Quantum Noise of Free-Carrier Dispersion in Semiconductor Optical Cavities

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This paper derives Langevin equations for an optical cavity where the dominant nonlinearity arises from free-carrier dispersion. We define a generalized Wigner function, compute a Fokker-Planck equation that approximates the master equation, and convert this to a system of stochastic differential equations. These equations are similar to the Wigner equations for an optical Kerr cavity, but have additional noise terms due to the incoherent carrier excitation and decay processes. We use these equations to simulate a phase-sensitive amplifier and latch and compare the results to a Kerr model.

Optical logic requires a platform that is fast, low-power and scalable to compete with electronics. In the past decade, nanophotonics has advanced to the point where optical cavities of size $\lesssim (\lambda/n)^3$ and $Q$ factors $\gtrsim 10^4$ can be fabricated with standard techniques \cite{1}. The hope is that these cavities can be used to amplify the optical nonlinearity of materials or defects and perform all-optical logic for communications and computing at speeds and energy scales comparable to electronics.

Free-carrier dispersion is a promising nonlinearity for low-power optical logic. The effect arises in all semiconductors. In a semiconductor, there is a filled valence band and an empty conduction band, and when photons are absorbed, they excite electrons from the valence band to the conduction band. Each absorption creates two free carriers – an electron and a hole – which evolve independently and decay on some timescale set by the material and its geometry. The carriers provide feedback to the optical field by altering the absorption of the material (free-carrier absorption) or its refractive index (free-carrier dispersion). On timescales long compared to the free-carrier lifetime, it acts as an effective optical nonlinearity and can be used to construct switches, amplifiers and other logic elements.

Accurate, semiclassical models for free-carrier effects already exist, and these are valid when the carrier and photon number are very large \cite{2, 3}. However, the real promise of free-carrier effects lie in their application to low-power photonic computing. In some materials, free-carrier effects are strong enough that switching can be achieved with as few as 100 photons per cavity. In this regime, quantum effects become important and place fundamental limits on device performance. For example, quantum fluctuations in the photon number add noise to quantum amplifiers \cite{4} and lead to spontaneous switching in optical memories \cite{5, 6}. This motivates the need to develop a quantum model for the free-carrier nonlinearity that works at low photon numbers, similar to the models that exist for cavity quantum electrodynamics (QED) and $\chi^{(3)}$ (Kerr) systems \cite{7, 8}.

In this paper, we derive a quantum-mechanical model for the free-carrier nonlinearity, following the standard open quantum systems formalism used for cavity QED, optical parametric oscillators (OPO’s) and $\chi^{(3)}$ systems \cite{9}. However, simulating even a single cavity in this model is not practical, since the large number of available carrier modes makes the full Hilbert space exponentially large. Using a method based on the Wigner function \cite{10, 11}, we can reduce the master equation to a set of c-number Langevin equations that are simple to simulate. These equations bear resemblance to semiconductor laser rate equations and Bloch equations found in the literature \cite{12, 13, 14}. An adiabatic elimination reduces the model further, giving a set of stochastic differential equations (SDE’s) for the field, electron number, and hole number in the cavity. The deterministic part of these equations matches the classical models found in the previous literature, but the noise terms are new – and have a quantum origin.

Section I introduces the quantum model for the free-carrier cavity. In Section II we introduce the Wigner formalism and apply it to this model, and in Section III simplify them in the weak-doping, fast-dephasing limit. Extensions of this model, incorporating two-photon absorption and free-carrier absorption, are treated in Section IV.

The free carrier model we derive, summarized an Equations (90-92) and (93-94), resembles the equations of motion for the Kerr cavity derived in \cite{10}, but there are extra noise terms. The steady-state behavior, and correspondence to the Kerr cavity, are treated in Section V. Next, the free-carrier SDE’s are applied to simulate two devices: a phase-sensitive amplifier in Section VI and an all-optical SR-latch in Section VII. For the amplifier, the free-carrier device does not show squeezing, whereas its Kerr analog does. For the latch, the spontaneous switching rate is larger for the free-carrier device, and the discrepancy grows as the latching becomes stronger.

I. QUANTUM MODEL

Consider a single-mode optical cavity fabricated from an undoped semiconductor. The optical degree of freedom can be represented as a harmonic oscillator, with the Hilbert space $\mathcal{H}_\text{cav} = \text{span}(\{|n\rangle\}_{n=0}^\infty)$ and the cre-
In band theory, the electronic degree of freedom is represented by many uncoupled fermion modes. For a two-band model, we have $H_{k,e} = \text{span}([0]_{k,e}, [1]_{k,e})$, $H_{k,h} = \text{span}([0]_{k,h}, [1]_{k,h})$ for electrons and holes, represented by the (fermionic) creation / annihilation operators $c_k, c_k^\dagger$ (electrons) and $\bar{c}_k, \bar{c}_k^\dagger$ (holes). The Hilbert space is:

$$ \mathcal{H} = \mathcal{H}_{\text{cav}} \otimes \prod_k \mathcal{H}_{k,e} \otimes \prod_k \mathcal{H}_{k,h} $$

(1)

The operator algebra is:

$$ [a, a^\dagger] = 1 $$

(2)

$$ \{c_k, c_k^\dagger\} = \delta_{kl} $$

(3)

$$ \{\bar{c}_k, \bar{c}_k^\dagger\} = \delta_{kl} $$

(4)

The operator algebra is key to the Wigner analysis in Section II. In short, one can define a generalized Wigner function for the quantum system if one can find a “closed” set of operators $\{X_i\}$, where the bath-averaged time derivatives $(dX_i/dt)_{\text{ad}}$, $(d(X_i X_j)/dt)_{\text{ad}}$ defined in Section II are always functions of the $\{X_i\}$. However, the Wigner function is only defined for bosonic operators, so we will need to perform some bosonization from the start. Define the following operators:

$$ \sigma_{-k} = c_k \bar{c}_k $$

(5)

$$ \sigma_{+k} = c_k^\dagger \bar{c}_k^\dagger $$

(6)

$$ n_k = c_k^\dagger c_k $$

(7)

$$ \bar{n}_k = \bar{c}_k^\dagger \bar{c}_k $$

(8)

$$ Q_k = n_k \bar{n}_k = \sigma_{+k} \sigma_{-k} = c_k^\dagger \bar{c}_k^\dagger \bar{c}_k c_k $$

(9)

This is similar to the operator algebra in an ensemble of two-level atoms, but there are some extra terms. Analogous to the atom ensemble, the electronic polarization is given by $\sigma_{-k}$. However, the free-carrier system contains two number operators $n_k, \bar{n}_k$ rather than one, as well as a pairing operator $Q_k$. These arise because the electrons and holes in the free-carrier system have more freedom of movement: in an ensemble of atoms, each electron is confined to its parent atom and $n_k = \bar{n}_k = Q_k$, while in a semiconductor these three quantities are no longer equal, since electrons and holes freely scatter between modes $k$.

The bosonized operators are closed under commutation, with the following nonzero commutators:

$$ [\sigma_{\pm k}, n_l] = \mp \delta_{kl} \sigma_{\mp k} $$

(10)

$$ [\sigma_{\pm k}, \bar{n}_l] = \mp \delta_{kl} \sigma_{\mp k} $$

(11)

$$ [\sigma_{\pm k}, \sigma_{\mp l}] = \pm \delta_{kl} (n_k + \bar{n}_k - 1) $$

(12)

$$ [\sigma_{\pm k}, Q_l] = \mp \delta_{kl} \sigma_{\mp k} $$

(13)

Note that these are all commutators, rather than anti-commutators, because the operators have been bosonized.

### A. Hamiltonian

The Hamiltonian consists an electronic part, an optical part, and an interaction part. It can be written as:

$$ H = H_{\text{ph}} + H_{\text{el}} + H_{\text{int}} $$

(14)

In this work I am ignoring electron-electron effects, which could give rise to excitons at low temperature.

