Torus quotients of homogeneous spaces of the general linear group and the standard representation of certain symmetric groups

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Dedicated to Professor C De Concini on the occasion of his 60th birthday

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Abstract. We give a stratification of the GIT quotient of the Grassmannian $G_{2,n}$ modulo the normaliser of a maximal torus of $SL_n(k)$ with respect to the ample generator of the Picard group of $G_{2,n}$. We also prove that the flag variety $GL_n(k)/B_n$ can be obtained as a GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for a suitable choice of an ample line bundle on $GL_{n+1}(k)/B_{n+1}$.

Keywords. GIT quotient; line bundle; simple reflection.

0. Introduction

Let $k$ be an algebraically closed field. Consider the action of a maximal torus $T$ of $SL_n(k)$ on the Grassmannian $G_{r,n}$ of $r$-dimensional vector subspaces of an $n$-dimensional vector space over $k$. Let $N$ denote the normaliser of $T$ in $SL_n(k)$. Let $L_r$ denote the ample generator of the Picard group of $G_{r,n}$. Let $W = N/T$ denote the Weyl group of $SL_n(k)$ with respect to $T$.

In [7], it is shown that the semi-stable points of $G_{r,n}$ with respect to the $T$-linearised ample line bundle $L_r$ is same as the stable points if and only if $r$ and $n$ are co-prime.

In this paper, we describe all the semi-stable points of $G_{r,n}$ with respect to $L_r$. In this connection, we prove the following result:

First, we introduce some notation needed for the statement of the theorem.

Let $h_j$ be a Cartan subalgebra of $sl_{j+1}$, $P(h_j)$ be the projective space and $R_j \subseteq h_j^*$ be the root system. Let $V_j$ be the open subset of $P(h_j)$ defined by

$$V_j := \{ x \in P(h_j); \alpha(x) \neq 0, \forall \alpha \in R_j \}.$$

Here, the Weyl group of $sl_{j+1}$ is $S_{j+1}$, and $h_j$ is the standard representation of $S_{j+1}$.

With this notation, taking $m = \lceil \frac{n-1}{2} \rceil$ (for this notation, see Lemma 1.6) and $t = \lceil \frac{n-3}{2} \rceil$ we have the following.

**Theorem.** $N \backslash G_{2,n}^+ (L_2)$ has a stratification $\bigcup_{j=m+1}^{n} C_i$ where $C_0 = s_{m+1} \backslash P(h_m)$, and $C_i = s_{i+m+1} \backslash V_{i+m}$.

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On the other hand, the GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for any ample line bundle on $GL_{n+1}(k)/B_{n+1}$ and $GL_n(k)/B_n$ are both birational varieties. So, it is a natural question to ask whether the flag variety $GL_n(k)/B_n$ can be obtained as a GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for a suitable choice of an ample line bundle on $GL_{n+1}(k)/B_{n+1}$. We give an affirmative answer to this question. For a more precise statement, see Theorem 5.2. In this connection, we also prove that the action of the Weyl group $S_{n+1}$ on the quotient is given by the standard representation. For a more precise statement, see Corollary 5.4.

Section 1 consists of preliminary notation and some combinatorial lemmas about minuscule weights. Let $G$ be a reductive Chevalley group over an algebraically closed field $k$. Let $T$ be a maximal torus of the commutator subgroup $[G, G]$, $B$ a Borel subgroup of $G$ containing $T$ and $U$ be the unipotent radical of $B$. Let $N$ be the normaliser of $T$ in $[G, G]$. Let $W = N/T$ be the Weyl group of $[G, G]$ with respect to $T$ and $R$ denote the set of roots with respect to $T$, $R^+$ positive roots with respect to $B$. Let $U_α$ denote the one dimensional $T$-stable subgroup of $G$ corresponding to the root $α$ and let $S = \{α_1, \ldots, α_l\} \subseteq R^+$ denote the set of simple roots. For a subset $I \subseteq S$ denote $W^I = \{w \in W | w(α) > 0, α \in I\}$ and $W_I$ is the subgroup of $W$ generated by $s_I, α \in I$. Then every $w \in W$ can be uniquely expressed as $w = w_I.w_J$, with $w_I \in W_I$ and $w_J \in W_J$. We recall the notation $R(w) = \{α \in R^+: w(α) < 0\}$ from p. 142 of [10]. Let $w_0$ denote the longest element of $W$ with respect to $S$. Let $X(T)$ (resp. $Y(T)$) denote the set of characters of $T$ (resp. one parameter subgroups of $T$). Let $E_1 := X(T) \otimes \mathbb{R}, E_2 = Y(T) \otimes \mathbb{R}$. Let $\langle \cdot, \cdot \rangle: E_1 \times E_2 \rightarrow \mathbb{R}$ be the canonical non-degenerate bilinear form. Choose $λ_j$’s in $E_2$ such that $\langle α_i, λ_j \rangle = δ_{ij}$ for all $i$. Let the Weyl chamber corresponding to $B$ be denoted by $C$. We recall that $C := \{λ \in E_2 | \langle λ, α \rangle \geq 0 \forall α \in R^+\}$. For more details, see p. 64 of [1]. Also, we recall that for each $α \in R$, there is a homomorphism $SL_2 \xrightarrow{φ_α} G$ (see p. 19 of [2]). We have $\hat{α} : G_m \rightarrow G$ defined by $\hat{α}(t) = φ_α(t \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix})$. We also have $s_α(χ) = χ - ⟨χ, \hat{α}⟩α$ for all $α \in R$ and $χ \in E_1$. Set $s_i = s_{α_i}$ $∀ i = 1, 2, \ldots, l$. Let $\{ω_i : i = 1, 2, \ldots, l\} \subset E_1$ be the fundamental weights; i.e. $⟨ω_i, α_j⟩ = δ_{ij}$ for all $i, j = 1, 2, \ldots, l$. For a reference, see p. 180 of [1]. A dominant weight $χ$ is said to be minuscule if $⟨χ, \hat{α}⟩ \leq 1$ $∀ α \in R^+$. In this section, we prove the following elementary properties of minuscule weights:

Let $ω$ be a minuscule weight. Let $I := \{α \in S | ⟨ω, \hat{α}⟩ = 0\}$. Then, we have the following:

1. Let $α \in S$ and $τ \in W$ such that $l(s_α τ) = l(τ) + 1$ and $s_α τ \in W^I$, then $τ \in W^I$ with $s_α τ(ω) = τ(ω) - α$.
2. For any $w \in W^I$, the number of times $s_i$, $1 \leq i \leq n - 1$ appearing in a reduced expression of $w$ is equal to $(\text{coefficient of } α_i \text{ in } ω) - (\text{coefficient of } α_i \text{ in } w(ω))$ and hence it is independent of the reduced expression of $w$.
3. Let \( w \in W^I \) and let \( w = s_{i_1}s_{i_2}\ldots s_{i_k} \in W^I \) be a reduced expression. Then \( w(\omega) = \omega - \sum_{j=1}^{k} a_{i_j} \) and \( l(w) = ht(\omega - w(\omega)) \).

4. Let \( w = s_{i_1}s_{i_2}\ldots s_{i_k} \in W \) such that \( ht(\omega - s_{i_1}s_{i_2}\ldots s_{i_k}(\omega)) = k \) then \( w \in W^I \) and \( l(w) = k \).

5. There is a unique minimal element \( w \in W^I \) such that \( w(n\omega) \leq 0 \) for some positive integer \( n \).

6. There is a unique maximal element \( \tau \in W^I \) such that \( \tau(n\omega) \geq 0 \) for some positive integer \( n \).

For the details of proof of the above properties, see Lemma 1.3, Corollary 1.4(1), Corollary 1.4(2), Lemma 1.5, Corollary 1.9 and Corollary 1.10 respectively.

For notation in this section, we refer to [9].

**Lemma 1.1.** Let \( I \) be any nonempty subset of \( S \), and let \( \mu \) be a weight of the form \( \sum_{\alpha \in I} m_i \alpha_i - \sum_{\alpha \in S I} m_i \alpha_i \), where \( m_i \in \mathbb{Q} \) for all \( i \), \( 1 \leq i \leq l \); \( m_i > 0 \) for all \( \alpha_i \in I \) and \( m_i \geq 0 \) for all \( \alpha_i \in S \setminus I \). Then there is an \( \alpha \in I \) such that \( s_{\alpha}(\mu) < \mu \).

**Proof.** Since \( s_{\alpha}(\mu) = \mu - \langle (\alpha_i, \tilde{\alpha}) \rangle \alpha_i \) we need to find an \( \alpha \in I \) such that \( \langle (\mu, \tilde{\alpha}) \rangle > 0 \). This follows because the Cartan matrix \( (\langle \alpha_i, \tilde{\alpha} \rangle)_{i,j} \) is positive definite, so we can find an \( \alpha \in I \) such that \( \langle \sum_{\alpha \in I} m_i \alpha_i, \tilde{\alpha} \rangle > 0 \). Now we know that for any \( \alpha_i, \alpha_j \in S, i \neq j \), \( \langle \alpha_i, \tilde{\alpha}_j \rangle \leq 0 \). Hence, \( \langle \sum_{\alpha \in I} m_i \alpha_i, \tilde{\alpha} \rangle \leq 0 \) for this \( \alpha \in I \). Thus \( \langle (\mu, \tilde{\alpha}) \rangle > 0 \). This proves the lemma. \( \square \)

**Lemma 1.2.** Let \( \lambda \) be any dominant weight and let \( I = \{ \alpha \in S : \langle \lambda, \tilde{\alpha} \rangle = 0 \} \). Let \( w_1, w_2 \in W^I \) be such that \( w_1(\lambda) = w_2(\lambda) \). Then \( w_1 = w_2 \).

**Proof.** See [2] and [4]. \( \square \)

In the rest of this section, \( \omega \) will denote a minuscule weight and \( I := \{ \alpha \in S : \langle \omega, \tilde{\alpha} \rangle = 0 \} \).

**Lemma 1.3.** Let \( \alpha \in S \) and \( \tau \in W \) such that \( l(s_{\alpha} \tau) = l(\tau) + 1 \) and \( s_{\alpha} \tau \in W^I \), then \( \tau \in W^I \) with \( s_{\alpha} \tau(\omega) = \tau(\omega) - \alpha \).

