THE FIELD THEORETICAL FORMULATION OF GENERAL RELATIVITY AND GRAVITY WITH NON-ZERO MASSES OF GRAVITONS

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Abstract

It is a review paper related to the following topics. General relativity (GR) is presented in the field theoretical form, where gravitational field (metric perturbations) together with other physical fields are propagated in an auxiliary either curved, or flat background spacetime. A such reformulation of GR is exact (without approximations), is equivalent to GR in the standard geometrical description, is actively used for study of theoretical problems, and is directed to applications in cosmology and relativistic astrophysics. On the basis of a symmetrical (with respect to a background metric) energy-momentum tensor for all the fields, including gravitational one, conserved currents are constructed. Then they are expressed through divergences of antisymmetrical tensor densities (superpotentials). This form permits to connect a necessity to consider local properties of perturbations, which are analyzed in application tasks, with the academic imagination on the quasi-local nature of the conserved quantities in GR. The gauge invariance is studied, and its properties allow to consider the problem of non-localization in exact mathematical expressions. The M/string considerations point out to possible modification of GR, for example, by adding “massive terms” including masses of spin-2 and spin-0 gravitons. A such original modification on the basis of the field formulation of GR is given by Babak and Grishchuk, and we present and discuss it here. They have shown that all the local weak-field predictions of the massive theory are in agreement with experimental data. Otherwise, the exact non-linear equations of the new theory eliminate the black hole event horizons and replace a permanent power-law expansion of the homogeneous isotropic universe with an oscillator behaviour. One variant of the massive theory allows “an accelerated expansion” of the universe.

1 Introduction

Frequently in general relativity (GR) investigations are carried out under assumption that perturbations of physical fields propagate in a background given (fixed) spacetime, flat or curved, which is a solution to the Einstein equations [1] - [3]. A majority of tasks in modern cosmology and astrophysics are just formulated as the study of generation, propagation, evolution and interaction on different backgrounds. Exact cosmological and black hole solutions, for example, are used as such backgrounds. But, linear approximation (without taking into account “back reaction”), and flat or strongly simplified backgrounds frequently only are considered. Additional assumptions are used, but it is not clear how results will be changed without these assumptions, etc. All of these creates a necessity in a generalized (united) description of perturbed systems in GR. It is more natural and desirable to to present them in the framework of a field theory in a fixed background spacetime with the following properties:
• Constructions are covariant and give a possibility to use an arbitrary curved background — a solution to GR.

• A perturbed system is defined by a) Lagrangian (action); b) perturbed (field) equations; c) conserved quantities, like energy, its density, etc.

• Gauge (inner) transformations are defined explicitly with well described properties convenient in applications.

• There are no restrictions in approximations, thus perturbed equations, gauge transformations and conservation laws are to be exact.

Both in theoretical analysis of perturbations and in applications, definitions of energy, momentum, angular momentum, their densities, and conservation laws for them turn out especially important. It is well known that conserved quantities, like energy, are not localized in GR. This means that it is impossible to construct unique covariant densities of these quantities in general. A reason is in physical foundations of GR, namely in the equivalence principle (see, e.g., the textbook [2]). In mathematical terms the problem is explained by a double role of a spacetime in GR. On the one hand, it is an arena, on which physical fields interact and propagate; on the other hand, the spacetime itself is a dynamical object. Because the reason is objective, sometimes an opinion arises that the study of the conserved quantities in GR is senseless. The author of the paper, the same as many other researches, do not support this point of view. The fact of the non-localization cannot suppress the notions, like energy, themselves. Without doubts, gravitational interaction gives a contribution into a total energy, momentum, etc. of gravitating systems [2]. For example, to describe a binary star system one has to include a notion of gravitational energy as an energy of a connection. Considering gravitational waves in a bounded domain of empty space one can show that this domain has a total positive energy, etc. Such examples show that conserved quantities in GR are defined in non-contradictive manner, at least, as non-local characteristics, and, of course, have to be examined.

Total energy-momentum, angular momentum of an asymptotically flat spacetime have been studied in details and are continuing to be considered. One of the main successes was the proof of the positive energy theorem [4]. This initiated a renewed interest to the problem. Energy-momentum and angular momentum became to be associated with finite spacetime domains. Such quantities are called as quasi-local ones and their examination during last two-three decades was very successful (see a recent nice review [5] and references there in). Returning to the cosmological problems, where conversely more frequently local properties of perturbations are studied, we accent a necessity to connect local and quasi-local derivations.

There are many of approaches in GR where both an evolution of perturbations and conservation laws for them are studied. In this paper we consider only one of approaches, which in a more measure satisfies the above requirements. The perturbed Einstein equations are rewritten as follows. The linear in metric perturbation terms are placed on the left hand side; whereas all nonlinear terms are transported to the right hand side, and together with a matter energy-momentum tensor are treated as a total (effective) energy-momentum tensor \( t_{\mu\nu}^{(tot)} \). This picture was developed in a form of a theory of a tensor field with
self-interaction in a fixed background spacetime, where \( t^{(tot)}_{\mu\nu} \) is obtained by variation of an action with respect to a background metric. Frequently it is called as a field theoretical formulation \([6]\) of GR, we will call it simply as the field formulation, and we just review it here. The active history of these studies was begun in 50-th of XX century. Deser \([7]\) has generalized previous works and suggested the more clear presentation without expansions and approximations in Minkowski spacetime. We \([8]\) developed the field formulation of GR on arbitrary curved backgrounds. Advantages of a such description was demonstrated in several applications. A closed Friedmann world was presented as a gravitational field configuration in Minkowski space \([9]\); trajectories of test particles at neighborhoods of event horizons of black holes were analyzed \([10]\); \( t^{(tot)}_{\mu\nu} \) and its gauge properties were used for a development of elements of quantum mechanics with gravitational self-interaction \([11]\), in the frame of which some of variants of an inflation scenario were examined \([12]\); a distribution of energy in black holes solutions was constructed \([13]\); it was studied so-called the weakest falloff conditions for asymptotically flat spacetime at spatial infinity \([14]\). A related bibliography of earlier works particularly can be found in \([6] - [8]\. The foundation of the field formulation of GR and some references can be also found in the review and discussion works \([15, 16]\). A more full, as we know, the modern bibliography can be found in the works \([17]\).

There are different possibilities to arrive at the field formulation of GR. Deser \([7]\) used a requirement: • for a linear massless field of spin 2 in a background spacetime it has to be a source in the form of a total symmetrical energy-momentum tensor of all the fields including gravitational one. Namely this principal was used as a basis for constructions in \([8]\). It is well known the other method, which was more clearly represented by Grishchuk \([6]\) and which briefly is formulated as a transition • from gravistatics (Newtonian law) to gravidynamics (Einstein’s equations). Keeping in mind gauge properties of the Einstein theory one obtains the field formulation of GR as a result of • a “localization” of Killing vectors of a background spacetime, which have a sense of parameters in an action of a gauge theory \([18]\). The way, which has an explicit connection with the standard geometrical formulation of GR is based on a simple • decomposition of usual variables of the Einstein theory onto dynamical and background quantities \([19]\).

In section 2, a construction of the field formulation is given on the basis of the last of the above methods. In section 3, using the results of the works \([20, 21]\) we present conservation laws in the field formulation of GR. The conserved currents are constructed on the basis of symmetrical energy-momentum tensor and express, thus, local characteristics of conserved quantities. At the same time the currents are derived as divergences of antisymmetrical tensor densities (superpotentials), integration of which just leads to surface integrals, which are quasi-local conserved quantities. In section 4, we give a generalization of the results of sections 2 and 3 for various definitions of metrical perturbations and resolve related ambiguities. The one of more desirable properties is that an energy-momentum complex of a theory has to be free of the second (highest) derivatives of the field variables. The energy-momentum tensor in \([8]\) does not satisfy this requirement. Babak and Grishchuk recently improved this situation \([22]\). Developing the approach \([8]\) they reformulated the field interpretation of GR satisfying the above property; in section 5 we outline their approach and details of the method. Gauge transformation properties in the field formulation of GR and their connection with the non-localization problem in GR are discussed in section 6. The original and perspective technique
developed in work [22] is naturally generalized for constructing a gravitational theory with gravitons of non-zero masses in the work [23]. In its framework Babak and Grishchuk have also found and examined static spherically symmetric solutions in vacuum, as well as homogeneous and isotropic cosmological solutions. All of these are included in section 7. We discuss some questions in the last section 8. Thus, the goal of the present review is a presentation of the field approach to GR with its development permitting to arrive at the Babak-Grishchuk massive gravity. This gives a possibility to understand better the properties of this version of the massive gravity and its connection with GR. Here, we do not consider other variants of theories with massive gravitons.

Below we give more general notations used in the paper.