In standard single-particle electrodynamics, to first order in the optical field the light-matter coupling goes as:

$$ H_{\text{int}} = \frac{e \vec{A} \cdot \vec{p}}{m_0} $$

(15)

This can be generalized to many-particle systems by “second-quantizing” the Hamiltonian in terms of fermionic creation / annihilation operators $f_k, f_k^\dagger$ [10]:

$$ H_{\text{int}} \rightarrow \frac{e}{m_0} \sum_{k,l \in \text{states}} f_k^\dagger (k | A \cdot \vec{p} | l) f_l $$

(16)

Consider a two-band model. The $f_k$ here represent both valence-band and conduction-band states. If the field $A$ is driving at optical frequencies, only transitions between the valence band and conduction band need be considered – for these, the Hamiltonian becomes:

$$ H_{\text{int}} \rightarrow \frac{e}{m_0} \sum_k \left[ c_{k}^\dagger c_{k} (k | \vec{A}(x, t) \cdot \vec{p} | k, v) + \bar{c}_{k} c_{k} (k, v | \vec{A}(x, t) \cdot \vec{p} | k, c) \right] $$

(17)
For a resonant structure, $E(x, t)$ and $B(x, t)$ depend on the normal-mode fields $E_\omega(x)$ and their time-dependent amplitude $a_\omega(t)$ (which becomes the photon annihilation operator when the system is quantized). Working in the Coulomb gauge $\phi(x, t) = 0$, $E = -\partial A/\partial t$ and $A(x, t)$ is given by:

$$A(x, t) = \text{Re} \left[ \sum \omega -i \sqrt{2\hbar/\omega_0} a_\omega \tilde{E}_\omega(x) e^{-i\omega t} \right]$$

(18)

Here $E_\omega$ is normalized so that $\int n(x)^2 |E_\omega|^2 d^3x = 1$, and $a_\omega$ is the photon annihilation operator. For a good resonator, typically only one frequency $\omega$ is relevant (though multiple frequencies is a simple extension of this work), so hereafter we replace $a_\omega \rightarrow a$. Going into the interaction picture and neglecting rotating-wave terms and adding an arbitrary phase shift to the $c_i$ to fix the sign of $g_k$, we find:

$$H_{\text{int}} = \sum_k i g_k \left( a^\dagger c_k \bar{c}_k - a c_k^\dagger \bar{c}_k^\dagger \right)$$

(19)

with coupling constant $g_k$ given by:

$$g_k = \frac{e}{m a} \sqrt{\frac{\hbar}{2 \omega_0}} | \tilde{E}_\omega(x) \cdot \langle k, c|\tilde{\beta}|k, v \rangle |$$

(20)

The electronic and photon parts to the Hamiltonian take their canonical forms. The end result is:

$$H = \Delta_e a^\dagger a + \sum_k \left[ \frac{\Delta_k n_k + \bar{n}_k}{2 \hbar_{\text{ph}}} + i g_k (a^\dagger \sigma_{-k} - a \sigma_{+k}) \right]$$

(21)

where $\Delta_e = \hbar (\omega_e - \omega_{\text{ph}})$ is the cavity resonance detuning, $\Delta_k = E_{k,c} - E_{k,v} - \hbar \omega_{\text{ph}}$ is the detuning of the transition, and $g_k$ is the atom-photon coupling, given above.

Hamiltonian (21) resembles the cavity QED Hamiltonian. This is because both systems contain an optical term, and electronic term, and a light-matter interaction of the form (15). Thus, it should not be surprising if free-carrier cavities exhibit many of the same phenomena observed in cavity QED, e.g. bistability, amplification, limit cycles (17).

B. External Interactions

In the cavity, the optical field is relatively well isolated from its environment. The two primary interactions are optical absorption, which gives rise to particle-hole pairs and is treated through Eq. (21), and coupling to the external waveguide. Because these couplings are usually quite weak, the optical field tends to retain its coherence in spite of them.

The same is not true for the carriers. Many forces act to dephase, thermalize, and scatter the free carriers on very quick timescales (typically around $\sim 10 - 100$ fs) (2, 15). Even for very poor cavities with $Q \lesssim 1000$, this is much faster than the photon lifetime. The practical upshot of this will be that, on optical timescales, the “coherent” part to the carrier fields $\sigma_{\pm k}$ can be adiabatically eliminated and only the “slowly-varying” carrier numbers $n_k, \bar{n}_k, Q_k$ remain relevant to the system.

For a bosonic, Markovian bath, external interactions can be treated by adding extra Lindblad terms to the Master equation (9). The main external processes are given in Figure 2 above. As Lindblad terms, they are:

- Cavity Loss, mediated by
  $$L_{\text{cav}} = \sqrt{\kappa} a$$
  (22)

- Recombination, mediated by
  $$L_{\text{rc,k}} = \sqrt{\gamma_{\text{rc,k}}} \sigma_{-k}$$
  (23)

- Nonradiative Decay, mediated by
  $$L_{\text{nr,k}} = \sqrt{\gamma_{\text{nr,k}}} c_k, \sqrt{\gamma_{\text{nr,k}}} \bar{c}_k$$
  (24)

- Scattering / Dephasing, mediated by
  $$L_{\text{sc,k}\rightarrow \text{l}} = \sqrt{\gamma_{\text{sc,k}}} c_k, \sqrt{\gamma_{\text{sc,k}}} \bar{c}_k$$
  (25)

Some materials also have significant free-carrier absorption and two-photon absorption. For simplicity, these are not treated presently, but are discussed in Section IVB and the appendix.

C. Identical Mode Assumption

To make the computation more tractable, we assume for now that all carrier modes $k$ are identical. This is not a realistic assumption, and later on in Section IVA we will extend the result to non-identical modes. But assuming identical modes for now, $\Delta_k, g_k$, and all the
\( \gamma_k \)'s become independent of \( k \). One can now define mode-summed operators:

\[
\sigma_\pm = \sum_k \sigma_{\pm k}, \text{ etc.} \tag{26}
\]

In terms of these, the Hamiltonian and interaction terms are:

\[
H = \Delta_e a^\dagger a + \frac{1}{2} \Delta_e (n + \bar{n}) + ig(a^\dagger \sigma_- - a \sigma_+) \tag{27}
\]

\[
L_{\text{cav}} = \sqrt{k} a \tag{28}
\]

\[
L_{\text{ic.k}} = \sqrt{\gamma_{\text{ic}}} \sigma_k \tag{29}
\]

\[
L_{\text{nr.km}} = \sqrt{\gamma_{\text{nr}}} \sigma_k \tag{30}
\]

\[
L_{\text{sc.k-1}} = \sqrt{\gamma_{\text{sc}}} \sigma_k \tag{31}
\]

II. WIGNER FUNCTION AND SDE'S

Under the quantum model described in Section I above, the state of the cavity is given by a density matrix \( \rho \) and evolves according to the master equation:

\[
\frac{d\rho}{dt} = -i[H, \rho] + \frac{1}{2} \sum_k \left( 2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k \right) \tag{32}
\]

Unfortunately, with an exponentially large Hilbert space, it is not practical to compute \( \rho \) or its evolution. To get around this problem, we express \( \rho \) in terms of a generalized Wigner function, compute the equations and show that they can be approximated by a Fokker-Planck equation, and solve the Fokker-Planck equation stochastically using SDE's.

This approach is derived by Carter \[20\] for optical fibers with a \( \chi^{(3)} \) interaction; the same approach can be applied to optical cavities \[10\]. In this case, there is an injective linear mapping between states and Wigner functions

\[
W(\alpha, t) = \frac{1}{\pi^2} \int d^2 \beta e^{-i(\alpha^* \beta + \alpha \beta^*)} \text{Tr} \left[ e^{i(\beta^* a^\dagger + \beta a)} \rho(t) \right] \tag{33}
\]

This mapping transforms the master equation into a PDE for \( W(\alpha, t) \). Since the mapping is linear and the master equation is first-order in time, the corresponding PDE is also linear and first-order in time. For weak optical nonlinearities, one can ignore the third-order field derivatives in the PDE, e.g. \( \partial^3 W/\partial \alpha^3 \) turning it into a Fokker-Planck equation. The distribution that solves the Fokker-Planck equation can be sampled with the corresponding SDE's \[21\].

Not only does the Wigner method work for single nonlinear cavities; it can also be applied to large circuits. In this case, the dimension of \( \rho \) scales exponentially with system size, while the complexity of the Wigner SDE's grows only linearly. The Wigner method agrees with the quantum master equation for Kerr systems with \( \gtrsim 20 \) photons per mode \[10\].

Gronchi and Lugiato \[11\] extended this method to weakly-coupled many-atom cavity QED. In this case, coupling between the optical field and the atoms drives the optical nonlinearity. However, if there are many atoms, this suffers from the same exponential Hilbert-space scaling problem as our carrier system. Lugiato's approach defined a generalized Wigner function in terms of average atomic quantities, \( \sigma_x, \sigma_y, \sigma_z \), and showed that the PDE for this function can be approximated by a Fokker-Planck equation when the coupling is weak and the number of atoms is large. This result works for atom traps, but it does not apply directly to free-carrier systems because carriers can scatter between modes, whereas trapped atoms hold onto their electrons.

Following Lugiato's approach, we first define a generalized Wigner function for the free-carrier system. Let \( X \) be a complete bosonic operator algebra for the system, as follows:

\[
X = [a, a^\dagger, \sigma_-, \sigma_+, n, \bar{n}, Q] \tag{34}
\]

This is a 7-dimensional, operator-valued vector. The Wigner function is a 7-dimensional distribution, defined over the c-number variables:

\[
x = [\alpha, \alpha^*, v, v^*, m, \bar{m}, q] \tag{35}
\]

It is given by a Fourier transform of the characteristic function, just like the optical case:

\[
W(x, t) = \int d^7 y e^{-i \sum_k x_k y_k} \text{Tr} \left[ e^{i \sum_k X_k y_k} \rho(t) \right] \tag{36}
\]

To compute the Wigner SDE's, first we need to evaluate the operator derivatives:

\[
\frac{dX_m}{dt} \bigg|_{\text{ad}}, \quad \frac{d(X_m X_n)}{dt} \bigg|_{\text{ad}} \tag{37}
\]

These are computed using the adjoint equation:

\[
\frac{dA}{dt} \bigg|_{\text{ad}} = -i[A, H] + \frac{1}{2} \sum_k 2L_k^\dagger A L_k - L_k^\dagger L_k A - A L_k^\dagger L_k \tag{38}
\]

which is defined so that \( \text{Tr}[(dA/dt)_{\text{ad}} \rho] = \text{Tr}(A(d\rho/dt)) \).

