Classical Zero-Point Radiation and Relativity: The Problem of Atomic Collapse Revisited

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Abstract

The physicists of the early 20th century were unaware of two aspects which are vital to understanding some aspects of modern physics within classical theory. The two aspects are: 1) the presence of classical electromagnetic zero-point radiation, and 2) the importance of special relativity. In classes in modern physics today, the problem of atomic collapse is still mentioned in the historical context of the early 20th century. However, the classical problem of atomic collapse is currently being treated in the presence of classical zero-point radiation where the problem has been transformed. The presence of classical zero-point radiation indeed keeps the electron from falling into the Coulomb potential center. However, the old collapse problem has been replaced by a new problem where the zero-point radiation may give too much energy to the electron so as to cause “self-ionization.” Special relativity may play a role in understanding this modern variation on the atomic collapse problem, just as relativity has proved crucial for a classical understanding of blackbody radiation.
I. INTRODUCTION

In 1911 when Rutherford used the data of scattering experiments to publish the nuclear model of the atom, the problem of atomic collapse immediately arose. Earlier experimental work, notably by J. J. Thomson, had measured the ratio of \( e/m \) for the electron. The work on the normal Zeeman effect by Zeeman and Lorentz placed electrons as part of the atom and as connected to spectral lines. But Rutherford’s model of the atom involving electrons in Kepler-like orbits around a small nucleus raised a profound question: the orbiting electrons were obviously accelerating, and, according to classical electromagnetic theory, the electrons must radiate away their energy; what prevented the collapse of the electron into the nucleus? This “problem of atomic collapse” is always mentioned in our classes in modern physics when presenting the need for a quantum description of atomic physics.

In 1913, Bohr “solved” the problem of atomic collapse by fiat. The old quantum theory claimed simply that in certain stationary orbits, the electrons did not obey the laws of classical electromagnetism and did not radiate. Currently, the stationary orbits of old quantum theory have been replaced by the stationary states of Schroedinger quantum mechanics.

However, the outlook introduced by Bohr in 1913 remains; a new quantum theory must replace the ideas of classical physics when dealing with the structure of the atom.

In contrast to the historical view presented in our modern physics courses, today the century-old mystery of atomic collapse within classical physics is taking a fascinating new twist. It turns out that there are two significant ideas which were absent from the thinking of the early 20th century physicists which might have helped the classical theoreticians of the period. These two ideas are 1) the presence of classical electromagnetic zero-point radiation in the universe, and 2) the importance of special relativity. When these two ideas are introduced into the old problem of atomic collapse, the problem is transformed. In recent years, there have been attempts to solve the problem of atomic collapse when classical zero-point radiation is present.

Relativistic classical electron theory with classical electromagnetic zero-point radiation (sometimes termed “stochastic electrodynamics”) is the classical theory which most closely approximates quantum physics. Clearly, physicists would like to have a clear picture of the areas of agreement and disagreement between classical and quantum theories. Recently Nieuwenhuizen and Liska have undertaken a heroic effort to extend the fascinating
numerical calculations of Cole and Zou\cite{2} regarding the ground state of hydrogen when
classical zero-point radiation is present. Here we wish to review these efforts as an interesting
attempt to strengthen our understanding of the classical aspects of atomic physics.

