Abstract. Fast linear solvers and preconditioners are well developed for symmetric positive definite (SPD) matrices. Linear or near-linear complexity ("fast") algorithms have been developed for many systems of interest and, in many cases, theoretical results have been established on the convergence. The nonsymmetric setting poses a number of unique challenges over SPD matrices, in theory and in practice. Developing fast and robust nonsymmetric linear solvers is an active area of research and, in particular, theoretical results on fast nonsymmetric solvers are limited.

Algebraic multigrid (AMG) is one of the fastest numerical methods to solve large sparse linear systems. For SPD matrices, convergence of AMG is well motivated in the $A$-norm [2, 6, 7, 23, 32], and AMG has proven an effective solver for many applications. Recently, several AMG algorithms have been developed that are effective on nonsymmetric linear systems [15, 16, 24, 34]. Although motivation was provided in each case, the convergence of AMG for nonsymmetric linear systems is still not well understood, and algorithms are based largely on heuristics or incomplete theory. Several works have delved into convergence of NS-AMG [3, 12, 15, 18, 20], but there has yet to be a thorough study on conditions for convergence and, in particular, the practical implications for solver development. Here, we present the first such work, discussing why SPD theory breaks down in the nonsymmetric setting, and developing a general framework for convergence of NS-AMG. Classical multigrid weak and strong approximation properties are generalized to a fractional approximation property, and conditions developed for two-grid and multigrid convergence in the $\sqrt{A^*A}$-norm.

Key words. Algebraic Multigrid, Nonsymmetric.

1. Introduction. Large, sparse, nonsymmetric linear systems arise in a number of applications involving directed graph Laplacians, Markov chains, and the discretization of partial differential equations (PDEs). Algebraic multigrid (AMG) is a multilevel iterative method to solve large sparse linear systems based on projecting the problem into progressively smaller subspaces. AMG is traditionally motivated for symmetric positive definite (SPD) linear systems and M-matrices [2, 23], and has shown to be a robust and scalable solver for many such problems. Consistent with other approximate direct solvers, iterative methods, and Krylov methods, convergence theory in the case of SPD matrices is relatively well-understood [2, 6, 7, 13, 21, 23, 31–33, 36]. Although AMG solvers have been developed that can be effective on nonsymmetric problems [15, 16, 24, 34], few results have been proven regarding convergence of nonsymmetric AMG (NS-AMG).

Markov chain transition matrices, though nonsymmetric, have a number of nice properties, and several works have studied iterative methods for Markov-chain stationary distribution systems [1, 26, 27, 29]. Theoretical convergence results for iterative aggregation/disaggregation methods (IAD) are well known [9–11, 14], and [10] showed that IAD is equivalent to two-grid NS-AMG [27]. In fact, all Markov-chain stationar-
ary distribution systems can be seen as a specific case of directed graph Laplacian systems, and nonsymmetric lean algebraic multigrid (NS-LAMG), developed in [8], is an effective AMG algorithm to solve directed graph Laplacian linear systems. For PDEs, nonsymmetric smoothed aggregation methods were developed in [3,24], evoking Petrov-Galerkin coarse-grid operators by constructing restriction and interpolation based on left and right near-null space vectors, respectively. A reduction-based method was developed in [15] that is effective on upwind discretizations of hyperbolic PDEs, and in [16] this was generalized to be an effective solver for a variety of advection-diffusion problems, from fully advective to fully diffusive. Various other efforts have been made to develop robust AMG solvers applicable to nonsymmetric linear systems [12,18,19,25,34,35]. Despite a number of effective AMG methods for nonsymmetric problems and some motivating heuristics, there remains a lack of rigorous motivation or theory explaining why and when methods converge.

Typically in AMG, simple relaxation schemes are used and the focus of theory and algorithm development is on effective and complementary coarse-grid correction. For a nonsingular matrix \( A \in \mathbb{R}^{n \times n} \), a coarse-grid problem is defined by projecting \( A \) into a subspace using restriction and interpolation operators, \( R, P \in \mathbb{R}^{n \times n_c} \), respectively. The coarse-grid operator is defined as \( A_c := R^* A P \), and error propagation of coarse-grid correction given as a projection onto the range of \( P \):

\[
I - \Pi := I - P A_c^{-1} R^* A. \tag{1}
\]

Coarse-grid correction approximates the action of \( A^{-1} \) with the operator \( P A_c^{-1} R^* \); that is, it restricts the problem to a subspace, inverts \( A_c \) in the subspace, and interpolates the result back to the fine grid. If \( A_c \) is too large to directly invert, AMG is called recursively on the coarse-grid problem. For SPD matrices, convergence is considered in the so-called energy-norm or \( A \)-norm, \( \|x\|_A^2 = \langle A x, x \rangle \). Then, if we let \( R = P \), coarse-grid correction is an orthogonal projection onto the range of \( P \) in the \( A \)-norm.

The focus of AMG for SPD problems is then on building a "good" \( P \). In the non-SPD setting, \( \langle A x, x \rangle \) is not well defined. A key implication of this is that coarse-grid correction in NS-AMG is generally a non-orthogonal projection in any known inner product, which means that it can increase error. This poses an interesting dichotomy: coarse-grid correction is a principle mechanism by which AMG reduces error, but, in this case, it may also increase error at times. This makes convergence theory difficult to develop, as any potential increase in error due to coarse-grid correction must be overcome by other means.

The simplest measure of NS-AMG convergence is the spectral radius of error propagation, which bounds asymptotic convergence [15,16,20,34]. Although the spectral radius can provide motivation in developing NS-AMG, it is not necessarily indicative of practical performance. Recently, it was suggested that the field of values is a more appropriate measure [22], consistent with previous work on nonsymmetric linear systems as early as [17]. A proof of two-grid convergence was given in [12] for nonsymmetric matrices with positive real parts in the form absolute value norm. A significant theoretical framework was used to develop the form absolute value as a generalization of the \( A \)-norm for nonsymmetric matrices. However, the norm is difficult to compute or interpret in practice and leaves open questions on the respective roles of interpolation and restriction in NS-AMG. In [3], the \( A \)-norm was generalized to the nonsymmetric setting by considering the \( \sqrt{A^* A} \)-norm, and sufficient conditions were derived for two-grid convergence. However, the conditions in [3] include an assumption that the non-orthogonal coarse-grid correction is bounded in norm by some small constant. This assumption is one of the fundamental difficulties with NS-AMG.
and, again, leaves open questions on how to build $R$ and $P$ in the nonsymmetric setting. Here, we build on the framework in [3], developing general conditions on $R$ and $P$ in the nonsymmetric setting. Here, we build on the framework in [3], developing general conditions on $R$ and $P$ for two-grid convergence in Section 2, and extending these results to the multilevel setting in Section 3. Although the conditions are relatively stringent, this is the first general result on convergence in norm of NS-AMG, offering new insight on how to develop AMG methods for nonsymmetric systems. A discussion on results and their relation to recently developed, effective NS-AMG solvers is given in Section 4.

2. Two-grid convergence.

2.1. Background. Multigrid originated in the geometric setting, applied to elliptic differential operators. There, the $A$-norm corresponds with the $H^1$-Sobolev norm, which enforces accuracy of solution values and derivatives. This avoids approximate solutions with large oscillations and non-physical behavior that can occur when minimizing, for example, the $l^2$-norm. Such behavior is desirable when considering nonsymmetric problems as well, motivating a $\sqrt{A^*A}$- or $\sqrt{AA^*}$- generalization of the $A$-norm [3]. Let $A \in \mathbb{R}^{n \times n}$ be nonsingular with singular value decomposition (SVD) $A = U \Sigma V^*$ and singular values ordered such that $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$.

Defining $Q := VU^*$, then $\sqrt{A^*A} = QA = V \Sigma V^*$ and $\sqrt{AA^*} = AQ = U \Sigma U^*$. Because $\sqrt{A^*A}$ and $\sqrt{AA^*}$ are SPD, we can solve $Ax = b$ by applying classical AMG techniques to the equivalent (SPD) linear systems

$$QAx = Qb,$$

$$AQy = b \text{ for } x = Qy.$$ 

Although $Q$ is difficult to form in practice, these systems provide a framework for convergence of NS-AMG. In particular, classical AMG approximation properties can be considered with respect to SPD matrices $QA$ and $AQ$, corresponding to the right and left singular vectors.

Given an interpolation operator $P$, defining $R := Q^*P$ gives a $QA$-orthogonal coarse-grid correction. In this case, classical AMG theory applies, and the optimal $P$ with respect to two-grid convergence is given by letting columns of $P$ be the first $n_c$ right singular vectors, where $n_c$ is the size of the coarse grid [7]. It follows that the optimal $R$ then consists of the first $n_c$ left singular vectors. Thus, in the nonsymmetric development that follows, we consider $P$ that satisfies some approximation property with respect to $QA$ and $R$ that satisfies some approximation property with respect to $AQ$. Approximation properties on $P$ with respect to $QA$ ensure that right singular vectors with small singular values are well represented in $R(P)$, and likewise for $R$, $AQ$, and left singular vectors. Formally, classical multigrid approximation properties are defined as follows:

DEFINITION 1 (WAP on $P$ with respect to SPD $A$). An interpolation operator, $P$, satisfies the weak approximation property (WAP) with respect to SPD matrix $A$, with constant $K_W$ if, for any $v$ on the fine grid $A$, there exists a $v_c$ on the coarse grid

\[ 1 \] A reduction-based NS-AMG method was developed simultaneously with this work in [15]. There, sufficient conditions are developed for two-grid $\ell^2$-convergence of the error and residual, and an outline provided for multilevel convergence. Results here take a more traditional AMG approach (as opposed to reduction based), and develop a more detailed analysis of the multilevel setting.

\[ 2 \] Note that (2) resembles a normal-equation formulation of the problem. However, AMG is typically applied to large, sparse, ill-conditioned matrices, and solving the normal equations squares the condition number. Because $Q$ is unitary, the condition number of $QA$ equals that of $A$. 

