Double Kelvin Wave Cascade in Superfluid Helium

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We study the double cascade of energy and wave action in a local model of superfluid vortex filaments. The model is obtained from a truncated expansion of the 2D Local Induction Approximation and it is shown to support six-wave interactions. We argue that, because of the uncertainty in the vortex core profile, this model has the same status of validity as the traditionally used Biot-Savart model with cutoff, but it has advantage of being much simpler. Our minimal model leads to a wave kinetic equation for which we predict existence of two distinct power-law scaling in the spectrum, corresponding to a direct cascade of energy and an inverse one of wave action. Direct numerical simulations confirm the theoretical predictions in the weak turbulence regime.

It is well known that a classical vortex filament can support linear waves. These were predicted by Kelvin more than one century ago and experimentally observed about 50 years ago in superfluid $^4$He. At very low temperature, where the friction induced by normal fluid component can be neglected, Kelvin waves can be dissipated only at very high frequencies by phonon emission \cite{footnote1}. Therefore at lower frequency energy is transferred among different wavenumbers by the nonlinear coupling. This is the mechanism at the basis of the Kelvin wave cascade which sustains superfluid turbulence \cite{footnote2, footnote3}.

In recent years, single vortex Kelvin wave cascade has attracted much theoretical \cite{footnote4}, numerical \cite{footnote5, footnotetwo} and experimental \cite{footnote6} attention. Even within the classical one-dimensional vortex model, different degrees of simplification are possible. For small amplitudes, the vortex configuration can be described by a two component vector field, made of the coordinates of the vortex line in the plane transverse to the direction of the unperturbed filament. These depend on the single coordinate that runs along the filament. This system of equations admits a Hamiltonian formulation, dubbed the two-dimensional Biot-Savart formulation (2D-BS), see \cite{footnote7} below. Another, more drastic, simplification is obtained by considering local interactions only. This leads to the local induction approximation (LIA) which was originally derived starting from the full 3D-BS \cite{footnote8}. The main limitation of LIA is that it generates an integrable system with infinite conserved quantities, as it is equivalent to the nonlinear Schrödinger equation \cite{footnote9}, and therefore the resonant wave interactions are absent (at all orders) and one cannot reproduce the phenomenology of the full system. For this reason LIA, despite its simplicity, is of little help for the study of Kelvin wave turbulence.

In this paper we consider the simplest model for vortex filament able to sustain a turbulent energy cascade. The model is obtained in the limit of small amplitudes by a Taylor expansion of the 2D-LIA. The truncation breaks the integrability of the Hamiltonian and therefore generates a dynamical system with two inviscid invariants (energy and wave action). For this class of systems, whose prototype is the two-dimensional Navier-Stokes turbulence \cite{footnote10}, we expect a dual cascade phenomenology in which one quantity flows to small scales generating a direct cascade while the other goes to larger scales producing an inverse cascade. The possibility of a dual cascade scenario for Kelvin waves turbulence has been recently suggested \cite{footnote11, footnotetwo} but never observed. Direct numerical simulations at high resolution confirm the dual cascade picture with spectral exponents consistent with the theoretical prediction based on a six-waves kinetic equation.

At a macroscopic level, the superfluid vortex filament is a classical object whose dynamics is often described by the Biot-Savart equation (BSE)

$$ \dot{r} = \frac{\kappa}{4\pi} \int \frac{ds \times (r-s)}{|r-s|^3} $$

(1)

which describes the self-interaction of vortex elements. The quantum nature of the phenomenon is encoded in the discreteness of circulation $\kappa = h/m$ \cite{footnote12}.

The BSE dynamics of the vortex filament admits a Hamiltonian formulation under a simple geometrical constraint: the position $r$ of the vortex is represented in two-dimensional parametric form as $r = (x(z, t), y(z, t))$, where $z$ is a given axis. From a geometrical point of view, this corresponds to small perturbations with respect to the straight line configuration, i.e. the vortex cannot form folds in order to preserve the single-valuedness of the $x$ and $y$ functions. In terms of the complex canonical coordinate $w(z,t) = x(z,t) + iy(z,t)$, BSE can be written in Hamiltonian form $i\dot{w} = \delta H[w]/\delta w^*$ with \cite{footnote7}

$$ H[w] = \frac{\kappa}{4\pi} \int dz_1 dz_2 \frac{1 + Re(w''(z_1)w'(z_2))}{\sqrt{(z_1 - z_2)^2 + |w'(z_1) - w'(z_2)|^2}} $$

(2)

where we have used the notation $w'(z) = \partial w/\partial z$. The geometrical constraint of small amplitude perturbation can be expressed in terms of a parameter $\epsilon(z_1, z_2) = |w'(z_1) - w'(z_2)|/|z_1 - z_2| \ll 1$.

