Existence of Kazdan-Warner equation with sign-changing prescribed function

Linlin Sun\textsuperscript{a,b}, Jingyong Zhu\textsuperscript{c,*}

\textsuperscript{a}School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
\textsuperscript{b}Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan, 430072, China
\textsuperscript{c}Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany

Abstract

In this paper, we study the following Kazdan-Warner equation with sign-changing prescribed function $h$

\[ -\Delta u = 8\pi \left( \frac{he^u}{\int h e^u} - 1 \right) \]

on a closed Riemann surface whose area equals one. The solutions are the critical points of the functional $J_{8\pi}$ which is defined by

\[ J_{8\pi}(u) = \frac{1}{16\pi} \int_{\Sigma} |\nabla u|^2 + \int_{\Sigma} u - \ln \left| \int_{\Sigma} h e^u \right|, \quad u \in H^1(\Sigma). \]

We prove the existence of minimizer of $J_{8\pi}$ by assuming

\[ \Delta \ln h^+ + 8\pi - 2K > 0 \]

at each maximum point of $2\ln h^+ + A$, where $K$ is the Gaussian curvature, $h^+$ is the positive part of $h$ and $A$ is the regular part of the Green function. This generalizes the existence result of Ding, Jost, Li and Wang [Asian J. Math. 1(1997), 230-248] to the sign-changing prescribed function case. We are also interested in the blow-up behavior of a sequence $u_\varepsilon$ of critical points of $J_{8\pi-\varepsilon}$ with $\int_{\Sigma} h e^{u_\varepsilon} = 1$, $\lim_{\varepsilon \to 0} J_{8\pi-\varepsilon}(u_\varepsilon) < \infty$ and obtain the following identity during the blow-up process

\[ -\varepsilon = \frac{16\pi}{(8\pi - \varepsilon)(p_\varepsilon)} \left[ \Delta \ln h(p_\varepsilon) + 8\pi - 2K(p_\varepsilon) \right] \lambda_\varepsilon e^{-\lambda_\varepsilon} + O\left(e^{-\lambda_\varepsilon}\right), \]

where $p_\varepsilon$ and $\lambda_\varepsilon$ are the maximum point and maximum value of $u_\varepsilon$, respectively. Moreover, $p_\varepsilon$ converges to the blow-up point which is a critical point of the function $2\ln h^+ + A$.

Keywords: Kazdan-Warner equation, sign-changing prescribed function, existence.

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1. Introduction

Let $\Sigma$ be a closed Riemann surface whose area equals one. Let $h$ be a nonzero smooth function on $\Sigma$ such that $\max h > 0$. For each positive number $\rho$, we consider the following functional

\[ J_\rho(u) = \frac{1}{2\rho} \int_{\Sigma} |\nabla u|^2 + \int_{\Sigma} u - \ln \left| \int_{\Sigma} h e^u \right|, \quad u \in H^1(\Sigma). \]

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*Corresponding author.

Email addresses: sunll@whu.edu.cn (Linlin Sun), jizhu@mis.mpg.de (Jingyong Zhu)
The critical points of \( J_\rho \) are solutions to the following mean field equation

\[
-\Delta u = \rho \left( \frac{he^u}{\int_\Sigma he^u} - 1 \right)
\]

(1.1)

where \( \Delta \) is the Laplace operator on \( \Sigma \).

Mean field equation has a strong relationship with Kazdan-Warner equation. Forty years ago, Kazdan and Warner [15] considered the solvability of the equation

\[
-\Delta u = he^u - \rho,
\]

where \( \rho \) is a constant and \( h \) is some smooth prescribed function. When \( \rho > 0 \), the equation above is equivalent to the mean field equation (1.1). The special case \( \rho = 8\pi \) is sometimes called the Kazdan-Warner equation. In particular, when \( \Sigma \) is the standard sphere \( S^2 \), it is called the Nirenberg problem, which comes from the conformal geometry. It has been studied by Moser [21], Kazdan and Warner [15], Chen and Ding [6], Chang and Yang [3] and others. The mean field equation (1.1) appears in various context such as the abelian Chern-Simons-Higgs models. The existence of solutions of (1.1) and its evolution problem has been widely studied in recent decades (see for example [1, 2, 4, 5, 9–12, 16, 19, 20, 22, 23] and the references therein).

In this paper, we consider the existence theory of Kazdan-Warner equation \((\rho = 8\pi)\) with sign-changing prescribed function. The key is to analyze the asymptotic behavior of the blow-up solutions \( u_\varepsilon \) (see (1.2)) and the functional \( J_{8\pi} \).

We prove the following identity near the blow-up point, whose analogue was proved by Chen and Lin in [4] when the prescribed function \( h \) is positive.

**Theorem 1.1.** Let \( h \) be positive somewhere on \( \Sigma \) and \( u_\varepsilon \) a blow-up sequence satisfying

\[
-\Delta u_\varepsilon = (8\pi - \varepsilon) (he^\varepsilon - 1), \quad \text{in} \ \Sigma
\]

and

\[
\lim_{\varepsilon \searrow 0} J_{8\pi - \varepsilon} (u_\varepsilon) < \infty.
\]

Then up to a subsequence, for \( p_\varepsilon \in \Sigma \) with

\[
\lambda_\varepsilon = \max_\Sigma u_\varepsilon = u_\varepsilon (p_\varepsilon),
\]

we have

\[
-\varepsilon = \frac{16\pi}{(8\pi - \varepsilon)h(p_\varepsilon)} \left[ \Delta \ln h^*(p_\varepsilon) + 8\pi - 2K(p_\varepsilon) \lambda_\varepsilon e^{-\lambda_\varepsilon} + O(e^{-\lambda_\varepsilon}) \right],
\]

where \( K \) denotes the Gaussian curvature of \( \Sigma \).

