Maximum degree and spectral radius of graphs in terms of size*

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Abstract: Research on the relationship of the (signless Laplacian) spectral radius of a graph with its structure properties is an important research project in spectral graph theory. Denote by $\rho(G)$ and $q(G)$ the spectral radius and the signless Laplacian spectral radius of a graph $G$, respectively. Let $k \geq 0$ be a fixed integer and $G$ be a graph of size $m$ which is large enough. We show that if $\rho(G) \geq \sqrt{m-k}$, then $C_4 \subseteq G$ or $K_{1,m-k} \subseteq G$. Furthermore, we prove that if $q(G) \geq m-k$, then $K_{1,m-k} \subseteq G$. Both these two results extend some known results.

Keywords: Spectral radius; Maximum degree; Size

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1 Introduction

Graphs considered in the paper are simple and undirected. For a graph $G$, let $\rho(G)$ be the spectral radius of its adjacency matrix $A(G)$, and $q(G)$ be the spectral radius of its signless Laplacian matrix $Q(G)$. From Perron–Frobenius theorem, for a connected graph $G$, the adjacency (resp., signless Laplacian) spectral radius of $G$ is the maximum modulus of its adjacency (resp., signless Laplacian) eigenvalues. In general, we call $\rho(G)$ the spectral radius of $G$, and $q(G)$ the signless Laplacian spectral radius of $G$.

It is well known that the structure properties and parameters of graphs have close relationship with eigenvalues of graphs. During this recent thirty years, the (signless

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Laplacian) spectral radius among graphs with described structures properties has attracted considerable attention.

A graph $G$ is defined to be $H$-free if $G$ does not contain $H$ as a subgraph. As a spectral version of extremal graph theory, Nikiforov [11] posed a spectral Turán type problem that what is the maximal spectral radius $\rho(G)$ among $H$-free graphs $G$ of order $n$? This problem is also known as the Brualdi–Solheid–Turán type problem and has been investigated in much literature for some special graphs $H$, for which one can refer to clique [9], book [20], friendship [2] and references therein.

Recently, replacing the order $n$ by the size $m$, a perspective to spectral Turán type problem in terms of the size has received much research. This problem asks that what is the maximal spectral radius $\rho(G)$ among $H$-free graphs $G$ of size $m$? To the knowledge of us, the history of studying this problem may be dated back at least to Nosal’s theorem [14] in 1970. Up to now, there is few graph $H$ such that the maximal spectral radius $\rho(G)$ among $H$-free graphs of size $m$ has been determined. Some relevant conclusions have been obtained in the past two decades. Nikiforov in [8] extended Nosal’s theorem from triangles to clique, and answered a conjecture by Zhai, Lin and Shu [21] about books in [13]. For more detailed results, we refer one to [6,21].

Here we pay our main attention to the spectral Turán type problem on quadrilaterals and stars in terms of the size. Let $K_{1,n-1}$ and $C_n$ be a star and a cycle on $n$ vertices, respectively. Denote by $K_{1,n-1} + e$ the graph by inserting an edge to the independent set of $K_{1,n-1}$, and denote by $K_{1,n-1}^e$ the graph by attaching a pendent vertex to a pendent vertex of $K_{1,n-1}$.

In [10], Nikiforov determined the maximum spectral radius among all $C_4$-free graphs of size $m$.

**Theorem 1.** [10] Let $G$ be a graph of size $m \geq 9$. If $\rho(G) > \sqrt{m}$, then $C_4 \subseteq G$.

Zhai and Shu [22] improved the result in Theorem 1 for a non-bipartite connected graph by showing the following theorem.

**Theorem 2.** [22] Let $G$ be a non-bipartite and connected graph of size $m \geq 26$. If $\rho(G) \geq \rho(K_{1,m-1} + e)$, then $C_4 \subseteq G$ unless $G$ is $K_{1,m-1} + e$.

Recently, Wang [17] provided a generalization of Theorems 1 and 2.

**Theorem 3.** [17] Let $G$ be a graph of size $m \geq 27$. If $\rho(G) \geq \sqrt{m-1}$, then $C_4 \subseteq G$ unless $G$ is one of these graphs (with possibly isolated vertices): $K_{1,m}$, $K_{1,m-1} + e$, $K_{1,m-1}^e$, or $K_{1,m-1} \cup P_2$.

