Isotonicity of the projection onto the monotone cone

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Abstract

A wedge (i.e., a closed nonempty set in the Euclidean space stable under addition and multiplication with non-negative scalars) induces by a standard way a semi-order (a reflexive and transitive binary relation) in the space. The wedges admitting isotone metric projection with respect to the semi-order induced by them are characterized. The obtained result is used to show that the monotone wedge (called monotone cone in regression theory) admits isotone projection.

1. Introduction

The metric projection onto convex cones is an important tool in solving problems in metric geometry, statistics, image reconstruction etc. The idea to relate the ordering induced by the convex cone and the metric projection onto the convex cone goes back to the paper [3] of G. Isac and A. B. Németh, where a convex cone in the Euclidean space which admits an isotone projection onto it (called by the authors isotone projection cone) was characterized. The isotonicity is considered with respect to the order induced by the convex cone. This notion was considered in the context of the complementarity theory where the isotonicity of the projection provides new existence results and iterative methods [4, 5, 7].

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It turns out that the isotonicity of the projection is a very strong requirement which implies the latticiality of the order induced by the convex cone. Thus, the investigation of the isotone projection cones becomes part of the theory of latticially ordered Euclidean and Hilbert spaces.

A simple finite method of projection onto isotone projection cones proposed by us (see [6]) has become important in the effective handling of all the problems involving projection onto these cones. Besides nonlinear complementarity, isotone projection cones have applications in other domains of optimization theory. The positive monotone convex cone used in the Euclidean distance geometry (see [2]) is an isotone projection one. Our method has become important in the effective handling of the problem of map-making from relative distance information e.g., stellar cartography (see www.convexoptimization.com/wikimization/index.php/Projection_on_Polyhedral_Cone and Section 5.13.2.4 in [2]).

Although we shall not consider projection methods in this note, some of the results developed in [6] will be useful in our proofs. The notion of the cone in the above cited papers is used in the sense of “closed convex pointed cone”. Confronted with the question if the so called monotone cone (which is in fact a wedge in our terminology) used in regression theory admits or not an isotonic metric projection onto it (where isotonicity is considered with respect to the semi-order the monotone cone introduces), we shall develop a general theory in order to apply it to this special case. This seems to be the simplest way to tackle this problem. By using this approach, it turns out that the monotone cone indeed admits an isotone metric projection onto it.

2. Projecting onto closed wedges in $\mathbb{R}^m$

If $C$ is a non-empty, closed convex set in $\mathbb{R}^m$, then for each $x \in \mathbb{R}^m$ there exists a unique nearest point $P_Cx \in C$, that is, a point with the property that

$$\|x - P_Cx\| = \inf\{\|x - c\| : c \in C\},$$

where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^m$ ([8]).

The mapping $P_C : \mathbb{R}^m \to C$ is called the nearest point mapping of $\mathbb{R}^m$ onto $C$ or simply the (metric) projection onto $C$.

Let $W$ be a wedge in $\mathbb{R}^m$, i.e., a closed nonempty set with (i) $W + W \subset W$ and (ii) $tW \subset W, \forall t \in \mathbb{R}_+ = [0, +\infty)$. If $W \cap (-W) = \{0\}$, then $W$ is called a cone.

Lemma 1 Suppose $L = W \cap (-W)$ (the maximal subspace contained in $W$) and let $L^\perp$ be its orthogonal complement. Denote $K = L^\perp \cap W$. Then $K$ is a cone in $L^\perp$,

$$W = K \oplus L$$

(1)

where $\oplus$ stands for the orthogonal sum, and

$$P_Wx = P_Kx_k + x_l$$

(2)

where $x = x_k + x_l$ with $x_l \in L$ and $x_k \in L^\perp$. 


Proof. The relation (1) follows directly from
\[ W = W \cap (L^\perp \oplus L). \]
It is known ([8]) that the projection \( P_W x \) of \( x \) onto the wedge \( W \) is characterized by the couple of relations:
\[ \langle x - P_W x, y \rangle \leq 0, \quad \forall y \in W; \tag{3} \]
and
\[ \langle x - P_W x, P_W x \rangle = 0. \tag{4} \]
Hence, we have to verify the above relations for \( P_K x_k + x_l \) instead of \( P_W x \).

By the relation (1) \( P_K x_k + x_l \in W \).

Take an arbitrary \( y \in W \) represented by (1) in the form
\[ y = y_k + y_l \]
with \( y_k \in K \) and \( y_l \in L \). Then we have
\[ \langle x_k + x_l - (P_K x_k + x_l), y_k + y_l \rangle = \langle x_k - P_K x_k, y_k \rangle \leq 0, \quad \forall y = y_k + y_l \in W, \]
because \( y_l \) is perpendicular to \( x_k - P_K x_k \in L^\perp \), and because of the relation similar to (3) characterizing the projection of \( x_k \) onto the cone \( K \) in \( L^\perp \). Thus, relation (3) holds for \( P_K x_k + x_l \) in place of \( P_W x \).

