On-Site Interaction Effects on Localization: Dominance of Non-Universal Contributions

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The influence of on-site (Hubbard) electron-electron interaction on disorder-induced localization is studied in order to clarify the role of electronic spin. The motivation is based on the recent experimental indications of a “metal-insulator” transition in two dimensional systems. We use both analytical and numerical techniques, addressing the limit of weak short-range interaction. The analytical calculation is based on Random Matrix Theory (RMT). It is found that although RMT gives a qualitative explanation of the numerical results, it is quantitatively incorrect. This is due to an exact cancellation of short range and long range correlations in RMT, which does not occur in the non-universal corrections to RMT. An estimate for these contributions is given.

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I. INTRODUCTION

The question considered in this paper is whether electron-electron interaction can reduce disorder-induced localization, thus enabling metallic behavior in two dimensional disordered systems.

The common view about the subject in the last 20 years has been based on the well known scaling theory of localization, according to which two dimensional systems will always be localized (i.e., insulating), no matter how weak the disorder is. Although the original scaling theory did not take interactions into account, it was shown that weak interaction (i.e., high electron-density) does not affect its results. On the other hand, in the limit of very strong interaction (i.e., very dilute systems) it is known that the electron liquid freezes into a Wigner lattice, which is pinned by disorder and therefore insulating. All these results have lead to the opinion that the repulsion between electrons can only further decrease the conductance, so that all two dimensional systems will show insulating behavior, regardless of the strength of interaction between the electrons.

A series of experiments performed in the last few years showed that even though in the limits of both very dense and very dilute systems we get the expected insulating behavior, for intermediate values of density (corresponding to r_s between 4 and 40, where r_s is the average inter-electron distance measured in the units of the Bohr radius) metallic-like temperature dependence is found. The transition from an insulating behavior to a metallic one as the density decreases was entitled “Two Dimensional Metal-Insulator Transition” (2DMIT). An important feature of these systems is that an application of an in-plane magnetic field, (which cannot affect the electrons’ orbital motion but can direct their spins) reduces the conductance in the metallic regime, until for high enough magnetic fields the conductance saturates as a function of the field, and the systems show the expected insulating behavior. This saturation field was estimated to be the field of full alignment of all the spins.

These results arouse much interest and many ideas where suggested for their explanation. A debate started in the question of whether there is really a metallic behavior and a phase transition, probably caused by electron-electron interaction, or the system is really insulating, but the experimentally accessible temperatures are high enough to exhibit temperature dependent scattering, thus causing the apparent metallic behavior.

Analytical and numerical calculations have shown that, as expected, for spinless electrons repulsion can only further localize the electrons, and does not lead to a metal-insulator transition. However, when taking spin into account, the situation is still unclear. In a recent numerical exact-diagonalization study, an Anderson model with both long range Coulomb interaction and short range Hubbard interaction was considered. It was shown that the Coulomb interaction, existing between any two electrons regardless of their spin, can only increase localization. On the other hand, not too-strong Hubbard interaction were seen to cause delocalization (Strong Hubbard interaction will lead to a Mott-Hubbard insulator). Since this interaction exists only between electrons with opposite spins, its effect is decreased by an in-plane magnetic field, and disappears when all the spins are aligned. This dependence of localization on interaction-strength and in-plane magnetic field thus mimics, at least qualitatively, the experimentally observed phenomena. Similar results were obtained recently using Quantum Monte-Carlo methods.

In this paper we wish to study further the weak short-range interaction regime, in which interaction-induced delocalization was observed. We will first address the problem analytically, using a Random Matrix Theory (RMT) approach, and then compare it to numerical simulations on an Anderson model. It will be shown that RMT can give only a qualitative but not a quantitative explanation for the numerical results, since RMT does not take into account non-universal correlations existing between wave functions in the diffusive regime. An estimate for the effect’s order of magnitude and its dependence on the parameters of the system in the diffusive regime will be given.
II. ANALYTICAL RESULTS - RANDOM MATRIX THEORY

