C-MINIMAL TOPOLOGICAL GROUPS

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Abstract. We study topological groups having all closed subgroups (totally) minimal and we call such groups c-(totally) minimal. We show that a locally compact c-minimal connected group is compact. Using a well-known theorem of Hall and Kulatilaka [20] and a characterization of a certain class of Lie groups, due to Grosser and Herfort [19], we prove that a c-minimal locally solvable Lie group is compact.

It is shown that if a topological group $G$ contains a compact open normal subgroup $N$, then $G$ is c-totally minimal if and only if $G/N$ is hereditarily non-topologizable. Moreover, a c-totally minimal group that is either complete solvable or strongly compactly covered must be compact. Negatively answering [9, Question 3.10(b)] of Dikranjan and Megrelishvili we find, in contrast, a totally minimal solvable (even metabelian) Lie group that is not compact. We also prove that the group $A \times F$ is c-(totally) minimal for every (respectively, totally) minimal abelian group $A$ and every finite group $F$.

1. Introduction

All topological groups in this paper are Hausdorff. A topological group $(G, \tau)$ is called minimal [14, 34] if there exists no group topology on $G$ that is strictly coarser than $\tau$. Equivalently, if every continuous isomorphism $f : G \to H$, where $H$ is an arbitrary topological group, is a topological isomorphism. If the same holds for every quotient of $G$, then $G$ is totally minimal [10]. This is exactly a group $G$ that satisfies the open mapping theorem. The topological semidirect product $\mathbb{R} \rtimes \mathbb{R}_+$, where the multiplicative group of positive reals $\mathbb{R}_+$ acts on $\mathbb{R}$ by multiplication, is a locally compact minimal group that is not totally minimal (see [6]). For more information on minimal groups we refer the reader to [3, 12, 15, 29] (see also the survey [9] and the book [11]).

In 1971, Stephenson proved that a minimal locally compact abelian group is compact ([34, Theorem 1]). This result was extended substantially by Prodanov and Stoyanov.

Fact 1.1. [30] Every minimal abelian group is precompact.

Recently, Banakh [4] provided a quantitative generalization of this theorem.

Dikranjan, Toller and the authors introduced the following two concepts in [35, Definition 1.2] and [36, Definition 1.5], respectively.

Definition 1.2. A topological group $G$ is said to be:

\begin{itemize}
  \item[\textit{c-minimal}] if every continuous isomorphism $f : G \to H$, where $H$ is an arbitrary topological group, is a topological isomorphism.
  \item[\textit{totally c-minimal}] if every quotient of $G$ is c-minimal.
\end{itemize}
(1) *hereditarily minimal* if every subgroup of $G$ is minimal;
(2) *densely minimal* if every dense subgroup of $G$ is minimal.

It is easy to see that a topological group $G$ is hereditarily minimal if and only if every closed subgroup of $G$ is densely minimal. Prodanov [29] proved that the $p$-adic integers are the only infinite (locally) compact hereditarily minimal abelian groups. Later, Dikranjan and Stoyanov [12] classified all hereditarily minimal abelian groups. In [33, Theorem D], all infinite locally compact solvable hereditarily minimal groups were classified.

The next theorem provides another extension of Prodanov’s theorem.

**Fact 1.3.** [36, Theorem C] For an infinite locally compact abelian group $K$, the following conditions are equivalent:
(a) $K$ is a hereditarily minimal group;
(b) $K$ is a densely minimal group;
(c) $K$ is isomorphic to $\mathbb{Z}_p$ for some prime $p$.

In this paper, we study the topological groups having all closed subgroups minimal.

**Definition 1.4.** A topological group $G$ is called *c-minimal* if every closed subgroup of $G$ is minimal.

Clearly, a compact group is c-minimal. It follows from Fact 1.1 that every closed abelian subgroup of a complete c-minimal group $G$ (e.g., the center $Z(G)$) is compact. Megrelishvili [25] proved that every topological group is a group retract of a minimal group. Hence, every topological group is a closed subgroup (and a quotient) of a minimal group.

Note that both the class of minimal groups and the class of totally minimal groups are closed under taking closed central subgroups (see [11, Proposition 7.2.5]). In particular, a minimal abelian group is c-minimal while this property does not hold even for a two-step nilpotent minimal group, as the next example shows.

**Example 1.5.** The Weyl-Heisenberg group $G = H(\mathbb{R})/Z(H(\mathbb{R}))$, where

$$H(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

is the classical Heisenberg group, is minimal (see [9, Theorem 5.11]). However, $G$ is not c-minimal since it contains a copy of $\mathbb{R}$ as a closed non-minimal subgroup.

The following diagram summarizes the interrelations between the classes of minimal groups defined above. Example 1.6 shows, among other things, that all inclusions are proper.
(1) minimal groups;
(2) c-minimal groups;
(3) compact groups;
(4) densely minimal groups;
(5) hereditarily minimal groups.

Example 1.6.