The most important thing is this procedure is the fact that, applying (38) to the Hamiltonian \[21\] and couplings from Sec. (13), the time derivatives of \( X_m, X_m X_n \) can be expressed in terms of the \( X_m \). This means that the operator algebra \( X \) is closed under time evolution, and the
Wigner function $W(x)$ provides a complete description of the system evolution.

Truncated to second order, the PDE for the generalized Wigner function is a Fokker-Planck equation, and its solution is a probability distribution that is sampled by solving a set of SDE’s:

$$dx_m = \mu_m \, dt + \sum_n R_{mn} \, dw_n$$

with $dw_n$ a Wiener process and

$$\mu_m = C^{(1)}_m(x) = \left( \frac{dX_m}{dt} \right)_p$$

$$\langle RR^T \rangle_{mn} = C^{(2)}_{mn}(x) = \left( \frac{dX_m X_n}{dt} \right)_p - x_m C^{(1)}_n - C^{(1)}_{nm}$$

where $\langle \cdots \rangle_p$ is defined so that normally ordered products return simple c-number polynomials, e.g. $(a)_{p} = \alpha$, $\frac{i}{2}(a^\dagger a + a a^\dagger)_{p} = \alpha^* \alpha$, $(a a^\dagger)_{p} = \alpha^* \alpha - \frac{1}{2}$, etc.

Computing the cumulant matrices $C^{(1)}$ and $C^{(2)}$ from the SLH model \textsuperscript{(27, 31)} is straightforward but very tedious, so we used Mathematica to derive the result. The details are given in Appendix A; the SDE’s are:

$$d\alpha = \left[ \left( -\frac{\Delta_e}{2} - i\Delta_e \right) \alpha + g v \right] \, dt + d\xi_1$$

$$d\alpha^* = \left[ \left( -\frac{\Delta_e}{2} + i\Delta_e \right) \alpha^* + g v^* \right] \, dt + d\xi_2$$

$$dv = \left[ -\alpha (N - m - \bar{m}) g + \left( -\frac{\gamma_{sc}}{2} - i\Delta_e \right) v \right] \, dt + d\xi_3$$

$$dv^* = \left[ -\alpha^* (N - m - \bar{m}) g + \left( -\frac{\gamma_{sc}}{2} + i\Delta_e \right) v^* \right] \, dt + d\xi_4$$

$$dm = \left[ -g(\alpha v^* + \alpha^* v) - \gamma_{nr} m - \gamma_{rc} q \right] \, dt + d\xi_5$$

$$d\bar{m} = \left[ -g(\alpha v^* + \alpha^* v) - \gamma_{nr} \bar{m} - \gamma_{rc} q \right] \, dt + d\xi_6$$

$$dq = -\gamma_{sc} (m - \bar{m}) \, dt + d\xi_7$$

where the noise processes $d\xi_i$ have the covariance matrix:

$$d\xi_i d\xi_j^T = C^{(2)} \, dt$$

where $C^{(2)} = \frac{1}{2} \gamma_{sc} + \frac{1}{2} \gamma_{rc} + \frac{1}{2} \gamma_{nr} + \frac{1}{2} \gamma_{ps} + \frac{1}{2} \gamma_{ph}$ is the sum of the terms in Eqs. (A1-A10). Note how Equations (42-48) resemble both the Maxwell-Bloch equations and the Langevin equations for many-atom cavity QED derived by Gronchi and Lugliato \textsuperscript{(11)}. However, because of scattering between carrier modes, we need to keep track of $m$, $\bar{m}$ and $q$ separately.

### III. APPROXIMATIONS

Two approximations make these equations much simpler: nondegenerate excitation and fast dephasing. Fast dephasing assumes that the scattering rate $\gamma_{sc}$ and detuning $\Delta_e$ are faster than any other timescale in the system.

$$\gamma_{nr}, \gamma_{rc}, \gamma_{fc} \ll \gamma_{sc}, \Delta_e$$

This will ultimately lead to an adiabatic elimination of the dipole terms $(v, v^*)$ and the pair density $q$. These variables will be replaced by their steady-state values, and a new set of SDE’s will be obtained for the reduced basis $(\alpha, \alpha^*, m, \bar{m})$.

Fast dephasing is almost always a valid approximation. It is related to the relaxation-time approximation that holds form most semiconductors \textsuperscript{(2)}. In useful, optimized free-carrier devices, all of the carrier timescales $-\gamma_{nr}, \gamma_{rc}, \gamma_{fc}$ are of order the optical timescale. To achieve strong carrier effects, we generally have cavities with $Q \gtrsim 1000$, giving an optical timescale of $\gamma_{ph} \gtrsim ps$. Ultrafast studies show that inter-mode scattering takes place on times of order $10-100$ fs \textsuperscript{(15, 19)}, giving scattering rates at least $10-100$ times faster than any other timescale in the system.

Nondegenerate excitation assumes that the number of carriers is much less than the number of carrier modes $- \text{in other words, the valence and conduction bands are far from degenerately filled with electrons or holes. }$

This approximation is invoked by setting $m, \bar{m}, q \ll N$

This simplifies the equations of motion by discarding effects like absorption saturation that are negligible for low-power, high-$Q$ optical logic devices.

#### A. Fast Dephasing

The adiabatic limit we take is $\epsilon \to 0$, where the detuning, scattering rate and coupling scale as follows:

$$\Delta_e \sim \gamma_{sc} \sim O(\epsilon^{-1}), \quad g \sim O(\epsilon^{-1/2})$$

and everything else goes as $O(1)$. In this limit, we can replace $(v, v^*, q)$ with their steady-state values:

$$v \, dt \to \frac{1}{\frac{1}{2} \gamma_{sc} + i \Delta_e} \left[ -\alpha (N - m - \bar{m}) g \, dt + d\xi_3 \right] + O(\epsilon)$$

$$v^* \, dt \to \frac{1}{\frac{1}{2} \gamma_{sc} - i \Delta_e} \left[ -\alpha^* (N - m - \bar{m}) g \, dt + d\xi_4 \right] + O(\epsilon)$$

$$q \, dt \to \frac{m \bar{m}}{N} \, dt + O(\epsilon^{1/2})$$
Thus, \( v, v^* \sim O(e^{1/2}) \) and \( q - m\bar{m}/N \sim O(e^{1/2}) \). Retaining only \( O(1) \) terms in the noise matrix from (A1), we obtain:

\[
C^{(2)} = \begin{pmatrix}
0 & \kappa/2 & 0 & 0 & 0 & 0 \\
\kappa/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma_{sc}N/2 & 0 & 0 & 0 \\
0 & 0 & 0 & m\gamma_{nr} + q\gamma_{rc} & q\gamma_{rc} & 0 \\
0 & 0 & 0 & 0 & \bar{m}\gamma_{nr} + q\gamma_{rc} & 0 \\
0 & 0 & 0 & 0 & 0 & (q + m\bar{m}/N)\gamma_{sc}
\end{pmatrix}
\]

(56)

Likewise, retaining only \( O(1) \) terms in the equations of motion (42-48), we obtain the adiabatically eliminated SDEs. In this case, terms proportional to \( v, v^* \) still matter, because even though \( v \sim O(e^{1/2}) \), it is are pre-multiplied in the equations of motion by \( g \sim O(e^{-1/2}) \), and \( O(e^{1/2})O(e^{-1/2}) = O(1) \). The equations of motion become:

\[
d\alpha = \left[ \left( -\frac{\kappa}{2} - i\Delta_e \right) \alpha - \frac{g^2(N - m - \bar{m})}{\frac{1}{2}\gamma_{sc} + i\Delta_e} \right] dt + d\xi_1 + \frac{g}{\frac{1}{2}\gamma_{sc} + i\Delta_e} d\xi_3
\]

(57)

\[
d\alpha^* = \left[ \left( -\frac{\kappa}{2} + i\Delta_e \right) \alpha^* - \frac{g^2(N - m - \bar{m})}{\frac{1}{2}\gamma_{sc} - i\Delta_e} \alpha^* \right] dt + d\xi_2 + \frac{g}{\frac{1}{2}\gamma_{sc} - i\Delta_e} d\xi_4
\]

(58)

\[
dm = \left[ \frac{g^2(N - m - \bar{m})\gamma_{sc}}{\left( \frac{1}{2}\gamma_{sc} \right)^2 + \Delta_e^2} \alpha^* \alpha - \gamma_{nr}m - \frac{m\bar{m}}{N}\gamma_{rc} \right] dt + \frac{-g\alpha^* d\xi_3}{\frac{1}{2}\gamma_{sc} + i\Delta_e} + \frac{-g\alpha d\xi_4}{\frac{1}{2}\gamma_{sc} - i\Delta_e} + d\xi_5 + \frac{\gamma_{rc}}{\gamma_{sc}} d\xi_7
\]

(59)

\[
d\bar{m} = \left[ \frac{g^2(N - m - \bar{m})\gamma_{sc}}{\left( \frac{1}{2}\gamma_{sc} \right)^2 + \Delta_e^2} \alpha^* \alpha - \gamma_{nr}\bar{m} - \frac{m\bar{m}}{N}\gamma_{rc} \right] dt + \frac{-g\alpha^* d\xi_3}{\frac{1}{2}\gamma_{sc} + i\Delta_e} + \frac{-g\alpha d\xi_4}{\frac{1}{2}\gamma_{sc} - i\Delta_e} + d\xi_6 + \frac{\gamma_{rc}}{\gamma_{sc}} d\xi_7
\]

