AN EXAMPLE OF A NON-COFIBRANTLY GENERATED MODEL CATEGORY

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Abstract. We show that the model category of diagrams of spaces generated by a proper class of orbits is not cofibrantly generated. In particular the category of maps between spaces may be given a non-cofibrantly generated model structure.

1. Introduction and formulation of results

Several examples of non-cofibrantly generated model categories have appeared recently (see [1], [2], [6]) in response to a question stated by Mark Hovey on his home page. In this note we introduce another family of such examples.

By the category of spaces, denoted by $\mathcal{S}$, we mean the category of simplicial sets (or compactly generated topological spaces). There are plenty of model structures on categories of diagrams of spaces, with different notions of weak equivalences. Some of them are cofibrantly generated, e.g. for weak equivalences and fibrations being objectwise and cofibrations obtained by the left lifting property with respect to trivial fibrations, the corresponding model category is cofibrantly generated.

Let us remind (from [3], [4], [5]) that a diagram $O$ of spaces is called an orbit if $\text{colim } O = \ast$. The weak equivalences which we would like to consider arise naturally from the relation of equivariant homotopy. By the generalized Bredon theorem [5], a map $f : X \to Y$ is an equivariant homotopy equivalence between diagrams which are both cofibrant and fibrant iff $\text{map}(O, f) : \text{map}(O, X) \to \text{map}(O, Y)$ is a weak equivalence of spaces for any orbit $O$. A model category, generated by the collection of orbits, on diagrams of spaces was constructed in [4] with a map $f$ being a weak equivalence (reps. fibration) iff $\text{map}(O, f)$ is a weak equivalence (resp. fibration) for any orbit $O$. In the sequel we consider only this model category on diagrams of spaces. The simplest example of a non-cofibrantly generated model category is given by the following

Theorem 1.1. If $J = (\bullet \to \bullet)$ is the category with two objects and only one non-identity morphism, then the functor category $\mathcal{M} = \mathcal{S}^J$ of maps of spaces with the model structure as above is not cofibrantly generated.

However, not every small category gives rise to a non-cofibrantly generated model category of diagrams. For example, if we take $G$ to be a group, then the above model structure on $\mathcal{S}^G$ is cofibrantly generated. We conclude the paper by using this example to produce many other examples of the same nature.

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2. Preliminaries

By an orbit over a point in the colimit of a diagram \( X \) we mean the pull back of the canonical map \( f: X \to \colim X \) over \( g: * \to \colim X \). Let \( D \) be any small category enriched over \( S \). We denote by \( O \) the collection of all orbits of \( D \). By collection we mean a set or a proper class with respect to some fixed universe \( U \).

The operator \( \operatorname{codom}(\cdot) \) applied to a collection of maps returns the collection of ranges. Given a set \( I \) of maps in \( M = S^P \), we denote by \( I \)-cell the collection of relative \( I \)-cellular complexes and by abs-\( I \)-cell the collection of (absolute) \( I \)-cellular complexes. See [4, 2.1.9] for precise definitions.

Definition 2.1. Let \( \mathcal{X} = \{ X_\alpha \}_{\alpha \in A} \) be a collection of \( D \)-shaped diagrams of spaces. The collection of orbits of \( \mathcal{X} \), denoted by \( \Omega(\mathcal{X}) \subset O \), consists of all orbits \( O \in O \) such that there exists \( \alpha \in A \) and a point \( x \in \colim X_\alpha \) with \( O \) being the orbit over \( x \).

Lemma 2.2. Let \( I \) be a set of cofibrations in the model category \( M \) of \( D \)-shaped diagrams of spaces. Then \( \Omega(\text{abs-}I\text{-cell}) \subset \Omega(\text{codom}(I)) \).

Proof. Let \( X \in M \) be any \( I \)-cellular complex. We proceed by transfinite induction on the \( I \)-cellular filtration of \( X \). \( X_{-1} = \emptyset \), hence \( X_0 \in \operatorname{codom}(I) \) and in particular \( \Omega(X_0) \subset \Omega(\text{codom}(I)) \).

Suppose \( X_\beta \) satisfies \( \Omega(X_\beta) \subset \Omega(\text{codom}(I)) \). We need to show that \( X_\beta+1 \), which is obtained from \( X_\beta \) by attaching a map \( I \ni f : A \hookrightarrow B \), satisfies \( \Omega(X_\beta+1) \subset \Omega(\text{codom}(I)) \).

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & X_\beta \\
\downarrow & & \downarrow f' \\
B & \xrightarrow{f} & X_{\beta+1}
\end{array}
\]

Let \( O_s \) be an orbit over a point \( s \in \colim X_{\beta+1} = \colim X_\beta \cup_{\colim A} \colim B \).

