Causality violation and singularities

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Abstract

We show that singularities necessarily occur when a boundary of causality violating set exists in a space-time under the physically suitable assumptions except the global causality condition in the Hawking-Penrose singularity theorems. Instead of the global causality condition, we impose some restrictions on the causality violating sets to show the occurrence of singularities.

1 Introduction

Space-time singularities have been discussed for a long time in general relativity. In 1970, Hawking and Penrose\[1\] showed that singularities, which mean causal geodesic incompleteness, could occur in a space-time under seemingly reasonable conditions in classical gravity. Their singularity theorem has an important implication that our universe has an initial singularity if we do not consider quantum effects. However, this theorem is physically unsatisfactory in the sense that the causality requirement everywhere in a space-time seems too restrictive. We can only experience local events and there is no guarantee that the causality holds in the entire universe. As is well known, Kerr type black holes have causality violating sets if the space-time is maximally extended. Therefore, it will be important to investigate occurrence of singularities in a space-time in which the global causality condition is violated.

There are some works on a causality violation concerned with the occurrence of singularity. Tipler\[2,3\] showed that any attempt to evolve closed timelike curves from an initial regular Cauchy data would cause singularities to form in a
space-time. He presented a singularity theorem in which the global causality condition in the Hawking-Penrose theorem is replaced by the weaker one and adding the stronger energy condition. In his theorem his stronger energy condition is essential to the occurrence of singularities.

Kriele presented his singularity theorems in which causality violating sets are restricted but with usual energy condition in the Hawking-Penrose theorem instead of Tipler’s energy condition. He showed that the causality violating set has incomplete null geodesics if its boundary is compact [4]. Kriele[5] also presented a generalization of the Hawking-Penrose singularity theorem. In his paper he showed that singularities would occur provided that causality holds at least in the future endpoints of the trapped set.

Newman[6] found a black hole solution which had no singularities. This black hole solution is obtained by a suitable conformal transformation of the G"odel universe: One might consider his conclusion would suggest that causality violating set can prevent singularities from occurring. However, his case seems too special and even unphysical, because causality is violated in the entire space-time. It is physically more acceptable to assume that at least there must be causality preserving regions in a space-time. One can pick up the Taub-NUT universe as an example which contains both causality violating and preserving sets. In this universe, there exist singularities on the boundary of causality violating sets. This suggests that the boundary generates a geodesic incompleteness.

In this paper we shall show that the boundary of causality violating sets are essential to occurrence of singularities. We also discuss relation between our theorems and the Hawking-Penrose theorem.

In the next section, we briefly review Tipler’s and Kriele’s singularity theorems. In section 3, the definitions and the lemmas for discussing causal structure and singularities are listed up. We present our singularity theorems for partially causality violating space-times in section 4. Section 5 is devoted to summary.

2 Tipler’s and Kriele’s theorems

We review Tipler’s and Kriele’s theorems in this section. In addition, we discuss how causality violation is related to singularities in these theorems.

First, we quote Tipler’s theorem.

Tipler’s theorem(1977)
A space-time \((M, g)\) cannot be null geodesically complete if
(1) \(R_{ab}K^aK^b \geq 0\) for all null vectors \(K^a\);
(2) there is a closed trapped surface in \(M\);
(3) the space-time is asymptotically deterministic, and the Einstein equations hold;
(4) the partial Cauchy surface defined by (3) is non-compact.
Here the asymptotically deterministic condition in the condition (3) is defined as follows.

**Definition**
A space-time \((M, g)\) is said to be **asymptotically deterministic** if 
(i) \((M, g)\) contains a partial Cauchy surface \(S\) such that 
(ii) either \(H(S) = H^+(S) \cup H^-(S)\) is empty, or if not, then 

\[
\lim_{s \to a} \inf T_{ab}K^aK^b > 0
\]

on at least one of the null geodesic generators \(\gamma(s)\) of \(H(S)\), where \(a\) is the past limit of the affine parameter along \(\gamma\) if \(\gamma \in H^+(S)\), and the future limit if \(\gamma \in H^-(S)\). \((K^a\) is the tangent vector to \(\gamma\).)

This condition has been introduced by following reasons. In the case that the formation of a Cauchy horizon \(H^+(S)\) is due to causality violation, one would expect that the region where \(H^+(S)\) begins would contain enough matter (the condition (ii)) which causes gravitational field sufficiently strong so as to tip over the light cones and eventually leads to causality violation. We can regard this condition as a special type of energy condition which dispenses with the causality condition in the Hawking-Penrose singularity theorem.

