M-SYSTEMS AND CLUSTER ALGEBRAS

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Abstract. The aim of this paper is three-fold: (1) introduce four systems of equations called M-systems and dual M-systems of types $A_n$ and $B_n$ respectively; (2) make a connection between M-systems (dual M-systems) and cluster algebras and prove that the Hernandez-Leclerc conjecture is true for minimal affinizations of types $A_n$ and $B_n$; (3) give a new algorithm to compute the $q$-characters of minimal affinizations of types $A_n$ and $B_n$.

Key words: M-systems; cluster algebras; quantum affine algebras; minimal affinizations; $q$-characters; monoidal categorification of cluster algebras

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1. Introduction

In the paper [FZ02], Fomin and Zelevinsky introduced the theory of cluster algebras to study canonical bases of quantum groups introduced by Lusztig [L90] and Kashiwara [K91] and total positivity for semisimple algebraic groups developed by Lusztig [L94]. It has exciting connections and applications to many areas of mathematics and physics including integrable systems, Poisson geometry, quiver representations, Teichmüller theory, and tropical geometry.

The aim of this paper is three-fold:

(1) introduce four systems of equations called M-systems and dual M-systems of types $A_n$ and $B_n$ respectively;

(2) make a connection between M-systems (dual M-systems) and cluster algebras and prove that the Hernandez-Leclerc conjecture (Conjecture 13.2 in [HL10] and Conjecture 9.1 in [Le10]) is true for minimal affinizations of types $A_n$ and $B_n$;

(3) give a new algorithm to compute the $q$-characters of minimal affinizations of types $A_n$ and $B_n$.

Let $\mathfrak{g}$ be a simple Lie algebra and $U_q\widehat{\mathfrak{g}}$ the corresponding quantum affine algebra. In a remarkable paper [HL10], Hernandez and Leclerc introduced the concept of monoidal categorifications of cluster algebras. They conjectured that the Grothendieck rings of some subcategories of the category of all finite dimensional representations of $U_q\widehat{\mathfrak{g}}$ have cluster algebra structures, real prime simple $U_q\widehat{\mathfrak{g}}$-modules correspond to cluster variables, and real simple $U_q\widehat{\mathfrak{g}}$-modules correspond to cluster monomials. Some special cases of the conjecture were proved in [HL10], [KQ14], and [Nak11]. In the paper [HL13], Hernandez and Leclerc apply the theory of cluster algebras to study the $q$-characters of a family of $U_q\widehat{\mathfrak{g}}$-modules called Kirillov-Reshetikhin modules and they give a new algorithm to compute the $q$-characters of these modules.

The family of minimal affinizations of quantum groups is an important family of simple modules of $U_q\widehat{\mathfrak{g}}$ which was introduced in [C95]. The celebrated Kirillov-Reshetikhin modules are examples of minimal affinizations. Minimal affinizations are studied intensively in recent years,
see for example, \cite{CMY13, CG11, Her07, LM13, M10, MP11, MY12a, MY12b, MY14, Nao13}.

The T-systems are functional relations which were defined in \cite{KNS94}. Hernandez proved that the $q$-characters of Kirillov-Reshetikhin modules satisfy the T-systems, see \cite{Her06}. Mukhin and Young introduced the extended T-systems in \cite{MY12a} and they showed that the extended T-systems of types $A_n$, $B_n$ are satisfied by the $q$-characters of a class of $U_q\hat{g}$-modules called snake modules of types $A_n$, $B_n$. The class of snake modules contains all minimal affinizations.

In this paper, we use a new approach to study minimal affinizations. The family of minimal affinizations for $U_q\hat{g}$ can be divided into two parts according to the highest weight monomials. For example, in type $A_n$, let (the notations will be explained in Section 2)

\begin{align*}
T^{(s)}_{k_1,k_2,...,k_n} &= \prod_{j=1}^{n} \left( \prod_{j=0}^{k_j-1} j + 2 \sum_{p=1}^{j-1} k_p + 2i_j + (j-1) \right), \\
\hat{T}^{(s)}_{k_1,k_2,...,k_n} &= \prod_{j=1}^{n} \left( \prod_{j=0}^{k_j-1} j - 2 \sum_{p=1}^{j-1} k_p - 2i_j - (j-1) \right).
\end{align*}

The first (resp. second) part of the family of minimal affinizations of type $A_n$ consists of minimal affinizations with highest weight monomials $T^{(s)}_{k_1,k_2,...,k_n}$ (resp. $\hat{T}^{(s)}_{k_1,k_2,...,k_n}$). The M-systems (resp. dual M-systems) introduced in this paper are systems of equations which are satisfied by the $q$-characters of first (resp. second) part of the family of minimal affinizations of $U_q\hat{g}$.

The extended T-system of type $A_n$ (resp. $B_n$) is closed within the family of snake modules of type $A_n$ (resp. $B_n$). The M-system of type $A_n$ (resp. $B_n$) is closed within the family of minimal affinizations of type $A_n$ (resp. $B_n$). We show that the $q$-characters of the first part of the family of minimal affinizations of type $A_n$ (resp. $B_n$) satisfy the equations in the M-system of type $A_n$ (resp. $B_n$) and the $q$-characters of the second part of the family of minimal affinizations of type $A_n$ (resp. $B_n$) satisfy the equations in the dual M-system of type $A_n$ (resp. $B_n$). We also show that the modules in the summands on the right hand side of each equation in the M-systems and dual M-systems are simple.

T-systems have many applications to mathematics and physics, see \cite{KNS11}. Since the equations in M-systems and dual M-systems have some nice properties and they are satisfied by $q$-characters of some family of $U_q\hat{g}$-modules, we expect that M-systems and dual M-systems will have applications to mathematics and physics like T-systems.

We show that the equations in the M-system of type $A_n$ (resp. $B_n$) correspond to mutations in some cluster algebra $\mathscr{A}$ (resp. $\mathscr{A}'$). The cluster algebra $\mathscr{A}$ (resp. $\mathscr{A}'$) is the same as the cluster algebra for type $A_n$ (resp. $B_n$) quantum affine algebra introduced in \cite{HL13}. Moreover, every minimal affinization in the M-system of type $A_n$ (resp. $B_n$) corresponds to a cluster variable in $\mathscr{A}$ (resp. $\mathscr{A}'$).

We define two more cluster algebras $\tilde{\mathscr{A}}$, $\tilde{\mathscr{A}}'$ such that every minimal affinization in the dual M-system of type $A_n$ (resp. $B_n$) corresponds to a cluster variable in $\tilde{\mathscr{A}}$ (resp. $\tilde{\mathscr{A}}'$).

We give a proof of the fact that minimal affinizations of types $A_n$ and $B_n$ are real. According to the results in \cite{CMY13}, minimal affinizations of all types are prime. Therefore minimal affinizations of type $A_n$ (resp. $B_n$) are simple, real, and prime and they correspond to cluster
variables in \( A \) (resp. \( A' \)). Thus we have shown that the Hernandez-Leclerc conjecture (Conjecture 13.2 in [HL10] and Conjecture 9.1 in [Le10]) is true for minimal affinizations of types \( A_n \) and \( B_n \).

By using the M-systems and dual M-systems, we give a new algorithm to compute the \( q \)-characters of minimal affinizations of types \( A_n \) and \( B_n \). We also have m-systems and dual m-systems of types \( A_n, B_n \) which are obtained from M-systems and dual M-systems of types \( A_n, B_n \) by restricting the modules in M-systems and dual M-systems to \( U_q \mathfrak{g} \)-modules.

The M-systems also exist for other Dynkin types of minimal affinizations. The M-system of type \( G_2 \) is studied in the paper [QL14]. Since the method of proving that the \( q \)-characters of minimal affinizations satisfy the M-systems of types \( C, D, E, F \) are different from the method used in this paper, we will write them in other papers.

The paper is organized as follows. In Section 2, we give some background information about cluster algebras and representation theory of quantum affine algebras. In Section 3 we describe cluster algebras and representation theory of quantum affine algebras. In Section 4, we study relations between M-systems and cluster algebras. In Section 5 we study the dual M-systems of types \( A \) and \( B \). In Section 6 we show that the Hernandez-Leclerc conjecture is true for minimal affinizations of types \( A_n \) and \( B_n \). In Section 7 we give a new algorithm to compute minimal affinizations of types \( A_n \) and \( B_n \). In Sections 8 and 9 we prove Theorems 3.1 and 3.3 given in Section 3. In the Appendix, we give some examples of mutation sequences.

2. Preliminaries

2.1. Cluster algebras. Cluster algebras are invented by Fomin and Zelevinsky in [FZ02]. Let \( Q \) be the rational field and \( F = \mathbb{Q}(x_1, x_2, \ldots, x_n) \) the field of rational functions. A seed in \( F \) is a pair \( \Sigma = (y, Q) \), where \( y = (y_1, y_2, \ldots, y_n) \) is a free generating set of \( F \), and \( Q \) is a quiver with vertices labeled by \( 1, 2, \ldots, n \). Assume that \( Q \) has neither loops nor 2-cycles. For \( k = 1, 2, \ldots, n \), one defines a mutation \( \mu_k \) by \( \mu_k(y, Q) = (y', Q') \). Here \( y_i' = (y_1', \ldots, y_{i-1}', y_{i+1}', \ldots, y_n'), y_i = y_i \), for \( i \neq k \), and

\[
y_k' = \frac{\prod_{i \to k} y_i + \prod_{k \to j} y_j}{y_k},
\]

where the first (resp. second) product in the right hand side is over all arrows of \( Q \) with target (resp. source) \( k \), and \( Q' \) is obtained from \( Q \) by

(i) adding a new arrow \( i \to j \) for every existing pair of arrow \( i \to k \) and \( k \to j \);
(ii) reversing the orientation of every arrow with target or source equal to \( k \);
(iii) erasing every pair of opposite arrows possible created by (i).

The mutation class \( C(\Sigma) \) is the set of all seeds obtained from \( \Sigma \) by a finite sequence of mutation \( \mu_k \). If \( \Sigma' = ((y_1', y_2', \ldots, y_n'), Q') \) is a seed in \( C(\Sigma) \), then the subset \( \{y_1', y_2', \ldots, y_n'\} \) is called a cluster, and its elements are called cluster variables. The cluster algebra \( A_\Sigma \) is the subring of \( F \) generated by all cluster variables. Cluster monomials are monomials in the cluster variables supported on a single cluster.

In this paper, the initial seed in the cluster algebra we use is of the form \( \Sigma = (y, Q) \), where \( y \) is an infinite set and \( Q \) is an infinite quiver.

Definition 2.1 (Definition 3.1, [GG14]). Let \( Q \) be a quiver without loops or 2-cycles and with a countably infinite number of vertices labelled by all integers \( i \in \mathbb{Z} \). Furthermore, for each vertex
of $Q$ let the number of arrows incident with $i$ be finite. Let $y = \{y_i \mid i \in \mathbb{Z}\}$. An infinite initial seed is the pair $(y, Q)$. By finite sequences of mutation at vertices of $Q$ and simultaneous mutation of the set $y$ using the exchange relation (2.1), one obtains a family of infinite seeds. The sets of variables in these seeds are called the infinite clusters and their elements are called the cluster variables. The cluster algebra of infinite rank of type $Q$ is the subalgebra of $\mathbb{Q}(y)$ generated by the cluster variables.

2.2. Quantum affine algebra. Let $\mathfrak{g}$ be a simple Lie algebra and $I = \{1, \ldots, n\}$ the indices of the Dynkin diagram of $\mathfrak{g}$ (we use the same labeling of the vertices of the Dynkin diagram of $\mathfrak{g}$ as the one used in [Car05]). Let $C = (C_{ij})_{i,j \in I}$ be the Cartan matrix of $\mathfrak{g}$, where $C_{ij} = \frac{2(a_i,a_j)}{(a_i,a_i)}$.

We define $D = \text{diag}(d_i \mid i \in I)$, where $d_i = 1$, $i \in I$, for type $A_n$ and $d_i = 2$, $i = 1, \ldots, n - 1$, $d_n = 1$, for type $B_n$. Then $B = DC = (b_{ij})_{i,j \in I}$ is a symmetric matrix. Let $t = \max\{d_i \mid i \in I\}$. Then $t = 1$ for type $A_n$ and $t = 2$ for type $B_n$.

The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ in Drinfeld’s new realization, see [Dri88], is generated by $x_{i,n}^\pm (i \in I, n \in \mathbb{Z})$, $k_i^{\pm 1}$ $(i \in I)$, $h_{i,n}$ $(i \in I, n \in \mathbb{Z} \setminus \{0\})$ and central elements $c^{\pm 1/2}$, subject to certain relations.

The subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by $(k_i^{\pm 1})_{i \in I}, (x_{i,0}^\pm)_{i \in I}$ is a Hopf subalgebra of $U_q(\hat{\mathfrak{g}})$ and is isomorphic as a Hopf algebra to $U_q(\mathfrak{g})$, the quantized enveloping algebra of $\mathfrak{g}$. In this way, $U_q(\hat{\mathfrak{g}})$-modules restrict to $U_q(\mathfrak{g})$-modules.

2.3. Finite-dimensional representations and $q$-characters. In this section, we recall the standard facts about finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$ and $q$-characters of these representations, see [CP94], [CP95], [FR98], [MY12].

A representation $V$ of $U_q(\hat{\mathfrak{g}})$ is of type 1 if $c^{\pm 1/2}$ acts as the identity on $V$ and

$$V = \bigoplus_{\lambda \in P} V_\lambda, \ V_\lambda = \{v \in V : k_i v = q^{(\alpha_i, \lambda)} v\}. \quad (2.2)$$

In the following, all representations will be assumed to be finite-dimensional and of type 1 without further comment. The decomposition (2.2) of a finite-dimensional representation $V$ into its $U_q(\mathfrak{g})$-weight spaces can be refined by decomposing it into the Jordan subspaces of the mutually commuting operators $\phi_i^{\pm}$, see [FR98]:

$$V = \bigoplus_{\gamma} V_{\gamma}, \ \gamma = (\gamma_{i,\pm r})_{i \in I, r \in \mathbb{Z}_{\geq 0}}, \ \gamma_{i,\pm r} \in \mathbb{C}, \quad (2.3)$$

where

$$V_{\gamma} = \{v \in V : \exists k \in \mathbb{N}, \forall i \in I, m \geq 0, (\phi_{i,\pm m} - \gamma_{i,\pm m})^k v = 0\}.$$

Here $\phi_{i,n}^{\pm}$’s are determined by the formula

$$\phi_{i,n}^{\pm}(u) = \sum_{n=0}^{\infty} \phi_{i,\pm n}^{\pm} u^{\pm n} = k_i^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} u^{\pm m} \right). \quad (2.4)$$

If $\dim(V_{\gamma}) > 0$, then $\gamma$ is called an $l$-weight of $V$. For every finite-dimensional representation of $U_q(\hat{\mathfrak{g}})$, the $l$-weights are known, see [FR98], to be of the form

$$\gamma_{i}^{\pm}(u) = \sum_{r=0}^{\infty} \gamma_{i,\pm r} u^{\pm r} = q_i^{\deg Q_i - \deg R_i} Q_i(uq_i^{-1}) R_i(uq_i) / Q_i(uq_i) R_i(uq_i^{-1}), \quad (2.5)$$

$\text{deg } Q_i$ and $\text{deg } R_i$ have been defined in (1.11).
where the right hand side is to be treated as a formal series in positive (resp. negative) integer powers of $u$, and $Q_i, R_i$ are polynomials of the form

$$Q_i(u) = \prod_{a \in \mathbb{C}^x} (1 - ua)^{w_{i,a}}, \quad R_i(u) = \prod_{a \in \mathbb{C}^x} (1 - ua)^{x_{i,a}},$$

(2.6)

for some $w_{i,a}, x_{i,a} \in \mathbb{Z}_{\geq 0}, i \in I, a \in \mathbb{C}^x$. Let $\mathcal{P}$ denote the free abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i \in I, a \in \mathbb{C}^x}$. There is a bijection $\gamma$ from $\mathcal{P}$ to the set of $l$-weights of finite-dimensional modules such that for the monomial $m = \prod_{i \in I, a \in \mathbb{C}^x} Y_{i,a}^{w_{i,a} - x_{i,a}}$, the $l$-weight $\gamma(m)$ is given by (2.5), (2.6).

For $m \in \mathcal{P}$, we write $V_m$ for $V_{\gamma(m)}$. Let $\mathcal{Z}\mathcal{P} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^x}$ be the group ring of $\mathcal{P}$. The $q$-character of a $U_q\hat{g}$-module $V$ is given by

$$\chi_q(V) = \sum_{m \in \mathcal{P}} \dim(V_m)m \in \mathcal{Z}\mathcal{P}.$$

Let $\text{Rep}(U_q\hat{g})$ be the Grothendieck ring of finite-dimensional representations of $U_q\hat{g}$ and $[V] \in \text{Rep}(U_q\hat{g})$ the class of a finite-dimensional $U_q\hat{g}$-module $V$. The $q$-character map defines an injective ring homomorphism, see [FR98].

$$\chi_q : \text{Rep}(U_q\hat{g}) \to \mathcal{Z}\mathcal{P}.$$

For any finite-dimensional representation $V$ of $U_q\hat{g}$, denote by $\mathcal{M}(V)$ the set of all monomials in $\chi_q(V)$. For each $j \in I$, a monomial $m = \prod_{i \in I, a \in \mathbb{C}^x} Y_{i,a}^{x_{j,a}}$, where $x_{j,a}$ are some integers, is said to be $j$-dominant (resp. $j$-anti-dominant) if and only if $x_{j,a} \geq 0$ (resp. $x_{j,a} \leq 0$) for all $a \in \mathbb{C}^x$. A monomial is called dominant (resp. anti-dominant) if and only if it is $j$-dominant (resp. $j$-anti-dominant) for all $j \in I$. Let $\mathcal{P}^+ \subset \mathcal{P}$ denote the set of all dominant monomials.

