SOME INTEGRAL TRANSFORMS OF THE GENERALIZED $k$-MITTAG-LEFFLER FUNCTION

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Abstract. We generalize the notion "$k$-Mittag-Leffler function", establish some integral transforms of the generalized $k$-Mittag-Leffler function, and derive several special and known conclusions in terms of the generalized Wright function and the generalized $k$-Wright function.

1. Preliminaries

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ denote respectively the sets of complex numbers, real numbers, non-negative numbers, positive numbers, non-positive integers, and positive integers.

The Pochhammer symbol $(\lambda)_\nu$ can be defined for $\lambda, \nu \in \mathbb{C}$ by $(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$, where

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{\prod_{k=1}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \mathbb{Z}_0^-$$

is called the classical gamma function and its reciprocal $\frac{1}{\Gamma}$ is analytic on the whole complex plane $\mathbb{C}$. See [14, Chapter 5], [16, Section 1], and [26, Section 1.1]. In particular, when $\nu \in \{0\} \cup \mathbb{N}$, the quantity

$$(\lambda)_n = \begin{cases} 1, & \nu = 0 \\ \lambda(\lambda+1) \cdots (\lambda+n-1), & n \in \mathbb{N} \end{cases}$$

is called the rising factorial. See [18] and closely-related references therein.

The $k$-Pochhammer symbol $(\lambda)_{n,k}$ was defined in [2] for $\lambda, \nu \in \mathbb{C}$ and $k \in \mathbb{R}$ by

$$(\lambda)_{\nu,k} = \frac{\Gamma_k(\lambda+\nu k)}{\Gamma_k(\lambda)}, \quad \nu \in \mathbb{C}$$

where

$$(\lambda)_{\nu,k} = \frac{1}{\Gamma_k(\lambda)} \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda)_{\nu,n}}{n!} (\nu k)^n$$

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is called the \(k\)-gamma function. In particular,

\[
(\lambda)_{n,k} = \begin{cases} 
1, & n = 0; \\
\lambda(\lambda + k) \cdots (\lambda + (n - 1)k), & n \in \mathbb{N}.
\end{cases}
\]

In 1903, Mittag-Leffler, a Swedish mathematician, introduced and investigated in [12][13] the so-called Mittag-Leffler function

\[
E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}
\]

for \(z \in \mathbb{C}\) and \(\alpha \in \mathbb{R}^+_0\). In 1905, Wiman [27] generalized \(E_\alpha(z)\) as

\[
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}
\]

where \(z \in \mathbb{C}, \alpha, \beta \in \mathbb{C}\), and \(\text{Re}(\alpha), \text{Re}(\beta) > 0\). In 1971, Prabhakar [15] introduced the function

\[
E_{k,\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\gamma)^n_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!},
\]

where \(\alpha, \beta, \gamma, \delta, \tau \in \mathbb{C}, k \in \mathbb{R}, \text{Re}(\alpha), \text{Re}(\beta), \text{Re}(\gamma) > 0\), and \((\gamma)^n_{n,k}\) is the \(k\)-Pochhammer symbol. In 2012, Dorrego and Cerutti [3] introduced the \(k\)-Mittag-Leffler function

\[
GE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)^n_{n,q,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}
\]

for \(\alpha, \beta, \gamma, q \in \mathbb{C}, k \in \mathbb{R}, \text{Re}(\alpha), \text{Re}(\beta), \text{Re}(\gamma) > 0\), and \(q \in (0,1) \cup \mathbb{N}\) was introduced and studied in [6]. For more information on generalizations of the Mittag-Leffler function, please refer to the papers [19][21][23] and closely-related references therein.

