Exact solution of the $p+i\rho$ Hamiltonian revisited: duality relations in the hole-pair picture

Jon Links$^1$, Ian Marquette and Amir Moghaddam

School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia

E-mail: jrl@maths.uq.edu.au, i.marquette@uq.edu.au and a.moghaddam1@uq.edu.au

Received 7 April 2015, revised 14 July 2015
Accepted for publication 17 July 2015
Published 20 August 2015

Abstract

We study the exact Bethe ansatz solution of the $p+i\rho$ Hamiltonian in a form whereby quantum numbers of states refer to hole-pairs, rather than particle-pairs used in previous studies. We find an asymmetry between these approaches. For an attractive system states in the strong pairing regime take the form of a partial condensate involving two distinct hole-pair creation operators. An analogous feature is not observed in the particle-pair picture of an attractive system.

Keywords: pairing Hamiltonian, Bethe ansatz, duality

1. Introduction

The $p_i+i\rho_i$-wave pairing (or simply $p+i\rho$) Hamiltonian rose to prominence through the influential work of Read and Green [1] in identifying topological properties of superconducting systems. An exact Bethe ansatz solution of the Hamiltonian appeared in [2], a result which subsequently generated several studies in the exactly solvable framework [3–9]. Unlike its counterpart the exactly solvable $s$-wave pairing Hamiltonian, also known as the Richardson model [10, 11], the $p+i\rho$ model exhibits quantum phase transitions. A key to gaining a complete understanding of the model’s properties is to understand the distribution of the roots of the Bethe ansatz equations. For example, it is well established [4, 5] that there is a duality between states in the regimes known as weak pairing and strong pairing. There are states in the weak pairing regime characterized by the presence of particle-pairs which carry zero energy, and corresponding dual states in the strong pairing regime with the same energy. Particular groupings of particle-pairs should be viewed as bound, and only arise when
certain constraints are satisfied. The existence of these zero-energy particle-pairs may be interpreted as a form of partial condensation.

Our objective is to revisit the Bethe ansatz solution of the $p+ip$ model from the hole-pair perspective rather than the particle-pair perspective. This involves working with a very closely related, yet distinct, second form of the Bethe ansatz solution. There are a couple of motivations for taking this approach. The first is that Hirsch has long advocated that ‘electron–hole asymmetry is the key to superconductivity’ [12]. As will be established, there is indeed an asymmetry between hole-pairs and particle-pairs which manifests in the character of the roots of the associated Bethe ansatz equations. The second motivation is that there have been some very interesting studies of the Bethe ansatz solutions for pairing Hamiltonians which exploit the existence of two forms of solutions. Pogosov and collaborators have determined sets of relations between the roots of the Bethe ansatz equations in the $s$-wave [13] and ‘Russian doll’ [14] models, which in particular has led to a formula for the ground-state energy in the $s$-wave case. Also Faribault and collaborators [15, 16] have used the existence of the two forms of the exact solution to facilitate the calculation of wavefunction overlaps and scalar products for a general class of systems, which includes the $s$-wave model. Subsequently Claeyss et al [9] have generalized this latter work to accommodate the $p+ip$ model.

When the hole-pairing picture is adopted, we find that zero energy hole-pairs arise which characterize the same duality that was mentioned above. We also find that in addition there exist states characterized by infinite energy hole-pairs, in a manner such that the sum of the infinite energies is finite. Such infinite energy solutions have been previously observed [6, 8], but so far a systematic investigation of them has not been undertaken. Our main finding indicates a second form of duality, which relates the energies of eigenstates of the attractive pairing Hamiltonian to energies of eigenstates of the repulsive pairing Hamiltonian. It also provides a new perspective on the different regimes of the system in terms of hole-pair partial condensation, which we will discuss later.

In section 2 we present the Hamiltonian and the known Bethe ansatz solution in terms of particle-pair quantum numbers. We then determine the second solution in terms of hole-pair quantum numbers. Section 3 is devoted to the discussion of dualities. We commence by recalling the duality previously discussed in [4, 5]. We then continue to establish a second form of duality, and ultimately a third which is a combination of the two. Section 4 examines these dualities in the framework of the Bethe ansatz equations. In section 5 we undertake numerical solution of the Bethe ansatz equations, which leads into an investigation of the dualities diagram in section 6. Section 7 briefly compares the results of the mean-field approximations with the results of the exact solution. Concluding remarks are given in section 8.

2. The Hamiltonian and exact solution

We first introduce the Hamiltonian of the pairing model. We take the canonical (i.e. particle-number-preserving) Hamiltonian whose mean-field approximation leads to the Bogoliubov–de Gennes equations with order parameter having $p_x + ip_y$-wave symmetry up to quadratic approximation. Letting $c_k$, $c_k^\dagger$ denote the annihilation and creation operators for two-dimensional fermions of mass $m$ with momentum $k = (k_x, k_y)$, the Hamiltonian reads [2]

$$H = \sum_k \frac{|k|^2}{2m} c_k^\dagger c_k - \frac{G}{4m} \sum_{k = \pm k'} \left( k_x + ik_y \right) \left( k'_x - ik'_y \right) c_k^\dagger c_{-k} c_{k'} c_{-k'},$$
where $G$ is a dimensionless coupling constant which is positive for an attractive interaction and negative for a repulsive interaction. For any unpaired fermions the action of the pairing interaction is zero and we can thus decouple the Hilbert space into a product of paired and unpaired fermions states, for which the action of the Hamiltonian on the space for the unpaired fermions is diagonal in the basis of the number operator eigenvectors. This is known as the blocking effect [11] and permits an analysis of a simplified version of the Hamiltonian.

We set $z_k = |k|$ and $k_x + i k_y = |k| \exp(i \phi_k)$. It is convenient to introduce the following phase-dependent Cooper pair (or hardcore boson) operators $b_k^\dagger = \exp(i \phi_k) c_k^\dagger c_{-k} = b_k^\dagger b_{-k}$, $b_k = \exp(-i \phi_k) c_{-k} c_k = b_{-k}^\dagger$, and set $N_k = b_k^\dagger b_k = N_{-k}$. Using integers to enumerate the unblocked pairs of momentum states, and setting $m = 1$, the Hamiltonian takes the form

$$H(G) = \sum_{k=1}^L z_k^2 N_k - G \sum_{l=1}^L z_l z_k b_k^\dagger b_k.$$  

The hardcore boson operators satisfy the following commutation relations

$$[b_j, b_k^\dagger] = \delta_{jk} (I - 2N_j), \quad [b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0 \quad (1)$$

where $I$ denotes the identity operator, as well as the relations $(b_j^\dagger)^2 = 0$, $N_j^2 = N_j$. The Hamiltonian may be expressed in the compact form

$$H(G) = (1 + G)H_0 - GQ\dagger Q \quad (2)$$

where

$$H_0 = \sum_{l=1}^L z_l^2 N_l, \quad Q^\dagger = \sum_{l=1}^L z_l b_l^\dagger, \quad Q = \sum_{l=1}^L z_l b_l. \quad (3)$$

For later use we note the commutation relation

$$[Q^\dagger, Q] = 2H_0 - \sum_{j=1}^L z_j^2 I \quad (4)$$

which follows from equation (1).

