An infinite family of congruences arising from a second order mock theta function

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Abstract. Let $\beta(q) = \sum_{n \geq 0} b(n)q^n$ be a second order mock theta function defined by

$$\sum_{n \geq 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}}.$$

In this paper, we obtain an infinite family of congruences modulo powers of 3 for $b(n)$.

Keywords. Mock theta function, Ramanujan-type congruence, Ramanujan’s $1\psi_1$ identity.

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1. Introduction

In his last letter to Hardy [4, pp. 220–223], Ramanujan defined 17 functions, which he called mock theta functions. These functions subsequently attract the interest of many mathematicians. The interested readers may refer to [8] for a nice survey.

Let the $q$-shifted factorials be

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k),$$

$$(a)_\infty = (a;q)_\infty := \prod_{k \geq 0} (1 - aq^k),$$

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty(a_2; q)_\infty \cdots (a_m; q)_\infty.$$

One of the second order mock theta functions is defined by

$$\beta(q) := \sum_{n \geq 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n \geq 0} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}. \tag{1.1}$$

According to Andrews [1], this function does not appear in Ramanujan’s Lost Notebook. However, Andrews [1] also claimed that $\beta(q)$ may connect with another second order mock theta function $\mu(q)$ shown in the Lost Notebook as well as Mordell integrals through identities such as

$$q^{-1/8} \sqrt{\frac{\pi}{2\alpha}} \mu(q) = \frac{2\pi}{\alpha} q_{1/2}^{1/2} \beta(q_1) + \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2 + \alpha x}}{1 + e^{2\alpha x}} \, dx,$$

where $q = e^{-\alpha}$, $q_1 = e^{-\alpha'}$, $\alpha\alpha' = \pi^2$, and

$$\mu(q) := \sum_{n \geq 0} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q^2;q^2)_n}.$$
Using Watson’s \(q\)-analog of Whipple’s theorem [12, p. 100, Eq. (3.4.1.5)], Andrews [1] showed that
\[
\beta(q) = \frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(n+1)}}{1 - q^{2n+1}}.
\]
From the above representation of \(\beta(q)\), Gordon and McIntosh proved (cf. [8, p. 136])
\[
\frac{\beta(q) + \beta(-q)}{2} = \frac{(q^4; q^4)^5_\infty}{(q^2; q^2)^4_\infty}.
\]
(1.2)
Let
\[
\sum_{n \geq 0} b(n) q^n = \beta(q).
\]
It follows that
\[
\sum_{n \geq 0} b(2n) q^n = \frac{(q^2; q^2)^5_\infty}{(q; q)^4_\infty}.
\]
(1.3)
We notice that many Ramanujan-type congruences for mock theta functions (or functions related to mock theta functions) are derived by dissecting the function to a \(q\)-series quotient (i.e. product or quotient of certain \(q\)-shifted factorials). The interested readers may refer to [2, 6, 13] for details. Since the r.h.s. of (1.3), which comes from the 2-dissection of \(\beta(q)\), is also a \(q\)-series quotient, it is natural to consider the arithmetic properties of \(\beta(q)\). In this paper, we shall show the following infinite family of congruences modulo powers of 3.

**Theorem 1.1.** For \(\alpha \geq 1\) and \(n \geq 0\), we have
\[
b \left(2 \cdot 3^{2\alpha - 1} n + \frac{3^{2\alpha} - 1}{2}\right) \equiv 0 \pmod{3^{2\alpha}}.
\]
(1.4)
Our approach is an elementary refinement of the Watson–Atkin style proof [3, 14] (which involves modular forms) of congruences modulo powers of a prime for some \(q\)-series quotient. On the one hand, with the aid of some 3-dissection identities for Ramanujan’s theta functions, we are able to provide elementary proofs of several initial relations (see our Theorems 2.3 and 2.4), and hence avoid using modular forms. On the other hand, the recurrence relation shown in the Watson–Atkin style proof merely relies on Newton’s identity. Combining the two ingredients together, we therefore arrive at a completely elementary proof of Theorem 1.1.

2. The infinite family of identities

2.1. Ramanujan’s theta functions and 3-dissections. For notational convenience, we write \(E(q) := (q; q)_\infty\) throughout this paper.

