ATOMIC DECOMPOSITION AND WEAK FACTORIZATION FOR BERGMAN-ORLICZ SPACES

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Abstract. For $B^n$ the unit ball of $\mathbb{C}^n$, we consider Bergman-Orlicz spaces of holomorphic functions in $L^\Phi_\alpha(B^n)$, which are generalizations of classical Bergman spaces. We obtain atomic decomposition for functions in the Bergman-Orlicz space $A^\Phi_\alpha(B^n)$ where $\Phi$ is either convex or concave growth function. We then prove weak factorization theorems involving the Bloch space and a Bergman-Orlicz space and also weak factorization theorems involving two Bergman-Orlicz spaces.

1. Introduction and main results

Let $B^n$ be the unit ball of $\mathbb{C}^n$. We denote by $d\nu$ the Lebesgue measure on $B^n$. The space $H(B^n)$ is the set of holomorphic functions on $B^n$.

For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, we let

$$\langle z, w \rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$$

so that $|z|^2 = \langle z, z \rangle = |z_1|^2 + \cdots + |z_n|^2$.

We say that a function $\Phi$ is a growth function if it is a continuous and non-decreasing function from $[0, \infty)$ onto itself.

For $\alpha > -1$, we denote by $d\nu_\alpha$ the normalized Lebesgue measure $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$, with $c_\alpha$ such that $\nu_\alpha(B^n) = 1$. For $\Phi$ a growth function, the weighted Orlicz space $L^\Phi_\alpha(B^n)$ is the space of measurable functions $f$ such that, there exists a $\lambda > 0$ such that

$$\int_{B^n} \Phi\left(\frac{|f(z)|}{\lambda}\right) d\nu_\alpha(z) < \infty.$$ 

We define on $L^\Phi_\alpha(B^n)$ the following Luxembourg (quasi)-norm

$$\|f\|_{\Phi,\alpha} := \inf\{\lambda > 0 : \int_{B^n} \Phi\left(\frac{|f(z)|}{\lambda}\right) d\nu_\alpha(z) \leq 1\}$$

which is finite for $f \in L^\Phi_\alpha(B^n)$ (see [7]). The weighted Bergman-Orlicz space $A^\Phi_\alpha(B^n)$ is the subspace of $L^\Phi_\alpha(B^n)$ consisting of holomorphic functions.

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When $\Phi(t) = t^p$, we recover the classical weighted Bergman spaces denoted by $A^p_\alpha(B^n)$ and defined by

$$\|f\|_{p,\alpha}^p := \int_{B^n} |f(z)|^p d\nu_\alpha(z) < \infty.$$ 

We say that a growth function $\Phi$ is of lower type $p > 0$ if there exists $C > 0$ such that, for $s > 0$ and $0 < t \leq 1$,

$$(1.2) \quad \Phi(st) \leq Ct^p \Phi(s).$$

We denote by $\mathcal{L}$ the set of growth functions $\Phi$ of lower type $p$ for some $p$, $0 < p \leq 1$, such that the function $t \mapsto \Phi(t)$ is non-increasing. We also denote by $\mathcal{L}_p, 0 < p \leq 1$, the subset of $\mathcal{L}$ consisting of growth functions of lower type $p$.

We say that a growth function $\Phi$ is of upper type $q > 0$ if there exists $C > 0$ such that, for $s > 0$ and $t \geq 1$,

$$(1.3) \quad \Phi(st) \leq Ct^q \Phi(s).$$

We denote by $\mathcal{U}$ the set of growth functions $\Phi$ of upper type $q$ for some $q$, $q \geq 1$, such that the function $t \mapsto \Phi(t)$ is non-decreasing. We also denote by $\mathcal{U}_q, q \geq 1$, the subset of $\mathcal{U}$ consisting of growth functions of upper type $q$.

We say that $\Phi$ satisfies the $\Delta_2$-condition if there exists a constant $K > 1$ such that, for any $t \geq 0$,

$$(1.4) \quad \Phi(2t) \leq K\Phi(t).$$

It is easy to see the equivalence between (1.3) and (1.4). Moreover, if the function $t \mapsto \frac{\Phi(t)}{t}$ is non-increasing, then $\Phi$ satisfies the $\Delta_2$-condition; this is the case when $\Phi \in \mathcal{L}$.

Recall that two growth functions $\Phi_1$ and $\Phi_2$ are said to be equivalent if there exists some constant $c$ such that

$$c\Phi_1(ct) \leq \Phi_2(t) \leq c^{-1}\Phi_1(c^{-1}t).$$

Such equivalent growth functions define the same Orlicz space. Note that we may always suppose that any $\Phi \in \mathcal{L}_p$ (resp. $\mathcal{U}_q$) is concave (resp. convex) and that $\Phi$ is a $C^1$ function with derivative $\Phi'(t) \simeq \frac{\Phi(t)}{t}$ (see [2] for the lower type functions).

**Remark 1.1.** Given a growth function $\Phi$, we recall that the upper and lower indices, $a_\Phi$ and $b_\Phi$ respectively, of $\Phi$ are defined by:

$$a_\Phi = \sup \left\{ p : \inf_{t \geq 1, \lambda > 0} \frac{\Phi(t\lambda)}{t^p \Phi(\lambda)} > 0 \right\}, \quad b_\Phi = \inf \left\{ q : \sup_{t \geq 1, \lambda > 0} \frac{\Phi(t\lambda)}{t^q \Phi(\lambda)} < \infty \right\}$$

We say that $\Phi$ is of finite lower (resp. upper) type if $a_\Phi < \infty$ (resp. $b_\Phi < \infty$). In this case, $\Phi$ is of lower type $p$ (resp. of upper type $q$) for every $p < a_\Phi$ (resp. for every $q > b_\Phi$).
Our first interest in this paper is to obtain atomic decomposition theorems for functions in $\mathcal{A}_\Phi^\alpha(\mathbb{B}^n)$. For $p > 0$, atomic decomposition for functions in $\mathcal{A}_p^\alpha(\mathbb{B}^n)$ is a well known result, see [10, Theorem 2.30]. Our first main result extends the atomic decomposition from classical Bergman spaces to Bergman-Orlicz spaces whose growth function belongs to $\mathcal{L}$.

**Theorem 1.2.** Let $\Phi \in \mathcal{L}_p$ and $b \in \mathbb{R}$ with $b > \frac{n+1+\alpha}{p}$. There exists a sequence $a = \{a_k\}_{k=1}^\infty$ in $\mathbb{B}^n$, such that $\mathcal{A}_\Phi^\alpha(\mathbb{B}^n)$ consists exactly of functions of the form

$$f(z) = \sum_{k=1}^\infty \frac{c_k}{(1 - \langle z, a_k \rangle)^b}, \quad z \in \mathbb{B}^n,$$

where $\{c_k\}_{k=1}^\infty$ is a sequence of complex numbers that satisfies the condition

$$\sum_{k=1}^\infty (1 - |a_k|^2)^{n+1+\alpha} \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^b} \right) < \infty$$

and the series converges in the norm topology of $\mathcal{A}_\Phi^\alpha(\mathbb{B}^n)$. Moreover, there exists a sequence $\{c_k\}$ such that

$$\int_{\mathbb{B}^n} \Phi(|f(z)|)d\nu_\alpha(z) \simeq \sum_{k=1}^\infty (1 - |a_k|^2)^{n+1+\alpha} \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^b} \right).$$

After some minor modifications, this result is still valid for Bergman-Orlicz spaces with convex growth function. For $\Phi$ a convex growth function, we recall that the complementary function, $\Psi : [0, \infty) \to [0, \infty)$, is defined by

$$\Psi(s) = \sup_{t \in \mathbb{R}_+} \{ts - \Phi(t)\}.$$ 

One easily checks that if $\Phi \in \mathcal{U}$, then $\Psi$ is a growth function of lower type such that the function $t \mapsto \frac{\Psi(t)}{t}$ is non-decreasing, but which may not satisfy the $\Delta_2$—condition. We say that the growth function $\Phi$ satisfies the $\nabla_2$—condition whenever both $\Phi$ and its complementary satisfy the $\Delta_2$—condition.

We shall also prove the following analogous of the previous theorem for growth functions belonging to $\mathcal{U}$.

