SOME GENERAL PROBLEMS IN QUANTUM GRAVITY II: THE THREE DIMENSIONAL CASE

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Abstract

The general problems of three-dimensional quantum gravity are re-catitulated here, putting the emphasis on the mathematical problems of defining the measure of the path integral over all three-dimensional metrics. This work should be viewed as an extension of a preceding one on the four dimensional case ([12]), where also some general ideas are discussed in detail. We finally put forward some suggestions on the lines one could expect further progress in the field.
1 Introduction

We have grown used to the notion of "euclidean quantum gravity" (cf., for example, ref. [2]), thanks, among others, to the work of Hawking and collaborators.

It should nevertheless be stressed that this path integral stands apart from other, superficially similar, expressions one could write for other quantum fields, in the sense that it is not a representation of a known unitary evolution of states in a Hilbert space.

It seems hardly arguable, however, that the problems encountered while treating to make sense of the euclidean path integral, being of a very general character, will also reappear in one guise or another in any physical formulation of the problem.

The purpose of the present work is to present some general observations of the main characteristics of a sum over three-dimensional riemannian geometries. Before doing that, however, we review in the first paragraph Witten’s solution to three-dimensional quantum gravity as a Chern-Simons theory (cf. [2][3][4][5]), and we discuss in what sense it can be considered as a complete solution of our problem. In the second paragraph we explain some mathematical details on the problem of the homeomorphic equivalence in $d = 3$, in order to determine in what sense gauge fixing is possible in the sum over all topologies. In the fourth paragraph we dwell upon Thurston’s geometrization programme, an attempt at characterizing all three-dimensional manifolds in geometrical terms. In the fifth paragraph, we expound some recently discovered new invariants, mostly related to knots, whose relation

\[^1\text{We will almost always reduce ourselves to compact, closed, oriented three-dimensional manifolds; that is, the simplest situation.}\]
to Thurston’s viewpoint is still mostly unclear. Finally, in the sixth and last paragraph, we conclude on a very speculative note, trying to determine the main characteristics of a well-defined measure, whose determination, if possible at all, will require some deep mathematical research, as well as great doses of physical intuition.

2 Three-dimensional gravity as a topological Chern-Simons theory

E. Witten ([2], following earlier suggestions by Achucarro and Townsend ([29]), was the first to work out the consequences of the point of view consisting in interpreting the spin connection $\omega$ as a gauge field for the Lorentz group, $SO(3,1)$, and the vierbein as another gauge field corresponding to the translation group, $T$.

In first quantized formalism, the Einstein-Hilbert action for a three-dimensional space-time manifold, diffeomorphic to $M = \Sigma \times R$, where $\Sigma$ is a two-dimensional Riemann surface, can be written as:

$$S_{EH} = 1/2 \int \epsilon^{ijk} \epsilon^{abc} e^a_i (\partial_j \omega^b_k - \partial_k \omega^b_j + [\omega_j, \omega_k])$$

It is not difficult to see that the preceding action is the Chern-Simons action for the gauge field

$$A_i = e^a_i P_a + \omega^a_i J_a$$

The gauge symmetries of the system are easily seen to be:

$$\delta e^a_i = -\partial_i \rho^a - \epsilon^{abc} e^a_b \tau_c - \epsilon^{abc} \omega^b_i \rho_c$$

$$\delta \omega^a_i = -\partial_i \tau^a - \epsilon^{abc} \omega^b_i \tau_c$$
Where the $\tau$ are equivalent to Lorentz transformations, whereas the $\rho$ transformations are (on shell) equivalent to diffeomorphisms (plus Lorentz transformations). The system of constraints implies that the phase space consists of all flat connections (modulo gauge transformations). Witten has shown that (when the cosmological constant is zero, as in the action above), one should better regard the connections $\omega$ themselves as coordinates, and, besides, that this is a renormalizable theory, by expanding around the ”unbroken” state, $\omega = e = 0$, and imposing the gauge condition

$$D_0^i e^a_i = D_0^i \omega^a_{ib} = 0$$

(5)

(where the fiducial metric $g_0$ is unrelated to $e$ and $\omega$). Owing to the topological character of the interaction, he also argued that the beta function should be zero.

