Weighted Frechet Means as Convex Combinations in Metric Spaces: Properties and Generalized Median Inequalities.

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Abstract

In this short note, we study the properties of the weighted Frechet mean as a convex combination operator on an arbitrary metric space \((Y, d)\). We show that this binary operator is commutative, non-associative, idempotent, invariant to multiplication by a constant weight and possesses an identity element. We also cover the properties of the weighted cumulative Frechet mean. These tools allow us to derive several types of median inequalities for abstract metric spaces that hold for both negative and positive Alexandrov spaces. In particular, we show through an example that these bounds cannot be improved upon in general metric spaces. For weighted Frechet means, however, such inequalities can solely be derived for weights equal or greater than one. This latter limitation highlights the inherent difficulties associated with abstract-valued random variables.

Keywords: Abstract-valued random variable, Barycentre, Convex combination, Convex operator, Frechet mean, Frechet cumulative mean, Generalized triangle inequality, Median inequality, Metric space.

1. Introduction

The core task of statistics is to summarize data, which is commonly done by identifying typical elements. As observed by Fréchet (1948), typical elements are the elements in the sample space that are as similar as possible to all the other elements in that space. If a notion of distance is defined on this space of elements, then it follows that the most typical element is the one that differs the least from all the others. Such an element is commonly referred to as the Frechet mean, barycentre or Karcher mean (Karcher, 1977). These typical elements have been studied in metric spaces at various levels of generality. The almostsure convergence of the Frechet sample mean to its theoretical analogue has been demonstrated for separable metric spaces with bounded metric (Ziezold, 1977), in compact spaces (Sverdrup-Thygeson, 1981), and when the Frechet mean is assumed to be unique (Bhattacharya and Patrangenaru, 2003). The concept of Frechet mean has also proved to be useful in different domains of applications, such as in image analysis (Thorstensen et al., 2009; Bigot and Charlier, 2011) or when studying phylogenetic trees (Balding et al., 2009).

One of the outstanding questions in this field is whether the classical median inequality can be recovered using the Frechet mean. If a similar result can be derived, it may then be possible to generalize standard results in probability and statistics to abstract metric spaces. In the Euclidean
plane, the median inequality states that for every triangle \( \Delta ABC \),
\[
EC \leq \frac{1}{2}(AC + BC),
\]
where \( E \) is the midpoint of \( \overline{AB} \). In this short note, we consider generalizations of this law to abstract metric spaces, whereby the midpoint of a segment in such spaces is defined as the Frechet mean of the endpoints of that segment. In particular, we explore the properties of the Frechet mean and its cumulative extension in arbitrary metric spaces for any type of Alexandrov curvature (see Herz, 1992, for a study of typical elements in negatively curved metric spaces). Our definition of the Frechet mean as a convex combination bears some resemblance with the convex combination operator introduced by Terán and Molchanov (2006), and we will draw some specific links with the work of these authors in the sequel.

2. Weighted Frechet Mean

The Frechet mean generalizes the arithmetic mean to abstract metric spaces. In general, this quantity may not be unique. When this is the case, the set of all minimizers is referred to as the Frechet mean set. Here, we will only consider a given element from the set of all such minimizers. One of the interesting properties of the Frechet mean is that it also allows the combination of subsets of a metric space \((Y, d)\). This constitutes another important generalization of the classical notion of arithmetic mean, where one solely combines elements of the Real line. We therefore define the Frechet mean with respect to subsets of \((Y, d)\). Here, the distance between a subset \( A \subseteq Y \) and a point \( y \in Y \) is
\[
d(\{y\}, A) := \inf\{d(y, a) : a \in A\}.
\]

**Definition 1.** On a given metric space \((Y, d)\), the Frechet mean of the \( r \)-th order is defined for any two subsets \( A, B \subseteq Y \) and real numbers \( \alpha, \beta \geq 0 \), as follows,
\[
\alpha A \oplus_{r} \beta B \in \arg\inf_{y \in Y} \{\alpha d(A, y)^r + \beta d(y, B)^r\},
\]
for every \( r \geq 1 \). Similarly, the Frechet cumulative mean operator of the \( r \)-th order is defined for any finite sequence of subsets of \( Y \), denoted \( A_1, \ldots, A_n \), and sequence of non-negative real numbers \( \alpha_1, \ldots, \alpha_n \), as follows,
\[
\bigoplus_{i=1}^{n} \alpha_i A_i \in \arg\inf_{y \in Y} \sum_{i=1}^{n} \alpha_i d(A_i, y)^r.
\]

We now study the properties of the Frechet mean of any order, in abstract metric spaces. The following lemma and corollary are true for all \( r \geq 1 \), and therefore this subscript is omitted.