Let $V$ be a representation of $U_q\hat{g}$ and $m \in \mathcal{M}(V)$ a monomial. A non-zero vector $v \in V_m$ is called a highest $l$-weight vector with highest $l$-weight $\gamma(m)$ if

$$x_{i,r}^+ \cdot v = 0, \quad \phi_{i,±t}^+ \cdot v = \gamma(m)^{±}_{i,±t} v, \quad \forall i \in I, r \in \mathbb{Z}, t \in \mathbb{Z}_{\geq 0}.$$

The module $V$ is called a highest $l$-weight representation if $V = U_q\hat{g} \cdot v$ for some highest $l$-weight vector $v \in V$.

It is known, see [CP94], [CP95a], that for each $m_+ \in \mathcal{P}^+$ there is a unique finite-dimensional irreducible representation, denoted $L(m_+)$, of $U_q\hat{g}$ that is highest $l$-weight with highest $l$-weight $\gamma(m_+)$, and moreover every finite-dimensional irreducible $U_q\hat{g}$-module is of this form for some $m_+ \in \mathcal{P}^+$. We call $m_+$ the highest monomial in $\chi_q(L(m_+))$. Also, if $m_+, m'_+ \in \mathcal{P}^+$ and $m_+ \neq m'_+$, then $L(m_+) \neq L(m'_+)$. For $b \in \mathbb{C}^x$, define the shift of spectral parameter map $\tau_b : \mathcal{Z}\mathcal{P} \to \mathcal{Z}\mathcal{P}$ to be a homomorphism of rings sending $Y_{i,a}^{±1}$ to $Y_{i,b}^{±1}$. Let $m_1, m_2 \in \mathcal{P}^+$. If $\tau_b\chi_q(L(m_1)) = \chi_q(L(m_2))$, $b \neq m'_+$.

A finite-dimensional $U_q\hat{g}$-module $V$ is said to be special if and only if $\mathcal{M}(V)$ contains exactly one dominant monomial. It is called anti-special if and only if $\mathcal{M}(V)$ contains exactly one anti-dominant monomial. It is called thin if and only if no $l$-weight space of $V$ has dimension greater than 1. It is said to be prime if and only if it is not isomorphic to a tensor product of two non-trivial $U_q\hat{g}$-modules, see [CP97]. Clearly, if a module is special or anti-special, then it is irreducible. A simple $U_q\hat{g}$-modules $M$ is called real if $M \otimes M$ is simple, see [Le03].

For simplicity, we use $\chi_q(m_+)$ to denote $\chi_q(L(m_+))$ and use $\chi_q(m_1) \leq \chi_q(m_2)$ to denote $\mathcal{M}(L(m_1)) \subseteq \mathcal{M}(L(m_2))$ for dominant monomials $m_+, m_1, m_2$. 
The elements \( A_{i,a} \in \mathcal{P}, i \in I, a \in \mathbb{C}^\times \), are defined by
\[
A_{i,a} = Y_{i,aq} Y_{i,aq}^{-1} \prod_{C_{j_i} = -1} Y_{j,a}^{-1} \prod_{C_{j_i} = -2} Y_{j,a}^{-1} Y_{j,aq}^{-1} \prod_{C_{j_i} = -3} Y_{j,a}^{-1} Y_{j,aq}^{-1},
\]
see Section 2.3 in [FM01]. Let \( Q \) be the subgroup of \( \mathcal{P} \) generated by \( A_{i,a}, i \in I, a \in \mathbb{C}^\times \). Let \( Q^\pm \) be the monoids generated by \( A_{i,a}^{\pm 1}, i \in I, a \in \mathbb{C}^\times \). There is a partial order \( \leq \) on \( \mathcal{P} \) in which
\[
m \leq m' \text{ if and only if } m'm^{-1} \in Q^+.
\] (2.7)

For all \( m_+ \in \mathcal{P}^+, \mathcal{M}(L(m_+)) \subset m_+ Q^- \), see [FM01].

We will need the concept right-negative to classify dominant monomials. Let \( m \) be a monomial. If for all \( a \in \mathbb{C}^\times \) and \( i \in I \), we have the property: if the power of \( Y_{i,a} \) in \( m \) is non-zero and the power of \( Y_{j,aq} \) in \( m \) is zero for all \( j \in I, k \in \mathbb{Z}_{>0} \), then the power of \( Y_{i,a} \) in \( m \) is negative, then the monomial \( m \) is called right-negative, see [FM01]. For \( i \in I, a \in \mathbb{C}^\times \), \( A_{i,a}^{-1} \) is right-negative. A product of right-negative monomials is right-negative. If \( m \) is right-negative and \( m' \leq m \), then \( m' \) is right-negative, see [FM01], [Her06].

2.4. Minimal affinizations of \( U_q\mathfrak{g}\)-modules. From now on, we fix an \( a \in \mathbb{C}^\times \) and denote \( i_s = Y_{i,aq^s}, i \in I, s \in \mathbb{Z} \). Let \( \lambda = k_1 \omega_1 + \cdots + k_n \omega_n, k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0} \), and let \( V(\lambda) \) be the simple \( U_q\mathfrak{g} \)-module with highest weight \( \lambda \). Without loss of generality, we may assume that in type \( A_n \) a simple \( U_q\mathfrak{g} \)-module \( L(m_+) \) is a minimal affinization of \( V(\lambda) \) if and only if \( m_+ \) is one of the following monomials:
\[
T_{k_1,k_2,\ldots,k_n}^{(s)} = \prod_{j=1}^{n} \left( \prod_{i_j=0}^{k_j-1} \left( j + \sum_{p=1}^{s} k_p + 2i_j + (j-1) \right) \right),
\] (2.8)
\[
\tilde{T}_{k_1,k_2,\ldots,k_n}^{(s)} = \prod_{j=1}^{n} \left( \prod_{i_j=0}^{k_j-1} \left( j - \sum_{p=1}^{s} k_p - 2i_j - (j-1) \right) \right),
\] (2.9)
where \( s \in \mathbb{Z} \), see [CP96a]. Similarly, we may assume that in type \( B_n \), a simple \( U_q\mathfrak{g} \)-module \( L(m_+) \) is a minimal affinization of \( V(\lambda) \) if and only if \( m_+ \) is one of the following monomials:
\[
T_{k_1,k_2,\ldots,k_n}^{(s)} = \prod_{j=1}^{n-1} \left( \prod_{i_j=0}^{k_j-1} \left( j + \sum_{p=1}^{s} k_p + 4i_j + 2j - 2 \right) \right) \prod_{i_n=0}^{k_n-1} \left( n + \sum_{p=1}^{s} k_p + 2i_n + 2n - 3 \right),
\] (2.10)
\[
\tilde{T}_{k_1,k_2,\ldots,k_n}^{(s)} = \prod_{j=1}^{n-1} \left( \prod_{i_j=0}^{k_j-1} \left( j - \sum_{p=1}^{s} k_p - 4i_j - 2j + 2 \right) \right) \prod_{i_n=0}^{k_n-1} \left( n - \sum_{p=1}^{s} k_p - 2i_n - 2n + 3 \right),
\] (2.11)
where \( s \in \mathbb{Z} \), see [CP95b]. We denote \( A_{i,a}^{-1} \) by \( A_{i,a}^{-1} \). We use \( T_{k_1,k_2,\ldots,k_n}^{(s)} \) (resp. \( \tilde{T}_{k_1,k_2,\ldots,k_n}^{(s)} \)) to denote the irreducible finite-dimensional \( U_q\mathfrak{g} \)-module with highest \( l \)-weight \( T_{k_1,k_2,\ldots,k_n}^{(s)} \) (resp. \( \tilde{T}_{k_1,k_2,\ldots,k_n}^{(s)} \)), where \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0} \).
2.5. $q$-characters of $U_q\hat{\mathfrak{sl}_2}$-modules and the Frenkel-Mukhin algorithm. The $q$-characters of $U_q\hat{\mathfrak{sl}_2}$-modules are well-understood, see [CP91], [FR98]. We recall some results which will be used in this paper.

Let $W_k$ be the irreducible representation $U_q\hat{\mathfrak{sl}_2}$ with highest weight monomial

$$X_k^{(a)} = \prod_{i=0}^{k-1} Y_{aq^{k-2i-1}},$$

where $Y_a = Y_{1,a}$. Then the $q$-character of $W_k^{(a)}$ is given by

$$\chi_q(W_k^{(a)}) = X_k^{(a)} \prod_{i=0}^{k-1} A_{aq^{k-2i}}^{-i},$$

where $A_a = Y_{aq^{-1}}Y_{aq}$.

For $a \in \mathbb{C}^\times$, $k \in \mathbb{Z}_{\geq 1}$, the set $\Sigma_k^{(a)} = \{aq^{k-2i-1}\}_{i=0,\ldots,k-1}$ is called a string. Two strings $\Sigma_k^{(a)}$ and $\Sigma_{k'}^{(a')}$ are said to be in general position if the union $\Sigma_k^{(a)} \cup \Sigma_{k'}^{(a')}$ is not a string or $\Sigma_k^{(a')} \subset \Sigma_{k}^{(a)}$ or $\Sigma_{k'}^{(a)} \subset \Sigma_k^{(a)}$.

Denote by $L(m_+)$ the irreducible $U_q\hat{\mathfrak{sl}_2}$-module with highest weight monomial $m_+$. Let $m_+ \neq 1$ and $m_+ \in \mathbb{Z}[Y_a]_{a \in \mathbb{C}^\times}$ be a dominant monomial. Then $m_+$ can be uniquely (up to permutation) written in the form

$$m_+ = \prod_{i=1}^{s} \prod_{b \in \Sigma_{k_i}^{(a_i)}} Y_b,$$

where $s$ is an integer, $\Sigma_{k_i}^{(a_i)}$, $i = 1, \ldots, s$, are strings which are pairwise in general position and

$$L(m_+) = \bigotimes_{i=1}^{s} W_{k_i}^{(a_i)}, \quad \chi_q(L(m_+)) = \prod_{i=1}^{s} \chi_q(W_{k_i}^{(a_i)}).$$

For $j \in I$, let

$$\beta_j: \mathbb{Z}[Y_{i,a}^{\pm1}]_{i \in I, a \in \mathbb{C}^\times} \to \mathbb{Z}[Y_a^{\pm1}]_{a \in \mathbb{C}^\times}$$

be the ring homomorphism such that for all $a \in \mathbb{C}^\times$, $Y_{k,a} \mapsto 1$ for $k \neq j$ and $Y_{j,a} \mapsto Y_a$.

Let $V$ be a $U_q\hat{\mathfrak{g}}$-module. Then $\beta_i(\chi_q(V))$, $i \in \{1, 2, \ldots, n\}$, is the $q$-character of $V$ considered as a $U_q\hat{\mathfrak{g}}$-module.

In some situation, we can use the $q$-characters of $U_q\hat{\mathfrak{sl}_2}$-modules to compute the $q$-characters of $U_q\hat{\mathfrak{g}}$-modules for arbitrary $\mathfrak{g}$, see Section 5 in [FM01]. The corresponding algorithm is called the Frenkel-Mukhin algorithm. The Frenkel-Mukhin algorithm recursively computes the minimal possible $q$-character which contains $m_+$ and is consistent when restricted to $U_q\hat{\mathfrak{sl}_2}$, $i \in \{1, 2, \ldots, n\}$. For example, if a module $L(m_+)$ is special, then the Frenkel-Mukhin algorithm applied to $m_+$, see [FM01], produces the correct $q$-character $\chi_q(L(m_+))$. 
2.6. Path description of \(q\)-characters of types \(A_n, B_n\). We will need the path description of \(q\)-characters of minimal affinizations of types \(A_n, B_n\) which are in Section 3 and Section 6 of [MY12a] to classify dominant monomials in tensor products.

The length of \(\mathcal{T}_{k_1, k_2, \ldots, k_n}^{(s)}\) is defined as \(k_1 + k_2 + \cdots + k_n\).

**Theorem 2.2** (Theorem 6.1, [MY12b]). Suppose that the length of the minimal affinization \(\mathcal{T}_{k_1, k_2, \ldots, k_n}^{(s)}\) is \(M\). Then in the cases of types \(A_n\) and \(B_n\), we have

\[
\chi_q(\mathcal{T}_{k_1, k_2, \ldots, k_n}^{(s)}) = \sum_{(p_1, \ldots, p_M) \in \mathcal{P}(i, k)} \prod_{t=1}^{M} m(p_t). \tag{2.12}
\]

Now we explain the notations in Theorem 2.2, see [MY12a]. A path is a finite sequence of points in the plane \(\mathbb{R}^2\). In the case of type \(A_n\), let

\[
\mathcal{C} = \{(i, k) \in I \times \mathbb{Z} : i - k \equiv 1 \pmod{2}\}.
\]

For all \((i, k) \in \mathcal{C}\), let

\[
\varphi_{i,k} = \{(0, y_0), (1, y_1), \ldots, (n + 1, y_{n+1}) : y_0 = i + k, \ y_{n+1} = n + 1 - i + k, \ \text{and} \ y_{i+1} - y_i \in \{1, -1\}, \ 0 \leq i \leq n\}.
\]

The sets \(C_p^+\) of upper and lower corners of a path \(p = ((r, y_r))_{0 \leq r \leq n+1} \in \varphi_{i,k}\) are defined as follows:

\[
C_p^+ = \{(r, y_r) : r \in I, \ y_{r-1} = y_r + 1 = y_{r+1}\},
\]

\[
C_p^- = \{(r, y_r) : r \in I, \ y_{r-1} = y_r - 1 = y_{r+1}\}.
\]

In the case of type \(B_n\), let

\[
\mathcal{C} = \{(n, 2k + 1) : k \in \mathbb{Z}\} \cup \{(i, k) \in I \times \mathbb{Z} : i < n \text{ and } k \equiv 0 \pmod{2}\}.
\]

It is written that \((j, l) \in p\) if \((j, l)\) is a point of the path \(p\).

Fix an \(\varepsilon, 0 < \varepsilon < 1/2\), \(\varphi_{n,l}\) for all \(l \in 2\mathbb{Z} + 1\) are defined as follows. For all \(l \equiv 3 \pmod{4}\),

\[
\varphi_{n,l} = \{(0, y_0), (2, y_1), \ldots, (2n - 4, y_{n-2}), (2n - 2, y_{n-1}), (2n - 1, y_n) : y_0 = l + 2n - 1, y_{i+1} - y_i \in \{-2, 2\}, \ 0 \leq i \leq n - 2, \ \text{and} \ y_n - y_{n-1} \in \{1 + \varepsilon, -1 - \varepsilon\}\}.
\]

For all \(l \equiv 1 \pmod{4}\),

\[
\varphi_{n,l} = \{(4n - 2, y_0), (4n - 4, y_1), \ldots, (2n + 2, y_{n-2}), (2n, y_{n-1}), (2n - 1, y_n) : y_0 = l + 2n - 1, y_{i+1} - y_i \in \{-2, 2\}, \ 0 \leq i \leq n - 2, \ \text{and} \ y_n - y_{n-1} \in \{1 + \varepsilon, -1 - \varepsilon\}\}.
\]

For all \((i, k) \in \mathcal{C}, i < n, \ \varphi_{i,k}\) are defined as follows:

\[
\varphi_{i,k} = \{(a_0, a_1, \ldots, a_n, \overline{a}_n, \ldots, \overline{a}_1, \overline{a}_0) : (a_0, a_1, \ldots, a_N) \in \varphi_{n,k-(2n-2i-1)}, \ (\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_n) \in \varphi_{n,k+(2n-2i-1)}, \ \text{and} \ a_n - \overline{a}_n = (0, y) \ \text{where} \ y > 0\}.
\]

For all \((i, k) \in \mathcal{C}\), the sets of upper and lower corners \(C_p^\pm\) of a path

\[
p = ((j_r, l_r))_{0 \leq r \leq |p|-1} \in \varphi_{i,k},
\]
where $|p|$ is the number of points in the path $p$, are defined as follows:
\[ C^+_p \equiv \tau^{-1}\{(j_r, l_r) \in p : j_r \not\in \{0, 2n - 1, 4n - 2\}, l_r - 1 > l_r, l_{r+1} > l_r \} \]
\[ C^-_p \equiv \tau^{-1}\{(j_r, l_r) \in p : j_r \not\in \{0, 2n - 1, 4n - 2\}, l_r - 1 < l_r, l_{r+1} < l_r \} \]
where $\tau$ is defined as follows:
\[ \tau(i, k) = \begin{cases} (2i, k), & \text{if } i < n \text{ and } 2n + k - 2i \equiv 2 \pmod{4}, \\ (4n - 2 - 2i, k), & \text{if } i < n \text{ and } 2n + k - 2i \equiv 0 \pmod{4}, \\ (2n - 1, k), & \text{if } i = n. \end{cases} \]

A map $m$ sending paths to monomials is defined by
\[ m : \bigsqcup_{(i, k) \in \mathcal{C}} \varphi_{i, k} \to \mathbb{Z}[Y], (j_l, l) \in \mathcal{C} : p \to m(p) = \prod_{(j_l) \in C^+_p} Y_{j_l,l} \prod_{(j_l) \in C^-_p} Y_{j_l,l}^{-1}. \]

Let $p, p'$ be paths. It is said that $p$ is strictly above $p'$ or $p'$ is strictly below $p$ if $(x, y) \in p$ and $(x, z) \in p' \implies y < z$. It is said that a $T$-tuple of paths $(p_1, \ldots, p_T)$ is non-overlapping if $p_s$ is strictly above $p_t$ for all $s < t$.