Working out the one loop partition function one gets:

$$Z(M) = \int D e D \omega \exp(i S_{EH}) = \int D \omega \prod_{ijax} \delta(F^a_{ij}(x))$$

(6)

characterizing the moduli space of all flat ISO(2,1) connections, which is given by a mathematical expression known as the Ray-Singer torsion. Witten went even further, and interpreted an infrared divergence stemming from the calculations as a signal of the appearance in the theory of a classical regime (because the only natural scale a priori for the theory is the Planck one and any macroscopic distance is essentially divergent with respect to it.).

When the cosmological constant is non-zero, the situation is clarified if we introduce the natural decomposition:

$$A^a_{i}^{\pm} = \omega^a_i \pm \sqrt{\lambda} e^a_i$$

(7)
where the group generators are given by:

\[ J_a^\pm = 1/2(J_a \pm \lambda^{-1/2} P_a) \]  

(8)

\[ [J_a^+, J_b^-] = \epsilon_{abc} J_c^+ \]  

(9)

\[ [J_a^+, J_b^-] = 0 \]  

(10)

There are then two actions of the Chern-Simons type, the "standard" one we considered at the beginning of this paragraph,

\[ S_{EH}(A) = 1/(4\sqrt{\lambda})(S_{EH}(A^+) - S_{EH}(A^-)) \]  

(11)

and an "exotic" possibility,

\[ S_{ex} = 1/2(S_{EH}(A^+) + S_{EH}(A^-)) \]  

(12)

In the physical situation we are interested with in the present paper, namely, euclidean signature (and negative cosmological constant), the most general action can be written, after rescalings, as:

\[ S = 1/\hbar S + ik/8\pi S_{ex}(\lambda = 1) \]  

(13)

(where \( k \in Z \) in order for the action to be gauge invariant). In the semiclassical regime (\( \hbar \to 0 \)), the partition function

\[ Z(M) = \int D\epsilon D\omega \exp -S \]  

(14)

will be dominated by the classical solution of largest (negative) action, and we would have the behavior:

\[ Z \sim \exp -1/\hbar V + 2\pi ik C_s \]  

(15)
where $V$ is the volume of the three-dimensional manifold, and $C_s$ is the corresponding Chern-Simons invariant. Witten’s results allow computations of some topology changing amplitudes; for example, the amplitude for a Riemann surface $\Sigma$ to evolve into another Riemann surface $\Sigma'$, will be given by the integral over the exponential of minus the action, over the solution of the corresponding cobordism: $\partial M = \Sigma' - \Sigma$. It was even argued that factorization was suggested by Johnson’s results; although no matter could be included if renormalizability was to be maintained. Incidentally, some computations made by Carlip and de Alwis [30] suggest that in this case at least, the sum over wormholes is not Borel summable, and the corresponding cosmological constant did not appear to be driven to zero.

To summarize: Witten’s clever ansatz allow a computation of the partition function for three-dimensional quantum gravity, for a fixed topology (or rather, reduces this problem to an equivalent, but sometimes non-trivial, mathematical question, namely, the computation of the corresponding Ray-Singer torsion).

In two-dimensional quantum gravity, as defined by string theories, it seems necessary, however, to perform the sum over all topologies. Can we implement this further in the three-dimensional case as well?

A necessary first step in this direction, is the classification of all possible three-dimensional topologies; for example, by introducing a complete set of topological invariants. In the two dimensional situation, there is such a complete set (in the closed, compact case), namely, the Euler characteristic, $\chi = 2 - 2g$. This allows for any functional integral in the two-dimensional situation to be written in terms of the fixed genus one as:

$$\int Dg = \sum_{g=0}^{\infty} \int_{\text{fixed genus}} Dg$$ (16)
The purpose of the following sections is to determine whether something similar can be written in the three-dimensional case.

3 The problem of diffeomorphic equivalence in $d = 3$

In a preceding work [12], we studied the problem of the classification of all topologies, up to diffeomorphic equivalence, in the physical dimension $d = 4$.