such that
\[ \| \mathbf{v} - P \mathbf{v}_c \|^2 \leq \frac{K_W}{\| \mathbf{A} \|} \| \mathbf{A} \| \| \mathbf{v} \|^2 = \frac{K_W}{\| \mathbf{A} \|} \langle \mathbf{A} \mathbf{v}, \mathbf{v} \rangle. \]

**Definition 2 (SAP on \( P \) with respect to SPD \( \mathbf{A} \)).** An interpolation operator, \( P \), satisfies the strong approximation property (SAP) with respect to SPD matrix \( \mathbf{A} \), with constant \( K_P \) if, for any \( \mathbf{v} \) on the fine grid, there exists a \( \mathbf{v}_c \) on the coarse grid such that
\[ \| \mathbf{v} - P \mathbf{v}_c \|^2 \leq K_P \| \mathbf{A} \| \| \mathbf{A} \mathbf{v} \|^2. \]

**Definition 3 (SSAP on \( P \) with respect to SPD \( \mathbf{A} \)).** An interpolation operator, \( P \), satisfies the super strong approximation property (SSAP) with respect to SPD matrix \( \mathbf{A} \), with constant \( K_S \) if, for any \( \mathbf{v} \) on the fine grid, there exists a \( \mathbf{v}_c \) on the coarse grid such that
\[ \| \mathbf{v} - P \mathbf{v}_c \|^2 \leq K_S \| \mathbf{A} \|^2 \| \mathbf{A} \mathbf{v} \|^2 = K_S \| \mathbf{A} \| \| \mathbf{A}^2 \mathbf{v}, \mathbf{v} \|. \]

**Lemma 4 (Equivalence of approximation properties).** Let \( \mathbf{A} \) be SPD.
1. If \( P \) satisfies the SSAP with respect to \( \mathbf{A} \) with constant \( K_S \), then \( P \) also satisfies the WAP with respect to \( \mathbf{A} \) with constant \( K_W = K_S \).
2. If \( P \) satisfies the SSAP with respect to \( \mathbf{A} \) with constant \( K_S \), then \( P \) also satisfies the SAP with respect to \( \mathbf{A} \) with constant \( K_P = K_S \).
3. If \( P \) satisfies the SAP with respect to \( \mathbf{A} \) with constant \( K_P \), then \( P \) also satisfies the SSAP with respect to \( \mathbf{A} \) with constant \( K_S = K_P^2 \).
4. If \( P \) satisfies the SAP with respect to \( \mathbf{A} \) with constant \( K_P \), then \( P \) also satisfies the WAP with respect to \( \mathbf{A} \) with constant \( K_W = K_P^2 \).

**Proof.** Proofs can be found in [6,32].

Motivated by the inner-product form of the WAP and SSAP, we introduce a fractional approximation property (FAP) with respect to SPD matrix \( \mathbf{A} \):

**Definition 5 (FAP on \( P \) with respect to SPD \( \mathbf{A} \)).** An interpolation operator, \( P \), satisfies the FAP with respect to SPD matrix \( \mathbf{A} \), with power \( \zeta > 0 \) and constant \( K_F \) if, for any \( \mathbf{v} \) on the fine grid, there exists a \( \mathbf{v}_c \) on the coarse grid such that
\[ \| \mathbf{v} - P \mathbf{v}_c \|^2 \leq \frac{K_F}{\| \mathbf{A} \|^\zeta} \langle \mathbf{A}^\zeta \mathbf{v}, \mathbf{v} \rangle. \]

It is clear that the WAP is a FAP with \( \zeta = 1 \) and the SSAP, equivalent to the SAP, is a FAP with \( \zeta = 2 \). It is easily verified that a FAP with some power always implies a FAP with a lower power.

In the SPD setting, satisfying the WAP is a necessary and sufficient condition for two-grid convergence [6], and satisfying the SAP/SSAP on all levels are sufficient conditions for multilevel convergence [23,32]. Nonsymmetric matrices lead to a nonorthogonal coarse-grid correction, which requires stronger conditions for convergence. In particular, it is important that coarse-grid correction be stable, that is, coarse-grid correction can only increase error by some small constant \( C_H \geq 1 \), independent of the problem size:
Definition 6 (Stability of $\Pi$ in $\mathcal{A}$-norm).

\begin{equation}
\|\Pi\|_\mathcal{A}^2 \leq C_\Pi,
\end{equation}

where $C_\Pi \geq 1$ is a small constant, independent of the problem size.

A natural idea for NS-AMG is to introduce approximation properties on both $P$ and $R$. However, a simple example shows that building $P$ and $R$ to both satisfy a SAP does not imply stability:

Example 7. Let $n_c$ be the size of the coarse-grid problem and $\ell < n_c$ some number such that $\sigma_\ell \sim O(1)$. Then, define

\[
P := [v_1, \ldots, v_{\ell-1}, v_\ell, v_{\ell+1}, \ldots, v_{n_c+1}], \quad R := [u_1, \ldots, u_{n_c}].
\]

Although $v_\ell \notin \mathcal{R}(P)$, because $\sigma_\ell \sim O(1)$, $P$ trivially satisfies the SAP for $v_\ell$ by interpolating the zero vector. Then, it is clear that $P$ satisfies a SAP with respect to $QA$ and $R$ satisfies a SAP with respect to $AQ$. However, for the $n_c$th canonical basis vector, $e_{n_c}$, $RAPe_{n_c} = 0$. That is, $RAP$ is singular, which implies $\|\Pi\|$ is not bounded.

Thus, more than two approximation properties are needed for convergence of NS-AMG. In [3], Theorem 8 is proven, showing that stability and the SAP on $P$ with respect to the $QA$-norm, along with additional relaxation to account for potential increases in error from coarse-grid correction, are sufficient conditions for two-grid convergence in the $QA$-norm. In particular, the required number of relaxation iterations scales like the square of the stability constant.

Theorem 8 (Two-grid $QA$-Convergence (Theorem 2.3, [3])). Let $G$ be the error-propagation operator for $\nu$ iterations of Richardson-relaxation on the normal equations $(A^*A)$, $G := (I - \frac{A^*A}{\|A\|^2})^\nu$, and $(I - \Pi)$ the (non-orthogonal) coarse-grid correction defined by interpolation and restriction operators, $P$ and $R$, respectively (1). If $P$ satisfies a SAP with respect to the $QA$-norm with constant $K_P$ and coarse-grid correction is stable with constant $C_\Pi$, then

\[
\|(I - \Pi)Ge\|_{QA} \leq \frac{16C_\Pi K_P}{25\sqrt{4\nu + 1}} \|e\|_{QA}.
\]

Two-grid convergence of NS-AMG in the $QA$-norm follows by performing sufficient iterations of relaxation, $\nu$, such that $\|(I - \Pi)Ge\|_{QA} < \|e\|_{QA}$.

Defining a stable coarse-grid operator is a crux of NS-AMG. However, stability is not a practical constraint. Approximation properties alone are not sufficient for stability, and stability by definition does not give useful information for building $R$, motivating further study on conditions for stability and two-grid convergence. In particular, we seek conditions on $R$ and $P$ that give insight to their respective roles in NS-AMG convergence.

While this section is concerned with two-grid convergence, Section 3 presents a multilevel convergence theory. Here, assumptions are established that facilitate this development. On each grid level, assume $A$ is scaled such that $\sigma_n = 1$, and the next coarser level is chosen sufficiently large that $\sigma_{n_c+1} \geq C_\sigma$ where $C_\sigma \sim O(1)$. A constant is said to be independent of grid level if it does not depend on $n$, $n_c$ or $C_\sigma$. Moving forward, we call two SPD operators, $A$ and $B$, spectrally equivalent and two general operators, $A$ and $B$, norm equivalent if there exist constants $\alpha_s, \beta_s$ and...
\[ \alpha_n, \beta_n, \text{respectively, such that} \]
\[ \alpha_n \leq \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} \leq \beta_n, \]
\[ \alpha_n \leq \frac{\langle Ax, Ax \rangle}{\langle Bx, Bx \rangle} \leq \beta_n, \]
denoted \( A \sim_s B \) and \( A \sim_n B \). For self-adjoint, compact operators, \( A \sim_n B \implies A \sim_s B \), with the same constants [5]. More results on the equivalence of operators in a Hilbert space can be found in [5]. Finally, the following lemma bounding the action of a \( 2 \times 2 \) block matrix above and below will come up regularly, for which a proof can be found in the Appendix.

**Lemma 9.** Consider the block matrix \( \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \). Suppose
\[ 0 < a_0 \|x\| \leq \|Ax\|, \quad \|Ax\| \leq a_1 \|x\|, \quad \|Bx\| \leq b \|x\|, \]
\[ 0 < d_0 \|x\| \leq \|Dx\|, \quad \|Dx\| \leq d_1 \|x\|, \quad \|Cx\| \leq c \|x\|, \]
for all \( x \). Further, assume \( a_0 d_0 > bc \). Then,
\[ 0 < \eta_0 \leq \frac{\left\| \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2}{\|x\|^2 + \|y\|^2}, \quad \text{and} \quad \frac{\left\| \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2}{\|x\|^2 + \|y\|^2} \leq \eta_1, \]
where
\[ \eta_0 = \frac{a_0^2 + b^2 + c^2 + d_0^2 - \sqrt{(a_0^2 + b^2 - c^2 - d_0^2)^2 + 4(a_0 c + b d_0)^2}}{2}, \]
\[ \eta_1 = \frac{a_1^2 + b^2 + c^2 + d_1^2 + \sqrt{(a_1^2 + b^2 - c^2 - d_1^2)^2 + 4(a_1 c + b d_1)^2}}{2}. \]

The paper proceeds as follows. A basis under which to consider convergence is developed in Section 2.2, followed by a proof of sufficient conditions for stability and two-grid convergence in Section 2.3. Theorem 12. Section 3 examines the multilevel case, proving the coarse-grid equivalence in Section 3.1, and introducing the proof of multilevel convergence in Section 3.2.