- Greek indexes mean 4-dimensional spacetime coordinates. Small Latin indexes from the middle of alphabet i, j, k, . . . , as a rule, mean 3-dimensional space coordinates; large Latin indexes A, B, C, . . . are used as generalized ones for an arbitrary set of tensor densities, like $Q^A$. Usually $x^0 = ct$, where $c$ is speed of light; $\kappa = 8\pi G/c^2$ is the Einstein constant; $(\alpha\beta)$ and $[\alpha\beta]$ are symmetrization and antisymmetrization in $\alpha$ and $\beta$.

- The dynamic metric in the Einstein theory, as usual, is denoted by $g_{\mu\nu}$ ($g = \det g_{\mu\nu}$), whereas $\overline{g}_{\mu\nu}$ ($\overline{g} = \det \overline{g}_{\mu\nu}$) is the background metric; $\eta_{\mu\nu}$ is a Minkowskian metric. A hat means that a quantity “$\hat{Q}$” is a density of the weight +1, it can be $\hat{Q} = \sqrt{-g}Q$, or $\hat{Q} = \sqrt{-\overline{g}}Q$, or independently from metric’s determinants, it will be clear from a context. A bar means that a quantity “$\overline{Q}$” is a background one. Particular derivatives are denoted by $(\partial_\xi)$ and $(\partial_\alpha)$; $(D_\alpha)$ is a covariant derivative with respect to $g_{\mu\nu}$ with the Chistoffel symbols $\Gamma^\alpha_{\beta\gamma}$; $(\overline{D}_\alpha)$ is a background covariant derivative with respect to $\overline{g}_{\mu\nu}$ with the Chirstoffel symbols $\overline{\Gamma}^\alpha_{\beta\gamma}$; $\delta/\delta Q^A$ — the Lagrangian derivative; $\mathcal{L}_\xi Q^A = -\xi^\alpha \overline{D}_\alpha Q^A + Q^A \overline{\nabla}_\alpha \xi^\beta$ — the Lie derivative of a generalized tensor density $Q^A$ with respect to the vector $\xi^\alpha$; for example for the contravariant vector $Q^\mu$ one has $\mathcal{L}_\xi Q^\mu = -\xi^\alpha \overline{\nabla}_\alpha Q^\mu + \overline{\nabla}_\alpha \xi^\mu$.

- $R^\alpha_{\mu\beta\nu}$, $R_{\mu\nu}$, $G_{\mu\nu}$, $T_{\mu\nu}$ and $R$ are the Riemannian, Ricci, Einstein and matter energy-momentum tensors and the curvature scalar for the physical (effective) spacetime; $\overline{R}^\alpha_{\mu\beta\nu}$, $\overline{R}_{\mu\nu}$, $\overline{G}_{\mu\nu}$, $\overline{T}_{\mu\nu}$ and $\overline{R}$ are the Riemannian, Ricci, Einstein and matter energy-momentum tensors and the curvature scalar for the background spacetime.

2 Exact perturbed Einstein equations on an arbitrary curved background

At first we define the Lagrangian for the perturbed system. Consider the usual action of GR:

$$S = \frac{1}{c} \int d^4 x \mathcal{L}^E \equiv - \frac{1}{2\kappa c} \int d^4 x \hat{R}(g_{\mu\nu}) + \frac{1}{c} \int d^4 x \hat{L}^M(\Phi^A, g_{\mu\nu})$$

(2.1)
where for the sake of simplicity one assumes that \( \hat{L}^M (\Phi^A, g_{\mu\nu}) \) depends on the first derivatives only. Let us write out the Einstein equations together with the matter ones in the form:

\[
\frac{\delta \hat{L}^E}{\delta g^{\mu\nu}} - \frac{1}{2\kappa} \frac{\delta R}{\delta g^{\mu\nu}} + \frac{\delta \hat{L}^M}{\delta g^{\mu\nu}} = 0, \tag{2.2}
\]

\[
\frac{\delta \hat{L}^E}{\delta \Phi^A} = \frac{\delta \hat{L}^M}{\delta \Phi^A} = 0. \tag{2.3}
\]

Now, define the metric and matter perturbations as

\[
\sqrt{-g} g^{\mu\nu} \equiv \hat{g}^{\mu\nu} \equiv \hat{g}^{\mu\nu} + \hat{\mu}^{\mu\nu}, \quad \Phi^A \equiv \hat{\Phi}^A + \phi^A. \tag{2.4}
\]

The background system is described by the action:

\[
S = \frac{1}{c} \int d^4x \hat{L}^E \equiv - \frac{1}{2\kappa} \int d^4x R + \frac{1}{c} \int d^4x \hat{L}^M, \tag{2.5}
\]

and the background quantities \( \hat{g}^{\mu\nu} \) and \( \hat{\Phi}^A \) satisfy the corresponding background equations:

\[
- \frac{1}{2\kappa} \frac{\delta R}{\delta g^{\mu\nu}} + \frac{\delta \hat{L}^M}{\delta \Phi^A} = 0, \quad \frac{\delta \hat{L}^M}{\delta \Phi^A} = 0. \tag{2.6}
\]

The perturbations \( \hat{\mu}^{\mu\nu} \) and \( \phi^A \) now are thought as independent dynamic variables. The perturbed system is to be described by a corresponding Lagrangian on the background of the system (2.5) and (2.6). Let us construct it. Substitute the decompositions (2.4) into the Lagrangian of the action (2.1), subtract zero’s and linear in \( \hat{\mu}^{\mu\nu} \) and \( \phi^A \) terms of the functional expansion, and add any divergence:

\[
\hat{\dot{L}}^{dyn} = \hat{L}^E (\hat{g} + \hat{\mu}, \hat{\Phi} + \phi) - \hat{\mu}^{\mu\nu} \delta \hat{L}^E \delta g^{\mu\nu} - \phi^A \delta \hat{L}^E \delta \Phi^A - \hat{L}^E - \frac{1}{2\kappa} \partial_{\alpha} \hat{k}^{\alpha} = - \frac{1}{2\kappa} \hat{\dot{L}}^g + \hat{\dot{L}}^m, \tag{2.7}
\]

see on functional expansions in the the book [3]. Lagrangians, like (2.7), are called as dynamical Lagrangians in the terminology of [19]. Zero’s term is the background Lagrangian, whereas the linear term is proportional to the l.h.s. of the background equations (2.6). However, one should not to use the background equations in \( \hat{\dot{L}}^{dyn} \) before its variation because really \( \hat{\dot{L}}^{dyn} \) is not less than quadratic in the fields \( \hat{\mu}^{\mu\nu} \) and \( \phi^A \) in functional expansions.

If one chooses the vector density

\[
\hat{k}^{\alpha} \equiv \hat{g}^{\alpha\mu} \Delta^\mu_{\nu\mu} - \hat{\mu}^{\mu\nu} \Delta^\alpha_{\nu\mu}, \tag{2.8}
\]

with the definition

\[
\Delta^\alpha_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} = \frac{1}{2} g^{\alpha\rho} \left( \overline{D}_\mu g_{\rho\nu} + \overline{D}_\nu g_{\rho\mu} - \overline{D}_\rho g_{\mu\nu} \right), \tag{2.9}
\]

where the decomposition (2.4) is used, then a pure gravitational part in the Lagrangian (2.7) is

\[
\hat{\dot{L}}^g = \hat{R}(g^{\mu\nu} + \hat{\mu}^{\mu\nu}) - \hat{\mu}^{\mu\nu} \hat{R}_{\mu\nu} - g^{\mu\nu} \hat{R}_{\mu\nu} + \partial_{\alpha} \hat{k}^{\alpha} = - (\Delta^\rho_{\mu\nu} - \Delta^\sigma_{\rho\mu} \delta^\rho_{\sigma}) \overline{\nabla}_\rho \hat{\mu}^{\rho\nu} + (g^{\mu\nu} + \hat{\mu}^{\mu\nu}) \left( \Delta^\rho_{\mu\nu} \Delta^\sigma_{\rho\sigma} - \Delta^\rho_{\mu\sigma} \Delta^\sigma_{\rho\nu} \right). \tag{2.10}
\]

