Statistical tools and precision of searching for the Higgs boson at the LHC

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Abstract
Statistical tools have been developed to quantify the agreement between a hypothesis or a model and the observed data for probabilistic phenomena. In this work, we define statistical methods used to measure the properties of the higgs boson in the LHC on the one hand, but also to evaluate the uncertainty and precision on these measurements. They are based on the definition of a likelihood function whose construction will be detailed below. The challenge of HEP is to generate tons of data and to develop powerful analyses to tell if the data indeed contains evidence for the new particle, and confirm it's the expected Higgs boson.

Keywords: Higgs boson, High energy physics (HEP), LHC, hypothesis tests, Likelihood function

Introduction
At the present state of knowledge, the physics of high energies is based on the probabilistic laws and fundamental principles of quantum mechanics. The operating principle of the measuring instruments constituted by the detectors is partly based on stochastic mechanisms, such as the development of the electromagnetic shower for a calorimeter for sampling. Their limited performance, the existence of non-instrumented or non-exploitable regions, noise, stacking, etc., lead to a partial knowledge of the phenomena resulting from the collision. Finally, many stochastic physical phenomena disturb the measurement, such as non-collision background noise: secondary interactions of beams, interactions of particles with residual molecules in the beam tube, exc. the eventual energy of the cosmic rays. For these reasons, statistical tools are used for the exploitation of data and the quantification of the results in terms of discovery or exclusion limit, for the study of a particular channel or of the combination of different channels. Various types of statistical questions may arise depending on the data: hypothesis tests, parameter estimation or confidence intervals, etc. Two main schools of statistical approach exist, giving comparable results in the case of a sample of high size:

- The classic, frequentist approach restricts itself to making conclusions in the form of agreement between the experimental data and a hypothesis that can mimic the phenomenon sought. Probability in this context, only associated with data, is the frequency of a result within the limit of an infinite number of observations.
- The Bayesian approach, establishing the degree of belief in the existence of a certain law of nature, from the observed data. The Bayes theorem:
Establishes the link between the probabilities of a hypothesis by knowing the distribution data, and that of data knowing the hypothesis. The Bayesian approach requires prior knowledge of the probability of the hypothesis. He does not exist of precise rule for the choice of the Prior Probability Functions (Hyp).

1. Hypothesis test

To determine the degree of compatibility of the data with a physics model, the first step is to establish a so-called statistical test if the hypothesis This model is contradictory to the model, the so-called null hypothesis, and is rejected by the experimental results. Thus, in the particular case of the search for a physics signal, in order to establish the existence of a signal, the null hypothesis of absence of signal therefore the hypothesis only background noise (respectively signal plus background noise) is tested in terms of rejection. The alternative hypothesis is the signal hypothesis plus background noise (respectively background noise). The hypothesis to be tested is characterized by the so-called signal strength parameter \( \mu \), corresponding to the signal fraction compared to the Standard Model. For example, the values \( \mu = 0 \) and \( \mu = 1 \) correspond respectively to the hypothesis of background noise alone and signal plus background noise of the Standard Model. The advantage of a signal strength parameter is that it can be factored in the presence of several categories or channels, characterized by a different selectivity and hence a number of different signals, but by a single signal strength. From the statistical test, a function is constructed to establish likelihood (likelihood) of the data with respect to the prediction, large if the modeling of the law that the data follow is correct and in the absence of significant fluctuation thereof. The data can be real: observed data, or virtual data, i.e. simulated pseudo-experiments, so-called expected data, to quantify the sensitivity of the data. expected from the analysis, therefore, to be able to moderate the conclusion of a measure that materially avoids it.

The prediction of the signal to be discovered, unknown by definition, can only be done by simulation. However, in rare cases, a signal already known, similar to the one not discovered and extracted from the data, can be used to extrapolate a knowledge of the behavior of the signal distribution to certain cuts. The prediction of the background noise can be obtained by a method derived from the data, either directly by an adjustment, which can exclude the region of the signal, when the signal is sufficiently spiky so that the background noise appears as a continuum, either by one or more regions of control by background noise component, extrapolated then in the region of the signal, or finally by simulation. In the absence of an adjustment, the function ‘establishing the probability of measuring \( k \) events from the data in a signal region (SR for Signal Region) or control (CR for Control Region) when it is predicted of \( \lambda \) is the Poisson’s law:

\[
(\lambda, k) = \frac{\lambda^k}{k!} e^{-\lambda}
\]

Once the Parameter Of Interest (POI) parameter has been determined, typically the number of signal events for testing a given mass of Higgs, expressed in the form of a signal strength, all other observables that are adjustable for the model, affecting the quality of the measurement,
such as the parameters of the signal model and the background noise, but also the number of background noise events, are grouped in a vector of so-called nuisance parameters \( \tilde{\theta} \). These parameters, corresponding to systemic uncertainties, are evaluated by auxiliary measures. The formulation of the likelihood function \( L(\mu, \tilde{\theta}) \) for a subchannel or a category of a channel, called profiled due to the dependence with \( \mu \), depends on the method and the parametric choice of distributions of the final discriminant variables.

