Towards a Theory of Domains for Harmonic Functions and its Symbolic Counterpart

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Abstract In this paper, we begin by reviewing the calculus induced by the framework of [10]. In there, we extended Polylogarithm functions over a subalgebra of noncommutative rational power series, recognizable by finite state (multiplicity) automata over the alphabet \( X = \{ x_0, x_1 \} \). The stability of this calculus under shuffle products relies on the nuclearity of the target space [32]. We also concentrated on algebraic and analytic aspects of this extension allowing to index polylogarithms, at non positive multi-indices, by rational series and also allowing to regularize divergent polyzetas, at non positive multi-indices [10]. As a continuation of works in [10] and in order to understand the bridge between the extension of this “polylogarithmic calculus” and the world of harmonic sums, we propose a local theory, adapted to a full calculus on indices of Harmonic Sums based on the Taylor expansions, around zero, of polylogarithms with index \( x_1 \) on the rightmost end. This theory is not only compatible with Stuffle products but also with the Analytic Model. In this respect, it provides a stable and fully algorithmic model for Harmonic calculus. Examples by computer are also provided.

Keywords Theory of domains · Harmonic sums · Polylogarithms · Summable.

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1 Introduction

Riemann’s zeta function is defined by the series
\[ \zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} \quad (1.1) \]
where \( s \) is a complex number. It is absolutely convergent for \( \Re(s) > 1 \) (for any \( s \in \mathbb{C} \), \( \Re(s) \) stands for the real part of \( s \)).

It can be extended to a meromorphic function on the complex plane \( \mathbb{C} \) with a single pole at \( s = 1 \). In fact, the story began with Euler’s works to find the solution of Basel’s problem. In these works, Euler proved that \[ \zeta(2) = \frac{\pi^2}{6} \quad (1.2) \]
Moreover, for suitable \( s_1, s_2 \), Euler gave an important identity as follows:
\[ \zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_1 + s_2) + \zeta(s_2, s_1), \quad (1.3) \]
where
\[ \zeta(s_1, s_2) := \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{s_1}n_2^{s_2}}. \quad (1.4) \]

The numbers \( \zeta(s_1, s_2) \) were called “double zeta values” at \( (s_1, s_2) \). More generally, for any \( r \in \mathbb{N}_+ \) and \( s_1, \ldots, s_r \in \mathbb{C} \), we denote
\[ \zeta(s_1, \ldots, s_r) := \sum_{n_1 > \ldots > n_r \geq 1} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}. \quad (1.5) \]

Then the results of K. Matsumoto [16] showed that the series \( \zeta(s_1, \ldots, s_r) \) converges absolutely for \( s \in \mathcal{H}_r \) where
\[ \mathcal{H}_r := \{ s = (s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r; \Re(s_1) + \ldots + \Re(s_m) > m \}. \quad (1.6) \]

In the convergent cases, \( \zeta(s_1, \ldots, s_r) \) were called “polyzeta values” at multi-index \( s = (s_1, \ldots, s_r) \). Indeed \( s \mapsto \zeta(s) \) is holomorphic on \( \mathcal{H}_r \) and has been extended to \( \mathbb{C}^r \) as a meromorphic function (see [17, 34]).

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1 Whence the famous sum \( \zeta(-1) = 1 + 2 + 3 + \cdots = -\frac{1}{12} \) by which, among other “results”, S. Ramanujan was noticed by G. H. H. Hardy (see [1]).

2 In fact, in Euler’s formula, \( s_1, s_2 \in \mathbb{N}_+ \). This identity appeared under the name “Prima Methodus...” (see [15] pp 141-144).
In fact, for any r-uplet \((s_1, \ldots, s_r) \in \mathbb{N}_+^r, r \in \mathbb{N}_+\), the polyzeta \(\zeta(s_1, \ldots, s_r)\) is also the limit at \(z = 1\) of the polylogarithmic function, defined by:

\[
\text{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}}
\]

(1.7)

for any \(z \in \mathbb{C}\) such that \(|z| < 1\). It is easily seen that, for any \(s_i \in \mathbb{N}_+, r > 1\),

\[
\frac{d}{dz} \text{Li}_{s_1,\ldots,s_r}(z) = \text{Li}_{s_1-1,\ldots,s_r}(z) \text{ if } s_1 > 1
\]

(1.8)

\[
(1 - z) \frac{d}{dz} \text{Li}_{1,s_2,\ldots,s_r}(z) = \text{Li}_{s_2,\ldots,s_r}(z) \text{ if } r > 1
\]

and this formulas will be ended at the “seed” \(\text{Li}_1(z) = \log \left( \frac{1}{1-z} \right)\).

Moreover, if \(X^*\) is the free monoid of rank two (generators, or the alphabet, \(X = \{x_0, x_1\}\) and the neutral \(1_{X^*}\)) then the polylogarithms indexed by a list

\((s_1, \ldots, s_r) \in \mathbb{N}_+^r\) can be reindexed by the word \(x_0^{s_1-1} x_1 \cdots x_0^{s_r-1} x_1 \in X^* x_1\)

(1.9)

In order to reverse the recursion introduced in Equations (1.8) (two equations), we introduce two differential forms

\[
\omega_0(z) = z^{-1} dz \text{ and } \omega_1(z) = (1 - z)^{-1} dz,
\]

(1.10)

on \(\Omega^3\). We then get an integral representation\(^4\) of the functions (1.7) as follows\(^5\) [20]

\[
\text{Li}_w(z) = \begin{cases} 
1_{\mathcal{H}(\Omega)} & \text{if } w = 1_{X^*} \\
\int_0^z \omega_1(s) \text{Li}_w(s) \text{ if } w = x_1u \\
\int_0^z \omega_0(s) \text{Li}_w(s) \text{ if } w = x_0u \text{ and } |u|_{x_1} = 0, i.e. w \in x_0^* \\
\int_0^z \omega_0(s) \text{Li}_w(s) \text{ if } w = x_0u \text{ and } |u|_{x_1} > 0, i.e. w \notin x_0^*,
\end{cases}
\]

(1.11)

the upper bound \(z\) belongs to \(\Omega\) (as \(\Omega = \mathbb{C} \setminus (\infty, 0] \cup [1, +\infty]\)) is a simply connected domain, the integrals, which can be proved to be convergent in all cases, depend only on their bounds). The neutral element of the algebra of analytic functions \(\mathcal{H}(\Omega)\), a constant function, will be here denoted \(1_{\mathcal{H}(\Omega)}\).

This provides not only the analytic continuation of (1.7) to \(\Omega\) but also extends the indexation to the whole alphabet \(X\), allowing to study the complete generating series

\[
L(z) = \sum_{w \in X^*} \text{Li}_w(z) w
\]

(1.12)

and show that it is the solution of the following first order noncommutative differential equation

\[
\begin{align*}
\text{d}(S) &= (\omega_0(z)x_0 + \omega_1(z)x_1)S, & (NCDE) \\
\lim_{z \in \Omega, z \to 0} S(z)e^{-x_0\log(z)} &= 1_{\mathcal{H}(\Omega)\langle X \rangle}, & \text{asymptotic initial condition,}
\end{align*}
\]

(1.13)

where, for any \(S \in \mathcal{H}(\Omega)\langle X \rangle\).

Through term by term derivation, one gets [13]

\[
\text{d}(S) = \sum_{w \in X^*} \frac{d}{dz}(\langle S \mid w \rangle)w.
\]

(1.14)

\(^3\) \(\Omega\) is the simply connected domain \(\mathbb{C} \setminus (\infty, 0] \cup [1, +\infty]\).

\(^4\) In here, we code the moves \(z \frac{d}{dz}\) (resp. \((1 - z) \frac{d}{dz}\)) - or more precisely sections \(\int_0^z f(s) ds\) (resp. \(\int_0^z f(s) / (1 - s) ds\)) - with \(x_0\) (resp. \(x_1\)).

\(^5\) Given a word \(w \in X^*\), we note \(|w|_{x_1}\) the number of occurrences of \(x_1\) within \(w\).
This differential system allows to show that \( L \) is a \( \omega \)-character\(^6\) [24], \textit{i.e.}
\[
\forall u, v \in X^*, \quad \langle L \mid u \cdot w \cdot v \rangle = \langle L \mid u \rangle \langle L \mid v \rangle \text{ and } \langle L \mid 1_{X^*} \rangle = 1_{\mathcal{H}(\Omega)}.
\] (1.15)

Note that, in what precedes, we used the pairing \( \langle \bullet \mid \bullet \rangle \) between series and polynomials, classically defined by, for \( T \in \mathcal{k}(\langle X \rangle) \) and \( P \in \mathcal{k}(X) \)
\[
\langle T \mid P \rangle = \sum_{w \in X^*} \langle T \mid w \rangle \langle P \mid w \rangle,
\] (1.16)
where, when \( w \) is a word, \( \langle S \mid w \rangle \) stands for the coefficient of \( w \) in \( S \) and \( \mathcal{k} \) any commutative ring (as here \( \mathcal{H}(\Omega) \)).

