To Smooth or Not? When Label Smoothing Meets Noisy Labels

A PREPRINT

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ABSTRACT

Label smoothing (LS) is an arising learning paradigm that uses the positively weighted average of both the hard training labels and uniformly distributed soft labels. It was shown that LS serves as a regularizer for training data with hard labels and therefore improves the generalization of the model. Later it was reported LS even helps with improving robustness when learning with noisy labels. However, we observed that the advantage of LS vanishes when we operate in a high label noise regime. Intuitively speaking, this is due to the increased entropy of $P(\text{noisy label}|X)$ when the noise rate is high, in which case, further applying LS tends to “oversmooth” the estimated posterior. We proceeded to discover that several learning-with-noisy-labels solutions in the literature instead relate more closely to negative/not label smoothing (NLS), which acts counter to LS and defines as using a negative weight to combine the hard and soft labels! We provide understandings for the properties of LS and NLS when learning with noisy labels. Among other established properties, we theoretically show NLS is considered more beneficial when the label noise rates are high. We provide extensive experimental results on multiple benchmarks to support our findings too.

1 Introduction

Label smoothing (LS) [Szegedy et al., 2016] is an arising learning paradigm that uses positively weighted average of both the hard training labels and the uniformly distributed soft label:

$$y_{LS}^{r} = (1 - r) \cdot y + \frac{r}{K} \cdot 1,$$

(1)

where we denote the one-hot vector form of hard label and an all one vector as $y, 1$ respectively. $K$ is the number of label classes, and $r$ is the smooth rate in the range of $[0, 1]$. It was shown that LS serves as a regularizer for the hard training data and therefore improves generalization of the model. The regularizer role of LS prevents the model from fitting overly on the target class. Empirical studies have demonstrated the effectiveness of LS in improving the model performance across various benchmarks [Pereyra et al., 2017] (such as image classification [Szegedy et al., 2016], machine translation [Vaswani et al., 2017], language modelling [Chorowski and Jaitly, 2017]) and model calibration [Müller et al., 2019]. Later it was reported LS even helps with improving robustness when learning with noisy labels [Lukasik et al., 2020]. However, we observed that the advantage of LS vanishes when we operate in a high label noise regime. In Figure 1, we present a set of experiments on some UCI datasets [Dua and Graff, 2017]. We highlight best two smooth rates (possible to have tied smooth rates) under each label noise rate. Indeed, non-negative smooth rates (circles colored in red) outperform negative ones when the label noise rates are low. Nonetheless, with the increasing of noise rates, negative smooth rates $r < 0$ (Eqn. (1), diamonds colored in green) appear to be more competitive when learning with noisy labels. Intuitively speaking, this is due to the increased entropy of $P(\text{noisy label}|X)$ when the noise rate is high, in which case, further applying LS tends to “oversmooth” the estimated posterior. Motivated by this observation, we aim to provide a more thorough understanding of whether should we adopt label smoothing or not when learning with noisy labels, specifically, how to make a choice between LS and negative/not label smoothing (NLS)?

With the presence of label noise, we theoretically demonstrate that there exists a phase transition when finding the optimal label smoothing rate for $r \in (-\infty, 1]$. Particularly, when the label noise rate is low, LS is able to uncover the optimal model while NLS is considered more beneficial in a high label noise regime. Discovering that NLS differs substantially from LS in their achieved model confidence, we then proceed to explain such a transition. We also bridge the gap between NLS and several learning-with-noisy-labels solutions in the literature, including Loss Correction [Patrini et al., 2017], NLNL [Kim et al., 2019] and Peer Loss [Liu and Guo, 2020], to further validate our results.

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We provide extensive experimental evidences to support our findings. For instance, on multiple benchmark datasets, we present the clear transition of the optimal smoothing rate going from positive to negative when we keep increasing noise rates. In particular, we show a negative smoothing rate elicits higher model confidence on correct predictions and lower confidence on wrong predictions compared with the behavior of a positive one on CIFAR-10 test data.

Our contributions summarize as follows:

- We provide understandings for the decision between LS and NLS, when learning with noisy labels. (An understanding paper rather than a method paper)
- With the presence of label noise, we demonstrate learning with a negative smooth rate can be more robust to label noise compared with a positive rate when label noise rates are high. And this is best explained by the fact that NLS improves the confidence of model prediction. (Section 3 and 4)
- We show that several robust loss functions in the label-noise literature correspond to learning with NLS, under certain noise rate models. (Section 5)
- Extensive experiment results validate our main theoretical conclusions. In the Appendix, we also discuss practical considerations to mitigate the impact of label noise, and empirically show how LS and NLS result in trade-offs in model confidence, bias and variance of the generalization error. (Appendices B, C)

We defer all proofs to Appendix F. Our work primarily contributes to the literature of learning with noisy labels [Scott et al., 2013, Natarajan et al., 2013, Liu and Tao, 2015, Patrini et al., 2017, Liu and Guo, 2020]. Our core results are contingent on recent works of understanding the effect of label smoothing when training deep neural network models, i.e., label smoothing improves model calibration [Müller et al., 2019], more complicated forms of label smoothing [Li et al., 2020, Yuan et al., 2020], and in particular when label noise presents [Lukasik et al., 2020, Liu, 2021]. Due to the space limit, we defer a more detailed discussion of related works to Appendix A.

2 Preliminaries

2.1 Learning with smoothed labels

For a \( K \)-class classification task, we denote by \( X \in \mathcal{X} \) a high-dimensional feature and \( Y \in \mathcal{Y} := \{1, 2, \ldots, K\} \) the corresponding label. Suppose \((X, Y) \in \mathcal{X} \times \mathcal{Y}\) are drawn from a joint distribution \( D \). Let \( y_i \) be the one-hot encoded vector form of \( y_i \) which generates according to \( Y \). The random variable of smoothed label \( Y_{LS}^{r} \) with smooth rate \( r \in [0, 1] \) generates \( y_i^{LS, r} \) as [Szegedy et al., 2016]:

\[
y_i^{LS, r} = (1 - r) \cdot y_i + \frac{r}{K} \cdot 1.
\]

For example, when \( r = 0.3 \), the smoothed label of \( y_i = [0, 1, 0]^T \) becomes \( y_i^{LS, r=0.3} = [0.1, 0.8, 0.1]^T \).

To enable ease of presentations (instead of highlighting a crucial concept), we unify LS [Szegedy et al., 2016] and NLS into the generalized label smoothing (GLS), i.e., \( r \in (-\infty, 1] \):

\[
y_i^{GLS, r} := (1 - r) \cdot y_i + \frac{r}{K} \cdot 1, \tag{2}
\]

where \( y_i^{GLS, r} \) is given by the random variable of generalized smooth label \( Y_{GLS}^r \). Name the scenario \( r < 0 \) as negative/not label smoothing (NLS). A negative \( r \) indicates that the smoothed label might be negatively related to the corresponding feature and should not be (positively) smoothed. For example, when \( r = -0.3 \), the smoothed label
The noisy label literature [Natarajan et al., 2013, Liu and Tao, 2015, Patrini et al., 2017] considers the setting where we

2.2 Learning with noisy labels

... handling label noise, though a bit counter-intuitive at first sight. To clarify, we do not assume a strict lower bound for r. If r → −∞, normalizing yGLS,i by 1 − r returns yGLS,i = yi − 1/K. We will show when imposing a negative smoothing parameter will be considered beneficial as compared to a positive one. In the main paper, we mainly focus on the binary classification task where yi ∈ {0, 1} and K = 2, although we do include the discussion of multi-class extensions in Section 2.3. Denote a deep neural network as f, f(xi) is the model prediction of xi ∈ X with element f(xi)y := P(Y = yi|X = xi, f). Given the sample x ∈ X and a hard label y ∈ Y, the binary Cross-Entropy loss is defined as ℓCE(f(x), y) := −log(f(x)y). Throughout this paper, we shorthand ℓCE as ℓ for a clean presentation.

2.3 Model confidence

We define a key quantity, model confidence, that plays an important role in later sections.

Definition 2.2 (Confidence of model f for sample (x, y)). Given a model f, a sample x with its target label y ∈ {0, 1}, the model confidence of f w.r.t. sample x is defined as MC(f; x, y) = f(x)y − f(x)1−y.

MC(f; x, y) in Definition 2.2 characterizes the difference of the predicted probability between the target class and the other class. MC(f; x, y) = 0 simply means f has no confidence on its predictions since the model can not identify the target class of x. MC(f; x, y) is negative when f gives a wrong prediction and is not confident to predict the label of x as the target label y. To dig into how GLS influences the model confidence on correct and wrong predictions in following sections, we separate the distribution D into

D+ := {(X, Y) ∼ D : MC(f; X, Y) > 0}, D− := {(X, Y) ∼ D : MC(f; X, Y) ≤ 0}.

3 To Smooth or Not? In the View of Risk Minimization

In this section, we target at the optimal candidates of r in the unified setting to distinguish the preferences for LS and NLS, when the label noise presents.

For r ≤ 1, let ỹ be the vector form of noisy label ỹ obtained from Ỹ, we define the r-smoothed label of ỹ as ỹGLS,r, where ỸGLS,r := (1 − r) · ỹ + (r/K) · 1 and is generated by the random variable ỸGLS,r. Risk minimization w.r.t. smoothed noisy label distribution ỸGLS,r is defined as:

min E(X,Ỹ)∼D[ ℓ(f(X),ỸGLS,r)]. (3)
In Figure 1, we have shown that given the unseen test data, learning with non-negative smooth rates may not always return the best outcome. Based on this observation, we delve into details to show when NLS is more favorable than LS and Vanilla Loss (VL, $r = 0$). We start with stating Assumption 3.1

**Assumption 3.1.** We assume learning with clean data distribution $\mathcal{D}$ with smooth rate $r^* \leq 1$ in GLS returns the best performance on the unseen clean test data distribution $\mathcal{D}_{test}$.

Assumption 3.1 simply offers us a view to initiate our analysis for the noisy label setting. To clarify, we don’t rule out the possibility that other methods outperform LS, VL or NLS with optimal smooth rate $r^*$. At the end of this section and Appendix D, we will empirically test what $r^*$ usually is on various benchmarks. We define the $r^*$ smoothed label distribution as $Y^*$: $Y^* := (1 - r^*) \cdot y_i + \frac{r^*}{r} \cdot 1$. Based on Assumption 3.1 $f_D^* := \arg\min_f \mathbb{E}_{(X,Y)\sim \mathcal{D}}[\ell(f(X), Y^*)]$ returns the best performance among all candidates of $r$ on $\mathcal{D}_{test}$ (i.e., label shift such that $\mathcal{D} \neq \mathcal{D}_{test}$).