It is easy to check that $\rho(H) < \rho(K_{1,m})$ if $H \in \{K_{1,m-1} + e, K_{1,m-1}^e, K_{1,m-1} \cup P_2\}$ and $m \geq 27$. This, together with Theorems 1 and 3, indicates that if $\rho(G) \geq \sqrt{m}$ for a graph $G$ of size $m \geq 27$, then $C_4 \subseteq G$ unless $G$ is $K_{1,m}$. Indeed, from Theorems 1 and 3, if $m \geq 27$ and $\rho(G) \geq \sqrt{m-k}$ for $k = 0$ or 1, then $C_4 \subseteq G$ unless $K_{1,m-k} \subseteq G$. Motivated by this, we hope to give a general result in terms of the value of $k$. 
Theorem 4. Let $k \geq 0$ be an integer and $G$ be a graph of size $m \geq \max\{(k^2+2k+2)^2 + k + 1, (2k+3)^2 + k + 1\}$. If $\rho(G) \geq \sqrt{m-k}$, then $K_{1,m-k} \subseteq G$ or $C_4 \subseteq G$.

Next we turn our attention to study the relation of the maximum degree and the signless Laplacian spectral radius of a graph. In a sense, the signless Laplacian matrix can significantly reveal the structure properties of graphs $G$ since $Q(G)$ consists of the adjacency matrix and the diagonal matrix of degree sequence. For more details, readers are referred to [7, 15], and a series of survey by Cvetković and Simić [3–5].

A signless spectral Turán type version of extremal graph theory has been extensively studied by researchers. Much literature studied the maximal signless Laplacian spectral $q(G)$ among $H$-free graphs in terms of order, including triangles [25], cycles [12, 19] and linear forests [1]. There is few investigation on signless spectral Turán type problem in terms of the size.

The topic we focus on is inspired from a theorem by Zhai, Xue and Lou [24], which can be viewed as signless spectral Turán type problem for stars in terms of the size.

Theorem 5. [24] Let $G$ be a graph of size $m \geq 4$. If $G$ is a graph without isolated vertices, then $q(G) \leq m+1$ with equality if and only if $G = K_{1,m}$.

Theorem 5 infers that if $q(G) \geq m+1$ for a graph of size $m$, then $K_{1,m} \subseteq G$ (in fact, $G = K_{1,m}$ when $G$ has no isolate vertex). We show the following result, which extends Theorem 5.

Theorem 6. Let $k \geq 0$ be an integer and $G$ be a graph of size $m \geq \max\{\frac{1}{2}k^2 + 6k + 3, 7k + 25\}$. If $q(G) \geq m - k + 1$, then $K_{1,m-k} \subseteq G$.

The rest of this paper is organized as follows. Notations are introduced in Section 2. Proofs of Theorems 4 and 6 are presented in Sections 2 and 3, respectively. In Section 4, we propose a conjecture on the relation of maximum degree and spectral radius of adjacency matrix in terms of order.

2 Proof of Theorem 4

Before stating details of the proof, we shall introduce some terminologies and notations. For a graph $G$, let $u$ be a vertex of $G$, and $S, T$ be two subsets of $V(G)$. Then let $N_{S}(u)$ denote the set of neighbor of $u$ in $S$, and $d_{S}(u)$ be the cardinality of $N_{S}(u)$, i.e., $d_{S}(u) = |N_{S}(u)|$. Specially, if $S = V(G)$ then we omit the subscript $S$. The minimum degree of $G$ is defined to be $\delta(G) = \min\{d(u) : u \in V(G)\}$. Let $G[S]$ be the subgraph of $G$ induced by $S$, then denote $E(S)$ by the set of edges in $G[S]$ and $e(S)$ by the cardinality of $E(S)$. Suppose that $S \cap T = \emptyset$. Then we denote $e(S, T)$ by the number of edges with one endpoint in $S$ and another endpoint in $T$. 
Proof of Theorem 4. We prove Theorem 4 by way of contradiction. Assume that there are graphs \( H \) of size \( m \geq \max\{(k^2 + 2k + 2)^2 + k + 1, (2k + 3)^2 + k + 1\} \) with \( \rho(H) \geq \sqrt{m - k} \), such that \( K_{1,m-k} \not\subseteq H \) and \( C_4 \not\subseteq H \). Let \( G \) be a graph with the maximum spectral radius among graphs satisfying the above conditions. Since adding/deleting isolated vertices to/from \( G \) not changes the value of \( \rho(G) \), we can let \( G \) contain no isolated vertices. For simplification, we write \( \rho \) by \( \rho(G) \).

