Robust optimal investment and risk control for an insurer with general insider information

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Abstract

In this paper, we study the robust optimal investment and risk control problem for an insurer who owns the insider information about the financial market and the insurance market under model uncertainty. Both financial risky asset process and insurance risk process are assumed to be very general jump diffusion processes. The insider information is of the most general form rather than the initial enlargement type. We use the theory of forward integrals to give the first half characterization of the robust optimal strategy and transform the anticipating stochastic differential game problem into the nonanticipative stochastic differential game problem. Then we adopt the stochastic maximum principle to obtain the total characterization of the robust strategy. We discuss the two typical situations when the insurer is ‘small’ and ‘large’ by Malliavin calculus. For the ‘small’ insurer, we obtain the closed-form solution in the continuous case and the half closed-form solution in the case with jumps. For the ‘large’ insurer, we reduce the problem to the quadratic backward stochastic differential equation (BSDE) and obtain the closed-form solution in the continuous case without model uncertainty. We discuss some impacts of the model uncertainty, insider information and the ‘large’ insurer on the optimal strategy.

1 Introduction

The optimal investment and risk control problem for insurers is a classical topic in actuarial research. An effective method for insurers to manage risk is reinsurance. Many works have been carried out concerning this topic including maximizing the expected utility from terminal wealth (see [1, 2]), the mean-variance criterion (see [3]), and minimizing the probability of ruin (see [4]).

[5] considered the risk control problem in another way. In their model, the insurer manage risk by selecting the number of insurance policies instead of taking reinsurance business. They used the the jump-diffusion process to model the risk process of the insurer, which is negatively correlated with the risky asset process in the financial market. The optimal strategy is based on the criterion of expected utility maximization. [6] extended the the risk control model in [5] by introducing an extra jump-diffusion process, which is negatively correlated with the capital returns. [7] also considered the above risk control problem based on the mean-variance criterion.

Recently, there is a growing emphasis on the robust optimal strategy for the investment and risk control problem. As pointed out by [8], the risk-based models that constitute the paradigm have well documented empirical failures. A robust optimal strategy is the optimal strategy under the worst-case probability assuming that the ambiguity aversion is considered. In other words, when the insurer is ambiguity averse, she might not believe the model is accurate by empirical statistics, which forces her to choose the different (robust) optimal strategy compared to the ambiguity-neutral insurer. Usually, this means giving up a certain degree of utility in case of following a seriously misleading strategy. [9] studied the optimal investment problem for an ambiguity-averse insurer in a jump-diffusion risk process. [10] studied the robust optimal investment-reinsurance problem under Heston’s stochastic volatility (SV) model, and [11] extended it to the multi-dimensional SV model. Under the mean-variance criterion, [12] studied the optimal proportional reinsurance-investment strategy; [3] studied the equilibrium strategy of a robust optimal reinsurance-investment strategy for an insurer. [13] considered the reinsurance and investment game problem between two insurance companies under model uncertainty.

Besides model uncertainty, the insider information is another important topic of the investment and risk control problem, which has been taken into account by many scholars in recent years. Most of the existing works in the related literature including the above articles suppose the insurer develops strategies based on the public information flow, which is generated by the market noises. However, in the real world, many insurers or insurance companies owns the extra private information flow about the financial risky asset and the insurance polices, which is also called
the insider information. They could gain certain extra profit by reconsidering their strategies based on their insider information. The optimal investment strategy problem was first studied by [14] using the technique of enlargement of filtration. The insider information is modeled by a random variable, i.e., the initial enlargement type, in [14]. Although many researchers have studied the insider trading problems in finance (see [15, 16, 17, 18, 19, 20, 21, 22]), there are few articles concerning the optimal investment and risk control problem for an insurer with insider information. [23] studied the optimal investment-reinsurance problem under insider information. Both financial asset process and the insurance risk process in [23] are modeled by the Brownian motion. Then main method is based on the technique of enlargement of filtration proposed in [14]. [24] generalized the model in [23] by introducing jumps in the risk process. [25] studied the investment-reinsurance game between two insurance companies under some extra insider information. [26] used another method based on the theory of forward integrals proposed in [15] to study the optimal investment and risk control problem under insider information, which extended the insider information to the most general form (not necessarily the initial enlargement type). They also introduced jumps in both financial asset process and insurance risk process.

When combing model uncertainty with insider information, the optimal investment and risk control problem turns to an anticipating stochastic differential game problem, which is very difficult to solve directly. The reason is that one can not apply the forward integral method ([15]) directly since there is no useful result with variation to the other controlling process. [27] developed a general stochastic maximum principle for the anticipating stochastic differential game problem by using Malliavin calculus. However, the result could only be applied to controlled Itô-Lévy processes due to the complexity of the Malliavin derivative.

To the best our knowledge, only [28] combined the model uncertainty with insider information in the optimal investment-reinsurance problem. Inspired by [20], they used the Donsker δ functional technique to transform the problem into a nonanticipative (i.e., adapted) stochastic differential game problem, and then use the stochastic maximum principle to solve the problem. However, they only considered the special case when the insider information is of the initial enlargement type. Moreover, no closed-form solution or general-form solution is obtained when both model uncertainty and insider information are considered. The reason is that the subjective probability measure Q′ constructed in their paper is not exactly a probability measure when v(y) = v(Y), which leads to the result that the final equation is a nested linear BSDE.

In this paper, we study the optimal investment and risk control problem under both model uncertainty and insider information (based on the criterion of expected utility maximization). Unlike [28], the insider information in our model is of the most general form instead of the initial enlargement type. Thus the Donsker δ functional technique in [28] is invalid. Inspired by [15, 17], we use the theory of forward integrals to get the half equations about the robust optimal strategy, and then transform the problem into a nonanticipative stochastic differential game problem to obtain the other half equations by the stochastic maximum principle. We discuss the two typical cases where the insider is ‘small’ (i.e., her strategy has no influence on the market) and ‘large’ (i.e., her strategy has certain influence on the market), and obtain the corresponding solutions. The main contributions of this paper are as follows:

- A new optimal investment and risk control problem is established dealing with the model uncertainty, the general insider information and jumps. The definition of the set of prior probability measures is first introduced by the semimartingale decomposition theorem in the Itô theory;

- The insider information is of the most general form, which extends the models for insider information considered in [28]. Thus a new approach by combining the theory of forward integrals with stochastic maximum principles is adopted to solve the anticipating stochastic differential game problem, which is different from the Donsker δ technique used in [28]. In the theory of forward integrals, to obtain the similar semimartingale property of noise processes under the insider information filtration in [17], we first introduce the predictable covariation process and the predictable version of the Girsanov theorem, which is different from the original method in [17]. With the help of the above method, we derive the equation of the robust optimal strategy (see equations (25), (26), (36) and (37));

- For the ‘small’ insurer, we reduce the problem to the non-nested linear BSDE instead of the nested linear BSDE in [28] by introducing the fixed point function. Thus we obtain the formula for the robust optimal strategy. When the utility function is of the logarithmic form and the insurer owns the future information about the financial market or the insurance market, we obtain the closed-form solution in the continuous case and half closed-form solution in the case with jumps by using the Donsker δ functional technique. This extends the results in [28], where the closed-form solution was obtained only when there is no model uncertainty;
• The case of ‘large’ insurer is also considered in this paper, which can also be viewed as an extension of [28]. When the utility function is of the logarithmic form and the insurer owns the future information about the financial market or the insurance market, we reduce the problem to the quadratic BSDE in the continuous case by using the Donsker δ functional technique. We also obtain the closed-form solution when there is no model uncertainty, which is also an extension of [26];

• For both cases of the ‘small’ insurer and the ‘large’ insurer, we obtain the analytic expression for the value function in the continuous case. By comparing value functions in different situations, we find that both insider information and ‘large’ insurer have a positive impact on the utility and ambiguity-aversion has a negative impact on the utility. Moreover, we obtain the critical future time for a ‘small’ insurer, which implies what kind of insider information is needed for an ambiguity-averse insurer to gain the same utility of an ambiguity-neutral insurer with no insider information.

This paper is organized as follows. In Section 2, we introduce the basic theory of forward integrals. In Section 3, we formulate the optimal investment and risk control problem for an insurer under both model uncertainty and insider information. We give the first characterization and the total characterization of the robust optimal strategy in Section 4 and Section 5, respectively. We give further characterizations (BSDEs) of the robust optimal strategy when considering the situation of small insurer and large insurer in Section 6 and Section 7, respectively, and derive closed-form solutions or half closed-form solutions in some particular cases. Finally, we summarize our conclusions in Section 8.

2 The forward integral

The forward integral was introduced by [29] and defined by [30]. This type of integral has been studied before and applied to insider trading in financial mathematics (see [31,15,17]). In this section, we follow [18] and give a short review of the forward integral with respect to the Brownian motion and the Poisson random measure.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a filtered probability space. \(W^1\) and \(W^2\) are two independent Brownian motions. \(\tilde{N}^i(\cdot, dz) = N^i(\cdot, dz) - G^i(\cdot)dz\) and \(\tilde{N}^2(\cdot, dz) = N^2(\cdot, dz) - G^2(\cdot)dz\) are two independent compensated Poisson random measures with Lévy measures \(G^1(dz)\) and \(G^2(dz)\), respectively. Then \(\eta_i^z := \int_0^t \tilde{N}^i(ds, dz)\) is a pure jump Lévy process, \(i = 1, 2\). We assume that the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) is the \(\mathbb{P}\)-augmentation of the filtration generated by random variables \(W^1_s, W^2_s, \tilde{N}^1(z, s), \tilde{N}^2(z, s), z \in [0 \infty)\), \(0 \leq s \leq t\), which satisfies the usual condition. We refer to [32,33,34] for the basic relative definitions.

Fix a constant \(T > 0\) as the time horizon. We denote by \(\mathcal{C}_{ucp}\) and \(\mathcal{P}_{ucp}\) the Fréchet spaces (i.e., a complete metrizable locally convex topological vector space, see [35]) of continuous processes and càdlàg processes (not necessarily adapted to the filtration \(\{\mathcal{F}_t\}_{t \geq 0 \leq T}\) equipped with the metrizable topology of the uniform convergence in probability (ucp) on \([0, T]\) (see [36,31,37]), respectively.

**Definition 2.1.** Let \(\varphi\) be a measurable process such that \(\int_0^T |\varphi_t| dt < \infty\). The forward integral of \(\varphi\) with respect to \(W^i\) is defined by

\[
\int_0^t \varphi_s dW^i_s := \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \int_0^t \varphi_s \left( W^i_{s+\varepsilon} - W^i_s \right) ds,
\]

if the limit exists in \(\mathcal{C}_{ucp}, i = 1, 2\). In this case \(\varphi\) is called forward integrable with respect to \(W^i, i = 1, 2\). If the limit exists in probability for every \(t \in [0, T]\), we say \(\varphi\) is forward integrable in the weak sense.

The following multiplication formula for the forward integral is an immediate consequence of Definition 2.1.

**Proposition 2.2.** Suppose \(\varphi\) is forward integrable with respect to \(W^i, i = 1, 2\), and \(F\) is a random variable. Then \(F \varphi\) is forward integrable and

\[
\int_0^T F \varphi_s dW^i_s = F \int_0^T \varphi_s dW^i_s, \quad i = 1, 2.
\]

The next result shows that the forward integral is also an extension of the Itô integral with respect to a semimartingale.
Proposition 2.3. Let \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \) be another filtration which satisfies the usual condition. Suppose that \( W^i \) is an \( \mathcal{F}_t \)-martingale and \( \varphi \) is an \( \mathcal{F}_t \)-progressively measurable process which is Itô integrable with respect to \( W^i \), \( i = 1, 2 \). Then \( \varphi \) is forward integrable and

\[
\int_0^T \varphi_t dW^i_t = \int_0^T \varphi_t dW^i_t, \quad i = 1, 2.
\]

Proof. Since \( \varepsilon^{-1} \int_0^T \varphi_t (W^i_{t+\varepsilon} - W^i_t) d\varepsilon = \int_0^T \varepsilon^{-1} \int_{(t-\varepsilon),0}^0 u_t dW^i_t \) by Fubini theorem, the conclusion follows from the convergence property for the Itô integral with respect to continuous local martingales (see [32] Proposition 3.2.26).

Let us now give the corresponding definition of the forward integral with respect to the compensated Poisson random measure, and some related properties, which are slightly different from the original versions in [18].

Definition 2.4. Let \( \zeta = \{ \zeta(t,z) \}_{(t,z) \in [0,T] \times \mathbb{R}_0} \) be a measurable random field such that

\[
\int_0^T \int_{\mathbb{R}_0} |\zeta(t,z)| 1_{U_n} G^i(dz) dt < \infty
\]
for all \( n \in \mathbb{N}_+ \), \( i = 1 \). The forward integral of \( \zeta \) with respect to \( \tilde{N}^i \), \( i = 1 \), \( 2 \). If the limit exists in probability for every \( t \in [0,T] \), we say \( \tilde{N}^i \) is forward integrable in the weak sense. Here, \( \{ U_n \}_{n=1}^\infty \) is an increasing sequence of compact sets \( U_n \subset \mathbb{R}_0 \) with \( G(U_n) < \infty \) such that \( \lim_{n \to \infty} U_n = \mathbb{R}_0 \), \( i = 1 \), \( 2 \).

Similar to the case of the Brownian motion, we also have the following properties about the forward integral with respect to the compensated Poisson random measure.

Proposition 2.5. Suppose \( \zeta \) is forward integrable with respect to \( \tilde{N}^i \), \( i = 1 \), \( 2 \), and \( F \) is a random variable. Then \( F \zeta \)

is forward integrable and

\[
\int_0^T \int_{\mathbb{R}_0} F \zeta(t,z) \tilde{N}^i(d\tau, dz) = F \int_0^T \int_{\mathbb{R}_0} \zeta(t,z) \tilde{N}^i(d\tau, dz), \quad i = 1, 2.
\]

Proposition 2.6. Let \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \) be another filtration which satisfies the usual condition. Suppose that the \( \mathcal{F}_t \)-compensator \( \tilde{N}^i \) \( (dr, dz) \) of \( N^i \) \( (dr, dz) \) is absolutely continuous in time, i.e., \( \tilde{N}^i \) \( (dr, dz) \) \( = G^i(t, dz) dr \) and \( \zeta \) is an \( \mathcal{F}_t \)-predictable random field which is \( \varphi \)-integrable with respect to \( \tilde{N}^i \), \( i = 1 \), \( 2 \). Then \( \tilde{N}^i \) is forward integrable and

\[
\int_0^T \int_{\mathbb{R}_0} \zeta(t,z) \tilde{N}^i(d\tau, dz) = \int_0^T \int_{\mathbb{R}_0} \zeta(t,z) \tilde{N}^i(d\tau, dz), \quad i = 1, 2.
\]

Proof. We observe that \( \zeta(t, z) \tilde{N}^i(d\tau, dz) \) is an immediate consequence of the convergence property for the Itô integral with respect to random measures (see [38] Proposition 3.39). Thus, we only consider the case when \( \tilde{N}^i = \tilde{N}^i \), \( i = 1 \), \( 2 \). By Theorem 11.22 in [39], \( \tilde{N}^i \) \( \tilde{N}^i \)-integrable (hence \( \tilde{N}^i \)-integrable), \( i = 1 \), \( 2 \), which ensures the validity of the two terms in the right hand of \( \tilde{N}^i \). The conclusion follows from Proposition 3.39 in [38].

The Itô formula for the forward integral with respect to the Brownian motion and the compensated Poisson random measure was first proved in [31] and [40], respectively. Here we give the Itô formula for the forward integral with respect to the Lévy process.

\[1\] In fact, by the Lévy theorem (see [32] Theorem 3.3.16), the continuous \( \mathcal{F}_t \)-martingale part of \( W \) is an \( \mathcal{F}_t \)-Brownian motion.

\[2\] Since \( N^i \) \( (dr, dz) \) is also the jump measure of the \( \mathcal{F}_t \)-adapted càdlàg process \( \eta_t^i \), \( N^i \) \( (dr, dz) \) is also an \( \mathcal{F}_t \)-integer-valued random measure, which has a unique \( \mathcal{F}_t \)-compensator (the Lévy system) \( \tilde{N}^i \) \( (dr, dz) \) with respect to \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \), \( i = 1 \) (see Definition 1.2 and Remark 1.3).

\[3\] When \( \tilde{N}^i \) \( \neq \tilde{N}^i \), \( i = 1 \), \( 2 \), we require the existence of the Itô integral \( \int_0^T \int_{\mathbb{R}_0} \zeta(t,z) \tilde{N}^i(dr, dz) := \int_0^T \int_{\mathbb{R}_0} \zeta(t,z) \tilde{N}^i(dr, dz) + \int_0^T \int_{\mathbb{R}_0} \zeta(t,z) \tilde{N}^i(dr, dz) - \int_0^T \int_{\mathbb{R}_0} \zeta(t,z) G^i(dz) dr \).
Theorem 2.7. Consider a process of the form

\[ Y_t = Y_0 + \int_0^T \alpha_s \, ds + \sum_{i=1}^{2} \int_0^T \phi_i(s) \, dW^i_s + \frac{1}{2} \sum_{i=1}^{2} \int_{\mathbb{R}_0} \zeta_i(s,z) \, dN^i(d^-s, dz), \]

where \( Y_0 \) is a constant, \( \alpha \) is a càglàd process, \( \phi_i \) is càglàd and forward integrable with respect to \( W^i \), and \( \zeta_i(t,z) \) is càglàd in \( t \) for fixed \( z \) and continuous in \( z \) around zero for a.a. \( (t, \omega) \) and forward integrable with respect to \( N^i, i = 1, 2 \). Moreover, suppose that

\[ \int_0^T |\alpha_t| \, dt + \gamma \sum_{i=1}^{2} \int_0^T |\phi_i(t)| \, dz + \frac{1}{2} \sum_{i=1}^{2} \int_{\mathbb{R}_0} |\zeta_i(t,z)|^2 \, G^i(dz) \, dt < \infty. \]

Then for any function \( f \in C^2(\mathbb{R}) \) we have (in the weak sense)

\[ f(Y_t) = f(Y_0) + \int_0^t f'(Y_s) \alpha_s \, ds + \sum_{i=1}^{2} \int_0^t f'(Y_s) \phi_i(s) \, dW^i_s + \frac{1}{2} \sum_{i=1}^{2} \int_0^t f''(Y_s) \phi_i^2(s) \, dW^i_s \]

\[ + \sum_{i=1}^{2} \int_0^t \int_{\mathbb{R}_0} [f(Y_{t-} + \zeta_i(s,z)) - f(Y_{t-})] \, dN^i(ds,dz) \]

\[ + \sum_{i=1}^{2} \int_0^t \int_{\mathbb{R}_0} [f(Y_{t-} + \zeta_i(s,z)) - f(Y_{t-}) - f'(Y_{t-}) \zeta_i(s,z)] \, G^i(dz) \, ds. \]

Proof. We refer to [31] for the proof in the Brownian motion case and to [40] for the compensated Poisson random measure case.

\[ \square \]

3 Model formulation

We assume that all uncertainties come from the fixed filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) defined in Section 2 in our model. Fix a terminal time \( T > 0 \). Suppose all filtrations given in this section satisfy the usual condition.

Consider an insurer who can invest in the financial market. Two assets are available for investment, one risk-free asset (bond) \( B \) and one risky asset (stock) \( S \), whose dynamics are governed by the following anticipating stochastic differential equations (SDEs)

\[ \begin{align*}
    dB_t &= r_t B_t \, dt, \quad 0 \leq t \leq T, \\
    dS_t &= S_{t-} \left( \mu(t, \pi_t) \, dt + \sigma_t d^-W^1_t + \int_{\mathbb{R}_0} \gamma_t(t,z) \, dN^1(d^-t, dz) \right), \quad 0 \leq t \leq T,
\end{align*} \]

with constant initial values \( B_0 = 1 \) and \( S_0 > 0 \), respectively. Suppose that there are large investors in the market and they have access to insider information which can be characterized by a larger filtration \( \{\mathcal{G}^1_t\}_{0 \leq t \leq T} \) (i.e., \( \mathcal{F}_t \subset \mathcal{G}^1_t, 0 \leq t \leq T \)). Assume the insurer is also a large investor and owns insider information characterized by another filtration \( \{\mathcal{H}^1_t\}_{0 \leq t \leq T} \) such that

\[ \mathcal{F}_t \subset \mathcal{G}^1_t \subset \mathcal{H}^1_t, \quad 0 \leq t \leq T. \]

The investment strategies they take influence the coefficients of the financial asset processes. Thus, we suppose that \( r_t, \mu(t, x), \sigma_t \) and \( \gamma(t, z), t \in [0, T], x \in \mathbb{R}, z \in \mathbb{R}_0 \) are all càglàd stochastic coefficients adapted to \( \{\mathcal{G}^1_t\} \) for fixed \( x \) and \( z \), and \( \mu(t, \cdot) \) is \( C^1 \) for every \( t \in [0, T] \). The mean rate of return \( r \) on the risky asset partly depends on the investment strategy \( \pi_t \) of the insurer (see [16, 18]). Here, the investment strategy \( \pi_t \) is defined as an \( \mathcal{H}^1_t \)-adapted càglàd process, which represents the proportion of the insurer’s total wealth \( X_t \) invested in the risky asset \( S_t \) at time \( t \).

The insurer’s risk (per policy) is given by

\[ dR_t = a_t \, dt + b_t d^-W_t + \int_{\mathbb{R}_0} \gamma_t(t,z) \, dN^2(d^-t, dz), \quad 0 \leq t \leq T, \]

with zero initial value. Here, we choose \( \bar{W}_t := \rho W^1_t + \sqrt{1 - \rho^2} W^2_t \) with \(-1 < \rho \leq 0\) to describe the correlation between the insurer’s liabilities and her capital income from financial investment (see [5]). This means that claims
are the required payments to the insured holders due to either defaults of the obligors or for collateral calls when the prices of the insured risky asset declines. Similarly, the coefficients might be influenced by the uncertain economic environment characterized by a larger filtration \( \mathcal{G}^2_t \) such that

\[
\mathcal{F}_t \subset \mathcal{G}^2_t \subset \mathcal{H}_t, \quad 0 \leq t \leq T.
\]

Thus, we suppose that \( a_t, b_t \) and \( \gamma_t(t, z), z \in \mathbb{R}_0 \) are all càdlàg stochastic coefficients adapted to \( \mathcal{G}^2_t \) for fixed \( z \) (see [41]).

Note that \( W^i \) and \( \eta^i, i = 1, 2, \) may not be semimartingales with respect to \( \mathcal{G}^1_t \) or \( \mathcal{G}^2_t \), and the forward integrals defined in Section 2 are the natural generalization of semimartingale Itô integrals (see Propositions 2.3 and 2.6). Thus, the integrals with respect to \( W^1, W^2, N^1 \) and \( N^2 \) in (9) and (11) are viewed as the forward integrals instead of the Itô integrals.

Denote by \( \bar{N}^i_{\omega'}(dr, dz), i = 1, 2 \) the \( \mathcal{H}_t \)-compensator of \( N^i(dr, dz), i = 1, 2 \). We make some assumptions on the coefficients as follows:

- \( \sigma \geq \varepsilon > 0 \) (for some positive constant \( \varepsilon \)) is forward integrable with respect to \( W^1, b \) is forward integrable with respect to \( W^1 \) and \( W^2 \), \( \ln(1 + \eta_t(z)) \) and \( \gamma_t(z) \geq \varepsilon > 0 \) (for some positive constant \( \varepsilon \)) are continuous in \( z \) around zero for a.a. \((t, \omega)\) and forward integrable with respect to \( \bar{N}^i \) and \( N^j \), respectively;
- For each investment strategy \( \pi \) given above,
  \[
  \int_0^T (|\mu(t, \pi_r)| + |\alpha_t|) dt + \int_0^T (\sigma_t^2 + b_t^2) dt + \int_0^T \int_{\mathbb{R}_0} \left( |\ln(1 + \gamma_t(z))|^2 + |\gamma_t(z)|^2 \right) G_1^1(dz)dt
  + \int_0^T \int_{\mathbb{R}_0} |\gamma_t(z)|^2 G_2^2(dz)dt + \int_0^T \int_{\mathbb{R}_0} |\ln(1 + \gamma_t(z)) - \gamma_t(z)| G_1^1(dz)dt
  + \int_0^T \int_{\mathbb{R}_0} |\gamma_t(z)|^k N^k_{\omega'}(dr, dz) + \int_0^T \int_{\mathbb{R}_0} |\gamma_t(z)|^k \bar{N}^k_{\omega'}(dr, dz) < \infty, \quad k = 1, 2.
  \]

Under the above conditions, we can solve the anticipating SDEs (9) and (11) by using the Itô formula for forward integrals (Theorem 2.7). Moreover, the process of the risky asset \( S \) is given by

\[
S_t = S_0 \exp \left\{ \int_0^t \left( \mu(s, \pi_r) - \frac{1}{2} \sigma_r^2 \right) ds + \int_0^t \sigma_r dW_r^1 + \int_0^t \int_{\mathbb{R}_0} \ln(1 + \gamma(s, z)) \bar{N}^1(dz, dS) \right\}, \quad 0 \leq t \leq T.
\]

Let \( \kappa \) be the liability ratio process of the insurer (see [5]). Then \( \kappa X \) is the total number of insurance policies the insurer choose at time \( t \). Since the insider information of the insurer also contains the insurance claim, the insurance strategy \( \kappa \) is defined as an \( \mathcal{H}_t \)-adapted càdlàg process as well.

Note that both the investment strategy \( \pi \) and the insurance strategy \( \kappa \) can take negative values, which is to be interpreted as short-selling the risky asset and buying insurance policies from other insurers, respectively. Denote by \( u = (\pi, \kappa) \) the total strategy for the insurer. Then her wealth process \( X^u \) corresponding to \( u \) is governed by the following anticipating SDE (see [32], page 372) for the deduction:

\[
\frac{dX^u_t}{X^u_t} = \left[ r_t + (\mu_t - \kappa_t) - r \right] dt + (\lambda_t - \alpha_t) d\bar{N}_r^1 + (\sigma_t \pi_r - \rho b_t \kappa_t) dW^1_t - \sqrt{1 - \rho^2} b_t \kappa dW^2_t
  + \int_{\mathbb{R}_0} \pi_t \gamma_t(z) \bar{N}^1(t^- dS) - \int_{\mathbb{R}_0} \kappa_t \gamma_t(z) \bar{N}^2(t^- dS), \quad 0 \leq t \leq T,
\]

with constant initial value \( X_0 > 0 \). Here, \( \lambda_t \) is defined as a \( \mathcal{G}^2_t \)-adapted càdlàg process such that \( \int_0^T |\lambda_t| dt < \infty \) and \( \lambda_t - \alpha_t > 0 \) (in the sense of \( \mathbb{P} \)-a.s.) for every \( t \in [0, T] \). It represents the premium per policy for the insurer at time \( t \).

We can apply the Itô formula for forward integrals to solve the anticipating SDE (14). Before that, we impose the following admissible conditions on \( u \).

**Definition 3.1.** We define \( \mathcal{A}_1 \) as the set of all above strategies \( u = (\pi, \kappa) \) satisfying the following conditions:
\( \pi \) is forward integrable with respect to \( W^1 \), \( b \kappa \) is forward integrable with respect to \( W^1 \) and \( W^2 \), \( \ln(1 + \pi_t \gamma_t(t,z)) \) for a.e. \((t,z)\) with respect to both \( \mathcal{N}_t^1 \) and \( \mathcal{N}_t^2 \) for some \( \varepsilon_\pi \in (0,1) \) depending on \( \pi \) and \( \ln(1 - \kappa_t \gamma_t(t,z)) \) for a.e. \((t,z)\) with respect to both \( \mathcal{N}_t^1 \) and \( \mathcal{N}_t^2 \) for some \( \varepsilon_\kappa \in (0,1) \) depending on \( \kappa \) are continuous in \( z \) around zero for a.a. \((t,\omega)\) and forward integrable with respect to \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \), respectively.

