Generalizations of Sylvester’s determinantal identity

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Abstract. In this paper we deal with the noteworthy Sylvester’s determinantal identity and some of its generalizations. We report the formulae due to Yakovlev, to Gasca, Lopez–Carmona, Ramirez, to Beckermann, Gasca, Mühlbach, and to Mulders in a unified formulation which allows to understand them better and to compare them. Then, we propose a different generalization of Sylvester’s classical formula. This new generalization expresses the determinant of a matrix in relation with the determinant of the bordered matrices obtained adding more than one row and one column to the original matrix. Sylvester’s identity is recovered as a particular case.

1 Introduction

Sylvester’s determinantal identity [22] is a well-known identity in matrix analysis which expresses a determinant composed of bordering determinants in terms of the original one. It has been extensively studied, both in the algebraic and in the combinatorial context, and it is usually written in the form used by Bareiss [3].

In the last years some generalizations of this identity have been proposed, with useful applications mainly in interpolation and extrapolation problems [10, 9]. One of the first generalizations seems to have been proposed by Yakovlev [23, 24] who used it in the control theory domain. In [15], Gasca, Lopez-Carmona, and Ramirez proved a generalization of Sylvester’s classical determinantal identity; an alternative proof based on elimination techniques was suggested in [20], while a short and simpler one than the earlier attempts was given in [6]. More recently, Mulders [21] formulated another generalization related to fraction free Gaussian elimination algorithms. Moreover, in order to simplify the reading when dealing with minors identities, Leclerc [17] proposed a unified notation to encode many algebraic relations between minors of a matrix, including Sylvester’s determinantal formula and the Mühlbach-Gasca generalization [20, 19].

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These generalizations (and sometimes only a rewriting of Sylvester’s formula as Yakovlev’s) are scattered in the literature. Moreover, the y used different notations which makes difficult to understand them as a whole and to see the connections between them. Therefore, one of the important aims of this paper is to present these identities in a unified notation. Also, for helping the reader to better understand these generalizations, they are illustrated by several examples.

Then, for showing the potentiality of our unified approach, a new generalization of Sylvester’s identity will be given. Sylvester’s determinantal identity can be recovered as a particular case of it. Such identities may have a potential impact for the development of new recursive algorithms for interpolation and extrapolation.

In Section 2, the classical Sylvester’s determinantal identity is reminded, and some particular cases are discussed. Section 3 is devoted to the generalizations, mentioned above, of the classical identity. These identities are presented in a unified formulation which allows to understand them better and to compare them. In Section 4, the new generalization is proposed.

2 The classical Sylvester’s determinantal identity

To introduce the classical Sylvester’s determinantal identity we consider a square matrix $M = (a_{ij})$, of order $n$ over $K$ (a commutative field). We denote its determinant by det $M$ and, for an integer $t$, $0 \leq t \leq n - 1$, and a given couple of integers $i, j$ with $t < i, j \leq n$, we define the determinant $a_{i,j}^{(t)}$ as

$$a_{i,j}^{(t)} \equiv \begin{vmatrix} a_{11} & \cdots & a_{1t} & a_{1j} \\ a_{21} & \cdots & a_{2t} & a_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{t1} & \cdots & a_{tt} & a_{tj} \\ a_{i1} & \cdots & a_{it} & a_{ij} \end{vmatrix}, \text{ for } 0 < t \leq n - 1 \text{ and } a_{i,j}^{(0)} = a_{ij}, \quad (1)$$

that is the determinant of order $(t + 1)$ obtained from the matrix $M$ by extending its leading principal submatrix of order $t$ with the row $i$ and the column $j$ of $M$. Given these assumptions we can easily write down, in the following Theorem, the Sylvester’s identity.

**Theorem 1 (Sylvester’s identity)** Let $M$ be a square matrix of order $n$ and $t$ an integer, $0 \leq t \leq n - 1$. Then, the following identity holds

$$\det M \cdot \left[a_{i,t}^{(t-1)}\right]^{n-t-1} = \begin{vmatrix} a_{t+1,t+1}^{(t)} & \cdots & a_{t+1,n}^{(t)} \\ \vdots & \vdots & \vdots \\ a_{n,t+1}^{(t)} & \cdots & a_{n,n}^{(t)} \end{vmatrix}, \quad (2)$$

with $a_{i,j}^{(-1)} = 1$.

For different proofs of this Theorem, we refer to [14, 3, 2]. We note also [10], where a combinatorial proof, together with several non-commutative extensions, has been presented.
Some particular cases of this Sylvester’s identity are of interest:

Case 1: For \( t = 1 \) and \( a_{11} \neq 0 \) we obtain the Chio pivotal condensation method \cite{13} which states

\[
\det M = \frac{\det B}{(a_{11})^{n-2}}
\]

where \( B = (b_{ij}) \), is the square matrix of order \((n - 1)\) with entries

\[
b_{ij} = a_{i+1,j+1}^{(1)} = \left| \begin{array}{cc} a_{11} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{array} \right|,
\]

for \( i, j = 1, \ldots, n - 1 \).

Case 2: For \( t = n - 2 \), we consider \( M \) partitioned like

\[
M = \begin{pmatrix} a & b^T & e \\ c & D & f \\ g & h^T & l \end{pmatrix}
\]

(3)

where \( a, e, g, l \) are scalars, \( b, c, f, h \) are \((n - 2)\)-length column vectors, and \( D \) is a square matrix of order \((n - 2)\). It is easy to apply the Sylvester’s identity to such a matrix. In fact, by interchanging rows and columns in order to put \( D \) in the upper left corner, leaves the sign of determinant unaltered, so

\[
\det M = \left| \begin{array}{ccc} D & c & f \\ b^T & a & e \\ h^T & g & l \end{array} \right|,
\]

and applying (2) we have

\[
\det M \cdot \det D = \left| \begin{array}{ccc} D & c & f \\ b^T & a & e \\ h^T & g & l \end{array} \right| = \left| \begin{array}{ccc} a & b^T & e \\ c & D & f \\ g & h^T & l \end{array} \right|.
\]

By setting

\[
A' = \begin{pmatrix} a & b^T \\ c & D \end{pmatrix}, \quad B' = \begin{pmatrix} b^T & e \\ D & f \end{pmatrix}, \quad C' = \begin{pmatrix} c & D \\ g & h^T \end{pmatrix}, \quad D' = \begin{pmatrix} D & f \\ h^T & l \end{pmatrix},
\]

(4)

we can write the very simple rule

\[
\det M \det D = \det A' \det D' - \det B' \det C',
\]

(5)

frequently used to obtain recursive algorithms in sequence transformations (see, for instance, \cite{7, 8, 11}).
3 Generalizations of Sylvester’s identity

Several authors have deepened the main property of classical Sylvester’s identity, which is very useful in several domains. Some of these authors have generalized the classical identity obtaining interesting new formulae. Yakovlev [23, 24] proved a generalization useful in control theory. After, Beckermann, Gasca, Mühlbach et al. [6, 15, 20] devoted some papers to generalizations and applications of Sylvester’s identity, while Mulders [21] proposed a particular generalization of practical use in linear programming and related topics. In this section, we present a brief overview of these results.

