THE CLASSIFICATION OF QUASI-ALTERNATING MONTESINOS LINKS

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ABSTRACT. In this note, we complete the classification of quasi-alternating Montesinos links. We show that the quasi-alternating Montesinos links are precisely those identified independently by Qazaqzeh-Chbili-Qublan and Champanerkar-Ordin. A consequence of our proof is that a Montesinos link $L$ is quasi-alternating if and only if its double branched cover is an L-space, and bounds both a positive definite and a negative definite 4-manifold with vanishing first homology.

1. INTRODUCTION

Quasi-alternating links were defined by Ozsváth-Szabó [OS05, Definition 3.1] as a natural generalisation of the class of alternating links.

Definition 1. The set $Q$ of quasi-alternating links is the smallest set of links satisfying the following:

- The unknot $U$ belongs to $Q$.
- If $L$ is a link with a diagram containing a crossing $c$ such that
  1. both smoothings $L_0$ and $L_1$ of the link $L$ at the crossing $c$, as in Figure 1, belong to $Q$,
  2. $\det(L_0), \det(L_1) \geq 1$, and
  3. $\det(L) = \det(L_0) + \det(L_1)$,
then $L$ is in $Q$. The crossing $c$ is called a quasi-alternating crossing.

![Figure 1. $L$ and its two resolutions $L_0$ and $L_1$ in a neighbourhood of $c$.](image)

Ozsváth-Szabó showed that the class of non-split alternating links is contained in $Q$ [OS05, Lemma 3.2]. Moreover, quasi-alternating links share a number of properties with alternating links; we list a few of these. For a quasi-alternating link $L$:

(i) $L$ is homologically thin for both Khovanov homology and knot Floer homology [MO08].
(ii) The double branched cover $\Sigma(L)$ of $L$ is an L-space [OS05, Proposition 3.3].
(iii) The 3-manifold $\Sigma(L)$ bounds a smooth negative definite 4-manifold $W$ with $H_1(W) = 0$ [OS05, Proof of Lemma 3.6].
For some further properties see [LO15], [QC15], [Ter15] and [ORS13, Remark after Proposition 5.2].

Due to their recursive definition, it is difficult in general to determine whether or not a link is quasi-alternating. For example, there still remain examples of 12-crossing knots with unknown quasi-alternating status [Jab14]. Champanerkar-Kofman [CK09] showed that the quasi-alternating property is preserved by replacing a quasi-alternating crossing with an alternating rational tangle. They used this to determine an infinite family of quasi-alternating pretzel links, which Greene later showed is the complete set of quasi-alternating pretzel links [Gre10].

Qazaqzeh-Chbili-Qublan [QCQ15] and Champanerkar-Ording [CO15] independently generalised the sufficient conditions on pretzel links to obtain an infinite family of quasi-alternating Montesinos links. This family includes all examples of quasi-alternating Montesinos links found by Widmer [Wid09]. Furthermore, it was conjectured by Qazaqzeh-Chbili-Qublan that this family is the complete set of quasi-alternating Montesinos links. We mention that Watson [Wat11] gave an iterative surgical construction for constructing all quasi-alternating Montesinos links.

Some necessary conditions to be quasi-alternating in terms of the rational parameters of a Montesinos link were obtained in [QCQ15] and [CO15] based on the fact that a quasi-alternating link is homologically thin. Further conditions are described in [CO15] coming from the fact that the double branched cover of a quasi-alternating link is an L-space. Some additional restrictions were found in [QC15].

Our main result is the following theorem which states that the quasi-alternating Montesinos links are precisely those found by Qazaqzeh-Chbili-Qublan [QCQ15] and Champanerkar-Ording [CO15]:

**Theorem 1.** Let \( L = M(e; t_1, \ldots, t_p) \) be a Montesinos link in standard form, that is, where \( t_i = \frac{\alpha_i}{\beta_i} > 1 \) and \( \alpha_i, \beta_i > 0 \) are coprime for all \( i = 1, \ldots, p \). Then \( L \) is quasi-alternating if and only if

1. \( e < 1 \), or
2. \( e = 1 \) and \( \frac{\alpha_i}{\alpha_i - \beta_i} > \frac{\alpha_j}{\beta_j} \) for some \( i, j \) with \( i \neq j \), or
3. \( e > p - 1 \), or
4. \( e = p - 1 \) and \( \frac{\alpha_i}{\alpha_i - \beta_i} < \frac{\alpha_j}{\beta_j} \) for some \( i, j \) with \( i \neq j \).

