Quasi-toposes as elementary quotient completions

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Abstract

The elementary quotient completion of an elementary doctrine in the sense of Lawvere was introduced in previous work by the first and third authors. It generalises the exact completion of a category with finite products and weak equalisers. In this paper we characterise when an elementary quotient completion is a quasi-topos. We obtain as a corollary a complete characterisation of when an elementary quotient completion is an elementary topos. As a byproduct we determine also when the elementary quotient completion of a tripos is equivalent to the doctrine obtained via the tripos-to-topos construction.

Our results are reminiscent of other works regarding exact completions and put those under a common scheme: in particular, Carboni and Vitale’s characterisation of exact completions in terms of their projective objects, Carboni and Rosolini’s characterisation of locally cartesian closed exact completions, also in the revision by Emmenegger, and Menni’s characterisation of the exact completions which are elementary toposes.

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1 Introduction

The study of constructions for completing a category with quotients is a central topic not only in mathematics but also in computer science. In category theory a well-known related notion is that of exact completion of a category with finite limits, and that of a regular category, which has been widely studied and applied; see [CC82, CV98].

In [MR13b] the first and third authors generalized the notion of exact completion on a category with weak finite limits to that of an elementary quotient completion of a Lawvere’s elementary doctrine [Law69, Law70] as an universal construction to close such a doctrine with respect to a suitable notion of quotient.

The exact completion of a category $C$ with finite products and weak limits is an instance of such a construction in the sense that its subobject doctrine is the elementary quotient completion doctrine of the doctrine of variations of $C$ [Gra00].

In the paper we study elementary quotient completions performed on the special class of Lawvere’s elementary doctrines called triposes, introduced in [HJP80], to build elementary toposes by means of what is now known as the tripos-to-topos construction; see [Fre15]. We then characterize those triposes whose elementary quotient completion has an arithmetic quasi-topos—i.e. a quasi-topos equipped with a natural number object—as a base category.

To obtain the characterization, we extend some known results about exact completions such as Carboni and Vitale’s characterization of exact completions in terms of its projective objects in [CV98], Menni’s characterization of the exact completions which are toposes in [Men03] and Carboni and Rosolini’s characterization of the locally cartesian closed exact completions [CR00]. In particular, we show that
• an elementary doctrine $P : \mathcal{C}^{op} \to \text{POS}$ closed under effective quotients is the elementary quotient completion of the doctrine determined by the restriction of $P$ to the full subcategory of $\mathcal{C}$ on its projective objects;

• the base category of the elementary quotient completion of $P$ turns weak universal properties of $\mathcal{C}$ into (strong) universal properties of the base of the elementary quotient completion. Those include binary coproducts, a natural number object, a parametrized list object, a subobject classifier, a cartesian closed structure, a locally cartesian structure.

• by using results in [MR16] we characterize when an elementary quotient completion is an elementary topos;

• by using results in [MPR17] we characterize when an elementary quotient completion is a tripos-to-topos construction.

We conclude by pointing out some relevant examples of arithmetic quasi-toposes arising as non-exact elementary quotient completions. Most notably they include the category of equilogical spaces of [Sco76, Sco96, BBS04], that of assemblies over a partial combinatory algebra (see [Hyl82, vO08]), and the category of total setoids, in the style of E. Bishop, over Coquand and Paulin’s Calculus of Inductive Constructions which is the theory at the base of the proof-assistant Coq.

2 Preliminary definitions on doctrines and completions

This section collects the necessary definitions to introduce the elementary quotient completion of an elementary doctrine and related properties of a doctrine. Recall the following notions from [MR13a]; see also [EPR20].

2.1 Definition. A primary doctrine is an indexed poset $P : \mathcal{C}^{op} \to \text{POS}$ such that

(i) $\mathcal{C}$ is a category with a terminal object $T$ and binary products

$$
\begin{array}{c}
C_1 \xrightarrow{\text{pr}_1} C_1 \times C_2 \xrightarrow{\text{pr}_2} C_2
\end{array}
$$

for each pair of objects $C_1$ and $C_2$;

(ii) each fibre $P(C)$ is an inf-semilattice

$$
\begin{array}{c}
1 \xrightarrow{T} P(C) \xrightarrow{\wedge} P(C) \times P(C)
\end{array}
$$

(iii) and the operations are natural.

As usual with an indexed category, the category $\mathcal{C}$ is often called base of the doctrine. We say that $\alpha$ is over $A$ if $\alpha$ is an element of $P(A)$. We write $A_1 \times A_2 \times \ldots \times A_n$ for an arbitrary finite product with factors $A_1, A_2, \ldots, A_n$. 

2.2 Examples. (a) An example of primary doctrine comes directly from first-order logic is the Lindenbaum–Tarski algebras of well-formed formulas of a theory $\mathcal{T}$ over a first-order language $\mathcal{L}$. The base category is the category $\mathcal{V}$ of lists of distinct variables and term substitutions, and the primary doctrine $LT: \mathcal{V}^{op} \to \mathsf{POS}$ on $\mathcal{V}$ is given on a list of variables $\vec{x}$ by taking $LT(\vec{x})$ as the Lindenbaum–Tarski algebra of well-formed formulas with free variables in $\vec{x}$. Meets in $LT(\vec{x})$ are given by conjunctions while the top element by any true formula; see [MR13a] for more details.

(b) Let $\mathcal{H}$ be an inf-semilattice. The functor $P: \mathsf{Set}^{op} \to \mathsf{POS}$ sending a set $A$ to $\mathcal{H}^A$ and a function $f: A \to B$ to $P(f) = - \circ f$ is a primary doctrine.

(c) If $\mathcal{C}$ has finite limits the functor $\mathsf{Sub}_C: \mathcal{C}^{op} \to \mathsf{POS}$ is a primary doctrine.

(d) Another categorical example is given by a category $\mathcal{C}$ with binary products and weak pullbacks, by defining the doctrine functor $\Psi_C: \mathcal{C}^{op} \to \mathsf{POS}$ which evaluates as the poset reflection of each comma category $\mathcal{C}/A$ at each object $A$ of $\mathcal{C}$, introduced in [Gra00].

2.3 Definition. Primary doctrines are the objects of the 2-category $\mathbf{PD}$ where the 1-morphisms in $\mathbf{PD}$ are pairs $(F, b)$ where $F: \mathcal{C} \to \mathcal{D}$ is a functor and $b: P \mathop{\longrightarrow}^{\scriptscriptstyle \circ R \circ F^{op}} \mathcal{D}^{op}$ is a natural transformation as in the diagram

\[
\begin{array}{c}
\mathcal{C}^{op} \\
\downarrow F^{op} \\
\mathcal{D}^{op} \\
\downarrow G^{op} \\
\mathsf{POS} \\
\end{array}
\]

where the functor $F$ preserves products.

The 2-morphisms are natural transformations $\theta: F \Rightarrow G$ such that

\[
\begin{array}{c}
\mathcal{C}^{op} \\
\downarrow F^{op} \\
\mathcal{D}^{op} \\
\downarrow G^{op} \\
\mathsf{POS} \\
\end{array}
\]

so that, for every $A$ in $\mathcal{C}$ and every $\alpha$ in $P(A)$, one has $b_A(\alpha) \leq_{F(A)} R_{\theta_A}(c_A(\alpha))$.

2.4 Example. A set-theoretic model $\mathfrak{M}$ for a first-order theory $\mathcal{T}$ determines a 1-arrow from the primary doctrine $LT: \mathcal{V}^{op} \to \mathsf{POS}$ to the primary doctrine $\mathsf{Sub}_{\mathsf{Set}}: \mathcal{Set}^{op} \to \mathsf{POS}$ in $\mathbf{PD}$. And a homomorphism $f: \mathfrak{M} \to \mathfrak{M}'$ between two set-theoretic models of $\mathcal{T}$ gives rise to a 2-arrow; see [MR13b].

Given a doctrine $P: \mathcal{C}^{op} \to \mathsf{POS}$, a category $\mathcal{D}$ with finite products and a functor $F: \mathcal{D} \to \mathcal{C}$ that preserves products, the composition of $P$ with $F^{op}: \mathcal{D}^{op} \to \mathcal{C}^{op}$ gives a doctrine $PF^{op}: \mathcal{D}^{op} \to \mathsf{POS}$ called change of base of $P$ along $F$.
2.7 Examples. (a) Consider a theory \( \mathcal{T} \) over a first-order language \( \mathcal{L} \) and the associated primary doctrine \( LT: \mathcal{C}^{op} \longrightarrow \text{POS} \) as in 2.2(a). The doctrine \( LT \) is elementary if and only if \( \mathcal{T} \) is definable in \( \mathcal{T} \).

(b) Let \( \mathcal{H} \) be a inf-semilattice. The primary doctrine \( \mathcal{P}_{\mathcal{H}}: \mathcal{C}^{op} \longrightarrow \text{POS} \) as in 2.2(b) is elementary if and only if \( \mathcal{H} \) has a bottom element (see [EPR20]). In this case \( \delta \) is the function that maps \((a, a')\) to \(\top\) if \(a = a'\) and to \(\bot\) otherwise. In this case \( \mathcal{P}_{\mathcal{H}} \) has comprehensive diagonals.

(c) Suppose \( \mathcal{C} \) has finite limits. The doctrine \( \text{Sub}_{\mathcal{C}}: \mathcal{C}^{op} \longrightarrow \text{POS} \) as in 2.2(c) is elementary with comprehensive diagonals where \( \delta \) is represented by the diagonal on \( \mathcal{C} \). (d) Suppose \( \mathcal{C} \) has finite limits and weak pullbacks. The doctrine \( \Psi_{\mathcal{C}}: \mathcal{C}^{op} \longrightarrow \text{POS} \) as in 2.2(d) is elementary with comprehensive diagonals where \( \delta \) is represented by the diagonal on \( \mathcal{C} \).

2.8 Definition. Elementary doctrines are the object of \( \text{ED} \), the 2-full subcategory \( \text{PD} \) whose 1-morphisms are those 1-morphisms \((F, b): P \longrightarrow R \) of \( \text{PD} \) such that for every object \( A \) in \( \mathcal{C} \), the functor \( b_A: P(A) \longrightarrow R(F(A)) \) commutes with the left adjoints

\[
\begin{array}{ccc}
P(X \times A) & \xrightarrow{b_{X \times A}} & R(F(X \times A)) \\
\downarrow \mathcal{I}_e & & \sim \\
P(X \times A \times A) & \xrightarrow{b_{X \times A \times A}} & R(F(X \times A \times A)) \\
\end{array}
\]

\[
\begin{array}{ccc}
R(F \mathcal{P}_{\mathcal{H}}, F_{\mathcal{P}_{\mathcal{H}}, \mathcal{F}}_1) & \longrightarrow & R(FX \times FA) \\
\end{array}
\]

\[
\begin{array}{ccc}
P(X \times A) & \xrightarrow{b_{X \times A}} & R(F(X \times A)) \\
\downarrow \mathcal{I}_{\mathcal{E}} & & \sim \\
P(X \times A \times A) & \xrightarrow{b_{X \times A \times A}} & R(F(X \times A \times A)) \\
\end{array}
\]

\[
\begin{array}{ccc}
R(F \mathcal{P}_{\mathcal{H}}, F_{\mathcal{P}_{\mathcal{H}}, \mathcal{F}}_1) & \longrightarrow & R(FX \times FA \times FA) \\
\end{array}
\]
where $e$ is $\langle \text{pr}_1, \text{pr}_2, \text{pr}_2 \rangle: X \times A \rightarrow X \times A \times A$ and $e'$ is $\langle \text{pr}_1, \text{pr}_2, \text{pr}_2 \rangle: FX \times FA \rightarrow FX \times FA \times FA$.

We recall from [MR13a] that it is possible to add comprehensive diagonals to an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{POS}$ as follows.

2.9 Definition. Consider the category $X_P$, the “extensional collapse” of $P$ whose objects are the objects of $\mathcal{C}$ and where an arrow $[f]: A \rightarrow B$ is an equivalence class of morphisms $f: A \rightarrow B$ in $\mathcal{C}$ such that $\delta_A \leq_{A \times A} P_f(\delta_B)$ in $P(A \times A)$ with respect to the equivalence which relates $f$ and $f'$ when $\delta_A \leq_{A \times A} P_f(\delta_B)$. Composition is given by that of $\mathcal{C}$ on representatives, and identities are represented by identities of $\mathcal{C}$. We then define the doctrine $P_x: X_P^{\text{op}} \rightarrow \mathbf{POS}$ on $X_P$ essentially as $P$ itself.

2.10 Definition. Let $\text{ExD}$ denote the full subcategory of $\mathbf{ED}$ on elementary doctrines with comprehensive diagonals.

2.11 Proposition. There is left biadjoint to the inclusion of $\text{ExD}$ into $\mathbf{ED}$ which on objects associates the doctrine $P_x: X_P^{\text{op}} \rightarrow \mathbf{POS}$ to an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{POS}$.

2.12 Definition. An elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{POS}$ is existential when, for $A_1$ and $A_2$ in $\mathcal{C}$, for any projection $\text{pr}_1: A_1 \times A_2 \rightarrow A_1$, $i = 1, 2$, the functor $P_{\text{pr}_1} : P(A_1) \rightarrow P(A_1 \times A_2)$ has a left adjoint $\mathcal{J}_{\text{pr}_1}$—we shall call such a left adjoint existential—and those left adjoints satisfy the

Beck-Chevalley Condition: for any pullback diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{pr'} & A' \\
\downarrow{f'} & & \downarrow{f} \\
X & \xrightarrow{pr} & A
\end{array}
\]

with $pr$ a projection (hence also $pr'$ a projection), for any $\beta$ in $P(X)$, the natural inequality $\mathcal{J}_{pr'} P_f(\beta) \leq P_f \mathcal{J}_{pr}(\beta)$ in $P(A')$ is an identity;

Frobenius Reciprocity: for $pr: X \rightarrow A$ a projection, $\alpha$ in $P(A)$, $\beta$ in $P(X)$, the natural inequality $\mathcal{J}_{pr}(P_{pr}(\alpha) \wedge \beta) \leq \alpha \wedge \mathcal{J}_{pr}(\beta)$ in $P(A)$ is an identity.

2.13 Examples. (a) Consider a theory $\mathcal{T}$ over a first-order language $\mathcal{L}$ and the associated primary doctrine $LT: \mathcal{L}^{\text{op}} \rightarrow \mathbf{POS}$ as in 2.2(a). The doctrine $LT$ is existential where left adjoints along projections are given by the existential quantification.
(b) Let $\mathcal{H}$ be an inf-semilattice. The primary doctrine $P_{\mathcal{H}}: \text{Set}^{\text{op}} \rightarrow \mathbf{POS}$ is existential if and only if $\mathcal{H}$ is a frame (see [EPR20]).
(c) Suppose $\mathcal{C}$ has finite limits. The doctrine $\text{Sub}_\mathcal{C}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{POS}$ as in 2.2(c) is existential if and only if $\mathcal{C}$ is regular (see [Jac99]).
(d) Suppose $\mathcal{C}$ has finite limits and weak pullbacks. The doctrine $\Psi_\mathcal{C}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{POS}$ as in 2.2(d) is existential where left adjoint are given by composition.
2.14 Remark. As shown by Lawvere, in an elementary existential doctrine 
\( P: C^{op} \to \text{POS} \) every map \( f \) of the form \( Pf \) has a left adjoint \( \mathcal{J}_f \). For \( f: A \to B \) the maps \( \mathcal{J}_f: P(A) \to P(B) \) is

\[
\mathcal{J}_f(\alpha) = \mathcal{J}_{pr_2}[Pf \times \text{id}_B (\delta_B) \land P_{pr_1}(\alpha)] 
\]

see also [Pit02]. Moreover, these left adjoints satisfy the Frobenius Reciprocity
in the sense that for every \( \beta \) in \( P(B) \) it holds \( \mathcal{J}_f(\alpha) \land \beta = \mathcal{J}_f(\alpha \land Pf(\beta)) \).

But they not necessarily satisfy the Beck-Chevalley condition (see [MT23] for a
contraexample).

This allows to give a quick description of the 2-category \( \text{EED} \), which is the
2-full subcategory of \( \text{ED} \) on those elementary doctrines that are existential with
comprehensive diagonals and whose 1-morphisms are those \((F,b): P \to R\) such
that for every \( f: X \to A \) in \( C \) the following commute

\[
\begin{align*}
  P(X) & \xrightarrow{b_X} R(FX) \\
  P(A) & \xrightarrow{b_A} R(FA) \\
  \mathcal{J}_f & \downarrow \downarrow \mathcal{J}_{Ff}
\end{align*}
\]

The full subcategory of \( \text{EED} \) on those existential elementary doctrines with
comprehensive diagonals is \( \text{EExD} \).

A primary doctrine \( P: C^{op} \to \text{POS} \) is \textit{implicational} if, for every object
\( A \) in \( C \), every \( \alpha \) in \( P(A) \), the functor \( \alpha \land -: P(A) \to P(A) \) has a right adjoint
\( \alpha \Rightarrow -: P(A) \to P(A) \). A primary doctrine \( P: C^{op} \to \text{POS} \) is \textit{disjunctive}
if every \( P(A) \) has finite distributive joins and every \( Pf \) preserves them. A
primary doctrine \( P: C^{op} \to \text{POS} \) is \textit{universal} if, for \( A_1 \) and \( A_2 \) in \( C \), for a(ny)
projection \( pr_i: A_1 \times A_2 \to A_i \), \( i = 1, 2 \), the functor \( P_{pr_i}: P(A_i) \to P(A_1 \times A_2) \)
has a right adjoint \( V_{pr_i} \), and these satisfy the Beck-Chevalley condition:
for any pullback diagram

\[
\begin{align*}
  X' & \xrightarrow{pr'} A' \\
  f' & \downarrow \downarrow f \\
  X & \xrightarrow{pr} A
\end{align*}
\]

with pr a projection (hence also pr' a projection), for any \( \beta \) in \( P(X) \), the
canonical arrow \( Pf_{pr}(\beta) \leq V_{pr'} Pf_{pr}(\beta) \) in \( P(A') \) is iso.

2.15 Definition. A \textit{first-order doctrine} (f.o.d.) on \( C \) is an existential ele-
mentary doctrine \( P: C^{op} \to \text{POS} \) which is also implicational, disjunctive and
universal.

2.16 Examples. (a) Consider a theory \( \mathcal{T} \) over a first-order language \( \mathcal{L} \)
and the associated primary doctrine \( LT\): \( \eta^{op} \to \text{POS} \) as in 2.2(a). If \( \mathcal{L} \)
then the doctrine \( LT \) is first-order. In each fibre joins are given by disjunctions while
the cartesian closure is provided by the implication. Right adjoints are given by universal quantification.

(b) Let $\mathcal{H}$ be a frame. Then it is a complete Heyting algebra. Hence the doctrine $\mathcal{P}_\mathcal{H}: \mathbf{Set}^{\mathcal{H}} \to \mathbf{POS}$ is first-order where in each fibre the Heyting algebra operations are computed point-wise and right adjoint are given by arbitrary infima (see [Pit02] for details).

(c) Suppose $\mathcal{C}$ has finite limits. The doctrine $\text{Sub}_\mathcal{C}: \mathcal{C}^{\mathcal{C}} \to \mathbf{POS}$ is first-order if and only if $\mathcal{C}$ is an Heyting category (or a logos) in the sense of [FS90].

(d) Suppose $\mathcal{C}$ has finite limits. If $\mathcal{C}$ has finite coproducts and is weakly locally cartesian closed, then $\Psi_\mathcal{C}: \mathcal{C}^{\mathcal{C}} \to \mathbf{POS}$ as in (2.2)-(d) is first-order. We shall see later how to generalise the description to the case in which $\mathcal{C}$ is assumed to have weak pullbacks.

We will often deal with different doctrines on the same base and in several situations one of these is $\Psi_\mathcal{C}$, thus we find it convenient to adopt a specific notation to distinguish operations between doctrines and in particular operations in $\Psi_\mathcal{C}$. We will use the following.

2.17 Notation. We write $\Psi_\mathcal{C}(f)$ as $f^*$. The left adjoint to $f^*$ will be denoted by $\Sigma_f$. If $f^*$ has a right adjoint this will be denoted by $\Pi_f$. The equality predicated over $A$ will be denoted by $[\text{id}_A, \text{id}_A]$. Binary meets in $\Psi_\mathcal{C}(A)$ will be denoted by $[f] \times_A [g]$ while the top element over $A$ will be denoted by $[\text{id}_A]$. Joins will be denoted by $[f] +_A [g]$. The bottom element will be $0_A$. We will freely confuse a class with any of its representatives.

2.18 Definition. Let $P: \mathcal{C}^{\mathcal{C}} \to \mathbf{POS}$ a primary doctrine. An object $\Omega$ in $\mathcal{C}$ is a weak predicate classifier if there is an element $\in_1$ over $\Omega$ such that for every $\phi$ in $P(A)$ there is a (not necessarily unique) morphism $\chi_\phi: A \to \Omega$ satisfying $P\chi_\phi(\in_1) = \phi$.

$P$ has a strong predicate classifier, or simply a predicate classifier if it has a weak predicate classifier and arrows of the form $\chi_{\phi}$ are unique.

$P$ has weak power objects if for every $\mathcal{C}$-object $A$ there exists an object $\mathbb{P}A$ in $\mathcal{C}$ and an object $\in_A$ in $P(A \times \mathbb{P}A)$ such that for every $Y$ and relation $\phi$ in $P(A \times Y)$ there is a (not necessarily unique) morphism $\chi_\phi: Y \to \mathbb{P}A$ satisfying $P\chi_\phi(\in_A) = \phi$. $P$ has a strong power object if it has a weak power objects and arrows of the form $\chi_{\phi}$ are unique.

Observe the following relation between power objects and predicate classifiers. This is well known when $P$ is $\text{Sub}_\mathcal{C}$ for a finite limit category $\mathcal{C}$ and it can be proved analogously for a generic $P$.

2.19 Proposition. If $P$ as weak) power objects, then it has also a (weak) predicate classifier (it suffices to choose $\mathbb{P}1$ as $\Omega$). Moreover, if the base $\mathcal{C}$ is weakly cartesian closed, then $P$ has a (weak) predicate classifier if and only if it has (weak) power objects, where $\mathbb{P}A$ can be chosen as the weak exponential of $\Omega$ to the power of $A$. 

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2.20 Examples. (a) The doctrine \( LT: V^\text{op} \rightarrow \text{POS} \) built out of a theory \( \mathcal{T} \) over a first-order language \( L \) has no predicate classifiers.
(b) Let \( H \) be an inf-semilattice. The underlying set of \( H \) is denoted \( |H| \) and is a strong predicate classifier of the functor \( P_H \) where \( 1 \) is the identity \( \text{id}_{|H|} \).
(c) Suppose \( C \) has finite limits. The doctrine \( \text{Sub}_C : C^{\text{op}} \rightarrow \text{POS} \) as a predicate classifier if and only if it has a weak proof classifier in the sense of [Men03].
(d) Suppose \( C \) has finite products and weak pullbacks. If \( C \) has a weak predicate classifier if and only if it has a weak proof classifier in the sense of [Men03].

2.21 Definition. A primary doctrine \( P: C^{\text{op}} \rightarrow \text{POS} \) is a tripos if and only if \( P \) is a first-order doctrine with weak power objects. \( P \) is a strong tripos if it is a tripos with strong power objects.