II. THE HYDROGEN GROUND STATE IN CLASSICAL PHYSICS

A. Classical Electromagnetic Zero-Point Radiation

Classical electrodynamics is a well-established theory which is the basis for much of
modern technology; it has a well-defined framework of differential equations and boundary
conditions. In addition to describing the charges and currents which appear as sources for
the electric and magnetic fields in Maxwell’s equations, one must also choose the homoge-
neous boundary conditions on the equations. The situation seems obvious if one considers
the electromagnetic radiation in a laboratory before the experimenter’s sources are turned
on. As far as the experimenter is concern, his sources did not provide the radiation in the lab
which existed when he entered. This radiation which does not arise from the experimenter’s
sources can be treated as boundary conditions appearing in the solution of the homogeneous
Maxwell equations. The thermal radiation present in the laboratory is an example of ran-
dom classical radiation which would be treated as a solution of the homogeneous Maxwell
equations. Every classical electromagnetic theory based upon Maxwell’s differential equa-
tions must make an assumption regarding the homogeneous boundary conditions on these
differential equations. In the early years of the 20th century, the assumption was made that
the homogeneous boundary condition corresponded to a complete absence of radiation before
the sources acted. Lorentz makes this assumption explicitly in his book on classical electron
theory.\cite{5} A different but also legitimate classical boundary condition on Maxwell’s equa-
tions involves the presence of random classical radiation with a Lorentz-invariant spectrum,
classical electromagnetic zero-point radiation.\cite{6} Classical electromagnetic zero-point radi-
atation is allowed by classical electromagnetic theory and provides a classical understanding
of some aspects of nature.

The spectrum of classical zero-point radiation is derived as the Lorentz-invariant spec-
trum of random classical electromagnetic radiation.\cite{7} The spectrum is unique up to a
multiplicative constant as an energy $\mathcal{E}$ per normal mode of frequency $\omega$ given by

$$\mathcal{E} = \text{const} \times \omega$$  \hspace{2cm} (1)

When this Lorentz-invariant spectrum is used to calculate Casimir forces, it is found that the scale constant for classical zero-point radiation must be chosen as $\text{const} = 1.05 \times 10^{-34}\text{J}\cdot\text{s}$ in order to fit the experimental data. Although the $\text{const}$ here has nothing to do with quanta, the value of $\text{const}$ clearly corresponds to $\hbar/2$ where $\hbar$ is Planck’s constant appearing in quantum physics. Thus Planck’s constant $\hbar$ enters classical theory simply as the scale factor of classical zero-point radiation. Today we regard classical physics with classical zero-point radiation as the closest classical approximation to quantum physics.\[9\]

**B. Review of the Basic Idea for Hydrogen**

It is easy to calculate the behavior of a linear dipole oscillator in classical zero-point radiation.\[10\]\[11\] The dipole oscillator radiates away its energy but it also picks up energy from the random forces of the classical zero-point radiation and so comes to a steady state probability distribution for energy, amplitude, and velocity. Using classical zero-point radiation, it turns out that Casimir forces, van der Waals forces, blackbody radiation, low-temperature specific heats of solids, and diamagnetism all come within the framework of this classical theory.\[9\]

Although harmonic oscillator systems and systems of free fields are easy to calculate in the presence of classical zero-point radiation, the hydrogen atom is a far more complicated system. In 1975, when the results of classical electrodynamics with classical zero-point radiation were reviewed, it was pointed out that there was a clear qualitative suggestion that the classical theory might lead to a stable hydrogen ground state.\[10\] It was reported that if one approximated the ground-state motion as analogous to that of a nonrelativistic, planar, rigid rotator where a particle of charge $e$ and mass $m$ is held at a fixed distance $r$ from the rotation center, then the power loss $d\mathcal{E}_{\text{loss}}/dt$ due to radiation emission was given by

$$\frac{d\mathcal{E}_{\text{loss}}}{dt} = \frac{2}{3} \frac{e^2}{c^3} \omega^4 r^2,$$  \hspace{2cm} (2)

whereas the average energy pick-up $d\mathcal{E}_{\text{gain}}/dt$ from the random classical zero-point radiation
was given by
\[ \frac{dE_{\text{gain}}}{dt} = \frac{e^2 \hbar \omega^3}{2mc^3}, \tag{3} \]

