Symmetric Self-Adjunctions and Matrices

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Abstract
It is shown that the multiplicative monoids of Brauer’s centralizer algebras generated out of the basis are isomorphic to monoids of endomorphisms in categories where an endofunctor is adjoint to itself, and where, moreover, a kind of symmetry involving the self-adjoint functor is satisfied. As in a previous paper, of which this is a companion, it is shown that such a symmetric self-adjunction is found in a category whose arrows are matrices, and the functor adjoint to itself is based on the Kronecker product of matrices.

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1 Introduction
This paper is a companion to [5], where it was shown that the category $\text{Mat}_F$ whose objects are dimensions of finite-dimensional vector spaces and whose arrows are matrices is an example of an adjoint situation where a functor is adjoint to itself. In $\text{Mat}_F$ this functor is based on the Kronecker product of matrices. This self-adjunction underlies the orthogonal group case of Brauer’s representation of Brauer’s algebras, which can be restricted to the Temperley-Lieb subalgebras of Brauer’s algebras (see [2], [19], [12]).

In this paper we show that this self-adjunction of $\text{Mat}_F$ is specific in that it involves a kind of symmetry (see Section 8). This symmetry is the reason why the self-adjunction of $\text{Mat}_F$ is of the $\mathcal{K}$ kind and not of the $\mathcal{L}$ kind, in the terminology of [5] (see the end of Section 8; cf. Section 2). Roughly speaking, in the diagrammatic representation, the $\mathcal{L}$ kind takes account of the regions where...
the circles occur, while the $K$ kind just takes account of the number of circles, and does not take account of where they occur. In the $J$ kind, one does not take account of circles at all.

The first part of the paper (Sections 2-7) is devoted to axiomatizing (presenting by generators and relations) monoids for which we will show later that they are engendered by categories involved in symmetric self-adjoint situations. These monoids are closely related to multiplicative monoids of Brauer’s algebras. We establish that our monoids are isomorphic to monoids of diagrams which differ from the tangles of knot theory by having crossings in which under and over are disregarded. These crossings correspond to the symmetry of self-adjunction mentioned above. In the first part of the paper (Section 7) we also establish a property of our monoids that we call maximality, on which we rely later for the interpretation in $\text{Mat}_F$. (There is an alternative to this approach based on [7] and [8].)

In the second part (Sections 8-10) we introduce formally and gradually our notion of symmetric self-adjunction, starting from more general notions—ultimately from the notion of adjunction. We favour this gradual approach in order to make clearer the articulation of this notion. In this part we also connect freely generated symmetric self-adjunctions with the monoids of the first part.

In the third part (Sections 11-13) we survey a number of notions of category related to symmetric self-adjunction, but more general. Such are the notions of symmetric monoidal closed category and compact closed category. The notion that corresponds exactly to symmetric self-adjunction is a notion we call subsided category. It is shown that the category of symmetric self-adjunction freely generated by a single object is isomorphic to the subsided category $S$ freely generated by a single object. In $S$ the objects may be conceived as functors and the tensor product as composition of functors. If in the generating set of $S$ we have more than one object, then we fall upon a generalization of our notion of symmetric self-adjunction, which we call multiple symmetric self-adjunction (see Section 13).

The final part of the paper (Section 14) is about the symmetric self-adjunction of $\text{Mat}_F$. In $\text{Mat}_F$ we have as subcategories isomorphic copies of freely generated symmetric self-adjunctions.

2 The monoids $SK_\omega$ and $SJ_\omega$

The monoid $SK_\omega$ has for every $k \in \mathbb{N}^+ = \mathbb{N} - \{0\}$ a generator $[k]$, called a cap term, a generator $[k]$, called a cap term, and a generator $\sigma_k$, called a crossing term. (We prefer the name crossing to the name transposition because of the pictorial interpretation of $\sigma_k$ in the next section.) The terms of $SK_\omega$ are defined inductively by stipulating that the generators and $1$ are terms, and that if $t$ and $u$ are terms, then $(tu)$ is a term. As usual, we will omit the outermost parentheses of terms, and in the presence of associativity we will omit all parentheses, since
they can be restored as we please.

The monoid \( SK_\omega \) is freely generated from the generators above so that the following equations hold between terms of \( SK_\omega \) for \( j \leq k \):

**monoid equations:**

\[
\begin{align*}
(1) \quad t1 &= t1 = t, \\
(2) \quad t(uv) &= (tu)v;
\end{align*}
\]

**cup-cap equations:**

\[
\begin{align*}
\text{(cup)} \quad [k][j] &= [j][k+2], & \text{(cap)} \quad [j][k] &= [k+2][j], \\
\text{(cup-cap 1)} \quad [k+2][j] &= [j][k], & \text{(cap-cup 1)} \quad [j][k+2] &= [k][j], \\
\text{(cup-cap)} \quad [i][i+1] &= 1;
\end{align*}
\]

**σ equations:**

\[
\begin{align*}
\text{(σ)} \quad σ_{k+2}σ_j &= σ_jσ_{k+2}, \\
\text{(σ2)} \quad σ_iσ_i &= 1, \\
\text{(σ3)} \quad σ_{i+1}σ_iσ_{i+1} &= σ_iσ_{i+1}σ_i;
\end{align*}
\]

**σ-cup equations:**

\[
\begin{align*}
\text{(σ-cup 1)} \quad [k+2]σ_j &= σ_j[k+2], & \text{(σ-cap 1)} \quad σ_j[k+2] &= [k+2]σ_j, \\
\text{(σ-cup 2)} \quad [j]σ_{k+2} &= σ_k[j], & \text{(σ-cap 2)} \quad σ_{k+2}[j] &= [j]σ_k, \\
\text{(σ-cup 3)} \quad [i]σ_i &= [i], & \text{(σ-cap 3)} \quad σ_i[i] &= [i], \\
\text{(σ-cup 4)} \quad [i+1]σ_i &= [i]σ_{i+1}, & \text{(σ-cap 4)} \quad σ_i[i+1] &= σ_{i+1}[i].
\end{align*}
\]

To understand these equations it helps to have in mind their diagrammatic interpretation of the next section. The monoid and cup-cap equations are the equations of the monoid \( L_\omega \) of [5] and [6], while the monoid and σ equations are the well known equations of symmetric groups (see [4], Chapter 6.2, p. 64).

The equation

\[
[k][k] = [k+1][k+1]
\]

holds in \( SK_\omega \) since we have

\[
[k][k] = [k]σ_{k+1}σ_{k+1}[k], \quad \text{with (σ2)}
\]

\[
= [k+1][k+1], \quad \text{with (σ-cup 4), (σ-cap 4) and (σ2)}.
\]

The monoid \( K_\omega \) of [5] and [6], in which \( σ_i \) is lacking, is a submonoid of \( SK_\omega \).

Let us call terms of the form \([k][k]\) circles. Then, by the equation we have just derived, we have \([k][k] = [l][l]\), which means that we have only one circle, which we designate by \( c \).
We can easily derive the equation

\[(c)\quad tc = ct\]

for \(t\) being \([k], \lceil k \rceil\) or \(\sigma_i\), which yields \((c)\) for any term \(t\).

The monoid \(SJ_\omega\) is defined as \(SK_\omega\) save that we have the additional equation

\[(c1)\quad c = 1.\]

Let \(c^0\) be the empty sequence, and let \(c^{n+1} = c^n c\). Then it is easy to show by induction on the length of derivation that if \(t = u\) in \(SJ_\omega\), then for some \(n, m \in \mathbb{N}\) we have \(c^n t = c^m u\) in \(SK_\omega\). (The converse implication holds trivially.) Later (after Section 6) we will be able to establish that if \(t = u\) holds in \(SJ_\omega\) but not in \(SK_\omega\), then we have \(c^n t = c^m u\) in \(SK_\omega\) for \(n \neq m\) (cf. [5], end of Section 12).

3 SK and SJ diagrams

A coupling diagram is a partition of \(\mathbb{Z} - \{0\}\) into two-element subsets. (Alternatively, we could define a coupling diagram by an involution of \(\mathbb{Z} - \{0\}\) without fixed-points; a coupling diagram may be conceived as a split equivalence in the sense of [7] and [8].) The two-element sets of integers that belong to a coupling diagram are called threads of the diagrams. When in a thread \(\{a, b\}\) the members \(a\) and \(b\) have a different sign, the thread is called transversal. A thread \(\{a, b\}\), which is not transversal, is a cup when \(a\) and \(b\) are both positive, and it is a cap when they are both negative.

We denote threads systematically as follows. A transversal thread is denoted by \(\{n, -m\}\) for \(n, m \in \mathbb{N}^+\), a cup is denoted by \(\{n, m\}\) with the assumption that \(0 < n < m\), and a cap is denoted by \(\{-n, -m\}\), again with the assumption that \(0 < n < m\).

Consider coupling diagrams such that for some \(n, m \in \mathbb{N}^+\) for every \(k \in \mathbb{N}\) we have a transversal thread \(\{n+k, -(m+k)\}\). Such coupling diagrams will be called \(SJ\) diagrams, and \((n, m)\) is a type of this diagram. The type of an \(SJ\) diagram is not unique, since an \(SJ\) diagram of type \((n, m)\) is also of type \((n+k, m+k)\). It is clear that an \(SJ\) diagram has finitely many cups and caps. A coupling diagram without cups and caps corresponds to a permutation of \(\mathbb{N}^+\), and an \(SJ\) diagram without cups and caps corresponds to a permutation of an initial segment of \(\mathbb{N}^+\). In a type \((n, m)\) of such an \(SJ\) diagram we must have \(n = m\).

An SK diagram is a pair \((D, n)\) where \(D\) is an \(SJ\) diagram and \(n \in \mathbb{N}\) (we imagine that \(n\) is the number of circular components of the diagram; see the picture of composition below). We draw an \(SJ\) diagram by putting positive integers on a line, called the top line, and negative integers in reverse order on a line below, called the bottom line, and by connecting for each thread \(\{a, b\}\),
the numbers $a$ and $b$ by a, not necessarily straight, line segment in between the top and bottom lines. For example, the $SJ$ diagram

$$\{\{1, -3\}, \{2, 3\}, \{4, -4\}, \{5, -1\}, \{-2, -6\}, \{-5, -7\}, \{6, -8\}, \ldots, \{6+k, -(8+k)\}, \ldots\}$$

is drawn as follows:

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (6,0); \draw (0,-1) -- (6,-1);
\foreach \x in {1,2,3,4,5,6} \draw (\x,0) -- (\x,-1); \draw (6,0) -- (6,-1);
\foreach \y in {-1,-2,-3,-4,-5,-6,-7,-8} \draw (0,\y) -- (6,\y);
\fill (1,0) circle (2pt); \fill (2,0) circle (2pt); \fill (3,0) circle (2pt); \fill (4,0) circle (2pt); \fill (5,0) circle (2pt); \fill (6,0) circle (2pt);
\end{tikzpicture}
\end{center}

This $SJ$ diagram is of type $(6,8), (7,9), \ldots$.

Let the unit $SJ$ diagram be the coupling diagram $\{\{n, -n\} | n \in \mathbb{N}^+\}$, which is drawn as follows:

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (5,0); \draw (0,-1) -- (5,-1);
\foreach \x in {1,2,3,4,5} \draw (\x,0) -- (\x,-1);
\end{tikzpicture}
\end{center}

and is denoted by $I$. The unit $SK$ diagram is $(I, 0)$.

The $SJ$ diagram that is the composition $D_2 \circ D_1$ of two $SJ$ diagrams is defined pictorially by identifying the bottom line of $D_1$ with the top line of $D_2$, so that $-n$ and $n$ are the same point. The top line of $D_2 \circ D_1$ is the top line of $D_1$, and the bottom line of $D_2 \circ D_1$ is the bottom line of $D_2$. For example, for $D_1$ being the $SJ$ diagram taken as an example above, and $D_2$ the $SJ$ diagram

$$\{\{1, -1\}, \{2, 7\}, \{3, -3\}, \{4, -2\}, \{5, 6\}, \{8, -4\}, \ldots, \{8+n, -(4+n)\}, \ldots\},$$

the composition $D_2 \circ D_1$ is

$$\{\{1, -3\}, \{2, 3\}, \{4, -2\}, \{5, -1\}, \{6, -4\}, \ldots, \{6+n, -(4+n)\}, \ldots\},$$

which is clear from the following pictures:
We have in the picture on the left a closed loop made of the two caps of $D_1$ and the two cups of $D_2$. We call such loops circular components, and we do not take account of them when we define composition of $SJ$ diagrams. When, however, we define the composition $(D_2, n_2) \circ (D_1, n_1) = (D_2 \circ D_1, n_1 + n_2 + k)$ of the $SK$ diagrams $(D_1, n_1)$ and $(D_2, n_2)$, then $k$ is the number of such circular components that arose as in the picture above. This number must be finite, because $D_1$ and $D_2$ have a finite number of cups and caps.

Composition of $SJ$ diagrams, as well as composition of $SK$ diagrams, is associative, and it is clear that the unit $SJ$ and $SK$ diagrams are unit elements for these compositions; namely we have

\[
\begin{align*}
(1J) & \quad I \circ D = D \circ I = D, \\
(1K) & \quad (I, 0) \circ (D, n) = (D, n) \circ (I, 0) = (D, n).
\end{align*}
\]

To prove strictly the associativity of composition, we would need a more formal definition of composition, which with the definition of $SJ$ and $SK$ diagrams we have given here would be rather involved (cf. [7] and [8]). The proof of associativity would be even more so. A simpler proof of associativity could be obtained with a topological definition of diagrams and their composition, on the lines of [5] and [6] (Section 5), but this other definition would be rather cumbersome too, and since we have dealt with these matters in detail in [5] and [6], we will not go again into them here.

