Properties of the Nonextensive Gaussian entropy

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Abstract

The present work investigates the Lesche stability (experimental robustness), the thermodynamic stability, the Legendre structure of thermodynamics, and derives the Maximum Entropy distribution of the one-parametric “nonextensive Gaussian” entropy. We show that this entropy definition fulfills both stability conditions for all values of its parameter ($q \in \mathbb{R}$). The entropy maximizer contains the Lambert $W$–function, which allows the preservation of the Legendre transformations.

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I. INTRODUCTION

A very important quantity in statistical mechanics is the entropy. In the last years there is a great effort in this field to generalize the concept of thermal equilibrium entropy, which is the Boltzmann–Gibbs (BG) entropy, given by

\[ S_{BG} = \sum_{i=1}^{W} p_i \ln \left( \frac{1}{p_i} \right). \]

(1)

\( W \) is the total number of the accessible microstates and \( p_i \) their associated probabilities. A variety of systems whose behavior can not be sufficiently described by the BG statistics caused the tendency in this direction. The candidate entropies can be categorized into two classes, the trace-form and the non-trace-form ones. The earlier present the structure \( \sum_i p_i \Lambda(p_i) \), where \( \Lambda(p_i) \) in general can be an arbitrary function, while the latter do not. The entropy that dominates as a possible generalization of the BG entropy in the first class is the nonextensive Tsallis entropy defined as

\[ S^T_q = \frac{1}{1-q} \left( \sum_{i=1}^{W} p_i^q - 1 \right), \]

(2)

and in the second class is the Rényi entropy defined as

\[ S^R_q = \frac{1}{1-q} \ln \left( \sum_{i=1}^{W} p_i^q \right). \]

(3)

Both definitions in the limit \( q = 1 \) tend to Eq. (1). In contrast to the BG entropy, whose maximization, according to the Maximum Entropy (MaxEnt) principle, leads to exponential distributions, the two latter entropy definitions may also lead to power-law distributions. With respect to the information theory Eq. (2) and Eq. (3) can be considered as a limit of the two–parametric Sharma–Mittal (SM) entropy, which is given by

\[ S^{SM}_{(r,q)} = \frac{1}{1-q} \left( \sum_{i=1}^{W} p_i^r \right)^{\frac{1-q}{1-r}} - 1. \]

(4)

From Eq. (4) one can see that for \( r = q \) the SM entropy reduces to the Tsallis one, and for \( q = 1 \) this reduces to the Rényi one.

However, for \( r = 1 \) the Sharma–Mittal entropy contains yet a third choice of a generalized entropy, which combines the nonextensivity of Tsallis and the non-trace-form of Rényi
The authors in Ref. [4] suggested to call Eq. (5) “nonextensive Gaussian” (NeG) entropy. One of these authors, Frank, in Ref. [5] used this definition to solve a nonlinear Fokker–Planck equation which describes an Ornstein–Uhlenbeck process, in order to obtain analytical expressions for the transition probability densities. In the present article we shall keep this denomination.

We can easily see that the main difference of the nonextensive Gaussian entropy to the Rényi and Tsallis entropy is that the probability functional is a product of a combination of probabilities instead of sum and hereafter clearly distinguishes from the trace-form entropies. Another point is that the entire probability functional is raised to a power, in contrast to the probability functionals in Eq. (2) and Eq. (3) where every state-probability separately is raised to a power. Written in a different way one can observe that NeG entropy contains the entire structure of the BG entropy.

The purpose of the current work is to explore some statistical properties of the nonextensive Gaussian entropy and to present the connection to thermodynamics, in order to complete the study of all three limits, of the Sharma–Mittal definition, which lead to independent entropy structures. We show that Eq. (5), satisfies the nonextensive thermodynamic stability condition and the Lesche stability criterion. The fulfillment of both conditions is valid for all \( q \in \mathbb{R} \). Thus it could be also a good alternative representative of generalized non-trace-form entropies in statistical thermodynamics. We also derive the maximum-entropy probability for \( S^G_q \) and present the connection to the thermodynamical structure.