where here \( \hbar \) is the scale factor setting the scale of the classical zero-point energy. At high frequencies \( \omega \), the particle lost more energy by radiation than it picked up from the zero-point radiation; on the contrary, at low frequencies, the oscillator picked up more energy than it radiated away. Thus one expected a ground state at the balance point corresponding
\[ \frac{dE_{\text{loss}}}{dt} = \frac{dE_{\text{gain}}}{dt} \quad \text{or} \quad mr^2\omega = J = \frac{3}{4} \hbar \tag{4} \]

where \( J \) is the angular momentum of the rotator. Old quantum theory chose the result \( J = \hbar \). Thus this qualitative model for hydrogen gives approximately the same result as that for the Bohr-model ground state or for the Schroedinger ground state. However, this rough, heuristic estimate is far from a real calculation.\[12\]

Within classical physics, the basic picture for the hydrogen ground state involves a separation between a mechanical system of a particle in a Coulomb potential \( Z e^2/r \) and a spectrum of random classical radiation with a Lorentz-invariant spectrum. Both these systems can be described by action-angle variables.\[13\] The slowly-changing action-angle variables provide the appropriate variables for a canonical perturbation theory between the mechanical system and the random radiation. The mechanical orbits of the particle in the Coulomb potential are perturbed by the loss of energy to radiation and also by the random forces of the zero-point radiation. The zero-point radiation field is increased by the energy radiated by the charged particle and is diminished by the absorption of radiation by the mechanical system.

III. CURRENT RESEARCH ON THE HYDROGEN GROUND STATE

A. Calculations of Cole and Zou

Although the basic physical picture for the classical hydrogen atom is clear, there is at present no full analytic calculation of the hydrogen ground state when zero-point radiation is present. Rather, the evaluation of the electron’s motion involves enormously difficult computer simulations. The calculations are extremely difficult because of the need to simulate the spectrum of random classical zero-point radiation and because of the need
to follow the perturbed electron through vast numbers of orbits. In 2003, Cole and Zou reported the first computer simulation for the hydrogen ground state.\textsuperscript{[2]} In order to simplify the calculation, they restricted the mechanical particle motion to a single plane. Their nonrelativistic calculations were quite favorable to the classical theory. In no case did the electron fall into the Coulomb center. Also, the numerical calculations suggested that the probability distribution for the radial distance of the electron from the hydrogen nucleus roughly approximated that given by the Schrödinger ground state of quantum theory. In situations where the numerical simulations suggested that the energy gain from zero-point radiation was sufficiently large as to ionize the atom, more accurate recalculations showed that the ionization actually did not occur.

B. Claverie and Soto Regarding Ionization

Indeed, the question of self-ionization of the hydrogen atom in classical zero-point radiation appears as a new aspect of classical atomic structure. Whereas the traditional classical theory (without zero-point radiation) of the early 20th century suggested atomic collapse because the accelerating electron would radiate away its energy, the new classical theory (which included zero-point radiation) raises the possibility that the electron might acquire sufficient energy from the zero-point radiation so as to spontaneously ionize. Work by Marshall, Claverie, Pesquera, and Soto\textsuperscript{[14]} (using nonrelativistic mechanics for the electron) suggested that there was no stable ground state for hydrogen; the electron would always be ionized by the zero-point radiation. Presumably the nonrelativistic calculations of Cole and Zou did not continue sufficiently long so as to show this self-ionization.

The ionization found in the nonrelativistic classical theory always involved the plunging orbits of low angular momentum. However, it was pointed out that these plunging orbits of small angular momentum are precisely the orbits which are strongly modified by using relativistic theory for the mechanical motion of the electron.\textsuperscript{[15]} Thus whereas analytic calculations using nonrelativistic mechanics for the electron indicated that the classical hydrogen atom would spontaneously ionize, use of relativistic mechanics for the electron might still lead to a stable hydrogen ground state within classical electrodynamics.
C. Calculations of Nieuwenhuizen and Liska

