DECOHERENCE OF FRIEDMANN-ROBERTSON-WALKER GEOMETRIES
IN THE PRESENCE OF MASSIVE VECTOR FIELDS WITH U(1) OR SO(3)
GLOBAL SYMMETRIES

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ABSTRACT

Retrieval of classical behaviour in quantum cosmology is usually discussed in the framework of minisuperspace models in the presence of scalar fields and the inhomogeneous modes corresponding either to gravitational or scalar fields. In this work, we propose an alternative model to study the decoherence of homogeneous and isotropic geometries where the scalar field is replaced by a massive vector field with a global internal symmetry. We study here the cases with $U(1)$ and $SO(3)$ global internal symmetries. The presence of a mass term breaks the conformal invariance and allows for the longitudinal modes of the spin-1 field to be present in the Wheeler-DeWitt equation. In the case of the $U(1)$ global internal symmetry, we have only one single “classical” degree of freedom while in the case of the $SO(3)$ global symmetry, we are led to consider a simple two-dimensional minisuperspace model. These minisuperspaces are shown to be equivalent to a set of coupled harmonic oscillators where the kinetic term of the longitudinal modes has a coefficient proportional to the inverse of the scale factor. The conditions for a suitable decoherence process and correlations between coordinate and momenta are established. The validity of the semi-classical Einstein equations when massive vector fields (Abelian and non-Abelian) are present is also discussed.

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1 Introduction

The emergence of classical properties from the quantum mechanics formalism is still largely an open problem. Some progress has, however, been achieved through the so-called decoherence approach. On fairly general grounds, the decoherence process takes place as one considers the system under study to be part of a more complex world that interacts with other subsystems, usually referred to as the “environment”. The latter usually consists in the set of unobserved or irrelevant degrees of freedom which are traced out, implying that, at least in an operational way, the wave-function evolves non-linearly and is lead to its collapse [1]. In this way, quantum interference effects among states of the system are suppressed by the interaction with the environment. This suppression comes about as one integrates out the irrelevant degrees of freedom. This coarse-graining procedure leads to an effective action of the original model and clearly generalizes the usual field fluctuation splitting that is adopted when carrying out a quantum loop expansion in the standard effective action calculation. From the operational point of view, decoherence in the context of quantum cosmology, implies that the Universe is essentially an open system as observers have necessarily to disregard large classes of variables in any relevant observation. In addition to the notion of decoherence, which strictly means that there is no interference among different quantum states, a further condition that a system must satisfy to be regarded as classical is, of course, that it is driven by classical laws, implying that a sharp correlation between configuration space coordinates and conjugate momenta should exist in the wave function.

These ideas have been developed in some depth in the context of quantum cosmological models [2]–[11], where considering the decoherence process is mandatory as the observable Universe behaves clearly in a classical fashion and one expects that this classical features arise from a more fundamental quantum description. Furthermore, although minisuperspace models can be justified on symmetry grounds, the truncation which turns the full quantum gravity problem with its infinite degrees of freedom into a problem with a finite number of degrees of freedom actually violates the uncertainty principle, as the amplitudes and momenta of inhomogeneous modes are set to zero and the non-linear interactions of those modes among themselves and with the minisuperspace degrees of freedom are disregarded. Moreover, the validity of the minisuperspace approximation (cf. [3] and references therein) is ensured only when the back-reaction of the inhomogeneous modes on the minisuperspace variables is shown to be small.

Most of the literature concerning the emergence of classical behaviour in the context of quantum cosmology considers scalar fields [2]–[10], where the environment corresponds to the inhomogeneous modes either of the gravitational or of the scalar fields [1]. This is a shortcoming of most of the decohering models discussed so far as prior to the inflationary epoch the Universe was dominated by radiation, i.e. Yang-Mills fields, and, after spontaneous symmetry breaking phenomena, by massive vector fields. Furthermore, a relevant issue concerning the decoherence approach is to establish to what extent some of its features are specifically related with scalar fields and to achieve this goal one has to consider other fields to play the role of environment.

In this paper we propose a model to study the decoherence of homogeneous and isotropic geometries, where the scalar field is replaced by a massive vector field with a global internal symmetry. Our aim is to assess if such models are on the same footing with the ones where the
quantum to classical transition is achieved via tracing out the higher modes of scalar fields, or in other words, if massive vector fields are equally effective in playing the role of an environment with respect to the observed “classical” degrees of freedom. Notice that the presence of a mass term is a crucial feature as it breaks the conformal symmetry of the spin-1 field action which leads to a Wheeler-DeWitt equation where the gravitational and matter degrees of freedom decouple \[12\], similarly to the case of a free massless conformally invariant scalar field \[13\]. Hence, the presence of a mass term allows for interaction between gravitational and matter degrees of freedom providing a scale at which the decoherence process can take place. Moreover, the breaking of conformal invariance makes it possible for the longitudinal modes of the spin-1 field to be present in the Wheeler-DeWitt equation. As we shall see, the resulting system is equivalent to a set of coupled harmonic oscillators where the kinetic energy terms of the longitudinal modes have a coefficient proportional to the inverse of the scale factor.

An important point related to the discussion of quantum cosmology with vector fields concerns the compatibility of the simple homogeneous and isotropic Friedmann-Robertson-Walker (FRW) geometries we shall consider and the matter content of the Universe. As far as the homogeneous modes of the vector fields are concerned, we shall use here the Ansatz formulated in Refs. \[12\], \[14\]–\[20\]. More specifically, since the physical observables have to be \(SO(4)\)-invariant, the isometry group of closed FRW cosmologies, the fields with internal degrees of freedom may be allowed to transform under \(SO(4)\) if these transformations can be compensated by an internal symmetry transformation. Fortunately, there is a large class of fields satisfying the above conditions, namely the so-called \(SO(4)\)-symmetric fields, i.e. fields which are invariant up to an internal symmetry transformation. However, such construction can only be made possible in the presence of non-Abelian symmetries and hence, for the the Abelian case, we shall require the homogeneous modes of the spatial components of the vector field to vanish. For a \(\mathbb{R} \times S^3\) topology, such minisuperspace constructions with homogeneous and isotropic metric and gauge fields with \(SO(N)\) and \(SU(M) \ (N \geq 2, \ M \geq 3)\) gauge groups have been recently carried out. For Euclidean FRW geometries Einstein-Yang-Mills wormhole solutions have been found in Ref. \[14\]. Classical Einstein-Yang-Mills solutions for closed geometries and \(SO(N)\) gauge groups were obtained in Ref. \[13\]. The ground-state wave-function of the Einstein-Yang-Mills system with \(SO(3)\) gauge group was found in Ref. \[12\]. Massive vector fields with \(SO(3)\) global symmetry in flat FRW universes were studied in Ref. \[16\]. Finally, in a \(\mathbb{R}^4\) topology, homogeneous and isotropic metric and Yang-Mills field configurations were considered in Ref. \[17\] and the inclusion of the dilaton in the context of string theories was carried out in Ref. \[18\].

In this work, we shall study the decoherence process for homogeneous and isotropic metrics in the presence of massive vector fields with \(U(1)\) and \(SO(3)\) global symmetries in a \(\mathbb{R} \times S^3\) topology. For simplicity, we begin by considering the case with \(U(1)\) global symmetry. The role of the environment will be played by the inhomogeneous modes of the \(U(1)\) field and the corresponding minisuperspace will be actually one-dimensional as it has only a single “classical” degree of freedom as physical observable, the scale factor. Although this choice of the matter content may seem rather restrictive, we shall see that some of our results may be extended to the richer and more interesting system with \(SO(3)\) global symmetry, which we analyse afterwards. In particular, we shall establish, for both cases, the conditions required for achieving correlation between coordinates and momenta and a satisfactory decoherence process. The validity of the semi-classical Einstein equations when massive vector fields are present will be also discussed.
The minisuperspace for the non-Abelian model hereby studied is two-dimensional due to the specific Ansatz for the homogeneous modes of the spin-1 field (see Refs. [14]–[20]).

Multi-dimensional minisuperspace models have of course, a much richer structure and are therefore far more interesting to consider in what regards the retrieval of classical behaviour [7, 8, 24, 25, 26]. In models containing one single “classical” degree of freedom, the Hamilton-Jacobi equation has only two solutions, generating the same trajectory in opposite directions. The semi-classical wave-function has two WKB components, each of which may be called a WKB branch, of the form

\[ \Psi[\mathcal{O}, \mathcal{E}] = \sum_{(n)} C_{(n)}[\mathcal{O}] e^{i M^P_S(n)}[\mathcal{O}] \psi_{(n)}[\mathcal{O}, \mathcal{E}] \].

Here the subindex \((n)\) labels the WKB branches (taking only two values, say \(\pm 1\), uniquely identifying the two possible solutions of the Hamilton-Jacobi equation) and \(\mathcal{O}, \mathcal{E}\) denote, respectively, the “classical” physical observables and the extra degrees of freedom corresponding to the environment. To achieve a proper decoherence, one needs not only that the reduced density matrix turns out to be diagonal, but also that the different WKB components in (1) have negligible interference among the diagonal terms. It is important to stress that the analysis of correlations should be done within each classical WKB branch (i.e. a diagonal term, \(n = n'\)). The interference effects between the two possibilities of moving along the one-dimensional classical trajectory (corresponding to the expanding and collapsing wave function components, respectively) were shown to be effectively suppressed [4], which is interpreted as particle creation [25].

