Double Gegenbauer expansion of $|s-t|^\alpha$

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Abstract

Motivated by the study of symmetry breaking operators for indefinite orthogonal groups, we give a Gegenbauer expansion of the two variable function $|s-t|^\alpha$ in terms of the ultraspherical polynomials $C_\lambda^\ell(s)$ and $C_\mu^m(t)$. Generalization, specialization, and limits of the expansion are also discussed.

KEYWORDS

Gegenbauer polynomial; Sobolev inequality; Hermite polynomial; Selberg integral

AMS CLASSIFICATION

2010 MSC. Primary 42C05; Secondary 33C45, 33C05, 53C35, 22E46.

1 Main results

Let $C_\lambda^\ell(s)$ be the Gegenbauer polynomial of degree $\ell$. In this article, we give an expansion of the power $|s-t|^\alpha$ by two Gegenbauer polynomials $C_\lambda^\ell(s)$ and $C_\mu^m(t)$ with independent parameters $\lambda$ and $\mu$.

For $\ell, m \in \mathbb{N}$, we set

$$b_{\ell,m}^{\lambda,\mu,\nu} := \frac{(-1)^m(\lambda + \ell)(\mu + m)\Gamma(\lambda + \mu + 2\nu + 1)\Gamma(\lambda)\Gamma(\mu)\Gamma(2\nu + 1)}{2^{2\nu} \prod_{\delta,\varepsilon \in \{\pm 1\}} \Gamma \left( \nu + 1 + \frac{\lambda + \mu}{2} + \delta \frac{\lambda + \ell}{2} + \varepsilon \frac{\mu + m}{2} \right)}. \quad (1.1)$$

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**Theorem 1.1** (cf. Kobayashi–Mano [1 Lem. 7.9.1]). Let $\lambda, \mu$ be positive numbers and $\nu \in \mathbb{C}$ satisfying $2 \Re \nu > \lambda + \mu + 4$. For $\varepsilon = 0, 1$, we have an expansion

$$|s-t|^{2\nu} \text{sgn}^\varepsilon (s-t) = \sum_{\ell, m \in \mathbb{N}} b_{\ell,m}^{\lambda,\mu,\nu} C_{\ell}^\lambda(s) C_{m}^\mu(t), \quad (1.2)$$

where the right-hand side converges absolutely and uniformly in $[-1, 1]^2$.

More generally, we find a double Gegenbauer expansion of the function $(s-xt)_{\alpha}^\alpha$ of two variables $s$ and $t$ with two parameters $x \in [-1, 1]$ and $\alpha \in \mathbb{C}$ with $\Re \alpha > 0$. Here we recall that both $\{|y|^\alpha, |y|^\alpha \text{sgn}(y)\}$ and $\{y_+^\alpha, y_-^\alpha\}$ span the space of continuous homogeneous functions on $\mathbb{R}$ of degree $\alpha$ when $\Re \alpha > 0$, and that the change of basis is given by

$$|y|^\alpha \text{sgn}^\varepsilon (y) = y_+^\alpha + (-1)^\varepsilon y_-^\alpha \quad (\varepsilon = 0, 1), \quad (1.3)$$

where we set

$$y_+^\alpha := \left\{ \begin{array}{ll} y^\alpha & (y > 0) \\ 0 & (y \leq 0) \end{array} \right., \quad y_-^\alpha := \left\{ \begin{array}{ll} 0 & (y \geq 0) \\ |y|^\alpha & (y < 0) \end{array} \right..$$

(The equation (1.3) can be understood as the identity of distributions with meromorphic parameter $\alpha$, see [2 Sect. 2] or [3 Chap. 1], although we do not need this viewpoint here.) Then, we prove the following integral formula (see also Proposition 5.1 for its variants):

**Theorem 1.2.** For $\ell, m \in \mathbb{N}$, $\lambda, \mu, \nu \in \mathbb{C}$ with $\Re \lambda, \Re \mu > -\frac{1}{2}, \Re \nu > 0$, and for $-1 \leq x \leq 1$, we set

$$B_{\ell,m}^{\lambda,\mu,\nu} (x) := \int_{-1}^{1} \int_{-1}^{1} (s-xt)_{\alpha}^{2\nu} u_\ell^\lambda(s) u_m^\mu(t) ds dt,$$

where

$$u_\ell^\lambda(s) := \frac{2^{2\lambda-1} \ell! \Gamma(\lambda)}{\Gamma(2\lambda + \ell)} (1 - s^2)^{\lambda - \frac{\ell}{2}} C_{\ell}^\lambda(s).$$

Then we have

$$B_{\ell,m}^{\lambda,\mu,\nu} (x) = \frac{(-1)^m \pi^2 \Gamma(2\nu + 1) x^m} {2^{2\nu + 1} \Gamma \left( \nu - \frac{\ell + m}{2} + 1 \right) \Gamma \left( \mu + m + 1 \right) \Gamma \left( \lambda + \nu + \frac{\ell - m}{2} + 1 \right)} F_1 \left( \begin{array}{c} -\nu + \frac{\ell + m}{2}, -\lambda - \nu + \frac{m - \ell}{2} \\ \mu + m + 1 \end{array} ; x^2 \right), \quad (1.4)$$
By the Sobolev-type estimate for the Gegenbauer expansion given in Proposition 4.1 and the elementary identity (1.3), Theorem 1.1 is deduced readily from the special case at $x = 1$ of Theorem 1.2. As another application of Theorem 1.2, we prove the following integral formula: we set $d\omega^{\alpha,\beta}(x) := x^\alpha(1 - x)^\beta dx$ and $d\omega^{\alpha}(x) := (1 - x^2)^\alpha dx$.