As a corollary of our proof we obtain the following characterisation of the Montesinos links \( L \) which are quasi-alternating in terms of their double branched covers \( \Sigma(L) \):

**Corollary 1.** A Montesinos link \( L \) is quasi-alternating if and only if

1. \( \Sigma(L) \) is an L-space, and
2. there exist a smooth negative definite 4-manifold \( W_1 \) and a smooth positive definite 4-manifold \( W_2 \) with \( \partial W_i = \Sigma(L) \) and \( H_1(W_i) = 0 \) for \( i = 1, 2 \).

Note that in Corollary 1 and throughout, we assume all homology groups have \( \mathbb{Z} \) coefficients.

In light of this corollary, Theorem 1 can also be seen as a classification of the L-space Seifert fibered spaces over \( S^2 \) which bound both positive and negative definite 4-manifolds with vanishing first homology. To what extent Corollary 1 generalises to non-Montesinos links remains an interesting question.
This work also gives a classification of the Seifert fibered space formal L-spaces. The notion of a formal L-space was defined by Greene and Levine [GL16] as a 3-manifold analogue of quasi-alternating links. In fact, the double branched cover of a quasi-alternating link is an example of a formal L-space. In [LS17], Lidman and Sivek classified the quasi-alternating links of determinant at most 7. In fact, they show that the formal L-spaces $M^3$ with $|H_1(M)| \leq 7$ are precisely the double branched covers of quasi-alternating links with determinant at most 7. In this same direction, as a consequence of Corollary 1, we have the following.

Corollary 2. A Seifert fibered space over $S^2$ is a formal L-space if and only if it is the double branched cover of a quasi-alternating link.

Corollary 1 also seems significant given the recent independent characterisations of alternating knots by Greene [Gre17] and Howie [How17]. A non-split link is alternating if and only if it bounds negative definite and positive definite spanning surfaces (which are the checkerboard surfaces). The double branched cover of $B^4$ over such a surface is a definite 4-manifold of the appropriate sign. Generalising this, a quasi-alternating link has the property that it bounds a pair of surfaces in $B^4$ with double branched covers a positive definite and a negative definite 4-manifold (these surfaces cannot be embedded in $S^3$ in general). Corollary 1 shows that among Montesinos links with double branched covers which are L-spaces, this property characterises those which are quasi-alternating.

Our approach to proving Theorem 1 follows that of Greene [Gre10] on the determination of quasi-alternating pretzel links. One of Greene’s main strategies is as follows. Suppose $L$ is a quasi-alternating Montesinos link such that $\Sigma(L)$ is the oriented boundary of the standard negative definite plumbing $X^4$. Since the property of being quasi-alternating is closed under reflection, by property (iii) above, $-\Sigma(L) = \Sigma(\overline{L})$ bounds a negative definite 4-manifold $W$ with $H_1(W) = 0$. By Donaldson’s theorem [Don87], the smooth closed negative definite 4-manifold $X \cup W$ has diagonalisable intersection form. Hence, $H_2(X)/\text{Tors} \hookrightarrow H_2(X \cup W)/\text{Tors}$ is an embedding of the intersection lattice of $X$ into the standard negative diagonal lattice. Moreover, using that $H_1(W)$ is torsion free, it is shown that if $A$ is a matrix representing the lattice embedding then $A^T$ must be surjective.

When $L$ is a pretzel link of a certain form, Greene analyses the possible embeddings of the intersection lattice of $X$ into a negative diagonal lattice and shows that the aforementioned surjectivity condition cannot hold, and hence the link cannot be quasi-alternating. Our main contribution is to argue for more general Montesinos links $L$ that there is no lattice embedding for which $A^T$ is surjective. Key to our argument are some results on lattice embeddings by Lecuona-Lisca [LL11]. The condition we obtain combined with an obstruction based on $\Sigma(L)$ being an L-space leads to the precise necessary conditions to complete the determination of quasi-alternating Montesinos links.

