The minimum number of vertices in uniform hypergraphs with given domination number

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Abstract

The domination number \( \gamma(H) \) of a hypergraph \( H = (V(H), E(H)) \) is the minimum size of a subset \( D \subseteq V(H) \) of the vertices such that for every \( v \in V(H) \setminus D \) there exist a vertex \( d \in D \) and an edge \( H \in E(H) \) with \( v, d \in H \). We address the problem of finding the minimum number \( n(k, \gamma) \) of vertices that a \( k \)-uniform hypergraph \( H \) can have if \( \gamma(H) \geq \gamma \) and \( H \) does not contain isolated vertices. We prove that

\[
n(k, \gamma) = k + \Theta(k^{1-1/\gamma})
\]

and also consider the \( s \)-wise dominating and the distance-\( l \) dominating version of the problem. In particular, we show that the minimum number \( n_{dc}(k, \gamma, l) \) of vertices that a connected \( k \)-uniform hypergraph with distance-\( l \) domination number \( \gamma \) can have is roughly \( \frac{k^2}{l} \).

1. Introduction

In this paper we establish basic inequalities involving fundamental hypergraph parameters such as order, edge size, and domination number.

Many problems in extremal combinatorics are of the following form: what is the smallest or largest size that a graph, hypergraph, set system can have, provided it satisfies a prescribed property? In most cases,
In the present paper we address the problem of finding the minimum number of vertices in a $k$-uniform hypergraph that has a large domination number. The domination number $\gamma(G)$ of a graph $G$, a widely studied notion (see [10], [11]), is the smallest size that a subset $D \subset V(G)$ of the vertices can have if every vertex $v \in V(G) \setminus D$ has a neighbor in $D$.

We will be interested in the hypergraph version of this notion, which was investigated first in [4] and later studied in [2] [3] [4] [13] [16]. Let $H = (V(H), E(H))$ be a hypergraph. The neighborhood$^3$ of a vertex $v \in V(H)$ is the set $N_v := \{v\} \cup \bigcup_{E \in E(H) : v \in E} E$, and the neighborhood of a set $S \subset V(H)$ is defined as $N(S) := \bigcup_{v \in S} N_v$. A set $D \subset V(H)$ is called a dominating set of $H$ if $D \cap N_v \neq \emptyset$ for all $v \in V(H)$. Equivalently we can say that $D$ is a dominating set if and only if $N(D) = V(H)$. The minimum size $\gamma(H)$ of a dominating set in a hypergraph $H$ is the domination number of $H$. As all isolated vertices always are contained in every dominating set, they can be eliminated in an obvious way, therefore we restrict our attention to hypergraphs without isolates.

Let $n(k, \gamma)$ be the minimum number of vertices that a $k$-uniform hypergraph with no isolated vertices must contain if its domination number is at least $\gamma$. Beyond the trivial case of $n(k, 1) = k$, the problem of determining $n(k, \gamma)$ is natural and seems to be interesting enough to be addressed on its own right; nevertheless, Gerbner et al. (Problem 17 in [8]) arrived from a combinatorial search-theoretic framework at the particular problem of deciding whether $n(k, 3) \geq 2k + 3$ holds or not. We answer this problem in the negative, determining the asymptotic behavior of $n(k, \gamma)$ as a function of $k$ for every fixed $\gamma$, up to the exact growth order of the second term. To state our result in full strength, we need to introduce two generalizations of domination. For an integer $s > 0$ we call $D \subset V(H)$ an $s$-dominating set of $H$ if $|D \cap N_v| \geq s$ for all $v \in V(H) \setminus D$ and we call $D$ an $s$-tuple dominating set if $|D \cap N_v| \geq s$ for all $v \in V(H)$. Note that dominating sets are exactly the 1-dominating sets and 1-tuple dominating sets. As introduced in [7] and [9], respectively, the minimum size $\gamma(H, s)$ of an $s$-dominating set in a hypergraph $H$ is the $s$-domination number of $H$ and the minimum size $\gamma_s(H, s)$ of an $s$-tuple dominating set in a hypergraph $H$ is the $s$-tuple domination number$^4$ of $H$. By definition, we have $\gamma(H, s) \leq \gamma_s(H, s)$. For every pair $\gamma, s$ of integers with $\gamma \geq s$, let $n(k, \gamma, s)$ denote the minimum number of vertices that a $k$-uniform hypergraph $H$ must have if $\gamma(H, s) \geq \gamma$ holds and there exist no isolated vertices in $H$ and let $n_s(k, \gamma, s)$ denote the minimum number of vertices that a $k$-uniform hypergraph $H$ must have if $\gamma_s(H, s) \geq \gamma$ holds and there exist no isolated vertices. From the above, we have $n_s(k, \gamma, s) \leq n(k, \gamma, s)$.

Our main theorem about $s$-domination is the following.

Theorem 1.1. For every $\gamma \geq 2$ and $s \geq 1$ with $\gamma > s$ we have

$$ k + k^{1-1/(\gamma-s+1)} \leq n_s(k, \gamma, s) \leq n(k, \gamma, s) \leq k + (4 + o(1))k^{1-1/(\gamma-s+1)}. $$

Another generalization of domination is distance-$l$ domination, which was introduced by Meir and Moon in [17]. This notion has been studied only for graphs so far. A good survey of the results until 1997 is [12]. For more recent upper and lower bounds on the distance-$l$ domination number of graphs see [19] and [6].

In distance-$l$ domination a vertex $v$ dominates all vertices that are at distance at most $l$ from $v$. As the definition of distance in graphs involves paths, and paths in hypergraphs can be defined in several ways,
distance-$l$ domination could be addressed with each of those definitions. But as we will remark in Section \[ \text{[Section]} \]
only so-called ‘Berge paths’ offer new problems in our context. A Berge path of length $l$ is a sequence $v_0, H_1, v_1, H_2, v_2, \ldots, H_l, v_l$ with $v_i \in V(H)$ for $i = 0, 1, \ldots, l$ and $v_{i-1}, v_i \in E(H)$ for $i = 1, 2, \ldots, l$. The distance $d_H(u, v)$ of two vertices $u, v \in V(H)$ is the length of a shortest Berge path from $u$ to $v$. The ball centered at $u$ and of radius $l$ consists of those vertices of $H$ which are at distance at most $l$ from $u$; it will be denoted by $B_l(u)$. We call $D \subset V(H)$ a distance-$l$ dominating set of $H$ if $\bigcup_{u \in D} B_l(u) = V(H)$. Equivalently we can say that $D \subset V(H)$ is a distance-$l$ dominating set if and only if $D \cap B_l(v) \neq \emptyset$ for all $v \in V(H)$. Note that distance-1 dominating sets are the usual dominating sets.

The problem becomes more interesting when disconnected hypergraphs get excluded. Hence, for $k \geq 2$ and $l, \gamma \geq 1$ let $n_{dc}(k, \gamma, l)$ denote the minimum number of vertices that a $k$-uniform connected hypergraph $H$ with no isolated vertices can contain if $\gamma_d(H, l) \geq \gamma$ holds. The next proposition shows that $n_d(k, \gamma, l)$ does not depend on $l$ once $l \geq 2$ is supposed.

