Maximal Regularity for Compressible Two-Fluid System

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Abstract. We investigate a compressible two-fluid Navier–Stokes type system with a single velocity field and algebraic closure for the pressure law. The constitutive relation involves densities of both fluids through an implicit function. We are interested in regular solutions in a $L^p - L^q$ maximal regularity setting. We show that such solutions exists locally in time and, under additional smallness assumptions on the initial data, also globally. Our proof rely on appropriate transformation of the original problem, application of Lagrangian coordinates and maximal regularity estimates for associated linear problem.

Keywords. Compressible Navier–Stokes system, Two-fluid model, Local and global regular solutions, Maximal regularity.

1. Introduction

In the present paper, we analyze a bi-fluid compressible system. It arises, in particular, as a result of averaging procedure applied to liquid-gas free boundary problem (see for instance [14]). We assume a common velocity field and pressure for both fluids (the so-called algebraic pressure closure), for justification of this reduction and presentation of related models we refer to [2].

Our system of equations reads:

$$\begin{align*}
\partial_t (\alpha^\pm \varrho^\pm) + \text{div}(\alpha^\pm \varrho^\pm \mathbf{u}) &= 0, \\
\partial_t ((\alpha^+ \varrho^+ + \alpha^- \varrho^-)\mathbf{u}) + \text{div}((\alpha^+ \varrho^+ + \alpha^- \varrho^-)\mathbf{u} \otimes \mathbf{u}) - \text{div} \mathbf{S}(\mathbf{u}) + \nabla p &= 0, \\
\alpha^+ + \alpha^- &= 1, \\
p &= p^+ = p^-.
\end{align*}$$

By $p^+$, $p^-$ we denote the internal barotropic pressures for each fluid with the explicit forms:

$$p^\pm = (\varrho^\pm)^{\gamma^\pm},$$

where $\gamma^\pm > 1$ are given constants, and we assume without loss of generality that

$$\gamma^+ \geq \gamma^-.$$  

The stress tensor obeys the Newton rheological law

$$\mathbf{S}(\mathbf{u}) = 2\mu \mathbf{D}(\mathbf{u}) + \nu \text{div} \mathbf{u} \mathbf{I},$$

where $\mathbf{D}(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ and $\mu$ and $\nu$ are the nonnegative viscosity coefficients.

We consider the system (1.1) on a domain $\Omega \subset \mathbb{R}^3$ supplied with the Dirichlet boundary condition for the velocity

$$\mathbf{u}|_{\partial \Omega} = 0 \quad \text{for} \quad t \in (0, T),$$

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and with the initial conditions
\[ \alpha^+|_{t=0} = R_0, \quad \alpha^-|_{t=0} = Q_0, \quad u|_{t=0} = u_0. \]  
(1.6)
We assume that the total initial density is separated away from zero, i.e.
\[ R_0 \geq 0, \quad Q_0 \geq 0, \quad R_0 + Q_0 \geq \kappa \text{ for some } \kappa > 0 \]  
(1.7)
and that the initial data
\[ \alpha^\pm|_{t=0} = \alpha_0^\pm, \quad \varrho^\pm|_{t=0} = \varrho_0^\pm \] satisfy the following compatibility conditions
\[ \alpha_0^+ + \alpha_0^- = 1, \quad \alpha_0^+ \geq 0, \quad p^+(\varrho_0^+) = p^-(\varrho_0^-). \]  
(1.8)
We immediately switch to a reformulation of the system (1.1). Introducing the notation
\[ R = \varrho + \alpha, \quad Q = \varrho - \alpha, \quad Z = \varrho, \quad \alpha = \alpha^+, \] we check that the pressure \( p \) is expressed in terms of \( R, Q \). For this purpose we observe that, by (1.1d) and (1.2),
\[ (\varrho^- \alpha^-)^\gamma^- = (\alpha^-)^\gamma^- (\varrho^+)^\gamma^+ = \left[ \frac{\varrho^+(1 - \alpha^-)}{\varrho^+} \right]^\gamma^- (\varrho^+)^\gamma^+. \]
Therefore we have
\[ p = P(R, Q) = Z^{\gamma^+}, \]  
(1.9)
for \( Z = Z(R, Q) \) such that
\[ Q = \left( 1 - \frac{R}{Z} \right) Z^{\gamma^-}, \quad \text{with} \quad \gamma = \frac{\gamma^+}{\gamma^-}, \]  
(1.10)
and
\[ R \leq Z. \]  
(1.11)
Note that by (1.11) we immediately deduce that \( 0 \leq \alpha \leq 1 \).

The system (1.1)–(1.5) can be therefore transformed to the following form
\[ \partial_t R + \text{div}(R u) = 0, \]  
(1.12a)
\[ \partial_t Q + \text{div}(Q u) = 0, \]  
(1.12b)
\[ \partial_t ((R + Q) u) + \text{div}((R + Q) u \otimes u) - \text{div} S(u) + \nabla Z^{\gamma^+} = 0, \]  
(1.12c)
\[ Z = Z(R, Q), \quad R|_{t=0} = R_0, \quad Q|_{t=0} = Q_0, \quad u|_{t=0} = u_0, \]  
(1.12d)
\[ u|_{\partial \Omega} = 0, \]  
(1.12f)
where \( Z \) is related to \( R \) and \( Q \) through the non-explicit formula (1.10).

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(1.12c)
\[ Z = Z(R, Q), \quad R|_{t=0} = R_0, \quad Q|_{t=0} = Q_0, \quad u|_{t=0} = u_0, \]  
(1.12d)
\[ u|_{\partial \Omega} = 0, \]  
(1.12f)
where \( Z \) is related to \( R \) and \( Q \) through the non-explicit formula (1.10).

Note that by (1.1d), (1.2) and (1.9) the triplet \((R, Q, Z(R, Q))\) uniquely determines \((\varrho^+, \varrho^-, \alpha^+, \alpha^-)\).

Therefore it is legitimate to consider the system (1.12) instead of (1.1).

1.1. Notation

We use standard notation \( L_p(\Omega) \) and \( W^k_p(\Omega) \) for Lebesgue and Sobolev spaces, respectively. Furthermore by \( L_p(I; X) \), where \( I \subset \mathbb{R}_+ \) and \( X \) is a Banach space, we denote a Bochner space.

Next, we recall that for \( 0 < s < \infty \) and \( m \) a smallest integer larger than \( s \) we define Besov spaces as intermediate spaces
\[ B^s_{q,p}(\Omega) = (L_q(\Omega), W^m_q(\Omega))_{s/m,p}, \]  
(1.13)
where \((\cdot, \cdot)_{s/m,p}\) is the real interpolation functor, see [1, Chapter 7]. In particular,
\[ B^{2(1-1/p)}_{q,p}(\Omega) = (L_q(\Omega), W^2_q(\Omega))_{1-1/p,p} = (W^2_q(\Omega), L_q(\Omega))_{1/p,p}. \]  
(1.14)
We shall not distinguish between notation of spaces for scalar and vector valued functions, i.e. we write $L_q(\Omega)$ instead of $L_q(\Omega)^3$ etc. However, we write vector valued functions in boldface.

We also introduce a brief notation for the regularity class of the solution. Namely, for a triple $(g,h,f)$ we define a norm
\[
\| (g,h,f) \|_{X(T)} = \| f \|_{L_p(0,T;W^2_q(\Omega))} + \| f_r \|_{L_p(0,T;L_q(\Omega))} + \| g,h \|_{W^1_p(0,T;W^1_q(\Omega))}
\] (1.15)
and a seminorm
\[
\| (g,h,f) \|_{\dot{X}(T)} = \| f \|_{L_p(0,T;W^3_q(\Omega))} + \| f_r \|_{L_p(0,T;L_q(\Omega))} + \| \nabla g, \nabla h \|_{L_p(0,T;L_q(\Omega))}
\] + \| \partial_t g, \partial_t h \|_{L_p(0,T;w^3_q(\Omega))}.
\] (1.16)

Obviously, we denote by $X(T)$ a space of functions for which the norm (1.15) is finite.

Next, for $0 < \varepsilon < \pi$ and $\lambda_0 > 0$ we introduce
\[
\Sigma_\varepsilon = \{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| \leq \pi - \varepsilon \},\quad \Sigma_{\varepsilon,\lambda_0} = \{ \lambda \in \Sigma_\varepsilon : |\lambda| \geq \lambda_0 \},
\] \[
\mathbb{C}_+ := \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq 0 \}.
\] (1.17)

Finally, by $E(\cdot)$ we shall denote a non-negative non-decreasing continuous function such that $E(0) = 0$. We use the notation $E(T)$ in particular to denote these constants in various inequalities, which can be made arbitrarily small by taking $T$ sufficiently small.

### 1.2. Main Results

The first main result of this paper gives local in time existence and uniqueness for problem (1.12):

**Theorem 1.1.** Assume that $\Omega$ is a uniform $C^2$ domain and $2 < p < \infty$, $3 < q < \infty$, $\frac{2}{p} + \frac{3}{q} < 1$. Assume moreover that $R_0, Q_0$ satisfy (1.7) and $u_0$ satisfies the compatibility condition
\[
u_0|_{\partial \Omega} = 0.
\] (1.18)

Then for any $L > 0$ there exists $T > 0$ such that if
\[
\| \nabla R_0 \|_{L_q(\Omega)} + \| \nabla Q_0 \|_{L_q(\Omega)} + \| u_0 \|_{B^{2-q/2}_p(\Omega)} \leq L
\] (1.19)
then (1.12) admits a unique solution $(R, Q, u)$ on $(0, T)$ with the estimate
\[
\| (R, Q, u) \|_{X(T)} \leq CL,
\] \[
\int_0^T \| \nabla u \|_{L^\infty(\Omega)} \leq \delta,
\] (1.20)
where $\| \cdot \|_{X(T)}$ is defined in (1.15) and $\delta$ is a small positive constant.

Next we show global well-posedness for (1.12) assuming additionally that $\Omega$ is bounded and initial data is close to some constants:

**Theorem 1.2.** Assume that $\Omega$ is a bounded $C^2$ domain, $p, q, R_0, Q_0$ satisfy the assumptions of Theorem 1.1 and let $R_s, Q_s$ be any positive constants. Assume moreover that $u_0$ satisfies the compatibility condition (1.18). Then there exists $\varepsilon > 0$ s.t. if
\[
\| R_0 - R_s \|_{W^3_q(\Omega)} + \| Q_0 - Q_s \|_{W^3_q(\Omega)} + \| u_0 \|_{B^{2-q/2}_p(\Omega)} \leq \varepsilon,
\] (1.21)
then the solution to (1.12) exists globally in time and satisfies the following decay estimate:
\[
\| e^{\beta t} (R, Q, u) \|_{\dot{X}(+\infty)} \leq C_\varepsilon,
\] \[
\int_0^{+\infty} \| \nabla u \|_{L^\infty(\Omega)} \leq \delta,
\] (1.22)
where $\beta$ is a positive constant, $\delta$ is a small positive constant, and $\| \cdot \|_{\dot{X}}$ is defined in (1.16).
1.3. Discussion

The existence of global weak solutions to (1.1) in a space-periodic setting was established by Bresch et al. in [3], however with additional capillarity effects. In a one dimensional case the existence of global weak solutions without capillarity was shown in [4], where, moreover, it is shown that after a finite time a vacuum is excluded and at least component corresponding to one phase becomes regular. More recently, the global finite energy weak solutions to a two-fluid Stokes system has been shown by Bresch et al. [5]. Soon after, similar result for full two-fluid compressible Navier–Stokes equations with single velocity was proven by Novotný and Pokorný in [23], and by Jin and Novotný [15] for the so-called Baer-Nunziato type of system.

The existence of regular solutions for a simplified version of (1.1a) where influence of one of the phases in the convective term is neglected has been shown in [11]. Existence of global strong solutions in $L_2$ framework in the whole space in $\mathbb{R}^3$ under certain smallness assumptions has been shown by Evje et al. [8]. In the one dimensional case, more analytical and numerical results can be found in [9,10]. Existence, uniqueness and stability of global weak solutions, still in one dimension, to system (1.1), has been shown by Li et al. [19] and some non-uniqueness results for the inviscid version of this system in multiple dimensions was shown by Li and Zatorska in [20]. Finally, let us also mention that very recently Wang, Wen and Yao proved global existence and uniqueness of solutions to the non-conservative two-fluid model with unequal velocities using the energy methods [31].