The Hamiltonian equation (2) is our principal object of study. For each solution of the coupled equations

$$G^{-1} + \frac{2M - L - 1}{y_k} + \frac{1}{\sum_{l=1}^L y_k^2 - z_l^2} = \frac{2}{\sum_{j=k}^M y_k - y_j}, \quad k = 1, \ldots, M \quad (5)$$

there is an eigenstate of equation (2) with energy eigenvalue given by

$$E = (1 + G) \sum_{k=1}^M y_k \quad (6)$$

The eigenstate has the form

$$|\Phi\rangle = \prod_{j=1}^M C(y_j) |0\rangle \quad (7)$$

where

$$C(y) = \sum_{j=1}^L \frac{z_j}{y - z_j} b_j^\dagger. \quad (8)$$
Above, \( M \) is a quantum number which denotes the number of particle-pairs (i.e. Cooper pairs) for the state \( |\Phi\rangle \), i.e.

\[
N |\Phi\rangle = M |\Phi\rangle.
\]  

(7)

The exact solution was reported in [2], and subsequently shown that it could be derived through a variety of means including use of the classical Yang–Baxter equation [3], the quantum inverse scattering method [4], or the Gaudin algebra [5]. A second form of exact solution can be obtained by a particle–hole transformation, denoted \( \Upsilon \), which can be defined by

\[
\Upsilon N |\rangle = I - N, \\
\Upsilon b_j |\rangle = b_j^\dagger, \\
\Upsilon b_j^\dagger |\rangle = b_j,
\]

where \( \Upsilon = \Upsilon^{-1} \) and \( I \) denotes the identity operator. The action of \( \Upsilon \) naturally extends to states. In particular if \( |\chi\rangle \) denotes the completely filled state of \( L \) particle-pairs then

\[
|\chi\rangle = \Upsilon |0\rangle
\]

and if equation (7) holds true then

\[
N \Upsilon |\Phi\rangle = (L - M) \Upsilon |\Phi\rangle.
\]

We then find that

\[
\Upsilon H (G) \Upsilon = -H (-G) + \sum_{j=1}^L z_j^2 I.
\]  

(8)

This provides a simple relationship between the spectrum of the attractive model with \( G > 0 \) and that of the repulsive model with \( G < 0 \).

Below, we will investigate in more detail the consequences of equation (8) in terms of the eigenstates. First note that it follows that a second exact solution exists whereby for each solution of the coupled equations

\[
-\frac{G^{-1} + 2P - L - 1}{\gamma_k} + \sum_{l=1}^L \frac{1}{\gamma_k - \gamma_l} = \sum_{j=k+1}^P \frac{2}{\gamma_k - \gamma_j}, \quad k = 1, \ldots, P
\]  

(9)

there is an eigenstate of equation (2) with energy eigenvalue given by

\[
E = \sum_{l=1}^L z_l^2 + (G - 1) \sum_{k=1}^P \gamma_k,
\]  

(10)

where \( P = L - M \) is the quantum number which denotes the number of hole-pairs. In this instance the eigenstate is of the form

\[
|\Psi\rangle = \prod_{j=1}^P B(\gamma_j) |\chi\rangle
\]

where

\[
B(\gamma) = \Upsilon C(\gamma) \Upsilon \\
= \sum_{j=1}^L \frac{z_j}{\gamma - z_j^2} b_j.
\]  

(11)

This result can also be obtained by a direct calculation following the methods of [19]. The details are provided in appendix A.
3. Duality

Previous studies \([4, 5]\) have identified an exact duality relation in the spectrum of the Hamiltonian. Here we will first recall the essential aspects of that result, before continuing to establish a second exact duality.

A direct commutator calculation leads to the result

\[
[H(G), C(0)] = (1 + G[H_0, C(0)] - GQ^\dagger[Q, C(0)]
\]

\[
= Q^\dagger(GL - 2GN - G - 1).
\]

Using proof by induction we have more generally for \(J \in \mathbb{N}\)

\[
[H(G), (C(0))^J] = JQ^\dagger(C(0))^{J-1}(GL - 2GN - GJ - 1)
\]

At this point it is important to make the observation that \(N\) is a conserved operator, which partitions the space of states according to its eigenvalues \(M = 0, 1, 2, \ldots\). Consider a state \(|\Psi'\rangle\) satisfying

\[
H(G)|\Psi'\rangle = E(G)|\Psi'\rangle,
\]

\[
N|\Psi'\rangle = M'|\Psi'\rangle.
\]

Choosing \(G^{-1} = L - 2M' - J\), which is necessarily integer-valued, and setting

\[
|\Psi\rangle = (C(0))^J|\Psi'\rangle
\]

it follows from equation (12) that

\[
H(G)|\Psi\rangle = E(G)|\Psi\rangle,
\]

\[
N|\Psi\rangle = (M' + J)|\Psi\rangle.
\]

We call \(|\Psi\rangle\) and \(|\Psi'\rangle\) dual states. Setting \(M = M' + J\) dual states are characterized by the relation

\[
M + M' = L - G^{-1}.
\]

We may view the state \(|\Psi\rangle\) as a partial condensate containing \(J\) particle-pairs, each of zero energy in accordance with expression (6).

In the hole-pair picture the duality relation (13) is maintained, and can be verified in a similar fashion. In this instance it is convenient to use equation (4) to express equation (2) as

\[
H = G \sum_{k=1}^{L} z_k^2 I + (1 - G)H_0 - GQQ^\dagger.
\]

It is then found that

\[
[H(G), B(0)] = (1 - G)[H_0, B(0)] - GQ[Q^\dagger, B(0)]
\]

\[
= Q(2GN - GL - G + 1).
\]

Using proof by induction we have, more generally,

\[
[H(G), (B(0))^J] = JQ(B(0))^{J-1}(2GN - GL - GJ + 1)
\]

Consider a state \(|\Phi'\rangle\) satisfying

\[
H(G)|\Phi'\rangle = E(G)|\Phi'\rangle,
\]

\[
N|\Phi'\rangle = M'|\Phi'\rangle.
\]
Choosing $G^{-1} = L - 2M' + J$ and setting
\[ |\Phi\rangle = (B(0))' |\Phi'\rangle \] (15)

it follows from equation (14) that
\[
H(G)|\Phi\rangle = E(G)|\Phi\rangle, \\
N|\Phi\rangle = (M' - J)|\Phi\rangle.
\]

Setting $M = M' - J$ the states $|\Phi\rangle$ and $|\Phi'\rangle$ are characterized by the same duality relation (13). Here we may view the state $|\Phi\rangle$ as a partial condensate containing $J$ hole-pairs, each of zero energy in accordance with expression (10).

### 3.1. A mixed duality

Next we turn the discussion toward a different form of duality relating eigenstates of the Hamiltonian $H(G)$ with eigenstates of $H(-G)$. We first note the preliminary lemma for $K \in \mathbb{N}$
\[
\left[ H_0, Q^K \right] = KQ^{K-1}[H_0, Q]
\] (16)

which is proved by induction. The calculation is straightforward so we omit the details. Expanding out the commutators in equation (16), this expression can be rewritten as
\[
H_0Q^K - Q^KH_0 = KQ^{K-1}H_0Q - KQ^{K-1}H_0
\]
\[
H_0Q^K + (K - 1)Q^KH_0 = KQ^{K-1}H_0Q.
\] (17)

Next we proceed to the following identity, which is also proved by induction:
\[
\left[ H_0, Q^K \right] - \left[ Q^i, Q^K \right]Q = -K\left( H_0Q^K + Q^KH_0 \right) + K\sum_{j=1}^L z_j^2Q^K
\] (18)

The case $K = 1$ may be verified directly using equation (4):
\[
\left[ H_0, Q \right] - \left[ Q^i, Q \right]Q = -(H_0Q + QH_0) + \sum_{j=1}^L z_j^2Q.
\] (19)