With this at hand, we extend at once the indexation of \( L_i \) from \( X^* \) to \( \mathbb{C}(X) \) by
\[
L_i \mathcal{P} := \sum_{w \in X^*} \langle P \mid w \rangle \text{Li}_i w = \sum_{n \geq 0} \left( \sum_{|w| = n} \langle P \mid w \rangle \text{Li}_i w \right).
\] (1.17)

In [10], it has been established that the polylogarithm, well defined locally by (1.7), could be extended to some series (with conditions) by the last part of formula (1.17) where the polynomial \( P \) is replaced by some series. A complete theory of global domains was presented in [10], the present work concerns the whole project of extending \( \mathcal{H}_x \) \([9,19]\) over stuffle subalgebras of rational power series on the alphabet \( Y \), in particular the stars of letters and some explicit combinatorial consequences of this extension.

In fact, we focus on what happens in (well chosen) neighbourhoods of zero (see section 3), therefore, the aim of this work is manifold.

(a) Use the extension to local Taylor expansions\(^7\) as in (1.7) and the coefficients of their quotients by \( 1 - z \), namely the harmonic sums, denoted \( \mathcal{H}_x \) and defined, for any \( w \in X^*x_1 \), as follows\(^8\) ([22] see also related literature [4,19])
\[
\text{Li}_w(z) = \sum_{N \geq 0} \mathcal{H}_x(w)(N)z^N,
\] (1.18)
by a suitable theory of local domains which assures to carry over the computation of these Taylor coefficients and preserves the stuffle indentity, again true for polynomials over the alphabet \( Y = \{y_n\}_{n \geq 1} \), \textit{i.e.}\(^9\)
\[
\forall S, T \in \mathbb{C}(Y), \quad \mathcal{H}_x(S \cdot \omega \cdot T) = \mathcal{H}_x(S) \mathcal{H}_x(T) \text{ and } \mathcal{H}_x(1_{\mathbb{C}(Y)}) = 1_{\mathbb{C}(Y)} = 1_{\mathbb{C}^N}, \quad (1.19)
\]

note that \( 1_{\mathbb{C}(Y)} \) is identified with \( 1_{Y^*} \) and \( 1_{\mathbb{C}^N} \) is the constant (to one) function\(^10\) \( \mathbb{N} \to \mathbb{C} \). This means that \( \mathcal{H}_x : (\mathbb{C}(Y), \ \omega, \ 1_{Y^*}) \to (\mathbb{C}[w_{w \in Y^*}, x, 1) \)
mapping any word \( w = y_{s_1} \ldots y_{s_r} \in Y^* \)
\[
\mathcal{H}_w = \mathcal{H}_{s_1, \ldots, s_r} = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}
\] (1.20)
is a \( \omega \) (unital) morphism\(^11\).

\^6\text{ Here, the shuffle product is denoted by} \omega. \text{ Its definition is classical and recalled in the equation (5.1) of section 5.}

\^7\text{ Around zero.}

\^8\text{ Here, the conc-morphism} \pi_X : (\mathbb{C}(Y), \ \text{conc}, \ 1_{Y^*}) \to (\mathbb{C}(X), \ \text{conc}, \ 1_{X^*}) \text{ is defined by} \pi_X(y_n) = x_0^{n-1}x_1 \text{ and} \pi_Y \text{ is its inverse on} \text{Im}(\pi_X). \text{ See} [6,10] \text{ for more details and a full definition of} \pi_Y.

\^9\text{ Here,} \omega \text{ stands for the stuffle product which will be recalled as in the section 5.}

\^10\text{ In fact, it could be} \mathbb{Q} \text{ but we will use afterwards} \mathbb{C}-\text{linear combinations.}

\^11\text{ In fact, it was shown that this morphism is into, see} [22].
(b) Extend these correspondences \( i.e. \text{Li}_*, H_* \) to some series (over \( X \) and \( Y \), respectively) in order to preserve the identity\(^{12}\)\(^{22}\)

\[
\frac{\text{Li}_{\pi_X(S)}(z)}{1-z} \odot \frac{\text{Li}_{\pi_X(T)}(z)}{1-z} = \frac{\text{Li}_{\pi_X(S \shuffle T)}(z)}{1-z}
\]

true for polynomials \( S, T \in \mathbb{C}[Y] \).

To this end, we use the explicit parametrization of the conc-characters obtained in \(^{10}\) and the fact that, under the stuffle products, they form a group.

### 2 Polylogarithms: from Global to Local Domains

Now we are facing the following constraint:

In order that the results given by symbolic computation reflect the reality with complex numbers (and analytic functions), we have to introduce some topology\(^{13}\).

Let \( \mathcal{H}(\Omega) = C^\omega(\Omega; \mathbb{C}) \) be the algebra (for the pointwise product) of complex-valued functions which are holomorphic on \( \Omega \). Endowed with the topology of compact convergence \(^{14}\), it is a nuclear space\(^{15}\).

**Definition 1**

(i) Let \( S \in \mathbb{C}[[X]] \) be a series decomposed in its homogeneous (w.r.t. the length) components

\[
S_n = \sum_{|w|=n} \langle S \mid w \rangle w
\]

(so that \( S = \sum_{n \geq 0} S_n \)) is in the domain of \( \text{Li} \) iff the family \( (\text{Li}_{S_n})_{n \geq 0} \) is summable in \( \mathcal{H}(\Omega) \) in other words, due to the fact that the space is complete (see \(^{32}\)), iff one has

\[
(\forall W \in \mathcal{B}_\mathcal{H}(\Omega)) (\exists F \subset \text{finite } \mathbb{N})(\forall F' \subset \text{finite } (\mathbb{N} \setminus F)), \left( \sum_{j \in F'} \text{Li}_{S_j} \in W \right)
\]

where \( \mathcal{B}_\mathcal{H}(\Omega) \) is the set of neighbourhoods of 0 in \( \mathcal{H}(\Omega) \).

(ii) The set of these series will be noted \( \text{Dom}(\text{Li}) \) and, for \( S \in \text{Dom}(\text{Li}) \), the sum \( \sum_{n \geq 0} \text{Li}_{S_n} \) will be noted \( \text{Li}_S \).

Of course, the criterium (2.1) is only a theoretical tool to establish properties of the domain of \( \text{Li} \). In further calculations (i.e. in practice), we will not use it but the stability of the domain under certain operations.

**Example 1**\(^{190}\) For example, the classical polylogarithms: dilogarithm \( \text{Li}_2 \), trilogarithm \( \text{Li}_3 \), etc... are defined and obtained through the coding (1.9) by

\[
\text{Li}_k(z) = \sum_{n \geq 1} \frac{z^n}{n^k} = \text{Li}_{x_0^{k-1} x_1}(z) = \langle L(z) \mid x_0^{k-1} x_1 \rangle
\]

(where \( L(z) \) is as in Equation (1.12)) but, one can check that, for \( t \geq 0 \) (real), the series \((tx_0)^*x_1 \) belongs to \( \text{Dom}(\text{Li}_k) \) (see Definition 1. (ii)) iff \( 0 \leq t < 1 \). In fact, in this case,

\[
\text{Li}_{(tx_0)^*x_1}(z) = \sum_{n \geq 1} \frac{z^n}{n-t}.
\]

This opens the door to Hurwitz polyzetas \(^{25}\).

\(^{12}\) Here \( \odot \) stands for the Hadamard product \(^{18}\).

\(^{13}\) Readers who are not keen on topology or functional analysis may skip the details of this section and hold its conclusions.

\(^{14}\) This topology is defined by the seminorms (where \( K \subset \Omega \) is compact) \( p_K(f) = \sup_{s \in K} |f(s)| \).

\(^{15}\) Space where commutatively convergent and absolutely convergent series are the same. This will allow the domain of the polylogarithm to be closed by shuffle products (i.e. the possibility to compute legal polylogarithms through shuffle products).
The map $\text{Li}_\bullet$ is now extended to a subdomain of $\mathbb{C} \langle \langle X \rangle \rangle$, called $\text{Dom}(\text{Li}_\bullet)$ (see also [6, 10]).