2. Preliminaries

We briefly recall some material on Montesinos links and plumbings. See [CO15] or [BZH14] for further detail on Montesinos links, and [NR78] for more on plumbings. The Montesinos link $M(e; t_1, \ldots, t_p)$, where $t_i = \frac{\alpha_i}{\beta_i} \in \mathbb{Q}$ with $\alpha_i > 1$ and $\beta_i$ coprime integers, and $e$ is an integer, is given by the diagram in Figure 2. In the figure, each box labelled $t_i$ represents the corresponding rational tangle. The 0 rational tangle is shown in Figure 3. Introducing an additional positive (resp. negative) half-twist to the bottom of an $a/b$ rational tangle produces a rational tangle represented by $a/b + 1$ (resp. $a/b - 1$), see Figure 3. Rotating (in either direction) a rational tangle represented by $t \in \mathbb{Q} \cup \{1/0\}$
by 90 degrees produces the rational tangle represented by $-1/t$. The rational tangle represented by any $a/b \in \mathbb{Q} \cup \{1/0\}$ can be obtained from the 0 rational tangle by a sequence of these two operations. See [Cro04] for a more thorough treatment of rational links. Note however that an $a/b$ rational tangle with our conventions corresponds to a $b/a$ rational tangle in [Cro04].

We also note that with our conventions for a Montesinos link $M(e; t_1, \ldots, t_p)$, the integer $e$ has opposite sign to that used by Champanerkar-Ording [CO15], and agrees with that of Qazaqzeh-Chbili-Qublan [QCQ15] and Greene [Gre10].

![Diagram](image1.png)

**Figure 2.** The Montesinos link $M(e; t_1, \ldots, t_p)$ where a box labelled $t_i$ represents a rational tangle corresponding to $t_i$. The crossing type of the $|e|$ crossings depends on the sign of $e$, with the two possibilities shown on the left.

![Diagram](image2.png)

**Figure 3.** From left to right: the 0 rational tangle, an abstract representation of a $a/b$ rational tangle, the $a/b + 1$ rational tangle, and the $-b/a$ rational tangle.

The Montesinos link $M(e; t_1, \ldots, t_p)$ is isotopic to $M(e + 1; t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_p)$ where $t'_i = \frac{\alpha_i}{\beta_i + \alpha_i}$, and is also isotopic to $M(e - 1; t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_p)$, where $t'_i = \frac{\alpha_i}{\beta_i - \alpha_i}$. Hence, a Montesinos link is isotopic to one in *standard form*, that is, of the form $M(e; t_1, \ldots, t_p)$ where $t_i > 1$ for all $i$. 
Let \( L = M(e; t_1, \ldots, t_p) \) where \( t_i < -1 \) for all \( i \). Note that any Montesinos link can be put into this form. For each \( i \), there is a unique continued fraction expansion

\[
t_i = \lbrack a_i^1, \ldots, a_i^{h_i} \rbrack := a_1^i - \frac{1}{a_2^i - \frac{1}{\ddots - \frac{1}{a_{h_i}^i}}},
\]

where \( h_i \geq 1 \) and \( a_j^i \leq -2 \) for all \( j \in \{1, \ldots, h_i\} \).

![Figure 4. The weighted star-shaped plumbing graph \( \Gamma \).](image)