For any $(i_t, k_t) \in \mathcal{C}, 1 \leq t \leq T, T \in \mathbb{Z}_{\geq 1}, \mathcal{T}_{(i_t, k_t)_{1 \leq t \leq T}}$ is defined by
\[ \mathcal{T}_{(i_t, k_t)_{1 \leq t \leq T}} = \{(p_1, \ldots, p_T) : p_t \in \varphi_{i_t, k_t}, 1 \leq t \leq T, (p_1, \ldots, p_T) \text{ is non-overlapping}\}. \]

By Theorem 2.2 the $q$-character of a minimal affinization $\mathcal{T}_{k_1, k_2, \ldots, k_n}$ of type $A_n$ or $B_n$ with length $M$ is given by a set of $M$-tuples of non-overlapping paths. The paths in each $M$-tuple in this set are non-overlapping, this property is called non-overlapping property.

We also need the following notations in this paper. For all $(i, k) \in \mathcal{C}$, let $p^+_{i, k}$ be the highest path which is the unique path in $\varphi_{i, k}$ with no lower corners and $p^-_{i, k}$ the lowest path which is the unique path in $\varphi_{i, k}$ with no upper corners.

3. M-systems of types $A_n$, $B_n$

In this section, we describe $M$-systems of types $A_n$, $B_n$.

3.1. M-systems of types $A_n$, $B_n$. When we write $[T_{0, \ldots, 0, k_i, 0, \ldots, 0, k_j + 1, k_{j+1}, \ldots, k_n}]$, we mean $k_i$ is in the $(i + 1)$-th position, $k_j + 1$ is in the $j$-th position. Our first main result in this paper is the following systems which we call $M$-systems of types $A_n$ and $B_n$ respectively.

**Theorem 3.1.** In the case of type $A_n$, for $s \in \mathbb{Z}, k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$, we have
\[ [T^{(s-2)}_{k_1, k_2, k_3, \ldots, k_n}] [T^{(s)}_{1, k_2, k_3, \ldots, k_n}] = [T^{(s-2)}_{k_1, k_2 + 1, k_3, \ldots, k_n}] [T^{(s)}_{1 - 1, k_2 + 1, k_3, \ldots, k_n}] + [T^{(s-2)}_{0, k_1 + k_2 + 1, k_3, \ldots, k_n}] [T^{(s+2k_1)}_{0, k_2, k_3, \ldots, k_n}], \] (3.1)
where \( k_1 > 0; \)

\[
[T_{0,...,0,k_i,1,k_{i+1},2,...,k_n}][T_{0,...,0,k_i,1,k_{i+1},k_{i+2},...,k_n}] = [T_{0,...,0,k_i,1,k_{i+1},k_{i+2},...,k_n}][T_{0,...,0,k_i-1,k_{i+1}+1,k_{i+2},...,k_n}]
+ [T_{0,...,0,k_i,1,k_{i+1}+1,k_{i+2},...,k_n}][T_{0,...,0,k_i,0,k_{i+1}+1,k_{i+2},...,k_n}],
\]

(3.2)

where \( k_i > 0, 1 < i \leq n - 1; \)

\[
[T_{k_1,0,...,0,k_i,k_{i+1},...,k_n}][T_{k_1,0,...,0,k_i,k_{i+1},k_{i+2},...,k_n}] = [T_{k_1+1,0,...,0,k_i,k_{i+1},...,k_n}][T_{k_1-1,0,...,0,k_i,k_{i+1},...,k_n}]
+ [T_{k_1,0,k_i,0,...,0,k_i,k_{i+1},...,k_n}][T_{0,...,0,k_i,k_{i+1},...,k_n}],
\]

(3.3)

where \( k_1 > 0; \)

\[
[T_{0,...,0,k_i,0,...,0,k_i,k_{i+1},...,k_n}][T_{0,...,0,k_i,0,...,0,k_i,k_{i+1},k_{i+2},...,k_n}] = [T_{0,...,0,k_i,0,...,0,k_i,k_{i+1},k_{i+2},...,k_n}][T_{0,...,0,k_i,1,...,k_n}]
+ [T_{0,...,0,k_i,1,...,k_n}][T_{0,...,0,k_i,0,...,0,k_i,k_{i+1},...,k_n}],
\]

(3.4)

where \( k_i > 0, 2 < i + 1 < n. \)

In the case of type \( B_n, \) for \( s \in \mathbb{Z}, k_1,...,k_n \in \mathbb{Z}_{\geq 0}, \) we have

\[
[T_{k_1,k_2,k_3,...,k_n}][T_{k_1,k_2+1,k_3,...,k_n}] = [T_{k_1+1,k_2,k_3,...,k_n}][T_{k_1-1,k_2+1,k_3,...,k_n}]
+ [T_{0,k_1+k_2+1,k_3,...,k_n}][T_{0,k_2,k_3,...,k_n}],
\]

(3.5)

where \( k_1 > 0; \)

\[
[T_{0,...,0,k_i,k_{i+1},1,k_{i+2},...,k_n}][T_{0,...,0,k_i,k_{i+1},k_{i+2},...,k_n}] = [T_{0,...,0,k_i,k_{i+1},k_{i+2},...,k_n}][T_{0,...,0,k_i-1,k_{i+1}+1,k_{i+2},...,k_n}]
+ [T_{0,...,0,k_i,k_{i+1}+1,k_{i+2},...,k_n}][T_{0,...,0,k_i,1,...,k_n}],
\]

(3.6)

where \( k_i > 0, 1 < i < n - 1; \)

\[
[T_{0,...,0,k_{n-1},1}][T_{0,...,0,k_{n-1}+1}] = [T_{0,...,0,k_{n-1}+1,k_n-1+1,k_n+2}][T_{0,...,0,k_{n-1}-1,k_n+2}]
+ [T_{0,...,0,k_{n-1}+1,k_n+2}][T_{0,...,0,k_{n-1}+1,k_n+2}],
\]

(3.7)

where \( k_{n-1} > 0; \)

\[
[T_{0,...,0,k_{n-1},1,0}][T_{0,...,0,k_{n-1},1,1}] = [T_{0,...,0,k_{n-1}+1,0}][T_{0,...,0,k_{n-1}-1,1}]
+ [T_{0,...,0,k_{n-1}+1,1}][T_{0,...,0,k_{n-1}-1,0}],
\]

(3.8)

where \( k_{n-1} > 0; \)

\[
[T_{k_1,0,...,0,k_j,k_{j+1},...,k_n}][T_{k_1,0,...,0,k_j+1,k_{j+1},...,k_n}] = [T_{k_1+1,0,...,0,k_j,k_{j+1},...,k_n}][T_{k_1-1,0,...,0,k_j+1,k_{j+1},...,k_n}]
+ [T_{0,k_1,0,...,0,k_j+1,k_{j+1},...,k_n}][T_{0,...,0,k_j,k_{j+1},...,k_n}],
\]

(3.9)
where \( k_1 > 0; \)

\[
[T^{(s)}_{k_1,0,...,0,k_1}][T^{(s-4)}_{k_1,0,...,0,k_2+2}] = [T^{(s-4)}_{k_1+1,0,...,0,k_1}][T^{(s)}_{k_1-1,0,...,0,k_2+2}] + [T^{(s-4)}_{0,k_1,0,...,0,k_2+2}][T^{(s+4)}_{0,...,0,k_2}];
\]  

where \( k_1 > 0; \)

\[
[T^{(s)}_{k_1,0,...,0,k_1}][T^{(s-4)}_{k_1,0,...,0,1}] = [T^{(s-4)}_{k_1+1,0,...,0}] [T^{(s)}_{k_1-1,0,...,0,1}] + [T^{(s-4)}_{0,k_1,0,...,0,1}];
\]  

where \( k_1 > 0; \)

\[
[T^{(s)}_{0,...,0,k_i,...,0,k_{i+1},...}] [T^{(s-4)}_{0,...,0,k_i,...,0,k_{i+1},...}] = [T^{(s-4)}_{0,...,0,k_i+1,...,0,k_{i+1},...}] [T^{(s)}_{0,...,0,k_i-1,...,0,k_{i+1},...}]
+ [T^{(s-4)}_{0,...,0,k_i,...,0,k_{i+1},...}] [T^{(s)}_{0,...,0,k_i,...,0,k_{i+1},...}],
\]  

where \( k_i > 0, \ 2 < i + 1 < n; \)

\[
[T^{(s)}_{0,...,0,k_i,...,0,k_i+1,...}] [T^{(s-4)}_{0,...,0,k_i,...,0,k_i+1,...}] = [T^{(s-4)}_{0,...,0,k_i+1,...,0,k_i+1,...}] [T^{(s)}_{0,...,0,k_i-1,...,0,k_i+1,...}]
+ [T^{(s-4)}_{0,...,0,k_i,...,0,k_i+1,...}] [T^{(s)}_{0,...,0,k_i,...,0,k_i+1,...}],
\]  

where \( k_i > 0, \ 2 \leq i < n - 1; \)

\[
[T^{(s)}_{0,...,0,k_i,...,0,k_i+1,...}] [T^{(s-4)}_{0,...,0,k_i,...,0,k_i+1,...}] = [T^{(s-4)}_{0,...,0,k_i+1,...,0,k_i+1,...}] [T^{(s)}_{0,...,0,k_i-1,...,0,k_i+1,...}]
+ [T^{(s-4)}_{0,...,0,k_i,...,0,k_i+1,...}] [T^{(s)}_{0,...,0,k_i,...,0,k_i+1,...}],
\]  

where \( k_i > 0, \ 2 \leq i < n - 1. \)

Theorem 3.3 will be proved in Section 8.

**Example 3.2.** The following are some equations in the M-system of type \(A_3.\)

\[
[1-1][1-320] = [1-31-1][20] + [2-20], \\
[1-31-1][1-51-320] = [1-320][1-51-31-1] + [2-42-20], \\
[2-2][2-43-1] = [2-42-2][3-1] + [1-3][3-33-1], \\
[2-42-2][2-62-43-1] = [2-43-1][2-62-42-2] + [1-51-3][3-53-33-1], \\
[1-3][1-53-1] = [1-51-3][3-1] + [2-43-1], \\
[1-51-3][1-71-53-1] = [1-53-1][1-71-51-3] + [2-62-43-1], \\
[1-53-1][1-72-43-1] = [1-71-53-1][2-43-1] + [2-62-43-1][3-1], \\
[1-71-53-1][1-91-72-43-1] = [1-72-43-1][1-91-71-53-1] + [2-82-62-43-1][3-1].
\]
The following are some equations in the $M$-system of type $B_2$.

\begin{align*}
[1_{-3}] [1_{-2} - 2_{20}] &= [1_{-1} - 3] [2_{20}] + [2_{6} 2_{-4} 2_{-20}], \\
[1_{-1}] [1_{-2} - 2_{20}] &= [1_{-2} - 2_{20}] [1_{-1} - 1_{-1} - 1] + [2_{10} 2_{-8} 2_{-2} - 4_{-2} 2_{-20}], \\
[1_{-1}] [1_{-5} 2_{0}] &= [2_{0}] [1_{-1} - 1] + [2_{-4} 2_{-2} 2_{0}], \\
[1_{-1}] [1_{-9} - 1_{-5} 2_{0}] &= [1_{-5} 2_{0}] [1_{-9} - 1_{-5} 1_{-1}] + [2_{8} 2_{-6} 2_{-4} 2_{-2} 2_{0}].
\end{align*}

Moreover, we have the following theorem.

**Theorem 3.3.** The modules in the summands on the right hand side of each equation in Theorem 3.1 are simple.

Theorem 3.3 will be prove in Section 9.

3.2. The $m$-systems of types $A_n, B_n$. For $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}$, let

\[ m_{k_1, \ldots, k_n} \] be the restriction of $T_{k_1, \ldots, k_n}^{(s)}$ to $U_q \mathfrak{g}$. It is clear that $\text{Res}(T_{k_1, \ldots, k_n}^{(s)})$ doesn’t depend on $s$. Let $\chi(M)$ be the character of a $U_q \mathfrak{g}$-module $M$. By replacing each $[T_{k_1, \ldots, k_n}^{(s)}]$ in the $M$-system of type $A_n$ (resp. $B_n$) in Theorem 3.1 with $\chi(m_{k_1, \ldots, k_n})$, we obtain a system of equations which we called the $m$-system of type $A_n$ (resp. $B_n$). The following is an equation in the $m$-system of type $A_n$.

\[
\chi(m_{0, \ldots, 0, k_i, 0, \ldots, 0, k_j, k_{j+1}, \ldots, k_n}) \chi(m_{0, \ldots, 0, k_i+1, 0, \ldots, 0, k_j, k_{j+1}, \ldots, k_n}) = \chi(m_{0, \ldots, 0, k_i, 0, \ldots, 0, k_j, k_{j+1}, \ldots, k_n})
\]

\[
+ \chi(m_{0, \ldots, 0, k_i, 0, \ldots, 0, k_j, k_{j+1}, \ldots, k_n}) \chi(m_{0, \ldots, 0, k_i, 0, \ldots, 0, k_j+1, k_{j+1}, \ldots, k_n}).
\]

(3.15)

where $k_i > 0, k_j \geq 0, 2 < i + 1 < j \leq n$.

4. Relation between $M$-systems and cluster algebras

In this section, we will show that the equations in the $M$-system of type $A_n$ (resp. $B_n$) correspond to mutations in some cluster algebra $\mathcal{A}$ (resp. $\mathcal{A}'$). Moreover, every minimal affinization in the $M$-system of type $A_n$ (resp. $B_n$) corresponds to a cluster variable in the cluster algebra $\mathcal{A}$ (resp. $\mathcal{A}'$).

4.1. Definition of cluster algebras $\mathcal{A}$ and $\mathcal{A}'$. Let $I = \{1, 2, \ldots, n\}$ and

\[ S_1 = \{-2i - 1 \mid i \in \mathbb{Z}_{\geq 0}\}, \quad S_2 = \{-2i \mid i \in \mathbb{Z}_{\geq 0}\}, \quad S_3 = \{2i + 1 \mid i \in \mathbb{Z}_{\geq 0}\}, \quad S_4 = \{2i \mid i \in \mathbb{Z}_{\geq 0}\}, \quad S_5 = \{-4i - 1 \mid i \in \mathbb{Z}_{\geq 0}\}, \quad S_6 = \{-4i - 3 \mid i \in \mathbb{Z}_{\geq 0}\}.
\]

Let

\[ V = ((S_3 \cap I) \times S_1) \cup ((S_4 \cap I) \times S_2), \]

\[ V' = ((\{n\} \times S_2) \cup (\{1, \ldots, n - 1\} \times S_3) \cup (\{1, \ldots, n - 1\} \times S_5).\]

A quiver $Q$ (resp. $Q'$) for $U_q \widehat{\mathfrak{g}}$ of type $A_n$ (resp. $B_n$) with vertex set $V$ (resp. $V'$) will be defined as follows. The arrows of $Q$ (resp. $Q'$) are given by the following rule: there is an arrow from
the vertex \((i, r)\) to the vertex \((j, s)\) if and only if \(b_{ij} \neq 0\) and \(s = r + b_{ij} - d_i + d_j\). The quiver \(Q\) (resp. \(Q'\)) of type \(A_n\) (resp. \(B_n\)) is the same as the quiver \(G^-\) (resp. \(G^-\)) of type \(A_n\) (resp. \(B_n\)) in [HL13]. In the case of type \(A_n\), let

\[
t = \{(t_{i}^{(-2k_{i},i+3)}_{0,\ldots,0,k_{i},0,\ldots,0} | i \text{ is even}, \ k_{i} \in \mathbb{Z}_{\geq 0}) \cup \{(t_{i}^{(-2k_{i},i+2)}_{0,\ldots,0,k_{i},0,\ldots,0} | i \text{ is odd}, \ k_{i} \in \mathbb{Z}_{\geq 0})\}. \tag{4.1}
\]

In the case of type \(B_n\), let \(t' = t_1 \cup t_2\), where

\[
t_1 = \{(t_{i}^{(-2n-2k_{n}+5)}_{0,\ldots,0,k_{n}} | k_{n} \in \mathbb{Z}_{\geq 0})\}, \tag{4.2}
\]

\[
t_2 = \{(t_{i}^{(-4k_{i}-2i+3)}_{0,\ldots,0,k_{i},0,\ldots,0} | i \in \{1, \ldots, n-1\}, \ k_{i} \in \mathbb{Z}_{\geq 0}\}. \tag{4.3}
\]

Let \(\mathcal{A}\) (resp. \(\mathcal{A}'\)) be the cluster algebra defined by the initial seed \((t, Q)\) (resp. \((t', Q')\)). The cluster algebra \(\mathcal{A}\) (resp. \(\mathcal{A}'\)) of type \(A_n\) (resp. \(B_n\)) is the same as the cluster algebra for \(U_{\varphi}\) of type \(A_n\) (resp. \(B_n\)) introduced in [HL13].

4.2. Mutation sequences: type \(A_n\) case. We use “\(C_{2i-1}\)” to denote the column of vertices \((2i-1, -1), (2i-1, -3), (2i-1, -5), \ldots \) in \(Q\), where \(i \in \mathbb{Z}_{\geq 1}\). We use “\(C_i\)” to denote the column of vertices \((2i, 0), (2i, -2), (2i, -4), \ldots \) in \(Q\), where \(i \in \mathbb{Z}_{\geq 1}\). Let \(C_1, \ldots, C_n\) be the columns of the quiver. By saying that we mutate the column \(C_i\), \(i \in \{1, \ldots, n\}\), we mean that we mutate the vertices of \(C_i\) as follows. First we mutate at the first vertex of \(C_i\), then the second vertex of \(C_i\), an so on until the vertex at infinity. By saying that the mutate \((C_i, C_{i_2}, \ldots, C_{i_m})\), where \(i_1, \ldots, i_m \in \{1, 2, \ldots, n\}\), we mean that we first mutate the column \(C_{i_1}\), then the column \(C_{i_2}\), an so on up to the column \(C_{i_m}\).

