A discrete leading symbol and spectral asymptotics for natural differential operators

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Dedicated to the memory of Irving Segal

Abstract

We initiate a systematic study of natural differential operators in Riemannian geometry whose leading symbols are not of Laplace type. In particular, we define a discrete leading symbol for such operators which may be computed pointwise, or from spectral asymptotics. We indicate how this can be applied to the computation of another kind of spectral asymptotics, namely asymptotic expansions of fundamental solutions, and to the computation of conformally covariant operators.

1 Introduction

In this paper, we would like to set the stage for a better understanding of natural differential operators in Riemannian (and Riemannian spin) geometry whose leading symbols are not simply powers of $|\xi|^2$. Such operators or leading symbols have been called, in various contexts, non-minimal, nonscalar, non-Laplace type, and even exotic. Among the potential applications are the computation of resolvent and heat operator asymptotics of elliptic operators with nonscalar leading symbol, and the computation of explicit formulas for conformally invariant differential operators. Our principal tool will be an assignment of a finite-dimensional commutative algebra $A(\lambda)$ to each irreducible $\text{SO}(n)$ or $\text{Spin}(n)$ bundle $V(\lambda)$. (The label $\lambda$ is explained below.) This algebra simultaneously encodes information on the spectrum of the leading symbol (an operator on a finite-dimensional space), and spectral asymptotics of the realization of a natural differential operator on the standard sphere $S^n$ (an operator in an infinite-dimensional space). Thus it relates global and pointwise information. A version of this viewpoint was used in [10] to get sharp improved Kato constants for solutions of natural first-order elliptic systems on Riemannian (or Riemannian spin) manifolds. These constants are essentially bottom eigenvalues of certain natural symbols. For other applications, for example to the computation of spectral invariants of natural differential operators with nonscalar leading symbol, the symbol’s complete spectral resolution is required, and this paper provides that information.
2 Principal symbols and representation theory

2.1 Foundations

Let $H(n)$ be $SO(n)$ or $Spin(n)$, and let $M$ be an $n$-dimensional $H(n)$ manifold. That is, if $H(n) = SO(n)$, we require $M$ to be oriented and Riemannian; if $H(n) = Spin(n)$, we require $M$ to a a Riemannian spin manifold. Let $\mathcal{V}(\lambda) = \mathcal{F} \times_{\lambda} V$ be the vector bundle canonically associated to a finite-dimensional irreducible representation of $H(n)$ and the bundle $\mathcal{F}$ of $H(n)$-frames (i.e. oriented orthonormal frames, or Riemannian spin frames). Note that if we have spin structure, we may take $H(n)$ to be $Spin(n)$ even for $SO(n)$ bundles, since we may always compose with the covering homomorphism $Spin(n) \rightarrow SO(n)$.

Let $(\tau, T)$ be the defining representation of $SO(n)$; then $\mathcal{V}(\tau)$ is the cotangent (or tangent) bundle. It is well known that irreducible $SO(n)$ bundles are $H(n)$-isomorphic to tensor bundles; i.e. direct summands of tensor powers of $\mathcal{V}(\tau)$; in fact, this is guaranteed by the faithfulness of the representation $\tau$. Similarly, because the spin representation $(\sigma, \Sigma)$ (which splits into two irreducible direct summands $(\sigma_{\pm}, \Sigma_{\pm})$ in even dimensions) is faithful, all irreducible $Spin(n)$ bundles are summands in some tensor power of $\mathcal{V}(\sigma)$. Since $\sigma \otimes \sigma$ and $\sigma_{\pm} \otimes \sigma_{\pm'}$ are $Spin(n)$ isomorphic to tensor bundles (in fact, differential form bundles), each proper $Spin(n)$ bundle (i.e., each $Spin(n)$ bundle which is not an $SO(n)$ bundle) is realizable as a direct summand of some $\sigma \otimes \tau \otimes \cdots \otimes \tau$. That is, each may be realized as a bundle of tensors with spinor coefficients. A given $\mathcal{V}(\lambda)$ may not have a distinguished real form, so we generally think of our sections as being complex. It is possible, however, to speak of real cotangent vectors, and this is important in analytic considerations like those of [16, 10].

The classical branching rule gives the direct sum decomposition an irreducible $H(n)$-module $(\lambda, V)$ when restricted to a copy of $H(n-1)$ which is imbedded in the standard way. (On the Lie algebra level, relative to some orthonormal basis of the defining module, the subalgebra should be that of matrices living in the upper left $(n-1) \times (n-1)$ block. To see that nonstandard imbeddings are possible, see [8], Sec. 3.c.) It is known (see Sec. 3.3 below) that the branching rule is multiplicity free:

$$\lambda|_{H(n-1)} \cong_{H(n-1)} \beta_1 \oplus \cdots \oplus \beta_{b(\lambda)},$$

(1)

where the $\beta_i$ are irreducible representations of $H(n-1)$, and $\beta_i \cong_{H(n-1)} \beta_j$ $\iff$ $i = j$. We put

$$B(\lambda) = \{\beta_1, \ldots, \beta_{b(\lambda)}\}.$$

In particular, the above defines a numerical invariant $b(\lambda)$, the cardinality of the set of branches $B(\lambda)$.

By Weyl's invariant theory and the above remarks on tensor and tensor-spinor realizations, an $H(n)$-equivariant differential operator on (sections of) $\mathcal{V}(\lambda)$ is built polynomially from the covariant derivative (with respect to the Levi-Civita or Levi-Civita spin connection $\nabla$), the Riemann curvature $R$, the metric $g$ and its inverse $g^2$, the volume form $E$, and, if applicable, the fundamental tensor-spinor $\gamma$. Of course, such objects are not really operators, but functors which assign operators to $H(n)$ manifolds. (Similarly, the $\mathcal{V}(\lambda)$
are not bundles, but functors assigning bundles.) To be of universal order \( p \), i.e. order \( p \) on every \( H(n) \)-manifold, such an operator must have the form

\[
D = D_{\text{princ}} + D_{\text{lower}},
\]

where \( D_{\text{princ}} \) is a sum of monomials in all the above ingredients except \( R \), each monomial containing \( p \) occurrences of \( \nabla \), and \( \text{ord}(D_{\text{lower}}) \leq p - 1 \) on each \( H(n) \)-manifold. \( R \) must be is missing from the list of ingredients in \( D_{\text{princ}} \), to avoid the vanishing of \( D_{\text{princ}} \) on flat manifolds (i.e., to avoid contradicting the universality of the order \( p \)). The decomposition (2) is not unique, as commutation of covariant derivatives produces lower order terms. Nevertheless, we may read off from this decomposition the fact that the leading symbol \( \sigma_p(D) \), when viewed as a section of \( \text{Hom}(\text{SYM}^p \otimes V(\lambda), V(\lambda)) \), is parallel; i.e., annihilated by the covariant derivative, since \( g \), \( E \), and \( \gamma \) are. Here \( \text{SYM}^p \) is the bundle of symmetric \( p \)-tensors. \( \sigma_p(D) \) is also \( H(n) \)-invariant; that is, it is actually in \( \text{Hom}_{H(n)}(\text{SYM}^p \otimes V(\lambda), V(\lambda)) \). It is universal, in the sense of being given by a consistent expression in \( g \), \( g^2 \), \( E \), and, if applicable, \( \gamma \), at all points of all \( H(n) \)-manifolds. In fact, we get such leading symbols by “promoting” to the bundle setting actions of the \( H(n) \) module \( \text{sym}^p \) of symmetric \( p \)-tensors on \( (\lambda, V) \); that is, elements of \( \text{Hom}_{H(n)}(\text{sym}^p \otimes \lambda, \lambda) \).

It will also be useful to speak of the reduced order of an \( H(n) \)-operator or leading symbol. Given an \( H(n) \)-symbol \( \theta(\xi) \) of homogeneous degree \( p \), write \( \theta(\xi) = |\xi|^{2k}\theta_{\text{red}}(\xi) \) for \( k \) as large as possible; the reduced symbol is \( \theta_{\text{red}}(\xi) \). The reduced homogeneous degree, \( p - 2k \), is well-defined, and may also be detected as the degree of the restriction of \( \theta \) to the unit \( \xi \)-sphere (i.e., the unit sphere in the module \( (\tau, T) \)); we shall denote this restriction by \( \tilde{\theta} \). In fact, under the identification of symmetric tensors and homogeneous polynomials, the decomposition into trace-free tensors of various degrees corresponds to the decomposition into terms of the form

\[
|\xi|^{2k} \cdot (\text{spherical harmonic}).
\]

If \( \theta \) has homogeneous order \( p \), then

\[
\tilde{\theta} = \tilde{\theta}_p + \tilde{\theta}_{p-2} + \cdots + \tilde{\theta}_{0 \text{ or } 1},
\]

where \( \tilde{\theta}_q \) corresponds to an element of \( \text{Hom}_{H(n)}(\text{tfs}^q \otimes \lambda, \lambda) \).

We define the algebra \( \mathcal{A}(\lambda) \) of \( H(n) \)-equivariant principal symbols as that generated by the \( \tilde{\theta} \) above. The difference between a principal and a leading symbol is that we allow ourselves to add symbols of different orders within the principal symbol algebra. Given a choice of a real \( \xi \) on the unit sphere in \( (\tau, T) \), we get a decomposition (4) of the module \( \lambda \) under the \( H(n-1) \) subgroup fixing \( \xi \). By Schur’s Lemma and the multiplicity free nature of (4), if \( \kappa \in \mathcal{A}(\lambda) \), then \( \kappa(\xi) \) acts as multiplication by some scalar \( \mu_i \) on each \( \beta_i \) summand. By the invariance of \( \kappa \) and the transitivity of \( H(n) \) on \( (\tau, T) \), the eigenvalue list \( \mu_i \) is independent of the choice of \( \xi \), and \( \mu_i \) is always attached to the \( \beta_i \) summand. Thus \( \mathcal{A}(\lambda) \) is isomorphic to the algebra of complex-valued functions on the finite set \( B(\lambda) \), and, in particular, is commutative.

**Definition 2.1** The map \( \beta_i \mapsto \mu_i \) defined above is the discrete leading symbol. That is, we define \( k : \mathcal{A}(\lambda) \to \text{maps}(B(\lambda), \mathbb{C}) \) by

\[
k(\kappa)(\beta_i) = \mu_i.
\]
Discrete leading symbol

If $D$ is an $H(n)$-invariant differential operator of order $p$ on $\mathbb{V}(\lambda)$, we put

$$K(D) = k(\tilde{\sigma}_p(D)).$$

(4)

In other words, the discrete leading symbol is the spectral resolution of a principal symbol, or of the leading symbol, at a given unit $\xi$. The eigenvalues, with multiplicities, are independent of $\xi$, and the eigenspaces move in a predictable way, according to the action of $H(n)$.

There is a grading of $\mathcal{A}(\lambda)$ by order, in which an action of $\text{tfs}^p$ falls into $\mathcal{A}^p(\lambda)$. In particular, we shall give the name $d(\lambda)$ to the maximal $p$ for which $\text{tfs}^p$ acts:

$$d(\lambda) := \max\{p \mid \text{Hom}_{H(n)}(\text{tfs}^p \otimes \lambda, \lambda) \neq 0\}.$$

The behavior of this grading under multiplication is somewhat involved; in fact, in view of (3), this is exactly the problem of decomposing products of spherical harmonics into sums of spherical harmonics. (3) does, however, allow a simple description of the behavior of multiplication under a coarser grading. Let $\mathcal{A}_0(\lambda)$ (resp. $\mathcal{A}_1(\lambda)$) be the direct sum of the $\mathcal{A}^p(\lambda)$ over even (resp. odd) $p$. Then

$$\mathcal{A}_i(\lambda)\mathcal{A}_j(\lambda) \subset \mathcal{A}_{i+j}(\lambda),$$

where the addition in the subscripts is modulo 2. It is often the case that the odd part vanishes (see Theorem 1.2 below). Since the reduction of any $|\xi|^{2k}$ is 1, any principal symbol which is purely even or odd, i.e. any symbol $\kappa$ in $\mathcal{A}_0(\lambda)$ or in $\mathcal{A}_1(\lambda)$, will be represented by an actual differential operator; that is, there will be some invariant $D$ of some order $p$ with $\tilde{\sigma}_p(D) = \kappa$. Indeed, to get such a $D$, first get an invariant homogeneous polynomial $\theta$ in $\xi$ with $\theta = \kappa$; then replace each $\xi$ by an $-\sqrt{-1}\nabla$ in the tensorial expression for $\theta$.

We summarize some of the above considerations in:

**Proposition 2.2** The algebra $\mathcal{A}(\lambda)$ of principal symbols on $\mathbb{V}(\lambda)$ is isomorphic to the algebra of complex-valued functions on the finite set $\mathcal{B}(\lambda)$. In particular, it is commutative, and is generated by $b(\lambda)$ fundamental projections. Each principal symbol in $\mathcal{A}_0(\lambda)$ or in $\mathcal{A}_1(\lambda)$ is the reduced leading symbol of an $H(n)$-invariant operator of order at most $d(\lambda)$.

To see the fundamental projections in a very familiar (but deceptively simple) case, consider the bundle $\Lambda^k$ of differential $k$-forms, $0 < k < (n - 2)/2$. The fundamental projections are $\iota(\xi)\varepsilon(\xi)$ and $\varepsilon(\xi)\iota(\xi)$, where $\varepsilon$ and $\iota$ are exterior and interior multiplication. Here

- there are 2 fundamental projections;
- each has degree 2 in $\xi$;
- each is represented by a differential operator (the Hodge operators $\delta d$ and $d\delta$);
- the orthogonality of the projections persists on the operator level: $\delta dd\delta = d\delta d\delta = 0$.

By way of contrast, in the general $\mathbb{V}(\lambda)$ case,
Discrete leading symbol

• there are \( b(\lambda) \) fundamental projections, and \( b(\lambda) \) can be arbitrarily large, depending on \( \lambda \);
• the maximal degree of a projection in \( \xi \) is \( d(\lambda) \), which can be arbitrarily large, depending on \( \lambda \);
• the fundamental projections need not be represented by differential operators;
• even if they are, there is generally no choice of such operators \( D_i \) with \( D_i D_j = 0 \) for \( i \neq j \).