It depends on only the first derivatives of the gravitational variables \( \hat{\mu}^{\mu\nu} \). In the case of a flat background the Lagrangian (2.10) transfers to the Rosen covariant Lagrangian [24]. The matter part of the dynamical Lagrangian (2.7) is

\[
\hat{\dot{L}}^m = \hat{L}^M (\hat{g} + \hat{\mu}, \hat{\Phi} + \phi) - \hat{\mu}^{\mu\nu} \delta \hat{L}^M \delta g^{\mu\nu} - \phi^A \delta \hat{L}^M \delta \Phi^A - \hat{L}^M. \tag{2.11}
\]
The variation of an action with the Lagrangian $\hat{\mathcal{L}}_{\text{dyn}}$ with respect to $\hat{\imath}^{\mu\nu}$ and some algebraic calculations give the field equations in the form:

$$\hat{G}^L_{\mu\nu} + \hat{\Phi}^L_{\mu\nu} = \kappa \left( \hat{\imath}^g_{\mu\nu} + \hat{\imath}^m_{\mu\nu} \right) \equiv \kappa \hat{\imath}^{(\text{tot})}_{\mu\nu}, \quad (2.12)$$

where the l.h.s. linear in $\hat{\imath}^{\mu\nu}$ and $\phi^A$ consists of the pure gravitational and matter parts:

$$\hat{G}^L_{\mu\nu}(\hat{\imath}) \equiv \frac{\delta}{\delta g_{\mu\nu}} \hat{\imath}^{\rho\sigma} \sqrt{-\hat{g}} \hat{R}^{\rho\sigma} \equiv \frac{1}{2} \left( \nabla_{\rho} \nabla_{\sigma} \hat{\imath}^{\rho\sigma} + \nabla_{\rho} \nabla_{\sigma} \hat{\imath}^{\rho\sigma} - \nabla_{\rho} \nabla_{\sigma} \hat{\imath}^{\rho\sigma} - \nabla_{\rho} \nabla_{\sigma} \hat{\imath}^{\rho\sigma} \right), \quad (2.13)$$

$$\hat{\Phi}^L_{\mu\nu}(\hat{\imath}, \phi) \equiv -2\kappa \frac{\delta}{\delta g_{\mu\nu}} \left( \hat{\imath}^{\rho\sigma} \frac{\delta \hat{\mathcal{L}}^M}{\delta g^{\rho\sigma}} + \phi^A \frac{\delta \hat{\mathcal{L}}^M}{\delta \phi^A} \right). \quad (2.14)$$

The r.h.s. of Eq. (2.12) is the total symmetrical energy-momentum tensor density

$$\hat{\imath}^{(\text{tot})}_{\mu\nu} \equiv 2 \frac{\delta \hat{\mathcal{L}}_{\text{dyn}}}{\delta g^{\mu\nu}} = 2 \frac{\delta}{\delta g^{\mu\nu}} \left( -\frac{1}{2\kappa} \hat{\mathcal{L}}^g + \hat{\mathcal{L}}^m \right) \equiv \hat{\imath}^g_{\mu\nu} + \hat{\imath}^m_{\mu\nu}. \quad (2.15)$$

In expansions, $\hat{\imath}^{(\text{tot})}_{\mu\nu}$ is not less than quadratic in $\hat{\imath}^{\mu\nu}$ and $\phi^A$ that follows from the form of the Lagrangian $\hat{\mathcal{L}}_{\text{dyn}}$. The explicit form of the gravitational part is

$$\hat{\imath}^g_{\mu\nu} = \frac{1}{\kappa} \left[ \left( -\delta_\mu^\rho \delta_\nu^\sigma + \frac{1}{2} \phi_\mu^\rho \phi_\nu^\sigma \right) \left( \Delta^\rho_\alpha^\beta \Delta^\sigma_\alpha^\beta - \Delta^\rho_\alpha^\beta \Delta^\sigma_\alpha^\beta \right) + \nabla_\rho \nabla_\sigma \right], \quad (2.16)$$

where

$$2 \hat{Q}_{\mu\nu} \equiv -\phi_\mu^\rho \phi_\nu^\sigma \Delta^\rho_\alpha^\beta \Delta^\sigma_\alpha^\beta + \phi_\sigma^\rho \Delta^\rho_\alpha^\beta \Delta^\sigma_\alpha^\beta - \phi_\rho^\sigma \Delta^\rho_\sigma^\alpha \Delta^\sigma_\alpha^\beta - \phi_\mu^\sigma \Delta^\rho_\alpha^\beta \Delta^\sigma_\alpha^\beta - \phi_\phi^\tau \left( \Delta^\rho_\mu^\beta \phi_\sigma^\alpha + \Delta^\rho_\sigma^\alpha \phi_\alpha^\beta \right) + \phi_\sigma^\mu \left( \Delta^\phi_\rho^\omega \phi_\sigma^\tau \phi_\alpha^\nu + \Delta^\phi_\sigma^\rho \phi_\tau^\omega \phi_\alpha^\nu \right). \quad (2.17)$$

The matter part is expressed through the usual matter energy-momentum tensor $T_{\mu\nu}$ of the Einstein theory as

$$\hat{\imath}^m_{\mu\nu} = \sqrt{-\phi} \left[ \left( \delta_\mu^\rho \delta_\nu^\sigma - \frac{1}{2} \phi_\mu^\rho \phi_\nu^\sigma \right) \left( T_{\rho\sigma} - \frac{1}{2} \phi_{\rho\tau} T_{\tau\lambda} \phi_{\lambda\sigma} \right) - T_{\mu\nu} \right] - \frac{\delta}{\delta g^{\mu\nu}} \left( \hat{\imath}^{\rho\sigma} \frac{\delta \hat{\mathcal{L}}^M}{\delta g^{\rho\sigma}} + \phi^A \frac{\delta \hat{\mathcal{L}}^M}{\delta \phi^A} \right), \quad (2.18)$$

and is also not less than quadratic in $\hat{\imath}^{\mu\nu}$ and $\phi^A$ in expansions. At the usual description of GR the definition of the energy-momentum tensor by $\delta \hat{\mathcal{L}}^E / \delta g^{\mu\nu}$ is senseless because it is vanishing on the Eq. (2.2), whereas $\hat{\imath}^{(\text{tot})}_{\mu\nu}$ defined in (2.15) is not vanishing on the field equations (2.12). A formal reason is that in the Lagrangian (2.7) the linear terms are subtracted.

By the definitions (2.14) and (2.18), the field equations (2.12) can be rewritten in the form:

$$\hat{\mathcal{G}}^L_{\mu\nu} = \kappa \left( \hat{\imath}^g_{\mu\nu} + \delta \hat{\imath}^M_{\mu\nu} \right) = \kappa \hat{\imath}^{(\text{eff})}_{\mu\nu}, \quad (2.19)$$

where $\delta \hat{\imath}^M_{\mu\nu} \equiv \hat{\imath}^M_{\mu\nu} - \bar{\imath}^M_{\mu\nu}$ is equal to $\hat{\imath}^M_{\mu\nu}$ in Eq. (2.18) without the second line. Thus $\delta \hat{\imath}^M_{\mu\nu}$ can be thought as a perturbation of $\hat{\imath}^M_{\mu\nu} = T_{\mu\nu}$, which includes even linear perturbations in the dynamic fields and does not follow from the Lagrangian (2.7). But now $\hat{\imath}^{(\text{eff})}_{\mu\nu}$ is the source of the linear gravitational field only without linear matter part (see Introduction).
Let us demonstrate the equivalence with the Einstein theory. Transfer $i^{(\text{tot})}_{\mu\nu}$ to the l.h.s. of Eq. (2.12) and use the definitions (2.13), (2.14) and (2.15) with (2.7):

\[
\hat{G}^L_{\mu\nu} + \hat{\Phi}^L_{\mu\nu} - \kappa i^{(\text{tot})}_{\mu\nu} \\
= -2\kappa \frac{\partial g^{\mu\nu}}{\partial g^{\rho\sigma}} \frac{\delta}{\delta l^{\rho\sigma}} \left[ -\frac{1}{2\kappa} \hat{R} \left( \hat{g}^{\alpha\beta} + \hat{l}^{\alpha\beta} \right) + \hat{L}^M \left( \hat{\phi}^A + \hat{\phi}^A ; \hat{g}^{\mu\nu} + \hat{l}^{\mu\nu} \right) \right] \\
+ 2\kappa \frac{\delta}{\delta g^{\rho\sigma}} \left( -\frac{1}{2\kappa} \hat{R} + \hat{L}^M \right).
\]

(2.20)

Because the third line is proportional to the operator of the background equations in (2.6), then Eq. (2.12), as seen, is the Einstein equations (2.2), only in the form with using the decompositions (2.4).

