Mass-centering characteristics of solids within quasi-rotation surfaces

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Abstract. The methods for generation of digital 3D-models of solids within one sheet of a quadrilateral surface of quasi-rotation are described. The resulting models are used to create solids by means of 3D-printing. When analysing the physical properties of solids, an assumption is made about the specificity of the location of their centres of mass. To verify this assumption, the capabilities of the automatic projection are applied. Based on the results obtained, it is found that there is a certain solid - an individual case of the studied solids, the centre of mass of which is located in the geometric centre of its outline, which is a circle. The error curve method is applied to determine the initial parameters of modeling the desired object. As a result of the search, a digital and solid-state model of a solid within a single sheet of the quasi-rotation surface, whose center of mass coincides with the geometric center of its outline, are obtained. The produced solid has properties that partially coincide with the properties of rotation solids. Such a coincidence is not obvious. In general, the center of mass of the studied solids is not located at the indicated point.

1. Introduction
In reports [1,2,3] the method for rotation of points around curved axes of the second order has been described. The report [4] provides theoretical background for the described method, for which the term "quasi-rotation" is applied. Using quasi-rotation, just as with rotation, one can create surfaces. Such surfaces were studied in [5]. By means of a graphical method for determining the order of the surface, it is found that with a quasi-rotation of the circle around the ellipse, a surface of the twentieth order is formed. This surface is self-intersecting and self-touching, which makes it impossible to create solids within the surface. However, quasi-rotation surfaces are quadrilateral. Some of its individual sheets do not have self-intersections and self-touches and can limit geometric solids.

2. Formulation of the problem
In the sources [4,5] it was reported that material models of solids limited to a single sheet of the quasi-rotation surface were created using 3D printing. The initial four-sheeted surface was obtained by quasi-rotation of a circle with a radius of 10 mm around the elliptic axis with semi-axes of 20 mm and 30 mm. The centre of the circle is located in the centre of the ellipse. The model of this solid has an internal shape of a through hole. In this regard, taking into consideration the capabilities of modern 3D printing, it is decided to perform two separate halves of the solid, and then glue them together. The model of the half of the solid is obtained by dividing it with ab a plane, in which the axis and the surface generatrix lie. The picture of this object is shown on figure 1.
Figure 1. Photograph of the half of the material solid model within a single sheet of quasi-rotation surface, based on a horizontal plane support.

Obviously, this object contacts with the horizontal plane of the support at a point that is determined by the location of its center of mass. The position of the solid is balanced. Visual check makes it possible to assume that the flat part of this object’s surface in its balanced position is parallel to the support plane. If so, then the center of mass of the whole solid is located not only on its axis of symmetry, but also in the center of its profile outline, which is a circle. Based on the above material, the following tasks are formulated:

- Investigate the mass-centering characteristics of solids within a single sheet of the quasi-rotation surface, specified by a circle generatrix, whose center coincides with the center of the elliptic axis.
- Determine, whether the assumption about the specific location of the center of mass of the studied solid is correct, namely, if the position of the center of mass of the solid coincides with the geometric center of its profile axial section.

3. Theory
Quasi-rotation is a geometric interrelation in which any given point, lying in the same plane as a conic, generally interrelates to four circles, whose planes are perpendicular to the plane of the conic, and are determined according to the constructions shown in figure 2. In detail, the theoretical foundations of the "quasi-rotation" interrelation are given in works [1,4]. The algorithm of the produced geometric constructions is described in the report [2].

Figure 2. Geometric construction of the circles’ projections corresponding to the point $A$, with the quasi-rotation of the point $A$ around to the elliptic axis $i_e$.

In report [2] a designation system for geometric objects within the constructive quasi-rotation algorithm is developed. All objects shown in figure 2 are marked according to this system. For example, $k_1''$ is a circle formed by the rotation of point $A$ around the elliptical axis $i_e$ regarding to the
focus $F_1$ and the far center of the quasi-rotation $S_1''$. Point $A$ in a quasi-rotation around the focus $F_1$ and the far center $S_1''$ will occupy the position of point $A_1''$. An arbitrary point $A$ lying in the plane of the axis $i_e$, is interrelated with four circles by means of the apparatus $QRT_e$ (e- ellipse, Q- quasi, R-rotation, T- transformation):

$$QRT_{ie}(A) = k_1', k_1'', k_2', k_2''$$

(1)

Quasi-rotation can be applied not only to individual points, but also to lines of various shapes. In this report surfaces formed by the quasi-rotation of a circle around an elliptical axis are discussed. Then, by analogy with expression (1) for point $A$, the following expression for the circle $l$ is written:

$$QRT_{ie}(l) = \alpha = \beta_1', \beta_1'', \beta_2', \beta_2''$$

(2)

where $\alpha$ is the four-sheeted surface of the quasi-rotation, and $\beta_1', \beta_1'', \beta_2', \beta_2''$ are its sheets, each of which is obtained by rotating the circle $l$ around the corresponding focus and center of the quasi-rotation. The surface determinant $\alpha$ has the form:

$$\alpha(l,i)[li = QRT_i(l)]$$

(3)

The general example of the shape and corresponding position of the generatrix of the circle $l$ and the elliptic axis $i$ is shown in figure 3.