We further have
\[ \langle x_k + x_l - (P_K x_k + x_l), P_K x_k + x_l \rangle = \langle x_k - P_K x_k, P_K x_k \rangle = 0 \]
because \( x_l \) is perpendicular to \( x_k - P_K x_k \) and because of the relation similar to (4) applied to \( x_k \in L^\perp \) and its projection onto \( K \).

The obtained relation is exactly (4) for \( P_K x_k + x_l \) instead of \( P_W x \). \( \square \)

3. The isotonicity of the projection onto a closed wedge in \( \mathbb{R}^m \)

By putting \( u \leq_W v \) whenever \( u, v \in \mathbb{R}^m \) and \( v - u \in W \), the wedge \( W \subset \mathbb{R}^m \) induces a semiorder \( \leq_W \) in \( \mathbb{R}^m \) which is translation invariant (i.e. \( u \leq_W v \) implies \( u + z \leq_W v + z \) for any \( z \in \mathbb{R}^m \)) and scale invariant (i.e. \( u \leq_W v \) implies \( tu \leq_W tv \) for any \( t \in \mathbb{R}_+ \)).

The projection \( P_W \) is said \( W \)-isotone if \( u, v \in \mathbb{R}^m \), \( u \leq_W v \) implies \( P_W u \leq_W P_W v \). If \( P_W \) is \( W \)-isotone, then \( W \) is called an isotone projection wedge. A cone \( K \) is called an isotone projection cone if it is an isotone projection wedge.

Theorem 1 Let \( W \subset \mathbb{R}^m \) be a wedge,
\[ W = K \oplus L \]
with \( L = W \cap (-W) \) and \( K = W \cap L^\perp \). Then \( W \) is an isotone projection wedge if and only if \( K \subset L^\perp \) is an isotone projection cone in \( L^\perp \).
Proof. Take \( u, v \in L^\perp \). Then, \( u \leq_K v \) is equivalent to \( u \leq_W v \). If \( P_W \) is \( W \)-isotone, then \( u \leq_W v \) implies by Lemma 1
\[
P_K v - P_K u = P_W v - P_W u \in W.
\]
Since \( P_K u, P_K v \in L^\perp \), it follows that
\[
P_K v - P_K u \in L^\perp \cap W = K.
\]
The obtained relation shows that \( P_K \) is \( K \)-isotone, concluding the proof of the necessity of the theorem.

Suppose now that \( P_K \) is \( K \)-isotone and take \( u, v \in \mathbb{R}^m \) with \( u \leq_W v \). If \( u = u_k + u_l \) and \( v = v_k + v_l \) with \( u_k, v_k \in L^\perp \), and \( u_l, v_l \in L \), then using formula (1)
\[
v - u = v_k - u_k + v_l - u_l \in K \oplus L
\]
and hence \( v_k - u_k \in K \), that is \( u_k \leq_K v_k \) and by the \( K \)-isotonicity of \( P_K \) it follows that
\[
P_K v_k - P_K u_k \in K.
\]
Hence, using formula (2) we have
\[
P_W v - P_W u = P_K v_k + v_l - P_K u_k - u_l = P_K v_k - P_K u_k + v_l - u_l \in K \oplus L = W.
\]
That is \( P_W u \leq_W P_W v \), which concludes the isotonicity of \( P_W \). \qed

A simple geometric characterization of the isotone projection cones was given in [3]. It uses the notion of the polar of a wedge.

If \( W \subset \mathbb{R}^m \) is a wedge, then the set
\[
W^\perp = \{ y \in \mathbb{R}^m : \langle x, y \rangle \leq 0, \forall x \in W \},
\]
is called the polar of the wedge \( W \). The set \( W^\perp \) is obviously a wedge. If the wedge \( W \) is generating in the sense that \( W - W = \mathbb{R}^m \), then the polar \( W^\perp \) is a cone.

We have the following easily verifiable result:

**Lemma 2** Suppose that \( W \) is a generating wedge. Using the notations introduced in the Theorem 1 and denoting the polar of the cone \( K \) in the subspace \( L^\perp \) by \( K^\perp \), we have the relation
\[
W^\perp = iK^\perp,
\]
where \( i \) is the inclusion mapping of \( L^\perp \) into \( \mathbb{R}^m \).

Putting together the main result in [3], Theorem 1 and Lemma 2, we have the following conclusion:

**Corollary 1** The generating wedge \( W \) is an isotone projection wedge if and only if its polar \( W^\perp \) is a cone generated by linearly independent vectors forming mutually non-acute angles.
4. Application: The isotonicity of the monotone wedge

Suppose that $\mathbb{R}^m$ is endowed with a Cartesian coordinate system, and $x \in \mathbb{R}^m$, $x = (x^1, ..., x^m)$ where $x^i$ are the coordinates of $x$ with respect to this reference system. The set

$$W = \{x \in \mathbb{R}^m : x^1 \geq x^2 \geq ... \geq x^m\}$$  \hspace{1cm} (5)

is called the monotone cone (see e.g. [1]). To be in accordance with our earlier terminology, we shall use for $W$ instead the term monotone wedge.