An enormous simplification, both for theoretical and numerical purposes, is obtained by means of the so called...
local induction approximation (LIA) \([11]\). This approximation is justified by the observation that \([11]\) is divergent as \(s \to r\) and is obtained by introducing a cutoff at \(a = |r - s|\) in the integral in \([11]\) which represents the vortex filament radius. However, the cutoff operation does not take into account the distribution of the vorticity inside the core of the vortex filament. Indeed, let us consider linear Kelvin waves on an infinite straight filament with wavevector \(k\) such that \(ka \ll 1\). Dispersion relation for the frequency of such linear waves can be found exactly from the Euler equations for several special types of the vortex core shapes:

\[
\omega_k = \frac{\kappa k^2}{4\pi} \left[ \ln \left( \frac{1}{ka} \right) + C \right],
\]

where \(C = -\gamma + \ln 2 + \frac{1}{2}\) for a vortex with uniform vorticity inside a cylinder of radius \(a\) \([9]\) (where \(\gamma\) is the Euler constant), \(C = -\gamma + \ln 2\) for a hollow core \([10]\), and \(C = -\gamma - \frac{3}{2}\) for the BSE model \([11]\) with cutoff (see Appendix A). From \([3]\) we see that the particular shape of the vortex core affects the immediate next order with respect to the LIA term (log) and, therefore, it affects the leading order of the nonlinear dynamics. On the other hand, the choice of the vortex core shape is difficult and uncertain, considering that the fluid description itself fails within the vortex core. Thus, in the following we will use the simplest model which arises naturally from the truncated expansion of the LIA Hamiltonian.

When applied to Hamiltonian \([2]\), the LIA procedure gives

\[
H[w] = \frac{2\kappa}{4\pi} \ln \left( \frac{\ell}{\xi} \right) \int dz \sqrt{1 + |w'(z)|^2} = 2\beta L[w]
\]

where \(\ell\) is the length of the order of the curvature radius (or intervortex distance when the considered vortex filament is a part a vortex tangle), \(\beta = (\kappa/4\pi) \ln(\ell/a)\). Here, it was taken into account that because \(a\) is much smaller than any other characteristic size in the system, \(\beta\) will be about the same whatever characteristic scale \(\ell\) we take in its definition. We remark that in the LIA approximation the Hamiltonian is proportional to the vortex length \(L[w] = \int dz \sqrt{1 + |w'(z)|^2}\) which is therefore a conserved quantity. The equation of motion from \([3]\) (we set \(\beta = 1/2\) without loss of generality, i.e. we rescale time as \(2\beta t \to t\))

\[
\dot{w} = \frac{i}{2} \left( \frac{w'}{\sqrt{1 + |w'|^2}} \right),
\]

As a consequence of the invariance under phase transformations, equation \([5]\) conserves also the total wave action (also called the kelvon number) \([4]\)

\[
N[w] = \int dz |w|^2
\]

In addition to these two conserved quantities, the 2DLIA model possesses an infinite set of invariants and is integrable, as it is the LIA of BSE (which can be transformed into the nonlinear Schrödinger equation by the Hasimoto transformation, see Appendix B). Integrability is broken if one considers a truncated expansion of the Hamiltonian \([3]\) in power of wave amplitude \(w'(z)\). Taking into account the lower order terms only one obtains:

\[
H_{exp}[w] = H_0 + H_1 + H_2 = \int dz \left( 1 + \frac{1}{2}|w'|^2 - \frac{1}{8}|w'|^4 \right)
\]

Neglecting the constant term, the Hamiltonian can be written in Fourier space as

\[
H_{exp} = \int \frac{dk}{2\pi} |w_k|^2 + \int dk_1dk_2dk_3dk_4\delta_{456}^{123}(k)w_1^*w_2^*w_3w_4
\]

with \(\omega = k^2/2\), \(W_{1234} = -\frac{1}{8}k_1k_2k_3k_4\) and we used the standard notation \(\delta_{456}^{123}(k) = \delta(k_1 + k_2 - k_3 - k_4)\) and \(dk_{1234} = dk_1dk_2dk_3dk_4\). For the dispersion relation \(\omega \sim k^2\) there is no non-trivial solution to the four-wave resonant conditions. There is no five-wave interaction either, because there are no odd-degree Hamiltonians. The first non-vanishing process appears to be six-wave. Introducing a near-identity (weakly nonlinear) canonical transformation \(w_k \to c_k\) using the standard strategy described in \([17]\), it is possible to transform \([8]\) into (See Appendix C)