The remarkable result, due to Markov, is that given two four-dimensional manifolds, there is no algorithmic way of deciding when they are homeomorphic. This implies, in particular, that there is no a (countable) set of topological invariants, such that two four-manifolds are homeomorphic if and only if they have the same values for all the invariants in the set.

In this case there is an additional problem, namely, that not every homeomorphism can be lifted to a diffeomorphism. We know, in particular, owing to the work of Donaldson, that there are manifolds (namely, those for which a certain unimodular form over the integers $\mathbb{Z}$, the intersection form, is even (that is, its diagonal elements are even ), and positive definite), which can not be smoothed.

In dimension $n \geq 5$, there is an invariant, the Kirby-Siebenmann class,(16 17) $e(M) \in H^4(M, \mathbb{Z}^2)$, such that if $e(M) \neq 0$, the topological manifold $M$ can not be given any piecewise linear structure,(cf. the physicist-oriented discussion in [11])

In low dimensions, a theorem of Rado guarantees that any topological manifold can be upgraded to a piecewise linear structure; and in dimension 3 the same result is due to Moise. On the other hand, another theorem of Kirby and Siebenmann guarantees that if the dimension is $n \leq 3$, any
piecewise structure can be promoted to a differentiable one. (cf. [17]).

In order to get Markov’s result, it is enough to consider one of the simplest topological invariants, namely, the fundamental group, $\pi_1(M)$. As this object will play a rather important place in all our subsequent discussions, let us pause now for a minute to remind the reader of its definition.

The object of interest is the set of all closed paths in the manifold, that is, mappings

$$I = (0,1) \to M$$

such that

$$x(1) = x(0)$$

The law of composition of two such loops is simply (in a somewhat symbolic, but otherwise evident, notation):

$$(x_1 \ast x_2)(u) = \theta(1/2 - u).x_1(2u) + \theta(u - 1/2).x_2(2u - 1)$$

and we are going to consider two such loops as equivalent for our present purposes, if there is a continuous deformation of one into the other (that is, if they are homotopic), which means that there is a continuous function, $f$, such that

$$f : I \times I \to M$$

$$f(t,0) = f(t,1)$$

$$f(0,u) = x_1(u)$$

$$f(1,u) = x_2(u)$$

It should be stressed that the fundamental group is, in general, non-abelian (while the higher homotopy groups, $\pi_n(M), n \geq 2$ are all abelian). There is a
close relationship (namely, the Hurewicz isomorphism) between the homotopy (the concept we have just defined) and the homology (which, roughly, counts the number of holes in the manifold). The first non-vanishing homotopy and homology groups of a given (path-connected) manifold, occur at the same dimension, and are isomorphic.

Actually, the homology is, in a well defined sense, the abelianization of the homotopy:
\[ H_1(M, \mathbb{Z}) \sim \pi_1(M)/[\pi_1(M), \pi_1(M)] \tag{24} \]

It is actually possible to enumerate all possible compact, connected closed 3-manifolds, in the following sense: one can write down a (countably infinite) list of sets of parameters, such that each set specifies a three-dimensional manifold, and we are guaranteed that every three-manifold is contained in the list. We have no control over which ones in the set are diffeomorphic to one another, so that in general there will be infinite overcounting as well.

In the four dimensional case, every (finitely presented) group can be realized as the fundamental group of a four-dimensional manifold; this is not true anymore in the three-dimensional case, so that Markov’s trick of putting a non-recognizable succession of groups in correspondence with manifolds does not apply here, which means that there is no proof that the problem of diffeomorphic equivalence in the three-dimensional case is undecidable (but there is no proof of the contrary either).

In spite of what has been said above, the fundamental group is here also the origin of some trouble. We know that all \( \pi_1(M_3) \) are finitely generated, but the problem of characterizing those groups which can be fundamental groups of some three-dimensional manifold is highly non-trivial. It can actually be proved that this subclass of all finitely presented groups is algo-
arithmically non-recognizable; while it is also known that not every finitely
generated group can be the fundamental group of some three-dimensional
manifold; the classic example being the group $\mathbb{Z}^4$.

