On Solving A Generalized Chinese Remainder Theorem in the Presence of Remainder Errors

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Abstract

In estimating frequencies given that the signal waveforms are undersampled multiple times, Xia and his collaborators proposed to use a generalized version of Chinese remainder Theorem (CRT), where the moduli are $d_{m_1}, d_{m_2}, \ldots, d_{m_k}$ with $m_1, m_2, \ldots, m_k$ being pairwise coprime. If the errors of the corrupted remainders are within $\frac{d}{4}$, their schemes are able to construct an approximation of the solution to the generalized CRT with an error smaller than $\frac{d}{4}$. One of the critical ingredients in their approach is the clever idea of accurately finding the quotients. In this paper, we present two treatments of this problem. The first treatment follows the route of Wang and Xia to find the quotients, but with a simplified process. The second treatment takes a different approach by working on the corrupted remainders directly. This approach also reveals some useful information about the remainders by inspecting extreme values of the erroneous remainders modulo $d$. Both of our treatments produce efficient algorithms with essentially optimal performance. This paper also provides a proof of the sharpness of the error bound $\frac{d}{4}$.

Keywords: Chinese Remainder Theorem, remainder errors, reconstruction, sharp bound

1 Introduction

The usual Chinese Remainder Theorem (CRT) concerns reconstructing an integer given its remainders with respect to a set of coprime moduli. More precisely, let $M = \{m_1, m_2, \ldots, m_k\}$ be a set of pairwise coprime positive integers and $M = \prod_{i=1}^{k} m_i$. For a set of integers $r_1, r_2, \ldots, r_k$ with

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0 \leq r_i < m_i$, the Chinese Remainder Theorem says that the system of congruences
\[
\begin{aligned}
& x \equiv r_1 \pmod{m_1} \\
& x \equiv r_2 \pmod{m_2} \\
& \vdots \\
& x \equiv r_k \pmod{m_k}
\end{aligned}
\]
has a unique solution $0 \leq x < M$. In fact, using the extended Euclidean algorithm, one finds integers $u_1, u_2, \ldots, u_k$ such that
\[
\sum_{i=1}^{k} u_i \frac{M}{m_i} = 1.
\]
This implies that for any integer $N$, we have $N = \sum_{i=1}^{k} N u_i \frac{M}{m_i}$. In particular, if $N \equiv r_i \pmod{m_i}$ for $i = 1, 2, \ldots, k$, then
\[
N = \sum_{i=1}^{k} r_i u_i \frac{M}{m_i} \pmod{M}
\]
i.e., the right hand side gives us the desired solution. For a greater efficiency, one can also use a randomized algorithm by Cooperman, Feisel, von zur Gathen, and Havasin in [1] to get the coefficients $\{u_i : i = 1, \ldots, k\}$ of (2), see [2] for a concrete description.

The Chinese Remainder Theorem is a classical tool that is of both theoretical and practical interests. A solution of the number theoretic version of CRT was given in [5] under the name of “method of DaiYan aggregate”. The key subroutine in this method is the “method of deriving 1 through DaiYan” (which is exactly the method of finding modulo inverse). We would like to remark that the CRT discussed in [5] is actually in a general form, namely, the moduli are not assumed to be pairwise coprime (and the book did describe a way of reducing the general case to the case of pairwise coprime moduli), see also [4, 7]. A form that is similar to (2) was discussed in [5] as well. It is also interesting to note that the proof of the ring theoretic version of CRT (which contains the finite discrete Fourier transform as a special case) uses the similar idea as that from [5].

