Complexity and (un)decidability of fragments of 
\langle \omega^\lambda; \times \rangle

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Abstract

We specify the frontier of decidability for fragments of the first-order theory of ordinal multiplication. We give a NEXPTIME lower bound for the complexity of the existential fragment of \langle \omega^\lambda; \times, \omega, \omega + 1, \omega^2 + 1 \rangle for every ordinal \lambda. Moreover, we prove (by reduction from Hilbert Tenth Problem) that the \exists^*\forall^6-fragment of \langle \omega^\lambda; \times \rangle is undecidable for every ordinal \lambda.

1 Introduction

The first-order theory of ordinals was studied under different signatures: linear order, addition, multiplication or some of their combinations. The purpose of this paper is to refine a result of the first author proving that the first-order theory of the multiplicative monoid of an ordinal is decidable if and only if it is less than \omega^\omega, [1, Thm 11]. We investigate the natural issue of trying to determine the boundary between decidability and undecidability for syntactic fragments of the theory. We first prove that the existential theory of \langle \mathbb{N}; +, | \rangle (which was shown to be decidable by Lipshitz [11]) is interpretable in the existential fragment of \langle \omega^\lambda; \times, \omega, \omega + 1, \omega^2 + 1 \rangle for every ordinal \lambda, where \omega, \omega + 1, \omega^2 + 1 correspond to constants. By [10], this yields a NEXPTIME lower bound for the complexity of the latter fragment. Then we prove, by reduction from Hilbert Tenth Problem, that the \exists^*\forall^6-fragment (an arbitrary number of existential variables followed by 6 universal variables) of \langle \omega^\lambda; \times \rangle is undecidable for every ordinal \lambda. Our results leave open the question of whether the existential theory of \langle \alpha; \times \rangle is decidable for \alpha \geq \omega^\omega. Another related open question (raised in [1]) is whether one can decide satisfiability of systems of multiplicative equations over ordinals with
constants. Both questions seem to be quite difficult. We give some insight about the difficulty in the final section of the paper.

We recall some historical background of the first-order theory of ordinal arithmetic (see e.g. [18]). The study of decidability and definability issues related to ordinal theories was initiated by Mostowski and Tarski who proved by means of quantifier-elimination that the class of well-ordered structures has a decidable elementary theory ([16, 6], see also [17]). From the undecidability of first-order arithmetic ([5]) one can deduce easily that for every ordinal $\xi$, the first-order theory of $\langle \omega^\xi; +, \times \rangle$ is undecidable. In the 1960’s, by means of automata, Büchi ([2], see also [14]) proved that for any ordinal $\alpha$, the weak monadic second-order theory of $\langle \alpha; < \rangle$ is decidable, from which he deduced decidability of the elementary theory of $\langle 2^{\alpha}; + \rangle$. This implies that the first-order theory of $\langle \omega^\xi; + \rangle$ is decidable for every ordinal $\xi$, since $\omega^\xi = 2^{\omega^\xi}$. On the other hand, the second author proved in [4] that the first-order theory of $\langle \omega^\xi; +, \times \rangle$ is undecidable.

Concerning the decidability of arithmetic without addition, i.e., of $\langle \omega; \times \rangle$, the result was announced by Skolem in [20]. Mostowski proved it in [15] as a direct consequence of his results on direct products of structures and Presburger’s decidability result for $\langle \omega; + \rangle$. Other proofs can be found in [3] and [8]. However, unlike the case of addition, decidability does not extend to all ordinals: in [1] the first author proved that the theory of $\langle \lambda; \times \rangle$ is decidable if and only if $\lambda < \omega^\omega$. The result still holds if we replace $\times$ with the two binary predicates $|_r$ and $|_l$ where $x|_r y$ (resp. $x|_l y$) means that $x$ is a right-hand (resp. left-hand) divisor of $y$ (both predicates are definable in $\langle \lambda; \times \rangle$).

We now give a brief outline of our paper. Section 2 contains the basic notions on ordinals such as the ordering and the two arithmetic operations of addition and multiplication which should suffice even for the reader with no strong background in the theory of ordinals. In Section 3 we show that three specific constants which are instrumental for our purpose can be expressed in $\langle \lambda; \times \rangle$ with formulas of low syntactic complexity. We denote by $\Omega$ the structure $\langle \lambda; \times \rangle$ enriched with these constants and we show in Section 4 that the integers with the addition and the divisibility can be interpreted in the existential fragment of $\Omega$ which gives us the above mentioned NEXPTIME lower bound. In Section 5 we reduce the problem of solving Diophantine equations in the nonnegative integers to the $\exists^*\forall^0$-fragment of $\langle \lambda; \times \rangle$ showing thus, via use Matijasevič result that this fragment is undecidable. In the last section 6 we observe that a simple proof of the decidability of the existential fragment of $\Omega$, would immediately offer an alternative proof of Makanin’s result for word equations.
2 Preliminaries

2.1 Ordinal arithmetic

We recall useful results about ordinal arithmetic. We refer the reader to the handbook of Sierpinski [19] for a more complete exposition of the topic.

The following definition of the Cantor normal form, abbreviated CNF, is actually a property.

**Definition 1.** Every ordinal \( \alpha > 0 \) has a unique form as a sum of decreasing \( \omega \)-powers with integer coefficients, namely

\[
\alpha = \omega^{\lambda_r} a_r + \cdots + \omega^{\lambda_1} a_1,
\]

where \( \lambda_r > \cdots > \lambda_1 \geq 0 \) are ordinals and \( a_r, \ldots, a_1 > 0 \) are integers. The ordinal \( \lambda_r \) is the degree of \( \alpha \), written \( \partial(\alpha) \), and \( \lambda_1 \) its valuation written \( v(\alpha) \). An ordinal is a successor if \( v(\alpha) = 0 \), otherwise it is a limit.

