Abstract

We present the Variational Adaptive Newton (VAN) method which is a black-box optimization method especially suitable for explorative-learning tasks such as active learning and reinforcement learning. Similar to Bayesian methods, VAN estimates a distribution that can be used for exploration, but requires computations that are similar to continuous optimization methods. Our theoretical contribution reveals that VAN is a second-order method that unifies existing methods in distinct fields of continuous optimization, variational inference, and evolution strategies. Our experimental results show that VAN performs well on a wide-variety of learning tasks. This work presents a general-purpose explorative-learning method that has the potential to improve learning in areas such as active learning and reinforcement learning.

1 Introduction

Throughout our life, we continue to learn about the world by sequentially exploring it. We acquire new experiences using old ones, and use the new ones to learn even more. How can we design methods that can perform such “explorative” learning to obtain good generalizations? This is an open question in artificial intelligence and machine learning.

One such approach is based on Bayesian methods. This approach has not only been used as a theoretical model of human cognitive development (Perfors et al., 2011) but also been applied to a wide variety of practical explorative-learning tasks, e.g., in active learning and Bayesian optimization to select informative data examples (Houlsby et al., 2011; Gal et al., 2017; Brochu et al., 2010; Fisher and Loeb, 2012), and in reinforcement learning to learn through interactions (Wyatt, 1998; Strens, 2000). Unfortunately, Bayesian methods are computationally demanding because computation of the posterior distribution is a difficult task, especially for large-scale problems. In contrast, non-Bayesian methods, such as those based on continuous optimization methods, are generally computationally much cheaper, but they cannot directly exploit the mechanisms of Bayesian exploration because they do not estimate the posterior distribution. This raises the following question: how can we design explorative-learning methods that compute a distribution just like Bayesian methods, but cost similar to optimization methods?

In this paper, we propose such a method. Our method can be used to solve generic unconstrained function-minimization problem\(^1\) that take the following form:

\[
\theta^* = \arg\min_{\theta} f(\theta), \quad \text{where } \theta \in \mathbb{R}^D. \tag{1}
\]

A wide variety of problems in supervised, unsupervised, and reinforcement learning can be formulated in this way. Instead of directly solving the above problem, our method solves it indirectly by first taking the expectation of \(f(\theta)\) with respect to an unknown probability distribution \(q(\theta|\eta)\), and then solving the following minimization problem:

\[
\min_{\eta} \mathbb{E}_{q(\theta|\eta)} [f(\theta)] := \mathcal{L}(\eta). \tag{2}
\]

where minimization is done with respect to the parameter \(\eta\) of the distribution \(q\). This approach is referred to as the Variational Optimization (VO) approach by Staines and Barber (2012) and can lead us to the minimum \(\theta^*\) because \(\mathcal{L}(\eta)\) is an upper bound on the minimum value of \(f\), i.e., \(\min_{\theta} f(\theta) \leq \mathbb{E}_{q(\theta|\eta)} [f(\theta)]\). Therefore minimizing \(\mathcal{L}(\eta)\) minimizes \(f(\theta)\), and when the distribution \(q\) puts all its mass on \(\theta^*\), we recover the minimum value. This type of function minimization is commonly used in many areas of stochastic search such as evolution strategies (Hansen and Ostermeier, 2001; Wierstra et al., 2008). In our problem context, this formulation is advantageous because it enables learning via exploration, where exploration is facilitated through the distribution \(q(\theta|\eta)\).

\(^1\)This is a black-box optimization problem in the sense that we may not have access to an analytical form of the function or its derivatives, but we might be able to approximate them at a point.
Our main contribution is a new method to solve (2) by using a mirror-descent algorithm. We show that our algorithm is a second-order method which solves the original problem (1), even though it is designed to solve the problem (2). Due to its similarity to Newton’s method, we refer to our method as the Variational Adaptive Newton (VAN) method. Figure 1 shows an example of our method for a one-dimensional non-convex function.

We establish connections of our method to many existing methods in continuous optimization, variational inference, and evolution strategies, and use these connections to derive new algorithms for explorative learning. Below, we summarize the contributions made in the rest of the paper:

- In Section 3, we derive VAN and establish it as a second-order method. In Section 4, we derive computationally-efficient versions of VAN and discuss their relations to adaptive-gradient methods.
- In Section 5 and 6, we show connections to variational inference methods and natural evolution strategy (Wierstra et al., 2008).
- In Section 7, we apply our method to supervised learning, unsupervised learning, active learning, and reinforcement learning. In Section 8, we discuss relevance and limitations of our approach.

This work presents a general-purpose explorative-learning method that has the potential to improve learning in areas such as active learning and reinforcement learning.

2 Variational Optimization

We will focus on solving the problem (2) since it enables estimation of a distribution that can be used for exploration. In this paper, we will use the Gaussian distribution \( q(\theta) \).

The problem (2) can then be rewritten as follows,

\[
\min_{\mu, \Sigma} \mathbb{E}_{\mathcal{N}(\theta|\mu, \Sigma)} [f(\theta)] := \mathcal{L}(\mu, \Sigma)
\]

where \( q \) is the Gaussian distribution \( q(\theta|\eta) := \mathcal{N}(\theta|\mu, \Sigma) \) with \( \mu \) being the mean and \( \Sigma \) being the covariance, and \( \eta = (\mu, \Sigma) \). The function \( \mathcal{L} \) is differentiable under mild conditions even when \( f \) is not differentiable, as discussed in Staines and Barber (2012). This makes it possible to apply gradient-based optimization methods to optimize it.

A straightforward approach to minimize \( \mathcal{L} \) is to use Stochastic Gradient Descent (SGD) as shown below,

\[
\text{V-SGD} : \quad \mu_{t+1} = \mu_t - \rho_t \left( \nabla_{\mu} \mathcal{L}_t \right) \\
\Sigma_{t+1} = \Sigma_t - \rho_t \left( \nabla_{\Sigma} \mathcal{L}_t \right)
\]

where \( \rho_t > 0 \) is a step size at iteration \( t \), \( \nabla \) denotes an unbiased stochastic-gradient estimate, and \( \mathcal{L}_t = \mathcal{L}(\mu_t, \Sigma_t) \).