3 Conservation laws

At the beginning we discuss differential conservation laws on Ricci-flat (including flat) backgrounds. One has to take into account $\Phi^A \equiv 0$, $\hat{L}^M \equiv 0$, $\hat{\Phi}^L_{\mu\nu} \equiv 0$ and use $\delta \hat{R}/\delta \hat{g}_{\mu\nu} = 0$ instead of the background equations (2.6). Then the Lagrangian (2.7) is simplified to

\[
\hat{L}^{\text{dyn}} = -\frac{1}{2\kappa} \hat{L}^g + \hat{L}^m = -\frac{1}{2\kappa} \hat{L}^g + \hat{L}^M \left( \phi^A ; \hat{g}^{\mu\nu} + \hat{l}^{\mu\nu} \right),
\]

(3.1)

and the field equations (2.12) transform into the form of Eqs. (2.20):

\[
\hat{G}^L_{\mu\nu} = \kappa \left( \hat{i}^g_{\mu\nu} + \hat{i}^m_{\mu\nu} \right) \equiv \kappa i^{(\text{tot})}_{\mu\nu}.
\]

(3.2)

Thus, for Ricci-flat backgrounds $i^{(\text{tot})}_{\mu\nu}$ and $i^{(\text{eff})}_{\mu\nu}$ coincide, and Eqs. (2.20) and (3.2) have the form announced in Introduction. Because in Eq. (3.2) $\partial_\nu \hat{G}^L_{\nu\mu} = 0$, a divergence of Eq. (3.2) leads to

\[
\partial_\nu i^{(\text{tot})}_{\nu\mu} = 0.
\]

(3.3)

A contraction of $i^{(\text{tot})}_{\mu\nu}$ with background Killing vectors $\lambda^\alpha$ gives a current $\hat{J}^\nu(\lambda) = i^{(\text{tot})\nu}_{\mu} \lambda^\mu$, which is also differentially conserved:

\[
\nabla_\nu \hat{J}^\nu(\lambda) \equiv \partial_\nu \hat{J}^\nu(\lambda) = 0.
\]

(3.4)

Integration of $\hat{J}^\nu$ leads to non-local conserved quantities. Consider a background 4-dimensional volume $V_4$, the boundary of which consists of timelike “surrounding wall” $S$ and two spacelike sections: $\Sigma_0 := t_0 = \text{const}$ and $\Sigma_1 := t_1 = \text{const}$. Because the conservation law (3.4) is presented by the scalar density it can be integrated through the 4-volume $V_4$: $\int_{V_4} \partial_\mu \hat{J}^\nu(\lambda) d^4x = 0$. The generalized Gauss theorem gives

\[
\int_{\Sigma_1} \hat{J}^0(\lambda) d^3x - \int_{\Sigma_0} \hat{J}^0(\lambda) d^3x + \oint_S \hat{J}^\nu(\lambda) dS_\mu = 0
\]

(3.5)

where $dS_\mu$ is the element of integration on $S$. If in Eq. (3.5) $\oint_S \hat{J}^\nu(\lambda) dS_\mu = 0$, then the quantity

\[
\mathcal{P}(\lambda) = \int_{\Sigma} \hat{J}^0(\lambda) d^3x
\]

(3.6)

is conserved on $\Sigma$ restricted by $\partial \Sigma$, intersection with $S$. In the converse case, Eq. (3.5) describes changing the quantity (3.6), that is its flux through $\partial \Sigma$. It can be also assumed $\partial \Sigma \to \infty$. 

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The differential conservation laws (3.3) have also a place for backgrounds presented by Einstein spaces in Petrov’s definition [25]: \( \mathcal{F}_{\mu\nu} = \Lambda g_{\mu\nu} \) where \( \Lambda \) is a constant (see [8, 26, 27]). For arbitrary curved backgrounds there are no conservation laws, like (3.3). Indeed, in the general case \( \mathcal{F}_{\mu} \left( \tilde{G}^{L\mu} + \tilde{\Phi}^{L\mu} \right) \neq 0 \) in (2.12), and \( \mathcal{D}_{\mu} \tilde{G}^{L\mu} \neq 0 \) in (2.19). The reason is that the system (2.7) interacts with a complicated background geometry determined by the background matter fields \( \tilde{\Phi} \). Many of cosmological solutions are just not of the Einstein spaces.

Conservation laws for arbitrary curved backgrounds and arbitrary displacement vectors \( \xi^\alpha \) were constructed in [21]. With using the technique of canonical Noether procedure developed in [28] and applied to the Lagrangian (2.7) it was obtained the identity:

\[
\frac{1}{\kappa} \hat{\mathcal{I}}_{\mu}^{\nu} \xi^\nu + \frac{1}{\kappa} \tilde{\mathcal{P}}_{\mu\lambda} \xi^\nu + \hat{\dot{\xi}}^\nu \equiv \mathcal{D}_{\nu} \hat{I}^{\mu\nu} \equiv \partial_{\nu} \hat{I}^{\mu\nu}. \tag{3.7}
\]

The superpotential has the form:

\[
\hat{I}^{\mu\nu} \equiv \frac{1}{\kappa} \tilde{\mathcal{P}}_{\mu\rho} \xi^\rho + \tilde{\mathcal{P}}^{\mu\nu} \xi^\lambda \equiv \frac{1}{\kappa} \left( \tilde{\mathcal{P}}^{|\mu|} \xi^\rho + \xi^{|\nu|} \xi^\rho - \tilde{\mathcal{P}}^{\mu\nu} \xi^\rho \right) \tag{3.8}
\]

and, thus, \( \partial_{\mu\nu} \hat{I}^{\mu\nu} \equiv 0 \). It generalizes the Papapetrou superpotential [29]; indeed for the translations in Minkowski space \( \xi^\lambda = \delta^\lambda_{(\rho)} \) in the Lorentzian coordinates one gets

\[
\hat{I}_{\mu}^{\nu} = \tilde{\mathcal{P}}_{\mu\rho} \xi^\rho = \frac{1}{2\kappa} \partial_{\sigma} \left( \delta_{(\rho}^{[\mu} \tilde{\mathcal{P}}^{\nu]\sigma] - \delta_{\sigma}^{(\nu} \tilde{\mathcal{P}}^{\mu]\rho] - \tilde{\mathcal{P}}^{\nu\sigma} \tilde{\mathcal{P}}^{\mu\rho} + \tilde{\mathcal{P}}^{\sigma\nu} \tilde{\mathcal{P}}^{\mu\rho} \right). \tag{3.9}
\]

The same superpotential (3.8) was constructed in [20] by the other way, namely, by the Belinfante symmetrization of the canonical system in [28]. The last term on the l.h.s. of (3.7) is

\[
2\kappa \hat{\dot{\xi}}^\nu = 2 \left( \tilde{\varepsilon}^{\rho\sigma} \mathcal{D}_{\rho} \hat{I}_{\sigma}^{\mu} - \tilde{\mathcal{P}}^{\nu\sigma} \mathcal{D}_{\rho} \hat{I}^{\rho\sigma}_{\sigma} \right) - \left( \tilde{\varepsilon}^{\rho\sigma} \mathcal{D}_{\rho} \hat{I}_{\sigma}^{\nu} - \tilde{\mathcal{P}}^{\nu\sigma} \mathcal{D}_{\rho} \hat{I}^{\rho\sigma}_{\sigma} \right) + \left( \hat{I}^{\nu\sigma} \mathcal{D}_{\rho} \tilde{\varepsilon} - \tilde{\mathcal{P}}^{\nu\sigma} \mathcal{D}_{\rho} \hat{I}^{\rho\sigma} \right) \tag{3.10}
\]

where \( 2 \tilde{\varepsilon}^{\rho\sigma} = - \mathcal{L}_{\xi} \tilde{\mathcal{F}}_{\rho\sigma} \), and, thus, disappears on the Killing vectors of the background.

To write out a physically sensible conservation laws from the identity (3.7) one has to use the field equations, which we substitute in the form of Eq. (2.19). Then the identity (3.7) transforms to the equation

\[
\hat{I}^{\nu} = \mathcal{T}_{\nu}^{\mu} \xi^\nu + \hat{\dot{\xi}}^\nu = \mathcal{D}_{\nu} \hat{I}^{\mu\nu} \equiv \partial_{\nu} \hat{I}^{\mu\nu}. \tag{3.11}
\]

The generalized total energy-momentum tensor density is

\[
\hat{\mathcal{T}}_{\nu}^{\mu} \equiv \hat{I}^{\nu\mu}_{\text{tot}} + \delta \hat{I}^{\nu\mu}_{\text{tot}} + \frac{1}{\kappa} \hat{\mathcal{P}}_{\nu\lambda} \xi^\lambda + \frac{1}{\kappa} \hat{\mathcal{I}}^{(\text{eff})}_{\nu\lambda} \xi^\lambda + \frac{1}{\kappa} \hat{\mathcal{I}}^{(\text{tot})}_{\nu\lambda} \xi^\lambda \tag{3.12}
\]

where \( \hat{I}^{(\text{tot})}_{\nu\mu} \) is exchanging with \( \hat{I}^{(\text{eff})}_{\nu\mu} \), and the interaction with the background geometry term, \( \hat{I}^{(\text{tot})}_{\nu\lambda} \), is adding. Thus, \( \hat{\mathcal{T}}_{\nu}^{\mu} \) plays the same role as \( \hat{I}^{(\text{tot})}_{\nu\mu} \) in Eq. (3.4) if Killing vectors exist. However, the current \( \hat{I}^{\nu} \) has a more general applicability, than \( \hat{J}^{\nu} \) in (3.4): it is conserved, \( \mathcal{D}_{\mu} \hat{I}^{\nu} = \partial_{\mu} \hat{I}^{\nu} = 0 \), on arbitrary backgrounds and for arbitrary \( \xi^\alpha \). It is important, for example, for models with cosmological backgrounds where not only the Killing vectors are used fruitfully (see, e.g., [30]).