Corollary 1.3. Assume $\lambda, \mu, \nu, b \in \mathbb{C}$ and $\ell, m \in \mathbb{N}$ satisfy $\ell + m \in 2\mathbb{N}$, $\Re \lambda, \Re \mu > -\frac{1}{2}$, $\Re \nu > 0$, and $\Re b > -1$. Then $\frac{1}{2}(m - \ell)$ is an integer and we have

$$
\int_{-1}^{1} \int_{-1}^{1} \int_{0}^{1} |s - t\sqrt{y}|^{2\nu} C^\lambda_m(s) C^\mu_m(t) d\omega^{\mu + \frac{m}{2}, b}(y) d\omega^{\lambda - \frac{1}{2}}(s) d\omega^{\mu - \frac{1}{2}}(t) = c \cdot \frac{(2\lambda)\ell(2\mu)m (-\nu)\frac{m-\ell}{\ell!m!} \Gamma \left( \lambda + \frac{1}{2} \right) \Gamma \left( \mu + \frac{1}{2} \right) \Gamma \left( \nu + \frac{1}{2} \right) \Gamma \left( \lambda + \mu + 2\nu + b + 2 \right) \Gamma (b + 1)}{\Gamma \left( \lambda + \mu + \nu + b + \frac{\ell+m}{2} + 2 \right) \Gamma \left( \nu + \lambda + \mu + 2\nu + b + \frac{\ell-m}{2} + 2 \right) \Gamma (\lambda + \nu + b - \frac{\ell-m}{2} + 2)},
$$

where $c$ is the nonzero constant $\frac{\pi^\frac{1}{2}(-1)^{m-\ell}}{\ell!m!}$.

Remark 1. As in (1.3), we may take another basis for the space of continuous homogeneous functions on $\mathbb{R}$ of degree $\alpha$ given by

$$(x \pm i0)^\alpha = x_+^\alpha + e^{\pm \pi i \alpha} x_-^\alpha. \quad (1.5)$$

By change of basis, we can derive easily closed formulæ of the double Gegenbauer expansion of $|s - t|^2 \nu$ and $|s - t \pm i0|^2 \nu$ from (1.1) in Theorem 1.1 and vice versa. Similarly, we can find readily integral formulæ for $(s - xt)^2 \nu$, $|s - xt|^2 \nu$, $|s - xt|^2 \nu \text{sgn}(s - xt)$, and $(s - xt \pm i0)^2 \nu$ analogous to Theorem 1.2. Likewise for Corollary 1.3.

Selberg-type integrals are related to (finite-dimensional) representation theory of semisimple Lie algebras, see [4, 5] and references therein. On the other hand, (an equivalent form of) Theorem 1.1 was given earlier by Kobayashi–Mano [1, Lem. 7.9.1], which was utilized in the study of the unitary inversion operator of the geometric quantization of the minimal nilpotent orbit. Furthermore, the precise location of the zeros and the poles of the meromorphic continuation of the formulæ given in Theorem 1.2 and Proposition 5.2 will be used in the study of symmetry breaking operators for infinite-dimensional representations when we extend the work [6] on the rank one group to indefinite orthogonal groups $O(p,q)$ of higher rank. This will be given in a subsequent paper.

The proof of Theorems 1.1 and 1.2 will be given in Sections 5 and 2, respectively. Corollary 1.3 is shown in Section 6. Special cases and the limit case of Theorem 1.2 will be discussed in Sections 7 and 8.
Notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$, $(x)_n = x(x+1)\cdots(x+n-1)$ (the Pochhammer symbol), and $[\lambda]$ denotes the greatest integer that does not exceed $\lambda \in \mathbb{R}$.

2 Proof of the main theorem

In this section we prove that Theorem 1.2 is deduced from the special case $\ell = m = 0$, namely, from the following integral formula (2.1). Proposition 2.1 will be proved in Section 3.

**Proposition 2.1.** Suppose $a, b, c \in \mathbb{C}$ satisfy $\Re a, \Re b > 0$ and $\Re c > \frac{1}{2}$. For $-1 \leq x \leq 1$, we have

$$
\int_{-1}^{1} \int_{-1}^{1} (s - xt)^{2c-1} (1 - s^2)^{a-1} (1 - t^2)^{b-1} dsdt = \frac{\sqrt{\pi} \Gamma(a) \Gamma(b) \Gamma(c)}{2 \Gamma(a + c) \Gamma(b + \frac{1}{2})} {}_{2}F_{1} \left( -c + \frac{1}{2}, -a - c + 1; b + \frac{1}{2}; x^2 \right). \tag{2.1}
$$

**Proof of Theorem 1.2.** The Rodrigues formula for the Gegenbauer polynomial (see [7, (6.4.14)] for instance) shows

$$
u_{\lambda}^{\ell}(s) = \frac{(-1)^{\ell + \mu} \sqrt{\pi}}{\Gamma(\lambda + \ell + \frac{1}{2})} \cdot \frac{d^\ell}{ds^\ell} (1 - s^2)^{\lambda + \ell - \frac{3}{2}}. \tag{2.2}
$$

By the definition of $B_{\ell,m}^{\lambda,\mu,\nu}(x)$, the left-hand side of (1.4) amounts to

$$
\frac{2^{-\ell - m (-1)^{\ell + m}} \prod_{\ell, \mu}}{\Gamma(\lambda + \ell + \frac{1}{2}) \Gamma(\mu + m + \frac{1}{2})} I_{\ell,m}(x),
$$

where we set

$$
I_{\ell,m}(x) \equiv I_{\ell,m}^{\lambda,\mu,\nu}(x) := \int_{-1}^{1} \int_{-1}^{1} (s - xt)^{2\nu} \frac{\partial^{\ell + m}}{\partial s^{\ell + m}} (1 - s^2)^{\lambda + \ell - \frac{3}{2}} \frac{\partial^m}{\partial t^m} (1 - t^2)^{\mu + m - \frac{1}{2}} dsdt.
$$