2.22 Examples. (a) The doctrine \( LT: V^\text{op} \rightarrow \text{POS} \) built out of a theory \( \mathcal{T} \) over a first-order language \( L \) is not a tripos as it has no predicate classifiers.
(b) Let \( H \) be an inf-semilattice. The doctrine \( P_H \) as in 2.21 is a tripos if and only if \( H \) is a frame.
(c) Suppose \( C \) has finite limits. The doctrine \( \text{Sub}_C : C^{\text{op}} \rightarrow \text{POS} \) is a strong tripos if and only if \( C \) is an elementary topos.
(d) Suppose \( C \) has finite products and pullbacks. If \( C \) has a weak predicate classifier if and only if it has a weak proof classifier in the sense of [Men03].

A proof of the following proposition is in [Pas16].

2.23 Proposition. If the doctrine \( P \) is a tripos then \( P_x \) is a tripos, too.

Elementary doctrines are the cloven Eq-fibrations of [Jac99] and, as explained in loc.cit. and [MR16], there is a deductive calculus associated to those which is that of the \( \land=\)-fragment over type theory with just a unit type and a binary product type constructor. To fix notation, we will use the following.

2.24 Notation. Let \( P: C^{\text{op}} \rightarrow \text{POS} \) be elementary. Write
\[
a_1:A_1, \ldots, a_k:A_k | \phi_1, \ldots, \phi_n \vdash \psi
\]
in place of
\[
\phi_1 \land \ldots \land \phi_n \leq \psi \quad \text{in} \ P(A_1 \times \ldots \times A_k)
\]
and call such an expression sequent. Note that, in line with loc.cit., \( \delta_A \) in \( P(A \times A) \) will be written as \( a:A, a':A | a =_A a' \). Also write \( a:A | a \vdash \beta \) to abbreviate \( a:A | \alpha \vdash \beta \) and \( a:A | \beta \vdash \alpha \). Say that \( \alpha \) in \( P(A) \) is true over \( A \) if \( \top_A \leq \alpha \). An element of \( P(1) \) will be called sentence. An arrow \( r:1 \rightarrow A \) will be called constant (of type \( A \)) and for \( \alpha \) in \( P(A) \) we write \( \alpha(r) \) in place of \( P_r(\alpha) \).
If \( P \) is existential and \( a:A, x:X | \phi, \text{i.e.} \ \phi \text{ is in } P(A \times X) \), write \( a:A | \exists x:X \phi \) in place of \( \exists_{pr,x} \phi \) in \( P(A) \). Similarly when \( P \) is first-order, for \( \alpha, \beta \) in \( P(A) \) and \( \phi \) in \( P(A \times X) \) write \( a:A | \alpha \lor \beta \) and \( a:A | \alpha \Rightarrow \beta \) and \( a:A | \forall x:X \phi \) in place of \( \alpha \lor \beta \) and \( \alpha \rightarrow \beta \) and \( \forall_{pr,x} \phi \) in \( P(A) \). If \( P \) is a tripos write \( a:A, U:PA | a \in_A U \) in place of \( \in_A \) in \( P(A \times \mathbb{P}A) \).

From now on we feel free to employ this logical language in our proofs or definitions whenever we feel that readability is improved.
2.1 Comprehensions and strong monomorphisms

2.25 Definition. A primary doctrine \( P: \mathcal{C}^\text{op} \to \text{POS} \) is said to have **weak comprehensions** if for every \( A \) in \( \mathcal{C} \) and every \( \alpha \) in \( P(A) \) there is an arrow \( \{ \alpha \}: X \to A \) with \( \top_X = P_{\{\alpha\}}(\alpha) \) such that for every arrow \( f: Y \to A \) with \( \top_Y \leq P_f(\alpha) \) there is a (not necessarily unique) arrow \( k: Y \to X \) with \( \{ \alpha \} k = f \).

We shall say that comprehensions are **strong** if the mediating arrow \( k \) is the unique such arrow. We shall say that comprehensions are **full** if for every \( A \) and every \( \alpha, \beta \) over \( A \) it holds that \( \alpha \leq \beta \) if and only if \( \top_X = P_{\{\alpha\}}(\beta) \).

It is an easy check that a doctrine \( P \) with weak comprehensions has strong comprehension if and only if arrows of the form are monic.

2.26 Examples. (a) The doctrine \( \mathcal{L}T: \mathcal{V}^\text{op} \to \text{POS} \) built out of a theory \( \mathcal{T} \) over a first-order language \( \mathcal{L} \) has no comprehensions.

(b) When \( \mathcal{H} \) is an inf-semilattice, the doctrine \( P_{\mathcal{H}} \) has strong comprehensions. Given \( \alpha \) in \( P_{\mathcal{H}}(A) \) the arrow \( \{ \alpha \} \) is the inclusion of \( \{ a \in A \mid \alpha(a) = \top \} \) into \( A \). This comprehension is not full as one can see taking \( \alpha' : A \to \mathcal{H} \) that agrees with \( \alpha \) only on those elements \( a \) of \( A \) such that \( \alpha(a) = \top \). Then \( \alpha \) and \( \alpha' \) have the same comprehension arrow, even if they are not necessarily equivalent in the fibre.

(c) Suppose \( \mathcal{C} \) has finite limits. The doctrine \( \text{Sub}_C: \mathcal{C}^\text{op} \to \text{POS} \) as full strong comprehensions. The comprehension \( \alpha \) in \( \text{Sub}_C(A) \) is any representative of \( \alpha \).

(d) Suppose \( \mathcal{C} \) has finite products and weak pullbacks. The doctrine \( \Psi_C: \mathcal{C}^\text{op} \to \text{POS} \) has full weak comprehensions. The comprehension \( \alpha \) in \( \text{Sub}_C(A) \) is any representative of \( \alpha \) (that need not be monic).

An arrow of the form \( \{ \alpha \} \) will be often called comprehension arrow of \( \alpha \).

We recall from [MR13a] the following definition.

2.27 Definition. An elementary doctrine \( P: \mathcal{C}^\text{op} \to \text{POS} \) has **comprehensive diagonals** if and only if diagonals in \( \mathcal{C} \) are the strong comprehension arrows of the corresponding fibred equalities, i.e. \( \langle \text{id}_A, \text{id}_A \rangle = \{ \delta_A \} \).

Comprehensive diagonals were introduced originally in [MR13b] with the name of “comprehensive equalizers” since the following holds.

2.28 Proposition. Let \( P \) be an elementary doctrine \( P: \mathcal{C}^\text{op} \to \text{POS} \). The following are equivalent.

(i) \( P \) has comprehensive diagonals.

(ii) \( P \) is extensional as in def. 2.6, i.e. for any \( f, g: X \to A, \top_X = P_{\langle f, g \rangle}(\delta_A) \) if and only \( f = g \).

**Proof.** See [MR13b] Proposition 4.6.

2.29 Proposition. If \( P: \mathcal{C}^\text{op} \to \text{POS} \) is elementary with comprehensive diagonals and strong (weak) comprehensions, then \( \mathcal{C} \) has (weak) finite limits.
Proof. The equalizer of \( f, g : X \to A \) is \( \{ P_{(f,g)}(\delta_A) \} \) which is only weak if \( P \) has only weak comprehensions. \( \square \)

### 2.30 Proposition

Suppose \( P \) is elementary existential with weak comprehensions. Weak comprehensions are full if and only if for every \( \{ \alpha \} : X \to A \) it holds \( \mathcal{A}_{\{\alpha\}}(\top_X) = \alpha \).

**Proof.** Let \( \mathcal{A}_{\{\alpha\}} \vdash P_{\{\alpha\}} \), from \( \top_X = P_{\{\alpha\}}(\alpha) \) it follows \( \mathcal{A}_{\{\alpha\}}(\top_X) \leq \alpha \). Instead \( \alpha \leq \mathcal{A}_{\{\alpha\}}(\top_X) \) follows by fullness from the adjunction unit \( \top_X \leq P_{\{\alpha\}}(\mathcal{A}_{\{\alpha\}}(\top_X)) \).

Conversely, if \( \mathcal{A}_{\{\alpha\}}(\top_X) = \alpha \) from \( \top_X = P_{\{\alpha\}}(\beta) \) it follows \( \alpha = \mathcal{A}_{\{\alpha\}}(\top_X) = \mathcal{A}_{\{\alpha\}} P_{\{\alpha\}}(\beta) \leq \beta \) by the adjunction counit of \( \mathcal{A}_{\{\alpha\}} \vdash P_{\{\alpha\}} \). \( \square \)

### 2.31 Proposition

If \( P \) has full weak comprehensions and comprehensive diagonals, then for every weak pullback \( f g = h k \) one has \( P_f \mathcal{A}_h = \mathcal{A}_g P_k \).

**Proof.** This is theorem 2.19 in [MPRI17]. \( \square \)

Suppose that \( P : C^{\text{op}} \to \text{POS} \) has weak comprehensions. For every \( f : Y \to B \) in \( C \) and every \( \alpha \) in \( P(B) \) there is a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{m} & A \\
\{P_f(\alpha)\} & \downarrow & \{\alpha\} \\
Y & \xrightarrow{f} & B
\end{array}
\]

where \( m \) is the mediating arrow coming from the universal property of \( \{\alpha\} \) as \( P_f P_{\{\alpha\}} (\alpha) = P_{\{P_f(\alpha)\}} (P_f(\alpha)) = \top_X \).

### 2.32 Proposition

If \( k : Z \to A \) and \( h : Z \to Y \) are such that \( \{\alpha\} k = f h \), then \( h \) factors through \( \{P_f(\alpha)\} \). If \( \{\alpha\} \) is a strong comprehension, the square above is a weak pullback. If both \( \{\alpha\} \) and \( \{P_f(\alpha)\} \) are strong comprehensions the square is a pullback.

**Proof.** Consider \( k \) and \( h \) such that \( \{\alpha\} k = f h \). Then \( P_h (P_f(\alpha)) = P_\alpha (P_{\{\alpha\}}(\alpha)) = \top_Z \). Weak universality of \( \{P_f(\alpha)\} \) guarantees the existence of \( u : Z \to X \) with \( \{P_f(\alpha)\} u = h \). If \( \{\alpha\} \) is strong, then it is also monic. Hence \( mu = k \) if and only if \( \{\alpha\} mu = \{\alpha\} k \), which is true as \( \{\alpha\} mu = f \{P_f(\alpha)\} u = fh = \{\alpha\} k \).

Finally if \( \{P_f(\alpha)\} \) is monic the mediating arrow \( u \) is necessarily unique. \( \square \)

If \( P \) is elementary, we say that an arrow \( f : X \to A \) of \( C \) is **\( P \)-injective** if \( P_{f \times f}(\delta_A) = \delta_X \). While if \( P \) is also existential we say that \( f \) is **\( P \)-surjective** if \( \mathcal{A}_{\{f\}}(\top_X) = \top_A \).

It is immediate to show that if \( P \) has comprehensive diagonals, if an arrow is \( P \)-injective, then it is also monic and if an arrow is \( P \)-surjective, then it is also epic. Observe the following property about monics.

### 2.33 Proposition

Suppose \( P \) is an elementary doctrine with full weak comprehension and comprehensive diagonals on \( C \). An arrow \( m : X \to A \) is **\( P \)-injective** if and only if it is monic.
Proof. If \( m: X \to A \) is \( P \)-injective then it is clearly monic thanks to comprehensive diagonals. Conversely, if \( m \) is monic the square \( \langle \text{id}_A, \text{id}_A \rangle m = (m \times m) \langle \text{id}_X, \text{id}_X \rangle \) is a pullback. Since \( P \) has comprehensive diagonals, and the equalizer of \( m \) with itself is \( \{P_{(m,m)}(\delta_A)\} \) by prop. \( 2.24 \), it follows that \( \langle \text{id}_X, \text{id}_X \rangle = \{P_{(m,m)}(\delta_A)\} \) and since \( \langle \text{id}_X, \text{id}_X \rangle = \{\delta_X\} \) by fullness of comprehensions we conclude \( P_{(m,m)}(\delta_A) = \delta_X \), i.e. \( m \) is \( P \)-injective. \( \square \)

Let \( P: \mathcal{C}^{\text{op}} \to \text{POS} \) be an elementary existential doctrine. Using the language in \( 2.24 \) a \( F \) in \( P(A \times B) \) is

**total (or entire)** if \( \exists_{b:B} F(a,b) \) is true over \( A \)

**single-valued (or functional)** if \( a:A, b:B, b':B : F(a,b) \land F(a,b') = B b' \)

2.34 Definition. We say that \( P \) satisfies the **rule of unique choice** (RUC) if for every total and single-valued \( F \) in \( P(A \times B) \) there is \( f: A \to B \) such that \( a:A, b:B : F(a,b) \lor f(a) = B b \).

2.35 Definition. We say that \( P \) satisfies the **rule of choice** (RC) if for every total \( F \) in \( P(A \times B) \) there is \( f: A \to B \) such that \( a:A \mid \exists_{b:B} F(a,b) \lor F(a,f(a)) \).

2.36 Proposition. Suppose \( P \) is an existential \( m \)-variational doctrine on \( C \). The following are equivalent.

(i) Every arrow which is both \( P \)-injective and \( P \)-surjective is an isomorphism the subobject doctrine Sub\(_C\).

(ii) \( P \) satisfies (RUC).

(iii) \( P \) is equivalent to the subobject doctrine Sub\(_C\).

Proof. To prove the equivalence between (1) and (2) we use the notation in \( 2.24 \). If (1) holds, take \( F \) in \( P(A \times B) \) and consider \( \{F\}: X \to A \times B \). If \( F \) is total and single-valued. Then \( \text{pr}_1 \{F\} : X \to A \) is \( P \)-injective and \( P \)-surjective so it has an inverse \( h \). The arrow \( \text{pr}_1 h: A \to B \) is the desired arrow. Conversely, suppose (RUC) holds and take a \( P \)-injective and \( P \)-surjective arrow \( f: A \to B \). The formula \( b:B, a:A : b = B f(a) \) is entire and single-valued. By (RUC) there is \( k:B \to A \) which is the inverse of \( f \). The equivalence between (2) and (3) follows from point (i) of theorem 2.7 in \( \text{MPRI19} \). \( \square \)

Recall also from theorem 5.9 in \( \text{MPRI17} \) the following

2.37 Proposition. Suppose \( P \) is an existential variational doctrine on \( C \). \( P \) satisfies (RC) if and only if \( P \) is \( \Psi_C \).

Furthermore, observe the following facts for elementary existential doctrines with weak full comprehension which will be instrumental in the next:

2.38 Proposition. Let \( P: \mathcal{C}^{\text{op}} \to \text{POS} \) be an elementary existential doctrine which admits weak full comprehension.
(i) If $P$ has comprehensive diagonals, and the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{g} & A \\
k & \downarrow & \downarrow \\
B & \xrightarrow{h} & C
\end{array}
\]

is a weak pullback, then $P|\exists_h = \exists_g P_k$.

(ii) If every reindexing $P_f$ has a right adjoint, then $P$ is implicative.

Proof. (i) See [MPR17, Theorem 2.19].

(ii) See [MR13b, Lemma 4.9]. □

2.39 Proposition. Suppose $P: C^{op} \rightarrow \text{POS}$ is elementary existential admitting weak full comprehension. If the variational doctrine $\Psi_C$ is universal and implicative, then $P$ is a hyperdoctrine.

Proof. Proposition 2.29 (ii) ensures that $C$ has weak pullbacks. Hence, from [MPR19, Proposition 2.3], as well as [MPR17, Remark 2.10], it follows immediately that the universal quantifier of $\alpha$ in $P(A)$ along $f: A \rightarrow B$ is $\exists_{\Pi_f(\{\alpha\})} X$ where $\Pi_f(\{\alpha\}): X \rightarrow A$ is the universal quantifier $\{\alpha\}$ along $f$ in $\Psi_C(B)$. The fact that $P$ is implicative follows from Proposition 2.38 (ii). □

Primary doctrines with full strong comprehensions form the category $\text{CED}$, the 2-full sub category of the category of elementary doctrines $\text{ED}$ whose arrows are those arrow $(F, b): P \rightarrow R$ in $\text{ED}$ that preserve comprehensions, i.e. for every $A$ and $\alpha$ in $P(A)$ the arrow $F\{\alpha\}$ is isomorphic to $\{bA\alpha\}$.

We recall from [MR13b] [MR13a] that there is a left biadjoint to the forgetful 2-functor from $\text{ED}$ to $\text{CED}$, which associates to an elementary doctrine $P: C^{op} \rightarrow \text{POS}$ the elementary doctrine with full strong comprehensions the doctrine $P_c: C_c^{op} \rightarrow \text{POS}$ whose description is as follows.

2.40 Definition. Let $P$ be an elementary doctrines on $C$. The doctrine $P_c: C_c^{op} \rightarrow \text{POS}$ obtained by freely adding full comprehensions to $P$ has a base $C_c$, whose objects are pairs $(A, \alpha)$ where $\alpha$ is in $P(A)$ and an arrow $f: (A, \alpha) \rightarrow (B, \beta)$ is an arrow $f: A \rightarrow B$ in $C$ such that $\alpha \leq P_f(\beta)$. $P_c$ maps each $(A, \alpha)$ to $P_c(A, \alpha) = \{\phi \in P(A) \mid \phi \leq \alpha\}$ and each $f: (A, \alpha) \rightarrow (B, \beta)$ to the function $P_c(f): P_c(B, \beta) \rightarrow P_c(A, \alpha)$ determined by the assignment $\psi \mapsto P_f(\psi) \land \alpha$. For $\phi$ in $P_c(A, \alpha)$ it is $\{\phi\}_A = \text{id}_A: (A, \phi) \rightarrow (A, \alpha)$.

Doctrines with comprehensive diagonals and full weak comprehensions form the class of doctrines which we will mainly concerned with throughout this paper. In [MPR17], taking the terminology from [Gra00], we called them variational. While we called $m$-variational doctrines those variational doctrines in which comprehension is strong. We aim at giving a characterisation of variational doctrines in 3.38. To do this we first need some instrumental definitions and propositions.
Existential variational doctrines will be the main mathematical tool that we will employ throughout the whole paper. They form the category \( \text{EV} \) which is the full subcategory of \( \text{EEExD} \) on those doctrines that are also existential. We denote by \( \text{EmV} \) the subcategory of \( \text{EV} \) on \( m \)-variational doctrines and on those morphisms of doctrine that preserves strong comprehension. The adjoint situation described in 2.40 and in 2.9 compose to give the following

\[
\text{EED} \quad \downarrow \quad \text{EmV}
\]

2.1.1 Factorization systems from doctrines

Suppose \( C \) is a category with weak pullbacks. A \textit{right weak factorization system} is a pair \((E, R)\) of classes of arrows of \( C \) such that

(i) for every \( f \) in \( C \) there is \( e \) in \( E \) and \( r \) in \( R \) with \( f = re \)

(ii) for every commutative square \( rf = ge \) with \( e \) in \( E \) and \( r \) in \( R \) there is \( k \) with \( rk = g \).

A right weak factorization system is stable if any weak pullback of an arrow of \( E \) is in \( E \). A proper factorisation system \((E, R)\) on a category \( C \) with pullbacks gives rise to an existential \( m \)-variational doctrine \( R_C: C^{\text{op}} \rightarrow \text{POS} \) where \( R_C \) is the sub-infsemilattice of \( \Psi_C(A) \) on those arrows represented by arrows in \( R \), which are monic and hence also in \( \text{Sub}_C(A) \) [HJ03]. Analogously, to what shown in loc.cit. and observed in [MR16] we can show that category of existential elementary doctrines with full strong comprehensions is equivalent to the category of proper stable factorisation systems. More precisely,

2.41 Proposition. In every \( m \)-variational doctrine comprehension arrows and \( P \)-surjective arrows form a factorization system. Moreover, the category of existential \( m \)-variational doctrines is equivalent to the category of proper stable factorisation systems where diagonals are in the right class.

2.2 The elementary quotient completion

We now recall the definition of \( P \)-\textit{equivalence relation} for any elementary doctrine \( P: C^{\text{op}} \rightarrow \text{POS} \) and the related notion of \( \textit{quotients} \) from [MR13a].

2.42 Definition. A \( P \)-equivalence relation \( \rho \) over an object \( A \) of \( C \) is an element in \( P(A \times A) \) such that

(i) \( \delta_A \leq \rho \)

(ii) \( \rho = P_{(\text{pr}_2, \text{pr}_1)}(\rho) \)

(iii) \( P_{(\text{pr}_1, \text{pr}_2)}(\rho) \wedge P_{(\text{pr}_2, \text{pr}_1)}(\rho) \leq P_{(\text{pr}_1, \text{pr}_2)}(\rho) \)
where in (ii) \( \text{pr}_1, \text{pr}_2: A \times A \to A \) are the first and the second projection, while in (iii) \( \text{pr}_i \), with \( i = 1, 2, 3 \), are projections from \( A \times A \times A \) to each of the factors.

When no confusion arises, we shall refer at \( P \)-equivalence relations simply as equivalence relations, without specifying the doctrine \( P \). Note that in every elementary doctrine, fibred equalities are equivalence relations.

2.43 Examples. (a) Recall from [22] (a) the syntactic doctrine \( LT: \mathcal{L}^{op} \to \text{POS} \) built out of a theory \( \mathcal{T} \) over a first-order language \( \mathcal{L} \). A \( LT \)-equivalence relation over \( x \) is a formula \( \phi \) of \( \mathcal{L} \) such that \( \mathcal{T} \vdash \forall x \phi(x, x) \) and \( \mathcal{T} \vdash \forall xy (\phi(x, y) \to \phi(y, x)) \) and \( \mathcal{T} \vdash \forall xyz (\phi(x, y) \& \phi(y, z) \to \phi(x, z)) \).

(b) When \( \mathcal{H} \) is an inf-semilattice, a \( \mathcal{P}_\mathcal{H} \)-equivalence relation over a set \( A \) is an \( \mathcal{H} \)-valued ultra-pseudodistance on \( A \) after inverted the order, i.e. a function \( \rho: A \times A \to \mathcal{H} \) such that for all \( a, a', a'' \in A \) it is \( \rho(a, a) = \top \) and \( \rho(a, a') = \rho(a', a'') \) and \( \rho(a, a') \land \rho(a', a'') \leq \rho(a, a'') \).

(c) Suppose \( \mathcal{C} \) has finite limits, then \( \rho \) is a Sub\( \mathcal{C} \)-equivalence relation over \( A \) if and only if \( \rho \) is an equivalence relation of \( \mathcal{C} \) in the usual categorical sense (see for example [MM92]).

(d) Suppose \( \mathcal{C} \) has finite products and weak pullbacks, then \( \rho \) is a \( \Psi \mathcal{C} \)-equivalence relation over \( A \) if and only if it is a pseudo-equivalence relation of \( \mathcal{C} \) in the sense of [Car93].

2.44 Definition. An elementary doctrine \( P: \mathcal{C}^{op} \to \text{POS} \) is said to have quotients if for every \( A \) in \( \mathcal{C} \) and every equivalence relation \( \rho \) over \( A \) there exists a morphism \( q: A \to A/\rho \) such that \( \rho \leq P_{q \times q}(\delta_{A/\rho}) \) and for every morphism \( f: A \to Y \) such that \( \rho \leq P_{f \times f}(\delta_Y) \) there exists a unique \( h: A/\rho \to Y \) with \( hq = f \). Maps of the form \( q: A \to A/\rho \) will be called quotient arrow of \( \rho \). The quotient \( q: A \to A/\rho \) is effective if \( \rho = P_{q \times q}(\delta_{A/\rho}) \).

