Some new characterizations of \textit{PST}-groups

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Abstract

Let $H$ and $B$ be subgroups of a finite group $G$ such that $G = N_G(H)B$. Then we say that $H$ is \textit{quasipermutable} (respectively \textit{S-quasipermutable}) in $G$ provided $H$ permutes with $B$ and with every subgroup (respectively with every Sylow subgroup) $A$ of $B$ such that $(|H|,|A|) = 1$. In this paper we analyze the influence of \textit{S}-quasipermutable and quasipermutable subgroups on the structure of $G$. As an application, we give new characterizations of soluble \textit{PST}-groups.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover $p$ is always supposed to be a prime and $\pi$ is a subset of the set $\mathbb{P}$ of all primes; $\pi(G)$ denotes the set of all primes dividing $|G|$.

A subgroup $H$ of $G$ is said to be \textit{quasinormal} or \textit{permutable} in $G$ if $H$ permutes with every subgroup $A$ of $G$, that is, $HA = AH$; $H$ is said to be \textit{S-permutable} in $G$ if $H$ permutes with every Sylow subgroup of $G$.

A group $G$ is called a \textit{PT-group} if permutability is a transitive relation on $G$, that is, every permutable subgroup of a permutable subgroup of $G$ is permutable in $G$. A group $G$ is called a \textit{PST-group} if $S$-permutability is a transitive relation on $G$.

As well as $T$-groups, $PT$-groups and $PST$-groups possess many interesting properties (see Chapter 2 in [1]). The general description of $PT$-groups and $PST$-groups were firstly obtained by Zacher [2] and Agrawal [3], for the soluble case, and by Robinson in [4], for the general case. Nevertheless,

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\end{itemize}
in the further publications, the authors (see for example recent papers \[5\]– \[16\]) have found out and described many other interesting characterizations of soluble PT and PST-groups.

In this paper we give new "Hall"-characterizations of soluble PST-groups on the basis of the following

**Definition 1.1.** We say that a subgroup \(H\) is **quasipermutable** (respectively **S-quasipermutable**) in \(G\) provided \(H\) permutes with \(B\) and with every subgroup (respectively with every Sylow subgroup) \(A\) of \(B\) such that \((|H|, |A|) = 1\).

Examples and some applications of quasipermutable subgroups were discussed in our papers \[17\] and \[18\] (see also remarks in Section 5 below). In this paper, we give the following result, which we consider as one more motivation for introducing the concept of quasipermutability.

**Theorem A.** Let \(D = G^N\) and \(\pi = \pi(D)\). Then the following statements are equivalent:

(i) \(D\) is a Hall subgroup of \(G\) and every Hall subgroup of \(G\) is quasipermutable in \(G\).

(ii) \(G\) is a soluble PST-group.

(iii) Every subgroup of \(G\) is quasipermutable in \(G\).

(iv) Every \(\pi\)-subgroup of \(G\) and some minimal supplement of \(D\) in \(G\) are quasipermutable in \(G\).

In the proof Theorem A we use the next three our results.

A subgroup \(S\) of \(G\) is called a **Gaschütz subgroup** of \(G\) (L.A. Shemetkov \[19, IV, 15.3\]) if \(S\) is supersoluble and for any subgroups \(K \leq H\) of \(G\), where \(S \leq K\), the number \(|H : K|\) is not prime.

**Theorem B.** The following statements are equivalent:

(I) \(G\) is soluble, and if \(S\) is a Gaschütz subgroup of \(G\), then every Hall subgroup \(H\) of \(G\) satisfying \(\pi(H) \subseteq \pi(S)\) is quasipermutable in \(G\).

(II) \(G\) is supersoluble and the following hold:

(a) \(G = DC\), where \(D = G^N\) is an abelian complemented subgroup of \(G\) and \(C\) is a Carter subgroup of \(G\);

(b) \(D \cap C\) is normal in \(G\) and \((p, |D/D \cap C|) = 1\) for all prime divisors \(p\) of \(|G|\) satisfying \((p - 1, |G|) = 1\).

(c) For any non-empty set \(\pi\) of primes, every \(\pi\)-element of any Carter subgroup of \(G\) induces a power automorphism on the Hall \(\pi'\)-subgroup of \(D\).

(III) Every Hall subgroup of \(G\) is quasipermutable in \(G\).

