THE $e$-POSITIVITY AND SCHUR POSITIVITY OF THE CHROMATIC SYMMETRIC FUNCTIONS OF SOME TREES

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Abstract. We investigate the $e$-positivity and Schur positivity of the chromatic symmetric functions of some spider graphs with three legs. We obtain the positivity classification of all broom graphs and that of most double broom graphs. The methods involve extracting particular $e$-coefficients of the chromatic symmetric function of these graphs with the aid of Orellana and Scott’s triple-deletion property, and using the combinatorial formula of Schur coefficients by examining certain special rim hook tabloids. We also propose some conjectures on the $e$-positivity and Schur positivity of trees.

1. Introduction

Stanley [35] introduced the chromatic symmetric function for a simple graph $G$ as

$$X_G = X_G(x_1, x_2, \ldots) = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

where $x = (x_1, x_2, \ldots)$ is a countable set of indeterminates, and the sum is over all proper colorings $\kappa$, namely colorings such that every each color class is an independent set. The chromatic polynomial $\chi_G(k)$ counts the number of proper colorings using $k$ colors, which is a classical graph invariant dating back to Birkhoff [2]. The chromatic symmetric function satisfies $X_G(1^k) = \chi_G(k)$. The chromatic symmetric function $X_G$ is a symmetric function. Common bases for the algebra $\Lambda(x_1, x_2, \ldots)$ of symmetric functions include the monomial symmetric functions $\{m_\lambda\}$, elementary symmetric functions $\{e_\lambda\}$, Schur symmetric functions $\{s_\lambda\}$ and so on, see [37, Chapter 7]. For any basis $\{b_\lambda\}$ of $\Lambda(x_1, x_2, \ldots)$, the graph $G$ is said to be $b$-positive if the expansion of $X_G$ in $b_\lambda$ has only nonnegative coefficients.

A strong motivation of studying the positivity of chromatic symmetric functions is Stanley and Stembridge’s conjecture posed in 1993, see Conjecture 1.1. The smallest connected non-$e$-positive graph is the claw, whose chromatic symmetric function is

$$X_{\text{claw}} = 4e_4 + 5e_{31} - 2e_{22} + e_{212} = s_{31} - s_{22} + 5s_{212} + 8s_{14}.$$ 

A graph is claw-free if it does not contain an induced subgraph which is isomorphic to the claw. The incomparability graph of a poset $P$ is the graph with vertex set $P$, in which two elements are adjacent if and only if they are incomparable.

Conjecture 1.1 (Stanley and Stembridge). Any claw-free incomparability graph is $e$-positive.

Wolfgang [42] provided a powerful criterion that any connected $e$-positive graph has a connected partition of every type, where a connected partition of a graph $G$ is a partition $\{V_1, \ldots, V_k\}$ of $V(G)$ such that each induced subgraph $G[V_i]$ is connected. Many graph classes are shown to be $e$-positive, including complete graphs, paths, cycles, triad-free graphs, generalized bull graphs, (claw, $K_3$)-free

2010 Mathematics Subject Classification. 05E05 05A17 05A15 05C15.

Key words and phrases. chromatic symmetric function, $e$-positivity, Schur positivity, Young tableau.

This paper is supported by the General Program of National Natural Science Foundation of China (Grant No. 12171034) and the Fundamental Research Funds for the Central Universities in China (Grant No. 2021CX11012).
graphs, (claw, co-$P_3$)-free graphs, $F$-free unit interval graphs for all 4-vertex graphs $F$ except the co-diamond, $K_4$, $4K_1$ and $2K_2$; (claw, paw)-free graphs, (claw, co-paw)-free graphs, (claw, diamond, co-diamond)-free graphs, (claw, triangle)-free graphs, (claw, co-P$^3_3$)-free graphs, (claw, co-diamond, $2K_2$)-free graphs, $2K_2$-free unit interval graphs, $K$-chains, lollipop graphs, triangular ladders, (claw, co-claw)-free graph except the net, Ferrers graphs; see [3–5, 7, 8, 11, 12, 17, 19, 22, 35, 39]. Graphs that are proved not to be $e$-positive include the dart, generalized nets, saltire graphs $SA_{n,n}$, augmented saltire graphs $AS_{n,n}$ and $AS_{n,n+1}$, triangular tower graphs $TT_{n,n,n}$; see [9, 10, 12, 13]. In 2020 Dahlberg et al. [9] gave an infinite number of families of non-$e$-positive graphs that are not contractible to the claw. Moreover, one such family is additionally claw-free, thus establishing that the $e$-positivity is in general not dependent on the existence of an induced claw or of a contraction to a claw.

A second motivation of studying the positivity of chromatic symmetric functions is the close relationship between the Schur positivity and representation theory. The Schur functions are considered to be the most important basis of the algebra $\Lambda(x_1, x_2, \ldots)$ from several perspectives, see Macdonald [24, 25], Sagan [32] and Stanley [34, 37]. Gasharov [15] showed that any claw-free incomparability graph is Schur positive. Every $e$-positive graph is Schur positive since the $s_{\lambda}$-coefficient in $e_{\mu}$ is the Kostka number $K_{\lambda, \mu}$, which is nonnegative, see Mendes and Remmel [27, Exercise 2.12]. A leading conjecture in this direction is due to Gasharov [16] and Stanley [36].

**Conjecture 1.2** (Gasharov, Stanley). *Every claw-free graph is Schur positive.*

Stanley [36, Proposition 1.5] proved that the set of types of stable partitions of any Schur positive $n$-vertex graph is an order ideal of the poset of integer partitions of $n$ with respect to the dominance order. The authors [40] gave a combinatorial formula for the Schur coefficients of chromatic symmetric functions. Graphs that are shown to be Schur positive include tadpole graphs, the graphs obtained from two cycles $C$ and $C'$ by adding a path linking a vertex on $C$ and a vertex on $C'$, claw-free incomparability graphs, edge 2-colorable hyperforests, the incomparability graph of the natural unit interval order; see [15, 16, 33?]. Graphs that are proved not to be Schur positive include connected unbalanced bipartite graphs and the complete bipartite graphs $K_{m,n}$ with $m,n \geq 3$, see [40] for more graphs that are not Schur positive. Kaliszewski [21] confirmed the positivity of the $s_{\lambda}$ coefficients when $\lambda$ is of a hook shape.

In this paper, we concentrate on the chromatic symmetric functions of trees. This is not only for the simplicity of trees as a particular graph class, but also for the following major conjecture in this field, which is called *Stanley’s isomorphism conjecture* by Loebl and Sereni [23] and the *tree isomorphism conjecture* by Crew and Spirkl [6].

**Conjecture 1.3.** *The chromatic symmetric function distinguishes trees.*

In fact, Conjecture 1.3 was inspired by Stanley [35, Page 170]’s remark “We do not know whether $X_G$ distinguishes trees”. See [1, 13, 14, 18, 20, 26, 28, 29, 39] for its research progress.

The problem of determining whether a given tree is $e$-positive and whether it is Schur positive also received attention. Dahlberg et al. [10] conjectured the existence of an $n$-vertex Schur positive tree of maximum degree $\lfloor n/2 \rfloor$, which is disproved by Rambeloson and Shareshian [30] with a counterexample. They [10] also proved that any $n$-vertex $e$-positive tree has degree at most $\log_2 n$, and further conjectured the maximum degree of any $e$-positive tree to be 3.

**Conjecture 1.4** (Dahlberg et al.). *Any tree with a vertex of degree at least 4 is not $e$-positive.*

Zheng [43] obtained a breakthrough towards Conjecture 1.4 by proving that any tree with a vertex of degree at least 6 is not $e$-positive.

A particular class of trees, the spiders, plays an essential role in the study of $e$-positivity of graphs. A *spider* is a tree consisting of some paths with one endpoint on each path identified. Precisely
speaking, for any partition
\[ \lambda = \lambda_1 \cdots \lambda_d \vdash n - 1 \]
with \( d \geq 3 \), the spider \( S(\lambda) \) is the \( n \)-vertex tree consisting of the paths \( P_{1+\lambda_1}, \ldots, P_{1+\lambda_d} \) such that all of them share a common endpoint of degree \( d \). Dahlberg et al. \([10, Lemma 13]\) showed that if a connected graph \( G \) has a connected partition of type \( \mu \), then the spider \( S(\lambda) \) has a connected partition of type \( \mu \), where \( \lambda \) is the partition consisting of the sizes of connected components that are obtained by removing a vertex of degree at least 3 from \( G \). Therefore, the \( e \)-positivity of a general graph implies the \( e \)-positivity of certain spider in view of Wolfgang's criterion.

