Abstract

We propose chiral analogues of some infinite complexes appearing in the description of the coherent derived categories for projective spaces.

Introduction

The aim of this note is to propose some examples of acyclic complexes unbounded in both directions.

One might ask why do we care about acyclic complexes, after all they are ignored when we define derived categories? One possible motivation is a remarkable but not very wellknown description, due to Beilinson and Bernstein, of the coherent derived category $\mathcal{D}(\mathbb{P}^n)$ over a projective space, given in [G]. Namely, the last category turns out to be equivalent to a category $\mathcal{A}_n$ whose objects are unbounded acyclic complexes of free modules over the Grassmann (exterior) algebra $\Lambda[\mathbb{C}^{n+1}]$, cf. op. cit. §5. Here it is important that the complexes are unbounded in both directions.

Examples which we discuss in §§1 - 3 of this note concern the simplest case $n = 0$. So $\mathbb{P}^0$ is just a point, and coherent sheaves over it are finite dimensional vector spaces. We consider a complex $\mathcal{Kos} \in \mathcal{A}_0$ representing a one-dimensional vector space. The Grassmann algebra $\Lambda[\mathbb{C}]$ is the algebra of dual numbers $\mathbb{C}[\xi], \xi^2 = 0$.

The complex $\mathcal{Kos}$ which we call a Janus Koszul complex (cf. [P], [S], [KS] for a discussion of Janus objects) is defined as follows in loc. cit. (In the main body of the paper we will use, following loc. cit., the notation $\Delta$ for this complex.)
Its right part $\mathcal{K}os_+$ is concentrated in degrees $\geq 0$ and is by definition the Koszul resolution for the $\mathbb{C}[x]$-module $\mathbb{C}$ where we assign to $x$ the degree $1$. As a vector space

$$\mathcal{K}os_+ = \mathbb{C}[\xi] \otimes \mathbb{C}[x],$$

and its $i$-th term is a two-dimensional space $\mathbb{C}[\xi]x^i$. It has only one nonzero cohomology, in degree $0$, which is 1-dimensional.

The left part, $\mathcal{K}os_-$, is concentrated in degrees $\leq -1$ and is a sort of dual to $\mathcal{K}os_-; it has a unique nonzero cohomology, in degree $-1$, which is 1-dimensional as well. Afterwards these two guys are docked into one complex, like Apollo - Soyuz.

The whole complex admits a décalage symmetry

$$\sigma : \mathcal{K}os \xrightarrow{\sim} \mathcal{K}os[1].$$

For details see §2.

We remark that $\mathcal{K}os_+$ may be identified with the cochain complex of a cellular complex homeomorphic to the infinite sphere $S^\infty$. Such complexes are discussed in §1, which is accessible to high school students.

In §3 we describe a chiral analogue $\text{chK}os$ of $\mathcal{K}os$.

The chiral counterpart of $\mathcal{K}os_+$ is a dg vertex algebra $\text{chK}os_+$, and the chiral counterpart $\text{chK}os_-$ of $\mathcal{K}os_-$ is vertex contragradient dual module for $\text{chK}os_+$, see 3.5 and [FBZ], 10.4.6. The whole $\text{chK}os$ is obtained by docking the two halves, similarly to the finite dimensional case.

In §4 the above construction is generalized to the case $n > 0$. In addition, we present in 4.3 an interpretation of the negative part of the chiral Janus Koszul as a local cohomology of its positive part.

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§1. An infinite sphere

1.1. Spheres. We fix a base commutative ring \( l \).

1.1.1. The circle \( S^1 \). Consider a cellular decomposition of 
\[
S^1 = \{ z \in \mathbb{C} | |z| = 1 \}
\]
having two 0-cells
\[
c^+_0 = 1, \ c^-_0 = -1.
\]
and two 1-cells
\[
c^+_1 = \{ e^{i\theta} | 0 \leq \theta \leq \pi \}, \ c^-_1 = \{ e^{i\theta} | \pi \leq \theta \leq 2\pi \},
\]
where \( c^+_1 \) is oriented clockwise, and \( c^-_1 \) - counterclockwise.

So we have defined a \( CW \)-complex, to be denoted \( K_{[0,1]} \), whose complex of chains is
\[
C_\bullet(K_{[0,1]}; l) : 0 \rightarrow C_1(K_{[0,1]}; l) \xrightarrow{d} C_0(K_{[0,1]}; l) \rightarrow 0
\]
where
\[
C_1(K_{[0,1]}; l) = l c^+_1 \oplus l c^-_1, \ C_0(K_{[0,1]}; l) = l c^+_0 \oplus l c^-_0,
\]
and the differential
\[
dc^+_1 = dc^-_1 = c^+_0 - c^-_0,
\]
i.e. \( d \) is given by matrix
\[
A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.
\]
(1.1.1)
We define a complex \( C^\bullet_{[0,1]} \) as \( C_\bullet(K_{[0,1]}; l) \) placed in degrees 0, 1.

Thus
\[
H^0(C^\bullet_{[0,1]}) \cong l; \ H^1(C^\bullet_{[0,1]}) \cong l
\]

1.1.2. The sphere \( S^2 \). Similarly we define a cellular decomposition \( K_{[0,2]} \) of the sphere \( S^2 \) having two cells \( c^+_i \) in dimensions \( i = 0, 1, 2 \). Namely the 2-cells \( c^+_2 \) are northern and southern hemispheres, and
\[
K_{[0,1]} \subset K_{[0,2]}
\]
as the equator. Its complex of chains will be of length 2
\[ C_* (K_{[0,2]}; l) : 0 \rightarrow C_2 (K_{[0,2]}; l) \xrightarrow{d_2} C_1 (K_{[0,2]}; l) \xrightarrow{d_1} C_0 (K_{[0,2]}; l) \rightarrow 0 \]
with
\[ C_i (K_{[0,2]}; l) = l c_i^+ \oplus l c_i^-, \]
and both differentials are given by the matrix \(A\) (we note that \(A^2 = 0\)).

