SYMPLECTICALLY HYPERBOLIC MANIFOLDS

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Abstract. A symplectic form is called hyperbolic if its pull-back to the universal cover is a differential of a bounded one-form. The present paper is concerned with the properties and constructions of manifolds admitting hyperbolic symplectic forms. The main results are:

- If a symplectic form represents a bounded cohomology class then it is hyperbolic.
- The symplectic hyperbolicity is equivalent to a certain isoperimetric inequality.
- The fundamental group of symplectically hyperbolic manifold is non-amenable.

We also construct hyperbolic symplectic forms on certain bundles and Lefschetz fibrations, discuss the dependence of the symplectic hyperbolicity on the fundamental group and discuss some properties of the group of symplectic diffeomorphisms of a symplectically hyperbolic manifold.

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1. Introduction and statements of the results

Let $\omega \in \Omega^k(M)$ be a closed differential form on a closed manifold $M$. Let $p : \tilde{M} \to M$ be the universal covering. Let $g$ be a Riemannian metric on $M$ and $\tilde{g}$ the induced metric on the universal cover.

Definition 1.1 (Gromov [5]). The form $\omega$ is called $\tilde{d}$-bounded if its pullback is a differential of a bounded form. That is $p^*\omega = d\alpha$ and there exist a constant $C \in \mathbb{R}$ such that $\sup_{x \in M} |\alpha(x)| \leq C$. The norm of a differential form is defined by $|\omega(x)| := \max\{\omega(x)(X_1, \ldots, X_k) \mid |X_i| = 1\}$.

Proposition 1.2. The $\tilde{d}$-boundedness does not depend on the choice of a Riemannian metric on $M$. Moreover, if $\omega$ is $\tilde{d}$-bounded then so is $\omega + d\xi$.

Proof. Since $M$ is compact, the first statement is clear. If $p^*\omega = d\alpha$ then $p^*(\omega + d\xi) = d(\alpha + p^*(\xi))$. Again, due to the compactness of $M$, $\alpha$ is bounded if and only if $\alpha + p^*(\xi)$ is bounded. □

Definition 1.3. Let $(M, \omega)$ be a closed symplectic manifold. If the symplectic form is $\tilde{d}$-bounded then it is called hyperbolic. A manifold $(M, \omega)$ is then called symplectically hyperbolic.

Aim. The present paper provides some constructions of symplectically hyperbolic manifold and investigates their geometric and topological properties.

History. The $\tilde{d}$-boundedness was defined by Gromov in [5], where he proved a version of the Lefchetz theorem for $L^2$-cohomology for a symplectically hyperbolic manifold $(M, \omega)$ which is Kähler. As a corollary he obtained that $(-1)^n \chi(M) > 0$, where $\dim M = 2n$. This is a particular case of a conjecture (attributed to Hopf) stating that if an even dimensional manifold $M$ is negatively curved then its Euler characteristic satisfies $(-1)^n \chi(M) > 0$.

Properties of $\tilde{d}$-bounded forms were also investigated by Sikorav [15], where he proved certain isoperimetric inequalities.

Polterovich in [14] proved a number of results about symplectic diffeomorphisms of symplectically hyperbolic manifolds. For example, he proved that there are strong restrictions on finitely generated groups which admit a Hamiltonian representation on a symplectically hyperbolic manifolds. For example $\text{SL}(n, \mathbb{Z})$ does not admit any such representation. A Hamiltonian representation of a group $G$ on a symplectic manifold $(M, \omega)$ is a homomorphism $G \to \text{Ham}(M, \omega)$.

Basic examples and properties.
Example 1.4. Let \((\Sigma, \omega)\) be a closed surface of genus at least 2. Let \(g\) be a hyperbolic metric (i.e. the sectional curvature equal to \(-1\)). The universal cover is the hyperbolic plane \(H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}\). Let \(\omega\) be chosen so that the induced form on the universal cover is equal to \(\frac{1}{y^2} \cdot dy \wedge dx\). We have that \(\bar{\omega} = d(\frac{dx}{y})\). The hyperbolic metric on \(H\) is given by \(\frac{1}{y^2} (dx^2 + dy^2)\) so we calculate

\[
\left| \frac{dx}{y} \right| = \max \left\{ \frac{dx}{y} \left( a \partial_x + b \partial_y \right) \mid a^2 + b^2 = y^2 \right\} = \max \left\{ \frac{a}{y} \mid a^2 + b^2 = y^2 \right\} = 1
\]

for any point \((x, y) \in H\).

Example 1.5. The product of symplectically hyperbolic manifolds is symplectically hyperbolic. Hence, it follows from the previous example that the product form on a product of surfaces \(\Sigma_1 \times \cdots \times \Sigma_k\) of genus at least 2 is hyperbolic.

Since such a product contains a lagrangian torus the product symplectic form can be slightly perturbed to a form which does not vanish on this torus. Thus it is not hyperbolic anymore, due to Proposition 1.9 (cf. Example 1.14).

Example 1.6. Let \((M, \omega)\) be a symplectically hyperbolic and let \(f : S \to M\) be a symplectic immersion. Then \(f^* \omega\) is hyperbolic. We have the diagram of the universal covers.