One of the natural applications of CRT is in parallel computing, as CRT can be interpreted as an isomorphism between a ring (of bigger size) and a direct product of rings (of smaller sizes). Since arithmetic operations on those smaller rings are closed and independent, an operation in the bigger ring may be decomposed into operations in those smaller rings in parallel [2, 11]. Applications of CRT in fast computation can be found in cryptography [6], e.g., fast decryption in RSA, solving the discrete logarithm with a composite modulus, and the fast point counting method for elliptic curves. In a series of work by Xia and his collaborators, a generalized CRT (the so called Robust CRT) has been used to solve problems of estimating frequencies given that the signal waveforms are undersampled multiple times, [3, 8, 9]. For example, in [8], Wang and Xia discussed the following form of CRT, which is to solve the system of congruences
\[
\begin{aligned}
& x \equiv r_1 \pmod{dm_1} \\
& x \equiv r_2 \pmod{dm_2} \\
& \vdots \\
& x \equiv r_k \pmod{dm_k}
\end{aligned}
\]
given that the remainders are corrupted. They design a clever scheme to solve the problem by first
finding the quotients \( q_i = \frac{x - r_i}{m_i} \). They proved that if the remainder errors are bounded by \( \frac{d}{4} \), the
quotients can be computed exactly.

This paper studies the same problem by giving alternative solutions and more efficient computa-
tional procedures. We shall present two treatments of solving this problem. The first treatment
follows the route of Wang and Xia to find the quotients, but with a simplified process. The second
treatment takes a different approach by working on the corrupted remainders directly. This ap-
proach also reveals some useful information about the remainders by inspecting extreme values of
the erroneous remainders modulo \( d \). The technical difficulties mentioned in [8] have been suc-
cessfully avoided by the method proposed in our second treatment. Both of our treatments produce
efficient algorithms that invoke CRT only once, and they are essentially optimal in this sense. It is
a natural question to ask whether the remainder error bound \( \frac{d}{4} \) is sharp. We provide an affirmative
answer to this question by constructing a simple counterexample.

Recently, Xiao, Xia and Wang [10] designed a robust CRT that accommodates more general
situation where the great common divisors of the pairs of moduli are not necessarily the same. We
remark that our first treatment can handle this case as well. We note that the multi-stage use of
robust CRT proposed in [10] provides an interesting way of lifting some restriction on error bound.

The rest of our discussion will be divided into two sections. Section 2 describes the problem
setup and provides solutions. Section 3 is the conclusion.

2 Problem Setup and Solutions

In this section, we discuss solving the problem of following generalized CRT

\[
\begin{aligned}
    x &\equiv r_1 \pmod{dm_1} \\
    x &\equiv r_2 \pmod{dm_2} \\
    \vdots \\
    x &\equiv r_k \pmod{dm_k}
\end{aligned}
\]  

(4)

in the presence of remainder errors.

It is clear that when \( d > 1 \), the existence of a solution \( 0 \leq x < dM \) must impose some condition
on the remainders. Such a condition is derived by the following simple argument, see, [8].

**Proposition 1** The system (4) has a unique solution \( 0 \leq N < dM \) if and only if

\[
r_i - r_j \equiv 0 \pmod{d} \text{ for all } i, j.
\]  

(5)

1The author would like to thank Professor Xia for informing him of this new development.
Furthermore, if (5) holds, then the solution is

\[ N = dN_0 + a \]

where \( a = r_1 \pmod{d} \) and \( 0 \leq N_0 < M \) is the solution of the CRT system

\[
\begin{cases}
  y \equiv \gamma_1 \pmod{m_1} \\
  y \equiv \gamma_2 \pmod{m_2} \\
  \vdots \\
  y \equiv \gamma_k \pmod{m_k}
\end{cases}
\]

(6)

where \( \gamma_i = \frac{r_i - a}{d} \), \( i = 1, 2, \ldots, k \).

If (5) holds, then the solution of (4) can also be written as

\[ N = \sum_{i=1}^{k} r_i u_i \frac{M}{m_i} \pmod{dM}, \]

where \( \sum_{i=1}^{k} u_i \frac{M}{m_i} = 1 \) is the relation (2). However, the solving method in the proposition is of certain interest because it reveals the fact that the parameter \( d \) introduces redundancy. Small changes of \( r_i \)'s will not affect the integral part of \( \frac{r_i - a}{d} \) for most cases. It is noted that \( a \) is another important parameter, and prior knowledge or estimation of this number may be useful in getting a better approximation of \( N \). We shall call \( a = r_1 \pmod{d} \) the common remainder modulo \( d \).

Given a corrupt set of remainders \( r_1, \ldots, r_k \) with errors \( \Delta r_i = r_i - r_i \), our task is to find a good approximation \( \overline{N} \) to the solution \( N \) of (4). We shall assume \( |\Delta r_i| < \frac{d}{4} \) for all \( i = 1, 2, \ldots, k \).

