The Role of $\Delta(1232)$ in Two-pion Exchange
Three-nucleon Potential

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Abstract

In this paper we have studied the two-pion exchange three-nucleon potential
$(2\pi E - 3NP)$ using an approximate $SU(2) \times SU(2)$ chiral symmetry of the
strong interaction. The off-shell pion-nucleon scattering amplitudes obtained
from the Weinberg Lagrangian are supplemented with the contributions from the
well-known $\sigma$-term and the $\Delta(1232)$ exchange. It is the role of the $\Delta$-resonance
in $2\pi E - 3NP$, which we have investigated in detail in the framework of the
Lagrangian field theory. The $\Delta$-contribution is quite appreciable and, more
significantly, it is dependent on a parameter $Z$ which is arbitrary but has the
empirical bounds $|Z| \leq 1/2$. We find that the $\Delta$-contribution to the important
parameters of the $2\pi E - 3NP$ depends on the choice of a value for $Z$, although
the correction to the binding energy of triton is not expected to be very sensitive
to the variation of $Z$ within its bounds.

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1 Introduction

It is well-known that two-nucleon potentials are not always adequate in explaining nuclear properties. For example, all realistic two-body potentials which fit the two-nucleon data quite well, fail to reproduce the binding energy of triton [1, 2]. The experimental binding energy of \(^3H\) is 8.48 MeV, while calculations with the well-known two-body local potentials fall short by 0.5 – 1.25 MeV. An obvious attempt to overcome the deficiency is to include the three-nucleon potential (3NP) in binding energy calculations. Computational techniques for trinuclear systems with the inclusion of three-nucleon potentials have become sufficiently mature to make such attempts worthwhile [1, 3]. Because of the short-range two-body repulsion between the nucleons tending to keep them apart, we expect that the two-pion exchange three-nucleon potential \((2\pi E - 3NP)\) will have a larger effect than the relatively shorter range contributions to the 3NP due to the exchange of heavier mesons.

To construct the \(2\pi E - 3NP\) we need the pion-nucleon scattering amplitudes with the pions off-mass-shell. An important mechanism in \(\pi N\) scattering is the formation of the \(\Delta\)-resonance. We study the effect of the \(\Delta(1232)\) by considering the most general form of the \(\pi N\Delta\) interaction Lagrangian [4] which has been applied extensively in low energy \(\pi N\) scattering [5 - 8] and photo- and electro-production of pions [9]. This Lagrangian contains a parameter \(Z\) whose value is arbitrary. However, low energy phenomenology [5-9] constrains \(Z\) to lie between \(-1/2\) and \(1/2\). The other pieces of the effective \(\pi N\) interaction Lagrangian have been obtained from the nonlinear chiral Lagrangian of Weinberg [10]. The Weinberg Lagrangian incorporates the nucleon-exchange effects in \(\pi N\) scattering and, in addition, there is either a direct \(\pi\pi NN\) interaction, or \(\pi N\) scattering via \(\rho\)-exchange. Furthermore, we have added in the pion-nucleon \(\sigma\)-amplitude, \(A_\sigma^{(+)}\), parametrized in an appropriate manner, to account for some well-known constraints [11 - 13] in the scattering amplitude \(A^{(+)}\), which follow from Current Algebra and Partial Conservation of Axial-vector Current. The parameters of \(A_\sigma^{(+)}\) have been adjusted by using the recent information on the amplitude \(F^{(+)}\) in the subthreshold region, obtained by analyzing the data from meson factories [14]. The model for pion-nucleon interaction so constructed is also compatible with low-energy \(\pi N\) data.

The two-pion exchange three-nucleon potential constructed from our model of \(\pi N\) interaction is dominated by the \(\Delta\)-resonance and hence depends on \(Z\). The nonlinear realization of chiral symmetry proposed by Weinberg [10] leads to a pseudovector \(\pi NN\) coupling which does not contribute to the \(2\pi E - 3NP\) in the appropriate non-relativistic limits. The contribution to the \(2\pi E - 3NP\) from the direct \(\pi\pi NN\) interaction or from the \(\rho\)-exchange is small compared to the contribution from \(\Delta\)-exchange.

