Scalar Field Theory in the Derivative Expansion

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Abstract

The quantum correlations of scalar fields are examined as a power series in derivatives. Recursive algebraic equations are derived and determine the amplitudes; all loop integrations are performed. This recursion contains the same information as the usual loop expansion. The approach is pragmatic and generalizable to most quantum field theories.
1 Introduction

Scalar field theories are studied for a large number of reasons. However, there are complications due to the complexity of multi-loop integrations. This "bottleneck" prohibits explicit results beyond several loop orders. Better techniques are required, and in this work we develop an expansion of loop graphs in numbers of derivatives. This approach has the feature that all integrals are performed, and the the computation of amplitudes becomes algebraic. This content has applications to high energy physics as well as in condensed matter. Derivative expansions have been applied in the context of M-theory [1], N=8 supergravity [2] and in scalar field theory [4] (containing some properties of the integrals encountered here). Prior works on simplifying the complexity of diagrammatics is contained in [3], and others referenced therein.

The method is to (re-)write the (loop) expansion in terms of derivatives. Consider massive $\phi^n$ theory in $d$ dimensions; adding group theory is possible. It is important to appreciate the fact that energy scales are of phenomenological importance; experiments are built with greater and greater energies through the course in history. From this point of view, derivative expansions are more natural than coupling expansions. Derivative expansions also commute with gauge invariance and the non-perturbative dualities that supersymmetric field and string theories possess. These reasons, and the algebraic reduction, motivate the use of derivative expansions in the general theory.

2 Vertices and Sewing

The lagrangian is,

$$\mathcal{L} = \frac{1}{2} \phi \Box \phi + \frac{1}{2} m^2 \phi^2 + \sum \frac{\lambda}{n!} \phi^n \Lambda^{4-n},$$

and in general we may have derivatives on the vertices if a renormalized improved action; we consider dimensions $d \leq 4$. In the improved action, all of the irrelevant operators may be added. The standard Feynman rules consist of the propagator and vertices.

The quantum generating functional of the S-matrix is

$$\prod_{i=1}^{n} \frac{\delta}{\delta \phi(x_i)} \mathcal{L}_{q.c.},$$

(2.2)
with,

\[ \mathcal{L}_{\text{q.c.}} = \frac{1}{2} \phi (\Box + m^2) \phi + \sum_{d,i_d} g_{i_d}^{i_d} \mathcal{O}_{(d)}^{i_d} . \]  

(2.3)

The operators span all scalar operators of dimension \( d \), which ranges up to infinity. At \( k^2 < \Lambda^2 \) (or \( m^2 \)), these are

\[ \prod_{i=1}^{\tilde{d}} \left[ \prod_{j=1}^{n_i} \partial_{\mu(i,j)} \phi \right] \frac{1}{\Lambda^{\tilde{d}}} , \] 

(2.4)

with \( \sum_{i=1}^{n} n_i + n = \tilde{d} \). The quantum generating functional may be thought of as the infinite number of Feynman diagrams expanded at small momenta.

The general vertex has the form,

\[ t^{\nu} \prod_{i=1}^{n} \left( \prod_{j=1}^{m^2} \partial_{\rho(i,j)} \phi \right) (m^2)^{-p} , \] 

(2.5)

where the legs are amputated, and \( p \) an integer in 4 dimensions. The tensor \( t \) contains the coupling dependence from the Lagrangian. In a momentum cutoff theory, as opposed to dimensional regularization, the cutoff \( \Lambda \) also appears on the right hand side of the equation. If there were color in the theory, then the vertex has multiple trace structures. The diagrams seen in the following are planar, but are not planar in the color; multi-traces appear in the vertices, and are generated from the usual non-planar Feynman diagrams.

Iterating these vertices generate the S-matrix. Self-consistency of the S-matrix through unitarity or through rewriting the usual Feynman diagrams generate the diagrams (and integrals) in figure (1), as well as the coefficients. All of the integrals are performed in the following, leaving only algebraic recursions for the coefficients.