We will consider an Anderson Hamiltonian with on-site Hubbard interaction:

\[ \hat{H} = \sum_{s,\sigma} \epsilon_s \hat{n}_{s,\sigma} - t \sum_{<s,s'>,\sigma} \hat{a}^\dagger_{s,\sigma} \hat{a}_{s',\sigma} + U_H \sum_s \hat{n}_{s,\uparrow} \hat{n}_{s,\downarrow}, \]  

where \( \hat{a}^\dagger_{s,\sigma} \), \( \hat{a}_{s,\sigma} \), and \( \hat{n}_{s,\sigma} \) denote electron creation, annihilation, and number operators, respectively, for a state on site \( s \) with spin projection \( \sigma \) on some axis. The first term is a random on-site potential, where \( \epsilon_s \) is chosen randomly from the range \([-W/2,W/2]\); the second is the hopping or kinetic term, where the sum is over nearest-neighbor sites \( s, s' \) and \( t \) is an overlap integral; the third is the Hubbard term, the electrostatic interaction between two electrons in the same site (which must have opposite spins), whose strength is determined by the parameter \( U_H \).

To quantify localization, we will calculate the Inverse Participation Ratio (IPR), defined by \( P^{-1} = \sum_s |\psi(s)|^4 \). This quantity is of order 1 for localized states, and of order \( N^{-1} \) for delocalized states, where \( N \) is the number of lattice sites. The IPR thus decreases when the single-particle wave function \( \psi \) becomes less localized, and gives us an estimation for the changes in the conductance of the system.

We assume here that without interaction the single electron energies and eigenvectors distributions for the ensemble of Anderson Hamiltonians are described by the corresponding distributions for an ensemble of Gaussian real symmetric matrices, i.e., the Gaussian Orthogonal Ensemble (GOE). This ensemble is defined by the well known distribution:

\[ P(H)\mu(H) = \exp(-\frac{\beta}{4\lambda^2} Tr(H^2))\mu(H), \]

where \( \beta = 1, \lambda \) is a constant energy parameter, and \( \mu(H) \) is a suitable measure. The eigenvectors are then a set of random orthogonal real normalized vectors. The average IPR without interaction for an electron in the n-th level with spin \( \sigma \) is thus:

\[ P_n^{-1} = \frac{1}{N(N+2)} \sum_s \langle |\psi_{n,s}^0(s)|^4 \rangle = \frac{3}{N+2}, \]

where the superscript \( (0) \) denotes the state without interaction, and double angular brackets denote ensemble average.

Now we add a weak Hubbard interaction, treating it in a self consistent way to first order in perturbation theory. Thus, the effect of spin-down electrons on the electrons with spin up will be the following effective potential (since only electrons with different spins interact, we have no exchange term):

\[ \hat{V} = U_H \sum_s \left| \psi_{m,t}^0(s) \right|^2 \hat{n}_{s,\uparrow}. \]

According to the familiar first order perturbation theory, the first order change in the IPR of a spin-up electron in the n-th state due to its interaction with a spin-down electron in the m-th state is:

\[ \Delta_m P_n^{-1} \sim \frac{\sum_{l \neq n} \langle \psi_{m,t}^0(s) \rangle^2 \psi_{n,t}^0(s) \psi_{l,t}^0(s) \langle \psi_{n,t}^0(s) \rangle^3 \psi_{l,t}^0(s)}{E_n^0 - E_l^0}. \]

Since the wave functions can be chosen to be real due to time reversal symmetry, we omitted absolute value and complex conjugate notations in this and the following expressions.

According to RMT, the eigenvectors distribution is independent of the eigenvalues distribution, so we can separate the averages of the numerator and denominator in the above expression.

As for the average of the numerator, its value can be found in the literature\textsuperscript{11,12}, and the results are summarized in Table I. We note that when \( s = s' \) we have an average of even powers of wave functions at different sites, which is expected to be positive and vary as \( N^{-4} \), since we have eight wave function values in the expression, each of which goes as \( N^{-1/2} \). On the other hand, when \( s \neq s' \), it may appear at first glance that since we have an average of odd powers of values of wave functions at different sites, which are uncorrelated, we should get zero. However, we get in this case a nonzero negative value, going as \( N^{-5} \). This result is due to correlations resulting from the orthogonality requirement on the eigenvectors.

