Geometric and viscosity solutions for the Cauchy problem of first order

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Abstract
There are two kinds of solutions of the Cauchy problem of first order, the viscosity solution and the more geometric minimax solution and in general they are different. The aim of this thesis is to show how they are related: iterating the minimax procedure during shorter and shorter time intervals one approaches the viscosity solution. This can be considered as an extension to the contact framework of the result of Q. Wei [19] in the symplectic case.

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This article is concerned with the study of weak solutions to the following Cauchy problem for the Hamilton-Jacobi equation:

\[ \begin{aligned}
\partial_t u(t, x) + H(t, x, \partial_x u(t, x), u(t, x)) &= 0, & t \in (0, T] \\
u(0, t) &= v(x), & x \in \mathbb{R}^k.
\end{aligned} \] (HJ)

In particular, we focus on the relation between the notion of viscosity solution (introduced by Crandall, Evans and Lions) and the more geometric one of minmax solution (due to Chaperon and Sikorav); this latter is based on a minmax procedure and relies on the existence of suitable generating families. It is known that when the Hamiltonian \( H(t, x, y, z) \) is not assumed to be convex in the momentum component \( y \), then these two notions do not (in general) lead to the same set of solutions. For example, see [20, prop. 3.8].

The main aim of this paper is to describe an iterated minmax procedure - performed on finer and finer partitions of the time interval - that, in the limit, allows one to recover viscosity solutions from the minmax ones.

A result in the very same spirit has been recently obtained by Qiaoling Wei [19], in the case of Hamiltonians that depend only on \((t, x, p)\), the so-called symplectic case. This work can be seen as a (technical) extension of Wei’s result and techniques to the contact case, namely for Hamiltonians depending on \((t, x, y, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k\).

Although the article is a technical adaptation - yet, not immediate - of Wei’s approach to this context, we believe that it might be useful as a possible reference for further applications and developments.

The classical method (section 3) to solve this problem, for \( v \in C^2 \) and a short time interval, consists in solving characteristics equations

\[ \begin{aligned}
\dot{x} &= \partial_y H \\
\dot{y} &= -\partial_z H - y\partial_z H \\
\dot{z} &= y\partial_y H - H,
\end{aligned} \] (H1-3)

to get the characteristic lines, that is, the trajectories

\[ \Phi(t, q_0) = (t, x(t, q_0), y(t, q_0), z(t, q_0)); \]

where \( x(\cdot), y(\cdot), z(\cdot) \) are the solutions to characteristic equations with initial conditions \( q_0 = (x_0, Dv(x_0), v(x_0)) \) at time \( t = 0 \) and then obtain the solution \( u(t, x) \) of the Cauchy problem as follows: setting \( u_t(x) = u(t, x) \), for fixed \( t \), and
its 1-jet \( j^1 u_t(x) = (x, du_t(x), u_t(x)) \), the image of \( j^1 u_t \) is the section at time \( t \) of \( c \)
\[
\mathcal{L}(\Lambda_0) = \bigcup_{t \in [0,T]} \Phi(t, \Lambda_0) = \{ \Phi(t) | t \in [0,T], \Phi(0) \in \Lambda_0 \}
\]
where \( \Lambda_0 = \{(0, j^1 v(x))\} \).

This procedure does not yield a global solution of the problem in the whole interval \([0,T]\), as the geometric solution \( L \) is not always the set \( \{(t, j^1 u_t(x))\} \) for a function \( u(t,x) \): According to [1, section 3.2] the wavefront
\[
\mathcal{F} = \pi(\mathcal{L}(\Lambda_0)),
\]
(where \( \pi(t,x,y,z) = (t,x,z) \)) obtained in \((t,x,y,z)\) space by solving the equation
\[
dz = -H(t,x,y,z)dt + ydx
\]
restricted to \((t,x,y,z) \in \mathcal{L}\) it is not the graph of a function: the projected characteristics
\[
\pi(\Phi(t,q_0)) = (t,x(t,q_0),z(t,q_0)), q_0 \in \Lambda_0
\]
may cross after some time. For example, characteristics for conservation laws [2] example 3.2.5]

Whereas in some applications, e.g. to geometrical optics, the wavefront \( \mathcal{F} \) can be considered as a solution of the physical problem, one is interested in a single-valued solution \( u(t,x) \). Assuming that the projection of \( \mathcal{F} \) into \((t,x)\) space is onto, one can construct such a solution as a section of the wavefront, selecting a single \( u \) over each \((t,x)\). When the function \( H \) is sufficiently convex with respect to \( y \) (and \( v \) is not too wild at infinity), such a “selector” consists in choosing for \( u(t,x) \) the smallest \( u \) with \((t,x,u) \in \mathcal{F}\).

This min solution happens to be the “viscosity solution” which was first introduced as the viscosity limit when \( \epsilon \to 0^+ \) of the solution of the Cauchy problem for the viscous equation
\[
\partial_t u(t,x) + H(t,x,\partial_x u,u) = \epsilon \triangle_x u(t,x),
\]
and afterwards got a general definition for general nonlinear first order partial differential equations in the work of Crandall, Evans and Lions [13][11][13].

In the non-convex case the viscosity solution may not be a section of the wavefront (see for example [6]). On the other hand, Chaperon introduced in [5] weak solutions whose graph is a section of the wavefront, obtained by a “minimax” procedure which generalizes the minimum considered in the convex case and relies on the existence of suitable generating families for the geometric solution.

Let’s explain in more detail this procedure: First, consider the \textit{legendrian submanifold} \( \Lambda = \varphi^t(j^1 v) \subset J^1 \mathbb{R}^k \), for some \( \mathbb{R}^k \), where \( \varphi^t \) is the flow generated by [11]-[13] and therefore \( \Phi(t) = (t, \varphi(t)) \). It turns out that there exist a so called \textit{generating function quadratic at infinity}
\[
S : [0,T] \times \mathbb{R}^k \times \mathbb{R}^q, (t,x,\xi) \to S^t_{H,v}(x,\xi)
\]
r^q, a family of parameters $\xi$) such that
\[ \Lambda = \left\{ (x, \partial_x S_{H,v}(x, \xi), S_{H,v}(x, \xi)) \mid \partial_\xi S_{H,v}(x, \xi) = 0 \right\}. \quad (1) \]

Now we can define a minimax selector such that
\[ u(t, x) = \inf_{\xi} \max_{S_{H,v}(x, \xi)} \]
is a generalized solution of Cauchy problem \[ \text{(HJ)}. \] It is called the minimax solution. But in general this is not a classical solution. Indeed $u_t \in C^{\text{Lip}}(\mathbb{R}^k)$. Although, we have been considering $v \in C^2(\mathbb{R}^k)$, in the more general setting of Clarke calculus, we can consider $v \in C^{\text{Lip}}(\mathbb{R}^k)$, and thus for a given $H \in C^2(J^1\mathbb{R}^k)$, we obtain an operator
\[ R^{t_i}_{t_{i+1}} : C^{\text{Lip}}(\mathbb{R}^k) \to C^{\text{Lip}}(\mathbb{R}^k), v(\cdot) \mapsto \inf_{\xi} S_{H,v}(\cdot, \xi). \]

Of course, we can take another initial initial time $t_0 = s \neq 0$, for a different boundary condition $u(s, x) = v(x)$.

One may try to get a solution as a limit obtained by dividing a given time interval into small pieces and iterating the minimax procedure step by step. Our goal is to show that when the size of the time intervals go to zero, one indeed gets the viscosity solution as the limit:

**Theorem 1 (Main theorem)** Suppose
\[
\begin{align*}
H &\in C^2([0, T] \times J^1(\mathbb{R}^k)) \\
v &\in C^{\text{Lip}}(\mathbb{R}^k)
\end{align*}
\]
Then the viscosity solution is the limit of iterated minimax solutions for problem \[ \text{(HJ)} \] on $[0, T]$.

Indeed, it is the aim of this dissertation to give a complete proof of this results. Let’s explain the meaning if theorem 1 in more detail. Take a partition of $[0, T]$:
\[ 0 = t_0 < t_1 < \ldots < t_N = T, \]
and for $i = 0, ..., N - 1$, define the so called \textit{iterated minimax solution}
\[
\begin{align*}
u_0(t, x) &= v(x) \\
u_{i+1}(t, x) &= R^{t_i}_{t_{i+1}}(u_i(t, x)).
\end{align*}
\]

Our main result result establishes that $u_N(t, x)$ converges uniformly on compacts to the viscosity solution of Cauchy problem \[ \text{(HJ)} \] when
\[ \sup \{t_{i+1} - t_i\} \to 0. \]

This extends the result obtained by Q. Wei \[ [19] \] in the symplectic framework to the contact one. Indeed, Prof. Wei was student of Prof. Marc Chaperon,
who in [5] had already defined generating families and minimax in the contact setting. But in the way that minimax solutions were defined in that paper, they weren’t suitable for constructing a generating family similar to Wei managed to construct.

However, in his Ph.D. thesis [2], Prof. Mohan Bhupal defined a generating function for contactomorphisms in a more convenient manner, and we took advantage of his work to develop a theory in contact framework which generalize the results obtained by Wei.

As Prof. Chaperon told us in a brief talk a couple years ago, this problem was important in order to understand better the geometry of viscosity solutions. Although the theorem in symplectic was already important because its relations with classical mechanics, our generalization allows to study a more general class of problems related with non-linear first order partial differential equations.

For convenience of the reader, we give a very brief guide to this thesis: At first, in part I we deduce an explicit formula to obtain a generating family for legendrian submanifolds of the form $\Phi(t, f^1v)$, and generalize this construction to a more general setting of Clarke calculus in order to construct iterated minimax solutions.

Next, in part II we define minimax selector, iterated minimax solutions and prove the relations established above with viscosity solutions. We followed closely the methods of Prof. Wei in order to get the desired generalization, and so is in this section where our contribution appears.

At the end of this work, three appendices are given: The first one is about contact topology, in particular, legendrian submanifolds and contactomorphisms; the second one is about Clarke calculus; and the last one about minimax principles. A more detailed treatment of these topics could be consulted in [8], [7] and [15], respectively.

Finally, let’s us remark, as Arnold did in his lectures “Contact Geometry and Wave Propagation” given at the University of Oxford

“The relations between symplectic and contact geometries are similar to those between linear algebra and projective geometry. First, the two related theories are formally more or less equivalent: every theorem in symplecticegeometry may be formulated as a contact geometry theorem, and any assertion in contact geometry may be translated into the language of symplectic geometry. Next, all the calculations look algebraically simpler in the symplectic case, but geometrically things are usually better understood when translated into the language of contact geometry.”

