Geometrical Bell inequalities for arbitrarily many qudits with different outcome strategies

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Received 5 June 2015, revised 25 September 2015
Accepted for publication 30 November 2015
Published 18 December 2015

Abstract

Greenberger–Horne–Zeilinger (GHZ) states are intuitively known to be the most nonclassical ones. They lead to the most radically nonclassical behavior of three or more entangled quantum subsystems. In the case of two-dimensional systems, it has been shown that GHZ states lead to an exponentially higher robustness of Bell nonclassicality against the white noise for geometrical inequalities than in the case of Weinfurter–Werner–Wolf–Zukowski–Brukner ones. We introduce geometrical Bell inequalities for collections of arbitrarily many systems of any dimensionality. We show that the violation factor of these inequalities grows exponentially with the number of parties and study their behavior in terms of dimensionality of subsystems and number of local measurements. We also investigate various strategies of assigning mathematical objects to events in the experiment, each leading to different violation ratios.

Keywords: Bell inequalities, nonclassicality, geometric Bell inequalities

(Some figures may appear in colour only in the online journal)

1. Introduction

The potency of various states to violate Bell inequalities (BIs) [1], apart from its fundamental consequences, distinguishes them as forms of a resource directly usable in quantum information processing. Not only can the violation ensure the security of a scheme of a cryptographic key generation [2], but it can also provide communication advantage in distributed computing [3], or

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increase security of secret sharing protocols [4]. It is hence an interesting and important question to investigate, in which situations such a violation can occur.

Various schemes of generating BIs for collections of qubits have been found (e.g. [5-9]). For larger subsystems, still very little is known about falsifying local hidden variable models in general. The most profound versions of the theorem are CGMLP inequalities [10] and their chained generalizations [11]. From the experience with qubits we know that the violation ratio of BIs can grow with the number of parties involved in the experiment, thus we expect it to be the same for qudits. We would also like to believe that the higher dimensionality can lead to stronger nonclassical effects. Similarly, one may also check if the contrast between quantum mechanics and local realistic (LR) models is more radical under a closer inspection of the system, i.e. with more measurement settings to choose from by each observer. Results for qubits lead to various conclusions [12].

In this work, we present a new Bell scheme in which these questions can be at least partially answered. Specifically, we formulate geometrical BIs for any number of subsystems \( N \geq 2 \), any number of local measurement settings \( L \geq 2 \) and any dimensionality of each local subsystem \( d \geq 2 \). Geometrical BIs have been introduced in [13]. They are based on an approach in which the correlation function is a vector with components described by measurement settings. A scalar product of the quantum correlation function with itself is compared with the product with all LR models. In the original version, they have utilized all possible measurements lying in a given plane. This approach led to an observation that among all correlation-based inequalities, they provide the strongest robustness of the violation against white noise. Subsequently, they were generalized to the case of a finite number of measurements.

Note that geometrical BIs were already formulated for qutrits in [14]. The estimates have shown that, indeed, the large number of parties provides a stronger violation. These inequalities, however, do not fit our scheme.

The elasticity of the model also gives us an opportunity to check what violation ratios are observed under various treatments of outcomes. We will compare three different strategies. In the first we will treat each local measurement as a dichotomic one. It will be said to yield one value if a single specific local outcome has occurred. Second, we will consider the case in which outcomes of local measurements are assigned integers, which are sent to a referee. The referee sums the results modulo the dimension of each subsystem. The third treatment is to associate local outcomes with complex root of unity and compute the correlation function by their multiplication. We are thus able to see the performance of considering each probability, a specific type of correlations, or a commonly used straight-forward generalization of Pauli matrices as a unitary operators based approach in the same Bell scheme (e.g. [15, 16]). This will be seen as a hint for future constructions of optimal BIs. In this way we want to emphasize various degrees of ignorance introduced in constructing the correlation function.