In order to have a common working framework, we first introduce here some general notations and definitions related to an $n \times m$ matrix $M$ over $\mathbb{K}$.

An index list $I$ of length $k \leq n$ is a $k$-tuple (with possible repetition) of integers from $\{1, 2, \ldots, n\}$, that is $I = (i_1, \ldots, i_k)$, with $1 \leq i_l \leq n$ for $l = 1, \ldots, k$. If we impose that all the elements are different, we have an index list without repetition. So, in this case, $\text{card}(I) = k$. If, in addition, $\alpha < \beta$ implies $i_\alpha < i_\beta$, then $I$ is called an ordered index list. The usual symbols $\in, \subset, \subseteq, \cup, \cap, \setminus$, used for the sets will be used for the ordered index lists and the result will be again an ordered index list. For any $n \in \mathbb{N}^+$ (the set of all positive integer), we define the ordered index list $N_n = (1, 2, \ldots, n)$.

In the sequel, if nothing is stated to the contrary, all index lists are assumed to be ordered and without repetition.

Let now $I = (i_1, \ldots, i_\alpha) \subseteq N_n$ and $J = (j_1, \ldots, j_\beta) \subseteq N_m$ two ordered index lists. We denote the $\alpha \times \beta$ submatrix, extracted from $M$, with rows labeled by $i \in I$ and columns labeled by $j \in J$ as

$$M \begin{pmatrix} I \\ J \end{pmatrix} = M \begin{pmatrix} i_1, \ldots, i_\alpha \\ j_1, \ldots, j_\beta \end{pmatrix} = \begin{pmatrix} a_{i_1j_1} & \cdots & a_{i_1j_\beta} \\ \vdots & & \vdots \\ a_{i_\alpha j_1} & \cdots & a_{i_\alpha j_\beta} \end{pmatrix}. \tag{6}$$

Let us mention that, in some papers, the indices for rows and columns are inverted. However, the notations we are using, seem to be the most common ones; see, for example, [1].

If $\alpha = \beta$, $n = m$, we denote the corresponding determinant as

$$M \begin{pmatrix} I \\ J \end{pmatrix} = M \begin{pmatrix} i_1, \ldots, i_\alpha \\ j_1, \ldots, j_\alpha \end{pmatrix} = \begin{vmatrix} a_{i_1j_1} & \cdots & a_{i_1j_\alpha} \\ \vdots & & \vdots \\ a_{i_\alpha j_1} & \cdots & a_{i_\alpha j_\alpha} \end{vmatrix}.$$

With these notations, $a_{ij}^{(t)} = M \begin{pmatrix} 1, \ldots, t, i \\ 1, \ldots, t, j \end{pmatrix}$, for $t < i, j \leq n$ and $0 < t \leq n - 1$.

3.1 Generalization of Yakovlev

In [23, 24], Yakovlev presented some determinant identities used to simplify several procedures in control theory. In particular, in [24] he was the first to give a simple
generalization of Sylvester’s identity, which can be viewed as a renaming of the row and column indices.

Assume that \( M = (a_{ij}) \) is a square matrix of order \( n \). Let \( 0 < t \leq n - 1 \), and consider the two ordered index lists \( I = (i_1, \ldots, i_t) \subset N_n \), \( J = (j_1, \ldots, j_t) \subset N_n \), and the complementary ordered index lists \( I' = (i'_1, \ldots, i'_n) \subset N_n \), \( J' = (j'_1, \ldots, j'_n) \subset N_n \) (that is \( I \cup I' = J \cup J' = N_n \)).

Starting from the well-known Laplace expansion formula for the determinant of a squared matrix, as proposed in [14], the author proved a determinantal identity and use it in the solution of certain applied problems as regulator synthesis for system of high order. This identity can be transformed in the following formula

\[
\det M \cdot \left( M \begin{bmatrix} i_1, \ldots, i_t \\ j_1, \ldots, j_t \end{bmatrix} \right)^{n-t-1} = \sum_{P(\alpha_{t+1}, \ldots, \alpha_n)} (-1)^\mu \prod_{\beta=t+1}^n M \begin{bmatrix} i_1, \ldots, i_t, \alpha_\beta \\ j_1, \ldots, j_t, \beta \end{bmatrix},
\]

where \( P(\alpha_{t+1}, \ldots, \alpha_n) \) represent the set of all permutations of \((t + 1, \ldots, n)\) and \( \mu \) is the number of inversions needed to pass from \((t + 1, \ldots, n)\) to a certain permutation \((\alpha_{t+1}, \ldots, \alpha_n)\).

Let us now take \( I = J = (1, \ldots, t) \). Thus, \( I' = J' = (t + 1, \ldots, n) \). Formula (7) becomes

\[
\det M \cdot \left( M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix} \right)^{n-t-1} = \sum_{P(\alpha_{t+1}, \ldots, \alpha_n)} (-1)^\mu \prod_{\beta=t+1}^n M \begin{bmatrix} 1, \ldots, t, \alpha_\beta \\ 1, \ldots, t, \beta \end{bmatrix}.
\]

This expression is one of the different ways to formulate the Sylvester’s determinantal identity. In fact

\[
M \begin{bmatrix} 1, \ldots, t, \alpha_\beta \\ 1, \ldots, t, \beta \end{bmatrix} = \delta_{(t)}^{(t)} \epsilon_{\alpha_\beta, \beta},
\]

and, by Leibniz formula,

\[
\sum_{P(\alpha_{t+1}, \ldots, \alpha_n)} (-1)^\mu \prod_{\beta=t+1}^n \delta_{(t)}^{(t)} \epsilon_{\alpha_\beta, \beta}
\]

is exactly the determinant of order \( n - t \) on the right hand side of (2).

In the following examples we consider \( P(\alpha_{t+1}, \ldots, \alpha_n) \) as an ordered set of \((n - t)!\) elements, and we denote by \( \mu = (\mu_1, \ldots, \mu_{(n-t)!}) \) the vector whose elements are the corresponding number of inversions.

