RESIDUAL AMENABILITY AND THE APPROXIMATION OF
$L^2$-INVARIANTS

BRYAN CLAIR

Abstract. We generalize Lück’s Theorem to show that the $L^2$-Betti numbers of a residually amenable covering space are the limit of the $L^2$-Betti numbers of a sequence of amenable covering spaces. We show that any residually amenable covering space of a finite simplicial complex is of determinant class, and that the $L^2$ torsion is a homotopy invariant for such spaces. We give examples of residually amenable groups, including the Baumslag-Solitar groups.

Introduction

In 1994, Wolfgang Lück [13] proved the beautiful theorem that if $X$ is a finite simplicial complex with residually finite fundamental group, the $L^2$-Betti numbers of the universal covering of $X$ can be approximated by the ordinary Betti numbers of a sequence of finite coverings of $X$. In fact, the question of approximation dates back to Kazhdan [10] (see also [9, Pg. 20]) but only an inequality was known. Dodziuk and Mathai [6] have shown a result analogous to Lück’s Theorem in the situation where the covering transformation group is amenable. Specifically, they show that the $L^2$-Betti numbers of an amenable covering $\tilde{X}$ of $X$ can be approximated by the ordinary Betti numbers of a sequence of Følner subsets of $\tilde{X}$. This paper generalizes Lück’s Theorem to the case where the cover of $X$ has residually amenable transformation group, a large class of groups that includes the residually finite groups of Lück’s Theorem and the amenable groups of Dodziuk and Mathai.

In this paper, we also consider $L^2$ torsion. For $L^2$ acyclic covering spaces, $L^2$ analytic torsion was first studied in [16] and [12], and $L^2$ Reidemeister-Franz torsion was first studied in [3], see also [14]. To define these $L^2$ torsions, one needs to establish decay of the $L^2$ spectral density function at 0. In the case of a residually finite covering, Lück [13], derives an elegant estimate on the spectral distribution functions for the finite covers, which in the limit gives the necessary decay for the combinatorial $L^2$ Laplacian. Lück also proves the homotopy invariance of $L^2$ combinatorial torsion in this case.

Recently, in [8], the combinatorial and analytic torsion invariants were defined more generally as volume forms on $L^2$ homology and $L^2$ cohomology respectively, the decay condition on the spectrum now replaced by a similar condition known as determinant class. This allowed interpretation of results of [8] as the equality of the combinatorial and analytic $L^2$ torsions.

Dodziuk and Mathai [6] show that coverings with amenable covering group are of determinant class, and Mathai and Rothenberg [17] have recently extended Lück’s results to prove the homotopy invariance of $L^2$ torsion in that case. In this paper

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we show that coverings with residually amenable covering group are of determinant class, and that $L^2$ torsion of such spaces is a homotopy invariant.

On a different note, Farber [7] has recently generalized Lück’s Theorem in a new direction, viewing it as a statement about flat bundles rather than finite coverings. In particular, he gives precise conditions for the convergence of $L^2$-Betti numbers of finite non-regular covers. A reasonable direction for future work would be to try and extend the results of this paper using his techniques.

We now formulate the main results of this paper. Let $Y$ be a connected simplicial complex. Suppose that a finitely generated group $\pi$ acts freely and simplicially on $Y$ so that $X = Y/\pi$ is a finite simplicial complex.

Suppose there is a nested sequence of normal subgroups $\pi = \Gamma_1 \supset \Gamma_2 \supset \ldots$ such that $\bigcap_{n=1}^{\infty} \Gamma_n = \{e\}$. Form $Y_n = Y/\Gamma_n$, so that $Y_1, Y_2, \ldots$ are a tower of covering spaces of $X$.

Say that $\pi$ is residually finite if there exist $\Gamma_n$’s so that the quotients $\pi/\Gamma_n$ are all finite. Then Lück’s Theorem [13] states that

$$b^{(2)}_j(Y : \pi) = \lim_{n \to \infty} \frac{1}{|\Gamma_n|} b^{(2)}_j(Y_n)$$

where $b^{(2)}_j(Y : \pi)$ is the $j$th $L^2$-Betti number of $Y$.

We generalize Lück’s Theorem to the situation where $\pi$ is residually amenable, meaning there exist $\Gamma_n$’s so that the quotients $\pi/\Gamma_n$ are all amenable. The first main result of this paper is

**Theorem 0.1** (Approximation Theorem). Suppose $Y$ is a simplicial complex, $\pi$ acts freely and simplicially on $Y$, and $X = Y/\pi$ is a finite simplicial complex. If $\pi$ is residually amenable, then

$$b^{(2)}_j(Y : \pi) = \lim_{n \to \infty} b^{(2)}_j(Y_n : \pi/\Gamma_n).$$

The next result gives more evidence for the determinant class conjecture, which states that any regular covering space of a finite simplicial complex is of determinant class. For $\pi$ residually finite this follows from [13], and it was shown for $\pi$ amenable in [5].

**Theorem 0.2** (Determinant Class Theorem). Suppose $Y$ is a simplicial complex, $\pi$ acts freely and simplicially on $Y$, and $X = Y/\pi$ is a finite simplicial complex. If $\pi$ is residually amenable, then $Y$ is of determinant class.

