An application of proof mining to nonlinear iterations

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Abstract
In this paper we apply methods of proof mining to obtain a highly uniform effective rate of asymptotic regularity for the Ishikawa iteration associated to nonexpansive self-mappings of convex subsets of a class of uniformly convex geodesic spaces. Moreover, we show that these results are guaranteed by a combination of logical metatheorems for classical and semi-intuitionistic systems.

1 Introduction

Proof mining is a paradigm of research concerned with the extraction of hidden finitary and combinatorial content from proofs that make use of highly infinitary principles. This new information is obtained after a logical analysis of the original mathematical proof, using proof-theoretic techniques called proof interpretations. In this way one obtains highly uniform effective bounds for results that are more general than the initial ones. While the methods used to obtain these new results come from mathematical logic, their proofs can be written in ordinary mathematics. We refer to Kohlenbach’s book [19] for a comprehensive reference for proof mining.

This line of research, developed by Kohlenbach in the 90’s, has its origins in Kreisel’s program of unwinding of proofs. Kreisel’s idea was to apply proof-theoretic techniques to analyze concrete mathematical proofs and unwind the information hidden in them; see for example [23] and, more recently, [28].

Proof mining has numerous applications to approximation theory, asymptotic behavior of nonlinear iterations, as well as (nonlinear) ergodic theory, topological dynamics and Ramsey theory. In these applications, Kohlenbach’s monotone functional interpretation [14] is crucially used, since it systematically transforms any statement in a given proof into a new version for which explicit bounds are provided.

Terence Tao [38] arrived at a proposal of so-called hard analysis (as opposed to soft analysis), inspired by the finitary arguments used by him and Green [10] in their proof that there are arithmetic progressions of arbitrary length in the prime numbers, as well as by him alone in a series of papers [37, 39, 40, 41]. As Kohlenbach points out in [17], Tao’s hard analysis could be viewed as carrying
out, using monotone functional interpretation, analysis on the level of uniform bounds.

For mathematical proofs based on classical logic, general logical metatheorems were obtained by Kohlenbach [16] for important classes of metrically bounded spaces in functional analysis and generalized to the unbounded case by Gerhardy and Kohlenbach [8]. They considered metric, hyperbolic and CAT(0)-spaces, (uniformly convex) normed spaces and inner product spaces also with abstract convex subsets. The metatheorems were adapted to Gromov $\delta$-hyperbolic spaces and $\mathbb{R}$-trees [24], complete metric and normed spaces [19] and uniformly smooth Banach spaces [20]. The proofs of the metatheorems are based on extensions to the new formal systems of Gödel’s functional interpretation combined with negative translation and parametrized versions of majorization. These logical metatheorems guarantee that one can extract effective uniform bounds from classical proofs of $\forall\exists$-sentences and that these bounds are independent from parameters satisfying weak local boundedness conditions. Thus, the metatheorems can be used to conclude the existence of effective uniform bounds without having to carry out the proof analysis: we have to verify only that the statement has the right logical form and that the proof can be formalized in our system.

Gerhardy and Kohlenbach [7] obtained similar logical metatheorems for proofs in semi-intuitionistic systems, that is proofs based on intuitionistic logic enriched with noneffective principles, such as comprehension in all types for arbitrary negated or $\exists$-free formulas. The proofs of these metatheorems use monotone modified realizability, a monotone version of Kreisel’s modified realizability [22]. A great benefit of this setting is that there are basically no restrictions on the logical complexity of mathematical theorems for which bounds can be extracted.

The goal of this paper is to present an application of proof mining to the asymptotic behavior of Ishikawa iterations for nonexpansive mappings.

Let $X$ be a normed space, $C \subseteq X$ a convex subset and $T : C \to C$. We shall denote with $Fix(T)$ the set of fixed points of $T$. The Ishikawa iteration starting with $x \in C$ was introduced in [13] as follows:

$$x_{0} = x, \quad x_{n+1} = (1 - \lambda_{n})x_{n} + \lambda_{n}T((1 - s_{n})x_{n} + s_{n}Tx_{n}),$$

where $(\lambda_{n}), (s_{n})$ are sequences in $[0, 1]$. The well-known Krasnoselski-Mann iteration [21, 29] is obtained as a special case by taking $s_{n} = 0$ for all $n \in \mathbb{N}$.

Ishikawa proved that for convex compact subsets $C$ of Hilbert spaces and Lipschitzian pseudocontractive mappings $T$, this iteration converges strongly towards a fixed point of $T$, provided that the sequences $(\lambda_{n})$ and $(s_{n})$ satisfy some assumptions.

In the following we consider the Ishikawa iteration for nonexpansive mappings and sequences $(\lambda_{n}), (s_{n})$ satisfying the following conditions:

$$\sum_{n=0}^{\infty} \lambda_{n}(1 - \lambda_{n}) \text{ diverges, } \limsup_{n \to \infty} s_{n} < 1 \text{ and } \sum_{n=0}^{\infty} s_{n}(1 - \lambda_{n}) \text{ converges.} \quad (1)$$

Tan and Xu [36] proved the weak convergence of the Ishikawa iteration in uniformly convex Banach spaces $X$ which satisfy Opial’s condition or whose norm is Fréchet differentiable, generalizing in this way a well-known result of Reich [32]
for the Krasnoselski-Mann iteration. Dhompongs and Panyanak [6] obtained the $\Delta$-convergence of the Ishikawa iteration in CAT(0) spaces. $\Delta$-convergence is a concept of weak convergence in metric spaces introduced by Lim [27].

One of the most important properties of any iteration associated to a non-linear mapping is asymptotic regularity, defined by Browder and Petryshyn [3] for the Picard iteration. The \textit{Ishikawa iteration} $(x_n)$ is said to be \textit{asymptotically regular} if \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \). A rate of convergence of $(\|x_n - Tx_n\|)$ towards 0 will be called a \textit{rate of asymptotic regularity} of $(x_n)$. Asymptotic regularity is the first property one gets before proving the weak or strong convergence of the iteration towards a fixed point of the mapping. Thus, the following asymptotic regularity result is implicit in the proof of Tan and Xu.

\textbf{Theorem 1.1.} Let $X$ be a uniformly convex Banach space, $C \subseteq X$ a convex subset and $T: C \to C$ be nonexpansive with $\text{Fix}(T) \neq \emptyset$. Assume that $(\lambda_n), (s_n)$ satisfy (1). Then \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \) for all $x \in C$.

In this paper we show that the proof of the generalization of Theorem 1.1 to a class of uniformly convex geodesic spaces (the so-called \textit{UCW-hyperbolic spaces}) can be analyzed using a combination of logical metatheorems for the classical and semi-intuitionistic setting. As we explain in Section 3, there are two main steps, the first one with a classical proof, analyzed using the combination of monotone functional interpretation and negative translation, while the second one has a constructive proof, analyzed more directly using monotone modified realizability.

As a consequence, the logical metatheorems guarantee that one can obtain a quantitative version for the generalization of Theorem 1.1 obtained by taking convex subsets of \textit{UCW-hyperbolic spaces} and by replacing the hypothesis of $T$ having fixed points with the weaker assumption that $T$ has approximate fixed points in a $b$-neighborhood of the starting point $x$ for some $b > 0$. In the last section of the paper (Theorem 4.1) we give a direct mathematical proof of this quantitative version, providing a uniform rate of asymptotic regularity for the Ishikawa iteration.

We point out that in [26] we computed, for the case when we assume that $T$ has fixed points, a rate of asymptotic regularity $\Phi$ for $(x_n)$. Applied to the analyzed proof in [26], the logical metatheorems guarantee a priori that the same rate $\Phi$ as in [26] holds when we assume only the existence of approximate fixed points instead. However, the rate we compute in Theorem 4.1 is slightly changed because we give a more readable mathematical proof of this result.

As we use both functional and modified realizability interpretations to give a logical explanation of our results, we think that an interesting direction of research could be to see if the hybrid functional interpretation [11, 30] can be used instead.

\textbf{Notation:} $\mathbb{N} = \{0, 1, 2, \ldots \}$ and $[m, n] = \{m, m + 1, \ldots, n - 1, n\}$ for any $m, n \in \mathbb{N}, m \leq n$. 

