Localized nonlinear matter waves in Bose-Einstein condensates with spatially and spatiotemporally modulated nonlinearities

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The novel phenomena arising from Bose-Einstein condensates with spatially and spatiotemporally modulated nonlinearities in external potential is reviewed from a theoretical viewpoint. We first present theoretical analysis and numerical studies of the localized nonlinear matter waves in one-dimensional single and two-component BECs with spatially and spatiotemporally modulated nonlinearities, respectively. It is shown that the spatially or spatiotemporally modulated nonlinearity can support stable novel localized nonlinear matter waves such as the breathing solitons and moving solitons. Then the quasi-two-dimensional BEC with spatially modulated nonlinearity is investigated, and we show that all of the BECs, similar to the linear harmonic oscillator, can have an arbitrary number of localized nonlinear matter waves with discrete energies. Their properties are determined by the principal quantum number and secondary quantum number. Moreover, we investigate the quantized vortices in a rotating BEC with spatiotemporally modulated interaction in harmonic and anharmonic potentials, respectively. The exact quantized vortex and giant vortex solutions are constructed explicitly by similarity transformation. Their stability behavior is examined by numerical simulation, which shows that a new series of stable vortex states which are defined by radial and angular quantum numbers, can be supported by the spatiotemporally modulated interaction in this system. We find that there exist stable quantized vortices with large topological charges in repulsive condensates with spatiotemporally modulated interaction.

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I. INTRODUCTION

Since the remarkable experimental realization of Bose-Einstein condensations in alkali metals [1–3], there has been an extensive explosion of the experimental and theoretical activity devoted to the nonlinear dynamical properties of atomic matter waves in dilute ultracold bosonic gases. Moreover, the investigation of rotating gases has been a central issue in the theory of superfluidity [4–7] since rotation can lead to the formation of quantized vortices which order into a vortex array, in close analogy with the behavior of superfluid helium. Under conditions of rapid rotation, when the vortex density becomes large, atomic Bose gases offer the possibility to explore the physics of quantized vortices in novel parameter regimes. During recent years, there have been advances in experimental discoveries [8–10] of rotating ultra-cold atomic Bose gases, and these developments have been reviewed in [11].

A natural starting point for studying the dynamics of Bose-Einstein condensates (BECs) is the theory of weakly interacting bosons which takes the form of the mean-field Gross-Pitaevskii (GP) equations [12–24] which provide a universal model for a study of the dynamics of the envelope waves in BECs. In particular, they reproduce typical properties exhibited by superfluid systems, like the propagation of collective excitations, Josephson effect and the interference effects originating from the phase of the order parameter. The GP equations are well suited to describing most of the effects of two-body interactions in the dilute atomic Bose gases at zero temperature and can be naturally generalized to also investigate thermal effects.

It is known that most properties of BECs including their shapes, collective nonlinear excitations are determined by the sign and magnitude of the s-wave scattering length [25]. In past decades, techniques for managing scattering lengths have attracted considerable attention. For example, a magnetically induced Feshbach resonance [26] for tuning the scattering length is a key technique for investigation of quantum degenerate gases of ultracold atoms. As mentioned by Marinescu and You [27], in the presence of a dc electric field the interatomic potential is changed due to the effective dipole-dipole interaction between the polarized atoms. Thus Feshbach resonance induced by electric field can also be used to tune the s-wave scattering length. Moreover, an alternative approach, the optical Feshbach resonance [28–30], utilizing optical coupling between bound and scattering states, is a promising technique for providing this possibility. With optical Feshbach resonance approach, the intensity and the detuning of the coupling laser are used to control the scattering length. More recently, Yamazaki et al. [30] demonstrated experimentally submicron spatial control of interatomic interactions in a BEC of ytterbium successfully by utilizing optical Feshbach resonance technique, which has opened the possibility to realize spatial modulation of the scattering length.

As stated above, by using the Feshbach resonance techniques [25–30], one can investigate atomic matter waves and
the nonlinear excitations in BECs with spatiotemporally modulated interactions. A number of theoretical studies are carried out to reveal the dynamical properties of solutions for GP equations with the spatially and spatiotemporally modulated nonlinearity coefficients [15–24]. Under special choice of interaction parameters, exact localized nonlinear matter wave solutions of the GP equations can be constructed explicitly. Belmonte-Beitia et al. [16, 19] use the Lie group theory and similarity transformations to construct explicit solutions of nonlinear Schrödinger equation (NLS) with spatially inhomogeneous nonlinearity and spatiotemporal inhomogeneous nonlinearity, respectively. Rodas-Verde et al. [23] demonstrate the controllable emission of matter-wave bursts from a BEC with spatially inhomogeneous nonlinearity in a shallow optical dipole trap. Salerno et al. [24] study the dynamics of matter waves in BEC in linear and nonlinear optical lattices and show that by properly designing the spatial dependence of the scattering length it is possible to induce long-living Bloch oscillations of gap-soliton matter waves in optical lattices. These studies [16, 19, 22–24] are only limited in the one dimensional (1D) and one-component BECs with spatially modulated nonlinearity. We have extended the similarity transformation to the two-component BECs with spatiotemporally modulated nonlinearities, the quasi-2D BEC with spatially modulated nonlinearity and rotating BEC with spatiotemporally modulated nonlinearity, respectively. Moreover, we revealed the quantum mechanics properties of BECs with spatially or spatiotemporally modulated nonlinearities, which are common in linear systems such as the linear harmonic oscillator.

The paper is organized as follows. In Sec. II, the localized nonlinear matter waves in the 1D single and two-component BECs with spatially and spatiotemporally modulated nonlinearities are investigated analytically and numerically. In Sec. III, we study the localized nonlinear matter waves in the quasi-2D BEC with spatially modulated nonlinearity in the harmonic potential, and the quantized vortices in a rotating BEC with spatiotemporally modulated nonlinearity, respectively. Finally, we conclude the main results of this paper in Sec. IV.

