TIME-VARYING INTEGRO-DIFFERENTIAL INCLUSIONS WITH CLARKE SUB-DIFFERENTIAL AND NON-LOCAL INITIAL CONDITIONS: EXISTENCE AND APPROXIMATE CONTROLLABILITY

YONG-KUI CHANG∗ AND XIAOJING LIU
School of Mathematics and Statistics
Xidian University, Xi’an 710071, Shaanxi, China

(Communicated by Cristina Pignotti)

Abstract. In this paper, we mainly consider a time-varying semi-linear integro-differential inclusion with Clarke sub-differential and a non-local initial condition. By a suitable Green function combined with a resolvent operator, we firstly formulate its mild solutions and show that it admits at least one mild solution which can exist in a well-defined ball with a radius big enough. Through constructing a proper functional, we then derive a useful characterization of the approximate controllability for its related linear system in Green function terms, and establish a sufficient condition for the approximate controllability of the time-varying semi-linear integro-differential inclusion. Lastly, we also consider the finite approximate controllability of the time-varying semi-linear integro-differential inclusion via variational method.

1. Introduction. In this paper, we consider the following time-varying semi-linear integro-differential inclusion with Clarke sub-differential and a non-local initial condition

\[
\begin{cases}
  u'(t) \in \mathcal{A}(t) \left[ u(t) + \int_0^t a(t,s)u(s)ds \right] + \mathcal{B}v(t) + \partial \mathcal{G}(t,u(t)), \quad t \in I, \\
  u(0) = \sum_{k=1}^{m} c_k u(t_k),
\end{cases}
\]

where \( I := [0,b] \), the operator \( \mathcal{A}(t) \) with its domain \( \mathcal{D} \) independent of \( t \), and \( a(t,s) \in \mathcal{L}(\mathcal{H}), 0 \leq s \leq t \leq b \), generate a integral resolvent operator \( \mathcal{R}(t,s) \) in a Hilbert space \( \mathcal{H} \) (see Definition 2.2), \( \mathcal{B} \) is a bounded linear operator from a Hilbert space \( \mathcal{V} \) to \( \mathcal{H} \), the notation \( \partial \mathcal{G}(t,\cdot) \) denotes the Clarke sub-differential of \( \mathcal{G}(t,\cdot) \), the state \( u(\cdot) \) takes its value in \( \mathcal{H} \), the control \( v(\cdot) \) is given in \( L^2(I,\mathcal{V}) \), and \( 0 < t_1 < t_2 < \cdots < t_m < b, \) \( m \) is a positive integer, \( c_k \neq 0, k = 1, 2, \cdots, m \) are real numbers.

Partial integro-differential equations are often used to deal with models of viscoelastic materials or materials with memory ([8, 32]). Under certain conditions, those partial integro-differential equations can be treated as abstract evolution
equations by resolvent operators without semi-group properties ([17, 18]). It is remarkable that the resolvent operator with fixed point method is an effective approach for the solvability of abstract integro-differential equations. For example, RaviKumar [37] proved the existence of semi-classical and mild solutions of nonlinear integro-differential equations with nonlocal conditions in Banach spaces via analytic resolvent operators and Schaefer’s fixed point theorem. Ezzinbi and Ghnimi [15] considered existence and regularity of solutions for a neutral partial functional integro-differential equation. Dieye, Diop and Ezzinbi [13] studied the existence and exponentially stability in $p$-mean square for some stochastic integro-differential equation with delays. Lizama and N’Guérékata [24] presented a general operator theoretical approach to study bounded mild solutions for a semi-linear integro-differential equation in Banach spaces. Chang and Ponce [6] investigated uniform exponential stability and the existence and uniqueness of bounded solutions to a semi-linear integro-differential equation. Sometimes, when treated some partial integro-differential equations in parabolic cases, they can usually be formulated as time-varying (or non-autonomous) abstract evolution equations. See for instance, RaviKumar [38] dealt with regularity of solutions of evolution integro-differential equations with deviating argument. Liu and Ezzinbi [21] obtained the existence and uniqueness of mild and classical solutions to a non-autonomous integro-differential equation with non-local initial conditions. Non-local initial conditions originate from physical phenomena. It is shown that, the non-local initial condition $u(0) = \sum_{k=1}^{m} c_k u(t_k)$ can be more suitable than the classical initial condition $u(0) = u_0$ in some physical models. For instance, Deng [12] applied the non-local condition $u(0) = \sum_{k=1}^{m} c_k u(t_k)$ to describe the diffusion phenomenon of a small amount of gas in a transparent tube. Under such circumstance, this non-local condition is involved in some additional measurements at $t_k, k = 1, 2, \cdots, m$, which is more precise than the measurement just at $t = 0$. It was also suggested by Byszewski in [5] that, if $c_k \neq 0, k = 1, 2, \cdots, m$, then the results could be used to kinematics to determine the location evolution $t \rightarrow u(t)$ of a physical object for which the specific positions $u(0), u(t_1), u(t_2), \cdots, u(t_m)$ may be unknown, but the non-local condition $u(0) = \sum_{k=1}^{m} c_k u(t_k)$ does hold. The importance of non-local initial conditions to other types of equations can be found in [25, 40, 4, 44, 19] and references therein.

The concept of controllability plays a crucial role in design and analysis of control systems. Exact controllability and approximate controllability are two important concepts in mathematical control theory [3]. Exact controllability means that the addressed system can be steered to arbitrary final state whereas approximate controllability enables us to steer the system to arbitrarily small neighborhood of final state. In general, it is difficult to realize exact controllability of control systems in infinite dimensional spaces or some special systems in finite dimensional spaces (see [1, 2, 26]). From a practical point of view, the approximate controllability becomes a more natural concept. Therefore, there are interesting works on approximate controllability of different systems modelled by evolution equations, integro-differential equations, functional differential equations, differential inclusions and fractional evolution equations ([45]) in infinite dimensional spaces. See for instance,
Abbas et al [31] established approximate controllability of sub-diffusion equation with impulsive condition. Sakthivel et al [39] considered approximate controllability of fractional stochastic differential inclusions with non-local conditions. Yan and Jia [43] studied approximate controllability of partial neutral stochastic functional integro-differential inclusions with state-dependent delay. Fu [16] discussed the approximate controllability of semi-linear non-autonomous evolution systems with state-dependent delay. Xiao and Zhu [41] considered approximate controllability for a second order semi-linear impulse functional differential inclusion in Hilbert spaces. Finite approximate controllability is a stronger property than that of approximate controllability. Lions and Zuazua [20] have proved that finite approximate controllability is a consequence of approximate controllability in the context of linear heat equation while one property may not be deduced as a consequence of the other one in the nonlinear context. Recently, Mahmudov [28] has developed the notion of finite approximate controllability to a semi-linear evolution equations in Hilbert spaces, which implies that the control can be selected such that the final state is not only approximately controllable but also satisfies simultaneously a finite number of exact constraints. The concept of finite approximate controllability has further been extended to some semi-linear fractional evolution systems in Hilbert spaces [29, 30].

The time-varying integro-differential inclusion problem (1) has a closed relation to the following H-variational inequality problem

\[
\begin{align*}
\{ & -u'(t) + \mathcal{A}(t) \left[ u(t) + \int_0^t a(t,s)u(s)ds \right] + \mathcal{B}v(t), \mu \} + G^0(t, u(t); \mu) \geq 0, \\
& u(0) = \sum_{k=1}^m c_k u(t_k),
\end{align*}
\]

for \( \forall \mu \in \mathcal{H}, t \in I \), where \( G^0(t, \cdot; \cdot) \) represents the Clarke directional derivative of \( G(t, \cdot) \) (see Sect. 2). In fact, if there exists a function \( g \in L^2(I, \mathcal{H}) \) such that \( g(t) \in \partial G(t, u(t)) \) and

\[
\begin{align*}
\{ & u'(t) = \mathcal{A}(t) \left[ u(t) + \int_0^t a(t,s)u(s)ds \right] + \mathcal{B}v(t) + g(t), \ t \in I, \\
& u(0) = \sum_{k=1}^m c_k u(t_k),
\end{align*}
\]

then for \( \forall \mu \in \mathcal{H} \) and a.e. \( t \in I \), we have

\[
\begin{align*}
\{ & -u'(t) + \mathcal{A}(t) \left[ u(t) + \int_0^t a(t,s)u(s)ds \right] + \mathcal{B}v(t), \mu \} + g(t, \mu)_{\mathcal{H}} = 0, \\
& u(0) = \sum_{k=1}^m c_k u(t_k).
\end{align*}
\]