For a system with \(N\) degrees of freedom, the Hamilton-Jacobi equation is expected to have an \(N-1\) parameter family of solutions, each one generating a \(N-1\) parameter family of classical trajectories in the minisuperspace. In the multi-dimensional case, a general solution of the Wheeler-DeWitt equation may contain an infinite superposition of semi-classical solutions of the form (1) with the subindex \((n)\) now corresponding to a set of parameters that uniquely identify a specific Hamilton-Jacobi solution. However, each WKB branch must actually be interpreted as describing a whole family of classical trajectories, i.e. a set of different universes and not a single one as for the \(N = 1\) case. Furthermore, the \(N = 2\) and \(N > 2\) cases are rather different as far as the diagonalization of the reduced density matrix is concerned. These differences seem to have gone unnoticed until they were discussed in Ref. [8]. An example of \(N = 2\) model was studied in Ref. [24], where the two “classical” degrees of freedom correspond to the scale factor and homogeneous mode of a minimally massless scalar field and the environment was identified with the inhomogeneous perturbations of another minimally massless scalar field. The multi-dimensional cases also allow one to better address the relation between the reduced density matrix formalism and the Feynman-Vernon influence functional [27], and to the Schwinger-Keldish or closed time path effective action [28], as pointed out in Ref. [29].

Some preliminary work performed in Ref. [19], with homogeneous and isotropic cosmologies in the presence of massive vector fields, has shown that the necessary ingredients for the process of decoherence to take place are present, although several aspects of the full hyperspherical harmonics expansion of the fields remained to be fully assessed. The Wheeler-DeWitt equation obtained in Ref. [19] is fairly similar to the one corresponding to a FRW minisuperspace model with a massive scalar field with a quartic self-coupling, \(\lambda \phi^4\), conformally coupled with gravity [7]. In addition, we shall compare the results of this paper with the ones obtained
previously in Refs. [7, 8]. Actually, there are many similarities and this will allows us to use some of the framework described in those references. We would like to mention that, in considering perturbations in the quantum Einstein-Yang-Mills model of Ref. [12], the authors of Ref. [22] consider also, as we do, the harmonic expansion of the fields involved on $SO(4)$ and $SO(4)/SO(3)$ (see above).

This paper is organized as follows. In Section 2 we present our model and, through general considerations, introduce an Ansatz for the vector fields with global symmetry as well as the expansion in $S^3$ hyperharmonics which will give rise to an effective action. We then proceed to a minisuperspace description of such a model, which will allow us to study both the $U(1)$ and $SO(3)$ global symmetry cases. In Section 3, we discuss the decoherence process and correlations within the $U(1)$ model and, in Section 4, we address the perturbed minisuperspace model with massive vector fields with $SO(3)$ global symmetry. Our conclusions are presented in Section 5.

2 FRW Minisuperspace with Massive Vector Fields

The action of our model consists of a Proca field coupled with gravity:

$$ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2k^2}(R - 2\Lambda) + \frac{1}{8e^2}\text{Tr}(F_{\mu\nu}^{(a)}F^{(a)\mu\nu}) + \frac{1}{2}m^2\text{Tr}(A_{\mu}^{(a)}A^{(a)\mu}) \right], $$

(2)

where $k^2 = 8\pi M_P^{-2}$, $M_P$ being the Planck mass, $e$ is a gauge coupling constant and $m$ the mass of the Proca field. To action (2) one adds the boundary action

$$ S_B = -\frac{1}{k^2} \int_{\partial M} d^3x \sqrt{h}K, $$

(3)

with $h_{ij}(i, j = 1, 2, 3)$ being the induced metric on the three-dimensional boundary $\partial M$ of $M$, $h = \det(h_{ij})$ and $K = K_{\mu}^\mu$ is the trace of the second fundamental form on $\partial M$.

In quantum cosmology one is concerned with spatially compact topologies and we will consider here the FRW Ansatz for the $\mathbf{R} \times S^3$ geometry

$$ ds^2 = \sigma^2a^2(\eta) \left[ -N(\eta)^2d\eta^2 + \sum_{i=1}^{3}\omega^i \omega^i \right], $$

(4)

where $\sigma^2 = 2/3\pi$, $\eta$ is the conformal time, $N(\eta)$ and $a(\eta)$ being the lapse function and the scale factor, respectively and $\omega^i$ are left-invariant one-forms in $SU(2) \simeq S^3$ which satisfy

$$ d\omega^k = -\epsilon_{kij}\omega^i \wedge \omega^j. $$

(5)

Aiming to obtain solutions of the Wheeler-DeWitt equation satisfied by the wave function $\Psi[h_{ij}, A_{\mu}^{(a)}]$:

$$ \left[ G_{ijkl} \frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + M_P^2\sqrt{h} \left( (3)R - 2\Lambda \right) + \frac{M_P^2}{2}\sqrt{h} T^{ii} \left[ A_{\mu}^{(a)}, -i\frac{\delta}{\delta A_{\mu}^{(a)}} \right] \right] \Psi[h_{ij}, A_{\mu}^{(a)}] = 0, $$

(6)

1The authors are grateful to B.L. Hu and J.P. Paz for pointing this out.

2Usually one makes the choice $\sigma^2 = 2/3\pi M_P^2$, but we shall keep the powers of $M_P^2$ explicitely in our effective action in order to compare our results with the ones of Refs. [7, 8, 12].
where the superspace metric is given by
\[ G_{ijk\ell} = \frac{\sqrt{h}}{2} (h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell}) , \]  
we shall expand the metric as
\[ h_{ij} = \sigma^2 a^2 (\Omega_{ij} + \epsilon_{ij}) , \]
with \( \Omega_{ij} \) being the metric on the unit \( S^3 \) and \( \epsilon_{ij} \) a perturbation that can be expanded in scalar harmonics \( D^{J}_{NM}(g) \), which are the usual \( (2J + 1) \)-dimensional \( SU(2) \) matrix representation and spin-2 hyperspherical harmonics \( Y^{2LJ}_{MN}(g) \) on \( S^3 \) as:
\[ \epsilon_{ij} = \Omega_{ij} \sum_{J} \sqrt{\frac{\pi}{3\pi^2}} a_{J} \sigma^{N}_{M}(\eta) D^{J}_{NM}(g) + \sigma^{m}_{i} \sigma^{n}_{j} (\frac{A}{2} \frac{1}{M} \frac{1}{N}) \epsilon_{A} , \]
where
\[ \epsilon_{A} = \sum_{L=J} \sqrt{\frac{32(\pi^2 - 4)}{15(\pi^2 - 1)}} b_{LMN}(\eta) Y^{2L}_{MN}(g) + \sum_{|J-L|=1} \sqrt{\frac{32(\pi^2 - 4)}{5}} c_{LMN}(\eta) Y^{2L}_{MN}(g) + \sum_{|J-L|=2} \sqrt{\frac{32}{5}} d_{LMN}(\eta) Y^{2L}_{MN}(g), \]
with \( \sigma^{m}_{i} \) described below, the coefficients \( a_{J}^{MN}, \ldots, d_{J}^{MN} \) depend only on the conformal time, \( (L, J, M, m) \) represent 3-J symbols or Clebsh-Gordon coefficients and \( \eta = J + L + 1 \).

The massive vector field
\[ \mathbf{A} = A_{m}^{ab} \omega^{m}_{s} \mathcal{T}_{ab} = A_{m}^{ab} \omega^{m}_{s} \mathcal{T}_{ab}, \]
where \( \omega^{m}_{s} \) denote the one-forms in a spherical basis with \( m = 0, \pm 1 \), \( \mathcal{T}_{ab} \) are the \( SO(3) \) group generators and
\[ \sigma^{m}_{i} = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix} , \]
can be expanded in spin-1 hyperspherical harmonics as [21]:
\[ A_{0}(\eta, x^{j}) = \sum_{JMN} \alpha^{ab}_{m} \mathcal{T}_{ab} = \sum_{JMN} \alpha^{ab}_{m} \mathcal{T}_{ab}, \]
\[ A_{i}(\eta, x^{j}) = \sum_{LJNM} \beta^{ab}_{LM} \mathcal{T}_{ab} = \sum_{LJNM} \beta^{ab}_{LM} \mathcal{T}_{ab}, \]
\[ = \frac{1}{2} \left[ 1 + \sqrt{\frac{2\pi}{3\pi}} \chi(\eta) \right] \epsilon_{ab} \mathcal{T}_{ab} + \sum_{LJNM} \beta^{ab}_{LM} \mathcal{T}_{ab}, \]
where \( \chi = e^{2}/4\pi \) and \( A_{0} \) is a scalar on each fixed time hypersurface, such that it can be expanded in scalar harmonics \( \mathcal{D}^{J}_{NM}(g) \). The coordinates \( x^{j} \) are written as an element of \( SU(2) \). The expansion of \( A_{i} \) is performed in terms of the spin-1 spinor hyperspherical harmonics, \( Y^{1LJ}_{MN}(g) \). Longitudinal and transversal harmonics correspond to \( L = J \) and \( L - J = \pm 1 \), respectively.
The $\alpha^{abJM}_{N}(\eta)$ and $\beta^{abMN}_{LJ}(\eta)$ are time-dependent functions and each one identifies a spin-1 mode from the $A_0$ and $A_i$ components, respectively. The expressions in the first equality, eqs. (13) and (14), represent the general expansion. In the $U(1)$ model, we choose the homogeneous modes of the spin-1 field spatial components to be identically zero. In this way, the homogenous modes will produce field strenght configurations compatible with FRW geometries, i.e. a diagonal energy-momentum tensor, vanishing for the $U(1)$ case. In addition, the r.h.s. of the same expressions correspond to a decomposition of the expansion in homogeneous (first term) and inhomogeneous modes for the non-Abelian case. There, we use a $SO(4)$-symmetric Ansatz for the homogeneous modes of the vector field which is compatible with the FRW geometry [16]. For the case of a $SO(3)$ global symmetry, we have, for the homogeneous modes

$$A_0^{ab}(\eta) = 0,$$

$$A_i(\eta) = \frac{1}{2} \left[ 1 + \sqrt{\frac{2\alpha}{3\pi}} \chi(\eta) \right] \epsilon_{bic} T_{bc},$$

where $\chi(\eta)$ is time-dependent scalar function. The idea underlying this Ansatz for the non-Abelian spin-1 field consists in defining an homorphism from the isotropy group $SO(3)$ to the gauge group. This homomorphism defines the internal transformation which, for the symmetric fields, compensates the action of a given $SO(3)$ space rotation. Hence, the above form for the gauge field, where the $A_0$ component is taken to be identically zero. By imposing the above mentioned symmetry conditions, we obtain a one-dimensional mechanical-type model depending only on time [14]. The resulting one-dimensional model has some residual symmetries from the ones of the full four-dimensional theory. In particular, the invariance under general coordinate transformations in four dimensions leads to an invariance under arbitrary time-reparametrizations. However, due to our choice of $SO(4)$-symmetry conditions for the spin-1 field, none of the local internal symmetries survive as all the available internal transformations are required to cancel out the action of a given $SO(3)$ space rotation.