(ii) Assume the following integrability:

\[\begin{align*}
\int_0^T |\mu(t,\pi_t) - r_t| \, d\pi_t + |\lambda_t - a_t| \, d\kappa_t + |\sigma_t b_t \pi_t \kappa_t| \, d\tau + \int_0^T (\sigma_t^2 \pi_t^2 + b_t^2 \kappa_t^2) \, d\tau \\
+ \int_0^T \int_{\mathbb{R}_0} \left( \ln(1 + \pi_t \gamma_t(t,z))^k |\pi_t \gamma_t(t,z)|^k \right) \mathcal{N}_t^1 (dz) \, d\tau + \int_0^T \int_{\mathbb{R}_0} \left( \ln(1 - \kappa_t \gamma_t(t,z))^k + |\kappa_t \gamma_t(t,z)|^k \right) \mathcal{N}_t^2 (dz) \, d\tau \\
+ \int_0^T \int_{\mathbb{R}_0} \left( \ln(1 + \pi_t \gamma_t(t,z))^k + |\pi_t \gamma_t(t,z)|^k \right) \mathcal{N}_t^1 (dz) \, d\tau + \int_0^T \int_{\mathbb{R}_0} \left( \ln(1 - \kappa_t \gamma_t(t,z))^k + |\kappa_t \gamma_t(t,z)|^k \right) \mathcal{N}_t^2 (dz) \, d\tau < \infty,
\end{align*}\]

Let \( u \in \mathcal{A}_1 \). By Theorem 2.7, the solution of (14) is given by

\[\begin{align*}
X_n^u &= X_0 \exp \left\{ \int_0^t \left[ r_s + (\mu(s,\pi_s) - r_s) \pi_s + (\lambda_s - a_s) \kappa_s - \frac{1}{2} \sigma_s^2 \pi_s^2 + \rho \sigma_s b_s \pi_s \kappa_s - \frac{1}{2} b_s^2 \kappa_s^2 \right] \, ds \\
&\quad + \int_0^t \left( \sigma_s \pi_s - \rho \kappa_s \right) \, dW^1_s - \int_0^t \sqrt{1 - \rho^2} \, b_s \kappa_s \, dW^2_s + \int_0^t \int_{\mathbb{R}_0} \ln(1 + \pi_s \gamma_s(t,z)) \, d\mathcal{N}_s^1 \,(dz) \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \ln(1 - \kappa_s \gamma_s(t,z)) \, d\mathcal{N}_s^2 \,(dz) \right\}, \quad 0 \leq t \leq T.
\end{align*}\]

Consider a model uncertainty setup. Assume that the insurer is ambiguity averse, implying that she is concerned about the accuracy of statistical estimation, and possible misspecification errors. Thus, a family of parametrized subjective probability measures \( \{ \mathcal{P}^\eta \} \) equivalent to the original probability measure \( \mathcal{P} \) is assumed to be exist in the real world (see [13]). However, since the insurer has insider information filtration \( \{ \mathcal{H}_n \} \) under which \( W^1 \) and \( \eta^1 \) might not be semimartingales, \( i = 1, 2 \), a generalization for the construction of \( \{ \mathcal{P}^\eta \} \) need to be considered by means of the forward integral.

**Definition 3.2.** We define \( \mathcal{A}_1 \) as the set of all \( \mathcal{H}_n \)-adapted càdlàg processes \( \nu_t = (\theta_1(t), \theta_2(t), \theta_3(t), \theta_4(t)), t \in [0, T] \), satisfying the following conditions:

(i) \( \theta_1(t), \theta_2(t), \theta_3(t), \theta_4(t) \) are all forward integrable with respect to \( W^1, W^2, \mathcal{N}^1, \mathcal{N}^2 \), respectively;

(ii) Assume the following integrability:

\[\begin{align*}
\mathbb{E} \left\{ \int_0^T \left[ \theta_1^2(s) + \theta_2^2(s) \right] \, ds + \int_0^T \int_{\mathbb{R}_0} \left[ \ln(1 + \theta_3(s))^k + |\theta_3(s)|^k \right] \mathcal{N}_t^1 (dz) \, d\tau \\
&\quad + \int_0^T \int_{\mathbb{R}_0} \left[ \ln(1 + \theta_4(s))^k + |\theta_4(s)|^k \right] \mathcal{N}_t^2 (dz) \, d\tau < \infty, \quad k = 1, 2;
\end{align*}\]

(iii) \( \int_0^T \theta_1(s) \, dW^1_s + \int_0^T \theta_2(s) \, dW^2_s + \int_0^T \int_{\mathbb{R}_0} \theta_3(s) \, d\mathcal{N}^1(s) (dz) + \int_0^T \int_{\mathbb{R}_0} \theta_4(s) \, d\mathcal{N}^2(s) (dz) \) is an \( \mathcal{H}_n \)-special semimartingale (see Appendix A for the definition), the local martingale part (in the canonical decomposition) of which satisfies the Novikov condition (see Theorem A.7).
For $v \in \mathcal{A}_2$, the Doléans-Dade exponential $\mathcal{E}^\nu_t$ is the unique $\mathcal{H}_t$-martingale governed by (see Theorem A.7)

$$\mathcal{E}^\nu_t = 1 + \mathcal{E}^\nu_t \left( \int_0^t \theta_1(s) d^-W^1_s + \int_0^t \theta_2(s) d^-W^2_s + \int_0^t \int_{\mathcal{B}_0} \theta_3(s) \tilde{N}^1(d^-s, dz) + \int_0^t \int_{\mathcal{B}_0} \theta_4(s) \tilde{N}^2(d^-s, dz) \right)^M, \quad 0 \leq t \leq T,$$

where $(\cdot)^M$ denotes the local martingale part of a special semimartingale. Thus, we have $\mathcal{E}^\nu_T > 0$ and $\int_0^T \mathcal{E}^\nu_t d\mathbb{P} = 1$, which induces a probability $\mathbb{Q}^\nu$ equivalent to $\mathbb{P}$ such that $\frac{d\mathbb{Q}^\nu}{d\mathbb{P}} = \mathcal{E}^\nu_T$. Then all such $\mathbb{Q}^\nu$ form a set of subjective probability measures $\{\mathbb{Q}^\nu\}_{v \in \mathcal{A}_2}$.

Taking into account the extra insider information and model uncertainty, the optimization problem for the insurer can be formulated as a (zero-sum) anticipating stochastic differential game. In other words, we need to solve the following problem.

**Problem 3.3.** Select a pair $(u^*, v^*) \in \mathcal{A}_1^* \times \mathcal{A}_2^*$ (see Definition 3.4 below) such that

$$V := J(u^*, v^*) = \sup_{u \in \mathcal{A}_1^*} \inf_{v \in \mathcal{A}_2^*} J(u, v) = \inf_{v \in \mathcal{A}_2^*} \sup_{u \in \mathcal{A}_1^*} J(u, v),$$

where the performance functional is given by

$$J(u, v) := \mathbb{E}_{\mathbb{Q}^u} \left[ U(X_T^u) + \int_0^T g(s, v_s) ds \right] = \mathbb{E} \left[ \mathcal{E}^\nu_T U(X_T^u) + \int_0^T \mathcal{E}^\nu_t g(s, v_s) ds \right],$$

the utility function $U : (0, \infty) \to \mathbb{R}$ is a strictly increasing and concave function with a strictly decreasing derivative, the penalty function $g : [0, T] \times \mathbb{R}^4 \times \Omega \to \mathbb{R}$ is a measurable function and Fréchet differentiable in $v$. The term $\mathbb{E}_{\mathbb{Q}^u} \left[ \int_0^T g(s, v_s) ds \right]$ is viewed as a step adopted to penalize the difference between $\mathbb{Q}^\nu$ and $\mathbb{P}$. We call $V$ the value (or the optimal expected utility under the worst-case probability) of Problem 3.3.

**Definition 3.4.** Define $\mathcal{A}_1^*$ as some subset of $\mathcal{A}_1$ with $\mathbb{E} \left[ U(X^u_T)^2 + |U'(X^u_T)X^u_T|^2 \right] < \infty$ for all $u \in \mathcal{A}_1^*$. Define $\mathcal{A}_2^*$ as some subset of $\mathcal{A}_2$ with $\mathbb{E} \left[ |\mathcal{E}^\nu_T|^2 + \int_0^T |g(s, v_s)|^2 ds \right] < \infty$ for all $v \in \mathcal{A}_2^*$.

**Remark 3.5.** For $v \in \mathcal{A}_2^*$, $|\mathcal{E}^\nu_t|^2$ is an $\mathcal{H}_t$-submartingale and $\mathbb{E} |\mathcal{E}^\nu_t|^2 \leq \mathbb{E} |\mathcal{E}^\nu_T|^2 < \infty$ for all $t \in [0, T]$ by the Jensen inequality.

**Remark 3.6.** Here, we suppose $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ are some subsets of $\mathcal{A}_1$ and $\mathcal{A}_2$ with the above integrability conditions, respectively, since we need more assumptions in Sections 4-6 to characterize the optimal pair $(u^*, v^*) \in \mathcal{A}_1^* \times \mathcal{A}_2^*$. If $u^* \in \mathcal{A}_1^*$ or $v^* \in \mathcal{A}_2^*$ do not satisfy those assumptions, respectively, we can impose those assumptions to $\mathcal{A}_1^*$ or $\mathcal{A}_2^*$ to narrow the two sets such that $(u^*, v^*) \in \mathcal{A}_1^* \times \mathcal{A}_2^*$ fits those assumptions (see Remark 4.5).

### 4 A half characterization of the robust optimal strategy

**Assumption 4.1.** If $(u^*, v^*) \in \mathcal{A}_1^* \times \mathcal{A}_2^*$ is robust optima for Problem 3.3 then for all bounded $\alpha \in \mathcal{A}_1^*$, there exists some $\delta > 0$ such that $u^* + \alpha \in \mathcal{A}_1^*$ for all $|\alpha| < \delta$. Moreover, the following family of random variables

$$\left\{ \mathcal{E}^\nu_t U'(X^u_t)^{\alpha \beta} \frac{d}{dy} X^u_t \right\}_{\gamma \in (-\delta, \delta)}$$

is $\mathbb{P}$-uniformly integrable, where $\frac{d}{dy} X^u_t$ means that $\frac{d}{dy} X^u_t \in \mathcal{L}^2$ and $\ln X^u_t$ exists and the interchange of differentiation and integral with respect to $\ln X^u_t$ in $\mathcal{L}^2$ is justified.

**Assumption 4.2.** Let $u = (\theta_1 1_{(t, t+h]}(s), \theta_2 1_{(t, t+h]}(s))$, $0 \leq s \leq T$, for fixed $0 \leq t < t + h \leq T$, where the random variable $\theta_i$ is of the form $1_A$, for any $\mathcal{H}_t$-measurable set $A$, $i = 1, 2$. Then $u \in \mathcal{A}_1^*$.

**Theorem 4.3.** Suppose $(u^*, v^*) \in \mathcal{A}_1^* \times \mathcal{A}_2^*$ is optimal for Problem 3.3 under Assumptions 4.1 and 4.2. Then the following stochastic processes

$$m_T^{u^*}(t) := \int_t^T \left( \mu(s, \pi_s^u) - r_s + \frac{d}{ds} \mu(s, \pi_s^u) \right) ds + \int_t^T \int_{\mathbb{B}_0} \gamma(s, z) \tilde{N}^1(d^-s, dz) + \int_t^T \int_{\mathbb{B}_0} \pi_s^u \gamma(s, z) \tilde{N}^2(d^-s, dz) - \int_t^T \int_{\mathbb{B}_0} \pi_s^u \gamma(s, z) G^1(dz) ds, \quad 0 \leq t \leq T,$$

are continuous.
and
\[
m_2^o(t) := \int_0^t (\lambda_s - a_s + \rho \sigma b_s \pi_s - b_s^2 \kappa_s^o) \, ds - \int_0^t \rho b_s dW_s^1 - \int_0^t \sqrt{1 - \rho^2} b_s dW_s^2
\]
\[- \int_0^t \int_{\mathbb{R}_0} \frac{\gamma_v(s, z)}{1 - \kappa_s^o \gamma_v(s, z)} N^2(d^- s, dz) - \int_0^t \int_{\mathbb{R}_0} \frac{\gamma_v s}{1 - \kappa_s^o \gamma_v(s, z)} G^2(dz) \, ds, \quad 0 \leq t \leq T,
\]

are \( (\mathcal{H}, Q_{\alpha^o, \gamma^r})\)-martingales, where \( F_{\alpha^o, \gamma^r}(s) := \mathbb{E} \left[ \mathcal{H}_t \right], s \in [0, T], \) and \( Q_{\alpha^o, \gamma^r} \) is an equivalent probability measure of \( \mathbb{P} \) on \( (\Omega, \mathcal{F}) \) defined by \( Q_{\alpha^o, \gamma^r}(d\omega) := \mathcal{F}_{\omega^o, \gamma^r}(T) \mathbb{P}(d\omega) \).

**Proof.** Suppose that the pair \((u^o, \gamma^r) \in \mathcal{A}_1' \times \mathcal{A}_2'\) is optimal. Then for any bounded \((\alpha_1, 0) \in \mathcal{A}_1'\) and \(|\gamma| < \delta\), we have \( J(u^o + y(\alpha_1, 0), \gamma^r) \leq J(u^o, \gamma^r) \), which implies that \( y = 0 \) is a maximum point of the function \( y \mapsto J(u^o + y(\alpha_1, 0), \gamma^r) \). Thus, we have \( \frac{d}{dy} J(u^o + y(\alpha_1, 0), \gamma^r) |_{y=0} = 0 \) once the differentiability is established. Thanks to Assumption 4.1, we can deduce by (15) that
\[
\frac{d}{dy} J(u^o + y(\alpha_1, 0), \gamma^r) |_{y=0} = 0.
\]

Now let us fix \( 0 \leq t < t + h \leq T \). By Assumption 4.2, we can choose \((\alpha_1, 0) \in \mathcal{A}_1'\) of the form
\[
\alpha_1(s) = \vartheta_1 1_{[t, t+h]}(s), \quad 0 \leq s \leq T,
\]
where \( \vartheta_1 = 1_{A_1} \) for an arbitrary \( \mathcal{H}\)-measurable set \( A_1 \). Then (21) gives
\[
0 = \mathbb{E}_{Q_{\alpha^o, \gamma^r}} \left\{ \vartheta_1 \left[ \int_t^{t+h} \left( \mu(s, \pi_s^o) - r_s + \frac{\partial}{\partial x} \mu(s, \pi_s^o) \pi_s^o - \sigma_s^o \pi_s^o + \rho \sigma b_s \kappa_s^o \right) \, ds + \int_t^{t+h} \sigma_s dW_s^1 \right.ight.
\]
\[
\left. + \int_t^{t+h} \int_{\mathbb{R}_0} \frac{\gamma_v(s, z)}{1 + \pi_s^o \gamma_v(s, z)} N^1(d^- s, dz) - \int_t^{t+h} \int_{\mathbb{R}_0} \pi_s^o \gamma_v(s, z) G^1(dz) \, ds \right\}
\]
by Propositions 2.2 and 2.3. Since this holds for all such \( \vartheta_1 \), we can conclude that
\[
0 = \mathbb{E}_{Q_{\alpha^o, \gamma^r}} \left[ m_2^o(t + h) - m_2^o(t) | \mathcal{H}_t \right].
\]

Hence, \( m_2^o(t), t \in [0, T] \), is an \( \mathcal{H}\)-martingale under the probability measure \( Q_{\alpha^o, \gamma^r} \). By similar arguments, we can deduce that \( m_2^o(t), t \in [0, T] \), is an \( \mathcal{H}\)-martingale under \( Q_{\alpha^o, \gamma^r} \) as well.

Moreover, we obtain the following result under the original probability measure \( \mathbb{P} \). Unless otherwise stated, all statements are back to \( \mathbb{P} \) from now on.

**Theorem 4.4.** Suppose \((u^o, \gamma^r) \in \mathcal{A}_1' \times \mathcal{A}_2'\) is optimal for Problem 2.3 under Assumptions 4.1 and 4.2. Then the following stochastic processes
\[
\hat{m}_1^{u^o, \gamma^r}(t) := m_1^u(t) - \int_0^t F_{u^o, \gamma^r}(s^-) \, d\langle F_{u^o, \gamma^r} \rangle^{-1}, m_1^u(t) \rangle_{Q_{u^o, \gamma^r}}, \quad 0 \leq t \leq T,
\]
and
\[
\hat{m}_2^{u^o, \gamma^r}(t) := m_2^u(t) - \int_0^t F_{u^o, \gamma^r}(s^-) \, d\langle F_{u^o, \gamma^r} \rangle^{-1}, m_2^u(t) \rangle_{Q_{u^o, \gamma^r}}, \quad 0 \leq t \leq T,
\]
are \( \mathcal{H}\)-local martingales, provided that the \( (\mathcal{H}, Q_{u^o, \gamma^r})\)-predictable covariation process \( \langle F_{u^o, \gamma^r} \rangle^{-1}, m_1^u(t) \rangle_{Q_{u^o, \gamma^r}}, t \in [0, T] \), exists (see Appendix A for the definition of \( \langle \cdot, \cdot \rangle_1 \)) and is absolutely continuous, i.e., \( u = 1, 2 \). Here, \( m_1^o(t) \) and \( m_2^o(t) \) are given in Theorem 4.3.
Proof. If \((u^*,v^*) \in \mathcal{A}_1^* \times \mathcal{A}_2^*\) is optimal, then by Theorem 4.3 we know that \(m_t^{u^*}(t), t \in [0,T]\), is an \((\mathcal{H}_t,Q_{u^*,v^*})\)-martingale, \(i = 1,2\). The conclusion is an immediate result from the Girsanov theorem (see Theorem A.3).

\[\]
tion of a special semimartingale, \( u^* \) solves the following equations

\[
0 = \int_0^t \left( \mu(s, \pi^*_s) - r_s + \frac{\partial}{\partial x} \mathbb{E}_0 \left[ \frac{\mu(x, \pi^*_s) - \sigma_x^2 \pi^*_s - \rho \sigma_x b_x \kappa_s^*}{1 + \pi^*_s \gamma_1(s, z)} G^1 dz \right] ds + \int_0^t \sigma_x \phi_1(s)ds - \int_0^t \int_{\mathbb{R}_0} \frac{\gamma_1(s, z)}{1 + \pi^*_s \gamma_1(s, z)} [G^{1, s, dz} - G^1(dz)] ds - \int_0^t F_{u^*, r^*}(s) \mathbb{E}_0 (\langle F_{u^*, r^*} \rangle^{-1}, m_{i_s}^{u^*, r^*}, 0 \leq t \leq T, \quad \text{and} \end{align}

\[
0 = \int_0^t (\lambda_x + \rho \sigma_x b_x \pi^*_s - b_x^2 \kappa_s^*) ds - \int_0^t \rho b_x \phi_2(s)ds - \int_0^t \sqrt{1 - \rho^2 b_x \phi_2(s)ds - \int_0^t \int_{\mathbb{R}_0} \frac{\gamma_2(s, z)}{1 - \kappa_s^* \gamma_2(s, z)} G^2(dz)ds - \int_0^t F_{u^*, r^*}(s) \mathbb{E}_0 (\langle F_{u^*, r^*} \rangle^{-1}, m_{i_s}^{u^*, r^*}, 0 \leq t \leq T, \end{align}

\]

**Remark 4.7.** Theorem 4.6 generalizes the main result (Theorem 4.2) in [28]. However, this is not a trivial generalization of [28]. Without further introducing the new probability measure \( Q_{u^*, r^*} \) in Theorem 4.3 and the predictable version of the Girsanov theorem in Theorem 4.4, the model uncertainty and more general utility functions could not be considered here, nor could an extra but important decomposition theorem be obtained in Theorem 4.6. Moreover, all coefficients here are all anticipating processes (i.e., the anticipating environments of the financial market and the insurance market are considered, or namely, the classical SDEs of the risky asset process and the insurer’s risk process in [26] are replaced by the anticipating SDEs [9] and [11], respectively), and a large insurer is considered here (i.e., the mean rate of return \( \mu \) on the risky asset is influenced by her investment strategy \( \pi \)).

Further, by Theorem 4.6 and Propositions 2.3 and 2.6, the dynamic of the \( \mathcal{H}_t \)-martingale \( \mathcal{E}_t^\nu \) (see (16)) can be rewritten as

\[
\mathcal{E}_t^\nu = 1 + \epsilon_t^\nu \left( \int_0^t \theta_1(s) dW_{\mathcal{H}_t}^1(s) + \int_0^t \theta_2(s) dW_{\mathcal{H}_t}^2(s) \right)
\]

\[
+ \int_0^t \int_{\mathbb{R}_0} \theta_3(s) \hat{N}_{\mathcal{H}_t}^1(ds, dz) + \int_0^t \int_{\mathbb{R}_0} \theta_4(s) \hat{N}_{\mathcal{H}_t}^2(ds, dz) \right), \quad 0 \leq t \leq T, \end{align}

for \( \nu = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathcal{A}_2^B \). By the Itô formula for Itô processes (see Theorem A.6), we have

\[
(e_t^\nu)^{-1} = 1 + \int_0^t (e_{s-}^\nu)^{-1} (\theta_1(s) \theta_2(s)^2) ds - \int_0^t (e_{s-}^\nu)^{-1} \theta_1(s) dW_{\mathcal{H}_t}^1(s) - \int_0^t (e_{s-}^\nu)^{-1} \theta_2(s) dW_{\mathcal{H}_t}^2(s)
\]

\[
- \int_0^t \int_{\mathbb{R}_0} (e_{s-}^\nu)^{-1} \frac{\theta_3(s)}{1 + \theta_3(s)} \hat{N}_{\mathcal{H}_t}^1(ds, dz) - \int_0^t \int_{\mathbb{R}_0} (e_{s-}^\nu)^{-1} \frac{\theta_4(s)}{1 + \theta_4(s)} \hat{N}_{\mathcal{H}_t}^2(ds, dz)
\]

\[
+ \int_0^t \int_{\mathbb{R}_0} (e_{s-}^\nu)^{-1} \frac{\theta_3(s)^2}{1 + \theta_3(s)} G_\mathcal{H}_t^1(ds, dz) ds + \int_0^t \int_{\mathbb{R}_0} (e_{s-}^\nu)^{-1} \frac{\theta_4(s)^2}{1 + \theta_4(s)} G_\mathcal{H}_t^2(ds, dz) ds.
\]

Moreover, by the Girsanov theorem (see Theorem A.9), the following terms

\[
W_{\mathcal{H}_t, \mathcal{D}_t}^i(t) = W_{\mathcal{H}_t}^i(t) - \int_0^t \theta_i(s) ds, \quad 0 \leq t \leq T,
\]

\[
\hat{N}_{\mathcal{H}_t, \mathcal{D}_t}^i(dr, dz) = (1 + \theta_{i+2}(t)) G_{\mathcal{H}_t}(t, dz) dr, \quad 0 \leq t \leq T, \quad z \in \mathbb{R}_0,
\]

are \( (\mathcal{H}_t, \mathcal{D}_t) \)-Brownian motion and \( (\mathcal{H}_t, \mathcal{D}_t) \)-compensator of \( N_i(dr, dz) \), respectively, \( i = 1, 2 \).

For the optimal pair \( (u^*, \nu^*) \), it is often difficult to obtain a concrete expression of the \( (\mathcal{H}_t, Q_{u^*, \nu^*}) \)-predictable covariation process \( \langle F_{u^*, \nu^*} \rangle^{-1}, m_{i_s}^{u^*, \nu^*}, i = 1, 2 \), as in Theorem 4.6. However, when the utility function of the logarithmic form, i.e., \( U(x) = \ln(x) \), we have \( F_{u^*, \nu^*}(t) = \mathcal{E}_t^\nu \) and \( Q_{u^*, \nu^*} = \mathcal{G}^\nu \). By (19), (20), (28) and (29), we can rewrite the \( \mathcal{H}_t \)-adapted processes \( m_{i_s}^{u^*, \nu^*}(t), m_{i_s}^{u^*, \nu^*}(t) \) and \( (e_t^\nu)^{-1} \) with respect to \( W_{\mathcal{H}_t, \mathcal{D}_t}^i(t) \) and \( \hat{N}_{\mathcal{H}_t, \mathcal{D}_t}^i(dr, dz) \) under the probability
measure $\mathbb{P}^v$, $i = 1, 2$. Then the $(\mathcal{H}_t, \mathbb{P}^v)$-predictable covariation process $\langle (e^{v^*})^{-1}, m_{1}^v \rangle_{t}^{\mathbb{P}^v}$ and $\langle (e^{v^*})^{-1}, m_{2}^v \rangle_{t}^{\mathbb{P}^v}$ can be calculated as follows

$$\langle (e^{v^*})^{-1}, m_{1}^v \rangle_{t}^{\mathbb{P}^v} = -\int_{0}^{t} (e_{t}^{v^*})^{-1} \theta_{1}^v(s) \sigma_{1} ds - \int_{0}^{t} \int_{\mathbb{R}_0} (e_{t}^{v^*})^{-1} \theta_{3}^v(s) \gamma_{1}(s, z) G_{\mathcal{H}}^1(s, dz) ds,$$

$$\langle (e^{v^*})^{-1}, m_{2}^v \rangle_{t}^{\mathbb{P}^v} = \int_{0}^{t} (e_{t}^{v^*})^{-1} \theta_{1}^v(s) \rho b_{1} ds + \int_{0}^{t} (e_{t}^{v^*})^{-1} \theta_{2}^v(s) \sqrt{1 - \rho^2} \rho b_{1} ds + \int_{0}^{t} \int_{\mathbb{R}_0} (e_{t}^{v^*})^{-1} \theta_{3}^v(s) \gamma_{2}(s, z) G_{\mathcal{H}}^2(s, dz) ds.$$ 

We obtain the following corollary by substituting (30) into (25) and (26) in Theorem 4.4 respectively.