Noticing that \( g(t) \in \partial G(t, u(t)) \) and \( g(t), g(t, \mu)_{\mathcal{H}} \leq G^0(t, u(t); \mu) \) (see also Sect. 2), we can see that for \( \forall \mu \in \mathcal{H} \), the H-variational inequality (2) holds true. So, we can study the H-variational inequality (2) through investigating the problem (1). H-variational inequality initiated by Panagiotopoulos, has been regarded as one of the most powerful tools to deal with non-smooth and non-convex energy superpotentials problems in mechanics [35, 36]. Solvability, well-posedness and optimal controls of some H-variational inequalities are also investigated under different conditions in mathematics, we can refer to Lu and Liu [27], Huang and Xiao [42], Motreanu and
Motreanu [34], Migórski and Sofonea [33], Chang and Pei [7], and references cited therein for more details. Approximate controllability of control systems represented by H-variational inequalities was first studied in [22] by Liu and Li, in which sufficient conditions were established for existence and approximate controllability of a semi-linear time-invariant system with classical initial conditions. Particularly, let $a(t, s) \equiv 0$ with a classical initial condition $u(0) = u_0$ in (2), it is reduced to a semi-linear evolution H-variational inequality problem, the approximate controllability of which was investigated in [23] via a two-parameter evolution system having a semi-group property. However, there is still little information on approximate controllability of time-varying system with a non-local initial condition described by the integro-differential H-variational inequality (2). Consequently, it is of interest to consider the problem (1).

Inspired by above mentioned works, the main purpose of this paper is to investigate approximate controllability of the time-varying system (1). By introducing a well-defined Green function combined with a resolvent operator, we firstly give the well-defined Green function and then establish a sufficient condition for the approximate controllability of (1). Finally, we also investigate the finite approximate controllability of (2) via variational method. The structure of this paper is as follows: Sect. 2 is Preliminaries on some basic definitions, lemmas and notations. Sect. 3 is focused upon solvability of the problem (1). Sect. 4 is devoted to approximate controllability of the system (1).

2. Preliminaries. This section is mainly concentrated on some basic facts which are used throughout this paper.

For a given Hilbert space $H$, $L(H)$ denotes the space of all bounded linear operators from $H$ into itself with uniform operator norm, and $C(I, H)$ is the Banach space of all continuous functions from $I$ to $H$ with sup-norm. For $p \in [1, +\infty)$, the space $L^p(I, H)$ is a set formed by all $p$-th $H$-valued Bochner integrable functions on $I$ with its norm $\| \cdot \|_{L^p} = \left( \int_I \| \cdot \|^p dt \right)^{\frac{1}{p}}$.

Let $Z$ be a Banach space with its dual $Z^*$, and the symbol $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $Z$ and $Z^*$. The notation $\mathcal{P}(Z)$ represents a class of nonempty subsets of $Z$. Denote by $\mathcal{P}_b(Z) = \{ \Omega \in \mathcal{P}(Z) : \Omega \text{ is bounded} \}$, and $\mathcal{P}_{cp,cv}(Z) = \{ \Omega \in \mathcal{P}(Z) : \Omega \text{ is compact and convex} \}$.

A multi-valued map $G : Z \to \mathcal{P}(Z)$ has convex (closed) values if $G(z)$ is convex (closed) for all $z \in Z$. $G$ admits a fixed point if there exists $z \in Z$ such that $z \in G(z)$. $G$ is said to be upper semicontinuous (u.s.c.) on $Z$ if for each $z_0 \in Z$, the set $G(z_0)$ is a nonempty, closed subset of $Z$, and if for each open set $\mathcal{O}$ of $Z$ containing $G(z_0)$, there exists an open neighborhood $\mathcal{N}$ of $z_0$ such that $G(\mathcal{N}) \subseteq \mathcal{O}$. Also, $G$ is called to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(Z)$. If $G$ is completely continuous with nonempty compact values, then $G$ is u.s.c. if and only if $G$ has a closed graph, i.e., $u_n \to u_*$, $g \to g_*$, $g \in G(u_n)$ imply $g_* \in G(u_*)$.

A multi-valued map $F : I \to \mathcal{P}(Z)$ is said to be measurable if $F^{-1}(\mathcal{C}) = \{ t \in I : F(t) \cap \mathcal{C} \neq \emptyset \} \in \Sigma$ for each closed set $\mathcal{C} \subseteq Z$. If $F : I \times Z \to \mathcal{P}(Z)$, then measurability of $F$ means that $F^{-1}(\mathcal{C}) \in \Sigma \otimes \mathcal{B}Z$, where $\Sigma \otimes \mathcal{B}Z$ is the $\sigma$-algebra...
of subsets in $I \times Z$ generated by the sets $A \times B$, $A \in \Sigma$, $B \in B_Z$, and $B_Z$ is the $\sigma$-algebra of the Borel sets in $Z$.

The Clarke directional derivative $h^0(x, d)$, of a locally Lipshitz function $h : Z \to \mathbb{R}$ at $x$ in the direction $d$, is given by

$$h^0(x, d) = \limsup_{y \to x, \; t \to 0} \frac{h(y + td) - h(y)}{t}.$$ 

The Clarke sub-differential $\partial h(x)$, of $h$ at $x$, is a subset of $Z^*$ given by

$$\partial h(x) = \{x^* \in Z^* : h^0(x, d) \geq \langle x^*, d \rangle, \; \forall d \in Z\}.$$ 

We list some fundamental properties below.

**Lemma 2.1.** [9] Let $h : O \to \mathbb{R}$ be a locally Lipshitz function on an open set $O$ of $Z$. Then the following results hold:

(i) For each $x, d \in Z$, one has $h^0(x, d) = \max\{(x^*, d) : \forall x^* \in \partial h(x)\}$.

(ii) For each $x \in O$, $\partial h(x)$ is a nonempty, convex, weak* compact subset of $Z^*$, and $\|x^*\|_{Z^*} \leq L$ for each $x^* \in \partial h(x)$ (where $L > 0$ is the Lipschitz constant of $h$ near $x$).

(iii) The graph of $\partial h$ is close in $Z \times Z^*_w$, topology, i.e., if $\{x_n\} \subset O$ and $\{x^n_\ast\} \subset Z^*$ are sequences such that $x^n_\ast \in \partial h(x_n)$ and $x_n \to x$ in $Z$, $x^n_\ast \to x^\ast$ weak* in $Z^*$, then $x^\ast \in \partial h(x)$ (where $Z^*_w$ denotes the Banach space $Z^*$ equipped with the $w^*$-topology).

(iv) The multifunction $O \ni x \to \partial h(x) \subseteq Z^*$ is u.s.c. from $O$ into $Z^*_w$.

For more detailed results on multi-valued analysis and Clarke sub-differential, we can refer to [45, 33, 9, 14]. Next, we introduce the concept of the resolvent operator related to the problem (1). We always suppose that hypotheses [38, (H1)-(H2)] or [21, (H1)-(H2)] are satisfied in order to generate a well-defined resolvent operator.

**Definition 2.2.** [21, 38] An operator valued function $R(t, s) \in \mathcal{L}(H)$ for $t, s \in I$ is called to be the resolvent operator of the equation

$$u'(t) = \mathcal{A}(t) \left[u(t) + \int_0^t a(t, s)u(s)ds\right],$$

if it satisfies the following properties:

1. $R(t, s)$ is strongly continuous in $s$ and $t$, $R(t, t) = I$ (the identity operator on $H$). And there exist some constants $M, \beta$ such that $\|R(t, s)\| \leq Me^{\beta(t-s)}$, $t, s \in I$.
2. $R(t, s)Y \subseteq Y$ (the Banach space formed from $D$ with the graph norm), and $R(t, s)$ is strongly continuous in $s$ and $t$ on $Y$.
3. For $y \in Y$, $R(t, s)y$ is continuously differentiable in $s$ and $t$, and

$$\frac{\partial}{\partial t} R(t, s)y = \mathcal{A}(t) \left[R(t, s)y + \int_0^t a(t, r)R(r, s)ydr\right], 0 \leq s \leq t \leq b.$$ 

The following facts are also crucial in proving our main results.

**Lemma 2.3.** [14] Let $Z$ be a Banach space. If $\Omega \subset Z$ is nonempty, close and convex with $0 \in \Omega$, and $\mathcal{F} : \Omega \to \mathcal{P}_{cp,cs}(\Omega)$ a u.s.c. multifunction that maps bounded sets into relatively compact sets, then one of the following statements is true:

(i) The set $U = \{x \in \Omega : x \in \lambda \mathcal{F}(x), \lambda \in (0, 1)\}$ is unbounded.

(ii) $\mathcal{F}$ admits a fixed point, i.e. there exists $x \in \Omega$ such that $x \in \mathcal{F}(x)$. 