**Example 1:**

Let \( n = 6 \) the order of \( M \), and set \( t = 4 \). We consider the ordered index lists \( I = (i_1, \ldots, i_4) = (1, 3, 5, 6) \), \( J = (j_1, \ldots, j_4) = (1, 2, 4, 6) \); then \( I' = (i'_5, i'_6) = (2, 4) \), \( J' = (j'_5, j'_6) = (3, 5) \). We have

\[
\det M \cdot M \begin{bmatrix} I \\ J \end{bmatrix} = \sum_{P(\alpha_5, \alpha_6)} (-1)^\mu \prod_{\beta=5}^6 M \begin{bmatrix} I, i'_\alpha \beta \\ J, j'_\beta \end{bmatrix}.
\]
Since $P(\alpha_5, \alpha_6) = \{(5, 6), (6, 5)\}$, and $\boldsymbol{\mu} = (0, 1)$, we obtain
\[
\det M \cdot \begin{bmatrix} I \\ J \end{bmatrix} = M \begin{bmatrix} I, 2 \\ J, 3 \end{bmatrix} \cdot M \begin{bmatrix} I, 4 \\ J, 5 \end{bmatrix} - M \begin{bmatrix} I, 4 \\ J, 3 \end{bmatrix} \cdot M \begin{bmatrix} I, 2 \\ J, 5 \end{bmatrix}.
\]

With the same values, but with $I = J = (1, \ldots, 4)$, $I' = J' = (5, 6)$, $P(\alpha_5, \alpha_6) = \{(5, 6), (6, 5)\}$, and $\boldsymbol{\mu} = (0, 1)$ we obtain
\[
\det M \cdot \begin{bmatrix} I \\ J \end{bmatrix} = M \begin{bmatrix} I, 5 \\ J, 5 \end{bmatrix} \cdot M \begin{bmatrix} I, 6 \\ J, 6 \end{bmatrix} - M \begin{bmatrix} I, 6 \\ J, 5 \end{bmatrix} \cdot M \begin{bmatrix} I, 5 \\ J, 6 \end{bmatrix},
\]
that is
\[
\det M \cdot \begin{bmatrix} 1, \ldots, 4 \\ 1, \ldots, 4 \end{bmatrix} = \begin{vmatrix} a_{5,5}^{(4)} & a_{6,6}^{(4)} \\ a_{6,5}^{(4)} & a_{6,6}^{(4)} \end{vmatrix},
\]
which is the classical Sylvester’s identity.

**Example 2:**

Let now $n = 8$ and $t = 5$, and consider the ordered index lists $I = (i_1, \ldots, i_5) = (2, 3, 5, 6, 8)$, $J = (j_1, \ldots, j_5) = (2, 4, 5, 6, 7)$, and the complementary $I' = (i'_6, i'_7, i'_8) = (1, 4, 7)$ $J' = (j'_6, j'_7, j'_8) = (1, 3, 8)$. We have
\[
\det M \cdot \left( M \begin{bmatrix} I \\ J \end{bmatrix} \right)^2 = \sum_{P(\alpha_6, \alpha_7, \alpha_8)} (-1)^\beta \prod_{\beta = 6}^8 M \begin{bmatrix} I, i'_\alpha \\ J, j'_\beta \end{bmatrix}.
\]

Since $P(\alpha_6, \alpha_7, \alpha_8) = \{(8, 7, 6), (8, 6, 7), (7, 8, 6), (7, 6, 8), (6, 7, 8), (6, 7, 8)\}$, and $\boldsymbol{\mu} = (3, 2, 1, 0, 1)$, we have
\[
\det M \cdot \left( M \begin{bmatrix} I \\ J \end{bmatrix} \right)^2 = -M \begin{bmatrix} I, 7 \\ J, 1 \end{bmatrix} \cdot M \begin{bmatrix} I, 4 \\ J, 3 \end{bmatrix} \cdot M \begin{bmatrix} I, 1 \\ J, 8 \end{bmatrix} + M \begin{bmatrix} I, 7 \\ J, 1 \end{bmatrix} \cdot M \begin{bmatrix} I, 4 \\ J, 3 \end{bmatrix} \cdot M \begin{bmatrix} I, 1 \\ J, 8 \end{bmatrix} + 
\begin{bmatrix} I, 4 \\ J, 1 \end{bmatrix} \cdot M \begin{bmatrix} I, 7 \\ J, 3 \end{bmatrix} \cdot M \begin{bmatrix} I, 1 \\ J, 8 \end{bmatrix} - M \begin{bmatrix} I, 4 \\ J, 1 \end{bmatrix} \cdot M \begin{bmatrix} I, 7 \\ J, 3 \end{bmatrix} \cdot M \begin{bmatrix} I, 1 \\ J, 8 \end{bmatrix} + 
\begin{bmatrix} I, 1 \\ J, 1 \end{bmatrix} \cdot M \begin{bmatrix} I, 4 \\ J, 3 \end{bmatrix} \cdot M \begin{bmatrix} I, 7 \\ J, 8 \end{bmatrix} - M \begin{bmatrix} I, 1 \\ J, 1 \end{bmatrix} \cdot M \begin{bmatrix} I, 7 \\ J, 3 \end{bmatrix} \cdot M \begin{bmatrix} I, 4 \\ J, 8 \end{bmatrix}.
\]

### 3.2 Generalization of Gasca, Lopez-Carmona, and Ramirez

In [15], Gasca, Lopez-Carmona, and Ramirez proved a first generalization of Sylvester’s determinantal identity; this generalized identity has been applied to the derivation of a recurrence interpolation formula for the solution of a general interpolation problem.

Assume that $M$ is a square matrix of order $n$, and that $n = t + q$, for some positive integers $t, q$. Given a set of $q$ ordered index lists $J_k = (j_1^k, \ldots, j_{t+1}^k) \subset N_n$, $k = 1, \ldots, q$, with
\[
\left\{ \begin{array}{c}
\text{card}(J_k) = t + 1, \\
\text{card}(J_k \cap J_{k+1}) = t,
\end{array} \right. \quad k = 1, \ldots, q-1,
\]
we set
\[
J^{(q)} = \bigcup_{k=1}^q J_k,
\]
\[
S_k = J_k \cap J_{k+1} = (s_1^k, \ldots, s_t^k), \quad k = 1, \ldots, q-1.
\]
Let $B$ the matrix with elements

$$b_{ik} = M \begin{bmatrix} 1, 2, \ldots, t, t + i \end{bmatrix}_{J_k} \quad 1 \leq i, k \leq q$$

and let $j^k_h$ be the element of $J_k$ such that

- $j^k_h \in J_k - J_{k-1} = J_k - S_{k-1}$, $k = 2, \ldots, q$
- $j^1_h \in J_1 - J_2 = J_1 - S_1$, $k = 1$.

With these notations the authors give the following generalization of Sylvester’s identity

$$\det B = c \cdot \det M \cdot \prod_{k=1}^{q-1} M \begin{bmatrix} 1, \ldots, t \end{bmatrix}_{S_k}$$

(8)

where $c$ is a sign factor which does not depend on the element $a_{ij}$ of $M$, but only on the set of $J_k$ according to

$$c = \begin{cases} 0 & \text{if } \text{card}(J^{(q)}) < t + q \\ (-1)^\mu, \text{ with } \mu = q(q-1)/2 + \sum_{k=1}^{q} (j^k_h - h_k) & \text{if } \text{card}(J^{(q)}) = t + q \end{cases}$$

with $J^{(q)} = \bigcup_{k=1}^{q} J_k$.

When $J_k = (1, \ldots, t, t + k)$, for $k = 1, 2, \ldots, q$, we have, for all $k$, $S_k = (1, \ldots, t)$, and the identity (8) gives

$$\det B = \det M \cdot \left( M \begin{bmatrix} 1, \ldots, t \end{bmatrix}_{[1, \ldots, t]} \right)^{q-1}$$

which is exactly (2), when $n = t + q$.