Let \(\phi(q)\) and \(\psi(q)\) be two of Ramanujan’s theta functions given by
\[
\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) := \sum_{n \geq 0} q^{n(n+1)/2}.
\]
It is well known that
\[
\phi(-q) = \frac{E(q)^2}{E(q^2)} \quad \text{and} \quad \psi(q) = \frac{E(q^2)^2}{E(q)}.
\]
We have the following 3-dissection identities.
Lemma 2.1. It holds that
\[
\frac{1}{\phi(-q)} = \frac{\phi(-q^3)^3}{\phi(-q^3)^4} (1 + 2qw(q^3) + 4q^2 w(q^3)^2),
\]
(2.1)
\[
\psi(q) = \psi(q^9) \left( \frac{1}{w(q^3)} + q \right),
\]
(2.2)
where
\[
w(q) = \frac{E(q) E(q^6)^3}{E(q^2) E(q^3)^3}.
\]

One may refer to [9, Eq. (14.3.5)] for (2.2). Let \( \omega \) be a cube root of unity other than 1. We substitute \( \omega q \) and \( \omega^2 q \) for \( q \) in the 3-dissection of \( \phi(-q) \) (cf. [9, Eq. (14.3.4)])
\[
\phi(-q) = \phi(-q^9) (1 - 2qw(q^3))
\]
and multiply the two results to get (2.1).

We further notice that \( qw(q)^3 \) can be represented as follows.

Lemma 2.2. It holds that
\[
qw(q)^3 = \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right).
\]
(2.3)

Proof. We have
\[
qw(q)^3 = q \left( \frac{E(q) E(q^6)^3}{E(q^2) E(q^3)^3} \right)^3 = q \frac{\phi(-q)^3}{\phi(-q^3)^3} \psi(q)^3.
\]
Recall the following identity (cf. [11, Eq. (3.2)])
\[
\frac{\psi(q^3)^3}{\psi(q)} = \frac{1}{8q} \left( \frac{\phi(-q^3)^3}{\phi(-q)} - \frac{\phi(-q)^3}{\phi(-q^3)} \right).
\]
Hence
\[
qw(q)^3 = q \frac{\phi(-q)}{\phi(-q^3)^3} \left( \frac{1}{8q} \left( \frac{\phi(-q^3)^3}{\phi(-q)} - \frac{\phi(-q)^3}{\phi(-q^3)} \right) \right)
\]
\[
= \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right).
\]
\( \square \)

2.2. Initial relations. Let \( q := \exp(2\pi i \tau) \) with \( \tau \in \mathbb{H} \), the upper half complex plane. We put
\[
X = X(\tau) := \frac{E(q^2)^4 E(q^3)^8}{E(q)^8 E(q^6)^4} = \frac{\phi(-q^3)^4}{\phi(-q)^2},
\]
\[
\xi = \xi(\tau) := q^{-2} \frac{E(q^2)^5 E(q^3)^4}{E(q)^4 E(q^6)^3} = q^{-2} \psi(q)^2 \frac{\phi(-q^3)}{\phi(-q) \psi(q^3)^2}.
\]
For a q-series expansion \( \sum_{n \geq n_0} a_n q^n \), we introduce the \( U \)-operator defined by
\[
U \left( \sum_{n \geq n_0} a_n q^n \right) := \sum_{3n \geq n_0} a_{3n} q^n.
\]
We shall show
Theorem 2.3. It holds that
\[ U(X) = 10X - 36X^2 + 27X^3, \] (2.4)
\[ U(X^2) = -8X + 306X^2 - 2160X^3 + 5508X^4 - 5832X^5 + 2187X^6, \] (2.5)
\[ U(X^3) = X - 360X^2 + 10566X^3 - 99144X^4 + 423549X^5 - 944784X^6 
+ 1141614X^7 - 708588X^8 + 177147X^9. \] (2.6)

Proof. We have
\[ X = \frac{\phi(-q^3)^4}{\phi(-q)^4} = \phi(-q)^4 \left( \frac{\phi(-q^9)^3}{\phi(-q)^4} (1 + 2qw(q^3) + 4q^2w(q^3)^2) \right)^4 \]
\[ = \frac{\phi(-q^3)^12}{\phi(-q)^12} \left( 1 + 8qw(q^3) + 40q^2w(q^3)^2 + 128q^3w(q^3)^3 + 304q^4w(q^3)^4 
+ 512q^5w(q^3)^5 + 640q^6w(q^3)^6 + 512q^7w(q^3)^7 + 256q^8w(q^3)^8 \right). \]

It therefore follows that
\[ U(X) = \frac{\phi(-q^3)^12}{\phi(-q)^12} \left( 1 + 128qw(q^3) + 640q^2w(q^6) \right). \]