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2 Logical metatheorems for UCW-hyperbolic spaces

A W-hyperbolic space is a structure \((X, d, W)\), where \((X, d)\) is a metric space and \(W : X \times X \times [0, 1] \to X\) is a convexity mapping satisfying the following axioms:

\[
\begin{align*}
(W1) & \quad d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y), \\
(W2) & \quad d(W(x, y, \lambda), W(x, y, \tilde{\lambda})) = |\lambda - \tilde{\lambda}| \cdot d(x, y), \\
(W3) & \quad W(x, y, \lambda) = W(y, x, 1 - \lambda), \\
(W4) & \quad d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w).
\end{align*}
\]

Takahashi \[35\] initiated in the 70’s the study of convex metric spaces as structures \((X, d, W)\) satisfying \((W1)\). The notion of W-hyperbolic space defined above was introduced by Kohlenbach \[16\]. We refer to \[19, \text{p.384}\] for a very nice discussion on these spaces and related structures. First examples of W-hyperbolic spaces are normed spaces; just take \(W(x, y, \lambda) = (1 - \lambda)x + \lambda y\). A very important class of W-hyperbolic spaces are Busemann’s non-positively curved spaces \[4, 5\], extensively studied in the monograph \[31\].

Uniform convexity can be defined in the setting of W-hyperbolic spaces following Goebel and Reich’s definition for the Hilbert ball \[9, \text{p.105}\]. A W-hyperbolic space \((X, d, W)\) is said to be uniformly convex \[25\] if there exists a mapping \(\eta : (0, \infty) \times (0, 2] \to (0, 1]\) such that for all \(r > 0, \varepsilon \in (0, 2]\) and all \(a, x, y \in X\),

\[
\begin{align*}
d(x, a) & \leq r \\
d(y, a) & \leq r \\
d(x, y) & \geq \varepsilon r
\end{align*}
\]  

\[\Rightarrow\]

\[
d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1 - \eta(r, \varepsilon))r.
\]

The mapping \(\eta\) is said to be a modulus of uniform convexity. We use the notation \((X, d, W, \eta)\) for a uniformly convex W-hyperbolic space with modulus \(\eta\).

Uniformly convex W-hyperbolic spaces \((X, d, W, \eta)\) with \(\eta\) being nonincreasing in the first argument are called UCW-hyperbolic spaces, following \[26\]. Obviously, uniformly convex Banach spaces are UCW-hyperbolic spaces with a modulus \(\eta\) that does not depend on \(r\) at all. Other examples of UCW-hyperbolic space are CAT(0) spaces, important structures in geometric group theory (see \[2\]). As the author remarked in \[25\], CAT(0) spaces have a modulus of uniform convexity \(\eta(\varepsilon) = \frac{\varepsilon^2}{8}\), quadratic in \(\varepsilon\).

In the following we give adaptations to UCW-hyperbolic spaces of general logical metatheorems for W-hyperbolic spaces proved by Gerhardy and Kohlenbach for classical systems in \[8\] and for intuitionistic systems in \[7\].

Let \(\mathcal{A}^\omega\) be the system of weakly extensional classical analysis, which goes back to Spector \[34\]. It is formulated in the language of functionals of finite types and consists of \(\text{WE} - \text{PA}^\omega\), the weakly extensional Peano arithmetic in all finite types, the axiom schema \(\text{QF} - \text{AC}\) of quantifier-free axiom of choice and the axiom schema \(\text{DC}^\omega\) of dependent choice in all finite types. Full second
order arithmetic in the sense of reverse mathematics [33] is contained in $\mathcal{A}^\omega$ if we identify subsets of $\mathbb{N}$ with their characteristic functions. We refer the reader to [19] for all the undefined notions related to the system $\mathcal{A}^\omega$, including the representation of real numbers in this system. As a consequence of this representation, the relations $=\mathcal{R}$, $\leq\mathcal{R}$ are given by $\Pi^0_1$ predicates, while $<\mathcal{R}$ is given by a $\Sigma^0_1$ predicate.

The theory $\mathcal{A}^\omega[X,d]_{-b}$ for metric spaces is defined in [8] by extending $\mathcal{A}^\omega$ to the set $\mathbb{T}^X$ of all finite types over the ground types 0 and $X$ and by adding two new constants $0_X$ of type $X$ and $d_X$ of type $X \rightarrow X \rightarrow 1$ together with axioms expressing the fact that $d_X$ represents a pseudo-metric. One defines the equality $=_{X}$ between objects of type $X$ as follows:

$$x =_{X} y := d_X(x,y) =_{\mathbb{R}} 0_{\mathbb{R}}.$$  

Then $d_X$ represents a metric on the set of equivalence classes generated by $=_{X}$.

We use the subscript $-b$ here and for the theories defined in the sequel in order to be consistent with the notations from [19].

The theory $\mathcal{A}^\omega[X,d,W]_{-b}$ for $W$-hyperbolic spaces results from $\mathcal{A}^\omega[X,d]_{-b}$ by adding a new constant $W_X$ of type $X \rightarrow X \rightarrow 1 \rightarrow X$ together with the formalizations of the axioms (W1)-(W4).

In order to define the theory associated to UCW$^+$-hyperbolic spaces, we prove the following lemma, giving equivalent characterizations for these spaces.

**Proposition 2.1.** Let $(X,d,W)$ be a $W$-hyperbolic space. The following are equivalent:

(i) $X$ is an UCW$^+$-hyperbolic space with modulus $\eta$.

(ii) there exists $\eta_1 : (0,\infty) \times \mathbb{N} \rightarrow \mathbb{N}$ nondecreasing in the first argument such that for any $r > 0, k \in \mathbb{N}$, and $x, y, a \in X$

$$
\begin{align*}
\{ & d(x,a) \leq r \\
& d(y,a) \leq r \\
& d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) > \left(1 - 2^{-\eta_1(r,k)}\right) r \}
\}
\Rightarrow
\{ & d(x,y) < 2^{-k}r. 
\end{align*}
$$

(iii) there exists $\eta_2 : (0,\infty) \times \mathbb{N} \rightarrow \mathbb{N}$ nondecreasing in the first argument such that for any $r > 0, k \in \mathbb{N}$, and $x, y, a \in X$

$$
\begin{align*}
\{ & d(x,a) < r \\
& d(y,a) < r \\
& d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) > \left(1 - 2^{-\eta_2(r,k)}\right) r \}
\}
\Rightarrow
\{ & d(x,y) \leq 2^{-k}r. 
\end{align*}
$$

(iv) there exists $\eta_3 : \mathbb{Q}^+_\times \mathbb{N} \rightarrow \mathbb{N}$ nondecreasing in the first argument such that for any $r \in \mathbb{Q}^+_\times, k \in \mathbb{N}$, and $x, y, a \in X$

$$
\begin{align*}
\{ & d(x,a) < r \\
& d(y,a) < r \\
& d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) > \left(1 - 2^{-\eta_3(r,k)}\right) r \}
\}
\Rightarrow
\{ & d(x,y) \leq 2^{-k}r. 
\end{align*}
$$

**Proof.** (i) $\Rightarrow$ (ii) Let $\eta$ be a modulus of uniform convexity nonincreasing in the first argument and define $\eta_1 : (0,\infty) \times \mathbb{N} \rightarrow \mathbb{N}$, $\eta_1(r,k) = \left\lfloor -\log_2 \eta(r,2^{-k}) \right\rfloor$. 

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(ii) ⇒ (i) Define \( \eta : (0, \infty) \times [0, 2] \to (0, 1] \), \( \eta(r, \varepsilon) = 2^{-\eta_1(r, \lceil -\log_2 \varepsilon \rceil)} \).

(ii) ⇒ (iii) Obviously, just take \( \eta_2 := \eta_1 \).

