Star Product Reduction
for Coisotropic Submanifolds of Codimension 1

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FR-THEP-98/10
20 April 1998

Abstract

We propose a reduction procedure that leads to a reduced star product on the reduced phase space of a “First Class”–constrained system, where no symmetries, group actions or the like are present. For the case that the coisotropic constraint submanifold has codimension 1, we establish a constructive method to compute the reduced star product explicitly. Concluding examples show that this method depends crucially on the constraint function singled out to describe the constraint submanifold and not only on this submanifold itself, and that two different constraint functions for the same constraint submanifold will generally result in not only different but inequivalent reduced star products.

1 Introduction

In 1978, Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer established in [3] the concept that we call deformation quantization. The idea is to replace the pointwise product on the algebra of phase space functions $C^\infty(P)$ of a classical dynamical system by a non–commutative so–called star product $\ast$ on $C^\infty(P)[\lambda]$. For two $f, g \in C^\infty(P)[\lambda]$, $f \ast g$ is again a formal power series in $\lambda$, is the pointwise product in 0th order, and $f \ast g - g \ast f$ is $i$ times the Poisson bracket $\{f, g\}$. In this way, the formal parameter $\lambda$ plays the rôle of Planck’s constant $\hbar$ and the algebra $C^\infty(P)$ can be addressed as the algebra of quantum observables, deformed from the classical observables $C^\infty(P)$ in the sense of Gerstenhaber [13]. The naturally arising question whether star products exist for arbitrary symplectic manifolds was answered in the affirmative 1983 by DeWilde and Lecomte [8], Omori, Maeda and Yoshioka [17], while Fedosov [10] gave a more geometric proof in 1994. In the more general setting of a Poisson manifold, the existence of star products was shown only recently by Kontsevich [15].

In this paper, we propose a kind of “quantum” or “deformed” reduction mechanism. It
shall serve to answer the following question: given an arbitrary star product $*$ on $(P, \omega)$, and a coisotropic submanifold $C \hookrightarrow P$ with Reduced Phase Space $(\hat{C}, \hat{\omega})$, how can we construct a star product $*$ on the symplectic manifold $(\hat{C}, \hat{\omega})$ which can be addressed as the result of some reduction procedure starting with $(P, *)$?

This problem has been dealt with in several publications: after the authors of [3] had discussed some basic examples, Fedosov [11] investigated reductions induced by certain $U(1)$–actions on the phase space. The case of $\mathbb{C}P^n$ and its noncompact dual were considered in [3] and [1] by Bordemann et al., even resulting in explicit formulae. Schirmer [18] gives a generalization to Grassmann manifolds. In [12], Fedosov introduces a reduction procedure for Hamiltonian group actions of arbitrary compact Lie groups.

But in these treatments it has been essential that the constraint submanifold $C$ was a level surface of a momentum mapping, that is, the phase space reduction in these cases was the outcome of some symmetry group acting on the phase space $P$; all reduction processes cited above consider a star product on the phase space $P$ such that the invariant functions on $P$ form a subalgebra. We will try a first step towards an algebraic reduction process for general coisotropic submanifolds, as they may occur for example as the result of a non–surjective fiber derivative (see [1], esp. 3.5). Symmetry groups, momentum mappings etc. will consequently play no part in our considerations, and we are free to choose any derivative (see [1], references therein. As general reference for this subject may serve [1], Chapter 5, esp. exercises and sections 3, 4 and 5 concentrate on the codimension 1 case and provide the necessary structures.

Before we sketch our methods and results, we briefly review the notion of classical phase space reduction, in order to introduce the algebraic structures we will deform in the ensuing section. As general reference for this subject may serve [3], Chapter 5, esp. exercises and references therein.

If a dynamical system on a phase space $P$ (equipped with symplectic form $\omega$) is forced to stay on a coisotropic submanifold $C \hookrightarrow P$ ("by first class constraints"; see [1] and [14]), we can perform the well-known procedure commonly called phase space reduction [12] to obtain a phase space which represents in some sense the true degrees of freedom of the physical system. The reduction process in its differential geometric picture consists essentially in pulling back $\omega$ on $C$ and then dividing $C$ by the foliation which is generated by the integrable distribution associated to the kernel of the pull–back of $\omega$. We come out with the Reduced Phase Space $(\hat{C}, \hat{\omega})$. On the algebraic side, this picture is reflected in the following way. The constraint manifold $C$ is characterized by its vanishing ideal $I := \{f \in C^\infty(P) \mid i^*f = 0\}$ with $i : C \hookrightarrow P$ the embedding. This algebra is a Poisson ideal in $B := \{f \in C^\infty(P) \mid \{f, f'\} \in I \forall f' \in I\}$ ($\{\cdot, \cdot\}$ denoting as usual the Poisson bracket that comes with $\omega$), which in turn is a Poisson subalgebra of $C^\infty(P)$. We therefore can define the quotient $B/I$ as the Reduced Algebra of the constrained system. It carries a Poisson structure inherited from that of $P$, and it turns out that $B/I$ is Poisson–isomorphic to $C^\infty(\hat{C}, \hat{\omega})$, the functions on the Reduced Phase Space (this is the case essentially because the Hamiltonian vector fields to functions in $I$ span the kernel of the pull–back of $\omega$). In this sense, the construction of $B/I$ is the algebraic form of phase space reduction.