This yields a uniform bound of minimizers as \( \varepsilon \searrow 0 \) provided that \( \Delta \ln h^* + 8\pi - 2K > 0 \) at all blow-up points.

Let \( G(q, p) \) be the Green function on \( \Sigma \) with singularity at \( p \), i.e.,

\[
\Delta G(\cdot, p) = 1 - \delta_p, \quad \int_\Sigma G(\cdot, p) = 0.
\]

Under a local normal coordinate \( x \) centering at \( p \), we have

\[
8\pi G(x, p) = -4 \ln |x| + A(p) + b_1 x_1 + b_2 x_2 + c_1 x_1^2 + 2c_2 x_1 x_2 + c_3 x_2^2 + O( |x|^3 )
\]

(1.4)

By Lemma 2.4, we know the blow-up point has to be a critical point of \( 2 \ln h^*(p) + A(p) \). Thus, we get an existence result.

**Corollary 1.2.** Let \( \Sigma \) be a compact Riemann surface and \( K(p) \) be its Gaussian curvature. Suppose \( h(p) \) is a smooth function which is positive somewhere on \( \Sigma \). If we have the following for all critical points of \( 2 \ln h^* + A \)

\[
\Delta \ln h^* + 8\pi - 2K > 0,
\]

then equation (1.1) has a solution for \( \rho = 8\pi \).
Furthermore, if \( u_\varepsilon \) is a minimizer of \( J_{8\varepsilon} \), we can show the blow-up point is actually the maximum point of \( 2 \ln h^+ + A \).

**Theorem 1.3.** If \( u_\varepsilon \) is a minimizer of \( J_{8\varepsilon} \) and blows up as \( \varepsilon \searrow 0 \), then the blow-up point \( p_0 \) is a maximum point of the function \( 2 \ln h^+ + A \). Moreover,

\[
\inf_{\rho \in H^1(\Sigma)} J_{8\varepsilon} = -1 - \ln \pi - \left( \ln h(p_0) + \frac{1}{2}A(p_0) \right),
\]

and there is a sequence \( \phi_\varepsilon \in H^1(\Sigma) \) such that

\[
J_{8\varepsilon}(\phi_\varepsilon) = -1 - \ln \pi - \left( \ln h(p_0) + \frac{1}{2}A(p_0) \right) - \frac{1}{4} \left( \Delta \ln h(p_0) + 8\pi - 2K(p_0) \right) \varepsilon \ln \varepsilon^{-1} + o(\varepsilon \ln \varepsilon^{-1}).
\]

Hence, we obtain a minimizing solution of the functional \( J_{8\varepsilon} \).

**Theorem 1.4.** Let \( \Sigma \) be a compact Riemann surface and \( K \) be its Gaussian curvature. Suppose \( h \) is a smooth function which is positive somewhere on \( \Sigma \). If the following holds at the maximum points of \( 2 \ln h^+ + A \)

\[
\Delta \ln h + 8\pi - 2K > 0,
\]

then equation (1.1) has a minimizing solution for \( \rho = 8\pi \).

**Remark 1.1.** The condition mentioned in Theorem 1.4 cannot hold on 2-sphere with arbitrary metric. Assume \( g = e^{2\phi}g_0 \) and solve

\[
-\Delta_{g_0} \psi = \frac{1}{|\Sigma|_{g_0}} - \frac{e^{2\phi}}{|\Sigma|_g}, \quad \int_\Sigma \psi \, d\mu_{g_0} = 0,
\]

where \(|\Sigma|_g\) stands for the area of \( \Sigma \) with respect to the metric \( g \). Set \( h_0 = he^{-2\phi + \rho \phi} \). Then

\[
J_{\rho, h_0}(u) = J_{\rho, h_0, g}(u - \rho\psi) - \frac{\rho}{2} \int_\Sigma |d\psi|^2_{g_0} \, d\mu_{g_0}.
\]

If the condition mentioned in Theorem 1.4 holds, then there is a minimizer of \( J_{8\varepsilon, h_0, g} \). Hence, there is also a minimizer of \( J_{8\varepsilon, h_0, g} \). If \( \Sigma \) is a 2-sphere, we choose \( g_0 \) such that the Gaussian curvature is constant, then \( h_0 \) must be a constant (see [14]). Thus \( h \) is a positive function and

\[
\Delta_{g_0} \ln h + \frac{8\pi}{|\Sigma|_g} - 2K_{g_0} = e^{-2\phi} \left( \Delta_{g_0} \ln h_0 + \frac{8\pi}{|\Sigma|_{g_0}} - 2K_{g_0} \right) = 0
\]

which is a contradiction.

**Remark 1.2.** Zhu [24] also obtained the infimum of the functional \( J_{8\varepsilon} \) if there is no minimizer (when \( h \) is non-negative). He pointed out the blow-up point must be the positive point of \( h \) and used the maximum principle to estimate the lower bound of the functional \( J_{8\varepsilon} \) when \( h \) is non-negative. In our case, the maximum principle does not work since \( h \) is sign-changed. We will use the method of energy estimate to give the lower bound of the functional \( J_{8\varepsilon} \). Such a method also can be used to consider the flow case (cf. [16, 23]) and the Palais-Smale sequence.