Although Markov’s proof does not work in $d = 3$, for precisely this reason,
(as we have just seen), this does not mean that the homeomorphy problem is
completely solved. There are no definitive results in this field, and the fact is
that there is no known procedure to determine when two three-dimensional
manifolds are diffeomorphic.

## 3.1 Topological invariants

Let us quickly review the usefulness of the known topological invariants in
the three-dimensional case (cf.[10]):

- The fundamental group.
  It is too complicated to be of any practical value.

- The homology.
  This is too simple. For example, it does not distinguish between $(S^3 - K)_{(1, q)}$ and $S^3 = (S^3 - K)_{(1, 0)}$. (cf.next paragraph for an explanation of the notation).

- Euler characteristic.
  Actually $\chi(M) = 0$ for all closed, compact three.-manifolds.

- The volume
  Strange as it may seem, the volume is a topological invariant for a cer-
tain type of manifolds, called "hyperbolic"; where, by Mostow’s the-
orem, different volumes imply non-homeomorphic manifolds.(More on
this later). The volume is, in some sense, a measure if the complexity of the manifold and, moreover, the set of manifolds with any given volume is finite. It is not however, a complete invariant: for example, the manifold $WL_{(1,1)}$ is not homeomorphic to $WL_{(5, -1)}$, even though they both have the same volume.

- The Chern-Simons and Eta invariants

They are actually both related (cf. 32).

$$3 \eta(M) = 2 Cs(M)$$

(25)

the $\eta$ invariant contains obviously more information than Chern-Simons, and besides, a theorem of Meyerhoff and Ruberman guarantees that, given any rational number in $R/Z$, there exist hyperbolic manifolds with equal volumes whose $Cs$ invariants differ by that rational number.

- Is this enough?

Unfortunately no, because another theorem of Meyerhoff and Ruberman implies the existence of certain "mutations" of closed hyperbolic manifolds, which leave the volume, the $Cs$ (mod.1), and the $\eta$ invariants unchanged, while P. Kirk has constructed explicit examples of non-homeomorphic mutants.

In some particular cases (namely, for homology 3-spheres, that is, when $H_1(M) = 0$), there are other invariants related to knots, which will be briefly commented upon later on. Let us stress, however, that not even Poincare’s conjecture (that every homology 3-sphere is actually a $S^3$) has been established in the three-dimensional case. (In the four-dimensional case this has been done in a classic work of Freedman: [19])
4 The geometrization conjecture

There is a very general form of characterizing a three-dimensional manifold (cf. [25]). Every closed 3-manifold can be obtained by a procedure called Dehn surgery along some link whose complement is hyperbolic, starting from the three-sphere, $S^3$. (A link is, by definition, a one-dimensional compact submanifold in a three-dimensional manifold, $M$). The intuitive concept of what Thurston calls Dehn surgery is quite simple: just remove a regular neighborhood of the link $K$, and glue it back after some new identification. In this way we can construct, for example, new manifolds, $(S^3 - K)_{p,q}$, by removing a solid torus as a regular neighborhood of a closed knot $K$, and gluing it back after having performed $p \times 2\pi$ rotations around the homology cycle $A$, and $q \times 2\pi$ rotations around the other homology cycle, $b$. The behavior of $M_{p,q}$ as $(p, q)$ get large is well understood.; they are called cusped hyperbolic three-manifolds, and are well described by a cartesian product of a torus times a half-interval $[0, \infty)$ affixed to the "belly" of the 3-manifold.

William Thurston ([6][7][10]), has put forward a conjecture, known as the geometrization conjecture, and has as well over the years produced an impressive amount of evidence in its support; although neither he nor anybody else has succeeded in proving it for the time being.

The basic idea is very simple, and stems from the uniformization theorem of Poincare, in which every two-dimensional Riemann surface is proved to be conformally equivalent to either the sphere $S^2$ (for genus zero), or to the quotient of the two-dimensional plane with a lattice, $C/\Gamma$ (for genus one), or else to the quotient of Siegel’s upper half plane with a fuchsian group of the second kind $H/G$ (for higher genus).