This paper is organized as follows. In Section 2 we give an overview for necessary notion and notation, as well as known results in the study of graph positivities that will be of use in the subsequent sections. Sections 3 and 4 are devoted to the positivity of spiders \( S(a, b, 1) \) and \( S(a, b, 2) \), respectively. We obtain some bounds of \( a \) in terms of \( b \) for the \( e \)-positivity of these spiders, and conjecture the Schur positivity of these spiders. In Section 5, we obtain the positivity classification of all broom graphs and the positivity classification of most double broom graphs. We end this paper with a conjecture that completes the positivity classification of double broom graphs, see Conjecture 5.4.

The chromatic symmetric functions of explicit graphs on a small number of vertices in this paper are computed by using Russell’s program \([31]\).

2. Preliminaries

Let \( n \) be a positive integer. A composition \( \kappa \) of \( n \) is a sequence \((\kappa_1, \ldots, \kappa_\ell)\) of integers that sum to \( n \). We write
\[
\kappa^! = \prod_{i \geq 1} \kappa_i^! \quad \text{and} \quad \kappa^! = \prod_{i \geq 1} k_i^!,
\]
where \( k_i \) is the number of occurrences of the part \( i \) in \( \kappa \). An integer partition \( \lambda \) of \( n \) is a composition \((\lambda_1, \ldots, \lambda_\ell)\) of \( n \) in non-increasing order, denoted \( \lambda \vdash n \). It can be recast as \( 1^{a_1}2^{a_2}\cdots \), where \( a_i \) is the multiplicity of \( i \) in \( \lambda \).

Let \( G = (V, E) \) be a graph with vertex set \( V \) and edge set \( E \). The order of \( G \) is the number \( |V| \) of vertices. A partition of \( G \) is a set partition \( \rho = V_1/\cdots/V_\ell \) of \( V \). It is said to be a bipartition if \( \ell = 2 \). A bipartition \( V_1/V_2 \) is balanced if \( |V_1| - |V_2| \in \{-1, 0, 1\} \). We call the sets \( V_i \) blocks of \( \rho \). We say that a partition \( \rho \) is semi-ordered if for any number \( m \), the blocks of order \( m \) in \( \rho \) are ordered. A block in \( \rho \) is stable if any two vertices in the block are not adjacent by an edge. A partition \( \rho \) is stable if its every block is stable. The type of \( \rho \) is the integer partition consisting of the block cardinalities, denoted \( \tau_\rho \). For any composition \( \kappa \) obtained by rearranging the parts of \( \tau_\rho \), without confusion, one may say that \( \rho \) is of type \( \kappa \).

For any partition \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \), the monomial symmetric function \( m_\lambda \) is defined by
\[
m_\lambda = \sum_\alpha x^\alpha,
\]
where \( \alpha \) runs over all distinct permutations of \( \lambda \); the augmented monomial symmetric function \( \tilde{m}_\lambda \) is defined by \( \tilde{m}_\lambda = \lambda^! m_\lambda \); the elementary symmetric function \( e_\lambda \) is defined to be
\[
e_\lambda = m_{1^{\lambda_1}} \cdots m_{1^{\lambda_\ell}};
\]
the Schur function \( s_\lambda \) is defined by
\[
s_\lambda = \sum_T x^T,
\]
where \( T \) ranges over all semistandard Young tableaux of shape \( \lambda \), and \( x^T \) is the monomial \( x_1^{i_1}x_2^{i_2} \cdots \) such that \( T \) contains exactly \( i_j \) cells with entry \( j \) for all \( j \).
Stanley [35, Propositions 2.3, 2.4, 5.3 and Theorem 2.5] gave some basic properties of chromatic symmetric functions.

**Proposition 2.1 (Stanley).** $X_{G \cup H} = X_G X_H$, where $G \cup H$ is the disjoint union of the graphs $G$ and $H$.

**Proposition 2.2 (Stanley).** The chromatic symmetric function $X_G$ of a graph $G = (V, E)$ can be computed by

$$X_G = \sum_{\lambda \vdash |V(G)|} a_{\lambda} \tilde{m}_{\lambda} = \sum_{E' \subseteq E} (-1)^{|E'|} p_{\lambda(E')}$$

where $a_{\lambda}$ is the number of stable partitions of $G$ of type $\lambda$, and $\lambda(E')$ is the integer partition consisting of the component orders of the spanning subgraph $(V, E')$.

**Proposition 2.3 (Stanley).** The chromatic symmetric functions $X_{P_n}$ for the $n$-vertex paths $P_n$ satisfy

$$\sum_{n \geq 0} X_{P_n} z^n = \frac{E(z)}{F(z)} = 1 + e_1 z + 2 e_2 z^2 + (3 e_3 + e_2) z^3 + \cdots,$$

where $E(z) = \sum_{n \geq 0} e_n z^n$ and $F(z) = E(z) - z E'(z)$.

Wolfgang [42, Proposition 1.3.3] derived a powerful criterion for the $e$-positivity of a graph.

**Theorem 2.4 (Wolfgang).** Any $e$-positive graph contains a connected partition of any type.

For any basis $\{b_\lambda\}$ of the algebra $\Lambda(x_1, x_2, \ldots)$ and any symmetric function $F \in \Lambda(x_1, x_2, \ldots)$, we use the notation $[b_\lambda]F$ to denote the coefficient of $b_\lambda$ in the $b$-expansion of $F$. By Proposition 2.3, Wolfe [41, Theorem 3.2] exhibited explicit formulas for the coefficients of $e_\lambda$ of paths.

**Proposition 2.5 (Wolfe).** Let $\lambda = 1^{a_1} 2^{a_2} \cdots d^{a_d} \vdash d$. Then

$$[e_\lambda] X_{P_d} = \left( \frac{\ell}{a_1, \ldots, a_d} \right) \prod_{a_i \geq 1} (j - 1)^{a_j} + \sum_{a_i \geq 1} \frac{\ell - 1}{a_1, \ldots, a_i - 1, \ldots, a_d} (i - 1)^{a_i - 1} \prod_{j \neq i, j \neq 2} (j - 1)^{a_j},$$

where $\ell = a_1 + \cdots + a_d$ is the length of $\lambda$.

Orellana and Scott [29, Theorem 3.1, Corollaries 3.2 and 3.3] established the beautiful triple-deletion property as follows.

**Theorem 2.6 (Orellana and Scott).** Let $G$ be a graph with a stable set $\{u, v, w\}$. Write $e_1 = uv$, $e_2 = uw$, and $e_3 = wu$. For any set $S \subseteq \{1, 2, 3\}$, denote by $G_S$ the graph with vertex set $V(G)$ and edge set $E(G) \cup \{e_j : j \in S\}$. Then

$$X_{G_{12}} = X_{G_1} + X_{G_{23}} - X_{G_3} \quad \text{and} \quad X_{G_{123}} = X_{G_{12}} + X_{G_{23}} - X_{G_2}.$$

Dahlberg and van Willigenburg [8, Proposition 5] generalized this to $k$-cycles, called the $k$-deletion property. Dahlberg et al. [10, Lemma 18 and Theorem 30] gave quick criteria for the $e$-positivity of spiders.

**Theorem 2.7 (Dahlberg et al.).** Let $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash n - 1$. If the spider $S(\lambda)$ is $e$-positive, then $\lambda_1 \geq \lfloor n/2 \rfloor$ and $d < \log_2 n + 1$.

Here are some contributions due to Zheng [43, Theorem 3.4, Lemma 4.4, Theorem 5.3] to the $e$-positivity of spiders.

**Lemma 2.8 (Zheng).** Let $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash n - 1$. Let $m \in \mathbb{Z}^+$ and $R_m = 1 + \sum_{i=1}^d r_i$, where $r_i$ is the least nonnegative residue of $\lambda_i$ modulo $m$. Suppose that $n = mq + r$, where $q, r \in \mathbb{Z}$ and $0 \leq r \leq m - 1$. If the spider $S(\lambda)$ is $e$-positive, then we have the following.
Theorem 3.2. Let \( R_m < 2m \), 

(2) if \( R_m \geq m \), then \( r_i \geq r \) for some \( i \in [d] \).

Lemma 2.9 (Zheng). Let \( \lambda = (\lambda_1, \ldots, \lambda_d) \vdash n-1 \). Suppose that 

\[
\{\lambda_1, \ldots, \lambda_d\} = \{2k_1 + 1, 2k_2 + 1, 2k_3, 2k_4, \ldots, 2k_d\}
\]
as multisets, where \( k_i \in \mathbb{Z} \). Then 

\[
[e_{32k_1+k_2+\cdots+k_d}]X_S(\lambda) = 4(k_1 + k_2 - k_3 - \cdots - k_d) + 2d - 1.
\]

Lemma 2.10 (Zheng). Suppose that \((a, b, c) \vdash n-1 \). Then the chromatic symmetric function of the spider \( S(a, b, c) \) can be computed by 

\[
X_{S(a,b,c)} = X_{P_n} + \sum_{i=1}^c (X_{P_{n-i}} - X_{P_{n-b_i}}X_{P_{n-b_{i-1}}}).
\]

For Schur positivity of graphs, the authors [40] obtained the following results.