**2.2. Building a basis.** We begin the discussion by demonstrating that coarse-grid correction, \( I - \Pi \), is invariant over any change of basis for \( P \) and \( R \). If we let \( B_P \) and \( B_R \) be nonsingular \( n_c \times n_c \) square matrices such that \( \tilde{P} := PB_P \) and \( \tilde{R} := RB_R \), then,
\[ \Pi = P(R^* AP)^{-1} R^* A = \tilde{P}(\tilde{R}^* \tilde{A} \tilde{P})^{-1} \tilde{R}^* A. \]
Convergence of nonsymmetric AMG will be proved by developing appropriate bases for \( P \) and \( R \) under which to consider convergence.

To prove two-grid and multilevel convergence using change-of-bases \( \tilde{P} := PB_P \) and \( \tilde{R} := RB_R \), there are several results we need to prove and will take into consideration when constructing a basis:

1. **Stability:** \( \|\Pi\|_{QA} \leq C \).
2. **Boundedness:** \( B_P^* B_P \sim_s I \) and \( B_R^* B_R \sim_s I \).
3. **Coarse-grid equivalence:** \( (A^*_c A_c)^{1/2} \sim_n P^* QAP \).
Stability is used to prove two-grid convergence, and is considered through a representation of \( P \) and \( R \) in terms of singular vectors. Define \( \mathcal{P} := V^*P \) and \( \mathcal{R} := U^*R \). Then,

\[
\| \Pi \|_{QA}^2 = \sup_{x \neq 0} \frac{\langle QA\Pi x, x \rangle}{\langle x, x \rangle} = \sup_{y \neq 0} \frac{\| \Sigma^{1/2}V^*P(R^*AP)^{-1}R^*U\Sigma y \|^2}{\| \Sigma^{1/2}y \|^2} = \| \Sigma^{1/2}\mathcal{P}(R^*\Sigma\mathcal{P})^{-1}R^*\Sigma^{1/2} \|^2.
\]

(6)

This representation will prove useful in bounding \( \| \Pi \|_{QA} \).

Boundedness of the change-of-basis operators ensures that if \( P \) is nicely bounded, \( P^*P \sim_s I \), then \( \tilde{P} \) is also nicely bounded, \( \tilde{P}^*\tilde{P} \sim_s I \). This is a subtle but important result for multilevel convergence. The coarse-grid equivalence is also important for multilevel convergence and is discussed in Section 3.

Let \( \Pi_{0_p} = P(P^*P)^{-1}P^* \) be the \( \ell^2 \)-orthogonal projection onto the range of \( P \), and define

\[
V_1 = [v_1, v_2, \ldots, v_k], \quad \Sigma_1 = \text{diag}[^{\sigma_1}, \sigma_2, \ldots, \sigma_k], \quad W_1 = \Pi_{0_p}V_1,
\]

\[
V_2 = [v_{k+1}, \ldots, v_n], \quad \Sigma_2 = \text{diag}[^{\sigma_{k+1}}, \ldots, \sigma_n], \quad N_1 = (I - \Pi_{0_p})V_1,
\]

where \( k \) will be chosen later such that \( \sigma_{k+1} \sim O(1) \). Let \( W_2 = [w_{k+1}, \ldots, w_n] \) be the \( \ell^2 \)-orthogonal complement of \( W_1 \) in \( \text{R}(P) \), normalized so that \( W_2^*\Sigma_2W_2 = I \). There are many choices for the basis of \( W_2 \). Below, a special basis will be constructed.

Assume that \( P \) satisfies a FAP with respect to \( QA \) with power \( 2\beta > 0 \) and constant \( K_{F_p} \). Choose \( k \) such that \( \delta_P := \sigma_k^{1/2}K_{F_p}^\beta < 1.0 \) (\( \delta_P \) will be chosen later). From the FAP,

\[
\| N_1x \|^2 = \|(I - \Pi_{0_p})V_1x\|^2 \leq K_{F_p} \langle QA \rangle^{2\beta} \| V_1x, V_1x \rangle = K_{F_p} \Sigma_1^\beta \| x \|^2 = \delta_P^2 \| x \|^2.
\]

(7)

Because \( \text{R}(W_2) \subset \text{R}(P) \), \( \Pi_{0_p}W_2 = W_2 \). By construction, \( 0 = W_1^*W_2 = V_1^*\Pi_{0_p}W_2 = V_1^*W_2 \), and \( N_2^*W_2 = V_1^*(I - \Pi_{0_p})W_2 = 0 \). Using this basis for \( P \), we can write

\[
\mathcal{P} = V^*P = [V_1, V_2]^*[W_1, W_2] = [V_1, V_2]^*[V_1 - N_1, W_2] = \begin{bmatrix} I - N_1 & 0 \\ -N_2 & W_2 \end{bmatrix},
\]

where

\[
N_1 := V_1^*N_1, \quad N_2 := V_2^*N_1, \quad W_2 := V_2^*W_2.
\]

Given \( V_1V_1^* + V_2V_2^* = I \), it follows that \( W_2^*N_2 = -W_2^*(I - V_1V_1^*)N_1 = 0 \).

Noting the orthogonal decomposition \( \| V^*N_1x \|^2 = \| V_1^*N_1x \|^2 + \| V_2^*N_1x \|^2 = \| N_1x \|^2 + \| N_2x \|^2 \) and plugging into (7),

\[
\| N_1x \|^2 + \| N_2x \|^2 \leq K_{F_p} \Sigma_1^\beta \| x \|^2,
\]

and, for some \( \theta \),

\[
\| N_1x \|^2 \leq \cos^2(\theta)K_{F_p} \Sigma_1^\beta \| x \|^2, \quad \| N_2x \|^2 \leq \sin^2(\theta)K_{F_p} \Sigma_1^\beta \| x \|^2.
\]

(8)

In the development below, we will replace \( x \) in (8) with \( \Sigma_1^{-\beta}x \).
By assumption of a FAP and an appropriate choice of $k$, $\|N_{11}\| = \|N_1\| \leq \delta_P < 1$. Then, $(I - N_{11})$ is invertible, and we can consider a further change of basis to obtain

\[
\hat{P} = P \begin{bmatrix}
(I - N_{11})^{-1} & 0 \\
0 & I
\end{bmatrix} = V \begin{bmatrix}
I & -\hat{N}_{21}(I - N_{11})^{-1}\Sigma_1^{-\beta}\Sigma_1^{-\beta} \\
-\hat{N}_{2}(I - N_{11})^{-1}\Sigma_1^{-\beta} I & 0
\end{bmatrix} W_2.
\]

Here, we denote $\hat{N}_2 = N_{21}(I - N_{11})^{-1}\Sigma_1^{-\beta}$, and $\hat{P}$ takes the form

\[
\hat{P} = V \begin{bmatrix}
I & 0 \\
-\hat{N}_{2}\Sigma_1^{-\beta} & W_2
\end{bmatrix}.
\]

It is reasonable to take pause and ask why we added a factor of $\Sigma_1^{-\beta}$ to the block $N_{21}(I - N_{11})^{-1}$ in (9). As a result of the FAP, it can be shown that $N_{21}(I - N_{11})^{-1}\Sigma_1^{-\beta}$ is nicely bounded for powers of $q \leq \beta$. In particular, we can write $\hat{N}_2 = N_{21}\Sigma_1^{-\beta}(I - \Sigma_1^\gamma N_1^\gamma)^{-1}$. Note that, from (8), $\|\Sigma_1^\gamma N_1^\gamma\| \leq \|\Sigma_1^\gamma\||N_1^\gamma\| \leq \sigma_k^\gamma \cos(\theta)K_{F_P} \leq \delta_P < 1$, and, thus, $I - \Sigma_1^\gamma N_1^\gamma$ is invertible. Again using (8),

\[
\|\hat{N}_2\| = \sup_{x \neq 0} \|N_{21}\Sigma_1^{-\beta}(I - \Sigma_1^\gamma N_1^\gamma)^{-1}x\| = \sup_{y \neq 0} \|\Sigma_1^\gamma N_1^\gamma y\| = \sup_{y \neq 0} \|\Sigma_1^\gamma N_1^\gamma y\| \leq \delta_P \sqrt{1 - \delta_P \cos \theta},
\]

where, recall, $\delta_P := \sigma_k^\gamma K_{F_P}^{1/2}$. The maximum over $\theta$ occurs when $\cos(\theta) = \delta_P$, leading to the bound

\[
\|\hat{N}_2\|^2 \leq \frac{K_{F_P}}{1 - \delta_P^2} := \hat{K}_{F_P}.
\]

The significance of this result is that the block in (10), $\hat{N}_2\Sigma_1^\gamma$, is now bounded when multiplied by $\Sigma_1^{-\ell}$ for $\ell \leq \beta$. In particular,

\[
\|\hat{N}_2\Sigma_1^{-\ell}\| \leq \sigma_k^{\beta-\ell} \hat{K}_{F_P}^{1/2} \leq \hat{K}_{F_P}^{1/2},
\]

for $\ell \leq \beta$. This is a stronger result than obtained by bounding $N_{21}(I - N_{11})^{-1}$ and considering its product with $\Sigma_1^{-\ell}$. Such a result highlights the significance of the order of FAP satisfied by $P$, and is important in proving stability and coarse-grid equivalence.

An equivalent approach can be used to develop a basis for $R$. The preceding discussion is summarized in the following lemma.