In two dimensions, besides, there is a constructive procedure, (the sewing)
for gluing some elementary structures (the pants, or sphere with three-
parametrized boundaries, $P(0,3)$, and the cylinder, or sphere with two parametrized
boundaries, $P(0,2)$) and obtain the generic Riemann surface (cf, for example,
[27]).

The geometrization conjecture just states that the interior of every com-
pact three-manifold has a canonical decomposition into pieces which have
geometric structures.

This decomposition proceeds in two stages: one first performs a "prime
decomposition", by cutting along two-spheres embedded in $M_3$, so that they
separate the manifold into two parts, neither of which is a 3-ball, and then
gluing 3-balls to the resulting boundary components, thus obtaining closed
3-manifolds which are simpler; and the second stage involves cutting along
tori in an adequate manner.

And what is a geometric structure? We shall demand, first of all, that it
admits a complete, locally homogeneous metric (That is, that for all points
$x, y \in M$, there should exist isometric neighborhoods, $U_x$ and $U_y$). If $M$ is
complete, and the space $X$ is simply-connected, then $M = X/\Gamma$, where $\Gamma$
is a discrete subgroup of the isometry group of $X$, $G$, without fixed points.
This concept clearly generalizes the three two-dimensional simply connected
Riemann surfaces.

One can show that the are precisely eight such homogeneous spaces,
$(X, G)$, needed for a geometric characterization of 3-manifolds. We can
uniquely characterize them by demanding that $X$ be simply-connected; and,
besides, that $G$ is a group of diffeomorphisms of $X$ such that the stabilizer of
an arbitrary point $x \in X$ is a compact subgroup of $G$. The group $G$ itself can
be proven to be unimodular, which implies, in particular, that there exists a
measure right as well as left invariant. We shall actually demand that $G$ is
a maximal group of homeomorphisms of $X$ with compact stabilizers.

Let us briefly list the eight geometries (cf. [13]):

## 4.1 The eight three-dimensional Geometries

**Spherical Geometry** Here $X = S^3$ and $G = SO(4)$. The identity component of the stabilizers at $x$, $G_x = SO(3)$. All three dimensional spherical manifolds have been classified.

**Euclidean Geometry** Here $X = \mathbb{R}^3$ and $G = \mathbb{R}^3 \times SO(3)$. The stabilizer is $G_x = SO(3)$. There are only ten non-homeomorphic 3-dimensional euclidean manifolds (of which 6 are orientable). All of them are, moreover finitely covered by the three-torus, $T^3$.

**Hyperbolic Geometry** Now $X$ is the hyperbolic three-space, $X = (x, y, z, z \geq 0)$, and $G = PSL(2, \mathbb{C})$. The stabilizer is $G_x = SO(3)$. According to Beltrami’s half space model, a good metric is

$$ds^2 = 1/z^2(dx^2 + dy^2 + dz^2) \quad (26)$$

and the points $z = 0$ are called points at infinity. A concrete example is the Seifert-Weber dodecahedral space. To construct it, we identify opposite faces, after a $3/10.2\pi$ clockwise twist. The 30 edges then reduce to 6, after identification; the 20 vertices reduce to only 1, and the 12 faces collapse to 6. The Euler characteristic is $\chi = V - E + F - C' = 1 - 6 + 6 - 1 = 0$, so that this construction represents a compact, closed three-dimensional space. This procedure goes through both in euclidean and in hyperbolic space. This case is, in some sense, the generic one.
The two-sphere cross the euclidean line $X = S^2 \times E^1 = G$. The stabilizer is $G_x = SO(2)$.

There are only two non-homeomorphic examples of compact manifolds with this geometry. It is however, an important case from the physical point of view, because it represents a two-manifold of spherical topology evolving in time.

Hyperbolic two-space cross the euclidean line $X = H^2 \times E^1, G = IS(H^2) \times IS(E^1)$. The stabilizer is $G_x = SO(2)$. Every manifold modelled on this geometry is finitely covered by the product of a surface and a circle. This case is also very important physically, because it represents a two-manifold homeomorphic to a generic Riemann surface, evolving in time.