Let us now turn to our model with a massive vector field. From actions (2) and (3) one can work out the effective Hamiltonian density. Upon substitution of the expansions (4)-(14) and after integrating over $S^3$, the canonical conjugate momenta of the dynamical variables are found to be

$$\pi_a = \frac{\partial L_{\text{eff}}}{\partial \dot{a}} = -\frac{\dot{a}}{N}, \quad \pi_{\chi} = \frac{\partial L_{\text{eff}}}{\partial \dot{\chi}} = \frac{\dot{\chi}}{N},$$

$$\pi_{\beta_{LM}^{ab}} = \frac{\partial L_{\text{eff}}}{\partial \dot{\beta}_{LM}^{ab}} = \frac{\dot{\beta}_{LM}^{ab}}{2N\pi \alpha},$$

$$\pi_{\beta_{JJ}^{ab}} = \frac{\partial L_{\text{eff}}}{\partial \dot{\beta}_{JJ}^{ab}} = \frac{\dot{\beta}_{JJ}^{ab}}{2N\pi \alpha} - \frac{1}{2\pi \alpha} N A_{abJM}^{abJM} (-1)^{2J} \sqrt{16\pi^2 J(J+1)} 2J + 1.$$

where $L_{\text{eff}}$ denotes the effective Lagrangian density associated with (2), which we omit here, and the dots represent derivatives with respect to the conformal time. To second order in the coefficients of the expansions and all orders in $a$, one finds that most of the gravitational degrees of freedom are gauge type and, as such, the wave function cannot depend on them; furthermore, we shall consider the gravitons in the ground state. For the case of $SO(3)$ global
symmetry, dropping the primes, it follows that the effective Hamiltonian density reads (where for the Abelian case\(^3\) one disregards the last four terms):

\[
\mathcal{H}_{\text{eff}} = -\frac{1}{2M_p^2} \pi_a^2 + M_p^2 \left( -a^2 + \frac{4A}{9\pi} a^4 \right) + \sum_{J,L} \frac{4}{3\pi} m^2 a^2 \beta_{abNM} \beta_{abLJ} \\
+ \sum_{J,L} \left[ \alpha \pi \beta_{abNM} J \pi_{\beta NM} \beta_{abLJ} + \beta_{abNM} \beta_{abLJ} (L + J + 1)^2 \right] \\
+ \sum_J \left\{ \alpha \pi \left[ (-1)^{4J} \left( \frac{16\pi^2 J(J+1)}{2J+1} \right) \frac{3\pi}{4m^2} \right] \left[ 1 + \frac{1}{a(t)} \right] \right\} \pi_{\beta NM} \pi_{\beta NM} \\
+ \alpha \pi \frac{\pi}{3\pi} \left[ \chi^2 - \frac{3\pi}{2\pi} \right]^2 \\
+ \sum_{J,L} \frac{\beta_{abNM}}{\alpha \pi} \left[ 1 + \sqrt{\frac{2\pi}{3\pi}} \chi \right] \beta_{abLJ} \beta_{abNM} \\
+ 4\pi a^2 m^2 \left[ 1 + \sqrt{\frac{2\pi}{3\pi}} \chi \right]^2,
\]

(20)

where, from now on, for brevity, we shall collectively denote \((J,L,M,N)\) by \(J\), being implicit the difference between longitudinal and transversal modes as well as the sum over contracted \(SO(3)\) group indexes. In the Appendix, neglecting the expansions (8)-(10), we present the complete effective Hamiltonian for the perturbed non-Abelian model.

The Hamiltonian constraint, \(\mathcal{H}_{\text{eff}} = 0\), gives origin to the Wheeler-DeWitt equation after promoting the canonical conjugate momenta (17),(19) into operators:

\[
\pi_a = -i \frac{\partial}{\partial a}, \quad \pi_\chi = -i \frac{\partial}{\partial \chi}, \quad \pi_{\beta NM} = -i \frac{\partial}{\partial \beta_{NM}}, \quad \pi_a^2 = -a^{-P} \frac{\partial}{\partial a} \left( a^P \frac{\partial}{\partial a} \right),
\]

(21)

where in (21) the last substitution parametrizes the operator order ambiguity with \(P\) being a real constant. The Wheeler-DeWitt equation is obtained imposing that the Hamiltonian operator annihilates the wave function \(\Psi[a, \chi, \beta_{NM}, \beta_{NM}]\).

Before we proceed in discussing the way the decoherence process takes place within the framework of our model let us comment on some features of the Hamiltonian density (20). The first line in (20) corresponds to the contribution from gravitational degrees of freedom in their ground state. The third line is associated with the transversal modes \((J - L = \pm 1)\) of the spin-1 field. The fourth line corresponds, on its hand, to the contribution of the longitudinal modes \((J = L)\). Notice that the presence of the mass term in the model entangles all modes of the spin-1 fields (longitudinal and transversal) with the metric as exhibited in the second line in (20). The mass of the vector field is also present in the longitudinal kinetic part, where there is also a term proportional to \(a(t)^{-1}\). In the fifth line, in (20), one has the kinetic piece of the homogeneous part of the spin-1 field as well as the quartic potential, typical from

\(^3\)For other presentations of an Hamiltonian formulation of systems involving the Proca field, see e.g. Refs. [35, 36].
the dimensional compactification procedure for treating the cosmological problem of coupling gravity with Yang-Mills and vector fields [12], [14]–[20]. The coupling between the homogeneous, and all the inhomogeneous modes of the vector field are shown in the sixth line. Finally in the seventh line in (20) one has the coupling of the gravitational ground state mode to the homogeneous part of the mass term of the spin-1 field. Thus, our system can be regarded as a set of coupled harmonic oscillators (gravitational and spin-1 field) where the kinetic term of the longitudinal modes has a coefficient proportional to $a(t)^{-1}$.

In what follows, we shall compare our models with the ones discussed in the literature when studying correlations between coordinates and momenta and the decoherence process.

3 Decoherence and Back-Reaction Processes in the Presence of Massive Abelian Vector Fields

In this Section, we discuss the process of decoherence in a closed FRW model with a massive vector field and global $U(1)$ symmetry and its relation to the conditions which support correlations between coordinates and canonical momenta. The validity of the semi-classical Einstein equations, i.e. the so-called back-reaction problem, is also discussed. As explained in the previous sections, we take the homogeneous modes of the spatial components of the vector field to be zero, whereas the environment is identified with the inhomogeneous modes, i.e. the $\beta_J$-functions (see eq. (22) below). Therefore, our minisuperspace will be one-dimensional and the FRW scale factor will correspond to the physical observable about which predictions can be made.

We rewrite the Wheeler-DeWitt equation (20),(21) for the $U(1)$ case as

$$H_{\text{eff}} \Psi[a, A_{\mu}] = \left[ \frac{1}{2M_P^2} \frac{\partial^2}{\partial a^2} - M_P^2 \left( a^2 - \frac{4\Lambda}{9\pi} a^4 \right) - \sum_J f_J(a) \left[ \frac{\partial^2}{\partial (\beta_J a^2)} - \Omega^2_J(a) \beta_J \beta_J \right] \right] \Psi[a, A_{\mu}] = 0, \quad (22)$$

where

$$f_J(a) = \begin{cases} \frac{\alpha\pi}{\alpha\pi + \left( -1 \right)^{4J} 4\pi^3 \left( J + 1 \right) \left( 2J + 1 \right)^2 / (2\pi)^4} & \text{if } J - L = \pm 1 \\ \frac{\left( 4m^2 a^2 / 3\pi \right) + \left( L + J + 1 \right)^2 / \alpha\pi}{
\alpha\pi + \left( -1 \right)^{4J} 4\pi^3 \left( J + 1 \right) \left( 2J + 1 \right)^2 / (2\pi)^4} & \text{if } J = L \end{cases}, \quad (23)$$

and

$$\Omega^2_J(a) = \begin{cases} \frac{\left( 4m^2 a^2 / 3\pi \right) + \left( L + J + 1 \right)^2 / \alpha\pi}{\alpha\pi + \left( -1 \right)^{4J} 4\pi^3 \left( J + 1 \right) \left( 2J + 1 \right)^2 / (2\pi)^4} & \text{if } J - L = \pm 1 \\ \frac{\left( 4m^2 a^2 / 3\pi \right) + \left( L + J + 1 \right)^2 / \alpha\pi}{\alpha\pi + \left( -1 \right)^{4J} 4\pi^3 \left( J + 1 \right) \left( 2J + 1 \right)^2 / (2\pi)^4} & \text{if } J = L \end{cases}. \quad (24)$$

following the notation of Ref. [4] and setting the ordering ambiguity factor to $P = 0$. Our massive $U(1)$ model and the $k = +1$ FRW minisuperspace model with a massive conformally coupled scalar field discussed in Ref. [4] share some similar features. Indeed, up to different constant coefficients and the $1 + 1/a$ factor, the Hamiltonian (20) is equivalent to the one of a massive conformally coupled scalar field in a closed FRW background. As far as the $1 + 1/a$ factor is concerned, if one considers expanding solutions, then the condition $1 + 1/a \rightarrow 1$ will be rapidly satisfied and, hence, we can apply the framework used in [4] and draw similar
conclusions. However, the case of contracting solutions as well as the interference between the two WKB branches have to be addressed differently.