**Example 2** For any $\alpha, \beta \in \mathbb{C}$, $(\alpha x_0)^* \ast (\beta x_1)^*$ and $(\alpha x_0 + \beta x_1)^* = (\alpha x_0)^* \cup (\beta x_1)^*$. We have

$$\text{Li}_{(\alpha x_0)^*}(z) = z^\alpha; \quad \text{Li}_{(\beta x_1)^*}(z) = (1 - z)^{-\beta}; \quad \text{Li}_{(\alpha x_0 + \beta x_1)^*}(z) = z^\alpha (1 - z)^{-\beta}$$

where $z \in \Omega$.

**Proposition 1** (i) The domain $\text{Dom}(\text{Li})$ is a shuffle subalgebra of $(\mathbb{C} \langle \langle X \rangle \rangle, \cup, 1_X^*)$.

(ii) The extended polylogarithm $\text{Li} : \text{Dom}(\text{Li}) \to \mathcal{H}(\Omega)$ is a shuffle morphism, i.e. $S, T \in \text{Dom}(\text{Li})$, we still have

$$\text{Li}_{S \cup T} = \text{Li}_S \text{Li}_T \quad \text{and} \quad \text{Li}_{1_{X^*}} = 1_{\mathcal{H}(\Omega)} \quad (2.2)$$

**Proof** This proof has been done in [10].

The picture about $\text{Dom}(\text{Li})$ within the algebra $(\mathbb{C} \langle \langle X \rangle \rangle, \cup, 1_X^*)$, the positioning of $\mathbb{C}^{\text{rat}} \langle \langle X \rangle \rangle$ (rational series, see [2, 6, 10]) and shuffle subalgebras as, for example, $\mathcal{A} = \mathbb{C} \langle X \rangle \cup \mathbb{C}^{\text{rat}} \langle x_0 \rangle \cup \mathbb{C}^{\text{rat}} \langle x_1 \rangle$ read as follows:

3 From Polylogarithms to Harmonic sums

Definition of $\text{Dom}(\text{Li})$ has many merits and can easily be adapted to arbitrary (open and connected) domains. However this definition, based on a global condition of a fixed domain $\Omega$, does not provide a sufficiently clear interpretation of the stable symbolic computations around a point, in particular at $z = 0$. One needs to consider a sort of “symbolic local germ” worked out explicitly. Indeed, as the harmonic sums (or MZV$^{17}$) are the coefficients of the Taylor expansion at zero of the convergent polylogarithms divided by $1 - z$, we only need to know locally these functions. In order to gain more indexing series and to describe the local situation at zero, we reshape and define a new domain of $\text{Li}$ around zero to $\text{Dom}^{\text{loc}}(\text{Li}_\bullet)$.

The first step will be to characterize the polylogarithms having a removable singularity at zero. The following proposition helps us characterize their indices.

**Proposition 2** Let $P \in \mathbb{C}(X)$ and $f(z) = \langle L \mid P \rangle = \sum_{w \in X^*} \langle P \mid w \rangle \text{Li}_w$.

1. The following conditions are equivalent

   (i) $f$ can be analytically extended around zero.

---

$^{16}$ As the fact that, due to special properties of $\mathcal{H}(\Omega)$ (it is a nuclear space [32]), one can show that $\text{Dom}(\text{Li})$ is closed by shuffle products.

$^{17}$ Multiple Zeta Values.
(ii) \( P \in \mathbb{C}(X)x_1 \oplus \mathbb{C}.1_{X^*} \).

2. In this case \( \Omega \) itself\(^{18}\) can be extended to \( \Omega_1 = \mathbb{C} \setminus (\{ \infty, -1 \} \cup [1, +\infty[) \).

**Proof** (Sketch) (ii) \( \Longleftrightarrow \) (i) being straightforward, it remains to prove that (ii) \( \Longleftrightarrow \) (i). Let then \( P \in \mathbb{C}(X) \) such that \( f(z) = \langle L | P \rangle \) has a removable singularity at zero. As a consequence of Radford’s results [30], one can write down a basis of any free shuffle algebra in terms of Lyndon words. This implies that our polynomial reads

\[
P = \sum_{k \geq 0} \alpha_k (P_k \uplus x_0^{\uparrow k}) \quad \text{with} \quad \alpha_k \in \mathbb{C}, \; P_k \in \mathbb{C}(X)x_1 \oplus \mathbb{C}.1_{X^*} \tag{3.1}
\]

the family \( (P_k)_{k \geq 0} \) being unique and finitely supported. Using (3.1) and (1.15), we get

\[
\text{Li}_P(z) = \sum_{k \geq 0} \alpha_k \text{Li}_{P_k}(z) \log(z)^k
\]

the result now follows easily using asymptotic scale \( x^n \log(x)^m \) along the axis \( ]0, +\infty[ \) (and for \( x \to 0_+ \)). \( \square \)

The second step will be provided by the following Proposition which says that, for appropriate series, the Taylor coefficients behave nicely.

**Proposition 3** Let \( S \in \mathbb{C}\langle\langle X \rangle\rangle x_1 \oplus \mathbb{C}.1_{X^*} \) such that \( S = \sum_{n \geq 0} [S]_n \) where

\[
[S]_n = \sum_{w \in X^*, |w|=n} \langle S | w \rangle w, \quad ([S]_n \text{ are the homogeneous components of } S),
\]

we suppose that \( 0 < R \leq 1 \) and that \( \sum_{n \geq 0} \text{Li}_{[S]_n} \) is unconditionally convergent (for the standard topology) within the open disk \( |z| < R \).\(^{19}\) Remarking that \( \frac{1}{1 - z} \sum_{n \geq 0} \text{Li}_{[S]_n}(z) \) is unconditionally convergent in the same disk, we set

\[
\frac{1}{1 - z} \sum_{n \geq 0} \text{Li}_{[S]_n}(z) = \sum_{N \geq 0} a_N z^N.
\]

Then, for all \( N \geq 0 \),

\[
\sum_{n \geq 0} H_{\pi_Y([S]_n)}(N) = a_N.
\]

**Proof** Let us recall that, for any \( w \in X^* \), the function \( (1 - z)^{-1} \text{Li}_w(z) \) is analytic in the open disk \( |z| < R \). Moreover, one has

\[
\frac{1}{1 - z} \text{Li}_w(z) = \sum_{N \geq 0} H_{\pi_Y(w)}(N) z^N.
\]

\(^{18}\) The domain, for \( z \) of \( \text{Li}_P \).

\(^{19}\) With the definition given later (3.2) this amounts to say that

\( S \in \mathbb{C}\langle\langle X \rangle\rangle x_1 \oplus \mathbb{C}.1_{X^*} \cap \text{Dom}_R(\text{Li}) \).
Since \([S]_n = \sum_{w \in X^*, |w| = n} \langle S \mid w \rangle w\) and \((1 - z)^{-1} \sum_{n \geq 0} \text{Li}_n\) absolutely converges (for the standard topology\(^{20}\)) within the open disk \(D_{<R}\), one obtains, for all \(|z| < R\)
\[
\frac{1}{1 - z} \sum_{n \geq 0} \text{Li}_n(z) = \frac{1}{1 - z} \sum_{n \geq 0} \sum_{w \in X^*, |w| = n} \langle S \mid w \rangle \text{Li}_w(z)
\]
\[
= \sum_{n \geq 0} \sum_{w \in X^*, |w| = n} \langle S \mid w \rangle \frac{\text{Li}_w(z)}{1 - z}
\]
\[
= \sum_{n \geq 0} \sum_{w \in X^*, |w| = n} \langle S \mid w \rangle \sum_{N \geq 0} \text{H}_{\text{Sy}(w)}(N) z^N
\]
\[
= \sum_{N \geq 0} \sum_{n \geq 0} \text{H}_{\text{Sy}(\{S\}_n)}(N) z^N,
\]
(*) being possible because \(\sum_{w \in X^*, |w| = n}\) is finite. This implies that, for any \(N \geq 0\),
\[
a_N = \sum_{n \geq 0} \text{H}_{\text{Sy}(\{S\}_n)}(N).
\]
\[
\square
\]
To prepare the construction of the “symbolic local germ” around zero, let us set, in the same manner as in [6,10],
\[
\text{Dom}_R(\text{Li}) := \{ S \in \mathbb{C}[[X]]x_1 \oplus \mathbb{C}X^* \mid \sum_{n \geq 0} \text{Li}_n\} \text{ is unconditionally convergent in } \mathcal{H}(D_{<R})
\]
and prove the following:

**Proposition 4** With the notations as above, we have:

1. The map given by \(R \mapsto \text{Dom}_R(\text{Li})\) from \([0, 1]\) to \(\mathbb{C}[[X]]\) (the target is the set of subsets\(^{21}\) of \(\mathbb{C}[[X]]\) ordered by inclusion) is strictly decreasing
2. Each \(\text{Dom}_R(\text{Li})\) is a shuffle (unital) subalgebra of \(\mathbb{C}[[X]]\).