**Proof.** The proof of the first part of the lemma is clear. Now \( s_{\alpha} \tau(\omega) = \tau(\omega) - \langle \tau(\omega), \tilde{\alpha} \rangle \alpha \). Since the pairing \( \langle \cdot, \cdot \rangle \) is \( W \)-invariant, \( \langle \tau(\omega), \tilde{\alpha} \rangle = \langle \omega, (\tau^{-1} \alpha) \rangle \). Again since \( l(s_{\alpha} \tau) = l(\tau) + 1 \), we have \( \tau^{-1} \alpha > 0 \). Let \( (\tau^{-1} \alpha) = \sum_{i=1}^{l} m_i \alpha_i, m_i \in \mathbb{Z}_{>0} \). Now, if \( \langle \omega, (\tau^{-1} \alpha) \rangle = 0 \), then \( m_i > 0 \Rightarrow \langle \omega, (\tau^{-1} \alpha) \rangle = 0 \) for \( 1 \leq i \leq l \). This gives a contradiction, since \( s_{\alpha} \tau \in W^I \) and \( s_{\alpha} \tau(\tau^{-1} \alpha) = s_{\alpha}(\alpha) < 0 \). Thus, \( \langle \omega, (\tau^{-1} \alpha) \rangle = 1 \). Hence the lemma is proved. \( \square \)

**COROLLARY 1.4**

1. For any \( w \in W^I \), the number of times \( s_i \), \( 1 \leq i \leq n - 1 \) appearing in a reduced expression of \( w = (\text{coefficient of } \alpha_i \text{ in } \omega) - (\text{coefficient of } \alpha_i \text{ in } w(\omega)) \) and hence it is independent of the reduced expression of \( w \).

2. Let \( w \in W^I \) and let \( w = s_{i_1}s_{i_2}\ldots s_{i_k} \in W^I \) be a reduced expression. Then \( w(\omega) = \omega - \sum_{j=1}^{k} \alpha_i \) and \( l(w) = ht(\omega - w(\omega)) \).

**Proof.** Follows from Lemma 1.3. \( \square \)

**Lemma 1.5.** Let \( w = s_{i_1}s_{i_2}\ldots s_{i_k} \in W \) such that \( ht(\omega - s_{i_1}s_{i_2}\ldots s_{i_k}(\omega)) = k \) then \( w \in W^I \) and \( l(w) = k \).
Lemma 1.6. Let \( \omega = \sum_{i=1}^{l} m_i \alpha_i \), \( m_i \in \mathbb{Q}_{\geq 0} \) be a minuscule weight. Let \( I = \{ \alpha \in S : \langle \alpha, \check{\alpha} \rangle = 0 \} \). Then, there exists a unique \( w \in W^I \) such that \( w(\omega) = \sum_{i=1}^{l} (m_i - [m_i]) \alpha_i \) where for any real number \( x \),

\[
[x] := \begin{cases} 
  x & \text{if } x \text{ is an integer} \\
  \lfloor x \rfloor + 1 & \text{otherwise} 
\end{cases}
\]

Proof. Using Lemma 1.1 and the fact that \( \omega \) is minuscule, we can find a sequence \( s_{i_k}, s_{i_{k-1}}, \ldots, s_{i_1} \) of simple reflections in \( W \) such that for each \( j, 2 \leq j \leq k + 1 \), coefficient of \( \alpha_j \) in \( s_{i_{j-1}} \cdot s_{i_{j-2}} \cdots s_{i_1} (\omega) \) is positive and \( \langle s_{i_k} \cdot s_{i_{k-1}} \cdots s_{i_1} (\omega) \rangle = \omega_r - \sum_{j=1}^{k} \alpha_j \). The existence part of the lemma follows from here. The uniqueness follows from Lemma 1.2. \( \square \)

Lemma 1.7. Let \( \omega = \sum_{i=1}^{l} m_i \alpha_i \), \( m_i \in \mathbb{Q}_{\geq 0} \) be a minuscule weight. Let \( I = \{ \alpha \in S : \langle \alpha, \check{\alpha} \rangle = 0 \} \). Then, there exists a unique \( \tau \in W^I \) such that \( \tau(\omega) = \sum_{i=1}^{l} (m_i - [m_i]) \alpha_i \).

Proof. Proof is similar to that of Lemma 1.6. \( \square \)

Now onwards, we say that for two elements \( w_1 \) and \( w_2 \) in \( W \), \( w_1 \leq w_2 \) if \( l(w_2) = l(w_1) + l(w_2 w_1^{-1}) \).

Lemma 1.8. Let \( \omega \) and \( I \) be as in Lemma 1.6 and \( w_1, w_2 \in W^I \). Then \( w_2(\omega) \leq w_1(\omega) \iff w_1 \leq w_2 \).

Proof. Only the implication \( \Rightarrow \) is to be proved. The proof is by induction on \( h_t(w_1(\omega) - w_2(\omega)) \) which is a non-negative integer. By Lemma 1.2, the height may be assumed to be positive.

\[
h_t(w_1(\omega) - w_2(\omega)) = 1; \text{This means } w_1(\omega) = w_2(\omega) + \alpha \text{ for some } \alpha \in S. \text{Applying } s_\alpha \text{ on both sides of this equation, we have}
\]

\[
s_\alpha w_1(\omega) = w_1(\omega) - \omega, (w_1^{-1} \alpha') \alpha = -2\alpha + w_2(\omega) - \omega, (w_2^{-1} \alpha') \alpha \\
\implies \omega, (w_2^{-1} \alpha') = 2 + \omega, (w_2^{-1} \alpha').
\]

Since \( \omega \) is minuscule, we get \( \omega, (w_1^{-1} \alpha') = 1 \) and \( \omega, (w_2^{-1} \alpha') = -1 \). This implies, by Lemma 1.5, that \( l(s_\alpha w_1) = l(w_1) + 1 \) and \( s_\alpha w_1 \in W^I \). Now, we have \( s_\alpha w_1(\omega) = w_2(\omega) \).

Hence, by Lemma 1.2, we get \( w_2 = s_\alpha w_1 \) with \( l(w_2) = l(w_1) + 1 \). Thus the result follows in this case.

Let us assume that the result is true for \( h_t(w_1(\omega) - w_2(\omega)) \leq m - 1 \).

\[
h_t(w_1(\omega) - w_2(\omega)) = m; \text{Let } w_1(\omega) - w_2(\omega) = \sum_{j \in J} m_j \alpha_j \text{ where } J \subseteq S \text{ and } m_j \text{'s are positive integers. Since } \langle \sum_{j \in J} m_j \alpha_j, \sum_{j \in J} m_j \alpha_j \rangle \geq 0, \text{there exists an } \alpha_j \in J \text{ such that } \langle w_1(\omega) - w_2(\omega), \alpha_j \rangle > 0. \text{Hence, either } \langle w_1(\omega), \alpha_j \rangle > 0 \text{ or } \langle w_2(\omega), \alpha_j \rangle < 0.
\]

Case I. Let us assume \( \langle w_1(\omega), \alpha_j \rangle > 0 \). Then \( l(s_{\alpha_j} w_1) = l(w_1) + 1 \) and \( s_{\alpha_j} w_1 \in W^I \).

Now \( h_t(s_{\alpha_j} w_1(\omega) - w_2(\omega)) = m - 1 \). Hence, by induction hypothesis \( w_2 = \phi_1 s_{\alpha_j} w_1 \) with \( l(w_2) = l(\phi_1) + l(s_{\alpha_j} w_1) \). Thus taking \( \phi = \phi_1 s_{\alpha_j} \) and noting that \( l(\phi) = l(\phi_1) + 1 \), we are done in this case.
Case II. Let us assume \( (w_2(\omega), \sigma_j) < 0 \). Then \( l(s_{a_j} w_2) = l(w_2) - 1 \) and \( s_{a_j} w_2 \in W^I \). Since \( w_1(\omega) - s_{a_j} w_2(\omega) = m - 1 \), by induction hypothesis \( s_{a_j} w_2 = \phi_2 w_1 \) with \( l(s_{a_j} w_2) = l(\phi_2) + l(w_1) \). Thus taking \( \phi = s_{a_j} \phi_2 \) and noticing that \( l(\phi) = 1 + l(\phi_2) \), we are done in this case also. This completes the proof. \( \square \)

**COROLLARY 1.9**

Let \( \omega, w \) and \( I \) be as in Lemma 1.6. Let \( \sigma \in W^I \) be such that \( \sigma(n\omega) \leq 0 \) for some positive integer \( n \). Then, we have \( w \leq \sigma \).

**Proof.** The proof follows from Lemmas 1.6, 1.8 and the fact that \( \omega \) is minuscule. \( \square \)

**COROLLARY 1.10**

Let \( \omega, \tau \) and \( I \) be as in Lemma 1.7. Let \( \sigma \in W^I \) be such that \( \sigma(n\omega) \geq 0 \) for some positive integer \( n \). Then, we have \( \sigma \leq \tau \).

**Proof.** The proof follows from Lemmas 1.7, 1.8 and the fact that \( \omega \) is minuscule. \( \square \)

2. **Description of Schubert varieties in the Grassmannian having semi-stable points**

In this section, we have the following notation. Let \( G = GL_n(k) \) where characteristic of \( k \) is either zero or bigger than \( n \). Let \( r \in \{2, \ldots, n-2\} \). Consider the action of a maximal torus \( T \) of \( SL_n(k) \) on the Grassmannian \( G_{r,n} \). Let \( B \) be a Borel subgroup of \( G \) containing \( T \). Let \( S = \{\alpha_1, \ldots, \alpha_{n-1}\} \) be the set of simple roots with respect to \( B \) arranged in the ordering of the vertices in the Dynkin diagram of type \( A_{n-1} \). Let \( L = S \backslash \{\alpha_r\} \). We first note that \( G_{r,n} \) is the homogeneous space \( GL_n(k)/P_r \) where \( P_r = BW_I \). \( B \) is the maximal parabolic subgroup of \( GL_n(k) \) containing \( B \) associated to the simple root \( \alpha_r \). Let \( \omega_r \) be the fundamental weight associated to the simple root \( \alpha_r \) and let \( \mathcal{L}_r \) denote the line bundle on \( GL_n(k)/P_r \) corresponding to \( \omega_r \). We describe all Schubert cells in \( GL_n(k)/P_r \) admitting semi-stable points for the above-mentioned action of \( T \) with respect to the line bundle \( \mathcal{L}_r \).

Some of the elementary facts about the combinatorics of \( W^I \) which are being used in this section can be found in [9]. For the convenience of the reader, we prove them here.

**Lemma 2.1.** Let \( w \in W^I \), \( w \neq id \). Then there exists an \( i \in \mathbb{N} \), \( i \leq r \) and a sequence of positive integers \( \{a_j\} \), \( j = 1, 2, \ldots, r \) such that the following holds.