2. Geometric approach

Consider a real vector \( \vec{V} \) and a set of real vectors \( S \). Any element of \( S \) will be denoted as \( \vec{S} \). If the norm of \( \vec{V} \) exceeds the scalar product with all elements of \( S \), then it cannot belong to this set, i.e. \( \vec{V} \) cannot be represented as a convex combination of elements of \( S \),

\[
\vec{V} \cdot \vec{V} > \vec{S} \cdot \vec{V} \Rightarrow \vec{V} \not\in S.
\]  

Note, however, that the converse statement is not true in general.

As the vector \( \vec{V} \) we take the quantum correlation function in the form of \( E_{QM}(\alpha_1, \alpha_2, ..., \alpha_N) \), where \( \alpha_j \) denotes a parameter of measurement observables for the \( j \)th party. This function is the average of the product of local results. On the other hand, LR
theories assume that the local results are predetermined before the measurements, contrary to the quantum mechanical description. Then, the correlation function can be simulated by
\[ \rho(\lambda) = \frac{1}{4} \sum_{i=1}^{N} (\alpha_i, \lambda) \] for all \( E_{LR} \), then the LR description cannot describe the quantum prediction. With the different outcome strategies, we numerically calculate the ratio between the quantum and classical description in the form of
\[ \frac{E_{QM}}{E_{LR}} \] we call it a quantum-to-classical ratio (QCR).

Here, we will exploit the principle known for qutrits as the 1-0-1 rule [14]. It states that for three squared orthogonal components of spin-1 the outcomes of measurements will be 1, 1 and 0 in some order. However, in noncontextual and local theories, the assignment of 0 to a specific state cannot change if a compatible measurement is performed. It hence lies at the heart of Kochen–Specker and Bell arguments. We are interested in such orbits of observables, which involve commuting operators, but with changed (permuted) eigenvalues.

Given the dimensionality of each subsystem \( d \), the permutation of these vectors is easily realized by the transformation
\[ U(\alpha) = \text{diag}(1, e^{i\alpha}, e^{2i\alpha}, \ldots, e^{(d-1)i\alpha}) F, \] where \( \text{diag}(\cdot) \) is a diagonal matrix and \( F \) is the Fourier transform,
\[ F = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \\ \vdots & \vdots & \vdots \end{pmatrix}, \] and \( \omega = \exp(2\pi i/d) \). Notice that \( U(\alpha) \) realizes cyclic permutations for \( \alpha = 2k\pi/d \) with integers \( k \).

3. Outcome strategies

3.1. Strategy I: multiplying real local outcomes

We first consider the case of traditional von Neumann measurements, where the outcomes are simply scalars and the correlation function is the expectation value of the local outcomes. We study these observables, for which one-dimensional subspace is distinguished from the rest by an eigenvalue
\[ J(\alpha) = U(\alpha) \text{diag} \left( \frac{d-1}{d}, \frac{-1}{d}, \ldots, \frac{-1}{d} \right) U^\dagger(\alpha). \] Let us remark that all strategies will involve outcomes, which sum up to 0. Only then can we associate the ratio between the quantum and the maximum LR value with a strength of violation and robustness against white noise. This is due to the fact that our local observables are traceless. Then, for the generalized Werner states in the form of
\[ \rho = p |\psi\rangle \langle \psi| + (1 - p) 1/d^N, \] a convex combination of an entangled pure state \( |\psi\rangle \) and white noise, the mean values are scaled by factor \( p \) (no contribution from the white noise due to the traceless character). This results in the left-hand side of the geometrical condition (1) scaling as \( p^2 \), while the right-hand side scales as \( p \).