### 2.1 Reconstruction by finding the quotients

A nice observation by Wang and Xia [8] is that if \( |\Delta r_i| < \frac{d}{4} \) for all \( i = 1, \ldots, k \), then one is able to reconstruct an approximation \( \overline{N} \) of the solution \( N \) such that

\[ |N - \overline{N}| < \frac{d}{4}. \]

The interesting strategy used in [8] is to consider the quotients \( q_i = \left\lfloor \frac{N}{m_i} \right\rfloor = \frac{N - r_i}{m_i} \) and prove that they are invariants when the remainder errors are bounded by \( \frac{d}{4} \). The procedure in [8] is summarized as

\[ \text{Throughout this paper, the expression } g = h \pmod{m} \text{ means that } g \text{ is the least nonnegative remainder of } h \text{ modulo } m, \text{i.e., } m|(g - h) \text{ and } 0 \leq g < m. \]

\[ \text{We assume that each of these observed remainder satisfies } 0 \leq \overline{r_i} < dm_i. \text{ Otherwise, errors can be reduced by simply setting } \overline{r_i} = dm_i - 1 \text{ (or } \overline{r_i} = 0) \text{ if } \overline{r_i} \geq dm_i \text{ (or } \overline{r_i} < 0). \]
Table 1: Algorithm of [8]

| Step 1. | For each $i = 2, \ldots, k$
| | find $\Gamma_{i,1} = m_i^{-1} \mod m_1$;
| | compute $\xi_{i,1} = \left[\frac{r_i - r_1}{d}\right] \mod m_i$.
| Step 2. | Use CRT to find the solution $q_1$:
| | $q_1 \equiv \xi_{2,1} \mod m_2$
| | $\ldots$
| | $q_1 \equiv \xi_{k,1} \mod m_k$
| Step 3. | For each $i = 2, \ldots, k$
| | $q_i = \frac{q_1 m_i - r_i}{m_i}$.
| Step 4. | For each $i = 1, 2, \ldots, k$
| | compute $N(i) = dm_i + \bar{r}_i$;
| | Compute $N = \left[\frac{1}{k} \sum_{i=1}^k N(i)\right]$.

Here $[x]$ denotes the rounding integer, i.e., $x - \frac{1}{2} \leq [x] < x + \frac{1}{2}$.

**Remark.** Under the conditions $|\Delta r_i| < \frac{d}{4}$, one has $\left[\frac{r_i}{d}\right] = \frac{r_i - r_1}{d}$ and hence $q_i = q_i$. It is also remarked that the computation in step 1 is equivalent to an invocation of CRT.

The main purpose of this subsection is to present a concise approach to this problem. For each $1 \leq j \leq k$, recall that the quotient $q_j$ is $q_j = \frac{N - r_j}{dm_j}$. Denote $M_j = \frac{M}{m_j}$. From (2) (i.e., $\sum_{i=1}^k u_i \frac{M}{m_i} = 1$), we have

$$q_j = q_j u_1 \frac{M}{m_1} + \ldots + q_j u_k \frac{M}{m_k} \equiv \sum_{i=1}^k \frac{r_i - r_j}{d} u_i \frac{M_j}{m_i m_j} \mod M$$

$$= \sum_{i=1}^k \frac{r_i - r_j}{d} u_i \frac{M_j}{m_i} \mod M_j. \quad (7)$$

Note that $q_j$ can be uniquely specified because $0 \leq q_j < M_j$.