In the next section, we propose a deformation of the classical algebras $I$ and $B$ into new algebras $I^*$ and $B^*$. The quotient algebra $B^*/I^*$ turns out to be an associative star algebra, so that the aim is to establish a linear isomorphism from $B^*/I^*$ to $B/I$ which then is declared to be a star product isomorphism, thereby providing $B/I$ with the desired star product addressed as "reduced from $(P, *)". We emphasize that – though the algebras $B/I$ and $B^*/I^*$ can always be formed – the construction of such an isomorphism is (of course) by no means natural and constitutes the essential task of our reduction process. Sections 3, 4 and 5 concentrate on the codimension 1 case and provide the necessary structures.
and proofs for a constructive method to compute this isomorphism. Sections 6, 7 and 8 give examples to show plausibility and feasibility of the reduction process. A simple $\mathbb{R}^n$ example is followed by a reduction of the Wick product to $\mathbb{C}P^n$, where the reduced star product is the same as in $\mathbb{R}^n$, although constructed in a completely different way. But, as mentioned before, we can choose our constraint function arbitrarily among all functions in $I$ and are in no way restricted to stick to the $U(1)$ momentum mapping that usually serves to the classical reduction $\mathbb{C}^{n+1}\setminus\{0\} \rightarrow \mathbb{C}P^n$. We make use of this our freedom in section 8 and reduce the Wick product again, this time by a different constraint function. The obtained reduced star product is different from the one derived before, and what is more, it is even inequivalent to this, as our concluding remarks will show. This may in turn be set in contrast to [19], where an inequivalent reduced star product on $\mathbb{C}P^n$ is obtained by deforming the classical momentum mapping, that is by adding terms of higher order in the formal parameter but leaving the classical momentum mapping untouched in 0th order.

But before we start our discussions, we fix our notion of a star product $*$: a star product of two formal power series $f, g \in C^\infty(P)[\lambda]$ be defined as $f * g := \sum_{n=0}^{\infty} \lambda^n M_n(f, g)$, where $M_0(f, g) = f \cdot g$ is the usual point product, $M_1(f, g) - M_1(g, f) = i\{f, g\}$ is $i$ times the Poisson bracket on $P$, $f * 1 = 1 * f = f$ and $\text{supp}(f * g) \subseteq \text{supp}(f) \cap \text{supp}(g)$, the latter condition saying that the star product is local. We will sometimes need the antisymmetric parts of the $M_n$’s (or rather two times this) and denote them by $\tilde{M}_n(f, g) := M_n(f, g) - M_n(g, f)$.

## 2 Translating the classical into deformed structures

To establish a star product phase space reduction, we define algebraic structures corresponding to the classical ones $I$ and $B$ reviewed in section 4. (We remark that from now on, $I$ and $B$ shall be considered as containing formal power series from $C^\infty(P)[\lambda]$ instead of simple functions from $C^\infty(P)$.) We begin with $I$, which we want to “deform” into an $I^\ast$. What properties should this have?

Firstly, since in a theory quantized by representation of the observables as operators, the Hilbert space of physical states may often be defined via the operators $\hat{J}_i$ corresponding to the constraint functions $J_i$, $i = 1 \ldots \text{codim} \mathcal{C}$, we make the ansatz that such a star product reduction incorporates a preferred choice of codimension–$C$–many first class constraints. This way, the star product reduction “sees” not only the constraint surface $\mathcal{C}$ itself, but a certain sandwich neighborhood $U \supset \mathcal{C}$, as different from the classical phase space reduction. So, we construct the new algebra $I^\ast$ to contain the first class constraints $J_i$, $i = 1 \ldots \text{codim} \mathcal{C}$ in an explicit way. We take a special set of constraint functions as given and refrain from discussing the reasons that could lend preference to this choice over possible other ones; the reasons may be found in symmetries of $(P, \omega)$, or there may be no reasons at all – the star product reduction should work with every set (but dependent on it).

Secondly, $I^\ast$ should contain the (noncommutative) star product and therefore be a one–sided star ideal.

Thirdly, the classical vanishing ideal $I$ should be regained by performing the limit $\lambda \rightarrow 0$, $\lambda$ being the formal deformation parameter.