**Remark 1.3.** The method in the proof of Theorem 1.4 can be used to prove the convergence of the Kazdan-Warner flow. In other words, under the same condition mentioned in Theorem 1.4, there exists an initial date \( u_0 \) such that the following flow

\[
\frac{\partial u}{\partial t} = \Delta u + 8\pi \left( \frac{he^u}{\int_\Sigma he^u} - 1 \right), \quad u(0) = u_0
\]

converges to a minimizer of \( J_{8\varepsilon} \). This gives a generalization of the previous results [16] (positive prescribed function case) and [23] (non-negative prescribed function case). Recently, Chen, Li, Li and Xu [8] consider another flow approach to the Gaussian curvature flow on sphere and reproved the existence result for sign-changing prescribed function which was obtained by Han [14].
2. Preliminary

Recall the strong Trudinger-Moser inequality (cf. [13, Theorem 1.7])

\[
\sup_{u \in H^1(\Sigma), \|\nabla u\|_1 \leq 1} \int_{\Sigma} \exp\left(4\pi u^2\right) < \infty,
\]

which implies the Trudinger-Moser inequality

\[
\ln \int_{\Sigma} e^u \leq \frac{1}{16\pi} \int_{\Sigma} |\nabla u|^2 + \int_{\Sigma} u + c \tag{2.1}
\]

where \(c\) is a uniform constant depends only the geometry of \(\Sigma\).

We may assume \(h\) is positive somewhere. If \(0 < \rho < 8\pi\), then applying the Trudinger-Moser inequality (2.1) Kazdan and Warner ([15, Theorem 7.2]) proved that the Kazdan-Warner equation (1.1) admits a solution \(u\) which minimizes the functional \(J_\rho\) and satisfies

\[
\int_{\Sigma} he^u = 1.
\]

We consider the critical case \(\rho = 8\pi\). For every \(\epsilon \in (0, 8\pi)\), let \(u_\epsilon\) be a minimizer of \(J_{8\pi-\epsilon}\) which satisfies

\[
\int_{\Sigma} he^{u_\epsilon} = 1.
\]

Thus \(u_\epsilon\) satisfies (1.2). It is clear that the function

\[
\rho \mapsto \inf_{u \in H^1(\Sigma)} J_\rho(u)
\]

is a decreasing function on \((0, +\infty)\). In particular, \(u_\epsilon\) satisfies (1.3). By the Trudinger-Moser inequality (2.1), we have

\[
J_{8\pi-\epsilon}(u_\epsilon) \geq \ln \int_{\Sigma} e^{u_\epsilon} - c. \tag{2.2}
\]

Thus (1.3) and (2.2) gives

\[
\int_{\Sigma} e^{u_\epsilon} \leq C, \quad \forall \epsilon \in (0, 4\pi). \tag{2.3}
\]

One can check that

\[
\lim_{\epsilon \to 0} J_{8\pi}(u_\epsilon) = \inf_{u \in H^1(\Sigma)} J_{8\pi}(u).
\]

If

\[
\limsup_{\epsilon \to 0} \max_{\Sigma} u_\epsilon < +\infty,
\]

then up to a subsequence \(u_\epsilon\) converges smoothly to a minimizer of \(J_{8\pi}\).

In the rest of this section, we only assume \(u_\epsilon\) is a solution to (1.2) and satisfies the condition (2.3).

Assume now \(\{u_\epsilon\}\) is a blow-up sequence, i.e.,

\[
\limsup_{\epsilon \to 0} \max_{\Sigma} u_\epsilon = +\infty.
\]

Without loss of generality, we may assume \(h^\epsilon e^{u_\epsilon} \, d\mu\Sigma\) converges to a nonzero Radon measure \(\mu^\epsilon\) as \(\epsilon \to 0\). Define the singular set \(S\) of the sequence \(\{u_\epsilon\}\) by

\[
S = \left\{ x \in \Sigma : |\mu|((x)) \geq \frac{1}{2} \right\},
\]
where \(|u| = \mu^+ + \mu^-\). It is clear that \(S\) is a finite set. Applying Brezis-Merle’s estimate [1, Theorem 1], one can obtain that for each compact subset \(K \subset \Sigma \setminus S\) (cf. [9, Lemma 2.8])

\[
\left\| u_\varepsilon - \int_\Sigma u \right\|_{L^q(K)} \leq C_K.
\]

Then one obtain a characterization of \(S\) by the blow-up sets of \(|u_\varepsilon|\) (cf. [1, Page 1240])

\[
S = \left\{ p \in \Sigma : \exists \ p_\varepsilon \in \Sigma, \ \text{s.t.} \ \lim_{\varepsilon \to 0} p_\varepsilon = p, \ \lim_{\varepsilon \to 0} u_\varepsilon (p_\varepsilon) = \infty \right\}
\]

Moreover, \(S\) is nonempty and

\[
\lim_{\varepsilon \to 0} \int_\Sigma u_\varepsilon = -\infty,
\]

which implies that \(u_\varepsilon\) goes to \(-\infty\) uniformly on each compact subsets \(K \subset \Sigma \setminus S\). Thus, \(|u|\) is a Dirac measure. By using blow-up analysis (cf. [18, Lemma 1]) together with the classification result of Chen-Li [7, Theorem 1], one can show that \(\mu^- = 0\) and

\[
S = \{ p \in \Sigma : \mu^+ (\{p\}) \geq 1, h(p) > 0 \}.
\]

Notice that \(h e^{\rho_\varepsilon} \, d\mu_\varepsilon\) converges to the nonzero Radon measure \(\mu^+\) as \(\varepsilon \to 0\). We conclude that \(S = \{ p_0 \}\) is a single point set and \(|u| = \mu^+ = \delta_{p_0}\). Thus

**Lemma 2.1** (cf. Lemma 2.6 in [9]), \(u_\varepsilon - \int u_\varepsilon\) converges to \(8\pi G(\cdot, p_0)\) weakly in \(W^{1,q}(\Sigma)\) and strongly in \(L^q(\Sigma)\) for every \(q \in (1, 2)\), and converges in \(C^2_{\text{loc}}(\Sigma \setminus \{p_0\})\).