Figure 1: Illustrative application of VAN on an example by Huszar (2017). The top figure shows the function \( f(\theta) = \text{sinc}(\theta) \) with a blue curve. The global minima is at \( \theta^* = 1 \) although there are many local minima and maxima as well. The second plot shows the VO objective \( \mathcal{L}(\mu, \sigma) = \mathbb{E}_q[f(\theta)] \) for the Gaussian \( q = \mathcal{N}(\theta|\mu, \sigma^2) \). The red points and arrows show the iterations of our VAN method initialized at \( \mu = -3.2 \) and \( \sigma = 1.5 \). The \( \theta \) values corresponding to the value of \( \mu \) are marked in the top figures where we see that these iterations converge to the global minima of \( f \) and avoids other local optima. When we use a Newton’s method initialized at the same point, i.e., \( \theta_0 = -3.2 \), it converges to the local minima at -3.5.

VAN can avoid such local minima because it optimizes in the space of \( \mu \) and \( \sigma^2 \). The progression of the distribution \( q \) is shown in the bottom figure, where darker curves indicate higher iterations. We see that the distribution \( q \) is flat in the beginning which enables more exploration which in turn helps the method to avoid local minima. As desired, the distribution peaks around \( \theta^* \) as iterations increase.

We refer to this approach as Variational SGD or simply V-SGD to differentiate it from the standard SGD that optimizes \( f(\theta) \) in the \( \theta \) space.

The V-SGD approach is simple and can also work well when used with adaptive-gradient methods to adapt the step-size, e.g., AdaGrad and RMSprop. However, as pointed by Wierstra et al. (2008), it has issues, especially when REINFORCE (Williams, 1992) is used to estimate
the gradients of $f(\theta)$. Wierstra et al. (2008) argue that the V-SGD update becomes increasingly unstable when the covariance is small, while becoming very small when the covariance is large. To fix these problems, Wierstra et al. (2008) proposed a natural-gradient method. Our method is also a natural-gradient method, but, as we show in the next section, its updates are much simpler and they lead to a second-order method which is similar to Newton’s method.

3 Variational Adaptive-Newton Method

VAN is a natural-gradient method derived using a mirror-descent algorithm. Due to this, the updates of VAN are fundamentally different from V-SGD. We will show that VAN adapts the step-sizes in a very similar spirit to the adaptive-gradient methods. This property will be crucial in establishing connections to Newton’s method.

VAN can be derived by making two minor modifications to the V-SGD objective. Note that the V-SGD update in (4) and (5) are solutions of following optimization problem:

$$\eta_{t+1} = \arg\min_{\eta} \left\langle \eta, \hat{\nabla}_\eta L_t \right\rangle + \frac{1}{2\rho_t} \| \eta - \eta_t \|^2.$$  

(6)

The equivalence to the update (4) and (5) can be shown by simply taking the derivative with respect to $\eta$ of (6) and setting it to zero:

$$\hat{\nabla}_\eta L_t + \frac{1}{\rho_t} (\eta_{t+1} - \eta_t) = 0,$$  

(7)

which results in the update (4) and (5). A simple interpretation of this optimization problem is that, in V-SGD, we choose the next point $\eta$ along the gradient but contain it within a scaled $L_2$-ball centered at the current point $\eta_t$. This interpretation enables us to obtain VAN by making two minor modifications to V-SGD.

The first modification is to replace the Euclidean distance $\| \cdot \|^2$ by a Bregman divergence which results in the mirror-descent method. Note that, for exponential-family distributions, the Kullback-Leibler (KL) divergence corresponds to the Bregman divergence (Raskutti and Mukherjee, 2015). Using the KL divergence results in natural-gradient updates which results in better steps when optimizing the parameter of a probability distribution (Amari [1998]).

The second modification is to optimize the VO objective with respect to the mean parameterization of the Gaussian distribution $m := \{ \mu, \mu \mu^2 + \Sigma \}$ instead of the parameter $\eta := \{ \mu, \Sigma \}$. We emphasize that this modification does not change the solution of the optimization problem since the Gaussian distribution is a minimal exponential family and the relationship between $\eta$ and $m$ is one-to-one (see Section 3.2 and 3.4.1 of Wainwright and Jordan [2008] on the basics of exponential family and mean-parameterization respectively).

The two modifications give us the following problem:

$$m_{t+1} = \arg\min_m \left\langle m, \hat{\nabla}_m L_t \right\rangle + \frac{1}{\beta_t} \mathbb{D}_{KL}[q || q_t],$$  

(8)

where $q := q(\theta|m)$, $q_t := q(\theta|m_t)$, and $\mathbb{D}_{KL}[q || q_t] = \mathbb{E}_q[\log(q/q_t)]$ denotes the KL divergence. The convergence of this procedure is guaranteed under mild conditions (Ghadimi et al., 2014).

As shown in Appendix A, a solution to this optimization problem is given by

$$\mu_{t+1} = \mu_t - \beta_t \Sigma_{t+1} \hat{\nabla}_\mu L_t$$  

(9)

$$\Sigma_{t+1}^{-1} = \Sigma_t^{-1} + 2\beta_t \left[ \hat{\nabla}_\Sigma L_t \right]$$  

(10)

The above update differs from those of V-SGD in two ways. First, here we update the precision matrix $\Sigma^{-1}$ while in V-SGD we update the covariance matrix $\Sigma$. However, both updates use the gradient with respect to $\Sigma$. Second, the step-size for $\mu_{t+1}$ are adaptive in the above update since $\beta_t$ is scaled by the covariance $\Sigma_{t+1}^{-1}$.

The above updates corresponds to a second-order method which is very similar to Newton’s method. We can show this using the following identities (Opper and Archambeau, 2009):

$$\nabla_\mu \mathbb{E}_q [f(\theta)] = \mathbb{E}_q [\nabla_\theta f(\theta)],$$  

(11)

$$\nabla_\Sigma \mathbb{E}_q [f(\theta)] = \frac{1}{2} \mathbb{E}_q [\nabla_{\theta \theta} f(\theta)].$$  

(12)

By substituting these into (9) and (10), we get the following updates which we call the VAN method:

$$\text{VAN:} \quad \mu_{t+1} = \mu_t - \beta_t P_{t+1}^{-1} \mathbb{E}_{q_t} \left[ \nabla_\theta f(\theta) \right]$$  

(13)

$$P_{t+1} = P_t + \beta_t \mathbb{E}_{q_t} \left[ \nabla_{\theta \theta} f(\theta) \right]$$  

(14)

where $P_t := \Sigma_t^{-1}$ is the precision matrix and $q_t := \mathcal{N} (\theta | \mu_t, \Sigma_t)$. The precision matrix $P_t$ contains a running-sum of the past averaged Hessians, and the search-direction for the mean is obtained by scaling the averaged gradients by the inverse of $P_t$. If we compare Eq. (9) to the following update of Newton’s method,

$$\theta_{t+1} = \theta_t - \rho_t \left[ \nabla_{\theta \theta} f(\theta_t) \right]^{-1} \left[ \nabla_\theta f(\theta_t) \right],$$  

(15)

we can see that the Hessian matrix is replaced by the Precision matrix in the VAN update. Due to this connection to Newton’s method and the use of an adaptive scaling matrix $P_t$, we call our method the Variational Adaptive-Newton (VAN) method.