Applying Lemma 2.2, we have
\[ U(X) = \frac{\phi(-q^3)^12}{\phi(-q)^12} \left( 1 + 128 \left( \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right) \right) + 640 \left( \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right) \right)^2 \right) \]
\[ = 10 \frac{\phi(-q^3)^4}{\phi(-q)^4} - 36 \left( \frac{\phi(-q^3)^4}{\phi(-q)^4} \right)^2 + 27 \left( \frac{\phi(-q^3)^4}{\phi(-q)^4} \right)^3 \]
\[ = 10X - 36X^2 + 27X^3. \]

Similarly, we have
\[ U(X^2) = \frac{\phi(-q^3)^24}{\phi(-q)^24} \left( 1 + 896qw(q^3) + 50176q^2w(q^6) + 520192q^3w(q^9) 
+ 1089536q^4w(q)^{12} + 262144q^5w(q^{15}) \right) \]
\[ = \frac{\phi(-q^3)^24}{\phi(-q)^24} \left( 1 + 896 \left( \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right) \right) + 50176 \left( \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right) \right)^2 \right) \]
\[ + 520192 \left( \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right) \right)^3 + 1089536 \left( \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right) \right)^4 \]
\[ + 262144 \left( \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right) \right)^5 \]
\[ = -8X + 306X^2 - 2160X^3 + 5508X^4 - 5832X^5 + 2187X^6. \]

The proof of (2.6) follows in the same way. \qed

We also have

Theorem 2.4. It holds that
\[ U(\xi) = 9X, \] (2.7)
\[ U(\xi X) = -9X + 252X^2 - 891X^3 + 729X^4, \] (2.8)
\[ U(\xi X^2) = X - 378X^2 + 8613X^3 - 54675X^4 + 138510X^5 - 150903X^6. \]
\[ U(\xi X^3) = 147X^2 - 14553X^3 + 312255X^4 - 2617839X^5 + 10764414X^6 - 23914845X^7 + 29288304X^8 - 18600435X^9 + 4782969X^{10}. \]  

(2.10)

**Proof.** We have

\[
\xi = q^{-2} \psi(q)^2 \phi(-q^9) \phi(-q)^4 \left( 1 + 2qw(q^3) + 4q^2w(q^3)^2 \right) \left( \frac{1}{w(q^3)} + q \right)^2
\]

\[= \phi(-q^9)^4 \left( \frac{1}{w(q^3)^2} + \frac{4}{qw(q^3)} + 9 + 10qw(q^3) + 4q^2w(q^3)^2 \right). \]

It follows that

\[ U(\xi) = 9 \phi(-q^3)^4 \phi(-q)^4 = 9X. \]

Similarly, we have

\[ U(\xi X) = \phi(-q^3)^{16} \phi(-q)^{16} \left( 81 + 3312qw(q^3) + 14400q^2w(q^3)^2 + 4608q^3w(q^3)^3 \right) \]

\[= \phi(-q^3)^{16} \left( 81 + 3312 \left( \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right) \right) \right) \]

\[+ 14400 \left( \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right) \right)^2 + 4608 \left( \frac{1}{8} \left( 1 - \frac{\phi(-q)^4}{\phi(-q^3)^4} \right) \right)^3 \]

\[= -9X + 252X^2 - 891X^3 + 729X^4. \]

The proofs of (2.9) and (2.10) follow in the same way. \(\square\)

**Remark 2.1.** Here (2.7) is also a direct consequence of a result of Chern and Tang; see [5, Corollary 2.7].

**2.3. Further relations.** We now define the following two matrices \((a_{i,j})_{i \geq 1, j \geq 1}\) and \((b_{i,j})_{i \geq 1, j \geq 1}\) (for all matrices defined here and below, we sometimes use \(m(i, j)\) to denote \(m_{i,j}\)):

(1). For \(1 \leq i \leq 3\), the entries \(a(i, j)\) and \(b(i, j)\) are respectively given by

\[ U(X^i) = \sum_{j \geq 1} a(i, j)X^j, \]

\[ U(\xi X^i) = \sum_{j \geq 1} b(i, j)X^j. \]

We refer to Theorems 2.3 and 2.4 for their exact values.