For a fixed small \(\delta_0 > 0\) and \(u_\varepsilon\) of \(J_{8\pi}\), we define \(\rho_\varepsilon\) to be

\[
\rho_\varepsilon = (8\pi - \varepsilon) \int_{B_{\delta_0}(p_0)} he^{\rho_\varepsilon}
\]

and

\[
\lambda_\varepsilon = u_\varepsilon(p_\varepsilon) = \max_{B_{\delta_0}(p_0)} u_\varepsilon \to +\infty.
\]

We may assume

\[
h|B_{\delta_0}(p_0) \geq \frac{1}{2} h(p_0) > 0, \quad \max_{\partial B_{\delta_0}(p_0)} u_\varepsilon - \min_{\partial B_{\delta_0}(p_0)} u_\varepsilon \leq C, \quad \int_{B_{\delta_0}(p_0)} e^{\rho_\varepsilon} \leq C.
\]

Li [17, Theorem 0.3] obtained the following local estimate

\[
\left| u_\varepsilon(p) - \ln \frac{e^{\rho_\varepsilon}}{1 + \frac{(8\pi - \varepsilon)\rho_\varepsilon}{8} e^{\rho_\varepsilon} |p - p_\varepsilon|^2} \right| \leq C
\]

for \(p \in B_{\delta_0}(p_0)\), where \(|p - p_\varepsilon|\) stands for the distance between \(p\) and \(p_\varepsilon\). Together with Lemma 2.1, the above local estimate (2.5) gives the following

**Lemma 2.2** (cf. Corollary 2.4 in [4]). *There exists a constant \(C > 0\) such that*

\[
|u_\varepsilon + \lambda_\varepsilon| \leq C \text{ in } \Sigma \setminus B_{\delta_0}(p_0).
\]

**Lemma 2.3** (cf. Estimate A in [4]). *Set \(\omega_\varepsilon\) to be the error term defined by*

\[
\omega_\varepsilon(q) = u_\varepsilon(q) - \rho_\varepsilon G(q, p_\varepsilon) - \tilde{u}_\varepsilon
\]

*on \(\Sigma \setminus B_{\delta_0}(p_0)\). Then we have*

\[
||\omega_\varepsilon||_{C^0(\Sigma \setminus B_{\delta_0}(p_0))} = O\left( \varepsilon^{-1/2} \right).
\]
Proof. Notice that $h$ maybe non-positive outside of $B_{\delta_0/2}(p_0)$ and in this case we also have the above estimate. We list a proof here. By Green representation formula, for every $q \in \Sigma \setminus B_{\delta_0}(p_0)$

\[
 u_e(q) - \bar{u}_e = (8\pi - \varepsilon) \int_{\Sigma} G(q, p) \left[ h(p) e^{u_e(p)} - 1 \right] \, d\mu_{\Sigma}(p) \\
= (8\pi - \varepsilon) \int_{\Sigma \setminus B_{\delta_0/2}(p_0)} (G(q, p) - G(q, p_\varepsilon)) \left[ h(p) e^{u_e(p)} - 1 \right] \, d\mu_{\Sigma}(p) \\
+ (8\pi - \varepsilon) \int_{B_{\delta_0/2}(p_0)} (G(q, p) - G(q, p_\varepsilon)) h(p) e^{u_e(p)} \, d\mu_{\Sigma}(p) \\
+ (8\pi - \varepsilon) G(q, p_\varepsilon) + O\left(e^{-\lambda_e/2}\right).
\]

Here we used estimate (2.4) and Li’s local estimate (2.5). By definition,

\[
 \rho_e = (8\pi - \varepsilon) - (8\pi - \varepsilon) \int_{\Sigma \setminus B_{\delta_0}(p_0)} h e^{u_e} = (8\pi - \varepsilon) + O\left(e^{-\lambda_e}\right).
\]

Thus

\[
 u_e(q) - \bar{u}_e - \rho_e G(q, p_\varepsilon) = O\left(e^{-\lambda_e/2}\right), \quad \forall q \in \Sigma \setminus B_{\delta_0}(p_0).
\]

Notice that

\[
 -\Delta (u_e - \bar{u}_e - \rho_e G(\cdot, p_\varepsilon)) = (8\pi - \varepsilon) h e^{u_e} + \rho_e - (8\pi - \varepsilon) = O\left(e^{-\lambda_e}\right), \quad \text{in } \Sigma \setminus B_{\delta_0}(p_0)
\]

and

\[
 u_e - \bar{u}_e - \rho_e G(\cdot, p_\varepsilon) = O\left(e^{-\lambda_e/2}\right), \quad \text{on } \partial B_{\delta_0}(p_0).
\]

The standard elliptic estimate gives

\[
 \|u_e - \bar{u}_e - \rho_e G(\cdot, p_\varepsilon)\|_{C^1(\Sigma \setminus B_{\delta_0}(p_0))} = O\left(e^{-\lambda_e/2}\right).
\]

\]

Based on these facts, we then have the following local estimates. The proofs are same as those in [4], so we omit them here.

