Numerical and analytical calculation of longitudinal bending of prismatic elastic rods under the action of axial compressive load taking into account the authorized weight

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Abstract. The article is devoted to the problem of calculating the stability of compressed rods taking into account their own weight. A resolving differential equation of the second and fourth order with respect to the dimensionless deflection is obtained. A technique for numerically analytical and numerical solution of the obtained equation using the finite difference method is proposed. As a result, the problem was reduced to a generalized eigenvalue problem. Comparison with the solution of other authors is given.

The problem of determining the critical values for the rods is posed as the problem of finding the domain of these values depending on the boundary conditions and the coefficients α and β. The objective function is a converging power series with unknown coefficients. The solution of the prismatic rod stability problem is carried out in the MatLab environment. The values of α and β, for which the critical load takes a minimum value, are found.

To test the methodology, a number of problems were solved and compared with known solutions.

The proposed technique, in contrast to analytical solutions, allows solving the problem with arbitrary fixation of the ends of the rod. It is also suggested to take into account the length-varying stiffness and heterogeneity of the rod. Test problems showed good agreement with literature data. In the future, it is planned to develop a calculation method taking creep into account.

The higher quality of the analytical solution is shown in comparison with the existing methods.

Introduction
The problems of calculating stability compressed rods are studied by many authors, and from the latest publications on this problem it is worth noting the following works [1-19]. In the articles [1-3], the solution of this problem is considered taking into account creep using the finite difference method. In the works [4-5] the energy method in the form of Ritz-Timoshenko is applied. The finite element method in the form of the Bubnov-Galerkin method [6-9] can also be used to calculate the compressed rods, including creep. The problems of dynamic stability of rods are discussed in articles [10-16]. In the articles [17-19] for calculating the rods stability, the apparatus of the probability theory is connected. However, none of these publications dealt with self-weight sustainability. The existing analytical solutions [20-24] are applicable only for specific fixings of the rod ends.
Consider a homogeneous, rectilinear, vertically positioned rod with arbitrary fixing of its ends, experiencing the specified load (compressive force and dead weight).

Let us select some infinitesimal section of the rod that lags behind the origin at a distance \( t \) within the interval \([0, x]\) and measuring \( \text{d}t \) (Figure 1).

![Figure 1. Element \( \text{d}t \) of the infinitely small rod section.](image)

Then within the considered element \( v(x) = \text{const} \). Here \( v(x) \) — is the amount of deflection at the center of an infinitesimal element \( \text{d}t \), and \([v(x) - v(t)]\) — defines the load arm \( q \), acting on an infinitesimal element \( \text{d}t \) relative to point \( A \). Then the moment relative to this point will be:

\[
M_A = q \cdot \int_0^x [v(x) - v(t)] \text{d}t = qv(x)x - q \int_0^x v(t) \text{d}t
\]

To take into account all the fixing options, we consider a rigid fixing, movable along the axis \( x \) (Figure 1).

In this case, in the section \( A \) moment will be equal:

\[
M(x) = F \cdot v(x) + \left\{ qv(x)x - q \int_0^x v(t) \text{d}t \right\} + Rx - M
\]

On the other hand, in the case of small deformations, the differential equation of the curved axis will be as follows:

\[
\frac{d^2v(x)}{dx^2} = -\frac{M(x)}{EИ_z}
\]
Here $E$ — is a Young’s modulus of the material, $I_z$ — is an axial moment of inertia, $M(x)$ — is a bending moment.