Let \(\mathcal{F}\) be a class of groups. If \(1 \in \mathcal{F}\), then we write \(G^\mathcal{F}\) to denote the intersection of all normal subgroups \(N\) of \(G\) with \(G/N \in \mathcal{F}\). The class \(\mathcal{F}\) is said to be a **formation** if either \(\mathcal{F} = \emptyset\) or \(1 \in \mathcal{F}\) and every homomorphic image of \(G/G^\mathcal{F}\) belongs to \(\mathcal{F}\) for any group \(G\). The formation \(\mathcal{F}\) is said to be **saturated** if \(G \in \mathcal{F}\) whenever \(G/\Phi(G) \in \mathcal{F}\). A subgroup \(H\) of \(G\) is said to be an \(\mathcal{F}\)-**projector** of \(G\) provided \(H \in \mathcal{F}\) and \(E = E^\mathcal{F}H\) for any subgroup \(E\) of \(G\) containing \(H\). By the Gaschütz’s theorem
VI, 9.5.4 and 9.5.6], for any saturated formation \( \mathcal{F} \), every soluble group \( G \) has an \( \mathcal{F} \)-projector and any two \( \mathcal{F} \)-projectors of \( G \) are conjugate.

**Theorem C.** Let \( \mathcal{F} \) be a saturated formation containing all nilpotent groups. Suppose that \( G \) is soluble and let \( \pi = \pi(C) \cap \pi(G^\mathcal{F}) \), where \( C \) is an \( \mathcal{F} \)-projector of \( G \). If every maximal subgroup of every Sylow \( p \)-subgroup of \( G \) is \( S \)-quasipermutable in \( G \) for all \( p \in \pi \), then \( G^\mathcal{F} \) is a Hall subgroup of \( G \).

**Theorem D.** Let \( \mathcal{F} \) be a saturated formation containing all supersoluble groups and \( \pi = \pi(F^*(G^\mathcal{F})) \). If \( G^\mathcal{F} \neq 1 \), then for some \( p \in \pi \) some maximal subgroup of a Sylow \( p \)-subgroup of \( G \) is not \( S \)-quasipermutable in \( G \).

In this theorem \( F^*(G^\mathcal{F}) \) denotes the generalized Fitting subgroup of \( G^\mathcal{F} \), that is, the product of all normal quasinilpotent subgroups of \( G^\mathcal{F} \).

The main tool in the proofs of Theorems C and D is the following our result.

**Proposition.** Let \( E \) be a normal subgroup of \( G \) and \( P \) a Sylow \( p \)-subgroup of \( E \) such that \( |P| > p \).

(i) If every number \( V \) of some fixed \( M_{\phi}(P) \) is \( S \)-quasipermutable in \( G \), then \( E \) is \( p \)-supersoluble.

(ii) If every maximal subgroup of \( P \) is \( S \)-quasipermutable in \( G \), then every chief factor of \( G \) between \( E \) and \( O_p(E) \) is cyclic.

(iii) If every maximal subgroup of every Sylow subgroup of \( E \) is \( S \)-quasipermutable in \( G \), then every chief factor of \( G \) below \( E \) is cyclic.

In this proposition we write \( M_{\phi}(G) \), by analogy with [21], to denote a set of maximal subgroups of \( G \) such that \( \Phi(G) \) coincides with the intersection of all subgroups in \( M_{\phi}(G) \).

Note that Proposition may be independently interesting because this result unifies and generalize many known results, and in particular, Theorems 1.1–1.5 in [21] (see Section 5). In Section 5 we discuss also some further applications of the results.

All unexplained notation and terminology are standard. The reader is referred to [19], [22], or [23] if necessary.

## 2 Basic Propositions

Let \( H \) be a subgroup of \( G \). Then we say, following [17], that \( H \) is *propermutable* (respectively \( S \)-propermutable) in \( G \) provided there is a subgroup \( B \) of \( G \) such that \( G = N_G(H)B \) and \( H \) permutes with all subgroups (respectively with all Sylow subgroups) of \( B \).

**Proposition 2.1.** Let \( H \leq G \) and \( N \) a normal subgroup of \( G \). Suppose that \( H \) is quasipermutable (\( S \)-quasipermutable) in \( G \).