Suppose $\Re \nu > \frac{\ell + m}{2}$, $\Re \lambda > \frac{1}{2}$ and $\Re \mu > \frac{1}{2}$. Then integration by parts gives

$$
I_{\ell,m}(x) = (-1)^{\ell + m} \int_{-1}^{1} \int_{-1}^{1} \left( \frac{\partial^{\ell + m}}{\partial s^{\ell + m}} (s - xt)^{2\nu} \right) (1 - s^2)^{\lambda + \ell - \frac{3}{2}} (1 - t^2)^{\mu + m - \frac{1}{2}} dsdt
$$

$$
= (-1)^{m} (-2\nu)^{\ell + m} \int_{-1}^{1} \int_{-1}^{1} (s - xt)^{2\nu - \ell - m} (1 - s^2)^{\lambda + \ell - \frac{3}{2}} (1 - t^2)^{\mu + m - \frac{1}{2}} dsdt,
$$
because
\[
\frac{\partial^{\ell+m}}{\partial s^{\ell} \partial t^{m}} (s-xt)^{2\nu} = (-1)^{\ell} (-2\nu)^{\ell+m} x^{m} (s-xt)^{2\nu-\ell-m}.
\]
Applying Proposition 2.1 with \((a,b,c) = (\lambda + \ell + \frac{1}{2}, \mu + m + \frac{1}{2}, \nu + \frac{1}{2} (1 - \ell - m))\), we see that the equation (1.4) holds in the domain of \((\lambda, \mu, \nu)\) that we treated. Now Theorem 1.2 follows by analytic continuation. □

3 Proof of Proposition 2.1

In this section we show Proposition 2.1. We use the following two lemmas.

Lemma 3.1. For \(a, b \in \mathbb{C}\) with \(\Re a, \Re b > 0\) and for \(-1 \leq x \leq 1\) we have
\[
\int_{-1}^{1} (1 - tx)^{a-1} (1 - t^{2})^{b-1} dt = B \left( \frac{1}{2}, b \right) _{2F1} \left( \frac{1 - a}{2}, \frac{2 - a}{2} ; x^{2} \right). \tag{3.1}
\]

Lemma 3.2. Let \(a, b, d \in \mathbb{C}\) with \(b + \frac{1}{2} \notin -\mathbb{N}\) and \(d \notin -\mathbb{N}\). Then the series
\[
G(a, b, d; \zeta) := \sum_{i=0}^{\infty} \frac{(a)_{i}(1-a)_{i}}{2^{i}i!(d)_{i}}_{2F1} \left( \frac{1 - d - i}{2}, \frac{2 - d - i}{2} ; \zeta \right)
\]
converges when \(|\zeta| < 1\), and we have the following closed formula:
\[
G(a, b, d; \zeta) = \frac{2^{1-d} \sqrt{\pi} \Gamma(d)}{\Gamma \left( \frac{a+d}{2} \right) \Gamma \left( \frac{1-a+d}{2} \right)}_{2F1} \left( \frac{1 - a + d}{2}, \frac{1 + a - d}{2} ; \zeta \right). \tag{3.2}
\]

Postponing the verification of Lemmas 3.1 and 3.2 we first show Proposition 2.1.

Proof of Proposition 2.1. By the change of variables \(s = 1 - (1-x)t\), the interval \(0 \leq t \leq 1\) is transformed onto \((-1 \leq x \leq s \leq 1\), and thus Euler’s integral representation of the hypergeometric function \(_{2F1}\) shows
\[
\int_{-1}^{1} (s-x)^{2c-1} (1-s^{2})^{a-1} ds = 2^{a-1} B(2c, a) (1-x^{2})^{2c+a} _{2F1} \left( \frac{1-a}{2c+a}, \frac{1-x}{2} \right).
\]
Therefore the left-hand side of (2.1) equals
\[
2^{a-1} B(2c, a) \int_{-1}^{1} (1-tx)^{2c+a-1} _{2F1} \left( \frac{1-a}{2c+a}, \frac{1-tx}{2} \right) (1-t^{2})^{b-1} dt.
\]
Fix $\varepsilon > 0$. Assume $|x| \leq 1 - 2\varepsilon$. Then $\left| \frac{1-tx}{2} \right| \leq 1 - \varepsilon$. Expanding the hypergeometric function as a uniformly convergent power series of $\frac{1}{2}(1-tx)$, we can rewrite the integral in the right-hand side as

$$
\sum_{i=0}^{\infty} \frac{(1-a)_i(a)_i}{2^i i!(2c+a)_i} \int_{-1}^{1} (1-tx)^{2c+a-1+i}(1-t^2)^{b-1}dt.
$$

Owing to Lemma 3.1, this is equal to

$$
B \left( \frac{1}{2}, b \right) \sum_{i=0}^{\infty} \frac{(1-a)_i(a)_i}{2^i i!(2c+a)_i} \, _2F_1 \left( \frac{1-2c-a-i}{2}, \frac{2-2c-a-i}{b+\frac{1}{2}} ; x^2 \right).
$$

Now (2.1) follows from Lemma 3.2 with $\zeta = x^2$ and $d = 2c + a$ and from the duplication formula of the Gamma function $\Gamma(2c) = \pi^{-\frac{1}{2}} 2^{2c-1} \Gamma(c) \Gamma \left( c + \frac{1}{2} \right)$ as far as $-1 < x < 1$. Finally, by the continuity, (2.1) holds for $x = \pm 1$ under the assumption on the parameters $a, b, c \in \mathbb{C}$.