Since $\left[\frac{r_i - r_j}{d}\right] = \frac{r_i - r_1}{d}$ for $i = 2, \ldots, k$, the following is true by (7)

$$q_1 = \bar{q}_1 = \sum_{i=2}^k \left[\frac{r_i - r_1}{d}\right] u_i \frac{M_j}{m_i} \mod M_1.$$ 

Therefore, we get the following procedure by replacing steps 1 and 2 of the algorithm in [8] with the above discussion.
Table 2: Algorithm 1

| Step 1 | Use the extended Euclidean algorithm to get $u_i$ such that $\sum_{i=1}^{k} u_i \frac{M}{m_i} = 1$. |
|--------|-------------------------------------------------------------------------------------------------|
| Step 2 | Compute $q_1$: $q_1 = \sum_{i=2}^{k} \left( \frac{\tau_i - \tau_1}{d} \right) u_i \frac{M_1}{m_i} \pmod{M_1}$ |
| Step 3 | For each $i = 2, \cdots, k$, $q_i = \frac{\tau_1 m_i - \left( \tau_i - \tau_1 \right)}{d}$ |
| Step 4 | For each $i = 1, 2, \cdots, k$, compute $N(i) = dm_i + \tau_i$; Compute $N = \left( \sum_{i=1}^{k} N(i) \right) / k$. |

**Remark.** 1. Steps 1 and 2 are equivalent to an invocation of CRT. In step 3, $q_i$ may be computed via (7).

**Remark.** 2. Recently Xiao, Xia, and Wang [10] extended the problem to approximate the solution $N$ of the following system:

$$\begin{align*}
   x &\equiv r_1 \pmod{n_1} \\
   x &\equiv r_2 \pmod{n_2} \\
   \cdots \\
   x &\equiv r_k \pmod{n_k}
\end{align*}$$

from erroneous remainders $\tau_1, \cdots, \tau_k$. Where $d_{ij} = \gcd(n_i, n_j)$ can be arbitrary, and the remainder errors are bounded by $\tau = \max_{1 \leq i \leq k} \min_{1 \leq j \leq k, j \neq i} d_{ij}$. Similar to proposition 1, we see that (8) has a solution if and only if $d_{ij} | r_i - r_j$ for all $i, j$. We remark that the procedure we just described can be adopted to this problem without much difficulty. In fact, assuming $\tau = \min_{2 \leq j \leq k} d_{1j}$ without loss of generality, we can recover the quotient $q_1 = \frac{N - r_1}{n_1}$. To this end, we let $M = \text{lcm}\{n_1, \cdots, n_k\}$. Since $\gcd(\frac{M}{n_1}, \cdots, \frac{M}{n_k}) = 1$, we get integers $v_1, \cdots, v_k$ such that $\sum_{i=1}^{k} v_i \frac{M}{n_i} = 1$. This yields $\sum_{i=1}^{k} q_i v_i \frac{M}{n_i} = q_1$. A routine manipulation gives

$$q_1 = \frac{r_2 - r_1}{d_{12}} v_2 \frac{d_{12} M}{n_1 n_2} + \cdots + \frac{r_k - r_1}{d_{1k}} v_k \frac{d_{1k} M}{n_1 n_k} \pmod{\frac{M}{n_1}}.$$

Since $\frac{r_j - r_1}{d_{1j}} = \left[ \frac{\tau_j - \tau_1}{d_{1j}} \right]$ for $j = 2, \cdots, k$, $q_1$ can be computed exactly from the corrupted remainders.
2.2 Extreme values of the erroneous remainders modulo $d$

In this subsection, we will take a different approach to solve the problem. We will use corrupted remainders directly in solving CRT. The main points we would like to make include

1. The extreme values such as $\max\{r_i \pmod{d}\}$ and $\min\{r_i \pmod{d}\}$ should be inspected to reveal useful information about the errors.

2. The common remainder $a$ maybe shifted so that more accurate estimation can be made.

Let $a = r_1 \pmod{d}$ be the common remainder of $r_1, \cdots, r_k$ modulo $d$. We define

$$\alpha = \max\{r_i \pmod{d} : i = 1, 2, \cdots, k\}$$
$$\beta = \min\{r_i \pmod{d} : i = 1, 2, \cdots, k\}$$
$$\mu = \min\{\overline{r}_i \pmod{d} : r_i \pmod{d} > \frac{d}{2}\}$$
$$\nu = \max\{\overline{r}_i \pmod{d} : r_i \pmod{d} < \frac{d}{2}\}$$

The numbers $\mu, \nu$ are defined only when the corresponding sets are nonempty.

