Scaling limit of critical random trees in random environment

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Abstract

We consider Bienaymé-Galton-Watson trees in random environment, where each generation $k$ is attributed a random offspring distribution $\mu_k$, and $(\mu_k)_{k \geq 0}$ is a sequence of independent and identically distributed random probability measures. We work in the “strictly critical” regime where, for all $k$, the average of $\mu_k$ is assumed to be equal to 1 almost surely, and the variance of $\mu_k$ has finite expectation. We prove that, for almost all realizations of the environment (more precisely, under some deterministic conditions that the random environment satisfies almost surely), the scaling limit of the tree in that environment, conditioned to be large, is the Brownian continuum random tree. The habitual techniques used for standard Bienaymé-Galton-Watson trees, or trees with exchangeable vertices, do not apply to this case. Our proof therefore provides alternative tools.

1 Introduction

In this paper, we consider Bienaymé-Galton-Watson trees in random environment, in the sense that each generation $k \geq 0$ is assigned a random offspring distribution $\mu_k$ sampled from a common measure $\Lambda$, independently over $k$. Throughout the paper, we work in the strictly-critical case, i.e. we assume that, for all $k \geq 1$, the average of $\mu_k$ is equal to 1 almost surely. However, we allow the variance $\sigma_k^2$ to be a non-degenerate random variable. See Section 1.1 for a precise definition of the model.

We prove that, if the expectation of $\sigma_k^2$ is finite, then a.s., the environment $\mu = (\mu_k)_{k \geq 0}$ is such that the scaling limit of the tree in environment $\mu$, conditioned to have at least $n$ vertices with $n \to \infty$, is Aldous’ continuum random tree, see Theorem 1.1.

The main technical challenge is that standard techniques, developed in the case of a constant environment, do not apply here, because the classical exploration algorithms become more difficult to analyse. For instance, the exploration process (in depth-first order) is a Markovian random walk when the environment is constant, whereas it becomes a non-Markovian process in our case. This is due to the fact that the offspring distribution of the $n$-th vertex (in depth-first order) depends on its height, which in turn depends on the whole exploration up to that step, see Section 2.2. Another challenge posed by the varying environment is that degrees are not exchangeable, as they do not appear with the same probability at different heights. Hence arguments resorting to exchangeability, that have in particular been developed in the literature of random trees with prescribed degrees (see e.g. [ABD22, Ald91a, BR21, BM14]), are not relevant in our setting. Other approaches developed in recent papers on the width of a tree in varying environment at a given large height and genealogical properties below that height (see CSC22, BFRS22 for the latter and Section 2.4 for more references) do not seem to be
of use when trying to characterise the structure of the entire tree, and thus its scaling limit. Consequently, we provide an alternative approach for proving scaling limits of random trees.

Our model is naturally related to Bienaymé-Galton-Watson trees in varying environment, which are a generalisation of Bienaymé-Galton-Watson trees in which the offspring distribution varies from generation to generation according to a deterministic sequence of distributions. These can be thought of as a quenched version of random trees in random environment. In Section 2.3, we provide deterministic conditions on the environment (which are a.s. satisfied by our random environment) for our main result to hold.

Both these models, where the offspring distribution depends on the generation, are natural extensions of the original Bienaymé-Galton-Watson trees. Indeed, they provide a framework that takes into account the possibility of a variation in the reproduction rate of a population depending, for example, on seasons (to simplify, one can think of a population reproducing faster in the summer and slower in the winter). Therefore, Bienaymé-Galton-Watson trees in random environment or in varying environment have been extensively studied in the literature. We refer the reader to Kersting and Vatutin’s book [KV17] for a comprehensive review of the state of the art on these processes; more discussions will follow in Section 2.4.

1.1 Definition of the model and notation

In this section, we define Bienaymé-Galton-Watson trees in varying environment and in random environment, abbreviated respectively BPVE and BPRE for branching processes in varying environment and in random environment.

All the probability measures and random variables below are defined on a common measurable space \((\Omega, \mathcal{F})\). An environment is a sequence \(\nu = (\nu_k)_{k \geq 0}\) of probability distributions on \(\mathbb{N} = \{0, 1, \ldots\}\). For all environments \(\nu\), we define a collection of offspring random variables \((\xi_k^{(i)}, i \geq 1, k \geq 0)\) under a product measure \(P_{\nu}\) so that these are independent and, for each \(k \geq 0\), \(\xi_k^{(i)}\) has distribution \(\nu_k\), for all \(i \geq 1\). Under the measure \(P_{\nu}\), we define a BPVE in the environment \(\nu\), or \(\nu\)-BPVE, recursively as follows: Let \(Z_0 = z_0 \in \mathbb{N}\) and, for all \(k \geq 0\),

\[
Z_{k+1} = \sum_{i=1}^{Z_k} \xi_k^{(i)}.
\]

Note that \((\xi_k^{(i)})_{i \geq 1}\) above is a sequence of i.i.d. random variables of distribution \(\nu_k\), independent of \(Z_k\). It is classical to interpret this process as the evolution of a population which has \(Z_k\) individuals at generation \(k\) and where each individual \(i, 1 \leq i \leq Z_k\), has \(\xi_k^{(i)}\) offspring in generation \(k+1\).

We now give the definition of a BPRE. Let \(\Lambda\) be a probability measure on \(\mathcal{P}(\mathbb{N})\), the set of probability measures on \(\mathbb{N}\), and define the product measure \(P = \Lambda^{\otimes \mathbb{N}}\). We choose the random environment \(\mu = (\mu_k)_{k \geq 0}\) to be distributed under \(P\), hence it consists of a sequence of i.i.d. probability measures. A branching process is a BPRE with environment distribution \(P\), or \(P\)-BPRE, if it is a BPVE in the environment \(\mu\) with \(\mu\) a random variable of distribution \(P\).

We let \(P_{\mu}\) denote the quenched measure and \(P\) the annealed measure of \(T^\mu\). I.e., \(P_{\mu}\) is the distribution of \(T^\mu\) given \(\mu\), while \(P\) is the unconditional distribution of \(T^\mu\). For all \(k \geq 0\), we define the average offspring and the variance at generation \(k\) by

\[
\bar{\mu}_k = E_{\mu}[\xi_k^{(1)}] \quad \text{and} \quad \sigma_k^2 = E_{\mu}\left(\xi_k^{(1)} - \bar{\mu}_k\right)^2.
\]

Note that if \(P\) is the distribution of \(\mu\), then \((\bar{\mu}_k, \sigma_k^2)_{k \geq 0}\) is a sequence of i.i.d. random variables taking values in \([0, +\infty)^2\). In our main result, we assume that \(\bar{\mu}_k = 1\) almost surely, and discuss this condition in Section 1.3.
Scaling limit theorems for random trees are classically expressed as convergence of metric spaces; therefore, we see a BPRE or a BPVE as a metric space, endowed with a measure. More precisely, we consider triplets of the form $(T^\mu, d^\mu, \pi^\mu)$, where $T^\mu$ is the set of all the nodes (individuals) created by that process, $d^\mu$ is the distance induced by the graph distance on the family tree of the branching process, and $\pi^\mu$ is the uniform distribution on the vertices, with total mass 1.

From now on, we may simply refer to $(T^\mu, d^\mu, \pi^\mu)$ or $T^\mu$ as a BPRE or a $\mu$-BPVE, often overlooking the metric space structure.

There is a natural metric on the Polish space of such compact measured metric spaces, called the Gromov-Hausdorff-Prokhorov metric, which is often used for scaling limits of random geometric objects, and in particular random trees. For a definition of the Gromov-Hausdorff-Prokhorov metric, we refer the reader to, e.g., [ABBGM17, Section 2], and the references therein.

1.2 Statement of the main result

In this section, we state our main theorem. We are interested in the family tree of a BPRE, seen as a measured metric space, when this tree is large. Let $\mu$ be a sequence of distributions drawn under $P$. On the environment $\mu$, for all $n \geq 1$, we define the tree $(T_n, d_n, \pi_n)$ as the $\mu$-BPVE distributed under the conditional measure $P^\mu_{\{|T^\mu| \geq n\}}$, where $|\cdot|$ denotes the cardinality of a set. As in the previous section, $T_n$ is the set of the $|T_n|$ individuals in the total population, $d_n$ is the distance induced by the graph distance on the family tree of the branching process, and $\pi_n$ is the uniform distribution on the vertices with total mass 1.

**Theorem 1.1.** Assume that $\bar{\mu}_0 = 1$ $P$-a.s. and that $\sigma^2 := E[\sigma_0^2] \in (0, \infty)$. Let $\mu$ be a sequence of i.i.d. distributions drawn under $P$ and let $(T_n, d_n, \pi_n)$ be a $\mu$-BPVE. For $P$-almost all $\mu$,
\[
\left( T_n, \frac{1}{\sigma \sqrt{|T_n|}} d_n, \pi_n \right) \xrightarrow{(d)} (T, d, \pi),
\]
as $n \to \infty$, on the set of compact metric spaces equipped with the Gromov-Hausdorff-Prokhorov metric, and where $(T, d, \pi)$ is Aldous’ continuum random tree.

Note that the result above is quenched, in the sense that it holds for almost all environment drawn under $P$. In fact, we prove the convergence for all environment $\mu$ satisfying the assumptions (I)-(V) presented in Section 2.4 and prove that these conditions are satisfied $P$-almost surely.

Moreover, observe that, in Theorem 1.1, we assume that $\bar{\mu}_0 = 1$ $P$-a.s., which puts us in the case called strictly critical. We discuss this assumption in Section 1.3 below. In particular, this implies that $P$-a.s., a $\mu$-BPVE is a.s. finite.

For the definition and properties of the continuum random tree (CRT), we refer to the seminal series of papers by Aldous [Ald91a, Ald91b, Ald93]. Recalling that $P = \Lambda \otimes N$, and choosing $\Lambda$ to be a Dirac measure putting mass on a single probability distribution with finite variance, it is straightforward to see that all of our proofs apply to critical Bienaymé-Galton-Watson trees, and one can thus recover the classical result of Aldous [Ald93] from Theorem 1.1. Although our results can be seen as a generalisation of the classical Bienaymé-Galton-Watson case, the classical proofs are not robust when the environment varies, and new ideas are needed. This is discussed in more details in Section 2.2.

1.3 Critical and strictly critical BPREs

In Theorem 1.1, we give a scaling limit of a critical random tree conditioned to be large. Nevertheless, contrary to the case of usual Bienaymé-Galton-Watson trees, the notion of criticality
for BPRE and BPVE is more delicate, as it is well-explained in [KV17]. The definition below provides two levels of criticality for a BPRE: in Theorem 1.1 we work in the strictly critical case. (Recall that, under $\mathbb{P}$, $\mu$ is a sequence of i.i.d. random distributions, and that $\bar{\mu}_k$ denotes the average of $\mu_k$.)

**Definition 1.** A BPRE of environment distribution $\mathbb{P}$ is called critical if $\mathbb{E} \left[ \log \bar{\mu}_0 \right] = 0$. It is called strictly critical if $\mu_0 = 1$ $\mathbb{P}$-almost surely.

Note that a critical BPRE is eventually almost surely extinct, see [KV17, Theorem 2.1] for the critical case, and [KV17, Section 2.5] for the strictly critical case. See [KV17] for a more properties of critical and strictly critical BPREs.

The assumption of strict criticality made in Theorem 1.1 is necessary. We now provide a short argument for the fact that the convergence to Aldous’ CRT does not hold in the non-strictly critical case: By [KV17, Theorem 2.6], the probability that a strictly-critical BPRE survives up to generation $n$ is of order $1/n$, which is consistent with critical Bienaymé-Galton-Watson trees. In contrast, by [KV17, Theorem 5.1], the probability that a non-strictly critical BPRE survives up to generation $n$ is of order $1/\sqrt{n}$. This extra large probability is due to the fact that the environment over the first $n$ generations can be very favorable to survival by looking like a super-critical environment, which happens with probability $1/\sqrt{n}$. Hence, one cannot expect that our theorem holds for non-strictly-critical BPREs.

### 2 Sketch of proof and discussion

In this section, we provide the structure of the proof of Theorem 1.1, define the key objects needed, and explain why the classical techniques used for scaling limits of standard Bienaymé-Galton-Watson trees do not apply in varying environment. We start by a technical section providing the necessary tools and notation. Then, we describe the plan of the proof in Section 2.2. We develop our ideas and detail our technical assumptions in Section 2.3. We discuss the existing literature in Section 2.4 and open questions in Section 2.5.

#### 2.1 Orders on forests and exploration processes

Throughout the paper, we work on forests instead of single trees, as it is usually easier to observe a large tree occurring naturally in a forest, rather than conditioning a single tree to be large.

A forest $\mathcal{F}$ is a collection $(T^{(i)})_{i \geq 1}$ of finite trees. For a finite tree $T$ we often let $(Z_m(T))_{m \geq 0}$ denote the sizes of its successive generations. We define the height of the tree as

$$h(T) = \max \{ m \geq 0 : Z_m(T) > 0 \},$$

and the width of the tree as

$$w(T) = \max_{m \geq 0} Z_m(T).$$

Similarly, if $\mathcal{F}$ is made of finitely many trees, say $\ell$, we define the height and width of $\mathcal{F}$ by

$$h(\mathcal{F}) = \max_{1 \leq i \leq \ell} h(T_i) \quad \text{and} \quad w(\mathcal{F}) = \max_{m \geq 0} \sum_{i=1}^{\ell} Z_m(T_i).$$

When applied to a vertex $x$ of a tree $T$, $h(x)$ will denote the height of $x$ in $T$, i.e. the graph distance from the root of $T$ to $x$.

Now, we define a lexicographical order on a forest $\mathcal{F}$, which is a labeling defined as follows. We label the trees one by one, starting with $T^{(1)}$. Start by calling $v_{1,0}$ the root of the first tree.
Figure 1: A forest with its lexicographical order \((v_{i,j})'s\) and depth-first order (labels 0 – 14, 17, and 20).

\(T^{(1)}\) of \(F\), and \(v_{i,1}, 1 \leq i \leq Z_1(T^{(1)})\), the children of this root. Let \(k \geq 0\) and assume that we have assigned a label to all the vertices at height at most \(k\). If the generation \(k + 1\) is empty then we have labelled the whole tree. If not, assign the labels \(v_{i,k+1}, 1 \leq i \leq Z_{k+1}(T^{(1)})\), to the vertices at height \(k+1\) of \(T^{(1)}\) in such a way that, for all \(i_1 < i_2\), if \(v_{j_1,k+1}\) and \(v_{j_2,k+1}\) are offspring of \(v_{i_1,k}\) and \(v_{i_2,k}\) respectively then \(j_1 < j_2\). For \(i \geq 1\), once we have fully labelled the trees \(T^{(j)}\) for \(1 \leq j \leq i\), we now label the tree \(T^{(i+1)}\) in a similar fashion, using the label \(\tilde{v}_{\ell,k} = v_{\theta_{i,k}(\ell),k}\) for \(\ell \geq 1\) and \(k \geq 0\), where \(\theta_{i,k}(\ell) = \ell + \sum_{j=1}^{i} Z_k(T^{(j)})\). This ordering is illustrated on Figure 1.

Remark 1. The lexicographical order gives access to a graphical construction of the forest \(F\). Indeed, in a discrete grid with coordinates \((i,k)\) for \(i \geq 1\) and \(k \geq 0\), put the node \(v_{i,k}\) at position \((i,k)\). Let \(\{\xi_{i,k} : i \geq 1, k \geq 0\}\) be a collection of independent random variables such that for all \(i \geq 1\) and \(k \geq 0\), \(\xi_{i,k}\) has distribution \(\mu_k\). We interpret \(\xi_{i,k}\) as the number of children of \(v_{i,k}\). Now, we construct the forest \(F\) by assigning \(\xi_{i,k}\) edges between \(v_{i,k}\) and nodes at height \(k+1\) in a planar manner (i.e. no two edges can cross each other), such that all nodes at height at least one have a parent (i.e. we use all of them) and such that each connected component is a tree. Consequently, \((v_{i,k})_{i,k}\) corresponds to the lexicographical labelling of that forest, see Figure 1 for another graphical representation.

We now define the Lukasiewicz path \((X_n)_{n \geq 0}\) and the height process \((H_n)_{n \geq 0}\) associated to the forest \(F\). For all \(n \geq 0\), let \(L_n\) be the number of children of the node \(n\), and let \(H_n\) be its
height, defined as the graph-distance from this node to the root of its tree. We let

$$X_n = \sum_{i=0}^{n-1} (L_i - 1).$$  \hfill (2.4)$$

Last, we define the forest explored at step $n$ as follows. Recall that we labelled the forest $F$ in depth-first order, starting from 0. For $n \geq 0$, let $i_n$ be the unique integer such that the node labelled $n$ belongs to $T(i_n)$. For all $n \geq 0$, we define the explored forest at step $n$ as the finite forest

$$F_n = \bigcup_{i=0}^{i_n} T(i).$$ \hfill (2.5)$$

In words, the explored forest $F_n$ consists of the trees of $F$ that have been explored or partially explored at step $n$. Note that by construction, the node $n$ is always included in $F_n$.

From now on, $F$ will denote an infinite sequence of i.i.d. $\mu$-BPVE, unless we state explicitly otherwise.

### 2.2 Backbone of the proof

As in the case of Bienaymé-Galton-Watson trees, our proof relies on the fact that convergence for the Gromov-Hausdorff-Prokhorov metric is implied by (and in fact equivalent to) convergence of the normalised height process to a Brownian motion (see, e.g., [CKG20, Equation (8)]). We prove the convergence in Theorem 1.1 in three main steps:

1. convergence of the Lukasiewicz path of the forest $F_n$ to a reflected Brownian motion;
2. convergence of the height process of $F_n$ to the same Brownian motion, up to a multiplicative constant;
3. deducing the convergence of one tree conditioned on having total population at least $n$ from the convergence of the forest.

Although these steps are identical to those taken in the classical proof, for constant environments, Steps 1. and 2. become much more involved when the environment is random. Let us now describe a few difficulties one encounters when dealing with random environments.

1. **Convergence of the Lukasiewicz path.** As explained above, it is classical to start by considering a forest, i.e. a sequence of unconditioned copies of the random tree we want to study, and explore it depth-first using the Lukasiewicz path defined in (2.4). The reason for introducing this process is that, in the case of a constant environment, this process is a random walk with i.i.d. steps. Hence the step 1 above is a direct consequence of Donsker’s invariance principle, and we obtain that the Lukasiewicz path, properly normalised, converges to a Brownian motion.

In the random environment framework, this step is far from trivial. Indeed, the Lukasiewicz path is still a random walk but its increments are no longer identically distributed or independent. More precisely, the distribution of the increment at a given step depends on the height of the vertex being explored, which depends on the whole past trajectory of the exploration. Thus the Lukasiewicz path is no longer Markovian.

To prove convergence of the rescaled Lukasiewicz path, we use a martingale convergence theorem from [Whi07, Theorem 2.1(ii)], which is an extension of [EK86, Theorem 7.1]. For this theorem to apply, we need to control the global shape of the explored forest: in Section 3 we
prove that the height and width of the explored forest at step $n$ are both of order $\Theta(\sqrt{n})$. We also need that, $\mu$-almost surely,

$$\frac{1}{n} \sum_{i=1}^{n} \sigma_{H_i}^2 \rightarrow \sigma^2,$$

(2.6)
in probability as $n \rightarrow \infty$, recalling that $\sigma_k^2$ is the variance of $\mu_k$, for all $k \geq 0$. For a non-varying environment, the limit above would actually be a straightforward equality, while in the case of a varying environment the sum above corresponds to the sum of the variances collected along a random path whose trajectory itself depends on those variances. Here, considering the explored forest is not enough anymore and we need to track the exact position in the forest of the node with depth-first label $n$. Establishing (2.6) is far from trivial and is the most tedious step of the proof.

2. Convergence of the height process. After having proved convergence of the normalised Lukasiewicz path to a reflected Brownian motion, one needs to prove that the height process also converges to a Brownian motion. Recall that the height process $H_n$ at step $n$ was defined above (2.4) as the height of the node $n$, i.e. its graph-distance to the root of its tree. In the classical case of non-varying environments, this is proved in [DLG02, Theorem 2.2.1] by noting that:

(a) the height of the $n$-th node is equal to the number of running minima up to time $n$ of the Lukasiewicz path $(X_k)_{k \geq 0}$;

(b) $(X_n - X_k)_{0 \leq k \leq n}$ has the same distribution as $(X_k)_{0 \leq k \leq n}$, for all $n \geq 0$;

(c) consequently, the height of the $n$-th node has the same law as the number of running maxima (starting from time 0) of $(X_n - X_k)_{0 \leq k \leq n}$;

(d) the pieces of trajectory of $(X_k)$ between two maxima are i.i.d.;

(e) the number of records of $(X_k)_{0 \leq k \leq n}$ is asymptotically proportional to $X_n$ with high probability by the Law of Large Numbers.

Crucially, the Steps (b) and (d) both fail in the case of a varying environment. Fortunately, there is another link between $X_n$ and $H_n$, as $X_n$ counts the number of unexplored children of each ancestor of the $n$-th vertex (we detail this in the next section, see Figure 2). It turns out that the offspring of the ancestor of the $n$-th vertex at height $k$ has a distribution close to the size-biased version of $\mu_k$, almost independently for different heights. From this, we can derive a Law of large numbers in the annealed environment - hence for almost all realisations of the environment. The combination of this link between $X$ and $H$ with a decomposition of the tree along a spine with size-biased degrees is used in [BM14] for trees with a prescribed degree sequence (and to some extent in [ABD13] for Galton-Watson trees, in particular in the proof of Theorem 1.2). Compared to both these works, a major additional difficulty in our case is that degrees are not exchangeable, due to the fact that the distribution of the degree of a vertex is conditioned by its height.

3. The tree conditioned to have at least $n$ vertices. So far, we have worked on the forest $F$ where trees of all sizes naturally appear, but now we would like to obtain information on a single tree conditioned to be large. This is done by extracting the largest excursions of the height process, corresponding to the largest trees of the forest $F_n$. The method developed in [Ald97] and [AL98] in the case of non-varying environments still applies in our case.
2.3 Auxiliary results and conditions for varying environments

In this section, we give more details on the steps 1 and 2 explained in Section 2.2 above. We start by giving deterministic conditions (I)-(V) on the varying environment, and we will then explain how they imply Theorem 1.1. We will show later in Lemma 3.2 that these conditions are satisfied \( P \)-a.s. by a strictly critical BPRE.

We need a few more definitions in order to state our conditions. Let \( \mu \) be a deterministic strictly critical environment, that is, for all \( k \geq 0 \), \( \mu_k \) has expectation 1 and variance \( \sigma_k^2 \). In that case, we define

\[
\sigma^2 = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \sigma_k^2.
\]

For all \( k \geq 0 \), let \( \xi_k \) be distributed according to the size-biased version of \( \mu_k \), i.e. for all \( i \geq 0 \),

\[
P_{\mu_k}(\xi_k = i) = i\mu_k(\{i\}).
\]

For all \( k \geq 0 \), let \( \zeta_k \) be a uniform random variable in \( \{0, \ldots, \bar{\xi}_k - 1\} \). Impose the independence of the family \( (\zeta_k)_{k \geq 0} \). Following [KV17], we define, for all \( k \geq 0 \) and \( s \in [0, 1) \),

\[
\varphi_k(s) = \frac{1}{1 - f_k(s)} - \frac{1}{1 - s},
\]

where \( f_k \) is the generating function of \( \mu_k \), and we set \( \omega_k(\epsilon) := \sup_{1 - \epsilon \leq s \leq t < 1} |\varphi_k(s) - \varphi_k(t)| \) for all \( \epsilon > 0 \).

We are now ready to define our five conditions on the environment \( \mu \):

(I) **Averaging of the variance along generations.** We have that \( \sigma^2 > 0 \) and

\[
\frac{\sigma_0^2 + \ldots + \sigma_{n-1}^2}{n} \to \sigma^2;
\]

(II) **Non-degeneracy of the offspring distributions.** There exists a constant \( c > 0 \) such that, for all \( 0 \leq a < b \),

\[
\liminf_{k \to +\infty} \frac{\mu(\{a\}) + \ldots + \mu(\{b\})}{(b-a)k} \geq c;
\]

(III) **Control of the large degrees I.** There exists a sequence \( (h_n)_{n \geq 1} \) such that we have \( \lim_{n \to +\infty} h_n = +\infty \) and such that, for all \( \epsilon > 0 \),

\[
\frac{1}{\sqrt{n}} \sum_{k=0}^{[h_n \sqrt{n}]} \mathbb{E}_{\mu}[\xi_k^2 1_{\{\xi_k \geq \epsilon n\}}] \to 0,
\]

where \( \xi_k \sim \mu_k \) for all \( k \geq 0 \);

(IV) **Position of the spine.** We have that, \( P_\mu \)-a.s.,

\[
\frac{1}{n} \sum_{k=0}^{n-1} \zeta_k \to \frac{\sigma^2}{2};
\]

where \( \sigma^2 \) is given by (I)

(V) **Control of the large degrees II.** We have that

\[
\lim\limsup_{\epsilon \to 0} \frac{1}{n} \sum_{k=0}^{n} \omega_k(\epsilon) = 0.
\]
We comment below on Conditions [I]–[V] and in particular explain how they imply our main result.