II. ONE-DIMENSIONAL BECS WITH SPATIALLY AND SPATIOTEMPORALLY MODULATED NONLINEARITIES

The properties of an atomic BEC are mainly determined by the two-body atom interaction, whose strength is proportional to the s-wave scattering length $a_s$. It is widely known that at zero temperatures the evolution of a BEC is governed by the GP equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V_{\text{ext}}(r)\psi + G|\psi|^2\psi,$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 = \partial^2/\partial r^2 + 1/r \times \partial/\partial r + 1/r^2 \partial^2/\partial \theta^2 + \partial^2/\partial z^2$ with $r^2 = x^2 + y^2$, $m$ is the atom mass, $\theta$ is the azimuthal angle, the wave function $\psi = \psi(r, t)$ is normalized by the total particle number $N = \int \, d^3 r \, |\psi|^2$, $V_{\text{ext}} = (m/2)(\omega_x^2 r^2 + \omega_y^2 z^2)$ is an external trapping potential with $\omega_x$ and $\omega_y$ the trapping frequencies in the transverse and longitudinal directions, respectively. Parameter $G = 4\pi\hbar^2a_s/m$ represents the strength of interatomic interaction characterized by the s-wave scattering length $a_s$, which is positive (negative) for repulsive (attractive) condensates consisting of, e.g., $^{23}$Na or $^{87}$Rb ($^{85}$Rb or $^7$Li) atoms. In this section, as in [16], we consider a collisionally space-inhomogeneous condensate, i.e., a spatially varying scattering length according to $a_s = a_s(x, t)$. Assuming a cigar-shaped trap with $\omega_x \gg \omega_y$, we can seek solutions of Eq. (1) in the form of

$$\psi(r, t) = \Psi(x, t)\Phi(r)e^{-iEt},$$

where $\Phi(r)$ is a solution of the eigenvalue problem for the 2D linear harmonic oscillator

$$-\frac{\hbar^2}{2m} \nabla^2 \Phi + \frac{1}{2}m\omega_\perp^2 r^2 \Phi = E\Phi.$$

The ground state solution of Eq. (3) is $\Phi(r) = \pi^{-1/2}\sqrt{r} \exp(-r^2/2\omega_\perp^2)$ with $\omega_\perp = \sqrt{\hbar/m\omega_\perp}$ the transverse harmonic oscillator length. Substituting Eq. (2) with Eq. (3) in Eq. (1) and then multiplying two sides by $\Phi(r)^*$ and integrating with respect to $r$, we finally have the 1D GP equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \bar{V}(x)\Psi + 2\hbar a_s\omega_\perp |\Psi|^2 \Psi,$$

where external potential is $\bar{V}(x) = m\omega_y^2 x^2/2$. We can then reduce the 1D GP equation (4) to a dimensionless form as

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi + g|\Psi|^2 \Psi,$$

where $g = m\omega_y^2/2\hbar a_s\omega_\perp$. This corresponds to the NLS equation with spatially inhomogeneous nonlinearity and spatiotemporal inhomogeneous nonlinearity, respectively.
where $V(x) = (1/2)\omega^2 x^2$ with $\omega = \omega_x/\omega_\perp$ determining the magnetic trap strength, the interaction parameter $g = 2a_s$ and the length and time are measured in units of $a_\perp = \sqrt{\hbar/m\omega_\perp}$ and $\omega_x^{-1}$, respectively.

### A. One-dimensional Bose-Einstein condensate with spatially modulated nonlinearity

As in [16], we investigate the localized stationary solutions of the 1D GP equation (5) by similarity transformation. To do so, we first assume

$$\psi(x, t) = \rho U(X)e^{-i\mu t},$$

where $\rho = \rho(x), X = X(x)$ and $\mu$ is chemical potential of the system. Substituting Eq. (6) into (5) and letting $U = U(X)$ solve the following ordinary differential equation (ODE)

$$\frac{\partial^2 U}{\partial X^2} + \mu_0 U + \mu_2 U^3 = 0,$$  \hspace{1cm} (7)

we have

$$\rho x_{xx} + 2\rho x_x = 0, \quad 2\mu_0 x_0 - 2\rho x_x - 2\mu_0 x_0 x^2 = 0.$$  \hspace{1cm} (8)

**Case 1.** If the external potential is harmonic potential $V = (1/2)\omega^2 x^2$, we have $\mu_0 = 0$ in Eq. (7) and the chemical potential $\mu = -\omega/2$ and

$$\rho = \rho_0 e^{-x^2/2}, \quad X = (\sqrt{\pi}/2)\text{erf}(\sqrt{\omega} x), \quad g = -\frac{\mu_2 e^{\omega x^2/2}}{2\rho_0^2},$$

and

$$U(X) = \sqrt{1/\mu_1}/\nu_1 \text{cn}(\nu_1 X, \sqrt{2}/2), \quad U(X) = \sqrt{1/(2\mu_1)}/\nu_2 \text{sd}(\nu_2 X, \sqrt{2}/2),$$

where $\text{erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-\tau^2} d\tau$ is called error function, $\text{sd}(\cdot, \cdot) = \text{sn}(\cdot, \cdot)/\text{dn}(\cdot, \cdot)$, $\text{cn}$ and $\text{sn}$ are Jacobi elliptic functions, and $\rho_0, \nu_1, \nu_2$ are nonzero constants.

In this case, the localized stationary solutions of the 1D GP equation (5) are

$$\psi_n(x, t) = \rho_0 \nu_1 \sqrt{1/\mu_1} e^{\nu_1 x^2/2} \text{cn}(\nu_1 X, \sqrt{2}/2) e^{\omega x^2/2},$$

$$\psi_m(x, t) = \rho_0 \nu_2 \sqrt{1/(2\mu_1)} e^{\nu_2 x^2/2} \text{sd}(\nu_2 X, \sqrt{2}/2) e^{\omega x^2/2},$$

where $X = (\sqrt{\pi}/2)\text{erf}(\sqrt{\omega} x)$. When imposing the boundary conditions for localized solutions $\lim_{x\to\pm\infty} \psi(x, t) = 0$, we have $\nu_1 = 2(2n + 1)K(\sqrt{2}/2)/\sqrt{\pi}$ and $\nu_2 = 2(2m)K(\sqrt{2}/2)/\sqrt{\pi}$, where $K(s) = \int_0^{\pi/2} [1 - s^2 \sin^2 \tau]^{-1/2} d\tau$ is the
When we investigate the dynamics of the localized stationary solutions of the 1D GP equation (5), the linear stability analysis is necessary. The linear stability analysis will lead to the linearization spectrum of the solitary wave, which is determined by the interaction strength and is negative corresponding to the attractive condensates.

From the properties of Jacobi elliptic functions, we know that the matter wave functions in Eq. (11) satisfy \( \psi_n(-x, t) = \psi_n(x, t) \), so they are even parity and are invariant under space inversion, and the matter wave functions in Eq. (12) satisfy \( \psi_m(-x, t) = -\psi_m(x, t) \), so they are odd parity. In physical units, to apply our results into the real experiments [2], we take the \(^7\)Li condensate in a pancake-shaped trap with radial frequency \( \omega_r = 2\pi \times 500 \) Hz and axial frequency \( \omega_z = 2\pi \times 5 \) Hz. Therefore, the ratio of trap frequency \( \omega_x/\omega_z \) in Eq. (5) is 0.01 which is determined by \( \omega = \omega_x/\omega_z \). In Fig. 1, we display the plots of wave functions in Eqs. (11) and (12) for different integers \( n \) and \( m \). Fig. 1(a), (c) and (e) demonstrate the profiles of the even parity wave functions (11) with \( n = 0, 1 \) and 2, respectively. Fig. 1(b), (d) and (f) demonstrate the profiles of the odd parity wave functions (12) with \( m = 1, 2 \) and 3, respectively. This is very similar to the quantum properties in the linear harmonic oscillator.