Let \( x \) be a nonnegative eigenvector of \( A(G) \) corresponding to \( \rho \) with coordinate \( x_i \) corresponding to the vertex \( v_i \) of \( G \). Let \( u^* \) be a vertex of \( G \) with \( x_{u^*} = \max\{x_i : v_i \in V(G)\} \), then we have a partition \( \{u^*\} \cup A \cup B \) of \( V(G) \) where \( A = N(u^*) \) and \( B = V(G) \setminus N[u^*] \). Thus,

\[
\rho^2 x_{u^*} = \sum_{u \in N(u^*)} \rho x_u = \sum_{u \in N(u^*)} \sum_{v \in N(u)} x_v = |A|x_{u^*} + \sum_{uv \in E(A)} (x_u + x_v) + \sum_{u \in B} d_A(u)x_u. \tag{1}
\]

Next we establish two necessary claims.

Claim 1. For a vertex \( u \) in \( B \), \( d_A(u) \leq 1 \).

**Proof.** This claim follows from the fact that \( C_4 \not\subseteq G \). \( \square \)

Following the partition of \( V(G) \), we give a refinement of \( B \). Let \( B = B_1 \cup B_2 \) be a partition of \( B \), such that \( B_1 = \{u \in B : d_B(u) = 0\} \) and \( B_2 = B \setminus B_1 \). Then for a vertex \( u \in B_1 \), we have \( d_A(u) = 1 \) from claim 1, and so \( d(u) = 1 \).

Claim 2. For a vertex \( u \) in \( B_1 \), \( x_u \leq \frac{1}{\rho} x_{u^*} \).

**Proof.** Let \( u \in B_1 \), then we have \( \rho x_u = \sum_{v \in N(u)} x_v \leq x_{u^*} \). The claim follows. \( \square \)

Thus, from claim 2, by (1) we have

\[
\rho^2 x_{u^*} \leq |A|x_{u^*} + \sum_{uv \in E(A)} (x_u + x_v) + \sum_{u \in B_1} \frac{1}{\rho} x_{u^*} + \sum_{u \in B_2} d_A(u)x_u. \tag{2}
\]

Note that

\[
e(A, B_2) \leq 2e(B_2) = 2e(B). \tag{3}
\]

Note that \( V(G) \) has the partition \( \{u\} \cup A \cup B_1 \cup B_2 \). Clearly \( P_3 \not\subseteq G[A] \) since \( C_4 \not\subseteq G \). Then, from claim 2, by (3) we obtain

\[
\rho \sum_{uv \in E(A)} (x_u + x_v) \leq 2e(A)x_{u^*} + \sum_{uv \in E(A)} (x_u + x_v) + \sum_{u \in B_1} d_A(u)x_u + \sum_{u \in B_2} d_A(u)x_u \leq 2e(A)x_{u^*} + \sum_{uv \in E(A)} (x_u + x_v) + \frac{e(A, B_1)}{\rho} x_{u^*} + e(A, B_2)x_{u^*}
\]

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Claim 2. For a vertex \( u \) in \( B_1 \), \( x_u \leq \frac{1}{\rho} x_{u^*} \).

**Proof.** Let \( u \in B_1 \), then we have \( \rho x_u = \sum_{v \in N(u)} x_v \leq x_{u^*} \). The claim follows. \( \square \)

Thus, from claim 2, by (1) we have

\[
\rho^2 x_{u^*} \leq |A|x_{u^*} + \sum_{uv \in E(A)} (x_u + x_v) + \sum_{u \in B_1} \frac{1}{\rho} x_{u^*} + \sum_{u \in B_2} d_A(u)x_u. \tag{2}
\]

Note that

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e(A, B_2) \leq 2e(B_2) = 2e(B). \tag{3}
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Note that \( V(G) \) has the partition \( \{u\} \cup A \cup B_1 \cup B_2 \). Clearly \( P_3 \not\subseteq G[A] \) since \( C_4 \not\subseteq G \). Then, from claim 2, by (3) we obtain

\[
\rho \sum_{uv \in E(A)} (x_u + x_v) \leq 2e(A)x_{u^*} + \sum_{uv \in E(A)} (x_u + x_v) + \sum_{u \in B_1} d_A(u)x_u + \sum_{u \in B_2} d_A(u)x_u \leq 2e(A)x_{u^*} + \sum_{uv \in E(A)} (x_u + x_v) + \frac{e(A, B_1)}{\rho} x_{u^*} + e(A, B_2)x_{u^*}
\]
\[ \leq 2e(A)x_{u^*} + \sum_{uv \in E(A)} (x_u + x_v) + \frac{e(A, B_1)}{\rho} x_{u^*} + 2e(B)x_{u^*}. \]