(a) By Bader and Gelander [1], the special linear group \( SL(n, \mathbb{F}) \) is totally minimal for every local field \( \mathbb{F} \) (see also [26] for a new independent proof). In particular, \( SL(3, \mathbb{C}) \) is totally minimal. By [26, Corollary 5.11(3)], its dense subgroup \( SL(3, \mathbb{Q}(i)) \) fails to be minimal, where \( \mathbb{Q}(i) := \{ a + bi : a, b \in \mathbb{Q} \} \) is the Gaussian rational field. Moreover, \( SL(3, \mathbb{C}) \) is not c-minimal as it contains \( \mathbb{C} \) as a closed subgroup. This shows that the union of (2) and (4) is properly contained in (1).

(b) The rational circle \( \mathbb{Q}/\mathbb{Z} \) is minimal precompact abelian group (see [9, Example 1.1(a)]). Being abelian it is also c-minimal. However, this group is neither compact nor hereditarily minimal (see [11] for the classification of all torsion abelian hereditarily minimal groups). This shows that the union of (3) and (5) is properly contained in (2). Using Example 3.9 below one can find a complete non-compact c-minimal group that is not hereditarily minimal.

(c) Let \( G \) be any infinite compact abelian group that is not isomorphic to \( \mathbb{Z}_p \) for any prime \( p \). By Fact 1.3, \( G \) is in (3) but not in (4).

(d) Any non-closed subgroup of \( \mathbb{Z}_p \) is in (5) but not in (3).

(e) By [24, Theorem 4.7(a)], the group \( K \ltimes K^* \) is minimal for every non-discrete locally retrobounded division ring \( K \), where \( K^* = (K \setminus \{0\}, \cdot) \) acts on \( K \) by multiplication. In particular, the group \( G = (\mathbb{Q}_p, +) \ltimes \mathbb{Q}_p^* \) is minimal. By [35, Corollary 2.11], \( G \) is even densely minimal. However, it is not c-minimal since it contains \( \mathbb{Q}_p \) as a closed non-minimal subgroup. This means that (4) is not contained in (2).

(f) By [36, Example 3.11], the compact two-step nilpotent group

\[
G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{Z}_p \right\}
\]

is densely minimal but not hereditarily minimal. Since every compact group is c-minimal, we deduce that a topological group that is simultaneously c-minimal and densely minimal need not be hereditarily minimal. So (5) is properly contained in the intersection of (2) and (4).

(g) By [35, Corollary 1.12], every infinite hereditarily minimal locally compact solvable group \( G \) is compact metabelian. In particular, it lies in the intersection of (3) and (5).

1.1. Notation and terminology. The fields of rationals, reals and complex numbers are denoted by \( \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \), respectively, and \( \mathbb{T} = \{ a \in \mathbb{C} : |a| = 1 \} \) denotes the unit circle group. For a prime number \( p \), \( \mathbb{Z}_p \) stands for the ring of \( p \)-adic integers and \( \mathbb{Q}_p \) is the field of \( p \)-adic numbers. We denote by \( \mathbb{Z} \) the group of integers, while \( \mathbb{N} \) and \( \mathbb{N}_+ \) are its subsets of non-negative integers and positive natural numbers, respectively.
An element \( x \) of a group \( G \) is \textit{torsion} if the subgroup of \( G \) generated by \( x \), denoted by \( \langle x \rangle \), is finite. Moreover, \( G \) is \textit{torsion} if every element of \( G \) is torsion. Let \( \mathcal{P} \) be an algebraic (or set-theoretic) property. A group is called \textit{locally} \( \mathcal{P} \) if every finitely generated subgroup has the property \( \mathcal{P} \). For example, in a locally finite group every finitely generated subgroup is finite. In particular, a locally finite group is torsion.

A group \( G \) is \textit{solvable} if there exist \( k \in \mathbb{N} \) and a subnormal series 
\[
G_0 = \{ e \} \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_k = G,
\]
where \( e \) denotes the identity element, such that the quotient group \( G_j/G_{j-1} \) is abelian for every \( j \in \{1, \ldots, k\} \). In particular, \( G \) is \textit{metabelian}, if \( k \leq 2 \). The subgroup \( Z(G) \) denotes the \textit{center} of \( G \), and we set \( Z_0(G) = \{ e \} \) and \( Z_1(G) = Z(G) \). For \( n > 1 \), the \textit{n-th center} \( Z_n(G) \) is defined as follows:
\[
Z_n(G) = \{ x \in G : [x, y] \in Z_{n-1}(G) \text{ for every } y \in G \},
\]
where \([x, y]\) denotes the commutator \( xyx^{-1}y^{-1} \). A group is \textit{nilpotent} if \( Z_n(G) = G \) for some \( n \in \mathbb{N} \). In this case, its nilpotency class is the minimum of such \( n \). We denote by \( G' \) the \textit{derived subgroup} of \( G \), namely, the subgroup of \( G \) generated by all commutators \([x, y]\), where \( x, y \in G \).

