AN INFINITE SURFACE WITH THE LATTICE PROPERTY II: DYNAMICS OF PSEUDO-ANOSOV

W. PATRICK HOOPER

ABSTRACT. We study the behavior of hyperbolic affine automorphisms of a translation surface which is infinite in area and genus that is obtained as a limit of surfaces built from regular polygons studied by Veech. We find that hyperbolic affine automorphisms are not recurrent and yet their action restricted to cylinders satisfies a mixing-type formula with polynomial decay. Then we consider the extent to which the action of these hyperbolic affine automorphisms satisfy Thurston’s definition of a pseudo-Anosov homeomorphism. In particular we study the action of these automorphisms on simple closed curves and on homology classes. These objects are exponentially attracted by the expanding and contracting foliations but exhibit polynomial decay. We are able to work out exact asymptotics of these limiting quantities because of special integral formula for algebraic intersection number which is attuned to the geometry of the surface and its deformations.

INTRODUCTION

Translation surfaces built from two copies of a regular polygon as depicted in Figure 1 were studied by Veech and proven to have beautiful properties [9]. Perhaps most surprisingly, these surfaces admit affine symmetries distinct from the obvious Euclidean symmetries. An understanding of these symmetries allowed Veech to prove his famed dichotomy theorem: In all but countably many directions every trajectory equidistributes, and the countably many exceptional directions are completely periodic. Veech also used this symmetry group to answer natural counting problems on these surfaces.

Let \( X_n \) denote the double regular \( n \)-gon surface. In [4], we showed that by choosing affine maps \( A_n \) of the plane which send three consecutive vertices of the regular \( n \)-gon to the points \((-1,1), (0,0) \) and \((1,1) \), the sequence of surfaces \( P_{\cos \frac{\pi}{n}} = A_n(X_n) \) converges to the infinite area surface \( P_1 \) built from two polygonal parabolas, the convex hulls of the sets \( \{(n,n^2): n \in \mathbb{Z}\} \) and \( \{(n,-n^2): n \in \mathbb{Z}\} \). The limiting surface \( P_1 \) is depicted in the center of Figure 2.

An affine automorphism of a translation surface \( S \) is a homeomorphism \( \phi : S \to S \) which is a real affine map in local coordinates. Using the natural identification between tangent spaces to non-singular points of \( S \) with the plane,
we see that the derivative \( D\phi \) of such a map must be constant and so we interpret \( D\phi \) as an element of \( GL(2,\mathbb{R}) \). The subgroup of \( GL(2,\mathbb{R}) \) consisting of derivatives of affine homeomorphisms of \( S \) is called the surface's Veech group. (We allow orientation reversing elements in the Veech group, which differs from conventions in some other articles.) For the surfaces we will consider, the Veech
groups are contained in \( SL^2(\mathbb{R}), \) the group of \( 2 \times 2 \) matrices with real entries and determinant \( \pm 1. \)

In [4], it was shown that the symmetries of \( P_{\cos \frac{x}{g}} \) persist to the limiting surface \( P_1 \) and beyond: There are surfaces \( P_c \) defined for \( c \geq 1 \) and all these surfaces have topologically conjugate affine automorphism group actions [4, Theorem 7]. The Veech groups of \( P_c \) vary continuously in \( c \) and lie in the image of the representation \( \rho_c : \mathcal{G} \to SL^2(\mathbb{R}) \) where \( \mathcal{G} = (C_2 * C_2 * C_2) \times C_2 \) with \( C_2 \) denoting the cyclic group of order two and

\[
\rho_c(\mathcal{G}) = \left\langle \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -c & c-1 \\ -c-1 & c \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle
\]

describes the images of the generators.

We call a matrix in \( SL^2(\mathbb{R}) \) hyperbolic if it has two eigenvalues with distinct absolute values and call an affine automorphism hyperbolic if its derivative is hyperbolic. A hyperbolic matrix in \( SL^2(\mathbb{R}) \) has two real eigenvalues \( \lambda^u \) and \( \lambda^s \) with \( |\lambda^u| > 1, |\lambda^s| < 1 \) and \( \lambda^u \lambda^s = \pm 1. \) The Veech group \( \rho_1(\mathcal{G}) \) of \( P_1 \) has numerous hyperbolic elements.

A significant goal of this paper is to address the extent to which hyperbolic affine automorphisms of infinite surfaces satisfy the defining properties of pseudo-Anosov homeomorphisms of closed surfaces. The example we study is very special which enables us to say more about these questions than we'd expect to be able to for a general surface, so in particular we expect this paper sets some limits on what we could hope to be true in general.

For closed surfaces, the dynamics of hyperbolic affine automorphisms are well known to be mixing since they admit Markov partitions [3, §10.5]. On the surface \( P_1 \) we observe that cylinders satisfy a mixing-type formula but with polynomial decay. A cylinder on a translation surface is a subset isometric to \( \mathbb{R} / c\mathbb{Z} \times [0,h] \) for some circumference \( c > 0 \) and height \( h > 0. \) Figure 3 depicts horizontal cylinders on \( P_1. \) We say two sequences \( a_n \) and \( b_n \) are asymptotic and write \( a_n \sim b_n \) if \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1. \)

**Theorem 1.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be cylinders in \( P_1 \) and \( \phi : P_1 \to P_1 \) be a hyperbolic affine automorphism with derivative \( D\phi = \rho_1(g). \) Then

\[
\text{Area}(\phi^n(\mathcal{A}) \cap \mathcal{B}) \sim \frac{1}{4\sqrt{2\pi}} \left( \frac{1}{g^n} \right)^{\frac{3}{2}} \text{Area}(\mathcal{A}) \text{Area}(\mathcal{B}), \quad \text{where} \quad \beta = \frac{1}{\lambda_c^u} \left[ \frac{d}{dc} \lambda_c^u \right]_{c=1}
\]

is a positive constant which can be computed using the formula above where we use \( \lambda_c^u \) to denotes the eigenvalue of \( \rho_c(g) \) with greatest absolute value.

Note that \( \lambda_c^u \) is real and varies analytically in a neighborhood of \( c = 1 \) because the entries of the matrix \( \rho_c(g) \) are real polynomials in \( c \) and \( \rho_1(g) \) is hyperbolic (and hyperbolicity is stable under perturbation of the matrix). This ensures that the quantity \( \frac{d}{dc} \lambda_c^u \) is a well-defined real number. It will follow from later work (Lemma 5) that the quantity \( \beta \) is positive.

As a consequence of this theorem, we note that no hyperbolic \( \phi \) is recurrent because of the \( n^{\frac{3}{2}} \) decay rate seen above:
COROLLARY 2. If $\phi : P_1 \to P_1$ is hyperbolic then its action on $P_1$ is totally dissipative: there is a countable collection $\mathcal{W}$ of Lebesgue-measurable subsets of $P_1$ so that

1. the Lebesgue measure of the complement $P_1 \setminus \bigcup_{W \in \mathcal{W}} W$ is zero, and
2. each $W \in \mathcal{W}$ is wandering in the sense that the collection \{${\phi^{-n}(W)} : n \geq 0$\} is pairwise disjoint.

Because of non-recurrence, we tend to think points as the wrong objects to act to study hyperbolic affine automorphisms of $P_1$. Fortunately, some famous observations of Thurston suggest that acting on simple closed curves might be more natural.

We briefly recall Thurston’s definition of a pseudo-Anosov homeomorphism of a closed surface $M$ following [8]. Let $\mathcal{S} = \mathcal{S}(M)$ be the collection of homotopically non-trivial simple closed curves up to isotopy on $M$. Letting $i : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ denote geometric intersection number (i.e., the minimum number of transverse intersections among curves from the isotopy classes), we have an induced map

$i_* : \mathcal{S} \to \mathbb{R}^{\mathcal{S}}$ defined by $i_*(\alpha)(\beta) = i(\alpha, \beta)$

so that the image is contained in the non-negative cone of $\mathbb{R}^{\mathcal{S}}$. We let $\mathbb{P}\mathbb{R}^{\mathcal{S}}$ denote the projectivization $\mathbb{R}^{\mathcal{S}}/(\mathbb{R} \setminus \{0\})$ and $P : \mathbb{R}^{\mathcal{S}} \to \mathbb{P}\mathbb{R}^{\mathcal{S}}$ denote the projectivization map. We endow $\mathbb{R}^{\mathcal{S}}$ with the product topology and $\mathbb{P}\mathbb{R}^{\mathcal{S}}$ with the
quotient topology. Thurston observed that for a compact surface $M$ the projectivized image $P \circ i_*(\Sigma(M))$ has compact closure which we will denote by $\mathcal{PF}(M)$. As long as the surface $M$ has negative Euler characteristic, the closure $\mathcal{PF}(M)$ is a sphere of dimension one less than the dimension of the Teichmüller space $\mathcal{T}(M)$ (and forms its *Thurston boundary*). The space $\mathcal{PF}(M)$ was identified with the space of projective measured foliations on $M$ which Thurston also introduced and gives a geometric meaning to $\mathcal{PF}$. Homeomorphisms of $M$ naturally act on $\mathcal{PF}(M)$ and isotopic homeomorphisms act in the same way. A *pseudo-Anosov* homeomorphism of $M$ is a homeomorphism $\phi : M \to M$ for which there are non-zero $\mu^u$ and $\mu^s$ in $\mathbb{R}^\Sigma$ such that their projections lie in $\mathcal{PF}(M)$ and so that there is a $\lambda > 0$ so that $\mu^u \circ \phi^{-1} = \lambda \mu^u$ and $\mu^s \circ \phi = \lambda^{-1} \mu^s$. The action of a pseudo-Anosov homeomorphism $\phi$ on $\mathcal{PF}(M)$ is analogous the action of a hyperbolic isometry on the boundary of hyperbolic space: For any $\alpha \in \mathcal{PF}(M)$,

$$
\lim_{n \to -\infty} P \circ i_*(\phi^n(\alpha)) = P(\mu^u) \quad \text{and} \quad \lim_{n \to +\infty} P \circ i_*(\phi^{-n}(\alpha)) = P(\mu^s),
$$

see [3, Corollary 12.3]. Thurston showed that more generally for any $P(\nu) \in \mathcal{PF}(M) \setminus \{P(\mu^u), P(\mu^s)\}$,

$$
\lim_{n \to -\infty} P(\nu \circ \phi^{-n}) = P(\mu^u) \quad \text{and} \quad \lim_{n \to +\infty} P(\nu \circ \phi^n) = P(\mu^s).
$$

This paper investigates the extent to which the above results hold for the surface $P_1$. We begin with trying to emulate the above definitions for $P_1$. Perhaps we have been slightly abusing terminology to call $P_1$ a “surface”. It has two infinite cone singularities coming from the vertices of the polygonal parabola. These singularities do not have neighborhoods locally homeomorphic to an open subset of the plane, so we define $P_1^*$ to be $P_1$ with these singularities removed. The space $P_1^*$ is an infinite genus topological surface. We define $\Sigma = \Sigma(P_1^*)$ to be the collection of simple closed curves in $P_1^*$ up to isotopy and define $i_* : \Sigma \to \mathbb{R}^\Sigma, \mathbb{R}P^\Sigma$ and $P : \mathbb{R}^\Sigma \to \mathbb{R}P^\Sigma$ as above. We define

$$
\mathcal{PF}(P_1^*) = P \circ i_*(\Sigma(P_1^*))
$$

only this time we note that $\mathcal{PF}$ is not compact. (If $\alpha_n$ is a sequence of simple closed curves exiting every compact subset of $P_1^*$ then $\lim i_*(\alpha_n) = 0 \in \mathbb{R}^\Sigma$ and no subsequence of $P \circ i_*(\alpha_n)$ converges in $\mathbb{R}P^\Sigma$.) We show:

**Theorem 3.** Fix a hyperbolic affine automorphism $\phi : P_1 \to P_1$ and let $\lambda_1^u \in \mathbb{R}$ denote the expanding eigenvalue of $D\phi$. Let $\mu^u$ and $\mu^s$ be the elements of $\mathbb{R}^\Sigma(P_1^*)$ corresponding to the transverse measures on $P_1$ to foliations parallel to the expanding and contracting eigenspaces of $D\phi$ respectively. Then:

1. We have $\mu^u \circ \phi^{-1} = |\lambda_1^u| \mu^u$ and $\mu^s \circ \phi^{-1} = |\lambda_1^s|^{-1} \mu^s$.

2. For any $\alpha \in \Sigma(P_1^*)$, equation (1) holds. In fact, in the space $\mathbb{R}^\Sigma(P_1^*)$ we have

$$
\lim_{n \to -\infty} \frac{\| \alpha \|^2}{|\lambda_1^u|^n} i_*(\phi^n(\alpha)) = \frac{\mu^s(\alpha) \mu^u}{4 \beta^2 \sqrt{2 \pi} |\mu^u \wedge \mu^s|}.
$$
and
\[
\lim_{n \to \infty} \frac{n^2}{|\lambda|^n} i_*(\phi^{-n}(\alpha)) = \frac{\mu^u(\alpha)}{4\beta^2 \sqrt{2\pi |u^u \wedge u^s|}} \mu^s,
\]
where \( \beta \) is given as in Theorem 1 and \( u^u \) and \( u^s \) denote expanding and contracting unit eigenvectors of \( D\phi \). In particular, we have \( P \circ i_*(\phi^u(\alpha)) \rightarrow P(\mu^u) \) and \( P \circ i_*(\phi^{-n}(\alpha)) \rightarrow P(\mu^s) \) so that \( P(\mu^u), P(\mu^s) \in \mathcal{P}(P_1^c) \).

On the other hand, it is not true that (2) holds for every \( P(v) \in \mathcal{P}(P_1^c) \setminus \{P(\mu^u), P(\mu^s)\} \). For every direction \( \theta \) of irrational slope and every \( c > 1 \), there is a direction \( \theta' \) and a homeomorphism \( h : P_1 \to P_c \) (in the isotopy class of a standard identification between these surfaces) which carries the foliation on \( P_1 \) in direction \( \theta \) to the foliation in direction \( \theta' \) on \( P_c \). Furthermore, if \( \theta \) was stabilized by a hyperbolic affine automorphism \( \phi : P_1 \to P_1 \) then \( \theta' \) is stabilized by an affine automorphism \( \phi' : P_c \to P_c \) which is the same up to the canonical identification and isotopy. We can pull back the transverse measure on \( P_c \) in direction \( \theta' \) to obtain another measured on the foliation of \( P_1 \) in direction \( \theta \) which is stabilized by \( \phi \). This was proved in a more general context in [5, Theorem 4.4]. This new measure corresponds to a distinct \( \phi \)-invariant element of \( \mathcal{H}(P_1^c) \).