(iii) ⇒ (ii) Define \( \eta_1(r, k) = \eta_2(r, k + 1) \). Let \( r > 0, k \in \mathbb{N} \) and \( a, x, y \in X \) be such that \( d(x, a) \leq r, d(y, a) \leq r \) and \( d\left(\frac{1}{2}x + \frac{1}{2}y, a\right) > (1 - 2^{-m(r, k)})r \) for all \( m \geq 1 \). Then \( d(x_n, a) = (1 - \frac{1}{n})d(x, a) < r \) and similarly \( d(y_n, a) < r \). Furthermore, \( d(x_n, x) = \frac{1}{n}d(x, a), d(y_n, y) = \frac{1}{n}d(y, a) \), so

\[
\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y.
\]

Since \( 0 \leq d(z_n, \frac{1}{2}x + \frac{1}{2}y) \leq \frac{1}{2}d(x_n, x) + \frac{1}{2}d(y_n, y) \), we get that \( \lim_{n \to \infty} z_n = \frac{1}{2}x + \frac{1}{2}y \). Applying the continuity of \( d \), it follows that

\[
\lim_{n \to \infty} d(x_n, y_n) = d(x, y) \quad \text{and} \quad \lim_{n \to \infty} d(z_n, a) = d\left(\frac{1}{2}x + \frac{1}{2}y, a\right).
\]

Hence, \( d(z_n, a) > (1 - 2^{-m(r, k + 1)})r \) for all \( n \) from some \( N \) on. We can then apply (iii) to get that for all \( n \geq N, d(x_n, y_n) \leq 2^{-k-1}r \). By letting \( n \to \infty \), it follows that \( d(x, y) \leq 2^{-k-1}r \). Since \( 0 \leq \eta_1(q, k) = \eta_2(q, k) \) for all \( q \in \mathbb{Q}_+, k \in \mathbb{N} \).

(iv) ⇒ (iii) For every \( r > 0 \), let \( (q_n^r)_{n \geq 1} \) be a nondecreasing sequence of positive rationals such that \( q_n^r \in \left(\frac{r}{1 + n}, r\right) \) for all \( n \geq 1 \). Define

\[
\eta_2 : (0, \infty) \times \mathbb{N} \to \mathbb{N}, \quad \eta_2(r, k) = \sup\{\eta_1(q, k) \mid n \geq 1\} \leq \eta_3([r], k).
\]

We define the theory \( A^r[X, d, UCW, \eta]_{-b} \) of \( UCW \)-hyperbolic spaces as an extension of \( A^r[X, d, W]_{-b} \) obtained by adding a constant \( \eta_X \) of type \( 0 \to 0 \to 0 \), together with three axioms expressing that \( \eta_X \) satisfies Proposition 2.1(iv).

(A1) \( \forall r^0, k^0. x^X, y^X, a^X \left( \lambda n^0, r \geq_{\mathbb{R}} 0_{\mathbb{R}} \wedge d_X(x, a) \leq_{\mathbb{R}} \lambda n^0, r \wedge d_X(y, a) \leq_{\mathbb{R}} \lambda n^0, r \wedge d_X(W(x, y, 1/2), a) \geq_{\mathbb{R}} 1_{\mathbb{R}} - 2^{-\eta_X(r, k)} \cdot \lambda n^0, r \right) \)

\( \to \left( d_X(x, y) \leq_{\mathbb{R}} 2^{-k}, \quad \lambda n^0, r \right) \),

(A2) \( \forall r^0, (r_1, r_2) \leq_{\mathbb{R}} 2_{\mathbb{R}} \to \eta_X(r_1, k) \leq \eta_X(r_2, k) \),

(A3) \( \forall n^0, k^0. (c(n, r) := \eta_X(c(r), k)) \),

where \( c(n) := \min p \leq n p = q \leq n \) is the canonical representative for rational numbers and, since \( r^0 \) codes a rational number \( q \), \( \lambda n^0, r \) represents \( q \) as a real number (see [15], Chapter 4 for details).

If \( X \) is a nonempty set, the full-theoretic type structure \( S^{\omega}X := \langle S^\rho \rangle_{\rho \in \mathbb{T}^X} \) over \( 0 \) and \( X \) is defined by \( S_0 := N, S_X := X, S_{n \to r} := S^r_{\omega}, \) where \( S^r_{\omega} \) is the set of all set-theoretic functions \( S_\rho \to S_r \).

Let \( (X, d, W, \eta) \) be a \( UCW \)-hyperbolic space. \( S^{\omega}X \) becomes a model of \( A^r[X, d, UCW, \eta]_{-b} \) by letting the variables of type \( \rho \) range over \( S_\rho \), giving the

\footnote{Corrections to the definition of the theory \( A^r[X, d, UCW, \eta]_{-b} \) in [24]: Proposition 2.1 is the corrected version of [24], Proposition 3.8]. Furthermore, the axiomatization (A1)-(A3) given in this paper corrects the one from [24].}
natural interpretations to the constants of $\mathcal{A}^\omega$, interpreting $0_X$ by an arbitrary element of $X$, the constants $d_X$ and $W_X$ as specified in [16] and $\eta_X$ by $\eta_X(r, k) := \eta(c(r), k)$. We say that a sentence in the language $L(\mathcal{A}^\omega[X, d, UCW, \eta]_{\leq})$ holds in a nonempty UCW-hyperbolic space $(X, d, W, \eta)$ if it is true in all models of $\mathcal{A}^\omega[X, d, UCW, \eta]_{\leq}$ obtained from $S^{\omega, X}$ as above.

For any type $\rho \in T^X$, we define the type $\hat{\rho} \in T$, obtained by replacing all occurrences of the type $X$ in $\rho$ by 0. We say that $\rho$ has degree $\leq 1$ if $\rho = 0$ or $\rho = 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow X$ and that $\rho$ has degree $1^*$ if $\hat{\rho}$ has degree $\leq 1$. Furthermore, $\rho$ has degree

(i) $(0, X)$ if $\rho = X$ or $\rho = 0 \rightarrow 0 \rightarrow 0 \rightarrow X$;
(ii) $(1, X)$ if $\rho = X$ or $\rho = \rho_1 \rightarrow \ldots \rightarrow \rho_n \rightarrow X$, where each $\rho_i$ has degree $\leq 1$ or $(0, X)$;
(iii) $(\cdot, 0)$ if $\rho = 0 \rightarrow 0 \rightarrow \ldots \rightarrow \rho_n \rightarrow 0$;
(iv) $(\cdot, X)$ if $\rho = X$ or $\rho = \rho_1 \rightarrow \ldots \rightarrow \rho_n \rightarrow X$.

From now on, in order to improve readability, we shall usually omit the subscripts $N, R, Q, X$ excepting the cases where such an omission could create confusions. We shall use $N$ instead of 0, $N^N$ or $N \rightarrow N$ instead of 1 and, moreover, write $n \in N$, $f : N \rightarrow N$, $x \in X$, $T : X \rightarrow X$ instead of $n^N$, $f^N$, $x^N$, $T^X$.

The notion of majorizability was originally introduced by Howard [12] and subsequently modified by Bezem [1]. Based on Bezem’s notion of strong majorizability $s$-maj, Gerhardy and Kohlenbach [8] defined, for every parameter $a$ of type $X$, an $a$-majorization relation $\geq^a_\rho$ between objects of type $\rho \in T^X$ and their majorsants of type $\hat{\rho} \in T$ as follows:

(i) $x^a \geq^a_\rho x \equiv x^a \geq^a_N x$ for $x, x^a \in N$;
(ii) $x^a \geq^a_X x \equiv (x^a)_X \geq^a d(x, a)$ for $x^a \in N, x \in X$;
(iii) $x^a \geq^a_X x \equiv \forall y^a(y^a \geq^a_N y \rightarrow x^a y^a \geq^a_N xy) \land \forall z^a, z(z^a \geq^a_\rho z \rightarrow x^a z^a \geq^a_X x^a z)$.

Restricted to the types $T$, the relation $\geq^a_\rho$ coincides with strong majorizability $s$-maj and, hence, for $\rho \in T$ one writes $s$-maj$_\rho$ instead of $\geq^a_\rho$, as in this case the parameter $a$ is irrelevant.

If $t^* \geq^a_\rho t$ for terms $t^*, t$, we say that $t^*$ $a$-majorizes $t$ or that $t$ is $a$-majorized by $t^*$. A term $t$ is said to be majorizable if it has an $a$-majorant for some $a \in X$. One can prove that $t$ is majorizable if and only if it has an $a$-majorant for all $a \in X$ (see, e.g., [19, Lemma 17.78]).

