}
\]

and satisfies the equation (1.3) in \( \Omega \).

Next, we claim that \( u_2 \equiv 0 \) in \( \{(t, x) \in [0, T] \times \mathbb{R} : x \geq b(t)\} \). Note that the process

\[
Y_s (t, x) := X_0 (t, x) - b_s \quad (b_s = b(s))
\]

satisfies the following stochastic differential equation:

\[
(3.2) \quad \begin{cases}
\quad \quad dY_s (t, x) = -Y_s (t, x) \, dw_s - b' (s) \, ds, \quad s \geq t, \\
Y_t (t, x) = x - b(t).
\end{cases}
\]

The solution to (3.2) is unique and has a representation

\[
Y_s = Y_t e^{w_t-w_s+\frac{1}{2} (t-s)} - \int_t^s e^{w_t-w_r+\frac{1}{2} (r-s)} b' (r) \, dr, \quad s \geq t.
\]
Therefore, from the assumption $b' \leq 0$, we conclude that $Y_s(t, x) \geq 0$ for all $s \geq t$ provided that $Y(t, x) = x - b(t) \geq 0$. In particular, we have $X_T(t, x) = X_T(t, x) - b(T) = Y_T(t, x) \geq 0$ if $x \geq b(t)$. Therefore, from (3.1) and the fact that $f_2 \equiv 0$ for $x \geq 0$, we find $u_0(x, t) = 0$ if $x \geq b(t)$. We have thus proved the claim that $u_2 \equiv 0$ in $\{(t, x) \in [0, T] \times \mathbb{R} : x \geq b(t)\}$.

Now, we will show that $u_2 \in C^{1,2}_loc(H_T)$. To comply with standard conventions in parabolic PDE theory, we make a change of variable $t \mapsto T - t$ and denote

\begin{equation}
\tag{3.3}
v(t, x) := u_2(T - t, x) \quad \text{and} \quad \psi(t) := b(T - t).
\end{equation}

By the observations made above, we have

\begin{equation}
\tag{3.4}
v \in C(H_T) \cap C^{1+\alpha/2, 2+\alpha}_loc(H_T \setminus \Gamma), \quad \text{where} \quad \Gamma := \{(t, x) \in H_T : x = \psi(t)\},
\end{equation}

and satisfies the equation

\[ v_t - \frac{1}{2}(x - \psi(t))^2v_{xx} = 0 \quad \text{in} \quad H_T \setminus \Gamma. \]

In order to show that $v \in C^{1,2}_loc(H_T)$, we need investigate the behavior of $v$ near $\Gamma$. By (1.5), we find that $\phi := \psi^{-1}$ is defined on $[0, \ell]$, where $\ell := \psi(T)$, and satisfies

\[ 1/m_2 \leq \phi'(x) \leq 1/m_1, \quad \text{for a.e.} \quad x \in (0, \ell). \]

In the rest of the proof, we use the following notation. For $z_0 = (t_0, x_0) \in \mathbb{R}^2$, we denote

\begin{align*}
C_r(z_0) &= \{(t, x) \in \mathbb{R}^2 : |t - t_0| < r, |x - x_0| < (m_1/2)r\}, \\
\mathcal{U}_r(z_0) &= C_r(z_0) \cap \{(t, x) \in H_T : x < \psi(t)\}, \\
\mathcal{U}'_r(z_0) &= \{(t, x) \in \mathcal{U}_r(z_0) : |x - x_0| < (m_1/4)r\}, \\
\Gamma_r(z_0) &= C_r(z_0) \cap \Gamma.
\end{align*}