In order to prove Theorem 1.1 we show a maximal regularity estimate for associated linear problem. For this purpose we apply the theory based on the concept of $\mathcal{R}$-boundedness. The underlying result for this approach is the famous Weis’ vector-valued Fourier multiplier theorem proved in [32], which we recall in the Appendix. In brief, this result allows to deduce maximal regularity for a time-dependent problem from $\mathcal{R}$-boundedness of a family of solution operators to associated resolvent problem. In [6] this approach was applied to prove maximal regularity for the heat equation using explicit solution formula. Further development of the theory in the context of equations of fluid mechanics was mostly due to Y. Shibata and his collaborators. In [29] the authors show a maximal regularity for the Stokes problem with Neumann boundary condition. For this purpose they prove several technical results enabling to show $\mathcal{R}$ boundedness for the resolvent problem. These results have been applied and extended in [7] to treat the compressible Navier–Stokes system with Dirichlet boundary condition. The result has been extended to slip boundary conditions in [21,22]. For further developments we can mention [27,28] for free boundary problems, [18] for a 2-phase compressible-incompressible flow or [12] for a compressible fluid-rigid body interaction problem. In a series of papers of Piasecki et al. [24–26] the above described approach was applied to treat a system describing multi-component compressible mixtures. Here we rely on the ideas we developed in these papers to treat quite complex nonlinearities resulting from Lagrangian transformation and linearization.

2. Lagrangian Coordinates

In order to define the Lagrangian transformation we start with a following simple observation:

Lemma 2.1. Let $p$ and $q$ satisfy the assumptions of Theorem 1.1. Then

(i) if $\|f\|_{L_p(0,T;W^2_q(\Omega))} \leq M$ for some $M > 0$, then

$$\int_0^T \|\nabla f(t, \cdot)\|_{L_\infty(\Omega)} dt \leq M E(T),$$

(ii) if $\|e^{\gamma t} f\|_{L_p(0,\infty;W^2_q(\Omega))} \leq M$ for some $M, \gamma > 0$ then

$$\int_0^\infty \|\nabla f(t, \cdot)\|_{L_\infty(\Omega)} dt \leq CM.$$
Proof. By the imbedding theorem and Hölder inequality we have
\[
\int_0^T \|\nabla f(t, \cdot)\|_{L^\infty(\Omega)} dt \leq C \int_0^T \|f(t, \cdot)\|_{W^2_q(\Omega)} dt \leq T^{1/p'} \int_0^T \left(\|f(t, \cdot)\|_{W^2_q(\Omega)}^{1/p} dt \leq ME(T),
\]
which proves the first assertion, and for the second we have
\[
\int_0^\infty \|\nabla f(t, \cdot)\|_{L^\infty(\Omega)} dt \leq C \int_0^\infty e^{-\gamma t} e^{\gamma t} \|f(t, \cdot)\|_{W^2_q(\Omega)} dt
\]
\[
\leq \left(\int_0^\infty e^{-\gamma t} dt \right)^{1/p'} \|e^{\gamma t} f\|_{L^p(0,\infty;W^2_q(\Omega))} \leq CM.
\]
\]

Now we can proceed with the transformation. Let \( v(y, t) \) be the velocity field in the Lagrange coordinates, so that the following change of variables is true:
\[
x = y + \int_0^t v(y, s) ds.
\]
Then for any differentiable function \( f \) we have
\[
\partial_t f(t, \phi(t, y)) = \partial_t f + v \cdot \nabla_x f.
\]
Moreover, we have
\[
\frac{\partial x_i}{\partial y_j} = \delta_{ij} + \int_0^t \frac{\partial v_i}{\partial y_j}(y, s) ds,
\]
where \( \delta_{ij} \) are Kronecker’s delta symbols. If \( v \) is the solution satisfying the regularity from Theorem 1.1, then by Lemma 2.1 we can assume
\[
\sup_{t \in (0, T)} \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty(\Omega)} ds \leq \delta
\]
with some small positive constant \( \delta \). Therefore by (2.5) the \( N \times N \) matrix \( \partial x/\partial y = (\partial x_i/\partial y_j) \) has the inverse
\[
\left(\frac{\partial x_i}{\partial y_j}\right)^{-1} = I + V^0(k_v)
\]
where
\[
k_v = \int_0^t \nabla v(y, s) ds,
\]
\( I \) is the \( 3 \times 3 \) identity matrix, and \( V^0(k) \) is the \( 3 \times 3 \) matrix of smooth functions with respect to \( k \in \mathbb{R}^{3 \times 3} \) defined on \( |k| < \delta \) with \( V^0(0) = 0 \), where \( k \) are independent variables corresponding to \( k_v \).

We have
\[
\nabla x = (I + V^0(k_v)) \nabla_y, \quad \frac{\partial}{\partial x_i} = \sum_{j=1}^N (\delta_{ij} + V^0_{ij}(k_v)) \frac{\partial}{\partial y_j}.
\]
Notice that adding (1.12a) to (1.12b) and using this equation to compute \( \partial_t (R+Q) \) we can rewrite (1.12c) as
\[
(R+Q) \partial_t u + (R+Q) u \cdot \nabla u - \text{div} S(u) + \nabla Z^+ = 0.
\]
Moreover, the map: \( x = \Phi(y, t) \) is bijection from \( \Omega \) onto \( \Omega \), and so setting
\[
v(y, t) = u(x, t), \quad r(y, t) = R(x, t), \quad q(y, t) = Q(x, t), \quad z(y, t) = Z(x, t),
\]
we can transform (1.12) to the following form
\[
\begin{align*}
\partial_t r + r \text{div} v &= O_1(U) \tag{1.2.1a} \\
\partial_t q + q \text{div} v &= O_2(U) \tag{1.2.1b} \\
(r + q) \partial_t v - \mu \Delta v - v \nabla \text{div} v + \gamma^+ (\gamma^{+1} \partial_{\dot{y}} \nabla r + \gamma^{+1} \partial_{\dot{y}} v) &= O_3(U), \tag{1.2.1c}
\end{align*}
\]
where
\[ U = (v, \tau, q). \]  
(2.13)

For consistency with notation (2.11) let us denote the initial data for the problem in Lagrangian coordinates as
\[ \tau_0(y) = R_0(y), \quad q_0(y) = Q_0(y), \quad v_0(y) = u_0(0). \]

We derive explicit form of the terms \( \partial_3 \), \( \partial_q \). Recall that \( \tau \leq 3 \). Let us now apply \( \partial_q \), \( \partial_3 \) to both sides of (2.12d), we obtain respectively:
\[ 1 = \gamma_3^{-1} \partial_3 q_3 - \tau (\gamma - 1) \gamma^{-2} \partial_q q_3, \]
\[ 0 = \gamma_3^{-1} \partial_3 q_3 - \gamma^{-1} - \tau (\gamma - 1) \gamma^{-2} \partial_q q_3, \]
therefore,
\[ \partial_q q_3 = \frac{1}{\gamma_3^{-1} - (\gamma - 1) \gamma^{-2}}, \quad \partial_3 q_3 = \frac{\gamma^{-1}}{\gamma_3^{-1} - (\gamma - 1) \gamma^{-2}}, \]
(2.14)
and so
\[ 3^{\gamma^{-1}} \partial_3 q_3 = \frac{3^{\gamma^+}}{\gamma_3^{-1} - (\gamma - 1) \gamma^{-1}}, \quad 3^{\gamma^{-1}} \partial_q q_3 = \frac{3^{\gamma^+}}{\gamma_3^{-1} - (\gamma - 1) \tau}. \]
(2.15)
Due to (1.3) and (1.11) we have
\[ \frac{1}{\gamma_3^{-1} - 1} \leq \partial_3 q_3 \leq \frac{1}{\gamma_3^{-1}}, \quad \gamma^{-1} \leq \partial_q q_3 \leq 1 \]
(2.16)
and
\[ \gamma^{-1} \gamma^{\gamma^{-1}} \leq 3^{\gamma^+} \partial_3 q_3 \leq 3^{\gamma^{-1}}, \quad \gamma^{-1} \gamma^{\gamma^{-1}} \leq 3^{\gamma^+} \partial_q q_3 \leq 3^{\gamma^+}. \]
(2.17)
We have
\[ \text{div}_x = \text{div}_y + \sum_{i,j=1}^{3} V_{ij}^0(k_v) \frac{\partial v_i}{\partial y_j}, \]
(2.18)
therefore by (2.4),(2.9) and (2.11), we obtain (2.12a) with
\[ O_1(U) = -\tau \sum_{i,j=1}^{3} V_{ij}^0(k_v) \frac{\partial v_i}{\partial y_j}. \]
(2.19)
Similarly,
\[ O_2(U) = -q \sum_{i,j=1}^{3} V_{ij}^0(k_v) \frac{\partial v_i}{\partial y_j}. \]
(2.20)
To find \( O_3(U) \) we need to transform the second order operators. By (2.9), we have
\[ \Delta u = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left( \frac{\partial u}{\partial x_k} \right) = \sum_{k,\ell,m=1}^{3} \left( \delta_{k \ell} + V_{k \ell}^0(k_v) \right) \frac{\partial}{\partial y_\ell} \left( (\delta_{km} + V_{km}^0(k_v)) \frac{\partial v}{\partial y_m} \right), \]
and so setting
\[ A_2 \Delta(k) \nabla^2 v = 2 \sum_{\ell,m=1}^{3} V_{k \ell}^0(k) \frac{\partial^2 v}{\partial y_\ell \partial y_m} + \sum_{k,\ell,m=1}^{3} V_{k \ell}^0(k)V_{km}^0(k) \frac{\partial^2 v}{\partial y_\ell \partial y_m}, \]
\[ A_1 \Delta(k) \nabla v = \sum_{\ell,m=1}^{3} (\nabla k V_{\ell m}^0)(k) \int_0^s (\partial t \nabla v) ds \frac{\partial v}{\partial y_m} + \sum_{k,\ell,m=1}^{3} V_{k \ell}^0(k)(\nabla k V_{km}^0)(k) \int_0^s \partial t \nabla v ds \frac{\partial v}{\partial y_m}, \]
we have
\[ \Delta u = \Delta v + A_{2\Delta}(k_v) \nabla^2 v + A_{1\Delta}(k_v) \nabla v. \]
Moreover, by (2.9), we have
\[ \frac{\partial}{\partial x_j} \text{div } u = \sum_{k=1}^3 (\delta_{jk} + V_{jk}^0(k_v)) \frac{\partial}{\partial y_k} \left( \text{div } v + \sum_{\ell,m=1}^3 V_{\ell m}^0(k_v) \frac{\partial u_\ell}{\partial y_m} \right), \]
therefore setting
\[ A_{2\text{div},j}(k) \nabla^2 v = \sum_{\ell,m=1}^3 V_{\ell m}^0(k) \frac{\partial^2 v_\ell}{\partial y_m \partial y_j} + \sum_{k=1}^3 V_{jk}^0(k) \frac{\partial}{\partial y_k} \text{div } v + \sum_{k,\ell=1}^3 V_{jk}^0(k) V_{\ell m}^0(k) \frac{\partial^2 v_\ell}{\partial y_k \partial y_m}, \]
\[ A_{1\text{div},j}(k) \nabla v = \sum_{\ell,m=1}^3 (\nabla_k V_{\ell m}^0)(k) \int_0^t \partial_j \nabla v \, ds \frac{\partial v_\ell}{\partial y_m} + \sum_{k,\ell,m=1}^3 V_{jk}^0(k) (\nabla_k V_{\ell m}^0)(k) \int_0^t \partial_k \nabla v \, ds \frac{\partial v_\ell}{\partial y_m}, \]
we obtain
\[ \frac{\partial}{\partial x_j} \text{div } u = \frac{\partial}{\partial y_j} \text{div } v + A_{2\text{div},j}(k_v) \nabla^2 v + A_{1\text{div},j}(k_v) \nabla v. \]
Since
\[ Z^{\gamma^+-1} \partial_R Z \nabla R + Z^{\gamma^+-1} \partial_Q Z \nabla Q = 3^{\gamma^+-1} \partial_3 (\nabla \tau + V^0(k_v) \nabla v) + 3^{\gamma^+-1} \partial_3 (\nabla q + V^0(k_v) \nabla q), \]
we have
\[ O_3(U) = \mu A_{2\Delta}(k_v) \nabla^2 v + \mu A_{1\Delta}(k_v) \nabla v + \nu A_{2\text{div}}(k_v) \nabla^2 v + \nu A_{1\text{div}}(k_v) \nabla v - 3^{\gamma^+-1} \partial_3 V^0(k_v) \nabla \tau - 3^{\gamma^+-1} \partial_3 V^0(k_v) \nabla q. \] (2.21)