Let \( G \) be a topological group and \( H \) be its subgroup. The closure of \( H \) in \( G \) is denoted by \( \overline{H} \), and \( c(G) \) is the connected component of \( G \). If \( c(G) = \{ e \} \), then \( G \) is \textit{totally disconnected}. It is known that the connected component of a Lie group is open. A topological group is called \textit{complete} if it is complete with respect to its two-sided uniformity, and it is \textit{precompact} if its completion is compact. Finally, \( G \) is \textit{balanced} if it admits a local base at the identity consisting of neighborhoods invariant under conjugations.

2. C-minimal Lie Groups

The minimality of Lie groups has been studied by many authors (see [16, 18, 28, 31]). By Omori [28], a connected nilpotent Lie group with compact center is minimal. By Remus and Stoyanov [31], a connected semi-simple Lie group is totally minimal if and only if its center is finite. In particular, the special linear groups \( SL(n, \mathbb{R}) \) are totally minimal.

The main goal of this section is to prove that a c-minimal locally solvable Lie group is compact (see Theorem 2.9 below). We start by recalling that a topological group \( G \) is \textit{compactly covered} if every element \( g \in G \) is contained in a compact subgroup of \( G \). It is easy to see that the class of compactly covered groups is closed under taking quotients and closed subgroups.

\textbf{Lemma 2.1.} A complete c-minimal group \( G \) is compactly covered.

\textit{Proof.} Take a non-torsion element \( g \in G \). As \( G \) is c-minimal group, it follows from Fact 1.1 that \( \langle g \rangle \) is precompact. The completeness of \( G \) implies that its subgroup \( \overline{\langle g \rangle} \) is compact. This proves that \( G \) is compactly covered. \hfill \Box

As noted above, a compact group is c-minimal. Using the next reformulated result, we shall prove in Proposition 2.3 that the converse is true for connected locally compact groups.

\textbf{Fact 2.2.} [2, Lemma 2.15] Let \( H \) be a connected locally compact group. If \( H \) is compactly covered, then it is compact.
Proposition 2.3. If $G$ is a locally compact c-minimal group, then $c(G)$ is compact. In particular, a connected locally compact c-minimal group is compact.

Proof. By Lemma 2.1, the closed subgroup $c(G)$ is compactly covered. Being also locally compact and connected, it must be compact by Fact 2.2. The last assertion now trivially follows. □

Lemma 2.4. Let $G$ be a locally compact group that is either balanced or Lie. If $G$ is c-minimal, then it contains a compact open normal subgroup.

Proof. Let $G$ be a locally compact c-minimal group. By Proposition 2.3, the connected component $c(G)$ is compact. In case $G$ is Lie, then its compact normal subgroup $c(G)$ is open.

In case $G$ is balanced, then $G/c(G)$ is a locally compact totally disconnected balanced group. Hence, the latter group has a local base at the identity consisting of compact open normal subgroups. Pick any $K$ from this local base. As $c(G)$ is compact normal subgroup of $G$, it follows that $N = q^{-1}(K)$ is a compact open normal subgroup of $G$, where $q: G \to G/c(G)$ is the quotient homomorphism. □

Proposition 2.5. Let $G$ be a c-minimal Lie group. Then $G$ satisfies the following properties:

1. $c(G)$ is compact;
2. every abelian subgroup of $G/c(G)$ is finite.

Proof. By [19, Theorem 3.2], a Lie group $G$ satisfies properties (1)-(2) if and only if all of its abelian subgroups are relatively compact. Now, if $H$ is an abelian subgroup of a c-minimal Lie group $G$, then $H$ is minimal. Being locally compact minimal abelian, $H$ must be compact in view of Fact 1.1. □

The next example, provided by D. Dikranjan and simplified on the advice of the referee, shows that a Lie group satisfying properties (1)-(2) of Proposition 2.5 need not be minimal. Recall that a group is called topologizable if it admits a non-discrete Hausdorff group topology and non-topologizable otherwise.

Example 2.6. Let $G = (T, \tau_d)$, where $T$ is a topologizable Tarski monster group (such groups can be found in [21, 27]) equipped with the discrete topology $\tau_d$. Then $G$ is a Lie group with $c(G) = \{e\}$ compact, and all abelian subgroups of $G/c(G) = G$ are finite cyclic. However, $G$ is not minimal since $T$ is topologizable.

The next fact, which deals with infinite discrete hereditarily minimal groups, will also be used in the sequel.

Fact 2.7. [33, Lemma 3.5] If $G$ is an infinite discrete hereditarily minimal group, then the abelian subgroups of $G$ are finite. In particular, the center of $G$ is finite, $G$ is torsion but it is neither locally finite nor locally solvable.

By Fact 2.7, a locally solvable discrete hereditarily minimal group is finite. One of the key ingredients in the proof of Theorem 2.9, which extends this result, is the following classical theorem of Hall and Kulatilaka.

Fact 2.8. [20] Every infinite locally finite group contains an infinite abelian subgroup.