(2). For \(i \geq 4\), both matrices satisfy the following recurrence relation (here \(m\) stands for a matrix):

\[ m(i, j) = 30m(i - 1, j - 1) - 108m(i - 1, j - 2) + 81m(i - 1, j - 3) \]

\[- 12m(i - 2, j - 1) + 9m(i - 2, j - 2) + m(i - 3, j - 1). \]

Note that all unspecified and undefined entries in a matrix are assumed to be 0.
Theorem 2.5. For $i \geq 1$, we have
\[
U(X^i) = \sum_{j \geq 1} a(i, j)X^j, \quad (2.11)
\]
\[
U(\xi X^i) = \sum_{j \geq 1} b(i, j)X^j. \quad (2.12)
\]

Proof. According to the definition of $(a_{i,j})$ and $(b_{i,j})$, the theorem holds for $1 \leq i \leq 3$. We now deal with the cases $i \geq 4$.

Let $x_t = X\left(\frac{\tau + t}{3}\right)$, $t = 1, \ldots, 3$.

Let $\sigma_t (t = 1, \ldots, 3)$ be the $t$-th elementary symmetric function of $x_1, \ldots, x_3$, i.e.,
\[
\sigma_1 = x_1 + x_2 + x_3, \\
\sigma_2 = x_1x_2 + x_2x_3 + x_3x_1, \\
\sigma_3 = x_1x_2x_3.
\]

Then each $x_t (t = 1, \ldots, 3)$ satisfies
\[
x_t^3 - \sigma_1x_t^2 + \sigma_2x_t - \sigma_3 = 0.
\]

For $i \geq 1$, we put $p_i = x_1^i + x_2^i + x_3^i$.

Note that for any integer $i$ we have
\[
3U(X^i) = \sum_{t=1}^{3} X\left(\frac{\tau + t}{3}\right)^i = p_i.
\]

By Theorem 2.3, we can write $p_1, \ldots, p_3$ as polynomials of $X$. It follows from Newton’s identities (cf. [10]) that
\[
\sigma_1 = p_1 = 30X - 108X^2 + 81X^3, \\
\sigma_2 = (\sigma_1p_1 - p_2)/2 = 12X - 9X^2, \\
\sigma_3 = (\sigma_2p_1 - \sigma_1p_2 + p_3)/3 = X.
\]

Let $u(\tau) : \mathbb{H} \to \mathbb{C}$ be a complex function. We put $u_t = u\left(\frac{\tau + t}{3}\right)$ for $t = 1, \ldots, 3$.

Then for $i \geq 4$,
\[
3U(uX^i) = \sum_{t=1}^{3} u_t x_t^i = \sum_{t=1}^{3} u_t(\sigma_1x_t^{i-1} - \sigma_2x_t^{i-2} + \sigma_3x_t^{i-3})
\]
\[
= \sigma_1 \sum_{t=1}^{3} u_t x_t^{i-1} - \sigma_2 \sum_{t=1}^{3} u_t x_t^{i-2} + \sigma_3 \sum_{t=1}^{3} u_t x_t^{i-3}
\]
\[
= 3\left(\sigma_1U(uX^{i-1}) - \sigma_2U(uX^{i-2}) + \sigma_3U(uX^{i-3})\right).
\]

At last, we respectively take $u = 1$ and $u = \xi$ to arrive at the desired result. \qed

Let $(d_\alpha)_{\alpha \geq 1}$ be a family of integer sequences with
\[
d_1 = (9, 0, 0, \ldots).
\]

For $\alpha \geq 2$, we recursively define
\[
d_\alpha(j) = \begin{cases} 
\sum_{k \geq 1} a(k, j)d_{\alpha-1}(k) & \text{if } \alpha \text{ is even}, \\
\sum_{k \geq 1} b(k, j)d_{\alpha-1}(k) & \text{if } \alpha \text{ is odd},
\end{cases}
\]
where \(d_\alpha(j)\) denotes the \(j\)-th element of sequence \(d_\alpha\).

Let

\[
\sum_{n \geq 0} g(n)q^n = \frac{E(q^2)^5}{E(q)^4}.
\]

(2.13)

We have

**Theorem 2.6.** For \(\alpha \geq 1\), we have

\[
\sum_{n \geq 0} g \left( 3^{2\alpha - 1}n + \frac{3^{2\alpha} - 1}{4} \right) q^n = \frac{E(q^6)^5}{E(q)^4} \sum_{j \geq 1} d_{2\alpha - 1}(j)X^j,
\]

(2.14)

\[
\sum_{n \geq 0} g \left( 3^{2\alpha}n + \frac{3^{2\alpha} - 1}{4} \right) q^n = \frac{E(q)^2}{E(q)^4} \sum_{j \geq 1} d_{2\alpha}(j)X^j.
\]