**Proof**

1. For \(0 < R_1 < R_2 \leq 1\) it is straightforward that \(\text{Dom}_{R_2}(\text{Li}) \subset \text{Dom}_{R_1}(\text{Li})\). Let us prove that the inclusion is strict. Take \(|z| < 1\) and let us, be it finite or infinite, evaluate the sum
\[
M(z) = \sum_{n \geq 0} |\text{Li}_n(z)| = \sum_{n \geq 0} (\langle S(t) \mid x_1^n \rangle \text{Li}_n(x_1))(z)
\]
then, by means of Lemma 1, with \(x_1^+ = x_1 x_1^* = x_1^* - 1\) and \(S(t) = \sum_{m \geq 0} t^m(x_1^+)^m\), we have
\[
M(z) = \sum_{n \geq 0} |\langle S(t) \mid x_1^n \rangle \text{Li}_n(x_1)(z)|
\]
\[
= \sum_{m \geq 0} t^m \sum_{n \geq 0} m! r^m \sum_{n \geq 0} S_2(n, m) \frac{|\text{Li}_n(x_1)(z)|}{n!} \leq \sum_{m \geq 0} t^m \sum_{n \geq 0} S_2(n, m) \frac{|\text{Li}_n(x_1)(z)|}{n!},
\]
due to the fact that \(|\text{Li}_n(x_1)(z)| \leq |\text{Li}_n(x_1)(z)|\) (Taylor series with positive coefficients). Finally, in view of equation (3.5), we get, on the one hand, for \(|z| < (t + 1)^{-1}\),
\[
M(z) \leq \sum_{m \geq 0} t^m (e^{\text{Li}_1(|z|)} - 1)^m = \sum_{m \geq 0} t^m \left(\frac{|z|}{1 - |z|}\right)^m m = \frac{1 - |z|}{1 - (t + 1)|z|}.
\]
\(^{20}\) For this topology, unconditional and absolute convergence coincide [32].
\(^{21}\) For any set \(E\), the set of its subsets is noted \(2^E\).
Towards a Theory of Domains for Harmonic Functions...

This proves that, for all \( r \in ]0, \frac{1}{t+1} [ \), \( \sum_{n \geq 0} \| L_1(S_{\lambda}(t))(z) \|_r < +\infty \).

On the other hand, if \( (t+1)^{-1} \leq |z| < 1 \), one has \( M(|z|) = +\infty \), and the preceding calculation shows that, with \( t \) chosen such that

\[
0 \leq \frac{1}{R_2} - 1 < t < \frac{1}{R_1} - 1,
\]

we have \( S(t) \in \text{Dom}_R_1(Li) \) but \( S(t) \notin \text{Dom}_R_2(Li) \) whence, for \( 0 < R_1 < R_2 \leq 1 \), \( \text{Dom}_R_2(Li) \subset \text{Dom}_R_1(Li) \).

2. One has (proofs as in [10])

(a) \( 1_X \in \text{Dom}_R(Li) \) (because \( 1_X \in \mathbb{C}(X) \)) and \( L_i = 1_{\mathcal{H}(\Omega)} \).

(b) Taking \( S, T \in \text{Dom}_R(Li) \) we have, by absolute convergence, \( S \cup T \in \text{Dom}_R(Li) \). It is easily seen that \( L_i S \cup T = L_i S \cup T \).

The combinatorial Lemma needed in the Theorem 1 is as follows:

**Lemma 1** For a letter \( "a" \), one has

\[
|a^+ \cup m|a^n = m!S_2(n, m),
\]

(\( S_2(n, m) \) being the Stirling numbers of the second kind). The exponential generating series of R.H.S. in equation (3.3) (w.r.t. \( n \)) is given by

\[
\sum_{n \geq 0} m!S_2(n, m) \frac{x^n}{n!} = (e^x - 1)^m.
\]

**Proof** The expression \( (a^+ \cup m) \) is the specialization of

\[
L_m = a_1^+ \cup a_2^+ \cup \ldots \cup a_m^+
\]

to \( a_j \to a \) (for all \( j = 1, 2 \ldots m \)). The words of \( L_m \) are in bijection with the surjections \([1 \ldots n] \to [1 \ldots m] \), therefore the coefficient \( (a^+ \cup m)|a^n \) is exactly the number of such surjections namely \( m!S_2(n, m) \). A classical formula\(^{23}\) says that

\[
\sum_{n \geq 0} m!S_2(n, m) \frac{x^n}{n!} = (e^x - 1)^m.
\]

In Theorem 1 below, we study, for series taken in \( \mathbb{C}(\{X\})x_1 \oplus \mathbb{C}1_X^* \), the correspondence \( L_i \) to some \( \mathcal{H}(D_{<R}) \), first (point 1) establishes its surjectivity (in a certain sense) and then (points 2 and 3) examine the relation between summability of the functions and that of their Taylor coefficients. For that, let us begin with a very general Lemma on sequences of Taylor series which adapts, for our needs, the notion of normal families as in [28].

**Lemma 2** Let \( \tau = (a_{n, N})_{n, N \geq 0} \) be a double sequence of complex numbers. Setting

\[
I(\tau) := \{ r \in ]0, +\infty[ | \sum_{n, N \geq 0} |a_{n, N}r^N| < +\infty \},
\]

one has

\(^{22}\) Proof by absolute convergence as in [10].

\(^{23}\) See [29], the twelvefold way, formula (1.94b)(pp. 74) for instance.
1. \( I(\tau) \) is an interval of \( [0, +\infty[ \), it is not empty iff there exists \( z_0 \in \mathbb{C} \setminus \{0\} \) such that
\[
\sum_{n,N \geq 0} |a_{n,N} z_0^N| < +\infty. \quad (3.6)
\]

In this case, we set \( R(\tau) := \sup(I(\tau)) \), one has

(a) For all \( N \), the series \( \sum_{n \geq 0} a_{n,N} \) converges absolutely (in \( \mathbb{C} \)). Let us note \( a_N \) - with one subscript - its limit

(b) For all \( n \), the convergence radius of the Taylor series \( T_n(z) = \sum_{N \geq 0} a_{n,N} z^N \) is at least \( R(\tau) \) and \( \sum_{n \in \mathbb{N}} T_n \) is summable for the standard topology of \( \mathcal{H}(D_{<R(\tau)}) \) with sum \( T(z) = \sum_{n,N \geq 0} a_{N} z^N \).

2. Conversely, we suppose that it exists \( R > 0 \) such that

(a) Each Taylor series \( T_n(z) = \sum_{N \geq 0} a_{n,N} z^N \) converges in \( \mathcal{H}(D_{<R}) \).

(b) The series \( \sum_{n \in \mathbb{N}} T_n \) converges unconditionally in \( \mathcal{H}(D_{<R}) \).

Then \( I(\tau) \neq \emptyset \) and \( R(\tau) \geq R \).

**Proof** 1. The fact that \( I(\tau) \subset [0, +\infty[ \) is straightforward from the Definition. If it exists \( z_0 \in \mathbb{C} \) such that
\[
\sum_{n,N \geq 0} |a_{n,N} z_0^N| < +\infty \text{ then, for all } r \in [0, |z_0|[, \text{ we have}
\]
\[
\sum_{n,N \geq 0} |a_{n,N} r^N| = \sum_{n,N \geq 0} \left| a_{n,N} z_0^N \right| \left( \frac{r}{|z_0|} \right)^N \leq \sum_{n,N \geq 0} |a_{n,N} z_0^N| < +\infty
\]
in particular \( I(\tau) \neq \emptyset \) and it is an interval of \( [0, +\infty[ \) with lower bound zero.