With the introduction of $r^*$ and $f_D^*$, our goal is then to recover the classifier $f_D^*$ using the noisy training labels. We define $\lambda_1, \lambda_2$ and offer Theorem 3.2.

\[ \lambda_1 := \left[ (e_0 - \frac{r^*}{2}) + (1 - 2e_0) \cdot \frac{r}{2} \right], \quad \lambda_2 := e_{\Delta} \cdot (1 - r). \]

**Theorem 3.2.** The risk minimization w.r.t. $Y^{GLS,r}$ in the noisy setting (Eqn. (3)) is equivalent to the risk w.r.t $Y^*$ defined on the clean data, with two additional bias terms:

\[
\min_{f} \mathbb{E}_{(X,Y)\sim \mathcal{D}}[\ell(f(X), Y^*)] + \lambda_1 \cdot \mathbb{E}_{(X,Y)\sim \mathcal{D}}[\ell(f(X), 1 - Y) - \ell(f(X), Y)] + \lambda_2 \cdot \mathbb{E}_{X,Y=1}[\ell(f(X), 0) - \ell(f(X), 1)].
\]

(4)

The True Risk is the risk w.r.t. clean optimal label distribution $Y^*$. Define $MC(f; X, Y) := \ell(f(X), 1 - Y) - \ell(f(X), Y)$, note that $MC(f; X, Y) = \log \left(\frac{1 - f(X)}{f(X)} \right)$ and $MC(f; X, Y) = 2 \cdot f(X) - 1$, both $\log \left(\frac{1 - x}{x} \right)$ and $2x - 1$ are monotonically increasing for $x \in (0, 1)$, model $f$ with a high $MC(f; X, Y)$ has high $MC(f; X, Y)$. The two extra bias terms explicitly affects the model confidence. Now we proceed to answer “what parameters are preferred in the noisy setting”.

### 3.1 Symmetric error rates with $e_{\Delta} = 0$

Symmetric error rates $e := e_0 = e_1$ indicates the probability of flipping to the other class is equal for both classes. In this case, $\lambda_2 = 0$, Term M-Inc2 is cancelled and Eqn. (4) reduces to

\[
\min_{f} \mathbb{E}_{(X,Y)\sim \mathcal{D}}[\ell(f(X), Y^*)] + \lambda_1 \cdot \mathbb{E}_{(X,Y)\sim \mathcal{D}}[\ell(f(X), 1 - Y) - \ell(f(X), Y)].
\]

(5)

**Noisy labels impairs model confidence on Vanilla Loss** In the unified framework, define the optimal $r$ that will cancel the impact of Term M-Inc1 as:

\[
\text{when } r_{\text{opt}} := \frac{r^* - 2e}{1 - 2e}, \quad \text{M-Inc1} = 0.
\]

(6)

The threshold $r_{\text{opt}}$ in Eqn. (6) implies:

**Theorem 3.3.** With Assumption 3.1 learning with smooth rate $r = r_{\text{opt}}$ under $(X, \tilde{Y}) \sim \tilde{D}$ yields $f_D^*$.

- **When error rate** $e < r^*/2$, $r = r_{\text{opt}} > 0$ (LS);
- **When error rate** $e = r^*/2$, $r = 0$ (VL);
- **When error rate** $e > r^*/2$, $r = r_{\text{opt}} < 0$ (NLS).

In Theorem 3.3 adopting NLS when noise rate $e < \frac{r^*}{2}$ induces $\lambda_1 < 0$, Term M-Inc1 makes $f$ overly-confident on its predictions compared with $Y^*$. In Figure 2 with the decreasing of $r^*$, LS is less tolerant of labels with high noise. Similarly, if $e \geq \frac{r^*}{2}$, with the decreasing of $r^*$, NLS is more robust in the high noise regime while LS makes the model $f$ become less-confident on its predictions. Clearly, NLS outperforms LS especially when noise rates are large and $r^*$ is small.
3.2 Asymmetric error rates with $e_\Delta \neq 0$

In this case, adopting $r = r^* - 2c_0$ removes the Term M-Inc1. However, when $r < 1$, Term M-Inc2 is not negligible due to asymmetric noise transition matrix. As a result, Term M-Inc2 becomes:

$$e_\Delta \cdot \frac{1 - r^*}{1 - 2c_0} \cdot \mathbb{E}_{X,Y=1} \left[ \ell(f(X), 0) - \ell(f(X), 1) \right],$$

with $e_\Delta \cdot \frac{1 - r^*}{1 - 2c_0} \geq 0$. Term M-Inc2 in the minimization increases the model confidence on $(X, Y = 0) \sim \mathcal{D}_i^+$. The model will then become overly-confident with the class that has a low noise rate $c_0$. Meanwhile, Term M-Inc2 decreases the model confidence on $(X, Y = 1) \sim \mathcal{D}_j^+$ (less-confident to the class with a high noise rate $e_1$).

3.3 Multi-class extension

As an extension to the binary classification task, we next show how Theorem 3.3 could be generalized to the multi-class setting under two broad families of noise transition model. We assume Assumption 3.1 holds in the multi-class setting. And for $Y, \hat{Y} \in [K]$, we extend the definition of model confidence to multi-class classification tasks as:

**Definition 3.4** (Model confidence of sample $(x, y)$ (K-class classification)). Given a model $f$, a sample $x$ with its target label $y \in [K]$, the model confidence score of $f$ w.r.t. sample $x$ is defined as

$$MC(f; x, y) = f(x)_y - \frac{1}{K-1} \sum_{i \neq y} f(x)_i.$$

**Sparse noise transition matrix** Sparse noise model [Wei and Liu, 2020] assumes $K$ is an even number. For $c \in [\frac{K}{2}]$, $i_c < j_c$, sparse noise model specifies $\frac{K}{2}$ disjoint pairs of classes ($i_c, j_c$) to simulate the scenario where particular pairs of classes are ambiguity and misleading for human annotators. The off-diagonal element of $T$ reads $T_{i_c, j_c} = c_0$, $T_{j_c, i_c} = c_1$. Suppose $c_0 + c_1 < 1$, the diagonal entries become $T_{i_c, i_c} = 1 - c_1$, $T_{j_c, j_c} = 1 - c_0$. Clearly, our conclusions in Theorem 3.3 extends directly to the sparse noise transition matrix by simply splitting the $K$-class classification task into $\frac{K}{2}$ disjoint binary ones.

**Symmetric noise transition matrix** Symmetric noise model [Kim et al., 2019] is a widely accepted synthetic noise model in the literature of learning with noisy labels. The symmetric noise model generates the noisy labels by randomly flipping the clean label to the other possible classes with probability $\epsilon$. $\forall i \neq j$, $T_{i, j} = \epsilon/(K - 1)$, and the diagonal entry is $T_{i,i} = 1 - \epsilon$. Define the optimal $r$ under the unified setting in the multi-class setting as $r_{\text{opt}} := \frac{(K-1)^2 - K\epsilon}{(K-1) - K\epsilon}$.

Theorem 3.3 can be extended to the multi-class setting as:

**Theorem 3.5.** Under Assumption 3.1, suppose the symmetric noise rate is not too large, i.e., $\epsilon < \frac{K-1}{K}$, learning with smooth rate $r = r_{\text{opt}}$ under $(X, Y) \sim \mathcal{D}$ yields $f^*_2$.

- **When error rate** $\epsilon < \frac{(K-1)r^*}{K}$, $r = r_{\text{opt}} > 0$ (LS);
- **When error rate** $\epsilon = \frac{(K-1)r^*}{K}$, $r = 0$ (VL);
- **When error rate** $\epsilon > \frac{(K-1)r^*}{K}$, $r = r_{\text{opt}} < 0$ (NLS).

![Figure 2: Decision between NLS, LS given $e$, $r^*$](image-url)
3.4 Clean empirical risk v.s. noisy empirical risk

Now we empirically verify Theorem 3.2 under symmetric noise setting, which relates the risk in the noisy setting to the clean ones. Assume the the noise label is generated through the symmetric noise transition matrix. We name the noisy risk as $E_{(X, \tilde{Y}) \sim \mathcal{D}} [\ell(f(X), \tilde{Y}^{GLS,r})]$, which is the objective in Eqn. (4).

We use a UCI dataset (Waveform) for illustration where the value of $r^*$ is approximately 0. When the noise rates are $0.1, 0.2, 0.3, 0.4$, the optimal smooth rate should be $-0.25, -0.67, -1.5, -4$ according to Eqn. (6). The estimated noisy risk of LS/VL/NLS on these noise settings can be summarized in Table 1. Clearly, when $e = 0.1$, $r = -0.25$ is closest to the estimated (clean) true risk (also returns the best test accuracy among these smooth rates). Similarly observations hold for all other $e$. Learning with $r_{opt}$ on the noisy data yields the closest risk to the corresponding clean risk with $r^*$!

Table 1: The difference between the empirical true risk of $Y^*$ on the clean data and empirical risk of LS/VL/NLS on noisy labels (UCI-Waveform data): $r^*$, empirical true risk, and empirical noisy risks of $r_{opt}$ under various noise levels are highlighted in purple.

| Smooth rate | Risk (clean) | Risk (e = 0.1) | Risk (e = 0.2) | Risk (e = 0.3) |
|-------------|--------------|----------------|----------------|----------------|
| $r = 0.8$   | 0.6773       | 0.6831         | 0.6873         | 0.6899         |
| $r = 0.6$   | 0.6295       | 0.6521         | 0.6689         | 0.6833         |
| $r = 0.4$   | 0.5437       | 0.5794         | 0.6408         | 0.6718         |
| $r = 0.2$   | 0.4134       | 0.5212         | 0.5956         | 0.6580         |
| $r^*$ = 0.0 | 0.1798       | 0.4057         | 0.5399         | 0.6314         |
| $r = -0.25$ | -36.8095     | 0.1983         | 0.4381         | 0.5957         |
| $r = -0.67$ | -33.1283     | -28.3508       | 0.2167         | 0.5132         |
| $r = -1.5$  | -97.4378     | -61892.8047    | -94.9509       | 0.1911         |

3.5 What is the practical distribution of $r^*$ and $r_{opt}$?

$r^*$ and $r_{opt}$ on UCI datasets [Dua and Graff, 2017] As for UCI datasets, we pick Twonorm and Splice for illustration. The noisy labels are generated by a symmetric noise transition matrix with noise rate $e_i = [0.1, 0.2, 0.3, 0.4]$. As highlighted in Table 1 (top of this page), $r_{opt}$ appears with positive values when the data is clean (same as $r^*$) or of a low noise rate. With the increasing of noise rates, the performance of LS results in a much larger degradation compared with NLS. We color-code different noise regimes where either VL/LS (redish) or NLS (greenish) outperforms the other. Clearly there is a separation of the favored smoothing rate for different noise scenarios (upper left & low noise for VL/LS, bottom right & high noise for NLS).