We are given two ordinals in their normal forms

\[
\alpha = \omega^{\lambda_r} a_r + \cdots + \omega^{\lambda_1} a_1, \quad \beta = \omega^{\mu_s} b_s + \cdots + \omega^{\mu_1} b_1,
\]

(1)

The order is defined by \( \alpha < \beta \) if one of the following conditions is satisfied.

1) \( r < s \) and for all \( i = 1, \ldots, r \) we have \( a_i = b_i \) and \( \lambda_i = \mu_i \).

2) for some \( t < \min\{r, s\} \) and for all \( i = 1, \ldots, t \) we have \( a_i = b_i \) and \( \lambda_i = \mu_i \), and either \( \lambda_{t+1} < \mu_{t+1} \) or \( \lambda_{t+1} = \mu_{t+1} \) and \( a_{t+1} < b_{t+1} \).

Furthermore, we define \( 0 < \alpha \) for every nonzero ordinal \( \alpha \).

We now recall the definition of the two arithmetic operations on the ordinals by use of their Cantor normal form

**Definition 2.** The sum \( \alpha + \beta \) of \( \alpha \) and \( \beta \) given by their CNF as in (1) is

\[
\begin{align*}
\omega^{\lambda_r} a_r + \cdots + \omega^{\lambda_1} a_1 + \omega^{\mu_s} b_s + \cdots + \omega^{\mu_1} b_1 & \quad \text{if } \lambda_1 > \mu_s > \lambda_{t-1}, \\
\omega^{\lambda_r} a_r + \cdots + \omega^{\lambda_1} a_{i+1} + \omega^{\mu_s} (a_i + b_r) + \cdots + \omega^{\mu_1} b_1 & \quad \text{if } \mu_s = \lambda_i, \\
\beta & \quad \text{if } \mu_s > \lambda_r.
\end{align*}
\]

Furthermore, we have \( 0 + \beta = \beta \) and \( \alpha + 0 = \alpha \). The sum is associative, has a neutral element 0 and is not commutative. It is left-cancellative (\( \alpha + \beta = \alpha + \gamma \) implies \( \beta = \gamma \)) but not right-cancellative.

**Definition 3.** The product \( \alpha \times \beta \) of \( \alpha \) and \( \beta \) given by their CNF as in (1) is

\[
\alpha \times \beta = \omega^{\lambda_r + \mu_s} b_s + \cdots + \omega^{\lambda_1 + \mu_1} b_1
\]

(2)

if \( \mu_1 > 0 \). If \( \mu_1 = 0 \) set \( \beta = \beta' + b_1 \) where \( 0 < b_1 < \omega \) and \( v(\beta') > 0 \). Then

\[
\alpha \times \beta = \alpha \times \beta' + \omega^{\lambda_r} (a_r \times b_1) + \omega^{\lambda_{r-1}} a_{r-1} + \cdots + \omega^{\lambda_1} a_1,
\]

3
yielding
\[ \alpha \times \beta = \omega^{\lambda_r + \mu_s} b_s + \omega^{\lambda_r + \mu_{s-1}} b_{s-1} + \cdots + \omega^{\lambda_r + \mu_2} b_2 + \omega^{\lambda_r} a_r b_1 + \omega^{\lambda_{r-1}} a_{r-1} + \cdots + \omega^{\lambda_1} a_1. \] (3)

Furthermore, we have \( 0 \times \beta = \alpha \times 0 = 0 \) for all \( \alpha, \beta \). The multiplication is associative, has a neutral element 1, is noncommutative, is left- (but not right-) cancellative (\( x \times y = x \times z \Rightarrow y = z \)) and left- (but not right-) distributes over the addition.

The next result is the unique combinatorial property of ordinals that we will need.

**Lemma 4.** ([19, page 352] The product of two successor ordinals commutes if and only if they are of the form \( \alpha^j \) and \( \alpha^n \) for some ordinal \( \alpha \) and some integers \( j, n \).

Observe that this property does not hold in general if the ordinals are limit, e.g., \( (\omega^2 + \omega)(\omega^3 + \omega^2) = (\omega^3 + \omega^2)(\omega^2 + \omega) \).

### 2.2 Prime factorization

We say that an ordinal \( \alpha > 0 \) is prime if it cannot be written as the product of two ordinals less than \( \alpha \); an equivalent definition is that \( \alpha \) admits exactly two right-hand divisors. One proves that there are three kinds of prime ordinals, [19, page 336]: natural primes less than \( \omega \) which are the usual primes, nonfinite successor primes which are of the form \( \omega^\mu + 1 \) for some \( \mu > 0 \), and limit primes which are of the form \( \omega^{\omega^\xi} \) for some ordinal \( \xi \).

The factorization of natural numbers is unique up to permutation of the factors. Here we require a stronger convention since, e.g., \( (\omega + 1) \cdot \omega = \omega \cdot \omega \). To this purpose Jacobsthal imposed a condition on the sequence of prime factors.

**Theorem 5 ([9]).** Every ordinal \( \alpha \) has a unique factorization of the form \( \alpha = \omega^{A_1} A_2 \) where \( A_1 \geq 0 \) and
\[
\begin{align*}
\omega^{A_1} &= (\omega^{\omega^{\xi_1}})^{a_1} \cdots (\omega^{\omega^{\xi_r}})^{a_r}, \\
A_2 &= a_0(\omega^{\mu_1} + 1)a_1(\omega^{\mu_2} + 1) \cdots a_{n-1}(\omega^{\mu_n} + 1)a_n
\end{align*}
\]
for some ordinals \( \xi_1 > \xi_2 > \cdots > \xi_r \), for some integers \( n \geq 0 \), \( a_0, a_1, \ldots, a_n > 0 \), and some ordinals \( \mu_1, \mu_2, \ldots, \mu_n \geq 1 \). We say that \( A_2 \) is the maximal successor right factor of the ordinal \( \alpha \).