Let \(k_1, k_2, \ldots, k_n \in \mathbb{Z}_{\geq 0}\) and let \(k_{r}\) be the first non-zero integer in \(k_1, k_2, \ldots, k_n\) from the right. We define some variables \(t_{k_1,k_2,\ldots,k_n}^{(s)}\), where

\[
s = -2\left(\sum_{i=1}^{r} k_i\right) - r - [r \mod 2] + 3,
\]

recursively as follows. The variables \(t_{0,\ldots,0,k,0,\ldots,0}^{(s)} (k \in \mathbb{Z}_{\geq 0})\) in \(4.1\) are already defined. They are cluster variables in the initial seed of \(\mathcal{A}\) define in Section 4.1.

We use \(\emptyset\) to denote the empty mutation sequence and use

\[
\prod_{k=1}^{\hat{r}} (C_{2k}, C_{2k-1}, C_{2k-2}, \ldots, C_{1})
\]

to denote the mutation sequence

\[
(C_2, C_1; C_4, C_3, C_2, C_1; \ldots; C_{n-3}, C_{n-4}, \ldots, C_1; C_{n-1}, C_{n-2}, \ldots, C_1).
\]

Let

\[
M^{(1)}_{r} = \begin{cases} 
\emptyset, & r = 1, 2, \\
\prod_{k=1}^{\hat{r}} (C_{2k}, C_{2k-1}, C_{2k-2}, \ldots, C_{1}), & r \equiv 1 \pmod{2}, r > 1, \\
\prod_{k=1}^{\hat{r}} (C_{2k}, C_{2k-1}, C_{2k-2}, \ldots, C_{1}), & r \equiv 0 \pmod{2}, r > 2.
\end{cases}
\]


Let $k_1, k_2, \ldots, k_n \in \mathbb{Z}_{\geq 0}$ and $k_i$ be the first non-zero integer in $k_1, k_2, \ldots, k_n$ from the right. Let Seq be the mutation sequence: first we mutate $M_r$ starting from the initial quiver $Q$, then we mutate $(C_{r-1}, C_{r-2}, \ldots, C_1)$ $k_r$ times, and then we mutate $(C_{r-2}, C_{r-3}, \ldots, C_1)$ $k_{r-1}$ times; continue this procedure, we mutate $(C_{t-1}, C_{t-2}, \ldots, C_1)$ $k_t$ times, $t = r-2, r-3, \ldots, 2$. If $k_t = 0$, then “we mutate $(C_{r-1}, C_{r-2}, \ldots, C_1)$ $k_t$ times” means “we do not mutate $(C_{t-1}, C_{t-2}, \ldots, C_1)$”.

We define

$$t(\sum_{i=1}^{r} k_i) - r \mod 2 + 1 = t(\sum_{i=1}^{r'} k_i) - r \mod 2 + 1 \quad (1 \leq p < q \leq n), \quad (4.4)$$

where

$$t(\sum_{i=1}^{r} k_i) - r \mod 2 + 3) \quad (4.5)$$

where $k_1 > 0$;

$$t(\sum_{i=1}^{r} k_i) - r \mod 2 + 1) \quad (4.6)$$

where $k_1 > 0$, $1 \leq i \leq n - 1$;

$$t(\sum_{i=1}^{r} k_i) - r \mod 2 + 1) \quad (4.7)$$

where $k_1 > 0$;

$$t(\sum_{i=1}^{r} k_i) - r \mod 2 + 1) \quad (4.8)$$

where $k_i > 0$, $2 < i + 1 < n$; are mutation equations which occur when we mutate Seq. The variables $[4.4]$ are defined in the order according to the mutation sequence Seq. In this order, every variable in $[4.4]$ is defined by an equation of $[4.5]$–$[4.8]$ using variables in $t$ and those variables in $[4.4]$ which are already defined.
4.3. Mutation sequences: type $B_n$ case. We use “$C_n$” to denote the column of vertices $(n, 0)$, $(n, -2)$, $(n, -4)$, ... in $Q'$. For $1 \leq i < n$, $n - i \equiv 1 \mod 2$, we use “$C_i$” to denote the column of vertices $(i, -1)$, $(i, -5)$, $(i, -9)$, ... in $Q'$; we use “$C_{2n-i}$” to denote the column of vertices $(i, -3)$, $(i, -7)$, $(i, -11)$, ... in $Q'$. For $1 \leq i < n$, $n - i \equiv 0 \mod 2$, we use “$C_i$” to denote the column of vertices $(i, -3)$, $(i, -7)$, $(i, -11)$, ... in $Q'$; we use “$C_{2n-i}$” to denote the column of vertices $(i, -1)$, $(i, -5)$, $(i, -9)$, ... in $Q'$.

Let $k_1, k_2, \ldots, k_n \in \mathbb{Z}_{\geq 0}$ and $k_r$ be the first non-zero integer in $k_1, k_2, \ldots, k_n$ from the right. We define

$$
\begin{align*}
N_n^{(1)} &= \begin{cases} 
\emptyset, & n = 2, \\
\prod_{k=0}^{n-3} (C_{2n-2k-1}, C_{2n-2k}, \ldots, C_{2n-1}), & n \equiv 1 \mod 2, n \geq 3, \\
\prod_{k=0}^{n-4} (C_{2n-2k-2}, C_{2n-2k-1}, \ldots, C_{2n-1}), & n \equiv 0 \mod 2, n > 2,
\end{cases} \\
N_n^{(2)} &= \begin{cases} 
\emptyset, & n = 2, 3, \\
\prod_{k=0}^{n-5} (C_{2k+2}, C_{2k+1}, \ldots, C_1), & n \equiv 1 \mod 2, n > 3, \\
\prod_{k=0}^{n-4} (C_{2k+1}, C_{2k}, \ldots, C_1), & n \equiv 0 \mod 2, n > 2,
\end{cases} \\
N_{n,r}^{(3)} &= \begin{cases} 
\emptyset, & r = 1, \\
\emptyset, & n \equiv 0 \mod 2, r = 2, \\
\prod_{k=0}^{r-1} (C_{2n-2k-1}, C_{2n-2k}, \ldots, C_{2n-1}), & n \equiv 1 \mod 2, r \equiv 1 \mod 2, r > 1, \\
\prod_{k=0}^{r-1} (C_{2n-2k-1}, C_{2n-2k}, \ldots, C_{2n-1}), & n \equiv 1 \mod 2, r \equiv 0 \mod 2, r \geq 2, \\
\prod_{k=0}^{r-1} (C_{2n-2k-2}, C_{2n-2k-1}, \ldots, C_{2n-1}), & n \equiv 0 \mod 2, r \equiv 0 \mod 2, r > 2, \\
\prod_{k=0}^{r-1} (C_{2n-2k-2}, C_{2n-2k-1}, \ldots, C_{2n-1}), & n \equiv 0 \mod 2, r \equiv 1 \mod 2, r > 1.
\end{cases}
\end{align*}
$$

For $k_n \neq 0$ is even and $n$ is odd (resp. even), let Seq be the mutation sequence: first we mutate $N_n^{(1)}$ starting from the initial quiver $Q'$, then we mutate $(C_{n+1}, C_{n+2}, \ldots, C_{2n-1})$ $k_n$ times, and then we mutation $(C_{n+2}, C_{n+3}, \ldots, C_{2r-1})$ $k_{n-1}$ times, continue this procedure, we mutate $(C_{2n-t+1}, C_{2n-t+2}, \ldots, C_{2n-1}) k_t$ times, $t = n-2, n-3, \ldots, 2$. For $k_n \neq 0$ is odd, let Seq be the mutation sequence: first we mutate $N_n^{(3)}$ starting from the initial quiver $Q'$, then
we mutate \( (C_{n-1}, C_{n-2}, \ldots, C_1) \) \( k_{n-1} \) times, and then we mutate \( (C_{n-2}, C_{n-3}, \ldots, C_1) \) \( k_{n-1} \) times; continue this procedure, we mutate \( (C_{t-1}, C_{t-2}, \ldots, C_1) \) \( k_t \) times, \( t = n - 2, n - 3, \ldots, 2 \). For \( k_n \), let \( k_r \) be the first non-zero integer from right in \( k_1, k_2, \ldots, k_n \) and let Seq be the mutation sequence: first we mutate \( \Lambda'_{n,r} \) starting from the initial quiver \( Q' \), then we mutate \( (C_{2n-r+1}, C_{2n-r+2}, \ldots, C_{2n-1}) \) \( k_r \) times, and then we mutation \( (C_{2n-r+2}, C_{2n-r+3}, \ldots, C_{2n-1}) \) \( k_{r-1} \) times; continue this procedure, we mutate \( (C_{2n-r+1}, C_{2n-r+2}, \ldots, C_{2n-1}) \) \( k_t \) times, \( t = r - 2, r - 3, \ldots, 2 \).

We define

\[
\begin{align*}
t^{(n-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} &= t_0^{(n-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1} + 1} \prod_{i=0}^{n-1} k_i \prod_{j=0}^{n-1} k_{j+1} \\
&= t_0^{(n-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} + t_0^{(n-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} \\
&= (n-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}
\end{align*}
\]

(4.9)

where

\[
\begin{align*}
t^{(-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} &= t^{(-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} \\
&= (n-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}
\end{align*}
\]

(4.10)

where \( k_1 > 0 \);

\[
\begin{align*}
t^{(-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} &= t^{(-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} \\
&= (n-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}
\end{align*}
\]

(4.11)

where \( k_i > 0, 1 < i < n - 1 \);

\[
\begin{align*}
t^{(-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} &= t^{(-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} \\
&= (n-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}
\end{align*}
\]

(4.12)

where \( k_{n-1} > 0 \);

\[
\begin{align*}
t^{(-4k_{n-1}-2n+7)} &= t_0^{(-4k_{n-1}-2n+7)} \\
&= (n-4k_{n-1}-2n+7)
\end{align*}
\]

(4.13)

where \( k_{n-1} > 0 \);

\[
\begin{align*}
t^{(-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} &= t^{(-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}} \\
&= (n-2) \sum d_i k_i - 1 - 2(n-r)_{mod 2+1}
\end{align*}
\]

(4.14)
where \( k_1 > 0; \)
\[
\begin{align*}
& \ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 5} \\
& = \frac{\ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 1}}{\ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 5}} \\
& = \frac{\ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 1} + (2 \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 1) \ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 5}}{0, 0, \ldots, 0, 0, 0}.
\end{align*}
\]

where \( k_1 > 0; \)
\[
\begin{align*}
& \ell^{(-4k_{n-1} - 2n + 7)} \\
& = \frac{\ell^{(-4k_{n-1} - 2n + 3)} + \ell^{(-4k_{n-1} - 2n + 3)} \ell^{(-4k_{n-1} - 2n + 7)}}{\ell^{(-4k_{n-1} - 2n + 7)}},
\end{align*}
\]

where \( k_1 > 0; \)
\[
\begin{align*}
& \ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 5} \\
& = \frac{\ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 1} + (2 \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 1) \ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 5}}{0, 0, \ldots, 0, 0, 0, 0}.
\end{align*}
\]

where \( k_1 > 0, 2 < i + 1 < n; \)
\[
\begin{align*}
& \ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 5} \\
& = \frac{\ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 1} + (2 \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 1) \ell^{(-2) \sum_{i=1}^s d_i k_i - 2(n-r) \mod 2 + 5}}{0, 0, \ldots, 0, 0, 0, 0}.
\end{align*}
\]

where \( k_1 > 0, 2 \leq i < n - 1; \)
\[
\begin{align*}
& \ell^{(-4k_i - 2n + 7)} \\
& = \frac{\ell^{(-4k_i - 2n + 3)} + \ell^{(-4k_i - 2n + 3)} \ell^{(-4k_i - 2n + 7)}}{\ell^{(-4k_i - 2n + 7)}},
\end{align*}
\]

where \( k_1 > 0, 2 \leq i < n - 1; \) are mutation equations which occur when we mutate \( \text{Seq} \). The variables \( (\text{4.9}) \) are defined in the order according to the mutation sequence \( \text{Seq} \). In this order, every variable in \( (\text{4.9}) \) is defined by an equation of \( (\text{4.10}) \)–\( (\text{4.19}) \) using variables in \( t \) and those variables in \( (\text{4.8}) \) which are already defined.

### 4.4. The equations in the \( M \)-system of types \( A_n \) (resp. \( B_n \)) correspond to mutations in the cluster algebra \( \mathcal{A} \) (resp. \( \mathcal{A}' \)).

By \( (\text{4.5}) \), we have
\[
\begin{align*}
& \ell^{(s-2)}_{k_1, k_2, \ldots, k_n} = \ell^{(s-2)}_{k_1, k_2, \ldots, k_n} \\
& \ell^{(s-2)}_{k_1, k_2, \ldots, k_n} = \ell^{(s-2)}_{k_1, k_2, \ldots, k_n} + \ell^{(s-2)}_{k_1, k_2, \ldots, k_n} + \ell^{(s+2k_1)}_{k_1, k_2, \ldots, k_n},
\end{align*}
\]
where \( s = -2(\sum_{i=1}^r k_i) - r - [r] \mod 2 + 3 \). Equations \( (\text{4.20}) \) correspond to Equations \( (\text{3.1}) \) in the \( M \)-system of type \( A_n \).
By (4.6), we have
\[ t_{0,...,0,k_1,k_{i+1},k_{i+2},...,k_n} = t_{0,...,0,k_1,k_{i+1},k_{i+2},...,k_n}^{(s-2)} + t_{0,...,0,k_1+k_{i+1},k_{i+2},...,k_n}^{(s-2)} t_{0,...,0,k_k,...,k_n} t_{0,...,0,k_{i+1},k_{i+2},...,k_n}^{(s)} t_{0,...,0,k_1,k_{i+1},k_{i+2},...,k_n}^{(s)}, \]
where \( s = -2(\sum_{i=1}^r k_i) - r - [r]_{\text{mod} 2} + 3 \). Equations (4.21) correspond to Equations (3.3) in the M-system of type \( A_n \).

By (4.7), we have
\[ t_{k_1,0,...,0,k_j+1,k_{j+1},...,k_n} = t_{k_1,0,...,0,k_j,k_{j+1},...,k_n}^{(s-2)} + t_{k_1+1,0,...,0,k_j,k_{j+1},...,k_n}^{(s-2)} t_{k_1-1,0,...,0,k_j+1,k_{j+1},...,k_n}^{(s-2)} t_{k_1,0,...,0,k_j,k_{j+1},...,k_n}^{(s+2k_1)}, \]
where \( s = -2(\sum_{i=1}^r k_i) - r - [r]_{\text{mod} 2} + 3 \). Equations (4.22) correspond to Equations (3.3) in the M-system of type \( A_n \).

By (4.8), we have
\[ t_{0,...,0,k_1,0,...,0,k_j+1,k_{j+1},...,k_n} = t_{0,...,0,k_1,0,...,0,k_j,k_{j+1},...,k_n}^{(s-2)} + t_{0,...,0,k_1+1,0,...,0,k_j+1,k_{j+1},...,k_n}^{(s-2)} t_{0,...,0,k_1,0,...,0,k_j,k_{j+1},...,k_n}^{(s)} t_{0,...,0,k_1,0,...,0,k_j,k_{j+1},...,k_n}^{(s)}, \]
where \( s = -2(\sum_{i=1}^r k_i) - r - [r]_{\text{mod} 2} + 3 \). Equations (4.23) correspond to Equations (3.4) in the M-system of type \( A_n \).

By (4.10), we have
\[ t_{k_1,k_2+1,k_3,...,k_n} = t_{k_1,k_2,k_3,...,k_n}^{(s-4)} + t_{k_1+1,k_2,k_3,...,k_n}^{(s-4)} t_{k_1-1,k_2+1,k_3,...,k_n}^{(s-4)} t_{k_1,k_2+k_3,...,k_n}^{(s+4k_1)}, \]
where \( s = -2(\sum_{i=1}^r d_i k_i) - 2r - 2[r - n]_{\text{mod} 2} + 5 \). Equations (4.24) correspond to Equations (3.5) in the M-system of type \( B_n \).

By (4.11), we have
\[ t_{0,...,0,k_1,k_{i+1},k_{i+2},...,k_n} = t_{0,...,0,k_1,k_{i+1},k_{i+2},...,k_n}^{(s-4)} + t_{0,...,0,k_1+1,k_{i+1},k_{i+2},...,k_n}^{(s-4)} t_{0,...,0,k_1,k_{i+1},k_{i+2},...,k_n}^{(s)} t_{0,...,0,k_1,k_{i+1},k_{i+2},...,k_n}^{(s)}, \]
where \( s = -2(\sum_{i=1}^r d_i k_i) - 2r - 2[n - r]_{\text{mod} 2} + 5 \). Equations (4.25) correspond to Equations (3.5) in the M-system of type \( B_n \).
where $s = -2 \left( \sum_{i=1}^{r} d_i k_i \right) - 2r - 2[n - r]_{\text{mod } 2} + 5$. Equations (4.26) correspond to Equations (3.6) in the M-system of type $B_n$.

By (4.12), we have

$$t^{(s-4)}_{0,0,0,k_{n-1},k_n+2} = t^{(s)}_{0,0,0,k_{n-1},k_n}$$

$$= t^{(s-4)}_{0,0,0,k_{n-1}+1,k_n} t^{(s)}_{0,0,0,k_{n-1}-1,k_n+2} + t^{(s-4)}_{0,0,0,2k_{n-1}+k_n+2} t^{(s)}_{0,0,0,k_{n-1},0,k_n}, \tag{4.26}$$

where $s = -2 \left( \sum_{i=1}^{r} d_i k_i \right) - 2r - 2[n - r]_{\text{mod } 2} + 5$. Equations (4.26) correspond to Equations (3.7) in the M-system of type $B_n$.