**Lemma 3.5** (Key lemma). Let $z_0 = (t_0, x_0) = (t_0, \psi(t_0)) \in \Gamma$ and $r \in (0, 1)$ be any number satisfying $\overline{C_r(z_0)} \subset D := (0, T) \times (0, \ell)$. Then, the function $v$ defined as in (3.3) satisfies

\begin{equation}
\tag{3.6}
|v|_{\partial \mathcal{U}_r(z_0)} \leq N_0 \phi^2 e^{-k_0 r}/|v|_{\partial \mathcal{D}},
\end{equation}

where $N_0 = N_0(m_1, m_2)$ and $k_0 = k_0(m_1, m_2) > 0$. Moreover, we have

\begin{equation}
\tag{3.7}
r^{3/2}|v_x(t_0 + r, x_0)| + r^3|v_{xx}(t_0 + r, x_0)| + r|v(t_0 + r, x_0)| \leq N_1 r^{1/2}e^{-k_0 r}/|v|_{\partial \mathcal{D}},
\end{equation}

where $N_1 = N_1(m_1, m_2)$.

**Proof.** Let $T$ be a linear mapping defined by

\begin{equation}
\tag{3.8}
T(t, x) := \left((t - t_0)/r, (x - x_0)/cr^{3/2}\right), \quad \text{where} \quad c := (m_1 + 2m_2)/\sqrt{8}.
\end{equation}

We shall denote $\Omega_r := T(\mathcal{U}_r(z_0))$, $\Sigma_r := T(\Gamma_r(z_0))$, and $Q_r := T(C_r(z_0)) = \{(t, x) \in \mathbb{R}^2 : |t| < 1, |x| < (m_1/2c)r^{-1/2}\}$.

We also define the functions $w(t, x)$ and $a(t, x)$ on $\overline{Q_r}$ by

\begin{align}
\tag{3.9}
w(t, x) &= v \circ T^{-1}(t, x) = v(t_0 + rt, x_0 + cr^{3/2}x), \\
\tag{3.10}
a(t, x) &= \frac{1}{2(c r^2)}(x_0 + cr^{3/2}x - \psi(t_0 + rt))^2.
\end{align}

Then $w \in C^{1+\alpha/2, 2+\alpha}_loc(\Omega_r) \cap C(\overline{\Omega_r})$ and satisfies

\begin{equation}
\tag{3.11}
\begin{cases}
Lw := w_t - a(t, x)w_{xx} = 0 \quad \text{in} \quad \Omega_r, \\
w = 0 \quad \text{on} \quad \Sigma_r,
\end{cases}
\end{equation}

where $L := \frac{\partial^2}{\partial x^2}$.
Note that \(a(t, x)\) satisfies the following inequalities in \(Q_r\).

(3.12) \[ 0 \leq a(t, x) = \frac{1}{2(c_r)^2} \left( x_0 + cr^{3/2} x - \psi(t_0 + rt) + \psi(t_0) - x_0 \right)^2 \leq \frac{1}{2(c_r)^2} \left( cr^{3/2}|x| + m_2 r|t| \right)^2 \leq \frac{1}{8c^2} (m_1 + 2m_2)^2 = 1. \]

Also, observe that \(\Sigma_r \subset \{ (t, x) \in \mathbb{R}^2 : |t| < 1/2 \}\). By (3.11) and (3.12), we may apply Lemma 2.1 to \(u(t, x) = w(t + 1, x)\) with \(R = (m_1/2c)r^{-1/2}\) to conclude that

(3.13) \[ |w|_{0, \Omega_r} \leq N_{r^{1/2}} e^{-k_0/r} |w|_{0, \Omega}, \]

where \(\Omega_r' = T(\mathcal{U}(z_0)), N_0 = 8(m_1 + 2m_2)/\sqrt{2}m_1, \) and \(k_0 = m_1^2/16(m_1 + 2m_2)^2\). It is obvious by (3.9) that (3.6) follows from (3.13).

