Quantum determinant revisited

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Abstract
Following [G] we define quantum determinants in certain quantum algebras, related to couples of compatible braidings. Also, we compare these quantum determinants with the highest elementary symmetric polynomials. Some properties of the quantum determinants are exhibited and their role for integrable systems theory is discussed.

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1 Introduction

In this paper we deal with analogs of the usual determinants defined in quantum matrix algebras (QMA) and some other close algebras. The best known QMA are the RTT algebras associated with Drinfeld-Jimbo quantum groups $U_q(sl(N))$. These algebras and the corresponding quantum determinant were introduced and intensively studied in the papers of L.D. Faddeev’s school (see, for instance, [KS]).

However, the definition of such a determinant can be easily generalized to the RTT algebras associated with other involutive or Hecke symmetries $R : V^\otimes 2 \rightarrow V^\otimes 2$ ($V$ is a finite dimensional vector space), provided the so-called $R$-skew-symmetric algebra $\Lambda_R(V)$ of the vector space $V$ is finite and its highest homogeneous component is 1-dimensional. We call such symmetries even.

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Let us introduce necessary definitions and notations. Given a finite dimensional vector space $V$, $\dim V = N$, a linear operator $R : V \otimes^2 \rightarrow V \otimes^2$ is called a braiding if it is a solution of the so-called braid relation
\[
(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R),
\]
where $I$ is the identity operator on $V$.

A braiding $R$ is called an involutive symmetry if it satisfies a supplementary condition
\[
R^2 = I,
\]
and $R$ is called a Hecke symmetry if it is subject to
\[
(R - qI)(R + q^{-1}I) = 0, \quad q \notin \{0, \pm 1\}.
\]
The parameter $q \in \mathbb{C}$ is assumed to be generic, i.e. such that
\[
k_q = \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0, \quad \forall k \in \mathbb{Z}.
\]

Given such an involutive or Hecke symmetry $R$, the corresponding RTT algebra is a unital associative algebra generated by a set of elements $t^i_j$, $1 \leq i, j \leq N$. The permutation relations on generators $t^i_j$ are usually written in the matrix form:
\[
R(T \otimes I)(I \otimes T) = (T \otimes I)(I \otimes T)R, \quad T = \|t^i_j\|_{1 \leq i, j \leq N}.
\]
Note that in the case related to $U_q(sl(N))$ the corresponding Hecke symmetry is $PR$, where $P$ is the usual flip and $R$ stands for the image of the universal quantum $R$-matrix in the space $\text{End}(V \otimes^2)$, where $V$ is the vector space of the $U_q(sl(N))$ first fundamental representation.

Also, note that in [G] a big family of even involutive and Hecke symmetries with unusual dimensions of homogenous components of the $R$-skew-symmetric algebras $\Lambda_R(V)$ was constructed and the quantum determinants in the corresponding RTT algebras were defined.

The objective of the present paper is three-fold. First, we extend the definition of the quantum determinant from [G] to the QMA different from RTT ones, namely, those associated with couples of compatible braidings as defined in [IOP], half-quantum algebras (HQA) as defined in [IO], and generalized (braided) Yangians as defined in [GS1, GS2].

Second, we compare this definition of the quantum determinant with another one, based on the notion of quantum elementary symmetric polynomials and study some properties of the quantum determinant. In particular, we present the characteristic polynomials for the generating matrices in some QMA.

Third, we want to draw the reader’s attention to the fact that contrary to popular belief the $R$-skew-symmetric algebra $\Lambda_R(V)$ of the space $V$ does not determine the corresponding quantum algebras and related determinants uniquely. We exhibit two symmetries — involutive and Hecke ones — with the same algebras $\Lambda_R(V)$ but with different quantum determinants in the corresponding QMA. This example shows that the so-called $q$-Manin matrices, arising from these symmetries (see [CFRS]), and the corresponding quantum determinants differ from each other.

Moreover, this fact also impacts the integrable systems theory. Indeed, though the algebras $\Lambda_R(V)$, corresponding to the mentioned symmetries, define the same quantization of the symmetric algebra $\text{Sym}(V)$ of the space $V$ (the so-called quantum planes), the related generalized Yangians are defined by the current braidings of different types. One of them is rational and the
other one — trigonometrical. Consequently, the Bethe subalgebras in these Yangian-like algebras are constructed via different formulae: the shifts of arguments in the generating matrices \( L(u) \) are additive in the first case and multiplicative in the second one.

The paper is organized as follows. In the next section following [G] we define the quantum determinants in the QMA associated with couples \((R, F)\) of compatible braidings, where \( R \) is an even involutive or a Hecke symmetry. Also, we exhibit the relation between the quantum determinant and the highest quantum elementary symmetric polynomial. In Section 3 we define the quantum determinants in the HQA and generalized Yangians. Also, we discuss the construction of the Bethe subalgebras in the latter algebras. In the last section we study some properties of the quantum determinants in certain QMA.