By (4.13), we have

$$t^{(s-4)}_{0,0,0,k_{n-1}-1} = t^{(s)}_{0,0,0,k_{n-1},0}$$

$$= t^{(s-4)}_{0,0,0,k_{n-1}+1,0} t^{(s)}_{0,0,0,k_{n-1}-1,0} + t^{(s-4)}_{0,0,0,2k_{n-1}+1} t^{(s)}_{0,0,0,0,k_{n-1},0}, \tag{4.27}$$

where $s = -4 k_{n-1} - 2n + 7$. Equations (4.27) correspond to Equations (3.8) in the M-system of type $B_n$.

By (4.14), we have

$$t^{(s-4)}_{k_1,0,0,k_{j+1},k_n} = t^{(s)}_{k_1,0,0,k_{j+1},k_n}$$

$$= t^{(s-4)}_{k_1+1,0,0,k_{j+1},k_n} t^{(s)}_{k_1-1,0,0,k_{j+1},k_n} + t^{(s-4)}_{0,0,k_{j+1},0,k_n+1} t^{(s)}_{0,0,0,k_{j+1},k_n}, \tag{4.28}$$

where $s = -2 \left( \sum_{i=1}^{r} d_i k_i \right) - 2r - 2[n - r]_{\text{mod } 2} + 5$. Equations (4.28) correspond to Equations (3.9) in the M-system of type $B_n$.

By (4.15), we have

$$t^{(s-4)}_{k_1,0,0,k_n+2} = t^{(s)}_{k_1,0,0,k_n}$$

$$= t^{(s-4)}_{k_1+1,0,0,k_n} t^{(s)}_{k_1-1,0,0,k_n+2} + t^{(s-4)}_{0,k_1,0,0,k_n+2} t^{(s)}_{0,0,0,k_n}, \tag{4.29}$$

where $s = -2 \left( \sum_{i=1}^{r} d_i k_i \right) - 2r - 2[n - r]_{\text{mod } 2} + 5$. Equations (4.29) correspond to Equations (3.10) in the M-system of type $B_n$.

By (4.16), we have

$$t^{(s-4)}_{k_1,0,0,0,1} = t^{(s)}_{k_1,0,0,0}$$

$$= t^{(s-4)}_{k_1+1,0,0,1} t^{(s)}_{k_1-1,0,0,1} + t^{(s-4)}_{0,k_1,0,0,0,1} t^{(s)}_{0,0,0,0}, \tag{4.30}$$

where $s = -4 k_1 - 2n + 7$. Equations (4.30) correspond to Equations (3.11) in the M-system of type $B_n$. 
By \((4.17)\), we have
\[
\begin{align*}
t_{0,...,0,k_i,0,...,0,k_j+1,...,k_n}^{(s-4)} &= t_{0,...,0,k_i,0,...,0,k_j+1,...,k_n}^{(s)} \\
t_{0,...,0,k_i+1,0,...,0,k_j+1,...,k_n}^{(s-4)} &= t_{0,...,0,k_i+1,0,...,0,k_j+1,...,k_n}^{(s)} + t_{i+1}^{(s-4)} + t_{i-1}^{(s)} \\
&= \frac{t_{0,...,0,k_i,0,...,0,k_j+1,...,k_n}^{(s)}}{t_{0,...,0,k_i,0,...,0,k_j+1,...,k_n}},
\end{align*}
\]

where \(s = -2(\sum_{i=1}^r d_i k_i) - 2r - 2[n - r]_{\text{mod } 2} + 5\). Equations \((4.31)\) correspond to Equations \((3.12)\) in the M-system of type \(B_n\).

By \((4.18)\), we have
\[
\begin{align*}
t_{0,...,0,k_i,0,...,0,k_n+2}^{(s-4)} &= t_{0,...,0,k_i,0,...,0,k_n}^{(s)} \\
t_{0,...,0,k_i+1,0,...,0,k_n}^{(s-4)} &= t_{0,...,0,k_i+1,0,...,0,k_n}^{(s)} + t_{i+1}^{(s-4)} + t_{i-1}^{(s)} \\
&= \frac{t_{0,...,0,k_i,0,...,0,k_n}^{(s)}}{t_{0,...,0,k_i,0,...,0,k_n}},
\end{align*}
\]

where \(s = -2(\sum_{i=1}^r d_i k_i) - 2r - 2[n - r]_{\text{mod } 2} + 5\). Equations \((4.32)\) correspond to Equations \((3.13)\) in the M-system of type \(B_n\).

By \((4.19)\), we have
\[
\begin{align*}
t_{0,...,0,k_i,0,...,0,1}^{(s-4)} &= t_{0,...,0,k_i,0,...,0}^{(s)} \\
t_{0,...,0,k_i+1,0,...,0,1}^{(s-4)} &= t_{0,...,0,k_i+1,0,...,0,1}^{(s)} + t_{i+1}^{(s-4)} + t_{i-1}^{(s)} \\
&= \frac{t_{0,...,0,k_i,0,...,0}^{(s)}}{t_{0,...,0,k_i,0,...,0}},
\end{align*}
\]

where \(s = -4k_i - 2n + 7\). Equations \((4.33)\) correspond to Equations \((3.14)\) in the M-system of type \(B_n\).

Therefore we have the following theorem.

**Theorem 4.1.** Every minimal affinization in the M-system of type \(A_n\) (resp. \(B_n\)) corresponds to a cluster variable in \(\mathcal{A}\) (resp. \(\mathcal{A}'\)) defined in Section 4.1.

5. **The dual M-systems of types \(A_n\) and \(B_n\)**

In this section, we study the dual M-systems of types \(A_n\) and \(B_n\).

5.1. **The dual M-systems of types \(A_n\) and \(B_n\).**

**Theorem 5.1** (Theorem 3.8, [Her07]). In the case of type \(A_n\) (resp. type \(B_n\)), the modules \(\tilde{T}_{k_1,...,k_n}^{(s)}\), \(s \in \mathbb{Z}, k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}\) are special and anti-special.

**Lemma 5.2.** In the case of type \(A_n\), let \(\iota : \mathbb{Z}P \to \mathbb{Z}P\) be a homomorphism of rings such that \(Y_1 a q^s \mapsto Y_{n-1,a q^{n-s+1}}, Y_2 a q^s \mapsto Y_{n-1,a q^{n-s+1}}, \ldots, Y_{n-1,a q^{n-s+1}}\) for all \(a \in \mathbb{C}^*, s \in \mathbb{Z}\). Then
\[
\chi_q(\tilde{T}_{k_1,...,k_n}^{(s)}) = \iota(\chi_q(T_{k_1,...,k_n}^{(s)})).\]
In the case of type \( B_n \), let \( \iota : \mathbb{Z}P \to \mathbb{Z}P \) be a homomorphism of rings such that \( Y_{1,aq^s} \mapsto Y_{1,aq^{s+1}}^{-1} \), \( Y_{2,aq^s} \mapsto Y_{2,aq^{s+1}}^{-1} \), \( Y_{n,aq^s} \mapsto Y_{n,aq^{s+1}}^{-1} \) for all \( a \in \mathbb{C}^\times \), \( s \in \mathbb{Z} \). Then

\[
\chi_q(T_{k_1,\ldots,k_n}) = \iota(\chi_q(T_{k_1,\ldots,k_n})).
\]

**Proof.** Let \( m_+ = T_{k_1,\ldots,k_n} \) and \( \tilde{m}_+ = \tilde{T}_{k_1,\ldots,k_n} \). By Theorem 5.3, the modules \( \tilde{T}_{k_1,\ldots,k_n} \), \( s \in \mathbb{Z} \), \( k_1,\ldots,k_n \in \mathbb{Z}_{\geq 0} \) are anti-special. Therefore \( \chi_q(L(m_+)) \) can be computed by the Frenkel-Mukhin algorithm starting from the lowest weight using \( A_{i,a} \) with \( i \in I, a \in \mathbb{C}^\times \). The procedure is dual to the computation of \( \chi_q(L(m_+)) \) which starts from \( m_+ \) using \( A_{i,a} \) with \( i \in I, a \in \mathbb{C}^\times \). The highest (resp. lowest) \( l \)-weight in \( \chi_q(L(m_+)) \) is sent to the lowest (resp. highest) \( l \)-weight in \( \chi_q(L(m_+)) \) by \( \iota \).

**Theorem 5.3.** Let \( s \in \mathbb{Z} \), \( k_1,\ldots,k_n \in \mathbb{Z}_{\geq 0} \). In the case of type \( A_n \), we have

\[
\tilde{T}_{k_1,0,0,\ldots,0,k_j+k_{j+1},\ldots,k_n} = \tilde{T}_{k_1,0,0,\ldots,0,k_j,k_{j+1},\ldots,k_n} \tilde{T}_{k_1,1,0,0,\ldots,0,k_j,k_{j+1},\ldots,k_n} = \tilde{T}_{k_1,1,0,0,\ldots,0,k_j,k_{j+1},\ldots,k_n} \tilde{T}_{k_1,0,0,\ldots,0,k_j+1,k_{j+1},\ldots,k_n} + \tilde{T}_{k_1,0,0,\ldots,0,k_j+1,k_{j+1},\ldots,k_n} \tilde{T}_{k_1,1,0,0,\ldots,0,k_j+1,k_{j+1},\ldots,k_n} \tilde{T}_{k_1,0,0,\ldots,0,k_j,k_{j+1},\ldots,k_n}\]

where \( k_1 > 0 \);
where $k_l > 0$, $1 < i < n - 1$;

\[
[\tilde{T}^{(s)}_{0,...,0,k_{n-1},k_n}][\tilde{T}^{(s-4)}_{0,...,0,k_{n-1},k_n+2}] = [\tilde{T}^{(s-4)}_{0,...,0,k_{n-1},k_n+1,k_n}][\tilde{T}^{(s)}_{0,...,0,k_{n-1}-1,k_n+2}]
+ [\tilde{T}^{(s-4)}_{0,...,0,2k_{n-1}+k_n+2}][\tilde{T}^{(s)}_{0,...,0,k_{n-1},0,k_n}],
\]

(5.7)

where $k_{n-1} > 0$;

\[
[\tilde{T}^{(s)}_{0,...,0,k_{n-1},0}][\tilde{T}^{(s-4)}_{0,...,0,k_{n-1}-1,1}] = [\tilde{T}^{(s-4)}_{0,...,0,k_{n-1}-1,1}][\tilde{T}^{(s)}_{0,...,0,k_{n-1},0,0}]
+ [\tilde{T}^{(s-4)}_{0,...,0,2k_{n-1}+1}][\tilde{T}^{(s)}_{0,...,0,0,0,0}],
\]

(5.8)

where $k_{n-1} > 0$;

\[
[\tilde{T}^{(s)}_{k_1,0,...,0,k_{j+1},...,k_n}][\tilde{T}^{(s-4)}_{k_1,0,...,0,k_{j+1},...,k_n}] = [\tilde{T}^{(s-4)}_{k_1+1,0,...,0,k_{j+1},...,k_n}][\tilde{T}^{(s)}_{k_1-1,0,...,0,k_{j+1},...,k_n}]
+ [\tilde{T}^{(s-4)}_{k_1,0,0,...,0,k_{j+1},...,k_n}][\tilde{T}^{(s)}_{k_1,0,...,0,k_{j+1},...,k_n}],
\]

(5.9)

where $k_1 > 0$;

\[
[\tilde{T}^{(s)}_{k_1,0,...,0,k_j}][\tilde{T}^{(s-4)}_{k_1,0,...,0,k_j+2}] = [\tilde{T}^{(s-4)}_{k_1+1,0,...,0,k_j}][\tilde{T}^{(s)}_{k_1-1,0,...,0,k_j+2}]
+ [\tilde{T}^{(s-4)}_{k_1,0,0,...,0,k_j+2}][\tilde{T}^{(s)}_{k_1,0,...,0,0}],
\]

(5.10)

where $k_1 > 0$;

\[
[\tilde{T}^{(s)}_{k_1,0,...,0}][\tilde{T}^{(s-4)}_{k_1,0,...,0,1}] = [\tilde{T}^{(s-4)}_{k_1+1,0,...,0}][\tilde{T}^{(s)}_{k_1-1,0,...,0,1}]
+ [\tilde{T}^{(s-4)}_{k_1,0,0,...,0,1}][\tilde{T}^{(s)}_{k_1,0,...,0,0}],
\]

(5.11)

where $k_1 > 0$;

\[
[\tilde{T}^{(s)}_{0,...,0,k_{j+1},...,k_n}][\tilde{T}^{(s-4)}_{0,...,0,k_{j+1},...,k_n}] = [\tilde{T}^{(s-4)}_{0,...,0,k_{j+1},...,k_n}][\tilde{T}^{(s)}_{0,...,0,1,0,...,0,k_{j+1},...,k_n}]
+ [\tilde{T}^{(s-4)}_{0,...,0,k_{j+1},...,k_n}][\tilde{T}^{(s)}_{0,...,0,0,1,0,...,0,k_{j+1},...,k_n}],
\]

(5.12)

where $k_i > 0$, $2 < i + 1 < n$;

\[
[\tilde{T}^{(s)}_{i,...,0,k_i,0,...,0,k_n}][\tilde{T}^{(s-4)}_{i,...,0,k_{i+1},0,...,0,k_n+2}] = [\tilde{T}^{(s-4)}_{i,...,0,k_{i+1},0,...,0,k_n}][\tilde{T}^{(s)}_{i,...,0,k_i-1,0,...,0,k_n+2}]
+ [\tilde{T}^{(s-4)}_{i,...,0,k_i,0,...,0,k_n+2}][\tilde{T}^{(s)}_{i,...,0,0,k_i,...,0,0,k_n}],
\]

(5.13)

where $k_i > 0$, $2 < i < n - 1$;

\[
[\tilde{T}^{(s)}_{i,...,0,k_i,0,...,0,1}][\tilde{T}^{(s-4)}_{i,...,0,k_{i+1},0,...,0,1}] = [\tilde{T}^{(s-4)}_{i,...,0,k_{i+1},0,...,0,1}][\tilde{T}^{(s)}_{i,...,0,k_i-1,0,...,0,1}]
+ [\tilde{T}^{(s-4)}_{i,...,0,k_i,0,...,0,1}][\tilde{T}^{(s)}_{i,...,0,0,k_i,...,0,0,1}],
\]

(5.14)

where $k_i > 0$, $2 < i < n - 1$. 

Proof. The case of type $A_n$. The lowest weight monomial of $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$ is obtained from the highest weight monomial of $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$ by the substitutions:

$$1_s \mapsto n_{n+s+1}^{-1}, \ 2_s \mapsto (n-1)_{n+s+1}^{-1}, \ldots, n_s \mapsto 1_{n+s+1}^{-1}.$$ 

After we apply $\iota$ to $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$, the lowest weight monomial of $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$ becomes the highest weight monomial of $\iota(\chi_q(T_{k_1,\ldots,k_n}^{(s)}))$. Therefore by Lemma 5.2, the highest weight monomial of $\iota(\chi_q(T_{k_1,\ldots,k_n}^{(s)}))$ is obtained from the lowest weight monomial of $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$ by the substitutions:

$$1_s \mapsto n_{n-s+1}^{-1}, \ 2_s \mapsto (n-1)_{n-s+1}^{-1}, \ldots, n_s \mapsto 1_{n-s+1}^{-1}.$$ 

It follows that the highest weight monomial of $\iota(\chi_q(T_{k_1,\ldots,k_n}^{(s)}))$ is obtained from the highest weight monomial of $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$ by the substitutions:

$$1_s \mapsto 1_{-s}, \ 2_s \mapsto 2_{-s}, \ldots, n_s \mapsto n_{-s}.$$ 

Therefore the dual M-system of type $A_n$ is obtained applying $\iota$ to both sides of every equation of the M-system of type $A_n$.

The case of type $B_n$. The lowest weight monomial of $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$ is obtained from the highest weight monomial of $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$ by the substitutions:

$$1_s \mapsto 1_{-4n+s-2}^{-1}, \ 2_s \mapsto 2_{-4n+s-2}^{-1}, \ldots, n_s \mapsto n_{-4n+s-2}^{-1}.$$ 

After we apply $\iota$ to $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$, the lowest weight monomial of $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$ becomes the highest weight monomial of $\iota(\chi_q(T_{k_1,\ldots,k_n}^{(s)}))$. Therefore by Lemma 5.2, the highest weight monomial of $\iota(\chi_q(T_{k_1,\ldots,k_n}^{(s)}))$ is obtained from the lowest weight monomial of $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$ by the substitutions:

$$1_s \mapsto 1_{-4n-s-2}^{-1}, \ 2_s \mapsto 2_{-4n-s-2}^{-1}, \ldots, n_s \mapsto n_{-4n-s-2}^{-1}.$$ 

It follows that the highest weight monomial of $\iota(\chi_q(T_{k_1,\ldots,k_n}^{(s)}))$ is obtained from the highest weight monomial of $\chi_q(T_{k_1,\ldots,k_n}^{(s)})$ by the substitutions:

$$1_s \mapsto 1_{-s}, \ 2_s \mapsto 2_{-s}, \ldots, n_s \mapsto n_{-s}.$$ 

Therefore the dual M-system of type $B_n$ is obtained applying $\iota$ to both sides of every equation of the M-system of type $B_n$.