**Corollary 4.8.** Assume that $U(x) = \ln(x)$. Suppose $(u^v, v^v) \in \mathcal{A}_1 \times \mathcal{A}_2$ is optimal for Problem 3.2 under the conditions of Theorem 4.4. Then $u^v$ solves the following equations

$$0 = \mu(t, \pi^v_t) - n + \frac{\partial}{\partial s} \mu(t, \pi^v_t) \pi^v - \sigma^2 \pi^v + \rho \sigma \sigma b_{1} \kappa^v + \sigma \phi_{1}(t) + \sigma \theta_{2}^v(t) - \int_{\mathbb{R}_0} \frac{\gamma_{1}(t, z)}{1 + \kappa_{1}^v \gamma_{1}(t, z)} G_{\mathcal{H}}^1(t, dz) - \int_{\mathbb{R}_0} \frac{\gamma_{1}(t, z)}{1 + \kappa_{1}^v \gamma_{1}(t, z)} G_{\mathcal{H}}^1(t, dz), \quad 0 \leq t \leq T,$$

and

$$0 = \lambda_{v} - n + \rho \sigma b_{1} \pi^v - b_{2} \kappa^v - \rho b_{1} \phi_{1}(t) - \rho b_{1} \theta_{2}^v(t) - \sqrt{1 - \rho^2} \rho b_{2} \phi_{2}(t) - \sqrt{1 - \rho^2} \rho b_{2} \theta_{3}^v(t) - \int_{\mathbb{R}_0} \frac{\kappa_{2}^v \gamma_{2}^2(s, z)}{1 - \kappa_{2}^v \gamma_{2}^2(s, z)} G_{2}^{2}(dz) - \int_{\mathbb{R}_0} \frac{\gamma_{1}(t, z)}{1 - \kappa_{2}^v \gamma_{2}^2(t, z)} (1 + \theta_{2}^v(t)) G_{2}^{2}(t, dz) + \int_{\mathbb{R}_0} \frac{\gamma_{1}(t, z)}{1 - \kappa_{2}^v \gamma_{2}^2(t, z)} G_{2}^{2}(dz), \quad 0 \leq t \leq T.$$ 

**Remark 4.9.** If we follow the method in [18] and use the ordinary version of the Girsanov theorem (see Theorem A.3), the last term of $\hat{m}_{i}^{v^*}$ in Theorem 4.4 reduces to $\int_{0}^{t} \epsilon_{i}^{v^*} d\left[\left( (e^{v^*})^{-1}, m_{i}^{v} \right) \right]_{s} \mathbb{P}^v$. The jump term of $\left[\left( (e^{v^*})^{-1}, m_{i}^{v} \right) \right]_{s}$ is of the form $\int_{0}^{t} \int_{\mathbb{R}_0} (e_{s}^{v^*})^{-1} \varphi_{i,1}^{v^*}(s, z) N_{\mathcal{H}}^{1}(ds, dz) + \int_{0}^{t} \int_{\mathbb{R}_0} (e_{s}^{v^*})^{-1} \varphi_{i,2}^{v^*}(s, z) N_{\mathcal{H}}^{2}(ds, dz)$ for some $\mathcal{H}_t$-predictable random field $\varphi_{i,j}^{v^*}(s, z)$, $j = 1, 2$, $i = 1, 2$, which is usually not continuous and contradicted with the absolute continuity assumption in Theorem 4.4. Thus, it leads to the triviality of $\mathcal{A}_2$. Moreover, the integral $\int_{0}^{t} \epsilon_{i}^{v^*} d\left[\left( (e^{v^*})^{-1}, m_{i}^{v} \right) \right]_{s} \mathbb{P}^v$ looks confusing and is usually not continuous as well, $i = 1, 2$, which could not lead to the decompositions in Theorem 4.6.

## 5 A total characterization of the robust optimal strategy

In the previous section, we give the characterization of $u^v$ for the optimal pair $(u^v, v^v)$ by using the maximality of $J(u^v, v^v)$ with respect to $u$. Thus, we obtain the relationship between $u^v$ and $v^v$ (see equations (25) and (26)). However, we have not used the minimality of $J(u^v, v^v)$ with respect to $v$. Thus, we need the other half characterization of $v^v$.

It is very difficult to give a characterization of $v^v$ directly due to the complexity of the other half controlled process $\epsilon_{i}^{v^*}$ (see the equation (16)). Fortunately, under the conditions of Theorem 4.4, we get the decompositions $W_{i}^{v} = W_{\mathcal{H}}^{v}(t) + \int_{0}^{t} \phi_{1}(s) ds$ and $N_{\mathcal{H}}^{1}(dr, dz) = G_{\mathcal{H}}^{1}(t, dz) dr, i = 1, 2$, with respect to the filtration $\{ \mathcal{H}_t \}$ by Theorem 4.6. Thus, we have a better expression of $\epsilon_{i}^{v^*}$ for $v \in \mathcal{A}_2$ (see (27)). Moreover, we can also rewrite the dynamic of the wealth process $X_{t}^{v^*}$ (see (33)) as follows:

$$\frac{dX_{t}^{v^*}}{X_{t}^{v^*}} = \left[ r_{v} + (\mu(t, \pi_{t}^{v}) - r_{v}) \pi_{t}^{v} + (\lambda_{v} - a_{v}) \kappa_{t}^{v} + (\sigma \pi_{t}^{v} - \rho b_{1} \kappa_{t}^{v}) \phi_{1}(t) - \sqrt{1 - \rho^2} \rho b_{2} \phi_{2}(t)ight] + \int_{\mathbb{R}_0} \pi_{t}^{v} \gamma_{l}(t, z) \left( G_{\mathcal{H}}^{1}(t, dz) - G_{2}^{2}(dz) \right) - \int_{\mathbb{R}_0} \kappa_{l} \gamma_{l}(t, z) \left( G_{\mathcal{H}}^{2}(t, dz) - G_{2}^{2}(dz) \right) dr$$

$$+ (\sigma \pi_{t}^{v} - \rho b_{1} \kappa_{t}^{v}) dW_{\mathcal{H}}^{v}(t) - \int_{\mathbb{R}_0} \sqrt{1 - \rho^2} \rho b_{2} \phi_{2}(t) dW_{\mathcal{H}}^{v}(t) + \int_{\mathbb{R}_0} \pi_{t}^{v} \gamma_{l}(t, z) N_{\mathcal{H}}^{1}(dr, dz) - \int_{\mathbb{R}_0} \kappa_{l} \gamma_{l}(t, z) N_{\mathcal{H}}^{2}(dr, dz).$$

Since (27) and (33) can be viewed as the usual SDEs with respect to the $\mathcal{H}_t$-Brownian motion $W_{\mathcal{H}}^{v}(t)$ and the $\mathcal{H}_t$-compensated random measure $\tilde{N}_{\mathcal{H}}^{i}(dr, dz), i = 1, 2$, Problem 3.3 turns to a nonanticipative stochastic differential game problem with respect to the filtration $\{ \mathcal{H}_t \}$. Thus, we can use the stochastic maximum principle to solve our
problem. Notice that since all coefficients of (27) and (33) are path-dependent stochastic processes, \( \varepsilon_t^x \) and \( X_t^\mu \) is not necessary a Markov process. Thus, we can not derive the corresponding HJB equation by the dynamic programming principle.

We make the following assumptions before our procedure.

**Assumption 5.1.** If \((u^*, v^*) \in \mathcal{A}_1 \times \mathcal{A}_2^*\) is optimal for Problem 5.3, then for all bounded \( \delta > 0 \) such that \( v^* + y^* \beta \in \mathcal{A}_2^* \) for all \( |y| < \delta \). Moreover, the following family of random variables

\[
\left\{ \frac{d}{dy} \varepsilon_t^{v^* + y^* \beta} U(X_t^{v^*}) \right\}_{y \in (-\delta, \delta)}
\]

is \( \mathbb{P} \)-uniformly integrable, and the following family of random fields

\[
\left\{ \frac{d}{dy} \varepsilon_t^{v^* + y^* \beta} g(s, v^* + y^* \beta) + \varepsilon_t^{v^* + y^* \beta} \nabla_v g(s, v^* + y^* \beta) \right\}_{y \in (-\delta, \delta)}
\]

is \( m \times \mathbb{P} \)-uniformly integrable, where \( m \) is the Borel-Lebesgue measure on \([0, T]\), \((\cdot)^\tau\) denotes the transpose of a vector, and \( \frac{d}{dy} \) means that \( \frac{d}{dy} \varepsilon_t^{v^* + y^* \beta} \) exists.

**Assumption 5.2.** If \((u^*, v^*) \in \mathcal{A}_1 \times \mathcal{A}_2^*\) is optimal for Problem 5.3 under the conditions of Theorem 4.4 and Assumption 5.1 then for all bounded \((\alpha, \beta) \in \mathcal{A}_1 \times \mathcal{A}_2^*\), we can define \( \psi^\alpha(t) := \frac{d}{dt} \varepsilon_t^{v^* + y^* \beta} |_{y=0} \) and \( \psi^\alpha(t) := \frac{d}{dt} \varepsilon_t^{v^* + y^* \beta} |_{y=0} \) by Assumptions 4.1 and 5.1 \( i = 1, 2, 3, 4 \).

**Assumption 5.3.** Let \( v = (\xi_1, \xi_2, \xi_3, \xi_4) \) be the Hamiltonian \( H : [0, T] \times \mathbb{X} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) by (see \([43, 44]\) for the construction)

\[
H(t, \varepsilon, u, v, q, \omega) := g(s, v) \varepsilon + x \left[ r_1 + (\mu(t, \pi) - r_1) \pi + (\lambda_t - a_t) \kappa + (\sigma, \pi - \rho h, \kappa) \phi_1(t) - \sqrt{1 - \rho^2 b, \kappa} \phi_2(t) \right] + \int_{B_0} \pi \gamma_1(t, z) \left( G_{1,\mathbb{X}}^1(t, dz) - G_1^1(dz) \right) - \int_{B_0} \kappa \gamma_2(t, z) \left( G_{2,\mathbb{X}}^2(t, dz) - G_2^2(dz) \right) \right] \right] p_1
\]

\[
+ x(\sigma, \pi - \rho h, \kappa) q_{11} - x \sqrt{1 - \rho^2 b, \kappa} q_{12} + \int_{B_0} x \pi \gamma_1(t, z) q_{13}(z) G_{1,\mathbb{X}}^1(t, dz) \right] - \int_{B_0} x \kappa \gamma_2(t, z) q_{14}(z) G_{2,\mathbb{X}}^2(t, dz) + \varepsilon \theta_1 q_{21} + \varepsilon \theta_2 q_{22} + \int_{B_0} \varepsilon \theta_3 q_{23}(z) G_{1,\mathbb{X}}^1(t, dz) + \varepsilon \theta_4 q_{24}(z) G_{2,\mathbb{X}}^2(t, dz),
\]

where \( u = (\pi, \kappa), v = (\theta_1, \theta_2, \theta_3, \theta_4), p = \left( \frac{p_1}{p_2} \right), q = \left( \frac{q_{11}}{q_{21}}, \frac{q_{12}}{q_{22}}, \frac{q_{13}}{q_{23}}, \frac{q_{14}}{q_{24}} \right) \), and \( \mathbb{A} \) represents the set of all the matrices \((q_{ij})_{2 \times 4}\) with the elements in the first two columns being real numbers and the elements in the last two columns being functions from \( B_0 \) to \( \mathbb{R} \) such that \( \int_{B_0} \left| q_{13}(z) \right|^2 + \left| q_{23}(z) \right|^2 \right) G_{1,\mathbb{X}}^1(t, dz) < \infty \) and \( \int_{B_0} \left| q_{14}(z) \right|^2 + \left| q_{24}(z) \right|^2 \right) G_{2,\mathbb{X}}^2(t, dz) < \infty \). It is obvious that \( H \) is differentiable with respect to \( x \) and \( \varepsilon \), and Fréchet differentiable with respect to \( u \) and \( v \). The associated BSDE system for the adjoint pair \((p_1, q(t, z))\) is given by (see \([43, 44]\))

\[
\begin{cases}
\begin{aligned}
dp_1(t) &= -\frac{\partial H}{\partial x}(t)dt + q_{11}(t)dW_{1,\mathbb{X}}(t) + q_{12}(t)dW_{2,\mathbb{X}}(t) \\
&+ \int_{B_0} q_{13}(t, z) N_{1,\mathbb{X}}^1(dr, dz) + \int_{B_0} q_{14}(t, z) N_{2,\mathbb{X}}^2(dr, dz), \quad 0 \leq t < T,
\end{aligned}
\end{cases}
\]

\[
p_1(T) = \varepsilon_T^\pi U'(X_T^\pi),
\]
and

\begin{align}
\begin{aligned}
dp_2(t) &= -\frac{\partial H}{\partial \psi}(t)dt + q_{21}(t)dW_{\gamma}(t) + q_{22}(t)dW_{\gamma}(t) \\
&\quad + \int_{B_0} q_{23}(t,z)\tilde{N}_{\gamma}(dr, dz) + \int_{B_0} q_{24}(t,z)\tilde{N}_{\gamma}(dr, dz), \quad 0 \leq t \leq T,
\end{aligned}
\end{align}

(35)

where \( p_i(t) \) is an \( \mathcal{F}_t \)-special semimartingale, and \( q_{ij}(t) \) is an \( \mathcal{F}_t \)-predictable process with the following integrability

\[
\int_0^T \left[ \frac{\partial H}{\partial \pi}(t) + |q_{13}(t)|^2 |q_{14}(t)|^2 \right] dt + \int_0^T \left[ |q_{13}(t,z)|^2 G_{\gamma}(t, dz) + |q_{14}(t,z)|^2 G_{\gamma}(t, dz) \right] dt < \infty,
\]

\( j = 1, 2, 3, 4, i = 1, 2 \). Here and in the following the abbreviated notation \( H(t) := H(t, X^u, \xi^u, u, \nu, p, q, \omega) \), etc., are taken.

We give a necessary maximum principle to characterize the optimal pair \( (u^*, v^*) \in \mathcal{A}' \times \mathcal{A}' \).

**Theorem 5.4.** Suppose \( (u^*, v^*) \in \mathcal{A}' \times \mathcal{A}' \) is optimal for Problem 5.3 under the conditions of Theorem 4.4 and Assumptions 5.1-5.3 and \( (p^*, q^*) \) is the associated adjoint pair satisfying BSDEs (34) and (35). Then \( (u^*, v^*) \) solves the following equations (the Hamiltonian system)

\[
\nabla_u H^*(t) = \left( \frac{\partial H^*}{\partial \pi}(t), \frac{\partial H^*}{\partial \psi}(t) \right) = (0, 0), \quad 0 \leq t \leq T,
\]

and

\[
\nabla_v H^*(t) = \left( \frac{\partial H^*}{\partial \theta_1}(t), \frac{\partial H^*}{\partial \theta_2}(t), \frac{\partial H^*}{\partial \theta_3}(t), \frac{\partial H^*}{\partial \theta_4}(t) \right) = (0, 0, 0, 0), \quad 0 \leq t \leq T,
\]

given the following integrability conditions

\[
\mathbb{E} \left\{ \int_0^T \left( \psi^\alpha_{\gamma}(t)^2 + \psi^\alpha_{\gamma}(t)^2 \right) dt \left[ p^\alpha_{\gamma} \right] + \int_0^T p^\gamma_{\gamma}(t)^2 dt \left[ \psi^\alpha_{\gamma} \right] + \int_0^T p^\gamma_{\gamma}(t)^2 dt \left[ \psi^\alpha_{\gamma} \right] \right\} < \infty,
\]

and

\[
\mathbb{E} \left\{ \int_0^T |\xi^\alpha_{\gamma}(t, z)|^2 dt \left[ p^\alpha_{\gamma} \right] + \int_0^T |\xi^\alpha_{\gamma}(t, z)|^2 dt \left[ \psi^\alpha_{\gamma} \right] + \int_0^T |\xi^\alpha_{\gamma}(t, z)|^2 dt \left[ \psi^\alpha_{\gamma} \right] \right\} < \infty,
\]

for all bounded \( (\alpha, \beta) \in \mathcal{A}' \times \mathcal{A}' \), \( i = 1, 2, 3, 4 \) and \( j = 3, 4 \). Here, \( H^*(t) := H(t, X^u, \xi^u, u, \nu, p, q, \omega) \), etc.

**Proof.** Suppose that the pair \( (u^*, v^*) \in \mathcal{A}' \times \mathcal{A}' \) is optimal. Then for any bounded \( (\beta_1, 0) \in \mathcal{A}' \) and \( |\eta| < \delta \), we have \( J(u^*, v^* + \eta(\beta_1, 0, 0, 0)) \geq J(u^*, v^*) \), which implies that \( \eta = 0 \) is a minimum point of the function \( y \mapsto J(u^*, v^* + \eta(\beta_1, 0, 0, 0)) \).
By Assumptions 5.1 and 5.2, Itô formula for Itô integrals (see [39]), and Remark A.1, we have

\[
\frac{d}{dy}f(u^*, v^* + y(b_1, 0, 0))|_{y=0} = \mathbb{E} \left[ \psi_i^r(T)U(X^r_T) + \int_0^T \psi_i^r(s)g(s, v^*_s)ds + \int_0^T \epsilon_s^r \frac{\partial}{\partial \theta_1}g(s, v^*_s)b_1(s)ds \right] \\
= \mathbb{E} \left[ \psi_i^r(T)p_2^*(T) + \int_0^T \psi_i^r(s)g(s, v^*_s)ds + \int_0^T \epsilon_s^r \frac{\partial}{\partial \theta_1}g(s, v^*_s)b_1(s)ds \right] \\
= \mathbb{E} \left[ \int_0^T \psi_i^r(s-d)p_2^*(s) + \int_0^T p_2^*(s-d)\psi_i^r(s) + [p_2^*, \psi_i^r]_T \right] \\
+ \int_0^T \psi_i^r(s)g(s, v^*_s)ds + \int_0^T \epsilon_s^r \frac{\partial}{\partial \theta_1}g(s, v^*_s)b_1(s)ds \\
= \mathbb{E} \left[ \int_0^T \frac{dH^i}{d\theta_1}(s)b_1(s)ds \right] \\
= 0.
\]

(38)

By Assumption 5.3 and the same procedure in Theorem 4.3, we can deduce that \( \frac{dH^i}{d\theta_1}(t) = 0, t \in [0, T] \). By similar arguments, we can conclude that \( \frac{dH^i}{d\theta_1}(t) = 0, t \in [0, T] \), i.e., there are no jumps in the risky asset process and the insurer’s risk process.

Remark 5.5. The integrability in Theorem 5.4 can be weakened using the localization technique, i.e., choosing a sequence of \( \mathcal{H}_i \)-stopping times such that Itô integrals of the stopped processes are \( \mathcal{H}_i \)-square-integrable martingales. We refer to [44] for more details.

Combining Theorem 5.4 with the conclusion in Section 4, we give the total characterization of the optimal pair \((u^*, v^*)\) as the following theorem.

Theorem 5.6. Suppose \((u^*, v^*) \in \mathcal{A}_1^1 \times \mathcal{A}_1^2\) is optimal for Problem 3.3 (with the associated pair \((p^*, q^*)\) satisfying BSDEs (24) and (25)) under the conditions of Theorem 5.4. Then \((u^*, v^*)\) solves equations (25), (26), (36) and (37).

Remark 5.7. In fact, equations (25), (26) and (36) are enough to obtain the optimal pair \((u^*, v^*)\). This combined method (rather than using the Hamiltonian system (25), (37)) can always give a better characterization of \((u^*, v^*)\). Moreover, when the mean rate of return \( \mu \) is dependent on \( \pi^* \), i.e., a large insurer is considered, it is very hard to obtain the solution \((u^*, v^*)\) by just using (36), (37) since the dynamic of \( X^u \) is not homogeneous in this situation (see Remark 6.3 in Section 6), while it is not by combining the Hamiltonian system with (25) and (26). We will give examples in Section 7 to illustrate this when the logarithmic utility is considered. However, since (25) and (26) are complicated for general utility, we turn to the Hamiltonian system (36), (37) for the optimal pair \((u^*, v^*)\) in general cases, which will be illustrated in Section 6.

6 The small insurer case: maximum principle

6.1 Without jumps

Suppose that \( G_1(dx) = G_2(dx) = 0 \), i.e., there are no jumps in the risky asset process and the insurer’s risk process. Assume that the mean rate of return \( \mu(t,x) = \mu_0(t) + px \) for some \( \mathcal{F}_t^1 \)-adapted càglàd processes \( \mu_0(t) \) and \( p \) with \( 0 \leq p < \frac{\bar{\sigma}^2}{4} \), and \( b \geq \varepsilon > 0 \) for some positive constant \( \varepsilon \). Put \( u_t = \frac{\mu_0(t) - \bar{\sigma}}{\bar{\sigma}}, \quad \tilde{\sigma} = \bar{\sigma} - \frac{2p}{\bar{\sigma}}, \quad \tilde{\rho}_t = u_t + \phi_t(t), \) and \( \tilde{\phi}_2(t) = \frac{\lambda - u_t + p_0 b_t}{\sqrt{1 - p^2 b_t^2}} - \phi_2(t). \) Assume further the penalty function \( g \) is of the quadratic form, i.e., \( g(s,v) = g(v) = \)
\[ \frac{1}{2}(\theta_1^2 + \theta_2^2). \] Then we have by the Girsanov theorem (see Theorem A.9) that

\[
E \left[ \int_0^T \epsilon_s^\nu g(v_s^\nu)ds \right] = E_{\mathcal{F}^s} \left[ \int_0^T g(v_s^\nu)ds \right] = E_{\mathcal{F}^s} \left[ \int_0^T \theta_1^*(s)dW_1^1(s) + \int_0^T \theta_2^*(s)dW_2^2(s) - \ln \epsilon_t^\nu \right] 
= E_{\mathcal{F}^s} \left[ \int_0^T (\theta_1^*(s)^2 + \theta_2^*(s)^2) ds - \ln \epsilon_t^\nu \right] 
= 2E_{\mathcal{F}^s} \left[ \int_0^T g(v_s^\nu)ds \right] - E_{\mathcal{F}^s} \left[ \ln \epsilon_t^\nu \right],
\]

which implies that

\[
E \left[ \int_0^T \epsilon_s^\nu g(v_s^\nu)ds \right] = E_{\mathcal{F}^s} \left[ \ln \epsilon_t^\nu \right] = E \left[ \epsilon_t^{n\nu} \ln \epsilon_t^{n\nu} \right].
\]

We make the following assumption before our procedure.

**Assumption 6.1.** Suppose the coefficients satisfy the following integrability

\[
\int_0^T (|\phi_1(t)|^2 + |\phi_2(t)|^2) dt < \infty.
\]

By the Hamiltonian system (37) in Theorem 5.4, we have

\[
\nabla_t H^*(t) = \left( (\theta_1^*(t) + q_{11}^*(t)) \epsilon_t^\nu, (\theta_2^*(t) + q_{22}^*(t)) \epsilon_t^\nu \right) = (0, 0),
\]

which implies that

\[
\theta_1^*(t) + q_{11}^*(t) = 0, \\
\theta_2^*(t) + q_{22}^*(t) = 0.
\]

Substituting (42) into the adjoint BSDE (35) with respect to \( p_2^*(t) \) we have

\[
\begin{align*}
\begin{cases}
    dp_2^*(t) = \frac{\theta_1^*(t)^2 + \theta_2^*(t)^2}{2} dt - \theta_1^*(t)dW_1^1(t) - \theta_2^*(t)dW_2^2(t), & 0 \leq t \leq T, \\
p_2^*(T) = U(X_T^{n\nu}).
\end{cases}
\end{align*}
\]

The SDE (27) of \( \epsilon^\nu_t \) combined with Theorem A.7 implies that

\[
d \ln \epsilon^\nu_t = -\frac{\theta_1^*(t)^2 + \theta_2^*(t)^2}{2} dt + \theta_1^*(t)dW_1^1(t) + \theta_2^*(t)dW_2^2(t).
\]

By comparing (43) with (44), the solution of the BSDE (43) can be expressed as

\[
p_2^*(t) = p_2^*(0) - \ln \epsilon_t^\nu.
\]

Denote the \( \mathcal{H}_0 \)-measurable random variable \( p_2^*(0) \) by \( c_2^* \). Substituting the terminal condition in (43), i.e., \( p_2^*(T) = U(X_T^{n\nu}) \), into (45) with \( t = T \), we have

\[
\ln \epsilon_T^{n\nu} + U(X_T^{n\nu}) = c_2^*.
\]

Since \( \epsilon^\nu_t \) is an \( \mathcal{H}_t \)-martingale, we have \( \epsilon^\nu_t = E \left[ \epsilon^\nu_T | \mathcal{H}_t \right] = E \left[ e^{\epsilon_T^{n\nu}-U(X_T^{n\nu})} | \mathcal{H}_t \right] \) by (46). Using that \( \epsilon_0^{n\nu} = 1 \) we obtain

\[
e^{\epsilon_T^{n\nu}} = \left( E \left[ e^{-U(X_T^{n\nu})} | \mathcal{H}_0 \right] \right)^{-1}.
\]

Thus, we can give the expression of \( \epsilon^\nu_t \) as follows

\[
\epsilon^\nu_t = E \left[ \left( E \left[ e^{-U(X_T^{n\nu})} | \mathcal{H}_0 \right] e^{U(X_T^{n\nu})} \right)^{-1} | \mathcal{H}_t \right].
\]
Moreover, substituting (47) into (46) we obtain

\[(49) \quad e_T^\pi = \left( \mathbb{E} \left[ e^{-U(X_T^\pi)} \big| \mathcal{F}_0 \right] e^{U(X_T^\pi)} \right)^{-1}. \]

On the other hand, by the Hamiltonian system (56) in Theorem 5.4, we have

\[(50) \quad \nabla_a H^\ast(t) = \left( X^\ast_t \left( \frac{\partial \mu}{\partial x}(t, \pi^\ast_t) + (\mu(t, \pi^\ast_t) - r_t) + \sigma_t \phi_t(t) \right) p_t^\ast(t) + X^\ast_t \sigma_t q_{11}^\ast(t), \right.

\[X^\ast_t \left( \lambda_t - \rho b_t \phi_t(t) - \sqrt{1 - \rho^2 b_t} \phi_2(t) \right) p_t^\ast(t) - X^\ast_t \rho b_t q_{11}^\ast(t) - X^\ast_t \sqrt{1 - \rho^2 b_t} q_{12}^\ast(t) \right) = (0, 0), \]

which implies that

\[(51) \quad \left( \frac{\partial \mu}{\partial x}(t, \pi^\ast_t) + (\mu(t, \pi^\ast_t) - r_t) + \sigma_t \phi_t(t) \right) p_t^\ast(t) + \sigma_t q_{11}^\ast(t) = 0,

\[(52) \quad \left( \lambda_t - \rho b_t \phi_t(t) - \sqrt{1 - \rho^2 b_t} \phi_2(t) \right) p_t^\ast(t) - \rho b_t q_{11}^\ast(t) - \sqrt{1 - \rho^2 b_t} q_{12}^\ast(t) = 0. \]

Substituting (31) into the adjoint BSDE (34) with respect to $p_t^\ast(t)$ yields

\[\begin{cases} dp_t^\ast(t) = - \left[ r_t - \frac{\sigma_t - \bar{\sigma}_t}{2} (1 + \pi_t^\ast) + \tilde{\phi}_1(t) \right] p_t^\ast(t) \text{d}t - \left[ \frac{\sigma_t - \bar{\sigma}_t}{2} (1 + \pi_t^\ast) + \tilde{\phi}_1(t) \right] p_t^\ast(t) \text{d}W^1_{\mathcal{F}_t}(t) \\
+ \left[ \frac{\phi_2(t) + \rho (\sigma_t - \bar{\sigma}_t)(1 + \pi_t^\ast)}{2 \sqrt{1 - \rho^2}} \right] p_t^\ast(t) \text{d}W^2_{\mathcal{F}_t}(t), \quad 0 \leq t \leq T, \\
p_t^\ast(T) = e_T^\pi U^\prime(X^\pi_T). \end{cases} \]

Suppose $\rho_t \equiv 0$, i.e., $\sigma_t - \bar{\sigma}_t \equiv 0$, then (52) degenerates to

\[\begin{cases} dp_t^\ast(t) = - r_t p_t^\ast(t) \text{d}t - \tilde{\phi}_1(t) p_t^\ast(t) \text{d}W^1_{\mathcal{F}_t}(t) + \phi_2(t) p_t^\ast(t) \text{d}W^2_{\mathcal{F}_t}(t), \quad 0 \leq t \leq T, \\
p_t^\ast(T) = e_T^\pi U^\prime(X^\pi_T), \end{cases} \]

which implies that all coefficients in (53) are independent of $\pi^\ast$. Then by Theorem A.7, the unique solution of (53) is given by

\[(54) \quad p_t^\ast(t) = c^\ast_t \Pi^\ast(0, t), \]

where $c^\ast_t := p_t^\ast(0)$ is an $\mathcal{F}_0$-measurable random variable, and $\Pi^\ast(0, t), t \in [0, T]$, is defined as

\[(55) \quad \Pi^\ast(0, t) := \exp \left\{ - \int_0^t r_s \text{d}s - \int_0^t \tilde{\phi}_1(s) \text{d}W^1_{\mathcal{F}_s}(s) + \int_0^t \phi_2(s) \text{d}W^2_{\mathcal{F}_s}(s) - \frac{1}{2} \int_0^t (\tilde{\phi}_1(s))^2 + \phi_2(s)^2 \text{d}s \right\}. \]

Since $U(x)$ has a strictly decreasing derivative $U'(x)$, we can denote the inverse function of $U'(x)$ by $I(x)$. Substituting (54) into (53) with $t = T$ we have

\[(56) \quad X_T^\pi = I \left( \frac{c^\ast_t \Pi^\ast(0, T)}{e_T^\pi} \right), \]

Combining (56) with (49) we have

\[(57) \quad X_T^\pi = I \left( c^\ast_t \Pi^\ast(0, T) e^{U(X_T^\pi)} \right), \]

where $c^\ast_t := c^\ast_t \mathbb{E} \left[ e^{-U(X_T^\pi)} \big| \mathcal{F}_0 \right]$ is also an $\mathcal{F}_0$-measurable random variable. Since $I$ is also strictly decreasing, there is a unique fixed point $x^\ast = I(y)$ of the equation $x = I(y e^{U(x)})$ for every $y > 0$. We also call $I(y), y > 0$, the fixed point function of $I$ (see (49)). Thus, the solution of (57) is given by

\[(58) \quad X_T^u = I \left( c^\ast_t \Pi^\ast(0, T) \right). \]
Remark 6.2. Here, we introduce the new $\mathcal{H}_0$-measurable random variable $c^*_t$ and use the technique of the fixed point function to separate $c^*_t$ from the terminal condition of $X^u_t$, which leads to the non-nested linear BSDE (60) of $X^u_t$ below (while the traditional method might lead to a nested linear BSDE, see [28,14]). Thus, we could solve the BSDE by traditional methods and obtain the formula for $X^u_t$ under mild conditions (see (63) below).