Then the integro-differential equation of the curved axis of the rod takes the form:

$$\frac{d^2 v(x)}{dx^2} = - \frac{1}{EI_z} \left[ F \cdot v(x) + \left\{ qv(x)x - q \int_0^x v(t)dt \right\} + Rx - M \right]$$

(3)

Further, it is convenient to consider the resolving equations in dimensionless quantities. Let us make the following replacement:

$$x = l \varphi; \quad t = l \xi; \quad v(x) = l v(\varphi)$$
$$v(t) = l v(\xi)$$

(4)

$$\frac{d^2 v(x)}{dx^2} = \frac{1}{l^2} \frac{d^2 v(x)}{d\varphi^2}$$
$$\frac{dv(x)}{dx} = \frac{1}{l} \frac{dv(x)}{d\varphi}$$

(5)

We rewrite the term in relation (3) in curly brackets with the integral in relative coordinates:

$$\frac{q}{EI_z} \int_0^x v(t)dt = \frac{q}{EI_z} \int_0^\varphi l v(\xi) \cdot l d\xi = \frac{q l^2}{EI_z} \int_0^\varphi v(\xi)d\xi$$

(6)

The expression (3), taking into account (5), can be rewritten in dimensionless coordinates:

$$\frac{1}{l} \frac{d^2 v(\varphi)}{d\varphi^2} + \frac{F l}{EI_z} v(\varphi) + \frac{q l v(\varphi)}{EI_z} l \cdot v(\varphi) - \frac{q l^2}{EI_z} \int_0^\varphi v(\xi)d\xi = - \left( R \frac{ql}{EI_z} - \frac{M}{EI_z} \right)$$

(7)

Multiplying the expression (7) by $l$, finally, in dimensionless coordinates, we represent:

$$\frac{d^2 v(\varphi)}{d\varphi^2} + \alpha^2 v(\varphi) + \beta^3 \varphi v(\varphi) - \beta^3 \int_0^\varphi v(\xi)d\xi = m - r \varphi$$

(8)

here

$$\alpha^2 = \frac{Fl^2}{EI_z}; \quad \beta^3 = \frac{ql^3}{EI_z}; \quad m = \frac{Ml}{EI_z}; \quad r = \frac{Rl^2}{EI_z}$$

(9)

Since the solution will also be presented numerically, for the integration convenience under various boundary conditions, we differentiate (8) twice:

$$\frac{d^4 v(\varphi)}{d\varphi^4} + \alpha^2 \frac{d^2 v(\varphi)}{d\varphi^2} + \beta^3 \left[ \frac{dv(\varphi)}{d\varphi} + \varphi \frac{d^2 v(\varphi)}{d\varphi^2} \right] = 0.$$  

(9a)

**Materials and methods**

The solution (8a) will be sought in the form of a power series:

$$v(\varphi) = \sum_{k=0}^n A_k \varphi^k;$$

(10)

Below are the first to fourth derivatives $v(\varphi)$:
\[
\frac{dv(\varphi)}{d\varphi} = \sum_{k=1}^{n} kA_k \varphi^{k-1}; \quad \frac{d^2v(\varphi)}{d\varphi^2} = \sum_{k=2}^{n} k(k-1)A_k \varphi^{k-2}; \quad \frac{d^3v(\varphi)}{d\varphi^3} = \sum_{k=3}^{n} k(k-1)(k-2)A_k \varphi^{k-3}; \quad \frac{d^4v(\varphi)}{d\varphi^4} = \sum_{k=4}^{n} k(k-1)(k-2)(k-3)A_k \varphi^{k-4};
\]

(10a)

For the same degree at \( \varphi^{k-4} \), as, for example, for the 4th derivative, we write (10a) differently:

\[
\frac{dv(\varphi)}{d\varphi} = \sum_{k=4}^{n+1} A_{k-3}(k-3)\varphi^{k-4}; \quad \frac{d^2v(\varphi)}{d\varphi^2} = \sum_{k=4}^{n+2} A_{k-2}(k-3)(k-2)\varphi^{k-4};
\]

\[
\varphi \frac{d^2v(\varphi)}{d\varphi^2} = \sum_{k=5}^{n+3} A_k k(k-1)\varphi^{k-1} = \sum_{k=5}^{n+3} A_{k-3} (k-4)(k-3)\varphi^{k-4};
\]

(11)

\[
\frac{d^4v(\varphi)}{d\varphi^4} = \sum_{k=4}^{n} k(k-1)(k-2)(k-3)A_k \varphi^{k-4};
\]