The averaged gradient and running sum of averaged Hessian allow VAN to avoid some types of local optima. Figure 1 shows such a result when minimizing $f(\theta) = \frac{1}{2} \|x - Ax\|^2$ with $A$.

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This example is discussed in a blog by Huszar (2017).
Variational Adaptive-Newton Method

\[ \text{sinc}(\theta) \text{ with initial solution at } \theta = -3.2. \text{ Clearly, both the gradient and Hessian at } \theta_0 \text{ suggest updating } \theta \text{ towards a local optimum at } \theta = -3.5. \text{ However, VAN computes an averaged gradient over samples from } q(\theta) \text{ which yields steeper descent directions pointing towards the global optimum. The adaptive scaling of the steps further ensures a smooth convergence.} \]

The averaging property of VAN is strikingly different from other second-order optimization methods. We expect VAN to be more robust due to averaging of gradients and Hessians. Averaging is particularly useful for optimization of stochastic objectives. For such objectives, application of Newton’s method is difficult because reliably selecting a step-size is difficult. Several variants of Newton’s method have been proposed to solve this difficulty, e.g., based on quasi-Newton methods (Byrd et al., 2016) or incremental approaches (Gürbüzbalaban et al., 2015), or by simply adapting mini-batch size (Mokhtari et al., 2016). VAN is most similar to the incremental approach of (Gürbüzbalaban et al., 2015) where a running sum of past Hessians is used instead of just a single Hessian. In VAN however the Hessian is replaced by the average of Hessians with respect to \( q \). For stochastic objectives, VAN differs substantially from existing approaches and it has the potential to be a viable alternative to them.

An issue with using the Hessian is that it is not always positive semi-definite, for example, for non-convex problems. For such cases, we can use a Gauss-Newton variant shown below (Bertsekas, 1999) which we call the Variational Adaptive Gauss-Newton (VAG) Method:

\[ \text{VAG: } P_{t+1} = P_t + \beta_t E_{q_t} \left[ (\nabla \theta f(\theta)) (\nabla \theta f(\theta))^T \right]. \tag{16} \]

4 VAN for Large-Scale Problems

Applying VAN to problems with large number of parameters is not feasible because we cannot compute the exact Hessian matrix. In this section, we describe several variants of VAN that scale to large problems. Our variants are similar to existing adaptive-gradient methods such as AdaGrad (Duchi et al., 2011) and AROW (Crammer et al., 2009). We derive these variants by using a mean-field approximation for \( q \). Our derivation opens up the possibility of a new framework for designing computationally efficient second-order methods by using structured distributions \( q \).

One of the most common way to obtain scalability is to use a diagonal approximation of the Hessian. In our case, this approximation corresponds to a distribution \( q \) with diagonal covariance, i.e., \( q(\theta | \eta) = \prod_{d=1}^{D} \mathcal{N}(\theta_d | \mu_d, \sigma_{d}^2) \), where \( \sigma_d^2 \) is the variance. This is a common approximation in variational inference methods and is called the mean-field approximation (Bishop, 2006). Let us denote the precision parameters by \( s_d = 1/\sigma_d^2 \) and a vector containing them by \( s \). Using this Gaussian distribution in the update (13) and (14), we get the following diagonal version of VAN, which we call VAN-D:

\[
\begin{align*}
\text{VAN-D: } & \mu_{t+1} = \mu_t - \beta_t \text{ diag}(s_{t+1})^{-1} E_{q_t} \left[ \nabla \theta f(\theta) \right] \\
& s_{t+1} = s_t + \beta_t E_{q_t} \left[ h(\theta) \right] \tag{17}
\end{align*}
\]

where \( \text{diag}(s) \) is a diagonal matrix containing the vector \( s \) as its diagonal and \( h(\theta) \) is the diagonal of the Hessian \( \nabla^2_{\theta \theta} f(\theta) \).

The VAN-D update requires computation of the expectation of the diagonal of the Hessian, which could still be difficult to compute. Fortunately, we can compute its approximation easily by using the reparameterization trick (Kingma and Bains, 2014). This is possible in our framework because we can express the expectation of the Hessian as gradients of an expectation, as shown below:

\[
\begin{align*}
E_q \left[ \nabla^2_{\theta \theta} f(\theta) \right] &= 2\nabla_{\theta} \sigma_d^2 \left[ E_q[f(\theta)] \right], \quad \tag{18} \\
&= 2\nabla_{\theta} \sigma_d^2 E_{N(\mu_d, \sigma_d^2)} \left[ f(\mu_d + \sigma_d \epsilon) \right], \quad \tag{19} \\
&= 2 E_{N(\mu_d, \sigma_d^2)} \left[ \nabla_{\theta} \sigma_d^2 f(\mu_d + \sigma_d \epsilon) \right]. \quad \tag{20}
\end{align*}
\]

where the first step is obtained using (11). In general, we can use the stochastic approximation by simultaneous perturbation (SPSS) method (Bhatnagar, 2007; Spall, 2000) to compute derivatives. A recent paper by Salimans et al. (2017) showed that this type of computation can also be parallelized which is extremely useful for large-scale learning. Note that these tricks cannot be applied directly to standard continuous-optimization methods. The presence of expectation with respect to \( q \) in VO enables us to leverage such stochastic approximation methods for large-scale learning.

Finally, when \( f(\theta) \) corresponds to a supervised or unsupervised learning problem with large data, we could compute its gradients by using stochastic methods. We use this version of VAN in our large-scale experiments and call it sVAN, which is an abbreviation for stochastic-VAN.