(a) \( a_j \geq j \) for all \( j \), \( i \leq j \leq r \).
(b) \( w = (s_{a_1} \cdot s_{a_1-1} \cdot s_i)(s_{a_1+1} \cdot s_{a_1+2} \cdot s_{a_2+1} \cdot s_{a_2+2} \cdot \ldots \cdot s_{a_r} \cdot s_{a_r-1} \cdot s_r) \) with \( l(w) = \sum_{j=1}^{r} (a_j - j + 1) \).

**Proof.** Let \( i \) be the least positive integer such that \( s_{a_i} \leq w \). The rest of the proof follows from braid relations in \( W \). \( \square \)

**Lemma 2.2.** Let \( w, \tau \in W^I \). Write \( w = (s_{a_1} \cdot s_{a_1-1} \cdot s_i)(s_{a_1+1} \cdot s_{a_1+1-1} \cdot s_{i+1}) \ldots (s_{a_r} \cdot s_{a_r-1} \cdot s_r) \) and \( \tau = (s_{b_1} \cdot s_{b_1-1} \cdot s_k)(s_{b_1+1} \cdot s_{b_1+1-1} \cdot s_{k+1}) \ldots (s_{b_r} \cdot s_{b_r-1} \cdot s_r) \) be as in Lemma 2.1. Then \( w \leq \tau \iff k \leq i \) and \( b_j \geq a_j \) for all \( j \), \( i \leq j \leq r \).
Proof. The proof follows from Lemma 1.8 and the fact that \( w(\omega_r) \geq \tau(\omega_r) \iff k \leq i \) and \( b_j \geq a_j \) for all \( j, i \leq j \leq r \).

Now, write \( n = qr + t \) with \( 1 \leq t \leq r \) and let \( \tau_r \in W_I^r \) be the unique element as in Lemma 1.6 for the case when \( \omega = \omega_r \). Then, \( \tau_r \) must be of the form

\[
\tau_r = (s_{a_1} \ldots s_1) \ldots (s_{a_r} \ldots s_r)
\]

where

\[
a_i = \begin{cases} i(q + 1) & \text{if } i \leq t - 1 \\ iq + (t - 1) & \text{if } t \leq i \leq r \end{cases}
\]

Let \( \tau^{(n-r)} \in W_{I-r}^r \) be the unique element as in Lemma 1.7 for the case \( \omega = \omega_r \). Let \( w_{I-r}^r \) denote the minimal representative of the longest element \( w_0 \) of \( W \) in \( W_{I-r}^r \). Then, \( \tau_r \) must be of the form

\[
(\tau^{(n-r)})^{-1} \tau_r = (s_{a_1} \ldots s_1) \ldots (s_{a_r} \ldots s_r)
\]

where \( a_i = \begin{cases} i(q + 1) & \text{if } i \leq t - 1 \\ iq + (t - 1) & \text{if } t \leq i \leq r \end{cases} \)

2. Let \( w \in W_I^r \) be such that \( w(n\omega_r) \leq 0 \). Then, we have

Lemma 2.3. \( \tau_r \leq w \) and \( w \tau_r^{-1} \leq (\tau^{(n-r)})^{-1} \).

Proof. Proof follows from Corollary 1.9 and Corollary 1.10.

For any such \( w \), we describe the set \( R(w_{I-r}^r) \).

Lemma 2.4. \( R(w_{I-r}^r) \) consists of roots of the form \( \alpha_j + \alpha_{j+1} + \cdots + \alpha_{a_i} \) for \( 1 \leq i \leq r \) where \( j \neq a_k + 1 \) for any \( k < i \).

Proof. We have \( w^{-1} = (s_r \ldots s_{a_1}) \ldots (s_2 \ldots s_{a_2})(s_1 \ldots s_{a_1}) \), which is a reduced expression. Thus the elements of \( R(w_{I-r}^r) \) are

\[
\beta_{i,j-i+1} = (s_{a_1} \ldots s_1) \cdot (s_{a_2} \ldots s_2) \ldots (s_{a_j} \ldots s_i)(\alpha_j)
\]

where \( i \leq j \leq a_i, 1 \leq i \leq r \). denotes omission of the symbols. We have

\[
(s_{a_1} \ldots s_{j+1} \cdot \hat{s}_j \cdot \hat{s}_{j-1} \ldots \hat{s}_i)(\alpha_j) = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{a_i}
\]

Since \( a_1 < a_2 < \cdots < a_r \), each \( \beta_{i,j} \) is of the form

\[
\alpha_j + \alpha_{j+1} + \cdots + \alpha_{a_i}
\]

Now \( j \neq a_k + 1 \) for any \( k < i \) follows from the fact that \( l(w) \) is the same as the cardinality of \( R(w_{I-r}^r) \).}

Remark 2.5. From the lemma it follows that the elements of \( R(w_{I-r}^r) \) can be written in an array as follows:

\[
\begin{array}{cccccccccccccc}
\beta_{1,1} & \beta_{1,2} & \ldots & \beta_{1,a_1} \\
\beta_{2,1} & \beta_{2,2} & \ldots & \beta_{2,a_1} & \beta_{2,a_1+1} & \beta_{2,a_1+2} & \ldots & \beta_{2,a_2-1} \\
\beta_{3,1} & \beta_{3,2} & \ldots & \beta_{3,a_1} & \beta_{3,a_1+1} & \beta_{3,a_1+2} & \ldots & \beta_{3,a_2-1} & \beta_{3,a_2} & \ldots & \beta_{3,a_3-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{r,1} & \beta_{r,2} & \ldots & \beta_{r,a_1} & \beta_{r,a_1+1} & \beta_{r,a_1+2} & \ldots & \beta_{r,a_2-1} & \beta_{r,a_2} & \ldots & \beta_{r,a_3-2} & \ldots & \beta_{r,a_r-r+1}
\end{array}
\]
where the array has $r$ rows, and the length of the $i$-th row is $a_i - (i - 1)$. Note that $\beta_1,a_1 = \alpha_{a_1}$, and for $2 \leq i \leq r$, $\beta_1,a_{i-1}+1 = \alpha_{a_i}$, only if $a_1 \geq a_i + 2$ (i.e. for all $i$). In this case, for all $j$, $i \leq j \leq r$, $\beta_j,a_{i-1}+l+2 = \beta_{i-1} - a_{i-1} - l+2 + \alpha_{a_{i-1}+l+2} + \cdots + \alpha_{a_j}$, and $\beta_j,a_{i-1}+l+3 = \alpha_{a_{i-1}+2} + \alpha_{a_{i-1}+3} + \cdots + \alpha_{a_j}$. If $a_i = a_i - 1 + 1$, then $a_i - i + 1 = a_i - (i - 1) + 1$, therefore, the $(i - 1)$-th and $i$-th rows have same length. In this case for all $j$, $i \leq j \leq r$, $\beta_j,a_{i-1}+1 = \beta_{i-1} - a_{i-1} + a_{a_{i-1}+2} + \cdots + \alpha_{a_j}$.

For any $w \in W^l$, let $X(w) := BwP_r / P_r$ denote the Schubert variety in $GL_n(k)/P_r$.

We recall $BwP_r / P_r = UwP_r$, where $U_w$ is the product $\prod_{\alpha \in R(w^{-1})} U_{\alpha}$ of the root groups $U_{\alpha}$, and we describe below the ordering of roots in which the product is taken.

Consider the open set

$$ V := \left\{ \prod_{\beta_i \in R(w^{-1})} u_{\beta_i}(x_{\beta_i}) wP_r : x_{\beta_i} \neq 0, \forall \beta_i \in R(w^{-1}) \right\} $$

of $X(w)$ in $GL_n(k)/P_r$, where the order in which the product is taken is as follows: Put a partial order on $R(w^{-1})$ by declaring $\beta_{ij} \leq \beta_{kl}$ if either $i = k$ and $j \geq l$ or if $i < k$. Now we take the product so that whenever $\beta_{ij} \leq \beta_{kl}$, $u_{\beta_{ij}}(x_{\beta_{ij}})$ appears on the right-hand side of $u_{\beta_{kl}}(x_{\beta_{kl}})$. Note that $u_{\beta_{ij}}(x_{\beta_{ij}})$'s commute with each other, since $\beta_{ij,kl}, \beta_{kl,ij} \in R(w^{-1})$ implies $\beta_{ij,kl} + \beta_{kl,ij}$ is not a root. This follows from the fact that no element of $R(w^{-1})$ starts or ends with $\alpha_{a_k+1}$, for any $k$, $1 \leq k \leq r - 1$ (i.e. for all $\beta_i,j \in R(w^{-1})$ and $1 \leq k \leq r - 1, \beta_i,j - \alpha_{a_k+1} \neq 0$ is not a root.)

Now the natural action of the maximal torus $T$ on $GL_n(k)/P_r$, induces an action of $T$ on $V$.

**Lemma 2.6.** Consider the torus $T' := \prod_{\beta \in R(w^{-1})} G_{m,\beta}$ where $G_{m,\beta} = G_m$ for each $\beta \in R(w^{-1})$. We have a natural action of $T$ on $T'$ through the homomorphism of algebraic groups $\Psi: T \to T'$ defined by $\Psi(t) = (t^\beta)$ for all $t \in T$. The map $V \to T'$ defined by $\prod u_{\beta}(x_{\beta})w \cdot P \mapsto (x_{\beta})_P$ is a $T$-equivariant isomorphism of varieties.

**Proof.** Proof is easy. \qed

Now, we recall the definition of the Hilbert–Mumford numerical function and definition of the semistable points from [5]. We also refer to [6] for notation in geometric invariant theory.

1. Let $X$ be a projective variety with an action of a reductive group $G$. Let $\lambda$ be a one-parameter subgroup of $G$. Let $L$ be a $G$-linearised very ample line bundle on $X$. Let $x \in \mathbb{P}(H^0(X,L^s))$ and $\hat{x}$ be a point in the cone over $X$ which lies on $x$. Write $\hat{x} = \sum_{i=1}^{r} v_i$, where each $v_i$ is a weight vector of $\lambda$ of weight $m_i$.

Then, we have

$$ \mu^L(x, \lambda) = - \min \{m_i | i = 1, \ldots, r \}. $$

2. A point $x \in X(w)$ is said to be semi-stable with respect to the $T$-linearised line bundle $L$ if there is a positive integer $m \in \mathbb{N}$, and a $T$-invariant section $s \in H^0(X(w),L^m)$ with $s(x) \neq 0$. We denote by $X(w)_T^s(L)$, the set of all points semi-stable points in $X(w)$ with respect to the $T$ linearised line bundle $L$.