**Lemma 2.2.** Let $T : X \rightarrow X$. The following are equivalent.

(i) $T$ is majorizable;
(ii) for all $x \in N$ there exists $\Omega : N \rightarrow N$ such that

$$\forall n \in N, y \in X \left( d(x, y) < n \rightarrow d(x, Ty) \leq \Omega(n) \right); \quad (2)$$
(iii) for all \( x \in \mathbb{N} \) there exists \( \Omega : \mathbb{N} \to \mathbb{N} \) such that
\[
\forall n \in \mathbb{N}, y \in X \left( d(x, y) \leq n \rightarrow d(x, Ty) \leq \Omega(n) \right).
\] (3)

Proof. \( T \) is majorizable if and only if \( T \) is \( x \)-majorizable for each \( x \in X \) if and only if for each \( x \in X \) there exists a function \( T^* : \mathbb{N} \to \mathbb{N} \) such that \( T^* \) is nondecreasing and satisfies
\[
\forall n \in \mathbb{N}, y \in X \left( d(x, y) \leq n \rightarrow d(x, Ty) \leq T^*(n) \right).
\]

(i) \( \Rightarrow \) (iii) is obvious: take \( \Omega = T^* \). For the implication (iii) \( \Rightarrow \) (i), given, for \( x \in X \), \( \Omega \) satisfying (3), define \( T^* n = \max_{k \leq n} \Omega(k) \).

(iii) \( \Rightarrow \) (ii) is again obvious. For the converse implication, given \( \Omega \) satisfying (2) define \( \tilde{\Omega}(n) = \Omega(n + 1) \). Then \( \tilde{\Omega} \) satisfies (3).

In the sequel, given a majorizable function \( T : X \to X \) and \( x \in X \), an \( \Omega \) satisfying (3) will be called a modulus of majorizability at \( x \) of \( T \); we say also that \( T \) is \( x \)-majorizable with modulus \( \Omega \). We give in Lemma 2.2 the equivalent condition (2) for logical reasons: since \( <_R \) is a \( \Sigma^0_1 \) predicate and \( \leq_R \) is a \( \Pi^0_1 \) predicate, the formula in (2) can be written in purely universal form.

The following lemma shows that natural classes of mappings in metric or \( W \)-hyperbolic spaces are majorizable; we refer to [19, Corollary 17.55] for the proof.

Lemma 2.3. Let \( (X, d) \) be a metric space.

(i) If \( (X, d) \) is bounded with diameter \( d_X \), then any function \( T : X \to X \) is majorizable with modulus of majorizability \( \Omega(n) := \lceil d_X \rceil \) for each \( x \in X \).

(ii) If \( T : X \to X \) is \( L \)-Lipschitz, then \( T \) is majorizable with modulus at \( x \) given by \( \Omega(n) := n + L^* b \), where \( b, L^* \in \mathbb{N} \) are such that \( d(x, Tx) \leq b \) and \( L \leq L^* \). In particular, any nonexpansive mapping is majorizable with modulus
\[
\Omega(n) = n + b.
\]

(iii) If \( (X, d, W) \) is a \( W \)-hyperbolic space, then any uniformly continuous mapping \( T : X \to X \) is majorizable with modulus \( \Omega(n) := n \cdot 2^{\alpha_T(0)} + 1 + b \) at \( x \), where \( d(x, Tx) \leq b \in \mathbb{N} \) and \( \alpha_T \) is a modulus of uniform continuity of \( T \), i.e. \( \alpha_T : \mathbb{N} \to \mathbb{N} \) satisfies
\[
\forall x, y \in X \forall k \in \mathbb{N} \left( d(x, y) \leq 2^{-\alpha_T(k)} \rightarrow d(Tx, Ty) \leq 2^{-k} \right).
\]

Whenever we write \( A(\mathcal{Y}) \) we mean that \( A \) is a formula in our language which has only the variables \( \mathcal{Y} \) free. A formula \( A \) is called a \( \forall \)-formula (resp. a \( \exists \)-formula) if it has the form
\[
A \equiv \forall x \mathcal{E} A_0(x, \mathcal{g}) \quad \text{(resp.} \quad A \equiv \exists x \mathcal{E} A_0(x, \mathcal{g})),
\]
where \( A_0 \) is a quantifier free formula and the types in \( \mathcal{E} \) are of degree 1* or \((1, X)\). We assume in the following that the constant \( 0_X \) does not occur in the
formulas we consider; this is no restriction, since $0_X$ is just an arbitrary constant which could have been replaced by any new variable of type $X$.

The following result is an adaptation of a general logical metatheorem proved by Kohlenbach [16] for bounded $W$-hyperbolic spaces and generalized to the unbounded case by Gerhardt and Kohlenbach [8].

**Theorem 2.4.** Let $P$ be $\mathbb{N}$, $\mathbb{N}^3$ or $\mathbb{N}^{\mathbb{N} \times \mathbb{N}}$, $K$ an $A^\omega$-definable compact metric space, $\rho$ of degree $1^*$, $B_\psi(u, y, z, n)$ a $\forall$-formula and $C_\Xi(u, y, z, N)$ a $\exists$-formula. Assume that $A^\omega[X, d, UCW, \eta]_{\leq b}$ proves that

$$\forall u \in P \forall y \in K \forall z^\omega \left( \forall n \in \mathbb{N} B_\psi \rightarrow \exists N \in \mathbb{N} C_\Xi \right).$$

Then one can extract a computable functional $\Phi : P \times \mathbb{N}^{(\mathbb{N} \times \mathbb{N})^k} \times \mathbb{N} \rightarrow \mathbb{N}$ such that the following statement is satisfied in all nonempty $UCW$-hyperbolic spaces $(X, d, W, \eta)$:

for all $z \in S_\rho$, $z^* \in \mathbb{N}^{(\mathbb{N} \times \mathbb{N})^k}$, if there exists $a \in X$ such that $z^* \succ a_z z$, then

$$\forall u \in P \forall y \in K \left( \forall n \in \Phi(u, z^*, \eta) B_\psi \rightarrow \exists N \leq \Phi(u, z^*, \eta) C_\Xi \right).$$

**Proof.** As $\leq_R$ is purely universal and $<_R$ is purely existential, one can easily see that the axioms (A1)-(A3) are universal. Furthermore, $\eta_X$ is strongly majorized by $\eta_X := \lambda \eta \min(n^0, m^0, \max\{\eta_X(i, j) : i \leq n, j \leq m\}$. Then the proofs from [8] extend immediately to our theory (see [8, Remark 4.13]).

**Remark 2.5.** (i) Instead of single premises $\forall u B_\psi$ and single variables $u, y, n$ we may have finite conjunctions of premises as well as tuples $u \in P, y \in K, \eta \in \mathbb{N}$ of variables.

(ii) We can have also $z^\rho = z_1^\rho, \ldots, z_k^\rho$ for types $\rho_1, \ldots, \rho_k$ of degree $1^*$. Then in the conclusion is assumed that $z_i^* \succ a_z z_i$ for one common $a \in X$ for all $i = 1, \ldots, k$. The bound $\Phi$ depends now on all the $a$-majorants $z_1^*, \ldots, z_k^*$.

The proof of Theorem 2.4 is based on an extension to $A^\omega[X, d, UCW, \eta]_{\leq b}$ of Spector’s [34] interpretation of classical analysis $A^\omega$ using $bar$ recursion, combined with $a$-majorization. Furthermore, the proof of the metatheorem actually provides an extraction algorithm for the functional $\Phi$, which can always be defined in the calculus of bar-recursive functionals. However, as we shall see also in this paper, for concrete applications usually small fragments of $A^\omega[X, d, UCW, \eta]_{\leq b}$ are needed to formalize the proof. As a consequence, one gets bounds of primitive recursive complexity and very often exponential or even polynomial bounds.

We give now a very useful corollary of Theorem 2.4.