**Theorem 1.3.** Let $\Phi \in \mathcal{U}$ and $b \in \mathbb{R}$ with $b > n+1+\alpha$. We suppose that $\Phi$ satisfy the $\nabla_2$—condition. There exists a sequence $a = \{a_k\}_{k=1}^\infty$ in $\mathbb{B}^n$, such that $\mathcal{A}_\Phi^\alpha(\mathbb{B}^n)$ consists exactly of functions of the form

$$f(z) = \sum_{k=1}^\infty \frac{c_k}{(1 - \langle z, a_k \rangle)^b}, \quad z \in \mathbb{B}^n,$$
where \( \{c_k\}_{k=1}^\infty \) is a sequence of complex numbers that satisfies the condition
\[
\sum_{k=1}^\infty (1 - |a_k|^2)^{n+1+\alpha} \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^{\beta}} \right) < \infty
\]
and the series converges in the norm topology of \( A_\alpha^p(B^n) \). Moreover, we have
\[
(1.10) \quad \int_{B^n} \Phi(|f(z)|) d\nu_\alpha(z) \simeq \sum_k (1 - |a_k|^2)^{n+1+\alpha} \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^{\beta}} \right)
\]
when the left hand side is bounded by 1.

It is well-known in the classical case that such atomic decompositions may be used to obtain weak factorization theorems for \( A_\alpha^p(B^n) \) for \( p \leq 1 \), in terms of products of functions in Bergman spaces \([10, \text{Corollary 2.33}]\). Recently, using the above atomic decomposition and their characterization of boundedness of Hankel operators (with loss) between two Bergman spaces, J. Pau and R. Zhao \([5]\) extended these weak factorization theorems for \( A_\alpha^p(B^n) \) with \( p > 1 \). So, for \( p > 0 \), each function \( f \in A_\alpha^p(B^n) \) can be decomposed as
\[
f(z) = \sum_k g_k(z)h_k(z), \quad z \in B^n,
\]
where each \( g_k \) is in \( A_\alpha^q(B^n) \) and each \( h_k \) is in \( A_\alpha^r(B^n) \), where \( \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \), with
\[
(1.11) \quad \sum_k \|g_k\|_{q,\alpha} \|h_k\|_{r,\alpha} \lesssim \|f\|_{p,\alpha}.
\]
This last inequality can be strengthened for \( p \leq 1 \) to obtain a weak factorization such that
\[
(1.12) \quad \sum_k \|g_k\|_{q,\alpha}^p \|h_k\|_{r,\alpha}^p \simeq \|f\|_{p,\alpha}^p.
\]
One may ask whether such weak factorizations may be obtained for Bergman-Orlicz spaces. This first proposition is an immediate corollary of an observation on Orlicz spaces given in \([9]\).

**Proposition 1.4.** Let \( \Phi_1 \) and \( \Phi_2 \) be two growth functions of finite lower type and let \( \Phi \) be a growth function such that
\[
(1.13) \quad \Phi^{-1} = \Phi_1^{-1} \times \Phi_2^{-1}.
\]
Then the product of two functions that are respectively in \( A_\alpha^{\Phi_1}(B^n) \) and \( A_\alpha^{\Phi_2}(B^n) \) is in \( A_\alpha^{\Phi}(B^n) \). Moreover
\[
\|fg\|_{\Phi,\alpha} \lesssim \|f\|_{\Phi_1,\alpha} \|g\|_{\Phi_2,\alpha}.
\]
Here for a growth function \( \Phi \), \( \Phi^{-1} \) is the inverse function of \( \Phi \).
One may ask whether one has a weak factorization of $A^\Phi_{\alpha}(B^n)$ in this context. Using the atomic decomposition obtained here and natural factorization together with good estimates that can be found in [8], we shall obtain for $\Phi \in \mathcal{L}$, weak factorization theorems for $A^\Phi_{\alpha}(B^n)$ in terms of products of functions in Bergman-Orlicz spaces. It is done in the last section of this paper. But we do not succeed in giving a critical equivalent of the norm, as in (1.12).

In view of applications to Hankel operators studied in [8], our second interest here is to obtain another type of weak factorization for functions in $A^\Psi_{\alpha}(B^n)$, $\Psi \in \mathcal{L}_p$, in terms of products of functions in $A^\Phi_{\alpha}(B^n)$ and in the Bloch space. We recall that given an holomorphic function $f$ on $B^n$, the radial derivative $Rf$ of $f$ is defined by

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

The Bloch class $\mathcal{B}$ is the space of holomorphic functions in $B^n$ such that

$$\sup_{z \in B^n} |Rf(z)|(1 - |z|^2) < \infty.$$  

The norm on $\mathcal{B}$ is given by $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in B^n} |Rf(z)|(1 - |z|^2)$. One has the following proposition for products of functions that are respectively in $A^\Phi_{\alpha}(B^n)$ and in $\mathcal{B}$.

**Proposition 1.5.** Let $\Phi$ be a growth function of finite lower type (resp. of finite upper type). The product maps continuously $A^\Phi_{\alpha}(B^n) \times \mathcal{B}$ into $A^\Psi_{\alpha}(B^n)$, where $\Psi(t) = \Phi \left( \frac{t}{\log(e+t)} \right)$. Moreover,

$$\|fg\|_{\Psi,\alpha} \lesssim \|f\|_{\Phi,\alpha} \|g\|_{\mathcal{B}}.$$  

The following is our second main result.

**Theorem 1.6.** Let $\Phi \in \mathcal{L}_p$ and let $\Psi(t) = \Phi \left( \frac{t}{\log(e+t)} \right)$. Every function $f \in A^\Psi_{\alpha}(B^n)$ may be written as the sum

$$f = \sum_{k=1}^{+\infty} f_k b_k,$$

with $f_k \in A^\Phi_{\alpha}(B^n)$ and $b_k \in \mathcal{B}$, with

$$\sum_{k=1}^{+\infty} \|f_k\|_{\Phi,\alpha} \|b_k\|_{\mathcal{B}} \lesssim \|f\|_{\Psi,\alpha}.$$
Moreover, if \( \|f\|_{\Psi,\alpha} \leq 1 \),

\[
(1.15) \quad \int_{\mathbb{B}^n} \Psi(|f(z)||) d\nu_\alpha(z) \simeq \sum_{k=1}^{+\infty} \int_{\mathbb{B}^n} \Psi(|f_k(z)b_k(z)||) d\nu_\alpha(z)
\]

\[
\simeq \sum_{k=1}^{+\infty} \int_{\mathbb{B}^n} \Phi(|f_k(z)||) d\nu_\alpha(z) \times \|b_k\|_B.
\]

These two estimates can be considered as the equivalent, in this context, of (1.12) and (1.11). Remark that, except when \( \Phi \) is equivalent to a homogeneous function, there is no way to pass from the Luxembourge norm to the quantity \( \int \Phi(|\cdot||) d\nu_\alpha \). In the previous statement only one of the two Bergman-Orlicz spaces involved can coincide with some \( A^p_\alpha(\mathbb{B}^n) \).

The same kind of statement has been considered for Hardy-Orlicz spaces and the class \( BMOA \) in [1]. But only the equivalent of (1.14) has been obtained. There is no equivalence as in the previous theorem. We have a better understanding of weak factorization in the context of Bergman-Orlicz spaces.

The paper is organized as follows. In section 2, we collect and establish some results that will be used later. In section 3, we give proofs of atomic decomposition theorems for functions in Bergman-Orlicz spaces. In particular, we establish Thorem 1.2 and Theorem 1.3. In section 4, we first prove Proposition 1.5; next, we prove weak factorization theorems for Bergman-Orlicz spaces, the first in terms of products of two factors, one in the Bloch space, the other in a Bergman-Orlicz space (Theorem 1.6), and the second in terms of products of two factors in two Bergman-Orlicz spaces (Theorem 4.4). We apply the first weak factorization theorem to recover a characterization result [8] of bounded small Hankel operators from a Bergman-Orlicz space \( A^p_\alpha(\mathbb{B}^n) \) to the weighted Bergman spaces \( A^1_\alpha(\mathbb{B}^n) \).

Finally, all over the text, \( C \) will be a constant not necessary the same at each occurrence. We will also use the notation \( C(k) \) to express the fact that the constant depends on the underlined parameter \( k \). Given two positive quantities \( A \) and \( B \), the notation \( A \lesssim B \) means that \( A \leq CB \) for some positive uniform constant \( C \). When \( A \lesssim B \) and \( B \lesssim A \), we write \( A \simeq B \).