Assuming that equation (18) holds true for $K = k - 1$, and using equation (19), we obtain
\[
\left[ H_0, Q^k \right] - \left[ Q^i, Q^k \right]Q = \left[ H_0, Q^{k-1} \right]Q + Q^{k-1}\left[ H_0, Q \right] \\
- \left[ Q^i, Q^{k-1} \right]Q^2 - Q^{k-1}\left[ Q^i, Q \right]Q \\
= \left( (k - 1)\left( H_0Q^{k-1} + Q^{k-1}H_0 \right) + (k - 1)\sum_{j=1}^L z_j^2Q^{k-1} \right)Q \\
+ Q^{k-1}\left( -(H_0Q + QH_0) + \sum_{j=1}^L z_j^2Q \right) \\
= (1 - k)H_0Q^k - kQ^{k-1}H_0Q - Q^{k-1}H_0Q + k\sum_{j=1}^L z_j^2Q^k.
\]
Appealing to equation (17) with $K = k$ then yields
\[
\left[ H_0, Q^k \right] - \left[ Q^\dagger, Q^k \right] Q = -k \left( H_0 Q^k + Q^k H_0 \right) + k \sum_{j=1}^{L} z_j^2 \]
which establishes that equation (18) holds true for $K = k$. The result follows by induction.

Finally,
\[
H(G)Q^k + Q^k H(-G) = ((1 + G)H_0 - GQ^k Q)Q^k + Q^k (1 - G)H_0 + GQ^k Q
\]
\[
= H_0 Q^k + Q^k H_0 + G \left[ H_0, Q^k \right] - G \left[ Q^\dagger Q, Q^k \right]
\]
\[
= H_0 Q^k + Q^k H_0 + G \left[ H_0, Q^k \right] - G \left[ Q^\dagger, Q^k \right] Q.
\]
Through use of equation (18), this leads to the following identity:
\[
H(G)Q^k + Q^k H(-G) = (1 - KG)(H_0 Q^k + Q^k H_0) + KG \sum_{j=1}^{L} z_j^2 Q^k.
\]

(20)

Consider a state $|\Theta\rangle$ satisfying
\[
H(-G)|\Theta\rangle = E'|\Theta\rangle,
\]
\[
N|\Theta\rangle = M'|\Theta\rangle.
\]
Choosing $G^{-1} = K$ which is necessarily integer-valued, and setting
\[
|\Theta\rangle = Q^k |\Theta\rangle,
\]
(21)
it follows from equation (20) that
\[
H(G)|\Theta\rangle = E |\Theta\rangle,
\]
\[
N|\Theta\rangle = M |\Theta\rangle,
\]
where
\[
E = \sum_{j=1}^{L} z_j^2 - E',
\]
\[
M = M' - K.
\]
(22)
We call $|\Theta\rangle$ and $|\Theta\rangle$ mixed dual states, which are characterized by the relation
\[
M' = M = G^{-1}.
\]
(23)
Note that
\[
Q = \lim_{y \to \infty} y B(y).
\]

In contrast to the earlier discussed duality in terms of zero-energy hole-pairs, the operator $Q$ is associated with an infinite energy hole-pair. However the energy of the state remains finite, as the sum of the diverging energies is finite. We prove this in section 4 through an analysis of the Bethe ansatz equations. Single infinite roots also occur in the case when $K = 1$ such that the limiting value of the energy through equation (10) remains finite as $G \to 1$.  

7
3.2. The combined duality

The two dualities described above can be combined to give the following result. Consider a state $|\Omega'\rangle$ satisfying
\[ H(-G)|\Omega'\rangle = E'|\Omega'\rangle, \]
\[ N|\Omega'\rangle = M'|\Omega'\rangle. \]
Setting
\[ |\Omega\rangle = Q^K(B(0))^{J'}|\Omega'\rangle \]
where
\[ K = G^{-1}, \quad J = 2M' - G^{-1} - L, \]
then
\[ H(G)|\Omega\rangle = E|\Omega\rangle, \]
\[ N|\Omega\rangle = M|\Omega\rangle, \]
where
\[ E = \sum_{l=1}^{L} y_l^2 - E', \]
\[ M = M' - K - J. \]
The last relation is equivalent to
\[ M + M' = L. \] (25)

One feature of the combined duality is that the dimension $d(M)$ of the sector for fixed $M$ is equal to $d(M')$. Explicitly, $d(M)$ is given by the binomial coefficient
\[ d(M) = \frac{L!}{M!(L - M)!}. \] (26)

It follows from equation (25) that $d(M) = d(M')$. This points to the possibility that the mapping from states $|\Omega'\rangle$ to $|\Omega\rangle$ through equation (24) is a bijection. Numerical results which support this view are discussed in section 5.

4. Compatibility of dualities and Bethe ansatz equations

In [4] it was shown that the duality characterized by equation (13) is compatible with the Bethe ansatz solution in equation (5). Here we extend that analysis to accommodate the second Bethe ansatz solution in equation (9).

Consider a generic splitting of the set of roots $Y$ of equation (9), with $|Y| = P$, into non-intersecting sets $Y'$ and $Z$ such that $Y = Y' \cup Z$. We can express equation (9) as
\[ -G^{-1} + 2P - L - 1 + \sum_{l=1}^{L} \frac{y_k}{y_l - z_l^2} = \sum_{y_k \in Y', y_l \neq y_k} \frac{2y_k}{y_k - y_l} + \sum_{y_k \in Z} \frac{2y_k}{y_k - y_l}, \]
\[ y_k \in Y' \] (27)
\[ -G^{-1} + 2P - L - 1 + \sum_{l=1}^{L} \frac{y_k}{y_l - z_l^2} = \sum_{y_k \in Z, y_l \neq y_k} \frac{2y_k}{y_k - y_l} + \sum_{y_k \in Y'} \frac{2y_k}{y_k - y_l}, \]
\[ y_k \in Z. \] (28)
Setting \(|Y'| = S\) and \(|Z| = T\) we take the sum in equation (28) over elements in \(Z\) to give

\[
T(-G^{-1} + 2P - L - 1) + \sum_{y \in Z} \sum_{k=1}^L \frac{y_k}{y_k - z_k} = \sum_{x, x' \in Z, y \in Y} \frac{2y_k}{y_k - y_j} + \sum_{y' \in Y'} \frac{2y_k}{y_k - y_j}
\]

\[
= T(T - 1) + \sum_{y \in Y', y' \in Z} \frac{2y_k}{y_k - y_j}.
\]

Equation (29) informs us that

\[
T = -G^{-1} + 2P - L,
\]

(30)

while from equation (27) we obtain

\[
-G^{-1} + 2P - L - 1 + \sum_{k=1}^L \frac{y_m}{y_m - z_k} = \sum_{y \in Y} \frac{2y_m}{y_m - y_j} + 2T, \quad y_m \in Y',
\]

\[
-G^{-1} + 2(P - T) - L - 1 + \sum_{k=1}^L \frac{y_m}{y_m - z_k} = \sum_{y' \in Y'} \frac{2y_m}{y_m - y_j}, \quad y_m \in Y'.
\]

(31)

Equation (31) is the set of Bethe ansatz equations for \(S = P - T\) roots of an attractive system. The above calculations indicate that given such a solution set \(Y'\), we can augment it with \(T\) additional roots which all have zero value to obtain the solution set \(Y\), provided that \(T\) is given by equation (30). Identifying

\[
M = L - P,
\]

\[
M' = L - S,
\]

then equation (30) is equivalent to the duality relation (13).