Considering two cases, \( s \in \colim X_\beta \subset \colim X_{\beta+1} \) and \( s \notin \colim X_\beta \), we find out that in the first case \( O_s \) equals the corresponding orbit of \( X_\beta \) and in the second case \( O_s \) is some orbit of \( B \). This follows immediately from the fact that the diagrams

\[
\begin{array}{ccc}
X_\beta & \xrightarrow{f'} & X_{\beta+1} \\
\downarrow & & \downarrow \\
\colim X_\beta & \xrightarrow{\cong} & \colim X_{\beta+1}
\end{array}
\quad \begin{array}{ccc}
B/A & \xrightarrow{\cong} & X_{\beta+1}/X_\beta \\
\downarrow & & \downarrow \\
\colim(B/A) & \xrightarrow{\cong} & \colim(X_{\beta+1}/X_\beta)
\end{array}
\]

are pull-backs. The first square is a pull-back by [4, 2.1] and the second by the observation that horizontal maps are isomorphisms. Hence \( \Omega(X_{\beta+1}) \subset \Omega(\text{codom}(I)) \).

Obviously, if \( \beta \) is a limit ordinal, then

\[
\Omega(X_\beta) = \bigcup_{\lambda < \beta} \Omega(X_\lambda) \subset \Omega(\text{codom}(I))
\]

Hence \( \Omega(\text{abs-}I\text{-cell}) \subset \Omega(\text{codom}(I)) \).
Let us prove first a slightly more general result.

**Proposition 3.1.** Let $D$ be a small category enriched over $S$ which admits a proper class of orbits $O_D$. Then the model category $M$ on the $D$-shaped diagrams of spaces generated by the orbits is not cofibrantly generated.

**Proof.** We argue by contradiction. Suppose the model category $M$ generated by the proper class of orbits $O_D$ is cofibrantly generated. Let $I$ be the set of generating cofibrations, then any cofibration is a retract of an $I$-cellular map. (This follows from Quillen’s small object argument; see [7, 2.1.15].) In particular, any orbit of $O_D$ is a retract of an $I$-cellular complex. Hence any orbit is a retract of some orbit of an $I$-cellular space. But by [2.2] the whole collection of orbits of $I$-cellular complexes form a set, hence the contradiction.

In particular, the model category $S_J$ is generated by the proper class of orbits $O_J = \{X \to \ast\}$, where $X$ runs through all the objects of $S$. Therefore, by Proposition 3.1, $S_J$ is not cofibrantly generated, hence the main result 1.1.

### 4. More examples

Let us conclude by giving more examples of non-cofibrantly generated model categories. Proposition 3.1 implies that $M = S^D$ is not cofibrantly generated iff $O_D$ is a proper class. We have already indicated in the introduction that if we take $D = G$ to be a group, then $O_G = \{G/H|H < G\}$ is a set, hence $S^G$ is cofibrantly generated. The same holds for groupoids. However, the following proposition provides us with a large family of examples.

**Proposition 4.1.** Let $D$ be a small category which admits a fully faithful functor $i : K \to D$, where $K$ is a category with two objects $k_1, k_2$, at least one arrow $f : k_1 \to k_2$ and no arrows in the opposite direction. Then $O_D$ consists of a proper class of orbits.

**Proof.** First define for each space $X \in S$ an orbit over $K$, i.e. a functor $T_X : K \to S$, by $T_X(k_1) = X$, $T_X(k_2) = \ast$, $T_X(g) = \text{id}_X$ for any $g \in \text{mor}(k_1, k_1)$ and $T_X$ on the elements of $\text{mor}(k_2, k_2)$, $\text{mor}(k_1, k_2)$ has a unique definition, since $\ast$ is the final object of $S$. Obviously $T_X$ is an orbit over $K$.

Next we define for each $T_X$ a $D$-orbit $O_X$ by extending the definition of $T_X$ to the whole $D$. More precisely, $O_X = \text{Lan}_i T_X$. We need to check that $O_X$ is an orbit. It follows from the fact that colimit is itself a left Kan extension along a functor into the trivial category. But any two left Kan extensions commute since they may be represented as coends, and for the coends there is a “Fubini” theorem. See [3, X] for the details.

The functor $i$ is taken to be fully faithful, hence $O_X(i(k_1)) = X$; therefore we obtain a proper class of $D$-orbits of the form $O_X$.

**Question 4.2.** Let $D$ be a monoid which is not a group. Is $S^D$ cofibrantly generated?

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