**Case 2.** If the external potential is double well potential \( V = (1/2) \omega^2 x^2 - (1/2) \mu_0 \omega \cos 2 \omega x^2 \) with \( \mu_0 \neq 0 \), we have the chemical potential \( \mu = -\omega/2 \) and the GP equation (5) becomes

\[
U(X) = \sqrt{(\nu_1^2 - \mu_0)/\mu_1 \text{cn}(\nu_1 X, \sqrt{2 \nu_1^2 - 2 \mu_0/2\nu_1})},
\]

\[
U(X) = \sqrt{(2 \nu_2^2 - 2 \mu_0)/\mu_1 \text{sn}(\nu_2 X, \sqrt{\mu_0 - \nu_2^2/\nu_2})},
\]

where \( \nu_1, \nu_2 \) are nonzero constants. In this case, the localized stationary solutions of the 1D GP equation (5) are

\[
\psi_n(x, t) = \rho_0 \sqrt{(\nu_1^2 - \mu_0)/\mu_1} e^{i\omega x^2/2} \text{cn}(\nu_1 X, \sqrt{2 \nu_1^2 - 2 \mu_0/2\nu_1}) e^{i\omega t/2},
\]

\[
\psi_m(x, t) = \rho_0 \sqrt{(2 \nu_2^2 - 2 \mu_0)/\mu_1} e^{i\omega x^2/2} \text{sn}(\nu_2 X, \sqrt{\mu_0 - \nu_2^2/\nu_2}) e^{i\omega t/2},
\]

where \( X = (\sqrt{\pi}/2) \text{erf}(\sqrt{\omega}x) \). When imposing the boundary conditions for localized solutions \( \lim_{|x|\to0} \psi(x, t) = 0 \), we have \( \nu_1 = 2(2n+1)K(k_1)/\sqrt{\pi} \) and \( \nu_2 = 2(2m)K(k_2)/\sqrt{\pi} \) with \( k_1 = \sqrt{2 \nu_1^2 - 2 \mu_0/2\nu_1} \) and \( k_2 = \sqrt{\mu_0 - \nu_2^2/\nu_2} \).

When we investigate the dynamics of the localized stationary solutions of the 1D GP equation (5), the linear stability analysis is necessary. The linear stability analysis will lead to the linearization spectrum of the solitary wave, which contains important information on the stability and other aspects of the solitary wave. In order to study the linear stability of the localized stationary solutions (11) into (12), we consider a perturbed solution of Eq. (5) as

\[
\psi(x, t) = \left\{ \psi_0(x) + [v(x) + w(x)] e^{i\lambda t} + [v^*(x) - w^*(x)] e^{-i\lambda t} \right\} e^{-i\mu t},
\]

where \( \psi_0(x) \) is the exact solution of the stationary NLS equation \( \frac{1}{2} \partial_x^2 \psi_0 + \mu \psi_0 - g\psi_0^3 - V \psi_0 = 0 \). Functions \( v, w \ll 1 \) are small perturbations to the exact solution and \( \lambda \) is called the linear-stability spectrum for the stationary solution. Inserting (17) into the 1D GP equation (5) and linearizing, we find the following linear eigenvalue problem

\[
L_1 v = \lambda w, \quad L_2 w = \lambda v,
\]
where $L_1 = \frac{1}{2} \partial_x^2 - 3 g \psi_0^2 - V + \mu$ and $L_2 = \frac{1}{2} \partial_x^2 - g \psi_0^2 - V + \mu$. If all the linear spectrums $\lambda$ are real, then the linear stability of the localized stationary solutions (11) and (12) is established. In contrast, if there is at least one linear spectrum $\lambda$ with nonzero imaginary part, then localized stationary solutions are unstable.

The eigenvalue problem (18) is solved numerically and the results show that the stationary solution (11) is linearly stable for $n = 0$ and unstable for $n \geq 1$. The stationary solution (12) is unstable for all integers $m \geq 1$, see Fig. 2.

**B. Two-component Bose-Einstein condensates with spatiotemporally modulated nonlinearities**

In this subsection, we investigate the dynamics of two-component BECs with spatiotemporally modulated nonlinearities. The dynamical properties of 1D two-component BECs is governed by the dimensionless GP equations

\[
\begin{align*}
\frac{i}{\hbar} \frac{\partial \psi_1}{\partial t} &= \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\gamma^2}{2} x^2 + b_{11} |\psi_1|^2 + b_{12} |\psi_2|^2 \right) \psi_1, \\
\frac{i}{\hbar} \frac{\partial \psi_2}{\partial t} &= \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\gamma^2}{2} x^2 + b_{12} |\psi_1|^2 + b_{22} |\psi_2|^2 \right) \psi_2,
\end{align*}
\]

where $\gamma = \omega_x / \omega_\perp$ with $\omega_\perp$ and $\omega_x$ the confinement frequencies in the transverse and axial directions, respectively, and $b_{11}, b_{22}, b_{12}$ are functions of $x$ and $t$. The units for length and time are $\sqrt{\hbar / (m \omega_\perp)}$ and $\omega_\perp^{-1}$, respectively.

We now consider the exact spatially localized solutions of Eq. (19) for which $\lim_{|x| \to \infty} \psi_{1,2}(x, t) = 0$. To do this, we take the following similarity transformation

\[
\psi_1(x, t) = \beta_1(x, t) e^{i \alpha_1(x, t) U[X(x, t)]},
\]

\[
\psi_2(x, t) = \beta_2(x, t) e^{i \alpha_2(x, t) V[X(x, t)]},
\]

(20) to reduce Eq. (19) to two coupled ODEs

\[
\begin{align*}
U_{XX} + g_{11} U^3 + g_{12} U V^2 &= 0, \\
V_{XX} + g_{22} V^3 + g_{12} U V^2 &= 0,
\end{align*}
\]

(21) where parameters $g_{11}, g_{12}$ and $g_{22}$ are constants, $\alpha_1, \alpha_2, \beta_1, \beta_2, X$ are functions of $x$ and $t$ to be determined. For brevity, we define variables $\sigma_1 = g_{12} - g_{11}, \sigma_2 = g_{12} - g_{22}$ and $\sigma_{12} = g_{22} - g_{11} g_{12}$. Substituting Eq. (20) into Eq. (19) and asking $U(X), V(X)$ to satisfy Eq. (21) we have a set of partial differential equations (PDEs). Solving this set of PDEs, we find that when

\[
b_{11} = g_{11} \theta(t, x), \quad b_{12} = g_{12} \theta(t, x), \quad b_{22} = g_{22} \theta(t, x),
\]