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2 Quantum determinants in QMA

With any Hecke symmetry \( R \) we associate the so-called \( R \)-symmetric and \( R \)-skew-symmetric algebras of the space \( V \) respectively defined by

\[
\text{Sym}_R(V) = T(V)/\langle \text{Im}(qI - R) \rangle, \quad \Lambda_R(V) = T(V)/\langle \text{Im}(q^{-1}I + R) \rangle.
\]

Here \( T(V) \) stands for the free tensor algebra of the space \( V \) and \( \langle J \rangle \) is a two-sided ideal generated by a set \( J \subset T(V) \).

Each homogenous component \( \text{Sym}^{(k)}_R(V) \) (resp., \( \Lambda^{(k)}_R(V) \)) can be identified with the image of the \( R \)-symmetrizer \( S^{(k)}(R) \) (resp., \( R \)-skew-symmetrizer \( A^{(k)}(R) \)) acting on the space \( V^\otimes k \). These projectors can be defined by the following recursion formulae

\[
S^{(1)} = I, \quad S^{(k)} = \frac{1}{k!} S^{(k-1)} \left( q^{-(k-1)}I + (k-1)qR_{k-1} \right) S^{(k-1)}, \quad k \geq 2,
\]

\[
A^{(1)} = I, \quad A^{(k)} = \frac{1}{k!} A^{(k-1)} \left( q^{-(k-1)}I - (k-1)qR_{k-1} \right) A^{(k-1)}, \quad k \geq 2.
\]

(2.1)

As usual, the bottom indices indicate the positions where matrices or operators are located. We also use a shorthand notation \( R_k = R_{k+1} \). These formulae can be deduced from the representation theory of the symmetric groups, provided \( R \) is involutive, or that of the Hecke algebras, provided \( R \) is a Hecke symmetry (see [GY]).

Let \( R \) and \( F \) be braidings. Following [IOP] we say that the ordered couple \((R, F)\) is compatible (or braidings \( R \) and \( F \) are compatible) if the following relations take place

\[
R_{12} F_{23} F_{12} = F_{23} F_{12} R_{23}, \quad R_{23} F_{12} F_{23} = F_{12} F_{23} R_{12}.
\]

Below we always assume \( R \) to be an involutive or a Hecke symmetry.

Let \( L = [l_{ij}]_{1 \leq i, j \leq N} \) be an \( N \times N \) matrix and \( L_1 = L \otimes I_{2^{p-1}} \), \( p \geq 2 \). (Thus, \( L_1 \) is an \( N^p \times N^p \) matrix.) We introduce the following notation:

\[
L = L_1, \quad L_{k+1} = F_k L_k F_k^{-1}, \quad k \leq p - 1
\]

(2.2)

Below, we mainly deal with Hecke symmetries. The corresponding results and formulae for involutive symmetries can be obtained by putting \( q = 1 \).
tensors $u$ and $v$ are defined up to a normalization understood. Thus, the element $u = \|l_j^i\|$ subject to the system of permutation relations:

$$R_{12}L_1L_2 = L_1L_2R_{12}.$$  \hspace{2cm} (2.3)

The matrix $L = \|l_j^i\|$ is called the generating matrix of the algebra $L(R, F)$. Using the compatibility of the braidings $R$ and $F$ one can show that the defining relations of the algebra $L(R, F)$ can be pushed forward to higher positions in the following sense

$$R_kL_kL_{k+1} = L_kL_{k+1}R_k, \quad k < p.$$

Note that each of the couples $(R, P)$ and $(R, R)$ is evidently compatible. The corresponding algebras $L(R, P)$ and $L(R, R)$ are respectively the well-known RTT algebra and Reflection Equation (RE) algebra. The defining relations of these QMA read

$$R_{12}L_1L_2 = L_1L_2R_{12}$$

for the RTT algebra $L(R, P)$ and

$$R_{12}L_1R_{12}L_1 = L_1R_{12}L_1R_{12}$$

for the reflection equation algebra $L(R, R)$.

**Remark 1** It should be emphasized that if a symmetry $R$ is a deformation of the usual flip the corresponding RTT and RE algebras are deformations of the commutative algebra $\text{Sym}(gl(N))$, i.e. dimensions of the homogenous components of these QMA are classical (if $R$ is a Hecke symmetry, the parameter $q$ must be generic).

However, if $R$ is a braiding coming from the quantum groups of the series $B_n, C_n, D_n$ this property fails. It can be seen from consideration of the corresponding quasiclassical term. This term, corresponding to the RTT algebra, is defined by the classical $r$-matrix, giving rise to the Poisson bracket, called the second Sklyanin bracket, on the function algebra $\text{Fun}(G)$, where $G$ is a Lie group from one of these series. Whereas, on the algebra $\text{Sym}(\mathfrak{g})$, where $\mathfrak{g}$ is the corresponding Lie algebra a similar bracket does not exist.