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{f} & \tilde{M} \\
\downarrow^{p_S} & & \downarrow^{p_M} \\
S & \xrightarrow{f} & M
\end{array}
\]

Let \(p^*_M \omega = d\alpha\), where \(\alpha\) is a bounded one-form. Let \(S\) be equipped with the Riemannian metric induced form \(M\). Then we have that \(p^*_S f^* \omega = \bar{f}^* p^*_M \omega = \bar{f}^* d\alpha = d(\bar{f}^* \alpha)\) and \(\bar{f}^* \alpha\) is bounded with respect to the above mentioned metric.

Example 1.7. Let \(f : (M, \omega) \to (W, \omega_W)\) be a smooth map between symplectic manifolds such that \(f^* [\omega_W] = [\omega]\). If \(\omega_W\) is hyperbolic then so is \(\omega\), according to Proposition 1.2. In particular, branched covers of symplectically hyperbolic manifolds are symplectically hyperbolic (see Gompf [3] for more detailed examples and calculations).

Example 1.8. The torus \(T^n\) equipped with the standard symplectic form is not symplectically hyperbolic. Let \(g\) be the standard flat metric. The universal cover is the standard flat \(\mathbb{R}^{2n}\). Let \(B(r) \subset \mathbb{R}^{2n}\)
denote the ball of radius \( r \). For any primitive \( \alpha \) we calculate
\[
\frac{(2\pi)^n}{2 \cdot 4 \cdots (2n)} \cdot r^{2n} = \text{vol}(B(r)) = \int_{B(r)} \tilde{\omega} = \int_{\partial B(r)} \alpha \\
\leq \text{vol}(\partial B(r)) \cdot \sup_{x \in B(r)} |\alpha(x)| \\
= \frac{(2\pi)^n}{2 \cdot 4 \cdots (2n - 2)} \cdot r^{2n-1} \cdot \sup_{x \in B(r)} |\alpha(x)|
\]
Hence we get that
\[
\sup_{x \in B(r)} |\alpha(x)| \geq \frac{r}{2n}
\]
which proves that every primitive of \( \tilde{\omega} \) is unbounded.

**Proposition 1.9.** If \( \omega \) is a hyperbolic symplectic form then
\[
\int_{f(\Sigma)} \omega = 0
\]
for any map \( f : \Sigma \to M \), where \( \Sigma \) is either a sphere or a torus. In particular, the same statement holds if \( [\omega] \) is bounded.

**Proof.** If \( \omega \) did not vanish on a sphere \( f : S^2 \to M \) then the induced form \( \tilde{\omega} \) would be non-zero in the cohomology of the universal cover. This contradicts the definition.

Let \( f : T^2 \to M \) be any smooth map. Since the pull-back \( f^* [\omega] \neq 0 \), the standard symplectic form is hyperbolic which contradicts Example \[1.8\] \( \square \)

**Question 1.10.** Suppose that a symplectic form vanishes on spheres and tori. Is it hyperbolic?

**A sufficient condition for symplectic hyperbolicity.** A cochain \( \alpha \in C^*(X) \) on a topological space \( X \) is called **bounded** if there exist a constant \( C > 0 \) such that \( \langle \alpha, s \rangle < C \) for every singular simplex \( s : \Delta \to X \). A cohomology class is called **bounded** if it is represented by a bounded cochain (see Gromov [6]). The following lemma is a particular case of Theorem 2.1 proved in Section 2.

**Lemma 1.11.** If a symplectic form represents a bounded cohomology class then it is hyperbolic. \( \square \)

Many of the results of the present paper use this lemma. That is, we construct symplectic forms which represent bounded cohomology classes. It is known that a cohomology class of degree at least two of a non-elementary hyperbolic group is bounded. Hence we get the following easy application of the above lemma.
Corollary 1.12. Let \((M, \omega)\) be a closed symplectic manifold. If \([\omega]\) is aspherical and \(\pi_1(M)\) is hyperbolic then \(\omega\) is hyperbolic. In particular, if \(M\) admits a Riemannian metric of negative sectional curvature then \(\omega\) is hyperbolic.

\(\square\)

Remark 1.13.

1. A cohomology class \(\alpha \in H^k(X)\) is called aspherical if \(\langle \alpha, f_*[S^k]\rangle = 0\) for any continuous map \(f : S^k \to X\). It is equivalent to the fact that \(\alpha = c_X^*(\Omega)\), where \(c_X : X \to K(\pi_1(X), 1)\) is the classifying map (see [8, 9] for more results about the topology of manifolds admitting aspherical symplectic forms).

2. A hyperbolic group is called non-elementary if it does not contain a finite index cyclic group. Since the fundamental group of a symplectic manifold with an aspherical symplectic form is of virtual cohomological dimension at least two, \(\pi_1(M)\) in the above corollary is automatically non-elementary.

3. There are no examples of closed manifolds of constant negative sectional curvature admitting a symplectic form. It is conjectured that such manifolds do not exist [10].

4. I do not know any example of a symplectic manifold \((M, \omega)\) such that \(\omega\) is hyperbolic and \([\omega]\) is not represented by a bounded cocycle.

Example 1.14. Let \(\mathbb{CH}^n\) be a complex hyperbolic space. Its sectional curvature is negative and pinched between \(-1\) and \(-1/4\). The isometry group if isomorphic to \(PU(n, 1)\) [2]. Thus any cocompact lattice in \(PU(n, 1)\) gives rise to a negatively curved closed Kähler manifold. We get this way aspherical symplectically hyperbolic manifolds whose fundamental groups are hyperbolic and have trivial the first Betti number [11]. Notice that any symplectic form is hyperbolic in this case according to Corollary 1.12 (cf. Example 1.5).