2.45 Definition. An elementary doctrine \( P \) is said to have stable quotients if for every pullback \( qp = fh \), if \( q \) is a quotient arrow of \( \rho \), then \( h \) is a quotient arrow of \( P_{p \times p}(\rho) \).

2.46 Definition. Given an elementary doctrine \( P: \mathcal{C}^{op} \to \text{POS} \) and a \( P \)-equivalence relation \( \rho \) over \( A \) in \( \mathcal{C} \), the inf-semilattice of descent data \( \mathcal{D}_{\rho} \) is the sub-inf-semilattice of \( P(A) \) on those \( \alpha \) such that

\[
P_{\text{pr}_1}(\alpha) \land \rho \leq P_{\text{pr}_2}(\alpha),
\]

where \( \text{pr}_1, \text{pr}_2: A \times A \to A \) are the projections.

For \( f: A \to B \) in \( \mathcal{C} \) the map \( P_f: P(B) \to P(A) \) takes values in \( \mathcal{D}_{\rho} \). We shall say that \( f \) is of effective descent if \( P_f: P(B) \to \mathcal{D}_{\rho} \) is an isomorphism. In particular this means that the functor \( P_f \) is of effective descent type as defined in [BW84].

2.47 Definition. An elementary doctrine \( P \) is said to have descent effective quotients if \( P \) has stable effective quotients and the quotient arrows are of effective descent.
2.48 Examples. (a) In general doctrines of the form $LT: \mathcal{V}^{op} \rightarrow \mathsf{POS}$ do not have quotients.

(b) For $\mathcal{H}$ an inf-semilattice, the doctrine $\mathcal{P}_{\mathcal{H}}$ has quotient. Let $\rho: A \times A \rightarrow \mathcal{H}$ be a $\mathcal{H}$-valued ultra-pseudodistance on $A$ as in 2.43(b) and define on $A$ the equivalence relation $a \sim a'$ generated by $\rho(a, a') = \top$. Then canonical surjection $q: A \rightarrow A/\sim$ is a quotient arrow. These quotients are not effective (unless $\rho$ is the boolean equality on $A$) and hence not of effective descent.

(c) If $C$ has finite limits, then $\text{Sub}_C$ has effective quotient if and only if $C$ is exact. In this case quotients are of effective descent (this follows also from the more general situation of 2.46 in the following).

(d) In general doctrines of the form $\Psi_C$ do not have quotients (unless $\Psi_C$ is equivalent to $\text{Sub}_C$ for $C$ exact).

2.49 Lemma. Let $P: C^{op} \rightarrow \mathsf{POS}$ an elementary and existential doctrine and $q: A \rightarrow A/\rho$ an effective descent arrow. Then, the inverse of $P_q: P(B) \rightarrow \mathcal{D}es_{\rho}$ is the restriction of $\mathcal{I}_q$ to $\mathcal{D}es_{\rho}$ and, hence, any $\phi$ of $P(A)$ is a descent data if and only if $\phi = P_q \mathcal{I}_q \phi$.

Proof. Let $P_q: P(A/\rho) \rightarrow \mathcal{D}es_{\rho}$ be an isomorphism with inverse the restriction of $\mathcal{I}_q$ to $\mathcal{D}es_{\rho}$ by lemma 2.40. Hence, for all descent data $\phi$, we have that $\phi = P_q \mathcal{I}_q \phi$ from which $\top_{A/\rho} = \top_A$. Conversely, if $q$ is $P$-surjective, observe, that for every $\psi$ in $P(A/\rho)$, by Frobenius condition $\mathcal{I}_q P_q \psi = \psi \land \mathcal{I}_q \top_B = \psi$ and hence $P_q$ is an isomorphism towards $\mathcal{D}es_{\rho}$ with inverse $\mathcal{I}_q$ since by lemma 2.40 we also know that every descent data satisfies $\phi = P_q \mathcal{I}_q \phi$.

2.50 Remark. For elementary and existential doctrines $P: C^{op} \rightarrow \mathsf{POS}$ we could have simply define an effective quotient arrow $q: A \rightarrow A/\rho$ of effective descent if and only if $P_q$ restricts to an isomorphisms toward its image in $P(A)$. Then, trivially an object $\phi$ is in the image of $P_q$ if and only if $\phi = P_q \mathcal{I}_q \phi$, which holds if and only if $\phi$ is a descent data in the sense of definition 2.48 by Beck-Chevalley conditions applied to the description of $D_q$ in remark 2.13 p.381 in [MR13b].

2.51 Lemma. If $P: C^{op} \rightarrow \mathsf{POS}$ is elementary and existential, an effective quotient arrow $q: A \rightarrow A/\rho$ is of effective descent if and only if the arrow $q$ is $P$-surjective, i.e. $\mathcal{I}_q \top_A = \top_{A/\rho}$.

Proof. If $q$ is of effective descent, then $P_q: P(A/\rho) \rightarrow \mathcal{D}es_{\rho}$ is an isomorphism with inverse the restriction of $\mathcal{I}_q$ to $\mathcal{D}es_{\rho}$ by lemma 2.40. Hence, for all descent data $\phi$, we have that $\phi = P_q \mathcal{I}_q \phi$ from which $\top_{A/\rho} = \top_A$. Conversely, if $q$ is $P$-surjective, observe, that for every $\psi$ in $P(A/\rho)$, by Frobenius condition $\mathcal{I}_q P_q \psi = \psi \land \mathcal{I}_q \top_B = \psi$ and hence $P_q$ is an isomorphism towards $\mathcal{D}es_{\rho}$ with inverse $\mathcal{I}_q$ since by lemma 2.40 we also know that every descent data satisfies $\phi = P_q \mathcal{I}_q \phi$.

2.52 Proposition. The quotient arrows of an existential m-variational doctrine $P: C^{op} \rightarrow \mathsf{POS}$ with stable effective quotients are of effective descent.

Proof. By 2.41 each quotient arrow $q: A \rightarrow A/\rho$ can be factored as $q = \{\mathcal{I}_q(\top_A)\} g$ where $q: A \rightarrow C$ is $P$-surjective. Since $\{\mathcal{I}_q(\top_A)\}$ is monic, it is also $P$-injective by 2.39 whence $\rho = P_q x q (\delta_A) = P_q x (P_q x (\mathcal{I}_q(\top_A)) (\delta_A)) = P_g x (\delta_C)$. The universal property of quotients implies the existence of an arrow $s: A/\rho \rightarrow C$ with $g = sq$ and hence $q = \{\mathcal{I}_q(\top_A)\} sq$, so $\{\mathcal{I}_q(\top_A)\} s = \text{id}_B$ as $q$ is epic. Since $\{\mathcal{I}_q(\top_A)\}$ is a monomorphism with a section it is an isomorphism. Therefore $q$ is isomorphic to $g$ and hence $P$-surjective. An application of Lemma 2.51 concludes the proof.
We recall from [MR13b] the construction called **elementary quotient completion** that freely adds descent effective quotients to any elementary doctrine.

### 2.53 Definition.
Given an elementary doctrine \( P : \mathcal{C}^{\text{op}} \to \text{POS} \) we call \( Q_P \) the category whose objects are pairs \((A, \rho)\) in which \( A \) is in \( \mathcal{C} \) and \( \rho \) is an equivalence relation over \( A \). An arrow \([f] : (A, \rho) \to (B, \sigma)\) is an equivalence class of arrows \( f : A \to B \) in \( \mathcal{C} \) such that \( \rho \leq P_f \times f(\sigma) \), with respect to the equivalence \( f \sim g \) if and only if
\[
\top_A = P_{(f,g)}(\sigma)
\]
The category \( Q_P \) has finite products: if \((A, \rho)\) and \((B, \sigma)\) are objects of \( Q_P \), their product is
\[
(A, \rho) (A \times B, \rho \boxtimes \sigma) \xrightarrow{\text{[pr}_1\text{]}} (B, \sigma)
\]
where \( \rho \boxtimes \sigma = P_{(\text{pr}_1, \text{pr}_2)}(\rho) \& P_{(\text{pr}_3, \text{pr}_4)}(\sigma) \). The elementary quotient completion of \( P \) is the doctrine \( \hat{P} : Q_P^{\text{op}} \to \text{POS} \) where
\[
\hat{P}(A, \rho) = \text{Des}_{\rho}, \quad \hat{P}_{[f]} = P_f
\]
It is proved in [MR13b] that the assignment on arrows does not depend on the choice of representatives. The doctrine \( \hat{P} \) is elementary with \( \delta_{(A, \rho)} = \rho \). It is immediate to see that \( \hat{P} \) as stable descent effective quotient: if \( \sigma \) is an equivalence relation over \((A, \rho)\), then its quotient arrow is \([\text{id}_A] : (A, \rho) \to (A, \sigma)\).
Moreover these quotients are stable.

### 2.54 Examples.
(a) The elementary quotient completion of doctrines of the form \( \mathcal{L}T : \mathcal{V}^{\text{op}} \to \text{POS} \) is connected to elimination of imaginaries as analysed in [EPR20].
(b) Assuming choice, in the sense that epimorphism in \( \text{Set} \) split, the base of \( \mathcal{P} : \text{Set}_P^{\text{op}} \to \text{POS} \) is equivalent to \( \text{UM}_H \), the category of \( H \)-valued ultrametric spaces. Indeed the functor that maps \( f : (A, \rho) \to (B, \sigma) \) in \( \text{Set}_P^{\text{op}} \) to \([f] : (A, \rho) \to (B, \sigma)\) in \( \text{Set}_P \) is full and faithful as Leibniz’s principle of identity of indiscernibles holds. For essential surjectivity, take \((A, \rho)\) in \( \text{Set}_P \) and consider the quotient \((A, \rho) \to (A/\sim, \delta_{A/\sim})\) where \( \sim \) is the equivalence relation described in 2.48(b). Any of its section represent an inverse in \( \text{Set}_P^{\text{op}} \).
(d) One the motivating examples for the study of the elementary quotient completion is given by doctrines of the form \( \Psi_C \) where \( \mathcal{C} \) has finite products and weak pullbacks. As proved in [MR13b] the doctrine \( \Psi_C \) is \( \text{Sub}_{\text{ex/lex}} \), i.e. the subobject doctrine of the exact completion \( C_{\text{ex/lex}} \) of \( \mathcal{C} \).

### 2.55 Remark.
It is quite evident that the elementary structure plays no role in the construction of \( \hat{P} \). We refer the reader to [Pas15][EPR20] for a detailed analysis of the situation. In particular, there is shown also that the elementary structure is necessary to embed \( \mathcal{C} \) into \( Q_P \).

The embedding is given by the functor \( J : \mathcal{C} \to Q_P \) that assigns to each \( f : X \to Y \) the arrow \([f] : (X, \delta_X) \to (Y, \delta_Y)\). This functor preserves binary
products, and it is full; it is faithful exactly when \( P \) has comprehensive diagonals (see [MR13b]). In this case it is immediate to check that \( P \) is the change of base of \( \hat{P} \) along \( \nabla_P \).

Denote by \( \text{QEExD} \) the subcategory of \( \text{EExD} \) on those doctrines with effective descent quotients and on those arrows that preserves quotients, i.e. on those \((F,b) \) from \( P: \mathcal{C}^{\text{op}} \longrightarrow \text{POS} \) to \( R: \mathcal{D}^{\text{op}} \longrightarrow \text{POS} \) such that the action of \( F \) on \( q: A \longrightarrow A/\rho \) is \( Fq: FA \longrightarrow FA/(b_{A\times A\rho}) \).

The following theorem is proved in [MR13b, MR13a].

**2.56 Proposition.** There is a left biadjoint to the inclusion of \( \text{QEExD} \) into \( \text{EExD} \) which associates the doctrine \( \hat{P}: \mathcal{Q}_P^{\text{op}} \longrightarrow \text{POS} \) to an elementary doctrine \( P \).

**2.57 Proposition.** Let \( P: \mathcal{C}^{\text{op}} \longrightarrow \text{POS} \) an elementary existential doctrine.

(i) \( P \) is implicational, respectively universal, if and only if \( \hat{P} \) is so.

(ii) \( P \) is existential if and only if \( \hat{P} \) is existential.

**Proof.** The sufficient condition follows from [MR13b, Proposition 6.7]. The necessary condition is immediate since \( \hat{P} \) restricts to \( P \) on objects \((A,\delta_A)\) where \( \delta_A \) is the equality predicate relative to \( P \). \( \square \)

**2.58 Proposition.** If \( P: \mathcal{C}^{\text{op}} \longrightarrow \text{POS} \) is elementary with comprehensive diagonals and weak comprehensions, then \( Q_P \) has finite limits.

**Proof.** If \( P \) has weak comprehensions, then \( \hat{P} \) has strong comprehension. The claim follows by 2.29. \( \square \)

We also recall from [MR16]

**2.59 Proposition.** An elementary and existential doctrine \( P \) satisfies (RC) if and only if its completion \( \hat{P} \) satisfies (RUC).

**2.60 Remark.** Let \( \text{EExD}^{\text{RUC}} \) and \( \text{EExD}^{\text{RC}} \) be the full subcategories of \( \text{EExD} \) on doctrines satisfying (RUC) and (RC) respectively and similarly \( \text{QExD}^{\text{RUC}} \) and \( \text{QExD}^{\text{RC}} \) are the full subcategories of \( \text{QExD} \) on doctrines satisfying (RUC) and (RC) respectively. These categories fit in the following diagram of inclusions.

\[
\begin{array}{ccc}
\text{EEExD}^{\text{RC}} & \longrightarrow & \text{EEExD} \\
\text{QEEExD}^{\text{RUC}} & \downarrow & \text{QEEExD} \\
\end{array}
\]

The vertical left adjoint in diagram (2) are given by the elementary quotient completion. The biadjoint to the inclusion of \( \text{QEEExD}^{\text{RUC}} \) into \( \text{QEEExD} \) is the functor that maps \( P: \mathcal{C}^{\text{op}} \longrightarrow \text{POS} \) to \( P_\mathcal{C}: \mathcal{E}^{\mathcal{C}^{\text{op}}} \longrightarrow \text{POS} \) where \( \mathcal{C}_P \) has the
objects of \( C \) and an arrow \( F: A \to B \) is a total and single-valued relation in \( P(A \times B) \). The functor \( P_F \) maps \( A \) to \( P_F(A) = P(A) \) and given \( \beta \) in \( P(B) \) it is \( P_F(\beta) \) is the formula \( a: A \mid \exists b. [F(a, b) \land \beta(b)] \) over \( A \) (see [Pas16]).

It is worth noting that if we start with an elementary doctrine \( P \) without comprehensive diagonals and just with weak full comprehensions we can get anyway a \( m \)-variational doctrine closed under effective quotients. Moreover, adding comprehensive diagonals to \( P \) before completing it with quotients does produce the same doctrine as that obtained by completing \( P \) itself.

2.6.1 Theorem. Let \( P \) an elementary doctrine with weak full comprehensions. Then \( \hat{P} \) is a \( m \)-variational doctrine with stable effective quotients. Moreover, the doctrine \( \hat{P}_x \) is equivalent to \( \hat{P} \).

3 Topology on a doctrine

Consider the following two diagrams in \( \text{PD} \).

\[
\begin{array}{ccc}
C^{\text{op}} & \xrightarrow{P} & \text{POS} \\
\text{Id}_C & \xrightarrow{j} & \text{POS} \\
C^{\text{op}} & \xrightarrow{P} & \text{POS} \\
\end{array}
\quad
\begin{array}{ccc}
C^{\text{op}} & \xrightarrow{P} & \text{POS} \\
\text{id}_C^{\text{op}} & \xrightarrow{l} & \text{id}^{\text{op}} \\
C^{\text{op}} & \xrightarrow{R} & \text{POS} \\
\end{array}
\]

where the adjointness symbols in the diagram on the right means that for every \( A \) in \( C \) and every \( \alpha \) in \( P(A) \) and every \( \beta \) in \( R(A) \) the inequalities \( l\alpha \leq \alpha \) and \( \beta \leq r\beta \) both hold.

3.1 Definition. Given two primary doctrines \( P, R \) We say that \( P \) and and \( R \) form an adjoint-retraction pair and we write \( P \ll R \) if they fit in a diagram as the one on the right and moreover \( lr = \text{id}_P \).

3.2 Definition. Let \( P \) a primary doctrine. An endomorphism of primary doctrine of the form \( (\text{id}_C, j): P \to P \) is called a topology on \( P \) if \( j \) is extensive and idempotent, i.e. for every \( A \) in \( C \) and every \( \alpha \) in \( P(A) \) it holds \( \alpha \leq j_A \alpha \) and \( j_A \alpha = j_A \alpha \). An element \( \alpha \) in \( P(A) \) is \( j \)-closed if \( \alpha = j_A(\alpha) \).

Every topology \( j \) on \( P \) determines a doctrine \( P_j: C^{\text{op}} \to \text{POS} \) as follows:

3.3 Definition. If \( j \) is a topology on the primary doctrine \( P \), we call doctrine of \( j \)-closed element of \( P \) the doctrine \( P_j: C^{\text{op}} \to \text{POS} \) where \( P_j(A) \) is the sub-insemilattice of \( P(A) \) on those \( \alpha \) such that \( j_A \alpha = \alpha \).

The following is proved in [Pas16]; remarks in [MPR19] also useful.

3.4 Proposition. If \( P \ll R \) then there is \( L: Q_R \to Q_P \) and \( R: Q_P \to Q_R \) with \( L \) left adjoint to \( R \) and \( R \) is full and faithful.

Proof. \( L \) maps \( [f]:(A, \rho) \to (B, \sigma) \) to \( [f]:(A, l_{A \times A}(\rho)) \to (B, l_{B \times B}(\sigma)) \). The functor \( R \) is built analogously. \( \square \)
3.5 Proposition. Suppose \( P: \mathcal{C}^{\text{op}} \to \text{POS} \) and \( R: \mathcal{C}^{\text{op}} \to \text{POS} \) are functors based on \( \mathcal{C} \). Then \( P \triangleleft R \) if and only if \( P \) is isomorphic to \( R_j \) for some topology \( j \) on \( R \).

Proof. Suppose \( P \triangleleft R \) as in the diagram above. The composition \( rl \) is clearly extensive and also idempotent as \( lr = \text{id}_P \). The morphism \((\text{Id}_C, r): P \to R \) is then the inverse of \((\text{id}_C, l): R \to P \), whence \( j = rl \) is the desired topology on \( R \). Conversely let \( j \) be a topology on \( R \). There is an morphism \((\text{Id}_C, \iota): R \to R \) where \( \iota \) is a family of inclusions and a morphism \((\text{Id}_C, j): R \to R_j \). It is immediate to see that \( R_j \triangleleft R \). \( \square \)

3.6 Proposition. Suppose \( P: \mathcal{C}^{\text{op}} \to \text{POS} \) is a primary doctrine and \( j \) is a topology on \( P \). We have the following:

(i) If \( P \) is elementary, then so is \( P_j \).

(ii) If \( P \) is existential, then so is \( P_j \).

(iii) If \( P \) is disjunctive, then so is \( P_j \).

(iv) If \( P \) is implicational, then so is \( P_j \).

(v) If \( P \) is universal, then so is \( P_j \).

(vi) If \( P \) has a weak predicate classifier, then so does \( P_j \).

Proof. Standard argument: for \( f: A \to B \) and \( \alpha, \beta \) in \( P_j(A) \) note that \( \alpha \to \beta \) determines the universal and the implicational structure in \( P_j \) (if they already exists in \( P \)) while for the left adjoints and the disjunctive structure one takes the closure of the corresponding ones of \( P \) so \( j_B(\exists_B(\alpha)) \) and \( j_A(\alpha) \lor j_A(\beta) \).

The weak power object of \( A \) is \( P A \) with membership predicate \( j_A \times P A(\epsilon_A) \). \( \square \)

3.7 Example. Given a first-order doctrine \( P \) a major example of topology is the double negated topology associating \( \neg\neg \alpha \) to \( \alpha \) of \( P \). Then \( P_{\neg\neg} \) is a boolean first-order doctrine by prop. 3.6 which is an algebraic rendering of Gödel-Gentzen double negation translation extended to first-order equality.

3.8 Proposition. Suppose \( P: \mathcal{C}^{\text{op}} \to \text{POS} \) is elementary existential and \( \mathcal{C} \) has weak pullbacks. The following are equivalent:

(i) \( P \) is isomorphic to \( \Psi_{C_j} \) for some topology \( j \) over \( \Psi_C \);

(ii) \( P \triangleleft \Psi_C \);

(iii) \( P \) has full weak comprehensions.

Proof. The equivalence between items 1 and 2 is a special case of 3.5, while 3\(\Rightarrow\)2 is proposition 2.3 of [MPR19] It remains 2\(\Rightarrow\)3. Take \( \alpha \) in \( P(A) \) and let \( \{\alpha\}: X \to A \) be any representative of \( r_A(\alpha) \). We first need prove that \( \top_X \leq P_{[\alpha]}(\alpha) \). Observe that
\[
\alpha = r_A(\alpha) = (\sum_{r_A(\alpha)} \text{id}_X) = \exists_{r_A(\alpha)} \text{id}_X = \exists_{r_A(\alpha)} r(\top_X) = \exists_{r(\alpha)} \top_X
\]
Hence $\top_X \leq P_{r_A(\alpha)}(\exists r_A(\alpha) \top_X) = P_{r_A(\alpha)}(\alpha) = P_{\{\alpha\}}(\alpha)$. Take $f : Y \to A$ with $\top_Y \leq P_f(\alpha)$. Then $id_Y = r_A(\top_Y) \leq r_A P_f(\alpha) = f^* r_A(\alpha) = f^* \{\alpha\}$ from which we conclude that $\{\alpha\}$ is a full weak comprehension of $\alpha$.

Since $id_X$ factors through the weak pullback of $\{\alpha\}$ along itself, it is

$$r_X(\top_X) = [id_X] \leq \{\alpha\}^* \{\alpha\} = \{\alpha\}^* r_A(\alpha) = r_X P_{\{\alpha\}}(\alpha)$$

The equality $rl = id_P$ implies that components of $r$ reflect the order, so $\top_X \leq P_{\{\alpha\}}(\alpha)$. Then $[id_Y] = l_Y(\top_Y) \leq l_Y P_f(\alpha) = f^* \{\alpha\}$. This means that $id_Y$ factors through the weak pullback of $\{\alpha\}$ along $f$, so $f$ factors through $\{\alpha\}$.

To show that $P$ has comprehensive diagonals take $h, k : X \to A$ and suppose $\top_X \leq P_{(h, k)}(\delta_A)$. Using again $r$ (and the fact that it is a morphism of elementary doctrines) it holds

$$[id_X] = r_X(\top_X) \leq r_X P_{(h, k)}(\delta_A) = \langle h, k \rangle^* r_A(\delta_A) = \langle h, k \rangle^* [\langle id_A, id_A \rangle]$$

so $h = k$. The second part of the theorem is analogous once one observes that a comprehension arrow is strong if and only if it is monic.

The following proposition is stated without a proof as its proof is perfectly analogous to the one of 3.8.

3.9 Proposition. Suppose $P : C^{\text{op}} \to \text{POS}$ is elementary existential and $C$ has pullbacks. Then $P$ is isomorphic to $\Psi C_j$ for some topology $j$ over $\text{Sub}_C$ if and only if $P \triangleleft \text{Sub}_C$ if and only if $P$ has full comprehensions.

Proof. It follows from proposition 3.6.

The elementary quotient completion presented in section 2.2 very well behaves with respect to comprehensions, in particular we have the following.