The state under consideration is the generalized Greenberger–Horne–Zeilinger (GHZ) state of \( N \) qudits,
Then the quantum correlation function reads

$$\psi_{\alpha}^{N} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle^{\otimes N}. \quad (5)$$

Now, let us consider the fixed part of $J(\alpha)$ and $F \text{diag} \left( \frac{d-1}{d}, -\frac{1}{d}, ..., -\frac{1}{d} \right) F^{\dagger}$, which has the matrix representation

$$F |0\rangle \langle 0| F^{\dagger} - \frac{1}{d} = \begin{pmatrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (7)$$

Because the GHZ state is correlated in the computational basis (all qudits always yield the same outcome of a measurement in this basis), the correlation function can be written as

$$E_{QM}(\alpha_{1}, ..., \alpha_{N}) = \frac{2}{d^{N+1}} \sum_{j=1}^{d-1} (d-j) \cos(j\bar{\alpha}), \quad (8)$$

where $\bar{\alpha} = \sum_{k=1}^{d-1} \alpha_{k}$.

On the other hand, an LR model implies that the local outcomes are predetermined before measurements, i.e. the model freely preassigns values $(d-1)/d$ and $-1/d$. The only requirement is that once one setting $\alpha_{i}$ has been ascribed value $(d-1)/d$, we demand that all other settings $\alpha_{i}$ are assigned to $-1/d$. The settings used by each observer are $\alpha_{i}(j_{i}, k_{i}) = 2\pi \left( \frac{k_{i}}{L} + \frac{j_{i}}{d} \right)$, with $L$ being the number of settings (i.e. different bases) for each observer. The variables $k_{i} \in \{0, 1, ..., L-1\}$ enumerate a basis for the $i$th observer measures in, and $j_{i}$ encodes the result the observer reports upon seeing one of his detectors triggered. Then the LR correlation function $E_{LR}(\alpha_{1}, ..., \alpha_{N})$ is simply a product of values assigned to specific settings. We numerically show the values of QCR for some combinations of $N$, $d$ and $L$ in figures 1(a)–(c), minimized over all LR models.

A few remarks ought to be made on these results. First is that, similarly to [14], we observe a sort of fluctuation of the values of QCR; the violation for $L = 3$ is the lowest one and for $L = 5$ is the second (see figure 1(b)). Let us mention that in [14] another LR model turned out to be more optimal occasionally, whereas this is not the case here. Second, we observe the effect known from [17]; the violation ratio is the highest for two settings for two and three qudits (see figures 1(a) and (b)), but for four (presumably more) parties the values grow with $L$, saturating at the limit of $L \to \infty$ (see figure 1(c)). Finally, we should remark that the case of $d = L = 2$ recovers the Mermin inequality [18].

We are also able to find the violation of the inequalities in the limit of $L \to \infty$. The quantum side of the inequality reads

$$E_{QM} \cdot E_{QM} = \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} E_{QM}^{2}(\alpha_{1}, ..., \alpha_{N}) d\alpha_{1} \cdots d\alpha_{N} = \frac{(2\pi)^{N} (2d-1)(d-1)}{3d^{2N-1}}. \quad (9)$$

On the other hand, since $E_{QM}$ in equation (8) shows a global maximum at $\bar{\alpha} = 2\pi k$ with integer $k$, the optimal LR model $E_{LR}$ can be obtained by integrating the correlation function over all $\alpha_{i}$ in intervals $(-\pi/d, \pi/d]$. As a result, the QCR reads
For a derivation of this formula, see appendix A.

For $N = d = 2$, this formula is in an agreement with the results of [19]. It also guarantees exponential growth of violation strength with the number of parties. For qubits, it behaves like $(\pi/2)^N \approx 1.571^N$, for qudits: $(2\pi/(\sqrt{d}))^N \approx 1.209^N$, for ququats: $(\pi/(2\sqrt{2}))^N \approx 1.111^N$, and for qudecs $(d = 6)$: $(\pi/3)^N \approx 1.047^N$. The violation growth factor tends to 1 as $d \to \infty$. Still, for any finite $d$ and $N$, we observe a firm violation.