The double branched cover \( \Sigma(L) \) of \( L \) is the oriented boundary of the 4-dimensional plumbing \( X_\Gamma \) of \( D^2 \)-bundles over \( S^2 \) described by the weighted star-shaped graph \( \Gamma \) shown in Figure 4. We call \( \Gamma \) the standard star-shaped plumbing graph for \( L \). The \( i \)th leg of \( \Gamma \) corresponding to \( t_i \) is the linear subgraph generated by the vertices labelled with weights \( a_1^i, \ldots, a_{h_i}^i \). The degree \( p \) vertex labelled with weight \( e \) is called the central vertex. Denote the vertices of \( \Gamma \) by \( v_1, v_2, \ldots, v_k \). The zero-sections of the \( D^2 \)-bundles over \( S^2 \) corresponding to each of \( v_1, \ldots, v_k \) in the plumbing together form a natural spherical basis for \( H_2(X_\Gamma) \). With respect to this basis, the intersection form of \( X_\Gamma \) is given by the weighted adjacency matrix \( Q_\Gamma \) with entries \( Q_{ij} \), \( 1 \leq i, j \leq k \) given by

\[
Q_{ij} = \begin{cases} 
w(v_i), & \text{if } i = j \\
1, & \text{if } v_i \text{ and } v_j \text{ are connected by an edge} \\
0, & \text{otherwise}
\end{cases}
\]

where \( w(v_i) \) is the weight of vertex \( v_i \). We call \((\mathbb{Z}^k, Q_\Gamma)\) the intersection lattice of \( X_\Gamma \) (or of \( \Gamma \)).

3. Results

Equivalent sufficient conditions for a Montesinos link to be quasi-alternating were given in [CO15, Theorem 5.3] and [QCQ15, Theorem 3.5]. The goal of this section is to prove Theorem 1 which states that these sufficient conditions for a Montesinos link to be quasi-alternating are also necessary conditions.

**Lemma 1.** Let \( L = M(e; t_1, \ldots, t_p) \), \( p \geq 3 \), be a Montesinos link in standard form, i.e. where \( t_i = \frac{\alpha_i}{\beta_i} > 1 \) and \( \alpha_i, \beta_i > 0 \) are coprime for all \( i \). Suppose that \( e \leq p - 2 \) and
$e - \sum_{i=1}^{p} \frac{1}{t_i} > 0$ (in particular $e \geq 1$). Then $\Sigma(L)$ is not an L-space, and therefore $L$ is not quasi-alternating.

Proof. The reflection of $L$ is given by $\overline{L} = M(e'; t'_1, \ldots, t'_p) = M(-e; -t_1, \ldots, -t_p)$. The space $\Sigma(\overline{L})$ is the oriented boundary of a plumbing $X_\Gamma$ corresponding to the standard star-shaped plumbing graph $\Gamma$ for $\overline{L}$. Since $e' - \sum_{i=1}^{p} \frac{1}{t'_i} = - \left( e - \sum_{i=1}^{p} \frac{1}{t_i} \right) < 0$, by [NR78, Theorem 5.2], $X_\Gamma$ has negative definite intersection form.

Since $X_\Gamma$ is negative definite and $\Gamma$ is almost-rational, by [Ném05, Theorem 6.3] we have that $\Sigma(\overline{L})$ is an L-space if and only if $X_\Gamma$ is a rational surface singularity (more generally, see [Ném15]). Note that $\Gamma$ is almost-rational since by sufficiently decreasing the weight of the central vertex we obtain a plumbing graph satisfying $-w(v) \geq \deg(v)$ for all vertices $v$, where $w(v)$ denotes the weight of $v$, and such a graph is rational (for details see [Ném05, Example 8.2(3)]).

Laufer’s algorithm [Lau72, Section 4] can be used to determine whether the negative definite plumbing $X_\Gamma$ is a rational surface singularity as follows. Let $v_1, \ldots, v_k$ be the vertices of $\Gamma$ and for $i \in \{1, \ldots, k\}$, let $[\Sigma_{v_i}] \in H_2(X_\Gamma)$ be the spherical class naturally associated to $v_i$. The algorithm is as follows (see [Sti08, Section 3] for a similar formulation).

1. Let $K_0 = \sum_{i=1}^{k} [\Sigma_{v_i}] \in H_2(X_\Gamma)$.
2. In the $i$th step, consider the pairings $\langle PD[K_i], [\Sigma_{v_j}] \rangle$, for $j \in \{1, \ldots, k\}$. Note that these pairings may be evaluated using the adjacency matrix $Q$. If for some $j$ the pairing is at least 2 then the algorithm stops and $X_\Gamma$ is not a rational surface singularity. If for some $j$, the pairing is equal to 1, then set $K_{i+1} = K_i + [\Sigma_{v_j}]$ and go to the next step. Otherwise all pairings are non-positive, the algorithm stops and $X_\Gamma$ is a rational surface singularity.