Now, assume the symmetry $R$ to be even. Let $\Lambda_R(x) = m \geq 2$ be the highest non-trivial homogenous component of the algebra $\Lambda_R(V)$. Since, by definition, the dimension of this component is 1, then there exist two tensors

$$u = \|u_i \cdots i_m\| \quad \text{and} \quad v = \|v_j \cdots j_m\|,$$

such that

$$A^{(m)}(x_{i_1} \otimes \cdots \otimes x_{i_m}) = u_{i_1 \cdots i_m} v_j^{i_1 \cdots j_m} x_{j_1} \otimes \cdots \otimes x_{j_k}, \quad \langle v, u \rangle := v^{j_1 \cdots j_m} u_{i_1 \cdots i_m} = 1.$$ 

Hereafter, $\{x_i\}_{1 \leq i \leq N}$ is a basis in the space $V$ and summation over repeated indices is always understood. Thus, the element $v^{j_1 \cdots j_m} x_{j_1} \otimes \cdots \otimes x_{j_m}$ is a generator of $\text{Im}(A^{(m)})$. Note that the tensors $u$ and $v$ are defined up to a normalization

$$u \mapsto au, \quad v \mapsto a^{-1}v, \quad a \in \mathbb{C}, \; a \neq 0.$$

According to \cite{G}, we introduce the following definition.

\footnote{Apart from these two examples, we can mention the only example of compatible braidings $(R, F)$ with an even Hecke symmetry $R$. These braidings are exhibited below in \cite{G}, where the second matrix plays the role of $R$. Nevertheless, the language of compatible couples is very useful, since it enables us to consider the RTT and RE algebras from the universal viewpoint.}

\footnote{In general, $m$ could be different from $N = \dim V$ (see \cite{G} \cite{GPS3}).}
Definition 2 The element of QMA $\mathcal{L}(R, F)$

$$\det_{\mathcal{L}(R,F)}(L) := \langle v | L_{T_{1}}...L_{m} | u \rangle := v^{i_1...i_m} (L_{T_{1}}...L_{m})^{j_1...j_m} u_{j_1...j_m}, \quad (2.4)$$

is called the quantum determinant of the generating matrix $L$.

Of course, the quantum determinant $\det_{\mathcal{L}(R,F)}(L)$ can be written in another explicit form, modulo the defining relations of the QMA $\mathcal{L}(R, F)$. The form (2.4) will be called canonical.

Let us consider two examples. In the case $N = 2$ we consider two symmetries, represented by the following matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \quad (2.5)$$

Each of these symmetries is a deformation of the usual flip $P$ and for both of them $m = N = 2$. The former symmetry is involutive, the latter one is Hecke.

For the involutive symmetry we have

$$u = (u_{11}, u_{12}, u_{21}, u_{22}) = \frac{1}{2}(0, 1, -q^{-1}, 0), \quad v = (v^{11}, v^{12}, v^{21}, v^{22}) = (0, 1, -q, 0).$$

For the Hecke symmetry we have

$$u = \frac{1}{2q}(0, q^{-1}, -1, 0), \quad v = (0, 1, -q, 0).$$

Observe that the tensors $v$ corresponding to these symmetries coincide with each other and, consequently, the algebras $\text{Sym}_R(V) = T(V)/\langle v \rangle = T(V)/\langle xy - qyx \rangle$, called the quantum planes, are the same for the involutive and the Hecke symmetries $R$ presented in (2.5). Nevertheless, the tensors $u$ are different. Consequently, the canonical forms of the corresponding determinants differ from each other. Let us calculate them in the corresponding algebras $\mathcal{L}(R, P)$.

Below, we denote $a = l_{11}^1, b = l_{12}^2, c = l_{11}^2, d = l_{22}^2$. Then the defining relations in the algebra $\mathcal{L}(R, P)$, corresponding to the first matrix from (2.5) are

$$ab = q^{-1}ba, \quad ac = qca, \quad ad = da, \quad bc = q^2cb, \quad bd = qdb, \quad cd = q^{-1}dc.$$

According to our definition, the quantum determinant in this algebra is

$$\det_{\mathcal{L}(R,P)}(L) = \frac{1}{2}(ad - qcb - q^{-1}bc + da) = ad - q^{-1}bc = da - qcb. \quad (2.6)$$

The defining relations in the algebra corresponding to the second matrix from (2.5) are (see [FRT])

$$ab = qba, \quad ac = qca, \quad ad - da = (q - q^{-1})bc, \quad bc = cb, \quad bd = qdb, \quad cd = qdc.$$

The corresponding quantum determinant is

$$\det_{\mathcal{L}(R,P)}(L) = \frac{1}{2q}(q^{-1}ad - bc - cb + qda) = ad - qbc = da - q^{-1}cb. \quad (2.7)$$
Note that the first expressions in formulae (2.6) and (2.7) are canonical.

Let us also exhibit the corresponding algebras \( \Lambda_R(V) \). If \( R \) is the first symmetry from (2.5), then we have
\[
\Lambda_R(V) = T(V) / (x^2, y^2, xy + qyx).
\]
If \( R \) is the second symmetry from (2.5), then in the above formula the last element in the set generating the ideal should be \( qxy + yx \).

Thus, we see that to the same quantum plane there correspond different RTT algebras and different quantum determinants in dependence of the tensor \( u \). Also, the skew-symmetric counterparts of the quantum plane differ from each other for different \( u \).