Theorem 2.9. If $G$ is a c-minimal locally solvable Lie group, then $G$ is compact.
Proof. As a quotient of a locally solvable group is still locally solvable (see page 2 of [13]), we deduce that $G/c(G)$ is locally solvable. By Proposition 2.5(2), the factor group $G/c(G)$ is torsion. So, it is locally finite according to [13, Proposition 1.1.5]. Since $c(G)$ is compact by Proposition 2.5(1), it suffices to prove that $G/c(G)$ is finite. Assume for a contradiction that $G/c(G)$ is infinite. In view of Fact 2.8, it must contain an infinite abelian subgroup. This contradicts Proposition 2.5(2). □

3. C-TOTALLY MINIMAL GROUPS

Definition 3.1. Call a topological group $G$ c-totally minimal if every closed subgroup of $G$ is totally minimal.

The main results of this section are Theorem 3.6 and Theorem 3.15. Using Theorem 3.6, one can produce many precompact groups which are c-(totally) minimal and are not (necessarily) abelian or compact. In Theorem 3.15, we characterize the c-totally minimal groups having a compact open normal subgroup. We start by recalling some related concepts.

Definition 3.2. A topological group $G$ is said to be:

1. [35, Definition 6.15] hereditarily totally minimal if every subgroup of $G$ is totally minimal;
2. [36, Definition 1.8] densely totally minimal if every dense subgroup of $G$ is totally minimal.

Now we give examples of densely totally minimal groups that are not c-minimal.

Example 3.3.

1. Consider the projective special linear group $\text{PSL}(n, \mathbb{R}) = \text{SL}(n, \mathbb{R})/Z$, where $\text{SL}(n, \mathbb{R})$ is the special linear group with center $Z = \{cI : c^n = 1\}$. By [31, Theorem 2.4], the group $\text{PSL}(n, \mathbb{R})$ is totally minimal. Being also simple (see [32, Corollary 3.2.9]), it must be densely totally minimal according to [36, Lemma 4.4]. However, since $\text{PSL}(n, \mathbb{R})$ is a connected non-compact Lie group, it is not c-minimal by Proposition 2.8.
2. Shelah [33] constructed a simple non-topologizable group $G$ under the assumption of CH. Using the same arguments from (1), one can see that $G$ is densely totally minimal when equipped with the (unique) discrete topology. Being also torsion free, it is not hereditarily minimal as discrete hereditarily minimal groups are torsion by Fact 2.7. Moreover, since discrete c-minimal groups are hereditarily minimal, we deduce that $G$ is not c-minimal.

Clearly, a compact group is c-totally minimal. A hereditarily totally minimal group is both densely totally minimal and c-totally minimal. As the following example shows, the converse is not true in general.

Example 3.4. By [36, Example 4.6], the special orthogonal group $G = \text{SO}(n, \mathbb{R})$, where $n \geq 5$, is densely totally minimal but not hereditarily minimal. Being compact, $G$ is also c-totally minimal.

The next fact was originally proved in [15] and we use it several times in the sequel.

Fact 3.5. [11, Theorem 7.3.1] If a topological group $G$ contains a compact normal subgroup $N$ such that $G/N$ is (totally) minimal, then $G$ is (resp., totally) minimal.
Recall that a subset $H$ of a group $G$ is called \textit{unconditionally closed} if $H$ is closed in every Hausdorff group topology of $G$ (see \[22\]). For example, the center $Z(G)$ is always unconditionally closed.

**Theorem 3.6.** Let $G = A \times F$ be a topological direct product, where $A$ is an abelian group and $F$ is a finite group.

(1) If $A$ is minimal, then $G$ is $c$-minimal.

(2) If $A$ is totally minimal, then $G$ is $c$-totally minimal.

**Proof.** (1) Let $H$ be a closed subgroup of $G$. If $H$ is abelian, then $H$ is a subgroup of $A \times p_2(H)$, where $p_2 : H \to F$ is the canonical projection on the second coordinate of $H$. By Fact 3.5, $A \times p_2(H)$ is minimal since $p_2(H)$ is a finite normal subgroup of $A \times p_2(H)$ and $A \cong (A \times p_2(H))/p_2(H)$ is minimal. Being a closed subgroup of a minimal abelian group, $H$ is also minimal. Now, let $H$ be a (not necessarily abelian) closed subgroup of $G$. To prove that $H$ is minimal let $\sigma \subseteq \tau|_H$ be a coarser Hausdorff group topology on $H$, where $\tau$ is the given product topology on $G$. Since $Z(H)$ is unconditionally closed in $H$, and $H$ is $\tau$-closed in $G$ we deduce that $Z(H)$ is a $\tau$-closed abelian subgroup of $G$. By the previous step, $Z(H)$ is minimal. In particular, $\sigma|Z(H) = \tau|Z(H) = (\tau|_H)|Z(H)$. Since $Z(H)$ is unconditionally closed in $H$ and

$$[H : Z(H)] \leq [H : Z(G) \cap H] = [HZ(G)/Z(G)] \leq [G/Z(G)] < \infty,$$

we deduce that the quotient topology on $H/Z(H)$ induced by either $\sigma$ or $\tau|_H$ is the discrete topology. By Merson’s Lemma (see \[11\], Lemma 7.2.3), $\sigma = \tau|_H$ and we establish the minimality of $H$.

