On the Non-relativistic Limit of Linear Wave Equations for Zero and Unity Spin Particles

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Abstract

The non-relativistic limit of the linear wave equation for zero and unity spin bosons of mass $m$ in the Duffin–Kemmer–Petiau representation is investigated by means of a unitary transformation, analogous to the Foldy–Wouthuysen canonical transformation for a relativistic electron. The interacting case is also analyzed, by considering a power series expansion of the transformed Hamiltonian, thus demonstrating that all features of particle dynamics can be recovered if corrections of order $1/m^2$ are taken into account through a recursive iteration procedure.

1 Introduction

Recently, in view of the increasing technical complexity of string theories as the best candidates for the unification of the fundamental interactions, there is a renewed interest in the quantum field theory of higher spins as a natural covariant formalism for accommodating the particle spectra in the Standard Model and quantum gravity theories, as well as in their supersymmetric counterparts. Thus, from the phenomenological standpoint, it is mandatory to investigate such theories in the low-energy regime, by examining their non-relativistic formal properties and taking into account the interaction with external electromagnetic and/or metric fields as a starting point.

In relativistic quantum mechanics, one must seek for a relation between irreducible representations of the Poincaré group and wave equations. In Wigner’s standard form, non-trivial wave equations can only be presented for wave functions with a large number of components, simultaneously expressing constraints
on redundant components and equations of motion for the physical ones. Considering general invariant equations, Gel’fand and Yaglom\textsuperscript{1} expressed relativistic wave functions in terms of linear differential operators, simultaneously determining both these operators and the finite-dimensional representations of the homogeneous Lorentz group, according to which the components of the wave functions transform. However, such a procedure is not applicable to non-relativistic wave equations whose solutions transform according to the homogeneous Galilei group. Following another approach, relying upon the Bargmann–Wigner method, Lévi-Leblond\textsuperscript{2} constructed a basis in a ten-dimensional representation space of the homogeneous Galilei group for free massive particles of spin 1, by taking a complete set of independent linear combinations of symmetrical tensor products of two-component wave functions which describe non-relativistic particles of spin 1/2, and arriving at a system of equations involving linear operators.

In order to investigate the physical properties of particles of zero and unity spin in the presence of electromagnetic external sources, instead of starting from Galilean-covariant wave equations, we start from a Lorentz-covariant linear wave equation in the Hamiltonian form and apply a canonical transformation, analogous to the Foldy–Wouthuysen (FW) transformation\textsuperscript{3} for Dirac fermions, to a suitable reference frame in which one can recognize the different couplings of charged bosons with the electromagnetic field. In this sense, the covariant linear representation\textsuperscript{4,5,6} of Duffin–Kemmer–Petiau (DKP) proves to be particularly useful, since all physical quantities are constructed from linear operators which obey convenient algebraic relations, in close similarity with the familiar Dirac operators. Notably, Darwin\textsuperscript{7} proposed a linear wave equation for the electromagnetic field some years before Petiau’s pioneering work, in close relation to the meson theory proposed by Kemmer, who referred to Dirac’s work\textsuperscript{8} on linear relativistic equations for particles with spins higher than one-half.

This work is organized as follows. In Section 2, we present the linear wave equation which describes bosons of spin zero and unity and the basic identities of the associated DKP algebra; we then rewrite this equation in the Hamiltonian form for non-interacting particles. In Section 3, we discuss the quantum canonical transformation for the free boson Hamiltonian, by analogy with the ordinary FW transformation. Next, in Section 4, we derive the non-relativistic limit of the Hamiltonian that describes charged bosons interacting with an external electromagnetic field. In Section 5, we make concluding remarks.

2 DKP Hamiltonian

Let us briefly review the DKP formalism for non-interacting bosons of spin zero and one. The relativistic wave equation in such a representation reads

\[(i\partial^\alpha - m)\psi = 0,\]

where \(\partial^\alpha \equiv \beta^\mu \partial^\mu\) and \(\psi\) is a five(ten)-row column associated with the zero (unity) spin field. The considerations of this work do not refer to any particular
representation for $\psi$.