**Proposition 1.2.** For any $k, l \geq 2$ and $\gamma \geq 1$ we have $n_d(k, \gamma, l) = k\gamma$, and the unique extremal hypergraph consists of $\gamma$ pairwise disjoint edges.

**Proof.** It is clear that the $k$-uniform hypergraph with just $\gamma$ disjoint edges yields the upper bound $n_d(k, \gamma, l) \leq k\gamma$.

We prove the lower bound by induction on $\gamma$. The case $\gamma = 1$ is trivial. So assume that $\gamma \geq 2$, and let $H = (V(H), \mathcal{E}(H))$ be a $k$-uniform hypergraph with $\gamma_d(H, l) \geq \gamma$. Consider an arbitrary $v \in V(H)$. Any vertex in $N(B_{l-1}(v))$ is distance-$l$ dominated by $v$, therefore the $k$-uniform hypergraph $H'$ induced by the edge set $\{H \in \mathcal{E}(H) : H \cap B_{l-1}(v) = \emptyset\}$ covers all vertices of $H$ not distance-$l$ dominated by $v$. The assumption $\gamma_d(H, l) \geq \gamma$ implies $\gamma_d(H', l) \geq \gamma - 1$ and thus using that $|B_{l-1}(v)| \geq k$ for $l \geq 2$ and by induction we obtain

$$|V(H)| = |B_{l-1}(v)| + |V(H')| \geq k + (\gamma - 1)k = \gamma k.$$

Strict inequality holds whenever $v$ has degree at least two. \hfill $\Box$

The problem becomes more interesting when disconnected hypergraphs get excluded. Hence, for $k \geq 2$ and $l, \gamma \geq 1$ let $n_{dc}(k, \gamma, l)$ denote the minimum number of vertices that a $k$-uniform connected hypergraph $H$ must contain if it has $\gamma_d(H, l) \geq \gamma$.

To state our main result concerning $n_{dc}(k, \gamma, l)$ we need to define the following function:

$$f(k, \gamma, l) := \begin{cases} \frac{1}{2}k\gamma + \max\{k, \gamma\} & \text{if } l \text{ is even,} \\ \frac{1}{2}(l+1)k\gamma & \text{if } l \text{ is odd.} \end{cases}$$

**Theorem 1.3.** (a) For any $k, l \geq 2$ we have

$$\frac{(2l+1)k}{2} \leq n_{dc}(k, 2, l) \leq \min \left\{ \left\lfloor \frac{(2l+1)(k+1)}{2} \right\rfloor, (l+1)k \right\}.$$

(b) For any $k \geq 2$, $l \geq 4$ and $\gamma \geq 3$ we have

$$k \left\lfloor \frac{l-1}{2} - 1 \right\rfloor \gamma < n_{dc}(k, \gamma, l) \leq f(k, \gamma, l).$$

(c) For any $k \geq 2$ and $\gamma \geq 3$ we have

$$k\gamma \leq n_{dc}(k, \gamma, 2) \leq k\gamma + \max\{k, \gamma\}.$$
(d) For any \( k \geq 2 \) and \( \gamma \geq 3 \) we have
\[
k\gamma \leq n_{dc}(k, \gamma, 3) \leq 2k\gamma.
\]

The remainder of the paper is organized as follows: we prove Theorem 1.1 in Section 2, and Theorem 1.3 in Section 3. Section 4 contains some final remarks, also including a general upper bound on \( \gamma_{dc}(\mathcal{H}, l) \) as a function of \( l \), the number of vertices, and the edge size.

2. Proof of Theorem 1.1

In this section we prove our bounds on \( n_\gamma(k, \gamma, s) \) and \( n(k, \gamma, s) \). First we verify the bound \( k + k^{1−1/(\gamma−s−1)} \leq n_\gamma(k, \gamma, s) \). Observe that it is enough to prove the statement for \( s = 1 \), since for any hypergraph \( \mathcal{H} \) we have \( \gamma_\gamma(\mathcal{H}, s) − (s − 1) \geq \gamma_\gamma(\mathcal{H}, 1) \) as for any \( s \)-tuple dominating set \( D \) of \( \mathcal{H} \) and a \( s' \)-subset \( D' \) of \( D \) the set \( D \setminus D' \) \((s − s')\)-tuple dominates \( \mathcal{H} \). Consequently
\[
n_\gamma(k, \gamma, s) \geq n_\gamma(k, \gamma − (s − 1), 1),
\]
which implies the statement.

To see \( n(k, \gamma, 1) \geq k + k^{1−1/\gamma} \) let \( \mathcal{H} \) be a \( k \)-uniform hypergraph with \( \gamma(\mathcal{H}) \geq \gamma \geq 2 \). Let \( G = (V(\mathcal{H}), E) \) be the graph with \((u, v) \in E\) if and only if no \( \mathcal{H} \in E(\mathcal{H}) \) contains both \( u \) and \( v \). The \( \gamma \geq 2 \) condition means that for any vertex \( v \in V(\mathcal{H}) \) there exists a \( u \) such that no edge \( \mathcal{H} \in E(\mathcal{H}) \) contains both \( u \) and \( v \), thus \( G \) does not contain any isolated vertices. Let us write \( n = |V(\mathcal{H})| = |V(G)| = k + x \) and let \( t \) be the number of edges in a largest matching \( M = (V(M), E(M)) \) of \( G \). Note that two distinct vertices \( u', v' \) outside \( V(M) \) cannot be adjacent to two distinct endpoints \( u, v \) of an edge \( e \in E(M) \) as the matching \((M \setminus \{e\}) \cup \{(u, u'), (v, v')\}) would contradict the maximality of \( M \). Then either just one of \( u \) and \( v \) has neighbors outside \( M \), or none of them have any, or they share their unique neighbor outside \( M \). We denote by \( e(v) \) the (or an) endpoint of \( e \) whose ‘outside’ neighborhood in this sense contains the ‘outside’ neighborhood of the other endpoint, and let \( d_{e(v)} \) denote the size of \( N_{e(v)} \setminus V(M) \).