A solution of (22)–(24) which corresponds to a “classical” behaviour of the $a$-variable on some region of minisuperspace will have an oscillatory WKB form (11) as

$$\Psi_{(n)}[a, A_\mu] = e^{iM_2 S_{(n)}(a)}C_{(n)}(a)\psi_{(n)}(a, A_\mu).$$

(25)

After expanding the functions in (25) in powers of $M_2$ and using (22)–(24) one finds that the lowest order action, $S_0$, satisfies the Hamilton-Jacobi equation,

$$-\frac{1}{2}S_0'^2 + V(a) = 0,$$

(26)

where $V(a) = -a^2 + \frac{4A}{a^4} a^4$ and the prime denotes derivative with respect to $a$. From (22)–(24) we see that the different modes do not interact among themselves. Thus, the wave function $\psi_{(n)}(a, A_\mu)$ can be factorized as

$$\psi_{(n)}(a, A_\mu) \equiv \psi_{(n)}(a, \{\beta_J\}) = \prod_J \psi_{(n),J}(\eta, \beta_J).$$

(27)

Defining the WKB time as

$$\frac{d}{d\eta} = \frac{\partial S}{\partial a} \frac{d}{da},$$

(28)

one obtains the Schrödinger equation satisfied by each wave function $\psi_{(n),J}(\eta, \beta_J)$:

$$\frac{1}{2} f_J(a) \left( -\frac{\partial^2}{\partial (\beta_J)^2} + \Omega_J^2 (\beta_J)^2 \right) \psi_{(n),J}(\eta, \beta_J) = i \frac{d}{d\eta} \psi_{(n),J}(\eta, \beta_J).$$

(29)

We stress that $\psi_{(n),J}(\eta, \beta_J)$ is actually $\psi_{(n),J}[a(\eta), \beta_J]$, dependent on the “classical” physical observable and $\beta_J$ as well. From (29), we can calculate, say, $\psi_{(n),J}(\eta, \beta_J)$ in $\eta'$ given the value of $\psi_{(n),J}(\eta, \beta_J)$ at $\eta'' < \eta'$.

In order to make predictions concerning the behaviour of $a$, one uses a coarse-grained description of the system working out the reduced density matrix associated with the WKB wave function of the form (23) to obtain (23)

$$\rho_R = \sum_{n,n'} e^{iM_2 [S_{(n)}(a_1) - S_{(n')} (a_2)]} C_{(n)}(a_1)C_{(n')}(a_2) \mathcal{I}_{n,n'}(a_2, a_1),$$

(30)

where

$$\mathcal{I}_{n,n'}(a_2, a_1) \equiv \int \psi^*_{(n')} (a_2, A_\mu) \psi_{(n)} (a_1, A_\mu) d[\mathcal{A}_\mu] = \Pi_J \int \psi^*_{(n')} (a_2, \beta_J) \psi_{(n)} (a_1, \beta_J) d[\beta_J].$$

(31)

The subindex $(n)$ labels the WKB branches. The term $\mathcal{I}_{n,n'}(a_2, a_1)$ describes the influence of the environment on the system. Notice that in (31) all modes must be included.

The analysis of correlations between minisuperspace coordinate and momenta is, in quantum cosmology, usually discussed using the Wigner function criterion (2, 30, 31): A strong sharp peak is likely to be located close to a classical trajectory defined by the Hamilton-Jacobi equation plus quantum-corrections. However, the Wigner function associated with the reduced density matrix (30) does not have a single sharp peak even for a pure WKB function as (25) or a linear
combination of them (cf. Refs. [24, 32, 33, 42]). Nevertheless, this problem can be overcome through the environment interaction with the “observed” system [24]. As explained in the Introduction, such interaction is at the origin of the loss of quantum-coherence or decoherence between different classical trajectories, i.e., WKB branches. More precisely, correlations between coordinates and momenta must be analysed within each classical branch (\( n = n' \)). This can be done by looking at the reduced density matrix associated with it or the corresponding Wigner functional:

\[
F_{W(n)}(a, \pi_a) = \int_{-\infty}^{+\infty} d\Delta [S'_{(n)}(a_1)S'_{(n)}(a_2)]^{-\frac{1}{2}} e^{-2i\pi_\alpha \Delta} e^{iM\Delta} I_{n,n}(a_2, a_1)
\]

where \( \Delta = (a_1 - a_2)/2 \). A correlation among variables will correspond to a strong peak about a classical trajectory in the phase space. Thus, there exists an important relation between correlation and decoherence as one needs the latter, i.e. fairly small off-diagonal terms in (30) such that quantum interference between alternative histories is negligible (\( I_{n,n'} \propto \delta_{n,n'} \)), in order to obtain the former. Hence, the decoherence process is rather crucial as it is only when the decoherence between different WKB branches is successful that correlations may be properly predicted.

Besides the decoherence between different WKB branches, the environment interaction also affects the correlations within a classically decohered branch; this is explicit in the functional \( I_{n,n}(a_2, a_1) \) in eq. (32). As pointed out by Zurek [41], the environment degrees of freedom \( \textit{continuously measure} \) the physical observables and this interaction not only suppresses the off-diagonal (\( n \neq n' \)) terms in (30) and (31), but also induces a “localizing” effect on the classical variables within each WKB branch. This corresponds to the back-reaction from the environment on the semi-classical evolution of the system. In particular, \( I_{n,n}(a_2, a_1) \) will be damped for \( |a_2 - a_1| \gg 1 \) and the reduced density matrix associated with (32) will be \textit{diagonal} with respect to \( a \). The sharpness and position of the peak will be determined by the behaviour of \( I_{n,n}(a_2, a_1) \). Furthermore, it has been shown in Refs. [4, 22] that the localization effect inside a classical branch is much more efficient than the decoherence between different WKB branches. It has been also remarked in Refs. [7, 8] that, if the conditions for achieving an effective localization (and diagonalization) of (32) are met, then the interference between the different WKB branches is also highly suppressed. Actually, the functional \( I_{n,n'}(a, a') \) has been usually identified as a measure of the decoherence between two different WKB histories, characterized by the parameters \( (n) \) and \( (n') \); an heuristic argument in support of that view was presented in Ref. [3].

Before proceeding, we point out that using a Gaussian Ansatz for the environment state as

\[
\psi_J(a, \beta_J) = D_J(t) e^{i\gamma_J(t) - B_J(t)\beta_J^2},
\]

for each \( \psi_J(\eta, \beta_J) \), where \( D_J, \gamma_J \) are real, \( B_J(t) = B_{r,J} + iB_{i,J} \), where \( B_{r,J} \) and \( B_{i,J} \) are also real and \( B_{r,J} > 0 \), and imposing the normalization condition

\[
\int \psi_J^*(t, [\beta_J]) \psi_J(t, [\beta_J]) d\beta_J = 1,
\]

the general form of \( I_{nn'}(a, a') \) for any mode is given by

\[
I_{nn'}(a, a') = \exp \left[ i \left( D_{(n')J}(a') - D_{(n)J}(a) \right) \right] \left[ \frac{4B_{(n')r,J}(a')B_{(n)r,J}(a)}{(B_{(n')J}(a') + B_{(n)J}(a))^2} \right]^{1/4}.
\]
The analysis of the decoherence between different WKB histories and correlations via the Wigner function (with $T_{(n,n),J}(a,a')$) requires the functions $B$ and $D$ to be found explicitly. The requirements for successful diagonalization and “localization” were generally established and discussed in Refs. [7, 8]. From the assumption that $|a_1 - a_2| \ll 1$ and the Gaussian Ansatz (33), (34) for $\psi_J(a, \beta_J)$, the above mentioned conditions read:

$$\left( \sum_J \frac{B'_{iJ}}{2B_{iJ}} \right)^2 \ll \sum_J \frac{|B'_{iJ}|^2}{4B_{rJ}^2},$$

$$2M_p^4 \left[ V(a) + \frac{1}{M_p^2} \sum_J \left( \frac{B_{rJ}^2 + B_{iJ}^2}{2B_{rJ}} + \frac{\Omega_J^2}{8B_{rJ}} \right) \right] \gg \sum_J \frac{|B'_{iJ}|^2}{4B_{rJ}^2},$$

$$\sum_J \frac{|B'_{iJ}|^2 \sigma^2}{4B_{rJ}^2} \gg 1,$$

where $\sigma = (a_1 + a_2)/2$ (the sums in these expressions and previously related ones have an implicit factor arising from the degeneracy of the $J,L,M,N$ mode). Expressions (36)–(38) are usually referred to as adiabaticity, strong decoherence and strong correlation conditions, respectively. Notice that the results (36)–(38) arise directly from (31),(32) and the Gaussian Ansatz for the state of the environment (33). Henceforth, the validity of these conditions has to be analysed using the quantities and parameters relevant to our particular models. The possibility that the massive spin-1 field models give rise to new conditions for the process of correlation and decoherence has to be properly considered. The adiabaticity condition warrants the validity of the zero-th order WKB evolution as its violation implies that the semiclassical Einstein equations are not valid due to contributions of high-order in the phase of $I_{(n,n)}(a_2, a_1)$. On the other hand, (37) reflects the fact that the peak in the Wigner function (shifted away from the expected classical trajectory by interaction with the environment) is sharp as far as the center of the peak is large when compared to the spread. Finally, expression (38) translates the condition of strong decoherence corresponding effectively to the requirement of diagonalization of the reduced density matrix associated with (32). It is important to mention that usually a compromise between decoherence and correlation is needed since if the later is too strong, then the peak in the Wigner function is actually broaden [23].