2.2 A calculation involving translation and tensor products of representations

A standard computation from the theory of induced representations actually leads immediately to the structure of the principal symbol algebra. Let \( \text{1} \) be the trivial representation of \( H(\text{n} - \text{1}) \), and let \( (\lambda, V) \) and \( (\mu, W) \) be finite-dimensional representations of \( H(\text{n}) \). We claim that

\[
\text{Hom}_{H(\text{n})}((\text{Ind}_{H(\text{n}-\text{1})}^{H(\text{n})} 1) \otimes \lambda, \mu) \cong \text{Hom}_{H(\text{n}-\text{1})}(\lambda|_{H(\text{n}-\text{1})}, \mu|_{H(\text{n}-\text{1})}).
\]  

(5)

Indeed,

\[
(\text{Ind}_{H(\text{n}-\text{1})}^{H(\text{n})} 1) \otimes \lambda \cong_{H(\text{n})} \text{Ind}_{H(\text{n}-\text{1})}^{H(\text{n})} (1 \otimes \lambda|_{H(\text{n}-\text{1})}) \cong_{H(\text{n})} \text{Ind}_{H(\text{n}-\text{1})}^{H(\text{n})} (\lambda|_{H(\text{n}-\text{1})});
\]

this is a form of the translation principle. By Frobenius Reciprocity,

\[
\text{Hom}_{H(\text{n})}(\text{Ind}_{H(\text{n}-\text{1})}^{H(\text{n})} (\lambda|_{H(\text{n}-\text{1})}), \mu) \cong \text{Hom}_{H(\text{n}-\text{1})}(\lambda|_{H(\text{n}-\text{1})}, \mu|_{H(\text{n}-\text{1})}),
\]
proving (5). In particular, if \( \lambda = \mu \), we get

\[
\text{Hom}_{H(\text{n})}((\text{Ind}_{H(\text{n}-\text{1})}^{H(\text{n})} 1) \otimes \lambda, \lambda) \cong \text{End}_{H(\text{n}-\text{1})}(\lambda|_{H(\text{n}-\text{1})}).
\]

(6)

This shows that the leading symbol algebra must isomorphic to \( \text{End}_{H(\text{n}-\text{1})} (\lambda) \), which, in view of the multiplicity-free nature of the branching rule, is another realization of the complex-valued functions on \( B(\lambda) \).
2.3 The selection rule and Stein-Weiss operators

An important class of $H(n)$-equivariant differential operators on $V(\lambda)$ are the generalized gradients of Stein and Weiss [16]. The starting point is the selection rule, which describes the $H(n)$-decomposition of $\tau \otimes \lambda$. (Recall that $\tau$ is the defining representation of $\mathfrak{so}(n).$)

As it happens, this decomposition, like the branching rule, is also multiplicity free:

$$\tau \otimes \lambda \cong_{H(n)} \mu_1 \oplus \cdots \oplus \mu_N(\lambda),$$

where the $\mu_u$ are irreducible representations of $H(n)$, and

$$\mu_u \cong_{H(n)} \mu_v \iff u = v.$$  

In particular, this defines a numerical invariant $N(\cdot)$ on $\hat{H}(n)$. On the bundle level,  

$$T^*M \otimes V(\lambda) \cong_{H(n)} V(\tau) \otimes V(\lambda) \cong_{H(n)} V(\sigma_1) \oplus \cdots \oplus V(\sigma_N(\lambda)).$$

The covariant derivative $\nabla$ carries sections of $V(\lambda)$ to sections of $T^*M \otimes V(\lambda)$. Because the selection rule is multiplicity free, we may project $\nabla$ onto the unique summand of covariance type $\mu_u$ to obtain our generalized gradient:

$$G_{\lambda\sigma_u} = G_u = \text{Proj}_u \circ \nabla.$$  

Up to normalization and up to $H(n)$-isomorphic realization of bundles, some examples of gradients or direct sums of gradients are the exterior derivative on forms, the conformal Killing operator on vector fields, the Dirac operator, the twistor operator, and the Rarita-Schwinger operator. In fact, every first-order Spin($n$)-equivariant differential operator is a direct sum of gradients [12]. In particular, the formal adjoint of a gradient is also a gradient. This immediately gives us access to a very important class of operators carrying $V(\lambda)$ to itself, namely the $G_u^*G_u$. For some $\lambda$ in odd dimensions, there is a self-gradient; that is, some $\mu_u$ is $H(n)$-isomorphic to $\lambda$ itself, so that there is a natural first-order operator $D_{\text{self}}$ carrying sections of $V(\lambda)$ to sections of $V(\lambda)$. (The most familiar examples are the Dirac operator on spinors, the Rarita-Schwinger operator on twistors, and the operator $*d$ on $(n-1)/2$-forms.) We shall show below that the leading symbols of the $G_u^*G_u$ and, when it exists, $D_{\text{self}}$, generate the principal symbol algebra $A(\lambda)$.

**Remark 2.3** In case $\lambda$ admits a self-gradient, that is if $\lambda$ itself is a selection rule target $\mu_u$ for $\lambda$, the target of $G_u$ is “born” as a subbundle $W$ of $T^*M \otimes V(\lambda)$ with $W \cong_{\text{Spin}(n)} V(\lambda)$. If we would like to use the same realization of $V(\lambda)$ as both source and target bundle for a realization $D_{\text{self}}$ of $G_u$, we need a choice of normalization. First, normalize the Hermitian inner product on $T^*M \otimes V$ so that $|\xi \otimes v|^2 = |\xi|^2|v|^2$; then define $D_{\text{self}}$ so that

$$D_{\text{self}}^2 = G_u^*G_u.$$  

(7)

Note that this determines $D_{\text{self}}$ only up to multiplication by $\pm 1$. This sign ambiguity is already apparent in the definition of the Dirac operator on spinors as $\gamma^a \nabla_a$, since changing $\gamma$ to $-\gamma$ does not disturb the Clifford relations, nor the relation $\nabla\gamma = 0$. 
As shown in [6], all second-order operators are linear combinations of the $G^*_u G_u$, modulo lower-order operators. The leading symbols of the $G^*_u G_u$, however, are generally not linearly independent. In fact, as shown in [6], Theorem 5.10, their leading symbols form a space of dimension

$$t(\lambda) := \left\lfloor \frac{(N(\lambda) + 1)}{2} \right\rfloor.$$  

The full set of linear relations among the leading symbols of the $G^*_u G_u$ is also given explicitly in [6], Theorem 5.10.

### 2.4 Numerical invariants of bundles

To recap, we have defined the following numerical invariants of an irreducible $H(n)$-bundle $V(\lambda)$. All of these are really invariants of the underlying representation $\lambda$.

- Number of selection rule summands: $N(\lambda)$
- Maximal degree in $\mathcal{A}(\lambda)$: $d(\lambda)$
- Dimension, thus number of fundamental projections, of $\mathcal{A}(\lambda)$: $b(\lambda)$
- Number of linearly independent $\sigma_2(G^*_u G_u)$: $t(\lambda)$.

Of these, $t(\lambda)$ and $N(\lambda)$ are related by (8). We might add the following, to take account of the even/odd grading:

- Maximal degree in $\mathcal{A}_i(\lambda)$ ($i = 0, 1$): $d_i(\lambda)$
- Dimension, thus number of fundamental projections, of $\mathcal{A}_0(\lambda)$: $b_0(\lambda)$
- Dimension, thus number of fundamental projections, of $\mathcal{A}_1(\lambda)$: $b_1(\lambda)$.

So far, we have done nothing explicit to actually compute these invariants, the fundamental projections in $\mathcal{A}(\lambda)$, or any discrete leading symbols. We shall now remedy this situation.

### 3 The weight game

#### 3.1 Parameterization by dominant weights

Irreducible representations of Spin($n$), and thus irreducible associated Spin($n$)-bundles, are parameterized by dominant weights $(\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}_\ell \cup (\mathbb{Z} + \mathbb{2Z})_\ell$, $\ell = \lfloor n/2 \rfloor$, satisfying the inequality constraint

$$\begin{align*}
\lambda_1 &\geq \ldots \geq \lambda_\ell \geq 0, & n \text{ odd}, \\
\lambda_1 &\geq \ldots \geq \lambda_{\ell-1} \geq |\lambda_\ell|, & n \text{ even}.
\end{align*}$$

The dominant weight $\lambda$ is the highest weight of the corresponding representation. The representations which factor through $\text{SO}(n)$ are exactly those with $\lambda \in \mathbb{Z}_\ell$.

Note that in the previous sections, we have used $\lambda$ as a notation for the representation itself. This is a standard abuse of notation, which we shall continue: the highest weight parameter of an irreducible representation will be used as a synonym for the representation. We shall also denote by $\chi(n)$ the set of dominant $H(n)$ weights.
3.2 The selection rule

We shall discuss several familiar examples of bundles and identify their highest weights below. One important highest weight is that of the defining representation \( \tau \), namely \((1, 0, \ldots, 0)\). With this, we can explicitly describe the selection rule mentioned above. If \( \lambda \) is an arbitrary irreducible representation of \( \text{Spin}(n) \), then

\[
\tau \otimes \lambda \cong_{H(n)} \sigma_1 \oplus \cdots \oplus \sigma_N(\lambda),
\]

where the \( \sigma_u \) are distinct, and a given \( \sigma \) appears if and only if \( \sigma \) is a dominant weight and

\[
\sigma = \lambda \pm e_a, \quad \text{some } a \in \{1, \ldots, \ell\}, \quad \text{or} \quad \begin{cases} n \text{ is odd, } \lambda_\ell \neq 0, & \sigma = \lambda. \end{cases}
\]  

Here \( e_a \) is the \( a \)-th standard basis vector in \( \mathbb{R}^\ell \). The selection rule follows immediately from the Brauer-Kostant formula; see \([2, 14]\). We shall use the notation

\[
\lambda \leftrightarrow \sigma
\]

for the selection rule: \( \lambda \leftrightarrow \sigma \) if and only if \( \sigma \) appears as a summand in \( \tau \otimes \lambda \). The notation \( \leftrightarrow \) is justified because the relation is symmetric. In fact, one can see a priori that \( \leftrightarrow \) must be symmetric: \( \tau \) is a real representation, and thus is self-contragredient.

3.3 The branching rule

The explicit branching rule for the restriction from \( H(n) \) to \( H(n-1) \) is as follows. By changing \( n \) to \( n-1 \) above we have a parameterization of the irreducible representations of \( H(n-1) \). The branching rule says that for a dominant \( H(n) \)-weight \( \beta \),

\[
\dim \text{Hom}_{H(n-1)}(\beta, \lambda_{|H(n-1)}) = 0 \text{ or } 1, \quad \text{with } \dim \text{Hom}_{H(n-1)}(\beta, \lambda_{|H(n-1)}) = 1 \text{ if and only if } \begin{cases} \lambda_1 - \beta_1 \in \mathbb{Z} \text{ and } \left\{ \begin{array}{ll} \lambda_1 \geq \beta_1 \geq \lambda_2 \geq \cdots \geq \beta_{\ell-1} \geq |\lambda_\ell|, & n \text{ even,} \\ \lambda_1 \geq \beta_1 \geq \lambda_2 \geq \cdots \geq \beta_{\ell-1} \geq \lambda_\ell \geq |\beta_\ell|, & n \text{ odd.} \end{array} \right. \end{cases}
\]  

\[
(12)
\]

We use \( \lambda \downarrow \beta \) or \( \beta \uparrow \lambda \) as an abbreviation for \((12)\). We shall actually also have use for the version of the branching rule that restricts from \( H(n+1) \) to \( H(n) \), so notations like \( \lambda(n \downarrow n-1) \beta \) and \( \alpha(n+1 \downarrow n) \lambda \) will sometimes be helpful. In this connection the following observation will be useful.

**Lemma 3.1** For \( \alpha \in \chi(n+1) \), let \( \check{\alpha} \) be the \([(n-1)/2]\)-tuple \((\alpha_2, \ldots, \alpha_{(n+1)/2})\); that is, \( \alpha \) without its first entry. Then \( \check{\alpha} \in \chi(n-1) \), and given \( \lambda \in \chi(n) \), \( \{ \check{\alpha} \mid \alpha(n+1 \downarrow n) \lambda \} = \{ \nu \mid \lambda(n \downarrow n-1) \nu \} \).

The proof comes directly upon examination of the branching rule. To paraphrase, the branching “offspring” of \( \lambda \in \chi(n) \) are the \( \check{\alpha} \) for the branching “parents” \( \alpha \) of \( \lambda \). The branching rule also allows us to give a formula for the dimension \( b(\lambda) \) of \( A(\lambda) \), based on its identification with \( \text{End}_{H(n-1)}(\lambda_{|H(n-1)}) \):
Lemma 3.2

\[ b(\lambda) = \dim \mathcal{A}(\lambda) = \begin{cases} (\lambda_{i-1} - |\lambda_i| + 1) \prod_{a=1}^{(n-4)/2}(\lambda_a - \lambda_{a+1} + 1), & n \geq 4 \text{ even}, \\ ([|\lambda_i| + 1) \prod_{a=1}^{(n-3)/2}(\lambda_a - \lambda_{a+1} + 1), & n \geq 3 \text{ odd}. \end{cases} \]

To paraphrase, \( B(\lambda) \) consists of lattice points in a rectangular box, whose various widths are determined by the spacing between successive entries of \( \lambda \). Thus “steeper” \( \lambda \) tend to produce larger \( b(\lambda) \). Familiar bundles, like spinors, differential forms, and trace-free symmetric tensors, tend to have relatively “flat” \( \lambda \).

We know that \( \mathcal{A}_p(\lambda) = 0 \) for large \( p \). A more quantitative form of this statement is:

Lemma 3.3 If \( n \geq 3 \), then \( p > 2\lambda_1 \Rightarrow \mathcal{A}_p(V) = 0 \). As a result, \( d(\lambda) \leq 2\lambda_1 \).

Proof. First note that \( \text{tfs}^p \cong_{H(n)} (p,0,\ldots,0) \) (see, e.g., [10]). If \( \text{Hom}_{H(n)}(\text{tfs}^p \otimes \lambda, \lambda) \) is to be nonzero, we must be able to realize \( \lambda \) as \( (p,0,\ldots,0) + \mu \), where \( \mu \) is a weight of \( V(\lambda) \). (This follows, for example, from the Brauer-Kostant formula [2, 14], which expresses the highest weights in summands of the tensor product in terms of the highest weight of one factor, together with all weights of the other factor.) All components of such a \( \mu \) must be \( \leq \lambda_1 \) in absolute value, since otherwise, we would have \( w \cdot \mu \) dominant and \( (w \cdot \mu)_1 > \lambda_1 \) for some element \( w \) of the Weyl group. \((w \cdot \mu)\) is a weight of \( V(\lambda) \) since the Weyl group permutes the weights of any finite-dimensional representation.) This gives

\[ \mu_1 = \lambda_1 - p, \quad |\mu_1| \leq \lambda_1, \]

whence \( p \leq 2\lambda_1 \). The bound on \( d(\lambda) \) is just a restatement, since \( d(\lambda) \) is the maximal \( p \) that occurs. \( \square \)

Remark 3.4 The basic principle of the proof of Lemma 3.3 may be used to get more refined information in special situations, without going through the full tensor product calculation. For example, if \( n \) is even and \( \lambda = (p,\ldots,p) \) with \( p > 0 \), then \( d(\lambda) < 2p \), since

\[ (p,\ldots,p) - (2p,0,\ldots,0) = (-p,p,\ldots,p) \]

is in the Weyl orbit of \( (p,\ldots,p,-p) \), a dominant weight not appearing in \( V(p,\ldots,p) \).

The leading symbol of any gradient \( G_u \) is a relatively familiar object from representation theory, namely the projection of a tensor product onto an irreducible summand. Let \( \xi \) be a vector from the defining representation, and consider

\[ \text{tens}(\xi) : \lambda \rightarrow \tau \otimes \lambda, \]

\[ \nu \mapsto \xi \otimes \nu. \]

Compose with the projection onto the \( \mu_u \) summand of \( \tau \otimes \lambda \) to get a map \( \zeta_u(\xi) \), and use the functoriality of the associated bundle construction to promote this to a bundle map

\[ \nabla(\lambda) \rightarrow \nabla(\sigma_u), \]
also denoted (in a slight abuse of notation) by \( \zeta_u(\xi) \). Then
\[
\sigma_1(G_u)(\xi) = \sqrt{-1} \zeta_u(\xi), \quad \sigma_1(G_u^*)(\xi) = -\sqrt{-1} \zeta_u(\xi)^*.
\]
\( \zeta_u \) may also be described by the formula \([G_u, m_f] = \zeta_u(df)\), where \( m_f \) is multiplication by the \( C^\infty \) function \( f \).

Let \( \sqrt{-1} \Upsilon_{\text{self}}(\xi) \) be the leading symbol of the self-gradient, when it exists. The following will be a consequence of Theorem 4.2 below:

**Theorem 3.5** \( \mathcal{A}(\lambda) \) is generated by the restrictions to the unit-\( \xi \) sphere (i.e., to the unit sphere bundle in the cotangent bundle) of the \( \zeta_u(\xi)^* \zeta_u(\xi) \) unless \( \lambda \leftrightarrow \lambda \), in which case the \( \zeta_u(\xi)^* \zeta_u(\xi) \) for \( \mu \neq \lambda \) together with \( \Upsilon_{\text{self}}(\xi) \) generate. \( \mathcal{A}_u(\lambda) \) is generated, in all cases, by the \( \zeta_u(\xi)^* \zeta_u(\xi) \).

Note that \( \zeta_{\lambda\lambda}(\xi)^* \zeta_{\lambda\lambda}(\xi) \) coincides, when it exists, with \( \Upsilon_{\text{self}}(\xi)^2 \), by (7).