Theorem 2.11 (Wang and Wang). Any Schur positive connected bipartite graph has a balanced stable bipartition.

Theorem 2.12 (Wang and Wang). For any graph \( G = (V, E) \) and any integer partition \( \lambda \) of \( |V| \), 

\[
|s_{\lambda}|X_G = \sum_{T \in T_\lambda} (-1)^{|W_T|} \tilde{a}_\kappa_T,
\]

where \( T_\lambda \) is the set of special rim hook tabloids \( T \) of shape \( \lambda \) such that \( G \) contains a stable partition of type \( \kappa_T \), \(|W_T|\) is the number of rim hooks of \( T \) that span an even number of rows, and \( \tilde{a}_\kappa \) is the number of semi-ordered stable partitions of \( G \) of type \( \kappa \).

3. The positivity of spiders \( S(a, b, 1) \)

This section is devoted to the \( e \)-positivity and Schur positivity of spiders \( S(a, b, 1) \). First of all, it is hardly true that a spider \( S(a, b, c) \) with odd \( b \) and odd \( c \) is \( e \)-positive.

Theorem 3.1. Let \((a, b, c) \vdash n-1 \). Suppose that \( b \) and \( c \) are odd. If the spider \( S(a, b, c) \) is \( e \)-positive, then \( a = b + c \).

Proof. Write \((a, b, c) = (\lambda_1, \lambda_2, \lambda_3) \). Suppose that \( \lambda_i = 2k_i + 1 \) for \( i \in \{2, 3\} \). Since \( S(\lambda) \) is \( e \)-positive, the part \( \lambda_1 \) must be even. Suppose that \( \lambda_1 = 2k_1 \). Then \( n = 2(k_1 + k_2 + k_3) + 3 \). By Lemma 2.9, 

\[
[e_{32k_1+k_2+k_3}]X_S(\lambda) = 4(k_2 + k_3 - k_1) + 5 = 2n - 1 - 4\lambda_1.
\]

Since \( X_{S(\lambda)} \) is \( e \)-positive, the non-negativity of the formula above implies that \( \lambda_1 \leq \lfloor n/2 \rfloor \). By Theorem 2.7 we know that \( \lambda_1 \geq \lfloor n/2 \rfloor \), i.e., \( \lambda_1 = \lambda_2 + \lambda_3 \). \( \square \)

Conversely, we do not know whether the spider \( S(b + c, b, c) \) is \( e \)-positive. The \( e \)-positivity of \( S(b+1, b, 1) \) was conjectured by Aliniaeifard, van Willigenburg, and Wang, which was a particular case of a more general conjecture, see Zheng [43, Conjecture 6.3]. It is direct to verify the \( e \)-positivity of the spider \( S(2m + 2, 2m + 1, 1) \) for \( m \leq 15 \).

We are able to prove that \( S(b+3, b, 3) \) for odd \( b \geq 7 \) are not \( e \)-positive.

Theorem 3.2. Let \( b \) be an odd positive integer. The spider \( S(b+3, b, 3) \) is \( e \)-positive if and only if \( b = 5 \).
Proof. Let $G = S(2m + 4, 2m + 1, 3)$. The order of $G$ is $N = 4m + 9$. Suppose that $m$ is an odd positive integer. By Lemma 2.10 and Proposition 2.5,

$$[e_{54m+1}]X_G = [e_{54m+1}]X_{P_{4m+9}} + [e_{54m+1}]\sum_{i=1}^{3}(X_{P_{5}X_{P_{4m+9-i}} - X_{P_{2m+1-i}}X_{P_{2m+8-i}}})$$

$$= [e_{54m+1}]X_{P_{4m+9}} - [e_{54m+1/2}]X_{P_{2m+2}}[e_{54m+1/2}]X_{P_{2m+7}} - [e_{54m-1/2}]X_{P_{2m+3}}[e_{54m-1/2}]X_{P_{2m+6}}$$

$$= 3^m(16m + 31) - 4 \cdot 3^{(m-1)/2} \cdot 3^{(m-1)/2}(8m + 23) - 3^{(m-3)/2}(8m + 7) \cdot 4 \cdot 3^{(m+1)/2}$$

$$= -3^{m-1}(16m + 27) < 0.$$

Now suppose that $m$ is an even positive integer. The positivity of $S(8, 5, 3)$ can be verified by direct computation with the aid of Lemma 2.10. Let $m \geq 4$. By Lemma 2.10 and Proposition 2.5,

$$[e_{5^4m-4}]X_G = [e_{5^4m-4}]X_{P_{4m+9}} - [e_{5^4m-4}]\sum_{i=1}^{3}X_{P_{2m+1+i}}X_{P_{2m+8-i}}$$

$$= [e_{5^4m-4}]X_{P_{4m+9}} - [e_{5^4m-2/2}]X_{P_{2m+2}}[e_{5^4m-2/2}]X_{P_{2m+7}} - [e_{5^4m-2/2}]X_{P_{2m+3}}[e_{5^4m-2/2}]X_{P_{2m+6}}$$

$$= (m + 1) \cdot 4^{m/2} \cdot 3^{m-5}(20 - \frac{4m + 9}{m + 1})$$

$$= \left(\frac{m/2}{2}\right) \cdot 4^{m/2-3} \cdot \left(20 - \frac{2m + 2}{m/2}\right) \cdot \left(\frac{m/2 + 1}{3}\right) \cdot 4^{m/2-3} \cdot \left(20 - \frac{2m + 7}{m/2 + 1}\right)$$

$$= \left(\frac{m/2}{3}\right) \cdot 4^{m/2-4} \cdot \left(20 - \frac{2m + 3}{m/2}\right) \cdot \left(\frac{m/2 + 1}{2}\right) \cdot 4^{m/2-2} \cdot \left(20 - \frac{2m + 6}{m/2 + 1}\right)$$

$$= 4 \cdot 3^{m/2} \cdot \left(\frac{m/2}{5}\right) \cdot 4^{m/2-6} \cdot \left(20 - \frac{2m + 5}{m/2}\right)$$

$$= \left(\frac{m/2}{4}\right) \cdot 4^{m/2-5} \cdot \left(20 - \frac{2m + 5}{m/2}\right) \cdot 3^{m/2-1}(20(m/2 + 1) - (2m + 5))$$

$$= -\frac{4}{3} \cdot 3^{m-7}(32m^5 - 300m^4 + 1475m^3 - 2970m^2 + 2048m - 240).$$

The positivity of $F(m) = 32m^5 - 300m^4 + 1475m^3 - 2970m^2 + 2048m - 240$ for even integers $m \geq 4$ can be seen by a direct check for $m \in \{4, 6, 8\}$ and by

$$F(m) = (32m - 300)m^4 + (1475m - 2970)m^2 + (2048m - 240) > 0$$

for $m \geq 10$. Therefore, $[e_{5^4m-4}]X_G < 0$ for even $m \geq 4$. This completes the proof. \qed

Now we concentrate on the positivity of $S(a, b, 1)$ for even $b$.

**Theorem 3.3.** Let $b$ be even and $a \geq b \geq 2$. If the spider $S(a, b, 1)$ is $e$-positive, then we have the following.

1. If $b \equiv 2 \pmod{3}$, then either
   - $a \equiv 0 \pmod{3}$ and $a \leq 2b + 2$, or
   - $a \equiv 1 \pmod{3}$ and $a \leq 2b - 3$.
2. If $b \not\equiv 2 \pmod{3}$, then $a \leq b^2 - 1$ or $a = b^2 + b$.
3. If $a$ is even, then $a > b + (1 + \sqrt{8b - 3})/2$. 
Proof. Let $G = S(a,b,1)$ and $a = (b+1)n + r$, where $r \in \{0,1,\ldots,b\}$. Let $N$ be the number of vertices of $G$. Then

$$N = (b+1)(n+1) + r + 1.$$  

Consider $G$ as obtained by adding a pending edge $v_{b+1}v_N$ to the path $v_1 \cdots v_{N-1}$. By Lemma 2.10, 

$$X_G = e_1X_{P_N} + X_{P_{N-1}} - X_{P_{a+1}}X_{P_{b+1}}.$$  

We will extract certain $e$-coefficient from both sides of Eq. (3.1) and show its negativity using Proposition 2.5.