2. Preliminaries

In this section, we recall some known results and establish some estimates that are needed in our study.

2.1. Some geometric properties in the unit ball. We recall the following facts for which details can be found in [10].
For $z \in \mathbb{B}^n$, let $\varphi_z$ be the involutive automorphism of $\mathbb{B}^n$ that interchanges $z$ and $0$. That is, $\varphi_z$ is a holomorphic function from $\mathbb{B}^n$ to itself that satisfies $\varphi_z \circ \varphi_z = id$ and $\varphi_z(0) = z$ and $\varphi_z(z) = 0$. Using the map $\varphi_z$, the Bergman metric, $d$ on $\mathbb{B}^n$, is defined by

$$d(z, w) = \frac{1}{2} \log \left( \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} \right).$$

For $r > 0$, we denote by $D(z, r)$ the Bergman ball, that is the ball with respect to the Bergman metric, of radius $r$ and centered at $z$. It is well-known that for $w \in D(z, r)$

\begin{equation}
\nu_\alpha(D(z, r)) \simeq |1 - \langle z, w \rangle|^{n+1+\alpha} \simeq (1 - |z|^2)^{n+1+\alpha} \simeq (1 - |w|^2)^{n+1+\alpha}.
\end{equation}

Here constants are uniform in $z$.

A sequence $\{a_k\}$ of points in $\mathbb{B}^n$ is a separated sequence (in Bergman metric) if there exists a positive constant $\delta > 0$ such that $d(a_k, a_j) \geq \delta$ for any $k \neq j$. A maximal $\delta-$ separated sequence $\{a_k\}$ in $\mathbb{B}^n$ has the property that

(i) $\mathbb{B}^n = \bigcup_k D(a_k, \delta),$

(ii) The sets $D(a_k, \frac{\delta}{2})$ are mutually disjoint,

(iii) Each point $z \in \mathbb{B}^n$ belongs to at most $N$ of the sets $D(a_k, 2\delta)$.

Here $N$ is an absolute constant, which does not depend of the sequence $\{a_k\}$. A sequence $\{a_k\}_{k=1}^\infty$ satisfying these conditions is called an $\delta-$lattice.

The following lemma will be useful.

**Lemma 2.1.** ([10] Lemma 2.28) Let $\{a_k\}_{k=1}^\infty$ be a $\delta-$lattice. There is a sequence of Borel sets $\{D_k\}_{k=1}^\infty$ in $\mathbb{B}^n$ satisfying the following conditions.

(i) $D(a_k, \frac{\delta}{2}) \subset D_k \subset D(a_k, \delta),$

(ii) The sets $D_k$ are mutually disjoint,

(iii) $\mathbb{B}^n = \bigcup_k D_k.$

It is classical that one can jointly construct one $1$-lattice $\{a_k\}$ and one $\eta-$ lattice $\{a_{kj}\}$ with remarkable properties. We have the following lemma (see [10] for more details).

**Lemma 2.2.** Let $\eta \in (0, 1)$ be small. There exists an integer $J$, which depends only on $\eta$, such that one can find simultaneously a $1$-lattice $\{a_k\}_{k=1}^\infty$ and an $\eta-$ lattice $\{a_{kj}\}$ with $k$ varying from $1$ to $\infty$ and $j$ from $1$ to $J$ with the supplementary property that

$$D(a_k, 1) \subset \bigcup_j D(a_{kj}, 2\eta).$$

Moreover, if $\{D_{kj}\}$ (resp. $\{D_k\}$) denotes the sequence of disjoint Borel sets corresponding to the $\eta-$lattice $\{z_{kj}\}$ (resp. to the $1-$lattice $\{z_k\}$) as described in Lemma 2.2, we have

$$D_k = \bigcup_{j=1}^J D_{kj}.\]
With these notations, for \( b > n \) and let \( \beta = b - n - 1 \), we define the following operator \( S \) by

\[
Sf(z) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{\nu_\beta(D_{kj}) f(a_{kj})}{(1 - \langle z, a_{kj} \rangle)^b},
\]

for \( f \in L^1(\mathbb{B}^n, d\nu_\beta(z)) \).

We will need the following lemma concerning \( S \).

**Lemma 2.3.** [10, Lemma 2.29] Let \( \{a_k\}_{k=1}^{\infty} \) and \( \{a_{kj}\} \) be as in Lemma 2.2. For any \( s > 0 \) and \( \alpha > -1 \) there exists a constant \( C > 0 \), independent of the separation constant \( \eta \), such that

\[
|f(z) - Sf(z)| \leq C \sigma + \sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)^b}{1 - \langle z, a_k \rangle} \left( \int_{D(ak,2)} |f(w)|^s \frac{d\nu_\alpha(w)}{\nu_\alpha(D(ak,2))} \right)^{\frac{1}{s}}
\]

for all \( z \in \mathbb{B}^n \), and \( f \in A^1_{\beta}(\mathbb{B}^n) \), where \( \sigma \) depends only on \( \eta \) and \( \sigma \to 0 \) as \( \eta \to 0 \).

2.2. Some useful estimates. We collect in this subsection some properties of growth functions we shall use later and establish some useful estimates.

For \( \Phi \) a \( C^1 \) growth function, the lower and the upper indices of \( \Phi \), defined in Remark 1.1, are respectively given by

\[
a_{\Phi} := \inf_{t > 0} t \Phi'(t) \quad \text{and} \quad b_{\Phi} := \sup_{t > 0} t \Phi'(t).
\]

We recall that when \( \Phi \) is convex, then \( 1 \leq a_{\Phi} \leq b_{\Phi} < \infty \) and, if \( \Phi \) is concave, then \( 0 < a_{\Phi} \leq b_{\Phi} \leq 1 \). We have the following simple but useful fact [8].

**Lemma 2.4.** Let \( \Phi \) be a \( C^1 \) growth function. Denote by \( p \) and \( q \) its lower and its upper indices respectively. Then the functions \( \frac{\Phi(t)}{t^p} \) and \( \frac{\Phi^{-1}(t)}{t^{1/q}} \) are non-decreasing.

**Remark 2.5.** One useful way to use Lemma 2.4 is to observe that it implies the following: if \( \Phi \in \mathcal{L}_p \) for some \( p \in (0,1) \), then the growth function \( \Phi_p(t) = \Phi(t^{1/p}) \), is in \( \mathcal{M}_q \) for some \( q \geq 1 \) (e.g. \( q = \frac{1}{p} \)). So we may assume that \( \Phi_p \) is convex.

We will make use very often of the following classical estimate.

**Theorem 2.6.** [10, Theorem 1.12] Let \( \alpha > -1 \) and \( c > 0 \). The following integral

\[
J_{c,\alpha}(z) = \int_{\mathbb{B}^n} \frac{(1 - |w|^2)^\alpha d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+c}}, \quad z \in \mathbb{B}^n,
\]

have the following asymptotic property.

\[
J_{c,\alpha}(z) \simeq (1 - |z|^2)^{-c}, \quad |z| \to 1^{-}.
\]
We use in particular this theorem for computations on atoms.

**Lemma 2.7.** Let \( \Phi \in \mathcal{L}_p \), \( a \in \mathbb{B}^n \) and a real \( b \) such that \( b > \frac{n+1+\alpha}{p} \). There exists a positive constant \( C \) such that for all \( \lambda \in \mathbb{C} \) and all functions \( f \) of the form \( f(z) = \frac{\lambda}{1-\langle z, a \rangle} \). There exists a positive constant \( C \) (depending only on fixed constants) such that:

\[
(2.4) \quad C^{-1} \Phi \left( \frac{|\lambda|}{(1-|a|^2)^{\frac{b}{p}}} \right) (1-|a|^2)^{n+1+\alpha} \leq \int_{\mathbb{B}^n} \Phi(|f(z)|) d\nu_\alpha(z) \leq C \Phi \left( \frac{|\lambda|}{(1-|a|^2)^{\frac{b}{p}}} \right) (1-|a|^2)^{n+1+\alpha}.
\]

**Proof.** Recall that \( \Phi \) is of lower type \( p \), so that \( \Phi(\tau s) \leq C\tau^p \Phi(s) \) for \( 0 < s \leq 1 \). Moreover, since \( \Phi \) satisfies the \( \Delta_2 \) condition, such an inequality is also valid for \( 1 \leq t \leq 2 \) (eventually after a modification of the constant). We can use it here for \( t = \left( \frac{1-|a|^2}{1-\langle z, a \rangle} \right)^{\frac{b}{p}} \), which is bounded by \( 2^{\frac{b}{p}} \). We write that

\[
\int_{\mathbb{B}^n} \Phi(|f(z)|) d\nu_\alpha(z) \leq C \Phi \left( \frac{|\lambda|}{(1-|a|^2)^{\frac{b}{p}}} \right) (1-|a|^2)^{n+1+\alpha}.
\]

**Remark 2.8.** Under mild modifications, if \( \Phi \in \mathcal{U} \), the estimate in \( (2.4) \) still holds for functions in \( \mathcal{A}_\Phi^{\alpha}(\mathbb{B}^n) \), provided that \( b > n+1+\alpha \).

