Quantization of Floreanini-Jackiw chiral harmonic oscillator

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Abstract

The Floreanini-Jackiw formulation of the chiral quantum-mechanical system oscillator is a model of constrained theory with only second-class constraints. in the Dirac’s classification. The covariant quantization needs infinite number of auxiliary variables and a Wess-Zumino term. In this paper we investigate the path integral quantization of this model using Güler’s canonical formalism. All variables are gauge variables in this formalism. The Siegel’s action is obtained using Hamilton-Jacobi formulation of the systems with constraints.

1 Introduction

Chiral bosons in two-dimensional space-time and (2 + 1) -dimensional Chern-Simons(CS) gauge theories are related problems which have been attracting much attention. These problems are important for the string program and for the development of the quantum Hall effect[1],[2]. Floreanini and Jackiw suggested an action suitable for the quantization of a two-dimensional chiral boson[3]. Siegel proposed an apparently unrelated action for the same system[4]. In[5] the connection between these two approaches was investigated. The quantization of Siegel action was investigated by Faddeev-Jackiw formalism in[6].

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The equivalent Lagrangian method is used to obtain the set of Hamilton-Jacobi partial differential equations \[7\]. In other words, the equations of motion are written as total differential equations in many variables.

The main aim of this paper is to investigate the quantization of the Floreanini-Jackiw chiral oscillator using Güler’s formalism.

The plan of our paper is the following:
In Section 2 we present the quantization of the field theories with constraints using Hamilton-Jacobi method. In Section 3 the path integral quantization for Floreanini-Jackiw chiral oscillator is given. The Siegel’s action was obtained using Güler’s formalism. In Section 4 we present our conclusions.

## 2 Hamilton-Jacobi quantization of the field theories with constraints

Starting from Hamilton-Jacobi partial-differential equation the singular systems were investigated using a formalism introduced by Güler (see for example Refs. \[7\], \[8\]).

The canonical formulation gives the set of Hamilton-Jacobi partial-differential equation as

\[
H'_\alpha(\chi_\beta, \phi_\alpha, \frac{\partial S}{\partial \phi_\alpha}, \frac{\partial S}{\partial \chi_\alpha}) = 0, \alpha, \beta = 0, n - r + 1, \ldots, n, a = 1, \ldots, n - r, \tag{1}
\]

where

\[
H'_\alpha = H_\alpha(\chi_\beta, \phi_\alpha, \pi_a) + \pi_\alpha \tag{2}
\]

and \(H_0\) is the canonical Hamiltonian. The equations of motion are obtained as total differential equations in many variables as follows

\[
d\phi_a = \frac{\partial H'_\alpha}{\partial \pi_a} d\chi_\alpha, d\pi_a = -\frac{\partial H'_\alpha}{\partial \phi_\alpha} d\chi_\alpha, d\pi_\mu = -\frac{\partial H'_\alpha}{\partial \chi_\mu} d\chi_\alpha, \mu = 1, \ldots, r \tag{3}
\]

\[
dz = (-H_\alpha + \pi_a \frac{\partial H'_\alpha}{\partial \pi_a}) d\chi_\alpha \tag{4}
\]

where \(z = S(\chi_\alpha, \phi_a)\). The set of equations \((3,4)\) is integrable if

\[
\frac{dH_0'}{d\phi_a} = 0, \frac{dH'_\mu}{d\phi_a} = 0, \mu = 1, \ldots r \tag{5}
\]

If conditions \((3,4)\) are not satisfied identically, one considers them as a new constraints and again tests the consistency conditions. Thus repeating this procedure one may obtain a set of conditions.

Let suppose that for a system with constraints we found all independent Hamiltonians \(H'_\mu\) using the calculus of variations \([7], [8]\). At this stage we will use Dirac’s procedure of quantization \([9]\). We have

\[
H'_\alpha \Psi = 0, \mu = 1, \ldots, r \tag{6}
\]
where $\Psi$ is the wave function. The consistency conditions are
\[
[H'_\alpha, H'_\beta] \Psi = 0, \alpha, \beta = 1, \cdots, r
\] (7)

If the hamiltonians $H'_\alpha$ satisfies
\[
[H'_\alpha, H'_\beta] = C_{\alpha\beta}H'_\gamma
\] (8)
they are of first class in the Dirac’s classification. On the other hand if
\[
[H'_\alpha, H'_\beta] = C_{\alpha\beta}
\] (9)
where $C_{\alpha\beta}$ do not depend of $\phi_i$ and $\pi_i$ then from(7) there arises naturally Dirac’s brackets and the canonical quatization will be performed taking Dirac’s brackets into commutators.

Güler’s formalism gives an action when all hamiltonians $H'_\alpha$ are in involution.Since in this formalism we work from the beginning in the extended space we suppose that variables $t_\alpha$ depend of $\tau$.Here $\tau$ is canonical conjugate with $p_0$.

If we are able, for a given system with constraints, to find the independent hamiltonians $H'_\alpha$ in involution then we can perform the quantization of this system using path integral quantization method with the action given by (4)
\[
z = \int \left( -H_\alpha + \pi_\beta \frac{\partial H'_\alpha}{\partial \pi_\beta} \right) \dot{\chi}_\alpha d\tau
\] (10)
where $\dot{\chi}_\alpha = \frac{d\chi_\alpha}{d\tau}$.