Table 2: Test accuracies of LS, VL, NLS on clean and noisy UCI Heart, Splice datasets with best two smooth rates (green: NLS; red: VL or LS). We adopt the two independent sample T-test (5 non-negative smooth rates V.S. the last 5 rows of reported negative smooth rates) to verify the overall performance comparisons between VL/LS and NLS. $p$-value is highlighted in green if NLS generally returns a higher accuracy (i.e., $t$-value $< 0$) than VL/LS, otherwise, in red. Results on more benchmark datasets are referred to Appendix D.

| Smooth Rate | $e_i = 0$ | $e_i = 0.1$ | $e_i = 0.2$ | $e_i = 0.3$ | $e_i = 0.4$ | $e_i = 0$ | $e_i = 0.1$ | $e_i = 0.2$ | $e_i = 0.3$ | $e_i = 0.4$ |
|-------------|-----------|-------------|-------------|-------------|-------------|-----------|-------------|-------------|-------------|-------------|
| $r = 0.8$   | 0.885     | 0.833       | 0.836       | 0.820       | 0.738       | 0.980     | 0.946       | 0.919       | 0.856       | 0.760       |
| $r = 0.6$   | 0.902     | 0.836       | 0.820       | 0.836       | 0.738       | 0.978     | 0.939       | 0.913       | 0.869       | 0.778       |
| $r = 0.4$   | 0.855     | 0.833       | 0.836       | 0.820       | 0.738       | 0.978     | 0.948       | 0.922       | 0.885       | 0.779       |
| $r = 0.2$   | 0.902     | 0.833       | 0.820       | 0.830       | 0.774       | 0.978     | 0.948       | 0.919       | 0.878       | 0.800       |
| $r = 0.1$   | 0.902     | 0.833       | 0.820       | 0.820       | 0.771       | 0.976     | 0.948       | 0.926       | 0.876       | 0.800       |
| $r = -0.4$  | 0.809     | 0.836       | 0.803       | 0.853       | 0.754       | 0.961     | 0.956       | 0.928       | 0.880       | 0.817       |
| $r = -0.6$  | 0.809     | 0.836       | 0.820       | 0.853       | 0.721       | 0.961     | 0.956       | 0.926       | 0.880       | 0.819       |
| $r = -1.0$  | 0.885     | 0.869       | 0.803       | 0.853       | 0.754       | 0.956     | 0.954       | 0.912       | 0.889       | 0.819       |
| $r = -2.0$  | 0.885     | 0.869       | 0.820       | 0.853       | 0.787       | 0.954     | 0.946       | 0.915       | 0.889       | 0.830       |
| $r = -4.0$  | 0.885     | 0.869       | 0.853       | 0.853       | 0.820       | 0.946     | 0.943       | 0.939       | 0.911       | 0.830       |
| $r = -8.0$  | 0.869     | 0.869       | 0.885       | 0.853       | 0.853       | 0.943     | 0.946       | 0.939       | 0.915       | 0.845       |

$p$-value = 0.020 0.136 0.549 0.002 0.243 0.001 0.332 0.002 0.015 0.005
When learning with a larger scale and more complex dataset, like CIFAR-10 and CIFAR-100, models are prone to converge on a local optimal solution rather than the global optimum. This phenomenon occurs frequently in NLS which ends up with performance degradation. Thus, in Table 3 when learning with noisy labels, we report the better performance of LS and NLS between direct training and loading the same warm-up model. We observe that the performance of NLS is more competitive than LS when learning with clean data. Clearly, NLS outperforms LS in CIFAR-10 and CIFAR-100 under various synthetic noise settings. The gap is larger when the noise rates are high. The results of two independent sample T-test further verify this conclusion.

Table 3: Test accuracy (mean±std) comparisons on synthetic noisy CIFAR-10, CIFAR-100 datasets. Best two smooth rates for each synthetic noise setting are highlighted for each ε (green: NLS; red: VL/LS).

| Smooth Rate | CIFAR-10 Symmetric | CIFAR-10 Asymmetric | CIFAR-100 Symmetric |
|-------------|---------------------|---------------------|---------------------|
| ε = 0.0     | 92.91±0.06          | 88.88±1.61          | 87.83±0.13          |
| ε = 0.2     | 90.45±2.06          | 87.83±0.13          | 89.56±0.07          |
| ε = 0.4     | 89.50±0.07          | 87.83±0.13          | 89.70±0.09          |
| ε = 0.6     | 89.70±0.07          | 87.83±0.13          | 89.70±0.09          |
| ε = 0.8     | 89.70±0.07          | 87.83±0.13          | 89.70±0.09          |
| ε = 1.0     | 89.70±0.07          | 87.83±0.13          | 89.70±0.09          |

*p-value = 0.0004 0.008 0.011 < 1e−14 < 1e−14 0.106 < 1e−14 < 1e−15

4 The Impacts on the Model Confidence

Continuing the discussion of differed model confidence in the previous section, we now empirically explore how such differences distinguish LS and NLS.

Remember that when the label is clean (ε₀ = ε₁ = 0), Eqn. (5) reduces to:

\[
\min \ E_{(X,Y) \sim D} \left( \ell(f(X), Y) \right) + \frac{r}{2} \cdot E_{(X,Y) \sim D} \left( \ell(f(X), 1 - Y) - \ell(f(X), Y) \right) .
\]

To clarify, we are not restricting D to have infinite samples, i.e., for the discrete distribution D = \{xᵢ, yᵢ\}ᵢ∈[N], Eqn. (7) becomes:

\[
\min \ \left[ \frac{1}{N} \sum_{i \in [N]} \ell(f(xᵢ), yᵢ) \right] + \left[ \frac{r}{2N} \sum_{i \in [N]} (\ell(f(xᵢ), 1 - yᵢ) - \ell(f(xᵢ), yᵢ)) \right] .
\]

The difference between LS and NLS lie in the weight of Term MCᵢ(f; X, Y) when learning with clean labels: NLS encourages high MCᵢ(f; X, Y) and MC(f; X, Y) while LS has an opposite effect.

4.1 Side-effects of over-confident

We adopt the generation of 2D (binary) synthetic dataset from [Amid et al. 2019] by randomly sampling two circularly distributed classes. The inner annulus indicates one class (blue), while the outer annulus denotes the other class (red). We hold 20% data samples for performance comparison. In Figure 3 the colored bands depict the different levels of prediction probabilities: light blue + orange bands indicate samples that satisfy MC < 0.4 (low model confidence). When learning with clean data, a non-positive smooth rate may yield over-confidence on the model prediction and a relatively low test accuracy.

4.2 Label noise reduces model confidence

Recent works [Liu 2021, Cheng et al. 2020] have demonstrated that with the presence of label noise, learning with noisy labels directly will eventually result in unconfident model predictions. Continuing the synthetic 2D dataset, we
Figure 3: Model confidence visualization of NLS, VL, and LS on synthetic data (Type 1) with the clean data. The optimal smooth rate falls in $[0, 0.4]$. (left: NLS; middle: Vanilla Loss; right: LS). The test accuracy is annotated above each plot.

Figure 4: Model confidence visualization of NLS, VL, and LS on synthetic data (Type 1) with noise rate $e_i = 0.4$. The optimal smooth rate is $-0.4$. (left: NLS; middle: Vanilla Loss; right: LS). The test accuracy is annotated above each plot.

4.3 Model confidence on CIFAR-10 test dataset

When trained on symmetric 0.2 noisy CIFAR-10 training dataset (see Figure 5), with the decreasing of smooth rates (from right to left), the model confidence on correct predictions gradually approach to its maximum, while for wrong predictions, the model confidence converges to its minimum value. We observe that NLS makes the model prediction become over-confident on correct predictions and in-confident on wrong predictions.

Figure 5: Model confidence distribution of correct and wrong predictions on CIFAR-10 test data. (From left to right: NLS ($r = -0.8, -0.4$), Vanilla Loss, LS ($r = 0.4$), trained on symmetric 0.2 noisy CIFAR-10 dataset).
5 Connection to Other Robust Methods

In this section, we aim to theoretically explore the connection between NLS and popular methods such as backward/forward loss correction [Natarajan et al., 2013, Patrini et al., 2017], NLNL [Kim et al., 2019] and peer loss [Liu and Guo, 2020], under the unified setting. We defer the corresponding empirical validations to Appendix B.

5.1 Loss correction

Loss correction [Patrini et al., 2017] studies two robust loss designs which are based on the knowledge of non-singular noise transition matrix $T$. The backward correction $\ell^\rightarrow(f(X), \tilde{Y})$ re-weights the loss $\ell(f(X), \tilde{Y})$ by $T_{\tilde{Y}, Y}^{-1}$ with $\tilde{Y}$ being the model predicted label, while the proposed forward correction $\ell^\leftarrow(f(X), \tilde{Y})$ multiplies the model predictions by $T$.

**Proposition 5.1.** For $r_{LC} := \frac{e_0}{2e_0-1} < 0$, $\lambda_{LC} := e_\Delta \cdot \frac{1}{1-2e_0}$, risk minimization of both backward and forward correction (with the knowledge of noise rates) are equivalent to the combination of NLS and an extra bias term Bias-LC

$$
\min E_{(X,Y)\sim D}[\ell^\rightarrow(f(X), \tilde{Y})] = \min E_{(X,Y)\sim D}[\ell^\rightarrow(f(X), \tilde{Y})] + \lambda_{LC} \cdot E_{X,Y=1}[\ell(f(X), 1) - \ell(f(X), 0)]
$$

The incurred Bias-LC controls the model confidence on $(X, Y = 1) \sim D_f$. Note that when the noise rate is not substantially high, i.e., $e_0 \in [0, \frac{1}{2})$, $\lambda_{LC} > 0$. Then, compared with loss correction, NLS with smooth rate $r_{LC}$ makes the model $f$ to be less confident on $(X, Y = 1) \sim D_f^+$ and more confident on $(X, Y = 1) \sim D_f^-$ (wrong predictions). However, the impact of term Bias-LC is diminishing when either $e_\Delta \to 0$ (symmetric noise rates) or $e_0 \to 0$ (low noise rates) as specified in Theorem 5.2.

**Theorem 5.2.** Assume the noise transition matrix is symmetric, i.e., $e_\Delta = 0$, backward and forward loss correction are a special form of NLS with smooth rate $r_{LC}$.

5.2 Learning from complementary labels

Complementary labels [Ishida et al., 2017] were firstly introduced to mitigate the cost of collecting data. Rather than encouraging the model to fit directly on the target, learning from complementary labels trains the model to not fit on the complementary label which differs from the target. Later, an indirect training method “Negative Learning” (NL) [Kim et al., 2019] was proposed to reduce the risk of providing incorrect information with the presence of noisy labels and is robust to label noise in multi-class classification tasks. A more generic unbiased risk estimator of learning with complementary labels was proposed [Ishida et al., 2019] and is defined as: $\ell_{CL}(f(X), \tilde{Y}) := \ell(f(X), Y) - \ell(f(X), 1 - \tilde{Y})$.