A consequence of Lemma 2.7 gives the Luxembourg quasi-norm estimate for such \( f \). We have

\[
(2.7) \quad \|f\|_{\Phi,\alpha} \simeq \frac{|\lambda|}{(1-|a|^2)^{\Phi^{-1} \left( \frac{1}{(1-|a|^2)^{n+1+\alpha}} \right)}}.
\]
DEFINITION 2.9. Functions $f$ of the form $f(z) = \frac{\lambda}{(1 - \langle z, a \rangle)^b}$ are called (non normalized) atoms of $A^\Phi_\alpha(B^n)$.

To finish this subsection, we recall elementary properties of norms and integrals. Remark first that the equivalence

$$\int_{B^n} \Phi(|f(z)|)d\nu_\alpha(z) \lesssim 1$$

is equivalent to $\|f\|_{\Phi,\alpha} \lesssim 1$. Moreover, when this condition is satisfied,

$$\|f\|_{\Phi,\alpha} \lesssim \int_{B^n} \Phi(|f(z)|)d\nu_\alpha(z) \lesssim \|f\|_{p,\alpha}^p$$

for $\Phi \in L_p$. One cannot reverse these inequalities.

We shall use the following results about Luxembourg norm estimates for bounded functions in $A^\Phi_\alpha(B^n)$. These are easy extensions of the same type of results in [4, Lemma 3.9].

**LEMMA 2.10.** Let $\alpha > -1$ and $\Phi \in L_p$. For any bounded holomorphic function $f$ in $B^n$, one has:

$$\|f\|_{\Phi,\alpha} \lesssim \|f\|_{\Phi,\alpha}^{-1} \left(\frac{\|f\|_{\Phi,\alpha,1}^2}{\|f\|_{1,\alpha}^2}\right).$$

**LEMMA 2.11.** Let $\alpha > -1$ and $\Phi \in U_q$. For any bounded holomorphic function $f$ in $B^n$, one has:

$$\|f\|_{\Phi,\alpha} \lesssim \|f\|_{\Phi,\alpha}^{-1} \left(\frac{\|f\|_{\Phi,1}^2}{\|f\|_{1,\alpha}^2}\right).$$

3. Atomic decomposition for Bergman-Orlicz spaces

In this section, we give the proofs of Theorem 1.2 and Theorem 1.3. Our proofs are adapted from the proofs in the classical weighted Bergman spaces.

3.1. Proof of Theorem 1.2. This subsection is devoted to the proof of Theorem 1.2. Let $\Phi \in L_p$. In one direction we have more than what is stated in the theorem. Namely, in the next proposition, $\{a_k\}$ is an arbitrary sequence of points of $\mathbb{B}^n$.

**PROPOSITION 3.1.** Suppose that $b > \frac{n+1+\alpha}{p}$. Let $\Phi \in L_p$ and let $\{a_k\}$ be a sequence of points in $\mathbb{B}^n$. Assume that $\{c_k\}$ satisfies the condition

$$\sum_k (1 - |a_k|^2)^{n+1+\alpha} \Phi \left(\frac{|c_k|}{(1 - |a_k|^2)^b}\right) < \infty.$$

Then the series $\sum_k \frac{c_k}{(1 - \langle z, a_k \rangle)^b}$ converges in $A^\Phi_\alpha(B^n)$ to a function $f$ and

$$\int_{B^n} \Phi \left(\sum_k \frac{c_k}{(1 - \langle z, a_k \rangle)^b}\right) d\nu_\alpha(z) \lesssim \sum_{k=1}^{+\infty} (1 - |a_k|^2)^{n+1+\alpha} \Phi \left(\frac{|c_k|}{(1 - |a_k|^2)^b}\right).$$
Proof. Let us define $f_k(z) = \frac{c_k}{(1-\langle z, a_k \rangle)^b}$. We use the concavity of the function $\Phi$ and Lemma 2.7 to obtain that for all finite sets of indices $K$ one has

$$\int_{\mathbb{B}^n} \Phi(|\sum_{k \in K} f_k|) d\nu_\alpha \leq \sum_{k \in K} \int_{\mathbb{B}^n} \Phi(|f_k|) d\nu_\alpha \leq \sum_{k \in K} C (1 - |a_k|^2)^{n+1+\alpha} \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^b} \right).$$

The space $A_{\alpha}^\Phi(\mathbb{B}^n)$ is a complete metric space for the distance defined by

$$(f, g) \mapsto \int_{\mathbb{B}^n} \Phi(|f(z) - g(z)|) d\nu_\alpha(z),$$

or, equivalently for the one defined by the Luxembourg quasi-norm. The condition on the sequence $\{c_k\}$ implies that the sequence of partial sums of the series is a Cauchy sequence in the space $A_{\alpha}^\Phi(\mathbb{B}^n)$. The function $f = \sum_{k=1}^\infty f_k$ is its limit. We conclude at once. □

Let $\{a_k\}_{k=1}^\infty$ and $\{a_{kj}\}$ be as in Lemma 2.2. The latter sequence is the sequence for which we will prove the representation of Theorem 1.2. For better understanding we keep a double index. We show now that every function $f \in A_{\alpha}^\Phi(\mathbb{B}^n)$ may be written as in (1.5), that is,

$$f(z) = \sum_{k=1}^\infty \sum_{j=1}^{+\infty} \frac{c_{kj}}{(1 - \langle z, a_{kj} \rangle)^b},$$

The constant $\eta$ will be chosen sufficiently small later on. We first prove that $f - Sf$ is small in $A_{\alpha}^\Phi(\mathbb{B}^n)$ when $\eta$ is small enough, where $S$ is defined in (2.2). Remark that, since $A_{\alpha}^\Phi(\mathbb{B}^n)$ is continuously embedded in $A_\beta^p(\mathbb{B}^n)$, for $f \in A_{\alpha}^\Phi(\mathbb{B}^n)$ the inequality

$$|f(z)|^p d\nu_\alpha(D(z, 1)) \leq C \int_{\mathbb{B}^n} \Phi(|f(z)|) d\nu_\alpha(z)$$

implies easily that $f$ belongs to $A_\beta^1(\mathbb{B}^n)$ when $b > \frac{n+1+\alpha}{p}$ and $\beta = b - n - 1$ as above. From Lemma 2.3, where we take $s = p$, the fact that $\Phi$ is of lower type $p$ and Proposition 3.1 there exists $C > 0$ such that