Observe that the condition on the exponents of limit primes is necessary if one wants to guarantee unicity: for instance, \( \omega \) and \( \omega^\omega \) are primes and \( \omega^\omega = \omega \times \omega^\omega \). The prime factorization of the product of two prime factorizations follows the rule
\[
\omega^{A_1} A_2 \times \omega^{B_1} B_2 = \begin{cases} 
\omega^{A_1} A_2 B_2 & \text{if } B_1 = 0 \\
\omega^{A_1 + \delta(A_2) + B_1} B_2 & \text{if } B_1 \neq 0
\end{cases}
\] (4)
2.3 Logic

Let us specify our logical conventions and notations. We work within first-order predicate calculus with equality. Given a signature $\mathcal{L}$, an $\mathcal{L}$-structure is denoted as $\langle D; \mathcal{L} \rangle$ where $D$ is the base set of the structure and $\mathcal{L}$ is a set of interpreted relation and function symbols. We will actually confuse formal symbols and their interpretations.

We recall that a predicate is $\Sigma_n$ (resp. $\Pi_n$) if it is defined by a formula that begins with some existential (resp. universal) quantifiers and alternates $n − 1$ times between blocks of existential and universal quantifiers. A fragment of a theory is the set of sentences of bounded complexity. E.g., the existential fragment is the set of sentences of complexity $\Sigma_1$. It is a general concern when dealing with an undecidable theory and a motivation for our work to investigate its fragment with lowest complexity which is undecidable.

We will, when possible or when relevant, consider not only the number of alternations of blocks, but also the sizes of the blocks by specifying the number of variables in each block. For example, if $\phi(x, y, z, t, u, \ldots)$ is a quantifier-free formula then $\forall x, y \exists z, t, u \phi(x, y, z, t, u, \ldots)$ has complexity $\Pi_2$ but we may say more precisely that its complexity is $\forall \exists \exists \exists$, also written $\forall^2 \exists^3$. When speaking of a set of formulas, not single formulas, we might say that their complexity is for example $\forall^* \exists^3$ meaning that for each formula there exists an integer $n$ such that its complexity is $\forall^n \exists^3$.

We assume the reader has some familiarity with computing the logical complexity. When evaluating the complexity we will skip some steps to keep the computation readable. The next definition is crucial.

**Definition 6.** Given an $\mathcal{L}$-structure $M = \langle D; \mathcal{L} \rangle$, an $n$-ary relation $R$ over $D$ is elementary definable (shortly: definable) in $M$ if there exists a $\mathcal{L}$-formula $\varphi$ with $n$ free variables such that $R = \{ (a_1, \ldots, a_n) : M \models \varphi(a_1, \ldots, a_n) \}$. Given a syntactic fragment $F$, we say that $R$ is $F$-definable if $R$ is definable by a formula which belongs to $F$.

**Example 7.** In the structure $\langle \omega^\omega; \times \rangle$ where $\lambda$ is an ordinal greater than 0, the formula $x \times x = x$ defines the set $\{0, 1\}$ and the formula $(x \times x = x) \land \exists y (x \times y \neq x)$ defines the singleton $\{1\}$.

3 The multiplicative monoid of ordinals

We shall consider logical structures with domain an ordinal $\alpha$, which is identified with the set of ordinals $\beta < \alpha$, and predicates and functions whose interpretation correspond to restrictions to $\alpha$ of relations and functions defined on the class of ordinals, such as the function $\times$. For simplicity we will use a single symbol for each restriction of a relation, e.g. we simply write $\langle \alpha; \times \rangle$. Note that [1] dealt with $\times$ as a ternary relational symbol, which
allows one to consider any ordinal $\alpha$ as the base set of the structure. In this paper we consider $\times$ as a function symbol, which imposes that the base set of the structure is closed under multiplication, which holds if and only if it is an ordinal of the form $\omega^\lambda$ for some ordinal $\lambda$, as can be readily verified from the basic definitions of the operations on ordinals.

### 3.1 Elementary predicates

We show how the four constants $1, \omega, \omega + 1$ and $\omega^2 + 1$ can be expressed in $\langle \omega^\lambda; \times \rangle$ for every ordinal $\lambda > 0$. The idea is that having these constants as free may lower the syntactical complexity of the predicates.

**Proposition 8.** For every ordinal $\lambda > 0$, the following predicates are definable in $\langle \omega^\lambda; \times \rangle$

1. $\text{Zero}(x) = \{0\}$ is $\exists$-and $\forall$-definable
2. $\text{One}(x) = \{1\}$ is $\exists$-and $\forall$-definable
3. $\text{Prime}(x) = \{\alpha \mid \alpha$ is a prime$\}$ is $\forall\forall$-definable.
4. $\text{LimPrime}(x) = \{\omega^\xi \mid \xi$ is an ordinal$\}$ is $\exists\forall$- and $\forall\exists$-definable.
5. $\text{Omega}(x) = \{\omega\}$ is $\exists\forall\forall$-definable.
6. $\text{OmegaPlusOne}(x) = \{\omega + 1\}$ is $\exists\forall\forall$-definable.
7. $\text{OmegaSquarePlusOne}(x) = \{\omega^2 + 1\}$ is $\exists\forall\forall$-definable.

**Proof.** We prove successively these assertions.

1. $\text{Zero}(x)$ can be defined by the formula $\forall y (x \times y = x)$ and by $\exists y (y \times y \neq y \land x \times y = x)$
2. $\text{One}(x)$ is definable by $\forall y (x \times y = y)$ and by $x \times x = x \land \exists y (x \times y \neq x)$.
3. $\text{Prime}(x)$ is definable by $\forall y, z A(x, y, z)$ where $A(x, y, z)$ is the formula $x \times x \neq x \land (y \times y \neq y \land z \times y = z \times y \Rightarrow y = x)$.

   Indeed, it suffices to say that $x$ has exactly two right-hand side divisors, namely 1 and $x$.