In fact we can see this element lives in \( \mathcal{P}(P_1^c) \) in many cases because the straight-line flow in many eigendirections of pseudo-Anosov homeomorphisms is ergodic [5, Theorem 4.5], so that we can obtain projective approximations by simple closed curves by flowing point forward until it returns close and closing it up. As we increase the flow time, convergence to \( P(\mu^u) \) is guaranteed for almost every starting point by the ratio ergodic theorem.

We have shown that elements in \( P(\mathcal{F}(P_1^c)) \) are attracted under \( \phi \) and \( \phi^{-1} \) orbits respectively by \( P(\mu^u) \) and \( P(\mu^s) \), but noted that this does not hold on the closure \( \mathcal{P}(P_1^c) \), so it is natural to wonder how attractive \( P(\mu^u) \) and \( P(\mu^s) \) are in other contexts.

Since \( P_1 \) is a translation surface, it is natural to orient our foliations in each direction and to consider our transverse measures to be signed measures. Informally, let \( H_1(P_1^c;\mathbb{R}) \) be real weighted finite sums of homology classes of closed curves in \( P_1^c \), and let \( H_1(P_1,\Sigma;\mathbb{R}) \) be real weighted finite sums of homology classes of closed curves and curves joining the singularities of \( P_1 \). Algebraic intersection number gives a weakly non-degenerate bilinear map
\[
\cap : H_1(P_1^c;\mathbb{R}) \times H_1(P_1,\Sigma;\mathbb{R}) \to \mathbb{R}.
\]
Let \( \mathcal{H} = \text{Hom}(H_1(P_1,\Sigma;\mathbb{R}),\mathbb{R}) \) be the collection of linear maps from \( H_1(P_1,\Sigma;\mathbb{R}) \) to \( \mathbb{R} \) and let \( \mathcal{PH} = \mathcal{H}/(\mathbb{R} \setminus \{0\}) \) be the projectivization of this space. In a parallel construction to geometric intersection number, we get a map induced by algebraic intersection and a projectivization map
\[
\cap_\ast : H_1(P_1^c;\mathbb{R}) \to \mathcal{H}, \quad P : \mathcal{H} \to \mathcal{PH}.
\]

Given an oriented arc on \( P_1 \) we can lift it to the universal cover and project it to the plane under the developing map. The holonomy vector of the arc is the difference of the developed end point and the starting point. Holonomy gives
rise to linear maps
\[ \text{hol}_1 : H_1(P^2_1; \mathbb{R}) \to \mathbb{R}^2 \quad \text{and} \quad \text{hol}_1 : H_1(P^1_1, \Sigma; \mathbb{R}) \to \mathbb{R}^2, \]
where we write \text{hol}_1 to indicate we are computing holonomy on \( P_1 \). Given a direction described by a unit vector \( u \in \mathbb{R}^2 \) we get an element of \( \mathcal{H} \) using the linear map
\[ (3) \quad H_1(P^1_1, \Sigma; \mathbb{R}) \to \mathbb{R}; \quad \sigma \to u \wedge \text{hol}_1(\sigma), \]
where \( \wedge : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) is the usual wedge product: \((a, b) \wedge (c, d) = ad - bc\).

If \( \phi : P_1 \to P_1 \) is an affine homeomorphism with hyperbolic derivative, then the choice of unit unstable and stable eigenvectors of \( D\phi \) give rise via (3) to respective elements \( \mu^u \) and \( \mu^s \) of \( \mathcal{H} \) satisfying
\[ \mu^u \circ \phi^{-1} = \lambda_1^u \mu^u \quad \text{and} \quad \mu^s \circ \phi^{-1} = \lambda_1^s \mu^s, \]
where \( \lambda_1^u \) and \( \lambda_1^s \) are the expanding and contracting eigenvalues of \( D\phi \). We show that the projectivized classes \( P(\mu^u) \) and \( P(\mu^s) \) respectively attract and repel every element of \( P \circ \cap_\ast \{ H_1(P^2_1; \mathbb{R}) \} \), but unlike in prior results the rate of polynomial decay depends on the chosen homology class.

**Theorem 4.** If \( \gamma \in H_1(P^2_1; \mathbb{R}) \) is non-zero then
\[ \lim_{n \to +\infty} P \circ \cap_\ast \{ \phi^n(\gamma) \} = P(\mu^u) \quad \text{and} \quad \lim_{n \to +\infty} P \circ \cap_\ast \{ \phi^{-n}(\gamma) \} = P(\mu^s). \]
Moreover, there is a descending sequence of subspaces indexed by \( \mathbb{N} \)
\[ H_1(P^2_1; \mathbb{R}) = S_0 \supset S_1 \supset S_2 \ldots \]
so that

1. Each \( S_{j+1} \) is codimension one in \( S_j \) (i.e., \( S_{j+1} \) is the kernel of a surjective linear map \( S_j \to \mathbb{R} \)).
2. The intersection \( \cap_{j=0} \) \( S_j \) is the zero subspace \( \{0\} \).
3. For any \( \gamma \in S_j \setminus S_{j+1} \) the sequence \( \frac{n^{1/2}}{\lambda_1^u} \cap_\ast \{ \phi^n(\gamma) \} \) converges in \( \mathcal{H} \) to a non-zero scalar multiple of \( \mu^u \).

We remark that statement (3) also holds in the opposite direction, i.e., \( \frac{n^{1/2}}{\lambda_1^u} \cap_\ast \{ \phi^{-n}(\gamma) \} \) converges to a multiple of \( \mu^s \), but the rate given by \( j \) is determined by a different sequence of nested subspaces. This theorem is a consequence of our Theorem 19 which is stronger in that it gives a formula for the constant appearing in the limit in statement (3). The proof of Theorem 4 contains a concrete description of the subspaces \( S_j \) and gives a slightly different formula for the limiting constant; see (34).

Again we could consider trying to extend this type of convergence to a larger space, and \( \mathcal{P} \mathcal{H} \) itself would be a natural candidate but this space also contains fixed points corresponding to measured foliations on \( P_c \). It is not clear to the author if there is a natural intermediate space between \( P \circ \cap_\ast \{ H_1(P^2_1; \mathbb{R}) \} \) and \( \mathcal{P} \mathcal{H} \) in which we see \( P(\mu^u) \) and \( P(\mu^s) \) as the global attractor and repeller.
We will now briefly discuss the method we use to prove these results. The main idea is to completely understand algebraic intersection numbers, and we provide a formula for computing the algebraic intersection numbers between curves on $P_1$. It first needs to be observed that the surfaces $P_c$ for $c \geq 1$ are canonically homeomorphic, with the homeomorphisms coming from viewing $P_c$ for $c \geq 1$ as a parameterized deformation of translation surfaces. Fixing a curve $\gamma \in P_1$ representing a homology class we can use these canonical homeomorphisms to obtain corresponding curves in $P_c$. We observe that the holonomies of these curves measured on $P_c$ denoted $\text{hol}_c \gamma \in \mathbb{R}^2$ depend polynomially in $c$. Thus this quantity makes sense for all $c \in \mathbb{R}$. In Lemma 12, we show that for any two curves $\gamma$ and $\sigma$ representing homology classes on $P_1$, their algebraic intersection number is given by

$$
\gamma \cap \sigma = \frac{1}{2\pi} \int_0^\pi \left( (\text{hol}_{\cos t} \gamma) \land (\text{hol}_{\cos t} \sigma) \right) (1 - \cos t) \, dt.
$$

(4)

This result gives a mechanism to reduce the Theorems above to questions involving the asymptotics of certain trigonometric integrals.

Organization of article.

- In §1 we provide a condensed description of main ideas we use from the theory of translation surfaces. We give more formal descriptions of the homological spaces mentioned above.
- In §2 we investigate the continuous family of representations that arises out of considering the Veech groups of the surfaces $P_c$. This family of representations is studied in the abstract and we prove results about eigenvalues that are crucial for our later arguments.
- Section 3 addresses the geometry and dynamics of $P_1$.
  - In §3.1 we review the construction of the surfaces $P_c$.
  - In §3.2 we compute generating sets for the homological spaces we work with.
  - In §3.3 we explain how holonomies of homology classes deform as we vary $c$ and formalize our intersection number formula described in (4).
  - In §3.4 we describe the affine automorphism groups of the surfaces $P_c$ for $c \geq 1$.
  - In §3.5 we prove Theorem 19, which is the most important result in the paper: it gives an asymptotic formula for algebraic intersections of the form $\phi^n(\gamma) \cap \sigma$. We use it to prove Theorem 4.
  - In §3.6 we prove our asymptotic formula for intersections of cylinder described by Theorem 1. The key is the observation that the area of intersection of two cylinders is largely governed by algebraic intersection numbers between the core curves.
  - In §3.7 we consider geometric intersection numbers and prove Theorem 3.
1. Background and Notation

A translation surface is a topological surface with an atlas of charts to the plane so that the transition functions are translations. Equivalently, an translation surface $S$ is a surface whose universal cover is equipped with a local homeomorphism called the developing map $\text{dev}$ from the universal cover $\tilde{S}$ to $\mathbb{R}^2$, so that for any deck transformation $\Delta: \tilde{S} \to \tilde{S}$, there is a translation $T: \mathbb{R}^2 \to \mathbb{R}^2$ so that $\text{dev} \circ \Delta = T \circ \text{dev}$. Such a surface should be considered equivalent to the surface obtained by post-composing the developing map with a translation. Note that this definition does not allow for cone singularities on the surface but they may be treated as punctures.

In this paper we will be considering surfaces of infinite genus, but our surfaces will be decomposable into countably many triangles. Each translation surface we consider $S$ will be a countable disjoint union of triangles with edges glued together pairwise by translations in such a way so that each point on the interior of an edge has a neighborhood isometric to an open subset of the plane. (Figure 6 depicts such a decomposition for $P_1$.) We will use $\Sigma$ to denote the singularities which are the equivalence classes of the vertices of the triangles in $S$. The surface $S \setminus \Sigma$ is a translation surface, while $S$ itself is not a surface if countably many triangles meet at a singularity.

Let $R$ be a ring containing $\mathbb{Z}$ such as $\mathbb{Z}$ or $\mathbb{R}$. For us, first relative homology over $R$ is $H_1(S, \Sigma; R)$ may be viewed as the $R$-module generated by oriented edges of the triangulation of $S$ and subject to the conditions that the sum of two copies of the same edge with opposite orientations is zero as is the sum of edges oriented as the boundary of a triangle. Less formally, $H_1(S, \Sigma; R)$ may be viewed as homology classes consisting of finite weighted sums of curves joining points in $\Sigma$ and closed curves subject to the condition that they pass through only finitely many triangles. Any such curve is homotopic to a union of edges and the resulting class in $H_1(S, \Sigma; R)$ is independent of the choice of such a homotopy.

Let $S^\circ$ denote $S \setminus \Sigma$. We define $H_1(S^\circ; \mathbb{Z})$ to be the abelianization of $\pi_1(S^\circ)$. Note that by compactness a closed curve in $S^\circ$ only intersects finitely many triangles defining $S$. We define $H_1(S^\circ; R)$ to be $H_1(S^\circ; \mathbb{Z}) \otimes_{\mathbb{Z}} R$. Again, we may consider $H_1(S^\circ; R)$ to represent all finite sums of closed curves in $S^\circ$ weighted by elements of $R$.

We use $\cap$ to denote the algebraic intersection number

$$\cap: H_1(S^\circ; \mathbb{Z}) \times H_1(S, \Sigma; \mathbb{Z}) \to \mathbb{Z},$$

which is bilinear. We follow the convention that $\alpha \cap \beta$ is positive if $\alpha$ is moving rightward and $\beta$ is moving upward. With $R$ as above, algebraic intersection extends to a bilinear map

$$\cap: H_1(S^\circ; R) \times H_1(S, \Sigma; R) \to R.$$

Let $\gamma: [0, 1] \to S$ be a curve and let $\tilde{\gamma}: [0, 1] \to \tilde{S}$ be a lift to the universal cover. The holonomy vector of $\gamma$ is
Observe that this quantity is independent of the choice of lift. Holonomy yields linear maps

\[ \text{hol}: H_1(S, \Sigma; \mathbb{R}) \to \mathbb{R}^2 \quad \text{and} \quad \text{hol}: H_1(S^0; \mathbb{R}) \to \mathbb{R}^2, \]

which send a weighted sum of curves to the corresponding weighted sum of holonomy vectors. We use \( \text{hol} \) for both maps because the following map diagram commutes

\[
\begin{array}{c}
H_1(S^0; \mathbb{R})
\end{array} \xrightarrow{\text{hol}} \begin{array}{c} H_1(S, \Sigma; \mathbb{R}) \\
\mathbb{R}^2 \\
\mathbb{R}^2
\end{array}
\]

where \( H_1(S^0; \mathbb{R}) \to H_1(S, \Sigma; \mathbb{R}) \) is induced by the inclusion \( S^0 \) of into \( S \).

Let \( S \) be a translation surface built as above by gluing together triangles in a countable set \( \mathcal{T} \). Let \( f: S \to S \) be a homeomorphism preserving the singular set \( \Sigma \) and satisfying the condition that for any \( T \in \mathcal{T} \), the image \( f(T \setminus \Sigma) \) intersects only finitely many triangles in \( \mathcal{T} \). Then \( f \) acts naturally on the homological spaces defined above. We denote this action by \( f_* \). Let \( \tilde{f}: \tilde{S} \to S \) be a lift to the universal cover. We say that \( f \) is an affine automorphism of \( S \) if there is an affine map \( A: \mathbb{R}^2 \to \mathbb{R}^2 \) so that \( \text{dev} \circ \tilde{f} = A \circ \text{dev} \).