4 Spectral asymptotics on the sphere

Let \( \alpha \) be an irreducible \( H(n + 1) \)-module (identified with its highest weight label when convenient). The \( H(n + 1) \)-finite section space \( \Gamma(\mathcal{V}(\lambda)) \) forms the space of the induced representation in the middle term of the following, and Frobenius Reciprocity supplies the second \( \cong \):

\[
\text{Hom}_{H(n+1)}(\Gamma(\mathcal{V}(\lambda)), \alpha) \cong \text{Hom}_{H(n+1)}(\text{Ind}_{H(n)}^H(n+1) \lambda, \alpha) \cong \text{Hom}_{H(n)}(\lambda, \alpha|_{H(n)}).
\]

Thus the \( H(n + 1) \)-finite section space of \( \mathcal{V}(\lambda) \) is
\[
\Gamma(\mathcal{V}(\lambda)) \cong_{H(n+1)} \bigoplus_{\chi(n+1) \ni \alpha \downarrow \lambda} \alpha. \tag{13}
\]

Let \( \mathcal{V}(\alpha; \lambda) \) be the subspace of \( \Gamma(\mathcal{V}(\lambda)) \) which is isomorphic to \( \mathcal{V}(\alpha) \); then
\[
\Gamma(\mathcal{V}(\lambda)) = \bigoplus_{\chi(n+1) \ni \alpha \downarrow \lambda} \mathcal{V}(\alpha; \lambda). \tag{14}
\]

For \( \lambda \in \chi(n) \), let \( \tilde{\lambda} \) be the rho-shift of \( \lambda \):
\[
\tilde{\lambda} = \lambda + \rho_n, \quad 2\rho_n = (n - 2, n - 4, \ldots, n - 2\ell).
\]

Notice that for the map
\[
\chi(n + 1) \to \chi(n - 1), \quad \alpha \mapsto \alpha^\circ,
\]
we have
\[
(\alpha^\circ)^\circ = (\tilde{\alpha})^\circ,
\]

since
\[
\rho_{n-1} = \tilde{\rho}_{n+1}.
\]

We would like to use the action of differential operators on these section spaces to define another discrete leading symbol, and establish its connection to the one already treated in Sec. 2.1.
Theorem 4.1 Let $D$ be an $H(n)$-invariant operator on $\mathbb{V}(\lambda)$ of the form

$$D = D_{\text{princ}} + D_{\text{lower}},$$

where $D_{\text{princ}}$ is a $p$-homogeneous polynomial in the $G_u^*G_u$ and, if applicable, $D_{\text{self}}$, and $\text{ord}(D_{\text{lower}}) < p$. (The homogeneity degree counts 2 for each $G_u^*G_u$ and 1 for each $D_{\text{self}}$.)

Then the realization $D_0$ of $D$ on the standard sphere $S^n$ has spectral asymptotics of the form

$$\text{eig}(D, \alpha) = J_p(D)(\alpha^2)j^p + O(j^{p-1}),$$

where $\alpha = (\lambda_1 + j, \alpha)$. 

Proof. The fact that $D$ has an eigenvalue on the $\alpha$ summand $\mathcal{V}(\alpha; \lambda)$ of $\Gamma(\mathbb{V}(\lambda))$ is guaranteed by the multiplicity free nature of the branching rule; i.e. by the fact that there is only one summand of covariance type $\alpha$. The asymptotics for $D_{\text{princ}}$ are guaranteed by [6], Theorem 4.1 and [8], Corollary 8.2. The standard elliptic estimate shows that the addition of $D_{\text{lower}}$ does not disturb things: indeed,

$$|\langle D_{\text{lower}} \varphi, \varphi \rangle_{L^2}| \leq \text{const}(\langle \nabla^* \nabla \rangle^{p/2} \varphi, \varphi)_{L^2}^{(p-1)/p}$$

and $\text{eig}(\nabla^* \nabla, \alpha) = j^2 + O(j)$ by [4], Theorem 1.1. \hfill \square

Let $G^p(\lambda)$ be the algebra generated by all $H(n)$-equivariant differential operators of the type described in Theorem 4.1.

Theorem 4.2 For sufficiently large $k$, the linear transformation $J_{2k} \oplus J_{2k+1} : G^{2k}(\lambda) \oplus G^{2k+1}(\lambda) \rightarrow \text{maps}(B(\lambda), \mathbb{C})$ is onto. If $n$ is even or $\lambda_\ell = 0$, the linear transformation $J_{2k} : G^{2k}(\lambda) \rightarrow \text{maps}(B(\lambda), \mathbb{C})$ is onto.

Proof. Note that the definitions immediately give

$$D \in G^p(\lambda), E \in G^q(\lambda) \Rightarrow J_{p+q}(DE) = J_p(D)J_q(E),$$

and if $L$ is $H(n)$-invariant of order $< 2k$,

$$J_{2k}((\nabla^* \nabla)^k + L) = 1,$$

where “1” is the function on $B(\lambda)$ which is identically 1.

By [3], Theorem 4.1, there are nonzero constants $c_u = c_{\lambda \mu_u}$ and $\tilde{c}_u = \tilde{c}_{\lambda \mu_u}$ such that

$$\text{eig}(G_u^*G_u, \alpha) = c_u \prod_{a=1}^{[(n+1)/2]} (\alpha^2_a - s_u^2) = \tilde{c}_u \prod_{a \in T(\lambda)} (\tilde{\alpha}^2_a - s_u^2),$$

where $T(\lambda)$ is the set of component labels $a \in \{1, \ldots, [(n+1)/2]\}$ for which $\tilde{\alpha}^2_a$ is allowed, by the interlacing condition $\alpha \downarrow \lambda$ from [12], to take on more than one value; and

$$s_u = \frac{1}{2}(|\tilde{\lambda}|^2 - |\tilde{\mu}_u|^2).$$
(The precise value of $\tilde{c}_u$ will become important below; it is given in [6], Theorem 5.2, and in Theorem 6.1 below.) By [6], the cardinality of $\mathcal{T}(\lambda)$ is $t(\lambda)$ (the same as the dimension of the space generated by the various $\zeta_u(\xi)^*\zeta_u(\xi)$).

By [6], Corollary 8.2, if $\lambda \leftrightarrow \lambda$, the constant $\tilde{c}_{\lambda\lambda}$ is positive, and the eigenvalue of $D_{self}$ on $\mathcal{V}(\alpha; \lambda)$ is

$$(\sgn \alpha_{\ell+1}) \sqrt{\text{eig}(D_{self}^2, \alpha)} = (\sgn \alpha_{\ell+1}) \sqrt{\tilde{c}_{\lambda\lambda}} \prod_{a \in \mathcal{T}(\lambda)} |\tilde{a}_a|.$$  

(Recall from Remark 2.3 that $D_{self}$ is well-defined only up to an overall sign.) Thus, since $\tilde{a}_1 = j + O(j^0)$,

$$J_2(G_* G_u)(\tilde{\alpha}) = \tilde{c}_u \prod_{1 < a \in \mathcal{T}(\lambda)} (\tilde{a}_a^2 - s_a^2),$$  

and if $\lambda \leftrightarrow \lambda$,

$$J_1(D_{self}) = (\sgn \alpha_{\ell+1}) \sqrt{\tilde{c}_{\lambda\lambda}} \prod_{1 < a \in \mathcal{T}(\lambda)} |\tilde{a}_a|. $$  

If $\beta \in \chi(n - 1)$, let

$$\bar{\beta} = \begin{cases} \beta & \text{when } n - 1 \text{ is odd}, \\ (\beta_1, \ldots, -\beta_k) & \text{when } n - 1 \text{ is even}. \end{cases}$$  

Note that $\lambda \downarrow \bar{\beta} \iff \lambda \downarrow \bar{\beta}$, and that there exist $H(n - 1)$-modules $\beta$ with $\lambda \downarrow \beta$ and $\bar{\beta} \neq \bar{\beta}$ if and only if $n$ is odd and $\lambda_\ell \neq 0$. This is exactly the case in which there is a self-gradient on $\mathcal{V}(\lambda)$. Note also that the $J_2(G_* G_u)$ are constant on sets $\{ \beta, \bar{\beta} \}$. We claim that the $G_* G_u$ separate the various sets $\{ \beta, \bar{\beta} \}$ in $B(\lambda)$, in the sense that

$$\gamma \notin \{ \beta, \bar{\beta} \} \Rightarrow \exists u : J_2(G_* G_u)(\gamma) \neq J_2(G_* G_u)(\beta).$$

If we assume the contrary, then by (17), the monic polynomials

$$\prod_{1 < a \in \mathcal{T}(\lambda)} (x^2 - \tilde{\beta}_a^2), \quad \prod_{1 < a \in \mathcal{T}(\lambda)} (x^2 - \tilde{\gamma}_a^2)$$

agree at the points $x = \pm s_u$ for all $u = 1, \ldots, N(\lambda)$. These polynomials have degree $2(t(\lambda) - 1)$, which is either $N(\lambda) - 2$ or $N(\lambda) - 1$. There are $N(\lambda)$ labels $u$. The $s_u$ comprise a set of cardinality $N(\lambda)$ except in the following cases:

- $n$ is even and $\lambda_{\ell-1} \neq 0 = \lambda_\ell$, or
- $n$ is odd and $\lambda_\ell \neq 0$, or
- $\lambda_\ell = \pm \frac{1}{2}$.

(Note that the last two cases overlap.) In these exceptional cases, the $s_u$ make up a set of cardinality $N(\lambda) - 1$. This shows that the two monic polynomials in (20) agree. Thus $(\tilde{\gamma}_a^2)_{1 < a \in \mathcal{T}(\lambda)}$ is a permutation of $(\tilde{\beta}_a^2)_{1 < a \in \mathcal{T}(\lambda)}$. By strict dominance of $\bar{\beta}$ and $\tilde{\gamma}$ (11) with all $\geq$ signs replaced by $>$ signs, since $(\rho_{n+1})_a > |(\rho_{n+1})_{a+1}|$, this can only be the identity permutation, and the claim is proved.
We now claim that for sufficiently large $k$, there are $2k$-homogeneous polynomials $P_\beta$ in the $G^*_uG_u$, one for each $\beta \in B(\lambda)$, with $J_{2k}(P_\beta)(\beta) = J_{2k}(P_\beta)(\bar{\beta}) = 1$ and $J_{2k}(P_\beta)(\gamma) = 0$ for $\gamma \notin \{\beta, \bar{\beta}\}$. Indeed, if $J_2(G^*_uG_u)$ separates $\{\beta, \bar{\beta}\}$ and $\{\gamma, \bar{\gamma}\}$, say

$$J_2(G^*_uG_u)(\beta) \neq J_2(G^*_uG_u)(\gamma) =: C,$$

then $Q_{\beta\gamma} := C \nabla^* \nabla - G^*_uG_u$ has

$$J_2(G^*_uG_u)(\beta) \neq 0 = J_2(G^*_uG_u)(\gamma) =: C.$$

Now let $P_\beta$ be the composition, in some order, of the $Q_{\beta\gamma}$ for the various $\gamma$ (one factor for each $\{\gamma, \bar{\gamma}\}$), and normalize. (Note that the $Q_{\beta\gamma}$ do not necessarily commute, but their realizations on the sphere commute. In particular, their leading symbols commute.)

We still need to separate $\beta$ from $\bar{\beta}$ when they are distinct. By (18), this is accomplished by composing the operators $P_\beta$ with $D_{\text{self}}$. ($D_{\text{self}}$ exists by the remarks following the definition (19). We may compose on either side; the difference is a lower-order operator, again by the multiplicity free nature of the $H(n+1)$-to-$H(n)$ branching rule applied to the realizations of the operators on the sphere.) After normalization, this gives us $(2k+1)$-homogeneous polynomial operators $P'_\beta$ with

$$J_{2k+1}(P'_\beta)(\beta) = -J_{2k+1}(P'_\beta)(\bar{\beta}) = 1, \quad J_{2k+1}(P'_\beta)(\gamma) = 0 \text{ for } \gamma \notin \{\beta, \bar{\beta}\}.$$

The first statement of the theorem is now established. For the second statement, we just note that the $P_\beta$ already separated points of $B(\lambda)$ in the case where there is no self-gradient. 

Thus by dimension count and order parity, we have:

**Corollary 4.3** The restrictions to the unit $\xi$-sphere of polynomials in the $\zeta_u(\xi)^*\zeta_u(\xi)$ generate $A_0(\lambda)$, and, when $D_{\text{self}}$ exists, the restrictions of $\Upsilon_{\text{self}}(\xi)$ times such polynomials generate $A_1(\lambda)$. If $n$ is even or $\lambda_\ell = 0$, we have $A(\lambda) = A_0(\lambda)$.

The last conclusion of the corollary can actually be seen by elementary means for $n$ even, since for each weight $m$ of the module $\lambda$, the numbers $m_1 + \cdots + m_\ell$ and $\lambda_1 + \cdots + \lambda_\ell$ in $\frac{1}{2}\mathbb{Z}$ have to agree mod 2.

**Remark 4.4** When realizing a given reduced symbol $\kappa(\xi)$ using the $\zeta_u(\xi)^*\zeta_u(\xi)$ and possibly $\Upsilon_{\text{self}}(\xi)$, it is a priori possible that the minimal homogeneous degree in the $\zeta$ and $\Upsilon$ quantities may exceed the degree of $\kappa$. It is probably reasonable to conjecture that this does not happen. However, in this paper, we only establish this for bundles $\nabla(\lambda)$ with $N(\lambda) \leq 4$ (Section 6).
5 The relation between the local and global discrete symbols

The estimate (13) shows that $J_p(D)$ depends only on the leading symbol of $D$. Thus the discrete symbol map $J_p$ factors through $\tilde{\sigma}_p(\mathcal{G}^p)$:

$$J_p(D) = j_p(\tilde{\sigma}_p(D)),$$

for some map $j_p$ on the space of restrictions to the unit $\xi$-sphere of $p$-homogeneous polynomials in the $\zeta_u(\xi)^*\zeta_u(\xi)$ and, if applicable, $\Upsilon_{self}(\xi)$. (Compare (4).) Let $j$ be the induced map on the space of all (not necessarily homogeneous) polynomials.

**Theorem 5.1** The maps $k$ and $j$ carrying $A(\lambda)$ to maps($B(\lambda), \mathbb{C}$) are identical.

**Proof.** By Theorem 4.2, it will be enough to show that $k$ and $j$ agree on the $\zeta_u(\xi)^*\zeta_u(\xi)$ and, if applicable, $\Upsilon_{self}$. Thus it will be enough to show that $K_2$ and $J_2$ agree on the $G_u^*G_u$ and, if applicable, $K_1$ and $J_1$ agree on $D_{self}$.

Let $L_{ij}$ be the standard generators of $\mathfrak{so}(n+1)$. By the spectral formula (16), the action of the differential operator $G_u^*G_u$ on the highest weight vector of each $\mathfrak{so}(n+1)$-type is given by the action, in the extension $S$ of the infinitesimal representation $\text{Ind}_{H(n)}H(n+1)\lambda$ to the enveloping algebra of $\mathfrak{so}(n+1)$, of

$$Q_u := \tilde{c}_u \prod_{a \in T(\lambda)} \left( -\sqrt{-1}L_{2a-1,2a} + \frac{n+1}{2} - a \right)^2 - s_u^2.$$

For a differential operator $D$ of order $p$ and a $C^\infty$ function $f$,

$$\sigma_p(D)(x, (df)_x)\varphi_x = \lim_{t \to \infty} t^{-p}D(e^{\sqrt{-1}tf}\varphi),$$

where $\varphi$ is any smooth extension of $\varphi_x \in V_x$ to a section of $V$. By the branching rule (12), the $H(n+1)$-types appearing in the section space where $f$ lives, i.e. that over the trivial representation, are the $(j,0,\ldots,0)$ for $j \in \mathbb{N}$. Choosing $f$ to depend on only one homogeneous coordinate, say $x^2$, is equivalent to choosing $f$, or any function of $f$, to be the sum of highest weight vectors. Choosing $\varphi$ to be a highest weight vector as above, we get

$$G_u^*G_u(e^{\sqrt{-1}tf}\varphi) = S(Q_u)(e^{\sqrt{-1}tf}\varphi)$$

$$= \left( -\sqrt{-1}L_{12} + \frac{n+1}{2} \right)^2 - s_u^2 \left( e^{\sqrt{-1}tf} \prod_{1 \leq a \leq T(\lambda)} \left( -\sqrt{-1}L_{2a-1,2a} + \frac{n+1}{2} - a \right)^2 - s_u^2 \right) \varphi$$

$$= \left( -\sqrt{-1}L_{12} + \frac{n+1}{2} \right)^2 - s_u^2 \left( e^{\sqrt{-1}tf} \sum_{\beta \in B(\lambda)} J(G_u^*G_u)(\beta)\varphi_\beta \right),$$

where $\varphi_\beta$ is the component of $\varphi$ in the direct sum of all $(\lambda_1 + j, \beta)$ summands in (13). Now realize $S^n$ as $H(n+1)/H(n)$, where the Lie algebra of $H(n)$ is the $\mathfrak{so}(n)$ stabilizing $e_1$. At the identity coset $o$, The right $H(n)$ covariance type of $\varphi_\beta$ at $o$ (but not necessarily
anywhere else) is the left $H(n)$ covariance type of $\varphi_b$, namely $\beta$. At $o$, the covector $df$ naturally picks a multiple of the unit covector $\xi$ corresponding to $e_2$.