Recently, Nieuwenhuizen and Liska published their much more extensive numerical calculations for the ground state of hydrogen within classical electrodynamics including classical zero-point radiation. Their numerical simulations involved full three-dimensional motion for the nonrelativistic electron and are extended to far longer times than those reported by Cole and Zou. According to the simulations, the balance between radiation energy loss and energy gain from the zero-point radiation gives millions of orbits for the electron in the approximate neighborhood of the Bohr orbit without any indication that the electron falls into the potential center. However, Nieuwenhuizen and Liska report that ionization of the electron always occurs in their simulations. Furthermore the distributions of radial position and energy are only in very rough agreement with the Schroedinger ground-state results. Once again, the ionization is reported as arising from the plunging orbits of low angular momentum. For these plunging orbits, the multiply periodic orbit expansions required high harmonics above the fundamental frequency. For these plunging orbits, both the energy loss at the higher harmonics will be large, and also the zero-point radiation energy gain from the higher harmonics will be large because the zero-point spectrum increases with frequency. Indeed, the figures of Nieuwenhuizen and Liska show spikes in energy which seem to occur at the same time as spikes to large values of orbital eccentricity, $\epsilon \to 1$, consistent with the idea of large energy changes for plunging orbits.

Apparently Nieuwenhuizen and Liska were aware of the proposal that use of relativistic trajectories for the electron would smooth the plunging trajectories and so modify the energy pick up and loss associated with the higher harmonics. In their second publication, they introduced the lowest-order relativistic correction to the nonrelativistic energy and carried out a new calculation using this corrected expression for the electron motion, where they also included terms involving electron spin. Their report is that these lowest-order relativistic corrections made little difference in their calculations. The ionization of the electron was still observed.

The question as to whether or not the presence of classical zero-point radiation leads to a stable ground state for hydrogen seems important. In order to understand nature, physicists need to know the areas of agreement and disagreement between classical and quantum theories. The calculations of Nieuwenhuizen and Liska present an important improvement
in understanding the classical description of the hydrogen ground state. Nevertheless, given the approximations in their calculations, one may wonder whether their conclusions regarding the ionization are indeed justified.

IV. ROLE OF RELATIVITY IN MODERN PHYSICS

A. When is Relativity Needed?

Contemporary physics regards all of nature as relativistic in its fundamental interactions. Indeed, electromagnetism is a relativistic theory. However, most physicists believe that relativistic physics is needed for mechanical systems only when dealing with particles whose speeds approach the speed of light relative to the laboratory. Thus for atomic electron speeds in hydrogen, we have \( v \approx \frac{e^2}{\hbar} \approx \frac{c}{137} \), so that nonrelativistic physics is usually deemed adequate, and the electron orbits in classical physics should correspond to the familiar conic sections: ellipse, parabola, and hyperbola. However, this confidence in the adequacy of nonrelativistic calculations is sometimes misplaced.

B. Relativistic Mechanical Kepler Orbits

Thus it comes as a shock to many physicists to learn that the plunging Kepler orbits of small angular momentum are radically different in relativistic versus nonrelativistic physics. This difference holds even for particles which have initial trajectories of arbitrarily speed, including very small speed. Indeed, using relativistic mechanical orbits and ignoring any radiation energy loss, if the orbital angular momentum \( J \) is less than \( Z e^2 / c \) (and irrespective of the particle energy), then the particle will plunge into the Coulomb center while conserving energy and angular momentum.\[18\] This behavior is totally different from the situation in nonrelativistic mechanics. In nonrelativistic physics, mechanical Kepler orbits never plunge into the potential center unless the angular momentum is exactly zero.