(22) with $\theta(t, x) = -\lambda^2 e^{-3 \lambda^2 x^2 - 6 \lambda \delta x - 2 \delta^2} / [2 \lambda^2 (t)]$, one has

\[
\begin{align*}
\alpha_1 &= \zeta_1(t) - \frac{\lambda x^2}{2 \lambda} - \frac{\delta x}{\lambda}, \\
\alpha_2 &= \zeta_2(t) - \frac{\lambda x^2}{2 \lambda} - \frac{\delta x}{\lambda}, \\
\beta_1 &= \beta_2 = \zeta_3(t) e^{\lambda x + \delta t}, \quad X = \frac{\sqrt{\pi}}{2} \text{erf} (\lambda x + \delta),
\end{align*}
\]

(23) where functions $\zeta_1(t) = \int (\lambda^4 - \delta^2 + \lambda^2 \delta^2)/(2 \lambda^2) dt + C_1, \zeta_2(t) = \int (\lambda^4 - \delta^2 + \lambda^2 \delta^2)/(2 \lambda^2) dt + C_2$ and $\zeta_3(t) = \sqrt{\pi} e^{\lambda^2 / 2}$, with $C_1$ and $C_2$ arbitrary constants and $\lambda, \delta$ satisfy

\[
\begin{align*}
\lambda^6 - \gamma^2 \lambda^2 - 2 \lambda \delta^2 + \lambda \delta \lambda &= 0, \\
2 \lambda^2 \delta^2 - 4 \delta \lambda &= 0.
\end{align*}
\]

(24) When setting $\sigma_1 / \sigma_{12} > 0$ and $\sigma_2 / \sigma_{12} > 0$, Eq. (21) has two families of exact solutions as

\[
U^{(1)}(X) = \sqrt{\sigma_2 / \sigma_{12}} \nu_1 \text{cn}(\nu_1 X, \sqrt{2}/2),
\]
\[ V^{(1)}(X) = \sqrt{\sigma_1/\sigma_2} \nu_1 \text{cn}(\nu_1 X, \sqrt{2}/2), \]  

and

\[ U^{(2)}(X) = \sqrt{2}/2 \sqrt{\sigma_2/\sigma_2} \text{sd}(\nu_2 X, \sqrt{2}/2), \]

\[ V^{(2)}(X) = \sqrt{2}/2 \sqrt{\sigma_1/\sigma_2} \text{sd}(\nu_2 X, \sqrt{2}/2), \]  

where \( \nu_1, \nu_2 \) are arbitrary constants. When imposing the bounded condition \( \lim_{|x| \to \infty} \psi_{1,2}(x, t) = 0 \), we have \( \nu_1 = 2(2n + 1)K(\sqrt{2})/\sqrt{\pi} \) for Eq. (25) and \( \nu_2 = 2(2m)K(\sqrt{2})/\sqrt{\pi} \) for Eq. (26), where \( n \) and \( m \) are integer numbers.

Eq. (24) has solution of the following form

\[ \delta(t) = c_1 e^{i \int \lambda^2 dt} + c_2 e^{-i \int \lambda^2 dt}, \]  

with \( c_1 \) and \( c_2 \) arbitrary constants.

Next we set \( \lambda = 1/\xi \) to rewrite the first equation in (24) in the form of the Ermakov-Pinney equation [31] as

\[ \xi_{tt} + \gamma^2 \xi = 1/\xi^3. \]  

According to the result in [31], to obtain the explicit solutions of Eq. (28) we choose \( \gamma \) to satisfy

\[ \gamma^2 = \gamma_0^2 + \epsilon \cos(\gamma_1 t), \]  

with \( \epsilon \in (-1, 1) \) and \( \gamma_0, \gamma_1 \in R \). Therefore, the general solution to Eq. (24) is

\[ \lambda = [A \xi_1^2 + B \xi_2^2 + 2C \xi_1 \xi_2]^{-1/2}, \]  

where \( A, B, C \) are constants satisfying \( AB - C^2 = 1/W^2 \), and the Wronskian \( W = \xi_1 \xi_{tt} - \xi_2 \xi_{tt} \) with \( (\xi_1, \xi_2) \) being two linearly independent solutions of homogeneous ODE

\[ \xi_{tt} + \gamma^2 \xi = 0. \]  

Combining Eqs. (20) (23) with (25) and (26), we have two families of exact solutions of equations (19) as

\[ \psi_{1}^{(j)}(x, t) = \sqrt{\lambda} e^{\lambda^2 \int x,t} e^{i\alpha_{1}(x,t)} U^{(j)}(X), \]

\[ \psi_{2}^{(j)}(x, t) = \sqrt{\lambda} e^{\lambda^2 \int x,t} e^{i\alpha_{2}(x,t)} V^{(j)}(X), \]  

where \( U^{(j)}(X) \) and \( V^{(j)}(X) \) are given by Eqs. (25) and (26), index \( j = 1, 2 \), \( \alpha_1, \alpha_2 \) satisfy Eq. (23), and \( X, \delta, \lambda \) satisfy Eqs. (23) (27) and (30), respectively.

We next discuss the dynamical properties and stability of the exact localized nonlinear wave solutions (32) of the coupled GP equations (19). Let us first provide the experimental parameters for producing the 1D two-component BECs composed of \( N_1 = N_2 = 5 \times 10^4 \) Rb atoms, confined in a cigar-shaped trap, with the ratio of the confining frequencies, \( \gamma = \omega_\perp/\omega_\parallel \) of order \( O(10^{-2}) \). Typically, we choose axial frequency \( \omega_\perp = 70\pi \) Hz and radial frequency \( \omega_\parallel = 800\pi \) Hz [32], so \( \gamma = 7/80 \). Moreover, within the safe region we will always take the parameters \( g_{ij} \) in Eq. (22) to be \( g_{11} = -1, g_{22} = -3, g_{ij} = 2 \).