Following that spirit, as far as possible, we has formulated our results as their symplectic counterpart are given on [19]. One more advice from Arnold’s lectures:

Hence one is advised to calculate symplectically but to think rather in contact geometry terms.
Part I
Generating Families

The main goal of this part is to construct generating families for Legendrian submanifolds $\phi^t(j^1v)$, where $\phi^t$ is the flow generated by (H1)-(H3). As a previous step, we have to demonstrate the existence of generating functions for contactomorphism $\phi^t$.

These constructions are fundamental for our work, because they will allow us to generalize in a very natural way those formulas given in [19]. Following that work, we will be able to define explicitly iterate minimax solutions in the next section, but in the more general setting of contact topology, that is, for Hamiltonians depending on $(t, x, \partial_x u, u)$.

At the end of the section, we extend the concept of generating families using Clarke calculus [7], allowing us to consider initial conditions $v \in C^{lip}$. This step is crucial because minimax solutions are not differentiable anymore but only Lipschitz, and we will iterate solutions of this type.

2 Generating Functions

Consider $J^1(M) = T^*M \times \mathbb{R}$ endowed with the natural contact structure $\ker \alpha$ given in local coordinates $(x, y, z)$ for $T^*M \times \mathbb{R}$ by $\alpha = dz - ydx$. We denote by $\pi : J^1(M) \to M$ the canonical projection. Recall that a submanifold $L$ of $J^1(M)$ is called Legendrian if $T_pL \subset \ker \alpha|_p$, for any $p \in L$ and $\dim L = \dim M$.

Suppose $S \in C^2(M \times \mathbb{R}^q)$ has fiber derivative $\partial_\xi S$ transversal to 0. Then

$$\Lambda := \{(x, \partial_\xi S(x,\xi), S(x,\xi)) | \partial_\xi S(x,\xi) = 0\}$$

is a Legendrian submanifold of $J^1M$ and we say that $S$ a generating family of $\Lambda$.

**Definition 2.1 (g.f.q.i.)** A generating family $S \in C^2(M \times \mathbb{R}^q)$ is quadratic at infinity (g.f.q.i. for short) if there exists a non degenerate quadratic form $Q$, such that for any compact $K \subset M$,

$$|\partial_\xi (S(x,\xi) - Q(\xi))|$$

is bounded on $K \times \mathbb{R}^q$.

Consider the sub-level sets $S^a_x := \{\xi : S(x,\xi) \leq a\}$, for $a$ large enough the homotopy type of $(S^a_x, S^{-a}_x)$ does not depend on $a$ and coincides with the homotopy type of $(Q^a, Q^{-a})$, so we may write it as $(S^\infty_x, S^{-\infty}_x)$. If the Morse index of $Q$ is $k$, then

$$H_i(S^\infty_x, S^{-\infty}_x; \mathbb{Z}_2) = H_i(Q^\infty, Q^{-\infty}; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & i = k, \\ 0, & i \neq k. \end{cases}$$
For more details, see [3] section 7.2.3

**Definition 2.2** The minimax function is defined as

\[ R_S(x) := \inf_{|\sigma| = A} \max_{\xi \in \sigma} S(x, \xi), \]

where \( A \) is a generator of the homology group \( H_k(S_x^{\infty}, S_x^{-\infty}; \mathbb{Z}_2) \) and \( |\sigma| \) denotes the image of the relative singular homology cycle \( \sigma \).

**Remark 2** In [21], Q. Wei proved that the minimax values of a function quadratic nondegenerate at infinity are equal when defined in homology or cohomology with coefficients in a field. However, by an example of F. Laudenbach, this is not always true for coefficients in a ring and, even in the case of a field, the minimax-maximin depends on the field.

Although this is the most common manner to describe the minimax function, we will give a more concrete description of the minimax in section II.

The function \( R_S \) is determined by \( A \) and does not depend on the particular choice of the g.f.q.i. \( S \) according to the following result

**Theorem 3** [16, prop 2.12] The g.f.q.i. of a Legendrian submanifold contact isotopic to the zero section is unique up to the following operations relating \( S_1 \) to \( S_2 \).

- **Stabilization** \( S_2(x, \xi, \eta) = S_1(x, \xi) + q(\eta) \) with \( q \) a non degenerate quadratic form.

- **Diffeomorphism** \( S_2(x, \xi) = S_1(x, \psi(x, \xi)) \) with \( \psi(x, \cdot) \) a diffeomorphism \( \forall x \in M \).

The following definition is common in the literature

**Definition 2.3 (strict g.f.q.i.)** A generating function is strictly quadratic at infinity if there is a non degenerate quadratic form \( Q \) such that \( S(x, \xi) = Q(\xi) \) for \( (x, \xi) \) outside some compact set (we will say strict g.f.q.i. for short).

This definition is more appropriate to work with hamiltonians

\[ H \in C^1([0,T] \times J^1 M), \]

for \( M \) a compact manifold. However under some restrictions, both definitions of g.f.q.i. are equivalent

**Proposition 4** [17, prop. 1.6]

Suppose there is a constant \( C \) such that

(a) \( \| \nabla(S - Q) \|_{C^0} < C \)

(b) \( \sup \{ |S - Q| \mid x \in \mathbb{R}^k, |\xi| \leq r \} < Cr \)
Lemma 7 (Cf. [19], lemmata 2.5, 2.6) If $L$ has a strict g.f., i.e.,

Recall that a diffeomorphism $\varphi : J^1 M \to J^1 M$ is called contactomorphism if $D\varphi(\ker \alpha) = \ker \alpha$ or equivalently $\varphi^*\alpha = g \alpha$ with $g \in C^1(J^1 M, \mathbb{R} - \{0\})$.

From here, we follow the construction given in [3, section 6].

Definition 2.4 Let $\varphi : J^1 \mathbb{R}^k \to J^1 \mathbb{R}^k$ be a contactomorphism. A generating function for $\varphi$ is a function $\Phi : \mathbb{R}^{2k+1} \to \mathbb{R}$ such that $1 - \partial_s \Phi(x, y, z)$ never vanishes and the set of equalities

\[
X - x = \partial_Y \Phi(x, y, z) \\
Y - y = -\partial_x \Phi(x, y, z) - y \partial_z \Phi(x, y, z) \\
Z - z = (X - x)Y - \Phi(x, y, z)
\]

is equivalent to $\varphi(x, y, z) = (X, Y, Z)$.

Remark 5 The contactomorphism $\varphi$ has compact support if and only if $\Phi$ does.

Proposition 6 [3, prop. 6.1]

(i) A contactomorphism $\varphi : J^1 \mathbb{R}^k \to J^1 \mathbb{R}^k$ with $\|1 - d\varphi(p)\| < \frac{1}{2}$ for all $p \in J^1 \mathbb{R}^k$ has a unique generating function.

(ii) If $\Phi \in C_\infty(\mathbb{R}^{2k+1})$ has sufficiently small first and second derivatives, there exists a unique contactomorphism $\varphi : J^1 \mathbb{R}^k \to J^1 \mathbb{R}^k$ having $\Phi$ as generating function.

3 Method of Characteristics

For $H \in C^2([0, T] \times J^1 \mathbb{R}^k)$ let $X_H$ be the associated time-dependent contact vector field given by (H1)-(H3), and $t \to \varphi_t(q)$ be the integral curve with $\varphi_0(q) = q$. One calls $\varphi^t$ the contact isotopy defined by $H$. We define $\varphi^{s,t} = \varphi_t \circ (\varphi^s)^{-1}$. Suppose that $H$ has compact support contained in the set

\[
\{(x, y, z) \in J^1 \mathbb{R}^k : |y| \leq a\}
\]

and let $c_H = \sup \{|DH_t(x, y, z)|, |D^2 H_t(x, y, z)|\}$, then $\max_t \|X_H\|_{\text{Lip}} \leq (2 + a)c_H$.

Lemma 7 (Cf. [19], lemmata 2.5, 2.6) If $\delta_H = \log 2/(2 + a)c_H$, for $0 < t - s < \delta_H$ there is a generating function $\Phi^{s,t} : J^1 \mathbb{R}^k \to \mathbb{R}$ for $\varphi^{s,t}$. Let $q = (x, y, z), r = (X, Y, Z)$ and for $s \leq \tau \leq t$, define $\varphi^{s,\tau}(q) = (x(\tau, q), y(\tau, q), z(\tau, q))$, $\varphi^{s,\tau}(r) = (\bar{x}(\tau, r), \bar{y}(\tau, r), \bar{z}(\tau, r))$. Then

\[
\Phi^{s,t}(x, y(t, q), z) = \int_s^t (\dot{x}(\tau, q)(y(t, q) - y(\tau, q)) + H(\tau, \varphi^{s,\tau}(q)))d\tau, \\
\partial_t \Phi^{s,t}(x, y(t, q), z) = H(t, \varphi^{s,t}(q)), \\
\partial_s \Phi^{s,t}(\bar{x}(s, r), Y, \bar{z}(s, r)) = H(t, \varphi^{s,t}(r))(\partial_x \Phi^{s,t}(\bar{x}(s, r), Y, \bar{z}(s, r)) - 1)
\]
Differentiating respect to $t$

On the other hand

Then

Differentiating respect to $t$

On the other hand

Comparing these expressions we obtain (3). Similarly we have

Comparing these expressions we obtain (4).
4 Generating families for Legendrian submanifolds

Let $H \in C^2_c([0, T] \times J^1 \mathbb{R}^k)$ and $\varphi^t$ be the contact isotopy defined by $H$. Following [2] we will construct a generating families for Legendrian submanifolds.

Let $s = t_0 < t_1 < \cdots < t_N = t$ be a partition such that $|t_{i+1} - t_i| < \delta_H$ so that

$$\varphi^{s,t} = \varphi^{t_{N-1},t} \circ \cdots \circ \varphi^{s_0,t_1}.$$

Proposition 8 (Cf. [19], corollary 2.8) Let $0 \leq s = t_0 < t_1 < \cdots < t_N = t \leq T$ be a partition such that $|t_{i+1} - t_i| < \delta_H$ and $\Phi^{t_i,t_{i+1}} : \mathbb{R}^{2k+1} \to \mathbb{R}$ be the generating function of $\varphi^{t_i,t_{i+1}}$ given by Proposition 6. For $v \in C^2(\mathbb{R}^k)$ we have

(a) One can define a generating family $S^{s,t} : \mathbb{R}^k \times \mathbb{R}^{2kN} \to \mathbb{R}$ of $\varphi^{s,t}(j^1 v)$ by

$$S^{s,t}(x; \xi) = v(x_0) + \sum_{j=1}^N y_j(x_j - x_{j-1}) - \Phi^{t_j-1,t_j}(x_{j-1}, y_j, z_{j-1}) \quad (6)$$

where $x = x_N$, $\xi = (x_0, \ldots, x_{N-1}, y_1, \ldots, y_N)$, and $z_0, \ldots, z_{N-1}$ are defined inductively by

$$z_0 = v(x_0)$$

$$z_j = z_{j-1} + (x_j - x_{j-1}) y_j - \Phi^{t_{j-1},t_j}(x_{j-1}, y_j, z_{j-1}) \quad 0 < j \leq N. \quad (7)$$

Notice that $z_j$ depends only on $(x_0, x_j, y_1, \ldots, y_j)$.