\[
H_c = \int \omega_k |c_k|^2 dk + \int dk_{123456}C_{123456}^{123}(k)c_1^*c_2^*c_3^*c_4c_5c_6
\]

where \(C_{123456} = -\frac{1}{8}k_1k_2k_3k_4k_5k_6\). This six-order interaction coefficient is obtained from coupling of two fourth-order vertices of \(H_2\). It is not surprising that the resulting expression coincides, with the opposite sign, the interaction coefficient of \(H_3\) in \([5]\). Indeed, \([5]\) is an integrable model which implies that if we retained the next order too, i.e. \(H_3\), then the resulting six-wave process would be nil, and the leading order would be an 8-wave process in this case. Thus, the existence of the six-wave process is a consequence of the truncation \([7]\) of the Hamiltonian.

Hamiltonian \([4]\) (or equivalently \([9]\)) constitutes the minimal model for Kelvin wave turbulence. It possesses the same scaling properties as the BSE system: it conserves the energy and the wave action, it gives rise to a six-wave system with an interaction coefficient with the same order of homogeneity as the one of the BSE. A slight further modification should be made in the time re-scaling factor as \(\beta = (\kappa/4\pi)\), - i.e. by dropping the large log factor from the original definition (to have the correct with respect to this log in rate of the six-wave process). This minimal model is much simpler than BSE...
and it has the same degree of validity as BSE because of the vortex core uncertainty discussed above.

Physical insight on Kelvin wave turbulence is obtained from the wave turbulence (WT) approach which yields the kinetic equation which describes the dynamics of the wave action density $n_k = \langle |c_k|^2 \rangle$.

The dynamical equation for the variable $c_k$ can be derived from Hamiltonian (9) by the relation $i \partial c_k / \partial t = \delta H_c / \delta c_k^\star$, and is

$$
i \frac{\partial c_k}{\partial t} - \omega_k c_k = \int dk' C_{k,k'} c_{k'} c_{k'}^\star \delta H_{23}(k). \tag{10}$$

Multiplying equation (10) by $c_k^\star$, subtracting the complex conjugate and averaging we arrive at

$$\frac{\partial (c_k c_k^\star)}{\partial t} = 6i \Im \int dk C_{k,k'} \delta H_{23}(k), \tag{11}$$

where $J_{k,k'} \delta H_{23}(k) = \langle c_k^\star c_{k'} c_{k'}^\star c_k \rangle$.

Assuming a Gaussian wave field, one can take $J_{k,k'}$ to the zeroth order $J_{k,k'}^{(0)}$, which is simplified via the Gaussian statistics to a product of three pair correlators,

$$J_{k,k'}^{(0)} = n_k n_{k'} n_{k''} \left[ \delta_{k,k'}^2 \delta_{k,k''}^2 + \delta_{k,k'}^2 \delta_{k,k''}^2 \right] + \delta_{k,k'}^4 \left( \delta_{k,k'}^2 \delta_{k,k''}^2 + \delta_{k,k'}^2 \delta_{k,k''}^2 \right) + \delta_{k,k'}^6 \left( \delta_{k,k'}^2 \delta_{k,k''}^2 + \delta_{k,k'}^2 \delta_{k,k''}^2 \right). \tag{12}$$

However, due to the symmetry of $C_{k,k'}$, this makes the right hand side of the kinetic equation (14) zero. To find a nontrivial answer we need to obtain a first order addition $J_{k,k'}^{(1)}$ to $J_{k,k'}^{(0)}$. To calculate $J_{k,k'}^{(1)}$ one takes the time derivative of $J_{k,k'}^{(0)}$, using the equation of motion (10) and inserting the zeroth order approximation for the tenth correlation function (this is similar to equation 12 but a product of five pair correlators involving ten wavevectors) $J_{k,k'}^{(1)}$ can then be written as

$$J_{k,k'}^{(1)} = B e^{i \Delta \omega t} + \frac{A_{k,k'} \delta H_{23}}{\Delta \omega}, \tag{13}$$

where $\Delta \omega = \omega_k + \omega_2 + \omega_3 - \omega_4 - \omega_5 - \omega_6$ and $A_{k,k'} = 3C_{k,k'} n_k n_{k'} n_{k''} \left[ \frac{1}{n_k} + \frac{1}{n_2} + \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5} - \frac{1}{n_6} \right]$. The first term of (13) is a fast oscillating function, its contribution to the integral (11) decreases with $z$ and is negligible at $z$ larger than $1 / \omega_k$, and as a result we will ignore the contribution arising from this term. The second term is substituted in equation (11), the relation $\Im (\Delta \omega) \sim -\pi \delta (\Delta \omega)$ is applied because of integration around the pole, and the kinetic equation (14) derived,