**Lemma 10 (Basis for $P$ and $R$).** Assume that $P$ satisfies a FAP with respect to $QA$, with power $2\beta > 0$ and constant $K_{F_P}$, and that $R$ satisfies a FAP with respect to $AQ$, with power $2\gamma > 0$ and constant $K_{F_R}$. Further, assume that $P^* P \sim I \sim R^* R$. Choose $k$ such that $\delta_P := \sigma_k^{\beta} K_{F_P}^{1/2} < 1/\sqrt{2}$ and $\delta_R := \sigma_k^{\gamma} K_{F_R}^{1/2} < 1/\sqrt{2}$. Then, there exist bases, $B_P$ for $P$ and $B_R$ for $R$:

\[
\hat{P} = PB_P = V^P = [V_1, V_2] \begin{bmatrix}
I_k & 0 \\
-N_2\Sigma_1^\gamma & \hat{W}_2
\end{bmatrix},
\]
\[ \hat{R} = RB_R = U\mathcal{R} = [U_1, U_2] \begin{bmatrix} I_k & 0 \\ -\hat{M}_2 \Sigma_1 & Z_2 \end{bmatrix}, \]

where
1. \( W_2^* \Sigma_2 W_2 = Z_2^* \Sigma_2 Z_2 = I, \)
2. \( \hat{W}_2^* \hat{N}_2 = \hat{Z}_2^* \hat{M}_2 = 0, \)
3. \( \|\hat{N}_2\| \leq \hat{K}_{F_1}^{1/2} := \left( \frac{K_{F_1}}{1 - \delta_1^2} \right)^{1/2}, \) and \( \|\hat{M}_2\| \leq \hat{K}_{F_2}^{1/2} := \left( \frac{K_{F_2}}{1 - \delta_2^2} \right)^{1/2}, \)
4. \( B_p^* B_p \sim_s I, \sim_s B_R^* B_R. \)

Furthermore, in these bases,
\[ Z_2^* \Sigma_2 W_2 = S_2 = \text{diag} \{s_1, s_2, \ldots, s_{n-k}\}, \]

with \( 0 \leq s_1 \leq \ldots \leq s_{n-k} \leq 1. \) These singular values are the cosines of the angles between the subspaces \( W_2 \) and \( QZ_2 \) in the QA inner product.

**Proof.** Results (1), (2), and (3) follow from the discussion above. It remains to show that \( B_p^* B_p \sim_s I. \) This is accomplished by observing that, by construction,
\[ \hat{P}^* \hat{P} = \begin{bmatrix} I + \Sigma_1^* \hat{N}_2^* \hat{N}_2 \Sigma_1 & 0 \\ 0 & W_2^* W_2 \end{bmatrix}. \]

By assumption, \( \delta_p < 1/\sqrt{2}, \)
and
\[ \|\Sigma_1^* \hat{N}_2^* \hat{N}_2 \Sigma_1\|^2 = \|\hat{N}_2\|^2 \|\Sigma_1\|^2 \leq \hat{K}_{F_1} \delta_1^2 = \frac{\delta_p^2}{1 - \delta_p^2} < 1. \]

This implies that
\[ \langle x, x \rangle \leq \langle (I + \Sigma_1^* \hat{N}_2^* \hat{N}_2 \Sigma_1) x, x \rangle \leq 2 \langle x, x \rangle. \]

Also,
\[ 1 = \frac{\langle \Sigma_2 W_2 x, W_2 x \rangle}{\langle x, x \rangle} \leq \frac{\langle W_2 x, W_2 x \rangle}{\langle x, x \rangle} \leq \frac{1}{\sigma_{n+1}} \frac{\langle W_2 x, W_2 x \rangle}{\langle x, x \rangle} = \frac{1}{\sigma_{n+1}}. \]

Recall the assumption \( \sigma_{n+1} \sim O(1). \) Thus, \( \|x\|^2 \leq \langle \hat{P} x, \hat{P} x \rangle \leq \max \{2, 1/\sigma_{n+1}\} \|x\|^2, \)
which implies \( \hat{P}^* \hat{P} \sim_s I. \) Together with the assumption \( P^* P \sim_s I, \) this implies \( B_p^* B_p \sim_s I. \) A similar result proves \( B_R^* B_R \sim_s I. \)

Next, let
\[ Z_2^* \Sigma_2 W_2 = \hat{U}_2 S_2 \hat{V}_2^*, \]
be a singular value decomposition. Replace \( W_2 \leftarrow W_2 \hat{V}_2 \) and \( Z_2 \leftarrow Z_2 \hat{U}_2. \) The fact that \( (\Sigma_2^{1/2} W_2) \) and \( (\Sigma_2^{1/2} Z_2) \) are orthonormal yields the bounds \( 0 \leq s_1 \leq s_{n-k} \leq 1. \)

To verify the last statement, recall \( W_2 = V_2^* W_2 \) and \( Z_2 = U_2^* Z_2 \) and, by construction, \( V_2^* W_2 = 0. \) Then,
\[ \langle \Sigma_2 W_2 x, Z_2 y \rangle = \langle \Sigma_2 V_2^* W_2 x, U_2^* Z_2 y \rangle = \langle U_2 \Sigma_2 V_2^* W_2 x, Z_2 y \rangle \]
\[ = \langle (U_1 \Sigma_1 V_1^* + U_2 \Sigma_2 V_2^*) W_2 x, Z_2 y \rangle = \langle AW_2 x, Z_2 y \rangle. \]
Noting that that the singular values are stationary values of the following inner product \[30\],
\[
\frac{\langle \Sigma_2 W_2 x, Z_2 y \rangle}{\|x\| \cdot \|y\|} = \frac{\langle \Sigma_2 W_2 x, Z_2 y \rangle}{\|W_2 x\| \cdot \|Z_2 y\|} = \frac{\langle A W_2 x, Z_2 y \rangle}{\|W_2 x\| A, \|Z_2 y\| A}\]
\[= \frac{\langle Q A W_2 x, Q Z_2 y \rangle}{\|W_2 x\| A, \|Q Z_2 y\| A} = \frac{\langle W_2 x, Q Z_2 y \rangle_{Q A}}{\|W_2 x\| A, \|Q Z_2 y\| A}.
\]
Equivalently, this defines the cosines of angles between \(R(W_2)\) and \(R(Q Z_2)\) in the \(Q A\)-inner product.

**Remark 11.** Here, \(s_1 = \cos \theta_{\text{max}}\), where \(\theta_{\text{max}}\) is the maximum angle between subspaces \(R(W_2)\) and \(R(Q Z_2)\) in the \(Q A\)-inner product. If \(R(W_2) = R(Q Z_2)\), then \(\theta_{\text{max}} = 0\) and \(s_1 = 1\). The less the spaces overlap, that is, the larger the opening angle between the spaces, the smaller \(s_1\) will be.

**2.3. Stability of \(\Pi\) and two-grid convergence.** We are now in position to prove stability of \(\Pi\) under appropriate hypotheses. This requires FAPs on \(P\) and \(R\), as well as an additional hypothesis relating the behavior of \(P\) and \(R\) on the singular vectors associated with larger singular values.

**Theorem 12.** Assume that \(P^* P \sim_s I\), and \(P\) satisfies a FAP with respect to \(Q A\), with power \(2 \beta \geq 1\) and constant \(K_{F_P}\). Similarly, assume that \(R^* R \sim_s I\), and \(R\) satisfies a FAP with respect to \(A Q\), with power \(2 \gamma \geq 1\) and constant \(K_{F_R}\), where \(\beta + \gamma > 1\). Assume there exists \(k \leq n_c\) such that:

1. \(\delta_P := \sigma_k^2 K_{F_P}^2 < 1/\sqrt{2}\), (Denote \(K_{F_P} := K_{F_P}/(1 - \delta_P^2)\))
2. \(\delta_R := \sigma_k^2 K_{F_R}^2 < 1/\sqrt{2}\), (Denote \(K_{F_R} := K_{F_R}/(1 - \delta_R^2)\))
3. \(\delta_{PR}^2 := \sigma_k^{2 + \gamma - 1} K_{F_P}^{1/2} K_{F_R}^{1/2} < 1/2\).

Finally, assume that
\[
s_1 > \frac{\delta_{PR}^2}{(1 - \delta_{PR}^2)}.
\]
Then, \(\|\Pi\|^2_{Q A} \leq C_{\Pi}\). A precise bound for \(C_{\Pi}\) appears in \((15)\) in the proof.

**Proof.** First note that the assumptions here satisfy those of Lemma 10. Using the decomposition of \(P\) and \(R\) developed in Lemma 10 and appealing to \((6)\), we have
\[
\|\Pi\|^2_{Q A} = \|\Sigma^{1/2} P (R^* \Sigma P)^{-1} R^* \Sigma^{1/2}\|^2
\]
\[= \left\| \begin{bmatrix} I & 0 \\ -\Sigma_2^{1/2} \Sigma_1^{\beta - 1/2} & \Sigma_2^{1/2} W_2 \\ I + \Sigma_1^{1/2} \tilde{M}_2^{\star} \Sigma_2 \Sigma_1^{\beta - 1/2} - \Sigma_1^{1/2} \tilde{M}_2^{\star} \Sigma_2 W_2 \\ -Z_2^{\star} \Sigma_2 \Sigma_1^{\beta - 1/2} & S_2^{-1} \end{bmatrix} \right\|^2.
\]
\[= \left\| \begin{bmatrix} I & -\Sigma_1^{1/2} \tilde{M}_2^{\star} \Sigma_1^{1/2} \\ 0 & \Sigma_1^{\beta - 1/2} \tilde{M} \Sigma_1 \end{bmatrix} \right\|^2.
\]
We will bound each of these three \(2 \times 2\) block matrices using Lemma 9. Nonzero off-diagonal blocks must be bounded from above in each case, which can be done using Lemma 10, the orthonormality of \(W_2^{\star} \Sigma_2 W_2 = Z_2^{\star} \Sigma_2 Z_2 = I\), and the scaling of \(A\) such that \(\sigma_k^2 \leq 1\) for all \(k\):
\[
\|\Sigma_2 \Sigma_2 \Sigma_1^{\beta - 1/2} \| \leq \|\Sigma_2^{1/2} \Sigma_1^{\beta - 1/2} \| \leq \sigma_k^{\beta - 1/2} \|\Sigma_2\| \leq \sigma_k^{\beta - 1/2} K_{F_P}^{1/2}.
\]
\[ \|\Sigma_1^{-1/2} \hat{M}_2 \Sigma_2 W_2 \| \leq \|\Sigma_1^{-1/2} \hat{M}_2 \Sigma_1^{-1/2} \| \leq \sigma_k^{1/2} \|\hat{M}_2\| \leq \sigma_k^{1/2} K_F^{1/2}. \]