Let us now address the issues of decoherence, correlations and back-reaction in our model with massive Abelian vector fields. Firstly, we shall assume that the decoherence between the two different WKB histories has occurred successfully and consider the correlation and localization effects within a classical branch. Afterwards, we shall comment on the decoherence between different WKB branches. The inclusion of transversal as well as longitudinal modes in eqs. (30), (31), (36)–(38) give rise to some difficulties for our $U(1)$ and $SO(3)$ models as far as the longitudinal modes are concerned. This will be discussed in the following. In particular, notice that eq. (30)–(32) and then (36)–(38) involve considering all modes, longitudinal and transversal. The same will apply to other equations as will be pointed out explicitly.

We start by considering the correlation and localization effects within each classical branch for the case of transversal modes (31) $(J \neq L)$. One can easily verify that up to a re-scaling of the $\beta_{iJ}$-modes by a factor of $\alpha \pi$ (and conversely for the corresponding canonical momenta),

---

The computations corresponding to each of the two WKB branches (expanding and contracting) are the same in the two cases ($n = \pm 1$, say).
This part of the model is equivalent to the one with a massive conformally coupled scalar field (cf. Refs. [7, 26]). Substituting (33) into the Schrödinger equation (29), we get the following equations

\[
\dot{\gamma}_J = - f_J(a) B_J, \\
\dot{B}_J = i f_J(a) [-2(B_J^2 - \Omega_J^2/4)],
\]

for \(D_J = \pi^{-1/4}(2B_J)^{1/4}\). With the above mentioned re-scaling, eq. (40) can be linearized via the choice \(B_J = -i \dot{\phi}_J / (2\varphi_J)\), to yield

\[
\ddot{\varphi}_J + \Omega_J^2 \varphi_J = 0.
\]

The initial state of the environment is associated with a particular choice of initial conditions when solving the preceding equation. In our present case, the Hamiltonian eq. (22) corresponds to a set of harmonic oscillators with a variable time-dependent mass. A convenient vacuum state can be defined assuming there exists an adiabatic zone such that the classical evolution emerging from the Hamilton-Jacobi equation satisfies the condition \(\dot{a}/a \to 0\) for large \(a\). This requires that \(V(a)\) be quadratic in \(a\), meaning that models with non-vanishing cosmological constant do not satisfy this adiabaticity condition as, in this case, \(V(a) \sim O(a^4)\) for large \(a\). Notice that the Hamilton-Jacobi equation for a generic \(V(a)\) has real solutions for \(a > a_0\) only if \(a_0\) is a single zero of \(V(a)\) for \(V(a) > 0\). Assuming a vanishing cosmological constant, one can identify a vacuum for the adiabatic out regime \((a \gg 1)\), being the out-modes of the form \([7, 8, 25, 26]\)

\[
\varphi_{J}^{\text{out}} = (2\Omega_J)^{-1/2} \exp \left[-i \int_{\eta}^{\eta'} \Omega_J(\eta') d\eta' \right],
\]

which diagonalize asymptotically the Hamiltonian for large values of \(a\). For small values of \(a\), a preferred initial state (an in vacuum state) may in some cases also be defined as the one which diagonalizes the Hamiltonian for \(a > a_0\) (in our model for \(\Lambda = 0\) and for other commonly used, \(da/d\eta \ll 1\), for small values of \(a\)). The relation between the in and out modes is given by the Bogolubov transformation

\[
\varphi_{J}^{\text{in}} = \hat{\alpha}_J \varphi_{J}^{\text{out}} + \hat{\beta}_J (\varphi_{J}^{\text{out}})^*.
\]

The \(\hat{\alpha}_J, \hat{\beta}_J\) are designated as Bogolubov coefficients and any particular choice for these determine different vacuum states for \(\varphi\). One obtains for a general potential \(V(a)\) inducing an adiabatic behaviour \([25]\)

\[
\hat{\alpha}_J \simeq 1; \quad \hat{\beta}_J \simeq \frac{i}{2} \exp \left\{-\frac{1}{2}\pi [m^2 a_0 V'(a_0)]^{-1/2} \left[d_J^2 + m^2 a_0^2 \right] \right\},
\]

where the coefficient \(d_J\) denotes the degeneracy associated to \(J, L, M, N\). Notice that for the case of a quadratic \(V(a)\) one obtains \(\hat{\beta}_J = 0\). If the condition \(\hat{\alpha}_J \simeq 1, \hat{\beta}_J \simeq 0\) is satisfied, which occurs when the evolution is indeed adiabatic, then such quantum state is usually identified as adiabatic vacuum and holds during all the evolution. Notice again that, in the case of a non-vanishing cosmological constant, the adiabaticity condition \((\dot{\Omega}_J/\Omega_J^2 \ll 1)\) cannot be satisfied.

Since the massive spin-1 transversal modes behave effectively as conformally coupled massive scalar fields, it comes as no surprise that the adiabaticity, strong decoherence and strong
correlation conditions (36)–(38) are indeed satisfied restricted to those environment modes. Notice then that
\[ \sum |B'_{jj}|^2 = \sum \left[ \frac{1}{4} \left( \Omega'_J \right)^2 + |\hat{\beta}_J|^2 \left( 4\Omega_J^2 \frac{1}{a} \right)^2 \right]. \] (45)
Within the adiabaticity evolution requirement, we take a quadratic \( V(a) \) and hence the first term in (45) will be dominant. Using eqs. (23) and (24), we see that the sum in eq. (43) implies (cf. Ref. [7]) that the strong decoherence condition (38) corresponds asymptotically to
\[ m^2 a^2 \gg 1, \] (46)
while the strong correlation condition (37) reduces asymptotically to
\[ |V(\pi)| \gg m^{-1} \pi, \] (47)
for large \( a \). As long as we restrict ourselves to assess the decoherence process and correlation analysis for the case of an expanding solution (as it is for the cases studied in most of the literature) the conditions (46),(47) are valid. In fact, that seems to be the right interpretation when explaining the quantum to classical transition of our Universe. As one can see, the mass of the of the Abelian vector field provides a scale at which the process of decoherence and the analysis of correlations take place. Indeed, for fairly small or negligible mass the process of decoherence is not completely achieved, which is consistent with the decoupling between the gravitational and the Yang-Mills field (\( m = 0 \)) [12]. The adiabaticity condition implies that the growth of \( V(a) \) must be slower than that of a quartic potential. For a quadratic \( V(a) \) this is immediate.

One could, for instance, consider an ad hoc potential from the start [7], although it would remain to be verified if such a potential would satisfy conditions (36)–(38) and could be derived from a realistic action (with anisotropy or even higher curvature terms). On its turn, the presence of a cosmological constant induces divergences in the decoherence factor (45) in addition to the ones from the back-reaction factor. If the latter were expected to correspond to the zero-point energy of the fields, the former may only be cured via the introduction of a fundamental cut-off since it cannot be removed by standard renormalization procedure [4, 8]. However, that seems rather unsatisfactory from the physical viewpoint. Considering the environment composed by modes whose physical wavelength is larger than the horizon is in disagreement with the expectation that the environment consists of small wavelength fluctuations.

Let us now return to the decoherence between different WKB branches. The quantity to analyse is \( \mathcal{I}_{nn'}(a, a) \) (see above remarks and Ref. [8]). We suppose once again that our minisuperspace model has an adiabatic out region for large values of \( a \). Then, up to second order for the adiabatic limit and \( \hat{\beta} \)-Bogolubov coefficients, one can find, in the case of the transversal modes of the massive Abelian field [8, 25, 26]:
\[ \mathcal{I}_{nn'}(a, a) \simeq \exp \left\{ \sum_J \left[ -\frac{1}{4} \left( |\hat{\beta}_{nJ}|^2 + |\hat{\beta}_{n'J}|^2 + 2 \hat{\beta}_{nJ} \hat{\beta}_{n'J} \cos \left( 2 \int (\Omega_{nJ} - \Omega_{n'J}) \right) \right] \right\}, \] (48)
where a sum over the environment transversal modes is understood (the longitudinal modes will be treated in the next paragraph). The case of a quadratic \( V(a) \) imposes \( \hat{\beta}_{L} = 0 \) and,
as mentioned previously, this means that the quantum state of the environment evolves as an adiabatic vacuum. In that case, for $\Omega_n = -\Omega_n'$ we have

$$I_{nn'}(a, a) \simeq \exp \left\{ -\frac{1}{16} \sum_J \frac{(\dot{\Omega}_{nJ})^2}{\Omega_J^4} \right\}, \quad (49)$$

and, since $\frac{\dot{\Omega}_{nJ}}{\Omega_J^2} \ll 1$, these terms are effectively destroyed for the two WKB histories. Another possibility is to consider that the Universe has undergone a static or quasi-static period for large $a$ such that $\dot{\Omega} \sim 0$. In that case, the framework established above for the $\varphi_J$ modes can be used for the environment and a natural in vacuum state can be defined, which is not equivalent to the out vacuum [25]. Associated particle creation takes place and, as a consequence, decoherence occurs and the relation between the amount of interference that is suppressed and the number of particles created is given by

$$I_{nn'}(a, a) \simeq \exp \left\{ -4 \sum_J d_J^2 \beta_{JJ}^2 \cos^2 \left[ 2 \int_0^a \Omega_{nJ} \right] \right\}. \quad (50)$$

Finally, one has to consider the longitudinal modes ($J = L$). Surely, they have to be included in eqs. (45) and (48)–(50) if the results (46), (47) and loss of quantum-interference among the expanding and contracting WKB branches are to be extended to a mode-complete massive Abelian vector field. As far as expanding solutions are concerned, namely the ones for which $a$ becomes much larger than 1 sufficiently fast, the previous remarks for the transversal modes can be extended to this case, up to the “mode by mode” re-scaling of the $\beta_{JJ}$ functions and the corresponding canonical momenta. Therefore, when analysing correlations and diagonalization of the reduced density matrix within the WKB branch corresponding to an expanding solution of the Hamilton-Jacobi equation, (26), the results obtained in the previous paragraphs can be extended to encompass the longitudinal modes as well. The conclusions concerning the conditions (36)–(38), i.e. (45)–(47), are thus valid in the case of expanding closed FRW models with massive Abelian vector fields. The same applies to (48)–(50). However, for contracting solutions of the Hamilton-Jacobi equation, there is a problem due to the $1 + 1/a$ factor. The equation corresponding to (11) is the following

$$\ddot{\varphi}_{JJ} + \left( \frac{-1 + f_{JJ}(a)}{\varphi_{JJ}} \right) \dot{\varphi}_{JJ}^2 + \Omega_{JJ}^2 \varphi_{JJ} = 0. \quad (51)$$

As it stands, we do not know of any exact or even adiabatic solution of this equation, which is of the type

$$\varphi_{JJ}^{in} = \dot{\alpha}_{JJ} \varphi_{JJ}^{out} + \beta_{JJ}(\varphi_{JJ}^{out})^*, \quad (52)$$

together with (42). It is therefore difficult to draw any conclusions about correlations as well as diagonalization (and localization) when the longitudinal modes are taken into consideration in the case of contracting WKB solutions.