To understand this, we may note that squaring the orthogonality relation \( \sum_s \psi_j(s)\psi_k(s) = 0 \) for \( j \neq k \) and averaging, using the known result:

\[ \langle (\psi_j(s))^2(\psi_k(s))^2 \rangle = \frac{1}{N(N+2)}, \]

we find that

\[ \langle (\psi_j(s)\psi_j(s')\psi_k(s')\psi_k(s')) \rangle = -\frac{1}{(N-1)N(N+2)}. \]

for \( s \neq s' \), i.e., if two different wave functions have the same sign on one site, from orthogonality they will tend to have opposite signs on another site and vice versa, hence the above nonzero negative average.

As for the average value of the energy denominator in Eq. 5, in principle it might be possible to calculate its value using RMT. However, to estimate the leading order we will assume the spectrum is composed of equidistant levels, with mean level spacing \( \Delta \).

Combining all those results together, we get, to the leading order in \( N \), the following result for the change in the IPR of a spin-up electron in the n-th level due to its interaction with a spin-down electron in the m-th level:

\[ \Delta_m P_n^{-1} = \frac{1}{(N-1)N(N+2)} \sum_{l \neq n} \langle \psi_{m,t}^0(s) \rangle^2 \psi_{n,t}^0(s) \psi_{l,t}^0(s) \langle \psi_{n,t}^0(s) \rangle^3 \psi_{l,t}^0(s) \left( E_n^0 - E_l^0 \right). \]
We observe that for \( m = n \) the correction will usually be positive, i.e., electrons in different levels repulse each other, resulting in further localization. As can be expected, the former effect is larger than the latter, due to the identity of the two interacting electrons’ wave functions in the former case. However, the order \( N \) difference between the case \( m = n \) and the case \( m \neq n \) is caused by an exact cancellation of the leading order dependence on \( N \) between the single short range \((s = s')\) term and all the \( N - 1 \) long range \((s \neq s')\) terms in the latter case, which doesn’t occur in the former. We will see below that this cancellation, together with the negative sign of the result for \( m \neq n \), is correct only in RMT.

Thus, if the lowest \( n_\downarrow \) levels are occupied by spin-down electrons, the total change in the IPR of a spin-up electron in the \( n \)-th level is:

\[
\Delta P_n^{-1} = -\frac{24}{N^3} U_H \left( 1 - \frac{n_\downarrow - 1}{N} \right) \left( \Phi(N - n) - \Phi(n - 1) \right)
+ \frac{48}{N^4} U_H \left( \Phi(n_\downarrow - n) - \Phi(n - 1) \right), \quad n \leq n_\downarrow; \\
-\frac{24n_\downarrow U_H}{N^3} \left( \Phi(N - n) - \Phi(n - 1) \right)
- \frac{48}{N^4} U_H \left( \Phi(n - 1) - \Phi(n - n_\downarrow - 1) \right), \quad n > n_\downarrow.
\]

The main features in the behavior of \( \Delta P_n^{-1} \) are as follows: For \( n \leq n_\downarrow \) the negative contribution of the spin-down electron at the same level \( n \) as the affected spin-up electron dominates the usually positive contribution of the other spin-down electrons. Therefore, \( \Delta P_n^{-1} \) is negative, but decreases in absolute value when \( n_\downarrow \) increases. For \( n > n_\downarrow \), there are spin-down electrons only in levels different from \( n \), thus \( \Delta P_n^{-1} \) is positive and increases when \( n_\downarrow \) increases. At \( n = n_\downarrow \) there is a discontinuous jump of \( \Delta P_n^{-1} \). In both cases, since \( \Delta \sim N^{-3} \) in real systems (although not in RMT), the effect is of order \( N^{-2} \), if we keep the concentration of spin-down electrons constant. (We neglect here the logarithmic factor coming from the function \( \Phi(n) \)). A plot of these formulas will be shown in the next section, where these expressions will be compared to numerical results.