We now describe all the Schubert varieties admitting semi-stable points.

Let $n = qr + t$, with $1 \leq t \leq r$ and let $w \in W^l$. 


Lemma 2.7. The following are equivalent:

1. $X(w)\mu_d(C_r)$ is non-empty.
2. $\tau_r \leq w$ and $w\tau_r^{-1} \leq (\tau^{(n-r)})^{-1}$.
3. $w = (s_{i_1} \cdots s_{i_l}) \cdots (s_{i_2} \cdots s_{i_r})$, where $\{a_i; i = 1, 2, \ldots, r\}$ is an increasing sequence of positive integers in $[1, 2, \ldots, n-1]$ such that $a_i \geq i(q+1) \forall i \leq t-1$ and $a_i = iq + (i-1) \forall i$ such that $t \leq i \leq r$.

Proof. By Hilbert–Mumford criterion (Theorem 2.1 of [5]) a point $x \in G/P_r$ is semi-stable if and only if $\mu_k(\sigma x, \lambda) \geq 0$ for all $\lambda \in \mathbb{C}$ and for all $\sigma \in W$. By Lemma 2.1 of [8], this statement is equivalent to $(-w_\sigma(\omega), \lambda) \geq 0$ for all $\lambda \in \mathbb{C}$ and for all $\sigma \in W$, where $w_\sigma \in W^h$ is such that $\sigma x \in Uw_\sigma P_r$. Thus, by Corollary 1.9 applied to the situation $\omega = 0$, a point $x$ is semi-stable if and only if $x$ is not in the $W$-translates of $U_i \tau P_r$ with $\tau \in W^h$ and $\tau_r \not\leq \tau$.

Now, for a $w \in W^h$, $X(w)$ is not contained in the finite union $\bigcup_{\tau \not\leq \tau} U_i \tau P_r$ if and only if $\tau_r \leq w$. The second condition $w\tau_r^{-1} \leq (\tau^{(n-r)})^{-1}$ is an immediate consequence when $w \geq \tau_r$. This completes the proof of (2). Proof of (3) follows from Corollary 1.9 and the discussion after Lemma 2.2. □

PROPOSITION 2.8

Let $X_{i,j}$ denote the regular function on $V$ defined by $\prod u_{\beta_i}(x_{\beta_i})w \cdot P \mapsto x_{\beta_j}$ for all $1 \leq i \leq r-1$ and $1 \leq j \leq a_i - i + 1$; and let $Y_{i,j} := \frac{X_{i,q^{-1}+X_{i+1,j}}}{X_{i,j+X_{i+1,q^{-1}+i}}}$, where $1 \leq j \leq a_i - i$, for each $i$, and $1 \leq i \leq r - 1$; $Y_{i,j}$ are algebraically independent.

Proof. Recall the map, $T \xrightarrow{\Psi} T'$ defined by $\Psi(t) = (t^\beta)$, $\beta \in R(w^{-1})$ as in Lemma 2.6. Proof of the proposition follows from the following claim.

Claim. $E_{i,j}q^{-1} - E_{i+1,j}q^{-1} - E_{i,j} - E_{i+1,j}; i = 1, 2, \ldots, r - 1$ and $j = 1, 2, \ldots, a_i - i$ forms a basis for $\text{Ker}(\Psi^*: X(T') \mapsto X(T))$, where $E_{i,k}$ is the matrix with 1 in the $(i, k)$-th place and 0 elsewhere.

Proof of the Claim. Now any character of $T'$ is of the form $(t^\beta) \mapsto \prod t^{m_\beta}$ where $m_\beta$ are integers. Now such a character is $T$-invariant iff the sum $\sum m_\beta \beta$ is zero. Plugging in the expression of $\beta$’s in terms of the simple roots $\alpha_k$’s and noting that they are linearly independent we get a set of linear equations over $\mathbb{Z}$, by equating to zero the coefficient of each $\alpha_k$. Let us denote by $R(p)$, $1 \leq p \leq r$ the set of roots appearing in $p$-th row of the array in Remark 2.5. Also, let $C(q)$, $1 \leq q \leq a_r - (r - 1)$ denote the set of roots appearing in the $q$-th column of the array in Remark 2.5.

Comparing the coefficient of $\alpha_1$, we have $\sum_{\beta \in C(1)} m_\beta = 0$.

Comparing the coefficient of $\alpha_2$, and using the above observation, we get $\sum_{\beta \in C(2)} m_\beta = 0$. Proceeding this way, we get

$$\sum_{\beta \in C(j)} m_\beta = 0 \quad \forall j, 1 \leq j \leq a_1.$$

Let $k$ be the least positive integer such that $\alpha_k + \cdots + \alpha_{a_k}$ is the first root in the column $C(a_1 + 1)$. Comparing the coefficient of $\alpha_k$, we get $\sum_{\beta \in C(a_1 + 1)} m_\beta = 0$. Proceeding this
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way, we get
\[ \sum_{\beta \in \mathcal{C}(j)} m_{\beta} = 0 \quad \forall j, \ 1 \leq j \leq a_r - r + 1. \]

Now, comparing the coefficient of \( \alpha_{a_r} \), we get
\[ \sum_{\beta \in \mathcal{R}(r)} m_{\beta} = 0. \]

Comparing the coefficient of \( \{ \alpha_j : j = a_r - 1, 2 + a_r - 1, \ldots, a_r \} \), we get
\[ \sum_{\beta \in \mathcal{R}(r-1)} m_{\beta} + \sum_{\beta \in \mathcal{R}(r)} m_{\beta} = 0. \]

Thus we have
\[ \sum_{\beta \in \mathcal{R}(r-1)} m_{\beta} = 0. \]

Proceeding this way, we get
\[ \sum_{\beta \in \mathcal{R}(i)} m_{\beta} = 0 \quad \forall i, \ 1 \leq i \leq r. \]

3. Description of the action of the Weyl group on the quotient \( T \backslash G^{ss}_{r,n}(L) \)

In this section, we describe the action of the Weyl group on the quotient \( T \backslash G^{ss}_{r,n}(L) \).

We recall the definition of \( \mu^L(x, \lambda) \) from [5].

We first write down the stabiliser of \( X(w) \) in \( W \). Let

\[ w = (s_{a_1} \ldots s_1)(s_{a_2} \ldots s_2) \ldots (s_{a_r} \ldots s_r) \in W^r \]

be such that \( w \geq \tau_r \). Then, we have as follows.

**Lemma 3.1.** The set \( \{ s_i : s_i(X(w)) \subseteq X(w), i = 1, 2, \ldots, n - 1 \} \) consists of:

1. \( \{ s_j : 1 \leq j \leq a_1 - 2 \} \).
2. \( \{ s_j : a_p + 2 \leq j \leq a_p + 1 - 1, p = 1, 2, \ldots, r - 1 \} \).
3. \( \{ s_j : a_r + 2 \leq j \leq n - 1 \} \).
4. \( \{ s_{a_p} : p = 1, 2, \ldots, r \} \).

**Proof.** We see that \( s_j w \leq w \) if and only if \( s_j(X(w)) \subseteq X(w) \).

1. For \( 1 \leq j \leq a_1 - 2 \), we have \( s_j(w) = (s_{a_1} \ldots s_1)s_{j+1}(s_{a_2} \ldots s_2) \ldots (s_{a_r} \ldots s_r) \),

\[ s_j s_{j+1}s_j = s_{j+1}s_js_{j+1}. \]

Using this for \( j + 1, \ldots, j + r - 1 \), we get \( s_j w = ws_{j+r} \).

Since \( j + r \in I_r \), we must get \( (s_j w)^{I_r} = w \).

2. If \( a_p + 2 \leq j \leq a_p + 1 - 1 \), then \( s_j w = ws_{j+r-p} \). Hence \( (s_j w)^{I_r} = w \) as \( j + r - p \in I_r \).

3. For \( a_r + 2 \leq j \leq n - 1 \), we have \( s_j w = ws_j \). Hence, \( (s_j w)^{I_r} = w \) as \( j \in I_r \).

4. If \( p = 1 \), there is nothing to prove.

If \( p \geq 2 \), we divide the proof into two cases.

**Case I.** If \( a_p = p \), then \( a_{p-1} = p - 1 \), and \( s_{a_p} w = ws_{p-1} \). Thus, \( (s_{a_p} w)^{I_r} = w \) as \( p - 1 \in I_r \).

**Case II.** If \( a_p \geq p + 1 \).
Subcase I. If \( a_p \geq a_{p-1} + 2 \), then we have

\[
s_{a_p} w = (s_{a_1} \ldots s_{a_1}) \ldots (s_{a_{p-1}} \ldots s_{p-1})(s_{a_p} - 1 \ldots s_p) \\
\times (s_{a_{p+1}} \ldots s_{p+1}) \ldots (s_{a_r} \ldots s_r) < w.
\]

Subcase II. If \( a_p = 1 + a_{p-1} \), then

\[
s_{a_p} = (s_{a_1} \ldots s_{a_1}) \ldots (s_{a_{p-2}} \ldots s_{p-2})(s_{a_p} s_{a_p-1} s_{a_p-1} \ldots s_{p-1}) \\
\times (s_{a_{p+1}} \ldots s_{p}) (s_{a_{p+1}} \ldots s_{p+1}) \ldots (s_{a_r} \ldots s_r) \\
= (s_{a_1} \ldots s_{a_1}) \ldots (s_{a_{p-1}} \ldots s_{p-1})(s_{a_p} \ldots s_p) s_{p-1}(s_{a_{p+1}} \ldots s_{p+1}) \\
\times \ldots (s_{a_r} \ldots s_r) = w s_{p-1}.
\]

Hence, \( (s_{a_p} w)^r = w \), as \( p - 1 \in I_r \). □

We now explicitly describe the action of the stabilisers.

