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To cite this version:
Gabriel Pallier. Large-scale sublinearly Lipschitz geometry of hyperbolic spaces. 2019. hal-01684826v3

HAL Id: hal-01684826
https://hal.science/hal-01684826v3
Preprint submitted on 15 Apr 2019

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LARGE-SCALE SUBLINEARLY LIPSCHITZ GEOMETRY OF HYPERBOLIC SPACES

GABRIEL PALLIER

ABSTRACT. Large-scale sublinearly Lipschitz maps have been introduced by Yves Cornulier in order to precisely state his theorems about asymptotic cones of Lie groups. In particular, Sublinearly biLipschitz Equivalences (SBE) are a weak variant of quasiisometries, with the only requirement of still inducing biLipschitz maps at the level of asymptotic cones. We focus here on hyperbolic metric spaces and study properties of boundary extensions of SBEs, reminiscent of quasiMöbius (or quasisymmetric) mappings. We give a dimensional invariant of the boundary that allows to distinguish hyperbolic symmetric spaces up to SBE, answering a question of Drutu.

INTRODUCTION

0.A. Main definitions and results. Sublinearly Lipschitz maps between metric spaces have been gradually made into an object of study by Y. Cornulier in a series of papers starting in 2008 [7, 8, 11]. Here is a short definition of a sublinearly biLipschitz equivalence (compare to Definition 1.4):

Definition 0.1. Let $X$ and $Y$ be pointed metric spaces. In $X$ and $Y$, denote the distances by $|\cdot - \cdot|$ and distances to the base-point by $|\cdot|$. A map $f : X \to Y$ is called a sublinearly biLipschitz equivalence (SBE) if there exists a nondecreasing, doubling function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ with $u(r) \ll r$ as $r \to +\infty$, and $(\lambda, \overline{\lambda}) \in \mathbb{R}_{>0}^2$ such that for any $x, x' \in X$ and $y \in Y$,

\begin{align}
\lambda |x - x'| - u(|x| \lor |x'|) &\leq |f(x) - f(x')| \leq \overline{\lambda}|x - x'| + u(|x| \lor |x'|), \\
\inf \{|y - y' : y' \in f(X)| &\leq u(|y|),
\end{align}

where $|x| \lor |x'|$ denotes $\sup \{|x|, |x'|\}$.

Note that while the function $u$ in the definition may depend up to an additive (or multiplicative, as $u$ takes values higher than 1) error on base-points, the large-scale Lipschitz and reverse Lipschitz data $(\lambda, \overline{\lambda})$ do not. The technical conditions on $u$ are required so that there is a well-behaved notion of $(\lambda, O(u))$-sublinearly biLipschitz equivalence (resp. $(\lambda, o(u))$-sublinearly biLipschitz equivalence) between non-pointed metric spaces; it is useful to retain only the class $O(u)$ or $o(u)$ for composition purposes, see Cornulier [11, Proposition 2.2] and section 1 below. When

Date: April 15, 2019.
2010 Mathematics Subject Classification. Primary 20F67, 30C65; Secondary 53C23, 20F69, 30L10.
Key words and phrases. hyperbolic groups, quasiconformal mappings, cross-ratios, asymptotic geometry, analysis on metric spaces.
The author thanks the support of project ANR-15-CE40-0018.
$u = 1$, $O(u)$-sublinearly biLipschitz equivalences are the more traditional quasiisometric maps.

Sublinearly Lipschitz maps were devised in the first place so that for any nonprincipal ultrafilter $\omega$ over $\mathbb{Z}_{\geq 0}$ or $\mathbb{R}_{\geq 0}$ and scaling sequence $(\lambda_j)$, $\text{Con}_\omega(\cdot, \lambda_j)$ (with fixed basepoint) defines a functor from the large-scale sublinearly Lipschitz category to the Lipschitz category [8, Proposition 2.9]. The asymptotic cone characterization of hyperbolicity (Gromov [21, 2.A], Drutu [13, 3.A.1.(iii)]) ensures that within the class of quasihomogeneous, geodesic metric spaces (such as finitely generated groups), hyperbolicity is preserved by sublinearly biLipschitz equivalences (see Cornulier [11, Proposition 4.2]). However, while asymptotic cones up to biLipschitz homeomorphisms are fine SBE invariants in order to distinguish, e.g., nilpotent groups, this is not the case in the hyperbolic setting, since all complete nonpositively curved Riemannian manifolds and nonelementary Gromov-hyperbolic groups share the same asymptotic cones, namely the universal $2^{\aleph_0}$-branched $\mathbb{R}$-tree, even defined up to isometry (see for instance Erschler and Polterovich [15, Theorem 1.1.3]). This suggests to study the effects of SBEs on other asymptotic invariants instead. In this direction, Cornulier proved that sublinearly biLipschitz equivalences induce biH"older homeomorphisms between geodesic boundaries of proper geodesic hyperbolic metric spaces equipped with visual distances [11, Theorem 1.7 and Theorem 4.3]. Restated within the spaces, this says that for pairs of triples of far apart points sent to each other by a sublinearly biLipschitz equivalence, Gromov products in the source and target are within linear control of each other, a feature which may be derived from the large scale biLipschitz behavior. Similarly to Gromov products, cross-differences, or positive logarithms of cross-ratios, have an incarnation as large distances within the space, so that one can hope that the same control remains between them, with a sublinear error term. This is our main result.

**Theorem 1** (Restatement of Theorem 4.2). Let $f : X \to Y$ be a $(\Delta, \overline{\lambda}, O(u))$-sublinearly biLipschitz equivalence between hyperbolic proper geodesic spaces. Then $f$ induces a map $\varphi$ between the geodesic boundaries with the property that for all distinct $(\xi_1, \ldots, \xi_4)$ on the geodesic boundary of $X$, all of them close enough,

$$\lambda \log^+(\xi_i) - v(\overline{\xi}(\xi_i)) \leq \log^+|\varphi(\xi_i)| \leq \overline{\lambda} \log^+(\xi_i) + v(\overline{\xi}(\xi_i)), \tag{0.3}$$

where $v = O(u)$ is a sublinear function, $\log^+(s) = \max(0, \log s)$ for all $s \in \mathbb{R}_{>0}$, $\overline{\xi}(\xi_i)$ denotes the supremum of all Gromov products over pairs in the four $\xi_i$'s, and the brackets $[\xi_i]$ denote the cross-ratios $[\xi_1, \ldots, \xi_4]$ (see 1.C for definitions).

The homeomorphisms as in (0.3) are given the name of sublinearly quasiM"obius (Definition 4.1). A distinctive feature of sublinearly quasiMöbius homeomorphisms is that their distortion of the moduli of small annuli (or “eccentricity” of small ellipsoids) is bounded at small, non-infinitesimal scale:

**Definition 0.2.** Let $\Xi$ be a metric space. An annulus $A$ of $\Xi$ is a difference of concentric balls $B(\xi, s) \setminus B(\xi, r)$ for some $\xi \in \Xi$ and $r, s \in \mathbb{R}_{>0}$. The real number $\mathfrak{M} = \log(s/r)$ is called a modulus\(^1\) for $A$.

**Proposition 0.3** (Restatement of Proposition 4.9). Let $\Xi$ and $\Psi$ be compact, uniformly perfect metric spaces and $\varphi : \Xi \to \Psi$ a $(\lambda^{-1}, \lambda, O(u))$-sublinearly quasiMöbius

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\(^1\)This would be an ill-defined function if applied to the set $A$ since $\xi, r, s$ may vary, nevertheless we write that $A$ is an annulus of modulus $\mathfrak{M}$. It mostly matters to bound moduli from above.
homeomorphism. Let \( A \) be an annulus of inner radius \( r \), outer radius \( R \) and modulus \( \mathcal{M} \). There exists \( w = O(u) \) such that if \( R \) is sufficiently small, \( \varphi(A) \) is contained in an annulus of modulus
\[
\mathcal{M}' = 2\lambda \mathcal{M} + w(-\log r).
\]

When \( u = 1 \) this is a characterization of power-quasisymmetric mappings, compare Mackay and Tyson, \cite[Lemma 1.2.18]{Mackay}. With their scale-sensitive moduli distortion, sublinearly quasiMöbius homeomorphisms may lack the analytic properties of quasisymmetric mappings, even between Euclidean spaces. Nevertheless we prove that they preserve the Hausdorff dimension of visual metrics in a favorable setting:

**Proposition 0.4** (Consequence of Proposition 5.9). Let \( \Xi^* \) and \( \Psi^* \) be punctured\(^2\) boundaries of purely real, normalized Heintze groups of Carnot type with homogeneous dimensions \( p \) and \( p' \) (see 5.B for definitions). Assume there exists a homeomorphism \( \varphi : \Xi^* \to \Psi^* \) which is sublinearly quasiMöbius over any compact subset (with respect to the visual metrics). Then \( p = p' \).

The Heintze groups of Carnot type form an intermediate class between hyperbolic symmetric spaces and simply connected negatively curved homogeneous spaces. The invariance of the topological dimension of the geodesic boundary is more generally granted by Cornulier’s theorem on biHölder continuity. Once combined, those two asymptotic invariants allow to distinguish all hyperbolic symmetric spaces, answering a question of Drută \cite[Question 1.16 (2)]{Druta}:

**Theorem 2.** Let \( X \) and \( Y \) be rank one Riemannian symmetric spaces of noncompact type. If there exists a sublinearly biLipschitz equivalence between \( X \) and \( Y \), then \( X \) and \( Y \) are homothetic.

In view of the Cornulier-Tessera characterization of hyperbolic connected Lie groups \cite[Corollary 3]{Cornulier}, this can be rephrased as: if there exists a large-scale sublinearly biLipschitz equivalence between two hyperbolic, non-amenable connected real Lie groups \( G \) and \( G' \), then \( G \) and \( G' \) are commable.

**0.B. Acknowledgements.** Yves Cornulier suggested to look for the main theorem, my advisor Pierre Pansu provided constant help and section 5 owes a lot to his ideas. I benefited from a conversation with Peter Haïssinsky about Lipschitz-Möbius constants, explanations by Prof. Viktor Schroeder on cross-ratios, translation by Aliaksandr Minets on parts of \cite{Insua}, and a key remark of Sébastien Gouëzel on the proof of the quantitative Morse lemma, see 2.C herein. I thank the anonymous referee for accurate and useful comments.

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\(^2\)The puncture is made at a distinguished point so that the remaining part of the boundary is transitively acted upon by the group; see 5.B.
1. Background

1.A. Large-scale Sublinearly Lipschitz maps. Here is a summary of Cor- mulier's definitions included for the reader's convenience. Call admissible any function \( u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1} \) with the following properties:

1. \( u \) is nondecreasing
2. \( u \) is doubling: \( \limsup_{r \to +\infty} u(2r)/u(r) < +\infty \)
3. \( u \) is strictly sublinear: \( \lim_{r \to +\infty} u(r)/r = 0 \).

It is not really restrictive, and in fact useful in statements, to allow such a function to be only eventually defined and conditions (1), (2) to hold only on a neighborhood of \(+\infty\) in \( \mathbb{R}_{\geq 0} \). However we will frequently work with a precise admissible function \( u \) while keeping track on explicit bounds, and where they become valid. To facilitate this we introduce the following notation:

- For all \( \varepsilon > 0 \), \( r_\varepsilon(u) = \sup\{r \in \mathbb{R}_{\geq 0} : u(r) > \varepsilon r\} \). This is finite by (3).
- Properties (1), (2) and the fact that \( \inf \tau, u(\tau r) > 0 \) ensure that for any \( \tau > 1 \), \( \sup_p u(\tau r)/u(r) \) is finite. We shall denote this number \( u \uparrow \tau \).

The following lemma is for our use only; it describes the way in which the constants \( r_\varepsilon(u) \) and \( u \uparrow \tau \) evolve when shifting function \( u \).

**Lemma 1.1.** Let \( u \) be an admissible function. For any \( p \in \mathbb{R}_{>0} \), define \( u_p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1} \) as \( u_p(t) = u(p + t) \). Then

1. For all \( \tau \in \mathbb{R}_{>1} \), \( u_p \uparrow \tau \leq u \uparrow \tau \).
2. For all \( \varepsilon \in \mathbb{R}_{>0} \), if \( p \geq r_\varepsilon(2u) \) then
   \[ r_\varepsilon(u_p) \leq \frac{u \uparrow 2}{\varepsilon} u(p). \]

**Proof.** Start with (1). By definition, \( u \) is nondecreasing, hence
   \[ u_p \uparrow \tau = \sup_r \frac{u(\tau r + p)}{u(r + p)} \leq \sup_r \frac{u(\tau r + \tau p)}{u(r + p)} = u \uparrow \tau. \]

As for (2), the hypothesis made on \( p \) means that for all \( p' \) greater than \( p \), \( u_p(p') \leq \frac{\varepsilon}{2}(p + p') \leq \varepsilon p' \), so \( r_\varepsilon(u_p) \leq p \), and then \( p + r_\varepsilon(u_p) \leq 2p \). Also note that since \( u_p \) is nondecreasing, \( \varepsilon r_\varepsilon(u_p) \) is equal to \( u(p + r_\varepsilon(u_p)) \), so that
   \[ \varepsilon r_\varepsilon(u_p) = u(p + r_\varepsilon(u_p)) \leq u(2p) \leq (u \uparrow 2) u(p). \]
Finally \( r_\varepsilon(u_p) \leq \varepsilon^{-1}(u \uparrow 2) u(p) \). \( \square \)

In the following, let \( u \) be an admissible function, and let \( X \) and \( Y \) be two pointed metric spaces. Recall that whenever \( r \) and \( s \) are real numbers, \( r \vee s \) denotes \( \sup\{r, s\} \) and \( r \wedge s \) denotes \( \inf\{r, s\} \).

**Definition 1.2.** A map \( f : X \to Y \) is called \( (\lambda, O(u)) \)-Lipschitz if there exists \( \lambda \in \mathbb{R}_{>0} \) (called a large-scale Lipschitz constant) and a nondecreasing function \( v = O(u) \) such that for all \( (x_1, x_2) \in X^2 \),
   \[ |f(x_1) - f(x_2)| \leq \lambda|x_1 - x_2| + v(|x_1| \vee |x_2|). \]
We may write that \( f \) is \((\lambda, v)\)-Lipschitz to put emphasis on \( v \), or on the contrary a \( O(u) \)-Lipschitz map if the actual Lipschitz constant and function \( v \) are not relevant.

**Definition 1.3.** \( f, g : X \rightarrow Y \) are \( O(u) \)-close if \( |f(x) - g(x)| = O(u(|x|)) \).

One checks that \( O(u) \)-Lipschitz maps can be composed (with a multiplicative effect on large-scale Lipschitz constants), in a way compatible with \( O(u) \)-closeness [11, Proposition 2.2], hence there is a well-defined category \( \mathcal{L}_{O(u)} \) with metric spaces as objects\(^3\) and large-scale \( O(u) \)-Lipschitz maps modulo \( O(u) \)-closeness as morphisms.

**Definition 1.4** (compare Definition 0.1), \( f : X \rightarrow Y \) is a \( O(u) \)-Sublinearly BiLipschitz Equivalence (SBE) if the \( O(u) \)-closeness, in a way compatible with \( O(u) \)-closeness, of \( f \) is an isomorphism in \( \mathcal{L}_{O(u)} \).

This can be metric-geometrically rephrased as follows [11, Proposition 2.4]:

1. \( f \) is \( O(u) \)-Lipschitz;
2. \( f \) is \( O(u) \)-expansive: there exists a nondecreasing \( v = O(u) \) and \( \lambda \in \mathbb{R}_{>0} \) such that
   \[ \forall (x_1, x_2) \in X^2, |f(x_1) - f(x_2)| \geq \lambda |x_1 - x_2| - v(|x| + |x'|); \]
3. \( f \) is \( O(u) \)-surjective: for \( y \in Y \),
   \[ d(y, f(X)) = O(u(|y|)). \]

Conditions (1) and (2) alone define the notion of a \( O(u) \)-Lipschitz embedding; precisely a \((\lambda, \lambda, v)\)-embedding is a map such that
\[
\forall (x_1, x_2) \in X^2, \lambda |x_1 - x_2| - v(|x| + |x'|) \leq |f(x_1) - f(x_2)| \leq \lambda |x_1 - x_2| + v(|x_1| + |x_2|).
\]

We will give an equivalent definition in subsection 3.A. If there exists an admissible \( u \) such that \( f \) is a \( O(u) \)-sublinearly biLipschitz equivalence (resp. embedding), then \( f \) is called a sublinearly biLipschitz equivalence (resp. embedding). In some occasion, we will abbreviate \((\lambda, \lambda, \lambda)\) into a single biLipschitz constant \( \lambda = \sup \{ \lambda, 1/\lambda \} \) and call \( f \) a \((\lambda, O(u))\)-sublinearly biLipschitz equivalence.

Two metric spaces \( X \) and \( Y \) such that there exists a sublinearly biLipschitz equivalence \( f : X \rightarrow Y \) are called asymptotically biLipschitz in Druțu and Kapovich’s book [14, 10.8].

**1.B. Gromov products and Cornulier’s estimates.** Let \( X \) be a metric space. Recall that for \( x_0, x_1, x_2 \in X \), the Gromov product of \( x_2 \) and \( x_3 \) seen from \( x_0 \) is by definition \( (x_2 \mid x_3)_{x_0} := \frac{1}{2} (|x_1 - x_0| + |x_2 - x_0| - |x_1 - x_2|) \), and that \( X \) is \( \delta \)-hyperbolic (as defined by M.Gromov [20, 1.1.C]) if there exists \( \delta \in \mathbb{R}_{>0} \) such that
\[
(1.1) \quad \forall (x_0, x_1, x_2, x_3) \in X^4, (x_1 \mid x_3)_{x_0} \geq \inf \{(x_1 \mid x_2)_{x_0}, (x_2 \mid x_3)_{x_0}\} - \delta.
\]

If \( X \) is \( \delta \)-hyperbolic and geodesic, then in addition, the Rips inequality is available: triangles in \( X \) are \( 4\delta \)-slim, [19, 2.21]. A Cauchy-Gromov sequence in \( X \) is a sequence \( (x_n)_{n \in \mathbb{Z}_{\geq 0}} \) such that \( (x_n \mid x_m) \rightarrow +\infty \) as \( n, m \rightarrow +\infty \). Two Cauchy-Gromov sequences \( \{x_n\}, \{y_n\} \) are equivalent, denoted \((x_n) \sim (y_n)\), if \( (x_n \mid y_n) \rightarrow +\infty \) as \( n \rightarrow +\infty \). This is an equivalence relation if \( X \) is hyperbolic thanks to (1.1), and the Gromov boundary of \( X \) is \( \partial_{\infty} X = \{ \text{Cauchy-Gromov sequences} \} / \sim \). If \( X \) is in addition proper and geodesic, this is also the visual boundary, or geodesic boundary that we will denote \( \partial_{\infty} X \). Though not stated by Cornulier in this form, the following is given by the proof of his theorem [11, 4.3].

\(^3\) More precisely, at first, objects are pointed metric spaces. Nevertheless the notion does not really depend on a given base-point.
**Proposition 1.5.** Let $u$ be an admissible function. Assume $X$ and $Y$ are hyperbolic, that $X$ is geodesic, and let $f : X \to Y$ be a $O(u)$-Lipschitz embedding. Then $f$ induces a (set-theoretic) boundary map $\partial_G f : \partial_G X \to \partial_G Y$. If $g$ is $O(u)$-close to $f$, then $\partial_G f = \partial_G g$. If $f$ is $O(u)$-surjective, then $\partial_G f$ is a bijection.

This can be expressed quantitatively; we restate below certain estimates from Cornulier’s proof, at the stage when the points that intervene still lie within the space. Whenever $\delta$ is a hyperbolicity constant, set a parameter

$$\mu = \begin{cases} 2^{1/\delta} & \delta > 0 \\ e & \delta = 0, \end{cases}$$

fix a base-point $o \in X$ and define a kernel $\rho_\mu : X \times X \to \mathbb{R}_{\geq 0}$, $\rho_\mu(x,y) := \mu^{-|x-y|_o}$. The $\delta$-hyperbolicity inequality (1.1) translates into a quasi-ultrametric inequality for $\rho_\mu : \rho_\mu(x_0,x_2) \leq \mu^\delta \rho(x_0,x_1) \lor \rho_\mu(x_1,x_2)$ for all $(x_0,x_1,x_2) \in X^3$. This $\rho_\mu$ can be made subadditive by the chain construction:

$$\rho_\mu(x,x') := \inf \left\{ \sum_{i=1}^n \mu^{-|x_{i-1}-x_i|_o} : n \in \mathbb{Z}_{\geq 1}, x = x_0, \ldots x_n = x' \right\}. \quad (1.3)$$

**Lemma 1.6** (Frink 1937, [18, Lemma 2]). Let $\mathcal{X}$ be a set and $\varrho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ be a $\mathbb{R}$-valued kernel on $\mathcal{X}$. Assume there exists $K \in \mathbb{R}_{\geq 1}$ such that for all $(x_0,x_1,x_2) \in \mathcal{X}^3$, $\rho(x_0,x_2) \leq K \rho(x_0,x_1) \lor \rho(x_1,x_2)$. Let $\tilde{\varrho}$ be associated to $\varrho$ by the chain construction (1.3). If $K \leq 2$, then $\tilde{\varrho} \leq \varrho \leq 4\tilde{\varrho}$.

This allows the construction of a true distance $d_\mu = \tilde{\rho}_\mu$ from $\rho_\mu$ on the visual boundary. Subadditivity of the resulting kernel for points within the space plays a key role in the following result.

**Theorem 1.7** (Cornulier). Let $v$ be an admissible function. Let $(\Delta, \varLambda)$ be large-scale expansion and Lipschitz constants. Let $f : (X,o) \to (Y,o)$ be a large-scale $(\Delta, \varLambda, v)$-sublinearly bi-Lipschitz embedding. Assume there exists $\delta \in \mathbb{R}_{\geq 0}$ such that $X$ and $Y$ are $\delta$-hyperbolic and that $X$ is geodesic. For all $\alpha \in (0, \varLambda)$ there exists a constant $M = M(\alpha, \delta) \in \mathbb{R}_{\geq 0}$ and $R = R(\alpha, \lambda, v, \delta, |f(o)|) \in \mathbb{R}_{\geq 0}$ such that for all $x, x' \in X$,

$$\frac{(x | x')_o}{R} \implies (f(x) | f(x'))_o \geq \alpha(x | x')_o - M(\alpha,\delta). \quad (1.4)$$

Especially, if $X$ and $Y$ are proper geodesic, then $\partial_G f = \partial_G \tilde{f}$ is $\alpha$-Hölder continuous for the metrics $d_\mu$ on the boundaries, where $\mu$ is set as in (1.2).

**Remark 1.8.** There is a dependence on $\mu$ in Cornulier’s version which disappears in (1.4) because $\mu$ depends on $\delta$ according to convention (1.2).