This notion does not depend on the lift \( \tilde{f} \). The derivative of an affine automorphism \( f: S \to S \) is the element \( Df \in \text{GL}(2, \mathbb{R}) \) which is given by \( A \) post-composed by a translation (so that the origin is fixed). We say \( f \) is hyperbolic if \( Df \) has distinct eigenvalues and their absolute values differ. Observe that if \( f \) is an affine automorphism then for any homology class \( \sigma \) we have

\[ \text{hol} f_*(\sigma) = Df \cdot \text{hol}(\sigma). \]

The collection of all affine automorphisms of \( S \) forms a group which we denote by \( \text{Aff}(S) \). In this paper, the Veech group of \( S \) is \( D\text{Aff}(S) \subseteq \text{GL}(2, \mathbb{R}) \). (Some papers consider the Veech group to be \( D\text{Aff}(S) \cap \text{SL}(2, \mathbb{R}) \).

2. A SUBGROUP DEFORMATION

For each real \( c \in \mathbb{R} \) let \( \Gamma_c \) denote the subgroup of \( \text{SL}^\pm(2, \mathbb{R}) \) generated by \(-I\) together with the involutions

\[ A_c = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_c = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad C_c = \begin{bmatrix} -c & c-1 \\ -c-1 & c \end{bmatrix}. \]

We will later see that \( \Gamma_c \) realizes the Veech group \( D\text{Aff}(P_c) \) for \( c \geq 1 \), but in this section we will be interested in proving some results about how the deformation \( \Gamma_c \) effects hyperbolic elements of \( \Gamma_1 \).

The group \( \text{PGL}(2, \mathbb{R}) \) is naturally identified with the isometry group of the hyperbolic plane, and each of these three matrices act by reflections in a line in the hyperbolic plane (called the axis of the reflection). Figure 4 depicts the axes
of reflections corresponding to the matrices above. Observe that the relationship between the lines $A$ and $C$ changes as $c$ varies. We will be concerned with the groups $\Gamma_c$ when $c \geq -1$. Observe that $C_{-1} = -B_{-1}$. When $-1 < c < 1$, the axes of reflections $A_c$ and $C_c$ intersect at an angle of $\theta$ where $\cos \theta = c$. When $\theta = \frac{2\pi}{n}$, the group $\Gamma_c$ is discrete with the triangle formed by the reflection axes forming a fundamental domain for the group action. When $c = 1$, these reflection axes become tangent at infinity, and $\Gamma_c$ becomes the congruence two subgroup of $SL^\pm(2, \mathbb{Z})$ whose fundamental domain is the ideal triangle enclosed by the three reflection axes. When $c \geq 1$, the group stays discrete but is no longer a lattice since the fundamental domain which consists of the enclosed ultra-ideal triangle has infinite area.

![Figure 4](image-url)

**Figure 4.** The fixed point sets of $A_c$, $B_c$ and $C_c$ in the upper half plane model depicted from left to right for the cases of $c = \cos \frac{\pi}{4}$, $c = 1$, and $c = \frac{3}{4}$ from left to right.

We will think of these groups as a continuous family of representations of the abstract group

$$\mathcal{G} = (C_2 \ast C_2 \ast C_2) \times C_2,$$

where $C_2 = \mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order two. We define the representation

$$\rho: \mathcal{G} \rightarrow SL^\pm(2, \mathbb{Z}[c])$$

sending the generators of $\mathcal{G}$ to $A_c$, $B_c$, $C_c$ and $-I$ respectively.

For any $z \in \mathbb{C}$, we define

$$\rho_z: \mathcal{G} \rightarrow SL^\pm(2, \mathbb{C}); \quad g \mapsto \rho(g)(z),$$

i.e., we evaluate $\rho(g)$ with $c = z$. Below and throughout this paper, we abuse notation by identifying the free variable $c$ temporarily with a constant, so we will simply write $\rho_c$ rather than $\rho_z$.

The following is the main result we will need about this family of representations.
**Lemma 5.** Let \( g \in \mathcal{G} \) be such that \( \rho_1(g) \) has a real eigenvalue \( \lambda^u_1 \) with \( |\lambda^u_i| > 1 \). For each \( c \in \mathbb{R} \), let \( \lambda^u_c \) denote the choice of an eigenvalue realizing the maximum value of the absolute value of an eigenvalue of \( \rho_c(g) \). Then:

1. We have \( |\lambda^u_c| < |\lambda^u_1| \) whenever \(-1 \leq c < 1\).
2. The function \( c \mapsto \lambda^u_c \) is differentiable at \( c = 1 \) and the constant
   \[
   \beta = \frac{1}{\lambda^u_1} \left[ \frac{d}{dc} \lambda^u_c \right]_{c=1}
   \]
   is positive.

We remark that the constant \( \beta \) appeared in Theorems 1 and 3.

The remainder of the section is devoted to a proof of the Lemma and the following consequence which involves the operator norm \( \| \cdot \| \) on real \( 2 \times 2 \) matrices defined using the Euclidean metric on \( \mathbb{R}^2 \).

**Proposition 6.** Let \( g \in \mathcal{G} \) be as in Lemma 5. Let \( c_0 \in \mathbb{R} \) be a number so that \(-1 \leq c_0 < 1\). Then there is a constant \( \xi \in \mathbb{R} \) with \( 1 < \xi < \lambda^u_1 \) so that
   \[
   \| \xi^{-n} \rho_c(g)^n \| \text{ converges to zero uniformly for } c \in [-1, c_0].
   \]

The group \( \text{PGL}(2, \mathbb{R}) \) is the isometry group of the hyperbolic plane. For a hyperbolic \( G \in \text{PGL}(2, \mathbb{R}) \) the eigenvalue \( \lambda^u \) of largest absolute value has the geometric significance:

\[
\inf_{x \in \mathbb{H}^2} \text{dist}(x, Gx) = 2 \log |\lambda^u|.
\]

Moreover, this infimum is achieved. The collection of points where this infimum is achieved is a geodesic in \( \mathbb{H}^2 \) called the axis of the hyperbolic isometry \( G \). The axis has a canonical orientation determined by the direction \( G \) translates the geodesic. If \( G \) belongs to a discrete group \( \Gamma \), then this axis projects to a curve of length \( 2 \log |\lambda^u| \) in the quotient \( \mathbb{H}^2/\Gamma \).

We will need to extend these ideas to triangular billiard tables which are not quotients of \( \mathbb{H}^2 \) by a discrete group. Let \( \Delta \) be any triangle in \( \mathbb{H}^2 \) (with the case of interest being the triangles shown in Figure 4). We label the edges by the \( \{A, B, C\} \) and we will use \( \ell_a, \ell_b, \) and \( \ell_c \) to denote the bi-infinite geodesics in \( \mathbb{H}^2 \) that contain the marked edges. Let \( \mathcal{G} := C_2 \ast C_2 \ast C_2 \) represent \( \mathcal{G} \) modulo the central \( C_2 \) in \( \mathcal{G} \). Given \( \Delta \) we get a representation \( \rho_\Delta : \mathcal{G} \to \text{PGL}(2, \mathbb{R}) \) where we send the generators \( g_A, g_B, g_C \in \mathcal{G} \) to the reflections in \( \ell_a, \ell_b, \) and \( \ell_c \) respectively.

Each conjugacy class \( [g] \) in \( \mathcal{G} \) has a representative of the form

\[
\gamma = g_{L_0} g_{L_1} \cdots g_{L_{n-1}} \quad \text{with each } L_i \in \{A, B, C\},
\]

where \( L_i \neq L_{i+1} \) for any \( i \in \{0, \ldots, n-1\} \) and addition taken modulo \( n \). Moreover, this representative is unique up to cyclic permutations. We define the orbit-class \( \Omega([g]) \) to be the collection of all closed rectifiable curves in \( \mathbb{H}^2 \) that visit the lines \( \ell_{L_0}, \ell_{L_1}, \ldots, \ell_{L_{n-1}} \) in that order. We define the orbit-length of \([g]\) to be

\[
\mathcal{L}(\Omega([g])) = \inf_{\gamma \in \Omega([g])} \text{length}(\gamma).
\]

We have the following lemma.
**Lemma 7.** Fix $\Delta$ and notation as above. Let $g \in \mathcal{G}'$ and let $[g]$ be its conjugacy class. Let $\lambda^u$ denote the eigenvalue of $\rho_{\Delta}(g)$ with largest absolute value. Then

$$2 \log |\lambda^u| \leq \mathcal{L}([g]).$$

The proof is essentially the “unfolding” construction from polygonal billiards:

**Proof.** The statement is certainly true unless $|\lambda^u| > 1$. Since eigenvalues are a conjugacy invariant, we may assume that $g$ is given as in (10). If the conclusion is false, there is a $\gamma \in \Omega(G)$ with length less than $2 \log |\lambda^u|$. We will draw a contradiction by construct a new path $\gamma'$ in $\mathbb{H}^2$ with length equal to that of $\gamma$ such that $\rho_{\Delta}(g)$ translates the starting point of $\gamma'$ to the ending point of $\gamma'$ violating (9).

We may assume $\gamma$ begins on the geodesic $\ell_{L_0}$ and then travels to $\ell_{L_1}$ and so on. For $i \in \{0, \ldots, n-1\}$, let $\gamma_i$ be the portion of $\gamma$ which travels from $\ell_{L_{i-1}}$ to $\ell_{L_i}$ with subscripts taken modulo $n$ so that $\gamma = \gamma_1 \cup \ldots \cup \gamma_{n-1} \cup \gamma_0$. Define $h_0, \ldots, h_n \in \mathcal{G'}$ inductively so that $h_0$ is the identity element of $\mathcal{G'}$ and $h_{i+1} = h(i)g_{L_i}$. This makes $h_n = g$ as given in (10). For $i \in \{0, \ldots, n-1\}$ set $\gamma_i' = \rho_{\Delta}(h_i)(\gamma_i)$ and set $\gamma_n' = \rho_{\Delta}(h_n)(\gamma_0)$. We observe

1. The collection $\gamma_0' \cup \ldots \cup \gamma_n'$ is a continuous curve.
2. We have $\gamma_n' = \rho_{\Delta}(h_n)(\gamma_0')$.

Statement (2) holds by construction since $\gamma_0 = \gamma_0'$ (as $h_0$ is the identity). To see statement (1) observe that $\gamma_i$ terminates at $\ell_{L_{i-1}}$ at the same point at which $\gamma_{i+1}$ starts. Thus, $\gamma_i' \cup \rho_{\Delta}(g_{L_i})(\gamma_{i+1})$ passes continuously through $\ell_{L_i}$. It follows that the arcs $\gamma_i' = \rho_{\Delta}(h_i)(\gamma_i)$ and $\gamma_{i+1}' = \rho_{\Delta}(h_i; g_{L_i})(\gamma_{i+1})$ are joined at some point on $\rho_{\Delta}(h_i)(\ell_{L_i})$. This proves (1).

From (1) we see that a point on $\gamma_i'$ is joined to the corresponding point on $\gamma_n'$ by a path of length equal to the length of $\gamma$, and from (2) we see that these points differ by an application of the hyperbolic isometry $\rho_{\Delta}(h_n)$. Recalling that $h_n = g$, we see this violates (9). \hfill $\Box$

We will need to introduce one more idea. The **Klein model** for the hyperbolic plane consists of $\mathbb{KH}^2 = \{(x^2, y^2) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. The boundary $\partial \mathbb{KH}^2 = \{(x^2, y^2) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Geodesics in the Klein model are Euclidean line segments. Distance between points in the Klein model may be computed in two ways. Let $P_1$ and $P_2$ be two points in $\mathbb{KH}^2$, and let $\overline{P_1P_2}$ be the Euclidean line through them. Let $Q_1$ and $Q_2$ be the two points of $\partial \mathbb{KH}^2 \cap \overline{P_1P_2}$, chosen so that $P_1$ is closest in the Euclidean metric to $Q_1$. Then the distance between $P_1$ and $P_2$ is given by the logarithm of the cross ratio,

$$\text{dist}_{\mathbb{KH}^2}(P_1, P_2) = \log \left( \frac{\text{dist}_{\mathbb{R}^2}(P_1, P_2)}{\text{dist}_{\mathbb{R}^2}(Q_1, Q_2)} \right). \tag{11}$$

Alternately, we can compute distance by using the metric tensor $ds$.

$$ds = \sqrt{dx^2 + dy^2 + (xdx + ydy)^2} \quad \frac{1}{1 - x^2 - y^2} \quad \frac{1}{1 - x^2 - y^2} \quad \text{.} \tag{12}$$

**Proof.**
The distance between two points can be computed by integrating the metric tensor over the geodesic path between them. See [1] or any hyperbolic geometry text for more details.

As discussed above the projectivization of \( \rho_c(\mathcal{G}) \) to a subgroup of \( PGL(2, \mathbb{R}) \) is generated by the reflections in the sides of a triangle \( \Delta_c \) in \( \mathbb{H}^2 \). See Figure 4. When \( c = \cos \theta \leq 1 \), the triangle has two ideal vertices and one vertex with angle \( \theta \). For our purposes, we will think of \( \Delta_c \subset \mathbb{K}\mathbb{H}^2 \). We define

\[
\Delta_c = \text{Convex Hull}(\{P_1, P_2, P_3\}) \subset \mathbb{K}\mathbb{H}^2,
\]

where \( P_1 = (-1, 0), P_2 = (1, 0) \), and \( P_3 = (0, \frac{\sqrt{1+c}}{\sqrt{2}}) \). It is an elementary check that this triangle is isometric to the triangle used to define the reflection group \( \rho_c(\mathcal{G}) \). The reflection lines of the elements \( A_c, B_c, \) and \( C_c \) are given by \( \ell_A = P_3P_1 \), \( \ell_B = P_1P_2 \), and \( \ell_C = P_2P_3 \) respectively.

Fixing \( c \) observe that \( \rho_c: \mathcal{G} \to SL^+(2, \mathbb{R}) \) projectivizes to a representation \( \rho_c': \mathcal{G}' \to PGL(2, \mathbb{R}) \). The generators \( g_A, g_B, g_C \in \mathcal{G}' \) are mapped under \( \rho_c' \) respectively to \( A_c, B_c \) and \( C_c \) of (5) viewed as elements of \( PGL(2, \mathbb{R}) \). Then \( \ell_A, \ell_B, \) and \( \ell_C \) are the fixed point sets of these hyperbolic reflections.

**Proof of Lemma 5, statement (1).** Choose \( g \in \mathcal{G} \) so that \( \rho_1(g) \) has a real eigenvalue \( \lambda^n \) with \( |\lambda^n| > 1 \). Let \( G_1 = \rho_1(g) \) which we think of as a hyperbolic transformation of \( \mathbb{H}^2 \). In the case of \( c = 1 \), the group \( \rho_1'(\mathcal{G}') \) is discrete and \( \Delta_1 \) is a fundamental domain for the action. Thus, the projection of the axis of \( G_1 \) to \( \mathbb{H}^2/\rho_1(\mathcal{G}) \) minimizes length in the orbit-class \( \Omega(g) \). In particular, this billiard path \( \gamma_1 \) realizes the infimum discussed in Lemma 7.