To understand the dynamics of the explicitly exact solutions (32), we take special parameters \( \gamma_0, \gamma_1 \) and \( \epsilon \) in Eq. (29). When the ratio of the confining frequencies \( \gamma \) of the harmonic potential is time independent, that is, parameters \( \gamma_1 = \epsilon_1 = 0 \), solving Eq. (31) we have \( \lambda = 1/\xi \) with \( \xi = [A - (A - B) \cos^2(\gamma_{0} t) + \sqrt{AB \gamma_0^2 - I \sin(2 \gamma_0 t) / \gamma_0}]^{1/2} \). Here \( \xi \) is the width of the explicitly exact solutions (32) and \( \sqrt{\xi} \) is its amplitude. Specially, we further suppose \( c_1 = c_2 = 0 \), i.e. \( \delta = 0 \). So from Eq. (29) the parameter \( \gamma_0 = \gamma = 7/80 \). In order to determine parameters \( A \) and \( B \) in Eq. (30), we further consider the initial condition of Eq. (28) as \( \xi(0) = \sqrt{\gamma} \) and \( \xi_{t}(0) = 1/80 \). The evolutions of condensate density of the wave functions \( \psi_{1}^{(n)} \) and \( \psi_{2}^{(n)} \) in (32) with the above parameters are shown in Fig. 3. Fig. 3(a)-(b) demonstrate the density profiles of the wave function \( \psi_{1}^{(n)} \) for \( n = 0, 1 \), respectively, and Fig. 3(c)-(d) demonstrate the density profiles of the wave function \( \psi_{2}^{(m)} \) for \( m = 1, 2 \), respectively. Fig. 3(e) demonstrates the width \( \xi = 1/\lambda \) and amplitude \( \sqrt{\lambda} \) of the wave functions. It is observed that the localized nonlinear matter waves are space-localized and time-periodic, which are usually called breathing solitons. Here \( n \) and \( m \) are the order of the breathing solitons. It is
FIG. 3: (color online). Dynamics of breathing solitons in two-component BECs with spatiotemporally modulated nonlinearities in quadratic potential. (a)-(b) are the evolution of condensate density $|\psi^{(1)}_1|^2$ for order $n = 0$ and 1, respectively. (c)-(d) are the examples of condensate density $|\psi^{(2)}_1|^2$ for order $m = 1$ and 2, respectively. (e) demonstrates the width $\xi(t) = 1/\lambda(t)$ (upper line) and amplitude $\sqrt{\lambda(t)}$ (lower line) of wave functions. The parameters are $\gamma_1 = \epsilon = c_1 = c_2 = 0$ and $\gamma_0 = 7/80$.

also observed that the amplitude and width of the localized nonlinear matter waves vary periodically with respect to time.

We next consider the case of $\delta \neq 0$, which demonstrate more wonderful nonlinear matter waves. The time dependent function $\delta$ can be determined by Eq. (27) with Eq. (30). In this case, the nonlinearities in Eq. (19) become more complicated, the amplitude of the nonlinear matter wave seems more complex, i.e. $\sqrt{\lambda_0 + \delta^2}$, and the center of the solitons can move following the time because of variable $X = \frac{x}{\sqrt{\pi}} \text{erf} (\lambda x + \delta)$. So for $\delta \neq 0$, our localized nonlinear matter waves are localized moving solitons. For simplicity, we assume $\gamma$ is time independent, i.e. restricting parameters $\gamma_0 = 7/80$ and $\gamma_1 = \epsilon = 0$. Fig. 4(a)-(b) describe the evolutions the density profiles for the wave function $\psi^{(1)}_1$ with orders $n = 0, 1$, respectively, and Fig. 4(c)-(d) describe the evolutions of the density profiles for the wave function $\psi^{(2)}_1$ with orders $m = 1, 2$, respectively. Fig. 4(e) demonstrates the shapes of the function $\sqrt{\lambda}$, the width $\xi = 1/\lambda$ and the amplitude $\sqrt{\lambda_0 + \delta^2}$ of the moving breathing solitons. It is found that the localized nonlinear matter waves are space localized and time periodically moving. The amplitude and width of the localized nonlinear matter waves vary periodically with respect to time.

The stability of the exact localized nonlinear wave solutions (32) in response to perturbation by initial stochastic noise is investigated by direct numerical simulations. The numerical results for the evolution of a breathing soliton solution (32) for $j = 1$ and $n = 0$ with initial Gaussian noise are described in Fig. 5(a)-(b). The other parameters
FIG. 4: (Color online). Dynamics of moving breathing solitons in two-component BECs with spatiotemporally modulated nonlinearities in quadratic potential. (a)-(b) describe the evolution of condensate density $|\psi_1(1)|^2$ for order $n = 0$ and 1, respectively. (c)-(d) describe the examples of condensate density $|\psi_1(2)|^2$ for order $m = 1$ and 2, respectively. (e) describes the width $\xi(t) = 1/\lambda(t)$ (upper line), amplitude $\sqrt{\lambda(t)} e^{i\phi}$ (middle line) and function $\sqrt{\lambda(t)}$ (lower line). The parameters are the same with that in Fig. 3 except for $c_1 = c_2 = 1/2$.

are $g_{11} = 1, g_{22} = -3, g_{12} = 5, \gamma_0 = 7/80, \gamma_1 = \epsilon = 0$ and $c_1 = c_2 = 0$, i.e. $\delta = 0$. Fig. 6(a)-(b) show the numerical results for the evolution of a moving breathing soliton solution (32) for $j = 1$ and $n = 0$ with also initial Gaussian noise. The other parameters are the same as Fig. 5 but $c_1 = c_2 = 1/2$, i.e. $\delta \neq 0$. Here the Gaussian noise is included by adding to the first component a Gaussian distributed random number with mean 1/2 and unit variance, and the second component with mean 2/3 and unit variance multiplied level 5%. It is observed that the exact solution (32) for $j = 1$ and $n = 0$ is dynamically stable for $\delta = 0$, i.e. one order breathing soliton is stable, and the exact solution (32) for $j = 1$ and $n = 0$ is dynamically unstable for $\delta \neq 0$, i.e. one order moving breathing soliton is unstable. We have tested the dynamical stability of the exact localized nonlinear wave solutions (32) for other quantum numbers. The results show that only for order $n = 0$ (ground state) is the breathing soliton solution (32) with $\delta = 0$ dynamically stable, while the moving soliton solution (32) with $\delta \neq 0$ is dynamically unstable for all $n$. 
FIG. 5: (Color online). Time evolution of a breathing soliton solution (32) for $j = 1$ and $n = 0$ with initial Gaussian noise of level 5%. The other parameters are $g_{11} = 1, g_{22} = -3, g_{12} = 5, \gamma_0 = 7/80, \gamma_1 = \epsilon = 0$ and $c_1 = c_2 = 0$, i.e. $\delta = 0$.

FIG. 6: (Color online). Time evolution of a moving breathing soliton solution (32) for $j = 1$ and $n = 0$ with initial Gaussian noise of level 5%. The other parameters are $g_{11} = 1, g_{22} = -3, g_{12} = 5, \gamma_0 = 7/80, \gamma_1 = \epsilon = 0$ and $c_1 = c_2 = 1/2$, i.e. $\delta = 0$. It is shown that the moving breathing solitons are unstable.

III. TWO-DIMENSIONAL BOSE-EINSTEIN CONDENSATES WITH SPATIALLY AND SPATIOTEMPORALLY MODULATED NONLINEARITIES

In the past years, the studies of BEC with spatially and spatiotemporally modulated nonlinearity are limited in the quasi-1D cases as above [16, 19, 22–24]. Moreover, in the study of nonlinear problems no one discusses their quantum properties which are common in linear systems such as the linear harmonic oscillator. In this section, we extend the similarity transformation to the quasi-2D BEC with spatially modulated nonlinearity in harmonic potential, and find a family of stable localized nonlinear matter wave solutions. In addition, we also perform theoretical analysis and numerical studies of the quantized vortices in a rotating BEC with spatiotemporally modulated nonlinearity in harmonic and anharmonic potentials, respectively.