3. Local Well-Posedness

3.1. Linearization Around the Initial Condition

We introduce new unknowns
\[ \sigma = \tau - r_0, \quad \eta = q - q_0. \] (3.1)
Denoting
\[ V = (v, \sigma, \eta) = U - (0, r_0, q_0) \] (3.2)
we obtain the following system with time-independent coefficients
\[ \partial_t \sigma + r_0 \text{div } v = f_1(V), \]
\[ \partial_t \eta + q_0 \text{div } v = f_2(V), \]
\[ (r_0 + q_0) \partial_t v - \mu \Delta v - \nu \nabla \text{div } v + \omega_1^0 \nabla \sigma + \omega_2^0 \nabla \eta = f_3(V), \]
\[ (\sigma, \eta, v)|_{t=0} = (0, 0, v_0), \quad v|_{\partial \Omega} = 0, \] (3.3)
where \( \delta_0 = \delta_0(t_0, q_0) \) is defined by
\[ q_0 = \left(1 - \frac{r_0}{\delta_0}\right) \delta_0 \]
and
\[ \omega_1^0 = \frac{\gamma^+ \delta_0^+}{\gamma \delta_0 - (\gamma - 1) r_0}, \quad \omega_2^0 = \frac{\gamma^+ \delta_0^+}{\gamma \delta_0 - (\gamma - 1) r_0 \delta_0^+}. \] (3.4)
The right hand side of (3.3) is given by
\[ f_1(V) = O_1(V + (0, t_0, q_0)) - \sigma \text{ div } v, \]  
\[ f_2(V) = O_2(V + (0, t_0, q_0)) - \eta \text{ div } v, \]  
\[ f_3(V) = O_3(V + (0, t_0, q_0)) - (\sigma + \eta) \partial_t v \]
\[ = \gamma^+ \gamma [\gamma_0 \gamma^{-1} (\gamma - 1)^{-1} + \gamma_0 (\gamma - 1)^{-1}] \nabla \sigma \]
\[ - \frac{\gamma^+ \gamma [\gamma_0 \gamma^{-1} (\gamma - 1)^{-1} + \gamma_0 (\gamma - 1)^{-1}]}{\gamma_3 \gamma - (\gamma - 1) \gamma_0 \gamma_0^{-1}} \nabla \eta \]
\[ - \frac{\gamma^+ \gamma [\gamma_0 \gamma^{-1} (\gamma - 1)^{-1} + \gamma_0 (\gamma - 1)^{-1}]}{\gamma_3 \gamma - (\gamma - 1) \gamma_0 \gamma_0^{-1}} \nabla q_0. \]  

3.2. Maximal Regularity

In this section we show the maximal regularity for the linear problem corresponding to (3.3), i.e.:
\[ \partial_t \sigma + t_0 \text{ div } v = f_1, \]
\[ \partial_t \eta + q_0 \text{ div } v = f_2, \]
\[ (t_0 + q_0) \partial_t v - \mu \Delta v - \nu \text{ div } v + \omega^0_1 \nabla \sigma + \omega^0_2 \nabla \eta = f_3, \]
\[ (\sigma, \eta, v)|_{t=0} = (\sigma_0, \eta_0, v_0), \quad v|_{\partial \Omega} = 0. \]

The main result of this section reads

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a uniform \( C^2 \) domain. Let \( 1 < p, q < \infty, \frac{2}{p} + \frac{1}{q} \neq 1 \). Assume that there exists constants \( a_1, a_2, \kappa > 0 \) and \( r > 3 \) such that
\[ 0 \leq t_0, q_0, \omega^0_1, \omega^0_2 \leq a_1, \quad t_0 + q_0 \geq \kappa, \]
\[ \| \nabla t_0, \nabla q_0, \nabla \omega^0_1, \nabla \omega^0_2 \|_{L_1(\Omega)} \leq a_2. \]  

Assume moreover that
\[ f_1, f_2 \in L_p(0, T, W^1_q(\Omega)), \quad f_3 \in L_p(0, T, L_q(\Omega)), \quad \sigma_0, \eta_0 \in W^1_q(\Omega), \quad v_0 \in B^{2-1/p}_q(\Omega). \]

Finally, for \( \frac{2}{p} + \frac{1}{q} < 2 \) assume that \( v_0 \) satisfy the compatibility condition (1.18). Then there exists positive constants \( b, C \) such that for any \( T > 0 \) problem (3.8) admits a unique solution \( (\sigma, \eta, v) \in X(T) \) with the estimate
\[ \| (\sigma, \eta, v) \|_{X(T)} \leq C e^{b T} \| v_0 \|_{B^{2-1/p}_q(\Omega)} + \| \sigma_0, \eta_0 \|_{W^1_q(\Omega)} \]
\[ + \| f_3 \|_{L_p(0, T; L_q(\Omega))} + \| f_1, f_2 \|_{L_p(0, T; W^1_q(\Omega))}. \]  

In order to prove Theorem 3.1 we prove \( R \)-boundedness for associated resolvent problem which is obtained applying Laplace transform to (3.8):
\[ \lambda \sigma + t_0 \text{ div } v = f_1, \]  

(3.11a)
In the Appendix we recall how \( R \)-boundedness for the resolvent problem implies maximal regularity due to Weis vector-valued Fourier Multiplier Theorem. Therefore in order to prove Theorem 3.1 it is sufficient to show the following result:

**Theorem 3.2.** Let \( 1 < q < \infty \) and \( 0 < \epsilon < \pi/2 \). Assume that \( \Omega, \tau_0, q_0, \omega_1^0, \omega_2^0 \) satisfy the assumptions of Theorem 3.1. Then, there exist a positive constant \( \lambda_0 \) and operator families \( \mathcal{A}_1(\lambda), \mathcal{A}_2(\lambda) \) in \( \text{Hol}(\Lambda_{\epsilon, \lambda_0}, \mathcal{L}(W^1_q(\Omega)^2 \times L_q(\Omega), W^1_q(\Omega))) \) and \( \mathcal{B}(\lambda) \in \text{Hol}(\Lambda_{\epsilon, \lambda_0}, \mathcal{L}(W^1_q(\Omega)^2 \times L_q(\Omega), W^2_q(\Omega)^N)) \), such that for any \( (f_1, f_2, f_3) \in (W^1_q(\Omega))^2 \times L_q(\Omega) \) and \( \lambda \in \Sigma_{\epsilon, \lambda_0} \), \( (\sigma_\lambda = \mathcal{A}_1(\lambda)(f_1, f_2, f_3), \eta_\lambda = \mathcal{A}_2(\lambda)(f_1, f_2, f_3), u_\lambda = \mathcal{B}(\lambda)(f_1, f_2, f_3)) \) is a unique solution of (3.11) and

\[
\begin{align*}
\mathcal{R}_{\mathcal{L}(W^1_q(\Omega)^2 \times L_q(\Omega), W^1_q(\Omega))}((\tau \partial_z)^{\ell} \mathcal{A}_1(\lambda) : \lambda \in \Sigma_{\epsilon, \lambda_0}) \leq M, \\
\mathcal{R}_{\mathcal{L}(W^1_q(\Omega)^2 \times L_q(\Omega), W^1_q(\Omega))}((\tau \partial_z)^{\ell} \mathcal{A}_2(\lambda) : \lambda \in \Sigma_{\epsilon, \lambda_0}) \leq M, \\
\mathcal{R}_{\mathcal{L}(L_q(\Omega) \otimes W^1_q(\Omega), W^{2-j}_{q-j}(\Omega)^N)}((\tau \partial_z)^{\ell/2} \mathcal{B}(\lambda) : \lambda \in \Sigma_{\epsilon, \lambda_0}) \leq M,
\end{align*}
\]

for \( \ell = 0, 1, 2 \) and some constant \( M > 0 \), where by \( \text{Hol} \) we denote the space of holomorphic operators.

**Remark 1.** We say that the operator valued function \( T(\lambda) \) is holomorphic if it is differentiable in norm for all \( \lambda \) in a complex domain. For more details, see [16, Chaper VII,§1.1].

Following the idea introduced already in [7] in context of the compressible Navier–Stokes equations, in order to prove Theorem 3.2 we transform (3.11) using the fact that continuity equation becomes an algebraic equation in the resolvent problem. Computing \( \eta_\lambda \) and \( \sigma_\lambda \) from (3.11a)–(3.11b) we obtain

\[
\sigma_\lambda = \lambda^{-1} (f_1 - \tau_0 \text{div} v), \quad \eta_\lambda = \lambda^{-1} (f_2 - q_0 \text{div} v).
\]

Plugging these identities into (3.11c) we obtain

\[
(t_0 + q_0) \lambda \nabla v - \nu \Delta v \lambda - [\nu + \lambda^{-1}(\omega_1^0 t_0 + \omega_2^0 q_0)] \nabla v = f_3, \quad (\tau_0 + q_0) \lambda \nabla v + \mu \Delta v \lambda - [\nu + \lambda^{-1}(\omega_1^0 t_0 + \omega_2^0 q_0)] \nabla v = f.
\]

The result is

**Lemma 3.3.** Let \( 0 < \epsilon < \pi/2 \). Assume that \( \Omega \) is a uniform \( C^2 \) domain in \( \mathbb{R}^N \). Then, there exists a positive constant \( \lambda_0 \) such that there exists an operator family \( \mathcal{C}(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(L_q(\Omega), W^2_q(\Omega))) \) such that for any \( \lambda \in \Sigma_{\epsilon, \lambda_0} \) and \( f \in L_q(\Omega), v = \mathcal{C}(\lambda) f \) is a unique solution of (3.13), and

\[
\mathcal{R}_{\mathcal{L}(L_q(\Omega), W^2-q(\Omega))}((\tau \partial_z)^{\ell} \mathcal{C}(\lambda) : \lambda \in \Sigma_{\epsilon, \lambda_0}) \leq M
\]

for \( \ell = 0, 1, 2 \) and some constant \( M > 0 \).

**Proof.** An analog of Proposition 3.3 has been shown in [7, Theorem 2.10] for a problem

\[
\begin{align*}
\lambda u - \mu \Delta u - (\nu + \gamma^2 \lambda^{-1}) \nabla q \text{div} q u = f, \quad u_0 |_{\partial \Omega} = 0
\end{align*}
\]

where \( \mu, \nu \) and \( \gamma \) are constants satisfying \( \mu + \nu > 0 \) and \( \gamma > 0 \). The proof requires only minor modifications in order to prove Proposition 3.3, therefore we present only a sketch. First we solve a problem with constant coefficients in the whole space

\[
(t_0^* + q_0^*) \lambda \nabla v - \nu \Delta v \lambda - [\nu + \lambda^{-1}(\omega_1^0 t_0 + \omega_2^0 q_0^*)] \nabla v = f \in \mathbb{R}^n, \quad (3.15)
\]
where $t_0^*, q_0^*, \omega_1^0, \omega_2^0$ are constants satisfying

$$r_0^*, q_0^*, \omega_1^0, \omega_2^0 \geq 0, \quad r_0^* + q_0^* > 0.$$  

$\mathcal{R}$-boundedness for (3.15) can be shown following the proof Theorem 3.1 in [7]. The latter is shown for problem (3.14) in the whole space. In order to adapt it to (3.15) we can divide (3.15) by $r_0^* + q_0^*$ using the fact that this constant is strictly positive. We obtain

$$\lambda v - \mu^* \Delta v - [\nu^* + \lambda^{-1} \omega^*] \nabla \text{div} v = f \in \mathbb{R}^n,$$

where

$$\mu^* = \frac{\mu}{r_0^* + q_0^*}, \quad \nu^* = \frac{\nu}{r_0^* + q_0^*}, \quad \omega^* = \frac{\omega_1^0 t_0^* + \omega_2^0 q_0^*}{r_0^* + q_0^*}.$$

Therefore the only difference is that now we have $\omega^* \geq 0$ instead of $\gamma^2$. However, strict positivity of this constant neither its square structure is not necessary, nonnegativity is sufficient.

Next we consider (3.15) in a half-space supplied with the boundary condition $v|_{\partial \mathbb{R}_+^n} = 0$. Here we can follow the proof of Theorem 4.1 in [7] which works without modifications for $\omega^* \geq 0$.

The third step consists in showing $\mathcal{R}$-boundedness in a bent half-space, the necessary result is Theorem 5.1 in [7], where replacing $\lambda^{-1} \gamma^2 > 0$ with $\omega^* \geq 0$ is again harmless.