Lemma 2.4 (cf. Estimate B in [4]). By using the local normal coordinate $x$ centering at $p_\varepsilon$, we set the regular part of Green function $G(x, p_\varepsilon)$ to be

\[
 \tilde{G}_e(x) = G(x, p_\varepsilon) + \frac{1}{2\pi} \ln |x|, 
\]

and set

\[
 G^*_e(x) = \rho_e \tilde{G}_e(x).
\]

Then we get

\[
 \|\nabla (\ln h^* + G^*_e) (p_\varepsilon)\| = O\left(e^{-\lambda_e/2}\right).
\]

Notice that the Green function is symmetric and we conclude that

\[
 \|\nabla \left(2 \ln h^* + \frac{8\pi - \varepsilon}{8\pi} A\right) (p_\varepsilon)\| = O\left(e^{-\lambda_e/2}\right).
\]
In $B_{\delta_0}(p_\varepsilon)$, we define the following function as in [4]

$$v_\varepsilon(p) = \ln \frac{e^{\lambda_\varepsilon}}{1 + \frac{8\pi - \varepsilon(h(p_\varepsilon))e^{\lambda_\varepsilon}}{8}|p - q_\varepsilon|^2},$$

where $q_\varepsilon$ is chosen to satisfy

$$\nabla v_\varepsilon(p_\varepsilon) = \nabla \ln h(p_\varepsilon),$$

which implies $|p_\varepsilon - q_\varepsilon| = O(e^{-\lambda_\varepsilon})$. We also set the error term as

$$\eta_\varepsilon(p) = u_\varepsilon(p) - v_\varepsilon(p) - (G_\varepsilon^*(p) - G_\varepsilon^*(p_\varepsilon))$$

and

$$R_\varepsilon = \left(\frac{(8\pi - \varepsilon)h(p_\varepsilon)e^{\lambda_\varepsilon}}{8}\right)^\frac{1}{2} \delta_0.$$

Then we have the following estimate for the scaled function $\tilde{\eta}_\varepsilon(z) = \eta_\varepsilon(\delta_0R_\varepsilon^{-\varepsilon}z)$ for $|z| \leq R_\varepsilon$.

**Lemma 2.5** (cf. Estimates C, D and E in [4]). For any $\tau \in (0, 1)$, there exists a constant $C = C_\tau$ such that

$$\eta_\varepsilon(p) = \left(4 - \frac{P_\varepsilon}{2\pi}\right)\ln |p - p_\varepsilon| + O\left(|\int\partial_\varepsilon e^{\frac{\varepsilon}{2}|\eta_\varepsilon|}\sup_{\|p - p_\varepsilon\| \leq \delta_0} |\eta_\varepsilon| + e^{-\frac{\varepsilon}{2}}\right)$$

and

$$|\tilde{\eta}_\varepsilon(z)| \leq C (1 + |z|)^{\tau} e^{-\tau|\varepsilon|_\varepsilon} + e^{-\frac{\varepsilon}{2}}|8\pi - P_\varepsilon|$$

hold for $p \in B_{\delta_0}(p_\varepsilon) \setminus B_{\delta_0/3}(p_\varepsilon)$ and $|z| \leq R_\varepsilon$.

The following lemma shows the relationship between $P_\varepsilon - 8\pi$ and $\eta_\varepsilon$.

**Lemma 2.6** (cf. Estimate F in [4]).

$$P_\varepsilon - 8\pi = -\int_{\partial B_{\delta_0}(p_\varepsilon)} \frac{\partial \eta_\varepsilon}{\partial v} d\sigma + O\left(e^{-\lambda_\varepsilon}\right),$$

where $v$ denotes the unit outer normal of $\partial B_{\delta_0}(p_\varepsilon)$.

### 3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 as in [4].

**Proof.** By Lemma 2.2, we have

$$P_\varepsilon = 8\pi - \varepsilon + O\left(e^{-\lambda_\varepsilon}\right).$$

This implies that we need to control $P_\varepsilon - 8\pi$, which is equivalent to compute $-\int_{\partial B_{\delta_0}(p_\varepsilon)} \frac{\partial \eta_\varepsilon}{\partial v} d\sigma$ by Lemma 2.6. To do so, we set

$$\psi = \frac{1 - a|x - y_\varepsilon|^2}{1 + a|x - y_\varepsilon|^2} \text{ for } x \in \mathbb{R}^2,$$

where $a = \frac{8\pi - \varepsilon(h(p_\varepsilon))e^{\lambda_\varepsilon}}{8}$. Then $\psi$ satisfies

$$\Delta_0 \psi + (8\pi - \varepsilon)h(p_\varepsilon)e^{\lambda_\varepsilon} \psi = 0,$$

where $\Delta_0$ is the standard Laplacian in $\mathbb{R}^2$. On the other hand, by (3.1), we have

$$\Delta_0 \eta_\varepsilon = \Delta_0 u_\varepsilon - \Delta_0 v_\varepsilon - \Delta_0 G_\varepsilon^* = -(8\pi - \varepsilon)h(p_\varepsilon)e^{\lambda_\varepsilon} H(x, \eta_\varepsilon) + O(e^{-\lambda_\varepsilon}),$$

(3.3)
Lemma 2.4, we have

\[ H(x, t) = \frac{h'(x)}{h(p_x)} e^{\epsilon G(x) - G(x)} - 1 \]

and \( h'(x) = h(x) e^{2\delta(x)} \), \( \phi(x) \) comes from the metric \( ds^2 = e^{2\delta(x)} dx^2 \) with \( \phi(0) = 0 \) and \( \nabla \phi(0) = 0 \). By using (3.2), (3.3) and integration by parts, we get

\[
\int_{\partial B_\delta(p_x)} \left( \psi \frac{\partial \eta_x}{\partial y} - \eta_x \frac{\partial \psi}{\partial y} \right) d\sigma = \int_{\partial B_\delta(p_x)} (\psi \Delta \eta_x - \eta_x \Delta \psi) d\sigma
= - \int_{\partial B_\delta(p_x)} \psi(x)(8\pi - \epsilon)h(p_x)e^{\epsilon\phi(x)}(H(x, \eta_x) - \eta_x(x)) + O(e^{-\lambda_1}).
\]