A particular instance of theorem 1.7 occurs when the source space is $\mathbb{R}_{\geq 0}$ or $\mathbb{Z}_{\geq 0}$. For the latter, constants $R$ and $M$ can be explicitly extracted from the beginning of Cornulier’s proof:

$$\forall s, t \in \mathbb{Z}_{\geq t_o}, (\bar{\gamma}(s) | \bar{\gamma}(t))_o \geq \alpha \inf \{s, t\} - \log \mu \left( \frac{2}{1-\mu^{-|\alpha+\Delta|^2/2}} \right), \quad (1.5)$$

where $\bar{\gamma}$ replaces $f$ of Lemma 1.7, and

$$t_o = \sup \{ s \in \mathbb{Z}_{\geq 0} : |\bar{\gamma}(0)| + v(s) \geq 4(\Delta - \alpha)s \} \quad (1.6)$$

\footnote{see also Bourbaki [2, IX.6, Proposition 2].}
replaces $R$ of 1.7. This form will be of special interest in subsection 3.B, especially the dependence of $t_\alpha$ on $|\gamma(0)|$ is important for us.

1.C. Metric invariants of 4 points at infinity. Let $(Y, o)$ be a pointed, proper geodesic hyperbolic space, and let $\partial_\infty^4 Y$ denote the space of distinct 4-tuples on $\partial_\infty Y$. For $(\eta_1, \eta_2, \eta_3, \eta_4) \in \partial_\infty^4 Y$, define
\[
\Xi \{\eta_1, \eta_2, \eta_3, \eta_4\} := \sup \{(\eta_i | \eta_j)_o : i \neq j\}, \text{ and}
\]
\[
\chi \{\eta_1, \eta_2, \eta_3, \eta_4\} := \inf \{(\eta_i | \eta_j)_o : i \neq j\}.
\]
More generally, let $(\Xi, \mu)$ be a metric space (to be thought of as a geodesic boundary with a visual distance) and let $(\xi_1, \ldots, \xi_4)$ be distinct points in $\Xi$. Define their metric cross-ratio as
\[
[\xi_1, \xi_2, \xi_3, \xi_4]^\mu = [\xi_1]^\mu := \frac{\rho(\xi_1, \xi_3)\rho(\xi_2, \xi_4)}{\rho(\xi_1, \xi_4)\rho(\xi_2, \xi_3)}.
\]
The superscript $\mu$ might be omitted if sufficiently clear. Observe that if $\rho$ has been obtained by the chain construction (1.3) from a quasi-distance $\hat{\rho}$ on $\Xi$ such that
\[
\exists K \in [1, 2], \forall (\xi_1, \xi_2, \xi_3) \in \Xi^3, \rho(\xi_1, \xi_3) \leq K \sup \{\rho(\xi_1, \xi_2), \rho(\xi_2, \xi_3)\},
\]
then by Frink’s theorem $\hat{\rho} \leq \rho \leq 4\hat{\rho}$, and
\[
(1.7) \quad \forall \nu \in \mathbb{R}_{>1}, \left| \log^{\nu}_{\mu}[\xi] - \log^{\nu}_{\nu}[\xi] \right| \leq \log^{\nu}_\mu 16.
\]
Especially, if $(\Xi, \rho) = (\partial_\infty X, \hat{\rho}_\nu)$ for a $\delta$-hyperbolic, proper geodesic, pointed space $(X, o)$ and a parameter $\nu \in (1, \mu(\delta)]$, then by (1.7), $\log^{\nu}_{\mu}[\xi]^{d\nu}$ depends on $\nu$ only up to an additive error: precisely for all $\nu, \nu' \in (1, \mu]$,
\[
|\log^{\nu}_{\mu}[\xi]^{d\nu} - \log^{\nu'}_{\mu}[\xi]^{d\nu'}| \leq |\log^{\nu}_{\mu}[\xi]^{d\nu} - (\xi, \xi_4)_o - (\xi_2, \xi_3)_o + (\xi_1, \xi_3)_o + (\xi_2, \xi_4)_o|
\]
\[
+ |\log^{\nu'}_{\mu}[\xi]^{d\nu'} - (\xi, \xi_4)_o - (\xi_2, \xi_3)_o + (\xi_1, \xi_3)_o + (\xi_2, \xi_4)_o|
\]
\[
\leq \log^{\nu}_\mu 16 + \log^{\nu'}_\mu 16.
\]
In the sequel we refer to $\log^{\nu}_{\mu}[\xi]^{d\nu}$ as $\log[\xi]^{\nu}$, where $\mu$ follows convention (1.2). If nonnegative, this logarithm has a geometric interpretation:

**Proposition 1.9.** Let $(X, o)$ be a proper geodesic, $\delta$-hyperbolic space. There exists a constant $C = C(\delta)$ in $\mathbb{R}_{\geq 0}$ such that for all $(\xi_1, \ldots, \xi_4) \in \partial_\infty X$,
\[
d_X(\chi_{14}, \chi_{23}) - C \leq \log^{+}[\xi] \leq d_X(\chi_{14}, \chi_{23}) + C.
\]
where $\chi_{ij}$ are geodesic lines between $\xi_i$ and $\xi_j$ (whose existence is provided by the visibility property of $X$, see Ghys and de la Harpe [19, 7.6]).

**Proposition 1.9** seems well-known, yet we could not locate a proof in the literature, so we include one in subsection 2.D. It is better understood as a statement about cross-differences, see Buyalo and Schroeder [5, 4.1].

2. Preliminaries from hyperbolic metric geometry

2.A. A lemma on right-angled quadrilaterals. Let $\delta \in \mathbb{R}_{\geq 0}$ be a constant, and let $X$ be a geodesic $\delta$-hyperbolic metric space. We shall work under the following convention. In the course of proofs or statements about $X$, one often needs to construct objects (e.g., a geodesic segment between two points). The rules of $\delta$-hyperbolic geometry only allow to locate such objects in $X$ up to a few multiples of $\delta$. For us, as soon as an object in $X$ has been constructed, it remains fixed until
the end of the statement or proof so that forthcoming objects can be attached to it. This means that, for instance, if a geodesic segment between two points has been previously defined, then the midpoint of these points will be understood as the midpoint of this geodesic segment. Especially, if $\gamma \subset X$ is a geodesically convex subspace and $b \in X$ is a point, $p_\gamma(b)$ is an orthogonal projection (closest point) of $b$ on $\gamma$. This is well defined up to $16\delta$, and $p_\gamma$ has a contracting behavior on distances expressed by the following lemma.

**Lemma 2.1** (See Shchur, [27, Lemma 1]) Let $\gamma$ be a geodesic, $b$ a point in $X$. Then for all $c \in \gamma$, $|c - p_\gamma(b)| \leq |b - c| - |b - p_\gamma(b)| + 16\delta$. In particular, for all $b, b' \in X$,

$$
|p_\gamma(b) - p_\gamma(b')| \leq |b - b'| + 16\delta.
$$

**Definition 2.2.** Let $\alpha \in \mathbb{R}_{>0}$. Say that a metric space $P$ is $\alpha$-connected if for any $\alpha' \in \mathbb{R}_{>\alpha}$, the equivalence relation generated by $[d(x,y) \leq \alpha']$ over $x, y \in P$ has a unique class.

**Lemma 2.3.** Let $\alpha \in \mathbb{R}_{>0}$ and let $S \subset X$ be an $\alpha$-connected subspace (for instance a quasigeodesic). Let $\gamma$ be a geodesic of $X$. Then any $p_\gamma(S)$ is $(\alpha + 16\delta)$-connected. In particular if $S$ is a geodesic then $p_\gamma(S)$ is $16\delta$-connected.

**Proof.** Let $S' = p_\gamma(S)$ and let $\alpha' \in \mathbb{R}_{>0}$ be such that $\alpha' > \alpha + 16\delta$. If there is $s_1' = p_\gamma(s_1)$ such that $d(s_1', S' \setminus \{s_1'\}) \geq \alpha'$, then for all $s_2' = p_\gamma(s_2)$, $|s_1' - s_2'| > \alpha'$ implies with (2.1), that $s_1 - s_2 > \alpha'$. Thus $S$ is not $\alpha$-connected. \hfill $\square$

**Definition 2.4.** Let $\eta \in \mathbb{R}_{>0}$ be a constant and let $X$ be a geodesic space. Say that an ordered list $x_1, \ldots, x_r$ of points in $X$ with $r \geq 3$ is $\eta$-almost lined up if there exists a geodesic segment $\sigma$ such that for all $i$, $x_i$ lies in the $\eta$-neighborhood $N_\eta(\sigma)$ of $\operatorname{im}(\sigma)$ and the $p_\sigma(x_i)$ are lined up in this order on $\sigma$.

**Lemma 2.5** (Gromov product of almost lined up points). Let $\eta \in \mathbb{R}_{>0}$ and let $x_1, x_2, x_3$ be three points in a geodesic metric space $X$ which are $\eta$-almost lined up. Then

$$
|(x_2 \mid x_3)_{x_1} - |x_1 - x_2|| \leq 5\eta.
$$

**Proof.** Let $\sigma$ be a geodesic segment achieving the almost-lined upness assumption. For $i \in \{1, 2, 3\}$, let $y_i = p_\sigma(x_i)$. By hypothesis $|x_i - y_i| \leq \eta$, so by the triangle inequality $||y_i - y_j| - |x_i - x_j|| \leq 2\eta$; then by definition of the Gromov product $|(x_2 \mid x_3)_{x_1} - (y_2 \mid y_3)_{y_1}| \leq 3\eta$. Finally, $y_1, y_2$ and $y_3$ are lined up, hence $(y_2 \mid y_3)_{y_1} = |y_1 - y_2|$. The conclusion follows from the triangle inequality in $\mathbb{R}$. \hfill $\square$

**Lemma 2.6** (Right-angled triangles degenerate). Let $\sigma$ be a geodesic of a geodesic hyperbolic space $X$, $b \in X$ and $a = p_\sigma(b)$ on $\sigma$. Let $c$ be a point of $\sigma$. Then there exists $t \in [bc]$ such that

1. $|a - t| \leq 28\delta$
2. $d(t, \sigma) \leq 4\delta$ and $d(t, [ba]) \leq 4\delta$.
3. for any $u$ in the subsegment $[tc]$ of $[bc]$, $d(u, \sigma) \leq 4\delta$.

---

5We will abusively write $\gamma$ when referring to $\operatorname{im}(\gamma)$ when $\gamma$ is a (quasi)geodesic.

6There is a $4\delta$ additive error term instead of our $16\delta$ in Shchur’s version, because Shchur defines a $\delta$-hyperbolic space via Rips inequality there.
In particular, if $|b-a|$, $|c-a|$ are large enough, then $b, a, c$ are $28\delta$-almost lined up in this order.

**Proof.** Let $\triangle$ be the geodesic triangle $abc$ with sides $[ba]$, $[bc]$ and the subsegment $[ac]$ of $\sigma$. Set $\ell = [b-c]$ and assume $\alpha : [0, \ell] \rightarrow X$ parametrizes $[bc]$ so that $\alpha(0) = c$, $\alpha(\ell) = b$. If $\sup \{d(\alpha(s), \sigma) : s \in [0, \ell] \} \leq 4\delta$, set $t = b$; then (2) is automatically true, while $|a-t| = d(t, \sigma) \leq 4\delta \leq 28\delta$ so that also (1) is true. Otherwise, define

$$t = \alpha(s), s = \inf \{ u \in [0, \ell], d(\alpha(u), \sigma) > 4\delta \}.$$ 

As $\triangle$ is $4\delta$-slim, $d(t, [ba]) \leq 4\delta$ while $d(t, \sigma) \leq 4\delta$ also. Let $t_b$, resp. $t_c$ be an orthogonal projection of $t$ on $\sigma$, resp. on $[ba]$. By the triangle inequality, $|t_c-t_b| \leq 4\delta + 4\delta = 8\delta$. Then $|t_b-b| \leq |t_c-b| + 8\delta \leq |b-a| + 8\delta$. By the contraction Lemma 2.1, $|t_b-a| \leq 8\delta + 16\delta = 24\delta$. By the triangle inequality, $|t-a| \leq 24\delta + 4\delta = 28\delta$.

**Lemma 2.7** (Quadrilaterals with two consecutive right-angles degenerate). Let $a_0, a_1, b_0, b_1$ be four points in $X$. For $i \in \{0, 1\}$, let $\gamma_i$ be a geodesic segment between $a_i$ and $b_i$. Assume that $138\delta \leq |a_0-a_1|$, and that one of the following holds:

1. Either, $a_i = p_\sigma(b_i)$ for all $i \in \{0, 1\}$, or
2. $a_i = p_{\gamma_i}a_{1-i}$ for all $i \in \{0, 1\}$.

Then for all $i \in \{0, 1\}$, $d(a_i, [b_0b_1]) \leq 56\delta$.

**Proof.** Let $\sigma$ be a geodesic segment between $a_0$ and $a_1$, and let $m$ be the midpoint of $\sigma$. By Lemma 2.6, there exists $t_0$ and $t_1$ on $[b_0m]$ and $[b_1m]$ respectively such that

$$\forall i \in \{0, 1\}, |a_i-t_i| \leq 28\delta.$$ 

Moreover, by (2) of Lemma 2.6 and the triangle inequality,

$$|p_\sigma(t_i) - a_i| \leq |p_\sigma(t_i) - t_i| + |t_i - a_i| \leq 4\delta + 28\delta = 32\delta.$$ 

Thus $a_i, p_\sigma(t_i), m$ and $a_{1-i}$ are lined up on $\sigma$ as below:

$$\begin{array}{ccccccc}
\sigma & 32\delta & 16\delta & 16\delta & 32\delta \\
a_0 & p_\sigma(t_0) & m & p_\sigma(t_1) & a_1
\end{array}$$

Next, we proceed to prove that $t_i$ is far from $[mb_{1-i}]$. Note that since the triangles $ma_i b_i$ are slim, one need only show that $t_i$ is far from $[a_{1-i}b_{1-i}]$ and $[ma_{1-i}]$.

- In case (1), for all $a'_{1-i} \in \gamma_i$, since $p_\sigma(a'_i) = a_i$ and by (2.4) and Lemma 2.1,

  $$|t_i-a'_{1-i}| \geq |p_\sigma(t_i)-a_{1-i}| - 16\delta \geq |a_i-a_{1-i}| - |p_\sigma(t_i)-a_i| - 16\delta \geq 138\delta - 48\delta = 90\delta,$$

  hence $d(t_i, \gamma_i) \geq 90\delta$.

- In case (2), as $a_{1-i} = p_{\gamma_i-a_i}$, $d(t_i, \gamma_i) \geq d(a_i, \gamma_{1-i}) - 28\delta \geq 110\delta$.

- In both cases, $d(t_i, [ma_{1-i}]) \geq 79\delta - 32\delta = 45\delta$.

Using the previous inequality together with the fact that the triangle $a_{1-i}mb_{1-i}$ is $4\delta$-slim,

$$d(t_i, [mb_{1-i}]) \geq 45\delta - 4\delta = 41\delta > 4\delta.$$ 

Finally, $b_1mb_2$ is $4\delta$-slim, hence $d(t_i, [b_1b_2]) \leq 4\delta$, and by the triangle inequality,

$$d(a_i, [b_0b_1]) \leq |a_i-t_i| + d(t_i, [b_0b_1]) \leq 28\delta + 4\delta \leq 56\delta.$$

$\square$
2.B. An estimate on geodesic projections. Let $X$ be as before a geodesic $\delta$-hyperbolic metric space, and fix a base-point $o \in X$.

**Lemma 2.8.** Let $\gamma, \gamma' : \mathbb{R} \to X$ be two geodesics; define $\xi_- = [\gamma]_{-\infty}$, $\xi_+ = [\gamma]_{+\infty}$, $\xi'_- = [\gamma']_{-\infty}$ and $\xi'_+ = [\gamma']_{+\infty}$ on the boundary at infinity $\partial_X X$ of $X$. Assume that the $\xi_\pm, \xi'_\pm$ are all distinct. Then

$$(2.5) \sup \{|p_\gamma(b) : b \in \gamma'\} \leq \overline{\mathcal{M}}(\xi_-, \xi_+, \xi'_-, \xi'_+) + 284\delta,$$

where we recall that $\overline{\mathcal{M}}(\xi_-, \xi_+, \xi'_-, \xi'_+)$ is an abbreviation for $\sup(\xi_1 | \xi_2)_o$ over distinct pairs $\{\xi_1, \xi_2\}$ in $\{\xi_\pm, \xi'_\pm\}$.

**Proof.** Change if necessary the parametrizations of $\gamma$ and $\gamma'$ in such a way that $\gamma(0) = p_\gamma(o)$, $\gamma'(0) = p_{\gamma'}(o)$. Let $b \in \gamma'$.

- Either $|p_\gamma(b) - \gamma(0)| < 138\delta$; then by the triangle inequality, $|p_\gamma(b)| < |\gamma(0)| + 138\delta$. Let $s \in \mathbb{R}$. Since $X$ is $\delta$-hyperbolic,

$$(2.6) (\gamma(s) | (\gamma(s))_o \geq \min \{|(\gamma(0) | \gamma(0))_o, (\gamma(0) | \gamma(s))_o\} - \delta.$$

By Lemma 2.6, when $s$ is large enough $o$, $\gamma(0)$ and $\gamma(s)$ (resp. $o$, $\gamma(0)$ and $\gamma(s)$) are $28\delta$-almost lined up in this order, so by Lemma 2.5, (2.6) becomes

$$|\gamma(s) | (\gamma(s))_o \geq |\gamma(0)| - 5 \cdot 28\delta - \delta = |\gamma(0)| - 114\delta.$$

Finally, $|p_\gamma(b)| < |\gamma(0)| + 138\delta \leq (\gamma(s) | (\gamma(s))_o + 138\delta + 141\delta$. Letting $s \to +\infty$,

$$|p_\gamma(b)| \leq \lim_{s \to +\infty} \inf (\gamma(s) | (\gamma(s))_o \leq (\xi_- | (\xi_+)_o + 279\delta$$

$$\leq (\xi_- | (\xi_+)_o + 284\delta.$$

- Or $|p_\gamma(b) - \gamma(0)| \geq 138\delta$ in which case Lemma 2.7 applies so that $o$, $\gamma(0)$ and $p_\gamma(b)$, $b$ are $56\delta$-almost lined up in this order. Let $s, s' \in \mathbb{R}$ be such that $\inf\{|s|, |s'|\} \geq \sup\{|p_\gamma(b) - \gamma(0)|, |b - \gamma'(0)|\}$. Then

$$(\gamma(s) | (\gamma'(s'))_o \geq \min \{|(\gamma(s) | (\gamma(0))_o, (\gamma(0), p_\gamma(b))_o, \gamma'(0))_o, (\gamma'(0) | (\gamma'(s'))_o - 4\delta.$$

Applying repeatedly Lemma 2.5,

$$(\gamma(s) | (\gamma'(s'))_o \geq \min \{||\gamma(0)| - 140\delta, |\gamma(0)| - 140\delta, |p_\gamma(b)| - 5 \cdot 56\delta, |\gamma'(0)| - 140\delta, |\gamma'(0)| - 140\delta\} - 4\delta.$$

\[\text{Figure 1. Main points occurring in the proof of Lemma 2.7.}\]
Figure 2. Configuration of Lemma 2.8 in the half-plane model of \( \mathbb{H}^2 \).

Now letting \( s, s' \to \pm \infty \),

\[
|p_\gamma(b)| \leq \mathbb{M} \{ \xi_-, \xi'_-, \xi_+, \xi'_+ \} + 5 \cdot 56 \delta + 4 \delta
\]

\[
= \mathbb{M} \{ \xi_-, \xi'_-, \xi_+, \xi'_+ \} + 284 \delta.
\]

□

2.C. Quantitative Morse stability. We prove here a version of the Morse lemma with an emphasis on the linear dependence of the tracking distance on the quasi-isometry additive error term.

Lemma 2.9 (Morse stability for quasigeodesics). Let \( c, \delta \in \mathbb{R}_{\geq 0} \), \( (\lambda, \lambda') \in \mathbb{R}^2_{>0} \) be constants. Let \( X \) be a geodesic, \( \delta \)-hyperbolic metric space. Let \( J = [a, b] \) be a closed bounded interval of \( \mathbb{R} \) and let \( \tilde{\gamma} : J \to X \) be \((\lambda, \lambda', c)\) quasigeodesic, i.e.

\[
\forall (s, t) \in J^2, A|s - t| - c \leq |\tilde{\gamma}(s) - \tilde{\gamma}(t)| \leq A|s - t| + c.
\]

Recall that \( \lambda = \sup \{ \lambda, 1/\lambda \} \), and assume that\(^7\) \( c \geq 6 \lambda^2 \delta \). There exist functions \( h, \tilde{h} : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) such that if \( \gamma : [0, |\tilde{\gamma}(b) - \tilde{\gamma}(a)|] \to X \) is any geodesic segment with same endpoints as \( \tilde{\gamma} \), then

\[
\forall t \in J, d(\tilde{\gamma}(t), \im(\gamma)) \leq h(\lambda)(\delta + c)
\]

\[
\forall s \in [0, |\tilde{\gamma}(b) - \tilde{\gamma}(a)|], d(\gamma(s), \im(\tilde{\gamma})) \leq \tilde{h}(\lambda)(\delta + c).
\]

Precisely, \( h \) and \( \tilde{h} \) can be taken as \( h(\lambda) = 12(1 + 8 \lambda^2) \) and \( \tilde{h}(\lambda) = 16(5 + 6 \lambda^2) \).

Remark 2.10. Our expression for \( \tilde{h}(\lambda) \) is certainly not optimal: Shchur [27, Theorem 2] claims that \( \tilde{h}(\lambda) = O(\log \lambda) \). For us in the following, only the linear dependence over the sum of additive errors \( \delta + c \) in (2.7) and (2.8) matters.