By the definition of \( \gamma = \gamma_\gamma(\mathcal{H}, 1) = \gamma(\mathcal{H}, 1) \) and \( G \) we have that for any set \( \Gamma \) of \( \gamma − 1 \) vertices in \( V(G) \) there is a vertex \( v \in V(G) \) which is connected by edges in \( E(G) \) to all the vertices of \( \Gamma \). If \( \Gamma \) is a subset of \( V(G) \setminus V(M) \), then the vertex which is adjacent to all vertices of \( \Gamma \) must be in \( V(M) \), since \( M \) is maximal. By this we obtain
\[
\sum_{e \in E(M)} \left( d_{e(v)} \right)^\gamma - 1 \geq \left( |V(G) \setminus V(M)| / \gamma - 1 \right).
\]

Writing \( d := \max_{e \in E(M)} d_{e(v)} \) the above inequality yields
\[
td^{\gamma - 1} \geq (k + x − 2t)^{\gamma - 1},
\]
and rearranging gives
\[
d \geq \frac{k + x - 2t}{t^{\gamma - 1}}.
\]
Let \( e \in E(M) \) be an edge with \( d_{e(v)} = d \), and let \( H \) be any hyperedge \( H \in E(\mathcal{H}) \) containing \( e(v) \). Just as any hyperedge, \( H \) must avoid an endpoint of each edge in \( M \), and \( H \) is disjoint from \( N_{e(v)} \setminus V(M) \). Therefore, we obtain \( k + x = n \geq d + t + k \) and thus \( x \geq d + t \). Plugging the previous inequality into this and rearranging yields:
\[
t^{\frac{1}{\gamma - 1}} (x - t + 2t^{\gamma - 2}) \geq k + x.
\]
Now using that \(x \geq t\) and \(t \geq \frac{2 \gamma - 2}{\gamma} + x\), we obtain that the left-hand side of the previous inequality is at most \(x \frac{\gamma}{\gamma} + x\) and therefore we have

\[
x \frac{\gamma}{\gamma} + x \geq k + x,
\]

which proves the required lower bound.

To prove the bound \(n(k, \gamma, s) \leq k + (4 + o(1))k^{1-1/(\gamma-s+1)}\) we need a construction. This involves projective geometries or linear vector spaces over finite fields. We will use the Gaussian or \(q\)-binomial coefficient \(\binom{n}{k}_q\) that denotes the number of \(k\)-dimensional subspaces of a vector space of dimension \(n\) over \(\mathbb{F}_q\), i.e.

\[
\binom{n}{k}_q := \frac{\prod_{i=1}^k (q^{n-i+1} - 1)}{\prod_{i=1}^k (q^i - 1)}
\]

and we will omit \(q\) from the subscript when it is clear from the context. Let \(q\) be a prime power, \(t\) be any positive integer and \(U\) be a \(\gamma\)-dimensional vector space over \(\mathbb{F}_q\). Let \(E_1, E_2, \ldots, E_m\) be the 1-dimensional subspaces of \(U\) and \(U_1, U_2, \ldots, U_m\) the \((\gamma - 1)\)-dimensional subspaces of \(U\), where \(m = \left[\frac{n}{q}\right] = \left[\frac{n-1}{q}\right] = q^\gamma + 1 + q \gamma - 2 + \ldots + 1\). Let \(A_1, A_2, \ldots, A_m, B\) be pairwise disjoint sets with \(B = \{b_1, b_2, \ldots, b_m\}\) and \(|A_i| = t\) for all \(1 \leq i \leq m\). Let us define \(H_{q, \gamma, s} = \{H_1, H_2, \ldots, H_m\}\) by

\[
H_i := \{b_i\} \cup \bigcup_{j : E_j \not\subseteq U_i} A_j.
\]

We claim that \(\gamma(H_{q, \gamma, s+1, t}, s) \geq \gamma\). Suppose not and let \(D = D_B \cup D_A\) be a minimal \(s\)-dominating set of \(H = H_{q, \gamma, s+1, t}\) with \(D_B = D - D_A\), \(D_A = D \setminus D_B\) and \(|D| < \gamma\). As every vertex \(d \in D_B\) is contained in exactly one hyperedge \(H_d\) of \(H\), each such \(d\) can be replaced by a vertex \(d' \in V(H) \setminus (D \cup B)\) to obtain an \(s\)-dominating set \(D'\) of \(D' \subseteq V(H) \setminus B\) and \(|D'| = |D| < \gamma\). Let \(D' = \{d_1, d_2, \ldots, d_p\}\) and \(D'' = \{d_1, d_2, \ldots, d_{\gamma-s}\}\). Then for \(Z = \bigcap_{j : \exists d \in D'' \cap A_j} U_j\) we obtain

\[
\dim(Z) \geq 1.
\]

If \(E\) is a 1-subspace of \(Z\), then the corresponding vertex \(b \in V(H)\) is not dominated by any vertex \(d \in D''\) and thus at most \((s - 1)\)-dominated by \(D'\), which is a contradiction.

Let \(C_1, C_2, \ldots, C_{q+1}\) be pairwise disjoint sets all of size \(\left\lceil \frac{e}{q-\gamma+s+1} \right\rceil\), all being disjoint from \(V(H_{q, \gamma, s+1, t})\). We number the subspaces \(U_1, U_2, \ldots, U_m\) in such a way that \(U_1, U_2, \ldots, U_{q+1}\) correspond to the dual of a \((q + 1)\)-arc in \(PG(\gamma - s, q)\), i.e. every 1-subspace \(E\) of \(V\) is contained in at most \(\gamma - s - 1\) subspaces among \(U_1, U_2, \ldots, U_{q+1}\). (For a general introduction to finite geometries, see [15].) Therefore, for any \(1 \leq i \leq m\), the sets \(I_i := \{j : E_i \not\subseteq U_j, 1 \leq j \leq q+1\}\) satisfy \(|I_i| \geq q - \gamma + s + 2\) and thus there exists a set \(T_i \subset \bigcup_{j \in I_i} C_j\) of size \(e\). Let us define

\[
H_i' := \{b_i\} \cup \bigcup_{j : E_j \not\subseteq U_j} A_j \cup T_i.
\]

By definition we have \(|H_i'| = k'\) for all \(i = 1, 2, \ldots, m\). The \(s\)-domination number of the new hypergraph is the same as that of the old one, as for any \(v \in C_j\) and \(u \in A_i\) we have \(N_u \subset N_v\). Moreover the number \(n'\) of vertices in the new hypergraph is

\[
n + \left\lceil \frac{e}{q-\gamma+s+1} \right\rceil(q+1) \leq k + 4k^{1-1/(\gamma-s+1)} + e + O_\gamma(e/q) \leq k' + (4 + o(1))k^{1-1/(\gamma-s+1)},
\]

as \(O_\gamma(e/q) = o(q^{\gamma-s})\) holds by \(e = o(q^{\gamma-s+1})\).

\[\square\]
3. Distance domination

In this section we prove Theorem 1.3, the lower and upper bounds on \( n_{dc}(k, \gamma, l) \).

3.1. The \( j \)-radius of trees

We start with some definitions and an auxiliary statement that we will use in the proof.

**Definition.** For positive integers \( a_1, a_2, ..., a_h \) the spider graph, denoted by \( S(a_1, a_2, ..., a_h) \), is the tree on \( 1 + \sum_{i=1}^{h} a_i \) vertices which is obtained from \( h \) paths of lengths \( a_1, a_2, ..., a_h \), respectively, by identifying the first vertices of those paths to a single vertex \( v \) of degree \( h \). Hence, \( S(a_1, a_2, ..., a_h) \{ v \} \) has \( h \) connected components, say \( C_1, C_2, ..., C_h \), where each \( C_i \) is a path \( P_{a_i} \) on \( a_i \) vertices (for \( i = 1, 2, ..., h \)).