In this paper, we consider a more general generalization

\[
E_{k,\alpha,\beta,\delta,\tau}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)^n_{n,\tau,k}}{\Gamma_k(\alpha n + \beta) (\delta)_{\tau}} \frac{z^n}{n!}
\]

where \(\alpha, \beta, \gamma, \delta, \tau \in \mathbb{C}, k \in \mathbb{R}, \text{Re}(\alpha), \text{Re}(\beta) > 0\), and \(\delta \neq 0, -1, -2, \ldots\). It is clear that \(E_{k,\alpha,\beta,1}^{\gamma,\delta,1}(z) = GE_{k,\alpha,\beta}^{\gamma,q}(z)\) and \(E_{k,\alpha,\beta,1}^{\gamma,1}(z) = E_{k,\alpha,\beta}^{\gamma,1}(z)\).

It is well known [14][19] that the generalized hypergeometric function can be defined by

\[
_{p}F_{q}[(\alpha_p); (\beta_q); z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n} \frac{z^n}{n!}
\]
for $|z| < 1$ and $p \leq q$ with $p = q + 1$ and that the generalized Wright hypergeometric function $\Psi_q(z)$ is given by the series

$$p\Psi_q(z) = p\Psi_q[(a_1, \alpha_1)_{1,p}; (b_j, \beta_j)_{1,q}; z] = \frac{\prod_{i=1}^p \Gamma(\beta_j)}{\prod_{i=1}^p \Gamma(\alpha_i)} \sum_{k=0}^\infty \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}$$

for $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}$ with $1 \leq i \leq p$ and $1 \leq j \leq q$. Asymptotic behavior of the function $p\Psi_q(z)$ for large values of argument of $z \in \mathbb{C}$ were studied in [5, 28, 29].

Now we are in a position to state and prove our main results.

The aim of this paper is to present the Euler, Laplace, Whittaker, and Fractional Fourier transforms of the generalized $k$-Mittag-Leffler function [10, 20]. From these conclusions, we can derive some known and new results.

2. Main results

Now we are in a position to state and prove our main results.
Theorem 2.1. If \( k \in \mathbb{R}, \alpha, \beta, \gamma, a, b, \sigma \in \mathbb{C}, \text{Re}(\alpha), \text{Re}(\beta) > 0, \delta \neq 0, -1, -2, \ldots, \) and \( q > 0, \) then

\[
(2.1) \quad \int_{0}^{1} z^{a-1}(1 - z)^{b-1} E_{k, \alpha, \beta, \delta}^{\gamma, \tau}(xz) \, dz = \frac{k^{1-\beta/k} \Gamma(b) \Gamma(\delta)}{\Gamma(\tau)} 2 \Psi_2 \left[ \left( \frac{\gamma}{k}, \tau \right), (a, \sigma); \left( \frac{\alpha}{k}, \frac{\beta}{k} \right), (a + b, \sigma); k^{-\alpha/k} x \right].
\]

Proof. Denote the left-hand side of the equation (2.1) by \( I_1. \) By definition of the generalized \( k-\)Mittag-Leffler function and (1.5), we have

\[
I_1 = \int_{0}^{1} z^{a-1}(1 - z)^{b-1} E_{k, \alpha, \beta, \delta}^{\gamma, \tau}(xz) \, dz = \int_{1}^{0} z^{a-1}(1 - z)^{b-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n \tau, k}}{\Gamma(k(n+\beta))} (xz)^n \, dz.
\]

By interchanging the order of the integration and summation, we obtain

\[
I_1 = \sum_{n=0}^{\infty} \frac{(\gamma)_{n \tau, k}}{\Gamma(k(n+\beta))} \Gamma(a + \sigma n) \Gamma(b) \Gamma(a + b + \sigma n) k^{\alpha n/k} \Gamma(\beta k + \alpha n/k).
\]

From (1.4) and (1.5), we acquire

\[
I_1 = k^{1-\beta/k} \Gamma(b) \Gamma(\delta) \sum_{n=0}^{\infty} \frac{\Gamma(\frac{n}{k} + n \tau) \Gamma(\sigma + n \tau) \Gamma(\sigma + n) \Gamma(\delta + n) \Gamma(\alpha n + \beta)}{\Gamma(\frac{n}{k} + \frac{n}{k}) \Gamma(\sigma + n \tau) \Gamma(\sigma + n) \Gamma(\delta + n) \Gamma(\alpha n + \beta)} k^{\alpha n/k}.
\]