Next, we turn to the proof of (3.7). Note that by a similar calculation as in (3.12), we have (recall \(0 < r < 1\))

(3.14) \[ |\partial_t a|_{L^\infty(Q_r)} \leq 4(m_1 + 2m_2), \quad |\partial_t a|_{L^\infty(Q_r)} \leq 4m_2/(m_1 + 2m_2). \]

Let us denote \(\Pi_r := (1 - \rho^2, 1) \times (-\rho, \rho)\) for \(\rho > 0\). Note that if \((t, x) \in \Pi_r\), then

(3.15) \[ a(t, x) \geq \frac{1}{2(c_r)^2} \left( |\psi(t_0) - \psi(t_0 + r)| - |\psi(t_0 + r) - \psi(t_0 + r)| - cr^{3/2}|x| \right)^2 \geq \frac{1}{2(c_r)^2} \left( m_1 r - m_2 r^2 - cr^{3/2} \right)^2 \geq \frac{1}{2c^2} \left( m_1 - m_2 r^2 - cr \right)^2. \]

Fix \(\rho_0 = \rho_0(m_1, m_2) \in (0, 1/2]\) such that

\[ m_1 - m_2 \rho_0^2 - cr \rho_0 \geq m_1/2 \quad \text{and} \quad \Pi_{\rho_0} \subset \Omega_r'. \]

Then by (3.12) and (3.15), we have

(3.16) \[ 2m_1/(m_1 + 2m_2)^2 \leq a(t, x) \leq 1, \quad \forall (t, x) \in \Pi_{\rho_0}. \]

By (3.14), (3.16), and the interior Schauder estimates, we have

(3.17) \[ |w_x(1, 0)| + |w_{xx}(1, 0)| + |w_x(1, 0)| \leq C|w|_{0, \Pi_{\rho_0}}, \]

where \(C = C(m_1, m_2); \) see e.g. [3]. Now, the estimate (3.7) follows from (3.9), (3.13), and (3.17). The lemma is proved. \(\square\)

We are ready to prove that \(v \in C^{1,2}_{\text{loc}}(H_T)\). We define \(v_x = 0\) (resp. \(v_{xx} = 0, v_0 = 0\)) on \(\Gamma\). By (3.4), it is enough to show that \(v_\delta\) (resp. \(v_{xx}, v_t\)) is continuous at each \(z_0 = (t_0, x_0) \in \Gamma\).

Fix an \(r_0 = r_0(z_0) \in (0, 1)\) such that \(C_{r_0}(z_0) \subset D = (0, T) \times (0, t)\). Note that for any \(z_1 \in \Gamma_{r_0/4}(z_0)\) and \(r < r_0/4\), we have \(C_{r}(z_1) \subset C_{r_0}(z_0)\). Therefore, by Lemma 3.3

(3.18) \[ |w(\phi(x) + r, x)| \leq N r^{-\beta} e^{-k_0/r} |w|_{0, D}, \quad \forall r \in (0, r_0/4) \quad \forall x \in (x_0 - r_0/4, x_0 + r_0/4), \]

where \(w := v_x\) (resp. \(w := v_{xx}, w := v_t\)) and \(\beta = -1\) (resp. \(\beta = -5/2, \beta = -1/2\)). On the other hand, note that there is some \(\delta = \delta(m_1, m_2) > 0\) such that

(3.19) \[ \mathcal{U}_{r_0}(z_0) \subset \{ (\phi(x) + r, x) \in \mathbb{R}^2 : 0 < r < r_0/4, |x_0 - x| < r_0/4 \}. \]

From (3.18) and (3.19), we find that \(\lim_{r \to 0} |w|_{0, C_{\delta}r_0(z_0)} = 0\). The theorem is proved.
4. Generalization of Key Lemma

Let \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) be a Lipschitz continuous function satisfying \( \| \nabla \phi \|_{L^\infty(\mathbb{R}^n)} \leq M_0 \) for some \( M_0 \in (0, \infty) \) and denote

\[
\Gamma := \{ (t, x) \in \mathbb{R} \times \mathbb{R}^n : t = \phi(x) \}.
\]