A sense of the change in perspective between relativistic and nonrelativistic mechanical orbits is given easily by considering a circular Kepler orbit at radius \( r \) for a particle of mass \( m \). In the relativistic case, we have Newton’s second law for a circular orbit

\[
m\gamma \frac{v^2}{r} = \frac{Ze^2}{r^2} \quad (5)
\]
where the angular momentum $J$ is given by

$$J = rm\gamma v. \quad (6)$$

Therefore, combing these two equations, we have

$$Jv = Ze^2 \quad \text{or} \quad v = Ze^2 / J. \quad (7)$$

This last equation holds also in the nonrelativistic case which we can calculate by omitting the relativistic factor of $\gamma = (1 - v^2/c^2)^{-1/2}$. However, for the relativistic orbit, the speed $v$ is limited by $c$, so that $v = Ze^2 / J < c$. But then for any circular orbit, we must have

$$Ze^2 / c < J. \quad (8)$$

This limit does not exist in nonrelativistic classical physics where circular Kepler orbits of arbitrarily small positive angular momentum are possible. As we have mentioned above, if $J \leq Ze^2 / c$, then no relativistic circular orbit exists, and the particle will plunge into the potential center while conserving energy and angular momentum. Indeed, the critical angular momentum $Ze^2 / c$ appears in a problem in Goldstein’s Classical Mechanics.\cite{19} The relativistic mechanical energy $E$ for a relativistic particle of mass $m$ in a Coulomb potential $Ze^2 / r$ when expressed in terms of action-angle variables $J_2$ and $J_3$ (which include angular momentum) is given as\cite{20}

$$E/mc^2 = \left( 1 + \left[ \frac{J_3 c}{Ze^2} - \frac{J_2 c}{Ze^2} + \left\{ \left( \frac{J_2 c}{Ze^2} \right)^2 - 1 \right\}^{1/2} \right]^{-2} \right)^{-1/2} \quad (9)$$

We notice that if $J_2 < Ze^2 / c$, then the energy expression involves the square-root of a negative quantity; this is the signal that the periodic behavior assumed in defining the action-angle variable $J_2$ no longer holds, because the trajectories plunge into the Coulomb center.\cite{21}

\section{C. Blackbody Radiation}

Within classical physics, the problem of atomic collapse is intimately bound up with the problem of the blackbody radiation spectrum. Thus the equilibrium spectrum of random classical radiation (blackbody radiation) must be stable under scattering by a classical scattering system. The hydrogen atom should provide an example of a scattering system in
nature. Thus the hydrogen atom should scatter the radiation, and radiation should in turn provide the structure of the hydrogen atom. From this perspective, zero-point radiation, which is the ground state of the random radiation, should be in equilibrium with the scattering provided by the ground state of hydrogen.

The classical blackbody spectrum provides an example of the crucial importance of special relativity. Understanding the Planck blackbody spectrum within classical physics requires the use of a relativistic analysis; use of nonrelativistic theory gives only the Rayleigh-Jeans low-frequency limit. For example, it is a familiar observation in modern physics classes that the use of the equipartition theorem from nonrelativistic statistical mechanics leads to the Rayleigh-Jeans spectrum. Furthermore, use of nonrelativistic scatterers to obtain the equilibrium spectrum of random classical radiation also leads to the Rayleigh-Jeans spectrum.

Since nonrelativistic scatterers give only the Rayleigh-Jeans spectrum as the stable spectrum under scattering, one wishes to consider relativistic scatterers. However, many physicists are unaware that relativistic physics places very strong restrictions on the allowed mechanical systems. Most mechanical systems have no relativistic extension consistent with electromagnetism. The “no-interaction theorem” of Currie, Jordan, and Sudarshan requires that any mechanical interaction beyond point collisions requires a field theory. It is precisely on account of this restriction that elementary classroom discussions of relativistic particle interactions always involve point collisions and never interactions through a potential. The Coulomb potential is the only classical mechanical potential which has been extended to a relativistic field theory, namely classical electrodynamics. The scattering of random electromagnetic radiation in the presence of a relativistic hydrogen atom indeed corresponds to a relativistic field-theory situation which has all the qualitative aspects allowing radiation equilibrium at the Planck spectrum. Thus the mass $m$ of the charged particle in orbit about the Coulomb center provides the only scale factor for length or frequency, while the action-angle variables are invariant under adiabatic transformation of the strength $Ze^2$ of the Coulomb potential. Thus any situation of classical radiation equilibrium in zero-point radiation for one mass $m$ and scale $Ze^2$ will also hold for other masses $m'$ or other choices of the constant $Z'$, while the action-angle variables will retain their values, provided that $Z$ is not too large.