A. Quantized quasi-2D Bose-Einstein condensate with spatially modulated nonlinearity

We consider a quantized quasi-2D BEC confined in a harmonic trap $V(r) = m(\omega_r^2 r^2 + \omega_z^2 z^2)/2$, where $m$ is atomic mass, $r^2 = x^2 + y^2$, and $\omega_r, \omega_z$ are the confinement frequencies in the radial and axial directions, respectively. If the trap is pancake-shaped, i.e. $\omega_z \gg \omega_r$, it is reasonable to reduce the 3D GP equation (1) for the condensate wave
function to a quasi-2D GP equation [15]

\[ i\psi_t = -\frac{1}{2}(\psi_{xx} + \psi_{yy}) + \frac{1}{2}\omega^2(x^2 + y^2)\psi + g(x, y)|\psi|^2\psi, \]

(33)

where \( \omega = \omega_{\perp}/\omega_z \), \( g(x, y) = 4\pi a_s(x, y) \) represents the strength of interaction characterized by the s-wave scattering length, and the length, time and wave function \( \psi \) are measured in units of \( a_h = \sqrt{\hbar/m \omega_z}, \omega_z^{-1} \) and \( \hbar^{-1} \), respectively.

Consider the spatially localized stationary solution \( \psi(x, y, t) = \phi(x, y)e^{-i\mu t} \) of Eq. (33) with \( \phi(x, y) \) a real function for \( \lim_{|x|, |y| \to \infty} \phi(x, y) = 0 \) and \( \mu \) the chemical potential. This maps Eq. (33) onto a stationary NLS equation

\[ \frac{1}{2}\phi_{xx} + \frac{1}{2}\phi_{yy} - \frac{1}{2}\omega^2(x^2 + y^2)\phi - g(x, y)\phi^3 + \mu\phi = 0. \]

Solving this stationary equation by similarity transformation [16, 19], we obtain a families of exact localized nonlinear wave solutions for Eq. (33) as

\[ \psi_n = \frac{(n + 1)K(k)\eta}{\sqrt{\nu}} \text{cn}(\eta, k)e^{-i\nu t}, \quad n = 0, 2, 4, \cdots \]

(34)

\[ \psi_n = \frac{(n + 1)K(k)\eta}{\sqrt{2\nu}} \text{sn}(\eta, k)e^{-i\nu t}, \quad n = 1, 3, 5, \cdots \]

(35)

where \( k = \sqrt{2}/2 \) is the modulus of elliptic function, \( \nu \) is a positive real constant, \( \eta, \theta \) and \( g \) are determined by

\[ \eta = e^{\omega x^2}K\text{ummerU}[-\mu/(2\omega), 1/2, \omega (x^2 - y^2)/2], \]

(36)

\[ \theta = (n + 1)K(k)\text{erf}[\sqrt{2\omega} (x + y)/2], \quad g(x, y) = -2\omega \nu/(\pi\eta^2)e^{-\omega (x+y)^2}, \]

and

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}dt \]

is error function, and KummerU\((a, c, s)\) is Kummer function of the second kind [33].

In the above construction, it is observed that the number of zero points of \( \eta \) in Eq. (36) is equal to that of function KummerU\([-\mu/(2\omega), 1/2, \omega (x^2 - y^2)/2]\), which strongly depends on \( \omega \) and the ratio \( \mu/\omega \). We assume the number of zero points in \( \eta \) along line \( y = -x \) is \( l \). In the following, we will see that integer \( n \) is associated with the energy levels of the atoms and integers \( n, l \) determine the topological properties of atom packets, so \( n \) and \( l \) are named the principal quantum number and secondary quantum number in quantum mechanics. In addition, the three free
parameters $\omega$, $\mu$ and $\nu$ are positive, so the dimensionless interaction function $g(x,y)$ is negative, which indicates an attractive interaction between atoms, such as the $^{85}$Rb [34] and $^7$Li condensates [2].

In order to translate our results into units relevant to the experiments [2], we take the $^7$Li condensate containing $10^3 \sim 10^5$ atoms in a pancake-shaped trap with radial frequency $\omega_\perp = 2\pi \times 10$ Hz and axial frequency $\omega_z = 2\pi \times 500$ Hz [35]. In this case, the ratio of trap frequency $\omega$ in Eq. (33) is 0.02 which is determined by $\omega_\perp/\omega_z$. The unit of length is 1.69 $\mu$m, the unit of time is 0.32 ms and the unit of chemical potential is $nK$.

Firstly, when fixing the secondary quantum number $l$, we can modulate the principal quantum number $n$ to analyze the matter waves in quasi-2D BEC. Fig. 7 shows the density profiles in quasi-2D BEC with spatially modulated nonlinearities in harmonic potential for $l = 0$. It is easy to see that the matter wave functions in Eq. (34) satisfy $\psi_n(-x,-y) = \psi_n(x,y)$, so they are even parity and are invariant under space inversion. Figs. 7(a)-7(c) demonstrate the density profiles of the even parity wave functions (34) with Eq. (36) for $n = 0, 2, 4$, which correspond to a low energy state and two highly excited states. The matter wave functions in Eq. (35) satisfy $\psi_n(-x,-y) = -\psi_n(x,y)$, which denotes that they are odd parity. Fig. 7(d)-7(f) demonstrate the density profiles of the odd parity wave functions (35) with Eq. (36) for $n = 1, 3, 5$, which correspond to three highly excited states. It is observed that when the secondary quantum number $l = 0$, the number of nodes along line $y = x$ for each quantum state is equal to the corresponding principal quantum number $n$, i.e. the nth level quantum state has $n$ nodes along $y = x$. And the number of density packets increases one by one along line $y = x$ when the $n$ increases. This is similar to the quantum properties in the linear harmonic oscillator.

Secondly, when fixing the principal quantum number $n$, we can tune the secondary quantum number $l$ to observe the novel quantum phenomenon in quasi-2D BEC. Figs. 8(a)-(d) show the density profiles of the even parity wave function (34) with Eq. (36) for $n = 0$, and $l = 0, 1, 2$ and 3, respectively. It is seen that the number of nodes for the density packets along line $y = -x$ is equal to the corresponding secondary quantum number $l$ which describes the topological patterns of the atom packets, and the number of density packets increases one by one when $l$ increases. Figs. 8(e)-(h) show the density profiles of the odd parity wave function (35) with Eq. (36) for $n = 1$ and $l = 0, 1, 2, 3$. We see that the number of density packets increases pair by pair when $l$ increases. The number of density packets for each quantum state is equal to $(n + 1) \times (l + 1)$, and all the density packets are symmetrical with respect to lines $y = \pm x$, as shown in Figs. 7-8.