$$\dot{n}_k = 18\pi \int dk \left| C_{k,k'} \right|^2 \delta H_{23}(k) \times \delta H_{23} \oint k \frac{f_{k,k'}}{f_{k,k'}}, \tag{14}$$

where we have introduced

$$n_k n_{k'} n_{k''} n_{k'''} n_{k'''} \left[ \frac{1}{n_k} + \frac{1}{n_2} + \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5} - \frac{1}{n_6} \right].$$

A simple dimensional analysis of (14) gives

$$\dot{n}_k \sim k^{14} n_k^5 \tag{15}$$

which is the same form obtained from the full BSE [4]. The energy flux at wavenumber $k$ is defined as $\Pi_{k}^{(H)} = \int dk' n_k \omega_k$ which, using (15), becomes $\Pi_{k}^{(H)} \sim k^{17} n_k^5$. By requiring the existence of a range of scales in which the energy flux is $k$-independent leads to the spectrum

$$n_k \sim k^{-17/5} \tag{16}$$

A similar argument can be applied to the wave action $\tilde{\Pi}$ whose flux is $\tilde{\Pi}_{k}^{(N)} = \int dk' \tilde{n}_k \sim k^{15} n_k^5$. Therefore a scale independent flux of wave action requires a spectrum

$$n_k \sim k^{-3} \tag{17}$$

The two spectra (16, 17) occur in different scale ranges and the two cascades develops in opposite directions, as in the case of two-dimensional turbulence [13]. Among the two conserved quantities, the largest contribution to energy comes from smaller scales than those that contribute to wave action (because the former contains the field derivatives). Therefore, according to the Fjørtoft argument [15], we expect to have a direct cascade of energy with $k^{-17/5}$ spectrum at large $k$ and an inverse cascade of wave action with spectrum $k^{-3}$ at small $k$.

In the following we will consider numerical simulations of the system (7) under the conditions in which a stationary turbulent cascade develops. Energy and wave action are injected in the vortex filament by a white-in-time external forcing $\phi(z,t)$ acting on a narrow band of wavenumbers around a given $k_f$. In order to have a stationary cascade, we need additional terms which remove $H$ and $N$ at small and large scales. The equation of motion obtained from (7) is therefore modified as

$$\dot{w} = \frac{i}{2} \left[ w' \left( 1 - \frac{1}{2} w'^2 \right) \right] - (-1)^p \nu \nabla^2 w + \alpha w + \phi \tag{18}$$

In (18) the small scale dissipative term (with $p > 1$) physically represents the radiation of phonons (at a rate proportional to $\nu$) and the large scale damping term can be interpreted as the friction induced by normal fluid at a rate $\alpha$.

Assuming the spectra (16, 17) a simple dimensional analysis gives the IR and UV cutoff induced by the dissipative terms. The direct cascade is removed at a scale $k_\nu \sim \nu^{-5/(10p-2)}$ while the inverse cascade is stopped at $k_\alpha \sim \alpha^{1/2}$. Therefore, in an idealized realization of infinite resolution one would obtain a double cascade by keeping $k_f = O(1)$ and letting $\nu, \alpha \to 0$. In order to have an extended inertial range, in finite resolution numerical simulations we will restrict to resolve a single cascade by putting $k_f \approx k_\alpha$ or $k_f \approx k_\nu$ for the direct and inverse cascade respectively.

We have developed a numerical code which integrates the equation of motion (18) by means of pseudospectral
method for a periodic vortex filament of length $2\pi$ with a resolution of $M$ points. Linear, dissipative terms are integrated explicitly while the nonlinear term is solved by a second-order Runge-Kutta time scheme. Vortex filament is initially a straight line ($w(z, t = 0) = 0$) and long time integration is performed until a stationary regime (indicated by the values of $H$ and $N$) is reached. The ratio between the two terms in the expansion is $H_1/H_2 \approx 20$, confirming a posteriori the validity of the perturbative expansion.

The first set of simulations is devoted to the study of the direct cascade. Energy fluctuations are injected at forcing wavenumber $k_f \approx 2$ and the friction coefficient $\alpha$ is set in order to have $k_\alpha \approx k_f$. Energy is removed at small scales by hyperviscosity of order $p = 4$ which restricts the range of dissipation on a narrow range of wavenumber close to $k_{max}$.