1. Convergence of the Lukasiewicz path. This part uses only Conditions [I], [II] and [III].
   Condition [I] ensures that a high tree should not have inhomogeneities between macroscopic slices of different heights (which might prevent convergence towards a self-similar object like the Continuum Random Tree). Without Condition [II] there could be too many offspring distributions close to a Dirac mass at 1. This would mean that the tree is likely to be long and thin as many of its vertices would have exactly one child. Condition [III] prevents large degrees, which would create jumps in the scaling limit of the Lukasiewicz path, in a rectangle of height and width of order at least $\sqrt{n}$ that encompasses $\mathcal{F}_n$. Note that Condition [III] is not implied by Conditions [I] and [II] in general.

As a first step in the proof of Theorem 1.1, we show that the Lukasiewicz path converges, after normalisation, to a Brownian motion.

**Theorem 2.1.** For any environment $\mu$ satisfying [I], [II] and [III], we have that, under $P_{\mu}$,
\[
\left( \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \right)_{0 \leq t \leq 1} \xrightarrow{(d)} (\sigma B_t)_{0 \leq t \leq 1}
\]
as $n \to \infty$ in the Skorokhod space $D([0,1])$, where $B$ is a standard Brownian motion.

We prove Theorem 2.1 in two steps, focussing on $\mathcal{F}_n$, the forest $\mathcal{F}$ restricted to the first $n$ vertices explored. We start by controlling the scaling of $\mathcal{F}_n$: we show in Proposition 3.4 that, under [I] and [II], $\mathcal{F}_n$ has height and width (the size of the largest generation) of order $\Theta(\sqrt{n})$. Then, we bootstrap this estimate. We use a martingale convergence theorem from [Whi07, Theorem 2.1 (ii)], which is an extension of [EK86, Theorem 7.1]. This theorem gives convergence to a Brownian motion under three relatively weak conditions: the first condition of [Whi07, Theorem 2.1 (ii)] holds if the largest degree in $\mathcal{F}_n$ is $o(\sqrt{n})$, ensured in our case by Condition [III]; the second condition holds if one can control the maximal variance of a degree, which is implied by Condition [I]; the third and last condition follows from (2.6), which is established in Lemma 4.7. In the proof, we divide the tree vertically and horizontally into mesoscopic boxes and control the behavior of the forest on those boxes. While this may sound as a discretized version of a proof that would be easier or more elegant on the scaling limit, we have not found such a proof (referring e.g. to the limit given by the generation sizes of a forest of $m$ trees, with $m \to \infty$, is a Feller diffusion, see the discussion in Section 2.4). In particular, one difficulty is that $\mathcal{F}_n$ cuts with high probability the last tree, and that the latter has a typical volume $\Theta(n)$, as the tree containing the $n$-th vertex has a much higher probability to be large than an unconditionned tree. In other words, we would need to cut a Feller diffusion on the right, along a curve that seems difficult to handle (corresponding to the ancestry line of the $n$-th vertex of $\mathcal{F}$ in the discrete setting), so that the left part has a fixed volume. This complicates significantly the matter. On the other hand, if we stop the exploration of $\mathcal{F}$ after $n$ trees have been fully seen (denote $t_n$ the corresponding stopping time), then many limit theorems, including the one of [Whi07], can not be used. Moreover, controlling the random variations of $t_n$ is not an easy task.

Below, we state the following corollary of Theorem 2.1 and the continuous mapping theorem, proved in Section 4. Recall that the law of a Brownian motion reflected above zero is that of its absolute value and let $I_n := \min_{j \leq n} X_j$, so that $X_n - I_n$ is the Lukasiewicz path reflected above its running minimum.
Corollary 2.2. Let $\mu$ be a strictly critical environment satisfying Conditions (I), (II) and (III). We have that, under $P_\mu$,
\[
\left( \frac{X_{\lfloor nt \rfloor} - I_{\lfloor nt \rfloor}}{1/\sqrt{n}} \right)_{0 \leq t \leq 1} \overset{(d)}{\rightarrow} (\sigma |B_t|)_{0 \leq t \leq 1}
\]
as $n \to \infty$ in Skorokhod topology, where $B$ is a standard Brownian motion.

2. Convergence of the height process. The second part of our proof is to deduce from Corollary 2.2 the joint convergence of the (reflected) normalised Lukasiewicz path and the height process to the same reflected Brownian motion, up to a multiplicative constant. For this purpose, we require all five Conditions (I)-(V).

Theorem 2.3. Let $\mu$ be a strictly critical environment satisfying Conditions (I)-(V). Then, under $P_\mu$,
\[
\frac{1}{\sqrt{n}} (X_{\lfloor nt \rfloor} - I_{\lfloor nt \rfloor}, H_{\lfloor nt \rfloor})_{0 \leq t \leq 1} \overset{(d)}{\rightarrow} \left( \sigma |B_t|, \frac{2}{\sigma} |B_t| \right)_{0 \leq t \leq 1}
\]
as $n \to \infty$ in Skorokhod topology for both coordinates, where $B$ is a standard Brownian motion.

As explained in Section 2.2, the classical approach used for Bienaymé-Galton-Watson trees does not apply, since the law of the $n$-th increment of the Lukasiewicz path for $n \geq 0$ depends potentially on the whole past of the trajectory. However, there is still one Markovian object in the exploration: the spine between the $n$-th vertex and the root of its tree, keeping track of the number of offspring of each of its ancestors that have not yet been explored (the number of red vertices at each generation in Figure 2). The sequence of successive spines for $n \geq 0$ is a discrete version of the Lévy snake introduced in [LG93].

The spine carries an important relation between $H_n$ and $X_n - I_n$, as illustrated in Figure 2. The idea is to show that the ancestors of the $n$-th vertex (blue vertices on Figure 2) have essentially independent offspring, with a size-biased distribution, as vertices with a larger
offspring are more likely to have more descendants. This defines a spine from the root to the $n$-th explored vertex. Intuitively, once a vertex $y$ at height $k$ is known to be along the spine, the next vertex in the spine is chosen uniformly among its children. In average, such a vertex $y$ has $\sigma^2_k + 1$ children, as its offspring is size-biased. The two precedent facts put together imply that, in average, among the children of $y$, $\sigma^2_k / 2$ will be on the left of it (black vertices) and $\sigma^2_k / 2$ will be on the right of it (red vertices). This is rigorously done in Section 5.1. Averaging over $k$ and using Condition (I), this heuristic explains why the limiting ratio between $X_n - I_n$ and $H_n$ is $\sigma^2 / 2$. For standard Galton-Watson tree, Note that, with $m = \beta(y)$, $\xi_m$ defined before Condition (IV) stands for the number of children of the ancestor of the $n$-th vertex at height $m$, and $\zeta_m$ stands for the number of these children that are on the right of the spine, that is, black vertices on Figure 2.

The delicate part of the proof is to replace the “essentially independent offspring” in the heuristic argument above by a rigorous claim. To do so, we show that the distribution of $T$ is close to that of $T^*$, the tree built by growing independent BPVE trees along an infinite spine with size-biased offspring (i.e. Kesten’s tree in the varying environment, see Section 5.1 for definition and details). On $T^*$, the ancestors of a given vertex have the desired distribution. This allows us to show that, for most vertices $x$ of $T$ at some large height (conditionally on $T$ reaching that height), the ratio $(X_x - I_x)/H_x$ between the reflected Lukasiewicz path and the height process at $x$ is arbitrarily close to $\sigma^2 / 2$ (Proposition 5.3).

2.4 Discussion of the existing literature on BPVEs and BPREs

We give a brief review of existing results on BPVEs and BPREs; we refer the reader to Kersting and Vatutin’s book [KV17] for more details.

BPVEs have been investigated for more than half a century. First works focussed on the survival or extinction, and on the long-term behaviour. Lindvall [Lin74] proved that, as long as
the varying environment is non-degenerate, i.e. \( P(\xi_n^{(1)} = 0) < 1 \) for all \( n \geq 0 \), then \( Z_n \) converges almost surely to a possibly infinite random variable \( Z_\infty \) (convergence in distribution was already known, see [AK71, Chu71, Jag74]). Lindvall also showed that, under some conditions on the varying environment, \( 0 < Z_\infty < \infty \), which means that a BPVE does not obey the usual dichotomy of extinction or divergence to infinity. Jagers [Jag74], and later Kersting [Ker20], are interested in regular BPVEs, i.e. when \( Z_\infty \in \{0, +\infty\} \) almost surely. They classify these regular BPVEs into sub-critical, critical and super-critical (see, e.g., [Ker20, Proposition 1]).

The first scaling limits to have been obtained for BPVEs were the limits of generation sizes: one starts with a large initial population and tracks the evolution of its size generation after generation. This does not provide information on the genealogical structure of the forest. Such limits are the most accessible. First results track back to Kurtz [Kur78] and Helland [Hel81]. They show that the scaling limit for the successive generation sizes is a Feller diffusion both for strictly critical BPREs and critical Bienaym´e-Galton-Watson trees. While a third moment assumption was required in [Kur78], these results have been considerably extended in the last decade, to cases with infinite variance, and bottlenecks (corresponding to a catastrophe in the environment killing almost all the population). See e.g., Bansaye and Simatos [BS15] and Fang, Li and Liu [FZ12]. Contrary to Bienaym´e-Galton-Watson trees, the limiting process is then no longer a Continuous State Branching Process and can be difficult to characterize.

In the last couple of years, finer scaling limits have been proved for regular, critical BPVEs conditioned to survive, showing that these trees behave somewhat similarly to critical Bienaym´e-Galton-Watson trees. On the one hand, Cardona-Tob´on and Palau [CTP21] give an asymptotic result for the size of the \( n \)-the generation of a regular, critical BPVE conditioned on non-extinction. On the other hand, Kersting [Ker21] shows that the height of the last common ancestor of all individuals at generation \( n \) is asymptotically uniform in \( [0,n] \).

Weeks before this preprint was finished, more results were published in the same direction. Harris, Palau and Pardo [CSC22] analyze the genealogy of an arbitrary fixed number \( k \in \mathbb{N} \) of individuals chosen uniformly in the \( n \)-th generation, conditionally on survival until that height. They draw \( k \) spines through the first \( n \) generations of the tree, extending the 2-spine decomposition in [CTP21]. Boenkost, Foutel-Rodier and Schertzer [BFRS22], using a spinal decomposition similar to [FRS22], obtain the genealogy in Gromov-Hausdorff-Prokhorov metric of the first \( n \) generations towards the (possibly time-changed) Brownian coalescent point process. The latter tracks where the spines meet, i.e. where pairs of individuals of generation \( n \) have their last common ancestor. All these results (for critical BPVEs in [CSC22], and for nearly-critical BPVEs in [BFRS22], both under a second moment assumption) are consistent with the fact that a strictly critical BPRE converges to the continuum random tree, as we prove in our main result.

Let us stress that our results and approach are conceptually different: in these papers, the idea is to sample several coalescing spines of fixed length (forward in time in [CSC22], backward in [BFRS22]) to analyze the genealogy of the tree. In our work, we track the evolution of a dynamic spine, with varying length, which follows the depth-first exploration of the tree. This is in the spirit of the works of Duquesne, Le Gall and Le Jan [LG93, GJ98, DLG02].

2.5 Open questions

**Deterministic conditions on the environment \( \mu \).** As mentioned in Section 2.3 Theorem 1.1 is valid under Conditions (I)-(V). These conditions are satisfied by the branching process in random environment we consider, but it would be nice to find necessary and sufficient conditions on the deterministic varying environment \( \mu \). Condition (I) ensures that the laws of successive generations are not too erratic, and Condition (II) rules out cases where the
tree would have long portions with vertices with only one offspring, which would take us out of
the Brownian regime. These conditions could perhaps be relaxed by normalizing the height as
a function of the cumulant sum of the variances on successive generations, as in the assumption
(2) of [BFRS22], which may accordingly stretch the limiting object. Condition (III) is essentially
necessary in order to not see vertices with large degrees in the limit, and it should be noted
that it is not a consequence of Conditions (I) and (II). Note that Condition (III) corresponds
to assumption (3') in [BFRS22]. On the other hand, Conditions (IV) and (V) seem to be of
technical nature. In fact, one can show that Condition (V) is not necessary once one already
has Conditions (I) (II) and (III) we only use it to estimate the probability that a $\mu$-BPVE
survives at least $m$ generations for large $m$, in (3.4). The latter can be shown by adapting
the proof of the Kolmogorov estimate in [BFRS22] (Theorem 4.1 - taking $m = \kappa N$, one can replace
$\sqrt{\kappa N}$ by $\kappa N$ in Lemma 4.3, using Condition (II) to get a lower bound on the $\varphi_k(0)$'s in the first
equation p.17).

Extension to other regimes. As mentioned in Section 1.3 in the non-strictly critical regime,
the structure of $T$ heavily depends on the environment $\mu$, in particular on the sequence
$(\log \mu_k)_{k \geq 0}$. Let $S_i := \log \mu_1 + \ldots + \log \mu_i$, for $i \geq 0$. In the strictly critical regime, $(S_i)$ is
a.s. identically equal to 0, while in the non-strictly critical case, $(S_i)$ is a random walk with
i.i.d. increments. Some finite but rather long portions of its trajectory (of order $i$ on $[0,i]$ if
the increments of $(S_i)$ have a finite second moment, for instance) look as if it were induced by
a super-critical environment, and a tree can typically exhibit super-polynomial growth on the
corresponding generations, so that the Continuum Random Tree cannot be its scaling limit.
One can ask what would be good conditions on $(S_i)$ in order to still have convergence to the
CRT, or a slightly modified version of it. For instance, the aforementioned result of [BFRS22]
holds for BPVEs in the near critical regime, i.e. when $(S_i)$ is bounded, hence one can conjecture
that the scaling limit of a large tree would be an untruncated version of a Coalescent Point
Process.
Moreover, one could investigate distributions with weaker moment assumptions, for instance
heavy-tailed offspring distributions having an infinite variance. For standard Bienaymé-Galton-
Watson trees, this has already been done in [DLG02, Duq03, Kor13]. In varying or random
environment, only the scaling limits the successive generation sizes are known. One intrigu-
ing question, asked by Bansaye and Simatos in [BS15] (Section 2.5.1), is what happens when
offspring distributions have different heavy tails.

Conditioning on the total population size to be exactly $n$ instead of at least $n$.
Generally speaking, when studying scaling limits of critical random trees, it is noticeably more
delicate to condition the total population size to be exactly $n$. For usual critical Bienaymé-
Galton-Watson trees with finite variance, this was proved by Aldous [Ald93, Section 5], and
later Markert and Mokkadem [MM03] proposed a simpler proof, but under the assumption
that the offspring distribution has exponential moments. The case of critical Bienaymé-Galton-
Watson trees with heavy-tailed offspring distributions was treated by Duquesne [Duq03] and
Kortchemski [Kor13]. The arguments in the works mentioned above rely heavily on exact
algebraic or distributional identities that do not hold in varying environment. Hence, one
would need here again to come up with alternative arguments in order to obtain this refined
conditioning.
To conclude, we believe that our result holds under this stronger conditioning, but its proof
would require significant effort, hence we choose to leave this question for further work.
2.6 Plan

In Section 3, we prove preliminary results whose counterparts for critical Bienaymé-Galton-Watson trees are either straightforward or classic, but not for a BPRE. In particular, Lemma 3.2 states that Conditions (I)-(V) hold \( P \)-a.s., and Proposition 3.4 states that \( \mathcal{F}_n \) has height and width \( \Theta(\sqrt{n}) \). In Section 4, we prove Theorem 2.1. Finally, in Section 5, we prove Theorem 2.3, and conclude the proof of Theorem 1.1.

3 Basic properties

The main results of this section are Lemma 3.2 and Proposition 3.4.

3.1 A strictly critical BPRE satisfies Conditions (I)-(V)

Before proving in Lemma 3.2 that a BPRE satisfies almost surely our deterministic conditions, we provide below a technical lemma which will be useful in different places.

Lemma 3.1. Let \((s_k)_{k \geq 0}\) be a sequence such that \( \ell := \lim_{k \to +\infty} \frac{s_1 + \ldots + s_k}{k} \) exists and is finite. For all \( 0 \leq a < b \),

\[
\frac{s_{\lfloor ak \rfloor} + 1 + s_{\lfloor ak \rfloor} + 2 + \ldots + s_{\lfloor bk \rfloor}}{(b-a)k} \xrightarrow{k \to +\infty} \ell.
\]

Moreover, for any \( c > 0 \) and \( M \in \mathbb{N} \),

\[
\max_{1 \leq j \leq M} \left| \frac{s_{\lfloor (j-1)ck \rfloor} + \ldots + s_{\lfloor jck \rfloor}}{ck} - \ell \right| \xrightarrow{k \to +\infty} 0.
\]

Proof. Let us fix \( 0 \leq a < b \). Fix \( \varepsilon > 0 \). There exists \( K_\varepsilon \in \mathbb{N} \) such that for all \( k \geq K_\varepsilon \), and for \( c \in \{a, b\} \), \( |s_1 + \ldots + s_{\lfloor ck \rfloor} - \ell ck| \leq \varepsilon (b-a)k/2 \), as can be shown by a straightforward computation, using the definition of \( \ell \). Using the triangle inequality, we obtain

\[
|s_{\lfloor ak \rfloor} + 1 + s_{\lfloor ak \rfloor} + 2 + \ldots + s_{\lfloor bk \rfloor} - \ell (b-a)k| \leq |s_1 + \ldots + s_{\lfloor ak \rfloor} - \ell ak| + |s_1 + \ldots + s_{\lfloor bk \rfloor} - \ell bk| \leq \varepsilon (b-a)k.
\]

The above implies that for all \( \varepsilon > 0 \), there exists \( K_\varepsilon \in \mathbb{N} \) such that, for all \( k \geq K_\varepsilon \), we have

\[
\left| \frac{s_{\lfloor ak \rfloor} + 1 + s_{\lfloor ak \rfloor} + 2 + \ldots + s_{\lfloor bk \rfloor}}{(b-a)k} - \ell \right| \leq \varepsilon.
\]

This proves the first statement. The second statement is a straightforward consequence of the first one by choosing \( a = (j-1)c, b = jc \), for \( 1 \leq j \leq M \).

Recall that, as defined in Section 1.1 under \( P \), \( \mu \) is a sequence of i.i.d. random distributions, and that \( \bar{\mu}_k \) denotes the average of \( \mu_k \) and \( \sigma_k^2 \) its variance, for \( k \geq 0 \).

Lemma 3.2. Consider a \( P \)-BPRE such that \( \bar{\mu}_0 = 1 \) \( P \)-almost surely, and \( \sigma^2 := E[\sigma_0^2] < \infty \) \( P \)-almost surely. Then, \( \mu \) satisfies Conditions (I)-(V) \( P \)-almost surely, where the constant \( c > 0 \) in Condition (III) is deterministic under \( P \).

Proof. Condition (I) Under \( P \), \( (\sigma_k^2)_{k \geq 0} \) is a sequence of i.i.d. random variables with mean \( \sigma^2 < \infty \), hence it satisfies the Law of Large Numbers and therefore Condition (I) holds \( P \)-almost surely.
**Condition (II)** We will prove a statement that is stronger than Condition (II) which is that there exists a constant $c > 0$ such that $P$-a.s., for $0 \leq a < b$,

$$\lim_{k \to +\infty} \frac{\mu_{[ak]}(\{0\}) + \ldots + \mu_{[bk]}(\{0\})}{(b-a)k} = c.$$  \hspace{1cm} (3.1)

First, note that it is enough to prove that this statement holds $P$-a.s. for all pairs $(a, b)$ of rationals. By union bound, it is enough to prove that for a pair of rationals $(a, b)$, (3.1) holds $P$-almost surely.

By definition of $P$ and $\mu$, we have that $\mu_0(\{0\}) \in [0, 1]$ $P$-almost-surely, and $(\mu_k(\{0\}))_{k \geq 0}$ is thus a sequence of bounded random variables. Hence, using the Law of Large Numbers, Lemma 3.1 and the fact that $\mu_{[ak]}(\{0\})$ is bounded by 1, we have that (3.1) is satisfied with $c = E[\mu_0(\{0\})].$

To conclude that Condition (II) holds, it remains to prove that $P[\mu_0(\{0\})] > 0$. For this purpose, assume by contradiction that $E[\mu_0(\{0\})] = 0$. As $\mu_0(\{0\}) \geq 0$ and $\mu_0 = 1$ $P$-a.s., this implies that $\mu_0(\{0\}) = 0$ $P$-a.s. and thus that $\mu_0(\{1\}) = 1$ $P$-almost surely by strict criticality of $\mu_0$.

This yields that $\sigma^2 = 0$ $P$-a.s., which contradicts that $\sigma^2 = E[\sigma_0^2] > 0$. This concludes the proof that Condition (II) holds almost surely.

**Condition (III).** Let us fix $\varepsilon > 0$. For all $k \geq 0$ and $n \geq 1$, let

$$e_{k,n} := E_{\mu}[\xi_1^{2} \mathbf{1}_{\{\xi_2 \geq \varepsilon n\}}].$$

We start by the following fact: since, for all $n \geq 0$, the random variables $e_{k,n}$, $k \geq 0$, are i.i.d. and since $\xi_0^2$ is $\mathbb{P}$-integrable, we have that

$$\lim_{n \to \infty} E[e_{k,n}] = \lim_{n \to \infty} E[e_{0,n}] = \lim_{n \to \infty} E[\xi_{0}^{2} \mathbf{1}_{\xi_2 \geq \varepsilon n}] = 0.$$ 

Hence, for all $m \geq 1$, there exists $n_m \in \mathbb{N}$ such that

$$E[e_{0,n_m}] \leq \frac{1}{m^2}.$$ 

Note again that $(e_{k,n})_{k \geq 0}$ is an i.i.d. sequence of integrable random variables, hence it satisfies the Law of Large Numbers. Moreover, for all $k \geq 0$ and for all $n \geq n_m$, we have $e_{k,n} \leq e_{k,n_m}$.

Therefore, we have that $P$-almost surely, there exists a random (but $\mu$-measurable) increasing sequence $(N_m)_{m \geq 1}$ of natural integers such that for all $m \geq 1$ and all $n \geq N_m$, we have

$$\inf_{h \geq 1} \frac{1}{h \sqrt{n}} \sum_{k=0}^{[h n / \pi]} e_{k,n} \leq \inf_{h \geq 1} \frac{1}{h \sqrt{n}} \sum_{k=0}^{[h n / \pi]} e_{k,N_m} \leq \frac{2}{m^2}.$$ 

For all $n \geq 1$, let $h_n$ be the unique index such that $N_{h_n} \leq n < N_{h_n+1}$, letting $N_0 = 0$. Note that, because $N_m < \infty$ $P$-a.s. for all $m \geq 0$, we have that $h_n$ goes to infinity with $n$ and $h_n \geq 1$ for all $n \geq N_1$. Hence, all the previous implies that, $P$-a.s., we have for all $n \geq N_1$,

$$\frac{1}{h_n \sqrt{n}} \sum_{k=0}^{[h_n \sqrt{n} / \pi]} e_{k,n} \leq \frac{2}{h_n^2}.$$ 

Altogether, we prove that for all $\varepsilon > 0$, there exists $P$-a.s. a sequence of real numbers $(h_n)_{n \geq 1}$ diverging to infinity such that

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{k=0}^{[h_n \sqrt{n} / \pi]} e_{k,n} \leq \lim_{n \to \infty} \frac{2}{h_n} = 0.$$ 

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which concludes the proof of that Condition (III) holds P-almost surely.

Condition (IV). Recall that, for $k \geq 0$, $\bar{\xi}_k$ denotes a random variable distributed under the size-biased version of $\mu_k$, and that $\zeta_k$ is uniform in $\{0, \ldots, \xi_k - 1\}$. Under $P_\mu$, the $\bar{\xi}_k$’s have different distributions, and we cannot apply the Law of Large Numbers. However, the $\bar{\xi}_k$’s are i.i.d. under $P$. We let $\xi_0$ be a random variable of distribution $\mu_0$. With this notation, for all $k \geq 0$,

$$E[\bar{\xi}_k] = E[\xi_0] = E[E_\mu[\bar{\xi}_0]] = E[E_\mu[\xi_0^2]] = E[\text{Var}_\mu(\xi_0) + E_\mu[\xi_0]^2] = E[\sigma_0^2 + 1] = \sigma^2 + 1,$$

so that, for all $k \geq 0$,

$$E[\zeta_k] = E[\bar{\xi}_k] - 1 = \frac{\sigma^2}{2}.$$

By the strong Law of Large Numbers applied to the sequence $(\zeta_k)_{k \geq 0}$,

$$1 = P_\mu\left(\lim_{n \to +\infty} n^{-1} \sum_{k=0}^{n-1} \zeta_k = \frac{\sigma^2}{2}\right) = E_\mu\left(P_\mu\left(\lim_{n \to +\infty} n^{-1} \sum_{k=0}^{n-1} \zeta_k = \frac{\sigma^2}{2}\right)\right).$$

Hence, $P_\mu(n^{-1} \sum_{k=0}^{n-1} \zeta_k \longrightarrow \sigma^2/2) = 1$ P-a.s., as required.