The stability of exact solutions is very important, because only stable localized nonlinear matter waves are promising for experimental observations and physical applications. To understand the stability of exact solutions (34)-(35) with Eq. (36), we consider a perturbed solution $\psi(x,y,t) = [\phi_n(x,y) + \Psi(x,y,t)]e^{-i\lambda t}$ of Eq. (33), where $\phi_n(x,y)$ are the exact solutions of the stationary NLS equation $\frac{1}{2}\phi_{xx} + \frac{1}{2}\phi_{yy} - \frac{1}{2}\omega^2(x^2 + y^2)\phi - g(x,y)\phi^3 + \mu \phi = 0$ and $\Psi(x,y,t) \ll 1$ is a small perturbation. $\Psi(x,y,t) = [R(x,y) + iI(x,y)]e^{i\lambda t}$ is decomposed into its real and imaginary parts. Substituting this perturbed solution to the quasi-2D GP equation (33) and neglecting the higher-order terms in
In rotating frame, the system can be described by the dimensionless GP equation as

$$B. \text{Quantized vortices in a rotating Bose-Einstein condensate with spatiotemporally modulated nonlinearity}$$

Numerical experiments show that when the frequencies of pancake-shaped trap is fixed, the stability of the exact solutions (34)-(35) with Eq. (36) rests only on the principle quantum number $n$. It is seen that when the frequencies of pancake-shaped trap is fixed, the stability of the exact solutions (34)-(35) with Eq. (36) linear stability. This suggests that for $\omega = 0.02$ the exact localized nonlinear matter wave solution (34) is linearly stable only for $n = 0$ and solution (35) is linearly unstable for all $n$ see Fig. 9. It is seen that when the frequencies of pancake-shaped trap is fixed, the stability of the exact solutions (34)-(35) with Eq. (36) rests only on the principle quantum number $n$.

B. Quantized vortices in a rotating Bose-Einstein condensate with spatiotemporally modulated nonlinearity

In this subsection, we study the quantized vortices in a rotating BEC with spatiotemporally modulated nonlinearity. In rotating frame, the system can be described by the dimensionless GP equation as

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \mathbf{\nabla}^2 \psi + V(r,t) \psi + g(r,t) |\psi|^2 \psi + \mu \frac{\partial \psi}{\partial \theta},$$

where the interaction strength is $g(r,t) = a(r,t) / a_0$ with $a(r,t)$ the $s$-wave scattering length and $a_0$ a constant, the radial trapping potential is $V(r,t) = \omega^2 (t) r^2 / 2$ with $\omega(t) = \omega_r(t) / \omega_z$ with $\omega_r$ and $\omega_z$ the confinement frequencies in the radial and axial directions, respectively, and $\Omega = \Omega_0 / \omega_z$ with $\Omega_0$ the rotating frequency around the z-axis. The length, time and wave function $\psi$ are measured in units of $\omega_h = (\hbar / m \omega_z)^{1/2}$, $1 / \omega_z$ and $1 / \omega_h \sqrt{\hbar \omega_0}$, respectively. In what follows, we consider not only the harmonic potential like this but also an anharmonic potential.

To get the exact vortex solutions to Eq. (37), we first assume

$$\psi(r,\theta, t) = e^{i S \theta + i \phi(r,t)} \rho(r,t) U[R(r,t)],$$

where $S$ is the topological charge related to the angular momentum of the condensate, $\rho(r,t)$ denotes the amplitude of wave function and $R(r,t)$ is an intermediate variable reflecting the changes of main wave function $U$. Substituting Eq. (38) into (37) and letting $U[R(r,t)]$ satisfy the following ODE

$$d^2 U / dR^2 + \mu_0 U + \mu_1 U^3 = 0,$$

where $\mu_0$ and $\mu_1$ are real constants, we can get a set of PDEs about $\rho(r,t)$, $R(r,t)$, $\phi(r,t)$, $V(r,t)$ and $g(r,t)$. If putting $V(r,t) = \omega(t) r^2 / 2 - \mu_0 R_t^2 / 2$, $\mu_0 = 0$ corresponds to harmonic potential and $\phi(r,t) = f_1 r^2 + f_2$ ($f_1$ and $f_2$ are time-dependent functions, and $f_1$ is frequency chirp and $f_2$ is phase), then solving the set of PDEs we have

$$\rho(r,t) = e^{- 2 \int f_1 dt} \Theta(r e^{- 2 \int f_1 dt}),$$

FIG. 9: (Color online). Eigenvalue for different principal quantum number $n$ with parameters $\omega = 0.02$, $\mu = 0.001$ and $\nu = 0.1$. It is shown that only for $n = 0$ are the localized nonlinear matter wave solutions (34)-(35) with Eq. (36) linear stability.
n=2

We get the exact solution of Eq. (39) as

\[ U = \mathrm{W}(\mu, \tau) \]

the exact vortex solution to Eq. (37) is

\[ \mu \]

where functions

\[ \lim_{\tau \to \infty} \frac{1}{\Theta(\tau)} \]

the vortex solution (44) of the attractive rotating BEC for topological charge \( S = 1 \) and various radial quantum numbers. The parameters are \( \Omega = 0.7, \mu_1 = 1000, \lambda_1 = 4, \lambda_2 = 2, c_1 = c_2 = 1, \epsilon = 0 \) and \( \omega_0 = 0.028 \).

\[ R(r, t) = \int_0^{r e^{-2/ f_1 dt}} 1/\Theta(\tau) d\tau, \]

\[ g(r, t) = -\mu_1 R_t^2/2 \rho^2, \]

where \( \mu_1 \) is a parameter controlling the sign of interaction parameter \( g(r, t), \Theta(\tau) \) is defined by Whittaker M and W functions [33], i.e. \( \Theta(\tau) = [c_1 M(\lambda_1/4 \lambda_2, S/2, \lambda_2 \tau^2) + c_2 W(\lambda_1/4 \lambda_2, S/2, \lambda_2 \tau^2)]/\tau \), where \( \lambda_1, \lambda_2, c_1, c_2 \) are nonzero constants and \( c_1 c_2 > 0 \). In particular, the above \( f_1 \) and \( f_2 \) satisfy the following two ODEs

\[ 2 \Omega S - \lambda_1 e^{-4 f_1 dt} - 2 d f_2/dt = 0, \]

\[ \omega^2(t) + 4 f_1^2 + 2 d f_1/dt - \lambda_2^2 e^{-8 f_1 dt} = 0. \]

When the parameter \( \mu_0 = 0 \), the external potential is just harmonic form \( V(r, t) = \omega^2(t) r^2/2 \), and we get explicit solution of Eq. (39) as \( U(R) = \nu_1 cn(\nu_1 R + \nu_0, \sqrt{2}/2)/\sqrt{\mu_1} \), where \( \nu_0 \) and \( \nu_1 \) are arbitrary constants and \( \mu_1 > 0 \). So the exact vortex solution to Eq. (37) is

\[ \psi = \frac{\nu_1}{\sqrt{\mu_1}} e^{i (S \theta + f_1 r^2 + f_2)} \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time
FIG. 11: (Color online) Time evolution of the density distributions $|\psi(x, y, 0)|^2$ and phase diagrams for the vortex solution (44) of the attractive rotating BEC for radial quantum number $n = 1$. (a) Stable vortex for topological charge $S = 1$. (b)-(c) Unstable vortex for $S = 2$ and 3, respectively. The other parameters are $\Omega = 0.7$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\mu_1 = 1000$, $c_1 = c_2 = 1$, $\epsilon = 0$ and $\omega_0 = 0.028$. Here the unit of time is 0.25 ms and the unit of length is 1.51 $\mu$m for $^7$Li atom.