$$\int_{\mathbb{B}^n} \Phi(|f(z) - Sf(z)|) d\nu_\alpha(z) \leq$$

$$C \sigma^p \int_{\mathbb{B}^n} \Phi \left( \sum_{k=1}^+ \frac{(1 - |a_k|^2)^b}{1 - \langle z, a_k \rangle} \left( \int_{D(a_k, 2)} |f(w)|^p \frac{d\nu_\alpha(w)}{\nu_\alpha(D(a_k, 2))} \right)^\frac{1}{p} \right) d\nu_\alpha(z) \leq C \sigma^p \sum_{k=1}^+ (1 - |a_k|^2)^{n+1+\alpha} \Phi \left\{ \left( \int_{D(a_k, 2)} |f(w)|^p \frac{d\nu_\alpha(w)}{\nu_\alpha(D(a_k, 2))} \right)^\frac{1}{p} \right\}. $$
For the first inequality, we took $\eta$ sufficiently small so that $C_\eta \leq 1$. Since $\Phi_p(t) = \Phi(t^{1/p})$ is convex (see Remark 2.5), we will make use of the following Jensen inequality
\[\Psi \left( \int_X g d\mu \right) \leq \int_X \Psi(g) d\mu,\]
valid for any convex function $\Psi$, nonnegative function $g$, and a probability measure $d\mu$ on $X$, to obtain
\[
\begin{align*}
\int_{\mathbb{B}^n} \Phi \left( |f(z) - Sf(z)| \right) d\nu_\alpha(z) &\leq \sigma^p \sum_{k=1}^{+\infty} \left( 1 - |a_k|^2 \right)^{n+1+\alpha} \Phi_p \left( \int_{D(a_k,2)} |f(w)|^p \frac{d\nu_\alpha(w)}{\nu_\alpha(D(a_k,2))} \right) \\
&\leq C \sigma^p \sum_{k=1}^{+\infty} \left( 1 - |a_k|^2 \right)^{n+1+\alpha} \int_{D(a_k,2)} \Phi_p \left( |f(w)|^p \right) \frac{d\nu_\alpha(w)}{\nu_\alpha(D(a_k,2))} \\
&\leq C \sigma^p \sum_{k=1}^{+\infty} \int_{D(a_k,2)} \Phi \left( |f(w)| \right) d\nu_\alpha(w).
\end{align*}
\]
We have used the fact that $\nu_\alpha(D(a_k,2)) \simeq (1 - |a_k|^2)^{n+1+\alpha}$.

By the finite overlapping property of a $1$–lattice (property (iii)) we have finally
\[
\int_{\mathbb{B}^n} \Phi \left( |f(z) - Sf(z)| \right) d\nu_\alpha(z) \leq CN \sigma^p \int_{\mathbb{B}^n} \Phi \left( |f(w)| \right) d\nu_\alpha(w).
\]
We choose $\eta$ small enough so that $CN \sigma^p \leq \frac{1}{2}$.

For $g \in A^\alpha_\Phi(\mathbb{B}^n)$, we deduce from the previous inequality that $\sum_{n \geq 0} \int \Phi((I-S)^ng) d\nu_\alpha \leq 2 \int \Phi(|g|) d\nu_\alpha$. We use again the concavity of $\Phi$ to deduce that the Neumann series $\sum_{n=0}^{+\infty} (I-S)^ng$ converges in $A^\alpha_\Phi(\mathbb{B}^n)$. As for Banach spaces, we obtain that the bounded operator $S$ on $A^\alpha_\Phi(\mathbb{B}^n)$ is invertible and its inverse $S^{-1}$ is given by $S^{-1}(g) = \sum_{n=0}^{+\infty} (I-S)^ng$. Therefore, every $f \in A^\alpha_\Phi(\mathbb{B}^n)$ admits a representation
\[f(z) = \sum_{k=1}^{+\infty} \sum_{j=1}^{J} \frac{c_{kj}}{(1 - \langle z, a_k \rangle)^b},\]
where
\[c_{kj} = \nu_\beta(D_{kj}) g(a_{kj}) \quad \text{and} \quad g = S^{-1}f \in A^\alpha_\Phi(\mathbb{B}^n).
\]

It remains to show that
\[
I := \sum_{k=1}^{+\infty} \sum_{j=1}^{J} \left( 1 - |a_{kj}|^2 \right)^{n+1+\alpha} \Phi \left( \frac{|c_{kj}|}{(1 - |a_{kj}|^2)^b} \right) \lesssim \int_{\mathbb{B}^n} \Phi(|f(z)|) d\nu_\alpha(z).
\]

We know that
\[\nu_\beta(D_{kj}) \leq \nu_\beta(D_k) \simeq (1 - |a_k|^2)^{n+1+\beta} = (1 - |a_k|^2)^b,
\]
and, by the mean value property \[10, \text{Lemma 2.24}\], we also have
\begin{equation}
|g(a)|^p \leq C \frac{1}{\nu_\alpha(D(a,1))} \int_{D(a,1)} |g(w)|^p \nu_\alpha(w).
\end{equation}

Using (3.1) and (3.2), the Jensen inequality as above and the finite overlapping property, we have
\begin{align*}
I & \lesssim \sum_{k=1}^{+\infty} \sum_{j=1}^{J} \left(1 - |a_{kj}|^2\right)^{n+1+\alpha} \Phi(|g(a_{kj})|) \\
& \lesssim \sum_{k=1}^{+\infty} \sum_{j=1}^{J} \left(1 - |a_{kj}|^2\right)^{n+1+\alpha} \Phi \left(\int_{D(a_{kj},1)} |g(w)|^p \frac{d\nu_\alpha(w)}{\nu_\alpha(D(a_{kj},1))}\right) \\
& \leq CJ \sum_{k=1}^{+\infty} \left(1 - |a_k|^2\right)^{n+1+\alpha} \int_{D(a_k,1)} \Phi(|g(w)|) \frac{d\nu_\alpha(w)}{\nu_\alpha(D(a_k,1))} \\
& \leq C J N \int \Phi(|g(w)|) d\nu_\alpha(w) < \infty,
\end{align*}
which is what we wanted to prove. The converse inequality has been given by Proposition 3.1. So we have completed the proof of Theorem 1.2.

We finish this subsection by a remark. Theorem 1.2 gives the integral \[\int \Phi(|f|)d\nu_\alpha\] in terms of the coefficients of a representation of \(f\). In order to deal with Luxembourg norms, we first give some definitions. Given a growth function \(\Phi\) and \(b > 0\), we define the space \(l^{\Phi,\alpha,b}_\alpha\) as the space of couple of sequences \(\{a_k\}_{k \in \mathbb{N}}\) in \(\mathbb{B}^n\), and \(\{c_k\}_{k \in \mathbb{N}}\) in \(\mathbb{C}\) such that, for some \(\lambda > 0\),
\begin{equation}
\sum_{k} \left(1 - |a_k|^2\right)^{n+1+\alpha} \Phi \left(\frac{|c_k|}{\lambda (1 - |a_k|^2)^{b/2}}\right) < \infty.
\end{equation}
An element of this space identifies with a sequence of non normalized atoms of the form \(\{f_{a_k,c_k}\}_k\), with
\begin{equation}
f_{a_k,c_k} = \frac{c_k}{(1 - \langle z, a_k \rangle)^{b/2}}.
\end{equation}
We define the quasi norm on this space by:
\begin{equation}
\|\{f_{a_k,c_k}\}\|_{l^{\Phi,\alpha,b}_\alpha} = \inf \{\lambda > 0 : \sum_{k} \left(1 - |a_k|^2\right)^{n+1+\alpha} \Phi \left(\frac{|c_k|}{\lambda (1 - |a_k|^2)^{b/2}}\right) \leq 1\}.
\end{equation}

With this definition it is straightforward to deduce from Theorem 1.2 that
\begin{equation}
\|f\|_{\Phi,\alpha} \simeq \inf_{\{a_k\},\{c_k\}} \left\{\|\{f_{a_k,c_k}\}\|_{l^{\Phi,\alpha,b}_\alpha} : f = \sum f_{a_k,c_k}\right\}.
\end{equation}
3.2. Proof of Theorem 1.3. This subsection is devoted to the proof of Theorem 1.3. Proposition 3.1 is no more valid in all generality, but we have the following proposition.