**Lemma 1.** For any \( A, B, C \subseteq Y \) in a metric space \((Y, d)\), and for any \( \alpha \geq 0 \), the Frechet mean operator satisfies the following:

(i.) **Commutativity:** \( A \oplus B = B \oplus A \);

(ii.) **Non-associativity:** \( A \oplus (B \oplus C) \neq (A \oplus B) \oplus C \);

(iii.) **Idempotency:** \( A \oplus A = A \);
(iv.) Proportionality: \( \alpha A \oplus \alpha B = A \oplus B \);

(v.) Identity element: \( A \oplus \emptyset = A \), for every \( A \).

\textbf{Proof.} Commutativity in (i) is immediate from the commutativity of addition on the real numbers. Invariance with respect to a constant weight in (iv) is a direct consequence of the definition of the Frechet mean operator. Idempotency in (iii) follows from the fact that metrics do not distinguish between identical elements, and therefore \( d(A, y) + d(y, A) = 2d(y, A) \), using the symmetry of \( d \) and invoking (iv). The existence of the identity element in (v) can be deduced by noting that \( d(A, y) + d(y, Y) \) is only null when \( y \in A \cap Y = A \). Finally, non-associativity in (ii) can be proved through a numerical counter-example in \( \mathbb{R} \) equipped with the Euclidean metric (i.e. take any three distinct real numbers). Therefore, associativity does not hold in general.

Equivalent properties can be immediately deduced from Lemma 1 for the case of the Frechet cumulative mean operator, as described in the following corollary.

\textbf{Corollary 1.} For any sequence of subsets, denoted \( \{A_1, \ldots, A_n\} \), in a metric space \( (Y, d) \), for any real number \( \alpha \geq 0 \), and any \( n \in \mathbb{N} \), the Frechet cumulative mean operator satisfies for any label permutation \( \nu : \mathbb{I} \mapsto \mathbb{I} \), where \( \mathbb{I} := \{1, \ldots, n\} \),

(i.) Commutativity: \( \bigoplus_{i=1}^{n} A_i = \bigoplus_{\nu(i)=1}^{n} A_{\nu(i)} \);

(ii.) Proportionality: \( \bigoplus_{i=1}^{n} \alpha A_i = \bigoplus_{i=1}^{n} A_i \);

(iii.) Idempotency: \( \bigoplus_{i=1}^{n} A = A \).

Note that property (i) of the Frechet cumulative mean corresponds to condition (i) in Terán and Molchanov (2006). These authors have studied the behavior of convex combination operators in metric spaces. However, the regrouping condition, denoted (ii) in Terán and Molchanov (2006) does not hold in general abstract spaces for the Frechet mean, due to its non-associativity. This lack of associativity will also lead to some difficulties when extending the generalized median inequality from the binary Frechet mean to its cumulative analogue.

\textbf{Definition 2.} A set \( A \subseteq (Y, d) \) is \( \oplus \)-convex if for every sequence \( y_1, \ldots, y_n \) of points in \( A \) and non-negative numbers \( \alpha_i \)'s, we have \( \bigoplus_{i=1}^{n} \alpha_i y_i \in A \).

The Frechet mean operator therefore allows the construction of \( \oplus \)-convex hulls in \( (Y, d) \), such that for every \( A \subseteq Y \), the \( \oplus \)-convex hull of \( A \) of the \( r \)-th order is defined as

\[ H^r(A) := \left\{ \bigoplus_{i=1}^{n} \alpha_i y_i \left| y_i \in A, \alpha_i \geq 0, \forall n \in \mathbb{N} \right. \right\}, \tag{3} \]

Here, although \( H^r(A) \) is \( \oplus \)-convex, \( A \) need not be convex in the classical sense. That is, if \( A \) is a subset of a vector space, for instance, it may not be convex with respect to vector addition. Nonetheless, given any metric on that vector space, one can construct a hull, which is convex with respect to the Frechet mean based on that particular metric.