Another way of introducing quantum analogs of the determinant is based on the notion of the quantum elementary symmetric polynomials defined via the quantum traces. Such a quantum trace is well known in the cases related to a quantum group. Nevertheless, the quantum trace can be associated with any skew-invertible (see below) braiding \( R \) by means of the following method belonging to V.Lyubashenko [L1, L2].

We say that a given braiding \( R: V^{\otimes 2} \to V^{\otimes 2} \) is skew-invertible if there exists an \( N \times N \) matrix \( \Psi \) such that
\[
\text{Tr}_{(2)} R_{12} \Psi_{23} = P_{13} = \text{Tr}_{(2)} \Psi_{12} R_{23} \quad \Leftrightarrow \quad R_{ij} \Psi_{jm} = \delta_m^i \delta^n_j = \Psi_{ij} R_{jm}.
\]

If \( R \) is a skew-invertible braiding, the corresponding \( R \)-trace \( \text{Tr}_R \) is defined by the formula
\[
\text{Tr}_R X = \text{Tr}(C^R X), \quad C^R := \text{Tr}_{(2)} \Psi.
\]
Here \( X \) is an arbitrary \( N \times N \) matrix (may be with noncommutative entries).

This \( R \)-trace possesses many remarkable properties. In particular, it meets the following relations
\[
R_{12} C^R_{12} = C^R_{12} R_{12}, \quad \text{Tr}_{R(2)} R_{12}^{\pm 1} X_1 R_{12}^{\mp 1} = I_1 \text{Tr}_R X. \tag{2.8}
\]

Consider a compatible couple of braidings \((R, F)\) and suppose the braiding \( F \) to be skew-invertible. Now, we define a quantum version of the elementary symmetric functions in the algebra \( \mathcal{L}(R, F) \) as follows
\[
e_0 = 1, \quad e_k = \text{Tr}_{(1\ldots k)} A^{(k)} L_T L_T \ldots L_T, \quad k \geq 1. \tag{2.9}
\]
Hereafter, \( \text{Tr}_{(1\ldots k)} = \text{Tr}_{(1)} \ldots \text{Tr}_{(k)} \).

By using the equality
\[
A^{(k)} L_T \ldots L_T = A^{(k)} L_T \ldots L_T A^{(k)},
\]
valid in any QMA (and even in any HQA, defined below), we get the following relation
\[
\text{Tr}_{F(1\ldots m)} A^{(m)} L_T \ldots L_T = \text{Tr}_{F(1\ldots m)} A^{(m)} L_T \ldots L_T A^{(m)} = (v \cdot_F u) (v|L_T \ldots L_T|u), \tag{2.10}
\]
where
\[
(v \cdot_F u) = v^{i_1 \ldots i_m} (C^F)^{j_1}_{j_1} (C^F)^{j_2}_{j_2} \ldots (C^F)^{i_m}_{j_m} u_{i_1 \ldots i_m}. \tag{11.1}
\]
Thus, the highest elementary symmetric polynomial \( e_m \) differs from the quantum determinant \( \det_{\mathcal{L}(R,F)}(L) \) by a numerical factor. In the particular case \( F = P \) these elements are equal to each other since in this case \( (v \cdot_P u) = 1 \) (note that \( C^P = I \)).

**Remark 3** The quantum elementary polynomials (2.9) can be also defined via the \( R \)-trace \( \text{Tr}_R \) instead of \( \text{Tr}_F \), provided \( R \) is a skew-invertible symmetry (see, for instance [GPS2]). Such a replacement leads to a change of the factor in (2.10).
3 Quantum determinants in HQA and generalized Yangians

Consider a compatible couple \((R, F)\), where \(R\) is a Hecke symmetry and \(F\) is a skew-invertible braiding. Introduce two systems of relations on generating matrix \(L = \|l^i_j\|\)

\[
S^{(2)} L^2 L = 0, \quad \tag{3.1}
\]

\[
A^{(2)} L^2 S^{(2)} = 0, \quad \tag{3.2}
\]

where \(R\)-symmetrizer \(S^{(2)}\) and \(R\)-skew-symmetrizer \(A^{(2)}\) are defined in (2.1). The matrices with overlined indices have the same meaning as above (see (2.2)).

The following claim is well known and can be checked straightforwardly.

**Proposition 4** The system (2.3) is equivalent to the union of the systems (3.1) and (3.2).

By imposing only one of the systems (3.1) or (3.2), we get a larger algebra than that \(\mathcal{L}(R, F)\). Nevertheless, even in such an algebra it is possible to develop some elements of linear algebra. We refer the reader to the paper [IO] for details where algebras defined by the system (3.1) are studied.

Following [IO], we call a half-quantum algebra (HQA) any algebra defined by the system (3.1). The HQA related to a couple of compatible braidings \((R, F)\) will be denoted \(\mathcal{H}(R, F)\).

Let us precise that if \(R\) is an even symmetry, we still define the quantum determinant in the algebra \(\mathcal{H}(R, F)\) by formula (2.4). Quantum elementary symmetric polynomials are also defined in [IO] by formula (2.9). The quantum determinant in the algebra \(\mathcal{H}(R, F)\) also differs from the highest quantum elementary symmetric polynomial by a factor.