(b) One can define a $C^2$ function $S^{s,t} : [s, t] \times \mathbb{R}^k \times \mathbb{R}^{2kN} \to \mathbb{R}$, such that each $S^{s,t}(\tau, \cdot)$ is a generating family of $\varphi^{s,t}_H(j^1 v)$, as follows: let $\tau_j = s + (\tau - s) \frac{t_j - s}{t - s}$ and

$$S^{s,t}(\tau, x; \xi) = v(x_0) + \sum_{j=1}^N y_j(x_j - x_{j-1}) - \Phi^{t_{j-1},\tau_j}(x_{j-1}, y_j, \tilde{z}_{j-1}). \quad (8)$$

where $x = x_N$, $\xi = (x_0, \ldots, x_{N-1}, y_1, \ldots, y_N)$, and $\tilde{z}_0, \ldots, \tilde{z}_{N-1}$ are defined inductively as before.

(c) For each critical point $\xi$ of $S^{s,t}(\tau, \cdot)$ we have

$$S^{s,t}(\tau, x, \xi) = \int_s^\tau \left( \dot{x}(\sigma, j^1 v(x_0)))y(\sigma, j^1 v(x_0)) - H(\sigma, \varphi^{s,\sigma}(j^1 v(x_0))) \right) d\sigma$$

where $\varphi^{s,\sigma}(p) = (x(\sigma, p), y(\sigma, p), z(\sigma, p))$, $x(\sigma, j^1 v(x_0)) = x$. \quad (9)
Proof. We have that the generating function \( \Phi_i := \Phi_i^{t, t+1} : \mathbb{R}^{2k+1} \to \mathbb{R} \) satisfies \( 1 - \partial_z \Phi_i \neq 0 \) and that conditions

\[
\begin{align*}
  x_{i+1} - x_i &= \partial_y \Phi_i (x_i, y_{i+1}, z_i) \\
  y_{i+1} - y_i &= -\partial_x \Phi_i (x_i, y_{i+1}, z_i) - y_i \partial_z \Phi_i (x_i, y_{i+1}, z_i) \\
  z_{i+1} - z_i &= (x_{i+1} - x_i) y_{i+1} - \Phi_i (x_i, y_{i+1}, z_i)
\end{align*}
\] (10)

hold if and only if \( \Phi_i^{t, t+1} (x_i, y_i, z_i) = (x_{i+1}, y_{i+1}, z_{i+1}) \). We have

\[
\partial_x S^{s,t} (x; \xi) = y_N.
\] (11)

Let \( i = 0, \ldots, N-1 \). For \( i < j - 1 \) we have

\[
\partial_x, z_j = \partial_x (z_{j-1} + (x_j - x_{j-1}) y_j - \Phi_{j-1} (x_{j-1}, y_{j}, z_{j-1}))
\]

\[
= \partial_x, z_{j-1} - \partial_{z_{j-1}} \Phi_{j-1} \partial_x, z_{j-1} = (1 - \partial_{z_{j-1}} \Phi_{j-1}) \partial_x, z_{j-1}
\]

and since \( \partial_x, z_i = y_i \) we get

\[
\partial_x, z_{i+1} = y_i - y_{i+1} - \partial_x, \Phi_i - y_i \partial_z, \Phi_i.
\]

As \( S^{s,t} (x, \xi) = z_N \), for \( 0 < i < N \) we obtain

\[
\partial_x, S^{s,t} (x, \xi) = (1 - \partial_{z_{N-1}} \Phi_{N-1}) \cdots (1 - \partial_{z_i} \Phi_{i+1})
\]

\[
\times (y_i - y_{i+1} - \partial_x, \Phi_i - y_i \partial_z, \Phi_i),
\] (12)

\[
\partial_{x_0} S^{s,t} (x; \xi) = (1 - \partial_{z_{N-1}} \Phi_{N-1}) \cdots (1 - \partial_{y} \Phi_1)
\]

\[
\times (d \partial v) - y_1 - \partial_{x_0} \Phi_0 - \partial_{z_0} \Phi_0 d \partial v (x_0)
\] (13)

For \( i < j \leq N \)

\[
\partial_{y_i}, z_j = \partial_{y_i} (z_{j-1} + (x_j - x_{j-1}) y_j - \Phi_{j-1} (x_{j-1}, y_{j}, z_{j-1}))
\]

\[
= \partial_{y_i}, z_{j-1} - \partial_{y_{j-1}} \Phi_{j-1} \partial_{y_i}, z_{j-1} = (1 - \partial_{y_{j-1}} \Phi_{j-1}) \partial_{y_i}, z_{j-1},
\]

\[
\partial_{y_i}, z_i = \partial_{y_i} (z_{i-1} + (x_i - x_{i-1}) y_i - \Phi_{i-1} (x_{i-1}, y_{i}, z_{i-1}))
\]

\[
= x_i - x_{i-1} - \partial_{y_i} \Phi_{i-1},
\]

so we get

\[
\partial_{y_i} S^{s,t} (x, \xi) = (1 - \partial_{z_{N-1}} \Phi_{N-1}) \cdots (1 - \partial_{y} \Phi_1) (x_i - x_{i-1} - \partial_{y} \Phi_{i-1}).
\] (14)

From (10), (11), (12) and equations (13), (14) we have that the system \( \partial_\xi S (x; \xi) = 0 \) is equivalent to

\[
\varphi^{s,t} (x_0, d \partial v (x_0), v (x_0)) = (x_1, y_1, z_1),
\]

\[
\varphi^{t, t+1} (x_i, y_i, z_i) = (x_{i+1}, y_{i+1}, z_{i+1}), \quad i = 1, \ldots, N - 2,
\]

\[
\varphi^{N-1} (x_{N-1}, y_{N-1}, z_{N-1}) = (x, \partial_x S^{s,t} (x; \xi), S^{s,t} (x; \xi)).
\] (15)
Letting \( q_i = (x_i, y_i, \tilde{z}_i) \) we have from Lemma 7

\[
\Phi^{\tau_1, \tau_{i+1}}(x_i, y_{i+1}, \tilde{z}_i) = y_i(x_{i+1} - x_i) - \int_{\tau_i}^{\tau_{i+1}} \dot{x}(\sigma, q_i) y(\sigma, q_i) - H(\sigma, \varphi_{\sigma, \tau}(q_i)) d\sigma
\]

from which item (c) follows. ■

Defining

\[
Q(\xi) = -y Nx_{N-1} + \sum_{i=1}^{N-1} y_i(x_i - x_{i-1}),
\]

(17)

\[
W^{s,t}(\tau, x, \xi) = v(x_0) + x \cdot y_N - \sum_{j=1}^{N} \Phi_{j-1, j}(x_{j-1}, y_j, \tilde{z}_{j-1}),
\]

(18)

we see that for \( v \in C^2, \text{Lip}(\mathbb{R}^k) \), \( S^{s,t}(\tau, x, \xi) \) is a g.f.q.i.

## 5 Generalized generating families

We consider the Cauchy problem \([11]\) with \( H \in C^2([0, T] \times J^1 \mathbb{R}^k) \) and \( v \in C^{\text{Lip}} \).

**Proposition 9 (Cf. [19], prop. 2.18)** Suppose that in the Cauchy problem \([11]\) \( v \) is locally Lipschitz and let \( \partial v = \{(x, y, v(x)) : y \in \partial v(x)\} \). The generating family \( S^{s,t} \) given by \([8]\) generated \( L^* = \varphi_H^{s,t}(\partial v) \) in the sense that

\[
L^* = \{(x, \partial_x S^{s,t}(\tau, x, \xi), S^{s,t}(\tau, x, \xi)) : 0 \in \partial_x S^{s,t}(\tau, x, \xi)\},
\]

(19)

where \( \partial \) denotes Clarke’s generalized derivative and

\[
\partial v = \{(x, y, z) \mid y \in \partial v(x), z = v(x)\}.
\]

**Proof.** The condition 0 \( \in \partial_x S(x, \xi) \) means that there exists \( y_0 \in \partial v(x_0) \) such that

\[
y_0 - y_1 = \partial_x \Phi_{s,t_1}(x_0, y_1, v(x_0)) + y_0 \partial_x \Phi_{s,t_1}(x_0, y_1, v(x_0)) \quad (c1)
\]

\[
y_i - y_{i+1} = \partial_x \Phi_{s,t_{i+1}}(x_i, y_{i+1}, z_i) + \partial_x \Phi_{s,t_{i+1}}(x_i, y_{i+1}, z_i)y_i, \quad 0 < i < N \quad (c2)
\]

\[
x_i - x_{i-1} = \partial_y \Phi_{s,t_{i-1}}(x_{i-1}, y_i, z_{i-1}), \quad 0 < i \leq N \quad (c3)
\]

Since \( \partial_x S^{s,t}(x; \xi) = y_N \), we have that \( \varphi_{s,t_1}(x_0, y_0, v(x_0)) = (x_1, y_1, z_1) \), \([15]\) and \([16]\) hold, and using \([6]\) give \( \varphi_{s,t}(x_0, y_0, v(x_0)) = (x, \partial_x S^{s,t}(x; \xi), S^{s,t}(x; \xi)) \). ■

**Proposition 10 (Cf. [19], prop. 2.21)** Let \( H \in C^2([0, T] \times J^1 \mathbb{R}^k) \), \( v \in C^{\text{Lip}}(\mathbb{R}^k) \).

Write \( S^{s,t} : [s, t] \times \mathbb{R}^k \times \mathbb{R}^d \rightarrow \mathbb{R} \) given by \([8]\) as

\[
S^{s,t}(\tau, x, \xi) = W^{s,t}(\tau, x, \xi) + Q(\xi),
\]

with \( Q, W^{s,t} \) defined in \([17]\), \([18]\) For each compact subset \( K \) of \( \mathbb{R}^k \), the family of functions \( \{W^{s,t}(\tau, x, \cdot)\}_{\tau \in [s, t], x \in K} \) is uniformly Lipschitz. Moreover for any
\( \theta \in C_c(\mathbb{R}^n, [0, T]) \) identically 1 in a neighborhood of the origin with \( \|D\theta\| < 1 \), there exists a constant \( a_K > 1 \) such that for \( \tau \in [s, t] \),

\[
(x, \xi) \mapsto S_{K}^{s,t}(\tau, x, \xi) = \theta\left(\frac{\xi}{a_K}\right)W^{s,t}(x, \xi) + Q(\xi)
\]

is a g.f.q.i. for

\[
L_{K}^\tau = L^\tau \cap \pi^{-1}(K) = \{(x, \partial_x S_{K}^{s,t}(\tau, x, \xi), S_{K}^{s,t}(\tau, x, \xi)|0 \in \partial_\xi S_{K}^{s,t}(\tau, x, \xi))\},
\]

where \( \pi : J^1 \mathbb{R}^k \to \mathbb{R}^k, (x, y, z) \to x \).