Lemma 2.4. [14] If \( \mathbb{D} \) is a compact subset of a Banach space \( Z \), then its convex closure \( \text{conv}(\mathbb{D}) \) is compact.

3. Existence of mild solutions. In order to obtain the approximate controllability result, it is needed to investigate the solvability of the problem (1). We first impose the following conditions on the problem (1).

(A1) The resolvent operator \( R(t,s)(t > s) \) is compact, and there exists a constant \( M > 0 \) such that \( \| R(t,s) \| \leq M \) for \( 0 < s < t < b \).

(A2) \( c^* := \sum_{k=1}^{m} |c_k| < \frac{1}{M} \).

(A3) The function \( G : I \times H \to \mathbb{R} \) satisfies the following conditions:

(a) For all \( u \in H \), \( G(\cdot, u) \) is measurable;

(b) For a.e. \( t \in I \), \( G(t, \cdot) \) is locally Lipschitz continuous;

(c) There exists a function \( \varphi(\cdot) \in L^2(I, \mathbb{R}^+) \) and a constant \( \rho > 0 \) such that for a.e. \( t \in I \) and all \( u \in H \)

\[
\| \partial G(t,u) \| := \sup \{ \| g(t) \| : g(t) \in \partial G(t,u) \} \leq \varphi(t).
\]

(A4) There exists a function \( \psi(\cdot) \in L^1(I, \mathbb{R}^+) \) such that

\[
\| B(\cdot) \| \leq \psi(t) \text{ for all } v \in L^2(I, \mathcal{V}) \text{ and } t \in I.
\]

Remark 1. Owing to \( I = [0, b] \) is a compact interval, \( \varphi(\cdot) \in L^2(I, \mathbb{R}^+) \) implies that \( \varphi(\cdot) \in L^1(I, \mathbb{R}^+) \). Thus \( \int_0^b \varphi(t) dt < +\infty \).

Define the multi-valued map \( S : L^2(I, H) \to \mathcal{P}(L^2(I, H)) \) as

\[
S(u) = \{ g \in L^2(I, H) : g(t) \in \partial G(t, u(t)) \text{ a.e. } t \in I \text{ for } u \in L^2(I, H) \}.
\]

According to [23, Lemmas 3.1, 3.2], we have the following results.

**Lemma 3.1.** If the condition (A3) holds, then for \( u \in L^2(I, H) \), the set \( S(u) \) has nonempty, convex, and weakly compact values.

**Lemma 3.2.** Assume that the condition (A3) holds. Let the operator \( S \) satisfy: \( z_n \to z \) in \( L^2(I, H) \), \( w_n \in S(z_n) \), and \( w_n \to w \) weakly in \( L^2(I, H) \), then we have \( w \in S(z) \).

In view of the assumption (A2), we can see that

\[
\left\| \sum_{k=1}^{m} c_k R(t_k, 0) \right\| \leq Mc^* < 1.
\]

From spectrum theory of operators, we know that

\[
\Delta := \left( I - \sum_{k=1}^{m} c_k R(t_k, 0) \right)^{-1}
\]

exists and bounded, where \( I \) is the identity operator. Moreover, the above defined operator \( \Delta \) can be expressed by Neumann expression

\[
\Delta = \sum_{n=0}^{\infty} \left( \sum_{k=1}^{m} c_k R(t_k, 0) \right)^n.
\]
Hence,
\[
\|\Delta\| \leq \sum_{n=0}^{\infty} \left\| \sum_{k=1}^{m} c_k R(t_k, 0) \right\|^n = \frac{1}{1 - \sum_{k=1}^{m} c_k R(t_k, 0)} \leq \frac{1}{1 - Mc^*}. \tag{3}
\]

For any \( g(t) \in C(I, \mathcal{H}) \), we know the mild solution to the following problem
\[
\begin{cases} 
  u'(t) = \mathcal{A}(t) \left[ u(t) + \int_0^t a(t, s)u(s)ds \right] + g(t), \ t \in I, \\
  u(0) = \tilde{u},
\end{cases}
\]
can be expressed by (see [21, 38])
\[
u(t) = R(t, 0)u(0) + \int_0^t R(t, s)g(s)ds, t \in I. \tag{4}
\]

By (4), we have
\[
u(t_k) = R(t_k, 0)u(0) + \int_0^{t_k} R(t_k, s)g(s)ds, k = 1, 2, \cdots, m. \tag{5}
\]

In view of the nonlocal initial condition \( u(0) = \sum_{k=1}^{m} c_k u(t_k) \) and (5), we obtain
\[
u(0) = \sum_{k=1}^{m} c_k R(t_k, 0)u(0) + \sum_{k=1}^{m} c_k \int_0^{t_k} R(t_k, s)g(s)ds. \tag{6}
\]

Considering (A2) and the definition of \( \Delta \), we further have
\[
u(0) = \sum_{k=1}^{m} c_k \Delta \int_0^{t_k} R(t_k, s)g(s)ds. \tag{7}
\]

Now taking (7) into (4), we obtain
\[
u(t) = \sum_{k=1}^{m} c_k R(t, 0)\Delta \int_0^{t_k} R(t_k, s)g(s)ds + \int_0^t R(t, s)g(s)ds, t \in I, \tag{8}
\]
which gives a mild solution to the following linear problem with non-local initial conditions
\[
\begin{cases} 
  u'(t) = \mathcal{A}(t) \left[ u(t) + \int_0^t a(t, s)u(s)ds \right] + g(t), \ t \in I, \\
  u(0) = \sum_{k=1}^{m} c_k u(t_k).
\end{cases}
\tag{9}
\]

In convenience, we introduce the following notion known as a Green function
\[
\Gamma(t, s) = \sum_{k=1}^{m} \chi_k(s)c_k R(t, 0)\Delta R(t_k, s) + \chi_t(s)R(t, s), \tag{10}
\]
where
\[
\chi_t(s) = \begin{cases} 
  1, & s \in [0, \tau), \\
  0, & s \in [\tau, b].
\end{cases}
\]
Now by (8) and (10), we deduce that the mild solution to (9) can be expressed simply by
\[ u(t) = \int_0^b \Gamma(t, s)g(s)ds. \]

Through the previous preparation, we have the following definition.

**Definition 3.3.** Given \( v \in L^2(I, V) \), a function \( u \in C(I, H) \) is called to be a mild solution to the problem (1) if there exists a function \( g \in L^2(I, H) \) such that \( g(t) \in \partial G(t, u(t)) \) a.e. \( t \in I \) and the following equation holds:
\[ u(t) = \int_0^b \Gamma(t, s)[\mathcal{B}v(s) + g(s)]ds. \]

For each \( r > 0 \), we define \( B_r := \{ u \in C(I, H) : \|u(t)\| \leq r, t \in I \} \). The existence result for the problem (1) can be stated as following.

**Theorem 3.4.** Assume that conditions (A1)-(A4) hold. Then the problem (1) admits at least one mild solution in a suitable ball \( B_r \) on \( I \), with its radius \( r \) satisfying
\[ r \geq MM_2(MM_1 + 1), \tag{11} \]
where
\[ M_1 = \frac{c^*}{1 - Mc^*}, \quad M_2 = \int_0^b [\varphi(s) + \psi(s)]ds. \tag{12} \]

**Proof.** Based upon Lemma 3.1, the multi-valued map \( F : C(I, H) \to \mathcal{P}(C(I, H)) \) defined by
\[ F(u) = \left\{ f \in C(I, H) : f(t) = \int_0^b \Gamma(t, s)[\mathcal{B}v(s) + g(s)]ds, g \in S(u) \right\}, \tag{13} \]
has nonempty values, where \( \Gamma(t, s) \) is given by (10). From Definition 3.3, seeking a mild solution to (1) is equivalent to finding a fixed point of \( F \) defined by (13). In the following, we shall prove that the operator \( F \) admits a fixed point via Lemma 2.3. Firstly, for each \( u \in C(I, H) \), \( F(u) \) is convex by the convexity of \( S(u) \). For the seek of convenience, the remainder of the proof is divided into several steps.

**Step 1.** \( F(B_r) \subseteq B_r \), with \( r \) satisfying (11).
Indeed, let \( r \) be a big enough number satisfy (11). Then for each \( u(\cdot) \in B_r \), in view of relations (3), (10), (11), (13), and assumptions (A1)-(A4), it follows that
\[ \|f(t)\| \leq \int_0^b \|\Gamma(t, s)\|[\mathcal{B}v(s) + g(s)]\|ds \]
\[ \leq M\|\Delta\| \int_0^b \sum_{k=1}^m |\chi_{t_k}(s)c_k|\|\mathcal{R}(t_k, s)\|[\mathcal{B}v(s) + g(s)]\|ds \]
\[ + \int_0^b |\chi_t(s)|\|\mathcal{R}(t, s)\|[\mathcal{B}v(s) + g(s)]\|ds \]
\[ \leq M^2M_1 \int_0^{t_k} [\varphi(s) + \psi(s)]ds + M \int_0^t [\varphi(s) + \psi(s)]ds \]
\[ \leq MM_2(MM_1 + 1), \]
which proves the assertion.