The final step is application of partition of unity and properties of a uniform $C^2$ domain. Here we have to deal with variable coefficients which is not in the scope of Theorem 2.10 in [7]. However, we can refer to a more recent result, Theorem 4.1 in [24] which gives $\mathcal{R}$-boundedness for a resolvent problem corresponding to more complicated system describing flow of a two-component mixture. It is sufficient to follow Sect. 6.3 of [24] with some obvious modifications. At this stage we need continuity of the coefficients $t_0, q_0, \omega_1^0, \omega_2^0$ which is assured by (3.9).

$$\square$$

### 3.3. Preliminary Estimates

We start with recalling two embedding results for Besov spaces. The first one is [1, Theorem 7.34 (c)]:

**Lemma 3.4.** Assume $\Omega \subset \mathbb{R}^n$ satisfies the cone condition and let $1 \leq p, q \leq \infty$ and $sq > n$. Then

$$B_{q,p}^s(\Omega) \subset C_B(\Omega),$$

where $C_B$ we denote the space of continuous bounded functions.

In particular $u \in B_{q,p}^{2-2/p}(\Omega)$ implies $\nabla u \in B_{q,p}^{1-2/p}(\Omega)$. Therefore Lemma 3.4 with $s = 1 - 2/p$ and $n = 3$ yields

**Corollary 3.5.** Assume $\frac{2}{p} + \frac{3}{q} < 1$ and let $\Omega$ satisfy the assumptions of Theorem 1.1. Then $B_{q,p}^{2-2/p}(\Omega) \subset W^1_{\infty}(\Omega)$ and

$$\|f\|_{W^1_{\infty}(\Omega)} \leq C\|f\|_{B_{q,p}^{2-2/p}(\Omega)}, \quad (3.16)$$

The next result is due to Tanabe (cf. [30, p.10]):
Lemma 3.6. Let $X$ and $Y$ be two Banach spaces such that $X$ is a dense subset of $Y$ and $X \subset Y$ is continuous. Then for each $p \in (1, \infty)$
\[ W^1_p((0, \infty), Y) \cap L_p((0, \infty), X) \subset C([0, \infty), (X, Y)_{1/p, p}) \]
and for every $u \in W^1_p((0, \infty), Y) \cap L_p((0, \infty), X)$ we have
\[ \sup_{t \in (0, \infty)} \|u(t)\|_{(X, Y)_{1/p, p}} \leq (\|u\|_{L_p((0, \infty), X)} + \|u\|_{W^1_p((0, \infty), Y)})^{1/p}. \]

This result allows to show the following imbedding.

Lemma 3.7. Assume $p, q$ satisfy the assumptions of Theorem 1.1. Let $f_1 \in L_p(0, T; L_q(\Omega))$, $f_0 \in L_p(0, T; W^2_q(\Omega))$, $f(0, \cdot) \in B_{q, p}^{2-2/p}(\Omega)$. Then
\[ \sup_{t \in (0, T)} \|f\|_{B_{q, p}^{2(1-1/p)}(\Omega)} \leq C(\|f_1\|_{L_p(0, T; L_q(\Omega))} + \|f\|_{L_p(0, T; W^2_q(\Omega))} + \|f(0)\|_{B_{q, p}^{2-2/p}(\Omega)}), \]
(3.17)
\[ \sup_{t \in (0, T)} \|f\|_{W^2_q(\Omega)} \leq C(\|f_1\|_{L_p(0, T; L_q(\Omega))} + \|f\|_{L_p(0, T; W^2_q(\Omega))} + \|f(0)\|_{B_{q, p}^{2-2/p}(\Omega)}), \]
(3.18)
where $C$ does not depend on $T$.

Proof. In order to prove (3.17) we introduce an extension operator
\[ e_T[f](\cdot, t) = \begin{cases} f(\cdot, t) & t \in (0, T), \\ f(\cdot, 2T - t) & t \in (T, 2T), \\ 0 & t \in (2T, +\infty), \end{cases} \]
(3.19)
If $f|_{t=0} = 0$, then we have
\[ \partial_t e_T[f](\cdot, t) = \begin{cases} 0 & t \in (2T, +\infty), \\ (\partial_t f)(\cdot, t) & t \in (0, T), \\ -(\partial_t f)(\cdot, 2T - t) & t \in (T, 2T) \end{cases}, \]
(3.20)
in a weak sense. Therefore obviously
\[ \|e_T f\|_{L_p(\mathbb{R}_+; W^2_q(\Omega))} + \|e_T f\|_{W^1_p(\mathbb{R}_+; L_q(\Omega))} \leq 2(\|f\|_{L_p(0, T; W^2_q(\Omega))} + \|f\|_{W^1_p(0, T; L_q(\Omega))}). \]
(3.21)
Now we construct an extension $f^0$ of the initial data such that
\[ \|f^0\|_{L_p(0, T; W^2_q(\Omega))} + \sup_{t \in \mathbb{R}_+} \|f^0\|_{B_{q, p}^{2-2/p}(\Omega)} \leq C(\|f(0)\|_{B_{q, p}^{2-2/p}(\Omega)}), \]
(3.22)
and
\[ f^0|_{\Omega \times \{t=0\}} = f(0, \cdot). \]
(3.23)
For this purpose we can for instance extend $f(0, \cdot)$ to $g^0 \in B_{q, p}^{2-2/p}(W)$ such that $\Omega \subset W$ and
\[ g^0|_{\partial V} = 0, \quad \|g^0\|_{B_{q, p}^{2-2/p}(W)} \leq C(\Omega, V, \|f(0, \cdot)\|_{B_{q, p}^{2-2/p}(\Omega)}). \]
(3.24)
Then we can find $f^0$ as a solution to the heat equation
\[ \partial_t f^0 - \Delta f^0 = 0 \quad \text{in} \quad W \times (0, T), \]
\[ f^0|_{\partial V} = 0, \quad f^0|_{t=0} = g^0. \]
(3.25)
Then the standard $L_p$-regularity theory of the heat equation yields
\[ \|f^0\|_{L_p(0, T; W^2_q(W))} + \sup_{t \in \mathbb{R}_+} \|f^0\|_{B_{q, p}^{2-2/p}(W)} \leq C\|g^0\|_{B_{q, p}^{2-2/p}(W)}, \]
which together with (3.23) imply (3.22).

Applying Lemma 3.6 with $X = W^2_q(\Omega)$, $Y = L_q(\Omega)$ and using (3.19) and (3.20) we have
By (3.22) we have
\[
\sup_{t \in (0,T)} \|f(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq \sup_{t \in (0,\infty)} \|E_t[f - f^0]\|_{B^{2(1-1/p)}_{q,p}(\Omega)}
\]
\[
\leq \left( \|E_t[f - f^0]\|_{L_p((0,\infty),W^2_q(\Omega))}^p + \|E_t[f - f^0]\|_{W^2_{\infty}((0,\infty),L_q(\Omega))}^p \right)^{1/p}
\]
\[
\leq C \left( \|f - f^0\|_{L_p(0,\infty;W^2_q(\Omega))} + \|\partial_t f\|_{L_p(0,T;L_q(\Omega))} + \|f(0)\|_{B^{2-2/p}_{q,p}(\Omega)} \right).
\]
This gives (3.17), which together with Corollary 3.5 implies (3.18).

A direct conclusion from Lemma 3.7 is

**Lemma 3.8.** Assume $p, q$ satisfy the assumptions of Theorem 1.1. Assume moreover that
\[
\|(z_1, z_2, w)\|_{X(T)} \leq M, \quad \|w(0)\|_{B^{2-2/p}_{q,p}(\Omega)} \leq L, \quad z_1(0) = z_2(0) = 0
\]
for some positive constants $M, L$. Then
\[
\|V^0(k'y), \nabla k'y V^0(k'y)\|_{L_{\infty}((0,T) \times \Omega)} \leq C(M, L)E(T),
\]
(3.25)
\[
\sup_{t \in (0,T)} \|z_i(\cdot,t)\|_{W^1_q(\Omega)} \leq C(M)E(T), \quad i = 1, 2,
\]
(3.26)
\[
\sup_{t \in (0,T)} \|w(\cdot,t) - w(0)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq C(M, L),
\]
(3.27)
\[
\|w\|_{L_{\infty}(0,T;W^1_{\infty}(\Omega))} \leq C(M, L),
\]
(3.28)
where $k'y$ is defined in (2.8).

**Proof.** (3.25) follows immediately from Lemma 2.1. Next, we have
\[
\|z(\cdot,t)\|_{W^1_q(\Omega)} \leq \int_0^t \|\partial_t z(\cdot,s)\|_{W^1_q(\Omega)} ds \leq T^{1/p'} \|\partial_t z\|_{L_p((0,T),W^1_q(\Omega))} \leq C(M)E(T),
\]
which implies (3.26). Finally, (3.27) and (3.28) follow from (3.17) and (3.18), respectively.

### 3.4. Estimate of the Right Hand Side of (3.3)

Before going to the essence of this section let is observe the following fact

**Lemma 3.9.** Let \( \mathcal{z} = \mathcal{z}(r, q) \) be as in (2.11) and satisfy (1.10) and (1.11), then the following is true:
\[
\mathcal{r} + q > 0 \implies \mathcal{z} > 0,
\]
(3.29)
\[
\mathcal{r} + q \geq \kappa > 0 \implies \mathcal{z} \geq \min \left\{ \frac{\kappa}{2}, \left( \frac{\kappa}{2} \right)^{1/\gamma} \right\},
\]
(3.30)
\[
\mathcal{z} \leq \max \{ (2q)^{1/\gamma}, 2r \}.
\]
(3.31)

**Proof.** If $q > 0$ then $\mathcal{z} > 0$ by (1.10), while if $r > 0$ then $\mathcal{z} > 0$ by (1.11), which proves (3.29).

If $\mathcal{r} + q \geq \kappa$ then $\mathcal{r} \geq \frac{\kappa}{2}$ or $q \geq \frac{\kappa}{2}$. In the first case we have $\mathcal{z} \geq \frac{\kappa}{2}$ by (1.11). In the second case we have by (1.10)
\[
\mathcal{z} = \left(1 - \frac{r}{\mathcal{z}}\right)^{-1} q \geq q \geq \frac{\kappa}{2},
\]
therefore we have (3.30).

In order to show (3.31) we consider separately the cases $r \geq \frac{\kappa}{2}$ and $r < \frac{\kappa}{2}$. In the first case $\mathcal{z} \geq 2r$. In the second we have $\left(1 - \frac{r}{\mathcal{z}}\right)^{-1} < 2$, which by (1.10) implies $\mathcal{z}^\gamma < 2q$. This completes the proof. \(\square\)
We are now ready to prove the following estimate for the RHS of (3.3):

**Proposition 3.10.** Assume $p, q$ satisfy the assumptions of Theorem 1.1. Let $ar{V} = (\bar{\sigma}, \bar{\eta}, \bar{v})$, where $\bar{\sigma} = \bar{r} - r_0$, $\bar{\eta} = \bar{q} - q_0$ satisfy

$$\|\bar{V}\|_{X(T)} \leq M, \quad \bar{v}|_{t=0} = v_0, \quad \bar{\sigma} + \bar{r}_0 + \bar{\eta} + q_0 \geq \delta,$$

(3.32)

for some constants $M, \delta > 0$. Then

$$\|f_1(\bar{V}), f_2(\bar{V})\|_{L_p(0,T;W^1_2(\Omega))} \leq C(M, L)E(T),$$

(3.33a)

$$\|f_3(\bar{V})\|_{L_p(0,T;L_2(\Omega))} \leq C(M, L)E(T),$$

(3.33b)

where $L$ is the constant from (1.19).

