Complementary Polynomials From Rodrigues’ Representations For Confluent And Hypergeometric Functions And More

H. J. Weber
Department of Physics
University of Virginia
Charlottesville, VA 22904, U.S.A.

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Abstract

Complementary polynomials of Legendre polynomials are briefly presented, as well as those for the confluent and hypergeometric functions, relativistic Hermite polynomials and corresponding new pre-Laguerre polynomials. The generating functions are all given in closed form and are much simpler than the standard ones. Some are simply polynomials in two variables. New recursions and addition formulas are derived.

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1 Introduction

The novel method introduced in [1] transforms the Rodrigues’ formula of given polynomials step by step thus defining the com-
plementary polynomials recursively. The most useful property of complementary polynomials is that their generating function can always be given in closed form. As a rule, it is simpler than the standard generating function of the polynomials defined by the Rodrigues’ formula.

Hence polynomials satisfying a Rodrigues’ formula are accompanied by their complementary polynomials. The classical polynomials that are important in mathematical physics are such cases: Legendre polynomials and the polynomial components of associated Legendre functions form such pairs; so do Laguerre and associated Laguerre polynomials, etc. Some are discussed below, along with with new and old results (with new simple derivations).

For associated Laguerre polynomials, it leads to so simple a generating function in closed form that the H-atom’s wave functions become truly accessible to undergraduates. This is now implemented in Chapt. 18 on pp. 892-894 of the textbook [2].

For complementary Legendre polynomials, which are the polynomial parts of associated Legendre functions [3], the generating function is simply a polynomial in two variables.

For Romanovsky polynomials their complementary polynomials are again Romanovski polynomials. This furnishes a generating function for them in closed form [4]. These polynomials have recently played a prominent role in physics [5].

Here we also consider generalized Rodrigues’ formulas without the frame-work of a hypergeometric-type ODE and Pearson’s ODE to make its operator self-adjoint. Yet, they generate polynomials along with their complementary ones that have a closed-form generating function.

The normalizations of complementary Legendre polynomials are revisited in Sect. 2. The generating function and some applications of the confluent hypergeometric ODE are dealt with in Sect. 3. The hypergeometric ODE and its complementary polynomials are the subject of Sect. 4. Relativistic harmonic oscillator polynomials are treated in Sect. 5. Similar pre-Laguerre polynomials are introduced in Sect. 6.
2 Complementary Polynomials For Legendre Polynomials

Definition 2.1 Complementary polynomials $\mathcal{P}_\nu(x, l)$ are defined recursively by Rodrigues’ formula \[3\]

\[(-1)^l l! P_l(x) = \frac{d^{l-\nu}}{dx^{l-\nu}} \left((1 - x^2)^{l-\nu}\mathcal{P}_\nu(x, l)\right), \quad \nu = 0, 1, 2, \ldots, l,\]

where $P_l(x)$ are the Legendre polynomials.

Theorem 2.2 The generating function has the closed form

\[\mathcal{P}(y, x, l) = (1 - y(1 + x))^l(1 + y(1 - x))^l.\]  \hspace{1cm} (1)

Proof. This is Eq. (12) in Sect. 3 of \[3\]. \bullet

Thus, $\mathcal{P}(y, x, l)$ is simply a product of two forms each linear in $x, y$ and raised to the $l$th power. In terms of the angular variable $x = \cos \theta$, common in physics applications,

\[\mathcal{P}(y, \cos \theta, l) = (1 - 2y \cos^2 \theta)^l(1 - 2y \sin^2 \theta)^l.\] \hspace{1cm} (2)

Binomially expanding Eq. (2) yields Eq. (14) in \[3\] in trigonometric form.

Corollary 2.3 The basic composition laws

\[[(1 - xy)^2 - y^2] \mathcal{P}(y, x, l) = \mathcal{P}(y, x, l + 1),\]

\[\mathcal{P}_\nu(x, l + 1) = \mathcal{P}_\nu(x, l) - 2x\nu \mathcal{P}_{\nu-1}(x, l) - (1 - x^2)\nu(\nu - 1)\mathcal{P}_{\nu-2}(x, l)\]

hold.

Proof. The generating function identity follows at once from Eq. (1). When expressed in terms of complementary polynomials this gives the recursion relation. \bullet

For many other composition laws we refer to \[3\].