Alternatively, suppose that at some limiting value of \(G\) we have

\[
y_k \neq 0 \quad \text{for all } y_k \in Y,
\]

\[
y_k = 0 \quad \text{for all } y_k \in Z.
\]

Taking note that \(S + T = P\) equation (29) informs us that

\[
T = -G^{-1} + 2P - L,
\]

(32)

while from equation (27) we obtain

\[
-G^{-1} + 2P - L - 1 + \sum_{k=1}^L \frac{y_m}{y_m - z_k} = \sum_{y \in Y} \frac{2y_m}{y_m - y_j}, \quad y_m \in Y',
\]

\[
G^{-1} + 2(P - G^{-1}) - L - 1 + \sum_{k=1}^L \frac{y_m}{y_m - z_k} = \sum_{y' \in Y'} \frac{2y_m}{y_m - y_j}, \quad y_m \in Y'.
\]

Equation (33) is the set of Bethe ansatz equations for \(S = P - G^{-1}\) roots of a repulsive system. The above calculations indicate that given such a solution set \(Y'\), we can augment it
with \( T \) additional roots which all have infinite value to obtain the solution set \( Y \), provided that \( T \) is given by equation (32), which simplifies to \( T = G^{-1} \). Identifying as before \( MLP \), then equation (32) is equivalent to the mixed duality relation (23).

Moreover, from equation (28) we find

\[
(-G^{-1} + 2P - L - 1) y_k + \sum_{i=1}^{L} \frac{y_i^2}{y_k - z_i^2} = \sum_{y_k \in Z} \frac{2y_k^2}{y_k - z_j^2} + \sum_{y_k \in V} \frac{2y_k^2}{y_k - y_j}, \quad y_k \in Z
\]

\[
(-G^{-1} + 2P - L - 1) \sum_{y_k \in Z} y_k + \sum_{y_k \in Z} \sum_{i=1}^{L} \frac{y_k y_i^2}{y_k - z_i^2}
\]

\[
= \sum_{y_k \in Z} \frac{2y_k^2}{y_k - z_j^2} + \sum_{y_k \in V} \frac{2y_k}{y_k - y_j}
\]

\[
(-G^{-1} + 2P - L - 1) \sum_{y_k \in Z} y_k + \sum_{y_k \in Z} \sum_{i=1}^{L} \frac{y_k y_i^2}{y_k - z_i^2}
\]

\[
= 2(S + T - 1) \sum_{y_k \in Z} y_k + \sum_{y_k \in Z} \sum_{y_j \in V} \frac{2y_k y_j}{y_k - y_j}
\]

Equation (34) then reduces to

\[
\sum_{k=1}^{L} z_k^2 = (1 - T^{-1}) \sigma + 2 \sum_{y_j \in V} y_j
\]

and consequently the energy is found from equation (10):

\[
E = \sum_{l=1}^{T} z_l^2 + \lim_{G^{-1} \to T} (G - 1) \sum_{k=1}^{P} y_k
\]

\[
= \sum_{l=1}^{T} z_l^2 + (T^{-1} - 1) \sum_{y_j \in V} y_j + (T^{-1} - 1) \sigma
\]

\[
= (1 + T^{-1}) \sum_{y_j \in V} y_j.
\]

The above expression is simply equation (6), with the set \( \{y_j\} \) a solution of equation (5). Proceeding further we have

\[
E = \sum_{l=1}^{T} z_l^2 - E'
\]
in agreement with equation (22), where
\[ E' = \sum_{l=1}^{L} z_l^2 - \left( 1 + T^{-1} \right) \sum_{y_j \in Y} y_j \]
is the energy of the dual state for the repulsive Hamiltonian \( H \left( -T^{-1} \right) \) as given by equation (10).

5. Numerical results

It has been established that the Bethe ansatz solution (9) has the property that for particular values of \( G \) there are roots which are zero, and others which are infinite. To numerically solve equation (9) near these values necessarily means that the elements of the solution set will vary across several orders of magnitude, potentially imposing a large computational cost. To alleviate this issue we perform a change of variables:
\[ y_j = \frac{1 + v_j}{1 - v_j}, \]
\[ z_l^2 = \frac{1 + \varepsilon_l}{1 - \varepsilon_l}, \]
such that
\[ y_j = 0 \iff v_j = -1, \]
\[ y_j = \infty \iff v_j = 1. \]
Under this change of variables the Bethe ansatz equations (9) become
\[ \frac{-2G^{-1} + 4P - 2L - 2}{1 - v_n^2} + \frac{L - 2P + 2}{1 - v_n} + \sum_{l=1}^{L} \varepsilon_l - \sum_{q=a}^{P} \frac{2}{v_q} = \sum_{q=a}^{P} \frac{2}{v_q - v_l}. \]

To numerically solve the above equations we adapt a technique described in [17].

The first case we consider is a system with \( L = 8 \) and \( P = 3 \). In this sector the dimension of the state space is 56. Consider \( G = 1/2 \). With respect to equation (23), the dual sector corresponds to \( P = 1 \), which has dimension 8. In this instance the mapping between sectors given by equation (21) cannot be a bijection.

We perturb the coupling by a small amount and numerically solve equation (37) for all roots, with the results displayed in figure 1. It can be seen that that there is a subset of roots close to the value 1. As \( G \to 1/2 \) they converge to 1. In table 1 the root sets are sorted according to increasing energy which is computed through equation (10). It is apparent that the eight lowest energy states each have two roots close to the value 1. Between the particular sectors \( P = 3 \) and \( P = 1 \), the operator \( Q^2 \) is an injection in the limit \( G \to 1/2 \). However since the dimension of the co-domain is larger than the dimension of the domain, not all states in the co-domain are of the form (24).

The second case we consider is a system with \( L = 8 \) and \( P = 6 \). In this sector the dimension of the state space is 28. Again consider \( G = 1/2 \). With respect to equation (25), the dual sector corresponds to \( P = 2 \) which also has dimension 28. In this instance the mapping between sectors given by equation (21) may possibly be a bijection.

We again perturb the coupling by a small amount to make it apparent that the solutions we obtain are not spurious, and that we can account for the full dimension of the sector. We
numerically solve equations (37), with the results displayed in table 2. The root sets are sorted according to increasing energy which is computed through equation (10). It is apparent that all energy states each have a complex conjugate pair of roots close to the value $-1$, and two real roots close to the value 1. As $G \to 1/2$ sixteen roots collapse to the point $\text{Re}(\gamma) = 1$, as illustrated in panel (b) and its inset for which $G = 1/2$.

6. Dualities diagram

We introduce the rescaled coupling parameter $g = GL$ and the filling fraction $x = M/L$. It is convenient to represent the duality relations in terms of the dualities diagram shown in figure 2, which depicts six regions in the $g^{-1} - x$ plane. For the attractive model with $g > 0$ the three regions denoted IV, V, and VI have been previously identified as the weak pairing, strong pairing, and weak coupling regimes respectively [2]. The weak pairing and strong pairing regimes are dual with respect to equation (13). The boundary between these regions is known as the Read–Green line and is given by the relation

$$x = \frac{1}{2} \left( 1 - g^{-1} \right).$$

This line extends into the repulsive region $g < 0$ and provides the boundary between regions II and III which are also dual with respect to equation (13). The boundary between the regions IV and VI is known as the Moore–Read line, and is given by

$$x = 1 - g^{-1}. $$
The ground state on the Moore–Read line is dual to the vacuum through equation (13) and has zero energy.

