A MAXIMAL INEQUALITY FOR STOCHASTIC CONVOLUTIONS IN
2-SMOOTH BANACH SPACES

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Submitted May 24, 2011, accepted in final form October 24, 2011

AMS 2000 Subject classification: Primary 60H05; Secondary 60H15
Keywords: Stochastic convolutions, maximal inequality, 2-smooth Banach spaces, Itô formula.

Abstract
Let \((e^{tA})_{t \geq 0}\) be a \(C_0\)-contraction semigroup on a 2-smooth Banach space \(E\), let \((W_t)_{t \geq 0}\) be a cylindrical Brownian motion in a Hilbert space \(H\), and let \((g_t)_{t \geq 0}\) be a progressively measurable process with values in the space \(\gamma(H, E)\) of all \(\gamma\)-radonifying operators from \(H\) to \(E\). We prove that for all \(0 < p < \infty\) there exists a constant \(C\), depending only on \(p\) and \(E\), such that for all \(T \geq 0\) we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t e^{(t-s)A} g_s \, dW_s \right\|^p \leq C \mathbb{E} \left( \int_0^T \|g_t\|^2_{\gamma(H, E)} \, dt \right)^{\frac{p}{2}}.
\]

For \(p \geq 2\) the proof is based on the observation that \(\psi(x) = \|x\|^p\) is Fréchet differentiable and its derivative satisfies the Lipschitz estimate \(\|\psi'(x) - \psi'(y)\| \leq C(\|x\| + \|y\|)^{p-2}\|x - y\|\); the extension to \(0 < p < 2\) proceeds via Lenglart’s inequality.

1 Introduction

Let \((e^{tA})_{t \geq 0}\) be a \(C_0\)-contraction semigroup on a 2-smooth Banach space \(E\) and let \((W_t)_{t \geq 0}\) be a cylindrical Brownian motion in a Hilbert space \(H\). Let \((g_t)_{t \geq 0}\) be a progressively measurable process with values in the space \(\gamma(H, E)\) of all \(\gamma\)-radonifying operators from \(H\) to \(E\) satisfying

\[
\int_0^T \|g_t\|^2_{\gamma(H, E)} \, dt < \infty \quad \mathbb{P}\text{-almost surely}
\]
for all \( T \geq 0 \). As is well known (see \([6, 15, 16]\)), under these assumptions the stochastic convolution process
\[
X_t = \int_0^t e^{(t-s)A}g_s\,dW_s, \quad t \geq 0,
\]
is well-defined in \( E \) and provides the unique mild solution of the stochastic initial value problem
\[
dX_t = AX_t\,dt + g_t\,dW_t, \quad X_0 = 0.
\]
In order to obtain the existence of a continuous version of this process, one usually proves a maximal estimate of the form
\[
\mathbb{E}\sup_{0 \leq t \leq T} \|X_t\|^p \leq C_p \mathbb{E}\left(\int_0^T \|g_t\|^2_{\gamma(H,E)}\,dt\right)^{\frac{p}{2}}.
\] (1.1)
The first such estimate was obtained by Kotelenez \([11, 12]\) for \( C_0 \)-contraction semigroups on Hilbert spaces \( E \) and exponent \( p = 2 \). Tubaro \([19]\) extended this result to exponents \( p \geq 2 \) by a different method of proof which applies Itô’s formula to the \( C^2 \)-mapping \( x \mapsto \|x\|^p \). The case \( p \in (0, 2) \) was covered subsequently by Ichikawa \([10]\). A very simple proof, still for \( C_0 \)-contraction semigroups on Hilbert spaces, which works for all \( p \in (0, \infty) \), was obtained recently by Hausenblas and Seidler \([9]\). It is based on the Sz.-Nagy dilation theorem, which is used to reduce the problem to the corresponding problem for \( C_0 \)-contraction groups. Then, by using the group property, the maximal estimate follows from Burkholder’s inequality. This proof shows, moreover, that the constant \( C \) in (1.1) may be taken equal to the constant appearing in Burkholder’s inequality. In particular, this constant depends only on \( p \).

The maximal inequality (1.1) has been extended by Brzeźniak and Peszat \([4]\) to \( C_0 \)-contraction semigroups on Banach spaces \( E \) with the property that, for some \( p \in [2, \infty) \), \( x \mapsto \|x\|^p \) is twice continuously Fréchet differentiable and the first and second Fréchet derivatives are bounded by constant multiples of \( \|x\|^{p-1} \) and \( \|x\|^{p-2} \), respectively. Examples of spaces with this property, which we shall call \( (C_p^2) \), are the spaces \( L^q(\mu) \) for \( q \in [p, \infty) \). Any \( (C_p^2) \) space is 2-smooth (the definition is recalled in Section 2), but the converse doesn’t hold:

**Example 1.1.** Let \( F \) be a Banach space. The space \( \ell^2(F) \) is 2-smooth whenever \( F \) is 2-smooth \([8, \text{Proposition 17}]\). On the other hand, the norm of \( \ell^2(F) \) is twice continuously Fréchet differentiable away from the origin if and only if \( F \) is a Hilbert space \([14, \text{Theorem 3.9}]\). Thus, for \( q \in (2, \infty) \), \( \ell^2(\ell^q) \) and \( \ell^2(L^q(0,1)) \) are examples of 2-smooth Banach spaces which fail property \( (C_p^2) \) for all \( p \in [2, \infty) \).

To the best of our knowledge, the general problem of proving the maximal estimate (1.1) for \( C_0 \)-contraction semigroups on 2-smooth Banach space remains open. The present paper aims to fill this gap:

**Theorem 1.2.** Let \((e^{tA})_{t \geq 0}\) be a \( C_0 \)-contraction semigroup on a 2-smooth Banach space \( E \), let \((W_t)_{t \geq 0}\) be a cylindrical Brownian motion in a Hilbert space \( H \), and let \((g_t)_{t \geq 0}\) be a progressively measurable process in \( \gamma(H,E) \). If
\[
\int_0^T \|g_t\|^2_{\gamma(H,E)}\,dt < \infty \quad \mathbb{P}\text{-almost surely},
\]
then the stochastic convolution process \( X_t = \int_0^t e^{(t-s)A}g_s\,dW_s \) is well-defined and has a continuous version. Moreover, for all \( 0 < p < \infty \) there exists a constant \( C \), depending only on \( p \) and \( E \), such that
\[
\mathbb{E}\sup_{0 \leq t \leq T} \|X_t\|^p \leq C_p \mathbb{E}\left(\int_0^T \|g_t\|^2_{\gamma(H,E)}\,dt\right)^{\frac{p}{2}}.
\]
For \( p \geq 2 \), the proof of Theorem 1.2 is based on a version of Itô’s formula (Theorem 3.1) which exploits the fact (proved in Lemma 2.1) that in 2-smooth Banach spaces the function \( \psi(x) = \|x\|^p \) is Fréchet differentiable and satisfies the Lipschitz estimate

\[
\|\psi'(x) - \psi'(y)\| \leq C(\|x\| + \|y\|)^{p-2}\|x - y\|.
\]

The extension to exponents \( 0 < p < 2 \) is obtained by applying Lenglart’s inequality (see (4.1)).

