Solvable supersymmetric algebraic model for descriptions of transitional even and odd mass nuclei near the critical point of the spherical to unstable shapes

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Exactly solvable solution for the spherical to gamma - unstable transition in transitional nuclei based on dual algebraic structure and nuclear supersymmetry concept is proposed. The duality relations between the unitary and quasispin algebraic structures for the boson and fermion systems are extended to mixed boson-fermion system. It is shown that the relation between the even-even and odd-A neighbors implied by nuclear supersymmetry in addition to dynamical symmetry limits can be also used for transitional regions. The experimental evidences are presented for even-even \([E(5)]\) and odd-mass \([E(5/4)]\) nuclei near the critical point symmetry.
Exactly solvable models play a key role in understanding some many-body problems in nuclear physics. A model may be solvable if its energy levels can be determined analytically and eigenstates are identified completely by the quantum numbers of a subgroup chain. There are many-body problems in nuclear physics that are exactly solvable, for example, three limits of dynamical symmetry of the interacting boson model. The IBM describes even-even nuclei in terms of correlated pairs of nucleons with \( L = 0, 2 \) treated as bosons (s, d bosons). The N-boson system space is spanned by the irreps \([N] \) of \( U(6)_\text{BF} \). The interacting boson-fermion model also explains odd-A nuclei in terms of correlated pairs, s and d bosons, and unpaired particles of angular momentum \( j \) fermions. The states of the boson-fermion system can be classified according to the irreducible representation \([N] \times [1] \) of \( U(6)_\text{BF} \times U(4) \) where \( M \) is the dimension of the single particle space. The IBM and IBFM can be unified into a supersymmetry (SUSY) approach that was discovered into nuclear structure physics in the early 1980s. The experiments performed at various laboratories have confirmed the predictions made using a SUSY scheme.

Originally, nuclear supersymmetry was considered as symmetry among pairs of nuclei consisting of an even-even and an odd-even nucleus. The supersymmetric representations \([N] \) of \( U(6)_\text{BF} \) spanned a space explaining the lowest states of an even-even nucleus with \( N \) bosons and an odd-A nucleus with \( N - 1 \) bosons and an unpaired fermion. The nuclear supersymmetry has been used successfully in description of the dynamical symmetry limits of the even-even and odd-A nuclei. Virtually simultaneously, with the introduction of the nuclear supersymmetry, the idea of spherical-deformed phase transitions at low energy in finite nuclei germinated. Studies of QPTs in odd-even nuclei with supersymmetric scheme had implicitly been initiated years before by A. Frank et al. They have studied successfully a combination of \( U(6)_\text{BF} \) and \( SO(6)_\text{BF} \) symmetry by using \( U(6/12) \) supersymmetry for the Ru and Rh isotopes. Iachello extended the concept of critical symmetry to critical supersymmetry and provided a benchmark for the study of phase transition in odd-even nuclei and also J. Jolie et al. studied QPTs in odd-even nuclei using a supersymmetric approach in interacting Boson-Fermion model.

The purpose of this letter is to point out that we have proposed a new solvable model for describing QPT for even and odd mass nuclei by using supersymmetry approach. In this paper, concept of supersymmetry and phase transitions are brought together by using the generalized quasi-spin algebra and Richardson - Gaudin method. We consider the state of fermions with spin \( j = 3/2 \) coupled to boson core, however, our method is applicable for the whole systems with other values of spin of the fermions. In order to obtaining an algebraic solution for transitional region, we have used of dual algebraic structures. The duality symmetries are a powerful tool in relating the Hamiltonians with the number-conserving unitary and number-nonconserving quasi-spin algebras for system with pairing interactions. These relations are obtained for both bosonic and fermionic systems. We have established the duality relations for mixed boson-fermion system. In this paper, we display that the relation between the even-even and odd-A neighbors implied by nuclear supersymmetry in addition to dynamical-symmetry limits can also be used for transitional regions. So, the testing SUSY in all nuclear regions( dynamical symmetry limits and transitional region ) are possible. We investigate the change in level structure induced by the phase transition by doing a quantal analysis. The experimental evidences are presented for even- even \([E(5)]\) and odd-mass \([E(5/4)]\) nuclei near the critical point symmetry.