(1) If either \( H \) is a Hall subgroup of \( G \) or for every prime \( p \) dividing \( |H| \) and for every Sy-
low \( p \)-subgroup \( H_p \) of \( H \) we have \( H_p \not\subseteq N \), then \( HN/N \) is quasipermutable (\( S \)-quasipermutable, respectively) in \( G/N \).

(2) If \( \pi = \pi(H) \) and \( G \) is \( \pi \)-soluble, then \( H \) permutes with some Hall \( \pi' \)-subgroup of \( G \).

(3) \( H \) permutes with some Sylow \( p \)-subgroup of \( G \) for every prime \( p \) dividing \( |G| \) such that \( (p, |H|) = 1 \).

(4) \(|G : N_G(H \cap N)|\) is a \( \pi \)-number, where \( \pi = \pi(N) \cup \pi(H) \).

(5) If \( H \) is propermutable (\( S \)-propermutable) in \( G \), then \( HN/N \) is propermutable (\( S \)-propermutable, respectively) in \( G/N \).

(6) If \( H \) is \( S \)-propermutable in \( G \), then \( H \) permutes with some Sylow \( p \)-subgroup of \( G \) for any prime \( p \) dividing \( |G| \).

(7) Suppose that \( G \) is \( \pi \)-soluble. If \( H \) is a Hall \( \pi \)-subgroup of \( G \), then \( H \) is propermutable (\( S \)-propermutable, respectively) in \( G \).

**Proof.** By hypothesis, there is a subgroup \( B \) of \( G \) such that \( G = N_G(H)B \) and \( H \) permutes with \( B \) and with all subgroups (with all Sylow subgroups, respectively) \( A \) of \( B \) such that \( (|H|, |A|) = 1 \).

(1) It is clear that

\[
G/N = (N_G(H)N/N)(BN/N) = N_{G/N}(HN/N)(BN/N).
\]

Let \( K/N \) be any subgroup (any Sylow subgroup, respectively) of \( BN/N \) such that \( (|HN/N|, |K/N|) = 1 \). Then \( K = (K \cap B)N \). Let \( B_0 \) be a minimal supplement of \( K \cap B \cap N \) to \( K \cap B \). Then \( K/N = (K \cap B)N/N = B_0(K \cap B \cap N)/N = B_0N/N \) and \( K \cap B \cap N \cap B_0 = N \cap B_0 \leq \Phi(B_0) \).

Therefore \( \pi(K/N) = \pi(K \cap B/K \cap B \cap N) = \pi(B_0) \), so \( (|HN/N|, |B_0|) = 1 \). Suppose that some prime \( p \in \pi(B_0) \) divides \( |H| \), and let \( H_p \) be a Sylow \( p \)-subgroup of \( H \). We shall show that \( H_p \not\subseteq N \).

In fact, we may suppose that \( H \) is a Hall subgroup of \( G \). But in this case, \( H_p \) is a Sylow \( p \)-subgroup of \( G \). Therefore, since \( p \in \pi(B_0) \subseteq \pi(G/N) \), \( H_p \not\subseteq N \). Hence \( p \) divides \( |HN/N| \), a contradiction. Thus \( (|H|, |B_0|) = 1 \), so in the case, when \( H \) is quasipermutable in \( G \), we have \( HB_0 = B_0H \) and hence \( HN/N \) permutes with \( K/N = B_0N/N \). Thus \( HN/N \) is quasipermutable in \( G/N \).

Finally, suppose that \( H \) is \( S \)-quasipermutable in \( N \). In this case, \( B_0 \) is a \( p \)-subgroup of \( B \), so for some Sylow \( p \)-subgroup \( B_p \) of \( B \) we have \( B_0 \leq B_p \) and \( (|H|, p) = 1 \). Hence \( K/N = B_0N/N \leq B_pN/N \), which implies that \( K/N = B_pN/N \). But \( H \) permutes with \( B_p \) by hypothesis, so \( HN/N \) permutes with \( K/N \). Therefore \( HN/N \) is \( S \)-quasipermutable in \( G/N \).