Let us finally mention that, for \( p = 2 \), (1.2) implies (1.1).

Let \( 1 < q \leq 2 \). A Banach space \( E \) is \( q \)-smooth if the modulus of smoothness

\[
\rho_{\|\cdot\|}(t) = \sup \left\{ \frac{1}{2}\|x + ty\| - \|x - ty\| : \|x\| = \|y\| = 1 \right\}
\]

satisfies \( \rho_{\|\cdot\|}(t) \leq Ct^q \) for all \( t > 0 \).

It is known (see [17, Theorem 3.1]) that \( E \) is \( q \)-smooth if and only if there exists a constant \( K \geq 1 \) such that for all \( x, y \in E \),

\[
\|x + y\|^q + \|x - y\|^q \leq 2\|x\|^q + K\|y\|^q.
\]  

(2.1)

**Lemma 2.1.** Let \( E \) be a Banach space and let \( 1 < q \leq 2 \) be given. For \( p \geq q \) set \( \psi_p(x) := \|x\|^p \).

1. \( E \) is \( q \)-smooth if and only if the Fréchet derivative of \( \psi_q \) is globally \( (q - 1) \)-Hölder continuous on \( E \).
2. If $E$ is $q$-smooth, then for $p > q$ the Fréchet derivative of $\psi_p$ is locally $(q - 1)$-Hölder continuous on $E$.

Moreover, for all $p \geq q$ and $x, y \in E$ we have

$$\|\psi'_p(x) - \psi'_p(y)\| \leq C(||x|| + ||y||)^{p-q}||x - y||^{q-1}, \tag{2.2}$$

where $C$ depends only on $p$, $q$ and $E$.

**Proof.** If the Fréchet derivative of $\psi_q$ is $(q - 1)$-Hölder continuous on $E$, then by the mean value theorem we can find $0 \leq \theta, \rho \leq 1$ such that for all $x, y \in E$,

$$||x + y||^q + ||x - y||^q - 2||x||^q = (||x + y||^q - ||x||^q) + (||x - y||^q - ||x||^q) \leq \|\psi'_q(x + \theta y) - \psi'_q(x - \rho y)\| ||y|| \leq L(||x + \theta y) - (x - \rho y)||^{q-1}||y|| \leq 2^{q-1}L||y||^q.$$

Hence the Banach space $E$ is $q$-smooth.

Suppose now that the norm of $E$ is $q$-smooth. Then for all $x, y \in E$ with $||x||, ||y|| = 1$ and all $t > 0$ we have

$$||x + ty|| + ||x - ty|| - 2||x|| \leq K|ty|^q. \tag{2.3}$$

Thus

$$\lim_{t \rightarrow 0} \frac{||x + ty|| + ||x - ty|| - 2||x||}{|ty|^q} = 0,$$

which by [7, Lemma I.1.3] means that $\| \cdot \|$ is Fréchet differentiable on the unit sphere. Hence, by homogeneity, $\| \cdot \|$ is Fréchet differentiable on $E \setminus \{0\}$. Let us denote by $f_x$ its Fréchet derivative at the point $x \neq 0$.

We begin by showing the $(q - 1)$-Hölder continuity of $x \mapsto f_x$ on the unit sphere of $E$, following the argument of [7, Lemma V.3.5]. We fix $x \neq y \in E$ such that $||x||, ||y|| = 1$ and $h \in E$ with $||h|| = ||x - y||$ and $x - y + h \neq 0$. Since the norm $\| \cdot \|$ is a convex function,

$$f_x(x - y) \leq ||x|| - ||y||.$$

Similarly, we have

$$f_x(h) \leq ||x + h|| - ||x||, \quad f_y(y - x - h) \leq ||2y - x - h|| - ||y||.$$

By using above inequalities and the linearity of the function $f_x$, we have

$$f_x(h) - f_y(h) \leq ||x + h|| - ||x|| - f_y(h)$$

$$= ||x + h|| - ||y|| - f_y(x + h - y) + ||y|| - ||x|| + f_y(x - y)$$

$$\leq ||x + h|| - ||y|| - f_y(x + h - y)$$

$$= ||x + h|| - ||y|| + f_y(y - x - h)$$

$$\leq ||x + h|| + ||2y - x - h|| - 2||y||$$

$$\leq ||x + h - y|| \cdot \frac{x + h - y}{||x + h - y||} + ||y - ||x + h - y|| \cdot \frac{x + h - y}{||x + h - y||} - 2||y||$$

$$\leq K||x + h - y||^q \leq K(||x - y|| + ||h||)^q = 2^q K||x - y||^q,$$
where we also used (2.3). Since the roles of \(x\) and \(y\) may be reversed in this inequality, this implies

\[
\|f_x - f_y\| = \sup_{|h| = |x-y|} \frac{|f_x(h) - f_y(h)|}{\|x-y\|} \leq 2qK\|x-y\|^{q-1}
\]

This proves the \((q-1)\)-Hölder continuity of the norm \(\|\cdot\|\) on the unit sphere.

We proceed with the proof of (2.2); the \((q-1)\)-Hölder continuity of \(\psi_q\) as well as the local \((p-1)\)-Hölder continuity of \(\psi_p\) follow from it. For all \(x, y \in E\) with \(x \neq 0\) and \(y \neq 0\) we have \(\psi'_p(x) = p\|x\|^{p-1} f_x\).