For a cut-and-count method, there is no form parameter: the expected background noise \( b \) plays the role of the nuisance parameter \( \tilde{\theta} \). The likelihood function of observing \( n \) events in a window of the final discriminant variable, typically the invariant mass, when the expected background noise and signal are respectively \( b \) and \( s \) and the signal strength \( \mu \), is written from the Poisson's law:

\[
L(\mu, \tilde{\theta}) = e^{-(\mu s + b)} \frac{(\mu s + b)^n}{n!}
\]

For a method using the form of one or more final discriminant variables, if the representation is made by measurement interval (bins), the exact value of the variable is not known for each event. Are known for each value \( i \) of the \( N \) measurement intervals, the numbers of events \( n_i \), measured, background noise \( b_i \) and signal \( s_i \), if expected, the latter two quantities depending on the form final discriminant variables \( \langle DV \rangle \):

\[
s_i = s_{tot} \int_{bin_i} f_s(\langle DV \rangle, \tilde{\theta}_s) d\langle DV \rangle, \quad b_i = b_{tot} \int_{bin_i} f_b(\langle DV \rangle, \tilde{\theta}_b) d\langle DV \rangle
\]

The likelihood function is written:

\[
L(\mu, \tilde{\theta}) = \prod_{i=1}^{N} \frac{e^{-(\mu_i s_i + b_i)} (\mu_i s_i + b_i)^{n_i}}{n_i!}
\]

For a method using a set of discriminant variables with a representation without measurement interval, that is to say parametrically, the probability densities \( f_s \) of the signal and \( f_b \) of background noise are introduced and weighted by their respective weights for each value of the set of discriminant variables, the signal \( s = s(\tilde{\theta}) \) and the background noise \( b = b(\tilde{\theta}) \) depending on nuisance parameters. An extension by a Poisson term takes into account the probability of obtaining \( n \) events when \( \mu s + b \) is expected. The likelihood function, called extended, is written as follows:

\[
L(\mu, \tilde{\theta}) = e^{-(\mu s + b)} \frac{(\mu s + b)^n}{n!} \prod_{i=1}^{n} \mu \frac{s}{\mu s + b} f_s(\langle DV_i \rangle, \tilde{\theta}) + \frac{b}{\mu s + b} f_b(\langle DV_i \rangle, \tilde{\theta})
\]

which can be written in a simplified way taking into account the term \( (\mu s + b)^n \) and \( \prod_{i=1}^{n} \frac{1}{\mu s + b} \).
The interval method has the advantage of allowing to directly use histogram models, is faster in terms of calculation cost since the number of intervals is typically less than the number of events. However, this method can be sensitive to the statistical fluctuations of the models of discriminant variables. The parametric method requires an analytical adjustment of the discriminant variables, which can sometimes be difficult for a complex form, and requires an error of the model. However, the adjustment makes it possible to suppress the effects of statistical fluctuations in a natural way.

The generalization of likelihood to multiple channels, categories, discriminating variables or even experiences (the combination of ATLAS and CMS) is described, assuming a model without a correlation, as the product of individual likelihoods:

$$L(\mu, \vartheta) = \prod_{i=1}^{N} L_i(\mu, \vartheta_i)$$

When the background noise is estimated by a control region, a multiplicative term appears in the likelihood function, adding information that takes into account a possible agreement between expected and measured background noise. For example, for one case per measurement interval, this term is of the form:

$$\prod_{l=1}^{\text{bins}} e^{-b_{l}^{\text{CR}}(\vartheta)} \frac{b_{l}^{\text{CR}}(\vartheta)^{N_{l}^{\text{CR}}}}{N_{l}^{\text{CR}}}$$

The extrapolation of the background noise measured in the region of control to the signal region is done by a corrective coefficient ($b_{l}^{\text{SR}} = \alpha b_{l}^{\text{CR}}$) that can be determined independently, by example by simulation. The signal and background noise terms are typically respectively

$$\mu s(\vartheta) = \mu \sigma \text{BRLE} \left( \prod_{\text{syst } k} (K_k(\vartheta_k)) \right) + \sigma_{\text{spurious}}$$

And

$$b(\vartheta) = b \prod_{\text{syst } k} (K_k(\vartheta_k))$$

The predefined number of signals introduces the signal strength $\mu$, the cross section $\sigma$, the branching ratio BR of the considered channel, the luminosity $L$ and the effectiveness $\varepsilon$ of the selection. At the statistical processing level, systemic uncertainties can be classified into two groups: those relating to the number of events and those relating to the form of distributions of the discriminant variables used. In the statistical processing described later, a problem would appear if the nuisance parameters of the signal number system uncertainties were adjusted since the fixed parameter $\mu$ of the report numerator likelihoods $L(\mu, \vartheta(\mu))$ would be artificially adjusted through nuisance parameters, making the ratio equal to 1, which would lose
the discriminating power of the statistical test. One solution is to introduce a probability density for the nuisance parameters, called the constraint (or penalty) function $K_k(\theta_k)$. For example, if an auxiliary measure determines a quantity $X_k$ and its systemic uncertainty $\sigma_k$, the systemic uncertainty is taken into account in statistical processing by replacing $X_k$ by typically $X_k(1+\sigma_k\theta_k)$. Where $\theta_k$ is the nuisance parameter of systemic uncertainty and follows an arbitrary random law motivated by physics principles. For example, for a Gaussian constraint function, it would be of the form $\frac{1}{\sqrt{2\pi}}e^{-\frac{(\theta_k-X_k)^2}{2}}$. The nuisance parameters adjusted directly to the data do not introduce a penalty function.

Different constraint functions can be used (Fig.1)

![Figure 1 - Examples of Probability Distributions for the Nuisance Parameter.](image)

A Gaussian has the advantage of being able to describe positive or negative uncertainties. On the other hand, for positive definite observables, such as cross sections, efficiency, luminosity, an alternative to the 0 truncation of the Gaussian is to use for example a log-normal pdf:

$$f(\theta, \theta_0) = \frac{1}{\theta \sqrt{2\pi \ln \kappa}} e^{-\frac{(\ln(\theta/\theta_0))^2}{2}}$$

Due to a poor modeling of the background noise, a discrepancy between the background noise model and the data may appear as a false signal, called spurious signal (Fig.2). Following the sign of this spurious signal, it gives rise to different consequences at the level of discovery or exclusion. Unlike other nuisance parameters, the spurious signal is not a factor that corrects the signal strength. The spurious term is obtained for example by a Monte-Carlo study integrating into a realism window of the signal, for the final discriminant variable (for example the invariant mass but not necessarily), the difference between the simulated data and the noise model background.
The uncertainty on the normalization of background noise is expressed as the quadratic sum of the statistical term \( B \) and the spurious term \( n_{\text{spurious}} \). From the formula of significance for a counting method, it comes

\[
S \xrightarrow{L \to \infty} S, \quad \sqrt{B + n_{\text{spurious}}^2} \to n_{\text{spurious}}
\]

that is the spurious signal generates a limit that can not be exceeded for significance.