By definition, \( H^r(A) \) is \( \oplus_r \)-convex for every \( r \geq 1 \). Similarly, observe that the closure of \( Y \) is trivially \( \oplus_r \)-convex. Although the definition in equation (3) appears to be the one of a convex cone, in fact, it defines a convex hull. That is, although in our adopted definition, we have not explicitly required the \( \alpha_i \)'s to sum to 1, it follows from the \textit{proportionality} of the Frechet cumulative

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mean that these weights can be normalized without altering the choice of the optimal elements in \( \mathcal{Y} \). Since by definition, the Frechet cumulative mean of a collection of points, \( \{y_1, \ldots, y_n\} \), is necessarily located in the convex hull of these points, it follows that the Frechet cumulative mean of any order can be regarded as a convex combination on \((\mathcal{Y}, d)\). Note, however, that this concept is here used in a more general sense than in Terán and Molchanov (2006).

3. Median Inequalities in Metric Spaces

We here state and prove the main results of this paper for the Frechet mean and cumulative mean of the first order. Hence, in this section, all Frechet operations will be assumed to be conducted with respect to \( r = 1 \). The more general case will be studied in section 4. Note also that, without loss of generality, we have formulated these results in terms of single elements in \((\mathcal{Y}, d)\). However, all of these results also hold for subsets of \( \mathcal{Y} \).

**Theorem 1.** For any abstract metric space \((\mathcal{Y}, d)\), and for every \( x, y, \xi \in \mathcal{Y} \),

\[
\text{d}(\xi, x \oplus y) \leq \text{d}(x, \xi) + \text{d}(y, \xi).
\]

**Proof.** Assume that the result does not hold, and that for some \( x, y, \xi \in \mathcal{Y} \), we have instead \( \text{d}(\xi, x \oplus y) > \text{d}(x, \xi) + \text{d}(y, \xi) \). By the triangle inequality with respect to \( y \), it follows that

\[
\text{d}(x, \xi) + \text{d}(\xi, y) < \text{d}(\xi, x \oplus y) \leq \text{d}(y, x \oplus y),
\]

which simplifies to \( \text{d}(x, \xi) < \text{d}(y, x \oplus y) \). Similarly, by invoking the triangle inequality with respect to \( x \), we obtain \( \text{d}(\xi, y) < \text{d}(x, x \oplus y) \). Now, combining these two strict inequalities and using the symmetry of \( d \), this gives

\[
\text{d}(x, \xi) + \text{d}(\xi, y) < \text{d}(x, x \oplus y) + \text{d}(x \oplus y, y),
\]

but this contradicts the minimality of \( x \oplus y \), and therefore proves the theorem.

Observe that the Euclidean median law does not hold in general metric spaces, as illustrated by figure [1] for a negatively curved Alexandrov space (see Burago et al., 2001). Therefore, the result in theorem 1 is tight in the sense that this inequality can be saturated for some metric spaces. By contrast, the inequality is strict in the Euclidean case. A similar inequality can be derived for the case of a weighted Frechet mean, albeit observe that such weights should be equal or greater than 1.

**Corollary 2.** For any abstract metric space \((\mathcal{Y}, d)\), for every \( x, y, \xi \in \mathcal{Y} \), and any \( \alpha, \beta \geq 1 \),

\[
\text{d}(\xi, \alpha x \oplus \beta y) \leq \alpha \text{d}(x, \xi) + \beta \text{d}(\xi, y).
\]

**Proof.** The proof is similar to the one of theorem 1, and also proceeds by contradiction. Assuming the reverse and invoking the triangle inequality and using the fact that \( \alpha \geq 1 \), we have

\[
\alpha \text{d}(x, \xi) + \beta \text{d}(\xi, y) < \text{d}(\xi, \alpha x \oplus \beta y) \leq \alpha \text{d}(x, \alpha x \oplus \beta y),
\]

which reduces to \( \beta \text{d}(\xi, y) < \alpha \text{d}(x, \alpha x \oplus \beta y) \). Through an analogous procedure, we may obtain \( \alpha \text{d}(\xi, x) < \beta \text{d}(y, \alpha x \oplus \beta y) \), and combining these inequalities this gives the desired contradiction.
Figure 1: Illustrative metric space with negative Alexandrov curvature (Burago et al., 2001). Albeit the distances between each point in this space satisfy the triangle inequality, we nonetheless have \( d(x, x @ y) > \frac{1}{2} (d(x, y) + d(y, ξ)) \), and therefore the classical Euclidean median inequality does not hold in this metric space. In addition, observe that, in this setting, \( ξ \) also constitutes a mean between \( x \) and \( y \), since we have \( d(x, ξ) + d(ξ, y) = d(x, x @ y) + d(x @ y, y) \).