Our purpose in this work is to examine in detail whether the parameter \(Z\) in the \(\pi N\Delta\) interaction Lagrangian introduces appreciable \(Z\)-dependence in the three-nucleon potential and, consequently, in the calculations of physical quantities like the binding energy of triton. The three-nucleon potential obtained from our model is of the same form as the Tucson-Melbourne (TM) potential [15] or the Brazil poten-
tional [10], which contains four parameters a, b, c and d. In the present case b and d are functions of \( Z \). The parameter b, which gives the dominant contribution to the binding energy correction of triton, is not very sensitive to \( Z \), although the \( \Delta \)-contribution to b, \( b_\Delta \), varies appreciably with \( Z \). The reason for the insensitivity of b to variations of \( Z \) is that, in our model, the amplitude \( A_\sigma^+ \) is also indirectly \( Z \)-dependent through the slope parameter \( \sigma' \) (Sec. 3.3). The \( \Delta \)-exchange and the amplitude \( A_\sigma^+ \) both contribute to the parameter b of the three-nucleon potential and their resultant contribution is such that b is more or less independent of \( Z \). On the other hand, the parameter d varies appreciably with \( Z \). However, the contribution from d to the binding energy correction \( B_3 \) of triton has been found to be much smaller compared with that from b [1, 3]. Therefore, the calculation of \( B_3 \) is not likely to be quite sensitive to the variation of \( Z \) as long as \( |Z| \leq 1/2 \).

The plan of the remaining portion of the paper is as follows: in Sec. 2 we review briefly the derivation of the \( 2\pi E - 3NP \) using the pion-nucleon off-shell scattering amplitudes as input, in Sec. 3 we discuss our model for pion-nucleon scattering and evaluate the nonrelativistic reduction of the amplitudes which are to be used in the \( 2\pi E - 3NP \). Finally, a detailed discussion of the results are given in Sec. 4.

2 Two-pion exchange three-nucleon potential

A three-nucleon potential means an irreducible potential energy function of the coordinates of the three nucleons — irreducible in the sense that the function cannot be written as a sum of functions involving fewer coordinates. The Feynman diagram corresponding to \( 2\pi E - 3NP \) is shown in figure 1. The amplitude for the process shown in figure 1 can be written as

\[
\langle p'_1p'_2p'_3 | S - 1 | p_1p_2p_3 \rangle = -i\delta^4 (P - P') \frac{1}{(2\pi)^5} \sqrt{\frac{m^6}{p_{10}p_{20}p_{30}p'_{10}p'_{20}p'_{30}}} T^{3N}_{123},
\]

where

\[
T^{3N}_{123} = \left[ \bar{u}(p'_2) \gamma^\mu q_\mu \gamma_5 \tau_a u(p_2) \right] \frac{f/\mu}{q^2 - \mu^2} \left\{ T^{ba}_{\pi N} \right\} \frac{f/\mu}{q'^2 - \mu^2} \left[ \bar{u}(p'_3) \gamma^\nu q'_\nu \gamma_5 \tau_b u(p_3) \right].
\]

The pseudovector coupling for the \( \pi NN \) vertex has been chosen in conformity with the results of the nonlinear realization of the chiral \( SU(2) \times SU(2) \) symmetry for pion-nucleon interaction [10].

In the expressions (1) and (2) \( P \) and \( P' \) are the total four momenta before and after scattering; \( q = p_2 - p'_2 \) and \( q' = p'_3 - p_3 \); a and b are the isospin indices of the pion, and \( \mu \) is the mass. The off-shell pion-nucleon T-matrix \( T^{ba}_{\pi N} \) describes the scattering process

\[
\pi^a(q) + N(p_1) = \pi^b(q') + N(p'_1).
\]
The pion-nucleon T-matrix is related to the S-matrix through the relation

\[ <q'p'_1 | S_{\pi N}^{\pi N} - 1 | qp> = -i(2\pi)^4\delta^4(q+p_1-q'-p'_1) \sqrt{\frac{m^2}{p_10p'_10q_0q'_0}} T^{ba}_{\pi N}(\nu, t, q^2, q'^2), \]

where \( \nu \) and \( t \) are defined as

\[ \nu = \frac{(q+q')(p_1+p'_1)}{4m}, \]
\[ t = (q-q')^2. \]