Applying Laufer’s algorithm to $X_\Gamma$, we claim that the algorithm terminates at the 0th step. To see this, note that for $v$ the central vertex of $\Gamma$, $\langle PD[K_0], [\Sigma_v] \rangle = p - e$ (each vertex adjacent to $v$ contributes 1, the central vertex contributes $-e$). By assumption $e \leq p - 2$, so $\langle PD[K_0], [\Sigma_v] \rangle = p - e \geq 2$. Hence, the algorithm terminates, we conclude that $X_\Gamma$ is not a rational surface singularity and hence $\Sigma(\overline{L})$ is not an L-space. Therefore $\Sigma(L)$ is not an L-space. □

The following lemma will provide an obstruction to a Montesinos link being quasi-alternating.

Lemma 2 ([Gre10, Lemma 2.1]). Suppose that $X$ and $W$ are a pair of 4-manifolds, $\partial X = -\partial W = Y$ is a rational homology sphere, and $H_1(W)$ is torsion-free. Express the map $H_2(X)/\text{Tors} \to H_2(X \cup W)/\text{Tors}$ with respect to a pair of bases by the matrix $A$. This map is an inclusion, and $A^T$ is surjective. In particular, if some $k$ rows of $A$ contain all the non-zero entries of some $k$ of its columns, then the induced $k \times k$ minor has determinant $\pm 1$.

The following two technical lemmas will be useful when we apply the obstruction to being quasi-alternating based on Lemma 2.

Lemma 3 ([LL11, Lemma 3.1]). Suppose $-1/r = [a_1, \ldots, a_n]$ and $-1/s = [b_1, \ldots, b_m]$ where $r + s = 1$. Consider a weighted linear graph $\Psi$ having two connected components, $\Psi_1$ and $\Psi_2$, where $\Psi_1$ consists of $n$ vertices $v_1, \ldots, v_n$ with weights $a_1, \ldots, a_n$ and $\Psi_2$ of $m$ vertices $w_1, \ldots, w_m$ with weights $b_1, \ldots, b_m$. Moreover, suppose that there is an embedding
of the lattice \((\mathbb{Z}^{n+m}, Q_\Psi)\) into \((\mathbb{Z}^k, -\text{Id})\), with basis \(e_1, \ldots, e_k\). For \(S\) a subset of vertices of \(\Psi\), define

\[ U_S = \{ e_i | e_i \cdot v \neq 0 \text{ for some } v \in S \} . \]

Suppose further that \(e_1 \in U_{v_1} \cap U_{v_2}\) and \(U_\Psi = \{ e_1, \ldots, e_k \}\). Then \(U_{v_1} = U_{v_2}\) and \(k = n + m\).

**Lemma 4** ([LL11, Lemma 3.2]). Let \(-1/r = [a_1, \ldots, a_n]\) and \(-1/s = [b_1, \ldots, b_m]\) be such that \(r + s \geq 1\). Then there exists \(n_0 \leq n\) and \(m_0 \leq m\) such that \(-1/r_0 = [a_1, \ldots, a_{n_0}]\) and \(-1/s_0 = [b_1, \ldots, b_{m_0}]\) satisfy \(r_0 + s_0 = 1\).

**Theorem 1.** Let \(L = M(e; t_1, \ldots, t_p)\) be a Montesinos link in standard form, that is, where the latter is written in standard form and \(1 \in \mathbb{Z}^n\). It may assume that both \(e\) and \(p\) are not quasi-alternating. If \(p \geq 2\), we need to prove that this implies that \(L\) is not quasi-alternating. Thus, assume none of the conditions are satisfied, in particular \(p \geq 2\).