With respect to the mixed duality relation (23) regions II and V are dual, as are regions III and IV. With respect to the combined duality governed by equation (25), regions III and V are dual. For a state in the strong pairing regime (region V) which may be expressed in the form (24), we define the fraction of zero energy hole-pairs as

\[ h_0 = \frac{J}{L} \]

and the fraction of infinite energy hole-pairs as

\[ h_\infty = \frac{K}{L} \]

Table 1. Numerical solution of the Bethe ansatz equations (37) with \( L = 8, P = 3, G = (1/2) + 0.0000001 \) and \( \varepsilon_j = j/10 \). The energies are calculated through (10) after transforming back to the variables \( y_j \) and \( z_i \) through equations (35) and (36).

| Energy   | Bethe roots                   | Energy   | Bethe roots                   |
|----------|-------------------------------|----------|-------------------------------|
| −7.54299 | 0.99997, 1.00003, 1.49644     | 22.36145 | 0.35540, 0.54943, 0.74665     |
| 1.94946  | 0.13030, 0.99994, 1.00006     | 22.43743 | 0.74865, 0.45717 ± 0.04099    |
| 2.44011  | 0.23860, 0.99994, 1.00006     | 22.55229 | 0.24377, 0.55370, 0.74773     |
| 3.08380  | 0.34552, 0.99994, 1.00006     | 22.70038 | 0.13294, 0.55556, 0.74847     |
| 3.97611  | 0.45327, 0.99994, 1.00006     | 22.92889 | 0.24797, 0.44046, 0.75208     |
| 5.30314  | 0.55903, 0.99993, 1.00007     | 23.05279 | 0.35360, 0.63967 ± 0.04101i   |
| 7.49744  | 0.66657, 0.99992, 1.00008     | 23.07422 | 0.75333, 0.32407 ± 0.03481i   |
| 11.87227 | 0.77566, 0.99989, 1.00011     | 23.07834 | 0.13439, 0.44448, 0.75282     |
| 12.52145 | 0.18033, 0.21200, 0.93364     | 23.12185 | 0.24304, 0.65007 ± 0.04141i   |
| 12.64936 | 0.13716, 0.33338, 0.93230     | 23.20321 | 0.13263, 0.65541 ± 0.04094i   |
| 12.83630 | 0.93013, 0.31915 ± 0.03170i   | 23.33978 | 0.13784, 0.33456, 0.75472     |
| 12.87624 | 0.13380, 0.44202, 0.92986     | 23.34662 | 0.48232, 0.53200, 0.62467     |
| 12.98062 | 0.24692, 0.43819, 0.92852     | 23.49317 | 0.18649, 0.20946, 0.75562     |
| 13.24971 | 0.13239, 0.55018, 0.92567     | 23.78106 | 0.54736, 0.54021 ± 0.11610i   |
| 13.31567 | 0.92413, 0.44485 ± 0.04048i   | 24.17544 | 0.24905, 0.44448, 0.63927     |
| 13.34510 | 0.24279, 0.54830, 0.92424     | 24.30757 | 0.64194, 0.33081 ± 0.03821i   |
| 13.47474 | 0.35936, 0.54410, 0.92226     | 24.32987 | 0.13496, 0.44873, 0.64022     |
| 13.99730 | 0.13162, 0.65940, 0.91663     | 24.60177 | 0.13850, 0.33606, 0.64421     |
| 14.07798 | 0.24104, 0.65843, 0.91494     | 24.63370 | 0.24842, 0.51582 ± 0.02981i   |
| 14.16919 | 0.91099, 0.57513 ± 0.03533i   | 24.73161 | 0.13481, 0.52553 ± 0.03537i   |
| 14.18711 | 0.34973, 0.65670, 0.91260     | 24.75667 | 0.19766, 0.20219, 0.64576     |
| 14.34237 | 0.45985, 0.65262, 0.90915     | 24.87852 | 0.53021, 0.35199 ± 0.04392i   |
| 17.62521 | 0.13125, 0.77844, 0.85271     | 25.18880 | 0.42933, 0.37736 ± 0.09555i   |
| 17.77665 | 0.24030, 0.77968, 0.84490     | 25.25408 | 0.13948, 0.33963, 0.53465     |
| 18.00125 | 0.34831, 0.78269, 0.83171     | 25.40724 | 0.53757, 0.20335 ± 0.01472i   |
| 18.38397 | 0.45676, 0.79787 ± 0.01289i   | 25.57900 | 0.13973, 0.40092 ± 0.00763i   |
| 19.32412 | 0.56759, 0.77091 ± 0.04229i   | 25.79494 | 0.42898, 0.21141 ± 0.02721i   |
| 20.41664 | 0.67105, 0.73644 ± 0.10065i   | 26.03673 | 0.31998, 0.22656 ± 0.05659i   |

The ground state on the Moore–Read line is dual to the vacuum through equation (13) and has zero energy.

With respect to the mixed duality relation (23) regions II and V are dual, as are regions III and IV. With respect to the combined duality governed by equation (25), regions III and V are dual. For a state in the strong pairing regime (region V) which may be expressed in the form (24), we define the fraction of zero energy hole-pairs as

\[ h_0 = \frac{J}{L} \]

and the fraction of infinite energy hole-pairs as

\[ h_\infty = \frac{K}{L} \]
Table 2. Numerical solution of the Bethe ansatz equations (37) with $L = 8$, $P = 6$, $G = (1/2) + 0.0000001$ and $\varepsilon_j = j/10$. The energies are calculated through equation (10) after transforming back to the variables $y_j$ and $z_l$ through (35) and (36).

| Energy       | Bethe roots                                                                 |
|--------------|------------------------------------------------------------------------------|
| $-11.83981209$ | $-0.99999981 \pm 0.00024393i, 0.99997107, 1.35407523, 9.17378076$            |
| $-3.65796172$  | $-0.9999946 \pm 0.00057037i, 0.99996003, 1.72872448, 0.13265362$            |
| $-2.80008720$  | $-0.9999569 \pm 0.00045582i, 0.99995624, 1.89428154, 0.34849627$            |
| $-1.74992090$  | $-0.99999963 \pm 0.00041512i, 0.99995320, 1.00048686, 0.24958685$          |
| $-0.41519412$  | $-0.99999967 \pm 0.00038118i, 0.99994857, 1.0005151, 0.51435633$           |
| $1.37437992$   | $-0.99999970 \pm 0.00035212i, 0.99994039, 2.8406296, 0.66839900$           |
| $4.02553678$   | $-0.99999662 \pm 0.00195582i, 0.99993382, 1.00066865, 0.12039371$          |
| $5.5935661$    | $-0.99999709 \pm 0.00175551i, 0.99993146, 1.00068685, 0.14110366$          |
| $5.89937520$   | $-0.9999774 \pm 0.00148676i, 0.99992647, 1.0007365, 0.13730723, 0.44668753$ |
| $6.01657120$   | $-0.9999807 \pm 0.00128240i, 0.99992250, 1.0007462, 0.3360194 \pm 0.03914953$ |
| $6.38806868$   | $-0.9999819 \pm 0.00123967i, 0.99992318, 1.0007694, 0.25126848, 0.44251302$ |
| $7.21930929$   | $-0.9999827 \pm 0.00124763i, 0.99991780, 1.0008233, 0.13570677, 0.55541293$ |
| $7.70384297$   | $-0.9999861 \pm 0.00104596i, 0.99991363, 1.0008651, 0.24697377, 0.55356595$ |
| $8.18565261$   | $-0.9999896 \pm 0.00084940i, 0.99990850, 1.0009165, 0.46637678, 0.03850363i$ |
| $8.33050601$   | $-0.9999890 \pm 0.00088637i, 0.99990761, 1.0009255, 0.35820837, 0.54928501$ |
| $8.82488231$   | $-0.9999977 \pm 0.00032685i, 0.99992267, 1.0007744, 2.86124262, 0.77673107$ |
| $9.40566610$   | $-0.9999862 \pm 0.00107506i, 0.99998980, 1.0010037, 0.13481277, 0.66436225$ |
| $9.88812763$   | $-0.9999887 \pm 0.00091253i, 0.99998934, 1.0010878, 0.24597898, 0.66345460$ |
| $10.51623429$  | $-0.9999908 \pm 0.00078718i, 0.99988379, 1.0011642, 0.35490601, 0.66184154$ |
| $11.36752125$  | $-0.9999923 \pm 0.00069725i, 0.99986777, 1.0013248, 0.46378746, 0.65830396$ |
| $11.72101631$  | $-0.9999930 \pm 0.00065231i, 0.99985675, 1.0014354, 0.59339982 \pm 0.01965751$ |
| $13.77136379$  | $-0.9999885 \pm 0.00095049i, 0.99985955, 1.0014067, 0.13423675, 0.77454140$ |
| $14.25217295$  | $-0.9999905 \pm 0.00081659i, 0.99984872, 1.0015152, 0.24402831, 0.77416049$ |
| $14.88038709$  | $-0.9999921 \pm 0.00071242i, 0.99985265, 1.0016762, 0.35222296, 0.77357060$ |
| $15.74513536$  | $-0.9999932 \pm 0.00063925i, 0.99980693, 1.0019339, 0.46018824, 0.77251429$ |
| $17.00197794$  | $-0.9999940 \pm 0.00058367i, 0.99976580, 1.0023458, 0.56880755, 0.77001849$ |
| $18.42439918$  | $-0.9999946 \pm 0.00054548i, 0.99971075, 1.0028967, 0.68226619, 0.74962317$ |
where as before $P = L - M$. The states (24) only exist for certain integer values of $G^{-1}$, however in the thermodynamic limit the values of $h_0$ and $h_\infty$ become dense. The thermodynamic limit is obtained by taking the limits