It follows that
\[ \sum_{uv \in E(A)} (x_u + x_v) \leq \left( \frac{2e(A) + 2e(B)}{\rho - 1} + \frac{e(A, B_1)}{\rho(\rho - 1)} \right) x_{u^*}. \]

This, together with (2), indicates that
\[
\begin{align*}
\rho^2 x_{u^*} &\leq |A| x_{u^*} + \left( \frac{2e(A) + 2e(B)}{\rho - 1} + \frac{e(A, B_1)}{\rho(\rho - 1)} \right) x_{u^*} + \frac{e(A, B_1)}{\rho} x_{u^*} + \sum_{u \in B_2} d_A(u)x_u \\
&\leq |A| x_{u^*} + \left( \frac{2e(A) + 2e(B)}{\rho - 1} + \frac{e(A, B_1)}{\rho(\rho - 1)} \right) x_{u^*} + \frac{e(A, B_1)}{\rho} x_{u^*} + e(A, B_2)x_{u^*} \\
&= \left( |A| + \frac{2e(A) + 2e(B)}{\rho - 1} + \frac{e(A, B_1)}{\rho(\rho - 1)} + e(A, B_2) \right) x_{u^*}.
\end{align*}
\]

That is, \( \rho^2 \leq |A| + \frac{2e(A) + 2e(B)}{\rho - 1} + \frac{e(A, B_1)}{\rho(\rho - 1)} + e(A, B_2). \)

On the other hand, we know that \( \rho^2 \geq m - k. \) Note that \( m = |A| + e(A) + e(B) + e(A, B) = |A| + e(A) + e(B) + e(A, B_1) + e(A, B_2). \) We have
\[ \rho^2 \geq |A| + e(A) + e(B) + e(A, B_1) + e(A, B_2) - k. \quad (4) \]

Hence,
\[ |A| + e(A) + e(B) + e(A, B_1) + e(A, B_2) - k \leq |A| + \frac{2e(A) + 2e(B)}{\rho - 1} + \frac{e(A, B_1)}{\rho(\rho - 1)} + e(A, B_2), \]

which implies that
\[ (\rho - 3)e(A) + (\rho - 3)e(B) + (\rho - 2)e(A, B_1) \leq k(\rho - 1). \]

Since \( m \geq (2k + 3)^2 + k + 1, \) we have \( \rho \geq \sqrt{m - k} > 2k + 3. \) Thus, \( e(B) \leq \frac{e-1}{\rho-3}k < k + 1, \) and so \( e(B) \leq k. \)

Therefore, for a vertex \( u \in B_2, \) we have \( d(u) \leq k + 1, \) and
\[ \rho x_u = \sum_{v \in N(u)} x_v \leq d(u)x_{u^*} \leq (k + 1)x_{u^*}, \]

which follows that \( x_u \leq \frac{k+1}{\rho} x_{u^*}. \) Furthermore, we obtain
\[
\begin{align*}
\rho \sum_{uv \in E(A)} (x_u + x_v) &\leq 2e(A)x_{u^*} + \sum_{uv \in E(A)} (x_u + x_v) + \sum_{u \in B_1} d_A(u)x_u + \sum_{u \in B_2} d_A(u)x_u \\
&\leq 2e(A)x_{u^*} + \sum_{uv \in E(A)} (x_u + x_v) + \frac{e(A, B_1)}{\rho} x_{u^*} + \frac{(k+1)e(A, B_2)}{\rho} x_{u^*}.
\end{align*}
\]
That is,
\[
\sum_{uv \in E(A)} (x_u + x_v) \leq \frac{1}{\rho - 1} \left( 2e(A) + \frac{e(A, B_1)}{\rho} + \frac{(k + 1)e(A, B_2)}{\rho} \right) x_u^*.
\]