In view of (1.4), we arrive at the desired result. \( \square \)

Remark 2.1. Taking \( \delta = 1 \) in Theorem 2.1 gives \[25, \text{Eq. (24)}\] which reads that

\[
\int_{0}^{1} z^{a-1}(1 - z)^{b-1} E_{k, \alpha, \beta, \delta}^{\gamma, \tau}(xz) \, dz = \frac{k^{1-\beta/k} \Gamma(b)}{\Gamma(\tau)} 2 \Psi_2 \left[ \left( \frac{\gamma}{k}, \tau \right), (a, \sigma); \left( \frac{\alpha}{k}, \frac{\beta}{k} \right), (a + b, \sigma); k^{-\alpha/k} x \right].
\]

Setting \( \delta = 1 \) and \( \tau = q > 0 \) in Theorem 2.1 leads to \[25, \text{Eq. (25)}\] which states that

\[
\int_{0}^{1} z^{a-1}(1 - z)^{b-1} E_{k, \alpha, \beta, \delta}^{\gamma, \tau}(xz) \, dz = \frac{k^{1-\beta/k} \Gamma(b)}{\Gamma(\tau)} 2 \Psi_2 \left[ \left( \frac{\gamma}{k}, q \right), (a, \sigma); \left( \frac{\alpha}{k}, \frac{\beta}{k} \right), (a + b, \sigma); k^{\alpha/q} x \right].
\]
Further letting $k = 1$ in the above equation derives \([25\text{ Eq. (26)}]\) which formulates that

$$
\int_0^1 z^{a-1}(1-z)^{b-1}E_{\alpha,q}^{\gamma,q}(xz^\alpha)dz = \frac{\Gamma(b)}{\Gamma(\gamma)^2} \Psi_2[(\gamma,q),(a,\sigma);(\beta,a),(a+b,\sigma);x].
$$

**Theorem 2.2.** If $k \in \mathbb{R}$, $\alpha, \beta, \gamma, a, \sigma \in \mathbb{C}$, $\text{Re}(\alpha), \text{Re}(\beta), \text{Re}(s) > 0$, $\tau \in \mathbb{C}$, $|\frac{\alpha}{k}| < 1$, and $\delta \neq 0, -1, -2, \ldots$, then

$$
\int_0^\infty z^{a-1}e^{-sz}E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\alpha)dz
= \frac{k^{1-\beta/k}\Gamma(\delta)}{s^{\alpha} \Gamma(\frac{k}{k})^{\frac{k}{k}}} \Psi_2\left[\left(\frac{\gamma}{k},\tau\right), (a,\sigma),(1,1);\left(\frac{\beta}{k},\frac{\alpha}{k}\right), (\delta,1); \frac{xk^{\tau-a/k}}{s^{\sigma}}\right].
$$

**Proof.** Denote the left-hand side of (2.2) by $I_2$. Applying definition of the generalized $k$-Mittag-Leffler function results in

$$
I_2 = \int_0^\infty z^{a-1}e^{-sz}E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\alpha)dz = \int_0^\infty z^{a-1}e^{-sz}\sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{(xz^\alpha)^n}{\delta_n}dz.
$$

Interchanging the order of the integration and summation leads to

$$
I_2 = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{x^n}{\delta_n} \int_0^\infty z^{a+\sigma n-1}e^{-sz}dz.
$$

In view of definition of the Laplace transform, we have

$$
I_2 = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{x^n}{\delta_n} \frac{\Gamma(\sigma n + a)}{s^{\sigma n + a}}.
$$

Utilizing \([141]\) and \([122]\) derives the required result. \(\square\)