3.2. Strategy II: summing local outcomes modulo $d$

In the second scenario, clicking of each detector corresponds to an integer outcome, ordered in an increasing manner. After the measurements have been performed, the outcomes associated with the detectors that have clicked are sent to a referee, who sums them modulo $d$. If the sum is 0 modulo $d$, then the value $(d - 1)/d$ is taken as an outcome of the measurement. Otherwise, $-1/d$. Then, for the generalized GHZ state (5) the quantum mechanical correlation function $E_{QM}$ is given by

\begin{equation}
QCR = \left(\frac{2\pi}{d}\right)^N \frac{(2d - 1)(d - 1)}{6 \sum_{k=1}^{d-1} (d - k) \left[ \frac{2 \sin(k\pi/d)}{k} \right]^N}.
\end{equation}

Figure 1. The values of QCR for the first and second strategies, assigning real values as outcomes, in the generalized GHZ state (5) with $N = 2$, 3 and 4, respectively. They show the same results (see the main text for details). The numerical data are given in table B1 in appendix B.
where \( \alpha = \sum_{k=1}^{N} \alpha_k \).

This result is equivalent to the one (8) for the first strategy except for the coefficient. The LR correlation function is obtained similarly as for the first strategy. We show in figure 2 that the values of QCR for the strategies I and II are equivalent for the generalized GHZ state for \( N = 2, 3 \) and \( 4 \). This can be explained in the following.

Notice that the span of the correlation function, which is the difference between its maximal and minimal values is 1. Also

\[
\int_{0}^{2\pi} E_{\text{QM}}(\alpha_1, \ldots, \alpha_N) d\alpha_i = 0
\]

allows us to shift the LR model so that it takes only values 0 and 1. In such a case, a product with the optimal model would consist of an integral over \( d^{N-1} \) boxes of dimension \( \frac{2\pi}{d} \times \cdots \times \frac{2\pi}{d} \), representing a sum of local outcomes \( \{ a_i; \sum_{i=1}^{N} a_i \mod d = 0 \} \). Each such box will be centered at the peak of \( E_{\text{QM}} \); that is, at \( \bar{\alpha} = 2\pi k \) with integer \( k \).

3.3. Strategy III: multiplying complex local outcomes

Another generalization of measurement outcomes on a qudit is \( d^\text{th} \) order roots of unity over the complex field, \( \omega = \exp(2\pi i/d) \). They combine the approach described above—multiplying outcomes—with those described below—assigning objects to their sums modulo \( d \). Again, upon an occurrence of a event (detector click) an observer reports outcome \( 0, \ldots, d-1 \). At each side, detectors are labeled in a natural manner. The referee computes \( \omega^{\bar{a}} \), where \( A \) is the sum of the submitted outcomes. Then, the quantum correlation function for the GHZ state (5) reads

\[
E_{\text{QM}}(\alpha_1, \ldots, \alpha_N) = \frac{1}{d} \left[ (d-1) e^{-i\bar{\alpha}} + e^{i(d-1)\bar{\alpha}} \right].
\]

where \( \bar{\alpha} = \sum_{k=1}^{N} \alpha_k \). In fact, we will compare \( E_{\text{QM}} \cdot E_{\text{QM}} \) with \( \text{Re}(E_{\text{LR}} \cdot E_{\text{QM}}) \). Note that the way that detector clicks are interpreted as outcomes implies a strong constrain on LR models. If a model assigns some outcome \( a_i \) to angle \( \alpha_i \), it must consequently assign \( a_i - 1 \) to \( \alpha_i + 2\pi/d \), etc.

The procedure is the same as in other strategies; the inequalities for a few low values of \( d \), \( L \) and \( N \) are studied case-by-case to give an idea about the structure of the optimal LR.
model. This turns out to assign a fixed outcome to an interval of angles, say, 0 to \(-\pi/d \leq \alpha_i < \pi/d\), which we will call a packed model. Interestingly, for optimal models, \((E_{LR} \cdot E_{QM})\) is strictly real. Violations for finite \(L\) for \(N = 2, 3\) are given in tables B1 and B2.