Let \( |\lambda^n| \) denote the greatest absolute value of an eigenvalue of \( G_c = \rho_c(g) \). We will use Lemma 7 to show that \( |\lambda^n| < |\lambda^m| \) for all \(-1 < c < 1\). In coordinates on the closure of \( \mathbb{K}\mathbb{H}^2 \) consider the linear map

\[
M_c : \mathbb{K}\mathbb{H}^2 \to \mathbb{K}\mathbb{H}^2 : (x, y) \mapsto \left( x, \frac{y\sqrt{1+c}}{\sqrt{2}} \right).
\]

Observe from our formula for the vertices that \( M_c(\Delta_1) = \Delta_c \). We claim that \( M_c \) shortens every line segment in \( \Delta_1 \) except line segments contained in the side \( P_1P_2 \). Consider a segment \( XY \) with finite length in \( \Delta_1 \). Let \( PQ \) be the geodesic containing \( XY \), so that \( PQ \in \partial\mathbb{K}\mathbb{H}^2 \) with \( P \) the closest to \( X \) as in the left side of Figure 5. Then by (11),

\[
dist_{\mathbb{K}\mathbb{H}^2}(X, Y) = \log \left( \frac{\text{dist}_{\mathbb{H}^2}(X, Y)\text{dist}_{\mathbb{H}^2}(P, Q)}{\text{dist}_{\mathbb{H}^2}(X, P)\text{dist}_{\mathbb{H}^2}(Y, Q)} \right).
\]

And, let \( P', Q' \in \partial\mathbb{K}\mathbb{H}^2 \) be the points where the geodesic \( M_c(P)M_c(Q) \) intersects the boundary. Then,

\[
dist_{\mathbb{K}\mathbb{H}^2}(M_c(X), M_c(Y)) = \log \left( \frac{\text{dist}_{\mathbb{H}^2}(M_c(X), M_c(Y))\text{dist}_{\mathbb{H}^2}(P', Q')}{\text{dist}_{\mathbb{H}^2}(M_c(X), P')\text{dist}_{\mathbb{H}^2}(M_c(Y), Q')} \right).
\]
It is a standard computation that $\text{dist}_{\mathbb{H}^2}(X, Y) > \text{dist}_{\mathbb{H}^2}(M_c(X), M_c(Y))$ so long as the line segment $P'Q'$ strictly contains the line segment $M_c(P)M_c(Q)$, i.e., as long as $M_c(P) \neq P'$ or $M_c(Q) \neq Q'$. In particular, the only time this inequality could be false is when $PQ \subset \{(x, y) | y = 0\}$. Our claim is proved.

To finish the proof, we apply the claim to the billiard path $\gamma_1$ constructed in the first paragraph. No finite length billiard path can have a segment contained in the line $y = 0$, therefore

$$\text{length}(M_c(\gamma_1)) < \text{length}(\gamma_1).$$

whenever $-1 \leq c < 1$. Then by Lemma 7, for all such $c$,

$$2 \log |\lambda^u_1| \leq \text{length}(M_c(\gamma_1)) < \text{length}(\gamma_1) = 2 \log |\lambda^u_1|.$$ 

Thus $|\lambda^u_c| < |\lambda^u_1|$.

\[ \square \]

**Proof of Lemma 5, statement (2).** Let $g \in \mathcal{G}$ be chosen so that $\rho_1(g)$ has an eigenvalue $\lambda^u_1$ so that $|\lambda^u_1| > 1$. By possibly replacing $\rho_1(g)$ with $-I\rho_1(g)$ we may assume $\lambda^u_1 > 1$. Let $\lambda^u_c$ be the largest eigenvalue of $G_c = \rho_c(g)$, which by continuity of $c \mapsto G_c$ satisfies $\lambda^u_c > 1$ in a neighborhood of 1. We must show

$$\frac{d}{dc} \lambda^u_c |_{c=1} > 0.$$ 

The quantity $\frac{d}{dc} \lambda^u_c$ exists at $c = 1$, since entries of $G_c$ vary polynomially in $c$. Thus we can afford to just look for a one-sided derivative and we will restrict attention to the case of $c < 1$ so that we can make use of the map $M_c : \Delta_1 \to \Delta_c$ of (14).

Let $m = \frac{\sqrt{1+x^2}}{\sqrt{2}}$. We have $\frac{d}{dc} m = \frac{1}{2\sqrt{2}+2c}$, and $\frac{d}{dc} m |_{c=1} = \frac{1}{4}$. By (12), the $\mathbb{H}^2$ length of the tangent vector $i = (1, 0)$ at the point $(x, y) \in \mathbb{H}$ is given by

$$I_1 = \frac{\sqrt{1-y^2}}{1-x^2-y^2}.$$
The length of the tangent vector \( M_c(i) = (1, 0) \) at the point \( M_c(x, y) = (x, my) \) is given by

\[
I_2 = \frac{\sqrt{1 - m^2 y^2}}{1 - x^2 - m^2 y^2}.
\]

We compute

\[
\frac{d}{dc} I_1 \bigg|_{c=1} = \frac{y^2(1 + x^2 - y^2)}{4(1 - y^2)(1 - x^2 - y^2)}
\]

which is positive on all of \( \mathbb{H}^2 \) other than those points where \( y = 0 \). Note, we are perturbing \( c \) in the negative direction. This says that off the line \( y = 0 \), \( M_c \) is compressing every horizontal vector enough to be detected by the first derivative. Let \( J_1 \) be the \( \mathbb{H}^2 \) length of the tangent vector \( J = (0, 1) \) at the point \( (x, y) \in \mathbb{H}^2 \) and \( J_2 \) be the \( \mathbb{H}^2 \) length of the vector \( M_c(j) = (0, m) \) at the point \( M_c(x, y) = (x, my) \). We have

\[
J_1 = \frac{\sqrt{1 - x^2}}{1 - x^2 - y^2} \quad \text{and} \quad J_2 = \frac{m \sqrt{1 - x^2}}{1 - x^2 - m^2 y^2}.
\]

We compute

\[
\frac{d}{dc} J_2 \bigg|_{c=1} = \frac{1 - x^2 + y^2}{4(1 - x^2 - y^2)} > 0.
\]

In this case, \( M_c \) is compressing every vertical vector enough to be detected by the first derivative.

The argument concludes in the same manner as the previous proof. Let \( \gamma_1 \) be the billiard path on \( \Delta_1 \) corresponding to \( G_1 \). The argument above tells us that \( \frac{d}{dc} \text{length}(M_c(\gamma_1)) = k > 0 \). But for \( c < 1 \),

\[
2\log \lambda^{c}_1 \leq \text{length}(M_c(\gamma_1)) = \text{length}(\gamma_1) - k(1 - c) + \text{higher order terms}.
\]

By taking a straightforward derivative, we get

\[
2\log \lambda^{u}_c = 2\log \lambda^{u}_1 - (1 - c) - \frac{2}{dc} \frac{\lambda^{u}_c}{\lambda^{u}_1} |_{c=1} + \text{higher order terms}.
\]

This pair of equations imply

\[
\frac{d}{dc} \lambda^{u}_c |_{c=1} \geq \frac{k \lambda^{u}_1}{2} > 0. \quad \square
\]

**Proof of Proposition 6.** Fix a \( c_0 < 1 \). The function \( c \rightarrow |\lambda^{u}_c| \) is continuous so attains its maximum in \([-1, c_0]\). By (1) of Lemma 5 we know this maximum is less than \( \lambda^{u}_1 \), so we can select \( \xi \) so that \( |\xi^{u}_c| < \xi < \lambda^{u}_1 \) for all \( c \in [-1, c_0] \). We will show that \( \xi^{-n} \rho_c(g)^n \) tends to the zero matrix uniformly in the operator norm.

First observe that it suffices to find an \( N' \in \mathbb{N} \) so that \( n \geq N' \) implies \( \|\rho_c(g)^n\| < \xi^n \) for \( c \in [-1, c_0] \). Suppose this statement is true and fix an \( c > 0 \). We will find an \( N \) so that \( n > N \) implies that the operator norm \( \|\xi^{-n} \rho_c(g)^n\| < c \). By continuity of \( c \rightarrow \|\rho_c(g)^N\| \) we can set

\[
\|\rho_c(g)^N\| = 2/3.
\]
\[ \xi' = \sup \{ \| \rho_c(g)^{N'} \| : c \in [-1, c_0] \} < \xi \quad \text{and} \]
\[ M = \sup \{ \xi^{-n} \| \rho_c(g)^{-n} \| : 0 \leq n < N' \text{ and } c \in [-1, c_0] \}. \]

Then choose \( K \) sufficiently large so that
\[ k > K \quad \text{implies} \quad M \left( \frac{\xi'}{\xi} \right)^{-kN'} < \epsilon. \]

Any \( n > (K + 1)N' \) can be written in the form \( n = kN' + n' \) for some \( k > K \) and some \( n' \) satisfying \( 0 \leq n' < N' \) so that by submultiplicativity of the operator norm we have
\[ \xi^{-n} \| \rho_c(g)^{n} \| \leq \xi^{-n'} \| \rho_c(g)^{-n'} \| \cdot \left( \xi^{-N'} \| \rho_c(g)^{N'} \| \right)^{k} \leq M \left( \frac{\xi'}{\xi} \right)^{-kN'} < \epsilon. \]

The main idea is to continuously diagonalize the matrix action of \( \rho_c(g)^{n} \). However there is a minor difficulty that for some values of \( c \) the matrix might not be diagonalizable. The matrices \( \rho_c(g) \) have constant determinant \( \pm 1 \) and so the eigenvalues are distinct except in the case that the determinant is one and the trace is \( \pm 2 \). There are only finitely many \( c \) at which the trace of \( \rho_c(g) \) is \( \pm 2 \) since the trace is polynomial in \( c \) and is not constant (by statement (2) of Lemma 5).

Let \( c_1, \ldots, c_k \) denote the finitely many values of \( c \) in \([-1, c_0]\) for which \( \rho_c(g) \) has trace \( \pm 2 \). Observe that if \( P \in SL(2, \mathbb{R}) \) is any matrix with determinant one and trace \( \pm 2 \), then the operator norm \( \| P^{n} \| \) grows asymptotically linearly. We conclude that for each \( i \in \{1, \ldots, k\} \) there is a \( N_i \) so that \( n > N_i \) implies that \( \| \rho_c(g)^{n} \| < \xi^n \). Then by continuity and submultiplicativity of the operator norm (similar to the argument above), we can find an \( r_i > 0 \) and a \( N'_i > N_i \) so that
\[ \| \rho_c(g)^{n} \| < \xi^n \quad \text{when} \quad n > N'_i \quad \text{and} \quad |c - c_i| < r_i. \]  \[ (15) \]

It remains to consider the complement \([-1, c_0] \setminus \bigcup_{i=1}^{k} (c_i - r_i, c_i + r_i) \). Let \( J \) be the closure of this complement which is a finite union of closed intervals. On each \( J \) the matrix \( \rho_c(g) \) is continuously diagonalizable. That is, we have a continuous functions \( \psi^u, \psi^s : J \to \mathbb{C}^2 \) and continuous \( \lambda^u, \lambda^s : J \to \mathbb{C} \) with \( \lambda^u_c \neq \lambda^s_c \) for all \( c \in J \) so that
\[ \rho_c(g) \psi^u_c = \lambda^u_c \psi^u_c \quad \text{and} \quad \rho_c(g) \psi^s_c = \lambda^s_c \psi^s_c \quad \text{for all} \quad c \in J. \]
(Note that at this point \( u \) and \( s \) are just being used to distinguish eigenvectors and do not necessarily correspond to expanding and contracting directions.) Let \( P^u_c \) be the projection matrix so that \( \psi^u \) is an eigenvector with eigenvalue 1 and so that \( \psi^s \) is an eigenvector with eigenvalue zero. Let \( P^s_c \) be the matrix with the \( \psi^u \) and \( \psi^s \) playing opposite roles. Then \( P^u_c \) and \( P^s_c \) vary continuously for \( c \in J \) and we have
\[ \rho_c(g)^n = (\lambda^u_c)^n P^u_c + (\lambda^s_c)^n P^s_c \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad c \in J. \]  \[ (16) \]
Letting $M$ denote the supremum of the operator norms of all matrices of the form $P_c^\mu$ and $P_c^\nu$ with $c \in [-1, c_0]$ we see that
\[ \| \rho_c(g)^n \| \leq M(\| \lambda_c^\mu \| + \| \lambda_c^\nu \|) \leq 2M(\xi')^n, \]
where $\xi'$ is the supremum of $|\lambda_c^\mu|$ and $|\lambda_c^\nu|$ over $c \in J$. Then by definition of $\xi'$ we have $\xi' < \xi$ and thus $2M(\xi')^n$ tends exponentially to zero. In particular, there is a $N_j$ so that $n > N_j$ implies that $\| \rho_c(g)^n \| < \xi^n$ for all $c \in J$.

Setting $N' = \max\{N'_1, \ldots, N'_k, N_J\}$ we see that for any $n > N'$ we have $\| \rho_c(g)^n \| < \xi^n$. From our first observation this proves the proposition. 

3. The Parabola Surface

3.1. Construction. We follow the construction of the parabola surface as an affine limit of Veech’s surfaces built from 2 regular $n$-gons as described in [4]. Consider the regular $n$-gon in $\mathbb{R}^2$ to be the convex hull of the orbit of $(1, 0)$ under the rotation matrix
\[ R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad \text{where} \ t = \frac{2\pi}{n}. \]

Three points on this orbit are given by $R_t^{-1}(1,0) = (\cos t, -\sin t), \ (1,0)$ and $R_t^{-1}(1,0) = (\cos t, \sin t)$.