PROPOSITION 3.2

Description of the action of the \( s_i \)'s:

1. \( s_i \) interchanges \( Y_{i,j} \) and \( Y_{i,j+1} \) for \( i = 1, 2, \ldots, r - 1 \), and keeps all other \( Y_{i,k} \)'s fixed.
2. \( s_i \) interchanges \( Y_{i,j} \) and \( Y_{i,j+1} \) for \( j + 1 \leq i \leq r - 1 \), and keeps all other \( Y_{i,k} \)'s fixed.
3a. If \( 2 \leq p \leq r \), then \( s_{a_p} \) fixes all the \( Y_{i,k} \), \( 1 \leq i \leq p - 1 \).
3b. If \( p \leq i \leq r - 1 \), and \( a_p - p = a_i - i \) and \( 1 \leq k \leq a_p - p \), then \( s_{a_p} (Y_{i,a_p} - p) = Y_{i,a_p} - p \), and \( s_{a_p}(Y_{i,k}) = Y_{i+k} \cdot Y_{i+a_p-p}^ {-1} \).
3c. If \( p + 1 \leq i \leq r - 1 \), and \( a_p - p = p - 1 \), then \( s_{a_p} (Y_{i,a_p} - p) = Y_{i,a_p} - p + 1 \), and keeps all other \( Y_{i,k} \)'s fixed.
4a. \( 2 \leq p \leq r - 1 \), and \( a_p = a_{p-1} + 1 \).
   i. If \( 3 \leq p \leq r \) and \( 1 \leq k \leq a_{p-2} - p + 2 \), then \( s_{a_p} (Y_{p-2,k}) = Y_{p-2,k} \cdot Y_{p-1,k} \cdot Y_{p-1,a_p-p+3} \).
   ii. If \( 1 \leq k \leq a_p - p \) then \( s_{a_p} (Y_{p-1,k}) = Y_{p-1,k} \cdot Y_{p-1,k} \).
   iii. \( Y_{i,k} \)'s are fixed for \( i \neq p - 2, p - 1, p \) and \( 1 \leq k \leq a_i - i \).
4b.
   i. If \( 1 \leq i \leq p - 1 \) or \( a_p - p + 1 \leq k \leq a_r \), \( Y_{i,k} \)'s are fixed.
   ii. If \( i = p \) and \( 1 \leq k \leq a_p - p \) then \( s_{a_p} (Y_{p,k}) = 1 - Y_{p,k} \).
   iii. If \( p + 1 \leq i \leq r - 1 \) and \( 1 \leq k \leq a_p - p \), then,
\[
\begin{align*}
s_{a_p} (Y_{i,k}) = & \frac{1 - 1}{\Pi_{p=1}^{a_p} (Y_{m,k}/Y_{m,a_p-p+1})} \\
& \times Y_{i,a_p-p+1}.
\end{align*}
\]
4c. Action of \( s_{a_r} \).
   i. If \( a_r = a_{r-1} + 1 \), then
\[
s_{a_r} (Y_{r-2,k}) = Y_{r-2,k} \cdot Y_{r-1,k} \cdot Y_{r-1,a_r-2-r+3} \text{ for } 1 \leq k \leq a_{r-2} - r + 2 \text{ and}
\]
\[
s_{a_r} (Y_{r-1,k}) = Y_{r-1,k} \text{ for } 1 \leq k \leq a_r - r.
\]
   ii. If \( a_r + 2 \leq a_r \), then \( Y_{r,k} \)'s are fixed for \( 1 \leq k \leq a_r - r + 1 \).
Proof. Proof is essentially based on the following properties of groups with $BN$-pair and commutator relations:

(i)

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 1 \\ 0 & 1 \end{pmatrix}.
\]

and

(ii)

\[
[u_\alpha(x_\alpha), u_\beta(x_\beta)] = \begin{cases} u_{\alpha+\beta}(x_\alpha \cdot x_\beta) & \text{if } \alpha = \epsilon_i - \epsilon_j \text{ and } \beta = \epsilon_j - \epsilon_k, i < j < k; \\
u_{\alpha+\beta}(-x_\alpha \cdot x_\beta) & \text{if } \alpha = \epsilon_i - \epsilon_j \text{ and } \beta = \epsilon_k - \epsilon_i, k < i < j. \end{cases}
\]

We first consider the action of $W$ on the $X_{j,k}$’s and then describe resulting action on the $Y_{j,k}$’s. If $1 \leq i \leq a_1 - 2$, then, $s_i$ interchanges $X_{j,i}$ and $X_{j,i+1}$ for all $j, 1 \leq j \leq r$. Therefore, it follows that $s_i$ interchanges $Y_{j,i}$ and $Y_{j,i+1}$ for all $j, 1 \leq j \leq r - 1$ and keeps all other $Y_{j,k}$’s fixed. Similarly for $p \geq 2$ and $a_p + 2 \leq a_{p+1}$, if $a_p + 2 \leq i \leq a_{p+1} - 2$, $s_i$ interchanges $X_{j,i-p}$ and $X_{j,i-p+1}$. Thus $s_i$ interchanges $Y_{j,i-p}$ and $Y_{j,i-p+1}$ for all $j, i + 1 \leq j \leq r - 1$ and keeps all other $Y_{j,k}$’s fixed. Now, we compute the actions of $s_{a_1-1}$, $s_{a_1}$, and $s_{a_1+1}$.

Action of $s_{a_1+1}$ for each $i$, $1 \leq i \leq r - 1$

Case I. $a_i + 2 \leq a_{i+1}$. In this case we have

\[
s_{a_1+1}\cdot w = s_{a_1+1} \cdot (s_{a_1} \ldots s_1) \cdot (s_{a_2} \ldots s_2) \ldots (s_{a_r} \ldots s_r) = (s_{a_1} \ldots s_1) \ldots (s_{a_1+1} \cdot s_{a_1} \ldots s_1) \ldots (s_{a_r} \ldots s_r)
\]

which is a reduced expression and $s_{a_1+1} \cdot w \in W^I$ by Lemma 1.12. Now Lemma 1.13 implies that $s_{a_1+1} \cdot w \geq w$. Hence, $X(w)$ is not stable under the action of $s_{a_1+1}$.

Case II. $a_i + 1 = a_{i+1}$. In this case $s_{a_1+1} = s_{a_1}$ and the action will be described in the later part of this paragraph. In fact we see that in this case $(s_{a_1+1}w)^I = w$. Hence $X(w)$ is stable under the action of $s_{a_1+1}$.

Action of $s_{a_1-1}$

In case $i = 1$, we may assume that $a_1 \neq 1$, and for $i \geq 2$, $a_{i-1} \neq a_i - 1$. Now $s_{a_1-1}$ interchanges the $(a_i - i)$-th and $(a_i - i + 1)$-th columns of each of the $j$-th row of the array of roots $R(w^{-1})$, for $i \leq j \leq r$; thus $s_{a_1-1}$ interchanges $X_{j,a_i-i}$ and $X_{j,a_i-i+1}$ for each $j, i \leq j \leq r$. Therefore, the action of $s_{a_1-1}$ is as follows:

1. $s_{a_1-1}$ fixes all the $Y_{j,k}$, for $1 \leq j \leq i - 1$, for $i \geq 2$.
2. For $j \geq i \leq r - 1$ and $a_i - i = a_j - j$, $Y_{j,a_i-i} \mapsto Y_{j,a_i-i}^{-1}$, and for $Y_{j,k} \mapsto Y_{j,k} \cdot Y_{j,a_i-i}^{-1}$ for $1 \leq k < a_i - i$.
3. For $i + 1 \leq j \leq r - 1$ if $a_j - j > a_i - i$, then $s_{a_1-1}$ interchanges $Y_{j,a_i-i}$ and $Y_{j,a_i-i+1}$ and keeps all other $Y_{j,k}$’s fixed.
Action of $s_{a_i}$ for $1 \leq i \leq r$

Let us show that $X(w)$ is stable under the action of each of the $s_{a_i}$. Let

$$w = (s_{a_1} \ldots s_1) \cdot (s_{a_2} \ldots s_2) \cdots (s_{a_r} \ldots s_r).$$

Thus

$$s_{a_i} w = (s_{a_1} \ldots s_1) \cdots (s_{a_{i-2}} \ldots s_{i-2}) \cdot s_{a_i} \cdot (s_{a_{i-1}} \ldots s_{i-1}) \cdots (s_{a_r} \ldots s_r).$$

**Case I.** $i = 1$ or $a_{i-1} + 2 \leq a_i$ for $i \geq 2$. In this case it is clear that

$$s_{a_i} w = (s_{a_1} \ldots s_1) \cdots (s_{a_{i-2}} \ldots s_{i-2}) \cdot (s_{a_{i-1}} \ldots s_{i-1}) \cdots (s_{a_r} \ldots s_r)$$

which, by Lemmas 2.1 and 2.2, is in $W^L$ and $s_{a_i} w \leq w$.

**Case II.** $a_{i-1} + 1 = a_i$. Note that,

$$w_1 = (s_{a_{i-1}} \ldots s_{i-1}) \cdot (s_{a_i} \ldots s_i) \in W^I,$$

where $J = S \setminus \{a_i\}$. Now,

$$w_1(\omega_i) = \omega_i - \sum_{j=1}^{a_{i-1}} \alpha_j - \sum_{j=i}^{a_i} \alpha_j,$$

which implies that $w_1(\omega_i) = w_1(\omega_i)$. Hence, by Lemma 1.3, we get $s_{a_i} w_1 = w_1 s_{a_i}$ for some $a \in J$. This gives $w^{-1}_1 s_{a_i} w_1 = w^{-1}_1(\omega_i) = s_{a_i}$. Now it follows that $w^{-1}_1(\omega_i) = a_{i-1}$. Hence, $s_{a_i} w_1 = w_1 \cdot s_{a_i}$. Therefore, in both the sub-cases $s_{a_i} \cdot w = w \cdot s_{a_i}$, in particular $(s_{a_i} \cdot w)^L = w$. Now we shall compute the action of $s_{a_i}$, for $1 \leq i \leq r$.

**Case I.** $2 \leq i \leq r - 1$ and $a_i = a_{i-1} + 1$. In this case, $s_{a_i}$ interchanges $X_{j,k}$ and $X_{j-1,k}$ for $1 \leq k \leq a_i - i + 1$ and keeps all other $X_{j,k}$’s fixed. Hence, the action of $s_{a_i}$ on $Y_{j,k}$’s is as follows:

1. If $i \geq 3$, $Y_{i-2,k} \leftrightarrow Y_{i-2,k} \cdot Y_{i-1,k} \cdot Y_{i-1,a_i-2,i+3}^{-1}$ for $1 \leq k \leq a_i - 2 - i + 2$
2. $Y_{i-1,k} \leftrightarrow Y_{i-1,k}^{-1}$ for $1 \leq k \leq a_i - i$
3. $Y_{i,k} \leftrightarrow Y_{i,k} \cdot Y_{i-1,k}$ for $1 \leq k \leq a_i - i$
4. $Y_{j,k}$ is fixed for $1 \leq k \leq a_j - j$ for each $j \neq i - 2, i - 1, i$.