Note that this is where the assumption of \( \gamma, \beta \geq 1/2 \) is important. Both diagonal blocks of the first term (11) and third term (13) are bounded above and below by one; the upper diagonal block in each case is the identity, and the lower diagonal blocks are given by \( \|\Sigma_2^{-1/2} W_2 \| = \|\Sigma_2^{-1/2} \| = 1 \). Diagonal blocks in the middle term can be bounded in a similar manner, noting that

\[(1 - \delta_{PR}^2) \|x\| \leq \left\| (I + \Sigma_1^{-1/2} \hat{M}_2 \Sigma_2 \Sigma_1^{-1/2}) x \right\| \leq (1 + \delta_{PR}^2) \|x\|,
\]

Then, the first term (11) and third term (13) are easily bounded above:

\[ \left\| \begin{bmatrix} I & 0 \\ -\Sigma_1^{1/2} \hat{M}_2 \Sigma_1^{-1/2} & \Sigma_2^{-1/2} W_2 \end{bmatrix} \right\|^2 \leq 1 + \frac{\sigma_k^{2\beta-1} K_{FP} + \sqrt{\sigma_k^{4\beta-2} K_{FP}^2 + 4\sigma_k^{2\beta-1} K_{FP}^2}}{2} < 2 + \sigma_k^{2\beta-1} K_{FP}, \]

\[ \left\| \begin{bmatrix} I & -\Sigma_1^{-1/2} \hat{M}_2 \Sigma_2 \Sigma_1^{-1/2} \\ 0 & \Sigma_2^{-1/2} W_2 \end{bmatrix} \right\|^2 \leq 1 + \frac{\sigma_k^{2\gamma-1} K_{FR} + \sqrt{\sigma_k^{4\gamma-2} K_{FR}^2 + 4\sigma_k^{2\gamma-1} K_{FR}^2}}{2} < 2 + \sigma_k^{2\gamma-1} K_{FR}. \]

To bound the middle term (12) from above, note that if \( \eta_0 \|x\|^2 \leq \|Ax\|^2 \), then

\[ \|A^{-1}\| = \sup_{x \neq 0} \frac{\|A^{-1}x\|^2}{\|x\|^2} = \sup_{y \neq 0} \frac{\|y\|^2}{\|Ay\|^2} \leq \frac{1}{\eta_0}. \]

In notation of Lemma 9, blocks of the middle term have bounds

\[ a_0 = 1 - \delta_{PR}^2, \quad d_0 = s_1, \quad b = \sigma_k^{(\gamma-1/2)} K_{FR}^{1/2}, \quad c = \sigma_k^{(\beta-1/2)} K_{FR}^{1/2}. \]

Lemma 9 applies when \( a_0d_0 > bc \). Plugging in, this constraint is satisfied when

\[ 1 \geq s_1 > \frac{\delta_{PR}^2}{1 - \delta_{PR}^2}. \]

Equation (14) is the final assumption above, which requires \( \delta_{PR}^2 < 1/2 \) (Assumption 3) and, in turn, \( \beta + \gamma > 1 \). Lemma 9 then yields

\[ \eta_0 = \frac{a_0^2 + b^2 + c^2 + d_0^2 - \sqrt{(a_0^2 + b^2 - c^2 - d_0^2)^2 + 4(a_0c + bd_0)^2}}{2} > 0, \]

Putting this all together, we have

\[ \|\Pi\|^2_{QA} \leq \frac{(1 + \sigma_k^{2\beta-1} K_{FP})(1 + \sigma_k^{2\gamma-1} K_{FR})}{\eta_0}. \]

Equation (15) provides clear separation of three measures of an AMG hierarchy: the two terms in the numerator reflect the approximation properties on \( P \) and \( R \), and the size of the denominator reflects the relation of the action of \( P \) and \( R \) on singular vectors associated with larger singular values. Note that the approximation
properties of $P$ and $R$ do not have to be equal. Stability requires at least a WAP on each, and together, hypotheses require the slightly stronger statement, $\beta + \gamma > 1$. Beyond satisfying a WAP, stronger approximation properties in $P$ or $R$ are reflected through larger $\beta$ and $\gamma$, both of which reduce the bound on $\|P\|_{QA}$.

For larger singular values, approximation properties hold trivially and, for SPD matrices, this means that one need only pay attention to singular vectors with small singular values. In the nonsymmetric setting, stability requires the additional constraint in hypothesis 4, which establishes a relationship between $\delta_{PR}$ and $s_1$. This hypothesis is derived from relating the action of $P$ and $R$ on singular vectors associated with larger singular values. From Lemma 10, we know that $Z_2^*\Sigma_2W_2 = \text{diag}[s_1,\ldots,s_{n-k}]$, where these values are the cosines of angles between subspaces $W_2$ and $QZ_2$. For example, suppose the $j$th right singular vector $v_j \subset \mathbb{R}(P)$ for $j > k$, but the $j$th left singular vector $u_j \not\subset \mathbb{R}(R)$. Then there exists a vector $x$ such that $\langle W_2x, QZ_2y \rangle = \langle UV^*v_j, Z_2y \rangle = \langle u_j, Z_2y \rangle = 0$ for all $y$. Then, $\theta_{\max} = \pi/2$, $s_1 = 0$, and we do not have stability (see Remark 11 and Example 7). Thus, $R$ and $P$ must have a similar action on left and right singular vectors associated with large singular values, respectively. How strong the constraint is depends on approximation properties. When $\delta_{PR} = 0$, the constraint is $s_1 > 0$; that is, $S_2$ need only be nonsingular. When $\delta_{PR} \geq 1/\sqrt{2}$, the restriction is $s_1 > 1$ which is not possible to satisfy. By choosing a smaller $k$, $\delta_{PR}$ can be made smaller. However, choosing $k$ smaller also makes the dimension of spaces $W_2$ and $Z_2$ larger, which makes $s_1$ smaller, and less likely to satisfy the constraint. The hypotheses hold only if there is some $k$ for which (14) holds.

Stronger approximation properties, through either smaller constants, $K_{F_p}$ and $K_{F_R}$, or larger $\beta$ and $\gamma$, make $\delta_{PR}$ smaller. This makes it easier to satisfy the hypotheses of Theorem 12 and, in particular, the constraint on $s_1$. It is also worth considering how accurate approximation properties must be. Suppose we assume equal approximation properties on $P$ and $R$ with power $\beta = \delta = \delta_p = \delta_R$ and $K_F := K_{F_p} = K_{F_R}$. Then, bounding $\delta_{PR}^2 < 1/2$ is equivalent to

$$K_F < \frac{1}{2\sigma_k^{2\beta-1} + \sigma_k^{2\gamma}}.$$ 

Here, we can see that more accurate approximation (smaller $K_F$) is required for weaker approximation properties (smaller $\beta$), while stronger approximation properties require a less accurate approximation. Large $\sigma_k$ also requires a more accurate approximation through smaller $K_F$.

Remark 13 (Grid independence). One of the important properties of multigrid methods is grid-independent convergence. This arises in the context of a family of discretizations, where one can expect the same convergence properties, regardless of mesh resolution. For results presented thus far, as well as for multilevel results in Section 3, this requires the assumption that constants, such as in the FAP, are independent of mesh and problem size. Grid-independent convergence can also refer to convergence independent of the number of levels in a multigrid hierarchy. In this case, constants are assumed to hold on all levels in the hierarchy. In the nonsymmetric setting, level-independent convergence may also require additional relaxation or multigrid cycling.

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3 A similar derivation under the stronger initial assumption that $V_1^*P$ and $U_1^*R$ are nonsingular leads to stability with similar assumptions on the action of $P$ and $R$ on singular vectors associated with larger singular values, and $\beta + \gamma = 1$, that is, $P$ and $R$ both satisfy a WAP. The stronger requirement in Theorem 12, $\beta + \gamma > 1$, may be a shortcoming of this line of proof.
on coarser levels, which is discussed in Section 3.

3. Multigrid convergence. Recall from (5) that coarse-grid correction is invariant under a change of basis, $P = PBP$ and $R = RB_R$, for change of basis matrices $B_P$ and $B_R$. Here, we use the basis developed in Lemma 10 to consider multilevel convergence in the nonsymmetric setting. There are two approximations that must be accounted for in considering multilevel error propagation of coarse-grid correction, which do not arise in the two-level setting. First, and consistent with SPD multigrid theory, we must account for an inexact coarse-grid solve given by recursively calling AMG on the coarse-grid problem. The nonsymmetric setting poses additional difficulties in this recursive call. Specifically, some correction is interpolated to the fine AMG on the coarse-grid problem. The nonsymmetric setting poses additional difficulties in this recursive call. Specifically, some correction is interpolated to the fine grid, which assumes an inner-product form along the lines of:

$$\langle PV_c e_c, PV_c e_c \rangle_{QA} = \langle V_c e_c, V_c e_c \rangle_{P^*QAP},$$

where $V_c$ is the error-propagation operator of the approximate coarse-grid solve. For SPD matrices, $P^*QAP = P^*APv = A_c$, which is exactly the coarse-grid operator formed in practice, on which a recursive assumption is made, $\|V_c\|_{P^*QAP} < 1$. In the nonsymmetric setting, the coarse-grid operator is defined as $A_c := R^*AP$, and the corresponding $QAP$ is equivalent to ($QAP$) norm that we are studying is no longer equal to $P^*QAP$. Then, the recursive assumption of coarse-grid convergence is with respect to the $\sqrt{A_c^*A_c}$-norm, as opposed to the $P^*QAP$-norm. Thus, a fundamental piece of proving AMG convergence in the nonsymmetric setting is to prove an equivalence between inner products in the orthogonal coarse-grid, $P^*QAP$, and the norm measured in practice, $(A_c^*A_c)^{1/2}$.

Conditions for equivalence between inner products are established in Section 3.1. Section 3.2 then combines all of the pieces developed so far to prove multilevel convergence of AMG in the nonsymmetric setting.

3.1. Equivalence of Innerproducts. Proving the necessary equivalence of inner products will be accomplished by proving a stronger statement, the norm equivalence of $A_c := R^*AP$ and $P^*QAP$. Notice that

$$\|A_c x\|^2 \leq \|P^*(QA)Px\|^2 = \|P^*(QA)x\|^2,$$

that is, $A_c \sim_p P^*QAP$ is equivalent to $(A_c^*A_c)^{1/2} \sim_n P^*QAP$. Given that $(A_c^*A_c)^{1/2}$ and $P^*QAP$ are both compact self-adjoint operators, norm equivalence implies spectral equivalence (with the same constants) [5]. Spectral equivalence, $(A_c^*A_c)^{1/2} \sim_s P^*QAP$, then gives bounds needed to prove multilevel convergence:

$$\gamma \geq \beta \geq 1.$$
Further, (19) yields

(20)

or, equivalently,

Then, there exist constants, \( 0 < c_0 \leq c_1, \) such that, \( \forall x \)

The constants are specified below.