There is an affine transformation $C_t : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane which carries these three points to $(-1,1), (0,0)$ and $(1,1)$, respectively and a calculation shows
\[ C_t(x, y) = \left( \frac{y}{\sin t}, \frac{x - 1}{\cos t - 1} \right). \]

The image of the regular polygon under $C_t$ is the polygon $Q_c^+$ whose vertices lie in the orbit of $(0,0)$ under $T_c = C_t \circ R_t \circ C_t^{-1}$ which we compute to be the affine map
\[ T_c : \mathbb{R}^2 \to \mathbb{R}^2; \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c & c - 1 \\ c + 1 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{where} \ c = \cos t. \]

Observe that because $T_c$ is an affine transformation defined with coefficients in $\mathbb{Z}[c]$, for any $k \in \mathbb{Z}$, the $k$-th vertex $T_c^k(0,0)$ of $Q_c^+$ has coordinates which are polynomial in $c$.

From the above it can be observed that when $c = \cos \frac{2\pi}{n}$ that $Q_c^+$ is an $n$-gon. When $c = 1$, the $T_c$-orbit is given by $T_c^n(0,0) = (n, n^2)$, and we interpret $Q_c^+$ as a polygonal parabola. Similarly, when $c > 1$, $Q_c^+$ should be interpreted as a polygonal hyperbola.

We let $Q_c^-$ be the image of $Q_c^+$ under rotation by $\pi$ about the origin. To form a surface $P_c$ for some $c = \cos \frac{2\pi}{n}$ or $c \geq 1$, we identify each edge $e$ of $Q_c^+$ by translation with the image of the edge of $Q_c^-$ obtained by applying this rotation. These surfaces are depicted in Figure 2.
3.2. Homological generators. We use $\Sigma$ to denote the collection of two singularities of $P_1$.

**Proposition 8.** The saddle connections in the common boundaries of the parabolas $Q^+_1$ and $Q^-_1$ generate $H_1(P_1, \Sigma; \mathbb{Z})$.

We denote these saddle connections by $\sigma_i$ for $i \in \mathbb{Z}$. See Figure 6.

**Proof.** The surface $P_1$ is triangulated by the saddle connections in the set $\{\sigma_i\}$ together with horizontal and slope one saddle connections. See Figure 6. Observe that by using the relation that the sum of edges around a triangle is zero, we can inductively write each horizontal and slope one saddle connection as a sum of the $\sigma_i$.

For each integer $j \neq 0$ we define $\gamma_j$ to be the closed geodesic which travels within $Q^+_1$ from the midpoint of $\sigma_0$ to the midpoint of $\sigma_j$ and then travel back within $Q^-_1$. See the right side of Figure 6. Let $P^*_1 = P_1 \setminus \Sigma$.

**Proposition 9.** The curves $\gamma_j$ generate $H_1(P^*_1; \mathbb{Z})$.

**Proof.** Consider the polygonal region $R$ in $\mathbb{R}^2$ formed by gluing together $Q^+_1$ and $Q^-_1$ (not including vertices) along the interior of the edge $\sigma_0$. The region $R$ is simply connected. To form $P^*_1$ we glue along the interiors of edges $\sigma_j$ for $j \neq 0$. Thus the fundamental group of $P^*_1$ is generated by curves within $R$ which cross over exactly one of these edges. These are our $\gamma_j$ curves, and this means that they also generate the abelianization of the fundamental group $H_1(P^*_1; \mathbb{Z})$. □
3.3. Deformation and holonomy. The surfaces \( P_c \) for \( c \geq 1 \) are all homeomorphic by homeomorphisms \( P_c \to P_{c'} \) respecting the decompositions \( P_c = Q^+_c \cup Q^-_c \) and \( P_{c'} = Q^+_c \cup Q^-_c \) and sending the orbit \( n \to T^c_n(0,0) \) of vertices of \( Q^+_c \) to the orbit \( n \to T^{c'}_n(0,0) \) of vertices of \( Q^+_c \). This uniquely characterizes the homeomorphism up to isotopy.

The paper [4] noted that the surfaces \( P_c \) for \( c \geq 1 \) are all naturally homeomorphic, and proved that the surfaces have the same geodesics in a coding sense and have affine automorphism groups which act in the same way (up to the natural homeomorphism on the surfaces).

For each \( c \geq 1 \) we have notion of holonomy on \( P_c \). Using the canonical homeomorphism \( P_1 \to P_c \) we can evaluate this holonomy on classes in \( P_1 \) giving us a family of holonomy maps

\[
\text{hol}_c : H_1(P_1, \Sigma; \mathbb{Z}) \to \mathbb{R}^2 \quad \text{defined for } c \geq 1.
\]

Observe that \( T_c \) has determinant one and entries which are polynomial in \( c \). Thus all vertices of \( Q^+_c \) have coordinates which are integer polynomials in \( c \). It follows that for any \( \sigma \in H_1(P_1, \Sigma; \mathbb{Z}) \) the map \( c \mapsto \text{hol}_c \sigma \) lies in \( \mathbb{Z}[c]^2 \). We define the deformation holonomy map of the family of surfaces \( P_c \) to be the map

\[
\widetilde{\text{hol}} : H_1(P_1, \Sigma; \mathbb{Z}) \to \mathbb{Z}[c]^2
\]

so that for \( c \geq 1 \) we have \( \widetilde{\text{hol}}(\sigma)(c) = \text{hol}_c(\sigma) \).

By tensoring with \( \mathbb{R} \) we extend \( \widetilde{\text{hol}} \) to

\[
(20) \quad \widetilde{\text{hol}} : H_1(P_1, \Sigma; \mathbb{R}) \to \mathbb{R}[c]^2.
\]

We have the following:

**Proposition 10.** The map \( \widetilde{\text{hol}} \) of (20) is an isomorphism of \( \mathbb{R} \)-modules.

*Proof.* For \( d \geq 0 \) let \( P_d \) denote the collection of polynomials of degree at most \( d \) with coefficients in \( \mathbb{R} \). Because we have normalized three vertices joined by \( \sigma_{-1} \) and \( \sigma_0 \) in our construction of \( Q^+_c \) we can see that

\[
(21) \quad \widetilde{\text{hol}}\sigma_0 = (1,1) \quad \text{and} \quad \widetilde{\text{hol}}\sigma_{-1} = (1,-1).
\]

Both these vectors lie in \( P^2_0 \). Recall that \( T_c \) carries each vertex of \( Q^+_c \) to the subsequent vertex. From the definition of \( T_c \) in (19) we see that for all \( j \in \mathbb{Z} \),

\[
\widetilde{\text{hol}}\sigma_{j+1} = \begin{pmatrix} c & c-1 \\ c+1 & c \end{pmatrix} \text{hol}_c\sigma_j \quad \text{and} \quad \widetilde{\text{hol}}\sigma_{j-1} = \begin{pmatrix} c & -c+1 \\ -c-1 & c \end{pmatrix} \text{hol}_c\sigma_j.
\]

Just considering the \( cs \) appearing in these matrices we can prove inductively that for \( k > 0 \),

\[
(22) \quad \widetilde{\text{hol}}\sigma_k \in ((2c)^k, (2c)^k) + P^2_{k-1} \quad \text{and} \quad \widetilde{\text{hol}}\sigma_{-k} \in ((2c)^k, -(2c)^k) + P^2_{k-1}.
\]

To prove the proposition, it suffices to show that for every \( n \geq 0 \), the restriction of \( \widetilde{\text{hol}} \) to \( \text{span}_\mathbb{R} \{\sigma_{-n-1}, \sigma_{-n}, \ldots, \sigma_{n-1}, \sigma_n\} \) is an isomorphism to \( P^2_n \). Noting that there are \( 2n+2 \) vectors in the list which matches the dimension of the space \( P^2_n \), we see it suffices to prove that

\[
\widetilde{\text{hol}}(\text{span}_\mathbb{R} \{\sigma_{-n-1}, \sigma_{-n}, \ldots, \sigma_{n-1}, \sigma_n\}) \to P^2_n.
\]
We prove this by induction. From (21) we see that the statement is true when \( n = 0 \). Now assuming the statement holds for \( n = k - 1 \), i.e.,

\[
\overline{\text{hol}}(\text{span}_\mathbb{R}\{\sigma_{-k-2}, \ldots, \sigma_{k-1}\}) \supset P_k^2,
\]

we see from (22) that by adding \( \sigma_k \) and \( \sigma_{k-1} \) to the list, the image of the new span contains \( P_k^2 \).

Now consider the related map

\[
\overline{\text{hol}}: H_1(P_1; \mathbb{R}) \to \mathbb{R}[c]^2.
\]

To understand it, consider the exact sequence of groups

\[
0 \to H_1(P_1; \mathbb{R}) \xrightarrow{i_*} H_1(P_1, \Sigma; \mathbb{R}) \xrightarrow{\delta_*} H_0(\Sigma; \mathbb{R}) \to H_0(P_1; \mathbb{R}) \to H_0(P_1, \Sigma; \mathbb{R}) \to 0.
\]

This is the standard relative homology long exact sequence in our setting, where we have used the observation that \( H_n(P_1; \mathbb{R}) \cong H_n(P_1^o; \mathbb{R}) \) since the two singularities are isolated and of infinite cone type guaranteeing that \( P_1 \) is homotopy equivalent to \( P_1^o \). The holonomy map factors through the inclusion \( i_* \), i.e., \( \overline{\text{hol}} \circ i_* = \overline{\text{hol}} \) and the above exact sequence says that \( i_* \) is an inclusion so we see that the holonomy map (23) is an isomorphism to a subspace of \( \mathbb{R}[c]^2 \). To figure out what that subspace is observe that \( i_*(H_1(P_1; \mathbb{R})) = \ker \partial_* \). We can enumerate the two singularities as \( s_0 \) representing the identified vertices \( \{(n, n^2)\} \) of the polygonal parabola with even coordinates and \( s_1 \) representing the identified vertices with odd coordinates. The image of \( \delta_* \) is the collection elements of \( H_0(\Sigma; \mathbb{R}) \) of the form \( x[s_1] - x[s_0] \) for some \( x \in \mathbb{R} \). We can recover this value of \( x \) as the image of \( H_1(P_1, \Sigma; \mathbb{R}) \) using the map

\[
e: H_1(P_1, \Sigma; \mathbb{R}) \to \mathbb{R}; \quad \sigma \mapsto \frac{1}{2}(\partial \sigma(s_1) - \partial \sigma(s_0))
\]

which allows us to define the short exact sequence

\[
0 \to H_1(P_1^o; \mathbb{R}) \xrightarrow{i_*} H_1(P_1, \Sigma; \mathbb{R}) \xrightarrow{e} \mathbb{R} \to 0.
\]

**Proposition 11.** For \( \sigma \in \text{hol}(P_1, \Sigma; \mathbb{R}) \) and \((x, y) = \overline{\text{hol}}(\sigma)\) the quantity \( y(-1) \) gives the value of \( e(\sigma) \). Thus we have the isomorphism

\[
\overline{\text{hol}}: H_1(P_1^o; \mathbb{R}) \to \{(x, y) \in \mathbb{R}[c]^2: \ y(-1) = 0\}.
\]

**Proof.** Since the map sending \( \sigma \) to \( y(-1) \) is linear, it suffices to check that \( e(\sigma) \) agrees with \( y(-1) \) on our basis \( \sigma_j \) for \( \text{hol}(P_1, \Sigma; \mathbb{R}) \). Observe that \( e(\sigma_j) = (-1)^j \).

Using formulas from the proof of Proposition 10 we can compute that

\[
\text{hol}_{-1}(\sigma_j) = (-1)^j(2j + 1, 1),
\]

and note that the \( y \)-coordinate matches \( e(\sigma_j) \). \( \square \)

We will now prove the integral formula mentioned in the introduction.
Lemma 12 (Intersection as integration). The algebraic intersection number of any \( \gamma \in H_1(P^1; \mathbb{R}) \) and any \( \sigma \in H_1(P_1, \Sigma; \mathbb{R}) \) is given by

\[
\frac{1}{2\pi} \int_0^\pi \left( (\overline{\text{hol}} \gamma) \wedge (\overline{\text{hol}} \sigma) \right) (1 - \cos t) \, dt,
\]

where the wedge product in the integral yields an element of \( \mathbb{Z}[c] \) interpreted as a function of \( t \) by setting \( c = \cos t \).

Proof. Observe that the integral expression is bilinear in \( \gamma \) and \( \sigma \). Algebraic intersection number is bilinear as well, so it suffices to check the formula on our generating sets for \( H_1(P^1; \mathbb{Z}) \) and \( H_1(P_1, \Sigma; \mathbb{Z}) \). Observe that \( \gamma_j \) (oriented to move from \( \sigma_1 \) to \( \sigma_j \) in \( Q^+_1 \)) intersects \( \sigma_j \) with positive sign and \( \sigma_0 \) with negative sign. Thus \( \gamma_j \cap \sigma_k = \delta_{j,k} - \delta_{0,k} \) where \( \delta_{a,b} \) equals 1 if \( a = b \) is zero otherwise. We will show that the integral evaluates to the same expression.

First we need to compute \( \overline{\text{hol}} \gamma_j \) and \( \overline{\text{hol}} \sigma_k \). Let \( v_n = T^R_c(0,0) \in \mathbb{Z}[c]^2 \) be the \( n \)-th vertex of \( Q^+_c \) viewed as a polynomial in \( c \). Observe

\[
\overline{\text{hol}} \gamma_j = v_{j+1} + v_j - v_1 - v_0 \quad \text{and} \quad \overline{\text{hol}} \sigma_k = v_{k+1} - v_k.
\]

Using the fact that \( T_c = C_t \circ R_t \circ C_t^{-1} \) and the definition of \( C_t \) in (18), we see \( v_n = C_t(\cos nt, \sin nt) \). Therefore

\[
(24) \quad \overline{\text{hol}} \gamma_j = C_t \left( \cos((j+1)t) + \cos(jt) - \cos(t) - 1, \sin((j+1)t) + \sin(jt) - \sin(t) \right),
\]

\[
\overline{\text{hol}} \sigma_k = C_t \left( \cos((k+1)t) - \cos(kt), \sin((k+1)t) - \sin(kt) \right).
\]

The transformation \( C_t \) is affine and scales signed area by a multiplicative constant. Namely,

\[
C_t(\mathbf{v}) \wedge C_t(\mathbf{w}) = \frac{1}{(1 - \cos t) \sin t} (\mathbf{v} \wedge \mathbf{w}) \quad \text{for any} \, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2.
\]

Letting \( \mathbf{v} = C_t^{-1}(\overline{\text{hol}} \gamma_j) \) and \( \mathbf{w} = C_t^{-1}(\overline{\text{hol}} \sigma_k) \) be the quantities in parenthesis enclosed in (24), we see that the quantity being integrated is

\[
\frac{\mathbf{v} \wedge \mathbf{w}}{\sin t} = 2 \cos((k - j)t) - 2 \cos(k t),
\]

where we have done significant simplifying using trigonometric identities. It follows that

\[
\frac{1}{2\pi} \int_0^\pi \frac{\mathbf{v} \wedge \mathbf{w}}{\sin t} \, dt = \delta_{j,k} - \delta_{0,k},
\]

as desired.