Question 1.15. What is the relation between bounded and \(\tilde{d}\)-bounded cohomology classes on a closed manifold?

Linear isoperimetric inequality. We characterize symplectically hyperbolic manifolds by a geometric condition inspired by a characterization of hyperbolic groups by certain isoperimetric inequality. More precisely, let \((M, \omega)\) be a symplectic manifold and \(g\) a Riemannian metric on \(M\). We say that the symplectic form \(\omega\) satisfies the linear isoperimetric inequality (with respect to \(g\)) if there exists a
constant $C > 0$ such that for all $f : D^2 \to M$ we have
\[
\int_{f(D^2)} \omega \leq C \cdot \text{Length}(f(\partial D^2)).
\]

**Theorem 1.16.** Let $(M, \omega)$ be a closed symplectic manifold. The symplectic form $\omega$ satisfies a linear isoperimetric inequality if and only if it is hyperbolic.

**Fundamental group.** In Section 5 we prove that symplectic hyperbolicity depends (in an appropriate sense) on the fundamental group only. More precisely, let $(M, \omega)$ and $(W, \omega_W)$ be closed symplectic manifolds with isomorphic fundamental groups. If $[\omega] = c_M^*(\Omega)$ and $[\omega_W] = c_W^*(\Omega)$ for a class $\Omega \in H^2(\pi_1(M); \mathbb{R})$ then $\omega$ is hyperbolic if and only if so is $\omega_W$. Here, $c_X : X \to K(\pi_1(X), 1)$ denotes the map classifying the universal cover.

Recall that the space $K(G,1)$ is usually not a manifold so the $d$-boundedness cannot be directly generalized. The above result shows that the notion of symplectic hyperbolicity makes sense for classes $\Omega \in H^2(G, \mathbb{R})$, where $G$ is finitely presented group (cf. the discussion in [5]).

**Properties of the fundamental group.** In Section 6 we prove that the fundamental group of a symplectically hyperbolic manifold is non-amenable. In particular, it has exponential growth. We give a direct proof of the last statement since it is essentially easier than the proof of non-amenability which relies on a nontrivial result of Gromov. Both results, however, follows from $d$-boundedness of the volume form rather than symplectic hyperbolicity.

**Groups of diffeomorphisms.** In Section 7 we prove that the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of symplectically hyperbolic $(M, \omega)$ is homotopy equivalent to $\text{Symp}_0(M, \omega)$, the component of the identity of the group of all symplectomorphisms.

**Further examples.**

1. Let $F$ be a closed surface of genus at least two. An oriented surface bundle $F \to E \to B$ over a symplectically hyperbolic manifold admits a hyperbolic symplectic form (see Corollary 2.2).
2. If $F \to E \to B$ is a flat symplectic bundle, where the base and the fibre are symplectically hyperbolic then the total space admits a hyperbolic symplectic form (see Theorem 3.2).
3. Certain Lefschetz fibrations admits hyperbolic symplectic forms. We give a precise statement and construction in Section 3.3.
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2. Bounded classes are $\tilde{d}$-bounded

Theorem 2.1. Let $M$ be a closed manifold. Let $\omega \in \Omega^k(M)$ be a closed differential form such that its cohomology class $[\omega] \in H^k(M; \mathbb{R})$ is represented by a bounded cochain $c \in C^k(M, \mathbb{R})$. Then $\omega$ is $\tilde{d}$-bounded.

Proof. Let $p : \tilde{M} \to M$ be the universal cover. We shall show that $p^*(\omega) = d\alpha$, where $\alpha$ is a form bounded with respect to the metric $\tilde{g}$ induced from a metric $g$ on $M$.

Let $c \in C^k(M, \mathbb{R})$ be a bounded singular cocycle representing $[\omega]$. Since $\tilde{M}$ is simply connected, $p^*(c) = \delta(b)$ for some bounded real cochain $b \in C^{k-1}(M, \mathbb{R})$. Here we use the fact that if the fundamental group of a space $X$ is amenable (e.g. solvable) then the bounded cohomology of $X$ is trivial in positive degrees [6]. Let us denote the bounding constant by $C \in \mathbb{R}$.

Let $K$ be a smooth triangulation of $M$ and $K'$ the induced triangulation of the universal cover $\tilde{M}$. Let $c' \in C^k(K, \mathbb{R})$ and $b' \in C^{k-1}(K', \mathbb{R})$ be simplicial cochains induced by $c$ and $b$ respectively. We have that $\delta(b') = p^*(c')$ and $b'$ is bounded. That is $\langle b', S \rangle < C$ for any simplex $S \in K'$. Let $\psi_S$ be a cochain which attains the value $\pm 1$ on $\pm S$ and zero on other simplices. We can express the cochain $b'$ as a sum $b' = \sum_S r_S \psi_S$, where $S$ ranges over $(k-1)$-simplices of $K'$ and $|r_S| < C$.

Let $\Phi_{K'} : C^*(K') \to \Omega^*(M)$ be a chain map which is a right inverse to the integration over the simplicial chains (see pages 148-149 in Singer-Thorpe [16] for the definition and details). It also has the following property (see Lemma 1 (4) on page 148 in [16]):

If $S$ is an oriented simplex of the triangulation $K'$ then $\Phi_{K'}(\psi_S)$ is supported in the star of $S$.