Proof. A sketch of proof for the part of lemma expressed by (2.7) can be found in Thurston’s exposition of the Mostow rigidity theorem. [28, 5.9.2] with non-explicit right-hand side bound; see also an early (and more explicit) proof by Efremovich and Tihomirova [16, p. 1142–1143], also taking place in \( \mathbb{H}^n_{\mathbb{R}} \). When projecting onto

---

\(^7\)This assumption could be dropped; we make it in order to simplify \( h \) and \( \tilde{h} \), and because \( c \) is to be replaced by an unbounded function \( v \) in the next section.
a geodesic line in hyperbolic space, the lengths of curves situated at a distance \( \eta \) are contracted with a factor depending exponentially\(^8\) on \( \eta \), so that the length of portions of quasigeodesics leaving a tube of thickness \( \eta \) around a geodesic can be bounded. This can be carried into a general argument in \( \delta \)-hyperbolic space, replacing length by a rough analogue; for this we build on Shchur’s work \cite{Shchur1990}. For \( \alpha \in \mathbb{R}_{>0}, I \subset \mathbb{R} \) a bounded interval and \( \sigma : I \to X \) a curve such that \( \sigma(I) \) is \( \alpha/2 \)-connected, define the length of \( \sigma \) at scale \( \alpha \) as

\[
\ell_\alpha(\sigma) = \sup_{(t_i) \in T_\alpha(\sigma)} \sum_i |\sigma(t_i + 1) - \sigma(t_i)|,
\]

where \((t_i) \in T_\alpha(\sigma)\) if there is \( r \in \mathbb{Z}_{\geq 0} \) such that \( \inf I = t_0 < \cdots < \sup I = t_r \) and if \( \{\sigma(t_i)\} \) is a \( \alpha \)-separated net in \( \text{im}(\sigma) \). If \( \sigma \) is a \( (\Delta, \lambda, c) \)-quasi-geodesic segment (e.g. a portion of \( \tilde{\gamma} \)) and \( \alpha \) is such that \( \alpha \geq 2c \), then

\[
\ell_\alpha(\sigma) \leq 2\Delta |I|,
\]

see Shchur \cite[Lemma 7]{Shchur1990}. Now let \( \eta \) be a positive real number (to be fixed later). Define \( N_{\eta} \gamma \) as the \( \eta \)-neighborhood of \( \text{im}(\gamma) \) in \( X \), and

\[
U_\eta = \{ t \in J, \tilde{\gamma}(t) \notin N_{\eta} \gamma \}.
\]

Let \( I = \pi_0(U_\eta) \), \( t = \inf I \) and \( t' = \sup I \). \( t \) and \( t' \) are both finite, since \( J \) is bounded and \( \tilde{\gamma} \) and \( \gamma \) have the same endpoints. Then \( \tilde{\gamma}_{[t,t']} \) is outside \( N_{\eta} \gamma \); by Shchur’s exponential contraction estimate\(^9\) \cite[Lemma 10]{Shchur1990}, there exists a constant \( S \in \mathbb{R}_{>0} \) such that, as soon as \( \eta \geq 2c + 12\delta \),

\[
|p_\gamma \tilde{\gamma}(t) - p_\gamma \tilde{\gamma}(t')| \leq \sup \left\{ \frac{6\delta}{c} e^{-S\eta \ell_{2\Delta} \tilde{\gamma}_{[t,t']}}, 24\delta \right\}
\]

\[\leq 24\delta + \frac{6\delta}{c} e^{-S\eta \ell_{2\Delta} \tilde{\gamma}_{[t,t']}},\]

(2.10)

On the other hand,

\[
\ell_{2\Delta} \tilde{\gamma}_{[t,t']} \leq 2\Delta |t' - t| \leq 2(\Delta/\lambda) ||\tilde{\gamma}(t') - \tilde{\gamma}(t)|| + c
\]

\[\leq 2\lambda^2 [2\eta + |p_\gamma \tilde{\gamma}(t) - p_\gamma \tilde{\gamma}(t')| + c],\]

(2.11)

\(^8\)It is useful to write the hyperbolic metric in cylindrical coordinates around \( \gamma \) to appreciate that the contraction factor is a hyperbolic cosine of \( \eta \).

\(^9\)Shchur’s lemma is actually stated in a slightly different form, namely our \( |p_\gamma \tilde{\gamma}(t) - p_\gamma \tilde{\gamma}(t')| \) is replaced by \( \text{diam} p_\gamma \tilde{\gamma}(t) \), and follows a different convention on the \( \delta \) hyperbolicity constant.
where we used the triangle inequality together with the fact that $\tilde{\gamma}(t), \tilde{\gamma}(t') \in \partial \mathcal{N}_\eta \gamma$ for the last inequality. Combining (2.10) and (2.11),

$$\frac{1}{2\lambda^2} \ell_{2c\tilde{\gamma}|[t,t']} \leq 2\eta + |p_\gamma \tilde{\gamma}(t) - p_\gamma \tilde{\gamma}(t')| + c$$

$$\leq 2\eta + 24\delta + \frac{6\delta}{c} e^{-S\eta} \ell_{2c\tilde{\gamma}|[t,t']} + c,$$

hence

(2.12) $$\left(\frac{1}{2\lambda^2} - \frac{6\delta}{c} e^{-S\eta}\right) \ell_{2c\tilde{\gamma}|[t,t']} \leq 2\eta + 24\delta + c.$$

Define $\eta_I = \sup_{u \in I} d(\tilde{\gamma}(u), \gamma)$. Then, as $c > 3\delta \lambda^2$ by hypothesis,

(2.13) $$\eta_I \leq \eta + \frac{1}{2} \ell_{2c\tilde{\gamma}|[t,t']} \vee 2c \leq \eta + 2c + \frac{2\eta + 24\delta + c}{1/\lambda^2 - (3\delta/c) \cdot e^{-S\eta}}.$$

It remains to set $\eta$ in order to explicit the bound on $\eta_I$ given by the last inequality. Actually, as $c \geq 6\delta \lambda^2$, if $\eta = 2c + 12\delta$ (remember that $\tilde{\gamma}_{|I}$ must be at least this far for the exponential contraction to operate),

$$\eta_I \leq \eta + 2c + \frac{2\eta + 24\delta + c}{1/(2\lambda^2)}$$

$$\leq 12\delta + 4c + \lambda^2 (4\eta + 48\delta + c)$$

$$= 12\delta + 4c + \lambda^2 (96\delta + 5c) \leq 12(1 + 8\lambda^2)(\delta + c).$$

Finally,

$$\sup \{d(\tilde{\gamma}(t), \gamma) : t \in J\} = \eta \vee \sup_{t \in \pi_0 U_n} \eta_I \leq 12(\delta + c) \vee 12(1 + 8\lambda^2)(\delta + c)$$

$$= 12(1 + 8\lambda^2)(\delta + c).$$

This is (2.7). Now, let $s \in [0, |\tilde{\gamma}(b) - \tilde{\gamma}(a)|)$. Because $\tilde{\gamma}$ is $c$-connected, by Lemma 2.3 $p_\gamma \tilde{\gamma}$ is $c + 16\delta$-connected, so there is $s' \in [0, |\tilde{\gamma}(b) - \tilde{\gamma}(a)|]$ such that $|s' - s| \leq c + 16\delta$ and $s' = p_\gamma (\tilde{\gamma}(t))$ for a $t \in J$. The triangle inequality in $X$ yields

$$d(\gamma(s), \im(\tilde{\gamma})) \leq |\gamma(s) - \tilde{\gamma}(t)| \leq |s - s'| + |\gamma(s') - \tilde{\gamma}(t)|$$

$$\leq 12(1 + 8\lambda^2)(\delta + c) + 16\delta + c \tag{2.7}$$

$$\leq 16(5 + \lambda^2)(\delta + c).$$

This is (2.8). \hfill \Box

Remark 2.11. V. Shchur [27, Theorem 1] claims a stronger result. However the proof in [27] has a gap, noticed by S. Gouëzel and recently fixed by Gouëzel and Shchur.

2.D. Proof for Proposition 1.9. Let $\xi_1, \ldots, \xi_4$ be as in the statement of Proposition 1.9 and assume that the geodesic lines $\chi_{14}$ and $\chi_{23}$ are parametrized in such a way that a common perpendicular geodesic segment $\sigma$ falls on $\chi_{14}(0)$ and $\chi_{23}(0)$, accordingly to figure 4. Let $H$ be the metric subspace of $X$ defined as $\chi_{14} \cup \chi_{23} \cup \sigma$ and denote by $|.|_H$ the path distance in $H$. By Lemma 2.7 (1), if $d(\chi_{14}, \chi_{23}) \geq 138\delta$ then for all $t \in \mathbb{R}$, whenever $(\chi, \chi') \in \{\chi_{14}, \chi_{23}\}^2$,

(2.14) $$|\chi(t) - \chi'(ct)| - |\chi(t) - \chi'(ct)|_H | \leq 4 \cdot 56\delta = 212\delta.$$
can be deduced

\[ v \]

See Bourdon’s remark [14].

For all \( t \in \mathbb{R} \) (compare Buyalo and Schroeder [5, p. 37]),

\[
\begin{align*}
2 \left\{ \begin{array}{l}
\langle \chi_{14}(-t) | \chi_{14}(t) \rangle_o \\
+ \langle \chi_{23}(t) | \chi_{23}(-t) \rangle_o \\
- \langle \chi_{14}(-t) | \chi_{23}(t) \rangle_o \\
- \langle \chi_{14}(t) | \chi_{23}(-t) \rangle_o
\end{array} \right\} = \left\{ \begin{array}{l}
-|\chi_{14}(-t)| + |\chi_{14}(t)| - 2t \\
+|\chi_{23}(-t)| + |\chi_{23}(t)| - 2t \\
-|\chi_{14}(-t)| + |\chi_{14}(t)| - \chi_{23}(t)| \\
-|\chi_{14}(t)| - |\chi_{23}(-t)| + \chi_{23}(t)|
\end{array} \right\}
\]

(2.15)

\[
-4t + |\chi_{14}(-t) - \chi_{23}(t)| + |\chi_{14}(t) - \chi_{23}(-t)|.
\]

By (2.14), there is \( \Delta \) with \( |\Delta| \leq 2 \cdot 212\delta = 424\delta \) such that

\[
\begin{align*}
-4t + |\chi_{14}(-t) - \chi_{23}(t)| + |\chi_{14}(t) - \chi_{23}(-t)| \\
= -4t + |\chi_{14}(-t) - \chi_{23}(t)|_H + |\chi_{14}(t) - \chi_{23}(-t)|_H + \Delta
\end{align*}
\]

(2.16)

\[
= 2d(\chi_{14}, \chi_{23}) + \Delta.
\]

On the other hand, by (1.7),

\[
\left| \log_{1/2} |\xi| - \lim_{t \to +\infty} \left\{ \langle \chi_{14}(-t) | \chi_{14}(t) \rangle_o + \langle \chi_{23}(t) | \chi_{23}(-t) \rangle_o \right\} \right| \leq 8\delta + \log \mu 16.
\]

If \( d(\chi_{14}, \chi_{23}) \) is large enough, letting \( t \to +\infty \) in (2.15) combined with the estimate (2.16), we reach the desired inequality of Proposition 1.9. This is valid for small values as well since \( \log^{-} \) then takes small values.

Remark 2.12. The right-hand side inequality of Proposition 1.9 can be deduced from the elementary case of a metric tree via tree approximation [19, Theorem 2.12]. See Bourdon’s remark [4, 2.3].

3. Sublinear tracking

Sublinearly biLipschitz embeddings of the real half-line, resp. of the real line admit tracking by geodesic rays, resp. lines; we prove this in 3.B, resp. 3.C. In the spirit of (2.7) and (2.8), the bound on the tracking distance can be expressed as a constant (denoted \( H, \bar{H} \)) times the additive error function \( v \), however at the cost of being valid only farther than a given tracking radius. The tracking constants and the tracking radii depend on \( v \), more precisely through its large-scale features \( v \uparrow \tau \), \( r_v(\tau) \) and \( \sup \{ r : v(r) \leq \text{cst}(\lambda, \delta, \ldots) \} \) described in 1.A. While the use of tracking
radii allow tracking estimates to take a particularly simple form when applied in 3.D, their dependence upon \( v \) must not be kept entirely implicit, especially it must be taken into account for later use in section 4 when \( v \) becomes a parameter, a task undertaken in 3.E.

3.A. Preliminaries. Unless otherwise stated, geodesic rays into a pointed metric space are assumed to have their origin at the base-point. This convention will not apply to the rougher \( O(v) \)-rays that we define hereafter.

**Definition 3.1.** Let \( u \) be an admissible function and \( X \) a metric space. A \( O(u) \)-geodesic, resp. a \( O(u) \)-ray in \( X \) is a \( O(u) \)-sublinearly biLipschitz embedding \( R \to X \), resp. \( R_{\geq} \to X \).

When \( u = 1 \), this is the classical notion of a quasi-geodesic, resp. of a quasi-geodesic ray. By definition, \( O(u) \)-geodesics, resp. \( O(u) \)-rays, are sent to \( O(u) \)-geodesics resp. \( O(u) \)-rays when one applies a \( O(u) \)-sublinearly biLipschitz embedding to the space. \( O(u) \)-geodesics behave like quasi-geodesic inside every ball, with an additive error parameter controlled by the radius; however the containing ball sits in the target space, so that the dependence of the additive error on radius only becomes apparent on the large scale. We turn this observation into a lemma, which may be considered as an alternative definition for large-scale Lipschitz embeddings, easier to handle through certain technical steps.

**Lemma 3.2.** Let \( u \) be an admissible function. Let \( (\lambda, \lambda) \) be Lipschitz constants, let \( v = O(u) \) be nondecreasing, and let \( f : (X, o) \to (Y, o) \) be a large-scale \( (\lambda, \lambda, v) \)-biLipschitz embedding. Then there exist \( \hat{\nu} = O(u) \), \( t_{\circ} \in R_{\geq} \) and \( R_{\circ} \in R_{\geq} \) (depending on \( f \) and \( v \)) such that for all \( x, x_1, x_2 \in X \)

1. If \( x \notin B(o, t_{\circ}) \) or \( f(x) \notin B(o, R_{\circ}) \) then
   \[
   \frac{1}{3\lambda}|x| \leq |x| \land |f(x)| \leq 3\lambda|x|.
   \]

2. If \( x_1, x_2 \in X \setminus B(o, t_{\circ}) \) or \( f(x_1), f(x_2) \in Y \setminus B(o, R_{\circ}) \), then
   \[
   \begin{cases}
   |f(x_1) - f(x_2)| \leq \lambda|x_1 - x_2| + \hat{\nu}(\{|x_1| \lor |x_2|\} \land (|f(x_1)| \lor |f(x_2)|)), \\
   |f(x_1) - f(x_2)| \geq \frac{1}{3\lambda}|x_1 - x_2| - \hat{\nu}(\{|x_1| \lor |x_2|\} \land (|f(x_1)| \lor |f(x_2)|)).
   \end{cases}
   \]

Moreover \( t_{\circ}, R_{\circ} \) and \( \hat{\nu} \) may be taken as:

1. \( t_{\circ}(|f(o)|, v) = \sup \{ r : v(r) \geq \frac{r}{3\lambda} \} \lor 3\lambda|f(o)| = r_{1/(3\lambda)}(v) \lor 3\lambda|f(o)|, \)
2. \( R_{\circ}(|f(o)|, v) = 4|f(o)| \lor 2(2\lambda + 1)t_{\circ}(|f(o)|, v) \), and
3. \( \hat{\nu} = (v \uparrow 3\lambda)v. \)

**Proof.** By definition of \( t_{\circ}(|f(o)|, v) \), for all \( x \in X \setminus B(o, t_{\circ}) \), \( |f(o)| \leq 1/(3\lambda)|x| \) and \( v(|x|) \leq \frac{1}{3\lambda}|x| \), so \( \frac{1}{3\lambda}|x| \leq |f(x)| \leq (\lambda + \frac{1}{3\lambda})|x| \leq 3\lambda|x| \); this is the first case in (I). Now assume that \( R_{\circ} \) is defined as in (3.2). Note that \( R_{\circ} \geq 2r_{1/(3\lambda)}(v) \geq r_{1/2}(v) \) so that if \( f(x) \in Y \setminus B(o, R_{\circ}(|f(o)|, v)) \) then
   \[
   |x| \geq \lambda^{-1}(|f(x)| - |f(o)| - v(|x|)) \geq \begin{cases}
   \lambda^{-1}(|f(x)| - |f(o)| - v(|f(x)|) & \text{if } |x| \leq |f(x)|, \\
   \lambda^{-1}(|f(x)| - |f(o)| - |x|/2) & \text{if } |x| \geq |f(x)|.
   \end{cases}
   \]
Lemma 3.3. Let $\gamma$ be a proper geodesic hyperbolic space, and $\gamma' : \mathbb{R} \to Y$ a $O(u)$-geodesic ray. Inequality (1.5) says in particular that $\{\gamma'(t)\}_{t \in \mathbb{Z}_{\geq 0}}$ is a Cauchy-Gromov sequence. Since $Y$ is proper and geodesic, its Gromov boundary is equal to $\partial_\infty Y$ and there exists a geodesic ray $\gamma : (\mathbb{R}_{\geq 0},0) \to (Y,o)$ such that $\eta := [\gamma] = \partial_\infty \gamma(+)$. We will prove that $\gamma$ actually tracks $\gamma'$, in the sense that the growth of distance between them is in the $O(u)$-class. We need a preliminary lemma.

Lemma 3.3. Let $\delta \in \mathbb{R}_{\geq 0}$, and let $(Y,o)$ be a proper geodesic $\delta$-hyperbolic space. Let $\gamma : \mathbb{R} \to Y$ be a geodesic ray into $Y$, and let $\gamma'$ be a non-pointed geodesic ray asymptotic to $\gamma$, i.e. $[\gamma] = [\gamma'] \in \partial_\infty Y$. Then for all $s \in \mathbb{R}_{\geq 0}$ such that $s \geq |\gamma'(0)| + 16\delta$,

$$d(\gamma'(s), \im(\gamma)) \leq 8\delta.$$

Proof. This is a classical result in hyperbolic metric geometry, use for instance the proof of (ii) $\implies$ (iii) in [19, Proposition 7.1] with appropriate changes of notation, and replace Ghys and de la Harpe’s $D$ with $\sup \{|\gamma'(0)|, 16\delta\}$. □

Lemma 3.4 (Sublinear tracking for rays). Let $v$ be an unbounded admissible function. Let $(Y,o)$ be a proper, geodesic, pointed metric space. Assume there exists $\delta \in \mathbb{R}_{\geq 0}$ such that $Y$ is $\delta$-hyperbolic. Let $(\underline{\lambda}, \lambda) \in \mathbb{R}_{\geq 0}^2$ be Lipschitz data, and let $\tilde{\gamma} : \mathbb{R}_{\geq 0} \to Y$ be a $(\underline{\lambda}, \lambda,v)$-ray. Let $\eta \in \partial_\infty Y$ be the endpoint of $\tilde{\gamma}$, and let $\gamma$ be any geodesic ray such that $[\gamma] = \eta$. Then there exist constants $H, \tilde{H} \in \mathbb{R}_{>0}$, $t_\infty, R_\infty \in \mathbb{R}_{\geq 0}$ such that for all positive real $t$ and $s$,

\begin{align*}
& t \geq t_\infty \implies d(\tilde{\gamma}(t), \gamma) \leq Hv(t) \\
& s \geq R_\infty \implies d(\gamma(s), \im(\tilde{\gamma})) \leq \tilde{H}v(s),
\end{align*}

where $H$ and $\tilde{H}$ depend on $\lambda$ and $v$ only, while $t_\infty$ and $R_\infty$ can be decomposed into

\begin{align*}
& t_\infty = t_\infty^0(\lambda, v, \delta) + 2\lambda|\gamma(0)| \\
& R_\infty = R_\infty^0(\lambda, v, \delta) + |\gamma(0)|.
\end{align*}

Remark 3.5. In view of Lemma 2.9, it does matter for us that $v$ be unbounded. If $v$ is bounded, though, $\tilde{\gamma}$ is a quasi-geodesic ray and the same result classically holds, see for instance Ghys and de la Harpe [19, 5.25], with extra additive terms in the estimates (3.4) and (3.5).

Remark 3.6. It is important to make the dependence of the tracking radius $R_\infty^0$ upon the function $v$ explicit, at least to some extent. However, in order not to overload the current proof, we reconstruct it separately (but along with other tracking radii) in subsection 3.E, and only keep record of the steps needed for its definition here, with enough details to ensure that it only depends on $\lambda, v$ and $\delta$. 
Sketch of proof for Lemma 3.4. For every $t \in \mathbb{R}_{\geq 0}$, set a real positive $T$ large enough according to $t$ so that (1.5) ensures the Gromov product $(\tilde{\gamma}(T), \eta)_o$ is significantly greater than $|\tilde{\gamma}(t)|$, and use the stability lemma 2.9 to prove that $\tilde{\gamma}(t)$ is not far from the geodesic segment $\gamma_T$ between $o$ and $\tilde{\gamma}(T)$. Here keeping an efficient inequality requires that $T$ stay within linear control of $t$, which can be done consistently with the antagonist constraint of (1.5). Further, show that the projection of $\tilde{\gamma}(t)$ on $\gamma_T$ is close to $\gamma$, using the slim triangle $o\tilde{\gamma}(T)\eta$, see Figure 5. Finally, (3.5) is deduced from (3.4) with a metric connectedness argument in the same way that (2.8) was deduced from (2.7) in the proof of Lemma 2.9.

Proof of lemma 3.4. Setting $\alpha = \frac{\Lambda}{2}$ in the Gromov product estimate (1.5) and letting $s \to +\infty$,

$$\forall T \in \mathbb{Z}_{\geq t_o}, \ (|\gamma| \ | \tilde{\gamma}(T))_o \geq \frac{\Lambda T}{2} - M'(\Lambda, \delta),$$

where $M'(\Lambda, \delta) = \log \mu \left( \frac{2}{1 - \mu^{-2\Lambda/\dagger}} \right)$. We will first prove the lemma in the case $|\tilde{\gamma}(0)| = 0$, i.e. $\tilde{\gamma}(0) = o$ (this is pictured on Figure 5), and then use 3.3 to extend the result to the general case.