In a connected graph \( G = (V(G), E(G)) \), the *excentricity* of a vertex \( v \in V(G) \) is defined as
\[
\text{exc}_G(v) := \max \{ d_G(u, v) : u \in V(G) \}
\]
and let the *radius* of \( G \) be
\[
r(G) := \min \{ \text{exc}_G(v) : v \in V(G) \}.
\]
More generally, for any \( \emptyset \neq W \subset V(G) \) let us write
\[
\text{exc}_G(W) := \max \{ \min \{ d_G(u, w) : w \in W \} : u \in V(G) \}
\]
and for an integer \( j \geq 1 \) let the *\( j \)-radius of \( G \)* be
\[
r_j(G) := \min \{ \text{exc}_G(W) : W \subset V(G), |W| \leq j \}.
\]
Certainly we have \( r(G) = r_1(G) \). Finally, let
\[
r_j(n) := \max \{ r_j(T) : |V(T)| = n, \ T \text{ is a tree} \}.
\]

The numerical bounds themselves in the next lemma concerning the radius of a tree are folklore; for later use, however, we need a more detailed assertion which describes some structural properties, too. Some bounds on the function \( r_j(n) \) can be derived also from results of Meir and Moon [17], but the following is a little sharper.

**Lemma 3.1.** Let \( n \geq j \) be positive integers. Then we have
\[
\left\lfloor \frac{n}{j+1} \right\rfloor \leq r_j(n) \leq \left\lceil \frac{n}{j+1} \right\rceil.
\]
Moreover, \( r_1(n) = \left\lfloor \frac{n-1}{2} \right\rfloor \) and

(i) if \( n \) is even, then the only tree with \( r_1(T) = \left\lfloor \frac{n-1}{2} \right\rfloor \) is the path \( P_n \) on \( n \) vertices.

(ii) If \( n \) is odd and \( r_1(T) = \left\lfloor \frac{n-1}{2} \right\rfloor \) holds, then \( T \) is a path \( P_{n-1} \) with a pendant edge. Furthermore, \( T \) contains two copies of \( P_{n-1} \) if and only if \( T \) is either a path \( P_n \) or a fork \( F_n \). Otherwise \( T \) contains just one copy of \( P_{n-1} \).
Proof. Let us first prove the statements about $r_l(n)$. Let $T$ be an arbitrary tree on $n$ vertices and let $v$ be a middle vertex of a longest path $P$ in $T$. If $P$ contains $l$ vertices, then any vertex is at distance at most $\left\lceil \frac{n-1}{j+1} \right\rceil$ from $v$. This implies all assertions of the lemma if $n$ is even. If $n$ is odd, this implies that $T$ must contain a path on $n-1$ vertices and thus $T$ is a path $P_{n-1}$ and a pendant edge.

Let us now prove the general lower bound. We claim that

$$\left\lceil \frac{n}{j+1} \right\rceil = r_j \left( S \left( \left\lceil \frac{n-1}{j+1} \right\rceil, \left\lceil \frac{n}{j+1} \right\rceil, \ldots, \left\lceil \frac{n+j-1}{j+1} \right\rceil \right) \right)$$

holds, which proves the lower bound by the definition of $r_j(n)$. To see that the claim is true, observe that any set $U \subset V(S(\left\lceil \frac{n-1}{j+1} \right\rceil, \left\lceil \frac{n}{j+1} \right\rceil, \ldots, \left\lceil \frac{n+j-1}{j+1} \right\rceil))$ of size $j$ is disjoint from at least one component $C$ of $S(\left\lceil \frac{n-1}{j+1} \right\rceil, \left\lceil \frac{n}{j+1} \right\rceil, \ldots, \left\lceil \frac{n+j-1}{j+1} \right\rceil) \setminus \{v\}$.

Thus if $v \notin U$, then the leaf of $S(\left\lceil \frac{n-1}{j+1} \right\rceil, \left\lceil \frac{n}{j+1} \right\rceil, \ldots, \left\lceil \frac{n+j-1}{j+1} \right\rceil)$ belonging to $C$ has distance at least

$$1 + \left\lceil \frac{n-1}{j+1} \right\rceil \geq \left\lceil \frac{n}{j+1} \right\rceil$$

from any vertex of $U$.

If $v \in U$ holds, then $U$ is disjoint from at least two components $C_1, C_2$ of $S(\left\lceil \frac{n-1}{j+1} \right\rceil, \left\lceil \frac{n}{j+1} \right\rceil, \ldots, \left\lceil \frac{n+j-1}{j+1} \right\rceil) \setminus \{v\}$, and the leaf of $S(\left\lceil \frac{n-1}{j+1} \right\rceil, \left\lceil \frac{n}{j+1} \right\rceil, \ldots, \left\lceil \frac{n+j-1}{j+1} \right\rceil)$ belonging to the larger path has distance at least $\left\lceil \frac{n}{j+1} \right\rceil$ from $v$ and thus from $U$. This completes the proof of the general lower bound.

To see the general upper bound, let $T$ be any tree on $n$ vertices. We will use the following claim repeatedly.

Claim 3.2. Let $m < n$ be two positive integers. Then in any tree $T$ on $n$ vertices there exists a vertex $v$ such that if $C_1, C_2, \ldots, C_s$ denote those components of $T \setminus \{v\}$ whose all vertices are at distance at most $m$ from $v$, then $\sum_{i=1}^{s} |C_i| \geq m$ holds.

Proof of Claim 3.2. Let $P$ be a longest path of $T$. If $P$ contains at least $m$ vertices, then any vertex can play the role of $v$. If $P$ contains at least $m+1$ vertices, then let $v$ be the $(m+1)$st vertex from one end of $P$.

For $t = 1, 2, \ldots, j - 1$ let $m_t = \left\lfloor \frac{n+j-1}{j+1} \right\rfloor$ and let $T_1 = T$. We apply Claim 3.2 to $T_t$ and $m_t$ for $t = 1, 2, \ldots, j - 1$ to obtain $v_t$, and then set

$$T_{t+1} := T_j \setminus \cup_{i=1}^{k_t} C_{i,t},$$

where the $C_{i,t}$ ($i = 1, 2, \ldots, k_t$) are the components of $T_t \setminus \{v_t\}$ whose vertices are at distance at most $m_t$ from $v_t$. By the claim we also have $\sum_{i=1}^{k_t} |C_{i,t}| \geq m_t$.

In this way we obtain a tree $T_j$ of at most $\left\lfloor \frac{2n}{j+1} \right\rfloor$ vertices. Let $v_j$ be a vertex of $T_j$ within distance $\left\lfloor \frac{|V(T_j)|-1}{2} \right\rfloor$ from all vertices of $T_j$. Such a vertex exists by the result on $r_1(n)$. Clearly, $U = \{v_1, v_2, \ldots, v_j\}$ is a set of vertices with $\text{exc}_T(U) \leq \left\lfloor \frac{n}{j+1} \right\rfloor$, which proves $r_j(T) \leq \left\lfloor \frac{n}{j+1} \right\rfloor$.\qed

3.2. Putting things together: the proof of Theorem 1.3

Let us first prove the upper bounds of Theorem 1.3. To do so we introduce two types of hypergraphs with distance-$l$ domination number $\gamma$. The second construction will prove the upper bounds of (b), (c), and (d). If $\gamma = 2$, then the construction giving the smaller number of vertices depends on the values of $k$ and $l$. This is why we have the minimum of two expressions in the upper bound of (a).
Construction 1:

For $i = 1, \ldots, 2l(\gamma - 1) + 1$ let $U_i$ be pairwise disjoint sets, and let $v_i$ and $w$ be distinct vertices which are not elements of $\bigcup_{i=1}^{2l(\gamma - 1) + 1} U_i$. During Construction 1 all the indices will be taken modulo $2l(\gamma - 1) + 1$, e.g. we then have $2l(\gamma - 1) + 2 = 1$.