Partial derivatives of the generating function, such as

\[\frac{\partial \mathcal{P}(y, x, l)}{\partial y} = -2l[x + y(1 - x^2)]\mathcal{P}(y, x, l - 1),\]

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and
\[ \frac{\partial \mathcal{P}(y, x, l)}{\partial x} = -2ly(1 - xy)\mathcal{P}(y, x, l - 1), \]
yield the identity
\[ y(1 - xy) \frac{\partial \mathcal{P}(y, x, l)}{\partial y} = [x + y(1 - x^2)] \frac{\partial \mathcal{P}(y, x, l)}{\partial x}. \quad (3) \]

**Corollary 2.4** The following three-term recursion is valid
\[ \mathcal{P}_\nu - 2(\nu - 1 - l)x\mathcal{P}_{\nu-1} = (\nu - 1)(\nu - 2l - 2)(1 - x^2)\mathcal{P}_{\nu-2}. \]

*Proof.* Equation (3), when expressed in terms of complementary polynomials, gives the recursion
\[ \nu \mathcal{P}_\nu - \nu(\nu - 1)x\mathcal{P}_{\nu-1} = x\mathcal{P}'_\nu + \nu(1 - x^2)\mathcal{P}'_{\nu-1}. \]
Using the basic differential recursion [1] Cor. 4.2
\[ \mathcal{P}'_\nu(x, l) = -\nu(2l - \nu + 1)\mathcal{P}_{\nu-1}(x, l) \quad (4) \]
allows us to eliminate the derivatives to obtain the 3-term recursion.

From a comparison of the ODE satisfied by the complementary polynomials with the ODE of the polynomial components \( \mathcal{P}_{l-m}(x) \) of associated Legendre functions, one finds [3]
\[ \mathcal{P}_{l-m}(x, l) = N_l^m \mathcal{P}_l^m(x); \ m = 0, 1, \ldots, l. \]

For even \( l - m \) the normalizations \( N_l^m \) (see Eq. (43) of [3] and Eq. (7) below) follow from the special values
\[ \mathcal{P}_{2\nu}(0, l) = (-1)^\nu(2\nu)! \binom{l}{\nu} \]
implied by the generating function \( \mathcal{P}(y, 0, l) = (1 - y^2)^l \) at \( x = 0 \).

For odd \( l - m \) we now derive a new formula for the normalizations.
Comparing the differential recursion \[1\] (Cor. 4.2) (see Eq. (4)
above) with the well known
\[ P_m l = (-1)^m \frac{d^m P_l(x)}{dx^m} = -\frac{d}{dx} P_{m-1}^l(x), \quad m = 0, 1, \ldots, l \quad (5) \]
yields, when expressed in associated Legendre polynomials,
\[ N_m^m P_m^m(x) = -(l - m)(l + m + 1) N_{m+1}^m P_{m+1}^m(x). \quad (6) \]
Comparing Eqs. (5) and (6), we obtain Eq. (8) for even \( l - m \):

**Theorem 2.5** The normalizations of complementary Legendre polynomials are given by
\[ N_l^m = (-1)^m \left( \frac{l}{(l-m)/2} \right) \frac{(l-m)!!(l-m)!}{(l+m-1)!!}, \quad l - m \text{ even} \quad (7) \]
\[ N_{l+1}^m = \frac{N_l^m}{(l-m)(l+m+1)}, \quad -l < m < l. \quad (8) \]
So, for \( N_{l+1}^m \) the index \( l - m - 1 \) is odd, and this simple recursion renders unnecessary the lengthy developments in \[3\] Sect. 4 after Example 4.1.

### 3 Confluent Hypergeometric ODE

We write the standard regular solution of the confluent hypergeo-
metric ODE
\[ xy'' + (c - x)y' - ay = 0 \quad (9) \]
as \( M(a, c, x) \), as usual. Then, in the notations of \[1\],
\[ \sigma(x) = x, \quad \tau(x) = c - x, \quad \sigma' = 1, \quad \tau - \sigma' = c - 1 - x. \]
Thus, Pearson’s ODE for the weight function \( w(x) \)
\[ xw' = (c - 1 - x)w(x) \]
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is solved by \( w(x) = x^{c-1}e^{-x}, x \in [0, \infty) \).

**Definition 3.1** The Rodrigues’ formula for the polynomial solutions \( R_l(x) \) of Eq. (2) is [1] (Eq. (3))

\[
R_l(x) = x^{1-c}e^{x} \frac{d^l}{dx^l}(x^{l+c-1}e^{-x}), \quad l = 0, 1, \ldots
\]

and for the complementary polynomials [1] (Eq. (5)) it is

\[
P_\nu(x, l) = x^{1-c+\nu-l}e^{x} \frac{d^\nu}{dx^\nu}(x^{c-1+l}e^{-x}), \quad l = 0, 1, \ldots
\]

so that \( R_l(x) = P_l(x, l) \). The \( R_l \) also depend on the parameters of the ODE, but these are suppressed for simplicity.