### III. NUMERICAL RESULTS

In this section we will examine results of numerical calculations and compare them to the analytical results discussed above. Two model Hamiltonians will be considered: an RMT Hamiltonian and an Anderson Hamiltonian. It will be shown that their results differ by an order of magnitude as well as in other characteristics. The theoretical predictions will be shown to agree with the former but not with the latter, and reasons for the discrepancy will be given.

#### A. Random Matrix Hamiltonian

We will first consider the change in the IPR for a true RMT Hamiltonian. Since we consider here only the weak interaction regime, instead of solving the exact many-body problem we simply diagonalize first the Hamiltonian without interaction, and then use the wave functions to construct the effective potential, given in Eq. (4). This potential is then used to calculate the wave functions and the IPR with interaction. The applicability of this one loop Hartree-Fock approximation is justified by the fact that the change in \( P_n^{-1} \) was found to be linear in \( U_H \), as required.

The matrix size chosen was 408 × 408, and the elements were chosen according to the distribution law in Eq. (2). We have chosen \( \lambda = 0.1t \), so that the mean level spacing...
is $\Delta = 0.0196t$, approximately equal to the spacing in the Anderson Hamiltonian, Eq. (11), used in the next section (0.022t to 0.025t for $W$ between 2.0t and 4.0t). The interaction strength $U_H$ was taken as 1.0t. The calculated quantities were averaged over an ensemble of $5 \times 10^4$ different realizations.

The numerical results for the change in the IPR vs. the level number of the affected spin-up electron due to its interaction with different numbers of spin-down electrons, are shown in Fig. 1(a) together with the theoretical formula, Eq. (9). The theoretical formula was corrected, taking into account that the mean level spacing is not constant across the spectrum, but varies according to the semicircle law

$$\Delta(E) = \frac{\rho(E)}{2\pi\lambda^2\beta} \sqrt{4\lambda^2\beta N - E^2}, \quad (10)$$

where $\rho(E)$ is the density of states.

As can be seen, there is a good agreement between the numerical and the theoretical results. All the main features discussed at the end of the previous section can be clearly seen in the numerical data.

### B. Anderson Hamiltonian

Now we will discuss the changes in the IPR for the Anderson Hamiltonian given in Eq. (1). The calculation was performed in the same method as was used for the random matrix Hamiltonian (i.e., one-loop Hartree-Fock approximation).

We have chosen a $17 \times 24$ lattice, corresponding to a $408 \times 408$ matrix. As for the RMT calculations, we took $U_H = 1.0t$, while four values of disorder were used: $W=2.0t$, $W=2.5t$, $W=3.0t$ and $W=4.0t$. The results were averaged over $10^4$ realizations of disorder.

First, in Fig. 2 the value of the IPR without interaction is shown for the four values of disorder, as well as the RMT value, Eq. (3). We can see a difference here, as the Anderson model gives higher values (more localized) of the IPR than RMT. The effect is caused by non-universal (i.e., beyond RMT) corrections to the IPR and is more pronounced for higher disorder. The corrections for the IPR were calculated using supersymmetry techniques\cite{13}, resulting in $P^{-1} - P_{RMT}^{-1} \sim g^{-1}N^{-1}$ (where $g$ is the dimensionless conductance). We can also see, as expected, that the levels near the band edge have higher IPR, and
The results are averages over an ensemble of 10^4 realizations of systems on a 17 × 24 lattice. The estimated error approximately equals the width of the numerical results. Further parameters are given in the text.

are thus more localized, than levels near the center of the band.

Now we move to interaction effects in the Anderson model. The results are shown in Fig. 2 with the same occupation numbers as those chosen in the previous RMT calculations, for the four values of the disorder. As in RMT, the change in the IPR is negative for n ≤ n_i, changes sharply (though not discontinuously) at n = n_i. Nevertheless, it doesn’t change its sign there. Moreover, the change in the IPR is larger by about an order of magnitude than the one found from RMT. Also, even in the range n ≤ n_i, it increases in absolute value, rather than decreases, when n_i increases. All this is in contrast with Eq. (9) and the discussion following it.