(60)

The band-filling carrier-dependent detuning can be defined as follows:

\[
\delta = \delta = \frac{g^2}{\Delta_e - \frac{1}{2}i\gamma_{sc}}
\]

(61)

Since \( \Delta_e = \omega_e - \omega \), this function has one pole (for \( \omega \)) in the lower half-plane, \( \omega = \omega_e - \frac{1}{2}i\gamma_{sc} \). As a result, its real and imaginary parts satisfy the Kramers-Kronig relations. The carrier-dependent dispersion and absorption are given by the real and imaginary parts of \( \delta \), respectively. Another important quantity is the linear absorption in the absence of carriers, related to the per-carrier absorption saturation as follows:

\[
\eta_{bf} = 2 \text{Im}[\delta] \Rightarrow \eta = N\eta_{bf}
\]

(62)

In terms of these, the SDE’s may be written in the following form:

\[
d\alpha = \left[ \left( -\frac{\kappa}{2} - i\delta(N - m - \bar{m}) \right) \alpha \right] dt + d\xi_{\alpha}
\]

(63)

\[
d\alpha^* = \left[ \left( -\frac{\kappa}{2} + i\delta^*(N - m - \bar{m}) \right) \alpha^* \right] dt + d\xi_{\alpha^*}
\]

(64)

\[
dm = \left[ \eta_{bf}(N - m - \bar{m})\alpha^* \alpha - \gamma_{nr}m - \frac{m\bar{m}}{N}\gamma_{rc} \right] dt + d\xi_{m}
\]

(65)

\[
d\bar{m} = \left[ \eta_{bf}(N - m - \bar{m})\alpha^* \alpha - \bar{\gamma}_{nr}\bar{m} - \frac{m\bar{m}}{N}\gamma_{rc} \right] dt + d\xi_{\bar{m}}
\]

(66)

The noises in (63-66) satisfy a new covariance matrix related to the full \( C^{(2)} \) by

\[
C^{(2)} = TC^{(2)}_{\text{full}} T^T
\]

(67)
where $T$ is the transformation matrix $d\xi = T d\xi_{\text{full}}$:

$$
T = \begin{bmatrix}
1 & 0 & -i\delta/g & 0 & 0 & 0 \\
0 & 1 & i\delta/g & 0 & 0 & 0 \\
0 & 0 & i\delta \alpha + g & 0 & 0 & 1 \\
0 & 0 & i\delta \alpha^* - g & -i\delta^* \alpha/g & 0 & 1 \\
\end{bmatrix}
$$

(68)

The covariance matrix is simple enough to factor by hand, writing the noises in terms of real and complex Wiener processes as follows:

$$
d\xi_\alpha = -\sqrt{\kappa} d\beta_\alpha - \sqrt{\eta} d\beta_\eta \\
d\xi_{\alpha^*} = -\sqrt{\kappa} d\beta_{\alpha^*} - \sqrt{\eta} d\beta_{\eta^*} \\
d\xi_m = \sqrt{\eta}(\alpha^* d\beta_\eta + \alpha d\beta_{\eta^*}) + \sqrt{\gamma} m d\omega_m + \sqrt{\gamma_{rc}/N} d\omega_q \\
d\xi_{\bar{m}} = \sqrt{\eta}(\alpha^* d\beta_{\bar{\eta}} + \alpha d\beta_{\bar{\eta}^*}) + \sqrt{\gamma_{rc}/N} d\omega_{\bar{m}} + \sqrt{\gamma_{rc}/N} d\omega_{\bar{q}}
$$

(69-72)

The $d\beta_\alpha$ is the vacuum noise of the input field and $d\beta_\eta$ is the noise due to linear absorption; each behaves as a vacuum Wiener process $d\beta = \frac{1}{2} dt$ [10]. The $d\omega_m$, $d\omega_{\bar{m}}$ and $d\omega_q$ are real-valued noises due to carrier loss and recombination, and go as $d\omega^2 = dt$.

The noise term for $\alpha$ is fairly standard for open quantum systems: a sum of two vacuum noises. The noise terms for $m$ and $\bar{m}$ have Poisson statistics: for each process with rate $R dt$, there is a corresponding noise term $\sqrt{R} dw$. Since carrier generation involves photon absorption, one should not be surprised by the Poisson noise on this signal. Likewise, since the carrier number is quantized and carrier decay is a random process, there should also be Poisson noise on the decay terms.

B. Nondegenerate Excitation

Equations (63-66) and (69-72) can be simplified further if we assume nondegenerate excitation. This is a good assumption in the absence of electrical pumping, because the optical powers needed to achieve degenerate doping are very large, taking us well outside the quantum regime of interest in this paper. (However, degenerate doping may still be relevant for confined systems like quantum wells, quantum wires or thin-film materials, where the carrier density is lower.) Nondegenerate excitation assumes that $m, \bar{m} \ll N$.

The carrier-dependent dispersion and absorption go as $\delta m, \delta \bar{m}$. To have interesting carrier effects, either the dispersive effect or the absorptive effect must be of order unity. Obviously the absorptive effect cannot be, since this would put us in the degenerate-excitation regime $m, \bar{m} \sim N$. On the other hand, the dispersive effect can still be relevant. Free-carrier dispersion will scale as $\delta(m + \bar{m})$, while absorption will scale as $\eta = 2 N \text{Im} \delta$. This means that:

$$
\frac{\text{Re}[\delta]}{\text{Im}[\delta]} \sim \frac{1/(m + \bar{m})}{1/N} \sim \frac{N}{m + \bar{m}} \gg 1
$$

(73)

So $\delta$ is sufficiently real that we can ignore its imaginary component. In this case, we can approximate

$$
\delta = \frac{g^2}{\Delta_e} = \frac{g^2}{\omega_e - \omega}
$$

(74)

All of the $N - m - \bar{m}$ terms can be replaced with $N$ in the SDE’s. The absorption coefficient becomes a constant $\eta$, the constant term in the dispersion can be absorbed into $\Delta_e$, and the SDE’s take the following form:

$$
da = \left[ -\frac{\kappa + \eta}{2} - i\Delta_e - i(\delta m + \delta \bar{m}) \right] \alpha dt + d\xi_\alpha \\
dm = \left[ \eta \alpha^* \alpha - \gamma_{rc} N \right] dm + d\xi_m \\
d\bar{m} = \left[ \eta \alpha^* \alpha - \gamma_{rc} N \right] d\bar{m} + d\xi_{\bar{m}}
$$

(75-77)

The noise terms still take the form of (69-72).

IV. RELATED MODELS

Equations (75-77) are the simplest free-carrier model: identical modes, no two-photon absorption, no interaction between carriers, no excitons. In many ways it is unrealistic. However, it forms the basis for generalized models that include these effects and better approximate the real system.

A. Non-Identical Modes

The most obvious generalization is to include many non-identical carrier modes. This means that, rather than grouping all of the modes together into $(m, \bar{m})$, they are binned into spectrum of modes $(m_\alpha, \bar{m}_\alpha)$. A similar binning technique is used in many-atom cavity QED when the atomic couplings are not equal [17]. The equations are a straightforward generalization of (75-77):
\[ d\alpha = \left[ -\frac{\kappa + \eta}{2} - i\Delta_{c} - i \sum_{a} (\delta_{a}\alpha a + \delta_{a} \bar{m}_{a}) \right] \alpha dt - d\xi_{a} \]  

(78)

\[ dm_{a} = \left[ \eta_{a} a^{*} a - \gamma_{nr,a} m_{a} - \gamma_{rc,a} \frac{m_{a} \bar{m}_{a}}{N_{a}} \right] dt + d\xi_{m,a} \]  

\[ + \sum_{b} \left( \gamma_{b \rightarrow a} m_{b} - \gamma_{a \rightarrow b} m_{a} \right) dt + d\xi_{m,a} \]  

(79)

\[ d\bar{m}_{a} = \left[ \eta_{a} a^{*} a - \gamma_{nr,a} \bar{m}_{a} - \gamma_{rc,a} \frac{m_{a} \bar{m}_{a}}{N_{a}} \right] dt + d\xi_{m,a} \]  

\[ + \sum_{b} \left( \gamma_{b \rightarrow a} \bar{m}_{b} - \gamma_{a \rightarrow b} \bar{m}_{a} \right) dt + d\xi_{m,a} \]  

(80)

The only change here is the introduction of indices and the cross-scattering terms \( \gamma_{a \rightarrow b} \). These terms, like the other carrier excitation / decay terms, have Poisson statistics. The Poisson statistics of different modes are, of course, correlated just as the flows are – this conserves total carrier number in the scattering processes.