**Lemma 3.2** The ODE for complementary polynomials is

\[
x \frac{d^2}{dx^2} P_\nu + (l - \nu + c - x) \frac{d}{dx} P_\nu = -\nu P_\nu(x, l), \quad l = 0, 1, \ldots
\]

so that

\[
P_\nu(x, l) = P_\nu(0, l) M(-\nu, l - \nu + c, x),
\]

\[
R_l(x) = P_l(0, l) M(-l, c, x), \quad P_\nu(0, l) = \nu! \binom{c-1+l}{\nu}.
\]

**Proof.** The ODE (11) follows from [1] Theor. 5.1. Since it is the confluent hypergeometric ODE again, this implies the other relations.

Thus, all polynomial solutions of the confluent hypergeometric ODE are summed in the following 1-parameter generating function in closed form, which is quite simple given the complexity of the regular solution of the confluent hypergeometric ODE.

**Theorem 3.3** The generating function of the complementary polynomials takes the form

\[
\mathcal{P}(y, x, l) = \frac{[x(1+y)]^{c-1}e^{-x(1+y)}(1+y)^{l}}{x^{c-1}e^{-x}}((1+y)^c-1+l e^{-xy}.
\]

\[
\mathcal{P}_\nu(x, l) = \sum_{\mu=0}^{\nu} \binom{c-1+l}{\nu-\mu} \frac{\nu!}{\mu!} (-x)^\mu.
\]
Proof. The generating function follows from [1] Theor. 3.2 by substituting Eq. (10) in \( P(y, x, l) = \sum_\nu y^\nu P_\nu(x, l) \) and recognizing the sum as the Taylor expansion of the closed form. Binomially expanding Eq. (12) yields the polynomial formula.

**Corollary 3.4** The partial differential equation (PDE) of the generating function

\[
\frac{\partial P}{\partial x} = -yP(y, x, l)
\]

is equivalent to the basic differential recursion

\[
P_\nu'(x, l) = -\nu P_\nu - 1(x, l),
\]

and

\[
(1 + y)\frac{\partial P}{\partial y} = [c - 1 + l - x(1 + y)]P(y, x, l)
\]

to the three-term recursion

\[
(c - 1 + l - \nu - x)P_\nu(x, l) = \nu xP_\nu - 1(x, l) + P_{\nu + 1}(x, l).
\]

Proof. Taking partial derivatives we obtain the PDEs (13), (14) and expanding them in complementary polynomials gives the recursions, of which the former also follows from Eq. (32) Cor. 4.2 [1].

**Theorem 3.5** The generating function obeys elegant composition laws

\[
P(y, x, l_1)P(y, x, l_2) = P(y, x, l_1 + l_2)P(y, x, 0)
\]

\[
= P(y, x, l_1 + l_2 - l)P(y, x, l), \ 0 \leq l \leq l_1 + l_2.
\]
Proof. The latter follows by expanding Eq. (15) binomially. •

**Theorem 3.6** (Addition Law)

\[
\sum_{\lambda=0}^{\nu} \binom{c - 1 + l}{\nu - \lambda} \frac{\nu!}{\lambda!} P_{\lambda}(x_1 + x_2, l) = \sum_{\nu_1=0}^{\nu} \binom{\nu}{\nu_1} P_{\nu_1}(x_1, l) P_{\nu - \nu_1}(x_2, l)
\]

\[
(1 + y)^{c-1+l} P(y, x_1 + x_2, l) = P(y, x_1, l) P(y, x_2, l).
\]

Proof. The polynomial addition law follows similarly from the addition law of the generating function. •

The sum on the lhs of the addition law may be inverted by expanding similarly

\[
P(y, x_1 + x_2, l) = (1 + y)^{1-c-l} P(y, x_1, l) P(y, x_2, l)
\]

yielding

\[
P_{\nu}(x_1 + x_2, l) = \sum_{\nu_1, \nu_2 \leq \nu_1 + \nu_2 \leq \nu} \binom{1 - c - l}{\nu - \nu_1 - \nu_2} \frac{\nu!}{\nu_1! \nu_2!} P_{\nu_1}(x_1, l) P_{\nu_2}(x_2, l).
\]

**Theorem 3.7** The full addition law

\[
P(y, x_1 + x_2, l_1 + l_2) = (1 + y)^{1-c-l} P(y, x_1, l_1) P(y, x_2, l_2)
\]

implies similarly

\[
P_{\nu}(x_1 + x_2, l_1 + l_2) = \sum_{\nu_1, \nu_2 \geq 0, \nu_1 + \nu_2 \leq \nu} \binom{1 - c}{\nu - \nu_1 - \nu_2} \frac{\nu!}{\nu_1! \nu_2!} P_{\nu_1}(x_1, l_1) P_{\nu_2}(x_2, l_2).
\]

Laguerre and Hermite polynomials are the usual special cases of Eq. (9) including their complementary polynomials and their composition and addition laws [1],[2].
4 Hypergeometric ODE