Substituting into the above, we have that

$$\sigma_2(G^*_uG_u)(o, e_2)\varphi_x = \sum_{\beta \in B(\lambda)} J(G^*_uG_u)(\beta)(\varphi_x)_{\beta},$$

where $(\varphi_x)_{\beta}$ is the projection of $\varphi_x \in V_\alpha$ to the summand of right $H(n-1)$ covariance type $\beta$. On the other hand, by definition,

$$\sigma_2(G^*_uG_u)(o, e_2)\varphi_x = \sum_{\beta \in B(\lambda)} K(G^*_uG_u)(\beta)(\varphi_x)_{\beta}.$$

By the generating property of Corollary 5.2, the transforms $k$ and $j$ agree on $A_0(\lambda)$, and thus on $A(\lambda)$ when there is no self-gradient. The self-gradient, when it exists, is handled by an entirely similar argument, using the action of the enveloping algebra element

$$\sqrt{c_{\lambda}} \prod_{a=1}^{(n+1)/2} \left( \sqrt{-1}L_{2a-1, 2a} + \frac{n + 1}{2} - a \right).$$

□

**Corollary 5.2** Let $D$ be natural differential operator of order $p$ on some $V(\lambda)$.

(a) $D$ has spectral asymptotics on the sphere of the form

$$\text{eig}(D_{S^n}, V((\lambda_1 + j, \beta); \lambda)) = F(D, \beta)j^p + O(j^{p-1}),$$

where the coefficient $F(D, \beta)$ is the pointwise-determined eigenvalue of $\sigma_p(D)(\xi)$ on the summand of $V(\lambda)$, transforming according to the representation $\beta$ of the $H(n-1)$ subgroup fixing $\xi$ (on any $H(n)$ manifold, for any point $x$, at any nonzero covector $\xi$).

(b) This natural differential operator is elliptic if and only if its asymptotics on the sphere satisfy $F(D, \beta) \neq 0$ for all $\beta \in B(\lambda)$. It has positive definite leading symbol if and only if $F(D, \beta) > 0$ for all $\beta \in B(\lambda)$.

(c) For some $k \geq 0$, $(\nabla^* \nabla)^kD$ is a $(p + 2k)$-homogeneous polynomial in the $G^*_uG_u$ and, if applicable, $D_{\text{self}}$, modulo operators of order at most $p + 2k - 1$.

**Remark 5.3** The following remarks are clear from the above:

- The range of $A_0(\lambda)$ (resp. $A_1(\lambda)$) under $k$ consists of those functions on $B(\lambda)$ which are even (resp. odd) under $\beta \mapsto \bar{\beta}$.

**Remark 5.4** The degree of an element $\theta$ of $A(\lambda)$ is the maximal $p$ for which the $A^p(\lambda)$ component of $\theta$ is nonzero. It is clear that

$$\deg \left( (\zeta_{u_1}(\xi)^*\zeta_{u_1}(\xi) \cdots \zeta_{u_{k+1}}(\xi)^*\zeta_{u_{k+1}}(\xi))^\top \right) \leq 2 + \deg \left( (\zeta_{u_1}(\xi)^*\zeta_{u_1}(\xi) \cdots \zeta_{u_k}(\xi)^*\zeta_{u_k}(\xi))^\top \right)$$

By the generating property of Corollary 5.3, the existence of nonzero, degree $p$ homogeneous polynomials (and thus monomials) guarantees the existence of monomials of degrees $p - 2, p - 4, \ldots, 0$ or 1. Thus

- $\text{Hom}_{H(n)}(\text{tfs}^2k \otimes \lambda, \lambda) \neq 0, \quad k = 0, \ldots, d_0(\lambda)/2$,

- $\text{Hom}_{H(n)}(\text{tfs}^{2k+1} \otimes \lambda, \lambda) \neq 0, \quad k = 0, \ldots, (d_1(\lambda) - 1)/2$. 

6 Low $\mathcal{N}(\lambda)$

There is an essential simplification of the above theory for bundles $\mathbb{V}(\lambda)$ whose $\mathcal{N}(\lambda)$ is at most 4. It is worthwhile to work this out in detail because these are the bundles one is most likely to meet “in real life”. For example, differential form bundles have $\mathcal{N}(\lambda) = 3$ (or $\mathcal{N}(\lambda) = 2$, for the half middle-form bundles in even dimensions). The trace-free symmetric bundles $\mathbb{V}(p,0,\ldots,0)$ have $\mathcal{N}(\lambda) = 3$ for $p > 0$. Spinor bundles have $\mathcal{N}(\lambda) = 2$, and twistor bundles have $\mathcal{N}(\lambda) = 4$. Bundles of algebraic Weyl tensors have $\mathcal{N}(\lambda) = 2$ (for $n = 4$), $\mathcal{N}(\lambda) = 3$ (for $n = 5$ and $n \geq 7$), or $\mathcal{N}(\lambda) = 4$ (for $n = 6$). What we find for $\mathcal{N}(\lambda) = 3, 4$ is the type of information that is typically computed case by case, using some explicit realization of the bundle and operator involved. What we get from the discrete leading symbol is a realization that all of these computations are special cases of universal results, and in fact may be obtained by substituting the weight parameter $\lambda$ into certain universal formulas.

Suppose $\mathcal{N}(\lambda)$ is 3 or 4. Then $t(\lambda)$, the number of linearly independent $G_u^*G_u$, is 2. This means that the restrictions of leading symbols of $\nabla^*\nabla$ and any single $G_u^*G_u$ (i.e., 1 and a single $\zeta_u(\eta)^*\zeta_u(\eta)$) generate $\mathcal{A}_0(\lambda)$. (We just need to verify that these symbols are linearly independent; this is clear from (17).) By finite dimensionality, the list

$$1, \ zeta_u(\eta)^*zeta_u(\eta), (zeta_u(\eta)^*zeta_u(\eta))^2, \ldots, (zeta_u(\eta)^*zeta_u(\eta))^k, \ldots$$

eventually becomes linearly dependent. Thus there is a minimal polynomial $M_{\lambda,u}(x)$ with

$$f(zeta_u(\eta)^*zeta_u(\eta)) = 0, \ f \in \mathbb{C}[x] \iff f|_{M_{\lambda,u}},$$

$$b_0(\lambda) = \deg M_{\lambda,u}. \quad (21)$$

The minimal polynomial must also be the product of the distinct $x-k(zeta_u(\eta)^*zeta_u(\eta))(\beta)$ over all $\beta \uparrow \lambda$. Thus (just counting degrees) each of the $b_0(\lambda)$ classes $\{\beta, \tilde{\beta}\}$ for $\beta \uparrow \lambda$ must take a different value under $k(zeta_u(\eta)^*zeta_u(\eta))(\beta)$. Since $t(\lambda) = 2$, there is just one value $a_0$ of $a$ for which $\tilde{\beta}_a^2$ is allowed more than one value by the branching condition $\lambda \downarrow \beta$. By (17) and Theorem 5.1,

$$k(zeta_u(\eta)^*zeta_u(\eta))(\beta) = \tilde{c}_u(\tilde{\beta}_a^2 - s_u^2).$$

As a result,

$$M_{\lambda,u}(x) = \prod_{m=0}^{b_0(\lambda)-1} \left(x - \tilde{c}_u \left(\left(\lambda_{a_0} - m + \frac{n-1}{2} - a_0\right)^2 - s_u^2\right)\right). \quad (22)$$

The minimal polynomials $M_{\lambda,u}$ for different $u$ are closely related: in terms of the $u$-independent data

$$h_m := \lambda_{a_0} - m + \frac{n-1}{2} - a_0, \quad M_\lambda(x) := \prod_{m=0}^{b_0(\lambda)-1} (x - h_m^2),$$

we have

$$M_{\lambda,u}(x) = \tilde{c}_u^{-b_0(\lambda)}M_\lambda(c_u^{-1}x + s_u^2).$$
In particular, the set of roots of \( M_{\lambda,u} \) is obtained from that of \( M_\lambda \) by an affine map on \( \mathbb{R} \). By (22), the fundamental projections on \( A_0(\lambda) \) are

\[
\Pi_m(\eta) := \prod_{m' \neq m} \frac{\zeta_u^*(\eta)\zeta_u(\eta) - \tilde{c}_u(h_{m'}^2 - s_u^2)}{\tilde{c}_u(h_m^2 - s_u^2) - \tilde{c}_u(h_{m'}^2 - s_u^2)} = \tilde{c}_u^{-b_0(\lambda)+1} \prod_{m' \neq m} \frac{\zeta_u(\eta)\zeta_u(\eta) - \tilde{c}_u(h_{m'}^2 - s_u^2)}{h_m^2 - h_{m'}^2}.
\]

They are represented by the differential operators

\[
D_{m,u} := \tilde{c}_u^{-b_0(\lambda)+1} \prod_{m' \neq m} \frac{G_u^* G_u - \tilde{c}_u(h_{m'}^2 - s_u^2)\nabla\ast\nabla}{h_m^2 - h_{m'}^2},
\]

which have order at most \( 2(b_0(\lambda) - 1) \). Note that the projections \( \Pi_m \) are independent of \( u \); that is, independent of which \( \zeta_u(\eta)^* \zeta_u(\eta) \) we have chosen as the preferred generator for \( A_0(\lambda) \). The \( D_{m,u} \) are in general not independent of \( u \); what one can say is that any \( D_{m,u} - D_{m',u} \) has order at most \( 2(b_0(\lambda) - 2) \). (This order is strictly less than \( 2(b_0(\lambda) - 1) \), but by invariant theory, each term must introduce a curvature, dropping the order by at least 2.) In fact, the product in (24), which is really a composition, is sensitive to the ordering of the factors, as \( G_u^* G_u \) and \( \nabla\ast\nabla \) commute only modulo second order operators. Thus \( D_{m,u} \) is really only well-defined modulo operators of order at most \( 2(b_0(\lambda) - 2) \).

Among other things, these results illuminate a typical experience in computing with leading symbols on a bundle: eventually, as the symbol order goes up, there are no new combinatorial interactions of symbol and bundle indices.

Since \( (\zeta_u(\xi)^* \zeta_u(\xi))^k \) can involve actions of at most symmetric \( 2k \)-tensors, we have in addition

\[
d_0(\lambda) \leq 2((\deg m_{\lambda,u}) - 1) = 2(b_0(\lambda) - 1) = \begin{cases} 2|\lambda_\ell|, & \text{n odd, } a_0 = \ell, \\ 2(\lambda_{a_0} - |\lambda_{a_0+1}|) & \text{otherwise}. \end{cases} (25)
\]

If there is no self-gradient, \( A_0(\lambda), b_0(\lambda), \) and \( d_0(\lambda) \) can be replaced by \( A(\lambda), b(\lambda), \) and \( d(\lambda) \) respectively in the above remarks. The estimate (23) becomes

\[
d(\lambda) \leq 2(\lambda_{a_0} - |\lambda_{a_0+1}|),
\]

a clear improvement on Lemma 3.3 unless \( a_0 = 1 \) and \( \lambda_{a_0+1} = 0 \).

Still in the case \( N(\lambda) = 3 \) or 4, if there is a self-gradient, its restricted symbol together with 1 generate \( A(\lambda) \). There is a minimal polynomial \( m_{\lambda,\text{self}}(x) \) with

\[
\begin{align*}
f(\Upsilon_{\text{self}}) &= 0, \ f \in \mathbb{C}[x] \iff f|m_{\lambda,\text{self}}, \\
A(\lambda) &\cong \mathbb{C}[x]/(m_{\lambda,\text{self}}), \\
b(\lambda) &= \deg m_{\lambda,\text{self}}.
\end{align*} (26)
\]

In fact, reasoning as above, if we let \( u \) be the index corresponding to the self-gradient (so that \( D_{\text{self}}^2 = G_u^* G_u \), then

\[
m_{\lambda,\text{self}}(x) = \prod_{m=0}^{b_0(\lambda)-1} \left( x - \sqrt{\tilde{c}_u}\left(\lambda_{a_0} - m + \frac{n-1}{2} - a_0\right) \right)
\]

\[
= \prod_{m=0}^{b_0(\lambda)-1} \left( x^2 - \tilde{c}_u h_m^2 \right), \ \lambda_\ell \in \frac{1}{2} + \mathbb{N}, \\
= \prod_{m=0}^{b_0(\lambda)-2} \left( x^2 - \tilde{c}_u h_m^2 \right), \ \lambda_\ell \in \mathbb{Z}^+.
\]

\[
M_{\lambda,u}(x^2) = \tilde{c}_u^{b_0(\lambda)} M_\lambda(\tilde{c}_u^{-1} x^2), \ \lambda_\ell \in \frac{1}{2} + \mathbb{N}, \\
M_{\lambda,u}(x^2)/x = c_u^{b_0(\lambda)} M_\lambda(\tilde{c}_u^{-1} x^2)/x, \ \lambda_\ell \in \mathbb{Z}.
\]
Note that $s_u = 0$ in this case. Note that if $a_0 = \ell$, the first expression for the minimal polynomial simplifies:

\[ m_{\lambda, \text{self}}(x) = \prod_{m=0}^{b(\lambda)-1} \left( x - \sqrt{c_u}(\lambda - m) \right) \quad \text{if} \quad a_0 = \ell. \]

The fundamental projections on $\mathcal{A}(\lambda)$ are

\[ \Pi_m = \prod_{m' \neq m} \frac{Y_{\text{self}}(\eta) - \sqrt{c_u} h_{m'}}{m' - m} \]

where $h_m$ is defined as before, in this new range of $m$. If $a_0 = \ell$, then $h_{m'}$ is just $\lambda - m'$. These projections generally have no operator representatives, since they mix even and odd orders.

If $N(\lambda) = 2$, then $\mathcal{A}_0(\lambda)$ is generated (and thus spanned) by 1. If there is a self-gradient, $\mathcal{A}_1(\lambda)$ is spanned by the odd function of absolute value 1, and thus $\mathcal{A}(\lambda)$ is generated and spanned by this function and 1. (In fact, if $N(\lambda) = 2$ and there is a self-gradient, then $n$ is odd and $\lambda = (\frac{1}{2}, \ldots, \frac{1}{2})$. If $N(\lambda) = 2$ and there is no self-gradient, then $n$ is even and $\lambda = (p, \ldots, p, \pm p) \text{ with } p \neq 0$.)

If $N(\lambda) = 1$, then there can be no self-gradient, and $\mathcal{A}(\lambda)$ is generated and spanned by the function 1. (In fact, the only $N(\lambda) = 1$ case is $\lambda = 0$.)

Summing up the assertions made and proved above, we have

\textbf{Theorem 6.1} \textbf{(a)} \quad \text{If } n \text{ is odd, } \lambda \neq 0, \text{ and } N(\lambda) \text{ is } 3 \text{ or } 4, \text{ then the principal symbol algebra } \mathcal{A}(\lambda) \text{ is generated by } 1 \text{ and the leading symbol of } D_{\text{self}}. \quad \textbf{(b)} \quad \text{In all other cases where } N(\lambda) \text{ is } 3 \text{ or } 4, \text{ } \mathcal{A}(\lambda) \text{ is generated by } 1 \text{ together with the leading symbol of any } G_u^*G_u. \quad \textbf{(c)} \quad \text{If } n \text{ is odd and } \lambda = (\frac{1}{2}, \ldots, \frac{1}{2}), \text{ then } N(\lambda) = 2, \text{ and } \mathcal{A}(\lambda) \text{ is generated by the leading symbol of the Dirac operator.} \quad \textbf{(d)} \quad \text{If } n \text{ is even and } \lambda = (p, \ldots, p, \pm p) \text{ with } p \neq 0, \text{ then } N(\lambda) = 2, \text{ and } \mathcal{A}(\lambda) \text{ is generated by } 1. \quad \textbf{(e)} \quad \text{If } \lambda = (0), \text{ then } N(\lambda) = 1, \text{ and } \mathcal{A}(\lambda) \text{ is generated by } 1.