Let

$$L = W \cap (-W) = \{x \in \mathbb{R}^m : x^1 = x^2 = ... = x^m\}.$$  \hspace{1cm} (6)

Then $L \subset W$, the maximal subspace contained in $W$, is of dimension one. We have also that

$$K = L^\perp \cap W$$

is an $m - 1$-dimensional cone in the hyperplane $L^\perp$ and

$$W = W \cap (L^\perp \oplus L) = K \oplus L.$$  \hspace{1cm} (6)

We will show that the cone $K$ given by (6) is an isotone projection cone in $L^\perp$. To do this, we have to introduce some notations.

Let us take the following base in $\mathbb{R}^m$:

$$e_1 = (1, 0, ..., 0)$$

$$e_2 = (1, 1, 0, ..., 0),$$

$$...$$

$$e_{m-1} = (1, ..., 1, 0),$$

$$e_m = (1, 1, ..., 1).$$

An arbitrary element $x = (x^1, ..., x^m) \in \mathbb{R}^m$ can be represented in the form

$$x = (x^1 - x^2)e_1 + (x^2 - x^3)e_2 + ... + (x^{m-1} - x^m)e_{m-1} + x^m e_m,$$  \hspace{1cm} (7)

the relation $x \in W$ being equivalent with

$$x^{j-1} - x^j \geq 0, \; j = 2, ..., m.$$  \hspace{1cm} (8)

Let us consider further the following base in $L^\perp$:

$$e'_1 = (m - 1, -1, -1, ..., -1).$$

$$e'_2 = (m - 2, m - 2, -2, ..., -2),$$

$$...$$
\( e'_{m-1} = (1, \ldots, 1, -(m-1)). \)

The following notation is standard in the convex geometry and ordered vector space theory: If \( M \subset \mathbb{R}^m \) is a non-empty set, then let

\[
\text{cone} M = \{ t^1 m_1 + \ldots + t^k m_k : m_i \in M, t^i \in \mathbb{R}_+ = [0, +\infty), i = 1, \ldots, k; k \in \mathbb{N} \}.
\]

(The set \( \text{cone} W \) is the minimal wedge containing the set \( M \) and it is called the wedge generated by \( M \).)

We will see next that \( K = \text{cone}\{e'_1, \ldots, e'_{m-1}\} \). \hfill (9)

Since \( e'_j \in W \cap L^\perp \), we have obviously that

\[
\text{cone}\{e'_1, \ldots, e'_{m-1}\} \subset K.
\]

Comparing the vectors \( e_i \) and \( e'_j \) we get

\[
\frac{1}{m-j+1}(e'_j + e_m) = e_j, \quad j = 1, \ldots, m-1.
\]

By substitution of \( e_j, \ j = 1, \ldots, m-1 \), the representation (7) of \( x \) becomes

\[
x = (x^1 - x^2)\frac{1}{m}(e'_1 - e_m) + (x^2 - x^3)\frac{1}{m-1}(e'_2 + e_m) + \ldots + (x^{m-1} - x^m)\frac{1}{2}(e'_{m-1} + e_m) + x^m e_m.
\]

Suppose now that \( x \in W \), that is, relations (8) hold. Then the coefficients of \( e'_j, \ j = 1, \ldots, m-1 \) in its representation (12) are non-negative. Thus, we have

\[
x \in W \iff x = \sum_{j=1}^{m-1} t^j e'_j + t^m e_m, \quad t^j \in \mathbb{R}_+, \ j = 1, \ldots, m-1, \ t^m \in \mathbb{R}.
\]

(13)

In particular, if \( x \in K \), then, by (6), we have \( x \in L^\perp \). Hence, by multiplying (13) scalarly by \( e^m \) and by using \( \langle x, e_m \rangle = 0 \) (which follows from \( x \in L^\perp \) and \( e_m \in L \)) and \( \langle e'_j, e_m \rangle = 0 \) (which follows from \( e'_j \in L^\perp \) and \( e_m \in L \)), we get \( t^m = 0 \). This reasoning shows that

\[
K \subset \text{cone}\{e'_1, \ldots, e'_{m-1}\}.
\]

inclusion which together with (10) proves (9).

We consider now the vectors

\[
u_1 = (-1, 1, 0, \ldots, 0),
\]

\[
u_2 = (0, -1, 1, 0, \ldots, 0),
\]

\[
\ldots
\]

\[
u_{m-1} = (0, \ldots, 0, -1, 1).
\]
Then $u_i \in L^\perp$, $i = 1, \ldots, m - 1$, and we have
\[
\langle u_i, e'_j \rangle = 0 \text{ if } i \neq j, \quad \langle u_i, e'_i \rangle < 0, \quad i, j = 1, \ldots, m - 1.
\]
(14)

According to the reasonings in [6] the relations (14) show that
\[
\text{cone}\{u_1, \ldots, u_{m-1}\}
\]
is the polar of $K$ in the subspace $L^\perp$. Further, we have
\[
\langle u_i, u_j \rangle \leq 0 \text{ if } i \neq j.
\]

By the main result in [3] this shows that $K$ is an isotone projection cone in $L^\perp$.

In conclusion, using Theorem 1 we have the

\begin{corollary}
The monotone wedge $W$ given by the formula (5) admits an isotone projection.
\end{corollary}

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