Furthermore, the full Planck spectrum can be obtained explicitly by use of scaling sym-
metry within a relativistic setting. The zero-point radiation spectrum is derived as the Lorentz-invariant spectrum in Minkowski spacetime.\textsuperscript{[7][8]} This spectrum is invariant under the scale transformation appropriate for electromagnetic theory.\textsuperscript{[28]} Thus electromagnetism is invariant under the scale transformation which simultaneously multiplies all lengths and times by a positive constant $\sigma$ (thus preserving $c = \text{length}/\text{time}$) and all energies by $1/\sigma$ (thus preserving $e^2 = \text{energy} \times \text{length}$). Under this scale transformation, thermal radiation at the temperature $T$ is transformed to thermal radiation at temperature $T/\sigma$; but zero-temperature zero-point radiation is unchanged in Minkowski spacetime, consistent with zero temperature divided by any positive constant $\sigma$ still being zero temperature. However, zero-point radiation, which is scale invariant in Minkowski spacetime, acquires a local scale in an accelerated Rindler coordinate frame.\textsuperscript{[29]} When a scale transformation is applied to zero-point radiation in a Rindler frame, the zero-point radiation spectrum is carried into thermal radiation at non-zero temperature. If we now imagine moving ever further away from the Rindler event horizon while applying a rescaling so as to keep our local temperature constant, then we move to a region where the acceleration vanishes, and we recover exactly the Planck spectrum at finite temperature in Minkowski spacetime.\textsuperscript{[29]} We note that this analysis requires a fully relativistic theory at every step.\textsuperscript{[30]}

V. CONCLUSION

Today our classes in modern physics still hear mention of the problem of atomic collapse as an example of the failure of classical theory to account for atomic structure. Furthermore, some quantum descriptions state that zero-point motion prevents the collapse. However, most classes receive no mention of the possibility of classical electromagnetic zero-point radiation causing the zero-point motion, despite the fact that classical zero-point radiation gives at least a heuristic classical idea regarding many phenomena and an exact classical account of some phenomena which are currently regarded as “quantum phenomena.” Thus Casimir forces, van de Waals forces, oscillator behavior, oscillator specific heats, blackbody radiation, and diamagnetism all have unimpeachable classical calculations which give results in exact agreement with the corresponding quantum results.

Regarding the problem of atomic collapse, we are currently in an interesting new regime. The classical problem of atomic collapse mentioned in our modern physics classes assumes
that there is no random classical radiation present to give energy to the radiating electron which is losing energy. When classical electromagnetic zero-point radiation (which gives us results for Casimir forces identical to those obtained from quantum electrodynamics) is applied to the classical hydrogen atom, the traditional problem of atomic collapse disappears, and the electron indeed orbits the Coulomb center near the Bohr radius for millions of orbits without falling into the potential center. However, the new numerical calculations suggest that the classical problem of atomic collapse has been replaced by the problem of the zero-point radiation delivering too much energy to the orbiting electron so as to cause self-ionization of the atom. But this self-ionization problem may be linked to the question of using a nonrelativistic analysis. An analytic calculation for the relativistic hydrogen atom in zero-point radiation has never been done. At the present time, the numerical simulations for the classical hydrogen atom are exceedingly difficult, even though they are not fully relativistic. We still do not know whether or not classical zero-point radiation provides an acceptable ground state or causes self-ionization of the classical hydrogen atom.

[1] See the discussion of any textbook of modern physics. For example, R. Eisberg and R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles 2nd ed*, (Wiley, New York 1985), Chapter 4, or K. S. Krane, *Modern Physics 2nd ed*, (Wiley, New York 1996), Chapter 6.