By [Sav02, Section 1.2.3] (see also [CO15, Proposition 4.1]), we have that

\[ \det(L) = |\alpha_1 \ldots \alpha_p (e - \sum_{i=1}^{p} \frac{\beta_i}{\alpha_i})| . \]

If \(p = 2\), since none of the conditions are satisfied we must have \(e = 1\) and \(\frac{\alpha_i}{\alpha_i - \beta_i} = \frac{\alpha_i}{\beta_i}\). Hence, \(\det(L) = |\alpha_1 \alpha_2 (1 - \frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2})| = 0\), and so \(L\) is not quasi-alternating (in fact \(L\) must be the two component unlink). For the remainder of the argument we assume that \(p \geq 3\), and \(\det(L) \neq 0\), that is, \(e - \sum_{i=1}^{p} \frac{\beta_i}{\alpha_i} \neq 0\).

First consider the case \(1 < e < p - 1\). The reflection of \(L\) is given by

\[ \overline{L} = M \left( -e, -\frac{\alpha_1}{\beta_1}, \ldots, -\frac{\alpha_p}{\beta_p} \right) = M \left( p - e, \frac{\alpha_1}{\alpha_1 - \beta_1}, \ldots, \frac{\alpha_p}{\alpha_p - \beta_p} \right) , \]

where the latter is written in standard form and \(1 < p - e < p - 1\). Moreover, we see that a reflection reverses the sign of \(e - \sum_{i=1}^{p} \frac{\beta_i}{\alpha_i}\), and thus by a reflection if necessary we may assume that \(e - \sum_{i=1}^{p} \frac{\beta_i}{\alpha_i} > 0\). Then by Lemma 1, \(\Sigma(L)\) is not an L-space, so \(L\) is not quasi-alternating.

It remains to consider the cases \(e = 1\) and \(e = p - 1\). By a reflection if necessary we may assume that \(e = 1\). Note that conditions (2) and (4) are equivalent under a reflection. We assume that condition (2) is not satisfied. We need to prove that this implies that \(L\) is not quasi-alternating. If \(e - \sum_{i=1}^{p} \frac{\beta_i}{\alpha_i} > 0\) then by Lemma 1, \(\Sigma(L)\) is not an L-space, and therefore \(L\) is not quasi-alternating.

Otherwise \(e - \sum_{i=1}^{p} \frac{\beta_i}{\alpha_i} < 0\). We have that

\[ L = M \left( 1; \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_p}{\beta_p} \right) = M \left( 1 - p; \frac{\alpha_1}{\beta_1 - \alpha_1}, \ldots, \frac{\alpha_p}{\beta_p - \alpha_p} \right) , \]

where \(\frac{\alpha_i}{\beta_i - \alpha_i} < -1\) for all \(i\).
The double branched cover $\Sigma(L)$ of $L$ is therefore the boundary of a plumbing 4-manifold $X_\Gamma$ on the standard star-shaped planar graph $\Gamma$ with central vertex of weight $-(p - 1)$ and legs corresponding to the fractions $\frac{\alpha_i}{\beta_i}$, $i \in \{1, \ldots, p\}$. Our assumption that $e - \sum_{i=1}^p \frac{\beta_i}{\alpha_i} < 0$ implies that $X_\Gamma$ is negative definite [NR78, Theorem 5.2]. Suppose for the sake of contradiction that $L$ is quasi-alternating. Then $L$ is quasi-alternating and $-\Sigma(L) = \hat{\Sigma(L)}$ bounds a negative definite 4-manifold $W$ with $H_1(W) = 0$ [OS05, Proof of Lemma 3.6]. By Donaldson’s theorem [Don87], the smooth closed negative definite 4-manifold $X_\Gamma \cup W$ has diagonalisable intersection form. Thus, the map $H_2(X_\Gamma)/\text{Tors} \rightarrow H_2(X_\Gamma \cup W)/\text{Tors}$ induced by the inclusion map is an embedding of the intersection lattice $(\mathbb{Z}^k, Q_\Gamma)$ of $X_\Gamma$ into the standard negative diagonal lattice $(\mathbb{Z}^n, -\text{Id})$ for some $n$. Denote by $e_1, \ldots, e_n$ a basis for $(\mathbb{Z}^n, -\text{Id})$.