**Proposition 3.2.** Let $\Phi \in \mathcal{U}^q$ and let $\{a_k\}_{k=1}^\infty$ be a sequence of $r$-separated points in $\mathbb{B}^n$. Assume that $\{c_k\}_{k=1}^\infty$ satisfies the condition

$$
\sum_{k=1}^\infty (1 - |a_k|^2)^{n+1+\alpha} \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^b} \right) \leq 1.
$$

Then the series $\sum_{k=1}^\infty c_k (1 - \langle z, a_k \rangle)^b$ converges in $A^\Phi_{\alpha}(\mathbb{B}^n)$ to a function $f$, which is such that $\int_{\mathbb{B}^n} \Phi(|f|) d\nu_\alpha \lesssim 1$. Furthermore, for a given $r$-separated sequence and $\{c_k\}$ such that $\{f_{a_k,c_k}\}$ is in $l^\Phi_{\alpha,b}$, the series $\sum f_{a_k,c_k}$ converges in $A^\Phi_{\alpha}(\mathbb{B}^n)$ and

$$
\left\| \sum f_{a_k,c_k} \right\|_{\Phi,\alpha} \lesssim \left\| \{f_{a_k,c_k}\} \right\|_{l^\Phi_{\alpha,b}}.
$$

**Proof.** Let $\Phi \in \mathcal{U}^q$. We assume that $\Phi$ is convex, so that the Luxembour-ignon norm is a norm. We will prove a little more, that is,

$$
\left\| \sum_{k=1}^\infty \frac{|c_k|}{|1 - \langle z, a_k \rangle|^b} \right\|_{\Phi,\alpha} \lesssim 1
$$

assuming that (3.6) holds. Let us assume that we succeeded in proving this. Then, by (2.10), the same inequality holds for $\int_{\mathbb{B}^n} \Phi(\sum_{k=1}^\infty \frac{|c_k|}{|1 - \langle z, a_k \rangle|^b}) d\nu_\alpha(z)$. As a consequence,

$$
\int_{\mathbb{B}^n} \Phi \left( \sum_{k=1}^\infty \frac{|c_k|}{|1 - \langle z, a_k \rangle|^b} \right) d\nu_\alpha(z)
$$

tends to 0 when $N$ tends to $\infty$ and the same is valid for the Luxembourg norm, so that the sequence of partial sums of the series $\sum f_{a_k,c_k}$ is a Cauchy sequence, which converges and its sum satisfies the required estimate.

So let us prove (3.8). We will make use of the operator

$$
T f(z) = \int_{\mathbb{B}^n} f(w) \frac{d\nu_\beta(w)}{|1 - \langle z, w \rangle|^b}
$$

(recall that $\beta = b - n - 1$). Since $\Phi$ satisfies the $\nabla_2$-condition, the operator $T$, defined in (3.9), is bounded on $L^\Phi_{\alpha}(\mathbb{B}^n)$ (see [3]). We also define the function $F$ as

$$
F(z) = \sum_{k=1}^{+\infty} |c_k|(1 - |a_k|^2)^{-b} \chi_D(a_k,r/2).
$$
The balls \( D(a_k, r/2) \) are disjoint because of the assumption that the sequence \( \{a_k\} \) is \( r \)-separated and so

\[
\int_{\mathbb{B}^n} \Phi(|F(z)|) \, d\nu_\alpha(z) = \sum_{k=1}^{+\infty} \Phi(|c_k|(1 - |a_k|^2)^{-b}) \nu_\alpha(D(a_k, r/2)) \lesssim 1.
\]

This shows that \( F \in L^\Phi_\alpha(\mathbb{B}^n) \). Moreover, because of (2.10), we have that \( \|F\|_{\Phi, \alpha} \lesssim 1 \). Applying \( T \) to \( F \), we obtain

\[
TF(z) = \sum_{k=1}^{+\infty} |c_k|(1 - |a_k|^2)^{-b} \int_{D(a_k, r/2)} \frac{1}{|1 - \langle z, w \rangle|^b} \, d\nu_\beta(w).
\]

Since for \( w \in D(a_k, r/2) \), we have \( 1 - |a_k|^2 \simeq 1 - |w|^2 \) and \( |1 - \langle z, a_k \rangle| \simeq |1 - \langle z, w \rangle| \),

\[
\sum_{k=1}^{+\infty} \frac{|c_k|}{|1 - \langle z, a_k \rangle|^b} \leq C TF(z), \quad z \in \mathbb{B}^n
\]

Using the continuity properties of \( T \) in the Banach space \( L^\Phi_\alpha(\mathbb{B}^n) \), we get (3.8), which we wanted to prove. The inequality (3.7) is obtained by homogeneity. This concludes the proof of the proposition. \( \square \)

It remains to adapt the remaining proof of Theorem 1.2 to the present situation. We now take \( \beta = \alpha \) in the definition of \( S \), and \( s = 1 \) when we use Lemma 2.3. We choose \( \eta \) so that \( I - S \) has a small norm as an operator on the Banach space \( \mathcal{A}^\Phi_\alpha(\mathbb{B}^n) \). We need to assume that \( \int \Phi(|f|) \, d\nu_\alpha \) is bounded by 1 to be able to use Proposition 3.2. We leave the details to the reader. By homogeneity we obtain the same condition on norms as in the concave case:

\[
\|f\|_{\Phi, \alpha} \simeq \inf_{\{a_k\}, \{c_k\}} \left\{ \|\{f_{a_k, c_k}\}\|_{\Phi, \alpha}^{s, b} : f = \sum f_{a_k, c_k} \right\}.
\]

This completes the proof of the theorem.

4. WEAK FACTORIZATION THEOREMS FOR BERGMAN-ORLICZ SPACES

In this section, we use atomic decomposition in order to obtain weak factorization theorems for functions in \( \mathcal{A}^\Phi_\alpha(\mathbb{B}^n) \). We give two types of weak factorization for functions in \( L^\Phi_\alpha(\mathbb{B}^n) \). The first one is in terms of Bloch space and Bergman-Orlicz space and the second one is in terms of two Bergman-Orlicz spaces.
4.1. Products of functions. The proof of Proposition 1.5. We suppose that \( \Phi \) is a growth function of lower type \( p \). Since \( A^{p}_{\alpha}(\mathbb{B}^{n}) \subset A^{p}_{\alpha}(\mathbb{B}^{n}) \), we know that the product of \( f \in A^{p}_{\alpha}(\mathbb{B}^{n}) \) and \( g \in B \) (Bloch space) is well defined as the product of a function in \( A^{p}_{\alpha}(\mathbb{B}^{n}) \) and a function in \( A^{s}_{\alpha}(\mathbb{B}^{n}) \) for all \( 1 < s < \infty \) (since \( B \subset A^{s}_{\alpha}(\mathbb{B}^{n}) \)). So it is a function of \( A^{q}_{\alpha}(\mathbb{B}^{n}) \) for \( q < p \). But this can be replaced by the sharp statement given by Proposition 1.5. The following key lemma shows that the Bloch space is contained in the exponential class in the unit ball.

**Lemma 4.1.** Let \( \Phi(t) = \exp(t) - 1 \). There exists a constant \( C \) such that for any \( f \in B \),

\[
\|f\|_{\Phi, \alpha} \leq C\|f\|_{B}.
\]

**Proof.** It is enough to show that there exists constants \( \lambda > 0 \) and \( C(\lambda) \) such that for any \( f \in B \), with \( \|f\|_{B} \neq 0 \),

\[
\int_{\mathbb{B}^{n}} \exp \left( \frac{|f(z)|}{\lambda\|f\|_{B}} \right) d\nu_{\alpha}(z) \leq C(\lambda).
\]

We know that for \( f \in B \), we have

\[
|f(z)| \leq \log \left( \frac{4}{1 - |z|^{2}} \right) \|f\|_{B}, \quad z \in \mathbb{B}^{n}.
\]

From (4.2), we have

\[
\int_{\mathbb{B}^{n}} \exp \left( \frac{|f(z)|}{\lambda\|f\|_{B}} \right) d\nu_{\alpha}(z) \leq \int_{\mathbb{B}^{n}} \exp \left( \frac{1}{\lambda} \log \left( \frac{4}{1 - |z|^{2}} \right) \right) d\nu_{\alpha}(z)
\]

\[
\leq C \int_{0}^{1} \exp \left( \log \left( \frac{4}{1 - r^{2}} \right)^{\frac{1}{\lambda}} \right) (1 - r^{2})^{\alpha}2rdr
\]

\[
= C4^{1/\lambda} \int_{0}^{1} (1 - r)^{- \frac{2}{4} + \alpha} dr.
\]

We easily obtain (4.1) by taking \( \lambda > \frac{1}{1+\alpha} \). This finishes the proof of the lemma. \( \square \)

Since Lemma 4.1 shows that a function \( f \in B \) is in the exponential class, Proposition 1.5 follows from the use of Hölder inequality for Orlicz spaces (see Proposition 1.4 and \([9]\)).