**Proof of Lemma 3.1** By Euler’s integral representation of $_2F_1$ again, the left-hand side of (3.1) amounts to

$$
2^{a-1}(1-x)^a \, _2F_1 \left( \frac{1-a, b}{2b} ; \frac{-2x}{1-x} \right). \tag{3.3}
$$

Applying the quadratic transformation of $_2F_1$ (cf. [7, Thm. 3.13]):

$$
_2F_1 \left( \frac{1-a, b}{2b} ; u \right) = \left( 1 - \frac{u}{2} \right)^{a-1} \, _2F_1 \left( \frac{1-a}{2}, \frac{2-a}{b+\frac{1}{2}} ; \left( \frac{u}{2-u} \right)^2 \right),
$$

with $u = \frac{-2x}{1-x}$, we get the desired result after a small computation using the duplication formula of the Gamma function.

**Proof of Lemma 3.2** We list some elementary identities for the Pochhammer symbol $(y)_j = \frac{\Gamma(y+j)}{\Gamma(y)}$:

$$
(y)_i(1-y-i) = (-1)^i \Gamma(1-y), \quad \left( \frac{y}{2} \right)_j \left( \frac{1+y}{2} \right)_j = 2^{-2j} (y)_{2j}, \quad (y)_i(1-y)_{2j} = (1-y-i)_{2j}(y-2j). \tag{3.6}
$$

To prove the equation (3.2), we first show the following expansion:

$$
G(a, b, d; \zeta) = \sum_{j=0}^{\infty} \frac{(1-d)_{2j}\zeta^j}{2^{2j} j! \left( b+\frac{1}{2} \right)_j} \, _2F_1 \left( \frac{a, 1-a}{d-2j} ; \frac{1}{2} \right). \tag{3.7}
$$
Indeed, by expanding the hypergeometric function as a power series and by using (3.5) with \( y = 1 - d - i \), we have

\[
G(a, b, d; \zeta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a)_i (1 - a)_i (1 - d - i)_{2j}}{2^{i+2j} i! j! (d)_i (b + \frac{1}{2})_j} \zeta^j,
\]

which is equal to the right-hand side of (3.7) by (3.6) with \( y = d \).

As \( \, _2F_1 \left( \begin{array}{c} a, 1-a \cr c \end{array} ; \frac{1}{2} \right) = \frac{2^{1-c} \sqrt{\pi} \Gamma(c)}{\Gamma \left( \frac{a+c}{2} \right) \Gamma \left( \frac{-a+1}{2} \right)} \) (see [7, Thm. 5.4] for instance), we can continue as

\[
\frac{1}{2} \, _2F_1 \left( \begin{array}{c} a, 1-a \cr c \end{array} ; \frac{1}{2} \right) = \frac{2^{1-d+c} \sqrt{\pi} \Gamma(1-a)}{\Gamma \left( \frac{a+d}{2} \right) \Gamma \left( \frac{1-a+d}{2} \right)} \zeta^j,
\]

where we have used (3.4) with \((y, i) = (1-d, 2j), (1-\frac{1}{2}(a+d), j), \) and \( (\frac{1}{2}(1+a-d), j) \) in the second equality. Hence Lemma 3.2 is proven. \( \square \)

4 Sobolev-type estimate for Gegenbauer expansion

In this section we formulate a Sobolev-type estimate for Gegenbauer expansion, by which Theorem 1.1 follows readily from the special value of the integral formula (Theorem 1.2), as we shall see in Section 5.

We begin with a basic setup. If \( \lambda > -\frac{1}{2} \) and \( \lambda \neq 0 \), then the Gegenbauer polynomials \( \{C_n^\lambda\}_{n \in \mathbb{N}} \) form an orthogonal basis in the Hilbert space \( L^2_\lambda := L^2 \left( (-1, 1), (1-x^2)^{\lambda-\frac{1}{2}} dx \right) \) with the norm

\[
v_n^\lambda := \|C_n^\lambda\|^2_{L^2_\lambda} = \frac{2^{1-2\lambda} \pi \Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}. \tag{4.1}
\]

This means that any \( f \in L^2_\lambda \), has an \( L^2 \)-expansion

\[
f(x) = \sum_{n=0}^{\infty} a_n(f) C_n^\lambda(x), \tag{4.2}
\]

where \( a_n(f) \in \mathbb{C} \) is given by

\[
a_n(f) = \frac{1}{v_n^\lambda} \int_{-1}^{1} f(x) C_n^\lambda(x) (1-x^2)^{\lambda-\frac{1}{2}} dx.
\]
Proposition 4.1. (Sobolev-type inequality for Gegenbauer expansion) Suppose $\lambda > 0$. Then there exists $D_\lambda > 0$ with the following property: let $N := \lceil \lambda \rceil + 2$, the integer satisfying $\lambda + 1 < N \leq \lambda + 2$. Then
\[
\|f\|_{L^\infty(-1,1)} \leq D_\lambda \left( \|f\|_{L_\lambda^2} + \|f^{(N)}\|_{L_\lambda^2} \right) \tag{4.3}
\]
for any $f \in L_\lambda^2$ such that the $N$-th derivative $f^{(N)}$ belongs to $L_\lambda^2$. Moreover, the Gegenbauer expansion (4.2) converges absolutely and uniformly in $[-1,1]$ for any such $f$.

Remark 2. Note that for $\lambda = \frac{1}{2}$, $L_\lambda^2 = L^2(0,1)$ and (4.3) follows from the classical Sobolev inequality.