Step 1: $\tilde{\gamma}(0) = o$. — Let $(t, T) \in \mathbb{R}_{\geq 0}$ be such that $t \leq T$. Since $v$ is nondecreasing and unbounded, there is $T_2 \in \mathbb{R}_{\geq 0}$ such if $T \geq T_2$, then $v(T) \geq 6\Lambda^2 \delta$. This is the condition required to apply Lemma 2.9. By inequality (2.7) of this lemma applied to $\gamma_T = [o \tilde{\gamma}(T)]$ and $\tilde{\gamma}_{[0, T]}$,

$$\text{(3.9)} \quad \text{if } T \geq T_2, \ d(\tilde{\gamma}(t), \im(\gamma_T)) \leq h(\lambda) (\delta + v(T)).$$

Similarly, by (2.8), if $T \geq T_2$ then

$$\text{(3.10)} \quad \forall S \in [0, |\tilde{\gamma}(T)|], \ d(\gamma_T(S), \im(\tilde{\gamma})) \leq \tilde{h}(\lambda) (\delta + v(T)).$$

By (1.6) and our definition of $\alpha$, $t_o = r(\alpha)(v)$; start assuming that $t \geq t_o \lor T_2$. We look for $T$ greater than $t$ (hence, greater than $t_o$ and $T_2$) such that $|\tilde{\gamma}(T) | \eta)_o \geq \delta \tilde{\gamma}(T))$. Thanks to (3.8) this holds when $T = [t] \lor \left( \frac{4}{\dagger} (|\tilde{\gamma}(t)| + M(\Lambda, \delta)) \right)$; we keep this dependence of $T$ with respect to $t$ from now on. Let $\triangle_T$ be a (geodesic, semi-ideal) triangle with vertices $o$, $\tilde{\gamma}(T)$ and $\eta$ (Recall that by convention, $\gamma_T$ is the side of $\triangle_T$ between $o$ and $\tilde{\gamma}(T)$). By (3.9) and the triangle inequality,

$$|p_{\gamma_T}(\tilde{\gamma}(t))| \leq h(\lambda) (\delta + v(T)) + |\tilde{\gamma}(t)|.$$
Again by the triangle inequality,
\[ d(p_{\gamma^T}\bar{\gamma}(t), [\bar{\gamma}(T)\eta]) \geq |p_{\bar{\gamma}(T)\eta}o| - |p_{\gamma^T}\bar{\gamma}(t)| \]
\[ \geq |p_{\bar{\gamma}(T)\eta}o| - h(\lambda)(\delta + v(T)) - |\bar{\gamma}(t)|. \]
By the triangle inequality \(|p_{[\bar{\gamma}(T)\eta]}o| \geq (\bar{\gamma}(T) \mid \eta)\), so that the previous inequality becomes
\[ d(p_{\gamma^T}\bar{\gamma}(t), [\bar{\gamma}(T)\eta]) \geq (\bar{\gamma}(T) \mid \eta)\eta o - h(\lambda)(\delta + v(T)) - |\bar{\gamma}(t)| \]
(3.11)
where we have replaced the Gromov product according to the definition of \( T \). Let us now bound \( v(T) \). By definition,
\[ T \leq 4\lambda|\bar{\gamma}(t)| + 4\lambda M'(\lambda, \bar{\delta}) + 1 \]
\[ \leq 4\lambda (\lambda t + v(t)) + 4\lambda M'(\lambda, \bar{\delta}) + 1, \]
hence for \( t \geq t_0 = \sup \{ t_0, T_2, 4\lambda M'(\lambda, \bar{\delta}) + 1 \} \), \( T \leq (1 + 8\lambda^2)t \), and
(3.12)
\[ v(T) \leq (v \uparrow 1 + 8\lambda^2)v(t). \]
Substituting this in inequality (3.11), for all \( t \) such that \( t \geq t_0 \),
\[ d(p_{\gamma^T}\bar{\gamma}(t), [\bar{\gamma}(T)\eta]) \geq |\bar{\gamma}(t)| - h(\lambda)(\delta + (v \uparrow 1 + 8\lambda^2)v(t)). \]
Applying Lemma 3.2 to \( \bar{\gamma} \), define
\[ t_1 := t_0 \lor \sup \{ s : v(s) \leq \delta \} \lor 3\lambda r_1/(2h(\lambda)\varepsilon_{1+8\lambda^2})(v) \lor 12\lambda \delta \lor t_0. \]
Then by definition of \( t_1 \),
\[ \forall t \in \mathbb{R}_{>t_1}, d(p_{\gamma^T}\bar{\gamma}(t), [\bar{\gamma}(T)\eta]) > 4\delta. \]
But \( \Delta_T \) is \( 4\delta \)-slim, so
(3.13)
\[ \forall t \in \mathbb{R}_{>t_1}, d(p_{\gamma^T}\bar{\gamma}(t), \gamma) \leq 4\delta. \]
By the triangle inequality,
\[ \forall t \in \mathbb{R}_{>t_1}, d(\bar{\gamma}(t), \im(\gamma)) \leq |\bar{\gamma}(t) - p_{\gamma^T}\bar{\gamma}(t)| + d(p_{\gamma^T}\bar{\gamma}(t), \im(\gamma)) \]
\[ \leq h(\lambda)(\delta + v(T)) + 4\delta \]
(3.9), (3.13)
\[ \leq h(\lambda)(\delta + (v \uparrow 1 + 8\lambda^2)v(t)). \]
(3.12)
Define \( t_3 = \sup \{ s : v(s) \leq h(\lambda)\delta \} \lor t_1 \). The last inequality implies
(3.14)
\[ \forall t \in \mathbb{R}_{>t_3}, d(\bar{\gamma}(t), \im(\gamma)) \leq (2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1) \lor t(\delta). \]
We have proved (3.4) in the special case \(|\bar{\gamma}(0)| = 0 \); set \( H_0(\lambda, v) = 2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1 \).

**Step 2: \( \bar{\gamma}(0) \) arbitrary.** — Let \( \gamma' \) be a non-pointed geodesic ray \([\bar{\gamma}(0)\eta]\). Apply (3.14) to \( \bar{\gamma} \) and \( \gamma' \). This gives the existence, for all \( t \in \mathbb{R}_{>t_3} \), of \( s' \in \mathbb{R}_{>0} \) such that \( d(\bar{\gamma}(t), \gamma'(s')) \leq H_0(\lambda, v)\varepsilon(t) \). Moreover \( s' = d(\bar{\gamma}(0), s) \geq d(\bar{\gamma}(0), \gamma(t)) - H_0(\lambda, v)\varepsilon(t) \geq \lambda \varepsilon - (1 + H_0(\lambda, v)) \varepsilon(t) \). Hence for all \( t \) such that \( t \geq t_4 := \sup \{ t_3, \lambda \varepsilon/(2+2H_0(\lambda, v))(\varepsilon(t)) \} \),
(3.15)
\[ s' \geq t/(2\lambda). \]
Set \( t_5 := t_4 \lor \sup \{ s : v(s) \leq 8\delta \} \), \( t_\infty := t_5 \lor 16\delta \) and then \( t_\infty = t_\infty + 2\lambda|\bar{\gamma}(0)| \). By (3.15), \( s' \geq |\bar{\gamma}(0)| + 16\delta \). Moreover \( \gamma' \) and \( \gamma \) are asymptotic, so that by Lemma 3.3
on asymptotic geodesic rays, $d(\gamma'(s'), \text{im}(\gamma)) \leq 8\delta$. By the triangle inequality and the definition of $t_\omega$, we conclude that

$$d(\tilde{\gamma}(t), \gamma) \leq |\tilde{\gamma}(t) - \gamma'(s')| + d(\gamma'(s'), \text{im}(\gamma)) \leq (1 + H_0(\lambda, v))v(t).$$

By construction, $t_\omega^0$ only depends on $\lambda, v, \delta$, so (3.4) is reached in the general case. From now on we proceed to attain (3.5). As before start by assuming $|\tilde{\gamma}(0)| = 0$. For all $t \in \mathbb{R}_{\geq 0}$, since $\tilde{\gamma}_{[0, t]}$ is $v(t)$-connected, $p_\gamma(\tilde{\gamma}_{[0, t]})$ is $v(t) + 16\delta$-connected by Lemma 2.3, in particular it is 2$v(t)$-connected as soon as $t \geq t_6 := \sup\{ r : v(r) \leq 16\delta \}$. On the other hand, by Lemma 3.2, if $t \geq t_7 := \sup\{ t_6, t_5 \}$, the convex hull of $p_\gamma(\tilde{\gamma}_{[0, t]})$ contains $\gamma([0, (\Lambda/3)t - Hv(t)])$ where $H$ is the constant from (3.4) (note that $t_7$ only depends on $v, \lambda, \delta$ since we are assuming $\tilde{\gamma}(0) = 0$).

Hence for all $t \geq t_8 = \sup\{ t_7, r_{\Lambda/6H}(v) \}$, every $s \in [0, (\Lambda/6)t]$ lies between two orthogonal projections of points of $\tilde{\gamma}_{[0, t]}$ on $\gamma$. Define $R_8 := t_8/(6\lambda)$. For all $s \in \mathbb{R}$ such that $s \geq R_8$, there is $t_s \in [0, 6\lambda s]$ such that

$$(3.16) \quad |\gamma(s) - p_\gamma(\tilde{\gamma}(t_s))| \leq 2v(6\lambda s) \leq 2(v + 6\lambda)v(s).$$

By the triangle inequality,

$$|\gamma(s) - \tilde{\gamma}(t_s)| \leq |\gamma(s) - p_\gamma(\tilde{\gamma}(t_s))| + |p_\gamma(\tilde{\gamma}(t_s)) - \tilde{\gamma}(t_s)| \leq H_0v(t_s) + 2(v + 6\lambda)v(s) \leq 2(v + 6\lambda)(H_0 + 1)v(s)$$

(3.17) for all $s \geq R_8$.

where we used that $v(t_s) \leq (v + 6\lambda)v(s)$ for the last inequality. Set $\bar{H}_0(\lambda, v) := 2(v + 6\lambda)(H_0 + 1)$, and assume from now that $\tilde{\gamma}(0)$ is arbitrary. Define $R_0^\omega = R_8 \vee \sup\{ \bar{H}_0v(r) \leq 8\lambda \} \vee 16\delta$ and $\bar{H} = 2\bar{H}_0$. Then by Lemma 3.3 applied to $\gamma = o\gamma'$ and $\gamma' = \tilde{\gamma}(0)\eta$, (3.17) and the triangle inequality, for all $s \geq R_0^\omega + |\tilde{\gamma}(0)|$, $d(\gamma(s), \text{im}(\tilde{\gamma})) \leq \bar{H}v(s)$. \hfill \Box

3.C. Geodesics. Our next aim consists in tracking $O(v)$-geodesics $\tilde{\gamma}$. For this we need two steps:

1. Control the Gromov product of ends $\partial_\infty \tilde{\gamma}(-\infty)$ and $\partial_\infty \tilde{\gamma}(+\infty)$ with respect to $|\tilde{\gamma}(0)|$. This is achieved by Lemma 3.7.

2. Track $\tilde{\gamma}$ near both ends, starting at a distance linearly controlled by their Gromov product, and interpolate in between using the classical version of the stability lemma. This strategy is set up in Lemma 3.8.

Beware that, in contrast to the situation with (quasi)geodesics, one cannot reparametrize a $(\lambda, v)$-geodesic (e.g. to assume that $\tilde{\gamma}(0)$ is the closest point $\bar{b}$ to $o$ in $\text{im}(\tilde{\gamma})$) without changing the function $v$). For this reason, and in order to simplify bounds on the tracking distance in step (2), we introduce an additional constant $L$ and, from Lemma 3.8 on, make the assumption that $|\tilde{\gamma}(0)| \leq Lb$.

**Lemma 3.7.** Let $\delta \in \mathbb{R}_{\geq 0}$, $\lambda \in \mathbb{R}_{\geq 1}$ be constants, and let $(Y, o)$ be a pointed proper geodesic $\delta$-hyperbolic space. Let $v$ be an admissible function. Let $\tilde{\gamma}$ be a $(\lambda, v)$-geodesic into $Y$. Denote $\eta_{\pm}$ in $\partial_\infty X$ its endpoints, precisely $\eta_{\pm} = \partial_\infty \tilde{\gamma}(\pm\infty)$. Then there exist $K = K(\lambda, v, \delta)$ and $R_\gamma = R_\gamma(\lambda, v, \delta)$, both in $\mathbb{R}_{\geq 0}$ such that if $|\tilde{\gamma}(0)| \geq R_\gamma$,

$$(3.18) \quad (\eta_- | \eta_+) \leq K|\tilde{\gamma}(0)|.$$ 

\footnote{Such a point $\bar{b}$ exists since $\tilde{\gamma}$ is proper.}
Proof. The proof uses that \(O(u)\)-geodesics cannot make large round trips; see figure 6. Assume by contradiction that \((\eta_\ell \mid \eta_\ell^+)_0 \geq 3(\ell_0^2 + |\gamma(0)|) + 4\delta = 3R_\gamma + 4\delta\) for \(\gamma(0)\) arbitrarily far. Track the rays \(\gamma_\ell = t \mapsto \gamma(-t)\) and \(\gamma_\ell^+ = t \mapsto \gamma(t)\) with geodesic rays \(\gamma_-\) and \(\gamma_+\). Let \(\gamma = (\eta_- \eta_+)^\prime\) be a geodesic line. Define \(p_\ell\) as the intersection point of \(\gamma_\ell\) and \(\partial B(o, 2R_\gamma)\), i.e. \(p_\ell = \gamma(2R_\gamma)\). The twice-ideal triangle \(\eta_- \eta_+\) is \(4\delta\)-thin, and by the triangle inequality
\[
|q_\ell - q_-| \leq |p_\ell - p_-| + 2Hv(2R_\gamma) \leq 8\delta + 2Hv(2R_\gamma).
\]
Let \(t_\ell, t_-\) in \(\mathbb{R}\) be such that \(q_\ell = \gamma(t_\ell)\), and write \(T = \sup\{|t_\ell|, |t_-|\}\). The portion of \(\gamma\) between \(t_-\) and \(t_\ell\) is a \((\lambda, v(T))\) quasi-geodesic segment. By a length-distance estimate for quasi-geodesics, for \(\alpha\) large enough,
\[
\ell_\alpha(\gamma[|t_-|, t_\ell]) \leq 2\lambda(t_\ell - t_-) \leq 2\lambda(\lambda|q_- - q_\ell| + v(T)) \leq 4\lambda^2 Hv(2R_\gamma) + 2\lambda v(T) + 16\lambda^2\delta.
\]
\(T\) can be bounded from above for \(|\gamma(0)|\) large enough:
\[
\Delta T - v(T) \leq \sup \{|\gamma(0) - q_-|, |\gamma(0) - q_\ell|\} \leq 2R_\gamma^0 + |\gamma(0)| + 2Hv(T),
\]
so that since \(v(T) \ll T\), there is a constant \(T_0\) depending on \(v, \lambda\) (explicitly \(T_0 = r_{v/(8Hv)}(v)\)) such that \(T \leq \inf\{T_0, \lambda(2R_\gamma^0 + |\gamma(0)|)\}\). On the other hand, \(\ell_\alpha(\gamma[|t_-|, t_\ell])\) is greater than \(|q_\ell - \gamma(0)| + |q_- - \gamma(0)|\), and
\[
|q_\ell - \gamma(0)| + |q_- - \gamma(0)| \geq |p_\ell - \gamma(0)| + |p_- - \gamma(0)| - 2Hv(2R_\gamma) \geq 2R_\gamma^0 + |\gamma(0)| - 2Hv(2R_\gamma).
\]
Substitute this in (3.20) and make all dependences over $|\tilde{\gamma}(0)|$ explicit:

$$2R_o^0 + |\tilde{\gamma}(0)| - 2H v(2R_o) \leq 4\lambda^2 H v(2R_o) + 2\lambda v(T) + 16\lambda^2 \delta.$$ 

$$\leq 4\lambda^2 H v(2R_o) + 2\lambda v(T_0) + 2\lambda v((\lambda/2)(2R_o)) + 16\lambda^2 \delta.$$ 

The last inequality rewrites under the form

$$|\tilde{\gamma}(0)| \leq [4\lambda^2 H + 2\lambda (v \uparrow \lambda)] [v \uparrow 2] v(R_o) + 2\lambda v(T_0) + 16\lambda^2 \delta + 2R_o^0,$$

(3.21) 

where $H_3 = [4\lambda^2 H + 2\lambda (v \uparrow \lambda)] [v \uparrow 2]$. If $|\tilde{\gamma}(0)| \geq 3R_o^0$ then (3.21) yields

$$|\tilde{\gamma}(0)| \leq 3(v \uparrow 2)H_3 v(|\tilde{\gamma}(0)|) + \frac{3\lambda}{4H} r_{1/(8\lambda H)}(v) + 4\lambda^2 \delta.$$ 

This inequality would lead to a contradiction for $|\tilde{\gamma}(0)|$ larger than

$$R_\gamma := 3R_o^0 \vee \left( \frac{3\lambda}{2H} r_{1/(8\lambda H)}(v) + 96\lambda^2 \delta \right) \vee r_{1/(6v \uparrow 2)H_3(\lambda, \delta, v)}(v),$$

(3.22) 

precisely, if $|\tilde{\gamma}(0)| \geq R_\gamma$, then $(\eta_-, | \eta_+)_o \leq 3(R_o^0 + |\tilde{\gamma}(0)|) + 4\delta \leq 5|\tilde{\gamma}(0)|$ as $R_\gamma \geq R_o^0 \vee 4\delta$. One may take $K = 5$. \hfill \Box

**Lemma 3.8** (Tracking for $O(u)$-geodesics). Let $\delta \in \mathbb{R}_{\geq 0}$, $\lambda \in \mathbb{R}_{\geq 1}$, let $u$ be an admissible function and let $v = O(u)$ be nondecreasing. Let $(Y,o)$ be a proper geodesic pointed $\delta$-hyperbolic space, and let $\tilde{\gamma} : \mathbb{R} \to Y$ be a $(\lambda, v)$-geodesic. Define $\tilde{b}$ as a closest point to $o$ in $\im(\tilde{\gamma})$. Let $L \in \mathbb{R}_{\geq 1}$ be a real constant and assume that the Gromov product $(\partial_\infty \tilde{\gamma}(+\infty) \mid \partial_\infty \tilde{\gamma}(−\infty))_o$ is larger than $60\delta$. There exist constants $\bar{R} = \bar{R}(\lambda, \delta, v, L)$, $H_2$ and $\bar{H}^2$ in $\mathbb{R}_{> 0}$ (depending on $\lambda, v$ and $L$) such that if

$$(3.23) \quad \bar{R} \leq |\tilde{\gamma}(0)| \leq L|\tilde{b}|,$$

then for any geodesic $\gamma : \mathbb{R} \to Y$ with $[\gamma]_{\pm \infty} = \partial_\infty \tilde{\gamma}(\pm \infty)$ and $\gamma(0) = p_o$, $\gamma(0) = p_o$,

$$(3.24) \quad \forall t \in \mathbb{R}, d(\tilde{\gamma}(t), \gamma) \leq H_2 v(|\tilde{\gamma}(t)|),$$

(3.25) 

$$\forall s \in \mathbb{R}, d(\gamma(s), \im(\tilde{\gamma})) \leq \bar{H}_2 v(|\gamma(s)|).$$

**Proof.** We divide the proof into 4 steps.

**Step 1: Setting.** As before, write $\eta_{\pm} = \partial_\infty \tilde{\gamma}(\pm \infty)$, cut $\tilde{\gamma}$ in two $(\lambda, v)$-geodesic rays $\tilde{\gamma}_{\pm}$ starting at $\tilde{\gamma}(0)$, and track $\tilde{\gamma}_{\pm}$ with geodesic rays $\gamma_{\pm}$. Let $\gamma = (\eta_−, \eta_+)$, parametrized in such a way that $p_o = \gamma(0)$. Define $\tilde{R}_0 = R_\gamma$ and start assuming $|\tilde{\gamma}(0)| \geq \tilde{R}_0$. Let $k$ be a real parameter whose value should be fixed later; only assume for now that $k \geq 2K + 1$, where $K$ is the constant from Lemma 3.7. Define

$$(3.26) \quad p_\pm := \gamma_{\pm} \left( k \left[ |\tilde{\gamma}(0)| + R_o^0 \right] \vee 2(\lambda + 1)r_{1/(8\lambda H)}(v) \vee r_{1/(2\bar{H})}(v) \right),$$

where $R_o^0$ is the constant from Lemma 3.4 applied to $\tilde{\gamma}_+$ or $\tilde{\gamma}_-$ and let $q_\pm$ be a closest point to $p_\pm$ on $\im(\tilde{\gamma}_{\pm})$. Since $k \geq 2K$ and $|\tilde{\gamma}(0)| \geq R_\gamma$, by Lemma 3.7, $k|\tilde{\gamma}(0)| \geq 2(\eta_− | \eta_+)_o$, and $|p_+| = |p_-| \geq 2(\eta_− | \eta_+)_o$ as well. Let $\sigma$ be
a geodesic segment from \( o \) to \( p, o \). By the triangle inequality, \( d(p_{\pm}, \sigma) \geq 2(\eta_+ | \eta_+) + \sup_{c \in o} |c| \geq (\eta_+ | \eta_+) - 56\delta \). As \((\eta_+ | \eta_+) > 60\delta\) by hypothesis,

\[
(3.27) \quad d(p_{\pm}, \sigma) > 60\delta - 56\delta = 4\delta, \text{ hence } d(p_{\pm}, \gamma) \leq 4\delta,
\]

where we used that the once-ideal right-angled triangles \( \eta_{\pm}(p, o) \) are \( 4\delta \)-slim (recall that \( p_{\pm} \) lies on the side \( \gamma_{\pm} \) by definition). Further, because \( k \geq 1 \), inequality (3.5) of the tracking lemma 3.4 allows to bound \( |q_\pm - p_-| \) and \( |q_\pm - p_+| \):

\[
(3.28) \quad |p_{\pm} - q_\pm| \leq \bar{H}v(|p_{\pm}|),
\]

so that by the triangle inequality and the definition (3.26) of \( p_{\pm} \),

\[
(3.29) \quad |q_{\pm}| \geq |p_{\pm}| - \bar{H}v(|p_{\pm}|) \geq 1/2|p_{\pm}|.
\]

**Step 2: Selection of \( k \).** — At this point, in order to control the quasi-geodesic additive error term of \( \breve{\gamma} \) between \( q_- \) and \( q_+ \) we need to select \( k \) large enough so that \( |q_{\pm}| \geq R_0 \), where \( R_0 \) is associated to \( \breve{\gamma} \) as in Lemma 3.2. Recall from the expression (3.2) of \( R_0 \) that \( R_0 = 4|\breve{\gamma}(0)| \sqrt{2(2\lambda + 1)(3\lambda|\breve{\gamma}(0)| \vee r_1/(3\lambda)v)} \). Thus from now on we fix \( k = (2K + 1) \vee 8 \vee 12\lambda(2\lambda + 1) \). By inequality (3.28), this is sufficient to ensure \( |q_{\pm}| \geq R_0 \), and then using the estimates and notation of Lemma 3.2, the portion of \( \breve{\gamma} \) situated between \( q_+ \) and \( q_- \) is a \((\lambda, c)\)-quasigeodesic segment, with \( c = \tilde{h}(q_+ \vee |q_-|) \).