For $k \in \mathbb{N}^+$ let the cup $SJ$ diagram $V_k$ be the $SJ$ diagram that is drawn as follows:

The cap $SJ$ diagram $\Lambda_k$ is the analogous $SJ$ diagram that is drawn as follows:
The \textit{crossing} \( SJ \) diagram \( X_k \) is the \( SJ \) diagram that is drawn as follows:

\[
\begin{array}{ccc}
1 & 2 & k-1 & k \\
\ldots & & \ldots & \\
-1 & -2 & -(k-1)-k & -(k+1)-(k+2)
\end{array}
\]

The cup, cap and crossing \( SK \) diagrams are \((V_k,0),(\Lambda_k,0)\) and \((X_k,0)\) respectively.

Let \( D \) be the set of \( SJ \) diagrams. We define inductively as follows a map \( \iota \) from the terms of \( SK_\omega \) into \( D \):

\[
\begin{align*}
\iota([k]) &= V_k, \\
\iota([k]) &= \Lambda_k, \\
\iota(\sigma_k) &= X_k, \\
\iota(1) &= I, \\
\iota(tu) &= \iota(t) \circ \iota(u).
\end{align*}
\]

By this definition, we have that

\[
\iota(c) = \iota([k][k]) = \iota([k]) \circ \iota([k]) = V_k \circ \Lambda_k = I.
\]

The map \( \kappa \) from the terms of \( SK_\omega \) into \( D \times N \) is defined by

\[
\begin{align*}
\kappa(t) &= (\iota(t),0), \text{ where } t \text{ is } [k], [k], \sigma_k \text{ or } 1, \\
\kappa(tu) &= \kappa(t) \circ \kappa(u).
\end{align*}
\]

By this definition we have that

\[
\kappa(c) = \kappa([k][k]) = \kappa([k]) \circ \kappa([k]) = (V_k,0) \circ (\Lambda_k,0) = (I,1).
\]
Then we can prove the following lemma.

**Soundness Lemma.**

- **(K)** If \( t = u \) in \( SK_\omega \), then \( \kappa(t) = \kappa(u) \).
- **(J)** If \( t = u \) in \( SJ_\omega \), then \( \iota(t) = \iota(u) \).

**Proof.** We have mentioned above that we have \((1J)\) and \((1K)\), and that \( J \) and \( K \) composition is associative, which takes care of the equations (1) and (2). It is straightforward to verify the remaining equations of \( SK_\omega \) and \( SJ_\omega \).

This lemma says that \( \kappa \) and \( \iota \) give rise to homomorphisms from the monoids \( SK_\omega \) and \( SJ_\omega \) to monoids \((D \times N, \circ, (I, 0))\) and \((D, \circ, I))\) respectively. Our goal in the following sections is to show that these homomorphisms are isomorphisms.

## 4 Normal forms in \( SK_\omega \) and \( SJ_\omega \)

Let \( i, j \in \mathbb{N}^+ \), and let \( j \leq i \). We define as follows the term \([j,i]\) of \( SK_\omega \), which we call a **block-cup**: \([j,j]\) is \([j]\), and if \( j < i \), then \([j,i]\) is \([j, i-1]\) \( \sigma_i \).

We define analogously the **block-cap** \([i,j]\): we have that \([j,j]\) is \([j]\), and if \( j < i \), then \([i,j]\) is \( \sigma_i[i-1,j] \).

For \( i \) and \( j \) as above, let the **block-crossing** be the term \([i,j]\) of \( SK_\omega \) defined as follows: \([j,j]\) is \( \sigma_j \), and if \( j < i \), then \([i,j]\) is \( \sigma_i[i-1,j] \).

It is easy to see that for \( j < i \) the \( SJ \) diagram \( \iota([j,i]) \), which we call the **block-cup \( SJ \) diagram**, is drawn as follows:

\[
\begin{array}{cccccc}
1 & \ldots & j-1 & j & j+1 \ldots & i & i+1 & i+2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Analogously, the **block-cap \( SJ \) diagram** \( \iota([i,j]) \) is drawn as follows:

\[
\begin{array}{cccccc}
-1 & \ldots & -(j-1) & -(j) \ldots & -(i) & -(i+1) & -(i+2) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]
And the block-crossing $SF$ diagram $i([i,j])$ is drawn as follows:

Consider terms of $SK_\omega$ of the form

$$c^l [i_p, j_p] \ldots [i_1, j_1] 1 [k_1, l_1] \ldots [k_q, l_q] |m_1, n_1| \ldots |m_r, n_r|$$

where $l, p, q, r \geq 0$, $j_1 < \ldots < j_p$, $k_1 < \ldots < k_q$ and $m_1 < \ldots < m_r$. The notation $c^l$ has been introduced in Section 2. If $p = 0$, then the sequence $[i_p, j_p] \ldots [i_1, j_1]$ is empty, and analogously if $q = 0$, or $r = 0$. Terms of this form will be said to be in normal form.

We can then prove the following lemma for $SK_\omega$.

**Normal Form Lemma.** Every term is equal to a term in normal form.

**Proof.** We will give a reduction procedure that transforms every term $t$ of $SK_\omega$ into a term $t'$ in normal form such that $t = t'$ in $SK_\omega$. This procedure will have three phases, but it might terminate after any of the phases (or before any phase, if the term we start from is in normal form).

Roughly speaking, in the first phase of our procedure we push caps to the left, where they are transformed into block-caps, which we arrange in the decreasing order of the normal form. Circles we might have at the beginning and circles that arise during the first phase are all pushed to the left in this phase. Similarly, unit terms $1$ that we might have at the beginning and that arise during the first phase are all deleted, except if the whole term is $1$. If the whole term is not $1$, we put one term $1$ at an appropriate place. This is the first phase of the procedure.

In the second phase we push cups to the right, where they are transformed into block-cups, which we arrange in the increasing order of the normal form. (We could as well have decided to push cups to the right in the first phase, and then caps to the left in the second phase.)

After the first and second phase are over, the crossings that may remain in between the block-caps on the left and the block-cups on the right are transformed into block-crossings, which we arrange in the increasing order of the normal form. This is the third, and last, phase of the procedure.
At the beginning of the first phase of the reduction procedure we replace every cap \([i]\) by the block-cap \([i,i]\). Then, according to the following equations of \(\mathcal{SK}_\omega\), which are derived in a straightforward manner, we replace every subterm of the form on the left-hand side by the term on the right-hand side of these equations:

**cup-block-cap equations:**

\[
\begin{align*}
\text{(cup-cap i)} & \quad [k][i,j] = [i-2,j-2][k] \quad \text{if } k+2 \leq j, \\
\text{(cup-cap ii)} & \quad \text{for } k \leq j \leq k+1, \\
\text{(cup-cap ii.1)} & \quad [k][i,j] = \sigma_{i-2} \ldots \sigma_k \quad \text{if } k+2 \leq i, \\
\text{(cup-cap ii.2)} & \quad [k][i,j] = 1 \quad \text{if } i = k+1, \\
\text{(cup-cap ii.3)} & \quad [k][i,j] = c \quad \text{if } i = k, \\
\text{(cup-cap iii)} & \quad \text{for } j \leq k-1, \\
\text{(cup-cap iii.1)} & \quad [k][i,j] = [i-2,j][k-1] \quad \text{if } k \leq i-1, \\
\text{(cup-cap iii.2)} & \quad \text{for } i \leq k \leq i+1, \\
\text{(cup-cap iii.2.1)} & \quad [k][i,j] = \sigma_j \ldots \sigma_{k-2} \quad \text{if } j < k-1, \\
\text{(cup-cap iii.2.2)} & \quad [k][i,j] = 1 \quad \text{if } j = k-1, \\
\text{(cup-cap iii.3)} & \quad [k][i,j] = [i,j][k-2] \quad \text{if } i+2 \leq k; \\
\end{align*}
\]

**1 and c equations:**

\[
\begin{align*}
\text{(1)} & \quad 1t = t, \quad t1 = t, \\
\text{(c)} & \quad tc = ct; \\
\end{align*}
\]

**σ-block-cap equations:**

\[
\begin{align*}
\text{(σ-cap i)} & \quad \sigma_k[i,j] = [i,j]\sigma_k \quad \text{if } k+2 \leq j, \\
\text{(σ-cap ii)} & \quad \sigma_k[i,j] = [i,j-1] \quad \text{if } j = k+1, \\
\text{(σ-cap iii)} & \quad \text{for } j = k, \\
\text{(σ-cap iii.1)} & \quad \sigma_k[i,j] = [i,j+1] \quad \text{if } j < i, \\
\text{(σ-cap iii.2)} & \quad \sigma_k[i,j] = [i,j] \quad \text{if } i = j, \\
\text{(σ-cap iv)} & \quad \text{for } j \leq k-1, \\
\text{(σ-cap iv.1)} & \quad \sigma_k[i,j] = [i,j]\sigma_{k-1} \quad \text{if } k+1 \leq i, \\
\text{(σ-cap iv.2)} & \quad \sigma_k[i,j] = [i-1,j] \quad \text{if } i = k, \\
\text{(σ-cap iv.3)} & \quad \sigma_k[i,j] = [i+1,j] \quad \text{if } i = k-1, \\
\text{(σ-cap iv.4)} & \quad \sigma_k[i,j] = [i,j]\sigma_{k-2} \quad \text{if } i \leq k-2 \text{ (this case is impossible if } j = k-1); \\
\end{align*}
\]
double block-cap equations:

\[
\text{for } j \leq l \]
\[
\begin{align*}
(cap \ i) \quad & [i,j][k,l] = [k+2,l+2][i,j] \quad \text{if } i \leq l, \\
(cap \ ii) \quad & [i,j][k,l] = [k+2,l+1][i-1,j] \quad \text{if } l+1 \leq i \leq k+1, \\
(cap \ iii) \quad & [i,j][k,l] = [k+1,l+1][i-2,j] \quad \text{if } k+2 \leq i.
\end{align*}
\]

Note that with the equation (c) we can push all circles to the extreme left. Note next that the cup-block-cap equations cover all cases for subterms of the form \([k][i,j]\), and the \(\sigma\)-block-cap equations cover all cases for subterms of the form \(\sigma_k[i,j]\). The double block-cap equations cover all cases for subterms of the form \(\lceil i,j \rceil \lceil k,l \rceil\) with \(j \leq l\), and they replace such subterms by \(\lceil k',l' \rceil \lceil i',j' \rceil\) with \(l' > j'\). So, when we can no longer make any replacement by one of the equations above, we are left with the term 1 or with a term of the form

\[
c^l[i_p,j_p] \ldots [i_1,j_1]u_1 \ldots u_n
\]

where \(l, p, n \geq 0, l + p + n \geq 1, j_1 < \ldots < j_p\), and the term \(u_1 \ldots u_n\), which is the empty sequence if \(n = 0\), is made exclusively of cup terms and crossing terms. In case we are not left with 1, we replace the term above with

\[
c^l[i_p,j_p] \ldots [i_1,j_1]1u_1 \ldots u_n.
\]

With that the first phase of the reduction procedure is over.

If \(n = 0\), then the whole procedure is over, and we are left with a term in normal form. If \(n \geq 1\), then we enter into the second phase of the procedure, and reduce \(u_1 \ldots u_n\). We first replace every cup \([i]\) in this term by the block-cup \([i,i]\), and then we make replacements of subterms of the form \([j,i]\sigma_k\) and \([l,k][j,i]\) with \(j \leq l\) by using equations exactly dual to the \(\sigma\)-block-cap equations and the double block-cap equations. When we can no longer make any of these replacements, we have instead of \(u_1 \ldots u_n\) a term of the form

\[
v_1 \ldots v_m[m_1,n_1] \ldots [m_r,n_r]
\]

where \(m, r \geq 0, m + r \geq 1, m_1 < \ldots < m_r\), and the term \(v_1 \ldots v_m\), which is the empty sequence if \(m = 0\), is made exclusively of crossing terms. With that the second phase of the reduction procedure is over.

If \(m = 0\), then the whole reduction procedure is over, and we are left with a term in normal form. If \(m \geq 1\), then we enter into the third phase of the procedure, and reduce \(v_1 \ldots v_m\). We first replace every crossing term \(\sigma_i\) in this term by the block-crossing \([i,i]\). Then, according to the following equations of \(SK_\omega\), which are derived in a straightforward manner from the monoid equations and the \(\sigma\) equations of Section 2, we replace every subterm of the form on the left-hand side by the term on the right-hand side of these equations:
block-$\sigma$ equations:

for $k \leq i$

\begin{align*}
(\sigma i) \quad [i, j][k, l] &= [k, l][i, j] \quad \text{if } k + 2 \leq j, \\
(\sigma ii) \quad [i, j][k, l] &= [i, l] \quad \text{if } j = k + 1, \\
(\sigma iii) \text{ for } j = k, \\
(\sigma iii.1) \quad [i, j][k, l] &= [k - 1, l][i, j + 1] \quad \text{if } j < i, l < k, \\
(\sigma iii.2) \quad [i, j][k, l] &= [i, j + 1] \quad \text{if } j < i, l = k, \\
(\sigma iii.3) \quad [i, j][k, l] &= [k - 1, l] \quad \text{if } j = i, l < k, \\
(\sigma iii.4) \quad [i, j][k, l] &= 1 \quad \text{if } j = i, l = k,
\end{align*}

\begin{align*}
(\sigma iv) \text{ for } j \leq k - 1, \\
(\sigma iv.1) \quad [i, j][k, l] &= [k - 1, l][i, j + 1] \quad \text{if } l \leq j, \\
(\sigma iv.2) \quad [i, j][k, l] &= [k - 1, l - 1][i, j] \quad \text{if } j < l;
\end{align*}

\begin{align*}
1 \text{ equations:} \\
(1) \quad 1t = t, \quad t1 = t.
\end{align*}

Note that the block-$\sigma$ equations cover all cases for subterms of the form $[i, j][k, l]$ with $k \leq i$, and they replace such subterms by $[k', l'][i', j']$ with $k' < i'$, or by single block-crossings, or by $1$. The terms $1$ are eliminated by the $1$ equations, except if the whole term is $1$. When we can no longer make any of these replacements, we have instead of $v_1 \ldots v_m$ either $1$ or a term of the form $[k_1, l_1] \ldots [k_q, l_q]$ where $q \geq 1$ and $k_1 < \ldots < k_q$. With that the third phase of the reduction procedure is over. The original term has now been reduced to normal form, except for one more possible application of the $1$ equations if $v_1 \ldots v_m$ has been replaced by $1$ and $l + p + r \geq 1$.