The recursion has the form in figure (1) and includes both the sum over the intermediate lines and the sum over the derivatives. The class of integrals appearing in the sewing are special in that they are free-field ones in composite operator calculations; these cage diagrams are easily performed in x-space and appear complicated in k-space. For this reason, we work in x-space when the integrals are manipulated; the Fourier transform is easily implemented. Another point is that at a given order in the external legs, an arbitrary number of internal lines are are involved (and arbitrary-point vertex); the lines are correlated with the loop order.
Figure 1: Integrals involved in generating the S-matrix. There are an arbitrary number of internal lines, from one to infinity.

The diagrams in figure (1) evaluate to,

\[ \sum_L t_{\mu\sigma,m_i\tilde{m}_i}^2 \prod_{i=1}^2 \frac{m_i^\phi}{\prod_{j=1} \partial_{\mu\sigma(i,j)} \phi(k_i)} \prod_{i=3}^4 \frac{\tilde{m}_i^\phi}{\prod_{j=1} \partial_{\mu\sigma(i,j)} \phi(k_i)} \]

(2.6)

\[ \times \frac{1}{L!} \left( \prod_{i=1}^L \prod_{j=1} m_j^\phi \prod_{j=1} \phi \prod_{a,b=1} \partial_{\mu\sigma(a,b)} \prod_{j=1} \phi \right) + (1 \leftrightarrow 3) + (1 \leftrightarrow 4) \]

(2.7)

with,

\[ \sum m_i^\phi = \sum \tilde{m}_i^\phi = L \]

(2.8)

The two quantities, the first equation (with coefficient t) and the second (product of two t’s), are equated; this is the recursion relation. It is possible to evaluate all of the integrals in the second equation.

3 Integrations

The integrals contain derivatives, and first we simplify these by extracting the derivatives. We simplify the form with the identity,
\[ \partial^\mu \Delta(m, x)^L = L \left[ (2 - d) + m^2 \partial_m^2 \right] \times \left( \frac{k^\mu}{k^2} \right) \Delta(m, x)^L. \quad (3.1) \]

The massive propagator is
\[ \Delta(m, x) = (x^2)^{-d/2+1} K_{d/2}(mx). \quad (3.2) \]
in terms of the modified Bessel function. The multiple iterations generate,
\[ \int e^{ix \cdot k} \prod_{j=1}^n \partial^\mu j \Delta(m, x)^L = \sum_{\sigma, \sigma'} \prod_{\eta} \eta_{\mu_a \nu_a} \prod \frac{k_{\mu'}}{k_{2n_1+2n_2}} \left( -2 \right)^{n_2-1} \quad (3.3) \]
\[ \times \left[ (2 - d) + m^2 \partial_m^2 \right]^n \Delta(m, x)^L \]
where the two products contain \( n_1 \) and \( n_2 \) terms; we sum over all combinations such that \( n_1 + n_2 = n \), found by successively applying the substitution rule in eqn (3.1).

Due to the simple topology of the diagram only one momenta \( k \) labels the external momenta. This procedure reduces all tensor integrals to scalar ones, similar to the Passarino-Veltman or Feynman-Brown one-loop integral reduction.

The tensor structure comes from the \( t = (k_i + \ldots + k_j)^2 \) invariants, after transforming the set of internal \( k_a \) momenta to x-space, \( k_a \to \partial_x \). We maintain the external lines in k-space.