Proof. Using Lemma 10 and an appropriate choice of \( k \) (by assumption), there are change of bases, \( \bar{P} = PB_P \) and \( \bar{R} = RB_R, \) such that

where

1. \( \bar{W}_2^* \Sigma_2 W_2 = Z_2^* \Sigma_2 Z_2 = I, \)
2. \( \bar{W}_2^* \bar{N}_2 = \bar{Z}_2^* M_2 = 0, \)
3. \( \|\bar{N}_2\| \leq \bar{K}_F^{1/2}, \) and \( \|\bar{M}_2\| \leq \bar{K}_R^{1/2}, \)
4. \( \bar{Z}_2^* \Sigma_2 W_2 = S_2 = \text{diag} [s_1, \ldots, s_{n-k} \] with \( 0 \leq s_1 \leq \cdots \leq s_{n-k} \leq 1. \)

By assumption, there also exist constants such that

Using the proof of Lemma 10,

Thus, it is sufficient to establish bounds on (18) with \( P \) and \( R \) replaced by \( \bar{P} \) and \( \bar{R}. \) Further, (19) yields

By transitivity of norm equivalence [5], it is then sufficient to show

By transitivity of norm equivalence [5], it is then sufficient to show

or, equivalently,

4 The factor of \( \Sigma^{-1} \) is what necessitates a stronger approximation property on \( P \) to establish bounds independent of problem size. A stronger approximation property is not required to show \( P^* Q A P \sim_n I, \) but is to show \( R^* A P \sim_n I, \) at least in this line of proof.
For ease of notation, we denote bounds using notation of Lemma 9: additional requirements on the bounds (see Lemma 9), which we verify are satisfied. In the case of the lower bound, there are the proof. In each case, the diagonal blocks must be bounded from above and below, and the off-diagonal blocks from above. In the case of the lower bound, there are additional requirements on the bounds (see Lemma 9), which we verify are satisfied. For ease of notation, we denote bounds using notation of Lemma 9:

\[
P^* \Sigma P \begin{bmatrix} \Sigma_1^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I + \Sigma_1^\beta \Sigma_2 \Sigma_1^{-1} & \Sigma_1^\beta \Sigma_2 W_2 \\ W_2 \Sigma_2 \Sigma_1^{-1} & I \end{bmatrix},
\]

\[
R^* \Sigma P \begin{bmatrix} \Sigma_1^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I + \Sigma_1^\gamma \Sigma_2 \Sigma_1^{-1} & \Sigma_1^\gamma \Sigma_2 W_2 \\ Z_2 \Sigma_2 \Sigma_1^{-1} & S \end{bmatrix}.
\]

Next, we will invoke Lemma 9 to bound the action of each of these operators from above and below. This will imply norm equivalence to the identity and complete the proof. In each case, the diagonal blocks must be bounded from above and below, and the off-diagonal blocks from above. In the case of the lower bound, there are additional requirements on the bounds (see Lemma 9), which we verify are satisfied. For ease of notation, we denote bounds using notation of Lemma 9:

\[
\begin{bmatrix} A & -B \\ -C & D \end{bmatrix} \rightarrow \begin{cases} a_0 \|x\| \leq \|Ax\| \leq a_1 \|x\|, & \|Bx\| \leq b \|x\|, \\ d_0 \|x\| \leq \|Dx\| \leq d_1 \|x\|, & \|Cx\| \leq c \|x\|. \end{cases}
\]

for \(a_0, d_0 > 0, a_1, b, c, d_1 \geq 0, \) and \(a_0 d_0 > b c\). Most of these bounds have been shown previously and, in all cases, follow naturally from the bases constructed in Section 2.2 and Lemma 10.

Equation (21): \(\|x\| \leq \|Ax\| \leq (1 + \delta_p^2) \|x\|, \quad \|Bx\| \leq \sigma_k^\beta K_{F_P}^{1/2} \|x\|, \quad \|Cx\| \leq \sigma_k^{\beta - 1} K_{F_P}^{1/2} \|x\|\).

Note that \(\|B\| \|C\| = \delta_p^2\). Here, \(\beta \geq 1\), all terms are bounded independent of \(\Sigma_1\), and the determinant bound \(a_0 d_0 - b c = 1 - \delta_p^2 > 0\) is satisfied. Application of Lemma 9 yields the result.\(^5\)

Equation (22): \((1 - \delta_{PR}^2) \|x\| \leq \|Ax\| \leq (1 + \delta_{PR}^2) \|x\|, \quad \|Bx\| \leq \hat{\delta}_R \|x\|, \quad s_1 \|x\| \leq \|Dx\| \leq \|x\|, \quad \|Cx\| \leq \frac{\delta_{PR}^2}{\hat{\delta}_R} \|x\|\).

Lemma 9 applies here if each term is bounded independent of \(\Sigma_1\) and \(s_1 > \frac{\delta_{PR}}{(1 - \delta_p^2)}\), which is ensured by Hypothesis 4 and the assumption that \(\beta + \gamma > 1\).

Constants \(\hat{c}_0, \hat{c}_1\) can be found by applying Lemma 9. Finally, (20) may be used to find \(c_0 = c_1\).

\[\text{COROLLARY 15. If all assumptions in Lemma 14 are independent of grid level, then } R^* AP \sim_n P^* QAP, \text{ with constants independent of grid level.}\]

With this line of proof, it is clear why \(P\) must have at least a SSAP for inner-product equivalence, that is, \(\beta \geq 1\). If not, then \(\|A\|\) and \(\|C\|\) are not bounded independent of \(\Sigma_1\). Also note that \(R\) plays a minor role. Although stronger approximation properties for \(R\) (larger \(\gamma\)) improve the equivalence constants, no constraint on the value of \(\gamma\) is required by the proof except \(\beta + \gamma > 1\). Of course, everything is made easier by choosing \(R\) to be close to \(Q^* P\), in which case \(R\) shares the same FAP power as \(P\).

\(^5\)Slightly better bounds can be obtained for \(P^* QAP\) by directly proving spectral equivalence; however, the proof is longer and is not significant to the final result.
Remark 16. The same relation between $\sigma_k$ and the constraint on $s_1$ discussed in Section 2.3 for stability applies here as well. The definitions of $B$ in (21) and (22) are slightly different, but satisfy the same properties. As $k$ is chosen smaller, $\delta_p$ and $\delta_p$ get smaller, which reduces the constraint on $s_1$. However, smaller $k$ leads to $W_2$ and $Z_2$ of larger dimensions, which likely makes $s_1$ smaller.

3.2. Multilevel convergence. So far we have considered the relation between the orthogonal coarse-grid operator and coarse grid attained in practice. To prove multilevel convergence, we will decompose error over the subspaces $R(\Pi)$ and $R(I - \Pi)$. For an orthogonal projection, say $\tilde{\Pi}$ with respect to norm $\| \cdot \|$, $\|e\|^2 = \|(I - \tilde{\Pi})e\|^2 + \|\tilde{\Pi}e\|^2$. Because $\Pi$ as used here is a non-orthogonal projection, this equality does not hold. However, bounds on the decomposition are closely related to stability as proved in Section 2.3, and the angle between the subspaces $R(\Pi)$ and $R(I - \Pi)$.

From a given level in the AMG hierarchy, denote the coarse-grid matrix $A_c$, and define $Q_c A_c := (A_c^{-1})^2$, where $Q_c A_c$ defines the norm we will consider on the coarse grid. Then, consider the difference between the exact projection, $\Pi$, and the inexact projection, $\tilde{\Pi} = PB_c^{-1} R^* A$, where $B_c^{-1}$ denotes the AMG cycle applied to the coarse-grid problem. This corresponds to the recursive application of a multilevel AMG cycle. Assume $B_c^{-1}$ is convergent, with bound

$$\|I - B_c^{-1} A_c\|_{Q_c A_c}^2 = \|(A_c^{-1} - B_c^{-1}) A_c\|_{Q_c A_c}^2 < \rho_c,$$

and let $G(\nu)$ denote the error-propagation operator corresponding to $\nu$ iterations of relaxation. Then, from (17) and Lemma 14,

$$\|(\Pi - \tilde{\Pi}) G(\nu) e^{(i)}\|_{Q_A}^2 \leq \frac{1}{c_0} \left\| (A_c^{-1} - B_c^{-1}) A_c (A_c^{-1} R^* A G(\nu)) e^{(i)} \right\|_{Q_c A}^2$$

$$\leq \frac{\rho_c}{c_0} \left\| (A_c^{-1} R^* A G(\nu)) e^{(i)} \right\|_{Q_c A}^2$$

$$\leq \frac{c_1 \rho_c}{c_0} \left\| P A_c^{-1} R^* A G(\nu) e^{(i)} \right\|_{Q_A}^2$$

$$= \frac{c_1 \rho_c}{c_0} \|\Pi G(\nu) e^{(i)}\|_{Q_A}^2.$$

Error can then be expanded as

$$\|e^{(i + 1)}\|_{Q_A}^2 \leq \|(I - \Pi) G(\nu) e^{(i)}\|_{Q_A}^2 + 2 \left\langle (I - \Pi) G(\nu) e^{(i)}, (\Pi - \tilde{\Pi}) G(\nu) e^{(i)} \right\rangle_{Q_A} + \|(\Pi - \tilde{\Pi}) G(\nu) e^{(i)}\|_{Q_A}^2.$$ 

(23)

In order to bound the middle inner product, we introduce the following result connecting the angle between subspaces of a Hilbert space, the norm of an oblique projection, and a strengthened Cauchy-Schwarz inequality.