$$M \to \infty,$$

$$L \to \infty,$$

$$G \to 0$$

such that $x$ and $g$ are finite [4, 5]. In this limit we have

$$h_0 = 1 - 2x - g^{-1},$$

$$h_\infty = g^{-1}$$

such that $h_0 + h_\infty$ is independent of $g$.

If we conduct an analogous analysis in the particle-pair picture, through use of the Bethe ansatz solution (5), we do not obtain a complementary portrait. There are no zero-energy particle-pairs, nor are there any infinite energy particle-pairs, in the strong pairing regime. This is a key result of this study, that there is a clear asymmetry between the hole-pair picture and the particle-pair picture.

6.1. Inversion

It is worth briefly mentioning that besides the duality relations discussed above, there exists another type of relation which we call inversion. Consider the Bethe ansatz equations (9) and set $\mu_k = \gamma_k^{-1}$. Then
\[
\begin{align*}
(-G^{-1} + 2P - L - 1)u_k + \sum_{l=1}^{L} \frac{u_k z_l^{-2}}{z_l^{-2} - u_k} &= \sum_{j \neq k} \frac{2u_k u_j}{u_j - u_k} \\
-G^{-1} + 2P - L - 1 + \sum_{l=1}^{L} z_l^{-2} - u_k + u_k &= \sum_{j \neq k} 2(u_j - u_k - u_k) \\
-G^{-1} + 2P - L - 1 + \sum_{l=1}^{L} \frac{u_k}{z_l^{-2} - u_k} &= 2P - 2 + \sum_{j \neq k} \frac{2u_k}{u_j - u_k} \\
\frac{G^{-1} - 1}{u_k} + \sum_{l=1}^{L} u_k - z_l^{-2} &= \sum_{j \neq k} 2 \\
-\tilde{G}^{-1} + 2P - L - 1 + \sum_{l=1}^{L} 1 &= \sum_{j \neq k} 2 
\end{align*}
\]

where
\[
\tilde{G}^{-1} = -G^{-1} + 2P - L.
\tag{38}
\]

If the momentum parameters are chosen such that
\[
\{ z_l; l = 1, \ldots, L \} = \{ z_l^{-1}; l = 1, \ldots, L \}
\tag{39}
\]

inversion maps roots for a Hamiltonian \(H(G)\) to a set of roots for the Hamiltonian \(H(\tilde{G})\). It provides an invertible mapping between solution sets in regions I and VI, and between solution sets in regions II and IV, while regions III and V are each stable under inversion.

It is worthwhile to note that when the condition (39) holds, the composition of the duality relation (13) and the inversion relation (38) is not equivalent to the mixed duality relation (23). This is seen through a counter example. Consider a state \(|\Psi\rangle\) on the boundary of the regions IV and V. The state \(|\Psi\rangle\) is self-dual with respect to equation (13), and under inversion maps to the boundary of regions II and V. Reversing the order of composition, \(|\Psi\rangle\) maps to the boundary of regions II and III under inversion, which is dual to the boundary of regions III and IV with respect to equation (13). However the mixed dual state of \(|\Psi\rangle\) lies on the boundary of regions II and III.

### 7. Mean-field approximation

Our final point of discussion concerns a mean-field approximation analysis, which is a standard technique applied to the analysis of pairing Hamiltonians in general. It has been previously shown in [4, 5] that the mean-field gap and chemical potential equations and the Bethe ansatz solution (5) for the ground state in the continuum limit are equivalent. Here we investigate the extension of that correspondence to include the Bethe ansatz equations (9).

Using a mean-field approach, in particular where products of operators \(A\) and \(B\) are approximated as
\[
AB \approx A \langle B \rangle + \langle A \rangle B - \langle A \rangle \langle B \rangle
\]
the Hamiltonian (2) may be approximated by
\[
\text{2 The coefficient } 1 + G \text{ for } H_0 \text{ has been approximated as 1 since } G = gL^{-1} \text{ and } L \text{ is considered to be large in the mean-field approximation.}
\[ \mathcal{H} = H_0 - \frac{1}{2} \hat{\Delta}^{\dagger} Q - \frac{1}{2} \hat{\Delta} Q^{\dagger} + \frac{\Delta^2}{4G} - \mu (N - M), \]  
(40)

where \( \hat{\Delta} = 2G \langle Q \rangle \), \( \Delta = |\hat{\Delta}| \), \( N \) is the particle-pair number operator, \( M = \langle N \rangle \) and \( \mu \) is a Lagrange multiplier which is introduced since the mean-field approximation does not conserve the particle number. Setting

\[ \mathcal{E}(z_j) = \sqrt{(z_j^2 - \mu)^2 + z_j^2 \Delta^2} \]

the ground-state energy is found to be

\[ E_{\min} = \frac{1}{2} \sum_{j=1}^{L} (z_j^2 - \mu) - \frac{1}{2} \sum_{j=1}^{L} \mathcal{E}(z_j) + \frac{\Delta^2}{4G} + \mu M \]  
(41)

associated to the mean-field ground state

\[ |\Psi_{\text{min}}\rangle = \prod_{j=1}^{L} (u_j I + v_j b_j^\dagger)|0\rangle = \prod_{j=1}^{L} (u_j b_j + v_j I)|\chi\rangle \]  
(42)

where

\[ |u_j|^2 = \frac{1}{2} \left( 1 + \frac{z_j^2 - \mu}{\mathcal{E}(z_j)} \right), \quad |v_j|^2 = \frac{1}{2} \left( 1 - \frac{z_j^2 - \mu}{\mathcal{E}(z_j)} \right). \]

Through the use of the Hellmann–Feynman theorem we may take partial derivatives of equations (40) and (41) to generate the following constraint equations:

\[ \frac{1}{G} = \sum_{j=1}^{L} \frac{z_j^2}{\mathcal{E}(z_j)}, \]  
(43)

\[ L - 2M = \sum_{j=1}^{L} \frac{z_j^2 - \mu}{\mathcal{E}(z_j)} \]  
(44)

which are known as the gap and chemical potential equations. It is apparent that equation (43) cannot admit a solution when \( G < 0 \). However equation (5) maps to (9) with the change \( G \rightarrow -G \) and changing the quantum number to count hole-pairs instead of particle-pairs, while the mean-field wavefunction (42) can be equally expressed in terms of particle creation operators acting on the vacuum or particle annihilation operators (i.e. hole creation operators) acting on the completely filled particle state. At first sight it appears there is a paradox.