![Figure 3](image)

**Figure 3.** The general example of the generatrix of the circle and the elliptic axis location within the geometric part of the surface determinant of the quasi-rotation.

The designations of the curves parameters seen in figure 3 can be used to write down the determinant (3) of the surface $\alpha$:

$$\alpha(l(R,y_0^1,x_0^1),i(a,b,y_0^0,x_0^0))[li = QRT_i(l)]$$

(4)

If we substitute in (4) the values of the parameters of the surface studied in this work, we get:

$$\alpha(l(R,0,0),i(20,30,0,0))[li = QRT_i(l)]$$

(5)

According to (5), the center of the circle $l$ coincides with the center of the ellipse $i_e$. Consequently, the geometric construction of the sheets $\beta_1'$ and $\beta_2'$ of the surface $\alpha$ will be symmetric to each other via Oy, as well as $\beta_1''$ and $\beta_2''$:

$$\beta_1' = AS_{oy}(\beta_2'), \beta_1'' = AS_{oy}(\beta_2'')$$

(6)
Figure 4 shows three projections of the digital model of the surface $\alpha$ obtained by computer mathematical modeling. This model is a three-dimensional graph. Also figure 4 shows the shape and interrelated location of the pair axis-generatrix.

The surface $\alpha$ is symmetric via each of the three coordinate planes. Its symmetry via the plane $xy$ explains the equalities (6). Four sheets ($\beta_1', \beta_1'', \beta_2', \beta_2''$) of the surface $\alpha$ limit four pairwise equal solids:

$$B_1' = B_2', \quad B_1'' = B_2''$$  \hspace{1cm} (7)

Taking into account equalities (7), let us consider the solids $B_1'$ и $B_2''$. In this report the determinants differ only in the value $R$ for all surfaces $\alpha$. In this connection, the following notation was used to symbolize the solids - $B'_1(RP)$, where $P$ is equal to the radius of the circle generatrix of the surface $\alpha$. Figure 5d shows that self-intersection of the solid $B_2''(R10)$ along the curve $g$. Consequently, this solid cannot be a whole material object. The properties of solids of the type $B_1'(R10)$ are studied in this work.
The constructive method for building the models of the studied solids was used. The constructive method for surface modeling is discussed in the reports [6,7]. The constructive description of the quasi-rotation apparatus is provided in [2]. 3D models of solids $B_1^1(R10)$ can be created by means of CAD. As the quasi-rotation surface is a set of circles, it is necessary to use CAD tool “surface from curve network” for creation of its 3D model. Generating curves are determined as a result of quasi-rotation of separately taken points of the initial generatrix (figure 5c). Figure 5b shows the ellipse $l_1$, which is the axis of the quasi-rotation, circle $l$ and the projection of the circle $k_1$. The method for constructing projections of the circle $k_1$ is described in reports [1,2,5]. The material of the part is the alloy GOST 4832-95.
The calculation shows the presence of the dislocation between the center of mass and the geometric center of the profile outline of the solid (figure 6a). The dislocation value is 1.5605 mm. The obtained results indicate that, generally, the solids \( B'(RP) \) have eccentricity in regards to \( j \) axis. Next, the dependence of the dislocation of the center of mass \( \Delta C \) on the radius of the generatrix of the circle \( l \) is studied. Figure 6b shows that the center of mass of the solid \( B_2(Q_{20}) \) is removed to the opposite direction from the geometric centre of its profile section than that of the solid \( B'_2(R10) \). This observation indicates that there exists the solid \( B'_2(RP) \) with zero eccentricity. In other words, for a given elliptical axis there exists a value of the radius of the circle \( l \) generatrix, at which the center of mass of the solid \( B'_2(RP) \) is located in the geometric centre of its profile axial section. This value is searched using the error curve method. For this purpose, the models of solids with the radius of the initial generatrix of the circle of 5, 12.5 and 15 mm are created. The rest of the parameters were not changed. The obtained mass-centering characteristics are used to construct the error curve \( q \) (figure 7). This curve illustrates the dependence of the eccentricity value \( \Delta C \) on the radius \( R \) of the generatrix of the circle \( l \).