For the hypergeometric ODE

\[ x(1 - x)y'' + [c - (a + b + 1)x]y' - aby(x) = 0, \quad (16) \]

in the notations of [1],

\[
\sigma(x) = x(1 - x), \quad \tau(x) = c - (a + b + 1)x, \\
\sigma' = 1 - 2x, \quad \tau - \sigma' = c - 1 - (a + b - 1)x
\]

so that Pearson’s ODE for the weight function \( w(x) \)

\[ x(1 - x)w' = [c - 1 - (a + b - 1)x]w(x) \]

has the solution

\[ w(x) = x^{c-1}(1 - x)^{a+b-c}; \quad x \in [0, 1]. \]

In the following we denote the standard regular solution of the hypergeometric ODE \((16)\) simply by \( F(a, b; c; x) \) (instead of \( 2F_1 \)).

**Definition 4.1** The Rodrigues’ formula for polynomial solutions \( R_l(x) \) of the hypergeometric ODE \((16)\) is [1] Eq. (3)

\[ R_l(x) = x^{1-c}(1 - x)^{c-a-b} \frac{d^l}{dx^l} \left( x^{l+c-1}(1 - x)^{l+a+b-c} \right), \]

with

\[ R_l(x) = R_l(0)F(-l, b; c; x), \]

and for the complementary polynomials it is [1] Eq. (5)

\[ P_\nu(x, l) = x^{\nu+1-c-l}(1 - x)^{\nu+l+c-a-b} \frac{d^\nu}{dx^\nu} \left( x^{\nu-1+l}(1 - x)^{l+a+b-c} \right), \]

so that \( P_l(x, l) = R_l(x) \). The \( R_l \) also depend on the parameters of the hypergeometric ODE, but these are suppressed for simplicity. (The \( R_l \) here are not to be confused with those in Sect. 3.)
Theorem 4.2 The generating function of the complementary polynomials is simply a polynomial in two variables; it has the closed form

\[ P(y, x, l) = [1 + y(1 - x)]^{l+c-1}(1 - xy)^{l+a+b-c} \quad (17) \]

with

\[ P_\nu(x, l) = \nu! \sum_{\lambda=0}^{\nu} \binom{l + a + b - c}{\lambda} \binom{l + c - 1}{\nu - \lambda} (1 - x)^{\nu-\lambda} (-x)^\lambda. \]

Proof. Eq. (17) follows from [1] (Theor.3.2). Binomially expanding the generating function yields the polynomial expression. •

Note that the generating function (17) is just a product of powers of two bilinear forms in two variables and amazingly simple, given the complexity of the regular solution of the hypergeometric ODE. It is certainly much simpler than the standard one for Jacobi polynomials which, in essence, are the complementary polynomials. The latter are the main ingredient of the rotation matrix elements \( d_{nmn}'(\theta) \) commonly used by physicists [6].

Taking partial derivatives leads to

\[ (1 - xy)[1 + y(1 - x)] \frac{\partial P}{\partial y} = [(l + c - 1)(1 - x)(1 - xy) \] 
\[ -x(l + a + b - c)(1 + y)(1 - x)]P(y, x, l) \quad (18) \]

and

\[ (1 - xy)[1 + y(1 - x)] \frac{\partial P}{\partial x} = -y[(l + c - 1)(1 - xy) \] 
\[ +(l + a + b - c)(1 + y)(1 - x)]P(y, x, l). \quad (19) \]

Corollary 4.3 The following recursions hold for the complementary polynomials

\[ P_{\nu+1} = [l + c - \nu - 1 - x(2l - 2\nu + a + b - 1)]P_\nu \] 
\[ -(2l + a + b - 1)x(1 - x)\nu P_{\nu-1} + \nu(\nu - 1)x(1 - x)P_{\nu-2}, \quad (20) \]
\[(\nu + 2l - 2)P_\nu + \nu[(\nu - a - b - 2)(1 - 2x) + l + a + b - c \\
-x(2l + a + b - 1)]P_{\nu - 1} - \nu(\nu - 1)(\nu - a - b - 3)x(1 - x) \\
\cdot P_{\nu - 2} = 0. \] (21)

**Proof.** Expanding the PDE (18) in terms of complementary polynomials gives the recursion (20), while the PDE (19) yields

\[P'_\nu + \nu(1 - 2x)P'_{\nu - 1} - \nu(\nu - 1)x(1 - x)P'_{\nu - 2} \]
\[= -(2l + a + b - 1)\nu P_{\nu - 1} - [l + a + b - c - x(2l + a + b - 1)] \\
\cdot \nu(\nu - 1)P_{\nu - 2}. \]

Using the basic recursive ODE (23) below allows eliminating the derivatives, thus yielding the recursion (21), which reflects the complexity of the regular solution of the hypergeometric ODE (16).