If scattering between bins \( m_{a} \) is fast compared to carrier excitation or decay, we can replace the mode occupancies by the thermal average \( m_{a} = f_{c,a}(T)m, \bar{m}_{a} = f_{h,a}(T)\bar{m} \), where \( f_{c,a}, f_{h,a} \) are normalized Boltzmann distributions. In terms of the total carrier numbers \( m = \sum_{a} m_{a} \) and \( \bar{m} = \sum_{a} \bar{m}_{a} \), we recover Equations (75-77), with the effective rates:

\[ \delta = \sum_{a} \delta_{a} f_{a}(T) \]  

\[ \tilde{\delta} = \sum_{a} \delta_{a} \bar{f}_{a}(T) \]  

\[ \gamma_{nr} = \sum_{a} \gamma_{nr,a} f_{a}(T) \]  

\[ \bar{\gamma}_{nr} = \sum_{a} \bar{\gamma}_{nr,a} \bar{f}_{a}(T) \]  

\[ \gamma_{rc} = N \sum_{a} \gamma_{rc,a} \frac{f_{a}(T) \bar{f}_{a}(T)}{N_{a}} \]  

(81)

\section{B. Other Processes: Kerr, TPA, FCA}

A host of additional processes may be relevant in semiconductor cavities: among the most important are the Kerr effect, two-photon absorption (TPA), and free-carrier absorption (FCA). Thermal effects and excitonic effects, while very important for some systems, are beyond the scope of this paper.

\subsection{1. TPA and Kerr}

In indirect-gap materials, like silicon, the linear absorption is not an effective pathway for carrier generation. Instead, two photon absorption is the dominant excitation process. Typically, two-photon absorption also comes with a dispersive (Kerr) effect. In other cases, the band gap is tuned to be very close to the photon energy, and both processes are important. Unlike linear absorption, which tends to create carriers very close to the band gap, two-photon absorption tends to create highly excited carriers with excess kinetic energy. After excitation, these carrier quickly thermalize and subsequently decay.

We can model this with the following Hamiltonian and decay process:

\[ H = \frac{1}{2} \Delta_{x}(n_{x} + \bar{n}_{x}) + ig \sum_{x} ((a^{\dagger})^{2}\sigma_{x} - a^{2}\sigma_{x}) \]  

(82)

\[ L_{x \rightarrow k} = \sqrt{\gamma_{k} \epsilon_{x} \epsilon_{k}^{\dagger}} \sqrt{\gamma_{k} \epsilon_{x} \epsilon_{k}} \]  

(83)

where the new modes \( \epsilon_{x}, \epsilon_{k}^{\dagger} \) defined for the highly excited carriers. Note that, as these modes are highly excited, there is no process \( L_{k \rightarrow x} \).

Since the excited state is so short-lived, it can be adiabatically eliminated. For on-resonant transitions \( \Delta_{x} = 0 \) this gives a two-photon absorption term \( \beta \); in the off-resonant case \( \Delta_{x} \neq 0 \), one finds two photon absorption plus a dispersive \( \chi^{(3)} \) (Kerr) term.

These effects add the following terms to the Wigner equations:

\[ \Delta(da) = (-i \chi - \beta)(\alpha^{*} a dt - 2\sqrt{\beta} \alpha^{*} \beta dt) \]  

(84)

\[ \Delta(dm) = \beta(\alpha^{*} a)^{2} dt + \sqrt{\beta} ((\alpha^{*})^{2} \beta dt + \alpha^{*} d\beta) \]  

(85)

\[ \Delta(d\bar{m}) = \beta(\alpha^{*} a)^{2} dt + \sqrt{\beta} ((\alpha^{*})^{2} \beta dt + \alpha^{*} d\beta) \]  

(86)

Note how Equations (84-86) predict that single electron-hole pair is created for every two photons absorbed, in contrast to linear absorption (eq:05-csde-4), where the ratio is one-to-one.

\subsection{2. Free-Carrier Absorption}

In some materials, including silicon, free carriers can increase the absorption of the medium, an effect known as free-carrier absorption. In addition, for indirect bandgap materials, the free-carrier dispersion is larger than as free-carrier absorption. In addition, for indirect bandgap materials, the free-carrier dispersion is larger than a linear medium, an effect known as free-carrier absorption. However, for indirect bandgap materials, the free-carrier dispersion is larger than the bandgap.

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\[ da = \left[ -\frac{K}{2} - i \Delta_c - i(\delta m + \delta \bar{m}) \right] \alpha dt + \left[ -\frac{\eta}{2} \alpha dt - \sqrt{\kappa_d \beta} \right] \alpha \beta_d \left[ -i \chi - \beta \right] (\alpha^* \alpha) \alpha dt - 2 \beta \alpha^* \kappa \beta_d \alpha \beta_d \left[ -\sqrt{\kappa_c} d \beta \right] \right] \]

\[ dm = \left[ -\gamma_{nr} m dt - \sqrt{\gamma_{nr} m} \frac{\bar{m}}{N} dw_{nr} \right] + \left[ -\gamma_{nc} \frac{m}{N} \frac{\bar{m}}{N} dt + \sqrt{\gamma_{nc} \frac{m}{N} \frac{\bar{m}}{N} dw_{nc}} \right] - \left[ \alpha^* (\alpha)_{\eta} + \alpha (\alpha)_{\eta} \right] - \frac{\alpha^* (\alpha)_{\beta} + \alpha (\alpha)_{\beta}^*}{2} \]

\[ d\bar{m} = \left[ -\gamma_{nr} \bar{m} dt - \sqrt{\gamma_{nr} \bar{m}} \frac{\bar{m}}{N} dw_{nr} \right] + \left[ -\gamma_{nc} \frac{\bar{m}}{N} \frac{\bar{m}}{N} dt + \sqrt{\gamma_{nc} \frac{\bar{m}}{N} \frac{\bar{m}}{N} dw_{nc}} \right] - \left[ \alpha^* (\alpha)_{\eta} + \alpha (\alpha)_{\eta} \right] - \frac{\alpha^* (\alpha)_{\beta} + \alpha (\alpha)_{\beta}^*}{2} \]

C. Single-Carrier Approximation

One can reduce equations (90-92) further if only one of the carrier species is relevant (for instance in silicon, where \( \delta \gg \delta \) due to the plasma effect [3]), if the recombination process \( \gamma_{nc} \) is dominant, or if the number of recombination sites is limited so that electrons and holes tend to decay together. In any of these cases, one can impose the constraint \( m = \bar{m} \equiv N_c \), set \( \delta + \delta \to \delta \), and the equations of motion become:

\[ da = \left[ -\frac{K}{2} - i \Delta_c - i \delta N_c \right] \alpha dt + \left[ -\frac{\eta}{2} \alpha dt - \sqrt{\kappa_d \beta} \right] \alpha \beta_d \left[ -i \chi - \beta \right] (\alpha^* \alpha) \alpha dt - 2 \beta \alpha^* \kappa \beta_d \alpha \beta_d \left[ -\sqrt{\kappa_c} d \beta \right] \right] \]

\[ dN_c = \left[ -\gamma_{nr} N_c dt - \sqrt{\gamma_{nr} N_c} d \alpha \right] \left[ -\gamma_{nc} \frac{N_c}{N} \frac{\alpha}{N} dt + \sqrt{\gamma_{nc} \frac{N_c}{N} \frac{\alpha}{N} d \alpha} \right] - \left[ \alpha^* (\alpha)_{\eta} + \alpha (\alpha)_{\eta} \right] - \frac{\alpha^* (\alpha)_{\beta} + \alpha (\alpha)_{\beta}^*}{2} \]

V. STEADY-STATE BEHAVIOR

A. Steady-State Limit

In the steady-state case, we set all noise terms to zero and solve for \( N = \bar{N} = 0 \). Solving for \( \bar{N} = 0 \), the steady-state internal field \( \bar{N} \) can be related to \( N_c \) and the input field \( \beta_{in} \) as follows:

\[ \alpha = \frac{-\sqrt{\kappa} \beta_{in}}{\kappa + \eta} + i(\Delta + \delta N_c) \]

This is the familiar formula for the field in a resonant cavity, where the detuning \( \Delta + \delta N_c \) depends on the free-carrier number. Solving the \( N = 0 \) equation gives \( N = (\eta/\gamma_{nc}) \alpha^* \alpha \). This can be rearranged into a polynomial equation for \( \alpha^2 \) as:

\[ \kappa \beta_{in}^2 = (\alpha^* \alpha) \left( (k/2)^2 + (\Delta + (\eta\delta/\gamma_{nc})(\alpha^* \alpha))^2 \right) \]

When the external power \( P = \beta_{in}^2 \beta_{in} \) is set, this is a cubic equation for the internal photon number \( N_{ph} = \alpha^* \alpha \). It is the same optical bistability cubic as the Kerr cavity [8] [23], with the effective Kerr nonlinearity:

\[ \chi_{eff} = \frac{\eta \delta}{\gamma_{nr}} \]
One can solve the cubic (96) to obtain $\alpha^*\alpha$; it is not always uniquely defined. Just like Kerr cavities and atom cavities, free-carrier cavities exhibit hysteresis and bistability, with both “low” and “high” intensity states being allowed for the same input power. Figure 4 shows the stable lower- and upper states, and an unstable middle-state, for varying values of $\Delta$.