Condition (V). Recall the definition of $\varphi_k$ and $\omega_k$ in (2.7) and below it. We have that, P-a.s., for all $k \geq 0$,

$$\varphi_k(0) = \mu_k(\{0\}), \quad \varphi_k(1) := \lim_{s \uparrow 1} \varphi_k(s) = \frac{\sigma_k^2}{2} \quad \text{and} \quad \frac{1}{2} \varphi_k(0) \leq \varphi_k(s) \leq 2\varphi_k(1), \quad \text{for all } s \in [0, 1],$$

where the second line is given by [KV17, Proposition 1.4], and the first line can be easily proved by a straightforward computation, and by using Taylor expansion together with the fact that $E_\mu[\xi_k] = 1$ P-a.s., see [KV17, Equation (1.9)] for a similar computation. Using the definition of $\omega_k(\cdot)$ and the triangle inequality, this implies that, P-a.s., for all $\varepsilon \in [0, 1]$ and for all $k \geq 0$, $0 \leq \omega_k(\varepsilon) \leq 2\sigma_k^2$ and thus $\omega_k(\varepsilon)$ is integrable.

For all $\varepsilon \in [0, 1]$, $(\omega_k(\varepsilon))_{k \geq 0}$ is a sequence of i.i.d. random variables, thus the strong Law of Large Numbers implies that, P-a.s.,

$$\lim_{n \to +\infty} \frac{\sum_{k=0}^{n-1} \omega_k(\varepsilon)}{n} = E[\omega_0(\varepsilon)] \leq 2\sigma^2.$$

Now, note that $\omega_0(\cdot)$ is non-decreasing, hence the sequence $(\omega_0(1/m))_{m \geq 1}$ is non-increasing and converging to 0 P-a.s. by (3.2). By the monotone convergence theorem, we have that $E[\omega_0(1/m)]$ goes to 0 as $m$ goes to infinity, and thus

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to +\infty} \frac{\sum_{k=0}^{n-1} \omega_k(\varepsilon)}{n} = 0.$$

This concludes the proof that Condition (V) holds P-almost surely.

3.2 Survival up to some height

The following lemma will be useful in Section 5.
Lemma 3.3. Let \( \mu \) be a strictly critical environment such that Conditions \( (I) \) \( (II) \) \( (V) \) hold. Let \( T = T^\mu \) be a \( \mu \)-BPVE. Then, for all \( \eta > 0 \), there exists \( \varepsilon_0 = \varepsilon_0(\mu, \eta) > 0 \) and \( M = M(\mu, \eta, \varepsilon_0) \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and \( m \geq M \),

\[
\frac{1}{2\sigma^2 m} \leq P_{\mu}(h(T) \geq m) \leq \frac{4}{cm}, \tag{3.3}
\]

\[
P_{\mu}(h(T) \geq (1 + \varepsilon)m) \geq (1 - \eta)P_{\mu}(h(T) \geq m), \tag{3.4}
\]

\[
P_{\mu}(Z_m(T) \leq \varepsilon m \mid h(T) \geq m) \leq \eta. \tag{3.5}
\]

Proof. Remark that for all \( m \geq 1, \varepsilon \mapsto P_{\mu}(h(T) \geq (1 + \varepsilon)m) \) is non-increasing and \( \varepsilon \mapsto P_{\mu}(Z_m(T) \leq \varepsilon m \mid h(T) \geq m) \) is non-decreasing, so that it is enough to prove that the statements hold for \( \varepsilon \) instead of \( \varepsilon \in (0, \varepsilon_0) \).

We start with the proof of (3.3). Fix \( \eta > 0 \). For all \( k, m \geq 1 \), we let \( q_{k,m} := P_{(\mu_{\varepsilon}, \varepsilon) \geq k}(h(T) < m - k) \) be the \( \mu_{\varepsilon} \)-probability that a given vertex at height \( k \) does not have descendants at height \( m \). By [KV17, Equation (2.11)], we have

\[
P_{\mu}(h(T) \geq m) = \frac{1}{1 + \sum_{k=0}^{m-1} \varphi_k(q_{k,m})}, \tag{3.6}
\]

where \( \varphi_k \) is defined in (2.7). Using (3.2) together with Conditions \( (I) \) and \( (II) \) we have that there exists \( M_1 = M_1(\mu) \) such that, for all \( m \geq M_1 \),

\[
\sum_{k=0}^{m-1} \varphi_k(q_{k,m}) \geq \frac{1}{2} \sum_{k=0}^{m-1} \varphi_k(0) \geq \frac{1}{2} \sum_{k=0}^{m-1} \mu_k(0) \geq \frac{c}{4} m,
\]

\[
1 + \sum_{k=0}^{m-1} \varphi_k(q_{k,m}) \leq 1 + 2 \sum_{k=0}^{m-1} \varphi_k(1) \leq 2 \sigma^2 m,
\]

where \( c \) is the constant provided by Condition \( (II) \). This implies that, for all \( m \geq M_1 \),

\[
\frac{1}{2\sigma^2 m} \leq P_{\mu}(h(T) \geq m) \leq \frac{4}{cm}
\]

and (3.3) follows. We now turn to the proof of (3.4). For all \( m \geq 0 \) and \( \varepsilon \in (0, 1/2) \), we define

\[
S_m = \sum_{k=0}^{m-1} \varphi_k(q_{k,m}), \quad S_{(1+\varepsilon)m} = \sum_{k=0}^{[(1+\varepsilon)m]-1} \varphi_k(q_{k,[(1+\varepsilon)m]}),
\]

\[
A_{\varepsilon,m} = \sum_{k=0}^{[(1+\varepsilon)m]-1} (\varphi_k(q_{k,[(1+\varepsilon)m]}) - \varphi_k(q_{k,m})), \quad \text{and}
\]

\[
B_{\varepsilon,m} = \sum_{k=m}^{[(1+\varepsilon)m]-1} \varphi_k(q_{k,[(1+\varepsilon)m]}).
\]

By (3.6), we have that

\[
P_{\mu}(h(T) \geq m) - P_{\mu}(h(T) \geq (1 + \varepsilon)m) \leq \frac{A_{\varepsilon,m} + B_{\varepsilon,m}}{(1 + S_m)(1 + S_{(1+\varepsilon)m})}.
\]

Moreover, (3.2) yields that

\[
\limsup_{m \to \infty} \frac{1}{m} B_{\varepsilon,m} \leq \limsup_{m \to \infty} \frac{1}{m} \sum_{k=m}^{[(1+\varepsilon)m]-1} \sigma_k^2 = \varepsilon \sigma^2, \tag{3.8}
\]
where we used Condition (I) and Lemma 3.1 for the last equality. Thus, there exists \( M_2 = M_2(\mu, \varepsilon) \) such that for all \( m \geq M_2, B_{\varepsilon, m} \leq 2\varepsilon\sigma^2 m \). Furthermore, by (3.7), we have that, for all \( m \geq M_1, \)

\[
(1 + S_m)(1 + S_{(1+\varepsilon)m}) \geq \frac{c^2 m^2}{16}.
\]

Therefore, for all \( m \geq M_1 \lor M_2 \), we have that

\[
P_\mu(h(T) \geq m) - P_\mu(h(T) \geq (1 + \varepsilon)m) \leq 16 \frac{A_{\varepsilon, m}}{c^4 m^2} + 32 \frac{\varepsilon \sigma^2}{c^2 m}.
\]

(3.9)

We now aim at bounding \( A_{\varepsilon, m} \) from above. For all \( k \geq 1 \) and \( m \geq 1 \), let

\[
A_{\varepsilon, m}^{(1)} := \sum_{k=0}^{[\frac{(1-\varepsilon)m}{1}]} (\varphi_k(q_{k,\lfloor(1+\varepsilon)m\rfloor}) - \varphi_k(q_{k,m})),
\]

\[
A_{\varepsilon, m}^{(2)} := \sum_{k=[(1-\varepsilon)m]}^{k_m} (\varphi_k(q_{k,\lfloor(1+\varepsilon)m\rfloor}) - \varphi_k(q_{k,m})),
\]

(3.10)

so that \( A_{\varepsilon, m} = A_{\varepsilon, m}^{(1)} + A_{\varepsilon, m}^{(2)} \). First, proceeding as for \( B_{\varepsilon, m} \) above, we obtain that there exists \( M_3 = M_3(\mu, \varepsilon) \) such that, for all \( m \geq M_3, \)

\[
A_{\varepsilon, m}^{(2)} \leq 2\varepsilon\sigma^2 m.
\]

(3.11)

Second, by (3.2) and (3.6), we have that for all \( k \leq (1-\varepsilon)m, \)

\[
q_{k,m} = 1 - \frac{1}{1 + \sum_{j=k}^{m-1} \varphi_j(q_{j,m})}.
\]

\[
q_{k,m} \geq 1 - \frac{1}{1 + \frac{1}{2} \sum_{j=[(1-\varepsilon)m]}^{m} \mu_j(\emptyset)}.
\]

Hence, by Condition (II) there exists \( M_4 = M_4(\mu, \varepsilon) \), such that, for all \( m \geq M_4, \)

\[
\min_{0 \leq k \leq (1-\varepsilon)m} q_{k,m} \geq 1 - \frac{4}{c\varepsilon m} \geq 1 - \varepsilon.
\]

(3.12)

Using similar arguments, one can obtain that for all \( m \geq M_4, \)

\[
\min_{0 \leq k \leq (1-\varepsilon)m} q_{k,\lfloor(1+\varepsilon)m\rfloor} \geq 1 - \varepsilon.
\]

(3.13)

Recall that \( \omega_k(\varepsilon) = \sup_{1-\varepsilon \leq s \leq t \leq 1} |\varphi_k(s) - \varphi_k(t)| \). Using (3.12), (3.13) and the triangle inequality, we obtain that, for all \( m \geq M_4, \) and for all \( k \leq (1-\varepsilon)m, \)

\[
|\varphi_k(q_{k,\lfloor(1+\varepsilon)m\rfloor}) - \varphi_k(q_{k,m})| \leq \omega_k(\varepsilon).
\]

(3.14)

Hence, for all \( m \geq M_4, \)

\[
A_{\varepsilon, m}^{(1)} \leq \sum_{k=0}^{m-1} \omega_k(\varepsilon).
\]

By Condition (V) there exists \( \varepsilon = \varepsilon(\mu, \eta) > 0 \) small enough and \( M_5 = M_5(\mu, \eta, \varepsilon) \), such that, for all \( m \geq M_5, \)

\[
A_{\varepsilon, m}^{(1)} \leq \frac{c^2 \eta}{48} m.
\]

(3.15)
Hence, putting (3.9), (3.11) and (3.15) together, we deduce the existence of \( M_6 = M_6(\mu, \eta, \varepsilon) \), such that, for all \( m \geq M_6 \),

\[
P_\mu(h(T) \geq m) - P_\mu(\varepsilon m) \leq (1 + \varepsilon) \leq 16 \frac{2 \varepsilon^2}{c^2m} + 32 \frac{\varepsilon^2}{c^2m} \leq \frac{\eta}{\varepsilon^2m}. \tag{3.16}
\]

Using the above together with (3.3), we obtain that there exists \( M_7 = M_7(\mu, \eta, \varepsilon) \) such that, for all \( m \geq M_7 \),

\[
P_\mu(h(T) \geq (1 + \varepsilon)m) \geq (P_\mu(h(T) \geq m) ) - P_\mu(h(T) \geq (1 + \varepsilon)m) \geq P_\mu(h(T) \geq m) - \frac{\eta}{\varepsilon^2m} \geq P_\mu(h(T) \geq m) \left(1 - \frac{\eta}{2}\right) \geq P_\mu(h(T) \geq m)(1 - \eta).
\]

since \( \eta > 0 \) was arbitrary, this concludes the proof of (3.4), taking \( \varepsilon_0 = \varepsilon \) and \( M = M_7 \).

We finally establish (3.5). Let \( \eta > 0 \) and write \( p_{\alpha, m} := P_\mu(h(T) < (1 + \varepsilon_0)m) \). By (3.4), there exists \( \varepsilon_0 = \varepsilon(\mu, \eta) > 0 \) and \( M = M(\mu, \eta, \varepsilon_0) \geq 0 \) such that for all \( m \geq M \) and \( \alpha > 0 \), we have that

\[
P_\mu(Z_m(T) \leq \alpha m \mid h(T) \geq m) \leq \frac{p_{\varepsilon_0, m} + p_\mu(Z_{(1+\varepsilon_0)m}(T) > 0 \text{ and } 0 < Z_m(T) \leq \alpha m)}{P_\mu(h(T) \geq m)} \leq \frac{\eta}{2} + \frac{p_\mu(Z_{(1+\varepsilon_0)m}(T) > 0 \text{ and } 0 < Z_m(T) \leq \alpha m)}{P_\mu(Z_m(T) > 0)} \leq \frac{\eta}{2} + \frac{\alpha m p_{(\mu)k \geq m}(h(T) \geq \varepsilon_0 m)}{P_\mu(h(T) \geq \varepsilon_0)} . \tag{3.18}
\]

wherer we used a union bound on the vertices of \( Z_m(T) \) for the last inequality. Using formulas similar to (3.6) and (3.7) together with Lemma 3.1, we obtain that there exists \( M_8 = M_8(\mu, \eta, \varepsilon_0) \) such that, for all \( m \geq M_8 \) and for all \( \alpha < \eta \varepsilon_0 / \varepsilon \),

\[
P_\mu(Z_m(T) \leq \alpha m \mid h(T) \geq m) \leq \frac{\eta}{2} + \frac{\alpha m}{1 + \sum_{k=m}^{\infty} \varphi_k(0)} \leq \frac{\eta}{2} + \frac{\alpha m}{1 + \sum_{k=m}^{\infty} \varphi_k(0)} \leq \frac{\eta}{2} + \frac{4 \alpha}{\varepsilon_0} \leq \eta. \tag{3.19}
\]

This concludes the proof. \(
\)

### 3.3 Coarse shape of the forest \( F_n \)

Recall the definition (2.3) of \( F_n \), the forest explored at step \( n \). Let \( t(F_n) \) be the number of trees in \( F_n \) (including the last tree that is possibly not fully explored). Recall that \( w(F_n) \) and \( h(F_n) \) denote the width and the height of \( F_n \), respectively, defined in (2.3).

The proposition below states that at step \( n \), the explored forest \( F_n \) is contained in a square of height and width both of order \( \sqrt{n} \): this means that the highest tree in \( F_n \) is of order \( \sqrt{n} \) and, that the total number of points of \( F_n \) in a given generation is at most of order \( \sqrt{n} \), see Figure 3 for an illustration of this.
Proposition 3.4. Let $\mu$ be a strictly critical environment satisfying Conditions (I) and (II). For all $\epsilon > 0$, there exist constants $a_1 = a_1(\epsilon), A_1 = A_1(\epsilon), a_2 = a_2(\epsilon), A_2 = A_2(\epsilon) > 0$ depending only on $\sigma^2, \epsilon$ and the constant $c$ from Condition (III) and there exists $N = N(\mu, \epsilon)$ such that, for all $n \geq N$,

$$P_{\mu}(a_1\sqrt{n} \leq t(F_n) \leq w(F_n) \leq A_1\sqrt{n} \text{ and } a_2\sqrt{n} \leq h(F_n) \leq A_2\sqrt{n}) \geq 1 - \epsilon.$$ 

The following lemma will be used in the proof of Proposition 3.4 and several times in the rest of the paper.

Lemma 3.5. Let $\mu$ be a strictly critical environment, let $k_0 \in \mathbb{N}$ and define $\nu := (\mu_{k_0} + k)_{k \geq 0}$. Let $z_0 \in \mathbb{N}$ and let $(Z_k)_{k \geq 0}$ be a $\nu$-BPVE with $Z_0 = z_0$. Then, for all $k_1 \geq 1$, we have that

$$\text{Var}_\nu(Z_{k_1}) = z_0(\sigma^2_{k_0} + \ldots + \sigma^2_{k_0+k_1-1}), \quad (3.20)$$

and, for all $K > 0$,

$$P_{\nu}(\max_{k \leq k_1} |Z_k - Z_0| > K) \leq \frac{4z_0(\sigma^2_{k_0} + \ldots + \sigma^2_{k_0+k_1-1})}{K^2}. \quad (3.21)$$

Proof. We show (3.20) by induction on $k_1$. The initialisation at $k_1 = 1$ is straightforward: indeed $Z_1$ is simply a sum of $z_0$ independent random variables with distribution $\mu_{k_0}$ hence $\text{Var}_\nu(Z_1) = z_0\sigma^2_{k_0}$. Suppose now that (3.20) holds for some $k_1 \geq 1$. Then by the law of total variance,

$$\text{Var}_\nu(Z_{k_1+1}) = \text{Var}_\nu(\mathbb{E}_\nu[Z_{k_1+1}|Z_{k_1}]) + \mathbb{E}_\nu[\text{Var}_\nu(Z_{k_1+1}|Z_{k_1})]. \quad (3.22)$$

We have $\mathbb{E}_\nu[Z_{k_1+1}|Z_{k_1}] = Z_{k_1}$ $\nu$-a.s., hence, using the induction hypothesis,

$$\text{Var}_\nu(\mathbb{E}_\nu[Z_{k_1+1}|Z_{k_1}]) = \text{Var}_\nu(Z_{k_1}) = z_0(\sigma^2_{k_0} + \ldots + \sigma^2_{k_0+k_1-1}). \quad (3.23)$$

Moreover, we have that $\text{Var}_\nu(Z_{k_1+1}|Z_{k_1}) = Z_{k_1}\sigma^2_{k_1}$ $\nu$-a.s., therefore, using that $\mu$ is strictly critical, we have that

$$\mathbb{E}_\nu[\text{Var}_\nu(Z_{k_1+1}|Z_{k_1})] = \mathbb{E}_\nu[Z_{k_1}\sigma^2_{k_0+k_1}] = z_0\sigma^2_{k_0+k_1}. \quad (3.24)$$

Combining (3.22), (3.23) and (3.24), we obtain that

$$\text{Var}_\nu(Z_{k_1+1}) = z_0(\sigma^2_{k_0} + \ldots + \sigma^2_{k_0+k_1}). \quad (3.25)$$

Hence, by induction, (3.20) holds for all $k_1 \geq 1$.

Finally, as $\nu$ is strictly critical, $(Z_k)_{k \geq 0}$ is a martingale and (3.21) is a straightforward consequence of (3.20) and Doob’s $L^2$-inequality. \[\square\]

We now turn to the proof of Proposition 3.4.

Proof of Proposition 3.4. Let us fix $\delta = \frac{\epsilon}{12}$ throughout the proof. We will proceed in five steps.

Step 1. Let us prove that there exists $N_0 \in \mathbb{N}$ such that, for all $n \geq N_0$,

$$P_{\mu}(t(F_n) \geq A_0\sqrt{n}) \leq \delta, \quad (3.26)$$

with $A_0 = \max\{32\sigma^2, \frac{\epsilon}{7}\}$.

For all $k \geq 0$, let $Z_k$ denote the number of vertices at height $k$ in the first $\lceil A_0\sqrt{n} \rceil$ trees of $F$, so
that \((Z_k)\) is a \(\mu\)-BPVE with \(Z_0 = |A_0 \sqrt{n}|\). By applying Lemma 3.5 with \(k_0 = 0, k_1 = |\delta \sqrt{n}|\) and \(K = A_0 \sqrt{n}/2\), we have

\[
P_{\mu} \left( \min_{k \leq \delta \sqrt{n}} Z_k \leq \frac{A_0 \sqrt{n}}{2} - 1 \right) \leq P_{\mu} \left( \max_{k \leq \delta \sqrt{n}} |Z_k - Z_0| > \frac{A_0 \sqrt{n}}{2} \right) \leq \frac{16 \left( \sigma_0^2 + \ldots + \sigma_{\delta \sqrt{n}}^2 \right)}{A_0 \sqrt{n}}.
\]

By Condition (I), there exists \(N'_0 = N'_0(\mu, \delta) \in \mathbb{N}\) such that, for all \(n \geq N'_0\), we have that \((\sigma_0^2 + \ldots + \sigma_{\delta \sqrt{n}}^2)/\sqrt{n} \leq 2\delta^2\) and thus

\[
P_{\mu} \left( \min_{k \leq \delta \sqrt{n}} Z_k \leq \frac{A_0 \sqrt{n}}{2} - 1 \right) \leq \frac{32 \delta^2 \sqrt{n}}{A_0 \sqrt{n}} \leq \delta. \quad (3.27)
\]

If \(\min_{k \leq \delta \sqrt{n}} Z_k > (A_0 \sqrt{n}/2) - 1\), then there are at least \((A_0 \sqrt{n}/2) - 1\) vertices on the first \(|\delta \sqrt{n}|\) generations of the first \(|A_0 \sqrt{n}|\) trees of \(F\). Hence, provided that \(n \geq \delta^2 \sqrt{2}\delta^{-2}\), these trees together have a total number of vertices exceeding

\[
\left( \frac{A_0 \sqrt{n}}{2} - 1 \right) \times |\delta \sqrt{n}| \geq \left( \frac{4 \sqrt{n}}{\sqrt{\delta}} - 1 \right) \times (\delta \sqrt{n} - 1) \geq \frac{2 \sqrt{n}}{\sqrt{\delta}} \times \frac{\delta \sqrt{n}}{2} \geq n,
\]

thus \(F_n\) is included in these first \(|A_0 \sqrt{n}|\) trees. This proves that

\[
\{ t(F_n) \geq A_0 \sqrt{n} \} \subset \left\{ \min_{k \leq \delta \sqrt{n}} Z_k \leq \frac{A_0 \sqrt{n}}{2} - 1 \right\}. \quad (3.28)
\]

Combining (3.27) and (3.28) proves (3.26) with \(N_0(\mu, \delta) = N'_0(\mu, \delta) \vee \delta^2 \sqrt{2}\delta^{-2}\).

**Step 2.** Let us prove that there exists \(N_2 = N_2(\mu, \delta) \in \mathbb{N}\) such that, for all \(n \geq N_2\),

\[
P_{\mu}(h(F_n) \geq A_2 \sqrt{n}) \leq 2\delta, \quad (3.29)
\]

where \(A_2 = 4A_0/(c\delta) = \max\{\frac{128\delta^2}{c^2}, \frac{32}{c^2}\}\).

Let \((Z_k)_{k \geq 0}\) be a \(\mu\)-BPVE with \(Z_0 = 1\). By (3.6), (3.2) and Condition (II) we have

\[
\limsup_{k \to +\infty} k P_{\mu}(Z_k > 0) \leq 2/c, \quad (3.30)
\]

where \(c > 0\) is given by Condition (II). Using (3.30) above and (3.26), there exists \(N_2 = N_2(\mu, \delta) \in \mathbb{N}\) such that, for all \(n \geq N_2\),

\[
P_{\mu}(h(F_n) \geq A_2 \sqrt{n}) \leq P_{\mu}(t(F_n) \geq A_0 \sqrt{n}) + P_{\mu}(\{t(F_n) < A_0 \sqrt{n}\} \cap \{h(F_n) \geq A_2 \sqrt{n}\})
\]

\[
\leq \delta + \frac{4A_0}{c} \times \frac{\sqrt{n}}{A_2 \sqrt{n}}
\]

\[
\leq 2\delta,
\]

where we applied a union bound for the second inequality, and used that \(A_2 = 4A_0/(c\delta)\) for the last inequality. This proves (3.29).

**Step 3.** Let us prove that there exists \(N_1 \in \mathbb{N}\) such that, for all \(n \geq N_1\),

\[
P_{\mu}(w(F_n) \geq A_1 \sqrt{n}) \leq 4\delta, \quad (3.31)
\]

with \(A_1 = A_0 + \sqrt{8A_0A_2\delta^2}/\delta\), where \(A_0\) and \(A_2\) are defined in (3.26) and (3.29).
As in Step 1, for all $k \geq 0$, let $Z_k$ denote the number of vertices at height $k$ in the first $|A_0 \sqrt{n}|$ trees of $F$, so that $(Z_k)_k$ is a $\mu$-BPVE with $Z_0 = |A_0 \sqrt{n}|$. By (3.26) and (3.29), we have that, for all $n \geq N_0 \lor N_2$,

$$P_\mu(w(F_n) \geq A_1 \sqrt{n}) \leq P_\mu(t(F_n) > A_0 \sqrt{n}) + P_\mu(h(F_n) \geq A_2 \sqrt{n}) + P_\mu\left(\max_{k \leq A_2 \sqrt{n}} Z_k \geq A_1 \sqrt{n}\right) \leq 3\delta + P_\mu\left(\max_{k \leq A_2 \sqrt{n}} |Z_k - Z_0| \geq (A_1 - A_0) \sqrt{n}\right).$$  \hfill (3.32)

Noting that $A_1 > A_0$ and applying Lemma 3.5, we obtain

$$P_\mu\left(\max_{k \leq A_2 \sqrt{n}} |Z_k - Z_0| \geq (A_1 - A_0) \sqrt{n}\right) \leq \frac{4A_0 \sqrt{n}}{(A_1 - A_0)^2 n} (\sigma_0^2 + \ldots + \sigma_{A_2 \sqrt{n}}^2).$$

By Condition (I), we know that $\lim_{n \to +\infty} (\sigma_0^2 + \ldots + \sigma_{A_2 \sqrt{n}}^2) / \sqrt{n} = A_2 \sigma^2$. Hence, there exists $N'_1 = N'_1(\mu, \delta) \in \mathbb{N}$ such that for all $n \geq N'_1$,

$$P_\mu\left(\max_{k \leq A_2 \sqrt{n}} |Z_k - Z_1| \geq (A_1 - A_0) \sqrt{n}\right) \leq \frac{8A_0 A_2 \sigma^2}{(A_1 - A_0)^2} = \delta. \hfill (3.33)$$

Combining (3.32) and (3.33) concludes the proof of (3.31) with $N_1(\mu, \delta) = \max\{N_0, N_2, N'_1\}$.

