Noise kernels of stochastic gravity in conformally-flat spacetimes

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Abstract

The central object in the theory of semiclassical stochastic gravity is the noise kernel, which is the symmetric two point correlation function of the stress–energy tensor. Using the corresponding Wightman functions in Minkowski, Einstein and open Einstein spaces, we construct the noise kernels of a conformally coupled scalar field in these spacetimes. From them we show that the noise kernels in conformally-flat spacetimes, including the Friedmann–Robertson–Walker universes, can be obtained in closed analytic forms by using a combination of conformal and coordinate transformations.

Keywords: stochastic gravity, noise kernel, conformally-flat spacetimes

1. Introduction

In semiclassical gravity (SCG) the effect of quantum fields on the background spacetime [1] is accounted for by solving the Einstein equation with the expectation value of the quantum matter field’s stress–energy tensor as the source, known as the semiclassical Einstein equation (SCE). To tackle this so-called ‘backreaction’ problem [2, 3], one needs first to find a finite expression of the expectation value of the stress–energy tensor by suitable regularization and renormalization procedures. The resulting expression for the stress–energy tensor is complicated and usually cannot be expressed in a closed analytic form. For example, even for the generic Schwarzschild black hole, it takes considerable work to get an approximate analytical expression of the stress–energy tensor which checks with the numerically accurate results [4].

One exception is for the conformally invariant fields, especially in conformally-flat spacetimes. It is possible to derive the stress–energy tensor in closed form by integrating the local expressions of the corresponding conformal anomaly [5, 6]. One subtlety in this procedure is in the choice of the vacuum state, even in flat space, be it the Minkowski or the Rindler vacua. This point has been discussed in some detail in the paper by Candelas and...
Dowker [7] where conformally-flat spacetimes are classified into two categories with the help of the Penrose diagrams. With the Minkowski vacuum there are the spatially flat de Sitter, the flat Friedmann–Robertson–Walker (FRW) Universe, the Einstein Universe, the global de Sitter, and the closed FRW Universe. While with the Rindler vacuum one has the open Einstein Universe, the Milne Universe, the open FRW Universe, and the static de Sitter. On the other hand, the two vacua are related by ‘thermalization’\(^3\), in the way that the Minkowski vacuum is the ‘thermalization’ of the Rindler vacuum. In the next section we shall explore this situation by looking at the Wightman functions in various conformally-flat spacetimes.

SCG based on the expectation value of the stress–energy tensor provides a mean field theory description. To take into account the effects of the matter field quantum fluctuations and the induced metric fluctuations of the spacetime (obtained as a solution of the SCE), one needs to invoke stochastic gravity \([8]\). In this theory the matter quantum fluctuations manifest as a stochastic force term in the Einstein equation, resulting in the so-called Einstein–Langevin equation (ELE) \([9]\)—a physical way of derivation and understanding is by means of the Brownian motion paradigm (see, e.g., \([10]\) and references therein). The correlator of the stochastic force is given by the noise kernel, which is also the symmetric two point correlation function of the stress–energy tensor. In \([11]\) a general formula for the noise kernel of a scalar field in terms of higher covariant derivatives of the Green functions is provided. In this paper we first find in section 2 the Wightman functions of conformal scalar fields and then use the expressions of \([11]\) to evaluate the noise kernels in Minkowski, Einstein, and open Einstein spacetimes.

It is proven in \([12]\) that the noise kernels of a conformally coupled scalar field in two conformally related spacetimes, \(\tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x)\), are related simply by \(\tilde{N}_{\mu\nu\rho\sigma}(x, x') = \Omega^{-2}(x)N_{\mu\nu\rho\sigma}(x, x')\Omega^{-2}(x')\) (here primes on indices denote tensor indices at the point \(x'\) and unprimed ones denote indices at the point \(x\)). Hence, from the noise kernels in Minkowski and Einstein spacetimes, it is possible to obtain the various FRW noise kernels by the appropriate conformal transformations. With additional coordinate transformations, the static de Sitter case can be dealt with. Lastly, the Rindler noise kernel will also be presented. These are the main contents in sections 3 and 4. Conclusions and discussions are given in section 5.

2. Wightman functions in Einstein universes

In this section we consider the Wightman functions of a conformally coupled scalar field in Minkowski, closed Einstein and open Einstein spacetimes. Using these functions we can derive the corresponding noise kernels in these spacetimes and also in the related FRW spacetimes. We shall do that in the next two sections.

In the Minkowski spacetime, \(ds^2_M = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\), the corresponding Wightman function of a scalar field is just

\[
G^+_M(x, x') = \frac{1}{4\pi^2 \left(-\Delta t^2 + \Delta s^2\right)},
\]

where \(\Delta t = t - t'\) and \(\Delta s = \sqrt{r'^2 + r^2 - 2rr' \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')}\).

\(^3\) This terminology used by \([7]\) refers to a transformation between states, one of which is thermal. It has nothing to do with the thermalization process in nonequilibrium systems.
Next, we look at the Einstein Universe \((R \times S^3)\) with the metric
\[
\mathrm{d}s_E^2 = -\mathrm{d}t^2 + a^2 \left( \mathrm{d}x^2 + \sin^2 \chi \ d\theta^2 + \sin^2 \chi \sin^2 \theta \ d\phi^2 \right),
\]
where \(\pi \geq \chi \geq 0\). Under the coordinate transformation
\[
t \pm r = a \tan \left( \frac{t_E/a \pm \chi}{2} \right),
\]
the Einstein Universe metric becomes
\[
\mathrm{d}s_E^2 = 4 \cos^2 \left( \frac{t_E/a + \chi}{2} \right) \cos^2 \left( \frac{t_E/a - \chi}{2} \right) \mathrm{d}s_M^2.
\]
This indicates that the Einstein Universe is conformally related to the Minkowski spacetime with the conformal factor
\[
\Omega(x) = 2 \cos \left( \frac{t_E/a + \chi}{2} \right) \cos \left( \frac{t_E/a - \chi}{2} \right).
\]