![Image](image.png)

**Figure 7.** The error curve

The parameters of the created models of solids, such as mass, surface area, volume, as well as the dislocation value of the center of mass of the solid from the geometric center of its profile axial section, are given in table 1.

| Radius of a circle generatrix (mm) | Dislocation value of the center of the mass (mm) | Solid mass (g) | Volume (mm\(^3\)) | Surface area (mm\(^2\)) |
|-----------------------------------|-----------------------------------------------|----------------|-------------------|-------------------------|
| 5                                 | 7.2808                                        | 616            | 78818             | 20171                   |
| 10                                | 1.5605                                        | 1874           | 239682            | 29658                   |
| 12.24                             | 0.0022                                        | 2501           | 319842            | 32218                   |
| 12.27                             | -0.0185                                       | 2509           | 320908            | 32233                   |
| 12.5                              | -0.1351                                       | 2581           | 330085            | 32512                   |
| 15                                | -1.3728                                       | 3324           | 425164            | 34937                   |
| 20                                | -3.0511                                       | 4915           | 628640            | 39342                   |

The data from the table 1 can be used when checking the calculation formulas for the corresponding values. Calculation formulas should be derived on the basis of the mathematical description of the studied method of shaping. It’s worth noting that the values of mass, volume and surface area are directly dependent on the radius of the generatrix of the circle.

### 4. Results

Based on the obtained results, using the error curve and the iteration method, the radius of the generatrix of the circle \( l \) with the eccentricity of 0.0022 mm is found (figure 7). Further approximations do not give lower eccentricity. As a result, the accuracy limit of the search method has
been reached. Note that the indicated eccentricity value is significantly lower than the accuracy that is available with 3D printing. The solid $B_1' (R \ 12.24)$ is shown in figure 8.

Figure 8. The position of the center of mass of the solid $B_1' (R \ 12.24)$.

Figure 9 shows halves of the material models of solids $B_1' (R \ 10), B_1' (R \ 12.24)$ and $B_1' (R \ 20)$ from left to right accordingly. The displacement of their centers of mass determines the angle of inclination of the plane that limits them to the plane of the horizontal support. The plane within half of the body $B_1'(R12.24)$ is parallel to the plane of the support.

Figure 9. Photos of the material models of the halves of the solids $B_1' (R \ 10), B_1' (R \ 12.24)$ and $B_1' (R \ 20)$ from left to right.

The curve $p$ in figure 6 is a spline passing through the centers of mass of horizontal plane sections of the solid $B_1'(R12.24)$. Section planes are spaced in 4 mm increments. One of these sections belongs to the plane $G$. Its center of mass $Q$ is moved to the right of the vertical axis of the solid. Such a removal creates a rotation moment. In this case, the centers of sections’ masses located in the area $L$ (figure 8)
are shifted to the left. The shape of the solid shows that the resulting moment of all flat sections is zero.

5. Discussion
If we omit the value of the eccentricity (ΔC = 0.0022 mm), then the center of mass of the solid $B_1'(R12.24)$ coincides with the centre of the circle which is its profile outline. It means that this solid installed in a horizontal flat support, will not have the moment of rotation. The touch point of the solid and support should lie on the circle $k_0$.

Figure 10. Photographs of the model of the solid $B_1'(R12.24)$ located on the flat horizontal support.

Figure 10 shows various stable positions of the solid $B_1'(R12.24)$ on a flat horizontal support. This solid is balanced not only regarding to the frontally projecting axis $v$, like the solids $B_2'(R10)$ and $B_1'(R20)$, but also regarding to any straight line lying in the plane $W$ and passing through the point $C$, such as $v'$ and $v''$ for any $\phi$ (figure 8). Figure 8 shows straight axis $j$ regarding to which the solid $B'(R12.24)$ is also balanced. So, the solid can roll along the horizontal surface touching it with a circle $k_0$, and not showing any beating. Note that the tangent plane $K$ to the surface of the solid intersects the axis $j$ at point $N$ and is not perpendicular to it. Consequently, this solid cannot be in the state of a balance, relying on point $N$ on a horizontal flat surface. Using the CAD tool “normal to a curve through a given point”, a straight line $CM$ and $CM'$ is generated. There are no other normals passing through $C$ and not lying in the plane $W$ to the horizontal outline of the solid $B'(R12.24)$. This observation indicates the existence of the curve $m$ passing through points $M$ and $M'$ and belonging to the surface of the studied solid. Any line passing through $C$ and intersecting $m$ will be a normal to the surface of the solid. If the touch point of the solid surface and the horizontal surface of the support lies on the curve $m$, then this position will be stable.

6. Conclusions
In this work, the mass-centring properties of a solid within one of the four layers of the quasi-rotation surface are studied. Under the consideration there is the surface whose centre of the initial generatrix of the circle coincides with the centre of its elliptical axis. A number of surfaces with given parameters is also considered with a variable radius of the initial generatrix. Based on the results of the research, there is a conclusion that for an arbitrary elliptic axis there exists a value of the radius of the generatrix, for which the solid obtained by quasi-rotation has specific mass-centring properties. With obvious asymmetry, such a solid presents properties inherent in rotation solids. The described unusual combination of the shape and object’s properties can be used in the design of parts of mechanisms, enclosures and frame constructions. The described geometric properties of quasi-rotation solids can be applied in recognizing them when analysing images of the objects of phenomena around us [8]. Possible technologies for generating of such objects will be described in the future reports. It is also necessary to describe methods for modeling solids limited by separate layers of the surface of a quasi-rotation in accordance with predetermined shape parameters and mass-centring characteristics.

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