Remark 3. We note that there is a continuous embedding
\[
L_\lambda^2 \hookrightarrow L_{\lambda+a}^2 \quad \text{for any } a > 0.
\]
As the proof shows, we may strengthen Proposition 4.1 by replacing (4.3) with
\[
\|f\|_{L^\infty(-1,1)} \leq D_\lambda \left( \|f\|_{L_\lambda^2} + \|f^{(N)}\|_{L_{\lambda+N}^2} \right)
\]
and $f^{(N)} \in L_\lambda^2$ with $f^{(N)} \in L_{\lambda+N}^2$.

The rest of this section is devoted to the proof of Proposition 4.1. We begin with the following.

Lemma 4.2. Suppose $\mathbb{N} \ni N > \lambda > 0$. Then there is a constant $d_{\lambda,N} > 0$ such that
\[
\|C^\lambda_n\|_{L^\infty(-1,1)} \leq d_{\lambda,N}\|C^{\lambda+N}_{n-N}\|_{L_{\lambda+N}^2} n^{\lambda-N} \quad \text{for all } n \geq N.
\]

Proof. We recall from [7, (6.4.11)] that
\[
|C^\lambda_n(x)| \leq C^\lambda_n(1) = \frac{\Gamma(n + 2\lambda)}{n!\Gamma(2\lambda)} \quad \text{for all } -1 \leq x \leq 1. \tag{4.4}
\]
By (4.1)
\[
\frac{C^\lambda_n(1)^2}{C_{n-N}^{\lambda+N}} = \frac{\Gamma(\lambda + N)^2}{\Gamma(2\lambda)^2 \cdot 2^{1-2\lambda-2N} \cdot \pi} \cdot \frac{\Gamma(n + 2\lambda)^2(n - N)!(n + \lambda)}{(n!)^2\Gamma(n + 2\lambda + N)}.
\]
The first term depends only on $\lambda$ and $N$, and the second term has the following asymptotics: $n^{2\lambda-2N}$ as $n$ tends to $\infty$ because
\[
\frac{\Gamma(n + a)}{\Gamma(n + b)} \sim n^{a-b} \quad \text{as } n \to \infty.
\]
Now Lemma 4.2 follows from (4.4). \qed
We are ready to complete the proof of Proposition 4.1.

**Proof of Proposition 4.1.** Let $N := [\lambda] + 2$ be as in Proposition 4.1. Iterating the differential formula

$$\frac{d}{dx} C_n^\lambda(x) = 2\lambda C_{n-1}^\lambda(x),$$

we get the following $L^2$-expansion:

$$f^{(N)}(x) = 2^N(\lambda) N \sum_{n=N}^{\infty} a_n(f) C_{n-N}^\lambda(x).$$

Thus, for all $n \geq N$, we have

$$|a_n(f)| \leq \frac{1}{2^N(\lambda) N} \frac{\|f^{(N)}\|_{L_{\lambda+N}^2}}{\|C_{n-N}^\lambda\|_{L_{\lambda+N}^2}}.$$

By Lemma 4.2

$$|a_n(f)||C_n^\lambda|_{L^\infty(-1,1)} \leq \frac{d_{\lambda,N}}{2^N(\lambda) N} \|f^{(N)}\|_{L_{\lambda+N}^2} n^{\lambda-N}.$$

Therefore the right-hand side of (4.2) converges uniformly in $[-1,1]$ because $\lambda - N < -1$.

For $0 \leq n < N$, we use $|a_n(f)| \sqrt{v_n^\lambda} \leq \|f\|_{L^2}$ to conclude

$$\left(\sum_{n=0}^{N-1} + \sum_{n=N}^{\infty} a_n(f)\right)\|C_n^\lambda\|_{L^\infty(-1,1)} \leq D_\lambda \left(\|f\|_{L^2} + \|f^{(N)}\|_{L_{\lambda+N}^2}\right),$$

where we set

$$D_\lambda := \max \left(\frac{d_{\lambda,N}}{2^N(\lambda) N} \sum_{n=N}^{\infty} n^{\lambda-N}, \left\{\frac{\|C_n^\lambda\|_{L^\infty(-1,1)}}{\sqrt{v_n^\lambda}}\right\}_{n=0,\ldots,N-1} \right).$$

Hence Proposition 4.1 is proved. \(\square\)

## 5 Proof of Theorem 1.1

We obtain from Theorem 1.2 the following.
Proposition 5.1. With the same assumption as in Theorem 1.2, we have

\[
\begin{align*}
\int_{-1}^{1} \int_{-1}^{1} (s-xt)^{2\nu} u_{c}^{\lambda} (s) u_{m}^{\mu} (t) ds dt &= (-1)^{\ell+m} B_{\ell,m}^{\lambda,\mu,\nu} (x), \\
\int_{-1}^{1} \int_{-1}^{1} |s-xt|^{2\nu} u_{c}^{\lambda} (s) u_{m}^{\mu} (t) ds dt &= (1 + (-1)^{\ell+m}) B_{\ell,m}^{\lambda,\mu,\nu} (x), \\
\int_{-1}^{1} \int_{-1}^{1} |s-xt|^{2\nu} \text{sgn} (s-xt) u_{c}^{\lambda} (s) u_{m}^{\mu} (t) ds dt &= (1 - (-1)^{\ell+m}) B_{\ell,m}^{\lambda,\mu,\nu} (x).
\end{align*}
\]

Proof. Taking into account that \( y_{a}^{-} = (-y_{a})_{a}^{-} \) and \( u_{c}^{\lambda} (-s) = (-1)^{\ell} u_{c}^{\lambda} (s) \), one derives the first integral formula from Theorem 1.2. In turn, the second and the third ones hold by the change of the basis \{ \| y_{a}^{-} \| , \| y_{a}^{-} \| \text{sgn} (y) \} \to \{ y_{a}^{\alpha} , y_{a}^{\alpha} \} \), see (1.3).