The quantity \( \nu \) is related to the Mandelstam variables \( s = (q+p_1)^2 \) and \( u = (p_1-q')^2 \) by \( \nu = (s-u)/4m \). Now the T-matrix \( T^{ba}_{\pi N} \) has the general isospin decomposition

\[ T^{ba}_{\pi N} = \bar{u}(p_1') \left\{ \left[ A^{(+)} + \frac{1}{2}(q' + q') B^{(+)} \right] \delta_{ba} + \left[ A^{(-)} + \frac{1}{2}(q' + q') B^{(-)} \right] i\epsilon_{bac} \tau_c \right\} u(p_1), \]

where \( A^{(\pm)} \) and \( B^{(\pm)} \) are the isospin-even(+) and the isospin-odd(-) invariant amplitudes. An alternative isospin decomposition of \( T^{ba}_{\pi N} \) is

\[ T^{ba}_{\pi N} = \bar{u}(p_1') \left\{ \left( F^{(+)} - \frac{1}{4m}(q' + q') B^{(+)} \right) \delta_{ba} + \left( F^{(-)} - \frac{1}{4m}(q' + q') B^{(-)} \right) i\epsilon_{bac} \tau_c \right\} u(p_1), \]

where

\[ F^{(\pm)} = A^{(\pm)} + \nu B^{(\pm)}. \]
Now, since the potential is a nonrelativistic concept, we need to take the nonrelativistic limit of Eq. (1) to define a three-nucleon potential. In the theory of nonrelativistic potential scattering the S-matrix and the T-matrix are related by

\[
\langle \vec{p}'_1 \vec{p}'_2 \vec{p}'_3 | s - 1 | \vec{p}_1 \vec{p}_2 \vec{p}_3 \rangle = -2\pi i \delta(E - E') \langle \vec{p}'_1 \vec{p}'_2 \vec{p}'_3 | t | \vec{p}_1 \vec{p}_2 \vec{p}_3 \rangle. \tag{10}
\]

In the first approximation the t-matrix in Eq. (10) can be equated to the three-nucleon potential \( W \). The nonrelativistic reduction of Eq. (1) can be compared to Eq. (10) to obtain the 3NP. Thus we find

\[
\langle \vec{p}'_1 \vec{p}'_2 \vec{p}'_3 | W(123) | \vec{p}_1 \vec{p}_2 \vec{p}_3 \rangle \approx -\frac{1}{(2\pi)^6} \delta^3(\vec{P} - \vec{P}') t_{123}^{3N}, \tag{11}
\]

where \( t_{123}^{3N} \) is the nonrelativistic reduction of \( T_{123}^{3N} \). Taking the appropriate limit, we finally obtain

\[
\langle \vec{p}'_1 \vec{p}'_2 \vec{p}'_3 | W(123) | \vec{p}_1 \vec{p}_2 \vec{p}_3 \rangle = -\frac{1}{(2\pi)^6} \delta^3(\vec{P} - \vec{P}') \left( f^+(\vec{q}^2) \frac{H(\vec{q}^2)}{\vec{q}^2 + \mu^2} H(\vec{q}'^2) \frac{H(\vec{q}'^2)}{\vec{q}'^2 + \mu^2} (\vec{\sigma}_2 \cdot \vec{q}) (\vec{\sigma}_3 \cdot \vec{q}') \tau_a \tau_b \tau_c \right) \times
\]

\[
\left\{ f^+(\vec{q}) \frac{1}{2m} \vec{\sigma}_1 \cdot (\vec{q} \times \vec{q}') b^+(\vec{q}) \right\} \delta_{ba} + \left\{ f^-(\vec{q}) \frac{1}{2m} \vec{\sigma}_1 \cdot (\vec{q} \times \vec{q}') b^-(\vec{q}) \right\} i \epsilon_{bac} \tau_c. \tag{12}
\]

where \( f^+(\vec{q}) \) and \( b^+(\vec{q}) \) are the nonrelativistic limits of \( F^+(\vec{q}) \) and \( B^+(\vec{q}) \); \( H(\vec{q}^2) \) and \( H(\vec{q}'^2) \) refer to the form factors which are introduced because the pions are off-shell. We take \( H(\vec{q}^2) \) as

\[
H(\vec{q}^2) = \left( \frac{\Lambda^2 - \mu^2}{\Lambda^2 + \vec{q}^2} \right)^2. \tag{13}
\]

3 Model for pion-nucleon interaction

3.1 The Weinberg Lagrangian

We begin with the Weinberg Lagrangian [10] which is based on a nonlinear realization of the chiral \( SU(2) \times SU(2) \) symmetry. The interaction Lagrangian relevant for pion-nucleon scattering can be written as

\[
\mathcal{L}_W = \mathcal{L}_{\pi NN} + \mathcal{L}_{\pi \pi NN} \tag{14}
\]

where

\[
\mathcal{L}_{\pi NN} = \left( \frac{f}{\mu} \right) \bar{\psi} \gamma_5 \gamma^\mu \tau_i \psi \partial_\mu \phi_i, \tag{15}
\]

\[
\mathcal{L}_{\pi \pi NN} = \left( \frac{i}{4f_\pi^2} \right) (\bar{\psi} i \gamma^\mu \tau_3 \psi) \epsilon_{ijk} \phi_j \partial_\mu \phi_k. \tag{16}
\]