Cases (a) - (e) exhaust all bundles with $N(\lambda) \leq 4$.

On manifolds of constant curvature, we use this to get a statement about generators of the algebra of invariant operators (as opposed to symbols). On a manifold of constant curvature, the Weyl and trace-free Ricci tensors vanish, along with all covariant derivatives of curvature; the only local tensorial invariants are polynomials in the (constant) scalar curvature.

\textbf{Theorem 6.2} \quad \text{On a manifold of constant curvature, the algebra } \mathcal{D}(\lambda) \text{ of natural differential operators on } \mathcal{V}(\lambda), \text{ for } N(\lambda) \leq 4, \text{ is generated by the identity operator together with:}

- $\nabla^*\nabla$ and $D_{\text{self}}$ in case (a) above;
- $\nabla^*\nabla$ and any single $G_u^*G_u$ in case (b);
Discrete leading symbol

- the Dirac operator in case (c);
- \( \nabla^* \nabla \) in cases (d) and (e).

**Proof.** By the constant curvature assumption, natural operators are polynomial in the metric, the covariant derivative, the volume form, and (if applicable) the fundamental tensor-spinor. Let \( D \in \mathcal{D}(\lambda) \). Theorem 6.1 shows that there is an operator \( P \) in the algebra generated by the putative generating set such that \( \text{ord}(D - P) < \text{ord}(D) \). Since \( D - P \) is natural, the result follows by induction on the order. \( \Box \)

What goes wrong with the attempt to have something like Theorem 6.2 in general is that the leading symbol of an operator like \( r_{ij} \nabla^i \nabla_j \) (where \( r \) is the Ricci tensor) does not necessarily induce an element of \( \mathcal{A}(\lambda) \).

As a corollary, we have:

**Corollary 6.3** On the hyperboloid \( G/K \), where \( G = \text{Spin}_0(n,1) \) and \( K = \text{Spin}(n) \), the \( G \)-invariant differential operators on \( \nabla(\lambda) \), for \( N(\lambda) \leq 4 \), are generated by the operators listed in Theorem 6.2. If \( \lambda \) is integral, we may replace “Spin” by “SO” in the definitions of \( G \) and \( K \).

### 7 Examples

In the following, let \( \eta \) be a vector on the unit sphere in the cotangent bundle. In doing explicit examples, it is helpful to know explicitly all the linear relations among the various \( \zeta_u(\xi)^* \zeta_u(\xi) \). These are given in [3], Theorem 5.10:

**Theorem 7.1** Given \( \lambda \in \chi(n) \),

\[
\sum_{u=1}^{N(\lambda)} b_u \zeta_u(\xi)^* \zeta_u(\xi) = 0
\]

if and only if

\[
\sum_{u=1}^{N(\lambda)} b_u \tilde{c}_{\lambda \sigma_u} s_u^{2j} = 0, \quad j = 0, 1, \ldots, t(\lambda) - 1.
\]

Here \( \tilde{c}_{\lambda \sigma_u} = \)\
\[
\begin{align*}
&\frac{(-1)^{t(\lambda)+1}}{\prod_{1 \leq v \leq N(\lambda)} (s_v - s_u)} & \text{if } N(\lambda) \text{ is odd;} \\
&\frac{(-1)^{t(\lambda)+1}}{\prod_{1 \leq v \leq N(\lambda)} (s_v - s_u)} & \text{if } n \text{ is even, } \lambda_{\ell} = 0 \neq \lambda_{\ell-1}, \sigma_u = \lambda \pm e_{\ell}; \\
&2 \prod_{u=1}^{N(\lambda)-2} (s_u + \frac{1}{2}) & \text{if } n \text{ is even, } \lambda_{\ell} = 0 = \lambda_{\ell-1}, \sigma_u = \lambda \pm e_{\ell}; \\
&\frac{(-1)^{t(\lambda)}}{\prod_{1 \leq v \leq N(\lambda)} (s_v - s_u)} & \text{otherwise.}
\end{align*}
\]
The values of $\chi_{\lambda\sigma_u}$ were given in [6], Theorem 5.2. (Recall from (16) that the $\tilde{c}_u$ appear in the spectral asymptotics of the $G^*G_u$ on the sphere.)

The parameter $t(\lambda)$, in addition to the roles it plays above, is also

$$t(\lambda) = \dim \text{Hom}_{H(n)}((\text{tfs}^0 \oplus \text{tfs}^2) \otimes \lambda, \lambda)$$

by, for example, [6], p.57.

We shall adopt the convention of ordering the selection rule targets in decreasing (lexicographical) order.

**Example 7.2** Let $\mathbb{V}(\lambda)$ be (Spin($n$)-isomorphic to) the spinor bundle $\Sigma$ for $n \geq 3$ odd. Then $\lambda = (\frac{1}{2}, \ldots, \frac{1}{2})$ and

$$N(\lambda) = 2, \quad t(\lambda) = 1, \quad b_0(\lambda) = b_1(\lambda) = 1.$$ 

There are actions of $\text{tfs}^0$ and (by the selection rule) $\text{tfs}^1$ on $\lambda$. These exhaust the total of $b(\lambda) = 2$ linearly independent actions, so

$$d_0(\lambda) = 0, \quad d_1(\lambda) = 1.$$ 

The $\text{tfs}^1$ action is, in fact, Clifford multiplication. The gradient targets are $\sigma = (\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $\lambda$ itself. $\mathbb{V}(\sigma)$ is the **twistor bundle**, and may be realized as spinor-one-forms $\varphi$ which are annihilated by interior Clifford multiplication: $\gamma^\alpha \varphi_\alpha = 0$. The fundamental projections on $\mathcal{A}(\lambda)$ are

$$\text{id} \pm \frac{\sqrt{-1}}{2} \gamma(\eta).$$

Since these mix even and odd orders (as will always be the case when there is a self-gradient), there are no differential operators representing these projections. There will always be differential operators representing the fundamental projections of $\mathcal{A}_0(\lambda)$, which in this case is just one-dimensional. Since

$$s_1 = -n/2, \quad s_2 = 0, \quad \tilde{c}_1 = (n - 1)/n, \quad \tilde{c}_2 = 1/n,$$

(31) is still good, so again $\mathcal{V}^2 = G_2^*G_2$.

**Example 7.3** Let $\mathbb{V}(\lambda)$ be the positive spinor bundle $\Sigma_+$ for $n \geq 4$ even. (The considerations for $\Sigma_-$ are entirely analogous.) Then $\lambda = (\frac{1}{2}, \ldots, \frac{1}{2})$ and

$$N(\lambda) = 2, \quad t(\lambda) = 1, \quad b(\lambda) = b_0(\lambda) = 1.$$ 

Thus there are no actions of trace-free symmetric tensors beyond that of $\text{tfs}^0$, and

$$d(\lambda) = 0.$$ 

The identity generates $\mathcal{A}(\Sigma)$; note that there is no analogue of the Dirac leading symbol, since the Dirac operator carries $\Sigma_+$ to $\Sigma_-$. (The fact that $\text{tfs}^1 \otimes \Sigma_+$ contains a copy of $\Sigma_-$ reflects the fact that Clifford multiplication carries $\Sigma_+$ to $\Sigma_-$. ) As for normalizations, (31) is still good, so again $\mathcal{V}^2 = G_2^*G_2$. 
Example 7.4 Let $V(\lambda)$ be the differential form bundle $\Lambda^k$ for $0 < k < (n - 2)/2$. Then $\lambda = (1_k)$, and the gradient targets are $(2, 1_{k-1})$, $(1_{k+1})$, and $(1_{k-1})$. There is no self-gradient, so $A(\lambda) = A_0(\lambda)$. We have

$$N(\lambda) = 3, \ t(\lambda) = 2, \ b(\lambda) = 2.$$ Combining this with (30), we deduce the existence of a tf$^2$ action, which together with the obvious tf$^0$ action exhausts the possible tf$^p$ actions:

$$d(\lambda) = 2.$$ (Lemma 3.3 already implies that $d(\lambda) \leq 2$.) The fundamental projections are $\iota(\eta)\varepsilon(\eta)$ and $\varepsilon(\eta)\iota(\eta)$, where $\varepsilon$ and $\iota$ are exterior and interior multiplication; their differential representatives are the familiar Hodge operators $\delta d$ and $d\delta$.

Example 7.5 Let $V(\lambda) = \text{TFS}^p$, and suppose that $p \geq 2$, $n \geq 5$. There is no self-gradient, so $A(\lambda) = A_0(\lambda)$. We have $\lambda = (p)$ and

$$N(\lambda) = 3, \ t(\lambda) = 2, \ b(\lambda) = p + 1.$$ There is an action of tf$^{2p}$ on tf$^p$, namely

$$\varphi_{a_1 \ldots a_p} \mapsto \Psi_{a_1 \ldots a_p} \varphi_{b_1 \ldots b_p}.$$ By Remark 5.4, there must therefore be actions of tf$^{2k}$ for $k = 0, \ldots, p$. This exhausts the available $b(\lambda) = p + 1$ actions. Thus we know that

$$d(\lambda) = 2p,$$ and in fact we know $\dim_{\text{SO}(n)}(\text{tf}^k \otimes \text{tf}^p, \text{tf}^p)$ for every $k$ and $p$. The selection rule targets are $(p + 1)$, $(p, 1)$, and $(p - 1)$, and we have

$$s_1 = -\frac{1}{2}(n + 2p - 1), \quad s_2 = -\frac{1}{2}(n - 3), \quad s_3 = \frac{1}{2}(n + 2p - 3), \quad \bar{c}_1 = -\frac{1}{(p + 1)(n + 2p - 2)}, \quad \bar{c}_2 = \frac{1}{(p + 1)(n + p - 3)}, \quad \bar{c}_3 = -\frac{1}{(n + 2p - 2)(n + p - 3)}.$$ By Theorem 7.1, the single linear relation among the $\zeta_u(\xi)^*\zeta_u(\xi)$ has coefficients $(b_u)$ (in the notation of the theorem), where $(b_u)$ is the unique (up to constant multiple) solution of the system

$$0 = \bar{c}_1 b_1 + \bar{c}_2 b_2 + \bar{c}_3 b_3$$ $$= \bar{c}_1 s_1^2 b_1 + \bar{c}_2 s_2^2 b_2 + \bar{c}_3 s_3^2 b_3.$$ Thus the linear relation is

$$-p\zeta_1(\xi)^*\zeta_1(\xi) + \zeta_2(\xi)^*\zeta_2(\xi) + (n + p - 2)\zeta_3(\xi)^*\zeta_3(\xi) = 0.$$
This allows us to write everything in terms $|\xi|^2$ and a single $\zeta_u(\xi)^*\zeta_u(\xi)$, say $\zeta_1(\xi)^*\zeta_1(\xi)$:

$$\zeta_2(\xi)^*\zeta_2(\xi) = \frac{1}{n + p - 3} \{(n + p - 2)|\xi|^2 - (n + 2p - 2)\zeta_1(\xi)^*\zeta_1(\xi)\},$$

$$\zeta_3(\xi)^*\zeta_3(\xi) = \frac{1}{n + p - 3} \{-|\xi|^2 + (p + 1)\zeta_1(\xi)^*\zeta_1(\xi)\}$$

Now consider the discrete leading symbols of the above operators. To avoid trivialities, we exclude the case $p = 0$. $B(\lambda)$ is the $(p + 1)$-point space of all $(q) \in \chi(n - 1)$ with $0 \leq q \leq p$. By (23) with $m = p - q,$

$$h_m = q + \frac{n - 3}{2},$$

and the minimal polynomial of $\zeta_1(\xi)^*\zeta_1(\xi)$ is

$$M_{\lambda,1}(x) = \prod_{q=0}^{p} \left(x + \frac{1}{(p + 1)(n + 2p - 2)}(h_m - \frac{1}{2}(n + 2p - 1))(h_m + \frac{1}{2}(n + 2p - 1))\right)$$

$$= \prod_{q=0}^{p} \left(x - \frac{(p - q + 1)(q + p + n - 2)}{(p + 1)(n + 2p - 2)}\right).$$

To see what is going on tensorially, we first need a formula for $G_1$. If $\psi_{a_0...a_p}$ is a section of $T^* \otimes \text{TFS}^p$, then its projection to $\text{TFS}^{p+1}$ is

$$(P\psi)_{a_0...a_p} := \frac{1}{p + 1} \sum_{s=0}^{p} \psi_{a_0a_0...a_s...a_p} - \alpha \sum_{s<t} g_{a_0a_1} \psi^b_{ba_0...a_s...a_t...a_p},$$

where the number $\alpha$ is determined by the condition that the $a_0a_1$ metric trace (and thus every other trace, by symmetry) vanishes. A short calculation gives

$$\alpha = \frac{2}{(k + 1)(n + 2k - 2)}.$$  

For example, if $p = 2$, then

$$(P\psi)_{a_0a_1a_2} = \frac{1}{3}(\psi_{a_0a_1a_2} + \psi_{a_1a_0a_2} + \psi_{a_2a_0a_1}) - \frac{2}{3(n + 2)}(g_{a_0a_1} \psi^b_{ba_0} + g_{a_0a_2} \psi^b_{ba_1} + g_{a_1a_2} \psi^b_{ba_0}).$$

$G_1\varphi$ is just $P(\nabla\varphi)$, and $\zeta_1(\xi)\varphi$ is just $P(\xi \otimes \varphi)$; each may be expanded according to the formula above. Furthermore, since $G_1^* G_1 = \nabla^* G_1$, we have

$$(G_1^* G_1)_{a_1...a_p} = -\frac{1}{p + 1} \nabla^{ab} \left(\sum_{s=0}^{p} \nabla_{a_s} \varphi_{a_0...a_s...a_p} - \frac{2}{n + 2p - 2} \sum_{s<t} g_{a_0a_1} \nabla^b \varphi_{ba_0...a_s...a_t...a_p}\right).$$

For example, if $p = 2$,

$$(G_1^* G_1)_{a_0a_1} = -\frac{1}{3} \nabla^c \left(\nabla_{c} \varphi_{ab} + \nabla_{a} \varphi_{bc} + \nabla_{b} \varphi_{ac}\right) - \frac{2}{n + 2} \left(g_{ca} \nabla^d \varphi_{db} + g_{cb} \nabla^d \varphi_{da} + g_{ab} \nabla^d \varphi_{dc}\right).$$  

(32)
Let us concentrate on the example $p = 2$ for a moment. The minimal polynomial with respect to the gradient target $\sigma_1 = (3)$ is

$$M_{\lambda,1}(x) = \left(x - \frac{n}{n+2}\right) \left(x - \frac{2(n+1)}{3(n+2)}\right) \left(x - \frac{1}{3}\right). \quad (33)$$

The fundamental projections are

$$\Pi_0 = \frac{\left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{2(n+1)}{3(n+2)}\right) \left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{1}{3}\right)}{\left(\frac{n}{n+2} - \frac{2(n+1)}{3(n+2)}\right) \left(\frac{n}{n+2} - \frac{1}{3}\right)} = \frac{9(n+2)^2}{2(3n-2)(n-1)} \left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{2(n+1)}{3(n+2)}\right) \left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{1}{3}\right),$$

$$\Pi_1 = \frac{\left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{n}{n+2}\right) \left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{1}{3}\right)}{\left(\frac{2(n+1)}{3(n+2)} - \frac{n}{n+2}\right) \left(\frac{2(n+1)}{3(n+2)} - \frac{1}{3}\right)} = \frac{9(n+2)^2}{n(3n-2)} \left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{n}{n+2}\right) \left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{1}{3}\right),$$

$$\Pi_2 = \frac{\left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{n}{n+2}\right) \left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{2(n+1)}{3(n+2)}\right)}{\left(\frac{1}{3} - \frac{n}{n+2}\right) \left(\frac{1}{3} - \frac{2(n+1)}{3(n+2)}\right)} = \frac{9(n+2)^2}{2n(n-1)} \left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{n}{n+2}\right) \left(\zeta_1(\xi)^*\zeta_1(\xi) - \frac{2(n+1)}{3(n+2)}\right),$$

and $\Pi_q$ is the projection onto the $(q) \in \chi(n-1)$ summand in the decomposition of $(p) \in \chi(n)$ under the $SO(n-1)$ subgroup fixing $\eta$.