**Step 4.** Let us prove that, for all $n \geq N_3$,

$$P_\mu(h(F_n) \leq a_2 \sqrt{n}) \leq 4\delta, \hfill (3.34)$$

with $a_2 = 1/(2A_1)$ and $N_3(\mu, \delta) = \max\{N_1, 4A_1^2\}$. Note that, by definition of $F_n$, $w(F_n)$ and $h(F_n)$, we have that

$$n \leq |F_n| \leq (1 + h(F_n)) \times w(F_n),$$

which implies that

$$w(F_n) \geq \frac{n}{1 + h(F_n)}.$$

We conclude the proof of (3.34) by noting that, for all $n \geq N_3$,

$$P_\mu(h(F_n) \leq a_2 \sqrt{n}) \leq P_\mu\left(\frac{w(F_n)}{1 + a_2 \sqrt{n}} \geq \frac{n}{1 + a_2 \sqrt{n}}\right) \leq P_\mu\left(\frac{w(F_n)}{A_1 \sqrt{n} + \frac{1}{2}} \geq \frac{n}{A_1 \sqrt{n} + \frac{1}{2}}\right) \leq P_\mu\left(w(F_n) \geq A_1 \sqrt{n}\right) \leq 4\delta,$$

where we used that $a_2 = 1/(2A_1)$ for the second inequality, that $N_3 \geq 4A_1^2$ for the third inequality and (3.31) for the last inequality.

**Step 5.** Let us prove that there exists $N_4 = N_4(\mu, \delta) \in \mathbb{N}$ such that, for all $n \geq N_4$,

$$P_\mu(t(F_n) \leq a_1 \sqrt{n}) \leq 2\delta, \hfill (3.36)$$

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where
\[ a_1 = \left( \frac{\delta c}{4 \left( 1 + \sqrt{32\sigma^2/\delta^2 c} \right)} \right)^{1/2}. \]

For all \( k \geq 0 \), let \( Z_k \) be the number of vertices at height \( k \) in the first \( \lfloor a_1 \sqrt{n} \rfloor \) trees of \( F \), so that \( (Z_k)_{k \geq 0} \) is a \( \mu \)-BPVE with \( Z_0 = \lfloor a_1 \sqrt{n} \rfloor \). Let \( F^{a_1} \) be the forest consisting of the first \( \lfloor a_1 \sqrt{n} \rfloor \) trees of \( F \). Using that \( |F^{a_1}| \leq (1 + h(F^{a_1})) \times w(F^{a_1}) \), we obtain that

\[
P_\mu \left( t(F_n) \leq \lfloor a_1 \sqrt{n} \rfloor \right) \leq P_\mu \left( |F^{a_1}| \geq n \right)
\leq P_\mu \left( h(F^{a_1}) \geq \frac{4a_1 \sqrt{n}}{\delta c} - 1 \right) + P_\mu \left( \max_{k \leq \lfloor a_1 \sqrt{n} \rfloor} |Z_k| \geq \frac{\delta c}{4a_1} \sqrt{n} \right). \tag{3.37}
\]

By (3.30), there exists \( N'_4 = N'_4(\mu, \delta) \in \mathbb{N} \) such that for all \( n \geq \max\{N'_4, c^2\delta^2/a_1^2\} \),

\[
P_\mu \left( h(F^{a_1}) \geq \frac{4a_1 \sqrt{n}}{\delta c} - 1 \right) \leq \lfloor a_1 \sqrt{n} \rfloor \frac{3}{c \left( \frac{4a_1 \sqrt{n}}{\delta c} - 1 \right)} \leq \delta, \tag{3.38}
\]

where we used a union bound on the \( \lfloor a_1 \sqrt{n} \rfloor \) trees of \( F^{a_1} \) for the first inequality. Moreover, we have that

\[
P_\mu \left( \max \left\{ Z_k : k \leq \frac{4a_1 \sqrt{n}}{\delta c} - 1 \right\} > \frac{\delta c}{4a_1} \sqrt{n} \right)
\leq P_\mu \left( \max \left\{ |Z_k - Z_0| : k \leq \frac{4a_1 \sqrt{n}}{\delta c} - 1 \right\} > \left( \frac{\delta c}{4a_1} - a_1 \right) \sqrt{n} \right)
\leq \frac{4a_1}{(\frac{\delta c}{4a_1} - a_1)^2} \sqrt{n} (\sigma_0^2 + \ldots + \sigma_{\frac{4a_1 \sqrt{n}}{\delta c}}^2),
\]

where we used Lemma 3.5 for the last inequality, recalling that \( Z_0 = \lfloor a_1 \sqrt{n} \rfloor \). By Condition [(I)] there exists \( N''_4 = N''_4(\mu, \delta) \in \mathbb{N} \) such that, for all \( n \geq N''_4 \),

\[
P_\mu \left( \max_{k \leq \lfloor a_1 \sqrt{n} \rfloor} |Z_k| \geq \frac{\delta c}{4a_1} \sqrt{n} \right) \leq \frac{32a_1^2\sigma^2}{\delta c(\frac{\delta c}{4a_1} - a_1)^2} \leq \delta, \tag{3.39}
\]

where the last inequality is obtained from the definition of \( a_1 \) by a straightforward computation. Combining (3.37), (3.38) and (3.39) concludes the proof of (3.36) with \( N_4(\mu, \delta) = \max\{N'_4, N''_4, c^2\delta^2/a_1^2\} \). Finally, this concludes the proof of the lemma by combining Steps 2 to 4, noting that the inequality \( t(F_n) \leq w(F_n) \) is trivial, recalling that \( \varepsilon = 12\delta \) and defining \( N(\mu, \varepsilon) = \max_{1 \leq i \leq 4} N_i(\mu, \delta) \).

\[ \Box \]

4 Proofs of Theorem 2.1 and Corollary 2.2

4.1 Proof of Theorem 2.1

In this section, we prove the convergence of the Łukasiewicz path, i.e. Theorem 2.1, which is a direct consequence of Proposition 4.2 and Lemmas 4.3, 4.4 and 4.7 below. We will consider a strictly critical environment \( \mu \) satisfying Conditions [(I), (II) and (III)]. To prove the theorem,
we will apply a criterion for the convergence of martingales given by [Whi07, Theorem 2.1 (ii)], which is a consequence of [EK86, Theorem 7.1]. Recall that we want to prove that the process 
\[(M_n(t))_{0 \leq t \leq 1} := \left( \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \right)_{0 \leq t \leq 1}\]
converges to a Brownian motion, where \((X_k)_{k \geq 0}\) is the Lukasiewicz path defined in (2.4). It is straightforward to check that, as soon as \(\mu\) is strictly critical, \((X_k)_{k \geq 0}\) is a martingale with respect to its natural filtration.

Due to technicalities to control the largest degree (see (4.3) below), we will in fact apply [Whi07, Theorem 2.1(ii)] to stopped martingales obtained from \((M_n)_{n \geq 1}\) by stopping the exploration processes when they exit a large rectangle. To make this more precise, let us prove the following easy consequence of Condition (III).

**Corollary 4.1.** Let \(\mu\) be an environment satisfying Condition (III). Then, there exist two sequences \((h_n)_{n \geq 1}\) and \((w_n)_{n \geq 1}\), both diverging to \(+\infty\), such that
\[h_n \leq n^{1/3}, \text{ for all } n \geq 1,\]
\[\frac{w_n}{\sqrt{n}} \sum_{k=0}^{\lfloor h_n \sqrt{n} \rfloor} \mathbb{E}_{\mu}[\xi_k^2 1_{\xi_k^2 \geq \varepsilon n}] \to 0,\]
where \(\xi_k \sim \mu_k\), for all \(k \geq 0\).

**Proof.** First, note that if Condition (III) holds for a sequence \((\tilde{h}_n)\), then it trivially holds for \((h_n)\) where, for all \(n \geq 1\), \(h_n = n^{1/3} \wedge \tilde{h}_n\). Consider such a sequence \((h_n)_{n \geq 1}\) so that Condition (III) holds and \(h_n \leq n^{1/3}\) for all \(n \geq 1\).

Define, for all \(n \geq 1\),
\[w_n = \left( \frac{1}{\sqrt{n}} + \frac{\sum_{k=0}^{\lfloor h_n \sqrt{n} \rfloor} \mathbb{E}_{\mu}[\xi_k^2 1_{\xi_k^2 \geq \varepsilon n}]}{\sqrt{n}} \right)^{-1/2}.\]

By Condition (III) we have that \(w_n \to +\infty\) and
\[\left| \frac{w_n}{\sqrt{n}} \sum_{k=0}^{\lfloor h_n \sqrt{n} \rfloor} \mathbb{E}_{\mu}[\xi_k^2 1_{\xi_k^2 \geq \varepsilon n}] \right| \leq \left| \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor h_n \sqrt{n} \rfloor} \mathbb{E}_{\mu}[\xi_k^2 1_{\xi_k^2 \geq \varepsilon n}] \right|^{1/2} \to 0,\]
which concludes the proof. \(\Box\)

For all \(n \geq 1\), define the random time
\[\tau_n := n \wedge \min\{k: h(\mathcal{F}_k) \geq h_n \sqrt{n} \text{ or } w(\mathcal{F}_k) \geq w_n \sqrt{n}\}.\] (4.1)

Recall that \(H_n\) and \(L_n\), \(n \geq 0\), denote respectively the height and the number of children of the \(n\)-th node along the depth-first exploration of the forest \(\mathcal{F}\), so that \(L_n = 1 + X_{n+1} - X_n\). Recall that \(\sigma_i^2\) is the variance of \(\mu_i\), for \(i \geq 0\).

We start by stating and proving a proposition providing three sufficient conditions for the convergence of the Lukasiewicz path. These three conditions are then proved to hold in Lemmas 4.3, 4.4 and 4.7 which altogether proves Theorem 2.1.
Proposition 4.2. Let \( \mu \) be a strictly critical environment satisfying Conditions (I), (II) and (III). Consider \((h_n)_{n \geq 1}\) and \((w_n)_{n \geq 1}\) provided by Corollary 4.1, and \((\tau_n)_{n \geq 1}\) defined in (4.1). The conclusion of Theorem 2.1 holds if the following three conditions are satisfied:

1. \[
\lim_{n \to +\infty} \max_{0 \leq i \leq h_n \sqrt{n}} \frac{\sigma_i^2}{n} = 0, \tag{4.2}
\]

2. \[
\lim_{n \to +\infty} \mathbb{E}_\mu \left[ \max_{0 \leq i \leq \tau_n - 1} \frac{L_i^2}{n} \right] = 0, \tag{4.3}
\]

3. \[
\lim_{n \to +\infty} \frac{n - 1}{n} \sum_{k=0}^{n-1} \frac{\sigma^2_{H_k}}{H_k} = \sigma^2, \text{ in } \mathbb{P}_\mu\text{-probability.} \tag{4.4}
\]

The first condition is a domination of the maximal jump of the quadratic variation of the Lukasiewicz path. The second condition provides a control of the maximal square degree encountered by the exploration process. The third condition gives the convergence of the quadratic variation of the Lukasiewicz path.

Proof. For all \( n \geq 1 \) and for all \( k \geq 0 \), define

\[ \tilde{X}^{(n)}_k := X_{k \wedge \tau_n}. \]

It is straightforward to check that \( \tau_n \) is a stopping time with respect to the natural filtration of \((X_k)_{k \geq 0}\). Hence, \( \tilde{X}^{(n)} \) is a stopped martingale and thus a martingale. For all \( n \geq 1 \), define the process

\[ \tilde{M}_n = \left( n^{-1/2} \tilde{X}^{(n)}_{\lfloor nt \rfloor} \right)_{0 \leq t \leq 1}. \]

By Proposition 3.4 and using that both \((h_n)\) and \((w_n)\) diverge to infinity, we have that, for all \( \varepsilon > 0 \), there exist \( N = N(\mu, \varepsilon) \in \mathbb{N} \) such that for all \( n \geq N \), \( w_n \geq A_1(\varepsilon) \) and \( h_n \geq A_2(\varepsilon) \) and thus

\[ P_\mu(\tau_n < n) \leq P_\mu(h(F_n) \geq h_n \sqrt{n} \text{ or } w(F_n) \geq w_n \sqrt{n}) < \varepsilon. \]

The above implies that

\[ \limsup_{n \to \infty} P_\mu(M_n \neq \tilde{M}_n) = \limsup_{n \to \infty} P_\mu(\exists k \leq n, X_k^{(n)} \neq \tilde{X}_k^{(n)}) = \limsup_{n \to \infty} P_\mu(\tau_n < n) = 0. \tag{4.5} \]

Therefore, it is enough to prove the convergence in law of \( \tilde{M}_n \) to \((\sigma B_t)_{0 \leq t \leq 1}\), as \( n \) goes to infinity.

Recalling that

\[ \tilde{X}^{(n)}_k = \sum_{i=0}^{k \wedge \tau_n - 1} (L_i - 1), \]

it is straightforward to check that the angle-bracket process associated with \( \tilde{M}_n \) is given, for all \( t \in [0, 1] \), by

\[ \langle M_n(t) \rangle = \sum_{k=0}^{\lfloor nt \rfloor \wedge \tau_n - 1} \frac{\sigma_{H_k}}{n}. \]
By [Whi07, Theorem 2.1 (ii)], the three following conditions are sufficient for this convergence to hold:

\[
\lim_{n \to \infty} E_{\mu} \left[ \max_{0 \leq i \leq \tau_n - 1} \frac{\sigma_i^2}{n} \right] = 0, 
\]

(4.6)

\[
\lim_{n \to \infty} E_{\mu} \left[ \max_{0 \leq i \leq \tau_n - 1} \frac{(L_i - 1)^2}{n} \right] = 0, 
\]

(4.7)

\[
\langle M_n(t) \rangle \Rightarrow \sigma^2 t \text{ in distribution, for all } t \in [0, 1]. 
\]

(4.8)

For all \( i \leq \tau_n - 1 \), using definition of \( \tau_n \), we have that \( 0 \leq H_i < h_n \sqrt{n} \), hence

\[
E_{\mu} \left[ \max_{0 \leq i \leq \tau_n - 1} \sigma_i^2 H_i \right] \leq \max_{i \leq h_n \sqrt{n}} \sigma_i^2, 
\]

and (4.2) implies (4.6). For all \( i \geq 0 \), \( (L_i - 1)^2 \leq L_i^2 + 1 \), thus we have that (4.3) implies (4.7). Finally, (4.4) implies that, for all \( t \in [0, 1] \),

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{\lfloor nt \rfloor - 1} \sigma_k^2 = \sigma^2 t 
\]

in \( P_\mu \)-probability. Using that, for all \( t \in [0, 1] \), \( P_\mu(\tau_n < \lfloor nt \rfloor) \to 0 \), we have that (4.4) implies that the limit in (4.8) holds in probability and thus in distribution. This concludes the proof of the proposition.

\[ \square \]

**Lemma 4.3.** Let \( \mu \) be a strictly critical environment satisfying Conditions \([I], [II] \text{ and } [III] \). Consider \( (h_n)_{n \geq 1} \) provided by Corollary 4.1. We have that

\[
\lim_{n \to +\infty} \max_{0 \leq i \leq h_n \sqrt{n}} \frac{\sigma_i^2}{n} = 0. 
\]

In other words, (4.2) is satisfied.

**Proof.** By Corollary 4.1, we know that Condition \([III] \) holds with a sequence \( (h_n)_{n \geq 1} \) such that \( h_n \leq n^{1/3} \) for all \( n \geq 1 \). By Condition \([I] \), we have that

\[
\limsup_{n \to \infty} \max_{0 \leq i \leq h_n \sqrt{n}} \frac{\sigma_i^2}{n} \leq \limsup_{n \to \infty} \frac{\sum_{i=0}^{\lfloor h_n \sqrt{n} \rfloor} \sigma_i^2}{n} \leq n^{-1/6} \limsup_{n \to \infty} \frac{\sum_{i=0}^{\lfloor h_n \sqrt{n} \rfloor} \sigma_i^2}{h_n \sqrt{n}} = 0. 
\]

\[ \square \]

**Lemma 4.4.** Let \( \mu \) be a strictly critical environment satisfying Conditions \([I], [II] \text{ and } [III] \). Consider \( (h_n)_{n \geq 1} \) and \( (w_n)_{n \geq 1} \) provided by Corollary 4.1 and \( (\tau_n)_{n \geq 1} \) defined in (4.1). We have that

\[
\lim_{n \to +\infty} E_{\mu} \left[ \max_{0 \leq i \leq \tau_n - 1} \frac{L_i^2}{n} \right] = 0. 
\]

In other words, (4.3) is satisfied.

In the proof below, we use the notation \( E_{\mu}[A; B] = E_{\mu}[A 1_B] \), for a random variable \( A \) and an event \( B \).
Proof. Fix $\varepsilon > 0$. For all $k \geq 0$, let $(\xi_{k,i})_{i \geq 1}$ be a sequence of i.i.d. random variables with distribution $\mu_k$. For all $n \geq 1$, we have that

$$E_{\mu_k} \left[ \max_{0 \leq i \leq \tau_n-1} \frac{L_i^2}{n} \right] \leq \varepsilon + E_{\mu_k} \left[ \max_{0 \leq i \leq \tau_n-1} \frac{L_i^2}{n} \cdot \max_{0 \leq i \leq \tau_n-1} L_i^2 \geq \varepsilon n \right]$$

$$\leq \varepsilon + E_{\mu_k} \left[ \max_{0 \leq k \leq h_n \sqrt{n}} \frac{\xi_{k,i}^2}{n} \cdot \max_{1 \leq i \leq w_n \sqrt{n}} \xi_{k,i}^2 \geq \varepsilon n \right]$$

$$\leq \varepsilon + \frac{1}{n} \sum_{0 \leq k \leq h_n \sqrt{n}} \sum_{1 \leq i \leq w_n \sqrt{n}} E_{\mu_k} \left[ \xi_{k,i}^2 ; \xi_{k,i}^2 \geq \varepsilon n \right]$$

$$\leq \varepsilon + \frac{1}{n} \sum_{0 \leq k \leq h_n \sqrt{n}} \sum_{1 \leq i \leq w_n \sqrt{n}} E_{\mu_k} \left[ \xi_{k,i}^2 1_{\{\xi_{k,i}^2 \geq \varepsilon n\}} \right]$$

$$\leq \varepsilon + \frac{w_n}{\sqrt{n}} \sum_{k=0}^{[h_n \sqrt{n}]} E_{\mu_k} \left[ \xi_{k,1}^2 1_{\{\xi_{k,1}^2 \geq \varepsilon n\}} \right],$$

where the third inequality follows from a union bound, decomposing the event

$$\{\max_{0 \leq k \leq h_n \sqrt{n}} \xi_{k,i}^2 \geq \varepsilon n\}$$

according to which index $(k, i)$ realizes the maximum. The above and Corollary 4.1 imply that, for all $\varepsilon > 0$,

$$\limsup_{n \to \infty} E_{\mu_k} \left[ \max_{0 \leq i \leq \tau_n-1} \frac{L_i^2}{n} \right] \leq \varepsilon,$$

which yields the conclusion by taking $\varepsilon$ to zero. \qed

In order to conclude the proof of Theorem 2.1 it remains to prove that (4.4) holds. This is done in Lemma 4.7 below, whose proof is a refinement of the reasoning in Section 3.3. Recall that the goal is to prove that the sum of the variances collected along the depth-first order exploration of $\mathcal{F}$ at step $n$ is roughly equal to $\sigma^2 n$.

The general idea of the proof is close, in spirit, to a coarse graining, as we will divide the forest $\mathcal{F}_n$ in mesoscopic blocks of width and height $\Theta(\sqrt{n})$, and show that the average value of $\sigma^2_{H_k}$ on most blocks is close to $\sigma^2$. Before stating and proving Lemma 4.7, we will need a few other lemmas.

Recall that, in Section 2.1, we defined the depth-first labeling of $\mathcal{F}$ together with the lexicographical order of $\mathcal{F}$.

Let us now define $(n, \delta, \gamma)$-blocks, or simply blocks, on the forest $\mathcal{F}$ with this lexicographical order. For this purpose, let us denote $\mathcal{T}(v)$ the set of descendants of $v$, that is the subtree rooted at $v$, for a vertex $v \in \mathcal{F}$. For $\gamma > 0$ and $\delta > 0$, for all $n \geq 1$, for all $i \geq 0$ and all $k \geq 0$, we define the $(n, \delta, \gamma)$-block by

$$B_{i,k}^n = \bigcup_{j=i\lfloor \delta \sqrt{n} \rfloor+1}^{(i+1)\lfloor \delta \sqrt{n} \rfloor} \left\{ v \in \mathcal{T}(v_{j,k\lfloor \gamma \sqrt{n} \rfloor}) : k\lfloor \gamma \sqrt{n} \rfloor \leq h(v) < (k+1)\lfloor \gamma \sqrt{n} \rfloor \right\}.$$

(4.9)
Figure 3: A representation of the infinite forest $\mathcal{F}_\infty$. The vertex $v_{k,\ell}$ is drawn in position $(k, \ell)$ on the plane. The different trees of the infinite forest have distinct colours. The vertices in red are the ones that belong to $\mathcal{F}_n$.

See Figure 4 for a representation of these blocks. In words, $B^n_{i,k}$ is the collection of the $\lceil \delta \sqrt{n} \rceil$ finite subtrees of height (at most) $\lceil \gamma \sqrt{n} \rceil - 1$ rooted at $v_{j,k}[\gamma \sqrt{n}]$, for $i \lceil \delta \sqrt{n} \rceil < j \leq (i + 1) \lceil \delta \sqrt{n} \rceil$. In this notation, we ignore the dependence on $\delta$ and $\gamma$, as we will quickly fix these constants.

For a block $B^n_{i,k}$, $i, k \geq 0$, we define the total variance of the block as

$$W_{i,k} = \sum_{v \in B^n_{i,k}} \sigma^2_{h(v)}.$$  \hfill (4.10)

In what follows, for $n \geq 1$, $i, k \geq 0$, and for $\varepsilon > 0$, we say that the block $B^n_{i,k}$ is $\varepsilon$-good if

$$\left| \frac{W_{i,k}}{|B^n_{i,k}| \sigma^2} - 1 \right| \leq 16\varepsilon.$$  \hfill (4.11)

Moreover, if the above is not satisfied, then we say that $B^n_{i,k}$ is not $\varepsilon$-good.

**Remark 2.** Note that, for all $n \geq 1$, $\delta > 0$ and $\gamma > 0$, the pairs of random variables $(|B^n_{i,k}|, W_{i,k})$, $i \geq 0$, $k \geq 0$ are independent under $P_\mu$ and their distributions depend on $k$ only, by definition of a BPVE. Therefore, the events $\{B^n_{i,k} \text{ is not } \varepsilon\text{-good}\}$, $i \geq 0$, $k \geq 0$ are independent.

The following lemma is the main ingredient for the proof of (4.4) in Lemma 4.7. It states that a given box has a large probability to be $\varepsilon$-good, and provides a control on the maximum volume and weight of a box (this will be used to prevent that a not $\varepsilon$-good box has too much importance in the sum in (4.4)).