Here \( \psi \) and \( \phi \) are the nucleon and the pion fields, \( f_\pi = 92.6 \text{ MeV} \) [17] is the pion decay constant and \( \mu \) the mass of the pion. The interaction Lagrangian \( \mathcal{L}_W \) consists of the usual derivative pion-nucleon coupling (Eq. [15]) and a direct interaction between a pion and a nucleon (Eq. [16]).
3.2 The $\pi N \Delta$ interaction Lagrangian

The most general form of the interaction Lagrangian $\mathcal{L}_{\pi N \Delta}$ can be written in the form \[4\]
\begin{align*}
\mathcal{L}_{\pi N \Delta} &= \frac{1}{\sqrt{2}} \left( f^* \mu \right) \left[ i\overline{\Psi}_\mu \Theta^{\mu\nu} T_i \psi \partial_\nu \Phi_i + h.c. \right], \\
\Theta_{\mu\nu} &= \left\{ g_{\mu\nu} + \left[ \frac{1}{2} (1 + 4Z) A + Z \right] \gamma_\mu \gamma_\nu \right\},
\end{align*}
(17)
where $\Psi_\mu$ is the Rarita-Schwinger field and the T’s are a set of matrices corresponding to the isospin-$\frac{3}{2}$. The propagator for the $\Delta(1232)$ is written as
\begin{equation}
\langle 0 | T(\psi_\mu(x) \overline{\psi}_\nu(y)) | 0 \rangle = i d_{\mu\nu}(\partial) \Delta_F(x - y)
\end{equation}
(19)
where
\begin{align*}
d_{\mu\nu}(\partial) &= \left( i \gamma^\lambda \partial_\lambda + M \right) \left[ g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{1}{3M} (\gamma_\mu i\partial_\nu - \gamma_\nu i\partial_\mu) + \frac{2}{3M^2} \partial_\mu \partial_\nu \right] + \\
&\frac{1}{3M^2} \left( \frac{A+1}{2A+1} \right) \left\{ \left[ -\frac{1}{2} \left( \frac{A+1}{2A+1} \right) i\gamma^\lambda \partial_\lambda + \left( \frac{A}{2A+1} \right) M \right] \gamma_\mu \gamma_\nu - \\
&\gamma_\mu i\partial_\nu - \left( \frac{A}{2A+1} \right) \gamma_\nu i\partial_\mu \right\} (\Box + M^2)
\end{align*}
(20)
and
\begin{equation}
\Delta_F(x - y) = \frac{1}{(2\pi)^4} \int d^4p \frac{\exp[-ip(x - y)]}{p^2 - M^2 + i\epsilon}.
\end{equation}
(21)
The interaction Lagrangian $\mathcal{L}_{\pi N \Delta}$ depends on two parameters $A$ and $Z$. The parameter $A$, which occurs also in the propagator, can assume any value except $-1/2$ \[4\]. However, $A$ drops out from the final expressions of the scattering amplitudes which therefore depend on $Z$ only. There is no consensus on the exact value of $Z$, although $Z = 1/2$ is preferred theoretically \[4\]. From phenomenological studies a reliable bound,
\begin{equation}
|Z| \leq \frac{1}{2},
\end{equation}
(22)
can be placed on the value of $Z$ \[5 - 9\]. The value of the $\pi N \Delta$ coupling constant is taken as $f^* / 4\pi = 0.3359$. As the $\Delta(1232)$ makes the dominant contribution to the $2\pi E - 3NP$, and as there is some confusion regarding the magnitude of the $\Delta(1232)$ contribution, we shall discuss in detail this aspect of the problem in section 4.