Taking an explicit tensorial viewpoint and proceeding from scratch in this example, there are three independent combinatorial interactions of $\xi$ and $\varphi \in \text{tfs}^2$, namely

$$X_0(\xi, \varphi) : = \varphi,$$

$$X_2(\xi, \varphi)_\alpha\beta : = \eta^\lambda \eta_\alpha(\varphi_\beta)\lambda - \frac{1}{n} g_{\alpha\beta} \eta^\lambda \eta^\mu \varphi_\lambda \mu,$$

$$X_4(\xi, \varphi)_{\alpha\beta} : = \eta_\alpha \eta_\beta \eta^\lambda \eta^\mu \varphi_\lambda \mu - \frac{1}{n} g_{\alpha\beta} \eta^\lambda \eta^\mu \varphi_\lambda \mu.$$

(Recall that $\eta_\alpha \eta^\alpha = 1$.) In fact, these formulas make explicit the actions of $\text{tfs}^0$, $\text{tfs}^2$, and $\text{tfs}^4$ on $\text{tfs}^2$. An alternative basis consists of the identity $X_0$ together with

$$\zeta_1(\eta)^*\zeta_1(\eta) = \frac{1}{3} X_0 + \frac{2n}{3(n+2)} X_2,$$

$$(\zeta_1(\eta)^*\zeta_1(\eta))^2 = \frac{1}{9} X_0 + \frac{2n(3n+4)}{9(n+2)^2} X_2 + \frac{2n(n-2)}{9(n+2)^2} X_4.$$

The fact that we have exhausted the combinatorial possibilities means that the cube of $\zeta_1(\eta)^*\zeta_1(\eta)$ will be a linear combination of previous powers, and indeed,

$$\begin{align*}
(\zeta_1(\eta)^*\zeta_1(\eta))^3 &= \frac{1}{27} X_0 + \frac{2n(7n^2 + 18n + 12)}{27(n+2)^3} X_2 + \frac{4n(n-2)(3n+2)}{27(n+2)^3} X_4 \\
&= \frac{2n(n+1)}{9(n+2)^2} - \frac{11n^2 + 18n + 4}{9(n+2)^2} \zeta_1(\eta)^*\zeta_1(\eta) + \frac{2(3n+2)}{3(n+2)} (\zeta_1(\eta)^*\zeta_1(\eta))^2.
\end{align*}
\quad (34)$$
But the difference of the extreme left and right sides of (34) is exactly the minimal polynomial of (33), applied to \( \zeta_1(\eta)^* \zeta_1(\eta) \). That is, substitution into our general machinery checks with the result of naive calculation.

**Example 7.6** An interesting and potentially useful example is the bundle \( \mathcal{W} \) of algebraic Weyl tensors; i.e. totally trace-free tensors with the symmetries

\[
Y_{\alpha\beta\lambda\mu} = Y_{\lambda\mu\alpha\beta} = -Y_{\alpha\lambda\mu\beta} = -Y_{\alpha\mu\beta\lambda}.
\]

If \( n \geq 7 \), these are a realization of \( \mathcal{V}(2, 2, 0, \ldots, 0) \) (see [17]), a bundle with no self-gradient, and selection rule targets

\[
\sigma_1 = (3, 2, 0, \ldots, 0), \quad \sigma_2 = (2, 2, 1, 0, \ldots, 0), \quad \sigma_3 = (2, 1, 0, \ldots, 0).
\]

Thus

\[
N(\lambda) = 3, \quad t(\lambda) = 2, \quad b(\lambda) = 3.
\]

There is one action of \( \text{tfs}^0 \), and there is \( t(\lambda) = 1 = 1 \) action of \( \text{tfs}^2 \). (Recall that in general, there are \( t(\lambda) \) actions of \( \text{tfs}^0 \oplus \text{tfs}^2 \) on any given \( \lambda \), by (30).) If \( n \) is even, the only other possible action is by \( \text{tfs}^4 \), by parity considerations and Lemma 3.3. The tensorial formula for this must continue to odd dimensions, so we have

\[
\dim \text{Hom}_{SO(n)}(\text{tfs}^p \otimes \mathcal{W}, \mathcal{W}) = \begin{cases} 1, & p = 0, 2, 4, \\ 0, & \text{otherwise}. \end{cases}
\]

Formulas for these actions will in fact emerge from the minimal polynomial calculations, much as in the previous examples.

We choose to compute the minimal polynomial and projections from the viewpoint of the third selection rule target \( (2, 1, 0, \ldots, 0) \). Straightforward computation yields

\[
s_3 = \frac{n - 1}{2}, \quad \tilde{c}_3 = -\frac{1}{(n + 1)(n - 3)},
\]

\[
h_m = \frac{n - 1}{2} - m, \quad m = 0, 1, 2.
\]

As a result the minimal polynomial of \( \zeta_3(\eta)^* \zeta_3(\eta) \) is

\[
M_{\lambda,3}(x) = \prod_{m=0}^2 \left( x - \frac{m(n-1-m)}{(n+1)(n-3)} \right) = x \left( x - \frac{n-2}{(n+1)(n-3)} \right) \left( x - \frac{2}{n+1} \right), \quad (35)
\]
and the fundamental projections on $A(\lambda)$ are

$$
\Pi_0 = \frac{(\zeta_3(\eta)^*\zeta_3(\eta) - \frac{n-2}{(n+1)(n-3)}) (\zeta_3(\eta)^*\zeta_3(\eta) - \frac{2}{n+1})}{(n-3)(n+1)^2 (\zeta_3(\eta)^*\zeta_3(\eta))^2 - \frac{(n+1)(3n-8)}{2(n-2)} \zeta_3(\eta)^*\zeta_3(\eta) + 1},
$$

$$
\Pi_1 = \frac{\zeta_3(\eta)^*\zeta_3(\eta) (\zeta_3(\eta)^*\zeta_3(\eta) - \frac{2}{n+1})}{(n+1)(n-3)^2 (\zeta_3(\eta)^*\zeta_3(\eta))^2 - \frac{(n+1)(n-3)}{2(n-2)}},
$$

$$
\Pi_2 = \frac{n+1}{2(n-4)} \{(n-3)(n+1)(\zeta_3(\eta)^*\zeta_3(\eta))^2 - (n-2)\zeta_3(\eta)^*\zeta_3(\eta)\}.
$$

(Here $\Pi_m$ is the projection on the branch $(2, 2 - m)$.)

To see what is going on tensorially, note that the symbol $\zeta_3(\eta)^*\zeta_3(\eta)$ is closely related to the symbol $\zeta_1'(\eta)$ of the top gradient $\mathbb{V}(2, 1) \to \mathbb{V}(2, 2)$:

$$
\zeta_3(\eta)^*\zeta_3(\eta) = C \cdot \zeta_1'(\eta)\zeta_1'(\eta)^*
$$

for some universal constant $C$. This constant is easily evaluated by the discrete principal symbol. In fact, more generally, by (14),

$$
c_{\sigma\lambda} k(\zeta_{\sigma\lambda}(\eta)^*\zeta_{\sigma\lambda}(\eta)) = c_{\lambda\sigma} k(\zeta_{\sigma\lambda}(\eta)\zeta_{\sigma\lambda}(\eta)^*).
$$

In particular, the quotient $c_{\lambda\sigma}/c_{\sigma\lambda}$ can be evaluated by computing at any $\beta \in \chi(n-1)$ having $\beta \uparrow \lambda$ and $\beta \uparrow \sigma$. Note that $c$ cannot be replaced by $\tilde{c}$ in this statement, since $\mathcal{T}(\lambda)$ and $\mathcal{T}(\sigma)$ need not be the same. In fact, by (14),

$$
\frac{\tilde{c}_{\lambda\sigma\lambda}}{\tilde{c}_{\sigma\lambda\lambda}} = \frac{c_{\lambda\sigma\lambda} \prod_{\alpha \in \mathcal{T}_{\sigma\lambda}} (\tilde{\alpha}_a^2 - s_u^2)}{c_{\sigma\lambda\lambda} \prod_{\alpha \in \mathcal{T}_{\lambda\sigma}} (\tilde{\alpha}_a^2 - s_u^2)}.
$$

In the present situation, with $\lambda = (2, 2)$ and $\sigma_u = (2, 1)$, we have

$$
\frac{\tilde{c}_{\lambda\sigma\lambda}}{\tilde{c}_{\sigma\lambda\lambda}} = \frac{c_{\lambda\sigma\lambda}}{c_{\sigma\lambda\lambda}} (\tilde{\alpha}_2^2 - s_u^2),
$$

where $\tilde{\alpha}_2^2 = \left(\frac{n+1}{2}\right)^2$ is the only admissible value of $\tilde{\alpha}_2^2$, so

$$
\frac{c_{\lambda\sigma\lambda}}{c_{\sigma\lambda\lambda}} = \frac{4(n-2)}{(n+1)(n-3)},
$$

$$
k(\zeta_{\lambda\sigma\lambda}(\eta)^*\zeta_{\lambda\sigma\lambda}(\eta)) = \frac{4(n-2)}{(n+1)(n-3)} k(\zeta_{\sigma\lambda\lambda}(\eta)\zeta_{\sigma\lambda\lambda}(\eta)^*).
One realization of $\nabla(2, 1)$ is as the bundle $H$ of 3-tensors that are totally trace free, antisymmetric in their last 2 indices, and Bianchi-like in all 3 indices. (For a different realization, replace “antisymmetric” with “symmetric” in the last sentence.) The natural operator $D$ defined by
\[
(D\psi)_{ijkl} = \frac{1}{4} (\nabla_i\psi_{jkl} - \nabla_j\psi_{ikl} + \nabla_k\psi_{ijl} - \nabla_l\psi_{ijk})
\]
carries $H$ to $W$; thus it is a constant multiple of the corresponding realization of $G_{\sigma_3}$. Writing $D\psi = Q\nabla\psi$, where $Q$ is a bundle map on $T^*M \otimes H$, one finds $Q^2 = Q$, so we have the correct normalization, and $D$ is exactly the realization of $G_{\sigma_3}$. Since
\[
(D^*\Psi)_{ijk} = -\nabla^i\Psi_{lijk},
\]
we have (in these realizations)
\[
(G^*G\Psi)_{ijkl} = \frac{4(n-2)}{(n+1)(n-3)} (DD^*\Psi)_{ijkl}
\]
\[
= -\frac{n-2}{(n+1)(n-3)} \left( \nabla_i \nabla^p \psi_{pijkl} - \nabla_j \nabla^p \psi_{piklj} + \nabla_k \nabla^p \psi_{plijk} - \nabla_l \nabla^p \psi_{pljik} \right)
\]
\[
+ \frac{2}{(n+1)(n-3)} \nabla^m \nabla^p \left\{ g_{ik}(\psi_{pjml} + \psi_{pmlj}) - g_{jk}(\psi_{piml} + \psi_{pmli}) 
- g_{il}(\psi_{pjmk} + \psi_{pmkj}) + g_{jl}(\psi_{pimk} + \psi_{pkmj}) \right\}.
\]
(36)
\[
\zeta_3(\eta)^*\zeta_3(\eta) \text{ is obtained, of course, by replacing each } \nabla_i \nabla_j \text{ with } -\eta_i \eta_j \text{ in the last expression:}
\]
\[
(\zeta_3(\eta)^*\zeta_3(\eta)\Psi)_{ijkl} = \frac{n-2}{(n+1)(n-3)} \left( \eta_i \eta^p \psi_{pijkl} - \eta_j \eta^p \psi_{piklj} + \eta_k \eta^p \psi_{plijk} - \eta_l \eta^p \psi_{pljik} \right)
\]
\[
- \frac{2}{(n+1)(n-3)} \eta^m \eta^p \left\{ g_{ik}(\psi_{pjml} - g_{jk}(\psi_{piml} - g_{il}(\psi_{pjmk} + g_{jl}(\psi_{pimk} \right) \right\}.
\]
To get an independent calculation of the minimal polynomial of $\zeta_3(\eta)^*\zeta_3(\eta)$ and of the fundamental projections, we just need to use (36) to compute the powers of $\zeta_3(\eta)^*\zeta_3(\eta)$. The results are
\[
(((\zeta_3(\eta)^*\zeta_3(\eta))^2\Psi)_{ijkl} = \frac{(n-2)^2}{(n-3)^2(n+1)^2} \left\{ \eta_i \eta^p \psi_{pijkl} - \eta_j \eta^p \psi_{piklj} + \eta_k \eta^p \psi_{plijk} - \eta_l \eta^p \psi_{pljik} \right\}
\]
\[
- \frac{4}{(n-3)(n+1)^2} \eta^m \eta^p \left\{ g_{ik}(\psi_{pjml} - g_{jk}(\psi_{piml} - g_{il}(\psi_{pjmk} + g_{jl}(\psi_{pimk} \right) \right\}.
\]
and
\[
(((\zeta_3(\eta)^*\zeta_3(\eta))^3\Psi)_{ijkl} = \frac{(n-2)^3}{(n-3)(n+1)^3} \left\{ \eta_i \eta^p \psi_{pijkl} - \eta_j \eta^p \psi_{piklj} + \eta_k \eta^p \psi_{plijk} - \eta_l \eta^p \psi_{pljik} \right\}
\]
\[
- \frac{8}{(n-3)(n+1)^2} \eta^m \eta^p \left\{ g_{ik}(\psi_{pjml} - g_{jk}(\psi_{piml} - g_{il}(\psi_{pjmk} + g_{jl}(\psi_{pimk} \right) \right\}.
\]
and
\[
(((\zeta_3(\eta)^*\zeta_3(\eta))^4\Psi)_{ijkl} = \frac{(n-2)^4}{(n-3)(n+1)^4} \left\{ \eta_i \eta^p \psi_{pijkl} - \eta_j \eta^p \psi_{piklj} + \eta_k \eta^p \psi_{plijk} - \eta_l \eta^p \psi_{pljik} \right\}
\]
\[
- \frac{16}{(n-3)(n+1)^3} \eta^m \eta^p \left\{ g_{ik}(\psi_{pjml} - g_{jk}(\psi_{piml} - g_{il}(\psi_{pjmk} + g_{jl}(\psi_{pimk} \right) \right\}.
\]
From this, one finds that for $A = \zeta_3(\eta)^* \zeta_3(\eta)$, the only linear relation among $A^3$, $A^2$, $A$, and 1 is, up to a constant multiple,

$$A^3 - \frac{3n - 8}{(n+1)(n-3)} A^2 + \frac{2(n-2)}{(n+1)^2(n-3)} A = 0. \quad (37)$$

But the left side of (37) factors to

$$A \left( A - \frac{n - 2}{(n+1)(n-3)} \right) \left( A - \frac{2}{n+1} \right),$$

exactly the minimal polynomial $M_{\lambda,3}(A)$ determined in (35) from abstract considerations.