It is easy to check that \(f_x = f_{\frac{x}{\|x\|}}\) and \(\|f_x\| = 1\). Following once more the argument of [7, Lemma V3.5], this gives

\[
\|\psi'_p(x) - \psi'_p(y)\| = p\|x\|^{p-1} f_x - \|y\|^{p-1} f_y
\]

\[
\leq p\|x\|^{p-1}(f_{\frac{x}{\|x\|}} - f_{\frac{y}{\|y\|}}) + p\|x\|^{p-1} - \|y\|^{p-1} f_{\frac{x}{\|x\|}}
\]

\[
\leq p2^qK\|x\|^{p-1}\|x\|^{-1}|x - y|\|x\|^{-q} + p\|x\|^{p-1} - \|y\|^{p-1}
\]

\[
= p2^qK\|x\|^{p-1}\|y\|^{1-q}|x - y| + y(\|y\| - \|x\|)^{q-1} + p\|x\|^{p-1} - \|y\|^{p-1}
\]

\[
\leq p2^qK\|x\|^{p-1}\|y\|^{1-q} (2\|y\|\|x - y\|)^{q-1} + p\|x\|^{p-1} - \|y\|^{p-1}
\]

\[
= p2^{q-1}K\|x\|^{p-1}\|y\|^{q-1} + p\|x\|^{p-1} - \|y\|^{p-1}.
\]

(2.4)

If \(q \leq p \leq 2\), then by the inequality \(|t^r - s^r| \leq |t - s|^r\), valid for \(0 < r \leq 1\) and \(s, t \in [0, \infty)\), we have

\[
\|x\|^{p-1} - \|y\|^{p-1} \leq \|x - y\| \leq \|x - y\|^{p-1} \leq \|x\| - \|y\|^{p-1} \leq (\|x\| + \|y\|)^{p-2}\|x - y\|^{q-1}.
\]

If \(p > 2\), by applying the mean value theorem, for some \(\theta \in [0, 1]\) we have

\[
\|x\|^{p-1} - \|y\|^{p-1} = (p-1)\|\theta x + (1-\theta)y\|^{p-2} f_{\theta x + (1-\theta)y}(x - y)
\]

\[
\leq (p-1)(\|x\| + \|y\|)^{p-2}\|x - y\|
\]

\[
\leq (p-1)(\|x\| + \|y\|)^{p-2}(\|x\| + \|y\|)^2\|x - y\|^{q-1}
\]

\[
= (p-1)(\|x\| + \|y\|)^{p-q}\|x - y\|^{q-1}.
\]

Also, since \(\psi'_p(0) = 0\), for \(y \neq 0\) we have

\[
\|\psi'_p(0) - \psi'_p(y)\| = p\|y\|^{p-1} = p\|y\|^{p-1} \frac{y}{\|y\|} \frac{y}{\|y\|} \leq p\|y\|^{p-1} \|y\|^{q-1} = p\|y\|^{p-1}\|y\|^{q-1}.
\]

\[
\square
\]

The above lemma will be combined with the next one, which gives a first order Taylor formula with a remainder term involving the first derivative only.
Lemma 2.2. Let $E$ and $F$ be Banach spaces, let $0 < \alpha \leq 1$, and let $\psi : E \to F$ be a Fréchet differentiable function whose Fréchet derivative $\psi' : E \to \mathcal{L}(E,F)$ is locally $\alpha$-Hölder continuous. Then for all $x, y \in E$ we have

$$\psi(y) = \psi(x) + \psi'(x)(y - x) + R(x, y),$$

where

$$R(x, y) = \int_0^1 \psi'(x + r(y - x))(y - x) - \psi'(x)(y - x) \, dr.$$  \hspace{1cm} (2.5)

Proof. Pick $w \in E$ such that $\|w\| \leq 1$ and consider the function $f : \mathbb{R} \to F$ by

$$f(\theta) := \psi(x + \theta w).$$

For all $x^* \in F^*$, $(f', x^*)$ is locally $\alpha$-Hölder continuous. To see this, note that for $|\theta_1|, |\theta_2| \leq R$ and $\|x\| \leq R$ we have $\|x + \theta_1 w\|, \|x + \theta_2 w\| \leq 2R$, so by assumption there exists a constant $C_{2R}$ such that

$$|\langle f'(\theta_1) - f'(\theta_2), x^* \rangle| = |\langle \psi'(x + \theta_1 w)w, x^* \rangle - \langle \psi'(x + \theta_2 w)w, x^* \rangle| \leq \|\psi'(x + \theta_1 w) - \psi'(x + \theta_2 w)\| \|x^*\| \leq C_{2R} |\theta_1 - \theta_2|^\alpha \|x^*\|.$$

Applying Taylor’s formula and [1, Lemma 1, Theorem 3] to the function $(f, x^*)$ we obtain

$$\langle f(t) - f(0), x^* \rangle = t \langle f'(0), x^* \rangle + \langle R_f(0, t), x^* \rangle,$$

where $R_f(0, t) = \int_0^1 t(f'(r t) - f'(0)) \, dr$. Now let $x, y \in E$ be given and set $t = \|y - x\|$ and $w = \frac{y - x}{\|y - x\|}$. With these choices we obtain

$$\langle \psi(y), x^* \rangle - \langle \psi(x), x^* \rangle - \langle \psi'(x)(y - x), x^* \rangle = \langle \psi(x + tw), x^* \rangle - \langle \psi(x), x^* \rangle - t \langle \psi'(x)w, x^* \rangle = \langle f(t) - f(0) - t f'(0), x^* \rangle = \int_0^1 t(f'(r t) - f'(0), x^*) \, dr = \int_0^1 \langle \psi'(x + r(y - x))(y - x) - \psi'(x)(y - x), x^* \rangle \, dr.$$

Since $x^* \in F^*$ was arbitrary, this proves the lemma. $\square$

3 An Itô formula for $\| \cdot \|^p$

From now on we shall always assume that $E$ is a 2-smooth Banach space. We fix $T \geq 0$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t \in [0,T]}$. Let $H$ be a real Hilbert space, and denote by $\gamma(H, E)$ the Banach space of all $\gamma$-radonifying operators from $H$ to $E$. We denote by $\mathcal{M}([0, T]; \gamma(H, E))$ the space of all progressively measurable processes $\xi : [0, T] \times \Omega \to \gamma(H, E)$ such that

$$\int_0^T \| \xi_t \|^2_{\gamma(H, E)} \, dt < \infty \quad \mathbb{P}\text{-almost surely.}$$
The space of all such $\xi$ which satisfy
\[
\mathbb{E}\left( \int_0^T \|\xi_t\|^2_{\gamma(H,E)} \, dt \right)^{\frac{p}{2}} < \infty
\]
is denoted by $M^p([0,T];\gamma(H,E))$, $0 < p < \infty$.