An alternative proof of this identity was suggested by Gasca and Mühlbach in [20]; this proof is based on the Laplacian expansion and on an elimination strategy. Roughly speaking, when a determinant is simplified by performing elementary operations leaving its value unchanged, then elimination strategies are applied, for example Gauss or Neville elimination.

### 3.2.1 Examples

**Example 1:**

Let $M$ be a nonsingular square matrix of order $n = 5$, choose $t = 2, q = 3$, and consider the ordered index lists $J_1 = (1, 3, 4), J_2 = (1, 4, 5), J_3 = (2, 4, 5)$ and $J^{(3)} =$
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(1, 2, 3, 4, 5) = N_5. We have S_1 = (1, 4) and S_2 = (4, 5). Let

\[
B = \begin{pmatrix}
M \begin{bmatrix} 1, 2, 3 \\ 1, 3, 4 \\ 1, 2, 4 \\ 1, 2, 5 \\ 1, 3, 4 \\ 1, 3, 4 \end{bmatrix} & M \begin{bmatrix} 1, 2, 3 \\ 1, 4, 5 \\ 1, 4, 5 \\ 1, 2, 5 \\ 1, 4, 5 \\ 1, 4, 5 \end{bmatrix} & M \begin{bmatrix} 1, 2, 3 \\ 2, 4, 5 \\ 2, 4, 5 \\ 1, 2, 5 \\ 2, 4, 5 \\ 2, 4, 5 \end{bmatrix} \\
\end{pmatrix}
\]

Since \(\text{card}(J^{(q)}) = n\), for computing the sign factor we first have to determine \(j_{h_k}^k\) for \(k = 1, \ldots, 3\). For \(k = 1\), \(j_{h_1}^1 = 3\), \(h_1 = 2\), for \(k = 2\), \(j_{h_2}^2 = 5\), \(h_2 = 3\), for \(k = 3\), \(j_{h_3}^3 = 2\), \(h_3 = 1\). So \(\mu = 3 + 4 = 7\) and \(c = -1\).

Finally we have

\[
\det B = -\det M \cdot M \begin{bmatrix} 1, 2 \\ 1, 4 \end{bmatrix} \cdot M \begin{bmatrix} 1, 2 \\ 4, 5 \end{bmatrix}.
\]

Example 2: Let again \(n = 5\), \(t = 2\), \(q = 3\), but consider the ordered index lists \(J_1 = (1, 2, 3), J_2 = (2, 3, 4), J_3 = (1, 2, 4)\) with \(\text{card}(J^{(3)} = (1, 2, 3, 4)) < 5\). So \(c = 0\), and the determinant of the matrix

\[
B = \begin{pmatrix}
M \begin{bmatrix} 1, 2, 3 \\ 1, 2, 3 \\ 1, 2, 4 \\ 1, 2, 3 \\ 1, 2, 3 \end{bmatrix} & M \begin{bmatrix} 1, 2, 3 \\ 2, 3, 4 \\ 1, 2, 4 \\ 1, 2, 3 \\ 1, 2, 3 \end{bmatrix} & M \begin{bmatrix} 1, 2, 3 \\ 1, 2, 4 \\ 1, 2, 4 \\ 1, 2, 3 \\ 1, 2, 3 \end{bmatrix} \\
\end{pmatrix}
\]

is equal to zero.

3.3 Generalization of Beckermann, Gasca and Mühlbach

Sylvester’s classical identity can be interpreted as an extension of Leibniz’s definition of a matrix determinant, known as the Muir’s law of extensible minors [18]. Muir first stated the Law of Complementaries, which was been already known to Cayley in 1878 and reads as

To every general theorem which takes the form of an identical relation between a number of the minors of a determinant or between the determinant itself and a number of its minors, there corresponds another theorem derivable from the former by merely substituting for every minor its cofactor in the determinant, and then multiplying any term by such a power of the determinant that will make the terms of the same degree.

Brualdi and Schneider [12] gave a formal treatment of determinantal identities of the minors of a matrix and then provided a careful exposition of the Law of Extensible Minors and of the Law of Complementaries, two methods for obtaining from a given determinantal identity another determinantal identity, as mentioned earlier. In [19], Mühlbach using the relation between Muir’s Law of Extensible Minors, Sylvester’s identity and the Schur complement, presented a new principle for extending determinantal identities which generalizes Muir’s Law. Mühlbach’s proof makes use of general elimination strategies and of generalized Schur complements. As applications of this
technique, he derived a generalization of Sylvester’s identity, which is the same as the one proposed by Gasca and Mühlbach in [20]. Later, Beckermann and Mühlbach [6] gave an approach, shorter and conceptually simpler than the earlier attempts [20, 19], for a general determinantal identity of Sylvester’s type. In the sequel we report these last results.

We consider a \( n \times m \) matrix \( M \) with rows and columns numbered in the usual way. Let \( I = (i_1, \ldots, i_\alpha) \subset N_n \) and \( J = (j_1, \ldots, j_\beta) \subset N_m \) two ordered index lists, and consider also the row vectors \( z_1, \ldots, z_q \), where \( z_k = (z_{k,j})_{j \in N_m} \).

We border the matrix \( M(I,J) \), extracted from \( M \), by the \( q \) row vectors \( z_k' = (z_{k,j_\lambda})_{\lambda=1,\ldots,\beta}, k = 1, \ldots, q \), extracted from the \( z_k \), that is we consider

\[
M(I_1, \ldots, I_q \cup J_1, \ldots, J_q) =
\begin{pmatrix}
  a_{i_1,j_1} & \cdots & a_{i_1,j_\beta} \\
  \vdots & \ddots & \vdots \\
  a_{i_\alpha,j_1} & \cdots & a_{i_\alpha,j_\beta} \\
  z_1,j_1 & \cdots & z_1,j_\beta \\
  \vdots & \ddots & \vdots \\
  z_q,j_1 & \cdots & z_q,j_\beta
\end{pmatrix}.
\]

Let \( q \in \mathbb{N}^+ \) be fixed, and consider the index lists

\[
I_1, \ldots, I_q \subset N_n \quad \text{and} \quad J_1, \ldots, J_q \subset N_m
\]

such that

\[
\text{card}(J_k) = \text{card}(I_k) + 1 \quad \text{for} \quad k = 1, \ldots, q.
\]

We set, for \( k = 1, \ldots, q \),

\[
I^{(k)} := \bigcap_{i=1}^k I_i \quad \text{and} \quad J^{(k)} := \bigcup_{j=1}^k J_j
\]

The authors give the following theorem.

**Theorem 2** Let

\[
I_0 \subseteq I^{(q)} \text{ such that } \text{card}(I_0) = \text{card}(J^{(q)}) - q.
\]

Then, for any matrix \( N_n \times N_m \) matrix \( M \), there exists an element \( c \in \mathbb{K} \) depending only on \( M(I_0,J^{(q)}) \) such that, from all row vectors \( z_1, \ldots, z_q \in \mathbb{K}^m \), and so for the vectors \( z_k' = (z_{k,j_\lambda})_{j_\lambda \in J^{(q)}}, k = 1, \ldots, q \), extracted from them,

\[
det B = c \cdot M[I_0, 1, \ldots, q, J^{(q)}],
\]

where \( B \) is the matrix with elements

\[
b_{ij} = M[I_i, j, J^{(q)}], \quad 1 \leq i, j \leq q,
\]

(9)
that is the matrix $M \begin{bmatrix} I_i \\ J_i \end{bmatrix}$ bordered with the row vector $z_j' = (z_{j,\lambda})_{\lambda \in J_i}$.