The $\beta$-matrices obey the algebra
\begin{equation}
\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = \beta_\mu g_{\nu \rho} + \beta_\rho g_{\nu \mu},
\end{equation}
which implies the following consequences:
\begin{align}
\beta_0 \beta_k \beta_0 &= 0, \quad k = 1, 2, 3, \\
\beta_0^2 &= \beta_0, \\
\beta \beta_\nu \beta &= \beta \beta_\nu, \\
(\vec{\beta} \cdot \vec{b}) \beta_0 (\vec{\beta} \cdot \vec{b}) &= 0,
\end{align}
where $b_\mu = (b_0, \vec{b})$ is a generic four-vector.

Multiplying (1) by $\partial /\beta_\mu$ and using (4),
\begin{equation}
(i \partial /\beta_\mu - m \partial /\beta_\mu) \psi = (i \partial_\mu - m \partial_\mu) \psi = 0,
\end{equation}
and then (1),
\begin{equation}
(m \partial_\mu - m \partial_\mu) \psi = 0,
\end{equation}
one obtains
\begin{equation}
\partial_\mu \psi = \partial /\beta_\mu \psi.
\end{equation}

Multiplying (1) by $\beta_0$ and taking the zero component of (6), times the imaginary unity, one obtains, upon adding the results,
\begin{equation}
\{ i [\partial_0 + \partial^k (\beta_0 \beta_k - \beta_k \beta_0)] - m \beta_0 \} \psi = 0,
\end{equation}
or
\begin{equation}
i \partial_0 \psi = H \psi,
\end{equation}
where
\begin{equation}
H = -i \vec{\alpha} \cdot \vec{\nabla} + \beta_0 m = \vec{\alpha} \cdot \vec{p} + \beta_0 m
\end{equation}
is the DKP Hamiltonian, and $\vec{\alpha}$ is defined by its spatial components:
\begin{equation}
\alpha_k \equiv \beta_0 \beta_k - \beta_k \beta_0, \quad k = 1, 2, 3.
\end{equation}

3 FW Transformation

As in the electron case, we now look for a unitary transformation,
\begin{align}
\psi' &= e^{iU} \psi, \\
H' &= e^{iU} H e^{-iU},
\end{align}
which should eliminate the term that involves the spatial components of the four-momentum. In case $H$ explicitly depends on time, equation (7) yields
\begin{equation}
i \partial_0 (e^{-iU} \psi') = H e^{-iU} \psi',
\end{equation}
so that
\[ e^{-iU} (i\partial_0 \psi') = (He^{-iU} - i\partial_0 e^{-iU}) \psi' , \]
or
\[ i\partial_0 \psi' = H' \psi' , \]
where
\[ H' = e^{iU} (H - i\partial_0) e^{-iU} . \tag{9} \]

Let us choose
\[ U = -\frac{\vec{\beta} \cdot \vec{p}}{|\vec{p}|} . \]
The \( \beta \)-algebra (2) implies the identity
\[ 2(\vec{\beta} \cdot \vec{p})^3 = \sum_{ijk} p_i p_j p_k (\beta_i \beta_j \beta_k + \beta_k \beta_j \beta_i) \]
\[ = - \sum_{ijk} p_i p_j p_k (\beta_i \delta_{jk} + \beta_k \delta_{ji}) , \]
so that
\[ (\vec{\beta} \cdot \vec{p})^3 = -|\vec{p}|^2 (\vec{\beta} \cdot \vec{p}) . \tag{10} \]

Representing (10) in the form
\[ [(\vec{\beta} \cdot \vec{p})^2 + |\vec{p}|^2] (\vec{\beta} \cdot \vec{p}) = 0 \]
and then, on the mass shell,
\[ (\vec{\beta} \cdot \vec{p}) = \beta_0 p_0 - \vec{p} = \beta_0 E + m , \]
we have
\[ [(\vec{\beta} \cdot \vec{p})^2 + |\vec{p}|^2] (\vec{\beta} \cdot \vec{p}) E + m \psi = 0 . \tag{11} \]