Note that the cup-cap equations of Section 2 are all instances of the block equations mentioned in the course of this proof. The equation $(\text{cap})$ is an instance of $(\text{cap i})$ for $i = j$ and $k = l$, and analogously for $(\text{cup})$ and the dual equation with block-cups, which we mentioned in the proof, but did not write down. The equation $(\text{cup-cap} 1)$ is an instance of $(\text{cup-cap iii.3})$ for $i = j$, the equation $(\text{cup-cap} \ 1)$ is an instance of $(\text{cup-cap i})$ for $i = j$, and the equations $(\text{cup-cap})$ are instances of $(\text{cup-cap ii.2})$ for $i = j$ and of $(\text{cup-cap iii.2.2})$ for $i = j$.

The same holds for the $\sigma$-cup and $\sigma$-cap equations of Section 2. The equation $(\text{sigma-cap} 1)$ is an instance of $(\text{sigma-cap i})$ for $i = j$, the equation $(\text{sigma-cap} 2)$ is an instance of $(\text{sigma-cap iv.4})$ for $i = j$, the equation $(\text{sigma-cap} 3)$ is $(\text{sigma-cap iii.2})$, and $(\text{sigma-cap} 4)$ is an instance of $(\text{sigma-cap ii})$ for $i = j$. We have an analogous situation for the $\sigma$-cup equations with respect to the equations dual to the $\sigma$-block-cap.
equations with block-cups, which we mentioned in the proof, but did not write down.

Finally, the \( \sigma \) equations of Section 2 are instances of the block-\( \sigma \) equations of the proof above. The equation \((\sigma)\) is an instance of \((\sigma i)\) for \(i = j\) and \(k = l\), the equation \((\sigma 2)\) is \((\sigma iii.4)\), and \((\sigma 3)\) is an instance of \((\sigma iv.2)\) for \(i = j + 1 = k = l\).

So if for presenting \( SK_\omega \) we had taken as generators block-cups, block-caps and block-crossings, instead of cups, caps and crossings, the block equations of the reduction procedure of our proof of the Normal Form Lemma would suffice for this presentation, though some equations are redundant. (In this block presentation, the equations \((\sigma \text{-} \text{cap iv.3})\) and \((\sigma ii)\) are, however essential, in order to justify the definition of block-caps and block-crossings in terms of caps and crossings.)

Note that if we restrict ourselves to the third phase of the proof, we have provided a normal form for terms of symmetric groups, and an alternative presentation, with block-crossings as generators, for these groups. This normal form for symmetric groups, which may be derived from [3] (Note C, pp. 464-469), is analogous up to a point to the Jones normal form for terms of the monoids of Temperley-Lieb algebras (see [5] and [6], Section 9). As we shall see later (Section 6), this normal form serves well to show the completeness of the presentation of symmetric groups with the \( \sigma \) equations of Section 2, or with the block-\( \sigma \) equations of our proof of the Normal Form Lemma. (This completeness proof is different that in [3], Note C; cf. [9], Section 5.2.)

If in the definition of normal form for terms of \( SK_\omega \) we require that \( l = 0 \), i.e. that \( c^l \) is the empty sequence (there are no circles at the beginning), we obtain the definition of normal form appropriate for \( SJ_\omega \), which we call \( J \)-normal form. Then our proof of the Normal Form Lemma suffices to prove this lemma for \( SJ_\omega \) with respect to the \( J \)-normal form. We just may have to apply at the end of our procedure the equation \((c1)\) of Section 2, together with the equation \((1)\), to get rid of circles.

5 Generating \( SK \) and \( SJ \) diagrams

A cup \( \{j, i\} \) of an \( SJ \) diagram \( D \) is maximal when there is no cup \( \{j', i'\} \) of \( D \) such that \( j < j' \). (Remember that according to our convention of Section 3 we have here \( j < i \) and \( j' < i' \).) Analogously, a cap \( \{-j, -i\} \) is maximal when there is no cap \( \{-j', -i'\} \) of \( D \) such that \( j < j' \). If there are cups in \( D \), then there must be a maximal one because there are only finitely many cups in \( D \), and analogously for caps.

Let \( P \) be an \( SJ \) diagram without cups and caps. Transversal threads \( \{l, -k\} \) of \( P \) such that \( l < k \) will be called falling. A falling thread of \( P \) is maximal when there is no falling thread \( \{l', k\} \) of \( P \) such that \( k < k' \). If there are falling threads in \( P \), then there must be a maximal one, because \( P \) is of type \((n, n)\) for some \( n \in \mathbb{N}^+ \).
Then we can prove the following lemma.

**Generating Lemma.** Every $SJ$ diagram is equal to a diagram obtained from cup $SJ$ diagrams, cap $SJ$ diagrams, crossing $SJ$ diagrams and the unit $SJ$ diagram with the help of composition.

*Proof.* Let $D$ be an $SJ$ diagram. If there are cups in $D$, then there must be a maximal cup $\{m, n+1\}$, with $m \leq n$. Then consider the partition of $\mathbb{Z} - \{0, m, n+1\}$ obtained from $D$ just by omitting the cup $\{m, n+1\}$. This partition gives rise to an $SJ$ diagram $D'$ by replacing every $x \in \mathbb{Z}$ such that $m+1 \leq x \leq n$ by $x-1$, and every $y \in \mathbb{Z}$ such that $n+2 \leq y$ by $y-2$. If $D$ is of type $(s_1, s_2)$, then $D'$ is of type $(s_1-2, s_2)$. Then we can check that $D = D' \circ \iota([m, n])$, and $D'$ has one cup less than $D$. For example, we have

\[\begin{array}{cccccccc}
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
D & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}\]

\[\begin{array}{cccccccc}
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
D' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}\]

\[\begin{array}{cccccccc}
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
D & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}\]

\[\begin{array}{cccccccc}
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
D' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}\]

We proceed analogously, pulling out maximal caps on the left, if $D$ has caps.

By repeated applications of this procedure we must end up with an $SJ$ diagram equal to $D$ of the form $\iota([i_p, j_p]) \circ \ldots \circ \iota([i_1, j_1]) = P \circ \iota([m_1, n_1]) \circ \ldots \circ \iota([m_r, n_r])$, where $p, r \geq 0$, $p+r \geq 0$, $j_1 < \ldots < j_p$, $m_1 < \ldots < m_r$, and there are no cups and caps in the $SJ$ diagram $P$. The $SJ$ diagram $P$ corresponds to a permutation of an initial segment of $\mathbb{N}^+$, and it has a type $(n, n)$ for some $n \in \mathbb{N}^+$.

If there are no falling threads in $P$, then $P$ is equal to the unit $SJ$ diagram $I$. If there are falling threads in $P$, then let $\{l_q, -(k_q+1)\}$ with $l_q \leq k_q$ be the maximal falling thread of $P$. Then consider the partition of $\mathbb{Z} - \{0, l_q, -(k_q+1)\}$ obtained from $P$ just by omitting the thread $\{l_q, -(k_q+1)\}$. This partition gives rise to an $SJ$ diagram $P'$ without cups and caps by replacing every $x \in \mathbb{Z}$ such that $l_q + 1 \leq x \leq k_q + 1$ by $x-1$, and by adding the thread $\{k_q+1, -(k_q+1)\}$. If there are falling threads in $P'$ (i.e., if $P'$ is not the unit $SJ$ diagram $I$), then its maximal falling thread $\{l, -(k+1)\}$ must have $k < k_q$ (otherwise, $\{l_q, -(k_q+1)\}$ would not be maximal in $P$). Then we can check that $P = P' \circ \iota([k_q, l_q])$. For example, we have

14
By repeated applications of this procedure we must end up with an \( \mathcal{SJ} \) diagram equal to \( P \) of the form

\[ I \circ \iota([k_1, l_1]) \circ \ldots \circ \iota([k_q, l_q]) \]

where \( q \geq 0 \) and \( k_1 < \ldots < k_q \).

By definition, block-cup, block-cap and block-crossing \( \mathcal{SJ} \) diagrams are generated by cup \( \mathcal{SJ} \) diagrams, cap \( \mathcal{SJ} \) diagrams and crossing \( \mathcal{SJ} \) diagrams with the help of composition. So the lemma is proved.

The last part of the proof, which involves block-crossing \( \mathcal{SJ} \) diagrams, is about a well-known fact. We have, nevertheless, preferred to go through it in order to show that it is of the same sort as the previous parts of the proof. This part of the proof is also a part of our proof of the completeness of the standard presentation of symmetric groups.

As a matter of fact, we need not have strived in the proof above to get \( j_1 < \ldots < j_p \), \( n_1 < \ldots < n_r \) and \( k_1 < \ldots < k_q \). However, if we do so, we shall end up with a composition of \( \mathcal{SJ} \) diagrams corresponding to a term of \( \mathcal{SK} \) in \( \mathcal{J} \)-normal form.

Strictly speaking, unit \( \mathcal{SJ} \) diagrams can be omitted in the formulation of the Generating Lemma since \( I = V_k \circ \Lambda_{k+1} = V_{k+1} \circ \Lambda_k = X_k \circ X_k \) for any \( k \in \mathbb{N}^+ \).

The Generating Lemma is provable also when in its formulation \( \mathcal{J} \) is replaced everywhere by \( \mathcal{K} \). For any \( \mathcal{SK} \) diagram \((D, n)\) we have

\[ (D, n) = (I, n) \circ (D, 0). \]

The \( \mathcal{SK} \) diagram \((I, n)\) can be generated from cup \( \mathcal{SK} \) diagrams, cap \( \mathcal{SK} \) diagrams and the unit \( \mathcal{SK} \) diagram with the help of composition since we have

\[ (I, n+1) = (I, n) \circ (I, 1) = (I, n) \circ (V_k, 0) \circ (\Lambda_k, 0). \]

For \((D, 0)\), we first go through our proof of the Generating Lemma for \( \mathcal{SJ} \) diagrams in order to check that in the composition involved in the proof circular components never arise; then we apply this lemma.

From the Generating Lemma for \( \mathcal{SJ} \) diagrams it follows that the homomorphism from \( \mathcal{SJ}_\omega \) to \((D, \circ, I)\) defined via \( \iota \) is onto, and from the Generating Lemma for \( \mathcal{SK} \) diagrams it follows that the homomorphism from \( \mathcal{SK}_\omega \) to
\((D \times N, \ast, (I, 0))\) defined via \(\kappa\) is onto. In the next section we show that these homomorphisms are also one-one.

6 \(SK_\omega\) and \(SJ_\omega\) are monoids of \(SK\) and \(SJ\) diagrams

It is easy to see that the following holds.

**Remark.** Suppose \(t\) is the following term of \(SK_\omega\) in normal form:
\[
e^t[i_p, j_p] \ldots [i_1, j_1][k_1, l_1] \ldots [k_q, l_q][m_1, n_1] \ldots [m_r, n_r].
\]
Then
(i) \(\kappa(t) = (\iota(t), l')\) iff \(l' = l\);
(ii) if \(p > 0\), then \(\{-j, -(i+1)\}\) is the maximal cap of \(\iota(t)\) iff \(j = j_p\)
and \(i = i_p\); if \(r > 0\), then \(\{m, n+1\}\) is the maximal cup of \(\iota(t)\) iff
\(m = m_r\) and \(n = n_r\);
(iii) if \(p = r = 0\) and \(q > 0\), then \(\{l, -(k+1)\}\) is the maximal falling
thread of \(\iota(t)\) iff \(l = l_q\) and \(k = k_q\).

In a cup \(\{l, k\}\) of an \(SJ\) diagram we call \(l\) (which according to our convention
of Section 3 is lesser than \(k\)) the left end point of this cup. Analogously, in a cap
\(\{-l, -k\}\) we call \(-l\) the left end point of this cap. Then under the assumption
for \(t\) of the Remark we have that

(iv) the numbers \(-j'_1, \ldots, -j'_p\) are the left end points of the caps of \(\iota(t)\)
iff \(p = p'\) and \(j'_1, \ldots, j'_p\) is \(j_1, \ldots, j_p\); the numbers \(m'_1, \ldots, m'_q\) are
the left end points of the cups of \(\iota(t)\) iff \(q = q'\) and \(m'_1, \ldots, m'_q\) is
\(m_1, \ldots, m_q\).

To prove (iv) it is enough to see that in \(\iota([i_p, j_p])\) we have a transversal
thread \(\{j_{p-1}, -j_{p-1}\}\), and analogously with cups. But we will have no use for
(iv) below. We can then prove the following lemma.

**Auxiliary Lemma.**

\((K)\) If \(t\) and \(u\) are terms of \(SK_\omega\) in normal form and \(\kappa(t) = \kappa(u)\), then \(t\)
and \(u\) are the same term.

\((J)\) If \(t\) and \(u\) are terms of \(SK_\omega\) in \(J\)-normal form and \(\iota(t) = \iota(u)\), then \(t\)
and \(u\) are the same term.