Now we turn to the problem of homotopy invariance of $L^2$ torsion. Let $M$ and $N$ be compact manifolds, $\tilde{M}$ and $\tilde{N}$ regular $\pi$-covering spaces. As in [17], a homotopy equivalence $f : M \to N$ induces a canonical isomorphism $\tilde{f}^* : \det \overline{H}(\tilde{N}) \to \det \overline{H}(\tilde{M})$ of determinant lines of $L^2$ cohomology.

Let $\phi_{\tilde{M}} \in \det \overline{H}(\tilde{M})$ denote the $L^2$ torsion of $M$.

**Theorem 0.3** (Homotopy Invariance of $L^2$ Torsion). Suppose $f : M \to N$ is a homotopy equivalence of compact odd dimensional manifolds, and $\tilde{M}$ and $\tilde{N}$ are regular covering spaces with residually amenable covering group. Then via the above identification of determinant lines of $L^2$ cohomology,

$$\phi_{\tilde{M}} = \phi_{\tilde{N}} \in \det \overline{H}(\tilde{M}).$$
This provides more evidence for the conjecture in [17] (see also [13]) that $L^2$ torsion is always a homotopy invariant when the covering spaces in question are of determinant class.

This paper is organized as follows. The first section covers preliminaries on residually amenable groups, and exhibits interesting examples. The second section proves the main technical theorem. It essentially states that the $L^2$-spectra of Laplacians on the $Y_n$ approximate the $L^2$-spectrum of Laplacian on $Y$. Finally, the third section proves the three main theorems.

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1. Preliminaries

1.1. Residual Properties.

**Definition 1.1.** Let $C$ be a nonempty class of groups (though possibly containing only one group). A group $\pi$ is residually $C$ if for any element $g \in \pi$, $g \neq e$, there exists a quotient group $\pi'(g)$ belonging to $C$ such that $g \mapsto g' \in \pi'(g)$ with $g' \neq e$.

If a countable group $\pi$ is residually $C$, then there is a nested sequence of normal subgroups $\pi = \Gamma_1 \supset \Gamma_2 \supset \cdots$ such that $\pi/\Gamma_n$ belongs to $C$ and $\bigcap_{n=1}^{\infty} \Gamma_n = \{e\}$. For other basic theorems concerning residual properties of groups, we refer to [15].

When $C = \{\text{finite groups}\}$ we say that $\pi$ is a residually finite group.

1.2. Amenability. Let $\pi$ be a finitely generated discrete group, with word metric $d$. We use the following characterization of amenability, due to Følner.

**Definition 1.2.** $\pi$ is amenable if there is a sequence of finite subsets $\{\Lambda_k\}_{k=1}^{\infty}$ such that for any fixed $\delta > 0$

$$\lim_{k \to \infty} \frac{\#(\partial_\delta \Lambda_k)}{\# \Lambda_k} = 0$$

where $\partial_\delta \Lambda_k = \{\gamma \in \pi : d(\gamma, \Lambda_k) < \delta \text{ and } d(\gamma, \pi - \Lambda_k) < \delta\}$ is a $\delta$-neighborhood of the boundary of $\Lambda_k$.

Examples of amenable groups include finite groups, abelian groups, nilpotent groups and solvable groups, and groups of subexponential growth. Amenability for discrete groups is preserved by the following five processes:

1. Taking subgroups;
2. Forming quotient groups;
3. Forming group extensions by amenable groups;
4. Forming upward directed unions of amenable groups;
5. Forming a direct limit of amenable groups.

Free groups with two or more generators, and fundamental groups of closed negatively curved manifolds are not amenable.

1.3. Residual Amenability.

**Definition 1.3.** If $\pi$ is residually $C$, where $C = \{\text{amenable groups}\}$, we say that $\pi$ is residually amenable.
Recall that the derived subroups \( \pi^{(i)} \) of a group \( \pi \) are defined by \( \pi^{(0)} = \pi \) and \( \pi^{(i+1)} = [\pi^{(i)}, \pi^{(i)}] \). Say that \( \pi \) is \textit{solvable} if \( \pi^{(i)} = \{e\} \) for some \( i \).

Contained in the class of residually amenable groups is the important class of residually solvable groups. Unlike residually finite groups, the class of residually solvable groups is closed under extensions. And in contrast to amenable groups, the free product of two residually solvable groups is residually solvable. The statement for extensions is Prop 1.2 below. Closure under free products follows from the fact that solvability is a root property as discussed in [15].

**Proposition 1.1.** The following are equivalent:

(i) \( \pi \) is residually solvable;

(ii) \( \bigcap_{i=1}^{\infty} \pi^{(i)} = \{e\} \);

(iii) \( \pi \) contains no nontrivial perfect subgroup. \( \Gamma \) is perfect if \( \Gamma = [\Gamma, \Gamma] \).

**Proof.** (ii) \( \implies \) (i) is clear. Since \( \bigcap_{i=1}^{\infty} \pi^{(i)} \) is a perfect subgroup of \( \pi \), (iii) implies (ii).