In the case of \(L \to \infty\), we were able to find a formula for violation

\[
\text{QCR} = \frac{(d-1)^2 + 1}{d \left[ 1 + (d-1)^{N+1} \right]} \left[ \frac{\pi (d-1)}{d \sin \frac{\pi}{d}} \right]^N.
\] (14)

Naturally, the violation is guaranteed for any \(N\) for \(d = 2\), as the result from \([19]\) must be recovered. However, for any finite \(N\), the inequality is violated up to a certain value of \(d\), and above that threshold, it is satisfied, tending to 1 as \(d \to \infty\). This is somewhat expected, since for large \(d\), various powers of \(\omega\) can be close to one another. Numerical evaluation of equation (14) reveals that violation is sustained up to \(d \approx 1.641N\).

4. Biased GHZ states

Having found the first and the second strategies to be equivalent in the Bell scenario given above, we want to find a difference between them by altering the design of the experiment. Particularly, we consider biased GHZ states (of Schmidt rank \(d - 1\)),

\[
|\psi\rangle = \frac{1}{\sqrt{d-1}} \sum_{i=1}^{d-1} |i\rangle^{\otimes N}.
\] (15)

There are a few instances (e.g., \(N = 3, d = 3\)) where finite \(L\) is optimal, but the inequality is not violated. First, let us consider the first strategy. In most cases, inequalities for \(L \to \infty\) become optimal, and the violation ratio reads

\[
\text{QCR} = \frac{(2d^2 - 7d + 6)(2\pi)^N}{6d^N \sum_{k=1}^{d-1} (d - k - 1) \left[ \frac{2 \sin(k\pi/d)}{k} \right]^N}.
\] (16)

Note that for \(N \leq 5\) the values of QCR increase with \(d\), while for \(N \geq 7\) they decrease. In the case of \(N = 6\), the QCR is the lowest at \(d = 7\) (see figure 3).

The results distinguish the first strategy as the one leading to the highest robustness against white noise. Once more, for low values of the particle number, \(N < 4\), it is optimal to refrain from measuring more than two bases at each side. For \(N \geq 4\) the QCR grows with \(L\). Unlike for biased GHZ states, QCR grows with the dimensionality of the subsystems. The QCR for scalar dichotomic outcomes is given in table B3 for \(N = 2, 3\) and \(L = 2\).

When we assign complex outcomes, the correlation function becomes factorizable, \(E_{QM} = (d - 2)/(d - 1)e^{-\alpha_i}\), and hence the inequality cannot be violated.

5. Conclusions

We have presented geometrical BIs for a collection of qudits and an arbitrarily high number of local measurements settings. Their violation for the GHZ states has been demonstrated. Interestingly, regardless of \(d\), in this state their violation is the strongest for \(L = 2\) distinct measurement bases with \(N = 2\) or 3 subsystems, and for \(L \to \infty\) in the case of \(N \geq 4\).

Within the same Bell scenario, we have also compared various strategies of treating the measurement outcomes. The fixed Bell experiment has guaranteed us the same amount of
nonclassicality in the raw data. Basically, the aim to compare these strategies was to establish a degree of negligence we can afford to maintain robustness against white noise. First, we investigated reporting one value by an observer when his/her specific detector clicks. Because of the symmetry of geometrical BIs, this revealed the full structure of the probability distribution. The other strategy was to sum up the outcomes of local measurements modulo $d$, while the last one was to represent this sum as one of complex roots of unity of degree $d$. The last of these strategies represented measurement outcomes as numbers quite close to each other (for large $d$) and resulted in weak or no violation of BIs. The second singled out only a specific kind of correlation, and performed as good as the first one for the full rank GHZ states, but dropped back for biased ones.

Then we have decided to partially break the symmetry of the state by rejecting one of its Schmidt modes. We have been able to distinguish the real local scalars as the outcome strategy providing the most robust violation. This suggests that we can focus on general types of correlations, rather than individual probabilities only for highly symmetric states. Also, we shall point out that complex scalars have led to a fully factorizable correlation function. While it is possible to formulate all-versus-nothing paradoxes, in practical applications, complex measure outcomes can witness only the strongest correlations.