On $(\Omega, \mathcal{F}, \mathbb{P})$, let $(W_t)_{t \in [0,T]}$ be an $(\mathcal{F}_t)_{t \in [0,T]}$-cylindrical Brownian motion in $H$. For adapted simple processes $\xi \in M([0,T];\gamma(H,E))$ of the form
\[
\xi_t = \sum_{i=0}^{n-1} 1_{(t_i,t_{i+1})}(t) \otimes A_i,
\]
where $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ is a partition of the interval $[0,T]$ and the random variables $A_i$ are $\mathcal{F}_t$-measurable and take values in the space of all finite rank operators from $H$ to $E$, we define the random variable $I(\xi) \in L^p(\Omega, \mathcal{F}_T; E)$ by
\[
I(\xi) := \sum_{i=0}^{n-1} A_i(W_{t_{i+1}} - W_{t_i})
\]
where $(h \otimes x)W_t := (W_t, h) \otimes x$. It is well known that
\[
\mathbb{E}\|I(\xi)\|^2 \leq C^2 \mathbb{E} \int_0^T \|\xi_t\|^2_{\gamma(H,E)} \, dt,
\]
where $C$ depends on $p$ and $E$ only. It follows that $I$ has a unique extension to a bounded linear operator $M^2([0,T];\gamma(H,E))$ to $L^2(\Omega, \mathcal{F}_T; E)$. By a standard localisation argument, $I$ extends continuous linear operator from $M([0,T];\gamma(H,E))$ to $L^1(\Omega, \mathcal{F}_T; E)$. In what follows we write
\[
\int_0^t \xi_s \, dW_s := I(1_{[0,t]} \xi), \quad t \in [0,T].
\]

This stochastic integral has the following properties:

1. For all $\xi \in M([0,T];\gamma(H,E))$ the process $t \to \int_0^t \xi_s \, dW_s$ is an $E$-valued continuous local martingale, which is a martingale if $\xi \in M^2([0,T];\gamma(H,E))$.

2. For all $\xi \in M([0,T];\gamma(H,E))$ and stopping times $\tau$ with values in $[0,T]$,
\[
\int_0^\tau \xi_s \, dW_s = \int_0^T 1_{[0,\tau]}(t) \xi_t \, dW_t \quad \mathbb{P}\text{-almost surely.} \tag{3.1}
\]

3. For all $\xi \in M^2([0,T];\gamma(H,E))$ and $0 \leq u < t \leq T$,
\[
\mathbb{E}\left( \int_u^t \|\xi_s\|^2_{\gamma(H,E)} \, ds \mid \mathcal{F}_u \right) \leq C \mathbb{E}\left( \int_0^t \|\xi_s\|^2_{\gamma(H,E)} \, ds \mid \mathcal{F}_u \right). \tag{3.2}
\]

4. (Burkholder's inequality [2, 6]) For all $0 < p < \infty$ there exists a constant $C$, depending only on $p$ and $E$, such that for all $\xi \in M^p([0,T];\gamma(H,E))$ and $t \in [0,T]$,
\[
\mathbb{E}\sup_{s \in [0,t]} \left\| \int_0^s \xi_u \, dW_u \right\|^p \leq C \mathbb{E}\left( \int_0^t \|\xi_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{p}{2}}. \tag{3.3}
\]
An excellent survey of the theory of stochastic integration in 2-smooth Banach spaces with complete proofs is given in Ondreját’s thesis [16], where also further references to the literature can be found.

In what follows we fix \( p \geq 2 \) and set \( \psi(x) := \psi_p(x) = \|x\|^p \). Since we assume that \( E \) is 2-smooth, this function is Fréchet differentiable. Following the notation of Lemma 2.2 we set

\[
R_\psi(x,y) := \int_0^1 (\psi'(x + r(y-x))(y-x) - \psi'(x)(y-x)) \, dr.
\]

We have the following version of Itô’s formula.

**Theorem 3.1 (Itô formula).** Let \( E \) be a 2-smooth Banach space and let \( 2 \leq p < \infty \). Let \( (a_t)_{t \in [0,T]} \) be an \( E \)-valued progressively measurable process such that

\[
\mathbb{E}\left( \int_0^T \|a_t\| \, dt \right)^p < \infty
\]

and let \( (g_t)_{t \in [0,T]} \) be a process in \( M^p([0,T];\gamma(H,E)) \). Fix \( x \in E \) and let \( (X_t)_{t \in [0,T]} \) be given by

\[
X_t = x + \int_0^t a_s \, ds + \int_0^t g_s \, dW_s.
\]

The process \( s \mapsto \psi'(X_s)g_s \) is progressively measurable and belongs to \( M^1([0,T];H) \), and for all \( t \in [0,T] \) we have

\[
\psi(X_t) = \psi(x) + \int_0^t \psi'(X_s)(a_s) \, ds + \int_0^t \psi'(X_s)(g_s) \, dW_s + \lim_{n \to \infty} \sum_{i=0}^{m(n)-1} R_\psi(X_{t_{i+1}}^t, X_{t_i}^t, \gamma(H,E))
\]

with convergence in probability, for any sequence of partitions \( \Pi_n = \{0 = t_0^n < t_1^n < \cdots < t_m(n) = T\} \) whose meshes \( \|\Pi_n\| := \max_{0 \leq i \leq m(n)-1} |t_{i+1}^n - t_i^n| \) tend to 0 as \( n \to \infty \). Moreover, there exists a constant \( C \) and, for each \( \epsilon > 0 \), a constant \( C_\epsilon \), both independent of \( a \) and \( g \), such that

\[
\mathbb{E}\liminf_{n \to \infty} \sum_{i=0}^{m(n)-1} |R_\psi(X_{t_{i+1}}^t, X_{t_i}^t, \gamma(H,E))| \leq \epsilon C \mathbb{E} \sup_{s \in [0,t]} \|X_s\|^p + C_\epsilon \mathbb{E} \left( \int_0^t \|g_s\|_{\gamma(H,E)}^2 \, ds \right)^{\frac{p}{2}}.
\]

The proof shows that we may take \( C_\epsilon = C'(\epsilon^{-1-\gamma} + 1) \) for some constant \( C' \) independent of \( a \), \( g \), and \( \epsilon \).

Before we start the proof of the theorem we state some lemmas. The first is an immediate consequence of Burkholder’s inequality (3.3).

**Lemma 3.2.** Under the assumptions of Theorem 3.1 we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} \|X_t\|^p \leq C \mathbb{E} \left( \int_0^T \|a_s\| \, ds \right)^p + C_\epsilon \mathbb{E} \left( \int_0^T \|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{p}{2}}.
\]

**Lemma 3.3.** Under the assumptions of Theorem 3.1, the process \( t \mapsto \psi'(X_t)(g_t) \) is progressively measurable and belongs to \( M^1([0,T];H) \).
Proof. By the identity \( \|\psi'(x)\| = p\|x\|^{p-1} \) and Hölder’s inequality,
\[
E\left( \int_0^T \|\psi(X_{s\gamma})(g_s)\|^2_H \, dt \right)^{\frac{1}{2}} \leq E\left( \int_0^T \|\psi(X_{s\gamma})\|^2 \|g_s\|^2_{\gamma(H,E)} \, dt \right)^{\frac{1}{2}} \\
\leq E\left( \sup_{r \in [0,T]} \|X_{r\gamma}\|^{p-1} \left( \int_0^T \|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{1}{2}} \right) \\
\leq C\left( E\left( \sup_{r \in [0,T]} \|X_{r\gamma}\|^p \right)^{\frac{p-1}{p}} \left( E\left( \int_0^T \|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} ,
\]
and the right-hand side is finite by the previous lemma. The progressively measurability is clear.