(1) Suppose that $b \equiv 2 \pmod{3}$. If $a \equiv 0 \pmod{3}$, then $N \equiv 1 \pmod{3}$ and 

$$[e_{43(n-4)/3}]X_G = [e_{43(n-4)/3}]X_{P_N} - [e_{43(n-3)/3}]X_{P_{a+1}}[e_{3(b+1)/3}]X_{P_{b+1}}$$

$$= 2^{(N-1)/3-3}(-3a + 6b + 7).$$

Thus the $e$-positivity of $G$ together with the residue of $a$ implies that $a \leq 2b + 2$. If $a \equiv 1 \pmod{3}$, then $N \equiv 2 \pmod{3}$ and 

$$[e_{3(n-2)/3}]X_G = [e_{3(n-2)/3}]X_{P_N} - [e_{3(n-1)/3}]X_{P_{a+1}}[e_{3(b+1)/3}]X_{P_{b+1}}$$

$$= 2^{(N-2)/3-2}(-a + 2b - 1).$$

Thus the $e$-positivity of $G$ implies $a \leq 2b - 3$. If $a \equiv 2 \pmod{3}$, then $N \equiv 0 \pmod{3}$ and 

$$[e_{3n/3}]X_G = [e_{3n/3}]X_{P_N} - [e_{3(n+1)/3}]X_{P_{a+1}}[e_{3(b+1)/3}]X_{P_{b+1}} = -3 \cdot 2^{N/3-2} < 0.$$

(2) We proceed according to the value of $r$. When $r = b$, we see that $G$ is not $e$-positive by taking $m = b+1$ in Lemma 2.8. If $1 \leq r \leq b-1$, then 

$$[e_{(b+1)^n(r+1)}]X_G = [e_{(b+1)^n(r+1)}]X_{P_N} - [e_{(b+1)^n(r+1)}]X_{P_{a+1}}[e_{b+1}]X_{P_{b+1}}$$

$$= b^{n-1}r[b^2 - (b+1)n] - b^n,$$

which is negative as if $n \geq b - 1$. If $r = 0$, then 

$$[e_{(b+2)(b+1)^n}]X_G = [e_{(b+2)(b+1)^n}]X_{P_N} - [e_{(b+2)(b+1)^n}]X_{P_{a+1}}[e_{b+1}]X_{P_{b+1}}$$

$$= b^{n-2}[b+1]^2(b-n) + 1],$$

which is negative as if $n \geq b + 1$. Summing up the results above yields either $a \leq b^2 - 1$ or $a = b^2 + b$.

(3) Now, we suppose that $a$ and $b$ are even. By Eq. (3.1), 

$$[e_{32n/2-3}]X_G = [e_{32n/2-3}]X_{P_N} - [e_{32n/2-1}]X_{P_{a+1}}[e_{32n/2-1}]X_{P_{b+1}}$$

$$= a^2 - (2b+1)a + (b^2 - b + 1).$$

Thus the $e$-positivity of $G$ implies that $a \geq b + (1 + \sqrt{8b - 3})/2$, in which the equality does not hold since $\sqrt{8b - 3}$ is not an integer. This completes the proof. ∎

Theorem 3.3 is sharp in the following sense: (1) The spiders $S(6,2,1)$, $S(18,8,1)$ and $S(30,14,1)$ are $e$-positive, as well as the the spiders $S(13,8,1)$, $S(25,14,1)$ and $S(37,20,1)$. (2) The spiders $S(15,4,1)$ and $S(35,6,1)$ are $e$-positive. (3) The spiders $S(6,2,1)$, $S(8,4,1)$ and $S(10,6,1)$ are $e$-positive.

Corollary 3.4. We have the following.

(1) The spider $S(a,2,1)$ is $e$-positive if and only if $a \in \{3,6\}$.

(2) The spider $S(a,4,1)$ is $e$-positive if and only if $a \in \{5,8,10,12,13,15,20\}$.

(3) The spider $S(a,6,1)$ is $e$-positive if and only if $a \in \{7,8,\ldots,35\} \cup \{42\} \setminus \{8,13,20,27,34\}$.

(4) The spider $S(a,8,1)$ is $e$-positive if and only if $a \in \{9,13,15,18\}$.
Proof. By Theorem 3.3, it suffices to check the \( e \)-positivity of a few number of spiders for each spider classes \( S(a, b, 1) \) with \( b \in \{2, 4, 6, 8\} \). One may compute the chromatic symmetric function of these spiders straightforwardly by using Eq. (3.1) and Proposition 2.3. \( \square \)

In view of the sporadic case \( a = b^2 + b \) in (2), we propose Conjecture 3.5, which is checked to be true for \( b \in \{4, 6\} \).

**Conjecture 3.5.** If \( b \) is even and \( b \not\equiv 2 \pmod{3} \), then the spider \( S(b^2 + b, b, 1) \) is \( e \)-positive.

We further conjecture that all spiders \( S(a, b, 1) \) that have been shown not to be \( e \)-positive in Theorem 3.3 are Schur positive.

**Conjecture 3.6.** Suppose that \( a \geq b \geq 2 \) and \( b \) is even. The spider \( S(a, b, 1) \) is Schur positive if one of the following is true.

1. \( b \equiv 2 \pmod{3}, a \equiv 0 \pmod{3} \) and \( a \geq 2b + 5 \).
2. \( b \equiv 2 \pmod{3}, a \equiv 1 \pmod{3} \) and \( a \geq 2b \).
3. \( b \equiv 2 \pmod{3}, a \equiv 2 \pmod{3} \) and \( a \geq 2b \).
4. \( b \equiv 2 \pmod{3} \) and \( a \geq b^2 \).
5. \( a \) is even and \( b \leq a \leq b + (1 + \sqrt{8b - 3})/2 \).

It is routine to check Conjecture 3.6 for the first few values of the pair \( (a, b) \): (1) is true for \( b = 2 \) and \( a \leq 30 \), as well as for \( b = 8 \) and \( a \leq 24 \); (2) is true for \( b = 2 \) and \( a \leq 28 \), as well as for \( b = 8 \) and \( a \leq 22 \); (3) is true for \( b = 2 \) and \( a \leq 29 \), as well as for \( b = 8 \) and \( a \leq 20 \), and \( b = a = 14 \); (4) is true for \( b = 4 \) and \( a \leq 25 \); (5) is true for \( b \leq 12 \).

**Conjecture 3.7.** Suppose that \( a \geq b \geq 2 \) and \( b \) is even. The spider \( S(a, b, 1) \) is Schur positive if one of the following is true.

1. \( b \equiv 2 \pmod{3}, a \equiv 0 \pmod{3} \) and \( a \leq 2b + 2 \).
2. \( b \equiv 2 \pmod{3}, a \equiv 1 \pmod{3} \) and \( a \leq 2b - 3 \).
3. \( b \equiv 2 \pmod{3} \) and \( b \leq a \leq b^2 - 1 \).
4. \( a \) is even and \( a > b + (1 + \sqrt{8b - 3})/2 \).

It is routine to check Conjecture 3.7 for the first few values of the pair \( (a, b) \): (1) and (2) are true for \( b \in \{2, 8\} \); (3) is true for \( b = 4 \), and for \( b = 6 \) with \( a \leq 23 \); (4) is true for ***.

4. The positivity of spiders \( S(a, b, 2) \)

This section is devoted to the \( e \)-positivity and Schur positivity of spiders \( S(a, b, 2) \).

**Theorem 4.1.** Let \( a \geq b \geq 2 \). Suppose that the spider \( S(a, b, 2) \) is \( e \)-positive. Let \( r_a \) and \( r_b \) be the remainders of \( a \) and \( b \) modulo 3, respectively. Then either \( (r_a, r_b) = (0, 1) \) or \( r_b = 0 \).

**Proof.** Write \( G = S(a, b, 2) \). By Lemma 2.8, we find \( r_a + r_b \leq 2 \). Before proceeding according to the values of \( r_a \) and \( r_b \), we need a recurrence to compute \( X_G \). The order of the graph \( G \), denoted \( N \), is \( a + b + 3 \). For any partition \( \lambda \vdash N \) such that every part of \( \lambda \) is at least 3, we can extract the coefficient of \( e_\lambda \) by Lemma 2.10 and obtain

\[
[e_\lambda]X_G = [e_\lambda]X_{P_N} - [e_\lambda]X_{P_{a+1}}X_{P_{b+2}} - [e_\lambda]X_{P_{a+2}}X_{P_{b+1}}.
\]
If \( r_a = r_b = 1 \), then \( a, b \geq 4 \) and \( N = a + b + 3 \geq 11 \). Setting \( \lambda = 53^{(N-5)/3} \) in Eq. (4.1) and using Proposition 2.5 we can deduce that

\[
\begin{aligned}
\left[e_{53^{(N-5)/3}}\right]X_G &= \left[e_{53^{(N-5)/3}}\right]X_{P_N} - \left[e_{3^{(a-4)/3}}\right]X_{P_{n+1}} - \left[e_{3^{(a-2)/3}}\right]X_{P_{n+2}} - \left[e_{3^{(a-1)/3}}\right]X_{P_{n+3}} \\
&= 2^{(N-5)/3}(13 - N).
\end{aligned}
\]

It is negative unless \( N \leq 13 \). Suppose that \( N \leq 13 \). Since \( a \geq b \geq 4 \) and \( N = a + b + 3 \), we find \( a = b = 4 \). In this case, the graph \( G \) is \( S(4,4,2) \). It is not \( e \)-positive by Theorem 2.7.