Two corollaries in [6] are also of interest.

**Corollary 1** If $\text{card}(I^{(k)}) > \text{card}(J^{(k)}) - k$, for some $k \in \{2, \ldots, q\}$, then (9) holds with $c = 0$.

**Corollary 2** If $\text{card}(J^{(q)}) < q$, then, for all choices of vectors $z_1, \ldots, z_q$, we have $\det B = 0$.

In some particular cases, the constant $c$ of Theorem 2 can be computed explicitly. This is the case of Sylvester’s classical determinantal identity. In fact, we consider a nonsingular matrix $M$ of order $n$, and we choose two integers $t, q \in \mathbb{N}^+$, such that $t + q = n$. Let

$J_k = (1, \ldots, t, t + k), \quad \text{for} \quad k = 1, \ldots, q$

$I_k = (1, \ldots, t), \quad \text{for} \quad k = 1, \ldots, q$

$I_0 = I^{(q)} = (1, \ldots, t), \quad J^{(q)} = N_n = (1, \ldots, n)$

$z_k' = M \begin{pmatrix} t + k \\ J_k \end{pmatrix}, \quad \text{for} \quad k = 1, \ldots, q.$

We consider the matrix $B$ with elements

$b_{ij} = M \begin{bmatrix} 1, \ldots, t \vert i \\ 1, \ldots, t, t + j \end{bmatrix}, \quad 1 \leq i, j \leq q.$

From [2] we have

$$\det B = c \cdot M \begin{bmatrix} 1, \ldots, t + q \\ 1, \ldots, t \end{bmatrix} = c \cdot \det M \quad (10)$$

where $c$ depends only on $M \begin{pmatrix} I_0 \\ N_n \end{pmatrix}$ and can be evaluated by choosing the particular row vectors $z_k = (\delta_{\lambda,t+k})_{\lambda \in N_n}$ for $k = 1, \ldots, q$, where $\delta$ is the usual Kronecker symbol. With this choice we have

$$\det B = \left( M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix} \right)^q,$$

while the right hand side becomes

$$c \cdot M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix}.$$

So, assuming that $M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix} \neq 0$,

$$c = \left( M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix} \right)^{q-1}.$$
and from (10) it follows
\[
\det B = \det M \cdot \left( M \begin{bmatrix} 1, \ldots, t \\ 1, \ldots, t \end{bmatrix} \right)^{q-1}.
\]
which is the classical Sylvester’s identity [2].

In a similar way [6], is also possible to obtain the identities of Schweins and Monge [1].

The relation (9) can be very useful for obtaining other determinantal identities, since \( c \) depends on \( M \) but not on the rows corresponding to indexes from \( N_n \setminus I_0 \). As an application of this general determinantal identity, in [6] old and new recurrence relations for the E-transforms [7] are derived. In the same paper, a more general Theorem, based on a chain of index lists (that is lists \( I_k \subset N_n \) and \( J_k \subset N_m \) such that, for all \( k = 1, \ldots, q \), \( \text{card}(J^{(k)}) = \text{card}(I^{(k)}) + k \)) is also provided.

3.3.1 Examples

Example 1:

Let \( M \) a \( n \times m \) matrix, with \( n = 6 \) and \( m = 7 \). Set \( q = 3 \), and consider the ordered index lists \( I_1 = (2, 3, 4), I_2 = (2, 4), I_3 = (1, 2, 4) \) and \( J_1 = (2, 3, 4, 5), J_2 = (2, 3, 4), J_3 = (2, 3, 4, 7) \). We have \( I^{(3)} = (2, 4) \) and \( J^{(3)} = (2, 3, 4, 5, 7) \). Choose any \( z_1, z_2, z_3 \), and \( I_0 = I^{(3)} \). Let

\[
B = \begin{pmatrix}
M \begin{bmatrix} 2, 3, 4, 1 \\ 2, 3, 4, 5 \end{bmatrix} & M \begin{bmatrix} 2, 3, 4, 2 \\ 2, 3, 4, 5 \end{bmatrix} & M \begin{bmatrix} 2, 3, 4, 3 \\ 2, 3, 4, 5 \end{bmatrix} \\
M \begin{bmatrix} 2, 4, 1 \\ 2, 3, 4 \end{bmatrix} & M \begin{bmatrix} 2, 4, 2 \\ 2, 3, 4 \end{bmatrix} & M \begin{bmatrix} 2, 4, 3 \\ 2, 3, 4 \end{bmatrix} \\
M \begin{bmatrix} 1, 2, 4, 1 \\ 2, 3, 4, 7 \end{bmatrix} & M \begin{bmatrix} 1, 2, 4, 2 \\ 2, 3, 4, 7 \end{bmatrix} & M \begin{bmatrix} 1, 2, 4, 3 \\ 2, 3, 4, 7 \end{bmatrix}
\end{pmatrix},
\]

where \( \det B \neq 0 \). If \( M \begin{bmatrix} 2, 4, 1, 2, 3 \\ 2, 3, 4, 5, 7 \end{bmatrix} \neq 0 \), there exists \( c \) (which cannot be computed a priori) such that (8) holds.

Example 2: Let again \( n = 6 \) and \( m = 7 \), but consider \( q = 5 \) and the ordered index lists \( I_1 = (1, 4), I_2 = (3, 4), I_3 = (2, 4), I_4 = (4, 5), I_5 = (4, 6) \) and \( J_1 = (1, 2, 4), J_2 = (1, 4, 7), J_3 = (1, 2, 7), J_4 = (1, 4, 7), J_5 = (1, 2, 7) \). We have \( I^{(5)} = (4) \) and \( J^{(5)} = (1, 2, 4, 7) \). Let

\[
B = \begin{pmatrix}
M \begin{bmatrix} 1, 4, 1 \\ 1, 2, 4 \end{bmatrix} & M \begin{bmatrix} 1, 4, 2 \\ 1, 2, 4 \end{bmatrix} & M \begin{bmatrix} 1, 4, 3 \\ 1, 2, 4 \end{bmatrix} & M \begin{bmatrix} 1, 4, 4 \\ 1, 2, 4 \end{bmatrix} & M \begin{bmatrix} 1, 4, 5 \\ 1, 2, 4 \end{bmatrix} \\
M \begin{bmatrix} 3, 4, 1 \\ 1, 4, 7 \end{bmatrix} & M \begin{bmatrix} 3, 4, 2 \\ 1, 4, 7 \end{bmatrix} & M \begin{bmatrix} 3, 4, 3 \\ 1, 4, 7 \end{bmatrix} & M \begin{bmatrix} 3, 4, 4 \\ 1, 4, 7 \end{bmatrix} & M \begin{bmatrix} 3, 4, 5 \\ 1, 4, 7 \end{bmatrix} \\
M \begin{bmatrix} 2, 4, 1 \\ 1, 2, 7 \end{bmatrix} & M \begin{bmatrix} 2, 4, 2 \\ 1, 2, 7 \end{bmatrix} & M \begin{bmatrix} 2, 4, 3 \\ 1, 2, 7 \end{bmatrix} & M \begin{bmatrix} 2, 4, 4 \\ 1, 2, 7 \end{bmatrix} & M \begin{bmatrix} 2, 4, 5 \\ 1, 2, 7 \end{bmatrix}
\end{pmatrix}.
\]