These considerations lead to the definition of the star–left–ideal

$$I^\ast := \{ f \in C^\infty(P)[\lambda] \mid f = \sum g^i \ast J_i \text{ for some } g^i \in C^\infty(P)[\lambda] ; i = 1 \ldots \text{codim} \mathcal{C} \}. \quad (1)$$

We proceed along these lines defining $B^\ast$ as “deformation” of $B$. $B^\ast$ has to be a star–subalgebra of $C^\infty(P)[\lambda]$, and $I^\ast$ has to be a two–sided star–ideal in $B^\ast$.

$$B^\ast := \{ f \in C^\infty(P)[\lambda] \mid f \ast g - g \ast f \in I^\ast \forall g \in I^\ast \} \quad (2)$$
fulfills these requirements, as can be seen by simple computations. So, $B^*/I^*$ is well-defined and carries by representant–wise definition $[f] \star [g] := [f \circ g]$ a well–defined star product, thereby forming an associative star algebra. We will use $B^*$ mostly in the equivalent form of

$$B^* = \{ f \in C^\infty(P)[\lambda] \mid J_i \circ f \in I^* \ \forall J_i, i = 1 \ldots \text{codim} C \},$$

which can be obtained from the definition in a rather direct way.

The general aim now is to establish a linear isomorphism between $B/I$ (enlarged, as we agreed, to obtain formal power series), and $B^*/I^*$. Then the isomorphisms

$$C^\infty(C)[\lambda] \cong B/I \cong (B^*/I^*, \star)$$

provides us with the desired reduced star product on the classical Reduced Phase Space.

In the next three sections, we will construct the isomorphism $B/I \cong B^*/I^*$ in an explicit way, but only for the codimension 1 case.

3 Sum decomposition of $C^\infty(P)[\lambda]$

Our first step consists in defining a prolongation prescription for series of functions on $C$, i. e. a mapping

$$\rho : C^\infty(C)[\lambda] \to C^\infty(U)[\lambda] \quad \text{with} \quad i^* \circ \rho = \text{Id}$$

$$(U \subset P \text{ is a sandwich neighborhood of the constraint surface } C).$$

This can be done arbitrarily; we could, for instance, establish a Riemannian metric $g$ on $P$ and use the gradient flow of the constraint function – remember that we are in the codimension 1 case. In many examples with symmetry, a preferred choice for this prescription will present itself. However, the reduction process works for every choice (but is dependent on it).

Let $I : C \hookrightarrow P$ be the imbedding of the constraint surface into $P$, we then define the prolongation of a series $f \in C^\infty(P)[\lambda]$ by

$$\text{prol} : C^\infty(P)[\lambda] \to C^\infty(U)[\lambda]$$

$$\text{prol}(f) := \rho(i^* f),$$

and we set

$$F := \{ f \in C^\infty(P)[\lambda] \mid f(p) = (\text{prol}(f))(p) \ \forall p \in U \},$$

calling such series “pure prolongations”. We agree that from now on, we do not distinguish between (series of) functions that are different just outside a sandwich neighborhood $U$.

“Uniqueness” will be understood in this sense in what follows.— Now, because clearly $f - \text{prol}(f) \in I$ for all $f \in C^\infty(P)[\lambda]$, “Hadamard’s trick” or any other form of the mean value theorem gives us a unique smooth series $h \in C^\infty(U)[\lambda]$ such that

$$f - \text{prol}(f) = h \cdot J.$$ 

We set $\pi_j(f) := h$ as the “component of $f$ along the constraint function $J$”. Remark that while $\text{prol}$ is a projection, $\pi_j$ is not. We end with a uniquely defined decomposition of $C^\infty(P)[\lambda]$ as a direct sum

$$C^\infty(P)[\lambda] = F \oplus I$$

$$f = \text{prol}(f) + \pi_j(f) \cdot J$$

In a second step, we inductively define the following formal power series of $C$–linear operators on $C^\infty(P)[\lambda]$, using the bilinear operators $M_r$ of our star product $\star$ on $P$, the $\pi_j$ just
introduced and the constraint function $J$:

$$T = \sum_{n=0}^{\infty} \lambda^n T_n$$

$$T_0 := \text{Id}$$

$$T_n(f):= -\sum_{k=1}^{n} T_{n-k}(M_k(\pi_J(f), J)) \text{ for } n \geq 1$$

(7)

**Lemma 1** The above defined operator $T$ has the following properties:

i) $T : I^* \to I$ is one–to–one and onto.

ii) $T(\text{prol}(f)) = \text{prol}(f)$ for any $f \in C^\infty(P)[\lambda]$, and $T(J) = J$.

iii) $T(f \ast J) = f : J$ for any $f \in C^\infty(P)[\lambda]$.

iv) for every $g \in C^\infty(P)$, $\text{supp}(T_n(g)) \cap C \subseteq \text{supp}(g) \cap C$ for all $n$.