3.13 Proposition. $P$ is variational if and only if $\hat{P}$ is $m$-variational.
2.24)

3.15 Definition. Let \( D \in \top \)\( P \in \top \) its comprehension is

\[ \{ \{ \alpha \} \} : (X, P_{\{\alpha\}} \times \{\alpha\} (\rho)) \rightarrow (A, \rho) \]

where \( \{ \alpha \} : X \rightarrow A \) is the weak comprehension of \( \alpha \) in \( P \). Since \( \hat{P} \) has comprehensive diagonals, the arrow \( [\{ \alpha \}] \) is monic. \( \Box \)

3.14 Proposition. If \( P \) is variational, then \( Q_P \) has finite limits.

Proof. By \( \Box \) \( \hat{P} \) is m-variational, then apply 2.25 \( \Box \)

Every topology \( j \) on a primary doctrine \( P \) determines a topology \( \hat{j} \) on the elementary quotient completion \( \hat{P} \). The topology \( \hat{j} \) is simply the restriction of \( j \) to the poset of descent data, i.e. for \( \rho \) a \( P \)-equivalence relation over \( A \) and \( \alpha \)

\[ \text{induced over } \hat{P}, \text{ it is } \hat{j}(A,\rho) (\alpha) = j_A (\alpha) \] (see also [Men01]). Indeed (using notation as in 2.24)

\[ a : A, a' : A \mid j \alpha (a) \land \rho (a, a') \vdash j \alpha (a) \land j \rho (a, a') \vdash j (\alpha (a) \land \rho (a, a')) \vdash j \alpha (a') \]

After proposition 3.8 we know that every existential variational doctrine \( P \) on a category \( C \) with weak pullbacks generates a topology on \( \Psi_C \).

3.15 Definition. Let \( P \) an existential variational doctrine. The canonical topology on \( \Psi_C \) is the topology induced on \( \Psi_C \) and denoted with the symbol \( \hat{\top}_P \), i.e. \( \hat{\top}_P (f) = \{ \exists \top \} \top \) for \( f : A \rightarrow B \) in \( C \).

3.16 Proposition. Suppose \( C' \text{ is the category of } \hat{\top}_P \text{-separated objects for the topology } \hat{\top}_P \text{ induced over } \text{Sub}_{\text{ex/lex}} \).

Proof. By \( \Box \) there is a full and faithful \( R : Q_P \rightarrow Q_{\Psi_C} \) which is right adjoint to \( L : Q_{\Psi_C} \rightarrow Q_P \). Take a \( \Psi_C \)-equivalence relation \([k : X \rightarrow A \times A]\) over \( A \). The object \((A,[k])\) in \( Q_{\Psi_C} \) is equivalent to one in \( Q_P \) if and only if \((A,[k]) \simeq RL(A,[k])\). From the construction of \( L \) and \( R \) as in 3.4 this happens if and only if

\[ \delta((A,[k])) = [k] = j_P A \times A \cdot [k] = \hat{j}_P A \times A \cdot (A,[k])(\delta(A,[k])) \]

Note that \( \hat{j}_P \) is a topology on \( \Psi_C \) which is \( \text{Sub}_{\text{ex/lex}} \), whence the claim. \( \Box \)

A first-order doctrine \( P : C^\text{op} \rightarrow \text{POS} \) is boolean if for every \( A \) and \( \alpha \) in \( P(A) \) it holds \( \top_A = \alpha \lor \neg \alpha \) where \( \neg \alpha \) is short for \( \alpha \rightarrow \bot \).

3.17 Proposition. A first-order variational doctrine \( P : C^\text{op} \rightarrow \text{POS} \) on a base \( C \) with weak pullbacks and an initial object \( 0 \) such that \( \{ \bot \} = un \) where \( un : 0 \rightarrow 1 \) in \( C \), is boolean if and only if \( P \) is isomorphic to \( \Psi_{C_{\rightarrow \neg \rightarrow}} \), the doctrine of \( \neg \neg \rightarrow \) closed elements of \( \Psi_C \). A first-order m-variational doctrine \( P : C^\text{op} \rightarrow \text{POS} \) on a base \( C \) is boolean if and only if \( P \) is isomorphic to \( \text{Sub}_{C_{\rightarrow \neg \rightarrow}} \), the doctrine of \( \neg \neg \rightarrow \) closed elements of \( \text{Sub}_C \).
Proof. We show the non-trivial direction. Suppose $P$ is boolean. By 3.8 there is a topology $j$ on $\Psi_C$ such that $P$ is isomorphic to $\Psi_C^j$ and for $[f: X \to A]$ in $\Psi_C(A)$ it is $j_A[f] = \{ \{ f \subseteq X \} \}$. It is $\{ \{ f \subseteq X \} \} = \{ \{ \neg f \subseteq X \} \} = \neg \neg \{ \{ f \subseteq X \} \} = \neg \neg \{ \{ \neg f \subseteq X \} \}$ and hence the claim as $\{ \{ \neg f \subseteq X \} \} = [f]$. \qed

4 Decomposition of the elementary quotient completion

In this section we recall from [MR13a] that the construction of the elementary quotient completion of an elementary doctrine introduced in [MR13b] is not primitive because it can be obtained by applying two other free constructions to $P$. First, we apply to $P$ the intensional quotient completion of an elementary doctrine which freely adds just effective descent quotients, second we apply the extensional collapse construction of an elementary doctrine which freely adds comprehensive diagonals. Both constructions were introduced in [MR13a] and the second had been already recalled in 2.9. We now recall the first construction.

4.1 Definition. Given a elementary doctrine $P: C^{\text{op}} \to \text{POS}$ we call $Q_{P_i}$ the category whose objects are pairs $(A, \rho)$ in which $A$ is in $C$ and $\rho$ is an equivalence relation over $A$. An arrow $f: (A, \rho) \to (B, \sigma)$ is an arrow $f: A \to B$ in $C$ such that $\rho \leq P_{f \times f}(\sigma)$.

Then we define an elementary doctrine called intensional quotient completion of $P$ $P_{iq}(A, \rho) = \{ \phi \in P(A) \mid P_{pr_1}(\phi) \land \rho \leq P_{pr_2}(\phi) \}$

where $pr_1$ and $pr_2$ are the first and the second projection form $A \times A$. It is proved in [MR13a] that the assignment on arrows does not depend on the choice of representatives.

4.2 Theorem. There is a left biadjoint to the forgetful 2-functor from the full 2-category of elementary doctrines with stable effective descent quotients into the 2-category $\text{ED}$ of elementary doctrines which associates the doctrine $P_{iq}: Q_{P_i}^{\text{op}} \to \text{POS}$ to an elementary doctrine $P: C^{\text{op}} \to \text{POS}$.

Note that the property that quotients are stable effective descent is crucial in the proof of the above theorem, including to show that a doctrine morphism from $P$ to any other doctrine lifted on $P_{iq}$ preserves binary products by proving that the binary product of two quotients $A/\rho \times B/\sigma$ is isomorphic to $A \times B/\rho \boxtimes \sigma$ in any elementary doctrine with stable effective descent quotients.

As shown in [MR13a] the intensional quotient completion of a doctrine has stable effective descent quotients: if $\sigma$ is an equivalence relation over $(A, \rho)$, then its quotient is given by $[\text{id}_A]: (A, \rho) \to (A, \sigma)$.
Observe that in the doctrine \( Q_{P_i} \), the equality predicate over \((A, \rho)\) is \( \rho \) itself \[ \MR13a \], i.e.
\[
\delta_{(A, \rho)} = \rho
\]

Note that for any elementary doctrine \( Q: \mathcal{C}^{\op} \to \POS \) with effective quotients, the doctrine \( Q_x \) has only a weak form of quotients. But when \( Q = Q_{P_i} \) of an elementary doctrine \( P \), the following result from \[ \MR13a \] applies.

4.3 Proposition. Let \( P: \mathcal{C}^{\op} \to \POS \) be an elementary doctrine. The morphism \((K, k): Q_{P_i} \to Q_{P_i^x}\) preserves quotients and therefore \( Q_{P_i^x} \) has effective descent quotients of \( Q_{P_i^x} \)-equivalence relations and coincides with the elementary quotient completion of \( P \).

Further properties and applications of the intensional quotient completion can be found in \[ \text{Pas15, EPR20} \].

5 A characterisation of elementary quotient completions

In this section we give a characterisation of those elementary doctrines with effective descent quotients that arise as elementary quotient completions by using the concept of regular projective relative to a doctrine. This characterization generalizes the well known characterization given in \[ \text{CV98} \] for the exact completion of a lex category. Indeed, recall that the ex/lex completion of a category \( \mathcal{C} \) with finite products and weak pullbacks is the base of the elementary quotient completion of the doctrine of variations of \( \mathcal{C} \). Then, our characterisation arises as a generalisation to the framework of doctrines of the fact that an exact category with enough regular projectives is equivalent to the ex/lex completion of its full subcategory on projective objects.

5.1 Definition. Suppose \( P: \mathcal{C}^{\op} \to \POS \) is an elementary doctrine. An object \( X \) of \( \mathcal{C} \) is said \( P \)-projective if for every diagram of the form

\[
\begin{array}{ccc}
X & \xrightarrow{k} & Y \\
\downarrow{f} & & \downarrow{q} \\
A & \xrightarrow{\rho} & A
\end{array}
\]

where \( q \) is a quotient arrow of \( P \), there is an arrow \( k: X \to Y \) with \( qk = f \).

5.2 Definition. Suppose \( P: \mathcal{C}^{\op} \to \POS \) is an elementary doctrine. We say that \( \mathcal{C} \) has enough \( P \)-projectives if for every \( A \) in \( \mathcal{C} \) there is a \( P \)-projective object \( X \) and a quotient arrow \( q: X \to A \), called \( P \)-cover of \( A \).

Suppose \( E \) is a class of morphisms of a category \( \mathcal{C} \).

5.3 Lemma. Suppose \( P: \mathcal{C}^{\op} \to \POS \) is an elementary doctrine. Denote by \( \mathcal{D} \) the full subcategory of \( \mathcal{C} \) consisting only of \( P \)-projective objects. If \( \mathcal{M} \) is a full subcategory of \( \mathcal{D} \) closed under binary products and such that every object of \( \mathcal{D} \) is covered by one in \( \mathcal{M} \), then \( \mathcal{D} \) is closed under binary products.
Proof. Suppose $A$ and $B$ are in $\mathcal{D}$ and consider the diagram

$$
\begin{array}{c}
X_A \times X_B \ar[rrr]^{s_A \times s_B} \ar[rr]_{q_A \times q_B} \ar@{-->}[dr] & & \ar[r]_f & A \times B \\
\ar@{-->}[ur] & & & \ar[rr]_q & & Q \\
Y & \ar[u] \ar[r]_q & \end{array}
$$

where $q$ is quotient map. Let $q_A: X_A \rightarrow A$ and $q_B X_B \rightarrow B$ be $P$-covers respectively of $A$ and $B$, i.e. $X_A$ and $X_B$ are in $\mathcal{M}$. Since both $A$ and $B$ $P$-projectives each cover has a section $s_A$ and $s_B$, i.e. $q_A s - A = \text{id}_A$ and $q_B s - A = \text{id}_B$. Since $X_A \times X_B$ is $P$-projective because in $\mathcal{M}$, then there is $\overline{f}: X_A \times X_B \rightarrow X$ with $\overline{f} = f(q_A \times q_B)$. Hence $\overline{f}(s_A \times s_B): A \times B \rightarrow X$ is such that $\overline{f}(s_A \times s_B) = f(q_A \times q_B)(s_A \times s_B) = f$ proving that $A \times B$ is $P$-projective and hence in $\mathcal{D}$. □

5.4 Lemma. Suppose $P: \mathcal{C}^{op} \rightarrow \text{POS}$ is an elementary doctrine. If $q: X \rightarrow X/\rho$ and $q': X' \rightarrow X'/\rho'$ are quotient arrows, and if $X$ is $P$-projective, then for every arrow $f: X/\rho \rightarrow X'/\rho'$ there is an arrow $g: X \rightarrow X'$ with $\sqcap X = P(fq,q'g)(\delta_{X/\rho})$. Obviously if the doctrine has comprehensive diagonals it is $fg = q'g$.

5.5 Theorem. Suppose $P: \mathcal{C}^{op} \rightarrow \text{POS}$ is an elementary doctrine with comprehensive diagonals and effective descent quotients. The following are equivalent.

(i) $P: \mathcal{C}^{op} \rightarrow \text{POS}$ is of the form $\overline{P_0}: \mathcal{Q}^{op}_{P_0} \rightarrow \text{POS}$ for some elementary doctrine $P_0: \mathcal{C}_0^{op} \rightarrow \text{POS}$ with comprehensive diagonals.

(ii) $P: \mathcal{C}^{op} \rightarrow \text{POS}$ has enough $P$-projectives and these are closed under binary products.

When one of the conditions holds, then $P$ is the intensional elementary quotient completion of its restriction to the full subcategory of $\mathcal{C}$ made of $P$-projectives.

Proof. i)$\Rightarrow$ ii) All quotient arrows in $\mathcal{Q}_{P_0}$ are of the form $[\text{id}_A]: (A, \rho) \rightarrow (A, \sigma)$, thus objects of the form $(A, \delta_A)$ are $P$-projective and they determine a full subcategory of $\mathcal{Q}_{P_0}$ which is closed under products. Hence $\mathcal{Q}_{P_0}$ has enough $P$-projectives and these are closed under binary products by 5.3.

ii)$\Rightarrow$ i) Denote by $\mathcal{C}_0$ the full subcategory of $\mathcal{C}$ on all its $P$-projectives and by $P_0$ the restriction of $P$ to $\mathcal{C}_0$ (i.e. the change of base of $P$ along the inclusion of $\mathcal{C}_0$ into $\mathcal{C}$). Since $\mathcal{C}_0$ is closed under products $P_0$ is an elementary doctrine with comprehensive diagonals. We need prove that $\mathcal{Q}_{P_0}$ is equivalent to $\mathcal{C}$. Consider $[f]: (A, \rho) \rightarrow (B, \sigma)$ in $\mathcal{Q}_{P_0}$. Every representative of $[f]$ determines a
commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{q} & A/\rho \\
\downarrow f & & \downarrow \overline{f} \\
B & \xrightarrow{e} & B/\sigma
\end{array}
\]

where \(\overline{f}\) is the map determined by the universal property of quotients. The diagram above extends a functor from \(Q_{P_0} \rightarrow C\). Which is faithful by effectiveness of quotients. Since \(A\) and \(B\) are projective every arrow \(A/\rho \rightarrow B/\sigma\) determines an arrow \(A \rightarrow B\), then the functor is also full. Essential surjectivity is a straightforward consequence of the hypothesis that \(C\) has enough \(P\)-projectives. Since quotients are of effective descent, for every \((A,\rho)\) in \(Q_{P_0}\) the poset \(\hat{P}_0(A,\rho)\) is isomorphic to \(P(A/\rho)\); this completes the proof. □

5.6 Theorem. Suppose \(P: C^{\text{op}} \rightarrow \text{POS}\) is an elementary doctrine with comprehensive diagonals, full comprehensions and effective descent quotients. The following are equivalent.

(i) \(P: C^{\text{op}} \rightarrow \text{POS}\) is of the form \(\hat{P}_0: Q_{P_0}^{\text{op}} \rightarrow \text{POS}\), i.e. an elementary quotient completion, for some elementary doctrine \(P_0: C_0^{\text{op}} \rightarrow \text{POS}\) with comprehensive diagonals and full comprehensions.

(ii) \(P: C^{\text{op}} \rightarrow \text{POS}\) has enough \(P\)-projectives and these are closed under finite limits.

When one of the conditions holds, then \(P\) is the intensional elementary quotient completion of its restriction to the full subcategory of \(C\) made of \(P\)-projectives.

Proof. Analogous to the proof of theorem 5.5. Just observe that for the direction \(i \Rightarrow ii\) \(P\)-projectives are closed under pullbacks, and hence finite limits by prop. 2.29 applied to \(P_0\) which has full comprehensions and comprehensive diagonals. □

In a similar way we obtain a characterization of intensional quotient completions.

5.7 Theorem. Suppose \(P: C^{\text{op}} \rightarrow \text{POS}\) is an elementary doctrine with effective descent quotients. The following are equivalent.

(i) \(P: C^{\text{op}} \rightarrow \text{POS}\) is the intensional elementary quotient of some elementary doctrine \(P_0: C_0^{\text{op}} \rightarrow \text{POS}\).

(ii) \(P: C^{\text{op}} \rightarrow \text{POS}\) has enough \(P\)-projectives and these are closed under binary products.

When one of the conditions holds, then \(P\) is the intensional elementary quotient completion of its restriction to the full subcategory of \(C\) made of \(P\)-projectives.
We now show how theorem 5.5 is a generalization of Carboni and Vitale’s characterization of exact completions of a lex category. To this purpose, we need some lemmas.

**5.8 Lemma.** In a category $\mathcal{C}$ with finite limits an object is projective with respect to the subobject doctrine $\text{Sub}_\mathcal{C}$ of $\mathcal{C}$ if and only if it is a regular projective.

*Proof.* The notion of $\text{Sub}_\mathcal{C}$-effective quotient coincide with that of categorical effective quotient. □

**5.9 Lemma.** Let $\mathcal{C}$ be an exact category. Denote by $\mathcal{C}_0$ the full subcategory of $\mathcal{C}$ on its regular projectives. If $\mathcal{C}$ has enough projectives closed under finite limits then the subobject doctrine $\text{Sub}_\mathcal{C}$ restricted to $\mathcal{C}_0$ is isomorphic to the doctrines of variations $\Psi_{\mathcal{C}_0}:\mathcal{C}_0^{\text{op}} \rightarrow \text{POS}$.

*Proof.* The doctrine of variations can be fully and faithfully embedded in $\text{Sub}_\mathcal{C}$ as follows: to any map $f:A \rightarrow B$ in $\mathcal{C}_0$ we associate the subobject $i_f:\text{Im}(f) \rightarrow B$ given by the image factorization of $f$ in an exact category.

Conversely, given any subobject $i:C \rightarrow B$ in $\mathcal{C}$ over a projective $B$, by hypothesis there exists a projective cover $q_C:X_C \rightarrow C$ of $C$ which gives rise to map $iq_C:X_C \rightarrow B$ which is in $\mathcal{C}_0$. The correspondence is bijective since the weak subobject given by $iq_C$ is the same as that of $f$ due to the projectivity of $X_{imf}$ and $A$, and the image of $iq_C$ is the subobject of $i$ by the uniqueness of the image factorization in an exact category. □

**5.10 Corollary.** Let $\mathcal{C}$ be an exact category. The following are equivalent.

(i) $\mathcal{C}$ is an ex/lex completion.

(ii) $\mathcal{C}$ has enough regular projectives closed under finite limits.

When one of the conditions holds, then $\mathcal{C}$ is the ex/lex completion of its full subcategory of regular projectives.

*Proof.* First, recall that the exact completion of a category with binary products and weak pullbacks is an instance of the elementary quotient completion (see 2.54(c)) and that the subobject doctrine of an exact category has effective descent quotients by lemma 2.52. Then, the claim is an instance of theorem 5.6 by lemmas 5.8 and 5.9. □

6 Structural properties in the elementary quotient completion

In this section and in the next ones we generalize well known facts concerning the categorical structure of the ex/lex completion to the elementary quotient completion.

It is well known that the ex/lex completion brings weak structures of $\mathcal{C}$ to strong structures in $\mathcal{C}_{\text{ex/lex}}$ and this holds for elementary quotient completions in an analogous way. In this section we shall focus on local cartesian closure,
disjoint stable coproducts and predicate classifier from a given variational elementary doctrine \( P \). First note the following.

6.1 Proposition. \( P \) is a first-order doctrine if and only if \( \tilde{P} \) is a first-order doctrine.

Proof. Immediate; see also [Pas16]. \( \square \)

A \( J \)-diagrams in \( \mathcal{C} \) is a functor of the form \( J \to \mathcal{C} \). We say that \( \mathcal{C} \) has \( J \)-indexed limits if every \( J \)-diagram has a limits and that \( \mathcal{C} \) has \( J \)-weak indexed limits if every \( J \)-diagram has the existence property of a limit but not the uniqueness condition. Accordingly we say that \( \mathcal{C} \) has \( J \)-indexed colimits if every \( J \)-diagram has colimits.

Recall that for a variational doctrine \( P \) on \( \mathcal{C} \) the \( \nabla_P: \mathcal{C} \to \mathcal{Q}_P \) (i.e. the functor that maps \( f: A \to B \) to \( [f]: (A, \delta_A) \to (B, \delta_B) \)) is full and faithfull.

6.2 Proposition. If \( P: \mathcal{C}^{op} \to \mathcal{POS} \) is a variational doctrine, then for every \( J \to \mathcal{C} \) it holds

(i) if \( \mathcal{Q}_P \) has \( J \)-indexed limits, then \( \mathcal{C} \) has \( J \)-indexed weak limits;

(ii) if \( \mathcal{Q}_P \) has \( J \)-indexed colimits of the form \( (W, \delta_W) \), then \( \mathcal{C} \) has \( J \)-indexed colimits.

Proof. 1. Let \( F \) be a \( J \)-diagram in \( \mathcal{C} \). If \( (W, \omega) \) is the limit of \( \nabla_P F \) in \( \mathcal{Q}_P \), then \( (W, \delta_W) \) is a weak limit of \( \nabla_P F \) in the image of \( \nabla_P \) within \( \mathcal{Q}_P \), and hence \( W \) is a weak limit of \( F \) in \( \mathcal{C} \). 2. Consider a diagram \( F: J \to \mathcal{C} \) and suppose \( (W, \delta_W) \) is the colimit for \( \nabla_P F \). \( W \) is easily seen to be a weak colimit for \( F \) in \( \mathcal{C} \). Suppose \( X \) is a cocone and let arrows \( q, p: W \to X \) be such that they make commute all the relevant triangles. So do \( [q], [p]: (W, \delta_W) \to (X, \delta_X) \) in \( \mathcal{Q}_P \). Universality of \( (W, \delta_W) \) ensures that \( [q] = [p] \), i.e. \( \top_W \leq P_{(q, p)}(\delta_X) \), whence \( q = p \) in \( \mathcal{C} \) as diagonals are comprehensive. \( \square \)

6.3 Proposition. Suppose \( P \) is a variational doctrine and \( f: X \to A, g: Y \to A \) are arrows of \( \mathcal{C} \). Suppose also that \( \rho \) and \( \sigma \) are \( P \)-equivalence relations over \( X \) and \( Y \) respectively such that \( \rho \leq P_{f \times f}(\delta_A) \) and \( \sigma \leq P_{g \times g}(\delta_A) \). Consider two arrows \( k: P \to X \) and \( h: P \to Y \) and the following two squares

If the left square is a weak pullback in \( \mathcal{C} \), the right square is a pullback in \( \mathcal{Q}_P \).
Proof. Take any two arrows \([a]: (C, \gamma) \to (X, \rho)\) and \([b]: (C, \gamma) \to (Y, \sigma)\) with \([f][a] = [g][b]\). That is to say \(\top_C = P_{fa \times gb}(\delta_A)\). Since diagonals are comprehensive we have \(fa = gb\). Hence there is \(u:C \twoheadrightarrow S\) with \(ku = a\) and \(hu = b\). Since \(\gamma \leq P_{a \times a}(\rho) = P_{a \times a}P_{k \times k}(\rho)\) and \(\gamma \leq P_{b \times b}(\rho) = P_{a \times a}P_{h \times h}(\sigma)\) one has \(\gamma \leq P_{a \times a}(P_{h \times h}(\sigma) \cap P_{k \times k}(\rho))\), showing that \([u]\) is an arrow in \(Q_P\), thus also \([k][u] = [a]\) and \([h][u] = [b]\). If \([u']\) is such that \([k][u'] = [a]\) and \([h][u'] = [b]\), then also \([k][u'] = [k][u]\) and \([h][u'] = [h][u]\), i.e.

\[
\top_C \leq P_{u \times u'}P_{k \times k}(\rho) \quad \top_C \leq P_{u \times u'}P_{h \times h}(\sigma)
\]

whence \(\top_C \leq P_{u \times u'}(P_{h \times h}(\sigma) \cap P_{k \times k}(\rho))\) showing that \([u] = [u']\). \(\square\)

6.4 Proposition. Suppose \(P:C^{op} \to \text{POS}\) is a variational doctrine and consider two squares of the form

\[
\begin{array}{ccc}
S & \xrightarrow{k} & X \\
\downarrow h & & \downarrow f \\
Y & \xrightarrow{g} & A
\end{array}
\quad \quad
\begin{array}{ccc}
(S, \theta) & \xrightarrow{[k]} & (X, \delta_X) \\
\downarrow [h] & & \downarrow [f] \\
(Y, \delta_Y) & \xrightarrow{[g]} & (A, \delta_A)
\end{array}
\]

If the right square is a pullback in \(Q_P\), the left square is a weak pullback in \(C\).