After setting the corresponding derivatives in the equation (8a) and canceling by \((k-3)\), we get:

\[
\sum_{k=4}^{n} k(k-1)(k-2)A_k \varphi^{k-4} = -\alpha^2 \sum_{k=4}^{n+2} A_{k-2} \varphi^{k-4} - \beta^3 \sum_{k=4}^{n+1} A_{k-3} \varphi^{k-4} + \sum_{k=5}^{n+3} A_{k-3}(k-4)\varphi^{k-4}
\]

\[
= 0
\]

Or, equating the coefficients at \( \varphi^{k-4} \), get:

\[
A_k = -\frac{[\alpha^2 A_{k-2}(k-2) + \beta^3 A_{k-3}(k-3)]}{k(k-1)(k-2)}
\]

(12)

Let us determine the coefficients at \( \varphi^0 \):

\[
v(\varphi) = A_0 + A_1 \varphi + A_2 \varphi^2 + A_3 \varphi^3 + \sum_{k=4}^{n} A_k \varphi^k;
\]

\[
\frac{dv(\varphi)}{d\varphi} = A_1 + 2A_2 \varphi + 3A_3 \varphi^2 + 4A_4 \varphi^3 + 5A_5 \varphi^4 + 6A_6 \varphi^5 \ldots;
\]

\[
\frac{d^2v(\varphi)}{d\varphi^2} = 2A_2 + 6A_3 \varphi + 12A_4 \varphi^2 + 20A_5 \varphi^3 + 30A_6 \varphi^4 \ldots;
\]

\[
\varphi \frac{d^2v(\varphi)}{d\varphi^2} = 2A_2 \varphi + 6A_3 \varphi^2 + 12A_4 \varphi^3 + 20A_5 \varphi^4 + 30A_6 \varphi^5 \ldots;
\]

\[
\frac{d^3v(\varphi)}{d\varphi^3} = 6A_3 + 24A_4 \varphi + 60A_5 \varphi^2 + 120A_6 \varphi^3 \ldots;
\]

\[
\frac{d^4v(\varphi)}{d\varphi^4} = 24A_4 \varphi + 120A_5 \varphi + 360A_6 \varphi^2 \ldots;
\]

(13)

We substitute the corresponding terms for \( \varphi^0 \) in (13):

\[
A_4 = -\frac{[\alpha^2 \cdot 2 \cdot A_2 + \beta^3 A_1]}{24}
\]

(14)

We substitute the corresponding terms for \( \varphi^1 \) in (2.13):
\[ A_5 = - \frac{[\alpha^2 \cdot A_3 + \beta^4 \cdot A_2]}{24} \]  

(15)

Let us consider the boundary conditions for various fixings of the rod:

1. **Hinge-Hinge**

\[ v(0) = 0; \quad \frac{d^2 v(0)}{d\phi^2} = 0; \quad v(1) = 0; \quad \frac{d^2 v(1)}{d\phi^2} = 0; \]  

(16)

2. **Hinge-Pinch**

\[ v(0) = 0; \quad \frac{d^2 v(0)}{d\phi^2} = 0; \quad v(1) = 0; \quad \frac{d v(1)}{d\phi} = 0; \]  

(17)

3. **Pinch-Hinge**

\[ v(0) = 0; \quad \frac{d v(0)}{d\phi} = 0; \quad v(1) = 0; \quad \frac{d^2 v(1)}{d\phi^2} = 0; \]  

(18)

4. **Pinch-Pinch**

\[ v(0) = 0; \quad \frac{d v(0)}{d\phi} = 0; \quad v(1) = 0; \quad \frac{d v(1)}{d\phi} = 0; \]  

(19)

### The study results. Model problems solution

**Task 1. Variant of the Hinge-Hinge rod fastening**

Below the schemes of tasks to be solved and in accordance with the graphics will be given.

Let us consider in more detail the problem for a rod, taking into account its own weight with the option of "Hinge-Hinge" fixation (Figure 3).