**Proof.** i) Injectivity follows from $T_0 = \text{Id}$ and surjectivity is clear once iii) is proven. ii) follows from the facts that $\pi_J \circ \text{prol} = 0$ and $\pi_J(J) = 1$. iii) It is sufficient to consider functions $f \in C^\infty(P)$. On one hand it is in $n$th order $(T(f \ast J))_n = T_n(fJ) + \sum_{k=1}^{n} T_{n-k}(M_k(f, J))$, $n \geq 1$. On the other hand, $T_n(fJ) = -\sum_{k=1}^{n} T_{n-k}(M_k(\pi_J(fJ), J))$ but $\pi_J(fJ) = f$, so $(T(f \ast J))_n = 0$ for $n \geq 1$. $(T(f \ast J))_0 = fJ$ is trivial. iv) We use $\text{supp}(\pi_J(g)) \cap C \subseteq \text{supp}(g) \cap C$ and the fact that the $M_k$ do not enlarge the supports of their arguments in a straightforward inductive reasoning.

This operator $T$ yields another sum decomposition of $C^\infty(P)[\lambda]$. Indeed, if we apply $T^{-1}$ to $T(f) = \text{prol}(T(f)) + \pi_J(T(f)) \cdot J$, keeping in mind properties ii) and iii) from the above lemma, we obtain for every $f \in C^\infty(P)[\lambda]$ the equation $f = \text{prol}(T(f)) + \pi_J(T(f)) \ast J$, the last summand being an element of $I^*$. So there is a further sum decomposition of $C^\infty(P)[\lambda]$ as

$$C^\infty(P)[\lambda] = F \oplus I^*$$

$$f = \text{prol}(T(f)) + \pi_J(T(f)) \ast J$$

(8)

4 **Isomorphism between $B \cap F$ and $B^* \cap F$**

With the help of the operator series $T$ just defined, we are able to construct a formal power series of $C$–linear operators $S_n$ which establishes a linear isomorphism $S : B \cap F \to B^* \cap F$, needed to map $B/I$ and $B^*/I^*$ on each other $C$–linearly and bijectively. We point out that we will now make an additional assumption, namely, we suppose there is a transversal section $C$ of the foliation on $C$ associated to the Hamiltonian vector field $X_J$ of the constraint function $J$, and with $p : C \to \sigma$ we denote the projection on the section along the leaves of this foliation. Let $\Phi_t^J$ be the Hamiltonian flow of $J$, with flow parameter $t$.

It may be remarked that, if a global transversal section is not at hand, neighbourhoods on that individual operators $S$ can be constructed in the manner described below can be put together to yield a common $S$ operator on the union of the neighbourhoods, as long as their intersection fulfills certain requirements; since it is the aim of the present discussion to outline the main ideas of the star product reduction presented here, we do not embark on giving the details of this problem.

**Lemma 2** Let $p \equiv \Phi_t^J(p(p))$ be any point on $C$ and $f \in C^\infty(P)[\lambda]$, and let $S = \sum_{n=0}^{\infty} \lambda^n S_n$ be inductively defined as

$$S_0 = \text{Id}$$
\( (S_n f)(p) := -i \int_0^{t(p)} \Phi^{++}_t(F_{n+1}[S_0, \ldots, S_{n-1}; T_0, \ldots, T_n](f))(p) \, dt \)

for \( n \geq 1 \), where \( (n \geq 2) \)

\[ F_n[S_0, \ldots, S_{n-2}; T_0, \ldots, T_{n-1}](f) := \sum_{k=2}^{n} \sum_{i=1}^{n-1} T_i(M_k(J, S_{n-k-1} f)) \]

and in the sandwich neighborhood \( U \supset C \) we set \( S_n f := \text{prol}(S_n f) \).

Then \( S : B \cap F \to B^* \cap F \) is a linear isomorphism.

Proof. By construction we have for \( n \geq 1 \) and \( f \in B \cap F \) that \( L_{X_n}(S_n f)(p) = -iF_{n+1}(f)(p), \ p \in C \), so \( \{ J, S_n f \} = -iF_{n+1}(f) \) on \( C \). But a direct computation shows \( \{ J, S_n f \} - iF_{n+1}(f) = (T(J \ast Sf - Sf \ast J))_{n+1} \), that is \( T(J \ast Sf - Sf \ast J) = 0 \) on \( C \) at order \( n \geq 2 \) with our \( S \).

At order 1, \( T(J \ast Sf - Sf \ast J) = i\{ J, S_0 f \} = 0 \) on \( C \) as \( f \in B \). All in all, \( T(J \ast Sf - Sf \ast J) = h \cdot J \) for some \( h \in C^\ast(P)[\lambda] \), because it is an element of \( I \). Applying \( T^{-1} \) to both sides and using iii) of lemma 3 yields \( J \ast Sf = g \ast J \) for some \( g \in C^\ast(P)[\lambda] \), showing that \( Sf \in B^* \) according to equation 4. \( Sf \in C \) is clear by construction, and injectivity of \( S : B \cap F \to B^* \cap F \) follows from \( S_0 = \text{Id} \). So it remains to show that \( S \) is onto. To this end, we fix an arbitrary \( f = \sum_{n=0}^\infty \lambda^n f_n \in B^* \cap F \). From this \( f \), we construct a sequence \( (g^{(k)})_{k \in \mathbb{N}} \) of power series in \( \lambda \), inductively defined by \( g^{(0)} := f \), \( g^{(k+1)} := f - g^{(k)} \ast S_{2k+1} \), and then in turn we can write down \( g := 1 + \lambda^2 \sum_{k=0}^\infty g^{(2k)} - \lambda^2 g^{(0)} \), picking always the 0th order term out of every series \( g^{(k)} \). It is not difficult to show now that \( g \) is a well-defined power series in \( \lambda \), that \( g \in B \cap F \) and \( Sg = f \). \( \blacksquare \)

For the locality of the future reduced star product, the following lemma is essential.