Similarly, the Wightman function \(G_E^+(x, x')\) of a conformally coupled scalar field in the Einstein Universe is related to the corresponding Wightman function in Minkowski spacetime \(G_M^+(x, x')\) by
\[
G_E^+(x, x') = \Omega^{-1}(x) G_M^+(x, x') \Omega^{-1}(x').
\]
Since the spatial manifold is homogeneous, one can take \(\theta = \theta'\) and \(\phi = \phi'\) without loss of generality. Then
\[
G_M^+ = \left( \frac{1}{4\pi^2} \right) \left[ a \tan \left( \frac{t_E/a + \chi}{2} \right) - a \tan \left( \frac{t_E/a - \chi}{2} \right) \right]^{-1}
\times \left[ -a \tan \left( \frac{t_E/a - \chi'}{2} \right) + a \tan \left( \frac{t_E/a - \chi'}{2} \right) \right]^{-1},
\]
and [13]
\[
G_E^+ = \frac{1}{8\pi^2 a^2} \left[ \cos \left( \frac{\Delta t_E}{a} \right) - \cos \left( \frac{\Delta s}{a} \right) \right]^{-1}.
\]
Here \(\Delta s = a \Delta \chi\). In general it is the geodesic distance between two points on the spatial \(S^3\), \(\Delta s = a (\Delta \chi)\) with
\[
\cos \Delta \chi = \cos \chi \cos \chi' + \sin \chi \sin \chi' (\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')).
\]

Finally we come to the open Einstein Universe \((R^1 \times H^3)\) with the metric
\[
\mathrm{d}s_E^2 = -\mathrm{d}t^2 + a^2 \left( \mathrm{d}x^2 + \sinh^2 \chi \ d\theta^2 + \sinh^2 \chi \sin^2 \theta \ d\phi^2 \right),
\]
where \(\chi \geq 0\). Since one could go from the sphere \(S^3\) to the hyperboloid \(H^3\) by just changing \(a \rightarrow ia\), naively one would assume that the Wightman function in this case can be obtained from the Einstein Universe one by
\[
G_E^+|_{a \rightarrow ia} = -\frac{1}{8\pi^2 a^2} \left[ \cosh \left( \frac{\Delta t_E}{a} \right) - \cosh \left( \frac{\Delta s}{a} \right) \right]^{-1}.
\]
However, this is in fact not the case. Open Einstein Universe is a static spacetime and therefore it has a unique vacuum. The corresponding Wightman function was derived exactly
by Bunch [14] giving

\[ G^0_0(x, x') = \frac{\Delta s/a}{4\pi^2 \sinh (\Delta s/a)} \left( -\Delta t_0^2 + \Delta s^2 \right), \]

where \( \Delta s = a\Delta \tau \) with \( \cosh \Delta \tau = \cosh \chi \cosh \chi' - \sinh \chi \sinh \chi' \left( \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \right) \), (13) is again the geodesic distance on \( H^3 \). This Wightman function is not the same as that in equation (11). Therefore the vacua of the closed and the open Einstein universes are not conformally related even though the spacetimes are.

Along this line the authors in [7] have classified the conformally-flat spacetimes into two classes, one with the conformal vacuum of the Minkowski spacetime and the other with that of the open Einstein spacetime. On the other hand, these two classes are related by ‘thermalization’. This can be seen using the Wightman functions as follows. From equation (12) the Wightman function of a thermal state in the open Einstein universe with temperature \( T = 1/\beta = 1/2\pi a \) can be written as

\[ G^+_0(x, x')_{\text{thermal}} = \sum_{n=-\infty}^{\infty} \frac{\Delta s/a}{4\pi^2 \sinh (\Delta s/a)} \left( -\Delta t_0 + in\beta \right)^2 + \Delta s^2 \]

which satisfies the KMS condition. The summation over \( n \) can be done using the formula [15]

\[ \sum_{n=-\infty}^{\infty} \frac{1}{(n + A)^2 + B^2} = \frac{\pi \sinh 2\pi B}{B(\cosh 2\pi B - \cos 2\pi A)}. \]

Then equation (14) becomes

\[ G^+_0(x, x')_{\text{thermal}} = -\frac{1}{8\pi^2 a^2} \left[ \cosh \left( \frac{\Delta t_0}{a} \right) - \cosh \left( \frac{\Delta s}{a} \right) \right]^{-1} \]

which is the same as equation (11) since \( t_0 = t_E \). This ‘thermalization’ relation between the open and the closed Einstein universes also holds between the static and the global or spatially flat de Sitter, as well as the Rindler and the Minkowski spacetimes.

3. Noise kernels related to flat and closed FRW spacetimes

Here we are dealing with conformally invariant scalar fields. Their noise kernels in conformally related spacetimes have the transformation property [12]

\[ \tilde{N}_{\mu \nu \alpha \beta}^\prime (x, x') = \Omega(x)^{-2} N_{\mu \nu \alpha \beta} (x, x') \Omega(x')^{-2}. \]

Since the FRW spacetimes are conformal to the Minkowski and the Einstein universes, we can obtain the noise kernels of various FRW spacetimes from the Einstein Universe ones.

In this section we shall concentrate on the flat and the closed universes and leave the discussions on the open FRW ones to the next section. To compute the noise kernels in the Einstein universes, we make use of the formula [11]

\[ N_{\mu \nu \alpha \beta} = \dot{K}_{\mu \nu \alpha \beta} + g_{\mu \nu} \dot{K}_{\alpha \beta} + g_{\alpha \beta} \dot{K}_{\mu \nu} + g_{\mu \nu} g_{\alpha \beta} \dot{K}, \]
where

\begin{align*}
9 \mathcal{K}_{\mu \nu \alpha \beta} &= 4 \left( G_{\alpha \beta ; \mu} G_{; \beta \nu} + G_{; \alpha \beta ; \nu} G_{; \mu} + G G_{; \alpha \beta ; \mu \nu} \right) \\
&\quad - 2 \left( G_{; \nu} G_{; \alpha ; \mu \beta} + G_{; \mu} G_{; \alpha ; \nu \beta} + G_{; \beta} G_{; \alpha ; \mu \nu} + G_{; \alpha} G_{; \mu \nu} \right) \\
&\quad + 2 \left( G_{; \nu} G_{; \alpha \beta \mu} + G_{; \alpha \beta \nu} G_{; \mu} \right) \\
&\quad - \left( G_{; \mu \nu} R_{\alpha \beta \mu} + G_{; \mu \nu} G_{; \alpha \beta} \right) G + \frac{1}{2} R_{\alpha \beta \mu} R_{\mu \nu} G^2 \tag{19}
\end{align*}

\begin{align*}
36 \mathcal{K}' &= 8 \left( -G_{; \rho ; \nu} G_{; \mu ; \rho} + G_{; \mu} G_{; \rho ; \nu} + G_{; \nu} G_{; \rho ; \mu} \right) \\
&\quad + 4 \left( G_{; \rho} G_{; \mu \nu} - G_{; \mu} G_{; \rho \nu} - G_{; \nu} G_{; \mu \rho} \right) \\
&\quad - 2 R \left( 2 G_{; \rho} G_{; \mu \nu} - G_{; \mu} \right) \\
&\quad - 2 \left( G_{; \rho} G_{; \nu} - 2 G_{; \nu} \right) R_{\mu \nu} - R' R_{\mu \nu} G^2 \tag{20}
\end{align*}