We use the lattice embedding to identify elements of $(\mathbb{Z}^k, Q_\Gamma)$ with their image in $(\mathbb{Z}^n, -\text{Id})$. For convenience, we will not distinguish between a vertex of $\Gamma$ and the vector it corresponds to in the lattice. The central vertex $v$ of $\Gamma$ has weight $-(p - 1)$, and so $v \cdot e_i \neq 0$ for at most $p - 1$ values of $i \in \{1, \ldots, n\}$. Thus, by applying an automorphism if necessary, we may assume that $v$ pairs non-trivially with precisely $e_1, \ldots, e_m$ where $m \leq p - 1$. Since there are $p$ legs, by the pigeonhole principle there must exist some $e_j$, where $j \in \{1, \ldots, m\}$, and two distinct vertices $v_1, v_2$ adjacent to $v$ with $v_1 \cdot e_j \neq 0$ and $v_2 \cdot e_j \neq 0$. Without loss of generality we assume that $j = 1$ and that for $i \in \{1, 2\}$, the vertex $v_i$ belongs to the $i$th leg of $\Gamma$, i.e. corresponding to the fraction $\frac{\alpha_i}{\beta_i}$.

Since we are assuming condition (2) does not hold, we have that $\frac{\alpha_i}{\alpha_i - \beta_i} \leq \frac{\alpha_i}{\beta_i}$ for all $i, j$ with $i \neq j$. In particular, we have $\frac{\alpha_1}{\alpha_1 - \beta_1} \leq \frac{\alpha_2}{\beta_2}$. Rearranging this gives $\frac{\alpha_1}{\alpha_1} + \frac{\beta_2}{\beta_2} \leq 1$. Note that the two legs correspond to the fractions $-1/r = -\frac{\alpha_1}{\alpha_1 - \beta_1} = [a_1^1, \ldots, a_1^{k_1}]$ and $-1/s := -\frac{\alpha_2}{\alpha_2 - \beta_2} = [a_2^1, \ldots, a_2^{k_2}]$, where $r, s \in \mathbb{Q}$, and where our notation is as in Section 2. Thus, we have that $r + s = 2 - \frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2} \geq 1$. Since $r + s \geq 1$, by Lemma 4 there exist $h'_1 \leq h_1$ and $h'_2 \leq h_2$ such that $-1/r_0 = [a_1^1, \ldots, a_1^{h_1'}]$ and $-1/s_0 = [a_2^1, \ldots, a_2^{h_2'}]$ with $r_0 + s_0 = 1$.

Let $\Psi$ be the union of the linear graph containing the first $h'_1$ vertices of the first leg (where we count vertices in a leg starting away from the central vertex), and the linear graph containing the first $h'_2$ vertices of the second leg. By restricting our embedding of $(\mathbb{Z}^k, Q_\Gamma)$, we have an embedding of the sublattice corresponding to $\Psi$ into $(\mathbb{Z}^n, -\text{Id})$. The image of this embedding is contained in a sublattice $(\mathbb{Z}^d, -\text{Id})$ of $(\mathbb{Z}^n, -\text{Id})$ spanned by $\{e_i \in \mathbb{Z}^n \mid e_i \cdot v \neq 0 \text{ for some vertex } v \text{ of } \Psi\}$. Hence $U_\Psi$ consists of $d$ elements (see Lemma 3 for definition of $U_\Psi$). Let $v_1, v_1$ be the two vertices of $\Psi$ adjacent to the central vertex in $\Gamma$. By our choice of the two legs of $\Gamma$ which contain the vertices of $\Psi$, we know that $e_j \in U_{v_1} \cap U_{v_1}$ for some $j \in \{1, \ldots, n\}$. This shows that the hypothesis of Lemma 3 are satisfied, hence we conclude that $d = h'_1 + h'_2$.