In order to be able to measure distinct couplings, for example the \( \text{gg} \to \text{H} \) processes, VBF, WH, ZH, ttH In the context of Higgs research, it is possible to separate the contributions of production processes, distinguishing or not the possible slight variations in probability densities:

\[
\mu s(\hat{\theta}) = \mu \sum_i \sigma_i BRL E_i \left\{ \prod_{k \text{ syst}} (K_k(\hat{\theta}_k)) \right\} + \sigma_{\text{spurious}} \theta_{\text{spurious}}
\]

The statistical test \( q \) (Fig.3) is constructed on the basis of likelihood functions, traditionally so that large values correspond to an interpretation that is unfavorable to the null hypothesis tested.
An estimator of the confidence level of the hypothesis is the p-value, representing the probability of obtaining, with the probability density \(s + b\) or \(b\), a value of the statistical test at less as inconsistent with the null hypothesis as the observed or expected result. The observed (respectively expected) p-value is calculated as the integer of the probability density \(f\), for the \(s + b\) or \(b\) hypothesis, of the statistical test \(q\) of the null hypothesis, noted \(q_{\mu}\) for the exclusion search and \(q_{0}\) for the observation search, where a bound corresponds to the observed value \(q_{\text{obs}}\) (respectively the expected value median \(q_{\text{med}}\) of the alternative hypothesis) of the statistical test and the other bound corresponds to the extreme value of the statistical test of the alternative hypothesis with respect to the tested probability density, which corresponds, for the signal plus background noise hypothesis and the exclusion search, or for the background noise hypothesis alone and the observation search, or for the background noise hypothesis alone and the observation search:

\[
\begin{align*}
    p_{\text{exclusion}}^{\mu_{\text{obs}}, s+b} &= \int_{q_{\text{obs}}}^{\infty} f(q_{\mu} | \mu) dq_{\mu} \\
    p_{\text{exclusion}}^{\mu_{\text{exp}}, s+b} &= \int_{q_{\text{exp}}}^{\infty} f(q_{\mu} | \mu) dq_{\mu} \\
    P_{\text{obs}}^{\text{med}} &= \int_{q_{\text{obs}}}^{\infty} f(q_{0} | 0) dq_{0} \\
    P_{\text{exp}}^{\text{med}} &= \int_{q_{\text{exp}}}^{\infty} f(q_{0} | 0) dq_{0} \\
    p_{\text{exp}}^{\text{med}} &= \int_{q_{\text{med}}}^{\infty} f(q_{0} | 0) dq_{0} \\
    P_{\text{exp}} &= \int_{q_{\text{exp}}}^{\infty} f(q_{0} | 0) dq_{0}
\end{align*}
\]

In the case of the expected p-value, the integers are also calculated with the median value of the statistical test varied by a percentage corresponding to that of the variation by a fixed number of standard deviations \(\sigma\) of a Gaussian: \(d_{\mu, \text{med}} \pm 1\sigma\), \(d_{\mu, \text{med}} \pm 2\sigma\), in order to establish the uncertainty bands \(\pm 1\sigma\) and \(\pm 2\sigma\). A low p-value corresponds to a value of the statistical test which is not compatible with the null hypothesis. For the hypothesis null signal plus background noise and the prescription CLs + b, if the result is \(p_{\mu, s+b} < 1-x\), the hypothesis of \(\mu\) considered is excluded at 100 \(\times\) \(x\)% confidence level, notex CL (CL for Confidence Level). For example, for the traditional choice \(x = 0, 95\), if \(p_{\mu} < 0.05\), the value of \(\mu\) considered is excluded at 95% CL. For the background noise hypothesis alone (respectively signal plus background noise), the convention
is that the weak (respectively large) confidence level values are favorable to the hypothesis. Thus, in the exclusion search frame and for the $\text{CL}_{s+b}$ prescription, the confidence level for the signal plus background noise hypothesis is $8 \text{CL}_{s+b} = p_{s+b}$. The confidence level $\text{CL}$ at which the $\mu$ hypothesis is excluded is $1 - p_{s+b}$. In the case of observation / discovery search, the confidence level for background noise is $\text{CL}_b = 1 - p_b$.

2. **Construction of the statistical test**

According to the Neyman-Pearson theorem, in the absence of systematic uncertainties, the most discriminating statistical test is the likelihood ratio $\frac{L(s+b)}{L(b)}$. A monotonic transformation is equivalent, optimal, in particular $\frac{L(s+b)}{L(s+b)+L(b)}$, which has the advantage of being bounded between 0 and 1. At the LHC, a variant is used, benefiting from asymptotic properties explained later: the statistical test uses a variant presented later by a so-called likelihood ratio

$$\lambda(\mu) = \frac{L(\hat{\mu}, \hat{\theta}(\mu))}{L(\hat{\mu}, \hat{\theta}(\mu))}$$

$$L(\hat{\mu}, \hat{\theta}(\mu)) \geq L(\hat{\mu}, \hat{\theta}(\mu))$$

bound by values 0 and 1. The term $L(\mu, \hat{\theta}(\mu))$ represents the conditional likelihood of a sample of real or predicted data under the fixed $\mu$ hypothesis, for which the nuisance parameters screw of a model are adjusted $\hat{\theta}(\mu)$. The term $L(\hat{\mu}, \hat{\theta}(\mu))$ represents the unconditional likelihood for which both the parameters $\mu$ and nuisance are adjusted $\hat{\mu}$ and $\hat{\theta}(\mu)$. By construction, the likelihood is higher when the two parameters are adjusted: $L(\hat{\mu}, \hat{\theta}(\mu)) \geq L(\hat{\mu}, \hat{\theta}(\mu))$.