Due to Corollary 2 since \( \text{card}(J^{(5)}) < 5 \), then, for any \( z_1, \ldots, z_5 \), \( \det B \) is equal to zero.
3.4 Generalization of Mulders

Another generalization of Sylvester’s identity is due to Mulders \[21\]. The author follows the idea of Bareiss \[3, 4\], where the Sylvester’s identity is used to prove that certain Gaussian elimination algorithms, transforming a matrix into an upper-triangular form, are fraction-free. Fraction-free algorithms also have applications in the computation of matrix rational approximants, matrix GCDs, and generalized Richardson extrapolation processes \[5\]. They are used for controlling, in exact arithmetic, the growth of intermediate results. The generalization of Sylvester’s identity, due to Mulders, was used to prove that also certain random Gaussian elimination algorithms are fraction-free. The fraction-free random Gaussian elimination algorithm, obtained in that way, has been used in the simplex method (to solve linear programming problems) and for finding a solution of $Ax = b$, $x \geq 0$, with $A$ a $m \times n$ matrix.

Given a $n \times m$ matrix $M$, we consider the index list $I = (i_1, \ldots, i_t) \subseteq N_n$ (without repetition but not necessarily ordered) and another index list $J = (j_1, \ldots, j_t) \subseteq N_m$, (with possible repetition and not necessarily ordered). The two lists have the same length $t$, $0 \leq t \leq \min(m, n)$. Notice that it is possible to have $t = 0$, so an empty list. With these assumptions, we can define an equivalence class $[(i_1, j_1), \ldots, (i_t, j_t)]$ of all possible permutations of the $t$ pairs and define the determinant

$$a^{[(i_1, j_1), \ldots, (i_t, j_t)]} = M \begin{bmatrix} i_1, \ldots, i_t \\ j_1, \ldots, j_t \end{bmatrix},$$

with $a^{[\emptyset]} = M \begin{bmatrix} \emptyset \\ \emptyset \end{bmatrix} = 1$ when $t = 0$. Note that, when $J$ has repeated elements, the determinant is equal to zero.

We also define the operation $\leftarrow$ on the pairs defining the above determinant as

$$[(i_1, j_1), \ldots, (i_t, j_t)] \leftarrow [(u_1, v_1), \ldots, (u_r, v_r)].$$

$I$ and $U = (u_1, \ldots, u_r)$ are index lists without repetition but not necessarily ordered, $J$ and $V = (v_1, \ldots, v_r)$ are index lists not necessarily ordered but with possible repetitions. This operation produces a new class of pairs by adding to the first class all the pairs $(u_k, v_k)$ of the second class when $u_k \notin I$, and by replacing the pair $(i_\ell, j_\ell)$ of the first class by the pair $(u_k, v_k)$, when $u_k = i_\ell$ for some $i_\ell \in I$. This operation is performed for $k = 1, \ldots, r$, where $r$ is an integer. This operation is associative and commutative if $I \cap U = \emptyset$. For example, the result of $[(2, 1), (3, 3), (1, 5), (5, 3)] \leftarrow [(1, 1), (2, 3), (4, 4)]$, is $[(2, 3), (3, 3), (1, 1), (5, 3), (4, 4)]$, since the pair $(4, 4)$ has been added, and the pairs $(1, 1), (2, 3)$ have replaced the pairs $(1, 5), (2, 1)$, respectively.

We consider now two integers $i$ and $j$, $1 \leq i \leq n, 1 \leq j \leq m$ and the determinant

$$a^{[(i_1, j_1), \ldots, (i_t, j_t)] \leftarrow [(i, j)]},$$

Due to the operation previously defined, the result is the following:

- When $i \notin I$, then we simply extend the matrix with the $i$th row and $j$th column of $M$, and we compute the determinant of the new square matrix of order $(t + 1)$, that is

$$a^{[(i_1, j_1), \ldots, (i_t, j_t)] \leftarrow [(i, j)]} = M \begin{bmatrix} i_1, \ldots, i_t, i \\ j_1, \ldots, j_t, j \end{bmatrix}.$$
• When it exists $1 \leq k \leq t$ such that $i_k = i$, we replace, in $a^{[(i_1,j_1),\ldots,(i_t,j_t)]}$, the pair $(i_k,j_k)$ by the pair $(i,j)$, that is we take the following determinant of a new matrix of order $t$
\[a^{[(i_1,j_1),\ldots,(i_t,j_t)]} = M^{[i_1,\ldots,i_k=i,\ldots,i_t]}_{j_1,\ldots,j}.
\]

Let us consider now the particular case where $i_k = k$, $j_k = k$, for all $k = 1,\ldots,t$, that is $I = J = N_t$. When $t < i,j \leq \min(n,m)$, we recover the usual definition $a_{i,j}^{(t)}$ given in (1), but not in the other cases. For simplicity, in the sequel, we set $a_{i,j}^{(t)} = a^{[(1,1),\ldots,(t,t)]}$. We have
\[
\begin{cases}
\text{If } t < i,j & a_{i,j}^{(t)} \text{ of order } t + 1 & a_{i,j}^{(t)} = a_{i,j}^{(t)} \\
\text{If } j \leq t < i & a_{i,j}^{(t)} \text{ of order } t + 1 & a_{i,j}^{(t)} = 0 \\
\text{If } i,j \leq t, i \neq j & a_{i,j}^{(t)} \text{ of order } t & a_{i,j}^{(t)} = 0 \\
\text{If } i = j \leq t & a_{i,j}^{(t)} \text{ of order } t & a_{i,j}^{(t)} = a^{[(1,1),\ldots,(t,t)]} \\
\text{If } i < t < j & a_{i,j}^{(t)} \text{ of order } t & a_{i,j}^{(t)} \text{ is a new determinant.}
\end{cases}
\]

By using this particular choice for $I$ and $J$, Mulders formulated the following generalized Sylvester’s determinantal identity that allows to consider the determinantal elements $a_{i,j}^{(t)}$ when $i$ and/or $j$ are $\leq t$. In the sequel, we state a more generalized identity.