This lemma implies that the stochastic integral in (3.4) is well-defined.

**Lemma 3.4.** Let \( 0 \leq u \leq t \leq T \) be arbitrary and fixed. Under the assumptions of Theorem 3.1, the process \( s \rightarrow \psi'(X_u)(g_s) \) is progressively measurable and belongs to \( M^1([0,T];H) \). Moreover, \( \mathbb{P} \)-almost surely,
\[
\psi'(X_u) \int_u^t g_s \, dW_s = \int_u^t \psi'(X_u)(g_s) \, dW_s.
\]

**Proof.** By similar estimates as in the previous lemma,
\[
E\left( \int_u^t \|\psi'(X_u)(g_s)\|^2_H \, ds \right)^{\frac{1}{2}} \leq C\left( E\|X_u\|^p \right)^{\frac{p-1}{p}} \left( E\left( \int_u^t \|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{1}{2}} \right)^{\frac{1}{2}},
\]
The progressively measurability is again clear. To prove the identity we first assume that \( g \) is a simple adapted process of the form
\[
g_s = \sum_{i=0}^{n-1} 1_{(t_i,t_{i+1})}(s)A_i,
\]
where \( \Pi = \{u = t_0 < t_1 < \cdots < t_n = t\} \) is a partition of the interval \([0,T]\) and the random variables are \( \mathcal{F}_i \)-measurable and take values in the space of all finite rank operators from \( H \) to \( E \). Then,
\[
\psi'(X_u) \int_u^t g_s \, dW_s = \psi'(X_u) \left( \sum_{i=0}^{n-1} A_i(W_{t_{i+1}} - W_{t_i}) \right) \\
= \sum_{i=0}^{n-1} \psi'(X_u)(A_i(W_{t_{i+1}} - W_{t_i})) = \int_u^t \psi'(X_u)(g_s) \, dW_s.
\]
For general progressively measurable \( g \in L^p(\Omega;L^2([0,T];\gamma(H,E))) \), the identity follows by a routine approximation argument.

**Proof of Theorem 3.1.** The proof of the theorem proceeds in two steps. All constants occurring in the proof may depend on \( E \) and \( p \), even where this is not indicated explicitly, but not on \( T \). The numerical value of the constants may change from line to line.
Step 1 – Applying Lemma 2.2 to the function $\psi(x) = \|x\|^p$ and the process $X$, we have, for every $t \in [0, T]$,

$$
\psi(X_t) - \psi(x) = \sum_{i=0}^{m(n)-1} \left( \psi(X_{t_{i+1}^n \wedge t}) - \psi(X_{t_i^p \wedge t}) \right)
$$

$$
= \sum_{i=0}^{m(n)-1} \psi'(X_{t_i^p \wedge t})(X_{t_{i+1}^n \wedge t} - X_{t_i^p \wedge t}) + \sum_{i=0}^{m(n)-1} R_i(X_{t_i^p \wedge t}, X_{t^p_{i+1} \wedge t}).
$$

We shall prove the identity (3.4) by showing that

$$
\lim_{n \to \infty} \sum_{i=0}^{m(n)-1} \psi'(X_{t_i^p \wedge t})(X_{t_{i+1}^n \wedge t} - X_{t_i^p \wedge t}) = \int_0^t \psi'(X_s)(a_s) \, ds + \int_0^t \psi'(X_s)(g_s) \, dW_s
$$

with convergence in probability. In view of the definition of $X_t$, it is enough to show that

$$
\lim_{n \to \infty} \sum_{i=0}^{m(n)-1} \psi'(X_{t_i^p \wedge t}) \left( \int_{t_i^p \wedge t}^{t_{i+1}^n \wedge t} a_s \, ds \right) - \int_0^t \psi'(X_s)(a_s) \, ds = 0 \quad \text{P-almost surely}
$$

and

$$
\lim_{n \to \infty} \sum_{i=0}^{m(n)-1} \psi'(X_{t_i^p \wedge t}) \left( \int_{t_i^p \wedge t}^{t_{i+1}^n \wedge t} g_s \, dW_s \right) - \int_0^t \psi'(X_s)(g_s) \, dW_s = 0 \quad \text{in probability. (3.6)}
$$

By (2.2), P-almost surely we have

$$
\lim_{n \to \infty} \sup_{x \in [0, T]} \|X_x\|^{p-2} \times \lim_{n \to \infty} \sup_{t \in [0, T]} \sum_{i=0}^{m(n)-1} \left( \int_{t_i^p \wedge t}^{t_{i+1}^n \wedge t} \|a_s\| \, ds \right)
$$

$$
\leq C \sup_{x \in [0, T]} \|X_x\|^{p-2} \times \lim_{n \to \infty} \sup_{t \in [0, T]} \int_{t_i^p \wedge t}^{t_{i+1}^n \wedge t} \|X_{t_i^p \wedge t} - X_{t_{i+1}^n \wedge t}\| \, ds
$$

$$
\leq C \sup_{x \in [0, T]} \|X_x\|^{p-2} \times \lim_{n \to \infty} \sup_{t \in [0, T]} \left( \sup_{0 \leq i \leq m(n)-1} \sup_{x \in [t_i^p \wedge t, t_{i+1}^n \wedge t]} \|X_{t_i^p \wedge t} - X_{t_{i+1}^n \wedge t}\| \right)
$$

$$
\times \left( \sum_{i=0}^{m(n)-1} \int_{t_i^p \wedge t}^{t_{i+1}^n \wedge t} \|a_s\| \, ds \right)
$$

$$
= 0,
$$

where we used the continuity of the process $X$ in the last line.

Next, by Lemma 3.4 and the inequalities (3.2) and (2.2),

$$
\sum_{i=0}^{m(n)-1} \psi'(X_{t_i^p \wedge t}) \left( \int_{t_i^p \wedge t}^{t_{i+1}^n \wedge t} g_s \, dW_s \right) - \int_0^t \psi'(X_s)(g_s) \, dW_s
$$
Step 2 – In this step we prove the estimate (3.5). By (2.2), for all $x \in E$ it suffices to observe that

$$
\sum \left| \psi'(X_{t_i^{n} \wedge t})(g_i) - \psi'(X_{t_i^{n} \wedge t})(g_i) \right| dW_i
$$

is $\mathbb{P}$-almost surely $\in L^2$. Hence, from (3.4) it follows that

$$
\lim_{n \to \infty} \sup_{s \in [0, t]} \sum_{i=0}^{m(n)-1} \psi'(X_{t_i^{n} \wedge t})(g_i) - \psi'(X_{t_i^{n} \wedge t})(g_i) = 0
$$

in probability.