Another point is that the effect increases with disorder. This is seen by comparing ΔP_{n}^{-1} for the same level n but different values of W; or by observing that, for the same value of W, levels near the band edge, which are more localized, show larger ΔP_{n}^{-1}.

The reason for these differences is the above mentioned cancellation between long range and short range wave-function correlations in RMT. As has been seen in our RMT calculations (Table I), the average of wave functions product appearing in the numerator of Eq. (5), is of order N^{-4} and positive when the two sites considered coincide, but are only of order N^{-5} and negative when the sites are different. Since there are N−1 terms of the latter type for each term of the former type, their total contributions are of the same order but their signs are opposite. Due to the equallity of the numerical coefficients of the two types of terms when the interacting electrons are in different levels, they cancel out exactly to the leading order in N, leaving behind a small negative term, of order N^{-5}. Therefore, in RMT interaction between electrons in different levels increases their localization, opposite to the situation for electrons in the same level. From this followed the decrease in the absolute value of ΔP_{n}^{-1} as n_i increases in the range n ≤ n_i, its positive value for n > n_i, and the overall N^{-2} dependence of the effect for constant density of spin-up electrons.

All this is correct when g is infinite. For finite g there exist non-universal corrections to the wave-function averages. Those corrections were not calculated before for the averages required here, but their behavior can be conjectured from known corrections for simpler averages (like those in Eqs. (10, 11)). We may expect them to have the same N dependence and sign as the RMT value, but to be smaller by a factor of g. The corrections for the short range (s = st) terms and long range (s ≠ st) terms will not, in general, have equal numerical coefficients, even when the interacting electrons are in different levels. Hence, after summation over st we are left with an order g^{-1}N^{-4} contribution instead of the order N^{-5} contribution in RMT. For this reason, although the non-universal corrections are of order g^{-1}, for most of the averaged terms they are about N times larger, so they will determine both the magnitude and the sign of the interaction-induced change in the IPR. Since the corrections for s = st will, in general, have a long range part, persisting for s ≠ st and having the same sign for neighboring sites (although for larger distances we may expect some oscillations), their sign will dominate the overall sign of the results. We will thus get a negative change in the IPR not only from interaction between electrons in the same level but also when the interacting electrons are in different levels. Hence, ΔP_{n}^{-1} will always be negative, as can be seen in the numerical results.

Moreover, repeating the calculations with the non-universal correction to the averages of wave functions product, we can estimate the dependence of the effect on the system parameters. We expect the total change in the IPR of a spin-up electron due to its interaction with n_i spin-down electrons to vary as

\[ \Delta P_{n}^{-1} \sim \frac{1}{g} \frac{U_{\mu} \, n_i}{\Delta \, N^3}. \] (11)

This expression does not include a factor coming from the sum over energy denominators, which has only a weak dependence on N and n_i (logarithmic for equidistant levels, a weak power law for a non-constant density of states). Because wave functions corresponding to neighboring levels are more correlated than wave-functions corresponding to far away levels, there is also a factor, which changes sharply (though not discontinuously) when we pass from n ≤ n_i to n > n_i, as seen in the numerical results. Since Δ ∼ N^{-1} in real systems (although not in RMT), the effect is of order g^{-1}N^{-1}, if we keep the concentration of spin-down electrons constant. This is in contrast to the N^{-2} dependence in RMT. Because N/g is much larger than unity in our numerical calculations, we can now understand the order of magnitude difference between RMT

![FIG. 2: The IPR for non-interacting electrons in the Anderson model. The IPR is plotted as a function of the level number. The lowest curve shows the RMT value, while the other ones are the Anderson model results for W=2.0t, W=2.5t, W=3.0t and W=4.0t, from lower to upper, respectively. The results are averages over an ensemble of 10^4 realizations of systems on a 17 × 24 sites lattice. The estimated error approximately equals the width of the numerical results. Further parameters are given in the text.](image-url)
and Anderson model results. Thus, all the features of the numerical data can be explained by taking non-universal corrections into account.