Due to antisymmetry of the superpotential (3.9) the conserved quantity, like (3.6), is expressed over a surface integral

\[
\mathcal{P}(\xi) = \oint_{\partial\Sigma} \hat{I}^{\nu}_{\text{tot}}(\xi) ds_k \tag{3.13}
\]
where $ds_k$ is the element of integration on $\partial \Sigma$. It is important expression because it connects a quantity $\mathcal{P}(\xi)$ obtained by integration of local densities with a surface integral playing a role of a quasi-local quantity (see discussion in Introduction).

4 Different definitions for perturbations

In GR, components of each of metrical densities

$$g^a = \{g^{\mu\nu}, g_{\mu\nu}, \sqrt{-g}g^{\mu\nu}, \sqrt{-g}g_{\mu\nu}, (-g)g^{\mu\nu}, \ldots\}$$

(4.1)

could be chosen as independent dynamic variables. In the terms of generalized variables (4.1) the action of GR (2.1) is rewritten as

$$S = \frac{1}{c} \int d^4x \hat{L}_E^{(a)} = -\frac{1}{2\kappa c} \int d^4x \hat{R}(g^a) + \frac{1}{c} \int d^4x \hat{L}_M(\Phi^a, g^a).$$

(4.2)

Variation with respect to $g^a$ gives the gravitational equations in a corresponding form instead of (2.2). The perturbations could be also defined for each of metric variables in Eq. (4.1) as

$$\{g^a = g^{\mu\nu} + h_{\mu\nu}, \hat{g}^{\mu\nu} = \sqrt{-g^{\mu\nu}} + \hat{l}^{\mu\nu}, g^{\mu\nu} = \bar{g}^{\mu\nu} + r^{\mu\nu}, \ldots\}.$$  

(4.3)

For the decomposition (4.3) following to the rules of constructing the Lagrangian (2.7) one gets

$$\hat{L}_E^{(a)} = -\frac{1}{2\kappa} \hat{R}(g^{a} + h^{a}) + \hat{L}_M(\Phi^a + \phi^a; g^a + h^a) - h^a \left( \frac{1}{2\kappa} \frac{\delta \hat{R}}{\delta g^a} + \frac{\delta \hat{L}_M}{\delta g^a} \right) - \phi^a \frac{\delta \hat{L}_M}{\delta \Phi^a} \left( -\frac{1}{2\kappa} \hat{R} + \bar{L}_M \right) - \frac{1}{2\kappa} \partial_\nu \hat{k}^{\nu}. $$

(4.4)

Its variation with respect to $h^a$ and some re-calculations give the Einstein equations in the form (2.12):

$$\hat{G}_{\mu\nu}^{L(a)} + \hat{\Phi}_{\mu\nu}^{L(a)} = \kappa \bar{t}_{\mu\nu}^{(total a)}.$$  

(4.5)

The total symmetrical energy-momentum tensor density is defined as usual:

$$\bar{t}_{\mu\nu}^{(total a)} = 2 \frac{\delta \hat{L}_E^{(a)}}{\delta g^{\mu\nu}}.$$  

(4.6)

In the l.h.s. of Eq. (4.5) the independent variables $h^a$ are replaced by the other variables

$$\hat{t}_{\mu\nu}^{(total a)} = h^a \frac{\partial g^{\mu\nu}}{\partial g^a},$$

(4.7)

which are also considered as independent ones due to the background equations. Thus, the same operators (2.13) and (2.14) are applied to $\hat{t}_{\mu\nu}^{(a)}$.

For some of different decompositions (4.3): $g_1^a = g_1^{\mu\nu} + h_1^{\mu\nu}$ and $g_2^a = g_2^{\mu\nu} + h_2^{\mu\nu}$ the variables (4.7) differ one from another in the second order in perturbations: $\hat{t}_{\mu\nu}^{(a2)} = \hat{t}_{\mu\nu}^{(a1)} + \hat{\beta}^{(a)12}$. Because differences inter the linear expressions of equations (4.5) the energy-momentum tensor densities $\bar{t}_{\mu\nu}^{(total a1)}$ and $\bar{t}_{\mu\nu}^{(total a2)}$ get the same differences too. Firstly, for the case of flat backgrounds this fact was noted by Boulware and Deser [31].
For the system (4.4) the identity
\[ \frac{1}{\kappa} G^L_{\nu}{}^{(a)\mu} \xi^\nu + \frac{1}{\kappa} \tilde{I}^\lambda_{(a)\nu} \mathcal{R}_{\lambda\nu} \xi^\nu + \tilde{c}^\mu_{(a)\nu} \equiv \partial_{\nu} \tilde{I}^\mu_{(a)} \] (4.8)
takes a place and has exactly the form of the identity (3.7) with replacing \( \tilde{I}^\mu_{(a)} \) with \( \tilde{I}^\mu_{(a)} \) only. Substituting Eq. (4.5) into the identity (4.8) we obtain the conservation law in the form:
\[ \tilde{I}^\mu_{(a)} = \left( \tilde{I}^\nu_{(a)\mu} + \delta \tilde{I}^M_{(a)\mu} + \kappa^{-1} \tilde{I}^\nu_{(a)\mu} \mathcal{R}_{\lambda\nu} \right) \xi^\nu + \tilde{c}^\mu_{(a)} = \tilde{\mathcal{T}}^\mu_{(a)\nu} \xi^\nu + \tilde{c}^\mu_{(a)} = \partial_{\nu} \tilde{I}^\mu_{(a)} \] (4.9)
analogous to (3.11) with (3.12). Thus a family of conservation laws (4.9) presents a corresponding family of superpotentials, which can be presented in the form of the Abbott-Deser type [26]:
\[ \tilde{I}^\mu_{(a)} = \frac{1}{\kappa} \left( \tilde{I}^\nu_{(a)\mu} \mathcal{T}_\rho \xi^\nu \right) + \tilde{c}^\nu_{(a)\mu} \mathcal{T}_\sigma \xi^\sigma \right) \]
(4.10)
It is exact the formula (3.8) with exchanging \( \tilde{I}^\mu_{(a)} \) by \( \tilde{I}^\mu_{(a)} \). Indeed, the known Abbott-Deser superpotential

In this section we present the results by Babak and Grishchuk [22]. Following them here and in next
sections we consider the Minkowski space with \( \mathcal{R}_{\lambda\nu} = 0 \) as the background spacetime. Our presentation
is technically simpler than that of Babak and Grishchuk [22], although, of course, is equivalent to their
one. Thus we repeat their calculations on the basis of formulae (2.7) - (2.20) simplified to the case of Eqs.
(3.1) - (3.3). Also, in the work [22] as independent variables it is used \( I^\nu_{(a)} = \tilde{I}^\nu_{(a)} / \sqrt{-g} \), whereas we use
\( \tilde{I}^\nu_{(a)} \). Leaving the field equations to be equivalent, this leads to a difference in the direct definitions of the energy-momentum tensors. However, with taking into account the field equations this difference disappears
and does not influence on results and conclusions.

With using the definition (2.4) we present the expression (2.9) through the gravitational variables \( \tilde{I}^\mu_{(a)} \):
\[ \Delta^\mu_{\nu \rho} = \frac{1}{2 \sqrt{-g}} \left[ g_{\nu \rho} \mathcal{T}_\mu \tilde{I}^\lambda_{\nu \rho} + g_{\nu \rho} \mathcal{T}_\mu \tilde{I}^\lambda_{\nu \rho} - g_{\mu \lambda} g_{\nu \beta} g^{\lambda \nu} \mathcal{T}_\mu \tilde{I}^\alpha_{\beta} \right. \\
+ \left. \frac{1}{2} \left( g_{\alpha \beta} \mathcal{T}_\mu \tilde{I}^\alpha_{\beta} + g_{\alpha \beta} \mathcal{T}_\mu \tilde{I}^\alpha_{\beta} - g_{\alpha \beta} g_{\mu \lambda} g^{\lambda \nu} \mathcal{T}_\rho \tilde{I}^\beta_{\alpha} \right) \right] \] (5.1)
where \( g_{\mu \nu} \), \( \tilde{g}^{\mu \nu} \) and \( \sqrt{-g} \) are thought as dependent on the definition (2.4). Substituting Eq. (5.1) into Eq.
(2.16) with (2.17) one finds that \( \tilde{I}^\nu_{(a)} \) depends on the second derivatives of \( \tilde{I}^\mu_{(a)} \). After using the field equations (3.2) the second derivatives are left, but only minimally, as
\[ \tilde{I}^\mu_{(g-red)} = Q^{\alpha \beta \mu \nu} \tilde{I}^\alpha_{\nu} - \frac{1}{2} \tilde{g}^{\alpha \beta} \tilde{I}^\rho_{\rho} \] (5.2)
and
\[ (\sqrt{-g})^2 Q^{\alpha \beta \nu} \equiv \tilde{I}^\mu_{(g-red)} + \tilde{I}^\mu_{(g-red)} - \frac{1}{2} \tilde{I}^\mu_{(g-red)} \] (5.3)
The reduced part with only the first derivatives is