**Proof.** (K) Suppose \(t\) and \(u\) are terms of \(SK_\omega\) in normal form and \(\kappa(t) = \kappa(u)\).
Since \(\kappa(t) = (\iota(t), l)\) and \(\kappa(u) = (\iota(u), l')\), we have \(\iota(t) = \iota(u)\) and \(l = l'\).
Suppose \(t\) is
\( c^l[i_p, j_p] \ldots [i_1, j_1] [k_1, l_1] \ldots [k_q, l_q][m_1, n_1] \ldots [m_r, n_r] \).

Then the weight \( w(t) \) of \( t \) is \( p+q+r \) if \( l = 0 \) and \( p+q+r+1 \) if \( l > 0 \). We prove \( (K) \) by induction on \( w(t) \).

If \( w(t) = 0 \), then \( t \) is \( 1 \), and by the Remark, \( u \) must be 1 too.

If \( l > 0 \), then, by Remark (i), we must have that \( u \) is of the form \( c^l u' \) where \( u' \) is without circles and in normal form: \( t \) is then of the form \( c^l t' \) for \( t' \) in normal form, and \( w(t') = w(t) - 1 \). It is easy to see that \( \iota(t') = \iota(t) \) and \( \iota(u') = \iota(u) \), so that we have \( \iota(t') = \iota(u') \). Hence \( \kappa(t') = (\iota(t'), 0) = (\iota(u'), 0) = \kappa(u') \), and we can apply the induction hypothesis.

If \( p > 0 \), then, by the Remark, we have that \( u \) is of the form \( c^l[i_p, j_p]u'' \), where \( c^l u'' \) is in normal form: \( t \) is then of the form \( c^l[i_p, j_p]t'' \), where \( c^l t'' \) is in normal form, and \( w(c^l t'') = w(t) - 1 \).

We have the following equations in \( SK_\omega \):

\[
[i-1, j][i+1][i, j] = 1, \quad \text{if } j < i \\
[i+1][i, j] = 1, \quad \text{if } j = i.
\]

If \( v \) stands for \( [i_p-1, j_p][i_p+1] \) or \( [i_p+1] \), depending on whether \( j_p < i_p \) or \( j_p = i_p \), then

\[
\kappa(v) \circ \kappa(t) = \kappa(v) \circ \kappa(u) \\
\kappa(vt) = \kappa(vu) \\
\kappa(c^l u'') = \kappa(c^l u''), \quad \text{by the Soundness Lemma},
\]

and we can apply the induction hypothesis. We proceed analogously if \( r > 0 \), using the following equations of \( SK_\omega \):

\[
[j, i][i+1] \sigma_j \ldots \sigma_{i-1} = 1, \quad \text{if } j < i \\
[j, i][i+1] = 1, \quad \text{if } j = i.
\]

If \( p = r = 0 \) and \( q > 0 \), then, by the Remark, we have that \( u \) is of the form \( u'''[k_q, l_q] \) for \( u''' \) in normal form: \( t \) is then of the form \( t'''[k_q, l_q] \) for \( t''' \) in normal form, and \( w(t''') = w(t) - 1 \).

We have the following equations in \( SK_\omega \):

\[
[k, l] \sigma_1 \ldots \sigma_k = \sigma_1 \ldots \sigma_k[k, l] = 1.
\]

From that we conclude that \( \kappa(t'''') = \kappa(u'''') \), and we can apply the induction hypothesis.

With that the proof of \( (K) \) is over. The proof of \( (J) \) is analogous, with the case \( l > 0 \) of the induction step omitted. \( \dashv \)
Finally, we can prove the following.

**Completeness Lemma.**

(K) If $\kappa(t) = \kappa(u)$, then $t = u$ in $SK_\omega$.

(J) If $\iota(t) = \iota(u)$, then $t = u$ in $SJ_\omega$.

**Proof.** (K) By the Normal Form Lemma of Section 4, for every term $t$ and every term $u$ of $SK_\omega$ there are terms $t'$ and $u'$ in normal form such that $t = t'$ and $u = u'$ in $SK_\omega$. By the Soundness Lemma (K) of Section 3, we obtain $\kappa(t) = \kappa(t')$ and $\kappa(u) = \kappa(u')$, and if $\kappa(t) = \kappa(u)$, it follows that $\kappa(t') = \kappa(u')$. Then, by the Auxiliary Lemma, $t'$ and $u'$ are the same term, and hence $t = u$ in $SK_\omega$. We proceed analogously for (J).

In the proof of the Auxiliary Lemma we can restrict ourselves to terms of $SK_\omega$ without cups and caps, and thereby obtain a Completeness Lemma for the standard presentation of symmetric groups.

So we have established that the homomorphism from $SK_\omega$ to $(D \times N, \circ, (I, 0))$ defined via $\kappa$ and the homomorphism from $SJ_\omega$ to $(D, \circ, I)$ defined via $\iota$ are isomorphisms. We can also conclude that for every term $t$ of $SK_\omega$ there is a unique term $t'$ in normal form such that $t = t'$ in $SK_\omega$. If $t = t'$ and $t = t''$ in $SK_\omega$, then $t' = t''$ in $SK_\omega$, and hence, by the Soundness Lemma, $\kappa(t') = \kappa(t'')$. If $t'$ and $t''$ are in normal form, we obtain by the Auxiliary Lemma that $t'$ and $t''$ are the same term. We have an analogous uniqueness for the $J$-normal form with respect to $SJ_\omega$.

Let $SK_n$ be the submonoid of $SK_\omega$ made of elements of $SK_\omega$ whose $SK$ diagrams are all the $SK$ diagrams of type $(n, n)$. This monoid corresponds to the multiplicative part of Brauer’s algebra $B_n$ generated out of the basis (see [2] and [19]). How the monoid $SK_n$ might be axiomatized (presented by generators and relations) can be gathered from [20] and [1], where, however, instead of symmetry one finds braiding.

7 The maximality of $SK_\omega$

Suppose $t$ and $u$ are terms of $SK_\omega$ such that $t = u$ does not hold in $SK_\omega$, and let $\chi$ be the monoid defined as $SK_\omega$ save that we assume the additional equation $t = u$. We will prove the following.

**Maximality of $SK_\omega$.** For some $k, l \in N$ such that $k \neq l$ we have $e^k = c^l$ in $\chi$.

**Proof.** Remember that we have seen in the proof of the Auxiliary Lemma of the preceding section that for every $i, j \in N^+$ such that $j \leq i$ we have terms $v_1$ and $v_2$ such that $v_1[i, j] = 1$ and $[j, i]v_2 = 1$. For terms made uniquely of
crossing terms we also have two-sided inverses. Let the normal forms of $t$ and $u$ be respectively
\[
c^{l}[i_p, j_p] \ldots [i_1, j_1]1[k_1, l_1] \ldots [k_q, l_q][m_1, n_1] \ldots [m_r, n_r],
\]
\[
c^{l'}[i_{p'}, j_{p'}] \ldots [i_{1}', j_{1}'][k_{1}', l_{1}'] \ldots [k_{q}', l_{q}'][m_{1}', n_{1}'] \ldots [m_{r}', n_{r}'].
\]

(1) Suppose these normal forms differ only by having $l \neq l'$. Then $c^l = c^{l'}$ holds in $\mathcal{X}$ by the remark in the preceding paragraph.

(2) Suppose now that these normal forms differ in some other way, while either $l \neq l'$ or $l = l'$. Assume $l' \leq l$. Let $t_1$ be an abbreviation for $[j_1, i_1] \ldots [j_p, i_p]$, while $t_2$ is an abbreviation for
\[
[n_r, m_r] \ldots [n_1, m_1](\sigma_{i_1} \ldots \sigma_{k_1}) \ldots (\sigma_{i_q} \ldots \sigma_{k_q}).
\]
It can then be shown that $t_1 t_2 = c^{l+p+r}$, while $t_1 u t_2 = c^{l'+k} u'$, where $k \leq p+r$ and $u'$ is in $\mathcal{J}$-normal form. If $k = p+r$, then we do not have that $u'$ is 1, because otherwise we would be in case (1), as can be seen from the corresponding $\mathcal{SK}$ diagrams.

So in $\mathcal{X}$ we have $c^{l'+k} u' = c^{l+p+r}$. If $u'$ is 1, then $k < p+r$, and so $l'+k < l+p+r$, and we have proved $c^{l+k} = c^{l+p+r}$ in $\mathcal{X}$. If, on the other hand, $u'$ is not 1, then we have the following cases.

(2.1) Let $u'$ be $[i, j] u''$. Then in $\mathcal{SK}_\omega$ we have
\[
[i, j] [j, i] u' = [i, j] [j, i] [i, j] u'' = cu'.
\]
So in $\mathcal{X}$ we have
\[
c^{l+p+r}[i, j] [j, i] = c^{l'} [i, j] [j, i] u' = c^{l'+k+1} u' = c^{l+p+r+1},
\]
and hence we have
\[
c^{l+p+r}[i+1, j] [j, i] [i+1] = c^{l+p+r+1}[i+1] [i+1] = c^{l+p+r+2}.
\]
We proceed analogously if $u'$ is of the form $u''[j, i]$.

(2.2) Let $u'$ be $1 u''[k, l]$. Then in $\mathcal{X}$ we have
\[
c^{l+p+r+1} = c^{l'+k} [k+1] u'[k+1].
\]
Let $u'''$ be the $\mathcal{J}$-normal form of $[k+1] u'[k+1]$; the term $u'''$ is made uniquely of crossing terms and 1. For $k \leq 2n-1$ we have in $\mathcal{X}$
\[ [1][3] \ldots [2n-1]c^{l+p+r+1} [2n-1] \ldots [3][1] \]
\[ = [1][3] \ldots [2n-1]c^{l'+k''}u''[2n-1] \ldots [3][1] \]
\[ cl+p+r+1+n = cl'+k+n' \]
for \( n' \leq n \).

The demonstration in (2.2) could, as a matter of fact, proceed in different ways. Another such way would be to rely on the following property of symmetric groups.

Let \( S_\omega \) be the group generated by crossing terms that satisfies the monoidal equations and \( \sigma \) equations of Section 2. This is the union of the chain \( S_1 \subset S_2 \subset S_3 \subset \ldots \) of all finite symmetric groups. Then we can establish the following.

**Maximality of \( S_\omega \).** If \( t = 1 \) does not hold in \( S_\omega \), and \( X \) is defined as \( S_\omega \) save that we assume the additional equation \( t = 1 \), then for every \( m, n \in \mathbb{N}^+ \) we have \( \sigma_m = \sigma_n \) in \( X \).

To prove this we have first the following lemma.

**Lemma 1.** If in \( X \) we have \( \sigma_i = \sigma_j \) for \( i < j \), then in \( X \) we have \( \sigma_i = \sigma_j = \sigma_{j+1} \).

**Proof.** We have

\[ \sigma_{j+1} = \sigma_j \sigma_{j+1} \sigma_j \sigma_{j+1} \sigma_j, \quad \text{by (1), (2) and (3)} \]
\[ = \sigma_i \sigma_{j+1} \sigma_j \sigma_{j+1} \sigma_i, \quad \text{since } \sigma_i = \sigma_j \]
\[ = \sigma_i, \quad \text{by (1), (2) and (1)}. \]

We prove analogously the following lemma.

**Lemma 2.** If in \( X \) we have \( \sigma_i = \sigma_j \) for \( i < j \), then in \( X \) we have \( \sigma_{i-1} = \sigma_i = \sigma_j \), provided \( i > 1 \).

Then we prove the Maximality of \( S_\omega \) as follows.

**Proof of the Maximality of \( S_\omega \).** Suppose \( t' \) is the \( J \)-normal form of \( t \), which must be different from \( 1 \). So \( t' \) is of the form \( u \sigma_i v \) where \( i \) is the maximal index of the crossing terms in \( t' \); so all indices of crossing terms in \( u \) and \( v \) are strictly smaller. From \( t' = 1 \) it follows that \( \sigma_i = u^{-1}v^{-1} \), where the maximal index in \( u^{-1}v^{-1} \) is strictly smaller than \( i \). Then we have in \( X' \)

\[ \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_{i+1} u^{-1}v^{-1} \sigma_{i+1} \]
\[ \sigma_i \sigma_{i+1} \sigma_i = u^{-1}v^{-1} \]
\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_i \]
\[ \sigma_i = \sigma_{i+1}. \]
It suffices now to apply Lemmata 1 and 2. Then we can infer that in the case (2.2) of the proof of the Maximality of $S\mathcal{K}_\omega$ we would have that $c^{+p+r}\sigma_1 = c^{'+k}\sigma_2$ holds in $\mathcal{X}$, from which we infer

$$c^{+p+r}\sigma_1[1] = c^{'+k}\sigma_2[1]$$

$$c^{'+p+r+1} = c^{'+k}.$$  

The maximality of $S\omega$ is related to questions concerning the existence of proper nontrivial subgroups of symmetric groups. Suppose $t$ and $u$ are terms of $S\mathcal{K}_\omega$ such that $t = u$ does not hold in $S\mathcal{J}_\omega$, and let $\mathcal{X}$ be the monoid defined as $S\mathcal{J}_\omega$ save that we assume the additional equation $t = u$. Then it is possible to prove the following.

**Maximality of $S\mathcal{J}_\omega$.** For every $k \in \mathbb{N}^+$ we have $[k][k] = 1$ in $\mathcal{X}$.

The proof of that would be an extension of a proof that may be found in [6], Section 10, in which we would rely also on the Maximality of $S\omega$. We will not go into this proof, since we will have no occasion to apply the Maximality of $S\mathcal{J}_\omega$ in this paper.

As a consequence of the Maximality of $S\mathcal{J}_\omega$ we have that in $\mathcal{X}$ we have $\sigma_k = 1$ for every $k \in \mathbb{N}^+$ (which entails $\sigma_m = \sigma_n$, but which need not hold in the $\mathcal{X}$ of the Maximality of $S\omega$). In $\mathcal{X}$ we also have $[k] = [k+1]$ and $[k] = [k+1]$, and $\mathcal{X}$ is isomorphic to the group $\mathbb{Z}/n$ for some $n \in \mathbb{N}$.