One can obtain the Tsallis entropy in a heuristic way by replacing the logarithm in the BG entropy with a generalized one. Here we present a heuristic way to obtain the entropy in Eq. (5) by generalizing also the BG entropy using actually the same generalized logarithmic function. In an integral form the BG entropy can be written as

\[
S_{BG} = \int_{1}^{f(p_i)} \frac{dx}{x}, \quad \text{with} \quad f(p_i) = \exp \left[ \sum_{i=1}^{W} p_i \ln \left( \frac{1}{p_i} \right) \right]. \tag{6}
\]

There are two possible ways to generalize the BG entropy using the concept of a generalized logarithmic function (see Section V). The one leads to the Tsallis entropy and the other
leads to the NeG entropy:

\[ S^T_q = \int_1^f_q(p_i) \frac{dx}{x}, \quad \text{with} \quad f_q(p_i) = \exp \left[ \sum_{i=1}^W p_i \ln_q(1/p_i) \right], \]

\[ S^G_q = \int_1^f(p_i) \frac{dx}{x^q}, \quad \text{with} \quad f(p_i) = \exp \left[ \sum_{i=1}^W p_i \ln(1/p_i) \right]. \]  

(7)

For equal probabilities \( p_i = 1/W \), both \( S^G_q \) and \( S^T_q \) tend to

\[ S^G_q = S^T_q = \ln_q(W) := \frac{W^{1-q} - 1}{1-q} \quad (\ln_1(W) = \ln(W)), \]

which is the generalized logarithm introduced by Tsallis and coworkers. One can verify that the additivity rule for \( S^G_q \) is the same as in the case of Tsallis entropy, namely

\[ S^G_q(A + B) = S^G_q(A) + S^G_q(B) + (1-q) S^G_q(A)S^G_q(B). \]  

(9)

Under the concept of stability of a state functional as Tsallis proposed in Ref. [9], or “experimental robustness” as Tsallis proposed in Ref. [11] in order to avoid confusions with the thermodynamic stability (see Section III), we understand the following: by making a measure, we obtain a distribution of probabilities \( \{p_i\}_{i=1,2,...,W} \). Repeating the same experiment we obtain a new distribution of probabilities \( \{p'_i\}_{i=1,2,...,W} \) which may be slightly different from the previous one. Now, if we use a statistical entropy \( S \), then we expect that its value should not change dramatically for these two slightly different distributions \( \{p_i\}_{i=1,2,...,W} \) and \( \{p'_i\}_{i=1,2,...,W} \). Then the entropy \( S \) is stable or experimentally robust and is of physical relevance. The essence of this kind of stability lies in the existence of an entropy associated observable. Lesche in 1982 [9] formulated a condition (Lesche stability), which reflects the property of experimental robustness, as follows:

\[ (\forall \varepsilon > 0) \ (\exists \delta_\varepsilon > 0) \ \left( \|p - p'\|_1 < \delta_\varepsilon \Rightarrow \left| \frac{S(p) - S(p')}{S_{\text{max}}} \right| < \varepsilon \right), \]

(10)

for any value of \( W \), where \( \|A\|_1 = \sum_{i=1}^W |A| \) and \( S_{\text{max}} \) is the maximum value of \( S \). In the same work he showed that Shannon entropy is stable, while the Rényi entropy does not fulfill this condition for \( q \neq 1 \). After this, several entropy definitions have been explored with regard to this criterion. Some of these, which passed this test, are the Abe entropy [10], Tsallis entropy [11] and Kaniadakis \( \kappa \)-entropy [12]. Common property of these entropies is their trace-form state functional. It is striking that non of the non-trace-form examined entropies like Rényi, Landsberg-Vedral and escort entropy fulfills the Lesche criterion.
Another very important condition that an entropy definition has to satisfy, is the thermodynamic stability condition, which is equivalent to the positivity of the heat capacity $C := \partial U/\partial T = [-T^2\{\partial^2 S_{BG}/\partial U^2\}]^{-1}$. $U$ is the internal energy and $T := (\partial S_{BG}/\partial U)^{-1}$ the temperature. It is well known that in the case of BG entropy the thermodynamical stability condition (TSC) and concavity are equivalent to each other:

$$\frac{\partial^2 S_{BG}(U)}{\partial U^2} \leq 0.$$  \hfill (11)