4.2. Weak factorization with one factor in the Bloch space. This subsection is devoted to the proof of Theorem 1.6.

**Proof.** Let \( \Phi \in \mathcal{L}_{p} \) and let \( \Psi(t) = \Phi \left( \frac{t}{\log(e+t)} \right) \). Since \( t \mapsto \frac{t}{\log(e+t)} \in \mathcal{L}_{p} \), we know that \( \Psi \in \mathcal{L}_{p} \). Let \( f \in A^{\Psi}_{\alpha}(\mathbb{B}^{n}) \). From Theorem 1.2 we know
that there exist a sequence of points \( \{a_k\} \) in \( \mathbb{B}^n \) and a sequence of complex numbers \( \{c_k\} \) such that
\[
f(z) = \sum_{k=1}^{+\infty} \frac{c_k}{(1 - \langle z, a_k \rangle)^b}, \quad z \in \mathbb{B}^n,
\]
with
\[
\int_{\mathbb{B}^n} \Psi(|f(z)|) d\nu_\alpha(z) \simeq \sum_k (1 - |a_k|^2)^{n+1+\alpha} \Psi \left( \frac{|c_k|}{(1 - |a_k|^2)^b} \right).
\]
We assume that \( \|f\|_{\Psi,\alpha} \leq 1 \) and prove (1.15). Let
\[
h := h_k = \frac{c_k}{(1 - \langle z, a_k \rangle)^b}.
\]
We write \( c \) and \( a \) without index for simplification. We want to write each \( h \) as a product \( g_\theta \), with
\[
(1 - |a|^2)^{n+1+\alpha} \Psi \left( \frac{|c|}{(1 - |a|^2)^b} \right) \simeq \left( \int_{\mathbb{B}^n} \Phi(|g(z)|) d\nu_\alpha(z) \right) \|\theta\|_B.
\]
Indeed, if we find such a factorization for each term, the expression of \( f \) as a sum of products that satisfies (1.15) follows at once. The choice of factors will depend on the quantity \( |c|(1 - |a|^2)^{-b} \).

- Assume that \( |c|(1 - |a|^2)^{-b} \leq 4 \). Then
\[
(1 - |a|^2)^{n+1+\alpha} \Phi \left( \frac{|c|}{(1 - |a|^2)^b} \right) \simeq (1 - |a|^2)^{n+1+\alpha} \Psi \left( \frac{|c|}{(1 - |a|^2)^b} \right).
\]

We can take \( g = h \) and \( \theta = 1 \) since the left hand side is equivalent to \( \int_{\mathbb{B}^n} \Phi(|g(z)|) d\nu_\alpha(z) \) by Lemma 2.7.

- Assume that \( |a|^2 \leq 1 - \eta \), for some \( \eta \in (0, 1) \) which will be chosen below. We can take the same choice of factors since we still have \( |c| \leq C(1 - |a|^2)^{b - \frac{n+1+\alpha}{p}} \).

- Assume that \( |c|(1 - |a|^2)^{-b} > 4 \) and \( |a|^2 > 1 - \eta \). Under the first condition, \( \log(e + \frac{|c|}{1 - |a|^2}) \simeq \log(\frac{|c|}{1 - |a|^2}) \). We use the inequality
\[
\sum_k (1 - |a_k|^2)^{n+1+\alpha} \Psi \left( \frac{|c_k|}{(1 - |a_k|^2)^b} \right) \lesssim 1
\]
and the fact that \( \Psi \) is of lower type \( p \) to remark that
\[
|c| \leq C(1 - |a|^2)^{b - \frac{n+1+\alpha}{p}}
\]
for some uniform constant \( C \). So, if we choose \( \eta \) small enough, we have \( |c| < 1 \). Let
\[
\delta := \frac{\log |c|}{b \log(1 - |a|^2)}.
\]
We have $0 < \delta < 1$ and
\[(1 - \delta) \log \left(\frac{4}{1 - |a|^2}\right) \simeq \log \left(\frac{|c|}{(1 - |a|^2)^b}\right).
\]
We choose
\[\theta(z) = 1 + (1 - \delta) \log \left(\frac{4}{1 - \langle a, z \rangle}\right).\]

It is easy to see and classical that $\theta$ is uniformly in the Bloch class, with $\|\theta\|_B \simeq 1$. So to conclude it is sufficient to prove the following lemma, which we use with $\lambda = \frac{c}{2}$. We suppressed the constant 1 before the logarithm for simplicity, which makes no harm for the bound above. The proof is identical for the bound below.

**Lemma 4.2.** Let $\Phi \in L_p$, $a \in B^n$ and $b > \frac{n+1+\alpha}{p}$. Then the function
\[g(z) = \frac{\lambda}{(1 - \langle z, a \rangle)^b \log\left(\frac{4}{1 - \langle a, z \rangle}\right)} \quad (\lambda > 0)
\]
satisfies the inequalities
\[(4.5) \int_{B^n} \Phi(|g(z)|) d\nu_a(z) \simeq (1 - |a|^2)^{n+1+\alpha} \Phi \left( \frac{|\lambda|}{(1 - |a|^2)^b \log\left(\frac{4}{1 - |a|^2}\right)} \right)
\]
uniformly in $a$ and $\lambda$.

**Proof.** This is the analog of Lemma 2.7 but with an extra logarithmic factor. Recall that $|1 - \langle z, a \rangle| \geq 1 - |a|$. It follows that for fixed $\eta > 0$, with $\eta < 1/8$, this factor is bounded below and above when $|a| \leq 1 - \eta$. So it remains to consider the case when $|a| > 1 - \eta$. For the lower bound we have a smaller quantity with the logarithm replaced by $\log\left(\frac{4}{1 - |a|^2}\right)$ which is equivalent to $\log\left(\frac{4}{1 - |a|^2}\right)$. We then use the lower estimate of Lemma 2.7 for the remaining function.

We now proceed to prove the upper bound in (4.5). We mimic the proof of Lemma 2.7 but have now the supplementary factor
\[A := \frac{\log\left(\frac{4}{1 - |a|^2}\right)}{\log\left(\frac{4}{1 - \langle z, a \rangle}\right)}.
\]
It follows from elementary properties of the logarithm that
\[A \leq C_{\varepsilon} \left(\frac{1 - |a|^2}{1 - \langle z, a \rangle}\right)^{-\varepsilon}
\]
for every $\varepsilon > 0$. We choose $\varepsilon$ so that $b - \varepsilon > \frac{n+1+\alpha}{p}$. From this point the proof is the same as for Lemma 2.7 using the fact that the lower type property (1.2) is valid for $t \leq 2^b C_{\varepsilon}$, which is a bound when $t = C_{\varepsilon} \left(\frac{1 - |a|^2}{1 - \langle z, a \rangle}\right)^{b-\varepsilon}$. 

This proves (1.15). To finish the proof of the theorem we need to prove (1.14). By homogeneity we may assume that $\|f\|_{\Psi, \alpha} = 1$. By (1.15), it is sufficient to prove that $\|f_k\|_{\Phi, \alpha} \lesssim \int_{\mathbb{B}^n} \Phi(|f_k(z)|)d\nu_\alpha(z)$ which is a consequence of (2.9).

□

4.3. Application to the characterization of bounded small Hankel operators. As a corollary of Theorem 1.6, we obtain the following characterization of bounded Hankel operators from $A^2_\alpha(\mathbb{B}^n)$ into $A^1_\alpha(\mathbb{B}^n)$. Recall that for $b \in A^2_\alpha(\mathbb{B}^n)$, the small Hankel operator with symbol $b$ is defined for a bounded holomorphic function $f$ by $h_b(f) := P_\alpha(bf)$. Here $P_\alpha$ is the orthogonal projection of the Hilbert space $L^2_\alpha(\mathbb{B}^n)$ onto its closed subspace $A^2_\alpha(\mathbb{B}^n)$, called the Bergman projection, and it is given by

\begin{equation}
(4.6)
P_\alpha(f)(z) = \int_{\mathbb{B}^n} K_\alpha(z, \xi)f(\xi)d\nu_\alpha(\xi),
\end{equation}

where

\begin{equation}
K_\alpha(z, \xi) = \frac{1}{(1 - \langle z, \xi \rangle)^{n+1+\alpha}}.
\end{equation}

Let $\gamma > 0$. We say that a growth function $\rho$ is of restricted upper type $\gamma$ on $[0, 1]$ if there exists a constant $C$ such that

\begin{equation}
(4.7)
\rho(st) \leq Cs^\gamma \rho(t),
\end{equation}

for $s > 1$ and $st \leq 1$. We will call $\rho$ a growth function which is of restricted upper type $\gamma$, for some $\gamma > 0$.