We define a complex \( C_* \) as \( C_* (K_{[0,2]}; l) \) placed in degrees 0, 1, 2.
Thus
\[ H^0 (C_* ) \cong l, \quad H^1 (C_* ) = 0, \quad H^2 (C_* ) \cong l. \]

1.1.3. The sphere \( S^n \). Similarly for an arbitrary \( n \) we define a cellular decomposition \( K_{[0,n]} \) of the sphere \( S^n \) having two cells \( c_i^\pm \) in dimensions \( i = 0, 1, \ldots, n \), with the embedding
\[ K_{[0,n-1]} \subset K_{[0,n]} \]
as the equator.

Its complex of chains will be of length \( n \)
\[ C_* (K_{[0,n]}; l) : 0 \rightarrow C_n (K_{[0,n]}; l) \xrightarrow{d_n} \ldots \xrightarrow{d_1} C_0 (K_{[0,n]}; l) \rightarrow 0 \]
with
\[ C_i (K_{[0,n]}; l) = l c_i^+ \oplus l c_i^-, \]
and both differentials given by the matrix \( A \).

We define a complex \( C_* \) as \( C_* (K_{[0,n]}; l) \) placed in degrees 0, \ldots, \( n \).
Thus
\[ H^0 (C_* ) \cong l, \quad H^i (C_* ) = 0, 0 \leq i \leq n - 1, \quad H^n (C_* ) \cong l. \]

1.2. As remarked M.Kapranov, we may consider \( K_{[0,n]} \) as a globular \( n\)-category (cf. for example [ManS]) having two objects, two 1-morphisms, two 2-morphisms, etc.

There exists a canonical functor of \( n\)-categories
\[ \phi_n : K_{[0,n]} \rightarrow HB_{n+2} \]
where \( HB_{n+2} \) is the globular \( n\)-category of higher Bruhat orders on the symmetric group \( \Sigma_{n+2} \). This follows from the main result of op. cit.
For example, $\phi_1$ takes two objects of $K_{[0,1]}$ to permutations $e = ()$ and $31$ from $\Sigma_3$, and two arrows to two possible paths from $e$ to $(13)$ for the weak Bruhat order in $\Sigma_3$.

1.3. The space $S^\infty$. We may pass to the limit $n \to \infty$, and define an infinite dimensional $CW$-complex

$$K_{[0,\infty]} = \bigcup_{n=1}^\infty K_{[0,n]}.$$ 

It is contractible.

The corresponding complex $$C^\bullet_+ = C^\bullet_{[0,\infty]}$$ will live in degrees $\geq 0$: namely $$C^i_+ = l_c^i_+ \oplus l_c^i_-,$$ $i \geq 0,$

with $d = A$, and will have the cohomology

$$H^0(C^\bullet_{[0,\infty]}) \cong \mathbb{I}, \ H^i(C^\bullet_{[0,\infty]}) = 0, \ i > 0.$$ 

Similarly, using the left shifts of complexes $C^\bullet(K_{[0,n]:1})$ we define a complex

$$C^\bullet_- = C^\bullet_{[-\infty,0]}$$

living in degrees $\leq 0$ with $$C^i_- = l_c^i_+ \oplus l_c^i_-, \ i \leq 0,$$

with $d = A$, having the cohomology

$$H^0(C^\bullet_{[-\infty,0]}) \cong \mathbb{I}, \ H^i(C^\bullet_{[-\infty,0]}) = 0, \ i < 0.$$ 

1.4. Double infinite, or Janus. We define a complex

$$C^\bullet = C^\bullet_{[-\infty,\infty]}$$

living in all degrees $i \in \mathbb{Z}$ with $$C^i = l_c^i_+ \oplus l_c^i_-,$$ $i \in \mathbb{Z},$

with $d = A$.

It is acyclic:

$$H^i(C^\bullet) = 0, \ i \in \mathbb{Z}.$$ 

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§2. Koszul complex

2.1. The following is a particular case of a construction due to A. Beilinson and J. Bernstein, see [G], §5.

Let $\Xi$ be a one-dimensional vector space over $\mathbb{C}$ with base $\xi$, $X = \Xi^\ast$ the dual space with the base $x$ dual to $\xi$. Let

$$\Lambda(\Xi) = \mathbb{C} \oplus \Xi$$

be the exterior algebra, $\Lambda(\Xi)[-1]$ the free $\Lambda(\Xi)$-module of rank 1 with a generator $e$ living in degree $-1$,

$$S^\ast(X) = \bigoplus_{i=0}^{\infty} \mathbb{C} x^i$$

the symmetric algebra.

2.1.1. Positive part. Let

$$\Delta_+^\bullet = \Lambda(\Xi)[-1] \otimes S^\ast(X)$$

be the shifted Koszul complex. By definition it is a complex of $\Lambda(\Xi)$-modules

$$\Delta_+^\bullet : 0 \longrightarrow \Lambda(\Xi)[-1] \otimes S^0(X) \xrightarrow{d_0} \Lambda(\Xi)[-1] \otimes S^1(X) \xrightarrow{d_1} \cdots$$

with a differential

$$d(\mu \otimes t) = \xi \mu \otimes x t$$

The only nonzero cohomology of this complex is

$$H^0(\Delta_+^\bullet) \cong \mathbb{C}.$$

2.1.2. Negative part. For a $\Lambda(\Xi)$-module $M$ denote

$$M^\ast = \text{Hom}_{\Lambda(\Xi)}(M, \Lambda(\Xi))$$

the dual module. The complex $\Delta_-^\bullet$ consists of dual $\Lambda(\Xi)$-modules and lives in degrees $\leq -1$:

$$\Delta_-^\bullet : \cdots \xrightarrow{d_-} \Delta_-^2 \xrightarrow{d_-} \Delta_-^1 \longrightarrow 0$$

where

$$\Delta_-^i = (\Delta_+^{i-1})^\ast = (\Lambda(\Xi)[-1])^\ast \otimes S^i(X)$$

and

$$d_-^i = (d_+^{i-2})^\ast.$$
The only nonzero cohomology of this complex is

\[ H^{-1}(\Delta^\bullet) \cong \mathfrak{l}. \]