The physical background of Eq. (11) is based on the combination of the entropy maximum principle and the additivity of (1). Thus the fulfillment of the condition (11) is a very important point in statistical thermodynamics. However, the demand of the concavity for a nonextensive entropy definition does not suffice to preserve the thermodynamic stability [13, 14]. This can be easily understood since the total entropy of two composed subsystems is not the sum of the partial entropies of each subsystem.

In Section II we present the proof of the fulfillment of the Lesche stability criterion. In Section III we derive in two different ways the thermodynamic stability condition for the nonextensive Gaussian entropy and show that this is satisfied by the latter. In Section IV we present the connection of $S_q^G$ to thermodynamics. In the final section we draw our main conclusions.

II. LESCHE STABILITY

The Lesche stability condition reflects the reproducibility of the values of any observable quantity. Here we prove this condition with regard to $S_q^G$. The exponential form of the $S_q^G$–state functional makes difficult, if not impossible, the usage of the formalism in Ref. [11]. In order to overcome this problem we shall try to find a probability functional which is greater than the expression $\exp[(1 - q)S_{BG}]$ and makes possible the application of the formalism we discussed above. We follow the next steps.

The Young Inequality for $x, y, p_1, p_2 > 0$ and $p_1 + p_2 = 1$ is expressed as

$$xy \leq p_1 x^{1/p_1} + p_2 y^{1/p_2}.$$  \hfill (12)

By substituting $x = p_1^{p_1(q-1)}$ and $y = p_2^{p_2(q-1)}$ we rewrite the last equation as

$$p_1^{p_1(q-1)} p_2^{p_2(q-1)} \leq p_1^q + p_2^q, \quad (q \in \mathbb{R}).$$  \hfill (13)
Now, if we use a finite set of variables \( \{x_i\}_{i=1,\ldots,w} \), Eq. (13) can be extended as follows
\[
p_1^{p_1(q-1)} \cdots p_w^{p_w(q-1)} \leq p_1^q + \cdots + p_w^q
\]
\[
\Rightarrow \prod_{i=1}^W p_i^{(q-1)p_i} \leq \sum_{i=1}^W p_i^q, \quad (q \in \mathbb{R}).
\]

From Eq. (14) one can observe two things. First, after applying the logarithmic function on both sides, we obtain the known inequality
\[
\ln \left( \prod_{i=1}^W p_i^{(q-1)p_i} \right) = (1-q)S_{BG} \leq \ln \left( \sum_{i=1}^W p_i^q \right) \quad \Rightarrow \quad S_{q>1}^R \leq S_{BG} \leq S_{q\leq1}^R. \quad (15)
\]

Second, we recall that a convex function \( \phi \) satisfies the relation
\[
\phi \left( \sum_{i=1}^W p_i g(x_i) \right) \leq \sum_{i=1}^W p_i \phi(g(x_i)), \quad \left( \sum_{i=1}^W p_i = 1 \right). \quad (16)
\]

For \( \phi(x) = \exp(x) \), \( g(x_i) = \ln(x_i) \) and \( x_i = p_i^{q-1} \) we obtain
\[
e^{(1-q)S_{BG}} = \prod_{i=1}^W p_i^{(q-1)p_i} = e^{\ln\left(\prod_{i=1}^W p_i^{q-1}\right)} \leq \left\langle e^{\ln\left(\prod_{i=1}^W p_i^{q-1}\right)} \right\rangle = \sum_{i=1}^W p_i^q. \quad (17)
\]