Now suppose \( \pi \) is residually solvable, and suppose that \( \Gamma \) is a perfect subgroup of \( \pi \). Then \( \Gamma \) is also residually solvable. If \( \Gamma \) is nontrivial, there is some nontrivial map \( f: \Gamma \to S \) with \( S \) solvable of rank \( k \). Then

\[
\{e\} = S^{(k)} \supseteq f(\Gamma)^{(k)} = f(\Gamma^{(k)}) = f(\Gamma)
\]

which contradicts nontriviality of \( f \). This shows (i) \( \implies \) (iii). \( \square \)

**Proposition 1.2.** If \( \Gamma_1 \) and \( \Gamma_2 \) are residually solvable, and \( \pi \) is an extension

\[
1 \to \Gamma_1 \xrightarrow{\iota} \pi \xrightarrow{\kappa} \Gamma_2 \to 1
\]

then \( \pi \) is residually solvable.

**Proof.** Suppose \( H \) is a perfect subgroup of \( \pi \). Then \( \kappa(H) \) is a perfect subgroup of \( \Gamma_2 \), hence trivial. Then \( H \subset \Gamma_1 \) and so \( H \) is trivial. Thus \( \pi \) is residually solvable. \( \square \)

**Example 1.1.** For nonzero integers \( p \) and \( q \), define the \textit{Baumslag-Solitar} group \( BS(p, q) \) by

\[
BS(p, q) = \langle a, b \mid a^{-1}b^pa = b^q \rangle.
\]

A group \( \pi \) is \textit{Hopfian} if \( \pi/\Gamma \cong \pi \) implies \( \Gamma = \{e\} \). The family of groups \( BS(p, q) \) were first defined in [3], where it was shown that \( BS(p, q) \) is Hopfian if and only if \( p \) and \( q \) are \textit{meshed}, which means \( p|q, q|p \), or \( p \) and \( q \) have exactly the same set of prime divisors. As any finitely generated residually finite group is Hopfian, the groups \( BS(p, q) \) are not residually finite when \( p \) and \( q \) are not meshed.

Kropholler shows in [11] that the second derived subgroup \( \pi^{(2)} \) is free when \( \pi = BS(p, q) \), for any \( p, q \). When \( p \) and \( q \) are not both \( \pm 1 \), \( \pi^{(2)} \) is free on two or more generators and therefore \( \pi \) is not amenable. However, \( \pi \) is residually solvable since it is an extension

\[
1 \to \pi^{(2)} \to \pi \to \pi/\pi^{(2)} \to 1
\]

with \( \pi^{(2)} \) residually solvable and \( \pi/\pi^{(2)} \) solvable.

More generally, let \( \pi \) be any non-cyclic group which is the fundamental group of a graph of infinite cyclic groups. Then from [11], \( \pi^{(2)} \) is free and the above argument shows \( \pi \) is residually solvable.
Example 1.2. We show that the amalgamated free product of two abelian groups is residually solvable. Slightly more generally, suppose we have two residually solvable groups $A$ and $B$, and two homomorphisms from a group $H$ into the centers of $A$ and $B$ given by $\alpha: H \to Z(A)$ and $\beta: H \to Z(B)$. Then the free product with amalgamation $A *_H B$ is residually solvable.

To see this, let $N = \{ (\alpha(h), \beta(h)^{-1}) | h \in H \} \subset A \times B$. Since $H$ is abelian, $N$ is a subgroup of $A \times B$. $N$ is normal because $H$ includes into the centers of both $A$ and $B$. Note that if $H$ is only known to be normal in $A$ and $B$, $N$ is unlikely to be normal in $A \times B$.

Let $K$ be the kernel of the natural map $A *_H B \to (A \times B)/N$. In $A *_H B$, $K$ has trivial intersection with all conjugates of $A$ and $B$, hence $K$ is free by a well known theorem of group actions on trees. Then $A *_H B$ is an extension of the residually solvable group $(A \times B)/N$ by the residually solvable group $K$, and so $A *_H B$ is residually solvable.

Example 1.3. It is shown in [18] that any HNN-extension of a finitely generated abelian group is residually solvable.

Example 1.4. R.J. Thompson, G. Higman, K. Brown, and E.A. Scott have demonstrated various classes of finitely presented infinite simple groups. (See for example, [18]). The example of Scott contains a free subgroup on two generators and is therefore not amenable and not residually amenable.

2. Main Technical Theorems

As in the introduction, suppose $Y$ is a simplicial complex, $\pi$ acts freely and simplicially on $Y$, and $X = Y/\pi$ is a finite simplicial complex. Suppose we have a nested sequence of normal subgroups $\pi = \Gamma_1 \supset \Gamma_2 \supset \cdots$ such that $\bigcap_{n=1}^{\infty} \Gamma_n = \{ e \}$. Define $Y_n = Y/\Gamma_n$.

Suppose $X$ has $a_j$ cells in dimension $j$, and choose a lift to $Y$ of each $j$-cell of $X$. These choices give a basis over $l^2(\pi)$ of the space $C^j(\Delta)(Y)$ of $j$ dimensional $l^2$-cochains on $Y$. The lifts also descend to give bases of $C^j(\Delta)(Y_n)$ over $l^2(\pi/\Gamma_n)$.