Different statistical test variants [1] exist for the exclusion search. In all cases, including for an observation search, from a likelihood ratio $\lambda$, it is convenient to introduce a statistical test such as $q = -2 \ln \lambda$ whose logarithm converts the products in sum, which simplifies the analytical calculation, but also the algorithmic cost of the operations.

For an exclusion search (the null hypothesis is such that $\mu \neq 0$), the simplest likelihood ratio is

$$\lambda(\mu) = \frac{L(\mu, \hat{\theta}(\mu))}{L(\hat{\mu}, \hat{\theta}(\mu))}$$

and the associated statistical test $t_\mu = -2 n \lambda(\mu)$. 

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A lower stress on the adjusted value \( 0 \leq \hat{\mu} \) is imposed by a physical criterion that the signal must be defined positive, which gives the likelihood ratio \( \tilde{\lambda}(\mu) \):

\[
\tilde{\lambda}(\mu) = \begin{cases} 
L(\mu,\hat{\theta}(\mu)) & \hat{\mu} < 0 \\
L(0,\hat{\theta}(0)) & 0 \leq \hat{\mu} \leq \mu \\
L(\mu,\hat{\theta}(\mu)) & \hat{\mu} > \mu 
\end{cases}
\]

and the associated statistical test \( t_\mu = -2 n \tilde{\lambda}(\mu) \).

In the specific case of exclusion search, an adjusted value \( \mu \) that is too high (Fig.4) corresponds to a configuration favoring a model with more signal than the hypothesis but paradoxically is quantitatively unfavorable to the force signal hypothesis. \( \mu \). In particular, in the event of too much positive local fluctuation of the background noise, there is a risk of erroneously excluding an actually existing signal.

Figure 4 - Distribution of a final discriminant variable (invariant mass) for different adjusted \( \mu \) values.

To prevent this artifact in the specific case of search for exclusion limits, the upper stress \( \hat{\mu} \leq \mu \) is imposed by setting arbitrarily to the maximum value 1 the likelihood ratio (0 for the associated statistical test) in the case where the value \( \mu \) is greater than \( \mu \), which gives, taking into account the likelihood ratio variants \( \lambda(\mu) \) and \( \tilde{\lambda}(\mu) \), the two statistical test variants:
\[
q_\mu = \begin{cases} 
-2 \ln \lambda(\mu) & \text{for } 0 \leq \hat{\mu} \leq \mu \\
-2 \ln \lambda \hat{1} = 0 & \text{for } \hat{\mu} > \mu
\end{cases}
\]

And

\[
\tilde{q}_\mu = \begin{cases} 
-2 \ln \tilde{\lambda}(\mu) & \text{for } 0 \leq \hat{\mu} \leq \mu \\
-2 \ln \hat{1} = 0 & \text{for } \hat{\mu} > \mu
\end{cases}
\]

Different statistical tests for exclusion have been used previously by the other experiments. Table 1 summarizes the different variants.

| Statistic test | LEP | Tevatron | LHC |
|----------------|-----|----------|-----|
| LEP            | $-2 \ln \frac{L(\hat{\mu}, \hat{\theta}(\mu))}{L(\hat{\mu}, \hat{\theta})}$ | $-2 \ln \frac{L(\hat{\mu}, \hat{\theta}(\mu))}{L(\hat{\mu}, \hat{\theta})}$ | $-2 \ln \frac{L(\mu, \hat{\theta}(\mu))}{L(\mu, \hat{\theta})}$ |

Table 1 - Variants of statistical tests used by different experiments for the calculation of limits.

By analogy with the exclusion approach, in the case of observational research, only positive fluctuations in the data are taken into account as a source of disagreement with the background noise hypothesis alone. The statistical test is written then:

\[
q_0 = \begin{cases} 
0 & \text{for } \hat{\mu} < 0 \\
-2 \ln \frac{L(\mu = \hat{\mu}, \hat{\theta}(0))}{L(\hat{\mu}, \hat{\theta})} & \text{for } \hat{\mu} \geq 0
\end{cases}
\]

The stress $\hat{\mu} \geq 0$ gives an accumulation of the statistical test at the value 0 for reasons with negative fluctuations. Equivalently, the p-value can not be greater than 0.5. A variant, used from
2012 is to allow negative fluctuations. The $p_0$ obtained, said "uncapped" [2] allows the negative values of $\mu$, and is written:

$$q_0 = \begin{cases} 
\frac{L(\mu=0, \hat{\theta}(0))}{L(\hat{\mu}, \hat{\theta})} - 2 \ln \frac{L(\mu=0, \hat{\theta}(0))}{L(\hat{\mu}, \hat{\theta})} & \hat{\mu} \geq 0 \\
\frac{L(\mu=0, \hat{\theta}(0))}{L(\hat{\mu}, \hat{\theta})} + 2 \ln \frac{L(\mu=0, \hat{\theta}(0))}{L(\hat{\mu}, \hat{\theta})} & \hat{\mu} < 0 
\end{cases}$$

Another possible statistical test is related to the particular case of disagreement with the Standard Model: $q_1 = -2 \ln \frac{L(\hat{\mu}, \hat{\theta})}{L(1, \hat{\theta}(1))}$.