**Case II.** $a_i \geq a_{i-1} + 2$ for $2 \leq i \leq r - 1$, or $i = 1$. In this case $s_{a_i}$ changes only the $i$-th row and the $(a_i - i + 1)$-th column of the array of roots $R(w^{-1})$. The resulting $i$-th row turns out to be

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{a_i-1}, \alpha_2 + \cdots + \alpha_{a_i-1}, \ldots, \alpha_{a_i} + \cdots + \alpha_{a_i-1},$$

$$\alpha_{a_i} + \alpha_{a_i+1} + \cdots + \alpha_{a_{i-1}}, \alpha_{a_i} + \cdots + \alpha_{a_{i-1}}, -\alpha_{a_i}$$
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and the transpose of the \((a_i - i + 1)\)-th column turns out to be

\[-\alpha_i, \alpha_{a_i+1}, \ldots, \alpha_{a_i+1}, \alpha_{a_i+1} + \cdots + \alpha_{a_i}, \ldots, \alpha_{a_i+1} + \cdots + \alpha_{a_i}.

Let \(\beta_{j,k}\) be any root which is fixed under the action of \(s_{a_i}\) and let \(\beta_{p,q}\) be any root of the \((a_i - i + 1)\)-th row or the \((a_i - i + 1)\)-th column, i.e. either \(p = i\) or \(q = a_i - i + 1\). We claim that \(u_{\beta_{j,k}}(X_{i,j})\) and \(u_{\beta_{p,q}}(X_{p,q})\) commute. This follows from the fact that \(\beta_{j,k} - \alpha_{a_i} \notin R(w^{-1})\) and the observation that for any root \(\beta \in R(w^{-1})\) and \(1 \leq m \leq r\), 

\[\beta - \alpha_{a_i+1} \notin R^+\].

Let us denote by \(M\) the sub-array consisting of \(\beta_{k,l}\) where \(k \geq i\) and \(1 \leq l \leq a_i - i + 1\). Then, by the above discussion, \(s_{a_i}\) acts trivially on the roots in \(M\).

We also have \(s_{a_i} \cdot u_{\beta_{j,k}}(X_{k,l}) = u_{s_{a_i}(\beta_{j,k})}(X_{k,l}).\)

Thus, the action of \(s_{a_i}\) in this case is as follows:

\[X_{i,a_i-i+1} \mapsto X_{i+1,a_i-i+1}^{-1} \cdot X_{i,i} \cdot X_{i,a_i-i+1} \quad \text{for} \quad k \leq a_i - i,
\]

\[X_{j,k} \mapsto X_{j,k} \cdot X_{i,a_i-i+1}^{-1} \cdot X_{i,k} \quad \text{for} \quad i + 1 \leq j \leq r \quad \text{and} \quad 1 \leq k \leq a_i - i,
\]

\[X_{j,a_i-i+1} \mapsto -X_{j,a_i-i+1} \cdot X_{i,a_i-i+1} \quad \text{for} \quad i + 1 \leq j \leq r.
\]

From this the resulting action on \(Y_{j,k}\) turns out to be as follows:

(1) \(s_{a_i}\) fixes \(Y_{j,k}\)’s provided \(j \leq i - 1\) or \(k \geq a_i - i + 1\). We now make the convention that \(Y_{j,k} = 1\) if \(k \geq a_j - j + 1\) or if \(j \geq r\).

(2) Here, for \(k \leq a_i - i - 1\),

\[\begin{align*}
Y_{j,k} &= \frac{X_{i,a_i-i+1} \cdot X_{i+1,k}}{X_{i+1,a_i-i+1} \cdot X_{i,k}}.
\end{align*}
\]

\[\therefore \quad s_{a_i}(Y_{j,k}) = \frac{X_{i,a_i-i+1}^{-1} \cdot (X_{i+1,k} - X_{i+1,a_i-i+1} \cdot X_{i,a_i-i+1})}{X_{i,k} \cdot X_{i,a_i-i+1}^{-1} \cdot (-X_{i+1,a_i-i+1} \cdot X_{i,a_i-i+1})}
\]

\[= 1 - Y_{j,k}.
\]

(3) For \(i + 1 \leq j \leq r - 1\) and \(1 \leq k \leq a_i - i\), define \(Y'_{j,k} = (X_{i,a_i-i+1} \cdot X_{j,k})/(X_{j,a_i-i+1} \cdot X_{j,k}).\) Then, we have \(s_{a_i}(Y_{j,k}) = 1 - Y_{j,k} \cdot Y_{j,k}^{-1}\). It follows that \(Y_{j,k} = Y'_{j+1,k} \cdot Y_{j,k}^{-1} \cdot\). Hence, \(s_{a_i}(Y_{j,k}) = Y_{j+1,k} \cdot Y_{j,k}^{-1}\).

\[Y'_{j,k} = \prod_{m=i}^{j-1} \frac{X_{m,a_m-i+1} \cdot X_{m+1,k}}{X_{m+1,a_m-i+1} \cdot X_{m,k}}
\]

\[= \prod_{m=i}^{j-1} \left\{ \left( \frac{X_{m,a_m-i+1} \cdot X_{m+1,k}}{X_{m+1,a_m-i+1} \cdot X_{m,k}} \right) \right\}^{-1}
\]

\[= \prod_{m=i}^{j-1} \left( \frac{Y_{m,k}}{Y_{m,a_m-i+1}} \right).
\]
Thus we have,

\[ s_{ar}(Y_{j,k}) = \frac{1 - \prod_{i=1}^{j}(Y_{m,k}/Y_{m,a_i-1})}{1 - \prod_{i=1}^{j-1}(Y_{m,k}/Y_{m,a_i-1})} \times Y_{j,a_i-1}. \]

**Case III.** Action of \( s_{ar} \):

1. If \( a_r = a_r - 1 + 1 \), then \( s_{ar} \) interchanges \( X_{r-1,k} \) and \( X_{r,k} \), \( 1 \leq k \leq a_r - r + 1 \).

   A straightforward checking proves as in Case I above, that in this case the action of \( s_{ar} \) is as follows:
   \[
   Y_{r-2,k} \mapsto Y_{r-2,k} \cdot Y_{r-1,k} \cdot Y_{r-1,a_r-2-r+3}^{-1}, \quad \text{for } 1 \leq k \leq a_r - 2 - r + 2
   
   Y_{r-1,k} \mapsto Y_{r-1,k}^{-1}, \quad \text{for } 1 \leq k \leq a_r - r.\]

2. If \( a_r \geq a_r - 1 + 2 \), \( s_{ar} \) changes only \( X_{r,k} \)'s for \( 1 \leq k \leq a_r - r + 1 \), as follows:
   \[
   X_{r,k} \mapsto X_{r,k} \cdot X_{r,a_r-r+1}^{-1} \quad \text{for } 1 \leq k \leq a_r - r
   
   X_{r,a_r-r+1} \mapsto X_{r,a_r-r+1}^{-1}.\]

   It can be easily checked from here that the \( Y_{i,j} \)'s are all fixed by \( s_{ar} \). \( \square \)

**4. A stratification of \( N\setminus\Gamma_{2,n}^*(L_2) \)**

In this section, we give a stratification of \( N\setminus\Gamma_{2,n}^*(L_2) \).

**Lemma 4.1.** Let \( w \in W \). Let \( x \in U_w w P_2 \) be such that \( x \) is not in the \( W \)-translate of \( X_{\tau}, \tau < w \). If \( \sigma(x) \in U_w w P_2 \), then \( \sigma \in \text{Stabiliser of } X(w) \) in \( W \).

**Proof.** If the lemma does not hold, then, there exists a \( \sigma \in W \) such that \( \sigma x \in U_w w P_2 \) with \( \sigma \) not in the stabilizer of \( X(w) \). Since \( X(w)^{\sigma \sigma} \) is nonempty, by Lemma 2.7, we can write \( w = (s_m \ldots s_1)(s_{a_1} \ldots s_2) \) with \( m \geq \lceil \frac{n-1}{2} \rceil \). Since \( \sigma \) is not in the stabiliser of \( X(w) \), \( \exists \) a positive integer \( t \in \mathbb{N} \) and \( a_1 \in W \) such that \( \sigma = \sigma_1 s_{m+t} s_{m+t-1} \ldots s_{m+1} \) with \( l(\sigma) = l(\sigma_1) + t \).

We proceed with the proof by considering two cases.

**Case 1.** \( t = 1 \). As \( l(\sigma) = l(\sigma_1) + 1, \sigma_1(a_{m+1}) > 0 \). Also, as \( \sigma x \in U_w w P_2 \), \( \sigma_1 \) must be of the form \( \sigma_1 = \phi s_{m+1} s_m \) with \( l(\sigma_1) = l(\phi) + 2 \). As \( \sigma \) is minimal for this choice of \( t \), \( \phi = id \).

Hence
\[
\sigma x = s_{m+1} s_m s_m x = s_{m+1} u' s_{m+1} w P_2
\]

with \( u'_{a_{m+1}} \neq id \). Thus, \( \sigma x \in U s_{m+1} w P_2 \), a contradiction.

**Case 2.** \( t \geq 2 \). Now, \( s_{m+t} \ldots s_{m+1} x \in U (s_{m+t} \ldots s_{m+1} w P_2) \). Hence, \( \sigma_1 \) must be of the form
\[
\sigma_1 \neq \phi s_{m+t} s_{m+t}, \text{ with } l(\phi) + 2 = l(\sigma_1).
\]

\[ \]
Thus,

\[ \sigma = \sigma_1(s_{m+t} \ldots s_{m+1}) \]

\[ = \phi(s_{m+t}s_{m+t-1}s_{m+t-2} \ldots s_{m+1}) \]

\[ = \phi(s_{m+t-1}s_{m+t-2} \ldots s_{m+1}). \]

Hence, we have \( l(\sigma) \leq l(\phi) + t = l(\sigma_1) + t - 2 < l(\sigma) \), which is a contradiction. Hence the lemma is proved.

The longest element of \( W^{I_2} \) is

\[ w^{I_2} = (s_{n-2} \cdot s_{n-3} \ldots s_1) \cdot (s_{n-1} \cdot s_{n-2} \ldots s_2) \]

and the unique minimal element \( \tau_2 \) of \( W^{I_2} \) such that \( \tau_2(n\omega_2) \leq 0 \) is

\[ \tau_2 = (s_{[\frac{n-1}{2}]} \cdot s_{[\frac{n+1}{2}]-1} \ldots s_1) \cdot (s_{n-1} \cdot s_{n-2} \ldots s_2). \]

Therefore any element \( w \in W^{I_2} \) such that \( X(w)^{I_2}(\mathcal{L}_2) \neq \emptyset \) is of the form

\[ w = (s_m \cdot s_{m-1} \ldots s_{\frac{n-1}{2}} \cdot s_{\frac{n+1}{2}} \ldots s_1) \cdot (s_{n-1} \cdot s_{n-2} \ldots s_2) \]

with \( m \geq \lceil \frac{n-1}{2} \rceil \).