The irreducibility of every module in the summands on the right hand side of every equation in the dual M-system of type $A_n$ (resp. $B_n$) follows from Theorem 3.3 and Lemma 5.2. \qed
Example 5.4. The following are some equations in the dual M-system of type $A_3$.

\[
\begin{align*}
[1_1|2_01_3] &= [1_11_3]|2_0| + [2_02_2], \\
[1_11_3]|2_01_31_5] &= [2_01_3][1_11_31_5] + [2_02_22_4], \\
[1_3][2_12_4] &= [2_22_4][3_1] + [1_3[3_13_3], \\
[2_02_4]|3_12_42_6] &= [3_24][2_22_42_6] + [1_31_5[3_13_33_5], \\
[1_3][3_11_5] &= [1_31_5][3_1] + [3_12_4], \\
[1_31_5][3_11_51_7] &= [3_11_5][3_151_7] + [3_12_42_6], \\
[3_11_5][3_12_41_7] &= [3_11_51_7][3_12_4] + [3_12_42_6][3_1], \\
[3_11_51_7][3_12_41_70] &= [3_12_41_7][3_11_51_70] + [3_12_42_62_8][3_1].
\end{align*}
\]

The following are some equations in the M-system of type $B_2$.

\[
\begin{align*}
[1_3][2_02_11] &= [1_31_7][2_02_2] + [2_02_22_42_6], \\
[1_31_7][2_02_211] &= [2_02_2111][1_31_7111] + [2_02_22_22_82_10], \\
[1_1][2_01_5] &= [2_0][1_11_5] + [2_02_22_4], \\
[1_11_5][2_01_51_9] &= [2_01_5][1_11_51_9] + [2_02_22_22_8].
\end{align*}
\]

5.2. The dual m-systems of types $A_n$, $B_n$. For $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$, let \( \tilde{m}_{k_1,\ldots,k_n} = \text{Res}(\tilde{T}_{k_1,\ldots,k_n}^{(s)}) \) be the restriction of \( \tilde{T}_{k_1,\ldots,k_n}^{(s)} \) to \( U_q \mathfrak{g} \). It is clear that \( \text{Res}(\tilde{T}_{k_1,\ldots,k_n}^{(s)}) \) doesn’t depend on \( s \). By replacing each \( [\tilde{T}_{k_1,\ldots,k_n}^{(s)}] \) in the M-system of type $A_n$ (resp. $B_n$) in Theorem 5.3 with \( \chi(\tilde{m}_{k_1,\ldots,k_n}) \), we obtain a system of equations which we called the dual m-system of type $A_n$ (resp. $B_n$). The following is an equation in the dual m-system of type $A_n$.

\[
\chi(\tilde{m}_{0,\ldots,0,k_i,0,\ldots,0,k_j,0,\ldots,0,k_{j+1},\ldots,k_n}) = \chi(\tilde{m}_{0,\ldots,0,k_i+1,0,\ldots,0,k_j,k_{j+1},\ldots,k_n}) + \chi(\tilde{m}_{0,\ldots,0,k_i,0,\ldots,0,k_j,0,\ldots,0,k_{j+1},\ldots,k_n}),
\]

where \( k_i > 0, k_j \geq 0, 2 < i + 1 < j < n \).

5.3. Relation between dual M-systems and cluster algebras. Let \( I = \{1, 2, \ldots, n\} \) and

\[
\begin{align*}
S_1 &= \{2i + 1 \mid i \in \mathbb{Z}_{\geq 0}\}, \quad S_2 = \{2i \mid i \in \mathbb{Z}_{\geq 0}\}, \\
S_3 &= \{4i + 1 \mid i \in \mathbb{Z}_{\geq 0}\}, \quad S_4 = \{4i + 3 \mid i \in \mathbb{Z}_{\geq 0}\}.
\end{align*}
\]

Let

\[
\begin{align*}
V &= ((S_1 \cap I) \times S_1) \cup ((S_2 \cap I) \times S_2), \\
V' &= (\{n\} \times S_2) \cup (\{1, \ldots, n - 1\} \times S_3) \cup (\{1, \ldots, n - 1\} \times S_4).
\end{align*}
\]

A quiver \( \tilde{Q} \) (resp. \( \tilde{Q}' \)) for \( U_q \tilde{\mathfrak{g}} \) of type $A_n$ (resp. $B_n$) with vertex set \( V \) (resp. \( V' \)) will be defined as follows. The arrows of \( \tilde{Q} \) (resp. \( \tilde{Q}' \)) are given by the following rule: there is an arrow from
the vertex \((i, r)\) to the vertex \((j, s)\) if and only if \(b_{ij} \neq 0\) and \(s = r - b_{ij} + d_i - d_j\). In the case of type \(A_n\), let
\[
\tilde{t} = \{t_0^{(-2k_i-i+3)} \mid i \text{ is even}, \ k_i \in \mathbb{Z}_{>0}\} \cup \{t_0^{(-2k_i-i+2)} \mid i \text{ is odd}, \ k_i \in \mathbb{Z}_{>0}\}. \tag{5.16}
\]
In the case of type \(B_n\), let \(\tilde{t}' = \tilde{t}_1 \cup \tilde{t}_2\), where
\[
\tilde{t}_1 = \{t_0^{(-2n-2k_n+5)} \mid k_n \in \mathbb{Z}_{>0}\}, \tag{5.17}
\]
\[
\tilde{t}_2 = \{t_0^{(-4k_i-2i+3)} t_0^{(-4k_i-2i+5)} \mid i \in \{1, \ldots, n-1\}, \ k_i \in \mathbb{Z}_{>0}\}. \tag{5.18}
\]

Let \(\widehat{\mathcal{A}}\) (resp. \(\widehat{\mathcal{A}}'\)) be the cluster algebra defined by the initial seed \((\tilde{t}, \tilde{Q})\) (resp. \((\tilde{t}', \tilde{Q}')\)). By similar arguments in Section 4.1 we have the following theorem.

**Theorem 5.5.** Every equation in the dual \(M\)-system of type \(A_n\) (resp. \(B_n\)) corresponds to a mutation equation in the cluster algebra \(\mathcal{A}\) (resp. \(\mathcal{A}'\)). Every minimal affinization in the dual \(M\)-system of type \(A_n\) (resp. \(B_n\)) corresponds to a cluster variable of the cluster algebra \(\mathcal{A}\) (resp. \(\mathcal{A}'\)).

### 6. Connection with the Hernandez-Leclerc Conjecture

In this section, we show that our results imply that the Hernandez-Leclerc conjecture (Conjecture 13.2 in [HL10] and Conjecture 9.1 in [Le10]) is true for minimal affinizations of types \(A_n\) and \(B_n\). We have the following theorem.

**Theorem 6.1.** Minimal affinizations of type \(A_n\) (resp. \(B_n\)) are simple, real, prime and they correspond to cluster variables in \(\mathcal{A}\), \(\widehat{\mathcal{A}}\) (resp. \(\mathcal{A}'\), \(\widehat{\mathcal{A}}'\)).

By Theorem 4.1 and Theorem 5.5, every minimal affinization in the \(M\)-system of type \(A_n\) (resp. \(B_n\)) corresponds to a cluster variable in \(\mathcal{A}\) (resp. \(\mathcal{A}'\)), every minimal affinization in the dual \(M\)-system of type \(A_n\) (resp. \(B_n\)) corresponds to a cluster variable in \(\widehat{\mathcal{A}}\) (resp. \(\widehat{\mathcal{A}}'\)). By the results in [CNY13], minimal affinizations of all Dynkin types are prime. By definition, minimal affinizations are simple. Therefore to prove Theorem 6.1 we only need to show that minimal affinizations of types \(A_n\) and \(B_n\) are real. We have the following theorem.

**Theorem 6.2.** Minimal affinizations of types \(A_n\) and \(B_n\) are real.

**Proof.** The theorem follows from the following facts:

1. \(\chi_q(T_{k_1,k_2,\ldots,k_n}^{(s)}) \chi_q(T_{k_1,k_2,\ldots,k_n}^{(s)})\), \(s \in \mathbb{Z}, \ k_1, \ldots, k_n \in \mathbb{Z}_{>0}\), has only one dominant monomial \(T_{k_1,k_2,\ldots,k_n}^{(s)}\);
2. \(\chi_q(T_{k_1,k_2,\ldots,k_n}^{(s)}) \chi_q(T_{k_1,k_2,\ldots,k_n}^{(s)})\), \(s \in \mathbb{Z}, \ k_1, \ldots, k_n \in \mathbb{Z}_{>0}\), has only one dominant monomial \(T_{k_1,k_2,\ldots,k_n}^{(s)}\).

We will prove Fact (1) in the case of type \(A_n\). The other cases are similar.

Let \(k_1, \ldots, k_n\) be the non-zero integers in \(k_1, \ldots, k_n\). Then the length of \(T_{k_1,k_2,\ldots,k_n}^{(s)}\) is \(M = k_{i_1} + k_{i_2} + \cdots + k_{i_v}\). Let \(m = \prod_{l=1}^M m(p_l)\) (resp. \(m' = \prod_{l=1}^M m'(p'_l)\)) be a monomial in the first (resp. the second) \(\chi_q(T_{k_1,k_2,\ldots,k_n}^{(s)})\) in \(\chi_q(T_{k_1,k_2,\ldots,k_n}^{(s)}) \chi_q(T_{k_1,k_2,\ldots,k_n}^{(s)})\), where \((p_1, \ldots, p_M) \in \mathcal{P}(c_1,d_1)_{1 \leq i \leq M}\).
(resp. $(p'_1, \ldots, p'_M) \in \mathcal{C}(d_1, d_2)_{1 \leq i \leq M}$) is a tuple of non-overlapping paths, $d_1, \ldots, d_M$ are some integers, and
\[
c_1 = c_2 = \cdots = c_{k_1} = 1, \ c_{k_1} + 1 = c_{k_1} + 2 = \cdots = c_{k_2} = i_2, \ \cdots, \\
\cdots, \ c_{k_1} + k_2 + \cdots + k_{r-1} + 1 = c_{k_1} + k_2 + \cdots + k_{r-1} + 2 = \cdots = c_{k_1} + k_2 + \cdots + k_{r-1} + k_r = i_r.
\]
Without loss of generality, we may assume that $s = 0$. We have
\[
T_{k_1, k_2, \ldots, k_n}^{(0)} = (i_1)_{i_1-1} + (i_1)_{i_1-1+2} + \cdots + (i_1)_{i_1-1+2(k_1-1)}(i_2)_{i_2-1} + 2(k_1-1) + (i_2-1) + 3(i_2)_{i_2-1+2(k_1-1)} + (i_2-1) + 5 \cdots \cdots \cdots (i_r)_{i_r-1} + 2 \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-1) + \sum_{p=1}^{r} (i_p - i_{p-1} - 1)
\]
\[
\text{(6.1)}
\]
If $p_M \neq p_{c_n, d_n}^+$ in $m$, then the monomial of $\chi_q(T_{k_1, k_2, \ldots, k_n}^{(s)}) \chi_q(T_{k_1, k_2, \ldots, k_n}^{(s)})$ is right-negative and hence not dominant. Therefore, $p_M = p_{c_n, d_n}^+$ by the non-overlapping property, we have $p_M = p_{c_n, d_n}$. Similarly, for $\sum_{p=1}^{r-1} k_{ip} < u \leq M$, we have $p'_u = p_{c_n, d_n}^+$.

Suppose that $p_{\sum_{p=1}^{r-1} k_{ip}} \neq p_{\sum_{p=1}^{r-1} k_{ip}}^+$. Then $p_1p_2 \cdots p_M$ in $m$ has some negative factor $h_{b_1}$, where $h_{b_1}$ is one of the following factors:
\[
1^{-1}_{i_1-1+2} \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-2) + \sum_{p=1}^{r-1} (i_p - i_{p-1} - 1) + i_{r-1} + 1 + \cdots + (i_r-1) \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-2) + \sum_{p=1}^{r-1} (i_p - i_{p-1} - 1) + 3
\]
\[
\text{(6.1)}
\]
By the Frenkel-Mukhin algorithm and the fact that $p_M = p_{c_n, d_n}$ for $\sum_{p=1}^{r-1} k_{ip} < u \leq M$, the factors
\[
1^{-1}_{i_1-1+2} \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-2) + \sum_{p=1}^{r-1} (i_p - i_{p-1} - 1) + i_{r-1} + 1 + (i_r-1) \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-2) + \sum_{p=1}^{r-1} (i_p - i_{p-1} - 1) + i_{r-1} + 1
\]
cannot be cancelled by $p_1p_2 \cdots p_M^+$ in $m'$. If $h_{b_1}$ (2 $\leq h \leq i_{r-1} - 1$) in $m$ is cancelled by some $h_b$ in $p_1p_2 \cdots p_M^+$ in $m'$, then $mm'$ will have one of the factors
\[
1^{-1}_{i_1-1+2} \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-2) + \sum_{p=1}^{r-1} (i_p - i_{p-1} - 1) + i_{r-1} + 1 + 2^{-1}_{i_1-1+2} \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-2) + \sum_{p=1}^{r-1} (i_p - i_{p-1} - 1) + i_{r-1} + 1
\]
\[
\cdots, (i_r-1) \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-2) + \sum_{p=1}^{r-1} (i_p - i_{p-1} - 1) + 3
\]
If $h_{b_1}$ (i_{r-1} + 1 $\leq h \leq i_r - 2$) in $m$ is cancelled by some $h_b$ in $p_1p_2 \cdots p_M^+$ in $m'$, then $mm'$ will have one of the factors
\[
(i_{r-1} + 1)^{-1}_{i_1-1+2} \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-2) + \sum_{p=1}^{r-1} (i_p - i_{p-1} - 1) + 3(i_{r-1})^{-1}_{i_1-1+2} \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-2) + \sum_{p=1}^{r-1} (i_p - i_{p-1} - 1) + i_{r-1} + 1
\]
\[
\cdots, (i_r-1) \sum_{p=1}^{r-1} (k_{ip} - 1) + 3(r-2) + \sum_{p=1}^{r-1} (i_p - i_{p-1} - 1) + i_{r-1} + 1
\]
If $h_b^{-1}$ in $m$ is cancelled by some $i_{r-1}$ in $p'_1p'_2\cdots p'_M = m'$, then $mm'$ will have one of the factors
\[
(1)^{-1}t_1-1+2\sum_{p=1}^{r-1}(i_{p-1}-1)+i_{r-1}+1,\ldots, (i_{r-1}-1)^{-1}t_1-1+2\sum_{p=1}^{r-1}(i_{p-1}-1)+3\quad \text{(7.1)}
\]
\[
\cdots, (i_{r-1})^{-1}t_1-1+2\sum_{p=1}^{r-1}(i_{p-1}-1)+3\quad \text{(7.1)}
\]

Therefore $mm'$ is not dominant which contradicts our assumption. Hence
\[
p_{t}^{1}d_{t}^{-1}k_{t} = p_{t}^{1}d_{t}^{-1}k_{t}.
\]

By the non-overlapping property, we have $p_{t} = p_{t}^{+}d_{t}^{-1}$, $\sum_{p=1}^{t-1}k_{p} < t \leq \sum_{p=1}^{t}k_{p}$. Similarly, we have $p_{t} = p_{t}^{+}d_{t}$ for $1 \leq t \leq \sum_{p=1}^{r-1}k_{p}$. Therefore the only dominant monomial in $\chi_{q}(T_{k_1,k_2,\ldots,k_n})$ is $T_{k_1,k_2,\ldots,k_n}^{(s)}$.

7. A NEW ALGORITHM TO COMPUTE $q$-CHARACTERS OF MINIMAL AFFINIZATIONS OF TYPES $A_n$ AND $B_n$

In this section, we give a new algorithm to compute $q$-characters of minimal affinizations of types $A_n$ and $B_n$.

The algorithm has four functions $qch_1$, $qch_2$, $qch_3$, $qch_4$, where $qch_1$ computes the $q$-characters of the modules $T_{k_1,k_2,\ldots,k_n}$ in type $A_n$ by using the M-system of type $A_n$, $qch_2$ computes the $q$-characters of the modules $\tilde{T}_{k_1,k_2,\ldots,k_n}$ in type $B_n$ by using the M-system of type $B_n$, $qch_3$ computes the $q$-characters of the modules $\tilde{T}_{k_1,k_2,\ldots,k_n}$ in type $A_n$ by using the dual M-system of type $A_n$, $qch_4$ computes the $q$-characters of the modules $\tilde{T}_{k_1,k_2,\ldots,k_n}$ in type $B_n$ by using the dual M-system of type $B_n$.

If $T_{k_1,k_2,\ldots,k_n}$ is a Kirillov-Reshetikhin module, then $\chi_{q}(T_{k_1,k_2,\ldots,k_n})$ can be computed by the Frenkel-Mukhin algorithm, see [Her06], [FM01], or the Hernandez-Leclerc algorithm given in [HL13]. We will describe the function $qch_1$ as follows.

Input: a minimal affinization $T_{k_1,k_2,\ldots,k_n}$ in type $A_n$.

Output: $\chi_{q}(T_{k_1,k_2,\ldots,k_n})$.

Algorithm (recursive):

$q$-character $qch_1$ (type $A_n$ module $T_{k_1,k_2,\ldots,k_n}$)

if $T_{k_1,k_2,\ldots,k_n}$ is a Kirillov-Reshetikhin module, then we use the Frenkel-Mukhin algorithm or the Hernandez-Leclerc algorithm to compute $\chi_{q}(T_{k_1,k_2,\ldots,k_n})$ and return $\chi_{q}(T_{k_1,k_2,\ldots,k_n})$;

else if $k_1, k_2 > 0$, then return

\[
qch_1(T_{k_1+1,k_2,k_3,\ldots,k_n}) = qch_1(T_{k_1-1,k_2+1,k_3,\ldots,k_n}) + qch_1(T_{0,k_2+1,k_3,\ldots,k_n})qch_1(T_{0,k_2,k_3,\ldots,k_n}),
\]

\[
qch_1(T_{k_1,k_2+1,k_3,\ldots,k_n}) = qch_1(T_{k_1,k_2,k_3,\ldots,k_n}) + qch_1(T_{k_1+1,k_2+1,k_3,\ldots,k_n}qch_1(T_{0,k_2+1,k_3,\ldots,k_n})),
\]

\[
qch_1(T_{k_1+1,k_2,k_3,\ldots,k_n}) = qch_1(T_{k_1,k_2,k_3,\ldots,k_n}) + qch_1(T_{0,k_2+1,k_3,\ldots,k_n}qch_1(T_{0,k_2,k_3,\ldots,k_n})),
\]

\[
qch_1(T_{k_1+1,k_2+1,k_3,\ldots,k_n}) = qch_1(T_{k_1+1,k_2,k_3,\ldots,k_n}) + qch_1(T_{0,k_2+1,k_3,\ldots,k_n}qch_1(T_{0,k_2,k_3,\ldots,k_n})),
\]

\[
qch_1(T_{k_1+1,k_2,k_3,\ldots,k_n}) = qch_1(T_{k_1,k_2,k_3,\ldots,k_n}) + qch_1(T_{0,k_2+1,k_3,\ldots,k_n}qch_1(T_{0,k_2,k_3,\ldots,k_n})),
\]
Since \( T \) we compute Example 7.2.