### 3. Exclusion limits

The concrete implementation of the limit calculation is carried out in different stages. Simulations by pseudo-experiments of background noise alone and signal plus background noise are generated separately according to the probability density expected or measured. Each sample is then adjusted twice with the common property that the nuisance parameters are adjusted in both cases: once by setting $\mu$ to the value of the hypothesis, contributing to the likelihood function $L(\mu, \hat{\theta}(\mu))$, again by adjusting it, contributing to the likelihood function $L(1, \hat{\theta}(1))$. From these two adjustments, the value of the statistical test is obtained. The $p$-value and the confidence level of the signal plus noise hypothesis are then calculated for each value of $\mu$ (Fig. 3.1a) in order to determine the lower limit of $\mu$, in the form a ratio $\mu/1$ excluded for a certain level of confidence, traditionally chosen 95%. The consequence of this definition is that in 5% of the experiments, the value $\mu$ is excluded by mistake. To reduce the error, either the confidence level criterion is reinforced or a value of $\mu$ more loose is considered for the exclusion assertion. The denominator in the expression $\mu/1$ corresponds to the value of the Standard Model. If the observed ratio is one, the main parameter of the model (typically the mass $m_H$ of the Higgs hypothesis) is excluded at the confidence level considered. In the case of the expected level of confidence, the bands of uncertainty $\pm 1\sigma$ and $\pm 2\sigma$ are calculated. The procedure is carried out for each parameter of the model, resulting in the exclusion curve (Fig. 5b), containing the observed limit (solid curve), the expected limit (dotted) and its uncertainty band at $\pm 1\sigma$ and $\pm 2\sigma$. 
Figure 5 - (a) Example of confidence level (CL) of a given prescription as a function of the value of μ. The red line is the 95% confidence level used to calculate the 95% CL limit for each value of the parameter. (b) Example of 95% exclusion limit CL based on a given parameter. The typical parameter is the mass of the wanted signal. The red horizontal line at 1 corresponds to the Standard Model.

The Figure 6 presents an example of an educational example, not using real data, observed and expected limits. The horizontal line corresponds to the limit for which the searched channel is excluded at 95% confidence level, in terms of observation or sensitivity, if the observed or expected limit is below this threshold. If, for a mass point, the observed and expected limits are excluded, the result reinforces the interpretation of the exclusion of the physics channel for this mass. Since such an exclusion is characterized by a certain degree of confidence (95% CL for the nominal choice), the addition of additional data can sometimes no longer exclude the mass considered, for example in the event that statistical fluctuations have been responsible for the result. Such a possibility of no longer excluding a region previously excluded is all the less likely that the expected and observed limits are below the threshold of $1 \times SM$. If an exclusion is observed but not expected, then there is an exclusion observed without real sensitivity, which can be interpreted for example by a negative fluctuation of the background noise. If an exclusion is expected but there is no exclusion observed, the result can be interpreted as a positive fluctuation of the background noise, a poor evaluation of it or systematic uncertainties, or even the existence a signal, the latter can also undergo a positive fluctuation. More generally, a limit observed less (respectively more) exclusive than an expected limit characterizes an excess (respectively a deficit).
Thus, if the background noise is underestimated, the model induces false signal excess which makes the limits less exclusive. If the background noise is overestimated, the model makes it more difficult to observe excess signal, which may incorrectly set exclusion limits. The use of a spurious signal previously calculated to estimate the residual between the actual data and the model makes it possible to reduce such an effect.

The confidence level estimator specifically using CL_{s+b} (not chosen for Figure.5) has limitations since the band at +2σ of the expected statistical test excludes at 95% confidence level the assumption of 0 × SM signal. Indeed, the expected p-value for the signal plus background distribution distribution corresponds by definition to the integral of this distribution from the threshold corresponding to the median value of the background noise assumption, for which CL_b = 0, 5 Similarly, the p-value of the signal plus noise hypothesis for the expected threshold +2σ corresponds for the background noise assumption alone at CL^{exp+2σ}_b = 0, 0228. Since by test construction statistic, dominated at low values by the signal+noise hypothesis and at large values by the background noise assumption, CL_b > CL_{s+b} so CL^{exp+2σ}_b <0,0228 <0,05 so that the Compound Standard Model background noise alone (0 × SM) is excluded for the band at +2σ of the statistical test expected for this confidence level requirement. To overcome this problem but
also to avoid a potential false exclusion due to a negative local fluctuation of the background noise, for which the analysis would have no sensitivity, two methods exist:

- The \( CL_s = CL_{s+b} / CL_b \) method, developed in the LEP [4] and used by the Tevatron for a large part of its analysis, consists in renormalizing the p-value by the confidence level of the noise of background \( CL_b \), that is to say using the effective p-value \( p'_\mu = \frac{p_\mu}{1 - p_b} \). Since \( CL_b \) is a positive quantity less than or equal to 1, \( CL_s \geq CL_{s+b} \), so that the limits obtained by this variant are conservative since it is the quantity \( CL_s \) that is required to be less than 0, 05 to exclude at 95% CL hypothesis;

- The PCL (Power Constraint Limit) [5] method consists, if the observed exclusion limit is less than the band at -1σ of that expected, to replace the limit observed by this threshold.

From the various possible choices of statistical tests and prescription of confidence level, several methods can be used to establish limits according to the choice of the estimator and the prescription of the level of confidence. In the case of the limits observed for the 2011 Moriond analysis of photon pair Higgs, the PCL method was used in place of \( CL \), with the statistical test \( q_\mu \). After, the primary, conservative, retained method for the LHC, is the CLs prescription. In the statistical test described previously related to a likelihood ratio, the low values are consistent with the signal plus background noise hypothesis. Large values are consistent with the background noise hypothesis. On the other hand, if one chooses for the statistical test a count of events, a large (respectively weak) number corresponds to a signal hypothesis plus background noise (respectively background noise), since the search for a signal corresponds to an excess of events compared to that predicted by the background noise alone. Thus, the p-value observed for the signal plus noise hypothesis, in this case of event counting, possesses discrete and non-discrete values.

