Symbiotic gap and semi-gap solitons in Bose-Einstein condensates

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Using the variational approximation (VA) and numerical simulations, we study one-dimensional gap solitons in a binary Bose-Einstein condensate trapped in an optical-lattice potential. We consider the case of inter-species repulsion, while the intra-species interaction may be either repulsive or attractive. Several types of gap solitons are found: symmetric or asymmetric; unsplittable or split, if centers of the components coincide or separate; intra-gap (with both chemical potentials falling into a single bandgap) or inter-gap, otherwise. In the case of the intra-species attraction, a smooth transition takes place between solitons in the semi-infinite gap, the ones in the first finite bandgap, and semi-gap solitons (with one component in a bandgap and the other in the semi-infinite gap).

PACS numbers: 03.75.Ss,03.75.Lm,05.45.Yv

I. INTRODUCTION

One of the milestones in studies of Bose-Einstein condensates (BECs) was the creation of bright solitons in $^7\text{Li}$ and $^{85}\text{Rb}$ in “cigar-shaped” traps \cite{1}, with the atomic scattering length made negative (which corresponds to the attraction between atoms) by means of the Feshbach-resonance (FR) technique \cite{2}. Normally, BEC features repulsion among atoms. In that case, it was predicted that an optical-lattice (OL) potential may support gap solitons (GSs) \cite{3}, whose chemical potential falls in finite bandgaps of the OL-induced spectrum. Although GSs, unlike ordinary solitons in self-attractive BEC, cannot realize the ground state of the condensate, it was demonstrated that they may easily be stable against small perturbations \cite{4}. A GS in $^{87}\text{Rb}$ was experimentally created in a cigar-shaped trap combined with an OL potential, pushing the BEC into the appropriate bandgap by acceleration \cite{5}. Other possibilities for the creation of GSs are offered by phase imprinting \cite{6}, or squeezing the system into a small region by a tight longitudinal parabolic trap, which is subsequently relaxed \cite{7}.

BEC mixture of two hyperfine states of the same atom are also available to the experiment \cite{8}. The sign and strength of the inter-species interaction may also be controlled by means of the FR \cite{9}, hence one may consider a binary condensate with intra-species repulsion combined with attraction between the species. It was proposed to use this setting for the creation of symbiotic solitons \cite{10}, in which the attraction overcomes the intrinsic repulsion.

In this work, we aim to study compact (tightly bound \cite{11}) symbiotic gap solitons in a binary BEC, which are trapped, essentially, in a single cell of the underlying OL potential. Unlike the situation dealt with in Refs. \cite{10}, we consider the case of inter-species repulsion, while the intra-species interactions may be repulsive or attractive. In Ref. \cite{11} it was already demonstrated that the addition of intra-species repulsion expands the stability region of symbiotic GSs supported primarily by the inter-species repulsion. The case of attraction between two self-repulsive species was recently considered in Ref. \cite{12}, where it was shown that the attraction leads to a counter- intuitive result – splitting between GSs formed in each species. This effect can be explained by a negative effective mass, which is a characteristic feature of the GS \cite{3}. Indeed, considering the interaction of two GSs belonging to different species, one may expect that the interplay of the attractive interaction with the negative mass will split the GS pair.

Using variational \cite{13} and numerical methods, we here construct families of stable GSs of two kinds: unsplittable (fully overlapping) and split (separated). The splitting border is predicted by the variational approximation (VA) in an almost exact form. In terms of chemical potentials of the two components, the solitons may be of intra- and inter-gap types \cite{11}, with the two components sitting, respectively, in the same gap or different gaps. In particular, the states with one component residing in the semi-infinite gap (which is possible in the case of intra-species attraction) will be called semi-gap solitons.

The paper is organized as follows. The formulation of the system and analytical results, obtained by the variational method \cite{13}, are given in Sec. II. Numerical findings are reported in Sec. III, including maps of GS families in appropriate parameter planes. Section IV summarizes the work.