**Proof.** For a fixed compact \( K \), let \( c_K = \max\{\|W^{s,t}(\tau, x, \cdot)\|_{\text{Lip}} : \tau \in [s, t], x \in K\} \). Writing \( Q(\xi) = \frac{1}{2}(B\xi, \xi) \)

\[
\partial_\xi S_{K}^{s,t}(x, \xi) \subset \frac{1}{a_K} D\theta\left(\frac{\xi}{a_K}\right)W^{s,t}(\tau, x, \xi) + \theta\left(\frac{\xi}{a_K}\right)\partial_\xi W^{s,t}(\tau, x, \xi) + B\xi.
\]

Defining \( b_K = \max\{|W(\tau, x, 0)| : \tau \in [s, t], x \in K\} \) we have

\[
|W^{s,t}(\tau, x, \xi)| \leq |W^{s,t}(\tau, x, 0)| + |W^{s,t}(\tau, x, \xi) - W^{s,t}(\tau, x, 0)| \leq b_K + c_K|\xi|.
\]

Thus, if \( a_K, b_K \) are sufficiently large, for \( |\xi| \geq b_K \) and any \( w \in \partial_\xi W^{s,t}(\tau, x, \xi) \) we have

\[
\left| \frac{1}{a_K} D\theta\left(\frac{\xi}{a_K}\right)W^{s,t}(\tau, x, \xi) + \theta\left(\frac{\xi}{a_K}\right)w \right| \leq \frac{1}{a_K}(b_K + c_K|\xi|) + c_K
\]

\[
\leq \frac{1}{2}\|B^{-1}\|^{-1}|\xi|
\]

\[
< \|B\xi\|.
\]

We can choose \( a_K \) sufficiently large so that \( \theta\left(\frac{\xi}{a_K}\right) = 1 \) if \( |\xi| \leq b_K \). Thus \( S = S_k \), for \( |\xi| \leq b_K \), and \( 0 \notin \partial_\xi S_k(x, \xi) \) for \( |\xi| \geq b_K \). Therefore

\[
L_K = \{(x, \partial_x S_{K}^{s,t}(\tau, x, \xi), S_{K}^{s,t}(\tau, x, \xi) : 0 \in \partial_\xi S_{K}^{s,t}(\tau, x, \xi)\}.
\]

**Part II**

**Generalized Solutions of the Cauchy problem**

In this last part, we will prove our main result, Theorem 1. First, we will define minimax solutions and give some of their basic properties. For this goal, we need a minimax principle, which basic definitions and results are given in [15].

\[13\]
Most of analytic results that we have obtain in this section are generalizations for those in [19]. Next, using those results, we finally give a demonstration of Theorem 1. Constructions obtained in previous section allows us to follow very closely methods in [19, section 3] to achieve our goal. At the end of this section, we present an example. Although it is very simple, this example shows that it is possible to use our results in order to study Hamiltonians with non-compact support.

6 Minimax Selector

Let $K \subset \mathbb{R}^k$ be a compact set, $S_{K}^{s,t} \in C^1([s,t] \times \mathbb{R}^k \times \mathbb{R}^q)$ be given as in (20) and $Q(\xi) = \frac{1}{2} \langle P\xi, \xi \rangle$ be the associated quadratic form. As $S_{K}^{s,t} = Q$ outside a compact set, the critical levels of $S_{K}^{s,t}$ are bounded. There is $R(K) < 0$ such that for $R' < R(K)$, the sub–level set

$$(S_{K}^{s,t})_{R',x} = \{\xi \in \mathbb{R}^q | S_{K}^{s,t}(\tau, x; \xi) < R'\}$$

is identical to the sub–level $Q_{R'}$.

Definition 6.1 Let $j$ be the Morse index of $Q$ and $a > 0$ large. We define $\Theta_a$ as the set of continuous maps $\sigma : B_j \to \mathbb{R}^q$, of the unit ball $B_j$ of dimension $j$, such that

$$\sigma(\partial B_j) \subset Q^{-1}(-\infty, -a).$$

Lemma 11 Let $v \in C^{L^p}(\mathbb{R}^k)$ $H \in C^2_c([0,T] \times J^1\mathbb{R}^k)$. Let $S_{K}^{s,t} \in C^1([s,t] \times \mathbb{R}^k \times \mathbb{R}^q)$ be as in (20). Let $K \subset \mathbb{R}^k$ be compact, $a > -R(K)$. The function

$$(\tau, x) \in [s,t] \times \mathbb{R}^k \mapsto R_{H,K}^{s,t} v(x) = \inf_{\sigma \in \Theta_a, x \in B_j} \max_{e \in B_j} S_{K}^{s,t}(\tau, x, \sigma(e)),$$ (21)

has the following properties

(a) $R_{H,K}^{s,t} v(x)$ is a critical value of $\xi \mapsto S_{K}^{s,t}(\tau, x, \xi)$;

(b) it is a Lipschitz function and therefore differentiable almost everywhere a.e.;

(c) $j^{1} R_{H,K}^{s,t} v$ is an a.e. section of the wave front

$$\{(\tau, x, \partial_S S_{K}^{s,t}(\tau, x, \xi), S_{K}^{s,t}(\tau, x, \xi)) : 0 \in \partial_S S_{K}^{s,t}(\tau, x, \xi)\}.$$

Proof. Since $S_{K}^{s,t}(\tau, x, \xi) = Q(\xi)$ outside a compact set and $Q$ is non-degenerate, we have that

$$\Sigma_{K}^{s,t} = \{(\tau, x, \xi) | \partial_S S_{K}^{s,t}(\tau, x, \xi) = 0\}$$

is compact and $S_{K}^{s,t}(x, \tau, \cdot) : \mathbb{R}^q \to \mathbb{R}$ satisfies the Palais-Smale condition.

Let $\pi : \Sigma_{K}^{s,t} \to \mathbb{R} \times \mathbb{R}^k$ be the projection $(\tau, x, \xi) \mapsto (\tau, x)$. By Sard’s Theorem the set of critical values of $\pi$ has null measure.
(a) Apply the minimax principle (theorem 12) with the positively-invariant family $\Theta_a$ of Definition 11.

(b) Given $\epsilon > 0$, there exists $\sigma_0 \in \Theta_a$ such that

$$R_{H,K}^{s,\tau} v(x_0) \geq \max_{e \in B_j} S_{K}^{s,t}(\tau_0, x_0, \sigma_0(e)) - \epsilon \geq S_{K}^{s,t}(\tau_0, x_0, \sigma_0(e)) - \epsilon,$$

for any $e \in B_j$. Let $\max S_{K}^{s,t}(\tau_1, x_1, \sigma_0(e)) = S_{K}^{s,t}(\tau_1, x_1, \xi_1)$, then

$$R_{H,K}^{s,\tau_1} v(x_1) - R_{H,K}^{s,\tau} v(\tau_0, x_0) \leq S_{K}^{s,t}(\tau_1, x_1, \xi_1) - S_{K}^{s,t}(\tau_0, x_0, \xi_1) + \epsilon$$

$$= \theta\left(\frac{\xi_1}{\alpha_0}\right) (W_{s,t}(\tau_1, x_1, \xi_1) - W_{s,t}(\tau_0, x_0, \xi_1)) + \epsilon$$

$$\leq A_K(\vert \tau_1 - \tau_0 \vert + \|x_1 - x_0\|) + \epsilon.$$

Letting $\epsilon \to 0$ and exchanging $(\tau_0, x_0)$ and $(\tau_1, x_1)$, we get

$$\left|R_{H,K}^{s,\tau_1} v(x_1) - R_{H,K}^{s,\tau} v(\tau_0, x_0)\right| \leq A_K(\vert \tau_1 - \tau_0 \vert + \|x_1 - x_0\|).$$

(c) Let $x_0 \in \mathbb{R}^k$ be a regular value of $\pi : \Sigma \to \mathbb{R}^k$. There is a neighborhood $U$ of $x$ and diffeomorphisms $\phi_i : V_i \to U, i = 1, \ldots, m$ such that

$$\pi^{-1}(U) = \bigcup_{i=1}^{m} \phi_i(U).$$

For each $x \in U$ there is $i = 1, \ldots, m$ such that

$$R_{H,K}^{s,\tau} v(x) = S_{K}^{s,t}(\tau, x; \phi_i(x)) \quad (22)$$

Let $x \in U$ be differentiability point of $R_{H,K}^{s,\tau} v$. Proving that there is $i = 1, \ldots, m$ such that

$$dR_{H,K}^{s,\tau} v(x) = dS_{K}^{s,t}(\tau, x; \phi_i(\cdot))(x) \quad (23)$$

will finish the proof. Indeed, as $\partial_\xi S_{K}^{s,t}(\tau, x, \phi_i(x)) = 0$ we have

$$dS_{K}^{s,t}(\tau, x; \phi_i(\cdot))(x) = \partial_\xi S_{K}^{s,t}(\tau, x; \phi_i(x)) + \partial_\xi S_{K}^{s,t}(\tau, x; \phi_i(x)) d\phi_i(x)$$

$$= \partial_\xi S_{K}^{s,t}(\tau, x; \phi_i(x)).$$

and so

$$j^1 R_{H,K}^{s,\tau} v(x) \in \{ (x, \partial_\xi S_{K}^{s,t}(\tau, x; \xi), S_{K}^{s,t}(\tau, x; \xi)) : \partial_\xi S_{K}^{s,t}(\tau, x; \xi) = 0 \}.$$
holds for a base of unit vectors. Now, there is \( \epsilon > 0 \) such that for any unit vector \( h \) and \( |s| < \epsilon x + sh \in U \) and so there is \( i = i(h, s) \) such that \( R^s_{H, K} v(x + sh) = S^i_K(x + sh; \phi_i(x + sh)) \). For \( h \) fixed there is \( i = i(h) \) for which and a sequence \( s_k \) converging to zero such that

\[
R^s_{H, K} v(x + s_k h) = S^i_K(\tau, x + s_k h; \phi_i(x + s_k h))
\]

which implies [24].