Lemma 3 At every order \( n \) and for every \( f \in B \cap F \), \( \text{supp}(S_n f) \subseteq \text{supp}(f) \).

Proof. Because \( f \in F \) and \( S_n f \in F \), it is sufficient to consider the intersection of the supports with \( C \); so we have to prove that \( \text{supp}(S_n f) \cap C \subseteq \text{supp}(f) \cap C \), \( n \in \mathbb{N} \), \( f \in B \cap F \).

Suppose we already had proved this for \( 1, \ldots, n-1 \). Then it follows from the construction of \( S_n \) (using lemma 4, iv) in the step \( \text{supp}(F_n(f)) \cap C \subseteq \text{supp}(f) \cap C \) with \( p : C \to \sigma \), that \( p(\text{supp}(S_n f) \cap C) \subseteq p(\text{supp}(f) \cap C) \). But \( f \in B \), that is \( f(p) = f(p(p)) \) for all \( p \in C \), so for every set \( A \subseteq C \), the implication \( p(A) \subseteq p(\text{supp}(f) \cap C) \Rightarrow A \subseteq \text{supp}(f) \cap C \) holds. \( \blacksquare \)

5 Construction of the Reduced Star Product

Lemma 4 The spaces \( B^*/I^* \) and \( B^* \cap F \) are linearly isomorphic through \( \text{prol} \circ T \). Likewise, \( B/I \) and \( B \cap F \) are isomorphic through \( \text{prol} \).

Proof. Let \( b \in B^* \), then \( b = f + i, f \in F, i \in I^* \). But \( i \in I^* \subset B^* \Rightarrow f = b - i \in B^* \Rightarrow f \in B^* \cap F \). So \( b \in B^* \cap F \cap I \cap I^* \). Conversely, \( B^* \subset C^\ast(P)[\lambda] \) is a subspace, so trivially \( B^* \cap F \cap B^* \cap I \cap I^* \subset B^* \) and with \( B^* \cap I^* \cap B^* \cap I \subset B^* \).

From \( B^* = B^* \cap F \cap I \) then, we see that (comparing this with the unique decomposition in equation 5 \( \text{prol}(T(f)) \in B^* \cap F \) for \( f \in B^* \), and furthermore \( \text{prol} \circ T \) is well-defined on \( B^*/I^* \), because for \( f \in B^*, i \in I^*, \text{prol}(T(f + i)) = \text{prol}(T(f)) \) since \( T(i) \in I \). \( \blacksquare \)

We now have the following chain of linear isomorphisms:

\[
C^\ast(C, \hat{\omega}) \cong B/I \cong B \cap F \cong B^* \cap F \cong (B^*/I^*, *)
\]
the latter space endowed with a star product inherited from that of \((P, \omega)\). Now let \(f, g \in B \cap F\), then \(Sf, Sg \in B^* \cap F\). Regarding them as representants in \(B^*/I^*\), we form \(Sg \ast Sf\) (this is in \(B^*\), but not in \(F\) anymore), so that \(\text{prol}(T(Sf \ast Sg)) \in B^* \cap F\). Finally, applying \(S^{-1}\) brings us back to \(B \cap F\).

**Lemma 5** and definition. Identifying \(B \cap F\) and \(C^\infty(\hat{\mathcal{C}}, \hat{\omega})\), we set

\[
f \ast g := S^{-1}(\text{prol}(T(Sf \ast Sg)))
\]

for \(f, g \in B \cap F\) and have a (local) star product on \(B \cap F \cong C^\infty(\hat{\mathcal{C}}, \hat{\omega})\). We call it “reduced from ∗” by the first class constraint \(J\).

**Proof.** The properties of a star product (basically clear by construction) can be checked one by one, remembering \(S_0 = (S^{-1})_0 = T_0 = \text{Id}\) and \(\text{prol}(f \cdot g) = \text{prol}(f) \cdot \text{prol}(g)\); to prove \(f \ast 1 = 1 \ast f = f\), we use that \(S\) vanishes on constants and \(\text{prol}(T(f)) = f\) for \(f \in F\) (lemma 5 ii), so \(f \ast 1 = S^{-1}(\text{prol}(T(Sf \ast 1))) = S^{-1}(\text{prol}(T(Sf))) = S^{-1}(Sf) = f\) (since \(Sf \in B^* \cap F \subset F\)). Locality follows from lemma 5 iv) and lemma 3.