By (2), we have
\[
\rho^2 x_u^* \leq |A|x_u^* + \frac{1}{\rho - 1} \left( 2e(A) + \frac{e(A, B_1)}{\rho} + \frac{(k + 1)e(A, B_2)}{\rho} \right) x_u^* + \frac{e(A, B_1)}{\rho} x_u^* + \sum_{u \in B_2} d_A(u)x_u
\]
\[
\leq |A|x_u^* + \frac{1}{\rho - 1} \left( 2e(A) + \frac{e(A, B_1)}{\rho} + \frac{(k + 1)e(A, B_2)}{\rho} \right) x_u^* + \frac{e(A, B_1)}{\rho} x_u^* + \frac{(k + 1)e(A, B_2)}{\rho} x_u^*
\]
\[
= \left( |A| + \frac{2e(A)}{\rho - 1} + \frac{e(A, B_1)}{\rho - 1} + \frac{(k + 1)e(A, B_2)}{\rho - 1} \right) x_u^*.
\]

Combining this inequality with (4), we obtain
\[
|A| + e(A) + e(B) + e(A, B_1) + e(A, B_2) - k \leq |A| + \frac{2e(A)}{\rho - 1} + \frac{e(A, B_1)}{\rho - 1} + \frac{(k + 1)e(A, B_2)}{\rho - 1},
\]
which implies that
\[
(\rho - 3)e(A) + (\rho - 1)e(B) + (\rho - 2)e(A, B_1) + (\rho - k - 2)e(A, B_2) \leq k(\rho - 1).
\]

Since $K_{1,m-k} \not\subseteq G$. Then $e(A) + e(B) + e(A, B_1) + e(A, B_2) = m - |A| \geq k + 1$. If $k = 0$, then $(\rho - 3)e(A) + (\rho - 1)e(B) + (\rho - 2)e(A, B_1) + (\rho - 2)e(A, B_2) \leq 0$ by (5). So $\rho \leq 3$. Hence, $3 \geq \rho \geq \sqrt{m - k} \geq \sqrt{(2k + 3)^2 + k + 1 - k} = \sqrt{10}$, a contradiction.

If $k \geq 1$, then by (5) we have
\[
e(A) + e(B) + e(A, B_1) + e(A, B_2) \leq \frac{\rho - 1}{\rho - k - 2} k < k + 1
\]
since $\rho \geq \sqrt{m - k} \geq \sqrt{(k^2 + 2k + 2)^2 + k + 1 - k} > k^2 + 2k + 2$. This is a contradiction.

This completes the proof.

\section{Proof of Theorem 6}

Notations appeared in this section are the same as those in section 2. For a graph $G$, if $x$ is a unit eigenvector of $Q(G)$ corresponding to $q(G)$ with coordinate
Let $x_i$ corresponding to the vertex $v_i$ of $G$, by the well-known Courant-Fisher theorem, then we have

$$q(G) = \max_{\|y\|_2=1} y^TQ(G)y = \sum_{v_i, v_j \in E(G)} (x_i + x_j)^2. \tag{6}$$

Note that the formulate $Q(G)x = q(G)x$ implies that $(q(G)I - D(G))x = A(G)x$. Then for a vertex $u \in V(G)$, we have

$$(q(G) - d(u))x_u = \sum_{v \in N(u)} x_v. \tag{7}$$

**Proof of Theorem 6.** We prove theorem 6 by way of contradiction. Suppose that $G$ is the extremal graph with the maximum signless Laplacian spectral radius among graphs $H$ of size $m \geq \max\{\frac{1}{2}k^2 + 6k + 3, 7k + 25\}$ and $K_{1,m-k} \not\subseteq H$. Then $q(G) \geq m - k + 1$. Let $x$ be a nonnegative unit eigenvector of $Q(G)$ corresponding to $q(G)$, and $u^*$ be a vertex of $G$ with $x_{u^*} = \max\{x_i : v_i \in V(G)\}$. For simplification, write $q$ by $q(G)$.

Denote by

$$W = \left\{ u \in V(G) : x_u \geq \frac{1}{2}x_{u^*} \right\}. $$

Note that $u^* \in W$, and so $|W| \geq 1$. We prove the following claim.

**Claim 3.** $|W| = 1$.

**Proof.** For a vertex $u \in W$, we know $x_u \geq \frac{1}{2}x_{u^*}$. Then, by (7),

$$(q - d(u))\frac{1}{2}x_{u^*} \leq (q - d(u))x_u = \sum_{v \in N(u)} x_v \leq d(u)x_{u^*},$$

which follows that $d(u) \geq \frac{1}{3}q$.