![FIG. 1: Wave number spectrum $n_k$ for a simulation of the direct cascade in stationary conditions at resolution $M = 2048$. Forcing is restricted to a range of wavenumbers $1 \leq k_f \leq 3$ and phonon dissipation is modeled with hyperviscosity of order $p = 4$. The line represents the kinetic equation prediction $n_k \approx k^{-\alpha/2}$. The inset shows the spectrum compensated with the theoretical prediction.

In Figure 1 we plot the wave action spectrum for the direct cascade run averaged over time in stationary conditions. A well developed power law spectrum very close to prediction is observed over more than one decade (see inset). This spectrum confirms the existence of nontrivial dynamics with six-wave process for the truncated Hamiltonian.

We remind that the direct cascade for the Hamiltonian was already discussed by Svistunov [2, 7] who also gave the dimensional prediction. Numerical simulations with a discrete version of the Hamiltonian [7], which breaks the integrability of the original system as well, confirmed the validity of the dimensional prediction.

We now turn to the simulations for the inverse cascade regime. To obtain an inverse cascade, forcing is concentrated at small scales, here $k_f = 683$. In order to avoid finite size effects and accumulation at the largest scale [19], the friction coefficient is chosen in such a way that wave action is removed at a scale $k_\alpha \approx 10$. Figure 2 shows the spectrum for this inverse cascade in stationary conditions. In the compensated plot, a small deviation from the power-law scaling at small scales, probably due to the presence of forcing, is observed. Nevertheless, a clear scaling compatible with the dimensional analysis of the kinetic equation is observed over about a decade.

In summary, we have introduced a minimal model for Kelvin wave turbulence, and presented an argument why this model should be preferential over BSE. Namely, this model is much simpler than BSE and it has the same degree of validity due to the vortex core shape sensitivity and uncertainty. We have used our model for numerical simulations of the direct and the inverse cascades and found spectra which are in very good agreement with the predictions of the WT theory.

**I. APPENDIX A - INTERACTION COEFFICIENTS OF 2D-BS**

In this Appendix we extend the work of Kozik and Svistunov on the Kelvin wave cascade (KS04) [4]. They consider the full Biot-Savart Hamiltonian in 2D, and simplify the denominator by Taylor expansion. The criterion for Kelvin-wave turbulence is that wave amplitude is small compared to wavelength, this is formulated as:

$$\epsilon(z_1, z_2) = \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|} \ll 1.$$ (19)
KS04 find the Biot-Savart Hamiltonian (2) expanded in powers $\epsilon$ ($H = H_0 + H_1 + H_2 + H_3$, here $H_0$ is just a number and is ignored) is represented as:

$$
H_1 = \frac{\kappa}{8\pi} \int \frac{dz_1 dz_2}{|z_1 - z_2|} \left[ 2 \text{Re} \left( w^{*}(z_1) w(z_2) \right) - \epsilon^2 \right],
$$

$$
H_2 = \frac{\kappa}{32\pi} \int \frac{dz_1 dz_2}{|z_1 - z_2|} \left[ 3\epsilon^4 - 4\epsilon^2 \text{Re} \left( w^{*}(z_1) w(z_2) \right) \right],
$$

$$
H_3 = \frac{\kappa}{64\pi} \int \frac{dz_1 dz_2}{|z_1 - z_2|} \left[ 6\epsilon^4 \text{Re} \left( w^{*}(z_1) w(z_2) \right) - 5\epsilon^6 \right].
$$

By taking the Fourier transformation $w(z) = \int dk w_k e^{ikz}$ of (20), invoking a cutoff at $|z_1 - z_2| \rightarrow 0$, KS04 derived the coefficients of $H_1$, $H_2$ and $H_3$ in terms of cosines in Fourier space [4]. Kozik and Svistunov however, left their work unfinished. We extend the work of KS04 by integrating the Fourier representations of $H_1$, $H_2$ and $H_3$ using integration by parts, and applying the following cosine identity [20]

$$
\int_a^\infty \frac{\cos(t)}{t} dt = -\gamma - \ln(a) - \int_0^a \frac{\cos(t)}{t} dt = -\gamma - \ln(a) - \sum_{k=1}^\infty \frac{(-a)^2}{2k(2k)!} = -\gamma - \ln(a) + O(a^2)
$$