**Theorem 4.4** The following composition laws are valid

\[P(y, x, l_1)P(y, x, l_2) = P(y, x, l_1 + l_2)P(y, x, 0) \]
\[= P(y, x, l_1 + l_2 - l)P(y, x, l), \ 0 \leq l \leq l_1 + l_2; \]

\[
\sum_{\nu_1=0}^{\nu} \binom{\nu}{\nu_1} P_{\nu_1}(x, l_1)P_{\nu-\nu_1}(x, l_2) \\
= \sum_{\nu_1=0}^{\nu} \binom{\nu}{\nu_1} P_{\nu_1}(x, l_1 + l_2 - l)P_{\nu-\nu_1}(x, l).
\]

**Proof.** Binomially expanding the generating function identities yields the polynomial composition laws.

**Theorem 4.5** The \(P_\nu(x, l)\) obey the hypergeometric ODE (16) with parameters

\[(a, b, c) \rightarrow (A = -\nu, B = 2l - \nu + a + b, C = l - \nu + c) \] (22)

so that

\[P_\nu(x, l) = P_\nu(0, l)F(-\nu, B; C; x), \ P_\nu(0, l) = \nu! \left(\binom{l + C - 1}{\nu}\right). \]
Proof. From [1] (Theor. 5.1) the $P_\nu(x, l)$ obey the ODE

$$x(1-x)\frac{d^2 P_\nu(x, l)}{dx^2} + [(l - \nu)(1 - 2x) + c - (a + b + 1)x]$$

$$\cdot \frac{dP_\nu(x, l)}{dx} = -\nu(2l - \nu + a + b)P_\nu(x, l),$$

where the rhs determines $A$ and $B$ in Eq. (22). Then one verifies that this ODE is the hypergeometric ODE by comparing the coefficient of $\frac{dP_\nu(x, l)}{dx}$ of this ODE with Eq. (16) after making the substitutions (22).

The generating function (Theor. 4.2) applies to the polynomial solutions of Eq. (16), albeit in conjunction with the parameters of Eq. (22). The complementary polynomials also satisfy the basic recursive ODE [1] (Cor. 4.2)

$$\frac{dP_\nu}{dx} = \nu(\nu - a - b - 2)P_{\nu-1}(x, l).$$

Displaying the parameters, the generating function obeys the symmetry

$$P(-y, 1-x, l, c-1, a+b-c) = P(y, x, l, a+b-c, c-1).$$

5 Relativistic Hermite Polynomials

Definition 5.1 Replacing $e^{-x^2} \rightarrow (1 + \frac{x^2}{N})^{-N}$, relativistic Hermite polynomials are defined by the Rodrigues’ formula [7]

$$H_N^x(x) = (-1)^n \left(1 + \frac{x^2}{N}\right)^{N+n} \frac{d^n}{dx^n} \left(1 + \frac{x^2}{N}\right)^{-N},$$

(24)

which is beyond the framework of [1].

Definition 5.2 The complementary polynomials are defined recursively by

$$H_n^N(x) = (-1)^{n-\nu} \left(1 + \frac{x^2}{N}\right)^{N+n} \frac{d^{n-\nu}}{dx^{n-\nu}} \frac{P_\nu(x, N)}{(1 + \frac{x^2}{N})^{N+\nu}}.$$
From $\nu = n$ in Def. 5.2 it follows that $P_\nu(x, N) = H_\nu^N(x)$, just as Hermite polynomials coincide with their complementary polynomials [1]. Substituting this in Eq. (25) yields the new relations

$$H_\nu^N(x) = (-1)^{n-\nu} \left(1 + \frac{x^2}{N}\right)^{N+n} \frac{d^{n-\nu}}{dx^{n-\nu}} \left(\frac{H_\nu^N(x)}{(1 + \frac{x^2}{N})^{N+\nu}}\right)$$  \hspace{1cm} (26)

For $\nu = n - 1$ this gives the useful differential recursion

$$H_n^N(x) - \frac{2x}{N}(N + n - 1)H_{n-1}^N(x) + \left(1 + \frac{x^2}{N}\right) \frac{dH_{n-1}^N}{dx} = 0.$$  

**Theorem 5.3** The generating function $H^N(y, x)$ is given in closed form by

$$H^N(y, x) = \sum_{\nu=0}^{\infty} \frac{y^\nu}{\nu!} H_\nu^N(x) = \left[\left(1 - \frac{xy}{N}\right)^2 + \frac{y^2}{N}\right]^{-N}.$$  