If the \(M_n\) are bidifferential operators of finite order, so are the \(\tilde{M}_n\) associated with ∗, as can be seen by confirming that neither \(T\) nor \(S\) can increase the number of derivatives.

6 Example: Moyal product on \(\mathbb{R}^{2n}\) to \(\mathbb{R}^{2n-2}\)

We first show that in a most simple example, the reduction formalism gives the expected result. To this purpose, we take \(\mathbb{R}^{2n}\) with the usual symplectic form and a global chart \((q^1, \ldots, q^n; p_1, \ldots, p_n)\). We feed the constraint function \(J(q; p) := p_n\) into the classical reduction formalism and get \(\mathbb{R}^{2n-2}\) as Reduced Phase Space, for which \((q^1, \ldots, q^{n-1}; p_1, \ldots, p_{n-1})\) may serve as a global chart. On \(\mathbb{R}^{2n}\), we suppose the **Moyal product** as given (for operator orderings see e.g. [2]): its explicit form is \(f \ast g = \sum_{r=0}^{\infty} \frac{1}{r!} \Lambda^{l_1} \ldots \Lambda^{l_r} \frac{\partial^r f}{\partial q^{r_1} \partial p^{r_2} \ldots \partial q^{r_2} \partial p^{r_3}}, \)

\(l_s, k_s = 1 \ldots 2n\), where \(\Lambda\) denotes the Poisson tensor to \(\omega = dq^i \wedge dp_i\).

For the star product reduction, we have to choose a prolongation prescription off the constraint surface \(\mathcal{C} = \{(q; p) \in \mathbb{R}^{2n} \mid p_n = 0\}\), and we do this in the simplest manner by setting \(\text{prol}(f)(q; p_1, \ldots, p_{n-1}) := f(q; p_1, \ldots, p_{n-1}, 0)\). In this case \(\pi_n(f)\) is nothing else but a difference quotient in the direction of \(p_n\). It turns out that the operator \(T: I \rightarrow I^*\) can in this example be written as \(T = \frac{1}{1 - \lambda K}\) with \(Kf := -M_1(\pi_n(f), J)\) being \(\frac{1}{2i}\) times the difference quotient of \(\frac{\partial f}{\partial q}\) in the direction of \(p_n\), and a short calculation shows that \(S: B \cap F \rightarrow B^* \cap F\) is equal to (the prolongation of) \(\text{Id} - \lambda K\). But \(B \cap F\) can be recognized as the space of series of functions not depending on \(q^n\) and \(p_n\) (which is clear because \(C^\infty(\mathbb{R}^{2n-2})[\lambda] \cong B/I \cong B \cap F\)), so \(S = \text{Id}: B \cap F \rightarrow B^* \cap F\); the equality of the spaces \(B \cap F\) and \(B^* \cap F\) can, of course, be established also in a direct way. But furthermore \(T(f \ast g) = f \ast g\) for \(f, g \in B \cap F\) on the basis of the Moyal product’s special form, as well as \(\text{prol}(f \ast g) = f \ast g\), by putting all this together we end with \(f \ast g = f \ast g\): the Reduced Star Product is just again the Moyal product, this time for functions on \(\mathbb{R}^{2n-2}\).

7 Example: Wick product from \(\mathbb{C}^{n+1}\) to \(\mathbb{C}P^n\)

Things look different if we reduce the Wick product (see for example again [2]) from \(\mathbb{C}^{n+1} \setminus \{0\}\) to \(\mathbb{C}P^n\). This has already been done, even resulting in an explicit formula [2], but taking into

7
account the symmetries of the problem — which we will ignore. We can therefore try our reduction mechanism on two different constraint functions, and we will obtain explicit formulæ in both cases, for two star products on $\mathbb{C}P^n$ that are not only different but inequivalent.

In the first case, we consider $\mathbb{C}^{n+1}\setminus\{0\}$ with usual symplectic form $\omega = \frac{i}{2} dz^i \wedge d\bar{z}^i$ and $J(z) = -\frac{1}{2} \bar{z}^i \partial_i - \mu \in \mathbb{R}^-$ (where $\bar{z}^i$ abbreviates $\sum_{i=1}^{n+1} z_i \bar{z}^i$). This $J$ is an ad$_e$–equivariant momentum mapping for the $U(1)$ group action $z \mapsto e^{i\mu} z$ on $\mathbb{C}^{n+1}\setminus\{0\}$, but we agreed to ignore these aspects altogether. We regard $J$ as first class constraint only and see that $\mathcal{C} := J^{-1}(0)$ is an immersed sphere $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}\setminus\{0\}$ with radius $\sqrt{-2\mu}$ and that $\hat{\mathcal{C}}$, the classical Reduced Phase Space, is just $\mathbb{C}P^n$. On $\mathbb{C}^{n+1}\setminus\{0\}$, let the **Wick product** be given: 

\[
 f \ast g = \sum_{\nu=0}^{\infty} \frac{2\lambda}{\nu!} \sum_{i_1=1}^{\nu} \cdots i_{\nu}=1 \frac{\partial f}{\partial z_{i_1}^{i_1}} \frac{\partial g}{\partial z_{i_{\nu}}^{i_{\nu}}}.
\]