**Remark 2.2.** If setting $\delta = 1$ in Theorem 2.2, then we deduce \([25\text{ Eq. (27)}]\) which formulates that

$$
\int_0^\infty z^{a-1}e^{-sz}E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\alpha)dz = \frac{k^{1-\beta/k}s^{-a}}{\Gamma(\gamma)} \Psi_2\left[\left(\frac{\gamma}{k},\tau\right), (a,\sigma),(\beta,\alpha), \frac{xk^{\tau-a/k}}{s^{\sigma}}\right].
$$

If taking $\tau = q > 0$, $k = 1$, and $\delta = 1$, then we acquire \([25\text{ Eq. (29)}]\) which reads that

$$
\int_0^\infty z^{a-1}e^{-sz}E_{\alpha,\beta}^{\gamma,q}(xz^\alpha)dz = \frac{s^{-a}}{\Gamma(\gamma)} 2\Psi_1\left[\left(\gamma,q\right), (a,\sigma),(\beta,\alpha), \frac{x}{s^{\sigma}}\right].
$$

If taking $k = q = 1$ and $\delta = 1$ in the above equation reduces to

$$
\int_0^\infty z^{a-1}e^{-sz}E_{\alpha,\beta}^{\gamma}(xz^\alpha)dz = \frac{s^{-a}}{\Gamma(\gamma)} 2\Psi_1\left[\left(\gamma,1\right), (a,\sigma),(\beta,\alpha), \frac{x}{s^{\sigma}}\right]
$$

which is the main result in \([24]\).
Recall that
\[(2.3) \int_{t=0}^{\infty} t^{\nu-1} e^{-t/2} W_{\lambda,\mu}(t) dt = \frac{\Gamma(\frac{1}{2} + \mu + v)\Gamma(\frac{1}{2} - \mu + v)}{\Gamma(1 - \lambda + v)}, \quad \text{Re}(v + \mu) > -\frac{1}{2},\]
where the Whittaker function
\[W_{\lambda,\mu}(t) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \lambda)} M_{\lambda,\mu}(t) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \lambda)} M_{\lambda,-\mu}(t)\]
and \(M_{\lambda,\mu}(t) = z^{\mu + 1/2} e^{-t/2} I_1(\frac{1}{2} + \mu + v; 2\mu + 1; t)\) are given in [11].

**Theorem 2.3.** If \(k \in \mathbb{R}, \alpha, \beta; \gamma, \delta, \tau, \eta \in \mathbb{C}, \text{Re}(\alpha), \text{Re}(\beta), \text{Re}(\rho) > 0, \delta \neq 0, -1, \ldots, \) and \(\text{Re}(\rho \pm \mu) > -\frac{1}{2}, \) then
\[
\int_{t=0}^{\infty} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\Gamma,\beta,\gamma}(wt^n) dt = \frac{k^{1-\beta / \rho} p^{-\rho} \Gamma(\delta)}{\Gamma(\frac{1}{2})} \Psi_2 \left[ \left( \frac{\gamma}{k}, \tau \right), \left( \frac{1}{2} \pm \mu + \rho, \eta \right); \left( \frac{\beta}{k}, \frac{\alpha}{k} \right), \left( 1 - \lambda + \rho, \eta \right); \frac{w^{\rho - \alpha / k}}{p^n} \right].
\]

**Proof.** Letting \(pt = v, \) interchanging the integration and summation, and using the formula for the Whittaker transform (2.3) yields
\[
\int_{t=0}^{\infty} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\Gamma,\beta,\gamma}(wt^n) dt = \sum_{n=0}^{\infty} e^{-v/2} \left( \frac{v}{p} \right)^{\rho-1} \frac{w^n}{\Gamma(k(\alpha + \beta) + \delta)} \frac{\delta n 1}{p^n} dv
\]
\[
= \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2} + \mu + v)\Gamma(\frac{1}{2} - \mu + v)}{\Gamma(1 - \lambda + v)} \frac{w^n}{\Gamma(k(\alpha + \beta) + \delta)} \frac{\delta n 1}{p^n} dv
\]
\[
= \frac{1}{p^n} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2} + \mu + \delta n + \rho)\Gamma(\frac{1}{2} - \mu + \delta n + \rho)}{\Gamma(k(\alpha + \beta) + \delta)} \frac{w^n}{\Gamma(1 - \lambda + \delta n + \rho)} \frac{\delta n 1}{p^n} dv
\]
\[
= \frac{1}{p^n} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2} + \mu + \delta n + \rho)\Gamma(\frac{1}{2} - \mu + \delta n + \rho)}{\Gamma(k(\alpha + \beta) + \delta)} \frac{w^n}{\Gamma(1 - \lambda + \delta n + \rho)} \frac{\delta n 1}{p^n} dv.
\]
In view of (1.1) and (1.2), we find the desired result. \(\square\)