**Corollary 6.1** If \( v \leq w \) then \( R^s_{H, K} v \leq R^s_{H, K} w \)

**Proof.** This is clear from (8), (20) and (21).

**Proposition 12** (Cf. [19], 2.22) Let \( K, K' \subset R^k \) be compact. If \( x \in K \cap K' \), \( \tau \in [s,t] \) then \( R^s_{H, K'} v(x) = R^s_{H, K} v(x) \).

**Proof.** This follows from the fact for \( a > -R(K), a' > -R(K') \), any \( \sigma \in \Theta_0, \sigma' \in \Theta_0' \) can be deformed into an \( \sigma'' \in \Theta_0'' \), \( a'' > a, a' \), with

\[
\max_{e \in B_j} S^s_{K'}(\tau, x, \sigma''(e)) = \max_{e \in B_j} S^s_{K'}(\tau, x, \sigma''(e)),
\]

by using the gradient flow de \( Q \), suitable truncated. Propositions 10 and 12 allow one to define

\[
R^s_{H} v(x) = \inf_{\sigma \in \Theta_0} \max_{e \in B_j} S^s_{K}(\tau, x, \sigma(e)).
\]  

From Lemma 11 we obtain

**Theorem 13** Function \( (\tau, x) \in [s,t] \times R^k \mapsto R^s_{H} v(x) \) has the following properties

(a) \( R^s_{H} v(x) \) is a critical value of \( \xi \rightarrow S^s_{K}(\tau, x, \xi) \);

(b) it is a Lipschitz function and therefore differentiable almost everywhere a.e..

(c) \( j^1 R^s_{H} v \) is an a.e. section of the wave front

\[
\{(\tau, x, \partial_x S^s_{K}(\tau, x, \xi), S^s_{K}(\tau, x, \xi)) : 0 \in \partial_x S^s_{K}(\tau, x, \xi)\}.
\]

**Proposition 14** The definition of \( R^s_{H} v(x) \) is independent of the partition of \([0, T] \) used to define \( S \).

**Proof.** First assume \( t - s < \delta_H \); and let \( \tau \in (s, t) \). Consider the family of partitions \( \zeta_\mu = \{s \leq s + \mu(\tau - s) < t\}, \mu \in [0, 1], \) and the corresponding generating families

\[
S^s_{\mu}(\tau, x; y_0, y_1, x_1, y_2) = v(x_0) + y_1(x_1 - x_0) - \Phi^{s,s+\mu(\tau-s)}(x_0, y_1, z_0) + y_2(x - x_1) - \Phi^{s+\mu(\tau-s),t}(x_1, y_2, z_1)
\]  

(26)
Function \( S_\mu \) is continuous in \( \mu \) and the minimax \( R^{s,t}_\mu(\tau, x) \) is a critical value of the map \( \eta \mapsto S^{s,t}_\mu(\tau, x, \eta) \). By \( \text{(3)} \) the set of such critical values is independent of \( \mu \), and by Sard’s Theorem, it has measure zero. Therefore \( R^{s,t}_\mu \) is constant for \( \mu \in [0, 1] \). Letting \( x' = x_0 - x_1 \) and \( y' = y_2 - y_1 \), we obtain

\[
S^{s,t}_0(\tau, x; x_0, y_1, x_1, y_2) = v(x_0) - \Phi^{s,t}(x_1, y_2, z_1) + (x - x_1) y_2 + x'y'.
\]

One gets this g.f.q.i., adding the quadratic form \( x'y' \) to the g.f.q.i.

\[
S^{s,t}(\tau, x; x_0, y_1, x_1, y_2) = v(x_0) - \Phi^{s,t}(x_1, y_2, z_1) + (x - x_1) y_2,
\]

so that

\[
R_{S}^{s,t}v(x) = R_{S_0}^{s,t}v(x) = R_{S_1}^{s,t}v(x).
\]

In general, given two partitions \( \zeta', \zeta'' \) of \([s, t]\) with \(|\zeta'|, |\zeta''| < \delta_H\), let

\[
\zeta = \zeta' \cup \zeta'' = \{s = t_0 < \cdots < t_n = t\},
\]

be the (smallest) common refinement of \( \zeta', \zeta'' \). If \( t_j \) does not belong to \( \zeta' \), consider the family of partitions

\[
\zeta_\mu(j) = \{t_0 < t_{j-1} \leq t_j - 1 + \mu(t_j - t_{j-1}) < t_{j+1} < \cdots < t_n = t\}, \mu \in [0, 1]
\]

The argument given at the beginning shows that the minimax relative to \( \zeta_0(j) \) and \( \zeta_1(j) \) coincide. Continuing this process, we obtain that the minimax relative to \( \zeta' \) and \( \zeta \) coincide, and so do the minimax relative to \( \zeta'' \) and \( \zeta \) as well as the minimax relative to \( \zeta' \) and \( \zeta'' \).

**Proposition 15 (Cf. [19], prop. 2.24)** The critical levels

\[
C(\tau, x) := \{\eta : 0 \in \partial_\eta S^{s,t}(\tau, x, \eta), S^{s,t}(\tau, x, \eta) = R^{s,t}_H v(x)\}
\]

are compact and the set-valued correspondence \((\tau, x) \mapsto C(\tau, x)\) is upper semicontinuous, i.e. for every convergent sequence \((\tau_j, x_j, \eta_j) \to (\tau, x, \eta)\) with \( \eta_j \in C(\tau_j, x_j) \), one has \( \eta \in C(\tau, x) \). In other words the graph \( \{(\tau, x, \eta)|\eta \in C(\tau, x)\} \) of the correspondence is closed.

**Proof.** Let \( (\tau_j, x_j, \eta_j) \to (\tau, x, \eta) \) with \( \eta_j \in C(\tau_j, x_j) \). Since \( S^{s,t} \) is \( C^1 \) with respect to \( x \), one has \( \partial S^{s,t} = \partial_x S^{s,t} \times \partial_\eta S^{s,t} \), which is upper semicontinuous (lemma [19 A.2]). It follows that the limit \((\partial_x S^{s,t}(\tau_j, x, \eta_j), 0)\) of the sequence \( \partial_x S^{s,t}(\tau_j, x, \eta_j) \in \partial S^{s,t}(\tau, x, \eta) \) belongs to \( \partial S^{s,t}(\tau, x, \eta) \), hence, \( 0 \in \partial S^{s,t}(\tau, x, \eta) \). As \( S^{s,t} \) and \( R^{s,t}_H \) are continuous, \( S^{s,t}(\tau_j, x_j, \eta_j) \to S^{s,t}(\tau, x, \eta) \), \( R^{s,t}_H v(x_j) \to R^{s,t}_H v(x) \), and therefore \( \eta \in C(\tau, x) \).

**Lemma 16 (Cf. [19], lemma 2.25)** Given \( \delta > 0 \), there exists \( \epsilon > 0 \) such that

\[
R^{s,t}_H v(x) = \inf_{\sigma \in \Sigma_\epsilon} \max\{S^{s,t}(\tau, x, \sigma(e)) : \sigma(e) \in C_\delta(x)\}
\]

where

\[
\Sigma_\epsilon = \left\{ \sigma \in \Theta_0 : \max_{e \in E_1} S^{s,t}(\tau, x, \sigma(e)) \leq R^{s,t}_H v(x) + \epsilon \right\}
\]

and \( C_\delta(x) = B_\delta(C(\tau, x)) \) denotes the \( \delta \)– neighborhood of the critical set \( C(\tau, x) \).
Proof. We apply to $S_{r,x}^\sigma(\cdot) = S^{r,t}(\tau, x, \cdot)$ the following result:

**Lemma 17 (Deformation Lemma)** \cite{15, Theorem 3.4] Suppose $f$ satisfies the Palais-Smale condition. If $c \in \mathbb{R}$ is a critical value of $f$ and $N$ any neighbourhood of $K_c := \text{Crit}(f) \cap f^{-1}(c)$, then there exist $\epsilon > 0$ and a bounded smooth vector field $V$ equal to 0 off $f^{c+\epsilon} \setminus f^{c-\epsilon}$, whose flow $\varphi_V^t$ satisfies $\varphi_V^t(f^{c+\epsilon} \setminus N) \subset f^{c-\epsilon}$.

For $\delta > 0$, and $c = R_H^{s,t} v(x)$, there exist $\epsilon > 0$ and $V$, a smooth vector field vanishing outside $(S_{r,x}^t)^{c+\epsilon} \setminus (S_{r,x}^t)^{c-\epsilon}$ such that

$$\varphi_V^t((S_{r,x}^t)^{c+\epsilon} \setminus C_\delta(x)) \subset (S_{r,x}^t)^{c-\epsilon}.$$  

For $\sigma \in \Theta_a$ we have $\sigma(B_j) \cap C_\delta(x) \neq \emptyset$, because otherwise

$$\max_{\epsilon \in B_j} S^{r,t}(\tau, x; \varphi_V^t(\sigma(\epsilon))) \leq R_H^{s,t} v(x) - \epsilon$$

which contradicts the definition of the minimax.

For any $r < c$, the complement of $(S_{r,x}^t)^\tau$ is a neighborhood of $C(\tau, x)$. By the same argument one has that $\sigma(B_j) \cap C_\delta(x) \setminus (S_{r,x}^t)^\tau \neq \emptyset$. Therefore, for any $r < c$ and $\sigma \in \Sigma_\epsilon$ one has

$$r \leq \max \{S^{r,t}(\tau, x, \sigma(\epsilon)) : \sigma(\epsilon) \in C_\delta(x)\} \leq \max_{\epsilon \in B_j} S^{r,t}(\tau, x, \sigma(\epsilon))$$

which implies (27).

**Proposition 18 (Cf. [19], prop. 2.27)** The generalized gradient of $R_H^{s,t} v$ satisfies

$$\partial R_H^{s,t} v(x) \subset \text{co} \left\{ \partial_z S^{s,t}(\tau, x, \eta) : \eta \in C(\tau, x) \right\},$$  

where $\text{co}$ denotes the convex envelope.