**Corollary 2.6.** Let $P$ be $\mathbb{N}$, $\mathbb{N}^3$ or $\mathbb{N}^{\mathbb{N} \times \mathbb{N}}$, $K$ an $A^\omega$-definable compact metric space, $B_\psi(u, y, x, x^*, T, n)$ a $\forall$-formula and $C_\Xi(u, y, x, x^*, T, N)$ a $\exists$-formula. Assume that $A^\omega[X, d, UCW, \eta]_{\leq b}$ proves that

$$\forall u \in P \forall y \in K \forall x, x^* \in X \forall T : X \rightarrow X \forall \Omega : N \rightarrow N \left( T \text{ is } x\text{-majorizable with modulus } \Omega \wedge \forall n \in \mathbb{N} B_\psi \rightarrow \exists N \in \mathbb{N} C_\Xi \right).$$
Then one can extract a computable functional $\Phi$ satisfying the following statement for all $u \in P$, $b \in \mathbb{N}$ and $\Omega : \mathbb{N} \to \mathbb{N}$:

$$\forall y \in K \forall x, x^* \in X \forall T : X \to X$$

$$T \text{ is } x\text{-majorizable with modulus } \Omega \land d(x, x^*) \leq b \land \forall n \leq \Phi(u, b, \Omega, \eta) B_\nu$$

$$\implies \exists N \leq \Phi(u, b, \Omega, \eta) C_3.$$ 

holds in all nonempty UCW-hyperbolic spaces $(X, d, W, \eta)$.

**Proof.** The premise "$T$ is $x$-majorizable with modulus $\Omega$" is a $\forall$-formula, by (2). Furthermore, $0$ is an $x$-majorant for $x^*$, since $d(x, x^*) \leq b$, and $T^* := \lambda n. \max_{k \leq n} \Omega(k)$ is $x$-majorizes $T$, by the proof of Lemma 2.2. Apply now Theorem 2.4. □

**Remark 2.7.** As in the case of Theorem 2.4, instead of single $n \in \mathbb{N}$ and a single premise $\forall n B_\nu$ we could have tuples $u = n_1, \ldots, n_k$ and a conjunction of premises $\forall n_1 B_\nu^1 \land \ldots \land \forall n_k B_\nu^k$. In this case, in the premise of the conclusion we shall have $\forall n \leq \Phi B_\nu^1 \land \ldots \land \forall n_k \leq \Phi B_\nu^k$.

If $(X, d)$ is a metric space, $C \subseteq X$ and $T : C \to C$ is a mapping, we denote with $Fix(T)$ the set of fixed points of $T$. For $x \in X$ and $b, \delta > 0$, let

$$Fix_\delta(T, x, b) = \{y \in C \mid d(y, x) \leq b \land d(y, Tx) < \delta\}.$$ 

If $Fix_\delta(T, x, b) \neq \emptyset$ for all $\delta > 0$, we say that $T$ has approximate fixed points in a $b$-neighborhood of $x$.

The following more concrete consequence of Theorem 2.4 shows that, under some conditions, the hypothesis of $T$ having fixed points can be replaced by the weaker one that $T$ has approximate fixed points in a $b$-neighborhood of $x$. Its proof is similar with the one of [8, Corollary 4.22].

**Corollary 2.8.** Let $P$ be $\mathbb{N}$, $\mathbb{N}^\mathbb{N}$ or $\mathbb{N}^{\mathbb{N} \times \mathbb{N}}$, $K$ an $A^\omega$-definable compact metric space, $B_\nu(u, x, T, n)$ a $\forall$-formula and $C_3(u, x, T, n)$ a $\exists$-formula. Assume that $A^\omega[X, d, UCW, \eta]_b$ proves that

$$\forall u \in P \forall y \in K \forall x \in X \forall T : X \to X \forall \Omega : \mathbb{N} \to \mathbb{N}$$

$$\bigl( T \text{ is } x\text{-majorizable with mod. } \Omega \land Fix(T) \neq \emptyset \land \forall n \in \mathbb{N} B_\nu \implies \exists N \in \mathbb{N} C_3 \bigr).$$

It follows that one can extract a computable functional $\Phi$ such that for all $u \in P$, $b \in \mathbb{N}$ and $\Omega : \mathbb{N} \to \mathbb{N}$,

$$\forall y \in K \forall x \in X \forall T : X \to X$$

$$\bigl( T \text{ is } x\text{-majorizable with modulus } \Omega \land \forall \delta > 0 (Fix_\delta(T, x, b) \neq \emptyset) \land \forall n \leq \Phi(u, b, \Omega, \eta) B_\nu \implies \exists N \leq \Phi(u, b, \Omega, \eta) C_3 \bigr).$$

holds in any nonempty UCW-hyperbolic space $(X, d, W, \eta)$.
Proof. The statement proved in \( \mathcal{A}^\omega[X,d,\text{UCW},\eta]_{-b} \) can be written as

\[
\forall x \in P^\forall y \in K \forall x,p \in X \forall T : X \rightarrow X \forall \Omega : N \rightarrow N \\
\left( \exists x,\forall \Omega \wedge \forall k \in \mathbb{N} \left( d(p,T_p) \leq_R 2^{-k} \right) \land \forall n \in NB_\eta \rightarrow \exists N \in \mathbb{N} C_3 \right).
\]

We have used the fact that \( \text{Fix}(T) \neq \emptyset \) is equivalent with \( \exists p \in X(Tp =_X p) \) that is further equivalent with \( \exists p \in X \forall k \in \mathbb{N} \left( d(p,T_p) \leq_R 2^{-k} \right) \), by using the definition of \( =_X \) and \( =_R \) in our system. As all the premises are \( \forall \)-formulas, we can apply Corollary 2.6 to extract a functional \( \Phi \) such that for all \( b \in \mathbb{N} \),

\[
\forall x \in P^\forall y \in K \forall x,p \in X \forall T : X \rightarrow X \forall \Omega : N \rightarrow N \\
\left( \exists x,\forall \Omega \wedge \forall k \in \mathbb{N} \left( d(p,T_p) \leq 2^{-k} \right) \land \forall n \in \mathbb{N} \left( \Phi(\Omega,b,\eta) \rightarrow \exists N \in \mathbb{N} C_3 \right) \right),
\]

that is

\[
\forall x \in P^\forall y \in K \forall x \in X \forall T : X \rightarrow X \forall \Omega : N \rightarrow N \\
\left( \exists x,\forall \Omega \wedge \exists p \in X \left( d(p,x) \leq b \land \forall k \leq \Phi(\Omega,b,\eta) \left( d(p,T_p) \leq 2^{-k} \right) \right) \land \forall n \in \mathbb{N} \left( \Phi(\Omega,b,\eta) \rightarrow \exists N \in \mathbb{N} C_3 \right) \right).
\]

Use the fact that the existence of \( p \in X \) such that \( d(x,p) \leq b \) and \( \forall k \leq \Phi(d(p,T_p) \leq 2^{-k}) \) is equivalent with the existence of \( p \in X \) such that \( d(x,p) \leq b \) and \( d(p,T_p) \leq 2^{-\delta} \) which is obviously implied by \( \forall \delta > 0 (\text{Fix}_\delta(T,x,b) \neq \emptyset) \).

We shall apply the above corollary in the next section for nonexpansive mappings \( T : X \rightarrow X \). In this case, as we have seen in Lemma 2.3, a modulus of majorizability at \( x \) is given by \( \Omega(n) = n+b \), where \( b \geq d(x,Tx) \), so the bound \( \Phi \) will depend on \( \Omega, \eta, b \) and \( b > 0 \) such that \( d(x,Tx) \leq b \).

For all \( \delta > 0 \) there exists \( y \in X \) such that \( \text{Fix}_\delta(T,x,b) \neq \emptyset \), hence

\[
d(x,Tx) \leq d(x,y) + d(y,Ty) + d(Ty,Tx) \leq 2d(x,y) + d(y,Ty) \leq 2b + \delta
\]

for all \( \delta > 0 \). It follows that \( d(x,Tx) \leq 2b \), so we can take \( \tilde{b} := 2b \). As a consequence, the bound \( \Phi \) will depend only on \( \Omega, b \) and \( \eta \).