[2] D. C. Cole and Y. Zou, “Quantum mechanical ground state of hydrogen obtained from classical electrodynamics,” Physics Letters A 317, 14-20 (2003).

[3] T. M. Nieuwenhuizen and M. T. P. Liska, “Simulation of the hydrogen ground state in Stochastic Electrodynamics,” Physica Scripta T165, 014006 (2015), (arXiv: 1502.06856v2).

[4] T. M. Nieuwenhuizen and M. T. P. Liska, “Simulation of the hydrogen ground state in Stochastic Electrodynamics-2,” Found. Phys. 45, 1190-1202 (2015), (arXiv: 1506.06787v1).

[5] Traditional classical electron theory is described by H. A. Lorentz, *The Theory of Electrons* (Dover, New York 1952). This volume is a republication of the second edition of 1915 based on Lorentz’s Columbia University lectures of 1909. Note 6, p. 240, gives Lorentz’s explicit assumption on the boundary conditions.

[6] In the textbooks, classical electromagnetism is treated from the point of view of electro-
magnetic technology where the charge and current sources provide overwhelmingly large electromagnetic fields; classical electromagnetism is not treated as a subject relevant for atomic physics where the sources may give fields comparable to the background fields of thermal radiation or zero-point radiation. Accordingly, the homogeneous boundary conditions on Maxwell’s equations are rarely if ever mentioned in textbooks of classical electromagnetism.

[7] T. W. Marshall, “Statistical Electrodynamics,” Proc. Camb. Phil. Soc. 61, 537-546 (1965).

[8] T. H. Boyer, “Derivation of the Blackbody Radiation Spectrum without Quantum Assumptions,” Phys. Rev. 182, 1374-11383 (1969). T. H. Boyer, “Conformal Symmetry of Classical Electromagnetic Zero-Point Radiation,” Found. Phys. 19, 349-365 (1989).

[9] See, for example, the discussion by T. H. Boyer, “Any classical description of nature requires classical electromagnetic zero-point radiation,” Am. J. Phys. 79, 1163-1167 (2011).

[10] See, for example, the review by T. H. Boyer, “Random electrodynamics: The theory of classical electrodynamics with classical electromagnetic zero-point radiation,” Phys. Rev. D 11, 790-808 (1975).

[11] A review of the work on classical electromagnetic zero-point radiation up to 1996 is provided by L. de la Pena and A. M. Cetto, The Quantum Dice - An Introduction to Stochastic Electrodynamics (Kluwer Academic, Dordrecht 1996).

[12] In 1975, I was well aware that use of a harmonic oscillator model gave exactly \( J = \hbar \). However, this suggested an exact agreement with the quantum ground-state results for hydrogen. The harmonic-oscillator model result was published by H. E. Puthoff, “Ground state of hydrogen as a zero-point-fluctuation-determined state,” Phys. Rev. D 35, 3266-3269 (1987). However, neither the rotator model nor the oscillator model corresponds to particle orbits in a Coulomb potential, and so these models are at best suggestive qualitative approximations.

[13] T. H. Boyer, “Blackbody Radiation and the Scaling Symmetry of Relativistic Classical Electron Theory with Classical Electromagnetic Zero-Point Radiation,” Found. Phys. 40, 1102-1116 (2010).

[14] T. Marshall and P. Claverie, “Stochastic electrodynamics of nonlinear systems. I: particle in a central field of force,” J. Math. Phys. 21, 1918-1925 (1980). P. Claverie, L. Pesquera and F. Soto, “Existence of a constant stationary solution for the hydrogen atom problem in stochastic electrodynamics,” Phys. Lett. A 80, 113-116 (1980). P. Claverie and F. Soto, “Nonrecurrence of the stochastic process for the hydrogen atom problem in stochastic electrodynamics,” J.
Math. Phys. 23, 753-759 (1982).