Put $z^*_t = (z^*_1(t),z^*_2(t)) = \left( (\sigma_t \pi_t^* - \rho b_t \kappa_t^*)X^u_t, -\sqrt{1 - \rho^2 b_t \kappa_t^*}X^u_t \right)$. Then we have

$$
\begin{align*}
\pi_t^* &= \frac{z^*_1(t)}{\sigma_t X^u_t} - \frac{\rho z^*_2(t)}{\sqrt{1 - \rho^2 \sigma_t X^u_t}}, \\
\kappa_t^* &= -\frac{z^*_2(t)}{\sqrt{1 - \rho^2 b_t X^u_t}}.
\end{align*}
$$

(59)

The SDE (63) of $X^u_t$ combined with (59) leads to the following linear BSDE

$$
\begin{align*}
\text{d}X^u_t &= -f_L(t,X^u_t;z^*_t,\omega)\text{d}t + z^*_t \text{d}W_{\mathcal{F}}(t), \quad 0 \leq t \leq T, \\
X^u_T &= I(c^*_T \Pi^*(0,T)),
\end{align*}
$$

(60)

where $W_{\mathcal{F}}(t) = (W^1_{\mathcal{F}}(t),W^2_{\mathcal{F}}(t))^T$, and the generator (or the driver) $f_L : [0,T] \times \mathbb{R} \times \mathbb{R}^2 \times \Omega \rightarrow \mathbb{R}$ is given by

$$
f_L(t,x,z,\omega) = -r_t x - \tilde{\phi}_1(t)z_1 + \tilde{\phi}_2(t)z_2.
$$

Remark 6.3. If $\rho_t \neq 0$, the terminal condition in BSDE (60) will depend on $z^*_t$ by (52), which makes the BSDE (60) irregular and very hard to solve. The reason is that the SDE (63) of $X^u_t$ is not homogeneous in this situation. However, once the relationship (equations (25) and (26) in Theorem 4.6 of $u^*$ and $v^*$ is solved, we can then overcome this situation by a combined method in Section 7.

By the Itô formula for Itô integrals, we have

$$
\begin{align*}
d \left( \Pi^*(0,t)X^u_t \right) &= \Pi^*(0,t)\text{d}X^u_t + X^u_t \text{d}\Pi^*(0,t) + \text{d}\langle X^u, \Pi^*(0,\cdot) \rangle_t, \\
&= \Pi^*(0,t) \left( z^*_1(t) - \tilde{\phi}_1(t)X^u_t \right) \text{d}W_{\mathcal{F}}(t) \\
&\quad + \Pi^*(0,t) \left( z^*_2(t) + \tilde{\phi}_2(t)X^u_t \right) \text{d}W_{\mathcal{F}}(t).
\end{align*}
$$

(61)

Suppose the following integrability condition holds

$$
\begin{align*}
\mathbb{E} \left[ \int_0^T \Pi^*(0,t)^2 \left( z^*_1(t) - \tilde{\phi}_1(t)X^u_t \right)^2 \text{d}t \right]^\frac{1}{2} + \left( \int_0^T \Pi^*(0,t)^2 \left( z^*_2(t) + \tilde{\phi}_2(t)X^u_t \right)^2 \text{d}t \right)^\frac{1}{2} + \left( X^u_T \right)^2 < \infty.
\end{align*}
$$

(62)

Then by the Burkholder-Davis-Gundy inequality (see (32)), $\Pi^*(0,t)X^u_t$, $t \in [0,T]$, is an $\mathcal{F}_t$-martingale. Thus we have

$$
X^u_t = \mathbb{E} \left[ \Pi^*(0,T)I(c^*_T \Pi^*(0,T)) | \mathcal{F}_t \right],
$$

(63)

where $\Pi^*(t,T) := \Pi^*(0,T)/\Pi^*(0,t)$. By (63) and the initial value condition $X^u_0 = X_0$, the $\mathcal{H}_0$-random variable $c^*_t$ can be (implicitly) determined by

$$
X_0 = \mathbb{E} \left[ \Pi^*(0,T)I(c^*_T \Pi^*(0,T)) | \mathcal{F}_0 \right].
$$

(64)

Remark 6.4. Here we could obtain the formula for $X^u_t$ without the traditional assumptions for the existence and the uniqueness of the solution to the linear BSDE (60) (see [28,14]). The reason is that there is already a priori hypothesis for the existence of $X^u_t$ in the necessary maximum principle (see Theorem 5.4), and the formula for $X^u_t$ is only based on the Itô formula and the integrability condition (62). Moreover, the existence and the uniqueness of the solution to the BSDE (60) need more assumptions (such as the assumption for the filtration $\{\mathcal{H}_t\}_{0 \leq t \leq T}$ below) to obtain the existence and the uniqueness of the solution to the BSDE (60). While those assumptions are very strong and can conversely ensures the condition (62) (see [46]).
Moreover, substituting (58) into (48) and (49) we obtain

\begin{equation}
\epsilon^*_v = E \left[ \left( E \left[ e^{-U(I(c_3^1 \Pi^0(t),T)))} \right] \right| \mathcal{H}_0 \right) e^{U(I(c_3^1 \Pi^0(t),T)))}^{-1} \left| \mathcal{H}_t \right],
\end{equation}

and

\begin{equation}
\epsilon^*_u = \left( E \left[ e^{-U(I(c_3^1 \Pi^0(t),T)))} \right] \right| \mathcal{H}_0 \right) e^{U(I(c_3^1 \Pi^0(t),T)))}^{-1}.
\end{equation}

Further, if the filtration \(\{\mathcal{H}_t\}_{0 \leq t \leq T}\) is the augmentation of the natural filtration of \(W^1_{\mathcal{H}}(t)\) and \(W^2_{\mathcal{H}}(t)\), which was also assumed in [13] in the continuous case, then \(c_3^1\) is a constant since \(\mathcal{H}_0\) is generated by the trivial \(\sigma\)-algebra and all \(\mathbb{P}\)-negligible sets. Suppose that \(E[I(c_3^1(\Pi^0(t),T)))^2 < \infty\), and \(r, \phi_1, \phi_2\) are bounded. Then by [44] Theorem 4.8, the linear BSDE (60) has a unique strong solution \(\epsilon\), where

\begin{equation}
\epsilon = \int_0^T E \left[ \left( E \left[ e^{-U(I(c_3^1 \Pi^0(t),T)))} \right] \right| \mathcal{H}_0 \right) e^{U(I(c_3^1 \Pi^0(t),T)))}^{-1} \left| \mathcal{H}_t \right] E \left[ e^U(I(c_3^1 \Pi^0(t),T))) \right] dW^1_{\mathcal{H}}(t)
\end{equation}

Moreover, \(\epsilon^*_v\) is an \(\mathcal{H}_t\)-square-integrable martingale by virtue of Remark 3.5. Thus, by (66) and the generalized Clark-Ocone representation theorem in Malliavin calculus (see [49]), we have

\begin{equation}
\epsilon^*_u = 1 + \int_0^T E \left[ D^1_t \left( E \left[ e^{-U(I(c_3^1 \Pi^0(t),T)))} \right] \right| \mathcal{H}_0 \right) e^{U(I(c_3^1 \Pi^0(t),T)))}^{-1} \left| \mathcal{H}_t \right] dW^1_{\mathcal{H}}(t)
+ \int_0^T E \left[ D^2_t \left( E \left[ e^{-U(I(c_3^1 \Pi^0(t),T)))} \right] \right| \mathcal{H}_0 \right) e^{U(I(c_3^1 \Pi^0(t),T)))}^{-1} \left| \mathcal{H}_t \right] dW^2_{\mathcal{H}}(t),
\end{equation}

where \(D_t = (D^1_t, D^2_t)\) is the Malliavin derivative operator from the Sobolev space \(D^{1,2}(\Omega)\) to \(L^2(\Omega; L^2([0,T]))\). We refer to [48] for the theory of Malliavin calculus and relevant definitions.

Remark 6.5. Under mild conditions, we can obtain the formulae for \(z_1^1(t)\) and \(z_2^1(t)\) as follows (see [47], Proposition 3.5.1)

\begin{equation}
\begin{align*}
z_1^1(t) &= D^1_t X^u_t, \\
z_2^1(t) &= D^2_t X^u_t,
\end{align*}
\end{equation}

where \(D_t = (D^1_t, D^2_t)\) is the Malliavin derivative operator from the Sobolev space \(D^{1,2}(\Omega)\) to \(L^2(\Omega; L^2([0,T]))\). We refer to [48] for more details. Comparing (68) with (27) we obtain

\begin{equation}
\begin{align*}
\theta^1_1(t) &= E \left[ D^1_t \left( E \left[ e^{-U(I(c_3^1 \Pi^0(t),T)))} \right] \right| \mathcal{H}_0 \right) e^{U(I(c_3^1 \Pi^0(t),T)))}^{-1} \left| \mathcal{H}_t \right] \\
\theta^2_1(t) &= E \left[ D^2_t \left( E \left[ e^{-U(I(c_3^1 \Pi^0(t),T)))} \right] \right| \mathcal{H}_0 \right) e^{U(I(c_3^1 \Pi^0(t),T)))}^{-1} \left| \mathcal{H}_t \right].
\end{align*}
\end{equation}

To sum up, we give the following theorem.

**Theorem 6.6.** Assume that \(G^1(dz) = G^2(dz) = 0, \mu(t,x) = \mu_0(t)\) for some \(\mathcal{G}^1\)-adapted càdlàg process \(\mu_0(t)\), \(b \geq 0\) for some positive constant \(\varepsilon\), and \(g(s,v) = \frac{1}{2}(\theta_1 + \theta_2)\). Suppose \((u^*, v^*)\) in \(\mathcal{Q}^1 \times \mathcal{Q}^2\) is optimal for Problem 3.3 under the conditions of Theorem 3.4 and Assumption 6.2 and the integrability condition (62) hold. Then \(u^*, v^*, X^u, X^v\) and \(\epsilon^*_v\) are given by (59), (62), (60), (63) and (65), respectively, where \(\Pi^0\) is given by (55), \(c_3^1\) is determined by (64), and \((X^u, z_1^1, z_2^1)\) solves the linear BSDE (60). Furthermore, if \(\{\mathcal{H}_t\}_{0 \leq t \leq T}\) is the augmentation of the natural filtration of \(W^1_{\mathcal{H}}(t)\) and \(W^2_{\mathcal{H}}(t)\), \(E[I(c_3^1(\Pi^0(t),T)))^2 < \infty\), and \(r, \phi_1, \phi_2\) are bounded, then the linear BSDE (60) has a unique strong solution, and \(v^*\) is given by (69).
Remark 6.7. If the filtration $\{\mathcal{H}_t\}_{0 \leq t \leq T}$ in Theorem 6.6 is not the augmentation of the natural filtration of $W^1_\mathcal{H}(t)$ and $W^2_\mathcal{H}(t)$, or the coefficients of the generator $f_t$ is not necessarily bounded, we refer to [50, 51, 52, 53] for further results. In those cases, the existence and uniqueness of the solution to the BSDE (60) still hold under mild conditions when a general martingale representation property was assumed, or a transposition solution was considered, or a stochastic Lipschitz condition was considered.

When the utility function is of the logarithmic form, i.e., $U(x) = \ln x$, we can easily calculate the fixed point function $I(y)$ as follows

$$I(y) = \frac{1}{\sqrt{y}}, \quad y > 0.$$  

Combining (64) with (65), we have

$$X_t^{u^*} = \frac{X_0}{\mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0]} \sqrt{\Pi^*(0,T)},$$  

and

$$X_T^{u^*} = \frac{X_0}{\mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0]} \sqrt{\Pi^*(0,T)}.$$  

Thus the BSDE (60) can be rewritten as

$$\begin{cases}
dX_t^{u^*} = -f_t(t,x^{u^*},z_t^*,\omega)dt + z_t^*dW_\mathcal{H}(t), & 0 \leq t \leq T, \\
X_T^{u^*} = \frac{X_0}{\mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0]} \sqrt{\Pi^*(0,T)},
\end{cases}$$  

where $f_t$ is given by

$$f_t(t,x,z,\omega) = -r_t x - \phi_1(t) z_1 + \phi_2(t) z_2.$$  

By (39), (40), (41) and (72), we can calculate the value of Problem 3.3 as follows

$$V = \mathbb{E} \left[ \mathcal{L} \frac{X_T^{u^*}}{\mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0]} \right] = \mathbb{E} \left[ \mathcal{L} X_T^{u^*} \right] = \mathbb{E} \left[ \mathcal{L} X_T^{u^*} \sqrt{\Pi^*(0,T)} \right]$$  

$$= \mathbb{E} \left[ \mathcal{L} \left( X_T^{u^*} \sqrt{\Pi^*(0,T)} \right) \right] = \mathbb{E} \left[ \sqrt{\Pi^*(0,T)} \mathcal{L} X_T^{u^*} \right] = \mathbb{E} \left[ \sqrt{\Pi^*(0,T)} \mathcal{L} X_T^{u^*} \right]$$  

$$= -\mathbb{E} \left[ \mathcal{L} \left( X_T^{u^*} \sqrt{\Pi^*(0,T)} \right) \right] = \mathbb{E} \left[ \mathcal{L} X_T^{u^*} \mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0] \right]$$  

Corollary 6.8. Assume that $U(x) = \ln x$, $G^1(dz) = G^2(dz) = 0$, $\mu(t,x) = \mu_0(t)$ for some $\mathcal{G}^1$-adapted càdlàg process $\mu_0(t)$, $b \geq \varepsilon > 0$ for some positive constant $\varepsilon$, and $g(x,v) = \frac{1}{2}(\theta_1^2 + \theta_2^2)$. Suppose $(u^*,v^*) \in \mathcal{A}_1^1 \times \mathcal{A}_2^1$ is optimal for Problem 3.3 under the conditions of Theorem 5.4 and Assumption 6.1 and the integrability condition (62) hold. Then $(u^*,v^*)$ and $V$ are given by (69), (71)-(72) and (74), respectively, where $\Pi^*$ is given by (65), and $(X^{u^*},z_1^{u^*},z_2^{u^*})$ solves the linear BSDE (73).

When the insurer has no insider information, i.e., $\mathcal{H}_t = \mathcal{F}_t$, we have $\phi_1 = \phi_2 = 0$. Assume further that all the parameter processes are deterministic functions. By (71) we have

$$X_t^{u^*} = \frac{X_0}{\mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0]} \sqrt{\Pi^*(0,T)},$$

By (57) and Malliavin calculus, we can easily obtain

$$\varepsilon_t^{u^*} = \frac{D_t X_t^{u^*}}{\mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0]} = \frac{X_0}{\mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0]} \sqrt{\Pi^*(0,T)} \frac{1}{2} \mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0]D_t \Pi^*(0,t)$$

$$= \frac{1}{2} \mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{H}_0]D_t \Pi^*(0,t).$$
and
\[ z_2^*(t) = D^*_T X^*_T = X_0 \mathbb{E} \left[ D^2_T \sqrt{\Pi^*(t,T)} \right] \mathbb{E} \sqrt{\Pi^*(0,T)} - \frac{1}{2} X_0 \mathbb{E} \left[ \sqrt{\Pi^*(t,T)} \right] \mathbb{E} \sqrt{\Pi^*(0,T)} D^*_T \Pi^*(0,t) \]
\[ = - \frac{1}{4} X_0 \mathbb{E} \left[ \sqrt{\Pi^*(t,T)} \right] \mathbb{E} \sqrt{\Pi^*(0,T)} \lambda - a_i + \rho b_i t \]
\[ \sqrt{1 - \rho^2 b_i^2}. \]

We can obtain the robust optimal strategy by (59) as follows
\[ \pi^*_i = \frac{t_i}{\sigma_i} + \frac{\rho (\lambda_i - a_i + \rho b_i t_i)}{2(1 - \rho^2) \sigma_i b_i}, \]
\[ \kappa^*_i = \frac{\lambda_i - a_i + \rho b_i t_i}{2(1 - \rho^2) b_i^2}. \]

By the equations (31) and (32) in Corollary 4.8, we can also obtain
\[ \theta_1^*(t) = \frac{\lambda_i}{2}, \]
\[ \theta_2^*(t) = \frac{\lambda_i - a_i + \rho b_i t_i}{2 \sqrt{1 - \rho^2 b_i^2}}. \]

The value of Problem 5.3 can be obtained by (74) as follows
\[ V = \ln X_0 \ln \left( \mathbb{E} \sqrt{\Pi^*(0,T)} \right) \]
\[ = \ln X_0 + \int_0^T r_i dt + \frac{1}{4} \int_0^T \left( t_i^2 + \frac{(\lambda_i - a_i + \rho b_i t_i)^2}{(1 - \rho^2) b_i^2} \right) dt. \]

Thus we obtain the following corollary.

**Corollary 6.9.** Assume that \( U(x) = \ln x, G^1(\xi) = G^2(\xi) = 0, \mu(t,x) = \mu_0(t) \) for some càdlàg function \( \mu_0(t) \), \( b \geq \varepsilon > 0 \) for some positive constant \( \varepsilon \), and \( g(s,v) = \frac{1}{2} (\theta_1^2 + \theta_2^2) \). Assume further that \( \mathcal{H}_t = \mathcal{F}_t \) and all parameter processes are deterministic functions. Suppose \((u^*, v^*) \in \mathcal{A} \times \mathcal{A}\) is optimal for Problem 6.10 under the conditions of Theorem 5.3. Then \((u^*, v^*)\) is given by (73) and (72), and the value \( V \) is given by (70).

Next, we concentrate on a special situation without model uncertainty. The stochastic differential game problem 5.2 degenerates to the following anticipating stochastic control problem.

**Problem 6.10.** Select \( u^* \in \mathcal{A}_t \) such that
\[ \bar{V} := \mathcal{F}(u^*) = \sup_{u \in \mathcal{A}_t} \mathcal{F}(u), \]
where \( \mathcal{F}(u) := \mathbb{E}[U(X_{T})] \). We call \( \bar{V} \) the value (or the optimal expected utility) of Problem 6.10.

We still assume that \( U(x) = \ln x, \mathcal{H}_t = \mathcal{F}_t \) and all parameter processes are deterministic functions. Then the terminal value condition of \( X_{T}^{u^*} \) in (73) is replaced by
\[ X_{T}^{u^*} = \frac{X_0}{\Pi^*(0,T)}. \]

By the similar procedure, we obtain the following result without insider information.

**Proposition 6.11.** Assume that \( U(x) = \ln x, G^1(\xi) = G^2(\xi) = 0, \mu(t,x) = \mu_0(t) \) for some càdlàg function \( \mu_0(t) \), \( b \geq \varepsilon > 0 \) for some positive constant \( \varepsilon \), and no model certainty is considered. Assume further that \( \mathcal{H}_t = \mathcal{F}_t \) and all parameter processes are deterministic functions. Suppose \( u^* \in \mathcal{A}_t \) is optimal for Problem 6.10 under the conditions of Theorem 5.4 (with \( \mathcal{A}_t = \{(0,0)\}) \). Then \( u^* \) and \( V \) are given by
\[ \pi^*_i = \frac{t_i}{\sigma_i} + \frac{\rho (\lambda_i - a_i + \rho b_i t_i)}{2(1 - \rho^2) \sigma_i b_i}, \]
\[ \kappa^*_i = \frac{\lambda_i - a_i + \rho b_i t_i}{2(1 - \rho^2) b_i^2}, \]
\[ \bar{V} = \ln X_0 + \int_0^T r_i dt + \frac{1}{4} \int_0^T \left( t_i^2 + \frac{(\lambda_i - a_i + \rho b_i t_i)^2}{(1 - \rho^2) b_i^2} \right) dt. \]
Remark 6.12. By comparing Corollary 6.9 with Proposition 6.11, the insurer should cut both investment and insurance in half once she is ambiguity averse. The reason is that ambiguity aversion makes her strategy more conservative. Moreover, the difference between the optimal expected utility under the worst-case probability and that under the original reference probability delivers aversion to such ambiguity, and can be characterized by $V - \bar{V} = -\frac{1}{2} \int_0^T \left( t^2 + \frac{(\lambda - \alpha + \rho \beta k)^2}{(1 - \rho^2)\beta^2} \right) dt$, which is nonpositive.

6.1.1 A particular case

Next, we give a particular case to derive the closed form of the robust optimal strategy. Assume that $U(x) = \ln x$, the insider information is related to the future value of risk in the insurance market, i.e.,

\[ \mathcal{H}_t = \bigcap_{s > t} (\mathcal{F}_s \vee Y_0) := \bigcap_{s > t} \left( \mathcal{F}_s \vee \int_0^{T_0} \varphi(s') d\tilde{W}_{s'} \right), \quad 0 \leq t \leq T, \tag{84} \]

for some $T_0 > T$, and all the parameter processes are assumed to be deterministic functions. Here, $\varphi_1$ is some deterministic function satisfying $\|\varphi\|_{C,[0,\infty)}^2 := \int_0^T \varphi^2(s') ds' < \infty$ for all $0 \leq s \leq T_0$, and $\|\varphi\|_{[T,T_0]}^2 > 0$.

In this situation, each $\mathcal{H}_t$-adapted process $x_t$, $t \in [0,T]$, has the form $x_t = x_1(t,Y_0,\omega)$ for some function $x_1 : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that $x_1(t,y)$ is $\mathcal{F}_t$-adapted for every $y \in \mathbb{R}$. For simplicity, we write $x$ instead of $x_1$ in the sequel. To solve the anticipating linear BSDE (60), we need to introduce some white noise techniques in Malliavin calculus (see [20, 21, 35, 18]).

Definition 6.13 (Donsker $\delta$ functional). Let $Y : \Omega \to \mathbb{R}$ be a random variable which belongs to the distribution space $(\mathcal{S})^{-1}$ (see [35] for the definition). Then a continuous linear operator $\delta(Y) : \mathbb{R} \to (\mathcal{S})^{-1}$ is called a Donsker $\delta$ functional of $Y$ if it has the property that

\[ \int_{\mathbb{R}} f(y) \delta(Y) dy = f(Y) \]

for all Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that the integral converges in $(\mathcal{S})^{-1}$.

The following lemma gives a sufficient condition for the existence of the Donsker $\delta$ functional. The proof can be found in [18].

Lemma 6.14. Let $Y : \Omega \to \mathbb{R}$ be a Gaussian random variable with mean $\bar{\mu}$ and variance $\bar{\sigma}^2 > 0$. Then its Donsker $\delta$ functional $\delta_{\phi}(Y)$ exists and is uniquely given by

\[ \delta_{\phi}(Y) = \frac{1}{\sqrt{2\pi}\bar{\sigma}^2} \exp \left\{ -\frac{(y - \bar{\mu})^2}{2\bar{\sigma}^2} \right\} \in (\mathcal{S})' \subset (\mathcal{S})^{-1}, \]

where $(\mathcal{S})'$ is the Hida distribution space, and $\odot$ denotes the Wick product. We refer to [35] for relevant definitions. Moreover, we can obtain the explicit expressions of $\delta_{\varphi_1}$ and $\delta_{\varphi_0}$ as the following lemma, which was first proved in [21, Theorem 2.2].

Lemma 6.15 (Enlargement of filtration). Suppose $Y$ is an $\mathcal{F}_{T_0}$-measurable random variable for some $T_0 > T$ and belongs to $(\mathcal{S})'$. The Donsker $\delta$ functional of $Y$ exists and satisfies $\mathbb{E}[\delta(Y)|\mathcal{F}_t] \in L^2(m \times \mathbb{P})$ and $\mathbb{E}[D_1\delta(Y)|\mathcal{F}_t] \in L^2(m \times \mathbb{P})$, where $D_1$ is the (extended) Hida-Malliavin derivative (see [18]), $i = 1, 2$. Assume further that $\mathcal{H}_t = \bigcap_{s > t} (\mathcal{F}_s \vee Y_0)$, and $W$ is an $\mathcal{H}_t$-semimartingale with the decomposition $\mathcal{H}_t = \mathcal{H}_t \oplus \mathcal{H}_t$. Then we have

\[ \phi_1(t) = \left. \frac{\mathbb{E}[D_1\delta(Y)|\mathcal{F}_t]|_{y=Y}}{\mathbb{E}[\delta(Y)|\mathcal{F}_t]|_{y=Y}} \right|_{y=Y}, \]

\[ \phi_2(t) = \left. \frac{\mathbb{E}[D_2\delta(Y)|\mathcal{F}_t]|_{y=Y}}{\mathbb{E}[\delta(Y)|\mathcal{F}_t]|_{y=Y}} \right|_{y=Y}. \]

By Lemma 6.14 and the Lévy theorem, the Donsker $\delta$ functional of $Y_0$ in (84) is given by

\[ \delta_\phi(Y_0) = \frac{1}{\sqrt{2\pi}\phi\|\phi\|_{[0,T_0]}^2} \exp \left\{ -\frac{(y - Y_0)^2}{2\phi\|\phi\|_{[0,T_0]}^2} \right\}, \tag{85} \]
Moreover, when the filtration \( \mathcal{F}_t \) is of the form \( \mathcal{F}_t \), we have by Lemma 6.15 that

\[
\phi_1(t) = \phi_1(t, Y_0) = \frac{Y_0 - \int_0^t \phi_0 \, dW_s}{\| \phi \|^2_{[s, t]}} \rho \phi,
\]

\[
\phi_2(t) = \phi_2(t, Y_0) = \frac{Y_0 - \int_0^t \phi_0 \, dW_s}{\| \phi \|^2_{[s, t]}} \sqrt{1 - \rho^2} \phi.
\]

Substituting (87) into (55) and using the Itô formula we can rewrite the expression of \( E \) and we have

\[
E \left[ \delta_t (Y_0) \big| \mathcal{F}_T \right] = \frac{1}{\sqrt{2\pi \| \phi \|^2_{[s, t]}}} \exp \left\{ - \frac{(y - \int_0^t \phi_0 \, dW_s)^2}{2\| \phi \|^2_{[s, t]}} \right\}, \quad 0 \leq t \leq T.
\]

Moreover, when the filtration \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \) is of the form (84), we have by Lemma 6.15 that

\[
\phi_1(t) = \phi_1(t, Y_0) = \frac{Y_0 - \int_0^t \phi_0 \, dW_s}{\| \phi \|^2_{[s, t]}} \rho \phi,
\]

\[
\phi_2(t) = \phi_2(t, Y_0) = \frac{Y_0 - \int_0^t \phi_0 \, dW_s}{\| \phi \|^2_{[s, t]}} \sqrt{1 - \rho^2} \phi.
\]

Substituting (87) into (55) and using the Itô formula we can rewrite the expression of \( \Pi^\ast (0, t) \) as follows

\[
\Pi^\ast (0, t) = \Pi^\ast (0, t, y) \big|_{y = y_0}
\]

\[
= \exp \left\{ - \int_0^t \phi_1(s, Y_0) \, dW_s - \int_0^t \phi_2(s, Y_0) \, dW_s - \frac{1}{2} \int_0^t \left( \phi_1^2(s, Y_0) + \phi_2^2(s, Y_0) \right) \, ds \right\} \Pi^\ast (0, t)
\]

\[
= \mathbb{E} \left[ \delta_t (Y_0) \big| \mathcal{F}_0 \right] \frac{\Pi^\ast (0, t)}{\mathbb{E} \left[ \delta_t (Y_0) \big| \mathcal{F}_0 \right] \Pi^\ast (0, T) \big|_{y = y_0}.
\]

where

\[
\Pi^\ast (0, t) := \exp \left\{ - \int_0^t \phi(s, dW_s) + \int_0^t \psi(s, dW_s) - \frac{1}{2} \int_0^t \left( \phi^2(s) + \psi^2(s) \right) \, ds \right\}
\]

is an \( \mathcal{F}_t \)-adapted process.