**Lemma 4.5.** Let $\mu$ be a strictly critical environment satisfying Conditions (I), (II) and (III). Fix $\varepsilon \in (0, 1/2)$, $\delta > 0$ and $\gamma > 0$. For all $k \geq 0$, there exists $N_k = N_k(\mu, \varepsilon, \delta, \gamma)$ such that, for all $n \geq N_k$, we have that

$$P_\mu(B^n_{0,k} \text{ is } \varepsilon\text{-good}) \geq 1 - 6\sigma^2\gamma\delta^{-1}\varepsilon^{-2}.$$  \hfill (4.12)
and, for all $K \geq 6\gamma\delta(1 + \sigma^2)$,
\[
P_\mu(\mid B_{0,k}^n \mid > Kn) \leq 6^3\gamma^3\delta\sigma^2K^{-2},
\]
\[
P_\mu(\mid W_{0,k} \mid > Kn) \leq 6^3\gamma^3\delta\sigma^6K^{-2}.
\]

**Proof.** We start by proving that a block has a sufficient size with large probability. For all $\ell$ such that $0 \leq \ell < \sqrt{\gamma\sqrt{n}}$, let $Z_\ell$ denote the number of vertices of $B_{0,k}^n$ with height $k\sqrt{\gamma\sqrt{n}} + \ell$. In particular, $(Z_\ell)_{0 \leq \ell < \sqrt{\gamma\sqrt{n}}}$ is a piece of $(\mu_j)_{j \geq k[\sqrt{\gamma\sqrt{n}}]}$BPVE with $Z_0 = [\delta\sqrt{n}]$. By Lemma 3.1, we have that, for $M > 0$,
\[
P_\mu\left(\max_{0 \leq \ell < \sqrt{\gamma\sqrt{n}}} |Z_\ell - Z_0| \geq M\right) \leq \frac{4[\delta\sqrt{n}] \left(\sigma^2_{k[\sqrt{\gamma\sqrt{n}}]} + \ldots + \sigma^2_{(k+1)[\sqrt{\gamma\sqrt{n}}]}\right)}{(M[\delta\sqrt{n}])^2} \leq 4\gamma\delta^{-1}M^{-2} \frac{\sigma^2_{k[\sqrt{\gamma\sqrt{n}}]} + \ldots + \sigma^2_{(k+1)[\sqrt{\gamma\sqrt{n}}]}}{\gamma\sqrt{n}}.
\]

By Condition (I) and Lemma 3.1, there exists $\tilde{N}_k = \tilde{N}_k(\mu, \epsilon, \gamma)$ such that, for all $n \geq \tilde{N}_k$,
\[
(1 - \epsilon)\sigma^2 \leq \frac{\sigma^2_{k[\sqrt{\gamma\sqrt{n}}]} + \ldots + \sigma^2_{(k+1)[\sqrt{\gamma\sqrt{n}}]}}{\gamma\sqrt{n}} \leq (1 + \epsilon)\sigma^2,
\]
and therefore, for all $n \geq \tilde{N}_k$, and for $M > 0$,
\[
P_\mu\left(\max_{0 \leq \ell < \sqrt{\gamma\sqrt{n}}} \left|\frac{Z_\ell}{Z_0} - 1\right| \geq M\right) \leq 4(1 + \epsilon)\gamma\sigma^2\delta^{-1}M^{-2} \leq 6\gamma\sigma^2\delta^{-1}M^{-2}.
\]
Note that, if $\max_{0 \leq \ell < \sqrt{\gamma\sqrt{n}}} \left|\frac{Z_\ell}{Z_0} - 1\right| < M$ then we have that $(1 - M)[\delta\sqrt{n}] \leq Z_\ell \leq (1 + M)[\delta\sqrt{n}]$, for all $0 \leq \ell < \sqrt{\gamma\sqrt{n}}$. Let $N_k = \max(\tilde{N}_k, (\epsilon\delta)^{-2}, (\epsilon\gamma)^{-2})$ and observe that, for all $n \geq N_k$,
\[
\left\{ \max_{0 \leq \ell < \sqrt{\gamma\sqrt{n}}} \left|\frac{Z_\ell}{Z_0} - 1\right| < M \right\} \subset \left\{ W_{0,k} \leq (1 + M)[\delta\sqrt{n}] \left(\sigma^2_{k[\sqrt{\gamma\sqrt{n}}]} + \ldots + \sigma^2_{(k+1)[\sqrt{\gamma\sqrt{n}}]}\right) \right\} 
\subset \left\{ W_{0,k} \leq (1 + M)(1 + \epsilon)\gamma\sigma^2(\delta + n^{-1/2})n \right\}
\subset \left\{ W_{0,k} \leq (1 + M)(1 + \epsilon)^2\gamma\delta\sigma^2n \right\}.
\]
Moreover, we have that, for all \( n \geq N_k \) and for \( M > 0 \),
\[
\left\{ \max_{0 \leq t < \gamma \sqrt{n}} \left| \frac{Z_t}{Z_0} - 1 \right| < M \right\} \subseteq \{ |B_{0,k}^n| \geq (1 - M)\gamma \delta n \} \cap \{ W_{0,k} \leq (1 + M)(1 + \varepsilon)^2\gamma \delta \sigma^2 n \}.
\]

(4.16)

Following similar steps, one can prove that for all \( n \geq N_k \) and for \( M > 0 \),
\[
\left\{ \max_{0 \leq t < \gamma \sqrt{n}} \left| \frac{Z_t}{Z_0} - 1 \right| < M \right\} \subseteq \{ W_{0,k} \geq (1 - M)(1 - \varepsilon)\gamma \delta \sigma^2 n \} 
\]
\[
\cap \{ |B_{0,k}^n| \leq (1 + M)(1 + \varepsilon)^2\gamma \delta n \}.
\]

(4.17)

Applying (4.16) and (4.17) to \( M = \varepsilon \), we obtain that, for all \( n \geq N_k \),
\[
\left\{ \max_{0 \leq t < \gamma \sqrt{n}} \left| \frac{Z_t}{Z_0} - 1 \right| < \varepsilon \right\} \subseteq \left\{ \frac{W_{0,k}}{|B_{0,k}^n|\sigma^2} \leq 1 + 7\varepsilon \right\} \subseteq \left\{ \frac{W_{0,k}}{|B_{0,k}^n|\sigma^2} \leq 1 + 16\varepsilon \right\},
\]
\[
\left\{ \max_{0 \leq t < \gamma \sqrt{n}} \left| \frac{Z_t}{Z_0} - 1 \right| < \varepsilon \right\} \subseteq \left\{ \frac{W_{0,k}}{|B_{0,k}^n|\sigma^2} \geq 1 - 9\varepsilon \right\}.
\]

Thus, by (4.14) applied with \( M = \varepsilon \) and using the definition (4.11) of \( \varepsilon \)-good, we obtain that, for all \( n \geq N_k \),
\[
P_\mu \left( B_{0,k}^n \text{ is } \varepsilon \text{-good} \right) \geq 1 - 6\sigma^2\gamma \delta^{-1}\varepsilon^{-2}.
\]

Now, combining (4.14) and (4.17) with \( M = K3^{-1}\gamma^{-1}\delta^{-1} - 1 \), a straightforward computation using that \( K \geq 6\gamma \delta \) (and thus \( M \geq K6^{-1}\gamma^{-1}\delta^{-1} \)) yields
\[
P_\mu \left( |B_{0,k}^n| > Kn \right) \leq 63\gamma^3 \delta \sigma^2 K^{-2}.
\]

Finally, by (4.14) and (4.15) with \( M = K3^{-1}\gamma^{-1}\delta^{-1}\sigma^{-2} - 1 \), a similar computation using that \( K \geq 6\gamma \delta \sigma^2 \) yields
\[
P_\mu \left( |W_{0,k}| > Kn \right) \leq 63\gamma^3 \delta \sigma^6 K^{-2}.
\]

This concludes the proof.

\[\square\]

Let us define, for \( \delta, \gamma > 0 \) and \( A_1, A_2 > 0 \), the set of indices
\[
\mathcal{I}(\delta, \gamma, A_1, A_2) = \left\{ (i, k) : 0 \leq i \leq \lfloor \delta^{-1} A_1 \rfloor, 0 \leq k \leq \lfloor \gamma^{-1} A_2 \rfloor \right\}.
\]

(4.18)

Note that, with this choice, the box
\[
B_n = \bigcup_{(i, k) \in \mathcal{I}(\delta, \gamma, A_1, A_2)} B_{i,k}^n
\]

is roughly of size \( A_1 \sqrt{n} \) by \( A_2 \sqrt{n} \). Later, we will choose \( A_1 > 0 \) and \( A_2 > 0 \) so that the depth-first exploration of the tree is contained in \( B_n \) with high probability. Hence, we start by controlling the behavior of the boxes \( B_{i,k}^n \) for \( (i, k) \in \mathcal{I}(\delta, \gamma, A_1, A_2) \).

Let us define, for \( \varepsilon \in (0, 1/2) \), \( \delta > 0 \), \( \gamma > 0 \), \( A_1 > 0 \), \( A_2 > 0 \) and \( n \geq 1 \),
\[
\text{Bad}_n(\delta, \gamma, \varepsilon, A_1, A_2) = \left\{ (i, k) \in \mathcal{I}(\delta, \gamma, A_1, A_2) : B_{i,k}^n \text{ is not } \varepsilon \text{-good} \right\}.
\]

(4.20)

Lemma 4.6. Let \( \mu \) be a strictly critical environment satisfying Conditions (I), (II) and (III). Fix \( \varepsilon \in (0, 1/2) \), \( \delta \in (0, 1) \), \( \gamma \in (0, 1) \), \( A_1 > 1 \) and \( A_2 > 1 \). Let \( \mathcal{I} = \mathcal{I}(\delta, \gamma, A_1, A_2) \) and \( \text{Bad}_n = \text{Bad}_n(\delta, \gamma, \varepsilon, A_1, A_2) \), recalling the definitions (4.18) and (4.20). Assume moreover that
\[
\delta < \frac{\varepsilon^3}{120 A_1 A_2 (1 + \sigma^2)}.
\]

(4.21)
There exists $N = N(\mu, \varepsilon, \delta, \gamma)$ such that, for all $n \geq N$, we have that, for all $K \geq 6\gamma\delta(1 + \sigma^2)$,

$\begin{align*}
\mathbb{P}_\mu \left( \max_{(i,k) \in I} |B^n_{i,k}| > Kn \right) & \leq 6^4 A_1 A_2 \sigma^2 \gamma^2 K^{-2}, \\
\mathbb{P}_\mu \left( \max_{(i,k) \in I} |W_{i,k}| > Kn \right) & \leq 6^4 A_1 A_2 \sigma^6 \gamma^2 K^{-2},
\end{align*}
(4.22, 4.23)

$\mathbb{P}_\mu \left( \text{Bad}_n > \delta^{-3} \right) \leq \frac{\varepsilon}{5}. 
(4.24)$

\textbf{Proof.} Let $N = \max_{0 \leq k \leq \lfloor \gamma^{-1} A_2 \rfloor} N_k(\mu, \varepsilon, \delta, \gamma)$, where the $N_k$’s are given by Lemma 4.5. First, note that (4.22) and (4.23) are straightforward consequences of Lemma 4.5 and a union bound over $I$, observing that $|I| \leq 4A_1A_2\delta^{-1}\gamma^{-1}$. Second, for $n \geq N$, by Lemma 4.5 and Remark 2, Bad$_n$ is stochastically dominated by a binomial random variable $Y$ with parameters $[4A_1A_2\delta^{-1}\gamma^{-1}]$ and $\min\{1, 6\sigma^2\gamma^2\delta^{-1}\varepsilon^{-2}\}$. We will use the following bound, which can be deduced from a Chernoff bound. For all $\alpha > 8$,

$$
\mathbb{P}[Y \geq (1 + \alpha)\mathbb{E}[Y]] \leq \exp\left(\left((\alpha - (1 + \alpha)\log(1 + \alpha))\mathbb{E}[Y]\right) \leq \exp\left(-\alpha\mathbb{E}[Y]\right) \leq \frac{1}{\alpha\mathbb{E}[Y]} \right)
(4.25)
$$

Let $\alpha = \delta^{-3}\mathbb{E}[Y]^{-1} - 1$, so that by (4.21),

$$
\delta^3\mathbb{E}[Y] \leq 24A_1A_2\sigma^2\varepsilon^{-2}\delta \leq \frac{\varepsilon}{5} < 1/9
$$

and $\alpha > 8$. Using (4.25) together with the fact that $Y$ stochastically dominates Bad$_n$, we have that, for $n \geq N$,

$$
\mathbb{P}_\mu \left( \text{Bad}_n \geq \delta^{-3} \right) \leq \frac{\delta^3\mathbb{E}[Y]^{n}}{\mathbb{E}[Y](1 - \delta^3\mathbb{E}[Y])} \leq 2\delta^3 \leq \frac{\varepsilon}{5}.
$$

This concludes the proof.

\textbf{Lemma 4.7.} Let $\mu$ be a strictly critical environment satisfying Conditions (I), (II) and (III). We have that

$$
\lim_{n \to +\infty} \frac{\sum_{k=0}^{n-1} \sigma^2_{H_k}}{n} = \sigma^2, \text{ in } \mathbb{P}_\mu\text{-probability.}
$$

In other words, (4.4) is satisfied.

\textbf{Proof.} For $n \geq 1$, let us denote, $V_n$ the set of vertices of $F_n$ (i.e. the vertices with depth-first label smaller than $n$), so that

$$
\sum_{k=0}^{n} \sigma^2_{H_k} = \sum_{v \in V_n} \sigma^2_{H(v)}.
$$

Now, for $\varepsilon \in (0, 1/2)$, $\delta \in (0, 1)$, $\gamma \in (0, 1)$, $A_1 > 1$ and $A_2 > 1$, recalling the definitions (4.18), (4.19) and (4.20), let us define

$$
\begin{align*}
I^0 & = \{(i,k) \in I : B^n_{i,k} \subset V_n, B^n_{i,k} \text{ is } \varepsilon\text{-good}\} \\
\partial I & = \{(i,k) \in I : B^n_{i,k} \cap V_n \neq \emptyset, B^n_{i,k} \text{ is } \varepsilon\text{-good}\} \setminus I^0 \\
I^c & = \{(i,k) \in I : B^n_{i,k} \text{ is not } \varepsilon\text{-good}\}.
\end{align*}
(4.26)
$$

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Let us define the events

\[ E_1 = \{ V_n \subset B_n \} , \]
\[ E_2 = \{ \text{Bad}_n \leq \delta^{-3} \} , \]
\[ E_3 = \bigg\{ \max_{(i,k) \in \mathcal{I}} |B^n_{i,k}| \leq \gamma^{1/2} n \bigg\} , \]
\[ E_4 = \bigg\{ \bigg\{ (i,k) \in \mathcal{I} : |B^n_{i,k}| > \gamma \delta^{1/2} n \bigg\} \leq \delta^{-2} \bigg\} , \]
\[ E_5 = \bigg\{ \max_{(i,k) \in \mathcal{I}} |W_{i,k}| \leq \delta^3 \varepsilon n \bigg\} . \]

\( E_1 \) allows to locate \( \mathcal{F}_n \) inside \( B_n \), \( E_2 \) bounds the number of not \( \varepsilon \)-good boxes. Then, to limitate the influence of those boxes, \( E_3 \) (resp. \( E_5 \)) bounds the maximal volume (resp. weight) of a box.

It turns out that the bound in \( E_3 \) is too rough, and \( E_4 \) bounds the number of boxes having some already large volume.

On the event \( E_1 \), using the definition (4.11) of \( \varepsilon \)-good blocks, we have that

\[ \sum_{k=0}^{n} \sigma^2_{H_k} \leq \sum_{(i,k) \in \mathcal{I}^o} W_{i,k} + \sum_{(i,k) \in \partial \mathcal{I}} W_{i,k} + \sum_{(i,k) \in \mathcal{I}^c} W_{i,k} \]
\[ \leq (1 + 16 \varepsilon) \sigma^2 \sum_{(i,k) \in \mathcal{I}^o} |B^n_{i,k}| + (1 + 16 \varepsilon) \sigma^2 \sum_{(i,k) \in \partial \mathcal{I}} |B^n_{i,k}| + \sum_{(i,k) \in \mathcal{I}^c} W_{i,k} \]
\[ \leq (1 + 16 \varepsilon) \sigma^2 n + (1 + 16 \varepsilon) \sigma^2 \sum_{(i,k) \in \partial \mathcal{I}} |B^n_{i,k}| + |\mathcal{I}^c| \max_{(i,k) \in \mathcal{I}} |W_{i,k}| , \]

and

\[ \sum_{k=0}^{n} \sigma^2_{H_k} \geq (1 - 16 \varepsilon) \sum_{(i,k) \in \mathcal{I}^o} |B^n_{i,k}| \]
\[ \geq (1 - 16 \varepsilon) \sigma^2 \left( n - \sum_{(i,k) \in \partial \mathcal{I}} |B^n_{i,k}| - |\mathcal{I}^c| \max_{(i,k) \in \mathcal{I}} |B_{i,k}| \right) \]
\[ \geq \sigma^2 n - 16 \sigma^2 \varepsilon n - \sigma^2 \left( \sum_{(i,k) \in \partial \mathcal{I}} |B^n_{i,k}| + |\mathcal{I}^c| \max_{(i,k) \in \mathcal{I}} |B_{i,k}| \right) . \]

Now, note that because the exploration is depth-first, we have that if \( (i, k) \in \partial \mathcal{I} \), then \( (j, k) \in \mathcal{I}^o \cup \mathcal{I}^c \) for all \( 0 \leq j < i \) and \( B^n_{j,k} \cap V_n = \emptyset \) for all \( \ell > i \). This implies that for each \( k \geq 0 \), there is at most one index \( i \) such that \( (i, k) \in \partial \mathcal{I} \) and therefore

\[ |\partial \mathcal{I}| \leq \gamma^{-1} A_2 + 1 \leq 2 \gamma^{-1} A_2 . \]

Hence, on the event \( E_3 \cap E_4 \), we have that, if

\[ \delta \leq \frac{\varepsilon^2}{4 A_2^2} , \]
\[ \gamma \leq \varepsilon^2 \delta^6 , \]

then

\[ \sum_{(i,k) \in \partial \mathcal{I}} |B^n_{i,k}| \leq \gamma^{1/2} n \times \delta^{-2} + \gamma \delta^{1/2} n \times 2 \gamma^{-1} A_2 = (\gamma^{1/2} \delta^{-2} + 2 \gamma^{1/2} A_2) n \leq 2 \varepsilon n . \]
Observe as well that $|\mathcal{I}| = \text{Bad}_n$, thus on the event $E_2 \cap E_3 \cap E_5$, we have that, if (4.35) holds then

$$
|\mathcal{I}| \max_{(i,k) \in \mathcal{I}} |W_{i,k}| \leq \delta^{-3} \times \delta^3 \varepsilon n = \varepsilon n,
$$

Putting (4.32), (4.33), (4.36) and (4.37) together and using that $\varepsilon < 1/2$, we obtain that, if (4.34) and (4.35) hold then

$$
\left\{ \bigcap_{1 \leq i \leq 5} E_i \right\} \subset \left\{ \left| \frac{1}{n} \sum_{k=0}^{n} \sigma H_k - \sigma^i \right| \leq 35(1 + \sigma^2)\varepsilon \right\}. \quad (4.38)
$$

To conclude the proof, we will show that for all $\varepsilon \in (0, 1/2)$, there exist $A_1 > 1$, $A_2 > 1$, $\delta \in (0, 1)$, $\gamma \in (0, 1)$ satisfying (4.34) and (4.35), and there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$
P_\mu \left( \left\{ \bigcap_{1 \leq i \leq 5} E_i \right\}^c \right) \leq \varepsilon. \quad (4.39)
$$

Note that, by (4.19) and (4.9), for all $A_1, A_2 > 1$, $\delta \in (0, 1)$, $\gamma \in (0, 1)$ and for all $n \geq 1$,

$$
\{ V_n \subset B_n \}^c \subset \{ w(\mathcal{F}_n) > A_1 \sqrt{n} \} \cup \{ h(\mathcal{F}_n) > A_2 \sqrt{n} \}.
$$

By Lemma 3.4, for all $\varepsilon \in (0, 1/2)$, there exists $A_1 = A_1(\varepsilon) > 1$, $A_2 = A_2(\varepsilon) > 1$ and $N_1 = N_1(\mu, \varepsilon)$ such that, for all $n \geq N_1$,

$$
P_\mu (E_1^c) \leq \frac{\varepsilon}{5}. \quad (4.40)
$$

Now, let us choose

$$
\delta = \frac{\varepsilon^3}{6^6 A_1 A_2^3 (1 + \sigma^2)^2},
$$

$$
\gamma = \frac{\varepsilon^3 \delta^6}{6^6 A_1 A_2^2 (1 + \sigma^2)^3},
$$

so that (4.34) and (4.35) are satisfied. Then (4.21) holds, so that we can apply Lemma 4.6 first, there exists $N_2 = N_2(\mu, \varepsilon)$ such that, for all $n \geq N_2$,

$$
P_\mu (E_2^c) \leq \frac{\varepsilon}{5}; \quad (4.41)
$$

second, using (4.22) with $K = \sqrt{\gamma} \geq 6\gamma \delta (1 + \sigma^2)$, we have that, for all $n \geq N_2$

$$
P_\mu (E_3^c) \leq 6^4 A_1 A_2^2 \gamma \leq \frac{\varepsilon}{5}; \quad (4.42)
$$

and third, using (4.23) with $K = \delta^3 \varepsilon \geq 6\gamma \delta (1 + \sigma^2)$, we obtain that, for all $n \geq N_2$,

$$
P_\mu (E_5^c) \leq 6^4 A_1 A_2^2 \gamma^2 \delta^{-6} \varepsilon^{-2} \leq \frac{\varepsilon}{5}. \quad (4.43)
$$

Now, by Lemma 4.5 with $K = \gamma \delta^{1/2} \geq 6\gamma \delta (1 + \sigma^2)$, there exists $N_3 = N_3(\mu, \varepsilon)$ such that, for all $(i, k) \in \mathcal{I}$,

$$
P_\mu \left( |B_{i,k}^n| > \gamma \delta^{1/2} n \right) \leq 6^3 \sigma^2 \gamma.
$$
Therefore, \( \{ (i, k) \in \mathcal{I} : |B_{i,k}^n| > \gamma \delta^{k/2} n \} \) is dominated by a binomial random variable \( Y \) with parameters \( 4A_1 A_2 \delta^{-1} \gamma^{-1} \) and \( \delta^2 \gamma \). Using (4.25) and the definition of \( \delta \), a straightforward computation yields that
\[
P_\mu (E_4) \leq \mathbb{P} (Y > \delta^{-2}) \leq \frac{\varepsilon}{5}.
\]

Finally, by (4.40), (4.41), (4.42), (4.44) and (4.43), for all \( \varepsilon \in (0, 1) \), choosing \( A_1, A_2, \delta, \gamma \) as above and letting \( N = \max\{N_1, N_2, N_3\} \), we have that (4.39) holds for all \( n \geq N \). Using (4.38), this yields that, for all \( \varepsilon \in (0, 1/2) \), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \),
\[
P_\mu \left( \left| \frac{1}{n} \sum_{k=0}^{n} \sigma^2_{H_k} - \sigma^2 \right| > 35(1 + \sigma^2)\varepsilon \right) \leq \varepsilon,
\]
which implies the required convergence in probability and concludes the proof.

\[\square\]

### 4.2 Proof of Corollary 2.2

Let \( \phi \) be the map from \( D([0, 1]) \) to itself such that for all \( f \in D([0, 1]) \) and \( t \in [0, 1] \), \( \phi(f)(t) := f(t) - \inf_{0 \leq s \leq t} f(s) \). One checks straightforwardly that \( \phi \) is continuous in Skorokhod space. Hence, Corollary 2.2 follows from Theorem 2.1 and the continuous mapping theorem applied to \( \phi \).

### 5 Proof of Theorem 1.1

As we explain in Section 5.3, Theorem 1.1 can be deduced from Theorem 2.3. Hence, most of this section is devoted to the proof of Theorem 2.3 stating the joint convergence of the Lukasiewicz path and the height process.

To prove this convergence, we introduce Kesten’s tree in varying environment \( T^* \), also known as the Geiger tree, which is the local limit of the tree \( T \) conditioned on having a large height. The Geiger tree consists of an infinite spine along which the offspring distribution is size-biased at each generation, and on which are attached trees with the non-biased offspring distributions.

The first step of the proof of Theorem 2.3 in Section 5.1 is to prove that for a typical vertex in \( T^* \) with depth-first label \( x \), the ratio \( (X_x - I_x)/H_x \) is close to \( \sigma^2/2 \), where we recall that \( (X_x - I_x) \) is the Lukasiewicz path reflected above its running minimum. The second step of the proof of Theorem 2.3 is to transpose this to \( T \) by comparing the distributions of \( T \) and \( T^* \), see Proposition 5.2. This allows us to show that, for most vertices \( x \) in \( \mathcal{F}_n \), the ratio \( (X_x - I_x)/H_x \) is also close to \( \sigma^2/2 \), see Proposition 5.3. We conclude the proof of Theorem 2.3 by a continuity argument: this is done following the proof of [11] Theorem 2.3.1 for the usual Bienaymé-Galton-Watson case, with more technicalities due to the varying environment.

Finally, in Section 5.3 we explain how to obtain Theorem 1.1 from Theorem 2.3.