On the other hand, due to the identity
\[ (\vec{\beta} \cdot \vec{p})^2 \beta_0 = \sum_{ij} p_i p_j (\beta_i \beta_j \beta_0) \]
\[ = - \sum_0 p_i p_j (\beta_0 \beta_j \beta_i + \beta_0 \delta_{ij}) = -\beta_0 [(\vec{\beta} \cdot \vec{p})^2 + |\vec{p}|^2] , \]
equation (11) implies
\[ (m - \beta_0 E) (\vec{\beta} \cdot \vec{p})^2 \psi = -|\vec{p}|^2 m \psi , \]
or
\[ (m^2 - \beta_0^2 E^2)(\vec{\beta} \cdot \vec{p})^2 \psi = -(m^2 + \beta_0 E m) |\vec{p}|^2 \psi . \]

Then, due to (3), we obtain
\[ (\vec{\beta} \cdot \vec{p})^2 = -|\vec{p}|^2 + (Em) \beta_0 + E^2 \beta_0^2 . \tag{12} \]
Eq. (8) does not contain the complete information about the system because of multiplication by the singular matrix $\beta_0$.

Multiplying (1) by $(1 - \beta_0^2)$, one finds the additional constraint
\[
[i \partial^k \beta_k \beta_0^2 - (1 - \beta_0^2) m] \psi = 0 ,
\]
or
\[
(\vec{\beta} \cdot \vec{p}) \beta_0^2 + (1 - \beta_0^2) m = 0 ,
\]
on the mass shell. Also, left-multiplying (12) by $(\vec{\beta} \cdot \vec{p})$, and using (10) and (13), one obtains
\[
(\vec{\beta} \cdot \vec{p}) \beta_0 = E (1 - \beta_0^2) .
\]
Now, multiplying (12) by $(\vec{\beta} \cdot \vec{p})^2$ and using (13), (14), one gets
\[
(\vec{\beta} \cdot \vec{p})^4 = -(\vec{\beta} \cdot \vec{p})^2 |\vec{p}|^2 .
\]

Then
\[
e^{iU} = e^{(\vec{\beta} \cdot \vec{p}) |\vec{p}| \theta} = 1 + \frac{(\vec{\beta} \cdot \vec{p})^2}{|\vec{p}|^2} (1 - \cos \theta) + \frac{(\vec{\beta} \cdot \vec{p})}{|\vec{p}|} \sin \theta ,
\]
where (10) and (15) have been used in the series expansion. Hence,
\[
H' = (\vec{\alpha} \cdot \vec{p}) \left( \cos \theta - \frac{m}{|\vec{p}|} \sin \theta \right) + \beta_0 \left( |\vec{p}| \sin \theta + m \cos \theta \right) .
\]
Choosing
\[
\sin \theta = \frac{|\vec{p}|}{E} , \ \cos \theta = \frac{m}{E} ,
\]
on one arrives at
\[
H' = \frac{\beta_0}{E} \left( \vec{p}^2 + m^2 \right) = \beta_0 E .
\]

4 DKP Interaction Hamiltonian

In order to have a better understanding of the particle content of the theory, let us examine the behavior of charged bosons in the presence of an external electromagnetic field, transformed to a reference frame where particles carry low momenta. The electromagnetic interaction is introduced by means of the covariant derivative, so that
\[
(iD - m) \psi = 0 ,
\]
where the covariant derivative
\[
D_\mu = \partial_\mu + ieA_\mu
\]
satisfies the commutation relation
\[
[D_\mu, D_\nu] = ieF_{\mu\nu} .
\]
Multiplying (16) by $D\beta_\mu$, one obtains, 

$$D_\mu \psi = D\beta_\mu \psi + \frac{e}{2m} F^{\rho\sigma} (\beta_\rho \beta_\mu \beta_\sigma - \beta_\rho g_{\mu\sigma}) \psi.$$  

(17)

Then, from equations (16) and (17),

$$i\partial_0 \psi = H\psi,$$

it follows that

$$H = H^{(0)} + H^{(1)},$$

where

$$H^{(0)} = \vec{\alpha} \cdot \vec{\pi} + m\beta_0 - eA_0,$$

$$H^{(1)} = \frac{ie}{2m} F^{\rho\sigma} (\beta_\rho \beta_0 \beta_\sigma - \beta_\rho g_{0\sigma}) ,$$

and

$$\vec{\pi} = \vec{p} - e\vec{A}.$$  

Using (19) and the Baker–Campbell–Hausdorf formula, one can write

$$H' = H + \frac{\partial U}{\partial t} + i \left[ U, H + \frac{1}{2} \frac{\partial U}{\partial t} \right] - \frac{1}{2!} \left[ U, \left[ U, H + \frac{1}{3} \frac{\partial U}{\partial t} \right] \right] + \ldots.$$  