Proof. Suppose \(a:C \twoheadrightarrow X\) and \(b:C \twoheadrightarrow Y\) are such that \(fa = gb\). Then \([a]: (C, \delta_C) \to (X, \delta_A)\) and \([b]: (C, \delta_A) \to (Y, \delta_Y)\) are such that \([f][a] = [g][b]\). So there is \([u]\) with \([k][u] = [a]\) and \([h][u] = [b]\), i.e.

\[
\top_C = P_{(ku, a)}(\delta_X) \quad \top_C = P_{(hu, b)}(\delta_Y)
\]

Since diagonals are comprehensive, \(u\) is such that \(ku = a\) and \(hu = b\) in \(C\). \(\square\)

6.1 Local cartesian closure

We introduce some technical notions which will be used in the proof of the characterisation Theorem 6.32.

6.5 Definition. Let \(\mathcal{C}\) be a category with finite products and let \(J: \mathcal{D} \to \mathcal{C}\) be an inclusion of a subcategory in it. We say that an object \(X\) is \textbf{weakly exponentiable relative to} \(\mathcal{D}\) if the functor

\[
X \times (-): \mathcal{D} \to \mathcal{C}
\]

is a weak left adjoint, in the sense of [Kai71]: for every object \(Y\) in \(\mathcal{C}\) there are an object \(W\) in \(\mathcal{D}\) and an arrow

\[
X \times J(W) \xrightarrow{ev} Y
\]

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in $\mathcal{C}$ such that for every $D$ in $\mathcal{D}$ and every arrow $f: X \times J(D) \to Y$ there is a commutative diagram

$$
\begin{array}{ccc}
X \times J(D) & \xrightarrow{f} & D \\
\downarrow{\text{id}_X \times J(f)} & & \downarrow{f} \\
X \times J(W) & \xrightarrow{\text{ev}} & Y \\
\end{array}
$$

where the dotted arrow indicates that the condition need not determine it uniquely.

**6.6 Remark.** The condition of weak left adjoint in Definition 6.5 provides a family of surjective functions

$$
\mathcal{D}(D,W) \longrightarrow \mathcal{C}(X \times J(D), Y)
$$

natural in $D$.

We shall be interested in weak relative exponentiability in slice categories of the form $Q\mathcal{P}/(A,\delta_A)$. They shall involve a specific kind of objects which will be introduced in Definition 6.8.

**6.7 Remark.** Consider a category $\mathcal{C}$ with finite products. An object $Y$ is weakly exponentiable in the usual sense if (and only if) it is weakly exponentiable relative to $\mathcal{C}$. So $\mathcal{C}$ is weakly cartesian closed if and only if every object is weakly exponentiable relative to $\mathcal{C}$.

**6.8 Definition.** Let $P: \mathcal{C}^{\text{op}} \to \text{POS}$ be an elementary doctrine, and let $Q\mathcal{P}$ be its elementary quotient completion. An arrow in $Q\mathcal{P}$ of the form $[f]: (X,\delta_X) \to (A,\delta_A)$ is called a dependent $P$-projective. We write as $\mathcal{D}_A$ the full subcategory of $Q\mathcal{P}/(A,\delta_A)$ on the dependent $P$-projectives in it.

In case $\mathcal{C}$ has (strong) pullbacks, local cartesian closure suffices to show that the doctrine $\Psi_{\mathcal{C}}$ is universal. In the weak case this need not happen, and motivates the following definition.

**6.9 Definition.** Let $P: \mathcal{C}^{\text{op}} \to \text{POS}$ be an elementary doctrine with comprehensive diagonals and weak full comprehension, then its base $\mathcal{C}$ has weak pullbacks and $Q\mathcal{P}$ has pullbacks by Proposition 2.29 and 3.14. We say that $P$ is slice-wise weakly cartesian closed when the following conditions are satisfied:

(i) the doctrine $P$ is implicational and universal;

(ii) for every object $A$ in $\mathcal{C}$, each dependent $P$-projective is weakly exponentiable in $Q\mathcal{P}/(A,\delta_A)$ relative to $\mathcal{D}_A$.

**6.10 Remark.** It may be useful to expand condition (ii) taking advantage of the full embedding $J: \mathcal{C} \to Q\mathcal{P}$ introduced in Remark 2.54—so, in particular,
$Ja = (A, \delta_A)$. Given objects $Jf: JX \rightarrow JA$ and $[g]: (Y, \rho) \rightarrow JA$ in the slice category $Q_P/JA$, there is a diagram of arrows in $C$

![Diagram](image)

where the inner square is a weak pullback. The arrow $ev: S \rightarrow A$ is the representative of an arrow $[ev]: Jf \times_JA Jw \rightarrow [g]$ in $Q_P/JA$ such that, for any arrow $w: U \rightarrow A$ in $C$ and any arrow $[k]: Jf \times_JA Ju \rightarrow [g]$ in $Q_P/JA$, there exists $\hat{k}: U \rightarrow W$ in $C$ such that the diagram

![Diagram](image)

commutes in $Q_P/JA$.

**6.11 Remark.** In [Cio23], it is shown that a category $Q_P/(A, \delta_A)$ is an example of elementary quotient completion of a suitable biased elementary doctrine for which dependent $P$-projectives $[f]: (X, \delta_X) \rightarrow (A, \delta_A)$ are covering projectives.

**6.12 Definition.** Let $P: C^{op} \rightarrow POS$ be an elementary doctrine with comprehensive diagonals whose base $C$ has weak pullbacks. We say that $Q_P$ is **slice-wise exponentiable on dependent projectives** if the following conditions are satisfied.

(i) The doctrine $P$ is implicational and universal;

(ii) For every object $A$ in $C$, each dependent $P$-projective is exponentiable in $Q_P/(A, \delta_A)$.

**6.13 Remark.** In his Ph.D. thesis [Cio22], Cipriano Jr. Cioffo introduced the notion of *extensional exponential*, which is equivalent to the universal property in remark 6.10. In the case of a variational doctrine it coincides with that of extensional exponential given by Jacopo Emmenegger in [Emm20].

Next we note a connection between the fibres $\Psi_C(A)$ of the variational doctrine on $C$ and $\Psi_{Q_P}(A, \delta_A)$ of that on the elementary quotient completion $Q_P$.

**6.14 Proposition.** Let $P: C^{op} \rightarrow POS$ be an elementary doctrine with comprehensive diagonals, and let $A$ be an object in $C$. The functor

$$\nabla_A: \Psi_C(A) \longrightarrow \Psi_{Q_P}(A, \delta_A)$$

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induced by the functor $J: C \longrightarrow Q_P$ of Remark 2.55 is a full coreflective embedding.

**Proof.** The functor $\nabla_A$ is full thanks to comprehensive diagonals. The right adjoint is

$$\Psi_{Q,P}(A, \delta_A) \xrightarrow{R_A} \Psi_C(A)$$

$$[f: (B, \sigma) \rightarrow (A, \delta_A)] \xrightarrow{\Psi_{Q,P}} [f: B \rightarrow A]$$

**6.15 Remark.** Note that, without an assumption about existence of weak pullbacks in $C$, the assignment $A \mapsto \Psi_C(A)$ does not extend to a doctrine. Proposition 6.14 does not have that hypothesis among the assumptions because one is hard put to prove that the family of functors $\nabla_A$ be natural in $A$.

But it is easy to prove that, if an arrow $f: A' \rightarrow A$ in $C$ is such that pullback of any arrow $g: X \rightarrow A$ along $f$ exists in $C$, then the reindexing functors commute with $\nabla$ as in the following diagram

$$\begin{array}{ccc}
\Psi_C(A) & \xrightarrow{\nabla_A} & \Psi_{Q,P}(A, \delta_A) \\
\Psi_C(f) & & \Psi_{Q,P}([f]) \\
\Psi_C(A') & \xrightarrow{\nabla_{A'}} & \Psi_{Q,P}(A', \delta_{A'}). \\
\end{array}$$

**6.16 Lemma.** Suppose that $P: C^{op} \longrightarrow POS$ is elementary with comprehensive diagonals, and $C$ has weak pullbacks. If $Q_P$ is locally cartesian closed, then

(i) the variational doctrine $\Psi_C: C^{op} \longrightarrow POS$ is universal and implicational;

(ii) if moreover $P$ is existential, then $P$ is a hyperdoctrine.

**Proof.** Note first of all that, since $Q_P$ is locally cartesian closed, the variational doctrine $\Psi_{Q,P}$ is universal and implicational. Write as $\forall_P^{Q,P}$ the right adjoint to reindexing along a projection $pr: D \rightarrow A$.

(i) Since pullbacks along a projection always exist in $C$, thanks to Proposition 6.14 the reindexing functor $\Psi_C(f)$ along any projection $f: D \rightarrow A$ has a right adjoint computed coreflecting the restriction of $\forall_P^{Q,P}$ along $\nabla_A$. The Beck–Chevalley condition for the universal quantification is satisfied since any reindexing of the doctrine $\Psi_C$ has a left adjoint.

(ii) By Proposition 2.57, $\hat{P}$ is existential since $P$ is existential. Since $\hat{P}$ admits full strong comprehension, by the previous point (i) and Proposition 2.59 it follows that $\hat{P}$ is universal and implicational. Applying again Proposition 2.57 yields that $P$ is universal and implicational.

**6.17 Lemma.** Suppose that $P: C^{op} \longrightarrow POS$ is elementary with comprehensive diagonals, and that $C$ has weak pullbacks. If $Q_P$ is locally cartesian closed, then for every object $A$ in $C$, a dependent $P$-projective over $A$ is weakly exponentiable relative to $D_A$.  

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Proof. Let \([f]: (X, \delta_X) \to (A, \delta_A)\) be a dependent \(P\)-projective and \(g: (Y, \tau) \to (A, \delta_A)\) any object in \(Q_P\). Consider the following diagram in \(Q_P\):

\[
\begin{array}{ccc}
(S', \theta') & \xrightarrow{[q_1]} & (W, \delta_W) \\
[q_2] & & [\text{id}_W] \quad [\text{ev}] \\
(S, \theta) & \xrightarrow{[p_2]} & (W, \xi) \quad (Y, \tau) \\
[p_1] & & [g][f] \\
(X, \delta_X) & \xrightarrow{[f]} & (A, \delta_A)
\end{array}
\]

where the two squares are pullbacks and \([\text{ev}]: (S, \theta) \to (Y, \tau)\) is the universal arrow of the exponential. Fix a representative \(w\) of the equivalence class \([g][f]\).

Then \(w: (W, \delta_W) \to (A, \delta)\) together with \([\text{ev}][q_2]: (S', \theta') \to (Y, \tau)\) is clearly a weak exponential of \([f]\) over \([g]\) relative to the full subcategory \(D_A\) on the \(P\)-dependent projectives.

6.18 Proposition. Suppose \(P: C^{op} \to \text{POS}\) is elementary with comprehensive diagonals, and \(C\) has weak pullbacks. Suppose also that \(Q_P\) is locally cartesian closed. If \(P\) is either universal or existential, then \(P\) is slice-wise weakly cartesian closed.

Proof. After Lemma 6.17 one needs only to invoke Lemma 6.16 to get that, in case \(P\) is existential, \(P\) is also universal and implicational.

We now aim at proving a partial converse to Proposition 6.18 as we shall replace the assumption of weak pullbacks in \(C\) with that of the doctrine \(P\) admitting full weak comprehension—we shall consider only the case when the elementary doctrine \(P\) is universal and implicational because of 6.16 (ii). To that purpose we produce an equivalent presentation of objects of \(C/A\), giving an algebraic presentation in line with the characterisation in [Mai09, Proposition 4.12]. For sake of simplicity we introduce some explicit notation for certain arrows related to constructions in the base \(Q_P\) of the elementary quotient completion.

6.19 Remark. Let \(P: C^{op} \to \text{POS}\) be an elementary doctrine which admits full weak comprehension. Let \(f: B \to A\) be a representative of an arrow \([f]: (B, \sigma) \to (A, \rho)\) in \(Q_P\), and write \(c_{\rho, f}: X \to B \times A\) a comprehending arrow for \(P_{f \times \text{id}_A}(\rho)\) in \(P(B \times A)\). We use the following notation for the compositions
in the diagram

\[
\begin{array}{c}
\begin{array}{c}
X \xleftarrow{f_\rho} B \times A \xrightarrow{\text{id}_B} B \\

\end{array}
\end{array}
\]

Write \(\rho \otimes f \sigma\) for the \(P\)-equivalence relation on \(X\) determined by the conjunction
\[P_{f_\rho \times f_\rho}(\rho) \land P_{f_\sigma \times f_\sigma}(\sigma)\] so that in the internal logic \(\rho \otimes f \sigma(x, x')\) abbreviates the formula \(\rho(f_\rho(x), f_\rho(x')) \land \sigma(f_\sigma(x), f_\sigma(x'))\).

6.20 Remark. It is immediate to see from the definition of the relation \(\rho \otimes f \sigma\) that the arrow \(f_\sigma: X \to B\) determines an arrow \([f_\sigma]: (X, \rho \otimes f \sigma) \to (B, \sigma)\) in the diagram (3) as well as the arrow \(f_\rho: X \to A\) gives an arrow \([f_\rho]: (X, \rho \otimes f \sigma) \to (A, \rho)\).

Also, there is a commutative diagram in \(C\)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B \xleftarrow{k} \times A \xrightarrow{\text{id}_A} A \\

\end{array}
\end{array}
\end{array}
\]

where the arrow \(k\) exists by weak universality of \(c_{p,f}: X \to B \times A\), since \(\rho\) is a \(P\)-equivalence relation and
\[P_{(\text{id}_B, f)}P_{f \times \text{id}_A}(\rho) = P_{(f, f)}(\rho) = \top_B.\]

In particular, it gives a retraction pair

\[\text{id}_B \circ k \circ f_\sigma^{-1} \circ X.\]

Moreover, \(\top_X = P_{(f f_\sigma, f_\rho)}(\rho)\) since \((f f_\sigma, f_\rho) = (f \times \text{id}_A)c_{p,f}\), i.e.
\[x: X \vdash \rho(f f_\sigma(x), f_\rho(x))\]

in the internal logic of the doctrine \(P\).
6.21 Proposition. In the notations of Remark 6.20, the following diagram

commutes in $Q_P$.

**Proof.** To complete the proof after Remark 6.20 one must show that $[k]: (B, \sigma) \rightarrow (X, \rho \otimes f \sigma)$, and $[k][f^\sigma] = [id_X]: (X, \rho \otimes f \sigma) \rightarrow (X, \rho \otimes f \sigma)$.

The simple argument to see that $[k]: (B, \sigma) \rightarrow (X, \rho \otimes f \sigma)$ performed in the internal logic is as follows:

$$x: B, x': B \mid \sigma(x, x') \vdash \rho(f_p(k(x)), f_p(k(x')))$$

because $f_p k = f$ and $[f]: (B, \sigma) \rightarrow (A, \rho)$, and

$$x: B, x': B \mid \sigma(x, x') \vdash \sigma(f^\sigma(k(x)), f^\sigma(k(x')))$$

because $f^\sigma k = id_B$. To prove that $[k][f^\sigma] = [id_X]$, again in the internal logic, the simple checks are as follows: recalling (4) and (3), one sees that

$$x: X \vdash \rho(f f^\sigma(x), f_p(x))$$

$$\vdash \rho(f_p(k f^\sigma(x)), f_p(x))$$

and

$$x: X \vdash \sigma(f^\sigma(x), f^\sigma(x))$$

$$\vdash \sigma(f^\sigma(k f^\sigma(x)), f^\sigma(x))$$

So conjoining the two conclusions gives the statement. □

6.22 Remark. Proposition 6.21 shows that $[f^\sigma]: [f_p] \rightarrow [f]$ in the slice category $Q_P/(A, \rho)$. Note, though, that $[f]$ and $[f_p]$ need not factor through each other in $\mathcal{C}$. Indeed, Remark 6.20 shows that $f$ factors through $f_p$ in $\mathcal{C}$, but nothing guarantees the other factorisation may occur. Since $Q_P$ is a category with finite limits, one can see that $[f_p]$ is isomorphic to $\Sigma_{[id_A]}[id_A]^*(f_p)$, where $\Sigma_{[id_A]}$ denotes the left adjoint to the pullback functor $[id_A]^*: Q_P/(A, \rho) \rightarrow Q_P/(A, \delta_A)$ along the map $[id_A]: (A, \delta_A) \rightarrow (A, \rho)$.

The following is the fundamental step toward the proof of the main result. It takes advantage of the iso $[f^\sigma]: [f_p] \rightarrow [f]$ in the slice category $Q_P/(A, \rho)$ to compute explicitly any product of $[f]$ in the slice category starting from a product in the slice category $\mathcal{C}/A$. Like before, we employ the notation introduced in 6.19.
6.23 Lemma. Let $P: C^{op} \rightarrow \text{POS}$ be an elementary existential doctrine with comprehensive diagonals and admitting weak comprehension. Consider two arrows

\[
(W, \theta) \quad \downarrow \quad [q] \\
(B, \sigma) \xrightarrow{[f]} (A, \rho)
\]

in $Q_P$, and consider the following diagrams

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\xymatrix{Z \ar[r]^{f_p'} & W} \\
q' \ar[d] \ar[r] & q \\
X \ar[r]_{f_p} & A
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\xymatrix{(Z, P_{q' \times q'}(\rho \otimes f \sigma) \land \varphi^*_p \times \varphi^*_p(\theta)) \ar[r]^{[f_p']} & (W, \theta)} \\
[q'] \ar[d] \ar[r] & [q] \\
(X, \varphi \otimes f \sigma) \ar[r]_{[f_p]} & (A, \rho).
\end{array}
\end{array}
\end{array}
\end{array}
\]

If \([3]\) is a weak pullback in $C$, then \([4]\) is a pullback in $Q_P$.

Proof. We write $\zeta$ for the $P$-equivalence relation $P_{q' \times q'}(\rho \otimes f \sigma) \land \varphi^*_p \times \varphi^*_p(\theta)$ on $Z$. Clearly, if diagram \([3]\) commutes in $C$, then so does \([4]\) in $Q_P$. Consider a commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{(C, \gamma) \ar[rd]^{[\ell]} \ar[d]_{[h]} & (W, \theta) \ar[l]_{[f_p']} \\
(Z, \zeta) \ar[r]^{[f_p]} & (A, \rho) \ar[d]_{[q]}
\end{array}
\end{array}
\end{array}
\]

in $Q_P$. So, in the internal logic of $P$, we have that

(a) $x, x': C \mid \gamma(x, x') \vdash \rho(f_p(h(x)), f_p(h(x')));$ 
(b) $x, x': C \mid \gamma(x, x') \vdash \sigma(f^\sigma(h(x)), f^\sigma(h(x')));$ 
(c) $x, x': C \mid \gamma(x, x') \vdash \theta(\ell(x), \ell(x'));$ 
(d) $x: C \vdash \rho(f_p h(x), q\ell(x)).$

Recall that $[f_p] = [ff^\sigma]$ by Proposition 6.21. So

\[
x: C \vdash \rho(ff^\sigma h(x), q\ell(x)).
\]
Hence weak universality of $c_{\rho,f}: X \to B \times A$ produces a filler in

$$
\begin{array}{c}
C \xrightarrow{(h, \ell)} X \\
\downarrow f_\sigma \downarrow q \downarrow X
\end{array}
\begin{array}{c}
\xrightarrow{c_{\rho,f}} \ B \times A \\
\downarrow f \times \text{id}_A \downarrow \ x \times \text{id}_A
\end{array}
\begin{array}{c}
\xrightarrow{f_\sigma} B \times A \\
\downarrow \text{pr1} \downarrow A \times A
\end{array}
\begin{array}{c}
\xrightarrow{\text{pr1}'} W f_\sigma \times q \times \text{id}_A \\
\downarrow f_\sigma \downarrow \ x \times \text{id}_A
\end{array}
\begin{array}{c}
\xrightarrow{f_\sigma} B \times A \\
\downarrow \text{pr1} \downarrow A \times A
\end{array}
\begin{array}{c}
\xrightarrow{\text{pr1}'} \ \text{pr1} \downarrow A \times A
\end{array}
$$

which shows that $[j]: (C, \gamma) \to (X, \rho \otimes_f \sigma)$ and it is equal to $[h]$. But also that the diagram

$$
\begin{array}{c}
C \xrightarrow{\ell} W \\
\downarrow j \downarrow q \downarrow X
\end{array}
\begin{array}{c}
\xrightarrow{\ f_\rho \ } A
\end{array}
\begin{array}{c}
\xrightarrow{\ f_\rho \ } A
\end{array}
\begin{array}{c}
\xrightarrow{\ f_\rho \ } A
\end{array}
$$

commutes in $C$. Therefore, since $[3]$ is a weak pullback, there is an arrow

$$
\begin{array}{c}
C \xrightarrow{\ell} W \\
\downarrow j \downarrow q \downarrow X
\end{array}
\begin{array}{c}
\xrightarrow{\ f_\rho \ } A
\end{array}
\begin{array}{c}
\xrightarrow{\ f_\rho \ } A
\end{array}
\begin{array}{c}
\xrightarrow{\ f_\rho \ } A
\end{array}
$$

Thanks to the definition of the $P$-equivalence relation $\zeta$, it is immediate to prove that that gives a unique arrow filling in the diagram [5].

6.24 Remark. It is possible to derive a moral from Lemma 6.23. Even though there are only weak pullbacks in $C$, each object in a slice category of $Q_P$ may be replaced by an isomorphic copy on which (strong) pullbacks can be computed as if pullbacks were strong also in $C$.

As obscure as that moral may be, it is going to be employed in the construction of exponentials in each slice category of $Q_P$.

We approach the main theorem of the section by first introducing the explicit construction of an exponential in the slice $Q_P/(A, \rho)$; the following remark presents the first steps of that construction by producing the relevant $P$-equivalence relation to be used then in the proof of Theorem 6.32.