In the following sections, we shall impose some conditions on causality violating sets to replace global causality condition in the Hawking-Penrose theorem instead of imposing this energy condition.

Next, we quote some definitions and Kriele’s theorems [4, 5].

**Definition**

• **focal point**
Let \(S\) be a locally spacelike surface (not necessary achronal surface) and let us consider a future directed null geodesic, \(\beta(t)\), from \(S\) parameterized by \(t\). If for any point \(\beta(t)\) such that \(t \geq t_1\), there exists an arbitrarily close timelike curve from \(S\) to the point \(\beta(t)\), then \(\beta\) is called a **focal point** to \(S\).

• **Generalized future horismos of \(S\)**
Generalized future horismos of \(S\), denoted by \(e^+(S, M)\), is a closure of all future null geodesics \(\beta\) from \(S\) which have no focal points. (The future end points of \(e^+(S, M)\) correspond to the focal points.)

• **cut locus: \(cl(S, M, +)\)**
The set of future end points of \(e^+(S, M)\).

• **almost closed causal curve**
Choose an arbitrary Riemannian metric $h$ of $M$. Let $\alpha$ be a curve and $\beta$ be a reparametrization of $\alpha$ with $h(\beta', \beta') = 1$. Then $\alpha$ is called almost closed if there exists an $X \in \beta'(t)$ such that for every neighbourhood $U$ of $X$ in the tangent bundle, $TM$, there exists a deformation $\gamma$ of $\beta$ in $\pi_{TM}(U)$ which yields a closed curve and satisfies $\gamma(t) \in \pi(U) \Rightarrow \gamma'(t) \in U$.

Kriele’s theorem
Theorem 1(1990)

$(M, g)$ is causal geodesically incomplete if:
1. $R_{ab}K^aK^b \geq 0$ for every causal vector $K^a$ and the generic condition is satisfied.
2. (a) there exists a closed locally spacelike but not necessarily achronal trapped surface $S$ or (b) there exists a point $r$ such that on every past (or every future) null geodesic from $r$ the divergence $\theta$ of the null geodesics from $r$ becomes negative or (c) there exists a compact achronal set $S$ without edge.
3. neither $\text{cl}(S, M, +)$ (respectively $\text{cl}(r, M, +)$) nor any $\text{cl}(D, M, -)$, where $D$ is a compact topological submanifold (possibly with boundary) with $D \cap S \neq \emptyset$ (respectively $r \in D$), contains any almost closed causal curve that is a limit curve of a sequence of closed timelike curves.

This theorem is the maximum generalization of the Hawking-Penrose theorem in the sense that causality may be violated in the almost all regions except the cut locus. In this theorem causality violation seems to play a role of keeping the space-time under consideration from having singularities.

In theorem 2 below it is shown that there exist singularities when the causality violating set is compact even if there is no trapped surface. However, one cannot see which causes singularities, the compactness of the causality violating set or causality violation itself.

Theorem 2(1989)

Let $(M, g)$ be a space-time with chronology violating set $V$ that satisfies
1. $R_{ab}K^aK^b \geq 0$ for every null vector $K^a$ and the generic condition is satisfied.
2. $V$ has a compact closure but $M - V \neq \emptyset$.

Then $V$ is empty or $\dot{V}$ is generated by almost closed but incomplete null geodesics.

3 Preliminaries

We consider a space-time $(M, g)$, where $M$ is a four-dimensional connected differentiable manifold and $g$ is a Lorentzian and suitably differentiable metric. In this section, we quote some definitions and useful lemmas from (HE)[1] for the
discussion of causal structure and space-time singularities.

**Definition (HE)**

A point \( p \) is said to be a *limit point* of an infinite sequence of non-spacelike curves \( l_n \) if every neighbourhood of \( p \) intersects an infinite number of the \( l_n \)'s.

A non-spacelike curve \( l \) is said to be a *limit curve* of the sequence \( l_n \) if there is a subsequence \( l'_n \) of the \( l_n \) such that for every \( p \in l \), \( l'_n \) converges to \( p \).

**Proposition 1 (HE 6.4.1)**

The chronology violating set \( V \) of \( M \) is the disjoint union of the form \( I^+(q) \cap I^-(q), q \in M \).

**Lemma 1 (HE 6.2.1)**

Let \( O \) be an open set and let \( l_n \) be an infinite sequence of non-spacelike curves in \( O \) which are future-inextendible in \( O \). If \( p \in O \) is a limit point of \( l_n \), then through \( p \) there is a non-spacelike curve \( l \) which is future-inextendible in \( O \) and which is a limit curve of the \( l_n \).