As we have mentioned before, the non-universal part of the IPR without interaction, i.e., the difference between the value of the IPR without interaction in the Anderson model and its value in RMT, varies as $g^{-1}N^{-1}$. According to our estimate, the change in the IPR due to interaction in the Anderson model also goes as $g^{-1}N^{-1}$. Thus, their ratio, $\Delta P_n^{-1}/(P^{-1} - P_{RMT}^{-1})$, should be independent of $g$, i.e. of the degree of disorder. It should also be independent of the number of lattice sites $N$ if the densities of spin-up and spin-down electrons are kept constant. Thus, this ratio may be used to test our conjecture for the parametric form of $\Delta P_n^{-1}$.

We first test the $g$ independence of the ratio $\Delta P_n^{-1}/(P^{-1} - P_{RMT}^{-1})$ by plotting it in Fig. 4 for systems with identical lattice sizes (taken to be $17 \times 24$, as in the previous calculations), but different values of disorder. We can clearly see that the differences between curves corresponding to different $W$ values are much smaller than the corresponding differences in Fig. 3. The only exception is the value $W=2.0$ (the lowest curve), which shows a marked difference from the other $W$ values. This is probably due to the fact that for $W=2.0$ disorder is not high enough, so the electrons’ motion is not fully diffusive, and ballistic boundary effects may be important.

We now test $N$ independence of the ratio $\Delta P_n^{-1}/(P^{-1} - P_{RMT}^{-1})$ by plotting it in Fig. 5 for systems with the same value of disorder (taken as $W=4.0$) but different lattice sizes – $8 \times 13$, $13 \times 19$, $17 \times 24$. In all the cases the densities of spin-up and spin-down electrons are approximately equal (the horizontal axis is not the level number of the affected spin-up electron as before, but the filling $\nu$, defined as the ratio of the number of spin-up electrons $n$ and the total number of lattice sites $N$). We can clearly see that the different curves are almost identical. The only exception is the small $8 \times 13$ lattice, whose slightly different behavior can again be attributed to ballistic boundary effects.

FIG. 3: Change in the IPR of a spin-up electron due to its interaction with spin-down electrons in the Anderson model. The change is plotted as a function of the level number of the affected spin-up electron for different numbers of spin-down electrons: (a) $n_{\downarrow} = 50$; (b) $n_{\downarrow} = 100$; (c) $n_{\downarrow} = 150$; (d) $n_{\downarrow} = 200$. In all the graphs the curves correspond to $W=4.0t$, $W=3.0t$, $W=2.5t$ and $W=2.0t$, from lower to upper, respectively. The results are averages over an ensemble of $10^4$ realizations of systems on a $17 \times 24$ sites lattice. The estimated error approximately equals the width of the numerical results. Further parameters are given in the text.
FIG. 4: Ratio between the change in the IPR of a spin-up electron due to its interaction with spin-down electrons in the Anderson model and the non-universal part of the IPR without interaction. The ratio is plotted as a function of the level number of the affected spin-up electron for different numbers of spin-down electrons: (a) \( n_\downarrow = 50 \); (b) \( n_\downarrow = 100 \); (c) \( n_\downarrow = 150 \); (d) \( n_\downarrow = 200 \). In all the graphs the curves correspond to \( W=2.0t \), \( W=2.5t \), \( W=3.0t \) and \( W=4.0t \), from lower to upper, respectively. The results are averages over an ensemble of \( 10^4 \) realizations of systems on a \( 17 \times 24 \) sites lattice. The estimated error approximately equals the width of the numerical results. Further parameters are given in the text.