We have considered the matters of this section so as to obtain the following consequence of the Maximality of $S\mathcal{K}_\omega$, which we will apply later. Let $h$ be a monoid homomorphism from $S\mathcal{K}_\omega$ to a monoid $M$. Then $S\mathcal{K}_\omega$ is isomorphic to the submonoid $h(S\mathcal{K}_\omega)$ of $M$ iff for every $k, l \in \mathbb{N}$ such that $k \neq l$ we have in $M$ that $h(c^k) \neq h(c^l)$.

**8 Symmetric endoadjunctions and self-adjunctions**

An *adjunction* is a sextuple $\langle A, B, F, G, \varphi, \gamma \rangle$ where, first, $A$ and $B$ are categories. (Throughout, we deal only with small categories.) This means that for $f: A \to B$, $g: B \to C$ and $h: C \to D$ arrows of $A$ we have the equations

\[(cat\ 1)\quad 1_B \circ f = f \circ 1_A = f,\]
\[(cat\ 2)\quad h \circ (g \circ f) = (h \circ g) \circ f,\]

and analogously for arrows of $B$. Next, $F$, the *left adjoint*, is a functor from $B$ to $A$, which means that we have the equations

\[(fun\ 1)\quad F1_B = 1_{FB},\]
\[(fun\ 2)\quad F(g \circ f) = Fg \circ Ff,\]
and $G$, the right adjoint, is a functor from $A$ to $B$. Next, $\varphi$, the counit of the adjunction, is a natural transformation from the composite functor $FG$ from $A$ to $A$ to $Id_A$, the identity functor on $A$, with members (components) $\varphi_A: FGA \rightarrow A$, and $\gamma$, the unit of the adjunction, is a natural transformation from $Id_B$ to the composite functor $GF$ from $B$ to $B$, with members $\gamma_B: B \rightarrow GFB$; this means that we have the equations

\[(nat \varphi) \quad f \circ \varphi_A = \varphi_B \circ FGF,\]
\[(nat \gamma) \quad GFF \circ \gamma_A = \gamma_B \circ f.\]

Finally, we have the triangular equations

\[(\varphi \gamma F) \quad \varphi_{FB} \circ F\gamma_B = 1_{FB},\]
\[(\varphi \gamma G) \quad G\varphi_A \circ \gamma_{GA} = 1_{GA}.\]

(For the notion of adjunction, and its importance in category theory, and mathematics in general, see [16], Chapter IV.)

An endoadjunction is a quintuple $\langle A, F, G, \varphi, \gamma \rangle$ such that $\langle A, A, F, G, \varphi, \gamma \rangle$ is an adjunction.

A symmetric endoadjunction is a sextuple $\langle A, F, G, \varphi, \gamma, \chi \rangle$ such that $\langle A, F, G, \varphi, \gamma \rangle$ is an endoadjunction, and with the definition

\[\chi_A = \text{df} \quad FF(\varphi_A \circ \chi_{GA}^{GF}) \circ F\chi_{FA}^{GF} \circ \gamma_{FA}^{GF} \circ \chi_{FA}^{GF} : FFA \rightarrow FFA,\]

we have that $\chi$ is a natural transformation from $FF$ to $FF$ whose members are self-inverse; this means that we have the following equations:

\[(nat \chi) \quad FFF \circ \chi_A = \chi_B \circ FFF,\]
\[(\chi \chi) \quad \chi_A \circ \chi_A = 1_{FFA}.\]

We assume, moreover, the equation

\[(\chi \chi \chi) \quad \chi_{FA} \circ \chi_{FA} \circ \chi_{FA} = F\chi_A \circ \chi_{FA} \circ F\chi_A.\]

and, finally, $\chi^{GF}$ is a family of arrows indexed by all the objects of $A$ such that for every such object $A$ the arrow $\chi_A^{GF}: GFA \rightarrow GFA$ is the inverse of

\[\chi_A^{FG} = \text{df} \quad G\varphi_A \circ G\chi_{GA} \circ \gamma_{FA} : FFA \rightarrow FFA,\]

which means that we have the equations

\[\chi_A^{GF} \circ \chi_A^{FG} = 1_{FGA}, \quad \chi_A^{FG} \circ \chi_A^{GF} = 1_{GFA}.\]

Instead of making our assumptions for $\chi$, we could alternatively make them for the natural transformation $\chi^{GG}$ from $GG$ to $GG$, whose members are defined by

\[\chi_A^{GG} = \text{df} \quad GG\varphi_A \circ G\chi_{GA}^{FG} \circ \gamma_{GA} : GGA \rightarrow GGA.\]
When \( \langle A, B, F, G, \varphi, \gamma \rangle \) is an adjunction such that \( F \) is left adjoint to \( G \) and \( \langle B, A, G, F, \varphi', \gamma' \rangle \) is an adjunction such that \( G \) is left adjoint to \( F \), then \( \langle A, B, F, G, \varphi, \gamma, \varphi', \gamma' \rangle \) will be called a \textit{bijunction}. It can then be checked that for every symmetric endoadjunction \( \langle A, F, G, \varphi, \gamma, \chi^{GF} \rangle \) we have a bijunction \( \langle A, F, G, \varphi, \gamma, \varphi', \gamma' \rangle \) where \( \varphi' \) and \( \gamma' \) are defined by

\[
\begin{align*}
\varphi'_A &= \text{df } \varphi_A \circ \chi_A^{GF}, \\
\gamma'_A &= \text{df } \chi_A^{GF} \circ \gamma_A.
\end{align*}
\]

Another alternative definition of symmetric endoadjunction is that it is a bi-
junction \( \langle A, A, F, G, \varphi, \gamma, \chi \rangle \) together with a natural transformation \( \chi \) from \( FF \) to \( FF \) such that the equations \((\chi \varphi)\) and \((\chi \gamma)\) are satisfied. With this
definition, we define \( \chi^{GF} \) in the following manner:

\[
\chi_A^{GF} = \text{df } \varphi_{FGA} \circ G\chi_{GA} \circ GF\gamma_A.
\]

This definition may be derived from \[17\] (see also \[18\], Chapter 1).

In every symmetric endoadjunction we have the equations

\[
\begin{align*}
(\chi \varphi) \quad \varphi_{FA} \circ F\chi_A^{FG} &= F\varphi_A \circ \chi_{GA}, \\
(\chi \gamma) \quad \chi_{FA}^{FG} \circ F\gamma_A &= G\chi_A \circ \gamma_{FA}.
\end{align*}
\]

A \textit{self-adjunction} is a quadruple \( \langle A, F, \varphi, \gamma \rangle \) such that \( \langle A, F, F, \varphi, \gamma \rangle \) is an endoadjunction. (This notion was considered in \[10\], Section 4.1, and \[5\].) We
call the functor \( F \) in a self-adjunction \textit{self-adjoint}. Note that in a self-adjunction
the equation \((\varphi \gamma G)\) can be replaced by

\[
(\varphi \gamma) \quad \varphi_{FA} \circ F\gamma_A = F\varphi_A \circ \gamma_{FA}.
\]

A \textit{symmetric self-adjunction} is a quintuple \( \langle A, F, \varphi, \gamma, \chi \rangle \) where \( \langle A, F, \varphi, \gamma \rangle \) is a self-adjunction, \( \chi \) is a natural transformation from \( FF \) to \( FF \) such that the equations \((\chi \chi)\) and \((\chi \chi \chi)\) are satisfied, and where, moreover, we have

\[
\begin{align*}
(\chi \varphi 1) \quad \varphi_A \circ \chi_A &= \varphi_A, \\
(\chi \gamma 1) \quad \chi_A \circ \gamma_A &= \gamma_A, \\
(\chi \varphi 2) \quad \varphi_{FA} \circ F\chi_A &= F\varphi_A \circ \chi_{FA}, \\
(\chi \gamma 2) \quad \chi_{FA} \circ F\gamma_A &= F\chi_A \circ \gamma_{FA}.
\end{align*}
\]

Alternatively, a symmetric self-adjunction may be defined as a symmetric endoadjunction where \( F \) and \( G \) are the same functor, and where

\[
\chi_A^{GF} = \chi_A^{FG} = \chi_A, \quad \varphi'_A = \varphi_A, \quad \gamma'_A = \gamma_A.
\]

The equations \((\chi \varphi 1)\) and \((\chi \gamma 1)\) are obtained from the definitions of \( \varphi' \) and \( \gamma' \),
while the equations \((\chi \varphi 2)\) and \((\chi \gamma 2)\) are instances of \((\chi \varphi)\) and \((\chi \gamma)\) above. Note
that \((\chi \varphi 2)\) is obtained from \((\varphi \gamma)\) by replacing \( \gamma \) by \( \chi \), while \((\chi \gamma 2)\) is obtained
from the same equation \((\varphi \gamma)\) by replacing \( \varphi \) by \( \chi \).
Let \( \kappa_A \) be an abbreviation for \( \varphi_A \circ \gamma_A : A \to A \). Then in every self-adjunction, for \( f : A \to B \) we have that
\[
f \circ \kappa_A = \kappa_B \circ f,
\]
and in every symmetric self-adjunction we have that
\[
F\kappa_A = \kappa_{FA}.
\]
This is derived as follows:
\[
F\kappa_A = F\varphi_A \circ \chi_{FA} \circ \chi_{FA} \circ F\gamma_A,
\]
with \((\text{fun } 2)\) and \((\chi\chi)\)
\[
= \varphi_{FA} \circ F\chi_A \circ F\chi_A \circ \gamma_{FA},
\]
by \((\chi\varphi 2)\) and \((\chi\gamma 2)\)
\[
= \kappa_{FA},
\]
with \((\text{fun } 2)\) and \((\chi\chi)\).
So every symmetric self-adjunction is a \( \mathcal{K} \)-adjunction in the sense of [5] (Section 14).

9 Free symmetric endoadjunctions and self-adjunctions

The *free symmetric endoadjunction* \( (\mathcal{K}', F, G, \varphi, \gamma, \chi_{GF}) \) generated by a single object, which we denote by 0, is defined as follows. The objects of the category \( \mathcal{K}' \) of this endoadjunction may be identified with finite, possibly empty, sequences of occurrences of the letters \( F \) and \( G \).

An *arrow-term* of \( \mathcal{K}' \) will be a word \( f \) that has as a *type* the ordered pair \( (A, B) \), where \( A \) and \( B \) are objects of \( \mathcal{K}' \). That \( f \) is of type \( (A, B) \) is expressed by \( f : A \to B \). We define the arrow-terms of \( \mathcal{K}' \) inductively as follows. We stipulate first for every object \( A \) of \( \mathcal{K}' \) that \( \mathbf{1}_A : A \to A \), \( \varphi_A : FGA \to A \), \( \gamma_A : A \to GFA \) and \( \chi_{FA} : GFA \to FGA \) are arrow-terms of \( \mathcal{K}' \). Next, if \( f : A \to B \) is an arrow-term of \( \mathcal{K}' \), then \( Ff : FA \to FB \) and \( Gf : GA \to GB \) are arrow-terms of \( \mathcal{K}' \), and if \( f : A \to B \) and \( g : B \to C \) are arrow-terms of \( \mathcal{K}' \), then \((g \circ f) : A \to C \) is an arrow-term of \( \mathcal{K}' \). As usual, we do not write parentheses in \((g \circ f)\) when they are not essential.

On these arrow-terms we impose the equations of symmetric endoadjunctions. Formally we take the smallest equivalence relation \( \equiv \) on the arrow-terms of \( \mathcal{K}' \) satisfying, first, congruence conditions with respect to \( F \), \( G \) and \( \circ \), namely,
\[
\text{if } f \equiv g, \text{ then } Ff \equiv Fg \text{ and } Gf \equiv Gg,
\]
\[
\text{if } f_1 \equiv f_2 \text{ and } g_1 \equiv g_2, \text{ then } g_1 \circ f_1 \equiv g_2 \circ f_2,
\]
provided \( g_1 \circ f_1 \) and \( g_2 \circ f_2 \) are defined, and, secondly, the conditions obtained from the equations of symmetric endoadjunctions by replacing the equality sign
= by ≡. Then we take the equivalence classes of arrow-terms as arrows, with
the obvious sources and targets, all arrow-terms in the same class having the
same type. On these equivalence classes we define 1, ϕ, γ, χ^GF, F, G and ◦
in the obvious way. This defines the category K', in which we have clearly a
symmetric endoadjunction.

The category K' has the following universal property. If s maps the object
0 to an arbitrary object of the category A of an arbitrary symmetric endoad-
djunction, then there is a unique functor S of symmetric endoadjunctions (de-
efined in the obvious way, so that the structure of symmetric endoadjunctions is
preserved) such that S maps 0 to s(0). This property characterizes K' up to
isomorphism with a functor of symmetric endoadjunctions. This justifies calling
free the symmetric endoadjunction of K'.

The free symmetric self-adjunction generated by a single object, whose cat-
egory we will call K, is defined analogously. The objects of K, which are finite,
possibly empty, sequences of occurrences of the letter F preceding 0, may be
identified with the natural numbers (cf. [5], Section 11).

10 K and SKω

Let F^0 be the empty sequence, and let F^{k+1} be F^kF. On the arrows of the
category K of the free symmetric self-adjunction, we define a total binary op-
eration * based on composition of arrows in the following manner. For f: m → n
and g: k → l,

\[ g * f = df \begin{cases} 
    g^*F^k-f & \text{if } n \leq k \\
    F^{n-k}G = f & \text{if } k \leq n.
\end{cases} \]

Next, let f ≡K g iff there are k, l ∈ N such that F^k f = F^l g in K. It is easy
to check that ≡K is an equivalence relation on the arrows of K, which satisfies
moreover

- (congr *) if f_1 ≡K f_2 and g_1 ≡K g_2, then g_1 * f_1 ≡K g_2 * f_2,
- (congr F) if f ≡K g, then F f ≡K F g.