**Proposition 2 (HE 4.5.10)**

If \( p \) and \( q \) are joined by a non-spacelike curve \( l(v) \) which is not a null geodesic they can also be joined by a timelike curve.

**Proposition 3 (HE 4.4.5)**

If \( R_{ab}K^aK^b \geq 0 \) everywhere and if at \( p = \gamma(v_1), K^eK^dK_{[a}R_{b]cd[e}K_{f]} \) is non-zero, there will be \( v_0 \) and \( v_2 \) such that \( q = \gamma(v_0) \) and \( r = \gamma(v_2) \) will be conjugate along \( \gamma(v) \) provided \( \gamma(v) \) can be extended to these values.

**Proposition 4 (HE 4.5.12)**

If there is a point \( r \) in \((q, p)\) conjugate to \( q \) along \( \gamma(t) \) then there will be a variation of \( \gamma(t) \) which will give a timelike curve from \( q \) to \( p \).

**Proposition 5 (HE 6.4.6)**

If \( M \) is null geodesically complete, every inextendible null geodesic curve has a pair of conjugate points, and chronology condition holds on \( M \), then the strong causality condition holds on \( M \).

**Proposition 6 (HE 6.4.7)**

If the strong causality condition holds on a compact set \( \varphi \), there can be no past-inextendible non-spacelike curve totally or partially past imprisoned in \( \varphi \).

*Prop. 5* physically means that the chronology condition is equivalent to the strong causality condition if energy conditions are satisfied.
4 The theorem

Generally, one can consider either of the following two cases in which causality violating sets and their boundaries exist. One is that there are closed null geodesic curves lying on the boundary or closed non-spacelike curves which pass through at least one point on the boundary. The other is that there is no closed non-spacelike curve which passes through a point on the boundary. Here we have used the word closed curve in a specific sense that the curve is closed and moreover one lap length of the curve does not diverge.

We will show in each case that such a space-time has singularities in what follows.

Theorem 1

If a space-time \((M, g)\) is null geodesically complete, then the following three conditions cannot be all satisfied together:

(a) There exists a chronology violating region \(V\) which does not coincide with the whole space-time, i.e. \(M - V \neq \emptyset\),

(b) every inextendible non-spacelike geodesic in \((M, g)\) contains a pair of conjugate points,

(c) there exists at least one point \(p\) on the boundary of \(V\) such that each closed timelike curve through a point in the \(V \cap \varepsilon\) can be entirely contained in some compact set \(K\). (\(\varepsilon\) is an arbitrary small neighbourhood of \(p\).)

As mentioned above, if the condition (c) is satisfied, roughly speaking, one can always pick out an infinite sequence such that the one lap length of each closed timelike curve does not diverge and their shape does not change abruptly when a point on each closed curve approaches to the boundary of \(V\).

This condition (c) is satisfied, for example, on the causality violating sets which cause compactly generated Cauchy horizons [7]. Causality violating sets of the Taub-NUT universe also satisfy the condition (c) because whose boundaries contain closed null geodesics. Therefore we can apply Theorem 1 to the Taub-NUT universe, which indeed has singularities.

This condition (c) does not require that the boundary \(\partial V\) is compact. Thus Theorem 1 is essentially different from the Kriele’s theorem 2.

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3The case that whole null generators of the boundary are closed or imprisoned is similar to the situation which Hawking considered [7]. When he discussed the chronology violating sets appearing in a bounded region of general space-time without curvature singularities, he introduced the notion of the compactly generated Cauchy horizon defined as a Cauchy horizon such that all the past directed null geodesic generators enter and remain within a compact set. This is analogous to the existence of closed null curves on the boundary of \(V\). He asserted that one cannot make such a Cauchy horizon while the weak energy condition is satisfied. This also supports our claims.
Proof.
The chronology violating set $V$ is an open set by Prop 1. If $V \neq \emptyset$, from the condition (a), we can find a boundary set $\bar{V}$ in $M - V$. Let us consider a sequence of points $q_n$ in $V \cap \varepsilon$ which converges to $p$ ($\lim_{n \to \infty} q_n = p$). By the definition of $V$ there is a closed timelike curve $l_n$ through $q_n$. From the condition (c), there exists a compact set $K$ such that each $l_n$ is entirely contained in $K \cap V$. Let $l$ be a limit curve of the sequence $l_n$ which passes through the limit point $p$. Choosing a suitable parameter of each $l_n$ so that $l_n$ is inextendible, the limit curve $l$ is also non-spacelike inextendible curve in $K \cap \tilde{V}$ by Lemma 1.