4. $\text{LimPrime}(x)$ is definable by $\forall y, z \exists t C(x, y, z, t)$ and $\exists t \forall y, z C(x, y, z, t)$ where $C(x, y, z, t) = A(x, y, z) \land B(x, t)$ with $B(x, t) = t \times t \neq t \land t \times x = x$.

   It suffices to say that $x$ is a prime and that $t \times x = x$ for some $t \neq 0, 1$.

   Indeed if $x$ is a limit prime then e.g. $2x = x$. On the other hand if $x$ is a successor prime, then by unicity of factorization the equation $yx = x$ implies $y = 1$.
5. Omega(x) is definable by ∃t∀y, z E(x, y, z, t)
where E(x, y, z, t) = C(x, y, z, t) ∧ D(x, y, z) and D(x, y, z) is the formula

\[ y \times x = x \land z \times x = x \Rightarrow (y \times z = z \times y) \]

It suffices to say that ω is a limit prime and that all its left divisors are integers, thus ordinals that commute pairwise. Indeed, a limit prime different from ω is of the form ω^αω with α > 0, but then 2 × ω^αω = ωω^αω = ω^αω and 2 × ω = ω ≠ ω × 2.

6. OmegaPlusOne(x) can be defined by

Prime(x) ∧ ∃u (Omega(u) ∧ x × u = u × u ∧ x × u ≠ u × x)
= ∀y∀z A(x, y, z) ∧ ∃u(∃tvw E(u, v, w, t) ∧ x × u = u × u ≠ u × x)
= ∃u∀t∀y∀z (A(x, y, z) ∧ (E(u, y, z, t) ∧ x × u = u × u ≠ u × x)))
= ∃u∀t∀y∀z F(x, y, z, u, t)

Indeed, it suffices to say that ω + 1 is the only prime p ≠ ω which satisfies p × ω = ω^2. Indeed, if p is finite then p × ω = ω. If it is a nonfinite successor ω^α + 1 then p × ω = ω^α+1 = ω^2 implies α = 1. If it is of the form ω^αω with α > 0 then ω^αω × ω = ω^α+1 ≠ ω^2

7. OmegaSquarePlusOne(x). Similarly to the above case we observe that the ordinal ω^2 + 1

is the only prime p ≠ ω^2 such that p × ω = ω^3. This leads to a formula of the form ∃u∀t∀y∀z G(x, y, z, u, t) for some suitable quantifier-free formula G (replacing the formula F of the above case).

4 Interpreting the existential fragment of \langle \mathbb{N}; +, | \rangle in the existential fragment of \langle ω^n×, ω, ω + 1, ω^2 + 1 \rangle

Let λ > 1 be an ordinal. The full theory of \langle ω^λ× \rangle is undecidable, cf. [1, Thm. 11]. Here we show that the existential fragment of \langle ω^n×, +, | \rangle, cf. [11], can be interpreted in the existential fragment of the structure Ω = \langle ω^λ×, ×, 1, ω, ω + 1, ω^2 + 1 \rangle (i.e. the expansion of \langle ω^λ× \rangle with constants 1, ω, ω + 1, ω^2 + 1). This gives a lower bound on the complexity of the latter fragment.

The idea is to take advantage of the fact that the products of the elements ω + 1 and ω^2 + 1 define a free monoid to which the elementary combinatorial property of Lemma 4 applies.
Lemma 9. For every integer \( n \geq 0 \), the non null solutions of the equation
\[ z(\omega + 1)^n(\omega^2 + 1) = (\omega + 1)^n(\omega^2 + 1)z \]
in \( \omega^\lambda \) are of the form \( ((\omega + 1)^n(\omega^2 + 1))^r \) for arbitrary integers \( r \).

Proof. By Theorem 5, \( z \) can be written as \( z = \omega z_1 z_2 \) where \( z_2 \) is a successor ordinal. Applying again Theorem 5 to each member of the equation \( z(\omega + 1)^n(\omega^2 + 1) = (\omega + 1)^n(\omega^2 + 1)z \) implies that \( z_1 = 0 \), i.e. that \( z \) is a successor ordinal. By Lemma 4 this implies that there exist \( \alpha \) and \( j, r < \omega \) such that \( (\omega + 1)^n(\omega^2 + 1) = \alpha j \) and \( z = \alpha r \). The former condition implies \( j = 1 \), thus \( z = ((\omega + 1)^n(\omega^2 + 1))^r \). \( \square \)

The interpretation of \( \langle \mathbb{N}; +, |, 0, 1 \rangle \) in \( \Omega \) consists of identifying any integer \( i \) with the ordinal \( (\omega + 1)^i \).

Lemma 10. The set \( \text{Dom} = \{ (\omega + 1)^i \mid 0 \leq i < \omega \} \) is definable in \( \Omega \) by the quantifier-free formula
\[ \theta(x) : \quad x(\omega + 1) = (\omega + 1)x \land x(\omega + 1) \neq x. \]

Proof. If \( x \in \text{Dom} \) then it is clear that \( \theta(x) \) holds. For the converse assume that \( \theta(x) \) holds. The condition \( x(\omega + 1) \neq x \) implies \( x \neq 0 \). By Theorem 5, \( x \) has a unique factorization of the form \( x = \omega A_1 A_2 \) where \( A_2 \) is a successor ordinal. The equality \( x(\omega + 1) = (\omega + 1)x \) implies \( \omega A_1 A_2 (\omega + 1) = (\omega + 1) \omega A_1 A_2 \). Using again Theorem 5, we can deduce that \( A_1 = 0 \), i.e. that \( x \) is a successor ordinal. Therefore we can apply Lemma 4 to the equation \( x(\omega + 1) = (\omega + 1)x \), which yields \( x \in \text{Dom} \). \( \square \)

Proposition 11. For every existential closed formula \( \exists x_1, \ldots, x_n \phi(x_1, \ldots, x_n) \)
of \( \langle \mathbb{N}; +, |, 0, 1 \rangle \) there exists an existential closed formula \( \exists x_1, \ldots, x_n \Phi(x_1, \ldots, x_n) \)
of \( \Omega \) such that the following properties hold:

1. For all \( i_1, \ldots, i_n \in \mathbb{N} \):
\[ \langle \mathbb{N}; +, |, 0, 1 \rangle \models \phi(i_1, \ldots, i_n) \iff \Omega \models \Phi((\omega + 1)^{i_1}, \ldots, (\omega + 1)^{i_n}) \]

2. For all ordinals \( \alpha_1, \ldots, \alpha_n \) the condition \( \Omega \models \Phi(\alpha_1, \ldots, \alpha_n) \) implies that all \( \alpha_i \)'s are powers of \( \omega + 1 \) with integer exponent.