Denote by $\Delta$ and $\Delta_n$ the Laplacian on $C^j(\Delta)(Y)$ and $C^j(\Delta_n)(Y_n)$ respectively. All arguments to follow will apply to a specific value of $j$, but this dependence will not be indicated.

Let $\{ P(\lambda) : \lambda \in [0, \infty) \}$ and $\{ P_n(\lambda) : \lambda \in [0, \infty) \}$ denote the right continuous family of spectral projections of $\Delta$ and $\Delta_n$. Since $\Delta$ is $\pi$-equivariant, so are $P(\lambda) = \chi_{[0,\lambda]}(\Delta)$ for $\lambda \in [0, \infty)$. Similarly, $P_n(\lambda)$ are $\pi/\Gamma_n$-equivariant. Let $F, F_n : [0, \infty) \to [0, \infty)$ denote the spectral density functions

$$F(\lambda) = \operatorname{Tr}_\pi P(\lambda)$$

$$F_n(\lambda) = \operatorname{Tr}_{\pi/\Gamma_n} P(\lambda).$$

We now set

$$\overline{F}(\lambda) = \limsup_{n \to \infty} F_n(\lambda); \quad \underline{F}(\lambda) = \liminf_{n \to \infty} F_n(\lambda)$$

$$\overline{F}^+(\lambda) = \lim_{\delta \to 0^+} \overline{F}(\lambda + \delta); \quad \underline{F}^+(\lambda) = \lim_{\delta \to 0^+} \underline{F}(\lambda + \delta).$$

With the above notation, we state the main technical results of this paper.
Theorem 2.1. For all $\lambda \in [0, \infty)$,
$$F(\lambda) = \overline{F}(\lambda) = F^+(\lambda)$$

Theorem 2.2. Suppose there is some right continuous function $s : [0, \varepsilon) \to [0, \infty)$, with $s(0) = 0$ and so that for all $n$ and for all $\lambda \in [0, \varepsilon)$ we have
$$F_n(\lambda) - F_n(0) \leq s(\lambda)$$
then
1. $\overline{F}(\lambda)$ and $\underline{F}(\lambda)$ are right continuous at zero and one has the equalities
$$\overline{F}(0) = \overline{F}(0) = F(0) = F^+(0).$$
2. For all $\lambda \in [0, \varepsilon)$,
$$F(\lambda) - F(0) \leq s(\lambda).$$

These theorems and their proofs are quite similar to Lück [13, Theorem 2.3], but here they are stated so as to require no conditions on the quotient groups $\pi/\Gamma_n$. The residual finiteness condition in Lück’s Theorem or the residual amenability condition of this paper are required to provide $s$, the uniform decay at zero of the spectral density functions for the covers $Y_n$.

To show the two technical theorems, we first prove a number of preliminary lemmas.

Lemma 2.1. There exists a number $K > 1$ such that the operator norms of $\Delta$ and $\Delta_n$ are smaller than $K^2$ for all $n$.

Proof. Choosing lifts of cells of $X$, we have identified the space of $l^2$-cochains on $Y$ with $\bigoplus_{i=1}^a l^2(\pi)$. The combinatorial Laplacian $\Delta$ is then described by an $a \times a$ matrix $B$ with entries in $\mathbb{Z}[\pi]$, acting by right multiplication. The Laplacian $\Delta_n$ is described by the same matrix $B$, now acting by right multiplication on $\bigoplus_{i=1}^b l^2(\pi/\Gamma_n)$.

For $u = \sum_{g \in \pi} \lambda_g \cdot g \in \mathbb{C}[\pi]$ define $|u|_1 = \sum_{g \in \pi} |\lambda_g|$. Choose $K > 1$ so that
$$K \geq a \cdot \max \{|B_{ij}|_1; i = 1, \ldots, a\}.$$  

The proof then proceeds exactly as in [13, Lemma 2.5].

Lemma 2.2. Let $p(\mu)$ be a polynomial. There is a number $n_0$, depending only on the system of groups $\pi = \Gamma_1 \supset \Gamma_2 \supset \ldots$ such that for all $n \geq n_0$
$$\text{Tr}_\pi p(\Delta) = \text{Tr}_{\pi/\Gamma_n} p(\Delta_n)$$

Proof. We identify $\Delta$ with an $a \times a$ matrix $B$ with entries in $\mathbb{Z}[\pi]$, as in the previous Lemma.