If \((r_a, r_b) = (0,2)\), then \( a \geq b + 4 \) by Theorem 2.7, and \( N = a + b + 3 \geq 11 \). Setting \( \lambda = 43^{(N-8)/3} \) in Eq. (4.1) and using Proposition 2.5 we can deduce that

\[
\begin{aligned}
\left[e_{43^{(N-8)/3}}\right]X_G &= \left[e_{43^{(N-8)/3}}\right]X_{P_N} - \left[e_{43^{(b-2)/3}}\right]X_{P_{n+1}} - \left[e_{43^{(b-1)/3}}\right]X_{P_{n+2}} \\
&= 2^{(N-17)/3}(-3b^2 - 12tb - 41b - 3t^2 - 13t - 22),
\end{aligned}
\]

where \( t = a - b - 4 \geq 0 \). It is negative. This completes the proof. \( \square \)

For the remaining possible \( e \)-positive spiders \( S(a,b,2) \), we first give an upper bound of \( a \) in terms of \( b \) in Theorem 4.3, for which we need Lemma 4.2.

**Lemma 4.2.** Let \( k \geq 2 \). The set of positive integers \( n \) for which the equation \( n = xk + y(k+1) \) has a solution \((x,y) \in \mathbb{N}^2\) is

\[
\{qk + r: 1 \leq q \leq k - 2, 0 \leq r \leq q\} \cup \{n: n \geq k(k-1)\}.
\]

**Proof.** The desired set is

\[
\{xk + y(k+1): x, y \in \mathbb{N}\} = \{(x+y)k + y: x, y \in \mathbb{N}\} = \{qk + r: q \geq r \geq 0, q \geq 1\}.
\]

Suppose that \( n = qk + r \), where \( q \in \mathbb{N} \) and \( 0 \leq r \leq k - 1 \). If \( n \geq k(k-1) \), then \((x,y) = (q-r,r)\) is a solution in \( \mathbb{N}^2 \) since \( q \geq k - 1 \geq r \). Otherwise \( n < k(k-1) \) and \( n = qk + r: 1 \leq q \leq k - 2, 0 \leq r \leq q\} \). This completes the proof. \( \square \)

**Theorem 4.3.** If the spider \( S(a,b,2) \) is \( e \)-positive, then

\[
a \in \bigcup_{q=1}^{b-2} \{m \in \mathbb{N}: (b+2)q \leq m \leq (b+1)q + b - 2\}.
\]

Moreover, if \((r_a, r_b) = (0,1)\), where \( r_a \) and \( r_b \) are the remainders of \( a \) and \( b \) modulo 3 respectively, then \( a \leq 2b + 4 \).

**Proof.** Let \( G = S(a,b,2) \). Then the number \( N \) of vertices in \( G \) is \( a + b + 3 \). Suppose that \( N \) has a partition \((b+2)^x(b+1)^y\) for some integers \( x, y \in \mathbb{N} \). By easy combinatorial arguments, the spider \( S(a,b,2) \) does not contain a connected partition whose blocks are of sizes \( b+1 \) or \( b+2 \), contradicting Theorem 2.4. Therefore, the equation \( N = x(b+1) + y(b+2) \) has no solution \((x,y) \in \mathbb{N}^2\). Taking the complement of the set in Lemma 4.2, we obtain the desired range of \( a \).

If \((r_a, r_b) = (0,1)\), then \( b \geq 4 \), \( a \geq 6 \) and \( N \geq 13 \). Setting \( \lambda = 43^{(N-4)/3} \) in Eq. (4.1) and using Proposition 2.5 we can deduce that

\[
\begin{aligned}
\left[e_{43^{(N-4)/3}}\right]X_G &= \left[e_{43^{(N-4)/3}}\right]X_{P_N} - \left[e_{43^{(a-3)/3}}\right]X_{P_{n+1}} - \left[e_{43^{(a-2)/3}}\right]X_{P_{n+2}} \\
&= 2^{(N-10)/3}(-3a + 6b + 13).
\end{aligned}
\]
Since \([e_{43(N-4)/3}]X_G \geq 0\), we find \(a \leq 2b + 4\). \(\Box\)

**Corollary 4.4.** The spiders \(S(a, b, 2)\) with \(b \leq 11\) satisfy the following.

1. No spider \(S(a, b, 2)\) with \(b \equiv 2 \) (mod 3) is e-positive.
2. When \(b \in \{3, 6, 7\}\), the spider \(S(a, b, 2)\) is e-positive if and only if \(a = b + 2\).
3. The spider \(S(a, 4, 2)\) is e-positive if and only if \(a \in \{6, 12\}\).
4. No spider \(S(a, 9, 2)\) or \(S(a, 10, 2)\) is e-positive.

**Proof.** By direct computation with the aid of Theorems 4.1 and 4.3. We take \(S(a, 9, 2)\) with \(a \in \{56, 57, 66, 67, 77\}\) for example. When \(a \equiv 1 \) (mod 5) and \(a \geq 11\),
\[
[e_{615(a-6)/5}]X_{S(a, 9, 2)} = [e_{615(a-6)/5}]X_{P_{a+2}} - [e_{52}]X_{P_{10}}[e_{635(a-16)/5}]X_{P_{a+2}} - [e_{65}]X_{P_{11}}[e_{625(a-11)/5}]X_{P_{a+1}}
\]
\[
= \frac{2^{(2a-37)/5}}{3}(5a^3 + 42a^2 - 3533a + 4206),
\]
which is negative for \(a \geq 26\). In particular, \([e_{63510}]X_{S(56,9,2)} < 0\) and \([e_{63512}]X_{S(66,9,2)} < 0\). When \(a \equiv 2 \) (mod 5) and \(a \geq 12\),
\[
[e_{513(a+8)/54}]X_{S(a, 9, 2)} = [e_{513(a+8)/54}]X_{P_{a+2}} - [e_{52}]X_{P_{10}}[e_{513(a-2)/54}]X_{P_{a+2}} = 2^{(2a-4)/5}(110 - 3a),
\]
which is negative for \(a \geq 37\). In particular, \([e_{5134}]X_{S(57,9,2)}, [e_{5154}]X_{S(57,9,2)}, [e_{5154}]X_{S(67,9,2)}, [e_{5174}]X_{S(77,9,2)}\) are negative. This completes the proof. \(\Box\)

When \(\{r_a, r_b\} = (0, 1)\), the upper bound \(2b + 4\) of \(a\) is sharp in the sense that the spider \(S(10, 4, 2)\) is e-positive. Now we give a lower bound of \(a\) for \(b\) that is divisible by 3 for e-positive spiders \(S(a, b, 2)\).

**Theorem 4.5.** Let \(a \geq b \geq 12\). Suppose that 3 divides \(b\) and the spider \(S(a, b, 2)\) is e-positive. Then
\[
a \geq \begin{cases} 
3b + 3, & \text{if } a \equiv 0 \pmod{3}, \\
\frac{b + 9}{4} + \frac{\sqrt{12b^2 - 180b + 565}}{4}, & \text{if } a \equiv 1 \pmod{3}, \\
\frac{b}{2} + \frac{1}{3} + \frac{\sqrt{27b^2 - 54b + 112}}{6}, & \text{if } a \equiv 2 \pmod{3}.
\end{cases}
\]

**Proof.** Let \(G = S(a, b, 2)\). Denote by \(N\) the number of vertices in \(G\). Then \(N = a + b + 3\). We proceed according to the remainder of \(a\) modulo 3.