The proof of Proposition 11 relies on the possibility to interpret divisibility of integers in \( \Omega \).

Lemma 12. The predicate \( \text{Div}(x, y) = \{ (\omega + 1)^i, (\omega + 1)^j) \mid i \text{ divides } j \} \)
is definable in \( \Omega \) by a formula of the form \( \exists z, t E(x, y, z, t) \) where \( E \) is quantifier-free.
Proof. Consider the following predicates
\[
A(x, y, t) : \quad \theta(x) \land \theta(y) \land x = t(\omega + 1)^2, \\
B(z, t) : \quad zt(\omega^2 + 1) = t(\omega^2 + 1)z, \\
C(y, z) : \quad y\omega = z\omega.
\]
where $\theta$ is the formula of Lemma 10.

- $A(x, y, t)$ is equivalent to the existence of $i, j \geq 0$ such that $x = (\omega + 1)i + 2$, $t = (\omega + 1)i$ and $y = (\omega + 1)j$.
- Via Lemma 9, $B(z, t)$ is equivalent to the existence of an integer $r$ such that $z = ((\omega + 1)i(\omega^2 + 1))^r$.
- $C(y, z)$ implies that $j = (i + 2)r$.

Set $D(x, y, z, t) = A(x, y, t) \land B(z, t) \land C(y, z)$. For all ordinals $\alpha, \beta$ we have
\[
\Omega \models \exists z, t \ D(\alpha, \beta, z, t) \iff \alpha = (\omega + 1)i + 2, \beta = (\omega + 1)j \text{ where } (i+2) \text{ divides } j.
\]
It remains the case $i = 1$ which divides whatever value of $j$ and the case $i = j = 0$. Then $\text{Div}(x, y)$ can be defined in $\Omega$ by the formula
\[
(x = \omega + 1 \land y(\omega + 1) = (\omega + 1)y) \lor (x = y = 1) \lor \exists z, t \ D(x, y, z, t) \\
\equiv \exists z, t \ E(x, y, z, t) \quad (5)
\]
where $E(x, y, z, t) : (x = \omega + 1) \lor (x = y = 1) \lor (D(x, y, z, t))$ is quantifier-free.

Now we can prove Proposition 11.

Proof. Any existential closed formula in $\langle \mathbb{N}; +, |, 0, 1 \rangle$ is equivalent to a disjunction of formulas of the form
\[
\exists x_1, \ldots, x_n \bigwedge_k \ell_k(x_1, \ldots, x_n) | r_k(x_1, \ldots, x_n)
\]
where $\ell_k(x_1, \ldots, x_n)$ and $r_k(x_1, \ldots, x_n)$ are linear expressions in the variables $x_1, \ldots, x_n$. Thus we can assume w.l.o.g. that $\phi(x_1, \ldots, x_n)$ has the form $\bigwedge_k \ell_k(x_1, \ldots, x_n) | r_k(x_1, \ldots, x_n)$.

We recall that the interpretation goes by identifying the integer $i$ with the ordinal $(\omega + 1)^i$. Any linear combination $i_1 + \cdots + i_p + a$ is thus identified with the product
\[
(\omega + 1)^{i_1} \cdots (\omega + 1)^{i_p}(\omega + 1)^a = (\omega + 1)^{i_1 + \cdots + i_p + a}.
\]
The formula $\Phi(x_1, \ldots, x_n)$ is defined as $\bigwedge_k \gamma_k(x_1, \ldots, x_n)$ where each $\gamma_k$ is a quantifier-free formula which translates in $\Omega$ the formula $\ell_k(x_1, \ldots, x_n)|\tau_k(x_1, \ldots, x_n)$. The latter formula has the form

$$(x_i_1 + \cdots + x_i_p + a)\,(x_j_1 + \cdots + x_j_q + b)$$

where $1 \leq i_1, \ldots, i_p, j_1, \ldots, j_q \leq n$ and $a, b$ are integers. We can define $\gamma_k$ as

$$\exists x, y, z, t \ (E(x, y, z, t) \land \bigwedge_{k=1}^p \theta(x_i_k) \land \bigwedge_{\ell=1}^q \theta(x_{j_\ell})$$

$$\land x = x_{i_1} \cdots x_{i_p}(\omega + 1)^a \land y = x_{j_1} \cdots x_{j_q}(\omega + 1)^b$$

where $\theta$ is the formula of Lemma 10. 

Proposition 11 gives a lower bound for the satisfiability of the existential fragment of the theory $\Omega$.

Corollary 13. The satisfiability of the existential fragment of the theory $\Omega$ is at least as hard as that of the satisfiability of the existential fragment of the theory $\langle \mathbb{N}; +, |, 0, 1 \rangle$ which is in NEXPTIME (see [10]).

5 Undecidability of the $\Sigma_2$-fragment of $\langle \omega^\omega; \times \rangle$

Our objective is to interpret the Diophantine fragment of the nonnegative integers in the simplest fragment of $\Omega$ as possible. This is achieved in two steps. First by expressing in $\Omega$ the least common multiple of two integers, abbreviated lcm, in terms of divisibility, and then by expressing the multiplication in terms of lcm.