3.3 The pion-nucleon $\sigma$-term

Current Algebra and PCAC impose certain constraints \[11 - 13\] on the isospin-even invariant amplitude $A^{(+)}$ at some unphysical values of the kinematical variables. To
satisfy these constraints and to account for the empirical information on the low-energy \( \pi N \) scattering, we include in our calculations an additional amplitude \( A_{\sigma}^{(+)} \), called the pion-nucleon \( \sigma \)-amplitude which is parametrized [6 - 8, 18] as follows:

\[
A_{\sigma}^{(+)}(\nu, \nu_B) = \frac{\sigma_{NN}(t = 2\mu^2)}{f_\pi^2} \left[ \frac{q^2 + q'^2 - \mu^2}{\mu^2} + \frac{\sigma'(4m\nu_B)}{\mu^2} \right],
\]

(23)

where

\[ \nu = (s - u)/4m, \quad \nu_B = (t - q^2 - q'^2)/4m; \]

\( s, t \) and \( u \) are the Mandelstam variables, \( q \) and \( q' \) are the momenta of the incoming and the outgoing pion respectively.

Recently the pion-nucleon scattering amplitudes in the subthreshold region have been recalculated by using the meson factory \( \pi N \) data and dispersion relation [14]. The important results relevant for our discussion are

\[
\bar{F}^{(+)}(\nu = 0, t = 2\mu^2, q^2 = \mu^2, q'^2 = \mu^2) \approx 1.35 \mu^{-1}
\]

\[
\bar{F}^{(+)}(\nu = 0, t = \mu^2, q^2 = \mu^2, q'^2 = \mu^2) \approx -0.08 \mu^{-1}
\]

(24)

which yield

\[
\sigma_{NN}(t = 2\mu^2) = 82 \text{ MeV}
\]

(25)

and

\[
\sigma' = \begin{cases} 
0.66 & \text{for } Z = 1/2 \\
0.50 & \text{for } Z = 1/4 \\
0.40 & \text{for } Z = 0 \\
0.36 & \text{for } Z = -1/4 \\
0.37 & \text{for } Z = -1/2 
\end{cases}
\]

(26)

The amplitude \( \bar{F}^{(+)} \) is the remainder of \( F^{(+)} \) after the (pseudovector) nucleon Born terms have been subtracted from it. It may be noted here that while the value of \( \sigma' \) is sensitive to the choice of the parameter \( Z \) in the \( \pi N \Delta \) interaction Lagrangian, \( \sigma_{NN}(t = 2\mu^2) \) is independent of \( Z \).

### 3.4 Nonrelativistic limits of pion-nucleon scattering amplitudes

The contributions to the pion-nucleon invariant amplitudes \( A^{(\pm)} \) and \( B^{(\pm)} \) due to nucleon-exchange, \( \Delta \)-exchange and direct \( \pi \pi NN \) interaction can be easily calculated and are quoted in different places [12, 19]. For our purpose we need only the nonrelativistic reductions \( f^{(\pm)} \) and \( b^{(\pm)} \) of the amplitudes \( F^{(\pm)} = A^{(\pm)} + \nu B^{(\pm)} \) and \( B^{(\pm)} \) respectively.

First, consider the nucleon-exchange contribution to pion-nucleon scattering. The invariant amplitudes \( A_N^{(\pm)} \) and \( B_N^{(\pm)} \) consists of the forward propagating Born term (FPBT) and the backward propagating Born term (BPBT). The FPBT is already
accounted for as the iterate of two-nucleon one-pion exchange potential. Therefore the FPBT has to be subtracted from the invariant amplitudes. If we take the nonrelativistic limit of what remains we obtain

\[ f_N^{(\pm)} = 0, \quad b_N^{(\pm)} = 0. \tag{27} \]

We thus see that the nucleon-exchange contribution to the \( \pi N \) amplitudes does not contribute to the \( 2\pi E - 3NP \). The results in Eq. (27) are correct only if we use the gradient coupling for \( \pi NN \) interaction.

For the \( \Delta \)-contribution to the \( \pi N \) amplitudes, we find

\[
F_\Delta^{(+)} \rightarrow f_\Delta^{(+)} = \alpha_\Delta^{(+)} \vec{q} \cdot \vec{q}',
\]

\[
F_\Delta^{(-)} \rightarrow f_\Delta^{(-)} = 0,
\tag{28}
\]

and

\[
B_\Delta^{(+)} \rightarrow b_\Delta^{(+)} = 0,
\]

\[
B_\Delta^{(-)} \rightarrow b_\Delta^{(-)} = \beta_\Delta^{(-)}(Z),
\tag{29}
\]

where

\[
\alpha_\Delta^{(+)} = \left( 2f^2 / 9\mu^2 \right) \left[ \frac{(4M^2 - Mm + m^2)}{(M - m)M^2} - \frac{4(M + m)Z}{M^2} - \frac{4(2M + m)Z^2}{M^2} \right]. \tag{30}\]

and

\[
\beta_\Delta^{(-)} = \left( f^2 / 9\mu^2 \right) \left[ \frac{2m(2M^2 + Mm - m^2)}{(M - m)M^2} + \frac{8m(M + m)Z}{M^2} + \frac{8m(2M + m)Z^2}{M^2} \right]. \tag{31}\]