IV. CONCLUSIONS

In conclusion, we have shown how a spin-dependent interaction can cause delocalization, at least for weak short-range interaction. Localized electrons highly repulse each other, especially if they have the same orbital wave function and thus a different spin. This results in a tendency for interaction-induced delocalization. The effect on an electron with a given orbital level and spin direction is stronger if the same orbital level is occupied by an electron with an opposite spin, and increases with the total number of electrons with opposite spin. The delocalization is thus reduced by an in-plane magnetic field.

All this is in accordance, at least qualitatively, with recent experimental findings and numerical simulations regarding the in-plane magnetoresistance.

We have also seen that the main difference in the influence of the Hubbard interaction between realistic finite \( g \) systems and the RMT stems from exact cancellation of the leading order long range and short range terms in the former. Thus, while in RMT a state is correlated only to the same state with an opposite spin (except for weak anti-correlations with all other states), for finite \( g \) correlations between different states lead to a stronger repulsion between these states resulting in a stronger delocalization due to the on-site interactions. Nevertheless, the order of magnitude and parametric dependence of the IPR can be calculated using RMT, once the non-universal corrections are properly taken into account.

Acknowledgments

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1 E. Abrahams, P. W. Anderson, D. C. Licciardello and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).

2 B. L. Altshuler, A. G. Aronov and P. A. Lee, Phys. Rev.
FIG. 5: Ratio between the change in the IPR of a spin-up electron due to its interaction with spin-down electrons in the Anderson model and the non-universal part of the IPR without interaction. The ratio is plotted as a function of the filling of the affected spin-up electron (i.e., the ratio of the number of spin-up electrons and the number of lattice sites) for different fillings of spin-down electrons: (a) \( \nu \approx 1/8 \); (b) \( \nu \approx 1/4 \); (c) \( \nu \approx 3/8 \); (d) \( \nu \approx 1/2 \). In each graph we use three different lattice sizes – 8 \( \times \) 13, 13 \( \times \) 19, 17 \( \times \) 24, but a constant value of disorder, \( W=4.0 \). The results are averages over an ensemble of \( 10^4 \) realizations. The estimated error approximately equals the width of the numerical results. Further parameters are given in the text.

\textsuperscript{Lett.} 44, 1288 (1980).
\textsuperscript{3} B. Tanatar and D. M. Ceperley, Phys. Rev. B \textbf{39}, 5005 (1989).
\textsuperscript{4} For a recent review see: E. Abrahams, S. V. Kravchenko and M. P. Sarachik, Rev. Mod. Phys. \textbf{73}, 251 (2001), and references cited therein.
\textsuperscript{5} A. M. Finkel’stein, Z. Phys. B \textbf{56}, 189 (1984); C. Castellani, C. DiCastro, P. A. Lee and M. Ma, Phys. Rev. B \textbf{30}, 527 (1984); A. Punnoose and A. M. Finkel’stein, Phys. Rev. Lett. \textbf{88}, 016802 (2002).
\textsuperscript{6} G. Zala, B. N. Narozhny and I. L. Aleiner, Phys. Rev. B \textbf{64}, 214204 (2001); \textit{ibid.} \textbf{65}, 020201 (2002).
\textsuperscript{7} R. Berkovits, J. W. Kantelhardt, Y. Avishai, S. Havlin and A. Bunde, Phys. Rev. B \textbf{63}, 085102 (2001).
\textsuperscript{8} R. Berkovits and J. W. Kantelhardt, Phys. Rev. B \textbf{65}, 125308 (2002).
\textsuperscript{9} P. J. H. Denteneer and R. T. Scalettar, Phys. Rev. Lett. \textbf{90}, 246401 (2003).
\textsuperscript{10} M. L. Mehta, \textit{Random Matrices} (Academic Press, New York, 1991).
\textsuperscript{11} N. Ullah, J. Math. Phys. \textbf{4}, 1279 (1963).
\textsuperscript{12} T. Gorin J. Math. Phys. \textbf{43}, 3342 (2002).
\textsuperscript{13} V. N. Prigodin and B. L. Altshuler, Phys. Rev. Lett. \textbf{80}, 1944 (1998).
\textsuperscript{14} A. D. Mirlin, Phys. Rep. \textbf{326}, 260 (2000).