\begin{align*}
36 \mathcal{K} &= 2 G_{; \mu \nu \alpha \beta} G_{; \rho \sigma} + 4 \left( G_{; \rho \sigma} G_{; \alpha \beta} + G_{; \alpha \beta} G_{; \rho \sigma} \right) \\
&\quad - 4 \left( G_{; \rho} G_{; \alpha \beta \sigma} + G_{; \alpha \beta} G_{; \rho \sigma} \right) \\
&\quad + R G_{; \rho} G_{; \alpha \beta} + R' G_{; \rho \alpha \beta} \\
&\quad - 2 \left( R G_{; \rho \alpha \beta} + R' G_{; \rho \alpha \beta} \right) G + \frac{1}{2} R R' G^2 . \tag{21}
\end{align*}

Note that the superscript + on $G$ has been omitted for notational simplicity. $R_{\mu \nu}$ and $R'_{\alpha \beta \mu}$ are the Ricci tensor evaluated at the points $x$ and $x'$, respectively; $R$ and $R'$ are the scalar curvature evaluated at $x$ and $x'$.

Since the expressions above are of the noise kernel of a conformally coupled scalar field, they should satisfy the traceless and the conservation conditions. For the traceless condition, we have

\begin{equation}
g^{\mu \nu} N_{\mu \alpha \nu \beta} = 0 \Rightarrow \left( \bar{\mathcal{K}}_{\mu \alpha \nu \beta} + 4 \bar{\mathcal{K}}_{\alpha \mu \nu \beta} \right) + g_{\alpha \beta} \left( \bar{\mathcal{K}}_{\mu \nu} + 4 \bar{\mathcal{K}} \right) = 0, \tag{22}
\end{equation}

and for the conservation condition, we have

\begin{equation}
N_{\mu \nu \alpha \beta ; \mu} = 0 \Rightarrow \left( \bar{\mathcal{K}}_{\mu \nu \alpha \beta ; \mu} + \bar{\mathcal{K}}_{\alpha \beta ; \mu} \right) + g_{\alpha \beta} \left( \bar{\mathcal{K}}_{; \mu} + \bar{\mathcal{K}}_{; \mu} \right) = 0. \tag{23}
\end{equation}

Indeed, with the expressions from equations (19) to (21) and using the fact that the Wightman function $G$ satisfies the equation $G_{\mu \rho} = RG/6$, one can check that $\bar{\mathcal{K}}_{\mu \rho \alpha \beta} + 4 \bar{\mathcal{K}}_{\alpha \rho \beta \mu}$, $\bar{\mathcal{K}}_{\rho \alpha \beta ; \mu} + \bar{\mathcal{K}}_{\mu \alpha \beta ; \rho}$, and $\bar{\mathcal{K}}_{\mu \rho ; \alpha} + \bar{\mathcal{K}}_{\mu} \alpha$ all vanish.

### 3.1. Flat FRW spacetimes

Here we evaluate the noise kernel of a flat FRW spacetime

\begin{equation}
\text{d} s^2 = a^2 (\eta) \left( -d\eta^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \right), \tag{24}
\end{equation}

which is conformal to the Minkowski spacetime. First by plugging the Minkowski Wightman function in equation (1) into the formula in equation (18), one can arrive at the Minkowski noise kernel. For the convenience of later presentation, we shall write the noise kernel in the following manner by defining the coefficient functions $C_{ij}$
\[ N_{\rho\eta'\rho'}(x, x') = C_{11} \]
\[ N_{\rho\eta'\eta'}(x, x') = C_{21}s_j' \]
\[ N_{\rho\eta'\rho'}(x, x') = C_{31}s_j s_k' + C_{32}g_{\rho'k'} \]
\[ N_{\eta\eta'\eta'}(x, x') = C_{41}s_j s_k' + C_{42}g_{\eta'k'} \]
\[ N_{g^0\rho'k'}(x, x') = C_{51}s_j s_k' + C_{52}s_j s_k s_l' + C_{53}\left(g_{\rho'k'} s_k' + g_{ik'}s_j\right) \]
\[ N_{g^i\rho'k'}(x, x') = C_{61}s_i s_j s_k s_l' + C_{62}\left(g_{g^i k'} s_k' + s_j s_k s_l'\right) + C_{63}\left(g_{ik'} s_j s_k s_l' + g_{ik'} s_j s_k s_l' + g_{ik'} s_j s_k s_l'\right) + C_{64}\left(g_{ik'} g_{jk'} + g_{ik'} g_{jk'}\right) + C_{65}g_{g' k'} \]

(25)

where \( s_i = V_i(\Delta s) \) and \( s_j' = V'_j(\Delta s) \) are the derivatives on the spatial geodesic distance \( \Delta s \) between \( x \) and \( x' \). Also, \( g_{ij} \) is the parallel transport bivector such that \( s_i = -g_i' s_j' \). \( C_{ij} \) are functions of \( \Delta \eta \) and \( \Delta s \).

In terms of these coefficient functions the traceless condition in equation (22) is given by

\[ C_{11} - C_{31} - 3C_{32} = 0 \quad ; \quad C_{21} + C_{31} + 3C_{32} - 2C_{53} = 0 \]
\[ C_{31} - C_{61} - 3C_{62} + 4C_{63} = 0 \quad ; \quad C_{32} - C_{62} - 2C_{64} - 3C_{65} = 0. \]

(26)