Since $|\Delta r_i| < \frac{d}{4}$ for all $i = 1, 2, \cdots, k$, we have the following five mutually exclusive cases:

(a) For all $i = 1, 2, \cdots, k$, $a + \Delta r_i < 0$.
   In this case, $\alpha - \beta < \frac{d}{4}$. We also have $a < \frac{d}{4}$.

(b) For all $i = 1, 2, \cdots, k$, $a + \Delta r_i \geq d$.
   In this case, $\alpha - \beta < \frac{d}{4}$.

(c) For all $i = 1, 2, \cdots, k$, $0 \leq a + \Delta r_i < d$.
   In this case, $\alpha - \beta < \frac{d}{2}$.

(d) There are $i_1$ and $j_1$ such that $a + \Delta r_{i_1} < 0$ and $a + \Delta r_{j_1} \geq d$.
   In this case, we must have $a < \frac{d}{4}$, $\alpha - \beta \geq \mu - \nu > \frac{d}{2}$

(e) There are $i_1$ and $j_1$ such that $a + \Delta r_{i_1} < d$ and $a + \Delta r_{j_1} \geq d$.
   In this case, $\alpha - \beta \geq \mu - \nu > \frac{d}{2}$. We also have $a > \frac{3d}{4}$.

We note that even though these five cases cannot be checked individually (due to the unknown parameters $\Delta r_i$ and $a$), we can still divide them into two verifiable situations. In fact, it is easy to see that the condition $\alpha - \beta < \frac{d}{4}$ covers cases (a), (b) and (c), while the condition $\alpha - \beta > \frac{d}{2}$ covers cases (d) and (e).

We shall first get good approximations of $\gamma_i = \frac{\Delta r_i - a}{d}$ based on these conditions.
Proposition 2  For \( i = 1, 2, \cdots, k \), set
\[
\overline{\gamma}_i = \begin{cases} 
\left\lfloor \frac{\gamma_i d}{d} \right\rfloor & \text{if } \alpha - \beta < \frac{d}{2}, \\
\left\lfloor \frac{\gamma_i + d - \mu}{d} \right\rfloor & \text{if } \alpha - \beta \geq \frac{d}{2}.
\end{cases}
\]

Then we have

1. For case (a), \( \overline{\gamma}_i = \gamma_i - 1 \) holds for all \( i = 1, 2, \cdots, k \).
2. For cases (c) and (d), \( \overline{\gamma}_i = \gamma_i \) holds for all \( i = 1, 2, \cdots, k \).
3. For cases (b) and (e), \( \overline{\gamma}_i = \gamma_i + 1 \) holds for all \( i = 1, 2, \cdots, k \).

Proof. The verification is fairly straightforward. We just consider the cases (b) and (e) here. We recall that \( r_i = \gamma_i d + a \).

For case (b), since \( a + \Delta r_i \geq d \) for all \( i \), so
\[
\overline{\gamma}_i = \left\lfloor \frac{\gamma_i d}{d} \right\rfloor = \left\lfloor \frac{\gamma_i d + a + \Delta r_i}{d} \right\rfloor = \gamma_i + 1.
\]

For case (e), we know that if \( \overline{r}_i \pmod{d} < \frac{d}{2} \), then \( a + \Delta r_i \geq d \). This implies that
\[
d \leq a + \Delta r_i + d - \mu < 2d.
\]
If \( \overline{r}_j \pmod{d} \geq \frac{d}{2} \), then \( a + \Delta r_j < d \) and \( \overline{r}_j \pmod{d} = a + \Delta r_j \). This, together with the minimality of \( \mu \), implies that
\[
d \leq a + \Delta r_j + d - \mu < 2d.
\]
Therefore,
\[
\overline{\gamma}_i = \left\lfloor \frac{\overline{r}_i + d - \mu}{d} \right\rfloor = \left\lfloor \frac{\gamma_i d + a + \Delta r_i + d - \mu}{d} \right\rfloor = \gamma_i + 1.
\]

Now let us present an algorithm to get an approximation \( \overline{N} \) of \( N \), based on the above discussion.