**Step 3: Tracking between \( q_- \) and \( q_+ \), and estimation of \( H_2 \).** — Let \( \gamma \) be a geodesic segment between \( q_+ \) and \( q_- \). Let \( t_{\pm} \in \mathbb{R} \) be such that \( \tilde{\gamma}(t_{\pm}) = \tilde{\gamma}_{\pm}(\pm t_{\pm}) = q_{\pm} \). By Lemma 2.9 \( \text{dist}_H(\tilde{\gamma}, \gamma|_{[t_-, t_+]}) \leq (h(\lambda) \vee \tilde{h}(\lambda))(\delta + c) \), and by hyperbolic geometry, letting \( s_{\pm} \) be such that \( \gamma(s_{\pm}) = p_{\mp}(q_{\pm}) \), \( \text{dist}_H(\tilde{\gamma}, \gamma|_{s_-, s_+}) \) cannot be much greater than the pairwise distance between the endpoints of these geodesic segments:

\[
(3.30) \quad \text{dist}_H(\tilde{\gamma}, \gamma|_{s_-, s_+}) \leq |q_{\pm} - \gamma(s_{\pm})| + 8\delta
\]

\[
(3.31) \quad \leq 4\delta + \bar{H}v(|p_{\pm}|) + 8\delta.
\]
where we combined (3.27) and (3.28) by means of the triangle inequality. Hence
\[\forall t \in [t_-, t_+], d(\tilde{r}(t), \gamma) \leq (h(\lambda) \vee \tilde{h}(\lambda))(\delta + (v \uparrow 3\lambda)v(|q_+| \vee |q_-|)) + 8\delta + 4\delta + \tilde{H}v(|q_+| \vee |q_-|)\]
\[\leq (12 + \tilde{h}(\lambda))\delta + (\tilde{h}(\lambda)(v \uparrow 3\lambda) + \tilde{H})v\left(|p_\pm| + \tilde{H}v(|p_\pm|)\right)\]  
(3.31)
(here \(\tilde{h}(\lambda)\) is used alone as it is equal to \(\tilde{h}(\lambda) \vee h(\lambda)\)). If \(|\tilde{r}(0)| \geq R_7\), then in view of the definition of \(p_\pm\) (3.26),
\[|p_\pm| = k \left[|\tilde{r}(0)| + R_0^0\right] \vee 2(2\lambda + 1)r_1/3\lambda(v) \vee r_1/(2\tilde{H})(v)\]
\[\leq \left( k + \frac{R_0^0}{R_7} \right) |\tilde{r}(0)| \vee 2(2\lambda + 1)r_1/3\lambda(v) \vee r_1/(2\tilde{H})(v)\]
(3.32)
where we used \(k \geq 1\) so that \(k + 1/3 \leq 2k\) in the last inequality. Define \(\tilde{R}_1 = \tilde{R}_0 \vee 2(2\lambda + 1)r_1/3\lambda(v) \vee r_1/(2\tilde{H})(v)\). By (3.32), if \(|\tilde{r}(0)| \geq \tilde{R}_1\),
\[v\left(|p_\pm| + \tilde{H}v(|p_\pm|)\right) \leq v\left(3k|\tilde{r}(0)|\right)\]
Recall that by the right-hand side of assumption (3.23), \(|\tilde{r}(0)| \leq L[\tilde{r}] = L \inf\{|\tilde{r}(t)| : t \in \mathbf{R}\}. Plugging (3.32) in (3.31), one obtains that for all \(t\) in \([t_-, t_+]\),
\[d(\tilde{r}(t), \gamma) \leq (12 + \tilde{h}(\lambda))\delta + (\tilde{h}(\lambda)(v \uparrow 3\lambda) + \tilde{H})v\left(3k|\tilde{r}(0)|\right)\]
\[\leq (12 + \tilde{h}(\lambda))\delta + 2\tilde{H}(v \uparrow 3\lambda)v(3Lk|\tilde{r}(t)|),\]
where we used that \(\tilde{H} \geq \tilde{h}(\lambda)\) on the second line; this is because by definition, \(\tilde{H} = 4(v \uparrow 6\lambda)(2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1) \geq 8h(\lambda) \geq \tilde{h}(\lambda)\). Define \(\tilde{R}_3 = \tilde{R}_2 \vee L^{-1} \sup\{r : v(r) \leq (12 + \tilde{h}(\lambda))\delta\}\). If \(|\tilde{r}(0)| \geq \tilde{R}_3\), the right-hand side of assumption (3.23) ensures that \(v(|\tilde{r}(t)|) \geq (12 + \tilde{h}(\lambda))\delta\) for all \(t\), so we have proved
(3.33)
\[\forall t \in [t_-, t_+], d(\tilde{r}(t), \gamma) \leq (1 + 2(v \uparrow 3\lambda)\tilde{H}(v \uparrow 3Lk))v(|\tilde{r}(0)|)\]
On the other hand, in view of the tracking lemma 3.4, for all \(t \in (-\infty, t_-)\),
\[d(\tilde{r}(t), \gamma) \leq (v \uparrow 3\lambda)Hv(|\tilde{r}(t)|)\]
and similarly for all \(t \in (t_+, +\infty)\),
\[d(\tilde{r}(t), \gamma) \leq (v \uparrow 3\lambda)Hv(|\tilde{r}(t)|)\]. Since the two-ideal triangle \(o_{\eta_-}q_+\) is 4\(\delta\)-slim, using the triangle inequality and the fact that \(v(|\tilde{r}(t)|) \geq 12\delta\) for all \(t\) provided \(|\tilde{r}(0)| \geq \tilde{R}_3\) by definition of \(\tilde{R}_3\),
(3.34)
\[\forall t \in \mathbf{R}\setminus [t_-, t_+], d(\tilde{r}(t), \gamma) \leq ((v \uparrow 3\lambda)H + 4\delta/(12\delta))v(|\tilde{r}(t)|)\].
Putting (3.33) and (3.34) together yields the expected tracking inequality (3.24) for the provisional \(\tilde{R}_3\). Precisely \(H_2\) may be taken as
(3.35)
\[H_2 = 4(v \uparrow 3Lk)(v \uparrow 3\lambda)\tilde{H} \vee 2(v \uparrow 3\lambda)H.\]

Step 4: Estimation of the tracking constant \(\tilde{H}_2\). — From here, one could deduce
(3.25) using (3.24) for \(\tilde{R}\) large enough by a metric connectedness argument as in Lemma 2.9 or Lemma 3.4, but let us rather use the former estimates from the current proof. Define \(\tilde{R}_4 = \tilde{R}_3 \vee r_1/(2LH_2)(v)\); then if \(|\tilde{r}(0)| \geq \tilde{R}_4\), it follows from
the tracking inequality just obtained for $\tilde{\gamma}$ that $|p_\gamma o| = |\gamma(0)| \geq (1/2)|\tilde{\gamma}(0)|$. Then for all $s \in [s_-, s_+]$, by (3.30),

$$d(\gamma(s), \tilde{\gamma}) \leq \tilde{H} v(|p_\pm|) + 12\delta \leq 2\tilde{H} v(2k|\gamma(s)|) \lor 24\delta.$$  

On the other hand, recall that by Lemma 2.9, for all $c \in \tilde{\gamma}$, $d(c, \tilde{\gamma}) \leq \tilde{h}(\lambda)(\delta + (v \uparrow 3\lambda))v(|q_+| \lor |q_-|)$. Combining this with (3.36) by means of the triangle inequality while remembering the bound on $|q_\pm|$ implied by (3.28), one obtains

$$\forall s \in [s_-, s_+], d(\gamma(s), \tilde{\gamma}) \leq (2\tilde{H} + \tilde{h}(\lambda)(\delta + (v \uparrow 3\lambda))v(3k|\gamma(s)|)).$$

Finally, if $s \in \mathbb{R}$ is such that $s \leq s_-$ or $s \geq s_+$, since $p_\gamma o$ and $\gamma(s)$ are $28\delta$-almost lined up, $|\gamma(s)| \geq |s| - |p_\gamma o| \geq |s|/2$. $\gamma(s)$ is at most $4\delta$ away from its orthogonal projection on $\gamma(t(s))$, where $t(s)$ is the sign of $s$. Given the definition of $p_\pm, p_{\gamma(t)}(\gamma(s))$ is at a distance at least $R_o = |s|$ from the origin, and inequality (3.5) from Lemma 3.4 bounds its distance to $\tilde{\gamma}$ so that

$$d(\gamma(s), \tilde{\gamma}) \leq |\gamma(s) - p_{\gamma(t)}(\gamma(s))| + d(p_{\gamma(t)}(\gamma(s)), \tilde{\gamma})$$

$$\leq 4\delta + \tilde{H} v(|p_{\gamma(t)}(\gamma(s))|) \leq 2\tilde{H} v(|\gamma(s)|).$$

Together with (3.37), this proves (3.25) with $\tilde{R} = \tilde{R}_4$ and

$$\tilde{H}_2 = \left(2\tilde{H} + \tilde{h}(\lambda)\right) (\delta + (v \uparrow 3\lambda)) \lor (v \uparrow 3k).$$

3.D. Distance between $O(u)$-geodesics.

**Lemma 3.9.** Let $\delta \in \mathbb{R}_{\geq 0}$ be a constant. Let $\gamma_1$ and $\gamma_2$ be geodesic lines into a $\delta$-hyperbolic space, with four pairwise distinct endpoints. Define $\Delta = d(\operatorname{im}(\gamma_1), \operatorname{im}(\gamma_2))$. Then for all $s_1, s_2 \in \mathbb{R}$,

$$|\gamma_1(s_1) - \gamma_2(s_2)| \geq \Delta + d(\gamma_1(s_1), p_{\gamma_1, \operatorname{im}(\gamma_2)}) \lor d(\gamma_2(s_2), p_{\gamma_2, \operatorname{im}(\gamma_1)}) - 56\delta.$$

**Proof.** The distance on the left is symmetric relatively to $\gamma_1(s_1)$, so it suffices to prove $|\gamma_1(s_1) - \gamma_2(s_2)| \geq \Delta + d(\gamma_1(s_1), p_{\gamma_1, \operatorname{im}(\gamma_2)}) - 56\delta$. The points $\gamma_1(s_1), p_{\gamma_2, \gamma_1(s_1)}$ and $\gamma_2(s_2)$ are the vertices of a right-angled hyperbolic triangle so that by Lemma 2.6, they are $28\delta$-almost lined up. By the triangle inequality,

$$d(\gamma_1(s_1), \gamma_2(s_2)) + 2 \cdot 28\delta \geq d(\gamma_1(s_1), p_{\gamma_2, \gamma_1(s_1)}) + d(p_{\gamma_2, \gamma_1(s_1)}, \gamma_2(s_2))$$

$$\geq \Delta + d(\gamma_1(s_1), p_{\gamma_1, \operatorname{im}(\gamma_2)}).$$

**Lemma 3.10.** Let $v^1$ and $v^2$ be admissible functions, and define $v = v^1 \lor v^2$. Let $L \in \mathbb{R}_{\geq 1}$ be a constant. Let $\delta$ be a hyperbolicity constant and let $\lambda = (\Delta, X) \in \mathbb{R}_{\geq 0}$ be expansion and Lipschitz constants. There exist $J = J(\lambda, v, L), R = R(\delta, \lambda, v, L)$ and, for $i \in \{1, 2\}$, $\tilde{R}^i = \tilde{R}^i(\delta, \lambda, v^i, L)$ in $\mathbb{R}_{\geq 0}$ such that for any proper geodesic, pointed $\delta$-hyperbolic space $(Y, o)$, if $(\gamma_1, \gamma_1)$ and $(\gamma_2, \gamma_2)$ are such that

(i) $\gamma_1, \gamma_2$ are geodesics $\mathbb{R} \to Y$ with four distinct endpoints $\eta^\pm_{i} = \gamma_i(\pm \infty)$, 
(ii) for $i \in \{1, 2\}$, $\tilde{\gamma}_i$ is a $(\lambda, v^i)$-geodesics $\mathbb{R} \to Y$, 
(iii) for $i \in \{1, 2\}$, $\partial_{\infty} \tilde{\gamma}_i(\pm \infty) = [\gamma_{i\pm}]$, 
(iv) $\mathbb{R} \{\eta^\pm_{1}, \eta^\pm_{2}\} \geq 60\delta$, and $\mathbb{R} \{\eta^\pm_{1}, \eta^\pm_{2}\} \geq R$, 
(v) for all $i \in \{1, 2\}$, $\tilde{R}^i \leq |\tilde{\gamma}_i(0)| \leq L \inf_{t \in \mathbb{R}} |\tilde{\gamma}_i(t)|$,

then

$$|d(\gamma_1, \gamma_2) - d(\tilde{\gamma}_1, \tilde{\gamma}_2)| \leq Jv \left(\mathbb{R} \left\{\eta^\pm_{1}, \eta^\pm_{2}\right\}\right).$$
Figure 8. Main points occurring in the proof of Lemma 3.10. Straight, resp. wavy lines depict geodesic, resp. $O(u)$-geodesic lines; the boundary is dashed.

Sketch of proof for Lemma 3.10. See figure 8. The main tool is the geodesic tracking lemma 3.8; however the tracking between $\bar{\gamma}_i$ and $\gamma_i$ becomes inefficient far from the origin. Thus we need to prove that shortest geodesic segments between $\gamma_1$ and $\gamma_2$ on the one hand, and between $\bar{\gamma}_1$ and $\bar{\gamma}_2$ on the other hand, are close to the origin (at most not significantly farther than the largest Gromov product). The part concerning $\gamma_1$ and $\gamma_2$ was already expressed by Lemma 2.8; as for the other part we show (inequality (3.44)) that letting $t_1$, $t_2$ be such that $d(\bar{\gamma}_1, \bar{\gamma}_2) = |\bar{\gamma}_1(t_1) - \bar{\gamma}_2(t_2)|$, $|\bar{\gamma}_1(t_1)| \vee |\bar{\gamma}_2(t_2)|$ is linearly controlled by $\mathbb{S}\{\eta^+_1, \eta^+_2\}$ on the large-scale. This uses the well-known behavior described by Lemma 3.9: geodesic rays spread apart linearly from each other after the Gromov products are reached; since they track $O(u)$-geodesics at a distance growing sublinearly, $\bar{\gamma}_1$ and $\bar{\gamma}_2$ also spread away from each other, which prevents $\bar{\gamma}_i(t_i)$ from being much farther than all the Gromov products.

Proof. For $i \in \{1, 2\}$, let $s_i \in \mathbb{R}$ be such that $|\gamma_1(s_1) - \gamma_2(s_2)| = d(\gamma_1, \gamma_2)$. As $\gamma_1(s_1) \in p_{\gamma_1}(\gamma_2)$, and similarly $\gamma_2(s_2) \in p_{\gamma_2}(\gamma_1)$, by the projection lemma 2.8, $\sup_i |\gamma_i(s_i)| \leq \mathbb{S}\{\eta^+_1, \eta^+_2\} + 284 \delta$. Further, set $\tilde{R} = \tilde{R}(\lambda, \delta, v, L)$ according to the tracking lemma 3.8. Note that by the assumptions (i) to (iii), the first inequality in assumption (iv) and the right-hand side inequality in assumption (v), applied to the pairs $(\gamma_i, \bar{\gamma}_i)$, by Lemma 3.8,

$$\forall i \in \{1, 2\}, d(\gamma_i(s_i), \bar{\gamma}_i) \leq \tilde{H}_2 v(|\gamma_i(s_i)|) \leq \tilde{H}_2 v(\mathbb{S}\{\eta^+_1, \eta^+_2\} + 284 \delta).$$

By the triangle inequality, setting $J^+ = 2\tilde{H}_2(v \uparrow 2)$ and $R_0 = \sup\{r : v(r) \leq 284 \delta\}$, as soon as $\mathbb{S}\{\eta^+_1, \eta^+_2\} \geq R_0$,

$$d(\bar{\gamma}_1, \bar{\gamma}_2) - d(\gamma_1, \gamma_2) \leq d(\gamma_1(s_1), \bar{\gamma}_1) + d(\gamma_2(s_2), \bar{\gamma}_2) \leq J^+ v(\mathbb{S}\{\eta^+_1, \eta^+_2\}).$$

This is one half of inequality (3.40).
For $i \in \{1, 2\}$ let $t_i \in \mathbb{R}$ be such that $d(\bar{\gamma}_1, \bar{\gamma}_2) = |\bar{\gamma}_1(t_1) - \bar{\gamma}_2(t_2)|$. Let $s_i$ be such that $\gamma_i(s_i) = p_{\gamma_i, \bar{\gamma}_i(t_i)}$. By the triangle inequality and the tracking lemma 3.8,

$$\tag{3.42} |\gamma_1(s_1) - \gamma_2(s_2)| \leq d(\bar{\gamma}_1, \bar{\gamma}_2) + 2H_2v(|\bar{\gamma}_1(t_1)| \vee |\bar{\gamma}_2(t_2)|).$$

Inequality (3.39) of Lemma 3.9 gives a lower bound on $|\gamma_1(s_1) - \gamma_2(s_2)|$, which can be plugged into (3.42) yielding

$$\tag{3.43} d(\gamma_1, \gamma_2) + d(\gamma_2(s_2), p_{\gamma_2, \gamma_1(s_1)}) \leq d(\gamma_1(s_1), \gamma_2(s_2)) \leq d(\bar{\gamma}_1, \bar{\gamma}_2) + 2H_2v(|\bar{\gamma}_1(t_1)| \vee |\bar{\gamma}_2(t_2)|) + 56\delta.$$

On the other hand, using twice the triangle inequality and Lemma 2.8,

$$d(\gamma_1(s_1), p_{\gamma_1, \gamma_2}) \vee d(\gamma_2(s_2), p_{\gamma_2, \gamma_1}) \geq |\gamma_1(s_1)| \vee |\gamma_2(s_2)| - 284\delta.$$

Reorganizing (3.43),

$$|\bar{\gamma}_1(t_1)| \vee |\bar{\gamma}_2(t_2)| \leq \max\{\eta_1^+, \eta_2^+\} + 340\delta + 3H_2v(|\bar{\gamma}_1(t_1)| \vee |\bar{\gamma}_2(t_2)|)$$

$$\leq \max\{\eta_1^+, \eta_2^+\} + 340\delta + 3H_2v(|\bar{\gamma}_1(t_1)| \vee |\bar{\gamma}_2(t_2)|)$$

$$+ J^+v(\max\{\eta_1^+, \eta_2^+\})$$

when $\max\{\eta_1^+, \eta_2^+\} \geq F_0$. Hence,

$$|\bar{\gamma}_1(t_1)| \vee |\bar{\gamma}_2(t_2)| \leq \inf\left\{r_{1/(6H_2)}(v), \quad 2\max\{\eta_1^+, \eta_2^+\} + 340\delta + J^+v(\max\{\eta_1^+, \eta_2^+\})\right\}.$$

Set $R_1 = \sup\{r : v(r) \geq 584\delta/J^+\}$ and $R_2 = \sup\{R_0, R_1, r_{1/(2J^+)}(v)\}$. Then if $\max\{\eta_1^+, \eta_2^+\} \geq R_2$,

$$\tag{3.44} |\bar{\gamma}_1(t_1)| \vee |\bar{\gamma}_2(t_2)| \leq \inf\left\{r_{1/(6H_2)}(v), 4\max\{\eta_1^+, \eta_2^+\}\right\}.$$

Thus if $R_3 = r_{1/(4H_2)}(v)$, and if $t_1, t_2 \in \mathbb{R}$ are such that $|\bar{\gamma}_1(t_1)| \vee |\bar{\gamma}_2(t_2)| \geq R_3$, then

$$H_2v(\bar{\gamma}_1(t_1) \vee |\bar{\gamma}_2(t_2)|) \leq H_2(v \uparrow 4)v(\max\{\eta_1^+, \eta_2^+\}).$$

Finally by the triangle inequality, writing $J^- = 2H_2(v \uparrow 4)$,

$$d(\gamma_1, \gamma_2) - d(\bar{\gamma}_1, \bar{\gamma}_2) \leq d(\gamma_1(s_1), \gamma_2(s_2)) - d(\gamma_1(t_1), \gamma_2(t_2)) \leq J^-v(\max\{\eta_1^+, \eta_2^+\}).$$

To reach the conclusion of Lemma 3.10, define $J = J^- \vee J^+$ and then combine (3.41) with (3.45). \hfill \Box

3.E. Tracking radii. While there are four relevant parameters ($\lambda$, $v$, $\delta$, $L$) to express $R_{s_{\ell}}$, $\bar{R}$, $\bar{R}$, and $R$, only the dependence on $v$ is of interest for what follows. Consequently, a constant depending on the remaining parameters $\lambda, \delta, L$ can be written as, e.g., $C(\lambda, \delta)$ or $C(\lambda, \delta, L)$. 

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Lemma 3.11. Let $v$ be an admissible function. Let $\lambda \in \mathbb{R}_{\geq 1}$ be a biLipschitz constant. Let $\delta$ be a hyperbolicity constant. There exist a positive integer $n$ and constants $C(\lambda)$, $C(\lambda, \delta)$, $C(\lambda, \delta, L)$ such that in Lemma 3.4, Lemma 3.7, Lemma 3.8 and Lemma 3.10, the tracking radii may be taken as

\begin{align*}
R^0_{\text{cl}} &= C(\lambda)(v_{\gamma 1+\lambda})^{-n} \vee C(\lambda, \delta)(1 + \sup \{ r : v(r) \leq C(\lambda, \delta) \}) \\
\bar{R} &= C(\lambda, \delta)C(\lambda)(v_{\gamma L})^{-1}(v_{\gamma 1+\lambda})^{-n} \vee (1 + \sup \{ r : v(r) \leq C(\lambda, \delta, L) \}) \\
R &= r_{C(\lambda, \delta, L)}(v_{\gamma 1+\lambda})^{-n} \vee C(\lambda, \delta)(1 + \sup \{ r : v(r) \leq C(\lambda, \delta, L) \})
\end{align*}

Proof. It will be used without further notice that $r_\alpha(v) \vee r_\beta(v) = r_{\alpha \wedge \beta}(v)$, for all $\alpha, \beta \in \mathbb{R}_{>0}$, and that $\lambda \geq 1$, especially $1/\lambda \leq \lambda \leq \lambda^2$. The bounds we obtain need not be excessively precise, and we allow losing multiplicative factors frequently. Start with (3.46), and notation as in the proof of Lemma 3.4.

\begin{align*}
t_1 &= r_{3\lambda}^2(v) \vee r_{1/(3\lambda)}(v) \vee 3\lambda r_{1/(2\lambda)(v_{\gamma 1+\lambda} + 8\lambda^2)}(v) \vee \sup \{ s : v(s) \leq \delta \} \\
&\quad \vee 12\lambda \delta \vee (4\lambda M'(\lambda, \delta) + 1) \vee \sup \{ r : v(r) \leq 6\lambda^2 \delta \} \\
&\leq 3\lambda r_{1/(2\lambda)(v_{\gamma 1+\lambda} + 8\lambda^2)}(v) \vee C(\lambda, \delta).
\end{align*}

Next, $t_3 = t_1 \vee \sup \{ r : v(r) \leq h(\lambda)\delta \}$ since $h(\lambda)\delta \geq 6\lambda^2 \delta$. After that, $t \geq t_4 := \sup \{ t_3, r_{\lambda/(2+2H_0(\lambda, v))}(v) \}$, where $H_0(\lambda, v) = 2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1$. From this and (3.49) we deduce

\begin{align*}
t_4 \leq r_{1/(3\lambda H_0)}(v) \vee \sup \{ r : v(r) \leq h(\lambda)\delta \} \vee C(\lambda, \delta)
\end{align*}

and then

\begin{align*}
t_5 &\leq t_4 \vee \sup \{ r : v(r) \leq 8\delta \} \\
&\leq r_{1/(3\lambda H_0)}(v) \vee \sup \{ r : v(r) \leq 8h(\lambda)\delta \} \vee C(\lambda, \delta); \\
t_6 &\leq t_5 \vee 16\delta = t_5 \vee C(\lambda, \delta); \\
t_6 &\leq r_{1/(3\lambda H_0)}(v) \vee \sup \{ r : v(r) \leq 16h(\lambda)\delta \} \vee C(\lambda, \delta)
\end{align*}

As $t_7 = t_6 \vee t_0$, $t_0 = r_{1/(3\lambda)}(v)$ and $H_0 \geq 1$, the same bound applies to $t_7$. Next,

\begin{align*}
t_8 &= t_7 \vee r_{\lambda/(6H)}(v) \leq r_{\lambda/(6H)}(v) \vee \sup \{ r : v(r) \leq 16h(\lambda)\delta \} \vee C(\lambda, \delta)
\end{align*}

(remember that $H = 1 + H_0$ by definition). Thus

\begin{align*}
R^0_{\text{cl}} &= t_8/(6\lambda) \vee \sup \{ r : 2(v \uparrow 6\lambda)(H_0 + 1)v(r) \leq 8\lambda \} \\
&\leq t_8 \vee \sup \{ r : 2(H_0 + 1)v(6\lambda r) \leq 8\lambda \} \\
&\leq t_8 \vee \sup \{ r : 2(2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1) v(6\lambda r) \leq 8\lambda \} \\
&\leq t_8 \vee \sup \{ r : 6h(\lambda)v(6\lambda(1 + 8\lambda^2)r) \leq 8\lambda \} \\
&\leq r_{1/(6\lambda H)}(v) \vee C(\lambda, \delta) (1 + \sup \{ r : v(r) \leq C(\lambda, \delta) \}).
\end{align*}