As a corollary we obtain undecidability of the $\Sigma_2$-fragment of $\Omega$ and finally of $\langle \omega^\omega; \times \rangle$ by removing the constants.

By using the same construction as in the proof of Proposition 11 (where we identify the integer $i$ with the ordinal $(\omega + 1)^i$) we are led to interpret the function lcm of integers in $\Omega$ and to investigate the complexity of this predicate. We use the $\Sigma_1$-predicate $\text{Div}(x, y)$ introduced in Lemma 12 and set

$$\text{LCM}(x, y, z) = \{((\omega + 1)^i, (\omega + 1)^j, (\omega + 1)^k) \mid k = \text{lcm}(i, j)\}.$$ 

Lemma 14. The predicate $\text{LCM}(x, y, z)$ is definable in $\Omega$ by a $\exists^4 \forall^6$-formula.

Proof. The predicate $\text{LCM}(x, y, z)$ is definable by the formula

$$\text{Div}(x, z) \land \text{Div}(y, z) \land \forall u((\text{Div}(x, u) \land \text{Div}(y, u) \land u\omega \neq z\omega)$$

$$\rightarrow \forall t(u\omega \neq z\omega)).$$

Indeed, this formula can be interpreted as saying that $x = (\omega + 1)^i, y = (\omega+1)^j$ and $z = (\omega+1)^k$ where $i$ and $j$ divide $k$ and all integers $\ell \neq k$ divisible
by $i$ and $j$ are greater than $k$. Substitute expression 5 for each occurrence of $\exists ! v$. The following sequence of equivalences leads to a complexity in $\exists^4 \forall^6$, namely

\[
\begin{align*}
\text{LCM}(x, y, z) &\equiv \\
&\exists u_1, u_2 \ E(x, z, u_1, u_2) \land \exists u_3, u_4 \ E(y, z, u_3, u_4) \\
&\land \forall v((\exists v_1, v_2 \ E(x, v, v_1, v_2) \land \exists v_3, v_4 \ E(y, v, v_3, v_4) \land v \omega \neq z \omega) \\
&\rightarrow \forall v_5 (v_5 \omega \neq z \omega)) \\
&\equiv \exists u_1, u_2 \ E(x, z, u_1, u_2) \land \exists u_3, u_4 \ E(y, z, u_3, u_4) \\
&\land \forall v (\forall v_1, v_2 (\lnot E(x, v, v_1, v_2)) \lor \forall v_3, v_4 (\lnot E(y, v, v_3, v_4)) \lor (v \omega = z \omega) \\
&\lor \forall v_5 (v_5 \omega \neq z \omega)) \\
&\equiv 3^4 \forall^6 \exists^6 \ F(x, y, z, \overrightarrow{u}, \overrightarrow{v})
\end{align*}
\]

where $F$ is quantifier-free.

\[
\Box
\]

Multiplication is expressible in terms of the least common multiple. Indeed, a simple computation shows

\[
x \times y = \frac{1}{2} (\text{lcm}(x + y, x + y + 1) - \text{lcm}(x, x + 1) - \text{lcm}(y, y + 1)).
\]

(6)

We are ready to show that the multiplication can be interpreted in $\Omega$ at a low cost. We set

\[
\text{MULT}(x, y, z) = \{(\omega + 1)^i, (\omega + 1)^j, (\omega + 1)^k) \mid i \times j = k\}.
\]

Lemma 15. The predicate $\text{MULT}(x, y, z)$ is definable in $\Omega$ by a $\exists^1 \forall^5 \exists^6$-formula.

Proof. In view of expression 6 we can define $\text{MULT}(x, y, z)$ as follows:

\[
\exists r_1 r_2 r_3 \ (z \times z \times r_1 \times r_2 = r_3 \land \text{LCM}(x \times y, x \times y (\omega + 1), r_3) \land \text{LCM}(x, x (\omega + 1), r_1) \\
\land \text{LCM}(y, y (\omega + 1), r_2)).
\]

Using the expression given in the proof of Lemma 14 we get

\[
\exists r_1 r_2 r_3 \exists u_1' \exists u_2' \exists u_3' \exists v_1' \exists v_2' \exists v_3' \exists^4 \forall^6 \exists^6 \forall^6 \exists^4 \forall^6 \exists^6 \forall^6
\]

\[
((z \times z \times r_1 \times r_2 = r_3) \land F(x \times y, x \times y (\omega + 1), r_3, u_1', v_1')) \\
\land F(x, x (\omega + 1), r_1, u_1', v_1') \land F(y, y (\omega + 1), r_2, u_2', v_2'))
\]

\[
\Box
\]

We wish to express in $\Omega$ all terms of $\langle \mathbb{N}; +, \times, 0, 1 \rangle$. We start with terms $W$ which are products of variables.

11
Definition 16. Let \( W(x_1, \ldots, x_k) \) be a term of \( \langle \mathbb{N}; +, \times, 0, 1 \rangle \) which consists of a product of occurrences among of the variables \( x_1, \ldots, x_k \). We set
\[
\text{TERM}_W(y, x_1, \ldots, x_k) = \{(\omega+1)^i, (\omega+1)^j, \ldots, (\omega+1)^k) \mid i = W(j_1, \ldots, j_k)\}.
\]

Lemma 17. For each term \( W(x_1, \ldots, x_k) \) which consists of a product of occurrences among of the variables \( x_1, \ldots, x_k \), the predicate \( \text{TERM}_W(y, x_1, \ldots, x_k) \) is definable in \( \Omega \) by a formula of the form
\[
\exists^m u \ \forall^6 v G(y, x_1, \ldots, x_k, u, v)
\]
where \( m \) depends on the term, and \( G \) is quantifier-free.