In the case of the logarithmic utility, the fixed point function \( f(y) \) is given by (70). Thus, the terminal value condition of \( X_T^\ast \) in (73) leads to

\[
X_T^\ast = X_T^\ast (y) \big|_{y = y_0} = \mathbb{E} \left[ \frac{x_0}{\Pi^\ast (0, T) \big| \mathcal{F}_0 \big| \mathcal{F}_T} \right] \mathbb{E} \left[ \frac{\delta_t (Y_0) \big| \mathcal{F}_T \big|} {\mathbb{E} \left[ \delta_t (Y_0) \big| \mathcal{F}_T \right] \Pi^\ast (0, T) \big|_{y = y_0}} \right] = c_2^3 (y) \mathbb{E} \left[ \frac{\delta_t (Y_0) \big| \mathcal{F}_T \big|} {\Pi^\ast (0, T) \big|_{y = y_0}} \right],
\]

where \( c_2^3 (y) := \mathbb{E} \left[ \frac{x_0}{\Pi^\ast (0, T) \big| \mathcal{F}_0 \big| \mathcal{F}_T} \right] \) is a Borel measurable function with respect to \( y \).

By Definition 6.13 the BSDE (73) can be rewritten as

\[
\int_R X_T^\ast (y) \delta_t (Y_0) \, dy = \int_R X_T^\ast (y) \delta_t (Y_0) \, dy + \int_0^T \left[ -r X_T^\ast (y) - t z_2^\ast (s, y) + \frac{\lambda u - a_x + \rho b_u}{1 - \rho^2} z_2^\ast (s, y) \right] \, ds \delta_t (Y_0) \, dy
\]

\[
- \int_0^T \int_0^T z_2^\ast (y) \, dW_s (Y_0) \, dy, \quad 0 \leq t \leq T,
\]

where \( W_t = (W_1 (t), W_2 (t))^\ast \). Since \( z_2^\ast (y) \) is \( \mathcal{F}_t \)-adapted for each \( y \), the forward integral in (91) is just the Itô integral. Then (73) holds if and only if we choose \( (X_T^\ast (y), z_2^\ast (y)) \) for each \( y \) as the solution of the following classical linear BSDE with respect to the filtration \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \)

\[
\begin{cases}
\frac{dx_T^\ast (y)}{dr} = -f_L (t, X_T^\ast (y), z_2^\ast (y)) \, dr + z_2^\ast (y) \, dW_t, & 0 \leq t \leq T,

X_T^\ast (y) = c_2^3 (y) \frac{\mathbb{E} \left[ \delta_t (Y_0) \big| \mathcal{F}_T \right]} {\Pi^\ast (0, T) \big|_{y = y_0}},
\end{cases}
\]

where the generator \( f_L : [0, T] \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is given by

\[
f_L (t, x, z) = -r x - t z_1 + \frac{\lambda u - a_x + \rho b_u}{1 - \rho^2} z_2.
\]
By [44, Theorem 4.8], the unique strong solution of (92) is given by

$$X_t^u(y) = E \left[ \Pi^u_t(y) \hat{c}_1^u(y) \sqrt{\frac{\mathbb{E}[\delta(\gamma_0)|\mathcal{F}_T]}{\Pi^u_T}} \left| \mathcal{F}_t \right. \right],$$

where $\Pi^u_t := \Pi^u_t(0, T)$ by the definition of $\Pi^u_T$. By (93) and the initial value condition $X_0^u(y) = X_0$, the Borel measurable function $\hat{c}_1^u(y)$ is given by

$$\hat{c}_1^u(y) = \frac{X_0}{E \sqrt{\Pi^u_0(0) E[\delta(y_0)|\mathcal{F}_T]|\mathcal{F}_T}} = \frac{X_0}{E \left[ \sqrt{\Pi^u_T(0, T)} E[\delta(y_0)|\mathcal{F}_T]|\mathcal{F}_T \right]}.$$

The last equation in (94) is by the definition of $\hat{c}_1^u(y)$. Substituting (94) into (93) we obtain

$$X_t^u(y) = \frac{X_0 E \left[ \sqrt{\Pi^u_t(0, T)} E[\delta(y_0)|\mathcal{F}_T]|\mathcal{F}_T \right]}{E \sqrt{\Pi^u_0(0)}}.$$

By [47, Proposition 3.5.1] and Malliavin calculus, we have

$$z_1^t(y) = D_t^1 X_t^u(y) = \frac{X_0 E D_t^1 \left[ \sqrt{\Pi^u_T(0, T)} E[\delta(y_0)|\mathcal{F}_T]|\mathcal{F}_T \right]}{E \sqrt{\Pi^u_0(0) E[\delta(y_0)|\mathcal{F}_T]|\mathcal{F}_T}} = - \frac{1}{2 E \sqrt{\Pi^u_0(0) E[\delta(y_0)|\mathcal{F}_T]|\mathcal{F}_T}} D_t^1 \Pi^u_t(0, t),$$

and

$$z_2^t(y) = D_t^2 X_t^u(y) = \frac{X_0 E D_t^2 \left[ \sqrt{\Pi^u_T(0, T)} E[\delta(y_0)|\mathcal{F}_T]|\mathcal{F}_T \right]}{E \sqrt{\Pi^u_0(0) E[\delta(y_0)|\mathcal{F}_T]|\mathcal{F}_T}} = \frac{1}{2 E \sqrt{\Pi^u_0(0) E[\delta(y_0)|\mathcal{F}_T]|\mathcal{F}_T}} \lambda_t - a_t + \rho b_t t.$$

Substituting (96) into (59) we obtain the robust optimal investment strategy

$$\pi_t^* = \frac{\lambda_t - a_t + \rho b_t t}{2(1 - \rho^2) \sigma_t},$$

Combining (59) with (97), the robust optimal insurance strategy can also be given as

$$\kappa_t^* = \kappa^*_t(y)|_{y = y_0}$$

$$= \frac{\lambda_t - a_t + \rho b_t t}{2(1 - \rho^2) b_t^2} \left[ \int_0^t \frac{b_s}{\sqrt{\Pi^u_s(0, T)}} D_s^1 \left[ \sqrt{\Pi^u_T(0, T)} E[\delta(y_0)|\mathcal{F}_T]|\mathcal{F}_T \right] \phi_s \right]|_{y = y_0},$$

where

$$\Pi^u_0(0, t) := \exp \left\{ - \int_0^t \frac{b_s}{2} dW_s^1 + \int_0^t \frac{\lambda_t - a_t + \rho b_t t}{2 \sqrt{1 - \rho^2}} dW_s^2 - \frac{1}{8} \int_0^t \left( \frac{\left( \lambda_t - a_t + \rho b_t t \right)^2}{(1 - \rho^2) b_t^2} \right) ds \right\}.$$
Then by the Girsanov theorem (see Theorem A.9), \( W^1_Q(t) := W^1_t + \int_0^t \frac{\lambda - a + \rho b_t}{2(1 - \rho^2)} ds \) and \( W^2_Q(t) := W^2_t - \int_0^t \frac{\lambda - a + \rho b_t}{2(1 - \rho^2)} ds \) are two independent \( \mathcal{F}_t \)-Brownian motions under the new equivalent probability measure \( \mathbb{Q} \) defined by \( d\mathbb{Q} = \Pi_t^\prime(0, T) d\mathbb{P} \). Thus, by the Bayes rule (see [32]), we can rewritten the robust optimal insurance strategy as follows

\[
\kappa^* = \frac{\lambda_t - a_t + \rho b_t}{2(1 - \rho^2)} \mathbb{E}_Q \left[ \mathbb{E}_Q \left[ \mathbb{E}_Q \left[ \left( y - \int_0^T \phi_t d\tilde{W}_t \right) \bigg| \mathcal{F}_T \right] \bigg| \mathcal{F}_t \right] \bigg| \mathcal{F}_0 \right] \right] \phi_t = \frac{\lambda_t - a_t + \rho b_t}{2(1 - \rho^2)} \mathbb{E}_Q \left[ \frac{\mathbb{E}_Q \left[ \mathbb{E}_Q \left[ \left( y - \int_0^T \phi_t d\tilde{W}_t \right) \bigg| \mathcal{F}_T \right] \bigg| \mathcal{F}_t \right] \bigg| \mathcal{F}_0 \right] \right] \phi_t,
\]

(101)

where \( s_t := -\frac{1}{2} \int_0^T \phi_t \frac{\lambda_t - a_t}{b_t} d\mathbb{Q}, \) and \( W_Q(t) := \rho W^1_t + \sqrt{1 - \rho^2} W^2_Q(t), t \in [0, T). \) On the other hand, the conditional \( \mathbb{Q} \) law of \( \int_0^T \phi_t d\tilde{W}_Q(s) \), given \( \mathcal{F}_t \), is normal with mean \( \int_0^T \phi_t d\tilde{W}_Q(s) \) and variance \( \left\| \phi_t \right\|^2_{\mathcal{F}_t} \) due to the Markov property of Itô diffusion processes (see [32]). Thus (101) leads to

\[
\kappa^* = \frac{\lambda_t - a_t + \rho b_t}{2(1 - \rho^2)} \mathbb{E}_Q \left[ \frac{\mathbb{E}_Q \left[ \mathbb{E}_Q \left[ \left( y - \int_0^T \phi_t d\tilde{W}_t \right) \bigg| \mathcal{F}_T \right] \bigg| \mathcal{F}_t \right] \bigg| \mathcal{F}_0 \right] \right] \phi_t = \frac{\lambda_t - a_t + \rho b_t}{2(1 - \rho^2)} \mathbb{E}_Q \left[ \frac{\mathbb{E}_Q \left[ \mathbb{E}_Q \left[ \left( y - \int_0^T \phi_t d\tilde{W}_t \right) \bigg| \mathcal{F}_T \right] \bigg| \mathcal{F}_t \right] \bigg| \mathcal{F}_0 \right] \right] \phi_t.
\]

(102)

By the equations (31) and (32) in Corollary 4.8, we can also obtain

\[
\theta_1^*(t) = -\frac{1}{2} \left[ \frac{Y_0 - \int_0^T \phi_t d\tilde{W}_t - \frac{1}{2} \int_0^T \phi_t \frac{\lambda - a}{b} ds}{\left\| \phi_t \right\|^2_{\mathcal{F}_T} + \left\| \phi_t \right\|^2_{\mathcal{F}_0}} - Y_0 - \int_0^T \phi_t d\tilde{W}_t \right] \rho \phi_t,
\]

(103)

\[
\theta_2^*(t) = \frac{\lambda_t - a_t + \rho b_t}{2\sqrt{1 - \rho^2}} \left[ \frac{\mathbb{E}_Q \left[ \mathbb{E}_Q \left[ \left( y - \int_0^T \phi_t d\tilde{W}_t \right) \bigg| \mathcal{F}_T \right] \bigg| \mathcal{F}_t \right] \bigg| \mathcal{F}_0 \right] \right] \phi_t = \frac{\lambda_t - a_t + \rho b_t}{2\sqrt{1 - \rho^2}} \mathbb{E}_Q \left[ \frac{\mathbb{E}_Q \left[ \mathbb{E}_Q \left[ \left( y - \int_0^T \phi_t d\tilde{W}_t \right) \bigg| \mathcal{F}_T \right] \bigg| \mathcal{F}_t \right] \bigg| \mathcal{F}_0 \right] \right] \phi_t.
\]

To sum up, we give the following theorem.

**Theorem 6.16.** Assume that \( U(x) = \ln x, G^1(dz) = G^2(dz) = 0, \mu(t, x) = \mu_0(t) \) for some càdlàg function \( \mu_0(t), b \geq 0 > 0 \) for some positive constant \( b \), and \( g(s, v) = \frac{1}{2} (\theta_1^* + \theta_2^*) \). Assume further that \( \{ \mathcal{F}_t^\prime \}_{0 \leq t \leq T} \) is given by [32] and all parameter processes are deterministic functions. Suppose \( (u^*, v^*) \in \mathcal{A}_1^* \times \mathcal{A}_2^* \) is optimal for Problem 5.3 under the conditions of Theorem 5.4. Then \( (u^*, v^*) \) is given by (28), (102), and (103).

Moreover, if \( \phi = 1, u^* \) is given by

\[
\xi^* = \frac{\lambda_t - a_t + \rho b_t}{2(1 - \rho^2)} \mathbb{E}_Q \left[ \mathbb{E}_Q \left[ \left( y - \int_0^T \phi_t d\tilde{W}_t \right) \bigg| \mathcal{F}_T \right] \bigg| \mathcal{F}_t \right] \bigg| \mathcal{F}_0 \right] \right] \phi_t
\]

(104)

\[
\kappa^* = \frac{\lambda_t - a_t + \rho b_t}{2(1 - \rho^2)} \mathbb{E}_Q \left[ \mathbb{E}_Q \left[ \left( y - \int_0^T \phi_t d\tilde{W}_t \right) \bigg| \mathcal{F}_T \right] \bigg| \mathcal{F}_t \right] \bigg| \mathcal{F}_0 \right] \right] \phi_t
\]

(105)

and \( v^* \) is given by

\[
\theta_1^*(t) = \frac{1}{2} \left[ \frac{W_0 - W_t - \frac{1}{2} \int_0^T \lambda - a ds}{T_0 - t + T_0 - T} - \frac{W_0 - W_t}{T_0 - t} \right]
\]

(106)

\[
\theta_2^*(t) = \frac{\lambda_t - a_t + \rho b_t}{2\sqrt{1 - \rho^2}} + \left[ \frac{W_0 - W_t - \frac{1}{2} \int_0^T \lambda - a ds}{T_0 - t + T_0 - T} - \frac{W_0 - W_t}{T_0 - t} \right]
\]
By (104), we have

\[
\mathbb{E}[\sqrt{\Pi^*(0,T)}|\mathcal{F}_0] = \left( \mathbb{E} \left[ \sqrt{\Pi^*_0(0,T)} \frac{\mathbb{E}[\delta(Y_0)|\mathcal{F}_T]}{\mathbb{E}[\delta(Y_0)|\mathcal{F}_0]} \right] \right)_{y=Y_0}.
\]

Substituting (106) into (74), we have by Girsanov theorem and tedious calculation that

\[
V = \ln X_0 - 2\mathbb{E} \left[ \ln \sqrt{\Pi^*_0(0,T)} \frac{\mathbb{E}[\delta(Y_0)|\mathcal{F}_T]}{\mathbb{E}[\delta(Y_0)|\mathcal{F}_0]} \right]_{y=Y_0}
= \ln X_0 + \int_0^T r_t dt + \frac{1}{4} \int_0^T \left( \frac{\lambda_t - \alpha_t + \rho b_t u_t}{1 - \rho^2 b_t^2} \right)^2 dt - 2\mathbb{E} \left[ \ln \sqrt{\mathbb{E}[\delta(Y_0)|\mathcal{F}_T]} \frac{\mathbb{E}[\delta(Y_0)|\mathcal{F}_0]}{\mathbb{E}[\delta(Y_0)|\mathcal{F}_0]} \right]_{y=Y_0}
= \ln X_0 + \int_0^T r_t dt + \frac{1}{4} \int_0^T \left( \frac{\lambda_t - \alpha_t + \rho b_t u_t}{1 - \rho^2 b_t^2} \right)^2 dt + \frac{1}{2} \ln \left( 1 - \frac{T^2}{(2T_0 - T)^2} \right) \cdot
\]

Thus we obtain the following corollary.

**Corollary 6.17.** Assume that \( U(x) = \ln x, \ G^1(dx) = G^2(dx) = 0, \mu(t,x) = \mu_0(t), \ b \geq \varepsilon > 0 \) for some càglàd function \( \mu_0(t), \ b \geq \varepsilon > 0 \) for some positive constant \( \varepsilon \), and \( g(x,v) = \frac{1}{2} (\theta_x + \theta_v^2) \). Assume further that \( \{\mathcal{F}_t\}_{0 \leq t < T} \) is given by (54) with \( \phi = 1 \) and all parameter processes are deterministic functions. Suppose \( (u^*, v^*) \in \mathcal{A}^1 \times \mathcal{A}^2 \) is optimal for Problem 3.3 under the conditions of Theorem 3.4 Then \( (u^*, v^*) \) is given by (104) and (105), and the value \( V \) is given by (107).

**Remark 6.18.** By comparing Corollary 6.17 with Corollary 6.9 we can see that, if the insurer captures insider information in the insurance market and \( \mathcal{W}_0 > \mathcal{W}_t > \frac{1}{2} \int_0^T \frac{\lambda_t - \alpha_t}{b_t} ds \) at time \( t \), she should reduce the liability ratio by \( \frac{\mathcal{W}_0 - \mathcal{W}_t}{b(t_0 - t + T_0 - T)} \) to maximize her utility under model uncertainty. Given \( \frac{\mathcal{W}_0 - \mathcal{W}_t}{b(t_0 - t + T_0 - T)} \), we can also see that the closer the future time \( T_0 \) of the insider information is, the more she should reduce her liability ratio. However, the robust optimal investment strategy is not affected since she has no insider information about the risky asset. Moreover, her optimal expected utility under the worst-case probability is gained by \( \Delta V = \frac{1}{2} \ln \left( 1 - \frac{T^2}{(2T_0 - T)^2} \right) \cdot \frac{(\int_0^T \frac{\lambda_t - \alpha_t}{b_t} ds)^2}{2(1 - \rho^2)} > 0 \), and \( \Delta V \) is greater as \( T_0 \) is closer (i.e., the insurer has ‘better’ insider information). On the other hand, by comparing Corollary 6.17 with Proposition 6.11 if \( T_0^* \) is the solution of \( V = \hat{V} \), then the optimal expected utility of the insurer under the worst-case probability with insider information equals that under the original reference probability without insider information. We call \( T_0^* \) the critical future time, which indicates that if the insurer who is ambiguity averse wants to exceed the optimal expected utility under the original reference probability, the cost is that she needs the insider information about the value of risk at time \( T_0 \) such that \( T < T_0 \). In particular, if all parameter processes are constant, \( \rho = 0 \), and \( T = 1 \), then \( T_0^* \) satisfies

\[
t^2 + c^2 - \frac{c^2 + 2}{2T_0 - 1} = -2 \ln \left( 1 - \frac{1}{1 - \frac{1}{(2T_0 - 1)^2}} \right),
\]

where \( c = \frac{\lambda - \alpha}{b} \). By mathematical analysis, equation (111) has a unique solution \( T_0^* \). Moreover, the critical future time \( T_0^* \) gets closer as \( t \) increases. The reason is that the increase of \( \mu \) improves the optimal expected utility of the insurer under the original reference probability more considerably. Thus she requires ‘better’ extra insider information to improve her optimal expected utility under the worst-case probability such that \( V = \hat{V} \).

**Remark 6.19.** By similar procedure, we can also obtain the robust optimal strategy when the insurer has insider information about the future value of the risky asset. For instance, if \( \mathcal{H}_t = \bigcap_{s \leq t} \left( \mathcal{F}_s \cup \mathcal{W}_{t_0}^1 \right) \), the robust optimal strategy is given by

\[
\pi^* = \frac{\lambda_t - \alpha_t + \rho b_t u_t}{2(1 - \rho^2) b_t^2}, \quad \kappa^* = \frac{\lambda_t - \alpha_t + \rho b_t u_t}{2(1 - \rho^2) b_t^2},
\]

where \( \lambda_t - \alpha_t + \rho b_t u_t \) is the solution of \( \mathcal{W}_t^1 = \mathcal{W}_0^1 + \frac{1}{2} \int_0^T \lambda_t - \alpha_t + \rho b_t u_t ds \).
which indicates that the insurer should increase the proportion of her total wealth invested in the risky asset by
\[
\frac{W_t^0 - W_t^0 + \frac{1}{2} t^T \sigma^2_t}{\delta_t(T_0 + t - T)}
\]
to maximize her utility under model uncertainty if she knows the future value of the risky asset and \( W_t^1 > W_t^1 \). Given \( W_t^1 - W_t^1 = x > -\frac{1}{2} t^T \sigma^2_t \), we can also see that the closer the future time \( T_0 \) is, the more she should increase her proportion with respect to the risky asset. However, the robust optimal insurance strategy is not affected since she has no information about the risk in the insurance market. Moreover, the value \( V \) can be given by

\[
V = \ln X_0 + \int_0^T r_t \, dt + \frac{1}{4} \int_0^T \left( t^2 + \frac{(\lambda_t - a_t + \rho \beta_t t)^2}{(1 - \rho^2) \sigma^2_t} \right) \, dt + \frac{1}{2} \ln \left( 1 - \frac{T^2}{(2T_0 - T)^2} \right)^{-1} + \frac{T}{2(2T_0 - T)}
\]

(110)

Thus the optimal expected utility under the worst-case probability is gained by \( \Delta V = \frac{1}{2} \ln \left( 1 - \frac{T^2}{(2T_0 - T)^2} \right)^{-1} + \frac{T}{2(2T_0 - T)} + \frac{1}{4(2T_0 - T)} \left( \int_0^T t^2 \, dt \right)^2 > 0 \), and \( \Delta V \) is greater as \( T_0 \) is closer (i.e., the insurer has ‘better’ insider information).

By comparing (110) with (109) we find that, if \( t_t = \frac{\lambda_t - a_t}{b_t} \), the insurer could derive the same optimal expected utility under the worst-case probability when she owns the insider information about the financial market or the insurance market. On the other hand, by Proposition 6.11, the critical future time \( T_0^* \) can be defined similarly. In particular, if all parameter processes are constant, \( \rho = 0 \), and \( T = 1 \), then \( T_0^* \) satisfies

\[
t^2 + c^2 - \frac{t^2 + 2}{2T_0^*} = -2 \ln \left( 1 - \frac{1}{(2T_0^* - 1)^2} \right),
\]

(111)

where \( c = \frac{\lambda - a}{b} \). The critical future time \( T_0^* \) gets closer as \( c \) increases since the increase of \( c \) improves the optimal expected utility of the insurer under the original reference probability more considerably. Thus she requires ‘better’ extra insider information to improve her optimal expected utility under the worst-case probability such that \( V = \bar{V} \).

Next, we concentrate on the special situation without model uncertainty, i.e., Problem 6.10. Then the terminal value condition of \( X^\pi_0 \) in (92) is replaced by

\[
X^\pi_T = \frac{X_0}{\Pi^\pi(0,T)} = \frac{X_0 \mathbb{E}[\delta_0(Y_0) | \mathcal{F}_T]}{\mathbb{E}[\delta_0(Y_0) | \mathcal{F}_0 \Pi^\pi(0,T)]} \bigg|_{y = Y_0}.
\]

(112)

By similar procedure, we obtain the following proposition.