(2.15)

**Proof.** Theorem 2.4 tells us \(U(\xi) = 9X = \sum_{j \geq 1} d_1(j)X^j\). On the other hand, we have

\[
U(\xi) = U \left( q^{-2} \frac{E(q^2)^5E(q^6)^4}{E(q)^4E(q^{18})^5} \right) = \frac{E(q^3)^4}{E(q^6)^5} U \left( \sum_{n \geq 0} g(n)q^{n-2} \right)
\]

\[
= \frac{E(q^3)^4}{E(q^6)^5} \sum_{n \geq 0} g(3n+2)q^n.
\]

Theorem 2.6 is therefore valid for \(\alpha = 1\).

We now assume that the theorem holds for some odd positive integer \(2\alpha - 1\). Then we have

\[
\sum_{j \geq 1} d_{2\alpha - 1}(j)X^j = \frac{E(q^6)^5}{E(q)^4} \sum_{n \geq 0} g \left( 3^{2\alpha - 1}n + \frac{3^{2\alpha} - 1}{4} \right) q^n.
\]

We now apply the \(U\)-operator to both sides of the above identity. Then the l.h.s. becomes

\[
U \left( \sum_{j \geq 1} d_{2\alpha - 1}(j)X^j \right) = \sum_{j \geq 1} d_{2\alpha - 1}(j)U(X^j) = \sum_{j \geq 1} d_{2\alpha - 1}(j) \sum_{\ell \geq 1} a(j, \ell)X^\ell
\]

\[
= \sum_{\ell \geq 1} \left( \sum_{j \geq 1} a(j, \ell) d_{2\alpha - 1}(j) \right) X^\ell = \sum_{\ell \geq 1} d_{2\alpha}(\ell)X^\ell.
\]

On the other hand, the r.h.s. is

\[
U \left( \frac{E(q^3)^4}{E(q^6)^5} \sum_{n \geq 0} g \left( 3^{2\alpha - 1}n + \frac{3^{2\alpha} - 1}{4} \right) q^n \right) = \frac{E(q^3)^4}{E(q^6)^5} \sum_{n \geq 0} g \left( 3^{2\alpha}n + \frac{3^{2\alpha} - 1}{4} \right) q^n.
\]

Combining them together, we have

\[
\sum_{j \geq 1} d_{2\alpha}(j)X^j = \frac{E(q)^4}{E(q^2)^5} \sum_{n \geq 0} g \left( 3^{2\alpha}n + \frac{3^{2\alpha} - 1}{4} \right) q^n.
\]
We next multiply by $\xi$ on both sides of the above identity and then apply the $U$-operator. Then the l.h.s. becomes

$$U \left( \sum_{j \geq 1} d_{2\alpha}(j) \xi X^j \right) = \sum_{j \geq 1} d_{2\alpha}(j) U(\xi X^j) = \sum_{j \geq 1} d_{2\alpha}(j) \sum_{\ell \geq 1} b(j, \ell) X^\ell$$

$$= \sum_{\ell \geq 1} \left( \sum_{j \geq 1} b(j, \ell) d_{2\alpha}(j) \right) X^\ell = \sum_{\ell \geq 1} d_{2\alpha+1}(\ell) X^\ell.$$  

On the other hand, the r.h.s. is

$$U \left( q^{-2} \frac{E(q^9)^4}{E(q^{18})^5} \sum_{n \geq 0} g \left( 3^{2\alpha} n + \frac{3^{2\alpha} - 1}{4} \right) q^n \right) = \frac{E(q^3)^4}{E(q^6)^5} \sum_{n \geq 0} g \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+2} - 1}{4} \right) q^n.$$  

Hence

$$\sum_{j \geq 1} d_{2\alpha+1}(j) X^j = \frac{E(q^3)^4}{E(q^6)^5} \sum_{n \geq 0} g \left( 3^{2\alpha+1} n + \frac{3^{2\alpha+2} - 1}{4} \right) q^n.$$  

The theorem therefore follows by induction. \hfill \Box

At last, we notice that the two infinite matrices $(a_{i,j})$ and $(b_{i,j})$ are row and column finite. Define $(t_{i,j})_{i \geq 1, j \geq 1}$ by

$$t_{i,j} = \sum_{k \geq 1} a_{i,k} b_{k,j}.$$  

Then the above sum is indeed a finite sum. From the recurrence relation for $d_{\alpha}$, one has, for $\alpha \geq 1$,

$$d_{2\alpha+1}(j) = \sum_{i \geq 1} t(i, j) d_{2\alpha-1}(i). \quad (2.16)$$

3. The 3-adic orders

3.1. The 3-adic orders. For any integer $n$, let $\pi(n)$ be the 3-adic order of $n$ with the convention that $\pi(0) = \infty$. Let $\lfloor x \rfloor$ be the largest integer not exceeding $x$.