Proposition 5.2. Let \( \varepsilon \in \{ 0, 1 \} \) and \( \ell, m \in \mathbb{N} \). Suppose \( \text{Re} \lambda, \text{Re} \mu > -\frac{1}{2} \) and \( \text{Re} \nu > 0 \).

\[
\begin{align*}
\int_{-1}^{1} \int_{-1}^{1} |s-t|^{2\nu} \text{sgn}^{\varepsilon} (s-t) (1 - s^{2})^{\lambda - \frac{1}{2}} (1 - t^{2})^{\mu - \frac{1}{2}} C_{\ell}^{\lambda} (s) C_{m}^{\mu} (t) ds dt \\
= \frac{1}{2} (1 + (-1)^{\ell+m+\varepsilon}) b_{\ell,m}^{\lambda,\mu,\nu} v_{c}^{\lambda} v_{c}^{\mu}.
\end{align*}
\]

(5.1)

Proof. Since \( _2F_1 \left( \begin{array}{c} a, b \\ c \end{array} ; 1 \right) = \frac{\Gamma (c-a-b) \Gamma (c)}{\Gamma (c-a) \Gamma (c-b)} \) if \( \text{Re} c > \text{Re} (a + b) \), the left-hand side of (5.1) amounts to

\[
\frac{((-1)^{m} + (-1)^{\ell+m}) \pi \frac{1}{\varepsilon} (2 \lambda)_{\ell} (2 \mu)_{m} \Gamma (\lambda + \frac{1}{2}) \Gamma (\mu + \frac{1}{2}) \Gamma (\nu + 1) \Gamma (\lambda + \mu + 2 \nu + 1)}{2 \Gamma (1 + \nu - \frac{\ell-m}{2}) ! \! m ! \Gamma (\lambda + \nu + \frac{\ell-m}{2}) + 1} \Gamma (\mu + \nu - \frac{\ell-m}{2}) + 1 \Gamma (\lambda + \mu + \nu + \frac{\ell+m}{2} + 1)
\]

from the second and third formulæ of Proposition 5.1 with \( x = 1 \). By the definition (1.1) of \( b_{\ell,m}^{\lambda,\mu,\nu} \) and the formula (4.1) of \( v_{c}^{\lambda} \), the proposition follows.

We are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Owing to Proposition 4.1 we can deduce Theorem 1.1 from Proposition 5.2 under the assumption on \( (\lambda, \mu, \nu) \) because for any \( m, n \in \mathbb{N} \) with \( m \leq \lambda + 2 \) and \( n \leq \mu + 2 \), we have

\[
\frac{\partial^{m+n}}{\partial s^{m} \partial t^{n}} |s-t|^{2\nu} \in L^{2}((-1, 1)^{2}, (1 - s^{2})^{\lambda+m} (1 - t^{2})^{\mu+n} ds dt).
\]

Hence Theorem 1.1 is proved.
Remark 4. Kobayashi–Mano obtained an analogous formula to (5.1) in [1, Lem. 7.9.1], from which Proposition 5.2 follows by the change of basis (1.3) and thus we could give an alternative proof of Theorem 1.1. Our proof of (5.1) is different from [1, Chap. 7], where they showed the following integral formula [1, (7.4.11)] as a first step: for Re $\lambda > -1$, Re $\nu > -\frac{1}{2}$ and $|x| < 1$,

$$x^{\lambda}_- \ast h^\nu_k(x) = q_k(\lambda, \nu)(1 - x^2)^{\frac{1}{2}(\lambda + \nu + \frac{1}{2})}P_{\nu + k - \frac{1}{2}}^{-(\lambda + \nu + \frac{1}{2})}(-x). \quad (5.2)$$

Here $h^\nu_k(x) := (1 - x^2)^{\nu - \frac{1}{2}}C^\nu_k(x)$ for $|x| < 1$; = 0 otherwise, $P_{\alpha}^\beta(x)$ denotes the associated Legendre function, and the coefficient $q_k(\lambda, \nu)$ is given explicitly by the Gamma functions. The integral formula (5.2) immediately implies closed formulæ for $x^{\lambda}_- \ast h^\nu_k$ and $(x \pm i0)^{\lambda} \ast h^\nu_k$ because $C^\nu_k(-x) = (-1)^kC^\nu_k(x)$. With the notation as in (4.1), the integral formula (5.2) implies an expansion

$$(x - y)^{\lambda}_- = \sum_{k=0}^{\infty} \frac{q_k(\lambda, \nu)}{v^\lambda_k} (1 - x^2)^{\frac{1}{2}(\lambda + \nu + \frac{1}{2})}P_{\nu + k - \frac{1}{2}}^{-(\lambda + \nu + \frac{1}{2})}(-x)C^\nu_k(y) \quad (5.3)$$

for any $\mathbb{R} \ni \nu > -\frac{1}{2}$ with $\nu \neq 0$, and similarly for $(x - y)^{\lambda}_+$ and $(x - y \pm i0)^{\lambda}_+$. Kobayashi–Mano’s work [1] appeared in arXiv:0712.1769. Afterwards, Cohl [8] and Szmytkowski [9] obtained similar results to (5.2) and (5.3), but not the double Gegenbauer expansion as in (5.1). To be more precise, Szmytkowski [9, (2.5), (2.7)] rediscovered the same formula with (5.2) by using from Cohl [8, Thm. 2.1]. We note that [8, Thm. 2.1] also follows from Kobayashi–Mano’s formula [1, Lem. 7.9.1] by change of basis (1.3) and analytic continuation.