Table 3: Algorithm 2

| Step 1. Compute } \alpha, } \beta and } \mu / / (if } \mu \text{ exists) |
|---|
| Step 2. For each } i = 2, \cdots, k, compute |
| \( \overline{\gamma}_i = \begin{cases} 
\left\lfloor \frac{\gamma_i d}{d} \right\rfloor & \text{if } \alpha - \beta < \frac{d}{2}, \\
\left\lfloor \frac{\gamma_i + d - \mu}{d} \right\rfloor & \text{if } \alpha - \beta \geq \frac{d}{2}.
\end{cases} \) |
| Step 3. Use CRT to find the solution } \overline{N}_0: |
| \( \overline{N}_0 \equiv \overline{\gamma}_1 \pmod{m_1} \) |
| \( \overline{N}_0 \equiv \overline{\gamma}_2 \pmod{m_2} \) |
| ... |
| \( \overline{N}_0 \equiv \overline{\gamma}_k \pmod{m_k} \) |

Continued on next page
Step 4. If \( \alpha - \beta < \frac{d}{2} \)

\[
\overline{N} = d\overline{N}_0 + \left\lceil \sum_{i=1}^{k} (\overline{r}_i \pmod{d}) \right\rceil
\]

Else (i.e., \( \alpha - \beta \geq \frac{d}{2} \))

\[
\overline{N} = d\overline{N}_0.
\]

Finally, we shall prove

**Proposition 3** Let \( N \) be the solution and \( \overline{N} \) be the output of Algorithm 2, we have

\[ |N - \overline{N}| < \frac{d}{4}. \]

**Proof.** Let \( N_0 \) be the solution of (6), then we know that \( N = dN_0 + a \).

For case (a). In this case, \( N_0 = N_0 - 1 \). Since all \( a + \Delta r_i < 0 \), we must have \( \overline{r}_i \pmod{d} = d + a + \Delta r_i \) and \( -\frac{d}{4} < a + \Delta r_i < 0 \). Thus

\[
\overline{N} = d\overline{N}_0 + \left\lceil \frac{\sum_{i=1}^{k} (\overline{r}_i \pmod{d})}{k} \right\rceil = dN_0 - d + \left\lceil \frac{\sum_{i=1}^{k} (d + a + \Delta r_i)}{k} \right\rceil
\]

\[
= dN_0 - d + \left\lceil d + a + \frac{\sum_{i=1}^{k} \Delta r_i}{k} \right\rceil = N + \left\lceil \frac{\sum_{i=1}^{k} \Delta r_i}{k} \right\rceil.
\]

For case (b). In this case, \( N_0 = N_0 + 1 \). Since all \( a + \Delta r_i \geq 0 \), we must have \( \overline{r}_i \pmod{d} = a + \Delta r_i - d \) and \( 0 \leq a + \Delta r_i - d < \frac{d}{4} \). Thus

\[
\overline{N} = d\overline{N}_0 + \left\lceil \frac{\sum_{i=1}^{k} (\overline{r}_i \pmod{d})}{k} \right\rceil = dN_0 + d + \left\lceil \frac{\sum_{i=1}^{k} (a + \Delta r_i - d)}{k} \right\rceil
\]

\[
= dN_0 + d + \left\lceil a - d + \frac{\sum_{i=1}^{k} \Delta r_i}{k} \right\rceil = N + \left\lceil \frac{\sum_{i=1}^{k} \Delta r_i}{k} \right\rceil.
\]

For case (c). In this case, \( N_0 = N_0 \). We know that in this case \( \overline{r}_i \pmod{d} = a + \Delta r_i \), so

\[
\overline{N} = d\overline{N}_0 + \left\lceil \frac{\sum_{i=1}^{k} (\overline{r}_i \pmod{d})}{k} \right\rceil = dN_0 + \left\lceil \frac{\sum_{i=1}^{k} (a + \Delta r_i)}{k} \right\rceil
\]

\[
= dN_0 + \left\lceil a + \frac{\sum_{i=1}^{k} \Delta r_i}{k} \right\rceil = N + \left\lceil \frac{\sum_{i=1}^{k} \Delta r_i}{k} \right\rceil.
\]

So in these three cases, \( |N - \overline{N}| < \frac{d}{4} \) holds.