Suppose that \(a \equiv 0 \pmod{3}\). By Theorem 2.7, we find \(a \geq b + 3 \geq 15\). Thus \(N \geq 30\). Setting \(\lambda = 4^{3}3^{(N-12)/3}\) in Eq. (4.1) and using Proposition 2.5 we can deduce that
\[
[e_{43(N-12)/3}]X_G = [e_{43(N-12)/3}]X_{P_N} - [e_{43(N-3)/3}]X_{P_{a+2}}[e_{43(N-6)/3}]X_{P_{b+2}} - [e_{43(N-9)/3}]X_{P_{b+1}}
\]
\[
= 2^{(N-21)/3}((9N - 87)b^2 - (9N - 87)(N - 3)b + 2N^3 - 35N^2 + 181N - 306).
\]

Consider the function
\[
F(b) = (9N - 87)b^2 - (9N - 87)(N - 3)b + 2N^3 - 35N^2 + 181N - 306.
\]
Since \(N = a + b + 3 \geq 2b + 6\) by Theorem 2.7, we find \(b \leq (N - 6)/2\). Note that \(F(b)\) is decreasing for \(b \leq (N - 3)/2\). If \(a \leq 2b + 3\), then \(N = a + b + 3 \leq 3b + 6\), that is, \(b \geq (N - 6)/3\). Since
\[
F\left(\frac{N - 6}{3}\right) = -\frac{2}{3}N^2 + 18N - 132 < 0,
\]
we deduce that \(F(b) < 0\), contradicting to the fact \([e_{43(N-12)/3}]X_G \geq 0\). This proves that \(a \geq 2b + 6\).
Below we suppose that $2b + 6 \leq a \leq 3b$. Then $a \geq 28$ and $N \geq 42$. Setting $\lambda = 4^63^{(N-24)/3}$ in Eq. (4.1) and using Proposition 2.5 we can deduce that

\[
\frac{e^{4g3(N-24)/3}}{XG} = \frac{e^{4g3(N-24)/3}}{XP_N} - \frac{e^{4g3(a-3)/3}}{XP_{a+1}} \frac{e^{4g3(2b-18)/3}}{XP_{a+5}} - \frac{e^{4g3(a-15)/3}}{XP_{a+11}} \frac{e^{4g3(b-6)/3}}{XP_{a+17}} - \frac{e^{4g3(a-10)/3}}{XP_{a+21}} \frac{e^{4g3(b-10)/3}}{XP_{a+23}} - \frac{e^{4g3(a-18)/3}}{XP_{a+25}} \frac{e^{4g3(b-3)/3}}{XP_{a+27}}
\]

where

\[
H(a) = 2a^6 - (6b + 124)a^5 + (-15b^2 + 35b + 3020)a^4 + (40b^3 + 110b^2 - 6640b - 37440)a^3 + (-15b^4 + 110b^3 - 11250b^2 + 125415b + 124938)a^2
\]
\[+ (-6b^5 + 355b^4 - 6460b^3 + 125415b^2 - 945468b + 408564)a + (2b^6 - 124b^5 + 3020b^4 - 37440b^3 + 124938b^2 + 408564b + 699840).
\]

First, it is routine to check that for $b \geq 3$,

\[
H(2b + 6) = -54b^6 + 2646b^5 - 43500b^4 + 184770b^3 + 194724b^2 - 2243376b + 2604960 < 0.
\]

In order to show that $H(a) < 0$ in the interval $[2b + 6, 3b]$, it suffices to show that the differential $H'(a)$ is negative for $a \in [2b + 6, 3b]$. In fact,

\[
H'(a) = 12a^5(30b + 650a^4 + (-60b^2 + 1420b + 1280)a^3 + (120b^3 + 330b^2 - 19920b - 112320)a^2
\]
\[+ (-30b^4 + 220b^3 - 22500b^2 + 250830b + 249876)a
\]
\[+ (-6b^5 + 355b^4 - 6460b^3 + 125415b^2 - 945468b + 408564).
\]

It is routine to check that for $b \geq 3$,

\[
H'(2b + 6) = -162b^5 + 1935b^4 - 42240b^3 + 256275b^2 - 392256b - 236628 < 0 \quad \text{and}
\]
\[H'(3b) = -150b^5 - 7895b^4 + 72740b^3 - 132975b^2 - 195840b + 408564 < 0.
\]

Therefore, it suffices to show that the second differential $H''(a)$ satisfies the following two properties:

- $H''(2b + 6) < 0$ for $b \geq 3$, and
- $H''(a)$ has at most one real root in the interval $[2b, 3b]$.

In fact,

\[
H''(a) = 60a^4 - (120b + 2480)a^3 + (-180b^2 + 4260b + 36240)a^2
\]
\[+ (240b^3 + 660b^2 - 39840b - 224640)a + (-30b^4 + 220b^3 - 22500b^2 + 250830b + 249876).
\]

It is routine to check that for $b \geq 3$,

\[
H''(2b + 6) = -270b^4 - 1260b^3 - 10140b^2 + 127710b - 251244 < 0.
\]

In order to prove that $H''(a)$ has at most one real root in $[2b, 3b]$, it suffices to show that $H''(a)$ has three distinct real roots less than $2b$. In fact, it is routine to check that for $b \geq 12$,

\[
H''(0) = -30b^4 + 220b^3 - 22500b^2 + 250830b + 249876 < 0,
\]
\[
H''(b/2) = \frac{135}{4}b^4 + 1305b^3 - 33360b^2 + 138510b + 249876 > 0.
\]

Note that the function $H''(a)$ is a polynomial of degree 4 with positive leading coefficient. By the intermediate value theorem, we derive that $H''(a)$ has a real root in the intervals

\[(-\infty, 0), \ (0, b/2), \ \text{and} \ (b/2, 2b),\]
respectively. This completes the proof for the fact $H(a) < 0$, contradicting $[e_{43(N-24)/3}]X_G \geq 0$. Hence $a \geq 3b + 3$.

If $a \equiv 1 \pmod{3}$, then $a \geq 16$ and $N \geq 31$. Setting $\lambda = 5^23^{(N-10)/3}$ in Eq. (4.1) and using Proposition 2.5 we can deduce that
\[
[e_{523(N-10)/3}]X_G = [e_{523(N-10)/3}]X_{PN} - [e_{53(a-4)/3}]X_{P_{a+1}}[e_{53(b-3)/3}]X_{P_{b+2}} - [e_{3(a+2)/3}]X_{P_{a+2}}[e_{523(b-9)/3}]X_{P_{b+1}} = 2(N-10)/3 (4a^2 - (4b + 18)a - 2b^2 + 54b - 121).
\]
The $e$-positivity of $G$ implies the desired inequality.

If $a \equiv 2 \pmod{3}$, then $a \geq 14$ and $N \geq 29$. Exchanging the letters $a$ and $b$ in Eq. (4.2), and sorting the terms according to the degree of $e$, we obtain
\[
[e_{423(N-8)/3}]X_G = 2(N-17)/3 (3b^2 + (11 - 6a)b + 6a^2 - 4a - 18) = 2(N-17)/3 (6a^2 - (6b + 4)a - 3b^2 + 11b - 18).
\]
The $e$-positivity of $G$ implies the desired inequality. $\square$

Note that in the formula in Theorem 4.5, the two discriminants $12b^2 - 180b + 565$ and $27b^2 - 54b + 112$ are positive for $b \geq 12$.

5. THE POSITIVITY OF BROOM GRAPHS AND DOUBLE BROOM GRAPHS

We call the spider $S(\lambda_1, 1^{d-1})$ a broom, denoted
\[ S(\lambda_1, 1^{d-1}) = \text{br}(\lambda_1, d - 1). \]
When $\lambda_1 \geq 2$, we call the path $P_{1+\lambda_1}$ the long leg of the broom $\text{br}(\lambda_1, d - 1)$. We have the following complete positivity classification for the family of brooms.

**Theorem 5.1.** The positivity classification of brooms $\{\text{br}(p, l) : p, l \geq 2\}$ is as follows.

1. $\text{br}(p, l)$ is $e$-positive if and only if $p = l = 2$.
2. $\text{br}(p, l)$ is Schur positive but not $e$-positive if and only if $p \in \{4, 6, 8, 10, 12\}$ and $l = 2$.
3. $\text{br}(p, l)$ is not Schur positive if $p \notin \{2, 4, 6, 8, 10, 12\}$ or $l \geq 3$.

**Proof.** Since $\text{br}(p, l)$ is a tree, it is bipartite. When $l \geq 3$ or $p$ is odd, the broom $\text{br}(p, l)$ is not balanced, and thus not Schur positive by Theorem 2.11. Suppose that $l = 2$ and $p$ is even. By using mathematical software, one may compute $X_{\text{br}(p, l)}$ and obtain that $\text{br}(2, 2)$ is $e$-positive, and that $\text{br}(p, 2)$ for $p \in \{4, 6, 8, 10, 12\}$ is Schur positive. Moreover, by using Lemma 2.9, we find $[e_{32}]X_G = 5 - 2p < 0$ for even $p \geq 4$.

Below we deal with the remaining brooms $G = \text{br}(2p, 2)$ with $2p \geq 14$. We shall show that $[s_\lambda]X_G = 6 - p$ by using Theorem 2.12, where $\lambda = (p + 1)^2\lambda$. We label the center of $G$ as $v_{2p+1}$, the long leg as $v_2v_{2p+1}$, and the remaining two vertices as $u$ and $w$, see Fig. 5.1.