If $k$ is odd, let $|U_i| = \frac{k-1}{2}$ for all $i$. We define a hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ in the following way. Let

$$V(\mathcal{H}) := \bigcup_{i=1}^{2l(\gamma - 1) + 1} (U_i \cup \{v_i\}),$$

and let the hyperedges of $\mathcal{H}$ be

$$H_i := U_i \cup U_{i+1} \cup \{v_i\}$$

for $i = 1, \ldots, 2l(\gamma - 1) + 1$. Then the size of $V(\mathcal{H})$ is

$$\frac{(2l(\gamma - 1) + 1)(k + 1)}{2}.$$

If $k$ is even, let $|U_{2i}| = \frac{k}{2}$ for $i = 1, \ldots, l(\gamma - 1)$ and $|U_{2i+1}| = \frac{k}{2} - 1$ for $i = 0, \ldots, l(\gamma - 1)$. We define $\mathcal{H}$ with the vertex set

$$V(\mathcal{H}) := \{w\} \cup \bigcup_{i=1}^{2l(\gamma - 1) + 1} (U_i \cup \{v_i\}),$$

and with the edge set $E(\mathcal{H}) := \{H_i \mid 1 \leq i \leq 2l(\gamma - 1) + 1\}$, where

$$H_i := U_i \cup U_{i+1} \cup \{v_i\}$$

if $i = 1, \ldots, 2l(\gamma - 1)$, and

$$H_i := U_i \cup U_{i+1} \cup \{v_i, w\}$$

if $i = 2l(\gamma - 1) + 1$. Then,

$$|V(\mathcal{H})| = \frac{(2l(\gamma - 1) + 1)(k + 1)}{2} + \frac{1}{2} = \left\lceil \frac{(2l(\gamma - 1) + 1)(k + 1)}{2} \right\rceil.$$

To see that $\gamma_d(\mathcal{H}, l) \geq \gamma$ holds in both cases, observe the following facts:

- vertex $v_i$ distance-$l$ dominates a vertex $v_j$ exactly for
  $$j \in \{i - l + 1, \ldots, i + l - 1\},$$

- vertex $w$ distance-$l$ dominates a $v_j$ exactly for
  $$j \in \{2l(\gamma - 1) - l + 2, \ldots, 2l(\gamma - 1) + l\},$$

- a vertex $u \in U_i$ distance-$l$ dominates a $v_j$ exactly for
  $$j \in \{i - l, \ldots, i + l - 1\}.$$
Figure 1: Construction 2 in case of even $l$

So, every vertex in $V(\mathcal{H})$ distance-$l$ dominates at most $2l$ vertices $v_i$. This yields $\gamma_d(\mathcal{H}, l) \geq \gamma$.

**Construction 2:**

This construction relies on the spider graph $S = S(a_1, a_2, \ldots, a_{\gamma})$ with all of the $a_i$ being equal to $\lfloor l/2 \rfloor$. Let $v$ be the only vertex of $S$ with degree $\gamma$. Let $u_1, u_2, \ldots, u_\gamma$ be the neighbors of $v$ in $S$, and let $u'_1, u'_2, \ldots, u'_\gamma$ be the vertices of $S$ that are at distance $\lfloor l/2 \rfloor$ from $v$.

Let $W$ be a set of size $\max\{k, \gamma\}$. Take a partition $(W_1, W_2, \ldots, W_\gamma)$ of $W$ such that $|W_i| = \lfloor \frac{|W|+i-1}{\gamma} \rfloor$. Finally, for every $u \in V(S) \setminus \{v\}$, let $U_u := U_{u,1} \cup U_{u,2}$ be a set of size $k$ such that

- $u \in U_{u,1}$ holds for all $u \in V(S) \setminus \{v\}$,
- $U_u \cap U_{u'} = \emptyset$ holds for all $u \neq u' \in V(S) \setminus \{v\}$,
- $U_u \cap W = \emptyset$ holds for all $u \in V(S) \setminus \{v\}$,
- $|U_{u,1}| = |W_i|$ for all those $u \in V(S) \setminus \{v\}$ which lie in the same component of $S \setminus \{v\}$ as $u_i$.

With the help of the previously defined sets we construct a $k$-uniform hypergraph $\mathcal{H}$ in the following way, depending on the parity of $l$:

**Case I: $l$ is even**

Let the vertex set of $\mathcal{H}$ be $V(\mathcal{H}) = W \cup \bigcup_{u \in V(S) \setminus \{v\}} U_u$. Thus we have

$$|V(\mathcal{H})| = \frac{kl\gamma}{2} + \max\{k, \gamma\}.$$  

The edge set $E(\mathcal{H})$ contains the following four types of hyperedges:
1. all $k$-subsets of $W$, i.e. \( \binom{W}{k} \subset \mathcal{E}(\mathcal{H}) \),
2. for all $u \in V(S) \setminus \{v\}$, we have $U_u \in \mathcal{E}(\mathcal{H})$,
3. for all $i = 1, 2, \ldots, \gamma$ let $W_i \cup U_{u_i,2} \in \mathcal{E}(\mathcal{H})$,
4. for every edge $(u, u') = e \in E(S)$ with $u, u' \neq v$ if $d_S(u, v) < d_S(u', v)$ holds, then let $U_{u,1} \cup U_{u',2} \in \mathcal{E}(\mathcal{H})$.

Clearly, $\mathcal{H}$ is connected due to $\binom{W}{k} \subset \mathcal{E}(\mathcal{H})$. We claim that $\gamma_d(\mathcal{H}, l) \geq \gamma$ holds. Indeed, if $D \subset V(\mathcal{H})$ has size at most $\gamma - 1$, then there exists an $i \leq \gamma$ such that
\[
D \cap (W_i \cup \bigcup_{u \in C_i} U_u) = \emptyset
\]
holds where $C_i$ is the component of $S \setminus \{v\}$ containing $u_i$. Then $u'_i$ is at distance at least $1 + 2\frac{l}{2} = l + 1$ from any vertex of $D$ and thus $u'_i$ is not distance $l$-dominated by $D$.

**Case II: $l$ is odd**

In addition to the sets defined above, let $Z_1, Z_2, \ldots, Z_\gamma$ be pairwise disjoint sets of size $k - |W_i|$, each of which is disjoint from all previously defined sets. Let the vertex set of $\mathcal{H}$ be
\[
V(\mathcal{H}) = W \cup \bigcup_{u \in V(S) \setminus \{v\}} U_u \cup \bigcup_{i=1}^{\gamma} Z_i.
\]
Thus we have
\[
|V(\mathcal{H})| \leq \left\lceil \frac{l}{2} \right\rfloor k\gamma.
\]
As for the edge set of $\mathcal{H}$, there is a fifth type of hyperedge:

5. for all $1 \leq i \leq \gamma$ let $U_{u'_i,1} \cup Z_i \in \mathcal{E}(\mathcal{H})$.