The construction of $\Phi_{K'}$ depends on the choice of a partition of unity of $\tilde{M}$ associated with the triangulation $K'$. We take this partition of unity to be induced from the partition of unity of $M$ used to define $\Phi_K : C^*(K) \to \Omega^*(M)$. Hence it follows (from the construction of $\Phi_K$) that if the triangulation $K$ of $M$ is fine enough then the maps $\Phi_K$ commute with the deck transormations. That is $\Phi_{K'}(h^*(\psi_S)) = h^*(\Phi_K(\psi_S))$, where $h : \tilde{M} \to \tilde{M}$ is a deck transformation. In particular, we get the following

$$p^* \circ \Phi_K = \Phi_{K'} \circ p^*.$$
Next we show that the form $\Phi_{K'}(b') = \sum S r S \Phi_{K'}(\psi_S)$ is bounded with respect to $\tilde{g}$. First notice that the forms $\Phi_{K'}(\psi_S)$ are uniformly bounded. To see this fix a fundamental domain in $\tilde{M}$ for the action of $\pi_1(M)$. For any $S$ there exists $h \in \pi_1(M)$ such that $h'\psi_S = \psi_F$, where $F$ is a simplex in the fundamental domain. Now the uniform boundedness follows from the fact that $\pi_1(M)$ acts on $(\tilde{M}, \tilde{g})$ by isometries. Then, since $|r_S| < C$, we get that the differential form $\Phi_{K'}(b') = \sum S r S \Phi_{K'}(\psi_S)$ is bounded with respect to $\tilde{g}$ as claimed.

We also claim that $d(\Phi_{K'}(b')) = p^*(\omega + d\beta)$, for some form $\beta \in \Omega^{k-1}(M)$. It is the following calculation.

\[
 d(\Phi_{K'}(b')) = \Phi_{K'}(\delta(b')) = \Phi_{K'}(p'(c')) = p'(\Phi_{K'}(c')) = p^*(\omega + d\beta).
\]

Finally, we have that $p^*(\omega) = d(\Phi_{K'}(b') - p^*(\beta))$ and $\alpha := \Phi_{K'}(b') - d(p^*(\beta))$ is clearly bounded.

\[\square\]

**Corollary 2.2.** Let $F$ be a closed oriented surface of genus at least 2 and let $(B, \omega_B)$ be a symplectically hyperbolic manifold. Then an oriented bundle $F \to M \to B$ admits a hyperbolic symplectic form.

**Proof.** Let $\omega_F$ be an area form on $F$. The Thurston construction (Theorem 6.3 in [12]) gives a symplectic form in the class $C \cdot p^* [\omega_B] + \Omega$, where $\Omega$ is any class in $H^2(M)$ such that $i^*(\Omega) = [\omega_F]$ and $C > 0$ is a constant large enough.

Let $\Omega$ be a constant multiple of the Euler class of the bundle $V := \ker dp \to M$ tangent to the fibers of $p$. According to Morita [13] this class is bounded and so is the class $p^*[\omega_B] + \Omega$. \[\square\]

3. Symplectic bundles and Lefschetz fibrations

In this section we give simple constructions of hyperbolic symplectic forms on certain bundles and Lefschetz fibrations.

3.1. Flat symplectic bundles. Recall that a symplectic form on the total space of a symplectic bundle is called compatible if its pull-back to each fibre is the symplectic form.

**Theorem 3.2.** Let $(M, \omega) \to E \to B$ be a symplectic flat bundle. If $\omega$ and $\omega_B$ are hyperbolic then $E$ admits a hyperbolic compatible symplectic form.

**Proof.** Since the bundle is flat the total space is of the form $E = (B \times M)/\pi_1(B)$, where the fundamental group of the base acts diagonally. The form $\pi^*_B(\omega_B) + \pi^*_M(\omega)$ is invariant under this action and it descend to a symplectic form $\omega_E$ on $E$. Hence on the universal
cover \( \tilde{E} \) the induced form \( \tilde{\omega}_E = \pi_B^*(\tilde{\omega}_B) + \pi_M^*(\tilde{\omega}) \) is clearly a differential of a bounded one-form. \( \square \)

### 3.3. Lefschetz fibrations

In this section we construct a hyperbolic symplectic form on certain Lefshetz fibrations. Let \( p : E \to B \) be a 4-dimensional Lefschetz fibration. According to Gompf and Thurston \([4, \text{Theorem 10.2.18}]\), if there exists \( \Omega \in H^2(E) \) such that it restricts to a non-trivial class of the fibre then the class \( \Omega + C \cdot p^*[\omega_B] \) admits a symplectic representative for \( C \in \mathbb{R} \) big enough. Here \( \omega_B \) is the symplectic form on the base. Thus if \( \Omega \) is \( \tilde{d} \)-bounded and the base is of genus at least two then the symplectic form is hyperbolic. In the next theorem we show that we can construct such forms with some control of the fundamental group. Let \( \Pi_g \) denote the fundamental group of a closed surface of genus \( g \).

#### Theorem 3.4
Let \( \Omega \in H^2(\Gamma; \mathbb{R}) \) be a bounded class. For any \( g \geq 2 \) there exists a 4-dimensional closed symplectic manifold \( (M, \omega) \) with \( \pi_1(M) = \Gamma \oplus \Pi_g \) such that the symplectic form is hyperbolic.