For this, in turn, it suffices to prove that

$$
\sup_{s \in [0, t]} \sum_{i=0}^{m(n)-1} \psi'(X_{t_i^{n} \wedge t})(g_i) - \psi'(X_{t_i^{n} \wedge t})(g_i) \to 0
$$

by the path continuity of $X$.

Step 2 – In this step we prove the estimate (3.5). By (2.2), for all $x, y \in E$ and $r \in [0, 1]$ we have

$$
|\psi'(x + r(y - x)) - \psi'(x)| \leq (|x|^p - 2|x - y|) + \|x - y\|^{p - 1}.
$$

Combining this with (2.5) we obtain

$$
|\psi'(X_{t_i^{n} \wedge t}, X_{t_i^{n} \wedge t})| \leq C|\psi(X_{t_i^{n} \wedge t}, X_{t_i^{n} \wedge t})|^{p - 2} + C|\psi(X_{t_i^{n} \wedge t}, X_{t_i^{n} \wedge t})|^2.
$$

We shall estimate the two terms on the right hand of (3.7) side separately.

For the first term, using the inequality $|a + b|^2 \leq 2|a|^2 + 2|b|^2$ we obtain

$$
\sum_{i=0}^{m(n)-1} \|X_{t_i^{n} \wedge t}\|^{p - 2} \|X_{t_i^{n} \wedge t} - X_{t_i^{n} \wedge t}\|^2
$$

$$
\leq 2 \sum_{i=0}^{m(n)-1} \|X_{t_i^{n} \wedge t}\|^{p - 2} \left( \int_{t_i^{n} \wedge t}^{t_i^{n} \wedge t} a_i ds \right)^2 + 2 \sum_{i=0}^{m(n)-1} \|X_{t_i^{n} \wedge t}\|^{p - 2} \left( \int_{t_i^{n} \wedge t}^{t_i^{n} \wedge t} g_i dW_i \right)^2
$$

$$
= I_1^n + I_2^n.
$$

For the first term we have

$$
I_1^n \leq 2C \sup_{s \in [0, t]} \|X_s\|^{p - 2} \times \sup_i \left( \int_{t_i^{n} \wedge t}^{t_i^{n} \wedge t} a_i ds \right) \times \sum_{i=0}^{m(n)-1} \left( \int_{t_i^{n} \wedge t}^{t_i^{n} \wedge t} a_i ds \right)
$$

$$
\leq 2C \sup_{s \in [0, t]} \|X_s\|^{p - 2} \times \sup_i \left( \int_{t_i^{n} \wedge t}^{t_i^{n} \wedge t} a_i ds \right) \times \int_{0}^{t} \|a_i\| ds.
$$
By letting \( n \to \infty \) we have \( \max_{0 \leq i \leq m(n)-1} (t_{i+1}^n - t_i^n) \to 0 \), so

\[
\sup_{0 \leq i \leq m(n)-1} \left\| \int_{t_i^n \wedge t}^{t_{i+1} \wedge t} a_s \, ds \right\| \to 0
\]
as \( n \to \infty \). Therefore,

\[
\lim_{n \to \infty} I_1^n = 0, \; \mathbb{P}\text{-almost surely.}
\]

To estimate \( I_2 \) we use (3.2) and Young’s inequality with \( \varepsilon > 0 \) to infer

\[
\mathbb{E} \liminf_n I_2^n \leq \liminf_n \mathbb{E} I_2^n = \liminf_n \mathbb{E} \sum_{i=0}^{m(n)-1} \|X_{t_i^n \wedge t}\|^{p-2} \left\| \int_{t_i^n \wedge t}^{t_{i+1} \wedge t} g_s \, dW_s \right\|^2
\]

\[
= \liminf_n \sum_{i=0}^{m(n)-1} \mathbb{E} \left( \|X_{t_i^n \wedge t}\|^{p-2} \left( \left\| \int_{t_i^n \wedge t}^{t_{i+1} \wedge t} g_s \, dW_s \right\|^2 \middle| \mathcal{F}_{t_i^n \wedge t} \right) \right)
\]

\[
\leq C \liminf_n \sum_{i=0}^{m(n)-1} \mathbb{E} \left( \|X_{t_i^n \wedge t}\|^{p-2} \left( \left\| g_s \right\|_{\gamma(H, E)}^2 \, ds \right) \right)
\]

\[
\leq C \liminf_n \mathbb{E} \left( \sup_{s \in [0, t]} \|X_s\|^{p-2} \int_0^t \left\| g_s \right\|_{\gamma(H, E)}^2 \, ds \right)
\]

\[
= CE \left( \sup_{s \in [0, t]} \|X_s\|^2 \int_0^t \left\| g_s \right\|_{\gamma(H, E)}^2 \, ds \right)^{\frac{1}{2}}
\]

Next we estimate the second term in (3.7). We have

\[
\sum_{i=0}^{m(n)-1} \left\| X_{t_{i+1} \wedge t} - X_{t_i \wedge t} \right\|^p \leq C \sum_{i=0}^{m(n)-1} \left\| \int_{t_i \wedge t}^{t_{i+1} \wedge t} a_s \, ds \right\|^p + C \sum_{i=0}^{m(n)-1} \left\| \int_{t_i \wedge t}^{t_{i+1} \wedge t} g_s \, dW_s \right\|^p
\]

\[
= I_3^n + I_4^n.
\]

A similar consideration as before yields

\[
\lim_{n \to \infty} I_3^n \leq C \lim_{n \to \infty} \sup_{0 \leq i \leq m(n)-1} \left\| \int_{t_i \wedge t}^{t_{i+1} \wedge t} a_s \, ds \right\|^{p-1} \times \int_0^t \left\| a_s \right\| \, ds = 0.
\]