We choose to prolongate every $f \in C^\infty(\mathbb{C}^{n+1}\setminus\{0\})[\lambda]$ off $S^{2n+1}$ in a radial way, that is we set $\text{prof}(f)(z) := f(p(z))$ where $p : \mathbb{C}^{n+1}\setminus\{0\} \rightarrow S^{2n+1}; z \mapsto (\sqrt{-2\mu/2}\bar{z})z$ projects radially on $S^{2n+1}$. From the general formula $\text{Res}$, we get $\pi_C(f)(z) = \frac{f(p(z)) - f(z)}{\frac{\sqrt{-2\mu}}{2} z}$. Onto $\mathcal{C} = S^{2n+1}$, this is continued as $\text{res}_C(f) = \frac{1}{2\mu}(E + \bar{E}) f$, where $E$ and $\bar{E}$ denote the Euler operators $E = z^k \partial_z$ and $\bar{E} = \bar{z}^k \partial_{\bar{z}}$. Even in this example, the inductive formula $\text{Res}$ for $T_n$ can be resolved in terms of the operator $K := \frac{1}{2} E \circ \pi_j$, yielding $T = \sum_{\nu=0}^{\infty} \lambda^\nu K^\nu = \frac{1}{1 - \lambda K}$. It is important that $K$, like $\pi_j$, can be expressed by the Euler operators $E$ and $\bar{E}$ when evaluated on $C$: it is $\text{Res}_C(K) = \frac{1}{2\mu}(\frac{1}{2}(E^2 - \bar{E}^2) + (E + \bar{E}) - (E + \bar{E}) ) f$. Another property of $K$ which will be of some importance later is that $K f \cdot h = K(f) \cdot h$ if $h \in C^\infty(\mathbb{C}P^n)[\lambda]$ is *homogeneous*, the latter meaning that $h(z) = h(\lambda z)$ for all $\lambda \in \mathbb{C}\setminus\{0\}$ or equivalently, that $h = \pi^* \eta$ with an $\eta \in C^\infty(\mathbb{C}P^n)[\lambda]$ and $\pi : \mathbb{C}^{n+1}\setminus\{0\} \rightarrow \mathbb{C}P^n$ the canonical projection.

The space of homogeneous series is equal to $B \cap F$, as an analysis of the conditions $f \in F$ and $f \in B$ will show (resulting in $f \in B \cap F \iff E f = 0$, $\bar{E} f = 0$) and like it should be because of $B \cap F \equiv C^\infty(\hat{\mathcal{C}})[\lambda]$. The operator $S$ has to be evaluated on $B \cap F$. Its general recursive definition $\text{Res}$ takes the form of $(E - \bar{E})(S_n f) = -\sum_{k=1}^{n} K^k ((E - \bar{E}) S_{n-k} f)$ in the present example, leading to $(E - \bar{E}) S_0 f = E - \bar{E}$ and $(E - \bar{E}) S_1 f = -K (E - \bar{E})$ with higher orders vanishing. Because on $\mathcal{C} = S^{2n+1}$, $K$ can be expressed in terms of the Euler operators, $S = \text{id} - \lambda K$ is a solution for $S$ on $\mathcal{C}$. But $K$ vanishes on $B \cap F$, so $S = \text{id} : B \cap F \rightarrow B^* \cap F$. The equality of $B \cap F$ and its “deformed” counterpart $B^* \cap F$ can of course be established by direct computations also. After putting all this together, lemma $\text{Res}$ gives us: let $f, g \in B \cap F$ be two homogeneous series of functions, $K$ as defined above and $p(z) = (\sqrt{-2\mu/2}\bar{z})z$. Then $(f \ast g)(z) := (\frac{1}{1 - \lambda K} f \ast g)(p(z))$ is homogeneous and $\ast$ thereby defines a star product on $\mathbb{C}P^n$, reduced from the Wick product $\ast$ on $\mathbb{C}^{n+1}\setminus\{0\}$.

The Reduced Star Product, though, can be considerably simplified by the following considerations. We define now bidirectional operators $M_r$ from the Wick product operators $M_r$ by $\hat{M}_r(f, g) := (z \bar{z})^r M_r(f, g)$ and observe that for $f, g$ homogeneous, $\hat{M}_r(f, g)$ is again homogeneous. We already mentioned that $K$, applied to a product of which one factor is homogeneous, this factor can be passed through, so in $\frac{1}{1 - \lambda K} \sum_{k=0}^{\infty} \frac{K^k}{(z \bar{z})^k} M_r(f, g)$, only terms of the form $K'(\frac{1}{z \bar{z}})(p(z))$ remain to be evaluated. The result of the ensuing computations is the star product on $\mathbb{C}P^n$, reduced from the Wick product $f \ast g = \sum_{k=0}^{\infty} \lambda^k M_k(f, g)$:

\[
 f \ast g = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\frac{1}{2\mu})^k}{k!} A_1^{(k)}(z \bar{z})^k M_k(f, g),
\]

\[
 A_1^{(k)} := (-1)^k \sum_{i_1=1}^{k} \sum_{i_2=1}^{k} \cdots \sum_{i_l=1}^{k} \cdot i_1 i_2 \cdots i_l
\]