While the conservation condition in equation (23) is given by

\[
\begin{align*}
\frac{\partial C_{11}}{\partial \Delta \eta} + \frac{\partial C_{21}}{\partial \Delta s} + 2\Delta C_{21} &= 0 \quad ; \quad \frac{\partial C_{21}}{\partial \Delta \eta} + \frac{\partial C_{31}}{\partial \Delta s} + \frac{\partial C_{32}}{\partial \Delta s} + 2\Delta C_{31} = 0 \\
\frac{\partial C_{21}}{\partial \Delta \eta} - \frac{\partial C_{41}}{\partial \Delta s} + \frac{\partial C_{42}}{\partial \Delta s} - 2\Delta C_{41} + 2(A + C)C_{42} &= 0 \\
\frac{\partial C_{51}}{\partial \Delta \eta} - \frac{\partial C_{51}}{\partial \Delta s} + \frac{\partial C_{53}}{\partial \Delta s} - 2\Delta C_{51} + 2(2A + 3C)C_{53} &= 0 \\
\frac{\partial C_{52}}{\partial \Delta \eta} - \frac{\partial C_{52}}{\partial \Delta s} - 2\Delta C_{52} - 2CC_{53} &= 0 \\
\frac{\partial C_{41}}{\partial \Delta \eta} + \frac{\partial C_{51}}{\partial \Delta s} + \frac{\partial C_{52}}{\partial \Delta s} - \frac{\partial C_{53}}{\partial \Delta s} + 2\Delta C_{51} + CC_{52} - (A + 2C)C_{53} &= 0 \\
\frac{\partial C_{42}}{\partial \Delta \eta} + \frac{\partial C_{52}}{\partial \Delta s} + CC_{52} + 3AC_{53} &= 0 \\
\frac{\partial C_{51}}{\partial \Delta \eta} - \frac{\partial C_{61}}{\partial \Delta s} - \frac{\partial C_{62}}{\partial \Delta s} - \frac{\partial C_{63}}{\partial \Delta s} + 2\Delta C_{61} - 2CC_{62} + 2(A + 3C)C_{63} &= 0 \\
\frac{\partial C_{52}}{\partial \Delta \eta} - \frac{\partial C_{62}}{\partial \Delta s} - \frac{\partial C_{65}}{\partial \Delta s} - 2\Delta C_{62} - 2CC_{63} + 2(A + C)C_{64} &= 0 \\
\frac{\partial C_{53}}{\partial \Delta \eta} - \frac{\partial C_{63}}{\partial \Delta s} + \frac{\partial C_{64}}{\partial \Delta s} - CC_{62} - 3AC_{63} + 3(A + C)C_{64} &= 0, \\
\end{align*}
\]

(27)

where \( A = 1/\Delta s \) and \( C = -1/\Delta s \) for \( R^3 \), \( A = \cot(\Delta s) \) and \( C = -\csc(\Delta s) \) for \( S^3 \), and \( A = \coth(\Delta s) \) and \( C = -\csch(\Delta s) \) for \( H^3 \) [16].

Now we go back to the consideration of the Minkowski spacetime. The corresponding coefficients are
\[ (C_{11})_M = \frac{3\Delta s^4 + 10\Delta s^2\Delta \eta^2 + 3\Delta \eta^4}{12\pi^4(-\Delta \eta^2 + \Delta s^2)^6}; \quad (C_{21})_M = \frac{2\Delta s\Delta \eta(\Delta s^2 + \Delta \eta^2)}{3\pi^4(-\Delta \eta^2 + \Delta s^2)^6} \]

\[ (C_{31})_M = \frac{4\Delta \eta^2\Delta s^2}{3\pi^4(-\Delta \eta^2 + \Delta s^2)^6}; \quad (C_{32})_M = \frac{1}{12\pi^4(-\Delta \eta^2 + \Delta s^2)^5} \]

\[ (C_{41})_M = -\frac{4\Delta s^2(3\Delta \eta^2 + \Delta s^2)}{3\pi^4(-\Delta \eta^2 + \Delta s^2)^6}; \quad (C_{42})_M = -\frac{\Delta \eta^2 + \Delta s^2}{6\pi^4(-\Delta \eta^2 + \Delta s^2)^5} \]

\[ (C_{51})_M = -\frac{4\Delta \eta \Delta s^3}{3\pi^4(-\Delta \eta^2 + \Delta s^2)^6}; \quad (C_{52})_M = 0; \quad (C_{53})_M = -\frac{\Delta \eta \Delta s}{3\pi^4(-\Delta \eta^2 + \Delta s^2)^5} \]

\[ (C_{61})_M = \frac{4\Delta s^4}{3\pi^4(-\Delta \eta^2 + \Delta s^2)^6}; \quad (C_{62})_M = 0; \quad (C_{63})_M = \frac{\Delta s^2}{3\pi^4(-\Delta \eta^2 + \Delta s^2)^5} \]

\[ (C_{64})_M = \frac{1}{6\pi^4(-\Delta \eta^2 + \Delta s^2)^4}; \quad (C_{65})_M = \frac{1}{12\pi^4(-\Delta \eta^2 + \Delta s^2)^4}. \quad (28) \]

It is easy to check that these coefficient functions do satisfy the traceless and the conservation conditions in equations (26) and (27).

Since the flat FRW spacetime is conformal to the Minkowski one with the conformal factor \( \Omega(x) = \alpha(\eta) \), the corresponding coefficients are just given by

\[ (C_{ij})_{\text{FRW}} = \alpha^{-2}(\eta) (C_{ij})_M \alpha^{-2}(\eta'). \quad (29) \]

The most prominent example here is the de Sitter spacetime in spatially flat coordinates with \( a(\eta) = -1/H\eta \), where \( H \) is the Hubble constant. The corresponding vacuum is the Bunch–Davies vacuum [17] which is conformal to the Minkowski one. Many applications in cosmology including those on inflation [18] invoke this metric.

This noise kernel has been considered previously in [19] in which it is coordinate transformed into de Sitter static coordinates. According to the ‘thermalization’ relation discussed earlier, the resulting noise kernel is the one in thermal vacuum with temperature \( H/2\pi \) with respect to the static vacuum. The behaviors of the noise kernel near the de Sitter horizon is then investigated and is compared with that near the black hole horizon in the Hartle–Hawking thermal vacuum [20].

### 3.2. Closed FRW spacetimes

Next, we look at the closed FRW spacetimes with the metric

\[ ds^2 = a^2(\eta)(-d\eta^2 + d\chi^2 + \sin^2\chi \, d\theta^2 + \sin^2\chi \, \sin^2\theta \, d\phi^2), \quad (30) \]

which is conformal to the Einstein Universe with the conformal factor \( a(\eta) \). Hence we need to first evaluate the noise kernel of the Einstein Universe. This can be done as in the last subsection by plugging the Wightman function in equation (8) into the formula in equation (18). The results are the following
(C_{11})_E = \frac{4 - \cos^2 \Delta \eta - 6 \cos \Delta \eta \cos \Delta s - \cos^2 \Delta s + 4 \cos \Delta \eta \cos^2 \Delta s}{192 \pi^4 (\cos \Delta \eta - \cos \Delta s)^6}

(C_{21})_E = \frac{\sin \Delta \eta \sin \Delta s (1 - \cos \Delta \eta \cos \Delta s)}{48 \pi^4 (\cos \Delta \eta - \cos \Delta s)^6}