Details of the group theoretical description will be omitted from this paper and will be given in a subsequent detailed publication. The quasi-spin algebras have been explained in detail in Refs. The generalized quasi-spin algebra contains both bosonic (B) and fermionic (F) operators defined as:

\[
S^0_{BF} = \frac{1}{2}(n_b + n_f) + \frac{1}{4}(N - M)
\]

\[
S^+_{BF} = \frac{1}{2} \sum_m (-1)^{2\tau m} b^+ m b^{- m} + \frac{1}{2} \sum_{m'} (-1)^{2\tau m'} a^+_{j m'} a^{- m'} = \frac{1}{2}(b^+ \cdot b^+) \mp (a^+ \cdot a^-)
\]

\[
S^-_{BF} = \frac{1}{2}(\tilde{b} \cdot \tilde{b}) \pm \frac{1}{2}(\tilde{a} \cdot \tilde{a})
\]

Where \( n_b \) and \( n_f \) are the boson and fermion number operators, respectively. Thus, the operators \( S^\pm_{BF}, S^0_{BF} \) form a generalization of the usual fermionic and bosonic quasi-spin algebras with commutation relations given as:

\[
[S^0_{BF}, S^\pm_{BF}] = \pm S^\pm_{BF}, \quad [S^+_{BF}, S^-_{BF}] = -2S^0_{BF}
\]

The bosonic quasi-spin operators satisfy the commutation relations of the quasi-spin SU(1,1) algebra while the fermionic quasi-spin operators satisfy the commutation relations of the quasi-spin SU(2) algebra. The irreps of generalized quasi-spin algebra (GQA) are given in terms of the eigenvalues of the quadratic Casimir invariant and the quasi-spin operator \( S^0_{BF} \). The basis states of an irreducible representation (irrep) GQA, \( [k, \mu] \), are determined by a single number \( k \), susceptible of any positive number and \( \mu = k, k + 1, \ldots \). Therefore,

\[
[k, \mu] = \left[ \frac{N - M}{4} + \frac{\nu_B + \nu_F}{2}, \frac{N - M}{4} + \frac{N_B + N_F}{2} \right]
\]
The basis states are determined considering the fact that fermionic part of the action of $S_{BF}^+$ is restricted by the Pauli Exclusion Principle and the action of $S_{BF}^-$ terminates when a state of $\nu$ unpaired particles is reached i.e $S_{BF}^+|\nu_B, \nu_F=0\rangle$. In order to investigate the phase transition in atomic nuclei according to IBM, we have considered two kinds of bosons with $L = 0, 2$ (s, d bosons)\cite{9}. The generators of SU$^d(1, 1)$ and SU$^s(1, 1)$ are designated by considering and putting $s$ and $d$ operators only in the left part of Eqs.[1-3] and SU$^s(1, 1)$ that is the $s$ and $d$ boson pairing algebras generated by \cite{15}

$$S^+(sd) = \frac{1}{2}(d^+d^\dagger \pm s^+s^\dagger), \quad S^-(sd) = \frac{1}{2}(\bar{d}\bar{d} \pm s^2), \quad S^0(sd) = \frac{1}{4} \sum_{\nu} (d^+_{\nu}d^+_{\nu} + d_{\nu}d^+_{\nu} + 1)(s^+s + ss^+)$$ (6)

Because of the duality relationships \cite{13, 15}, it is known that in even-odd nuclei the basis of U(5) $\supset$ SO(5) and SO(6) $\supset$ SO(5) are simultaneously the basis of SU$^d(1, 1) \supset$ U(1) and SU$^d(1, 1) \supset$ U(1), respectively. By the use of duality relations \cite{13, 15}, the Casimir operators of SO(5) and SO(6) can also be expressed in terms of the Casimir operators of SU$^d(1, 1)$ and SU$^s(1, 1)$, respectively

$$\hat{C}_2(SU^d(1, 1)) = \frac{5}{16} + \frac{1}{4}\hat{C}_2(SO^B(5))$$ (7)

$$\hat{C}_2(SU^s(1, 1)) = \frac{3}{4} + \frac{1}{4}\hat{C}_2(SO^B(6))$$ (8)

The correspondence between the basis vectors in this case was shown in Ref.[15]. For a mixed boson-fermion system, the chain of subalgebras of unitary superalgebras U(6/M ) for $j=3/2$ is shown in Fig.1 and also two-level pairing system has two dynamical symmetries defined with respect to the generalized quasispin algebras, corresponding to either the upper or lower subalgebra chains in Eq.(9).

$$GQA^s_{1f} \otimes GQA^d_{2f} \supset \left\{ \begin{array}{lc} GQA^s_{1f} & \otimes U^s_{1f}(1) \otimes U^d_{2f}(1) \end{array} \right\} \supset U^s_{1f, 2}$$ (9)