(a) Take \( r \in I(\tau) \) (hence \( r \neq 0 \)) and \( N \in \mathbb{N} \), then we get the expected result as
\[
r^N \sum_{n \geq 0} |a_{n,N}| = \sum_{n \geq 0} |a_{n,N} r^N| \leq \sum_{n,N \geq 0} |a_{n,N} r^N| < +\infty.
\]

(b) Again, take any \( r \in I(\tau) \) and \( n \in \mathbb{N} \), then \( \sum_{N \geq 0} |a_{n,N} r^N| < +\infty \) which proves that \( R(T_n) \geq r \), hence the result\(^{24}\). We also have
\[
|\sum_{N \geq 0} a_{N} r^N| \leq \sum_{N \geq 0} r^N |a_{n,N}| \leq \sum_{n,N \geq 0} |a_{n,N} r^N| < +\infty
\]
and this proves that \( R(T) \geq r \), hence \( R(T) \geq R(\tau) \).

2. Let \( 0 < r < r_1 < R \) and consider the path \( \gamma(t) = r_1 e^{2i \pi t} \), we have
\[
|a_{n,N}| = \left| \frac{1}{2i \pi} \int_\gamma \frac{T_n(z)}{z^{N+1}} dz \right| \leq \frac{2\pi}{2\pi} \frac{r_1 \|T_n\|_K}{r_1^{N+1}} \leq \frac{\|T_n\|_K}{r_1^{N}}
\]
with \( K = \gamma([0, 2\pi]) \), hence
\[
\sum_{n,N \geq 0} |a_{n,N} r^N| \leq \sum_{N \geq 0} \|T_n\|_K \left( \frac{r}{r_1} \right)^N \leq \frac{r_1}{r_1 - r} \sum_{n \geq 0} \|T_n\|_K < +\infty.
\]

\( \square \)

**Remark** 1 (i) First point says that every function analytic at zero can be represented around zero as \( \text{Li}_S(z) \) for some \( S \in \mathbb{C} \langle x_1 \rangle \).

\(^{24}\) For a Taylor series \( T \), we note \( R(T) \) the radius of convergence of \( T \).
(ii) In point 2, the arithmetic functions $H_{πY}(S) ∈ Q^N$, for $S ∈ Dom(Li)$ are quickly defined (and in a way extending the old definition) and we draw a very important bound saying that, in this condition, for some $r > 0$ the array $(H_{πY}(S)rn)_n$ converges (then, in particular, horizontally and vertically).

(iii) Point 3 establishes the converse.

**Theorem 1** 1. Let $T(z) = \sum_{N≥0} a_N z^N$ be a Taylor series converging on some non-empty disk centered at zero i.e. such that lim sup$_{N→+∞} |a_N|^N = B < +∞$, then the series

$$S = \sum_{N≥0} a_N (-(-x_1)^+)\ldots N$$

is summable in $C⟨⟨X⟩⟩$ (with sum in $C⟨⟨x_1⟩⟩$), $S ∈ Dom(Li)$ with $R = (B + 1)^{-1}$ and $Li_{S} = T$.

2. Let $S ∈ Dom_{R}(Li)$ and $S = \sum [S]_n$ (homogeneous decomposition), we define $N ↦→ H_{πY}(S)(N)$ by

$$Li_{S}(z) = \sum_{N≥0} H_{πY}(S)(N)z^N.$$  

(3.8)

Then,

$$∀r ∈ ]0, R[ , \sum_{n,N≥0} |H_{πY}(S)rn| < +∞.$$  

(3.9)

In particular, for all $N ∈ N$, the series (of complex numbers), $\sum_{n≥0} H_{πY}(S)rn$ converges absolutely to $H_{πY}(S)(N)$.

3. Conversely, let $Q ∈ C⟨⟨Y⟩⟩$ with $Q = \sum Q_n$ (decomposition by weights), we suppose that there exists $r ∈ ]0, 1]$ such that

$$\sum_{n,N≥0} |H_{Q_n}(N)r^N| < +∞,$$  

(3.10)

in particular, for all $N ∈ N$, $\sum H_{Q_n}(N) = ℓ(N) ∈ C$ unconditionally. Under such circumstances, $πX(Q) ∈ Dom_{r}(Li)$ and, for all $z ∈ C$, $|z| ≤ r$,

$$\frac{Li_{S}(z)}{1 − z} = \sum_{N≥0} ℓ(N)z^N.$$  

(3.11)

**Proof** 1. The fact that the series (3.7) is summable comes from the fact that

$$ω(a_N(-(-x_1)^+\ldots N) ≥ N.$$

Now from the Lemma 1, we get

$$(S)_n = \sum_{N≥0} (a_N(-(-x_1)^+\ldots N)n = (-1)^{N+n}a_NN!S_2(n, N)x_1^n.$$  

Then, with $r = sup_{z∈K} |z|$ (we have indeed $r = ||Id||_K$) and taking into account that

$$||Li_{x_1}||_K ≤ log\left(\frac{1}{1 − r}\right),$$

25 This definition is compatible with the old one when $S$ is a polynomial.
we have
\[
\sum_{n \geq 0} \| \text{Li}_{(S)n} \|_{K} \leq \sum_{n \geq 0} \sum_{N \geq 0} |a_{N}| N! S_{2}(n, N) \| \text{Li}_{x_{1}} \|_{K}
\]
\[
\leq \sum_{n \geq 0} \sum_{N \geq 0} |a_{N}| N! S_{2}(n, N) \| \text{Li}_{x_{1}} \|_{K}^{n} n!
\]
\[
\leq \sum_{N \geq 0} |a_{N}| \sum_{n \geq 0} N! S_{2}(n, N) \| \text{Li}_{x_{1}} \|_{K}^{n} n!
\]
\[
\leq \sum_{N \geq 0} |a_{N}| (e^{\log(1-r_{1})} - 1)^{N}
\]
\[
= \sum_{N \geq 0} |a_{N}| \left( \frac{r}{1-r} \right)^{N}.
\]

Now, if we suppose that \( r \leq (B + 1)^{-1} \), we have \( r(1-r)^{-1} \leq \frac{1}{B} \) and this shows that the last sum is finite.

2. This point and next point are consequences of Lemma 2. Now, considering the homogeneous decomposition \( S = \sum [S]_{n} \in \text{Dom}_{r}(\text{Li}) \). We first establish inequation (3.9). Let \( 0 < r < r_{1} < R \) and consider the path \( \gamma(t) = r_{1} e^{2i\pi t} \), we have
\[
|H_{\pi}([S]_{n})(N)| = \left| \frac{1}{2i\pi} \int_{\gamma} \frac{\text{Li}_{[S]_{n}}(z)}{(1-z)z^{N+1}} dz \right| \leq \frac{2\pi}{2\pi} \| \text{Li}_{[S]_{n}} \|_{K}
\]
\[
K = \gamma([0, 1]) \text{ being the circle of center 0 and radius } r_{1}. \text{ Taking into account that, for } K \subset_{\text{compact}} D_{< R}, \text{ we have a decomposition } \sum_{n \in \mathbb{N}} |\text{Li}_{[S]_{n}}|_{K} = M < +\infty, \text{ we get}
\]
\[
\sum_{n \geq 0} |H_{\pi}([S]_{n})(N)r_{1}^{N}| = \sum_{n \geq 0} |H_{\pi}([S]_{n})(N)r_{1}^{N}| \left( \frac{r}{r_{1}} \right)^{N}
\]
\[
= \sum_{N \geq 0} \left( \frac{r}{r_{1}} \right)^{N} \sum_{n \geq 0} |H_{\pi}([S]_{n})(N)r_{1}^{N}|
\]
\[
= \sum_{N \geq 0} \left( \frac{r}{r_{1}} \right)^{N} \frac{M}{1 - r_{1}r_{1}}
\]
\[
= \frac{M}{1 - r_{1}(r_{1} - r)} < +\infty.
\]
The series \( \sum_{n \geq 0} \text{Li}_{[S]_{n}}(z) \) converges to \( \text{Li}_{S}(z) \) in \( \mathcal{H}(D_{< R}) \) (\( D_{< R} \) is the open disk defined by \( |z| < R \)). For any \( N \geq 0 \), by Cauchy’s formula, one has,
\[
H_{\pi}([S])(N) = \frac{1}{2i\pi} \int_{\gamma} \frac{\text{Li}_{S}(z)}{(1-z)z^{N+1}} dz
\]
\[
= \frac{1}{2i\pi} \int_{\gamma} \frac{\sum_{n \geq 0} \text{Li}_{[S]_{n}}(z)}{(1-z)z^{N+1}} dz
\]
\[
= \frac{1}{2i\pi} \sum_{n \geq 0} \int_{\gamma} \frac{\text{Li}_{[S]_{n}}(z)}{(1-z)z^{N+1}} dz
\]
\[
= \sum_{n \geq 0} H_{\pi}([S]_{n})(N)
\]
the exchange of sum and integral being due to the compact convergence. The absolute convergence comes from the fact that the convergence of \( \sum_{n \geq 0} \text{Li}_{[S]_{n}}(z) \) is unconditional [32].
3. Fixing \( N \in \mathbb{N} \), from inequation (3.10), we get \( \sum_{n \geq 0} |H_{Q_n}(N)| < +\infty \) which proves the absolute convergence.