**Example 7.7** A good example of a bundle with a self-gradient is the **twistor bundle** $\mathbb{V}(\lambda)$ for $\lambda := (\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ in odd dimensions. This bundle is realized in spinor-one-forms $\varphi$ with $\gamma^i \varphi_i = 0$. Suppose $n \geq 5$; then the selection rule targets are

$$\sigma_1 = \lambda + e_1, \quad \sigma_2 = \lambda + e_2, \quad \sigma_3 = \lambda, \quad \sigma_4 = \lambda - e_1,$$

Thus

$$N(\lambda) = 4, \quad t(\lambda) = 2, \quad b(\lambda) = 4, \quad b_0(\lambda) = 2. \quad (38)$$

The self-gradient is sometimes called the **Rarita-Schwinger operator**; normalized to have square $G_3^* G_3$, it is

$$(\mathcal{S}\varphi)_i = \left( \frac{n}{(n+2)(n-2)} \right)^{1/2} \left\{ \gamma^j \nabla_j \varphi_i - \frac{2}{n} \gamma_i \nabla^j \varphi_j \right\}. \quad (39)$$

in the twistor realization. We denote the leading symbol of $\mathcal{S}$ by $\Upsilon$. $\Upsilon(\eta)^2$ will have a minimal polynomial $M_{\lambda,3}(x)$ of degree 2 by (21), and $\Upsilon(\eta)$ will have a minimal polynomial $m_\lambda(x)$ of degree 4 by (26). Counting degrees, we conclude that

$$m_\lambda(x) = M_{\lambda,3}(x^2).$$

In particular, $m_\lambda$ is an even polynomial. It is clear that there is one tfs$^0$ action and one tfs$^1$ action on $\lambda$. As a result, by (38), there is also one tfs$^2$ action and one tfs$^3$ action, and these exhaust the possible tfs$^p$ actions. That is,

$$\dim \text{Hom}_{\text{Spin}(n)}(\text{tfs}^p \otimes \lambda, \lambda) = \begin{cases} 1, & p = 0, 1, 2, 3, \\ 0, & p > 3. \end{cases}$$

Since

$$\tilde{c}_3 = \frac{4}{n(n+2)(n-2)}, \quad h_m = \frac{n}{2} - m, \quad m = 0, 1, \quad (40)$$
our minimal polynomials and fundamental projections on $A_0(\lambda)$ are

$$M_{\lambda,3}(x) = \left( x - \frac{n}{(n+2)(n-2)} \right) \left( x - \frac{n-2}{n(n+2)} \right),$$

$$\Pi_0 = \frac{n}{\eta(n+2)(n-2)} - \frac{n-2}{n(n+2)} = \frac{(n-2)\{n(n+2)\eta^2 - (n-2)\}}{4(n-1)},$$

$$\Pi_1 = \frac{n}{\eta(n+2)(n-2)} - \frac{n-2}{n(n+2)(n-2)} = \frac{n\{(n+2)(n-2)\eta^2 - n\}}{4(n-1)}.$$ (41)

$\Pi_m$ is the projection onto the $(\frac{3}{2} - m, \frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2})$ branches; $\Pi_0(\eta)$ and $\Pi_1(\eta)$ are represented by the second-order operators

$$\frac{(n-2)\{n(n+2)\mathcal{S}^2 - (n-2)\nabla^*\nabla\}}{4(n-1)}, \quad \frac{-(n+2)(n-2)\mathcal{S}^2 - n\nabla^*\nabla}{4(n-1)}.$$

To see what is going on tensorially, consider the (unnormalized) leading symbol of the Rarita-Schwinger operator,

$$(\rho(\sqrt{-1}\eta)\varphi)_i = \sqrt{-1} \left( \gamma^j_\eta \varphi_i - \frac{2}{n} \gamma_i \eta^j \varphi_j \right).$$

We have

$$(\rho(\sqrt{-1}\eta)^2\varphi)_i = \varphi_i + \frac{4(n-1)}{n^2} \eta_i \eta^j \varphi_j + \frac{4}{n^2} \alpha_{ij} \eta^j \varphi_j,$$

$$(\rho(\sqrt{-1}\eta)^3\varphi)_i = \sqrt{-1} \left( \gamma^j_\eta \varphi_i + \frac{4(n-2)}{n^2} \gamma^k \eta_i \eta^j \eta^k \varphi_j - \frac{2(n^2 - 2n + 4)}{n^3} \gamma_i \eta^j \varphi_j \right),$$

$$(\rho(\sqrt{-1}\eta)^4\varphi)_i = \varphi_i + \frac{8(n-1)(n^2 - 2n + 2)}{n^4} \eta_i \eta^j \varphi_j + \frac{8(n^2 - 2n + 2)}{n^4} \alpha_{ij} \eta^j \varphi_j,$$

where $\alpha$ is the antisymmetric Clifford symbol,

$$\alpha^{ij} = \frac{1}{2} (\gamma^i \gamma^j - \gamma^j \gamma^i).$$

It is clear that these expressions exhaust the combinatorial possibilities (subject to the Clifford relations, the twistor condition, and $|\eta|^2 = 1$), that the $\rho(\sqrt{-1})^p \varphi$ for $p = 0, 1, 2, 3$ are linearly independent, and that

$$\rho(\sqrt{-1})^4 = \frac{2(n^2 - 2n + 2)}{n^2} \rho(\sqrt{-1})^2 - \frac{(n-2)^2}{n^2}.$$ That is, $\rho(\sqrt{-1})^2$ satisfies the polynomial

$$x^2 - \frac{2(n^2 - 2n + 2)}{n^2} x + \frac{(n-2)^2}{n^2}.$$

The normalized symbol (from (39)) thus satisfies

$$\frac{(n+2)^2(n-2)^2}{n^2} x^2 - 2(n+2)(n-2)(n^2 - 2n + 2) x + \frac{(n-2)^2}{n^2}.$$
But this is exactly
\[
\left( \frac{(n+2)(n-2)}{n} \right)^2 M_{\lambda,3}(x),
\]
for the \( M_{\lambda,3}(x) \) predicted by (\[\Pi\]).

8 Applications

8.1 Computation of Green’s functions

The discrete leading symbol allows us to compute Green’s functions, or fundamental solutions, of natural elliptic differential operators \( D \) with nonscalar leading symbol, as follows. Suppose \( D \) acts on sections of \( \mathcal{V}(\lambda) \). By ellipticity and Corollary 5.2(b), \( K \) is a nonzero function on \( B(\lambda) \). Let \( \Pi_i \) be the fundamental projection corresponding to the branch \( \beta_i \), and let \( P_i \) be a natural differential operator with leading symbol \( \Pi_i \). Then

\[
E := \sum_{i=1}^{b(\lambda)} \frac{1}{K(\beta_i)} P_i
\]
is a natural differential operator for which the function \( K(DE) \) is identically 1; that is,

\[
DE = \Delta^k + \text{(lower order)}
\]

for some \( k \). The computation of a fundamental solution for \( D \), that is, a distribution \( G(x,y) \) for which \( D_x G(x,y) = \delta_y(x)\text{Id}_{\mathcal{V}(\lambda)} \), is now reduced to a similar computation for the operator \( DE \), whose leading symbol is less exotic. For if \( H \) is a fundamental solution for \( DE \), then

\[
D_x(E_x H(x,y)) = (DE)_x H(x,y) = \delta_y(x)\text{Id}_{\mathcal{V}(\lambda)}.
\]

That is, \( E_x H(x,y) \) is a fundamental solution for \( D \).

The problem of computing asymptotic expansions and parametrices for operators with principal part \( \Delta^k \) is considerably more straightforward than the same problem for operators with arbitrary natural principal part. With the fundamental projections in hand, we have an effective procedure for getting quasi-inverses \( E \) for natural elliptic \( D \), and thus for effecting this reduction to the case of scalar leading symbol.

The corresponding problem for the heat operator \( \exp(-tD) \) is not so greatly simplified by the computation of a quasi-inverse. However, as shown in \( \Pi \), it may also be attacked using the \( \Pi_i \).

8.2 Conformally covariant operators

The discrete spectral calculus allows a computation of the principal part of any conformally covariant operator on sections of \( \mathcal{V}(\lambda) \). As a result, it also gives a complete formula for any such operator in the conformally flat case.

To describe this, we need a short summary of the state of knowledge about conformally covariant operators. All conformally covariant differential operators in the conformally
flat case appear in Bernstein-Gelfand-Gelfand (BGG) resolutions; see [11] and references therein. Operators $D$ that exist in the conformally flat case and are not longest arrows in even-dimensional BGG resolutions have conformally curved generalizations; that is, operators $\tilde{D}$ that exist and are covariant generally (without the assumption of conformal flatness), and which generalize the conformally flat operator $D$. The exceptional case, longest arrows in even dimensions, consists exactly of operators

$$\nabla^{r-n/2}(\lambda) \to \nabla^{-r-n/2}(\lambda)$$

for which $(r, \tilde{\lambda})$ is a strictly dominant integral or half-integral $\mathfrak{so}(n + 2)$ weight, and $n$ is even. It is known [13] that the operator

$$\nabla^1(0) \to \nabla^{-5}(0),$$

which exists in the conformally flat case, has no conformally curved generalization. Since BGG resolutions are completely understood in the conformally flat case, we know exactly when conformal covariants exist in the conformally flat case.

A conformal covariant is a natural differential operator $D$ carrying sections of some $\mathbb{V}^s(\lambda)$ to sections of some $\mathbb{V}^t(\mu)$ with

$$\bar{g} = \Omega^2 g, \quad 0 < \Omega \in C^\infty \Rightarrow \bar{D} = \Omega^t D \Omega^{-s}.$$

(The power of $\Omega$ on the far right is to be understood as a multiplication operator.) If the volume form $E$ and/or fundamental tensor-spinor $\gamma$ are involved, they are assumed to scale compatibly:

$$E = \Omega^n E, \quad \bar{\gamma} = \Omega^{-1}\gamma.$$

(The latter scaling is enforced by the Clifford relations.)

The discrete leading symbol may be viewed as a tool for converting spectral information on differential operators into tensorial formulas. In [8], Sec. 3.a, a formula is given for the spectrum, on $S^n$, of intertwining operators for the conformal group $\text{Spin}(n + 1, 1)$. A conformal covariant automatically gives rise to an intertwinor, and the intertwinor with given weight parameters, if it exists, is unique ([8], Sec. 6). Thus we have a formula for the spectrum of each conformal covariant on each $\mathbb{V}(\lambda)$. The problem, a priori, is that we have a formula for much more – “most” of the operators are only pseudo-differential, not differential, operators. The spectral formula, from [8], (3.3), is

$$Z(r, \lambda) = \prod_{a=1}^{[(n+1)/2]} \frac{\Gamma(\tilde{\alpha}_a + \frac{1}{2} + r)}{\Gamma(\tilde{\alpha}_a + \frac{1}{2} - r)},$$

as long as

$$n \text{ is odd or } \lambda_\ell \neq 0.$$

The spectral function is to be viewed as a meromorphic function of $r$. If a formula of this type, as it stands, is identically zero or undefined, we renormalize by an $\alpha$-independent
meromorphic function of $r$ to bring out the information. With this in mind, we could also consider

$$
\tilde{Z}(r, \lambda) = \prod_{a \in \mathcal{T}(\lambda)} \frac{\Gamma(\tilde{\alpha}_a + \frac{1}{2} + r)}{\Gamma(\tilde{\alpha}_a + \frac{1}{2} - r)},
$$

(43)

to be the spectral function, since the factors contributed by $\mathcal{F}(\lambda)$ just provide a meromorphic renormalization. The order of the resulting operator as a pseudo-differential operator is $2r$, and the operator is intertwining between the spaces of $[12]$.

In particular, to get a differential operator, it is necessary (but not sufficient) that $2r$ be a nonnegative integer. Given that $2r \in \mathbb{N}$, one way to test whether our operator $D$ is differential is to try to write its spectrum on $S^n$ as a polynomial in the spectra of $G^*G$ (for gradients $G$), and, if applicable, the self-gradient. If we happen to know (by BGG methods, say) that there is a differential operator with these weight parameters, we can do something easier than realizing the spectrum as a polynomial in spectra of low-order operators – we can simply realize the discrete leading symbol as a polynomial in discrete leading spectra of low-order operators. If one of these processes succeeds, we have, in particular, a formula for the principal part of $D$, and thus for $D$ on standard $\mathbb{R}^n$. We can then use the formula for the conformal change of the Ricci tensor to write $D$ for any conformally flat metric. In doing this, we can either (1) check BGG resolutions to see whether there should be a covariant differential operator with the given parameters, or (2) simply go ahead with the procedure and see whether a covariant operator results.

More precisely, suppose we find that on $S^n$,

$$
D = P(D_u),
$$

where $D_u$ is $D_{\text{self}}$ when $G_u$ is a self-gradient, and is $G_u^*G_u$ otherwise. Now consider the conformal covariance relations for $G_u$, $G_u^*$, and $D_{\text{self}}$: if $\tilde{g} = e^{2\omega}g$, then

$$
\tilde{D}_{\text{self}} = \exp \left( -\frac{n+1}{2} \omega \right) D_{\text{self}} \exp \left( \frac{n-1}{2} \omega \right),
$$

$$
\tilde{G}_u = \exp \left( -\left( \frac{n+1}{2} + s_u \right) \omega \right) G_u \exp \left( \frac{n-1}{2} + s_u \right) \omega),
$$

$$
\tilde{G}_u^* = \exp \left( -\left( \frac{n+1}{2} - s_u \right) \omega \right) G_u^* \exp \left( \frac{n-1}{2} - s_u \omega \right).
$$

Since

$$
\tilde{D} = P(D_u),
$$

this gives us a formula, involving $\omega$, for $D$ at any conformally flat metric in terms of the formula for $D$ at the standard flat metric.

On the other hand, $\tilde{D}$ is natural, so there should be a formula for it, in terms of covariant derivatives in the metric $\tilde{g}$, which does not explicitly mention $\omega$. An efficient way to arrive at this formula is as follows. Suppose $g_0$ is a flat metric, let $g_\omega = e^{2\omega}g_0$, and affix the subscript $\omega$ to all quantities computed in the metric $g_\omega$. Then

$$
D_\omega = \exp \left( -r - \frac{n}{2} \right) \omega \right) D_0 \exp \left( -r + \frac{n}{2} \right) \omega)
$$

$$
= \exp \left( -r - \frac{n}{2} \right) \omega \right) P(\left( G_u^* \right)_0 (G_u)_0 \right) \exp \left( -r + \frac{n}{2} \right) \omega)
$$

$$
= \exp \left( -r - \frac{n}{2} \right) \omega \right) \exp \left( \frac{n+1}{2} - s_u \omega \right) \omega \right) \exp \left( 2s_u + 1 \right) \omega \right) (G_u) \omega \exp \left( -\left( \frac{n-1}{2} + s_u \right) \omega \right),
$$

$$
\exp \left( \frac{n+1}{2} \omega \right) (D_{\text{self}}) \omega \exp \left( -\frac{n-1}{2} \omega \right) \right) \exp \left( -r + \frac{n}{2} \right) \omega).
$$
Now all covariant derivatives and curvatures involved in the expression on the far right are those of $g_\omega$. Applying the Leibniz rule to move $\omega$ to the left, we obtain a natural differential operator with coefficients that are polynomial in (in addition to the usual ingredients) iterated covariant derivatives of $\omega$, of order at least 1. (The overall power of $e^{\omega}$ in front is 0, since each $G^*G$ contributes $-2$, each $D_{\text{self}}$ contributes $-1$, and the homogeneity degree of $P$ is $2r$.)

Now consider the formula for the conformal change of the Ricci tensor, as applied to $g_0$ and $g_\omega$.

$$\omega_{ij} = -V_{ij} - \omega_i \omega_j + \frac{1}{2} \omega_k \omega^k (g_\omega)_{ij}. \quad (44)$$

Here all covariant derivatives and curvatures are in the metric $g_\omega$, and $J$ and $V$ are the normalizations of the scalar curvature $K$ and Ricci tensor $r$ that are best adapted to conformal geometry:

$$J = \frac{K}{2(n-1)}, \quad V_{ij} = \frac{r_{ij} - J g_{ij}}{n-2}.$$  

In (44), we have also employed the usual notational abuse: for a scalar function, $\omega_{j...i} := \nabla^i \cdots \nabla^j \eta$. Using (44) and its iterated covariant derivatives, we may reduce the dependence of the coefficients to just $\nabla \omega$. The condition that this dependence also disappears is equivalent to the conformal covariance of $D$.

For example, consider the problem of finding a fourth-order conformal covariant on trace-free symmetric 2-tensors. We are assured of the existence of such an operator $S$, in the conformally flat case, by BGG considerations. Substituting into the spectral function, the discrete leading symbol of $S$ is

$$K(S)(q) = \left(q + \frac{n}{2}\right) \left(q + \frac{n}{2} - 1\right) \left(q + \frac{n}{2} - 2\right) \left(q + \frac{n}{2} - 3\right), \quad q \in \{0, 1, 2\}. \quad (45)$$

In this case, the parameter $\alpha_2$ has just one variable entry, namely $\alpha_2$, which we have renamed $q$ for simplicity.

Note that by Corollary 5.2, (45) shows that $S$ is elliptic whenever the dimension is not 2, 4, or 6. It has positive definite leading symbol when $n > 6$, positive semidefinite leading symbol when $n$ is 2, 4, or 6, and indefinite leading symbol when $n$ is 3 or 5.