### 2.1.3. Janus complex \( \Delta^\bullet \)

We glue the above two guys together using a map of \( \Lambda(\Xi) \)-modules

\[ d_{-1} : \Delta_{-1} = (\Lambda(\Xi)[-1])^* = \Lambda(\Xi) \to \Lambda(\Xi)[-1] = \Delta^0_+, \]

\[ d_{-1}(1) = \xi, \]

and get an infinite in both sides acyclic complex

\[ \Delta^\bullet : \ldots \to \Delta_{-2} \xrightarrow{d_{-}} \Delta_{-1} \xrightarrow{d_{-}} \Delta^0_+ \xrightarrow{d_{+}} \Delta^1_+ \xrightarrow{d_{+}} \ldots \]

### 2.2. Localized Koszul

The Koszul of 2.1.1, \( \Delta^\bullet_+ \), is often realized slightly differently. Given an \( \mathfrak{I} \)-algebra \( R \) with an element \( f \in R \), the Koszul complex can be defined to be the algebra \( \mathfrak{I}[\xi] \otimes R \) with differential defined to be the following derivation

\[ d_K = \partial_\xi \otimes f, \]

where

\[ \partial_\xi(\xi) = 1, \quad \partial_\xi(1) = 0. \]

Denote this object by \( K(R, f) \).

If \( R = \mathfrak{I}[x] \), then we give it a grading by letting \( K(\mathfrak{I}[x], f)^i = \mathfrak{I}[\xi]x^i \).

It is easy to see that \( K(\mathfrak{I}[x], x) \) is isomorphic with \( \Delta^\bullet_+ \) as a complex, the isomorphism

\[ K(\mathfrak{I}[x], x) \to \Delta^\bullet_+ \]

being defined by the assignment

\[ 1 \otimes x^n \mapsto \xi \otimes x^n; \quad \xi \otimes x^n \mapsto 1 \otimes x^n. \]

In the case where \( R = \mathfrak{I}[x, x^{-1}] \) (the grading being similarly defined by the powers of \( x \)), the Koszul complex \( K(\mathfrak{I}[x, x^{-1}], x) \) is obviously acyclic and is in fact naturally isomorphic, as a complex, with \( \Delta^\bullet_+ \), the Janus complex of 2.1.3. To define this isomorphism choose a basis of \( \mathfrak{I}[\xi] \otimes \mathfrak{I}[x] \) to be the set of monomials \( \{1 \otimes x^n, \xi \otimes x^n, \ n \geq 0\} \) and consider a map

\[ ([\xi] \otimes \mathfrak{I}[x^{-1}]x^{-1} \to ([\xi] \otimes \mathfrak{I}[x])^* \]
defined by the assignment:

\[ 1 \otimes x^{-n-1} \mapsto (\xi \otimes x^n)^{\vee}, \quad \xi \otimes x^{-n-1} \mapsto (1 \otimes x^n)^{\vee}, \]

where \( \{ (1 \otimes x^n)^{\vee}, (\xi \otimes x^n)^{\vee}, n \geq 0 \} \) is the basis dual to \( \{ 1 \otimes x^n, \xi \otimes x^n, n \geq 0 \} \).

Using the inverse dual to the above defined isomorphism \( K(\mathcal{I}[x], x) \rightarrow \Delta^*_+ \) we obtain the composite map

\[ \mathcal{I}[\xi] \otimes \mathcal{I}[x^{-1}] \rightarrow (\mathcal{I}[\xi] \otimes \mathcal{I}[x])^* \rightarrow (\Delta^*_+)^* = \Delta_-, \]

cf. 2.1.2, which is easily seen to be an isomorphism of complexes in degrees -2,-3,-4, etc. Taking the direct sum we obtain a vector space isomorphism

\[ K(\mathcal{I}[x^{-1}], x) = \mathcal{I}[\xi] \otimes \mathcal{I}[x^{-1}] \oplus \mathcal{I}[\xi] \otimes \mathcal{I}[x] \rightarrow \Delta^-_+ + \Delta^- + = \Delta^-. \]

Under this isomorphism the gluing map, \( \Delta^-_+ \rightarrow \Delta^0_+ \), which in 2.1.3 had to be defined by hand, is on the left simply the restriction of the standard Koszul differential to \( \mathcal{I}[\xi] \otimes x^{-1} \):

\[ \partial_\xi \otimes x(\xi \otimes x^{-1}) = 1 \otimes 1, \quad \partial_\xi \otimes x(1 \otimes x^{-1}) = 0. \]

Note that, unlike \( \Delta^*_+ \) or \( \Delta^-_+ \), \( K(R, f) \), \( K(\mathcal{I}[x], x) \), \( K(\mathcal{I}[x^{-1}], x) \) are commutative dg-algebras.