This inequality implies (3.46) (one may take $n = 4$ there), since $H = 2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 2 \leq C(\lambda)(v \uparrow 1 + \lambda)^2$. Let us turn to (3.47). Start establishing a similar bound for $R_{\text{cl}}$, with notation as in the proof of Lemma 3.7. By (3.22),

\begin{align*}
R_{\text{cl}} &= 3R^0_{\text{cl}} \vee \frac{3\lambda}{2H} r_{1/(8\lambda H)}(v) \vee 192\lambda^2 \delta \vee r_{1/(6(v_{\gamma 2}) H_0(\lambda, v, \delta))}(v),
\end{align*}
where \( H_3 = (4\lambda^2 H + 2\lambda (v \uparrow \lambda)) (v \uparrow 2) \leq 6\lambda^2 H (v \uparrow 2) \leq 6\lambda^2 \tilde{H} \), hence by (3.52),
\[
R \leq 3\lambda r_{1/(8\lambda^2 \tilde{H}(v \uparrow 2))} (v) \lor C(\lambda, \delta) \left( 1 + \sup \{ r : v(r) \leq C(\lambda, \delta) \} \right).
\]
In the proof of Lemma 3.8, \( \tilde{R} \) was defined as a supremum of four terms:
\[
\tilde{R} = R \lor 2(2\lambda + 1) r_{1/(3\lambda)} (v) \lor r_{1/(2\tilde{H})} (v) \lor r_{1/(2LH_2)} (v) \lor L^{-1} \sup \{ r : v(r) \leq (12 + \tilde{h}(\lambda)) \delta \}
\]
\[
\leq R \lor 5\lambda^2 r_{1/(3\lambda \lambda_1 / (2\tilde{H}) \lambda_1 / (2LH_2)} (v) \lor C(\lambda, \delta).
\]
We need to bound the tracking constants \( \tilde{H} \) and \( H_2 \). By definition of \( \tilde{H} \) in the proof of Lemma 3.4, \( \tilde{H} = 4(v \uparrow 6\lambda) (2\tilde{h}(\lambda)(v \uparrow 1 + 8\lambda^2) + 1) \leq C(\lambda) (v \uparrow 1 + \lambda)^n \), where \( n_0 \) is large enough, and by (3.35) with \( k = 11 \lor 12\lambda(2\lambda + 1) \leq 36\lambda^2 \),
\[
H_2 \leq 4 (v \uparrow 3Lk)(v \uparrow 3\lambda) \tilde{H} \lor 2(v \uparrow 3\lambda) H \leq C(\lambda)(v \uparrow L)(v \uparrow 1 + \lambda)^n,\]
where \( n_1 \) is large enough. By (3.53) and the previous bounds, \( \tilde{R} \) may be taken as
\[
\tilde{R} = 5\lambda^2 r_{1/(C(\lambda)(v \uparrow L)(v \uparrow 1 + \lambda)^n_1)} \lor C(\lambda, \delta) (1 + \sup \{ r : v(r) \leq C(\lambda, \delta) \}).
\]
This is a precise form of (3.47). Finally, we must prove (3.48). With notation as in the proof of Lemma 3.10, \( R = r_{1/(2J^+)} (v) \lor \sup \{ r : v(r) \leq 284\delta \lor 584\delta / J^+ \} \), where \( J^+ = 2\tilde{H}_2 (v \uparrow 2) \) so that
\[
R \leq r_{1/(4\tilde{H}_2 (v \uparrow 2))} (v) \lor \sup \{ r : v(2r) \leq 584\delta / (2\tilde{H}_2) \},
\]
and we need to bound \( \tilde{H}_2 \). With notation as in the proof of Lemma 3.8, recall from (3.35) that \( \tilde{H}_2 \) can be bounded by
\[
\tilde{H}_2 = (2\tilde{H} + \tilde{h}(\lambda))(\delta + (v \uparrow 3\lambda)) \leq C(\lambda) \tilde{H} C(\lambda, \delta) (v \uparrow 3\lambda) (v \uparrow 3k)
\]
\[
\leq C(\lambda, \delta) (v \uparrow 1 + \lambda)^7 (v \uparrow 3\lambda) (v \uparrow 3 \cdot 36\lambda^2) (v \uparrow 3\lambda).
\]
Plugging this inequality in (3.54) yields the expected (3.48). \( \square \)

**Lemma 3.12** (Sublinear growth of tracking radii). *Let \( w \) be an admissible function. For all \( p \in \mathbb{R}_{\geq 0} \), define \( w_p(r) = w(p + r) \), and then denote by \( \tilde{R}_p \), resp. \( \tilde{R}_p \) the constants \( R(\lambda, \delta, w_p) \) and \( \tilde{R}(\lambda, \delta, w_p) \) of Lemma 3.10. There exist \( \tilde{K} = \tilde{K}(\lambda, \delta, w, L) \) and \( K = K(\lambda, \delta, w, L) \) in \( \mathbb{R}_{\geq 0} \) such that
\[
\tilde{R}_p \leq \tilde{K} w(p), \text{ and }
\]
\[
R_p \leq K w(p).
\]

**Proof.** By Lemma 3.11, there exists a positive integer \( n \) such that \( \tilde{R}_p \) and \( R_p \) may be taken as
\[
\tilde{R}_p = C(\lambda, \delta) r_{C(\lambda)(w_p \uparrow L)}^{-1} (w_p \uparrow 1 + \lambda)^{-n} (w_p)
\]
\[
\lor C(\lambda, \delta) (1 + \sup \{ r : w_p(r) \leq C(\lambda, \delta, L) \})
\]
\[
R_p = r_{C(\lambda)(w_p \uparrow 1 + \lambda)^{-n}} (w_p) \lor C(\lambda, \delta) (1 + \sup \{ r : w_p(r) \leq C(\lambda, \delta, L) \}).
\]
The rightmost terms \( C(\lambda, \delta) (1 + \sup \{ r : w_p(r) \leq C(\lambda, \delta, L) \}) \) are nonincreasing functions of \( p \), since \( \{ w_p \} \) is a nondecreasing sequence of functions, so that their dependence over \( p \) can be removed. Further, \( w_p \uparrow 1 + \lambda \) is a nonincreasing function
of \( p \) by Lemma 1.1 (1), hence \((w_p \uparrow 1 + \lambda)^{-n}\) is a nondecreasing function of \( p \). Thus (3.57) and (3.58) may be simplified as
\[
\overline{R}_p = C(\lambda, \delta) r_{C(\lambda) (w_p \uparrow 1 + \lambda)^{-n}} (w_p) \vee C(\lambda, \delta, L, w)
\]
\[
R_p = r_{C(\lambda) (w \uparrow 1 + \lambda)^{-n}} (w_p) \vee C(\lambda, \delta, L, w).
\]
Then by Lemma 1.1 (2), \( \overline{R}_p \leq C(\lambda, \delta, \lambda, w) \vee C(\lambda)(w \uparrow 2)(w \uparrow 1 + \lambda)^n w(p) \). This proves (3.55) for a constant \( \overline{K} = K(\lambda, \delta, L, w) \), and similarly there exists \( K = K(\lambda, \delta, L, w) \) such that \( R_p \leq K w(p) \), which is (3.56).

\[\square\]

4. On the sphere at infinity

4.A. Sublinearly quasiMöbius homeomorphisms. With geodesic boundaries of hyperbolic spaces in mind, we abstractly define sublinearly quasi\( \lambda \)-Möbius homeomorphisms between compact metric spaces:

**Definition 4.1.** Let \( u \) be an admissible function. Let \((\underline{\omega}, \overline{\omega}) \in \mathbb{R}^2 \) be a couple of constants. Let \((\Xi, \varrho)\) and \((\Psi, \vartheta)\) be metric spaces and let \( \varphi : \Xi \to \Psi \) be a homeomorphism. \( \varphi \) is a \((\underline{\omega}, \overline{\omega}, O(u))\)-sublinearly quasi\( \lambda \)-Möbius homeomorphism if there exist \( v = O(u), \varphi \in \mathbb{R}_{>1} \) and \( \mathcal{E} \in \mathbb{R}_{>0} \) such that for all \( \{\xi_1, \ldots, \xi_j\} \in \Xi^j \) with \( 0 < \inf_{i \neq j} \varrho(\xi_i, \xi_j) \leq \sup_{i \neq j} \varrho(\xi_i, \xi_j) < \mathcal{E} \),
\[
\underline{\omega} \log^+ \varrho(\xi_i) - v \left( \sup_{i \neq j} \left[ - \log \varrho(\xi_i, \xi_j) \right] \right) \leq \log^+ \varphi(\xi_i) \leq \overline{\omega} \log^+ \varrho(\xi_i) + v \left( \sup_{i \neq j} \left[ - \log \varrho(\xi_i, \xi_j) \right] \right)
\]

Note that one would only need a change of function \( v \) within the \( O(u) \)-class to compensate a different choice of \( \nu \). We call \( \underline{\omega}, \overline{\omega} \) and \( \alpha = \sup \{ \overline{\omega}, 1/\underline{\omega} \} \) the Lipschitz-Möbius constants of \( \varphi \).

Although this is not a direct consequence of Definition 4.1, sublinearly quasi\( \lambda \)-Möbius homeomorphisms between uniformly perfect spaces are stable under composition; we postpone the proof to subsection 4.B. Also note that in the definition one could replace the source and target distances with any equivalent real-valued kernels \( \varrho \) and \( \vartheta \), or even, if no special attention is required on precise Lipschitz-Möbius constants, with kernels such that \( \varrho^{\gamma_1} \) and \( \vartheta^{\gamma_2} \) are equivalent to \( \varrho \) and \( \vartheta \) for a pair of exponents \( \gamma_1, \gamma_2 \in \mathbb{R}_{>0} \). This occurs on geodesic boundaries when \( \varrho \) and \( \vartheta \) are visual quasimetrics while \( \varrho \) and \( \vartheta \) are visual distances.

Recall that, by Proposition 1.5, any large-scale sublinearly Lipschitz embedding \( f \) between proper geodesic Gromov-hyperbolic spaces induces a boundary map, which only depends on the \( O(u) \)-closeness class of \( f \) so that it can be denoted \( \partial_{\infty}[f]_{O(u)} \).

**Theorem 4.2.** Let \( u \) be an admissible function. Let \((\underline{\lambda}, \overline{\lambda}) \in \mathbb{R}^2 \) be expansion and Lipschitz constants. Let \( f : X \to Y \) be a \((\underline{\lambda}, \overline{\lambda}, O(u))\)-sublinearly biLipschitz equivalence between proper, geodesic hyperbolic spaces. Then \( \partial_{\infty}[f]_{O(u)} \) is a \((\underline{\lambda}, \overline{\lambda}, O(u))\)-sublinearly quasi\( \lambda \)-Möbius homeomorphism.

**Sketch of proof for Theorem 4.2.** Our argument is inspired from the lecture notes by Bourdon [3, Theorem 2.2] on Mostow rigidity and Tukia’s theorem; the main
ingredient is Lemma 3.10, which ensures that the geometric interpretation of the cross-difference (see Proposition 1.9 and Figure 4) subsists with a sublinear error when applying a sublinearly biLipschitz equivalence and measuring distances between $O(u)$-geodesics in the target space. Lemma 3.10 must be applied with care, though, since the control functions and tracking radii deteriorate as the Gromov products of endpoints grow. This is where Lemma 3.12 intervenes and certifies that the growth of tracking radii is sublinear with respect to Gromov products, so that the tracking estimates and their consequences are ultimately valid.

Proof. Fix basepoints $o$ in $X$ and $Y$, and let $w = O(u)$ be an admissible function such that $f$ is a $(\Delta X, w)$-sublinearly biLipschitz equivalence from $(X, o)$ to $(Y, o)$. For any quadruple $(\xi_1, \ldots, \xi_4) \in \partial^2_\infty X$, write for short $\eta_i = \partial f(\xi_i)$ for all $i \in \{1, \ldots, 4\}$, and for all $\varepsilon, \mathcal{E} \in \mathbb{R}_{> 0}$ such that $\varepsilon < \mathcal{E}$, let $F(\varepsilon, \mathcal{E})$ be the subspace of $\partial X$ defined by

$$\begin{cases}
\|\bar{\mathbb{X}}(\xi_i)\| > -\log_\mu \varepsilon, \\
\|\bar{\mathbb{X}}(\xi_i)\| > -\log_\mu \mathcal{E}.
\end{cases}$$

Note that, since $\partial_\infty X$ is compact the space defined by the first inequality is a neighborhood of the ends in $\partial^2_\infty X$, hence it suffices to prove the inequality

$$\bar{\Delta} \{\xi_i \} - v(\bar{\mathbb{X}}(\xi_i)) \leq [\eta_i] \leq \bar{\Delta} \{\xi_i \} + v(\bar{\mathbb{X}}(\xi_i))$$

for all $(\xi_i) \in F(\varepsilon, \mathcal{E})$, for some small $\varepsilon$ and $\mathcal{E}$ and $v = O(u)$. For any pair $(i, j) \in \{\{1, 4\}, \{2, 3\}\}$ let $\gamma_{ij}$ be a geodesic in $X$ with endpoints $\xi_i$ and $\xi_j$, resp. $\gamma_{ij}$ a geodesic in $Y$ with endpoints $\eta_i$ and $\eta_j$ such that $\gamma_{ij}(0) = p_{\gamma_{ij}}(o)$ and $\gamma_{ij}(0) = \gamma_{ij}(0)$. Finally, write $\gamma_{ij}(t) = f \circ \gamma_{ij}(t)$, and observe that $\gamma_{ij}$ is a $(\lambda, w^\prime)$-geodesic, where $\lambda = \lambda_{ij}^\prime(t) := w(|\chi_{ij}(r)| \vee |\chi_{ij}(r)|) \leq w(|\xi_1 | \xi_4 \rangle \vee (|\xi_2 | \xi_3 \rangle + r)$. Especially, $\gamma_{ij}$ is a $(\lambda, w_p)$ geodesic, where

$$w_p(r) := w(p + r),$$

and $p = (\xi_i | \xi_j)_o$. We shall apply Lemma 3.10 with $v^1 = w(\xi_i | \xi_j)_o$, $v^2 = w(\xi_j | \xi_j)_o$ and $v = w(\xi_1 | \xi_2)_o$. Assumptions (i), (ii) and (iii) follow from the definitions of $\gamma_{ij}$ and $\gamma_{ij}$. Then, recall from inequality (3.1) in Lemma 3.2 that if $|\chi_{ij}(0)| > t_\infty(|f(o)|, w)$, then for all $t \in \mathbb{R}$, $\frac{1}{3|\chi_{ij}(0)|} |\chi_{ij}(t)| \leq |\xi_{ij}(t)| \leq 3\lambda|\chi_{ij}(t)|$, and then

$$\forall t \in \mathbb{R}, |\tilde{\gamma}_{ij}(0)| \leq 3\lambda|\chi_{ij}(0)| \leq 3\lambda|\chi_{ij}(t)| \leq 9\lambda^2|\tilde{\gamma}_{ij}(t)|.$$

This is the right-hand side inequality of (v) with $L = 9\lambda^2$, which we fix for the rest of the proof. Observe that the lower bound needed on the radii $|\chi_{ij}(0)|$ is guaranteed as soon as $\mathbb{X}(\xi_i) \geq t_\infty(|f(o)|, w) = r_{1/3\lambda}(\lambda) \vee 3\lambda|f(o)|$. On the other hand, by Cornulier’s theorem 1.7 $\partial_\infty f$ is uniformly continuous on $\partial_\infty X$, so there exists $R_\in \mathbb{R}_{> 0}$ such that $\mathbb{X}(\xi_i) \geq R_\in \implies \mathbb{X}(\eta_i) \geq 60\delta$. Let $K = K(\lambda, w, \delta, L)$ be the constant from Lemma 3.12, and define

$$\mathcal{E} = \mu - R_\in \wedge t_\infty(|f(o)|, w) \vee r_{1/(3\lambda K)}(w).$$

Then as soon as $\mathbb{X}(\xi_i) > -\log_\mu \mathcal{E}$,

$$\begin{cases}
\mathbb{X}(\eta_i) \geq R_\in, \\
|\gamma_{ij}(0)| \leq L \inf_{\xi \in \mathbb{X}} |\gamma_{ij}(t)| \geq \mathbb{X}(\eta_i) \geq t_\infty(|f(o)|, w) \wedge \lambda, \\
|\tilde{\gamma}_{ij}(0)| \leq \mathbb{X}(\eta_i) \leq 9\lambda^2|\tilde{\gamma}_{ij}(t)| \geq t_\infty(|f(o)|, w) \wedge r_{1/(3\lambda K)}(w).
\end{cases}$$

The first line is the first condition in (iv), the second and third one are the assumption (v); we used (3.55) from Lemma 3.12 in the third line. By the conclusion
of Cornulier’s theorem 1.7 applied to both \( \partial_{\infty}[f] \) and to \( \partial_{\infty}[f]^{-1} \), there exists \( \varepsilon_0 \in \mathbb{R}_{>0} \) such that

\[
\overline{\mathbb{X}}(\xi_i) > -\log_\mu \varepsilon_0 \implies 2\lambda \overline{\mathbb{X}}(\xi_i) \geq \overline{\mathbb{X}}(\eta_i) \geq \frac{1}{2\lambda} \overline{\mathbb{X}}(\xi_i).
\]

Let \( K \) be the constant from Lemma 3.12. Define \( \varepsilon = \varepsilon_0 \wedge E \wedge \mu^{-2\lambda\tau_{1/3\lambda K}(w)} \). Then by (3.56) of Lemma 3.12, \( \overline{\mathbb{X}}(\xi_i) > -\log_\mu \varepsilon \implies \overline{\mathbb{X}}(\eta_i) \geq R\overline{\mathbb{X}}(\xi_i) \). Thus if \( (\xi_i) \in F(\varepsilon, E) \) then Lemma 3.10 applies to \((\gamma_{ij}, \tilde{\gamma}_{ij})\), and

\[
|d_Y(\gamma_{23}, \gamma_{14}) - d_Y(\tilde{\gamma}_{23}, \tilde{\gamma}_{14})| \leq Ju(\overline{\mathbb{X}}(\xi_i))(\overline{\mathbb{X}}(\eta_i)) \leq J(w \uparrow 2\lambda)w(\overline{\mathbb{X}}(\eta_i)).
\]

(4.1)

Thanks to Proposition 1.9, there exists \( C = C(\delta) \) in \( \mathbb{R}_{>0} \) such that

\[
\begin{aligned}
\begin{cases}
d_X(\chi_{14}, \chi_{23}) - C(\delta) & \leq \log^+ [\xi_i] \leq d_X(\chi_{14}, \chi_{23}) + C(\delta), \\
d_Y(\gamma_{14}, \gamma_{23}) - C(\delta) & \leq \log^+ [\eta_i] \leq d_Y(\gamma_{14}, \gamma_{23}) + C(\delta).
\end{cases}
\end{aligned}
\]

In view of (4.1) and the previous set of inequalities, it suffices to prove

(4.2) \[ \Delta d_X(\chi_{14}, \chi_{23}) - v(\overline{\mathbb{X}}(\xi_i)) \leq d_Y(\gamma_{14}, \gamma_{23}) \leq \Delta d_X(\chi_{14}, \chi_{23}) + v(\overline{\mathbb{X}}(\xi_i)) \]

for some function \( v = O(u) \). Start with the left-hand side inequality. Letting \( \tilde{s}_1, \tilde{s}_2 \in \mathbb{R} \) be such that \( |f \circ \chi_{14}(\tilde{s}_1) - \chi_{23}(\tilde{s}_2)| = d(\tilde{\gamma}_{14}, \tilde{\gamma}_{23}) \),

\[
\begin{aligned}
\Delta d(\chi_{14}, \chi_{23}) - w(\overline{\mathbb{X}}(\xi_i)) & \leq \Delta |\chi_{14}(\tilde{s}_1) - \chi_{23}(\tilde{s}_2)| - w(|\chi_{14}(\tilde{s}_1)| \lor |\chi_{23}(\tilde{s}_2)|) \\
& \leq |f \circ \chi_{14}(\tilde{s}_1) - f \circ \chi_{23}(\tilde{s}_2)| \\
& = d(\tilde{\gamma}_{14}, \tilde{\gamma}_{23}) \\
& \leq d(\gamma_{14}, \gamma_{23}) + J(w \uparrow 2\lambda) \overline{\mathbb{X}}(\eta_i),
\end{aligned}
\]

(4.1)

hence

(4.3) \[ \Delta d(\chi_{14}, \chi_{23}) \leq d(\gamma_{14}, \gamma_{23}) + (1 + J(w \uparrow 2\lambda)) w(\overline{\mathbb{X}}(\eta_i) + \overline{\mathbb{X}}(\xi_i)). \]

Let us proceed in the same way for the right-hand side of (4.2). By Lemma 3.10, letting \( s_1, s_2 \in \mathbb{R} \) be such that \( |\chi_{14}(s_1) - \chi_{23}(s_2)| = d(\chi_{14}, \chi_{23}) \),

\[
\begin{aligned}
d(\gamma_{14}, \gamma_{23}) & \leq d(\tilde{\gamma}_{14}, \tilde{\gamma}_{23}) + J(w \uparrow 2\lambda)(\overline{\mathbb{X}}(\eta_i)) \\
& \leq |\tilde{\gamma}_{14}(s_1) - \tilde{\gamma}_{23}(s_2)| + J(w \uparrow 2\lambda)(\overline{\mathbb{X}}(\eta_i)) \\
& \leq \Delta d(\chi_{14}, \chi_{23}) + (1 + (w \uparrow 2\lambda)^2) w(\overline{\mathbb{X}}(\xi_i)).
\end{aligned}
\]

(4.4)

Setting \( v = (1 + (w \uparrow 2\lambda)^2) \) this proves (4.2) and the theorem. \( \square \)

4B. Properties of sublinearly quasiMöbius homeomorphisms. After simplifying the cross-ratio estimates when two, resp. one points are far away, one obtains that sublinearly quasiMöbius homeomorphisms between appropriate spaces are Hölder, resp. almost quasisymmetric, see figure 9. Precisely we work under the following assumption (Buyalo and Schroeder [5, 7.2] or Mackay and Tyson [23, 1.3.2])); see however Remark 4.8.

Definition 4.3. Let \( \Xi \) be a metric space. Then \( X \) is uniformly perfect if there exists \( \tau \in (0, 1) \) such that for every ball \( B \subseteq \Xi \), the annulus \( B \setminus \tau B \) is non-empty. Note that in the definition, for any positive integer \( k \), up to replacing \( \tau \) with \( \tau^k \) one can assume for free that \( B \setminus \tau B \) has \( k \) points. Uniform perfectness is granted for boundaries of non-elementary hyperbolic groups, or for connected spaces.
Figure 9. Hyperbolic ideal tetrahedra. On the left two points are far away from the remaining pair; on the right one point is far from the remaining triple.

Proposition 4.4 (“almost” Hölder continuity). Let $(\Xi; \rho)$ and $(\Psi, \vartheta)$ be compact uniformly perfect metric spaces and let $\varphi : \Xi \to \Psi$ be a $(\lambda, O(u))$-sublinearly quasiMöbius homeomorphism. Then $\varphi$ admits a modulus of continuity

$$\omega(t) = \exp(\lambda \log t + \nu(-\log t)),$$

with $\nu = O(u)$.