Here \( M = 1232 \text{ MeV} \) is the mass of \( \Delta(1232) \) and \( m = 938.9 \text{ MeV} \) is the nucleon mass. The quantities \( \alpha_\Delta^{(+)} \) and \( \beta_\Delta^{(-)} \) are obtained from the expressions for the \( \Delta \)-contribution to the amplitudes \( A^{(\pm)} \) and \( B^{(\pm)} \) as given in Ref. [19]. These results (Eqs. 30, 31) are the same as derived earlier by Coelho, Das and Robilotta [16]. However, they chose \( Z = -1/2 \) for the detailed discussions on the \( \Delta \)-contribution to the three-nucleon potential.

Next, for the direct \( \pi\pi NN \) interaction, only the amplitude \( B_d^{(-)} = 1/2 f_\pi^2 \) is nonzero. We therefore have

\[ b_d^{(+)} = 0, \quad b_d^{(-)} = 1/2 f_\pi^2, \tag{32} \]

and

\[ f_d^{(+)} = 0, \quad f_d^{(-)} = \nu / 2 f_\pi^2 \approx 0. \tag{33} \]
Finally, the nonrelativistic limit of the $\sigma$-contribution to the $\pi N$ amplitude is also simple. We have

$$f^{(+)}_\sigma = a^{(+)}_\sigma = \frac{\sigma_{NN}(t = 2\mu^2)}{f^2_\pi} \left[-1 + \frac{2\sigma' \cdot \vec{q}' \cdot \vec{q}'}{\mu^2} - \frac{(\vec{q}'^2 + \vec{q}'')^2}{\mu^2}\right]. \quad (34)$$

Since the pion is off-shell, each of the pion-nucleon amplitudes have to be multiplied by the form factor given in Eq. (13). Now, inserting the separate contributions to $f^{(+)}$ and $b^{(\pm)}$ in Eq. (12) we can write the $2\pi E - 3NP$ as

$$< \vec{p}_1' \vec{p}_2' \vec{p}_3' \mid W(123) \mid \vec{p}_1 \vec{p}_2 \vec{p}_3 > = \frac{1}{(2\pi)^6} \delta^3(\vec{p} - \vec{p}') \left( \frac{f}{\mu} \right)^2 \left( \frac{H(\vec{q}^2)}{(\vec{q}^2 + \mu^2)} + \frac{H(\vec{q}'^2)}{(\vec{q}'^2 + \mu^2)} \right) (\vec{\sigma}_2 \cdot \vec{q}) (\vec{\sigma}_3 \cdot \vec{q}') \times$$

$$\left\{ \vec{\tau}_2 \cdot \vec{\tau}_3 \left[ a + b \cdot \vec{q}' + c (\vec{q}'^2 + \vec{q}'') \right] - d (\vec{\tau}_1 \cdot \vec{\tau}_2 \times \vec{\tau}_3)(\vec{\sigma}_1 \cdot \vec{q} \times \vec{q}') \right\}, \quad (35)$$

where

$$a = \sigma_{NN}(t = 2\mu^2)/f^2_\pi$$

$$b = -\alpha^{(+)}_\Delta(Z) - 2\sigma' \sigma_{NN}(t = 2\mu^2)$$

$$c = \frac{\sigma_{NN}(t = 2\mu^2)}{f^2_\pi \mu^2}$$

$$d = -\frac{\beta^{(-)}_\Delta(Z)}{2m} = \frac{1}{4mf^2_\pi}. \quad (36)$$

The $2\pi E - 3NP$ given in Eq. (35) is of the same form as that derived by the Tucson-Melbourne (TM) group [15] except that the coefficients $a, b, c$ and $d$ in the TM potential are:

$$a = 1.130 \mu^{-1}$$

$$b = -2.580 \mu^{-3}$$

$$c = 1.000 \mu^{-3}$$

$$d = -0.753 \mu^{-3}. \quad (37)$$

### 4 Results and conclusions

The parameters $a$, $b$, $c$ and $d$ in Eqs. (36) receive contributions from $A^{(+)}_\sigma$, the $\Delta$-exchange and the direct term for $\pi N$ scattering. The $\Delta$-exchange contributes to $b$ and $d$, while the direct $\pi N$ interaction only to $d$. The parameters $a$ and $c$ receive contributions from $A^{(+)}_\sigma$ only; $A^{(+)}_\sigma$ contributes also to $b$.