Let us consider the case that the limit point $p \in J^+(q)$, $q \in V$ without loss of generality. This limit curve must also be contained in $K \cap \tilde{V}$ because of the condition (c). Therefore $l$ is totally past and future imprisoned in $K \cap \tilde{V}$. If some point $p'$ of $l$ which is in the past of $p$ is contained in $V$, there exists a closed non-spacelike curve but not null geodesic through $p$. Because one can connect the limit point $p$ to some point $c$ in $V$ in the future of the $p$ with some non-spacelike curve $\lambda$, one can always find a closed non-spacelike curve but not a null geodesic one such that $p \rightarrow c \rightarrow q \rightarrow p' \rightarrow p$ as depicted in Figure 1. This curve can be varied to a closed timelike curve through $p$ by Prop.2. This contradicts with the achronality of the boundary $\tilde{V}$ in which $p$ is contained. Therefore any point of $l$ in the past of $p$ is not contained in $V$, but in the compact set $\check{J}^+(q) \cap K$. If the null geodesic generator $l$ of $\check{J}^+(q)$ through $p$ is closed, this generator has no future and past end points. Then $l$ has pair conjugate points from Prop.3 if $l$ is complete. This contradicts with the achronality of the boundary $\tilde{V}$. Therefore this null geodesic generator $l$ is not closed but past imprisoned in the compact set $\check{J}^+(q) \cap K$. Let $p_n \in \{K \cap (M - \tilde{V})\}$ be an infinite sequence which converges to $p \in l$ and $r_n \in \{K \cap (M - \tilde{V})\}$ be another infinite sequence such that $r_n \in \check{J}^-(p_n)$ and converges to the point $r(\neq p)$ on $l$. Then one can take an infinite sequence of curves $\lambda_n$ such that each of which is an inextendible null geodesic through $p_n$ and $r_n$. If $M$ is null geodesically complete, each $\lambda_n$ can be extended into the open region $\{\check{J}^-(p_n) \cap \check{I}^-(K)\}$ because each $\lambda_n$ is entirely contained in $M - \tilde{V}$ where the strong causality condition holds by using Prop.5. Therefore, the limit curve $\lambda$ of $\lambda_n$, which is an inextendible null geodesic curve through $p$ and $r$ from Lemma 1, is not imprisoned in the compact set $\{K \cap \check{J}^+(q)\} \subset \{K \cap (M - \tilde{V})\}$. However, this contradicts the fact that $\lambda$ coincides with $l$ by reparametrization of affine parameter since both of them are null geodesics through the two points $p$ and $r$. Otherwise, there exists a null curve broken at $p$ and $r$ which is lying on $\tilde{V}$, and it can be deformed to a timelike curve. This contradicts the achronality of $\tilde{V}$. 

Combining Theorem 1 and the Hawking-Penrose theorem [1], we immediately get the following corollary.

Corollary
If a space-time $(M, g)$ is causally complete, then the following conditions cannot
all hold:
(1) every inextendible non-spacelike geodesic contains a pair of conjugate points,
(2) the chronology condition holds everywhere on \((M, g)\) or even if chronology
condition is violated somewhere, such a region satisfies the condition \((c)\),
(3) there exists a future-(or past-)trapped set \(S\).

We have considered the case that a chronology violating set satisfies the con-
dition \((c)\). However, the causality violating sets in the Kerr black hole do not
satisfy the condition \((c)\). So we cannot apply our Theorem 1 to the Kerr solution.
However, we could still prove the existence of singularities if a given s-
pace-time satisfies the condition below.

\textbf{Condition \((c')\)}

Let each chronology violating set be \(V_i\). Any \(V_i\) is causally separated from \(V_j \neq i\),
i.e. \((\dot{J}^+(q) \cup \dot{J}^-(q)) \cap V_{j \neq i} = \emptyset\) for all \(q \in V_i\).