**Theorem 5.3.** Learning from complementary labels with $\ell_{CL}$ is equivalent to NLS with smooth rate $r_{CL} \to -\infty$:

$$
\min E_{(X,\tilde{Y})\sim \tilde{D}}[\ell_{CL}(f(X), \tilde{Y})] = \min E_{(X,\tilde{Y})\sim \tilde{D}}[\ell(f(X), \tilde{Y}) + \ell_{CL}(f(X), \tilde{Y})].
$$

5.3 Peer loss functions

Peer loss functions [Liu and Guo, 2020] proposed a family of robust loss measures which do not require the knowledge of noise rates. The mathematical representation of peer loss functions is $\ell_{PL}(f(X), \tilde{Y}) := \ell(f(X), \tilde{Y}) - \ell(f(X), \tilde{Y}_1, \tilde{Y}_2)$, where $(X_i, \tilde{Y}_i) \sim \tilde{D}$. The second term of the peer loss evaluates on randomly paired data samples and labels to punish $f$ from overly fitting on noisy labels.

**Proposition 5.4.** For $r_{PL} := 2 \cdot \mathbb{P}(\tilde{Y} = 1)$, $\lambda_{PL} := 1 - r_{PL}$, risk minimization of the peer loss is equivalent to negative label smoothing regularization with an extra term Bias-PL, i.e.,

$$
\min E_{(X,\tilde{Y})\sim \tilde{D}}[\ell_{PL}(f(X), \tilde{Y})] = \min E_{(X,\tilde{Y})\sim \tilde{D}}[\ell(f(X), \tilde{Y}) - \ell(f(X), \tilde{Y}_{GLS,pl})] + \lambda_{PL} \cdot E_{X,\tilde{Y}=1}[\ell(f(X), 1) - \ell(f(X), 0)].
$$
The incurred term Bias-PL controls the model confidence on \((X, \tilde{Y} = 1) \sim \tilde{D}\) and has a diminishing effect as \(\mathbb{P}(\tilde{Y} = 1) \to 1/2\). Generally, the peer loss relates to the unified setting (GLS) as the negatively weighted GLS term appears to be a regularizer. Note that we have access to the \(\mathbb{P}(\tilde{Y} = 1)\), we can bridge gap by adding an estimable term Bias-PL. With some derivations, we further show in Theorem 5.5 when noisy priors are equal, the peer loss has an exact NLS form.

**Theorem 5.5.** When the noisy labels have equal prior, i.e., \(\mathbb{P}(\tilde{Y} = 0) = \mathbb{P}(\tilde{Y} = 1)\), the peer loss is a special form of NLS regularization with the smooth rate \(r_{PL}\). Besides,

\[
\min_{(X, \tilde{Y}) \sim \tilde{D}} \mathbb{E}(\ell_{PL}(f(X), \tilde{Y})) = \min_{(X, \tilde{Y}) \sim \tilde{D}} \mathbb{E}(\ell(f(X), \tilde{Y}^{GLS, r \to -\infty})).
\]

5.4 Performance comparisons

In Table 4, we compare VL(CE), LS and NLS with several robust methods in synthetic noisy CIFAR datasets. We emphasize that we are not proposing a new method to compete with state-of-the-art methods. Instead, we hope to help readers understand how the LS and NLS fare when label noise presents. Clearly, LS and NLS can definitely be viewed as competitive and efficient robust loss functions which outperform Cross Entropy, Bootstrap [Reed et al., 2014], SCE [Wang et al., 2019], APL [Ma et al., 2020] and Forward correction [Patrini et al., 2017] in most settings. More detailed comparisons between LS/NLS and more methods on real-world annotated CIFAR datasets are available at [Wei et al., 2021].

| Method               | CIFAR-10, Symmetric | CIFAR-10, Asymmetric | CIFAR-100, Symmetric |
|----------------------|---------------------|---------------------|----------------------|
|                      | \(\epsilon = 0.2\) | \(\epsilon = 0.4\) | \(\epsilon = 0.6\)  | \(\epsilon = 0.2\) | \(\epsilon = 0.3\) | \(\epsilon = 0.4\) | \(\epsilon = 0.6\) |
| Cross Entropy        | 86.45               | 82.72               | 74.04                | 88.59               | 86.14               | 48.20               | 38.27               |
| Bootstrap            | 86.06               | 81.65               | 75.26                | 87.69               | 85.51               | 47.28               | 35.81               |
| Forward correction   | 84.85               | 84.98               | 73.97                | 89.42               | 88.25               | 53.04               | 41.59               |
| SCE                  | 89.39               | 80.31               | 75.28                | 88.07               | 85.93               | 49.34               | 38.87               |
| APL                  | 88.42               | 81.27               | 76.62                | 88.75               | 87.41               | 51.63               | 42.31               |
| Peer Loss            | 90.21               | 86.40               | 79.64                | 91.38               | 89.65               | 62.16               | 53.72               |
| ELR                  | 92.57               | 91.32               | 88.86                | 93.48               | 92.21               | 68.03               | 60.49               |
| AUM                  | 91.52               | 87.85               | 81.71                | 92.17               | 90.63               | 59.29               | 44.05               |

Table 4: Performance comparisons on synthetic noisy CIFAR datasets: we adopt the same model architecture for all methods (ResNet 34 [He et al., 2016]), best achieved test accuracy is reported.

6 Conclusion

In this paper, we provide understandings for whether should we adopt label smoothing or not when learning with noisy labels. We show that learning with negatively smoothed labels explicitly improves the confidence of model prediction. This key property acts as a significant role when the confidence of model prediction drops. In contrast to existing works that promote the use of positive label smoothing, we show both theoretically and empirically the advantage of a negative smooth rate when the label noise rate increases. We also bridge the gap between negative label smoothing and existing learning with noisy label solutions, which further demonstrates the importance of negative/not label smoothing. In a nutshell, our observations provide new understanding for the effects of label smoothing, especially when the training labels are imperfect.
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Appendix

The Appendix is organized as follows.

- Section A presents the full version of related works.
- Section B includes empirical validations of theoretical conclusions in Section 5.
- Section C discusses practical considerations of the robustness for LS and NLS.
- Section D shows additional experiments on synthetic dataset and UCI datasets.
- Section E illustrates the bias and variance trade-off when learning with LS and NLS from clean data.
- Section F includes omitted proofs for theoretical conclusions in the main paper.

A Full Version of Related Works

Our work supplements to two lines of related works.

Learning with noisy labels Annotated labels from human labelers usually consists of an non-negligible amount of mis-labeled data samples. Making deep neural nets perform robust training on “noisily” labeled datasets remains a challenge. Classical approaches of learning with noisy labels assume the noisy labels are independent to features. They firstly estimate the noise transition matrix [Liu and Tao, 2015, Menon et al., 2015, Harish et al., 2016, Patrini et al., 2017], then proceed with a loss correction [Natarajan et al., 2013, Patrini et al., 2017, Liu and Tao, 2015] to mitigate label noise. Recent works propose robust loss functions [Xu et al., 2019, Kim et al., 2019, Liu and Guo, 2020] to train deep neural nets directly without the knowledge of noise rates, or design a pipeline which dynamically select and train on “clean” samples with small loss [Jiang et al., 2018, Han et al., 2018, Yu et al., 2019, Yao et al., 2020]. More recently, several approaches target at addressing more challenging noise settings, such as instance-dependent label noise [Cheng et al., 2020, Berthon et al., 2021].

Understanding the effect of label smoothing Learning with one-hot labels is prone to over-fitting, soft label learning then naturally draws attentions of machine learning researchers. Successful applications of soft label learning include the label distribution learning [Geng, 2016] which provides an instance with description degrees of all the labels. Label smoothing (LS) [Szegedy et al., 2016] is another arising learning paradigm that uses positively weighted average of both the hard training labels and uniformly distributed soft labels. Empirical studies have demonstrated the effectiveness of LS in improving the model performance [Pereyra et al., 2017, Szegedy et al., 2016, Vaswani et al., 2017, Chorowski and Jaitly, 2017] and model calibration [Müller et al., 2019]. However, knowledge distilling a teacher network (trained on smoothed labels) into a student network is much less effective [Müller et al., 2019]. Later, generalization effects of more advanced forms of label smoothing was studied, such as structural label smoothing [Li et al., 2020] and non-uniform label smoothing [Chen et al., 2020]. More recently, it was shown that an appropriate label smoothing regularizer with reduced label variance boosts the convergence [Xu et al., 2020]. When label noise presents, [Liu, 2021] gives theoretical justifications for the memorizing effects of label smoothing. And the effectiveness of label smoothing in mitigating label noise is investigated in [Lukasik et al., 2020].

B Empirical Validations of Main Theorems

In this section, we empirically validate our main theoretical conclusions in Section 5, i.e., the connection between LS/NLS and popular methods.

We compare the unified setting (GLS) with backward correction [Natarajan et al., 2013], forward correction [Patrini et al., 2017] and peer loss [Liu and Guo, 2020] on CIFAR-10 dataset. To approximate the performance of backward/forward Loss Correction, we adopt GLS with smooth rate \( \epsilon = \frac{1}{(r - 1)} \). As for the approximation of peer loss, we choose \( \ell(f(X), \tilde{Y}) - \ell(f(X), \tilde{Y}^{\text{GLS}, r=0.5}) \) which is equivalent to NLS when \( r \rightarrow -\infty \). Experiment results in Table 5 on CIFAR-10 under symmetric noise settings demonstrate that the equivalent forms of GLS are robust to label noise.

Explanation of the performance gap In practice, we adopt the same hyper-parameter setting as used for all other smooth rates for GLS form (VL, LS and NLS). Loss corrections will firstly warm-up with the cross-entropy loss, estimate the noise transition matrix with this pre-trained model, and then proceed to train with the backward/forward corrected loss. Peer loss functions adopt a dynamical adjustment for learning rate. The warming up, estimation error of noise transition matrix as well as the special hyper-parameter settings explain performance gaps.
Table 5: Comparison of test accuracies on CIFAR-10 under symmetric label noise.

| Method      | CIFAR-10, Symmetric ε = 0.2 | ε = 0.4 | ε = 0.6 |
|-------------|-----------------------------|---------|---------|
| Backward T  | 84.79                       | 83.40   | 71.52   |
| Forward T   | 84.85                       | 83.98   | 73.97   |
| GLS form    | 87.33                       | 81.73   | 75.80   |
| Peer Loss   | 90.21                       | 86.40   | 79.64   |
| GLS form    | 88.98                       | 85.05   | 76.66   |

C Practical Consideration of LS and NLS

In the main paper, we theoretically show when we should adopt NLS and LS. In this section, we discuss more practical considerations, including the optimal smoothing parameter, how to reduce the impacts of bias terms, and multi-class extensions.