**Proof.** This known result follows at once and in a new way from using the Rodrigues’ formula (24) for complementary polynomials in Theor. 3.2 [1], then recognizing the sum as the Taylor expansion of the closed form:

$$\sum_{\nu=0}^{\infty} \frac{y^\nu}{\nu!} H_\nu^N(x) = \left(1 + \frac{x^2}{N}\right)^N \left(1 + \frac{X^2}{N}\right)^{-N}, \quad X = x - y \left(1 + \frac{x^2}{N}\right).$$  

### 6 Pre-Laguerre Polynomials

**Definition 6.1** Replacing the weight function $e^{-x} \to (1 + \frac{x}{N})^{-N}$ we introduce the pre-Laguerre polynomials by the Rodrigues’ formula

$$L_l^N(x) = (1 + \frac{x}{N})^{N+l} \frac{d^l}{dx^l} \frac{x^l}{(1 + \frac{x}{N})^N},$$  \hspace{1cm} (27)

which is beyond the framework of [1].

**Proposition 6.2** The complementary polynomials $P_\nu^N(x, l)$ are defined recursively by

$$L_l^N(x) = (1 + \frac{x}{N})^{N+l} \frac{d^{l-\nu}}{dx^{l-\nu}} \frac{x^{l-\nu}P_\nu^N(x, l)}{(1 + \frac{x}{N})^{N+\nu}}, \quad \nu = 0, 1, \ldots, l.$$  \hspace{1cm} (28)
Proof. This follows by induction on $\nu$ provided they obey the linear differential recursion
\[
P_{\nu+1}(x, l) = \left[(1 + \frac{x}{N})(l - \nu) - (1 + \frac{\nu}{N})x\right]P_{\nu}(x, l) \\
+ \ x(1 + \frac{x}{N}) \frac{d}{dx}P_{\nu}(x, l)
\] (29)
upon carrying out the innermost differentiation in Eq. (28). •

For $\nu = 0$ we get $P_0^N(x, l) = 1$, for $\nu = 1$ : $P_1^N(x, l) = l(1 + x/N) - x$; for $\nu = l : P_l^N(x) = L_l^N(x, l)$.

Proposition 6.3 The complementary polynomials obey their own Rodrigues’ formula

\[
P_{\nu}(x, l) = x^{\nu-l} \left(1 + \frac{x}{N}\right)^{N+\nu} \frac{d^{\nu}}{dx^{\nu}} \frac{x^l}{(1 + \frac{x}{N})^N}.
\] (30)

Proof. This follows upon plugging Eq. (30) into Eq. (28) of Prop. 6.2, the recursive definition of complementary polynomials, implying the Rodrigues formula for the $L_l^N(x)$, Def. 6.1, and vice versa:

\[
(1 + \frac{x}{N})^{N+l} \frac{d^{l-\nu}}{dx^{l-\nu}} \frac{d^{\nu}}{dx^{\nu}} \frac{x^l}{(1 + \frac{x}{N})^N} = L_l^N(x).
\]

More generally we obtain similarly

Corollary 6.4

\[
P_{\nu}(x, l) = x^{\nu-l}(1 + \frac{x}{N})^{N+\nu} \frac{d^{\nu-\mu}}{dx^{\nu-\mu}} \left[\frac{x^{l-\mu}P_{\mu}(x, l)}{(1 + \frac{x}{N})^{N+\mu}}\right].
\]

Theorem 6.5 The generating function of the complementary polynomials is given in the simple closed form by

\[
P_N(y, x, l) = \frac{[1 + y(1 + \frac{x}{N})]^l}{(1 + \frac{x}{N})^N}.
\] (31)
Proof. This follows upon applying the proof of Theor. 3.2 [1] using Prop. 6.3:

\[
P_N(y, x, l) = \sum_{\nu=0}^{\infty} y^\nu \frac{P_N^\nu(x, l)}{\nu!} \frac{\nu!}{x^l} \frac{dx^\nu}{(1 + \frac{x}{N})^N}
\]

\[
= (1 + \frac{x}{N})^N \sum_{\nu=0}^{\infty} \frac{[xy(1 + \frac{x}{N})]^\nu}{\nu!} \frac{dx^\nu}{(1 + \frac{x}{N})^N}
\]

recognizing the sum as a Taylor expansion, then canceling \(x^l\) and thereby transforming it into Eq. (31).

**Proposition 6.6** In general, complementary polynomials are given by

\[
P_N^\nu(x, l) = \nu! \sum_{\mu=0}^{l} \binom{l}{\mu} \left( \frac{1}{\nu - \mu} \right) \left( \frac{x}{N} \right)^{\nu - \mu} (1 + \frac{x}{N})^{\mu}.
\]

Proof. This follows from expanding numerator and denominator powers in Eq. (28) binomially

\[
\sum_{\nu=0}^{\infty} \frac{1}{\nu!} P_N^\nu(x, l) = \left[ \sum_{\mu=0}^{l} \binom{l}{\mu} y^\mu(1 + \frac{x}{N})^{\mu} \right] \left[ \sum_{\lambda=0}^{\infty} \binom{-N}{\lambda} \left( \frac{xy}{N} \right)^{\lambda} \right]
\]

and collecting terms with \(\nu = \lambda + \mu\).