The upper subalgebra chain is corresponding to strong-coupling dynamical symmetry limit while lower chain is weak-coupling limit. Therefore, for odd-A nuclei, we have obtained the dual relation between the Casimir operators Spin$^{BF}_{(5)}$ and GQA$^{d f}$ (generalized quasispin algebra of d bosons and single fermion with $j=3/2$) as

If $\tau_1 = \nu_d - \frac{1}{2}$ and $\tau_2 = \frac{1}{2}$

$$\hat{C}_2(GQA^{d f}) = \frac{1}{4}\hat{C}_2(Spin^{BF}_{(5)}) - \frac{1}{4}(\tau_1 + \frac{3}{4})$$ (10)

If $\tau_1 = \nu_d + \frac{1}{2}$ and $\tau_2 = \frac{1}{2}$

$$\hat{C}_2(GQA^{d f}) = \frac{1}{4}\hat{C}_2(Spin^{BF}_{(5)}) - \frac{1}{4}(3\tau_1 + \frac{7}{4})$$ (11)

By the use of duality relations, the correspondence between the basis vectors Spin$^{BF}_{(5)}$ and GQA$^{d f}$ is

$$\left| N; [N_B = N], N_F = 1, \nu_d, (\tau_1 = \nu_d - \frac{1}{2}, \tau_2), n_DJM \right\rangle = \left| N; [N_B = N], N_F = 1, \nu_d, (\tau_1 = \nu_d + \frac{1}{2}, \tau_2), n_DJM \right\rangle$$ (12)

$$\left| N; [N_B = N], N_F = 1, \nu_d, (\tau_1 = \nu_d + \frac{1}{2}, \tau_2), n_DJM \right\rangle = \left| N; [N_B = N], N_F = 1, \nu_d, (\tau_1 = \nu_d - \frac{1}{2}, \tau_2), n_DJM \right\rangle$$ (13)

The Casimir operator of Spin$^{BF}_{(6)}$ and GQA$^{s d f}$ (generalized quasispin algebra of d and s bosons with fermion $j=3/2$) has the following correspondence

If $\sigma_1 = \sigma - \frac{1}{2}$ and $\sigma_2 = |\sigma_3| = \frac{1}{2}$

$$\hat{C}_2(GQA^{s d f}) = \frac{1}{4}\hat{C}_2(Spin^{BF}_{(6)}) - \frac{3}{4}(\sigma_1 + \frac{3}{4})$$ (14)

If $\sigma_1 = \sigma + \frac{1}{2}$ and $\sigma_2 = |\sigma_3| = \frac{1}{2}$

$$\hat{C}_2(GQA^{s d f}) = \frac{1}{4}\hat{C}_2(Spin^{BF}_{(6)}) - \frac{1}{4}(\sigma_1 + \frac{3}{2})$$ (15)
By use of duality relations, the correspondence between the basis vectors SpinBF(6) and GQA^self is

\[ |N; [N_B = N], N_F = 1, \sigma, (\sigma_1, \sigma_2, \sigma_3), (\tau_1, \tau_2), n_{\Delta J}M = |N; k_{sdf} = \frac{1}{2}(\sigma_1 + \frac{3}{2}), \mu_{sdf} = \frac{1}{4} + \frac{1}{2}(n_s + n_d + n_f), n_{\Delta J}M \]  
\[ (16) \]

\[ |N; [N_B = N], N_F = 1, \sigma, (\sigma_1, \sigma_2, \sigma_3), (\tau_1, \tau_2), n_{\Delta J}M = |N; k_{sdf} = \frac{1}{2}(\sigma_1 + \frac{5}{2}), \mu_{sdf} = \frac{1}{4} + \frac{1}{2}(n_s + n_d + n_f), n_{\Delta J}M \]  
\[ (17) \]

These relations have been used to effect simplifications of the calculations for two-level and multi-level systems. The infinite dimensional generalized quasispin algebra is generated by the use of [15] and GQA and Casimir operators of subalgebras, the following Hamiltonian for transitional region between U(6) and U(5) Hamiltonian if c_s = 0 and c_d = c_f = 0.  So, the c_s \neq c_d \neq c_f \neq 0 situation just corresponds to U^0(5) ↔ O^0(6) transitional region. Hamiltonian Eq. (21) in even-even nuclei is equivalent to O^0(6) Hamiltonian when c_s = c_d = c_f and with U^0(5) Hamiltonian if c_s = 0 and c_d \neq c_f \neq 0.  So, the c_s \neq c_d \neq c_f \neq 0 situation just corresponds to U^0(5) ↔ O^0(6) transitional region. Hamiltonian Eq. (21) in even-even nuclei is equivalent to O(6) Hamiltonian when c_s = c_d and with U(5) Hamiltonian if c_s = 0 and c_d \neq 0 and Hamiltonian in transitional region with c_s \neq c_d \neq 0.  In our calculation, we take c_d(= 1) constant value and c_s and c_f change between 0 and c_d.