*Proposition 6.20.* Assume that \( U(x) = \ln x \), \( G^1(\,dz\,) = G^2(\,dz\,) = 0 \), \( \mu(t, x) = \mu_0(t) \) for some càglârd function \( \mu_0(t) \), \( b \geq c > 0 \) for some positive constant \( c \), and no model certainty is considered. Assume further that \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) is given by (54) with \( \varphi = 1 \) and all parameter processes are deterministic functions. Suppose \( u^* \in \mathcal{A}^1_\nu \) is optimal for Problem 6.10 under the conditions of Theorem 5.4. Then \( u^* \) and \( \bar{V} \) are given by

\[
\pi^*_t = \frac{b_t}{\sigma_t} + \frac{\rho(\lambda_t - a_t + \rho \beta_t t)}{(1 - \rho^2) \sigma_t b_t},
\]

\[
\kappa^*_t = \frac{\lambda_t - a_t + \rho \beta_t t}{(1 - \rho^2) \sigma_t b_t} - \frac{W_t - \bar{W}_t}{b_t(T_0 - t)},
\]

(113)

\[
V = \ln X_0 + \int_0^T r_t \, dt + \frac{1}{2} \int_0^T \left( t^2 + \frac{(\lambda_t - a_t + \rho \beta_t t)^2}{(1 - \rho^2) \sigma_t^4} \right) \, dt + \frac{1}{2} \ln \left( 1 - \frac{T^2}{(2T_0 - T)^2} \right)^{-1} + \frac{T}{2(2T_0 - T)}.
\]

Remark 6.21. By comparing Corollary 6.17 with Proposition 6.20 when the insurer owns the insider information about the future value of risk in the insurance market, the difference between her optimal expected utility under the worst-case probability and that under the original reference probability is given by

\[
V - \bar{V} = \left[ \frac{1}{2} \ln \left( 1 - \frac{T^2}{(2T_0 - T)^2} \right)^{-1} + \frac{T}{2(2T_0 - T)} - \frac{1}{2} \ln \frac{T_0}{T_0^* - T} \right] + \left[ \frac{1}{4(2T_0 - T)} \left( \int_0^T \frac{\lambda_t - a_t}{b_t} \, dt \right)^2 - \frac{1}{4} \int_0^T \left( t^2 + \frac{(\lambda_t - a_t + \rho \beta_t t)^2}{(1 - \rho^2) \sigma_t^4} \right) \, dt \right] \leq 0.
\]

(114)
due to the Hölder inequality. On the other hand, suppose there is no model certainty, i.e., the insurer is not ambiguity averse or believes that her model is accurate. By comparing Proposition 6.20 with Proposition 6.11, we see that, if the insurer captures insider information in the insurance market and \( \bar{W}_{t_0} - \bar{W}_t \) at time \( t \), she should reduce the liability ratio by \( \frac{\bar{W}_{t_0} - \bar{W}_t}{\bar{G}(t_0 - t)} \) to maximize her utility. Given \( \bar{W}_{t_0} - \bar{W}_t = x > 0 \), we can also see that the closer the future time \( T_0 \) is, the more she should reduce her liability ratio. However, the optimal investment strategy is not affected. Moreover, her optimal expected utility is gained by

\[
\text{Remark 6.22.} \quad \text{By similar procedure, we can also obtain the optimal strategy without model uncertainty when the insurer owns the insider information about the future value of the risky asset. For instance, if } \mathcal{H}_t^i = \bigcap_{t > T} \left( \mathcal{F}_s \vee W_{t_0}^i \right), \text{ the optimal strategy is given by}
\]

\[
\pi^*_t = \frac{\bar{t}_t}{\bar{\sigma}_t} + \rho \frac{(\bar{\lambda}_t - \alpha_t + \rho b_t \bar{t}_t)}{(1 - \rho^2) \bar{\sigma}_t} \frac{W_{t_0}^i - W_t^i}{\bar{\sigma}_t (T_0 - t)}
\]

\[
\delta^*_t = \frac{\bar{\lambda}_t - \alpha_t + \rho b_t \bar{t}_t}{(1 - \rho^2) \bar{\sigma}_t^2}.
\]

Moreover, the value \( V \) can be given by

\[
\tilde{V} = \ln X_0 + \int_0^T r_t \, dt + \frac{1}{2} \int_0^T \left( \overline{\theta}_t^2 + \frac{(\overline{\lambda}_t - \alpha_t + \rho b_t \overline{t}_t)^2}{(1 - \rho^2) \overline{\sigma}_t^2} \right) \, dt + \frac{1}{2} \ln \frac{T_0}{T_0 - T},
\]

which is consistent with the fact that the difference between the optimal expected utility of the insurer under the worst-case probability and that under the original reference probability is nonpositive when she owns the insider information \( \mathcal{H}_t^i \) by Remark 6.19. On the other hand, suppose there is no model uncertainty. By Proposition 6.11, the insurer should increase the proportion of her total wealth invested in the risky asset by \( \frac{\bar{W}_{t_0} - \bar{W}_t^i}{\bar{\sigma}_t (T_0 - t)} \) if she knows the future value of the risky asset and \( \bar{W}_{t_0}^i > \bar{W}_t^i \) at time \( t \). Given \( \bar{W}_{t_0}^i - \bar{W}_t^i = x > 0 \), we can also see that the closer the future time \( T_0 \) is, the more she should increase her proportion with respect to the risky asset. However, the optimal insurance strategy is not affected. Moreover, her optimal expected utility is gained by \( \Delta \tilde{V} = \frac{1}{2} \ln \frac{T_0}{T_0 - T} > 0 \), and \( \Delta \tilde{V} \) is greater as \( T_0 \) is closer (i.e., she has ‘better’ insider information). By comparing (116) with (113), we find that, the insurer could derive the same optimal expected utility when she owns the insider information about the financial market or the insurance market.

### 6.2 With Poisson jumps

Now we focus on the situation where Possion jumps might happen on the insurance market. More specifically, we suppose that \( G^j(dz) = 0, G^2(dz) = \tilde{\lambda} \delta_1(dz), b = \tilde{\theta}_2 = \tilde{q}_2 = 0, \gamma_2(t, z) = \gamma_2(t) \), and \( q_{24}(t, z) = q_{24}(t) \), where \( \tilde{\lambda} > 0 \), and \( \delta_1 \) is the unit point mass at 1. Assume that the mean rate of return \( \mu(t, x) = \mu_0(t) \) for some \( G^1 \)-adapted cáglad process \( \mu_0(t) \). Put \( t_0 = \frac{\mu_0(t) - \gamma_2}{\sigma}, \tilde{\phi}_1(t) = t_0 + \phi_1(t) \), and \( \tilde{\phi}_4(t) = \frac{\lambda_t - \alpha_t}{G^j_2(t, dz) - G^j_1(dz)} \frac{\int_{\tilde{\phi}_1(t) G^j_2(t, dz) - G^j_1(dz)}}{\int_{\tilde{\phi}_1(t) G^j_2(t, dz) - G^j_1(dz)}}.

Assume further the penalty function \( g \) verifies

\[
\int_0^T |\tilde{\phi}_1(t)|^2 \, dt + \int_0^T \int_{R_0} \left( |\ln(1 + \tilde{\phi}_4(t))| + |\tilde{\phi}_4(t)| \right) G^j_2(t, dz) \, dt < \infty, \quad k = 1, 2.
\]

by the Girsanov theorem. Then we have

\[
g(s, v) = \frac{1}{2} \theta_2^2 + \int_{R_0} [(1 + \theta_4) \ln(1 + \theta_4) - \theta_4] G^j_2(s, dx).
\]

We make the following assumption before our procedure.

### Assumption 6.23

Suppose the coefficients satisfy the following integrability

\[
\int_0^T |\tilde{\phi}_1(t)|^2 \, dt + \int_0^T \int_{R_0} |\tilde{\phi}_4(t)|^2 G^j_2(t, dz) \, dt < \infty.
\]

Moreover, \( \tilde{\phi}_4(t) > -1 \) is \( \mathcal{H}_t^i \)-predictable.
By the Hamiltonian system (36)-(37) in Theorem 5.4 and the similar procedure in Section 6.1, we have
\[
\ln e^{\frac{\varepsilon I}{T}} + U(X^I_T) = c_2^*,
\]
(120)
\[
e^{\frac{\varepsilon I}{T}} = E \left[ \left( E \left[ e^{-U(X^I_T)} \left| \mathcal{H}_0 \right. \right] e^{U(X^I_T)} \right)^{-1} | \mathcal{H}_I \right],
\]
\[
e^{\frac{\varepsilon I}{T}} = E \left[ \left( E \left[ e^{-U(X^I_T)} \left| \mathcal{H}_0 \right. \right] e^{U(X^I_T)} \right)^{-1} \right],
\]
where \( c_2^* = p_2^*(0) \) is an \( \mathcal{H}_0 \)-measurable random variable. We also have
(121)
\[
X^I_T = I \left( \frac{c_1^* \Pi_T(0, T)}{e^{\frac{\varepsilon I}{T}}} \right) = \tilde{I} \left( c_3^* \Pi_T(0, T) \right),
\]
where \( c_1^* = p_1^*(0) \) and \( c_3^* = c_1^* E \left[ e^{-U(X^I_T)} \left| \mathcal{H}_0 \right. \right] \) are all \( \mathcal{H}_0 \)-measurable random variables, and \( \Pi_T(0, T) \) is defined by
\[
\Pi_T(0, t) := \exp \left\{ - \int_0^T r_s ds - \int_0^T \tilde{\phi}_s(s) dW_{\mathcal{H}_T}(s) - \frac{1}{2} \int_0^T \tilde{\phi}_s(s)^2 ds + \int_0^T \int_{\mathbb{R}_0} \ln(1 + \tilde{\phi}_s(s)) N^2_{\mathcal{H}_T}(ds, dz) \right\}
\]
(122)
\[+ \int_0^T \int_{\mathbb{R}_0} (\ln(1 + \tilde{\phi}_s(s)) - \tilde{\phi}_s(s)) G^2_{\mathcal{H}_T}(ds, dz) ds \right\}.
\]
Put \( z^*_I = (z^*_I(t), z^*_I(t)) = (\sigma X_{m_1 -}, -\kappa_3^* t X_{m_1 -}^{2}) \). Then we have
(123)
\[
\pi^*_I = \frac{z^*_I(t)}{\sigma X_{m_1 -}^{2}},
\]
\[
\kappa^*_3 = -\frac{z^*_I(t)}{\gamma(t) X_{m_1 -}^{2}}.
\]
Then SDE (33) leads to the following linear BSDE
(124)
\[
\begin{cases}
\frac{dX^I_t}{dt} = -f_{1,1}(t, X^I_t, z^*_I(t), \omega) \frac{dH}{dt} + z^*_I(t) dW_{\mathcal{H}_T}(t) + \int_{\mathbb{R}_0} z^*_I(t) N^2_{\mathcal{H}_T}(ds, dz), \quad 0 \leq t \leq T, \\
X^I_T = I \left( c_3^* \Pi_T(0, T) \right),
\end{cases}
\]
where the generator \( f_{1,1}: [0, T] \times \mathbb{R} \times \mathbb{R}^2 \times \Omega \rightarrow \mathbb{R} \) is given by
\[
f_{1,1}(t, x, z, \omega) = -r_x \tilde{\phi}_1(t) z_1 + \int_{\mathbb{R}_0} \tilde{\phi}_4(t) z_4 G^2_{\mathcal{H}_T}(t, dz).
\]
Moreover, when the following condition holds,
(125)
\[
\mathbb{E} \left[ \left( \int_0^T \Pi_T(0, t) \left( z^*_I(t) - \tilde{\phi}_1(t) X^I_t \right)^2 dt \right)^{\frac{1}{2}} \right] < \infty,
\]
we have
(126)
\[
X^I_t = E\left[ \Pi_T(0, T) I(c_3^* \Pi_T(0, T)) | \mathcal{H}_I \right],
\]
where \( \Pi_T(0, T) := \Pi_T(0, T) / \Pi_T(0, t) \). The \( \mathcal{H}_0 \)-random variable \( c_3^* \) can be determined by
(127)
\[
X_0 = E\left[ \Pi_T(0, T) I(c_3^* \Pi_T(0, T)) | \mathcal{H}_0 \right].
\]
Substituting (121) into (120) we obtain
(128)
\[
e^{\frac{\varepsilon I}{T}} = E \left[ \left( E \left[ e^{-U(I(c_3^* \Pi_T(0, T)))} \left| \mathcal{H}_0 \right. \right] e^{U(I(c_3^* \Pi_T(0, T)))} \right)^{-1} | \mathcal{H}_I \right],
\]
\[
e^{\frac{\varepsilon I}{T}} = E \left[ \left( E \left[ e^{-U(I(c_3^* \Pi_T(0, T)))} \left| \mathcal{H}_0 \right. \right] e^{U(I(c_3^* \Pi_T(0, T)))} \right)^{-1} \right],
\]
To sum up, we have the following result for the robust optimal strategy with jumps.
Theorem 6.24. Assume that \( G^1 (dz) = 0, G^2 (dz) = \lambda \delta_1 (dz), b = \theta_2 = q_{12} = q_{22} = 0, \gamma_2 (t, z) = \gamma_2 (t), q_{24} (t, z) = q_{24} (t), \mu (t, x) = \mu_0 (t) \) for some \( \mathcal{F}^3_t \)-adapted càdlàg process \( \mu_0 (t) \), and \( g(s, v) \) is given by (118). Suppose \( (u^*, v^*) \in \mathcal{A}^1_t \times \mathcal{A}^2_t \) is optimal for Problem 3.3 under the conditions of Theorem 5.4 and Assumption 6.23 and the integrability condition (123) hold. Then \( u^*, v^*, X^u_t, \) and \( e_t^u \) are given by (123), (23), (26), (126), and (128), respectively, where \( \Pi^t \) is given by (122), \( c_t^3 \) is determined by (122), and \( (X^u_t, z^1_t, z^2_t) \) solves the linear BSDE (124).

Remark 6.25. Similar to Theorem 6.6, the existence and uniqueness of the linear BSDE (124) hold under some mild conditions. We refer to [44, 50, 51, 52, 53] for more details.

When the utility function is of the logarithmic form, i.e., \( U(x) = \ln x \), the fixed point function is given by (10), i.e.,

\[
\overline{I}(y) = \frac{1}{\sqrt{y}}, \quad y > 0.
\]

Combining (126) with (127), we have

\[
X^u_t = \frac{X_0 E \left[ \sqrt{\Pi^t (0, T)} | \mathcal{H}_0 \right]}{E \left[ \sqrt{\Pi^t (0, T)} | \mathcal{H}_0 \right]}.
\]

and

\[
X^{\overline{\Gamma}}_t = \frac{X_0}{E \left[ \sqrt{\Pi^t (0, T)} | \mathcal{H}_0 \right]} \sqrt{\Pi^t (0, T)}
\]

Thus the BSDE (124) can be rewritten as

\[
\begin{aligned}
&\{ \frac{dX^u_t}{\sqrt{\Pi^t (0, T)} | \mathcal{H}_0}} \sqrt{\Pi^t (0, T)} \} = -f_{LJ} (t, x^u_t, z^1_t, \omega) dt + z^1_t \left( t \right) dW^1_t (t) + \int_{\mathbb{R}^0} z^2_u (t) d\mathcal{N}_t (dr, dz), \quad 0 \leq t \leq T, \\
&X^u_0 = X_0
\end{aligned}
\]

where \( f_{LJ} \) is given by

\[
f_{LJ} (t, x, z, \omega) = -r_t x - \overline{\phi}_1 (t) z_1 + \int_{\mathbb{R}^0} \overline{\phi}_4 (t) z_4 G^2 (t, dz).
\]

We can also calculate the value of Problem 6.3 as follows

\[
V = \ln X_0 - 2E \left[ \ln E \left[ \sqrt{\Pi^t (0, T)} | \mathcal{H}_0 \right] \right].
\]

Corollary 6.26. Assume that \( U(x) = \ln x, G^1 (dz) = 0, G^2 (dz) = \lambda \delta_1 (dz), b = \theta_2 = q_{12} = q_{22} = 0, \gamma_2 (t, z) = \gamma_2 (t), q_{24} (t, z) = q_{24} (t), \mu (t, x) = \mu_0 (t) \) for some \( \mathcal{F}^3_t \)-adapted càdlàg process \( \mu_0 (t) \), and \( g(s, v) \) is given by (118). Suppose \( (u^*, v^*) \in \mathcal{A}^1_t \times \mathcal{A}^2_t \) is optimal for Problem 3.3 under the conditions of Theorem 5.4 and Assumption 6.23 and the integrability condition (123) hold. Then \( (u^*, v^*) \) and \( V \) are given by (123), (23), (26), and (133), respectively, where \( \Pi^t \) is given by (122), and \( (X^u_t, z^1_t, z^2_t) \) solves the linear BSDE (124).

When the insurer has no insider information, i.e., \( \mathcal{H}_t = \mathcal{F}_t \), we have \( \overline{\phi}_1 (t) = \overline{\phi}_4 (t) = \frac{\lambda_t - a_t}{\lambda_2 (t)} \). Assume further that all parameter processes are deterministic functions. Similar by procedure in Section 6.1, we have the following corollary.

Corollary 6.27. Assume that \( U(x) = \ln x, U(x) = \ln x, G^1 (dz) = 0, G^2 (dz) = \lambda \delta_1 (dz), b = \theta_2 = q_{12} = q_{22} = 0, \gamma_2 (t, z) = \gamma_2 (t), q_{24} (t, z) = q_{24} (t), \mu (t, x) = \mu_0 (t) \) for some càdlàg function \( \mu_0 (t) \), and \( g(s, v) \) is given by (118). Assume further that \( \mathcal{H}_t = \mathcal{F}_t \), and all parameter processes are deterministic functions. Suppose \( (u^*, v^*) \in \mathcal{A}^1_t \times \mathcal{A}^2_t \) is optimal for Problem 3.3 under the conditions of Theorem 5.4. Then \( (u^*, v^*) \) is given by

\[
\begin{aligned}
\pi_t^* &= \frac{\gamma^2_t}{2 \sigma^2_t}, \\
\kappa_t^* &= \frac{\lambda_t - a_t}{(\lambda_t - a_t) \gamma^2_t + \lambda_2 \gamma^2_t (t) + \lambda_2 \gamma^2_t (t) + \lambda_2 \gamma^2_t (t)}, \\
\theta^*_t (t) &= -\frac{\gamma^2_t}{2}, \\
\theta^*_t (t) &= \frac{\lambda_t - a_t}{\lambda_2 \gamma^2_t (t) (1 - \kappa_t^2 \gamma^2_t (t) - \kappa_t^2 \gamma^2_t (t))}.
\end{aligned}
\]
Remark 6.28. Different from the continuous case in Section 6.1, the value $V$ of Problem 5.3 is hard to be calculated analytically by using the Girsanov theorem directly when jumps are considered.

When there is no model uncertainty, we can also obtain the following proposition.

**Proposition 6.29.** Assume that $U(x) = \ln x$, $U(x) = \ln x$, $G^1(\{d\varepsilon\}) = 0$, $G^2(\{d\varepsilon\}) = \tilde{\lambda}_1(\varepsilon)$, $b = \theta_2 = 42 = 42 = 0$, $\gamma_1(t, z) = \gamma_2(t, z)$, $q_{24}(t, z) = q_{24}(t, z)$, $\mu(t, x) = \mu_0(t)$ for some càglàd function $\mu_0(t)$, $g(x, \nu)$ is given by (116), and no model uncertainty is considered. Assume further that $\mathcal{H} = \mathcal{F}_t$ and all parameter processes are deterministic functions. Suppose $u^* \in \mathcal{A}'_1$ is optimal for Problem 6.10 under the conditions of Theorem 3.3 (with $\mathcal{A}'_2 = \{0, 0\}$). Then $u^*$ is given by

$$\pi^*_t = \frac{t}{\gamma_t},$$

$$\kappa^*_t = \frac{\lambda_t - a_t}{(\lambda_t - a_t) \gamma_t(t) + \lambda \gamma^2_t(t)},$$

### 6.2.1 A particular case

Next, we give a particular case. Assume that $U(x) = \ln x$, the insider information filtration is given by

$$\mathcal{H}_t = \bigcap_{s \geq t} (\mathcal{F}_s \vee \mathcal{Y}_0) := \bigcap_{s \geq t} \left( \mathcal{F}_s \vee \int_0^{T_0} \phi(s')dW^1_t \right), \quad 0 \leq t \leq T,$$

for some $T_0 > T$, and all the parameter processes are assumed to be deterministic functions. Here, $\phi$ is some deterministic function satisfying $\|\phi\|^2_{\mathcal{F}_t} := \int_0^T \phi^2(s')ds' < \infty$ for all $0 \leq t \leq T_0$, and $\|\phi\|^2_{\mathcal{F}_t} < \infty$.

By the Donsker $\delta$ functional $\delta_t(\mathcal{Y}_0)$ and similar procedure in Section 6.1.1, we have

$$\mathbb{E}[\delta_t(\mathcal{Y}_0) \mid \mathcal{F}_t] = \frac{1}{\sqrt{2\pi\|\phi\|^2_{\mathcal{F}_t}}} \exp \left\{ -\frac{(y - \int_0^T \phi_0 dW^1_t)^2}{2\|\phi\|^2_{\mathcal{F}_t}} \right\}, \quad 0 \leq t \leq T,$$

and

$$\phi_1(t) = \phi_1(t, y)\big|_{y = \mathcal{Y}_0} = \frac{y - \int_0^T \phi_0 dW^1_t}{\|\phi\|^2_{\mathcal{F}_t}} \big|_{y = \mathcal{Y}_0}$$

$$G^2_\mathcal{F}(t, d\varepsilon) = G^2(\{d\varepsilon\}) = \lambda \tilde{\delta}_1(\{d\varepsilon\}).$$

Substituting (138) into (122) and using the Itô formula we can rewrite the expression of $\Pi^*_j(0, t)$ as follows

$$\Pi^*_j(0, t) = \Pi^*_j(0, t, y)\big|_{y = \mathcal{Y}_0}$$

$$= \exp \left\{ -\int_0^t \phi_1(s, \mathcal{Y}_0) dW^1_t + \frac{1}{2} \int_0^t \phi_1^2(s, \mathcal{Y}_0) dy \right\} \Pi^*_j(0, t)$$

$$= \mathbb{E}[\delta_t(\mathcal{Y}_0) \mid \mathcal{F}_t] \bigg|_{y = \mathcal{Y}_0},$$

where

$$\Pi^*_j(0, t) := \exp \left\{ -\int_0^t \rho_1 ds - \int_0^t \tau_1 dW^1_t + \frac{1}{2} \int_0^t \tau_1^2 ds + \int_0^t \int_{\mathcal{R}_0} \ln(1 + \frac{\lambda_t - a_t}{\lambda \gamma^2_t(s)}) N^2(ds, dz) + \int_0^t \int_{\mathcal{R}_0} \left( \ln(1 + \frac{\lambda_t - a_t}{\lambda \gamma^2_t(s)}) - \frac{\lambda_t - a_t}{\lambda \gamma^2_t(s)} \right) G^2(dz) ds \right\}$$

is an $\mathcal{F}_t$-adapted process.
By similar procedure in Section 6.1.1, the BSDE (132) is equivalent to the following classical linear BSDE with respect to the filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\)

\[
\begin{aligned}
 dz^\pi_t(y) &= -\tilde{f}_{LJ}(t, X^\pi_t(y), z^\pi_t(y)) dt + z^\pi_t(t, y) dW^1_t + \int_{\mathbb{R}_0} z^\pi_t(t, y) N^2(dr, dz), \quad 0 \leq t \leq T, \\
 X^\pi_T(y) &= \mathcal{E}_T[y], \\
\end{aligned}
\]  

(141)

where the generator \(\tilde{f}_{LJ} : [0, T] \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}\) is given by

\[
\tilde{f}_{LJ}(t, x, z) = -r_t x - \lambda_t y - a_t \frac{d_t}{\lambda_t y} z^2 G^2(dz),
\]

and \(c^3(y) := \mathbb{E} \left[ \mathcal{E}_T[y] \right] \).

By [44, Theorem 4.8], the unique strong solution of (141) is given by

\[
X^\pi_t(y) = \mathcal{E} \left[ \Pi^*_y(t, T) \mathcal{E}_T[y] \right],
\]

(142)

where \(\Pi^*_y(t, T) := \Pi^*_y(0, T) / \Pi^*_y(0, t)\). By (92) and the initial value condition \(X^\pi_0(y) = X_0\), the Borel measurable function \(c^3(y)\) is given by

\[
c^3(y) = \frac{X_0}{\mathbb{E} \left[ \mathcal{E}_T[y] \right]} = \frac{X_0}{\mathbb{E} \left[ \mathcal{E}_T[y] \right]}.
\]

(143)

Substituting (143) into (142) we obtain

\[
X^\pi_t(y) = \frac{X_0 \mathbb{E} \left[ \mathcal{E}_T[y] \right]}{\mathbb{E} \left[ \mathcal{E}_T[y] \right]}.
\]

(144)

By [47, Proposition 3.5.1] and Malliavin calculus, we have

\[
z^\pi_1(t, y) = \frac{1}{2} \mathbb{E} \left[ \frac{\Pi^*_y(t, T) \mathcal{E}_T[y]}{\Pi^*_y(0, T)} \right] \left( y - \int_0^T f \phi dW^1_s \right) \|\phi\|^2_{L^2(T, \mathcal{F}_T)} \\
= \frac{1}{2} \mathbb{E} \left[ \frac{\Pi^*_y(t, T) \mathcal{E}_T[y]}{\Pi^*_y(0, T)} \right] \phi_t + \frac{1}{2} \mathbb{E} \left[ \frac{\Pi^*_y(t, T) \mathcal{E}_T[y]}{\Pi^*_y(0, T)} \right] \phi_t = b_t,
\]

(145)

and

\[
z^\pi_2(t, y) = -X^\pi_t(y) \frac{\lambda_t - a_t}{(\lambda_t - a_t) \gamma_2(t) + \lambda_t^2 \gamma_2^2(t)}.
\]

(146)

Substituting (145) and (146) into (123) we have

\[
\pi^*_t = \frac{1}{2} \mathbb{E} \left[ \frac{\Pi^*_y(t, T) \mathcal{E}_T[y]}{\Pi^*_y(0, T)} \right] \left( y - \int_0^T f \phi dW^1_s \right) \|\phi\|^2_{L^2(T, \mathcal{F}_T)} \phi_t + \frac{1}{2} \mathbb{E} \left[ \frac{\Pi^*_y(t, T) \mathcal{E}_T[y]}{\Pi^*_y(0, T)} \right] \phi_t
\]

\[
= b_t
\]

(147)

By Corollary [4.8] we have

\[
\theta^*_t = \frac{\lambda_t - a_t}{(\lambda_t - a_t) \gamma_2(t) + \lambda_t^2 \gamma_2^2(t)}.
\]

(148)

To sum up, we give the following theorem.
Theorem 6.30. Assume that $U(x) = \ln x$, $U(x) = \ln x$, $G^1(dx) = 0$, $G^2(dx) = \tilde{\lambda} \delta_1(dx)$, $b = \theta_2 = q_{12} = q_{22} = 0$, $\gamma(t, z) = \gamma_2(t)$, $q_2(t, z) = q_2(t)$, $\mu(t, x) = \mu_0(t)$ for some càdlàg function $\mu_0(t)$, and $g(s, v)$ is given by (118). Assume further that $\{\mathcal{H}_i\}_{0 \leq t \leq T}$ is given by (136) and all parameter processes are deterministic functions. Suppose $(u^*, v^*) \in \mathcal{A}_t^* \times \mathcal{A}_t^*$ is optimal for Problem 5.3 under the conditions of Theorem 5.4. Then $(u^*, v^*)$ is given by (147) and (148), where $\Pi_{j,a}$ is given by (140).

Remark 6.31. When the insider information is about the future value of the risk in the insurance market, i.e., $\mathcal{H}_i = \cap_{s \geq t} \left( \mathcal{F}_s \wedge \eta_{i_0}^2 \right)$, we have by (21) that

\[
\mathbb{E}[\delta_t(\eta_{i_0}^2)|\mathcal{F}_t] = \frac{1}{2\pi} \int_{\mathbb{R}} \exp \left\{ i x \eta_t^2 + \tilde{\lambda} (T_0 - t) (e^{ix} - 1 - ixy) \right\} dx,
\]

\[
\phi_1(t) = 0,
\]

\[
G^2_{xw}(t, dz) = G^2(dx) + \frac{1}{T_0 - t} \int_t^{T_0} N^2(dr, dz).
\]

By similar procedure, we obtain the robust optimal strategy as follows

\[
\pi^*_t = \frac{\mu}{\sigma^*},
\]

\[
\kappa^*_t = \frac{\lambda - a_t}{(\lambda - a_t) \gamma_2(t) + \tilde{\lambda} \gamma_2^2(t) + \frac{\lambda^2 \gamma_1^2(t)}{\sqrt{\lambda} \mathbb{E}[\Pi_{j,a}(t, T) Q(\eta)|\mathcal{F}_t]} - \sqrt{\lambda} \mathbb{E}[\Pi_{j,a}(t, T) Q(\eta)|\mathcal{F}_t]}{\sqrt{\lambda} \mathbb{E}[\Pi_{j,a}(t, T) Q(\eta)|\mathcal{F}_t]} \bigg|_{\eta = \eta_{i_0}^2},
\]

where $\Pi_{j,a}$ is given by (140), and $Q(\eta)$ is given by

\[
Q(\eta) = \sqrt{\frac{1}{2\pi}} \int_{\mathbb{R}} \exp \left\{ i x \eta_t^2 + \tilde{\lambda} (T_0 - T) (e^{ix} - 1 - ixy) \right\} e^{ix} dx
\]

\[
- \sqrt{\frac{1}{2\pi}} \int_{\mathbb{R}} \exp \left\{ i x \eta_t^2 + \tilde{\lambda} (T_0 - T) (e^{ix} - 1 - ixy) \right\} dx.
\]

Next, we concentrate on the special situation without model uncertainty, i.e., Problem 6.10. Then the terminal value condition of $X^*_T$ in (92) is replaced by

\[
X^*_T = \frac{X_0}{\Pi_{j,a}(0, T)} = \frac{X_0 \mathbb{E}[\delta_t(Y_0)|\mathcal{F}_T]}{\mathbb{E}[\delta_t(Y_0)|\mathcal{F}_0 \Pi_{j,a}(0, T)]}_{\eta = \gamma^2_{i_0}}.
\]

By similar procedure, we obtain the following proposition.