The above logical metatheorems were obtained for classical proofs in metric, W-hyperbolic or \( \text{UCW} \)-hyperbolic spaces. Gerhardy and Kohlenbach [7] considered similar metatheorems for semi-intuitionistic proofs, that is proofs in intuitionistic analysis enriched with some non-constructive principles. Let \( \mathcal{A}^\omega_\eta := \text{HA}^\omega + \text{AC} \), where \( \text{E} - \text{HA}^\omega \) is the extensional Heyting arithmetic in all finite types and \( \text{AC} \) is the full axiom of choice. The theories \( \mathcal{A}^\omega_\eta[X,d]_{-b} \), \( \mathcal{A}^\omega_\eta[X,d,W]_{-b} \) and \( \mathcal{A}^\omega_\eta[X,d,\text{UCW}]_{-b} \) are obtained as above as extensions of \( \mathcal{A}^\omega_\eta \); we refer to [7] for details.

Precomprehension for negated formulas is the following principle:

\[
\text{CFA}^\omega_\eta := \exists_n \Phi \leq \exists_n \beta \forall \beta \left( \Phi(y) \rightarrow \neg \text{A}(y) \right),
\]

where \( \rho = \rho_1, \ldots, \rho_k \) and \( y = y_{\rho_1}^{\beta_1}, \ldots, y_{\rho_k}^{\beta_k} \).

The following result is an adaptation to \( \text{UCW} \)-spaces of [7, Corollary 4.9].
Theorem 2.9. Let $P$ be $\mathbb{N}$, $\mathbb{N}^\mathbb{N}$, or $\mathbb{N}^{\mathbb{N} \times \mathbb{N}}$, $K$ an $\mathcal{A}_i^\omega$-definable compact Polish space and let $B(u,y,T,x)$, $C(u,y,T,x,N)$ be arbitrary formulas.
Assume that $\mathcal{A}_i^\omega [X,d,UCW,\eta]_b + CA_\omega$ proves that
$$\forall u \in P \forall y \in K \forall x \in X \forall T : X \rightarrow X \forall \Omega : N \rightarrow N \left( T \text{ is } x\text{-majorizable with modulus } \Omega \land \neg B \rightarrow \exists N \in \mathbb{N}C \right).$$
Then one can extract a Gödel primitive recursive functional $\Phi$ such that for all $u \in P$ and $\Omega : N \rightarrow N$,
$$\forall y \in K \forall x \in X \forall T : X \rightarrow X \exists N \leq \Phi(u,\Omega,\eta) \left( T \text{ is } x\text{-majorizable with modulus } \Omega \land \neg B \rightarrow C \right).$$
holds in any nonempty UCW-hyperbolic space $(X,d,W,\eta)$.
As before, instead of a single premise $B$, we may have a finite conjunction of premises.

3 Logical discussion of the asymptotic regularity proof
Throughout this section $(X,d,W,\eta)$ is a UCW-hyperbolic space, $C \subseteq X$ a convex subset and $T : C \rightarrow C$ is a nonexpansive mapping. The Ishikawa iteration starting with $x \in C$ is defined similarly with the case of normed spaces:
$$x_0 = x, \quad x_{n+1} = (1 - \lambda_n)x_n \oplus \lambda_nT((1 - s_n)x_n \oplus s_nTx_n),$$
where $(\lambda_n), (s_n)$ are sequences in $[0,1]$.
As we explain in the sequel, the logical metatheorems presented in the previous section guarantee that one can extract an effective uniform rate of asymptotic regularity for the Ishikawa iteration from the proof of the generalization of Theorem 1.1 to UCW-hyperbolic spaces.

Theorem 3.1. Assume that $\text{Fix}(T) \neq \emptyset$ and that $(\lambda_n), (s_n)$ satisfy (1). Then
$$\lim_{n \rightarrow \infty} d(x_n,Tx_n) = 0 \text{ for all } x \in C.$$

By an inspection of the proof of Theorem 3.1, one can see that it consists of two important steps. One proves first the following result.

Proposition 3.2. Assume that $\text{Fix}(T) \neq \emptyset$, \sum_{n=0}^{\infty} \lambda_n(1 - \lambda_n)$ diverges and
$$\limsup_{n \rightarrow \infty} s_n < 1.$$ 
Then \liminf_{n \rightarrow \infty} d(x_n,Tx_n) = 0 \text{ for all } x \in C.
A first remark is that the proof of Proposition 3.2 is by contradiction, hence it is ineffective. Secondly, it is enough to consider nonexpansive mappings $T : X \rightarrow X$, as convex subsets of UCW-hyperbolic spaces are themselves UCW-hyperbolic spaces.

The assumption that \sum_{n=0}^{\infty} \lambda_n(1 - \lambda_n)$ diverges is equivalent with the existence of a rate of divergence $\theta : \mathbb{N} \rightarrow \mathbb{N}$ for the series, that is a mapping $\theta$ satisfying
$\sum_{k=0}^{\infty} \lambda_k (1 - \lambda_k) \geq n$ for all $n \in \mathbb{N}$. As $(s_n)$ is a sequence in $[0, 1]$, the assumption that $\limsup_{n \to \infty} s_n < 1$ is equivalent with the existence of $L, N_0 \in \mathbb{N}, L \geq 1$ such that $s_n \leq 1 - \frac{1}{L}$ for all $n \geq N_0$.

Furthermore, since $d(x_n, Tx_n) \geq 0$, the following statements are equivalent:

(i) $\liminf_{n \to \infty} d(x_n, Tx_n) = 0$.

(ii) For all $k, l \in \mathbb{N}$ there exists $N \geq k$ such that $d(x_N, Tx_N) < 2^{-l}$.

By a modulus of liminf $\Delta$ for $(d(x_n, Tx_n))$ we shall understand a mapping $\Delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ satisfying

$$\forall k, l \in \mathbb{N} \exists N \leq \Delta(l, k) (N \geq k \land d(x_N, Tx_N) < 2^{-l}).$$

One can easily conclude that $\mathcal{A}^\omega[X, d, UCW, \eta] - b$ proves the following formalized version of Proposition 3.2:

$$\forall k, l, N_0, L \in \mathbb{N} \forall \theta : \mathbb{N} \to \mathbb{N} \forall \lambda^N \to (\mathbb{N} \to \mathbb{N}), s^N \to (\mathbb{N} \to \mathbb{N}) \forall x \in X \forall T : X \to X 

\left(\text{Fix}_T(\delta) \neq \emptyset \land B_{\varphi} \Rightarrow \exists N \in \mathbb{N} \left(N \geq k \land d_X(x_N, Tx_N) < 2^{-l}\right)\right),$$

where $\lambda^N \to (\mathbb{N} \to \mathbb{N}), s^N \to (\mathbb{N} \to \mathbb{N})$ represent elements of the compact Polish space $[0, 1]^\infty$ with the product metric and

$$B_{\varphi} \equiv T \text{ nonexpansive} \land \forall n \in \mathbb{N} \left(\frac{\theta(n)}{\sum_{i=0}^{\infty} \lambda_i (1 - \lambda_i) \geq n}\right) \land \left(L \geq 1 \land \forall n \in \mathbb{N} \left(n \geq N_0 \rightarrow s_n \leq R \left(1 - \frac{1}{L}\right)\right)\right).$$

Using the representation of real numbers in our system, one can see immediately that $B_{\varphi}$ is a universal formula. Corollary 2.8 and the discussion afterwards yield the extractability of a functional $\Delta := \Delta(l, \eta, b, k, N_0, L, \theta)$ such that for all $b, N_0, L, \theta, (\lambda_n), (s_n)$,

$$\forall x \in X \forall T : X \to X \left(\forall \delta > 0(\text{Fix}_T(T, x, b) \neq \emptyset) \land B_{\varphi} \Rightarrow 

\forall k, l \in \mathbb{N} \exists N \leq \Delta (N \geq k \land d_X(x_N, Tx_N) < 2^{-l})\right)$$

holds in any nonempty UCW-hyperbolic space $(X, d, W, \eta)$. As a consequence, $\Delta$ is a modulus of liminf for $(d(x_n, Tx_n))$.

The second step of the proof is the following result.