**Step 2.** \( F \) is equicontinuous on \( B_r \).
For each \( u \in B_r \), and \( f \in F(u) \), there exists \( g \in S(u) \) such that (13) holds for \( t \in I \). Take \( 0 \leq t' < t'' \leq b \). Observe that

\[
\|f(t'') - f(t')\| = \left\| \int_0^b [\Gamma(t'', s) - \Gamma(t', s)][A(t', s) + B(t', s)] ds \right\|
\leq \left\| [R(t'', 0) - R(t', 0)] \int_0^b \sum_{k=1}^m \chi_{t_k}(s) \Delta R(t_k, s)[B(t, s) + g(s)] ds \right\|
+ \left\| \int_0^b [\chi_{t'}(s) - \chi_{t''}(s)] R(t', s) [B(t, s) + g(s)] ds \right\|
+ \left\| \int_0^b \chi_{t''}(s) [R(t'', s) - R(t', s)] [B(t, s) + g(s)] ds \right\|
:= I_1 + I_2 + I_3.
\]

For the term \( I_1 \), by (A1)-(A4), (3), (10) and (12), we have

\[
\left\| \int_0^b \sum_{k=1}^m \chi_{t_k}(s) \Delta R(t_k, s)[B(t, s) + g(s)] ds \right\| \leq M M_1 M_2,
\]

and thus by the strong continuity of \( R(t, s) \), we can deduce that \( I_1 \to 0 \) as \( t'' \to t' \).

For the term \( I_2 \), by (A1), (A3) and (A4), we have

\[
I_2 \leq M \int_{t'}^{t''} [\varphi(s) + \psi(s)] ds \to 0, \text{ as } t'' \to t'.
\]

For \( t' = 0, 0 < t'' \leq b \), it is obvious that \( I_2 = 0 \). For \( 0 < t' < b \) and arbitrarily small \( 0 < \delta < t' \), by (A1), (A3)-(A4), we obtain

\[
I_3 \leq \int_0^{t'-\delta} \|R(t'', s) - R(t', s)\| [\varphi(s) + \psi(s)] ds
+ \int_{t'-\delta}^{t'} \|R(t'', s) - R(t', s)\| [\varphi(s) + \psi(s)] ds
\leq \int_0^{t'-\delta} \|R(t'', s) - R(t', s)\| [\varphi(s) + \psi(s)] ds
+ 2M \int_{t'-\delta}^{t'} [\varphi(s) + \psi(s)] ds.
\]

Now taking into account that

\[
\|R(t'', s) - R(t', s)\| [\varphi(s) + \psi(s)] \leq 2M [\varphi(s) + \psi(s)] \in L^1([0, t' - \delta], \mathbb{R}^+),
\]

and the compactness of \( R(t, s)(t - s > 0) \) (see (A1)) implies the continuity of \( R(t, s) \) in the uniform operator topology, we conclude by the Lebesgue dominated convergence theorem that

\[
I_3 \to 0 \text{ as } t'' \to t' \text{ and } \delta \to 0.
\]

As a result of above arguments, \( \|f(t'') - f(t')\| \) approaches zero independently of \( u \in B_r \) as \( t'' \to t' \). Thus, \( F \) is equicontinuous on \( B_r \).

**Step 3.** The set \( \Omega(t) := \{(Fu)(t) : u \in B_r\} \) is relatively compact for every \( t \in I \).
At first, for \( t = 0, \Omega(0) := \{(Fu)(0) : u \in B_r \} \), we see that
\[
    f(0) = \int_0^b \Gamma(0, s)[Bv(s) + g(s)]ds = \int_0^b \sum_{k=1}^m \chi_{t_k} c_k R(0, 0) \Delta R(t_k, s)[Bv(s) + g(s)]ds = \sum_{k=1}^m c_k \Delta \int_0^{t_k} R(t_k, s)[Bv(s) + g(s)]ds.
\]

Let \( \delta \) be a real number satisfying \( 0 < \delta < t_k \), we further introduce
\[
    (F^\delta u)(0) := \left\{ f^\delta \in C(I, H) : f^\delta(0) = \sum_{k=1}^m c_k \Delta \int_0^{t_k-\delta} R(t_k, s)[Bv(s) + g(s)]ds \right\}.
\]

The compactness of \( R(t, s)(t - s > 0) \) implies that the set
\[
    O_\delta := \{ R(t_k, s)[Bv(s) + g(s)] : 0 < s < t_k - \delta, u \in B_r \}
\]
is compact for all \( \delta > 0 \). Then \( \text{conv}(O_\delta) \) is also a compact set according to Lemma 2.4. By mean value theorem for Bochner integrals, we have \( f^\delta(0) \in (t_k - \delta)\text{conv}(O_\delta) \) for all \( t \in I \). Thus the set
\[
    \Omega_\delta(0) := \{(F^\delta u)(0) : u \in B_r \}
\]
is relatively compact in \( H \) for every \( \delta, \ 0 < \delta < t_k \). Moreover, for \( u \in B_r \),
\[
    \| (Fu)(0) - (F^\delta u)(0) \|
    \leq \left\| \sum_{k=1}^m c_k \Delta \int_{t_k-\delta}^{t_k} R(t_k, s)[Bv(s) + g(s)]ds \right\|
    \leq MM_1 \int_{t_k-\delta}^{t_k} \| Bv(s) + g(s) \| ds
    \leq MM_1 \left[ \int_{t_k-\delta}^{t_k} \| \varphi(s)ds + \int_{t_k-\delta}^{t_k} \psi(s)ds \right]
    \to 0 \quad \text{as } \delta \to 0.
\]

Thus, there are relatively compact sets \( \Omega_\delta(0) \) arbitrarily close to the set \( \Omega(0) \), and \( \Omega(0) \) is also relatively compact in \( H \).

Next, let \( 0 < t \leq b \) and \( \eta \) be a real number satisfying \( 0 < \eta < t \), we introduce
\[
    (Fu) = (F_1 u) + (F_2 u),
\]
\[
    (F_1 u) = \left\{ f_1 \in C(I, H) : f_1(t) = \sum_{k=1}^m c_k R(t, 0) \Delta \int_0^{t_k} R(t_k, s)[Bv(s) + g(s)]ds, \right\}
\]
\[
    (F_2 u) = \left\{ f_2 \in C(I, H) : f_2(t) = \int_0^t R(t, s)[Bv(s) + g(s)]ds \right\},
\]
\[
    (F_2^n u) = \left\{ f^n_2 \in C(I, H) : f^n_2(t) = \int_0^{t-\eta} R(t, s)[Bv(s) + g(s)]ds \right\}.
\]

By the compactness of \( R(t, s)(t - s > 0) \), the set
\[
    Q_\delta := \{ R(t, s)[Bv(s) + g(s)] : 0 \leq s < t - \eta, u \in B_r \}
\]
is compact for all \( \eta > 0 \). It is again by Lemma 2.4 that the set \( \text{conv}(Q_{\delta}) \) is also compact. From mean value theorem for Bochner integrals, we have \( f^{0}_{2}(t) \in (t - \eta)\text{conv}(Q_{\delta}) \) for all \( t \in I \). Therefore, the set
\[
\Omega_{2}^{0}(t) := \{(F^{0}_{2}u)(t) : u \in B_{r}\}
\]
is relatively compact in \( \mathcal{H} \) for every \( \eta, 0 < \eta < t \). Furthermore, for \( u \in B_{r} \),
\[
\|f_{2}(t) - f^{0}_{2}(t)\| \leq M \left[ \int_{t-\eta}^{t} \varphi(s)ds + \int_{t_{k}-\eta}^{t_{k}} \psi(s)ds \right] \to 0 \text{ as } \eta \to 0.
\]
Hence, there are relatively compact sets \( \Omega_{2}^{0}(t) \) arbitrarily close to the set
\[
\Omega_{2}(t) := \{(F_{2}u)(t) : u \in B_{r}\},
\]
and \( \Omega_{2}(t) \) is also relatively compact in \( \mathcal{H} \). Together with the compactness of \( \mathcal{R}(t, 0)(t > 0) \), the set \( \Omega(t) := \{(Fu)(t) : u \in B_{r}\} \) is relatively compact in \( \mathcal{H} \) for all \( t \in (0, b] \). From all above arguments, we conclude that the set \( \Omega(t) := \{(Fu)(t) : u \in B_{r}\} \) is relatively compact in \( \mathcal{H} \) for every \( t \in I \).