FIG. 12: (Color online) Time evolution of the density distributions $|\psi(x, y, 0)|^2$ and phase diagrams for the vortex solution (45) of the repulsive rotating BEC for topological charge $S = 1$ and radial quantum numbers $n = 1$ and 2, respectively, with an initial Gaussian noise of level 0.5%. The parameters are $\delta = 7.4$ (top) and $\delta = 14.9$ (bottom), $\Omega = 0.7$, $\lambda_1 = 4$, $\lambda_2 = 2$, $c_1 = c_2 = 1$, $\mu_1 = -10$, $\epsilon = 0$ and $\omega_0 = 0.028$, the unit of time is 0.25 ms.

where functions $\rho$ and $R$ are given above. When imposing the boundary conditions for vortex solution as $\lim_{|r|\to 0} \psi(r, \theta, t) = \lim_{|r|\to \infty} \psi(r, \theta, t) = 0$, we can get $\delta = 2nK(\sqrt{\mu_0/\delta^2 - 1})/R(\infty, 0)$.

The structures of the exact vortex solutions (44) and (45) can be controlled by modulating the frequency of the trapping potential and the spatiotemporal inhomogeneous s-wave scattering length as seen from Eq. (43). Taking into account the feasibility of the experiment, we first consider the case of harmonic potential ($\mu_0 = 0$) which corresponds to the attractive BEC as explained above. In real experiment, we assume the attractive condensates are trapped in an axisymmetric disk-shaped potential, where the axial confinement energy $\hbar \omega_z$ is much larger than the radial confinement and interaction energies, and the radial frequency of the trap is time-dependent which can be written as

$$\omega(t) = \omega_z(t)/\omega_z = \omega_0 + \epsilon \cos(\omega_1 t), \quad (46)$$

with $0 \leq \epsilon < \omega_0$. For $\epsilon = 0$, the radial frequency of the trap is time-independent and here, we choose the time-
FIG. 13: (Color online) Time evolution of the density distributions $|\psi(x, y, 0)|^2$ and phase diagrams for the vortex solution (45) of the repulsive rotating BEC for topological charge $S = 5$ and radial quantum numbers $n = 1$ and 2, respectively, with an initial Gaussian noise of level 0.5%. The parameters are $\delta = 1172.4$ (top) and $\delta = 2344.9$ (bottom). The other parameters are the same as Fig. 12.

independent part of radial frequency $\omega_r = (2\pi) \times 18$ Hz and axial frequency $\omega_z = (2\pi) \times 628$ Hz, so $\omega(t) = \omega_0 = 0.028$.

We demonstrate in Fig. 10 the density distributions for different radial quantum number $n$ with fixed topological charge $S = 1$ at $t = 0$, based on the exact vortex solution (44). The Fig. 10(a) corresponding to $n = 1$ is a lowest energy state and Figs. 10(b)-10(d) corresponding to $n = 2, 3, 4$ are three excited states. In the Fig. 10(bottom), we show the radial wave profiles of the exact vortex solution (44) at $t = 0$. It is clear to see that the number of ring structure of vortex solution increases by one with changing the radial quantum number $n$ by one, which is similar to the quantum states of harmonic oscillator.

It has been shown that attractive Bose condensates like $^{85}$Rb and $^7$Li become mechanically unstable and collectively collapse [2, 26] when the number of atoms in the condensate exceeds critical value $N_c$. So it is important to produce the stable states in attractive Bose condensates. In order to elucidate the dynamical stability of the exact vortex solutions proposed above, we conduct numerical experiments by solving Eq. (37) and take the exact vortex solutions (44) and (45) at $t = 0$ as initial data. To begin with, we consider the attractive rotating BEC with harmonic potential at $\epsilon = 0$ in (46), which has exact vortex solution (44). In Fig. 11, we show the density evolutions and phase diagrams of vortex solution (44) as initial condition with radial quantum number $n = 1$ and different topological charge $S$ or angular momentum quantum numbers based on numerical simulation of Eq. (37). It is shown that only when topological charge $S = 1$, vortex solution (44) is stable against perturbation with an initial Gaussian noise of level 0.5%, but for topological charge $S \geq 2$, giant vortex solution (44) will be unstable and split into single charge vortices and so destruct the ring structures.

Finally, we consider the repulsive rotating BEC in anharmonic potential $V(r, t) = \omega^2(t)r^2/2 - \mu_0 R_0^2/2$ with $\mu_0 \neq 0$, which has exact vortex solution (45). In Fig. 12 and 13, we demonstrate the density evolutions and phase diagrams of vortex solution (45) as initial condition with different radial quantum numbers $n = 1, 2$ and fixed angular momentum quantum numbers $S = 1$ and $S = 5$, respectively. It is very interesting to note that when the radial quantum number $n = 1$, the exact vortex solution (45) is always stable even for very large topological charge $S = 5$, which is very different from the attractive rotating BEC with harmonic potential where a stable region for vortex solution (44) was found only for $S = 1$ as shown in Fig. 11. Our results have provided a very promising method for stabilizing the giant vortex having very large topological charge $S \geq 2$ which has been conjectured unstable [36] by tuning the external potential and nonlinear interaction simultaneously in time. Numerical simulation shows that for the radial quantum number $n > 1$, the vortex solution (45) is always unstable for any topological charge $S$.

IV. CONCLUSIONS

In summary, we have reviewed the recent studies of the dynamical properties of BECs with spatially and spatiotemporally modulated nonlinearities in various external potential. Some exact localized nonlinear matter wave
solutions for the one-dimensional and two-dimensional Gross-Pitaevskii equations are obtained explicitly by similarity transformation. It is interesting to see that the spatially or spatiotemporally modulated nonlinearity can support stable novel localized nonlinear matter waves defined by quantum numbers. In real experiments, the spatially and spatiotemporally modulated nonlinearities can be realized by optical Feshbach resonance approach proposed by Yamazaki et al. [30]. In addition, the dynamics of the 3D BECs with spatially and spatiotemporally modulated nonlinearities is still open and we will study this issue in the near future. We hope that these results will stimulate further research on exploring localized nonlinear matter waves in future BEC experiments and help to understand the behavior of nonlinear excitation in physical systems with spatially and spatiotemporally modulated interactions.

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