4.1 Relationship with Adaptive-Gradient Methods

The VAN-D update given in (17) is closely-related to an adaptive-gradient method called AdaGrad (Duchi et al., 2011) which uses the following updates:

\[
\begin{align*}
\text{AdaGrad: } & \theta_{t+1} = \theta_t - \rho t \text{ diag}(s_{t+1})^{-1/2} g(\theta_t), \quad \tag{21} \\
s_{t+1} &= s_t + g(\theta_t) \odot g(\theta_t) \quad \tag{22}
\end{align*}
\]

Comparing these updates to (17), we can see that both AdaGrad and VAN-D compute the scaling vector \( s_t \) using a moving average. However, there are some major differences between their updates: 1. VAN-D uses average gradients instead of a gradient at a point. 2. VAN-D does not raise the scaling matrix to the power of 1/2. 3. The update of \( s_t \) in VAN-D uses the diagonal of the Hessian instead of
Table 1: Variants of VAN used in our experiments.

| Method     | Description                                      |
|------------|--------------------------------------------------|
| VAN        | Variational Adaptive-Newton Method using (13) and (14) |
| sVAN       | Stochastic VAN (gradient estimated using mini-batches) |
| sVAN-Exact | Stochastic VAN with no MC sampling               |
| sVAN-D     | Stochastic VAN with diagonal covariance $\Sigma$  |
| sVAN-Active| Stochastic VAN using active learning              |
| sVAG       | Stochastic Variational Adaptive Gauss-Newton Method. |
| sVAG-D     | Stochastic VAG with diagonal covariance $\Sigma$. |

the squared gradient values used in AdaGrad. It is possible to use the squared gradient in VAN-D but since we can compute the Hessian using the reparameterization trick, we do not have to make further approximations.

VAN can be seen as a generalization of an adaptive method called AROW (Crammer et al., 2009). AROW uses a mirror descent algorithm that is very similar to ours, but has been applied only to problems which use the Hinge Loss. Our approach not only generalizes AROW but also brings new insights connecting ideas from many different fields.

5 VAN for Variational Inference

Variational inference (VI) enables a scalable computation of the Bayesian posterior and therefore can also be used for explorative learning. In fact, VI is closely related to VO. In VI, we compute the posterior approximation $q(\theta|\eta)$ for a model $p(y, \theta)$ with data $y$ by minimizing the following objective:

$$\min_{\eta} E_{q(\theta|\eta)} \left[ -\log \frac{p(y, \theta)}{q(\theta|\eta)} \right] := \mathcal{L}_{VI}(\eta) \quad (23)$$

We can perform VI by using VO type of method on the function inside the square bracket. A small difference here is that the function to be optimized also depends on parameters $\eta$ of $q$. Conjugate-computation variational inference (CVI) is a recent approach for VI by Khan and Lin (2017). We note that by applying VAN to the variational objective, one recovers CVI. VAN however is more general than CVI since it applies to many other problems other than VI. A direct consequence of our connection is that CVI, just like VAN, is also a second-order method to optimize $f_{VI}(\theta)$, and is related to adaptive-gradient methods as well. Therefore, using CVI should give better results than standard methods that use update similar to V-SGD, e.g., black-box VI method of Ranganath et al. (2014).

6 VAN for Evolution Strategies

VAN performs natural-gradient update in the parameter-space of the distribution $q$ (as discussed earlier in Section 3). The connection to natural-gradients is based on a recent result by Kaskuti and Mukherjee (2015) that shows that the mirror descent using a KL divergence for exponential-family distributions is equivalent to a natural-gradient descent. The natural-gradient corresponds to the one obtained using the Fisher information of the exponential family distribution. In our case, the mirror-descent algorithm (8) uses the Bregman divergence that corresponds to the KL divergence between two Gaussians. Since the Gaussian is an exponential-family distribution, mirror descent (8) is equivalent to natural-gradient descent in the dual Riemannian manifold of a Gaussian. Therefore, VAN takes a natural-gradient step by using a mirror-descent step.

Natural Evolution Strategies (NES) (Wierstra et al., 2008) is also a natural-gradient algorithm to solve the VO problem in the context of evolution strategies. NES directly applies natural-gradient descent to optimize for $\mu$ and $\Sigma$ and this yields an infeasible algorithm since the Fisher information matrix has $O(D^4)$ parameters. To overcome this issue, Wierstra et al. (2008) proposed a sophisticated reparameterization that reduces the number of parameters to $O(D^2)$. VAN, like NES, also has $O(D^2)$ parameters, but with much simpler updates rules due to the use of mirror descent in the mean-parameter space.

7 Applications and Experimental Results

In this section, we apply VAN to a variety of learning tasks to establish it as a general-purpose learning tool, and also to show that it performs comparable to continuous optimization algorithm while extending the scope of their application. Our first application is supervised learning with Lasso regression. Standard second-order methods such as Newton’s method cannot directly be applied to such problems because of discontinuity. For this problem, we show that VAN enables stochastic second-order optimization which is faster than existing second-order methods such as iterative-Ridge regression. We also apply VAN to supervised learning with logistic regression and unsupervised learning with Variational Auto-Encoder, and show that stochastic VAN gives comparable results to existing methods such as AdaGrad. Finally, we show two applications of VAN for explorative learning, namely active learning for logistic regression and parameter-space exploration for deep reinforcement learning.

Table 1 summarizes various versions of VAN compared in our experiments. The first method is the VAN method which implements the update shown in (13) and (14). Stochastic VAN implies that the gradients in the updates are estimated by using minibatches of data. The suffix ‘Exact’ indicates that the expectations with respect to $q$ are computed exactly, i.e., without resorting to Monte Carlo (MC) sampling. This is possible for the Lasso objective as shown in Staines and Barber (2012) and also for logistic regression as shown in Marlin et al. (2011). The suffix ‘D’ indicates that a diagonal covariance with the update (17) is used. The suffix ‘Active’ indicates that minibatches are selected using...
an active learning method. Final, VAG corresponds to the Gauss-Newton type update shown in (15). For all methods, except ‘sVAN-Exact’, we use MC sampling to approximate the expectation with respect to $q$. In our plot, we indicate the number of samples by adding it as a suffix, e.g., sVAN-10 is the stochastic VAN method with 10 MC samples.