Eq. (17), because of Eq. (14), is always valid. Accordingly, the expression \( \exp[(1-q)S_{BG}] \) is a convex function for all \( q \)'s and thus its second derivative is positive. Then we get the relation
\[
(1-q)^2 \left( \frac{\partial^2 S_{BG}(p)}{\partial p} \right)^2 + (1-q) \frac{\partial^2 S_{BG}(p)}{\partial p} > 0, \quad (q \in \mathbb{R}). \quad (18)
\]

Eq. (18) is interesting for \( q < 1 \) since the first additive term is positive and the second negative. For \( q > 1 \) the validity of (18) is trivial. By applying in Eq. (14) the exponential function and after making some manipulations we can easily show that
\[
\left| \prod_{i=1}^W p_i^{(q-1)p_i} - \prod_{i=1}^W p_i^{(q-1)p_i}' \right| \leq \left| \sum_{i=1}^W p_i^q - \sum_{i=1}^W p_i^q' \right|, \quad (19)
\]
\[
\Rightarrow \quad |S_{q}^G(p) - S_{q}^G(p')| \leq |S_{q}^T(p) - S_{q}^T(p')|. \quad (20)
\]

Taking into account Eq. (8) we extend Eq. (19) to
\[
\left| \frac{S_{q}^G(p) - S_{q}^G(p')}{S_{q,\text{max}}^G} \right| \leq \left| \frac{S_{q}^T(p) - S_{q}^T(p')}{S_{q,\text{max}}^T} \right|. \quad (20)
\]

In Ref. [11] it has been proved that the Tsallis entropy is Lesche stable for \( q > 0 \). Thus from Eq. (20) we obtain that the nonextensive Gaussian entropy is also Lesche stable for
$q > 0$. Now, we still have to check this criterion for negative values of $q$, since for $q < 0$ the exponential state functional does not lead to any singularity, as in the case of the trace-form entropies (see Eq. (17)). But this is already done, because the proof steps in Ref. [11] can be extended without need of modifications into two regions, one for $q < 1$ and one for $q > 1$. Thus we finally obtain

$$\begin{align*}
\left| \frac{S_{q}(p) - S_{q}(p')}{S_{q,max}} \right| & \leq \left| \frac{S_{q}^{T}(p) - S_{q}^{T}(p')}{S_{q,max}} \right| \\
& \leq \begin{cases} 
(\|p - p'\|_{1})^{q} & (q < 1) \\
q\|p - p'\|_{1} & (q > 1)
\end{cases},
\end{align*}$$

(21)

in the limit $W \to \infty$. Therefore, taking $\|p - p'\|_{1} < \delta_{\varepsilon} \leq \varepsilon^{1/q}$ for $q < 1$ or $\|p - p'\|_{1} < \delta_{\varepsilon} \leq \varepsilon/q$ for $q > 1$ we see that the condition (11) is satisfied. Consequently the nonextensive Gaussian entropy is Lesche stable for all $q \in \mathbb{R}$.

III. THERMODYNAMIC STABILITY

It is well known that BG entropy is composable and additive. From the entropy maximum principle and the additivity of $S_{BG}$ one can derive the thermodynamic stability condition, which is expressed as follows:

$$\frac{\partial^{2}S_{BG}(U)}{\partial U^{2}} \leq 0.$$  \hspace{1cm} (22)

Let us consider an isolated system composed of two identical subsystems in equilibrium. The total entropy would be $S_{BG}(U, U) = 2S_{BG}(U)$. Transferring now an amount of energy $\Delta U$ from the one subsystem to the other the total entropy changes as $S_{BG}(U + \Delta U, U - \Delta U) = S_{BG}(U + \Delta U) + S_{BG}(U - \Delta U)$. According to the maximum entropy principle the final value of the entropy can not be larger than the initial one, consequently

$$2S_{BG}(U) \geq S_{BG}(U + \Delta U) + S_{BG}(U - \Delta U).$$

(23)