**Proof.** First we consider a point $\bar{x}$ where $R_H^{s,t} v$ is differentiable and prove that

$$dR_H^{s,t} v(\bar{x}) \subset \text{co} \left\{ \partial_z S^{s,t}(\bar{x}, \eta) : \eta \in C(\tau, \bar{x}) \right\}. $$  

Take $\delta, \epsilon > 0$ for $\bar{x}$ as in Lemma 16. Consider $K = \overline{B_1(\bar{x})}$ and $S_{K}^{s,t}$ as in (20). Choose $B = B_\rho(\bar{x})$ with $\rho \in (0, 1)$ sufficiently small such that for $x \in B$

$$\left| S_{K}^{s,t}(\tau, x, \cdot) - S_{K}^{s,t}(\tau, \bar{x}, \cdot) \right|_{C^0} < \frac{\epsilon}{4}. $$  

Let $y \in \mathbb{R}^d, \lambda < 0$ such that $x_\lambda = \bar{x} + \lambda y \in B$, and $\lambda^2 < \frac{\epsilon}{4}$. By definition of $R_H^{s,t} v$, for each $x_\lambda$, there exists $\sigma_\lambda \in \Theta_a$ such that

$$\max_{\epsilon \in B_j} S^{r,t}(\tau, x_\lambda, \sigma_\lambda(\epsilon)) \leq R_H^{s,t} v(x_\lambda) + \lambda^2,$$  

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then,

\[
\max_{e \in B_j} S^{s,t}(\tau, \bar{x}, \sigma_\lambda(e)) \leq \max_{e \in B_j} S^{s,t}(\tau, x_\lambda, \sigma_\lambda(e)) + \frac{\varepsilon}{4} \\
\leq R_H^{s,t} v(x_\lambda) + \frac{\varepsilon}{2} \leq R_H^{s,t} v(\bar{x}) + \frac{3\varepsilon}{4}
\]

On the other hand, there exists \( \eta_\lambda \in \sigma_\lambda(B_j) \cap C_\delta(\bar{x}) \) such that

\[
R_H^{s,t} v(\bar{x}) \leq \max\{S^{s,t}(\tau, \bar{x}, \sigma_\lambda(e)) : \sigma_\lambda(e) \in C_\delta(\bar{x})\} = S^{s,t}(\tau, \bar{x}, \eta_\lambda),
\]

that implies

\[
\left(R_H^{s,t} v(x_\lambda) + \lambda^2\right) - R_H^{s,t} v(\bar{x}) \geq S^{s,t}(\tau, x_\lambda, \eta_\lambda) - S^{s,t}(\tau, \bar{x}, \eta_\lambda),
\]

since \( \lambda < 0 \),

\[
\frac{1}{\lambda} \left(R_H^{s,t} v(x_\lambda) - R_H^{s,t} v(\bar{x})\right) \leq \frac{1}{\lambda} \left(S^{s,t}(\tau, x_\lambda, \eta_\lambda) - S^{s,t}(\tau, \bar{x}, \eta_\lambda)\right) - \lambda \\
\in \langle \partial_x S^{s,t}(\tau, x_\lambda, \eta_\lambda), y \rangle - \lambda,
\]

where the last belonging follows from the Mean Value Theorem for some \( x_\lambda' \) in the line segment between \( \bar{x} \) and \( x_\lambda \).

Take \( \lim \sup \) in (33) and let \( \lambda \to 0 \), we get for all \( y \in \mathbb{R}^d \):

\[
\langle dR_H^{s,t} v(\bar{x}), y \rangle \leq \max_{\eta \in C(\bar{x})} \langle \partial_x S^{s,t}(\bar{x}, \eta), y \rangle.
\]

Considering the convex function \( f(y) = \max_{\eta \in C(\bar{x})} \langle \partial_x S^{s,t}(\bar{x}, \eta), y \rangle \), inequality (34) implies

\[
dR_H^{s,t} v(\bar{x}) \in \partial f(0) = \co \{ \partial_x S^{s,t}(\bar{x}, \eta) : \eta \in C(\tau, \bar{x})\},
\]

In the general case

\[
\partial R_H^{s,t} v(x) = \co \left\{ \lim_{x' \to x} dR_H^{s,t} v(x') \right\}
\subset \co \left\{ \co \left\{ \lim_{x' \to x} \partial_x S^{s,t}(\tau, x', \eta') : \eta' \in C(\tau, x') \right\} \right\}
\subset \co \left\{ \partial_x S^{s,t}(\tau, x, \eta) : \eta \in C(\tau, x) \right\}
\]

by the upper semicontinuity of \( (\tau, x) \to C(\tau, x) \) and the continuity of \( \partial_x S \). 

\section{Viscosity solutions and iterated minimax}

We recall the definition of \textit{viscosity solution}

\textbf{Definition 7.1} Let \( V \subset \mathbb{R}^k \) be open
(a) A function $u \in C([0,T] \times V)$ is called a viscosity subsolution (respectively supersolution) of
\[ \partial_t u + H(t,x,\partial_x u, u) = 0, \]
if for any $\phi \in C^1(V \times [0,T])$ and any $(t_0,x_0) \in [0,T] \times V$ at which $u - \phi$ has a maximum (respectively minimum) one has
\[ \partial_t \phi(t_0,x_0) + H(t_0,x_0,\partial_x \phi(t_0,x_0), u(t_0,x_0)) \leq 0 \text{ (respectively } \geq 0). \]

(b) The function $u$ is a viscosity solution if it is both a viscosity subsolution and a supersolution.

**Theorem 19 (\cite{12})** If $v : \mathbb{R}^k \to \mathbb{R}$ is uniformly continuous and $H \in C^2_c([0,T] \times J^1 \mathbb{R}^k)$, then there exists a unique uniformly continuous viscosity solution of the Cauchy problem \cite{14}.

**Proposition 20** (Cf. \cite{19}, prop. 3.14) Suppose that
\[ H \in C^2_c([0,T] \times J^1 \mathbb{R}^k), \]
then the minimax operator $R^{s,t}_H : C^{\text{Lip}}(\mathbb{R}^k) \to C^{\text{Lip}}(\mathbb{R}^k)$ satisfies
(i) For $v \in C^{\text{Lip}}(\mathbb{R}^k)$,
\[ \|\partial(R^{s,t}_H v)\| \leq (\|\partial v\| + |t-s| \|\partial_x H\|) e^{(t-s)\|\partial_x H\|} \]
(ii) There is a constant $C(H) > 0$ such that for any $v \in C^{\text{Lip}}(\mathbb{R}^k)$,
\[ \|R^{s,t}_H v - R^{s,t}_H v\| \leq |t-s| C(H) \|H\|. \]
(iii) If $v^0, v^1 \in C^{\text{Lip}}(\mathbb{R}^k)$, $K \subset \mathbb{R}^k$ compact, there exists a bounded $\tilde{K} \subset \mathbb{R}^k$, depending on $K$ and $\|\partial v\|$, such that for all $0 \leq s < t \leq T$:
\[ \|R^{s,t}_H v^0 - R^{s,t}_H v^1\|_K \leq \|v^0 - v^1\|_K. \]

**Proof.** First we assume that $|t-s| < \delta_H$ so that
\[ S(\tau,x;x_s,y) = v(x_s) + (x - x_s)y - \Phi^{s,t}(x_s,y,v(x_s)) \]
is a g.t.q.i. for $\Phi^{s,t}_H(\partial v)$.

(i) For $(x_s,y_t) \in C(t,x)$, there is $y_s \in \partial v(x_s)$ such that
\[ \Phi^{s,t}_H(x_s,y_t,v(x_s)) = (x_t,y_t,S(t,x;x_s,y_t)) = (x_t,y_t,R^{s,t}_H v(x)). \]
As $\partial_x S(t,x;x_s,y_t) = y_t$, by \cite{28} on has
\[ \partial_x R^{s,t}_H v(x) \in \text{co} \{y_t : y_s \in \partial v(x_s), \Phi^{s,t}_H(x_s,y_s,v(x_s)) = (x_t,y_t,S(t,x;x_s,y_t))\}. \]

20
Let \( \gamma(\tau) = (x(\tau), y(\tau), z(\tau)) = \varphi_H^{s, t}(x_s, y_s, v(x_s)), y_s \in \partial v(x_s) \), then
\[
y_t - y_s = \int_s^t \dot{y}(\tau) d\tau = \int_s^t (-\partial_x H(\gamma(\tau)) - y(\tau) \partial_y H(\gamma(\tau))) d\tau,
\]
\[
|y_t| \leq |y_s| + \int_s^t |\partial_x H(\gamma(\tau))| d\tau + \int_s^t |y(\tau)| |\partial_y H(\gamma(\tau))| d\tau.
\]
Hence, by Grönwall’s inequality
\[
|y_t| \leq \left( |y_s| + \int_s^t |\partial_x H(\gamma(\tau))| d\tau \right) \exp \int_s^t |\partial_y H(\gamma(\tau))| d\tau
\leq (\|v\| + |t - s| \|\partial_x H\|) e^{t - s} \|\partial_y H\|
\]
(ii) For \((x_s, y_s) \in C(\tau, x)\), there is \(y_s \in \partial v(x_s)\) such that
\[
\varphi_H^{s, t}(x_s, y_s, v(x_s)) = (x, y_t, S(\tau, x; x_s, y_s)).
\]
By (3) we have
\[
\partial_x S(\tau, x; x_s, y_s) = -H(\tau, x, y_t, S(\tau, x; x_s, y_s)).
\]
Hence
\[
\partial_x R_H^{s, t} v(x) \subset \{ -H(\tau, x, y_t, S(\tau, x; x_s, y_s)) : y_s \in \partial v(x_s) \},
\varphi_H^{s, t}(x_s, y_s, v(x_s)) = (x, y_t, S(t, x; x_s, y_s))
\]
By the Mean Value Theorem
\[
|R_H^{s, t} v(x) - R_H^{s, t} v(x)| \leq |\tau - t| \|H\|.
\]
(iii) Consider \(v^\lambda = (1 - \lambda)v^0 + \lambda v^1\), \(\lambda \in [0, 1]\) and let \(S^{s, t}_\lambda\) be the corresponding generating family, then \(\partial_x S^{s, t}_\lambda(t, x; x_s, y_t) = v^1(x_s) - v^0(x_s)\). For \((x_s, y_t) \in C^\lambda(t, x)\), there is \(y_s \in \partial v(x_s)\) such that
\[
\varphi_H^{s, t}(x_s, y^\lambda_s, v(x_s)) = (x, y_t, S^{s, t}_\lambda(t, x; x_s, y_t))
\]
By a similar argument to the proof of (28) we have
\[
\partial_x R_H^{s, t} v^\lambda(x) \subset \{ \partial_x S^{s, t}_\lambda(t, x; x_s, y_t) : (x_s, y_t) \in C^\lambda(t, x) \}
\subset \{ v^1(x_s) - v^0(x_s) : x_s \in \tilde{K} \}
\]
where
\[
\tilde{K} = \left\{ x_s \in \mathbb{R}^k : \|x_s\| \leq \max_{x \in K} \|x\| + |t - s| \sup_{y \in Y} \|\partial_y H\| \right\},
\]
\[
Y = \left\{ y \in \mathbb{R}^k : \|y\| \leq (\|v^0\| + |t - s| \|\partial_y H\|) e^{t - s} \|\partial_y H\| \right\}.
\]
Thus we obtain
\[
|R_H^{s, t} v^0 - R_H^{s, t} v^1|_K \leq |v^0 - v^1|_{\tilde{K}}.
\]
For the general cases fix \(N > T/\delta_H\) and take the partition \(t_0 < \cdots < t_N\), with \(t_i = s + i(t - s)/N\).
(i) We have the generating family $S^{s,t}(x; \xi)$ given in (6) where

$$\xi = (x_0, \ldots, x_{N-1}, y_1, \ldots, y_N).$$

For $\xi \in C(t,x)$ there exists $y_0 \in \partial v(x_0)$ such that equations (c1)-(c3) are satisfied. Then we have that $\varphi^{s,t}(x_0, y_0, v(x_0)) = (x_i, y_i, z_i), i < N, y_N = \partial_x S^{s,t}(x; \xi), \varphi^{s,t}(x_0, y_0, v(x_0)) = (x, y_N, S^{s,t}(x; \xi))$. By Proposition [13]

$$\partial_x R^{s,t}_H v(x) \subset \text{co} \{ y_N : y_0 \in \partial v(x_0), \varphi^{s,t}_H(x_0, y_0, v(x_0)) = (x, y_N, S^{s,t}(x; \xi)) \}.$$