Neglecting terms of order $\sim a^2$ and higher we calculate the frequency and interaction coefficients of equations (20)

$$
H_1 = \int \omega_k |w_k|^2 dk,
$$

$$
H_2 = \int dk_{1234} W_{1234} w_1 w_2 w_3 w_4 \delta_{1234}(k),
$$

$$
H_3 = \int dk_{123456} C_{123456} w_1 w_2 w_3 w_4 w_5 w_6 \delta_{123456}(k).
$$

where

$$\omega_k = \frac{\kappa}{4\pi} k^2 \left[ \ln \left( \frac{1}{ka} \right) - \gamma - \frac{3}{2} \right],
$$

$$W_{1234} = \frac{\kappa}{64\pi} k_1 k_2 k_3 k_4 \left[ 1 + 4\gamma - 4 \ln \left( \frac{1}{k_{\text{eff}} a} \right) \right] + F_{1234},
$$

$$C_{123456} = \frac{\kappa}{128\pi} k_1 k_2 k_3 k_4 k_5 k_6 \left[ 1 - 4\gamma + 4 \ln \left( \frac{1}{k_{\text{eff}} a} \right) \right] + G_{123456}.
$$

We use the notation that $k_{\text{eff}}$ is the mean value of wavenumbers, $\gamma = 0.5772\ldots$ is the Euler Constant and $F_{1234}$ and $G_{123456}$ are logarithmic terms of order one shown below

$$F_{1234} = \frac{\kappa}{16\pi} \left[ 6 \sum_{N \in I} \frac{N^4}{24} \ln \left( \frac{N}{k_{\text{eff}}} \right) \right] - \sum_{N \in J} \left( \frac{N^2}{2} \ln \left( \frac{N}{k_{\text{eff}}} \right) \right)
$$

$$G_{123456} = \frac{\kappa}{16\pi} \left[ 3 \sum_{N \in K} \frac{k_0 k_2 N^4}{24} \ln \left( \frac{N}{k_{\text{eff}}} \right) \right] - 5 \sum_{N \in L} \frac{N^6}{720} \ln \left( \frac{N}{k_{\text{eff}}} \right)
$$

$$I = \{-[1], -[2], -[3], -[4], [2], [43], [2]\}
$$
$$J = \{-[4], [1], -[43], -[2]\}
$$
$$K = \{-[2], -[3], -[2], [45], [2], [6], -[6], -[56], -[3]\}
$$
$$L = \{-[4], -[6], -[46], [6], -[1], -[5], [45], [5], -[45], [65], -[3], -[6], -[1], -[3], [4], -[13], [3], -[36], -[5], -[3], -[65], -[12], [2], -[2]\}

The notation used for the logarithmic terms is: for $N \in K = \pm[56]$ then the corresponding term in $G_{123456}$ is

$$\pm \frac{3\kappa}{16\pi} k_0 k_2 (k_0 + k_2 - k_1)^4 \ln \left( \frac{k_0 + k_2 - k_1}{k_{\text{eff}}} \right).
$$

As you can see, the logarithmic terms $F_{1234}$ and $G_{123456}$ are of order $\sim 1$ and are extremely messy and complex. Applying the parametrization of Zakharov and Schulman [21] does not help in this case (see Appendix C). We know that LIA is fully integrable, i.e. that after applying the canonical transformation all terms of order $\ln(ka)$ will cancel. However, the presence of the order one additions to the dispersion and interaction coefficients, will break this integrability and result in some six-wave interactions. Due to us being unable to simplify the expression after the canonical transformation because of the non-canceling order one terms, we make the assumption of the interaction coefficients being products of wavenumbers, this is equivalent to the truncated LIA model suggested in this letter.

II. APPENDIX B - INTEGRABILITY OF 2D-LIA

We show the details of the derivation that the LIA of BSE yields the 2D-LIA model [5], i.e. that the cutoff
operation commutes with making the 2D reduction. In the view of integrability of the original LIA, this amounts in a proof of integrability of the 2D-LIA model.

The LIA representation of BSE can be written as

$$\mathbf{r} = \beta \mathbf{r}' \times \mathbf{r}''$$  \hspace{1cm} (34)

where the notation for the differentiation \( ' \) is \( d/dl \) where \( l = (1 + |\mathbf{w}|^2)^{-1/2} \) is the arc length. The two-dimensional representation of a vortex line can be represented by a vector \( \mathbf{r} = z \hat{z} + \mathbf{w} \). The vector \( \mathbf{w} = (x(z), y(z)) \) being a function of \( z \), orientated in the xy-plane.