7.1 Supervised Learning: Lasso Regression

Given $N$ example pairs $\{y_i, x_i\}_{i=1}^N$ with $x_i \in \mathbb{R}^D$, in lasso regression, we minimize the following loss that contains an $\ell_1$-regularization:

$$f(\theta) = \sum_{i=1}^N (y_i - \theta^T x_i)^2 + \lambda \sum_{d=1}^D |\theta_d|,$$

(24)

Because the function is non-differentiable, we cannot directly apply gradient-based methods to solve the problem. For the same reason, it is also not possible to use second-order methods such as Newton’s method. VAN can be applied to this method since expectation of $|\theta_d|$ is twice differentiable. We use the gradient and Hessian expression given in [Staines and Barber] (2012).

We compare VAN and sVAN with the iterative-ridge method (iRidge), which is also a second-order method. We compare on two datasets: Bank32nh ($N = 8192, D = 256, N_{\text{train}} = 7290, \lambda = 104.81$) and YearPredictionMSD ($N = 513545, D = 90, N_{\text{train}} = 448350, \lambda = 5994.84$). We set $\lambda$ values using a validation set where we picked the value that gives minimum error over multiple values of $\lambda$ on a grid. The iRidge implementation is based on minFunc implementation by Mark Schmidt. For sVAN, the size of the mini-batch used to train Bank32nh and YearPredictionMSD are $M = 30$ and $M = 122$ respectively. We report the absolute difference of parameters, $\theta - \theta_*$ where $\theta$ is the parameters estimated by a method and $\theta_*$ is the parameters optimal value (found by iRidge). For VAN the estimated value is equal to the mean $\mu$ of the distribution. Results are shown in Figure 2(a) and (b), where we observe that VAN and iRidge perform comparably, but sVAN is more data-efficient than them in the first few passes. Results on multiple runs show very similar trends.

In conclusion, VAN enables application of a stochastic second-order method to a non-differentiable problem where existing second-order method and their stochastic versions cannot be applied directly.

7.2 Supervised Learning: Logistic Regression

In logistic regression, we minimize the following:

$$f(\theta) = \sum_{i=1}^N \left[ \log(1 + e^{y_i(\theta^T x_i)}) \right] + \lambda \sum_{d=1}^D \theta_d^2,$$

(25)

where $y_i \in \{-1, +1\}$ is the label. We compare VAN to Newton’s method and AdaGrad both of which standard algorithms for batch and stochastic learning, respectively, on convex problems. We use VAN, its stochastic version sVAN, and the diagonal version sVAN-D. We use three real-world datasets from the libSVM database (Chang and Lin, 2011): ‘breast-cancer-scale’ ($N = 683, D = 10, N_{\text{train}} = 341, \lambda = 1.88$), ‘USPS’ ($N = 1,540, D = 256, N_{\text{train}} = 770, \lambda = 6.21$), and ‘a1a’ ($N = 32,561, D = 123, N_{\text{train}} = 1,605, \lambda = 67.23$). We compare the log-loss on the test set computed as follows: $\sum_{i=1}^D \log(1+\exp(y_i(\theta^T x_i)))/N_{\text{test}}$ where $\theta$ is the parameter estimate and $N_{\text{test}}$ is the number of examples in the test set. For sVAN and sVAN-D, we use a mini-batch size of 10 for ‘breast-cancer-scale’ dataset and a mini-batch of size 100 for the rest of the datasets.

Results are shown in Figure 2(d)-(j). The first row, with plots (d)-(f), shows comparison of Batch methods, where we see that VAN converges at a comparable rate to Newton’s method. The second row, with plots (h)-(j), shows the performance of the stochastic learning. Since sVAN uses the full Hessian, it converges faster than sVAN-D and AdaGrad which use a diagonal approximation. sVAN-D shows overall similar performance to AdaGrad. The main advantage of sVAN over AdaGrad is that sVAN maintains a distribution $q(\theta)$ which can be used to evaluate the uncertainty of the learned solution, as shown in the next subsection on active learning.

In conclusion, VAN and sVAN give comparable results to Newton’s method while sVAN-D gives comparable results to AdaGrad.

7.3 Active Learning for using VAN

An important difference between active and stochastic learning is that an active learning agent can query for its training data examples in each iteration. In active learning for classification, examples in a pool of input data $\{x_i\}_{i=1}^N$ are ranked using an acquisition score which measures how informative an example is for learning. We pick the top $M$ data examples as the mini-batch. Active learning is expected to be more data efficient than stochastic learning since the learning agent focuses on the most informative data samples.

In our experiments, we use the entropy score [Schein and Ungar, 2007] as the acquisition to select data example for binary logistic regression:

$$H(x) = - \sum_{c \in \{-1, 1\}} \hat{p}(y = c|x) \log \hat{p}(y = c|x),$$

(26)

where $\hat{p}(y = c|x)$ is the estimated probability that the label for input $x$ takes a value $y = c$. Within our VAN framework, we estimate these probabilities using distributions $q(\theta) = N(\theta|\mu_t, \Sigma_t)$ at iteration $t$. We use the following
approximation that computes the probability using samples from $q$:

$$
\hat{p}_t(y = c|x) \approx \int p(y = c|x, \theta) q(\theta | \mu_t, \Sigma_t) d\theta \tag{27}
$$

$$
\approx \frac{1}{S} \sum_{s=1}^{S} p(y = c|x, \theta^{(s)}_t), \tag{28}
$$

where $S$ is the number of MC samples, $\theta^{(s)}_t$ are sample from $q$, and $p(y|x, \theta)$ is the logisitic likelihood.

Figure 3 compares the performance on on the USPS dataset with active learning by VAN for mini-batch of 10 examples. The result clearly shows that VAN with active learning is much more data efficient and stable than VAN with stochastic learning.