**Remark 2.3.** Taking \(\delta = 1\) in Theorem 2.3 gives [25] Eq. (30)] which reads that
\[
\int_{t=0}^{\infty} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\Gamma,\beta,\gamma}(wt^n) dt = \frac{k^{1-\beta / \rho} p^{-\rho}}{\Gamma(\frac{1}{2})} \Psi_2 \left[ \left( \frac{\gamma}{k}, \tau \right), \left( \frac{1}{2} \pm \mu + \rho, \eta \right); \left( \frac{\beta}{k}, \frac{\alpha}{k} \right), \left( 1 - \lambda + \rho, \eta \right); \frac{w^{\rho - \alpha / k}}{p^n} \right].
\]
Setting \(\tau = q > 0, k = 1, \) and \(\delta = 1\) results in [25] Eq. (32)] which states that
\[
\int_{t=0}^{\infty} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{\alpha,\beta,\gamma}(wt^n) dt
\]
= \frac{1}{\mu^{\beta/k}} \binom{\alpha + \beta}{\beta/k} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{\alpha + \beta}{\beta/k} + n\right)}{\Gamma\left(\frac{\alpha + \beta}{\beta/k} + n\right)} \left(\frac{\lambda}{\alpha + \beta}\right)^n \lambda^n \Gamma\left(\frac{\alpha + \beta}{\beta/k} + n\right).$

Letting $q = 1$ in the above equation derives a result given in [24].

**Theorem 2.4.** If $k \in \mathbb{R}$, $\alpha, \beta, \gamma, a, b, \sigma \in \mathbb{C}$, $\text{Re}(\alpha), \text{Re}(\beta) > 0$, $\tau \in \mathbb{C}$, and $\delta \neq 0, -1, -2, \ldots$, then

$$\text{Im}_\sigma [E_{k, a, b, \delta}^{\gamma, \tau} (t)](w) = \frac{k^{1-\beta/k}(1/n(k+\beta)\alpha)\Gamma(\frac{\alpha + \beta}{\beta/k} + n\tau)\lambda^{n-1}w^{-(n+1)/\sigma}}{\Gamma(\frac{\alpha + \beta}{\beta/k} + n\tau + \frac{\alpha}{k})}.$$  

**Proof.** Using definitions of the generalized $k$-Mittag-Leffler function and the fractional Fourier transform and interchanging the integration and summation give

$$\text{Im}_\sigma [E_{k, a, b, \delta}^{\gamma, \tau} (t)](w) = \int_0^1 \exp(iw^{1/\sigma})E_{k, a, b, \delta}^{\gamma, \tau} (t) dt$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + \beta)(\delta)_n}{\Gamma(k(\alpha + \beta)(\delta)_n)} \int_R \exp(iw^{1/\sigma})t^n dt.$$  

Letting $iw^{1/\sigma} = -\eta$ reduces to

$$\text{Im}_\sigma [E_{k, a, b, \delta}^{\gamma, \tau} (t)](w) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + \beta)(\delta)_n}{\Gamma(k(\alpha + \beta)(\delta)_n)} \int_0^0 \exp(\eta)(\eta^{1/\sigma}) \eta^{-n} \eta^{d\eta}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + \beta)(\delta)_n}{\Gamma(k(\alpha + \beta)(\delta)_n)} \int_0^\infty e^{-\eta^n} \eta^n d\eta$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + \beta)(\delta)_n}{\Gamma(k(\alpha + \beta)(\delta)_n)} \int_0^\infty e^{-\eta^n} \eta^n d\eta.$$  