This is the thermodynamic stability condition for the BG entropy. In the limit $\Delta U \to 0$ Eq. (23) tends to Eq. (22). However, in the case of nonextensive entropies the concavity condition (22) does not correspond to thermodynamic stability. Considering again the maximum entropy principle ($S(U, U) \geq S(U + \Delta U, U - \Delta U)$ for any entropy $S$) as in Eq. (23) and taking this time into account the pseudo-additivity Eq. (9), the TSC of $S_{q}^{G}$ is written as
\[2S_q^G(U) + (1 - q) [S_q^G(U)]^2 \geq S_q^G(U + \Delta U) + S_q^G(U - \Delta U)\]  
\[+ (1 - q) S_q^G(U + \Delta U) S_q^G(U - \Delta U).\]  

In the limit \( \Delta U \to 0 \) this condition can be rewritten in the differential form

\[\frac{\partial^2 S_q^G(U)}{\partial U^2} + (1 - q) \left\{ S_q^G(U) \frac{\partial^2 S_q^G(U)}{\partial U^2} - \left( \frac{\partial S_q^G(U)}{\partial U} \right)^2 \right\} \leq 0.\]  

The TSC (25) is valid for every entropy, whose composability rule is given by Eq. (9). Replacing Eq. (5) in Eq. (25) we obtain for the nonextensive Gaussian entropy

\[e^{2(1-q)S_{BG}(U)} \frac{\partial^2 S_{BG}(U)}{\partial U^2} \leq 0.\]  

(26)

Since the exponential term is always positive and the second derivative of the BG entropy is always negative, Eq. (26) is satisfied for every \( q \). Accordingly, \( S_q^G \) is thermodynamically stable for all values of \( q \in \mathbb{R} \). A different way to obtain the TSC specific for the nonextensive Gaussian entropy is the condition Eq. (22) itself, since the BG entropy can be written as

\[S_{BG} = \frac{1}{1 - q} \ln[1 + (1 - q) S_q^G].\]  

(27)

Then from both Eqs. (22) and (27) we obtain

\[X \times \left\{ \frac{\partial^2 S_q^G(U)}{\partial U^2} + (1 - q) \left[ S_q^G(U) \frac{\partial^2 S_q^G(U)}{\partial U^2} - \left( \frac{\partial S_q^G(U)}{\partial U} \right)^2 \right] \right\} \leq 0,\]  

(28)

\[X = \left[ 1 + (1 - q) S_q^G(U) \right]^{-2}.

The first multiplicative term \( X \) is always positive and accordingly Eq. (28) reduces to Eq. (25).

IV. MAXENT DISTRIBUTIONS

In this section we want to explore the structure of the distributions which maximize \( S_q^G \) under appropriate constraints. For the internal energy \( U \) the constraint we shall use is the so called escort mean value [15] expressed as

\[\langle U \rangle_q = \frac{\sum_{\ell=1}^{W} p^q U_{\ell}}{\sum_{\ell=1}^{W} p^q}.\]  

(29)
where $U_\ell$ describes the energy levels of the system under consideration. We consider now the generalized canonical ensemble described by the entropy $S^G_q$ under the energy constraint (29) and the normalization constraint $\sum_{\ell=1}^W p_\ell = 1$. To derive the maximum-entropy (MaxEnt) distribution of this ensemble, we introduce the Lagrange multipliers $\alpha$ and $\beta$ and the function

$$I(\{p_\ell\}) = S^G_q(\{p_\ell\}) + \alpha \left(1 - \sum_{\ell=1}^W p_\ell\right) + \beta \left(\langle U \rangle_q - \frac{\sum_{\ell=1}^W p_\ell^q U_\ell}{\sum_{\ell=1}^W p_\ell^q}\right).$$

We require that the variation $\delta I$ vanishes for all perturbations $\delta p_i$ of the MaxEnt distribution, accordingly

$$\frac{\delta I(\{p_\ell\})}{\delta p_i} = -e^{(q-1)(\ln(p_i))}(1 + \ln(p_i)) - \beta q p_i^q \sum_{\ell=1}^W p_\ell^q \left(U_i - \langle U \rangle_q\right) - \alpha = 0 \quad (31)$$