Moreover, by Burkholder’s inequality (3.3),

\[
\mathbb{E} \liminf_n I_4^n \leq \liminf_n \mathbb{E} I_4^n = C \lim_{n \to \infty} \mathbb{E} \sum_{i=0}^{m(n)-1} \mathbb{E} \left\| \int_{t_i \wedge t}^{t_{i+1} \wedge t} g_s \, dW_s \right\|^p
\]
Stochastic convolutions in 2-smooth Banach spaces

\[ \begin{align*}
&\leq C \liminf_{n} \sum_{i=0}^{m(n)-1} \mathbb{E}\left( \int_{t_{i}^{n} \wedge t}^{t_{i+1}^{n} \wedge t} \|g_{s}\|_{\gamma(H,E)}^{2} \, ds \right)^{\frac{p}{2}} \\
&\leq C \liminf_{n} \mathbb{E}\left( \sum_{i=0}^{m(n)-1} \int_{t_{i}^{n} \wedge t}^{t_{i+1}^{n} \wedge t} \|g_{s}\|_{\gamma(H,E)}^{2} \, ds \right)^{\frac{p}{2}} \\
&= C \mathbb{E}\left( \int_{0}^{t} \|g_{s}\|_{\gamma(H,E)}^{2} \, ds \right)^{\frac{p}{2}}.
\end{align*} \]

Collecting terms, for any \( \varepsilon > 0 \) we obtain the estimate

\[ \mathbb{E}\liminf_{n \to \infty} \sum_{i=0}^{m(n)-1} |R_{\psi}(X_{t_{i}^{n} \wedge t}, X_{t_{i+1}^{n} \wedge t})| \leq C \mathbb{E}\left( \sup_{s \in [0,t]} \|X_{s}\|^{2} \right) + C(\varepsilon^{-1} - \varepsilon + 1) \mathbb{E}\left( \int_{0}^{t} \|g_{s}\|_{\gamma(H,E)}^{2} \, ds \right)^{\frac{p}{2}}. \]

In the proof of Theorem 1.2 we will also need the following simple observation.

**Lemma 3.5.** \( \mathcal{F} \)-Almost surely we have

\[ \liminf_{n \to \infty} \sup_{t \in [0,1]} \sum_{i=0}^{m(n)-1} |R_{\psi}(X_{t_{i}^{n} \wedge t}, X_{t_{i+1}^{n} \wedge t})| \leq \liminf_{n \to \infty} \sum_{i=0}^{m(n)-1} |R_{\psi}(X_{t_{i}^{n}}, X_{t_{i+1}^{n}})|. \]  

(3.8)

**Proof.** Fix \( t \in (0, T] \) and let \( k(n) \) be the unique index such that \( t \in (t_{k(n)}^{n}, t_{k(n)+1}^{n}] \). Then

\[ \begin{align*}
&\sum_{i=0}^{m(n)-1} |R_{\psi}(X_{t_{i}^{n} \wedge t}, X_{t_{i+1}^{n} \wedge t})| \\
&= \sum_{i=0}^{k(n)-1} |R_{\psi}(X_{t_{i}^{n}}, X_{t_{i+1}^{n}})| + |R_{\psi}(X_{t_{k(n)}}, X_{t_{k(n)+1}^{n}})| \\
&\leq \sum_{i=0}^{k(n)-1} |R_{\psi}(X_{t_{i}^{n}}, X_{t_{i+1}^{n}})| + |R_{\psi}(X_{t_{k(n)}}, X_{t_{k(n)+1}^{n}})| \\
&\leq \sum_{i=0}^{m(n)-1} |R_{\psi}(X_{t_{i}^{n}}, X_{t_{i+1}^{n}})| + C \|X_{t_{k(n)}}\|^{p-2} \|X_{t_{k(n)+1}^{n}}\|^{2} + C \|X_{t_{k(n)+1}^{n}}\|^{p} \\
&\leq \sum_{i=0}^{m(n)-1} |R_{\psi}(X_{t_{i}^{n}}, X_{t_{i+1}^{n}})| + C \sup_{s \in [0,T]} \|X_{s}\|^{p-2} \|X_{t_{k(n)+1}^{n}} - X_{t_{k(n)}}\|^{2} + C \|X_{t_{k(n)+1}^{n}}\|^{p}. \end{align*} \]

Now (3.8) follows by taking the limes inferior for \( n \to \infty \) and using path continuity. \( \square \)

4 Proof of Theorem 1.2

We proceed in four steps. In Steps 1 and 2 we establish the estimate in the theorem for \( g \in M^{p}([0, T]; \gamma(H, E)) \) with \( 2 \leq p < \infty \). In order to be able to cover exponents \( 0 < p < 2 \) in Step 3,
we need a stopped version of the inequalities proved in Steps 1 and 2. For reasons of economy of presentations, we therefore build in a stopping time \( \tau \) from the start. In Step 4 we finally consider the case where \( g \in M([0,T];\gamma(H,E)) \).

We shall apply (a special case of) Lenglart's inequality [13, Corollaire II] which states that if \((\xi_t)_{t \in [0,T]}\) and \((a_t)_{t \in [0,T]}\) are continuous non-negative adapted processes, the latter non-decreasing, such that \( \mathbb{E}\xi_t \leq \mathbb{E}a_t \) for all stopping times \( \tau \) with values in \([0,T]\), then for all \( 0 < r < 1 \) one has

\[
\mathbb{E} \sup_{0 \leq t \leq T} \xi_t^r \leq \frac{2 - r}{1 - r} \mathbb{E}a_t^r. \tag{4.1}
\]

Step 1 – Fix \( p \geq 2 \) and suppose first that \( g \in M^p([0,T];\gamma(H,D(A))) \). As is well known (see [16]), under this condition the process \( X_t = \int_0^t e^{(t-s)A}g_s \, dW_s \) is a strong solution to the equation

\[
\frac{dX}{t} = AX \, dt + g_t \, dW_t, \quad t \geq 0; \quad X_0 = 0.
\]

In other words, \( X \) satisfies

\[
X_t = \int_0^t AX_s \, ds + \int_0^t g_s \, dW_s \quad \forall t \in [0,T] \quad \mathbb{P}\text{-almost surely.}
\]

Hence if \( \tau \) is a stopping time with values in \([0,T]\), then by (3.1),

\[
X_{t \wedge \tau} = \int_0^t 1_{(0,\tau]}(s)AX_s \, ds + \int_0^t 1_{(0,\tau]}(s)g_s \, dW_s \quad \forall t \in [0,T], \quad \mathbb{P}\text{-almost surely.}
\]

Let us check next that \( a_t := 1_{(0,\tau]}(t)AX_t \) satisfies the assumptions of Theorem 3.1. Indeed, with \( h_t := 1_{(0,\tau]}(t)Ag_t \) we have, using the contractivity of the semigroup \( S \) and Burkholder's inequality (3.3),

\[
\mathbb{E}\left( \int_0^T \|a_t\| \, dt \right)^p \leq \mathbb{E}\left( \int_0^T \|\int_0^t e^{(t-s)A}h_s \, dW_s\| \, dt \right)^p \\
\leq C T^{p-1}\mathbb{E}\int_0^T \left( \int_0^t \|e^{(t-s)A}h_s\|_{\gamma(H,E)}^2 \, ds \right)^{\frac{p}{2}} \, dt \\
\leq C T^p \mathbb{E}\left( \int_0^T \|h_s\|_{\gamma(H,E)}^2 \, ds \right)^{\frac{p}{2}} < \infty.
\]