For every arrow f of K, let [f] be \{g \mid f ≡K g\}, and let K^* be \{[f] \mid
f is an arrow of K\}. With

\[ 1 = df [1_0], \]
\[ [g][f] = df [g * f], \]

we can check that K^* is a monoid. We will show that this monoid is isomorphic
to the monoid SKω.

Consider the map ψ from the arrow-terms of K to the terms of SKω defined
inductively by
\[ \psi(1_n) \] is \( 1 \),
\[ \psi(\varphi_n) \] is \( [n+1] \),
\[ \psi(\gamma_n) \] is \( [n+1] \),
\[ \psi(\chi_n) \] is \( \sigma_{n+1} \),
\[ \psi(Ff) \] is \( \psi(f) \),
\[ \psi(g \circ f) \] is \( \psi(g)\psi(f) \).

We can easily establish the following by induction on the length of \( f \).

**Remark I.** For every arrow-term \( f : m \to n \) of \( \mathcal{K} \), in the \( SK \) diagram \( \kappa(\psi(f)) = (D,k) \) the \( SJ \) diagram \( D \) is of type \((m,n)\).

We also have the following.

**Remark II.** If in the \( SK \) diagram \( \kappa(t) = (D,k) \) the \( SJ \) diagram \( D \) is of type \((m,n)\), then \( \kappa(t[\xi + 1]) = \kappa([n+1]t), \kappa(t|\xi + 1]) = \kappa([n+1]t) \) and \( \kappa(t\sigma_{m+1}) = \kappa(\sigma_{n+1}t) \).

Then we can prove the following lemma.

**Lemma.** If \( f = g \) in \( \mathcal{K} \), then \( \psi(f) = \psi(g) \) in \( \mathcal{SK}_{\omega} \).

**Proof.** We proceed by induction on the length of the derivation of \( f = g \) in \( \mathcal{K} \). All the cases are quite straightforward except when \( f = g \) is an instance of \( (nat \varphi), (nat \gamma) \) or \( (nat \chi) \), where we use Remarks I and II. In case \( f = g \) is an instance of a triangular equation, we use the equation \( (cup-cap) \) of Section 2; for \( (\chi \chi) \) we use \( (\sigma 2) \), for \( (\chi \chi \chi) \) we use \( (\sigma 3) \), for \( (\chi \varphi 1) \) we use \( (\sigma-cap 3) \), for \( (\chi \varphi 2) \) we use \( (\sigma-cap 4) \), for \( (\gamma 1) \) we use \( (\sigma-cap 3) \), for \( (\gamma 2) \) we use \( (\sigma-cap 4) \). ⊢

As an immediate corollary we have that if \( f \equiv \mathcal{K} g \), then \( \psi(f) = \psi(g) \) in \( \mathcal{SK}_{\omega} \). Hence we have a map from \( \mathcal{K}^* \) to \( \mathcal{SK}_{\omega} \), which we also call \( \psi \), defined by \( \psi([f]) = \psi(f) \). Since

\[ \psi([1]) = \psi(1) = 1, \]
\[ \psi([g \ast f]) = \psi(\varphi(g)\psi(f)), \]

this map is a monoid homomorphism.

Consider next the map \( \xi \) from the terms of \( \mathcal{SK}_{\omega} \) to the arrow-terms of \( \mathcal{K} \) defined inducively by

\[ \xi(1) \] is \( 1_0 \),
\[ \xi([k]) \] is \( \varphi_{k-1} \),
\[ \xi([k]) \] is \( \gamma_{k-1} \),
\[ \xi(\sigma_k) \text{ is } \chi_{k-1}, \]
\[ \xi(tu) \text{ is } \xi(t) \cdot \xi(u). \]

Then we establish the following lemmata.

**ξ Lemma.** If \( t = u \) in \( SK_\omega \), then \( \xi(t) \equiv_K \xi(u) \).

**Proof.** We proceed by induction on the length of the derivation of \( t = u \) in \( SK_\omega \).

The cases where \( t = u \) is an instance of (1) and (2) are quite straightforward. For \((\cup)\) or \((\cup \cap 1)\), by \((nat \varphi)\) we have
\[ \varphi_{k-1} \cdot F^{k-j+2} \varphi_{j-1} = F^{k-j} \varphi_{j-1} \cdot \varphi_{k+1}, \]
\[ \varphi_{k+1} \cdot F^{k-j+2} \gamma_{j-1} = F^{k-j} \gamma_{j-1} \cdot \varphi_{k+1}. \]

We proceed analogously for \((\cap)\) and \((\cap \cup 1)\) by using \((nat \gamma)\). For \((\cup-cap)\), by the triangular equations we have
\[ F \varphi_{i-1} \cdot \gamma_i = 1_i, \]
\[ \varphi_{i-1} \cdot F \gamma_{i-2} = 1_{i-1}. \]

For \((\sigma)\), by \((nat \chi)\) we have
\[ \chi_{k+1} \cdot F^{k-j+2} \chi_{j-1} = F^{k-j+2} \chi_{j-1} \cdot \chi_{k+1}, \]
and for \((\sigma 2)\) and \((\sigma 3)\) we use \((\chi \chi)\) and \((\chi \chi \chi)\) respectively. For \((\sigma-cap 1)\), by \((nat \varphi)\) we have
\[ \varphi_{k+1} \cdot F^{k-j+2} \chi_{j-1} = F^{k-j} \chi_{j-1} \cdot \varphi_{k+1}, \]
and for \((\sigma-cap 2)\), by \((nat \chi)\) we have
\[ F^{k-j+2} \varphi_{j-1} \cdot \chi_{k+1} = \chi_{k-1} \cdot F^{k-j+2} \varphi_{j-1}. \]

We proceed analogously for \((\sigma-cap 1)\) and \((\sigma-cap 2)\) by using \((nat \gamma)\) and \((nat \chi)\) respectively. Finally, for \((\sigma-cap 3)\) we use \((\chi \varphi 1)\), for \((\sigma-cap 4)\) we use \((\chi \varphi 2)\), for \((\sigma-cap 3)\) we use \((\chi \gamma 1)\), and for \((\sigma-cap 4)\) we use \((\chi \gamma 2)\).

We have already established that \( \equiv_K \) is an equivalence relation that satisfies \((congr *)\) and \((congr F)\). So the lemma follows.

**ξψ Lemma.** For every arrow-term \( f \) of \( K \) we have \( \xi(\psi(f)) \equiv_K f \).

**Proof.** We proceed by induction on the length of \( f \). We have
\[ \xi(\psi(1_n)) \text{ is } 1_0 \equiv_K 1_n, \]
\[ \xi(\psi(\varphi_n)) \text{ is } \varphi_n, \]
\[ \xi(\psi(\gamma_n)) \text{ is } \gamma_n. \]
\[ \xi(\psi(\chi_n)) \text{ is } \chi_n, \]
\[ \xi(\psi(Ff)) \equiv_{\mathcal{K}} \xi(f), \text{ by the induction hypothesis} \]
\[ \equiv_{\mathcal{K}} Ff, \]
\[ \xi(\psi(g \circ f)) \equiv_{\mathcal{K}} g \circ f, \]
by the induction hypothesis, (congr *) and the definition of *.

By a straightforward induction we can prove also the following lemma.

**ψξ Lemma.** For every term \( t \) of \( \mathcal{SK}_\omega \) we have that \( \psi(\xi(t)) = t \).

This establishes that \( \mathcal{K}^* \) and \( \mathcal{SK}_\omega \) are isomorphic monoids.

Next we prove the following lemma.

**\( \mathcal{K} \) Cancellation Lemma.** In \( \mathcal{K} \), if \( Ff = Fg \), then \( f = g \).

**Proof.** If for \( f, g: m \to n \) we have \( m > 0 \), then
\[ \varphi_n \circ FFf \circ \gamma_{m-1} = \varphi_n \circ FFg \circ \gamma_{m-1} \]
\[ f \circ \varphi_m \circ \gamma_{m-1} = g \circ \varphi_m \circ \gamma_{m-1}, \text{ by (nat } \varphi) \]
\[ f = g, \text{ by (\( \varphi \gamma F \)).} \]

If \( n > 0 \), then
\[ F\varphi_{n-1} \circ FFf \circ \gamma_n = F\varphi_{n-1} \circ FFg \circ \gamma_n \]
\[ F\varphi_{n-1} \circ \gamma_n \circ f = F\varphi_{n-1} \circ \gamma_n \circ g, \text{ by (nat } \gamma) \]
\[ f = g, \text{ by (\( \varphi \gamma G \)).} \]

If \( m = n = 0 \), then \( f \) is equal either to \( 1_0 \) or to \( \varphi_0 \circ f' \circ \gamma_0 \), and \( g \) is equal either to \( 1_0 \) or to \( \varphi_0 \circ g' \circ \gamma_0 \).

If \( f = g = 1_0 \), we are done. If \( f = 1_0 \) and \( g = \varphi_0 \circ g' \circ \gamma_0 \), then \( Ff = Fg \) in \( \mathcal{K} \) is excluded. From \( Ff = Fg \) in \( \mathcal{K} \) we obtain \( \kappa(\psi(f)) = (I, 0) = \kappa(\psi(g)) \), which is excluded because, by Remark 1, in \( \kappa(\psi(g')) = (D, k') \) the \( \mathcal{S} \mathcal{F} \) diagram \( D \) must be of type \((2, 2)\), from which we obtain that in \( \kappa(\psi(g)) = (I, k) \) we have \( k \geq 1 \). We deal analogously with the case when \( f = \varphi_0 \circ f' \circ \gamma_0 \) and \( g = 1_0 \).

If \( f = \varphi_0 \circ f'' \) and \( g = \varphi_0 \circ g'' \), where \( f'' \) is \( f' \circ \gamma_0 \) and \( g'' \) is \( g' \circ \gamma_0 \), then from \( \kappa(\psi(f)) = \kappa(\psi(g)) \) we conclude \( \kappa(\psi(f'')) = \kappa(\psi(g'')) \). We can do this because \( \kappa(\psi(f)) = \kappa(\psi(g)) = (I, k) \) for some \( k \geq 1 \), while \( \kappa(\psi(f'')) = \kappa(\psi(g'')) = (\Lambda_k, l-1) \). So \( \psi(f'') = \psi(g'') \), and, by the \( \xi \psi \)-Lemma, \( f'' \equiv_{\mathcal{K}} g'' \). Since \( f'' \) and \( g'' \) are both of type \( 0 \to 2 \), for some \( l \geq 0 \) we have \( Ff'' = Fg'' \) in \( \mathcal{K} \), and, by reasoning as at the beginning of the proof (in the case when \( n > 0 \)), we obtain \( f'' = g'' \) in \( \mathcal{K} \), and hence \( f = g \in \mathcal{K} \).

The \( \mathcal{K} \) Cancellation Lemma implies that \( f \equiv_{\mathcal{K}} g \) for \( f: m \to n \) and \( g: k \to l \) could be defined by \( F^{k-n}f = g \) in \( \mathcal{K} \) when \( n \leq k \), and by \( F^n-kg \) in \( \mathcal{K} \) when

28
\[ k \leq n. \] So for arbitrary \( f, g : m \to n \) we have established that \( f = g \) in \( \mathcal{K} \) iff \( f \equiv_{\mathcal{K}} g \). This establishes that \( \mathcal{K} \) is isomorphic to a category of \( \mathcal{SK} \) diagrams indexed by all their types.

We define a \( \mathcal{J} \)-symmetric self-adjunction as a symmetric self-adjunction that satisfies moreover \( \varphi_A \cdot \gamma_A = 1_A \), i.e., \( \kappa_A = 1_A \), according to the abbreviation introduced at the end of Section 8. In other words, a \( \mathcal{J} \)-symmetric self-adjunction is a symmetric self-adjunction that is a \( \mathcal{J} \)-adjunction in the terminology of [5] (Section 10). Let \( \mathcal{J} \) be the free \( \mathcal{J} \)-symmetric self-adjunction generated by a single object, which is defined analogously to the category \( \mathcal{K} \), and let \( \mathcal{J}^* \) be the monoid defined out of \( \mathcal{J} \) as \( \mathcal{K}^* \) is defined out of \( \mathcal{K} \). Then we can establish, by imitating what we had above, that \( \mathcal{J}^* \) and \( \mathcal{SJ}_\omega \) are isomorphic monoids. We can also establish that \( \mathcal{J} \) is isomorphic to a category of \( \mathcal{SJ} \) diagrams indexed by all their types.

### 11 Subsided categories

A *monoidal category* \( \mathcal{M} \) is a category that has a special object \( I \), a bifunctor \( \otimes \) on \( \mathcal{M} \) (i.e. a functor from \( \mathcal{M} \times \mathcal{M} \) to \( \mathcal{M} \)), and natural isomorphisms whose members (components) are

\[
\begin{align*}
a_{A,B,C} : & \quad A \otimes (B \otimes C) \to (A \otimes B) \otimes C, \\
\sigma_A : & \quad I \otimes A \to A, \\
\delta_A : & \quad A \otimes I \to A,
\end{align*}
\]

which satisfy the usual coherence equations (see [16], Section VII.1).

A *symmetric monoidal* category is a monoidal category that has in addition a natural isomorphism whose members are

\[
s_{A,B} : A \otimes B \to B \otimes A,
\]

which satisfy the usual coherence equations (see [16], Section VII.7).