**Proof.** The construction is the same as in Kędra-Rudyak-Tralle \([9]\) and relies on the construction of Amoros et al \([1]\). First, we construct a symplectic Lefschetz fibration \( F \to X \to S^2 \) such that:

1. \( \pi_1(X) = \Gamma \).
2. The pull-back \( c^*(\Omega) \) restricts nontrivially to the fibre \( F \); here \( c : X \to K(\Gamma, 1) \) is the classifying map.
3. The fibration has a section.

A construction of such Lefschetz fibration is provided in \([1]\).

Next take \( F \times \Sigma_g \) equipped with the product symplectic structure. Let \( M := X \#_g (F \times \Sigma_g) \) be the symplectic fibre sum. Let \( r : M \to X \) be a retraction. The cohomology class \( r^*(c^*(\Omega)) \) restricts nontrivially to the fibers hence the Gompf-Thurston construction gives a symplectic form \( \omega \) in the class \( r^*(c^*(\Omega)) + C \cdot p^*[\omega_g] \). This class is bounded hence the symplectic form \( \omega \) is hyperbolic. The calculation of the fundamental group is a direct application of the van-Kampen theorem. \( \square \)

#### Corollary 3.5
If \( \Gamma \) is a hyperbolic group with nontrivial \( H^2(\Gamma; \mathbb{R}) \) then for any \( g \geq 2 \) there exists a closed symplectically hyperbolic manifold \( (M, \omega) \) with \( \pi_1(M) = \Gamma \oplus \Pi_g \). \( \square \)

### 4. Isoperimetric inequality for symplectic forms

Let \( M \) be a manifold, \( g \) a Riemannian metric and \( \omega \in \Omega^2(M) \) a closed differential two-form. We say that the form \( \omega \) satisfies the
**linear isoperimetric inequality** (with respect to $g$) if there exists a constant $C > 0$ such that for every smooth map $f : D^2 \to M$ we have

$$\left| \int_{f(D^2)} \omega \right| \leq C \cdot \text{Length}(f(\partial D^2)).$$

**Theorem 4.1.** Let $M$ be a closed manifold. A closed 2-form $\omega$ satisfies the linear isoperimetric inequality if and only if it is $\tilde{d}$-bounded.

*Proof.* Suppose that $\omega$ is $\tilde{d}$-bounded. Let $f : D^2 \to M$ be a smooth map. It admits a lift $\tilde{f} : D^2 \to \tilde{M}$ to the universal cover $p : \tilde{M} \to M$. We calculate:

$$\left| \int_{D^2} f^* \omega \right| = \left| \int_{D^2} \tilde{f}^*(p^* \omega) \right| = \left| \int_{D^2} \tilde{f}^* d\alpha \right|
= \left| \int_{S^1} \tilde{f}^* \alpha \right| \leq C \cdot \text{Length}(\tilde{f}(\partial D^2))
= C \cdot \text{Length}(f(\partial D^2)).$$

Here, the constant $C$ bounds from above the norm of the primitive $\alpha$, that is $\|\alpha(x)\| \leq C$ for any $x \in \tilde{M}$. Thus $\omega$ satisfies an isoperimetric inequality.

The converse is a direct application of the following result due to Sikorav (Theorem 1.1 in [15]).

**Theorem 4.2** (Sikorav). Let $M$ be a Riemannian manifold with a triangulation of bounded geometry. Let $\omega \in \Omega^q(M)$ be a closed form, and let $f \in C^0(M, \mathbb{R}_+)$ be such that

$$\left| \int_T \omega \right| \leq \int_{\partial T} f$$
for every simplicial chain $T \in C_q(K)$. Then $\omega$ has a primitive $\alpha$ such that

$$\alpha(x) \leq C_1 \max_{B(x, C_2)} (|\alpha| + f),$$
for some constants $C_1, C_2$. $\square$

**Remark 4.3.** We explain the notions in the formulation of the Sikorav theorem (see [15] for more details).

1. The **triangulation of bounded geometry** is a triangulation which satisfies the following two properties:
   a. There exists a number $s \in \mathbb{N}$ such that the link of every simplex contains at most $s$ simplices.
(b) There exists a number \( l > 0 \) such that for any \( q \)-simplex \( S \) of the triangulation there exist a diffeomorphism \( \psi_S : S \to \Delta^q \) such that the norm of the differential of \( \psi_S^{-1} \) is bounded by \( l \), \( |d\psi_S^{-1}| < l \). Moreover, \( \psi \) can be extended (with the norm condition preserved) to a neighborhood of \( S \) sending it to a fixed neighborhood of \( \Delta^q \subset \mathbb{R}^n \), \( n = \dim M \).

Clearly, the triangulation of the universal cover of a closed manifold has bounded geometry.

(2) If \( S = \sum a_i \sigma_i \) is a \( q \)-chain, then
\[
\int |S| f := \sum_i |a_i| \int_{\Delta^q} \sigma_i^*(f \cdot \text{vol}_q),
\]
where \( \text{vol}_q \) is the \( q \)-dimensional volume induced by the Riemannian metric.

Now we can get back to the proof of Theorem 4.1. First observe that the isoperimetric inequality immediately implies the asphericity of \( \omega \). Moreover, the universal cover \((\tilde{M}, \tilde{\omega})\) also satisfies the isoperimetric inequality with the same constant. We shall show that this implies the hypothesis of the Sikorav theorem.