**Proposition 6.7** A recursive formula for the complementary polynomials is

\[
\sum_{\lambda=0}^{\nu} \binom{N}{\nu - \lambda} \left( \frac{x}{N} \right)^{\nu - \lambda} \frac{P_N^\nu(x, l)}{\lambda!} = \left\{ \begin{array}{ll}
\binom{l}{\nu} (1 + \frac{x}{N})^{\nu}, & \nu \leq l; \\
0, & \nu > l.
\end{array} \right.
\]

Proof. This follows by expanding binomially the numerator and denominator of Eq. (31) in the form

\[
(1 + \frac{x y}{N})^N P_N(y, x, l) = [1 + y(1 + \frac{x}{N})]^l
\]
of Eq. (31). •

Taking derivatives and then expanding binomially the form
\[
(1 + \frac{xy}{N})^N \frac{\partial P_N(y, x, l)}{\partial x} = \frac{ly}{N} (1 + \frac{x}{N}) \left[1 + y \left(1 + \frac{x}{N}\right)\right]^{l-1} - y \left[1 + y \left(1 + \frac{x}{N}\right)\right] \left[1 + y \left(1 + \frac{x}{N}\right)\right]^{l-1}
\]

of Eq. (31) yields a general formula and recursions for the derivatives of the complementary polynomials.

Upon applying Pearson’s ODE
\[
(\sigma w)' = \tau w, \ \sigma = x, \ w = (1 + \frac{x}{N})^{-N}
\]
gives \(\tau(x) = 1 - \frac{x}{1 + \frac{x}{N}}\). For \(N \to \infty\) the standard Laguerre ODE form results, \(\tau = 1 - x\). Thus, ODEs for \(L_i, P_\nu\) will take the forms
\[
x \frac{d^2}{dx^2} L_i^N + \left(1 - \frac{x}{1 + \frac{x}{N}}\right) \frac{dL_i^N}{dx} + V_i^N(x) L_i^N = \lambda_i L_i^N
\]
\[
x \frac{d^2}{dx^2} P_\nu^N + \left(1 - \frac{x}{1 + \frac{x}{N}}\right) \frac{dP_\nu^N}{dx} + \nu_i^N(x) P_\nu^N = \Lambda_i P_\nu^N.
\]

If such ODEs are to be the radial Schrödinger equation of a quantum mechanical problem, then the derivative term \((x - \frac{x}{1 + \frac{x}{N}}) \frac{d}{dx}\) has to be considered as part of the potential making it non-local.

**Corollary 6.8 (Recursion)** The PDE
\[
\left(1 + \frac{xy}{N}\right) \left[1 + y \left(1 + \frac{x}{N}\right)\right] \frac{\partial P_N}{\partial y} = \{l(1 + \frac{x}{N})(1 + \frac{xy}{N} - x[1 + y \left(1 + \frac{x}{N}\right)])\} P_N(y, x, l) \quad (33)
\]
is equivalent to the recursion
\[
P_{\nu+1}^N + [\nu - l + x(\frac{2\nu - l}{N} + 1)] P_\nu^N \\
+ \nu(\frac{\nu - l - 1}{N} + 1) x(1 + \frac{x}{N}) P_{\nu-1}^N = 0. \quad (34)
\]
Proof. Taking the partial derivative of $\mathcal{P}^N$ yields the PDE (33), and expanding it in complementary polynomials gives the recursion (34).

The elegant angular momentum addition identity

$$\mathcal{P}^N(y, x, l_1 + l_2) = (1 + \frac{xy}{N})^N \mathcal{P}^N(y, x, l_1) \mathcal{P}^N(y, x, l_2)$$

and in the form

$$\mathcal{P}^N(y, x, l_1) \mathcal{P}^N(y, x, l_2) = (1 + \frac{xy}{N})^{-N} \mathcal{P}^N(y, x, l_1 + l_2)$$

yield, upon expanding the generating functions into their complementary polynomials:

Theorem 6.9 (Composition Laws)

$$\mathcal{P}^N_\nu(x, l_1 + l_2) = \sum_{\nu_1=0}^{\nu} \frac{\nu!}{\nu_1!} \left( \frac{x}{N} \right)^{\nu - \nu_1} \mathcal{P}^N_{\nu_1}(x, l_1) \mathcal{P}^N_{\nu - \nu_1}(x, l_2)$$

and

$$\sum_{\nu_1=0}^{\nu} \binom{\nu}{\nu_1} \mathcal{P}^N_\nu(x, l_1) \mathcal{P}^N_{\nu - \nu_1}(x, l_2) = \sum_{\lambda=0}^{\nu} \frac{\nu!}{\lambda!} \left( \frac{-N}{\nu - \lambda} \right) \left( \frac{x}{N} \right)^{\nu - \lambda} \mathcal{P}^N_\lambda(x, l_1 + l_2).$$