For our later discussion of geometric intersection numbers, it will be useful for us to extend our notion of algebraic intersection number to a bilinear map

\[
\cap : H_1(P_1, \Sigma; \mathbb{R}) \otimes H_1(P_1, \Sigma; \mathbb{R}); \sigma_1 \cap \sigma_2 = \frac{1}{2\pi} \int_0^\pi \left( (\overline{\text{hol}} \sigma_1) \wedge (\overline{\text{hol}} \sigma_2) \right) (1 - \cos t) \, dt.
\]

This extension has geometric meaning for saddle connections.
Lemma 13. Let $\sigma_1$ and $\sigma_2$ be saddle connections and let $i(\sigma_1, \sigma_2)$ denote the number of (unsigned) intersections of $\sigma_1$ and $\sigma_2$ not counting those that occur at the singularities. Then

$$|i(\sigma_1, \sigma_2) - |\sigma_1 \cap \sigma_2|| \leq 1.$$ 

Proof. There two cases to consider. First suppose one of the curves joins a singularity to itself, say $\sigma_1$ has two endpoints at the singularity $s_*$. Then $\sigma_1$ is really a loop and we can apply a homotopy only deforming $\sigma_1$ in a small neighborhood of the $s_*$ which makes $\sigma_1$ into a new closed loop $\hat{\sigma}_1$. We will detail a way to obtain $\hat{\sigma}_1$ so that it is fairly easy to see what is happening. For $r \in (0, \sqrt{2})$ observe that the ball of radius $r$ about $s_*$ contains no complete saddle connections. By choosing $r \in (0, \sqrt{2})$ sufficiently small we can arrange that the ball $B_r(s_*)$ only intersects $\sigma_1$ and $\sigma_2$ in segments of length $r$ at the start and end of the saddle connections. (The ball will only intersect $\sigma_2$ if it starts or ends at the same singularity.) Chop off the segments of $\sigma_1$ that are within the ball. Since this singularity is an infinite cone singularity, the boundary $\partial B_r(s_*)$ is homeomorphic to the real line, so we can join the points of where $\sigma_1$ hits the ball by a unique arc $\partial B_r(s_*)$. Call the resulting loop $\hat{\sigma}_1$. The added arc may cross $\sigma_2$ if the saddle connection $\sigma_2$ starts or ends at $s_*$. If $\sigma_2$ has one endpoint at $s_*$, we have introduced at most one new crossing so that the result holds in this case. If $\sigma_2$ both starts and ends at $s_*$, then the added arc may cross twice, but if it does then the signs ascribed to the intersections are opposite. Again we have shown that $|\sigma_1 \cap \sigma_2|$ is within one of $i(\sigma_1, \sigma_2)$.

If we can not arrange to be in the first case, then both $\sigma_1$ and $\sigma_2$ join distinct singularities. Orientation is irrelevant for the statement we are trying to prove, so we can assume $\sigma_1$ moves from singularity $s_0$ to singularity $s_1$ and $\sigma_2$ moves from $s_1$ to $s_0$. Let $\gamma_1$ be the curve formed by concatenating $\sigma_1$ and $\sigma_2$. Because our extended definition of $\cap$ is bilinear and alternating, we have

$$\sigma_1 \cap \sigma_2 = \gamma_1 \cap \sigma_2.$$ 

We can again make $\gamma_1$ into a closed curve $\hat{\gamma}_1$ following the method of the previous paragraph. This time we choose $r$ small enough so that the two balls $B_r(s_0)$ and $B_r(s_1)$ do not intersect and so that the balls only intersect the saddle connections in initial and terminal segments. So $\hat{\gamma}_1$ follows $\sigma_1$ outside of the two balls then wraps around the boundary of $B_r(s_1)$, then follows a $\sigma_2$ until it hits $B_r(s_1)$ and closes up following the boundary of $B_r(s_0)$. Actually, it is better not to follow $\sigma_2$ exactly, instead we follow a parallel arc which stays on one side of $\sigma_2$. Again the arcs in the boundary of the balls may have introduced one or two new crossings, but if we introduce two then they occur with opposite signs. Again we have shown that $|\sigma_1 \cap \sigma_2|$ is within one of $i(\sigma_1, \sigma_2)$. \qed

3.4. Affine automorphisms. The affine automorphism group was investigated carefully in [4].

Theorem 14 (Theorem 3, [4]). For $c \geq 1$, the group $D(Aff(P_c))$ is generated by $A_c$, $B_c$, $C_c$ and $-I$ as defined in (5).
Since \( P_c \) never has translation automorphisms, we have the following:

**Proposition 15** (Proposition 5 [4]). For \( c \geq 1 \), the homomorphism \( D : \text{Aff}(P_c) \to \text{GL}(2, \mathbb{R}) \) is one-to-one.

As a consequence we see that for each \( c \geq 1 \), there is a canonical isomorphism
\[
\Phi_c : \mathcal{G} \to \text{Aff}(P_c)
\]
so that \( D \circ \Phi_c(g) = \rho_c(g) \) for all \( g \in \mathcal{G} \),

where \( \mathcal{G} \) and \( \rho_c \) are defined as in (6) and (8) respectively. The actions are essentially the same in a topological sense:

**Theorem 16** (Theorem 7, [4]). For \( c, c' \geq 1 \) and any \( g \in \mathcal{G} \), the automorphisms \( \Phi_c(g) : P_c \to P_c \) and \( \Phi_{c'}(g) : P_{c'} \to P_{c'} \) are the same up to conjugation by the canonical homomorphism \( P_c \to P_{c'} \) and isotopy.

This has the following consequence for our deformation holonomy map.

**Proposition 17.** Let \( \phi = \Phi_1(g) \in \text{Aff}(P_1) \). Then the induced actions of \( \phi \) on \( H_1(P_1; \mathbb{Z}) \) and \( H_1(P_1, \Sigma; \mathbb{Z}) \) satisfy
\[
\overline{\text{hol}} \circ \phi(\gamma) = \rho(g) \overline{\text{hol}}(\gamma),
\]
where \( \rho : \mathcal{G} \to SL^\pm(2, \mathbb{Z}[c]) \) was defined in (7).

**Proof.** Fixing a \( c \geq 1 \), we see that \( \overline{\text{hol}}_c \circ \Phi_c(g) = \rho_c(g) \cdot \overline{\text{hol}}_c(\gamma) \) since \( D \circ \Phi_c(g) = \rho_c(g) \). This expression represents the equation in the proposition evaluated at a specific \( c \geq 1 \), but we have verified it for uncountably many values (all \( c \geq 1 \)). Fixing any \( \gamma \), the expression claims equality of two elements of \( \mathbb{Z}[c]^2 \). The entries are polynomial and we have verified the equation on infinitely many values, so the equation holds for all \( c \).

\( \square \)

### 3.5. Asymptotics of algebraic intersections.

In this subsection we state our main result involving asymptotic algebraic intersection numbers of homology classes. The main ideas involve using \( \overline{\text{hol}} \) to convert the homology classes to elements of \( \mathbb{R}[c]^2 \), use Proposition 17 to convert an affine automorphism’s action to the action of an element \( \rho(g) \in SL^\pm(2, \mathbb{Z}[c]) \), and to use our integral formula to evaluate the intersection numbers.

It is useful to notice that since \( \rho_1(g) \) is hyperbolic, the matrices \( \rho_c(g) \) are diagonalizable in a neighborhood of \( c = 1 \). We can express this diagonalization in the form
\[
\rho_c(g) = \lambda_e^u P_c^u + \lambda_e^s P_c^s,
\]
where \( \lambda_e^u \) and \( \lambda_e^s \) are analytic real valued functions of \( c \) corresponding to the expanding and contracting eigenvalues of \( \rho_c(g) \) and \( P_c^u \) and \( P_c^s \) denote the projection matrices (matrices with one eigenvalue equal one and one equal zero) with the same eigenvectors as \( \rho_c(g) \). Again we just interpret these quantities as defined and analytic in a neighborhood of \( c = 1 \).

Let \( v, w \in \mathbb{R}[c]^2 \). We define
\[
v \wedge w = (P_c^u v) \wedge w,
\]
which is a real valued analytic function defined in a neighborhood of $c = 1$. This quantity relates to constants in our asymptotics and it will be important to know that this function can not be identically zero.

**Lemma 18.** Let $\rho_1(g)$ be hyperbolic and define $P_u^c$ and $\tilde{\Sigma}$ as above. Then, so long as $v, w \in \mathbb{R}[c]^2$ are both non-zero, the real analytic function $v \tilde{\wedge} w$ is not identically zero.

**Proof.** First observe that $P_u^c + P_c^c = I$ so that

$$v \tilde{\wedge} w = (P_u^c v) \wedge (P_c^c w + P_c^c w) = (P_u^c v) \wedge (P_c^c w).$$

Since the functions are analytic and the eigendirections are transverse, the only way $v \tilde{\wedge} w$ could be identically zero is if either $P_u^c v$ was identically zero or $P_c^c w$ was identically zero.

We will approach this by contradiction and without loss of generality we may assume $P_u^c v \equiv 0$. This says that $v(c)$ lies in the stable eigenspace of $\rho_c(g)$ for all $c$ sufficiently close to one. That is,

$$\rho_c(g)v = \lambda^c v.$$ 

The entries of both $\rho_c(g)v$ and $v$ lie in $\mathbb{R}[c]$. Thus it follows that $\lambda^c$ is a rational function, i.e., there are polynomials $p, q \in \mathbb{R}[c]$ so that $\lambda^c = \frac{p}{q}$ in a neighborhood of $c = 1$ wherever $q(c) \neq 0$ (which holds an open set since $v$ has non-zero polynomial entries). Furthermore, we can assume that $\frac{p}{q}$ is reduced in the sense that $p$ and $q$ share no roots in $\mathbb{C}$. Since $\det \rho_c(g) \equiv 1$, we know that $\lambda^c = \pm \frac{q}{p}$. The sum of the eigenvalues is the trace of $\rho_c(g)$ which we will denote by $t \in \mathbb{R}[c]$. Thus we have the identity

$$(28) \quad p^2 + q^2 = tpq.$$ 

By Lemma 5, we know $\frac{d}{dc} \lambda^c_{c=1} \neq 0$, so that at least one of $p$ and $q$ is non-constant. This means that one of those, say $p$, two has a root $z \in \mathbb{C}$. Then $z$ is also a root of $tpq$. Thus it follows from (28) that $z$ must be a root of $q^2$ and therefore also of $q$, but this contradicts our assumption that $p$ and $q$ do not share a common root. \qed

The following is our main technical result. It describes the asymptotics of homology classes under a hyperbolic affine automorphism.

**Theorem 19.** Suppose $\phi : P_1 \to P_1$ is hyperbolic affine automorphism with derivative $D\phi = \rho_1(g)$. Let $\gamma \in H_1(P_1^c, \mathbb{R})$ and $\sigma \in H_1(P_1, \Sigma; \mathbb{R})$ be non-zero classes. Define quantities as above in (26) and (27). Let $k \geq 0$ be an integer and $\kappa \in \mathbb{R}$ be nonzero so that the Taylor expansion of $\text{hol}\tilde{\gamma}\text{hol}\sigma$ about $c = 1$ is of the form

$$\text{hol}\tilde{\gamma}\text{hol}\sigma = \kappa(c-1)^k + O((c-1)^{k+1}).$$

(Note that this quantity is not identically zero by Lemma 18.) Then the sequence of algebraic intersection numbers $\phi^c_\gamma(\gamma) \cap \sigma$ is asymptotic to a constant times
\[ n^{-k-\frac{3}{2}}(\lambda_1^n)^n, \text{ and, in fact,} \]
\[ \lim_{n \to \infty} \frac{n^{k+\frac{3}{2}}}{(\lambda_1^n)^n} (\phi_n^p(\gamma) \cap \sigma) = \frac{(-1)^k \Gamma(k + \frac{3}{2}) \Lambda \sqrt{2}}{4 \pi \beta^{k+\frac{3}{2}}}, \]

where \( \Gamma \) denotes the gamma function and \( \beta = \frac{1}{\lambda} \left| \frac{d}{dc} \lambda_c \right|_{c=1} \) as in Theorem 1 of the introduction. This result also holds for \( \gamma, \sigma \in H_1(\mathbf{P}_1, \Sigma; \mathbb{R}) \) with algebraic intersection numbers computed as in (25).

**Proof.** Fix \( \gamma \) and \( \sigma \). We will compute intersections using the integral in Lemma 5 with this being the definition in the case \( \gamma, \sigma \in H_1(\mathbf{P}_1, \Sigma; \mathbb{R}) \). Let \( c = \cos t \) throughout this proof. Let \( v_c = \bar{h} \alpha \gamma \) and \( w_c = \bar{h} \alpha \sigma \) which are both elements of \( \mathbb{R}[c]^2 \).

Determine \( k \) and \( \Lambda \) as stated in the theorem. By our integral formula for intersection numbers and Proposition 17,

\[ \phi_n^p(\gamma) \cap \sigma = \frac{1}{2 \pi} \int_0^\pi \left( (\rho_c(g)^n v_c) \wedge w_c \right) (1 - \cos t) \, dt. \]

Here, the quantity \( (\rho_c(g)^n v_c) \wedge w_c \) lies in \( \mathbb{R}[c] \) and we integrate with respect to \( t \) while taking \( c = \cos t \).

We will be demonstrating that

\[ \frac{n^{k+\frac{3}{2}}}{2 \pi (\lambda_1^n)^n} \int_0^\pi \left( (\rho_c(g)^n v_c) \wedge w_c \right) (1 - \cos t) \, dt \]

is asymptotic to the quantity on the right side of the equation in the theorem.