Let \( K' \) be a smooth triangulation of \( M \). Then the triangulation \( K \) induced on the universal cover \( \tilde{M} \) has bounded geometry. Let \( T \in C_2(K) \) be a simplicial chain. Its boundary is a cycle so it is a sum of loops \( \partial T = \sum \gamma_i \). According to the simple connectivity of \( \tilde{M} \) there exists a chain \( T_i \) such that \( \partial T_i = \gamma_i \) and \( T_i \) is the image of a triangulation of the disc \( D^2 \). Since the sum \( T - \sum T_i \) is a cycle its symplectic area is zero, \( \int_{T - \sum T_i} \tilde{\omega} \). This is true because \( \tilde{M} \) is simply connected so any homology class of degree two is represented by a sum of spheres and \( \tilde{\omega} \) is aspherical.

In other words, we have \( \int_{\sum T_i} \tilde{\omega} = \int_T \tilde{\omega} \) and we calculate
\[
\left| \int_T \tilde{\omega} \right| = \left| \int_{\sum T_i} \tilde{\omega} \right| \leq \sum \left| \int_{T_i} \tilde{\omega} \right| \leq \sum C \cdot \text{Length}(\gamma_i) = \int_{\partial \sum T_i} C = \int_{\partial T} C
\]
Now according to Sikorav's theorem there exists a primitive \( \alpha \) such that
\[
|\alpha(x)| \leq C_1 \max_{B(x,C_2)} (|\omega| + C).
\]
That is \( \alpha \) is bounded since \( \omega \) is bounded.
5. Symplectic hyperbolicity and the fundamental group

In this section we prove that the symplectic hyperbolicity depends in a sense on the fundamental group only. More precisely, we have the following result.

**Theorem 5.1.** Let \((M, \omega)\) be a closed symplectically hyperbolic manifold with \([\omega] = f_M^*(\Omega)\), where \(\Omega \in H^2(\pi_1(M); \mathbb{R})\). If \((W, \omega_W)\) is a symplectic manifold with \(\pi_1(W) = \pi_1(M)\) and \([\omega_W] = f_W^*(\Omega)\) then \((W, \omega_W)\) is symplectically hyperbolic.

**Proof.** Let \(K_n\) denotes the \(n\)-skeleton of \(K(\pi_1(M), 1)\).

**Lemma 5.2.** There exists a manifold \(M_n\) such that \(K_n \subset M_n \hookrightarrow K(\pi_1(M), 1)\) and \(i^*(\Omega)\) is hyperbolic.

The classifying map \(f_W : W \to K(\pi_1(M), 1)\) factorizes through some finite dimensional skeleton \(K_n\) hence through \(M_n\) as well. The hyperbolicity of \([\omega_W]\) follows since \(i^*(\Omega)\) is hyperbolic.

**Proof of Lemma 5.2.** Regard \(K(\pi_1(M), 1)\) constructed from \(M\) by attaching cells. We want to perform this construction so that at every stage it is a manifold. This is done as follows. Let \(\sigma : S^k \to M\) represent a homotopy class \([\sigma] \in \pi_k(M)\) we are going to kill. Take a product \(M \times D^m\) so that \([\sigma]\) is represented by an embedding \(s : S^k \to M \times S^{m-1} = \partial(M \times D^m)\). Then attach a handle \(D^{k+1} \times D^{2n+m-k}\) along \(s\) (after choosing any framing of \(s\)). Call the resulting manifold \(M_\sigma\). The classifying map \(f_M : M \to K(\pi_1(M), 1)\) clearly extends to \(M_\sigma\). We shall show that the class \(f_{M_\sigma}^*(\Omega)\) is hyperbolic.

Denote by \(\omega \in \Omega^2(M \times D^m)\) the pull-back of the symplectic form on \(M\) under the projection. This form extends to a form \(\omega_\sigma\) on \(M_\sigma\) representing \(f_{M_\sigma}^*(\Omega)\). Choose a Riemannian metric \(g\) on \(M \times D^m\) so that \(\tilde{\omega} = da\) on the universal covering \(\tilde{M} \times D^m\) and \(a\) is bounded with respect the the metric induced from \(g\). Extend the metric \(g\) to \(g_\sigma\) on \(M_\sigma\). We need to show that the induced form \(\tilde{\omega}_\sigma\) on the universal cover of \(M_\sigma\) is a differential of a bounded one-form.

We have that \(\tilde{\omega}_\sigma = dB\) for some one-form \(B\). The universal cover \(\tilde{M}_\sigma\) is \(\tilde{M} \times D^m\) with infinitely handles \(H_\gamma := D^k \times D^{2n+m-k}\) attached. The handles correspond to elements \(\gamma\) of the fundamental group \(\pi_1(M)\). The induced form \(\tilde{\omega}_\sigma\) is the same when restricted to every handle.
Choose its primitive \( \tilde{\alpha} \). It is bounded due to the compactness of the handle.

Let \((R, S)\), where \( R \in [0, 1] \) and \( S \in S^k \) be polar coordinates on \( D^{k+1} \).

For \( R \in [1 - \varepsilon, 1] \) we have \( \tilde{\alpha}_R = d\alpha = d\tilde{\alpha} \) on the handle \( H_\gamma \). In this neighborhood we take the convex combination (in the affine space \( \alpha + \ker d \)) interpolating between them. This operation preserves the boundedness. Hence we obtain a bounded primitive of \( \tilde{\omega} \).

The rest of the proof is to apply the induction on the handles attached in order to obtain \( K(\pi_1(M), 1) \) from \( M \). This finishes the proof of Lemma 5.2.