**2.3. Comparison.** The matrix of \( d_+ \) is

\[ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]

The matrices \( A \) and \( B \) are conjugated: if

\[ \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

then

\[ A\alpha = \beta, \quad A\beta = 0. \]

Thus a base change gives rise to an isomorphism of complexes

\[ \Delta^*_+ \cong \mathcal{E}^*_+. \]

Similarly

\[ \Delta^-_+ \cong \mathcal{E}^-_. \]
\[ \Delta^* \cong \mathbb{C}^*. \]

§3. Chiral Koszul: one variable

3.1. From now on our ground ring I will be \( \mathbb{C} \) (or any commutative ring containing \( \mathbb{Q} \)).

Consider \( \mathbb{T}^* \mathbb{C} \) as an algebraic variety. Its ring of regular functions, \( \mathbb{C}[\mathbb{T}^* \mathbb{C}] \), is a polynomial ring on 2 variables, \( \mathbb{C}[x, \partial_x] \), where \( x \) is a coordinate on \( \mathbb{C} \) and \( \partial_x \) is the vector field s.t. \( \partial_x(x) = 1 \). In fact, \( \mathbb{C}[\mathbb{T}^* \mathbb{C}] \) is naturally a Poisson algebra with Poisson bracket uniquely determined by the condition \( \{ \partial_x, x \} = 1 \), \( \{ \partial, \partial \} = \{ x, x \} = 0 \).

Now consider the jet-space, \( J_\infty \mathbb{T}^* \mathbb{C} \). Its ring of regular functions \( \mathbb{C}[J_\infty \mathbb{T}^* \mathbb{C}] \) is a polynomial ring on infinitely many variables, \( \mathbb{C}[x_{(-n)}, \partial_{x_{(-n)}}, n > 0] \), with derivation, which is uniquely determined by its action on the generators, \( x_{(-n)} \mapsto nx_{(-n-1)} \), \( \partial_{x_{(-n)}} \mapsto n\partial_{x_{(-n-1)}} \).

\( \mathbb{C}[J_\infty \mathbb{T}^* \mathbb{C}] \) is not a Poisson algebra in any natural way; it is instead a coisson or vertex Poisson algebra. The coisson, or \( \text{Lie}^* \), bracket is determined by the assignment

\[ \{ \partial_{x_{(-n)}}, x_{(-m)} \} = \frac{1}{(n-1)!(m-1)!} \partial_x^{n-1} \partial_z^{m-1} \delta(z - w), \]

where we use some standard notation without explanation, referring the reader to the inspired and inspirational Introduction to [BD].

This coisson algebra structure is easy to quantize and the result of this quantization is well known. As a vector space, it is still the ring \( \mathbb{C}[x_{(-n)}, \partial_{x_{(-n)}}, n > 0] \) and the state-field correspondence is determined by the assignment

\[
\begin{align*}
1 & \mapsto \text{Id}, \\
x_{(-n)} & \mapsto \frac{1}{(n-1)!} \partial_z^{n-1} x(z), \\
\partial_{x_{(-n)}} & \mapsto \frac{1}{(n-1)!} \partial_z^{n-1} \partial_x(z),
\end{align*}
\]

where the only nontrivial OPE among the fields \( \partial_x(z), x(x) \) is as follows:

\[ \partial_x(z)x(w) = \frac{1}{z-w} + \cdots \]
This vertex algebra is often referred to as the “βγ-system.” We called it an algebra of chiral differential operators (CDO) and denoted it $D_{ch}^C$, cf. [GMS].

3.2. This construction has an obvious superalgebra extension. The underlying jet-space is $J_\infty T^*\Pi^TC$. The corresponding ring of functions is the supercommutative algebra $\mathbb{C}[x(-n), \partial x(-n), \xi(-n), \partial \xi(-n), n > 0]$ with derivation $x(-n) \mapsto nx(-n-1)$, $\partial x(-n) \mapsto n\partial x(-n-1)$, $\xi(-n) \mapsto n\xi(-n-1)$, $\partial \xi(-n) \mapsto n\partial \xi(-n-1)$; the variables $\xi(-n)$ and $\partial \xi(-n)$ are odd.

This is a supercoisson algebra, and its quantization is similar to the one described in sect. 3.1: the space is $\mathbb{C}[x(-n), \partial x(-n), \xi(-n), \partial \xi(-n), n > 0]$, and the state-field correspondence is obtained by extending the assignment at the end of 3.1 as follows:

$$\xi(-n) \mapsto \frac{1}{(n-1)!} \partial_z^{n-1} \xi(z),$$

$$\partial \xi(-n) \mapsto \frac{1}{(n-1)!} \partial_z^{n-1} \partial \xi(z),$$

The fields $\partial \xi(z)$, $\xi(z)$ are odd, and the only nontrivial OPE they satisfy is this:

$$\partial \xi(z)\xi(w) = \frac{1}{z-w} + \cdots$$

This is a super CDO, to be denoted $D_{ch}^{C|1}$. 

3.3. One advantage of dealing with a super CDO is that it contains various differentials. For example, consider the map

$$d_{K}^{ch} : D_{C|1}^{ch} \rightarrow D_{C|1}^{ch}$$

defined by

$$d_{K}^{ch} = \int \partial \xi(z)x(z) \, dz.$$ 

If we let the degree of $x(n)$ be one, of $\partial x(n)$ be -1, and that of $\xi(n)$ and $\partial \xi(n)$ be 0, for any $n$, then one easily verifies that $d_{K}^{ch}$ is of square 0 and degree 1, and so the pair ($D_{C|1}^{ch}$, $d_{K}^{ch}$) is a complex infinite in both directions – in fact a differential graded vertex algebra as the integral of a field, such as $d_{K}^{ch}$, is well known to be a derivative of a vertex algebra structure. One also notices that

$$\mathbb{C}[x(-1), \xi(-1)] = \mathbb{C}[x(-1)] \oplus \mathbb{C}[\xi(-1)] \subset D_{C|1}^{ch}$$

is a subcomplex, and as such it is rather naturally identified with the Koszul complex $K([x], x)$ of 2.2 if we think of $x(-1)$ as the coordinate $x$ on $\mathbb{C}$, and of $\xi(-1)$ as $\xi$. For example, the restriction of $d_{K}^{ch}$ to $\mathbb{C}[x(-1), \xi(-1)]$ is this:

$$d_{K}^{ch} = x(-1) \frac{d}{d\xi(-1)}.$$
which of course coincides with $d_K$ of 2.2.