The intuition behind this bistability is that, when the cavity is off resonance and a sufficiently large number of carriers are injected, it will shift back on resonance. If there is a strong enough input, then a large power builds up inside the cavity and this large carrier population can be maintained through absorption, giving rise to the high state. On the other hand, if there are no carriers to begin with, the cavity stays off resonance and there is never enough power in the cavity to raise the carrier number – hence the low state. Analytically, one can show that the bifurcation sets in when:

$$\Delta < -\sqrt{\frac{3}{4}}(\kappa + \eta)$$  \hspace{1cm} (98)

Much of our intuition behind free-carrier nonlinearities comes from this steady-state picture. It does not include any quantum effects or even any dynamics, but the shapes of the curves in Figure 4 suggest that the device could be used as an amplifier or a switch. We will show in a later paper that free-carrier cavities can do much more than this, but that will build on the fundamentals discussed here.

The steady-state picture has been amply discussed in the literature \cite{8,17,23}, so it is not worth describing in more detail here. Rather, we now proceed to look at the quantum noise and dynamics of these systems.

### B. Effective $\chi^{(3)}$ Model

Next, we go from the steady-state picture to the limit of short carrier lifetime. In this opposite limit, $\tau_{ph} \gg \tau_c$. Typical devices do not realize this limit, but it is useful because it enables an apples-to-apples comparison between the free-carrier and Kerr effects.

To adiabatically eliminate the carrier number, one replaces $N_c$ with its steady-state value:

$$N_c \rightarrow \int \frac{\eta(\alpha^*\alpha)dt + d\xi_N}{\gamma_{nr}}$$  \hspace{1cm} (99)
earizing equations of motion (93-94) around the steady-state take over. The gain and noise can be predicted by linearized between phase shifters and displacements $G = L(\beta') \triangleleft e^{i\phi} \triangleleft (\text{Cav}) \triangleleft e^{i\phi} \triangleleft L(\beta)$. Right: Symmetric two-cavity amplifier.

This gives the following SDE for the relevant dynamical variable, $\alpha$:

$$d\alpha = \left[ -\frac{\kappa + \eta}{2} - i \left( \Delta + \frac{\eta d}{\gamma_{nr}} (\alpha^* \alpha) \right) \right] \alpha dt - \sqrt{2d} \beta_{in} + d\xi' \tag{100}$$

where the $d\xi'$ is a new noise term that depends both on the $d\alpha$ and $d\xi_N$. As before, the analogy to the Kerr model is clear: Equation (100) is very close to the Wigner equations for the Kerr cavity [10], but the noise term is different. The effect of this noise term will be discussed in the following sections, where the performance of Kerr- and free-carrier based amplifiers and switches is analyzed.

VI. AMPLIFIER

Figure 4 shows that, for certain detunings, the state of the cavity changes very rapidly with a change in input power. One can imagine using such a device to amplify differential signals; if the input signal is perturbed, that perturbation will be multiplied by some gain factor in the output.

The real picture is actually a bit more complicated, since the input field has two quadratures. In the Kerr cavity, one of the quadratures is amplified while the other is de-amplified [23]. This gives rise to phase-sensitive amplification which, since there is no additional noise in the Kerr system, also squeezes the quantum noise of one quadrature below the vacuum level.

Key to an optical amplifier are its gain $G(\omega)$, its noise spectrum $S(\omega)$, and the scale on which nonlinear effects take over. The gain and noise can be predicted by linearizing equations of motion (93-94) around the steady-state value. This takes the general form:

$$d\bar{x} = A\bar{x} dt + B d\beta_{in} + \bar{F} dw \tag{101}$$

$$d\tilde{\beta}_{out} = C\bar{x} dt + \tilde{D} d\beta_{in} \tag{102}$$

where $\bar{x}$ and $\tilde{\beta}$ are doubled-up state vectors, which include the complex field operators and their conjugates [24], as well as the (real) carrier number: $x = (\delta\alpha, \delta\alpha^*, \delta N_c)$, $d\tilde{\beta} = (d\beta, d\beta^*)$ (removing any constant coherent input), and $\alpha, N$ are the steady-state values.

Linearization is key because many general results of stochastic systems theory only apply to linear or approximately linear systems [25]. For example, in a linearized system, the output squeezing spectrum can be computed exactly for Gaussian inputs [21, 26]. Many results in quantum feedback control theory are also restricted to linear systems [27, 28].

With a linearized model in hand, it is a simple matter to compute the internal state covariance $\bar{\sigma}$, the transfer and noise matrix $T(\omega), N(\omega)$, and the frequency-domain input-output relation [24, 29]:

$$\bar{A}\sigma + \bar{\sigma} A^\dagger + \frac{1}{2} BB^\dagger + \bar{F} F^\dagger = 0 \tag{103}$$

$$\beta_{out,\omega} = \left[ \bar{D} + \frac{C}{-i\omega - A} B \right] \beta_{in,\omega} + \frac{1}{-i\omega - A} \bar{F} w_\omega \tag{104}$$

Unfortunately, because the doubled-up matrices here are 3-by-3 rather than 2-by-2, the analytic results are rather cumbersome and therefore not reproduced here. Instead, in this section I compute these quantities numerically and compare the results to the Kerr system. The results here are compared against a Kerr cavity with the same effective nonlinearity, $\chi = \eta d / \gamma_{nr}$.

A. Gain

The gain is computed from the singular values of the doubled-up transfer function $T(\omega)$. If both singular values are the same, the device is a phase-insensitive amplifier. Both the Kerr and free-carrier cavities, however, only amplify one quadrature. As Figure VI A shows, they de-amplify the other quadrature as well.

At and below the ideal input $\beta_{in} \approx 9$, the Kerr and free-carrier cavities seem to amplify in the same way. For over-driven cavities, the behavior is very different. The free-carrier cavity becomes very efficient at amplifying off-resonance, whereas the Kerr cavity hardly amplifies at all.

Gain is maximized when the system is very close to instability – that is, when at least one of the eigenvalues of $A$ is very close to the imaginary axis. From (104), an eigenvalue decomposition of $A$ gives the transfer function the following form:

$$T = \bar{D} + \sum_i \frac{v_i u_i^T}{-i\omega - \lambda_i} \tag{105}$$
where \( v_i, u_i \) are related to \( B, C \) and the eigenvectors and \( \lambda_i \) are the eigenvalues of \( A \). Since \( B, C \sim O(\sqrt{\kappa}) \), the numerator term is proportional to \( \kappa \). Near the resonance, the sum is dominated by the eigenvalue closest to zero, \( \lambda_{\text{max}} \). The maximum gain should intuitively take the form:

\[
G \sim \frac{O(\kappa)}{|-i\omega - \lambda_{\text{max}}|} \quad (106)
\]

This is a Lorentzian with a peak at \( \text{Im}(\lambda_{\text{max}}) \) and bandwidth of \( \Delta \omega = -\text{Re}(\lambda_{\text{max}}) \). The peak gain is thus \( G_{\text{max}} = -O(\kappa)/(\text{Re}(\lambda_{\text{max}})) \). This gives us a gain-bandwidth relation:

\[
G_{\text{max}} \Delta \omega = O(\kappa) \quad (107)
\]

The greater the amplifier gain, the slower it responds and the narrower its bandwidth.

### B. Internal State

The internal state is computed using the Lyapunov equation \( (103) \). This time, the Wigner equations contain additional noise terms, which make the state noisier than the state of an equivalent Kerr cavity. This is plotted in the Figures 7-8. The state remains roughly Gaussian, but the size of the Gaussian is larger than in the Kerr case, especially above the inflection point.

Unlike in the Kerr case, the mode in the free-carrier cavity is never squeezed. As seen in Figure 7, the eigenvalues of the covariance matrix \( \sigma \) are always \( \geq \frac{1}{2} \), ensuring that the state is always “classical” in the sense that it has a valid \( P \) representation. Given that the carrier excitation and decay process is highly incoherent, it should not be too surprising that the cavity always remains in a classical state. But it is a clear departure from the Kerr model, and this classicality could conceivably be used to distinguish between the two in an experiment.

Also note that the noise grows linearly with the input field at high powers. This happens because the free-carrier number is constantly fluctuating, being driven by excitation and decay events that mimic a Poisson process. At high carrier numbers, this means that the cavity detuning and consequently the cavity field become very noisy. This does not happen in the Kerr cavity, where the nonlinearity is mediated by virtual transitions which do not add any noise to the system. It is a peculiar consequence of the incoherence of the free-carrier mechanism.
C. Output Noise Spectrum

Given a linearized input-output model, we can compute the squeezing spectrum (noise spectrum) for the cavity output field [21, 30]. The squeezing spectrum for quadrature $\theta$ is defined as the power spectral density of a homodyne measurement of $\beta_{\text{out}}(t)$. That is, for the following homodyne signal,

$$j_\theta(t) = e^{-i\theta} \beta_{\text{out}}(t) + e^{i\theta} \beta^*_{\text{out}}(t)$$  \hspace{1cm} (108)

the squeezing spectrum is:

$$S_\theta(\omega) = \sqrt{2P_\theta(\omega)}, \quad P_\theta(\omega) = \frac{\langle j_\theta(\omega)^* j_\theta(\omega') \rangle}{2\pi \delta(\omega - \omega')}$$  \hspace{1cm} (109)

$S_\theta(\omega)$ is normalized so that the coherent state has $S_\theta(\omega) = 1$. For general states, $S_\theta(\omega)$ depends on $\theta$. The maximum and minimum of $S_\theta(\omega)$, with respect to $\theta$, are denoted $S_+(\omega)$ and $S_-(\omega)$, respectively.