If on the other hand we calculate the highest-energy state of the approximation (40) we find

\[ E_{\max} = \frac{1}{2} \sum_{j=1}^{L} (z_j^2 - \mu) + \frac{1}{2} \sum_{j=1}^{L} \mathcal{E}(z_j) + \frac{\Delta^2}{4G} + \mu M \]  
(45)
associated to the mean-field highest-energy state
\[ |\psi_{\max}\rangle = \prod_{j=1}^{L} (v_j^* I - u_j^* b_j^\dagger) |0\rangle, \]
\[ = \prod_{j=1}^{L} (v_j^* b_j - u_j^* I) |\chi\rangle. \quad (46) \]

where * denotes complex conjugation. Through use of the Hellmann–Feynman theorem we may take partial derivatives of equations (40) and (45) to generate the following constraint equations:
\[ -\frac{1}{G} = \sum_{j=1}^{L} \frac{z_j^2}{E(z_j)}, \quad (47) \]
\[ 2 M - L = \sum_{j=1}^{L} \frac{z_j^2 - \mu}{E(z_j)}. \quad (48) \]

Equations (47) and (48) are equivalent to equations (43) and (44) via the particle-hole transformation and the change \( G \rightarrow -G \). This basically asserts that the mean-field approach is justified in calculating the low energy spectrum of the attractive model or the high energy spectrum of the repulsive model. The observation is entirely consistent with equation (8).

Note that in either case we can project the mean-field states (42) and (46) onto the sector with fixed \( M \), which leads to the following unnormalized states
\[ |\psi_{\min}\rangle \rightarrow \left( \sum_{j=1}^{L} v_j b_j^\dagger \right)^M |0\rangle \propto \left( \sum_{j=1}^{L} v_j^* b_j \right)^{L-M} |\chi\rangle, \quad (49) \]
\[ |\psi_{\max}\rangle \rightarrow \left( \sum_{j=1}^{L} u_j b_j^\dagger \right)^M |0\rangle \propto \left( \sum_{j=1}^{L} u_j^* b_j \right)^{L-M} |\chi\rangle. \quad (50) \]

It is known [4] that equation (49) is exactly the ground state on the boundary between regions IV and VI (Moore–Read line). Both forms shown in (49) are obtainable by the Bethe ansatz solutions (5) and (9) respectively. By the same methods it can be shown that equation (50) is exactly the highest-energy state on the boundary between regions I and II. These boundary lines provide the most mean-field-like states. In the hole-pair picture they arise from mappings of the form (21) where the domain is the one-dimensional space with basis \( \{|\chi\rangle\} \).

What is not apparent is how a mean-field approach may be implemented to observe the structure of states with the form (24) in region V, even at the level of an approximation. Conversely there are well-established methods (e.g. see [18]) which in principle permit the calculation of the ground state energy of the attractive system from the Bethe ansatz solution (9) in the continuum limit. Whether or not this approach simply reproduces the continuum limit of equations (43) and (44), or produces some new insights, presents an interesting open question.

8. Conclusion

We presented an alternative form of the Bethe ansatz equations, based on the hole-pair picture, in order to re-examine the \( p+ip \) model. One of the main results we discovered is an
intrinsic asymmetry between the particle-pair and the hole-pair perspectives of a given system with a fixed coupling $G$, in contrast to the $s$-wave paring model on which such a symmetry can be imposed [13, 14]. In particular for the attractive pairing system there are instances of diverging roots of the Bethe ansatz equations in the hole-pair picture which can be precisely identified and counted. It leads us to conjecture that, at particular values of $G$, all states in the strong pairing regime have the form of a partial condensate with the same number of zero-energy pairs, and infinite energy pairs whose energy sum is finite. Significantly, diverging roots do not occur in the Bethe ansatz solution of the attractive model in the particle-pair picture. Our findings are summarized in the dualities diagram in figure 2. A notable feature of the dualities diagram is that the boundary lines, which were determined by exact calculation without approximation, are all independent of the parameters $z_l$ implicit in the Hamiltonian (2).

Acknowledgments

Jon Links and Amir Moghaddam were supported by the Australian Research Council through Discovery Project DP110101414. Jon Links also received support through Discovery Project DP150101294. Ian Marquette was supported by the Australian Research Council through Discovery Early Career Researcher Award DE130101067.

Appendix A. Direct calculation of the exact solution

We start with the observation that

$$H|\chi\rangle = \sum_{j=1}^{L} z_j^A |\chi\rangle.$$ 

To determine exact eigenstates of the Hamiltonian by way of a Bethe ansatz solution, we follow the approach of [19]. Define generic states of the form

$$|\psi\rangle = \prod_{k=1}^{M} B(y_k)|\chi\rangle, \quad |\psi_j\rangle = \prod_{k=j}^{M} B(y_k)|\chi\rangle,$$

$$|\psi_{j\ell}\rangle = \prod_{k=j}^{M} B(y_k)|\chi\rangle$$

where $B(y)$ is given by equation (11). Noting the commutation relations

$$[H_0, B(y)] = \sum_{j=1}^{L} z_j^A b_j = Q - yB(y)$$

$$[Q^\dagger, B(y)] = \sum_{j=1}^{L} \frac{z_j^A}{y - z_j^A} (2N_j - I)$$

3 Some time after submission of the manuscript a proof was obtained. It is somewhat technical and cannot be concisely presented here, so will be deferred for a later publication.
it is then found that