Furthermore, one notices that thus defined embedding $K(\mathfrak{l}[x], x) \hookrightarrow D_{\mathbb{C}^1|1}^{ch}$ is a quasiisomorphism. This is because of a part of an $\mathbb{N}=2$ superconformal structure that $D_{\mathbb{C}^1|1}^{ch}$ carries. Consider 2 more fields

$$L(z) = (x_{(-2)}\partial_{x_{(-1)}})(z) + (\xi_{(-2)}\partial_{\xi_{(-1)}})(z), \; G(z) = (\xi_{(-2)}\partial_{\xi_{(-1)}})(z).$$

One readily verifies that $L(z)$ is the Virasoro field and, in particular, the component $L(1) = \int L(z) \, dz$, as an operator acting on $D_{\mathbb{C}^1|1}^{ch}$, defines the conformal grading,

$$D_{\mathbb{C}^1|1}^{ch} = \bigoplus_{n=0}^{\infty} D_{\mathbb{C}^1|1}^{ch},$$

where $D_{\mathbb{C}^1|1}^{ch},n$ is the eigenspace of $L(1)$ of eigenvalue $n$. A routine and familiar computation will show that this grading is, in fact, a grading on the underlying polynomial ring,

$$D_{\mathbb{C}^1|1}^{ch} = \mathbb{C}[x_{(-n)}; \partial_{x_{(-n)}}, \xi_{(-n)}; \partial_{\xi_{(-n)}}, n > 0],$$

where

$$\deg x_{(-n)} = n - 1; \; \deg \partial_{x_{(-n)}} = n; \; \deg \xi_{(-n)} = n - 1; \; \deg \partial_{\xi_{(-n)}} = n.$$ 

This grading is preserved by the differential $d_K^{ch}$. For example, the conformal weight zero subspace, $D_{\mathbb{C}^1|1}^{ch,0}$ is exactly the ordinary Koszul complex $K(\mathfrak{l}[x], x)$ in the form of $\mathbb{C}[x_{(-1)}; \xi_{(-1)}]$, which we considered above, $x_{(-1)}; \xi_{(-1)}$, being the only generators of conformal weight 0, and the differential $d_K^{ch}$ preserves this grading.

Furthermore,

$$[d_K^{ch}, \int G(z) \, dz] = L(1),$$

which means that the restriction of the differential $d_K^{ch}$ to each nonzero conformal weight subspace is homotopic to identity. This implies that the embedding $K(\mathfrak{l}[x], x) \hookrightarrow D_{\mathbb{C}^1|1}^{ch}$ is indeed a quasiisomorphism.

Therefore, the cohomology $H_{d_K^{ch}}(D_{\mathbb{C}^1|1}^{ch})$ is 1-dimensional and is spanned by the class of $1 \in D_{\mathbb{C}^1|1}^{ch}$.

3.4. We can localize and consider

$$D_{\mathbb{C}^1|1}^{ch}[x_{(-1)}^{-1}] \overset{\text{def}}{=} \mathbb{C}[x_{(-1)}; x_{(-1)}^{-1}, x_{(-n-1)}; \partial_{x_{(-n)}}, \xi_{(-n)}; \partial_{\xi_{(-n)}}, n > 0].$$

It is still a vertex algebra, [MSV,GMS], and it shares many features with the above analyzed $D_{\mathbb{C}^1|1}^{ch}$.
(1) the pair \((D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}],d_{K}^{ch})\) is a differential graded vertex algebra with differential \(d_{K}^{ch}\);

(2) it is conformally graded by the eigenvalues of \(L_{(1)}\):

\[
D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}] = \bigoplus_{n=0}^{\infty} D_{C^{1|1},n}^{ch}[x_{(-1)}^{-1}],
\]

the grading being determined in an obvious manner by that of the polynomial ring \(\mathbb{C}[x_{(-n)},\partial x_{(-n)},\xi_{(-n)},\partial \xi_{(-n)}, n > 0]\), see 3.3;

(3) localization of the identification \(K([l[x],x) = D_{C^{1|1},0}^{ch}\), see 3.3, gives an identification \(K([l[x,x^{-1}],x) = D_{C^{1|1},0}[x_{(-1)}^{-1}]\); and, finally,

(4) the restriction of \(d_{K}^{ch}\) to \(D_{C^{1|1},n}[x_{(-1)}^{-1}]\) with \(n > 0\) is homotopic to identity (thanks to the relation \([d_{K}^{ch}, \int G(z)zdz = L_{(1)}]\) and, therefore, the embedding (as the conformal weight zero subspace, see point (3))

\[
K([l[x,x^{-1}],x) \hookrightarrow D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]
\]

is a quasismorphism.

It is in this sense that we assert \(D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]\) is a chiralization of \(K([l[x,x^{-1}],x)\), hence of the Janus \(\Delta^{\bullet}\), 2.1.3.

3.5. The chiral Janus complex. An obvious embedding \(D_{C^{1|1}}^{ch} \hookrightarrow D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]\) is a vertex algebra morphism; therefore, the quotient \(D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]/D_{C^{1|1}}^{ch}\) is a \(D_{C^{1|1}}^{ch}\)-module - in fact a differential graded \(D_{C^{1|1}}^{ch}\)-module as \(d_{K}^{ch}\) tautologically acts on any \(D_{C^{1|1}}^{ch}\)-module.