Writing $\gamma(\tau) = \varphi^{s,t}_H(x_0, y_0, v(x_0))$ we have

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} (-\partial_x H(\gamma(\tau)) - y(\tau)\partial_x H(\gamma(\tau)))d\tau.$$

By Grönwall’s inequality we have as before

$$|y_{i+1}| \leq \left( |y_i| + \int_{t_i}^{t_{i+1}} |\partial_x H| \right) \exp \left( \int_{t_i}^{t_{i+1}} |\partial_x H| \right)$$

which imply by induction that

$$|y_i| \leq \left( |y_0| + \int_{t_0}^{t_i} |\partial_x H| \right) \exp \left( \int_{t_i}^{t_i} |\partial_x H| \right) .$$

Indeed, the inductive step is given by the inequalities

$$|y_{i+1}| \leq \left( |y_0| + \int_{t_0}^{t_i} |\partial_x H| \right) \exp \left( \int_{t_i}^{t_{i+1}} |\partial_x H| \right) \exp \left( \int_{t_i}^{t_{i+1}} |\partial_x H| \right)$$

$$\times \exp \left( \int_{t_i}^{t_{i+1}} |\partial_x H| \right)$$

$$= \left( |y_0| + \int_{t_0}^{t_i} |\partial_x H| \right) \exp \left( \int_{t_0}^{t_{i+1}} |\partial_x H| \right)$$

$$+ \left( \int_{t_i}^{t_{i+1}} |\partial_x H| \right) \exp \left( \int_{t_i}^{t_{i+1}} |\partial_x H| \right)$$

$$\leq \left( |y_0| + \int_{t_0}^{t_{i+1}} |\partial_x H| \right) \exp \left( \int_{t_0}^{t_{i+1}} |\partial_x H| \right).$$

Therefore we have

$$|y_N| \leq (||\partial v|| + |t - s| ||\partial_x H||) e^{|t - s||\partial_x H||}.$$

(ii) Set $D = \sup \{ \|g^{r-r'}\| : |t - \tau| < \delta_H \}$ where $g^{r-r'} \alpha = g^{r-r'} \alpha$. We have the generating family $S^{s,t}(\tau, x; \xi)$ given in (5) with $z_0 = v(x_0), z_1, \ldots, z_{N-1}$ defined inductively as in (7). Thus

$$\frac{d\tilde{z}_j}{d\tau} = - (\partial_{r_{j-1}} \Phi^{r_{j-1} - r_j}(x_{j-1}, y_j, \tilde{z}_{j-1}) + \partial_r \Phi^{r_{j-1} - r_j}(x_{j-1}, y_j, \tilde{z}_{j-1})) \frac{t_j - s}{t - s}$$

$$- \partial_x \Phi^{r_{j-1} - r_j}(x_{j-1}, y_j, \tilde{z}_{j-1}) \frac{d\tilde{z}_{j-1}}{d\tau}.$$
Using (3) and (4) one proves by induction that for \( j = 1, \ldots, N \) one has

\[
\frac{d\bar{z}_j}{dt} = \sum_{k=1}^{j-1} \prod_{i=k}^{j-1} (1 - \partial_x \Phi^{\tau_i,\tau_{i+1}}(x_i, y_{i+1}, \bar{z}_i))(H(x_k, y_{k+1}, \bar{z}_k) - H(x_{k-1}, y_k, \bar{z}_{k-1})) \frac{t_k - s}{t - s} - H(x_j - 1, y_j, \bar{z}_{j-1}) \frac{t_j - s}{t - s}
\]

For \( \xi \in C(\tau, x) \) there exists \( y_0 \in \partial v(x_0) \) such that equations (c1)-(c3) are satisfied. Then we have that \( \varphi^{s,t}(x_0, y_0, v(x_0)) = (x_i, y_i, z_i), i \leq N, y_N = \partial_x S^{s,t}(\tau, x; \xi) \), \( \bar{z}_N = S^{s,t}(\tau, x; \xi) \). We recall that

\[
g^{\tau_i,\tau_{i+1}}(x_i, y_i, \bar{z}_i) = (1 - \partial_x \Phi^{\tau_i,\tau_{i+1}})(x_i, y_i, \bar{z}_i)
\]

so that \( \|1 - \partial_x \Phi^{\tau_i,\tau_{i+1}}\| \leq D \) for \( i = 0, \ldots, N - 1 \) and then

\[
|\partial_x S^{s,t}(\tau, x; \xi)\| \leq \frac{D^N - 1}{D - 1} \|H\|
\]

Since

\[
\partial_x R_H^{s,t} v(x) \subset \operatorname{co} \{ \partial_x S^{s,t}(\tau, x; \xi) : \varphi_H^{s,t}(x_0, y_0, v(x_0)) = (x, \partial_x S^{s,t}(\tau, x; \xi), S^{s,t}(t, x; \xi)) \}
\]

by the Mean Value Theorem [35]

\[
|R_H^{s,\tau} v(x) - R_H^{s,t} v(x)| \leq |\tau - t| \frac{D^N - 1}{D - 1} \|H\|.\]

The proof of (iii) does not present changes in the general case. \( \blacksquare \)

Given a subdivision \( \zeta = \{ 0 = t_0 < t_1 < \cdots < t_n = T \} \) of \([0, T]\), we define its norm as \( |\zeta| = \max |t_{i+1} - t_i| \), and the step function

\[
\zeta(s) = \max \{ t_i : t_i \leq s \}, s \in [0, T]
\]

**Definition 7.2** The iterated minimax operator for the Cauchy problem (HJ) (with respect to \( \zeta \)) is defined as follows: for \( 0 \leq s' < s \leq T \),

\[
R_{H, \zeta}^{s', s} v(x) = R_{H, \zeta}^{s', s} \circ \cdots \circ R_{H, \zeta}^{s', t_j+1} v(x) \tag{41}
\]

where \( t_j = \zeta(s), t_i = \zeta(s') \). When \( H \) is fixed, we omit the corresponding subscript.

**Lemma 21** (Cf. [19], lemma 3.17) Suppose that \( (\zeta_n) \) is a sequence of partitions of \([0, T]\) such that \( |\zeta_n| \to 0 \). For \( v \in C^{\text{Lip}}(\mathbb{R}^k) \), the sequence of functions \( u_n(s, x) := R_{\zeta_n}^{s, \tau} v(x) \) is equi-Lipschitz and uniformly bounded on \([0, T] \times K \) for any compact \( K \subset \mathbb{R}^k \).
Proof. It follows from (36) that
\[ \| \partial R_{\zeta_n}^{i,s} \| \leq (\| \partial v \| + |s| \| \partial_s H \|)e^{\| \partial_s H \|}, \]
and from (37) that
\[ | R_{\zeta_n}^{0,t} v(x) - R_{\zeta_n}^{0,s} v(x) | \leq |t - s| C(H) \| H \|, \]
so that in particular
\[ \| R_{\zeta_n}^{0,s} v \|_K \leq \| v \|_K + TC(H) \| H \|. \]
The Lemma follows from these inequalities. ■

Proposition 22 (Cf. [19], prop. 3.18) For any sequence \( (\zeta_n) \) of subdivisions of \([0,T] \) with \( |\zeta_n| \to 0 \), and any compact set \( K \subset \mathbb{R}^k \), the sequence \( u_n := R_{\zeta_n}^{0,s} v(x) \) has a subsequence converging uniformly on \([0,T] \times K \) to the viscosity solution of the Cauchy problem (HJ).

Proof. By lemma 21 we can apply Arzela-Ascoli Theorem to \( (u_n) \subset C^0([0,T] \times K) \) to get a subsequence \( (u_n_k) \) converging uniformly to a function \( \tilde{R}^{0,s} v \). Define \( \tilde{K} \) as in (39)-(40).

Claim 23 For \( 0 \leq s' < s \leq T \) one has
\[ \tilde{R}^{0,s} v(x) = \lim_{n \to \infty} R_{\zeta_n}^{s,s} \circ \tilde{R}^{0,s'} v(x) \]
(42)

Proof of claim 23.
Applying Arzela-Ascoli Theorem to \( (u_n_k) \subset C^0([0,T] \times \tilde{K}) \), we can extract a subsequence converging uniformly in \([0,T] \times \tilde{K} \). To easy notation, when \( s' = 0 \) we omit this superscript, and for the iterated minimax with respect to the partition \( \zeta_n \), we use the subscript \( n \) instead of \( (s)_n \) instead of \( \zeta_n(s) \).