Remark now that \( (\pi X(Q))_n = \pi X(Q_n) \) and \( \pi Y(\pi X(Q_n)) = Q_n \), one has, for all \( |z| \leq r \),

\[
|Li_{\pi X(Q_n)}(z)| = \left| \sum_{N \in \mathbb{N}} H_{Q_n}(N)z^N \right| \leq \left| \sum_{N \in \mathbb{N}} H_{Q_n}(N)r^N \right|
\]

in other words

\[
\|Li_{\pi X(Q_n)}\|_{D \leq r} \leq \left\| \sum_{N \in \mathbb{N}} H_{Q_n}(N)r^N \right\|
\]

and

\[
\sum_{n \in \mathbb{N}} \|Li_{\pi X(Q_n)}\|_{D \leq r} \leq \left\| \sum_{n,N \in \mathbb{N}} H_{Q_n}(N)r^N \right\| < +\infty
\]

which shows that \( \pi X(Q) \in \text{Dom}_r(Li) \). The equation (3.11) is a consequence of point 2, taking \( S = \pi X(Q) \).

Now, we have have a better understanding of what can (and will) be the domain, \( \text{Dom}(H_\bullet) \), of harmonic sums.

**Definition 2** We set \( \text{Dom}^{\text{loc}}(Li) = \bigcup_{0 < R \leq 1} \text{Dom}_R(Li) \); \( \text{Dom}(H_\bullet) = \pi Y(\text{Dom}^{\text{loc}}(Li)) \) and, for \( S \in \text{Dom}^{\text{loc}}(Li) \),

\[
Li_S(z) = \sum_{n \geq 0} Li_{[S]_n}(z) \quad \text{and} \quad \frac{Li_S(z)}{1 - z} = \sum_{N \geq 0} H_{\pi Y(S)}(N)z^N.
\]

4 Applications

We remark that formula (1.7), i.e.,

\[
Li_{s_1, \ldots, s_r}(z) := \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}},
\]

still makes sense for \( |z| < 1 \) and \( (s_1, \ldots, s_r) \in \mathbb{C}^r \) so that we will freely use the indexing list to get index lists with \( s_i \in \mathbb{Z} \) for any \( i = 1, \ldots, r \) and \( r \in \mathbb{N}^+. \)

Recall that for any \( s_1, \ldots, s_r \in \mathbb{N} \), we can express \( Li_{-s_1, \ldots, -s_r}(z) \) as a polynomial of \( \frac{1}{1 - z} \) with integer coefficients. Then, using (2.2) and \( (kx_1)^* = [(x_1)^*]^k \), we get

\[
\frac{1}{(1 - z)^k} = Li_{(kx_1)^*}(z), \quad \forall k \in \mathbb{N}^+
\]

and we obtain a polynomial \( P \in \text{Dom}(Li) \cap \mathbb{C}[x_1^*] = \mathbb{C}[x_1^*] \) such that \( Li_{-s_1, \ldots, -s_r} = Li_P \) (see [10]). Using Theorem 1, we have

\[
\frac{Li_P(z)}{1 - z} = \sum_{N \geq 0} H_{\pi Y(P)}(N)z^N.
\]

This means that we can provide a class of elements of \( \text{Dom}(H_\bullet) \) (as in Definition 2) relative to the set of indices of harmonic sums at negative integer multiindices. Here are some examples.
**Example 3** For any $|z| < 1$, we have

\[
\text{Li}_{x}^*(z) = \frac{1}{1 - z}; \quad \text{Li}_{x-1}^*(z) = \frac{z}{1 - z} = \text{Li}_0(z); \quad \text{Li}_{(2x_1)^* - x_1^*}^*(z) = \frac{z}{(1 - z)^2} = \text{Li}_{-1}(z);
\]

\[
\text{Li}_{(2x_1)^* - 2x_1^* + 1}^*(z) = \frac{z^2}{(1 - z)^2} = \text{Li}_{0,0}(z);
\]

\[
\frac{z^4 + 7z^3 + 4z^2}{(1 - z)^5} = \text{Li}_{-2, -1}(z);
\]

\[
\frac{z^5 + 14z^4 + 21z^3 + 4z^2}{(1 - z)^6} = \text{Li}_{-2, -2}(z);
\]

\[
\frac{z^7 + 64z^6 + 424z^5 + 584z^4 + 179z^3 + 8z^2}{(1 - z)^8} = \text{Li}_{-3, -3}(z);
\]

\[
\frac{z^5 + 6z^4 + 3z^3}{(1 - z)^6} = \text{Li}_{-1, 0, -2}(z);
\]

\[
\frac{z^2 + 34z + 133z^5 + 100z^4 + 12z^3}{(1 - z)^8} = \text{Li}_{-1, -2, -2}(z).
\]

Thus, for any $N \in \mathbb{N}$, for readability, below 1 stands for $1_{x^*}$

\[
\text{H}_{\pi_N(x^*)}^1(N) = N + 1,
\]

\[
\text{H}_{\pi_N(x^*)^2}(N) = N^2 + \frac{1}{2}N = \sum_{n=1}^{N} n,
\]

\[
\text{H}_{\pi_N((2x_1)^* - x_1^*)}^1(N) = \frac{1}{2}N^2 - \frac{1}{2}N = \sum_{n=1}^{N} n - 1
\]

\[
\text{H}_{\pi_N((2x_1)^* - 2x_1^* + 1)}^1(N) = \frac{1}{2}N^2 - \frac{1}{2}N = \sum_{n=1}^{N} n - 1
\]

\[
\frac{1}{10}N^5 + \frac{1}{8}N^4 - \frac{1}{12}N^3 - \frac{1}{60}N - \frac{1}{8}N^2
\]

\[
\frac{1}{15}N^5 + \frac{1}{18}N^6 - \frac{5}{72}N^4 + \frac{1}{72}N^2
\]

\[
\frac{1}{60}N + \frac{1}{12}N^3 + \sum_{n=1}^{N} n^2
\]

\[
\frac{1}{60}N + \frac{1}{12}N^3 + \sum_{n=1}^{N} n^2
\]

\[
\frac{1}{10}N^5 + \frac{1}{72}N^6 - \frac{1}{36}N^4 + \frac{1}{72}N^2
\]

\[
\frac{1}{144}N^8 - \frac{7}{240}N^6 + \frac{1}{24}N^4 - \frac{7}{360}N^2 + \frac{23}{720}N^5 + \frac{1}{210}N - \frac{19}{720}N^3 = \sum_{n=1}^{N} n^2
\]

\[
\frac{13}{1260}N^7
\]

\[
\frac{1}{144}N^8 - \frac{7}{240}N^6 + \frac{1}{24}N^4 - \frac{7}{360}N^2 + \frac{23}{720}N^5 + \frac{1}{210}N - \frac{19}{720}N^3 = \sum_{n=1}^{N} n^2
\]
Observe that, from Definition 2, Theorem 2 will show us that Dom(H_ullet) is a shuffle subalgebra of \( \mathbb{C}[\!(\!Y\!)] \). Let us however remark that some series are not in this domain as shown below.

(i) The series \( T = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y_n}{n} \in \mathbb{C}[\!(\!Y\!)] \) is not in Dom(H_ullet) because we see that its decomposition by weights

\[
(T = \sum_{n=1}^{\infty} T_n \text{ as in (3.10)}) \text{ provides } T_n = \frac{(-1)^{n-1}}{n} y_n \text{ for } n \geq 1 \text{ and } T_0 = 0. \text{ Direct calculation gives, for } n \geq 1,
\]

\[
H_{y_n}(N) = \sum_{k=1}^{N} \frac{1}{k^n},
\]

so that we have \( H_{y_n}(N) \geq 1, \forall n \in \mathbb{N}^+; N \in \mathbb{N}^+ \), because \( H_{y_n}(0) = 0 \), for all \( 0 < r < 1 \), one has

\[
\sum_{n,N} |H_{T_n}(N)r^N| = \sum_{N \geq 0} \sum_{n \geq 1} \frac{1}{n} H_{y_n}(N)r^n \geq \left( \sum_{n \geq 0} \frac{1}{n} \right) \frac{r}{1-r} = +\infty. \quad (4.1)
\]

However one can get unconditional convergence using a summation by pairs (odd + even).