Further using formulas [1.1] and [1.2] arrives at the required result. \qed

**Remark 2.4.** If taking $\delta = 1$ in Theorem 2.3 then the equation

$$\text{Im}_\sigma [E_{k, a, b, \delta}^{\gamma, \tau} (t)](w) = \frac{k^{1-\beta/k}(1/n(k+\beta)\alpha)\Gamma(\frac{\alpha + \beta}{\beta/k} + n\tau)\lambda^{n-1}w^{-(n+1)/\sigma}}{\Gamma(\frac{\alpha + \beta}{\beta/k} + n\tau + \frac{\alpha}{k})},$$

in [25] Eq. (33)] follows readily.

If setting $\delta = 1$ and $\tau = q$ in Theorem 2.3 then the equation

$$\text{Im}_\sigma [E_{k, a, b, \delta}^{\gamma, \tau} (t)](w) = \frac{k^{1-\beta/k}(1/n(k+\beta)\alpha)\Gamma(\frac{\alpha + \beta}{\beta/k} + nq)\lambda^{n-1}w^{-(n+1)/\sigma}}{\Gamma(\frac{\alpha + \beta}{\beta/k} + nq + \frac{\alpha}{k})},$$

in [25] Eq. (34)] can be derived immediately.

If letting $\delta = k = 1$ and $\tau = q$ in Theorem 2.3 then

$$\text{Im}_\sigma [E_{k, a, b, \delta}^{\gamma, \tau} (t)](w) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} (-1)^n \Gamma(\gamma + n\tau)\lambda^{n-1}w^{-(n+1)/\sigma}$$

in [25] Eq. (34)] can be deduced straightforwardly.

**Remark 2.5.** In [8] Section 3], the quantity $t^k$ for $k \in \mathbb{N}$ was computed generally by three approaches.
3. Concluding remarks

Some integral transforms of the generalized $k$-Mittag-Leffler function are established and the results are expressed in terms of the generalized Wright function. By taking $\delta = 1$ and using formulas (1.1) and (1.2), we express Theorems 2.1 to 2.3 in terms of the generalized $k$-Wright function as follows.

It is noted that, using the appropriate formulas mentioned in Section 1, one can easily express the Euler integral in terms of the $k$-Wright function as

$$\int_{0}^{1} \frac{z^{a-1}(1-z)^{b-1}}{E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma)} \, dz = \frac{\Gamma(b)k^b}{\Gamma_k(\gamma)} \Psi_2^{b}[(\gamma, \tau k), (ak, \sigma k); (\beta, \alpha), ((a+b)k, \sigma k); x].$$

By applying suitable formula for the $k$-gamma function, Theorem 2.2 can be expressed in terms of the $k$-Wright function as

$$\int_{0}^{1} e^{-xz} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma) \, dz = \frac{k^{2-\gamma/k}}{(sk)\Gamma_k(\gamma)} \Psi_2^{b}[(\gamma, \tau k), (ak, \sigma k); (\beta, \alpha), \frac{x}{(ks)^\sigma}].$$

Theorem 2.3 can be expressed in terms of the $k$-Wright function as

$$\int_{0}^{1} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\alpha,\beta}^{\gamma,\tau}(wt^\eta) \, dt = \frac{p-\rho k^{1-\rho-\lambda}}{\Gamma_k(\gamma)} \times \Psi_2^{b}[(\gamma, \tau k), \left(\frac{1}{2} \pm \mu + \rho\right)k, \eta k]; (\beta, \alpha), ((1-\lambda+\rho)k, \eta k); \frac{xk^{\tau-\eta-\alpha/k}}{p^{\eta}}].$$

Remark 3.1. This paper is a slightly revised version of the preprint [17].

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