Let \( \lambda_c^n \) be an eigenvalue for \( \rho_c(g) \) realizing the maximum absolute value of an eigenvalue. From Lemma 5 we know that \( \frac{d}{dc} |\lambda_c^n| \big|_{c=1} > 0 \) and thus we can find an interval \( [0, \epsilon] \) on which \( t \to |\lambda_c^n| \) is decreasing and takes values larger than one. We can split the integral of (29) into two pieces at \( \epsilon \). The contribution of the interval \( [\epsilon, \pi] \) to (29) can be written as

\[ \frac{n^{k+\frac{3}{2}}}{2 \pi (\lambda_1^n)^n} \int_0^\pi \left( (\xi^n \rho_c(g)^n v_c) \wedge w_c \right) (1 - \cos t) \, dt, \]

where \( \xi \) is some quantity so that \( 1 < \xi < |\lambda_1^n| \) so that \( \xi^{-n} \rho_c(g)^n \) tends to zero uniformly for \( t \in [\epsilon, \pi] \) as obtainable from Proposition 6. Observe that the fraction in front of (30) decays to zero because of the exponential growth of \( \left( \frac{|\lambda_1^n|}{\epsilon} \right)^n \).

The integral in (30) also tends to zero because \( v_c, w_c, 1 - \cos t \) are continuous in \( t \) and therefore bounded in absolute value by a constant. On the other hand, from our use of Proposition 6, the quantity \( \xi^{-n} \rho_c(g)^n \) decays to zero uniformly.

Now consider the interval \( [0, \epsilon] \). As in (26), on this interval we can write

\[ \rho_c(g)^n = (\lambda_c^n)^n p_c^u + (\lambda_c^n)^n p_c^s \]

for \( n \geq 0 \). Thus we can split the contribution of \( [0, \epsilon] \) to (29) into unstable and stable parts. The stable part has the form

\[ \frac{n^{k+\frac{3}{2}}}{2 \pi (\lambda_1^n)^n} \int_0^\epsilon \left( (\lambda_c^n)^n p_c^s v_c \wedge w_c \right) (1 - \cos t) \, dt, \]
which tends to zero exponentially since $|\lambda^u_c| = |\lambda^u_\infty|^{-1} < 1$ so that the integral is uniformly bounded while the fraction in front decays exponentially.

The unstable part is more interesting and using the fact that $\lambda^u_c$ has constant sign for $t \in [0,\epsilon]$ we can write it as

$$
\int_0^\epsilon e^{-np(t)} q(t) \, dt,
$$

where $\tilde{\alpha}$ is used as in (27). We will use a theorem of Erdélyi following [6, Ch. 3 §8] to study the asymptotics of the sequence of integrals (temporarily ignoring the fraction in front). The integral can be written in the form

$$
\int_0^\epsilon e^{-np(t)} q(t) \, dt,
$$

where $p(t) = -\ln|\lambda^u_c|$ and $q(t) = (v_c, \tilde{\alpha}w_c)(1 - \cos t)$.

To apply Erdélyi’s theorem we need to know the first few terms of a series expansion for $p(t)$ and $q(t)$ about $t = 0$. By the chain rule, in a neighborhood of $t = 0$ we have

$$
p'(t) = -\frac{1}{|\lambda^u_c|} \left( \frac{d}{dc}|\lambda^u_c| \right)(\sin t).
$$

Thus we have $p(0) = -\ln|\lambda^u_\infty|$, $p'(0) = 0$, and $p''(0) = \frac{1}{|\lambda^u_\infty|} \cdot [\frac{d}{dc}|\lambda^u_c|]_{c=1} = \beta$, which is positive by statement (2) of Lemma 5. In addition, we know that $t = 0$ is the location of the minimum of $p(t)$ on $[0,\epsilon]$. Recalling that a Taylor series expansion for $v_c, \tilde{\alpha}w_c = \tilde{\alpha}c \gamma \gamma c \sigma$ in terms of $c$ was given as a hypothesis of the theorem, we have

$$
q(t) = \frac{(-1)^k \kappa}{2^{k+1}} t^{2k+2} + O(t^{2k+4}).
$$

By [6, Theorem 8.1], the sequence of values of integrals has the asymptotic form

$$
e^{-np(0)} \left[ \Gamma \left( k + \frac{3}{2} \right) \cdot n^{-k-\frac{3}{2}} \cdot \frac{(-1)^k \kappa}{2^{k+1}} \cdot \frac{1}{2} \left( \frac{p''(0)}{2} \right)^{-k-\frac{3}{2}} + O(n^{-k-2}) \right].
$$

Plugging these values in for the integral in (31), we obtain the limit stated in the theorem.

**Proof of Theorem 4.** It suffices to prove the second statement, since the statement on convergence of projective classes to $P(\mu^u)$ and $P(\mu^\gamma)$ follows from statement (3) in the theorem.

Fix $\phi$ and notation as above. Given a non-zero $\gamma \in H_1(S, \Sigma; \mathbb{R})$ observe that $P_c^u \tilde{\alpha}c \gamma$ has coordinates which are real analytic functions of $c$. As a consequence of Lemma 18, $P_c^u \tilde{\alpha}c \gamma$ is not identically zero. Then by analyticity, the following constant is defined

$$
k = \min \left\{ k \in \mathbb{Z} : \ k \geq 0 \ \text{and} \ \left[ \frac{d}{dc}^k P_c^u \tilde{\alpha}c \gamma \right]_{c=1} \neq 0 \right\}.
$$

The value of $k$ can be interpreted as a function $k : H_1(S, \Sigma; \mathbb{R}) \to \mathbb{Z} \cup \{+\infty\}$ where we assign $k(\gamma)$ as in (32) except if $\gamma = 0$ in which case we assign $+\infty$. Then we can define the subsets in the theorem to be $S_j = \{ \gamma : k(\gamma) \geq j \}$. Fixing
we observe that the map $D_k : \gamma \mapsto \frac{d^k}{dc^k} P_k^u \widehat{\text{holy}}$ is linear and consequently the $S_j$ are subspaces and each $S_{j+1}$ is codimension at most one in $S_j$. Recalling Proposition 11, we see that we can construct pairs of polynomials $(p, q)$ in the image of $\widehat{\text{hol}}(H_1(S, \Sigma; \mathbb{R}))$ so that the minimal $k \geq 0$ such that $\frac{d^k}{dc^k} P_k^u (p, q)|_{c=1} \neq 0$ is arbitrary: Such $(p(c), q(c))$ need to approximate the stable direction of $\rho_c(g)$ at $c = 1$ to order $k - 1$ but not order $k$. It follows that $S_{j+1}$ is always codimension one in $S_j$. Finally because of the previous paragraph we know that if $\gamma \neq 0$ then $k(\gamma) \neq 0$ so that $\cap_{j \geq 0} S_j = \{0\}$. We have proved statements (1) and (2) of the theorem.

To see (3) fix a non-zero $\gamma$ and let $k$ be as in (32). We need to show that there is a constant $L$ so that for any $\sigma \in H_1(S, \Sigma; \mathbb{R}),$

$$\lim_{n \to +\infty} \frac{n^{k+\frac{3}{2}}}{(\lambda_1^n)^{\frac{3}{2}}} (\phi^n(\gamma) \cap \sigma) = L \mu^u(\sigma),$$

and by definition $\mu^u(\sigma) = u^u \wedge \text{hol}(\sigma)$, where $u^u$ denotes a choice of a unit unstable eigenvector of $\rho_1(g)$. It suffices to prove this for elements of the form $\sigma_1$ which generate $H_1(S, \Sigma; \mathbb{R})$ by Proposition 9. Theorem 19 tells us that

$$(33) \lim_{n \to +\infty} \frac{n^{k+\frac{3}{2}}}{(\lambda_1^n)^{\frac{3}{2}}} (\phi^n(\gamma) \cap \sigma) = \frac{(-1)^k \Gamma(k + \frac{3}{2}) \kappa \sqrt{2}}{4\pi \beta^{k+\frac{3}{2}}}$$

with $\kappa$ chosen so that

$$\widehat{\text{holy}} \wedge \text{holo} \sigma = \kappa (c-1)^k + O((c-1)^{k+1}).$$

Let $u_1^u$ denote the continuous choice of unstable unit eigenvectors of $\rho_c(g)$ defined in a neighborhood of $c = 1$, which extends the choice of $u_1^u$ used in the definition of $\mu^u$. Then we can define the real-analytic function $f(c)$ so that

$$\widehat{\text{holy}} \wedge \text{holo} \sigma_1 = f(c) (u_1^u \wedge \text{holo} \sigma_1).$$

Here $f(c) = \pm \|P_k^u \widehat{\text{holy}}\|$ and may be changing signs as $c$ passes through 1. To compute $\kappa$ we just need to look at the lowest order terms in this expression computed at the point $c = 1$. Observe that $\text{holo} \sigma_1 \in \mathbb{Z}^2$ while $u_1^u$ has quadratic irrational slope so we know that $u_1^u \wedge \text{holo} \sigma_1$ is a non-zero constant. On the other hand using our definition of $k$ we know that $\widehat{\text{holy}} \wedge \text{holo} \sigma_1$ vanishes to order $k-1$ and so we must have $f = \alpha (c-1)^k + O((c-1)^{k+1})$ and with this definition of $\alpha$ we have $\kappa = \alpha (u_1^u \wedge \text{holo} \sigma_1) = \alpha \mu^u(\sigma_1)$ so we see by plugging in to (33) that

$$\lim_{n \to +\infty} \frac{n^{k+\frac{3}{2}}}{(\lambda_1^n)^{\frac{3}{2}}} (\phi^n(\gamma) \cap \sigma_1) = \frac{(-1)^k \Gamma(k + \frac{3}{2}) \alpha \sqrt{2}}{4\pi \beta^{k+\frac{3}{2}}} \mu^u(\sigma_1).$$

The constant in front of $\mu^u(\sigma_1)$ is now independent of $l$ so we see that

$$(34) \lim_{n \to +\infty} \frac{n^{k+\frac{3}{2}}}{(\lambda_1^n)^{\frac{3}{2}}} \cap_* \phi^n(\gamma) = \frac{(-1)^k \Gamma(k + \frac{3}{2}) \alpha \sqrt{2}}{4\pi \beta^{k+\frac{3}{2}}} \mu^u.$$
3.6. **Asymptotics of cylinder intersections.** The goal of this section is to prove Theorem 1 and Corollary 2 of the introduction.

We first need to make a general remark about cylinders intersection on a translation surface. Let \( C \) be a cylinder on a translation surface \( S \). A **core curve** \( \gamma_C \) of \( C \) is a closed geodesic in \( C \). Such closed geodesic must wind once around the circumference of \( C \).

**Proposition 20.** Let \( A \) and \( B \) be cylinders on a translation surface \( S \) with core curves \( \gamma_A \) and \( \gamma_B \). Assuming \( A \) and \( B \) are not parallel (i.e., \( \text{hol}(\gamma_A) \wedge \text{hol}(\gamma_B) \neq 0 \)), we have

\[
\text{Area}(A \cap B) = \frac{|\gamma_A \cap \gamma_B| \cdot \text{Area}(A) \cdot \text{Area}(B)}{|\text{hol}(\gamma_A) \wedge \text{hol}(\gamma_B)|},
\]

where \( \gamma_A \cap \gamma_B \) denotes algebraic intersection number.

**Proof.** Cylinders on a translation surface intersect in a union of parallelograms, which are isometric and differ only by parallel translation. The number of these parallelograms is the absolute value of the algebraic intersection number between the core curves. Thus, we need to show that the area of one such parallelogram is given by

\[
\text{Area}(A \cap B) = \frac{|\gamma_A \cap \gamma_B| \cdot \text{Area}(A) \cdot \text{Area}(B)}{|\text{hol}(\gamma_A) \wedge \text{hol}(\gamma_B)|},
\]

Develop the two cylinders into the plane from an intersection as in Figure 7. Define the vectors \( a, b, c, \text{ and } d \) as in the figure. Then the area of the two cylinders is given by the quantities

\[
\text{Area}(A) = |a \wedge (b + d)| \quad \text{and} \quad \text{Area}(B) = |(a + c) \wedge b|.
\]

Since the pair of vectors \( a \) and \( c \) are parallel as are the pair \( b \) and \( d \), we may write the product of areas as

\[
\text{Area}(A) \cdot \text{Area}(B) = |a \wedge b||a + c) \wedge (b + d)|.
\]
The wedge of the holonomies may be written as

\[ \text{hol}(\gamma_A) \land \text{hol}(\gamma_B) = |(a + c) \land (b + d)|. \]

Thus, the quotient given in (35) is \(|a \land b|\), the area of the parallelogram formed by the intersection.

**Proof of Theorem 1.** Let \(A\) and \(B\) be cylinders on \(P_1\) and let \(\gamma_A\) and \(\gamma_B\) be their core curves and let \(\nu, \omega \in \mathbb{Z}^2\) be their holonomies. Let \(\phi\) be a hyperbolic automorphism of \(P_1\). Then \(D\phi = \rho_1(g)\) is a hyperbolic element of \(SL^2(2, \mathbb{Z})\). As a consequence the eigenvalues of \(D\phi\) are quadratic irrationals and so \((P_1^n\nu) \land \omega\) is non-zero. In the context of Theorem 19 this implies that \(k = 0\) and \(\kappa = (P_1^n\nu) \land \omega\) and so the theorem gives that

\[ \phi^n(\gamma_A) \land \gamma_B \sim (\lambda_1^n)^n n^{\frac{3}{2}} \frac{\Gamma(\frac{3}{2}) \kappa}{\pi \beta^2 \sqrt{2}} = (\lambda_1^n)^n n^{\frac{3}{2}} \frac{\kappa}{4 \beta^2 \sqrt{2\pi}}. \]

For sufficiently large values of \(n\), the cylinders \(\phi^n(A)\) and \(B\) are not parallel, so we may apply Proposition 20 to obtain that

\[ \text{Area}(\phi^n(A) \cap B) \approx \frac{|\phi^n(\gamma_A) \land \gamma_B| \text{Area}(A) \text{Area}(B)}{|\rho_1(g)^n \nu \land \omega|} \sim \frac{n^{\frac{3}{2}} |\kappa| \text{Area}(A) \text{Area}(B)}{4 \beta^2 \sqrt{2\pi}|(\lambda_1^n)^n \rho_1(g)^n \nu \land \omega|}. \]

The quantity \(|(\lambda_1^n)^n \rho_1(g)^n \nu \land \omega|\) converges to \(|\kappa|\), so this gives us the expression in the statement of the theorem.