(ii) For all \( s \in ]1, +\infty[ \), the series \( T(s) = \sum_{n=1}^{\infty} (-1)^{n-1} y_n n^{-s} \in \mathbb{C}[\!(\!Y\!)] \) is in Dom(H_ullet).

We can now state the

**Theorem 2** Let \( S, T \in \text{Dom}^{\text{loc}}(\mathfrak{L}_i) \), then \( S \shuffle T \in \text{Dom}^{\text{loc}}(\mathfrak{L}_i), \pi_X(\pi_Y(S) \shuffle \pi_Y(T)) \in \text{Dom}^{\text{loc}}(\mathfrak{L}_i) \) and for all \( N \geq 0 \),

\[
\begin{align*}
\mathsf{Li}_S \shuffle T = \mathsf{Li}_S \mathsf{Li}_T; & \quad \mathsf{Li}_{1\chi^*} = 1\mathsf{Li}_{\mathsf{t}(\mathcal{O})}, \\
H_{\pi_Y(S) \shuffle \pi_Y(T)}(N) = H_{\pi_Y(S)}(N)H_{\pi_Y(T)}(N). & \quad (4.3)
\end{align*}
\]

\[
\frac{\mathsf{Li}_S(z)}{1-z} \odot \frac{\mathsf{Li}_T(z)}{1-z} = \frac{\mathsf{Li}_{\pi_Y(S) \shuffle \pi_Y(T)}(z)}{1-z}. \quad (4.4)
\]

**Proof** For equation (4.2), we get, from Lemma 4 that \( \text{Dom}^{\text{loc}}(\mathfrak{L}_i) \) is the union of an increasing set of shuffle subalgebras of \( \mathbb{C}[\!(\!X\!)] \). It is therefore a shuffle subalgebra of the latter.

For equation (4.3), suppose \( S \in \text{Dom}_{R_1}(\mathfrak{L}_i) \) (resp. \( T \in \text{Dom}_{R_1}(\mathfrak{L}_i) \)). By [18] and Theorem 1, one has

\[
\frac{\mathsf{Li}_S(z)}{1-z} \odot \frac{\mathsf{Li}_T(z)}{1-z} \in \text{Dom}_{R_1 R_2}(\mathfrak{L}_i) \text{ where } \odot \text{ stands for the Hadamard product [18]. Hence, for } |z| < R_1 R_2, \text{ one has}
\]

\[
f(z) = \frac{\mathsf{Li}_S(z)}{1-z} \odot \frac{\mathsf{Li}_T(z)}{1-z} = \sum_{N \geq 0} H_{\pi_Y(S)}(N)H_{\pi_Y(T)}(N)z^N \quad (4.5)
\]

and, due to Theorem 1 point (3.8), for all \( N, \sum_{p \geq 0} H_{\pi_Y(S_p)}(N) = H_{\pi_Y(S)}(N) \) and \( \sum_{q \geq 0} H_{\pi_Y(T_q)}(N) = H_{\pi_Y(T)}(N) \) (absolute convergence) then, as the product of two absolutely convergent series is absolutely convergent (w.r.t. the Cauchy product), one has, for all \( N, \)

\[
H_{\pi_Y(S)}(N)H_{\pi_Y(T)}(N) = \left( \sum_{p \geq 0} H_{\pi_Y(S_p)}(N) \right) \left( \sum_{q \geq 0} H_{\pi_Y(T_q)}(N) \right)
\]

\[
= \sum_{p,q \geq 0} H_{\pi_Y(S_p)}(N)H_{\pi_Y(T_q)}(N) = \sum_{n \geq 0} \sum_{p+q=n} H_{\pi_Y(S_p) \shuffle \pi_Y(T_q)}(N)
\]

\[
= \sum_{n \geq 0} H_{(\pi_Y(S) \shuffle \pi_Y(T)_n)}(N). \quad (4.6)
\]
Remains to prove that condition of Theorem 1, i.e. inequation (3.10) is fulfilled. To this end, we use the well-known fact that if \( \sum_{m \geq 0} c_m z^m \) has radius of convergence \( R > 0 \), then \( \sum_{m \geq 0} |c_m| z^m \) has the same radius of convergence (use \( 1/R = \limsup_{m \geq 1} |c_m|^{-m} \)), then from the fact that \( S \in \text{Dom}_{R_1}(\text{Li}) \) (resp. \( T \in \text{Dom}_{R_2}(\text{Li}) \)), we have (3.9) for each of them and, using the Hadamard product of these expressions, we get

\[
\forall r \in ]0, R_1 R_2[, \sum_{p,q,N \geq 0} |H_{\pi Y(S_p)}(N)H_{\pi Y(T_q)}(N) r^N| < +\infty,
\]

and this assures, for \( |z| < R_1 R_2 \), the convergence of

\[
f(z) = \sum_{n,N \geq 0} H_{(\pi Y(S) \mathcal{U} \pi Y(T))_n}(N) z^N.
\]  

(4.7)

applying Theorem 1 point (3) to \( Q = \pi Y(S) \mathcal{U} \pi Y(T) \) (with any \( r < R_1 R_2 \)), we get \( \pi X(Q) = \pi X(\pi Y(S) \mathcal{U} \pi Y(T)) \in \text{Dom}^{\text{loc}}(\text{Li}) \) and

\[
f(z) = \sum_{N \geq 0} \left( \sum_{n \geq 0} H_{(\pi Y(S) \mathcal{U} \pi Y(T))_n}(N) \right) z^N = \frac{\text{Li}_{\pi X(\pi Y(S) \mathcal{U} \pi Y(T))}(z)}{1 - z}.
\]

hence we obtain (4.3). \( \square \)

Recall that, as in Example 3, for any \( s_1, \ldots, s_r \in \mathbb{N} \), we can find an elements \( P \in \text{Dom}(\text{Li}) \) such that

\[
\frac{\text{Li}_P(z)}{1 - z} = \sum_{N \geq 0} H_{\pi Y(P)}(N) z^N.
\]

Theorem 2 proves that \( H \) is a stuffle character on \( \text{Dom}(H) \). Then for any mixed multiindices \( s \), we can find the elements \( P \in \text{Dom}(\text{Li}) \) satisfying

\[
\frac{\text{Li}_s(z)}{1 - z} = \sum_{N \geq 0} H_{\pi Y(P)}(N) z^N.
\]
Example 4

\[ H_{\pi_Y(\frac{1}{2}(x_1)^* - x_1^*) + \gamma(N)} = \frac{1}{4} N^2 - \frac{1}{4} N = \sum_{n_1=1}^{N} \frac{1}{n_1} \sum_{n_2=1}^{n_1-1} n_2, \]

hence

\[ H_{\pi_Y(\frac{1}{2}(x_1)^* - x_1^*) - \frac{1}{2} \pi_Y((x_1)^* - 1)}(N) = \sum_{n_1=1}^{N} \frac{1}{n_1} \sum_{n_2=1}^{n_1-1} \frac{1}{n_2}, \]

\[ H_{\pi_Y(\frac{1}{3}(x_1)^* - \frac{3}{2} (x_1)^* + x_1^*) - \frac{1}{6} \pi_Y((x_1)^* - \frac{1}{6})}(N) = \frac{1}{9} N^3 - \frac{1}{12} N^2 - \frac{1}{36} N = \sum_{n_1=1}^{N} \frac{1}{n_1} \sum_{n_2=1}^{n_1-1} n_2^2, \]

then

\[ H_{\pi_Y(\gamma(x_1)} \pi_Y(2(x_1)^* - 3(x_1)^* + x_1^*) - \pi_Y(\frac{1}{3}(x_1)^* - \frac{1}{6}(x_1)^* - \frac{1}{6} x_1^*)}(N) = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} \frac{1}{n_2}, \]

\[ H_{\pi_Y(\frac{1}{2}(x_1)^* - \frac{5}{6} x_1^* + \frac{1}{6} x_1^*)}(N) = \sum_{n_1=1}^{N} \frac{1}{n_1^2} \sum_{n_2=1}^{n_1-1} n_2^2, \]

which entails

\[ H_{\pi_Y(\chi_0 x_1)} \pi_Y(2(x_1)^* - 3(x_1)^* + x_1^*) - \pi_Y(\frac{1}{3}(x_1)^* + \frac{1}{6} x_1^* - \frac{1}{6} x_1^*) = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} \frac{1}{n_2^2}, \]