$$\hat{t}^\mu_\nu_{(g-red)} = \frac{1}{4\kappa\sqrt{-g}} \left[ 2\hat{D}_\rho \hat{t}^\mu_\nu_{D\sigma} \hat{i}^\rho\sigma - 2\hat{D}_\alpha \hat{t}^\mu_\nu_{D\beta} \hat{i}^\rho\beta \right. + g^\alpha\beta \left( 2g^\rho\sigma \hat{D}_\rho \hat{t}^\mu_\nu_{D\alpha} \hat{i}^\rho\beta + g^\mu\nu \hat{D}_\sigma \hat{i}^\rho\rho \hat{D}_\mu \hat{i}^\rho\sigma \right) - 4g^\beta\rho g^\alpha\delta \hat{D}_\sigma \hat{i}^\rho\rho \hat{D}_\alpha \hat{i}^\rho\sigma \left. + \frac{1}{4}(2g^\mu\delta \hat{g}^\nu\nu - g^\mu\nu g^\omega\beta)(2g_{\rho\alpha}g_{\sigma\beta} - g_{\rho\beta}g_{\sigma\rho}) \hat{D}_\delta \hat{i}^\rho\rho \hat{D}_\omega \hat{i}^\rho\omega \right]. \tag{5.4}$$

The matter part in (5.2) has appeared due to using the field equations (3.2).

In [22] it was suggested the original way to exclude the second derivatives from the energy-momentum tensor without changing the field equations. The Lagrangian (2.10) was modified as follows

$$\hat{\mathcal{L}}_{(mod)} = \hat{\mathcal{L}} + \hat{\Lambda}^{\alpha\beta\rho\sigma} \mathcal{R}_{\alpha\beta\rho\sigma}. \tag{5.5}$$

This is a typical way of incorporating constraints (because $\mathcal{R}_{\alpha\beta\rho\sigma} = 0$) by means of the undetermined Lagrange multipliers. The multipliers $\hat{\Lambda}^{\alpha\beta\rho\sigma}$ form a tensor which depends on $\hat{\mathcal{L}}$ and $\hat{t}^\mu_\nu$ (without their derivatives) and satisfy $\hat{\Lambda}^{\alpha\beta\rho\sigma} = -\hat{\Lambda}^{\rho\delta\alpha\sigma} = -\hat{\Lambda}^{\alpha\rho\beta\sigma} = \hat{\Lambda}^{\beta\alpha\sigma\rho}$. Thus, the field equations (3.2) do not change. Then, in a correspondence with the modified Lagrangian (5.5), the modified energy-momentum tensor density is

$$\kappa\hat{t}^\mu_\nu_{(mod)} = \kappa\hat{t}^\mu_\nu - \hat{D}_\alpha \left( \hat{\Lambda}^{\mu\nu\alpha\beta} + \hat{\Lambda}^{\nu\mu\alpha\beta} \right) \tag{5.6}$$

instead of (2.16). The originally undetermined multipliers $\hat{\Lambda}^{\mu\nu\alpha\beta}$ will now be determined. They can be chosen in such a way that the remaining second derivatives in (5.2) can now be removed. The unique possibility is $\hat{\Lambda}^{\mu\nu\alpha\beta} = \left( \hat{t}^\mu_\nu \hat{t}^\alpha_\beta - \hat{t}^\alpha_\nu \hat{t}^\mu_\beta \right)/4\sqrt{-g}$. Thus the equations (3.2) are not changed, but they have to be rewritten in the form

$$\hat{G}_{\mu\nu}^{\mu\nu}_{L(mod)} = \hat{G}_{\mu\nu}^L - 2\hat{D}_{\alpha\beta} \hat{\Lambda}^{(\mu\nu)(\alpha\beta)}$$

\[= \frac{1}{\sqrt{-g}} \hat{D}_\alpha \left[ \left( \hat{g}^{\mu\nu} + \hat{t}^\mu_\nu \right) \hat{g}^{\alpha\beta} \hat{t}^{\rho\beta} + \hat{i}^{\rho\beta} \right] - \left( \hat{g}^{\mu\alpha} + \hat{t}^{\mu\alpha} \right) \left( \hat{g}^{\nu\rho} + \hat{t}^{\nu\rho} \right) \]

\[= \kappa \left( \hat{t}^\mu_\nu_{(mod)} + \hat{t}^\mu_\nu_{(mod-tot)} \right) \equiv \kappa \hat{t}^\mu_\nu_{(mod-tot-red)}. \tag{5.7}\]

Here, the r.h.s. defined as a symmetrical (metric) energy-momentum tensor for the system (5.5) is the source for the generalized d’Alembert operator (general wave operator). Thus the l.h.s. in (5.7) is not more linear in $\hat{t}^\mu_\nu$. Because on the flat background the divergence of the l.h.s. in (5.7) is identical equal to zero, then $\hat{D}_\nu \hat{t}^\mu_\nu_{(mod-tot-red)} = 0$. In Eqs. (5.7), $\hat{t}^\mu_\nu_{(mod-tot)}$ can be reduced by the equations of motion, then they are rewritten as

$$\hat{G}_{\mu\nu}^{\mu\nu}_{L(mod)} = \kappa \left[ \hat{t}^\mu_\nu_{(g-red)} + \hat{Q}_{\alpha\beta}^{\mu\nu} (\hat{t}_{\alpha\beta}^m - \frac{1}{2} \hat{g}^{\alpha\beta} \hat{t}^{\rho\rho} + \hat{i}^{\rho\rho}) \right] \equiv \kappa \hat{t}^\mu_\nu_{(mod-tot-red-red)} \tag{5.8}$$

Thus, indeed on the equations of motion the energy-momentum tensor density in (5.8) is only with the first derivatives of gravitational variables. Again $\hat{D}_\nu \hat{t}^\mu_\nu_{(mod-tot-red-red)} = 0$. Let us show that Eq. (5.8) is equivalent to the usual Einstein equations. Multiplying it by $\sqrt{-g}$, and using the identification (2.4), the definition (2.18) for the flat background and the definition (5.3), in the Lorentzian coordinates, one easily gets

$$\frac{1}{2} \partial_{\alpha\beta} \left[ (-g)(g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta}) \right] = \kappa (-g) \left( \hat{t}_{LL}^\mu_\nu + T_{\mu\nu}^\nu \right).$$
After substituting the Einstein equations $\kappa T^{\mu\nu} = G^{\mu\nu}$ this equation transfers to the identity. Thus, indeed Eq. (5.8) is equivalent to the Einstein equations, and one finds that $(-g)\tilde{T}^{\mu\nu}_{LL}$ is the Landau-Lifshitz’s pseudotensor [1]. After all, one concludes that $\tilde{T}^{\mu\nu}_{(g-red)}$ is the covariantized pseudotensor $(-g)T^{\mu\nu}_{LL}/\sqrt{-g}$.

## 6 Gauge invariance properties

Properties of the field formulation of GR under gauge transformations follow from the usual covariant invariance properties of GR in the geometrical description. We demonstrate it briefly (for details see [8, 19]).

Consider the same solution to GR $\hat{g}^{\mu\nu}(x)$ and $\hat{g}^{\mu\nu}(x')$ presented in two different coordinate systems: $\{x\}$ and $\{x'\}$ connected by the coordinate transformation $x' = x(x)$. Now let us do a decomposition of the type (2.4) in both the cases: $\hat{g}^{\mu\nu}(x) = \tilde{g}^{\mu\nu}(x) + \hat{\mu}^{\mu\nu}(x)$ and $\hat{g}^{\mu\nu}(x') = \tilde{g}^{\mu\nu}(x') + \tilde{\mu}^{\mu\nu}(x')$, with the same form of the background metric $\tilde{g}^{\mu\nu}$. For the solution in primed coordinated from the points with coordinate quantities $x'$ one displaces to the points with coordinate quantities $x$. After that one has to compare both solutions. An analogous procedure has to be carried out under the matter variables. Assuming the coordinate transformation in the form:

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha} + \frac{1}{2!}\xi^{\beta}\xi_{,\beta}^{\alpha} + \frac{1}{3!}\xi^{\rho}(\xi^{\beta}\xi_{,\beta})_{,\rho} + \ldots,$$

where $\xi^{\mu}$ are assumed as enough smooth, we obtain the transformation [8, 19]:

$$\hat{\mu}^{\mu\nu} = \tilde{\mu}^{\mu\nu} + \sum_{k=1}^{\infty} \frac{1}{k!} L_{\xi}^{k} (\tilde{g}^{\mu\nu} + \hat{\mu}^{\mu\nu})^{k}, \quad \phi^{\prime A} = \phi^{A} + \sum_{k=1}^{\infty} \frac{1}{k!} L_{\xi}^{k} (\Phi^{A} + \phi^{A})^{k},$$

which are called as gauge (inner) transformations in the field formulation of GR. Indeed, they do not affect both the coordinates and the background quantities.