Theorem 2 can be generalized for the Frechet cumulative mean operator. This result essentially states that the cumulative mean operator is countably additive. When extending these inequalities to the case of the cumulative mean operator, observe that the non-associativity of the Frechet mean does not allow a direct proof by induction, and therefore other arguments have to be deployed in order to prove that an equivalent result holds in this general setting.

**Theorem 2 (Countable Additivity).** For any abstract metric space \((\mathcal{Y}, d)\), for every sequence \(\{y_1, \ldots, y_n\}\) and \(ξ \in \mathcal{Y}\), and for every \(n \in \mathbb{N}\),

\[
\sum_{i=1}^{n} d(y_i, ξ) \leq d\left(\bigoplus_{i=1}^{n} y_i, ξ\right).
\]

**Proof.** Again, seeking a contradiction, assume that \(\sum_{i=1}^{n} d(y_i, ξ) < d\left(\bigoplus_{i=1}^{n} y_i, ξ\right)\). It then follows that through \(n\) applications of the triangle inequalities, we obtain the following system of strict inequalities,

\[
\sum_{i=1}^{n} d(y_i, ξ) < d\left(\bigoplus_{i=1}^{n} y_i, ξ\right) \leq d(ξ, y_k) + d\left(y_k, \bigoplus_{i=1}^{n} y_i\right),
\]

for every \(k = 1, \ldots, n\). Each of these inequalities can be expanded using the positivity of \(d\), as follows,

\[
\sum_{i=1}^{n} d(y_i, ξ) < d(ξ, y_k) + d\left(y_k, \bigoplus_{i=1}^{n} y_i\right) + \sum_{i \in I(k, l)} d(y_i, ξ),
\]

where \(I := \{1, \ldots, n\}\), and where \(l := k + 1\) if \(k < n\) and \(l := 1\) if \(k = n\). Since the latter inequality holds for every \(k = 1, \ldots, n\), it suffices to sum these \(n\) inequalities in order to obtain

\[
\sum_{k=1}^{n} \left(n \sum_{i=1}^{n} d(y_i, ξ) - \sum_{k=1}^{n} d(ξ, y_k) - \sum_{k=1}^{n} \sum_{l \in I(k, l)} d(y_i, ξ)\right) < \sum_{k=1}^{n} d\left(y_k, \bigoplus_{i=1}^{n} y_i\right).
\]

which leads to \(\sum_{k=1}^{n} d(y_i, ξ) < \sum_{k=1}^{n} d(y_k, \bigoplus_{i=1}^{n} y_i)\). However, this contradicts the minimality of \(\bigoplus_{i=1}^{k} y_i\), as desired. \(\square\)
Corollary 3. For any abstract metric space \((\mathcal{Y}, d)\), for every sequence \(\{y_1, \ldots, y_n\}\) and \(\xi\) in \(\mathcal{Y}\), and for every sequence of real numbers \(\{\alpha_1, \ldots, \alpha_n\}\), satisfying \(\alpha_i \geq 1\),

\[
d\left(\bigoplus_{i=1}^{n} \alpha_i y_i, \xi\right) \leq \sum_{i=1}^{n} \alpha_i d(y_i, \xi),
\]

for every \(n \in \mathbb{N}\).

**Proof.** The proof strategy is similar to the one of theorem 2, but using the argument described in the proof of corollary 2. That is, using the same notation as in the proof of theorem 2 since \(\alpha_i \geq 1\) for all \(\alpha_i\), one can derive the following system of \(n\) inequalities,

\[
\sum_{i=1}^{n} \alpha_i d(y_i, \xi) < \alpha_1 d(y_i, \xi) + \alpha_i d\left(\bigoplus_{i=1}^{n} \alpha_i y_i, \xi\right) + \sum_{i \in \{2, \ldots, n\}} \alpha_i d(y_i, \xi), \quad (4)
\]

Combining these inequalities gives \(\sum_{i=1}^{n} \alpha_i d(y_i, \xi) < \sum_{k=1}^{n} \alpha_i d(y_k, \bigoplus_{i=1}^{k} \alpha_i y_i)\), which provides the required contradiction. \(\Box\)

4. Median Inequalities of the \(r^{th}\) Order

More generally, one may be interested in considering whether analogues of the above median inequalities also hold for Frechet means and cumulative means of arbitrary orders, i.e. for which \(r \geq 1\). The following two results state such generalized versions of the median inequality.

