SOME ASPECTS OF QUANTUM GRAVITY
IN THE CAUSAL APPROACH

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Abstract

We describe the construction of quantum gravity, i.e. of a theory of self-interacting massless spin-2 quantum gauge fields, the ‘gravitons’, on flat space-time, in the framework of causal perturbation theory.

Talk given at the 4th Workshop on Quantum Field Theory under the Influence of External Condition, Leipzig, Germany, 14-18 Sep 1998

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1 Introduction

The central aspect of this work is the construction of the $S$-matrix by means of causality in the quantum field theoretical (QFT) framework. This idea goes back to Stückelberg, Bogoliubov and Shirkov and the program was carried out correctly by Epstein and Glaser [1] for scalar field theory and subsequently applied to QED by Scharf [2] and to non-abelian gauge theories by Dütsch et al. [3, 4]. We now apply this scheme to ‘quantum gravity’ (QG), i.e. a QFT of self-interacting massless spin-2 quantum gauge fields on flat space-time. For this purpose, two main tools will be used: the Epstein-Glaser inductive construction of the perturbation series for the $S$-matrix with the related causal renormalization scheme and perturbative quantum gauge invariance [3, 4]. The first method provides an elegant way of dealing with the UV problem of QG and the second one ensures gauge invariance at the quantum level, formulated by means of the ‘gauge charge’ $Q$, in each order of perturbation theory. A detailed exposition will be found in forthcoming papers by the author.

2 Causal Perturbation Theory

We consider the $S$-matrix, a formal power series in the coupling constant, as a sum of smeared operator-valued distributions of the following form [1]:

$$S(g) = 1 + \sum_{n=1}^{\infty} \int d^4x_1 \ldots d^4x_n T_n(x_1, \ldots, x_n) g(x_1) \cdot \ldots \cdot g(x_n), \quad (1)$$

where $g$ is a Schwartz test function which plays the rôle of adiabatic switching of the interaction and provides a natural infrared cut off in the long-range part of the interaction. The $T_n$, n-point operator-valued distributions, are well-defined ‘renormalized time-ordered products’ and can be expressed in term of Wick monomials of free fields. They are constructed inductively from the first order $T_1(x)$, which defines the theory, by means of Poincaré covariance and causality; the latter, if correctly incorporated, leads directly to the finite perturbation series for the $S$-matrix. The construction of $T_n$ requires some care: if it were simply given by the usual time-ordering $T_n(x_1, \ldots, x_n) = T\{T_1(x_1) \ldots T_1(x_n)\}$, then UV-divergences would appear. If the arguments $x_1, \ldots, x_n$ are all time-ordered, i.e. if we have $x_1^0 > x_2^0 > \ldots > x_n^0$, then $T_n$ is rigorously given by $T_n(x_1, \ldots, x_n) = T_1(x_1) \ldots T_1(x_n)$; since $T_n$ is totally
symmetric in $x_1, \ldots, x_n$, we obtain $T_n$ everywhere except for the complete diagonal $\Delta_n = \{x_1 = x_2 = \ldots = x_n\}$. After performing Wick expansion of $T_n$, we can extend the c-number distributions from $R^{4n} \setminus \Delta_n$ to $R^{4n}$, so that we obtain

$$T_n(x_1, \ldots, x_n) = \sum_k : O_k(x_1, \ldots, x_n) : t^k_n(x_1, \ldots, x_n),$$

where $: O_k(x_1, \ldots, x_n) :$ represents a normally ordered product of free field operators and $t^k_n(x_1, \ldots, x_n)$ a well-defined c-number distribution which is not unique: it is ambiguous up to distributions with local support $\Delta_n$ which depend on the power counting degree of the distribution. This normalization freedom has to be restricted by further physical conditions. In momentum space, $\tilde{t}^k_n(x_1, \ldots, x_n)$ is best obtained by means of dispersion-like integrals which correspond to the splitting of causal distributions into retarded and advanced parts with respect to the last argument $x_n$ (see Sec. 5).