A *symmetric monoidal closed* category is a symmetric monoidal category \( \mathcal{M} \) such that for every object \( A \) there is a functor \( [A, \_] \) from \( \mathcal{M} \) to \( \mathcal{M} \) right-adjoint to the functor \( A \otimes \_ \), where for an arrow \( f \) of \( \mathcal{M} \) the arrow \( A \otimes f \) is \( 1_A \otimes f \). The counit of this adjunction has the members

\[
\varepsilon_{A,B} : A \otimes [A,B] \to B,
\]

and the unit has the members

\[
\eta_{A,B} : B \to [A, A \otimes B].
\]

In every symmetric monoidal closed category \( \mathcal{M} \) we have a functor \( [\_, \_] \) from \( \mathcal{M}^{op} \times \mathcal{M} \) to \( \mathcal{M} \) where for \( f : A \to B \) and \( g : C \to D \)
\[ f, g = [A, g] \circ [A, \varepsilon_{B,C}] \circ [A, f \otimes 1_{[B,C]}] \circ \eta_{A,[B,C]} \]

A compact closed category is a symmetric monoidal closed category such that for all objects \( A \) and \( B \) we have the arrow
\[
\nu_{A,B} : [A, B] \rightarrow [A, I] \otimes B
\]
as the inverse of
\[
[1_A, \sigma_B \circ (\varepsilon_{A,I} \otimes 1_B) \circ a_{A,[A,I],B}] \circ \eta_{A,[A,I] \otimes B} : [A, I] \otimes B \rightarrow [A, B].
\]
Alternatively, we could assume the inverse of
\[
[1_A, (\varepsilon_{A,C} \otimes 1_B) \circ a_{A,[A,C],B}] \circ \eta_{A,[A,C] \otimes B} : [A, C] \otimes B \rightarrow [A, C \otimes B],
\]
or of
\[
[1_A, (1_B, \varepsilon_{A,C}) \circ a_{B,A,[A,C]}^{-1} \circ (s_{A,B} \otimes 1_{[A,C]}) \circ a_{A,B,[A,C]} \circ \eta_{A,B \otimes [A,C]} : B \otimes [A, C] \rightarrow [A, B \otimes C].
\]
As a matter of fact (as noted in [15], p. 194) it would be enough to have the arrows \( \nu_{A,A} \), because \( \nu_{A,B} \) and all the inverses above can be defined in terms of \( \nu_{A,A} \).

A monoidal category is strictly monoidal when for all objects \( A, B \) and \( C \)
\[
A \otimes (B \otimes C) = (A \otimes B) \otimes C,
\]
\[
I \otimes A = A \otimes I = A,
\]
while \( a_{A,B,C} = 1_{A \otimes (B \otimes C)} \) and \( \sigma_A = \delta_A = 1_A \).

A subsided category is a compact closed category which is strictly monoidal, in which
\[
[A, B] = [A, I] \otimes B = A \otimes B,
\]
and where
\[
\begin{align*}
(\nu) & \quad \nu_{A,B} = 1_{A \otimes B}, \\
(\varepsilon 1) & \quad \varepsilon_{A,B} = \varepsilon_{A,B} \circ (s_{A,A} \otimes 1_B), \\
(\varepsilon 2) & \quad \varepsilon_{A \otimes B, C} = \varepsilon_{B,C} \circ \varepsilon_{A,B \otimes B \otimes C} \circ (1_A \otimes s_{B,A} \otimes 1_{B \otimes C}).
\end{align*}
\]
It can be shown that in every subsided category we have the equation
\[
(\eta 1) \quad \eta_{A,B} = (s_{A,A} \otimes 1_B) \circ \eta_{A,B},
\]
dual to \((\varepsilon_1)\), which could alternatively be used for defining subsided categories. We also have

\[
(\eta_2) \quad \eta_{A \otimes B, C} = (1_A \otimes s_{A, B} \otimes 1_{B \otimes C}) \ast \eta_{A, B \otimes B \otimes C} \ast \eta_{B, C};
\]

which could be used instead of \((\varepsilon_2)\).

A compact closed category may be defined as a symmetric monoidal category \(\mathcal{M}\) that has a functor \(\ast\) from \(\mathcal{M}^{op}\) to \(\mathcal{M}\) and dinatural transformations whose members are

\[
\begin{align*}
    f_A & : A \otimes A^* \to I, \\
    g_A & : I \to A^* \otimes A,
\end{align*}
\]

which satisfy the \textit{compact triangular equations}:

\[
\begin{align*}
    \delta_{A^*} \circ \sigma_A \circ (f_A \otimes 1_A) & = \delta_A^{-1}, \\
    \delta_{A^*} \circ (1_{A^*} \otimes f_A) & = \delta_A^{-1}.
\end{align*}
\]

That \(f\) and \(g\) are dinatural means that for every arrow \(h : A \to B\) we have the equations

\[
\begin{align*}
    f_B \circ (h \otimes 1_{B^*}) & = f_A \circ (1_A \otimes h^*), \\
    (1_{A^*} \otimes h) \circ g_A & = (h^* \otimes 1_B) \circ g_B
\end{align*}
\]

(see [16], Section IX.4).

With the definition of compact closed category given previously, \(A^*\) is defined as \([A, I]\), the arrow \(f_A\) is defined as \(\varepsilon_{A, I}\), and the arrow \(g_A\) is defined as \(\nu_{A, A^*} \circ [1_A, \delta_A] \circ \eta_{A, I}\). With the new definition, we define \([A, B]\) as \(A^* \otimes B\), we define \([A, f]\) as \(1_{A^*} \otimes f\), the arrow \(\varepsilon_{A, B}\) is defined as \(\sigma_B \circ (f_A \otimes 1_B) \circ a_{A^*, A^*, B}\), the arrow \(\eta_{A, B}\) is defined as \(\sigma_B^{-1} \circ (g_A \otimes 1_B) \circ \sigma_B\), and \(\nu_{A, B}\) is defined as \(\delta_{A^*}^{-1} \otimes 1_B\).

The notion of compact closed category was introduced in [14], and more precisely in [15] (see also [13], Section 7). A subsided category may alternatively be defined as a symmetric monoidal category which is strictly monoidal, in which for every object \(A\) the functor \(A \otimes _\_\) is self-adjoint, and in this adjunction we have the equations \((\varepsilon_1)\) and \((\varepsilon_2)\). Categories related to subsided categories, but which instead of symmetry have braiding, were investigated in [20] and [17] (see also [10], Section 4.1).

### 12 \(\mathcal{K}\) as a subsided category

To define the free subsided category \(\mathcal{S}\) generated by a single object, which we denote by \(p\), we rely on the alternative definition given at the end of the
the equations

The **arrow-terms** of \( S \) are defined inductively as follows. For all objects \( A \) and \( B \) of \( S \) we have that \( 1_A : A \to A \), \( s_{A,B} : A \otimes B \to B \otimes A \) (note that here \( A \otimes B = B \otimes A \)), \( \varepsilon_{A,B} : A \otimes A \otimes B \to B \) and \( \eta_{A,B} : B \to A \otimes A \otimes B \) are arrow-terms of \( S \). Next, if \( f : A \to B \) and \( g : C \to D \) are arrow-terms of \( S \), then \( f \otimes g : A \otimes C \to B \otimes D \) is an arrow-term of \( S \), and if \( f : A \to B \) and \( g : B \to C \) are arrow-terms of \( S \), then \( (g \cdot f) : A \to C \) is an arrow-term of \( S \).

On these arrow-terms we impose the equations of subsided categories; namely, the equations

\[
\begin{align*}
\text{(cat 1)} & \quad 1_B \cdot f = f \cdot 1_A = f, \\
\text{(cat 2)} & \quad h \cdot (g \cdot f) = (h \cdot g) \cdot f, \\
\text{(bfun 1)} & \quad 1_A \otimes 1_B = 1_{A \otimes B}, \\
\text{(bfun 2)} & \quad (g_2 \cdot g_1) \otimes (f_2 \cdot f_1) = (g_2 \otimes f_2) \cdot (g_1 \otimes f_1), \\
\text{(nat s)} & \quad s_{B,D} \cdot (f \otimes g) = (g \otimes f) \cdot s_{A,C}, \\
\text{(s2)} & \quad s_{B,A} \cdot s_{A,B} = 1_{A \otimes B}, \\
\text{(s3)} & \quad s_{A \otimes B,C} = (s_{A,C} \otimes 1_B) \cdot (1_A \otimes s_{B,C}), \\
\text{(nat \( \varepsilon \))} & \quad f \cdot \varepsilon_{A,B} = \varepsilon_{A,C} \cdot (1_{A \otimes A} \otimes f), \\
\text{(nat \( \eta \))} & \quad \eta_{A,C} \cdot f = (1_{A \otimes A} \otimes f) \cdot \eta_{A,B}, \\
\text{(\( \varepsilon \eta \))} & \quad \varepsilon_{A,A \otimes B} \cdot (1_A \otimes \varepsilon_{A,B}) = (1_A \otimes \varepsilon_{A,B}) \cdot \eta_{A,A \otimes B} = 1_{A \otimes B},
\end{align*}
\]

plus the equations (\( \varepsilon 1 \)) and (\( \varepsilon 2 \)). Formally, to get the arrows of \( S \) we take equivalence classes as in Section 9.

We will now show that the category \( K \) of the free symmetric self-adjunction generated by a single object (see Section 9) is isomorphic to \( S \). We define first in \( K \) the structure of a subsided category. The objects of \( K \) are natural numbers, the object \( I \) is 0, and \( \otimes \) on objects is addition, which is associative. We define \( \otimes \) on the arrows of \( K \) as follows. For \( f : m \to n \)

\[ 1_k \otimes f =_{df} F^k f, \]

while \( f \otimes 1_k \) is defined inductively as follows:

\[
\begin{align*}
\alpha_n \otimes 1_k & = \alpha_{n+k}, \text{ for } \alpha \text{ being } 1, \varphi, \gamma \text{ and } \chi, \\
F f \otimes 1_k & = F(f \otimes 1_k), \\
(g \cdot f) \otimes 1_k & = (g \otimes 1_k) \cdot (f \otimes 1_k).
\end{align*}
\]

For \( f : m \to n \) and \( g : k \to l \) we define \( f \otimes g : m+k \to n+l \) by

\[ f \otimes g =_{df} (1_n \otimes g) \cdot (f \otimes 1_k). \]
It remains to check that with these definitions $K$ is a strictly monoidal category, which is easy to do with the help of $SK$ diagrams. Hence $K$ satisfies $(bifun \, 1)$ and $(bifun \, 2)$, and $f \otimes (g \otimes h) = (f \otimes g) \otimes h$.

The arrows $s$ are defined inductively as follows:

$$s_{0,n} = s_{n,0} = 1_n,$$
$$s_{n+1,1} = (s_{n,1} \otimes 1_1) \circ (1_n \otimes \chi_0),$$
$$s_{n,m+1} = (1_m \otimes s_{n,1}) \circ (s_{n,m} \otimes 1_1).$$

With this definition $K$ is a symmetric monoidal category; it satisfies, namely, the equations $(nat \, s)$, $(s2)$ and $(s3)$ which is easy to check with the help of $SK$ diagrams.

In $K$ the object $[A,B]$ is $A \otimes B$, while the arrows $\varepsilon$ and $\eta$ are defined inductively as follows:

$$\varepsilon_{0,n} = 1_n,$$
$$\varepsilon_{m+1,n} = \varphi_n \circ (1_1 \otimes \varepsilon_{m,n} \otimes 1_1) \circ (s_{m,1} \otimes 1_{n+m+1}),$$
$$\eta_{0,n} = 1_n,$$
$$\eta_{m+1,n} = (s_{1,m} \otimes 1_{n+m+1}) \circ (1_1 \otimes \eta_{m,n} \otimes 1_1) \circ \gamma_n.$$

With these definitions, and with the help of $SK$ diagrams, we check that the equations $(nat \, \varepsilon)$, $(nat \, \eta)$, $(\varepsilon \eta)$, $(\varepsilon1)$ and $(\varepsilon2)$ are satisfied, which shows that $K$ is a subsided category. So we have a functor $\Theta$ from the free subsided category $S$ to $K$ that maps the generating object $p$ of $S$ into the number 1, i.e. $F0$ (which is not the generating object of $K$). It is clear that this functor is a bijection on objects: if objects in both categories are conceived as natural numbers, then this bijection is identity. It remains to show that it is a bijection on arrows too.

Now we define in $S$ the structure of a symmetric self-adjunction. The functor $F$ is $p \otimes \underline{\underline{\ldots}}$, the arrow $\varphi_A$ is $\varepsilon_{p,A}$, the arrow $\gamma_A$ is $\eta_{p,A}$, and the arrow $\chi_A$ is $s_{p,p} \otimes 1_A$. Since $K$ is free, we have a functor $\Theta^{-1}$ from $K$ to $S$ that maps the generating object 0 into $I$. On objects $\Theta^{-1}$ is identity, as $\Theta$ was. It remains to check that $\Theta^{-1}(\Theta(f)) = f$ and $\Theta(\Theta^{-1}(g)) = g$.

It can be verified that the category $K'$ of the free symmetric self-adjunction generated by a single object (see Section 9) is compact closed. It is not, however, isomorphic to the free compact closed category generated by a single object; for this isomorphism to hold, the latter category would need to be further strictified.