By virtue of the nonrelativistic limit $\theta \sim \sin \theta \sim |\vec{p}|/m$, one can choose, in the first approximation, by analogy with the free case,

$$U = -\frac{\vec{\beta} \cdot \vec{\pi}}{m}.$$  

From the commutation relations (A.1)–(A.7) and the vector identities (A.8), (A.9), listed in the Appendix, one obtains

$$[U, H^{(1)}] = -\frac{e}{m^2} [\vec{\beta} \cdot \vec{\pi}, (\vec{\beta} \cdot \vec{E}) \beta_0^2] + \frac{e}{m^2} [\vec{\beta} \cdot \vec{\pi}, \vec{\beta} \cdot \vec{E}] = -\frac{e}{2m^2} [\vec{\beta} \cdot \vec{\pi}, F^{ij} \beta_i \beta_j] .$$

In addition,

$$[\vec{\beta} \cdot \vec{\pi}, (\vec{\beta} \cdot \vec{E}) \beta_0^2] = i\vec{S} \cdot [\vec{\pi} \times \vec{E}] \beta_0^2 + (\vec{\beta} \cdot \vec{E}) [2(\vec{\beta} \cdot \vec{\pi}) \beta_0^2 - \vec{\beta} \cdot \vec{\pi}] ,$$

so that one arrives at

$$H' = m\beta_0 - eA_0 + \frac{\vec{\pi}^2}{2m} \left( \vec{\alpha} \cdot \vec{\pi} - \beta_0 \right) + \frac{e}{2m} (\vec{S} \cdot \vec{H}) \beta_0 + \frac{e}{2m} (\vec{\beta} \times \vec{\alpha}) \cdot \vec{H} + \frac{e}{2m^2} (\vec{S} \cdot (\vec{\pi} \times \vec{E})) (1 + 2\beta_0^2) + \frac{ie}{2m^2} [\vec{\beta} \cdot \vec{\pi}, (\beta_0 \vec{S} + \vec{\beta} \times \vec{\alpha}) \cdot \vec{H}]$$

$$- \frac{ie}{m} (\vec{\beta} \cdot \vec{E}) \beta_0^2 - \frac{ie}{m^2} (\vec{\beta} \cdot \vec{E}) [2(\vec{\beta} \cdot \vec{\pi}) \beta_0^2 - \vec{\beta} \cdot \vec{\pi}] + \mathcal{O} \left( m^{-3} \right) ,$$

(18)
where relation (A.10) has been used. In the above expression, $\vec{S}$ corresponds to the spin operator of bosons,

$$S_{ij} = i(\beta_i\beta_0\beta_j - \beta_j\beta_0\beta_i), \quad i,j = 1,2,3,$$

with eigenvalues 0 or 1, while $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields, respectively.

Expression (18) is analogous to the Hamiltonian of the Pauli equation for spin-1/2 fermions in the case of charged bosons of spin 0 and 1 on the background of an external electromagnetic field. In (18), we can recognize each term individually. For example, the second term is related to the electrostatic potential, while the third one corresponds to the kinetic term of the non-relativistic interaction Hamiltonian. In fact, taking the same steps that led to equation (14) on the mass shell, and using the definition of the matrices $\alpha_k$, one can rewrite the kinetic term in the transformed Hamiltonian as the expression

$$\vec{\pi}^2 \left[ \frac{\pi_0}{m}(2\beta_0^2 - 1) \right],$$

which is indeed diagonal and non-singular in the matrix realization of the DKP $\beta$-algebra, as one should expect by analogy with the disentangling property of the FW transformation.

In this approach, the most essential result is the appearance of the spin and orbital angular momentum couplings with the external magnetic field (the fourth and fifth terms, respectively), as well as the diagonal spin-orbital coupling (the sixth term) via the electric field; the last two terms may be interpreted as being similar to the Darwin term for spin-1/2 fermions in the presence of an electric field; the remaining terms represent higher-order corrections to such effects, as well as the (non-diagonal) corrections to the rest-energy (the first term).