Since \( \psi \) satisfies

\[ \psi(x) = -1 + \frac{2}{1 + a|x - y_x|^2} = -1 + O(e^{-\lambda_1}) \]

and \( |\nabla \psi(x)| = O(e^{-\lambda_1}) \),

for \( x \in \partial B_\delta(p_x) \), we have

\[ -\int_{\partial B_\delta(p_x)} \frac{\partial \eta_x}{\partial y} d\sigma = - \int_{\partial B_\delta(p_x)} \psi(x)(8\pi - \epsilon)h(p_x)e^{\epsilon\phi(x)}(H(x, \eta_x) - \eta_x(x)) + O(e^{-\lambda_1}). \]

Recall

\[
H(x, \eta_x) - \eta_x(x) = \frac{h'(x)}{h(p_x)} \phi' + G(x) - G(x) - 1 - \eta_x(x)
= H(x, 0) + H(x, 0)\eta_x + O(1)\eta_x^2,
\]

where

\[
H(x, 0) = \frac{h'(x)}{h(p_x)} e^{G(x) - G(0)} - 1
= \frac{1}{h(p_x)} e^{2\delta(x) + \ln h(x) + G(x) - G(p_x)} - 1
= \langle b_x, x \rangle + \langle B_x, x \rangle + O(1)|x|^{2+\delta},
\]

where \( b_x \) and \( B_x \) are the gradient and Hessian of \( H(x, 0) \) at \( x = 0 \). By Lemma 2.4, we have \( |b_x| = O(e^{-\lambda_1/2}) \).

Let \( z \) and \( z_\epsilon \) satisfy

\[
\begin{cases}
  x = e^{-\frac{h(p_x)(8\pi - \epsilon)}{8}} z_x \\
  y_x = e^{-\frac{h(p_x)(8\pi - \epsilon)}{8}} z_\epsilon
\end{cases}
\]

Then we get

\[
\left| \int_{B_\delta(p_x)} e^{\epsilon\langle b_x, x \rangle} dx \right| \leq Ce^{-\lambda_1} \int_{|z| \leq R_0} \left( 1 + |z - z_\epsilon|^2 \right)^{\frac{1}{2}} |dz| = O(e^{-\lambda_1}).
\]

\[
\int_{B_\delta(p_x)} e^{\epsilon|z|^{1+\delta}} dx \leq Ce^{-\frac{2\lambda_1}{1+\delta}} \int_{|z| \leq R_0} \left( 1 + |z|^2 \right)^{\frac{1}{2}} |z|^{2+\delta} dz = O(e^{-\lambda_1})
\]

and

\[
\int_{B_\delta(p_x)} e^{\epsilon\langle x_a - p_x, a \rangle} dx = \left( \frac{8\pi - \epsilon}{8} \right)^{2} e^{-\lambda_1} \int_{|z| \leq R_0} \left( 1 + |z - z_\epsilon|^2 \right)^{\frac{1}{2}} |z_\epsilon| dz
= \left( \frac{8\pi - \epsilon}{8} \right)^{2} e^{-\lambda_1} \pi |\delta_{\epsilon}| \ln R_x + O(\epsilon^{\frac{\lambda_1}{2}}),
\]

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where \( x_\alpha \) stands for the \( \alpha \)-th coordinate of \( x \) and \( 1 \leq \alpha, \beta \leq 2 \). Putting those estimates above together, we have

\[
\int_{B_{\delta_0}(p_e)} (8\pi - \varepsilon) h(p_e) e^{\varepsilon} H(x, 0) dx = \frac{32\pi}{(8\pi - \varepsilon) h(p_e)} \left( B_{e}^{11} + B_{e}^{22} \right) e^{-\lambda_e} \lambda_e + O(1) e^{-\lambda_e}.
\]

Note that \( \Delta_{0} G_{e}^\varepsilon(0) = \rho_{e} = (8\pi - \varepsilon) + O\left( e^{-\lambda_e} \right) \) and \( -\Delta_{0} \phi(0) = K(p_e) \). By Lemma 2.4, we know

\[
B_{e}^{11} + B_{e}^{22} = \frac{1}{2} \Delta_{0} H(0, 0)
\]

\[
= \frac{1}{2} (\Delta \ln h(p_e) + 8\pi - \varepsilon - 2K(p_e)) + O\left( e^{-\lambda_e} \right).
\]

For the remainder terms, we use Lemma 2.5 to get

\[
\int_{B_{\delta_0}(p_e)} e^{\varepsilon} H(x, 0) \eta_{e}(x) dx = O\left( e^{-\lambda_e} \right)
\]

\[
\int_{B_{\delta_0}(p_e)} e^{\varepsilon} \eta_{e}^{2}(x) dx = O\left( e^{-\lambda_e} + e^{-\tau E} |8\pi - \rho_{e}| \right).
\]

Therefore,

\[
\rho_{e} - 8\pi = \frac{16\pi}{(8\pi - \varepsilon) h(p_e)} \left[ \Delta \ln h(p_e) + 8\pi - 2K(p_e) \right] \lambda e^{-\lambda_e} + O\left( e^{-\lambda_e} \right)
\]

and this completes the proof. \( \square \)

4. Proof of Theorem 1.3

Proof. On one hand, checking the proof in [23, Theorem 1.2] step by step, we have

\[
\inf_{u \in H^1(\Omega)} J_{8\pi}(u) = \lim_{\varepsilon \to 0} J_{8\pi}(u_{e}) \geq -1 - \ln \left( \ln h(p_0) + \frac{1}{2} A(p_0) \right)
\]

\[
\geq -1 - \ln \pi - \max_{p \in \Sigma} \left( \ln \lambda(p) + \frac{1}{2} A(p) \right).
\]  