C.1 The optimal smoothing parameter

In practice, we don’t have access to noise rates $e_i$. Our work does not intend to particularly focus on the noise rate estimation. For readers interested in the noise rate estimation, please refer to [Liu and Tao, 2015; Menon et al., 2015; Harish et al., 2016; Patrini et al., 2017; Yao et al., 2020b; Zhu et al., 2021]. To estimate $r_{opt} = r^* - 2e_1 - 2e_0$, one can simply assume $r^* \rightarrow 0$. And the noise rate $e$ is estimable by a large family of noise estimation methods mentioned above. Our practical observations show that NLS with a CE warm-up is not sensitive to the negative smooth rate, for example, on CIFAR-10 and CIFAR-100 synthetic noisy datasets, $r < -1.0$ frequently achieves best results (see Table 3 in the main paper). Our current contribution focuses on understanding the generalized label smoothing, and we prefer leaving the task of identifying the optimal smooth rate to future works.

C.2 Making LS and NLS more robust to label noise

There is a line of related works targeting at distinguishing clean labels from the noisy labels. Current literature in selecting clean samples from noisily labeled dataset is based on the empirical evidence that samples with noisy/wrong labels have a larger loss than clean ones. For interested readers, please refer to [Han et al., 2018; Jiang et al., 2018; Yu et al., 2019; Yao et al., 2020a; Wei et al., 2020; Northcutt et al., 2021]. Compared with the risk minimization over the clean data distribution $(X, Y) \sim D$, learning directly with GLS on the noisy distribution $(X, \tilde{Y}) \sim \tilde{D}$ will result in an extra term $(e_1 - e_0) \cdot (1 - r) \cdot E_{E(X,Y=1) \sim D}[\ell(f(X),0) - \ell(f(X),1)]$ compared to the clean scenario. Empirically, we can estimate the bias term, perform a bias correction by subtracting the estimated bias term from the objective function in Eqn. (3).

Suppose we have access to a clean distribution $D_{clean}$ which consists of selected clean samples. Denote the estimated noise rates as $\hat{e}_i$, when $e_\Delta \neq 0$, in order to make LS/NLS be more robust to label noise and fit on the optimal distribution $Y^*$, we improve by performing a model confidence correction on the dominating class through:

$$\min_{E(X, \tilde{Y}) \sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}^{\text{GLS},r}) \right]$$

$$- (\hat{e}_1 - \hat{e}_0) \cdot (1 - r) \cdot E_{E(X,Y=1) \sim D_{clean}} \left[ \ell(f(X),0) - \ell(f(X),1) \right].$$

D Additional Experiment Results and Details

In this section, we include more experiment results, observations and details for learning with LS/NLS.

D.1 Experiment details on CIFAR-10, CIFAR-100

We firstly introduce experiment details on CIFAR-10 dataset adopted in our experiment designs.
Training settings of clean CIFAR-10 dataset [Krizhevsky et al., 2009]. We adopted ResNet34 [He et al., 2016], trained for 200 epochs with batch-size 128, SGD [Robbins and Monro, 1951] optimizer with Nesterov momentum of 0.9 and weight decay 1e-4. The learning rate of first 100 epochs is 0.1. Then it multiplies with 0.1 for every 50 epochs.

Generating noise labels on CIFAR datasets We adopt symmetric noise model which generates noisy labels by randomly flipping the clean label to the other possible classes with probability $\epsilon$. And we set $\epsilon = 0.2, 0.4, 0.6$ for CIFAR-100. We also make use of asymmetric noise model. The asymmetric noise is generated by flipping the true label to the next class with probability $\epsilon$. We set $\epsilon = 0.2, 0.3$ for CIFAR-10.

Training settings of synthetic noisy CIFAR datasets The generation of symmetric noisy dataset is adopted from Cheng et al., 2020. The symmetric noise rates are $[0.2, 0.4, 0.6]$. We choose two methods to train LS and NLS.

- **Direct training:** this setting is the same as training on clean CIFAR-10 dataset.
- **Warm-up:** in this case, we firstly train a ResNet34 model with Cross-Entropy loss for 120 epochs. For this warm-up, the only difference in hyper-parameter setting is the learning rate, where the initial learning rate is 0.1 and it multiplies 0.1 for every 40 epochs. After the warm-up, LS/NLS loads the same pre-trained model and trains for 100 epochs with learning rate 1e-6.

D.2 Why NLS is overlooked?

When learning from a relative large scale dataset, NLS tends to push the model become overly confident early in the training. The poor performances of NLS (direct-train) in Table 6 explain why NLS is neglected. When there is no warm-up, training NLS directly without warming up will reach a $88\% - 92\%$ test accuracy on the clean data. The performance will degrade much more significantly than LS when the noise level is high or $|r|$ is large. In Table 6 we provide the comparisons between direct-train and warm-up in several settings. The improvement bring by a warm-up procedure becomes much more significantly in the high noise regime. NLS makes the classifier be overly confident at the early training which results in converging to a bad local optimum (without CE warm-up, NLS frequently results in a worse performance in CIFAR-10 and CIFAR-100). Since the model will usually fit on the clean data first, then over-fits on the noisy ones [Liu et al., 2020], a large number of approaches (such as Loss corrections [Patrini et al., 2017], Peer Loss [Liu and Guo, 2020], etc) adopt a CE warm-up firstly. Note that there is no difference in the computing costs between NLS (with CE warmup) and CE loss, proceeding with NLS to enhance the model confidence makes NLS much more competitive in the high noise regime, also gives practical insights on how to make NLS work better when learning with clean data.

Table 6: Test accuracies of GLS on assymetric noisy CIFAR-10 and symmetric CIFAR-100 (left/right denotes direct train / warm-up).

| Smooth Rate | CIFAR-10 Asymmetric | CIFAR-100 Symmetric |
|-------------|---------------------|---------------------|
|             | $\epsilon = 0.2$    | $\epsilon = 0.3$    | $\epsilon = 0.4$ | $\epsilon = 0.6$ |
| $r = 0.8$   | 87.89 / 90.51       | 86.38 / 87.97       | 54.78 / 51.27    | 40.21 / 39.80    |
| $r = 0.6$   | 89.14 / 90.55       | 85.97 / 88.01       | 52.83 / 52.88    | 39.64 / 40.57    |
| $r = 0.4$   | 88.23 / 90.61       | 86.95 / 88.04       | 51.40 / 54.36    | 38.29 / 41.63    |
| $r = -0.4$  | 19.71 / 89.60       | 21.86 / 88.42       | 40.30 / 56.97    | 31.35 / 43.91    |
| $r = -0.8$  | - / 89.02           | - / 88.28           | 22.63 / 57.45    | 26.75 / 44.19    |
| $r = -1.0$  | - / 88.68           | - / 88.29           | - / 57.53        | - / 44.59        |
| $r = -2.0$  | - / 88.86           | - / 88.13           | - / 58.21        | - / 45.47        |
| $r = -4.0$  | - / 89.80           | - / 88.20           | - / 58.47        | - / 46.86        |
| $r = -6.0$  | - / 90.02           | - / 88.18           | - / 57.87        | - / 47.18        |

D.3 Experiment details on synthetic datasets and UCI

We introduce experiment details on synthetic datasets and UCI datasets adopted in our experiment designs.

Generation of synthetic dataset In the synthetic (Type 1) dataset, we generate 500 points for both classes. Class +1 distributes inside the circle with radius 0.25. Class -1 generates by randomly sampling 500 data points in the annulus with inner radius 0.28 and outer radius 0.45. As for synthetic (Type 2) dataset, we uniformly assign labels for 50% samples in the annulus (with inner radius 0.22, outer radius 0.31) based on Type 1 dataset.
Generating noise labels on synthetic datasets and UCI datasets  
Note that these datasets are all binary classification datasets, each label in the training and validation set is flipped to the other class with probability $e$, and we set $e = 0.1, 0.4$ for synthetic Type 1 dataset, $e = 0.1, 0.3$ for synthetic Type 2 dataset.

Training settings of synthetic datasets  
For both types of synthetic datasets, we adopted a three-layer ReLU Multi-Layer Perceptron (MLP), trained for 200 epochs with batch-size 128 and Adam [Kingma and Ba, 2014] optimizer. The initial learning rate is 0.1, and it multiplies 0.1 for every 40 epochs.

Training settings of UCI datasets [Dua and Graff, 2017]  
We adopted [Liu and Guo, 2020] a two-layer ReLU Multi-Layer Perceptron (MLP) for classification tasks on multiple UCI datasets, trained for 1000 episodes with batch-size 64 and Adam [Kingma and Ba, 2014] optimizer. We report the best performance for each smooth rate under a set of learning rate settings, $[0.0007, 0.001, 0.005, 0.01, 0.05]$.

D.4 Additional experiment on $r^*$ and $r_{opt}$  

$r^*$ and $r_{opt}$ on synthetic dataset  
We generate 2D (binary) synthetic dataset by randomly sampling two circularly distributed classes. The inner annulus indicates one class (blue), while the outer annulus denotes the other class (red). Clearly, the generated synthetic dataset is well-separable (Type 1) and we hold 20% data samples for performance comparison. The noise transition matrix takes a symmetric form with noise rate $e_j$ for both classes. To simulate the scenario where the clean data may not be perfectly separated due to a non-negligible amount of uncertainty samples clustering at the decision boundary, we flip the label of 50% samples near the intersection of two annulus to the other class (Type 2). As specified in Table 7, $r^* = [0.1, 0.4]$ for Type 1 data and $r^* = [0.0, 0.2]$ for Type 2 data. With the presence of label noise, the distribution of $r_{opt}$ shifts from non-negative ones to negative values. Even though NLS fails to outperform LS on clean data, we observe that NLS is less sensitive to noisy labels. Data with high level noise rates clearly favor NLS with a low smooth rate!