Continuous, given that the values that are inconsistent with the signal plus background hypothesis are distributed at values lower than the observation, is

\[
p - value = \sum_{k=s+b=0}^{n_{obs}=0} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda}
\]

A peculiarity of an analysis characterized by a weak statistic is that the absence of events passing the cuts of selection allows nevertheless to establish a limit of exclusion, of 3 signal events. Indeed, the p-value for the \( s + b \) hypothesis is the probability under this assumption that the result is at least as inconsistent as the (0 event) observation. Indeed, if no event is observed, \( b = 0 \), then the expected value of \( \lambda \) of the Poisson distribution is \( \lambda = s + b = s \), which gives \( p = 0, 05 = e^{-s} \) therefore \( s = 3 \) event events are excluded at 95% CL. Thus, if the shadow of expected signal events is 3, \( \mu = 1 \) is excluded at 95% CL.

More generally, if the number of signal events expected is \( n_{\text{sig}}^{\text{exp}} \) and no event is observed, then an exclusion limit of 95% CL is of \( \mu = \frac{3}{n_{\text{sig}}^{\text{exp}}} \). The consequence is that the evolution of the limit for a channel without event observation improves, as long as this condition is respected, in a way
that is proportional to the luminosity, therefore faster than by the evolution with the square root of the luminosity.

4. Observation and significance research

If the signal plus background noise hypothesis cannot be rejected by the limit calculation, two reasons are possible: either the amount of data is insufficient, leading to a sensitivity that is too low or the data contains a signal, naturally leading to a failure to reject the signal plus noise hypothesis. The presence of a signal, that is, the deviation of the consistency of the data with the background noise hypothesis, is quantified by a p-value constructed on the statistical test $q_\alpha$. If the background noise is underestimated, the model induces false signal excess in the computation of $p_0$, that is to say quantitatively peaks of low value of $p_0$. If the background noise is overestimated, the model makes it more difficult to observe excess signal in the calculation of $p_0$, that is, quantitatively makes it more difficult to observe peaks local low value of $p_0$.

The significance of the excess is defined as the deviation $Z$, typically expressed in multiples of the standard deviation, for which a reduced normalized Gaussian distribution ($\sigma = 1$) and centered in 0 would have a unilateral area (Fig. 4.1) equal to the value, by introducing the inverse function $\Phi^{-1}$ of the Gaussian distribution function.

$$
\Phi(x) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{x}{\sqrt{2}} \right) \right] : Z = \Phi^{-1}(1 - p)
$$

The table / figure.8 establishes the numerical correspondence between the significance and the p-value. The convention is that a significance of $3\sigma$ corresponds to a proof while $5\sigma$ corresponds to a discovery, equivalent to three chances out of 10 million that the effect is due to a fluctuation of the noise of background. A high threshold for a discovery is useful to account for possible underestimated systemic uncertainties in the analysis, the look-elsewhere effect, defined later,
not determined with certainty since it depends on the field of mass explored, and the scientific cost too high of the announcement of a false discovery.

| \( Z \) | test unilatéral | test bilatéral |
|-------|----------------|----------------|
| 0     | 0, 50          | 1, 00          |
| 1     | 0, 1587        | 0, 3173        |
| 2     | 2, 28 \( 10^{-2} \) | 4, 55 \( 10^{-2} \) |
| 3     | 1, 3 \( 10^{-3} \) | 2, 7 \( 10^{-3} \) |
| 4     | 3, 2 \( 10^{-5} \) | 6, 3 \( 10^{-5} \) |
| 5     | 2, 9 \( 10^{-7} \) | 5, 7 \( 10^{-9} \) |
| 1, 64 | 0, 05          | 0, 10          |
| 1, 96 | 0, 025         | 0, 05          |
| 1, 4  | 0, 116         | 0, 2320        |

Figure 8 - Relationships between p-value and significance. The p-value of 0.05 corresponding to 95% CL is equivalent to a significance of 1.64 \( \sigma \). The exclusion at 95% CL by the CLs prescription corresponds to 1.96 \( \sigma \), since the classical p-value must be divided by \( CL_b = 0.5 \) for the median value.

In the context of the search for a new particle, which appears as an excess compared to the background noise, the unilateral test is used. When the compatibility with a prediction is considered, to take into account the excesses and deficits, the bilateral test is used. In some areas of physics, such as the physics of the CKM matrix, for which the measurement of an angle or a phase can introduce a trigonometric ambiguity (\( \Phi \) ou \( \pi - \Phi \), etc), a combination of Bilateral cases may need to be considered.

A low value of \( p_0 \) expected and observed reinforces the interpretation of the existence of a signal to a certain significance. A low value of \( p_0 \) expected, corresponding to a good sensitivity of the analysis, associated with a great value observed, confirms a non-detection of the results compared to the hypothesis of background noise alone. Great value The expected value of \( p_0 \) is low, but a low observed value reinforces the interpretation of an unwanted fluctuation of the background noise or the existence of a signal. In the asymptotic behavior defined later, the significance can be approximated by \( Z_0 = \sqrt{q_{0,A}} \) and \( Z_\mu = \sqrt{q_{\mu, A}} \). For a counting method, the significance can be approximated as \( \sqrt{2((s+b)\ln(1+s/b)-s)} \) [1] for an observation/discovery search and as \( \sqrt{2s-2b\ln(1+s/b)} \) for an exclusion search. In the presence of systematic uncertainties characterized by an error \( \sigma_b \) the significance for an observation/discovery search is written [6]:

\[
\sqrt{2 \left( (s+b)\ln\left( \frac{(s+b)(b+\sigma_b^2)}{b^2+(s+b)\sigma_b^2} \right) - \frac{b^2}{\sigma_b^2} \ln\left[ 1 + \frac{s\sigma_b^2}{b(b+\sigma_b^2)} \right] \right)}
\]