For evaluating the eigenvalues of Hamiltonian Eq. (21) the eigenstates are considered as

\[ |k; \nu_s, \nu_d; n_{\Delta J}LM) = NS^+_{BF}(x_1)S^+_{BF}(x_2)S^+_{BF}(x_3)...S^+_{BF}(x_k) |w\rangle^{BF} \]  
\[ (22) \]

\[ S^+_{x_i} = \frac{c_s}{1 - c_s x_i} S^+_B (s) + \frac{c_d}{1 - c_d x_i} S^+_B (d) + \frac{c_f}{1 - c_f x_i} S^+_F (f) \]  
\[ (23) \]

The lowest weight state, |w\rangle^{BF} , is defined as

\[ |w\rangle^{BF} = |N = N_B + N_F, k_d = \frac{1}{2}(\nu_d + \frac{5}{2}), \mu_d = \frac{1}{2}(n_d + \frac{5}{2}), k_s = \frac{1}{2}(\nu_s + \frac{1}{2}), \mu_s = \frac{1}{2}(n_s + \frac{1}{2}), k_f = \frac{1}{2} \]

\[ (\nu_f - \frac{2j + 1}{2}), \mu_f = \frac{1}{2}(n_f - \frac{2j + 1}{2}), J, M) \]  
\[ (24) \]

\[ S^n_u |w\rangle^{BF} = \Lambda^n_u |w\rangle^{BF} , \quad \Lambda^n_u = c_s^n(\nu_u + \frac{1}{2}, \mu_u + \frac{5}{2}) + c_d^n(\nu_d + \frac{1}{2}) + c_f^n(\nu_f - \frac{2j + 1}{2}, \mu_f - \frac{2j + 1}{2}, \mu_f - \frac{1}{2}) \]  
\[ (25) \]

The eigenvalues of Hamiltonian Eq. (21) can then be expressed as

\[ E^{(k)} = h^{(k)} + \alpha \Lambda^0_1 + \beta (\tau_1(\tau_1 + 1) + \tau_2(\tau_2 + 1)) + \gamma J(J + 1) \quad h^{(k)} = \sum_{i=1}^{k} \frac{\alpha}{x_i} \]  
\[ (26) \]

In order to obtain the numerical results for energy spectra (E\(^{(k)}\)) of the considered nuclei, a set of non-linear Bethe-ansatz equations (BAE) with k- unknowns for k-pair excitations must be solved. Also constants of Hamiltonian with
The least square fitting processes to experimental data are obtained. To achieve this aim, we have changed variables as

\[ C_s = \frac{c_s}{c_d} \leq 1, \quad C_f = \frac{c_f}{c_d} \leq 1, \quad y_i = c_i^2 x_i \]

\[ \alpha = \frac{C_s^2 (\nu_s + \frac{3}{2})}{1 - C_s^2 y_i} + \frac{C_f^2 (\nu_f - \frac{2j+1}{2})}{1 - C_f^2 y_i} - \sum_{j \neq i} \frac{2}{y_i - y_j} \]

The quantum number \( k \) is related to \( \mathcal{N} \) by \( \mathcal{N} = 2k + \nu_s + \nu_f + \nu_t \). The quality of the fits is specified by the values of \( \sigma = \left( \frac{1}{N_{\text{tot}}} \sum_i |E_{\text{exp}}(1) - E_{\text{Cal}}(1)|^2 \right)^{1/2} \) (keV) and \( \phi = \frac{\sum |E_{\text{exp}}^\text{obs} - E_{\text{exp}}^\text{th}|}{\sum E_{\text{exp}}^\text{th}} \) \( (\%) \) (\( N_{\text{tot}} \) the number of energy levels where included in the fitting processes) [4, 15].

The complete study of the properties of quantum phase transitions comprises both the classical and quantal analyses. In this study, we focus only on the quantal analysis and present the calculated phase transition observables such as the level crossing, the expectation value of the d-boson number operator and the expectation value of the fermion number operator.