The fact that $\gamma_d(\mathcal{H}, l) \geq \gamma$ follows similarly as in the previous case, because for any $(\gamma - 1)$-set $D \subset V(\mathcal{H})$ there exists an $i$ such that any vertex $z \in Z_i$ is at distance at least $l + 1$ from $D$.

Let us now turn our attention to the lower bounds. We prove first that of (a). Consider a connected $k$-uniform hypergraph $\mathcal{H}$ with $\gamma_d(\mathcal{H}, l) \geq 2$. Let $\mathcal{M}$ be a maximal matching in $\mathcal{H}$ obtained in the following way. Let
\[
\mathcal{M}_1 := \{H_1\}, \ I_1 := \{H \in \mathcal{E}(\mathcal{H}) \setminus \{H_1\} : H \cap H_1 \neq \emptyset\} \text{ and } \mathcal{R}_1 := \mathcal{E}(\mathcal{H}) \setminus (\mathcal{M}_1 \cup \mathcal{I}_1).
\]
Then for $s \geq 2$ we define a sequence $\mathcal{M}_s, \mathcal{I}_s, \mathcal{R}_s$ of partitions of $\mathcal{E}(\mathcal{H})$ such that:

1. $\mathcal{M}_s$ is a matching,
2. every hyperedge in $\mathcal{I}_s$ meets at least one hyperedge in $\mathcal{M}_s$, and
3. all hyperedges in $\mathcal{R}_s$ are disjoint from all hyperedges in $\mathcal{M}_s$.

If $\mathcal{M}_s, \mathcal{I}_s, \mathcal{R}_s$ are defined with $\mathcal{R}_s \neq \emptyset$, then let $H_{s+1} \in \mathcal{R}_s$ be a hyperedge such that $H_{s+1} \cap I_s \neq \emptyset$ for some $I_s \in \mathcal{I}_s$. The existence of such $H_{s+1}$ follows from the assumption that $\mathcal{H}$ is connected. Set
\[
\mathcal{M}_{s+1} := \mathcal{M}_s \cup \{H_{s+1}\}, \ \mathcal{I}_{s+1} := \mathcal{I}_s \cup \{R \in \mathcal{R}_s \setminus \{H_{s+1}\} : R \cap H_{s+1} \neq \emptyset\}
\]
\[
\mathcal{R}_{s+1} := \mathcal{R}_s \setminus \{H_{s+1}\}.
\]
and

\[ R_{s+1} := \mathcal{E}(\mathcal{H}) \setminus (\mathcal{M}_{s+1} \cup \mathcal{I}_{s+1}) . \]

For the smallest positive \( t \) with \( R_t = \emptyset \), we let \( \mathcal{M} := \mathcal{M}_t \). Thus the size of \( \mathcal{M} \) is \( t \).

Now let us consider the auxiliary graph \( G_{\mathcal{M}} \) with vertex set \( \mathcal{M} \) and \( e = \{ H_i, H_j \} \in E(G_{\mathcal{M}}) \) if and only if there exists \( H \in \mathcal{H} \) with \( H \cap H_i \neq \emptyset \) and \( H \cap H_j \neq \emptyset \). By the definition of \( \mathcal{M} \), the graph \( G_{\mathcal{M}} \) is connected. For a vertex \( v \in \bigcup_{H \in \mathcal{M}} H \) let \( H_v \) denote the only element of \( \mathcal{M} \) containing \( v \).

Suppose that for a pair \( H, H' \in \mathcal{M} \) we have \( d_{G_{\mathcal{M}}}(H, H') = r \). Then for any pair of vertices \( u, v \in H \) we have \( d_{H}(u, v) \leq 1 + 2r \). To see this, consider the sequence \( H, H_{e_1}, H_{e_2}, \ldots, H_{e_r}, H' \), where \( e_s \) is the \( s \)th edge in a shortest path from \( H \) to \( H' \) and \( H_{e_s} \) is the \( s \)th vertex (i.e., a hyperedge in \( \mathcal{H} \)) in the same path. By the maximality of \( \mathcal{M} \), for every vertex \( w \) of \( \mathcal{H} \) there exists an edge \( H_w \) containing \( w \) and an edge \( H \in \mathcal{M} \) with \( H_w \cap H \neq \emptyset \), therefore by the observation above we have

\[ d_H(u, w) \leq 2 + 2r_{G_{\mathcal{M}}}(u) \]

for every \( u \in \bigcup_{H \in \mathcal{M}} H \) and \( w \in V(\mathcal{H}) \).

If \( t \geq l + 1 \) holds, then \( |V(\mathcal{H})| \geq kt \geq k(l + 1) \), proving the desired lower bound.

Now suppose that \( t \leq l - 2 \) or \( t = l - 1 \) with \( t \) being odd. As we have noted, \( G_{\mathcal{M}} \) is connected and thus by Lemma 3.1 we obtain

\[ r(G_{\mathcal{M}}) \leq \left\lfloor \frac{t - 1}{2} \right\rfloor . \]

Therefore, there exists an \( H^* \in \mathcal{M} = V(G_{\mathcal{M}}) \) such that \( r_{G_{\mathcal{M}}}(H^*) \leq \left\lfloor \frac{t - 1}{2} \right\rfloor \) holds and so, by the above, for a vertex \( v \in H^* \) we have

\[ d_H(v, v') \leq 2 + 2 \left\lfloor \frac{t - 1}{2} \right\rfloor \]

for any vertex \( v' \in V(\mathcal{H}) \). So in this case a vertex \( v \in H^* \) distance-\( l \) dominates \( \mathcal{H} \), contradicting \( \gamma_d(\mathcal{H}, l) \geq 2 \).

If \( t = l - 1 \) and \( t \) is even, then let \( T \) be a spanning tree of \( G_{\mathcal{M}} \). By Lemma 3.1 we obtain that \( T \) is a path on \( t \) vertices. So we may assume that \( E(G_{\mathcal{M}}) \supset \{(H_i, H_{i+1}) : i = 1, \ldots, t - 1\} \).

Let \( e = (H_{t/2}, H_{t/2+1}) \) and consider a vertex \( v \in H_{t/2} \cap H_e \). As for vertices \( v' \) with \( H_{w'} \cap H_i \neq \emptyset \) for some \( i > t/2 \), a shortest path in \( \mathcal{H} \) between \( v \) and \( v' \) need not contain \( H_{t/2} \). Thus we obtain that \( v \) distance-\( l \) dominates \( \mathcal{H} \), contradicting \( \gamma_d(\mathcal{H}, l) \geq 2 \).