**Lemma 17 (Strengthened Cauchy Schwarz).** Define the minimal canonical angle between $R(\Pi)$ and $R(I - \Pi)$ in the $Q_A$ inner product by

$$\cos \left( \theta_{\min}^{(\Pi)} \right) := \sup_{\substack{x \in R(\Pi), \|x\|_{Q_A} = 1, \ y \in R(I - \Pi), \|y\|_{Q_A} = 1}} |\langle x, y \rangle_{Q_A}|.$$

Then, for all \( x \in \mathcal{R}(I) \) and \( y \in \mathcal{R}(I - \Pi) \),
\[
\| \Pi \|_{QA} = \| I - \Pi \|_{QA} = \frac{1}{\sin \left( \theta_{\text{min}}^{(I)} \right)},
\]

\[
| \langle x, y \rangle_{QA} | \leq \cos \left( \theta_{\text{min}}^{(I)} \right) \| x \|_{QA} \| y \|_{QA}.
\]

Proof. See [4, 28].

Applying Lemma 17 and an \( \epsilon \)-inequality to (23) yields
\[
\| e^{(i+1)} \|^2_QA \leq \| (I - \Pi)G^\nu e^{(i)} \|^2_QA + \frac{c_1 \rho_1}{c_0} \| \Pi G^\nu e^{(i)} \|^2_QA
\]
\[
+ 2 \cos \left( \theta_{\text{min}}^{(I)} \right) \| (I - \Pi)G^\nu e^{(i)} \|_{QA} \sqrt{\frac{c_1 \rho_1}{c_0} \| \Pi G^\nu e^{(i)} \|_{QA}}
\]
\[
\leq \left( 1 + \frac{\cos^2 \left( \theta_{\text{min}}^{(I)} \right)}{\epsilon} \right) \| (I - \Pi)G^\nu e^{(i)} \|^2_QA + \frac{(1 + \epsilon)c_1 \rho_1}{c_0} \| \Pi G^\nu e^{(i)} \|^2_QA,
\]
for any \( \epsilon > 0 \) and angle \( \theta_{\text{min}}^{(I)} \) between \( \mathcal{R}(\Pi) \) and \( \mathcal{R}(I - \Pi) \). Here, the first term corresponds to error that is not in the range of interpolation and must be attenuated by relaxation, while the second term is the error that is in the range of interpolation, but has not been eliminated by the inexact coarse-grid correction. For ease of notation, and because the change in constants is not significant for the final result, let \( \epsilon = 1 \).

Then, note that
\[
C_{\Pi} \left( 1 + \cos^2 \left( \theta_{\text{min}}^{(I)} \right) \right) = C_{\Pi} \left( 2 - \sin^2 \left( \theta_{\text{min}}^{(I)} \right) \right) = 2C_{\Pi} - 1.
\]

Let \( G^\nu \) correspond to \( \nu \) iteration of Richardson relaxation on the normal equations. By Theorem 2.3 in [3] (see Theorem 8) and (24),
\[
\| e^{(i+1)} \|^2_QA \leq \frac{16K_{FP}(2C_{\Pi} - 1)}{25\sqrt{4\nu + 1}} \| e^{(i)} \|^2_QA + \frac{2C_{\Pi} c_1 \rho_1}{c_0} \| e^{(i)} \|^2_QA.
\]

Now, suppose we have some desired convergence factor, \( \hat{\rho} \). The coarsest level in the hierarchy, level \( \ell - 1 \), is solved exactly. Then on level \( \ell - 2 \), \( \rho_\ell = 0 \), and we can pick the number of relaxation steps, say \( \nu_{\ell-2} \), to be such that \( \frac{16K_{FP}(2C_{\Pi} - 1)}{25\sqrt{4\nu_{\ell-2} + 1}} \leq \hat{\rho} \).

Thus, the AMG preconditioner corresponding to the inexact solve of level \( \ell - 2 \) has convergence factor \( \| I - B_{\ell-2}^{-1}A_{\ell-2} \|^2_{Q_{\ell-2}A_{\ell-2}} \leq \hat{\rho} \).

Moving up the hierarchy, on level \( \ell - 3 \), let \( \nu_{\ell-3} \) denote the number of relaxation steps and \( \mu_{\ell-3} \) the number of AMG cycles applied as an inexact solve. Choose these values such that
\[
\frac{16K_{FP}(2C_{\Pi} - 1)}{25\sqrt{4\nu_{\ell-3} + 1}} \leq \frac{\hat{\rho}}{2},
\]
\[
\frac{2C_{\Pi} c_1 \hat{\rho}_{\ell-3}}{c_0} \leq \frac{\hat{\rho}}{2}.
\]

Then, \( \| I - B_{\ell-3}^{-1}A_{\ell-3} \|^2_{Q_{\ell-3}A_{\ell-3}} \leq \hat{\rho} \). By induction, this process proceeds to the finest level in the hierarchy, resulting in an AMG solver with error-propagation operator, \( E \), bounded in the \( QA \)-norm: \( \| E \|^2_QA \leq \hat{\rho} < 1 \). The result is summarized in the following theorem.
THEOREM 18 (Multilevel convergence). Consider an AMG hierarchy with \( \ell \) levels, and assume the conditions for Lemma 14 hold on each level. Let the constants, including \( c_1, K_{F_P}, \) and \( C_{H} \) denote the maximum corresponding values over all levels in the hierarchy, and \( c_0 \) the minimum value over all levels. For any given \( \hat{\rho} < 1 \), define

\[
\mu = \left[ 1 + \frac{\log \left( \frac{c_0}{c_1 C_{H}} \right)}{\log(\hat{\rho})} \right],
\]

\[
\nu = \left[ \frac{(32K_{F_P})^2 (2C_{H} - 1)^2 - (25\hat{\rho})^2}{(50\hat{\rho})^2} \right].
\]

Let \( \mathcal{E} \) denote the error propagation operator for an AMG hierarchy, with \( \nu \) iterations of Richardson relaxation on the normal equations during each multigrid cycle, and \( \mu \) multigrid cycles on each level in the hierarchy. Then, \( \mathcal{E} \) is bounded in the QA norm, \( \| \mathcal{E} \|_{QA} \leq \hat{\rho} < 1 \), independent of the number of levels in the hierarchy.

Proof. The proof follows from solving for \( \nu \) in (25), \( \mu \) in (26), and the discussion above.

REMARK 19. Note that \( \hat{\rho} \) can be chosen. That choice should be made to control the size of \( \mu \geq 2 \), where \( \mu = 2 \) corresponds to a W-cycle. The choice of \( \hat{\rho} \) will then dictate \( \nu \). In practice, it is also possible that F-cycles are sufficient to overcome the non-orthogonal coarse-grid correction [15], at a cheaper cost than W- or \( \mu \)-cycles. However, F-cycles do not fit naturally into a recursive-based proof.

Theorem 18 proves the existence of a convergent, multilevel \( \mu \)-cycle, with convergence independent of the problem size and number of levels in the hierarchy. Note that the constants \( \mu \) and \( \nu \) derived in Theorem 18 are theoretical in nature and, from the perspective of computational expense, probably not optimal. However, these constants are relevant in the broader question of whether the computational expense of the \( \mu \)-cycle is also scalable, which depends on the coarsening ratio. Let \( r \) be the coarsening ratio. Then, the number of matrix-vector operations for one AMG cycle is approximately given by

\[
C \approx (C + \nu) \sum_{k=1}^{\ell-1} (r\mu)^k,
\]

for some small constant \( C \) that accounts for computing residuals, restriction, and interpolation. In aggregation-based AMG, \( r \) is typically 8–10 in 2d, or higher in 3d or with aggressive coarsening schemes, suggesting that scalable convergence, in terms of floating-point operations with respect to problem size, is also possible. Recall that \( \hat{\rho} \) may be chosen, which determines \( \mu \). If \( r < 1/2 \), then \( \hat{\rho} \) can be chosen so that \( r\mu < 1 \), which yields a complexity bounded independent of of problem size. Note that smaller \( \mu \) comes at the expense of larger \( \nu \). If \( r \geq 1/2 \), then the bound on complexity grows like \( \log(n) \). However, this may be another shortcoming of this line of proof.

4. Discussion. Here, conditions have been established on \( R \) and \( P \) for two-grid and multigrid convergence of NS-AMG in the \( \sqrt{A^*A} \)-norm. Results indicate that it is not enough for \( R \) and \( P \) to include low-energy left and right singular vectors in their range (classical approximation-property-based AMG approach). For a stable coarse-grid correction, the action of \( R \) and \( P \) must also lead to a non-singular (and...
reasonably conditioned) coarse-grid operator. Sufficient conditions for this are that $R$ and $P$ accurately interpolate singular vectors associated with small singular values, and, additionally, $R$ and $P$ have a similar action on all left and right singular vectors, including those associated with large singular values. Furthermore, multilevel convergence of NS-AMG may require additional iterations of relaxation or multigrid cycles on coarser levels of the hierarchy to converge. However, Theorem 18 indicates that, with the appropriate AMG cycle, scalable convergence with respect to the number of levels in the hierarchy and problem size is possible if the coarsening ratio is less than $1/2$.

In general, conditions developed here may not be necessary for NS-AMG convergence. A reduction-based convergence framework was also developed recently, and sufficient conditions derived for $\ell^2$-convergence of the error and residual [15]. Conditions for convergence in [15] are different in that a SSAP with respect to QA is not necessarily required on both $R$ and $P$. Rather, in [15], a SSAP with respect to $QA$ (or, equivalently, a WAP with respect to $A^*A$) is required on at least one of $R$ or $P$. The other operator then must satisfy an additional assumption on approximating the ideal restriction or interpolation operator with some level of accuracy. Nevertheless, several takeaways of the two analyses are consistent. For a robust NS-AMG solver, it is best to consider $R \neq P$. Furthermore, both approaches indicate that classical AMG approaches to interpolation — building the range of $P$ to contain error associated with small eigenvalues — are applicable in the nonsymmetric setting, when coupled with an appropriate restriction operator. However, care must be taken to build $R$ and $P$ in a “compatible” sense, leading to a stable correction. Furthermore, due to the non-orthogonal nature of NS-AMG, both analyses indicate that modified cycles with additional relaxation or cycling on coarser grids may be necessary for scalable convergence. The reduction-based NS-AMG algorithms developed in [15,16] have shown promising results on highly nonsymmetric matrices resulting from the discretization of hyperbolic PDEs. Development of a robust a NS-AMG solver based on theory developed here is ongoing work.