(2) Let $H$ be a closed subgroup of $G = A \times F$ and suppose that $A$ is totally minimal and $F$ is finite. As in the proof of (1), one can show that $M = A \times p_2(H)$ is totally minimal using Fact 3.5. This implies that the quotient group $M/M'$ is totally minimal. Being abelian, the latter group is even $c$-totally minimal. As $A$ is abelian, we deduce that $M' = H'$. Being a closed subgroup of a $c$-totally minimal group, $H/M' = H/H'$ must be totally minimal. By Fact 3.6, $H$ is totally minimal since $H'$ is finite. \hfill $\square$

Note that by Fact 1.1 the group $A$ from Theorem 3.6 must be precompact, while the group $F$ need not be abelian.

**Example 3.7.**

(1) Let $A$ be a proper subgroup of $\mathbb{Q}/\mathbb{Z}$ containing $\text{soc}(\mathbb{Q}/\mathbb{Z})$, where the socle $\text{soc}(\mathbb{Q}/\mathbb{Z})$ is the subgroup generated by all subgroups of $\mathbb{Q}/\mathbb{Z}$ of prime order. By \[9\], Example 3.7, $A$ is minimal but not totally minimal. By Theorem 3.6(1), $A \times F$ is $c$-minimal for every finite group $F$.

(2) Consider the topological group $A = (\mathbb{Z}, \tau_p)$, where $\tau_p$ is the $p$-adic topology for some prime $p$. It is known that $A$ is totally minimal (see \[11\], Example 4.3.5). Theorem 3.6(2) implies that $A \times F$ is $c$-totally minimal for every finite group $F$. In case $F$ is also abelian, then $A \times F$ is hereditarily totally minimal if and only if $F$ is either trivial or a $p$-group (see \[12\]).

**Proposition 3.8.** Let $G$ be a topological group. If there exists a compact open normal subgroup $N$ of $G$ such that the quotient group $G/N$ is hereditarily (totally) minimal, then $G$ is $c$- (totally) minimal.
Proof: Let $H$ be a closed subgroup of $G$. Since $G/N$ is a hereditarily (totally) minimal group, its subgroup $HN/N$ is (totally) minimal. The discrete groups $HN/N$ and $H/(H \cap N)$ are isomorphic. So $H/(H \cap N)$ is also (totally) minimal group with $H \cap N$ compact. By Fact 3.5, $H$ is (totally) minimal. \qed

We now provide an example of locally compact c-totally minimal groups that are neither compact nor discrete. It is still open whether there exists a locally compact hereditarily minimal group that is neither compact nor discrete (see [21, Question 7.3(1)]). On the other hand, we shall see in Proposition 3.11 below that a complete solvable c-totally minimal group must be compact.

Example 3.9. Recall that a group is called hereditarily non-topologizable when it is hereditarily totally minimal in the discrete topology. It is known that such groups exist (see [21]). Let $G = N \times T$, where $N$ is a compact group and $T$ is a discrete hereditarily non-topologizable group. Then $G$ is c-totally minimal by Proposition 3.8.

The following lemma will be used in the proof of Proposition 3.11.

Lemma 3.10. Let $G_1, G_2$ be two subgroups of a topological group $G$.

(1) If $G_1$ is normal in $G_2$, then $G_1$ is normal in $G_2$, where $G_i$ denotes the closure of $G_i$ in $G$.

(2) If, in addition, $G_2/G_1$ is abelian, then so is $G_2/G_1$.

Proof. (1) Let us see that if $y \in G_2$ and $x \in G_1$, then $yxy^{-1} \in G_1$. Take nets $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ in $G_1$ and $G_2$, respectively, such that $\lim x_i = y$ and $\lim x_i = x$. Then for every $i \in I$ we have $y_i x_i y_i^{-1} \in G_1$ since $G_1$ is normal in $G_2$. As $yxy^{-1} = \lim y_i x_i y_i^{-1}$, we deduce that $yxy^{-1} \in G_1$.

(2) It suffices to show that the commutator $[x, y] \in G_1$, whenever $x \in G_2, y \in G_1$. If the nets $(x_i)_{i \in I} \subseteq G_2$ and $(y_i)_{i \in I} \subseteq G_1$ converge to $x$ and $y$, respectively, then $[x_i, y_i] \in G_1$ for every $i \in I$ since $G_2/G_1$ is abelian. Being the limit of $[x_i, y_i]$, the commutator $[x, y]$ belongs to $G_1$. \qed

Proposition 3.11. If $G$ is a solvable c-totally minimal group, then $G$ is precompact. In particular, a complete solvable c-totally minimal group is compact.

Proof. As $G$ is solvable, there exist $k \in \mathbb{N}$ and a subnormal series

$$G_0 = \{e\} \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_k = G$$

such that the quotient group $G_j/G_{j-1}$ is abelian for every $j \in \{1, \ldots, k\}$. By Lemma 3.10, we may assume without loss of generality that the subgroups $G_j$ are closed in $G$.