\( \text{Algorithm 7.1.} \) \textit{Input:} a minimal affinization \( \mathcal{T} \) of type \( A_n \) or \( B_n \).
\textit{Output:} \( \chi_q(\mathcal{T}) \).

\textit{Algorithm:}

\( q \)-character \( \text{qch}(\text{type } A_n \text{ or } B_n \text{ module } \mathcal{T}_{k_1,k_2,\ldots,k_n}) \)

if \( \mathcal{T} \) is of the form \( \mathcal{T}_{k_1,k_2,\ldots,k_n}^{(s)} \) in type \( A_n \), then return \( \text{qch}_1(\mathcal{T}) \);

\( \text{else if} \) \( \mathcal{T} \) is of the form \( \mathcal{T}_{k_1,k_2,\ldots,k_n}^{(s)} \) in type \( A_n \), then return \( \text{qch}_2(\mathcal{T}) \);

\( \text{else if} \) \( \mathcal{T} \) is of the form \( \mathcal{T}_{k_1,k_2,\ldots,k_n}^{(s)} \) in type \( B_n \), then return \( \text{qch}_3(\mathcal{T}) \);

\( \text{else} \) \( \mathcal{T} \) is of the form \( \mathcal{T}_{k_1,k_2,\ldots,k_n}^{(s)} \) in type \( B_n \), then return \( \text{qch}_4(\mathcal{T}) \).

The functions \( \text{qch}_2, \text{qch}_3, \text{qch}_4 \) are described in a similar way. We omit the details. We have the following algorithm.

\textbf{Algorithm 7.1.} \textit{Input:} a minimal affinization \( \mathcal{T} \) of type \( A_n \) or \( B_n \).
\textit{Output:} \( \chi_q(\mathcal{T}) \).

\textit{Algorithm:}

\begin{equation}
\text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s)}) = \frac{\text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)}) + \text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)})}{\text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)})},
\end{equation}

(7.2)

\begin{equation}
\text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)}) = \frac{\text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)}) + \text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)})}{\text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)})},
\end{equation}

(7.3)

\begin{equation}
\text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)}) = \frac{\text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)}) + \text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)})}{\text{qch}_1(\mathcal{T}_{0,0,\ldots,0,k_1,0,\ldots,0,k_2,\ldots,k_n}^{(s+2)})},
\end{equation}

(7.4)

\textbf{Example 7.2.} We compute \( \chi_q(\mathcal{T}_{1,1,1}^{(-7)}) = \chi_q(1-7-2-43-1) \) in type \( A_3 \) by using Algorithm 7.1.
Since \( \mathcal{T}_{1,1,1}^{(-7)} \) is of the form \( \mathcal{T}_{k_1,k_2,k_3}^{(s)} \) in type \( A_3 \), it is computed by using the function \( \text{qch}_1 \). We have

\begin{align}
\text{qch}_1(\mathcal{T}_{1,1,1}^{(-7)}) &= \frac{\text{qch}_1(\mathcal{T}_{2,0,1}^{(7)}) \text{qch}_1(\mathcal{T}_{0,1,1}^{(5)}) + \text{qch}_1(\mathcal{T}_{0,2,1}^{(7)}) \text{qch}_1(\mathcal{T}_{0,0,1}^{(3)})}{\text{qch}_1(\mathcal{T}_{1,0,1}^{(-5)})},
\end{align}

(7.5)
where all modules on the right hand side are Kirillov-Reshetikhin modules and they are computed by using the Frenkel-Mukhin algorithm or the Hernandez-Leclerc algorithm; for example, \( q\text{ch}_{1}(\mathcal{T}_{0,2,0}^{(-5)}) = \chi_{q}(\mathcal{T}_{0,2,0}^{(-5)}) \) is computed by the Frenkel-Mukhin algorithm or the Hernandez-Leclerc algorithm;

\[
q\text{ch}_{1}(\mathcal{T}_{1,0,1}^{(-5)}) = \frac{q\text{ch}_{1}(\mathcal{T}_{2,0,0}^{(-5)}) q\text{ch}_{1}(\mathcal{T}_{0,0,1}^{(-3)}) + q\text{ch}_{1}(\mathcal{T}_{0,1,1}^{(-5)})}{q\text{ch}_{1}(\mathcal{T}_{1,0,0}^{(-3)})};
\]

\[
q\text{ch}_{1}(\mathcal{T}_{0,2,1}^{(-7)}) = \frac{q\text{ch}_{1}(\mathcal{T}_{0,1,1}^{(-5)}) q\text{ch}_{1}(\mathcal{T}_{0,3,0}^{(-7)}) + q\text{ch}_{1}(\mathcal{T}_{2,0,0}^{(-5)}) q\text{ch}_{1}(\mathcal{T}_{0,0,3}^{(-7)})}{q\text{ch}_{1}(\mathcal{T}_{0,2,0}^{(-5)})};
\]

\[
q\text{ch}_{1}(\mathcal{T}_{2,0,1}^{(-7)}) = \frac{q\text{ch}_{1}(\mathcal{T}_{1,0,1}^{(-5)}) q\text{ch}_{1}(\mathcal{T}_{3,0,0}^{(-7)}) + q\text{ch}_{1}(\mathcal{T}_{0,2,1}^{(-7)}) q\text{ch}_{1}(\mathcal{T}_{1,0,3}^{(-5)})}{q\text{ch}_{1}(\mathcal{T}_{2,0,0}^{(-5)})}.
\]

In Example 7.2, we first compute the \( q \)-characters of the modules in Level 1 in Figure 1, then we compute the \( q \)-characters of the modules in Level 2. We continue this procedure, in the end we compute the \( q \)-character of the module in Level 5 which is \( \mathcal{T}_{1,1,1}^{(-7)} \).

**Figure 1**

8. **Proof of Theorem 3.1**

In this section, we will prove Theorem 3.1.

8.1. **Classification of dominant monomials.** First we classify all dominant monomials in each summand on the left and right hand sides of every equation in Theorem 3.1. We have the following lemma.

**Lemma 8.1.** The dominant monomials in each summand on the left and right hand sides of every equation in the M-system of type \( A_n \) (resp. \( B_n \)) are given in Table 1 (resp. Table 2).

We will prove Lemma 8.1 in Section 8.3.

In Table 1 and Table 2 \( M \prod_{0 \leq j \leq r} A_{r,s}^{-1} = M \) for \( r = -1, s \in \mathbb{Z} \).
| Equations | Summands in the Equations | $M$ | Dominant Monomials |
|-----------|--------------------------|-----|-------------------|
| $\chi T_{y_1} a_{y_2} y_3 \ldots a_{y_n}$ | $M = \prod_{i=1}^{r} (a_{y_i} \pm 1)$ | $M \prod_{i=1}^{r} (a_{y_i} \pm 1)$ |
| $\chi T_{y_1} a_{y_2} y_3 \ldots a_{y_n}$ | $M = \prod_{i=1}^{r} (a_{y_i} \pm 1)$ | $M \prod_{i=1}^{r} (a_{y_i} \pm 1)$ |
| $\chi T_{y_1} a_{y_2} y_3 \ldots a_{y_n}$ | $M = \prod_{i=1}^{r} (a_{y_i} \pm 1)$ | $M \prod_{i=1}^{r} (a_{y_i} \pm 1)$ |
| $\chi T_{y_1} a_{y_2} y_3 \ldots a_{y_n}$ | $M = \prod_{i=1}^{r} (a_{y_i} \pm 1)$ | $M \prod_{i=1}^{r} (a_{y_i} \pm 1)$ |
| $\chi T_{y_1} a_{y_2} y_3 \ldots a_{y_n}$ | $M = \prod_{i=1}^{r} (a_{y_i} \pm 1)$ | $M \prod_{i=1}^{r} (a_{y_i} \pm 1)$ |
| $\chi T_{y_1} a_{y_2} y_3 \ldots a_{y_n}$ | $M = \prod_{i=1}^{r} (a_{y_i} \pm 1)$ | $M \prod_{i=1}^{r} (a_{y_i} \pm 1)$ |
| $\chi T_{y_1} a_{y_2} y_3 \ldots a_{y_n}$ | $M = \prod_{i=1}^{r} (a_{y_i} \pm 1)$ | $M \prod_{i=1}^{r} (a_{y_i} \pm 1)$ |
| $\chi T_{y_1} a_{y_2} y_3 \ldots a_{y_n}$ | $M = \prod_{i=1}^{r} (a_{y_i} \pm 1)$ | $M \prod_{i=1}^{r} (a_{y_i} \pm 1)$ |
| $\chi T_{y_1} a_{y_2} y_3 \ldots a_{y_n}$ | $M = \prod_{i=1}^{r} (a_{y_i} \pm 1)$ | $M \prod_{i=1}^{r} (a_{y_i} \pm 1)$ |

Table 1. Classification of dominant monomials in the $M$-system of type $A_n$. 
| M^{i,s} | summands in the equations | M | dominant monomials |
|---|---|---|---|
| $\sum_{\chi} T^{(i,s)}_f T^{(i,s)}_s (\chi, \chi)$ | $M T^{(i,s)}_f T^{(i,s)}_s (\chi, \chi)$ | $M \Pi_{\text{BS-SS}} A^+_1 A^+_2 q_{4n-1}$ | $-15 \leq s_{4n-1} \leq 1$ |
| $\sum_{\chi} T^{(i,s)}_f T^{(i,s)}_s (\chi, \chi)$ | $M T^{(i,s)}_f T^{(i,s)}_s (\chi, \chi)$ | $M \Pi_{\text{BS-SS}} A^+_1 A^+_2 q_{4n-1}$ | $-15 \leq s_{4n-1} \leq 1$ |

Table 2. Classification of dominant monomials in the M-system of type $B_n$.  

31
8.2. Proof of Theorem 3.1 By Table 1 and Table 2, the dominant monomials in the q-characters of the left hand side and of the right hand side of every equation in Theorem 3.1 are the same. Therefore Theorem 3.1 is true.

8.3. Proof of Lemma 8.1 We will prove the case of the 10-th line in Table 1 and the case of 7-th line in Table 2. The other cases are similar.

Proof of the case of the 10-th line in Table 1. In this case, we have $2 \leq i+1 < j \leq n$. Let $M = k_1 + k_j + \ldots + k_n$. Let $m = \prod_{i=1}^{M} m(p_i)$ be a monomial in $\chi_q(T_0^{(s)}_{\mathbb{Z}})$, where $(p_1, \ldots, p_M) \in \mathcal{P}(c_i, d_i)_{1 \leq i \leq M}$ is a tuple of non-overlapping paths, $d_1, \ldots, d_M$ are some integers, and

$$c_1 = c_2 = \cdots = c_k = i, \quad c_{k+1} = \cdots = c_{k+j} = j, \cdots,$$

$$\cdots, c_{k+j+1} = \cdots = c_{k+j+2} = \cdots = c_{k+j+k_n} = n.$$

Let $m' = \prod_{i=1}^{M+1} m(p_i')$ be a monomial in $\chi_q(T_0^{(s+2)}_{\mathbb{Z}})$, where $(p_1', \ldots, p_{M+1}') \in \mathcal{P}(c_i', d_i)_{1 \leq i \leq M+1}$ is a tuple of non-overlapping paths, $d_1', \ldots, d_{M+1}'$ are some integers, and

$$c_1' = c_2' = \cdots = c_k' = i, \quad c_{k+1}' = \cdots = c_{k+j}' = j, \cdots,$$

$$\cdots, c_{k+j+1}' = \cdots = c_{k+j+k_n-1}' = \cdots = c_{k+j+k_n} = n.$$

Suppose that $mm'$ is dominant. We will prove that $p_t = p_{c_i, d_i}^+$ for all $1 \leq t \leq M$, and there exists $R$, $1 \leq R \leq k_1$, such that $p_u' = mp_{c_i, d_i}^+(A_{k_i+1-1})$ for $R \leq u \leq k_1$ and $p'_u = p_{c_i, d_i}^+$ for $1 \leq u < R$. Without loss of generality, we may assume that $s = 0$. We have

$$T_0^{(0)}_{\mathbb{Z}}$$

$$\cdots, j_i+j_{i+1}+2(k_i-1)+(j-i-1)+3j_i+2(k_i-1)+(j-i-1)+5 \cdots$$

(8.1)

$$T_0^{(-2)}_{\mathbb{Z}}$$

$$\cdots, j_i+j_{i+1}+2(k_i-1)+(j-i-1)+3j_i+2(k_i-1)+(j-i-1)+3 \cdots$$

(8.2)

The length of (8.1) is $M$ and the length of (8.2) is $M+1$. If $p_{M+1}' \neq p_{c_i, d_i}^+$, then $mm'$ is right-negative and hence not dominant. Therefore, $p_{M+1}' = p_{c_i, d_i}^+$. By the non-overlapping property, we have $p_u' = mp_{c_i, d_i}^+(A_{k_i+1-1})$, $k_i + k_j + 1 < u < M + 1$. By the same reason, we have $p_t = p_{c_i, d_i}^+$ for $k_i + k_j < t \leq M$.

Suppose that $p_{k_i, k_j} \neq p_{c_i, d_i}^+$ and let $k_v$ be the first non-zero integer such that $v > j$. Then $p_1p_2 \cdots p_M$ has some negative factor $h_{y}^{-1}$, where $h_{y}^{-1}$ is one of the following factors:

$$1^{j-1}_{i-1+2(k_i-1)+(j-i-1)+j+1}, \cdots, (j-1)^{j-1}_{i-1+2(k_i-1)+(j-i-1)+j+1}, j_i^{j-1}_{i-1+2(k_i-1)+(j-i-1)+j+1},$$

$$1^{(j+1)-1}_{i-1+2(k_i-1)+(j-i-1)+j+1}, \cdots, (v-1)^{1}_{i-1+2(k_i-1)+(j-i-1)+v+j+4}$$
By the Frenkel-Mukhin algorithm and the fact that $p'_u = p'_{c_{ki}d_{ki}}$ for $k_i + k_j + 1 < u \leq M + 1$, the factors

\[ \frac{1}{i-1+2(k_i-1)+(j-i-1)+j+4} \quad (v-1)^{-i-1+2(k_i-1)+2(k_j-1)+(j-i-1)+v-j+4} \]

cannot be cancelled by $p'_1 p'_2 \cdots p'_{M+1} = m'$. If $h_b^{-1} (2 \leq h \leq j - 1)$ in $m$ is cancelled by some $h_b$ in $p'_1 p'_2 \cdots p'_{M+1} = m'$, then $mm'$ will have one of the factors

\[ \frac{1}{i-1+2(k_i-1)+(j-i-1)+j+4} \quad 2^{-i-1+2(k_i-1)+(j-i-1)+j+3} \cdots \quad (j-1)^{-1} \frac{1}{i-1+2(k_i-1)+2(k_j-1)+(j-i-1)+j-6} \]

If $h_b^{-1} (j + 1 \leq h \leq v - 2)$ in $m$ is cancelled by some $h_b$ in $p'_1 p'_2 \cdots p'_{M+1} = m'$, then $mm'$ will have one of the factors

\[ (j+1)^{-1} \frac{1}{i-1+2(k_i-1)+2(k_j-1)+(j-i-1)+h-j+6} \quad (j+2)^{-i-1+2(k_i-1)+2(k_j-1)+(j-i-1)+h-j+7} \cdots \]
\[ \cdots, \quad (v-1)^{-i-1+2(k_i-1)+2(k_j-1)+(j-i-1)+v-j+4} \]

If $h_b^{-1}$ in $m$ is cancelled by some $j_b$ in $p'_1 p'_2 \cdots p'_{M+1} = m'$, then $mm'$ will have one of the factors

\[ \frac{1}{i-1+2(k_i-1)+j-1+4} \cdots \quad (j-1)^{-1} \frac{1}{i-1+2(k_i-1)+2(k_j-1)+(j-i-1)+6} \quad j_b^{-1} \frac{1}{i-1+2(k_i-1)+2(k_j-1)+(j-i-1)+5} \]
\[ (j+1)^{-i-1+2(k_i-1)+2(k_j-1)+(j-i-1)+6} \cdots \quad (v-1)^{-i-1+2(k_i-1)+2(k_j-1)+(j-i-1)+v-j+4} \]

Therefore $mm'$ is not dominant which contradicts our assumption. Hence $p_{k_i+k_j} = p_{c_{k_i}d_{k_i}}$. By the non-overlapping property, we have $p_t = p_{c_{k_i}d_{k_i}}$, $k_i < t \leq k_i + k_j$. By the same reason, we have $p'_t = p'_{c_{k_i}d_{k_i}}$, $k_i < u \leq k_i + k_j + 1$.