(C_{31})_E = \frac{\sin^2 \Delta \eta \sin^2 \Delta s}{48 \pi^4 (\cos \Delta \eta - \cos \Delta s)^6}

(C_{32})_E = \frac{1}{192 \pi^4 (\cos \Delta \eta - \cos \Delta s)^4}

(C_{41})_E = \frac{(1 + \cos \Delta \eta)(1 - \cos \Delta s)(2 - \cos \Delta \eta + \cos \Delta s - 2 \cos \Delta \eta \cos \Delta s)}{96 \pi^4 (\cos \Delta \eta - \cos \Delta s)^6}

(C_{42})_E = \frac{1 - \cos \Delta \eta \cos \Delta s}{96 \pi^4 (\cos \Delta \eta - \cos \Delta s)^3}

(C_{51})_E = \frac{-\sin \Delta \eta \sin \Delta s (1 + \cos \Delta \eta)(1 - \cos \Delta s)}{48 \pi^4 (\cos \Delta \eta - \cos \Delta s)^6}; \quad (C_{52})_E = 0

(C_{53})_E = \frac{-\sin \Delta \eta \sin \Delta s}{96 \pi^4 (\cos \Delta \eta - \cos \Delta s)^3}

(C_{61})_E = \frac{(1 + \cos \Delta \eta)^2 (1 - \cos \Delta s)^2}{48 \pi^4 (\cos \Delta \eta - \cos \Delta s)^6}; \quad (C_{62})_E = 0;

(C_{63})_E = \frac{(1 + \cos \Delta \eta)(1 - \cos \Delta s)}{96 \pi^4 (\cos \Delta \eta - \cos \Delta s)^3}

(C_{64})_E = \frac{1}{96 \pi^4 (\cos \Delta \eta - \cos \Delta s)^2}; \quad (C_{65})_E = \frac{1}{192 \pi^4 (\cos \Delta \eta - \cos \Delta s)^2}.

As a consistent check on these coefficients one can verify readily that they satisfy the traceless and the conservation conditions in equations (26) and (27). Moreover, in the short distance limit, $\Delta \eta \to 0$ and $\Delta s \to 0$, these coefficients reduce to the Minkowski ones in equation (28).

Here the coefficients for the closed FRW spacetimes are given by

\[ (C_{ij})_{\text{cFRW}} = a^{-2}(\eta) (C_{ij})_E a^{-2}(\eta'). \]  

For $a(\eta) = \alpha / \sin \eta$, where $\alpha$ is a constant, we have the de Sitter spacetime in the global coordinates.

4. Noise kernels related to open FRW spacetimes

In this section we consider the noise kernels related to the open FRW spacetimes which are conformal to the open Einstein Universe. Thus we first work out the noise kernel in the open Einstein Universe and then obtain those in the open FRW spacetimes by conformal transformations. We need to make a further coordinate transformation to arrive at the noise kernel in static de Sitter spacetime. On the other hand, to derive the noise kernel of the Rindler space from the open Einstein, it is necessary to make some combination of conformal and coordinate transformations. This is rather complicated to work through. Hence we would choose another route by working directly with the Rindler Wightman function [21, 22] to obtain the noise kernel in the last subsection.
4.1. Open FRW spacetimes and static de Sitter space

Consider the open FRW spacetimes with the metric

\[ ds^2 = a^2(\eta)(-d\eta^2 + d\chi^2 + \sinh^2\chi d\theta^2 + \sinh^2\chi \sin^2\theta d\phi^2), \]  

which is conformal to the open Einstein Universe. Starting with the Wightman function of the open Einstein Universe [14], one can obtain the corresponding noise kernel again using the formula in equation (18)

\begin{align*}
(C_{11})_O &= \frac{G^2}{9\Delta s^2} \left[ 1 + \left(3 + 2\Delta s^2\right) \text{csch}^2(\Delta s) - 6\Delta s \coth(\Delta s) \text{csch}^2(\Delta s) + 3\Delta s^2 \text{csch}^4(\Delta s) \right] \\
&\quad - \frac{8\pi^2 G^3 \sinh(\Delta s)}{9\Delta s^2} \left[ 5\Delta s - 6 \coth(\Delta s) + 12\Delta s \text{csch}^2(\Delta s) \right] \\
&\quad - \frac{6\Delta s^2 \coth(\Delta s) \text{csch}^2(\Delta s)}{3\Delta s^2} \\
&\quad + \frac{64\pi^4 G^4 \sinh^2(\Delta s)}{3\Delta s^2} \left[ \left(3 + \Delta s^2\right) - 2\Delta s \coth(\Delta s) + 2\Delta s^2 \text{csch}^2(\Delta s) \right] \\
&\quad - \frac{4096\pi^6 G^5 \sinh^3(\Delta s)}{3\Delta s} + \frac{16384\pi^8 G^6 \sinh^4(\Delta s)}{3}
\end{align*}

\begin{align*}
(C_{21})_O &= -\frac{16\pi^2 \Delta \eta G^3 \sinh(\Delta s)}{9\Delta s^3} \left[ \Delta s - \coth(\Delta s) + 2\Delta s \text{csch}^2(\Delta s) \right] \\
&\quad - \frac{\Delta s^2 \coth(\Delta s) \text{csch}^2(\Delta s)}{3\Delta s^2} \\
&\quad + \frac{64\pi^4 \Delta \eta G^4 \sinh^2(\Delta s)}{3\Delta s^2} \left[ \Delta s - 2 \coth(\Delta s) + 2\Delta s \text{csch}^2(\Delta s) \right] \\
&\quad - \frac{2048\pi^6 \Delta \eta G^5 \sinh^3(\Delta s)}{3\Delta s^2} + \frac{16384\pi^8 \Delta \eta G^6 \sinh^4(\Delta s)}{3\Delta s}
\end{align*}