II. ANALYTICAL CONSIDERATIONS

We consider a binary BEC loaded into a cigar-shaped trap combined with an OL potential acting in the axial direction. Starting with the system of coupled 3D Gross-Pitaevskii equations (GPEs) for wave functions of the two components, $\phi_1$ and $\phi_2$, one can reduce them to 1D equations \cite{14}. In the scaled form, they are

\begin{equation}
\begin{aligned}
&i(\phi_{1,2})_t = -(1/2)(\phi_{1,2})_{xx} + g|\phi_{1,2}|^2\phi_{1,2} \\
&\quad + g_{12}|\phi_{2,1}|^2\phi_{1,2} - V_0 \cos(2x) \phi_{1,2},
\end{aligned}
\tag{1}
\end{equation}

where the OL period is fixed to be $\pi$, and the wave functions are normalized to numbers of atoms in the two species, $\int_{-\infty}^{\infty} |\phi_{1,2}(x)|^2 dx = N_{1,2}$. In Eq. (1), time,
the OL strength, and nonlinearity coefficients are related to their counterparts measured in physical units as follows: \( t \equiv (\pi/L)^2 (h/m) t_{\text{phys}} \), \( V_0 \equiv (L/\hbar)^2 m (V_0)_{\text{phys}} \), \( \{ g, g_{12} \} \equiv (2L\omega_\perp/\hbar) \{ a, a_{12} \} \), where \( m \) is the atomic mass, \( L \) the OL period, \( a \) and \( a_{12} \) scattering lengths accounting for collisions between atoms belonging to the same or different species, and \( \omega_\perp \) the transverse confinement frequency. As said above, we assume repulsive inter-species interactions, with \( g_{12} > 0 \), while the intra-species nonlinearity may be both repulsive (\( g > 0 \)) and attractive (\( g < 0 \)).

While the model assumes equal intra-species scattering lengths, they are, in general, different for two hyperfine states \([8]\). Therefore, using a FR, one cannot modify both intra-species nonlinearities to keep exactly equal values of coefficient \( g \) in equations for both components ([cf. Eq. (1)]), running from negative to positive values (hence, strictly speaking, different cases considered in this work cannot be realized in a single mixture, but should be rather considered as a collection of situations occurring in different mixtures). However, we will consider asymmetric configurations, with \( N_1 \neq N_2 \), which give rise to a much stronger difference in the effective interaction strengths in the two components than a small difference in their intrinsic scattering lengths.

Stationary solutions to Eqs. (1) are looked for in the usual form, \( \phi_{1,2}(x, t) = (\exp(-i\mu_{1,2}t)) u_{1,2}(x) \), with chemical potentials \( \mu_{1,2} \) and functions \( u_{1,2}(x) \) obeying

\[
\mu_{1,2} u_{1,2} + u_{1,2}''/2 - g u_{1,2} u_{1,2}^2 - g_{12} u_{1,2} u_{1,2}^2 + V_0 \cos(2x) u_{1,2} = 0,
\]

with \( \int_{-\infty}^{+\infty} u_{1,2}^2(x) dx = N_{1,2} \). In the GS solutions constructed below, \( \mu_1 \) and \( \mu_2 \) belong to the first two finite bandgaps and/or the semi-infinite gap in the spectrum induced by potential \( -V_0 \cos(2x) \).

**Variational approximation for unsplit solitons:** Equation (2) can be derived from Lagrangian

\[
L = \int_{-\infty}^{+\infty} \left[ \mu_1 u_1^2 + \mu_2 u_2^2 - \frac{1}{2} \left( u_1'' \right)^2 + \frac{1}{2} \left( u_2'' \right)^2 \right] + V_0 \cos(2x) \left( u_1^2 + u_2^2 \right) - g_{12} u_1 u_2 u_1^2 + V_0 \cos(2x) u_1 u_2 + \left[ \text{other terms} \right]
\]