Fix elements $g_0, g_1, \ldots, g_r \in \pi$ and $\lambda_0, \lambda_1, \ldots, \lambda_r \in \mathbb{R}$ such that $g_0 = e$, $g_i \neq e$, and $\lambda_i \neq 0$ for $1 \leq i \leq r$ so that
$$\sum_{j=1}^a (p(B))_{j,j} = \sum_{i=0}^r \lambda_i g_i.$$  

Then
$$\text{Tr}_\pi p(\Delta) = \lambda_0.$$
The Laplacian $\Delta_n$ on $Y_n$ is also described by the matrix $B$, now acting on $\bigoplus_{i=1}^r l^2(\pi_i/\Gamma_n)$ by right multiplication. Then

$$\text{Tr}_{\pi/\Gamma_n} p(\Delta_n) = \sum_{i=1}^r \lambda_i \text{Tr}_{\pi_i/\Gamma_n} R(g_i)$$

where $R(g_i) : l^2(\pi_i/\Gamma_n) \to l^2(\pi_i/\Gamma_n)$ is right multiplication with $g_i$.

Since the intersection of the $\Gamma_i$'s is trivial, there is a number $n_0$ such that for $n \geq n_0$ none of the elements $g_i$ for $1 \leq i \leq r$ lies in $\Gamma_n$. Since $\Gamma_n$ is normal, we conclude for $n \geq n_0$ and $i \neq 0$

$$\text{Tr}_{\pi/\Gamma_n} R(g_i) = 0.$$

Then for $n \geq n_0$

$$\text{Tr}_\pi p(\Delta) = \lambda(0) = \text{Tr}_{\pi/\Gamma_n} p(\Delta_n).$$

\[\blacksquare\]

**Lemma 2.3.** Let $\{p_k(\mu)\}_{k=1}^\infty$ be a sequence of polynomials, uniformly bounded on $[0, ||\Delta||]$, such that for the characteristic function $\chi_{[0,\lambda]}(\mu)$ of the interval $[0, \lambda]$,

$$\lim_{k \to \infty} p_k(\mu) = \chi_{[0,\lambda]}(\mu)$$

holds for each $\mu \in [0, ||\Delta||]$. Then

$$\lim_{k \to \infty} \text{Tr}_\pi p_k(\Delta) = F(\lambda).$$

**Proof.** This lemma and its proof are identical to [13, Lemma 2.7] \[\blacksquare\]

We now prove Theorem 2.1. Fix $\lambda \geq 0$. Define for $k \geq 1$ a continuous function $f_k : \mathbb{R} \to \mathbb{R}$ by

$$f_k(\mu) = \begin{cases} 1 + \frac{1}{k} & \mu \leq \lambda \\ 1 + \frac{1}{k} - k(\mu - \lambda) & \lambda \leq \mu \leq \lambda + \frac{1}{k} \\ \frac{1}{k} & \lambda + \frac{1}{k} \leq \mu \end{cases}$$

Clearly $\chi_{[0,\lambda]}(\mu) < f_{k+1}(\mu) < f_k(\mu)$ and $f_k(\mu)$ converges to $\chi_{[0,\lambda]}(\mu)$ for all $\mu \in [0, \infty)$. For each $k$ choose a polynomial $p_k$ such that $\chi_{[0,\lambda]}(\mu) < p_k(\mu) < f_k(\mu)$ holds for all $\mu \in [0, K^2]$, where $K$ is as in Lemma 2.1. The polynomials $p_k$ satisfy the conditions of Lemma 2.3.

Because $\chi_{[0,\lambda]}(\mu) \leq p_k(\mu)$ for all $\mu \in [0, ||\Delta||]$, we have

$$F_n(\lambda) = \text{Tr}_{\pi/\Gamma_n} \left( \chi_{[0,\lambda]}(\Delta_n) \right)$$

$$\leq \text{Tr}_{\pi/\Gamma_n} \left( p_k(\Delta_n) \right).$$

(2.1)

On the other hand, we have $p_k(\mu) \leq 1 + \frac{1}{k}$ for $\mu \in [0, \lambda + \frac{1}{k}]$ and $p_k(\mu) \leq \frac{1}{k}$ for $\mu \in [\lambda + \frac{1}{k}, K^2]$. So

$$\text{Tr}_{\pi/\Gamma_n} \left( p_k(\Delta_n) \right) \leq \text{Tr}_{\pi/\Gamma_n} \left( (1 + \frac{1}{k})\chi_{[0,\lambda+\frac{1}{k}]}(\Delta_n) \right)$$

$$+ \text{Tr}_{\pi/\Gamma_n} \left( \frac{1}{k}\chi_{[\lambda+\frac{1}{k}, K^2]}(\Delta_n) \right)$$

(2.2)

$$= (1 + \frac{1}{k})F_n(\lambda + \frac{1}{k})$$

$$+ \frac{1}{k}(F_n(K^2) - F_n(\lambda + \frac{1}{k}))$$

$$= F_n(\lambda + \frac{1}{k}) + \frac{1}{k}F_n(K^2)$$
Now notice $F_n(K^2) = \text{Tr}_{\pi/\Gamma_n}(\chi_{[0,K^2]}(\Delta_n))$. But $\chi_{[0,K^2]}(\Delta_n)$ is the identity on the space $C^j(\Delta(Y_n))$, which is identified with $\bigoplus_{i=1}^{\alpha_n} I^2(\pi/\Gamma_n)$. Thus

\[(2.3)\]

By Lemma 2.2, there is a number $n_0(k)$ for each polynomial $p_k$ such that for $n \geq n_0(k)$

\[
\text{Tr}_{\pi} p_k(\Delta) = \text{Tr}_{\pi/\Gamma_n}(p_k(\Delta_n)).
\]