3 Quantization of Gravity

For the causal construction we need the equation of motion of the free graviton after fixing the gauge, the commutation relation between free fields at different space-time points and the first-order graviton self-coupling $T^h_1(x)$. Since we are interested in a quantum theory of Einstein’s general relativity, we therefore start from the Hilbert-Einstein Lagrangian density $L_{HE}$ written in terms of the Goldberg variable $\tilde{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ and by expanding it into a power series in the coupling constant $\kappa = 32\pi G$, by introducing the ‘graviton’ field $h^{\mu\nu}$ defined through $\kappa h^{\mu\nu} = \tilde{g}^{\mu\nu} - \eta^{\mu\nu}$, where $\eta^{\mu\nu}$ is the flat space-time metric tensor:

$$L_{HE} = -\frac{2}{\kappa^2} \sqrt{-g} R = \sum_{j=0}^{\infty} \kappa^j L^{(j)}_{HE},$$

and $L^{(j)}_{HE}$ represents an ‘interaction’ involving $j + 2$ gravitons. From this formulation of general relativity we extract the ingredients for the perturbative construction of causal QG. We stress however the fact that we consider the classical Lagrangian density Eq. (3) only as a ‘source’ of information about
the fields, the couplings and the gauge which we work with: causal perturbation theory doesn’t rely on any quantum Lagrangian. By considering the Euler-Lagrange variation of $L^{(0)}_{hE}$ from Eq. (3) in the Hilbert-gauge $h_{\alpha\beta} = 0$ we obtain the equation of motion for the free graviton field $\Box h_{\alpha\beta}(x) = 0$, and quantize it covariantly by imposing the commutation rule

$$[h_{\alpha\beta}(x), h_{\mu\nu}(y)] = -i \frac{1}{2} \left( \eta_{\alpha\mu} \eta_{\beta\nu} + \eta_{\alpha\nu} \eta_{\beta\mu} - \eta_{\alpha\beta} \eta_{\mu\nu} \right) D_0(x - y),$$

where $D_0(x)$ is the Jordan-Pauli causal distribution. The first order coupling among gravitons, being linear in the coupling constant $\kappa$, can be derived from Eq. (3) by taking the normally ordered product of $L^{(1)}_{hE}$:

$$T_1^h(x) = i \kappa : L^{(1)}_{hE}(x) := i \kappa \left\{ : h_{\alpha\beta}(x) h_{\mu\sigma}(x), u_{\rho\sigma}(x), \ldots \right\}. \quad (5)$$

4 Perturbative Quantum Gauge Invariance

The classical gauge transformations $h_{\alpha\beta} \rightarrow h_{\alpha\beta} + u_{\alpha,\beta} + u_{\beta,\alpha} - \eta_{\alpha\beta} u_{\sigma}$ can be quantum mechanically implemented in the following way by means of the ‘gauge charge’ $Q$:

$$h'^{\alpha\beta}(x) = e^{-i\lambda Q} h_{\alpha\beta}(x) e^{+i\lambda Q},$$

$$Q = \int_{x^\beta = t} d^3x \, h^{\alpha\beta}(x), \partial_0 u_{\alpha}(x),$$

which leads to the infinitesimal gauge variation of the asymptotic free graviton field

$$d_Q h^{\alpha\beta}(x) = [Q, h^{\alpha\beta}(x)] = -iu^{\alpha\beta\rho\sigma} u_\rho(x),$$

where $u^\alpha$ are c-number fields satisfying $\Box u^\alpha(x) = 0$. S-matrix gauge invariance: $\lim_{g \uparrow 1} (S'(g) - S(g)) = \lim_{g \uparrow 1} (-i \lambda [Q, S(g)] + \text{higher com.}) = 0$ is reached, if we can show that the ‘perturbative quantum gauge invariance’ condition

$$d_Q T_n(x_1, \ldots, x_n) = [Q, T_n(x_1, \ldots, x_n)] = \text{divergence} \quad (8)$$