From the boundary conditions:

\[ v(0) = 0; \quad \rightarrow A_0 = 0; \quad \frac{d^2 v(0)}{d\phi^2} = 0; \quad \rightarrow A_2 = 0; \]

\[ v(1) = 0; \quad \rightarrow \sum_{k=0}^{n} A_k = 0 \rightarrow k_{11} A_1 + k_{13} A_3 = 0; \]

To define a series \( k_{11} \), we use \( \sum_{k=0}^{n} A_k \), but we assume in this amount \( A_1 = 1, A_3 = 0 \);

To determine \( k_{13} \), we use \( \sum_{k=0}^{n} A_k \), but we assume in this amount \( A_1 = 0, A_3 = 1 \);

\[ \frac{d^2 v(1)}{d\phi^2} = 0; \quad \rightarrow \sum_{k=4}^{n+2} A_{k-2} (k-3)(k-2) = 0 \rightarrow k_{31} A_1 + k_{33} A_3 = 0; \]

To determine \( k_{31} \), we use \( \sum_{k=4}^{n+2} A_{k-2} (k-3)(k-2) \), but we assume in this amount \( A_1 = 1, A_3 = 0 \);

To determine \( k_{33} \), we use \( \sum_{k=4}^{n+2} A_{k-2} (k-3)(k-2) \), but we assume in this amount \( A_1 = 0, A_3 = 1 \);

We write the obtained relations

\[ \{k_{11} A_1 + k_{13} A_3 = 0 \]

\[ \{k_{31} A_1 + k_{33} A_3 = 0 \]

The condition for the nonzero solutions existence
The following sequence of numerical implementation is as follows: we assume \( \alpha = 0; 0.1; 0.2; \ldots \pi \) from the condition \( D = 0 \), and determine the values \( \beta \) by solving the nonlinear equation in Matlab using the function `fzero`. The results are shown in Table 1.

**Table 1. Dependence of the coefficients \( \beta \) from \( \alpha \) «Hinge-Hinge»**

| \( \alpha \) | \( \beta \) | \( \alpha \) | \( \beta \) | \( \alpha \) | \( \beta \) |
|---|---|---|---|---|---|
| 0  | 2.648057 | 1.1 | 2.540819 | 2.2 | 2.135201 |
| 0.1 | 2.647209 | 1.2 | 2.519243 | 2.3 | 2.071442 |
| 0.2 | 2.644669 | 1.3 | 2.495281 | 2.4 | 1.999977 |
| 0.3 | 2.640422 | 1.4 | 2.468781 | 2.5 | 1.919087 |
| 0.4 | 2.634448 | 1.5 | 2.439619 | 2.6 | 1.826341 |
| 0.5 | 2.626722 | 1.6 | 2.407531 | 2.7 | 1.71802 |
| 0.6 | 2.617205 | 1.7 | 2.372278 | 2.8 | 1.587946 |
| 0.7 | 2.605857 | 1.8 | 2.333558 | 2.9 | 1.424479 |
| 0.8 | 2.592618 | 1.9 | 2.291001 | 3.0 | 1.200214 |
| 0.9 | 2.577424 | 2.0 | 2.244158 | 3.1 | 0.803247 |
| 1.0 | 2.560195 | 2.1 | 2.196462 | 3.141593 | 0.0 |

**Figure 3.** Design scheme “Hinge-Hinge”

**Figure 4.** Design scheme “Hinge-Pinch”
By $\alpha = \pi$ and $\beta = 0$, according to (2.9), we obtain L. Euler’s formula [25], and for $\alpha = 0$ and $\beta = 2.648057$, according to (2.9), we obtain the formula of A.N. Dinnik [22]

$$l_{kp} = 2.65 \sqrt[3]{\frac{EIz}{q}}$$  \hspace{1cm} (20)

The dependence (Figure 7) can be approximated by the following curve [24]:

$$\alpha^2 + a\beta^3 = b$$  \hspace{1cm} (21)

Having defined constant $a$ and $b$ at the values $\alpha$ and $\beta$ on the axes, we get:

$$\alpha^2 + 0.532\beta^3 = \pi^2$$  \hspace{1cm} (22)

or taking into consideration (9)

$$F + 0.532ql = \frac{\pi^2EIz}{l^2}$$  \hspace{1cm} (23)
Figure 7. Stability region for a rod hinged at the ends

Figure 8. Stability region for a rod, the upper end of which is hinged and the lower end is rigidly clamped.