To predict solitons with a compact symmetric profile, which corresponds to numerical results displayed below, we adopt the Gaussian ansatz [12],

\[
u_{1,2}^{(\text{unsplit})}(x) = \pi^{-1/4} \sqrt{N_{1,2}/w_{1,2}} \exp \left( -\frac{x^2}{2w_{1,2}^2} \right),
\]

where variational parameters are widths \( w_{1,2} \), reduced norms \( N_{1,2} \), and \( \mu_{1,2} \). The substitution of the ansatz in Eq. (4) yields an effective Lagrangian, \( L = L(\mu_{1,2}, w_{1,2}) \). Then, the first pair of the variational equations, \( \partial L/\partial u_1 = 0 \), gives \( N_{1,2} = 1 \), which is substituted below, after performing the variation with respect to \( N_{1,2} \). Thus, the remaining equations, \( \partial L/\partial w_{1,2} = 0 \), take the form

\[
1 + \frac{g N_{1,2} w_{1,2}}{\sqrt{2\pi}} + \frac{2g_{12} N_{1,2} w_{1,2}^2}{\sqrt{2\pi}(w_1^2 + w_2^2)^{3/2}} = 4V_0 w_{1,2}^4 e^{-w_{1,2}^2}, \quad (5)
\]

\[
\mu_{1,2} = \frac{1}{4w_{1,2}^2} + \frac{g N_{1,2}}{2\sqrt{\pi} w_{1,2}} + \frac{g_{12} N_{1,2}}{\sqrt{2\pi}(w_1^2 + w_2^2)} - V_0 e^{-w_{1,2}^2}. \quad (6)
\]

Using Eqs. (5) and (6) we can predict borders between intra-gap and inter-gap soliton families of different types. To this end, we take \( \mu_{1,2} \) from Eqs. (5) and, referring to the spectrum of the linearized equation (6), identify curves in plane \( (N_1, N_2) \) which correspond to boundaries between different gaps in the two components.

**Variational approximation for split solitons:** Two-component solitons different from those considered above feature splitting between the two components. An issue of obvious interest is to predict the splitting threshold by means of the VA, for the symmetric case, with \( N_1 = N_2 = N \). For this purpose, we use the following ansatz,

\[
u_{1,2}^{(\text{split})}(x) = \pi^{-1/4} \sqrt{N/w} \left[ 1 \pm bx + C \frac{w^2 b^2}{4} \right]
\]

\[
-\frac{1}{2} (1 + C) b^2 x^2 \exp \left( -\frac{x^2}{2w^2} \right), \quad (7)
\]

with infinitesimal splitting parameter \( b \), the objective being to find a point at which a solution with \( b \neq 0 \) emerges. At small \( b \), the two components of expression (7) feature maxima shifted to \( x = \pm b/a + O(b^2) \), and up to order \( b^3 \), it satisfies the normalization conditions, \( \int_{-\infty}^{\infty} u_{1,2}^2(x) dx = N \). Unlike \( b \), constant \( C \), to be defined below, is not a variational parameter.

The substitution of ansatz (7) in Lagrangian (3) yields, at orders \( b^0 \) and \( b^2 \),

\[
L = -\frac{N}{2w^2} + 2V_0 e^{-w^2} - \frac{g + g_{12}}{\sqrt{2\pi}w} N^2 - b^2 N \left[ 1 + \frac{C}{2} \right]
\]

\[
-2CV_0 w^4 e^{-w^2} + \frac{g (C + 2) + (C - 2) g_{12}}{2\sqrt{2\pi}} \frac{w N}{N}. \quad (8)
\]

At order \( b^0 \) (i.e., for the unsplit soliton), variational equation \( \partial L/\partial w = 0 \) reduces to Eq. (3) with \( N_1 = N_2 = N \) and \( w_1 = w_2 = w \):

\[
1 + \frac{g N w}{\sqrt{2\pi}} = 4V_0 w^4 e^{-w^2}. \quad (9)
\]

At order \( b^2 \), equation \( \partial L/\partial (b^2) = 0 \) yields the splitting condition,

\[
\frac{C + 2}{4} \left[ \frac{g N w}{\sqrt{2\pi}} + \frac{C - 2 g_{12} N w}{\sqrt{2\pi}} - CV_0 w^4 e^{-w^2} \right] = 0. \quad (10)
\]

Obviously, the splitting should not occur if \( g_{12} = 0 \), i.e., Eq. (10) must only yield the trivial solution, \( w = 0 \),
in this case. This condition selects the value of $C$ which was arbitrary hitherto: $C = -2$, hence Eq. (10) takes the form $N_{g_{12}} = 2\sqrt{2\pi}V_0w^2e^{-w^2}$. Combining this with Eq. (9), we obtain $w = \sqrt{2\pi} / \sqrt[N_{g_{12}} - g]{N}$. Here and in all other figures, $V_0 = 5$. For the unsplit soliton, the variational profile is included too.