**Proof.** By (2.19) and (3.5) we have using summation convention:

$$f_1(\bar{V}) = -(\bar{\sigma} + r_0)V_{ij}^0(k_\nu)\frac{\partial v_i}{\partial y_j} - \sigma \text{div } v,$$

(3.34)

therefore by Lemma 3.8 we immediately get

$$\|f_1(\bar{V})\|_{L_p(0,T;L_2(\Omega))} \leq C(M, L)E(T).$$

(3.35)

Differentiating (3.34) we obtain

$$\nabla f_1(\bar{V}) = -\nabla(\bar{\sigma} + r_0)V_{ij}^0(k_\nu)\frac{\partial v_i}{\partial y_j} - (\bar{\sigma} + r_0)\left[\nabla k_\nu V_{ij}^0(k_\nu)\frac{\partial v_i}{\partial y_j} + V_{ij}^0(k_\nu)\frac{\partial^2 v_i}{\partial y_j \partial y_k}\right]$$

$$\quad - \text{div } \nabla \bar{\sigma} - \bar{\sigma} \text{div } v.$$

By Lemma 3.8 we see that in each component one term is bounded in $L_p(0,T;L_2(\Omega))$ while all remaining terms are small in $L_\infty((0,T) \times \Omega)$, therefore

$$\|\nabla f_1(\bar{V})\|_{L_p(0,T;L_2(\Omega))} \leq C(M, L)E(T).$$

(3.36)

As $f_2(\bar{V})$ has the same structure we obtain (3.33a). By (7) we have

$$f_3(\bar{V}) = O_3(\bar{V} + (0, r_0, q_0)) - (\bar{\sigma} + \bar{\eta})\partial_1 v - I_1(\bar{\tau}, \bar{q}, \bar{z}) - I_2(\bar{\tau}, \bar{q}, \bar{z}) - I_3(\bar{\tau}, \bar{q}, \bar{z}) - I_4(\bar{\tau}, \bar{q}, \bar{z}),$$

(3.37)

where $\bar{z}$ is defined by $\bar{q} = \left(1 - \frac{\bar{\tau}}{\bar{r}}\right)\bar{z}$. Recalling (2.21), by Lemma 3.8 we have

$$\|O_3(\bar{V} + (0, r_0, q_0)), (\bar{\sigma} + \bar{\eta})\partial_1 v\|_{L_p(0,T;L_2(\Omega))} \leq C(M, L)E(T).$$

(3.38)

In order to estimate the remaining parts observe first that all the denominators are bounded from below by positive powers of $\bar{z}$ and $\bar{z}_0$. More precisely,

$$\gamma \bar{z}_0 - (\gamma - 1)r_0 \geq \bar{z}_0, \quad \gamma \bar{z} - (\gamma - 1)\bar{r} \geq \bar{z},$$

$$\gamma \bar{z}_0 - (\gamma - 1)r_0 \bar{z}_0^{-1} \geq \bar{z}_0^{-1}, \quad \gamma \bar{z}^{-1} - (\gamma - 1)\bar{r} \bar{z}^{-1} \geq \bar{z}^{-1}.$$  

(3.39)

We have

$$\|\bar{z}(t) - \bar{z}_0\|_{L_\infty(\Omega)} \leq \int_0^t \|\partial_3 \bar{z}(s)\|_{L_\infty(\Omega)} ds = \int_0^t \|\partial_3 \bar{z}_0\|_{L_\infty(\Omega)} ds$$

$$\leq C(\|\bar{z}_0, \bar{z}_0, \bar{z}_0^{-1}, \bar{z}_0^{-1}\|_{L_\infty(\Omega)})^{1/\nu'} \|\bar{V}\|_{X(T)} \leq C(M, \|\bar{z}_0, \bar{z}_0, \bar{z}_0^{-1}, \bar{z}_0^{-1}\|_{L_\infty(\Omega)})E(T).$$

Similarly we get for any $\beta \in \mathbb{R}$

$$\|\bar{z}^\beta(t) - \bar{z}_0^\beta\|_{L_\infty(\Omega)} \leq \int_0^t \|\bar{z}^{\beta-1}\partial_3 \bar{z}(s)\|_{L_\infty(\Omega)} ds \leq C(M, \|\bar{z}_0, \bar{z}_0, \bar{z}_0^{-1}, \bar{z}_0^{-1}\|_{L_\infty(\Omega)})E(T).$$

(3.41)

Now we can estimate $I_1$-$I_4$ defined in (3.7). By (3.26),(3.39) and (3.40) we have

$$\|I_1(\bar{\tau}, \bar{q}, \bar{z})(t)\|_{L_p(\Omega)} \leq C(\|\bar{z}_0^\gamma + \bar{z}_0^{-\gamma} - \bar{z}_0, \bar{r} - r_0)\|\nabla \bar{\sigma}\|_{L_p(0,T;L_2(\Omega))},$$

$$\|I_4(\bar{\tau}, \bar{q}, \bar{z})(t)\|_{L_p(\Omega)} \leq C(M, \|\bar{z}_0, \bar{z}_0, \bar{z}_0^{-1}, \bar{z}_0^{-1}\|_{L_\infty(\Omega)})E(T).$$

(3.42)
Similarly, using also (3.41) with $\beta \in \{\gamma, \gamma - 1, \gamma^+\}$ we obtain
\[ ||I_2(\bar{r}, \bar{q}, \bar{j}), I_3(\bar{r}, \bar{q}, \bar{j}), I_4(\bar{r}, \bar{q}, \bar{j})||_{L^p(0,T;L^q(\Omega))} \leq C(M, ||\delta_0^{-1}, \delta^{-1}, \delta_3^{-1}, \delta_0, \delta||_{L^\infty(\Omega \times (0,T))} E(T)), \tag{3.43} \]
where smallness in time in the estimate for $I_4$ results from the fact that $\nabla q_0$ and $\nabla r_0$ are time independent.

Now it is enough to observe that by Lemma 3.9 and (3.32) we have
\[ ||\delta||_{L^\infty(\Omega \times (0,T))} \leq C(||\bar{r}_0 + t_0, \bar{q}_0||_{L^\infty(\Omega \times (0,T))}) = C(M, L), \]
\[ ||\delta_0||_{L^\infty(\Omega \times (0,T))} \leq C(||t_0, q_0||_{L^\infty(\Omega \times (0,T))}) = C(M, L), \]
\[ ||\delta_0^{-1}, \delta^{-1}||_{L^\infty(\Omega \times (0,T))} \leq C(M, L). \tag{3.44} \]
Combining these bounds with (3.38), (3.42) and (3.43) we obtain (3.33b).

\[ \square \]

3.5. Contraction Argument—Proof of Theorem 1.1

Let us define a solution operator
\[ (\sigma, \eta, v) = S(\bar{\sigma}, \bar{\eta}, \bar{v}) \]
which gives the assertion.

By (3.10) and (3.33) we have
\[ \text{Proof.} \]
\[ \text{so by (3.4) and (3.39) we see that} \]
\[ ||\nabla \omega^0||_{L^q(\Omega)} \leq C ||\nabla \delta_0||_{L^q(\Omega)} \leq C ||\nabla \omega^0, \nabla q_0||_{L^q(\Omega)}; \]

Therefore, by Theorem 3.1 and Proposition 3.10, $S$ is well defined on $X(T)$. First we show that for sufficiently small times $S$ maps certain bounded set in $X(T)$ into itself. Due to definition of $S(\cdot)$ we can restrict ourselves to functions satisfying the initial conditions of (3.3), this will be necessary for proving contractivity as we want to take advantage of smallness of time. Therefore we define
\[ B_M = \{ (\bar{\sigma}, \bar{\eta}, \bar{v}) \in X(T) : ||(\bar{\sigma}, \bar{\eta}, \bar{v})||_{X(T)} \leq M, (\bar{\sigma}, \bar{\eta}, \bar{v})_{|t=0} = (0, 0, v_0) \}. \tag{3.45} \]

**Lemma 3.11.** For any $L > 0$ there exists sufficiently large $M > 0$ and sufficiently small $T > 0$ such that $S(B_M) \subset B_M$, where $B_M$ is defined in (3.45).

**Proof.** By (3.10) and (3.33) we have
\[ ||(\bar{\sigma}, \bar{\eta}, \bar{v})||_{X(T)} \leq M \implies ||S(\bar{\sigma}, \bar{\eta}, \bar{v})||_{X(T)} \leq C[L + C(M, L)E(T)], \]
which gives the assertion. \[ \square \]

Now we show that $S$ is a contraction on $B_M$. For this purpose set
\[ V_i = (\sigma_i, \eta_i, v_i) = S(V_i), \quad V_i = (\bar{\sigma}_i, \bar{\eta}_i, \bar{v}_i), \quad i = 1, 2, \]
and define
\[ \bar{\sigma}_i = t_0 + \sigma_i, \quad \bar{q}_i = q_0 + \eta_i, \quad \bar{q}_i = \left(1 - \frac{\bar{r}_i}{\delta_0}\right) \frac{i}{\delta_0^i}, \quad i = 1, 2. \]

Then the difference $(\delta \sigma, \delta \eta, \delta v) = V_1 - V_2$ satisfies
\[ \partial_t \delta \sigma + t_0 \text{ div } \delta v = f_1(\bar{V}_1) - f_1(\bar{V}_2), \]
\[ \partial_t \delta \eta + q_0 \text{ div } \delta v = f_2(\bar{V}_1) - f_2(\bar{V}_2), \]
\[ (t_0 + q_0) \partial_t \delta v - \mu \Delta \delta v - \nu \nabla \delta v = f_3(\bar{V}_1) - f_3(\bar{V}_2), \]
\[ (\delta \sigma, \delta \eta, \delta v)|_{t=0} = 0, \quad \delta v|_{\partial \Omega} = 0. \tag{3.46} \]
By (3.10), in order to prove that $S$ is a contraction on $B_M$ it is enough to show

**Lemma 3.12.** Assume $\tilde{V}_1, \tilde{V}_2 \subset B_M$. Then

\[
\|f_1(\tilde{V}_1) - f_1(\tilde{V}_2), f_2(\tilde{V}_1) - f_2(\tilde{V}_2)\|_{L_p(0,T;W^s_q(\Omega))} \leq C(M, L)E(T)\|\tilde{V}_1 - \tilde{V}_2\|_{X(T)} \tag{3.47a}
\]

\[
\|f_3(\tilde{V}_1) - f_3(\tilde{V}_2)\|_{L_p(0,T;L_q(\Omega))} \leq C(M, L)E(T)\|\tilde{V}_1 - \tilde{V}_2\|_{X(T)}. \tag{3.47b}
\]

**Proof.** (3.47a) can be shown analogously to (3.33a). I order to show (3.47b) we can follow the proof of (3.33b). We directly get

\[
\|O_3(\tilde{V}_1) - O_3(\tilde{V}_2), (\tilde{\sigma}_1 + \tilde{\eta}_1)\partial_t \tilde{v}_1 - (\tilde{\sigma}_2 + \tilde{\eta}_2)\partial_t \tilde{v}_2\|_{L_p(0,T;L_q(\Omega))} \leq C(M, L)E(T)\|\tilde{V}_1 - \tilde{V}_2\|_{X(T)}.
\]

With the remaining terms we shall go into some details due to more complicated structure. Let us focus on the estimate for $I_1(\tilde{r}_1, \tilde{q}_1, \tilde{z}_1) - I_1(\tilde{r}_2, \tilde{q}_2, \tilde{z}_2)$ (recall the definition (3.7)). Let us denote

\[
H_0 = \frac{1}{\gamma_30 - (\gamma - 1)\tilde{r}_0}, \quad H_i = \frac{1}{\gamma_3i - (\gamma - 1)\tilde{r}_i}, \quad i = 1, 2.
\]

Then we have

\[
I_1(\tilde{r}_1, \tilde{q}_1, \tilde{z}_1) - I_1(\tilde{r}_2, \tilde{q}_2, \tilde{z}_2) = \delta I^1_1 + \delta I^2_1 + \delta I^3_1 + \delta I^4_1,
\]

where

\[
\delta I^1_1 = \gamma_30H_0 \left[ (\tilde{\sigma}_1 + \tilde{\eta}_1)\partial_t \tilde{v}_1 - (\tilde{\sigma}_2 + \tilde{\eta}_2)\partial_t \tilde{v}_2 \right] - (\tilde{\sigma}_1 - \tilde{\sigma}_2) + (\tilde{\eta}_1 - \tilde{\eta}_2)H_1 - H_2
\]

\[
\delta I^2_1 = \gamma_30^+H_0 \left[ (\tilde{\sigma}_1 + \tilde{\eta}_1)\partial_t \tilde{v}_1 - (\tilde{\sigma}_2 + \tilde{\eta}_2)\partial_t \tilde{v}_2 \right] - (\tilde{\sigma}_1 - \tilde{\sigma}_2) + (\tilde{\eta}_1 - \tilde{\eta}_2)H_1 - H_2
\]

\[
\delta I^3_1 = (\gamma - 1)\tilde{r}_0H_0 \left[ (\tilde{\sigma}_1 + \tilde{\eta}_1)\partial_t \tilde{v}_1 - (\tilde{\sigma}_2 + \tilde{\eta}_2)\partial_t \tilde{v}_2 \right] - (\tilde{\sigma}_1 - \tilde{\sigma}_2) + (\tilde{\eta}_1 - \tilde{\eta}_2)H_1 - H_2
\]

\[
\delta I^4_1 = (\gamma - 1)\tilde{r}_0H_0 \left[ (\tilde{\sigma}_1 + \tilde{\eta}_1)\partial_t \tilde{v}_1 - (\tilde{\sigma}_2 + \tilde{\eta}_2)\partial_t \tilde{v}_2 \right] - (\tilde{\sigma}_1 - \tilde{\sigma}_2) + (\tilde{\eta}_1 - \tilde{\eta}_2)H_1 - H_2
\]