**PROPOSITION 4.2**

Let \( r = 2, w = (s_m \ldots s_1)(s_{n-1} \ldots s_2), \lceil \frac{n-1}{2} \rceil \leq m \leq n-2 \). We can arrange the \( Y_{ij} \)'s as \( Y_1, Y_2, \ldots, Y_{m-1} \) with

\[ s_i(Y_i) = Y_{i+1}, \]

\[ s_i(Y_j) = Y_j \text{ if } j \neq i, \text{ } i+1 \text{ and } i = 1, 2, \ldots, m-2, \]

\[ s_{m-1}(Y_i) = Y_i \cdot Y_{m-1}^{-1}, \text{ if } i \leq m-2, \]

\[ s_{m-1}(Y_{m-1}) = Y_{m-1}^{-1}, \]

\[ s_m(Y_j) = 1 - Y_i \text{ for } i = 1, 2, \ldots, m-1. \]

Further, we have

\[ s_i(Y_j) = Y_j \quad \forall i = m + 2, \ldots, n-1, \text{ when } m \leq n-3 \]

and

\[ s_{n-1}(Y_j) = Y_j^{-1} \quad \forall j \text{ when } m = n-2. \]

**Proof.** Proof follows from Proposition 3.2.

Let \( w \) be as in Proposition 4.2. Now, let \( T_{m-1} \) be a maximal torus of \( \mathbb{P}GL_m, R_m \) is the root system of \( \mathbb{P}GL_m \). Here, the Weyl group is \( S_m \), the symmetric group on \( m \) symbols.

Let \( U = \{ t \in T; e^\alpha(t) \neq 1, \alpha \in R_m \} \). Clearly, \( U \) is \( S_m \)-stable. On the other hand, \( S_m \) stabilises \( (U_w P_2 / P_2)^{I_2}_T (\mathcal{L}_2) \). Let \( Y(w) = T \setminus (U_w P_2)^{I_2}_T (\mathcal{L}_2) \). Then, we have
COROLLARY 4.3
There is a $S_m$-equivariant isomorphism $\Psi_1: Y(w) \sim U$ such that

$$\Psi_1^*(e^{\alpha_1 + \cdots + \alpha_{m-1}}) = Y_i, \quad 1 \leq i \leq m-1.$$  

Proof. Proof follows from Proposition 4.2.

Let $h_m$ be a Cartan subalgebra of $sl_{m+1}$, $P(h_m)$ be the projective space and $R_m \subseteq h_m^*$ be the root system. Let $V_m$ be the open subset of $P(h_m)$ defined by

$$V_m := \{ x \in P(h_m) : \alpha(x) \neq 0, \quad \forall \alpha \in R_m \}.$$  

Clearly $V_m$ is $S_{m+1}$-stable. For a recent study of quotients of flag varieties modulo a maximal torus, see [3].

COROLLARY 4.4
Let $w = (s_{m} \cdots s_1)(s_{n-1} \cdots s_2), \left[ \frac{n-1}{2} \right] \leq m \leq n - 2$. Then, there is a $S_{m+1}$-equivariant isomorphism $\Psi_2: Y(w) \sim V_m$ of affine varieties.

Proof. For $i = 1, 2, \ldots, m-1$, take $Z_i = \frac{\alpha_1 + \cdots + \alpha_m}{\alpha_1}$ and define $\Psi_2$ such that $\Psi_2^*(Z_i) = Y_i$.

With notations as above and taking $t = \left[ \frac{n-3}{2} \right]$ and $m = \left[ \frac{n-1}{2} \right]$ we have the following.

Theorem. $N \setminus \{ G_2 \} \setminus (C_2)$ has a stratification $\bigcup_{i=0}^{t} C_i$, where $C_0 = s_{m+1} \setminus P(h_m)$, and $C_i = s_{m+1} \setminus V_i$.

Proof. Proof follows from Lemma 4.1, Proposition 4.2 and Corollary 4.4.

5. Flag variety as a GIT quotient of flag variety of higher dimension

Let $G = GL_{n+1}(k)$. Let $T$ be a maximal torus of $SL_{n+1}(k)$. Let $B_{n+1}$ be a Borel subgroup of $G$ containing $T$. Let $S = \{ \alpha_i : i = 1, 2, \ldots, n \}$ denote the set of simple roots with respect to $B_{n+1}$ and let $W = S_{n+1}$ be the Weyl group. Let $s_i$ be the simple reflection corresponding to the simple root $\alpha_i$. Let $I := S \setminus \{ \alpha_n \}$, and let $W_I$ be the subgroup of $W$ generated by $\{ s_i : i \in I \}$ and $w_0$, $T$ denote the longest element of $W_I$.

Lemma 5.1. Let $\chi = \sum_{i=1}^{n} m_i \alpha_i$ be a regular dominant character, where $m_i \in \mathbb{N}, m_i > m_1$ for $1 \leq i \leq n - 1$. Let $w \in W$. Then $w(\chi) \leq 0 \iff w = s_1 \cdot s_2 \cdots s_n \cdot \tau$ for some $\tau \in W_I$.

Proof. Since $\chi$ is dominant and $\tau \leq w_0$, for all $\tau \in W_I$, we have $\tau(\chi) \geq w_0(\tau(\chi))$; using the fact that $w_0(\tau)(\alpha_i) = -\alpha_{n-i}$ for $i = 1, \ldots, n - 1$ and $w_0, I(\alpha_n) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ we have $w_0, I(\chi) = \sum_{i=1}^{n-1} (m_n - m_{n-i})\alpha_i + m_n \cdot \alpha_n$. Therefore, $\tau(\chi) = \sum_{i=1}^{n} a_i \alpha_i + n \cdot \alpha_n$, $\alpha_i > 0$. Now, let $w = \phi \tau$ with $\phi \in W^I$, $\tau \in W_I$. Therefore, $w(\chi) = \phi(\tau(\chi)) = \phi(\sum_{i=1}^{n-1} a_i \alpha_i + n \cdot \alpha_n)$. Thus, $w(\chi) \leq 0$ implies that $\phi = s_1 \cdot s_2 \cdots s_n$. 

Let \( w = s_1 \cdot s_2 \ldots s_n \cdot \tau, \tau \in W_I \). Now,

\[
w(\chi) = s_1 \cdot s_2 \ldots s_n \tau(\chi) = s_1 \cdot s_2 \ldots s_n \left( \sum_{i=1}^{n-1} a_i \alpha_i + m_n \alpha_n \right) = -m_n \alpha_1 + \sum_{i=2}^{n} (a_{i-1} - m_n) \alpha_i.
\]

Since \( \chi \) is a dominant weight we have \( \chi - \tau(\chi) \geq 0 \). Hence we have \( a_i \leq m_i \leq m_n \). Thus \( w(\chi) \leq 0 \). This completes the proof.

Consider \( GL_n(k) \) as a subgroup of \( GL_{n+1}(k) \) given by the inclusion \( g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \). Let \( B_n = B_{n+1} \cap GL_n(k) \) as a Borel subgroup with \( I \) as the simple roots.

Let \( \chi \) be a regular dominant character as in Lemma 5.1.

**Theorem 5.2.** We have an isomorphism

\[
\Psi: T \setminus \left( (GL_{n+1}(k)/B_{n+1})^{ss}(L_\chi) \right) \sim GL_n(k)/B_n.
\]

**Proof.** Proof uses cellular decomposition of both homogeneous spaces \( GL_{n+1}(k)/B_{n+1} \) and \( GL_n(k)/B_n \). First, we fix a total order on the set of positive roots of \( B_{n+1} \) such that \( \sum_{i=1}^{n} q_i > \sum_{i=1}^{n-1} a_i > \cdots > a_1 > \sum_{i=2}^{n} a_i > \cdots > a_2 > \sum_{i=3}^{n} a_i > \cdots > a_3 > \cdots > a_{n-1} + a_n > a_n \). Now any \( GL_{n+1}/B_{n+1} \) (resp. \( GL_n/B_n \)) is the union of cells \( U_\alpha w B_{n+1} \) (resp. \( U_\alpha \tau B_n \)) with \( w \in W \) (resp. \( \tau \in W_I \)). Using the total order above we can write each element \( x \in U_\alpha \) as a product of \( u_\alpha \) in the decreasing order from left to right.

Let \( X_\alpha \) (resp. \( Y_\alpha \)) be the co-ordinate function on \( U_\alpha w B_{n+1} \) (resp. \( U_\alpha \tau B_n \)) corresponding to the root \( \alpha \) (resp. \( \beta \)).

With these notations we proceed with the proof.

Let \( \tau \in W_I \). Let \( w := s_1 s_2 \ldots s_n \tau \) and let \( V_\tau^0 := \{ x \in U_w w B_{n+1}: X_\alpha(x) \neq 0 \forall \alpha \geq \alpha_1 \} \).

Set \( V_\tau^0 := \bigcup_{\tau \in W_I} V_\tau^0 \).

**Step 1.** We prove that \( (GL_{n+1}(k)/B_{n+1})^{ss}(L_\chi) \subset V_0^0 \).

This can be seen as follows:

By Hilbert–Mumford criterion (see Theorem 2.1 of [5]), a point \( x \in GL_{n+1}/B_{n+1} \) is semi-stable \( \Leftrightarrow \mu^I(x, \lambda) \geq 0 \) for all 1-parameter subgroups \( \lambda \) of \( T \) \( \Leftrightarrow \mu^I(\sigma x, \lambda) \geq 0 \) for all one-parameter subgroups \( \lambda \in \mathbb{C} \) and for all \( \sigma \in W \). By Lemma 2.1 of [8], this statement is equivalent to \( -w_\sigma \chi, \lambda \geq 0 \) for all \( \lambda \in \mathbb{C} \) where \( \sigma x \in U_{w_\sigma w} \). But this is equivalent to \( w_\sigma(\chi) \leq 0 \). By Lemma 5.1, this is equivalent to \( w_\sigma \) being of the form \((s_1 \ldots s_n) \cdot \tau_1\) for some \( \tau_1 \in W_I \).