(2) By \([20]\), VI, 4.6], there are Hall \( \pi' \)-subgroups \( E_1 \), \( E_2 \) and \( E \) of \( N_G(H) \), \( B \) and \( G \), respectively, such that \( E = E_1E_2 \). Then \( H \) permutes with all Sylow subgroups of \( E_2 \) by hypothesis, so

\[
HE = H(E_1E_2) = (HE_1)E_2 = (E_1H)E_2 = E_1(HE_2) = E_1(E_2H) = (E_1E_2)H = EH
\]
by [22] A, 1.6).

(3) See the proof of (2).

(4) Let $p$ be a prime such that $p \notin \pi$. Then by (3), there is a Sylow $p$-subgroup $P$ of $G$ such that $HP = PH$ is a subgroup of $G$. Hence $HP \cap N = H \cap N$ is a normal subgroup of $HP$. Thus $p$ does not divide $|G : N_G(H \cap N)|$.

(5) See the proof of (1).

(6) See the proof of (2).

(7) Since $G$ is $\pi$-soluble, $B$ is $\pi$-soluble. Hence by [20] VI, 1.7, $B = B_\pi B_\pi'$ where $B_\pi$ is a Hall $\pi$-subgroup of $B$ and $B_\pi'$ is a Hall $\pi'$-subgroup of $B$. By [20] VI, 4.6, there are Hall $\pi$-subgroups $N_\pi$, $B_\pi$ and $G_\pi$ of $N_G(H)$, $B$ and $G$, respectively, such that $G_\pi = N_\pi B_\pi$. But since $H \leq N_\pi$, $N_\pi$ is a Hall $\pi$-subgroup of $G$. Therefore $G_\pi = N_\pi B_\pi = N_\pi$, so $B_\pi \leq N_\pi$. Hence $G = N_G(H)B = N_G(H)B_\pi B_\pi' = N_G(H)B_\pi'$, so $H$ is propermutable ($S$-propermutable, respectively) in $G$.

A group $G$ is said to be a $C_\pi$-group provided $G$ has a Hall $\pi$-subgroup and any two Hall $\pi$-subgroups of $G$ are conjugate.

On the basis of Proposition 2.1 the following two results are proved.

**Proposition 2.2.** Let $H$ be a Hall $S$-quasipermutable subgroup of $G$. If $\pi = \pi(|G : H|)$, then $G$ is a $C_\pi$-group.

**Proposition 2.3.** Let $E$ be a normal subgroup of $G$ and $H$ a Hall $\pi$-subgroup of $E$. If $H$ is nilpotent and $S$-quasipermutable in $G$, then $E$ is $\pi$-soluble.

### 3 Groups with a Hall quasipermutable subgroup

A group $G$ is said to be $\pi$-separable if every chief factor of $G$ is either a $\pi$-group or a $\pi'$-group. Every $\pi$-separable group $G$ has a series

$$1 = P_0(G) \leq M_0(G) < P_1(G) < M_1(G) < \ldots < P_t(G) \leq M_t(G) = G$$

such that

$$M_i(G)/P_i(G) = O_{\pi'}(G/P_i(G))$$

$(i = 0, 1, \ldots, t)$ and

$$P_{i+1}(G)/M_i(G) = O_\pi(G/M_i(G))$$

$(i = 1, \ldots, t)$

The number $t$ is called the $\pi$-length of $G$ and denoted by $l_{\pi}(G)$ (see [34] p. 249]).

One more result, which we use in the proof of our main results, is the following
**Theorem 3.1.** Let $H$ be a Hall subgroup of $G$ and $\pi = \pi(H)$. Suppose that $H$ is quasipermutable in $G$.

(I) If $p > q$ for all primes $p$ and $q$ such that $p \in \pi$ and $q$ divides $|G : N_{G}(H)|$, then $H$ is normal in $G$.

(II) If $H$ is supersoluble, then $G$ is $\pi$-soluble.

(III) If $H$ is $\pi$-separable, then the following hold:

(i) $H' \leq O_{\pi}(G)$. If, in addition, $N_{G}(H)$ is nilpotent, then $G' \cap H \leq O_{\pi}(G)$.

(ii) $l_{\pi}(G) \leq 2$ and $l_{\pi'}(G) \leq 2$.

(iii) If for some prime $p \in \pi'$ a Hall $\pi'$-subgroup $E$ of $G$ is $p$-supersoluble, then $G$ is $p$-supersoluble.