Proof. We proceed by induction on the length of the product. If the product is reduced to a single variable then by using dummy variables it is always possible to put the equality \( y = x \) in the appropriate form. For the induction step, consider two terms \( W_1(x_1, \ldots, x_k) \) and \( W_2(x_1, \ldots, x_k) \), and assume that \( \text{TERM}_{W_1}(y, x_1, \ldots, x_k) \) and \( \text{TERM}_{W_2}(y, x_1, \ldots, x_k) \) are definable in \( \Omega \) by the formulas
\[
\exists^{m_1} u_1 \ \forall^6 v_1 G_1(y, x_1, \ldots, x_k, u_1, v_1)
\]
and
\[
\exists^{m_2} u_2 \ \forall^6 v_2 G_2(y, x_1, \ldots, x_k, u_2, v_2),
\]
respectively, where the bound variables which are the components of \( u_1 \) and \( u_2 \) are pairwise different. Then \( \text{TERM}_{W_1 \times W_2}(y, x_1, \ldots, x_k) \) can be defined by the formula:
\[
\exists y_1, y_2 ((y = y_1 y_2) \land \exists^{m_1} u_1 \ \forall^6 v_1 G_1(y_1, x_1, \ldots, x_k, u_1, v_1) \land \exists^{m_2} u_2 \ \forall^6 v_2 G_2(y_2, x_1, \ldots, x_k, u_2, v_2)). \tag{7}
\]

By applying the rule \( \exists x f(x, \ldots) \land \exists y g(y, \ldots) \equiv \exists x f(x, \ldots) \land g(x, \ldots) \) provided \( x \) and \( y \) are different, \( y \) does not occur in \( f \) and \( x \) in \( g \), and the rule \( \forall x f(x, \ldots) \land \forall y g(y, \ldots) \equiv \forall x (f(x, \ldots) \land g(x, \ldots)) \), the formula 7 is equivalent to
\[
\exists y_1, y_2 \exists^{m_1} u_1 \exists^{m_2} u_2 \forall^6 v_1 ((y = y_1 y_2) \land G_1(y_1, x_1, \ldots, x_k, u_1, v_1) \land G_2(y_2, x_1, \ldots, x_k, u_2, v_1)). \tag{8}
\]

The ultimate goal is to apply Matjasevich undecidability result for the Diophantine fragment of the integers with addition and multiplication [13] to show that the \( \exists^* \forall^6 \)-fragment of \( \Omega \) is undecidable.

Definition 18. Let \( E(x_1, \ldots, x_p) \) be an atomic formula of \( \langle \mathbb{N}; +, \times, 0, 1 \rangle \) (i.e. a Diophantine equation in \( \mathbb{N} \)). We set
\[
\text{EQ}_E(x_1, \ldots, x_p) = \{(\omega+1)^i_1, \ldots, (\omega+1)^i_p) \mid \langle \mathbb{N}; +, \times, 0, 1 \rangle \models E(j_1, \ldots, j_p)\}.
\]
Theorem 19. For each atomic formula $E(x_1, \ldots, x_p)$ of $(\mathbb{N}; +, \times, 0, 1)$, the predicate $E(x_1, \ldots, x_p)$ is definable in $\Omega$ by a $\exists^\forall^6$-formula.

Proof. The equation $E(x_1, \ldots, x_p)$ can be written as

$$W_1(x_1, \ldots, x_p) + \cdots + W_n(x_1, \ldots, x_p) = W_{n+1}(x_1, \ldots, x_p) + \cdots + W_{n+m}(x_1, \ldots, x_p)$$

where each $W_i$ is a product of occurrences of variables among $x_1, \ldots, x_p$. By Lemma 17 for $i = 1, \ldots, n + m$ there exists a formula

$$\Phi_i \equiv \exists^n v_i \forall^6 v_i \phi_i(y_i, x_1, \ldots, x_p, \overrightarrow{u}, \overrightarrow{v})$$

which defines $\text{TERM}_{W_i}(y_i, x_1, \ldots, x_p)$ in $\Omega$. Set $\Phi(x_1, \ldots, x_p)$ as the formula

$$\exists y_1, \ldots, y_{n+m}( \bigwedge_i \exists y_i \forall^6 v_i \Phi_i(y_i, x_1, \ldots, x_p, \overrightarrow{u}, \overrightarrow{v}) ) \land (y_1 \times \cdots \times y_n = y_{n+1} \times \cdots \times y_{n+m}). \quad (9)$$

It is clear that $\Phi(x_1, \ldots, x_p)$ defines $E(x_1, \ldots, x_p)$ in $\Omega$. By routine rewriting on quantifiers and renaming of bound variables, the above formula is equivalent to

$$\exists y_1, \ldots, y_{n+m} \exists \overrightarrow{u} \cdots \exists \overrightarrow{u}_{n+m} \forall \overrightarrow{v} \left( \bigwedge_i \Phi_i(y_i, x_1, \ldots, x_p, \overrightarrow{u}_i, \overrightarrow{v}) \right) \land (y_1 \times \cdots \times y_n = y_{n+1} \times \cdots \times y_{n+m})$$

which is a formula of complexity $\exists^* \forall^6$. \qed

Corollary 20. The $\exists^\forall^6$-fragment of $\langle \omega^{\omega^\lambda}; x \rangle$ is undecidable for every ordinal $\lambda \geq 1$.