We would also like to stress that these BIs are relevant for analyzing the bright squeezed vacuum (BSV) state. The structure of each $n$-pair component of BSV is identical to the one of two-qudit singlets, and the unitary transformation can be conveniently realized with a polarization-dependent phase shift. However, an experimental challenge is to realize a projection on an unbiased superposition of all polarization states. Still, it might be possible to find similar inequalities utilizing projections more feasible in an experiment. In any case, BIs described here are yet another way to analyze BSV theoretically.

**Acknowledgments**

This work is a part of the project BRISQ2 financed by the European Commission. The authors acknowledge support from European funds distributed by the Foundation for Polish Science (FNP). MW was initially supported within the program HOMING PLUS, AD within MPD. MW was at a later stage supported by the Polish National Science Centre under grant DEC-2013/11/D/ST2/02638 and 2012/04/M/ST2/00789. JR is also supported by the TEAM project of FNP and by an ERC grant QOLAPS.
Appendix A. Derivation of formulae (10) and (14)

Let us begin with deriving the violation ratio for $L \to \infty$ for the first strategy, as given by equation (10). The numerator of this fraction has already been given in equation (9). Local observables have eigenvalues $(d - 1)/d$ (unique) and $-1/d$ (degenerated). This means that any deterministic local model can be written as a product of the functions

$$I_i(\alpha_i) = \chi_i(\alpha_i) - \frac{1}{d},$$  \hspace{1cm} (A.1)

where $\chi_i(\alpha_i) = 0, 1$ is a characteristic function and, following from the 1–0-1 rule, $\int_0^{2\pi} \chi_i(\alpha_i) = 2\pi/d$. When we calculate

$$\max_{E_{LR}} \left( E_{QM} \cdot E_{LR} \right) = \max_{\{\alpha_i\}} \int_0^{2\pi} \int_0^{2\pi} d\alpha_1 d\alpha_2 \cdots \times E_{QM}(\alpha_1, \alpha_2, \ldots)I_1(\alpha_1)I_2(\alpha_2),$$  \hspace{1cm} (A.2)

we can neglect the constant part of each LR models. Thus we have

$$\max_{E_{LR}} \left( E_{QM} \cdot E_{LR} \right) = \max_{\{\alpha_i\}} \int_0^{2\pi} \int_0^{2\pi} d\alpha_1 d\alpha_2 \cdots \times E_{QM}(\alpha_1, \alpha_2, \ldots)\chi_1(\alpha_1)\chi_2(\alpha_2).$$  \hspace{1cm} (A.3)

The $E_{QM}$ always has a distinctive peak at $\tilde{\alpha} = 0$, thus it is optimal to choose

$$\chi_i(\alpha_i) = \begin{cases} 1 & \frac{\pi}{d} \leq \alpha_i < \frac{\pi}{d} \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (A.4)

which straight-forwardly leads to equation (10). In a similar fashion, we find the same formula for strategy II.

We therefore pass to equation (14). Numerical case-by-case studies show that the optimal LR model takes the form

$$E_{LR,\text{opt}}(\alpha_i) = \begin{cases} 1 & \frac{-\pi}{d} \leq \tilde{\alpha} < \frac{\pi}{d} \\ \omega^{-1} & \frac{\pi}{d} \leq \tilde{\alpha} < \frac{3\pi}{d} \\ \omega^{-2} & \frac{3\pi}{d} \leq \tilde{\alpha} < \frac{5\pi}{d} \\ \vdots & \vdots \end{cases}$$  \hspace{1cm} (A.5)