Hence we may apply Theorem 3.1 and infer that

\[
\|X_{t \wedge \tau}\|^p = \int_0^t 1_{(0,\tau]}(s)\psi'(X_s)(AX_s) \, ds \\
+ \int_0^t 1_{(0,\tau]}(s)\psi'(X_s)(g_s) \, dW_s + \lim_{n \to \infty} \sum_{i=0}^{m(n)-1} R_{\psi}(X_{t^{i}_n \wedge \tau}, X_{t^{i+1}_n \wedge \tau}) \\
\leq \int_0^t 1_{(0,\tau]}(s)\psi'(X_s)(g_s) \, dW_s + \lim_{n \to \infty} \sum_{i=0}^{m(n)-1} R_{\psi}(X_{t^{i}_n \wedge \tau}, X_{t^{i+1}_n \wedge \tau}).
\]
since $\psi'(x)(Ax) \leq 0$ for all $x \in D(A)$ by the contractivity of $e^{tA}$ (see [3, Lemma 4.2]).

Hence, by Lemma 3.5,
\[
\mathbb{E} \sup_{t \in [0,T]} \|X_{t\wedge \tau}\|^p \\
\leq \mathbb{E} \sup_{t \in [0,T]} \int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(g_s) \, dW_s + \mathbb{E} \sup_{t \in [0,T]} \liminf_{n \to \infty} \sum_{i=0}^{m(n)-1} |R_{\psi}(X_{t_i \wedge \tau}, X_{t_{i+1} \wedge \tau})|
\]
\[
\leq \mathbb{E} \sup_{t \in [0,T]} \int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(g_s) \, dW_s + \mathbb{E} \liminf_{n \to \infty} \sum_{i=0}^{m(n)-1} |R_{\psi}(X_{t_i \wedge \tau}, X_{t_{i+1} \wedge \tau})|
\]
\[
\leq CE \sup_{t \in [0,T]} \int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(g_s) \, dW_s + \varepsilon \mathbb{E} \sup_{t \in [0,T]} \|X_{t\wedge \tau}\|^p + C_\varepsilon \mathbb{E} \left( \int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{p}{2}}.
\]

By Burkholder's inequality (3.3) and the identity $\|\psi'(y)\| = p\|y\|^{p-1},$
\[
\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t 1_{[0,\tau]}(s)\psi'(X_s)(g_s) \, dW_s \right|
\leq CE \left( \int_0^T 1_{[0,\tau]}(s)\|\psi'(X_s)\|^2 \|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{1}{2}}
\leq CE \left( \int_0^T 1_{[0,\tau]}(s)\|X_s\|^{2(p-1)} \|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{1}{2}}
\leq CE \left( \sup_{t \in [0,T]} \|X_{t\wedge \tau}\|^{p-1} \left( \int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{1}{2}} \right)
\leq Cp^p \left( \mathbb{E} \sup_{t \in [0,T]} \|X_{t\wedge \tau}\|^p \right)^{\frac{p-1}{p}} \left( \mathbb{E} \left( \int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}
\leq CE \mathbb{E} \sup_{t \in [0,T]} \|X_{t\wedge \tau}\|^p + C_\varepsilon \mathbb{E} \left( \int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{p}{2}},
\]
where we also used the Hölder's inequality and Young's inequality.

Combining these estimates and taking $\varepsilon > 0$ small enough, we infer that
\[
\mathbb{E} \sup_{t \in [0,T]} \|X_{t\wedge \tau}\|^p \leq CE \left( \int_0^T 1_{[0,\tau]}(s)\|g_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{p}{2}}.
\]

**Step 2** – Now let $g \in M^p([0,T];\gamma(H,E))$ be arbitrary. Set $g^n = n(nI - A)^{-1}g$, $n \geq 1$. These processes satisfy the assumptions of Step 1 and we have $\|g^n\|_{\gamma(H,E)} \leq \|g\|_{\gamma(H,E)}$ pointwise. Define $X^n = \int_0^t e^{(t-s)A}g^n_s \, ds$. From Step 1 we know that for any stopping time $\tau$ in $[0,T]$ we have
\[
\mathbb{E} \sup_{t \in [0,T]} \|X^n_{t\wedge \tau}\|^p \leq CE \left( \int_0^T 1_{[0,\tau]}(s)\|g^n_s\|^2_{\gamma(H,E)} \, ds \right)^{\frac{p}{2}}.
\]
In particular, as \( n, m \to \infty \),

\[
E \sup_{t \in [0,T]} \| X^n_t - X^m_t \|^p \to 0.
\]

In these circumstances there is a process \( \bar{X} \) such that

\[
E \sup_{t \in [0,T]} \| \bar{X}_t \|^p = 0 \quad \text{and} \quad E \sup_{t \in [0,T]} \| \bar{X}_t \wedge \tau \|^p \leq C \int_0^T \| g_s \|_{\gamma(H,E)}^2 ds.
\]

(4.2)

Also, notice that for every \( t \in [0,T] \), we have

\[
E \| X^n_t - X_t \|^p = E \left( \int_0^t e^{(t-s)A} g^n_s ds - \int_0^t e^{(t-s)A} g_s ds \right)^p \leq C \int_0^T \| g^n_s - g_s \|^2_{\gamma(H,E)} ds.
\]

Hence \( X^n_t \to X_t \) in \( L^p(\Omega;E) \). Therefore, \( \bar{X} \) is a modification of \( X \). This concludes the proof for \( p \geq 2 \).

**Step 3** – In this step we extend the result to exponents \( 0 < p < 2 \). First consider the case where \( g \in M^2([0,T];\gamma(H,E)) \). By (4.2), for all stopping times \( \tau \) in \([0,T]\) we have

\[
E \| X^n_\tau \|^2 \leq C \int_0^\tau \| g_s \|^2_{\gamma(H,E)} ds.
\]

It then follows from Lenglart’s inequality (4.1) that for all \( 0 < p < 2 \),

\[
E \sup_{t \in [0,T]} \| X_t \|^p \leq C \sup_{t \in [0,T]} \left( \int_0^\tau \| g_s \|^2_{\gamma(H,E)} ds \right)^{\frac{p}{2}}.
\]

For \( g \in M^p([0,T];\gamma(H,E)) \) the result follows by approximation.

**Step 4** – Finally, the existence of a continuous version for the process \( X \) under the assumption \( g \in M([0,T];\gamma(H,E)) \) follows by a standard localisation argument.

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