Remark 4.5. As a consequence, under the same assumptions and for all $\alpha \in (0, \lambda)$, $\varphi$ is $\alpha$-Hölder continuous. Thus Theorem 4.2 may be seen as a strengthening of Cornulier’s theorem (with the restriction made on spaces).

Proof. Let $E$ be the constant from Definition 4.1 associated to $\varphi$, and let $\tau$ be such that $\Xi$ is $\tau$-uniformly perfect. Define

$$D_1 := \frac{\tau^4}{4} \left( E \wedge \frac{\text{diam} \Xi}{3} \right)$$

and

$$D'_1 = \inf \left\{ \vartheta(\varphi(\xi_1), \varphi(\xi_2)) : \xi_1, \xi_2 \in \Xi, \vartheta(\varphi(\xi_1), \varphi(\xi_2)) \geq (\tau^{-1} - 1) D_1 \right\}.$$

Let $\xi_1$ and $\xi_2$ in $\Xi$ be such that $\rho(\xi_1, \xi_2) < D_1$. The ball $B := \tau^{-4} B(\xi_1, D_1)$ is not equal to $\Xi$ (this would indeed contradict the definition of $D_1$), so there exists $\alpha \in B \setminus \tau B$ and $\beta \in \tau^2 B \setminus \tau^3 B$. By the triangle inequality

$$\vartheta(\alpha, \beta) \geq (\tau^{-3} - \tau^{-2}) D_1 \quad \text{and} \quad \vartheta(\beta, \xi_2) \geq (\tau^{-1} - 1) D_1,$$

for short

$$\inf_i \vartheta(\alpha, \xi_i) \wedge \vartheta(\beta, \xi_i) \geq (\tau^{-1} - 1) D_1.$$

Further, by definition of $D'_1$, a similar inequality holds in the target space:

$$\inf_i \vartheta(\varphi(\alpha), \varphi(\xi_i)) \wedge \vartheta(\varphi(\beta), \varphi(\xi_i)) \geq D'_1.$$

By definition of the metric cross ratios,

$$\frac{\tau^{-1} - 1}{\text{diam}(\Xi)} D_1^2 \leq [\alpha, \xi_1, \xi_2, \beta] \leq \frac{\text{diam}(\Xi)^2}{D_1} \frac{1}{\vartheta(\xi_1, \xi_2)},$$

$$\frac{\tau^{-1} - 1}{\text{diam}(\Psi)} D'_1^2 \leq [\alpha', \varphi(\xi_1), \varphi(\xi_2); \beta'] \leq \frac{\text{diam}(\Psi)^2}{D'_1} \frac{1}{\vartheta(\varphi(\xi_1), \varphi(\xi_2)).$$
thus $\log^+[\alpha, \xi_1, \xi_2, \beta] - \log^+ \frac{1}{\bar{g}(\xi_1, \xi_2)}$ and $[\alpha', \varphi(\xi_1), \varphi(\xi_2); \beta'] - \log^+ \frac{1}{\bar{g}(\varphi(\xi_1), \varphi(\xi_2))}$ are bounded by

$$L = 2 \left( \left| \log \frac{1}{\tau} \right| + | \log D_1 | + | \log \text{diam}(\Xi) \right)$$

and $L' = (| \log \mathcal{D}' | + | \log \text{diam}(\Psi) |)$ respectively. Now by hypothesis $\varphi$ is $(\underline{\lambda}, \overline{\lambda}, v_0)$-sublinearly quasiMöbius for some $v_0 = O(u)$. By definition, setting $v = v_0 + L$, for all $\xi_1, \xi_2$ such that $\varphi(\xi_1, \xi_2) < D_1 \wedge 1$ (note that $\{\xi_1, \xi_2\}$ is the closest pair among $\xi_1, \xi_2, \alpha, \beta$),

$$\log \vartheta(\varphi(\xi_1), \varphi(\xi_2)) \leq \overline{\lambda} \left( - \log \varrho(\xi_1, \xi_2) \right) + v \left( - \log \varrho(\xi_1, \xi_2) \right),$$

$$\log \vartheta(\varphi(\xi_1), \varphi(\xi_2)) \geq \underline{\lambda} \left( - \log \varrho(\xi_1, \xi_2) \right) - v \left( - \log \varrho(\xi_1, \xi_2) \right).$$

In particular the conclusion (4.5) is equivalent to the second inequality. □

The Hölder continuity (4.9) intervenes in the following analog of Lemma 3.2, a technical refinement of definition 4.1.

**Lemma 4.6.** Let $u$ be an admissible function. Let $(\underline{\alpha}, \overline{\alpha})$ be Lipschitz-Möbius data. Let $\varphi$ be a $(\underline{\alpha}, \overline{\alpha}, O(u))$ sublinearly quasiMöbius homeomorphism between compact uniformly perfect spaces $(\Xi, \varrho)$ and $(\Psi, \vartheta)$. There exist $\bar{v} = O(u)$ and $\varepsilon_2 \in \mathbb{R}_{>0}$ such that for all $(\xi_1, \ldots, \xi_4) \in \Xi^4$ with $0 < \inf_{i \neq j} \varrho(\xi_i, \xi_j) \leq \sup_{i \neq j} \varrho(\xi_i, \xi_j) < \varepsilon_2$,

$$\underline{\alpha} \log \varrho(\xi_i) - \bar{v} \left( \sup_{i \neq j} \left[ - \log \varrho(\xi_i, \xi_j) \right] \right) \leq \log^+ \left[ \varrho(\xi_i) \right]$$

$$\overline{\alpha} \log \varrho(\xi_i) + \bar{v} \left( \sup_{i \neq j} \left[ - \log \varrho(\xi_i, \xi_j) \right] \right) \geq \log^+ \left[ \varrho(\xi_i) \right],$$

where $\bar{v} = O(u)$.

**Proof.** Let $v = O(u)$ be such that $\varphi$ is a $(\underline{\alpha}, \overline{\alpha}, v)$-sublinearly quasiMöbius homeomorphism. Then by Proposition 4.4 and the fact that $v$ is sublinear, there is $\varepsilon_H \in \mathbb{R}_{>0}$ such that for all $(\xi_1, \ldots, \xi_4) \in \Xi$ distinct and such that $\inf \varrho(\xi_1, \xi_2) \leq \varepsilon_H \wedge e^{-\log v} \underline{\alpha} \varrho(v)$,

$$\sup_{i \neq j} \left[ - \log \varrho(\varphi(\xi_i), \varphi(\xi_j)) \right] \geq (\underline{\lambda}/2) \sup_{i \neq j} \left[ - \log \varrho(\varphi(\xi_i), \varphi(\xi_j)) \right].$$

The conclusion follows, with $\bar{v} = \left( v \uparrow \frac{2}{\alpha} \right) v$. □

**Proposition 4.7.** Let $u$ be an admissible function. $O(u)$-sublinearly quasiMöbius homeomorphisms form a groupoid $\mathcal{M}_{O(u)}$ with uniformly perfect compact metric spaces as objects. Composition in $\mathcal{M}_{O(u)}$ has a multiplicative effect on Lipschitz-Möbius and reverse Lipschitz-Möbius constants.

**Proof.** Let $\Omega, \Xi, \Psi$ be compact metric spaces, and let $\varphi: \Xi \to \Psi$ and $\psi: \Omega \to \Xi$ be $O(u)$-quasiMöbius homeomorphisms, with respective parameters $(\underline{\alpha}_\varphi, \overline{\alpha}_\varphi, v_\varphi)$ and $(\underline{\alpha}_\psi, \overline{\alpha}_\psi, v_\psi)$. Let $(\omega_1, \ldots, \omega_4)$ be a 4-tuple of distinct points in $\Omega$; for $i \in \{1, \ldots, 4\}$ set $\xi_i = \psi(\omega_i)$ and $\eta_i = \varphi(\xi_i)$. Set

$$w = \overline{\alpha}_\psi v_\varphi + \left( v \uparrow \frac{2}{\alpha_\varphi} \right) v_\psi.$$

Then by the previous lemma, $\varphi \circ \psi$ is a $(\underline{\alpha}_\psi, \underline{\alpha}_\varphi, \overline{\alpha}_\psi, \overline{\alpha}_\varphi, w)$-sublinearly quasiMöbius homeomorphism. □
Remark 4.8. The assumption of uniform perfectness (Definition 4.3) could be dropped in Proposition 4.7 if one adopts the heavier form of Definition 4.1 given by the inequalities of Lemma 4.6. It follows from the proof of Theorem 4.2 that this more restrictive definition is still valid for boundary maps of sublinearly biLipschitz equivalences.

We now turn to the scale-sensitive moduli distortion property of sublinearly quasiMöbius homeomorphisms. Recall that for any $\xi$ in a metric space $\Xi$, the annulus $A = B(\xi, s) \setminus B(\xi, r)$ is said to have a modulus $\mathfrak{M} = \log(s/r)$.

Proposition 4.9. Let $\varphi$ be a $((\underline{\lambda}, \overline{\lambda}, \rho(u))-sublinearly\ quasi\ M{"o}bius$ homeomorphism between spaces $(\Xi, g)$ and $(\Psi, \vartheta)$. Assume that $\Xi$ is uniformly perfect. There exist $D_1 \in \mathbb{R}_{>0}$ and $w = O(u)$ such that the following holds: for any annulus of inner radius $r \in \mathbb{R}_{>0}$ and outer radius $R \in (r, D_1]$. Then $\varphi(A)$ is contained in an annulus of modulus $2\lambda \mathfrak{M} + w(-\log r)$.

Proof. Define $D_1$ and $D'_1$ as in the proof of Proposition 4.4. For any triple $(\xi_1, \xi_2, \xi_3) \in \Xi^3$ such that $\{\xi_1, \xi_2, \xi_3\}$ has diameter less than $D_1$ one can find $\omega \in \tau^{-4}B(\xi_1, D_1) \setminus \tau^{-3}B(\xi_1, D_1)$. Define $\omega' = \varphi(\omega)$. By the triangle inequality and the definition of $D'_1$

$$\inf_i \vartheta(\omega, \xi_i) \geq (\tau^{-3} - 1)D_1 \geq \frac{1 - \tau}{\tau} D_1,$$

where $\eta_i = \varphi(\xi_i)$ for $i \in \{1, 2, 3\}$. Define $D_1 = \frac{1}{1 - \tau} D_1$. Applying the definition of the metric cross-ratio we deduce from the previous inequalities

$$(4.10) \quad \left| \log[\omega, \xi_1, \xi_2, \xi_3] - \log \frac{\vartheta(\xi_1, \xi_3)}{\vartheta(\xi_1, \xi_2)} \right| \leq 2 |\log \text{diam}(\Xi)| \lor |\log D_2|$$

$$(4.11) \quad \left| \log[\omega', \eta_1, \eta_2, \eta_3] - \log \frac{\vartheta(\eta_1, \eta_3)}{\vartheta(\eta_1, \eta_2)} \right| \leq 2 |\log \text{diam}(\Xi)| \lor |\log D'_1|.$$

Denote by $\mathcal{L}$, resp. $\mathcal{L}'$ the right-hand side bounds of (4.10), resp. (4.11). Let $r \in \mathbb{R}_{>0}$ and $2\mathfrak{M} \in \mathbb{R}_{>0}$ be such that $R = r \exp(2\mathfrak{M}) \leq D_1$. Fix $\xi_1$ and write $B = B(\xi_1, r)$. Fix $\xi_2$ in $\tau B \setminus \tau^2 B$. For any $\xi_3 \in A = B(R) \setminus B(r)$ the triangle inequality gives $\vartheta(\xi_1, \xi_2) \land \vartheta(\xi_1, \xi_3) \land \vartheta(\xi_3, \xi_2) \leq (1 - \tau) \land \tau^2 r$.

Let $v_0$ be such that $\varphi$ is $(\underline{\lambda}, \overline{\lambda}, v_0)$-quasiMöbius. Define $v_1 = v_0 + \mathcal{L} \lor \mathcal{L}'$ and then $v_2 = (v_1 \uparrow (1 - \tau) \land \tau^2) v_1$. Applying Definition 4.1 to $\varphi$ for $(\omega, \xi_1, \xi_2, \xi_3)$ together with (4.10) and (4.11), one obtains the set of inequalities

$$\log \frac{\vartheta(\eta_1, \eta_3)}{\vartheta(\eta_1, \eta_2)} \leq \log \frac{\vartheta(\eta_1, \eta_3)}{\vartheta(\eta_1, \eta_2)} \leq \underline{\lambda} \log + \frac{\vartheta(\xi_1, \xi_3)}{\vartheta(\xi_1, \xi_2)} + v_1 (1 - \log (1 - \tau) \land \tau^2) r$$

$$\leq \underline{\lambda} \mathfrak{M} + v_2 (\log r) \leq 2\underline{\lambda} \mathfrak{M} + v_2 (\log r).$$

Hence for any $\xi_3, \xi_3' \in A$, by the triangle inequality in $\mathbb{R}$, using $\vartheta(\eta_1, \eta_2)$ as an intermediate point,

$$(4.12) \quad \left| \log \frac{\vartheta(\eta_1, \eta_3)}{\vartheta(\eta_1, \eta_3')} \right| \leq 2\underline{\lambda} \mathfrak{M} + 2v_2 (\log r) - 4\lambda \log r \leq 2\underline{\lambda} \mathfrak{M} + w(-\log r),$$
where \( w = O(u) \). The proposition follows from the last statement. The expansion constant \( \Lambda \) would intervene in lower bounds on \( \frac{\delta(\varphi(\xi_1), \varphi(\xi_2))}{\delta(\varphi(\xi_1), \varphi(\xi_2))} \) for \( \xi_2 \) in the internal ball, and \( \xi_3 \) outside the external ball, centered at \( \xi_1 \). \( \square \)

This last property of sublinearly Möbius maps will be of use in section 5 where we implement some measure theory on the boundary. There is still a need to reformulate it slightly, however, since we will be then working with balls rather than annuli, and quasiisometrics rather than true distances. In that purpose, we introduce the following terminology: for any \( s \in \mathbb{R}_{>0} \), if \( B \) is a quasiball \( B = B^\beta(\Xi, r) \) where \( \hat{\vartheta} \) is a kernel equivalent to the distance in \( X \), then \( sB \) is \( B^\beta(\Xi, sr) \). If \( \Xi \) is \( \tau \)-uniformly perfect for every \( \tau \in (0, 1) \) with respect to \( \hat{\vartheta} \) (for instance, if it is connected) this is a continuous operation of \( \mathbb{R}_{>0} \) on the space of quasiballs.

**Proposition 4.10.** Assume that \( (\Xi, \varrho) \) and \( (\Psi, \vartheta) \) are compact connected topological manifolds, and that \( \varphi : \Xi \to \Psi \) is a \( (\Lambda, \varrho, O(u)) \)-sublinearly quasiMöbius homeomorphism. Let \( Q \in \mathbb{R}_{>1} \) be a constant. Let \( \hat{\varrho} \), resp. \( \hat{\vartheta} \) be an equivalent kernel on \( \Xi \), resp. on \( \Psi \). Then for any \( \alpha \in (0, \Lambda) \) and \( \beta \in (\varrho, +\infty) \) there exists \( w = O(u) \) (depending on \( Q \)) such that for any \( \hat{\varrho} \)-quasiball \( B \subset \Xi \) with center \( \xi \) and small enough radius \( r \) there exists a \( \vartheta \)-quasiball \( B' \) in \( \Psi \), and

\[
\begin{align*}
\left\{ \begin{array}{l}
r^\beta \leq \text{radius}(B') \leq r^\alpha \\
B' \subseteq \varphi(Q^{-1}B) \subseteq \varphi(B) \subseteq Q^\lambda e^{w(-\log r)}B'.
\end{array} \right.
\end{align*}
\]

**Remark 4.11.** Though this would be valid, we do not include in the conclusion that \( B \) have center \( \varphi(\xi) \), since it will not be required in section 5.

**Proof.** The statement for any equivalent kernel follows from the particular case when \( \hat{\varrho} = \varrho \) and \( \hat{\vartheta} = \vartheta \). Let \( B'' \setminus B' \) be an annulus containing \( \varphi(B'' \setminus B') \). Since \( \varphi \) is a homeomorphism, images of balls, resp. spheres by \( \varphi \) are topological balls, resp. spheres. By the Jordan-Brouwer separation theorem, \( \varphi(Q^{-1}B) \) is one of the two connected components of \( \Psi \setminus \varphi(\partial(Q^{-1}B)) \), and by Proposition 4.4, if \( r \) is small enough its diameter is bounded by \( r^\alpha \). Since \( \varphi(\partial(Q^{-1}B)) \subset B'' \setminus B' \), \( B'' \subseteq \varphi(Q^{-1}B) \) and \( \text{radius}(B') \leq r^\alpha \). By Proposition 4.9 \( B'' \) can be written \( Q^\lambda e^{w(-\log r)}B' \). Finally, by Proposition 4.4, for all \( \beta \in (\varrho, \beta) \), \( \text{diam}(B'') \geq \text{diam}(\varphi(B)) \geq r^\beta \) if \( r \) is small enough. This implies the lower bound on \( \text{radius}(B') \) for \( r \) small enough. \( \square \)

5. **RIEMANNIAN NEGATIVELY CURVED HOMOGENEOUS SPACES**

5.A. **Setting.** Simple Lie groups of real rank one with left invariant metrics are mentioned early in Gromov’s essay as important examples of \( \delta \)-hyperbolic spaces [20, 1.5(2)] and it is natural to ask to which extent they – or their quasiisometrically related symmetric spaces of noncompact type – differ on the large scale. Beyond these examples, it was proved in 1974 by E. Heintze [22, § 2] that any connected homogeneous negatively curved Riemannian manifold is the principal space of a solvable Lie group \( S = N \rtimes \alpha \mathbb{R} \), where \( N \) is nilpotent with Lie algebra \( \mathfrak{n} \) and \( \alpha \in \text{Der}(\mathfrak{n}) \) is such that for any compact neighborhood \( K \) of 1 in \( N \), \( \bigcup_{t \geq 0} \exp(t\alpha)K = N \). Such an \( S \) is called a Heintze group.

For a principal space \( X \) of the Heintze group \( S \), denote by \( \omega \) the endpoint on \( \partial_X \) (in positive time) of the orbits of the \( \mathbb{R} \) factor, and by \( \partial_X^* \) the punctured boundary \( \partial_X \setminus \{ \omega \} \). Any choice of a basepoint \( o \in X \) will determine a chart \( \Phi : \partial_X^* \to N \)
by letting $(\omega \xi)$ be the $\Phi(\xi)$-left translate of the $R$ factor in $(X, o) \simeq (S, 1)$, and a horofunction $-t : X \to R$ from $\omega$ and such that $t(o) = 0$.

5.B. Quasimetrics and measures on the punctured boundary. From now on, we make an assumption that $S$ is purely real, i.e. $\alpha$ has only positive real eigenvalues. This is not restrictive as far as large-scale properties are concerned, due to the following fact:

**Proposition 5.1.** Any Heintze group is quasihomothetic to a purely real Heintze group.

Proposition 5.1 follows from a special case of a result by D.Alekseevski [1, Theorem 3.3]. See also Cornulier, [9, Corollary 5.16] for a generalized form.

For any $s \in R_{>0}$ there is a homomorphism $N \times_\alpha R \to N \times_\alpha R$, $(n, t) \mapsto (n, ts)$. Up to rescaling the operation of $R$, we will work under a normalization assumption:

**Definition 5.2.** A purely real Heintze group $N \times_\alpha R$ is normalized if the smallest eigenvalue of $\alpha$ is equal to 1. In this case, the eigenvalues are ordered in increasing order, $1 = \lambda_1, \ldots, \lambda_r$ and one defines $p = \text{tr} \alpha$.

**Lemma 5.3.** Choose a horofunction $\beta$ from $\omega$ in $X$, and let $\hat{\omega}$ be the visual quasimetric on $\partial^*_\infty X$ with parameter $e$ with respect to $\beta$. $\hat{\omega}$ is an $N$-invariant, $S$-equivariant adapted kernel on $\partial^*_\infty X$; precisely

\[ \forall \xi_1, \xi_2 \in \partial^*_\infty X, \hat{\omega}(s\xi_1, s\xi_2) = e^t \hat{\omega}(\xi_1, \xi_2), \]

if $s = (n, t)$ in the semidirect product decomposition $N \rtimes R$.

**Proof.** Applying $s$ is equivalent to removing $t$ to the horofunction $\beta$. $\square$

We refer to $\hat{\omega}$ as the homogeneous quasimetric on the punctured boundary; it is indeed a quasimetric (see e.g. Buyalo and Schroeder [5, 3.3]). Different, equally natural choices for $\hat{\omega}$ are possible; under the constraint of satisfying (5.1) and a quasiultrametric inequality they would lead to equivalent kernels. We shall give later (Lemma 5.5) a sufficient condition for $\hat{\omega}$ to be equivalent to a true distance. For the moment however we only draw measure-theoretic conclusions.

By definition, $N$ operates on $\partial^*_\infty X$, and then on the space of measures on $\partial^*_\infty X$; the invariant subspace is an affine line $\mathcal{L}$, by uniqueness of the Haar measure of $N$ up to scaling. This operation extends to $S \rtimes \mathcal{L}$ via its modular function: for any $\mu \in \mathcal{L}$, for any $\hat{\omega}$-quasiball $B$,

\[ \forall s \in S, \mu(sB) = \Delta(s)\mu(B), \]

where $\Delta(s) = \exp(t \cdot \text{tr} \alpha) = e^{pt}$ if $s = (n, t)$, and we recall that $p$ is the trace of $\alpha$.