(4.1)

We sketch the proof here. Without loss of generality, up to a conformal change of the metric, we may assume that the metric is the Euclidean metric around \( p_0 \) and we also assume \( p_0 \) is the origin \( o \in \mathbb{B} \subset \Sigma \). Choose \( p_{e} \to p_0 \) such that

\[
\lambda_{e} = u_{e}(p_{e}) = \max_{\Sigma} u_{e} \to +\infty.
\]

Set \( r_{e} = e^{-\lambda_{e}/2} \) and

\[
\tilde{u}_{e} = u_{e}(p_{e} + r_{e} x) + 2 \ln r_{e}, \quad |x| < r_{e}^{-1}(1 - |p_{e}|).
\]

Then \( \tilde{u}_{e} \) converges to \( w \) in \( C_{loc}^{\infty}\left( \mathbb{R}^2 \right) \) where

\[
w(x) = -2 \ln \left( 1 + \pi h(p_0) |x|^2 \right).
\]

We denote by \( o_{\varepsilon}(1) \) (resp. \( o_{\rho}(1), o_{\delta}(1) \)) the terms which tends to zero as \( \varepsilon \to 0 \) (resp. \( R \to \infty, \delta \to 0 \)). Moreover, \( o_{\varepsilon}(1) \) may depend on \( R, \delta \), while \( o_{\rho}(1) \) may depend on \( \delta \). We have

\[
\frac{1}{16\pi} \int_{B_{\delta_0}(p_e)} |\nabla H_{e}|^2 = \frac{1}{16\pi} \int_{B_{\delta}} |\nabla \tilde{u}_{e}|^2 = \ln \left( \pi h(p_0) R^2 \right) - 1 + o_{\varepsilon}(1) + o_{\rho}(1).
\]

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According to Lemma 2.1, a direct calculation yields
\[ \frac{1}{16\pi} \int_{\Sigma_{p}(\delta)} |\nabla u_\delta|^2 = -2 \ln \delta + \frac{1}{2} A(p_0) + o_\epsilon(1) + o_\delta(1). \]

Under polar coordinates \((r, \theta)\), set
\[ u_\epsilon'(r) = \frac{1}{2\pi} \int_0^{2\pi} u_\epsilon(p_e + r e^{i\theta}) d\theta. \]

Then
\[ u_\epsilon(\delta) = \int_\Sigma u_\epsilon - 4 \ln \delta + A(p_0) + o_\epsilon(1) + o_\delta(1), \]

\[ u_\epsilon'(r_e R) = -2 \ln r_e - 2 \ln \left(\pi h(p_0)R^2\right) + o_\epsilon(1) + o_R(1). \]

Solve
\[
\begin{cases}
-\Delta \xi_\epsilon = 0, & \text{in } B_\delta(p_\epsilon) \setminus B_{r_e R}(p_\epsilon), \\
\xi_\epsilon = u_\epsilon', & \text{on } \partial (B_\delta(p_\epsilon) \setminus B_{r_e R}(p_\epsilon)).
\end{cases}
\]

We have
\[
\frac{1}{16\pi} \int_{B_\delta(p_\epsilon) \setminus B_{r_e R}(p_\epsilon)} |\nabla u_\epsilon|^2 \geq \frac{1}{16\pi} \int_{B_\delta(p_\epsilon) \setminus B_{r_e R}(p_\epsilon)} |\nabla u_\epsilon'|^2 \]
\[ \geq \frac{1}{16\pi} \int_{B_\delta(p_\epsilon) \setminus B_{r_e R}(p_\epsilon)} |\nabla \xi_\epsilon|^2 = \frac{(u_\epsilon'(\delta) - u_\epsilon'(r_e R))^2}{8 (\ln \delta - \ln (r_e R))}. \]

Thus
\[
\frac{1}{16\pi} \int_{B_\delta(p_\epsilon) \setminus B_{r_e R}(p_\epsilon)} |\nabla u_\epsilon|^2 \geq \frac{(u_\epsilon'(\delta) - u_\epsilon'(r_e R))^2}{8 (\ln \delta - \ln (r_e R))} \left(1 + \frac{\ln (R/\delta)}{-8 \ln r_e}\right)
\]
\[ = \frac{(\tau_\epsilon + \frac{1}{8} u_\epsilon - 2 \ln r_e)^2}{-8 \ln r_e} + \frac{\tau_\epsilon u_\epsilon}{-8 \ln r_e} + \frac{1}{8} \left(2 + \frac{\tau_\epsilon}{\ln r_e} + \frac{u_\epsilon}{\ln r_e}\right)^2 \ln(R/\delta)
\]
\[ - \int \xi_\epsilon - 4 \ln (R/\delta) - A(p_0) - 2 \ln \left(\pi h(p_0)\right) + o_R(1) + o_\delta(1), \]

where
\[ \tau_\epsilon = u_\epsilon'(\delta) - u_\epsilon'(r_e R) - \int \xi_\epsilon + 2 \ln r_e
\]
\[ = 4 \ln (R/\delta) + A(p_0) + 2 \ln \left(\pi h(p_0)\right) + o_\epsilon(1) + o_\delta(1) + o_R(1). \]

Hence, we get
\[
C \geq J_{\delta R}(u_\epsilon) \geq -1 - \ln \pi - \ln h(p_0) - \frac{1}{2} A(p_0)
\]
\[ + \frac{(\tau_\epsilon + \frac{1}{8} u_\epsilon - 2 \ln r_e)^2}{-8 \ln r_e} + \frac{1}{8} \left(2 + \frac{\tau_\epsilon}{\ln r_e} + \frac{u_\epsilon}{\ln r_e}\right)^2 \ln(R/\delta)
\]
\[ + o_\epsilon(1) + o_R(1) + o_\delta(1) \]
which implies
\[ \int_\Sigma u_\varepsilon = -\lambda_\varepsilon + O(\sqrt{\lambda_\varepsilon}) \]
and we obtain (4.1).