6 Proof of Corollary 1.3

It is sufficient to prove the following.

Lemma 6.1. Suppose $\lambda, \mu, \nu, \beta \in \mathbb{C}$ and $\ell, m \in \mathbb{N}$ satisfy Re $\beta > -1, \Re (\mu + m) > -1$, and $\Re (\lambda + \mu + 2\nu + \beta + 2) > 0$. Then we have

$$\int_{0}^{1} x^{2\mu + m + 1}(1 - x^2)^{\beta} B_{\ell, m}^{\lambda, \mu, \nu}(x) dx = \frac{(-1)^m 2^{-2\nu - 2}\pi^2 \Gamma(2\nu + 1)\Gamma(\beta + 1)\Gamma(\lambda + \mu + 2\nu + \beta + 2)}{\Gamma \left( \nu - \frac{\ell - m}{2} + 1 \right) \Gamma \left( \lambda + \nu + \frac{\ell - m}{2} + 1 \right) \Gamma \left( \mu + \nu + \beta + \frac{m - \ell}{2} + 2 \right) \Gamma \left( \lambda + \mu + \nu + \beta + \frac{m + \ell}{2} + 2 \right)}.$$

11
To prove Lemma 6.1 we use [10, 20.2 (4)]:

\[
\int_0^1 y^{\gamma-1}(1-y)^{\rho-1} \binom{\alpha, \beta}{\gamma} \, dy = \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}, \tag{6.1}
\]

if \(\text{Re}\, \gamma > 0, \text{Re}\, \rho > 0, \text{Re}\,(\gamma+\rho-\alpha-\beta) > 0\).

**Proof.** By the change of variables \(x = y^2\), the formula (6.1) shows

\[
\int_0^1 x^{2\mu+2m+1}(1-x^2)^{\beta} \binom{-\nu + \frac{\ell+m}{2}, -\lambda - \nu - \frac{\ell-m}{2}}{\mu + m + 1} \, dx = \frac{\Gamma(\mu + m + 1)\Gamma(\beta + 1)\Gamma(\lambda + \mu + 2\nu + \beta + 2)}{2\Gamma(\mu + \nu + \beta - \frac{\ell-m}{2} + 2)\Gamma(\lambda + \mu + \nu + \beta + \frac{\ell+m}{2} + 2)}.
\]

Now the lemma follows from Theorem 1.2.

\[\square\]

7 Special values and Selberg-type integrals

In this section, we examine the relationship between Theorem 1.2 and some known integral formulae by Selberg, Dotsenko, Fateev, Tarasov, Varchenko and Warnaar among others when the parameters take special values. The hierarchy of the formulae treated here is summarized in Figure 1.

For this, we limit ourselves to the special case of Theorem 1.2 with \((\ell, m, x) = (0, 0, 1)\), or equivalently, of Proposition 5.2 with \((\ell, m) = (0, 0)\):

\[
\int_0^1 \int_{-1}^1 |s-t|^{2\nu}(1-s^2)^{\lambda-\frac{\ell}{2}}(1-t^2)^{\mu-\frac{\ell}{2}} \, dsdt = \frac{\pi^\frac{\ell}{2}\Gamma(\lambda + \frac{\ell}{2})\Gamma(\nu + \frac{\ell}{2})\Gamma(\lambda + \mu + 2\nu + 1)}{\Gamma(\lambda + \nu + 1)\Gamma(\mu + \nu + 1)\Gamma(\lambda + \mu + \nu + 1)}.	ag{7.1}
\]

**Example 7.1.** (Selberg integral [11]) The Selberg integral

\[
\int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{\alpha_i-1}(1-t_i)^{\beta_i-1} \left| \prod_{1 \leq i < j \leq n} (t_i - t_j) \right|^{2\nu} \, dt_1 \cdots dt_n = \prod_{j=1}^n \frac{\Gamma(\alpha + (j-1)\nu)\Gamma(\beta + (j-1)\nu)\Gamma(1+j\nu)}{\Gamma(\alpha + \beta + (n+j-2)\nu)\Gamma(1+\nu)}\tag{7.2}
\]
is a generalization of the Euler beta integral. The special case of Theorem 1.2 with $(\ell, m, x, \mu) = (0, 0, 1, \lambda)$, namely, (7.1) with $\lambda = \mu$ reduces to the special case of (7.2) with $(n, \alpha, \beta) = (2, \lambda + \frac{1}{2}, \lambda + \frac{1}{2})$, namely,

$$
\int_{-1}^{1} \int_{-1}^{1} (1 - s^2)^{\lambda-\frac{1}{2}} (1 - t^2)^{\lambda-\frac{1}{2}} |s - t|^{2\nu} dsdt = \frac{2^{4\lambda+2\nu} \Gamma (\lambda + \frac{1}{2})^2}{\Gamma (2\lambda + 1 + \nu)} \frac{\Gamma \left( \lambda + \nu + \frac{1}{2} \right)^2 \Gamma (1 + 2\nu)}{\Gamma (2\lambda + 2\nu + 1) \Gamma (1 + \nu)},
$$

(7.3)

after a change of variables $(t_1, t_2) = (\frac{1+t}{2}, \frac{1-t}{2})$.