**Proposition 3.3.** Assume furthermore that $\sum_{n=0}^{\infty} s_n (1 - \lambda_n)$ converges. Then

$$\lim_{n \to \infty} d(x_n, Tx_n) = 0.$$
Let us denote \( \alpha_n := \sum_{i=0}^{n} s_i (1 - \lambda_i) \). The proof of Proposition 3.3 is fully constructive and one can easily see that \( A^\alpha_\eta[X, d, UCW, \eta]_{-b} \) proves that

\[
\forall l \in \mathbb{N} \forall \gamma : \mathbb{N} \to \mathbb{N} \forall \lambda(\cdot), s(\cdot) \forall x, T \ (A \land \exists \Delta : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) B \to \exists N \in \mathbb{N} C),
\]

where

\[
C \equiv \forall n \in \mathbb{N} \left( d_X(x_{n+N}, Tx_{n+N}) \leq R \ 2^{-l} \right), \\
A \equiv T \text{ nonexpansive } \land \gamma \text{ Cauchy modulus for } (\alpha_n) \\
\equiv T \text{ nonexpansive } \land \forall p, n \in \mathbb{N} \left( \alpha_{\gamma(p)+n} - R \alpha_{\gamma(p)} \leq 2^{-p} \right) \text{ and } \\
B \equiv \forall k, l \in \mathbb{N} \exists N \leq \Delta(l, k) (N \geq k \land d_X(x_N, Tx_N) < R \ 2^{-l}).
\]

Let us consider the following universal formula

\[
D \equiv \forall k, l \in \mathbb{N} \exists N \leq \Delta(l, k) (N \geq k \land d_X(x_N, Tx_N)(l) < Q 2^{-l+1}),
\]

where we refer again to [15, Chapter 4] for details on the construction \( f^1 \mapsto \hat{f} \). Since for every \( l \in \mathbb{N} \), \( d_X(x_N, Tx_N)(l) \) is a rational \( 2^{-l} \)-approximation of \( d_X(x_N, Tx_N) \), it is easy to see that \( B \) implies \( D \). For the same reason, \( D \) implies

\[
\forall k, l \in \mathbb{N} \exists N \leq \Delta(l + 2, k) (N \geq k \land d_X(x_N, Tx_N) < R \ 2^{-l}).
\]

Thus, \( \exists \Delta : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) B \) is equivalent to \( \exists \Delta : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) D \), hence \( A^\alpha_\eta[X, d, UCW, \eta]_{-b} \) proves that

\[
\forall l \in \mathbb{N} \forall \gamma : \mathbb{N} \to \mathbb{N} \forall \Delta : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \forall \lambda(\cdot), s(\cdot) \forall x, T \ (A \land D \to \exists N \in \mathbb{N} C).
\]

We can apply Theorem 2.9 for \( T \) nonexpansive to conclude that we can extract a functional \( \Phi := \Phi(l, \eta, b, \Delta, \gamma) \) such that for all \( x \in X \), \( T : X \to X \),

\[
\exists N \leq \Phi(d_X(x, Tx)) \leq R \ b \land A \land D \to C
\]

holds in any nonempty \( UCW \)-hyperbolic space \((X, d, W, \eta)\). Thus, there exists \( N \leq \Phi \) such that \( d(x_{n+N}, Tx_{n+N}) \leq 2^{-l} \) for all \( n \in \mathbb{N} \), hence \( d(x_n, Tx_n) \leq 2^{-l} \) for all \( n \geq \Phi \). It follows that \( \Phi \) is a rate of asymptotic regularity for the Ishikawa iteration, whose extraction is guaranteed by logical metatheorems. The rate \( \Phi \) is also highly uniform, since it does not depend on \( X, C, T, x \) except for \( b \) and the modulus \( \eta \) of uniform convexity.

That we get a full rate of asymptotic regularity \( \Phi \) is a consequence of the fact that we treat the constructive proof of Proposition 3.3 directly, by applying Theorem 2.9. This also needs as input a Cauchy modulus \( \gamma \) for \( (\alpha_n) \). Alternatively, one can analyze the proof as a classical one and apply Theorem 2.4 and its corollaries. Since the fact that \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \) is a \( \forall \exists \forall \)-statement, one gets in this case only a rate of metastability (as defined by Tao [38]) for the sequence \((d(x_n, Tx_n))\), i.e. a mapping \( \Psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) satisfying for all \( k \in \mathbb{N} \) and all \( g : \mathbb{N} \to \mathbb{N} \),

\[
\exists N \leq \Psi(k, g) \forall i, j \in [N, N + g(N)] \ (|d(x_i, Tx_i) - d(x_j, Tx_j)| < 2^{-k}).
\]

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In order to get such a rate of metastability $\Psi$, one only needs a rate of metastability for $(\alpha_n)$. Furthermore, one can easily see that induction is used in the proof of Theorem 3.1 only to get inequalities on $(x_n)$ (see Proposition 4.3.(i)). However, these inequalities are universal lemmas, hence one can add them as axioms, since their proofs have no contribution to the extraction of the bounds. The rest of the proof uses only basic arithmetic, so it can be formalized in a small fragment of $\mathcal{A}^*\{X, d, UCW, \eta\}_{b_0}$. As a consequence, the logical metatheorems guarantee that the bound $\Phi$ is a simple polynomial in the input data and the unwinding of the proof, given in the next section, produces such a bound.

4 The quantitative asymptotic regularity result

In the following we give a proof of the quantitative version of Theorem 3.1.

**Theorem 4.1.** Let $(X, d, W, \eta)$ be a UCW-hyperbolic space, $C \subseteq X$ a convex subset and $T : C \to C$ a nonexpansive mapping.

Assume that $(\lambda_n), (s_n)$ are sequences in $[0, 1]$ satisfying the following properties

(i) $\sum_{n=0}^{\infty} \lambda_n(1 - \lambda_n)$ with rate of divergence $\theta : \mathbb{N} \to \mathbb{N}$;

(ii) $\lim\sup_{n} s_n < 1$ with $L, N_0 \in \mathbb{N}$ satisfying $s_n \leq 1 - \frac{1}{L}$ for all $n \geq N_0$;

(iii) $\sum_{n=0}^{\infty} s_n(1 - \lambda_n)$ converges with Cauchy modulus $\gamma$.

Let $x \in C, b > 0$ be such that for any $\delta > 0$ there is $y \in C$ with

$$d(x, y) \leq b \text{ and } d(y, Ty) < \delta.$$  \hfill (4)

Let $(x_n)$ be the Ishikawa iteration starting with $x$. Then $\lim_{n \to \infty} d(x_n, Tx_n) = 0$ and moreover

$$\forall \epsilon > 0 \forall n \geq \Phi(\epsilon, \eta, b, N_0, L, \theta, \gamma) \left( d(x_n, Tx_n) < \epsilon \right),$$  \hfill (5)

where

$$\Phi := \Phi(\epsilon, \eta, b, N_0, L, \theta, \gamma) = \theta(P + \gamma_0 + 1 + N_0),$$  \hfill (6)

with $\gamma_0 = \gamma \left( \frac{\epsilon}{8b} \right)$ and $P = \left[ \frac{L(b + 1)}{\epsilon \cdot \eta \left( \frac{b + 1}{L(b + 1)} \right)} \right]$.

We recall a very useful property of UCW-hyperbolic spaces.

**Lemma 4.2.** [26] Let $(X, d, W, \eta)$ be a UCW-hyperbolic space. Assume that $r > 0, \epsilon \in (0, 2]$ and $a, x, y \in X$ are such that $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq \epsilon r$. Then for any $\lambda \in [0, 1]$ and for all $s \geq r$,

$$d((1 - \lambda)x \oplus \lambda y, a) \leq (1 - 2\lambda(1 - \lambda)\eta(s, \epsilon)) r.$$
For simplicity, we use the notation $y_n := (1 - s_n)x_n \oplus s_nTx_n$. The following lemma collects properties of the Ishikawa iteration, which will be needed in the sequel.