The squeezing spectrum of the Kerr cavity can be computed analytically [23]. By contrast, since the free-carrier squeezing spectrum involves the inverse of a $3 \times 3$ matrix, it is unlikely that a simple expression can be found. However, it is not difficult to compute numerically.

In Figure 9, the noise spectrum is obtained in two separate ways: first, simulating the full system in the time domain and taking the Fourier transform of the homodyned output (blue, green curves); and second, from the analytic predictions of the linearized ABCD model. These agree everywhere except for very large pump powers, where the system approaches a bifurcation.

Figure 10 displays the noise spectrum for the whole range $0 \leq \beta_{\text{in}} \leq 20$, for both Kerr and free-carrier devices. Two things are obvious. First, the noise curve (at least for the $S_+$ component) matches the general form of the gain curve in Figure VI A. This is of course necessary because there must be noise wherever there is gain. The free-carrier cavity, unlike the Kerr cavity, amplifies not only at $\omega = 0$ near the point of maximum gain, but also for $\omega \neq 0$ for $\beta_{\text{in}}$ above that point.

Unlike the Kerr cavity, the free-carrier cavity does not squeeze the output field. Regardless of the parameters, regardless of the pump power, both $S_+$ and $S_-$ are always above the vacuum level, indicating that this is a classical field with no squeezing. The Kerr cavity, on the other hand, squeezes light over a broad range of the spectrum. This is in agreement with the results of the previous section, which showed that the internal field of the free-carrier cavity was classical. If the input and intracavity field are in a classical state, so is the output.

VII. LATCH

It is also possible to construct a switching device using only amplifiers, provided the amplification is large enough [31]. The circuit in Figure 11 uses two identical amplifiers in a feedback loop. Suppose that each amplifier has a gain $G$. Consider the fate of a perturbation in the top amplifier. An input $\delta \beta$ is amplified to $G \delta \beta$. This amplifier has a fan-out of 2, so $(G/\sqrt{2}) \delta \beta$ passes to the right and exits the system, while $(G/\sqrt{2}) \delta \beta$ passes to the lower amplifier.
In the lower amplifier, it grows to \((G^2/2)\delta\beta\), is fed back into the original amplifier. After passing through this loop, the signal strength has grown to \((G^2/2)\delta\beta\). This leads to a latching instability if the gain is sufficiently large:

\[ G > \sqrt{2} \]  

(110)

Symmetry gives the latch some very desirable properties. Unlike the single-cavity switch, the two states here are symmetric. Thus, there is less worry about finding the right bias field to “balance” the low and high state, and transitions between the states look the same. But this comes at the coast of the added complexity of two cavities, plus the extra connections.

Figure 12 shows a latch simulation for the same parameters used in the previous section. Here, the detuning is set to \(\Delta = -0.7\), large enough to realize a large gain, but not large enough make an individual cavity bistable. The symmetry between the two states is very clear.

Externally driven switching in the latch is good, because it gives allows the user to set the state of the latch, which becomes a memory element. But thanks to quantum noise, Kerr and free-carrier devices and also undergo spontaneous switching. This is generally bad, because it limits the lifetime of a carrier-based memory.

In the Kerr case, spontaneous switching is driven by vacuum fluctuations [10]. In the free-carrier case, vacuum fluctuations combine with stochastic carrier excitation and decay to drive the switching process. Because there are more fluctuations, we naturally expect the free-carrier cavity to spontaneously switch at a higher rate than the Kerr cavity.

When the switching rate is low, the switching process is well described by a two-state Markov chain. In a two-state Markov chain, there are two states \(a\) and \(b\), with jump probabilities

\[ P(a \rightarrow b) = \gamma_a dt, \quad P(b \rightarrow a) = \gamma_b dt \]  

(111)

In the latch, the states are symmetric, so \(\gamma_a = \gamma_b \equiv \gamma_{sw}\). The probability of being in a given state evolves as:

\[ \frac{dP_a}{dt} = -\gamma_{sw}P_a + \gamma_{sw}P_b \]

\[ \frac{dP_b}{dt} = \gamma_{sw}P_a - \gamma_{sw}P_b \]  

(112)
Solving this linear system, one finds that the system reverts to its equilibrium distribution with a characteristic time \( \tau_{sw} = 1/(2\gamma_{sw}) \). This time can be measured from simulations of the latch by looking at the autocorrelation function \( R(\tau) \), which decays exponentially for the Markov process:

\[
R(\tau) = \frac{\langle \alpha(t)\alpha(t-\tau)^* \rangle}{\langle \alpha(t)^*\alpha(t) \rangle} \rightarrow e^{-\tau/\tau_{sw}} \quad (113)
\]

Figure 13 shows time traces of the asymmetric field \( \alpha_1 - \alpha_2 \) as the latch detuning is varied from \(-0.50\) to \(-0.54\), about where the latching transition occurs. Larger negative detunings correspond to higher gain (see Fig. 11), and likewise stronger latching. However, for a fixed detuning, the free-carrier cavity has a shorter spontaneous switching lifetime.

This is also seen in Figure 14 which plots \( \tau_{sw} \) for the free-carrier and equivalent Kerr latches. Because the free-carrier cavity has more quantum noise than the Kerr cavity, its spontaneous switching rate is higher. The effect becomes noticeable once the latching transition sets in, and grows as the latching grows stronger.

VIII. CONCLUSION

In this paper, we introduced a method to simulate optical cavities where free-carrier dispersion is the dominant nonlinearity. This method is based on deriving an approximate Fokker-Planck equation for the Wigner function, the approximation being valid in the weak-coupling limit where the detuning per carrier is much smaller than the cavity linewidth and the mean photon number is large. Importantly, this allows us to keep track of the dominant quantum effects (vacuum noise in the optical field, Poisson noise in the carrier excitation and decay) without running a full quantum simulation. This method was then applied to simulate an optical amplifier and an SR-latch.

Because the semiclassical properties of these devices are well known, in this paper we focused on the quantum noise in the free-carrier amplifier and latch. Since the free-carrier dispersion creates an effective \( \chi^{(3)} \) nonlinearity, in both cases the free-carrier simulations were compared to simulations of an analogous Kerr-based device, for which the quantum model is well known. In the semiclassical, steady-state limit, the Kerr and free-carrier devices behave the same.

For the phase-sensitive amplifier, we find that the free-carrier and Kerr devices have the same gain near the amplification maximum. However, the noise in the amplification direction is larger for most frequencies. In the other quadrature, the Kerr cavity squeezes the field, while the free-carrier cavity does not produce any squeezing. In the SR-latch, we notice a difference in the the spontaneous switching rate: the free-carrier latch has switching rates of \( \gtrsim 5 \) larger than the Kerr latch, a discrepancy that grows as the latching becomes stronger.

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Appendix A: Full Wigner SDE’s

The cumulants \( C^{(1)} \) and \( C^{(2)} \) are computed from Eqs. \([40,41]\) using Mathematica. The terms are separated by physical origin in the sections below.

1. Uncoupled Cavity, Carrier Terms

In this case, \( H = \Delta_e a^\dagger a + \Delta_c (n + \bar{n})/2 \) and \( L + \sqrt{n} a \). It is easy to show that:

\[
C^{(1)} = \begin{bmatrix}
(-i\Delta_e - \frac{\gamma_e}{2})\alpha^* \\
(i\Delta_e - \frac{\gamma_e}{2})\alpha \\
-i\Delta_e v \\
i\Delta_e \bar{v}
\end{bmatrix}, \quad C^{(2)} = \begin{bmatrix}
0 & \frac{\gamma_e}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{\gamma_e}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(A1)

2. Photon-Carrier Interaction

Here, \( H = ig(a^\dagger \sigma_- - a \sigma_+) \), and there are no environment couplings. There is no noise term here.

\[
C^{(1)} = \begin{bmatrix}
gv \\
gv^* \\
-ga(N - m - \bar{m}) \\
-ga^*(N - m - \bar{m}) \\
-g(\alpha v^* + \alpha^* v) \\
-g(\alpha v^* + \alpha^* v)
\end{bmatrix}, \quad C^{(2)} = 0
\]

(A2)

3. Free-Carrier Dispersion / Absorption

Here, \( L_k = \sqrt{n_{fca}} n_k a, \sqrt{n_{fca}} n_k a \) and \( H = a^\dagger a (\delta_{fca} n_k + \delta_{fca} \bar{n}_k) \). Define \( \delta_{fc} = \delta_{fca} - i\gamma_{fca}/2 \). Then the cumulants become:

\[
\]
and the fermionic bosonic. However, if there are many more recombination sites than carriers, this mode’s dynamics are not relevant. Nonradiative decay is mediated through a term of the form \( L \). Recombination only takes place when an electron and hole occupy the same state \( k \), so the rate goes as the pair density \( q \), not as the carrier density \( m + \bar{m} \).

Equation (A3) has a nontrivial noise matrix. However, this can be greatly simplified in the non-degenerate, fast-dephasing limit usually taken.