For a counting method, in the case of an additional approximation using a Gaussian likelihood distribution, the significance can be written as \( s/\sqrt{s+b} \) for an exclusion search, and \( s/\sqrt{b} \) for an observation/discovery search. In the presence of systematic uncertainties characterized by an error \( \sigma_b \) the significance for an observation/discovery search is written [6]: \( s/\sqrt{(s+\sigma_b^2)} \).
A particular property of the objects or topology between the objects makes it possible to remove the background noise considerably, but also contributes to reducing the signal.

One solution is to define subsets of selection, so-called categories, naturally leading to subsets with different S/B purities. The gain in significance by this categorization is only possible if the various purities S/B differ. The strategy of categorization evolves with the state of knowledge of the desired signal: to maximize the significance as long as the discovery is not established, then to favor the study of its properties, for example, couplings, with different selectivity in each category. The importance of a particular category, or more generally of a channel in a combination of different analyzes, can be quantified by a relative weight for each channel $i$ in the context of an exclusion (or observation) search following:

$$\omega_i = \frac{(s_i / \sqrt{s_i + b_i})^2}{\sum (s_i / \sqrt{s_i + b_i})^2} \quad \text{(respectively,} \quad \omega_i = \frac{(s_i / \sqrt{b_i})^2}{\sum (s_i / \sqrt{b_i})^2} \text{)}$$

The weighting of the discriminant variable (Fig.9) makes it possible to bring out the resonance. A weighting variant is introduced by the probability weight of the signal, leading to the concept of sPlot [7].

![Figure.9 - Invariant mass of photons in the H → γγ channel without (a) and with (b) weighting Pn (1 + Si/Bi). [8]](image)

Figure.9 - Invariant mass of photons in the H → γγ channel without (a) and with (b) weighting Pn (1 + Si/Bi). [8]

The significance of an excess for a given mass hypothesis is based on the probability of having a localized fluctuation such that the local p-value at that mass is less than one. certain threshold. However, the probability of having at least one such fluctuation in the whole of the mass domain covered by the analysis is higher, typically higher. Great by the number of ways to achieve this effect. This freedom to look elsewhere, called the look-elsewhere effect (LEE) [9], generates an overall p-value related to local p-value by a corrective factor, called trial factor. In the case of the asymptotic regime explained later, the relation is written according to the value $u$ of the statistical test for the local excess and the number of crossings (Fig.10) to the top $n_0$ of the statistical test $q_0$ with a given threshold $u_0$, or equivalently because of the direct link between the statistical test and the adjusted value of the signal strength, from the local significance $Z$ and the number of crossings to the high $N_0$ of the adjusted signal strength $\mu$ with a given threshold $Z_0$: 
For example, in the framework of the \( H \to \gamma \gamma \) analysis of the Cern Council of December 2011 [10], the local p-value is \( p_{0 \text{ local}}^{\text{local}} = 0.27\% \), corresponding to a significance \( Z = 2.8 \sigma \). Count of the look-elsewhere effect with a number of positive crossings of the signal strength with the line \( \hat{\mu} = 0 \) of 4 in the domain in mass considered (Fig. 4.4b [11]), the global p-value is \( p_{0 \text{ global}}^{\text{global}} = 7\% \), corresponding to a significance \( Z = 1.5 \sigma \), taking into account the fact that the reference values are rounded in the presentation of the results.

\[
P_{0 \text{ global}}^{\text{global}} \approx p_{0 \text{ local}}^{\text{local}} + n_0 e^{-\frac{1}{2}(u-u_0)}
\]

\[
P_{0 \text{ global}}^{\text{global}} \approx p_{0 \text{ local}}^{\text{local}} + N_0 e^{-\frac{1}{2}(u^2-u_0^2)}
\]

In the presence of a signal, the number of crossings may be biased by one unit. A look-elsewhere approximation is obtained from the number of possibilities to have a local effect, introducing the number of bins as the ratio between the mass domain covered by the analysis and the number of bins. Signal resolution:

\[
P_{0 \text{ global}}^{\text{global}} = 1 - (1 - P_{0 \text{ local}}^{\text{local}})^{N_{\text{bins}}} = N_{\text{bins}} \times P_{0 \text{ local}}^{\text{local}}
\]

5. Asymptotic behavior

The need for sufficient precision in statistical studies necessitates the simulation of a large number of experiments, which may not be realistic in view of the computation time. The distribution of the statistical test can be obtained by an asymptotic approximation [1] corresponding to a high number of events. The asymptotic formula for the statistical test \(-2 \ln \lambda(\mu)\) is \( \frac{1}{2} f_X(t) + \frac{1}{2} \delta(t)\), according to Wilks’ theorem [12], while the asymptotic form for the test statistic \( q_{\mu} \) is:
\[ f(\tilde{q}_\mu | \mu) = \frac{1}{2} \delta(\tilde{q}_\mu) + \begin{cases} 
\frac{1}{2^{1/2} \pi^{1/2} \sqrt{q_\mu}} e^{-\tilde{q}_\mu^2 / 2}, & \text{if } 0 < \tilde{q}_\mu < \mu^2 / \sigma^2, \\
\frac{1}{\sqrt{2\pi}(2\mu/\sigma)} \exp \left[ -\frac{1}{2} \frac{(\tilde{q}_\mu + \mu^2 / \sigma^2)^2}{(2\mu/\sigma)^2} \right], & \text{if } \tilde{q}_\mu > \mu^2 / \sigma^2 
\end{cases} \]

with \( \sigma^2 = \frac{\mu^2}{q_{\mu,A}} \), where \( q_{\mu,A} \) is the statistical test for the so-called Asimov sample corresponding to the expected background noise with the nominal nuisance parameters.