\(D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]/D_{C^{1|1}}^{ch}\) is conformally graded by the eigenvalues of \(L_{(1)}\), because \(D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]\) and \(D_{C^{1|1}}^{ch}\) are, 3.3, 3.4, and the grading is preserved by \(d_{K}^{ch}\), which implies, as in 3.3, that the cohomology \(H_{d_{K}^{ch}}(D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]/D_{C^{1|1}}^{ch})\) is 1-dimensional, too, and is spanned by the cocycle \(\xi_{(-1)}/x_{(-1)}\), just as \(H_{d_{K}^{ch}}(D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]/D_{C^{1|1}}^{ch})\) is spanned by the cocycle 1, the last sentence of 3.3.

We can now mimic 2.1.3 and form a bouquet of the 2 complexes, \(D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]/D_{C^{1|1}}^{ch}\) and \(D_{C^{1|1}}^{ch}\) as follows. Define

\[
D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]/D_{C^{1|1}}^{ch} \oplus D_{C^{1|1}}^{ch}
\]

to be

\[
D_{C^{1|1}}^{ch}[x_{(-1)}^{-1}]/D_{C^{1|1}}^{ch} \oplus D_{C^{1|1}}^{ch}
\]
as a vector space. To define the differential we, first, stipulate that in nonzero conformal weight subspaces it is the direct sum of the original differentials in nonzero conformal
weight subspaces. More formally, on

\[ D_{C_{1|1}, n}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch} \oplus D_{C_{1|1}, n}^{ch}, \quad n > 0, \]

the differential is

\[ d_{K}^{ch} \oplus d_{K}^{ch} : D_{C_{1|1}, n}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch} \oplus D_{C_{1|1}, n}^{ch} \to D_{C_{1|1}, n}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch} \oplus D_{C_{1|1}, n}^{ch}. \]

Now focus on the 0 conformal weight subspace,

\[ D_{C_{1|1}, 0}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch} \oplus D_{C_{1|1}, 0}^{ch}. \]

By construction, \( D_{C_{1|1}, 0}^{ch} \) is the ordinary Koszul, \( K(\mathbb{Z}[x], x) \), see an explanation of this in 3.3, and \( D_{C_{1|1}, 0}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch} \) is its dual, see 2.2. This places us entirely in the situation of 2.2, where, in particular \( D_{C_{1|1}, 0}^{ch} \) is graded by positive powers of \( x_{-1} \),

\[ D_{C_{1|1}, 0}^{ch} = D_{C_{1|1}, 0}^{ch}[0] \oplus D_{C_{1|1}, 0}^{ch}[1] \oplus D_{C_{1|1}, 0}^{ch}[2] \oplus \cdots, \]

\( D_{C_{1|1}, 0}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch} \) by negative,

\[ D_{C_{1|1}, 0}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch} = \cdots \oplus (D_{C_{1|1}, 0}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch})[-2] \oplus (D_{C_{1|1}, 0}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch})[-1]. \]

In either case, the \( n \)-th graded component is 2-dimensional with basis \( \{ \xi_{-1} x_{-1}^{n}, x_{-1}^{-1} \} \).

We now mimic 2.1.3, as we did in 2.2, to join these complexes by using the map

\[ (D_{C_{1|1}, 0}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch})[-1] \to D_{C_{1|1}, 0}^{ch}[0], \]

which sends

\[ (D_{C_{1|1}, 0}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch})[-1] \ni \xi_{-1} x_{-1}^{-1} \mapsto 1 \in D_{C_{1|1}, 0}^{ch}[0], \]

\[ (D_{C_{1|1}, 0}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}, 0}^{ch})[-1] \ni 1 \mapsto x_{-1}^{-1} \in D_{C_{1|1}, 0}^{ch}[0]. \]

The result is the bouquet \( D_{C_{1|1}}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}}^{ch} \oplus D_{C_{1|1}}^{ch} \). It is acyclic by construction.

3.6. Remark. \( D_{C_{1|1}}^{ch} \) is a conformal vertex algebra; as such it defines a sheaf on \( \mathbb{P}^1 \).

In this situation, one defines a duality functor on the category of \( D_{C_{1|1}}^{ch} \)-modules, see [FBZ], 10.4.6. \( D_{C_{1|1}}^{ch}[x_{-1}^{-1}] / D_{C_{1|1}}^{ch} \) is naturally a \( D_{C_{1|1}}^{ch} \)-module and, in fact, it is the dual to \( D_{C_{1|1}}^{ch} \).
3.7. Let us explore the chiral Janus complex $D_{\text{ch}}^{\text{i}\i\i}\{x_{(-1)}^{-1}\}/D_{\text{ch}}^{\text{i}\i\i} \oplus D_{\text{ch}}^{\text{i}\i\i}$ in some detail. Its one essential feature is that although infinite in both directions, as a complex, it is a direct sum of conformal weight subspaces, 3.3, 3.4, and each such subspace is a subcomplex bounded in both directions. We shall now examine these subcomplexes, starting with the subcomplex $D_{\text{ch}}^{\text{i}\i\i}$.

It is convenient to shift the subindices of the generators in order to make sure they respect the conformal grading. Therefore we re-denote

$$x_{-n+1} = x_{(-n)}, \partial x_{-n} = \partial x_{(-n)}, \xi_{-n+1} = \xi_{(-n)}, \partial \xi_{-n} = \partial \xi_{(-n)}, n > 0.$$  

A moment’s thought will show that indeed the conformal weight of either of the generators $x_{-n}, \partial x_{-n}, \xi_{-n}, \partial \xi_{-n}$ is $n$.