We begin with the 3-adic orders of $(a_{i,j})$ and $(b_{i,j})$.

**Theorem 3.1.** For $i$ and $j \geq 1$, it holds that

$$\pi(a(i, j)) \geq \left\lfloor \frac{3j - i - 1}{2} \right\rfloor, \quad (3.1)$$

$$\pi(b(i, j)) \geq \left\lfloor \frac{3j - i}{2} \right\rfloor. \quad (3.2)$$

**Proof.** One may first check (3.1) directly for $i = 1, \ldots, 3$. Assume that (3.1) holds for $1, \ldots, i - 1$ with some $i \geq 4$. We have (for some undefined entries like $a(i, 0)$, since we assign its value to be 0, its 3-adic order is therefore $\infty$):

$$\pi(a(i-1, j-1)) + 1 \geq \left\lfloor \frac{3(j-1) - (i-1) - 1}{2} \right\rfloor + 1 \geq \left\lfloor \frac{3j - i - 1}{2} \right\rfloor,$$
Corollary 3.2. For \( m \) of the l.h.s. of the above 6 inequalities. We hence obtain (3.1) by induction.

Proof. Since \( \alpha \) theorem is valid for some \( a \). It follows from the recurrence relation of (3.2) that \( \alpha \) follows in the same way. □

As a direct consequence, we have

Corollary 3.2. For \( i \) and \( j \geq 1 \), it holds that

\[
\pi(t(i, j)) \geq \min_{k \geq 1} \{ \pi(a(i, k)) + \pi(b(k, j)) \} \geq \min_{k \geq 1} \left\{ \left\lfloor \frac{3k - i - 1}{2} \right\rfloor + \left\lfloor \frac{3j - k}{2} \right\rfloor \right\}.
\]

(3.3)

On the other hand, one may compute the values of \( (\pi(t(i, j)))_{i \geq 1, j \geq 1} \) for some small \( i \) and \( j \), which are listed below

\[
\begin{pmatrix}
2 & 2 & 4 & \cdots \\
3 & 2 & 4 & \cdots \\
2 & 2 & 4 & \cdots \\
0 & 3 & 3 & \cdots \\
0 & 2 & 3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}_{i \geq 1, j \geq 1}
\]

We next show

Theorem 3.3. For \( \alpha \) and \( j \geq 1 \), it holds that

\[
\pi(d_{2\alpha - 1}(j)) \geq 2\alpha + \left\lfloor \frac{2j - 2}{3} \right\rfloor.
\]

(3.4)

Proof. Since \( d_1 = (9, 0, 0, \ldots) \), we see that (3.4) holds for \( \alpha = 1 \). Assume that the theorem is valid for some \( \alpha \geq 1 \). We want to show

\[
\pi(d_{2\alpha + 1}(j)) \geq 2\alpha + 2 + \left\lfloor \frac{2j - 2}{3} \right\rfloor.
\]

Then the theorem follows by induction.

It follows from (2.16) that

\[
\pi(d_{2\alpha + 1}(j)) \geq \min_{i \geq 1} \{ \pi(d_{2\alpha - 1}(i)) + \pi(t(i, j)) \}.
\]

We further know from (3.3) that

\[
\pi(d_{2\alpha - 1}(i)) + \pi(t(i, j)) \geq 2\alpha + \left\lfloor \frac{2i - 2}{3} \right\rfloor + \min_{k \geq 1} \left\{ \left\lfloor \frac{3k - i - 1}{2} \right\rfloor + \left\lfloor \frac{3j - k}{2} \right\rfloor \right\}
\]

It follows from the recurrence relation of \( (a_{i, j}) \) that \( \pi(a(i, j)) \) is at least the minimum of the l.h.s. of the above 6 inequalities. We hence obtain (3.1) by induction. (3.2) follows in the same way. □
\[ \geq 2\alpha + \left\lfloor \frac{2i - 2}{3} \right\rfloor + \min_{k \geq 1} \left\lfloor \frac{3j + 2k - i - 3}{2} \right\rfloor \]
\[ = 2\alpha + \left\lfloor \frac{2i - 2}{3} \right\rfloor + \frac{3j - i - 1}{2} \]
\[ \geq 2\alpha + 2 + \left\lfloor \frac{2j - 2}{3} \right\rfloor + \frac{5j + i - 21}{6}. \]

Note that \( i \geq 1 \). Hence for \( j \geq 4 \), we have \( 5j + i \geq 21 \). Hence we merely need to check the cases \( j = 1, \ldots, 3 \).