The advantage is to reduce the calculation time very significantly. With increasing integrated brightness, the asymptotic calculation is expected to be as accurate as the use of pseudo-experiments that require heavy computations. For a given campaign (ATLAS combination of the Cern Council meeting in December 2011 corresponding to a brightness of 1.0 to 4.9 fb\(^{-1}\) depending on the channel), a comparison of the evaluation of the compatibility of the data with the background noise hypothesis is presented in figure.11 between the asymptotic method and that with the calculation by multiple pseudo-experiments. For each mass, at least 5000 experiments were considered, up to 12000 experiments in the region of the excess (very low for this analysis), showing a good agreement between the two methods.

![Figure.11 - Comparison between the asymptotic method and the method of multiple pseudoexperiments. Source: [11].](image)

6. **Generalization of the statistical test**

So far, the statistical test has been introduced in the context of signal search or exclusion. When a signal has been established, its properties become the parameters of interest. The statistical test of different studies is presented in Table.2.
| POI                                      | \(-2\ln \frac{L(\text{POI}, \hat{\theta})}{L(\text{POI}, \hat{\theta})}\) |
|-----------------------------------------|---------------------------------------------------------------------|
| signal strength \(\mu\)                | \(-2\ln \frac{L(\hat{\mu}, \theta)}{L(\mu, \theta)}\)            |
| cross-section                           | deduced from \(\mu\)                                                |
| couplings \(\bar{\mu}\)                | \(-2\ln \frac{L(\hat{\bar{mu}}, \theta)}{L(\bar{\mu}, \hat{\theta})}\) |
| mass \(m\)                              | \(-2\ln \frac{L(\hat{m}, \mu, \theta)}{L(m, \mu, \theta)}\)       |
| Compatibility of masses \(\Delta m\), with common mass \(m\) | \(-2\ln \frac{L(\Delta m, \mu_1, \mu_2, m, \theta)}{L(\Delta m, \mu_1, \mu_2, m, \hat{\theta})}\) |
| Width \(\Gamma\)                        | \(-2\ln \frac{L(\hat{\Gamma}, \theta)}{L(\Gamma, \hat{\theta})}\)  |
| Spin (e.g.: fraction \(\varepsilon_0\) of spin 0 wrt spin 2) | \(-2\ln \frac{L(\varepsilon_0, \hat{\theta})}{L(\varepsilon_0, \theta)}\) |

Table 2 - Examples of statistical tests for different contexts.

In the compatibility of masses with two distinct channels, two distinct \(\mu\)s are considered to account for each channel. In the case of a single channel, mass compatibility can be explored between categories, in which case, to improve the statistical error, typically a single, inclusive, signal force is considered. ee, which gives rise to the statistical test:

\[-2\ln \prod_{i=1}^{N_{\text{cat}}} L(\Delta_i, \hat{\mu}, \hat{m}, \hat{\theta}) \prod_{i=1}^{N_{\text{cat}}} L(\hat{\Delta}_i, \mu_i, m, \hat{\theta})\]

The \(\mu\) can be integrated into the nuisance parameters.

The compatibility of a category \(j\) with the combination of categories gives rise to the statistical test:

\[-2\ln \frac{\prod_{i \neq j}^{N_{\text{cat}}} L_i(\hat{\mu}, m, \hat{\theta})}{\prod_{i \neq j}^{N_{\text{cat}}} L_i(\hat{\mu}, m, \hat{\theta})} \cdot \frac{L_j(\hat{\Delta}, \mu, m, \hat{\theta})}{L_j(\hat{\Delta}, \mu, m, \hat{\theta})}\]
For the same channel, the compatibility of mass between two years can be written:

\[-2\ln \frac{L_{2011}(\hat{\Delta}_i, \hat{\mu}, \hat{m}, \hat{\theta})}{L_{2011}(\hat{\Delta}_i, \hat{\mu}, \hat{m}, \hat{\theta})} - 2\ln \frac{L_{2012}(\hat{\Delta}_i, \hat{\mu}, \hat{m}, \hat{\theta})}{L_{2012}(\hat{\Delta}_i, \hat{\mu}, \hat{m}, \hat{\theta})}\]

7. IT environment

The Roostats library [13] provides the main tools for the statistical analysis of LHC data, through C++ classes and interfaces for use on arbitrary models and data. This library is built as a layer superior to the RooFit [14] and Root [15] libraries. One of the classes of RooStats is the workspace, defining all the necessary parameters for the statistical procedure: probability densities of the discriminant variables, list of variables and categories, parameters of interest \((\mu, m_H, \text{etc.})\), nuisance (slopes, system uncertainties, etc.), real or virtual data sets (pseudo-data). The advantage of the workspace is to facilitate the exchange of information for statistical processing: reproducibility of the results, but also to facilitate the combination of the channels, within the same experiment, and between experiences.

Conclusion:

The quantitative evaluation of physics hypotheses and the measurement of their parameters is based on a rigorous statistical formalism, of which a software interface is the library Roostats. The search on a host of other types of New Physics has not yet resulted in any discovery claim and maximum limits have been set for production rates of possible particles or relevant effects. This has often resulted in the exclusion of various physics models beyond Standard Model (SM) in some regions of their parameter space. Perhaps, over time, some of these models will be abandoned. To date, CMS collaboration has produced about 200 articles on the maximum limits of various processes. We conclude with a list of useful resources to understand and/or implement statistical techniques for research.
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