This problem becomes somewhat more acute when we try to analyse the decoherence between the two WKB histories for $n = \pm 1$. Once again, we do not know how to obtain a form similar to (48) for $I_{nn'}(a, a)$ as that would involve adiabatic solutions of the type (42) and that seems, for the moment, difficult to obtain for the longitudinal modes in a WKB contracting solution.
A possible alternative to deal with the longitudinal modes and construct WKB solutions, may involve computing the influence functional (and the functions $B_i$) assuming another approach. Namely, that the vector field mass may be almost negligible and therefore the presence of longitudinal modes could be treated using a perturbative scheme [8], assuming that the massive transversal modes are dominant as they are anyway present in the massless case. In Ref. [4], it was considered that the interaction between the system and the environment is such that there is decoherence between different WKB branches. This has been shown for specific models in Refs. [4,25,26] for $N = 1$ and $N > 1$ minisuperspace cases. However, the use of massive Abelian vector field models raises the problem that for some situations that cannot be so easily achieved and demonstrated. Nevertheless, the factor $1 + 1/a$ indicates that some divergences will be present in the computation of the influence functional. One should try to cure them by some type of renormalization procedure, eg. via a cut-off, although its nature seems to indicate that one has to go beyond quantum cosmological models arising from Einstein theory plus matter, possibly considering an effective model arising from higher-derivative theories of gravity or even string theories.

4 Decoherence and Back-Reaction Processes in the Presence of Massive Vector Fields with $SO(3)$ Global Symmetry

In this section we extend the analysis of decoherence and correlations to massive vector fields with non-Abelian $SO(3)$ global symmetry. The corresponding Wheeler-DeWitt equation (20),(21) can be rewritten as:

$$\mathcal{H}^{\text{eff}} \Psi[a, A_{\mu}] = \left[ \frac{1}{2M_p^2} \frac{\partial^2}{\partial a^2} - M_p^2 \left( a^2 - \frac{4\Lambda}{9\pi} a^4 \right) - \frac{\partial^2}{\partial \chi^2} + \frac{\alpha}{3\pi} \left[ \chi^2 - \frac{3\pi}{2\alpha} \right]^2 + 4\pi a^2 m^2 \left[ 1 + \sqrt{\frac{2\pi}{3\pi}} \right]^2 \right] - \sum_{J} f_J(ab) \left( \frac{\partial^2}{\partial (\beta_J(ab))^2} - \Omega_J(ab) \alpha^{(ab)} \right) \Psi[a, A_{\mu}] = 0, \quad (53)$$

where

$$f_J(ab)(a) = \left\{ \begin{array}{ll} \frac{\alpha}{3\pi} & \text{if } J = L = \pm 1 \\
\pi^\xi a_{J(ab)}^{(a)} a_{J(ab)}^{(b)} [(J+1)/(2J+1)4m^2] [1+(1/a(t))] & \text{if } J = 0 \\
\alpha & \text{if } J = L = \pm 1 \\
\end{array} \right. \quad (54)$$

and

$$\Omega_J(ab)(a) = \left\{ \begin{array}{ll} \{(4m^2a^2/3\pi) + (L + J + 1)^2 + 4 \left[ 1 + \sqrt{\frac{2\pi}{3\pi}} \right]^2 /3\pi \}/\alpha & \text{if } J = L = \pm 1 \\
\alpha^\xi a_{J(ab)}^{(a)} a_{J(ab)}^{(b)} [(J+1)/(2J+1)4m^2] [1+(1/a(t))] & \text{if } J = L \\
\end{array} \right. \quad (55)$$

This hypothesis should, however, be considered with some care in view of eqs. (46),(47) and their implications regarding the presence of a massive vector field as providing a scale relatively to the decoherence process and correlation analysis.
and, furthermore, we have set the ambiguity factor to vanish, $P = 0$. Our minisuperspace is now two-dimensional, the scale factor and function $\chi(\eta)$ (parametrizing the homogeneous modes of the non-Abelian massive vector fields) being the classical observable degrees of freedom. The environment corresponds, as before, to the inhomogeneous modes, i.e., the $\beta_{j}^{(ab)}$-functions.

Actually, only a few particular multi-dimensional minisuperspace models have been considered from the point of view of decoherence and correlations between coordinates and momenta. The Kantowski-Sachs model ($N = 2$) with a cosmological constant and massive inhomogeneous conformally coupled scalar field modes was studied in Ref. [24], the Bianchi type-I ($N = 3$) with massless inhomogeneous conformally coupled scalar field modes was studied in Ref. [8] and in Ref. [24], an $N = 2$ model has been analysed where the two classical degrees of freedom were the scale factor and homogeneous mode of a minimally massless scalar field, the environment being associated with the inhomogeneous perturbations of another minimally massless scalar field. The analysis of multi-dimensional minisuperspace models is, in particular, also relevant as it provides a possible relation between the notion of decoherence between WKB branches and the decoherence between histories in the so-called Consistent Histories approach [6] in terms of space-time histories (see section V. of Ref. [8]). However, no arguments have yet been put forward to relate the diagonalization of the reduced density matrix within a given WKB branch to the notion of decoherence of different histories. Nevertheless, as pointed out in Ref. [4], the functional $\mathcal{I}_{nn}(a, \chi, \ldots; a', \chi', \ldots)$ can be related somehow to the notion of decoherence between histories for the cases where $N > 1$. A particular solution of the Hamilton-Jacobi equation generates a $N$-1 parameter family of trajectories, but there will be only one classical trajectory passing through each point of the minisuperspace generated by that solution of the Hamilton-Jacobi equation. In this sense, $\mathcal{I}_{nn}(a, \chi, \ldots; a', \chi', \ldots)$ will strengthen the suppression of interference between histories belonging to a given WKB branch as it produces a more efficient diagonalization of the reduced density matrix. However, calculations of Ref. [24] have shown that a successfull decoherence between histories associated to $\mathcal{I}_{nn}$ has not been achieved for any of the models considered so far, possibly due to their simplicity [7].

As far as our case is concerned, we can see from eqs. (53)– (55) that the typical quadratic potential of massive conformally coupled scalar field with homogeneous modes is now replaced by the double-well quartic potential $\frac{\alpha}{3\pi} \left[ \chi^2 - \frac{3\pi^2}{2\alpha} \right]^2$. The remarks made above concerning the $1+1/a$ factor still apply here. Notice, however, that one needs to consider the $(ab)$-$SO(3)$ group indexes. It follows in particular, that eqs.(27)–(32) (and subsequent ones) remain valid provided we include the $SO(3)$ group indexes (eg. $\Pi_{J} \rightarrow \Pi_{J(ab)}$) and the $\chi(\eta)$ function together with the scale factor. Generally speaking, all our working hypothesis, considerations and arguments presented in the last section can be extended to the non-Abelian case. The Hamilton-Jacobi equation is now

$$- \frac{1}{2} \left( \frac{\partial S}{\partial a} \right)^2 + V(a) + \frac{1}{2} \left( \frac{\partial S}{\partial \chi} \right)^2 + \frac{\alpha}{3\pi} \left[ \chi^2 - \frac{3\pi^2}{2\alpha} \right]^2 + 4\pi a^2 m^2 \left[ 1 + \sqrt{\frac{2\pi}{3\pi}} \right] = 0 ,$$

where $V(a) = -a^2 + \frac{4\Lambda}{9\pi} a^4$.

As our minisuperspace is now two-dimensional, the Hamilton-Jacobi equation (56) is expected to have a one-parameter family of solutions, each one generating a family of classical trajectories in minisuperspace, each WKB branch interpreted as describing a whole family of classical trajectories, i.e. a set of different universes (and not a single one as for the $N = 1$
From the Hamilton-Jacobi, eq. (56), it follows that the trajectories in our $N = 2$ minisuperspace are far more complicated than those of the $N = 1$ case (Section 3). Moreover, the WKB time is now defined as

$$\frac{d}{d\eta} = G^{AB} \frac{\partial S}{\partial q_A} \frac{\partial}{\partial q_B},$$

(57)

with $A, B = 1, 2$, $q_1 = a, q_2 = \chi$ and $G^{AB} = \text{diag}(-1, 1)$. We shall have as many $\eta$-affine parameters as different values of the $(n)$-parameter. Hence, different values of $(n)$ will lead to different definitions of time for the Schrödinger equation (29). This implies that the influence functional in (30),(31) is actually a functional of two histories. The state $\psi_{J(n)}(a, \chi, \{J_{(ab)}\})$ can be interpreted, not as being simply a function of a point in minisuperspace, but instead as a function of the whole history, which corresponds to the only trajectory that belongs to the $(n)$-WKB branch and goes through that particular point $(a, \chi)$. Such description is fairly similar to the one of the Feynman-Vernon influence functional [27].