**Step 4.** \( F \) has a closed graph.

Let \( u_{n} \to u_{*} \) in \( C(I, \mathcal{H}) \) and \( f_{n} \to f_{*} \) in \( C(I, \mathcal{H}) \) with \( f_{n} \in F(u_{n}) \). We need to show that \( f_{*} \in F(u_{*}) \). The fact \( f_{n} \in F(u_{n}) \) implies that there exists \( g_{n} \in \mathcal{S}(u_{n}) \) satisfying
\[
f_{n}(t) = \sum_{k=1}^{m} c_{k} \mathcal{R}(t, 0) \Delta \int_{0}^{t_{k}} \mathcal{R}(t_{k}, s) \mathcal{S}(s)ds + \int_{0}^{t} \mathcal{R}(t, s) \mathcal{S}(s)ds
\]
\[
\quad + \sum_{k=1}^{m} c_{k} \mathcal{R}(t, 0) \Delta \int_{0}^{t_{k}} \mathcal{R}(t_{k}, s)g_{n}(s)ds + \int_{0}^{t} \mathcal{R}(t, s)g_{n}(s)ds.
\]
(14)

From (A3), we know that \( \{g_{n}\}_{n \geq 1} \subseteq L^{2}(I, \mathcal{H}) \) is bounded. Thus, we may assume, passing to a subsequence if necessary, that
\[
g_{n} \to g_{*} \text{ weakly in } L^{2}(I, \mathcal{H}).
\]
(15)

From (14), (15) and the compactness of \( \mathcal{R}(t, s)(t - s > 0) \), we have
\[
f_{n}(t) \to \sum_{k=1}^{m} c_{k} \mathcal{R}(t, 0) \Delta \int_{0}^{t_{k}} \mathcal{R}(t_{k}, s) \mathcal{S}(s)ds + \int_{0}^{t} \mathcal{R}(t, s) \mathcal{S}(s)ds
\]
\[
\quad + \sum_{k=1}^{m} c_{k} \mathcal{R}(t, 0) \Delta \int_{0}^{t_{k}} \mathcal{R}(t_{k}, s)g_{*}(s)ds + \int_{0}^{t} \mathcal{R}(t, s)g_{*}(s)ds.
\]
(16)

By Lemma 3.2 and (15), we know \( g_{*} \in \mathcal{S}(u_{*}) \). Considering (16), and \( f_{n} \to f_{*} \) in \( C(I, \mathcal{H}) \) with \( f_{n} \in F(u_{n}) \), we can infer that \( f_{*} \in F(u_{*}) \).

By Steps 1-4 together with Arzelà-Ascoli theorem, we conclude that the multivalued map \( F \) is completely continuous, u.s.c. with convex values. Conducted as the proof of Step 1, it is easily shown that the assertion (i) in Lemma 2.3 is not true. Thus, \( F \) admits a fixed point \( u \) in \( B_{r} \), which in turn is a mild solution to the problem (1). The whole proof is finished. \( \square \)

4. **Approximate controllability.** In this section, we mainly establish the approximate controllability of the system (1). Define the set
\[
\mathcal{R}_{0}(\mathcal{G}) = \{u(b) \in \mathcal{H} : u(\cdot)
\]
is a mild solution to system (1) related to a control \( v \in L^{2}(I, \mathcal{V}) \),
which is known as the reachable set of system (1) at terminal time $b$. Denote by $\overline{R_b(G)}$ the closure of $R_b(G)$ in $H$. Let $E$ be the finite dimensional subspace of $H$, and $P_E$ is the orthogonal projection from $H$ to $E$. We introduce the following definitions (see also [23, 28]).

**Definition 4.1.** The system (1) is called to be approximately controllable on $I$ if $\overline{R_b(G)} = H$.

**Definition 4.2.** For given $\epsilon > 0$ and $u_b \in H$, the system (1) is said to be finite approximately controllable on $I$, if there exists a control $v_c \in L^2(I, V)$ such that the corresponding solution $u_c(t)$ of (1) satisfies the conditions: (a). $\|u_c(b) - u_b\| \leq \epsilon$, and (b). $P_E u_c(b) = P_E u_b$.

From Definition 4.2, we can see that the control $v_c$ can be selected such that the final state $u_c(b)$ is approximately controllable (i.e. the condition (a) holds), and satisfies simultaneously a finite number of exact constraints (i.e. the condition (b) holds). We first consider the following linear system related to (1).

$$
\begin{cases}
    u'(t) = A(t)u(t) + \int_0^t a(t, s)u(s)ds + B\nu(t), & t \in I, \\
    u(0) = \sum_{k=1}^m c_k u(t_k).
\end{cases}
$$

(17)

Let us introduce the following operators defined on the Hilbert space $H$ by

$$
\Pi_b^b := \int_0^b \Gamma(b, s)\mathcal{B}\Gamma^s(b, s)ds, \quad R(\epsilon, \Pi_b^b) := (\epsilon I + \Pi_b^b)^{-1}, \quad \epsilon > 0,
$$

(18)

where $\Gamma^s(b, s) = \sum_{k=1}^m \chi_{t_k}(s)\mathcal{R}^s(b, s) + \chi_t(b)\mathcal{R}^s(b, s)$, $s \in I$, and $\mathcal{R}, \Delta, \mathcal{B}$, $\mathcal{B}^*$ denote the adjoint operators of $\mathcal{R}, \Delta, \mathcal{B}$, respectively.

Let $u^c$ be a mild solution to the system (17) corresponding to the control $v \in L^2(I, V)$. Then the system (17) is said to be approximately controllable on $I$, if for every desired final state $u_b \in H$ and $\epsilon > 0$, there exists a control $v \in L^2(I, V)$ such that $\|u^c(b) - u_b\| \leq \epsilon$. Next, we give a useful characterization of the approximate controllability for (17) in Green function terms. Define the functional

$$
\mathcal{L}(v) = \|u^c(b) - u_b\|^2 + \epsilon \int_0^b \|v(t)\|^2 dt.
$$

(19)

**Lemma 4.3.** For given $\epsilon > 0$ and $u_b \in H$, the above defined functional (19) admits a unique optimal control $v_c(\cdot) \in L^2(I, V)$ such that

$$
v_c(t) = \mathcal{B}\Gamma^s(b, t)R(\epsilon, \Pi_b^b)u_b,
$$

with

$$
u^c_c(b) - u_b = -\epsilon R(\epsilon, \Pi_b^b)u_b.
$$

Proof. Clearly, the functional $\mathcal{L}$ is convex and differentiable. Thus, it attains a minimum $v_c$ which satisfies $\mathcal{L}'(v_c) = 0$. Then, for all $\rho \in L^2(I, V)$, we have

$$
0 = \langle \mathcal{L}'(v), \rho \rangle_V = \frac{d\mathcal{L}(v_c + \lambda \rho)}{d\lambda} \bigg|_{\lambda=0}
$$

$$
= 2 \left[ \langle u^c(b) - u_b, \int_0^b \Gamma(b, t)\mathcal{B}\rho(t)dt \rangle_H + \epsilon \int_0^b \langle v_c(t), \rho(t) \rangle_V dt \right]
$$
\[
\begin{align*}
\int_0^b \langle \mathcal{B}^* \Gamma^* (b, t) [u^\varepsilon (b) - u_b], \rho(t) \rangle_V dt + \epsilon \int_0^b \langle v_{\varepsilon}(t), \rho(t) \rangle_V dt \\
= 2 \int_0^b \langle \mathcal{B}^* \Gamma^* (b, t) [u^\varepsilon (b) - u_b] + \epsilon v_{\varepsilon}(t), \rho(t) \rangle_V dt.
\end{align*}
\]
Therefore,
\[
\int_0^b \langle \mathcal{L}'(v_{\varepsilon}), \rho \rangle_V dt = 0 = \int_0^b \langle \mathcal{B}^* \Gamma^* (b, t) [u^\varepsilon (b) - u_b] + \epsilon v_{\varepsilon}(t), \rho(t) \rangle_V dt.
\]
Owing to the arbitrariness of \( \rho \in L^2(I, V) \), we obtain that for a.e. \( t \in I \)
\[
v_{\varepsilon}(t) = - \frac{1}{\epsilon} \mathcal{B}^* \Gamma^* (b, t) [u^\varepsilon (b) - u_b].
\]
Therefore, we further have
\[
u^\varepsilon (b) = \int_0^b \Gamma (b, t) \mathcal{B}v_{\varepsilon}(t) dt
\]
\[
= - \frac{1}{\epsilon} \int_0^b \Gamma (b, t) \mathcal{B}^* \Gamma^* (b, t) [u^\varepsilon (b) - u_b] dt
\]
\[
= - \frac{1}{\epsilon} \Pi_0^b [u^\varepsilon (b) - u_b],
\]
which implies that
\[
\epsilon [u^\varepsilon (b) - u_b] = - \Pi_0^b [u^\varepsilon (b) - u_b] - \epsilon u_b.
\]
Taking into account (18), we have
\[
u^\varepsilon (b) - u_b = - \epsilon R(\epsilon, \Pi_0^b) u_b.
\]
The existence and uniqueness of an optimal control can be deduced from the general theorem on linear regulator problem (see [10]).