Since $G_0$ is compact, it suffices to show that if $G_{j-1}$ is precompact for some $j \in \{1, \ldots, k\}$, then so is $G_j$. As $G$ is c-totally minimal, the abelian factor group $G_j/G_{j-1}$ is minimal. Using Fact 1.1, we deduce that $G_j/G_{j-1}$ is precompact. As precompactness is a three space property (see [6] for example), the subgroup $G_j$ is also precompact, as needed. \qed

Remark 3.12. Dikarnjan and Megrelishvili asked (see [9, Question 3.10(b)]) whether solvable (at least metabelian) totally minimal groups must be precompact. Answering this question in the negative, we provide in the next example a metabelian
totally minimal Lie group that is not compact. In particular, it follows that Proposition 3.11 cannot be extended to solvable totally minimal groups. Note that a totally minimal nilpotent group is precompact (see [9, Theorem 3.11]).

Example 3.13. Mayer [23, Examples 2.6(i)] proved that the Euclidean motion group \( \mathbb{R}^n \times \text{SO}(n, \mathbb{R}) \) is totally minimal for every \( n > 1 \), where the special orthogonal group \( \text{SO}(n, \mathbb{R}) \) acts on \( \mathbb{R}^n \) by matrix multiplication. Fixing \( n = 2 \) one obtains the Lie group of orientation-preserving isometries of the complex plane

\[
G := \text{Iso}_+(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \ a \in \mathbb{T}, b \in \mathbb{C} \right\} \cong \mathbb{C} \rtimes \mathbb{T},
\]

where \( \mathbb{T} \) acts on \( \mathbb{C} \) by multiplication. Note that the total minimality of the non-compact metabelian Lie group \( G \) can be established in a different way by Banakh [3, Theorem 13].

The following lemma was proved in [35, Lemma 6.19] only for hereditarily totally minimal groups. We provide a similar proof here for the sake of completeness.

**Lemma 3.14.** If \( G \) is a c-totally minimal group, then all quotients of \( G \) are c-totally minimal.

**Proof.** Let \( N \) be a closed normal subgroup of \( G \) and \( q : G \to G/N \) be the quotient map. Take a closed subgroup \( D \) of \( G/N \) and let \( D_1 = q^{-1}(D) \). Now we prove that \( D \) is totally minimal. Consider the restriction \( q' : D_1 \to D \). Clearly, \( q' \) is a continuous surjective homomorphism. Since \( D_1 \) is totally minimal by the hypothesis on \( G \), we obtain that \( q' \) is open. Moreover, being a quotient group of \( D_1 \), we deduce that \( D \) is totally minimal. \( \square \)

**Theorem 3.15.** Let \( N \) be a compact open normal subgroup of a topological group \( G \). Then the following conditions are equivalent:

1. \( G \) is c-totally minimal;
2. \( G/N \) is hereditarily totally minimal (i.e., hereditarily non-topologizable).

**Proof.** (1) \( \Rightarrow \) (2): As \( G \) is c-totally minimal, Lemma 3.14 implies that the quotient \( G/N \) is c-totally minimal. Being discrete, \( G/N \) is hereditarily totally minimal.

(2) \( \Rightarrow \) (1): See Proposition 3.8. \( \square \)

**Corollary 3.16.** Let \( G \) be a locally compact group that is either balanced or Lie. Then the following conditions are equivalent:

1. \( G \) is c-totally minimal;
2. there exists a compact open normal subgroup \( N \) of \( G \) such that \( G/N \) is hereditarily totally minimal.

**Proof.** (1) \( \Rightarrow \) (2): By Lemma 3.14, there exists a compact open normal subgroup \( N \) of \( G \). Using Theorem 3.15, we deduce that \( G/N \) is hereditarily totally minimal.

(2) \( \Rightarrow \) (1): Use Theorem 3.15. \( \square \)

We now provide another sufficient condition for the compactness of a c-totally minimal group. A topological group \( G \) is called *strongly compactly covered* (see [17]) if every element of \( G \) is contained in a compact open normal subgroup of \( G \).

**Corollary 3.17.** Let \( G \) be a strongly compactly covered group. If \( G \) is c-totally minimal, then it is compact.
Proof. By [17, Proposition 1.1], $G$ contains a compact open normal subgroup $N$ such that $G/N$ is a torsion FC-group (i.e., every element has finitely many conjugates). By Theorem 3.13, this discrete locally finite group is also hereditarily totally minimal. According to Fact 2.7, which says that a locally finite discrete hereditarily minimal group is finite, we have that $G/N$ must be finite. The finiteness of $G/N$ implies that $G$ is compact, as needed. \[\square\]

4. OPEN QUESTIONS AND CONCLUDING REMARKS

In view of Theorem 3.6 a natural question arises:

**Question 4.1.** Let $A$ be a (totally) minimal abelian group and $K$ be a compact group. Is $G = A \times K$ c-(totally) minimal?

Note that by Fact 5.5 $G$ is (totally) minimal whenever $A$ has the same property.