7.4 Unsupervised Learning with Variational Auto-Encoder

We apply VAN to optimize the parameters of variational auto-encoder (VAE) (Kingma and Welling, 2013). Given observations $\{y_i\}_{i=1}^N \sim p(y)$, VAE models the data points using a neural-network decoder $p(y_i|z_i, \theta_d)$ with input $z_i$ and parameters $\theta_d$. The input $z_i$ are probabilistic and follow a prior $p(z_i)$. The encoder is also parameterized with a neural-network but follows a different distribution $q(z_i|y_i, \theta_e)$ with parameters $\theta_e$. The goal is to learn $\theta := \{\theta_d, \theta_e\}$ by minimizing the following.

$$
f(\theta) = -\sum_{i=1}^{N} E_{q(z_i|y_i, \theta_e)} \left[ \log p(y_i|z_i, \theta_d) \right] + D_{KL}[q(z_i|y_i, \theta_e) \| p(z_i)] \tag{29}
$$

where $z_i$ is the latent vector, and $\theta = \{\theta_1, \theta_2\}$ are parameters to be learned. Similar to previous work, we assume $p(z_i)$ to be a standard Gaussian, and use a Gaussian en-
We compare our methods to adaptive-gradient methods, namely AdaGrad (Duchi et al., 2011) and RMSprop (Tieleman and Hinton, 2012). For all the methods, we tune the step-size using a validation set and report the test log-loss after convergence. Figure 2 (c) shows the results of 5 runs of all methods. We see that all methods perform similar to each other. RMSprop has high variance among all methods. sVAN-D-1 is slightly worse than sVAG-D-1, which is expected since for nonconvex problems Gauss-Newton is a better approximation than using the Hessian.

### 7.5 Parameter Based Exploration in Deep Reinforcement Learning

Exploration is extremely useful in reinforcement learning (RL), especially in environment where obtaining informative feedback is rare. Unlike most learning tasks where a training dataset is readily available, RL agents need to explore the state-action space to collect data. An explorative agent that always chooses random actions might never obtain good data for learning the optimal policy. On the other hand, an exploitative agent that always chooses the current optimal action(s) may never try suboptimal actions that can lead to learning a better policy. Thus, striking a balance between exploration and exploitation is a must for an effective learning. However, finding such a balance is an open problem in RL.

In this section, we apply VAN to enable efficient exploration by using the parameter-based exploration (Rücksieß et al., 2010). In standard RL setup, we wish to learn a parameter $\theta$ of a parametric policy $\pi$ for action $a = \pi(s; \theta)$ given state $s$. We seek an optimal $\theta$ such that a state-action sequence $(s_1, a_1, s_2 \ldots)$ maximizes the expected returns where $s_t$ and $a_t$ are state and action at time $t$, respectively. To facilitate exploration, a common approach is to perturb the action by a random noise $\epsilon$, e.g., we can simply add it to the action $a = \pi(s; \theta) + \epsilon$. In contrast, in parameter-based exploration, exploration is facilitated in the parameter space as the name suggest, i.e., we sample the policy parameter $\theta$ from a distribution $\mathcal{N}(\theta | \mu, \Sigma)$. Our goal therefore is to learn the distribution parameters $\mu$ and $\Sigma$.

The parameter-based exploration is better than the action-space exploration because in the former a small perturbation of a parameter result in significant explorative behaviour in the action space. However, to get a similar behaviour through an action-space exploration, the extent of the noise needs to be very large which leads to instability (Rücksieß et al., 2010).

The existing methods for parameter-based exploration independently learn $\mu$ and $\Sigma$ (similar to V-SGD) (Rücksieß et al., 2010; Plappert et al., 2017; Fortunato et al., 2017). However, this method can be increasingly unstable as learning progresses (as discussed in Section 2). Since VAN exploits the geometry of the Gaussian distribution to jointly learn $\mu$ and $\Sigma$, we expect it to perform better than the existing methods that use V-SGD.

We consider the deep deterministic policy gradient (DDPG) method (Silver et al., 2014; Lillicrap et al., 2015).
where a local optimal $\theta$ is obtained by minimizing
\[
f(\theta) = -\mathbb{E}_{p(a)} \left[ \hat{Q}(s, \pi(a, \theta)) \right],
\]
where $\hat{Q}(s, a)$ is an estimated of the expected return and $p(a)$ a state distribution. Both $\pi$ and $\hat{Q}$ are neural networks. We compare the performance of sVAN and sVAG against two baseline methods denoted by SGD and V-SGD. SGD refers to DPG where there is no exploration at all, and V-SGD refers to an extension of DPG where the mean and covariance of the parameter distribution are learned using the update (4) and (5). Due to the large number of neural networks parameters, we use diagonal approximation to the Hessian for all methods. More details of experiments are given in Appendix B.

The result in Figure 4 shows the performance on Half-Cheetah from OpenAI Gym [Brockman et al., 2016] for 5 runs, where we see that VAN based methods significantly outperform existing methods. sVAN-D-1 and sVAG-D-1 both perform equally well. This suggests that both Hessian approximations obtained by using the reparameterization trick shown in (20) and the Gauss-Newton approximation shown in (16), respectively, are equally accurate for this problem. We can also see that sVAN-D-10 has better data efficiency than sVAN-D-1 especially in the early stages of learning. V-SGD is able to find a good policy during learning but has unstable performance that degenerates over time, as mentioned previously. On the other hand, SGD performs very poorly and learns a suboptimal solution. This strongly suggests that good exploration strategy is crucial to learn good policies of Half-Cheetah.

We also tried these comparisons on the 'pendulum' problem in OpenAI Gym where we did not observe significant advantages from explorations. We believe that this is because this problem does not benefit from using exploration and pure-exploitative methods are good enough for these problems. More extensive experiments are required to validate the results presented in this section.

8 Discussion and Conclusions

We proposed a general-purpose explorative-learning method called VAN. VAN is derived within the variational-optimization problem by using a mirror-descent algorithm. We showed that VAN is a second-order method and is related to existing methods in continuous optimization, variational inference, and evolution strategies. We proposed computationally-efficient versions of VAN for large-scale learning. Our experimental results showed that VAN works reasonably well on a wide-variety of learning problems.

For problems with high-dimensional parameters $\theta$, computing and inverting the full Hessian matrix is computationally infeasible. One line of possible future work is to develop versions of VAN that can deal with issue without making a diagonal approximation.

It is straightforward to extend VAN to non-Gaussian distributions. Our initial work, not discussed in this paper, suggests that student-t distribution and Cauchy distribution could be useful candidate. However, it is always possible to use other types of distributions, for example, to minimize discrete optimization problem within a stochastic relaxation framework [Geman and Geman, 1984].

Another venue is to evaluate the impact of exploration on fields such as active learning and reinforcement learning. In this paper, we have provided some initial results. An extensive application of the methods developed in this paper to real-world problems is required to further understand their advantages and disadvantages compared to existing methods.

The main strength of VAN lies in exploration, using which it can potentially accelerate and robustify optimization. Compared to Bayesian methods, VAN offers a computationally cheap alternative to perform explorative learning. Using such cheap explorations, VAN has the potential to solve difficult learning problems such as deep reinforcement learning, active learning, and life-long learning.