Universal covering of the special group $X = \overline{SL(2, R)}, G = R \times ISOM(H^2)$. The universal covering of $SL(2, R)$. The stabilizer is $G_x = SO(2)$. The space of unit tangent vectors to any hyperbolic surface is an example of a manifold with this geometry.

Heisemberg group Here $X = Nil, G = H \times S^1$. The stabilizer is $SO(2)$. Any oriented circle bundle over a 2-torus, $T^2$, which is not the three-torus, $T^3$, has this kind of geometric structure. A concrete realization consists of the upper triangular matrices with unit diagonal,

$$
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1 \\
\end{pmatrix}
$$

which can be interpreted as $R^3 = (x, y, z)$, with the composition law:

$$(x, y, z) . (x', y', z') = (x + x', y + y', z + z' + xy')$$

(28)
and the line element given by:

\[ ds^2 = dx^2 + dy^2 + (dz - xdy)^2 \]  \hspace{1cm} (29)

**Sol.** Here \( X = \text{Sol} \) (a soluble group), and \( G \) is an extension of \( X \) by \( Z_2 \). The stabilizer is now trivial \( G_x = 1 \). Any torus bundle over \( S^1 \) whose monodromy is a linear map with distinct, real eigenvalues has a geometric structure of this form. Actually, here also the space can be taken as \( R^3 = (x, y, z) \), with the composition law:

\[(x, y, z), (x', y', z') \rightarrow (x + \exp^{-zx'}, y + \exp^{-zy'}, z + z') \]  \hspace{1cm} (30)

and the metric given by:

\[ ds^2 = \exp 2zdx^2 + \exp -2zdy^2 + dz^2 \]  \hspace{1cm} (31)

Starting from a description of the (supposedly hiperbolic) manifold using Dehn surgery along a link, to compute the geometrical decomposition one must first of all, calculate representations of \( \pi_1(M) \rightarrow PSL(2, C) \) (the group of isometries of \( H^3 \)). Then one has to check whether a given representation is discrete and faithful, and finally one ends up with a hyperbolic manifold whose fundamental group is \( \pi_1(M) \). The procedure makes clear the fact that we have not control over homeomorphic equivalence in this process.

We have mentioned earlier on Mostow’s rigidity theorem. Stated somewhat more precisely, it says that if two hyperbolic manifolds of finite volume have isomorphic fundamental groups, they must necessarily be isometric to each other. This clearly implies that if a closed, orientable three-dimensional manifold possesses a hyperbolic structure, then this structure is unique (up to isometry).
In the non-compact case there is a curious non-rigidity theorem due to Thurston, which states that if \( M = H^3/\Gamma \) is orientable and non-compact, with finite volume, \( V \), then there is a succession of manifolds, \( M_j = H^3/h_j(\Gamma) \), with volumes strictly smaller than \( V \), \( V_j < V \), and such that \( \lim_{j \to \infty} M_j = M \), in some precise sense.

These structures can sometimes be characterized as (Seifert) bundles \( \eta \) over orbifolds \( Y \).

Classifying them according to the Euler numbers of the orbifold, \( \chi(Y) \), and the Euler number of the bundle, \( e(\eta) \), we get:

\[
\begin{align*}
& e(\eta) = 0 & \chi(Y) > 0 & \chi(Y) = 0 & \chi(Y) < 0 \\
\end{align*}
\]

\[
\begin{align*}
& \quad S^2 \times R & E^3 & H^2 \times R \\
& e(\eta) \neq 0 & S^3 & Nil & SL_2
\end{align*}
\]

\[
\text{(32)}
\]