From above lemma, we can easily deduce the following result.

**Lemma 4.4.** The linear system (17) corresponding to the system (1) is approximately controllable on \( I \) if \( \epsilon R(\epsilon, \Pi_0^b) \to 0 \) as \( \epsilon \to 0^+ \) in the strong operator topology.

Therefore, we assume further that
\[ (A5) \quad \epsilon R(\epsilon, \Pi_0^b) \to 0 \quad \text{as} \quad \epsilon \to 0^+ \quad \text{in the strong operator topology.} \]
Now we shall prove the approximate controllability of the system (1).

**Theorem 4.5.** Assume that conditions (A1)-(A5) are satisfied. Then the system (1) is approximately controllable on \( I \).

**Proof.** For all \( \epsilon > 0 \) and \( u_b \in H \), by Theorem 3.4, the problem (1) has at least one mild solution \( u_\epsilon \in B \) on \( I \). Thus, in view of Lemma 4.3, there exists \( g_\epsilon \in S(u_\epsilon) \) such that
\[
u_\epsilon (t) = \int_0^b \Gamma (t, s) [g_\epsilon (s) + \mathcal{B}v_\epsilon (s)] ds,
\]
with
\[
\epsilon v_\epsilon (t) = \mathcal{B}^* \Gamma^* (b, t) R(\epsilon, \Pi_0^b) p(u_\epsilon (\cdot)),
\]
and
\[
p(u_\epsilon (\cdot)) = u_b - \int_0^b \Gamma (b, s) g_\epsilon (s) ds.
\]
Through (10), (18) and (20)-(22), we can easily deduce that

\[
u(\epsilon) = \int_0^b \Gamma(b, s)[g_{\epsilon}(s) + \mathcal{B}v_{\epsilon}(s)]\,ds
\]

\[
= u_b - p(u_{\epsilon}(\cdot)) + \int_0^b \Gamma(b, s)\mathcal{B}\Gamma^*(b, s)R(\epsilon, \Pi_0^b)p(u_{\epsilon}(\cdot))\,ds
\]

\[
= u_b - p(u_{\epsilon}(\cdot)) + \Pi_0^b R(\epsilon, \Pi_0^b)p(u_{\epsilon}(\cdot))
\]

\[
= u_b - (\epsilon I + \Pi_0^b) R(\epsilon, \Pi_0^b)p(u_{\epsilon}(\cdot)) + \Pi_0^b R(\epsilon, \Pi_0^b)p(u_{\epsilon}(\cdot))
\]

\[
= u_b - \epsilon R(\epsilon, \Pi_0^b)p(u_{\epsilon}(\cdot)).
\]

(23)

It follows from (A3) that

\[
\left(\int_0^b \|g_{\epsilon}(s)\|^2\,ds\right)^{\frac{1}{2}} \leq \left(\int_0^b \varphi^2(s)\,ds\right)^{\frac{1}{2}}.
\]

We see that \(\{g_{\epsilon}\}\) is bounded in \(L^2(I, \mathcal{H})\), and there exists a subsequence converging weakly to \(g \in L^2(I, \mathcal{H})\). We can know that \(\Gamma(t, s)(t - s > 0)\) is also compact by the compactness of \(\mathcal{R}(t, s)(t - s > 0)\) and (10). Let

\[
W := u_b - \int_0^b \Gamma(b, s)g(s)\,ds.
\]

(24)

From (22), (24) and the compactness of \(\Gamma(t, s)(t - s > 0)\), we have

\[
\|p(u_{\epsilon}) - W\| \leq \left\|\int_0^b \Gamma(b, s)[g_{\epsilon}(s) - g(s)]\,ds\right\| \to 0, \epsilon \to 0^+.
\]

(25)

From (23), (25) and (A5), it follows that

\[
\|u_b - u_{\epsilon}(b)\| \leq \|\epsilon R(\epsilon, \Pi_0^b)p(u_{\epsilon}(\cdot))\|
\]

\[
\leq \|\epsilon R(\epsilon, \Pi_0^b)W\| + \|\epsilon R(\epsilon, \Pi_0^b)\|\|p(u_{\epsilon}(\cdot)) - W\| \to 0,
\]

as \(\epsilon \to 0^+\). In view of Lemma 4.4 and above arguments, the problem (1) is approximately controllable on \(I\). This ends of proof.

In the final, we investigate the finite approximate controllability of the system (1). We define the following functional

\[
\mathcal{L}_\epsilon(\Psi; u) = \epsilon\|(I - \mathcal{P}_\mathcal{E}) R(\epsilon, \Pi_0^b)\Psi\| + \frac{1}{2} \int_0^b \|\mathcal{B}\Gamma^*(b, s)R(\epsilon, \Pi_0^b)\Psi\|^2\,ds
\]

\[
- \langle p(u(\cdot)), R(\epsilon, \Pi_0^b)\Psi \rangle,
\]

(26)

where \(\epsilon > 0, u \in B_\mathcal{F}, \Psi : \mathcal{H} \to \mathcal{H}\), and

\[
p(u(\cdot)) = u_b - \int_0^b \Gamma(b, s)g(s)\,ds,
\]

\[
v(s) = \mathcal{B}\Gamma^*(b, s)R(\epsilon, \Pi_0^b)\Psi,
\]

\[
u(t) = \int_0^b \Gamma(t, s)[g(s) + \mathcal{B}v(s)]\,ds.
\]

with \(g \in \mathcal{S}(u)\) and \(u_b \in \mathcal{H}\). We now have the following fact.
Lemma 4.6. Assume that condition (A1)-(A5) are satisfied. Then, for any ball $B_r$, there exists $\varepsilon = \|R(\varepsilon, \Pi_0^b)\|$ such that the functional $\mathcal{L}_\varepsilon$ given by (26) satisfying

$$\lim_{\|\Psi\| \to \infty} \inf_{w \in B_r} \frac{\mathcal{L}_\varepsilon(\Psi; u)}{\|\Psi\|} \geq \varepsilon.$$  \hfill (27)

Proof. The proof can be conducted similarly as [28, Lemma 3]. If the assertion of the lemma is not true, then for any $\varepsilon > 0$, there exist sequences $\{\Psi_n\} \subset \mathcal{H}, \{u_n\} \subset B_r$ with $\|\Psi_n\| \to \infty$, such that

$$\lim_{n \to \infty} \frac{\mathcal{L}_\varepsilon(\Psi_n; u_n)}{\|\Psi_n\|} < \varepsilon.$$  \hfill (28)

As Step 3 in the proof of Theorem 3.4, we can see that the set $Q := \{p(u(\cdot)) : u \in B_r\}$ is relatively compact in $\mathcal{H}$. Thus, without loss of generality, for some $p(u(\cdot))$, we may assume by selection of a subsequence if necessary that

$$p(u_n(\cdot)) \to p(u(\cdot)),$$  \hfill (29)

We now normalize that $\tilde{\Psi}_n = \frac{\Psi_n}{\|\Psi_n\|}$, then $\|\tilde{\Psi}_n\| = 1$. In this case, we can select a subsequence (still denoted by $\tilde{\Psi}_n$) such that $\tilde{\Psi}_n \to \tilde{\Psi}$ weakly in $\mathcal{H}$. From the compactness of $\Gamma(t, s)(t - s > 0)$, it follows that

$$\mathcal{B}\Gamma^*(b, s)R(\varepsilon, \Pi_0^b)\tilde{\Psi}_n \to \mathcal{B}\Gamma^*(b, s)R(\varepsilon, \Pi_0^b)\tilde{\Psi},$$  \hfill (30)

By virtue of (26), we have

$$\frac{\mathcal{L}_\varepsilon(\Psi_n; u_n)}{\|\Psi_n\|} = \varepsilon\|(I - \mathcal{P}_\mathcal{E})R(\varepsilon, \Pi_0^b)\tilde{\Psi}_n\| + \frac{\|\Psi_n\|}{2} \int_0^b \|\mathcal{B}\Gamma^*(b, s)R(\varepsilon, \Pi_0^b)\tilde{\Psi}_n\|^2 ds$$

$$- \left\langle p(u_n(\cdot)), R(\varepsilon, \Pi_0^b)\tilde{\Psi}_n \right\rangle.$$