Note that HQA corresponding to the case \(F = R = P\) were considered in [CFR] under the name of Manin matrices.

Also, the so-called \(q\)-Manin matrices were introduced in [CFRS] via the symmetries (2.5).

We point out that these symmetries give rise to different HQA and, consequently, to different quantum determinants.

**Remark 5** Initially such type algebras were considered by Yu.Manin in [M]. Their definition is motivated by the following consideration. Let us endow the space \(V\) with the following coaction \(x_i \to t^i_k \otimes x_j\) and extend it to the space \(V^\otimes 2\) by assuming that the generators \(x_i\) and those \(t^i_k\) commute with each other. Then the relation (3.1) (resp., (3.2)), where we assume that \(F = P\) (i.e. the indexes are not overlined), means that the subspace \(\text{Im}S^{(2)}\) (resp., \(\text{Im}A^{(2)}\)) is preserved under this coaction.

Let us consider the HQA, corresponding to the symmetries (2.5) in more detail. The defining relations in the algebra \(\mathcal{H}(R, P)\), corresponding to the first matrix from (2.5) are

\[
ab = q^{-1}ba, \quad cd = q^{-1}dc, \quad ad - da = q^{-1}bc - qcb.
\]

The quantum determinant in this algebra is given by the same formulae as above:

\[
\det_{\mathcal{H}(R, P)}(L) = \frac{1}{2}(ad - qcb - q^{-1}bc + da) = ad - q^{-1}bc = da - qcb.
\]

In the algebra \(\mathcal{H}(R, P)\) corresponding to the second matrix from (2.5) we have

\[
ab = qba, \quad cd = qdc, \quad ad - da = qbc - q^{-1}cb.
\]
The corresponding quantum determinant is

$$\det_{\mathcal{H}(R,P)}(L) = \frac{1}{2q}(q^{-1}ad - bc - cb + qda) = ad - qbc = da - q^{-1}cb.$$ 

Though the latter forms of these quantum determinants look similar, they are different: a passage from the first quantum determinant to the second one is performed by replacing $q$ for $q^{-1}$.

Now, we pass to generalized (braided) Yangians defined in [GS1] [GS2]. Below, we discuss the relations between HQA and generalized Yangians.

First, describe the Baxterization procedure which enables us to construct current braidings via involutive and Hecke symmetries (see [J, Jo]). Let us precise that by a current braiding $R(u,v)$ we mean an operator depending on parameters and subject to the braid relation of the form:

$$R_{12}(u,v)R_{23}(u,w)R_{12}(v,w) = R_{23}(v,w)R_{12}(u,w)R_{23}(u,v).$$

(3.3)

Given an involutive symmetry $R$, we associate with it a current braiding by the rule

$$R(u,v) = R - \frac{I}{u-v},$$

(3.4)

whereas for a Hecke symmetry $R$ the corresponding current braiding reads

$$R(u,v) = R - \frac{(q - q^{-1})uI}{u-v}.$$ 

(3.5)

By a straightforward calculation one can verify that these operators do satisfy relation (3.3). The current braidings (3.4) and (3.5) (and all corresponding algebras) will be respectively called the rational and trigonometric ones.

Introduce a countable set of elements $l_{ij}[k], k \in \mathbb{Z}_{\geq 0}, 1 \leq i,j \leq N$, and consider a formal power series

$$L(u) = \sum_{k \geq 0} L[k]u^{-k}, \quad L[k] = \|l_{ij}[k]\|_{1 \leq i,j \leq N},$$

that is $L(u)$ is an $N \times N$ matrix and its entries are inverse power series in $u$ with coefficients $l_{ij}[k]$.

A generalized Yangian $\mathbf{Y}(R,F)$ is an associative unital algebra generated by elements $l_{ij}[k]$ subject to the (countable) set of relations

$$R_{12}(u,v)L_\mathcal{T}(u)L_\mathcal{T}(v) - L_\mathcal{T}(v)L_\mathcal{T}(u)R_{12}(u,v) = 0,$$

(3.6)

where $L_\mathcal{T}(u) = L_1(u)$ and $L_\mathcal{T}(u) = F_{12}L_\mathcal{T}(u)F_{12}^{-1}$. Thus, elements of the algebra $\mathbf{Y}(R,F)$ are polynomials in generators $l_{ij}[k]$.

In the case $F = R$ the algebra $\mathbf{Y}(R,R)$ with a supplementary condition $L[0] = I$ is called a braided Yangian of RE type (see [GS1] for detail). Note that the condition $L[0] = I$ is motivated by the evaluation morphism, which in this case is similar to that in the Drinfeld’s Yangian $\mathbf{Y}(gl(N))$. Also, note that the Drinfeld’s Yangian $\mathbf{Y}(gl(N))$ is a particular case of the generalized Yangians, corresponding to the rational braiding with $R = F = P$.

As was pointed out in [IO], for a special value of the ratio $u/v = q^2$ the system (3.6) in the trigonometric case can be treated in terms of the HQA. A similar treatment is possible in the rational case if $u - v = 1$. 