On the other hand, checking the proof in [9, Theorem 1.2] step by step, for each \( p \) with \( h(p) > 0 \), there exists a sequence \( \phi_\varepsilon \in H^1(\Sigma) \) such that
\[
J_{8\pi}(\phi_\varepsilon) = -1 - \ln \pi - \left( \ln h(p) + \frac{1}{2} A(p) \right) - \frac{1}{4} \left( \Delta \ln h(p) + 8\pi - 2 K(p) + \left| \nabla \left( \ln h + \frac{1}{2} A \right) (p) \right|^2 \right) \varepsilon \ln \varepsilon^{-1} + o \left( \varepsilon \ln \varepsilon^{-1} \right).
\]
Here we used the fact that the Green function \( G \) is symmetric. These test functions \( \phi_\varepsilon \) can be constructed as following: without loss of generality, assume \( p = 0 \) and
\[ 8\pi G(x, 0) = -2 \ln |x| + A(p) + b_1 x_1 + b_2 x_2 + \beta(x), \]
and take
\[
\phi_\varepsilon(x) = \begin{cases} 
-2 \ln (|x|^2 + \varepsilon) + b_1 x_1 + b_2 x_2 + \ln \varepsilon, & |x| < a_\varepsilon \sqrt{\varepsilon}, \\
8\pi G(x, 0) - \eta \left( a_\varepsilon \sqrt{\varepsilon} |x| \right) \beta(x) + C_\varepsilon + \ln \varepsilon, & a_\varepsilon \sqrt{\varepsilon} \leq |x| < 2 a_\varepsilon \sqrt{\varepsilon}, \\
8\pi G(x, 0) + C_\varepsilon + \ln \varepsilon, & |x| \geq 2 a_\varepsilon \sqrt{\varepsilon},
\end{cases}
\]
where \( \eta \) is a cutoff function supported in \([0, 2]\) and \( \eta = 1 \) on \([0, 1]\) and the positive constants \( a_\varepsilon \) and \( C_\varepsilon \) are chosen carefully. The assumption \( h \) is positive in [9] is used only to ensure that
\[ \lim_{\varepsilon \to 0} \int_\Sigma h \phi_\varepsilon \rightarrow 0. \]

If \( p \) is a critical point (e.g., a maximum point) of the function \( 2 \ln h^+ + A \), then
\[
J_{8\pi}(\phi_\varepsilon) = -1 - \ln \pi - \left( \ln h(p) + \frac{1}{2} A(p) \right) - \frac{1}{4} \left( \Delta \ln h(p) + 8\pi - 2 K(p) \right) \varepsilon \ln \varepsilon^{-1} + o \left( \varepsilon \ln \varepsilon^{-1} \right).
\]
This gives
\[
\inf_{u \in H^1(\Sigma)} J_{8\pi}(u) = -1 - \ln \pi - \max_{p \in \Sigma} \left( \ln h^+(p) + \frac{1}{2} A(p) \right) = -1 - \ln \pi - \left( \ln h(p_0) + \frac{1}{2} A(p_0) \right).
\]
In particular, the blow-up point \( p_0 \) must be a maximum point of the function \( \ln h^+ + A \).

\[ \square \]

Remark 4.1. One can write down the \( o_\varepsilon(1) \) as follows. By Lemma 2.3 and (1.4), direct computations give us
\[
\frac{1}{16\pi} \int_{\Sigma \setminus B(p_0)} |\nabla u_\varepsilon|^2 = \left( 1 - \frac{e}{4\pi} + \frac{e^2}{64\pi^2} + O(e^{-\delta}) \right) \left( -2 \ln \delta + \frac{1}{2} A(p_\varepsilon) + O(e^{-\delta}) + o_\delta(1) \right) + O(e^{-\delta})
\]
\[ = -2 \ln \delta + \frac{1}{2} A(p_\varepsilon) - \frac{e}{4\pi} \left( -2 \ln \delta + \frac{1}{2} A(p_\varepsilon) + O(e^{-\delta}) + o_\delta(1) \right)
\]
\[ + O(e^2) + O(e^{-\delta}) + o_\delta(1). \]
From the proof of Theorem 1.1, we also get the following.
\[
\int_{B_\delta(p_0)} |\nabla \eta_\varepsilon|^2 = O(\varepsilon^2 \delta) + O(e^{-\lambda \varepsilon}),
\]

\[
\frac{1}{16\pi} \int_{B_{\varepsilon}(p_0)} |\nabla v_\varepsilon|^2 = \ln(\pi h(p_0)R^2) - 1 + o_R(1),
\]

\[
\int_{B_\delta(p_0)} |\nabla G^\ast|^2 = O(\delta^2)
\]

and

\[
\frac{1}{16\pi} \int_{B_{\varepsilon}(p_0)} |\nabla G^\ast|^2 = O(r^2) = O(e^{-\lambda \varepsilon}).
\]

These imply that

\[
\frac{1}{16\pi} \int_{B_{\varepsilon}(p_0)} |\nabla u_\varepsilon|^2 = \ln(\pi h(p_0)R^2) - 1 + O\left(e^2 - \frac{\pi}{4}\right) + O(e^{-\lambda \varepsilon}) + o_R(1).
\]

On the neck, \(o_\varepsilon(1)\) are the convergent rates in Lemma 2.1 and \(\bar{u}_\varepsilon \to w\).

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