Thus, as $\|\Psi_n\| \to \infty$, it follows from (28)-(30) and the Fatou lemma that

$$\int_0^b \|\mathcal{B}\Gamma^*(b, s)R(\varepsilon, \Pi_0^b)\tilde{\Psi}_n\|^2 ds \leq \lim_{n \to \infty} \int_0^b \|\mathcal{B}\Gamma^*(b, s)R(\varepsilon, \Pi_0^b)\tilde{\Psi}_n\|^2 ds = 0.$$

In view of (A5) and Lemma 4.4, the linear system (17) is approximately controllable, which in turn implies that $\tilde{\Psi} = 0$ (see [11]). Therefore $\tilde{\Psi}_n \to 0$ weakly in $\mathcal{H}$. Considering that $\mathcal{E}$ is finite dimensional and $\mathcal{P}_\mathcal{E}$ is compact, we have $\mathcal{P}_\mathcal{E}\tilde{\Psi}_n \to 0$ in $\mathcal{H}$ and $\|\|I - \mathcal{P}_\mathcal{E}\|R(\varepsilon, \Pi_0^b)\tilde{\Psi}_n\| \to \|R(\varepsilon, \Pi_0^b)\|$. Now, we can deduce that

$$\lim_{n \to \infty} \frac{\mathcal{L}_\varepsilon(\Psi_n; u_n)}{\|\Psi_n\|} \geq \lim_{n \to \infty} \left( \varepsilon\|(I - \mathcal{P}_\mathcal{E})R(\varepsilon, \Pi_0^b)\tilde{\Psi}_n\| - \left\langle p(u_n(\cdot)), R(\varepsilon, \Pi_0^b)\tilde{\Psi}_n \right\rangle \right)$$

$$= \|R(\varepsilon, \Pi_0^b)\|\varepsilon,$$

which contradicts (28). The proof is complete. \hfill \Box

Theorem 4.7. The system (1) is finite approximately controllable on $I$ provided that conditions (A1)-(A5) are satisfied.

Proof. For any $\varepsilon > 0$, the system (1) admits a mild solution $u_\varepsilon \in B_r$ for a sufficient large $r$ according to Theorem 3.4. From Lemma 4.6, the functional $\mathcal{L}_\varepsilon(\Psi; u)$ is coercive on this ball. Then $\mathcal{L}_\varepsilon(\Psi; u_\varepsilon)$ has a unique critical point $\Psi_\varepsilon^0$ minimizing $\mathcal{L}_\varepsilon(\Psi; u_\varepsilon)$, i.e.

$$\mathcal{L}_\varepsilon(\Psi_\varepsilon^0; u_\varepsilon) = \min_{\Psi \in \mathcal{H}} \mathcal{L}_\varepsilon(\Psi; u_\varepsilon).$$
Thus for any $\Phi \in \mathcal{H}$, we have
\[
0 = \langle L_\epsilon (\Psi^\circ_{\epsilon}, u_{\epsilon}), \Phi \rangle = \left. \frac{d}{d\lambda} \langle L_\epsilon (\Psi^\circ_{\epsilon} + \lambda \Phi; u_{\epsilon}) \rangle \right|_{\lambda = 0}
\]
\[
= \epsilon \|(I - P_E)R(\epsilon, \Pi_0)\Phi\| + \int_0^b \langle \mathcal{B}^* (b, s) R(\epsilon, \Pi_0) \Psi^\circ_{\epsilon}, \mathcal{B}^* (b, s) R(\epsilon, \Pi_0) \Phi \rangle \, ds
\]
\[
- \langle p(u_{\epsilon}(\cdot)), R(\epsilon, \Pi_0) \Phi \rangle
\]
\[
= \epsilon \|(I - P_E)R(\epsilon, \Pi_0)\Phi\| + \int_0^b \langle \Gamma(b, s) \mathcal{B}^* (b, s) R(\epsilon, \Pi_0) \Psi^\circ_{\epsilon}, R(\epsilon, \Pi_0) \Phi \rangle \, ds
\]
\[
- \langle p(u_{\epsilon}(\cdot)), R(\epsilon, \Pi_0) \Phi \rangle
\]
\[
= \epsilon \|(I - P_E)R(\epsilon, \Pi_0)\Phi\|
\]
\[
+ \int_0^b \langle \Gamma(b, s) \mathcal{B} v_{\epsilon}(s), R(\epsilon, \Pi_0) \Phi \rangle \, ds - \langle p(u_{\epsilon}(\cdot)), R(\epsilon, \Pi_0) \Phi \rangle,
\]
where $v_{\epsilon}(s) = \mathcal{B}^* (b, s) R(\epsilon, \Pi_0) \Psi^\circ_{\epsilon}$. Now, we have
\[
\langle p(u_{\epsilon}(\cdot)), R(\epsilon, \Pi_0) \Phi \rangle = \epsilon \|(I - P_E)R(\epsilon, \Pi_0)\Phi\| + \int_0^b \langle \Gamma(b, s) \mathcal{B} v_{\epsilon}(s), R(\epsilon, \Pi_0) \Phi \rangle \, ds.
\]
In view of the definition of $p(u(\cdot))$ and $u(t)$, we have
\[
u_{\epsilon}(b) = \int_0^b \Gamma(b, s)[g_{\epsilon}(s) + \mathcal{B} v_{\epsilon}(s)] \, ds,
\]
and
\[
p(u_{\epsilon}(\cdot)) = u_b - u_{\epsilon}(b) + \int_0^b \Gamma(b, s) \mathcal{B} v_{\epsilon}(s) \, ds.
\]
Therefore,
\[
\langle u_b - u_{\epsilon}(b), R(\epsilon, \Pi_0) \Phi \rangle = \epsilon \|(I - P_E)R(\epsilon, \Pi_0)\Phi\| \leq \epsilon \|R(\epsilon, \Pi_0)\Phi\|
\]
holds for any $\Phi \in \mathcal{H}$. That is,
\[
\|u_{\epsilon}(b) - u_b\| \leq \epsilon, \quad P_E u_{\epsilon}(b) = P_E u_b.
\]
The proof is complete. 

**Example 4.1.** We finally give a simple example to illustrate the validity of our main results. Let $\mathcal{H} = \mathcal{V} := L^2([0, \pi], \mathbb{R}), \mathcal{U} := L^2(I, L^2([0, \pi], \mathbb{R}))$ and $I := [0, 1]$. Consider the following problem
\[
\begin{cases}
\frac{\partial}{\partial t} u(t, x) &\in \frac{\partial^2}{\partial x^2} \left[ u(t, x) + \int_0^t \kappa(t, s) u(s, x) \, ds \right] \\
&+ \left\{ t^2 \cos 2\pi t, t^2 \sin 2\pi t \right\} + g(t, x), \\
u(t, 0) &= u(t, \pi) = 0, \quad t \in [0, 1], \\
u(0, x) & = \sum_{k=1}^m c_k u(t_k, x), \quad x \in [0, \pi], c_k \in \mathbb{R}, k = 1, \ldots, m.
\end{cases}
\]
(31)

Define $A(t) := \frac{\partial^2}{\partial x^2}$ with the domain
\[
\mathcal{D}(A) := \left\{ z \in BU(\mathcal{H}) : \frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2} \in BU(\mathcal{H}), z(0) = z(\pi) = 0 \right\},
\]
where $BU(\mathcal{H})$ is given in [21, 38], and the function $\kappa(t, s)$ satisfies hypotheses [21, (H1)-(H2)] or [38, (H1)-(H2)]. It is known from [38, Example 5.1] that there exists
a resolvent operator $R(t,s)$, which can be extracted from the semigroup operator, satisfying $\|R(t,s)\| \leq e^{-\gamma t}$. Let $q > 0$, $u(t) := u(t, \cdot)$, $v(t) := v(t, \cdot)$, and $\partial G(t, u(t)) := \left\{ \frac{t^2 \cos 2\pi t}{1+|u(t,x)|}, \frac{t^2 \sin 2\pi t}{1+|u(t,x)|} \right\}$. We further define the bounded linear operator $B : U \to H$ by $Bu(t) := \rho v(t)$. Now we can see that the problem (31) can be turned into the problem (1) with a non-local initial condition. Note that in conditions (A3) and (A4), we can take $\phi(t) := \sqrt{\pi} t^2$ and $\psi(t) := \rho v(t)$. Suppose further that (A1) and $\sum_{k=1}^{m} |c_k| < 1$, then the problem (31) admits at least one mild solution in a suitable ball on $[0,1]$ by Theorem 3.4, and it is also (finite) approximate controllability via Theorems 4.5-4.7 under the condition (A5).