First of all, we note that the independence number $\alpha$ of $G$ is $\alpha = p + 2$, since the path $v_1 \cdots v_{2p}$ has independence number $p$ and the path $uv_{2p+1}w$ has independence number 2. In view of Theorem 2.12, only 3 rim hook tabloids of shape $\lambda$ need consideration, which are illustrated in Fig. 5.2 with their contents $\tau$ respectively. Now we compute their contributions to the coefficient $[s_\lambda]X_G$ independently.
The graph \( G \)

Let \((A, B, C)\) be a stable partition of type \((p + 2)p1\), where \(|A| = p + 2\), \(|B| = p\) and \(|C| = 1\). Then the set \(A\) must be of the form

\[
T_j = \{v_1, v_3, \ldots, v_{2j-1}, v_{2j+2}, v_{2j+4}, \ldots, v_{2p}, u, w\} \quad \text{where} \quad j \in \{0, 1, \ldots, p\}.
\]

If \(j = 0\), then \(C\) consists of an arbitrary vertex in \(V(G) \setminus T_0\). If \(1 \leq j \leq p\), then \(C\) consists of \(v_{2j}\) or \(v_{2j+1}\). Therefore, the contribution of such partitions is \(- (p + 1) - 2p = -(3p + 1)\).

Let \((A, B, C)\) be a stable partition of type \((p + 1)^21\), where \(|A| = |B| = p + 1\) and \(|C| = 1\).

(a) If \(v_{2p+1} \in A\), then \(A = \{v_1, v_3, \ldots, v_{2p+1}\}\). Since \(V(G) \setminus A\) is stable, there are \(p + 2\) possibilities for the partition \((B, C)\).

(b) If \(v_{2p+1} \in C\), then \(V(G) \setminus \{v_{2p+1}\}\) is partitioned into two stable sets. If \(v_1 \in A\), then

\[
\{v_1, v_3, \ldots, v_{2p-1}\} \subset A \quad \text{and} \quad \{v_2, v_4, \ldots, v_{2p}\} \subset B.
\]

Thus one of the vertices \(u\) and \(w\) belongs to \(A\) and the other to \(B\).

In summary, the contribution of this kind of partitions is \(2(p + 2 + 2) = 2p + 8\), where the factor 2 comes from the ordering of \(A\) and \(B\).

Hence \([s_\lambda] X_G = -1 - (3p + 1) + (2p + 8) = 6 - p\), which is negative for \(p \geq 7\). \(\square\)

Denote by \(br'(l, p, l')\) the \((l + p + l' + 1)\)-vertex graph obtained by identifying the center of the star \(S(1^l)\) and the leaf of the broom \(br(p, l')\) on its long leg. We have the following complete positivity classification for the family of \(br'(l, p, l')\).

**Theorem 5.2.** In the graph family

\[
\mathcal{G} = \{br'(l, p, l') : l' \geq l \geq 2, l' \geq 3, p \geq 1\},
\]

no one is \(e\)-positive. Moreover, only the following 10 graphs in \(\mathcal{G}\) are Schur positive:

\[
\begin{align*}
br'(2, 1, 3), & \quad br'(2, 5, 3), & \quad br'(2, 7, 3), & \quad br'(2, 9, 3), & \quad br'(2, 11, 3), \\
br'(3, 1, 3), & \quad br'(3, 1, 4), & \quad br'(4, 1, 4), & \quad br'(4, 1, 5), & \quad br'(5, 1, 5).
\end{align*}
\]
Proof. Suppose that $\text{br}'(l, p, l')$ is Schur positive. Since it is bipartite, it is balanced by Theorem 2.11. It follows that $p$ is odd and $l' \in \{l, l + 1\}$. For avoiding fractions, we consider the graph

$$G = \text{br}'(l, 2p - 1, l')$$

for $p \geq 1$. We regard $G$ as consisting of the path $v_1v_2 \cdots v_{2p}$, and the stars with edge sets $\{v_1x : x \in X\}$ and $\{v_{2p}y : y \in Y\}$, where $X$ and $Y$ are disjoint vertex sets of orders $l$ and $l'$ respectively, see Fig. 5.3. Let $W = \{v_1, \ldots, v_{2p}\}$. Then $V(G) = X \sqcup W \sqcup Y$ and $|V(G)| = l + l' + 2p$.

![Figure 5.3. The graph $\text{br}'(l, 2p-1, l')$.](image)

Our first goal is to show that $l \leq 5$. Suppose to the contrary that $l \geq 6$ and consider the partition

$$\lambda = (l' + p + 1)(l + p - 2)1.$$

Let $T$ be a special rim hook tabloid of shape $\lambda$ and some content $\tau$. By definition, we know that

$$\ell(\tau) \leq \ell(\lambda) = 3,$$

and the maximum part $\tau_1$ in $\tau$ satisfies

$$\tau_1 \geq \lambda_1 = l' + p + 1.$$

Let $A$ be a stable set in a stable partition of type $\tau$ such that $|A| = \tau_1$. The two inequalities above imply the following further results.

- $\{v_1, v_{2p}\} \cap A = \emptyset$. Otherwise, one of the sets $X$ and $Y$ would have empty intersection with $A$, which implies that $|A| \leq l' + p$, contradicting Ineq. (5.2).
- $\ell(\tau) = 3$. Otherwise, one would have $\ell(\tau) = 2$ by Ineq. (5.1). It follows that the vertices on the path $v_1 \cdots v_{2p}$ are partitioned into two stable sets. Therefore, the set $A$ must contain exactly one of the endpoints $v_1$ and $v_{2p}$, contradicting the previous result.

Thus a feasible type $\tau$ to form a special rim hook tabloid of shape $\lambda$ must be $(l' + p + 2)(l + p - 3)1$ or $(l' + p + 1)(l + p - 2)1$, see Fig. 5.4. In view of Theorem 2.12, we need to consider stable partitions

![Figure 5.4. The special rim hook tabloids in the set $T_t(l' + p + 1)(l + p - 2)1$ for the graph $\text{br}'(l, 2p-1, l')$.](image)

$(A, B, C)$ such that

$$|B| \in \{l + p - 3, l + p - 2\} \quad \text{and} \quad |C| = 1.$$
It follows that $|A| > |B| > |C|$ and $|B| \geq p + 3$. If $\{v_1, v_{2p}\} \subseteq B$, then the stability of $B$ implies that $B \subseteq W$. Thus $|B| \leq p$, a contradiction. Hence one of the vertices $v_1$ and $v_{2p}$ is in $B$, and the other forms the singleton $C$.

Suppose that $v_1 \in B$ and $C = \{v_{2p}\}$. Since each of the vertices $v_2, \ldots, v_{2p-1}$ is in $A$ or $B$, $\{v_1, v_3, \ldots, v_{2p-1}\} \subseteq B$ and $B \setminus \{v_1, v_3, \ldots, v_{2p-1}\} \subseteq Y$.

Thus there are $\binom{p'}{p-1}$ possibilities for the partition $(A, B, C)$. For the other case that $v_{2p} \in B$ and $C = \{v_1\}$, one may derive by symmetry that there are $\binom{p'}{p-1}$ possibilities for the partition. Note that the inequalities $|A| > |B| > |C|$ guarantee that the ordering of stable sets of the same order can be ignored. Hence

$$[s_\lambda]X_G = \left(\binom{l}{l-2} - \binom{l}{l-3}\right) + \left(\binom{l'}{l'-2} - \binom{l'}{l'-3}\right),$$

which is negative for $l \geq 6$. This contradiction implies that $l \leq 5$.

The second goal of ours is to show that $p \leq l$ if $l \geq 3$. Suppose to the contrary that $p \geq l + 1$ and $l \geq 3$. Consider the partition $\lambda = (l + l' + p - 2)(p + 1)1$.

Let $T$ be a special rim hook tabloid of shape $\lambda$ and content $\tau$. Since $l \geq 3$, we find

$$\tau_1 \geq \lambda_1 = l + l' + p - 2 \geq l' + p + 1.$$

Let $A$ be a stable set in a stable partition of type $\tau$ such that $|A| = \tau_1$. Along the same lines for the first goal, we can derive that $\{v_1, v_{2p}\} \cap A = \emptyset$ and $\ell(\tau) = 3$. It follows that $\tau$ must be

$$(l + l' + p - 1)p1 \text{ or } (l + l' + p - 2)(p + 1)1.$$

Consider stable partitions $(A, B, C)$ such that

$$|A| \in \{l + l' + p - 1, l + l' + p - 2\} \text{ and } |C| = 1.$$

It follows that $|A| > |B| > |C|$. If $|A| = l + l' + p - 1$, then

$$A = X \cup Y \cup \{v_2, v_4, \ldots, v_{2j}, v_{2j+3}, v_{2j+5}, \ldots, v_{2p-1}\} \text{ for some } j \in \{0, 1, \ldots, p - 1\}.$$

For each $j \in \{0, 1, \ldots, p - 1\}$, the singleton $C$ must be $\{v_{2j+1}\}$ or $\{v_{2j+2}\}$. Thus the negative part in $[s_\lambda]X_G$ by Theorem 2.12 is $-2p$.