Applying the chain rule to rewrite all derivatives to be with respect to \( z \) (from this point on \( ' \) will refer to \( d/dz \)), LIA can be represented as

$$\dot{\mathbf{r}} = \beta \left[ (1 + |\mathbf{w}'|^2)^{-3/2} (\dot{z} \times \mathbf{w}'' + \mathbf{w}' \times \mathbf{w}'') \right].$$  \hspace{1cm} (35)

With a little geometrical intuition, one can show \( \dot{w} = \hat{r} - (\hat{r} \cdot \hat{z}) (\hat{z} + \mathbf{w}') \) \[2\]. Both \( \mathbf{w}' \) and \( \mathbf{w}'' \) are perpendicular to \( \hat{z} \) direction, thus, one can represent \( \mathbf{w}' \times \mathbf{w}'' = ((\mathbf{w}' \times \mathbf{w}'') \cdot \hat{z}) \hat{z} = A \hat{z} \). Then equation (35) reduces to

$$\dot{\mathbf{w}} = \beta (1 + |\mathbf{w}'|^2)^{-3/2} [\dot{z} \times \mathbf{w}'' - A \mathbf{w}].$$  \hspace{1cm} (36)

We will now show that equation (36) is equivalent to equation (39). First we must change our representation of \( \mathbf{w} \) from a complex variable \( w(z) = x(z) + iy(z) \) to vector notation \( \mathbf{w} = (x(z), y(z)) \). Equation (6) is equivalent to

$$\dot{w} = \frac{1}{2} \dot{z} \times \frac{\partial}{\partial z} \left( \frac{\mathbf{w}'}{\sqrt{1 + |\mathbf{w}'|^2}} \right).$$  \hspace{1cm} (37)

Expanding, keeping track of \( \beta \) and applying the vector identity \( (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{a}) \mathbf{c} \), equation (37) can be re-written as

$$\dot{\mathbf{w}} = \beta (1 + |\mathbf{w}'|^2)^{-3/2} [\dot{z} \times \mathbf{w}'' - A \mathbf{w}].$$  \hspace{1cm} (38)

This is exactly the same result as Equation (36). The 2D-LIA model is equivalent to the LIA of BSE and so the 2D-LIA model is indeed integrable.

III. APPENDIX C - CANONICAL TRANSFORMATION

In Wave Turbulence, the near-identity transformation allows one to eliminate “unnecessary” lower orders of nonlinearity in the system if corresponding order of the wave interaction process is nil \[17\]. For example, if there is no three-wave resonances, then one can eliminate the cubic Hamiltonian. (The quadratic Hamiltonian corresponding to the linear dynamics, of course, stays). This process can be repeated recursively, in a way similar to the KAM theory, until the lowest order of the non-trivial resonances is reached. If no such resonances appear in any order, one has an integrable system.

In our case, there is no four-wave resonances (there are no non-trivial solution for the resonance conditions in for \( \omega \sim k^2 \) if \( x > 1 \)). There is also no five-wave resonances because the original terms in the Hamiltonian are of the even orders. However, there are nontrivial solutions of the six-wave resonant conditions. Thus, one can use the near-identity transformation to convert our system into the one with the lowest order interaction Hamiltonian to be of degree six. Let us do this.

A trick for finding a shortcut derivation of such a transformation was found in \[17\]. It relies on the fact that the time evolution operator is a canonical transformation. Taking the Taylor expansion of \( w(k, t) \) around \( w(k, 0) = c(k, 0) \) we get a desired transformation, that is by its derivation, canonical. The coefficients of each term can be calculated from an auxiliary Hamiltonian \( H_{aux} \). Similar procedure was done in Appendix A3 of \[17\] to eliminate the cubic Hamiltonian in cases when the three-wave interaction is nil, and here we apply a similar approach to eliminate the quadric Hamiltonian. The transformation is represented as

$$w_k = c(k, 0) + t \left( \frac{\partial c(k, t)}{\partial t} \right)_{t=0} + \frac{t^2}{2} \left( \frac{\partial^2 c(k, t)}{\partial t^2} \right)_{t=0} + \cdots$$  \hspace{1cm} (39)

The transformation is canonical for all \( t \), so for simplicity we set \( t = 1 \). The coefficients of (39) can be calculated from the followinf formulae,

$$\left( \frac{\partial c(k, t)}{\partial t} \right)_{t=0} = -i \frac{\delta H_{aux}}{\delta c^*},$$

$$\left( \frac{\partial^2 c(k, t)}{\partial t^2} \right)_{t=0} = -i \frac{\partial}{\partial t} \frac{\delta H_{aux}}{\delta c^*}. \hspace{1cm} (40)$$