For case (d). In this case, \( N_0 = N_0 \), and \( \overline{N} = dN_0 \). We know that in this case \( a < \frac{d}{4} \), so

\[ |N - \overline{N}| = a < \frac{d}{4}. \]
For case (e). In this case, \( \overline{N}_0 = N_0 + 1 \), and \( \overline{N} = dN_0 + d \). We know that in this case \( a > \frac{3d}{4} \), so

\[
|N - \overline{N}| = d - a < \frac{d}{4}.
\]

2.3 The sharpness of the condition \( \max_{1 \leq i \leq k} |\Delta r_i| \leq \frac{d}{4} \)

In this subsection, we shall discuss some issues about the error bound. It is easy to see that if we are given \( a = \left\lfloor \frac{d}{2} \right\rfloor \), then the case where the error bound is as big as \( \left\lfloor \frac{d}{2} \right\rfloor \) can be handled. However, we have no prior knowledge about \( a \) in general. To the best of our knowledge, \( \frac{d}{4} \) is the largest error bound available in literature. It is thus interesting to ask whether the bound \( \frac{d}{4} \) can be improved. Recall that in our algorithm 2, we need to deal with the cases \( \alpha - \beta < \frac{d}{4} \) and \( \alpha - \beta > \frac{d}{4} \) differently. This is very suggestive. Pushing this consideration further, we see that if an error is at least \( \frac{d}{4} \) in magnitude, then such distinction is no longer available. In fact, we are able to show that the bound \( \frac{d}{4} \) cannot be improved in general (i.e., it is a sharp bound) by the following simple counterexample.

**Example.** Let \( p, q \) be distinct primes, and \( d \) be a positive integer divisible by 4. We consider solving the system of congruence equations

\[
\begin{align*}
  x &\equiv r_1 \pmod{dp} \\
  x &\equiv r_2 \pmod{dq}
\end{align*}
\]

in the presence of remainder errors.

Assume that the remainder errors are allowed to be \( |\Delta r_1| \leq \frac{d}{4} \) and \( |\Delta r_2| \leq \frac{d}{4} \) (these are equivalent to \( |\Delta r_i| < \frac{d}{4} + 1 \)). We will show that for some corrupted remainders, it is impossible determine an approximation of the true solution. To this end, suppose we have corrupted remainders

\[
\overline{r}_1 = d, \quad \overline{r}_2 = \frac{3d}{2}.
\]

First, we consider the system (9) with \( r_1 = \frac{3d}{4}, r_2 = \frac{7d}{4} = d + \frac{3d}{4} \) (i.e., in this case \( a = \frac{3d}{4} \)). We get the corresponding solution

\[
N_1 = dvp + \frac{3d}{4}
\]

where \( v = p^{-1} \pmod{q} \). Since \( \overline{r}_1 = r_1 + \frac{d}{4}, \overline{r}_2 = r_2 - \frac{d}{4} \), the erroneous remainders \( \overline{r}_1, \overline{r}_2 \) are legitimate for this case.

Next, we consider the system (9) with \( r_1 = r_2 = \frac{5d}{4} = d + \frac{d}{4} \) (i.e., in this case \( a = \frac{d}{4} \)). We get the corresponding solution

\[
N_2 = \frac{5d}{4} = d + \frac{d}{4}.
\]

Since \( \overline{r}_1 = r_1 - \frac{d}{4}, \overline{r}_2 = r_2 + \frac{d}{4} \), the erroneous remainders \( \overline{r}_1, \overline{r}_2 \) are legitimate for this case too.
If we choose \( p \) to be large, then \( N_1, N_2 \) are far apart. Therefore, no approximation based on \( r_1, r_2 \) can be close to both of them. In other words, if the error bound is bigger than \( \frac{4}{7} \), the problem of solving (9) with erroneous remainders is not identifiable in general. We also note that the equality \( \alpha - \beta = \frac{d}{7} \) holds in both cases.

3 Conclusion

Efficient procedures of constructing an approximate solution for the generalized Chinese Remainder Theorem based on corrupted remainders are proposed in this paper. These are extensions of interesting schemes proposed by Xia and his collaborators. We also provide a proof of the sharpness for the error bound \( \frac{d}{7} \). The ideas in our second treatment might be of some independent interest.

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