Now let \( x \in U_w w B_{n+1} \) with \( w = (s_1 \ldots s_n) \tau, \tau \in W_I \).

Now, let \( X_\alpha(x) = 0 \) for some \( \alpha \geq \alpha_1 \). Let \( \alpha = \sum_{j=1}^{n} \alpha_j \). Then, we have \( s_1 s_2 \ldots s_n x = u \phi B_n + 1 \) with \( \phi \neq s_1 \ldots s_n \tau \) for any \( \tau \in W_I \). Hence, by the above discussion, \( x \) is not semi-stable.

**Step 2.** \( (GL_{n+1}(k)/B_{n+1})^{ss}(L_\chi) = V_0^0 \). This can be seen by the above discussion and from the following Claim.
Proof of Claim. Let $\tau \in W_1$, $\tau' = s_1 s_2 \ldots s_n \tau$ and $x \in U_1 \tau' B_{n+1}$, with $X_\alpha(x) \neq 0$ for all $\alpha \geq \alpha_1$. Then, we have $s_1 x \in U_1 \tau' B_{n+1}$ with $X_\alpha(s_1 x) = -\frac{X_\alpha(x)}{X_{\alpha_1}(x)}$ for $\alpha > \alpha_1$, and $X_{\alpha_1}(s_1 x) = \frac{1}{X_{\alpha_1}(x)}$. Hence, $s_1 x \in V^0$.

Now, let $i \neq 1$. If $X_{\alpha_1}(x) = 0$, then, $s_i x = u's_1 s_2 \ldots s_{n-1} \tau B_{n+1}$ with $X_\alpha(s_i x) = X_{\alpha_1}(x)$. Hence, $s_i x \in V^0$. Otherwise, we must have $s_i x \in U_1 \tau' B_{n+1}$ with $X_\alpha(s_i x) = X_{\alpha}(x)$ for all $\alpha$ such that $s_i(x) = \alpha$, $X_\alpha(s_i x) = X_{\alpha}(x)$ for all $\alpha$ of the form $\alpha = \sum_{j=k}^i \alpha_j$ such that $k < i$. $X_{\alpha_1}(s_i x) = \frac{1}{X_{\alpha_1}(x)}$, and $X_\alpha(s_i x) = \frac{X_{\alpha}(x)}{X_{\alpha_1}(x)}$ for all $\alpha$ of the form $\alpha = \sum_{j=k}^i \alpha_j$ such that $k > i$.

Hence, $s_i V^0 \subseteq V^0$ for all $i = 1, \ldots, n$. Thus, the Claim follows from the fact that $W$ is generated by $s_i$'s.

**Step 3.** Now, for each $\tau \in W_1$, we exhibit an isomorphism

$$\Psi_\tau: T \setminus V^0_\tau \sim \sim U_1 \tau B_n / B_n.$$ 

Let $\tau \in W_1$, consider the map $\pi_\tau : V^0_\tau \longrightarrow (U_1 \tau B_n) / B_n$ defined by $\pi_\tau(x) = y$ with $Y_{x_{n-1}}(y) = \left( -\frac{X_{\beta}(x)}{X_{\beta}(\pi_\tau(x))} \right)$ where for each $\beta \in R(w^{-1})$ with $\beta \geq \alpha_1$, $\beta'$ is the unique element of $R^+$ with $\beta' \geq \alpha_1$ such that $\beta + \beta' \in R^+$. Clearly this map is $T$-invariant. Thus the morphism $\pi_\tau$ gives rise to a morphism

$$\Psi_\tau: T \setminus V^0_\tau \longrightarrow U_1 \tau B_n / B_n.$$ 

Clearly, $\Psi_\tau$ is surjective. We now prove that $\Psi_\tau$ is injective:

$\pi_\tau$ is injective for each $w \in W$ of the form $w = s_1 \cdot s_2 \ldots s_n \tau$, for some $\tau \in W_1$.

Let $x_1$ and $x_2$ be two points of $V^0_\tau$ such that $\pi_\tau(x_1) = \pi_\tau(x_2)$. Hence, $\frac{X_{\beta}(x_1)}{X_{\beta}(\pi_\tau(x_1))} = \frac{X_{\beta}(x_2)}{X_{\beta}(\pi_\tau(x_2))}$. Let $t \in T$ be such that $(\alpha_1 + \cdots + \alpha_i) = \frac{X_{\alpha_1} + \cdots + X_{\alpha_i}}{X_{\alpha_1} + \cdots + X_{\alpha_i}(x)}$ for all $i$, $1 \leq i \leq n$. Then, it is easy to check that $t \cdot x_1 = x_2$. Thus $\Psi_\tau$ is injective for each $\tau \in W_1$.

**Step 4.** $\Psi$ puts together to give an isomorphism

$$\Psi: T \setminus V^0 \sim \sim G L_n(k) / B_n.$$ 

Since the $W$-translates of $V^0_{w_0, \tau}$ is the whole of $V^0$, and $W_1$-translates of $U_{w_0, \tau} B_n$ is the whole of $G L_n / B_n$, and there is an isomorphism from $W_3 \setminus \{\alpha_1\}$ to $W_1$ taking $s_j$ to $s_{j-1}$ for each $j = 2, \ldots, n$, to prove the theorem, it is sufficient to prove that the $T$-invariant morphisms $\pi_\tau : V^0_\tau \longrightarrow U_1 \tau B_n / B_n$, and $\pi_\tau(x) = \pi_\tau(s_{j-1} \tau) \cdot V^0_{(s_{j-1} \tau)} \rightarrow U_{(s_{j-1} \tau)} B_n / B_n$ satisfy the following:

$$Y_\alpha(\pi_\tau(x)) = Y_\alpha(s_{j-1} \pi_\tau(s_{j-1} \tau) \cdot (s_{j-1} \tau)) \text{ for all } \alpha \in R(\tau^{-1}).$$

(Here, the notation $(s_{j-1} \tau)^{-} = \tau$ if $s_{j-1} \tau < \tau$ and $(s_{j-1} \tau)^{-} = s_{j-1} \tau$ otherwise.)
We make use of the following observations using commutator relations:

\[
X_\alpha(s_i(x)) = \begin{cases} 
-\frac{X_{\alpha_i}(x)}{X_{\alpha_i}(\tau)} & \text{if } \alpha = \alpha_i, \\
\frac{1}{X_{\alpha_i}(\tau)} & \text{if } \alpha = \alpha_i + \cdots + \alpha_k \quad i \leq k \\
X_{\alpha_i}(x) & \text{otherwise.}
\end{cases}
\]

Let \( \alpha \in R(\tau^{-1}) \).

**Case 1.** \( \alpha = \alpha_{k-1} + \cdots + \alpha_{i-1}, \quad k < i, \quad w^{-1}(\alpha_{k-1} + \cdots + \alpha_i) = \tau^{-1}(\alpha_{k-1} + \cdots + \alpha_{i-1}) > 0 \) and \((s_{i-1}\tau)^{-1} = \tau\).

In this case,

\[
Y_\alpha(s_i(\pi_\tau(x))) = \frac{X_{\alpha_i} + \cdots + \alpha_{k-1}(x)X_{\alpha_{k-1} + \cdots + \alpha_i}(x)}{X_{\alpha_i} + \cdots + \alpha_{k-1}}(x) = Y_\alpha(\pi_\tau(s_i x)).
\]

**Case 2.** \( \alpha = \alpha_{i-1} + \cdots + \alpha_{k-1}, \quad i < k \quad \text{and} \quad w^{-1}(\alpha_{i-1} + \cdots + \alpha_{k}) = \tau^{-1}(\alpha_{i-1} + \cdots + \alpha_{k-1}) > 0 \) and \((s_{i-1}\tau)^{-1} = \tau\).

In this case,

\[
Y_\alpha(s_i(\pi_\tau(x))) = \frac{-X_{\alpha_i} + \cdots + \alpha_{k-1}(x)X_{\alpha_{k-1} + \cdots + \alpha_i}(x)}{X_{\alpha_i} + \cdots + \alpha_{k-1}}(x) = Y_\alpha(\pi_\tau(s_i x)).
\]

**Case 3.** \( \alpha = \alpha_{i-1} \).

\[
Y_\alpha(s_i(\pi_\tau(x))) = \frac{X_{\alpha_i} + \cdots + \alpha_{k-1}(x)X_{\alpha_{k-1} + \cdots + \alpha_i}(x)}{X_{\alpha_i} + \cdots + \alpha_{k-1}}(x) = Y_\alpha(\pi_\tau(s_i x)).
\]

In all other cases, we have

\[
Y_\alpha(s_i(\pi_\tau(x))) = \frac{X_{\alpha_i} + \cdots + \alpha_{k-1}(x)X_{\alpha_{k-1} + \cdots + \alpha_i}(x)}{X_{\alpha_i} + \cdots + \alpha_{k-1}}(x) = Y_\alpha(\pi_\tau(x)),
\]

where \( \beta' \) is the unique root such that \( \beta' \geq \alpha_1 \) and \( s_1 \ldots s_n(\alpha) + \beta' \) is a root.

This completes the proof. \( \square \)

With \( Y_\alpha \)'s as in the proof of Theorem 5.2, we have the following.

**COROLLARY 5.3**

\[
s_1(Y_\alpha) = \begin{cases} 
-(1 + Y_\alpha) & \text{if } \alpha \geq \alpha_1 \\
Y_\alpha & \text{otherwise.}
\end{cases}
\]

**Proof.** Proof follows from the fact that

\[
X_\alpha(s_{1,x}) = \begin{cases} 
X_{\alpha_1}X_{\alpha}(x) + X_{\alpha_1+\alpha}(x) & \text{if } \alpha = \alpha_1 + \cdots + \alpha_i, \quad i \geq 2, \\
-\frac{X_{\alpha_1}(x)}{X_{\alpha_1}(\tau)} & \text{if } \alpha = \alpha_1 + \cdots + \alpha_i, \quad i \geq 2, \\
X_\alpha(x) & \text{if } \alpha = \alpha_3 + \cdots + \alpha_i, \quad i \geq 3.
\end{cases}
\]

\( \square \)
COROLLARY 5.4

Let $h_n$ be a Cartan subalgebra of $sl_{n+1}(k)$. Let $\chi$ be a regular dominant character as in Theorem 5.2. Then, the action of $W$ on the GIT quotient

$$T\backslash(GL_{n+1}(k)/B_{n+1})^{\chi} \simeq GL_n(k)/B_n$$

is given by the $n$-dimensional representation $h_n$ of $W$.

Proof. Proof follows from Theorem 5.2 and Corollary 5.3.

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