For $q \to 1$ (31) reduces to the usual condition for Shannon’s maximizer. We multiply now both sides by $p_i$ and sum over $i$. Taking into account the normalization condition we obtain

$$\alpha = -(1 + \langle \ln(p_\ell) \rangle) e^{(q-1)(\ln(p_i))} \quad (32)$$

Replacing result (32) in (31) we get

$$\alpha \frac{\langle \ln(p_\ell) \rangle}{1 + \langle \ln(p_\ell) \rangle} = \frac{\alpha}{1 + \langle \ln(p_\ell) \rangle} \ln(p_i) - \frac{\beta q p_i^q}{\sum_{\ell=1}^W p_\ell^q} \left(U_i - \langle U \rangle_q\right). \quad (33)$$

With the following three substitutions

$$\kappa = \frac{1}{\langle \ln(p_\ell) \rangle}, \quad E_i(q) = \left(U_i - \langle U \rangle_q\right), \quad \beta_q = \beta \frac{q}{\sum_{\ell=1}^W p_\ell^q} \quad (34)$$

we receive the final form

$$1 = \kappa \ln(p_i) + p_i^q - \beta_q \kappa e^{(1-q)/\kappa} E_i(q). \quad (35)$$

A possible way to solve this equation is by using the Lambert $W$–function [16], given as

$$W(x)e^{W(x)} = x. \quad (36)$$

Accordingly, the distribution acquires the following structure

$$p_i = \frac{1}{Z} \exp \left(\frac{W[-(1-q)\beta_q E_i(q)]}{1-q}\right). \quad (37)$$

with $Z$ given as

$$Z = \sum_{\ell=1}^W \exp \left(\frac{W[-(1-q)\beta_q E_\ell(q)]}{1-q}\right). \quad (38)$$
Here we can define a new generalized exponential function
\[
e^\xi_q(x) := \begin{cases} 
\exp \left( \frac{W[(1-q)x]}{1-q} \right), & (1-q)x > 0 \\
0, & (1-q)x \leq 0 
\end{cases},
\]
which we call \(q\xi\)-exponential. Eq. (39) is symmetric with respect to \(1-q\) and because of Eq. (36) it can be written in the following different ways:
\[
e^\xi_q(x) = \left[ \frac{W[(1-q)x]}{(1-q)x} \right]^{1/(1-q)} = \left[ \frac{W[-(1-q)x]}{-(1-q)x} \right]^{1/(1-q)} = e^{\frac{W[(1-q)x]}{(1-q)}} = e^{\frac{W[-(1-q)x]}{-(1-q)}}.
\]

For \(q = 1\) the generalized exponential function in (39) and (40) tends to the ordinary one.

As we can see the form of the probability distribution (37) is not very familiar and clearly distinguishes from the Rényi/Tsallis ones. However, in the asymptotic limit, which is, for \(E_i(q) \gg 1/\beta_q(q-1)\), Eq. (37) tends to a power-law distribution function
\[
p_i \propto [E_i(q)]^{1/(1-q)} = \left( U_i - \langle U \rangle_q \right)^{1/(1-q)},
\]
same as the Rényi/Tsallis maximum entropy distribution \(p_i\).

V. CONNECTION TO THERMODYNAMICS

In Refs. [6] and [17] it has been shown that the entire Legendre structure of thermodynamics is \(q\)-invariant with regard to Tsallis and Rényi entropy respectively. In this section we explore whether the Legendre structure is also invariant with respect to NeG entropy. Using Eq. (40) we can express the Lambert function in Eq. (37) in the following two ways
\[
W[(q-1)\beta_q E_i(q)] = \begin{cases} 
(p_i Z)^{q-1}(q-1)\beta_q E_i(q), & \\
(1-q) \ln (p_i Z)
\end{cases}.
\]

Accordingly, we have
\[
(p_i Z)^{q-1}\beta_q E_i(q) = - \ln (p_i Z).
\]