Proof. We show how we can remove the constants $1, \omega, (\omega + 1)$ and $(\omega^2 + 1)$. By Proposition 8, there exist quantifier-free predicates $\alpha, \beta, \gamma, \delta$ such that

$$\{1\} \text{ is definable in } \langle \omega^{\omega^\lambda}; x \rangle \text{ by } \exists y \alpha(x, y)$$

$$\{\omega\} \text{ is definable by } \exists y \forall z, t \beta(x, y, z, t)$$

$$\{\omega + 1\} \text{ is definable by } \exists y_1 \exists y_2 \forall z, t \gamma(x, y, z, t)$$

$$\{\omega^2 + 1\} \text{ is definable by } \exists y_1 \exists y_2 \forall z, t \delta(x, y, z, t)$$

Thus every formula

$$\exists \overrightarrow{u} \forall \overrightarrow{v} \phi(x_1, \ldots, x_p, \overrightarrow{u}, \overrightarrow{v})$$

over $\Omega$, where $\overrightarrow{v} = (v_1, \ldots, v_6)$ and $\phi$ is quantifier-free, is equivalent in $\langle \omega^{\omega^\lambda}; x \rangle$ to the formula

$$\exists a, b, c, d (\exists y \alpha(a, y) \land \exists y \forall z, t \beta(b, y, z, t) \land \exists y_1 \exists y_2 \forall z, t \gamma(c, y, z, t) \land \exists y_1 \exists y_2 \forall z, t \delta(d, y, z, t) \land \exists \overrightarrow{u} \forall \overrightarrow{v} \Phi(x_1, \ldots, x_p, \overrightarrow{u}, \overrightarrow{v}))$$
where $\Phi$ is obtained from $\phi$ by substituting $a$, $b$, $c$ and $d$ for every occurrence of the constants $1$, $\omega$, $\omega + 1$ and $\omega^2 + 1$ and where $a$, $b$, $c$, $d$, $y$, $z$, $t$ are pairwise different variables which are also different from the components of $\overrightarrow{u}$ and $\overrightarrow{v}$. By introducing four new variables $z_1$, $z_2$, $z_3$, $z_4$ this formula is equivalent to

$$
\exists a, b, c, d, z_1, z_2, z_3, z_4, \overrightarrow{u}
\begin{array}{c}
(\alpha(a, z_1) \land \forall z, t \beta(b, z_2, z, t) \land \forall z, t \gamma(c, z_3, z, t) \land \forall z, t \delta(d, z_4, z, t) \\
\land \forall \overrightarrow{v} \overrightarrow{\Phi}(x_1, \ldots, x_p, \overrightarrow{u}, \overrightarrow{v})
\end{array}
$$

which is equivalent to

$$
\exists a, b, c, d, z_1, z_2, z_3, z_4, \overrightarrow{u}
\begin{array}{c}
\forall \overrightarrow{v} (\alpha(a, z_1) \land \beta(b, z_2, v_1, v_2) \land \gamma(c, z_3, v_1, v_2) \land \delta(d, z_4, v_1, v_2) \\
\land \overrightarrow{\Phi}(x_1, \ldots, x_p, \overrightarrow{u}, \overrightarrow{v})
\end{array}
$$

6 Final observation

Our results leave open the question of whether the existential fragment of $\Omega$ is decidable. By Section 4, a proof of decidability for the existential fragment of $\Omega$ would provide a new proof of decidability for the existential fragment of $\langle \mathbb{N} ; +, \mid \rangle$. We show that it would also provide a new proof of Makanin’s result of decidability for word equations with constants [12]. This suggests that a conceptually simple proof of decidability for the existential fragment of $\Omega$ may be difficult to obtain.

The relation between the problem we tackled and that of word equations with constants comes from the fact that the multiplicative monoid of the structure $\Omega = \langle \alpha; \times, 1, \omega, \omega + 1, \omega^2 + 1 \rangle$ (where $\alpha = \omega^\lambda$ for some $\lambda > 0$) has an (infinitely generated) free submonoid, namely that generated by the infinite successor primes less than $\alpha$.

Given a word equation with constants over a binary alphabet $\{a, b\}$ (which is no loss of generality because the unary case is trivial and the general case with more than one letter reduces to the binary case)

$$
L(x_1, \ldots, x_n, a, b) = R(x_1, \ldots, x_n, a, b), \quad (10)
$$

we define the following conditions over $\Omega$

$$
L'(x_1, \ldots, x_n, (\omega + 1), (\omega^2 + 1)) = R'(x_1, \ldots, x_n, (\omega + 1), (\omega^2 + 1)) \land (\omega + 1)x_1 \neq \omega x_1 \land \ldots \land (\omega + 1)x_n \neq \omega x_n \quad (11)
$$

where $L'$ (resp. $R'$) is obtained from $L$ (resp. $R$) by substituting $(\omega + 1)$ and $(\omega^2 + 1)$ for $a$ and $b$. Every solution $\theta(x_i) = u_i \in \{a, b\}^*$ of (10) yields a solution for (11), namely $\theta'(x_i) = f(u_i) \in \alpha$ where $f$ substitutes $(\omega + 1)$.
and \((\omega^2 + 1)\) for every occurrence of \(a\) and \(b\), respectively. Conversely, if \(\theta'(x_i) = \alpha_i \in \alpha\) is a solution for 11, then each inequality \((\omega + 1)x_i \neq \omega x_i\) implies that \(x_i\) is a successor. Now observe that the successors define a submonoid of \(\alpha\). On this submonoid consider the mapping \(g\) which maps every
\[
x = a_0(\omega^{\mu_1} + 1)a_1(\omega^{\mu_2} + 1) \cdots a_n-1(\omega^{\mu_n} + 1)a_n
\]
where \(\mu_i\)'s are non null ordinals and the \(a_i\)'s are positive integers, to
\[
g(x) = (\omega^{\nu_1} + 1)(\omega^{\nu_2} + 1) \cdots (\omega^{\nu_n} + 1)
\]
with \(\nu_i = \min\{\mu_i, 2\}\). It is clearly a morphism and if \(\theta'\) is a solution of 11, so is \(g \circ \theta'\). Composing with the morphism \(h : \{(\omega + 1), (\omega^2 + 1)\}^* \to \{a, b\}\) defined by \(h((\omega + 1)) = a, h((\omega^2 + 1)) = b\), yields a solution of 10 by setting \(\theta(x_i) = h(g(\theta'(x_i)))\).

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