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More precisely, for the indicated relations between the parameters \( u \) and \( v \), the current braiding \( R(u, v) \) becomes equal (up to a numerical factor) to the \( R \)-skew-symmetrizer \( A^{(2)} \). Thus, in the rational (respectively, trigonometric) case we arrive at the relations

\[
A^{(2)} L_T(u) L_T(u - 1) = L_T(u - 1) L_T(u) A^{(2)},
\]

\[
A^{(2)} L_T(u) L_T(q^{-2} u) = L_T(q^{-2} u) L_T(u) A^{(2)}.
\]

Consequently, we have

\[
A^{(2)} L_T(u) L_T(u - 1) S^{(2)} = 0,
\]

or, respectively,

\[
A^{(2)} L_T(u) L_T(q^{-2} u) S^{(2)} = 0.
\]

Using the formal Taylor series expansion

\[
L(u - 1) = e^{-\partial_u} L(u) e^{\partial_u}, \quad L(q^{-2} u) = q^{-2u\partial_u} L(u) q^{2u\partial_u},
\]

where \( \partial_u = \frac{d}{du} \), we can cast the above relations under the following forms

\[
A^{(2)} (e^{-\partial_u} L_T(u))(e^{-\partial_u} L_T(u)) S^{(2)} = 0,
\]

\[
A^{(2)} (q^{-2u\partial_u} L_T(u))(q^{-2u\partial_u} L_T(u)) S^{(2)} = 0.
\]

Thus, the operator \( e^{-\partial_u} L(u) \) (respectively, \( q^{-2u\partial_u} L(u) \)) plays the role of the generating matrix of a HQA.

Now, define the quantum determinants in the generalized Yangians (respectively, rational and trigonometric) by setting

\[
\det Y_{(R,F)}(L(u)) = \langle v | L_T(u) L_T(u - 1) \ldots L_T(u - m + 1) | u \rangle,
\]

\[
\det Y_{(R,F)}(L(u)) = \langle v | L_T(u) L_T(q^{-2} u) \ldots L_T(q^{-2(m-1)} u) | u \rangle.
\]

Here we assume \( R \) to be an even symmetry of bi-rank \((m|0)\).

As for the quantum elementary symmetric polynomials, we respectively define them in the rational and trigonometric cases as follows

\[
e_0(u) = 1, \quad e_k(u) = \text{Tr}_{F(1\ldots k)} A^{(k)} L_T(u) L_T(u - 1) \ldots L_T(u - k + 1), \quad k \geq 1,
\]

\[
e_0(u) = 1, \quad e_k(u) = \text{Tr}_{F(1\ldots k)} A^{(k)} L_T(u) L_T(q^{-2} u) \ldots L_T(q^{-2(k-1)} u), \quad k \geq 1.
\]

Note that the quantum determinant differs from the highest elementary symmetric polynomial \( e_m(u) \) by a factor in full analogy with \((2.10)\).

As for a quantum version of other symmetric polynomials, some of them can be also defined in all considered algebras. Thus, in the QMA the quantum versions of all symmetric polynomials (complete symmetric polynomials, Schur polynomials, power sums) are known. In the HQA there are different versions of quantum power sums (see \[10]). In the generalized Yangians quantum power sums can be defined too (see \[GS1, GS2\]). Thus, in the generalized Yangian \( Y(R,F) \) the quantum power sums have the following form

\[
p_0(u) = 1, \quad p_k(u) = \text{Tr}_{F(1\ldots k)} L_T(u - k + 1)L_T(u - k + 2) \ldots L_T(u) R_{k-1} \ldots R_2 R_1, \quad k \geq 1.
\]

for the rational case and

\[
p_0(u) = 1, \quad p_k(u) = \text{Tr}_{F(1\ldots k)} L_T(q^{-2(k-1)} u)L_T(q^{-2(k-2)} u) \ldots L_T(u) R_{k-1} \ldots R_2 R_1, \quad k \geq 1,
\]
for the trigonometric case.

Note that in the generalized Yangians of RE type (i.e. for $F = R$) these formulae can be simplified. Namely, we respectively have

$$p_k(u) = \text{Tr}_R L(u-k+1)L(u-k+2)\ldots L(u),$$

$$p_k(u) = \text{Tr}_R L(q^{-2(k-1)}u)L(q^{-2(k-2)}u)\ldots L(u).$$

So, in the generalized Yangians of RE type the quantum power sums look similar to the usual ones $\text{Tr}_u L(u)^k$ but the classical trace is replaced by its $R$-analog and the parameters of the factors are shifted. It should be emphasized that in the rational generalized Yangians the shifts of the argument $u$ are additive while in the trigonometric generalized Yangians — multiplicative. Note that in all considered algebras there are analogs of formulae expressing all known quantum symmetric polynomials via elementary ones (Newton and Wronski relations).

In the QMA $\mathcal{L}(R, F)$ all quantum symmetric polynomials commute with each other and consequently the subalgebras generated by these elements are commutative. It is also valid in the generalized Yangians. Namely, this property is used for constructing quantum integrable systems. We call these subalgebras characteristic. In integrable systems theory they are called the Bethe subalgebras.