6.25 Remark. Let $P: C^{\text{op}} \to \text{POS}$ be a fixed slice-wise weakly cartesian closed doctrine which admit weak comprehension. Let $[f]: (B, \sigma) \to (A, \rho)$ and $[g]: (C, \tau) \to (A, \rho)$ be two objects in the slice category $Q_P/(A, \rho)$. Consider then arrows $f_\rho: X \to A$ and $g_\rho: Y \to A$ in $C$, as given in 6.19, as well as the corresponding $P$-equivalence relations $\rho \otimes_f \sigma$ on $X$ and $\rho \otimes_g \tau$ on $Y$. Then, in
the slice category \( Q_P / (A, \delta_A) \) take the \( P \)-dependent projective \([f_\rho]: (X, \delta_X) \to (A, \delta_A)\) and the arrow \([g_\rho]: (Y, P_{g_\rho \times g_\rho} (\delta_A) \land P_{g^\tau \times g^\tau} (\tau)) \to (A, \delta_A)\) obtained by pulling back \([g_\rho]: (Y, \rho \otimes g \tau) \to (A, \rho)\) along \([\text{id}_A]: (A, \delta_A) \to (A, \rho)\) as in the following commutative diagram

\[
\begin{array}{ccc}
(Y, P_{g_\rho \times g_\rho} (\delta_A) \land P_{g^\tau \times g^\tau} (\tau)) & \xrightarrow{[\text{id}_Y]} & (Y, \rho \otimes g \tau) \\
\downarrow{[g_\rho]} & & \downarrow{[g_\rho]} \\
(A, \delta_A) & \xrightarrow{[\text{id}_A]} & (A, \rho).
\end{array}
\]

which is a pullback thanks to Lemma 6.23 because

\[
P_{g_\rho \times g_\rho} (\delta_A) \land P_{g^\tau \times g^\tau} (\tau) = P_{g_\rho \times g_\rho} (\delta_A) \land P_{g^\tau \times g^\tau} (\tau).
\]

Consider a weak exponential \([p]: (V, \delta_V) \to (A, \delta_A)\) of \([f_\rho]: (X, \delta_X) \to (A, \delta_A)\) and \([g_\rho]: (Y, P_{g_\rho \times g_\rho} (\delta_A) \land P_{g^\tau \times g^\tau} (\tau)) \to (A, \delta_A)\) which gives, in \( C \), the following arrows

\[
\begin{array}{ccc}
S & \xrightarrow{q_2} & V \\
\downarrow{q_1} & & \downarrow{q} \\
X & \xrightarrow{f_\rho} & A
\end{array} \\
\xRightarrow{ev'}
\]

where the inner square is a weak pullback. For a variable \( v: V \), write \( \xi(v) \) for the formula

\[
\forall s, s' : S \left[ [(q_2(s) = v \land q_2(s') = v) \land \rho \otimes f \sigma(q_1(s), q_1(s'))] \Rightarrow \rho \otimes g \tau(ev'(s), ev'(s)) \right]
\]

—note that the antecedent of the implication yields that the pair \( \langle s, s' \rangle \) is in the \( P \)-equivalence relation imposed on the upper left vertex in the diagram 4 of Lemma 6.23.

Consider the comprehending arrow \([\xi]: W \to V\). Take the weak pullback of \([\xi]\) along \( q_2 \) and paste it with that in diagram 6 to obtain another weak pullback and the composition \( ev = ev' u: V \to Y \), which will eventually be part of the evaluation arrow:

\[
\begin{array}{ccc}
Z & \xrightarrow{p_2} & W \\
\downarrow{p_1} & & \downarrow{p} \\
S & \xrightarrow{q_2} & V \\
\downarrow{q_1} & & \downarrow{q} \\
X & \xrightarrow{f_\rho} & A
\end{array} \\
\xRightarrow{ev} \\
\xRightarrow{ev'}
\]

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The necessary final piece of data is the appropriate $P$-equivalence relation on $W$: consider variables $w, w': W$ and write $\theta(w, w')$ for the formula

$$\rho(p(w), p(w')) \land \neg \forall z, z': Z \left[ \left( p_2(z) = W \land p_2(z') = W \right) \land \rho \otimes f \sigma(q_1(u(z)), q_1(u(z'))) \Rightarrow \rho \otimes g \tau(ev(z), ev(z')) \right]$$

so that $\theta$ is in $\mathcal{P}(W \times W)$—the same comment as for the formula $\xi(v)$ above, applies here with the pair $(z, z')$.

It is easy to show that $\theta$ is a $P$-equivalence relation over $W$ such that

$$z: Z, z': Z \mid \theta(p_2(z), p_2(z')) \land \sigma_f(p_1(z), p_1(z')) \vdash \rho \otimes g \tau(ev(z), ev(z'))$$

Write $\eta(z, z')$ for the equivalence relation $\theta(p_2(z), p_2(z')) \land \sigma_f(p_1(z), p_1(z'))$.

**6.26 Lemma.** Suppose $P$ on $\mathcal{C}$ is slice-wise weakly cartesian closed, then for every $A$ the slice $\mathcal{Q}_P/(A, \delta_A)$ is cartesian closed.

**Proof.** Suppose that $P$ is slice-wise weakly cartesian closed. Let $[f]: (B, \sigma) \rightarrow (A, \delta_A)$ and $[g]: (C, \tau) \rightarrow (A, \delta_A)$ be objects in $\mathcal{Q}_P/(A, \delta_A)$. Here, we align to the notation used in Remark 6.25. Consider $ev: Z \rightarrow Y$ defined as in (7) and $\eta$ as in (9). By definition of $\eta$, the arrow $ev$ determines an arrow

$$[ev]: (Z, \eta) \rightarrow (Y, \delta_A \otimes g \tau)$$

in $\mathcal{Q}_P/(A, \delta_A)$ from $[f_{\delta_A} p_1]$ to $[g_{\delta_A}]$. Moreover, Lemma 6.23 ensures that $(Z, \eta)$ is the pullback of $[p]$ along $[f_{\delta_A}]$.

Thanks to Proposition 6.21, it suffices to show that $[p]$ is the exponential in $\mathcal{Q}_P/(A, \delta_A)$ of $[g_{\delta_A}]$ and $[f_p]$ with evaluation $[ev]: [f_{\delta_A}] \times (A, \delta_A) \rightarrow [g_{\delta_A}]$. Consider an arbitrary object $[h]: (D, \nu) \rightarrow (A, \delta_A)$ in $\mathcal{Q}_P/(A, \delta_A)$, and let $[m]: [f_{\delta_A}] \times (A, \delta_A) \rightarrow [g_{\delta_A}]$. By Lemma 6.25 we can assume

$$\begin{array}{c}
\left(\mathcal{Q}, P_1 \times d_1, (A, \delta_A) \otimes f \sigma \land P_2 \times p_2(\nu)\right) \\
\xrightarrow{[m]} \\
\left(\mathcal{Q}_P/(A, \delta_A), (A, \delta_A) \otimes \tau\right) \\
\xleftarrow{[f_{\delta_A} d_1]} \\
\left(\mathcal{Q}_P/(A, \delta_A), \delta_{\delta_A}\right)
\end{array}$$

depicting an arrow in $\mathcal{Q}_P/(A, \delta_A)$ for an appropriate weak pullback in $\mathcal{C}$

$$\begin{array}{ccc}
Q & \xrightarrow{p_2} & D \\
d_1 & \downarrow & \downarrow h \\
X & \xrightarrow{f_{\delta_A}} & A.
\end{array}$$
Consider the commutative diagram

\[
\begin{array}{c}
(Q, P_{d_1 \times d_1} (\delta \otimes f \sigma) \wedge P_{p_2 \times p_2} (\nu)) \\
\downarrow^{[\text{id}_Q]} \\
(Q, P_{d_1 \times d_1} (\delta X) \wedge P_{p_2 \times p_2} (\delta D)) \\
\downarrow^{[m]} \\
(Y, P_{g \wedge g \delta} (\delta \otimes g \tau) \wedge P_{g \wedge g \tau} (\tau)) \\
\downarrow^{[g \delta \wedge]} \\
(A, \delta) \\
\end{array}
\]

where the square on the right face is a pullback. Since \(P\) is slice-wise weakly cartesian closed,

\[
[m]: (Q, P_{d_1 \times d_1} (\delta X) \wedge P_{p_2 \times p_2} (\delta D)) \rightarrow (Y, P_{g \wedge g \delta} (\delta \otimes g \tau) \wedge P_{g \wedge g \tau} (\tau))
\]
determines a commutative triangle

\[
\begin{array}{c}
D \\
\downarrow^{h} \\
A \\
\downarrow^{q} \\
V \\
\end{array}
\]

in \(C\) where \(V\) is a weak exponential, and a commutative diagram

\[
\begin{array}{c}
J f_{\delta \wedge} \times_{(A, \delta)} J h \\
\downarrow^{[\text{id}_X \times (A, \delta) \ J \widehat{m}]} \\
J f_{\delta \wedge} \times_{(A, \delta)} J q \\
\downarrow^{[\text{ev}']} \\
J f_{\delta \wedge} \times_{(A, \delta)} J [g \delta \wedge]
\end{array}
\]
in \(Q_{/P(A, \delta)}\) where \([g \delta \wedge]: (Y, P_{g \wedge g \delta} (\delta \otimes g \tau) \wedge P_{g \wedge g \tau} (\tau)) \rightarrow (A, \delta)\). By the weak universal property of comprehension, there is an arrow \(\mu: X \rightarrow W\) such that \(\widehat{m} = \mu [\xi]\). Thus the arrow \(\mu\) determines the required arrow \([\mu]: (D, \pi) \rightarrow (W, \theta)\) in \(Q_{/P(A, \delta)}\). Uniqueness is a direct consequence of the definition of \(\theta\). \(\square\)

After 6.26 it remains to prove that every slice of \(Q_P\) is cartesian closed. Proposition 6.26 says that for every \((A, \delta)\) and every \([f]\) in \(Q_{/P(A, \delta)}\) there is a right adjoint to the functor \(\times_{(A, \delta)} [f]: Q_{/P(A, \delta)} \rightarrow Q_{/P(A, \delta)}\). We aim at proving that these right adjoints exist for all \((A, \rho)\).
To this purpose we follow the line of the proof in [Emm20] by relying on the existence of right adjoints as a consequence of one of Barr’s tripos ability theorems in [BW84].

We first need some instrumental propositions.

6.27 Proposition. In a m-variational doctrine $P: C^{op} \to \text{POS}$ with stable effective quotients each quotient arrow is the coequalizer of its kernel pairs and hence it is a regular epimorphism. Moreover $C$ is regular.

Proof. In [MR13b] it is proved that the kernel pairs of each map in $C$ has a coequalizer which is an effective quotient arrow and hence $C$ is regular. □

6.28 Corollary. In a m-variational doctrine $P: C^{op} \to \text{POS}$ with stable effective quotients each coequalizer is a quotient arrow and it is stable under pullback.

Proof. Every coequalizer $e: A \to B$ is isomorphic to $q: A \to A/P_{exe}(\delta_B)$. □

6.29 Proposition. Let $P: C^{op} \to \text{POS}$ be an existential variational doctrine and $P: Q^{op}_P \to \text{POS}$ be its elementary quotient completion. Then, for every quotient arrow $q: (A, \delta_A) \to (A, \rho)$ in $Q_P$ the functor $q^*: Q_P/(A,\rho) \to Q_P/(A,\delta_A)$ is monadic.

Proof. The proof is the same as that for exact categories given in [JM95]. Alternatively, just observe that $Q_P$ is regular and that any quotient map is a regular epimorphism by lemma 6.27. Therefore it is well known that $q^*$ is of descent type in the sense of [BW84].

Moreover, any map $f: (B,\tau) \to (A,\rho)$ is isomorphic over $(A,\rho)$ to $\{P(id,f)(\rho)\}: (X, P_{(pr_1,pr_3)}(\rho) \land P_{(pr_1,pr_3)}(\tau)) \to (A,\rho)$ which comes from a descent datum represented by $pr_1 \cdot \{P(id,f)(\rho)\}: (X,\delta_X) \to (A,\delta_A)$ with the obvious action. □

Theorem 6.29 actually holds even in the more general situation of a generic existential variational doctrine with effective stable quotients with the same proof.

We recall from theorem 3.7.2 in [BW84].

6.30 Proposition. In a situation like the following

\[
\begin{array}{ccc}
B & \xrightarrow{W} & B' \\
\downarrow{U} & & \downarrow{U'} \\
F & \xleftarrow{C} & F'
\end{array}
\]

where $F \dashv U$ and $F' \dashv U'$, if furthermore

(i) $WF$ is naturally isomorphic to $F'$,

(ii) $U$ is monadic, and
(iii) \( W \) preserves co-equalizers of \( U \)-contractible pairs, then \( W \) has a right adjoint.

We apply this theorem to our context as follows.

**6.31 Proposition.** If an \( m \)-variational elementary doctrine \( P: \mathcal{D}^{op} \to \text{POS} \) has stable effective quotients, then for every quotient arrow \( q: A \to A/\rho \), if the slice \( \mathcal{D}/A \) is cartesian closed also the slice \( \mathcal{D}/(A/\rho) \) is cartesian closed.

**Proof.** The cartesian closure of \( \mathcal{D}/A \) ensures that for every \( \alpha \) in \( \mathcal{D}/(A/\rho) \) the functor \( - \times_A q^* \alpha: \mathcal{D}/A \to \mathcal{D}/A \) has a right adjoint \( R_{q^* \alpha} \). Consider the diagram

\[
\begin{array}{ccc}
\mathcal{D}/(A/\rho) & \xrightarrow{- \times (A/\rho) \alpha} & \mathcal{D}/(A/\rho) \\
\downarrow \Sigma_{q^*} & \downarrow R_{q^*} & \downarrow \Sigma_{q^*}(- \times_A q^* \alpha) \\
\mathcal{D}/A & \xrightarrow{q^*} & \mathcal{D}/A
\end{array}
\]

Since \( \Sigma_{q^*} \dashv q^* \) and \( \Sigma_{q^*}(- \times_A q^* \alpha) \dashv R_{q^*} \) (by composition of adjoints), the claim is proved if we show that this situation meets the three conditions in proposition \[.30\]

Condition 1 holds since, for any \( \beta \) in \( \mathcal{C}/A \), Frobenius reciprocity precisely says that \( \Sigma_{q^*} \beta \times (A/\rho) \alpha \) is isomorphic to \( \Sigma_{q^*} \beta \times_A q^* \alpha \).

Condition 2 holds since every quotient arrow \( q: A \to A/\rho \) induces a monadic functor \( q^* \) by \[.29\]

Condition 3 is fulfilled as well, since in \( \mathcal{C}/(A/\rho) \) coequalizers of \( q^* \)-contractible pairs are preserved by \( - \times_A q^* \alpha \) thanks to stability of coequalizers in corollary \[.28\] \( \square \)

Now we are ready to prove the main theorem of the section.

**6.32 Theorem.** Suppose \( P: \mathcal{C}^{op} \to \text{POS} \) is an elementary existential variational doctrine. The following are equivalent:

(i) the base \( \mathcal{C} \) is \( P \)-slice-wise weakly cartesian closed;

(ii) \( Q_P \) is slice-wise exponentiable on dependent projectives;

(iii) \( Q_P \) is locally cartesian closed.

**Proof.** 3. \( \Rightarrow \) 2. Condition (ii) of the definition of slice-wise exponentiable on dependent projectives follows immediately, while condition (i) follows from prop \[.16\]

3. \( \Rightarrow \) 1. follows from lemma \[.18\] and lemma \[.29\]

To show 1. \( \Rightarrow \) 2. first note that \( P \) is universal and implicational by lemma \[.39\] Therefore, we can apply lemma \[.26\] to deduce that every slice \( \mathcal{Q}_P/(A, \delta_A) \) is cartesian closed.

2. \( \Rightarrow \) 3. It follows from the fact that, for every object \( (A, \rho) \) in \( Q_P \) the arrow \( [id_A]: (A, \delta_A) \to (A, \rho) \) is a quotient arrow. The claim follows by prop \[.31\] when \( \mathcal{D} \) there is \( Q_P \). \( \square \)

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Recall that the exact completion of a category with binary products and weak pullbacks is an instance of the elementary quotient completion. More specifically $\Psi_C \simeq \text{Sub}_{\text{ex/lex}}$, which has as corollary $Q\Psi_C \equiv \text{C ex/lex}$.

An important instance of Theorem 6.32 is when the doctrine $P$ is the variational doctrine $\Psi_C$. In this case, we deduce that $C$ is slice-wise weakly cartesian closed if and only if $C_{\text{ex/lex}}$ is locally cartesian. Hence we recover the characterization in [CR00] and [Emm20] supposing $C$ with finite products and weak equalizers.

6.33 Corollary. Suppose $C$ a category with finite products and weak equalizers. Then $C$ is slice-wise weakly cartesian closed with respect to the weak subobject doctrine $\Psi_C$ if and only if $C_{\text{ex/lex}}$ is locally cartesian closed.

Proof. It follows from 6.32 when applied to the doctrine functor of variations $\Psi_C: C^{\text{op}} \rightarrow \text{POS}$ knowing that $C_{\text{ex/lex}}$ is equivalent to $Q\Psi_C$, that $\Psi_C$ is elementary existential, and it admits full weak comprehension and has comprehensive diagonals, since $C$ has finite products and weak pullbacks.

6.34 Remark. The proof of theorem 6.32 can be considered a generalization of Carboni and Rosolini’s characterization of locally cartesian closed exact completions $C_{\text{ex/lex}}$ in [CR00] only in the case the category has finite products. It can be generalized to the case of weak finite products in [Cio22]. Related proofs of locally cartesian closure for the elementary quotient completion of a syntactic category out of specific type theories are in [Pal19] and in [Mai09].

6.2 Finite disjoint coproducts

In this section we establish the necessary and sufficient conditions under which an elementary quotient completion has stable finite coproducts.

We are interested in studying those elementary doctrines with comprehensive diagonals whose elementary quotient completion has coproducts. After 6.2 we know that if $Q_P$ has coproducts then $P: C^{\text{op}} \rightarrow \text{POS}$ must have coproducts.

For the rest of the section, unless specified otherwise, $P$ is a variational f.o.d. on a category $C$ with binary distributive coproducts.

6.35 Proposition. Canonical injections are $P$-injective, i.e. $P_{i_A \times i_A} (\delta_{A+B}) = \delta_A$ and $P_{i_B \times i_B} (\delta_{A+B}) = \delta_B$.

Proof. The idea of the proof is simple when formulated in the internal language of doctrines. Given an element $a: A$ then we can define a projection $p: A + B \rightarrow A$ as follows

$p_z(z) = \begin{cases} w \text{ if } z = i_A(w) \\ a \text{ if } z = i_B(w') \end{cases}$

Hence if $i_A(x) = A+B i_A(y)$ then $x = A p_z(i_A(x)) =_A p_z(i_A(y)) =_A y$.

We, now, give its algebraic version, whose calculations are more involved.
Suppose \( i_A : A \rightarrow A + B \) is a canonical injection and consider the commutative diagram

\[
\begin{array}{ccc}
A \times A & \xrightarrow{pr_2} & A \\
\downarrow{\Delta_A \times id_A} & & \downarrow{id_A \times i_A} \\
(A \times A) \times (A \times B) & \xrightarrow{id_A \times i_A} & (A \times A) \times B
\end{array}
\]

where \( j \) is the isomorphism that comes from distributivity. Denote by \( pr_1, pr_2 \) the projections from \( Z = A \times (A + B) \times A \times (A + B) \). By Beck-Chevalley conditions on equality (see [MR13b]) one has both

\[
\delta_Z = P_{[pr_1, pr_3]}(\delta_A) \land P_{[pr_2, pr_4]}(\delta_{A+B})
\]

\[
\delta_Z \leq P_j \circ j P_{[pr_2, pr_1] \times [pr_2, pr_1]}(\delta_A)
\]

Evaluating both sides on \( P_e \) for

\[
e = (pr_1, i_A pr_1, pr_1, i_A pr_2) : A \times A \rightarrow Z
\]

where \( pr_1, pr_2 : A \times A \rightarrow A \) are projections, we obtain

\[
P_e(\delta_Z) = P_{[pr_1, pr_4]}(\delta_A) \land P_{i_A \times i_A} (\delta_{A+B}) = P_{i_A \times i_A} (\delta_{A+B})
\]

\[
P_e(\delta_Z) \leq P_j \circ j P_{[pr_2, pr_1] \times [pr_2, pr_1]}(\delta_A)
\]

Note that \( e \) can be obtained by the following composition

\[
\begin{array}{ccc}
A \times (A \times A) & \xrightarrow{(id_A, i_A) \times (id_A \times i_A)} & (A \times (A + B)) \times (A \times (A + B)) \\
\downarrow{\Delta_A \times id_A} & & \downarrow{e} \\
(A \times A) \times A & \xleftarrow{(id_A \times i_A) \times (id_A \times i_A)} & A \times A
\end{array}
\]

Therefore \( ([pr_2, pr_1] \times [pr_2, pr_1])(j \times j)e \) is equal to

\[
([pr_2, pr_1]j)(id_A \times i_A) \times ([pr_2, pr_1]j)(id_A \times i_A)(\Delta_A \times id_A)
\]

since \([pr_2, pr_1]j(id_A \times i_A) = pr_2 : A \times A \rightarrow A \) one has

\[
([pr_2, pr_1]j)(id_A \times i_A) \times ([pr_2, pr_1]j)(id_A \times i_A)(\Delta_A \times id_A) = id_{A \times A}
\]

whence \( P_{i_A \times i_A} (\delta_{A+B}) = P_e(\delta_Z) \leq \delta_A \). \( \square \)

We now aim at proving that the equality on a coproduct is given by the formula

\[
\delta_{A+B} = \delta_{i_A \times i_A} (\delta_A) \lor \delta_{i_B \times i_B} (\delta_B)
\]

To this purpose we give the following definition.

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6.36 Definition. Let $P$ be a variational f.o.d. on a category $C$ with binary distributive coproducts. Two arrows $h: A \to Y$ and $k: X \to Y$ are \textit{jointly $P$-surjective} if
\[
\top_Y = \mathcal{I}_h(\top_A) \vee \mathcal{I}_k(\top_X)
\]

6.37 Proposition. Canonical injections are jointly $P$-surjective.

Proof. Suppose $A$ and $B$ are objects of $C$ and consider their coproduct with canonical injections $i_A: A \to A + B$ and $i_B: B \to A + B$. Abbreviate by $k: X \to A + B$ the weak comprehension
\[
\mathcal{I}_{i_A}(\top_A) \vee \mathcal{I}_{i_B}(\top_B): X \to A + B
\]
Clearly
\[
\top_A \leq P_{i_A}(\mathcal{I}_{i_A}(\top_A)) \leq P_{i_A}(\mathcal{I}_{i_A}(\top_A) \vee \mathcal{I}_{i_B}(\top_B))
\]
and analogously for $i_B$. By the universal property of comprehensions there are arrows $t_A: A \to X$ and $t_B: B \to X$ with $kt_A = i_A$ and $kt_B = i_B$. These induce an arrow $[t_A, t_B]: A + B \to X$ which is a section of $k$. Thus $\mathcal{I}_{i_A}([t_A, t_B]) = \text{id}_{A + B}$ factors through $\mathcal{I}_{i_A}(\top_A) \vee \mathcal{I}_{i_B}(\top_B)$. Fullness of comprehensions completes the proof. \hfill \Box

6.38 Proposition. For every pair of $P$-injective arrows $h: A \to Y$ and $k: X \to Y$, for every reflexive relation $\rho$ over $A$ and every reflexive relation $\theta$ over $X$, the following relation over $Y$
\[
\mathcal{I}_{h \times k}(\rho) \vee \mathcal{I}_{k \times k}(\theta)
\]

is reflexive if and only if $h$ and $k$ are jointly surjective.