Then for $n \geq n_0(k)$, the equations (2.4), (2.2), and (2.3) give

\[F_n(\lambda) \leq \text{Tr}_{\pi} p_k(\Delta) \leq F_n(\lambda + \frac{1}{k}) + \frac{1}{k}a_j\]

Taking $\lim_{n \to \infty}$,

\[
F(\lambda) \leq \text{Tr}_{\pi} p_k(\Delta) \leq F(\lambda + \frac{1}{k}) + \frac{1}{k}a_j
\]

Taking $\lim_{k \to \infty}$, and using Lemma 2.3,

\[
F(\lambda) \leq F(\lambda) \leq F(\lambda)
\]

We have for all $\epsilon > 0$

\[
F(\lambda) \leq F(\lambda) \leq F(\lambda + \epsilon) \leq F(\lambda + \epsilon)
\]

and since $\lim_{\epsilon \to 0^+} F(\lambda + \epsilon) = F(\lambda)$ we get

\[
F(\lambda) = F^+(\lambda) = F^-(\lambda).
\]

This finishes the proof of Theorem 2.1.

Next we show Theorem 2.2.1 and Theorem 2.2.2. We suppose there is some right continuous function $s : [0, \epsilon) \to [0, \infty)$, with $s(0) = 0$ and so that for all $n$ and for all $\lambda \in [0, \epsilon)$ we have

\[
F_n(\lambda) - F_n(0) \leq s(\lambda).
\]

Taking the limit inferior and limit superior for $n \to \infty$ gives:

\[
F(\lambda) \leq F(0) + s(\lambda) \quad \text{and} \quad \overline{F}(\lambda) \leq F(0) + s(\lambda).
\]

Taking the limit for $\lambda \to 0$ gives

\[
F^+(0) \leq F(0) \quad \text{and} \quad \overline{F}^+(0) \leq F(0).
\]

And finally, since $F$ and $\overline{F}$ are increasing,

\[
F^+(0) = F(0) \quad \text{and} \quad \overline{F}^+(0) = F(0).
\]

We already know $\overline{F}^+(0) = F(0) = F^+(0)$ from Theorem 2.1, and this proves Theorem 2.2.1. Since $s$ is right continuous, we conclude:

\[
\overline{F}^+(\lambda) \leq F(0) + s(\lambda)
\]

and Theorem 2.2.2 follows from Theorem 2.1. This finishes the proof of Theorem 2.2.

The following lemma is needed in the proof of Theorem 1.2 in the last section.

**Lemma 2.4.**

\[
\int_0^{K^2} \frac{F(\lambda) - F(0)}{\lambda} d\lambda \leq \liminf_{n \to \infty} \int_0^{K^2} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda
\]

**Proof.** The proof of this lemma is identical to that of Lemma 3.3.1 in [13]. It follows from Theorem 2.1 and the monotone convergence theorem.
3. Proofs Of The Main Theorems

In this section, we prove the approximation theorem and determinant class theorem for residually amenable groups.

3.1. The Approximation Theorem.

Proof of Theorem 0.1 (Approximation Theorem).

Observe that the $j$th $L^2$ Betti numbers of $Y$ and $Y_n$ are given by

$$b_{(2)}(Y : \pi) = F(0); \quad b_{(2)}(Y_n : \pi/\Gamma_n) = F_n(0).$$

Therefore Theorem 0.1 will follow directly from Theorem 2.2.1, if we can establish a uniform decay of $F_n$ near zero.

Since $\pi/\Gamma_n$ is amenable, [6, Theorem 2.1.3] applies, and we have a constant $K > 1$ so that

$$F_n(\lambda) - F_n(0) \leq a_j \log K^2 - \log \lambda = s(\lambda)$$

for all $0 < \lambda < 1$. The constant $K$ can be any number larger than $\max(\|\Delta_n\|, 1)$, and therefore can be chosen independently of $n$, by Lemma 2.1.

Remark. In the case of Lück’s Theorem, the groups $\pi/\Gamma_n$ are finite, so the Laplacian on $Y_n$ is a finite, self-adjoint matrix. Lück then proves, for any self-adjoint matrix, an elegant estimate on the number of eigenvalues which are less than a fixed $\lambda$. The estimate is weakened if the product of the nonzero eigenvalues is small, but this product must be at least one as the Laplacian has integer entries.

Dodziuk and Mathai make use of the same fundamental estimate when proving the theorem for amenable groups.

3.2. Results For Manifolds. Suppose now $M$ is a compact Riemannian manifold, $\tilde{M}$ is a regular covering space for $M$ with residually amenable transformation group, and let $\tilde{\Delta}$ denote the Laplacian on $L^2$ $j$-forms on $\tilde{M}$. Following arguments in [3], one can investigate the analytic spectral density function $\tilde{F}(\lambda)$ of $\tilde{\Delta}$. To relate this analytic Laplacian to the combinatorial situation in the previous sections, choose $X$ to be a triangulation of $M$ and lifting to a triangulation $Y$ of $\tilde{M}$.