Let $\mathcal{M}$ and $\mathcal{N}$ be non-empty formations. Then the product $\mathcal{M}\mathcal{N}$ of these formations is the class of all groups $G$ such that $G^{\mathcal{N}} \in \mathcal{M}$. It is well-known that such an operation on the set of all non-empty formations is associative (Gaschütz). The symbol $\mathcal{M}^{t}$ denotes the product of $t$ copies of $\mathcal{M}$.

We shall need following well-known Gaschütz-Shemetkov’s theorem [26, Corollary 7.13].

**Lemma 3.2.** The product of any two non-empty saturated formations is also a saturated formation.

In the proof of Theorem 3.1 we use the following

**Lemma 3.3.** The class $\mathcal{F}$ of all $\pi$-separable groups $G$ with $l_{\pi}(G) \leq t$ is a saturated formation.

**Proof.** It is not difficult to show that for any non-empty set $\omega \subseteq \mathfrak{P}$ the class $\mathcal{G}_{\omega}$ of all $\omega$-groups is a saturated formation and that $\mathcal{F} = (\mathcal{G}_{\pi'}\mathcal{G}_{\pi})^{t}\mathcal{G}_{\pi'}$. Hence $\mathcal{F}$ is a saturated formation by Lemma 3.2.

**Lemma 3.4.** Suppose that $G$ is separable. If Hall $\pi$-subgroups of $G$ are abelian, then $l_{\pi}(G) \leq 1$.

**Proof.** Suppose that this lemma is false and let $G$ be a counterexample of minimal order. Let $N$ be a minimal normal subgroup of $G$. Since $G$ is $\pi$-separable, $N$ is a $\pi$-group or a $\pi'$-group. It is clear that the hypothesis holds for $G/N$, so $l_{\pi}(G/N) \leq 1$ by the choice of $G$. By Lemma 3.3, the class of all $\pi$-soluble groups with $l_{\pi}(G) \leq 1$ is a saturated formation. Therefore $N$ is a unique minimal normal subgroup of $G$, $N \nsubseteq \Phi(G)$ and $N$ is not a $\pi'$-group. Hence $N$ is a $\pi$-group and $N = C_{G}(N)$ by [22, A, 15.2]. Therefore $N \leq H$, where $H$ is a Hall $\pi$-subgroup of $G$. But since $H$ is abelian, $N = H$ is a Hall $\pi$-subgroup of $G$. Hence $l_{\pi}(G) \leq 1$.

A group $G$ is called $\pi$-closed provided $G$ has a normal Hall $\pi$-subgroup.

**Lemma 3.5.** Let $H$ be a Hall $\pi$-subgroup of $G$. If $G$ is $\pi$-separable and $H \leq Z(N_{G}(H))$, then $G$ is $\pi'$-closed.

**Proof.** Suppose that this lemma is false and let $G$ be a counterexample of minimal order. Then $G \neq H$. The class $\mathcal{F}$ of all $\pi'$-closed groups coincides with the product $\mathcal{G}_{\pi'}\mathcal{G}_{\pi}$. Hence $\mathcal{F}$ is a saturated formation by Lemma 3.2. Let $N$ be a minimal normal subgroup of $G$. Since $G$ is $\pi$-separable, $N$ is a $\pi$-group or a $\pi'$-group. Moreover, $G$ is a $C_{\pi}$-group by [34, 9.1.6]), so the hypothesis holds for
G/N. Hence G/N is π′-closed by the choice of G. Therefore N is the only minimal normal subgroup of G, N \not\leq \Phi(G) and N is a π-group. Therefore N \leq H and N = C_G(N) by [22, A, 15.2]. Since H \leq Z(N_G(H)) and H is a Hall π-subgroup of G, N = H. Therefore N \leq Z(G), which implies that N = H = G. This contradiction completes the proof of the lemma.

4 Proof of Theorem A

Recall that G is a PST-group if and only if G = D \rtimes M, where D = G^N is abelian Hall subgroup of G and every element x \in M induces a power automorphism on D [3]. Therefore the implication (i) \Rightarrow (ii) is a direct corollary of Theorem B.

Now suppose that G = D \rtimes M, where D = G^N, is a soluble PST-group. Let H be any subgroup of G and S a Hall π′-subgroup of H. Since G is soluble, we may assume without loss of generality that S \leq M. Hence H = (D \cap H)(M \cap H) = (D \cap H)S and D \cap H is normal in G. Let π_1 = π(S).