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A Derivation of VAN

Denote the mean parameters of \( q_t(\theta) \) by \( \mathbf{m}_t \) which is equal to the expected value of the sufficient statistics \( \phi(\theta) \), i.e., \( \mathbf{m}_t := \mathbb{E}_{q_t}[\phi(\theta)] \). The mirror descent update at iteration \( t \) is given by the solution to

\[
\mathbf{m}_{t+1} = \arg\min_{\mathbf{m}} \left\{ \mathbf{m}, \hat{\nabla}_m \mathcal{L}_t \right\} + \frac{1}{\beta_t} \mathcal{D}_{KL}[q \parallel q_t]
\]

\[
= \arg\min_{\mathbf{m}} \mathbb{E}_q \left[ \phi(\theta), \hat{\nabla}_m \mathcal{L}_t \right] + \log \left( \frac{q_t^{1/\beta_t}}{q_t} \right)
\]

\[
= \arg\min_{\mathbf{m}} \mathbb{E}_q \left[ \log \left( \frac{q_{t}^{1/\beta_t}}{q_t \exp \left( \phi(\theta), -\hat{\nabla}_m \mathcal{L}_t \right)} \right) \right]
\]

\[
= \arg\min_{\mathbf{m}} \frac{1}{\beta_t} \mathbb{E}_q \left[ \log \left( \frac{q}{q_t \exp \left( \phi(\theta), -\beta_t \hat{\nabla}_m \mathcal{L}_t \right)} \right) \right]
\]

where \( \mathcal{Z} \) is the normalizing constant of the distribution in the denominator which is a function of the gradient and step size. Minimizing this KL divergence gives the update

\[
q_{t+1}(\theta) \propto q_t(\theta) \exp \left( \phi(\theta), -\beta_t \hat{\nabla}_m \mathcal{L}_t \right).
\]

By rewriting this, we see that we get an update in the natural parameters \( \lambda_t \) of \( q_t(\theta) \), i.e.

\[
\lambda_{t+1} = \lambda_t - \beta_t \hat{\nabla}_m \mathcal{L}_t.
\]

Recalling that the mean parameters of a Gaussian \( q(\theta) = \mathcal{N}(\theta; \mu, \Sigma) \) are \( \mathbf{m}^{(1)} = \mu \) and \( \mathbf{M}^{(2)} = \Sigma + \mu \mu^T \) and using the chain rule, we can express the gradient \( \hat{\nabla}_m \mathcal{L}_t \) in terms of \( \mu \) and \( \Sigma \),

\[
\hat{\nabla}_m^{(1)} \mathcal{L} = \hat{\nabla}_\mu \mathcal{L} - 2 \left[ \hat{\nabla}_\Sigma \mathcal{L} \right] \mu
\]

\[
\hat{\nabla}_m^{(2)} \mathcal{L} = \hat{\nabla}_\Sigma \mathcal{L}.
\]

Finally, recalling that the natural parameters of a Gaussian \( q(\theta) = \mathcal{N}(\theta; \mu, \Sigma) \) are \( \lambda^{(1)} = \Sigma^{-1} \mu \) and \( \lambda^{(2)} = -\frac{1}{2} \Sigma^{-1} \), we can rewrite the VAN updates in terms of \( \mu \) and \( \Sigma \),

\[
\Sigma_t^{-1} = \Sigma_t^{-1} + 2\beta_t \left[ \hat{\nabla}_\Sigma \mathcal{L}_t \right]
\]

\[
\mu_t = \mu_t - \beta_t \left( \hat{\nabla}_\mu \mathcal{L}_t - 2 \left[ \hat{\nabla}_\Sigma \mathcal{L}_t \right] \mu_t \right)
\]

\[
= \Sigma_t \left( \Sigma_t^{-1} + 2\beta_t \left[ \hat{\nabla}_\Sigma \mathcal{L}_t \right] \right) \mu_t - \beta_t \Sigma_t \Sigma_t^{+1} \left[ \hat{\nabla}_\mu \mathcal{L}_t \right]
\]

\[
= \mu_t - \beta_t \Sigma_t^{+1} \left[ \hat{\nabla}_\mu \mathcal{L}_t \right].
\]

B Details on the RL experiment

In this section, we give details of the parameter-based exploration task in reinforcement learning (RL). An important open question in reinforcement learning is how to efficiently explore the state and action space. An agent always acting greedily according to the policy results in a pure exploitation. Exploration is necessary to visit inferior state and actions once in while to see if they might really be better. Traditionally, exploration is performed in the action space by, e.g., injecting...
noise to the policy output. However, injecting noise to the action space may not be sufficient in problems where the reward is sparse, i.e., the agent rarely observes the reward of their actions. In such problems, the agent requires a rich explorative behaviour in which noises in the action space cannot provide. An alternative approach is to perform exploration in the parameter space (Rückstieß et al. 2010). In this section, we demonstrate that variational distribution \( q(\theta) \) obtained using VAN can be straightforwardly used for such exploration in parameter space, \( \theta \).

### B.1 Background

First, we give a brief background on reinforcement learning (RL). RL aims to solve the sequential decision making problem where at each discrete time step \( t \) an agent observes a state \( s_t \) and selects an action \( a_t \) using a policy \( \pi \), i.e., \( a_t \sim \pi(a|s_t) \). The agent then receives an immediate reward \( r_t = r(s_t, a_t) \) and observes a next state \( s_{t+1} \sim p(s'|s_t, a_t) \). The goal in RL is to learn the optimal policy \( \pi^* \) which maximizes the expected returns \( \mathbb{E} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} r_t \right] \) where \( \gamma \) is the discounted factor and the expectation is taken over a sequence of densities \( \pi(a|s_t) \) and \( p(s'|s_t, a_t) \).

A central component of RL algorithms is the state-action value function or the Q-function \( Q^\pi(s, a) \) gives the expected return after executing an action \( a \) in a state \( s \) and following the policy \( \pi \) afterwards. Formally, it is defined as follows:

\[
Q^\pi(s, a) = \mathbb{E} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} r_t | s_1 = s, a_1 = a \right].
\]  

The Q-function also satisfies a recursive relation also known as the Bellman equation:

\[
Q^\pi(s, a) = r(s, a) + \gamma \mathbb{E}_{p(s'|s, a), \pi(a'|s')} \left[ Q^\pi(s', a') \right].
\]  

Using the Q-function, the goal of reinforcement learning can be simply stated as finding a policy which maximizes the expected Q-function, i.e.,

\[
\pi^* = \arg \max_{\pi} \int \int p(s) \pi(a|s) Q^\pi(s, a) ds da.
\]  

In practice, the policy is represented by a parameterized function such as neural networks with policy parameter \( \theta \) and the goal is to instead find the optimal parameters \( \theta^* \).