**Theorem 3.** For any abstract metric space \((\mathcal{Y}, d)\), for every \(x, y, \xi \in \mathcal{Y}\), and for every \(r \geq 1\), the Frechet mean of the \(r^{th}\) order satisfies,

\[
d(\xi, x \oplus_r y)^\rho \leq 2^{r-1} \left(d(x, \xi)^\rho + d(\xi, y)^\rho\right).
\]

**Proof.** Using the same argument described in the proof of theorem 1, we assume for contradiction that \(d(\xi, x \oplus_r y)^\rho > 2^{r-1} \left(d(x, \xi)^\rho + d(\xi, y)^\rho\right)\) holds. Here, we will require a result due to Fréchet (1948), which states that

\[
d(a, b)^\rho \leq 2^{r-1} \left(d(a, c)^\rho + d(c, b)^\rho\right),
\]

for every \(a, b, c \in \mathcal{Y}\) and every \(r \geq 1\). See equation (5) on page 228 of Fréchet (1948). This equation will be referred to in the sequel as the triangle inequality of the \(r^{th}\) order. By using this result, it immediately follows that

\[
2^{r-1} \left(d(x, \xi)^\rho + d(\xi, y)^\rho\right) < d(\xi, x \oplus_r y)^\rho \leq 2^{r-1} \left(d(x, \xi)^\rho + d(x, x \oplus_r y)^\rho\right)
\]

which reduces to \(d(\xi, y)^\rho < d(x, x \oplus_r y)^\rho\). Similarly, we have \(d(x, \xi)^\rho < d(y, x \oplus_r y)^\rho\). As before, combining these results contradicts the minimality of \(x \oplus_r y\), and therefore the result is true for the Frechet mean of the \(r^{th}\) order. But \(r\) was arbitrary and thus the theorem holds for any \(r \geq 1\). \(\Box\)

It is straightforward to generalize this result to the weighted Frechet cumulative mean of the \(r^{th}\) order. In its most general form, we therefore have the following median inequality.
**Corollary 4.** For any abstract metric space \((\mathcal{Y}, d)\), for every sequence \(\{y_1, \ldots, y_n\}\) of elements in \(\mathcal{Y}\), and for every sequence of real numbers \(\{\alpha_1, \ldots, \alpha_n\}\), satisfying \(\alpha_i \geq 1\),

\[
d\left(\bigoplus_{i=1}^{n} \alpha_i y_i, \xi \right)^r \leq 2^{r-1} \sum_{i=1}^{n} \alpha_i d(y_i, \xi)^r,
\]

for every \(n \in \mathbb{N}\), for every \(r \geq 1\).

**Proof.** Using the arguments invoked in the proofs of the aforementioned results, we proceed by contradiction and assume that the reverse of the conclusion of corollary 4 holds. We have by the triangle inequality of the \(r\)th order, for every \(k = 1, \ldots, n\),

\[
2^{r-1} \sum_{i=1}^{n} \alpha_i d(y_i, \xi)^r < d\left(\bigoplus_{i=1}^{n} \alpha_i y_i, \xi \right)^r \leq 2^{r-1} \left[d\left(\xi, y_k\right)^r + d\left(\bigoplus_{i=1}^{n} \alpha_i y_i, y_k\right)^r\right].
\]

This can be expanded using the fact that \(\alpha_i \geq 1\), for every \(i = 1, \ldots, n\), and simplified by dividing both sides by \(2^{r-1}\) in order to obtain the analogue of equation (4) but where all metrics are elevated to the \(r\)th power and the Fréchet cumulative mean is of the \(r\)th order. As in the proof of theorem 2, combining this system of \(n\) strict inequalities contradicts the minimality of the Fréchet cumulative mean of the \(r\)th order, and this completes the proof.

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