By multiplying Eq. (43) with \(p_i\) and taking the sum over all \(i\)'s the left hand side of Eq. (43) vanishes because of the constraint \(\sum_{i=1}^W p_i^q U_i = \langle U \rangle_q \sum_{i=1}^W p_i^q\). Then we obtain
\[
Z = e^{-\langle \ln(p_i) \rangle} = \prod_{i=1}^W p_i^{-p_i}.
\]
Consequently we can express the entropy $S_q^G$ in dependence on $Z$ as

$$S_q^G = \ln_q(Z).$$  \hspace{1cm} (45)

With the introduction of a temperature $1/T = \partial S_q^G / \partial \langle U \rangle_q$ [18], where $T$ is connected with the Lagrange multiplier $\beta$ as $\beta := 1/T$, and after defining the partition function $\tilde{Z}$ as

$$\ln_q(\tilde{Z}) := \ln_q(Z) - \beta \langle U \rangle_q,$$  \hspace{1cm} (46)

one can show that the escort mean energy $\langle U \rangle_q$ can be expressed as

$$\langle U \rangle_q := -\frac{\partial}{\partial \beta} \ln_q(\tilde{Z}).$$  \hspace{1cm} (47)

Then, the free energy $F_q$, which is defined as

$$F_q := \langle U \rangle_q - T S_q^G = \langle U \rangle_q - \frac{1}{\beta} S_q^G,$$  \hspace{1cm} (48)

can be written as

$$F_q = -\frac{1}{\beta} \ln_q(\tilde{Z}),$$  \hspace{1cm} (49)

for the maximum entropy distribution (37). We can also verify that

$$C_q := T \frac{\partial S_q^G}{\partial \beta} = \frac{\partial \langle U \rangle_q}{\partial \beta} = -T \frac{\partial^2 F_q}{\partial T^2},$$  \hspace{1cm} (50)

where $C_q$ is the generalized specific heat. In other words, the NeG entropy under the constraint of the internal energy (29) and the normalization constraint, preserves the Legendre structure of thermodynamics.

Next we shall present the relation between the generalized temperature and specific heat with the ordinary ones. In the BG case the temperature and the specific heat are given by

$$\frac{1}{T} = \frac{\partial S_{BG}}{\partial U}, \quad \frac{1}{C} = \frac{\partial T}{\partial U} = -T^2 \frac{\partial^2 S_{BG}}{\partial U^2}, \quad (U = \langle U \rangle_1).$$  \hspace{1cm} (51)

By replacing the BG entropy in Eq. (51) with Eq. (27) we obtain

$$T = \left\{1 + (1 - q)S_q^G\right\} T_q \quad \text{with} \quad \frac{1}{T_q} := \frac{\partial S_q^G}{\partial U},$$

$$\frac{1}{C} = \frac{1}{C_q} + (1 - q) \left(1 + \frac{S_q^G}{C_q}\right) \quad \text{with} \quad \frac{1}{C_q} := -T_q^2 \frac{\partial^2 S_q^G}{\partial U^2}.$$  \hspace{1cm} (52)
These two expressions for $T$ and $C$ are the same with those derived for the Tsallis entropy. Wada in Ref. [13] computed the relations in (52) from the composition rule (9). Accordingly, they are valid for every entropy that satisfies Eq. (9).

Finally, in Eq. (7) we showed, that by using the $q$-logarithm (8) there two possible ways to generalize the BG entropy. Here we explore the essence of this result and show that for an arbitrary generalized logarithm the BG entropy can be generalized actually in three different ways.

Therefore, we consider an isolated system composed by $N$ independent particles, with their energy levels characterized by the occupation numbers $n_1, n_2, \ldots, n_W$ and their respective probabilities $p_1, p_2, \ldots, p_W$. Then the number of all possible configurations of the particles is given by the multinomial coefficient $M$:

$$M := \frac{N!}{(n_1)!(n_2)\cdots(n_W)!} = \frac{N!}{(Np_1)!(Np_2)\cdots(Np_W)!}. \quad (53)$$