Now for $\alpha > -1$ and a weight $\rho$ (of restricted upper type $\gamma$), we define the weighted Lipschitz space $\Gamma_{\alpha, \rho}(\mathbb{B}^n)$ as the space of holomorphic functions $f$ in $\mathbb{B}^n$ satisfying the following property: for some integer $k > \gamma(n + 1 + \alpha)$, there exists a positive constant $C > 0$ such that

\begin{equation}
|R^k f(z)| \leq C(1 - |z|^2)^{-k}\rho \left((1 - |z|^2)^{n+1+\alpha}\right).
\end{equation}

The Lipschitz space $\Gamma_{\alpha, \rho}(\mathbb{B}^n)$ is a Banach space under the following norm

\begin{equation}
\|f\|_{\Gamma_{\alpha, \rho}(\mathbb{B}^n)} = |f(0)| + \sup_{z \in \mathbb{B}^n} \frac{|R^k f(z)|(1 - |z|^2)^k}{\rho \left((1 - |z|^2)^{n+1+\alpha}\right)}.
\end{equation}

It was proved in [8] that, as in the classical Lipschitz spaces, these spaces are independent of $k$ and they are duals of Bergman-Orlicz spaces with concave Orlicz functions. More precisely, $\Gamma_{\alpha, \rho}(\mathbb{B}^n)$ can be identified as the dual of the Bergman-Orlicz space $A^2_\alpha(\mathbb{B}^n)$ with $\Psi^{-1}(t) = \frac{1}{\rho(t)}$.

Using Theorem 1.6 we recover the following result proved in [8] using a different approach.
**Corollary 4.3.** Let \( \Phi \in L_p \). A Hankel operator \( h_b \) extends to a continuous operator from \( \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \) to \( \mathcal{A}_1^1(\mathbb{B}^n) \) if and only if \( b \in \Gamma_{\alpha, \rho} \), where \( \rho(t) = \frac{1}{\log(e+t)} \) with \( \Psi(t) = \Phi \left( \frac{t}{\log(e+t)} \right) \).

4.4. Weak factorizations with Bergman-Orlicz factors. We give here a weak factorization theorem for functions in \( \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \), with \( \Phi \in L_p \) in terms of products of functions in Bergman-Orlicz spaces.

**Theorem 4.4.** Let \( \Phi \in L_p \cup U_q \). Let \( \Phi_1 \) and \( \Phi_2 \) be two growth functions in either \( L_p \) or \( U_q \) such that

\[
\Phi^{-1} = \Phi_1^{-1} \times \Phi_2^{-1}.
\]

Every function \( f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \) admits a decomposition

\[
f(z) = \sum_{k=1}^{+\infty} g_k(z)h_k(z), \quad z \in \mathbb{B}^n,
\]

where each \( g_k \) is in \( \mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n) \) and each \( h_k \) is in \( \mathcal{A}_\alpha^{\Phi_2}(\mathbb{B}^n) \). Furthermore, if \( \Phi \in L_p \) then

\[
\sum_{k=1}^{+\infty} \|g_k\|_{\Phi_1, \alpha}\|h_k\|_{\Phi_2, \alpha} \leq C\|f\|_{\Phi, \alpha},
\]

where \( C \) is a positive constant independent of \( f \).

**Proof.** Let \( f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \). We know from Theorem 1.2 and Theorem 1.3, that there exists a sequence \( \{a_k\} \) in \( \mathbb{B}^n \) such that every \( f \in \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \) admits the following representation

\[
f(z) = \sum_k \frac{c_k}{(1 - \langle z, a_k \rangle)^\Phi}.
\]

where \( \{a_k\}, \{c_k\} \) belongs to the space \( l_\alpha^\Phi \) and the series converges in the norm topology of \( \mathcal{A}_\alpha^\Phi(\mathbb{B}^n) \). We have

\[
\sum_{k=1}^{+\infty} (1 - |a_k|^2)^{n+1+\alpha} \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^\Phi} \right) < \infty.
\]

Now take, for non zero \( c_k \),

\[
g_k(z) = \frac{(1 - |a_k|^2)^{bs}}{(1 - \langle z, a_k \rangle)^{bs} \Phi_1^{-1}} \left( \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^\Phi} \right) \right) e^{i\text{Arg}(c_k)}
\]

and

\[
h_k(z) = \frac{(1 - |a_k|^2)^{bt}}{(1 - \langle z, a_k \rangle)^{bt} \Phi_2^{-1}} \left( \Phi \left( \frac{|c_k|}{(1 - |a_k|^2)^\Phi} \right) \right)
\]

where \( s, t > 0 \) with \( s + t = 1 \). It is clear, using (4.8), that (4.9) holds. Using Lemma 2.7 or Remark 2.8, we easily see that \( g_k \in \mathcal{A}_\alpha^{\Phi_1}(\mathbb{B}^n) \) and \( h_k \in \mathcal{A}_\alpha^{\Phi_2}(\mathbb{B}^n) \).
It remains to prove (4.10). Let \( \Phi \in \mathcal{L}_p \). By homogeneity, we may suppose \( \|f\|_{\Phi, \alpha} = 1 \). We then have to show that there exists a constant \( C \), independent of \( f \), so that

\[
\sum_k \|g_k\|_{\Phi_1, \alpha} \|h_k\|_{\Phi_2, \alpha} \leq C.
\]

Using Lemma 2.10 or Lemma 2.11 and Theorem 2.6 again, with \( b \) large enough, we have, for \( \Phi_1, \Phi_2 \in \mathcal{L}_p \cup \mathcal{U}_q \)

\[
\|g_k\|_{\Phi_1, \alpha} \leq C \frac{\Phi_1^{-1} \left( \frac{|c_k|}{(1-|a_k|^2)^b} \right)}{\Phi_1^{-1} \left( \frac{1}{(1-|a_k|^2)^{n+1+\alpha}} \right)} \tag{4.12}
\]

and

\[
\|h_k\|_{\Phi_2, \alpha} \leq C \frac{\Phi_2^{-1} \left( \frac{|c_k|}{(1-|a_k|^2)^b} \right)}{\Phi_2^{-1} \left( \frac{1}{(1-|a_k|^2)^{n+1+\alpha}} \right)} \tag{4.13}
\]

Now, using (4.12), (4.13) and (4.8), we have

\[
\sum_k \|g_k\|_{\Phi_1, \alpha} \|h_k\|_{\Phi_2, \alpha} \leq C + \sum_k d_k = C \sum_k \Phi_1^{-1} \left( \frac{d_k}{(1-|a_k|^2)^{n+1+\alpha}} \right),
\]

where \( d_k = (1-|a_k|^2)^{n+1+\alpha} \Phi_1^{-1} \left( \frac{|c_k|}{(1-|a_k|^2)^b} \right) \). Since \( \|f\|_{\Phi, \alpha} = 1 \), there exists a uniform constant \( C \) such that \( \int_{\mathbb{R}^n} \Phi(|f(z)|) d\nu(z) \leq C \) (see (2.9)). By (4.11) the series \( \{d_k\} \) converges in \( l^1 \). This implies, without loss of generality that we may assume \( \{d_k\} \) is bounded by 1. Since \( u \mapsto \frac{\Phi(u)}{u} \) is non-increasing, we have that \( u \mapsto \frac{\Phi^{-1}(u)}{u} \) is non-decreasing, hence

\[
\Phi^{-1}(uv) \leq u\Phi^{-1}(v), \quad 0 \leq u \leq 1, \quad v \geq 0.
\]

From this, we have

\[
\sum_k \|g_k\|_{\Phi_1, \alpha} \|h_k\|_{\Phi_2, \alpha} \leq C \sum_{k=1}^{+\infty} d_k = C \sum_{k=1}^{+\infty} (1-|a_k|^2)^{n+1+\alpha} \Phi_1^{-1} \left( \frac{|c_k|}{(1-|a_k|^2)^b} \right) \leq C.
\]

This finishes the proof. \( \square \)
Remark 4.5. Weak factorization theorems with Bergman-Orlicz factors in the case where $\Phi \in \mathcal{U}$ are considered in an upcoming paper by the third author and R. Zhao.

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