In order to show the relative importance of the various contributions, we refer to table 1 where we have shown the values of $a$, $b$ and $d$ corresponding to five different values of $Z$, namely $Z = 1/2, 1/4, 0, -1/4$ and $-1/2$. Note that $a$ and $c$ are independent of $Z$, but $b$ and $d$ are not. Regarding the parameter $b$, as $Z$ is decreased from $1/2$ to $-1/4$, the contribution $b_\Delta$ from the $\Delta$-exchange decreases, while the contribution
$b_\sigma$ from $A^{(+)}$ increases. However, this trend is reversed somewhat at $Z = -1/2$. As a result, the net $b$ is not very sensitive to the variation of $Z$ within its acceptable bounds, $|Z| \leq 1/2$. Note that the $Z$-dependence of $b_\sigma$ is due to $\sigma'$ which depends on $Z$ (Eq. 26). Next, the parameter $d$ receives a small $Z$-independent contribution from the direct $\pi N$ scattering term, while the dominant contribution to $d$ comes from the $\Delta$-exchange which depends on $Z$. The value of $d$ increases steadily as $Z$ is decreased from 1/2 to -1/4, then it decreases slightly at $Z = -1/2$ (table 1).

Also shown in table 1 is the ratio $b_\Delta/d_\Delta$ which ranges from 1.26 to 4.0 as $Z$ is varied from 1/2 to -1/2. This contradicts the often quoted result that $b_\Delta/d_\Delta$ should be equal to four as a rule [20]. In our calculations this ratio is four only if $Z = -1/2$. However, there is no a priori justification for choosing this value of $Z$. In fact, in the theory of spin-3/2 field [4, 7], $Z = -1/2$ corresponds to calculations with a $\pi N \Delta$ vertex and a $\Delta$-propagator taking the $\Delta$ on-mass-shell in both cases. More explicitly, if we take $A = -1$ in Eqs. (18, 20) and then $Z = -1/2$ in Eq. (18), the off-mass-shell parts of $\Delta$ are eliminated from both the propagator and the interaction Lagrangian. Peccei [21] obtained a special form for the interaction Lagrangian $\mathcal{L}_{\pi N \Delta}$ which would correspond to $Z = -1/4$ in our formalism. For this value of $Z$, the ratio $b_\Delta/d_\Delta$ is 4.23. However, if we take $Z = 1/2$, the theoretically preferred value [4], this ratio is 1.26, much smaller than 4. It has already been noted in section (3.2) that, while $A$ may be assigned any value except $A = -1/2$, the empirical bounds on $Z$ is $|Z| \leq 1/2$.

Our main purpose in this paper is to see in what way and to what extent the $Z$-dependence of the $\pi N \Delta$ interaction Lagrangian effects the two-pion-exchange three-nucleon potential and whether any sensitive $Z$-dependence is likely to appear in calculations of relevant physical quantities, for example, the binding energy of triton.

A first-order perturbation calculation for the correction $E_3$ to the energy of triton due to the Tucson-Melbourne potential with the parameters $a$, $b$, $c$ and $d$ as in Eqs. (37) was done by Ishikawa et al [3]. The zeroth-order triton wave function was obtained by solving the Faddeev equations with a variety of two-nucleon potentials. Ishikawa et al used the dipole form factor at the vertices with several values of the cut-off parameter $\Lambda$. For $\Lambda = 800$ MeV and the Reid soft-core two-nucleon potential they found that the contributions to $E_3$ from the individual terms of the TM potential corresponding to the parameters $a$, $b$, $c$, and $d$ (Eqs. 37) are 0.05 MeV, -0.97 MeV, 0.25 MeV and -0.22 MeV respectively. In the first-order calculations of Ishikawa et al, $E_3$ is linear in $a$, $b$, $c$ and $d$. Since the two-pion exchange three-nucleon potential in our model is of the same form as the TM potential, we can easily estimate the binding energy correction $B_3(= -E_3)$ for triton due to our $\pi \pi$-exchange three-nucleon potential, simply by scaling Ishikawa et al’s results (table 2). This is done solely to examine the possible $Z$-dependence of $B_3$, although a first-order perturbative calculation is not expected to be very accurate. We see that as $Z$ decreases from 1/2 to -1/2, $B_3^\Delta$ increases initially and then remains more or less constant for negative $Z$, while $B_3^\sigma$ decreases and then becomes negligible for $Z \leq 0$. Therefore, $B_3 = B_3^\Delta + B_3^\sigma + B_3^d$, is not very sensitive to the variation of $Z$. This conclusion would not change appreciably if the direct $\pi N$ interaction is replaced by the $\rho$-mediated interaction.