The results analogous to those of [3] are:

Theorem 3.1. (Gap Criterion) The spectrum of $\tilde{\Delta}$ has a gap at zero if and only if there is a $\lambda > 0$ such that

$$\lim_{n \to \infty} F_n(\lambda) - F_n(0) = 0.$$ 

Theorem 3.2. (Spectral Density Estimate) There are constants $C > 0$ and $\varepsilon > 0$ independent of $\lambda$, such that for all $\lambda \in (0, \varepsilon)$

$$\tilde{F}(\lambda) - \tilde{F}(0) \leq \frac{C}{-\log \lambda}.$$ 

3.3. The Determinant Class Theorem. A covering space $Y$ of a finite simplicial complex $X$ is said to be of determinant class if, for each $j$,

$$-\infty < \int_{0^+}^{1} \log \lambda dF(\lambda),$$ 

where $F(\lambda)$ denotes the von Neumann spectral density function of the combinatorial Laplacian $\Delta_j$ on $L^2$ $j$-cochains.
For a regular cover of a compact Riemannian manifold $M$, there is a corresponding notion of analytic determinant class. We can also ask if $\tilde{M}$ is of determinant class in the combinatorial sense, by choosing a triangulation of $M$. This question turns out to be independent of triangulation, and equivalent to analytic determinant class. For a more detailed discussion, see [6].

We will prove that every residually amenable covering of a finite simplicial complex is of determinant class. The appendix of [2] contains a proof that every residually finite covering of a compact manifold is of determinant class. Their proof is based on Lück’s approximation of von Neumann spectral density functions. Since an analogous approximation holds in the setting of this paper, we can apply the argument of [2] to prove Theorem 0.2. The fact that our coverings are infinite necessitates some modifications.

**Proof of Theorem 0.2 (Determinant Class Theorem).**

As with the rest of this paper, this proof will proceed for a fixed $j$ which will be suppressed in the notation.

Denote by $\text{Det}'_{π/Γ_n} \Delta_n$ the modified Fuglede-Kadison [8] determinant of $\Delta_n$, that is, the Fuglede-Kadison determinant of $\Delta_n$ restricted to the orthogonal complement of its kernel. It is given by the following Lebesgue-Stieltjes integral,

$$\log \text{Det}'_{π/Γ_n} \Delta_n = \int_{0^+}^{K^2} \log \lambda dF_n(\lambda)$$

with $K$ as in Lemma 2.1. That is, $\|\Delta_n\| \leq K^2$.

Integration by parts yields

$$\log \text{Det}'_{π/Γ_n} \Delta_n = (\log K^2)(F_n(K^2) - F_n(0))$$

$$+ \lim_{ε \to 0^+} \left\{ -(\log ε)(F_n(ε) - F_n(0)) - \int_{ε}^{K^2} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda \right\}.$$

Now $Y_n$ is an amenable cover of $X$, so we are in the situation of Dodziuk-Mathai [6]. We require two results contained in their proof of the Determinant Class Theorem for amenable coverings [6, Thm 0.2]. The first is that

$$\log \text{Det}'_{π/Γ_n} \Delta_n \geq 0,$$

which is stronger than simply determinant class. The second is the existence of the integral

$$\int_{0^+}^{K^2} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda.$$

It is worth remarking that their proof and the corresponding proof in [6] require similar statements for the Laplacians on finite approximations. In the finite case, the Laplacian is an integer matrix, so the determinant is a positive integer and therefore at least 1. The existence of the integral (3.3) is clear in the finite case, as the spectrum is discrete.

From (3.2) and the existence of (3.3), $\lim_{ε \to 0^+} (\log ε)(F_n(ε) - F_n(0))$ exists. In fact, for the integral (3.3) to exist, we must have

$$\lim_{ε \to 0^+} (\log ε)(F_n(ε) - F_n(0)) = 0.$$
Then \((3.2)\) becomes
\[
\log \det' \pi \Delta_n = (\log K^2)(F_n(K^2) - F_n(0)) - \int_{0^+}^{K^2} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda
\]
which gives
\[
\int_{0^+}^{K^2} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda \leq (\log K^2)(F_n(K^2) - F_n(0)).
\]

Now from Lemma 2.4,
\[
\int_{0^+}^{K^2} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda \leq \liminf_{n \to \infty} \int_{0^+}^{K^2} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda
\leq \liminf_{n \to \infty} (\log K^2)(F_n(K^2) - F_n(0)).
\]
Since we have uniform decay \((3.1)\) of the \(F_n\) near 0, Theorem 2.2.1 applies and
\[
\lim_{n \to \infty} F_n(0) = F(0).
\]

Since \(K^2 \geq \|\Delta_n\|\) for all \(n\) and \(K^2 \geq \|\Delta\|\),
\[
F_n(K^2) = F(K^2) = a_j
\]
for all \(n\), so \((3.4)\) becomes
\[
\int_{0^+}^{K^2} \frac{F(\lambda) - F(0)}{\lambda} d\lambda \leq (\log K^2)(F(K^2) - F(0)).
\]