3 Chiral Oscillator

We consider the Lagrangian
\[
L_0 = \omega \dot{q}_i^{(0)} \epsilon_{ij} q_j^{(0)} + \omega^2 q_i^{(0)} q_i^{(0)}, i, j = 1, 2
\] (11)

From(11) we found the constraints
\[
\Omega_i = p_i^{(0)} - \omega \epsilon_{ij} q_j^{(0)}, i = 1, 2
\] (12)
and the canonical hamiltonian
\[
H_c = \left( p_i^{(0)} - \omega \epsilon_{ij} q_j^{(0)} \right) \dot{q}_i^{(0)} - \omega^2 q_k^{(0)} q_k^{(0)}
\] (13)

Then in the Güler’s formalism we have following hamiltonians
\[
H'_0 = p_0 + H_c, H'_i = p_i^{(0)} - \omega \epsilon_{ij} q_j^{(0)}, i = 1, 2
\] (14)
and all the variables \(q_i^{(0)}\) are gauge variables. The hamiltonians are not in involution because

\[
[H'_i, H'_j] = -2\omega \epsilon_{ij}
\]  

(15)

In order to obtain the hamiltonians in involution we will extend the space with new variable \(p_i^{(1)}\) and \(q_i^{(1)}\). The new expressions for the hamiltonians \(H''_i\) in involution are

\[
H''_i = H'_i - \omega \epsilon_{ij}q_j^{(1)} - p_i^{(1)}
\]  

(16)

but we get a new set of constraints

\[
H'_i = p_i^{(1)} - \omega \epsilon_{ij}q_i^{(1)}
\]  

(17)

If we repeat the procedure after \(N\) steps we get \(N+1\) hamiltonians in involution and the hamiltonians \(H^{N'}_i\) fulfilling

\[
[H^{N'}_i, H^{N'}_j] = -2\omega \epsilon_{ij}
\]  

(18)

The final form of the canonical hamiltonian obtained after an infinite repetition of the conversion process is

\[
H^{(\infty)}_c = \sum_{k=0}^{\infty} (p_i^{(k)} - \omega \epsilon_{ij}q_i^{(k)} - \omega^2 q_i^{(k)}q_i^{(k)} - 2\sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \omega \epsilon_{ij} q_j^{(m)}(q_i^{(m)} + \omega \epsilon_{il}q_i^{(m)}))
\]  

(19)

Then in the Güler's formalism we have infinite numbers of hamiltonians in involution

\[
H'_0 = p_0 + H^{(\infty)}_c, H'_k = p_i^{(k)} - \omega \epsilon_{ij}q_j^{(k)}, k = 1, \ldots, \infty
\]  

(20)

Using (10) we found after some calculations that the action has the form

\[
z = \int L d\tau
\]  

(21)

where \(L\) is given by

\[
L = \sum_{k=0}^{\infty} (\omega \epsilon_{ij} q_i^{(k)} q_j^{(k)} + \omega^2 q_i^{(k)} q_i^{(k)}) + 2\sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \omega \epsilon_{ij} q_j^{(m)}(q_i^{(m)} + \omega \epsilon_{il}q_i^{(m)})
\]  

(22)

This result is in agreement with those from [10].

### 3.1 Siegel’s action

Siegel was first one who proposed an action for the chiral-boson problem [1]. The Lagrangean density is

\[
L = \partial_- \phi \partial_+ \phi + \lambda (\partial_- \phi)^2
\]  

(23)

and the canonical hamiltonian becomes

\[
H_c = \frac{1}{2}(1 + \lambda)^{-1}(\pi + \lambda \phi')^2 + \frac{1}{2}(1 - \lambda)(\phi')^2
\]  

(24)
where $\pi$ is the canonical momentum conjugate to $\phi$. On the other hand we observe that

$$\pi_\lambda = 0$$

(25)

In G"uler's formalism we have the following hamiltonians

$$H'_0 = p_0 + H_c, \ H'_1 = \pi_\lambda$$

(26)

Imposing $dH'_0 = 0$ and $dH'_1 = 0$ we generate another hamiltonian $H'_2$ as

$$H'_2 = \pi_\phi - \phi'$$

(27)

From (26) and (27) we conclude that $\lambda$ and $\phi$ are gauge variables in this formalism.

Now we are interested in performing the quantization of the system. Using Dirac's procedure we have

$$H'_0 \Psi = 0, \ H'_1 \Psi = 0, \ H'_2 \Psi = 0$$

(28)

Because

$$[H'_1, H'_0] = \frac{1}{2} (1 + \lambda)^{-1} (\pi - \phi')^2$$

(29)

and

$$[H'_2, H'_2] = \partial_x \delta(x - y)$$

(30)

we conclude that the system has second class constraints in the Dirac's classification and we can quantize it using path integral quantization method \[11\]. Because we have obtained the same constraints $H'_0, H'_1$ and $H'_2$ as in \[5\] we found the same result after performing path integral quantization.

4 Concluding remarks

Using Hamilton-Jacobi formalism for systems with constraints we found that Floreanini-Jackiw chiral harmonic oscillator is a theory with an infinite number of Hamiltonians in involution. The path integral quantization, using the action given by G"uler's formalism, was performed and the results are in agreement with those obtained by others authors.

For Floreanini-Jackiw chiral harmonic oscillator we found that all the fields are gauge fields in the G"uler's formalism. Because all hamiltonians $H'_\alpha$ are constraints in the extended space in this formalism we have no first and second class constraints in the Dirac's classification at the classical level.

However the first and second class constraints become important in this formalism in the process of quantization. For the Siegel's action we found only three independent hamiltonians $H'_0, H'_1, H'_2$. This set of hamiltonians give us the correct result if the path integral quantization for this system is performed.
In the Güler's formalism all the variations of constraints $H'_0$, $H'_1$ and $H'_2$ do not give us new constraints.

The problem if the constraint $(H'_2)^2$(for more details see Ref.[5]) is of first or second class does not arise in the Güler's formalism. In this case we found that all variables are gauge variables.

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