When \( j = 1 \) or \( 2 \), we need to show
\[ \pi(d_{2\alpha + 1}(j)) \geq 2\alpha + 2. \]

Note that
\[ \pi(d_{2\alpha - 1}(i)) + \pi(t(i, j)) \geq 2\alpha + \left\lfloor \frac{2i - 2}{3} \right\rfloor + \pi(t(i, j)). \]

We have \( \left\lfloor \frac{(2i - 2)/3} \right\rfloor \geq 0 \) and \( \pi(t(i, j)) \geq 2 \) when \( 1 \leq i \leq 3 \) and \( j = 1 \) or \( 2 \), and \( \left\lfloor \frac{(2i - 2)/3} \right\rfloor \geq 2 \) and \( \pi(t(i, j)) \geq 0 \) when \( i \geq 4 \). Hence for \( j = 1 \) or \( 2 \)
\[ \pi(d_{2\alpha + 1}(j)) \geq \min_{i \geq 1} \{ \pi(d_{2\alpha - 1}(i)) + \pi(t(i, j)) \} \geq 2\alpha + 2. \]

When \( j = 3 \), we need to show
\[ \pi(d_{2\alpha + 1}(3)) \geq 2\alpha + 3. \]

Note again that
\[ \pi(d_{2\alpha - 1}(i)) + \pi(t(i, 3)) \geq 2\alpha + \left\lfloor \frac{2i - 2}{3} \right\rfloor + \pi(t(i, 3)). \]

We have \( \left\lfloor \frac{(2i - 2)/3} \right\rfloor \geq 0 \) and \( \pi(t(i, 3)) \geq 3 \) when \( 1 \leq i \leq 5 \), and \( \left\lfloor \frac{(2i - 2)/3} \right\rfloor \geq 3 \) and \( \pi(t(i, 3)) \geq 0 \) when \( i \geq 6 \). Hence
\[ \pi(d_{2\alpha + 1}(3)) \geq \min_{i \geq 1} \{ \pi(d_{2\alpha - 1}(i)) + \pi(t(i, 3)) \} \geq 2\alpha + 3. \]

We arrive at the desired result. \( \square \)

3.2. Proof of the main theorem. It follows from Theorems 2.6 and 3.3 that
\[ \sum_{n \geq 0} g \left( 3^{2\alpha - 1}n + \frac{3^{2\alpha} - 1}{4} \right) q^n = \frac{E(q^6)^5}{E(q^3)^4} \sum_{j \geq 1} d_{2\alpha - 1}(j) X^j \equiv 0 \pmod{3^{2\alpha}}, \]
since \( \pi(d_{2\alpha - 1}(j)) \geq 2\alpha \) for all \( j \). Hence we have

**Theorem 3.4.** For \( \alpha \geq 1 \) and \( n \geq 0 \), we have
\[ g \left( 3^{2\alpha - 1}n + \frac{3^{2\alpha} - 1}{4} \right) \equiv 0 \pmod{3^{2\alpha}}. \] (3.5)

At last, we notice from (1.3) that
\[ \sum_{n \geq 0} b(2n)q^n = \frac{E(q^2)^5}{E(q)^3} = \sum_{n \geq 0} g(n)q^n. \]

The desired congruence (1.4) therefore follows.
4. An alternative proof of (1.2)

We end this paper with a more direct proof of (1.2). Recall that Ramanujan’s bilateral $1_{\psi_{1}}$ identity [7, Eq. (5.2.1)] tells us

$$\sum_{n=-\infty}^{\infty} \frac{(a; q)_{n}z^{n}}{(b; q)_{n}} = \frac{(q, b/a, az, q/az; q)_{\infty}}{(b, q/a, z, b/az; q)_{\infty}}, \quad |b/a| < |z| < 1. \quad (4.1)$$