Let A be a Hall π_1-subgroup of M and E a complement to A in M. Then E \leq C_G(S). Therefore G = DM = DAE = N_G(H)(DA) and every subgroup L of DA satisfying (|H|, |L|) = 1 is contained in D. Thus H is quasipermutable in G. Thus (ii) \Rightarrow (iii).

(iv) \Rightarrow (ii) By Theorems C and D, G is supersoluble and D is a Hall subgroup of G. Therefore G = D \rtimes W, where W is a Hall π′-subgroup of G. By hypothesis, W is quasipermutable in G. Now arguing similarly as in the proof of Theorem B one can show that D is abelian and every subgroup of D is normal in G. Therefore G is a PST-group.

5 Final remarks

1. The subgroup S_3 is quasipermutable, S-propermutable and not propermutable in S_4. If H is the subgroup of order 3 in S_3, then H is S-quasipermutable and not quasipermutable in S_4.

2. Arguing similarly to the proof of Theorem A one can prove the following fact.

**Theorem 5.1.** Suppose that G is soluble and let π = π(G^N). Then G is a PST-group if and only if every subnormal π-subgroup and a Hall π′-subgroup of G are propermutable in G.

3. If G is metanilpotent, that is G/F(G) is nilpotent, then for every Hall subgroup E of G we have G = N_G(E)F(G). Therefore, in this case, every characteristic subgroup of every Hall subgroup of G is S-propermutable in G. In particular, every Hall subgroup of every supersoluble group is S-propermutable. This observation makes natural the following question: What is the structure of G under the hypothesis that every Hall subgroup of G is propermutable in G? Theorem B gives an answer to this question.
4. Every maximal subgroup of a supersoluble group is quasipermutable. Therefore, in fact, Theorem A shows that the class of all soluble groups in which quasipermutability is a transitive relation coincides with the class of all soluble PST-groups.

5. We say that $G$ is a SQT-group if $S$-quasipermutability is a transitive relation in $G$. Arguing similarly to the proof of Theorem A one can prove the following fact.

**Theorem 5.2.** A soluble group $G$ is an SQT-group if and only if $G = D \rtimes M$ is supersoluble, where $D$ and $M$ are Hall nilpotent subgroups of $G$ and the index $|G : DN_G(H \cap D)|$ is a $\pi(H)$-number for every subgroup $H$ of $G$.

6. A subgroup $H$ of $G$ is called SS-quasinormal (semi-normal) in $G$ provided $G$ has a subgroup $B$ such that $HB = G$ and $H$ permutes with all Sylow subgroups ($H$ permutes with all subgroups, respectively) of $B$.

It is clear that every SS-quasinormal subgroup is $S$-propermutable and every semi-normal subgroup is propermutable. Moreover, there are simple examples (consider, for example, the group $C_7 \rtimes \text{Aut}(C_7)$, where $C_7$ is a group of order 7) which show that, in general, the class of all $S$-propermutable subgroups of $G$ is wider than the class of all its SS-quasinormal subgroups and the class of all propermutable subgroups of $G$ is wider than the class of all its semi-normal subgroups. Therefore Proposition covers main results (Theorems 1.1–1.5) in [21].

7. Theorem 3.1 is used in the proof of Theorem B. From this result we also get

**Corollary 5.3** (See [35] Theorem 5.4]). Let $H$ be a Hall semi-normal subgroup of $G$. If $p > q$ for all primes $p$ and $q$ such that $p$ divides $|H|$ and $q$ divides $|G : H|$, then $H$ is normal in $G$.

**Corollary 5.4** (See [36] Theorem 3]). Let $P$ be a Sylow $p$-subgroup of $G$. If $P$ is semi-normal in $G$, then the following statements hold:

(i) $G$ is $p$-soluble and $P' \leq O_p(G)$.

(ii) $l_p(G) \leq 2$.

(iii) If for some prime $q \in \pi'$ a Hall $p'$-subgroup of $G$ is $q$-supersoluble, then $G$ is $q$-supersoluble.

**Corollary 5.5** (See [37] Theorem 3]). If a Sylow $p$-subgroup $P$ of $G$, where $p$ is the largest prime dividing $|G|$, is semi-normal in $G$, then $P$ is normal in $G$.

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