5 New invariants

There has been recent progress in defining new invariants, which, are only useful, however, in some simplifying situations. Floer, in particular, defines invariants of homology 3-spheres (that is, oriented, closed, 3-dimensional smooth manifolds such that \( H_1(M, Z) = 0 \)). Other (finer) invariants, defined by Casson, are, in a sense, one half the Euler characteristic of this homology groups. To be specific, in order to define the Casson invariants, one has to study the representations of the fundamental group into \( SU(2) \), and introduce a precise way of counting them. One then has:

\[
\lambda(M_3) = 1/2(Irreps : \pi_1(M_3) \to SU(2))
\]

\[
\text{(33)}
\]

\[\text{This means that there exists a decomposition of } M \text{ into disjoint circles (called fibres) such that each circle has a neighborhood in } M \text{ which is a union of fibres and is isomorphic to a fibred solid torus or Klein bottle.}\]
The precise form of the relationship with Floer homology is then,

\[ \lambda(M_3) = \sum_{i=0}^{7} (-1)^i \dim F_i^+(M_3) \quad (34) \]

The celebrated Jones polynomials, originally devised to characterize knots and links, can be generalized as well to some three-dimensional compact manifolds.

The intuitive way of constructing them (cf. [33]; [23]; [20] [21]) is to start with the known representation of a three-dimensional manifold as the result of performing surgery on a link on a three-dimensional sphere \( S^3 \). The actual computations performed by Reshetikin and Turaev, and later on, by Turaev and Viro, rely on quantum group techniques to build up appropriate averages of link polynomials. The two invariants are deeply related, and, for example, if \( M_q \) denotes the Turaev-Viro invariant, and \( Z_q(M) \) denotes the Reshetikin-Turaev invariant for \( SU(2)_q \), one has the relationship

\[ M_q = Z_q(M)Z_q(M) \quad (35) \]

Many new relations among all these invariants are constantly being unveiled, and it is a topic of current research among mathematicians - and also among physicists, thanks to the topological quantum field theories of Atiyah and Witten. (cf. Atiyah’s lucid reviews in [14][15][16]). As he likes to put it, one of the most important

open questions in three-dimensional topology is the relationship between this type of invariants and the geometrization programme of the fourth paragraph.
6 Some provisional conclusions

Let us imagine that Thurston’s conjecture is true. This means that we can represent every compact, closed, three-dimensional manifold as

$$M_3 = \bigcup_{i=1}^{\infty} G_{n_i}/\Gamma_{n_i}$$

where $n_i \in (1, \ldots, 8)$ represents one of the eight geometries, and $\Gamma$ is a subgroup of the isometry group of the corresponding geometry. The gluing in the preceding formula will be represented by a complicated (although computable in principle) set of moduli.

If each gluing is characterized by a different coupling constant (in the same way that in string theory the gluing of the pants and the cylinder is characterized by the single coupling constant $\kappa$, later to be identified with Newton’s constant) then we would have 28 different coupling constants $\kappa_i$, whose physical interpretation is perhaps possible without solving all problems of definition of the path integral.

It is also possible that, in the same way as in the open string case, in which two apparently unrelated coupling constants turn out to be the same due to consisteney requirementees, that here also not all of the $\kappa_i$ are independent, and consistenmcy forces to implement nontrivial relations between them.

The (symbolic) form of the measure would then be:

$$Dg \sim \sum_{\text{combinations}} D(\text{IsometryGroups})D(\text{Moduli})$$

When one thinks that even in the two-dimensional case, the integration region is not explicitly known for genus $g \geq 4$, (and, besides, the action of the mapping class group on the Fenchel-Nielsen coordinates is very complicated, so that one has in practice to use other coordinates on moduli space, much
less naturally related to the pants decomposition), one can easily realize that both measures in the preceding formula are very difficult mathematical problems.

Perhaps this can be done, however, with a bit of luck. But then, we have to face the main problem, and that is that nowhere in the preceding discussion we have taken into account that we have to sum in the path integral over non-homeomorphic manifolds only. This means that there will be infinite overcounting, and it seems impossible to overcome this problem unless some numerable set of topological invariants is devised which characterize completely a manifold up to diffeomorphisms.

Please note that this problem will still be there, even though the fundamental theory of gravity turns out to be topological (at least in the absence of matter). And this is because, although in this case one does not need to solve the homeomorphy problem, one needs, however, to be able at least to "count" and enumerate all topologies. Besides, the topological phase has to be broken at long distances, and there, in the description of the symmetry breaking, one has to face again all the problems of the homeomorphic equivalence.
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