where

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holds true for all \( n \geq 1 \). Already for \( n = 1 \), Eq. (8) is non-trivial, because \( d_Q \mathcal{T}^h_1(x) \neq \text{div} \); this requires the introduction of ghost and antighost fields, \( u^\alpha \) and \( \tilde{u}^\beta \), coupled to the graviton field through the ghost coupling \[5\]

\[
T_1^u = i \kappa \left( : \tilde{u}_\nu(x),_\mu h^{\mu\nu}(x),_\rho u^\rho(x) : + \ldots \right), \tag{9}
\]

and quantized as free fermionic vector fields \( \{ u^\mu(x), x, \tilde{u}^\nu(y) \} = i \eta^{\mu\nu} D_0(x - y) \) with infinitesimal gauge variations \( d_Q u^\mu(x) = 0 \) and \( d_Q \tilde{u}^\nu(x) = i h^{\nu\sigma}(x),_\sigma \), so that we obtain

\[
d_Q \left( \mathcal{T}_1^h(x) + T_1^u(x) \right) = \partial^\nu x, \mathcal{T}_1^{\nu 1}(x) = \text{divergence}. \tag{10}\]

The fermionic quantization is also necessary to have \( Q \) nilpotent, \( Q^2 = 0 \). The ghost fields, usually called Faddeev-Popov ghosts, are introduced in the causal construction as a consequence of perturbative gauge invariance Eq. (8) for \( n = 1 \). In the path-integral framework, the ghost fields appears as a ‘by-product’ of the quantization after gauge fixing, but it was already noticed by Feynman \[6\] that without ghost fields a unitarity breakdown occurs in 2nd order at the loop level.

5 Pure Quantum Gravity in 2nd Order

We now investigate the graviton self-energy contribution (graviton and ghost loops) in 2nd order. The inductive construction of \( T_2(x_1, x_2) \) can be accomplished in two steps: first we construct the following causal distribution from Eq. (11), (5) and (9) and apply Wick expansion

\[
D_{SE}^{h+u}(x_1, x_2) = [\mathcal{T}_1^{h+u}(x_1), \mathcal{T}_1^{h+u}(x_2)]|_{SE} = : h^{\alpha\beta}(x_1) h^{\mu\nu}(x_2) : d_2(x_1 - x_2)_{\alpha\beta\mu\nu}. \tag{11}
\]

because of translation invariance the c-number distribution \( d_2 \) depends only on the relative coordinate \( x = x_1 - x_2 \). In momentum space we get for the self-energy tensor-valued distribution

\[
\hat{d}_2(p)_{\alpha\beta\mu\nu} = \hat{P}^{(4)}(p)_{\alpha\beta\mu\nu} \Theta(p^2) \text{sgn}(p^0) \tag{12}\]

where \( \hat{P}^{(4)}(p)_{\alpha\beta\mu\nu} \) is a covariant polynomial of degree 4. Then, in order to obtain \( T_2(x) \), we split \( d_2(x) \), which has causal support, \( \text{supp}(d_2(x)) \subseteq V^+(x) \cup V^-(x) \), into a retarded and an advanced part. This splitting must be accomplished according to the correct singular order \[2\] \( \omega(d_2) \) which shows
intuitively the behaviour of $d_2$ near the coincidence point $x = 0$ or, in momentum space, the UV behaviour. In this case we find $\omega(d_2) = 4$. Thus, admitting free normalization terms with coefficients $c_0$, $c_2$ and $c_4$, we obtain

\[ \hat{t}_2(p)_{\alpha\beta\mu\nu} = \frac{i}{2\pi} \frac{\hat{P}^{(4)}(p)_{\alpha\beta\mu\nu}}{p^4} \left\{ p^4 \log \left( \frac{-(p^2 + i0)}{M^2} \right) + c_0 + c_2 p^2 + c_4 p^4 \right\} . \tag{13} \]