Let us estimate $\delta I^1_1$, it has three components. First observe that estimates from the previous section imply

\[
\|\tilde{\sigma}_0, H_0, H_1, H_2\|_{L_{\infty}(\Omega \times (0,T))} \leq C(M, L). \tag{3.49}
\]

By (3.41) we have

\[
\|\tilde{\sigma}_1 - \tilde{\sigma}_2\|_{L_{\infty}(\Omega \times (0,T))} \leq C(M, L)E(T),
\]

therefore

\[
\|\tilde{\sigma}_1^+ - \tilde{\sigma}_0^+\|_{L_{\infty}(\Omega \times (0,T))} \leq C(M, L)E(T)\|\tilde{V}_1 - \tilde{V}_2\|_{X(T)}. \tag{3.50}
\]

For the second term we have

\[
\|(\tilde{\sigma}_1^+ - \tilde{\sigma}_0^+)(H_1 - H_2)\|_{L_p(0,T;L_q(\Omega))} \leq C(M, L)E(T)\|\tilde{V}_1 - \tilde{V}_2\|_{X(T)}. \tag{3.51}
\]

We have

\[
H_1 - H_2 = \frac{\gamma(\tilde{\sigma}_2 - \tilde{\sigma}_1) - (\gamma - 1)(\tilde{v}_2 - \tilde{v}_1)}{[\gamma\tilde{\sigma}_1 - (\gamma - 1)\tilde{r}_1][\gamma\tilde{\sigma}_2 - (\gamma - 1)\tilde{r}_2]}.
\]

Therefore, by (3.45) and the estimates from the previous section,

\[
\|H_1 - H_2\|_{L_{\infty}(\Omega \times (0,T))} \leq C(M)\|\tilde{\sigma}_1 - \tilde{\sigma}_2, \tilde{v}_1 - \tilde{v}_2\|_{L_{\infty}(\Omega \times (0,T))} \leq C(M, L)E(T)\|\tilde{V}_1 - \tilde{V}_2\|_{X(T)},
\]

which together with (3.51) implies

\[
\|(\tilde{\sigma}_1^+ - \tilde{\sigma}_0^+)(H_1 - H_2)\|_{L_p(0,T;L_q(\Omega))} \leq C(M, L)E(T)\|\tilde{V}_1 - \tilde{V}_2\|_{X(T)}. \tag{3.52}
\]
And finally the third term. By (3.45) we have \((\hat{3}_1 - \hat{3}_2)|\epsilon = 0 = 0\), therefore
\[
\| (\hat{3}_1^+ - \hat{3}_2^+) H_2 \nabla \hat{\sigma}_2 \|_{L_p(0,T;L_q(\Omega))} \leq C \| \hat{\sigma}_1^+ - \hat{\sigma}_2^+ \|_{L^\infty(\Omega \times (0,T))} \| \nabla \hat{\sigma}_2 \|_{L_p(0,T;L_q(\Omega))}
\]
\[
\leq C(M) \int_0^t \| \partial_t (\hat{3}_1^+ - \hat{3}_2^+) (s) \|_{L^\infty(\Omega)} ds
\]
\[
\leq C(M) \int_0^t \left[ \| (\hat{3}_1^{\gamma+1} - \hat{3}_2^{\gamma+1}) \partial_t \hat{\sigma}_2 \|_{L^\infty(\Omega)} + \| \hat{\sigma}_1 - \hat{\sigma}_2 \|_{L^\infty(\Omega \times (0,T))} \right] ds
\]
\[
\leq C(M) \| \hat{\sigma}_1 - \hat{\sigma}_2, \bar{\eta}_1 - \bar{\eta}_2 \|_{L^\infty(\Omega \times (0,T))} (E(T) + C(M) E(T) \| \hat{V}_1 - \hat{V}_2 \|_{X(T)}).
\]

But \((\hat{\sigma}_1 - \hat{\sigma}_2, \bar{\eta}_1 - \bar{\eta}_2)|\epsilon = 0 = 0\), therefore
\[
\| \sigma_1 - \sigma_2, \eta_1 - \eta_2 \|_{L^\infty(\Omega \times (0,T))} \leq E(T) \| \hat{V}_1 - \hat{V}_2 \|_{X(T)},
\]
so altogether
\[
\| (\hat{3}_1^+ - \hat{3}_2^+) H_2 \nabla \hat{\sigma}_2 \|_{L_p(0,T;L_q(\Omega))} \leq C(M) E(T) \| \hat{V}_1 - \hat{V}_2 \|_{X(T)}.
\] (3.53)

Combining (3.49), (3.50), (3.52) and (3.53) we obtain
\[
\| \delta I_1 \|_{L_p(0,T;L_q(\Omega))} \leq C(M) E(T) \| \hat{V}_1 - \hat{V}_2 \|_{X(T)}.
\] (3.54)

The terms \(\delta I_2 - \delta I_4\) have similar structure to \(\delta I_1\) therefore we estimate them in the same way obtaining altogether
\[
\| \delta I_1 \|_{L_p(0,T;L_q(\Omega))} \leq C(M) E(T) \| \hat{V}_1 - \hat{V}_2 \|_{X(T)}.
\]

Now, if we would like to write precise form of the terms \(I_2(\bar{\tau}_1, \bar{\bar{q}}_1, \bar{\bar{q}}_1), I_2(\bar{\tau}_2, \bar{\bar{q}}_2, \bar{\bar{q}}_2), I_3(\bar{\tau}_1, \bar{\bar{q}}_1, \bar{\bar{q}}_1) - I_3(\bar{\tau}_2, \bar{\bar{q}}_2, \bar{\bar{q}}_2)\) and \(I_4(\bar{\tau}_1, \bar{\bar{q}}_1, \bar{\bar{q}}_1) - I_4(\bar{\tau}_2, \bar{\bar{q}}_2, \bar{\bar{q}}_2)\) it would be convenient to define
\[
H_{0,\gamma} = \frac{1}{\gamma \bar{\bar{q}}_0 - (\gamma - 1) \bar{\bar{q}}_0 \bar{\bar{q}}_0^{-1} - 1}, \quad H_{i,\gamma} = \frac{1}{\gamma \bar{\bar{q}}_i - (\gamma - 1) \bar{\bar{q}}_i \bar{\bar{q}}_i^{-1} - 1}, \quad i = 1, 2.
\]

Then we obtain expressions with structure similar to (3.48) with functions \(H_{0,\gamma}, H_{1,\gamma}, H_{2,\gamma}\) instead of \(H_0, H_1, H_2\). Now it is enough to observe that estimates from the previous section imply
\[
\| H_{0,\gamma}, H_{1,\gamma}, H_{2,\gamma} \|_{L^\infty(\Omega \times (0,T))} \leq C(M, L)
\] (3.55)
and
\[
\| H_{1,\gamma} - H_{2,\gamma} \|_{L^\infty(\Omega \times (0,T))} \leq C(M) E(T) \| \hat{V}_1 - \hat{V}_2 \|_{X(T)},
\]

since
\[
H_{1,\gamma} - H_{2,\gamma} = \frac{\gamma(\bar{\bar{q}}_1 - \bar{\bar{q}}_1) - (\gamma - 1)(\bar{\bar{q}}_2 - \bar{\bar{q}}_1)\bar{\bar{q}}_1^{-1} + \bar{\bar{q}}_1(\bar{\bar{q}}_2^{-1} - \bar{\bar{q}}_1^{-1})}{[\gamma \bar{\bar{q}}_1 - (\gamma - 1) \bar{\bar{q}}_1 \bar{\bar{q}}_1^{-1} - 1][\gamma \bar{\bar{q}}_2 - (\gamma - 1) \bar{\bar{q}}_2 \bar{\bar{q}}_2^{-1} - 1]}.
\]

Therefore we prefer not to bother the reader with repeating estimates similar to those leading to (3.54) and notice only that finally we obtain
\[
\| I_2(\bar{\tau}_1, \bar{\bar{q}}_1, \bar{\bar{q}}_1) - I_1(\bar{\tau}_2, \bar{\bar{q}}_2, \bar{\bar{q}}_2), I_3(\bar{\tau}_1, \bar{\bar{q}}_1, \bar{\bar{q}}_1) - I_3(\bar{\tau}_2, \bar{\bar{q}}_2, \bar{\bar{q}}_2), I_4(\bar{\tau}_1, \bar{\bar{q}}_1, \bar{\bar{q}}_1) - I_4(\bar{\tau}_2, \bar{\bar{q}}_2, \bar{\bar{q}}_2) \|_{L_p(0,T;L_q(\Omega))}
\]
\[
\leq C(M, L) E(T) \| \hat{V}_1 - \hat{V}_2 \|_{X(T)},
\]
so altogether we have (3.47b).

Now, applying Theorem 3.1 and Lemma 3.12 to (3.46) we obtain
\[
\| \hat{V}_1 - \hat{V}_2 \|_{X(T)} \leq C(M, L) E(T) \| \hat{V}_1 - \hat{V}_2 \|_{X(T)},
\]
and Theorem 1.1 follows from the Banach contraction principle.
4. Global Well-Posedness

4.1. Linearization Around the Constant State

In order to prove Theorem 1.2 we linearize (2.12) around the positive constants $R_*, Q_*$ to which the initial condition is assumed to be close. For convenience let us denote

$$ q_* = Q_*, \quad r_* = R_* $$

Then we define

$$ \sigma = r - r_*, \quad \eta = q - q_* $$

and

$$ V = (v, \sigma, \eta) = U - (0, r_*, q_*), $$

where $U$ is defined in (2.13) (notice that we use the same notation for perturbations as in the previous section, the reason is we prefer to avoid introducing additional notation). We obtain

\begin{equation}
\begin{aligned}
&\partial_t \sigma + r_* \text{ div } v = g_1(V), \\
&\partial_t \eta + q_* \text{ div } v = g_2(V), \\
&(r_* + q_*) \partial_t v - \mu \Delta v - \nu \nabla \text{ div } v + \omega_1 \nabla \sigma + \omega_2 \nabla \eta = g_3(V), \\
&(\sigma, \eta, v)|_{t=0} = (r_0 - r_*, q_0 - q_*, v_0), \quad v|_{\partial \Omega} = 0,
\end{aligned}
\end{equation}

where $\gamma_3 = \gamma_3(r_*, q_*)$ is a constant defined by

$$ q_* = \left(1 - \frac{r_*}{\gamma_3}\right) \gamma_3 $$

and, analogously to (3.4),

\begin{equation}
\begin{aligned}
\omega_0 &= \frac{\gamma^+ \gamma_3^+}{\gamma_3^+ - (\gamma - 1)r_*}, \\
\omega_2 &= \frac{\gamma^+ \gamma_3^+}{\gamma_3^+ - (\gamma - 1)r_* \gamma_3^+ - 1}.
\end{aligned}
\end{equation}

The right hand side of (4.1) is analogous to (3.5)–(3.7) with $(r_*, q_*, \gamma_3)$ instead of $(r_0, q_0, \gamma_3)$ and without terms with $\nabla r_0$ and $\nabla q_0$ in $g_3$:

\begin{equation}
\begin{aligned}
g_1(V) &= O_3(V + (0, r_*, q_*)) - \sigma \text{ div } v, \\
g_2(V) &= O_3(V + (0, r_*, q_*)) - \eta \text{ div } v, \\
g_3(V) &= O_3(V + (0, r_*, q_*)) - (\sigma + \eta)\partial_t v \\
&- \frac{\gamma^+ \gamma_3^+ + 1}{\gamma_3^+ - (\gamma - 1)r_*} \nabla \sigma \\
&- \frac{\gamma^+ \gamma_3^+ + 1}{\gamma_3^+ - (\gamma - 1)r_* \gamma_3^+ - 1} \nabla \eta \bigg|_{J_1(r,q,3)} \\
&- \frac{\gamma^+ \gamma_3^+ + 1}{(\gamma_3^+ - (\gamma - 1)r_3^+ - 1)(\gamma_3^+ - (\gamma - 1)r_* \gamma_3^+ - 1)} \nabla \eta \bigg|_{J_2(r,q,3)} \\
&- \frac{\gamma^+ \gamma_3^+ + 1}{(\gamma_3^+ - (\gamma - 1)r_3^+ - 1)(\gamma_3^+ - (\gamma - 1)r_* \gamma_3^+ - 1)} \nabla \eta \bigg|_{J_3(r,q,3)}.
\end{aligned}
\end{equation}
4.2. Exponential Decay