Firstly let us consider the field formulation of the section 2. It is not difficult to see that the Lagrangian (2.7) is invariant under the transformation (6.1) up to a divergence on the background equations (2.6). The expression (2.20) gives a possibility to understand that equations (2.12) are gauge invariant on themselves. Even on the field equations one has

$$\kappa^{\mu\nu}_{(total)} = \kappa^{\mu\nu}_{(total)} + \hat{G}_{\mu\nu}^{L}(l' - l) + \tilde{\Phi}_{\mu\nu}^{L}(l' - l, \phi' - \phi),$$

and

$$\kappa^{\mu\nu}_{(eff)} = \kappa^{\mu\nu}_{(eff)} + \hat{G}_{\mu\nu}^{L}(l' - l).$$

The transformations (6.1) with $\Phi^{A} = 0$, as it has to be for $\Phi_{\mu\nu\alpha\beta}^{(a)} = 0$, are also gauge transformations for the formulation in section 5. The Babak-Grishchuk Lagrangian is also gauge invariant up to a divergence, and the equations (5.8) are gauge invariant on themselves. Concerning the energy-momentum tensor, on the field equations one has

$$\kappa^{\mu\nu}_{(mod-tot-red)} = \kappa^{\mu\nu}_{(mod-tot-red)} + \hat{G}_{\mu\nu}^{L(mod)}(l' - l).$$

The non-localization problem of energy and other quantities in GR became evident from the moment of constructing GR beginning from the original Einstein’s works. During prolonged time it was illustrated
by a non-covariance of pseudotensors. The use of an auxiliary background spacetime permitted to consider covariant conserved quantities, but the non-localization became to be explained by an ambiguity in a choice of a background. However, on this level there was no suggested an unique mechanism for description of this ambiguity. The use of the gauge transformation properties in the field formulation of GR closes this gap in GR. At the beginning of this section we just express the connection between different choices of backgrounds explicitly. A gauge non-invariance in the energy-momentum tensors (6.2) and (6.3) just expresses the non-localization of energy, momentum, etc. in GR in exact (without approximations) and explicit mathematical expressions. It is a one of advantages in using the field formulation of GR.

7 Gravity with non-zero masses of gravitons

Babak and Grishchuk using their technique [22] have constructed a variant of theory of gravity with non-zero masses of gravitons with interesting properties [23]. Following to [23] independent variables \( \tilde{\ell}_{\mu \nu} = \tilde{\ell}_{\mu \nu} / \sqrt{-\tilde{g}} \) are used. It is natural to assume that the Lagrangian may also include an additional term similar to the one in Eq. (5.5), but where the quantity \( \tilde{\Lambda}_{\alpha \rho \beta \sigma} \) is the curvature tensor of an abstract spacetime with a constant non-zero curvature:

\[
\tilde{\Lambda}_{\alpha \rho \beta \sigma} = K (\tilde{g}_{\alpha \beta} \tilde{g}_{\rho \sigma} - \tilde{g}_{\alpha \rho} \tilde{g}_{\beta \sigma})
\]

where \( K \) is with the dimensionality of \([\text{length}]^{-2}\). If one adds \( \tilde{\Lambda}_{\mu \nu} \tilde{\Lambda}_{\alpha \rho \beta \sigma} \) with \( \tilde{\Lambda}_{\mu \nu} = (4 \sqrt{-\tilde{g}})^{-1} \left( \tilde{\ell}^{\mu \nu} \tilde{\ell}^{\alpha \beta} - \tilde{\ell}^{\alpha \beta} \tilde{\ell}^{\mu \nu} \right) \), changing \( \tilde{g}^{\mu \nu} \rightarrow \tilde{g}^{\mu \nu} \), then the additional term in the Lagrangian (5.5) is \( \frac{1}{2} \sqrt{-\tilde{g}} K \left( l_{\alpha \beta} l_{\alpha \beta} - l_{\alpha} l_{\alpha} \right) \). Clearly, the new theory is not GR, but one recognizes in this term the Fierz-Pauli mass-term [32]. Thus, noting that the structure (5.5) generates mass terms and finding that only two independent quadratic combinations of \( l^{\mu \nu} \) exist, Babak and Gishchuk arrive at a 2-parametric family of theories with the additional mass terms in the gravitational Lagrangian (5.5):

\[
\hat{L}^a_{(mass)} = \hat{L}^a_{(mod)} + \sqrt{-\tilde{g}} \left[ k_1 (l_{\alpha \beta} l_{\alpha \beta}) + k_2 (l_{\alpha} l_{\alpha}) \right],
\]

where \( k_1 \) and \( k_2 \) have a dimensionality of \([\text{length}]^{-2}\).

Of course, the additional term in (7.1) gives a contribution both into the r.h.s. and into the l.h.s. of Eq. (5.8), and the equations of the new gravity theory symbolically could be rewritten as

\[
\hat{G}^\mu_{(mass)} = \kappa l^\mu_{(tot-mass)}.
\]

These equations are, of course, covariant, however, unlike Eq. (3.2) and Eq. (5.8), the new field equations (7.2) are not gauge invariant. There are no transformations, like (6.2). Thus, there is no a problem with a localization of \( \hat{G}^\mu_{(tot-mass)} \) — it is localized!

To have a direct comparison with GR effects it is more convenient to present Eq. (7.2) in the quite equivalent quasi-geometrical form:

\[
G_{\mu \nu} + M_{\mu \nu} = \kappa T_{\mu \nu}
\]

where the massive term is

\[
M_{\mu \nu} \equiv (2 \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} - g^{\alpha \beta} g_{\mu \nu}) \left( k_1 l_{\alpha \beta} + k_2 \tilde{g}_{\alpha \beta} l_{\mu}^{\rho} \right).
\]

Notice that the Bianchi identity \( D_{\nu} G_{\mu}^{\nu} \equiv 0 \) takes a place in effective spacetime. Besides, with taking into account the matter equations (2.3) one has \( D_{\nu} T_{\mu}^{\nu} = 0 \), as usual. Thus, after differentiation of Eq. (7.3)
one obtains \( D^\nu M_{\mu\nu} = 0 \). Although these equations are merely the consequences of the full system (7.3), and therefore contain no new information, it proves convenient to use them instead of some members of the original set (7.3).

To give a physical interpretation of \( k_1 \) and \( k_2 \), following to the analysis by Ogievetsky and Polubarinov [33], and by van Dam and Veltman [34], one considers the linearization of the Eqs. (7.3):

\[
\frac{1}{2} \left( \overline{\mathcal{D}_\mu} \overline{\mathcal{D}^\nu} l_{\mu\nu} + \overline{g}_{\mu\nu} \overline{\mathcal{D}_\rho} \overline{\mathcal{D}^\sigma} l^{\rho\sigma} - \overline{\mathcal{D}_\rho} \overline{\mathcal{D}_\nu} l_\mu^\rho - \overline{\mathcal{D}_\rho} \overline{\mathcal{D}_\mu} l_\nu^\rho \right) + 2k_1 l_{\mu\nu} - (k_1 + 2k_2) \overline{\mathcal{g}_{\mu\nu}} l_\alpha^\alpha = 0. \tag{7.4}
\]

The divergence of this equation is

\[
\overline{\mathcal{D}_\nu} [2k_1 l_{\mu\nu} - (k_1 + 2k_2) \overline{\mathcal{g}^{\mu\nu}} l_\alpha^\alpha] = 0, \tag{7.5}
\]

which is the linearized version of the equation \( D^\nu M_{\mu\nu} = 0 \).

Consider the first case with \( k_1 \neq k_2 \). The full system (7.4) is equivalent to

\[
\Box H^{\mu\nu} + \alpha^2 H^{\mu\nu} = 0, \tag{7.6}
\]
\[
\Box l_\alpha^\alpha + \beta^2 l_\alpha^\alpha = 0, \tag{7.7}
\]

together with Eq. (7.5). Here, \( \Box \equiv \overline{g}^{\alpha\beta} \overline{\mathcal{D}_\alpha} \overline{\mathcal{D}_\beta} \),

\[
H^{\mu\nu} \equiv h^{\mu\nu} - \frac{k_1 + k_2}{3k_1} \overline{g}^{\mu\nu} l_\alpha^\alpha - \frac{k_1}{6k_1^2} \overline{\mathcal{D}_\rho} \overline{\mathcal{D}^{\rho\nu}} l_\alpha^\alpha + \frac{k_1 + k_2}{12k_1^4} \overline{\mathcal{g}^{\mu\nu}} \Box l_\alpha^\alpha \tag{7.8}
\]

with \( \overline{g}_{\mu\nu} H^{\mu\nu} = 0 \) and \( \overline{\mathcal{D}_\nu} H^{\mu\nu} = 0 \). Thus, parameters in the wave-like equations (7.6) and (7.7) are

\[
\alpha^2 = 4k_1, \quad \beta^2 = \frac{-2k_1(k_1 + 4k_2)}{k_1 + k_2}. \tag{7.9}
\]

They can be thought as inverse Compton wavelengths of the spin-2 graviton with the mass \( m_2 = \alpha \hbar/c \) associated with the field \( H^{\mu\nu} \) and of spin-0 graviton with mass \( m_0 = \beta \hbar/c \) associated with the field \( l_\alpha^\alpha \).