#### 5.1 The height process via spine decomposition

In this section, we study the height process of a BPVE and the main result is Proposition 5.2. We start by defining the distribution of the Geiger tree \( T^* \) on an environment \( \mu \). Let \( (\xi_m)_{m \geq 0} \) be a sequence of independent variables such that for all \( m \geq 0 \), \( \xi_m \) is distributed as a size-biased version of \( \mu_m \). Formally, \( P_\mu(\xi_m = i) = i P_\mu(\xi_m = i)/E_\mu[\xi_m] \) for all \( i \geq 0 \), where \( \xi_m \) is random variable of distribution \( \mu_m \). Let \( (T_{i,m})_{i \geq 1, m \geq 0} \) be a family of independent trees, independent of \( (\xi_m)_{m \geq 0} \), where for all \( m \geq 0 \) and \( i \geq 1 \), \( T_{i,m} \) is a BPVE in environment \( (\mu_{m+k})_{k \geq 0} \). We will define recursively the pair \( (T^*, (v_m)_{m \geq 0}) \) consisting of the Geiger tree and a sequence of marked
vertices, called the spine. We start with a pair $(T_0, v_0)$ where the tree $T_0$ consists of the single vertex $v_0$. Assume that for some $m \geq 0$, we have constructed the pair $(T_m, (v_i)_{1 \leq i \leq m})$ where $T_m$ is a tree of height $m+1$ and $v_1, \ldots, v_m$ are spine vertices so that $v_{i+1}$ is an offspring of $v_i$, for all $i \geq 0$. Then, we construct $(T_{m+1}, (v_i)_{1 \leq i \leq m+1})$ as follows:

- assign $\xi_m$ children to the spine vertex $v_m$,
- choose a new spine vertex $v_{m+1}$ uniformly at random among the children of $v_m$,
- attach $T_{i,m+1}$ to the $i$-th non-spine child of $v_m$, in lexicographical order.

As $m \to \infty$, this procedure produces the pair $(T^*, (v_m)_{m \geq 0})$ where $T^*$ is the Geiger tree and $(v_m)_{m \geq 0}$ is its spine. Note that, if $\mu$ is strictly critical, then the trees $T_{i,m}$ are finite almost surely and $T^*$ contains a single infinite path. This implies that knowing $T^*$ is enough to recover the spine $(v_m)_{m \geq 0}$.

We want to study the Lukasiewicz path on this infinite tree and compare it to the Lukasiewicz path of a finite tree conditioned to be large. A difficulty is that the depth-first labeling and the Lukasiewicz path are not well-defined on the Geiger tree $T^*$, as the depth-first exploration will never be able to cross the spine. Nevertheless, the value of the Lukasiewicz path reflected above its running minimum can be recovered from partial information. This value coincides with $\tilde{X}_x$ introduced below, which is well-defined on finite, infinite trees or forests. At the beginning of Section 5.2, we will provide details on how to relate $\tilde{X}_x$ and the Lukasiewicz path on the explored forest $F_n$.

Let $F$ be a forest (possibly just a tree, infinite or finite) and recall the lexicographical order defined in Section 2.1. We want to define the quantity $\tilde{X}_x$ for a vertex $x$ of $F$. For a vertex $x$ at height $m \geq 1$, there exists a unique $m$-uplet $(s_0, \ldots, s_m)$ such that $x = v_{s_m,m}$ and, for all $i \in \{0, \ldots, m - 1\}$, $v_{s_i,i}$ is the ancestor of $x$ at height $i$. Now, for each $i \in \{0, \ldots, m - 1\}$, let $\zeta_i(x)$ be the number of offspring of $v_{s_i,i}$ with lexicographical label strictly larger than that of $v_{s_{i+1},i+1}$, i.e. with label $v_{j,i+1}$ for some $j > s_{i+1}$. We define

$$\tilde{X}_x = \zeta_0(x) + \cdots + \zeta_{m-1}(x). \quad (5.1)$$

For a vertex $x \in F$, denote $L(x)$ the depth-first label of $x$, as explained in Section 2.1. By definition of the height process, we have that $H_{L(x)} = h(x)$, where $h(x)$ is the height of $x$ in $T$, as defined in Section 2.1. Finally, note that, when applied to the spine vertex $v_m$, we have that

$$\tilde{X}_{v_m} = \zeta_0 + \cdots + \zeta_{m-1}, \quad (5.2)$$

where the random variables $\zeta_i$ are independent, uniform in $\{0, \ldots, \xi_i - 1\}$ with $\xi$ has the size-biased distribution of $\mu_i$, as defined above (2.7). These are the random variables used in Condition [IV].

For any tree $T$ and any $\varepsilon > 0$, we say that $x$ is an $\varepsilon$-bad vertex if

$$\left| \frac{\tilde{X}_x}{h(x)} - \frac{\sigma^2}{2} \right| \geq \varepsilon, \quad (5.3)$$

where $h(x)$ is the height of $x$ in $T$. For $m \in \mathbb{N}$, say that $T$ is $(\varepsilon, m)$-bad if there are at least $\varepsilon Z_m(T)$ $\varepsilon$-bad vertices among the $Z_m(T)$ vertices at generation $m$ of $T$.

**Lemma 5.1.** Let $\mu$ be a strictly critical environment satisfying Condition [IV]. For all $\varepsilon > 0$ and for all $\delta > 0$, there exists $M = M(\mu, \varepsilon, \delta)$ such that, for all $m \geq M$,

$$\mathbb{P}_\mu (T^* \text{ is } (\varepsilon, m)-\text{bad}) \leq \delta.$$
Proof. As above, we let \((v_m)_{m \geq 0}\) be the spine of \(T^*\), which is the only infinite path in the tree. We will use the lexicographical order defined in Section 2.1. By [KV17, Lemma 1.2], for \(m \geq 1\), for a tree \(T\) with height at least \(m\), and for \(k \in \{1, \ldots, Z_m(T)\}\), we have that
\[
P_{\mu} \left( T^* = T, v_m = v_{k,m} \right) = \frac{1}{Z_m(T)} P_{\mu} \left( T^* = T \right),
\]
where the notation \(m\) means that the two trees are identical up to height \(m\) included.

Let \(E_B^m\) be the countable set of trees of height exactly \(m\) that are \((\varepsilon, m)\)-bad. For a tree \(T \in E_B^m\), we denote \(V_B^m(T)\) the set of its vertices at height \(m\) that are \(\varepsilon\)-bad. Hence, for \(T \in E_B^m\), we have that \(|V_B^m(T)| \geq \varepsilon Z_m(T)\). Using the above, recalling that \(v_m\) denotes the spine vertex of \(T^*\) at height \(m\), we have that
\[
P_{\mu} \left( T^* \text{ is } (\varepsilon, m)\text{-bad} \right) = \sum_{T \in E_B^m} P_{\mu} \left( T^* = T \right) = \sum_{T \in E_B^m} \frac{Z_m(T)}{|V_B^m(T)|} \sum_{v \in V_B^m(T)} P_{\mu} \left( T^* = T, v_m = v \right) \leq \frac{1}{\varepsilon} \sum_{T \in E_B^m} \sum_{v \in V_B^m(T)} P_{\mu} \left( T^* = T, v_m = v \right) \leq \frac{1}{\varepsilon} P_{\mu} \left( v_m \text{ is } \varepsilon\text{-bad} \right). \tag{5.4}
\]

Hence, using (5.3), it is enough to prove that, for \(m\) large enough
\[
P_{\mu} (v_m \text{ is } \varepsilon\text{-bad}) \leq \varepsilon \delta. \tag{5.5}
\]

By (5.2), \(\bar{X}_{v_m}\) is a sum of \(m\) independent random variables as in Condition [IV]. Hence, by Condition [IV], we have that \(P_{\mu}\)-almost surely,
\[
\frac{\bar{X}_{v_m}}{m} \rightarrow \sigma^2 \frac{2}{\varepsilon},
\]
and thus this convergence holds in \(P_{\mu}\)-probability as well. Hence, noting that \(h(v_m) = m\), this implies that, for all \(\varepsilon > 0\) and for all \(\delta > 0\), there exists \(M = M(\mu, \varepsilon, \delta)\) such that, for all \(m \geq M\),
\[
P_{\mu} (v_m \text{ is } \varepsilon\text{-bad}) = P_{\mu} \left( \left| \frac{\bar{X}_{v_m}}{m} - \frac{\sigma^2}{2} \right| \geq \varepsilon \right) \leq \varepsilon \delta.
\]

This concludes the proof. \(\Box\)

**Proposition 5.2.** Let \(\mu\) be a strictly critical environment that satisfies the four Conditions [I],[II],[IV] and [V]. For all \(\varepsilon > 0\) and \(\delta > 0\), there exists \(M = M(\mu, \varepsilon, \delta)\) such that, for all \(m \geq M\) and for all \(k \in \{0, \ldots, m\}\),
\[
P_{\mu} (T \text{ is } (\varepsilon, m)\text{-bad} | h(T) \geq k) \leq \delta.
\]
Proof. Fix \( \varepsilon > 0 \) and \( \delta > 0 \). Start by noting that, for all \( m \geq 0 \) and all \( k \in \{0, \ldots, m\} \), using that a tree needs to be of height at least \( m \) to be \((\varepsilon, m)\)-bad, we have that

\[
P_{\mu}(T \text{ is } (\varepsilon, m)\text{-bad } | h(T) \geq k) = P_{\mu}(T \text{ is } (\varepsilon, m)\text{-bad } | h(T) \geq m) \frac{P_{\mu}(h(T) \geq m)}{P_{\mu}(h(T) \geq k)}
\]

hence it is enough to prove the statement for \( k = m \).

By Lemma 3.3, there exists \( \delta' = \delta'(\mu, \delta) > 0 \) and \( M_0 = M_0(\mu, \delta) \) such that, for all \( m \geq M_0 \), we have that

\[
P_{\mu}(Z_m(T) \leq \delta' m \mid h(T) \geq m) \leq \frac{1}{2\sigma^2 m}.
\] (5.6)

By Lemma 5.1, there exists \( M_1 = M_1(\mu, \varepsilon, \delta) \), such that, for all \( m \geq M_1 \), we have that

\[
P_{\mu}(T^* \text{ is } (\varepsilon, m)\text{-bad}) \leq \frac{\delta' \delta}{4\sigma^2}.
\] (5.7)

As in the previous proof, for \( m \geq 1 \) and two trees \( T \) and \( T' \), we write \( T \equiv_{m}^{\varepsilon,m} T' \) when both trees are identical up to height \( m \). We will denote again \( E_m^B \) the set of trees of height exactly \( m \) that are \((\varepsilon, m)\)-bad, and we will denote \( \tilde{E}_m^B = \{ T \in E_m^B : Z_m(T) > \delta' m \} \), which are both countable.

By [KV17 Lemma 1.2], for all \( m \geq 1 \) and for all \( T \in \tilde{E}_m^B \), we have that

\[
P_{\mu}(T \equiv_{m}^{\varepsilon} T) = \frac{P_{\mu}(T^* \equiv_{m}^{\varepsilon} T)}{Z_m(T)}.
\] (5.8)

Define \( M = M(\mu, \varepsilon, \delta) = \max\{M_0, M_1\} \). Using (5.4), (5.6), (5.7) and (5.8), we conclude that for all \( m \geq M \),

\[
P_{\mu}(T \text{ is } (\varepsilon, m)\text{-bad } | h(T) \geq m) \leq P_{\mu}(Z_m(T) \leq \delta' m \mid h(T) \geq m) + \frac{P_{\mu}(T \text{ is } (\varepsilon, m)\text{-bad }, Z_m(T) > \delta' m)}{P_{\mu}(h(T) \geq m)}
\]

\[
\leq \frac{\delta}{2} + \frac{P_{\mu}(T \text{ is } (\varepsilon, m)\text{-bad }, Z_m(T) > \delta' m)}{P_{\mu}(h(T) \geq m)}
\]

\[
\leq \frac{\delta}{2} + \sum_{T \in \tilde{E}_m^B} \frac{P_{\mu}(T^* \equiv_{m}^{\varepsilon} T)}{P_{\mu}(h(T) \geq m)}
\]

\[
\leq \frac{\delta}{2} + \frac{2\sigma^2}{\delta'} \sum_{T \in \tilde{E}_m^B} \frac{P_{\mu}(T^* \equiv_{m}^{\varepsilon} T)}{P_{\mu}(h(T) \geq m)}
\]

\[
\leq \frac{\delta}{2} + \frac{2\sigma^2}{\delta'} \sum_{T \in \tilde{E}_m^B} P_{\mu}(T^* \equiv_{m}^{\varepsilon} T)
\]

\[
\leq \frac{\delta}{2} + \frac{2\sigma^2}{\delta'} P_{\mu}(T^* \text{ is } (\varepsilon, m)\text{-bad})
\]

\[
\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\]

\[\square\]
Using the above, we can now prove that the proportion of bad vertices in the explored forest is arbitrarily small.

**Proposition 5.3.** Let \( \mu \) be a strictly critical environment satisfying Conditions [(I), (II), (IV)] and [(V)]. For every \( \varepsilon \in (0, 1/2) \) and all \( \delta \in (0, 1) \), there exists \( N = N(\mu, \varepsilon, \delta) \) such for all \( n \geq N \),

\[
P_\mu(\mathcal{F}_n \text{ has at most } \varepsilon n \text{ bad vertices}) \geq 1 - \delta.
\]

**Proof.** Let us fix \( \varepsilon \in (0, 1/2) \) and \( \delta \in (0, 1) \). We start by bounding the size of the explored forest \( \mathcal{F}_n \) and the number of tall trees. By Proposition 3.4 and Lemma 3.3, there exist \( A_1 = A_1(\delta) \geq 1 \), \( A_2 = A_2(\delta) \geq 1 \), \( M = M(\delta) \geq 1 \) and \( N_1 = N_1(\mu, \delta) \) such that, for all \( n \geq N_1 \), we have that

\[
P_\mu \left( \left\{ t(\mathcal{F}_n) \leq w(\mathcal{F}_n) \leq A_1 \sqrt{n}, \ h(\mathcal{F}_n) \leq A_2 \sqrt{n} \right\}^c \right) \leq \frac{\delta}{6}, \quad (5.9)
\]

and

\[
P_\mu \left( \left\{ i \in \{1, \ldots, A_1 \sqrt{n}\} : h \left( \mathcal{T}^{(i)} \right) \geq A^{-1} \varepsilon^2 \sqrt{n} \right\} \right) \leq \frac{\delta}{6}. \quad (5.10)
\]

For the second inequality above, we used the fact that the number of tall trees is stochastically dominated by a binomial with parameters \( 4c^{-1} A_1 \varepsilon^{-2} n^{-1/2} \) and \( \lfloor A_1 \sqrt{n} \rfloor \), together with Hoeffding’s inequality.

Hence, we have that, for all \( n \geq N_1 \),

\[
P_\mu \left( E_{n,1}^c \right) \leq \frac{\delta}{4}, \quad (5.11)
\]

where

\[
E_{n,1} = \left\{ t(\mathcal{F}_n) \leq w(\mathcal{F}_n) \leq A_1 \sqrt{n}, \ h(\mathcal{F}_n) \leq A_2 \sqrt{n} \right\}
\cap \left\{ \left\{ i \in \{1, \ldots, \lfloor A_1 \sqrt{n}\} : h \left( \mathcal{T}^{(i)} \right) \geq A^{-1} \varepsilon^2 \sqrt{n} \right\} \leq M \right\}.
\]

(5.12)

The strategy of the rest of the proof is as follows. Ideally, we would like to control the number of bad vertices by using Proposition 5.2 and a union bound over all generations up to \( A_2 \sqrt{n} \). We encounter two problems in doing so. First, Proposition 5.2 does not apply to small generations, hence we distinguish below between the generations at a relatively small height, and the other generations. Second, a union bound would not be sufficient because we have a number of generations of order \( \sqrt{n} \). To solve this second issue, we apply Proposition 5.2 only to a bounded number of generations (the \( m_j \)'s defined below (5.19)), regularly distributed over \( A_2 \sqrt{n} \) generations. Then, using that \( \tilde{X} \) is non-decreasing from a vertex to its descendants, we show that if a lot of vertices \( x \) at some intermediate generation \( i \) are bad by having a value too large for \( \tilde{X}_x \), then a lot of vertices at the next generation \( m_j \) are bad, see (5.21). Similarly, if many vertices at generation \( i \) are bad by having a value too low for \( \tilde{X}_x \), then many vertices at the previous generation \( m_{j-1} \) are bad, see (5.22).

Let us split the vertices depending on whether their height is larger than \( A_1^{-1} \varepsilon^2 \sqrt{n} \) or not. Observe that, on the event \( E_{n,1} \), we almost surely have that

\[
\left| \left\{ v \in \mathcal{F}_n : h(v) \leq A_1^{-1} \varepsilon^2 \sqrt{n} \right\} \right| \leq A_1^{-1} \varepsilon^2 \sqrt{n} \times A_1 \sqrt{n} = \varepsilon^2 n < \frac{\varepsilon}{2} n. \quad (5.13)
\]

The above deals with the vertices with low height, thus now we need to deal with the vertices with large height. For this purpose, recall that

\[
\mathcal{F}_n = \mathcal{T}^{(1)} \cup \ldots \cup \mathcal{T}^{(t(\mathcal{F}_n))},
\]

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hence on the event $E_{n,1}$, we have that
\[
F_n \subset \bigcup_{i=1}^{\lfloor A_1 \sqrt{n} \rfloor} \mathcal{T}^{(i)}.
\] (5.14)

Let us define the event
\[
E_{n,2} = \bigcap_{i=1}^{\lfloor A_1 \sqrt{n} \rfloor} \left\{ \left\{ w \left( \mathcal{T}^{(i)} \right) > A_1 \sqrt{n} \right\} \cup \bigcup_{m=\lfloor A_1^{-1} \varepsilon^2 \sqrt{n} \rfloor}^{\lfloor A_2 \sqrt{n} \rfloor} \left\{ \left\{ v \in \mathcal{T}^{(i)} : h(v) = m, v \text{ is } \varepsilon\text{-bad} \right\} \right\} \right\}.
\] (5.15)

Note that by (5.13), (5.15) and the fact that on $E_{n,1}$, the trees in $F_n$ have a width smaller than $A_1 \sqrt{n}$ and less than $M$ of them have height larger than $A_1^{-1} \varepsilon^2 \sqrt{n}$, we have that
\[
E_{n,1} \cap E_{n,2} \subset \{|\{v \in F_n : v \text{ is } \varepsilon\text{-bad}\}| < \varepsilon n\}.
\] (5.16)

We now want to prove that
\[
P_\mu((E_{n,1} \cap E_{n,2})^c) \leq \delta.
\] (5.17)

For this purpose, let us define $\gamma = \varepsilon \delta/(2^{13} A_1 A_2 (1 + \sigma^2))$, $\varepsilon_1 = \delta \gamma^2 \varepsilon^2/(2^{12} A_1^4 A_2^3 M^3 (1 + \sigma^2))$ and, for all $i \geq 1$, $m \geq 0$ and all $k \geq 0$,
\[
\text{Bad}^{(-,i)}_m \left( v \in \mathcal{T}^{(i)} : h(v) = m, \frac{\widetilde{X}_v}{h(v)} < \frac{\sigma^2}{2} - \varepsilon_1 \right),
\]
\[
\text{Bad}^{(-,i)}_m (k) = \left\{ v \in \mathcal{T}^{(i)} : h(v) = m + k, v \text{ has an ancestor in } \text{Bad}^{(-,i)}_m \right\},
\]
\[
Z^{(-,i,m)}_k = \left| \text{Bad}^{(-,i)}_m (k) \right|,
\]
as well as, using the lexicographical order,
\[
\text{Bad}^{(+)}_m = \left\{ i \in \{1, \ldots, \lfloor A_1 \sqrt{n} \rfloor \} : \frac{\widetilde{X}_{v_{i,m}}}{h(v_{i,m})} < \frac{\sigma^2}{2} + \varepsilon \right\}.
\] (5.19)

Write $m_j = \lfloor j \gamma \sqrt{n} \rfloor$ for $j \geq 0$. For a set $S$ of vertices at a same height $h_S$, we denote $Z(S)$ the number of descendants at generation $m_j$ of the vertices in $S$, where $j_S$ is the smallest index $j$ such that $m_j \geq h_S$. Let us define the events
\[
E_{n,3} = \bigcap_{i=1}^{\lfloor A_1 \sqrt{n} \rfloor} \bigcap_{j=\lfloor \gamma^{-1} A_1 \rfloor}^{\lfloor \gamma^{-1} A_2 \rfloor} \left\{ \mathcal{T}^{(i)} \text{ is } (\varepsilon_1, m_j)\text{-bad} \right\}^c
\] (5.20)
\[
E_{n,4} = \bigcap_{i=1}^{\lfloor A_1 \sqrt{n} \rfloor} \bigcap_{j=\lfloor \gamma^{-1} A_1 \rfloor}^{\lfloor \gamma^{-1} A_2 \rfloor} \left\{ Z_{m_j} \left( \mathcal{T}^{(i)} \right) > A_1 \sqrt{n} \right\}
\]
\[
\cup \left\{ \max_{k \in \{0, \ldots, \lfloor 2 \gamma \sqrt{n} \rfloor \}} \left\{ \left( Z^{(-,i,m_j)}_k - Z^{(-,i,m_j)}_0 \right) \leq \frac{\varepsilon}{8 M A_2 \sqrt{n}} \right\} \right\}
\] (5.21)
\[
E_{n,5} = \bigcap_{i=1}^{\lfloor A_2 \sqrt{n} \rfloor} \left\{ |\text{Bad}^{(+)}_j| < \frac{\varepsilon \sqrt{n}}{8 A_2} \right\} \cup \left\{ Z(\text{Bad}^{(+)}_j) > \varepsilon_1 A_1 \sqrt{n} \right\}
\] (5.22)
Now, note that for \( \lfloor \gamma^{-1} A_1^{-1} \epsilon^2 \rfloor \leq j \leq \lceil \gamma^{-1} A_2 \rceil \), and for all \( 0 \leq k \leq \lfloor 2 \gamma \sqrt{m} \rfloor \), if \( v \) is a vertex at height \( m_j \) which is not \( \epsilon_1 \)-bad and \( x \) is a vertex with \( h(x) = k + m_j \) and \( y \) such that \( h(y) = m_j - k \), then we have that, for all \( n \geq N_2(\mu, \delta, \epsilon) = N_1 + 16\epsilon^{-2} \),

\[
\tilde{X}_x \leq \frac{\tilde{X}_v}{h(v)} \cdot \frac{m_j}{h(v)} = \frac{m_j}{m_j + 2\gamma \sqrt{n}}
\geq \frac{\tilde{X}_v}{h(v)} \cdot \left( 1 - \frac{8\gamma}{A_1^{-1} \epsilon^2} \right)
\geq \left( \frac{\sigma^2}{2} - \epsilon_1 \right) \times \left( 1 - \frac{8\gamma}{A_1 \epsilon^2} \right)
\geq \frac{\sigma^2}{2} - 4\epsilon^{-2}\sigma^2\gamma - \epsilon_1
\geq \frac{\sigma^2}{2} - \frac{\epsilon}{4} - \epsilon_1
\geq \frac{\sigma^2}{2} - \frac{\epsilon}{2},
\]

and

\[
\tilde{X}_y \leq \frac{\tilde{X}_v}{h(v)} \cdot \frac{m_j}{h(v)} = \frac{m_j}{m_j - 2\gamma \sqrt{n}}
\leq \frac{\tilde{X}_v}{h(v)} \cdot \left( 1 + \frac{8\gamma}{A_1 \epsilon^2} \right)
\leq \left( \frac{\sigma^2}{2} + \epsilon_1 \right) \times \left( 1 + \frac{8\gamma}{A_1 \epsilon^2} \right)
\leq \frac{\sigma^2}{2} + 4\epsilon^{-2}\sigma^2\gamma + 2\epsilon_1
\leq \frac{\sigma^2}{2} + \frac{\epsilon}{4} + \epsilon_1
\leq \frac{\sigma^2}{2} + \frac{\epsilon}{2}.
\]

Thus, by (5.23) and (5.24), for \( n \geq N_2 \), if there is a \( \epsilon \)-bad vertex at some given generation \( k \in \{ \lfloor A_1^{-1} \epsilon^2 \sqrt{n} \rfloor, \ldots, \lfloor A_2 \sqrt{n} \rfloor \} \), then there exists \( j \) such that \( m_j \leq k \leq m_{j+1} \), and either it has an ancestor at generation \( m_j \) that is \( \epsilon_1 \)-bad or all of its descendants (if any) at generation \( m_{j+1} \) are \( \epsilon_1 \)-bad.