**Theorem 3** For $0 \leq t \leq \min(n,m)$, $0 \leq p,q \leq t$ and $1 \leq s \leq \min(n-p,m-q)$, the following identity holds
\[
a^{[(1,1),\ldots,(t,t)]} = \left[\begin{array}{c}
a_{p+1,q+1}^{(t)} & \cdots & a_{p+1,q+s}^{(t)} \\
\vdots & \ddots & \vdots \\
a_{p+s+1,q+1}^{(t)} & \cdots & a_{p+s+1,q+s}^{(t)}
\end{array}\right]^{s-1} = a_{t,t}^{(t-1)}.
\]

For $m = n$, $p = q = t$ and $s = n - t$ we obtain the classical Sylvester’s identity (2).

### 3.4.1 Examples

**Example 1:**
Let $M$ a $n \times m$ matrix, with $n = 7$ and $m = 8$. Set $t = 5$, and $I = J = N_5$. Assume $p = 3$, $q = 4$, and $s = 3$, so in this case $p+s \geq t$. Then we have $a^{[(1,1),\ldots,(5,5)]} = M^{[1,2,3,4,5,6]}$ and $a_{5,5}^{(4)} = M^{[1,\ldots,5]}$. The right hand side of (12) is the following determinant of order 3
\[
\begin{vmatrix}
a_{4,5}^{(5)} & a_{4,6}^{(5)} & a_{4,7}^{(5)} \\
a_{5,5}^{(5)} & a_{5,6}^{(5)} & a_{5,7}^{(5)} \\
a_{6,5}^{(5)} & a_{6,6}^{(5)} & a_{6,7}^{(5)}
\end{vmatrix}.
\]
that is, thanks to the operation defined,
\[
\begin{vmatrix}
0 & M & 1,2,3,4,5 \\
\end{vmatrix} \begin{vmatrix}
1,2,3,6,5 \\
\end{vmatrix} \begin{vmatrix}
1,2,3,4,5 \\
\end{vmatrix} \begin{vmatrix}
1,2,3,7,5 \\
\end{vmatrix} \begin{vmatrix}
1,2,3,4,5 \\
\end{vmatrix} \begin{vmatrix}
1,2,3,4,7 \\
\end{vmatrix} \begin{vmatrix}
1,2,3,4,5 \times 6 \\
\end{vmatrix} \begin{vmatrix}
1,2,3,4,5 \times 7 \\
\end{vmatrix}
\]

and \((12)\) holds.

**Example 2:**
Let again \(M\) a \(7 \times 8\) matrix. Set now \(t = 6\), and \(I = J = N_6\). Assume \(p = 2, q = 3\), and \(s = 3\). In this case \(p + s < t\), so we have \(a^{(1,1),\ldots,(6,6)} \times \cdots \times (3,4),(4,5),(5,6) = M \times \cdots \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time
Theorem 4 (Generalized Sylvester’s identity) Let $M$ be a matrix of dimension $n \times m$, and $M_0$ its square leading principal submatrix of fixed order $t$, $0 < t \leq \min(n, m)$. Then, chosen $s, q \in \mathbb{N}$, $1 \leq s \leq \min(n-t, m-t)$, such that $\min(n, m) = t + q$ $s$, the following identities hold for $0 \leq k \leq q$

\[
\frac{\det M_k}{\det M_0^{(s-1)k}} = \frac{\prod_{i=1 \atop i \text{ odd}}^{k-1} \left[ \det B_i \right]^{(s-1)k-1-i}}{\prod_{j=0 \atop j \text{ even}}^{k-2} \left[ \det B_j \right]^{(s-1)k-1-j}}, \quad \text{for } k \text{ even,} \tag{15}
\]

\[
\det M_k \cdot \det M_0^{(s-1)k} = \frac{\prod_{i=0 \atop i \text{ even}}^{k-1} \left[ \det B_i \right]^{(s-1)k-1-i}}{\prod_{j=1 \atop j \text{ odd}}^{k-2} \left[ \det B_j \right]^{(s-1)k-1-j}}, \quad \text{for } k \text{ odd.} \tag{16}
\]

Proof. We make the proof by mathematical induction over $k$.

For $k = 0$, (15) is trivially satisfied since, in the right hand side, we have a ratio of empty products.

For $k = 1$, since the product in the denominator of the right hand side is an empty product, we have

\[
\det M_1 \cdot \det M_0^{s-1} = \det B_0,
\]

where $M_1 = M \begin{pmatrix} 1, \ldots, t + s \end{pmatrix}$, $\det M_0 = a_{t,t}^{(t-1)}$, and $B_0$ is the matrix with elements $b_{ij}^{(0)} = M \begin{pmatrix} 1, \ldots, t, t + i \end{pmatrix} = a_{t+i,t+j}^{(t)}$, for $1 \leq i, j \leq s$. So it is exactly the Sylvester’s identity (2) applied to $M_1$, and (16) holds.

In the inductive step we shall prove that, if (16) holds for $k < q$ odd, then (15) holds for $k$ even, moreover if (15) holds for $k$ even, then (16) holds for $k + 1$ odd.

Assume that (16) is satisfied for $k < q$ odd. We apply (2) to the matrix $M_{k+1}$, of order $(m_k + s)$, with $\det M_k = a_{m_k,m_k}^{(m_k-1)}$ fixed, and we have

\[
\det M_{k+1} \cdot \det M_k^{s-1} = \det B_k \tag{17}
\]

where $B_k$ is the matrix with elements as in (14). Substituting in (17) the expression for $\det M_k$ obtained from (16), leads to

\[
\det M_{k+1} \cdot \det M_0^{(s-1)k+1} = \det B_k,
\]

for $0 \leq k \leq q$. The proof is complete.
and then we get

\[
\det M_{k+1} \cdot [\det M_0]^{(s-1)k+1} = \prod_{j=1}^{k} \frac{[\det B_j]^{(s-1)k-j}}{[\det B_j]^{(s-1)k-i} \prod_{i=0}^{k-1} i \text{ even}}
\]

which is (15) for \( k + 1 \) even.

Assume, now, that (15) holds for \( k < q \) even. By using (2), the relation (17) holds again. Substituting in (17) the expression for \( \det M_k \), obtained from (15), gives now

\[
\det M_{k+1} \cdot [\det M_0]^{(s-1)k+1} \cdot \left[ \prod_{i=0}^{k-2} \frac{[\det B_i]^{(s-1)k-1-i}}{[\det B_j]^{(s-1)k-1-j} \prod_{j=0}^{k-1} j \text{ even}} \right]^{s-1} = \det B_k,
\]

and then

\[
\det M_{k+1} \cdot [\det M_0]^{(s-1)k+1} = \prod_{j=0}^{k} \frac{[\det B_j]^{(s-1)k-j}}{[\det B_i]^{(s-1)k-i} \prod_{i=1}^{k-1} i \text{ odd}}
\]

which proves (16) for \( k + 1 \) odd.

\[\square\]

So, now, we have established a relation between determinants obtained by bordering a leading principal submatrix of order \( t \) of a matrix \( M \), with blocks of \( s \) rows and \( s \) columns. \( m = n \) (i.e. \( s = n - t \)), (16) becomes exactly the classical Sylvester’s determinantal identity (2).