**Lemma 4.3.**

(i) For all $n \in \mathbb{N}$ and $x, z \in X$, the following hold

\[
(1 - s_n)d(x_n, Tx_n) \leq d(x_n, Ty_n) \leq d(x_n, Tz_n) + d(z, Tz) \tag{7}
\]

\[
d(x_{n+1}, Tz_{n+1}) \leq (1 + 2s_n(1 - \lambda_n))d(x_n, Tz_n) \tag{8}
\]

\[
d(y_n, z) \leq d(x_n, z) + d(z, Tz) \tag{9}
\]

\[
d(Ty_n, z) \leq d(x_n, z) + 2d(z, Tz) \tag{10}
\]

\[
d(x_{n+1}, z) \leq d(x_n, z) + 2\lambda nd(z, Tz) \tag{11}
\]

\[
d(x_n, z) \leq d(x, z) + 2\sum_{i=0}^{n-1} \lambda_i d(z, Tz) \leq d(x, z) + 2nd(z, Tz). \tag{12}
\]

(ii) Assume that $x \in C, b > 0$ are such that $T$ has approximate fixed points in a $b$-neighborhood of $x$. Then $d(x_n, Tx_n) \leq 2b$ for all $n \in \mathbb{N}$.

**Proof.** (i) (7) and (8) are proved in [26, Lemma 4.1]. For the proofs of (9)-(11), just remark that $d(y_n, z) \leq (1 - s_n)d(x_n, z) + s_n d(Tx_n, z) \leq d(x_n, z) + s_n d(Tz, z)$, $d(Ty_n, z) \leq d(y_n, z) + d(z, Tz)$ and $d(x_{n+1}, z) \leq (1 - \lambda_n)d(x_n, z) + \lambda nd(Ty_n, z)$. An easy induction gives us (13).

(ii) Let $n \in \mathbb{N}$. We shall prove that $d(x_n, Tx_n) \leq 2b + \varepsilon$ for all $\varepsilon > 0$. Applying the hypothesis with $\delta_n := \frac{\varepsilon}{4n + 1}$, we get $z \in X$ such that $d(x, z) \leq b$ and $d(z, Tz) \leq \delta_n$. It follows that

\[
d(x_n, Tx_n) \leq d(x_n, z) + d(Tx_n, z) \leq d(x_n, z) + d(Tx_n, Tz) + d(z, Tz)
\]

\[
\leq 2d(x_n, z) + d(z, Tz) \leq 2d(x, z) + (4n + 1)d(z, Tz) \text{ by (13)}
\]

\[
\leq 2b + (4n + 1)\delta_n = 2b + \varepsilon.
\]

We prove first the quantitative version of Proposition 3.2.

**Proposition 4.4.** For all $\varepsilon > 0$ and all $k \in \mathbb{N}$,

\[
\exists N \in [k, \Delta] \left( d(x_N, Tx_N) < \varepsilon \right),
\]

where $\Delta := \Delta(\varepsilon, k, \eta, b, N_0, L, \theta) = \theta(P + k + N_0)$.

**Proof.** Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Using (7) and the hypothesis (ii) of Theorem 4.1, one can easily see that it suffices to prove that

\[
\exists N \in [k + N_0, \Delta] \left( d(x_N, Ty_N) < \frac{\varepsilon}{L} \right). \tag{14}
\]

Assume by contradiction that (14) does not hold, hence $d(x_n, Ty_n) \geq \frac{\varepsilon}{L}$ for all $n \in [k + N_0, \Delta]$. Let $\delta := \frac{1}{4(\Delta + 1)}$ and $z \in Fix(T, x, b)$. We shall use in
the sequel the notation $a_n := d(x_n, z) + 2d(z, Tz)$. As a consequence of (13), we get that for all $n \in [k + N_0, \Delta]$,

$$a_n \leq d(x, z) + (2n + 2)d(z, Tz) \leq b + 2(\Delta + 1)\delta < b + 1.$$  

Remark that $d(x_n, z) \leq a_n, d(Ty_n, z) \leq a_n$ (by (10)), $d(x_n, Ty_n) \geq \frac{\varepsilon}{L} \geq \frac{\varepsilon}{L(b + 1)} \cdot a_n$ and $0 < \frac{\varepsilon}{L(b + 1)} < \frac{d(x_n, Ty_n)}{b + 1} \leq \frac{2a_n}{b + 1} \leq 2$. Hence, we can apply Lemma 4.2 with $r := a_n, s := b + 1$ and $\varepsilon := \frac{\varepsilon}{L(b + 1)}$ to obtain that

$$d(x_{n+1}, z) = d((1 - \lambda_n)x_n \oplus \lambda_n Ty_n, z)$$

$$\leq \left(1 - 2\lambda_n(1 - \lambda_n)\eta \left(b + 1, \frac{\varepsilon}{L(b + 1)}\right)\right)a_n$$

$$= d(x_n, z) + 2d(z, Tz) - 2\lambda_n(1 - \lambda_n)\eta \left(b + 1, \frac{\varepsilon}{L(b + 1)}\right)a_n.$$  

As $a_n \geq \frac{d(x_n, Ty_n)}{2} \geq \frac{\varepsilon}{2L}$, we get that for all $n \in [k + N_0, \Delta]$,

$$d(x_{n+1}, z) \leq d(x_n, z) + 2d(z, Tz) - \frac{\varepsilon}{L}\lambda_n(1 - \lambda_n)\eta \left(b + 1, \frac{\varepsilon}{L(b + 1)}\right)\lambda_n(1 - \lambda_n).$$  

Adding (15) for $n = k + N_0, \ldots, \Delta$, it follows that

$$d(x_{\Delta+1}, z) \leq d(x_{k+N_0}, z) + 2(\Delta - k - N_0 + 1)d(z, Tz) -$$

$$\frac{\varepsilon}{L}\eta \left(b + 1, \frac{\varepsilon}{L(b + 1)}\right)\sum_{n=k+N_0}^{\Delta} \lambda_n(1 - \lambda_n).$$

Since

$$\sum_{n=k+N_0}^{\Delta} \lambda_n(1 - \lambda_n) = \sum_{n=0}^{\theta(P+k+N_0)} \lambda_n(1 - \lambda_n) - \sum_{n=0}^{k+N_0-1} \lambda_n(1 - \lambda_n)$$

$$\geq (P + k + N_0) - (k + N_0) = P,$$

we get that

$$d(x_{\Delta+1}, z) \leq d(x_{k+N_0}, z) + 2(\Delta - k - N_0 + 1)d(z, Tz) -$$

$$\frac{P\varepsilon}{L}\eta \left(b + 1, \frac{\varepsilon}{L(b + 1)}\right)$$

$$\leq d(x, z) + 2(\Delta + 1)d(z, Tz) - \frac{P\varepsilon}{L}\eta \left(b + 1, \frac{\varepsilon}{L(b + 1)}\right) \text{ by (13)}$$

$$\leq b + \frac{1}{2} - (b + 1) < 0,$$

that is a contradiction. \[\square\]

The proof of Theorem 4.1 follows now exactly like the one of [26, Theorem 4.7]. However, for the sake of completeness, we sketch it in the sequel. Let us
denote \( \alpha_n := \sum_{i=0}^{n} s_i (1 - \lambda_i) \). Since, by Lemma 4.3.(ii), \( d(x_n, Tx_n) \leq 2b \), we get, as an application of (8), that for all \( m, n \in \mathbb{N} \),

\[
d(x_{n+m}, Tx_{n+m}) \leq d(x_n, Tx_n) + 4b(\alpha_{n+m-1} - \alpha_{n-1}).
\]

Apply now Proposition 4.4 to get \( N \in \mathbb{N} \) such that \( d(x_N, Tx_N) < \frac{\varepsilon}{2} \) and \( \gamma_0 + 1 \leq N \leq \Phi \). For \( n \geq \Phi \) it follows that

\[
d(x_n, Tx_n) \leq d(x_N, Tx_N) + 4b(\alpha_{n+\ell-1} - \alpha_{n-1}), \quad \text{where } \ell = n - N
\]

\[
= d(x_N, Tx_N) + 4b(\alpha_{\gamma_0 + q + \ell} - \alpha_{\gamma_0 + q}), \quad \text{where } q = N - 1 - \gamma_0
\]

\[
< \frac{\varepsilon}{2} + 4b(\alpha_{\gamma_0 + q + \ell} - \alpha_{\gamma_0}) \leq \varepsilon,
\]

since \( \gamma \) is a Cauchy modulus for \( (\alpha_n) \).

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