Let us now address the issues of correlations and decoherence within each WKB branch and compute the relevant influence functional for our $N = 2$ minisuperspace model. We briefly describe the main framework and consider, as in Section 3, the transversal and longitudinal modes separately. We adopt the terminology of Ref. [8] here as well. Defining new variables $q_A^{1,2} = x_A \pm \frac{1}{2}y_A$ and assuming the Gaussian Ansatz (33) for each $J_{(ab)}$-mode, we can write

$$I_{(n,n),J}(q_A^1, q_A^2) = \exp\left[-\epsilon^{AB}_{(J)} y_A y_B\right] \exp\left[i\tilde{\epsilon}^A_{(J)} y_A\right],$$

(58)

where $\epsilon^{AB}$ and $\tilde{\epsilon}^A$ are designated as decoherence matrix and phase vector respectively, and

$$\epsilon^{AB}_{(J)} = \frac{1}{4B_{rJ}}[(B_{rJ})^TA(B_{rJ})^TB + (B_{iJ})^TA(B_{iJ})^TB],$$

(59)

$$\tilde{\epsilon}^A_{(J)} = (D_J)^TA - (B_{iJ})^TA / 4(B_{rJ})^A.$$

(60)

Let us again consider the minisuperspace with a region, say for large values of $a$, for which an adiabatic solution of eq. (41) of the type (42) and (43) can be established. The quantum state of the environment is determined by choosing the Bogolubov coefficients. We then obtain [8]

$$\epsilon^{AB}_{(J)} = \Omega^{A}_{(J)} \Omega^{B}_{(J)} + \eta^{A}_{(J)} \eta^{B}_{(J)} |\hat{\beta}_{J}|^2,$$

(61)

$$\tilde{\epsilon}^A_{(J)} = \eta^{A}_{(J)} \left[\frac{\Omega_{J}}{2} + |\hat{\beta}_{J}|\right] + \frac{1}{2\Omega_{J}} \left[\frac{\Omega_{J}}{4\Omega_{J}}\right]^{TA}.$$

(62)

The first term in (61) corresponds to the adiabatic vacuum contribution ($\hat{\beta}_{J} = 0$) while the second one is related to particle creation. It is interesting to notice that for $N = 2$ one can always generate a diagonalization along two independent directions of minisuperspace by coupling to a variable mass.

For $N > 2$ the situation is however, different because $\epsilon^{AB}$ is positive definite and also degenerate whenever $N > 2$ as the directions of the $(B_{rJ})^A$ and $(B_{iJ})^A$ vectors diagonalize the reduced density matrix and therefore any other orthogonal direction to these is an eigenvector with null eigenvalues.
Correlations between each minisuperspace coordinate $a$ and $\chi$ and their canonical momenta can be analysed by examining peaks in the reduced Wigner function

$$F_{W1}(q^A, \pi_{q1}) = \int d\pi_{q2} F_{W(n)}(q^A; \pi_{qA})$$

which is equivalent to

$$F_{W1}(q^A, \pi_{q1}) = \exp \left[ \epsilon^{11} \left[ \pi_{q1} - M_{P}^2 \frac{\partial S}{\partial q^1} - \epsilon^1 \right]^2 \right],$$

where $\pi_{q1}$ is the momentum conjugate to $q^A$, $A = 1$. The strong correlation condition translates as

$$\epsilon^{AA} \ll \bar{\pi}_A^2$$

where $\bar{\pi}_A$ is a typical value of the momentum along the trajectory) and the strong decoherence condition is

$$\epsilon^{AA} \gg 1/q^A.$$

Notice that eqs. (63), (64) must include the summation over the $SO(3)$-group indexes and all modes, using eqs. (58) to (62). The same holds for eqs. (48)–(50) when analysing decoherence between different WKB branches.

The above construction is valid for the transversal modes as those behave similarly to massive conformally coupled scalar fields. Let us take the case for which $\beta_J = 0$, i.e., we choose the quantum state of the environment modes to be the adiabatic vacuum. In this case we must put $\Lambda = 0$. The relevant quantities to analyse the decoherence and correlation will be $\epsilon^{11}_J$ and $\epsilon^{22}_J$

$$\epsilon^{11}_J \simeq \frac{\left(\frac{4m^2}{3\pi}\right)^2 a^2}{\frac{4m^2a^2}{3\pi} + (L + J + 1)^2 + 4 \left[ 1 + \sqrt{\frac{2\pi}{3\pi} \chi} \right]^2},$$

$$\epsilon^{22}_J \simeq \frac{\frac{2\pi}{3\pi} \left[ 1 + \sqrt{\frac{2\pi}{3\pi} \chi} \right]^2}{\frac{4m^2a^2}{3\pi} + (L + J + 1)^2 + 4 \left[ 1 + \sqrt{\frac{2\pi}{3\pi} \chi} \right]^2}.$$  

The sum in $J$ implies that, similarly to Ref. [7], $\epsilon^{11}$ and $\epsilon^{22}$ behave, for large $a$, proportionally to $a$ and $a^{-1}$, respectively. Notice that one expects the $\chi$-field to evolve towards one of the minima of the potential $\frac{\pi}{3\pi} \left[ \chi^2 - \frac{3\pi}{2\pi} \chi \right]^2$ for large values of $a$, following Ref. [16], and neglecting curvature terms as $a \to \infty$. From the Hamilton-Jacobi equation we obtain that a typical value along a WKB trajectory for the canonical conjugate momenta to $a$ and $\chi$ are $\pi_a = \frac{\partial S}{\partial a} \sim a$ and $\pi_\chi = \frac{\partial S}{\partial \chi} \sim \chi^2$, assuming a small mass in order to neglect the interaction terms. Such approach has already been discussed in Section 3 regarding the longitudinal modes. As far as WKB expanding solutions are concerned, conditions (65) and (66) with (67) and (68) seem to indicate that, for the $\chi$-field, the strong correlation condition will be satisfied but not the strong decoherence one. For the scale factor, the strong correlation and strong decoherence conditions will be satisfied in the very sense of Ref. [8]. We shall discuss the apparent failure in fulfilling the strong decoherence condition for the $\chi$-field in Section 5.

Once again, the longitudinal modes are well behaved concerning diagonalization and correlation within a suitable expanding WKB branch and the above results are equally suited here. However, the difficulty associated to contracting branches still remains.

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Equations (48)–(50), including the \((ab) - SO(3)\) indexes, are valid for the transversal modes. The same conclusion holds for longitudinal modes within expanding WKB branches. If we consider two different WKB expanding branches in our N-2 dimensional minisuperspace with an adiabatic vacuum, eq. (48) can be reduced to

\[
\mathcal{I}_{nn'}(a, a') \simeq \exp \left\{ \sum_{J(ab)} \left[ -\frac{1}{64} \frac{\nabla S(n) - \nabla S(n')}{\Omega_{J}^{(ab)}} (\nabla \Omega_{J}^{(ab)})^2 \right] \right\},
\]

with \(\nabla S(n) \equiv G^{AB} \frac{\partial S}{\partial q_A} \frac{\partial }{\partial q_B}\). Hence, we find that the interference between terms with different \((n)\) is exponentially suppressed, depending on how different is the WKB time variation of \(\Omega_{J}^{(ab)}\) along the two trajectories (cf. comment after eq. (57)). However, when there are two branches and one corresponds to an expanding solutions while the other to contracting ones, the issues raised in Section 3 with respect to longitudinal modes equally will apply in this case as well. Namely, one cannot use (48)–(50) unless some consistent perturbative scheme to treat the longitudinal modes when \(m \ll 1\) is available and justifiable.

5 Conclusions and Discussion

We have discussed the quantum cosmology of a massive vector field coupled with gravity and we have shown that the resulting model possesses interesting properties in what concerns the decoherence of the scale factor of a closed FRW geometry. In the presence of a massive vector field with \(U(1)\) global symmetry, the scale factor is the only decohered quantity while for the non-Abelian case with \(SO(3)\) global symmetry, the scale factor and the homogeneous mode \(\chi(t)\) are the decohered variables expected to behave classically.

As far as we consider expanding semiclassical solutions, the models we propose can be regarded, to a certain extent, on the same footing as the ones where decoherence of degrees of freedom of the metric is achieved via tracing out higher modes of self-interacting scalar fields. The inhomogeneous modes of massive vector fields, which were expanded in spin-1 hyperspherical harmonics, represent an interesting alternative to play the role of environment for the metric in the Abelian case and for the metric and the homogeneous mode of the non-Abelian massive vector field. However, in the latter case, we find that the strong decoherence condition, eq. (66), for the \(\chi\)-field (parametrizing the non-Abelian massive vector field homogeneous modes) is not satisfied. Unfortunately, since the literature on \(N > 1\) minisuperspace models is rather scarce, we could not contrast our results with the existing ones. We can mention nevertheless, that, for instance, in Ref. [8], in a diagonal Bianchi type-I model where the environment consists of modes of a massless conformally coupled scalar field, a problem of similar nature to ours is encountered. One could speculate whether the decoherence and correlation conditions would be satisfied using solutions that already account for the back-reaction, rather than the classical histories or, instead, considering higher-derivative terms as done in Ref. [8]. A self-consistent approach along these lines could be seen as a way to implement our massive non-Abelian vector field model.

We have also shown that, for the transversal modes, known techniques and the associated discussion on decoherence and localization within WKB branches, namely using the Gaussian Ansatz for the wave function and computing the Wigner functional in the adiabatic limit, still
holds for both cases we have considered (Abelian and non-Abelian). The same can be said for the longitudinal modes at least in what concerns expanding models. Issues related with the problem of treating the longitudinal modes in contracting models, in particular regarding the decoherence between different WKB branches, have been discussed and some possible fixes have been suggested.