Let us resume. As our examples show, the same $R$-symmetric algebra of the space $V$ could give rise to different integrable systems both in the corresponding QMA and generalized Yangians.

As for the HQA, the property of the quantum elementary symmetric polynomials to commute with each other fails. Thus, the HQA are somewhat useless for constructing integrable systems.

4 Some properties of quantum determinants in QMA

In this section we consider some properties of the quantum determinant in QMA.

It is well known that the RTT algebra $\mathcal{L}(R, P)$ is a bi-algebra with the coproduct $\Delta$ defined by the rule:

$$\Delta(1) = 1 \otimes 1, \quad \Delta(l^i_j) = l^k_i \otimes l^j_k. \quad (4.1)$$

Recall that the summation over repeated indices is always understood.

The quantum determinant is a group-like element with respect to the coproduct $\Delta$, that is

$$\Delta(\det_{\mathcal{L}(R, P)}(L)) = \det_{\mathcal{L}(R, P)}(L) \otimes \det_{\mathcal{L}(R, P)}(L).$$

A similar structure mutatis mutandis is valid in all algebras $\mathcal{L}(R, F)$. We keep the same coproduct on the generating matrix $L$ in these algebras. Observe that for any $k$ we have $\Delta(L^k) = L^k \otimes L^k$. Now, by permuting the middle terms in the expression

$$(L^\tau \otimes L^\tau)(L^\tau \otimes L^\tau)$$

by means of the operator $L^\tau L^\tau \mapsto L^\tau L^\tau$ we get a “braided bi-algebra structure” in the algebra $\mathcal{L}(R, F)$ (see [GPS3] for detail). The corresponding quantum determinant is also a group-like element with respect to this coproduct and braided multiplication in $\mathcal{L}(R, F) \otimes \mathcal{L}(R, F)$.

Now, we consider the problem of centrality of the quantum determinants. It is well known that if a Hecke symmetry $R$ comes from the quantum group $U_q(sl(N))$, the quantum determinant $\det_{\mathcal{L}(R, P)}(L)$ is central (see [FRT]). If in a given RTT algebra $\mathcal{L}(R, P)$ the quantum determinant is central, then, by imposing the condition $\det_{\mathcal{L}(R, P)}(L) = 1$, we can define a Hopf algebra structure in the quotient algebra.
However, in general the quantum determinant is not central in the algebras $\mathcal{L}(R, P)$. The problem of its centrality was studied in [G]. It was shown that the quantum determinant $\det_{\mathcal{L}(R,P)}(L)$ is central if and only if the matrix

$$M = \|M_i^j\|, \quad \text{where} \quad M_i^j = u_{i_1\ldots i_m} v^{i_2\ldots i_m} j_i,$$

is scalar.

Let us consider centrally of the quantum determinants arising from the symmetries (2.5). For the second matrix from (2.5) the corresponding matrix $M$ is scalar:

$$\begin{pmatrix}
q & 0 & 0 & 0 \\
0 & q - q^{-1} & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & q
\end{pmatrix} \Rightarrow M = -\frac{1}{2q} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Whereas, for the first symmetry from (2.5) the matrix $M$ is only diagonal, but not scalar (provided $q \neq \pm 1$):

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & q & 0 \\
0 & q^{-1} & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \Rightarrow M = -\frac{1}{2} \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

As a consequence, the quantum determinant (2.6) is not central in the corresponding QMA $\mathcal{L}(R, P)$.

By contrast, in the RE algebras $\mathcal{L}(R, R)$ the quantum determinant is always central. Therefore, by imposing the condition $\det_{\mathcal{L}(R,R)}(L) = 1$, it is possible to define a braided Hopf algebra structure in the quotient algebra $\mathcal{A}$.

Now, we discuss the problem of constructing the characteristic polynomial for the matrix $L$ by assuming the bi-rank of a symmetry $R$ to be $(m|0)$. We say that a monic polynomial $ch(t)$ of degree $m$ is characteristic if $ch(L) = 0$. It is well-known that in the classical case $R = F = P$ (in this case the algebra $\mathcal{L}(P, P)$ is commutative) the characteristic polynomial reads $ch(t) = \det_{\mathcal{L}(P,P)}(L - t I)$ (the Cayley-Hamilton theorem). Consider a similar polynomial in the algebras $\mathcal{L}(R, R)$.

**Proposition 6** In the algebras $\mathcal{L}(R, R)$, where $R$ is a Hecke symmetry, the following relation holds

$$\det_{\mathcal{L}(R,R)}(L - t I) = \sum_{0 \leq k \leq m} (-t)^{m-k} \alpha_k e_k, \quad (4.2)$$

where $\alpha_k = q^{mk} \frac{m!}{m_k q^l} (m-k)!$.

**Proof.** Observe that, if $F = R$, the quantity (2.11) equals

$$(v \cdot_R u) = q^{-m^2}.$$

It results from the following formula (see [GS2])

$$\operatorname{Tr}_{R(k+1\ldots m)} A^{(m)}_{1\ldots m} = q^{-m(m-k)} \frac{k_{q}^l(m-k)q^l}{m_q^l} A^{(k)}_{1\ldots k} \quad (4.3)$$

$^4$Note that for an involutive $R$ such braided Hopf algebras were considered in [G] under the name of monoidal groups.
for \( k = 0 \).