Proposition 6.32. Assume that $U(x) = \ln x$, $U(x) = \ln x$, $G^1(dx) = 0$, $G^2(dx) = \tilde{\lambda} \delta_1(dx)$, $b = \theta_2 = q_{12} = q_{22} = 0$, $\gamma(t, z) = \gamma_2(t)$, $q_2(t, z) = q_2(t)$, $\mu(t, x) = \mu_0(t)$ for some càdlàg function $\mu_0(t)$, $g(s, v)$ is given by (118), and no model certainty is considered. Assume further that $\{\mathcal{H}_i\}_{0 \leq t \leq T}$ is given by (136) and all parameter processes are deterministic functions. Suppose $u^* \in \mathcal{A}_t^*$ is optimal for Problem 6.10 under the conditions of Theorem 5.4. Then $u^*$ is given by

\[
\pi^*_t = \frac{\mu}{\sigma^*},
\]

\[
\kappa^*_t = \frac{\lambda - a_t}{(\lambda - a_t) \gamma_2(t) + \tilde{\lambda} \gamma_2^2(t)}.
\]

Remark 6.33. When the insider information is about the future value of the risk in the insurance market, i.e., $\mathcal{H}_i = \cap_{s \geq t} \left( \mathcal{F}_s \wedge \eta_{i_0}^2 \right)$, the optimal strategy can be calculated by

\[
\pi^*_t = \frac{\mu}{\sigma^*},
\]

\[
\kappa^*_t = \frac{\lambda - a_t}{(\lambda - a_t) \gamma_2(t) + \tilde{\lambda} \gamma_2^2(t)} - \frac{\eta_{i_0}^2 - \eta_t^2}{(T_0 - t) [(\lambda - a_t) + \tilde{\lambda} \gamma_2(t)]}.
\]
7 The large insurer case: combined method

Now we consider the special case when the utility function of the insurer is characterized by the logarithmic utility, i.e., \( U(x) = \ln(x) \), and the insurer is ‘large’, which means that the mean rate of return \( \mu \) on the risky asset could be influenced by her investment strategy \( \pi \).

In this situation, the degree of the influence on \( \mu \) could not change the robust optimal insurance strategy \( \kappa^* \) if the correlation coefficient \( \rho = 0 \) intuitively. In fact, this can be verified by the equation (172) below when the insurer is not ambiguity averse. Thus we will always suppose \( b \neq 0 \) when the insurer is ‘large’. For simplicity, we only consider the continuous case in this section, that is, \( G^1(dx) = G^2(dx) = 0 \).

Just as in Section 6.1, we assume that the mean rate of return \( \mu(t,x) = \mu_0(t) + \rho x \) for some \( \mathcal{F}_t \)-adapted càdlàg processes \( \mu_0(t) \) and \( \rho \), with \( 0 \leq \rho < \frac{4}{9} \sigma^2 \), and \( b \geq \varepsilon > 0 \) for some positive constant \( \varepsilon \). Put \( t_* = \frac{\mu_0(t) - \tau_0}{\sigma} \), \( \tilde{\sigma}_t = \sigma - \frac{2 \rho}{\sigma} \), \( \tilde{\phi}_1(t) = t + \phi_1(t) \), and \( \tilde{\phi}_2(t) = \frac{\kappa_1 - \kappa_0 b}{\sqrt{1 - \rho^2 \sigma_0}} - \phi_2(t) \). Assume further the penalty function \( g \) is given by \( g(s,v) = g(v) = \frac{1}{2}(\theta_1^2 + \theta_2^2) \). Then we have

\[
\mathbb{E} \left[ \int_0^T \epsilon_s^\mu(\pi_s^*) ds \right] = \mathbb{E} \left[ \epsilon_0^\mu \ln \epsilon_0^\mu \right].
\]

In this continuous setting, the equations (31) and (32) in Corollary 4.3 can be reduced to

\[
\begin{align*}
\theta_1^* (t) &= \tilde{\sigma}_t \pi_1^* - \rho b_t \kappa_1^* - \tilde{\phi}_1(t), \\
\theta_2^* (t) &= -\sqrt{1 - \rho^2 b_t \kappa_1^* + \rho \frac{\sigma_t - \tilde{\sigma}_t}{\sigma_t + \tilde{\sigma}_t} \pi_1^* + \tilde{\phi}_2(t).}
\end{align*}
\]

By a similar procedure in Section 6.1 with respect to the Hamiltonian system (37), we have (see [46])

\[\ln (\epsilon_0^\mu X_0^\mu) = c^2 \]

where \( c^2 = p_2^*(0) \) is an \( \mathcal{F}_0 \)-measurable random variable.

The Itô formula for Itô integrals combined with the expressions of \( \epsilon_0^\mu \) and \( X_0^\mu \) (see Theorem A.7) yields the following SDE:

\[
\begin{align*}
d \ln (\epsilon_0^\mu X_0^\mu) &= \left[ r_t + (\mu_0(t) - r_t) \pi_1^* + (\lambda_t - \mu_t) \kappa_1^* + (\sigma_t \pi_1^* - \rho b_t \kappa_1^*) \phi_1(t) - \sqrt{1 - \rho^2 b_t \kappa_1^*} \phi_2(t) \right] dt \\
&\quad - \frac{1}{2} \left[ (\sigma_t \pi_1^*)^2 - 2 \rho \sigma_t b_t \pi_1^* \kappa_1^* + (b_t \kappa_1^*)^2 + \theta_1^*(t)^2 + \theta_2^*(t)^2 - \sigma_t (\sigma_t - \tilde{\sigma}_t) (\pi_1^*)^2 \right] dt \\
&\quad + (\sigma_t \pi_1^* - \rho b_t \kappa_1^* + \theta_1^*(t)) dW_1^\mathcal{Y}(t) - \left( \sqrt{1 - \rho^2 b_t \kappa_1^*} - \theta_2^*(t) \right) dW_2^\mathcal{Y}(t).
\end{align*}
\]

Put \( L_t^* = \ln (\epsilon_0^\mu X_t^\mu) \) and \( z_t^* = (z_1^*(t), z_2^*(t)) \)

\[
= \left( \sigma_t \pi_1^* - \rho b_t \kappa_1^* + \theta_1^*(t), -\sqrt{1 - \rho^2 b_t \kappa_1^*} + \theta_2^*(t) \right). \]

Suppose that \( \sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t \neq 0 \). Then by (154) we have

\[
\begin{align*}
\pi_1^* &= \frac{1 - \rho^2}{\sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t} (z_1^*(t) + \tilde{\phi}_1(t)) - \frac{\rho \sqrt{1 - \rho^2}}{\sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t} (z_2^*(t) - \tilde{\phi}_2(t)), \\
\kappa_1^* &= -\frac{\sqrt{1 - \rho^2} (\sigma_t + \tilde{\sigma}_t)}{2(\sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t)} (z_2^*(t) - \tilde{\phi}_2(t)) + \frac{\rho (\sigma_t - \tilde{\sigma}_t)}{2(\sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t)} (z_1^*(t) + \tilde{\phi}_1(t)),
\end{align*}
\]

and

\[
\begin{align*}
\theta_1^*(t) &= \frac{2 \tilde{\sigma}_t - 3 \rho^2 \sigma_t}{2(\sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t)} z_1^*(t) - \frac{2 \sigma_t - 3 \rho^2 \sigma_t}{2(\sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t)} \tilde{\phi}_1(t) + \frac{\rho \sqrt{1 - \rho^2} (\sigma_t - \tilde{\sigma}_t)}{2(\sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t)} (z_2^*(t) - \tilde{\phi}_2(t)), \\
\theta_2^*(t) &= \frac{\sigma_t + \tilde{\sigma}_t - 3 \rho^2 \sigma_t}{2(\sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t)} z_2^*(t) + \frac{(1 - \rho^2) (\sigma_t + \tilde{\sigma}_t)}{2(\sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t)} \tilde{\phi}_2(t) + \frac{\rho \sqrt{1 - \rho^2} (\sigma_t - \tilde{\sigma}_t)}{2(\sigma_t + \tilde{\sigma}_t - 2 \rho^2 \sigma_t)} (z_1^*(t) + \tilde{\phi}_1(t)).
\end{align*}
\]
The SDE (156) combined with (157), (158) and (155) leads to the following quadratic BSDE

\[
\begin{aligned}
dL_t &= -f_0(t, z_t, \omega) \, dt + z_t \, dW(t), \quad 0 \leq t \leq T, \\
L_T^* &= c_2^*,
\end{aligned}
\]

where the generator \( f_0 : [0, T] \times \mathbb{R}^2 \times \Omega \to \mathbb{R} \) is given by

\[
f_0(t, z, \omega) = \frac{z_1^2 + z_2^2}{4} - \frac{\hat{\theta}_1(t)}{2} z_1 + \frac{\hat{\phi}_1(t)^2}{2} - r_t - \frac{\phi_1(t)^2 + \phi_2(t)^2}{4} - \frac{\sigma_t - \sigma_1}{4(1 - \rho^2)} \left[ (1 - \rho^2) (z_1 + \phi_1(t)) - \rho \sqrt{1 - \rho^2} (z_2 - \phi_2(t)) \right].
\]

By (153) and (155), the value \( V \) can be calculated by

\[
V = \mathbb{E} \left[ \varepsilon_T^* \ln \left( \varepsilon_T^* X_T^* \right) \right] = \mathbb{E} c_2^* = \mathbb{E} L_T^*.
\]

Remark 7.1. Suppose that \( c_2^*, \phi_1, \phi_2, r, \sigma, \bar{\sigma} \) and \( 1/\sigma_1 + \bar{\sigma} - 2\rho^2\sigma_1 \neq 0 \) are bounded. Then, by (159), the quadratic BSDE (159) has a unique strong solution \( (L^*, z_1^*, z_2^*) \) and \( \varepsilon_T^* \) is a \( \mathcal{H}_T \)-adapted \( \mathcal{F}_t \)-progressively measurable process with \( \mathbb{F} = \mathbb{F}_t \) and \( \varepsilon_T^* \) is given by (157), (158) and (160), where \( (L^*, z_1^*, z_2^*) \) solves the quadratic BSDE (159), and the \( \mathcal{H}_0 \)-measurable random variable \( c_2^* \) can be determined by the initial value condition \( L_0^* = \ln X_0^* \). Furthermore, if \( \{ \mathcal{H}_t \}_{0 \leq t \leq T} \) is the augmentation of the natural filtration of \( W^1_{\mathcal{H}}(t) \) and \( W^2_{\mathcal{H}}(t) \), and \( c_2^*, \phi_1, \phi_2, r, \sigma, \bar{\sigma} \) and \( 1/\sigma_1 + \bar{\sigma} - 2\rho^2\sigma_1 \neq 0 \) are bounded, then the quadratic BSDE (159) has a unique strong solution, where \( c_2^* \) can be determined by traversing all constants such that the condition \( L_0^* = \ln X_0^* \) holds.

Remark 7.3. If the filtration \( \{ \mathcal{H}_t \}_{0 \leq t \leq T} \) is not the augmentation of the natural filtration of \( W^1_{\mathcal{H}}(t) \) and \( W^2_{\mathcal{H}}(t) \), or the coefficients of the generator \( f_0 \) is not necessarily bounded, we refer to [20, 50, 51, 52, 53] for further results. Meanwhile, the \( \mathcal{H}_0 \)-measurable random variable \( c_2^* \) can be determined by traversing all \( \mathcal{H}_0 \)-measurable random variable such that the condition \( L_0^* = \ln X_0^* \) holds. Moreover, if \( \mathcal{H}_0 \) is generated by a random variable \( Y \) and all \( \mathbb{P} \)-negligible sets, then \( c_2^* = c_2^*(Y) \) with some Borel measurable function \( c_2^*(y) \). Thus, \( c_2^* \) can be determined by traversing all Borel measurable functions \( c_2^*(y) \) such that the initial value condition \( L_0^* = \ln X_0^* \) holds.

In particular, we assume that the insider information is related to the future value of risk in the insurance market and is of the initial enlargement type, i.e.,

\[
\mathcal{H}_t = \bigcap_{s \leq t} \left( \mathcal{F}_t \vee Y_0 \right) := \bigcap_{s \leq t} \left( \mathcal{F}_s \vee \int_0^{T_0} \phi(s') \, d\bar{W}_{s'} \right), \quad 0 \leq t \leq T,
\]

for some \( T_0 > T \), and all parameter processes are deterministic functions. Here, \( \phi \) is some deterministic function satisfying \(|\phi|_{[s, t]} := \int_s^t \varphi(s') \, ds' < \infty \) for all \( 0 \leq s \leq t \leq T_0 \), and \(|\phi|_{[T_0, t]} > 0 \).

In fact, integrating (159) from \( t \) to \( T \) yields \( L_T^* - L_t^* = -\int_t^T f_0(s, z_s, \omega) \, ds + \int_t^T z_s \, dW(s) \). Taking conditional expectation and assuming the Itô integrals are \( L^2 \)-martingales, we get \( L_T^* = \mathbb{E} \left[ \int_0^T f_0(s, z_s, \omega) \, ds + L_T^* \mid \mathcal{H}_t \right] \). Taking \( t = 0 \) and using the initial value condition we have \( c_2^* = \ln X_0^* - \mathbb{E} \left[ \int_0^T f_0(s, z_s, \omega) \, ds \mid \mathcal{H}_0 \right] \).
By the Donsker \( \delta \) functional \( \delta_t(Y_0) \) and similar procedure in Section 6.1.1, we have

\[
\phi_1(t) = \phi_1(t, \gamma)|_{\gamma = y_0} = \frac{y - \int_0^t \phi_0 \, dw_t}{\| \phi \|^2_{[0, t]} - \rho \phi} \bigg|_{y = y_0},
\]

(163)

\[
\phi_2(t) = \phi_2(t, \gamma)|_{\gamma = y_0} = \frac{y - \int_0^t \phi_0 \, dw_t}{\| \phi \|^2_{[0, t]} - \rho^2 \phi} \bigg|_{y = y_0}.
\]

Thus the BSDE (159) is equivalent to the following classical quadratic BSDE with respect to the filtration \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \)

\[
\begin{align*}
\left\{ \begin{array}{l}
dL^*_T(y) = -\tilde{f}_Q(t, z^*_T(y), y, \omega) \, dt + z^*_T(y) \, dW_t, & 0 \leq t \leq T, \\
L^*_T(y) = c^*_2(y),
\end{array} \right.
\end{align*}

(164)

where the generator \( \tilde{f}_Q : [0, T] \times \mathbb{R}^2 \times \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) is given by

\[
\tilde{f}_Q(t, z, y, \omega) = \frac{z_1^2 + z_2^2}{4} - \frac{t - \phi_1(t, y)}{2} z_2 + \frac{c_1 + \phi_2(t, y)}{2} z_2 - r_t - \frac{(t + \phi_1(t, y))^2 + (c_1 - \phi_2(t, y))^2}{4} - \frac{(\sigma_1 + \sigma_2 - 2 \rho^2 \sigma_1)(\sigma_1 - \sigma_2)}{4(1 - \rho^2)} \left[ \frac{1 - \rho^2}{\sigma_1 + \sigma_2 - 2 \rho^2 \sigma_1} (z_1 + t + \phi_1(t, y)) - \frac{\rho \sqrt{1 - \rho^2}}{\sigma_1 + \sigma_2 - 2 \rho^2 \sigma_1} (z_2 - c_1 + \phi_2(t, y)) \right]^2,
\]

and \( c_t = \frac{\lambda_t - \mu_t + \rho h_2(t)}{\sqrt{1 - \rho^2 h_2}} \).

Moreover, the value can be calculated by

\[
V = \mathbb{E}(L^*_T(y)|_{y = y_0}).
\]

(165)

Thus we have the following result.

**Theorem 7.4.** Assume that \( U(x) = \ln x \), \( G^1(dz) = G^2(dz) = 0 \), \( \mu_t(x) = \mu_0(t) + \rho_t x \) for some càglàd functions \( \mu_0(t) \) and \( \rho_t \), with \( 0 \leq \rho_t < \frac{1}{\pi} \sigma_1^2 \), \( b \geq \varepsilon > 0 \) for some positive constant \( \varepsilon \), and \( g(s, v) = \frac{1}{2}(\theta_1^2 + \theta_2^2) \). Assume further that \( \{ \mathcal{H}_t \}_{0 \leq t \leq T} \) is given by (162) and all parameter processes are deterministic functions. Suppose \( (u^*, v^*) \in \mathcal{A}_1 \times \mathcal{A}_2 \) is optimal for Problem 3.3 under the conditions of Theorem 5.4 and \( \sigma_1 + \sigma_2 - 2 \rho^2 \sigma_1 \neq 0 \). Then \( (u^*, v^*) \) and \( V \) are given by (157), (158), and (165), where \( (L^*(y), z^*_1(y), z^*_2(y)) \) solves the classical quadratic BSDE (164), \( \phi_1(t, y) \) and \( \phi_2(t, y) \) are given by (163), and the \( \mathcal{A}_0 \)-measurable random variable \( c^*_2 \) can be determined by traversing all Borel measurable functions \( c^*_2(y) \) such that \( L^*_0 = \ln X_0 \).

**Remark 7.5.** We can also give a classical quadratic BSDE like (164) to characterize the optimal pair \( (u^*, v^*) \) when the insurer has insider information about the future value of the risky asset, i.e., \( \mathcal{H}_t = \cap_{b, \varepsilon} \left( \mathcal{F}_t \vee \int_0^t \phi(x) \, dW^*_t \right) \). In this case, we have \( \phi_1(t) = \phi_1(t, y)|_{y = y_0} = \frac{y - \int_0^t \phi_0 \, dw_t}{\| \phi \|^2_{[0, t]} - \rho \phi} \bigg|_{y = y_0} \) and \( \phi_2(t) = \phi_2(t, y)|_{y = y_0} = 0 \).

Since the quadratic BSDE (159) or (164) has no closed-form solution and can only be solved by numerical methods, we concentrate on the special situation without model uncertainty, that is, \( \mathcal{A}_2 = \{(0, 0)\} \). Then Problem 3.3 degenerates to the following anticipating stochastic control problem.

**Problem 7.6.** Select \( u^* \in \mathcal{A}_1^* \) such that

\[
\tilde{V} := \tilde{J}(u^*) = \sup_{u \in \mathcal{A}_1^*} \tilde{J}(u),
\]

(166)

where \( \tilde{J}(u) := \mathbb{E} \left[ \ln X_T^u \right] \). We call \( \tilde{V} \) the value (or the optimal expected utility) of Problem 7.6.

Since \( \mathcal{A}_2 = \{(0, 0)\} \), which implies that \( \theta_1^* = \theta_2^* = 0 \) in (154). Thus, the optimal strategy \( u^* \in \mathcal{A}_1^* \) is given by

\[
\begin{align*}
\pi^*_1 &= \frac{(1 - \rho^2) \phi_1(t) + \rho \sqrt{1 - \rho^2} \phi_2(t)}{(\sigma_1 - \rho^2) \sigma_t}, \\
\kappa^*_1 &= \frac{\sqrt{1 - \rho^2} \phi_2(t) + \rho (\sigma_1 - 1) \phi_1(t)}{(1 - \rho^2 \sigma_1 \sigma_t) b_t},
\end{align*}
\]

(167)
Assume that $U$ in Theorem 7.8. Theorem 7.8 covers the result of the continuous case in [26]. However, in Theorem 7.8, the coefficients of the model (29)-(31) in Section 3 are all anticipating processes influenced by uncertain economic environment (i.e., the classical SDEs in [26] are replaced by the anticipating SDEs (9)-(11)), the mean rate of return $\mu$ on the risky asset is influenced by the strategy $\pi$ of the large insurer, and the filtration $\{\mathcal{H}_t\}_{0 \leq t \leq T}$ in Theorem 7.8 is more general than $\{\mathcal{H}_t\}_{0 \leq t \leq T}$ introduced in [26]. These make Theorem 7.8 more general than [26].

Suppose all parameter processes are deterministic functions. When $\mathcal{H}_t = \mathcal{F}_t$ (i.e., the insurer has no insider information) and $\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{W}_{t_0}$ (i.e., the insurer owns the insider information $\mathcal{W}_{t_0}$ about the future risk in the insurance market), we have the following results.

\begin{align}
\bar{V} = \ln X_0 + \mathbb{E} \int_0^T r_t dt + \frac{1}{2} \mathbb{E} \int_0^T \left( \frac{1}{1 - \rho^2} \frac{1}{\sigma_t} \left[ \left( 1 - 2\rho^2 \right) \frac{\sigma_t}{\bar{\sigma}_t} + \rho^2 \right] \phi_t(t)^2 + 2\rho \sqrt{1 - \rho^2 \left( \sigma_t / \bar{\sigma}_t - 1 \right) \left( \phi_t(t) \phi_t(t) + (1 - \rho^2) \phi_t(t)^2 \right) dt.}
\end{align}

Assume further that the insider filtration $\{\mathcal{H}_t\}_{0 \leq t \leq T}$ is given by

\begin{align}
\mathcal{F} \subset \mathcal{H} \subset \bigcap_{s \geq t} (\mathcal{F}_s \vee W_{t_0}^{1} \vee \tilde{W}_{t_0}) =: \mathcal{H}_t, \quad 0 \leq t \leq T,
\end{align}

for some $T_0 > T$. Then the enlargement of filtration technique can be applied to give the concrete expression of $\phi_t$ in (167), $i = 1, 2$. We give the following lemma to characterize the decomposition of the $\mathcal{H}_t$-semimartingale $W^i_t$ in Theorem 4.6 $i = 1, 2$. The proof can be found in [18, page 327].

Lemma 7.7 (Enlargement of filtration). The process $W^i_t$, $t \in [0, T]$, is a semimartingale with respect to the filtration $\{\mathcal{H}_t\}_{0 \leq t \leq T}$ given by (169), $i = 1, 2$. Its semimartingale decomposition is

\begin{align}
W^i_t = W^i_{t^0} + \int_0^T \mathbb{E} \left[ \frac{W^i_{t_0} | \mathcal{H}_s} {W^i_{t_0}} - W^i_{t_0} \bigg| \mathcal{H}_s \right] ds, \quad 0 \leq t \leq T,
\end{align}

where $W^i_{t^0}$ is an $\mathcal{H}_0$-Brownian motion, $i = 1, 2$. Combined with Lemma 7.7, the $\mathcal{H}_t$-progressively measurable process $\phi^i_t$ in Theorem 4.6 is of the form

\begin{align}
\phi^i_t = \frac{\mathbb{E} \left[ W^i_{t_0} | \mathcal{H}_s \right] - W^i_{t_0}} {t_0 - t}, \quad 0 \leq t \leq T,
\end{align}

$i = 1, 2$. Thus, we can give a concrete characterization of the optimal strategy $u^*$ as follows

\begin{align}
\pi^i_t = \frac{\left( 1 - \rho^2 \right) \left( u_t + \frac{\mathbb{E} \left[ W^i_{t_0} | \mathcal{H}_s \right] - W^i_{t_0}} {t_0 - t} \right) + \rho \sqrt{1 - \rho^2} \left( \frac{\lambda_t - u_t + \rho b_t} {\sqrt{1 - \rho^2 b_t}} \right) - \frac{\mathbb{E} \left[ W^i_{t_0} | \mathcal{H}_s \right] - W^i_{t_0}} {t_0 - t} \right)} {\left( \sigma_t / \bar{\sigma}_t - \rho^2 \right) \sigma_t},
\end{align}

\begin{align}
\kappa^i_t = \frac{\sqrt{1 - \rho^2} \left( \frac{\lambda_t - u_t + \rho b_t} {\sqrt{1 - \rho^2 b_t}} \right) - \frac{\mathbb{E} \left[ W^i_{t_0} | \mathcal{H}_s \right] - W^i_{t_0}} {t_0 - t} \right) + \rho \left( \sigma_t / \bar{\sigma}_t - 1 \right) \left( u_t + \frac{\mathbb{E} \left[ W^i_{t_0} | \mathcal{H}_s \right] - W^i_{t_0}} {t_0 - t} \right)} {\left( 1 - \rho^2 \sigma_t / \bar{\sigma}_t \right) b_t}.
\end{align}

To sum up, we give the following theorem for this special case.

Theorem 7.8. Assume that $U(x) = \ln x$, $G^1(\sigma_x) = G^2(\sigma_x) = 0$, $\mu(t, x) = \mu_0(t) + \rho x$ for some $\mathcal{G}^1$-adapted càdlàg processes $\mu_0(t)$ and $\rho$ with $0 \leq \rho_t < \frac{1}{4} \sigma_t^2$, $b \geq \varepsilon > 0$ for some positive constant $\varepsilon$, and no model uncertainty is considered. Suppose $u^* \in \mathcal{A}_1$ is optimal for Problem 7.6 under the conditions of Theorem 4.4 and $\bar{\sigma}_t \neq \rho^2 \sigma_t$. Then $u^*$ is given by (167), and $\bar{V}$ is given by (168). Furthermore, if the filtration $\{\mathcal{H}_t\}_{0 \leq t \leq T}$ is of the form (169), then $u^*$ is given by (172).
**Corollary 7.10.** Assume that $U(x) = \ln x$, $G_1^1(x) = G_2^2(x) = 0$, $\mu(t, x) = \mu_0(t) + \rho(t, x)$ for some deterministic càdlàg functions $\mu_0(t)$ and $\rho(t, x)$, with $0 \leq \rho(t, x) < \frac{1}{2} \sigma^2(t)$, $b \geq \varepsilon > 0$ for some positive constant $\varepsilon$, and no model uncertainty is considered. Assume further that $\mathcal{F}_t = \tilde{\mathcal{F}}_t$ and all parameter processes are deterministic functions. Suppose $u^* \in \mathcal{A}'_t$ is optimal for Problem 7.6 under the conditions of Theorem 4.4 and $\tilde{\sigma}_t \neq \rho^2 \sigma_t$. Then $u^*$ and $\tilde{V}$ are given by

\begin{align}
\pi^*_t &= \frac{(1 - \rho^2) ut}{\sigma_t/\sigma_t - \rho^2} + \frac{\rho(\lambda - a_t + \rho b_t) u_t}{\sigma_t/\sigma_t - \rho^2} \\
\kappa^*_t &= \frac{\lambda_t - a_t + \rho b_t u_t}{(1 - \rho^2 \sigma_t/\sigma_t) b_t} + \frac{\rho(\sigma_t/\tilde{\sigma}_t - 1) u_t}{(1 - \rho^2 \sigma_t/\tilde{\sigma}_t) b_t} + \frac{\rho(\sigma_t/\tilde{\sigma}_t - 1) u_t}{(1 - \rho^2 \sigma_t/\tilde{\sigma}_t) b_t} \\
\tilde{V} &= \ln X_0 + \int_0^T r_t dt + \frac{1}{2} \int_0^T \left[ \frac{1}{1 - \rho^2 \sigma_t/\tilde{\sigma}_t} \left( (1 - 2 \rho^2) \sigma_t/\tilde{\sigma}_t + \rho^2 \right) \right] (\lambda_t - a_t + \rho b_t u_t)^2 b_t^2 + \frac{(1 - \rho^2)^2}{T_0 - t} \right] dt.
\end{align}

Note that if $\tilde{\sigma} = \sigma$ (i.e., the insurer is ‘small’), the optimal strategy $u^*$ and the value $\tilde{V}$ in Corollary 7.10 are consistent with those in Proposition 6.11.