\begin{align*}
(C_{31})_O &= \frac{G^2}{9\Delta s^2} \left[ 1 - \left(3 + 4\Delta s^2\right) \text{csch}^2(\Delta s) + 6\Delta s \coth(\Delta s) \text{csch}^2(\Delta s) - 3\Delta s^2 \text{csch}^4(\Delta s) \right] \\
&\quad + \frac{8\pi^2 G^3 \sinh(\Delta s)}{9\Delta s^3} \left[ \left(3 - 5\Delta s^2\right) + 9\Delta s \coth(\Delta s) - 6\Delta s^2 \text{csch}^2(\Delta s) \right] \\
&\quad - \frac{6\Delta s^3 \coth(\Delta s) \text{csch}^2(\Delta s)}{3\Delta s^2} \\
&\quad + \frac{64\pi^4 G^4 \sinh^2(\Delta s)}{3\Delta s} \left[ \Delta s - 2 \coth(\Delta s) + 2\Delta s \text{csch}^2(\Delta s) \right] \\
&\quad - \frac{4096\pi^6 G^5 \sinh^3(\Delta s)}{3\Delta s} + \frac{16384\pi^8 G^6 \sinh^4(\Delta s)}{3}
\end{align*}
\[
(C_{32})_0 = \frac{2G^2}{9\Delta s^2} \left[ (1 + \Delta s^2) \, \text{csch}^2(\Delta s) - 2\Delta s \, \text{coth} (\Delta s) \, \text{csch}^2(\Delta s) + \Delta s^2 \, \text{csch}^4(\Delta s) \right]
- \frac{8\pi^2G^3 \sinh (\Delta s)}{9\Delta s^3} \left[ 1 + \Delta s \, \text{coth} (\Delta s) + 2\Delta s^2 \, \text{csch}^2(\Delta s) \right]
- \frac{4\Delta s^3 \, \text{coth} (\Delta s) \, \text{csch}^2(\Delta s)}{3\Delta s^2} + \frac{64\pi^4G^4 \sinh^2(\Delta s)}{3\Delta s^2}
\]

\[
(C_{41})_0 = -\frac{8\pi^2G^3 \sinh (\Delta s)}{9\Delta s^3} \left[ (2 - 5\Delta s^2) + 10\Delta s \, \text{coth} (\Delta s) + 7\Delta s \, \text{csch}(\Delta s) \right]
- \frac{7\Delta s^2 \, \text{coth} (\Delta s) \text{csch}(\Delta s) - 12\Delta s^2 \, \text{csch}^2(\Delta s)}{3\Delta s^2}
- \frac{64\pi^4G^4 \sinh^2(\Delta s)}{3\Delta s^2} \left[ (1 + \Delta s^2) - 2\Delta s \, \text{coth} (\Delta s) - 4\Delta s \, \text{csch}(\Delta s) \right]
+ \frac{2\Delta s^2 \, \text{coth} (\Delta s) \text{csch}(\Delta s) + 3\Delta s^2 \, \text{csch}^2(\Delta s)}{3\Delta s^2}
+ \frac{1024\pi^8G^6 \sinh^4(\Delta s)}{3}\left[ 4 - \Delta s \, \text{csch}(\Delta s) \right] - \frac{16384\pi^8G^6 \sinh^4(\Delta s)}{3}
\]

\[
(C_{42})_0 = -\frac{56\pi^2G^3}{9\Delta s^2} \left[ 1 - \Delta s \, \text{coth} (\Delta s) \right] + \frac{128\pi^4G^4 \sinh (\Delta s)}{3\Delta s} \left[ 2 - \Delta s \, \text{coth} (\Delta s) \right]
- \frac{1024\pi^8G^6 \sinh^2(\Delta s)}{3}
\]

\[
(C_{51})_0 = \frac{8\pi^2\eta G^3 \sinh (\Delta s)}{9\Delta s^3} \left[ 2\Delta s - 5 \, \text{coth} (\Delta s) - 4 \, \text{csch}(\Delta s) + 2 \Delta s \, \text{coth} (\Delta s) \text{csch}(\Delta s) \right]
+ \frac{\Delta s \, \text{csch}^2(\Delta s) + 4 \Delta s^2 \, \text{coth} (\Delta s) \text{csch}^2(\Delta s) + 2 \Delta s^2 \, \text{csch}^4(\Delta s)}{3\Delta s^2}
+ \frac{64\pi^4\eta G^4 \sinh^2(\Delta s)}{3\Delta s^2} \left[ (3 - \Delta s^2) + 2\Delta s \, \text{coth} (\Delta s) + 4\Delta s \, \text{csch}(\Delta s) \right]
- \frac{4\Delta s^2 \, \text{coth} (\Delta s) \text{csch}(\Delta s) - 5\Delta s^2 \, \text{csch}^2(\Delta s)}{3\Delta s^2}
+ \frac{2048\pi^8\eta G^6 \sinh^4(\Delta s)}{3\Delta s^2} \left[ 1 - \Delta s \, \text{csch}(\Delta s) \right] - \frac{16384\pi^8\eta G^6 \sinh^4(\Delta s)}{3\Delta s}
\]

\[
(C_{52})_0 = \frac{8\pi^2\eta G^3 \sinh (\Delta s)}{9\Delta s^3} \left[ \text{coth} (\Delta s) + \Delta s \, \text{csch}^2(\Delta s) - 2\Delta s^2 \, \text{coth} (\Delta s) \text{csch}^2(\Delta s) \right]
- \frac{64\pi^4\eta G^4 \sinh^2(\Delta s)}{3\Delta s^3} \left[ 1 - \Delta s^2 \, \text{csch}^2(\Delta s) \right]
\]