**Proof of Corollary 2.** Fix \(\phi : P_1 \to P_1\) hyperbolic. We will explain how to produce the wandering sets \(W_{i,k}\) so that \(P_1 \setminus \bigcup_{i,k} W_{i,k}\) has zero measure. Let \(A_i\) be the \(i\)-th horizontal cylinder as depicted in Figure 3. By Theorem 1 applied to \(\phi^{-1}\) we have

\[ \sum_{n=0}^{\infty} \text{Area}(\phi^{-n}(A_i) \cap A_i) < \infty. \]

This means that the set of points in \(A_i\) which return infinitely often to \(A_i\) has measure zero. For \(k \geq 0\), let \(W_{i,k}\) to be the set of points in \(A_i\) which return exactly \(k\) times to \(A_i\) under the action of \(\phi^{-1}\). From the remarks above we see that \(A_i \setminus \bigcup_{k \geq 0} W_{i,k}\) is measure zero. Since \(P_1 = \bigcup_i A_i\) we see that \(W = \{W_{i,k}\}\) satisfies the statements in Corollary 2.

**3.7. Geometric intersection numbers.** The space \(P_1\) is non-positively curved in the sense that its universal cover is \(\text{CAT}(0)\). This guarantees that all homotopically non-trivial closed curves have geodesic representatives in the metric sense. In particular every such curve has a realization as a sequence of saddle connections \(\sigma_1, \ldots, \sigma_k\) so that the endpoint singularity of \(\sigma_i\) coincides with the start singularity of \(\sigma_{i+1} \mod k\) and the angle made between the two saddle connections at this singularity is at least \(\pi\). A closed metric geodesic may also be a closed non-singular straight-line trajectory, but by moving to the boundary of
the corresponding cylinder we can find a metric geodesic representative of this homotopy class consisting of a sequence of saddle connections.

We will now describe a way to compute the geometric intersection numbers between two non-trivial homotopy classes of closed curves in the punctured surface $P_1$. First we may find metric geodesic representatives $\alpha = \alpha_1 \cup \ldots \cup \alpha_k$ and $\gamma = \gamma_1 \cup \ldots \cup \gamma_l$ for the curves in $P_1$ where the $\alpha_i$ and $\gamma_j$ are saddle connections. There are two types of intersections between $\alpha$ and $\gamma$: those that occur at singularities and those that do not. The unit tangent bundle space $T_1 s^*$ at an infinite cone singularity $s^*$ is naturally homeomorphic to a line and we can make it a metric line using angle coordinates, identifying it with the universal cover of $\mathbb{R}/2\pi \mathbb{Z}$. Two saddle connections meeting at a singularity $s^*$ thus determine an interval $I$ in $T_1 s^*$. Our metric geodesics $\alpha$ and $\gamma$ thus determine two sequences of intervals $I_1, \ldots, I_k$ and $J_1, \ldots, J_l$ in the pair of lines $T_1 s_0 \cup T_1 s_1$. We say two intervals $I$ and $J$ are linked if $I$ contains an endpoint of $J$ and $J$ contains an endpoint of $I$.

**Proposition 21.** Assume the metric geodesics

$$\alpha = \alpha_1 \cup \ldots \cup \alpha_k \quad \text{and} \quad \gamma = \gamma_1 \cup \ldots \cup \gamma_l$$

are transverse in $P_1$ (i.e., none of the saddle connections coincide). Then the geometric intersection number $i(\alpha, \gamma)$ is given by the sum of the number of intersections in $P_1$ and the number of linked pairs of intervals $(I, J)$ with $I \in \{I_i\}$ and $J \in \{J_j\}$.

**Proof.** We will show that we can find representatives of the classes $\alpha$ and $\gamma$ which realize the claimed intersection number. Then we will argue that the representatives realize the intersection number.

There is an $r > 0$ so that the open balls $B_r(s_0)$ and $B_r(s_1)$ do not intersect and only intersect the saddle connections in the set $\{\alpha_i\} \cup \{\gamma_j\}$ in segments of length $r$ where the saddle connections enter and exit the singularities. Let $\mathbb{H}$ be the hyperbolic upper half plane. Because the singularities are infinite cone singularities, the balls are homeomorphic to $\mathbb{H} \cup \{\infty\}$ via a sending radial lines in the ball to vertical lines in $\mathbb{H}$ and sending the singularity to $\infty$. Consecutive arcs of saddle connections of $\alpha$ and $\gamma$ intersecting the balls $B_r(s_s)$ are then sent to a vertical geodesic headed to up infinity followed by a vertical geodesic back down to the real line in $\partial \mathbb{H}$. We can straighten such an arcs to a geodesic in $\mathbb{H}$ joining the places where the two arcs pass through the real axis. Do this for all the visits of the saddle connections to the singularities. Observe that two geodesics with distinct endpoints in $\mathbb{H}$ joining points in $\mathbb{R}$ intersect if and only if the corresponding intervals in $\mathbb{R}$ link. Transversality guarantees this distinct endpoint condition for the geodesics constructed as above. Thus performing this action results in a pair of curves $\hat{\alpha}$ and $\hat{\gamma}$ that intersect in precisely the number of times stated in the theorem.

We must argue that the number of intersections between $\hat{\alpha}$ and $\hat{\gamma}$ is minimal among curves in their homotopy classes. For this it suffices to show that we can not find a simple closed curve formed from an arc of $\hat{\alpha}$ and an arc of $\hat{\gamma}$ which is
homotopically trivial; see [3, Proposition 3.10]. Such arcs must join together at a pair of intersections for \( \hat{\alpha} \) and \( \hat{\gamma} \). Existence of such a curve is ruled out by the Gauss-Bonnet theorem which promises that for a closed polygon in the plane, the total exterior angle is \( 2\pi \). Indeed suppose we had two such arcs of \( \hat{\alpha} \) and \( \hat{\gamma} \) which formed a simple closed homotopically trivial loop \( \hat{\eta} \). Then \( \hat{\eta} \) bounds a topological disk. By deforming back to the flat geodesics \( \alpha \) and \( \gamma \) we get a curve \( \eta \) that bounds a flat polygonal disk. By transversality, the interior angles at the intersections are positive, so the exterior angles at these intersection points are each strictly less than \( \pi \). The other vertices of \( \eta \) must come from visits of \( \alpha \) or \( \gamma \) to the singularities, and since they are metric geodesics the interior angles are at least \( \pi \) and so the exterior angles are negative. We conclude that the total exterior angle of \( \eta \) is less than \( 2\pi \), which contradicts the existence of \( \eta \). \( \square \)

Lemma 13 tells us how to estimate (within 1) the number of interior intersections of saddle connections. Combining this lemma with the above result yields:

**Proposition 22.** Let \( \alpha = \alpha_1 \cup \ldots \cup \alpha_k \) and \( \gamma = \gamma_1 \cup \ldots \cup \gamma_l \) be transverse closed metric geodesics in \( \mathbb{P}_1 \). Then the geometric intersection number \( i(\alpha, \gamma) \) is within \( 2kl \) of

\[
\sum_{i=1}^{k} \sum_{j=1}^{l} |\alpha_i \cap \gamma_j| = \sum_{i=1}^{k} \sum_{j=1}^{l} \left| \frac{1}{2\pi} \int_0^\pi \left( (\tilde{\text{hol}} \alpha_i) \wedge (\tilde{\text{hol}} \gamma_j) \right) (1 - \cos t) \, dt \right|.
\]

Note that the identity in the equation above holds by definition see (25).

**Proof.** We get an error of up to \( kl \) from possible linked intervals in Proposition 21 and another error of up to one from comparing each integral to the number of interior intersections of the corresponding saddle connections (see Lemma 13). \( \square \)

Now consider an affine automorphism \( \phi : \mathbb{P}_1 \rightarrow \mathbb{P}_1 \). Let \( \alpha = \alpha_1 \cup \ldots \cup \alpha_k \) be a closed metric geodesic on \( \mathbb{P}_1 \). Observe that \( \phi(\alpha) = \phi(\alpha_1) \cup \ldots \cup \phi(\alpha_k) \) is also a closed metric geodesic because this image satisfies the angle condition on consecutive saddle connections. As a consequence we see that when considering the geometric intersection number \( i(\phi^\alpha(\alpha), \gamma) \) we can use the integral formula above and the only effect is that we introduce a uniformly bounded error.

Fix \( \phi : \mathbb{P}_1 \rightarrow \mathbb{P}_1 \) with hyperbolic derivative \( D\phi \). Let \( u^u \) and \( u^s \) be unstable and stable unit eigenvectors for \( D\phi \). Then the stable and unstable elements \( \mu^u, \mu^s \in \mathbb{R}^{\mathbb{P}_1} \) corresponding to the transverse measures on \( \mathbb{P}_1 \) to foliations parallel to the expanding and contracting directions are defined by evaluating them on a simple closed curve with metric geodesic representative \( \alpha = \alpha_1 \cup \ldots \cup \alpha_k \) by

\[
\mu^u(\alpha) = \sum_{i=1}^{k} |u^u \wedge \text{hol}_1 \alpha_i| \quad \text{and} \quad \mu^s(\alpha) = \sum_{i=1}^{k} |u^s \wedge \text{hol}_1 \alpha_i|.
\]

**Proof of Theorem 3.** Statement (1) is standard: In the unstable direction we have
\mu^u \circ \phi^{-1}(\alpha) = \sum_{i=1}^k |u^i \wedge D(\phi)^{-1}(\text{hol}_1 \alpha_i)| = \sum_{i=1}^k |D(\phi)(u^i) \wedge \text{hol}_1 \alpha_i|

= \sum_{i=1}^k |\lambda^u_i u^i \wedge \text{hol}_1 \alpha_i| = |\lambda^u_1| \mu^u(\alpha).

A similar argument works in the stable direction.

Now fix a homotopically non-trivial simple closed curve and let \( \alpha = \alpha_1 \cup \ldots \cup \alpha_k \) be a metric geodesic representative. We will prove that

\[
\frac{n^3}{|\lambda^u_1|^n} i \circ \phi^n(\alpha)
\]

converges to a constant times \( \mu^u \) with the constant as given in the theorem. To prove this let \( \gamma \) be another homotopically non-trivial simple closed curve and let \( \gamma = \gamma_1 \cup \ldots \cup \gamma_k \) be a metric geodesic representative. Fixing an \( i \) and a \( j \), and observe that \( \text{hol}_1 \alpha_i \) and \( \text{hol}_1 \gamma_j \) are non-zero vectors in \( \mathbb{R}^2 \) so that \((P^1_i \text{hol}_1 \alpha_i) \wedge \text{hol}_1 \gamma_k \neq 0\). This tells us that \( k = 0 \) in Theorem 19 and

\[
|\kappa| = |(P^1_i \text{hol}_1 \alpha_i) \wedge \text{hol}_1 \gamma_k| = \left| \frac{u^i \wedge \text{hol}_1 \alpha_i}{u^i \wedge u^i} \right| \wedge \text{hol}_1 \gamma_k = \frac{\mu^u(\alpha_i) \mu^u(\gamma_j)}{|u^i \wedge u^i|},
\]

where we are slightly abusing notation by applying (36) to saddle connections (but this is justified if we think of these functions as determining measured foliations.) Then from Theorem 19 we get

\[
\lim_{n \to \infty} \frac{n^3}{|\lambda^u_1|^n} |\phi^n(\alpha) \cap \gamma_j| = \frac{\mu^u(\alpha_i) \mu^u(\gamma_j)}{4\beta^2 \sqrt{2\pi} |u^i \wedge u^i|}.
\]

Then it follows from Proposition 22 that

\[
\lim_{n \to \infty} \frac{n^3}{|\lambda^u_1|^n} i(\phi^n(\alpha), \gamma) = \sum_{i=1}^k \sum_{j=1}^l \frac{\mu^u(\alpha_i) \mu^u(\gamma_j)}{4\beta^2 \sqrt{2\pi} |u^i \wedge u^i|} = \frac{\mu^u(\alpha) \mu^u(\gamma)}{4\beta^2 \sqrt{2\pi} |u^i \wedge u^i|},
\]

This is equivalent to the first limiting statement in statement (2) of the theorem.

The second limit can be obtained by switching \( \phi \) for \( \phi^{-1} \) and stable for unstable.

**Software used.** Software was used in several ways in this paper. SageMath [7] and the FlatSurf SageMath Module [2] were used to experimentally check the results in this paper. FlatSurf was also used to generate the figures of translation surfaces.

**Acknowledgments.** Theorem 1 was first proved while the author was a post-doc at Northwestern and the author would like to thank John Franks and Amie Wilkinson for helpful conversations at the time. The author would also like to thank Barak Weiss for some more recent conversations, and the anonymous referee for suggesting a number of improvements. This article is based upon work supported by the National Science Foundation under Grant Number DMS-1500965 as well as a PSC-CUNY Award (funded by The Professional Staff Congress and The City University of New York).
REFERENCES

[1] J. W. Cannon, W. J. Floyd, R. Kenyon and W. R. Parry, Hyperbolic geometry, in Flavors of Geometry, Math. Sci. Res. Inst. Publ., 31, Cambridge Univ. Press, Cambridge, 1997, 59–115.
[2] V. Delecroix and W. P. Hooper, Flat Surfaces in Sage, https://github.com/videlec/sage-flatsurf, accessed Feb. 22, 2018.
[3] A. Fathi, F. Laudenbach and V. Poénaru, Thurston’s Work on Surfaces, Mathematical Notes, 48, Princeton University Press, Princeton, NJ, 2012.
[4] W. Patrick Hooper, An infinite surface with the lattice property. I: Veech groups and coding geodesics, Trans. Am. Math. Soc., 366 (2014), 2625–2649.
[5] W. Patrick Hooper, The invariant measures of some infinite interval exchange maps, Geometry & Topology, 19 (2015), 1895–2038.
[6] F. W. J. Olver, Asymptotics and Special Functions, Academic Press, New York-London, 1974.
[7] The Sage Developers, Sagemath, the Sage Mathematics Software System (Version 8.1), 2018, http://www.sagemath.org.
[8] W. P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. (N.S.), 19 (1988), no. 2, 417–431.
[9] W. A. Veech, Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards, Invent. Math., 97 (1989), 553–583.

W. PATRICK HOOPER <whooper@ccny.cuny.edu>: Department of Mathematics, The City College of New York, 160 Convent Ave, New York, NY 10031, and Department of Mathematics, The Graduate Center, CUNY, 365 5th Ave, New York, NY 10016, USA