**Example 7.2.** (Warnaar integral) The special case of Theorem 1.2 with $(\ell, m, x, \nu) = (0, 0, 1, -\frac{\lambda+\mu}{2})$, namely, (7.1) with $\lambda+\mu+2\nu = 0$ reduces to a special case of Warnaar’s integral formula [12, (1.4)] with $(k_1, k_2, \alpha_1, \beta_1, \alpha_2, \beta_2, \gamma) = (1, 1, \lambda + \frac{1}{2}, \lambda + \frac{1}{2}, \mu + \frac{1}{2}, \mu + \frac{1}{2}, \lambda + \mu)$, namely,

$$
\left( \int \int_{0 \leq s < t \leq 1} + \cos (\pi \lambda) \int \int_{0 \leq t < s \leq 1} \right) t^{\lambda-\frac{1}{2}} (1-t)^{\lambda-\frac{1}{2}} s^{\mu-\frac{1}{2}} (1-s)^{\mu-\frac{1}{2}} |s - t|^{-\lambda-\mu} dsdt = \frac{\Gamma (\lambda + \frac{1}{2}) \Gamma (\mu + \frac{1}{2})^2}{\Gamma (1 + \mu - \lambda) \Gamma (\lambda + 1 - \mu)}.
$$

(7.4)

**Example 7.3.** (St3 Selberg integral of Tarasov and Varchenko) The special case of Theorem 1.2 with $(\ell, m, x, \mu) = (0, 0, 1, \frac{1}{2})$, namely, (7.1) with $\mu = \frac{1}{2}$ reduces to a special case of Tarasov–Varchenko’s integral formula [31, (3.4)] with $(k_1, k_2, \alpha, \beta_1, \beta_2, \gamma) = (1, 1, \lambda + \frac{1}{2}, \lambda + \frac{1}{2}, 1, -2\nu)$, namely,

$$
\int_{-1}^{1} \int_{-1}^{1} (1 - s^2)^{\lambda-\frac{1}{2}} (1 - t^2)^{\mu-\frac{1}{2}} |s - t|^{-2\nu} dsdt = \frac{2^{2\lambda+2\nu} \Gamma (\lambda + \frac{1}{2}) \Gamma (\mu + \frac{1}{2})}{(1 + 2\nu) \Gamma (2\lambda + 2\nu)}.
$$

(7.5)

**Example 7.4.** (Dotsenko–Fateev integral) The special case of Theorem 1.2 with $(\ell, m, x, \nu) = (0, 0, 1, -1)$, namely, (7.1) with $\nu = -1$ reduces to a special case of Dotsenko–Fateev’s integral formula [13] (A1) = (A35)] with $(n, m, \alpha, \beta, \rho) = (1, 1, \mu - \frac{1}{2}, \mu - \frac{1}{2}, -\frac{\mu-\frac{1}{2}}{\lambda-\frac{1}{2}})$, namely,

$$
\int_{-1}^{1} \int_{-1}^{1} (1 - s^2)^{\lambda-\frac{1}{2}} (1 - t^2)^{\mu-\frac{1}{2}} |s - t|^{-2\nu} dsdt = \frac{2^{2\lambda-2\nu} \Gamma (\lambda + \frac{1}{2}) \Gamma (\mu + \frac{1}{2})^2}{(1 - \lambda - \mu) \Gamma (2\lambda) \Gamma (2\mu)}.
$$

(7.6)

The hierarchy of the integral formulae in Examples 7.1, 7.4 and Theorem 1.2 is summarized as follows:
8 Limiting case

In this section we discuss the limiting case of our integral formula. Taking the limit in (1.2) as both $\lambda$ and $\mu$ tend to be zero, we obtain

Corollary 8.1. For $\rho \in \mathbb{C}$ with $\text{Re} \rho > 0$ and $\gamma \in \{0, 1\}$,

$$
|\cos \varphi + \cos \psi|^\rho \text{sgn}^\gamma (\cos \varphi + \cos \psi)
= 2^{-\rho} \Gamma(\rho + 1)^2 \sum_{\ell, m \in \mathbb{Z}} \frac{\cos \ell \varphi \cos m \psi}{\prod_{\delta, \varepsilon \in \{\pm 1\}} \Gamma \left(1 + \frac{1}{2}(\rho + \delta \ell + \varepsilon m)\right)}.
\quad (8.1)
$$

On the other hand, taking the limit in (5.1) with $\varepsilon = 1$ as $\lambda$ tends to infinity, we can deduce the following integral formula of the Hermite polynomial $H_n(x)$ from Proposition 5.2.

Corollary 8.2. Suppose $\ell, m \in \mathbb{N}$ with $\ell + m$ even.
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |s - xt|^{2\nu} e^{-s^2 - t^2} H_\ell(s) H_m(t) ds dt
= (-\nu) \frac{\ell + m}{2\nu} 2^{\ell + m} \pi^{\frac{1}{2}} \Gamma \left( \frac{1}{2} + \nu \right) (x^2 + 1)^{\nu - \frac{\ell + m}{2}} x^m. \tag{8.2}
\]

**Proof.** Use the limit formula

\[ H_n(x) = n! \lim_{\lambda \to \infty} \lambda^{-\frac{n}{2}} C_n^\lambda \left( \frac{x}{\sqrt{\lambda}} \right). \]

\[ \Box \]

**Example 8.3.** (Mehta integral \[14\]) The Mehta integral

\[
\frac{1}{(2\pi)^\frac{n}{2}} \int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-\frac{1}{2}t_i^2} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\nu} dt_1 \cdots dt_n = \prod_{j=1}^{n} \frac{\Gamma(1 + j\nu)}{\Gamma(1 + \nu)}
\]

in the special case \( n = 2 \) implies the following equation

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(s^2 + t^2)} |s - t|^{2\nu} ds dt = \frac{\Gamma(1 + 2\nu)}{\Gamma(1 + \nu)}. \tag{8.3}
\]

This coincides with the special case of Corollary 8.2 with \((\ell, m, x) = (0, 0, 1)\).

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