Table 7: Test accuracies of GLS on clean and noisy synthetic data. We report best test accuracy for each method. $r_{opt}$ and the corresponding test accuracy are highlighted (green: NLS; red: CE or LS).

| Method            | Synthetic data (Type 1) | Synthetic data (Type 2) |
|-------------------|-------------------------|-------------------------|
|                   | $e_i = 0$ | $e_i = 0.2$ | $e_i = 0.4$ | $e_i = 0$ | $e_i = 0.2$ | $e_i = 0.4$ |
| LS                | 0.896 | 0.878 | 0.786 | 0.894 | 0.848 | 0.842 |
| Vanilla Loss      | 0.889 | 0.882 | 0.806 | 0.894 | 0.875 | 0.868 |
| NLS               | 0.893 | 0.885 | 0.825 | 0.885 | 0.864 | 0.875 |
| $r_{opt} = (0.1, 0.4)$ | -0.2 | -0.4 | -0.3 | -0.2 | -0.3 | -0.5 |

Table 8: Test accuracy comparisons on clean and noisy UCI datasets (Image, Waveform, Heart, Banana) with best two smooth rates (green: NLS; red: CE or LS).

| Smooth Rate | $e_i = 0$ | $e_i = 0.1$ | $e_i = 0.2$ | $e_i = 0.3$ | $e_i = 0.4$ | $e_i = 0$ | $e_i = 0.2$ | $e_i = 0.3$ | $e_i = 0.4$ |
|-------------|-----------|-------------|-------------|-------------|-------------|-----------|-------------|-------------|-------------|
| r = 0.8     | 0.999 | 0.983 | 0.973 | 0.946 | 0.875 | 0.939 | 0.935 | 0.931 | 0.972 | 0.885 |
| r = 0.6     | 0.993 | 0.987 | 0.970 | 0.939 | 0.859 | 0.943 | 0.943 | 0.943 | 0.929 | 0.901 |
| r = 0.4     | 0.997 | 0.983 | 0.970 | 0.939 | 0.846 | 0.941 | 0.937 | 0.943 | 0.929 | 0.901 |
| r = 0.2     | 0.993 | 0.993 | 0.966 | 0.936 | 0.875 | 0.941 | 0.935 | 0.933 | 0.931 | 0.913 |
| r = 0.0     | 0.990 | 0.976 | 0.963 | 0.929 | 0.865 | 0.945 | 0.935 | 0.937 | 0.933 | 0.911 |
| r = 0.2     | 0.912 | 0.936 | 0.953 | 0.919 | 0.872 | 0.937 | 0.939 | 0.939 | 0.933 | 0.907 |
| r = 0.1     | 0.882 | 0.923 | 0.953 | 0.916 | 0.872 | 0.925 | 0.937 | 0.939 | 0.933 | 0.917 |
| r = 0.0     | 0.842 | 0.882 | 0.926 | 0.933 | 0.872 | 0.921 | 0.925 | 0.939 | 0.931 | 0.923 |
| r = 0.1     | 0.832 | 0.889 | 0.909 | 0.929 | 0.882 | 0.921 | 0.923 | 0.933 | 0.929 | 0.907 |
| r = 0.0     | 0.818 | 0.815 | 0.889 | 0.909 | 0.906 | 0.911 | 0.913 | 0.921 | 0.927 | 0.911 |

| Smooth Rate | $e_i = 0$ | $e_i = 0.1$ | $e_i = 0.2$ | $e_i = 0.3$ | $e_i = 0.4$ | $e_i = 0$ | $e_i = 0.2$ | $e_i = 0.3$ | $e_i = 0.4$ |
|-------------|-----------|-------------|-------------|-------------|-------------|-----------|-------------|-------------|-------------|
| r = 0.8     | 0.990 | 0.990 | 0.966 | 0.982 | 0.968 | 0.896 | 0.893 | 0.876 | 0.847 | 0.790 |
| r = 0.6     | 0.990 | 0.989 | 0.987 | 0.981 | 0.971 | 0.903 | 0.881 | 0.876 | 0.855 | 0.811 |
| r = 0.4     | 0.990 | 0.990 | 0.987 | 0.983 | 0.971 | 0.900 | 0.887 | 0.874 | 0.859 | 0.807 |
| r = 0.2     | 0.990 | 0.990 | 0.986 | 0.985 | 0.969 | 0.896 | 0.894 | 0.876 | 0.856 | 0.810 |
| r = 0.0     | 0.990 | 0.990 | 0.987 | 0.985 | 0.973 | 0.897 | 0.881 | 0.875 | 0.859 | 0.833 |
| r = 0.4     | 0.990 | 0.990 | 0.988 | 0.986 | 0.972 | 0.844 | 0.874 | 0.874 | 0.853 | 0.840 |
| r = 0.2     | 0.986 | 0.986 | 0.987 | 0.994 | 0.974 | 0.845 | 0.864 | 0.861 | 0.859 | 0.837 |
| r = 0.0     | 0.986 | 0.986 | 0.988 | 0.995 | 0.977 | 0.796 | 0.812 | 0.852 | 0.854 | 0.811 |
| r = 0.1     | 0.986 | 0.986 | 0.986 | 0.986 | 0.978 | 0.759 | 0.764 | 0.819 | 0.852 | 0.819 |
| r = 0.0     | 0.986 | 0.986 | 0.986 | 0.986 | 0.986 | 0.718 | 0.722 | 0.758 | 0.78 | 0.813 |
| r = 0.1     | 0.986 | 0.986 | 0.986 | 0.986 | 0.986 | 0.703 | 0.706 | 0.709 | 0.735 | 0.735 |
We further test the performance of generalized label smoothing on 7 more UCI datasets (Heart, Breast 1, Breast 2, Diabetes, German, Image and Waveform). Our observation remains unchanged: there exists a general trend that with the increasing of noise rates, NLS becomes much more competitive than LS. Here, we attach the results of 4 additional UCI datasets for illustration.

The noisy labels are generated by a symmetric noise transition matrix with noise rate $e_i = [0.1, 0.2, 0.3, 0.4]$. As highlighted in Table, $r_{opt}$ appears with positive values when the data is clean (same as $r^*$) or of a low noise rate. With the increasing of noise rates, NLS becomes more competitive than LS. We color-code different noise regimes where either LS (red-ish) or NLS (green-ish) outperforms the other. Clearly, there is a separation of the favored smoothing rate for different noise scenarios (upper left & low noise for LS, bottom right & high noise for NLS).

**D.5 Additional experiment results on model confidence**

**NLS improves model confidence on Synthetic Type 2 dataset** In this case, the clean data that are close to decision boundary distributes randomly. In Figure, the colored bands depict the different levels of prediction probabilities. When the smooth rate increases from negative to positive, more samples fall in the orange and light blue band which indicates uncertain predictions. When the smooth rate increases from negative to positive, learning with smoothed labels will result in more uncertain predictions. With the increasing of noise rates ($e_i = 0 \rightarrow 0.4$), Learning with a fixed smooth rate generally becomes less confident on its predictions. Thus, a smaller smooth rate is required when the noise rate increases.

![Model confidence visualization](image)

Figure 6: Model confidence visualization of NLS, VL and LS on synthetic data (Type 2) with the clean data. $r^* \in [0, 0.2]$. (left: NLS; middle: Vanilla Loss; right: LS).

![Model confidence visualization](image)

Figure 7: Model confidence visualization of NLS, VL and LS on synthetic data (Type 2) with noise rate $e_i = 0.3$. $r_{opt} = -0.5$. (left: NLS; middle: Vanilla Loss; right: LS).

**D.6 Effect of LS and NLS on pre-logits**

We visualise the pre-logits of a ResNet-34 for three classes on CIFAR-10. We adopt the method from [Müller et al., 2019] which illustrates how representations differ between penultimate layers of networks trained with different smooth rates in GLS. In Figure, NLS makes the model $f$ be confident on her predictions and the distances between three clusters are clearly larger than those appeared in Vanilla Loss and LS.
Figure 8: Effect of GLS on pre-logits (left: NLS; middle: Vanilla Loss; right: LS; trained with symmetric 0.2 noisy CIFAR-10 training dataset).

E Bias and Variance Trade-off of Learning with Smoothed Labels

Denote \( \hat{f}_H \), \( \hat{f}_S \) as pre-trained models on the training dataset \( D \) w.r.t. hard labels and soft labels, respectively. The vector form of the prediction w.r.t. sample \( x \) given by \( \hat{f}_H \) and \( \hat{f}_S \) are \( \hat{f}_H(x; D) \) and \( \hat{f}_S(x; D) \). For the ease of presentation, we relate notations with subscript H/S to hard/soft labels without further explanation. Given the sample \( x \) and the one-hot label \( y \), we denote the averaged model prediction by:

\[
\bar{f}_H(x; D) := \frac{1}{Z_H} \exp \left[ \mathbb{E}_D \log(\hat{f}_H(x; D)) \right], \quad \bar{f}_S(x; D) := \frac{1}{Z_S} \exp \left[ \mathbb{E}_D \log(\hat{f}_S(x; D)) \right]
\]

where \( Z_H, Z_S \) are normalization constants. The bias of model prediction is defined as the KL divergence \( D_{\text{KL}} \) between target distribution (one-hot encoded vector form) \( y \) and the averaged model prediction.

\[
\text{Bias}_H := \mathbb{E}_{x,y} \left[ y \log \frac{y}{\bar{f}_H(x; D)} \right], \quad \text{Bias}_S := \mathbb{E}_{x,y} \left[ y \log \frac{y}{\bar{f}_S(x; D)} \right]
\]

While the variance of model prediction measures the expectation of KL divergence between the averaged model prediction and model prediction over \( D \):

\[
\text{Var}_H := \mathbb{E}_D \left[ \mathbb{E}_{x,y} \left[ \bar{f}_H(x; D) \log \frac{\bar{f}_H(x; D)}{\bar{f}_H(x; D)} \right] \right], \quad \text{Var}_S := \mathbb{E}_D \left[ \mathbb{E}_{x,y} \left[ \bar{f}_S(x; D) \log \frac{\bar{f}_S(x; D)}{\bar{f}_S(x; D)} \right] \right]
\]

Empirical observation from [Zhou et al., 2021] shows that the variance brought by learning with positive soft labels given by a teacher’s model [Hinton et al., 2015] is less than the direct training w.r.t hard labels. As an extension, we are interested in how LS/NLS interferes with the bias and variance of model prediction.

Figure 9: Bias and variance of pre-trained LS/NLS models on clean CIFAR-10 test dataset.
Bias and variance of LS/NLS on clean dataset  We introduce our empirical observation regarding the role of LS/NLS in bias and variance trade-off in Figure 9. We select nine smooth rates of LS/NLS for illustration. Each smooth rate setting of LS/NLS trains on the CIFAR-10 dataset for 5 times with different data augmentations. To estimate the variance and bias of pre-trained models, we adopt the implementation in [Yang et al., 2020]. Empirical results show that learning directly with a larger positive smooth rate typically results in lower variance and higher bias. In Figure 9, we can observe almost constant bias values and very low variance for NLS. This is best explained by the warm-up of pre-trained models and the fact that NLS pushes the classifier to give confident predictions. As for LS, with the increase of smooth rate, the overall bias has an increasing tendency while the variance has the decreasing pattern. Especially when the smooth rate approaches to 1, i.e., \( r = 0.9 \), the variance is close to 0.