Consider a linear problem corresponding to (4.1)
\[
\begin{align*}
\partial_t \sigma + \mathbf{r}_* \cdot \nabla \mathbf{v} &= f_1, \\
\partial_t \eta + \mathbf{q}_* \cdot \nabla \mathbf{v} &= f_2, \\
(\mathbf{r}_* + \mathbf{q}_*) \partial_t \mathbf{v} - \mu \Delta \mathbf{v} - \nu \nabla \cdot \mathbf{v} + \omega_1 \nabla \sigma + \omega_2 \nabla \eta &= f_3, \\
(\sigma, \eta, \mathbf{v})|_{t=0} = (\sigma_0, \eta_0, \mathbf{v}_0), & \quad \mathbf{v}|_{\partial \Omega} = 0.
\end{align*}
\]

For the above problem we have exponential decay result

**Theorem 4.1.** Let \( p, q, \sigma_0, \eta_0 \) and \( \mathbf{v}_0 \) satisfy the assumptions of Theorem 3.1. Assume moreover that
\[
e^{\beta t} f_1, e^{\beta t} f_2 \in L_p(\mathbb{R}^+, W^1_q(\Omega)), e^{\beta t} f_3 \in L_p(\mathbb{R}^+, L_q(\Omega))
\]
for some constant \( \beta > 0 \). Then solution to (4.6) exists globally in time and satisfies
\[
\begin{align*}
\|e^{\beta t} \sigma, e^{\beta t} \eta, e^{\beta t} \mathbf{v}\|_{\mathcal{X}(\infty)} & \leq C_{p,q} \left[ \|\mathbf{v}_0\|_{B^2_{q,p}(\Omega)} + \|\sigma_0, \eta_0\|_{W^1_q(\Omega)} \right] \\
& \quad + \|e^{\beta t} f_3\|_{L_p(0,T;L_q(\Omega))} + \|(e^{\beta t} f_1, e^{\beta t} f_2)\|_{L_p(0,T;W^{1}_q(\Omega))}. \\
\end{align*}
\]

The keynote ideas necessary to prove Theorem 4.1 were developed by Enomoto and Shibata [7, Theorem 2.9], where analogous result for the compressible Stokes system was shown, however with additional zero mean condition for the right hand side of the continuity equation \((f_1, f_2)\) in our case. This condition was removed in [24, Theorem 5.1], where a system describing a mixture of two constituents is considered and its linearization consists of two parabolic equations coupled with the continuity equation. It is therefore slightly more complicated then the compressible Stokes system, but the essential ideas from [7] apply. In fact, the zero mean condition is used at the first stage of the proof [24, Theorem 7.1] to guarantee the unique solvability of the resolvent problem for \( \lambda = 0 \), but then the general case is reduced to the zero mean case. The details can be found in Sect. 7 of [24].

Analogous result for the compressible Stokes system was proved recently in [17, Proposition 3.2], also without the zero mean assumption. The proof is based on the ideas from [24] but the system is simpler, so we recommend also this proof for more details.

Theorem (4.1) can be proved following line by line the proof of Proposition 3.2 from [17], therefore we omit the proof here.

4.3. Bounds for Nonlinearities

We start with the following counterpart of Lemma 3.8:

**Lemma 4.2.** Let \((e^{\beta t} z_1, e^{\beta t} z_2, e^{\beta t} w) \in \mathcal{X}(\infty)\) and
\[
\|z_1(0) - \mathbf{r}_* , z_2(0) - \mathbf{q}_* \|_{W^1_q(\Omega)} + \|w(0)\|_{B^2_{q,p}(\Omega)} \leq \varepsilon.
\]

Then
\[
\begin{align*}
\|V^0(k_w), \nabla_{k_w} V^0(k_w)\|_{L^\infty((0,T) \times \Omega)} & \leq C \|e^{\beta t} (z_1, z_2, w)\|_{\mathcal{X}(\infty)}, \\
\sup_{t \in \mathbb{R}_+} \|z_1(\cdot, t) - \mathbf{r}_* \|_{W^1_q(\Omega)} & \leq \varepsilon + \|e^{\beta t} (z_1, z_2, w)\|_{\mathcal{X}(\infty)}, \\
\sup_{t \in \mathbb{R}_+} \|z_2(\cdot, t) - \mathbf{q}_* \|_{W^1_q(\Omega)} & \leq \varepsilon + \|e^{\beta t} (z_1, z_2, w)\|_{\mathcal{X}(\infty)}, \\
\sup_{t \in \mathbb{R}_+} \|w(\cdot, t) - w(0)\|_{B^2_{q,p}(\Omega)} & \leq C \|e^{\beta t} (z_1, z_2, w)\|_{\mathcal{X}(\infty)}, \\
\|w\|_{L^\infty(0,T;W^1_q(\Omega))} & \leq C \|e^{\beta t} (z_1, z_2, w)\|_{\mathcal{X}(\infty)},
\end{align*}
\]

where \(k_w\) is defined as in (2.8).
Proof. We have

$$\|V^0(k_w), \nabla_k w^0(k_w)\|_{L^\infty((0, T) \times \Omega)} \leq C(V^0) \int_0^\infty \|\nabla g w\|_\infty dt \leq \left( \int_0^\infty e^{-\beta tp'} dt \right)^{1/p'} \left( \int_0^\infty e^{\beta tp} \left\|V^0_2(\Omega_0) dt \right\|^p \right)^{1/p}, \quad (4.13)$$

which implies (4.8). Next,

$$\|z_1(\cdot, t) - \tau_*\|_{W^1_\eta(\Omega_0)} \leq \|z_1(0) - \tau_*\|_{W^1_\eta(\Omega_0)} + \int_0^t \|\partial_t z_1(s, \cdot)\|_{W^1_\eta(\Omega_0)} dt \leq \varepsilon + C \left( \int_0^t e^{-\beta s p'} ds \right)^{1/p'} \left( \int_0^\infty e^{\beta sp} \|z_t\|_{W^1_\eta(\Omega_0)} ds \right)^{1/p},$$

and similarly for $z_2$ and $q_*$, which gives (4.9) and (4.10). Finally, (4.11) and (4.12) follows from Lemma 3.7. □

We are now in a position to prove the following estimate for the RHS of (3.3), which will enable us to apply Theorem 4.1:

**Proposition 4.3.** Let $g_1(V), g_2(V), g_3(V)$ be defined by (4.3)–(4.5). Then

$$\|e^{\beta t} g_1(V), e^{\beta t} g_2(V)\|_{L_p(\mathbb{R}^+, W^1_\eta(\Omega))} + \|e^{\beta t} g_3(V)\|_{L_p(\mathbb{R}^+, L_q(\Omega))} \leq C\left( \|e^{\beta t} V\|_{\mathcal{X}(+\infty)}^2 + \varepsilon \left( \|e^{\beta t} V\|_{\mathcal{X}(+\infty)} + 1 \right) \right), \quad (4.14)$$

where $\varepsilon$ is from (1.21).

**Proof.** By (2.19) and (4.3) we have

$$g_1(V) = -(\sigma + \tau_*) V^0_{ij}(k_v) \frac{\partial v_i}{\partial y_j} - \sigma \div v, \quad (4.15)$$

differentiating (4.15) we obtain

$$\nabla g_1(V) = -\nabla \left( \sigma V^0_{ij}(k_v) \frac{\partial v_i}{\partial y_j} \right) - \sigma \frac{\partial v_i}{\partial y_j} + \left[ \nabla k_v V^0_{ij}(k_v) \nabla k_v \frac{\partial v_i}{\partial y_j} + V^0_{ij}(k_v) \frac{\partial^2 v_i}{\partial y_j \partial y_j} \right]$$

$$- \div v \nabla \sigma - \sigma \div \div v.$$ 

By Lemma 4.2 we easily obtain

$$\|e^{\beta t} \nabla g_1(V)\|_{L_p(0, T; L_q(\Omega))} \leq C\|e^{\beta t} V\|_{\mathcal{X}(+\infty)}^2.$$ 

As $g_2$ has the same form with $\eta, q_*$ instead of $\sigma, \tau_*$, altogether we get

$$\|e^{\beta t} g_1(V), e^{\beta t} g_2(V)\|_{L_p(\mathbb{R}^+, W^1_\eta(\Omega))} \leq C\|e^{\beta t} V\|_{\mathcal{X}(+\infty)}^2.$$ 

It remains to estimate $g_3$. This is more involved, however we can take advantage of ideas from the proof of local well posedness, roughly speaking replacing smallness of time by smallness of initial data. Firstly, by (2.21) and (4.2) we have

$$\|e^{\beta t} \mathcal{O}_3(V + (0, \tau_*, q_*))\|_{L_p(0, T; L_q(\Omega))} \leq C\|e^{\beta t} V\|_{\mathcal{X}(+\infty)}^2.$$ 

Now we have to estimate $J_1, J_2, J_3$. Taking into account (3.39) with $(r_*, q_*, \xi_*)$ instead of $(t_0, q_0, \xi_0)$, it is enough to find bounds on the numerators of the last three terms of (4.5). These contains many terms but of a similar structure. First representative case is

$$\|e^{\beta t} (r - \tau_*) \nabla \eta\|_{L_p(\mathbb{R}^+, L_q(\Omega))} \leq C\|r - \tau_*\|_{L_{\infty}(\Omega \times \mathbb{R}^+)} \|e^{\beta t} \nabla \eta\|_{L_p(\mathbb{R}^+, L_q(\Omega))}.$$
\[ \leq C \| e^{\beta t} V \| \dot{\chi}(+\infty) (\| e^{\beta t} V \| \dot{\chi}(+\infty) + \varepsilon), \]

where we have used (4.9). The second case is when we have increment of \( z \). Then we use
\[ \| z(t) - z^\ast \|_{L^\infty(\Omega)} \leq \| z_0 - z^\ast \|_{+\infty} + \| z - z_0 \|_{L^\infty(\Omega)} \]
\[ \leq \| (r_0 - r_\ast) \partial_\theta z, (q_0 - q_\ast) \partial_\gamma z \|_{L^\infty(\Omega)} + \int_0^t \| \partial_t z(s) \|_{W^1_q} ds \leq C(\varepsilon + \| e^{\beta t} V \| \dot{\chi}(t)) \]
to obtain
\[ \| e^{\beta t} z^{\ast\ast} z^\ast \nabla \sigma \|_{L_p(\mathbb{R}^+; L^q(\Omega))} \leq C \| z - z^\ast \|_{L^\infty(\Omega \times \mathbb{R}^+)} \| e^{\beta t} \nabla \sigma \|_{L_p(\mathbb{R}^+; L^q(\Omega))} \]
\[ \leq C(\varepsilon + \| e^{\beta t} V \| \dot{\chi}(+\infty)) \| e^{\beta t} V \| \dot{\chi}(+\infty). \]

Other terms containing \( z^\ast - z \) can be treated similarly. Terms with \( z^\beta - z^\beta \) for \( \beta \in \{ \gamma^+, \gamma - 1, \gamma - 1 \} \) can also be estimated in a similar way. Altogether we obtain
\[ \sum_{k=1}^3 \| e^{\beta t} J_k(r, q, z) \|_{L_p(\mathbb{R}^+; L^q(\Omega))} \leq C(\varepsilon + \| e^{\beta t} V \| \dot{\chi}(+\infty)) \| e^{\beta t} V \| \dot{\chi}(+\infty). \] (4.18)

Putting together (4.16), (4.17) and (4.18) we obtain (4.14).

### 4.4. Proof of Theorem 1.2

We proceed in a standard way, first we show

**Lemma 4.4.** Assume \((\sigma, \eta, v)\) is solution to (3.3) with \( r_0, q_0 \) and \( u_0 \) satisfying the assumptions of Theorem 1.2. Then
\[ \| e^{\beta t} (\sigma, \eta, v) \|_{\dot{\chi}(+\infty)} \leq E(\varepsilon). \] (4.19)

**Proof.** Applying Theorem 4.1 and Proposition 4.3 to (3.3) we obtain
\[ \| e^{\beta t} (\sigma, \eta, v) \|_{\dot{\chi}(t)} \leq C \left( \varepsilon + \| e^{\beta t} (\sigma, \eta, v) \|_{\dot{\chi}(t)}^2 \right) \quad \forall 0 < T \leq +\infty. \] (4.20)

Notice that we derived this inequality for \( T = \infty \), but the same arguments allow to obtain it for any \( T > 0 \). Now consider the equation
\[ x^2 - \frac{x}{C} + \varepsilon = 0 \]
with roots
\[ x_1(\varepsilon) = \frac{1}{2C} - \sqrt{\frac{1}{4C^2} - \varepsilon}, \quad x_2(\varepsilon) = \frac{1}{2C} + \sqrt{\frac{1}{4C^2} - \varepsilon}. \]

Observe that inequality (4.20) implies either \( \| e^{\beta t} (\sigma, \eta, v) \|_{\dot{\chi}(T)} \leq x_1(\varepsilon) \) or \( \| e^{\beta t} (\sigma, \eta, v) \|_{\dot{\chi}(T)} \geq x_2(\varepsilon) \). However,
\[ \lim_{T \to 0} \| e^{\beta t} (\sigma, \eta, v) \|_{\dot{\chi}(T)} = 0, \]
therefore
\[ \| e^{\beta t} (\sigma, \eta, v) \|_{\dot{\chi}(t)} \leq x_1(\varepsilon) \] (4.21)
for small times. However, \( \| e^{\beta t} (\sigma, \eta, v) \|_{\dot{\chi}(T)} \) is continuous in time and therefore (4.21) holds for \( 0 < T \leq +\infty \).