\[ H_{\pi_Y(\gamma(\frac{20}{6}(x_1)^* - \frac{3}{2}(x_1)^* + \frac{1}{6}(x_1)^* - \frac{1}{6} x_1^*)}(N) = \sum_{n_1=1}^{N} \frac{1}{n_1} \sum_{n_2=1}^{n_1-1} n_2^2 \sum_{n_3=1}^{n_2-1} n_3^2, \]

\[ H_{\pi_Y(\gamma(\frac{40}{6}(x_1)^* - \frac{3}{2}(x_1)^* + \frac{1}{6}(x_1)^* - \frac{1}{6} x_1^*)}(N) = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} \frac{1}{n_2} \sum_{n_3=1}^{n_2-1} n_3^2, \]

thus

\[ H_{\pi_Y(\gamma(x_1)} \pi_Y(40(x_1)^* - 132(x_1)^* + 161(x_1)^* - 87(x_1)^* + 19(2x_1)^* - x_1^*)}(N) = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} \frac{1}{n_2} \sum_{n_3=1}^{n_2-1} n_3^2 - \sum_{n_1=1}^{N} \frac{1}{n_1^2} \sum_{n_2=1}^{n_1-1} n_2^2 - \sum_{n_1=1}^{N} \frac{1}{n_1} \sum_{n_2=1}^{n_1-1} \frac{1}{n_2^2} \sum_{n_3=1}^{n_2-1} n_3^2. \]

5 Some Remarks about Stuffle Product, Stuffle Characters and their Symbolic Computations.

For the some reader’s convenience, we recall here the definitions of shuffle and stuffle products. As regards shuffle, the alphabet \( \mathcal{X} \) is arbitrary and \( \omega \) is defined by the following recursion (for \( a, b \in \mathcal{X} \) and \( u, v \in \mathcal{X}^* \))

\[ u \omega 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \omega u = u; \quad au \omega bv = a(u \omega bv) + b(au \omega v). \quad (5.1) \]

As regards stuffle, the alphabet is \( Y = \mathcal{Y}_{\mathbb{N}^+} = \{ y_s \}_{s \in \mathbb{N}^+} \), and \( \omega \) is defined by the following recursion

\[ u \omega 1_{\mathcal{Y}^*} = 1_{\mathcal{Y}^*} \omega u = u, \quad (5.2) \]

\[ y_s u \omega y_t v = y_s (u \omega y_t v) + y_t (y_s u \omega v) + y_{s+t} (u \omega v). \quad (5.3) \]

Be it for stuffle or shuffle, the noncommutative \(^{26}\) polynomials equipped with this product form an associative commutative and unital algebra namely \(( \mathbb{C}(\mathcal{X}), \omega, 1_{\mathcal{X}^*} \) (resp. \(( \mathbb{C}(\mathcal{Y}), \omega, 1_{\mathcal{Y}^*} \))\).

\(^{26}\) For concatenation.
Example 5 As examples of characters, we have already seen

- \( \text{Li}_* \) from \((\text{Dom}^{\text{loc}}(\text{Li}_*), \omega, 1_{\chi^*}) \) to \( \mathcal{H}(\Omega) \)
- \( \mathbb{H}_* \) from \((\text{Dom}(\mathbb{H}_*), \omega, 1_{1_{\chi^*}}) \) to \( \mathbb{C}^{[\mathbb{N}]} \) (arithmetic functions \( \mathbb{N} \to \mathbb{C} \))

In general, a character from a \( k \)-algebra\(^{27} \) \( (\mathcal{A}, *, 1_{\mathcal{A}}) \) with values in \( (\mathcal{B}, *_2, 1_{\mathcal{B}}) \) is none other than a morphism between the \( k \)-algebras \( \mathcal{A} \) and a commutative algebra\(^{28} \) \( \mathcal{B} \). The algebra \( (\mathcal{A}, *, 1_{\mathcal{A}}) \) does not have to be commutative, for example characters of \( (\mathcal{C}(\mathcal{X}), \text{conc}, 1_{\chi^*}) \) - i.e. \text{conc}-characters - were all proved to be of the form

\[
\left( \sum_{x \in \mathcal{X}} \alpha_x x \right)^* \tag{5.4}
\]

i.e. Kleene stars of the plane \([6,10]\). They are closed under shuffle and stuffle and endowed with these laws, they form a group. Expressions like the infinite sum within brackets in (5.4) (i.e. homogeneous series of degree 1) form a vector space noted \( \widehat{\mathcal{C}} \).

As a consequence, given \( P = \sum_{i \geq 1} \alpha_i y_i \) and \( Q = \sum_{j \geq 1} \beta_j y_j \), we know in advance that their stuffle is a \text{conc}-character i.e. of the form \( \sum_{n \geq 1} c_n y_n \). Examining the effect of this stuffle on each letter (which suffices), we get the identity

\[
\left( \sum_{i \geq 1} \alpha_i y_i \right)^* \text{conc} \left( \sum_{j \geq 1} \beta_j y_j \right)^* = \left( \sum_{i \geq 1} \alpha_i y_i + \sum_{j \geq 1} \beta_j y_j + \sum_{i,j \geq 1} \alpha_i \beta_j y_{i+j} \right)^* \tag{5.5}
\]

This suggests to take an auxiliary variable, say \( q \), and code “the plane” \( \widehat{\mathcal{C}} \), i.e. expressions like (5.4), in the style of Umbral calculus by

\[
\pi_{\mathcal{Y}}^\text{Umbra} : \sum_{n \geq 1} \alpha_n q^n \longmapsto \sum_{n \geq 1} \alpha_n y_n
\]

which is linear and bijective\(^{29} \) from \( \mathbb{C}[[q]] \) to \( \widehat{\mathcal{C}} \). With this coding at hand and for \( S, T \in \mathbb{C}[[q]] \), identity (5.5) reads

\[
(\pi_{\mathcal{Y}}^\text{Umbra}(S))^* \text{conc} \left( \pi_{\mathcal{Y}}^\text{Umbra}(T) \right)^* = \left( \pi_{\mathcal{Y}}^\text{Umbra}((1+S)(1+T)-1) \right)^*. \tag{5.6}
\]

This shows that if one sets, for \( z \in \mathbb{C} \) and \( T \in \mathbb{C}[[x]] \), \( G(z) = (\pi_{\mathcal{Y}}^\text{Umbra}(e^z T - 1))^* \), we get a one-parameter stuffle group\(^{30} \) such that every coefficient is polynomial in \( z \). Differentiating it we get

\[
\frac{d}{dz}(G(z)) = (\pi_{\mathcal{Y}}^\text{Umbra}(T)) G(z) \tag{5.7}
\]

and (5.7) with the initial condition \( G(0) = 1_{\mathcal{Y}} \) integrates as

\[
G(z) = \exp_{\text{conc}}(z \pi_{\mathcal{Y}}^\text{Umbra}(T)) \tag{5.8}
\]

where the exponential map for the stuffle product is defined, for any \( P \in \mathbb{C}[[\mathcal{Y}]] \) such that \( \langle P | 1_{\mathcal{Y}} \rangle = 0 \), is defined by

\[
\exp_{\text{conc}}(P) := 1_{\mathcal{Y}} + \frac{P}{1!} + \frac{P \text{conc} P}{2!} + \ldots + \frac{P \text{conc}^n P}{n!} + \ldots
\]

In particular, from (5.8), one gets, for \( k \geq 1 \),

\[
(z y_k)^* = \exp_{\text{conc}} \left( -\sum_{n \geq 1} y_{nk} \frac{(-z)^n}{n} \right).
\]

\( ^{27} \) Here we will use \( k = \mathbb{Q} \) or \( \mathbb{C} \).

\( ^{28} \) In this context all algebras are associative and unital.

\( ^{29} \) Its inverse will be naturally noted \( \pi_{\mathcal{Y}}^\text{Umbra} \).

\( ^{30} \) i.e. \( G(z_1 + z_2) = G(z_1) \text{conc} G(z_2) \); \( G(0) = 1_{\mathcal{Y}} \).
6 Conclusion

Noncommutative symbolic calculus allows to get identities easy to check and to implement. With some amount of complex and functional analysis, it is possible to bridge the gap between symbolic, functional and number theoretic worlds. This was the case already for polylogarithms, harmonic sums and polyzetas. This is the project of this paper and will be pursued in the forthcoming works.

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