For \( z = (t, x) \in \mathbb{R} \times \mathbb{R}^n \) and \( r > 0 \), we shall denote

\[
C_r(z) := \{(s, y) \in \mathbb{R} \times \mathbb{R}^n : |s - t| < r, \max_{1 \leq k \leq n} |y_k - x_k| < (1/2M_0)r\},
\]

\[
\mathcal{U}_r(z) := C_r(z) \cap \{(s, y) \in \mathbb{R} \times \mathbb{R}^n : s > \phi(y)\},
\]

\[
\mathcal{U}_r'(z) := \{(s, y) \in \mathcal{U}_r(z) : \max_{1 \leq k \leq n} |y_k - x_k| < (1/4M_0)r\},
\]

\[
\Gamma_r(z) := C_r(z) \cap \Gamma.
\]

**Theorem 4.1.** Let \( z_0 \in \Gamma \) and \( r > 0 \) be given. Assume that there are numbers \( \mu > 1 \) and \( \Lambda > 0 \) such that the coefficients \( (a_{ij}(t, x))_{i,j=1}^n \) satisfy

\[
0 \leq a_{ij}(t, x)\xi_i\xi_j \leq \Lambda |\phi(x) - t\mu| |\xi|^2, \quad \forall (t, x) \in C_r(z_0), \quad \forall \xi \in \mathbb{R}^n.
\]

Let \( u \in C^{1,2}_0(\mathcal{U}_r(z_0)) \cap C(\overline{\mathcal{U}_r(z_0)}) \) satisfy

\[
\begin{cases}
Lu := u_t - a_{ij}D_{ij}u = 0 & \text{in } \mathcal{U}_r(z_0), \\
u = 0 & \text{on } \Gamma_r(z_0).
\end{cases}
\]

Then the following estimate holds.

\[
|u|_{0;\mathcal{U}_r(z_0)} \leq N_0 r^{(\mu-1)/2} e^{-kr^{1-\mu}} |u|_{0;\mathcal{U}_r(0)},
\]

where \( N_0 = N_0(n, \mu, \Lambda, M_0) \) and \( k_0 = k_0(\mu, \Lambda, M_0) > 0 \).

**Proof.** The proof is a slight modification of that in Lemma 2.1. By renormalizing \( u \) to \( u/|u|_{0;\mathcal{U}_r(z_0)} = 1 \), let \( T \) be a linear mapping defined by

\[
T(t, x) := \left( (t - t_0)/r, (x - x_0)/c r^{(1+\mu)/2},\right), \quad \text{where } c := \Lambda^{1/2}(3/2)^{\mu/2}.
\]

Denote \( \Omega_r := T(\mathcal{U}_r(z_0)), \Omega'_r := T(\mathcal{U}_r'(z_0)), \Sigma_r := T(\Gamma_r(z_0)) \), and

\[
Q_r := T(\mathcal{C}_r(z_0)) = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : |t| < 1, \max_{1 \leq k \leq n} |x_k| < (1/2cM_0)r^{(1-\mu)/2}\}.
\]

Define the functions \( w(t, x) \) and \( \tilde{a}_{ij}(t, x) \) on \( \overline{\Omega}_r \) and \( Q_r \), respectively, by

\[
w(t, x) := u \circ T^{-1}(t, x) = u(t_0 + rt, x_0 + cr^{(1+\mu)/2}x),
\]

\[
\tilde{a}_{ij}(t, x) := (c^2 r^{\mu})^{-1} a_{ij}(t_0 + rt, x_0 + cr^{(1+\mu)/2}x).
\]

Then \( w \in C^{1,2}_0(\Omega_r') \cap C(\overline{\Omega}_r) \) and satisfies

\[
\begin{cases}
\tilde{L}w := w_t - \tilde{a}_{ij}(t, x)D_{ij}w = 0 & \text{in } \Omega_r', \\
w = 0 & \text{on } \Sigma_r.
\end{cases}
\]

By (4.2) and (4.6), for all \( (t, x) \in Q_r \) and \( \xi \in \mathbb{R}^n \), we have

\[
0 \leq \tilde{a}_{ij}(t, x)\xi_i\xi_j \leq \frac{\Lambda}{c^2 r^{\mu}} \left( |\phi(x_0) + cr^{(1+\mu)/2}x| - |\phi(x_0)| + r|t| \right)^\mu |\xi|^2
\]

\[
\leq \frac{\Lambda}{c^2 r^{\mu}} \left( M_0 cr^{(1+\mu)/2} |x| + r|t| \right)^\mu |\xi|^2 \leq \frac{\Lambda}{c^2} (3/2)^\mu |\xi|^2 = |\xi|^2.
\]