#### B.1.1 Deterministic Policy Gradients

Our parameter-based exploration via VAN can be applied to any reinforcement learning algorithms which rely on gradient ascent to optimize the policy parameter \( \theta \). For demonstration, we focus on a simple yet efficient algorithm called the deterministic policy gradients algorithm (DPG) (Silver et al. 2014). Simply speaking, DPG aims to find a deterministic policy that maximizes the action-value function by gradient ascent. Since in practice the action-value function is unknown, DPG learns a function approximator \( Q_\psi(s, a) \) with a parameter \( \psi \) such that \( Q_\psi(s, a) \approx Q^\pi(s, a) \). Then, DPG finds \( \theta^* \) which locally minimize an objective \( f(\theta) = -\mathbb{E}_{p(s)} \left[ Q_\psi(s, \pi_\theta(s)) \right] \) by gradient ascent where the gradient is given by

\[
\nabla_\theta f(\theta) = -\nabla_\theta \int p(s) Q_\psi(s, \pi_\theta(s)) ds
\]

\[
= -\int p(s) \nabla_\theta Q_\psi(s, \pi_\theta(s)) ds
\]

\[
= -\int p(s) \nabla_a Q_\psi(s, a) |_{a=\pi_\theta(s)} \nabla_\theta \pi_\theta(s) ds.
\]  

The parameter \( \psi \) of \( Q_\psi(s, a) \) may be learned by any policy evaluation methods. Here, we adopted the approach proposed by (Lillicrap et al. 2015) which minimizes the squared Bellman residual to the slowly moving target action-value function. More precisely, \( \psi \) is updated by

\[
\min_{\psi} \mathbb{E} \left[ \left( Q_\psi(s, a) - r(s, a) - \gamma \tilde{Q}_\psi(s', \tilde{\pi}_\theta(s')) \right)^2 \right],
\]  

where the expectation is taken over \( p(s) \) and \( p(s'|s, a) \). The \( \tilde{Q}_\psi \) and \( \tilde{\pi}_\theta \) denote target networks which are separate function approximators that slowly tracks \( Q_\psi \) and \( \pi_\theta \), respectively. The target networks help at stabilizing the learning procedure (Lillicrap et al. 2015).
Algorithm 1 Parameter-based exploration DPG via VAN

1: while Not converged do
2: Observe state $s_t$, sample parameter $\theta \sim \mathcal{N}(\theta|\mu, \Sigma)$, take action $a_t = \pi_\theta(s_t)$, observe reward $r_t$ and next state $s'_{t}$.
3: Add $(s_t, a_t, r_t, s'_{t})$ to a replay buffer $\mathcal{D}$.
4: for $i = 1, \ldots, K$ do
5: Draw $N$ minibatch samples $\{(s_i, a_i, r_i, s'_{i})\}_{i=1}^{N}$ from $\mathcal{D}$.
6: Update $Q_v(s, a)$ by gradient descent:
$$v \leftarrow v + \alpha_v \nabla_v \frac{1}{N} \sum_{i=1}^{N} \left[ (Q_v(s_i, a_i) - y_i)^2 \right],$$
where $y_i = r(s_i, a_i) + \gamma \tilde{Q}_v(s', \pi_\mu(s'))$.
7: Update parameter of $q(\theta)$ by sVAN in Eq.(17) with
$$\nabla_\sigma \mathcal{L}(\mu_i, \sigma_i^2) = -\frac{1}{NM} \sum_{i=1}^{N} \sum_{j=1}^{M} \left[ \nabla_\sigma \pi_{\theta_j}(s_i) \nabla_a Q_v(s_i, a)|_{a=\pi_{\theta_j}(s_i)} \right], \quad (51)$$
$$\nabla_\mu \mathcal{L}(\mu_i, \sigma_i^2) = -\frac{1}{NM} \sum_{i=1}^{N} \sum_{j=1}^{M} \left[ \nabla_\mu \pi_{\theta_j}(s_i) \nabla_a Q_v(s_i, a)|_{a=\pi_{\theta_j}(s_i)} \right], \quad (52)$$
where $\{\theta_j\}_{j=1}^{M} \sim q(\theta)$.
8: Update target network parameters $\bar{v}$ and $\bar{\mu}$ by moving average with, e.g., step size $\tau = 0.001$:
$$\bar{v} \leftarrow (1 - \tau)\bar{v} + \tau v,$$
$$\bar{\mu} \leftarrow (1 - \tau)\bar{\mu} + \tau \mu.$$
9: end for
10: end while

Overall, DPG is an actor-critic algorithm that iteratively update the critic (action-value function) by taking gradient of Eq.(49) and update the actor (policy) by the gradient Eq.(48). However, the crucial issue of DPG is that it uses a deterministic policy and does not perform exploration by itself. In practice, exploration is done for DPG by injecting a noise to the policy output, i.e., $a = \pi_\theta(s) + \epsilon$ where $\epsilon$ is a noise from some random process such as Gaussian noise. However, as discussed about, action-space noise may be insufficient in some problems. Next, we show that VAN can be straightforwardly applied to DPG to obtain parameter-based exploration DPG.

B.1.2 Parameter-based Exploration DDPG

To perform parameter-based exploration, we can relax the policy-gradient objective $f(\theta)$ by assuming that the parameter $\theta$ is sampled from a distribution $q(\theta) := \mathcal{N}(\theta|\mu, \Sigma)$, and solve the following optimization problem:
$$\min_{\mu, \Sigma} \mathbb{E}_{\mathcal{N}(\theta)|\mu, \Sigma}[f(\theta)] \quad (50)$$
This is exactly the VO problem of [2]. The stochasticity of $\theta$ through $q(\theta)$ allows the agent to explore the state and action space by varying its policy parameters. This exploration strategy is advantageous since the agent can now exhibits much more richer explorative behaviours when compared with exploration by action noise injection.

Algorithm 1 outlines parameter-based exploration DPG via VAN.