Acknowledgments. We would like to thank anonymous referees for carefully reading this manuscript and giving valuable suggestion for improvements.

REFERENCES

[1] K. Balachandran and J. P. Dauer, Controllability of nonlinear systems in Banach spaces: A survey, J. Optim. Theory Appl., 115 (2002), 7–28.
[2] K. Balachandran and J. H. Kim, Remarks on the paper “Controllability of second order differential inclusion in Banach spaces”, J. Math. Anal. Appl., 324 (2006), 746–749.
[3] A. E. Bashirov and N. I. Mahmudov, On concepts of controllability for deterministic and stochastic systems, SIAM J. Control Optim., 37 (1999), 1808–1821.
[4] M. Benchohra and M. S. Souid, $L^1$-Solutions for implicit fractional order differential equations with nonlocal conditions, Filomat, 30 (2016), 1485–1492.
[5] L. Byszewski, Existence and uniqueness of a classical solutions to a functional differential abstract nonlocal Cauchy problem, J. Math. Appl. Stoch. Anal., 12 (1999), 91–97.
[6] Y.-K. Chang and R. Ponce, Uniform exponential stability and its applications to bounded solutions of integro-differential equations in Banach spaces, J. Integral Equations Appl., 30 (2018), 347–369.
[7] Y.-K. Chang and Y. Pei, Degenerate type fractional evolution hemivariational inequalities and optimal controls via fractional resolvent operators, Int. J. Control, (2018). Available from: https://doi.org/10.1080/00207179.2018.1479540.
[8] R. M. Christensen, The Theory of Linear Viscoelasticity: An Introduction, Academic Press, New York, 1982.
[9] F. H. Clarke, Optimization and Nonsmooth Analysis, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1983.
[10] R. F. Curtain and A. J. Pritchard, Infinite Dimensional Linear Systems Theory, Lecture Notes in Control and Information Sciences, 8. Springer-Verlag, Berlin-New York, 1978.
[11] R. F. Curtain and H. J. Zwart, An Introduction to Infinite Dimensional Linear Systems Theory, Texts in Applied Mathematics, 21. Springer-Verlag, New York, 1995.
[12] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, J. Math. Anal. Appl., 179 (1993), 630–637.
[13] M. Dieye, M. A. Diopa and K. Ezzinbi, On exponential stability of mild solutions for some stochastic partial integrodifferential equations, Stat. Prob. Lett., 123 (2017), 61–76.
[14] S. Djebali, L. Gorniewicz and A. Ouahab, Solutions Set for Differential Equations and Inclusions, De Gruyter Series in Nonlinear Analysis and Applications, 18. Walter de Gruyter & Co., Berlin, 2013.
[15] K. Ezzinbi and S. Ghanmi, Existence and regularity of solutions for neutral partial functional integrodifferential equations, Nonlinear Anal. RWA, 11 (2010), 2335–2344.
[16] X. L. Fu, Approximate controllability of semilinear non-autonomous evolution systems with state-dependent delay, Evol. Equ. Control Theor., 6 (2017), 517–534.
[17] R. C. Grimmer, Resolvent operators for integral equations in a Banach space, Trans. Amer. Math. Soc., 273 (1982), 333–349.
[18] R. C. Grimmer and A. J. Pritchard, Analytic resolvent operators for integral equations in a Banach space, J. Differential Equations, 50 (1983), 234–259.
[19] Y. R. Jiang, N.-J. Huang and J.-C. Yao, Solvability and optimal control of semilinear nonlocal fractional evolution inclusion with Clarke subdifferential, *Appl. Anal.*, 96 (2017), 2349–2366.

[20] J.-L. Lions and E. Zuazua, The cost of controlling unstable systems: Time irreversible systems, *Rev. Mat. UCM*, 10 (1997), 481–523.

[21] J. H. Liu and K. Ezzinbi, Non-autonomous integrodifferential equations with nonlocal conditions, *J. Integral Equations Appl.*, 15 (2003), 79–93.

[22] Z. H. Liu and X. W. Li, Approximate controllability for a class of hemivariational inequalities, *Nonlinear Anal. RWA*, 22 (2015), 5452–5464.

[23] Z. H. Liu, X. W. Li and D. Motreanu, Approximate controllability for nonlinear evolution hemivariational inequalities in Hilbert spaces, *SIAM J. Control Optim.*, 53 (2015), 3228–3244.

[24] C. Lizama and G. M. N’Guérékata, Bounded mild solutions for semilinear integrodifferential equations in Banach spaces, *Integr. Equ. Oper. Theory*, 68 (2010), 207–227.

[25] C. Lizama and G. M. N’Guérékata, Mild solutions for abstract fractional differential equations, *Appl. Anal.*, 92 (2013), 1731–1754.

[26] Q. Lü and E. Zuazua, On the lack of controllability of fractional in time ODE and PDE, *Math. Control Signals Systems*, 28 (2016), Art. 10, 21 pp.

[27] L. Lu, Z. H. Liu, W. Jiang and J. L. Luo, Solvability and optimal controls for semilinear fractional evolution hemivariational inequalities, *Math. Methods Appl. Sci.*, 39 (2016), 5452–5464.

[28] N. I. Mahmudov, Finite-approximate controllability of evolution equations, *Appl. Comput. Math.*, 16 (2017), 159–167.

[29] N. I. Mahmudov, Finite-approximate controllability of fractional evolution equations: Variational approach, *Fract. Calc. Appl. Anal.*, 21 (2018), 919–936.

[30] N. I. Mahmudov, Variational approach to finite-approximate controllability of Sobolev-type fractional systems, *J. Optim. Theory Appl.*, 184 (2020), 671–686.

[31] M. A. Meyers and K. K. Chawla, *Mechanical Behavior of Materials*, Cambridge University Press, Cambridge, 2009.

[32] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics, 26. Springer, New York, 2013.

[33] D. Motreanu, V. V. Motreanu and N. S. Papageorgiou, Positive solutions and multiple solutions at non-resonance, resonance and near resonance for hemivariational inequalities with p-Laplacian, *Trans. Amer. Math. Soc.*, 360 (2008), 2527–2545.

[34] P. D. Panagiotopoulos, *Nonconvex superpotentials in sense of F. H. Clarke and applications*, *Mech. Res. Comm.*, 8 (1981), 335–340.

[35] P. D. Panagiotopoulos, *Hemivariational Inequalities: Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, 1993.

[36] R. Ravi Kumar, Nonlocal Cauchy problem for analytic resolvent integrodifferential equations in Banach spaces, *Appl. Math. Comput.*, 204 (2008), 352–362.

[37] R. Ravi Kumar, Regularity of solutions of evolution integrodifferential equations with deviating argument, *Appl. Math. Comput.*, 217 (2011), 9111–9121.

[38] R. Sakthivel, Y. Ren, A. Debbouche and N. I. Mahmudov, Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions, *Appl. Anal.*, 95 (2016), 2361–2382.

[39] R.-N. Wang, K. Ezzinbi and P. X. Zhu, Non-autonomous impulsive Cauchy problems of parabolic type involving nonlocal initial conditions, *J. Integral Equations Appl.*, 26 (2014), 275–299.

[40] J.-Z. Xiao and X.-H. Zhu, Approximate controllability for abstract semilinear impulsive functional differential inclusions based on Hausdorff product measures, *Topol. Method Nonlinear Anal.*, 52 (2018), 353–372.

[41] Y.-B. Xiao and Y.-H. Zhu, Approximate controllability for abstract semilinear impulsive functional differential inclusions based on Hausdorff product measures, *Topol. Method Nonlinear Anal.*, 52 (2018), 353–372.

[42] Y.-B. Xiao, X. M. Yang and N.-J. Huang, Some equivalence results for well-posedness of hemivariational inequalities, *J. Global Optim.*, 61 (2015), 789–802.

[43] Z. M. Yan and X. M. Jia, Approximate controllability of partial fractional neutral stochastic functional integro-differential inclusions with state-dependent delay, *Collect. Math.*, 66 (2015), 93–124.
[44] M. Yang and Q. R. Wang, Existence of mild solutions for a class of Hilfer fractional evolution equations with nonlocal conditions, *Fract. Calc. Appl. Anal.*, 20 (2017), 679–705.

[45] Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control*, Elsevier/Academic Press, London, 2016.

Received May 2019; revised January 2020.

E-mail address: lzchangyk@163.com, ykchang@xidian.edu.cn
E-mail address: xjliu_1@stu.xidian.edu.cn