Let \( v \) be given as in (2.4) with \( R = (1/2cM_0)r^{(1-\mu)/2} \) and define

\[
V(t, x) := V(t, x_1, \ldots, x_n) := \sum_{k=1}^n v(t + 1, x_k).
\]
Then, since \( v_{xx} \geq 0 \) by (2.6) and \( \tilde{a}_{ik} \leq 1 \), for all \( k = 1, \ldots, n \), by (4.9), we have

\[
LV = V_1 - \tilde{a}_{ij}D_{ij}V = \sum_{k=1}^{n} (v_{ij}(t, x_k) - \tilde{a}_{ik}v_{xx}(t, x_k)) \geq \sum_{k=1}^{n} (v_{ij}(t, x_k) - v_{xx}(t, x_k)) = 0 \quad \text{in} \quad Q_r.
\]

Note that by (2.5), \( V \geq 1 \) on \( \partial_0 Q_r := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : |t| < 1, |x_k| = R, \forall k = 1, \ldots, n \}. \) Also, observe that \( \Sigma_r \subset \{(t, x) \in Q_r : |t| < 1/2 \}. \) Therefore, we have \( V \geq |w| \) on the parabolic boundary \( \partial_0 \Omega_r \). Then, by the comparison principle, we obtain

\[
|w(t, x)| \leq V(t, x), \quad \forall (t, x) \in \Omega_r.
\]

On the other hand, by (2.6), (2.7), and (4.10), we have (recall (4.16) \( \tilde{R} \))

\[
(4.11) \quad V(t, x) \leq (32ncM_0/\sqrt{2\pi})^{(1-\mu)/2}e^{-r^{-\mu}/128c^2M_0^2} \quad \forall (t, x) \in \Omega_r.
\]

We obtain (4.3) by combining (4.6), (4.11), and (4.12). The theorem is proved. \( \square \)

**Theorem 4.13.** Let \( \tilde{z}_0 \in \Gamma \) and \( R > 0 \) be given. Assume that there are numbers \( \mu > 1 \) and \( \Lambda, \Lambda, M_1 > 0 \) such that the coefficients \( (a_{ij}(t, x))_{ij} \) satisfy

\[
(4.14) \quad \lambda |\phi(x) - t|^{\mu} |\xi|^2 \leq a_{ij}(t, x)\xi_i\xi_j \leq \Lambda |\phi(x) - t|^{\mu} |\xi|^2, \quad \forall (t, x) \in C_R(\tilde{z}_0), \quad \forall \xi \in \mathbb{R}^n,
\]

\[
(4.15) \quad |\nabla_x a_{ij}(t, x)| \leq M_1 |\phi(x) - t|^{\mu - 1}, \quad \text{for a.e. } (t, x) \in C_R(\tilde{z}_0).
\]

Suppose \( u \in C^{1+a/2/\alpha}_{loc}(U_R(\tilde{z}_0)) \cap C(\overline{U_R(\tilde{z}_0)}) \), for some \( a \in (0, 1) \), and satisfies

\[
\left\{ 
\begin{array}{ll}
Lu := u_{tt} - a_{ij}D_{ij}u = 0 & \text{in } U_R(\tilde{z}_0), \\
u = 0 & \text{on } \Gamma_0(\tilde{z}_0).
\end{array}
\right.
\]

Then if we extend \( u \equiv 0 \) in \( C_R(\tilde{z}_0) \setminus U_R(\tilde{z}_0), \) we have \( u \in C^{1,2}_{loc}(C_R/2(\tilde{z}_0)) \)