Suppose that $p_{k_i} \neq p_{c_{k_i}d_{k_i}}$. Then $p_1 p_2 \cdots p_M$ has some negative factor $h_b^{-1}$, where $h_b^{-1}$ is one of the following factors:

\[ \frac{1}{i-1+2(k_i-1)+i+1} \cdots \quad (i-1)^{-i-1+2(k_i-1)+3} \quad i^{-1} \frac{1}{i-1+2(k_i-1)+2} \]
\[ (i+1)^{-i-1+2(k_i-1)+3} \cdots \quad (j-1)^{-i-1+2(k_i-1)+(j-i)+1} \]

By the Frenkel-Mukhin algorithm and the fact that $p'_u = p'_{c_{k_i}d_{k_i}}$ for $k_i + k_j + 1 < u \leq M + 1$, the factors

\[ \frac{1}{i-1+2(k_i-1)+i+1} \cdots \quad (i-1)^{-i-1+2(k_i-1)+3} \quad i^{-1} \frac{1}{i-1+2(k_i-1)+2} \]
\[ (i+1)^{-i-1+2(k_i-1)+3} \cdots \quad (j-1)^{-i-1+2(k_i-1)+(j-i)+1} \]

cannot be cancelled by $p'_1 p'_2 \cdots p'_{M+1} = m'$. Therefore $mm'$ is not dominant which contradicts our assumption. Hence $p_{k_i} = p_{c_{k_i}d_{k_i}}$. For $1 \leq t \leq k_i$, we have $p_t = p_{c_{k_i}d_{k_i}}$ by the non-overlapping property.

If $p'_{k_i} = p'_{c_{k_i}d_{k_i}}$, then $p'_{k_i-\ell} = p'_{c_{k_i-\ell}d_{k_i-\ell}}$ $(1 \leq \ell \leq k_i - 1)$. Therefore

\[ mm' = M = T^{(0)}_{0,0,0,k_i,0,0,k_j,1,0,0,k_j+1,1,0,\ldots,k_j+1,0,0,0,0,0} \frac{T(-2)}{ii} \]

If $p'_{k_i} = m(p'_{c_{k_i}d_{k_i}}) A_{i-1+2(k_i-1)}^{-1}$, then $p'_{k_i-\ell} \in \{ m(p'_{c_{k_i-\ell}d_{k_i-\ell}}) A_{i-1+2(k_i-\ell-1)}^{-1} \}$ $(1 \leq \ell \leq k_i - 1)$. Therefore $mm'$ is one of the dominant monomials $M \Pi_{0 \leq \ell \leq r} A_{i-1+2(k_i-\ell-1)}^{-1}$.
exists $≤ \text{mm}$ Suppose that $\exists \text{monomial in } \chi_a M$ is one of the following factors: $\prod_{i=1}^{M} (p_i)^{c_i}$. By the non-overlapping property, these factors cannot be cancelled by $p_1 p_2 \cdots p_M = m$. It follows that $mm'$ is not dominant which contradicts our assumption. □

Proof of the case of the 7-th line in Table 2. Let $M = k_{n-1} + k_n$. Let $m = \prod_{i=1}^{M} (p_i)$ be a monomial in $\chi_0 (T_{0, \ldots, 0, k_{n-1}, k_n})$, where $(p_1, \ldots, p_M) \in \mathfrak{T}(c_i, d_i)_{i \leq M}$ is a tuple of non-overlapping paths, $d_1, \ldots, d_M$ are some integers, and $c_1 = c_2 = \cdots = c_{k_{n-1}} = n - 1$, $c_{k_{n-1}+1} = c_{k_{n-1}+2} = \cdots = c_{k_{n-1}+k_n} = n$.

Let $m' = \prod_{u=1}^{M+1} (p'_u)$ be a monomial in $\chi_0 (T_{0, \ldots, 0, k_{n-1}, k_n+2})$, where $(p'_1, \ldots, p'_{M+2}) \in \mathfrak{T}(c'_i, d'_i)_{1 \leq u \leq M+2}$ is a tuple of non-overlapping paths, $d'_1, \ldots, d'_{M+1}$ are some integers, and $c'_1 = c'_2 = \cdots = c'_{k_{n-1}} = n - 1$, $c'_{k_{n-1}+1} = c'_{k_{n-1}+2} = \cdots = c'_{k_{n-1}+k_n+2} = n$.

Suppose that $mm'$ is dominant. We will prove that $p_t = p_{c_t, d_t}$ for all $1 \leq t \leq M$, and there exists $R$, $1 \leq R \leq k_i$, such that $p'_u = m(p'_{c_t, d_t}) A_{n-1, s+4(u-1)-2}$ for $R \leq u \leq k_i$ and $p'_u = p'_{c'_t, d'_t}$ for $1 \leq u < R$. Without loss of generality, we may assume that $s = 0$. We have

\begin{align*}
T^{(0)}_{0, \ldots, 0, k_{n-1}, k_n} &= (n-1)^{2n-4} (n-1)^{2n-4+4} \cdots (n-1)^{2n-4+4k_{n-1}-4n^2n-4+4k_{n-1}+1n^2n-4+4k_{n-1}+3} n^2n-4+4k_{n-1}+2k_{n-1}, \\
T^{(-4)}_{0, \ldots, 0, k_{n-1}, k_n+2} &= (n-1)^{2n-4} (n-1)^{2n-4} \cdots (n-1)^{2n-4+4k_{n-1}-8n^2n-4+4k_{n-1}-3n^2n-4+4k_{n-1}-1} n^2n-4+4k_{n-1}+2k_{n-1}.
\end{align*}
By the Frenkel-Mukhin algorithm and the fact that $p'_u = p'_{c_{i+1}d_{i+1}}$ for $k_{n-1} < u \leq M + 2$, the factors cannot be cancelled by $p'_1p'_2 \cdots p'_{M+2} = m'$. Therefore $mm'$ is not dominant which contradicts our assumption. Hence, $p_{k_{n-1}} = p_{c_{k_{n-1}}d_{k_{n-1}}}'$. By the non-overlapping property, for $1 \leq t \leq k_{n-1}$, we have $p_t = p_{c_{i+1}d_{i+1}}$. If $p'_{k_{n-1}} = p'_{c_{k_{n-1}}d_{k_{n-1}}}$, then $p'_{k_{n-1}-\ell} = p'_{c_{k_{n-1}-\ell}d_{k_{n-1}-\ell}} (1 \leq \ell \leq k_{n-1} - 1)$. Therefore

$$mm' = M = T_0^{(0)}, T_0^{(-2)}, m = T_0^{(-4)}.$$ 

If $p'_{k_{n-1}} = m(p_{c_{k_{n-1}}d_{k_{n-1}}})^{-1}n^{-1}2_{n-4+4(k_{n-1}-1)-2},$ then

$$p'_{k_{n-1}-\ell} \in \{p_{c_{k_{n-1}-\ell}d_{k_{n-1}-\ell}}^{-1}, m(p_{c_{k_{n-1}-\ell}d_{k_{n-1}-\ell}}^{-1})n^{-1}2_{n-4+4(k_{n-1}-1)-2}, (1 \leq \ell \leq k_{n-1} - 1).$$

Therefore $mm'$ is one of the dominant monomials $M \prod_{0 \leq j \leq r} A_n^{-1}2_{n-4+4(k_{n-1}-1)-4j-6}, 0 \leq r \leq k_{n-1} - 1$. If $p'_{k_{n-1}} \notin \{p_{c_{k_{n-1}}d_{k_{n-1}}}, m(p_{c_{k_{n-1}}d_{k_{n-1}}})A_n^{-1}2_{n-4+4(k_{n-1}-1)-2},$ then $p'_{k_{n-1}}$ has some negative factor $h^{-1}$, where $h^{-1}$ is one of the following factors:

$$1_{n-4+4(k_{n-1}-1)+2}2_{n-4+4(k_{n-1}-1)+2}, \cdots, (n-2)2_{n-4+4(k_{n-1}-1)+2}. \quad (8.6)$$

Since $p_t = p_{c_{i+1}d_{i+1}}$ (1 \leq t \leq M), the factors in $[8.6]$ cannot be cancelled by $p_1p_2 \cdots p_M = m$. By the non-overlapping property, these factors cannot be cancelled by $p'_1p'_2 \cdots p'_{M+2} = m'$. It follows that $mm'$ is not dominant which contradicts our assumption. 

\[\square\]

9. Proof of Theorem 3.3

In this section, we prove Theorem 3.3.

By Lemma 3.3, we have the following result.

**Corollary 9.1.** The modules in the second summand on the right hand side of every equation of the M-system are special. In particular, they are simple.

Therefore in order to prove Theorem 3.3, we only need to prove that the modules in the first summand on the right hand side of every equation of the M-system are simple. We will prove that in the case of type $A_n$,

$$\chi_q(T_{i}^{(s-2)}_{0,\ldots,0,k_{i+1},\ldots,0,k_{j},k_{j+1},\ldots,k_{n}}) \chi_q(T_{i}^{(s)}_{0,\ldots,0,k_{i-1},\ldots,0,k_{j}+1,k_{j+1},\ldots,k_{n}}) \quad (9.1)$$

where $2 < i + 1 < j \leq n$, is simple, and in the case of type $B_n$,

$$\chi_q(T_{i}^{(s-4)}_{0,\ldots,0,k_{n-1}+1,k_{n}}) \chi_q(T_{s}^{(s)}_{0,\ldots,0,k_{n-1}-1,k_{n}+2}) \quad (9.2)$$

is simple. The other cases are similar.
The following is the proof of the fact that (9.1) is simple. Without loss of generality, we may assume that $s = 0$. We have

$$\begin{align*}
\mathcal{T}_{0\ldots,0,k_i-1,0\ldots,0,k_j+1,k_{j+1}\ldots,k_n}^{(0)} &= i \cdot i + i + 2 \cdot i + 1 + 2(k_i - 2) + j \cdot j + 1 + 2(k_i - 2) + (j - i - 1) + 3 \cdot j - i + 1 + 2(k_i - 2) + (j - i - 1) + 5 \cdot \ldots \\
&\quad \cdot n = i + 2(k_i - 1) + 3(n - j + 1) + (j - i - 1) + 2 \prod_{m=j}^{n} (k_{m-1}),
\end{align*}$$

(9.3)

$$\begin{align*}
\mathcal{T}_{0\ldots,0,k_i-1,0\ldots,0,k_j+1,k_{j+1}\ldots,k_n}^{(-2)} &= i \cdot i - 2 \cdot i + 1 \cdot i + 1 + 2(k_i - 2) + j \cdot j - 1 + 2(k_i - 2) + (j - i - 1) + 5 \cdot \ldots \\
&\quad \cdot n = i + 1 + 2(k_i - 1) + 3(n - j + 1) + (j - i - 1) + 2 \prod_{m=j}^{n} (k_{m-1} - 1).
\end{align*}$$

(9.4)

By Lemma 8.1, the dominant monomials in (9.1) are

$$M_r = M \prod_{0 \leq j \leq r} A_{i,j-1+2k_i-2j-3}^{-1}, \quad -1 \leq r \leq k_i - 2.$$ 

We need to show that $\chi_q(M_r) \not\subseteq \chi_q(M)$ for $0 \leq r \leq k_i - 2$. We will prove the case of $r = 0$. The other cases are similar.

$$M_0 = MA_{i,1+2k_i-3}^{-1}$$

$$= i \cdot i + 1 + 2(k_i - 2) + 1 \cdot i + 1 + 2(k_i - 2) + 1 \cdot i + 1 + 2(k_i - 2) + 1 \cdot i + 1 + 2(k_i - 2) + 1 \cdot i + 1 + 2(k_i - 2) + 1 \cdot i + 1 + 2(k_i - 2) + 1 \cdot i + 1 + 2(k_i - 2) + 1 \cdot i + 1 + 2(k_i - 2) + \ldots \ldots $$

$$= (i - 1) \cdot i - 1 + 2(k_i - 2) + (i - 1) \cdot i - 1 + 2(k_i - 2) + (i - 1) \cdot i - 1 + 2(k_i - 2) + \ldots \ldots $$

$$= (i - 1) \cdot i + 1 + 2(k_i - 2) + (i + 1) \cdot i + 1 + 2(k_i - 2) + (i + 1) \cdot i + 1 + 2(k_i - 2) + \ldots \ldots $$

We will show that $\chi_q(M_0) \not\subseteq \chi_q(M)$. By $U_3 s t_2$ argument, the monomial

$$n_1 = (i - 1) \cdot i - 1 + 2(k_i - 2) + (i + 1) \cdot i + 1 + 2(k_i - 2) + \ldots \ldots $$

$$= M_0 A_{i,1+2k_i-3}^{-1}$$

$$= MA_{i,1+2k_i-3}^{-2}$$

is in $\chi_q(M_0)$. 

Suppose that $n_1 \in \chi_q(T_{0,0,k_i-1,0,...,0,k_j+1,k_{j+1}...k_n})$. Then $n_1 = m_1m_2$, where

\[ m_1 \in \chi_q(T_{0,0,k_i-1,0,...,0,k_j+1,k_{j+1}...k_n}) \]

\[ m_2 \in \chi_q(T_{0,0,k_i+1,0,...,0,k_j,k_{j+1}...k_n}) \]

Since $n_1 = MA_{i-1+2k_i-3}$, by the expressions (9.3) and (9.4) we must have

\[ m_1 = T_{0,0,k_i-1,0,...,0,k_j+1,k_{j+1}...k_n}A^{-1}_{i-1+2k_i-3} \]

It follows that $m_2 = T_{0,0,k_i+1,0,...,0,k_j,k_{j+1}...k_n}A^{-1}_{i-1+2k_i-3}$. But by the Frenkel-Mukhin algorithm and (9.3), $T_{0,0,k_i+1,0,...,0,k_j,k_{j+1}...k_n}A^{-1}_{i-1+2k_i-3}$ is not in $\chi_q(T_{0,0,k_i-1,0,...,0,k_j,k_{j+1}...k_n})$. This is a contradiction. Hence $\chi_q(M_0) \not\subseteq \chi_q(M)$.

The following is the proof of the fact that (9.2) is simple. We have

\[ T_{0,0,...,0,k_{n-1},k_{n+2}}^{(0)} = (n-1)_{2n-4}(n-1)_{2n-4+4...}(n-1)_{2n-4+4k_{n-1}+8n2n-4+4k_{n-1}+3n2n-4+4k_{n-1}+1} \]

\[ \cdots \]

(9.5)

\[ T_{0,...,0,k_{n-1}+1,k_{n}}^{(-1)} = (n-1)_{2n-4+4...}(n-1)_{2n-4+4k_{n}+4n2n-4+4k_{n-1}+1}n2n-4+4k_{n-1}+3 \cdots n2n-4+4k_{n-1}+1 \]

By Lemma 8.1 the dominant monomials in (9.2) are

\[ M_r = M \prod_{0 \leq j \leq r} A^{-1}_{n-1,2n-4+4k_{n-1}+4j-6} \]

\[ -1 \leq r \leq k_{n-1} - 2. \]

We need to show that $\chi_q(M_r) \not\subseteq \chi_q(M)$ for $0 \leq r \leq k_{n-1} - 2$. We will prove the case of $r = 0$. The other cases are similar.

\[ M_0 = MA_{n-1,2n-4+4k_{n-1}+6}^{-1} \]

\[ = (n-1)^{-1}_{2n-4+4k_{n-1}+2+4}(n-2)_{2n-4+4k_{n-1}+2+2n2n-4+4k_{n-1}+2+1n2n-4+4k_{n-1}+2+3(n-1)_{2n-4+4k_{n-1}+2}} \]

\[ \cdots \]

(9.6)
We will show that $\chi_q(M_0) \not\subseteq \chi_q(M)$. By $U_q\mathfrak{sl}_2$ argument, the monomial

\[
n_1 = (n - 2)^2 n_2 + 4 n_4 + 4(k_n - 1) + 2 n_2 \in \chi_q(T^{(0)}_{0, \ldots, 0, k_n-1, k_n+2}) \chi_q(T^{(-4)}_{0, \ldots, 0, k_n-1, k_n+2}).
\]

Suppose that $n_1 \in \chi_q(T^{(0)}_{0, \ldots, 0, k_n-1, k_n+2}) \chi_q(T^{(-4)}_{0, \ldots, 0, k_n-1, k_n+2})$. Then $n_1 = m_1 m_2$, where

\[
m_1 \in \chi_q(T^{(0)}_{0, \ldots, 0, k_n-1, k_n+2}), \quad m_2 \in \chi_q(T^{(-4)}_{0, \ldots, 0, k_n-1, k_n+2}).
\]

Since $n_1 = MA_{n-1, 2n-4+4k_n-1-6}^2$ by the expressions (9.5) and (9.6) we must have

\[
m_1 = T^{(0)}_{0, \ldots, 0, k_n-1, k_n+2} A_{n-1, 2n-4+4k_n-1-6}^{-1}.
\]

It follows that $m_2 = T^{(-4)}_{0, \ldots, 0, k_n-1+1, k_n} A_{n-1, 2n-4+4k_n-1-6}^{-1}$. But by the Frenkel-Mukhin algorithm and (9.6), $T^{(-4)}_{0, \ldots, 0, k_n-1+1, k_n} A_{n-1, 2n-4+4k_n-1-6}^{-1}$ is not in $\chi_q(T^{(-4)}_{0, \ldots, 0, k_n-1+1, k_n})$. This is a contradiction. Hence $\chi_q(M_0) \not\subseteq \chi_q(M)$.

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**Appendix**

In this section, we give some examples of mutation sequences. We draw a box at a vertex to indicate that a mutation has been performed at the vertex. Figure 2 and Figure 3 are examples of mutation sequences of type $A_3$. Figure 4 and Figure 5 are examples of mutation sequences of type $B_2$. 
Figure 2. The mutation sequence $(C_1, C_1)$. 
Figure 3. The mutation sequence \((C_2, C_1, C_2, C_1, C_1)\).

Figure 4. The mutation sequence \((C_1, C_1)\).
Figure 5. The mutation sequence \((C_3, C_3)\).

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