### 13 Multiple symmetric self-adjunctions

Consider a family $\{\langle A, F^i, \varphi^i, \gamma^i \rangle \mid i \in I\}$ of self-adjunctions, all in the same category $A$, such that for every $i, j \in I$ and every object $A$ of $A$ we have an arrow $\chi^i_{A}: F^i F^j A \rightarrow F^j F^i A$, with the following equations being satisfied:
(nat \chi^i) \quad F^jF^if \circ \chi^j_A = \chi^j_B \circ F^jF^i f,

(\chi^i\chi^i) \quad \chi^j_A \circ \chi^j_A = 1_{F^jF^j A},

(\chi^i\chi^i) \quad \chi^j_A \circ \chi^j_A = 1_{F^jF^j A},

(\chi^i\chi^i) \quad \chi^j_A \circ \chi^j_A = 1_{F^jF^j A},

\varphi^i_A \circ \chi^j_A = \varphi^j_A,

\gamma^i_A \circ \chi^j_A = \gamma^j_A,

\varphi^i_{F^j A} \circ F^j \chi^j_A = F^j \varphi^i_{A} \circ \chi^j_A,

\chi^j_{F^j A} \circ F^j \gamma^j_A = F^j \chi^j_A \circ \gamma^j_{F^j A}.

It is clear that for every i we have that \langle A, \varphi^i, \gamma^i, \chi^i \rangle is a symmetric self-adjunction. We call our family a multiple symmetric self-adjunction indexed by \mathcal{I}.

Note that every symmetric self-adjunction is a multiple symmetric self-adjunction indexed by the set of natural numbers \mathbb{N}. The functor \mathcal{F}^0 is the identity functor, and \mathcal{F}^{n+1} is the composite functor \mathcal{F}^n \mathcal{F}. It can be shown that the category of the multiple symmetric self-adjunction generated by a single object and the index set \mathcal{I} is isomorphic to the free subsided category generated by the set of objects \mathcal{I}. We have treated above this isomorphism in the case where \mathcal{I} is a singleton, and what should be done for the general case is easily inferred from that.

An arbitrary subsided category \mathcal{C} gives rise to a multiple symmetric self-adjunction indexed by the set of objects of \mathcal{C}. For that we proceed as we did in the preceding section for \mathcal{S}, putting instead of \mathcal{p} an arbitrary object of \mathcal{C}.

Consider the arrow-terms of the category \mathcal{M} of the multiple symmetric self-adjunction generated by a single object and the index set \mathcal{I}. These arrow-terms can be put in a normal form analogous to the normal form for terms of \mathcal{SK}_\omega.

This normal form looks as follows:

\[ e_{i_1} \ldots e_{i_n} a_p \ldots a_1 d_1 \ldots d_q b_1 \ldots b_r \]

where

1. \( i_j \in \mathcal{I}, 1 \leq j \leq s, i_j \neq i_k, e_{i_j} \) is \( \varphi^i_{B} \circ \chi^j_B \circ \ldots \circ \varphi^i_{B} \circ \gamma^j_B \); these arrow-terms correspond to circles, which are now indexed by members of \mathcal{I};

2. \( a_j, 1 \leq j \leq p, \) is an arrow-term corresponding to a block-cap;

3. \( b_j, 1 \leq j \leq r, \) is an arrow-term corresponding to a block-cup;

4. \( d_j, 1 \leq j \leq q, \) is an arrow-term corresponding to a block-crossing.

This normal form is unique up to the order of the indices \( i_1 \ldots i_s \).

The category \mathcal{M} can be shown isomorphic to a category of diagrams analogous to \mathcal{SK} diagrams indexed by types, but whose types are now not (n,m) with \( n, m \in \mathbb{N}^+ \), but instead \( (A, B) \) where \( A \) and \( B \) are finite sequences of elements of \mathcal{I}. These new diagrams have circular components indexed by elements of \mathcal{I}. The proof of this isomorphism is analogous to the proof of isomorphism.
of the category \( \mathcal{K} \) of the free symmetric self-adjunction with a category of \( \mathcal{SK} \) diagrams indexed by their types (see Section 10).

We can establish that \( \mathcal{M} \) is maximal in the following sense. Suppose that \( t \) and \( u \) are arrow-terms of \( \mathcal{M} \) such that \( t = u \) does not hold in \( \mathcal{M} \), and let \( \mathcal{X} \) be the category defined as \( \mathcal{M} \) save that we assume the additional equation \( t = u \). We can then prove the following.

**Maximality of \( \mathcal{M} \).** *For two arrow-terms \( v \) and \( v' \) of \( \mathcal{M} \) in normal form such that \( v \) is*

\[
e_{i_1} \cdots e_{i_s} a_p \cdots a_1 d_1 \cdots d_q b_1 \cdots b_r,
\]

*\( v' \) is*

\[
e_{i_1'} \cdots e_{i_{s'}} a_{p'} \cdots a_1' d_1' \cdots d_q' b_1' \cdots b_{r'},
\]

*and \( \{e_{i_j} \mid 1 \leq j \leq s\} \neq \{e_{i_j'} \mid 1 \leq j \leq s'\}, \) we have that \( v = v' \) holds in \( \mathcal{X} \).*

*Proof.* If the normal forms of \( t \) and \( u \) differ in their \( e \)-parts as above, then we are done.

Suppose the normal forms of \( t \) and \( u \) do not differ in their \( e \)-parts, but they differ either in their \( a \)-parts or in their \( b \)-parts, and suppose the number of \( a \)-terms and \( b \)-terms in the normal form of \( t \) is greater than or equal to the corresponding number in the normal form of \( u \). Then with the arrow-terms \( a \) and \( b \), whose diagrams are mirror images of the diagrams corresponding respectively to the \( b \) and \( a \) parts of the normal form of \( t \), we have

This means that the normal forms of \( b \cdot t \cdot a \) and \( b \cdot u \cdot a \) differ in their \( e \)-parts, because in the diagram of \( b \cdot u \cdot a \) we must have less circular components.

If the normal forms of \( t \) and \( u \) differ only in their \( d \)-parts, then in the diagram corresponding to \( t \) we have a transversal thread

\[
\begin{array}{cccccccc}
i_1 & \cdots & i_2 & i & i_3 & \cdots & i_4 \\
i_5 & \cdots & i_6 & 2 & i_7 & \cdots & i_8
\end{array}
\]
which is missing in the diagram corresponding to $u$, but in the latter diagram
out of the $i$'s above and below go transversal threads, since the $a$-parts and
$b$-parts of the two diagrams do not differ. Then we have

This means that the normal forms of $b \cdot F_i \cdot a$ and $b \cdot F_i \cdot u \cdot a$ differ in the $c$-parts,
since the diagram of $b \cdot F_i \cdot a$ has an additional circular component.

In this section we have not been very precise, because the matters considered
are just complications of matters considered previously. More precision would
involve a considerable number of these complications, while the ideas would not
be essentially new.

14 The category $\text{Mat}_F$

Let $\text{Mat}_F$ be the skeleton of the category of finite-dimensional vector spaces
over a number field $F$ with linear transformations as arrows. (A skeleton of a
category $\mathcal{C}$ is any full subcategory $\mathcal{C}'$ of $\mathcal{C}$ such that each object of $\mathcal{C}$ is isomorphic
in $\mathcal{C}$ to exactly one object of $\mathcal{C}'$; any two skeletons of $\mathcal{C}$ are isomorphic categories,
so that, up to isomorphism, we may speak of the skeleton of $\mathcal{C}$.)

More precisely, the objects of the category $\text{Mat}_F$ are natural numbers (the
dimensions of our vector spaces), an arrow $M: m \to n$ is an $n \times m$ matrix,
composition of arrows $\circ$ is matrix multiplication, and the identity arrow $1_n : n \to n$ is the identity matrix (i.e. the $n \times n$ matrix with 1 on the diagonal and
0 elsewhere). If either $m$ or $n$ is 0, then there is only one matrix $M: m \to n$,
called the empty matrix, which is the empty map $\emptyset$ from $\emptyset$ to $F$ indexed by
$(m, n)$.

Consider the functor $\otimes$ from $\text{Mat}_F \times \text{Mat}_F$ to $\text{Mat}_F$ that is product on
objects and that on arrows, i.e. matrices, is the Kronecker product (see [11],
Chapter VII.5, pp. 211-213). Let the special object $I$ in $\text{Mat}_F$ be the number 1. With $\otimes$ and $I$ the category $\text{Mat}_F$ is a strictly monoidal category.

Let $S_{m,n}$ be the $nm \times mn$ matrix that for $1 \leq i \leq n$ and $1 \leq j \leq m$ has the
entries

$$S_{m,n}((i-1)m+j, k) = \delta(k, (j-1)n+i),$$

36
where $\delta$ is the Kronecker delta. If either $m$ or $n$ is 0, then $S_{m,n}$ is the empty matrix $1_0$. For example, $S_{3,2}$ is the matrix

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

Then we can check that with $\otimes$, $I$ and $S_{m,n}$ so defined, $\text{Mat}_F$ is a symmetric monoidal category.

Let $E_{m,1}$ be the $1 \times m^2$ matrix that for $1 \leq i, j \leq m$ has the entries $E_{m,1}(1, (i-1)m+j) = \delta(i,j)$. We define $E_{m,n}$ as the $n \times m^2$ matrix $E_{m,1} \otimes 1_n$. For example, $E_{2,2}$ is

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

If $n = 0$, then $E_{m,n}$ is $1_0$, and if $m = 0$, then $E_{m,n}$ is the empty matrix $M: 0 \to n$.

Let $H_{m,n}$ be the transpose of $E_{m,n}$. Taking that $E_{m,n}$ is $\varepsilon_{m,n}$ and $H_{m,n}$ is $\eta_{m,n}$, we can check that for every $m$ the functor $m \otimes -$ (where $m \otimes -$ on arrows is $1_m \otimes -$) is self-adjoint. Moreover, the equations $(\varepsilon 1)$ and $(\varepsilon 2)$ are satisfied.

So we can conclude that $\text{Mat}_F$ is a subsided category.

As a matter of fact, we can show that in $\text{Mat}_F$ we have as a subcategory an isomorphic copy of the skeleton $\mathcal{M}_s$ of the category $\mathcal{M}$ of the multiple symmetric self-adjunction freely generated by a single object $0$ and the index set $\mathbb{N}^+$. The objects of $\mathcal{M}_s$ may be identified with multisets of positive natural numbers.

We have a functor $F'$ from $\mathcal{M}_s$ to $\mathcal{M}$ (which are equivalent categories), and we have a functor $F''$ from $\mathcal{M}$ to $\text{Mat}_F$ such that $F''(0) = 1$, and, for $p_i$ being the $i$-th prime number, $F''(F_i A) = p_i F''(A)$. A functor $F$ from $\mathcal{M}_s$ to $\text{Mat}_F$ is obtained as the composite functor $F'' F'$. The functor $F$ establishes a one-to-one correspondence between the objects of $\mathcal{M}_s$ and the objects of $\text{Mat}_F$ with $0$ omitted.

That the functor $F$ is one-one on arrows is shown as follows. We verify first that $F''$ is a faithful functor. This is a consequence of the Maximality of $\mathcal{M}$ of the preceding section, as we will now show.

Suppose $t = u$ does not hold in $\mathcal{M}$. If we had $F''(t) = F''(u)$ in $\text{Mat}_F$, then by the Maximality of $\mathcal{M}$ we would have in $\text{Mat}_F$ that $F''(v) = F''(v')$ for $v$ and $v'$ as in the statement of this Maximality. The matrices

$$
F''(a_1 \ldots a_1 d_1 \ldots d_q b_1 \ldots b_r) \text{ and } F''(a'_{1'} \ldots a'_{1'} d'_{1'} \ldots d'_{q'} b'_{1'} \ldots b'_{r'})
$$

are not equal in $\text{Mat}_F$. This would contradict the faithfulness of $F''$. Hence $t = u$ in $\mathcal{M}$.
are 0-1 matrices, because \( F''(a_j) \) is a matrix with at most one 1 in each row, \( F''(b_j) \) is a matrix with at most one 1 in each column, and \( F''(d_j) \) is a matrix with at most one 1 in each row and each column. These matrices are not 0 matrices, because they all have 1 in the (1,1) entry. The matrix \( F''(e_i \ldots e_i) \) is different from \( F''(e_i' \ldots e_i') \), because for \( p_{i+1} \) the \( i+1 \)-th prime number \( F''(\varphi_B \cdot \gamma_B) = p_{i+1}1_{F''(B)} \). So \( F''(t) = F''(u) \) does not hold in \( \text{Mat}_F \), and hence \( F'' \) is faithful.

Since \( F' \) is faithful by definition, we have that the functor \( F \), which is \( F'' F' \), is faithful, and since it is one-one on objects, it is one-one on arrows.

That in \( \text{Mat}_F \) we have as a subcategory an isomorphic copy of the category \( K \) of the symmetric self-adjunction freely generated by a single object is a consequence of the fact that in \( \text{Mat}_F \) we have as a subcategory an isomorphic copy of \( M \), and of the fact that in \( M \) we have denumerably many isomorphic copies of \( K \). However, this can also be shown directly, and more easily, by relying on the Maximality of \( SK_\omega \) of Section 7. An alternative proof can be based on [7] and [8].

We just said that in \( M \) we have denumerably many isomorphic copies of \( K \); namely, there are denumerably many embeddings \( E_i \) of \( K \) in \( M \), such that \( E_i(FA) = F^i E_i(A) \). Conversely, there is also a functor \( E \) from \( M \) to \( K \) such that \( E(F^i A) = F E(A) \). This functor is not faithful because it maps \( \varphi_i \cdot \gamma_i \) and \( \varphi_j \cdot \gamma_j \) for \( i \neq j \) into \( \varphi_{E(A)} \cdot \gamma_{E(A)} \). If, however, the diagrams corresponding to \( f \) and \( g \) have no circular components, and \( E(f) = E(g) \) in \( K \), then \( f = g \) in \( M \).

A category analogous to \( \text{Mat}_F \) in which we would have an isomorphic copy of the category \( J \) would have as objects natural numbers and as arrows 0-1 matrices, whose composition is defined like composition of relations. This is like matrix multiplication together with \( 1+1 = 1 \).

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39
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