Hence the integral can be divided into $d^N$ blocks of dimension $\frac{2\pi}{d} \times \cdots \times \frac{2\pi}{d}$. Each of them equally contributes to the integral, e.g.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha_1 d\alpha_2 \cdots E_{QM}(\alpha_1, \ldots, \alpha_N)E_{LR,\text{opt}}(\alpha_1, \ldots, \alpha_N)$$

$$= 2^N \left[ 1 + (d - 1)^N \right] \sin \left( \frac{2\pi}{d} \right)^N \frac{d^N}{d(d-1)^N},$$  \hspace{1cm} (A.6)

which straight-forwardly leads to equation (14).
Appendix B. The values of QCR for geometrical BIs

Table B1. The values of QCR for the first and second strategies (assigning real outcomes) in the generalized GHZ state (5) with $N = 2$, 3 and 4, respectively. As previously stated in the main text, they show the same results.

| $d$ | 2    | 3    | 4    | 5    | 6    |
|-----|------|------|------|------|------|
|     |      |      |      |      |      |
| For $N = 2$ |      |      |      |      |      |
| 2   | 1.414| 1.299| 1.268| 1.255| 1.248|
| 3   | 1.170| 1.116| 1.101| 1.094| 1.090|
| 4   | 1.119| 1.077| 1.064| 1.059| 1.056|
| 5   | 1.098| 1.061| 1.050| 1.045| 1.043|
| 6   | 1.087| 1.053| 1.043| 1.038| 1.036|
| For $N = 3$ |      |      |      |      |      |
| 2   | 2.828| 2.923| 2.971| 2.996| 3.010|
| 3   | 1.658| 1.692| 1.707| 1.714| 1.718|
| 4   | 1.470| 1.493| 1.503| 1.508| 1.510|
| 5   | 1.397| 1.416| 1.424|      |      |
| For $N = 4$ |      |      |      |      |      |
| 2   | 2.000| 1.688| 1.941| 1.844| 1.939| 1.938|
| 3   | 1.277| 1.289| 1.356| 1.351| 1.373| 1.387|
| 4   | 1.056| 1.086| 1.109| 1.113| 1.119| 1.128|
| 5   | 0.988| 1.010| 1.022| 1.026| 1.029| 1.034|
| 6   | 0.962| 0.978| 0.986| 0.988| 0.990| 0.994|

Table B2. The values of QCR for the third strategy (assigning complex values as outcomes) in the generalized GHZ state (5) with $N = 2$ and 3, respectively.

| $d$ | 2    | 3    | 4    | 5    | 6    | $\infty$ |
|-----|------|------|------|------|------|----------|
|     |      |      |      |      |      |          |
| For $N = 2$ |      |      |      |      |      |          |
| 2   | 1.414| 1.299| 1.268| 1.255| 1.248|          |
| 3   | 1.170| 1.116| 1.101| 1.094| 1.090|          |
| 4   | 0.975| 0.982| 0.986| 0.988| 0.989| 0.991    |
| 5   | 0.939| 0.948| 0.951| 0.953| 0.954| 0.956    |
| 6   | 0.929| 0.936| 0.939| 0.939| 0.940| 0.942    |
| For $N = 3$ |      |      |      |      |      |          |
| 2   | 2.000| 1.688| 1.941| 1.844| 1.939| 1.938    |
| 3   | 1.277| 1.289| 1.356| 1.351| 1.373| 1.387    |
| 4   | 1.056| 1.086| 1.109| 1.113| 1.119| 1.128    |
| 5   | 0.988| 1.010| 1.022| 1.026| 1.029| 1.034    |
| 6   | 0.962| 0.978| 0.986| 0.988| 0.990| 0.994    |
Table B3. The values of QCR for scalar dichotomic outcomes in the biased GHZ states (15) with measurement setting $d = 2$.

| $N$ | 2 | 3 |
|-----|---|---|
| 3   | 0.770 | 0.889 |
| 4   | 0.863 | 0.976 |
| 5   | 0.911 | 1.020 |
| 6   | 0.940 | 1.047 |
| 7   | 0.959 | 1.064 |
| 8   | 0.973 | 1.077 |

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