Recall that a \( \pi_1 \)-cobordism between manifolds \( M_1 \) and \( M_2 \) is a cobordism \( W \) such that the inclusions induce isomorphisms on the fundamental group.

**Corollary 5.3.** Let \((M_1, \omega_1)\) and \((M_2, \omega_2)\) be closed symplectic manifolds and \( W \) a \( \pi_1 \)-cobordism between them. Suppose that there exist \( \Omega \in H^2(W; \mathbb{R}) \) such that it pulls back to the classes of the symplectic forms under the inclusions \( \iota_i : M_i \to W \). Then \( \omega_1 \) is hyperbolic if and only if \( \omega_2 \) is hyperbolic.

6. Properties of the fundamental group of a symplectically hyperbolic manifold

6.1. Infiniteness.

**Proposition 6.2.** Let \((M, \omega)\) be a closed symplectically hyperbolic manifold. Then

1. The fundamental group of a \((M, \omega)\) is infinite.
2. There exists \( \Omega \in H^2(K(\pi_1(M)); \mathbb{R}) \) such that \( c_M^*(\Omega) = [\omega] \), where \( c_M : M \to K(\pi_1(M), 1) \) is the classifying map.
3. If \( h : \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(M) \) is a homomorphism then \( h^* \Omega = 0 \)

**Proof.**

1. If \( \pi_1(M) \) were finite the the universal cover \( \tilde{M} \) would be closed and hence the induced symplectic form would not vanish on some sphere. The same would hold for \( \omega \) which contradicts Proposition 1.9.
2. Consider the homotopy fibration

\[
\tilde{M} \to M \to K(\pi_1(M), 1)
\]

and the associated spectral sequence. Since \( \tilde{M} \) is simply connected and \( [\omega] = 0 \), we get that \([\omega] = c_M^*(\Omega) \) for some \( \Omega \in H^2(K(\pi_1(M), 1), \mathbb{R}) \).
Suppose that \( h : \mathbb{Z}^2 \to \pi_1(M) \) is a homomorphism. It induces a continuous map \( h' : T^2 = K(\mathbb{Z}^2, 1) \to K(\pi_1(M), 1) \). Since \( \tilde{M} \) is simply connected and \( \dim T^2 = 2 \), the map \( h' \) admits a lift to \( H : T^2 \to M \). We have that \( c_M \circ H = h' \). Due to Proposition 1.9, we know that the symplectic form vanishes on tori. Thus \( 0 = H^*[\omega] = H^*(c^*_M \Omega) = h^*(\Omega) \) which finishes the proof.

\[ \square \]

**Remark 6.3.** The first two statements hold for aspherical symplectic forms.

### 6.4. Exponential growth.

**Proposition 6.5.** Let \((M, \omega)\) be closed symplectic manifold. If the symplectic form \( \omega \) is hyperbolic then the fundamental group \( \pi_1(M) \) has exponential growth.

**Proof.** It is known that the fundamental group of a manifold has exponential growth if and only if balls in the universal cover grow exponentially with respect to the radius. We shall prove the latter.

Let \((\tilde{M}, \tilde{\omega})\) be the universal cover of \((M, \omega)\) and let \( \tilde{g} \) be the Riemannian metric induced from a metric \( g \) compatible with \( \omega \). Let \( \alpha \in \Omega^1(\tilde{M}) \) be a primitive of \( \tilde{\omega} \) and let \( X \) be the corresponding vector field. That is \( \iota_X \tilde{\omega} = \alpha \). Since \( \tilde{g} \) is compatible with \( \tilde{\omega} \), the norm of \( X \) is uniformly bounded by the constant bounding the norm of \( \alpha \). Let vol \( = \tilde{\omega}^n \) denote the volume form. We have that \( L_X \text{vol} = L_X \tilde{\omega}^n = n \cdot \tilde{\omega}^n = n \cdot \text{vol} \).

Let \( B := B(x, 1) \) denote the ball of radius 1 with center at \( x \in \tilde{M} \). Let \( \psi : \mathbb{R} \to \text{Diff}(M) \) be the flow corresponding to the vector field \( X \). We claim that the volume of the image \( \psi_t(B) \) grows exponentially with \( t \). This is the following calculation.

\[
\frac{d}{dt} \bigg|_{t=s} \text{vol}(\psi_t(B)) = \frac{d}{dt} \bigg|_{t=s} \int_{\psi_t(B)} \text{vol} \\
= \frac{d}{dt} \bigg|_{t=s} \int_B \psi_t^* \text{vol} \\
= \int_B \frac{d}{dt} \bigg|_{t=s} \psi_t^* \text{vol} \\
= \int_B \psi_s^*(L_X \text{vol}) \\
= \int_{\psi_s(B)} n \cdot \text{vol} = n \cdot \text{vol}(\psi_s(B))
\]

Hence we get that \( \text{vol}(\psi_t(B)) = e^{nt} \cdot \text{vol}(B) \).
On the other hand $\psi_t(B) \subset B(\psi_t(x), 2Ct + 1)$ which implies that $\text{vol}(B(\psi_t(x), 2Ct + 1)) \geq e^{nt}\text{vol}(B(x, 1))$. Since the deck transformations are isometries of $\tilde{\mathcal{g}}$, we get that

$$\text{vol}(B(\psi_t(x), 2Ct + \text{diam}(M) + 1)) \geq e^{nt}\text{vol}(B(x, 1)).$$

The diameter constant is added because the deck transormation might not move $\psi_t(x)$ to $x$. This proves that the volume of balls centered at $x$ grow exponentially with respect to the radius and hence it proves the exponential growth of $\pi_1(M)$. $\square$

6.6. Non-amenability.

**Theorem 6.7.** Let $(M, \omega)$ be a closed symplectically hyperbolic manifold. Then its fundamental group $\pi_1(M)$ is non-amenable.