Using the above, for \( n \geq N_2 \) and on the event \( E_{n,1} \cap E_{n,3} \cap E_{n,4} \cap E_{n,5} \), for all \( 1 \leq i \leq \lfloor A_1 \sqrt{n} \rfloor \), for all \( \lfloor A_1^{-1} \epsilon^2 \sqrt{n} \rfloor \leq m \leq \lfloor A_2 \sqrt{n} \rfloor \), we have that, if \( \omega(T^{(i)}) \leq A_1 \sqrt{n} \) then

\[
\left| \{ v \in T^{(i)} : h(v) = m, \ v \text{ is } \epsilon \text{-bad} \} \right| \leq \epsilon_1 A_1 \sqrt{n} + \frac{\epsilon}{4MA_2} \sqrt{n} < \frac{\epsilon}{2MA_2} \sqrt{n},
\]

thus, we obtain that

\[
E_{n,1} \cap E_{n,3} \cap E_{n,4} \cap E_{n,5} \subset E_{n,1} \cap E_{n,2}.
\]

Hence, we have that, for \( n \geq N_4 \),

\[
P_\mu ((E_{n,1} \cap E_{n,2})^c) \leq P_\mu(E_{n,1}^c) + P_\mu(E_{n,1} \cap E_{n,3}^c) + P_\mu(E_{n,1} \cap E_{n,3} \cap E_{n,4}^c) + P_\mu(E_{n,5}^c).
\]
and, by (5.11), in order to prove (5.17) and conclude the proof of the proposition, it is enough to prove that
\[ P_\mu(E_{n,3}^c) \leq \frac{\delta}{4}, \]
(5.26)
\[ P_\mu(E_{n,3} \cap E_{n,4}^c) \leq \frac{\delta}{4}, \]
(5.27)
\[ P_\mu(E_{n,5}^c) \leq \frac{\delta}{4}. \]
(5.28)
By a union bound, Lemma 3.5 and Proposition 5.2, there exists \( N_3 = N_3(\mu, \epsilon, \delta, \gamma, A_1, A_2) \) such that, for all \( n \geq N_3 \),
\[ P_\mu(E_{n,3}^c) \leq A_1 \sqrt{n} \times \gamma^{-1} A_2 \max_{A_1^{-1} \epsilon^2 \sqrt{n}/2 \leq m \leq 2A_2 \sqrt{n}} P_\mu(T \text{ is } (\epsilon, m)\text{-bad}) \]
\[ \leq A_1 A_2 \gamma^{-1} \sqrt{n} \times P_\mu(\mu(h(T) \geq A_1^{-1} \epsilon^2 \sqrt{n}/2) \times (\delta c A_1^{-1} A_2^{-1} \gamma \epsilon^2/24) \]
\[ \leq A_1 A_2 \gamma^{-1} \sqrt{n} \times \frac{8}{c A_1^{-1} \epsilon^2 \sqrt{n}} \times (\delta c A_1^{-1} A_2^{-1} \gamma \epsilon^2/24) \]
\[ \leq \frac{\delta}{4}, \]
(5.29)
where \( c \) is the constant from Condition (II). This proves (5.26).

Now, note that by Lemma 3.1 and Condition (I) there exists \( N_4 = N_4(\mu, \gamma, A_1, A_2) \) such that, for all \( n \geq N_4 \),
\[ \max_{[\gamma^{-1} A_1^{-1} \epsilon^2] \leq j \leq [\gamma^{-1} A_2]} \frac{\sigma_{m_j}^2 + \ldots + \sigma_{m_j + \lfloor 2\gamma \sqrt{n} \rfloor - 1}^2}{2\gamma \sqrt{n}} \leq 2\sigma^2. \]
(5.30)
Observe that, for \([\gamma^{-1} A_1^{-1} \epsilon^2] \leq j \leq [\gamma^{-1} A_2] \), on \( E_{n,3} \cap \{Z_{[\gamma \sqrt{n}]} \leq A_1 \sqrt{n}\} \), we have that
\[ Z_0^{(i,[\gamma \sqrt{n}])} \leq \epsilon_1 A_1 \sqrt{n}, \]
for all \( 1 \leq i \leq A_1 \sqrt{n} \). Hence, on \( E_{n,3} \cap E_{n,4}^c \), for \([\gamma^{-1} A_1^{-1} \epsilon^2] \leq j \leq [\gamma^{-1} A_2] \) and \( 1 \leq i \leq A_1 \sqrt{n} \), we have that \((Z_k^{(i,[\gamma \sqrt{n}])})_{k \geq 0} \geq 0\) is stochastically dominated by a \( \mu_j\text{-BPVE} \) \((Z_k^j)_{k \geq 0} \) with \( Z_0^j = [\epsilon_1 A_1 \sqrt{n}] \) and where \( \mu_j = (\mu_{[\gamma \sqrt{n}]+k})_{k \geq 0} \). Thus, we obtain that, using Lemma 3.5 and (5.30), for all \( n \geq N_4 \),
\[ P_\mu(E_{n,3} \cap E_{n,4}^c) \]
\[ \leq A_1 \sqrt{n} \times 2\gamma^{-1} A_2 \]
\[ \times \max_{[\gamma^{-1} A_1^{-1} \epsilon^2] \leq j \leq [\gamma^{-1} A_2]} P_\mu(\max_{k \leq 2\gamma \sqrt{n}} |Z_k^{(1,[\gamma \sqrt{n}])} - Z_0^{(1,[\gamma \sqrt{n}])}| > \frac{\epsilon}{4 M A_2 \sqrt{n}}) \]
\[ \leq \epsilon_1 \times \frac{2^8 A_1^2 A_2^2 M^2 \gamma^{-2}}{\epsilon^2} \max_{[\gamma^{-1} A_1^{-1} \epsilon^2] \leq j \leq [\gamma^{-1} A_2]} \frac{\sigma_{j[\gamma \sqrt{n}]}^2 + \ldots + \sigma_{j[\gamma \sqrt{n]}+\lfloor 2\gamma \sqrt{n} \rfloor-1}^2}{2\gamma \sqrt{n}} \]
\[ \leq \epsilon_1 \times 2^9 A_1^2 A_2^2 M^2 \gamma^{-2} \sigma^{-2} \epsilon^{-2} \leq \frac{\delta}{4}. \]
(5.31)
This proves (5.27).
Finally, to prove (5.28), we explore, for a generation \( j \geq 0 \), the offspring of the vertices \( \{v_{i,j} : 1 \leq i \leq A_1 \sqrt{n} \} \), generation by generation, so that we can define the stopping time
\[
\tau_B = \min \left\{ j \in \{ \left[ A_1^{-1} \varepsilon^2 \sqrt{n} \right], \ldots, \left[ A_2 \sqrt{n} \right] \} : \left| \text{Bad}^{(+)} \right| \geq \frac{\varepsilon \sqrt{n}}{8A_2} \right\}.
\]
We let \( k_B \) be the smalled integer \( i \) such that \( \tau_b + i = m_j \) for some \( j \). Note that \( k_B \) is \( \tau_B \)-measurable. By applying the strong Markov property at time \( \tau_B \), proceeding as in (5.31) by letting \( Z_k \) be a BPVE with \( Z_0 = [\varepsilon \sqrt{n}/(8A_2)] \) and by using again Lemma 3.5 together with (5.30), we have that, for \( n \geq N_4 \),
\[
P_{\mu} (E_{n,5}^c) \leq E_{\mu} \left[ P_{\mu_B} \left( Z_{k_B} < \varepsilon_1 A_1 \sqrt{n} \right) \right] \leq E_{\mu} \left[ \frac{2^{12} A_2 \varepsilon^{-1}}{8 \sqrt{n}} (\sigma_{\tau_B}^2 + \cdots + \sigma_{\tau_B+k_B-1}^2) \right] \leq 2^{10} A_2 \varepsilon^{-1} \gamma \max_{|\gamma^{-1} A_1 |^2 \leq j \leq |\gamma^{-1} A_2 |} \frac{\sigma_{j}^2 + \cdots + \sigma_{|j \gamma \sqrt{n}| + |2 \gamma \sqrt{n}| - 1}^2}{2 \gamma \sqrt{n}} \leq 2^{11} A_2 \varepsilon^{-1} \sigma_2^2 \gamma \leq \frac{\delta}{4}.
\]
This concludes the proof. \( \square \)

### 5.2 Proof of Theorem 2.3

We are ready for the proof of Theorem 2.3 which states that the height process is proportional to the Lukasiewicz path reflected above its running minimum. In the next section, we will prove Theorem 1.1 by extracting the largest tree from the explored forest. We start by establishing a relation between the quantity \( \tilde{X}_x \) from the previous section and the value of the Lukasiewicz path on the explored forest \( F_n \). We then give an outline of the proof.

Consider the forest \( F_n \) and recall that it consists of a sequence of finite trees \( T^{(i)}, i \geq 1 \). For a vertex \( x \in F_n \), denote again \( L(x) \) the depth-first label of \( x \), as explained in Section 2.1. From the definition (2.4), one can observe that, if \( x \in T^{(1)} \), then \( \tilde{X}_x \) is precisely the value of the Lukasiewicz path right before revealing the offspring of \( x \), i.e. \( X_{L(x)} \) (this is the number of red vertices in Figure 2). Note however that every time the Lukasiewicz path has fully explored a tree, this creates an increment \(-1\) (corresponding in fact to the difference between the number of edges and the number of vertices of the tree). More precisely, if we denote \( n_i \) the depth-first label of the root of the tree \( T^{(i)} \) for \( i \geq 1 \), we have that
\[
\min_{n_i \leq k < n_{i+1}} (X_k - X_{n_i}) \geq 0, \text{ and } X_{n_{i+1}} - X_{n_i} = -1.
\]
From (5.32) and the definition of the Lukasiewicz path, we have that, for a vertex \( x \in F_n \),
\[
\tilde{X}_x = X_{L(x)} - I_{L(x)}.
\]
where we recall that \( I_n = \min \{X_k : 0 \leq k \leq n\} \).

Now, we explain the proof strategy, which is similar to that of [DLG02, Theorem 2.3.1]. From Proposition 5.3, we know that for every \( \varepsilon > 0 \), with high probability, every interval of length \( \varepsilon n \) in \([0, n]\) contains at least one integer \( k \) such that
\[
|(X_k - I_k) / H_k - \sigma^2 / 2| \leq \varepsilon.
\]
Let $k_1, \ldots, k_{K+1}$ be a sequence of such integers, for some $K \geq 1$ (we drop the dependency on $n$ to lighten the notation) such that $k_{i+1} - k_i \leq \varepsilon n$ for all $i \leq K$. Using Corollary 2.2 and the fact that the scaling limit of $X - I$ is continuous (and thus uniformly continuous on $[0,1]$), we can bound $\Delta := \max_{1 \leq i \leq K} \sup_{k_i \leq j < \ell \leq k_{i+1}} |(X_j - I_j) - (X_\ell - I_\ell)|$ by some constant $\gamma > 0$ (which can be chosen arbitrarily close to 0 as $\varepsilon \to 0$) with arbitrarily large probability. Therefore, it remains to control the variations of $H$ on the intervals $[k_i, k_{i+1}]$.

Controlling the negative variations of $H$ (i.e. $\min_j \{H_j - H_{k_i}\}$) does not seem obvious at all. In fact, the depth-first walk could in principle have large downwards jumps, when a vertex and a large number of its nearest ancestors are the last explored among their siblings (in Figures 2 and 3 this would correspond to many consecutive blue vertices having no red neighbours). Fortunately, it is enough to control only the positive variations of $H$: Indeed, combining the upper bound on $\Delta$ and (5.34), via the triangle inequality, allows us to control $\max_{i \leq K} |H_{k_{i+1}} - H_{k_i}|$ appropriately. Hence, negative and positive variations of $H$ in $[k_i, k_{i+1}]$ must be of the same order.

To handle these variations, we bound the height of the largest subtree built during $\varepsilon n$ steps of the exploration, using the estimates of Lemma 3.3. This is the purpose of Lemma 5.4 below. We first establish this lemma, and then proceed to the proof of Theorem 2.3.

**Lemma 5.4.** Let $\mu$ be a strictly critical environment satisfying Conditions $[I]$ to $[V]$. For all $\gamma > 0$ and all $\delta > 0$, there exist $\varepsilon = \varepsilon(\gamma, \delta) > 0$ and $N = N(\mu, \delta)$ such that, for all $n \geq N$, we have that

$$
P_\mu \left( \max_{0 \leq k \leq n} \max_{0 \leq i \leq \varepsilon n} (H_{k+i} - H_k) > \gamma \sqrt{n} \right) < \delta.
$$

(5.35)

**Proof.** We start by bounding the maximal height of the first $n$ vertices explored. By Proposition 3.4 there exist $A_2 = A_2(\delta) > 0$ and $N_1 = N_1(\mu, \delta)$ such that, for all $n \geq N_1$,

$$
P_\mu \left( \max_{0 \leq k \leq n} H_k > A_2 \sqrt{n} \right) = P_\mu(h(F_n) > A_2 \sqrt{n}) < \frac{\delta}{4}.
$$

(5.36)

Now, we will control the increments of $(H_k)_{0 \leq k \leq n}$ on vertices with height smaller than $A_2 \sqrt{n}$. Fix $\gamma > 0$ and $\delta > 0$. For all $n \geq 0$, let us define the sequence of stopping times $(\tau_k^{(n)})_{k \geq 0}$ by

$$
\tau_k^{(n)} := \inf \{ t > \tau_{k-1}, H_t \geq \min_{\tau_{k-1} \leq s \leq t} H_s + \gamma \sqrt{n} \}.
$$

(5.37)

Recall that, for the vertex with depth-first label $n$ and for a height $m \in \{0, \ldots, H_n - 1\}$, we have, from (5.1), that $\zeta_m(n)$ is the number of vertices at height $m + 1$ which are strictly on the right of the spine from the vertex $n$ to the root of its tree, and whose parent is on this spine.

For $n \geq 0$, let us define the $\sigma$-algebra

$$
G_n = \sigma \left( X_0, \ldots, X_n \right).
$$

In particular, note that $H_n$ and $\{\zeta_m(n), m \in \{0, \ldots, H_n - 1\}\}$ are $G_n$-measurable. Finally, we will use the notation $G_k^{(n)}$ for $G_{\tau_k^{(n)}}$. Our next goal is to prove that for all $\delta > 0$ there exists a constant $\eta = \eta(\gamma, \delta) > 0$ and there exists $N_2 = N_2(\mu, \gamma, \delta)$ such that, for all $n \geq N_2$ and for all $k \geq 0$,

$$
1 \left\{ H_{\tau_k^{(n)}} \leq A_2 \sqrt{n} \right\} \cdot P_\mu \left( \tau_{k+1}^{(n)} - \tau_k^{(n)} < \eta \sqrt{n} \right) \leq \frac{\delta}{2} \text{ a.s.}
$$

(5.38)

We will first prove the above, and then show how this implies the conclusion.
Figure 5: Black vertices are roots of fully explored trees. The yellow vertex is the root of the
tree being explored. Blue vertices are on the spine of the vertex with label \( n \). Red vertices are
parents of subtrees (\( v_1 \) to \( v_7 \)) and trees (from \( v_8 \) onwards) to be explored after the vertex \( n \).

Fix integers \( n \geq 0, k \geq 0 \) and suppose that we know \( \mathcal{G}_k^{(n)} \), i.e. we have explored the forest
until the node with depth-first label \( \tau_k^{(n)} \). Recall the definition (5.1) of \( \tilde{X}_n \) and let us define the
shorthand notation

\[
M = 1 + \tilde{X}_k^{(n)} = 1 + \zeta_0(\tau_k^{(n)}) + \cdots + \zeta_{H_{\tau_k^{(n)}}-1}(\tau_k^{(n)}),
\]

(5.39)

which counts \( \tau_k^{(n)} \) plus the total number of vertices attached to the spine from the root of the
tree of \( \tau_k^{(n)} \) to \( \tau_k^{(n)} \), and strictly on the right of this spine, see Figure 5. Let us denote these
vertices \( v_1, \ldots, v_M \) in depth-first order. There exists an index \( i_{k,n} \) such that
\( \tau_k^{(n)} \) belongs to the tree \( T_{i_{k,n}} \) in \( F \). We will denote, for \( j \geq 1 \), \( v_{M+j} \) the root of the tree \( T_{i_{k,n}+j} \) (which has not
yet been seen by the exploration).

The offspring of these nodes \( v_j \) for \( j \geq 1 \) have not yet been explored. Hence, given \( \mathcal{G}_k^{(n)} \),
we can attach to each node \( v_j \) a tree \( T_j \) distributed like a BPVE in environment
\( \mu_{h(v_j)} := (\mu_k)_{k \geq h(v_j)} \), such that the trees \( T_j \) are independent.
Note that the sequence \( (H_{v_j})_{j \geq 1} \) is non-increasing and that
\( \max_{j \geq 1} H_{v_j} \leq H_{\tau_k^{(n)}} \) (because the exploration is depth-first), hence the node \( \tau_{k+1}^{(n)} \) belongs to the first tree \( T_j \), \( j \geq 1 \) whose height
is at least \( \gamma \sqrt{n} \). Let us define the constant

\[
\kappa = \frac{\delta \gamma c}{2^3},
\]

(5.40)

where \( c \) is the constant from Condition [II].

Using the previous observations, we have that conditionally on \( \mathcal{G}_k^{(n)} \), for all \( \eta > 0, \)

\[
\left\{ \max_{1 \leq j \leq \kappa \sqrt{n}} h(T_j) < \gamma \sqrt{n} \right\} \cap \left\{ \sum_{j=1}^{\lfloor \kappa \sqrt{n} \rfloor} |T_j| \geq \eta n \right\} \subset \left\{ \tau_{k+1}^{(n)} - \tau_k^{(n)} \geq \eta n \right\}.
\]

(5.41)
Therefore, in order to prove (5.38), it is enough to show that there exist \( \eta = \eta(\gamma, \delta) > 0 \) and \( N_2 = N_2(\mu, \delta) \) such that for all \( n \geq N_2 \) and for all \( k \geq 0 \), the following holds almost surely:

\[
1 \left\{ H_{k(n)}^{(n)} \leq A_2 \sqrt{n} \right\} \cdot P_{\mu} \left( \max_{1 \leq j \leq \left\lfloor \kappa \sqrt{n} \right\rfloor} h(T_j) \geq \gamma \sqrt{n} \right| G_k^{(n)} \right) \leq \frac{\delta}{4},
\]

(5.42)

\[
1 \left\{ H_{k(n)}^{(n)} \leq A_2 \sqrt{n} \right\} \cdot P_{\mu} \left( \sum_{j=1}^{\left\lfloor \kappa \sqrt{n} \right\rfloor} |T_j| < \eta n \right| G_k^{(n)} \right) \leq \frac{\delta}{4}.
\]

(5.43)

Let us prove the first inequality above. For an integer \( 0 \leq i \leq A_2 \sqrt{n} \), we denote \( \mu_i := (\mu_k)_{k \geq i} \), and let us write simply \( T \) for a \( \mu_i \)-BPVE. By (3.2) and (3.6), we have that

\[
P_{\mu_i} (h(T) \geq \gamma \sqrt{n}) \leq \frac{1}{\sum_{m=i}^{i+1} \mu_m(\{0\})}.
\]

(5.44)

By Condition (II), there exists \( N_3 = N_3(\mu, A_2, \gamma) \) such that, for all \( n \geq N_3 \),

\[
\min_{0 \leq j \leq 2A_2 \gamma^{-1} + 2} \frac{\sum_{m=j-\left\lfloor \kappa \sqrt{n} / 2 \right\rfloor}^{j+1 - \left\lfloor \kappa \sqrt{n} / 2 \right\rfloor} \mu_m(\{0\})}{\sqrt{n}} \geq \frac{\gamma c}{4},
\]

hence, by (5.43), for all \( n \geq N_3 \),

\[
\max_{0 \leq i \leq A_2 \sqrt{n}} P_{\mu_i} (h(T) \geq \gamma \sqrt{n}) \leq \frac{4}{\gamma c \sqrt{n}}.
\]

(5.45)

Therefore, using the fact that if \( H_{k(n)}^{(n)} \leq A_2 \sqrt{n} \) then \( h(v_j) \leq A_2 \sqrt{n} \) for all \( j \geq 1 \), applying a union bound and using (5.40), we obtain that, for \( n \geq N_3 \) and for all \( k \geq 0 \),

\[
1 \left\{ H_{k(n)}^{(n)} \leq A_2 \sqrt{n} \right\} \cdot P_{\mu} \left( \max_{1 \leq j \leq \left\lfloor \kappa \sqrt{n} \right\rfloor} h(T_j) \geq \gamma \sqrt{n} \right| G_k^{(n)} \right) \leq \frac{4 \kappa}{\gamma c} \leq \frac{\delta}{4}.
\]

(5.46)

This proves the first inequality of (5.42) and we now turn to the proof of the second inequality. Note that (5.36) and (5.45) do not depend on \( \eta \). To prove this second inequality, define the constants

\[
\alpha = \frac{e^2 \kappa}{2^{15} \sigma^6 \ln(4/\delta)}, \quad \beta = \frac{\alpha c}{2^{10} \cdot \sigma^2}, \quad \eta = \frac{\alpha c}{2^6} = \frac{e^4 \delta \gamma}{2^{14} \sigma^6 \ln(4/\delta)}.
\]

(5.47)

For \( 0 \leq i \leq A_2 \sqrt{n} \), by (3.2) and (3.6), we have that

\[
\frac{1}{1 + \sum_{m=i}^{i+\left\lfloor \alpha \sqrt{n} \right\rfloor + 1} \sigma_m^2} \leq P_{\mu_i} (h(T) \geq \alpha \sqrt{n}) \leq \frac{1}{\sum_{m=i}^{i+\left\lfloor \alpha \sqrt{n} \right\rfloor} \mu_m(\{0\})}.
\]

(5.48)

By Condition (I) and Lemma 3.1 for the lower bound, and Condition (II) for the upper bound, there exists \( N_4 = N_4(\mu, A_2, \alpha) \) such that, for all \( n \geq N_4 \),

\[
\max_{0 \leq j \leq 2A_2 \alpha^{-1} / 2 + 1} \frac{\sum_{m=j-\left\lfloor 2 \alpha \sqrt{n} / 2 \right\rfloor}^{j+1 - \left\lfloor 2 \alpha \sqrt{n} / 2 \right\rfloor} \sigma_m^2}{\sqrt{n}} \leq 4 \alpha \sigma^2,
\]

(5.49)

\[
\min_{0 \leq j \leq 2A_2 \alpha^{-1} + 2} \frac{\sum_{m=j-\left\lfloor \alpha \sqrt{n} / 2 \right\rfloor}^{j+1 - \left\lfloor \alpha \sqrt{n} / 2 \right\rfloor} \mu_m(\{0\})}{\sqrt{n}} \geq \frac{\alpha c}{4}.
\]

(5.50)
hence, by (5.47), for all \( n \geq N_4 \),
\[
\frac{1}{8\alpha^2 \sqrt{n}} \leq \min_{0 \leq i \leq A_2 \sqrt{n}} P_{\mu_i} \left( h(T) \geq \alpha \sqrt{n} \right) \leq \max_{0 \leq i \leq A_2 \sqrt{n}} P_{\mu_i} \left( h(T) \right) \leq \frac{4}{\alpha \sqrt{n}}, \tag{5.49}
\]
where we assumed w.l.o.g. that \( N_4 \geq 1/(4\alpha^2)^2 \). To lighten the following equations, denote \( Z_{[\alpha \sqrt{n}]} := Z_{[\alpha \sqrt{n}]}(T) \) the number of vertices at height \( [\alpha \sqrt{n}] \) in \( T \). By Paley-Zygmund’s inequality, we have that for all \( 0 \leq i \leq A_2 \sqrt{n} \),
\[
P_{\mu_i} \left( Z_{[\alpha \sqrt{n}]} \geq \frac{\alpha \sqrt{n}}{32} \right) \left| h(T) \geq [\alpha \sqrt{n}] \right\rangle \geq E_{\mu_i} \left[ Z_{[\alpha \sqrt{n}]} \bigg| Z_{[\alpha \sqrt{n}]} > 0 \right\rangle^2
\]
\[
\times \left( 1 - \frac{\alpha \sqrt{n}}{32 E_{\mu_i} \left[ Z_{[\alpha \sqrt{n}]} \bigg| Z_{[\alpha \sqrt{n}]} > 0 \right\rangle \right)^2 \tag{5.50}
\]
By (5.48) and (5.49), for \( n \geq N_4 \) and for all \( 0 \leq i \leq A_2 \sqrt{n} \), we have that
\[
E_{\mu_i} \left[ Z_{[\alpha \sqrt{n}]} \bigg| Z_{[\alpha \sqrt{n}]} > 0 \right\rangle \geq \frac{\alpha \sqrt{n}}{4}, \tag{5.51}
\]
\[
E_{\mu_i} \left[ Z_{[\alpha \sqrt{n}]}^2 \bigg| Z_{[\alpha \sqrt{n}]} > 0 \right\rangle \leq 8\alpha \sigma^2 \sqrt{n} \left( 1 + \sum_{m=1}^{m+1} \sigma_m^2 \right) \leq 2^5 \alpha \sigma^4 n. \tag{5.52}
\]
By (5.50), we have that, for all \( n \geq N_4 \), for all \( 0 \leq i \leq A_2 \sqrt{n} \)
\[
P_{\mu_i} \left( Z_{[\alpha \sqrt{n}]} \geq \frac{\alpha \sqrt{n}}{32} \bigg| h(T) \geq [\alpha \sqrt{n}] \right\rangle \geq \frac{c^2}{512 \sigma^4} \times \frac{1}{15} \geq \frac{c^2}{2^{10} \sigma^4}. \tag{5.53}
\]
Now, for all \( 0 \leq i \leq A_2 \sqrt{n} \), letting \( (Z_k') \) be a BPVE in environment \( \mu_{i+1} \alpha \sqrt{n} \) with \( Z_0' = \lfloor \alpha \sqrt{n}/32 \rfloor \), we have by Lemma \[3.5\]
\[
P_{\mu_i} \left( Z_{[\alpha \sqrt{n}]} + [\beta \sqrt{n}] (T) < \eta \sqrt{n} \bigg| Z_{[\alpha \sqrt{n}]} \geq \frac{\alpha \sqrt{n}}{32} \right\rangle \leq P_{\mu_{i+1} \alpha \sqrt{n}} \left( Z'_{[\beta \sqrt{n}]} < \frac{\alpha c}{64 \sqrt{n}} \right\rangle
\]
\[
\leq P_{\mu_{i+1} \alpha \sqrt{n}} \left( Z'_{[\beta \sqrt{n}]} - Z_0' > Z_0' \right\rangle \frac{128 \left( \sum_{m=1}^{m+1} \sigma_m^2 \right)}{\alpha \sqrt{n}} \leq \frac{128 \left( \sum_{m=j+1}^{m+1} \sigma_m^2 \right)}{\alpha \sqrt{n}} \frac{\max_{0 \leq j \leq (A_2 + \alpha) \beta^{-1}} \left( \sum_{m=j+1}^{m+1} \sigma_m^2 \right)}{\sqrt{n}}. \tag{5.54}
\]
Again, by Condition \[1\] and Lemma \[3.1\] there exists \( N_5 = N_5(\mu, \beta) \) such that, for all \( n \geq N_5 \), we have that
\[
P_{\mu_i} \left( Z_{[\alpha \sqrt{n}]} + [\beta \sqrt{n}] (T) < \eta \sqrt{n} \bigg| Z_{[\alpha \sqrt{n}]} \geq \frac{\alpha \sqrt{n}}{32} \right\rangle \leq \frac{512 \beta \sigma^2}{\alpha c} \leq \frac{1}{2}. \tag{5.55}
\]
where we used the definition (5.46) of $\beta$ in the second inequality. Using (5.49), (5.51) and (5.54) together with (5.46), we have that, for all $n \geq N_4 \lor N_5$ and for all $0 \leq t \leq A_2 \sqrt{n}$,

$$
P_{\mu_i} \left( |T| \geq \eta \sqrt{n} \right) \geq P_{\mu_i} \left( h(T) \geq \alpha \sqrt{n} \right) \times P_{\mu_i} \left( \left| \frac{\alpha \sqrt{n}}{32} \right| h(T) \geq \left| \alpha \sqrt{n} \right| \right) \times P_{\mu_i} \left( Z_{[\alpha \sqrt{n}]}(T) \geq \eta \sqrt{n} \mid Z_{[\alpha \sqrt{n}]}(T) \geq \frac{\alpha \sqrt{n}}{32} \right) \geq \frac{1}{8\alpha^2 \sqrt{n}} \times \frac{c^2}{210 \sigma^2} \frac{1}{2} \geq \frac{2 \ln(4/\delta)}{\kappa \sqrt{n}}. \tag{5.55}$$

Using the above, we obtain that, for all $n \geq N_4 \lor N_5$ and if $n \geq 4\kappa^{-2}$, then

$$
I \left\{ H_{(n)} \leq A_2 \sqrt{n} \right\} \cdot P_{\mu} \left( \left| T_j \right| < \eta n \mid G_k^{(n)} \right) \leq \left( 1 - \frac{2 \ln(4/\delta)}{\kappa \sqrt{n}} \right) \left| \kappa \sqrt{n} \right| \leq \delta/4. \tag{5.56}
$$

This concludes the proof of (5.42) and thus of (5.38), with $N_2 = N_3 \lor N_4 \lor N_5 \lor (4\kappa^{-2})$ and $\eta = \eta(\gamma, \delta)$ as in (5.46).