F  Omitted Proofs

We observe that NLS connects to a special case of label smoothing regularization [Szegedy et al., 2016]. We highlight this in Theorem F.1.

**Theorem F.1.** \( \forall r \in [0, 1], \) NLS with smooth rate \(-r\) is a special form of label smoothing regularization:

\[
\min_E \mathbb{E}_{(X, \tilde{Y}) \sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}^{GLS,-r}) \right] = \min_E \mathbb{E}_{(X, \tilde{Y}) \sim \tilde{D}} \left[ 2 \cdot \ell(f(X), \tilde{Y}) - \ell(f(X), \tilde{Y}^{GLS,r}) \right].
\]

**F.1  Proof of Theorem F.1**

Before we prove Theorem F.1, we first introduce Lemma F.2.

**Lemma F.2.** \( \forall (x, y^{GLS,r}), \) \( \ell(f(x), y^{GLS,r}) = \left( 1 - \frac{r}{2} \right) \cdot \ell(f(x), y) + \frac{r}{2} \cdot \ell(f(x), 1 - y) \).

**Proof of Lemma F.2**

**Proof.** For CE loss, due to its linear property w.r.t. the label, we directly have:

\[
\ell(f(x), y^{GLS,r}) = \ell(f(x), y \cdot (1 - r) + \frac{r}{2}) = \left( 1 - \frac{r}{2} \right) \cdot \ell(f(x), y) + \frac{r}{2} \cdot \ell(f(x), 1 - y).
\]

\( \square \)

**Proof of Theorem F.1**

**Proof.** Based on Lemma F.2 with a bit of math, for NLS, we have:

\[
\min_E \mathbb{E}_{(X, \tilde{Y}) \sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}^{GLS,-r}) \right] = \min_E \mathbb{E}_{(X, \tilde{Y}) \sim \tilde{D}} \left[ \left( 1 + \frac{r}{2} \right) \cdot \ell(f(X), \tilde{Y}) - \frac{r}{2} \cdot \ell(f(X), 1 - \tilde{Y}) \right] = \min_E \mathbb{E}_{(X, \tilde{Y}) \sim \tilde{D}} \left[ \left( (1 + \frac{r}{2}) + (1 - \frac{r}{2}) \right) \cdot \ell(f(X), \tilde{Y}) - \left[ (1 - \frac{r}{2}) \cdot \ell(f(X), \tilde{Y}) + \frac{r}{2} \cdot \ell(f(X), 1 - \tilde{Y}) \right] \right] = \min_E \mathbb{E}_{(X, \tilde{Y}) \sim \tilde{D}} \left[ 2 \cdot \ell(f(X), \tilde{Y}) - \ell(f(X), \tilde{Y}^{GLS,r}) \right].
\]

\( \square \)
F.2 Proof of Theorem 5.2

Proof.

\[
\begin{align*}
\text{Eqn.} \ 3 & = \min_{E(X, \tilde{Y})} \mathbb{E} \left[ \left( 1 - \frac{r}{2} \right) \cdot \ell(f(X), \tilde{Y}) + \frac{r}{2} \cdot \ell(f(X), 1 - \tilde{Y}) \right] \\
& = \min_{E(X, Y) = 0} \mathbb{P}(\tilde{Y} = 0 | Y = 0) \cdot \left( c_1 \cdot \ell(f(X), 0) + c_2 \cdot \ell(f(X), 1) \right) \\
& \quad + \mathbb{P}(\tilde{Y} = 1 | Y = 0) \cdot \left( c_1 \cdot \ell(f(X), 1) + c_2 \cdot \ell(f(X), 0) \right) \\
& \quad + \mathbb{E}_{X, Y = 1} \left[ \mathbb{P}(\tilde{Y} = 0 | Y = 1) \cdot \left( c_1 \cdot \ell(f(X), 0) + c_2 \cdot \ell(f(X), 1) \right) \\
& \quad + \mathbb{P}(\tilde{Y} = 1 | Y = 1) \cdot \left( c_1 \cdot \ell(f(X), 1) + c_2 \cdot \ell(f(X), 0) \right) \right] \\
& = \min_{E(X, Y) = 0} \left[ (1 - e_0) \cdot c_1 + e_0 \cdot c_2 \cdot \ell(f(X), 0) + (1 - e_0) \cdot c_2 + e_0 \cdot c_1 \cdot \ell(f(X), 1) \right] \\
& \quad + \mathbb{E}_{X, Y = 1} \left[ (1 - e_0) \cdot c_1 + e_0 \cdot c_2 \cdot \ell(f(X), 1) + (1 - e_0) \cdot c_2 + e_0 \cdot c_1 \cdot \ell(f(X), 0) \right] \\
& = \min_{E(X, Y) = 0} \left[ (1 - e_0) \cdot c_1 + e_0 \cdot c_2 \cdot \ell(f(X), Y) + (1 - e_0) \cdot c_2 + e_0 \cdot c_1 \cdot \ell(f(X), 1 - Y) \right] \\
& \quad - e_\Delta \cdot (c_1 - c_2) \cdot \mathbb{E}_{X, Y = 1} \left[ \ell(f(X), 1) - \ell(f(X), 0) \right] \\
& = \min_{E(X, Y) = 0} \left[ (c_1 + c_2) \cdot \ell(f(X), Y) \right] \\
& \quad + \left[ (1 - e_0) \cdot c_2 + e_0 \cdot c_1 \cdot \mathbb{E}(X, Y) \cdot \ell(f(X), 1 - Y) - \ell(f(X), Y) \right] \\
& \quad - e_\Delta \cdot (c_1 - c_2) \cdot \mathbb{E}_{X, Y = 1} \left[ \ell(f(X), 1) - \ell(f(X), 0) \right] \\
& = \min_{E(X, Y) = 0} \left[ (c_1 + c_2) \cdot \ell(f(X), Y^*) \right] \\
& \quad + \left[ - \frac{r}{2} + (1 - e_0) \cdot c_2 + e_0 \cdot c_1 \cdot \mathbb{E}(X, Y) \cdot \ell(f(X), 1 - Y) - \ell(f(X), Y) \right] \\
& \quad - e_\Delta \cdot (c_1 - c_2) \cdot \mathbb{E}_{X, Y = 1} \left[ \ell(f(X), 1) - \ell(f(X), 0) \right] \\
& = \min_{E(X, Y) = 0} \left[ \ell(f(X), Y^*) \right] + \lambda_1 \cdot \mathbb{E}(X, Y) \cdot \ell(f(X), 1 - Y) - \ell(f(X), Y) \\
& \quad + \lambda_2 \cdot \mathbb{E}_{X, Y = 1} \left[ \ell(f(X), 0) - \ell(f(X), 1) \right] \\& \quad + \mathbb{E}(X, Y) \cdot \ell(f(X), Y) \\
& \quad + \lambda_1 \cdot \mathbb{E}(X, Y) \cdot \ell(f(X), 1 - Y) - \ell(f(X), Y) \\
& \quad + \lambda_2 \cdot \mathbb{E}_{X, Y = 1} \left[ \ell(f(X), 0) - \ell(f(X), 1) \right] \\& \quad + \lambda_2 \cdot \mathbb{E}_{X, Y = 1} \left[ \ell(f(X), 0) - \ell(f(X), 1) \right]
\end{align*}
\]

F.3 Proof of Proposition 5.1

Proof. The risk minimization of backward correction is equivalent to:

\[
\mathbb{E}(X, \tilde{Y}) \cdot \ell^*(f(X), \tilde{Y}) = \mathbb{E}(X, Y) \cdot \ell(f(X), Y) \quad \text{(By Theorem 1 in [Patrini et al., 2017])}
\]

The risk minimization of forward correction is equivalent to:

\[
\mathbb{E}(X, \tilde{Y}) \cdot \ell^+(f(X), \tilde{Y}) = \mathbb{E}(X, Y) \cdot \ell(f(X), Y) \quad \text{(By Theorem 2 in [Patrini et al., 2017])}
\]

20
Theorem 1 and 2 in [Patrini et al., 2017] demonstrate that forward and backward corrected losses equal the original loss \( \ell \) computed on the clean data in expectation. Thus, for \( r_{LC} = \frac{2\epsilon_0}{2\epsilon_0 - r} \), by Theorem 3.2 (adopt \( r^* = 0 \)), we have:

\[
\min E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}^{GLS, r_{LC}}) \right] + \lambda_{LC} \cdot E_{X,Y=1} \left[ \ell(f(X), 1) - \ell(f(X), 0) \right] = \min E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), Y) \right] + e_\Delta \cdot \left( \frac{1}{1 - 2\epsilon_0} - \frac{1}{1 - 2\epsilon_0} \right) \cdot E_{X,Y=1} \left[ \ell(f(X), 0) - \ell(f(X), 1) \right] = \min E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), Y) \right]
\]

Thus,

\[
\min E_{(X,Y)\sim \tilde{D}} \left[ \ell^{-}(f(X), \tilde{Y}) \right] = \min E_{(X,Y)\sim \tilde{D}} \left[ \ell^{+}(f(X), \tilde{Y}) \right] = \min E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}^{GLS, r_{LC}}) \right]
\]

\[\square\]

**F.4 Proof of Theorem 5.2**

*Proof.* Based on Proposition 5.1, when \( e_\Delta = 0, \lambda_{LC} = 0 \), we directly have:

\[
\min E_{(X,Y)\sim \tilde{D}} \left[ \ell^{-}(f(X), \tilde{Y}) \right] = \min E_{(X,Y)\sim \tilde{D}} \left[ \ell^{+}(f(X), \tilde{Y}) \right] = \min E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}^{GLS, r_{LC}}) \right]
\]

\[\square\]

**F.5 Proof of Theorem 5.3**

*Proof.* Note that

\[
\min E_{(X,Y)\sim \tilde{D}} \left[ \ell_{CL}(f(X), \tilde{Y}) \right] = \min E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}) - \ell(f(X), 1 - \tilde{Y}) \right]
\]

We have:

\[
\min E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}^{GLS, r_{CL}}) \right] = \min E_{(X,Y)\sim \tilde{D}} \left[ \left( 1 - \frac{r_{CL}}{2} \right) \cdot \ell(f(X), \tilde{Y}) + \frac{r_{CL}}{2} \cdot \ell(f(X), 1 - \tilde{Y}) \right] \iff \min E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}) + \frac{r_{CL}}{2} \cdot \ell(f(X), 1 - \tilde{Y}) \right]
\]

When \( r_{CL} \to -\infty \), we have \( \frac{r_{CL}}{2 - r_{CL}} \to -1 \). Thus,

\[
\min E_{(X,Y)\sim \tilde{D}} \left[ \ell_{CL}(f(X), \tilde{Y}) \right] = \min E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}^{GLS, r_{CL}}) \to -\infty) \right]
\]

\[\square\]