Due to the original Hamiltonian \[H_{exp} \] having \( U(1) \) symmetry we have no odd wave-interactions, this simplifies the canonical transformation considerably, because the absence of odd interaction in \( H_{exp} \) automatically fixes the arbitrary odd interaction coefficients within the auxiliary Hamiltonian \( H_{aux} \). Transformation (39) reduces to

$$w_k = c_k - \frac{i}{2} \int dk_{234} \tilde{W}_{k_{234}} c_{2} c_{3} c_{4} \delta_{k_{234}}(k)$$

$$-3i \int dk_{23456} \tilde{C}_{k_{23456}} c_{2} c_{3} c_{4} c_{5} c_{6} \delta_{k_{23456}}(k)$$

$$+ \frac{1}{8} \int dk_{234567} \tilde{W}_{k_{234567}} c_{2} c_{3} c_{4} c_{5} c_{6} c_{7} \delta_{k_{234567}}(k)$$

$$-2 \tilde{W}_{k_{2347}} \tilde{W}_{k_{23456}} c_{2} c_{3} c_{4} c_{5} c_{6} c_{7}. \hspace{1cm} (41)$$

Here we have used tildes to represent interaction coefficients of the auxiliary Hamiltonian \( H_{aux} \). To eliminate the nonresonant interactions we substitute transformation (39) into Hamiltonian (8), this will yield a new representation in variable \( c_k \) for Hamiltonian (8) where the
nonresonant terms (more specifically the four-wave interaction terms) will involve both $W_{1234}$ and $\tilde{W}_{1234}$. Arbitrariness of $\tilde{W}_{1234}$ implies that we can select this to eliminate the total four-wave nonresonant interaction term, $H_2$ present within the Hamiltonian. In our case,

$$\tilde{W}_{1234}^* = \frac{4iW_{1234}}{\omega_1 + \omega_2 - \omega_3 - \omega_4}. \quad (42)$$

This choice is valid as the denominator will not vanish due to the nonresonance of four-wave interactions. Hamiltonian (8) expresses in variable $c_k$, $H_c$ becomes

$$H_c = \int \omega_k c_k c_k^* dk + \int \left( C_{123456} - i(\omega_1 + \omega_2 + \omega_3 - \omega_4 - \omega_5 - \omega_6) \times \tilde{C}_{123456} \right) c_1^* c_2^* c_3 c_4 c_5 c_6 dk_{123456}. \quad (43)$$

$\tilde{C}_{123456}$ is the arbitrary six-wave interaction coefficient from the auxiliary Hamiltonian. This term does not contribute to the six-wave resonant dynamics as the factor in front will vanish on the resonant manifold that appears in the kinetic equation. However, we can select $i(\omega_1 + \omega_2 + \omega_3 - \omega_4 - \omega_5 - \omega_6) \tilde{C}_{123456}$ to equal $C_{123456}$ off the resonant manifold. This enables us to write $H_c$ as equation (44), where the explicit formula for $C_{123456}$ stemming from the transformation is

$$C_{123456} = \frac{1}{18} \sum_{i,j,k=1}^{3} \sum_{p,q,r=1}^{6} W_{p+q-i+j}W_{r+k-k+r}$$

$$\times \frac{W_{r+k-r+i}W_{k+i-k+i}W_{i+j-j+i}}{(\omega_{p+q-i+j} + \omega_{r+k-r+i} - \omega_{i+j-i+j})}.$$  \quad (44)

Zakharov and Schulman discovered a parametrisation [21] for the six-wave resonant condition with $\omega_k \sim k^2$,

$$k_1 = P + R \left[ u + \frac{1}{u} - \frac{1}{v} + 3v \right],$$

$$k_2 = P + R \left[ u + \frac{1}{u} + \frac{1}{v} - 3v \right],$$

$$k_3 = P - \frac{2R}{u} - 2Ru,$$

$$k_4 = P + \frac{2R}{u} - 2Ru,$$

$$k_5 = P + R \left[ u - \frac{1}{u} + \frac{1}{v} + 3v \right],$$

$$k_6 = P + R \left[ u - \frac{1}{u} - \frac{1}{v} - 3v \right].$$  \quad (45)

This parametrisation allows us to explicitly calculate $C_{123456}$ on the resonant manifold. This is important because the wave kinetics take place on this manifold, that corresponds to the delta functions within the kinetic equation. When this parametrisation is used with equation (44) and $W_{1234} = -\frac{1}{8}k_1k_2k_3k_4$ we find that the resonant six-wave interaction coefficient simplifies to $C_{123456} = -\frac{1}{16}k_1k_2k_3k_4k_5k_6$. Note, that this is indeed the identical to the next term, $H_3$ in the LIA expansion with opposite sign.

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