Since $c_4$ can be absorbed into $M^2$ and mass and coupling constant normalizations fix unambiguously $c_0 = c_2 = 0$, we are left with the new parameter $M$ which defines a mass scale in the theory. We emphasize the fact that, in virtue of the causal splitting prescription, all expressions are finite and Eq. (13) agrees exactly with the finite part obtained using ad-hoc regularization schemes. As a consequence it is not necessary to add counterterms to renormalize the theory. Besides, Eq. (13) satisfies the Slavnov-Ward identity for the 2-points connected Green function:

\[ p^\alpha p^\mu \left\{ b_{\alpha\beta\gamma\delta} \hat{t}_2(p)^{\gamma\delta\rho\sigma} b_{\rho\sigma\mu\nu} \right\} = 0, \tag{14} \]

as well as perturbative gauge invariance Eq. (8): $d_Q T_2^{SE}(x_1, x_2) = \nabla$. For the tree graphs we quote briefly the result of Schorn: perturbative gauge invariance in 2nd order, Eq. (8), ‘generates’ the 4-graviton couplings through local normalization terms $N_2(x_1, x_2)$ of tree graphs, in agreement with the expansion of the Hilbert-Einstein Lagrangian Eq. (3): $N_2(x_1, x_2) = i \kappa^2 : \mathcal{L}_{HE}^{(2)}(x_1) : \delta(x_1 - x_2)$. For the sake of completeness, we discuss also the vacuum graphs in 2nd order: in the causal perturbation theory they cannot be ‘divided away’ as in the GML series for connected Green functions, but this is not problematic since they are finite. The corresponding $T_2$ distribution has singular order $\omega = 6$ and reads $\hat{T}_2(p) = i p^6 \log \left( \frac{-(p^2 + i0)}{M^2} \right) + \sum_{i=0}^{3} c_{2i} (p^2)^i$. It is possible to show that the adiabatic (infrared) limit of vacuum graphs exists: $\lim_{g \uparrow 1} (\Omega, S_2(g)\Omega) = 0$, where $\Omega$ is the Fock vacuum of free asymptotic fields, as a consequence of the ‘bad’ UV behaviour of QG and, at the same time, free vacuum stability forces the free normalization constants to vanish.

### 6 Outlook

An interesting feature is the explicit construction of the physical Hilbert space of QG. In order to decouple the ghosts and the unphysical degrees of free-
dom of the graviton from the theory, we could apply the Gupta-Bleuler \[10\] formalism with indefinite metric, but we prefer to realize the free fields representations on a Fock space $\mathcal{F}$ with positive definite metric. Lorentz covariance requires then the introduction of the Krein structure \[11\] in $\mathcal{F}$ and we can characterize the physical subspace $\mathcal{F}_{phys}$ by

$$\mathcal{F}_{phys} = \ker \{ Q, Q^\dagger \} \cap \{ \Phi \in \mathcal{F} | \eta_{\alpha\beta} h^{\alpha\beta}(x)^{(+)\Phi} = 0 \}.$$ (15)

**Acknowledgments**

The author would like to thank the organizers of the workshop ‘Quantum Field Theory under the Influence of External Conditions’, Leipzig, 14-18 September 1998; Prof. G. Scharf and the members of the Institute for Theoretical Physics at the Zürich University. This work was partially supported by the Swiss National Science Foundation.

**References**

[1] H. Epstein and V. Glaser, *Ann. Inst. Poincaré* 19, 211 (1973).

[2] G. Scharf, *Finite QED*, 2nd ed., Springer Verlag (1995).

[3] M. Dütsch et al., *Nuovo Cimento A* 106, 1029 (1993); 107, 375 (1994).

[4] A. Aste et al., hep-th/9803011 and references therein.

[5] I. Schorn, *Class. Quant. Gravity* 14, 653 and 671 (1997).

[6] R.P. Feynman, *Acta Phys. Pol.* 24, 697 (1963).

[7] D.M Capper et al., *Phys. Rev. D* 8, 4320 (1973).

[8] S.A.A. Zaidi, *J. Phys. A: Math. Gen.* 24, 4325 (1991).

[9] G. t’Hooft and M. Veltman, *Ann. Inst. Poincaré* 20, 69 (1974).

[10] G. Scharf, *Nuovo Cimento A*, 109, 1605 (1996).

[11] F. Krahe, hep-th/9502094.