Figure 9. The stability region for a rod, the upper end of which is rigidly clamped, and the lower end is hinged.

Figure 10. The stability region for a rod with rigidly clamped ends

Summarizing the above-mentioned, we note that the values $\alpha$ and $\beta$, determined by the formula (22) i.e.

$$\beta = 3 \sqrt{\frac{\pi^2 - \alpha^2}{0.532}}$$

lie inside the stability region, which goes into the stability margin. This dependence in (Figure 3) is shown with a dotted line.
We find the section where the deflection takes the greatest value. It is worth recalling that we are considering the hinge-hinge case taking into account its own weight.

The border conditions:

\[ v(1) = 0; \quad \sum_{k=0}^{n} A_k k_{11} A_1 + k_{13} A_3 = 0; \]

To define a series, we use \( \sum_{k=0}^{n} A_k \), but we suppose \( A_1 = 1, A_3 = 0 \) in it;

To determine \( k_{13} \), we use \( \sum_{k=0}^{n} A_k \), but we suppose \( A_1 = 0, A_3 = 1 \) in it;

\[ \frac{d v(\mu)}{d \varphi} = 0; \quad \sum_{k=4}^{n+1} A_{k-3}(k-3)\varphi^{k-4}; \quad k_{31}^* A_1 + k_{33}^* A_3 = 0; \]

To determine \( k_{31}^* \), we use \( \sum_{k=4}^{n+1} A_{k-3}(k-3)\varphi^{k-4} \), but we suppose \( A_1 = 1, A_3 = 0 \) in it;

To determine \( k_{33}^* \), we use \( \sum_{k=4}^{n+1} A_{k-3}(k-3)\varphi^{k-4} \), but we suppose \( A_1 = 0, A_3 = 1 \) in it;

From it:

\[ D_1 = \begin{bmatrix} k_1 & k_r \\ k_{31}^* & k_{33}^* \end{bmatrix} = 0 \]

We carry out a numerical search using the found relationship between \( \alpha \) and \( \beta \) (Figure 2.3), i.e., at \( \alpha = 0 \) and \( \beta = 2,648057 \) we find \( \mu = 0,542750 \), and for and \( \beta = 0 \) \( \rightarrow \mu = 0,5 \)

So, the maximum deflection lies within:

\[ 0,5l < x_0 < 0,542750l \quad (25) \]

where \( x_0 = 0,5l \) can be when \( q = 0 \), a \( x_0 = 0,542750l \) by \( F = 0 \)

It should be noted that the problem was solved under the assumption that the material of the rod does not receive plastic deformations under compression, i.e., stresses arising at critical loads are less than the proportional limit. Hence, with the flexibility of the rod \( \lambda \) greater \( \lambda_{lim} \) formula (23) is applicable and vice versa.

In case of plastic deformations, the formula (23) takes the following form:

\[ F + 0,532ql = \frac{\pi^2 E_{lim} l_z}{l^2} \quad (26) \]

where

\[ E_{lim} = \frac{I_z E_k + I_2 E}{I_z} \quad (27) \]

Here \( I_z \) and \( I_2 \) — are the moments of reloading and unloading zones inertia relative to the zero line, \( I_z \) — is a moment of inertia of the entire section about the central axis, \( E_k \) — is a tangent module.

**Task 2. Variant of fastening the rod "Hinge-Pinch"**

Let us consider a rod with an articulated upper end and a pinched lower end (Figure 4). According to boundary conditions (17), we determine the coefficients \( A_0, A_2 \), and the numeric search is similar to the previous one. Dependency between \( \alpha \) and \( \beta \) shown in Figure 8.