By Proposition 3.11 a complete c-totally minimal solvable group is compact. Can this result be extended to locally solvable groups? In other words:

**Question 4.2.** Let $G$ be a complete c-totally minimal locally solvable group. Is $G$ compact?

The next proposition provides a positive answer to Question 4.2 in case $G$ has a compact open normal subgroup.

**Proposition 4.3.** Let $G$ be a c-totally minimal locally solvable group. If $G$ has a compact open normal subgroup $N$, then $G$ is compact.

**Proof.** By Theorem 3.15 the locally solvable discrete group $G/N$ is hereditarily totally minimal. By Fact 2.7, $G/N$ is finite. This implies that $G$ is compact. \[\square\]

We proved in Theorem 2.9 that a locally solvable c-minimal Lie group is compact. In view of Lemma 2.4 we ask:

**Question 4.4.** Let $G$ be a locally compact, locally solvable, c-minimal balanced group. Is $G$ compact?

Let $G$ be a locally compact balanced group satisfying the following conditions:

1. every abelian subgroup of $G$ has compact closure;
2. $G$ is topologically locally finite (i.e., each precompact subset of $G$ generates a precompact subgroup).

Grosser and Herfort conjectured that such a topological group is either totally disconnected or compact (see Remark on page 222 of [19]). Even though we cannot provide a positive answer to Question 4.4 at this point, we shall see in Corollary 4.6 below that a locally compact, locally solvable, c-minimal balanced group satisfies the compactness conditions (1)-(2). The next proposition extends [8, Proposition 2.2] from the abelian case.

**Proposition 4.5.** Let $G$ be a compactly covered, locally compact, locally solvable balanced group. Then every compact subset of $G$ is contained in a compact open subgroup of $G$.

**Proof.** Let $K$ be a compact subset of $G$. Since $G$ is locally compact there exists a compact neighborhood $V$ of the identity $e$. Replacing $K$, if necessary, with its superset $(K \cup \{e\}) \cdot V$ we may assume, without loss of generality, that $K$ itself
is a neighborhood of \( e \). The rest of the proof is very similar to the proof of [8, Proposition 2.2], and we give it here for the sake of completeness.

Consider the quotient homomorphism \( q : G \to G/c(G) \). As \( G \) is a locally compact compactly covered group, so is its closed subgroup \( c(G) \). By Fact 2.2, \( c(G) \) is compact. Since \( G/c(G) \) is a locally compact totally disconnected balanced group, it has a local base at the identity consisting of compact open normal subgroups. Let \( U \) be a compact open normal subgroup of \( G/c(G) \) contained in \( q(K) \). Being discrete and compactly covered, the group \( (G/c(G))/U \) is torsion. Using [12, Proposition 1.1.5], we have that the locally solvable group \( (G/c(G))/U \) is locally finite. Hence, there exists a finite subgroup \( N \) of \( (G/c(G))/U \) containing the finite set \( r(q(K)) \).

\[
r : G/c(G) \to (G/c(G))/U
\]

is the quotient homomorphism. Since the groups \( U, c(G) \) and \( N \) are all compact, it follows that \((r \circ q)^{-1}(N)\) is a compact open subgroup of \( G \) containing \( K \), as needed.

**Corollary 4.6.** Let \( G \) be a locally compact, locally solvable, \( c \)-minimal balanced group. Then \( G \) is topologically locally finite and every abelian subgroup of \( G \) has compact closure.

**Proof.** By Lemma 2.2, \( G \) is compactly covered. It follows from Proposition 4.5 that \( G \) is topologically locally finite. By Fact 1.1, every abelian subgroup of \( G \) has compact closure.

Let \( N \) be a compact open normal subgroup of a topological group \( G \). By Proposition 3.8, if \( G/N \) is hereditarily minimal, then \( G \) is \( c \)-minimal. Moreover, by Theorem 3.15, \( G \) is \( c \)-totally minimal if and only if \( G/N \) is hereditarily totally minimal.

**Question 4.7.** Let \( G \) be a \( c \)-minimal group with a compact open normal subgroup \( N \). Is \( G/N \) hereditarily minimal?