Suppose that $|A| = l + l' + p - 2$. Since $|A \cap W| \leq p - 1$ and $|X \cup Y| = l + l'$, we find $|A \cap W| \in \{p-1, p-2\}$. Denote $W' = W \setminus A$. Since $l \geq 2$, we find $\{v_1, v_{2p}\} \subseteq W'$. Suppose that $|A \cap W| = p - 2$. Then $X \cup Y \subseteq A$. The number of components of the induced subgraph $G[W']$ is $|A \cap W| + 1$. Since the graph $G[W']$ is partitioned into the stable set $B$ and the singleton $C$, at most one of its components is not an isolated vertex, and that component (if it exists) must be the path $P_2$ or $P_3$. Since $|A \cap W| = p - 2$, we find $|W'| \leq p + 1$. It follows that $|W| \leq 2p - 1$, which is absurd. This proves

$$|A \cap W| = p - 1.$$

As a consequence, the set $X \cup Y \setminus A$ is a singleton, say, $\{u\}$. Again, the induced subgraph $G[W' \cup \{u\}]$ is partitioned into the stable set $B$ and the singleton $C$, and consists of several isolated vertices and the path $P_2$ or $P_3$.

- If $u \in X$, then $G[W' \cup \{u\}]$ has a subgraph $uv_1$. Thus every component of $G[W' \cup \{u\}]$ which does not contain $u$ must be a singleton. It follows that $A \cap W = \{v_{2p-1}, v_{2p-2}, \ldots, v_3\}$ and $B = \{v_1\}$. 

Therefore, this case gives \( l \) possibilities for the vertex \( u \), which determines the partition \((A, B, C)\) as a consequence.

- The other case \( u \in Y \) gives \( l' \) possibilities for the partition \((A, B, C)\), by symmetry.

In summary,

\[
[s_{\lambda}]X_G = -2p + l + l' \leq -2(l + 1) + l + (l + 1) < 0.
\]

This contradiction proves \( p \leq l \).

Thirdly, we consider the graphs \( G = br'(2, 2p - 1, 3) \) and will show that \( p \leq 6 \). Note that \( G \) has independence number \( p + 4 \). Suppose to the contrary that \( p \geq 7 \). For the partition \( \lambda = (p + 3)(p + 1)1 \), the set \( T_\lambda \) consists of 3 tabloids, see Fig. 5.5.

![Figure 5.5. The special rim hook tabloids in the set \( T_{(p+3)(p+1)1} \) for the graph \( br'(2, 2p - 1, 3) \).](image)

For the type \((p + 3)(p + 2)\), \( G \) has a unique stable partition

\[
\{(v_1, v_3, \ldots, v_{2p-1}) \cup Y, \{v_2, v_4, \ldots, v_{2p}\} \cup X\}.
\]

This partition contributes \(-1\) to \( [s_{\lambda}]X_G \). For stable partitions of types \((p + 4)p1 \) and \((p + 3)(p + 1)1 \), we can suppose that \((A, B, C)\) is such a partition with \(|A| > |B| > |C|\).

Suppose that \(|A| = p + 4\). Then \( \{v_1, v_{2p}\} \cap A = \emptyset \). It follows that \(|A \cap W| \leq p - 1 \) and

\[
A = X \cup Y \cup \{v_2, v_4, \ldots, v_{2j}, v_{2j+3}, v_{2j+5}, \ldots, v_{2p-1}\} \quad \text{for some } j \in \{0, 1, \ldots, p - 1\}.
\]

For each \( j \in \{0, 1, \ldots, p - 1\} \), the singleton \( C \) is either \( \{v_{2j+1}\} \) or \( \{v_{2j+2}\} \). Therefore, such partitions contribute \(-2p \) to \( [s_{\lambda}]X_G \).

Suppose that \(|A| = p + 3 \) and \(|B| = p + 1 \). Then \(|A \cap W| \in \{p - 2, p - 1, p\} \).

- If \(|A \cap W| = p - 2\), then \( X \cup Y \subset A \). It follows that \( B \subset W \) and thus \(|B| \leq p\), a contradiction.
- If \(|A \cap W| = p - 1\), then there exists \( u \in X \cup Y \) such that \( X \cup Y \setminus A = \{u\} \), and the set \( B \) is contained in the subgraph \( G[\{u\} \cup W] \).
  - Suppose that \( u \in X \). Since \( B \) is stable and of order \( p + 1 \), we derive that \( B = \{u, v_2, v_4, \ldots, v_{2p}\} \).
    Since the vertex in \( X \setminus \{u\} \) is in \( A \), we find \( v_1 \notin A \), and \( C = \{v_1\} \). Thus the contribution of the case \( u \in X \) is 2 for \(|X| = 2\).
  - For the same reason, the case \( u \in Y \) contributes 3, for \(|Y| = 3\).

The total contribution is 5.

- If \(|A \cap W| = p\), then \( \{v_1, v_{2p}\} \cap A \neq \emptyset \). Since \(|A| = p + 3\), we find
  \[
  A = \{v_1, v_3, \ldots, v_{2p-1}\} \cup Y.
  \]

Since the singleton \( C \) may consist of any single vertex of the \((p + 2)\)-set \( X \cup \{v_2, v_4, \ldots, v_{2p}\} \), the contribution of this case is \( p + 2 \).

Summing up all contributions above, we obtain

\[
[s_{\lambda}]X_G = -1 - 2p + 5 + (p + 2) = 6 - p < 0.
\]
This contradiction proves that \( p \leq 6 \).

Now, the remaining graphs in the family \( \text{br}'(l, p, l') \) are listed as follows.

- \( \text{br}'(5, 2p - 1, l') \) with \( l' \in \{5, 6\} \) and \( p \in [5] \).
- \( \text{br}'(4, 2p - 1, l') \) with \( l' \in \{4, 5\} \) and \( p \in [4] \).
- \( \text{br}'(3, 2p - 1, l') \) with \( l' \in \{3, 4\} \) and \( p \in [3] \).
- \( \text{br}'(2, 2p - 1, 3) \) with \( p \in [6] \).

We compute the chromatic symmetric function of each of these graphs by using mathematical software, and obtain the desired classification.

**Proposition 5.3.** For any integer \( b \geq 1 \), the graph \( \text{br}'(2, b, 2) \) is not \( e \)-positive.

**Proof.** For even \( b \), the graph \( \text{br}'(2, b, 2) \) is not balanced and thus not Schur positive by Theorem 2.11. Let \( G = \text{br}'(2, 2p - 1, 2) \) where \( p \geq 1 \). We will show that \( [e_{(2p+2)}]X_G < 0 \).

Consider \( G \) as consisting of the path \( v_1 \cdots v_{2p} \) and the stars with edge sets \( \{v_1x, v_1x'\} \) and \( \{v_{2p}y, v_{2p}y'\} \). Taking \( e_1 = v_2v_1, e_2 = v_1x \) and \( e_3 = xv_2 \) in Theorem 2.6, we obtain

\[
X_G = e_1X_{G-x} + X_{\text{br}(2p+1, 2)} - 2e_2X_{\text{br}(2p-1, 2)}.
\]

By using Theorem 2.6 in the same way, we obtain

\[
X_{\text{br}(2a-1, 2)} = e_1X_{P_{2a+1}} + X_{P_{2a+2}} - 2e_2X_{P_{2a}}, \quad \text{for } a \geq 1.
\]

These two relations are illustrated in Fig. 5.6, in which the triple \((A, B, C)\) is \((\text{br}(2p - 2, 2), K_2, K_1)\) for Eq. (5.3) and \((P_{2a-2}, K_2, K_1)\) for Eq. (5.4). Extracting the coefficients of \( e_{2a+2} \) and \( e_2e_2 \) from each side of Eq. (5.4), we can compute by Proposition 2.5 as

\[
\begin{align*}
[e_{2a+2}]X_{\text{br}(2a-1, 2)} &= [e_{2a+2}]X_{P_{2a+2}} = 2a + 2 \quad \text{and} \\
[e_2e_2]X_{\text{br}(2a-1, 2)} &= [e_2e_2]X_{P_{2a+2}} - [2e_2]X_{P_{2a}} = (6a - 2) - 2 \cdot 2a = 2a - 2.
\end{align*}
\]

Extracting the coefficient of \( e_{(2p+2)} \) from both sides of Eq. (5.3), with the aid of the two formulas above, we obtain

\[
[e_{(2p+2)}]X_G = [e_{2p+2}e_2]X_{\text{br}(2p+1, 2)} - 2[e_{2p+2}]X_{\text{br}(2p-1, 2)} = 2p - 2(2p + 2) = -2p - 4 < 0.
\]

This completes the proof.

**Conjecture 5.4.** For any integer \( p \geq 1 \), the graph \( \text{br}'(2, 2p - 1, 2) \) is Schur positive.

We checked that Conjecture 5.4 is true up to \( 2p - 1 = 19 \).
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