**Remark 7.11.** When there is no insider information and $\rho = 0$ (i.e., there is little uncertainty in the insurance markets, see [5]), we can see from Corollary 7.10 that, the insurer should invest more in the risky asset if her investment has more influence on the mean rate of return. The reason is that she introduces a higher appreciate rate in the risky asset price when investing. However, her optimal insurance strategy is not affected. Moreover, the optimal expected utility increases with her influence on the mean rate of return.

**Corollary 7.12.** Assume that $U(x) = \ln x$, $G_1^1(x) = G_2^2(x) = 0$, $\mu(t, x) = \mu_0(t) + \rho(t, x)$ for some deterministic càdlàg functions $\mu_0(t)$ and $\rho(t, x)$, with $0 \leq \rho(t, x) < \frac{1}{2} \sigma^2(t)$, $b \geq \varepsilon > 0$ for some positive constant $\varepsilon$, and no model uncertainty is considered. Assume further that $\mathcal{F}_t = \tilde{\mathcal{F}}_t$ and all parameter processes are deterministic functions. Suppose $u^* \in \mathcal{A}'_t$ is optimal for Problem 7.6 under the conditions of Theorem 4.4 and $\tilde{\sigma}_t \neq \rho^2 \sigma_t$. Then $u^*$ and $\tilde{V}$ are given by

\begin{align}
\pi^*_t &= \frac{(1 - \rho^2) ut}{\sigma_t/\sigma_t - \rho^2} + \frac{\rho(\lambda - a_t + \rho b_t) u_t}{\sigma_t/\sigma_t - \rho^2} \\
\kappa^*_t &= \frac{\lambda_t - a_t + \rho b_t u_t}{(1 - \rho^2 \sigma_t/\sigma_t) b_t} + \frac{\rho(\sigma_t/\tilde{\sigma}_t - 1) u_t}{(1 - \rho^2 \sigma_t/\tilde{\sigma}_t) b_t} + \frac{\rho(\sigma_t/\tilde{\sigma}_t - 1) u_t}{(1 - \rho^2 \sigma_t/\tilde{\sigma}_t) b_t} \\
\tilde{V} &= \ln X_0 + \int_0^T r_t dt + \frac{1}{2} \int_0^T \left[ \frac{1}{1 - \rho^2 \sigma_t/\tilde{\sigma}_t} \left( (1 - 2 \rho^2) \sigma_t/\tilde{\sigma}_t + \rho^2 \right) \right] \left( \frac{(\lambda_t - a_t + \rho b_t u_t)^2}{b_t^2} + \frac{(1 - \rho^2)^2}{T_0 - t} \right] dt.
\end{align}

Note that if $\tilde{\sigma} = \sigma$ (i.e., the insurer is ‘small’), the optimal strategy $u^*$ and the value $\tilde{V}$ in Corollary 7.12 are consistent with those in Proposition 6.20.

**Remark 7.13.** When the insurer has insider information about the insurance market and $\rho = 0$, we can see from Corollary 7.12 that, the insurer should invest more in the risky asset if her investment has more influence on the mean rate of return. However, her optimal insurance strategy is not affected. Moreover, the optimal expected utility increases with her influence on the mean rate of return. On the other hand, by comparing Corollary 7.12 with Corollary 7.10, the insurer should reduce the liability ratio by $\frac{\tilde{W}_0 - \tilde{W}_T}{b_t(T_0 - t)}$ if she captures insider information in the insurance market and $\tilde{W}_0 > \tilde{W}_T$. Given $\tilde{W}_0 - \tilde{W}_T = x > 0$, we can also see that the closer the future time $T_0$ is, the more she should reduce her liability ratio. Moreover, her optimal expected utility (compared with the case she has no insider information) is gained by $\Delta V = \frac{1}{2} \ln x > 0$ if $\rho = 0$, and $\Delta V$ becomes greater as $T_0$ gets closer (i.e., the insurer has ‘better’ insider information).

**Remark 7.14.** By similar procedure, we can also obtain the optimal strategy when the insurer has insider information.
about the future value of the risky asset. For instance, if \( \mathcal{H}_t = \bigcap_{t \geq T} \left( \mathcal{F}_t \lor W^1_{T_0} \right) \), the optimal strategy is given by

\[
\pi_t^* = \frac{(1 - \rho^2)t}{(\bar{\sigma}_t / \sigma_t - \rho^2)\sigma_t} + \frac{\rho(\lambda_t - a_t + \rho b_t t)}{(\bar{\sigma}_t / \sigma_t - \rho^2)\sigma_t b_t} + \frac{(1 - \rho^2)}{(\bar{\sigma}_t / \sigma_t - \rho^2)\sigma_t(T_0 - t)},
\]

\[
\kappa_t^* = \frac{\lambda_t - a_t + \rho b_t t}{(1 - \rho^2)\sigma_t / \bar{\sigma}_t b_t^2} + \frac{\rho(\bar{\sigma}_t / \sigma_t - 1)\sigma_t}{(1 - \rho^2)\sigma_t / \bar{\sigma}_t b_t} + \frac{(1 - \rho^2)}{(1 - \rho^2)\sigma_t / \bar{\sigma}_t b_t(T_0 - t)}.
\]

Then the insurer should invest more in the risky asset if her investment has more influence on the mean rate of return whereas her optimal insurance strategy is not affected given that \( \rho = 0 \). Moreover, the value \( \bar{V} \) can be given by

\[
(176) \quad \bar{V} = \ln X_0 + \int_0^T r_t dt + \frac{1}{2} \int_0^T \frac{1}{1 - \rho^2 \sigma_t / \bar{\sigma}_t} \left[ \left( (1 - 2\rho^2) \sigma_t / \bar{\sigma}_t + \rho^2 \right) \left( \frac{t^2}{T_0 - t} \right) + 2\rho \left( \sigma_t / \bar{\sigma}_t - 1 \right) \frac{\lambda_t - a_t + \rho b_t t}{b_t} + \frac{\left( \lambda_t - a_t + \rho b_t t \right)^2}{b_t^2} \right] dt.
\]

The optimal expected utility increases with her influence on the mean rate of return. On the other hand, given that \( \rho = 0 \), the insurer should increase the proportion of her total wealth invested in the risky asset by \( \frac{W^1_{T_0} - W^1_{T_0}}{\sigma_t(T_0 - t)} \) if she knows the future value of the risky asset and \( W^1_{T_0} > W^1_{T_0} \). Given \( W^1_{T_0} - W^1_{T_0} = x > 0 \), we can also see that the closer the future time \( T_0 \) is, the more she should increase her proportion with respect to the risky asset. Moreover, her optimal expected utility (compared with the case she has no insider information) is gained by \( \Delta \bar{V} = \frac{1}{2} \int_0^T \frac{1}{\sigma_t(T_0 - t)} > 0 \) if \( \rho = 0 \), and \( \Delta \bar{V} \) becomes greater as \( T_0 \) gets closer (i.e., the insurer has ‘better’ insider information). By comparing (176) with (174), we find that, if \( \rho = 0 \), the insurer information about the financial market (compared with that about the insurance market) will provide the insurer more optimal expected utility if she can really influence the mean rate of return on the risky asset.

### 8 Conclusion

In this paper, we investigate the optimal investment and risk control problem for an insurer under both model uncertainty and insider information. The insurer’s risk process and the financial risky asset process are assumed to be correlated jump diffusion processes with very general Lévy forms. The insider information is about the financial risky asset and the insurance polices, and is of the most general form rather than the initial enlargement type. We establish the corresponding anticipating stochastic differential game problem and give the characterization of robust optimal strategy by combining the theory of forward integrals with the stochastic maximum principle. The two typical situations when the insurer is ‘small’ and ‘large’ are discussed and the corresponding optimal strategies are presented. Our results generalize some results in the literature. We also discuss the impacts of the model uncertainty, insider information and the ‘large’ insurer on the optimal strategy.

For further work, the quadratic BSDE corresponding to the robust optimal strategy for a large insurer need to be studied. Moreover, the some constraints on the strategy can be imposed, which is also a subject of ongoing research.

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Appendix A Basic theories of Itô integrals with jumps

In this section, we give basic definitions and theories of Itô integrals in discontinuous cases, i.e., Itô integrals with jumps. For Itô integrals in continuous cases, we refer to [32].

We begin with a filtered probability space \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P}) \), where \( \{ \mathcal{F}_t \} \) satisfies the usual condition, and we identify two stochastic processes if they are indistinguishable (see [32]).

Denote by \( \mathcal{M}_2, \mathcal{M}_{2, \text{loc}} \) and \( \mathcal{M}_{2, \text{loc}} \) the spaces of all \( \mathcal{F}_t \)-square-integrable martingales, \( \mathcal{F}_t \)-locally square-integrable martingales and \( \mathcal{F}_t \)-local martingales with zero initial values, respectively (see [32] for definitions). By the decomposition theorem for local martingales (see [39]), we can define \( \mathcal{M}^*_\text{loc} \) and \( \mathcal{M}^*_\text{loc} \) the subspaces of continuous local martingales and purely discontinuous local martingales, respectively (similar statements hold for \( \mathcal{M}_2 \) and \( \mathcal{M}_{2, \text{loc}} \)).

Given \( M, N \in \mathcal{M}_{\text{loc}} \), we can define the covariance process \( [M, N]_t \) of \( M \) and \( N \) (see [39]), which is an \( \mathcal{F}_t \)-adapted bounded variation process. If \( [M, N]_t \) is locally integrable, we denote by \( [M, N]_t \) the \( \mathcal{F}_t \)-compensator of \( [M, N]_t \), which is called the \( \mathcal{F}_t \)-predictable covariance process of \( M \) and \( N \). Note that if \( M \) and \( N \) both belong to \( \mathcal{M}_{2, \text{loc}} \), the \( \mathcal{F}_t \)-predictable covariance process \( [M, N]_t \) exists. If \( M = N \), we write \( [M]_t = [M, M]_t \) (or \( \langle M \rangle_t = \langle M, M \rangle_t \)) for simplicity, which is called the quadratic variation process (or \( \mathcal{F}_t \)-predictable quadratic variation process) of \( M \).

For fixed \( M \in \mathcal{M}_{\text{loc}} \), we say a process \( \varphi \) is Itô integrable with respect to \( M \) if \( \varphi \in \mathcal{M}^*_{\text{loc}}([M]) \), i.e., \( \varphi \) is \( \mathcal{F}_t \)-predictable and \( \int_0^t \varphi^2 DS_M \) is locally integrable. We can define the Itô integral \( I^M_t(\varphi) = \int_0^t \varphi_s dM_s \) of \( \varphi \) with respect to \( M \) (see [39]). Then the Itô integral \( I^M_t \) is a linear operator from \( \mathcal{L}^\infty_{\text{loc}}([M]) \) to \( \mathcal{M}_{\text{loc}} \). If \( M \in \mathcal{M}_{2, \text{loc}} \), \( I^M_t \) maps from \( \mathcal{L}^\infty_{\text{loc}}([M]) \) to \( \mathcal{M}_{2, \text{loc}} \), where \( \mathcal{L}^\infty_{\text{loc}}([M]) \) is the space of all \( \mathcal{F}_t \)-predictable processes \( \varphi \) such that \( \int_0^t \varphi^2_s d\langle M \rangle_s \) is locally integrable. If \( M \in \mathcal{M}_{2, \text{loc}} \), \( I^M_t \) maps from \( \mathcal{L}^\infty([M]) \) to \( \mathcal{M}_2 \), where \( \mathcal{L}^\infty([M]) \) is the space of all \( \mathcal{F}_t \)-predictable processes \( \varphi \) such that \( \mathbb{E} [\int_0^t \varphi^2_s d\langle M \rangle_s ] < \infty \), \( \forall t \geq 0 \).

We say a martingale \( M \) is square-integrable if \( \mathbb{E} [\int_0^\infty \varphi^2_s d\langle M \rangle_s ] < \infty \), which is different from definitions in some literature (see [39]).

We say an \( \mathcal{F}_t \)-adapted increasing process \( A_t \) is locally integrable, if there exists an \( \mathcal{F}_t \)-stopping times \( \{ T_n \}_{n=1}^\infty \) such that \( T_n \uparrow \infty \) and \( A_{T_n} \) is integrable for all \( n \in \mathbb{N} \). If an \( \mathcal{F}_t \)-adapted bounded variation process \( V_t \) can be represented by the difference of two \( \mathcal{F}_t \)-adapted locally integrable increasing processes, we say \( V_t \) is locally integrable. In this case we can define the \( \mathcal{F}_t \)-compensator (or the \( \mathcal{F}_t \)-dual predictable) of \( V_t \), which is an \( \mathcal{F}_t \)-adapted predictable bounded variation process.
Remark A.4. Let $\mathcal{F}_t$-predictable process such that $\int_0^t \varphi_s^2 d|M|_s$ is locally integrable if and only if $\varphi \in L^2_{\text{loc}}([M])$ and if $M^f(f) \in \mathcal{M}_{\text{loc}}$ is an $\mathcal{F}_t$-predictable process such that $\mathbb{E} \int_0^t \varphi_s^2 d|M|_s < \infty$ for all $t \geq 0$ if and only if $\varphi \in L^2_{\text{loc}}([M])$ and if $M^f(f) \in \mathcal{M}_2$.

We say an $\mathcal{F}_t$-adapted process $Z$ is an $\mathcal{F}_t$-semimartingale if $Z$ has a decomposition $Z_t = Z_0 + M_t + V_t$ for some $M \in \mathcal{M}_{\text{loc}}$ and $\mathcal{F}_t$-adapted bounded variation process $V$. If $V$ is $\mathcal{F}_t$-predictable, we say $Z$ is an $\mathcal{F}_t$-special semimartingale, in which case the above decomposition is unique such that $V$ is $\mathcal{F}_t$-predictable, and we call it the canonical decomposition of $Z$. We say a process $\varphi$ is Itô integrable with respect to a semimartingale $Z$ if $Z$ has a decomposition $Z_t = Z_0 + M_t + V_t$ such that $\varphi$ is Itô integrable with respect to $M$ and pathwise Lebesgue-Stieltjes integrable with respect to $V$, which is well-defined. The definition of the covariation and the $\mathcal{F}_t$-predictable covariation can also be extended to the case of semimartingales as well. We refer to [33, 55] for more Itô theories of semimartingales.

In particular, we can obtain finer results when we consider the two special types of $\mathcal{F}_t$-martingales, namely, the $\mathcal{H}^1$-martingale and the $\mathcal{B}_M \mathcal{O}$-martingale. The properties of them also play a crucial role in the theory of the quadratic BSDE (see [34]). Here, we only give the definitions. We refer to [33, 54] for more details.

Definition A.2. Denote by $\mathcal{H}^1$ the space of all local martingales $M$ such that $\|M\|_{\mathcal{H}^1} := \mathbb{E} \sqrt{\mathbb{E}[M]_t} < \infty$. Denote by $\mathcal{B}_M \mathcal{O}$ the space of all $L^2$-bounded martingales such that $\|M\|_{\mathcal{B}_M \mathcal{O}} := \text{sup}_{\mathcal{T} \in \mathcal{F}} \mathbb{E} \left(\frac{E(M_{\mathcal{T}} - M_{\mathcal{T}_0})(1_{\mathcal{T} < \infty})^2}{\mathbb{P}(\mathcal{V} < \infty)}\right) < \infty$, where $\mathcal{T}$ is the set of all $\mathcal{F}_t$-stopping times. Each element in $\mathcal{H}^1$ (resp. $\mathcal{B}_M \mathcal{O}$) is called an $\mathcal{H}^1$-martingale (resp. $\mathcal{B}_M \mathcal{O}$-martingale). Moreover, $\mathcal{B}_M \mathcal{O}$ is the dual space of the Banach space $\mathcal{H}^1$ (see [33]).

Now we turn to the Itô integral with respect to a compensated random measure. Before that, we give the definition of the integer-valued random measure.

Definition A.3. We say a family $\mu = (\mu(\omega; \cdot, d) : \omega \in \Omega)$ is a $\sigma$-finite random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ if $\mu(\omega, d) \in \mathcal{F}_t$ is a $\sigma$-finite measure on $\mathbb{R}_+ \times \mathbb{R}^d$ for fixed $\omega \in \Omega$ and $\mu(\cdot, d) = \mu(\cdot, A)$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ for fixed $A \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^d)$. Moreover, we say a $\sigma$-finite random measure $\mu$ is an $\mathcal{F}_t$-integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ if it satisfies the following extra conditions:

(i) $\mu(\omega, d) = \{0\}$ for any $d \in \Omega$;

(ii) $\mu(\omega, \{0\} \times \mathbb{R}^d) = 0$ for any $d \in \Omega$;

(iii) $\mu(\omega, \{t\} \times \mathbb{R}^d) \leq 1$ for any $d \in \Omega, t \geq 0$;

(iv) $\mu$ is $\mathcal{F}_t$-optional in the sense that $\mu(\cdot, [0, t] \times B)$ is an $\mathcal{F}_t$-optional process for any $B \in \mathcal{B}(\mathbb{R}^d)$;

(v) $\mu$ is $\mathcal{F}_t$-predictably $\sigma$-finite in the sense that there exists some strictly positive $\mathcal{F}_t$-predictable random field $V$ on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$ such that $\int_0^\infty \int_{\mathbb{R}^d} V(s, z) \mu(ds, dz)$ is integrable.

Denote by $\hat{\mu}(dt, dz)$ the $\mathcal{F}_t$-compensator (or the $\mathcal{F}_t$-dual predictable projection) of an $\mathcal{F}_t$-integer-valued random measure $\mu(dt, dz)$ (see 33 for the definition), which is a $\sigma$-finite random measure on $\mathbb{R}_+ \times \mathbb{R}^d$. Define $\hat{\mu}(dt, dz) := \mu(dt, dz) - \hat{\mu}(dt, dz)$, which is called the compensated random measure of $\mu(dt, dz)$.

Remark A.4. Let $Z$ be an $\mathcal{F}_t$-adapted càdlàg $\mathbb{R}^d$-valued process. Then

$$\mu^Z(dt, dz) := \sum_{s > 0} \delta_{(s, \Delta Z_s)}(dt, dz) 1_{\Delta Z_s \neq 0}(s)$$

defines an $\mathcal{F}_t$-integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, where $\delta_a$ denotes the Dirac measure at point $a$. We call $\mu^Z$ the jump measure of $Z$ and $\hat{\mu}^Z$ the Lévy system of $Z$.

In the rest of this section, we consider an $\mathcal{F}_t$-integer-valued random measure $\mu$, and suppose the $\mathcal{F}_t$-compensator $\hat{\mu}$ of $\mu$ is absolutely continuous in time, i.e., $\hat{\mu}(dt, dz) = K(t, dz)dt$ for some random transition measure $K(t, dz)$, the definition of which is similar to Definition A.3 (see 33, page 37) for the definition of transition measures).

Define the following three linear spaces:

(i) $L^1_{\text{loc}}(\mu)$ is the space of all $\mathcal{F}_t$-predictable random fields $\xi(t, z)$ such that $\mathbb{E} \int_0^t \xi^2(s, z) \mu(ds, dz)$ is locally integrable;

(ii) $L^2_{\text{loc}}(\hat{\mu})$ is the space of all $\mathcal{F}_t$-predictable random fields $\xi(t, z)$ such that $\int_0^t \mathbb{E} \xi^2(s, z) K(s, dz)ds < \infty, \forall t \geq 0$;
(iii) \( \mathcal{L}^*(\tilde{\mu}) \) is the space of all \( \mathcal{F}_t \)-predictable random fields \( \zeta(t, z) \) such that \( \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \zeta^2(s, z)K(s, dz)ds < \infty, \forall t \geq 0. \)

We have \( \mathcal{L}^*(\tilde{\mu}) \subset \mathcal{L}^*_{\text{loc}}(\tilde{\mu}) \subset \mathcal{L}^*_B(\mu) \). We say a random field \( \zeta \) is \( \text{Itô} \) integrable with respect to \( \tilde{\mu} \) if \( \zeta \in \mathcal{L}^*_B(\mu) \). In this case we can define the \( \text{Itô} \) integral \( \mathcal{I}^B(\zeta) := \int_0^T \int_{\mathbb{R}^d} \zeta(s, z)N(ds, dz) \) (see [39]), which belongs to \( \mathcal{M}^d_{\text{loc}} \). If \( \mu \) is induced by some \( \mathcal{F}_t \)-adapted càdlàg process \( Z \), then we have \( \Delta^B(\zeta) = \zeta(s, \Delta Z_t)1_{\{\Delta Z_t = 0\}} \). Moreover, \( f^B(\zeta) \) maps from \( \mathcal{L}^*_{\text{loc}}(\tilde{\mu}) \) to \( \mathcal{M}^d_{2,\text{loc}} \) and from \( \mathcal{L}^*(\tilde{\mu}) \) to \( \mathcal{M}^d_{2,\text{loc}} \). We refer to [38, Propositions 3.37 and 3.39], [55, Proposition II.1.14] and [39, Theorem 11.23] for more properties of the stochastic integral with respect to a compensated random measure in our settings.

**Remark A.5.** If an \( \mathcal{F}_t \)-predictable random field \( \zeta(t, z), t \in \mathbb{R}_+ \), satisfies that \( \int_0^T \int_{\mathbb{R}^d} |\zeta(s, z)|K(t, dz)ds < \infty \), then \( \zeta \in \mathcal{L}^*_{\text{loc}}(\tilde{\mu}) \) and \( \int_0^T \int_{\mathbb{R}^d} \zeta(s, z)N(ds, dz) = \int_0^T \int_{\mathbb{R}^d} \zeta(s, z)N(ds, dz) - \int_0^T \int_{\mathbb{R}^d} \zeta(s, z)K(t, dz)ds \).

The well-known \( \text{Itô} \) formula for general semimartingale or the \( \text{Lévy-Itô} \) process can be found in many literature (see [39]). Here we do some extension in the one-dimensional case to fit our settings in Section [7](similar statements hold for the multi-dimensional case).

**Theorem A.6.** Suppose \( \mu \) is induced by a one-dimensional càdlàg process \( Z \). We say the following \( \mathcal{F}_t \)-adapted process \( Y \) is an \( \text{Itô} \) process

\[
Y_t := Y_0 + \int_0^t \alpha(s)ds + \int_0^t \varphi(s)dw_s + \int_0^t \int_{\mathbb{R}^d} \zeta(s, z)\tilde{\mu}(ds, dz),
\]

where \( W \) is an \( \mathcal{F}_t \)-Brownian motion, \( \alpha \) and \( \varphi \) are \( \mathcal{F}_t \)-progressively measurable processes such that \( \int_0^T \int_{\mathbb{R}^d} (|\alpha| + \varphi^2)ds < \infty \), and \( \zeta \) is an \( \mathcal{F}_t \)-predictable random field such that \( \int_0^T \int_{\mathbb{R}^d} \zeta^2(s, z)K(t, dz)ds < \infty \). Then for any function \( f \in C^2(\mathbb{R}) \), \( f(Y_t) \) is also an \( \text{Itô} \) process and we have

\[
f(Y_t) = f(Y_0) + \int_0^t f'(Y_s)\alpha(s)ds + \int_0^t f'(Y_s)\varphi(s)dw_s + \frac{1}{2} \int_0^t f''(Y_s)\varphi^2(s)ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} [f(Y_{s-} + \zeta(s, z)) - f(Y_{s-})]\tilde{\mu}(ds, dz)
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} [f(Y_{s-} + \zeta(s, z)) - f(Y_{s-}) - f'(Y_{s-})\zeta(s, z)]K(s, dz)ds.
\]

**Proof.** We may assume \( f \) is a \( C^2 \) function with compact support without loss of generality. Then we have \( \sum_{s \leq t} [f(Y_t) - f(Y_{s-}) - f'(Y_{s-})\Delta Y_s] = \sum_{s \leq t} [f(Y_{s-} + \zeta(s, \Delta Z_s)) - f(Y_{s-}) - f'(Y_{s-})\zeta(s, \Delta Z_s)]1_{\{\Delta Z_s \neq 0\}} = \int_0^t \int_{\mathbb{R}^d} [f(Y_{s-} + \zeta(s, z)) - f(Y_{s-}) - f'(Y_{s-})\zeta(s, z)]\tilde{\mu}(ds, dz) \) then the formula is an immediate consequence from the \( \text{Itô} \) formula for general semimartingales (see [39]) and the Taylor expansion.

\[
\square
\]

The following theorems are important results in the \( \text{Itô} \) theory. The proofs can be found in [36, 56, 39, 55, 38].

**Theorem A.7.** Suppose \( Z \) is a \( \mathcal{F}_t \)-semimartingale. Let

\[
\mathcal{E}(Z)_t := \exp \left\{ Z_t - Z_0 - \frac{1}{2}(Z_t^2), t \right\} \Pi_{0<t\leq T}(1 + \Delta Z_t)e^{-\Delta Z_t}
\]

be the Doléans-Dade exponential of \( Z \). Then \( \mathcal{E}(Z)_t \) is the unique semimartingale satisfying the SDE \( d\mathcal{E}(Z)_t = \mathcal{E}(Z)_t dZ_t \) with initial value 1. Moreover, if \( Z \in \mathcal{M}^d_{2,\text{loc}} \), \( \Delta Z_t > -1 \), and \( \mathbb{E} \left[ e^{\frac{1}{2}X_t^2} + |X_t^d|_t \right] < \infty \), then \( \mathcal{E}(Z) \) is an \( \mathcal{F}_t \)-martingale.

**Theorem A.8.** Suppose \( Q \) is another probability measure which is equivalent to \( P \). Let \( Z_t := \mathbb{E} \frac{dQ}{dP}|\mathcal{F}_t \), which can be proved to be an \( (\mathcal{F}_t, P) \)-martingale. Assume further that \( M \in \mathcal{M}^d_\text{loc}(\mathbb{P}) \) and \( [Z, M] \) is locally integrable for \( P \). Then \( L_t := M_t - \int_0^t \frac{1}{2}d\mathcal{E}(Z, M)_t \in \mathcal{M}^d_\text{loc}(Q) \).

**Theorem A.9.** Fix \( T > 0 \). Let \( Y_t := \int_0^t \varphi(s)dw_s + \int_0^t \int_{\mathbb{R}^d} \zeta(s, z)\tilde{\mu}(ds, dz), t \in [0, T], \) where \( W \) is an \( \mathcal{F}_t \)-Brownian motion, \( \varphi, \zeta \in \mathcal{L}^*_{\text{loc}}(\mathbb{W}) \times \mathcal{L}^*_B(\mu), \) and \( \zeta > -1 \). If the Doléans-Dade exponential \( \mathcal{E}(Y)_t \), \( t \in [0, T] \), is an \( \mathcal{F}_t \)-martingale, then \( Q_T := \mathcal{E}(Y)_T d\mathbb{P} \) is a probability measure equivalent to \( P \). \( W_{Q_T}(t) := W_t - \int_0^t \varphi(s)ds, t \in [0, T] \), is the Brownian motion under \( Q_T \), and \( \tilde{\mu}_{Q_T}(dr, dz) := \mu(dr, dz) - \zeta(t, z)K(t, dz)dr, t \in [0, T], z \in \mathbb{R}^d \), is the compensated random measure of \( \mu(dx, dz) \) under \( P \). Moreover, \( \varphi' \in \mathcal{L}^*_{\text{loc}}(\mathbb{W}) \Rightarrow \varphi' \in \mathcal{L}^*_{\text{loc}}(\mathbb{W}_{Q_T}; Q_T) \), and \( \zeta' \in \mathcal{L}^*_B(\tilde{\mu}; Q_T) \Rightarrow \zeta' \in \mathcal{L}^*_{\text{loc}}(\tilde{\mu}; Q_T) \).