[15] T. H. Boyer, “Comments on Cole and Zou’s calculation of the hydrogen ground state in classical physics,” Found. Phys. Lett. 16, 613-617 (2003).

[16] Nieuwenhuizen and Liska define ionization “as the moment when the electron stays above $E = -0.05$ for a duration for at least $10^7 t_0$.” Here $E$ is given in Bohr units and $t_0$ is the Bohr period.

[17] See Figs. 1 and 2 of reference 3.

[18] T. H. Boyer, “Unfamiliar trajectories for a relativistic particle in a Kepler or Coulomb potential,” Am. J. Phys. 75, 992-997 (2004).

[19] H. Goldstein, Classical Mechanics 2nd ed, (Addison-Wesley, Reading, MA 1981), p. 498, problem 28.

[20] Here we have chosen action-angle variables differing by a factor of $1/(2\pi)$ from those of problem 28 of Goldstein’s text and have written the strength of the potential as $k = Ze^2$.

[21] As another example of the interesting connections between classical and quantum theories, we note that the quantum mechanical problem of atomic collapse for large values of $Z \gtrsim 137$ is mirrored in the relativistic classical situation. When zero-point radiation is present, we find that $\langle J \rangle \approx \hbar$. Thus, when $Z$ exceeds 137, then $Ze^2/(hc) > 1$, and the classical relativistic energy in Eq. (9) becomes complex, indicating trajectories spiraling into the Coulomb center. Apparently recent experimental work with graphene suggests confirmation of the ideas of quantum mechanical atomic collapse for large $Z$. See Y. Wang et al., “Observing Atomic Collapse Resonances in Artificial Nuclei on Graphene,” Science 340, 734-737 (2013).

[22] See, for example, Eisberg and Resnick in Ref. 1, p. 12 or Krane in Ref 1, p. 80.

[23] T. H. Boyer, “Equilibrium of random classical electromagnetic radiation in the presence of a nonrelativistic nonlinear electric dipole oscillator,” Phys. Rev. D 13, 2832-2845 (1976) and T. H. Boyer, “Statistical equilibrium of nonrelativistic multiply periodic classical systems and random classical electromagnetic radiation,” Phys. Rev. A 18, 1228-1237 (1978).

[24] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, “Relativistic Invariance and Hamiltonian theories of interacting particles,” Rev. Mod. Phys. 34, 350-375 (1963). The no-interaction theorem is referred to in Goldstein’s mechanics text Ref. 19, on pages 332, 334, and 362.

[25] See, for example, T. H. Boyer, “Illustrating some implications of the conservation laws in relativistic mechanics,” Am. J. Phys. 77, 562-569 (2009).
[26] T. H. Boyer, “Blackbody radiation and the scaling symmetry of relativistic classical electron theory with classical electromagnetic zero-point radiation,” Found. Phys. 40, 1102-1116 (2010).

[27] This adiabatic invariance corresponds to the familiar mechanical problem of slowly pulling a string through a hole in a frictionless table, with the string attached to a rotating block. The angular momentum $J$ (an action variable) is conserved during the slow change of the strength of the central force.

[28] T. H. Boyer, “Scaling symmetries of scatterers of classical zero-point radiation,” J. Phys. A: Math. Theor. 40, 9635-9642 (2007).

[29] T. H. Boyer, “The blackbody radiation spectrum follows from zero-point radiation and the structure of relativistic spacetime in classical physics,” Found. Phys. 42, 595-614 (2012).

[30] Related discussions of blackbody radiation are given by T. H. Boyer, “Classical physics of thermal scalar radiation in two spacetime dimensions,” Am. J. Phys. 79, 644-656 (2011) and T. H. Boyer, “Derivation of the Planck spectrum for relativistic classical scalar radiation from thermal equilibrium in an accelerating frame,” Phys. Rev. D 81, 105024 (2010).