We now replace $q$ by $q^{2}$ and take $a = -q$, $b = q^{3}$, $z = q$ in (4.1). Then

$$\sum_{n=-\infty}^{\infty} \frac{(-q; q^{2})_{n}q^{n}}{(q^{3}; q^{2})_{n}} = \frac{(q^{2}, -q^{2}, -q^{2}, -1; q^{2})_{\infty}}{(q^{3}, -q, q, -q; q^{2})_{\infty}}. \quad (4.2)$$

Note that

$$\sum_{n=-\infty}^{\infty} \frac{(-q; q^{2})_{n}q^{n}}{(q^{3}; q^{2})_{n}} = \sum_{n \geq 0} \frac{(-q; q^{2})_{n}q^{n}}{(q^{3}; q^{2})_{n}} + \sum_{n \geq 1} \frac{(-q; q^{2})_{-n}q^{-n}}{(q^{3}; q^{2})_{-n}}$$

$$= (1 - q) \left( \sum_{n \geq 0} \frac{(-q; q^{2})_{n}q^{n}}{(q^{3}; q^{2})_{n+1}} + \sum_{n \geq 0} \frac{(q; q^{2})_{n}(-q)^{n}}{(-q; q^{2})_{n+1}} \right).$$

On the other hand,

$$\frac{(q^{2}, -q^{2}, -q^{2}, -1; q^{2})_{\infty}}{(q^{3}, -q, q, -q; q^{2})_{\infty}} = 2(1 - q) \frac{(q^{4}; q^{4})_{\infty}^{5}}{(q^{2}; q^{2})_{\infty}^{4}}. \quad (4.3)$$

It follows that

$$\sum_{n \geq 0} \frac{(-q; q^{2})_{n}q^{n}}{(q^{3}; q^{2})_{n+1}} + \sum_{n \geq 0} \frac{(q; q^{2})_{n}(-q)^{n}}{(-q; q^{2})_{n+1}} = 2 \frac{(q^{4}; q^{4})_{\infty}^{5}}{(q^{2}; q^{2})_{\infty}^{4}},$$

which is essentially

$$\beta(q) + \beta(-q) = 2 \frac{(q^{4}; q^{4})_{\infty}^{5}}{(q^{2}; q^{2})_{\infty}^{4}}.$$ 

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References

1. G. E. Andrews, Mordell integrals and Ramanujan’s “lost” notebook, Analytic number theory (Philadelphia, Pa., 1980), pp. 10–18, Lecture Notes in Math., 899, Springer, Berlin-New York, 1981.
2. G. E. Andrews, D. Passary, J. Seller, and A. J. Yee, Congruences related to the Ramanujan/Watson mock theta functions $\omega(q)$ and $\nu(q)$, Ramanujan J. 43 (2017), no. 2, 347–357.
3. A. O. L. Atkin, Proof of a conjecture of Ramanujan, Glasgow Math. J. 8 (1967) 14–32.
4. B. C. Berndt and R. A. Rankin, Ramanujan: Letters and commentary, American Mathematical Society, London Mathematical Society, History of Mathematics Series, Vol. 9, 1995, 347 pp.
5. S. Chern and D. Tang, On certain weighted 7-colored partitions, Ramanujan J., in press. doi: 10.1007/s11139-017-9978-2.
6. S.-P. Cui, N. S. S. Gu, and L.-J. Hao, Congruences for some partitions related to mock theta functions, Int. J. Number Theory, in press. doi: 10.1142/S1793042118500641.
7. G. Gasper and M. Rahman, Basic hypergeometric series, Encyclopedia of Mathematics and its Applications, 35. Cambridge University Press, Cambridge, 1990. xx+287 pp.
8. B. Gordon and R. J. McIntosh, A survey of classical mock theta functions, Partitions, $q$-series, and modular forms, 95–144, Dev. Math., 23, Springer, New York, 2012.
9. M. D. Hirschhorn, *The power of q. A personal journey*, Developments in Mathematics, 49. Springer, Cham, 2017. xxii+415 pp.
10. D. G. Mead, Newton’s identities, *Amer. Math. Monthly* 99 (1992), no. 8, 749–751.
11. L. C. Shen, On the modular equations of degree 3, *Proc. Amer. Math. Soc.* 122 (1994), no. 4, 1101–1114.
12. L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge 1966 xiii+273 pp.
13. L. Wang, New congruences for partitions related to mock theta functions, *J. Number Theory* 175 (2017), 51–65.
14. G. N. Watson, Ramanujans Vermutung über Zerfallungszahlen, *J. Reine Angew. Math.* 179 (1938), 97–128.

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