Finally, it remains to prove the lower bound of (a) in case of \( t = l \) and thus it is enough to prove that \( |V(\mathcal{H})| \cup \bigcup_{H \in \mathcal{M}} H \geq k/2 \) holds. We may and will assume that the radius of \( G_{\mathcal{M}} \) is \( \left\lfloor \frac{t + 1}{2} \right\rfloor \). Let \( T \) be a spanning tree of \( G_{\mathcal{M}} \). By Lemma 3.1 we know that \( T \) is a path if \( l \) is even, and \( T \) contains a path on \( l - 1 \) vertices if \( l \) is odd. We claim that even if \( l \) is odd, \( T \) must be a path on \( t \) vertices. Indeed, otherwise any vertex \( v \in H_e \) distance-\( l \) dominates \( \mathcal{H} \) where \( e \) is the middle edge of a path on \( l - 1 \) vertices that is contained in \( T \). This would contradict \( \gamma_d(\mathcal{H}, l) \geq 2 \). By this we may assume that \( E(G_{\mathcal{M}}) \supset \{(H_i, H_{i+1}) : i = 1, \ldots, l - 1\} \).

**Claim 3.3.** We have the following:

(i) For any pair of edges \( e, e' \) in \( T \) we have \( H_e \cap H_{e'} = \emptyset \).

(ii) There exist \( w, w' \in V(\mathcal{H}) \setminus \bigcup_{H \in \mathcal{M}} H \) and \( H_w, H_{w'} \in \mathcal{E}(\mathcal{H}) \) with

\[ w \in H_w \text{ and } w' \in H_{w'}, \]

such that \( H_w \) meets only \( H_i \) and \( H_{w'} \) meets only \( H_i \). Moreover \( H_w \) and \( H_{w'} \) are disjoint from all the other \( H \in \mathcal{M} \) and also from \( H_e \) for all \( e \in E(T) \).
Proof of Claim. We have two cases depending on the parity of $l$.

Case I: $l$ is even.

Now we prove (i) in this case. Suppose that $H_{e_i} \cap H_{e_j} \neq \emptyset$ with $e_i = (H_i, H_{i+1}), e_j = (H_j, H_{j+1})$. If $i < j \leq l/2$, then a vertex $v \in H_{e_{i+1}} \cap H_{e_{j+1}}$ distance-$l$ dominates $H$, contradicting $\gamma_{\text{dist}}(H, l) \geq 2$. Similarly, if $i < j$ and $j \leq l/2$, then a vertex $v \in H_{e_{i+1}} \cap H_{e_{j+1}}$ distance-$l$ dominates $H$, contradicting $\gamma_{\text{dist}}(H, l) \geq 2$. Also, if $i < l/2 < j$, then if $l/2 - i \leq j - l/2$, then a vertex $v$ from $H_{e_{i+1}} \cap H_{e_{j+1}}$ distance-$l$ dominates $H$, while if $l/2 - i \geq j - l/2$, then a vertex $v$ from $H_{e_{i+1}} \cap H_{e_{j+1}}$ distance-$l$ dominates $H$, contradicting $\gamma_{\text{dist}}(H, l) \geq 2$. We are done with (i) in Case I.

To see (ii) suppose that, for every $w \in V(H) \setminus \bigcup_{H \in \mathcal{M}} H$ and $w$ containing $w$, the hyperedge $w$ meets $H$ for some $e \in E(T)$ or $H_{w}$ meets some $H_{z}$ with $z \geq 2$. Then a vertex in $H_{e_{[n/2]}} \cap H_{[n/2]}$ distance-$l$ dominates $H$, contradicting $\gamma_{\text{dist}}(H, l) \geq 2$. The existence of $w'$ and $H_{w'}$ can be shown analogously. This proves (ii) in Case I.

Case II: $l$ is odd.

The proof of this case is very similar to the previous one. Let us just show (ii). Suppose that, for every $w \in V(H) \setminus \bigcup_{H \in \mathcal{M}} H$ and $H_{w}$ containing $w$, the hyperedge $H_{w}$ meets $H_{e}$ for some $e \in E(T)$ or $H_{w}$ meets some $H_{z}$ with $z \geq 2$. Then a vertex in $H_{e_{[n/2]}} \cap H_{[n/2]}$ distance-$l$ dominates $H$, contradicting $\gamma_{\text{dist}}(H, l) \geq 2$.

Note that $H_{w} \cap H_{w} \subset V(H) \setminus \bigcup_{H \in \mathcal{M}} H$ and also $H_{w} \cup H_{w} \cup \bigcup_{e \in E(T)} H_{e} \subset V(H)$, and thus writing $I = |H_{w} \cap H_{w}|$ we obtain $|V(H)| = \max\{lk + I, (l + 1)k - I\} \geq lk + k/2$. This finishes the proof of the lower bound of (a).

Next we prove the lower bound of (b). We will need the following lemma.

Lemma 3.4. For any $\gamma, l \geq 2$, let $t^*$ denote the smallest $t$ with $r_{\gamma-1}(t) \geq \frac{l-1}{2}$. Then we have

$$n_{dc}(k, \gamma, l) \geq t^* k.$$

Proof. Let $\mathcal{H}$ be a connected $k$-uniform hypergraph with $\gamma_{\gamma}(H, l) \geq \gamma$. Let $\mathcal{M}$ be a maximal matching in $\mathcal{H}$ obtained as in the proof of the lower bound of part (a), and let us consider the auxiliary graph $G_{\mathcal{M}}$. For a vertex $v \in \bigcup_{H \in \mathcal{M}} H$ let $H_v$ denote the only element of $\mathcal{M}$ containing $v$. Let the size of $\mathcal{M}$ be $t$. We assume first that $t < t^*$, what means $r_{\gamma-1}(t) \leq \frac{l-1}{2}$.

Suppose that for a pair $H, H' \in \mathcal{M}$ we have $d_{G_M}(H, H') = r$. Then for any pair of vertices $u \in H, v \in H'$ we have $d_H(u, v) \leq 1 + 2r$. To see this, consider the sequence $H, H_{e_1}, H_{e_2}, H_{e_3}, \ldots, H_{e_s}, H'$, where $e_s$ is the $s$th edge in a shortest path from $H$ to $H'$ and $H_{e_s}$ is the $s$th vertex (i.e. a hyperedge in $\mathcal{H}$) in the same path. Let $U \subset \mathcal{M}$ be a subset of size $\gamma - 1$ with $r_{G_{\mathcal{M}}}(U) = r_{\gamma-1}(G_{\mathcal{M}}) \leq r_{\gamma-1}(t)$, and let $L \subset V(\mathcal{H})$ be a set containing one vertex from each $U \in \mathcal{U}$.

By the maximality of $\mathcal{M}$, for every vertex $w \in \mathcal{H}$ there exist an edge $H_w$ containing $w$ and an edge $H \in \mathcal{M}$ with $H_w \cap H \neq \emptyset$. Therefore by the observation above and by the definition of $U$, there exist a $U \in \mathcal{U}$ and a vertex $u \in U$ for which we have

$$d_H(u, w) \leq 2 + 2r_{G_M}(U) \leq 2 + 2r_{\gamma-1}(t) < 2 + 2\frac{l - 1}{2} = l + 1.$$

This means that if $t < t^*$ holds, then the $(\gamma - 1)$-subset $L$ distance-$l$ dominates $\mathcal{H}$. Therefore $\mathcal{M}$ consists of at least $t^*$ hyperedges and thus $|V(\mathcal{H})| \geq t^* k$ holds.