**F.6 Proof of Proposition 5.4**

*Proof.* Note that:

\[
E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}) \right] - E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}^{GLS, r}) \right] = E_{(X,Y)\sim \tilde{D}} \left[ 1 - \left( 1 - \frac{r}{2} \right) \cdot \ell(f(X), \tilde{Y}) - \frac{r}{2} \cdot \ell(f(X), 1 - \tilde{Y}) \right] = \frac{r}{2} \cdot E_{(X,Y)\sim \tilde{D}} \left[ \ell(f(X), \tilde{Y}) - \ell(f(X), 1 - \tilde{Y}) \right]
\]
And we have:

\[
E_{(X,Y) \sim \hat{D}} \left[ \ell(f(X), \hat{Y}) \right] \\
= E_X \left[ P(\hat{Y} = 0) \cdot \ell(f(X), 0) + (1 - P(\hat{Y} = 0)) \cdot \ell(f(X), 1) \right] \\
= E_{X,\hat{Y}=0} \left[ P(\hat{Y} = 0) \cdot \ell(f(X), 0) + (1 - P(\hat{Y} = 0)) \cdot \ell(f(X), 1) \right] \\
+ E_{X,\hat{Y}=1} \left[ (1 - P(\hat{Y} = 0)) \cdot \ell(f(X), 0) + P(\hat{Y} = 0) \cdot \ell(f(X), 1) \right] \\
+ (1 - 2 \cdot P(\hat{Y} = 0)) \cdot E_{X,\hat{Y}=1} \left[ \ell(f(X), 1) - \ell(f(X), 0) \right]
\]

Thus,

\[
\min_f E_{(X,Y) \sim \hat{D}} \left[ \ell_p(f(X), \hat{Y}) \right] = \min_f E_{(X,Y) \sim \hat{D}} \left[ \ell(f(X), \hat{Y}) - \ell(f(X_1), \hat{Y}_2) \right] \\
= \min_f E_{(X,Y) \sim \hat{D}} \left[ \ell(f(X), \hat{Y}) - E_{(X,Y) \sim \hat{D}} \left[ \ell(f(X_1), Y_2) \right] \right] \\
= \min_f E_{X,\hat{Y}=0} \left[ \ell(f(X), 0) \right] + E_{X,\hat{Y}=1} \left[ \ell(f(X), 1) \right] \\
- E_{X,\hat{Y}=0} \left[ P(\hat{Y} = 0) \cdot \ell(f(X), 0) + (1 - P(\hat{Y} = 0)) \cdot \ell(f(X), 1) \right] \\
- E_{X,\hat{Y}=1} \left[ (1 - P(\hat{Y} = 0)) \cdot \ell(f(X), 0) + P(\hat{Y} = 0) \cdot \ell(f(X), 1) \right] \\
- (1 - 2 \cdot P(\hat{Y} = 0)) \cdot E_{X,\hat{Y}=1} \left[ \ell(f(X), 1) - \ell(f(X), 0) \right]
\]

Thus, for \( r_{PL} = 2 \cdot P(\hat{Y} = 1) \), \( \lambda_{PL} = 1 - r_{PL} \), we have:

\[
E_{(X,Y) \sim \hat{D}} \left[ \ell_{PL}(f(X), \hat{Y}) \right] - E_{(X,Y) \sim \hat{D}} \left[ \ell(f(X), \hat{Y}) \right] - E_{(X,Y) \sim \hat{D}} \left[ \ell(f(X), Y^{GLS,r_{PL}}) \right] \\
= E_{(X,Y) \sim \hat{D}} \left[ (1 - P(\hat{Y} = 0)) \cdot [\ell(f(X), \hat{Y}) - \ell(f(X), 1 - \hat{Y})] \right] \\
- (1 - 2 \cdot P(\hat{Y} = 0)) \cdot E_{X,\hat{Y}=1} \left[ \ell(f(X), 1) - \ell(f(X), 0) \right] \\
- \frac{r_{PL}}{2} \cdot E_{(X,Y) \sim \hat{D}} \left[ \ell(f(X), \hat{Y}) - \ell(f(X), 1 - \hat{Y}) \right] \\
= E_{(X,Y) \sim \hat{D}} \left[ (1 - P(\hat{Y} = 0) - P(\hat{Y} = 1)) \cdot [\ell(f(X), \hat{Y}) - \ell(f(X), 1 - \hat{Y})] \right] \\
- (2 \cdot P(\hat{Y} = 1) - 1) \cdot E_{X,\hat{Y}=1} \left[ \ell(f(X), 1) - \ell(f(X), 0) \right] \\
= \lambda_{PL} \cdot E_{X,\hat{Y}=1} \left[ \ell(f(X), 1) - \ell(f(X), 0) \right]
\]
And we can conclude that:

\[
\min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell_{\mathcal{PL}}(f(X), Y) \right] = \min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell(f(X), Y) - \ell(f(X), \hat{Y}^{\text{GLS}, r_{\mathcal{PL}}}) \right] \\
+ \lambda_{PL} \cdot E_{X, Y = 1} \left[ \ell(f(X), 1) - \ell(f(X), 0) \right]
\]

\[\text{Bias-PL}\]

\[\square\]

**F.7 Proof of Theorem 5.5**

**Proof.** When \(\mathbb{P}(\hat{Y} = 0) = \mathbb{P}(\hat{Y} = 1)\), according to Proposition 5.4, we have \(\lambda_{PL} = 0\) and:

\[
\min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell_{\mathcal{PL}}(f(X), Y) \right] = \min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell(f(X), Y) - \ell(f(X), Y^{\text{GLS}, r_{\mathcal{PL}}}) \right]
\]

\[\leftarrow \min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell(f(X), Y) - \ell(f(X), 1 - \hat{Y}) \right]
\]

When \(r_{\mathcal{PL}} \to -\infty\), we further have:

\[
\min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell(f(X), Y^{\text{GLS}, r_{\mathcal{PL}}}) \right] \leftarrow \min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell(f(X), Y) + \frac{r_{\mathcal{PL}}}{2 - r_{\mathcal{CL}}} \cdot \ell(f(X), 1 - Y) \right]
\]

\[\leftarrow \min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell(f(X), Y) - \ell(f(X), 1 - \hat{Y}) \right]
\]

Thus, Theorem 5.5 is proved. \(\square\)

**F.8 Proof of Theorem 3.3**

**Proof.** Note that the optimal \(r\) that will cancel the impact of Term M-Inc1 is:

\[
r_{opt} := \frac{r^* - 2e}{1 - 2e}
\]

- When \(e < \frac{r^*}{2}, r_{opt} > 0\). In this case, learning LS with smooth rate \(r_{opt}\) results in:

\[
\min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell(f(X), Y^{\text{GLS}, r_{opt}}) \right] = \min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell(f(X), Y^*) \right]
\]

which yields \(f^*_D\);

- When \(e = \frac{r^*}{2}, r_{opt} = 0\). Learning with the Vanilla Loss yields \(f^*_D\) since:

\[
\min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell(f(X), Y) \right] = \min_{\mathcal{X}, \mathcal{Y}} E_{(X, Y) \sim \mathcal{D}} \left[ \ell(f(X), Y^*) \right]
\]

- Similarly, when \(e > \frac{r^*}{2}\), learning NLS with \(r = r_{opt} < 0\) yields \(f^*_D\).

\[\square\]
F.9 Proof of Theorem 3.5

Proof. Denote \( p_i = \mathbb{P}(Y = i) \) as the clean label distribution, \( \tilde{p}_i = \mathbb{P}(\tilde{Y} = i) \) as the clean label distribution. Let \( \epsilon' = \frac{K \epsilon}{K-1} \), we have:

\[
\mathbb{E}_{(X,\tilde{Y}) \sim \tilde{D}} ((1 - r) \cdot \ell(f(X), \tilde{Y})) + \mathbb{E}_X \left[ \sum_{i \in [K]} \frac{r}{K} \cdot \ell(f(X), i) \right]
\]

\[
= \sum_{i \in [K]} \mathbb{E}_{(X,\tilde{Y}) \sim \tilde{D}, Y = i} \left[ (1 - r) \cdot \ell(f(X), \tilde{Y}) \right] + \mathbb{E}_X \left[ \sum_{i \in [K]} \frac{r}{K} \cdot \ell(f(X), i) \right]
\]

\[
= (1 - r) \cdot \sum_{i \in [K]} \mathbb{E}_{X,Y} \left[ (1 - r) \cdot \ell(f(X), i) + \sum_{j \in [K]} \frac{r}{K} \cdot \ell(f(X), j) \right] + \mathbb{E}_X \left[ \sum_{i \in [K]} \frac{r}{K} \cdot \ell(f(X), i) \right]
\]

\[
= (1 - r) \cdot \sum_{i \in [K]} \mathbb{E}_{X,Y} \left[ (1 - r) \cdot \ell(f(X), i) \right] + \mathbb{E}_X \left[ \frac{(1 - r) \cdot \ell'}{K} + \frac{r}{K} \sum_{j \in [K]} \ell(f(X), j) \right]
\]

\[
= (1 - r) \cdot \underbrace{\mathbb{E}_{(X,Y) \sim D} \left[ \ell(f(X), Y) \right]}_{c_3} + \mathbb{E}_X \left[ \frac{(1 - r) \cdot \ell'}{K} + \frac{r}{K} \sum_{j \in [K]} \ell(f(X), j) \right]
\]

\[
= \underbrace{\frac{c_3}{1 - r^*} \cdot \mathbb{E}_{(X,Y) \sim D} \left[ \ell(f(X), Y^*) \right]}_{\text{True Risk}} + \left[ c_4 \cdot \mathbb{E}_X \left[ \sum_{j \in [K]} \ell(f(X), j) \right] \right]
\]

Adopting \( r_{\text{opt}} = \frac{\epsilon' - \epsilon}{1-r^*} \), with a bit of math, the weight of Term M-Inc1 becomes 0 and

\[
\mathbb{E}_{(X,Y) \sim D} \left[ \ell(f(X), Y^{GLS,r_{\text{opt}}}) \right]
\]

\[
= \mathbb{E}_{(X,Y) \sim D} \left[ (1 - r_{\text{opt}}) \cdot \ell(f(X), \tilde{Y}) \right] + \mathbb{E}_X \left[ \sum_{i \in [K]} r_{\text{opt}} \cdot \ell(f(X), i) \right]
\]

\[
= \left[ \frac{c_3}{1 - r^*} \cdot \mathbb{E}_{(X,Y) \sim D} \left[ \ell(f(X), Y^*) \right] \right] \iff \mathbb{E}_{(X,Y) \sim D} \left[ \ell(f(X), Y^*) \right]
\]

\[
\square
\]