By Sec. 6, we know that the discrete leading symbol of any natural differential operator on TFS$^2$ may be written as a polynomial in the discrete leading symbols of $G^*_1 G_1$ and $\nabla^* \nabla$, which are the functions

$$x(q) := \frac{(3-q)(q+n)}{3(n+2)} \text{ and } 1 \text{ respectively.}$$

We thus obtain a formula for the principal part of $S$ as a polynomial in $G^*_1 G_1$ and $\nabla^* \nabla$, that is,

$$S = a (G^*_1 G_1)^2 + b G^*_1 G_1 \nabla^* \nabla + c (\nabla^* \nabla) + (\text{lower order}),$$

by simultaneously solving

$$ax(q)^2 + bx(q) + c = \left(q + \frac{n}{2}\right) \left(q + \frac{n}{2} - 1\right) \left(q + \frac{n}{2} - 2\right) \left(q + \frac{n}{2} - 3\right).$$
for $q = 0, 1, 2$. The result is

\begin{align*}
a &= 9(n + 2)^2, \\
b &= -\frac{3}{2}(n + 2)(n^2 + 6n + 4), \\
c &= \frac{1}{16}n(n + 2)(n + 4)(n + 6).
\end{align*}

In view of (32), we have an operator with principal part

\begin{equation}
B_{ij} \mapsto \frac{1}{16}(n - 2)n(n + 2)(n + 4)B_{ij[k}k_l^l - \frac{1}{2}(n - 2)n(n + 2)B_{iklj}j^l_l \\
- \frac{1}{2}(n - 2)n(n + 2)B_{ijkl}k^l l + 2(n - 2)nB_{ijkl}k^l l \\
+ (n - 2)nB_{ijkl}k^l l m g_{ij}.
\end{equation}

Applying the procedure described above to parlay the principal symbol into a precise formula in the conformally flat case, we have:

\begin{equation}
(SB)_{ij} = \frac{1}{16}(n - 2)n(n + 2)(n + 4)B_{ij[k}k_l^l - \frac{1}{2}(n - 2)n(n + 2)B_{iklj}j^l_l \\
- \frac{1}{2}(n - 2)n(n + 2)B_{ijkl}k^l l + 2(n - 2)nB_{ijkl}k^l l \\
+ (n - 2)nB_{ijkl}k^l l m g_{ij} - \frac{1}{16}(n - 4)(n - 2)n(n + 2)(n + 6)B_{ij[k}J^l j_l \\
+ \frac{1}{4}(n - 2)n(n^2 + 2n - 16)B_{ij[k}J^l j_l + \frac{1}{2}(n - 2)n(n^2 + 2n - 16)B_{iklj}J^l l \\
- \frac{1}{2}(n - 2)nB_{ijkl}g_{ij}J^l l + \frac{1}{32}(n - 2)n(n + 2)(n^2 - 48)B_{ijkl}J^l l \\
- \frac{1}{2}(n - 2)nB_{ijkl}g_{ij}J^l l - (n - 4)(n - 2)nB_{ijkl}kl V_{ij} \\
- 2(n - 6)(n - 2)nB_{ijkl}kl V_{ij} - n(n^2 - 10n + 4)B_{ijkl}kl V_{ij} \\
+ \frac{1}{4}(n - 2)(n + 4)B_{ijkl}kl V_{ij} - \frac{1}{8}(n - 2)^2n(n^2 - 24)B_{ijkl}kl V_{ij} \\
+ \frac{1}{3}(n - 4)(n - 2)n(n + 2)B_{ijkl}kl V_{ij} - \frac{1}{4}(n - 6)(n - 2)nB_{ijkl}kl V_{ij} \\
- 2(n - 3)(n - 2)nB_{ijkl}kl V_{ij} + \frac{1}{2}(n - 4)(n - 2)nB_{ijkl}kl V_{ij} \\
- \frac{1}{8}(n - 2)n(n^2 + 2n^2 - 2n - 96)B_{ijkl}kl V_{ij} + \frac{1}{2}(n - 2)n(n^2 - 2n - 16)B_{ijkl}kl V_{ij} \\
- (n - 10)(n - 2)nB_{ijkl}kl V_{ij} + 2(n - 2)nB_{ijkl}kl V_{ij} \\
+ \frac{1}{4}(n - 6)(n - 2)n(n + 4)B_{ijkl}kl V_{ij} - \frac{1}{2}(n - 5)(n - 2)nB_{ijkl}kl V_{ij} \\
+ (n + 2)n(n^2 + 8n + 20)B_{ijkl}kl V_{ij} + \frac{1}{4}(n - 2)n(n + 2)(n + 4)B_{ijkl}kl V_{ij} \\
- \frac{1}{2}(n - 2)n(n^2 + 6n + 16)B_{ijkl}kl V_{ij} - \frac{1}{4}(n - 2)n(n^2 + 6n + 16)B_{ijkl}kl V_{ij} \\
+ (n + 2)n(n + 8)B_{ijkl}kl V_{ij} - \frac{1}{2}(n - 2)nB_{ijkl}kl V_{ij} \\
+ \frac{1}{3}(n - 4)(n - 2)n(n + 4)B_{ijkl}kl V_{ij} - \frac{1}{2}(n - 2)(n + 8)B_{ijkl}kl V_{ij} \\
+ \frac{1}{3}(n - 4)(n - 2)n(n + 4)B_{ijkl}kl V_{ij} - \frac{1}{3}(n - 4)(n - 2)n(n + 8)B_{ijkl}kl V_{ij} \\
+ \frac{1}{8}(n - 2)n(3n^3 + 4n^2 - 24n - 32)B_{ijkl}kl V_{ij} \\
+ \frac{1}{8}(n - 2)n(3n^3 + 4n^2 - 24n - 32)B_{ijkl}kl V_{ij} \\
- \frac{1}{16}(n - 2)(n + 10)(n - 4)n - 16)B_{ijkl}kl V_{ij} - (n - 2)^2nB_{ijkl}kl V_{ij} \\
+ 4(n - 2)nB_{ijkl}kl g_{ij}V_{ij} - \frac{1}{4}(n - 2)n(3n^2 - 6n - 16)B_{ijkl}kl g_{ij}V_{ij} \\
+ 4(n - 2)nB_{ijkl}kl g_{ij}V_{ij} - \frac{1}{4}(n - 2)n(3n^2 - 6n - 16)B_{ijkl}kl g_{ij}V_{ij}.
\end{equation}

(The transition from (46) to (47) was accomplished via an automated computation using Jack Lee’s Ricci package [15].)

Computing in the not necessarily conformally flat case (again using Ricci), one finds that the formula (47) is not conformally covariant in general. More precisely, the confor-
mal variation
\[ \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \left[ \exp \left( \left( \frac{n}{2} + 2 \right) \varepsilon \eta \right) S_{\exp(2\varepsilon\eta)} g \exp \left( - \left( \frac{n}{2} - 2 \right) \varepsilon \eta \right) \right] \]

(\text{where } \eta \text{ is an arbitrary smooth function) does not vanish identically. Schematically, this variation has three types of terms: } (\nabla C)(\nabla \eta)B, C(\nabla \eta)\nabla B, \text{ and } C(\nabla \nabla \eta)B, \text{ where } C \text{ is the Weyl conformal curvature tensor. Using the conformally invariant calculus of tractors} \footnote{3}, \text{ however, Branson and Gover} \footnote{7} \text{ have been able to get a formula for tractors } C \text{ which is conformally covariant in the general conformally curved case, and which reduces to } (47) \text{ for conformally flat metrics.}

As another example, consider the bundle } T \text{ of twistors, the subbundle of the spinor-one-forms } \varphi_a \text{ with } \gamma^a \varphi_a = 0. \text{ If } n \text{ is odd, this is (isomorphic to) the irreducible bundle } \nabla(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}); \text{ if } n \text{ is even, } T \text{ is isomorphic to}
\begin{equation}
\nabla \left( \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \right) \oplus \nabla \left( \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2} \right). \tag{48}
\end{equation}

Recall that the odd-dimensional case is worked out in detail above (Example \footnote{4,7}); the self-gradient } \mathcal{R} \text{ is the } \text{Rarita-Schwinger operator} \footnote{3}. \text{ By BGG considerations, we are led to expect differential intertwinors } \mathcal{R}_r, \text{ carrying}
\begin{equation}
T^{r-n/2} \rightarrow T^{-r-n/2}
\end{equation}

for each positive, properly half-integral } r. \text{ That is, there is an operator of each positive odd order. By the spectral formula } (43), \text{ the spectrum of } \mathcal{R}_r \text{ on } S^n \text{ is (up to constant multiples)}
\begin{equation}
\left\{ (\tilde{a}_1 - \frac{1}{2} + r)(\tilde{a}_1 - \frac{3}{2} + r) \cdots (\tilde{a}_1 + \frac{1}{2} - r) \right\},
\left\{ (\tilde{a}_2 - \frac{1}{2} + r)(\tilde{a}_2 - \frac{3}{2} + r) \cdots (\tilde{a}_2 + \frac{1}{2} - r) \right\},
\left\{ (\tilde{a}_L - \frac{1}{2} + r)(\tilde{a}_L - \frac{3}{2} + r) \cdots (\tilde{a}_L + \frac{1}{2} - r) \right\},
\end{equation}

where } L = (n+1)/2. \text{ In particular, the discrete leading symbol is}
\begin{equation}
\left\{ (\tilde{a}_2 - \frac{1}{2} + r)(\tilde{a}_2 - \frac{3}{2} + r) \cdots (\tilde{a}_2 + \frac{1}{2} - r) \right\},
\left\{ (\tilde{a}_L - \frac{1}{2} + r)(\tilde{a}_L - \frac{3}{2} + r) \cdots (\tilde{a}_L + \frac{1}{2} - r) \right\},
\end{equation}

(In each \cdots, the factors decrease by 1 each time.) The possible values for } \tilde{a}_2 \text{ are } \frac{q}{2} - 1 + q, \text{ where } q \in \{0, 1\}, \text{ and the possible values for } \tilde{a}_L \text{ are } \frac{1}{2} \varepsilon, \text{ where } \varepsilon = \pm 1. \text{ Thus the discrete leading symbol is}
\begin{equation}
x(q, \varepsilon) := \left\{ (\frac{q}{2} - \frac{3}{2} + q + r) \cdots (\frac{q}{2} - \frac{1}{2} + q - r) \right\},
\left\{ (\frac{1}{2} \varepsilon - \frac{1}{2} + r) \cdots (\frac{1}{2} \varepsilon + \frac{1}{2} - r) \right\}. \tag{49}
\end{equation}

If we match } \varepsilon\text{-dependent factors symmetric about the middle one to form differences of squares,
\begin{equation}
\left( \frac{1}{2} \varepsilon - \frac{1}{2} + r - p \right) \left( \frac{1}{2} \varepsilon + \frac{1}{2} - r + p \right) = \frac{1}{4} \varepsilon^2 - \left( \frac{1}{2} - r + p \right)^2 = \frac{1}{4} - \left( \frac{1}{2} - r + p \right)^2,
\end{equation}
for $p$ running from 0 to $(2r - 3)/2$, we get a polynomial in $r$ which does not depend on $\alpha$; thus, a renormalization in the sense described above. The discrete leading symbol of the renormalized operator is

$$x(q, \varepsilon) = \frac{1}{2} \varepsilon \left( \frac{n}{2} - \frac{3}{2} + q + r \right) \cdots \left( \frac{n}{2} - \frac{1}{2} + q - r \right)$$

$$= \frac{1}{2} \varepsilon \left( \frac{n}{2} - 1 + q \right) \prod_{p=1}^{(2r-1)/2} \left\{ \left( \frac{n}{2} - 1 + q + p \right) \left( \frac{n}{2} - 1 + q - p \right) \right\}$$

$$= \frac{1}{2} \varepsilon \left( \frac{n}{2} - 1 + q \right) \prod_{p=1}^{(2r-1)/2} \left[ \left( \frac{n}{2} - 1 + q \right)^2 - p^2 \right].$$

The self-gradient, being a conformal covariant, should be $\mathcal{R}_1$, up to a constant factor, and thus, by (19), should have discrete leading symbol $\frac{1}{2} \varepsilon \left( \frac{n}{2} - 1 + q \right)$. This agrees with (18). If $\sigma$ is the discrete leading symbol of $\mathcal{R}_1$, then (19) shows that the discrete leading symbol of $\mathcal{R}_{2r}$ is

$$\sigma = \prod_{p=1}^{(2r-1)/2} (4\sigma^2 - p^2),$$

indicating that $\mathcal{R}_{2r}$ has principal part

$$\mathcal{R}_1 \prod_{p=1}^{(2r-1)/2} (4\mathcal{R}_1^2 - p^2 \nabla^* \nabla).$$

To see the precise normalization of $\mathcal{R}_1$ in tensorial terms, note that by (18), (39), and (40),

$$\frac{2}{\sqrt{n(n+2)(n-2)}} (\mathcal{R}_1 \varphi)_i = \left( \frac{n}{(n+2)(n-2)} \right)^{1/2} \left\{ \gamma^j \nabla_j \varphi_i - \frac{2}{n} \gamma^i \nabla^j \varphi_j \right\},$$

so

$$(\mathcal{R}_1 \varphi)_i = \frac{n}{2} \left\{ \gamma^j \nabla_j \varphi_i - \frac{2}{n} \gamma^i \nabla^j \varphi_j \right\}.$$

Let us apply the procedure described above to get a formula for $\mathcal{R}_3$ in the general conformally flat case. We may re-express the principal part as a homogeneous polynomial in

$$(\mathcal{R} \varphi)_i = \gamma^j \nabla_j \varphi_i - (2/n) \gamma_i \nabla^j \varphi_j.$$

and $\mathbb{T}^* \mathbb{T}$, where $\mathbb{T}$ is the twistor operator carrying spinors to twisters:

$$(\mathbb{T} \psi)_i = \nabla_i \psi + (1/n) \gamma_i \gamma^j \nabla_j \psi,$$

since there is one linear relation among $\nabla^* \nabla$, $\mathcal{R}^2$, and $\mathbb{T}^* \mathbb{T}$. (The formal adjoint of $\mathbb{T}$ is $\mathbb{T}^* \mathbb{A} = - \nabla^j A_j$.) The result is:

$$(\mathcal{R}_3 \varphi)_i = \frac{n(n+2)}{4} (\mathcal{R}^3 \varphi)_i - \frac{4}{n-2} (\mathbb{T}^* \mathcal{R} \varphi)_i$$

$$- \frac{n+2}{n} \mathbb{J} \gamma_i \nabla^j \nabla_j \varphi_i + \mathbb{V}^j \gamma^k \nabla_k \nabla_j \varphi_j + (n+2) \mathbb{V}_i^k \gamma_k \nabla_j \varphi_j + (n+1) \mathbb{V}_i^j \gamma_k \nabla_k \varphi_j$$

$$- \frac{n(n+2)}{n} \mathbb{V}_i^j \gamma_k \nabla_j \varphi_i + (n-1) \mathbb{V}_i^j \gamma_k \nabla_i \varphi_j + \mathbb{V}_i^j \gamma^k \nabla_k \varphi_j + \frac{n}{2} \nabla^j \mathbb{J} \gamma_i \varphi_j$$

$$- \frac{n(n+2)}{4} (\nabla^j \mathbb{J}) \gamma_i \varphi_i + n(\nabla^k \mathbb{V}_i^j) \gamma_k \varphi_j,$$
where
\[ \alpha_{ijk} := \gamma_i \gamma_j \gamma_k \]
is the antisymmetrized iterated Clifford symbol. Again, the calculations were automated using \textit{Ricci}.

The operator and its conformal covariance relation may be considered as polynomial identities in the dimension \( n \). Since the covariance relation holds for an infinite number of \( n \) (all odd \( n \)), it may be continued to even dimensions. In the even dimensional case, each \( R_{2r} \) interchanges the summands in (18), because \( R_1 \) does.

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The operator and its conformal covariance relation may be considered as polynomial identities in the dimension \( n \). Since the covariance relation holds for an infinite number of \( n \) (all odd \( n \)), it may be continued to even dimensions. In the even dimensional case, each \( R_{2r} \) interchanges the summands in (18), because \( R_1 \) does.

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