5.C. Horizontal lines and horizontal curves. Let $n_1(\alpha) = \ker(\alpha - 1)$. In the tangent space of $\partial^*_\infty X$, the distribution $\Phi^* n_1(\alpha)$ does not depend on the chart $\Phi : \partial^*_\infty X \to N$. We refer to it as the horizontal distribution, and denote it by $\tau$ (not forgetting the left action of $N$). For any $N$-invariant line $L$ in $\tau$, denote by $\Gamma_L$ the family of horizontal $L$-lines in $\partial^*_\infty X$, that is, smooth horizontal curves $\gamma$ tangent to $L$. The space $\Gamma_L$ can be identified with the homogeneous space $N/R$, where $R$ is a one-parameter subgroup of $N$ whose infinitesimal generator represents $\Phi_L$. Since $N$ is a nilpotent Lie group it is unimodular, especially $\Delta_N$ is constant along $R$ so that $\Gamma_L$ possesses a Haar measure $\rho$, following A.Weil [29, § 9].
**Lemma 5.4.** Let $L$ be as above, and let $\mu$ be a $N$-invariant measure on $\partial^\infty X$. Then for any $Q \in \mathbb{R}_{>0}$ there exists $c \in \mathbb{R}_{>0}$ (depending on $\mu$, $Q$ and $L$) such that for any $\widehat{\rho}$-quasiball $B$,

$$
(5.3) \quad \rho \{ \gamma \in \Gamma_L : \gamma \cap B \neq \emptyset \} = c\mu(Q^{-1}B)^{(p-1)/p}.
$$

**Proof.** $S$ operates simply transitively on the space of $\widehat{\rho}$-quasiballs, while $N$ operates simply transitively on their centers preserving radii, and $\theta : B \mapsto \{ \gamma : \gamma \cap B \neq \emptyset \}$ defines a $S$-equivariant map. Hence it suffices to show $(5.3)$ for a one-parameter family of balls $\{ e^tB \}_{t \in \mathbb{R}}$. Let $v \in n_1(\alpha)$ be a nonzero vector such that $[\Phi^*v] = L$. Since $v \in n_1(\alpha)$, the linear map $\alpha$ operates on $n/(\mathbb{R}v)$ with trace $p-1$, and $\rho \{ \gamma \in \Gamma_L : \gamma \cap e^tB \neq \emptyset \}$ is proportional to $e^{t(p-1)}$. On the other hand, by Lemma 5.3 and (5.2), $\mu(Q^{-1}tB)$ is proportional to $e^{tp}$, hence $\mu(Q^{-1}tB)^{(p-1)/p}$ is proportional to $e^{(p-1)t}$ as well. \qed

**Lemma 5.5.** Assume that $S$ is normalized (Definition 5.2) and that the operation of the derivation $\alpha$ on $n^{ab}$ is scalar, hence the identity. Then

1. $(N, \alpha)$ is a Carnot graded group, i.e.
   a. $n$ admits a grading $(n_i)$ by $\mathbb{Z}_{>0}$ such that $n_i = \ker(\alpha - i)$
   b. $n$ is generated by $n_1$.
2. Let $\| \cdot \|$ be a norm on $n_1$, and let $\Phi : \partial^\infty X \to N$ be a chart. Then $\widehat{\rho}$ is equivalent to a subRiemannian Carnot-Caratheodory metric

$$
d_{CC}(\xi_1, \xi_2) = \inf \{ \ell(\gamma) : \gamma \in \Gamma(\xi_1, \xi_2) \},
$$

where $\Gamma(\xi_1, \xi_2)$ denotes the space of absolutely continuous curves $[0,1] \to \partial^\infty X$ between $\xi_1$ and $\xi_2$ with derivative almost everywhere in the horizontal distribution $\tau$, and $\ell(\gamma) = \int_{[0,1]} \| \Phi_*\gamma' \|$ is the length of $\gamma$.

In this case, $X$ is said to be of Carnot type (following Cornulier’s terminology).

If $X$ is of Carnot type, condition (1b) ensures that $\Gamma(\xi_1, \xi_2)$ is never empty and $d_{CC}$ takes finite values.

**Proof.** See the survey of Cornulier [10, 2.G.1 and 2.G.2] for (1). Further, $s = (n, t) \in S$ acts on the space of horizontal curves sending $\Gamma(\xi_1, \xi_2)$ on $\Gamma(s\xi_1, s\xi_2)$ and multiplying lengths by $e^t$, hence

$$
(5.4) \quad d_{CC}(s\xi_1, s\xi_2) = e^td_{CC}(\xi_1, \xi_2)
$$

for all $\xi_1, \xi_2 \in \partial^\infty X$. Select $\xi \in \partial^\infty X$. Since $d_{CC}$ and $\widehat{\rho}$ are both quasi-metrics, they are continuous, hence bounded, over unit quasiballs of each other centered at $\xi$. Finally, $S$ operates transitively on the spaces of quasiballs of $\widehat{\rho}$ and $d_{CC}$. Hence $\widehat{\rho}$ and $d_{CC}$ are equivalent (the control constants depend on $\| \cdot \|$). \qed

5.D. Volumes of quasiballs and intersecting horizontal lines.

**Lemma 5.6.** Let $q \in \mathbb{R}_{\geq 1}$ be a constant and let $X$ be a proper metric space. Let $\widehat{\rho}$ be an equivalent kernel on $X$ with quasi-ultrametric constant $q$. There exists a constant $Q$ depending on $q$, such that for any countable covering $\mathcal{B}$ of $X$ by $\widehat{\rho}$-quasiballs, there exists an extraction $\mathcal{B}'$ of $\mathcal{B}$ whose elements are disjoint and such that $\{QB\}_{B \in \mathcal{B}'}$ is a covering of $X$.

**Proof.** See A.P. Morse, [24, Theorem 3.4], or Federer [17, 2.8.4-2.8.6]. \qed
In the following, whenever \( q \in \mathbb{R}_{\geq 1} \) is a constant, \( Q \) is another constant depending on \( q \) defined by the previous lemma.

**Lemma 5.7** (adapted from P. Pansu, [26, Lemme 6.3]). Let \( \mathcal{X} \) be a proper metric space, and let \( \Gamma \) a measured space of curves on \( \mathcal{X} \) (denote its measure by \( \rho \)). Let \( p \in \mathbb{R} > 1 \) and \( q \in \mathbb{R}_{\geq 1} \) be constants. Let \( \hat{d} \) be a kernel on \( \mathcal{X} \), equivalent to the original distance and with a \( q \)-quasiumetric inequality. Let \( U \) be an open, bounded subset of \( \mathcal{X} \), endowed with Borel measures \( \mu \) and \( \nu \), such that for any \( \hat{d} \)-quasiball \( B \) contained in \( U \),

\[
\rho \{ \gamma \in \Gamma : \gamma \cap B \neq \emptyset \} \leq \mu \left( Q^{-1} B \right)^{(p-1)/p}.
\]

For all \( \gamma \in \Gamma \) and for all \( r > 0 \), set

\[
\Phi_r^1(\gamma) = \inf_{F} \sum_{B \in F} \phi(B),
\]

where \( \phi(B) := \nu \left( Q^{-1} B \right)^{1/p} \), the infimum taken over countable coverings \( F \) of \( \gamma \cap U \) with balls of radius \( r \) exactly, contained in \( U \). Then

\[
\int_{\Gamma} \Phi_r^1(\gamma) d\rho \leq \nu(U)^{1/p} \mu(U)^{(p-1)/p}.
\]

For Lemma 5.7, Pansu’s proof can be reproduced almost verbatim [26, Lemme 6.3], with the only differences of using Lemma 5.6 instead of the covering lemma used by Pansu, having \( r \) fixed and not going to the limit at the end. The argument is based on the Hölder inequality; in a more general setting it is aimed at bounding a discretized version of the conformal modulus, and then to obtain lower bounds for the conformal dimension, [25, § 2 and 3].

**Lemma 5.8** (compare [26, Proposition 6.5]). Let \((N, \alpha)\) and \((N', \alpha')\) be Carnot groups with grading derivations \( \alpha, \alpha' \), normalized, with positive eigenvalues, of traces \( p \) and \( p' \). Let \( X \) and \( X' \) be principal spaces of \( N \rtimes_{\alpha} \mathbb{R} \) and \( N' \rtimes_{\alpha'} \mathbb{R} \) respectively, and assume there exists a homeomorphism \( \partial^\infty_{\alpha} X \rightarrow \partial^\infty_{\alpha'} X' \) which is sublinearly quasiMöbius over every compact subset. Then \( p \leq p' \).

**Proof sketch.** Define \( \tau := p' / p \) and let \( \Gamma_L \) be a family of horizontal lines in the boundary of \( X \). We follow the lines of Pansu [26, Proposition 6.5], despite losing strength in the conclusion. Precisely this amounts to comparing two facts:

1. Without any assumption on \( N \) and \( \alpha \), for any \( \sigma \in (\tau, +\infty) \), the image of almost every horizontal curve \( \gamma \in \Gamma_L \) has locally finite \( \sigma \)-dimensional \( \hat{d} \)-Hausdorff measure. Hence almost every curve has \( \hat{d} \)-Hausdorff dimension less than \( \tau \).
2. Since \( X \) is of Carnot type, \( \hat{d} \) is equivalent to the subRiemannian distance \( d_{CC} \) by Lemma 5.5, hence any nonconstant curve should have \( \hat{d} \)-Hausdorff-dimension greater than 1.

This proves that \( \tau \geq 1 \), i.e. \( p \leq p' \).

**Proof.** Let \( U \) be a open, relatively compact subset of \( \partial^\infty_{\alpha} X \). Define \( U' = \varphi(U) \). Let \( \Gamma'_L \) be the (non-empty) set \( \{ \gamma \cap U : \gamma \in \Gamma_L \} \) measured with

\[
(\cap_U)_* \rho_\{ \gamma \in \Gamma_L : \gamma \cap U \neq \emptyset \},
\]
where $\rho$ has been defined in 5.6, and $\cap_U(\gamma) = U \cap \gamma$. We still denote this measure $\rho$. Let $\mu$, resp. $\mu'$ be a $N$-invariant measure on $\partial^*_\infty X$, resp. on $\partial^*_\infty X'$, restricted to $U$, resp. to $U'$. Define a measure $\nu$ on $U$ as

$$\nu(B) = \mu'(\varphi(B))$$

for any Borel subset $B \subset U$. Let $\hat{\varphi}$ be the homogeneous quasimetric on $\partial^*_\infty X$, let $q$ be its ultrametric constant and define $Q$ accordingly (see Lemma 5.6). Let $r \in \mathbf{R}_{>0}$ be a radius that will be repeatedly assumed as small as needed. Choose $\gamma \in \Gamma^U$, and let $\mathcal{F}$ be any covering of $\gamma$ with quasiballs of the same $\hat{\varphi}$-radius $r$ (we emphasize that all quasiballs must have radius $r$). By assumption, the quasiballs $\{\varphi(B), B \in \mathcal{F}\}$ cover $\varphi(\gamma)$. By Theorem 4.2 and Proposition 4.10, there exists $\nu = O(u)$, and if $r$ is small enough, a collection $\mathcal{F}'$ of quasiballs and $\mathcal{F} \rightarrow \mathcal{F}'$, $B \mapsto B'$ such that

$$\forall B \in \mathcal{F}, B' \subset \varphi(Q^{-1}B) \subset \varphi(B) \subset Q^{2\lambda}e^{v(-\log r)}B' =: B''.$$ 

Define $\mathcal{F}'' = \{B''\}$ together with a map $\mathcal{F} \rightarrow \mathcal{F}'', B \mapsto B''$. This is a quasiball covering of $\varphi(\gamma)$.

Next, define a gauge function $\phi(B) := \nu(Q^{-1}B)^{1/p} = \mu'(\varphi(Q^{-1}B))^{1/p}$. There exists a constant $c_0 \in \mathbf{R}_{>0}$, not depending on $r$ and such that

$$\phi(B) \geq \mu'(B')^{1/p} = c_0\tau \text{diam} (B')^\tau$$

$$\left(\frac{c_0}{Q^{2\lambda}e^{v(-\log r)}}\right)^\tau \text{diam} (B'')^\tau.$$ (5.7)

Define $r'' = r^{1/(2\lambda)}$. Using Cornulier’s theorem 1.7, if $r$ is small enough, then

$$\forall B \in \mathcal{F}, \text{diam } B'' \leq e^{v(-\log r)}Q^{2\lambda} \text{diam } B' \leq e^{v(-\log r)}Q^{2\lambda} \text{ diam } \varphi(B) \leq e^{v(-\log r)}Q^{2\lambda}r^{2/(3\lambda)} \leq r''.$$ 

where we used $v(s) \ll s$ and took $r$ small enough in the last line. On the other hand, using (4.8) from the proof of Proposition 4.4, one obtains a reverse inequality:

$$\forall B'' \in \mathcal{F}, \text{ log diam } B'' \geq 2\lambda \log r = 4\lambda^2 \log r''.$$ 

One can rewrite $Q^{2\lambda}e^{v(-\log r)}$ as $e^{w(-\log r''')}$ with $w = O(u)$. Taking logarithms in (5.7),

$$\log \phi(B) \geq \tau \log c_0 - w(-\log r') + \tau \log \text{diam } B'' \geq \log c_0 - w\left(\frac{1}{4\lambda^2} \log \text{diam } B''\right) + \tau \log \text{diam } B''.$$ (5.8)

The function $w$ is strictly sublinear, so for any $\sigma \in (\tau, +\infty)$, there is $r_{\sigma} \in \mathbf{R}_{>0}$ such that

$$\forall r \in (0, r_{\sigma}), \forall B \in \mathcal{F}, \phi(B) \geq (r'')^\sigma \geq (\text{diam } B'')^\sigma.$$ (5.9)

Recall that for all $\mathcal{F}$ the quasiballs $B'' \in \mathcal{F}''$ cover $\varphi(\gamma)$. By definition of the $\hat{\varphi}$-Hausdorff premeasure at scale $r''$,

$$\Phi^1_\gamma(\varphi) = \inf_{\mathcal{F}} \sum_{B \in \mathcal{F}} \phi(B) \geq \sum_{B''} \text{diam}(B'')^\tau \geq H^\tau_{\mathcal{F}}(\varphi(\gamma)).$$ (5.10)
By Lemma 5.4, the hypothesis \((H)\) of Lemma 5.7 is fulfilled. Hence, for all \(r \in (0, r_\sigma)\),
\[
\int_{\Gamma^\nu_{\L}} \Phi^r_1(\gamma) d\rho \leq \nu(U)^{1/p} \mu(U)^{(p-1)/p}.
\]
By monotone convergence, for \(\rho\)-almost every \(\gamma\), \(\sup_{\nu} \Phi^r_1(\gamma)\) is finite, and then by \((5.10)\), \(\mathcal{H}^\nu \varphi(\gamma)\) is finite. Considering this fact for all terms of a decreasing sequence \(\{\sigma_j\}\) converging to \(\tau\), one deduces that, still for \(\rho\)-almost every \(\gamma\),
\[
(5.11) \quad \dim_H \varphi(\gamma) \leq \inf_j \sigma_j = \tau.
\]
Finally, \(X\) has been assumed of Carnot type, hence \(\hat{\varphi}\) is equivalent to the Carnot-Caratheodory metric \(d_{CC}\) by Lemma 5.5. By the triangle inequality, the 1-dimensionnal \(d_{CC}\)-Hausdorff measure of any nonconstant curve is nonzero, in particular its \(d_{CC}\)-Hausdorff dimension must be greater than 1. This dimension does not change when replacing \(d_{CC}\) with the equivalent quasimetric \(\hat{\varphi}\). By \((5.11)\) there exists \(\gamma \in \Gamma^\nu_{\L}\) such that \(1 \leq \dim_H \varphi(\gamma) \leq \tau\). Hence \(1 \leq \tau\).

Lemma 5.8 is applied to show that \(p\) is a SBE invariant between spaces of Carnot type. In fact this can be made slightly more general:

**Proposition 5.9.** Let \(X_1\) and \(X_2\) be principal spaces of purely real, normalized Heintze groups \(N_1 \rtimes_{\alpha_1} \mathbb{R}\) and \(N_2 \rtimes_{\alpha_2} \mathbb{R}\). Assume that for all \(i \in \{1, 2\}\) the operation defined by \(\alpha_i\) on \(n^{\text{ab}}\) is unipotent. If there exists a sublinearly biLipschitz equivalence between \(X_1\) and \(X_2\), then \(\text{tr}(\alpha_1) = \text{tr}(\alpha_2)\).

**Proof.** For every \(i \in \{1, 2\}\), decompose \(\alpha_i\) into \(\alpha'_i + \alpha''_i\), where \(\alpha''_i\) is semi-simple and \(\alpha'_i\) is nilpotent. By hypothesis, \(n^\nu_i\) operates as the identity on \(n^{\text{ab}}\), hence \(N_i\) are Carnot gradable groups, and \(\alpha_i\) are grading derivations of their Lie algebra. A particular instance of a theorem by Cornulier implies that there exists \(O(\log)\)-sublinearly biLipschitz equivalences \(\psi_i : N \rtimes_{\alpha''_i} \mathbb{R} \rightarrow N \rtimes_{\alpha_i} \mathbb{R}\) (see \([8, \text{Theorem 4.4}]\): in our very special case the exponential radical is \(N\), and the Cartan subgroup is \(\mathbb{R}\)). The groups \(N \rtimes_{\alpha''_i} \mathbb{R}\) are of Carnot type, so by Theorem 4.2 and Lemma 5.8, \(\text{tr}(\alpha'_1) = \text{tr}(\alpha'_2)\). Finally, \(\text{tr}(\alpha_1) = \text{tr}(\alpha''_1) = \text{tr}(\alpha''_2) = \text{tr}(\alpha).\) \(\square\)

Note that if sublinearly biLipschitz equivalences are replaced by quasimetrics in the last statement, known invariants are much finer than the trace. In this direction, M. Carrasco Piaggio and E. Sequeira obtained that for normalized purely real Heintze groups, resp. for normalized purely real Heintze groups with a fixed Heisenberg group as exponential radical \(N\), the characteristic polynomial, resp. the full Jordan form of \(\alpha\), are quasimetric invariants \([6, \text{Theorem 1.1}, \text{resp. Theorem 1.3}]\). By contrast, the Jordan form of the normalized derivation is not a SBE invariant, precisely it is not a \(O(\log)\)-SBE invariant by Cornulier’s theorem \([8, \text{Theorem 4.4}]\).

5.E. **Proof of Theorem 2.** Notation is as before. When \(X\) is a rank one symmetric space of noncompact type, several restrictions appear (see Heintze, \([22, \text{Proposition 4 and Corollary}]\)):

1. \(X\) is of Carnot type.
2. The Lie algebra \(n\) is two-step, \(n = n_1 \oplus n_2\) where \(n_2\) is possibly zero.
(3) Save for one case, namely the Cayley hyperbolic plane, there exists a division algebra structure on $\mathbb{R} \oplus n_2$, and $n_1$ is a module over this division algebra. The structure of $n$ is completely determined by these data.

The Frobenius classification of division algebras over $\mathbb{R}$ reduces considerably the list of candidates thanks to (3): the two relevant parameters are the division algebra $K \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \}$ and a positive integer, the rank of $n_1$ over $K$. The Cayley hyperbolic plane fits in this list, setting $n_1 = \mathbb{O}$. The homogeneous dimension is computed as

$$\text{tr}(\alpha) = \dim n_1 + 2 \dim n_2 = \dim n + \dim n_2 = \dim X - 1 + \dim \text{Im}(K),$$

and $K$ is completely determined by $\dim \text{Im}(K) \in \{0, 1, 3, 7\}$. By Theorem 1.7 and Proposition 5.9, $K$ is a SBE invariant, as

$$\dim \text{Im}(K) = \dim_H(\partial^\infty X, \varrho) - \dim \partial_\infty X.$$

The rank $n$ of $n_1$ over $K$ is a SBE invariant as well, since it can be computed by the formula

$$(1 + n) \dim K = 1 + \dim \partial_\infty X.$$

REFERENCES

1. D. Alekseevski, Homogeneous Riemannian spaces of negative curvature, *Mat. Sb. (N.S.*) vol. 96 (138) (1975), 93–117. Translated by A. West.

2. N. Bourbaki, *Eléments de mathématiques*, Topologie générale, Chapitres 5 à 10, Hermann, Paris (1974).

3. M. Bourdon, Quasi-conformal geometry and Mostow rigidity, in *Géométries courbure négative ou nulle, groupes discrets et rigidités*, Sémin. Congr. 18, 201-212 (Soc. Math. France, Paris, 2009).

4. M. Bourdon, Mostow type rigidity theorems, to appear in Handbook of Group Actions, *Advanced lectures in Math*.

5. S. Buyalo, V. Schroeder, *Elements of asymptotic geometry*, EMS Monogr. Math. (European Mathematical Society, Zürich, 2007).

6. M. Carrasco Piaggio, E. Sequeira, On quasi-isometry invariants associated to the derivation of a Heintze group, *Geom. Dedicata* 189 (2017), 1-16.

7. Y. Cornulier, Dimension of asymptotic cones of Lie groups, *J. Topol.* 1 (2008) no. 2, 342–361.

8. Y. Cornulier, Asymptotic cones of Lie groups and cone equivalences, *Illinois J. Math.* 55 (2012), no. 1, 237–259.

9. Y. Cornulier, Commability and focal locally compact groups, *Indiana Univ. Math. J.* (1) 64 (2015), 115–150.

10. Y. Cornulier, On the quasi-isometric classification of locally compact groups, in *New directions in locally compact groups*, P.-E. Caprace, N. Monod eds, London Math. Soc. Lecture Notes Ser. 447 (2018), 275–342.

11. Y. Cornulier, On sublinearly Bilipschitz Equivalence of Groups, arXiv:1702.06618, 2017.

12. Y. Cornulier and R. Tessera, Contracting automorphisms and $L^p$-cohomology in degree one, *Ark. Mat.* 49 (2011), no.2, 295–324.

13. C. Druţu, Quasi-isometry invariants and asymptotic cones, *Int. J. Alg. Comp.* 12 (2002), 99–135.

14. C. Druţu, M. Kapovich, *Geometric Group Theory*, AMS Colloquium Publications 63 (2017).

15. A. Dyubina (Erschler), I. Polterovich, Explicit constructions of universal $\mathbb{R}$-trees and asymptotic geometry of hyperbolic spaces, *Bull. London Math. Soc.* 33 (2001), no. 6, 727–734.

16. V. A. Efremovich and E. S. Tikhomirova, Equimorphisms of hyperbolic spaces, *Izv. Akad. Nauk SSSR Ser. Mat.* 28 (1964), issue 5, 1139–1144.

17. H. Federer, *Geometric Measure Theory*, Grundlehren Math. Wiss. 129 (Springer-Verlag, Berlin 1969).

18. A. H. Frink, Distance functions and the metrization problem, *Bull. Amer. Math. Soc.* 43 (1937), 133–142.
19. E. Ghys and P. de la Harpe, *Sur les Groupes Hyperboliques d’après Mikhael Gromov*, Progr. Math. 83, Birkhäuser (1990).
20. M. Gromov, *Hyperbolic Groups*, in S.M. Gersten (eds), *Essays in Group Theory*, Math. Sci. Res. Inst. Publ., vol 8. (Springer, New York, 1987).
21. M. Gromov, *Asymptotic invariants of infinite groups*, in *Geometric group theory*, vol. 2, eds. A. Niblo and Martin A. Roller, London Math. Soc. Lecture Note Ser. 182 (Cambridge Univ. Press, Cambridge, 1993).
22. E. Heintze, On homogeneous manifolds of negative curvature, *Math. Ann.* (1) 211 (1974) 23–34.
23. J. M. Mackay and J.T. Tyson, *Conformal dimension: Theory and Application* Univ. Lecture Ser. 54 (Amer. Math. Soc., 2010).
24. A. P. Morse, A Theory of Covering and Differentiation, *Trans. Amer. Math. Soc.* 55 (1944), No. 2, 205–235.
25. P. Pansu, Dimension conforme et sphère à l’infini des variétés à courbure négative, *Ann. Acad. Sci. Fenn.*, Ser. A I, 14 (1989), 177–212.
26. P. Pansu, Métriques de Carnot-Carathéodory et quasimétries des espaces symétriques de rang un, *Ann. of Math.* 129 (1989), No.1, 1–60.
27. V. Shchur, A quantitative version of the Morse lemma and quasi-isometries fixing the ideal boundary, *J. Funct. Anal.* 264 (2013), Issue 3, 815–836.
28. W. Thurston, *The geometry and topology of three-manifolds*, (Lecture notes, Princeton, 1978-1979).
29. A. Weil, *L’intégration dans les groupes topologiques et ses applications*, Actual. Sci. Ind. no. 869, Hermann, Paris (1940).

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