\[
(C_{53})_0 = \frac{8\pi^2\eta G^3}{9\Delta s^3} \left[ 2 - \Delta s \, \text{coth} (\Delta s) - \Delta s^2 \, \text{csch}^2(\Delta s) \right]
+ \frac{128\pi^4\eta G^4 \sinh (\Delta s)}{3\Delta s^2} \left[ 1 - \Delta s \, \text{coth} (\Delta s) \right] - \frac{1024\pi^8\eta G^6 \sinh^2(\Delta s)}{3\Delta s}
\]
\[
(C_{61})_O = \frac{G^2}{9\Delta s^2} \left[ 1 - 16 \coth(\Delta s) \operatorname{csch}(\Delta s) - \left( 23 - 14\Delta s^2 \right) \operatorname{csch}^2(\Delta s) \\
- 11\Delta s \coth(\Delta s) \operatorname{csch}^2(\Delta s) - 16\Delta s \operatorname{csch}^3(\Delta s) \\
+ 32\Delta s^2 \coth(\Delta s) \operatorname{csch}(\Delta s) + 34\Delta s^2 \operatorname{csch}^2(\Delta s) \right] \\
+ \frac{8\pi^2 G^3 \sinh(\Delta s)}{9\Delta s^3} \left[ \left( 24 - 5\Delta s^2 \right) + 12\Delta s \coth(\Delta s) + 12\Delta s \operatorname{csch}(\Delta s) \\
- 28\Delta s^2 \coth(\Delta s) \operatorname{csch}(\Delta s) - 47\Delta s^2 \operatorname{csch}^2(\Delta s) \\
+ 11\Delta s^3 \coth(\Delta s) \operatorname{csch}^2(\Delta s) + 16\Delta s^3 \operatorname{csch}^3(\Delta s) \right] \\
- \frac{64\pi^4 G^4 \sinh^2(\Delta s)}{3\Delta s^2} \left[ \left( 3 - \Delta s^2 \right) + 2\Delta s \coth(\Delta s) + 16\Delta s \operatorname{csch}(\Delta s) \\
- 8\Delta s^2 \coth(\Delta s) \operatorname{csch}(\Delta s) - 13\Delta s^2 \operatorname{csch}^2(\Delta s) \right] \\
- \frac{4096\pi^6 G^5 \sinh^2(\Delta s)}{3\Delta s^3} \left[ 1 - 8 \Delta s \operatorname{csch}(\Delta s) \right] + \frac{16384\pi^8 G^6 \sinh^4(\Delta s)}{3}
\]
\[
(C_{62})_O = \frac{G^2 \operatorname{csch}^2(\Delta s)}{9\Delta s^2} \left[ 2 \left( 6 + \Delta s^2 \right) - 9\Delta s \coth(\Delta s) - 3\Delta s^2 \operatorname{csch}^2(\Delta s) \right] \\
- \frac{8\pi^2 G^3 \sinh(\Delta s)}{9\Delta s^3} \left[ 7 + 4\Delta s \coth(\Delta s) \\
+ \Delta s^2 \operatorname{csch}^2(\Delta s) - 9\Delta s^3 \coth(\Delta s) \operatorname{csch}^2(\Delta s) \right] \\
+ \frac{64\pi^4 G^4 \sinh^2(\Delta s)}{3\Delta s^2} \left[ 1 - 2 \Delta s^2 \operatorname{csch}^2(\Delta s) \right]
\]
\[
(C_{63})_O = -\frac{G^2 \operatorname{csch}(\Delta s)}{9\Delta s^2} \left[ 4 \coth(\Delta s) - 2 \left( 2 + 3\Delta s^2 \right) \operatorname{csch}(\Delta s) \\
+ 11\Delta s \coth(\Delta s) \operatorname{csch}(\Delta s) + 4\Delta s \operatorname{csch}^2(\Delta s) \\
- 8\Delta s^2 \coth(\Delta s) \operatorname{csch}(\Delta s) - 7\Delta s^2 \operatorname{csch}^2(\Delta s) \right] \\
+ \frac{8\pi^2 G^3}{9\Delta s^3} \left[ 3 - 7\Delta s \coth(\Delta s) - 11\Delta s \operatorname{csch}(\Delta s) \\
+ 11\Delta s^2 \coth(\Delta s) \operatorname{csch}(\Delta s) + 4\Delta s^2 \operatorname{csch}^2(\Delta s) \right] \\
- \frac{128\pi^4 G^4 \sinh(\Delta s)}{3\Delta s} \left[ 2 - \Delta s \coth(\Delta s) - \Delta s \operatorname{csch}(\Delta s) \right] \\
+ \frac{1024\pi^6 G^5 \sinh^2(\Delta s)}{3}
\]
\[
(C_{64})_O = \frac{G^2 \operatorname{csch}^2(\Delta s)}{9\Delta s^3} \left[ 2 \left( 2 + 3\Delta s^2 \right) - 11\Delta s \coth(\Delta s) + 7\Delta s^2 \operatorname{csch}^2(\Delta s) \right] \\
- \frac{88\pi^2 G^3 \operatorname{csch}(\Delta s)}{9\Delta s} \left[ 1 - 4 \Delta s \coth(\Delta s) \right] + \frac{128\pi^4 G^4}{3}
\]
\[
\left( C_{65} \right)_{O} = -\frac{G^2 \text{csch}^2(\Delta s)}{9\Delta s^2} \left[ 2 \left( 3 + 2\Delta s^2 \right) - 9\Delta s \coth(\Delta s) + 3\Delta s^2 \text{csch}^2(\Delta s) \right] \\
+ \frac{8\pi^2 G^2 \sinh(\Delta s)}{9\Delta s^3} \left[ 2 + 7\Delta s^2 \text{csch}^2(\Delta s) - 9\Delta s \coth(\Delta s) \text{csch}^2(\Delta s) \right] \\
- \frac{64\pi^2 G^4}{3},
\]

(34)

where \( G \) is the Wightman function in equation (12).

Again these coefficients satisfy the traceless and the conservation conditions in equations (26) and (27). As in the Einstein Universe case, they will reduce to the Minkowski ones in the short distance limit, \( \Delta \eta \to 0 \) and \( \Delta s \to 0 \). On the other hand, when \( \Delta s \to \infty \), these coefficients all decay exponentially but with different rates

\[
\left( C_{11} \right)_{O}, \quad \left( C_{21} \right)_{O}, \quad \left( C_{31} \right)_{O}, \quad \left( C_{32} \right)_{O}, \quad \left( C_{41} \right)_{O}, \quad \left( C_{51} \right)_{O}, \\
\left( C_{52} \right)_{O}, \quad \left( C_{61} \right)_{O}, \quad \left( C_{62} \right)_{O} \sim \left( e^{-\Delta s} \right)^2 \\
\left( C_{42} \right)_{O}, \quad \left( C_{53} \right)_{O}, \quad \left( C_{63} \right)_{O} \sim \left( e^{-\Delta s} \right)^3 \\
\left( C_{64} \right)_{O}, \quad \left( C_{65} \right)_{O} \sim \left( e^{-\Delta s} \right)^4.
\]

(35)

Therefore, the noise kernel components will all decay like \( \left( e^{-\Delta s} \right)^2 \) as \( \Delta s \to \infty \).

Lastly, as in the flat and the closed FRW cases, the coefficients for the open FRW spacetime is given by the conformal transformation

\[
\left( C_{ij} \right)_{O,FRW} = a^{-2}(\eta) \left( C_{ij} \right)_{O} a^{-2}(\eta').
\]

(36)

For \( a(\eta) = \alpha e^\eta \), where \( \alpha \) is a constant, we have the Milne Universe [23].

Next, we look at the noise kernel in static de Sitter spacetime. Going from the open Einstein metric, it is necessary to make one conformal and another coordinate transformations to arrive at this metric. From the metric in equation (10), we perform a conformal transformation with the conformal factor \( \Omega = 1/\cosh \chi \), that is

\[
ds^2 = \left( \frac{1}{\cosh^2 \chi} \right) \left( -d\eta^2 + d\chi^2 + \sinh^2 \chi \, d\theta^2 + \sin^2 \chi \, \sin^2 \theta \, d\phi^2 \right).
\]

(37)

Then making a coordinate transformation, \( r = \tanh \chi \), we arrive at the static de Sitter metric

\[
ds^2 = -\left( 1 - r^2 \right) d\eta^2 + \left( 1 - r^2 \right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right)
\]

(38)

with the horizon at \( r = 1 \).