5 Concluding Remarks

In the preceding sections, we have investigated the non-relativistic limit of the Lorentz-invariant wave equation which describes scalar and vector mesons in the so-called Duffin–Kemmer–Petiau representation. By constructing unitary operators involving the spatial components of the relativistic 4-momentum and those belonging to the associated DKP algebra, both for free particles and for charged bosons in an electromagnetic background, we performed a quantum canonical transformation to a reference frame where we succeeded in identifying the coupling terms with the electric and magnetic fields, in close similarity with the non-relativistic behaviour of interacting fermions described by the Pauli equation.

Our approach differs from that of Lévi-Leblond in the sense that he derived non-relativistic linear wave equations for particles of arbitrary spins which obey the Galilean invariance by construction, where the electromagnetic multipole moments are introduced on dimensional grounds. At the same time, in
the case of massive particles of spin 1 he settled the corresponding wave equations by employing the Bargmann–Wigner construction, not referring to the algebraic properties of the quantities involved, which we have done explicitly in our treatment.

In the context of the present work, it is relevant to mention the series of papers \cite{9,10,11,12} by Fushchych, Nikitin et al., who first introduced non-relativistic Duffin–Kemmer–Petiau equations: in \cite{9}, the authors presented a discussion on the Galilean-invariant equations for free particles with arbitrary spins, with particular emphasis on spin 0 and 1/2; however, in the presence of external fields \cite{10,11} their transformation operator, which partially diagonalizes the total Hamiltonian, differs from that of Foldy and Wouthuysen for spin 1/2 particles, as they pointed out in \cite{12}. Consequently, the transformation operator of \cite{10} is not the same as ours, since we follow the steps of the Foldy–Wouthuysen original algebraic construction (see, e.g., eqs. (5.2), (5.12)–(5.14) of \cite{11}), instead of appealing to formal group theoretical reasonings.

Yet in the framework of the DKP theory, other authors have recently investigated the non-relativistic wave equation for spinless bosons, also via the Galilean covariance, by introducing an extra degree of freedom into the free Lagrangian density\cite{13}, thus recovering the Schrödinger equation for a free particle. However, the introduction of electromagnetic potentials spoils the original structure of the associated Lie algebra on which the reasoning\cite{13} is grounded.

An interesting issue related to the present work is a possible generalization of the above procedure to theories of higher spins, as well as to their non-Abelian counterparts\cite{14}.

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A Appendix

Below, we present some useful commutation and vector relations derived from the algebra (2) of the $\beta$-matrices:

\begin{align}
[U, \vec{\alpha} \cdot \vec{\pi}] &= \frac{i}{m} \beta_0 \vec{\pi}^2, \\
[U, \beta_0] &= \frac{i}{m} \vec{\alpha} \cdot \vec{\pi}, \\
[U, A_0] &= -\frac{i}{m} \vec{\beta} \cdot \vec{\nabla} A_0, \\
[U, \partial U/\partial t] &= \frac{ie}{m^2} \vec{S} \cdot \left( \vec{\pi} \times \partial \vec{A}/\partial t \right), \\
[U, [U, \vec{\alpha} \cdot \vec{\pi}]] &= -\frac{\vec{\pi}^2}{m^2} (\vec{\alpha} \cdot \vec{\pi}), \\
[U, [U, \beta_0]] &= -\frac{1}{m^2} \beta_0 \vec{\pi}^2, \\
[U, [U, A_0]] &= -\frac{1}{m^2} \vec{S} (\vec{\pi} \times \vec{\nabla} A_0), \\
F^{\rho\sigma} \beta_\rho \beta_0 \beta_\sigma &= -2(\vec{E} \cdot \vec{\beta}) \beta_0^2 + \vec{E} \cdot \vec{\beta} + F^{ij} \beta_i \beta_0 \beta_j, \\
F^{\rho\sigma} \vec{\beta}_\rho \gamma_0 \gamma_\sigma &= -\vec{E} \cdot \vec{\beta}, \\
F^{ij} \beta_i \beta_0 \beta_j &= -i \beta_0 \vec{S} \cdot \vec{H} - i(\vec{\beta} \times \vec{\alpha}) \cdot \vec{H}.
\end{align}

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