The general integral is,
\[ \sum_{L} \prod_{i=1}^2 \left( \prod_{j=1}^\nu \partial_{\mu_a(i,j)} \right) \phi(k_i) \prod_{i=3}^4 \left( \prod_{j=1}^\nu \partial_{\mu_a(i,j)} \right) \phi(k_i) \quad (3.5) \]
\[ \times \sum_{\sigma, \sigma'} \prod \eta_{\mu_a \nu_a} \prod (k_1 + k_2)_{\mu_a} (k_1 + k_2)^{-2(n_1+n_2)} (-2)^{n_2-1} \quad (3.6) \]
\[ \times \sum_{\sigma, \sigma'} \prod \eta_{\mu_a \nu_a} \prod (k_3 + k_4)_{\mu_a} (k_3 + k_4)^{-2(n_1+n_2)} (-2)^{n_2-1} \quad (3.7) \]
\[ \times \left[ (2 - d) + m^2 \partial_m^2 \right]^n \prod_{\mu_a \nu_a} \frac{1}{m_1! \ldots m_i!} \int e^{ix \cdot k} \Delta(m, x)^L \quad (3.8) \]
The integrand reduces to scalar ones, via the contraction of the Pfaffian,
\[
\langle \prod_i \phi(x) \prod_j \phi(y) \rangle .
\] (3.9)

There is a delta function in \( m_i \) and \( m^\phi_i \).

The integrals over the products of Bessel functions are evaluated. With \( k^2 = (k_1 + k_2)^2 = (k_3 + k_4)^2 \) we have,

\[
\int d^d x \ e^{ix \cdot k} \Delta(m, x)^L = (k^2)^{-d/2-L(d/2-1)} \sum_n \left( \frac{k^2}{m^2} \right)^n \alpha_n^{(L)}
\] (3.10)

\[
= \sum_{n_1 \ldots + n_m = n} \prod \beta_{n_i} \left( \frac{L!}{n_1! \ldots n_m!} \right) \int d^d x \ e^{ik \cdot x} x^{-L-(d/2-1)L}
\] (3.11)

with \( \beta \) the expansion coefficients of the Bessel function, and (3.11) in dimensional regularization, and

\[
(k^2)^{-d/2-L(d/2-1)} \sum_{m,n} \left( \frac{k^2}{\Lambda^2} \right)^m \left( \frac{k^2}{m^2} \right)^n \alpha_m^{(L)},
\] (3.12)

in a momentum cutoff scheme. Both regulators can be applied simultaneously. The integrals follow from a power series expansion of the Bessel functions. Note that the dimension \( d = 2 \) is special for two reasons: the integrals are convergent and the factor in the substitution rule is nullify, except for the mass differential. Exact solution in \( d = 2 \) via recursion is promising.

As the integrals are performed and the recursion is algebraic, now it is straightforward to find higher coupling terms from lower coupling. The coefficients \( t \) have the expansion

\[
t = \sum_m \lambda^m a_m
\] (3.13)

with the \( a_m \) generated from the usual Feynman diagrams expanded at loop order \( m \). It is necessary to resum an infinite number of derivatives to rebuild the loop graphs; likewise an infinite number of derivative expanded Feynman diagrams are required to rebuild the individual terms in the derivative expansion. In order to begin the iteration the classical (or quantum improved) vertices are required, i.e. couplings
\[ \lambda_3 \phi^3 + \lambda_4 \phi^4, \quad (3.14) \]

and higher-point, or terms with derivatives.

To show that the derivative expanded graphs reproduce the usual coupling ones is straightforward and follows from the Schwinger-Dyson or Feynman-Mandelstam tree theorem. An explicit map to three loops is a simple exercise, including color structures.

4 Discussion

The scattering amplitudes of massive \( \Phi^n \) theory are examined and expressed in the derivative expansion. Unitarity and equivalence with the coupling loop expansion is clear. All integrals are performed, and due to the algebraic nature, the recursive construction of the coefficients is well suited to be implemented on a computer; integrals, not tensorial algebra, plague numerical calculations. The scattering amplitudes have a form similar to a set of matrix theory calculations.

These sewing relations are generalizable to gauge theory with matter, in particular, \( N = 4 \) supersymmetric and quantum chromodynamics, and any dynamical system.

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References

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