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Appendix

The full effective Hamiltonian density is given by:

$$
\mathcal{H}^{\text{eff}} = N\left\{ -\frac{1}{2} \pi^2 a^2 + \frac{4\Lambda}{9\pi M^2} a^4 + 4\pi a^2 \frac{m^2}{M^2} \left[ 1 + \sqrt{\frac{2\tilde{\alpha}}{3\pi}} \right]^2 + \frac{4m^2}{3\pi M^2} \right\}
$$

$$
+ \frac{1}{2\pi\tilde{\alpha}} \left[ 1 + \sqrt{\frac{2\tilde{\alpha}}{3\pi}} \right] \left[ \beta_{\Delta NM} \beta_{LJ, M'} + \beta_{\Delta NM} \beta_{LJ, M'} \right] \sigma_a^m \sigma_b^m S_5 \left[ N M \ N' M' \ m \right]
$$

$$
+ \frac{1}{2\pi\tilde{\alpha}} \left[ 1 + \sqrt{\frac{2\pi}{3\pi}} \chi \right] ^2 \left[ \beta_{\Delta NM} \beta_{LJ, M'} + \beta_{\Delta NM} \beta_{LJ, M'} \right] \sigma_a^m \sigma_b^m S_5 \left[ N M \ N' M' \ m \right]
$$

$$
- \frac{1}{\pi\tilde{\alpha}} \beta_{LJ} \beta_{LJ} \epsilon_{dab} \sigma_a^m \sigma_b^m \epsilon_{dab} \sigma_a^m \sigma_b^m S_6 \left[ L J \ M N \ M' N' \ N'' M'' \ m' m'' \right]
$$

$$
+ \frac{2}{\pi\tilde{\alpha}} \left( \beta_{LJ} \beta_{LJ} \epsilon_{dab} \sigma_a^m \sigma_b^m \epsilon_{dab} \sigma_a^m \sigma_b^m S_7 \right) \left[ L J \ M N \ N' M' \ m' \right]
$$

$$
+ \frac{2}{\pi\tilde{\alpha}} \beta_{LJ} \beta_{LJ} \epsilon_{dab} \sigma_a^m \sigma_b^m \epsilon_{dab} \sigma_a^m \sigma_b^m S_8 \left[ L J \ M N \ N' M' \ N'' M'' \ m' m'' \right]
$$

$$
+ \frac{2}{\pi\tilde{\alpha}} \left( \beta_{dMN} \epsilon_{dab} \sigma_a^m \sigma_b^m \epsilon_{dab} \sigma_a^m \sigma_b^m S_9 \right) \left[ L J \ N M \ N' M' \ m' \right]
$$

$$
+ \pi^2 + \frac{\tilde{\alpha}}{3\pi} \left[ \chi^2 - \frac{3\pi}{2\tilde{\alpha}} \right]^2 + \frac{1}{\pi\tilde{\alpha}} \beta_{dLJ} \beta_{dLJ} (L + J + 1)^2
$$

$$
+ \frac{4}{\pi\tilde{\alpha}} \left[ 1 + \sqrt{\frac{2\pi}{3\pi}} \right] \beta_{dLJ} \beta_{dLJ} + \pi\tilde{\alpha} \left( \pi_{\tilde{\alpha}LJ} + \pi_{\tilde{\alpha}LJ} \right)
$$

$$
+ \frac{\sqrt{2}}{\pi\sqrt{\pi\tilde{\alpha}}} \alpha_{M'} \beta_{dLJ} \epsilon_{bac} \sigma_a^m S_1 \left[ J J \ M' \ N' \ m \right]
$$

$$
+ \frac{2}{\pi\tilde{\alpha}} \left[ 1 + \sqrt{\frac{2\pi}{3\pi}} \right] \alpha_{M'} \beta_{dLJ} \epsilon_{bac} \sigma_a^m S_1 \left[ J J \ N M \ J' \ m \right]
$$

$$
+ \frac{4}{\pi\tilde{\alpha}} \alpha_{M'} \beta_{dLJ} \epsilon_{bac} \sigma_a^m S_2 \left[ N M \ N' M' \ J'' \ m \right]
$$
\[\begin{align*}
+(-1)^{2J} a \, \alpha_{M}^{b_e J N} \, \pi_{j_{b_e J N}} \sqrt{\frac{J(J + 1) 16\pi^2}{2J + 1}}
- \frac{4}{3\pi M_{N}^2} a_{4}^{b_e J M} \frac{\alpha_{M}^{b_e J N}}{N}
\end{align*}\]
\[ + \frac{\alpha^2}{\pi \tilde{\alpha}} \left[ \frac{\alpha^{\ell e N M} \beta_{I L} \beta_{E \tilde{N}}}{N} \right] \times S_3 \left[ N M \quad \bar{\bar{J}}_m \quad \bar{\bar{M}}_N \quad \bar{\bar{J}}' \bar{\bar{J}}' \right], \]

where

\begin{align*}
S_1 \left[ \frac{L J}{N M} \quad m' \quad m \right] & = \int d^3 x \sqrt{s^3} g Y_{m' N M}^{1 NJ} D_j \bar{J}_{M' N'}, \\
S_2 \left[ \frac{N M}{L J} \quad J'' N'' \quad m \quad m' \right] & = \int d^3 x \sqrt{s^3} g Y_{m' L J}^{1 NM} Y_{m'' N''}^{1 NJ} D_j \bar{J}_{M' N''}, \\
S_3 \left[ \frac{N J}{L J} \quad J'' N'' \quad m'' \right] & = \int d^3 x \sqrt{s^3} g Y_{m' L J}^{1 NM} D_j \bar{J}_{M' N''} D_j \bar{J}_{M'' N''}, \\
S_4 \left[ \frac{N M}{L J} \quad m'' \quad J'' N'' \quad M'' m'' \right] & = \int d^3 x \sqrt{s^3} g Y_{m' L J}^{1 NM} D_j \bar{J}_{M' N''} D_j \bar{J}_{M'' N''}, \\
S_5 \left[ \frac{N M}{L J} \quad J'' N'' \quad m'' \right] & = \int d^3 x \sqrt{s^3} g Y_{m' L J}^{1 NM} V_{m'' N''}, \\
S_6 \left[ \frac{L J}{N M} \quad J'' N'' \quad m'' \quad m'' ight] & = \int d^3 x \sqrt{s^3} g V_{m' L J}^{1 LM} Y_{m'' N''}, \\
S_7 \left[ \frac{L J}{M N} \quad m'' \quad m'' \right] & = \int d^3 x \sqrt{s^3} g \frac{\partial V_{m' L J}^{1 LM}}{\partial x^i} L^i_{m'' N''}, \\
S_8 \left[ \frac{L J}{M N} \quad J'' N'' \quad m'' \quad m'' \right] & = \int d^3 x \sqrt{s^3} g \frac{\partial V_{m' L J}^{1 LM}}{\partial x^i} L^i_{m'' N''}, \\
S_9 \left[ \frac{L J}{N M} \quad J'' N'' \quad m'' \quad m'' \right] & = \int d^3 x \sqrt{s^3} g V_{m' L J}^{1 LM} Y_{m'' N''}, \\
Y_{m' L J}^{1 LM} & = \sqrt{(2L+1)(2J+1)} \frac{1}{16\pi^2} D_{L N'}^{N} \left( \frac{L}{N'} \quad J \quad m \right),
\end{align*}

and

\[ Y_{m' L J}^{1 LM} = \sqrt{(2L+1)(2J+1)} \frac{1}{16\pi^2} D_{L N'}^{N} \left( \frac{L}{N'} \quad J \quad m \right), \]

where \( D_{L N'}^{N} \) is a representation for the scalar harmonics

\[ Q_{\ell m}^{n} = \pi_{\ell}^{n}(\chi) Y_{m}^{n}(\theta, \phi), \]

\[ \pi_{\ell}^{m}(\chi) = \sin^\ell \chi \frac{d^{\ell+1}(\cos n \chi)}{d(\cos \chi)^{\ell+1}} \]

are Fock harmonics, \( Y_{m}(\theta, \phi) \) are spherical harmonics on \( S^2 \) and \( \left( \frac{L}{N'} \quad J \quad m \right) \) are \( 3 - j \) symbols. \( \sqrt{s^3} g \) denotes the square root of the determinant of the metric over the unitary
3-sphere and $L^i_a$ represents the transformation between the left-invariant basis on $S^3$ and a coordinate basis. We have used

$$\int d^3x \sqrt{s^3} g Y^1_{mL} Y^{mN'}_{1L'} = \delta^N_M \delta^M_M' \delta^L_{L'}$$

and

$$\sigma^a_m L^i_a \partial_i D^J_N M = (-1)^{2J} \sqrt{\frac{J(J+1)16\pi^2}{2J+1}} Y^1_{mnJ} ,$$

and also that

$$s^3 = g_{ab} \omega^a \otimes \omega^b = c_{mn} \omega^p_m \otimes \omega^n_p$$

with

$$c_{mn} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

Furthermore, we have made use of the relation provided by the equation of motion:

$$\left( D^i F^{(bc)}_{i0} + m^2 A^{(bc)}_0 \right) T_{bc} = 0$$

i.e.,

$$\left( \partial^i F^{(bc)}_{i0} + m^2 A^{(bc)}_0 \right) + \left[ A^i, F^{(bc)}_{i0} \right] = 0,$$

where $\langle \rangle_{(bc)}$ means the $(bc)$-component projection. Using the expansion of $A = A^{(bc)}_\mu \omega^\mu T_{bc}$ in the above equation, multiplying by $D^J M_N$ and integrating over $S^3$ we get after integration by parts (cf. ref. [36]):

$$\alpha^{bcM}_{JN} + \frac{(-1)^{2J}}{\sigma^2 a} \sqrt{\frac{J(J+1)16\pi^2}{2J+1}} \frac{2N\pi\alpha}{m^2} \pi_{j\j'N'M}$$

$$+ \int_{s^3} d^3x \sqrt{s^3} g \left[ A^i, F^{(bc)}_{i0} \right] \cdot D^J_{M N} = 0 .$$

From the last term of the last equation we get integrals of the type $S_1, S_2, S_3, S_4$; the first two terms are valid only for the Abelian case (where the $(ab)$-SO(3) group indices have been obviously disregarded).