When \( \alpha = 4,493407 \) and \( \beta = 0 \), taking into account (9), we obtain the F.S. Yasinsky formula [23],

\[ F_{kp} = \frac{20,19 El_z}{l^2} \quad (28) \]
When $\alpha = 0$ and $\beta = 3.744452$, taking into account (9), we obtain the formula of A.N. Dinnik [22],

$$l_{kp} = 3.74 \sqrt[3]{\frac{EI_z}{q}}$$

(29)

Having defined the constant $a$ and $b$ at the values $\alpha$ and $\beta$ on the axes in (8), the approximate dependence in this case has the following form:

$$\alpha^2 + 0.385\beta^3 = 20.19$$

or taking into account (9)

$$F + 0.385ql = \frac{20.19EI_z}{l^2}$$

(30)

Let us note that the values $\alpha$ and $\beta$, determined by an approximate dependence, also lie within the stability region (Figure 8), and the greatest deflection at

$$0,4l < x_0 < 0,46l$$

(31)

where $x_0 = 0,4l$ can be when $q = 0$, and $x_0 = 0,46l$ when $F = 0$

Task 3. Variant of fastening the rod "Pinch-Hinge"

Let us consider a rod which upper end is pinched and the lower end is hinged (Figure 5). According to the boundary conditions 2.18, we determine the coefficients $A_0, A_2$, and the numeric search is similar to the previous one.

The dependency between $\alpha$ and $\beta$ is shown graphically in Figure 9. When $\alpha = 4.4934070$ and $\beta = 0$, taking into account (9), we obtain the formula (28) of F.S. Yasinsky [23].

When $\alpha = 0$ and $\beta = 3.107555$, taking into account (9), we refine the formula of A.N. Dinnik [22], which coefficient is 3.09,

$$l_{kp} = 3.11 \sqrt[3]{\frac{EI_z}{q}}$$

(32)

The approximate dependence (21) in such a fixing sequence has the following form:

$$\alpha^2 + 0.673\beta^3 = 20.19$$

or taking into account (9)

$$F + 0.673ql = \frac{20.19EI_z}{l^2}$$

(33)

Let us note that the values $\alpha$ and $\beta$, determined by an approximate relationship, also underestimate the exact values. The greatest deflection (Figure 5) at

$$0,6l < x_0 < 0,63l$$

(34)

where $x_0 = 0,6l$ can be when $q = 0$, and $x_0 = 0,63l$ obtained with $F = 0$

Task 4. Variant of fastening the rod "Pinch-Pinch"

Let us consider a rod in which the upper and lower ends are pinched (Figure 6). According to the boundary conditions 19, we determine the coefficients $A_0, A_2$, and the numeric search is similar to the previous one. The dependency between $\alpha$ and $\beta$ is shown in Figure 10.

When $\alpha = 6.283186$ and $\beta = 0$, taking into account (2.9), we obtain the F.S. Yasinsky formula [23],
\[ F_{kp} = \frac{4\pi^2 EI_x}{l^2} \]  

(35)

When \( \alpha = 0 \) and \( \beta = 4,210,175 \), taking into account (2.9), we correct the formula of A.N. Dinnik [22], which coefficient is 4.19.

\[ l_{kp} = 4,21 \sqrt{\frac{EI_x}{q}} \]  

(36)

The approximate dependence (2.21) in this fixing sequence has the following form:

\[ \alpha^2 + 0,529\beta^3 = 4\pi^2 \]

or taking into account (2.9)

\[ F + 0,529ql = \frac{4\pi^2 EI_x}{l^2} \]

(37)

It is necessary to note that the values \( \alpha \) and \( \beta \), determined by an approximate dependence, also lie within the stability region (Figure 10), and the greatest deflection (Figure 6) at

\[ 0,5l < x_0 < 0,545l \]  

(38)

where \( x_0 = 0,5l \) can be when \( q = 0 \), and only force acts \( F \), and at \( x_0 = 0,545l \) is obtained from the action of its own weight.