Once the eigenvalues have been obtained, we can display how the energy levels change within the whole range of the \( C_s \) and \( C_f \) control parameters. Fig. 2 shows the energy surfaces of Hamiltonian of Eq. (21) for the neighboring even-even (left panel) and odd-A nuclei (right panel). The calculations are performed by considering the same fit parameters for these nuclei, where the parameters are \( \alpha = 1000 \text{keV}, \beta = -1.29 \text{keV}, \gamma = 6.05 \text{keV}, N=10 \). Fig. 2 shows how the energy levels as a function of the control parameter \( C_s \) and \( C_f \) evolve from one dynamical symmetry limit to the other. It can be seen from Figs that numerous level crossings occur. The crossings are due to the fact that \( \nu_f \), O(5) quantum number called seniority, is preserved along the whole path between O(6) and U(5) [12, 17].

The other quantum order parameters that we consider here are the expectation values of the d-boson number operator and the expectation values of the fermion number operator. The expectation values of the d-boson number operator and fermion number operator are obtained as

\[ \langle n_d \rangle = \frac{\langle \psi | n_d | \psi \rangle}{N} = \frac{2C_s^2 C_f^2 (\Lambda_0^0 - k(1 - y_i^{-1})) - 2(C_s^2 + C_f^2)(\Lambda_0^0 - k(1 - y_i^{-1})) + 2(\Lambda_0^0 - k(1 - y_i^{-1}))}{N(1 - C_s^2)(1 - C_f^2)} + \frac{2j + 1}{2N} \]
Table 1. Parameters of Hamiltonian Eq.(20) used in the calculation of
the Ba and Xe nuclei. All parameters are given in keV.

| Nucleus | $N$ | $C_a$ | $C_f$ | $\alpha$ | $\beta$ | $\gamma$ | $\sigma$ | $\phi$ |
|---------|-----|------|------|-------|-----|-----|-----|-----|
| $^{134}$Ba $\rightarrow ^{135}$Ba | 5 | 0.52 | 0.8 | 333.12 | 1.945 | 0.73 | 155.42 | 10.73% |
| $^{130}$Xe $\rightarrow ^{131}$Xe | 5 | 0.55 | 0.9 | 187.84 | 0.9564 | 10.98 | 143.29 | 12.4% |

of parameters which reproduce these complete spectra with minimum variations. It means that our suggestion to use
this transitional Hamiltonian for the description of the Ba and Xe isotopes would not have any contradiction with
other theoretical studies done with special hypotheses about mixing of intruder and normal configurations. So, we
conclude from the values of control parameter which has been obtained and $R_2$ value, that $^{134-135}$Ba and $^{130-131}$Xe
isotopes are the best candidates for U(5)-O(6) transition in U(6/4) supersymmetry scheme.

Although, studies of QPTs in odd-even and even-even nuclei were extensively done, our results are novel, since (1)
we have proposed exactly-solvable supersymmetry Richardson-Gaudin (R-G) model for transitional region by which
we can investigate the phase transition observables in both of nuclei by using of the concept of supersymmetry
(2) The experimental evidences have been presented for E(5) and E(5/4) nuclei and have been analyzed them by
supersymmetry scheme. The important new result of the present paper is to have employed the nuclear supersymmetry
approach for description of the transitional region between spherical and gamma -unstable phase shape in addition
to dynamical symmetry limits in one chain isotopic.

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FIG. 1. The lattice of algebras in the U(6/4) supersymmetry scheme.

FIG. 2. Energy levels as a function of $C_s$ control parameter for a even-even nuclei (left panel) and for odd-A nuclei as a function of the $C_s$ and $C_f$ control parameters(right panel) in the Hamiltonian (20) for $N=10$ bosons with $\alpha = 1000\text{keV}, \beta = -1.29\text{keV}, \gamma = 6.05\text{keV}$.

FIG. 3. The expectation values of the d-boson number operator for the lowest states as a function of $C_s$ control parameter for an even-even nuclei (left panel) and for odd-A nuclei as a function of the $C_s$ and $C_f$ control parameters(right panel).
FIG. 4. The expectation values of the fermion number operator for odd-A nuclei for the lowest states as a function of $C_s$ and $C_f$ control parameters.

FIG. 5. Comparison between calculated and experimental spectra of positive parity states in $^{134}$Ba (left panel) and $^{135}$Ba(right panel). The parameters of the calculation are given in Table 1. The experimental spectra, is taken from [21].

FIG. 6. Comparison between calculated and experimental spectra of positive parity states in $^{130}$Xe(top panel) and $^{131}$Xe(bottom panel). The parameters of the calculation are given in Tables 1. The experimental spectra, is taken from [21].