The lower bound of (b) follows by applying Lemma 3.1 with $j = \gamma - 1$ together with Lemma 3.4 noting that $\left\lceil \frac{l}{\gamma} \right\rceil \geq \frac{l-1}{2}$ implies $\frac{l}{\gamma} > \frac{l-1}{2} - 1$. 

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Finally, we prove the lower bound of (c) and (d). This will follow from the claim that any maximal matching in the edge set $E(H)$ of a connected hypergraph $H$ with $\gamma_d(H, 2) \geq \gamma$ has size at least $\gamma$. To see this suppose that $M = \{H_1, H_2, \ldots, H_m\}$ is a maximal matching in $E(H)$ and for any $i = 1, 2, \ldots, m$ let $v_i$ be a vertex of $H_i$. As any vertex $v \in V(H)$ is contained in a hyperedge $H_v$ which, by maximality of $M$, intersects some $H_i \in M$, the set $D = \{v_i : i = 1, 2, \ldots, m\}$ distance-2 dominates $H$. Therefore $m \geq \gamma$ must hold as claimed.

\[\square\]

4. Final remarks and open problems

We addressed the problem of finding the minimum number of vertices that a connected $k$-uniform hypergraph with high domination number must contain, and we considered two main variants of the problem. For the original notion of domination and for $s$-wise domination we found general lower and upper bounds on $n(k, \gamma, s)$ in which even the order of magnitude of the second term matches. The natural open problem occurs: it can be of interest to find the constant coefficient of this second term.

Theorem 1.3, our main result concerning distance domination determines the asymptotics of $n_{dc}(k, \gamma, l)$ if $k$ and $\gamma$ are fixed and $l$ tends to infinity, or if all three parameters tend to infinity. Closing the gap of roughly $2k\gamma$ between the upper and lower bounds remains an interesting open problem.

We had a good reason to choose the notion of Berge paths in the definition of distance-$l$ domination. The most common other definitions of a path in hypergraphs are linear paths, where two consecutive hyperedges of the path must share exactly one vertex (an even more restrictive notion is a loose path) and tight paths where the vertices $v_1, v_2, \ldots, v_{k+l-1}$ of the path should be chosen in such a way that the $i$th hyperedge of the path is $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$ for all $i = 1, 2, \ldots, l$. This implies that consecutive hyperedges of a tight path share $k-1$ vertices. Note that in the construction showing the upper bound of Theorem 1.1 no pair of hyperedges has intersection size 1 or $k-1$, therefore the construction does not contain linear or tight paths of length larger than 1 and thus distance domination would not differ from ordinary domination, had we used these notions of hypergraph paths to define distance.

There are various results on different domination numbers of a hypergraph in the literature: on the $s$-domination number in [2], on the inverse domination number in [16], on the total domination number in [5], and on the connection of the domination number with the transversal number in [3], [4]. Let us finish with the following theorem that can be obtained simply by rearranging the lower bound of Theorem 1.3. In the style of Meir and Moon [17], it uses only the size of the vertex set, the prescribed distance bound $l$, and the uniformity of $H$.

Theorem 4.1. If $H$ is a connected $k$-uniform hypergraph with $|V(H)| = n$, then

$$\gamma_{dc}(H, l) \leq \begin{cases} \frac{n}{k} & \text{if } l = 2, 3 \text{ or } 4, \\ \frac{n}{k} \cdot \frac{2}{l-3} & \text{if } l > 4. \end{cases}$$

It remains an open problem to make these upper bounds tight.

References

[1] B. D. Acharya, Domination in hypergraphs, AKCE International Journal of Graphs and Combinatorics, 4 (2007), pp. 117–126.

[2] B. D. Acharya, Domination in hypergraphs II, New directions, Proceedings of ICDM, Mysore, India, 2008, Ramanujan Mathematical Society Lecture Notes Series, 13 (2010), pp. 1–18.
[3] S. Arumugam, B. K. Jose, Cs. Bujtáš, and Zs. Tuza, Equality of domination and transversal numbers in hypergraphs, Discrete Applied Mathematics, 161 (2013), pp. 1859–1867.

[4] Cs. Bujtáš, M. A. Henning, and Zs. Tuza, Transversals and domination in uniform hypergraphs, European Journal of Combinatorics, 33 (2012), pp. 62–71.

[5] Cs. Bujtáš, M. A. Henning, Zs. Tuza, and A. Yeo, Total transversals and total domination in uniform hypergraphs, The Electronic Journal of Combinatorics, 21(2014), #P2.24.

[6] R. Davila, C. Fast, M. A. Henning, and F. Kenter, Lower bounds on the distance domination number of a graph, arXiv:1507.08745

[7] J. F. Fink and M. S. Jacobson, On n-domination, n-dependence and forbidden subgraphs, In: Graph Theory with Applications to Algorithms and Computer Science, Wiley, New York (1985), pp. 301–311.

[8] D. Gerbner, B. Keszegh, D. Pálvölgyi, B. Patkós, M. Vizer, and G. Wiener, Finding a majority ball with majority answers, arXiv:1509.08276

[9] F. Harary and T. W. Haynes, Nordhaus-Gaddum inequalities for domination in graphs, Discrete Mathematics, 155 (1996), pp. 99-105.

[10] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), Fundamentals of Domination in Graphs, Marcel Dekker, Inc. New York, 1998.

[11] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), Domination in Graphs: Advanced Topics, Marcel Dekker, Inc. New York, 1998.

[12] M. A. Henning, Distance domination in graphs, Domination in Graphs: Advanced Topics, T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds), Marcel Dekker, Inc. New York, (1998), pp. 335–365.

[13] M. A. Henning, N. Lichiardopol, Distance domination in graphs with given minimum and maximum degree, manuscript.

[14] M. A. Henning, C. Löwenstein, Hypergraphs with large domination number and with edge sizes at least three, Discrete Applied Mathematics, 160 (2012), pp. 1757–1765.

[15] J. W. P. Hirschfeld, Projective geometries over finite fields, Clarendon Press, Oxford, 1979, 2nd edition, 1998.

[16] B. K. Jose, Zs. Tuza, Hypergraph domination and strong independence, Applicable Analysis and Discrete Mathematics, 3 (2009), pp. 347–358.

[17] A. Meir and J. W. Moon, Relations between packing and covering number of a tree, Pacific Journal of Mathematics, 61 (1975), pp. 225–233.

[18] Z.L. Nagy, B. Patkós, On the number of maximal intersecting k-uniform families and further applications of Tuzas set pair method, The Electronic Journal of Combinatorics, 22 (2015), #P1.83.

[19] Zs. Tuza, Critical hypergraphs and intersecting set-pair systems, Journal of Combinatorial Theory, Series B, 39 (1985), pp. 134–145.

[20] Zs. Tuza, Inequalities for two set systems with prescribed intersections, Graphs and Combinatorics, 3 (1987), pp. 75–80.