Since $q \geq m - k + 1$ and $m \geq 7k + 24$. Then

$$2m \geq \sum_{u \in W} d(u) \geq \frac{1}{3}q|W| \geq \frac{1}{3}(m - k + 1)|W|,$$

that is, $|W| \leq \frac{6m}{m-k+1} < 7$. Thus, $|W| \leq 6$.

Now we can improve the lower bound that $d(u) \geq \frac{1}{3}q$ for $u \in W$. By (7), we obtain that

$$(q - d(u^*))x_{u^*} = \sum_{v \in N(u^*)} x_v = \sum_{v \in N(u^*) \cap W} x_v + \sum_{v \in N(u^*) \setminus W} x_v \leq (|W| - 1)x_{u^*} + (d(u^*) - |W| + 1)\frac{1}{2}x_{u^*} = \frac{1}{2}(d(u^*) + |W| - 1)x_{u^*},$$
which follows that
\[
d(u^*) \geq \frac{2}{3} q - \frac{1}{3}|W| + \frac{1}{3} \geq \frac{2}{3} q - \frac{5}{3}, \tag{8}
\]

Assume that \(|W| \geq 2\). For a vertex \(u \in W \setminus \{u^*\}\), we obtain that
\[
(q - d(u))x_u = \sum_{v \in N(u)} x_v = \sum_{v \in N(u) \cap W} x_v + \sum_{v \in N(u) \setminus W} x_v
\leq (|W| - 1)x_{u^*} + (d(u) - |W| + 1)\frac{1}{2}x_{u^*}
= \frac{1}{2}(d(u) + |W| - 1)x_{u^*}.
\]
On the other hand, we have \((q - d(u))x_u \geq \frac{1}{2}(q - d(u))x_{u^*}\). Hence, \(\frac{1}{2}(d(u) + |W| - 1) \geq \frac{1}{2}(q - d(u))\), that is,
\[
d(u) \geq \frac{q}{2} - \frac{5}{2}.
\tag{9}
\]

Combining (8) and (9), we have
\[
m + 1 \geq d(u^*) + d(u) \geq \frac{2}{3} q - \frac{5}{3} + \frac{q}{2} - \frac{5}{2} = \frac{7}{6} q - \frac{25}{6} \geq \frac{7}{6}(m - k + 1) - \frac{25}{6}.
\]
Hence, \(m \leq 7k + 24\), which contradicts the fact that \(m \geq 7k + 25\). Thus, \(|W| \leq 1\), and so \(|W| = 1\) since \(|W| \geq 1\).

From claim 3, we have \(W = \{u^*\}\). Thus, for two vertices \(u, v \in V(G) \setminus \{u^*\}\), it has \(x_u + x_v < x_{u^*}\).

We assert that \(d(u^*) = m - k - 1\). On the contrary, suppose that \(d(u^*) \leq m - k - 2\). Then there is an edge, says \(u_1u_2 \in E(G)\), such that \(u \notin \{u_1, u_2\}\). Let \(G'\) be the graph obtained from \(G\) by deleting the edge \(u_1u_2\) and attaching a pendent vertex \(u_0\) to \(u^*\), and \(\mathbf{x}'\) be a vector with
\[
x'_w = \begin{cases} x_w, & \text{if } w \in V(G); \\ 0, & \text{if } w = u_0. \end{cases}
\]

Note that \(\|\mathbf{x}'\|_2 = 1\). By (6), we have
\[
q(G') - q(G) \geq \sum_{uv \in E(G')} (x'_u + x'_v)^2 - \sum_{uv \in E(G)} (x_u + x_v)^2
= (x_{u^*} + 0)^2 - (x_{u_1} + x_{u_2})^2 > 0.
\]
Since \(K_{1,m-k} \not\subseteq G'\). This deduces a contradiction to the maximality of \(G\). Thus, we have \(d(u^*) = m - k - 1\).