Appendix.

**Lemma 20.** Consider the block matrix \( \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \). Suppose

\[
0 < a_0 \|x\| \leq \|Ax\|, \quad \|Ax\| \leq a_1 \|x\|, \quad \|Bx\| \leq b \|x\|, \\
0 < d_0 \|x\| \leq \|Dx\|, \quad \|Dx\| \leq d_1 \|x\|, \quad \|Cx\| \leq c \|x\|,
\]

for all $x$. Further, assume $a_0 d_0 > b c$. Then, we can bound

\[
0 < \eta_0 \leq \frac{\left\| \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2}{\|x\|^2 + \|y\|^2}, \quad \text{and} \quad \frac{\left\| \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2}{\|x\|^2 + \|y\|^2} \leq \eta_1,
\]

where

\[
\eta_0 = \frac{a_0^2 + b^2 + c^2 + d_0^2 - \sqrt{(a_0^2 + b^2 - c^2 - d_0^2)^2 + 4(a_0 c + bd_0)^2}}{2},
\]
\[
\eta_1 = \frac{a_1^2 + b^2 + c^2 + d_1^2 + \sqrt{(a_1^2 + b^2 - c^2 - d_1^2)^2 + 4(a_1 c + bd_1)^2}}{2}.
\]
Proof. Starting with the lower bound, assume positive constants: \(a_0, b, c, d_0 > 0\). An \(\epsilon\)-inequality can be used to bound below in norm:

\[
\left\| \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 = \|Ax - By\|^2 + \|Cx - Dy\|^2
\]

\[
= \|Ax\|^2 - 2\langle Ax, By \rangle + \|By\|^2 + \|Cx\|^2 - 2\langle Cx, Dy \rangle + \|Dy\|^2
\]

\[
\geq (1 - \epsilon_1)\|Ax\|^2 - (1/\epsilon_1 - 1)\|By\|^2 + (1 - \epsilon_2)\|Dy\|^2 - (1/\epsilon_2 - 1)\|Cx\|^2
\]

\[
\geq \left[a_0^2(1 - \epsilon_1) - c^2(1/\epsilon_2 - 1)\right]\|x\|^2 + \left[d_0^2(1 - \epsilon_2) - b^2(1/\epsilon_1 - 1)\right]\|y\|^2
\]

for any \(\epsilon_1, \epsilon_2 \in (0, 1]\). Note that the upper bound on \(\epsilon_1\) and \(\epsilon_2\) is necessary to keep the leading constants on \(\|Ax\|^2\) and \(\|Dy\|^2\) positive because we bounded these from below, and vice versa for \(\|By\|^2\) and \(\|Cx\|^2\). This leads to a system of constraints

\[
\begin{align*}
C_1(\epsilon_1, \epsilon_2) &:= a_0^2(1 - \epsilon_1) - c^2(1/\epsilon_2 - 1) > 0, \\
C_2(\epsilon_1, \epsilon_2) &:= d_0^2(1 - \epsilon_2) - b^2(1/\epsilon_1 - 1) > 0,
\end{align*}
\]

for some \(\epsilon_1, \epsilon_2 \in (0, 1]\). The boundary of these constraints in the \((\epsilon_1, \epsilon_2)\)-plane is given by the functions

\[
\tilde{\epsilon}_2(\epsilon_1) = \frac{c^2}{c^2 + a_0^2(1 - \epsilon_1)}, \quad \tilde{\epsilon}_2(\epsilon_1) = 1 + \frac{b^2}{d_0^2} - \frac{b^2}{d_0^2 c_1} + \frac{b^2}{a_0^2}
\]

with the region of points satisfying the constraints bounded below by \(\tilde{\epsilon}_2\) and above by \(\tilde{\epsilon}_2\). A little algebra shows that \(\tilde{\epsilon}_2\) is concave up, \(\tilde{\epsilon}_2\) concave down, and both functions are monotonically increasing over \((0, 1]\) with a crossover point at \(\tilde{\epsilon}_2'(1) = \tilde{\epsilon}_2'(1) = 1\). It follows that there exists some region within \((0, 1) \times (0, 1)\) (constraints on \(\epsilon_1\) and \(\epsilon_2\)) that satisfies (27) if and only if \(\tilde{\epsilon}_2'(1) > \tilde{\epsilon}_2'(1)\), which reduces to \(a_0 d_0 > b c\).

The maximum bound is obtained by setting the leading constants on \(\|x\|^2\) and \(\|y\|^2\) equal. Thus we will consider a constrained maximization over \(C_1\) such that \(C_1 = C_2\) (or vice versa). Since we are maximizing the intersection of two convex functionals, which is also convex, the maximum is unique. Thus consider \(\epsilon_2(\epsilon_1)\) and denote \(\epsilon'_2 := \frac{\partial \epsilon_2}{\partial \epsilon_1}\). Then, at the maximum, we must have \(\frac{\partial}{\partial \epsilon_1} C_1(\epsilon_1, \epsilon_2(\epsilon_1)) = \frac{\partial}{\partial \epsilon_1} C_1(\epsilon_1, \epsilon_2(\epsilon_1)) = 0\):

\[
\begin{align*}
-a_0^2 \cdot \epsilon_1' + \frac{c^2}{\tilde{\epsilon}_2} &= 0 \quad \implies \quad \epsilon'_2 = \frac{a_0^2}{c_2^2} \tilde{\epsilon}_2' \\
-d_0^2 \cdot \epsilon_1' + \frac{b^2}{\tilde{\epsilon}_1} &= 0 \quad \implies \quad \epsilon'_2 = \frac{b^2}{d_0^2 \epsilon_1}.
\end{align*}
\]

Setting the functions for \(\epsilon'_2\) equal leads to the constraint \(\epsilon_2 = \frac{bc}{a_0 d_0 c_1}\), and plugging into \(C_1\) and \(C_2\) gives

\[
\begin{align*}
C_1(\epsilon_1) &= a_0^2 + c^2 - \epsilon_1 \left(a_0^2 + \frac{a_0 c d_0}{b}\right), \\
C_2(\epsilon_1) &= d_0^2 + b^2 - \frac{1}{\epsilon_1} \left(\frac{b c d_0}{a_0} + b^2\right).
\end{align*}
\]

Setting \(C_1 = C_2\) leads to a quadratic function in \(\epsilon_1\):

\[
\epsilon_1^2 \left(a_0^2 + \frac{a_0 c d_0}{b}\right) + \epsilon_1 \left(b^2 + d_0^2 - a_0^2 - c^2\right) - b \left(\frac{c d_0}{a_0} + b\right) = 0.
\]
Because $a_0, b, c, d_0 > 0$, we have $-b \left( \frac{cd_1}{a_0} + b \right) < 0$ and, thus, there exists exactly one positive root, given by

$$
\epsilon_1 = \frac{(a_0^2 + c^2 - b^2 - d_0^2) + \sqrt{(a_0^2 + c^2 - b^2 - d_0^2)^2 + 4(a_0b + cd_0)^2}}{2 \left( a^2 + \frac{a_0cd_1}{b} \right)}.
$$

Plugging into $C_1$ gives

$$
C_1(\epsilon_1) = C_2(\epsilon_1) = \frac{a_0^2 + b^2 + c^2 + d_0^2 - \sqrt{(a_0^2 + c^2 - b^2 - d_0^2)^2 + 4(a_0b + cd_0)^2}}{2},
$$

where $\eta_0 := C_1(\epsilon_1)$. Setting $b = 0$ or $c = 0$ and repeating the above process leads to a lower bound consistent with setting $b = 0$ or $c = 0$ in (28).

A similar derivation can be used for an upper bound. Let us start by assuming positive bounds, $a_1, b, c, d_1 > 0$. We bound in norm from above, again using an $\epsilon$-inequality, and seek to minimize the intersection of

$$
C_3(\epsilon_1, \epsilon_2) := a_1^2(1 + \epsilon_1) + c^2(1 + 1/\epsilon_2),
$$

$$
C_4(\epsilon_1, \epsilon_2) := d_1^2(1 + \epsilon_2) + b^2(1 + 1/\epsilon_1).
$$

Each of these are concave up, convex functionals in the positive $(\epsilon_1, \epsilon_2)$-plane (note, there are no constraints on the constants for this region to exist), and a minimum is attained when $\frac{\partial}{\partial \epsilon_1} C_3(\epsilon_1, \epsilon_2(\epsilon_1)) = \frac{\partial}{\partial \epsilon_1} C_4(\epsilon_1, \epsilon_2(\epsilon_1)) = 0$. This leads to a quadratic functional in $\epsilon_1$:

$$
\epsilon_1^2 \left( a_1^2 + \frac{a_1cd_1}{b} \right) + \epsilon_1 \left( a_1^2 + c^2 - b^2 - d_1^2 \right) - b \left( \frac{cd_1}{a_1} + b \right) = 0,
$$

with one positive root by Descartes’ rule of signs and the assumption $a_1, b, c, d_1 > 0$. The root is given by

$$
\epsilon_1 = \frac{(b^2 + d_1^2 - a_1^2 - c_1^2) + \sqrt{(a_1^2 + c^2 - b^2 - d_1^2)^2 + 4(a_1b + cd_1)^2}}{2 \left( a_1^2 + \frac{a_1cd_1}{b} \right)},
$$

which we can plug into $C_3$ and $C_4$ to solve for an upper bound

$$
\eta_1 = \frac{a_1^2 + b^2 + c^2 + d_1^2 + \sqrt{(a_1^2 + c^2 - b^2 - d_1^2)^2 + 4(a_1b + cd_1)^2}}{2}.
$$

In the case that some of $a_1, b, c$, or $d_1$ are equal to zero, it is straightforward to use a single $\epsilon$-inequality to derive an upper bound, and verify that this bound is equivalent to plugging the appropriate zeros into (29).

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