With studying the weak gravitational waves in the massive gravity one finds certain modifications of GR. Thus the spin-0 gravitational waves, presented by the trace \( l_\alpha^\alpha = l_\alpha^\beta \eta_{\alpha\beta} \), and the polarization state of the spin-2 graviton presented by the spatial trace \( H^{\alpha\beta} \eta_{\alpha\beta} \) both, unlike GR, become essential. They provide additional contributions to the energy-momentum flux carried by the gravitational wave, and the extra components of motion of the test particles. However, gravitational wave solutions, their energy-momentum characteristics, and observational predictions of GR are fully recovered in the massless limit \( \alpha \to 0, \beta \to 0 \).

For the case with the mass term of Fierz-Pauli type, \( k_1 + k_2 = 0 \), that corresponds \( \beta^2 \to \infty \) (see (7.9)), the full set of equations (7.4) is equivalent to

\[
l_\alpha^\alpha = 0, \quad \Box l^{\mu\nu} + 4k_1 l^{\mu\nu} = 0, \quad \overline{\mathcal{D}_\nu} l^{\mu\nu} = 0.
\]

This case is interpreted as unacceptable [23]. Even in the limit \( \alpha \to 0 \), there remains a nonvanishing “comoving mode” motion of test particles in the plane of the wave front. The extra component of motion is accounted for the corresponding additional flux of energy from the source, typically, of the same order of magnitude as the GR flux. This, at least, is in a conflict with already available indirect gravitational-wave observations of binary pulsars [35]. Such theories probably have to be rejected.
In [23], the full non-linear equations (7.3) were analyzed from the point view of the black hole and the cosmological solutions. Thus, searching for static spherically-symmetric solutions in vacuum it is necessary to consider three independent equations from (7.3), unlike GR where there are two ones only. The consideration is simplified if one assumes $\alpha = \beta$, however all the qualitative conclusions remain valid for $\alpha \neq \beta$. Combining analytical and numerical technique Babak and Grishchuk have demonstrated that the solution of the massive theory is practically indistinguishable from that of GR for all $2M \ll R \ll 1/\alpha$, where $R$ and $M$ are the radial and mass parameters of the Schwarzschild solution. For $R$ larger than $1/\alpha$ the solution takes the form of the Yukawa-type potentials; therefore they call this massive theory as finite-range gravity. The solution of new theory is also deviate strongly from that of GR in the vicinity of $R = 2M$ that is the location of the globally defined event horizon of the Schwarzschild black hole in GR. In the massive gravity the event horizon does not form at all, and the solution smoothly continues to the region $R < 2M$ and terminates at $R = 0$ where the curvature singularity develops. Since the $\alpha M$ can be extremely small, the redshift of the photon emitted at $R = 2M$ can be extremely large, but it remains finite in contrast with GR solutions. Infinite redshift is reached only at the singularity $R = 0$. In the astrophysical sense, all conclusions that rely specifically on the existence of the black hole even horizon, are likely to be abandoned. It is very remarkable and surprising that the phenomena of black hole should be so unstable with respect to the inclusion of the tiny mass-terms, whose Compton wavelength can exceed, say, the present-day Hubble radius.

It was also considered homogeneous isotropic solutions in the framework of the massive gravity. Matter sources were taken in the simplest form of a perfect fluid with a fixed equation of state. There are two independent field equations from the set (7.3), unlike GR where there is only one in the same case. First, if the mass of the spin-0 graviton is zero, $\beta^2 = 0$, the cosmological solutions are exactly the same as those of GR, independently of the mass of the spin-2 graviton, i.e., independently of the value of $\alpha^2$. This result is expected due to the highest spatial symmetry: the spin-2 degrees of freedom have no chance to reveal themselves. Then, for $\beta^2 \neq 0$ it was considered technically more simple case $4\beta^2 = \alpha^2$ which was studied in full details. Qualitative results are valid for $4\beta^2 \neq \alpha^2$. Again, combining analytical approximations and numerical calculations it was demonstrated that the massive solution has a long interval of evolution where it is practically indistinguishable from the Friedmann solution of GR. The deviation from GR are dramatic at very early times and very late times. The unlimited expansion is being replaced by a regular maximum of the scale factor, whereas the singularity is being replaced by a regular minimum of the one. The smaller $\beta$, the higher maximum and the deeper minimum, i.e., the arbitrary small term in the Lagrangian (7.1) gives rise to the oscillatory behaviour of the cosmological scale factor.

Following the logic of interpretation that $\alpha^2$ and $\beta^2$ define the masses, they are thought as positive. However, the general structure of the Lagrangian (7.1) does not imply this. Then, if one allows $\alpha^2$ and $\beta^2$ to be negative, the late time evolution of the scale factor presents an “accelerated expansion” that is similar to the one governed by a positive cosmological $\Lambda$-term. The development of this point could be useful in the light of the modern cosmological observational data [36].
8 Concluding remarks

Due to the non-localization problem/property a definition and a study of conserved quantities in GR are not trivial tasks. Then rather one has not to follow unconditionally to some one unique method. However, to restrict such methods, one has to examine their possibilities to satisfy known natural tests. As a rule, in applications of expressions of such approaches it is required: a) the energy density for the weak gravitational waves has to be positive [1]; b) ratio of mass to angular momentum in the Kerr solution has to be standard [28]; c) one has to obtain the standard conserved quantities both at spatial and at null infinity for asymptotically flat solutions [2]. The usual field formulation (UFF) of GR presented in sections 2 and 3, and the Babak-Grishchuk modification presented in section 5, they both satisfy all these evident requirements, and one cannot do a choice on this basis.

In Introduction it was presented reasons why it is important to study perturbations on arbitrary curved backgrounds, and to present conserved currents as divergences of superpotentials. In particular, just these requirements initiated a development of UFF. Conversely, the opinion of Babak and Grishchuk is that it is enough to use only the background Minkowski space. This has its own convincing foundation. All the modern "direct" experiments, as well as a aforementioned theoretical restrictions a) - c), use as a background a flat spacetime. Besides, the field formulation even can describe arbitrary curved and topologically non-trivial solutions of GR as a field configuration in Minkowski space (see [9, 10]). Paying a necessary opinion to the position of these authors, we note that there are no principal arguments against a following generalization of the modified Babak-Grishchuk formulation of GR onto arbitrary curved backgrounds and constructing conservation laws with the use of superpotentials.

Let us return to the requirement to have only the first derivatives in the energy-momentum tensor, one of the main reasons of which is a correct formulation of the initial problem. However, at least, from this point of view second derivatives, which appear in the framework of UFF do not initiate a criticism. One needs to consider the energy-momentum tensor density $\hat{\mathcal{T}}_{\mu\nu}$ and the conserved current $\hat{I}^\mu$ (see (2.10) and (3.11) with (3.12)). In [21] we have shown that the symmetrized quantities constructed in [20] coincide with the ones presented here and included into the conservation law (3.11). Due to this, and using the dynamic and background equations one concludes that zero's component $\hat{I}^0$ of the conserved current in (3.11), based on $\hat{i}_{\mu\nu}$, contains only the first time derivatives of $\hat{l}_{\mu\nu}$. Therefore $\hat{I}^0$ has the normal behaviour with respect to initial conditions with the definition of the integral quantities analogous to (3.6). Therefore this requirement, at least, may be unnecessarily restrictive.

Babak and Grishchuk give and discuss a wide bibliography of works, where a possibility to consider gravitons with non-zero masses is studied [23]. Here, we note only the Visser paper [37]. It is clear that an inclusion of non-zero masses of gravitons leads to a non-Einstein gravity. Visser, analyzing foundations of GR, find that a more economical way turns out an inclusion of a background metric, and he realizes this possibility. Thus a philosophy of the works [23] and [37] coincide. An advantage of the approach in [23] is that, using the field formulation of GR, it permits to study the problem from more general positions. Thus, the linear Visser’s equations are a particular case of Babak and Grishchuk’s linear equations, when
one sets $\alpha = \beta$. Beginning from the second order, equations in [23] and [37] are different even for $\alpha = \beta$. The reason is that Visser defines perturbations as $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$, whereas Babak and Grishchuk use $\nu_{\mu\nu}$ in correspondence with our above definitions. It is interesting that the same reason initiates differences in definitions of energy-momentum tensors (see section 4). In spite of these differences both of approaches, [23] and [37], qualitatively give the same predictions.

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