**Proof.** Let \( \tilde{z}_0 = (t_0, x_0) = (\phi(x_0), x_0) \in \Gamma_R/2(\tilde{z}_0) \) and let \( 0 < r < \min(1, R/2) \) so that \( r < 1 \) and \( C_r(\tilde{z}_0) \subset C_R(\tilde{z}_0) \). Then, by (4.3) of Theorem 4.1 we find

\[
|u_0|_{C_0(\tilde{z}_0)} \leq N_0r^{(\mu-1)/2}e^{-kr^{\mu-\nu}} |u_0|_{C_0(\tilde{z}_0)}.
\]

Let \( T, Q_r, w(t, x), \) and \( \tilde{a}_{ij}(t, x) \) be defined as in (4.4) - (4.7). Then, by (4.15) we have

\[
|\nabla_{x,a} \tilde{a}_{ij}|_{L^\infty(\Omega_r)} \leq CM_1, \quad \text{where } C = C(\Lambda, \mu).
\]

Denote \( \Pi_r := (1 - \rho^2, 1) \times (-\rho, \rho) \). Note that if \( (t, x) \in \Pi_r \), then we have

\[
|\phi(x_0 + cr^{(1+\mu)/2}x) - (t_0 + rt)| \geq |r(1 - \rho^2) - M_0cr^{1+\mu/2}/\rho| \geq |1 - \rho^2 - M_0c\rho|.
\]

Let us fix a number \( \rho_0 = \rho_0(\mu, \Lambda, M_0) \in (0, 1/2) \) such that \( |1 - \rho^2 - M_0c\rho| \geq (1/2)^{1/\mu} \) and \( \Pi_{\rho_0} \subset \Omega_r \). Then, it follows from (4.7) and (4.14) that

\[
(4.18) \quad \tilde{a}_{ij}(t, x)\xi_i\xi_j \geq (\lambda/2c^2)|\xi|^2 = (\lambda/\Lambda)2^{\mu-1}3^{-\mu}|\xi|^2.
\]

Then by (4.17), (4.9), (4.18), and the interior Schauder estimate, we have

\[
(4.19) \quad |D_{x}w(1, 0)| + |D^2_{x}w(1, 0)| + |w_t(1, 0)| \leq C|w_{0}|_{C^{1,\mu}_{\Pi_{\rho_0}}},
\]

where \( C = C(n, \alpha, \mu, \lambda, \Lambda, M_0, M_1) \). Therefore, by using (4.6) and (4.16), we conclude

\[
(4.20) \quad \mu^{(1+\mu)/2}|D_{x}u(t_0 + r, x_0)| + \mu^{1+\mu} |D^2_{x}u(t_0 + r, x_0)| + |u_t(t_0 + r, x_0)| \leq N_1\mu^{(\mu-1)/2}e^{-kr^{1+\mu}} |w_{0}|_{C^{1,\mu}_{\Pi_{\rho_0}}},
\]

\( \square \)
where \( N_1 = N_1(n, \alpha, \mu, \lambda, \Lambda, M_0, M_1) \). Finally, by using (4.20) instead of (3.7) and proceeding similarly as in the proof of Theorem 1.8 we see that \( u \in C^{1,2}_{\text{loc}}(C_{R/2}(\bar{z}_0)) \). This completes the proof.

**Acknowledgments** The author is grateful to Mikhail Safonov and Jan Večer for very helpful discussion.

**References**

[1] Hörmander, L. *Hypoelliptic second order differential equations*. Acta Math. 119 (1967) 147–171.
[2] Krylov, N. V. *Introduction to the theory of diffusion processes*. American Mathematical Society, Providence, RI, 1995.
[3] Lieberman, G. M. *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
[4] Večer, J. *A new PDE approach for pricing arithmetic average Asian options*. Journal of Computational Finance 4 (2001), no. 4, 105–113.
[5] Večer, J. *Unified Asian Pricing*. Risk 15 (2002), no. 6, 113–116

Department of Mathematics, Yonsei University, 262 Seongsanno, Seodaemun-gu, Seoul 120-749, Korea

E-mail address: kimseick@yonsei.ac.kr