For a vertex \(u \in V(G) \setminus \{u^*\}\), we have \(d(u) \leq k + 2\). Then, from (7), we have
\[
(q - d(u))x_u = \sum_{v \in N(u)} x_v \leq x_{u^*} + (d(u) - 1)\frac{1}{2}x_{u^*},
\]
which follows that
\[ x_u \leq \frac{d(u) + 1}{2(q - d(u))} x_{u^*} \leq \frac{k + 3}{2(q - k - 2)} x_{u^*}. \] (10)

We can further improve the lower bound in (10). Similarly, by (10) we have
\[ (q - d(u)) x_u = \sum_{v \in N(u)} x_v \leq x_{u^*} + (d(u) - 1) \frac{k + 3}{2(q - k - 2)} x_{u^*}, \]
which implies that
\[ x_u \leq \left( \frac{1}{q - d(u)} + \frac{d(u) - 1}{q - d(u)} \frac{k + 3}{2(q - k - 2)} \right) x_{u^*} \leq \left( \frac{1}{q - k - 2} + \frac{(k + 1)(k + 3)}{2(q - k - 2)^2} \right) x_{u^*}. \] (11)

Recall that \( q \geq m - k + 1 \) and \( d(u^*) = m - k - 1 \). By (7), we obtain
\[ 2x_{u^*} \leq (q - d(u^*)) x_{u^*} = \sum_{u \in N(u^*)} x_u \leq d(u^*) \left( \frac{1}{q - k - 2} + \frac{(k + 1)(k + 3)}{2(q - k - 2)^2} \right) x_{u^*}. \]
So
\[ (m - k - 1) \left( \frac{1}{q - k - 2} + \frac{(k + 1)(k + 3)}{2(q - k - 2)^2} \right) \geq 2. \]

On the other hand, we may check that
\[ (m - k - 1) \left( \frac{1}{q - k - 2} + \frac{(k + 1)(k + 3)}{2(q - k - 2)^2} \right) \leq (m - k - 1) \left( \frac{1}{m - 2k - 1} + \frac{(k + 1)(k + 3)}{2(m - 2k - 1)^2} \right) \]
\[ = (m - 2k - 1 + k) \left( \frac{1}{m - 2k - 1} + \frac{(k + 1)(k + 3)}{2(m - 2k - 1)^2} \right) \]
\[ = 1 + \frac{(k^2 + 6k + 3)m - (k^3 + 9k^2 + 9k + 3)}{2(m - 2k - 1)^2} \]
\[ < 2, \]
where the last inequality holds due to the fact that \( m \geq \frac{1}{2} k^2 + 6k + 3 \). This deduces a contradiction.

This completes the proof. \( \blacksquare \)

4 Concluding remarks

Nikiforov [9] showed that if \( G \) contains no \( C_4 \) then \( \rho(G) \leq \rho(F_n) \), where \( F_n \) is the friendship graph of odd order \( n \), with equality if and only if \( G = F_n \). In the same paper (also see [10]), Nikiforov posed a conjecture that for even \( n \), if \( G \) contains no \( C_4 \) then \( \rho(G) \leq \rho(F'_n) \), where \( F'_n \) is obtained from \( F_{n-1} \) by attaching a new vertex to the unique vertex of maximum degree, with equality if and only if \( G = F'_n \). The conjecture was confirmed by Zhai and Wang in [23].
It is easy to check that
\[
\rho(F_n) = \frac{1 + \sqrt{4(n-1) + 1}}{2}.
\]

Due to a well-known fact that \(\rho(G) \geq \frac{2m}{n}\) for a graph \(G\) of order \(n\) and size \(m\), if \(G\) contains no \(C_4\), then we have
\[
\frac{2m}{n} \leq \rho(G) \leq \rho(F_n) = \frac{1 + \sqrt{4(n-1) + 1}}{2}.
\]

That is,
\[
m \leq \frac{n \left(1 + \sqrt{4(n-1) + 1}\right)}{4},
\]
which is a classic upper bound of the Turán number for \(C_4\) by Reiman [16].

One can see that from Nikiforov’s result on odd \(n\) (resp., Zhai-Wang’s result on even \(n\)), if \(\rho(G) \geq \rho(F_n)\) (resp., \(\rho(G) \geq \rho(F'_n)\)), then \(C_4 \subseteq G\) or \(K_{1,n-1} \subseteq G\). Motivated by this property, we provide a natural conjecture in terms of the maximum degree as following.

**Conjecture 4.1.** Let \(s \geq 1\) be an integer and \(n \geq f(s)\), where \(f(s)\) is a function on \(s\). If \(G\) is a graph of order \(n\) and
\[
\rho(G) \geq \frac{1 + \sqrt{4(n-s) + 1}}{2},
\]
then \(K_{1,n-s} \subseteq G\) or \(C_4 \subseteq G\).

Nikiforov’s theorem confirmed Conjecture 4.1 for \(s = 1\). Indeed, Conjecture 4.1 provides a spectral method to pursue the Turán number for \(C_4\).

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