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**References**

[1] U. Bader and T. Gelander, *Equicontinuous actions of semisimple groups*, Group Geom. Dynam. 11 (2017), 1003–1039.
[2] R. W. Bagley, T. S. Wu and J. S. Yang, *On the structure of locally compact topological groups*, Math. Scand. 71 (1992), 145–160.
[3] T. Banakh, *Categorically closed topological groups*, Axioms 6:3 (2017), 23.
[4] T. Banakh, *A quantitative generalization of Prodanov–Stoyanov Theorem on minimal Abelian topological groups*, Topology Appl. 271 (2020), 106983.
[5] B. Banaschewski, *Minimal topological algebras*, Math. Ann. 211 (1974), 107–114.
[6] M. Bruguera and M. Tkachenko, *The three space problem in topological groups*, Topology Appl. **153** (2006), 2278–2302.
[7] S. Dierolf and U. Schwanengel, *Examples of locally compact non-compact minimal topological groups*, Pacific J. Math. **82** (1979), 349–355.
[8] D. Dikranjan and A. Giordano Bruno, *The Bridge Theorem for totally disconnected LCA groups*, Topology Appl. **169** (2014), 21–32.
[9] D. Dikranjan and M. Megrelishvili, *Minimality conditions in topological groups*, Recent progress in general topology III, Atlantis Press, Paris, (2014), 229–327.
[10] D. Dikranjan and Iv. Prodanov, *Totally minimal topological groups*, Annuaire Univ. Sofia Fut. Math. Méc. **69** (1974/75), 5–11.
[11] D. Dikranjan, Iv. Prodanov and L. Stoyanov, *Topological Groups: Character, Dualities and Minimal Group Topologies*, Pure Appl. Math. **130**, Marcel Dekker, New York-Basel, 1989.
[12] D. Dikranjan and L. Stoyanov, *Criterion for minimality of all subgroups of a topological abelian group*, C. R. Acad. Bulgare Sci. **34** (1981), 635–638.
[13] M. R. Dixon, *Sylow theory, formations and Fitting classes in locally finite groups*, World Scientific Publishing, River Edge, NJ, 1994.
[14] D. Doichinov, *Produits de groupes topologiques minimaux*, Bull. Sci. Math. **96** (1972), 59–64.
[15] V. Eberhardt, S. Dierolf and U. Schwanengel, *On products of two (totally) minimal topological groups and the three-space-problem*, Math. Ann. **251** (1980), 123–128.
[16] W. T. van Est, *Dense imbeddings of Lie groups*, Indag. Math. **54** (1951), 321–328.
[17] A. Giordano Bruno, M. Shlissberg and D. Toller, *Algebraic entropy on strongly compactly covered groups*, Topology Appl. **263** (2019), 117–140.
[18] M. Goto, *Absolutely closed Lie groups*, Math. Ann. **204** (1973), 337–341.
[19] S. Grosser and W. Herfort, *Abelian subgroups of topological groups*, Trans. Amer. Math. Soc. **283** (1984), 211–223.
[20] P. Hall and C. R. Kulatilaka, *A property of locally finite groups*, J. London Math. Soc. **39** (1964), 235–239.
[21] A. A. Klyachko, A. Yu. Ol’shanskij and D. V. Osin, *On topologizable and non-topologizable groups*, Topology Appl. **160** (2013), 2104–2120.
[22] A. A. Markov, *On unconditionally closed sets*, Mat. Sbornik **18** (1946), 3–28 (in Russian). English translation in: *Three papers on topological groups: I. On the existence of periodic connected topological groups, II. On free topological groups, III. On unconditionally closed sets*, Amer. Math. Soc. Transl. **1950** (1950), 120 pp.
[23] M. Mayer, *Asymptotics of matrix coefficients and closures of Fourier–Stieltjes algebras*, J. of Functional Analysis **143** (1997), 42–54.
[24] M. Megrelishvili, *G-minimal topological groups*, Proc. of Orsatti Conference ‘Abelian groups, module theory, and topology’ (Padua, 1997), Lecture Notes in Pure and Appl. Math., Dekker, New York, **201** (1998), 289–299.
[25] M. Megrelishvili, *Every topological group is a group retract of a minimal group*, Topology Appl. **155** (2008), 2105–2127.
[26] M. Megrelishvili and M. Shlissberg, *Minimality properties of some topological matrix groups*, arXiv:1912.12088v2 (2020), 1-21, submitted.
[27] S. Morris and V. Obraztsov, *Embedding free amalgams of discrete groups in non-discrete topological groups*, Geometric group theory down under (Canberra, 1996), 203–223, de Gruyter, Berlin, 1999.

[28] H. Omori, *Homomorphic images of Lie groups*, J. Math. Soc. Japan. 18 (1966), 97–117.

[29] Iv. Prodanov, *Precompact minimal group topologies and p-adic numbers*, Annaire Univ. Sofia Fac. Math. Méc. 66 (1971/72), 249–266.

[30] Iv. Prodanov and L. Stoyanov, *Every minimal abelian group is precompact*, C. R. Acad. Bulgare Sci. 37 (1984), 23–26.

[31] D. Remus and L. Stojanov, *Complete minimal and totally minimal groups*, Topology Appl. 42 (1991), 57–69.

[32] J. S. Robinson, *A course in the theory of groups*, Springer, Berlin, 1982.

[33] S. Shelah, *On a problem of Kurosh, Jónsson groups, and applications*, In: S. I. Adian, W. W. Boone and G. Higman, Eds., Word Problems II, North-Holland, Amsterdam, (1980), 373–394.

[34] R. M. Stephenson, *Minimal topological groups*, Math. Ann. 192 (1971), 193–195.

[35] W. Xi, D. Dikranjan, M. Shlossberg and D. Toller, *Hereditarily minimal topological groups*, Forum Math. 31 (2019), 619–646.

[36] W. Xi, D. Dikranjan, M. Shlossberg and D. Toller, *Densely locally minimal groups*, Topology Appl. 266 (2019), 106846.

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