We now want to prove that the sequence $(\tau_k^{(n)})_{k \geq 1}$ does not have too small increments before $\tau_k^{(n)}$ becomes larger than $n$.

By (5.38), there exists $u > 0$ and $N_6 = N_6(\mu)$ such that, for all $n \geq N_6$ and all $k \geq 0$,

$$
I \left\{ H_{(n)} \leq A_2 \sqrt{n} \right\} \cdot P_{\mu} \left( \tau_k^{(n)} - \tau_k^{(n)} < un \mid G_k^{(n)} \right) \leq \frac{1}{2}. \tag{5.57}
$$

Note that as long as $H_{(n)} \leq A_2 \sqrt{n}$, the random variable $\tau_k^{(n)} - \tau_k^{(n)}$ stochastically dominates $(un) \cdot B_k$, where $(B_k)_{k \geq 0}$ is a sequence of i.i.d. Bernoulli random variables with parameter 1/2. Therefore, for all $K \geq 0$, for all $n \geq N_6$,

$$
P_{\mu} \left( \tau_K^{(n)} \leq n, \max_{0 \leq k \leq n} H_k \leq A_2 \sqrt{n} \right) \leq P_{\mu} \left( \sum_{k=0}^{K-1} B_k \leq \frac{1}{u} \right).$$

Because the sequence $(B_k)_{k \geq 0}$ satisfies the law of large numbers, there exists $K$ (depending on $u$ only, and $u$ does not depend on other parameters), such that, for all $n \geq N_2$,

$$
P_{\mu} \left( \tau_K^{(n)} \leq n, \max_{0 \leq k \leq n} H_k \leq A_2 \sqrt{n} \right) \leq \frac{\delta}{8}. \tag{5.58}
$$

Using (5.38) again, there exists $\varepsilon = \varepsilon(\delta, \gamma) > 0$ and $N_7 = N_7(\mu, \delta) \geq N_6$ such that, for all $n \geq N_7$ and all $k \geq 0$,

$$
I \left\{ H_{(n)} \leq A_2 \sqrt{n} \right\} \cdot P_{\mu} \left( \tau_k^{(n)} - \tau_k^{(n)} < \varepsilon n \mid G_k^{(n)} \right) \leq \frac{\delta}{8K}. \tag{5.59}
$$
Thus, we obtain that for all \( n \geq N_7 \),

\[
P_\mu \left( \{ \max_{0 \leq k \leq n} H_k \leq A_2 \sqrt{n} \} \cap \bigcup_{k \geq 0} \{ \tau_k^{(n)} \leq n, \tau_{k+1}^{(n)} - \tau_k^{(n)} < \varepsilon n \} \right)
\]

\[
\leq P_\mu \left( \max_{0 \leq k \leq n} H_k \leq A_2 \sqrt{n}, \tau_k^{(n)} \leq n \right) + \sum_{k=0}^{K-1} P_\mu \left( H_k \leq A_2 \sqrt{n}, \tau_k^{(n)} < \varepsilon n \right)
\]

\[
\leq \frac{\delta}{8} + K \times \frac{\delta}{8K} \leq \frac{\delta}{4}.
\]

To conclude, define the events

\[
E_{1,n} = \left\{ \max_{0 \leq k \leq n} H_k \leq A_2 \sqrt{n} \right\},
\]

\[
E_{2,n} = \bigcap_{k \geq 0} \left( \{ \tau_k^{(n)} > n \} \cup \{ \tau_{k+1}^{(n)} - \tau_k^{(n)} \geq \varepsilon n \} \right).
\]

Observe that

\[
E_{1,n} \cap E_{2,n} \subset \left\{ \max_{0 \leq k \leq n} \max_{0 \leq i \leq \varepsilon n} (H_{k+i} - H_k) \leq \gamma \sqrt{n} \right\}
\]

Hence, using (5.36) and (5.60), we have that, for all \( n \geq N_1 \lor N_7 \),

\[
P_\mu \left( \max_{0 \leq k \leq n} \max_{0 \leq i \leq \varepsilon n} (H_{k+i} - H_k) > \gamma \sqrt{n} \right) \leq P_\mu \left( E_{1,n}^c \right) + P_\mu \left( E_{1,n}, E_{2,n}^c \right)
\]

\[
\leq \frac{\delta}{4} + \frac{\delta}{4} < \delta.
\]

This proves the statement of the lemma, with \( N = N_1 \lor N_7 \).

We are now ready to prove Theorem 2.3.

\textbf{Proof of Theorem 2.3.} Let \( \mu \) be a strictly critical environment satisfying Conditions (I)-(V). By Corollary 2.2, it is enough to prove that for all \( \gamma > 0 \), and all \( \delta > 0 \), there exists \( N = N(\mu, \gamma, \delta) \) such that, for all \( n \geq N \),

\[
P_\mu \left( \sup_{1 \leq k \leq n} \frac{1}{\sqrt{n}} \left| H_k - \frac{2(X_k - I_k)}{\sigma^2} \right| > \gamma \right) \leq \delta.
\]

For \( \varepsilon \in (0, 1/2) \), define \( E_{1,n}(\varepsilon) \) the event that there exist (random) integers \( K \in \mathbb{N} \), and \( 0 = k_0 < k_1 < \ldots < k_K < k_{K+1} = n \) such that:

(i) for all \( 0 \leq i \leq K \), \( k_{i+1} - k_i \leq 1 + \varepsilon n \),

(ii) for all \( 1 \leq i \leq K \), \( \left| \frac{X_{k_i} - I_{k_i}}{H_n} - \frac{\sigma^2}{2} \right| \leq \varepsilon \).

Recall the definition (5.3) of a \( \varepsilon \)-bad vertex and recall the relation (5.33). Using these two, note that if an index \( k \) is not \( \varepsilon \)-bad, then it satisfies the item (i) above, and if there are less than \( \varepsilon n \) \( \varepsilon \)-bad vertices, then we can find a collection of indices that are not \( \varepsilon \)-bad such that there are at most \( \varepsilon n \) \( \varepsilon \)-bad vertices between any two indices. Hence, we have that

\[
\{ \mathcal{F}_n \text{ has at most } \varepsilon n \ \varepsilon \text{-bad vertices} \} \subset E_{1,n}(\varepsilon).
\]

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Thus, by Proposition 5.3, there exists $N_1 = N_1(\mu, \varepsilon, \delta)$ such that for all $n \geq N_1$,

$$\mathbb{P}_\mu(\mathcal{E}_{1,n}^c(\varepsilon)) \leq \frac{\delta}{10}. \quad (5.61)$$

For all $n \geq 0$, $\varepsilon > 0$ and $\gamma > 0$, let us define the event

$$\mathcal{E}_{2,n}(\varepsilon, \gamma) = \bigcap_{|j-k| \leq 2n} \left\{ \frac{2(X_k - I_k)}{\sigma^2 \sqrt{n}} < \frac{1}{\varepsilon^{1/2}}, \quad \frac{1}{\sqrt{n}} \left| \frac{2(X_k - I_k)}{\sigma^2} - \frac{2(X_j - I_j)}{\sigma^2} \right| < \gamma \right\}. \quad (5.62)$$

To estimate the probability of the event above, we will use Corollary 2.2. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion under some measure $\mathbb{P}$. It is a.s. uniformly continuous (hence bounded) on $[0, 1]$. Thus there exists $\varepsilon_0 = \varepsilon_0(\delta, \gamma) \in (0, 1/2)$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbb{P} \left( \sup_{0 \leq t \leq 1} |B_t| \geq \frac{1}{2 \varepsilon^{1/2}} \right) + \mathbb{P} \left( \sup_{0 \leq t \leq 1} |B_t - B_s| \geq \frac{\gamma}{20} \right) \leq \frac{\delta}{10}. \quad (5.63)$$

By Corollary 2.2, for all $\varepsilon \in (0, \varepsilon_0)$, there exists $N_2 = N_2(\mu, \varepsilon, \delta, \gamma) \geq N_1(\mu, \varepsilon, \delta)$ such that for all $n \geq N_2$,

$$\mathbb{P}_\mu(\mathcal{E}_{1,n}^c(\varepsilon) \cup \mathcal{E}_{2,n}^c(\varepsilon, \gamma)) \leq \frac{\delta}{5}. \quad (5.64)$$

On $\mathcal{E}_{1,n}(\varepsilon) \cap \mathcal{E}_{2,n}(\varepsilon, \gamma)$, for all $1 \leq i \leq K$, we have that

$$\left| \frac{X_k - I_k}{H_{k_i}} - \frac{\sigma^2}{2} \right| \leq \varepsilon, \quad (5.65)$$

and thus

$$H_{k_i} \leq \frac{2(X_k - I_k)}{\sigma^2 + 2\varepsilon}. \quad (5.66)$$

Putting the two displays above together, on $\mathcal{E}_{1,n}^c(\varepsilon) \cap \mathcal{E}_{2,n}^c(\varepsilon, \gamma)$, we have for all $1 \leq i \leq K$

$$\left| \frac{2(X_k - I_k)}{\sigma^2} - H_{k_i} \right| \leq \left| \frac{X_k - I_k}{H_{k_i}} - \frac{\sigma^2}{2} \right| \cdot \frac{2H_{k_i}}{\sigma^2} \leq \frac{4\varepsilon(X_k - I_k)}{\sigma^2(\sigma^2 + 2\varepsilon)} \leq \frac{4\varepsilon^{1/2}}{\sigma^2(\sigma^2 + 2\varepsilon)} \sqrt{n}. \quad (5.67)$$

Let us define the event

$$\mathcal{E}_{3,n}(\varepsilon, \gamma) = \left\{ \max_{0 \leq k \leq n} \max_{0 \leq i \leq m_n} (H_{k+i} - H_k) \leq \frac{\gamma}{10 \sqrt{n}} \right\}. \quad (5.68)$$

By Lemma 5.4, we have that for all $\gamma > 0$ and all $\delta > 0$, there exists $\varepsilon_1 = \varepsilon_1(\gamma, \delta) \in (0, \varepsilon_0(\gamma, \delta))$, such that for all $\varepsilon \in (0, \varepsilon_1)$, there exists $N_3 = N_3(\mu, \varepsilon, \delta, \gamma) \geq N_2(\mu, \varepsilon, \delta, \gamma)$ such that if $n \geq N_3$, then

$$\mathbb{P}_\mu(\mathcal{E}_{3,n}^c(\varepsilon, \gamma)) < \frac{\delta}{10}. \quad (5.69)$$

Using (5.66) and (5.67), we have that, on $\mathcal{E}_{1,n}(\varepsilon) \cap \mathcal{E}_{2,n}(\varepsilon, \gamma) \cap \mathcal{E}_{3,n}(\varepsilon, \gamma)$, for all $j \in \{0, \ldots, n\}$, there exists $i(j) \in \{1, \ldots, K\}$ such that $|j - k_{i(j)}| \leq \varepsilon n$, hence, we have that

$$\left| H_j - \frac{2(X_j - I_j)}{\sigma^2} \right| \leq \left| H_j - H_{k_{i(j)}} \right| + \left| H_{k_{i(j)}} - \frac{2(X_{k_{i(j)}} - I_{k_{i(j)}})}{\sigma^2} \right| \quad (5.70)$$

$$+ \left| \frac{2(X_{k_{i(j)}} - I_{k_{i(j)}})}{\sigma^2} - \frac{2(X_j - I_j)}{\sigma^2} \right| \leq \frac{\gamma}{10 \sqrt{n}} + \frac{4\varepsilon^{1/2}}{\sigma^2(\sigma^2 + 2\varepsilon)} \sqrt{n} + \frac{\gamma}{10 \sqrt{n}}. \quad (5.71)$$
Choosing $\varepsilon$ small enough, depending on $\gamma$ and $\sigma^2$, the RHS of the last line can be made smaller than $\gamma \sqrt{n}$.

Hence, for all $\gamma > 0$ and $\delta > 0$, using the above together with (5.65) and (5.68), there exists $\varepsilon = \varepsilon(\gamma, \delta)$ and $N = N(\mu, \gamma, \delta)$ such that, for all $n \geq N$

$$
P_{\mu} \left( \max_{0 \leq j \leq n} \frac{1}{\sqrt{n}} H_j - \frac{2(X_j - I_j)}{\sigma^2} > \gamma \right) \leq P_{\mu} \left( \mathcal{E}^c_{1,n}(\varepsilon) \cup \mathcal{E}^c_{2,n}(\varepsilon, \gamma) \cup \mathcal{E}^c_{3,n}(\varepsilon, \gamma) \right) < \delta,$$

which concludes the proof of the theorem. 

### 5.3 Extracting large trees from the forest

In this section, we prove Theorem 1.1 as a consequence of Theorem 2.3.

Before doing so, we derive from Theorem 2.3 the convergence in distribution of the ordered excursions of the height process, following a standard reasoning originally due to Aldous [AL98].

For all $n \geq 1$ and $i \geq 1$, let $e_i^{(n)}$ be the $i$-th longest excursion above 0 of $n^{-1/2}(H_{[nt]})_{0 \leq t \leq 1}$ that ends before time 1. For $n \geq 1$ and $i \geq 1$, let $t_i^{(n)}$ be the starting time of $e_i^{(n)}$, and $r_i^{(n)}$ its ending time. Similarly, for $n \geq 1$ and $i \geq 1$, define $e_i^B$, $t_i^B$ and $r_i^B$ the $i$-th longest excursion, its starting time and ending time, of $\frac{2}{\sigma}(B_t)_{0 \leq t \leq 1}$, where $B$ is a standard Brownian motion.

**Lemma 5.5.** Let $\mu$ be a strictly critical environment satisfying Condition [H][V]. We have that

$$(e_i^{(n)}, t_i^{(n)})_{i \geq 1} \overset{(d)}{\rightarrow} (e_i^B, t_i^B)_{i \geq 1},$$

as $n$ goes to infinity, in Skorokhod topology for the first coordinate and in product topology for the second.

**Proof.** We are going to apply Proposition 6 stated in the appendix, as it is a slight modification of Lemma 5.8 in [CKG20]. To apply it, we need to have a better separation of the excursions of $(H_n)$.

Recall that, in (5.32), we established that every time $(X_n)$ has explored a whole tree, it reaches a new strict minimum that beats the previous minimum by $-1$. In other words, $(I_n)_n$, the running minimum of $(X_n)$, is equal to minus the number of trees fully explored at time $n$. On the other hand, $(H_n)$ starts at $0$ and returns to $0$ only when a tree has just been fully explored. Therefore, the process $(\sigma^2/2 H_n + I_n)_n$ achieves a new strict minimum (by $-1$) every time a tree has been fully explored.

Let us define, for all $n \geq 1$ and all $t \in [0, 1]$,

$$\tilde{H}_n(t) = \frac{1}{\sqrt{n}} \left( \frac{\sigma^2}{2} H_{[nt]} + I_{[nt]} \right).$$

By Theorem 2.1, Theorem 2.3, and the continuous mapping theorem, we have that $(\tilde{H}_n(t))_{t \in [0, 1]}$ converges to $(\sigma B_t)_{t \in [0, 1]}$.

Now we can apply Proposition 6 with $f = \sigma B$ and, for all $n \geq 1$, $f_n = \tilde{H}_n(\cdot)$. Moreover, let us define $J_n = -I_n + 1$ (i.e. the number of trees in $F_n$), for all $1 \leq i \leq J_n$, $\nu_{n,i} = j(i)/n$ where $j(i)$ is the depth-first label of the root of the $i$-th tree, i.e. the $i$-th time (multiplied by $n$) $\tilde{H}_n(\cdot)$ is at a new strict minimum.

The assumptions on $f$ are obtained by well known properties of Brownian motion. The facts that $t_{n,i} \in [0, 1]$, that $f_n(t_{n,i}) < f_n(s)$ for all $s < t_{n,i}$ and that $f_n(s) > f_n(t_{n,J_n})$ for all $s > t_{n,J_n}$ are easily seen to be satisfied by construction. In addition, remark that $\sqrt{n} \max_{1 \leq i < J_n} (f_n(t_{n,i}) - f_n(t_{n,i+1})) = 1$, thus we have that $\max_{1 \leq i < J_n} (f_n(t_{n,i}) - f_n(t_{n,i+1}))$ goes to $0$ almost surely.
Therefore, we have that the starting times (the $t_{n,i}$'s) and lengths of the excursions of $\tilde{H}_n(\cdot)$ above its running minimum converge to those of $\sigma B$.

Now, remark that the starting times and lengths of the excursions of $\tilde{H}_n(\cdot)$ are the starting times and lengths of the excursions of $(H_{[nt]}/\sqrt{n})$ strictly above 0. This yields the convergence of $(l_i^n, r_i^n - l_i^n)$ towards $(I_i^B, r_i^B - I_i^B)$. Finally, the convergence of $H_{[nt]}/\sqrt{n}$ towards $\frac{2}{\sigma} B$ on $[0, 1]$ imply the convergence on the excursion intervals (which are non-empty, as $r_i^B > I_i^B$ for all $i \geq 1$), and this concludes the proof of the lemma.

**Proof of Theorem 1.1.** By Skorokhod’s representation theorem, we can assume that this convergence holds $P_{\mu}$-almost surely. Then, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ small enough such that, with probability at least $1 - \varepsilon$, the first excursion $e^B(\delta)$ above 0 of $(|B_t|)_{0 \leq t \leq 1}$ of length at least $\delta$ is entirely comprised in $[0, 1/2]$. Moreover, there exists $\delta' = \delta'(\delta, \varepsilon)$ small enough such that with probability at least $1 - \varepsilon$, no other excursion starting in $[0, 1/2 + \delta']$ has length at least $\delta - 2\delta'$. This is due to the fact that for every $\delta > 0$, if $e$ is an excursion above 0 of a standard Brownian motion, then the map $x \mapsto P(\ell(e) > x | \ell(e) > \delta/2)$ is continuous on $(\delta, \infty)$, where $\ell(e)$ denotes the length of $e$.

Denote $e^{(n)}(\delta, \delta')$ the first excursion of $n^{-1/2}(H_{[nt]})_{0 \leq t \leq 1}$ having length at least $\delta - \delta'$. Then Lemma 5.5 yields that, on these events,

$$e^{(n)}(\delta, \delta') \rightarrow_{n \to \infty} e^B(\delta)$$

in Skorokhod space, as $e^B(\delta)$ is the only excursion long enough starting in $[0, 1/2 + \delta']$ towards which $e^{(n)}(\delta, \delta')$ can converge. But the distribution of $e^{(n)}(\delta, \delta')$ coincides with that of the height process of $T_{(\delta-\delta')/n}$, on the event that $n^{-1/2}(H_{[nt]})_{0 \leq t \leq 1}$ has such an excursion before time 1 (which has probability at least $1 - 2\varepsilon$ for $n$ large enough). Hence Theorem 1.1 follows from this, from the construction of the CRT (by [CKG20, Equation (8)]), convergence of the excursions in Skorokhod space implies that of the underlying trees in Gromov-Hausdorff-Prokhorov metric) and from the usual scaling property of the Brownian motion.

**Proposition 6.** Let $f : [0, 1] \to \mathbb{R}$ be a continuous function, let $E(f)$ be the set of non-empty intervals $e = (l, r) \subset [0, 1]$ such that $f(l) = \inf_{s \leq l} f(s) = f(r)$ and $f(s) > f(l)$ for all $s \in (l, r)$. We call such an interval $e$ an excursion of $f$ above its running minimum, and we call $r - l$ its length. We will assume furthermore that

- for all $\varepsilon > 0$, $E(f)$ contains only finitely many excursions of length greater than or equal to $\varepsilon$;
- the set $[0, \sup_{e \in E(f)} s \in e] \setminus \bigcup_{e \in E(f)} e$ has Lebesgue measure $0$;
- if $(l_1, r_1), (l_2, r_2) \in E(f)$ and $l_1 < l_2$, then $f(l_1) > f(l_2)$.

Let $(f_n)_{n \geq 1}$ be a sequence of càdlàg functions such that $f_n \to f$ in the Skorokhod sense. For each $n \geq 1$, let $J_n \geq 1$, $(t_{n,i})_{1 \leq i \leq J_n}$ be a collection of strictly increasing numbers in $[0, 1]$, with $t_{n,1} = 0$, such that $f_n(t_{n,i}) < f_n(s)$ for all $s < t_{n,i}$, $f_n(s) \geq f_n(t_{n,J_n})$ for all $s > t_{n,J_n}$, and $\lim_{n \to \infty} \max_{1 \leq i \leq J_n} (f_n(t_{n,i}) - f_n(t_{n,i+1})) = 0$.

Then, we have that

$$\{(t_{n,i}, t_{n,i+1} - t_{n,i}); 1 \leq i < J_n\} \to \{(l, r-l); (l, r) \in E(f)\},$$

as $n$ goes to infinity and where the convergence holds in the topology of vague convergence of counting measures on $[0, 1] \times (0, 1]$. 

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Proof. This is a slight modification of Lemma 5.8 in [CKG20], whose proof can be easily adapted. The main difference is that we assume here that $f$ is continuous and supported on $[0, 1]$. We assume in our statement that conditions (4), (5) and (6) of [CKG20] are satisfied. Conditions (1), (3) of [CKG20] are easily checked, by continuity, while condition (2) of [CKG20] can be dropped because $f$ is supported on $[0, 1]$. Similarly, we do not need to assume the second part of condition (i) of [CKG20] on the sequences $(t_{n,i})$. The proof of Lemma 5.8 in [CKG20] can be followed line by line, with simple adaptations to our case. \hfill ∎

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