### 4. Recombination

This is mediated by the term \( L = \sqrt{\gamma_{rc}} a_\gamma \). Recombination only takes place when an electron and hole occupy the same state \( k \), so the rate goes as the pair density \( q \), not as the carrier density \( m + \bar{m} \).

\[
C^{(1)} = \begin{bmatrix}
0 \\
-\frac{1}{2} v \\
-\frac{1}{2} v^* \\
q \\
-q \\
-q
\end{bmatrix}, \quad C^{(2)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} (N + 2q - m - \bar{m}) & 0 & \frac{1}{2} v & \frac{1}{2} v & \frac{1}{2} v \\
0 & 0 & \frac{1}{2} v & \frac{1}{2} v & \frac{1}{2} v & q & q & q \\
0 & 0 & \frac{1}{2} v & \frac{1}{2} v & \frac{1}{2} v & q & q & q \\
0 & 0 & \frac{1}{2} v & \frac{1}{2} v & \frac{1}{2} v & q & q & q
\end{bmatrix}
\]  

(A4)

### 5. Nonradiative Decay / Excitation

Nonradiative decay is mediated through a term of the form \( L = \sqrt{\gamma_{nr}} c_k \). Strictly speaking, one must include write \( L = \sqrt{\gamma_{nr}} c_k v_1 \), etc. where \( v_1 \) is the electronic mode into which the carrier decays, to make the \( L \) operator bosonic. However, if there are many more recombination sites than carriers, this mode’s dynamics are not relevant and the fermionic \( L \) gives the right result.

\[
C^{(1)} = \begin{bmatrix}
0 & 0 & \frac{1}{2} (\gamma_{nr} + \bar{\gamma}_{nr}) v \\
0 & \frac{1}{2} (\gamma_{nr} + \bar{\gamma}_{nr}) v \\
-\gamma_{nr} m & -\bar{\gamma}_{nr} \bar{m} \\
-(\gamma_{nr} + \bar{\gamma}_{nr}) q
\end{bmatrix}, \quad C^{(2)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} \gamma_{nr} (N - \bar{m}) + \frac{1}{2} \bar{\gamma}_{nr} (N - m) & \frac{1}{2} \gamma_{nr} v & \frac{1}{2} \bar{\gamma}_{nr} v & \frac{1}{2} (\gamma_{nr} + \bar{\gamma}_{nr}) v \\
0 & 0 & \frac{1}{2} \gamma_{nr} v & \frac{1}{2} \bar{\gamma}_{nr} v & \frac{1}{2} (\gamma_{nr} + \bar{\gamma}_{nr}) v & \gamma_{nr} m & 0 & \gamma_{nr} q & \bar{\gamma}_{nr} \bar{m} & \bar{\gamma}_{nr} q \\
0 & 0 & \frac{1}{2} \gamma_{nr} v & \frac{1}{2} \bar{\gamma}_{nr} v & \frac{1}{2} (\gamma_{nr} + \bar{\gamma}_{nr}) v & \gamma_{nr} q & \bar{\gamma}_{nr} \bar{m} & \bar{\gamma}_{nr} q & (\gamma_{nr} + \bar{\gamma}_{nr}) q
\end{bmatrix}
\]  

(A5)
6. Scattering

The scattering terms are \( L = \sqrt{\gamma_{sc}/2N} \hat{e}_k \hat{c}_l \). \( \sqrt{\gamma_{sc}/2N} \hat{e}_k \hat{c}_l \). Defining an average scattering rate by \( \frac{1}{2}(\gamma_{sc} + \gamma_{sc}) \rightarrow \gamma_{sc} \), we have:

\[
C^{(1)} = \gamma_{sc} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}v & 0 & 0 & \frac{1}{2}e_1 & 0 & 0 \\ -\frac{1}{2}v^* & 0 & 0 & \frac{1}{2}e_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \overline{m} - q & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
C^{(2)} = \gamma_{sc} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}v & 0 & \frac{1}{2}e_1 & 0 \\ 0 & 0 & \frac{1}{2}v^* & 0 & \frac{1}{2}e_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[\text{where } e_1 = 1 - (m + \tilde{m})/N + 2m\tilde{m}/N^2, \quad e_2 = 1 + (2q - m - \tilde{m} - 2/3)/N, \quad \text{and } e_3 = q(q - 2(m + \tilde{m}))/N. \]

In Section III B we take the limit \( m, \tilde{m}, q \ll N \). In this limit, \( e_1 = e_2 = 1, e_3 = 0 \).

Appendix B: Carrier Detuning in terms of Material Properties

It is useful to express the carrier-dependent detuning in [61] in terms of actual material properties. The carrier-dependent detuning is what fundamentally limits the performance of a free-carrier device – it sets the minimum number of carriers needed to switch by one linewidth, the energy figure of merit for a photonic switch. It is given by:

\[
\Delta(m, \tilde{m}) \approx \sum_k \delta_k (m_k + \tilde{m}_k)
\]

\[
\approx \sum_k \frac{\hbar^2 e^2 |\hat{E}_\omega(x_k) \cdot \hat{p}_{cv}|^2}{m_0^2 \epsilon_0 E_g^2} \frac{1}{2x(1 - x)} (m_k + \tilde{m}_k)
\]

\[
= \frac{\hbar^2 e^2 |\hat{E}_\omega(x_k) \cdot \hat{p}_{cv}|^2}{m_0^2 \epsilon_0 E_g^2} \frac{1}{2x(1 - x)} (m_k + \tilde{m}_k)
\]

\[\text{where } m_0 \text{ is the bare electron mass and } \hat{p}_{cv} \text{ is the matrix element } \langle \mathbf{k}, e | \hat{p} | \mathbf{k}, v \rangle \text{ between conduction- and valence-band states.} \]

This is a two-band calculation, which only includes transitions from a single valence band. Adding a second valence band doubles the effect of the electrons – since each electron “blocks” two transitions, one from each valence band, its bandfilling effect is doubled (Fig. 15). This does not happen for holes, since each hole transitions from a single valence band. Adding a second valence band does not affect the electrons.

This section considers two common cases: a III-V semiconductor near the band gap, where band filling is dominant, and silicon far from the band gap, where the plasma effect dominates. These effects are well studied in bulk materials; the point of this section is to translate them to the optical resonator picture used in this paper.

1. III-V Semiconductor near Band Gap

Here, the dominant effect comes from band-filling dispersion. We assume that all modes have roughly the same energy, \( E_g \), and that the optical field is at \( \hat{E} = \hat{E}_g \), where \( x < 1 \). If \( x \approx 1 \), then one can show that the carrier-dependent detuning takes the form:

\[
\Delta(m, \tilde{m}) \approx \sum_k \frac{3 \hbar^2 e^2 |\hat{E}_\omega(x_k) \cdot \hat{p}_{cv}|^2}{m_0^2 \epsilon_0 E_g^2} \frac{|\hat{E}_\omega(x_k) \cdot \hat{p}_{cv}|^2}{2x(1 - x)}
\]

\[\text{To get a sense of scaling, we replace } |E_\omega|^2 \rightarrow |\hat{E}_\omega|^2/(n_0^3 V). \]

Here, \( \hat{E}_\omega \) is designed to have near-unit amplitude within the cavity, and \( V \) is the mode volume. Unlike \( E_\omega \), \( \hat{E}_\omega \) is not normalized (its integral is not one), but having near unit-amplitude is what matters here. The mode volume is defined in terms of a normalized quantity, \( V = V/(\lambda/n)^3 \), which is \( O(1) \) for photonic crystals and \( O(10) \) for rings. Instead of looking at \( \Delta \), we look at \( \Delta/\omega \), since this is unitless, and we are well aware that
In the limit $x \approx 1$, this is consistent with previous derivations of the band-filling dispersion \[ [3, 32], \] under the replacements $|p_{cv}|^2 \rightarrow E_g m_0^2/2m_e$ and $|\langle \tilde{E}(x) \rangle^2|_k \rightarrow 1$ (this is always $O(1)$ and the equality can be imposed by scaling $\tilde{V}$). Because of the rotating-wave approximation taken in this paper, it will not be valid when $x$ deviates far from 1. However, optimized devices exploiting band-filling always operate near the band gap.

2. FCD in Silicon

In silicon, the indirect band gap makes the band-filling effect very weak. Instead, free-carrier dispersion is dominated by the plasma effect \[ [3]. \] Consider a simple Drude model with a carrier density $N$. The index of refraction is modified as follows:

$$\Delta n = -\frac{\hbar^2 e^2}{2n_0\epsilon_0 \tilde{V}^2} \left[ \frac{n}{m_e} + \frac{p}{m_h} \right]$$  \hspace{1cm} (B7)

Assuming the carriers are confined to a volume $V$, and defining the dimensionless $\tilde{V} = V/(\lambda/n)^3$ as above, and using standard coupled-mode theory to convert $\Delta n$ to a detuning, we find:

$$\Delta(m, \tilde{m}) = \frac{e^2 n_0}{16\pi^3 \epsilon_0 V^3} \left[ \frac{m}{m_e} + \frac{\tilde{m}}{m_h} \right]$$  \hspace{1cm} (B8)

A more detailed treatment shows that the dependence is linear for electrons, but nonlinear for holes \[ [33]. \] This 3 nonlinearity can be treated phenomenologically in \[ [63, 66] \]; the quantum noise terms derived in this section do not change.

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