**Proof.** Let $\alpha \in \Omega^1(\tilde{M})$ be a bounded primitive of $\tilde{\omega}$. Then the form $\alpha \wedge \tilde{\omega}^{n-1}$ is a bounded primitive of the volume form $\tilde{\omega}^n$. This implies that $(\tilde{M}, \tilde{\mathcal{g}})$ satisfies an isomerimetric inequality:

$$\int_X \tilde{\omega}^n = \int_{\partial X} \alpha \wedge \tilde{\omega}^{n-1} \leq C \cdot \text{vol}_{2n-1}(\partial X),$$

where $X \subset \tilde{M}$ is any domain and $\text{vol}_{2n-1}$ denotes the $2n-1$-dimensional volume induced by the Riemannian metric $\tilde{\mathcal{g}}$. It follows from Theorem 6.19 in [7] that $\pi_1(M)$ is not amenable. $\square$

**Example 6.8.** Let $(M, \omega) = G/\Gamma$ be a closed symplectic solvmanifold. That is it is a homogeneous space of a simply connected solvable Lie group $G$ and $\pi_1(M) = \Gamma \subset G$ is a lattice. Since $\Gamma$ is amenable, $\omega$ is not hyperbolic. If $\Gamma$ does not contain a nilpotent subgroup of finite index then it as exponential growth.

7. Groups of diffeomorphisms

Recall that the **flux group** $\Gamma_\omega$ is the image of the flux homomorphism

$$\text{Flux} : \pi_1(\text{Symp}_0(M, \omega)) \to H^1(M; \mathbb{R}).$$

It is defined by

$$\langle \text{Flux}[\xi_t], [A] \rangle = \int_{T^2} \xi_A^*\omega,$$

where $\xi_A(s, t) = (\xi_t)(A(s))$ and $\xi : S^1 \to \text{Symp}_0(M, \omega)$ is a loop based at the identity and $A : S^1 \to M$ is a 1-cycle in $M$ (see Section 10.2 in [12] for more details). Notice that if the flux homomorphism is non-trivial then the symplectic for does not vanish on some torus.
The groups of symplectic and Hamiltonian diffeomorphisms form the following extension (Theorem 10.18 in [12]).

\[ \text{Ham}(M, \omega) \to \text{Symp}_0(M, \omega) \to H^1(M; \mathbb{R})/\Gamma_{\omega} \]

The next proposition is an immediate consequence of the definition of the flux homomorphism.

**Proposition 7.1.** Let \((M, \omega)\) be a closed symplectic manifold such that the symplectic form vanishes on tori. Then \(\text{Symp}_0(M, \omega) \simeq \text{Ham}(M, \omega)\). In other words, the flux group of \((M, \omega)\) is trivial. In particular, the statement holds for symplectically hyperbolic \((M, \omega)\), due to Proposition 1.9. □

Polterovich proved (among others very interesting and beautiful results) in [14, 1.6.C] that if \(G \subset \text{Ham}(M, \omega)\) is a finitely generated subgroup and \((M, \omega)\) is symplectically aspherical then cyclic subgroups are undistorted in \(G\) with respect to the word metric. More precisely, if \(\text{Id} \neq g \in G\) then \(\|g^n\| \geq C \cdot |n|\) for some \(C > 0\), where \(\|g\|\) is the norm (the distance from the identity) given by the fixed finite set of generators. The property of being undistorted does not depend on the choice of a finite set of generators.

**Example 7.2.**

1. Any cyclic subgroup of a free or free abelian group is undistorted. Indeed, it is easy to check that \(\|g^n\| = n \cdot \|g\|\) in this case.
2. Torsion groups are not undistorted.
3. Let 1 < \(p < q\) be integers. The subgroup of the Baumslag-Solitar group \(G := \langle x, t \mid x^q = tx^p t^{-1} \rangle\) generated by \(x\) is not undistorted.
4. Let \(h : G \to \mathbb{Z}^k\) be a surjective homomorphism. If \(h(g) \neq 0\) then \(g\) generates an undistorted subgroup in \(G\). Indeed, let \(g_1, \ldots, g_n \in G\) be generators of \(G\) such that \(h(g_1), \ldots, h(g_n)\) are the standard generators of \(\mathbb{Z}^k\). With respect to this sets of generators, we have that \(\|h(g)\| \leq C \cdot \|g\|\) for some \(C > 0\). Since \(\|(h(g))^n\| = |n| \cdot \|h(g)\|\), we get that \(\|g^n\| \geq c \cdot |n|\), for some \(c > 0\).

The next proposition is a slight generalization of the above mentioned Polterovich’s result.

**Proposition 7.3.** Let \((M, \omega)\) be a symplectically hyperbolic manifold. If \(G \subset \text{Symp}_0(M, \omega)\) is a finitely generated subgroup then its cyclic subgroups are undistorted.
Proof. If \( G \subset \text{Symp}_0(M, \omega) \) is a subgroup then we have a morphism of extensions (the vertical arrows are inclusions):

\[
\begin{align*}
K & \longrightarrow \quad G \quad \longrightarrow \quad Q \\
\downarrow & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
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