Now, in the expansion of the element

\[
q^m \det_{L(R,R)} (L - tI) = \text{Tr}_{R(1...m)} A_{1...m}^{(m)} (L - tI)_1 \ldots (L - tI)_m
\]

we put together the terms containing \( k \) factors \( L_i \) on some places and the identity matrices on other positions. The number of such terms is \( \frac{m!}{k!(m-k)!} \) and they are equal to each other. This property is due to the fact that

\[
\text{Tr}_{R(1...m)} A_{1...m}^{(m)} L_{i_1} L_{i_2} \ldots L_{i_k} = \text{Tr}_{R(1...m)} A_{1...m}^{(m)} L_{\bar{i}_1} L_{\bar{i}_2} \ldots L_{\bar{i}_k},
\]

for any subset of ordered indices \( 1 \leq i_1 < i_2 < \ldots < i_k \leq m \). Now, it suffices to apply formula (4.3). Finally, we find

\[
\text{Tr}_{R(1...m)} A_{1...m}^{(m)} L_{1} \ldots L_{k} = q^{-m(m-k)} \frac{k!}{m!} e_k(L).
\]

The proof is completed. \( \blacksquare \)

If \( R \) is an involutive symmetry, then by setting \( q = 1 \) in (4.2) we get the following claim.

**Proposition 7** If \( R \) is an involutive symmetry, the polynomial \( \det_{L(R,R)} (L - tI) \) is characteristic.

If \( R \) is a Hecke symmetry, the polynomial \( \det_{L(R,R)} (L - tI) \) is not a characteristic one. However, we get the characteristic polynomial upon replacing \( \alpha_k \) by \( q^k \) in the right hand side of (4.2), that is we have:

\[
\text{ch}(t) := t^m - q e_1 t^{m-1} + q^2 e_2 t^{m-2} + \ldots + (-q)^{m-1} e_{m-1} t + (-q)^m e_m.
\]

Thus, on substituting \( t = L \) in this polynomial we get the Cayley-Hamilton identity for the matrix \( L \):

\[
L^m - q e_1 L^{m-1} + q^2 e_2 L^{m-2} + \ldots + (-q)^{m-1} e_{m-1} L + (-q)^m e_m I = 0.
\]

Note that the first prof of this identity in the algebras \( L(R, R) \) was given in [GPS1]. Also, note that this CH identity provides us with a method of inverting the generating matrix \( L \). Indeed, upon dividing the CH identity by the element \( (-q)^m L e_m \) we can express the matrix \( L^{-1} \) as a linear combination of the powers sums \( L^k, 0 \leq k \leq m-1 \) with coefficients belonging to the algebra \( L(R, R)[e_m^{-1}] \) (recall that the element \( e_m \) is central).

**Remark 8** The method of computing \( L^{-1} \) on the base of the quantum Cayley-Hamilton identity is also valid for the generating matrix of any algebra \( L(R, R) \) even if the symmetry \( R \) is not necessary even. If a given symmetry \( R \) is of bi-rank \( (m|n) n \neq 0 \), the generating matrix \( L \) also meets the Cayley-Hamilton identity

\[
a_{m+n} L^{m+n} + a_{m+n-1} L^{m+n-1} + \ldots + a_0 I = 0,
\]

where all the coefficients \( a_k \) belong to the center of the algebra \( L(R, R) \). Their explicit form can be found in [GPS2]. Note that in this case the leading coefficient \( a_{m+n} \) does not equal 1.

\(^{\text{Emphasize, that if } F \neq R \text{ this property fails.}}\)
Upon dividing this relation by \( a_0 \), we can express the matrix \( L^{-1} \) as a linear combinations of the matrices \( L^k \), \( 0 \leq k \leq m + n - 1 \) with the coefficients \( -a_k/a_0 \).

In this case we attribute the role of the quantum determinant (up to a sign) to the ratio \( a_0/a_{m+n} \). Note that the numerator and the denominator of this fraction are quantum analogs of some Schur polynomials. Also, note that this quantum determinant differs from quantum Berezinian which is associated with another fraction of two other quantum Schur polynomials. We refer the reader to [GPS2] for details.

In conclusion we would like to point out that the case \( F = R \) is exceptional in all algebras under consideration \( \mathcal{L}(R,R) \), \( \mathcal{H}(R,R) \) and \( \mathcal{Y}(R,R) \). First, in all these algebras the Cayley-Hamilton and Newton identities are closer to the classical ones. Second, the characteristic subalgebra of \( \mathcal{L}(R,R) \) is always central. As for the generalized Yangian \( \mathcal{Y}(R,R) \), its quantum determinant \( \det\mathcal{Y}(R,R)(L) \) is central. By contrast, in the generalized Yangian \( \mathcal{Y}(R,P) \) of RTT type the quantum determinant is central iff it is so in the QMA \( \mathcal{L}(R,P) \).

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