3.7.1. Consider the conformal weight 0 subspace, $D_{\text{ch}}^{\text{i}\i\i,0}$. It is equal to the Laurent superpolynomial ring, $\mathbb{C}[x_0, \xi_0]$, hence as a complex it is identified with the de Koszul complex $K(\mathbb{C}[x_0], x_0)$, cf. 3.3. In fact, it is a direct sum of 2-dimensional subcomplexes $C_n$, $n \geq 0$, $C_n$ being spanned by $\xi_0 x_0^n, x_0^{n+1}$, and one 1-dimensional, $C_{-1}$, spanned by 1. The matrix of the differential restricted to $C_n$, $n \geq 0$, in the indicated basis is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 
\end{pmatrix},
$$

cf. (2.2.1). The element $1 \in C_{-1}$ is a cocycle and spans the cohomology.

3.7.2. The conformal weight 1 component is a $\text{rk} = 4$ free module over the conformal weight 0 component, $\mathbb{C}[x_0, \xi_0]$, with basis $x_{-1}, \partial x_{-1}, \xi_{-1}, \partial \xi_{-1}$. Denote the 4-dimensional vector space spanned by these 4 vectors by $V_1$. We have

$$D_{\text{ch}}^{\text{i}\i\i,1} = V_1 \otimes \mathbb{C}[x_0, \xi_0],$$

a tensor product of 2 complexes, the action of the differential on $V_1$ being given by

$$\begin{align*}
\xi_{-1} &\mapsto x_{-1}, \ x_{-1} \mapsto 0, \\
\partial x_{-1} &\mapsto -\partial \xi_{-1}, \ \partial \xi_{-1} \mapsto 0.
\end{align*}$$

Of course this creates two more blocks of the familiar type

$$
\begin{pmatrix}
0 & 0 \\
1 & 0 
\end{pmatrix}
$$

along with yet another way of proving the acyclicity in conformal weight 1.

3.7.3. In an arbitrary conformal weight $N$ subspace, the situation is similar. We have a tensor product of complexes

$$D_{\text{ch}}^{\text{i}\i\i,N} = V_N \otimes \mathbb{C}[x_0, \xi_0],$$

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where $V_N$ is a finite dimensional vector space with basis consisting of monomials $x_{-n_1}^a \partial_{x_{-n_2}} \partial_{x_{-n_3}} \partial_{x_{-n_4}}$, where the subindices $n_j$ are all strictly negative and satisfy

$$bn_1 + cn_2 + dn_3 + en_4 = -N.$$ 

The action of the differential is determined by the following conditions:

- it is a derivation;
- it sends

$$\xi_{-n} \mapsto x_{-n}, \quad x_{-n} \mapsto 0,$$

$$\partial_{x_{-n}} \mapsto -\partial_{\xi_{-n}}, \quad \partial_{\xi_{-n}} \mapsto 0,$$

which makes the acyclicity of $V_N$, hence that of $D^{ch}_{C[1], N}$, $N > 0$, manifest.

3.7.4. The subcomplex $D^{ch}_{C[1], [x^{-1}_{(-1)}]}/D^{ch}_{C[1]}$, which we have just analyzed. In particular, its conformal weight 0 component is dual to $D^{ch}_{C[1], 0}$, also has 1-dimensional cohomology, and the formation of the Janus $D^{ch}_{C[1], 0}[x^{-1}_{(-1)}]/D^{ch}_{C[1], 0} \oplus D^{ch}_{C[1]0}$ was discussed in 3.5.

Being isomorphic to $K(\mathbb{C}[x_0], x_0)$, $D^{ch}_{C[1], 0}$ is also isomorphic to $\Delta^*_1$, see 2.2, therefore $D^{ch}_{C[1], 0}[x^{-1}_{(-1)}]/D^{ch}_{C[1], 0} \oplus D^{ch}_{C[1], 0}$ is nothing but the Janus complex of 2.1.3.

On the other hand, the higher conformal weight components of $D^{ch}_{C[1], [x^{-1}_{(-1)}]}/D^{ch}_{C[1]}$ are acyclic and enter the Janus $D^{ch}_{C[1], [x^{-1}_{(-1)}]}/D^{ch}_{C[1]} \oplus D^{ch}_{C[1]}$ simply as direct summands, 3.5.

§4. Chiral Koszul: several variables

4.1. If $V_i$, $i \in I$, is a family of vertex algebras, then the tensor product over the ground ring, $\otimes_I V_i$, is canonically a vertex algebra: if we write $v_i(z)$ for the field attached to $v_i \in V_i$, then we define the field attached to $\otimes_I v_i$ to be $\otimes_I (v_i(z))$.

Just as the coordinate ring $\mathbb{C}[C^I] = \otimes_I \mathbb{C}[C]$ or the ring of differential operators $D_{C^I} = \otimes_I D_{C}$, the CDO $D^{ch}_{C[1], I} = \otimes_I D^{ch}_{C[1]}$.

As a practical matter, what for $D^{ch}_{C[1]}$ was a quadruple of fields, $x(z), \partial_x(z), \xi(z), \partial_\xi(z)$ with the OPEs recorded in 3.1, 3.2, for $D^{ch}_{C[1], I}$ is an $I$-family of quadruples, $x_i(z), \partial_x(z), \xi_i(z), \partial_\xi(z), i \in I$, which pairwise commute if the corresponding indices $i$ are distinct, and satisfy the old OPE if they are the same.

Whatever we know about $D^{ch}_{C[1]}$ tends to carry over to $D^{ch}_{C[1], I}$ immediately. $D^{ch}_{C[1]}$ being the Koszul complex with differential $d^ch_K = \int x(z) \partial_\xi(z) dz$, $D^{ch}_{C[1], I}$ is the tensor product complex with differential $d^{ch}_{K} = \sum_I \int x_i(z) \partial_\xi_i(z) dz$. 