Remark: When \( s = 1 \), we have \( \det M_k = \det B_{k-1} = a_{t+k,t+k}^{(t+k-1)} \), for \( 0 \leq k \leq \min(n - t, m - t) \), and the \( B_{k-1} \) are scalars.

4.1 Applications

Obviously, from the computational point of view, if \( s \) is large, the formulae given in the Theorem 4 are not very interesting. So, let \( M \) a square matrix of order \( n \), and consider the particular case where \( s = 2 \), chosen \( t \) such that \( n - t \) is even, \( q = (n - t)/2 \), even or odd. We set

\[
M = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21}^{T} & A_{22}
\end{pmatrix},
\]

where \( A_{11} \) is a square matrix of order \( t \), \( A_{22} \) is a square matrix of order \( (n - t) = 2q \), \( A_{12}, A_{21} \) matrices of dimension \( t \times (n - t) = t \times 2q \).
Formulae \((15)\) and \((16)\), for \(k = q\), become
\[
\det M = \prod_{i=1}^{q-1} \det B_i / \prod_{j=0}^{q-2} \det B_j, \quad q \text{ even,} \tag{18}
\]
\[
\det M \cdot \det A_{11} = \prod_{i=0}^{q-1} \det B_i / \prod_{j=1}^{q-2} \det B_j, \quad q \text{ odd,} \tag{19}
\]
where the \(B_i\) matrices, computed as in \((14)\), are all of order 2.

Consider the particular case where \(q = 1\). Let \(D = A_{11}\) be a square matrix of order \(n - 2\),
\[
A_{22} = \begin{pmatrix} a & e \\ g & l \end{pmatrix}, \quad A_{12} = \begin{pmatrix} c & f \end{pmatrix}, \quad A_{21} = \begin{pmatrix} b & h \end{pmatrix},
\]
with \(a, e, g, l\) scalars, and \(b, c, f, h\) column vectors of length \((n - 2)\). Formula \((19)\) gives
\[
\det M_1 \cdot \det D = \det B_0,
\]
and since, in this case,
\[
\det B_0 = \begin{vmatrix} D & c & D & f \\ b^T & a & b^T & e \\ c & D & f \\ h^T & g & h^T & l \end{vmatrix},
\]
we recover exactly the rule \((5)\).

Let now \(M'\) a square matrix of order \(n\), partitioned as
\[
M' = \begin{pmatrix} A & B^T & E \\ C & D & F \\ G^T & H^T & L \end{pmatrix}, \tag{20}
\]
where the block \(D\) is a square matrix of order \(t\), \(A, E, G, L\) are square matrices of order 2, and \(B, C, F, H\) are of dimension \(t \times 2\), and \(n = t + 4\). Setting
\[
M = \begin{pmatrix} D & C & F \\ B^T & A & E \\ H^T & G^T & L \end{pmatrix},
\]
we have \(\det M' = \det M\). So considering \(s = q = 2\), \((18)\) gives
\[
\frac{\det M}{\det D} = \frac{\det B_1}{\det B_0}, \tag{21}
\]
with \(B_0\) and \(B_1\) of order 2.

For the above matrices we denote
\[
B = \begin{pmatrix} b_1 & b_2 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & c_2 \end{pmatrix}, \quad E = \begin{pmatrix} e_1 & e_2 \end{pmatrix}, \quad F = \begin{pmatrix} f_1 & f_2 \end{pmatrix}, \\
G = \begin{pmatrix} g_1 & g_2 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & h_2 \end{pmatrix}, \\
A = (a_{ij}), \quad L = (l_{ij}), \quad \text{for } i, j = 1, 2,
\]
where \( b_i, c_i, e_i, f_i, g_i, h_i, i = 1, 2 \), are column vectors. We have

\[
\det B_0 = \begin{vmatrix} D & c_1 & D & c_2 \\ b_1^T & a_{11} & b_1^T & a_{12} \\ b_2^T & a_{21} & b_2^T & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & b_1^T & a_{12} & b_1^T \\ c_1 & D & c_2 & D \end{vmatrix},
\]

\[
\det B_1 = \begin{vmatrix} D & C & f_1 & D & C & f_2 \\ B^T & A & e_1 & B^T & A & e_2 \\ h_1^T & g_1^T & l_{11} & h_1^T & g_1^T & l_{12} \\ B^T & A & e_1 & B^T & A & e_2 \\ h_2^T & g_2^T & l_{21} & h_2^T & g_2^T & l_{22} \end{vmatrix} = \begin{vmatrix} A & B^T & e_1 & A & B^T & e_2 \\ C & D & f_1 & C & D & f_2 \\ g_1^T & h_1^T & l_{11} & g_1^T & h_1^T & l_{12} \\ g_2^T & h_2^T & l_{21} & g_2^T & h_2^T & l_{22} \end{vmatrix}.
\]

So this application is a generalization of the Case 2 of the Section 2.

4.1.1 Examples

Example 1: Let \( n = 10, t = s = 2, q = (n - t)/2 = 4 \) even. Thanks to (18), we have

\[
\frac{\det M}{\det A_{11}} = \frac{\det B_1 \cdot \det B_3}{\det B_0 \cdot \det B_2}.
\]

If \( n = 10, t = 6, s = 2, q = (n - t)/2 = 3 \) odd, thanks to (19), we have

\[
\det M \cdot \det A_{11} = \frac{\det B_0 \cdot \det B_2}{\det B_1}.
\]

Example 2: We consider a square matrix \( M' \) of order \( n = 8 \) partitioned like in (20), with \( D = M'(3,\ldots,6,3,\ldots,6) \). Then \( t = 4, s = q = 2, \) and

\[
A = M'(1,2,1,2), \quad E = M'(1,2,7,8), \quad G = M'(7,8,1,2), \quad L = M'(7,8,7,8),
\]

\[
B^T = M'(1,2,3,\ldots,6), \quad H^T = M'(7,8,3,\ldots,6), \quad C = M'(3,\ldots,6,1,2), \quad F = M'(3,\ldots,6,7,8).
\]

So we can apply (21), and we obtain

\[
\frac{\det M'}{\det D} = \frac{M'(1,\ldots,6,7,1,\ldots,6,7) \cdot M'(1,\ldots,6,8,1,\ldots,6,8) - M'(1,\ldots,6,7,1,\ldots,6,7) \cdot M'(1,\ldots,6,8)}{M'(1,3,\ldots,6,1,3,\ldots,6) \cdot M'(2,3,\ldots,6,2,3,\ldots,6) - M'(1,3,\ldots,6,2,3,\ldots,6) \cdot M'(1,3,\ldots,6)}.
\]

5 Conclusions

In this paper we presented several generalizations of the Sylvester’s determinantal identity, proposed from various authors, and described here in a unified way. We started to
study those formulae because the classical formula is a very important tool for finding extrapolation algorithms [10, 9] for accelerating the convergence of scalar and vector sequences, and we need to have at our disposal generalizations able to be used in building new extrapolation algorithms for vector sequences (a case more difficult) [8]. During our study, we had the idea for the new generalization proposed.

Remark

This is an updated version of a work presented in two congresses in 2008.

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