In further we introduce the quantity $X := M^{1/N}$. For $N \to \infty$ and taking into account the relation $\lim_{N \to \infty} N! \approx (\frac{N}{e})^N$, we can easily show that

$$X = \prod_{i=1}^{W} p_i^{-p_i} = e^{\langle \ln (1/p_i) \rangle}. \quad (54)$$

Now, the BG entropy is defined in thermal equilibrium as the application of the logarithmic function on Eq. (54):

$$S_{BG} := \ln (X) = \ln \left( \prod_{i=1}^{W} p_i^{-p_i} \right) = \langle \ln (1/p_i) \rangle = - \langle \ln (p_i) \rangle. \quad (55)$$

Although all expressions in Eq. (55) are equal, it is obvious that the replacement of a generalized logarithmic function $L_{\vec{q}}$ with

$$\lim_{\vec{q} \to \vec{q}_0} L_{\vec{q}}(a) = \ln (a), \quad (a > 0), \quad (56)$$

and a set of parameters $\vec{q} := \{q_i\}_{i=1,\ldots,m}$, leads to different generalized entropy structures. These are the following:

$$S_{\vec{q}}^{(1)} = L_{\vec{q}} \left( \prod_{i=1}^{W} p_i^{-p_i} \right), \quad S_{\vec{q}}^{(2)} = \langle L_{\vec{q}} (1/p_i) \rangle, \quad S_{\vec{q}}^{(3)} = - \langle L_{\vec{q}} (p_i) \rangle. \quad (57)$$

There are three things to notice. First, the Rényi definition does not correspond to any of the three entropy generalizations in Eq. (57). Second, $S_{\vec{q}}^{(2)}$ and $S_{\vec{q}}^{(3)}$ have the same
structure. The small differences between them can be referred to a transformation with respect to $\vec{q}$ ($\vec{q}^{(2)} \rightarrow f(\vec{q}^{(3)}))$. Thus they represent actually the same quantity. Third, the maximization of $S^{(1)}_q$ under consideration of the constraint \[29\] leads always to Lambert exponential distributions, independent from the choice of the $\vec{q}$-logarithm, because of the following relation

$$\frac{\partial S^{(1)}_q}{\partial \vec{p}} = \frac{\partial L_{\vec{q}}(S_{BG})}{\partial S_{BG}} \frac{\partial S_{BG}}{\partial \vec{p}}.\tag{58}$$

Using the one-parametric generalized logarithm \[8\] we identify $S^{(1)}_q = S^G_q$, $S^{(2)}_q = S^T_q$ and $S^{(3)}_q$ is the $S^T_{2-q}$ transformed Tsallis entropy ($q \rightarrow 2 - q$).

VI. CONCLUSIONS

We have studied some statistical properties of the nonextensive Gaussian entropy \[5\]. $S^G_q$ is Lesche stable (or experimentally robust) for all values of $q \in \mathbb{R}$. We have shown that the Lesche stability of $S^G_q$ is a consequence of the Lesche stability of the Tsallis entropy. We found the same thermodynamical stability condition as in the case of Tsallis entropy with regard to the ordinary internal energy, using the concavity condition of the Boltzmann–Gibbs entropy, since $S^G_q$ can be expressed as functional of the entire $S_{BG}$. The condition is satisfied for all values of $q \in \mathbb{R}$. We derived the distribution that maximizes the nonextensive Gaussian entropy. This is based on the Lambert $W$-function. A new generalized $q_{LC}$-exponential function is defined. For $q = 1$ it returns to the ordinary one. In the thermodynamic limit it tends to a pure power-law function. The connection of $S^G_q$ to thermodynamics is presented. We showed that the Legendre structure is preserved through a convenient definition of a generalized partition function and the relation between the temperature and specific heat with the generalized ones is the same as in the case of Tsallis entropy. Finally, we demonstrated that by replacing the ordinary logarithm in the equilibrium Boltzmann–Gibbs entropy with a generalized one, we obtain three possible entropy structures, in which one can identify the nonextensive Gaussian entropy and the Tsallis entropy. The Rényi entropy, since it is not based on the concept of a generalized logarithmic function, does not belong to any of these three cases.
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