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$D_{\mathcal{C}^I}$ is a conformal vertex algebra, with Virasoro field $L(z) = \sum_i L^{(i)}(z)$, $L^{(i)}(z)$ being the copy of the $L(z)$ from 3.3. that acts on the $i$-th tensor factor of $D_{\mathcal{C}^I}$. The $(1)$-component of this field defines on $D_{\mathcal{C}^I}$ a conformal grading, and its 0-th component is the classical Koszul complex of the ring $\mathbb{C}[\mathcal{C}^I]$ w.r.t. the regular sequence $\{x_i, i \in I\}$, to be denoted, as usual, by $K(\mathbb{C}[\mathcal{C}^I], \{x_i, i \in I\})$, cf. 2.2.

This structure implies: the embedding of complexes

$$K(\mathbb{C}[\mathcal{C}^I], \{x_i, i \in I\}) \hookrightarrow D_{\mathcal{C}^I}$$

is a quasiisomorphism, and

$$H(\mathbb{C}[\mathcal{C}^I], \{x_i, i \in I\}) \cong H_dK(D_{\mathcal{C}^I}) = \mathbb{C}$$

spanned by the class of the cocycle $1 = \otimes_1 1$, similarly to 3.3.

4.2. The other half of the chiral Janus complex is again the tensor product

$$\otimes_1(D_{\mathcal{C}^I}[x_{-1}^{-1}] / D_{\mathcal{C}^I})$$. This notation is too cumbersome to be tolerated, and for the lack of a better one we shall write

$$(D_{\mathcal{C}^I})^d = \otimes_1(D_{\mathcal{C}^I}[x_{-1}^{-1}] / D_{\mathcal{C}^I})$$

and call it the *vertex dual* of $D_{\mathcal{C}^I}$. The right hand side of this equality is indeed the dual of $D_{\mathcal{C}^I}$ in the sense of [FBZ], 10.4.6, which has already been cited, 3.6, but since we are avoiding the discussion of this duality functor, let us take the right hand side of this equality for the definition of its left hand side.

The properties of $D_{\mathcal{C}^I}$ are parallel to those of $D_{\mathcal{C}^I}[x_{-1}^{-1}] / D_{\mathcal{C}^I}$ to the same extent those of $D_{\mathcal{C}^I}$ are parallel to those of $D_{\mathcal{C}^I}$. As a complex it is the dual of $D_{\mathcal{C}^I}$, and so its cohomology is 1 dimensional and spanned by the class of cocycle

$$\prod_1 \xi_i x_i, x_{-1}^{-1}$$

see the beginning of 3.5.

Analogously to 3.5, we define a chiral Janus, $(D_{\mathcal{C}^I})^d \oplus D_{\mathcal{C}^I}$, to be the direct sum $(D_{\mathcal{C}^I})^d \oplus D_{\mathcal{C}^I}$ as a vector space (and as a $D_{\mathcal{C}^I}$-module) with differential obtained by tweaking $d_K \oplus d_K$ so as to make sure that it sends the cocycle

$$\prod_1 \xi_i x_i, x_{-1}^{-1} \in (D_{\mathcal{C}^I})^d$$

to $\otimes_1 1 \in D_{\mathcal{C}^I}$,

thereby making $(D_{\mathcal{C}^I})^d \oplus D_{\mathcal{C}^I}$ an acyclic complex.
4.3. The negative (dual) chiral Koszul as a local cohomology of the positive one. The modules \((D_{C/I}^{ch})^d\) afford the following geometric interpretation.

The polynomial nature of \(D_{C/I}^{ch}\) has already been used to define localization \(D_{C/I}^{ch}[x_{(-1)}]^{-1}\) in 3.4. One key observation made in [MSV] is that one can extend a vertex algebra structure to any localization \(D_{C/I}^{ch}[S^{-1}], S\) being a multiplicative subset of \(\mathbb{C}[C/I]\). This makes \(D_{C/I}^{ch}\) into a sheaf on \(C/I\), to be denoted \(D_{C/I}^{ch}\).

4.3.1. Lemma. Let \(U = \mathbb{C}^I - \{0\}\), the “principal affine space,” \(Z = \{0\} \subset \mathbb{C}^I\). We have

(i) For any \(I\),
\[
(D_{C/I}^{ch})^d = H_Z^{|I|}(\mathbb{C}^I, D_{C/I}^{ch}).
\]

(ii) If \(|I| > 1\), then
\[
(D_{C/I}^{ch})^d = H_Z^{|I|}(\mathbb{C}^I, D_{C/I}^{ch}) = H^{|I|-1}(U, D_{C/I}^{ch}).
\]

Proof. Point (i) is of course well known, but we shall sketch the proof. Note that the point \(Z = \{0\}\) is defined by the ideal \((x_i, i \in I)\). It is known, [H, Theorem 2.3], that the local cohomology \(H_Z^{|I|}(\mathbb{C}^I, D_{C/I}^{ch})\) is computed by the direct limit of the Koszul complexes:

\[
H_Z^{|I|}(\mathbb{C}^I, D_{C/I}^{ch}) = \lim_{\{m_i\}} H^{|I|}(D_{C/I}^{ch}, \{(x_i, (-1))^{m_i}, i \in I\}).
\]

As the sequence \(\{(x_i, (-1))^{m_i}, i \in I\}\) is regular for any collection of exponents \(\{m_i\}\), the cohomology is concentrated in the \(|I|-th\) group. That it equals the space of “purely singular parts,” \(\otimes_I(D_{C/I}^{ch}[x_{(-1)}]^{-1})/D_{C/I}^{ch}\), is rather easy to see.

In view of (i), (ii) amounts to the isomorphism

\[
H_Z^{|I|}(\mathbb{C}^I, D_{C/I}^{ch}) = H^{|I|-1}(U, D_{C/I}^{ch}),
\]

which is even better known than (i) and constitutes part of [H, Prop.2.2]. \(\square\)

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