This shows in particular that the left hand integral exists, and arguing as with \(\Delta_n\) we have
\[
\log \det' \pi \Delta = (\log K^2)(F(K^2) - F(0)) - \int_{0^+}^{K^2} \frac{F(\lambda) - F(0)}{\lambda} d\lambda
\]
which is non-negative by \((3.3)\). Since this is true for all \(j\), \(Y\) is of determinant class.

\[\square\]

3.4. **Homotopy Invariance of \(L^2\) Torsion.** The Fuglede-Kadison determinant \(\det\) induces a homomorphism from the Whitehead group of \(\pi\),
\[
\Phi_\pi : \text{Wh}(\pi) \to \mathbb{R}^+
\]
which was defined in \([14]\) and \([13]\). Given a homotopy equivalence \(f : M \to N\) of compact manifolds, we choose cell decompositions for \(M\) and \(N\), choose \(f\) to be a cellular homotopy equivalence, and let \(M_f\) be the cellular mapping cone. Then the cochain complex \(C^\ast(M_f)\) is an acyclic complex over the group ring \(\mathbb{Z}[\pi]\), and defines the Whitehead torsion \(T(f) \in \text{Wh}(\pi)\). Then \(\Phi_\pi(T(f)) \in \mathbb{R}^+\).

Now suppose \(\pi\) is residually amenable, so \(M\) and \(N\) are of determinant class. Let \(\phi_M, \phi_N\) denote the \(L^2\) torsion of \(M\) and \(N\). The homotopy equivalence \(f\) canonically identifies the determinant lines of \(L^2\) cohomology of \(\tilde{M}\) and \(\tilde{N}\), so \(\phi_M \otimes \phi_N^{-1} \in \mathbb{R}^+\).

Then from \([14]\) Prop. 2.1, one has
\[
\phi_M \otimes \phi_N^{-1} = \Phi_\pi(T(f)) \in \mathbb{R}^+.
\]
Therefore, Theorem 0.3 will follow from the following
Proposition 3.1. Suppose that $\pi$ is a finitely presented residually amenable group. Then the homomorphism
\[ \Phi_\pi : \text{Wh}(\pi) \to \mathbb{R}^+ \]

is trivial.

Proof. We can represent an arbitrary element of $\text{Wh}(\pi)$ as the Whitehead torsion of a homotopy equivalence $f : L \to K$ of finite CW-complexes, which without loss of generality is an inclusion. Let $\tilde{L}$ and $\tilde{K}$ denote the corresponding regular $\pi$ covering complexes. The relative cochain complex $C^*(\tilde{K}, \tilde{L})$ is acyclic, and so is its $L^2$ completion $C^*_\Lambda(\tilde{K}, \tilde{L})$. In particular, the combinatorial Laplacian $\Delta_{\tilde{K}, \tilde{L}}^j$ is invertible and we see that
\[ \Phi_\pi(T(f)) = \prod_{j=0}^{n} \text{Det}_\pi(\Delta_{\tilde{K}, \tilde{L}}^j) \left(\frac{i\pi}{2}\right)^j > 0. \]

We claim that $\text{Det}_\pi(\Delta_{\tilde{K}, \tilde{L}}^j) = 1$ for each $j$. The $j$ will be suppressed in the notation.

Form the covering spaces $\tilde{L}_n$ and $\tilde{K}_n$ with amenable covering group $\pi/\Gamma_n$, and let $\Delta_{\tilde{K}_n, \tilde{L}_n}$ denote the Laplacian on $C^*_\Lambda(\tilde{K}_n, \tilde{L}_n)$. As $\Delta_{\tilde{K}, \tilde{L}}$ is invertible we can choose a common bound $K$ on $\|\Delta_{\tilde{K}, \tilde{L}}\|$ and $\|\Delta_{\tilde{K}, \tilde{L}}^{-1}\|$, as in Lemma 2.1. $K$ will also bound $\|\Delta_{\tilde{K}_n, \tilde{L}_n}\|$ and $\|\Delta_{\tilde{K}_n, \tilde{L}_n}^{-1}\|$.

It then follows from the proof of [17, Prop. 2.6] that for all $n$,
\[ \text{Det}_{\pi/\Gamma_n}(\Delta_{\tilde{K}_n, \tilde{L}_n}) = 1 \]
and $\Delta_{\tilde{K}_n, \tilde{L}_n}$ has a spectral gap at zero of size at least $K^{-2}$. Then, analogous to [13, Thm 3.4.3],
\[ \text{Det}_\pi(\Delta_{\tilde{K}, \tilde{L}}) = \lim_{n \to \infty} \text{Det}_{\pi/\Gamma_n}(\Delta_{\tilde{K}_n, \tilde{L}_n}) = 1. \]

This finishes the proof of Theorem 0.3. As remarked earlier, in both [13] and [17], the approximating spaces are finite. The approximating Laplacians have determinant 1 and spectral gap because they are finite invertible integer matrices, and the previous proposition essentially boils down to this fact.

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Department Of Mathematics, University Of Chicago. CHICAGO, IL 60615.

E-mail address: bryan@math.uchicago.edu