All the problems presented above are also solved numerically, in particular, by the finite difference method. To implement the grid method, the interval \( \xi \in [0; 1] \) breaks down into \( N_\xi \) parts with the step \( \Delta \xi \), and the derivatives in the equation (8a) and in the boundary conditions are replaced by the grid derivatives.

The problems are reduced to a system of linear algebraic equations of the form:

\[ ([A] + \alpha^2[B] + \beta^3[C])\{X\} = 0 \]  

(39)

where \( \{X\} = \{v_1 \ v_2 \ v_3 \ldots \ v_{n-1} \ v_n\} \) — is the nodes’ dimensionless displacements vector. Here the corresponding matrices\([A],[B],[C]\) have the forms:

\[
\begin{bmatrix}
 a_{zz} & -4 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -4 & 6 & -4 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & -4 & 6 & -4 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & -4 & 6 & -4 & 1 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 a_{nn} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{bmatrix}
\]

\[
\begin{bmatrix}
 -2 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 & 0 \\
 \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
 0 & 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 & 1 \\
 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & -2 
\end{bmatrix}
\]
where $q_i = (i - 1)\Delta \phi$

The coefficient $a_{zz}$ in the matrix $[A]$ equals to 5 if the upper end is hinged and 7 if it is rigidly clamped. The coefficient for $a_{nn}$ for the lower end is calculated in a similar way.

The system (39) is homogeneous and has a nonzero solution if its determinant is zero:

$$\begin{vmatrix}
-2\varphi_2 & \varphi_2 + \frac{\Delta \varphi}{2} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\varphi_3 - \frac{\Delta \varphi}{2} & -2\varphi_3 & \varphi_3 + \frac{\Delta \varphi}{2} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & \varphi_4 - \frac{\Delta \varphi}{2} & -2\varphi_4 & \varphi_4 + \frac{\Delta \varphi}{2} & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 - 2\varphi_{n-2} & \varphi_{n-2} - \frac{\Delta \varphi}{2} & 0 \\
0 & 0 & 0 & 0 & \varphi_{n-2} - \frac{\Delta \varphi}{2} & \varphi_{n-1} - \frac{\Delta \varphi}{2} & -2\varphi_{n-1} & \varphi_n + \frac{\Delta \varphi}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \varphi_n - \frac{\Delta \varphi}{2} & -2\varphi_n \\
\end{vmatrix}$$

$$\begin{vmatrix}
-2\varphi_2 & \varphi_2 + \frac{\Delta \varphi}{2} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\varphi_3 - \frac{\Delta \varphi}{2} & -2\varphi_3 & \varphi_3 + \frac{\Delta \varphi}{2} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & \varphi_4 - \frac{\Delta \varphi}{2} & -2\varphi_4 & \varphi_4 + \frac{\Delta \varphi}{2} & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 - 2\varphi_{n-2} & \varphi_{n-2} - \frac{\Delta \varphi}{2} & 0 \\
0 & 0 & 0 & 0 & \varphi_{n-2} - \frac{\Delta \varphi}{2} & \varphi_{n-1} - \frac{\Delta \varphi}{2} & -2\varphi_{n-1} & \varphi_n + \frac{\Delta \varphi}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \varphi_n - \frac{\Delta \varphi}{2} & -2\varphi_n \\
\end{vmatrix}$$

For the known $\alpha$, the problem of finding $\lambda = \beta^3$ is a generalized eigenvalue problem. In the same way, knowing $\beta$, it is easy to find $\alpha$. The discrepancy with the numerical analytical solution is less than 0.01%.

The proposed technique, in contrast to analytical solutions, allows solving the problem with arbitrary fixation of the rod ends. It is also possible to take into account the length-varying stiffness and heterogeneity of the rod. Test problems showed good agreement with the literature data. In the future, it is planned to develop a calculation method taking into account creep deformations.

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