The numbers $A_1^{(k)}$ fulfill a variety of inductive relations that in turn can be used to gain
another direct formula, namely \( A^{(k)}_i = \frac{1}{(k-1)!} \sum_{n=1}^{k} \binom{k-1}{n-1} (-1)^{k+l-n} n^{k+l-1} \). It is \( A^{(k)}_1 = (-1)^l \), \( A^{(k)}_0 = 1 \) and \( A^{(k)}_1 = -\frac{1}{2} k(k+1) \). This direct formula shows that these numbers are the same as the numbers just so called in [3] where they were obtained in an altogether different way, thereby proving that the star product on \( \mathbb{C}P^n \) constructed in [3] and ours are identical.

8 Example: Wick product from \( \mathbb{C}^{n+1} \) to \( \mathbb{C}P^n \), inequivalently

But because our constraint function \( J \) need not necessarily be a momentum mapping, we can repeat the whole reduction process with a different \( J \), for example \( J(z) := \frac{1}{2}(z \bar{z})^2 - \mu^2 \), \( \mu \in \mathbb{R}^- \). The classical Reduced Phase Space is of course \( \mathbb{C}P^n \) in both cases, the constraint submanifold being the same. The Reduced Star Products, however, turn out to be not just different but inequivalent, as we will see. The calculations proceed along the lines already followed, so there is no need to go into the details. Let it be sufficient to mention that \( T_n = \sum_{j=0}^{[2]} \frac{1}{2} \frac{d^2}{d\nu^2} (P + \nu R)^{n-j} \bigg|_{\nu=0} \) for \( (P f)(z) := \frac{1}{2} z \bar{z} E(\pi_j(f))(z) \) and \( (R f)(z) := -\frac{1}{2} E^2(\pi_j(f))(z) \left( \frac{n}{2} \right) ^{\frac{1}{2}} \) denoting the integer part of \( \frac{n}{2} \), that \( B \cap F \) is – of course! – again the space of homogeneous series of functions (remark that we did not touch the prolongation description), that \( S \) can be chosen as identity, and that both \( P \) and \( R \) do not “see” the homogeneous factors in their arguments. So, for \( f \) and \( g \) homogeneous,

\[
\begin{align*}
  f \ast g &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{-2 \mu}{2 \alpha} \right)^{k+l} B^{(k)}_l (z \bar{z})^k M_k(f, g) \\
  B^{(k)}_l := (-2\mu)^{k+l} \frac{1}{j!} \frac{d^j}{d\nu^j} (-\frac{1}{2} z \bar{z} E + \frac{1}{2} \nu E^2) \circ \pi_{j-l} \left( \frac{1}{(z \bar{z})^l} \right) \bigg|_{\nu=0}^{\nu=\frac{1}{2} \bar{z} \mu} ,
\end{align*}
\]

is another star product on \( \mathbb{C}P^n \) reduced from the Wick product on \( \mathbb{C}^{n+1} \) \( \setminus \{0\} \).

Let us denote the product of formula 10 with \( \ast \), then we obtain by a straightforward computation \( (f \ast g - f \ast \hat{g})_2 - (g \ast f - g \ast \hat{f})_2 = \frac{1}{2} i \left( \frac{1}{2 \alpha} \right)^2 z \bar{z} \{f, g\} \) to the second order. But \( \frac{1}{2} i \left( \frac{1}{2 \alpha} \right)^2 z \bar{z} \{f, g\} = \frac{1}{2 \alpha} \left( f \ast g - g \ast f \right)_1 = \frac{1}{2 \alpha} \{f, g\}_{\mathbb{C}P^n} \), where the Poisson bracket \( \{\cdot, \cdot\}_{\mathbb{C}P^n} \) belongs to the symplectic Fubini-Study form \( \omega_{\mathbb{C}P^n} \) on \( \mathbb{C}P^n \), and \( \omega_{\mathbb{C}P^n} \) is not exact. On the other hand, a result in [8] (see also [10]) says that two equivalent star products, equal up to the order \( k \), have necessarily an exact two–form as the antisymmetric part of their difference at order \( k+1 \). Because \( (f \ast g)_1 = (f \ast \hat{g})_1 \) in our examples, this theorem applies and we conclude that our two reduced star products cannot be equivalent. Roughly speaking, two different constraint functions, though inducing the same classical Reduced Phase Space, may lead to inequivalent quantum systems.

Acknowledgement

The author is greatly indebted to M. Bordemann, who suggested to the author the ansatz to define the algebras \( I^* \) and \( B^* \) and accompanied the ensuing work critically.

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