Finally we get to the special values. For $x = 0$ in Eq. (31) $\mathcal{P}^N(y, 0, l) = (1 + y)^l$ of Theor. 6.4 we find

$$\mathcal{P}^N_\nu(0, l) = \begin{cases} \nu! \binom{l}{\nu}, & \nu \leq l; \\ 0, & \nu > l; \end{cases}$$

while for $x = -N$ we find from $\mathcal{P}^N(y, -N, l) = (1 - y)^{-N}$

$$\mathcal{P}^N_\nu(-N, l) = \nu!(-1)^{\nu} \left( \frac{-N}{\nu} \right).$$
7 More General Framework

Summarizing and generalizing Sects. 5 and 6, for given polynomials \( \sigma(x), w(x) \) the Rodrigues’ relation

\[
S_l(x) = w(x)^{l+N} \frac{d^l \sigma(x)^l}{dx^l w(x)^N}
\]  

(35)

defines polynomials \( S_l \) with \( S_0 = 1 \). Complementary polynomials are defined recursively by

\[
S_l(x) = w(x)^{l+N} \frac{d^{l-\nu} \sigma^{l-\nu} P_\nu(x, l)}{dx^{l-\nu} w^{N+\nu}},
\]

(36)

which satisfy their Rodrigues’ relation

\[
P_\nu(x, l) = \sigma^{l-\nu} w^{N+\nu} \frac{d^\nu \sigma^l}{dx^\nu w^N}.
\]

(37)

Substituting Eq. (37) in Eq. (36) reproduces Eq. (35). Using Eq. (37) in the generating function sum yields its closed form

\[
P(y, x, l) = \left( \frac{w(x)}{w(x + y \sigma(x) w(x))} \right)^N \left( \frac{\sigma(x + y \sigma(x) w(x))}{\sigma(x)} \right)^l.
\]

Generalizing similarly Sect. 4 on the hypergeometric ODE, the Rodrigues’ formula

\[
S_l(x) = \frac{1}{w^a(x) \sigma^b(x)} \frac{d^l}{dx^l} (w^{a+l} \sigma^{b+l})
\]

(38)

defines polynomials \( S_l \) and their complementary set

\[
P_\nu(x, l) = \frac{1}{w^{a+l} \sigma^{b+l}} \frac{d^\nu}{dx^\nu} (w^a \sigma^b).
\]

For \( w(x) = x, \sigma(x) = 1 - x \) and appropriate parameters \( a, b \) these formulas reduce to those of the hypergeometric ODE. The
$l$-dependence $(w \sigma)^l$ in Eq. (38) can be removed by differentiating

$$w^a \sigma^b \lambda \text{-fold}$$

yielding an index translation formula, which may be resummed as a translation formula

$$w^a(x + h) \sigma^b(x + h) S_l(x + h) = \sum_{\lambda=0}^{\infty} \frac{h^\lambda}{\lambda!} \frac{d^\lambda}{dx^\lambda} (w^a \sigma^b S_l(x)) = w^a \sigma^b \sum_{\lambda=0}^{\infty} S_{l+\lambda}(x).$$

The generating function $S(y, x, l)$ for the complementary polynomials of the $S_l$ is

$$S(y, x, l) = \left( \frac{w(x + y \sigma(x) w(x))}{w(x)} \right)^{a+l} \left( \frac{\sigma(x + y \sigma(x) w(x))}{\sigma(x)} \right)^{b+l}.$$

The complementary polynomials are defined as

$$S_{\nu}(x, l) = \frac{1}{w^{a-\nu} \sigma^{b-\nu}} \frac{d^\nu}{dx^\nu} (w^{a+l} \sigma^{b+l}).$$

For Legendre polynomials $w(x) = 1 + x, \sigma(x) = 1 - x, a = b = 0$

and

$$P_l(x + h) = \sum_{\lambda=0}^{\infty} (-2h)^\lambda \left( \frac{l + \lambda}{l} \right) P_{l+\lambda}(x).$$

Starting from a pair of polynomials this generalized framework provides sets polynomials and their complementary sets that are summed up in closed form.

8 Summary And Conclusions

The recursion relation for the normalizations of the complementary Legendre polynomials completes their theory [3].

For the confluent and hypergeometric ODEs the result of central importance is that their complementary polynomials obey the same ODE, albeit with more general parameters, so that their generating functions in their simple closed form contain them all.
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