Proof. Suppose $h$ and $k$ are jointly surjective and consider the relation $\mathcal{I}_{h \times k}(\rho)$. By Beck-Chevalley conditions it is
\[
P_{\Delta_Y}(\mathcal{I}_{h \times k}(\rho)) = \mathcal{I}_h(P_{\Delta_A}(\rho)) = \mathcal{I}_h(\top_A)
\]
Analogously $P_{\Delta_Y}(\mathcal{I}_{k \times k}(\theta)) = \mathcal{I}_k(\top_X)$ whence
\[
P_{\Delta_Y}(\mathcal{I}_{h \times k}(\rho) \vee \mathcal{I}_{k \times k}(\theta)) = \mathcal{I}_h(\top_A) \vee \mathcal{I}_k(\top_X)
\]
Therefore, reflexivity holds, i.e. $\top_Y = P_{\Delta_Y}(\mathcal{I}_{h \times k}(\rho) \vee \mathcal{I}_{k \times k}(\theta))$ if and only if $h$ and $k$ are jointly surjective, i.e. $\top_Y = \mathcal{I}_h(\top_A) \vee \mathcal{I}_k(\top_X)$. \hfill \Box

6.39 Proposition. For every $A$ and $B$ in $C$ it is
\[
\delta_{A + B} = \mathcal{I}_{i_A \times i_A} \delta_A \vee \mathcal{I}_{i_B \times i_B} \delta_B
\]
where $i_A: A \to A + B$ and $i_B: B \to A + B$ are canonical injections.

Proof. From $\delta_A \leq P_{i_A \times i_A}(\delta_{A + B})$ and $\delta_B \leq P_{i_B \times i_B}(\delta_{A + B})$ one obtains a canonical inequality
\[
\mathcal{I}_{i_A \times i_A} \delta_A \vee \mathcal{I}_{i_B \times i_B} \delta_B \leq \delta_{A + B}
\]
This is actually an equality, since injections are $P$-injective \hfill \Box

6.37 \hfill \Box
We leave to the readers the proof of the following instrumental lemma before the main statement of the section.

**6.40 Proposition.** If $\rho$ in $P(A \times A)$ is transitive and if $h: A \rightarrow B$ is $P$-injective, then $\mathcal{I}_{h \times h}(\rho)$ is transitive.

*Proof.* Immediate.

We can finally focus on the elementary quotient completion of a doctrine whose base has distributive binary coproducts. First recall the following definition.

**6.41 Definition.** Suppose $C$ has finite coproducts. Coproducts are said **distributive** if the canonical arrow $(A \times B) + (A \times C) \rightarrow A \times (B + C)$ is an isomorphism.

**6.42 Theorem.** Suppose $P: C^{\text{op}} \rightarrow \text{POS}$ is a variational first-order doctrine. $C$ has distributive binary coproducts if and only if $Q_P$ has distributive binary coproducts.

*Proof.* Suppose $C$ has binary distributive coproducts. Consider $(A, \rho)$ and $(B, \sigma)$ in $Q_P$ and the coproduct $A + B$ with canonical injections $i_A$ and $i_B$. Define

$$
\rho \uplus \sigma = \mathcal{I}_{i_A \times i_A} \rho \lor \mathcal{I}_{i_B \times i_B} \sigma
$$

by lemmas [6.38] and [6.37] the relation $\rho \uplus \sigma$ is reflexive. It is trivially symmetric. Transitivity follows from [6.40] and [6.35]. Thus

$$(A, \rho) \stackrel{[i_A]}{\rightarrow} (A + B, \rho \uplus \sigma) \stackrel{[i_B]}{\rightarrow} (B, \sigma)$$

is a diagram in $Q_P$. We claim that it is a coproduct diagram of $(A, \rho)$ and $(B, \sigma)$. It is immediate to see that it is a weak coproduct. We now prove that it is a strong coproduct. Suppose that $f: A \rightarrow T$ and $g: B \rightarrow T$ represent two arrows $[f]: (A, \rho) \rightarrow (T, \theta)$ and $[g]: (B, \sigma) \rightarrow (T, \theta)$. If $k$ and $l$ represents two arrows $[k], [l]: (A + B, \rho \uplus \sigma) \rightarrow (T, \theta)$ with

$$[k][i_A] = [f] \text{ and } [k][i_B] = [g]$$

$$[l][i_A] = [f] \text{ and } [l][i_B] = [g]$$

i.e. $k$ and $l$ are such that

$$\top_A \leq P_{[k][i_A],f}(\theta) \text{ and } \top_B \leq P_{[k][i_B],g}(\theta)$$

$$\top_A \leq P_{[l][i_A],f}(\theta) \text{ and } \top_B \leq P_{[l][i_B],g}(\theta)$$

then we have also $\rho \leq P_{k \times f}(\theta) \land P_{l \times f}(\theta) \leq P_{i_A \times i_A} P_{k \times l}(\theta)$ and similarly $\sigma \leq P_{i_B \times i_B} P_{k \times l}(\theta)$. Therefore

$$\mathcal{I}_{i_A \times i_A} \rho \leq P_{k \times l}(\theta) \text{ and } \mathcal{I}_{i_B \times i_B} \sigma \leq P_{k \times l}(\theta)$$
from which $\rho \Box \sigma \leq P_{k \times l}(\theta)$, i.e. $[k] = [l]$.

For the converse, consider $A$ and $B$ in $C$. The coproduct in $Q_P$ of $(A, \delta_A)$ and $(B, \delta_B)$ is $(A + B, \delta_A \Box \delta_B)$. By 6.42 this is $(A + B, \delta_{A+B})$. The claim follows by 6.2. \[ \square \]

6.43 Definition. Suppose $P : C^{op} \to \text{POS}$ is a variational first-order doctrine whose base has finite coproducts. For objects $A$ and $B$ in the base, we say that $A + B$ is $P$-disjoint if

$$P_{i_A \times i_B}(\delta_{A+B}) = \bot_{A \times B}$$

6.44 Proposition. $C$ has $P$-disjoint distributive binary coproducts if and only if $Q_P$ has $\hat{P}$-disjoint distributive binary coproducts.

Proof. After 6.42 we only have to prove that coproducts in $C$ are $P$-disjoint if and only if coproduct in $Q_P$ are $\hat{P}$-disjoint. The necessary condition is immediate after 6.39. Consider the coproduct diagram in $Q_P$

$$(A, \rho) \xrightarrow{[i_A]} (A + B, \rho \Box \sigma) \xleftarrow{[i_B]} (B, \sigma)$$

it holds

$$P_{i_A \times i_B}(\rho \Box \sigma) = P_{i_A \times i_B}(\mathcal{I}_{i_A \times i_B}\rho \lor \mathcal{I}_{i_B \times i_B}\sigma)$$

but, denoting by $\text{pr}_i$ the projections from $A \times B \times (A + B) \times (A + B)$, $P_{i_A \times i_B}(\mathcal{I}_{i_A \times i_B}\rho)$ is equal to

$$\mathcal{I}_{(\text{pr}_1, \text{pr}_2)}[P_{(\text{pr}_1, \text{pr}_3)}(P_{i_A \times i_A}(\delta_{A+B})) \land P_{(\text{pr}_2, \text{pr}_4)}(P_{i_A \times i_B}(\delta_{A+B})) \land P_{(\text{pr}_3, \text{pr}_4)}(\rho)]$$

which is equal to $\bot_{A \times B}$ under the assumption that $P_{i_A \times i_B}(\delta_{A+B})$ is so. Analogously $P_{i_A \times i_B}(\mathcal{I}_{i_B \times i_B}\sigma)$ is $\bot_{A \times B}$ whence the claim. \[ \square \]

6.45 Corollary. Suppose $C$ is such that for every $A$ the domain of $\bot_A \bot$ is an initial object. Then $C$ has disjoint distributive binary coproducts if and only if $Q_P$ has disjoint distributive binary coproducts.

6.3 Classifiers

In this section we show how the elementary quotient completion of a suitable doctrine $P$ inherits a predicate classifier for the doctrine of strong monomorphisms which coincides with the doctrine $\hat{P}$.

We first need to establish the following characterisation of epimorphisms as $P$-surjective arrows.

6.46 Lemma. Suppose $P$ is an existential variational doctrine with a strong predicate classifier as defined in 2.18. An arrow $e : X \to A$ is epic if and only if it is $P$-surjective.
Proof. We have already observed that $P$-surjective arrows are epimorphism. So let $e: X \to A$ be epic. Consider the arrows $\chi_{\exists e}(\top_X), \chi_{\top A}: A \to \Omega$. Then

$$P_e(P_{\exists e}(\top_X)(\epsilon)) = P_e(\exists e(\top_X)) = \top_X = P_e(\top_A) = P_e(P_{\chi_{\top A}}(\epsilon))$$

Since the classifier is strong $\chi_{\top A}e = \chi_{\exists e}(\top_X)e$ and then $\chi_{\top A} = \chi_{\exists e}(\top_X)$ as $e$ is epic, whence $\top_A = P_{\chi_{\top A}}(\epsilon) = P_{\chi_{\exists e}(\top_X)}(\epsilon) = \exists e(\top_X)$. \hfill \Box

Recall that given two factorization systems $(E, M)$ and $(E', M')$ on the same category $C$, we have that $E = E'$ if and only if $M = M'$. Thus, the following is an immediate corollary of 6.46 and 2.41.

6.47 Proposition. In every $m$-variational existential doctrine with a strong predicate classifier comprehension arrows are the strong monomorphisms of the base.

Proof. By 2.41 comprehension arrows are the class of arrows which are right orthogonal to $P$-surjective arrows, but these coincide with epimorphisms by 6.46, whence the claim. \hfill \Box

6.48 Proposition. In every $m$-variational existential doctrine with a strong predicate classifier the domain $T$ of $\{ | \in | : T \to \Omega$ is a terminal object.

Proof. Let $1$ be terminal. The arrow $\chi_{\top 1}: 1 \to \Omega$ is such that $P_{\chi_{\top 1}}(\epsilon) = \top_1$. The universal property of $\{ | \in |$ produces an arrow $k: 1 \to T$ with $\{ | \in | k = \chi_{\top 1}$. The universal property of $1$ ensures that $\top k = \id_1$. Moreover from

$$\{ | \in |(\epsilon) = \top T = P_{\top}(\top_1) = P_{\top}P_{\chi_{\top 1}}(\epsilon)$$

we have $\chi_{\top 1}! = \{ | \in |$, so $\{ | \in | k!_T = \{ | \in |$, whence $k!_T = \id_T$ as $\{ | \in |$ is monic. \hfill \Box

6.49 Corollary. If $P: C^{op} \to \text{POS}$ is a $m$-variational existential doctrine, then $P$ has a strong predicate classifier if and only if $C$ has a classifier of strong monomorphism.

Proof. Suppose $\Omega$ is a strong predicate classifier in $P$. After 6.47 it suffices to show that $\Omega$ is a classifier for the class of comprehension arrows. Note that every $\alpha$ in $P(A)$ the following is a pullback

$$\begin{array}{ccc}
X & \xrightarrow{!} & T \\
\{ \alpha \} \downarrow & & \downarrow \{ \in \} \\
A & \xrightarrow{\chi_{\alpha}} & \Omega
\end{array}$$

where $!: X \to T$ is the arrow produced by the universal property of $\{ | \in |$ as $\top_X = P_{\{ \alpha \}}(\alpha) = P_{\{ \alpha \}}P_{\chi_{\alpha}}(\epsilon) = P_{\chi_{\alpha}}(\{ \epsilon \})$. The object $T$ is a terminal if $\{ | \in |$ is monic. If $f: A \to \Omega$ makes the square $\{ | \in | f = \{ \alpha \}$ a pullback, then $\{ \alpha \}$ is isomorphic to $\{ P_f(\epsilon) \}$ by lemma 2.62. By fullness of comprehension $P_f(\epsilon) = \alpha = P_{\chi_{\alpha}}(\epsilon)$, hence $f = \chi_{\alpha}$.
Conversely suppose \( t : 1 \rightarrow \Omega \) is a strong monomorphism classifier. Define \( \varepsilon = \mathcal{I}(\top_1) \). For \( \alpha \) in \( P(A) \) the arrow \( \{\alpha\} : X \rightarrow A \) is a strong monic by 6.47. Thus there is a unique \( \chi_\alpha : A \rightarrow \Omega \) that makes the following diagram a pullback. By 2.31 and 2.30 it is \( P_{\chi_\alpha}(\varepsilon) = P_{\chi_\alpha}\mathcal{I}(\top_1) = \mathcal{I}_{\{\alpha\}} P_\chi(\top_1) = \mathcal{I}_{\{\alpha\}}(\top_X) = \alpha \). Suppose \( f : A \rightarrow \Omega \) is such that \( P_f(\varepsilon) = \alpha \). Since \( t : 1 \rightarrow \Omega \) is a strong monomorphism classifier its pullback along \( f \) classifies a strong monomorphism. By 6.47 this is of the form \( \{\beta\} : X' \rightarrow A \). Moreover, by 2.31 \( \alpha = P_f(\varepsilon) = P_f(\mathcal{I}(\top_1)) = \mathcal{I}_{\{\beta\}}(\top_X) = \beta \). So \( \{\alpha\} = \{\beta\} \) and hence \( \chi_\alpha \) and \( f \) classifies the same strong monomorphisms showing that \( f = \chi_\alpha \). □

6.50 Proposition. Let \( P \) be a variational first-order doctrine on \( C \). The following are equivalent:

(i) \( P \) has a weak predicate classifier;

(ii) \( \hat{P} \) has a strong predicate classifier;

(iii) \( Q_P \) has a classifier for strong monomorphisms.

Proof. 1 \( \iff \) 2. The necessary condition is immediate, while if \( \Omega \) is a predicate weak classifier in \( C \), then \( (\Omega, \lambda) \) is a predicate classifier in \( Q_P \) where

\[
\lambda = P_{\text{pr}_1}(\varepsilon) \leftrightarrow P_{\text{pr}_2}(\varepsilon)
\]

2 \( \iff \) 3. By 3.13 the doctrine \( \hat{P} \) is m-variational. Then apply 6.49 □

7 The quasi-topos construction from a hyper-tripos

In this section we are going to show how the elementary quotient completion of a suitable tripos gives rise to a quasi-topos completion.

We start by recalling the definition of quasi-topos.

7.1 Definition. A quasi-topos is a category \( C \) with the following properties:

(i) it has finite limits;

(ii) it has finite co-limits;

(iii) it is locally cartesian closed;

(iv) there is a classifier for strong monomorphisms.
Then recall from [Wyl91] that every quasi-topos has effective quotients of strong equivalence relations, namely a Sub$_C$-equivalence relation represented by a strong monomorphism. Therefore it makes sense to try to characterise those quasi-toposes which arise as elementary quotient completions.

In the following when we refer to an equivalence relation in a category $C$ we mean a Sub$_C$-equivalence relation, while for a strong equivalence relation we mean a Sub$_C$-equivalence relation represented by a strong monomorphism.

If $C$ is a quasi-topos, then both $Ψ_C$ and Sub$_C$ are first-order doctrines. Another first-order doctrine is the doctrine of strong subobjects of $C$ denoted by $Stg_C^{op} \longrightarrow \text{POS}.$

7.2 Definition. A intensional hyper-tripos is an elementary existential doctrine $P: C^{op} \longrightarrow \text{POS}$ with a weak predicate classifier and full weak comprehensions such that $C$ has weak pullbacks, is slice-wise weakly cartesian closed and has finite distributive coproducts. A hyper-tripos is a intensional hyper-tripos with comprehensive diagonals.

7.3 Proposition. A intensional hyper-tripos is a tripos.

Proof. Suppose $P: C^{op} \longrightarrow \text{POS}$ is a intensional hyper-tripos. Then it has full weak comprehensions. By prop. [3.3] there is a topology $j$ on $Ψ_C$ such that $P$ is $Ψ_Cj.$ Since $C$ has weak pullbacks, is slice-wise weakly cartesian closed and has finite distributive coproducts, the doctrine $Ψ_C$ is first-order, so is $P$ by prop. [3.3]. $C$ is weakly cartesian closed and $P$ has a weak classifier, so $P$ has weak power objects by prop. [2.19].

7.4 Proposition. A hyper-tripos has $P$-disjoint coproducts.

Proof. Take two objects $A, B$ and consider $χ_+: A \rightarrow Ω$ and $χ\bot: B \rightarrow Ω$. Abbreviate with $d: A + B \rightarrow Ω$ the arrow $[χ_+, χ\bot]$. From $δ_{A+B} \leq P_{d×d}(δ_Ω)$ we get $P_{iA×iB}(δ_{A+B}) \leq P_{d_{A×B}}(δ_Ω) = χ_+ × χ\bot(δ_Ω).$ Finally observe that $δ_Ω \leq P_{pr_1}(ε) \leftrightarrow P_{pr_2}(ε)$ from which we deduce $χ_+ × χ\bot(δ_Ω) \leq P_{pr_1}(P_{χ_+}(ε)) \leftrightarrow P_{pr_2}(P_{χ_+}(ε)) = P_{pr_1}(Ω) \leftrightarrow P_{pr_2}(⊥B) = Ω \leftrightarrow ⊥A × B = ⊥A × B.$ One concludes that $P_{iA×iB}(δ_{A+B}) \leq ⊥A × B.$

7.5 Proposition. Let $P: C^{op} \longrightarrow \text{POS}$ be a tripos. For every $A$ and every $ρ$ in $P(A × A)$ there is a $P$-equivalence relation $\overline{ρ}$ over $A$ such that $ρ \leq \overline{ρ}$ and for every $P$-equivalence relation $μ$ over $A,$ if $ρ \leq σ$ then $μ \leq \overline{σ}.$

Proof. We shall employ the language introduced in [2.24]. Take $A$ in $C$ and define the formulas in $U: ⊕A × A$

\[
\begin{align*}
  r(U): &= \forall a:A(a, a) \in A U \\
  s(U): &= \forall a:A\forall a':A[(a, a') \in A U \Rightarrow (a', a) \in A U] \\
  t(U): &= \forall a:A\forall a':A\forall a'':A[((a, a') \in A U \land (a', a'') \in A U) \Rightarrow (a', a'') \in A U] \\
  eq(U): &= r(U) \land s(U) \land t(U)
\end{align*}
\]

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For every formula $\rho$ over $A \times A$ define $\overline{\rho}$ to be the following

\[ a: A, a': A \mid \forall U: P(A \times A) [\text{eq}(U) \land \forall x: A \forall x': A (\rho(x, x') \Rightarrow (x, x') \in A U) \Rightarrow (a, a') \in A U] \]

It is an easy exercise in first-order logic to check that $\overline{\rho}$ is a $P$-equivalence relation over $A \times A$. Then, take any $\mu$ in $P(A \times A)$ and consider $\chi_{\mu}: 1 \rightarrow P(A \times A)$. Recall that $\chi_{\mu}$ has the property that $(a, a') \in A \chi_{\mu} \models \mu(a, a')$, so in $a: A, a': A$ it holds

\[ \overline{\rho}(a, a') \vdash \text{eq}(\chi_{\mu}) \land \forall x: A \forall x': A (\rho(x, x') \Rightarrow \mu(x, x')) \Rightarrow \mu(a, a') \]

If $\mu$ is a $P$-equivalence relation over $A$ then $\text{eq}(\chi_{\mu})$ is a true sentence. If moreover $\rho \leq \mu$ also $\forall x: A \forall x': A (\rho(x, x') \Rightarrow \mu(x, x'))$ is a true sentence. Whence the sequent above reduces to $\overline{\rho}(a, a') \vdash \mu(a, a')$, which proves the claim. \hfill \Box

7.6 Lemma. The base of a tripos with effective quotients and comprehensive diagonals has coequalizers.

Proof. Let $P: \text{C}^{\text{op}} \rightarrow \text{POS}$ be a tripos with quotients. We shall employ the language introduced in 2.24. Take two arrows $f, g: Y \rightarrow A$ in $C$ and define $\rho$ in $P(A \times A)$ to be

\[ a: A, a': A \mid \exists_{\mu Y} (f(y) =_A a \land g(y) =_A a') \]

By 2.33 there is the smallest $P$-equivalence relation $\overline{\rho}$ over $A$ that contains $\rho$. Consider the quotient $q: A \rightarrow A/\overline{\rho}$. It is clear that

\[ y': Y \mid \exists_{\mu Y} f(y) = A f(y') \land g(y) = A g(y) \]

is a true formula, that is to say $\top_Y \leq P_{(f, g)}(\rho)$. So also $\top_Y \leq P_{(f, g)}(\overline{\rho})$ and by effectiveness of quotient $\top_Y \leq P_{(qf, qg)}(\delta_{A/\overline{\rho}})$. So $qf = qg$ as $P$ has comprehensive diagonals.

Suppose now $k: A \rightarrow Z$ is such that $kf = kg$, then in $y: Y, a: A, a': A$ it holds

\[ f(y) =_A a \land g(y) = A a' \vdash kf(y) =_A k(a) \land kg(y) = A k(a') \vdash k(a) =_A k(a') \]

Therefore $\exists_{\mu Y} (f(y) =_A a \land g(y) =_A a' \vdash k(a) =_A k(a')$. That is to say that $\rho \leq P_{k \times k}(\delta_A)$. By 2.33 also $\overline{\rho} \leq P_{k \times k}(\delta_A)$. By the universal property of quotients there is $h: A/\overline{\rho} \rightarrow Z$ with $hq = k$. \hfill \Box

7.7 Definition. Let $C$ be a quasi-topos. We denote with $\text{Stg}_C: \text{C}^{\text{op}} \rightarrow \text{POS}$ the doctrine of strong subobjects of $C$, namely equivalence classes up to isomorphisms of those monic orthogonal to epimorphisms in $C$.

From [Wyl91] we can easily deduce that:

7.8 Lemma. The doctrine of strong subobjects $\text{Stg}_C: \text{C}^{\text{op}} \rightarrow \text{POS}$ of a quasi-topos $C$ is a tripos.

7.9 Proposition. Let be $C$ a quasi-topos. The following are equivalent:

\[ 51 \]
(i) the quasi-topos $C$ is a topos;

(ii) the doctrine of strong subobjects $Stg_C$ satisfies (RUC);

(iii) the doctrine strong subobjects $Stg_C$ is the subobject doctrine $Sub_C$.

Proof. A quasi-topos $C$ is a topos if and only if it is balanced. By prop. 2.33 a monic arrow in $C$ is $Stg_C$-injective and by 6.40 epimorphisms are $Stg_C$-surjective. Then, the equivalences follow by prop. 2.36.

7.10 Theorem. A doctrine $P: C^{op} \to \text{POS}$ is a hyper-tripos if and only if $Q_P$ is a quasi-topos and $\hat{P}$ is the doctrine of strong subobjects.

Proof. In the def. 7.1 of quasi-topos, point (i) comes from prop. 3.14, point (ii) from prop. 6.32 while point (iv) comes from prop. 6.50. Finite coproducts come directly from prop. 6.44 and the existence of co-equalizers from lemma 7.6. In particular the initial object in $Q_P$ is the initial object of $C$ with the total relation. By prop. 3.13 the doctrine $\hat{P}: Q_P^{op} \to \text{POS}$ is m-variational. Hence $\hat{P}$ is the doctrine of strong monomorphisms of $Q_P$ by prop. 6.47.