It may be noted here that the numerical values for $B_3$ will depend on the choice
Table 1: The parameters $a$, $b$, $c$ and $d$ in the two-pion exchange three-nucleon potential displayed for five values of $Z$ within the bounds $|Z| \leq 1/2$. Note that $a$ and $c$ are independent of $Z$.

| $a = a_\sigma$ | $c = c_\sigma$ | $Z$ | $b_{\Delta}$ | $b_{\sigma}$ | $b$ | $d_{\Delta}$ | $d_{d}$ | $d$ | $b_{\Delta}/d_{\Delta}$ |
|---------------|---------------|-----|-------------|-------------|-----|-------------|--------|-----|----------------|---|
| (\mu^{-1})    | (\mu^{-3})    |     | (\mu^{-3})  | (\mu^{-3})  |     | (\mu^{-3})  | (\mu^{-3}) |     |                  |
| 1.341         | 1.341         | 1/2 | -1.038      | -1.770      | -2.808 | -0.821      | -0.056   | -0.877 | 1.26             |
|               |               | 1/4 | -1.445      | -1.341      | -2.786 | -0.618      | -0.674   |        | 2.34             |
|               |               | 0   | -1.706      | -1.073      | -2.779 | -0.487      | -0.543   |        | 3.50             |
|               |               | -1/4| -1.820      | -0.966      | -2.786 | -0.430      | -0.486   |        | 4.23             |
|               |               | -1/2| -1.787      | -0.992      | -2.779 | -0.447      | -0.503   |        | 4.0              |

Table 2: The correction, $B_3 = B_{\Delta}^3 + B_{\sigma}^3 + B_{d}^3$, to the binding energy of triton due to the two-pion exchange three-nucleon potential. The contributions $B_{\Delta}^3$ and $B_{\sigma}^3$ are $Z$-dependent. The table also shows the binding energy $B_2$ of triton for the two-nucleon Reid soft-core potential, the total binding energy $B = B_2 + B_3$ and $B_{\text{exp}} - B$. All the binding energies are in MeV.

| $B_2$ | $B_2^\sigma$ | $Z$ | $B_{\Delta}^3$ | $B_{\sigma}^3$ | $B_3$ | $B_{\text{exp}}$ | $B_{\text{exp}} - B$ |
|-------|-------------|-----|---------------|---------------|-------|-----------------|------------------|
| 7.24  | 0.02        | 1/2 | 0.63         | 0.27          | 0.92  | 8.16            | 8.48             | 0.32             |
|       |             | 1/4 | 0.72         | 0.11          | 0.85  | 8.09            |                 | 0.39             |
|       |             | 0   | 0.78         | 0.01          | 0.81  | 8.05            |                 | 0.43             |
|       |             | -1/4| 0.81         | -0.03         | 0.80  | 8.04            |                 | 0.44             |
|       |             | -1/2| 0.80         | -0.02         | 0.80  | 8.04            |                 | 0.44             |
of the two-nucleon potential and the value of the cut-off parameter $\Lambda$. In particular, the binding energy correction is quite sensitive to $\Lambda$ [1, 3]. The dependence on $\Lambda$ may be reduced somewhat if one includes the $\rho\pi$-exchange, $\rho\rho$-exchange three-nucleon forces in addition to the $\pi\pi$-exchange three-nucleon force [4]. However, our purpose is to investigate in detail the effect of $\Delta(1232)$ on the parameters of the two-pion exchange three-nucleon potential. The resonance $\Delta(1232)$ makes large contributions to the parameters $b$ and $d$, and these contributions depend on $Z$. We find that $b_\Delta$ is sensitive to the variation of $Z$, although $b = b_\Delta + b_\sigma$ does not change appreciably with $Z$. The parameter $d$, however, depends substantially on $Z$.

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