$$
\begin{aligned}
\left( H - \sum_{j=1}^{L} z_{j}^{2} I \right) |\Psi\rangle &= \left( (1 - G) H_{0} + (G - 1) \sum_{j=1}^{L} z_{j}^{2} I \right) |\Psi\rangle - GQQ |\Psi\rangle \\
&= (1 - G) \sum_{j=1}^{P} \left( \left. G |\Psi_{j}\rangle - y_{j} |\Psi\rangle \right) \\
&\quad - GQQ \sum_{j=1}^{P} \sum_{p=1}^{L} B(\chi_{j}) \left( \frac{z_{j}^{2}}{y - z_{j}^{2}} \left( 2N_{p} - 1 \right) \right) B(\chi_{p}) |\chi\rangle \\
&= (1 - G) \sum_{j=1}^{P} \left( \left. G |\Psi_{j}\rangle - y_{j} |\Psi\rangle \right) + GQQ \sum_{p=1}^{L} \sum_{j=1}^{P} \frac{z_{p}^{2}}{y_{p} - z_{p}^{2}} |\Psi_{j}\rangle \\
&\quad + 2GQQ \sum_{j=1}^{P} \sum_{p=1}^{L} \sum_{l=1}^{L} \frac{z_{p}^{2}}{y_{l} - z_{p}^{2}} \left( y_{l} - z_{p}^{2} \right) b_{j}^{\dagger} |\Psi_{j}\rangle \\
&= (1 - G) \sum_{j=1}^{P} \left( \left. G |\Psi_{j}\rangle - y_{j} |\Psi\rangle \right) - GQQ \sum_{j=1}^{L} \sum_{p=1}^{P} \frac{z_{j}^{2}}{y_{j} - z_{j}^{2}} |\Psi_{j}\rangle \\
&\quad + GQQ \sum_{j=1}^{P} \sum_{l=1}^{L} \left( \frac{y_{j} z_{l}^{2}}{(y_{j} - y_{l}) (y_{j} - z_{l}^{2})} + \frac{y_{l} z_{j}^{2}}{(y_{l} - y_{j}) (y_{l} - z_{j}^{2})} \right) b_{l} |\Psi_{l}\rangle \\
&= (1 - G) \sum_{j=1}^{P} \left( \left. G |\Psi_{j}\rangle - y_{j} |\Psi\rangle \right) - GQQ \sum_{j=1}^{L} \sum_{p=1}^{P} \frac{z_{j}^{2}}{y_{j} - z_{j}^{2}} |\Psi_{j}\rangle \\
&\quad + GQQ \sum_{j=1}^{L} \sum_{l=1}^{L} \left( \frac{y_{j}}{y_{j} - y_{l}} |\Psi_{l}\rangle + \frac{y_{l}}{y_{l} - y_{j}} y_{l} |\Psi_{l}\rangle \right) \\
&= (1 - G) \sum_{j=1}^{P} \left( \left. G |\Psi_{j}\rangle - y_{j} |\Psi\rangle \right) \\
&\quad - GQQ \sum_{j=1}^{P} \sum_{l=1}^{L} \frac{z_{j}^{2}}{y_{j} - z_{j}^{2}} |\Psi_{j}\rangle + 2GQQ \sum_{j=1}^{P} \sum_{l=1}^{L} \frac{y_{j}}{y_{j} - y_{l}} |\Psi_{l}\rangle
\end{aligned}
$$

The terms proportional to $|\Psi_{j}\rangle$ cancel provided

$$
G - 1 + G \sum_{l=1}^{L} \frac{z_{j}^{2}}{y_{j} - z_{j}^{2}} = 2G \sum_{l=1}^{P} \frac{y_{j}}{y_{j} - y_{l}}, \quad k = 1, \ldots, P
$$

which can be equivalently written as equation (9). For each solution of those coupled equations, $|\psi\rangle$ is an eigenstate of the Hamiltonian with energy eigenvalue given by

$$
E = \sum_{l=1}^{L} z_{l}^{2} + (G - 1) \sum_{k=1}^{P} y_{k}.
$$
Appendix B. Proof of an inequality

Here we show that when $M + M' = L - G^{-1}$ with $G^{-1} > 0$, and $L \geq M' \geq M$, that
\[ d(M') \geq d(M) \quad (51) \]
where $d(M)$ is given by equation (26). Noting that
\[ L \geq M' \geq L - M' - G^{-1}, \]
this is equivalent, in view of equation (26), to establishing that for $A - C \geq B \geq C$,
\[ \frac{C!(A - C)!}{B!(A - B)!} \geq 1. \]

When $X > Y$ we have from the definition of the factorial that
\[ \frac{X!}{Y!} \leq X^{X-Y}, \]
\[ \frac{X!}{Y!} \geq (Y + 1)^{X-Y}. \]
Then
\[ \frac{(A - C)!}{B!(A - B)!} \geq (B + 1)^{A-B-C}, \]
\[ \frac{C!(A - C)!}{(A - B)!} \geq \frac{1}{(A - B)^{A-B-C}}. \]
This establishes that
\[ \frac{C!(A - C)!}{B!(A - B)!} \geq 1 \]
whenever $2B + 1 \geq A$. Alternatively,
\[ \frac{(A - C)!}{(A - B)!} \geq (A - B + 1)^{B-C}, \]
\[ \frac{C!}{B!} \geq \frac{1}{B^{B-C}}. \]
This establishes that
\[ \frac{C!(A - C)!}{B!(A - B)!} \geq 1 \]
whenever $A + 1 \geq 2B$. It then follows that equation (51) is true.

References

[1] Read N and Green D 2000 Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect Phys. Rev. B 61 10267
[2] Ibañez M, Links J, Sierra G and Zhao S-Y 2009 Exactly solvable pairing model for superconductors with $p_x + ip_y$-wave symmetry Phys. Rev. B 79 180501
[3] Skrypnyk T 2009 Non-skew-symmetric classical r-matrices and integrable cases of the reduced BCS model J. Phys. A: Math. Theor. 42 472004
[4] Dunning C, Ibañez M, Links J, Sierra G and Zhao S-Y 2010 Exact solution of the $p + ip$ pairing Hamiltonian and a hierarchy of integrable models J. Stat. Mech.: Theor. Exp. P08025
[5] Rombouts S M A, Dukelsky J and Ortiz G 2010 Quantum phase diagram of the integrable $p_x + ip_y$ fermionic superfluid Phys. Rev. B 82 224510
[6] Ortiz G, Nussinov Z, Dukelsky J and Seidel A 2013 Repulsive interactions in quantum Hall systems as a pairing problem Phys. Rev. B 88 165303
[7] Rodríguez-Laguna J, Ibáñez Berganza M and Sierra G 2014 Energy space entanglement spectrum of pairing models with $s$-wave and $p$-wave symmetry Phys. Rev. B 90 041103
[8] Van Raemdonck M, De Baerdemacker S and Van Neck D 2014 Exact solution of the $p_x + ip_y$ pairing Hamiltonian by deforming the pairing algebra Phys. Rev. B 89 155136
[9] Claeyss P W, de Baerdemacker S, van Raemdonck M and van Neck D 2015 Eigenvalue-based method and form factor determinant representations for integrable XXZ Richardson–Gaudin models Phys. Rev. B 91 155102
[10] Richardson R W 1963 A restricted class of exact eigenstates of the pairing-force Hamiltonian Phys. Lett. 3 277
[11] von Delft J and Ralph D C 2001 Spectroscopy of discrete energy levels in ultrasmall metallic grains Phys. Rep. 345 61
[12] Hirsch J E 2003 Electron-hole asymmetry is the key to superconductivity Int. J. Mod. Phys. B 17 3236
[13] Pogosov W V, Lin N S and Misko V R 2013 Electron-hole symmetry and solutions of Richardson pairing model Eur. Phys. J. B 86 235
[14] Pogosov W V and Bork L V 2015 Particle-hole duality, integrability, and Russian Doll BCS model Nucl. Phys. B 897 405
[15] Faribault A and Schuricht D 2012 On the determinant representations of Gaudin models’ scalar products and form factors J. Phys. A: Math. Theor. 45 485202
[16] Tschirhart H and Faribault A 2014 Algebraic Bethe ansätze and eigenvalue-based determinants for Dicke–Jaynes–Cummings–Gaudin quantum integrable models J. Phys. A: Math. Theor. 47 405204
[17] Marquette I and Links J 2012 Generalized Heine-Stieltjes and Van Vleck polynomials associated with two-level, integrable BCS models J. Stat. Mech. P08019
[18] Amico L, di Lorenzo A, Mastellone A, Osterloh A and Raimondi R 2002 Electrostatic analogy for integrable pairing force Hamiltonians Ann. Phys. 299 228
[19] Birrel A, Isaac P S and Links J 2012 A variational approach for the quantum inverse scattering method Inverse Prob. 28 035008