Packing nearly optimal Ramsey $R(3, t)$ graphs

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Joint work with Lutz Warnke
Construct pseudo-random triangle-free subgraphs of dense graphs

- Previous results only make such construction in complete graphs
- Construct via polynomial time randomized algorithm
  - Self-stabilization mechanism built into algorithm to control errors
- Approximately decompose complete graph into such Δ-free graphs
  - Solve a Ramsey theory conjecture by Fox, Liebenau, Person, Szabo et al
Erdős (1961) + Spencer (1977) + Krivelevich (1994)

All find an $n$-vertex graph $G \subseteq K_n$ such that $G$ is $\Delta$-free with independence number $\alpha(G) \leq C\sqrt{n \log n}$

- Construct $G$ in the binomial random graph $G_{n,p}$

Kim (1995) + Bohman (2008): one nearly optimal $R(3, t)$ graph

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- Construct $G$ by (semi-random variation of) $\Delta$-free process: greedily add random edges that do not create a $\Delta$
Review of previous results

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$\Delta$-free process: add one random edge in each step
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![Diagram of a graph showing open and closed edges]
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[Diagram with edges labeled open (can add) and closed (can not add)]
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**Semi-random variation:** add many random edges in each step

![Graph representation](image)
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- Construct \( G \) by (semi-random variation of) \( \Delta \)-free process:
  greedily add random edges that do not create a \( \Delta \)
  * Tight up to the constant: Ajtai-Komlós-Szemerédi (1980)
  * Lead to the right order of magnitude of Ramsey number \( R(3, t) \)
    - Kim received Fulkerson Prize in 1997
Main Result: nearly optimal $R(3, t)$ graphs

Kim (1995) + Bohman (2008): one nearly optimal $R(3, t)$ graph

Both find an $n$-vertex graph $G \subseteq K_n$ such that $G$ is $\Delta$-free with independence number $\alpha(G) \leq C\sqrt{n \log n}$

- Using (semi-random variation of) $\Delta$-free process:
  greedily add random edges that do not create a $\Delta$

G., Warnke (2020): almost packing of nearly optimal $R(3, t)$ graphs

Given $\varepsilon > 0$, we find edge-disjoint graphs $(G_i)_{i \in I}$ with $G_i \subseteq K_n$ such that
(a) each $G_i$ is $n$-vertex $\Delta$-free with $\alpha(G_i) \leq C\varepsilon\sqrt{n \log n}$
(b) the union of the $G_i$ contains $\geq (1 - \varepsilon)\binom{n}{2}$ edges

- Using simple polynomial-time randomized algorithm:
  sequentially choose $G_i$ via semi-random variation of $\Delta$-free process
  - Start with $H_0 = K_n$
  - Find $G_i \subseteq H_i$ and set $H_{i+1} = H_i \setminus G_i$ and repeat
Glimpse of the proof

**Main-Technical-Result:** find pseudo-random $\Delta$-free subgraph $G \subseteq H$

Let $\varrho := \sqrt{\beta (\log n) / n}$ and $s := C\varepsilon \sqrt{n \log n}$. If $H \subseteq K_n$ is such that

$$e_H(A, B) \geq \varepsilon |A||B|$$

for all disjoint sets $A, B$ of size $s$, then we can find $\Delta$-free $G \subseteq H$ with

$$e_G(A, B) = (1 \pm \delta) \varrho e_H(A, B)$$

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- Pseudo-randomness of $G_i$ ensures all local parts of $H_i$ behave similarly

**Implies packing result:**
- Start with $H_0 = K_n$
- Sequentially choose $G_i \subseteq H_i$ and set $H_{i+1} = H_i \setminus G_i$
  $$e_{H_i}(A, B) = (1 - (1 \pm \delta)\varrho)^i |A||B|$$
- Stop when $e_{H_i}(A, B) \approx \varepsilon |A||B|$ holds
Main-Technical-Result: find pseudo-random \( \Delta \)-free subgraph \( G \subseteq H \)

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e_G(A, B) = (1 \pm \delta) \varrho e_H(A, B)
\]

for all disjoint \( A, B \) of size \( s \).

Proof based on semi-random variation of \( \Delta \)-free process:

- Do not require degree/codegree regularity of \( H \)
- ‘Self-stabilization’ mechanism built into process (to control errors)
- Tools: Bounded-Differences-Ineq. and Upper-Tail-Ineq. of Warnke
Summary

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Remarks
- Can find the $(G_i)_{i \in I}$ via polynomial time randomized algorithm
- Applications in Ramsey theory: solve a conjecture of Fox, Liebenau, Person, Szabo et al

Open problem
Other applications of ‘self-stabilization’ in design of randomized algorithms?

Reference
He Guo, Lutz Warnke, *Packing nearly optimal Ramsey $R(3, t)$ graphs*, Combinatorica 40, 63–103 (2020)