For the converse, if $Q_P$ is a quasi-topos, by lemma 7.8 $Stg_C$ is a tripos and it coincides with $\hat{P}$ by prop. 6.47. So by prop 6.42 and theorem 6.32 $P$ is first-order on a slice-wise weakly cartesian closed category with a weak predicated classifier by prop. 6.50. Whence $P$ is a tripos. Coproducts follows from prop. 6.44.

7.11 Corollary. If $P$ is a intensional hyper-tripos then $Q_P$ is a quasi-topos and also $P_x$ is a hyper-tripos.

Proof. $Q_P$ is a quasi-topos with the same proof in 7.10 as comprehensive diagonals in $P$ play no role in the proof. Moreover, from theorem 2.61 we know that $\hat{P}$ is equivalent to $\hat{P}_x$, and hence we conclude that $P_x$ is hyper-tripos and hence from theorem 7.10.

As a corollary we also get Menni’s characterization of toposes as exact completions in [Men03] as follows.

7.12 Corollary. For a finite product category $C$ with weak pullbacks, $C$ is slice-wise cartesian closed and has a weak proof classifier if and only if $C_{ex/lex}$ is a topos.

Proof. It follows by theorem 7.11 and prop 7.9 when $P = \Psi_C$ after recalling that $\Psi_C$ is a hyper-tripos precisely when $C$ is slice-wise cartesian closed and has a weak proof classifier as remarked in ex. (d) of 2.22 and that $C_{ex/lex}$ is equivalent to $Q\Psi_C$ as remarked in ex. (d) of 2.54.

Denote by $T_P$ the topos that comes from the tripos $P$ under the tripos to topos construction. If $P: C^{op} \to \text{POS}$ is a hyper-tripos (intensional hyper-tripos), then $T_P \equiv \mathcal{EF}\hat{P}$ (this is theorem 3.5 in [MPR17]). Thus for every hyper-tripos $P: C^{op} \to \text{POS}$ the topos $T_P$ is the topos of coarse objects of $Q_P$. 52
Note that from the proof of theorem 7.10 that comprehensive diagonals are not necessary to get a quasi-topos out of the elementary quotient completion.

Theorem 7.10 can be extended to produce arithmetic quasi-toposes.

7.13 Definition. A quasi-topos is arithmetic if it has a natural number object, and hence a parameterized natural numbers object.

7.14 Proposition. P is an intensional hyper-tripos with a natural numbers object if and only if Q_P is an arithmetic quasi-topos.

Proof. From theorem 7.10 and lemma 3.5. of [MPR19] stating we know that if P has a parameterized natural number objects if and only if Q_P has a parameterized natural number object.

Recall from [Joh02] that

7.15 Definition. An object A in a category C is coarse if for every morphism f: C \rightarrow B which is both monic and epic and every g: C \rightarrow A there is a unique t: B \rightarrow A such that g = tf.

7.16 Proposition. Every quasi-topos C contains a full reflective subcategory Crs_C which is a topos.

Proof. The topos Crs_C is reflective and the reflector is build as follows. For any object A note that the diagonal δ_A in Stg_C(A \times A) is represented by the diagonal (being the diagonal a strong monomorphism). Its classifying arrow χ_δ_A factors as a strong monic followed by an epimorphism as 

\[
\begin{array}{ccc}
A & \xrightarrow{\chi_\delta_A} & PA \\
\downarrow{\eta_A} & & \downarrow{\exists \chi_\delta_A \top_A} \\
S_A & \xrightarrow{} & \\
\end{array}
\]

The object S_A will be called object of A-singletons and we call η_A: A \rightarrow S_A the singleton arrow of A. Note that χ_δ_A is monic, whence the singleton arrow of A is both epic and monic. Strong monic are strong comprehension arrows in Stg_C so it is easy to see that every arrow f: A \rightarrow B determines a unique arrow S_f: S_A \rightarrow S_B with S_f i_A = i_B f. This determines a functor S: Crs_C \rightarrow C where Crs_C is the full subcategory of C on coarse objects.

As shown in [Joh02] prop 2.6.12] the functor S is left adjoint to the inclusion of Crs_C into C with singleton arrows as unite. So in particular an object A is coarse if and only if it is isomorphic to its own singletons, i.e. if A ∼ S_A.

Recall from remark 2.60 that the construction that maps an existential elementary doctrine P: C^{op} \rightarrow POS to the existential elementary doctrine P_P: E_{F^{op}} \rightarrow POS satisfying (RUC).

7.17 Proposition. Let C be a quasi-topos. The doctrine of strong subobjects Stg_C: C^{op} \rightarrow POS along the inclusion of the topos of coarse objects Crs_C is equivalent to Sub_C: E_{F^{op}}_{Stg_C} \rightarrow POS.
Proof. Any arrow \( m \) in \( C \) can be written as the composite \( m = se \) where \( s \) is strong monic and \( e \) is epic. If \( m \) is monic then \( e \) is monic too. So if \( m \) is monic in \( \text{Crs}_C \) then \( e \) is an isomorphism. So every monic in \( \text{Crs}_C \) along the inclusion of \( \text{Crs}_C \) into \( C \) is \( \text{Sub}_{\text{Crs}}: \text{Crs}_C \rightarrow \text{POS} \). So it suffices to show that \( \text{Crs}_C \) is equivalent to \( \mathcal{E}(Stg)_C \). We want to find a functor \( S' : \mathcal{E}(Stg)_Q \rightarrow \text{Crs}_Q \) naturally inverse to the composition

\[
\text{Crs}_C \xrightarrow{\mathcal{E} \text{Stg}_C} C \xrightarrow{\Gamma \text{Stg}_C} \mathcal{E}(\text{Stg})_C
\]

where \( \Gamma \text{Stg}_C \) acts as the identity on objects and maps \( f : A \rightarrow B \) to the formula \( a : A, b : B \mid f(a) =_B b \) in \( \text{Stg}_C(A \times B) \). On the object we define \( S' \) as the action of the reflector \( S : Q \rightarrow \text{Crs}_Q \) in lemma 7.17 i.e for any object \( A \) we put \( S' A = SA \). To define the inverse on morphisms take a total and single-valued relation \( F \in \text{Stg}_Q(A \times B) \), this is a strong monic \( F = \langle F_1, F_2 \rangle : X \rightarrow A \times B \) in \( Q_P \) where \( F_1 : X \rightarrow A \) is monic and epic. Thus \( SF_1 : SX \rightarrow SA \) is an isomorphism. Finally define \( S' F = SF_2(SF_1)^{-1} \).

We can characterize when an hyper-tripos leads to an elementary quotient completion which is a topos.

7.18 Theorem. Let \( P : C^{op} \rightarrow \text{POS} \) be a hyper-tripos. Then the following are equivalent.

(i) \( P \) satisfies (RC).

(ii) \( Q_P \) is a topos and coincides with the exact completion of the base of \( P \) as a finite product category.

Proof. By prop. 2.37 \( P \) is \( \Psi_C \) if and only if (1) holds. Hence the equivalence follows by 7.12.

Therefore, examples of toposes arising in this way are exactly those obtained as exact completions of a category \( C \) as in \([\text{Men03}]\) by taking \( \Psi_C \) for \( P \).

Furthermore, we characterize those elementary quotient completions which arise as tripos-to-topos constructions originally introduced in \([\text{HJP80}]\).

To this purpose, recall from \([\text{MPR17}]\) the following definition.

7.19 Definition. We say that an elementary existential doctrine \( P : C^{op} \rightarrow \text{POS} \) is equipped with \( \epsilon \)-operators if for any object \( A \) in \( C \) and any \( \alpha \) in \( P(A \times B) \) there exists an arrow \( \epsilon_\alpha : A \rightarrow B \) such that

\[
\exists \alpha_{pr_1} = P_{(id_A,\epsilon_\alpha)}(\alpha)
\]

holds in \( P(A) \), where \( pr_1 : A \times B \rightarrow A \) is the first projection.

7.20 Definition. Given a tripos \( P \), let us denote with \( \tau_P \) the tripos-to-topos construction the category \( \mathcal{T}_P \) consists of

objects: pairs \( \langle A, \rho \rangle \) such that \( \rho \) is in \( P(A \times A) \) and satisfies symmetry and transitivity as in (ii) and (iii) of 2.42

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arrows: an arrow $\phi: \langle A, \rho \rangle \to \langle B, \sigma \rangle$ is an object $\phi$ in $P(A \times B)$ such that

(i) $\phi \leq P_{(pr_1,pr_1)}(\rho) \wedge P_{(pr_2,pr_2)}(\sigma)$;

(ii) $P_{(pr_1,pr_2)}(\rho) \wedge P_{(pr_2,pr_2)}(\phi) \leq P_{(pr_1,pr_2)}(\phi)$ in $P(A \times A \times B)$
where the $pr_i$’s are the projections from $A \times A \times B$;

(iii) $P_{(pr_1,pr_2)}(\phi) \wedge P_{(pr_2,pr_2)}(\sigma) \leq P_{(pr_1,pr_2)}(\phi)$ in $P(A \times B \times B)$
where the $pr_i$’s are as in (ii);

(iv) $P_{(pr_1,pr_2)}(\phi) \wedge P_{(pr_1,pr_3)}(\phi) \leq P_{(pr_2,pr_3)}(\sigma)$ in $P(A \times B \times B)$
where the $pr_i$’s are as in (iii);

(v) $P_{(id_A,id_A)}(\rho) \leq P_{pr_1}(\phi)$ in $P(A)$
where the $pr_i$’s are the projections from $A \times B$.

7.21 Theorem. Let $P: C^{op} \to \text{POS}$ be a tripos. Then the following are equivalent.

(i) $P$ is equipped with $\epsilon$-operators.

(ii) $P_c$ satisfies (RC).

(iii) $Q_{P_c}$ is a topos and coincides with the tripos-to-topos construction $\tau_P$ of the tripos $P$.

Proof. (i) and (ii) are equivalent by theorem 5.15 in [MPR17] where the category of predicates $\text{Prd}_P$ denotes $X_P$. To show that (ii) implies (iii) observe that from theorem 5.5 in [MPR17] we know that $P_c$ satisfies the rule of choice iff $P_{c_X}$ satisfies the rule of choice. By theorem 7.18 we also know that $P_{c_X}$ satisfies the rule of choice iff $Q_{P_{c_X}}$ is a topos and coincides with the exact completion of the base of $P_{c_X}$. Moreover $Q_{P_{c_X}}$ is the exact completion of the base of $P_{c_X}$ iff $Q_{P_{c_X}}$ is equivalent to $\tau_P$ by corollary 6.3 in [MPR17]. From [MR13a] we know that $Q_{P_{c_X}}$ is equivalent to $Q_{P_c}$ and this concludes the proof.

7.22 Theorem. Suppose $P: C^{op} \to \text{POS}$ is a intensional hyper-tripos. Its tripos-to-topos construction $\mathcal{I}_P$ is a reflective subcategory of the quasi-topos $Q_P$ and coincides with the category of coarse objects of $Q_P$.

$$Q_P \begin{array}{c} \bot \end{array} \mathcal{I}_P$$

Proof. It follows by theorem 7.17, theorem 7.10 and theorem 4.7 of [MPR17].

8 Applications

Suppose $P: C^{op} \to \text{POS}$ is a intensional hyper-tripos. The category $Q_P$ is a quasi-topos by 7.11. Its reflective subcategory on coarse objects $\text{Crs}_{Q_P}$ is the topos $\mathcal{I}_P$ obtained from $P$ via the tripos-to-topos construction by theorem 7.22. Moreover, by prop. 3.5 and prop. 3.4, we know that the category $Q_P$ is also
a full reflective subcategory of \( Q \), and that \( Q \) is equivalent to \( C_{\text{ex/lex}} \) as summarized in this picture

\[
\begin{array}{c}
Q \Psi C \\
\downarrow \quad \downarrow \\
Q P \quad TP \\
\end{array}
\]

(11)

By prop. 3.5 the intensional hyper-tripos \( P \) generates a topology \( j_P \) on \( \Psi C \) whose extension \( \widehat{j_P} \) is a topology on \( \text{Sub}_{\text{ex/lex}} \). Moreover, \( Q \) is equivalent to \( \text{Sep}(\widehat{j_P}) \) by prop. 3.16. So picture (11) can be equivalently described as follows:

\[
\begin{array}{c}
Q \Psi C \\
\downarrow \quad \downarrow \\
Q P \quad \text{Sep}(\widehat{j_P}) \\
\downarrow \quad \downarrow \\
C_{\text{ex/lex}} \quad \text{Crs}_{Q_P}
\end{array}
\]

We now instantiate this picture on different choices of \( P \).

Recall triposes of the form \( P_H \) as 2.22-(b). As observed in prop. 3.5 these triposes have comprehensions but they are not full. We can then consider the completion \( P_H \to \text{POS} \) described in 3.5. Note that \( \text{Set}_c \) is Goguen’s category \( \text{Fuz}(H) \) of \( H \)-valued fuzzy sets [Gog74] for a local \( H \). Whence \( \text{Set}_c \) is a quasitopos and therefore it is slice-wise cartesian closed with finite coproducts. It can be equivalently described as the category \( H^+ \) obtained by freely adding coproducts to \( H \) [Men03]. The ex/lex completion of \( H_a \) is the topos \( \text{PreShv}(H) \) of presheaves of \( H \), while the tripos-to-topos construction applied to \( P \) gives the topos \( \text{Shv}(H) \) of sheaves over \( H \). Thus, when \( P \) is \( P_H \), picture (11) becomes

\[
\begin{array}{c}
\text{PreShv}(H) \\
\downarrow \quad \downarrow \\
\text{Sep}(\widehat{j_P}_{H'}) \\
\downarrow \quad \downarrow \\
\text{Shv}(H)
\end{array}
\]

The change of base of triposes of the form \( P_H \) along the forgetful functor \( \text{Top} \to \text{Set} \) is again a tripos as it suffices to endow \( H^A \) with the indiscrete topology to have the power objects (see [Pas18]). In the special case of \( H = \{0,1\} \), the tripos \( P_H \) reduces to the contravariant powerset functor \( P : \text{Set}^{op} \to \text{POS} \) and its change of base along \( \text{Top} \to \text{Set} \) is a tripos that we call \( T \). The tripos \( T \) has full strong comprehensions given by subspace topologies. Top has coproducts and is slice-wise weakly cartesian closed [CR00]. So \( T : \text{Top}^{op} \to \text{POS} \) is a hyper-tripos. The category of generalised equilogical spaces \( \text{Gequ} \) is equivalent to the base \( Q_P \) of the elementary quotient completion of \( P \). Since \( T \) is boolean the topology \( j_T \) is the double negation topology of example 3.7, whence also \( \widehat{j_T} \). So picture (11) becomes.

\[
\begin{array}{c}
\text{Top}_{\text{ex/lex}} \\
\downarrow \quad \downarrow \\
\text{Gequ} \quad \text{Set}
\end{array}
\]

As a byproduct we have that \( \text{Gequ} \) is the category of \( \neg\neg \)-separated objects of \( \text{Top}_{\text{ex/lex}} \) as shown in [Ros00].

The category \( \text{Asm} \) of assemblies has as objects are pair \((A, \alpha)\) where \( A \) is a set \( \alpha : A \to \mathbb{P}N \) is a function from \( A \) to non-empty subsets of natural numbers. An arrow \( f : (A, \alpha) \to (B, \beta) \) is a function \( f : A \to B \) such that there is \( n \in \mathbb{N} \) such that for all \( a \in A \) and all \( p \in \alpha(a) \) the application \( n.p \) is defined and it belongs
to $\beta(f(a))$. The category $\text{Pasm}$ of partitioned assemblies is the full subcategory of $\text{Asm}$ on those $(A, \alpha)$ such that each $\alpha(a)$ is a singleton, i.e. $\alpha$ can be seen as a function from $A$ to $\mathbb{N}$. The change of base of $\mathcal{P}$ along the forgetful functor $\text{Pasm} \to \text{Set}$ is a tripos as it suffice to chose has weak power object of $(A, \alpha)$ the partitioned assembly $(\mathcal{P}A, k_0)$ where $k_0$ is the constant map to 0. We call such a tripos $\mathcal{R} : \text{Pasm}^{op} \to \text{POS}$. It is easy to see that $\mathcal{R}$ has full strong comprehensions where for a partitioned assembly $(A, \alpha)$ and a subset $X \subseteq A$ the inclusion of of $X$ into $A$ determines a morphism of partitioned assembly $\{X\} : (X, \alpha|_X) \to (A, \alpha)$ which is the desired comprehension arrow. Since $\text{Pasm}$ is slice-wise cartesian closed, the tripos $\mathcal{R}$ is a hyper-tripos. Whence $\mathcal{Q}_\mathcal{R}$ is a quasi-topos by [10]. Recall that $\text{Pasm}_{ex/lex}$ is $\text{Eff}$. And the quasitopos of $\neg\neg$-separated objects of $\text{Eff}$ is $\text{Asm}$. So picture [10] becomes

$$
\begin{array}{ccc}
\text{Eff} & \hookrightarrow & \text{Asm} \\
\downarrow \vphantom{\text{Eff}} & & \downarrow \vphantom{\text{Asm}} \\
\downarrow & & \downarrow \\
\text{Set} & & \text{Set}
\end{array}
$$

As byproduct we have that $\mathcal{Q}_\mathcal{R}$ is equivalent to $\text{Asm}$. A different proof of this is given in [MPR19].

Another remarkable example is in type theory with the construction of the so called setoid models over Coquand-Huet’s Calculus of Inductive Constructions CoC [Coq90]. The setoid model of functional relations over CoC is the tripos-to-topos construction $T_{\text{FCoC}}$ with $\text{FCoC}$ the doctrine of propositions over CoC mentioned in [MR13b]. The topos $T_{\text{FCoC}}$ coincides with the topos of coarse objects within the quasi-topos $\mathcal{Q}_{\text{FCoC}}$ which was one of the inspiring examples to the introduction of the elementary quotient completion in [MR13b].

9 Conclusions

We have introduced the notion of quasi-topos construction of an hypertripos by employing the machinery of the elementary quotient completion introduced in [MR13b], [MR13a].

In doing so we have generalized three theorems regarding exact completions by adopting the approach of elementary quotient completions: Carboni-Vitali’s characterization of exact completions in terms of projectives in [CV98], Carboni-Rosolini’s characterization of locally cartesian closed exact completions of a category with finite products and weak pullbacks in [CR00], and Menni’s characterization of topoi as exact completions in [Men03]. These relevant examples of elementary quotient completions which are not exact completions like the category of assemblies of realizability topos.

In the future we intend to generalize the quasitopos construction to include examples like the syntactic models obtained from predicative theories such as the extensional level of the Minimalist Foundation in [Mai09].

References

[BBS04] A. Bauer, L. Birkedal, and D.S. Scott. Equilogical spaces. Theoret. Comput. Sci., 315(1):35–59, 2004.
[BW84] M. Barr and C. Wells. *Toposes Triples and Theories*. Springer-Verlag, 1984.

[Car95] A. Carboni. Some free constructions in realizability and proof theory. *J. Pure Appl. Algebra*, 103:117–148, 1995.

[CC82] A. Carboni and R. Celia Magno. The free exact category on a left exact one. *J. Aust. Math. Soc.*, 33(A):295–301, 1982.

[Cio22] C.Jr. Cioffo. *Homotopy setoids and generalized quotient completions*. Phd thesis, University of Milan, 2022.

[Cio23] C.Jr. Cioffo. Biased elementary doctrines and quotient completions, 2023.

[Coq90] T. Coquand. Metamathematical investigation of a calculus of constructions. In P. Odifreddi, editor, *Logic in Computer Science*, pages 91–122. Academic Press, 1990.

[CR00] A. Carboni and G. Rosolini. Locally Cartesian closed exact completions. *J. Pure Appl. Algebra*, 154(1-3):103–116, 2000.

[CV98] A. Carboni and E.M. Vitale. Regular and exact completions. *J. Pure Appl. Algebra*, 125:79–117, 1998.

[Emm20] J. Emmenegger. On the local cartesian closure of exact completions. *J. Pure Appl. Algebra*, 224(11):106414, 25, 2020.

[EPR20] J. Emmenegger, F. Pasquali, and G. Rosolini. Elementary doctrines as coalgebras. *J. Pure Appl. Algebra*, 224(12):106445, 16, 2020.

[FK72] P.J. Freyd and G.M. Kelly. Categories of continuous functors I. *J. Pure Appl. Algebra*, 2:169–191, 1972.

[Fre15] J. Frey. Tripods, q-toposes and toposes. *Ann. Pure Appl. Logic*, 166(2):232 – 259, 2015.

[FS90] P.J. Freyd and A. Scedrov. *Categories, Allegories*. ISSN. Elsevler Science, 1990.

[Gog74] J. A. Goguen. Concept representation in natural and artificial languages: Axioms, extensions and applications for fuzzy sets. *International Journal of Man-Machine Studies*, 6(5):513 – 561, 1974.

[Gra00] M. Grandis. Weak subobjects and the epi-monic completion of a category. *J. Pure Appl. Algebra*, 154(1-3):193–212, 2000.

[HJ03] J. Hughes and B. Jacobs. Factorization systems and fibrations: Toward a fibred Birkhoff variety theorem. *Electron. Notes Theor. Comput. Sci.*, 69:156–182, 2003.
[MR13b] M.E. Maietti and G. Rosolini. Quotient completion for the foundation of constructive mathematics. *Log. Univers.*, 7(3):371–402, 2013.

[MR16] M.E. Maietti and G. Rosolini. Relating quotient completions via categorical logic. In Dieter Probst and Peter Schuster (eds.), editors, *Concepts of Proof in Mathematics, Philosophy, and Computer Science*, pages 229–250. De Gruyter, 2016.

[MT23] M.E. Maietti and D. Trotta. A characterization of generalized existential completions. *Annals of Pure and Applied Logic*, 174(4):103234, 2023.

[Pal19] E. Palmgren. From type theory to setoids and back. https://arxiv.org/abs/1804.08585, 2019.

[Pas15] F. Pasquali. A co-free construction for elementary doctrines. *Appl. Categ. Structures*, 23(1):29–41, Feb 2015.

[Pas16] F. Pasquali. Remarks on the tripos to topos construction: Comprehension, extensionality, quotients and functional-completeness. *Appl. Categ. Structures*, 24(2):105–119, Apr 2016.

[Pas18] F. Pasquali. On a generalization of equilogical spaces. *Logica Universalis*, 12(1):129–140, May 2018.

[Pit02] A.M. Pitts. Tripos theory in retrospect. *Math. Structures Comput. Sci.*, 12(3):265–279, 2002.

[Ros00] G. Rosolini. Equilogical spaces and filter spaces. *Rend. Circ. Mat. Palermo (2) Suppl.*, 64:157–175, 2000. Categorical studies in Italy (Perugia, 1997).

[Sco76] D.S. Scott. Data types as lattices. *SIAM J. Comput.*, 5(3):522–587, 1976.

[Sco96] D.S. Scott. A new category? Domains, spaces and equivalence relations. Available at http://www.cs.cmu.edu/Groups/LTC/, 1996.

[vO08] J. van Oosten. *Realizability: An Introduction to its Categorical Side*, volume 152 of *Studies in Logic and the Foundations of Mathematics*. North Holland, 2008.

[Wy191] O. Wyler. *Lecture notes on Topoi and Quasitopoi*. World Scientific, 1991.