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Vertical stripes are the Bloch bands between the gaps (the account for direct transitions between different types of asymmetric \( N_1 \neq N_2 \) solitons accurately predicts borders between their different types of the unsplit ones, as predicted by the VA.

FIG. 3: (Color online) The number of atoms in the symmetric soliton, \( N_1 = N_2 \equiv N \), versus the common chemical potential of both components, at several values of \( g \) for \( g_{12} = 0.002 \). Vertical stripes are the Bloch bands between the gaps (the solution branch with \( g = -0.001 \) suffers a discontinuity when it hits the band separating the semi-infinite and first finite gaps).

FIG. 4: (Color online) Typical profiles of unsplit and split asymmetric \( (N_1 \neq N_2) \) solitons. The examples represent solitons of inter-gap types, as indicated in the panels. For the unsplit soliton, the profiles predicted by the VA are shown too.

cf. Fig. 2

Asymmetric solitons: Typical examples of solitons with \( N_1 \neq N_2 \) are displayed in Fig. 4. Similar to their symmetric counterparts, cf. Fig. 1, they feature both unsplit and split shapes (the former ones are well approximated by the VA), which are again confined to a single cell of the OL potential.

The entire family of asymmetric and symmetric GSs is mapped in the \( (N_1, N_2) \) plane, at fixed values of the interaction coefficients \((g_{12}, g)\), in Fig. 3. In these diagrams, the border between intra-gap solitons of different types shrink to a point belonging to the diagonal line \((N_1 = N_2)\), which corresponds to symmetric solitons that account for direct transitions between different types of intra-gap solitons. In Fig. 5 the VA for the unsplit solitons accurately predicts borders between their different varieties.

If none of the nonlinearities is attractive [Figs. 3(a) and (b)], no chemical potential may fall in the semi-infinite gap. Three types of GSs are possible if both nonlinearities are repulsive [Fig. 3(a)]: intra-gap ones,
in the two finite bandgaps, and the inter-gap species, combining them. If the intra-species nonlinearity exactly vanishes [Figs. 4b]], the inter-species repulsion cannot push both components into the second finite bandgap, which leaves us with two species: intra-gap in the first bandgap, and the one mixing the two finite bandgaps. The interplay of the attractive intra-species nonlinearity with the inter-species repulsion supports two intra-gap and two inter-gap types, as seen in Fig. 4c). Note that one of them skips the first bandgap, binding together components sitting in the semi-infinite and in second finite gaps. A notable feature of the map in Fig. 5(c) is the smooth transition from ordinary solitons, with both components in the semi-infinite gap, to ones of the semi-gap type.

IV. CONCLUSION

In this work, we have considered the interplay of the repulsion between two species of bosonic atoms with intra-species repulsion or attraction in a binary BEC mixture loaded into the OL potential. Families of stable solitons found in this setting are classified as symmetric/asymmetric, split/unsplit, and intra/inter-gap. Three varieties of intra-gap solitons, and another three types of inter-gap ones are identified, if the consideration is limited to the two lowest finite bandgaps of the OL-induced spectrum. Varying the atom numbers in the two components, \( N_{1,2} \), we have plotted maps of various states. Although different intra- and inter-gap species are separated by Bloch bands, transitions between them are continuous in the \((N_1,N_2)\) plane. In particular, a solution branch which connects the solitons (of the split type), populating the semi-infinite gap, and unsplit solitons in the first finite bandgap, features the turning point at the border between the two varieties. Other varieties revealed by the analysis represent semi-gap solitons, with one component belonging to the semi-infinite gap, and the other one falling into a finite bandgap.

A considerable part of the numerical findings reported in this work was accurately predicted by variational approximation. These include the shape of unsplit solitons (both symmetric and asymmetric ones), borders between their varieties, and the splitting border for the symmetric solitons.

We appreciate support from FAPESP and CNPq (Brazil), and Israel Science Foundation (Center-of-Excellence grant No. 8006/03).

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