Let $A$ be the matrix representing the embedding $(\mathbb{Z}^k, Q_\Gamma)$ into $(\mathbb{Z}^n, -\text{Id})$. Then the $h'_1 + h'_2$ columns of $A$ corresponding to the vertices of $\Psi$ are supported in $d = h'_1 + h'_2$ rows of $A$ corresponding to the $d$-dimensional sublattice of $(\mathbb{Z}^n, -\text{Id})$. Denote this $d \times d$ minor by $B$. Then $-B^T B$ is a matrix for the intersection form of the plumbing corresponding to $\Psi$. Hence $-B^T B$ is a presentation matrix for $H_1(Y)$ where $Y$ is the boundary of the (disconnected) plumbing corresponding to $\Psi$. The 3-manifold $Y$ is the disjoint union of two lens spaces, each given by surgery on the unknot with framings $-1/r_0 < -1$ and $-1/s_0 < -1$ respectively. Therefore $|\det(B)|^2 = |H_1(Y)| > 1$ contradicting Lemma 2. Thus, $L$ is not quasi-alternating.

\textbf{Corollary 1.} A Montesinos link $L$ is quasi-alternating if and only if
(1) $\Sigma(L)$ is an L-space, and
(2) there exist a smooth negative definite 4-manifold $W_1$ and a smooth positive definite 4-manifold $W_2$ with $\partial W_i = \Sigma(L)$ and $H_1(W_i) = 0$ for $i = 1, 2$.

Proof. This is a corollary of the proof of Theorem 1. Suppose first that $L$ is quasi-alternating. By [OS05, Proposition 3.3], $\Sigma(L)$ is an L-space. Furthermore, $\Sigma(L)$ must bound a negative definite 4-manifold $W_1$ with $H_1(W_1) = 0$ [OS05, Proof of Lemma 3.6]. Applying this to the reflection of $L$ which is also quasi-alternating, we get that $\Sigma(L)$ also bounds a positive definite 4-manifold $W_2$ with $H_1(W_2) = 0$. For the converse, note that these two necessary conditions are the only conditions used to obstruct a Montesinos link from being quasi-alternating in the proof of Theorem 1. 

As a consequence, we obtain a classification of the Seifert fibered spaces which are formal L-spaces. Before stating it, we recall the definition of a formal L-space. We say that a triple $(Y_1, Y_2, Y_3)$ of closed, oriented 3-manifolds form a triad if there is a 3-manifold $M$ with torus boundary, and three oriented curves $\gamma_1, \gamma_2, \gamma_3 \subset \partial M$ at pairwise distance 1, such that $Y_i$ is the result of Dehn filling $M$ along $\gamma_i$, for $i = 1, 2, 3$.

**Definition 2.** The set $\mathcal{F}$ of formal L-spaces is the smallest set of rational homology 3-spheres such that

(1) $S^3 \in \mathcal{F}$, and
(2) if $(Y, Y_0, Y_1)$ is a triad with $Y_0, Y_1 \in \mathcal{F}$ and

$$|H_1(Y)| = |H_1(Y_0)| + |H_1(Y_1)|,$$

then $Y \in \mathcal{F}$.

**Corollary 2.** A Seifert fibered space over $S^2$ is a formal L-space if and only if it is the double branched cover of a quasi-alternating link.

Proof. Let $L$ be a quasi-alternating Montesinos link. Then the double branched cover of $L$ is a Seifert fibered space over $S^2$. Ozsváth and Szabó show that the double branched cover of a quasi-alternating link is an L-space [OS05, Proposition 3.3]. Their proof in fact shows that the double branched cover of a quasi-alternating link is a formal L-space. Hence $\Sigma(L)$ is a formal L-space Seifert fibered space over $S^2$.

Now let $M$ be a formal L-space Seifert fibered space over $S^2$. Then $M$ is the double branched cover of a Montesinos link $L$. Ozsváth and Szabó’s in [OS05, Proof of Lemma 3.6] show that the double branched cover of a quasi-alternating link bounds both a positive definite, and a negative definite 4-manifold with vanishing first homology. However, their proof in fact shows this for all formal L-spaces. Hence $M = \Sigma(L)$ is a formal L-space bounding positive and negative definite 4-manifolds with vanishing first homology. Thus, Corollary 1 implies that $L$ is quasi-alternating. 

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