Hence, to obtain the noise kernel in static de Sitter we need to make the corresponding conformal and coordinate transformations on the open Einstein space noise kernel in equations (25) and (34). The coefficient functions \( \left( C_{ij} \right)_{sds} \) of the noise kernel in static de Sitter spacetime are given by

\[
\left( C_{ij} \right)_{sds} = \left( 1 - r^2 \right)^{-1} \left( C_{ij} \right)_{O} \left( 1 - r^2 \right)^{-1},
\]

(39)
where the geodesic distance is
\[
\Delta s = \cosh^{-1} \left\{ \left(1 - r^2\right)^{-1/2} \left(1 - r'^2\right)^{-1/2} \left[1 - rr' \left(\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')\right)\right] \right\}. \tag{40}
\]
Note that this is the de Sitter noise kernel in the static vacuum which is conformal to the open Einstein or the Rindler vacuum. By the virtue of the ‘thermalization’ relation we discussed earlier, the noise kernels in spatially flat de Sitter [19] or the global de Sitter cases can be understood as taken in respect to the thermal static vacuum.

4.2. Rindler space

In this subsection we would like to consider the noise kernel in the Rindler spacetime. The Rindler space is flat and its vacuum is conformal to the one in open Einstein Universe. Therefore, in principle using a combination of conformal and coordinate transformations, as we have done above, it is possible to obtain the noise kernel in Rindler spacetime from that in open Einstein Universe. However, the procedure is quite complicated. To see this we look at the metric. With the coordinate transformation
\[
t \pm r = \tanh \left( \frac{n \pm \chi}{2} \right), \tag{41}
\]
the open Einstein Universe metric becomes
\[
d s^2 = -dt^2 + d\chi^2 + \sinh^2 \chi \left[d\theta^2 + \sin^2 \theta \left(\sin \phi \left(\sin \phi' + \sin \phi' \sin \phi \right) + \cos \phi \left(\cos \phi' + \cos \phi' \cos \phi \right)\right]\right], \tag{42}
\]
Thus with a further conformal transformation one can arrive at the Minkowski metric. Going to Cartesian coordinates and then making another coordinate transformation
\[
t = \xi \sinh \tau ; \quad x = \xi \cosh \tau \tag{43}
\]
we can get the Rindler metric
\[
d s^2 = -\xi^2 d\tau^2 + d\xi^2 + dy^2 + dz^2. \tag{44}
\]
It is nevertheless quite a complicated procedure to implement this series of transformations on the open Einstein noise kernel to obtain the Rindler one. To avoid this we have chosen to work on the Wightman function in Rindler spacetime directly. This Wightman function has been given in [21, 22]
\[
G^+ = \frac{1}{4\pi^2} \left( \frac{\alpha}{\xi \xi'} \sinh \alpha \right) \left( \frac{1}{-(\tau - \tau^{'})^2 + \alpha^2} \right), \tag{45}
\]
where
\[
cosh \alpha = \frac{\xi^2 + \xi'^2 + (y - y')^2 + (z - z')^2}{2\xi \xi'}. \tag{46}
\]
Plugging this into equation (18) one can obtain the noise kernel components in the Rindler spacetime. However, the expressions for these components are much lengthier than the ones in FRW cases so we just display one of them.
The other components of the Rindler noise kernel can be derived analogously from equation (18). This completes our consideration on the noise kernels of a conformal scalar field in conformally-flat spacetimes.

5. Conclusions and discussion

In this paper we have considered the noise kernels of a conformal scalar field in conformally-flat spacetimes. First, conformally-flat spacetimes can be classified into two main categories according to the conformal vacuum they admit, namely, the Minkowski or the Rindler vacuum. We explicated this point with the help of the corresponding Wightman functions, from which we derive the noise kernels in three static spacetimes, namely, the Minkowski, Einstein, and open Einstein spacetimes. Basically the conformal transformations between noise kernels in conformally-related spacetimes are quite simple. Therefore, we are able to obtain the noise kernels of a conformal scalar field in flat, closed, and open FRW spacetimes with arbitrary time-dependent scale factors in closed analytic form from the ones in Minkowski, Einstein and open Einstein spacetimes, respectively.

Below we comment brieﬂy on how analytic expressions for the noise kernel are useful for the study of backreaction and fluctuation problems in the early Universe and black hole dynamics. Two structural levels are involved: first, at the level of SCG, solutions of the SCE provide a description of the self-consistent evolution of both the spacetime and the quantum matter field. The second level is that of the stochastic gravity. Here the inquiry can be carried out also at two sublevels. (A) The noise kernel itself describes the fluctuations of the quantum matter field. Finding out how these fluctuations behave near the black hole’s event horizon...
or near the cosmological singularity can already provide interesting physical information. In [19] the behaviors of the noise kernel near the de Sitter horizon was studied and then compared with that in the Schwarzschild spacetime [12]. Similar study can be carried out for the noise kernels found here. We also note related recent work of noise kernel for inhomogeneous spacetimes [25] and Riemann curvature correlators [26].

The second level of studies in stochastic gravity is (B) the backreaction of quantum matter fields including their fluctuations manifested as stochastic forces whose correlations are given by the noise kernel. Their backreaction is obtained by solving the ELE. For example, the correlation of the quantized gravitational perturbations can be found in the graviton noise kernel which is used in the stochastic gravity approach to structure formation [27].

We end with a comment of the dissipation kernel in the ELE and the fluctuation-dissipation relation (FDR) in SCG. Note that the ELE also contains dissipative terms in balance of the stochastic force. We believe that a FDR should exist in the ELE (and even in a broader setting for all quantum open systems obtained from coarse-graining a closed quantum system). An example was given in a paper by Hu and Sinha [28], which demonstrates the existence of this FDR relation based on earlier backreaction calculations (from [29] to [30]) and including the noise term derived from the fluctuations of the quantum field as source in the ELE. Once the explicit form of the noise kernel is computed one can in principle use this relation to obtain the dissipation kernel.

Without assuming the existence of a FDR, the dissipation kernel is derived in three steps, (1) for a specified class of classical background spacetime and quantum field, solving the SCE, (2) deriving the explicit form of the noise kernel, then (3) using that solution to the SCE as the background spacetime and the noise kernel for the quantum field one can derive the ELE and then explicitly identify the dissipation kernel. This belongs to work we plan to undertake in the next stage of our program.

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