For a space-time \((M, g)\) which satisfies this condition \((c')\) but not the condition
\((c)\), we can apply Kriele’s theorem 1, taking a set \(S\) in his theorem 2 as \(\dot{J}^+(q) \cap \dot{J}^-(q)\).
In usual, we expect that the set \(\dot{J}^+(q) \cap \dot{J}^-(q)\) is compact, which may have
the topology \(S^2\). However, there is a case that \(\dot{J}^+(q) \cap \dot{J}^-(q)\) has non-compact
topology. For example, in the case that \(\dot{J}^+(q) \cap \dot{J}^-(q)\) has topology \(S^1 \times \mathbb{R}\), we
can regard the quotient space \(e^+(\dot{J}^+(q) \cap \dot{J}^-(q))/\mathbb{R}\) as the \(e^+(S)\) in the Kriele’s
theorem 1, because the relevant thing in his theorem is that \(e^+(S)\) is compact.

We obtain the following theorem.

\textbf{Theorem 2}

A space-time \((M, g)\) which satisfies the conditions (a), (b), and either \((c)\) or \((c')\)
is null or timelike geodesically incomplete.

As easily verified from the Penrose diagram of the Kerr solution, the condi-
tion \((c')\) is satisfied for the Kerr solution. This theorem is applicable to the Kerr
solution which indeed has singularities.

\textbf{proof.}

We suppose that \((M, g)\) is null or timelike geodesically complete. We only have
to prove the case that the condition \((c')\) hold but the condition \((c)\) does not. In
such a space-time \((M, g)\), every null geodesic generator on \(\dot{V}\) is not closed.

Now we consider a non-closed null geodesic on \(\dot{V}\). This null geodesic belongs
to \(\dot{J}^+(q)\) or \(\dot{J}^-(q)\), as \(V\) is \(I^+(q) \cap I^-(q)\) \((q \in V)\). Let this null geodesic belong
to \(\dot{J}^+(q)\) without losing generality. If this null geodesic has a past end point, it
must be \(q\). Let us take a point \(p(\neq q)\) on this null geodesic and also let it be on
the \(\dot{V}\). Because \(q \in V\), there is a closed timelike curve through \(q\). This means
that a timelike curve from \(q\) to \(p\) exists by Prop. 2. Therefore, \(p\) belongs to \(I^+(q)\).
This contradicts \( p \in \dot{V} \). If this null geodesic has no past end point, it is inextendible in the past. If the boundary of \( V \) contains this null geodesic entirely, from the condition (b), this boundary can be connected by timelike curves by Prop.4. This is also contradiction to the achronality of \( \dot{V} \). Hence, let us consider the case that the boundary of \( V \) does not contain the whole segment of this null geodesic, that is, the null geodesic has an end point on the compact surface \( S := \mathring{J}^+(q) \cap \mathring{J}^-(q) \). Extending this null geodesic beyond the future end point, we obtain an inextendible null geodesic lying on \( \mathring{J}^+(q) \) and call it outgoing. We also obtain an inextendible null geodesic belongs to \( \mathring{J}^-(q) \) and call it ingoing. The outgoing null geodesic has a pair of conjugate points from the condition (b). One of the conjugate points is on the segment lying on the \( \dot{V} \). The other is on the segment lying on the \( \mathring{J}^+(q) - \dot{V} \). The ingoing null geodesic on the \( \mathring{J}^-(q) \) also has a pair of conjugate points in the same way as the outgoing case. Thus, \( S \) plays the same role as the trapped surface in the Kriele’s theorem 1. From the condition \((c')\), the condition (3) of Kriele’s theorem 1 is satisfied in the cut locuses of \( S \), intersections of outgoing and ingoing null geodesics, because the condition \((c)\) is not satisfied (if the condition \((c)\) is satisfied, there exists an almost closed causal curve.). Therefore we can show the existence of singularities from Kriele’s theorem 1. \( \square \)

5 Conclusions and discussion

We have shown that the boundaries of causality preserving and violating regions cause singularities in a physical space-time.

We would like to emphasize that Theorem 1 supplements the Hawking-Penrose theorem in the sense that the global causality condition is relaxed to some degree and instead, the condition \((c)\) or the condition \((c')\) is imposed on chronology violating sets. Roughly speaking, it is possible for observers to talk about the existence of singularities assuming that our space-time has a causality preserving region, which conforms to our experience.

Whether the quasi-global condition \((c')\) is removable or not is still an open question.

As well as the Hawking-Penrose theorem, our theorem cannot predict where singularities exist and how strong they are, which are left for future investigations.

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Figure 1: In the case that the past points of the limit curve $l$ go into the $V$, we can find a closed non-spacelike non-geodesic curve like a $p \rightarrow c \rightarrow q \rightarrow p' \rightarrow p$, which is the union of $\lambda$ and a segment $p' \rightarrow p$. 
Figure 2: Examples of the space-time in which the limit curve of infinite sequence of closed timelike curves do not close.