Now we can prolong the local solution for arbitrarily large times. For this purpose observe that if the initial data satisfies (1.21) then the time of existence from Theorem 1.1 satisfies \( T > C(\varepsilon) > 0 \). Therefore, for arbitrarily large \( T^\ast \) we obtain a solution on \((0, T^\ast)\) in a finite number of steps. By the estimate (4.19) this solution satisfies (1.22).
Appendix

Here we recall the definition of $\mathcal{R}$-boundedness:

**Definition 1.** Let $X$ and $Y$ be two Banach spaces, and $\| \cdot \|_X$ and $\| \cdot \|_Y$ their norms. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$ if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$ and $\{f_j\}_{j=1}^n \subset X$, the inequality

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u)T_j f_j \right\|_Y^p du \leq C \int_0^1 \left\| \sum_{j=1}^n r_j(u)f_j \right\|_X^p du,$$

where $r_j : [0,1] \to \{-1,1\}$, $j \in \mathbb{N}$, are the Rademacher functions given by $r_j(t) = \text{sign}(2^j \pi t)$. The smallest such $C$ is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$ which is written by $\mathcal{R}_{\mathcal{L}(X,Y)} \mathcal{T}$.

Next we quote Weis’ vector valued Fourier Multiplier Theorem [32]:

**Theorem 4.5.** Let $X$ and $Y$ be UMD spaces and $1 < p < \infty$. Let $M \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X,Y))$. Let us define the operator $T_M : \mathcal{F}^{-1} \mathcal{D}(\mathbb{R},X) \to \mathcal{S}'(\mathbb{R},Y)$:

$$T_M \phi(\tau) = \mathcal{F}^{-1}[M \mathcal{F}[\phi](\tau)].$$

Assume that

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{M(\tau) : \tau \in \mathbb{R} \setminus \{0\}\}) = \kappa_0 < \infty, \quad \mathcal{R}_{\mathcal{L}(X,Y)}(\{\tau M'(\tau) : \tau \in \mathbb{R} \setminus \{0\}\}) = \kappa_1 < \infty.$$ (4.23)

Then, the operator $T_M$ defined in (4.22) is extended to a bounded linear operator $L_p(\mathbb{R},X) \to L_p(\mathbb{R},Y)$ and

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \leq C(\kappa_0 + \kappa_1),$$

where $C = C(p,X,Y) > 0$.

**Remark 2.** For definitions and properties of UMD spaces we refer the reader for example to Chapter 4 in [13]. Here let us only note that $L_p$ spaces and $W^k_p$ spaces are UMD for $1 < p < \infty$.

We recall that the Fourier transform and its inverse are defined as

$$\mathcal{F}[f](\tau) = \int_{\mathbb{R}} e^{-it\tau} f(t) dt, \quad \mathcal{F}^{-1}[f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\tau} f(\tau) d\tau$$ (4.24)

while the Laplace transform and its inverse are

$$\mathcal{L}[f](\lambda) = \int_{\mathbb{R}} e^{-\lambda t} f(t) dt, \quad \mathcal{L}^{-1}_\beta[f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\lambda t} f(\lambda) d\lambda, \quad \text{where} \quad \lambda = \beta + i\tau.$$ (4.25)

The following result (Theorem 2.17 in [7]) explains how to obtain $L_p$-maximal regularity using Theorem 4.5 and Laplace transform:

**Theorem 4.6.** Let $X$ and $Y$ be UMD Banach spaces and $1 < p < \infty$. Let $0 < \varepsilon < \frac{\pi}{2}$ and $\beta_1 \in \mathbb{R}$. Let $\Phi_\lambda$ be a $C^1$ function of $\tau \in \mathbb{R} \setminus \{0\}$ where $\lambda = \beta + i\tau \in \Sigma_{\varepsilon,\beta_1}$ with values in $\mathcal{L}(X,Y)$. Assume that

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{\Phi_\lambda : \lambda \in \Sigma_{\varepsilon,\beta_1}\}) \leq M, \quad \mathcal{R}_{\mathcal{L}(X,Y)} \left( \left\{ \frac{\partial}{\partial \tau} \Phi_\lambda : \lambda \in \Sigma_{\varepsilon,\beta_1} \right\} \right) \leq M$$

for some $M > 0$. Let us define

$$\Psi f(t) = \mathcal{L}^{-1}_\beta[\Phi_\lambda \mathcal{L}[f](\lambda)](t) \quad \text{for} \quad f \in \mathcal{F}^{-1} \mathcal{D}(\mathbb{R},X),$$

(4.26)

where $\mathcal{L}$ and $\mathcal{L}^{-1}_\beta$ are the Laplace transform and its inverse defined in (4.25). Then

$$\|e^{-\beta t}\Psi f\|_{L_p(\mathbb{R},Y)} \leq C(p,X,Y)M\|e^{-\beta t}\|_{L_p(\mathbb{R},X)} \quad \forall \beta \geq \beta_1.$$ (4.27)
Proof. For $\lambda = \beta + i\tau$ we have the following relation between Laplace and Fourier transforms defined in, respectively, (4.25) and (4.24):
\[
L[f](\lambda) = \int_{\mathbb{R}} e^{-\lambda t} f(t) dt = \mathcal{F}[e^{-\beta t} f](\tau),
\]
\[
L^{-1} \beta[f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\lambda t} f(\lambda) d\lambda = e^{\beta t} \mathcal{F}^{-1}[f](t).
\]
Therefore by (4.26) we have
\[
e^{-\beta t} \Psi f(t) = \mathcal{F}^{-1}[\Psi_{\beta+i\tau} \mathcal{F}[e^{-\beta t}](\tau)](t).
\]
Applying Theorem 4.5 to the above formula we conclude (4.27).

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Declarations

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References

[1] Adams, R.A., Fournier, J.F.: Sobolev Spaces. Pure and Applied Mathematics (Amsterdam), vol. 140, 2nd edn. Elsevier/Academic Press, Amsterdam (2003)
[2] Bresch, D., Desjardins, B., Ghidaglia, J.-M., Grenier, E., Hillairet, M.: Multifluid models including compressible fluids. In: Giga, Y., Novotný, A. (eds.) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, pp. 1–52. Springer, Berlin (2017)
[3] Bresch, D., Desjardins, B., Ghidaglia, J.-M., Grenier, E.: Global weak solutions to a generic two-fluid model. Arch. Ration. Mech. Anal. 196(2), 599–629 (2010)
[4] Bresch, D., Huang, X., Li, J.: Global weak solutions to one-dimensional non-conservative viscous compressible two-phase system. Commun. Math. Phys. 309(3), 737–755 (2012)
[5] Bresch, D., Mucha, P.B., Zatorska, E.: Finite-energy solutions for compressible two-fluid Stokes system. Arch. Ration. Mech. Anal. 232(2), 987–1029 (2019)
[6] Denk, R., Hieber, M., Prüß, J.: $R$-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type, No. 788, vol. 166. Memoirs of AMS, Providence (2003)
[7] Enomoto, Y., Shibata, Y.: On the $R$-sectoriality and the initial boundary value problem for the viscous compressible fluid flow. Funkcialaj Ekvacioj 56, 441–505 (2013)
[8] Evje, S., Wang, W., Wen, H.: Global well-posedness and decay rates of strong solutions to a non-conservative compressible two-fluid model. Arch. Ration. Mech. Anal. 221(3), 1285–1316 (2016)
[9] Evje, S., Wen, H., Zhu, C.: On global solutions to the viscous liquid–gas model with unconstrained transition to single-phase flow. Math. Models Methods Appl. Sci. 27(2), 323–346 (2017)
[10] Qiao, Y., Wen, H., Evje, S.: Viscous two-phase flow in porous media driven by source terms: analysis and numerics. SIAM J. Math. Anal. 51(6), 5103–5140 (2019)
[11] Guo, Z.H., Yang, J., Yao, L.: Global strong solution for a three-dimensional viscous liquid–gas two-phase flow model with vacuum. J. Math. Phys. 52, 093102 (2011)
[12] Hieber, M., Murata, M.: The $L^p$-approach to the fluid-rigid body interaction problem for compressible fluids. Evol. Equ. Control Theory 4(1), 69–87 (2015)
[13] Hytönen, T., van Neerven, J., Veraar, M., Weis, L.: Analysis in Banach Spaces. Vol. I. Martingales and Littlewood-Paley Theory. A Series of Modern Surveys in Mathematics, vol. 63. Springer, Cham (2016)
[14] Ishii, M.: Thermo-Fluid Dynamic Theory of Two-Phase Flow. Eyrolles, Paris (1975)
[15] Jin, B.J., Novotný, A.: Weak-strong uniqueness for a bi-fluid model for a mixture of non-interacting compressible fluids. J. Differ. Equ. 268, 204–238 (2019)

[16] Kato, T.: Perturbation Theory for Linear Operators. Springer-Verlag, Berlin (1995)

[17] Kreml, O., Nečasová, Š, Piašek, T.: Compressible Navier–Stokes system on a moving domain in the $L_p – L_q$ framework. In: Bodnár, T., Galdi, G.P., Nečasová, Š (eds.) Waves in Flows. Advances in Mathematical Fluid Mechanics, pp. 127–158. Birkhäuser/Springer, Cham (2021)

[18] Kubo, T., Shibata, Y., Soga, K.: On the $\mathcal{R}$-boundedness for the two phase problem: compressible–incompressible model problem. Bound. Value Probl. 2014(1), 141 (2014)

[19] Li, Y., Sun, Y., Zatorska, E.: Large time behavior for a compressible two-fluid model with algebraic pressure closure and large initial data. Nonlinearity 33, 4075–4094 (2020)

[20] Li, Y., Zatorska, E.: On weak solutions to the compressible inviscid two-fluid model. J. Differ. Equ. 299, 33–50 (2021)

[21] Murata, M.: On a maximal $L_p – L_q$ approach to the compressible viscous fluid flow with slip boundary condition. Nonlinear Anal. 106, 86–109 (2014)

[22] Murata, M., Shibata, Y.: On the global well-posedness for the compressible Navier–Stokes equations with slip boundary condition. J. Differ. Equ. 260(7), 5761–5795 (2016)

[23] Novotný, A., Pokorný, M.: Weak solutions for some compressible multicomponent fluid models. Arch. Ration. Mech. Anal. 235, 355–403 (2020)

[24] Piašek, T., Shibata, Y., Zatorska, E.: On strong dynamics of compressible two-component mixture flow SIAM. J. Math. Anal. 51(4), 2793–2849 (2019)

[25] Piašek, T., Shibata, Y., Zatorska, E.: On the isothermal compressible multi-component mixture flow: the local existence and maximal $L_p – L_q$ regularity of solutions. Nonlinear Anal. 189, 111571 (2019)

[26] Shibata, Y.: On the global well-posedness of some free boundary problem for a compressible barotropic viscous fluid flow. In: Recent Advances in Partial Differential Equations and Applications. Contemporary Mathematics, vol. 666, pp. 341–356. American Mathematical Society, Providence, RI (2016)

[27] Shibata, Y., Shimizu, S.: On some free boundary problem for the Navier–Stokes equations. Differ. Int. Equ. 20, 241–276 (2007)

[28] Shibata, Y., Shimizu, S.: On the $L_p–L_q$ maximal regularity of the Neumann problem for the Stokes equations in a bounded domain. J. Reine Angew. Mat. 615, 157–209 (2008)

[29] Tanabe, H.: Functional Analytic Methods for Partial Differential Equations. Monographs and Textbooks in Pure and Applied Mathematics, vol. 204. Marcel Dekker, Inc., New York (1997)

[30] Wang, Y., Wen, H., Yao, L.: On a non-conservative compressible two-fluid model in a bounded domain: global existence and uniqueness. J. Math. Fluid Mech. 23, 4 (2021)

[31] Weis, L.: Operator-valued Fourier multiplier theorems and maximal $L_p$-regularity. Math. Ann. 319, 735–758 (2001)