We first notice that for \( 0 \leq s \leq T, x \in \tilde{K} \):
\[ R^s v(x) = \lim_{n \to \infty} R_{n}^{s_n} v(x) \]
(43)
because
\[ | R_{n}^{s} v(x) - R_{n}^{s_n} v(x) | = | R_{n}^{s} \circ R_{n}^{s_n} v(x) - R_{n}^{s_n} v(x) | \]
\[ \leq \| H \| (s - (s)_n) \leq \| H \| |\zeta_n| |. \]
Then for any \( \epsilon > 0 \), there exists \( N \) such that if \( i, j > N \), then
\[ \forall s \in [0,T]: \| R_i^{s} v - R_j^{s} v \|_K < \epsilon. \]
Therefore,
\[ \| R_i^{s} \circ R_j^{0,s'} v - R_i^{s} v \|_K \]
\[ \leq \| R_j^{s'} v - R_j^{s} v \|_K < \epsilon \]
(44)
Letting \( j \to \infty \), we get
\[
\| R_i^{(s'),(s)} \circ \tilde{R}^s v - R_i^s v \|_K < \epsilon,
\]
thus, for any \( x \in K \)
\[
\lim_{i \to \infty} R_i^{(s'),(s)} \circ \tilde{R}^s v(x) = \tilde{R}^s v(x).
\]
Similarly, we conclude that
\[
\lim_{i \to \infty} R_i^{s',s} \circ \tilde{R}^s v(x) = \lim_{i \to \infty} R_i^{s',s} \circ \tilde{R}^s v(x).
\]

We now prove that \( \tilde{R}^t v(x) \) is a viscosity subsolution of \((35)\). Let \( \psi \) be a \( C^2 \) function with bounded second derivative defined in a neighborhood of \((t, x) \in \mathbb{R} \times K \), such that for \( s \) is close enough to \( t \), \( \psi(s, y) = \psi_s(y) \geq \tilde{R}^s v(y) \), with equality at \((t, x)\).

Suppose that \( \tau \leq t \) is close enough \( t \), so that the projections of the characteristics originating from
\[
j^1(\psi_\tau)(x_\tau) = (x_\tau, d\psi_\tau(x_\tau), \psi_\tau(x_\tau))
\]
do not intersect. Hence, the map \( x_\tau \to x_t \) is a diffeomorphism.

We conclude that
\[
\psi_t(x) = \tilde{R}^t v(x) = \lim_{k \to \infty} R_{\tau,n}^{\tau,t} \circ \tilde{R}^\tau v(x) \leq \lim_{k \to \infty} R_{\tau,n}^{\tau,t} \psi_\tau(x) = R^{\tau,t} \psi_\tau(x).
\]
The inequality is consequence of Corollary 6.1.

Also, when \( \tau \) is close enough to \( t \), iterated minimax will be the minimax \((N = 1)\) which is a \( C^2 \) solution of \((HJ)\) with initial condition \( \psi_\tau \), and thus
\[
R^{\tau,t} \psi_\tau(x) = \psi_\tau(x) - \int_\tau^t H(\theta, j^1(\psi_\theta(x)))d\theta.
\]

Subtracting \((44)\) from \((45)\), \( \psi_t(x) \) to the right side, dividing both sides by \( t - \tau \) and letting \( \tau \to t \), we get
\[
0 \leq -\partial_t \psi_t(x) - H(t, j^1(\psi_t(x))).
\]

One proves that \( \tilde{R}^t v(x) \) is a viscosity supersolution of \((35)\) in a similar way. ■

Given \( H \in C^2([0, T] \times J^1(\mathbb{R}^k)) \), \( v \in C^{Lip}(\mathbb{R}^k) \), we say that a function \( w : [s, t] \times \mathbb{R}^k \to \mathbb{R} \) is the limit of iterated minimax solutions for \((HJ)\) on \([s, t] \times \mathbb{R}^k \), if for any sequence of subdivisions \( \{\zeta_n\}_{n \in \mathbb{N}} \) of \([s, t] \times \mathbb{R}^k \) such that \( |\zeta_n| \to 0 \) as \( n \to \infty \), the corresponding sequence of iterated minimax solutions
\[
\left\{ R_{H,\zeta_n}^{\tau,s} v(x) \right\}, (\tau, x) \in [s, t] \times \mathbb{R}^k
\]
converges uniformly on compact subsets to \( w \). We denote \( w(\tau, x) := \tilde{R}_{H}^{\tau,s} v(x) \).
We can now prove our main result.

**Proof of Main Theorem** Let $K \subset \mathbb{R}^k$ and $(\zeta_n)$ be any subsequence of subdivisions of $[0, T]$ such that $|\zeta_n| \to 0$. Denote $u_n(t, x) = R_{\zeta_n}^{\partial_t} v(x)$ and $u(t, x)$ the viscosity solution of the (HJ) problem. If $u_n$ does not converge uniformly on $[0, T] \times K$, there exists $\epsilon > 0$ and a subsequence $n_k$ such that $|u_{n_k} - u| > \epsilon$. Note that $\zeta_{n_k}$ is itself a sequence of subdivisions, this contradicts Proposition.

8 Example

Now we consider an example from [14]. This example shows that we can ask for weaker conditions, for example, $H$ is not bound to have compact support. Also, this example comes from a very different area than mechanics, namely control theory. Further directions of this work could be investigating on which conditions we can resemble this result and some applications to other areas like control.

Consider $H(x, y, z) = z + h(y)$ with $h$ of compact support. The characteristics equations are

\[
\begin{cases}
\dot{x} = dh(y) \\
\dot{y} = -y \\
\dot{z} = ydh(y) - z - h(y).
\end{cases}
\]

which can be integrated to obtain the flow

\[
\varphi^t(x_0, y_0, z_0) = (x_0 + \int_0^t dh(y_0 e^{-s}) ds, y_0 e^{-t}, -h(y_0 e^{-t}) + e^{-t}(h(y_0) + z_0)).
\]

Since that the map $(x_0, y_0, z_0) \mapsto (x_0, y_0 e^{-t}, z_0)$ is invertible, we can define a generating function of $\varphi^t$.

\[
\begin{align*}
\Phi^t(x_0, y, z_0) &= y \int_0^t dh(y e^{t-s}) ds + h(y) - e^{-t} h(e^t y) + z_0(1 - e^{-t}) \\
&= \int_0^t e^{-s} h(e^s y) ds + z_0(1 - e^{-t})
\end{align*}
\]

Thus the minimax solution of

\[
\begin{cases}
\partial_t u(t, x) + u(t, x) + h(\partial_x u(t, x)) = 0 \\
u(0, x) = v(x)
\end{cases}
\]

is given by

\[
u(t, x) = \inf \max S_t(x, x_0, y),
\]

where the generating function

\[
S_t(x, x_0, y) = (x - x_0) y - \int_0^t e^{-s} h(e^s y) ds + e^{-t} v(x_0)
\]
is quadratic at infinity because \( h \) has compact support. Indeed, \( Q(x_0, y) = -x_0y \) so that

\[
S_t(x, x_0, y) = xy - \int_0^t e^{-s} h(e^s y) ds + e^{-t} v(x_0).
\]

Since \( S_t(x, x_0, y) \) is \( C^1 \) with respect to \( y \), we have

\[
\partial_{(x_0, y)} (S_t(x, x_0, y) - Q(x_0, y)) = e^{-t} \partial v(x_0) \times \left\{ x - \int_0^t dh(e^s y) ds \right\}
\]

\[
= \left\{ (e^{-t} p, x - \int_0^t dh(e^s y) ds) : p \in \partial v(x_0) \right\}.
\]

As \( h \) is compactly supported and \( v \) is a Lipschitz function,

\[
\| \partial_{(x_0, y)} (S(t, x) - Q) \| = \max_{(x_0, y)} \left\{ \left\| (e^{-t} p, x - \int_0^t dh(e^s y) ds) \right\| : p \in \partial v(x_0) \right\}
\]

is bounded, and therefore \( S_t \) is a \( g\text{f}\text{q}\text{i} \).

Had we assumed instead that \( h \) was convex we would still had obtained a minimax

\[
u(t, x) = \inf_{x_0} \max_y \left( (x - x_0)y - \int_0^t e^{-s} h(e^s y) ds \right) + e^{-t} v(x_0)
\]

with the \( \max_y \) a Legendre transform being achieved when

\[
x - x_0 = \int_0^t dh(e^s y) ds.
\]

Letting \( l \) to be the Legendre transform of \( h \), it is not hard to prove that

\[
u(t, x) = \min_y \int_0^t e^{-s} l(dh(e^s y)) ds + v \left( x - \int_0^t dh(e^s y) ds \right),
\]

(cf. [14] (4)], Hopf-Lax formula with discount.)

Part III

Appendix

9 Contact topology

As noticed in [18], due to a more complex geometry than ordinary differential equations (\textit{ode}'s), there not exists a unified theory for partial differential equations (\textit{pde}'s).
For an ode, we can always consider locally integrable vector fields, that is, there are always integral curves for them. However, even an hyperplane field on $\mathbb{R}^3$ is not always integrable.

For example, let us examine the hyperplane field given by equation

$$\alpha := dz - ydx = 0.$$ 

Now, consider $v \in \ker \alpha \mid (x,y,z)$, $v = (v_x, v_y, v_z)$. Thus, $\alpha \mid (x,y,z)$ is a linear form induced with associated matrix $R_\alpha = [-y, 0, 1]$. The last equation states that $v_z = yv_x$.

This hyperplane field is not integrable (that is, there is no submanifold $N$ such that at each $p \in N$, $T_pN$ is on the hyperplane field at $p$) because of Frobenius integrability condition $(\alpha \wedge d\alpha = 0)$ does not hold at $\alpha$ inasmuch as

$$\alpha \wedge d\alpha = (dz - ydx) \wedge (-dy \wedge dx) = dx \wedge dy \wedge dz \neq 0,$$

that is, $\alpha \wedge d\alpha$ is a volume form on $\mathbb{R}^3$.

In this part, we shall give definitions and results from contact geometry and theory of first-order p.d.e’s in order to make clear statements of subsequent results in this thesis.

Any pde of first order can be written as

$$F(x_1, ..., x_n, \partial_{x_1} u(x), ..., \partial_{x_n} u(x), u(x)) = 0,$$

where $x = (x_1, ..., x_n)$. So any pde of first order can be regarded as a hypersurface $\Theta$ in $J^1(\mathbb{R}^n) \simeq T^*\mathbb{R}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$, and a solution of this pde as a function $u : M \to \mathbb{R}$ whose $1$-graph

$$j^1u = \{j^1_x u \mid x \in M\}.$$ 

(where $j^1_x u := (x, \partial_x u(x), u(x))$) lies in $\Theta$.

Indeed, a $1$-graph is a section over $M$. Therefore $J^1M$ is also a vector bundle over $M$ with projection

$$\pi_M : J^1(M) \to M, (x, y, z) \mapsto x.$$ 

$J^1(M)$ is not just a differentiable manifold, but has an analogous structure to the symplectic structure of the cotangent bundle $T^*M$. Now, we will explain some details about this contact structure. At the end of the section, we will define a very important concept, namely contactomorphism, that is, transformations preserving this structure. For example, every solution $\Theta$ of $(F=0)$ is a legendrian submanifold of $J^1M$ and the flow $\varphi^t$ generated by the characteristic equations consists of contactomorphisms.

**Definition 9.1** A contact structure on $\Xi$ is a field of hyperplanes $\xi = \ker \alpha \subset T\Xi$ where $\alpha$ is a $1$-form satisfying $\alpha \wedge (d\alpha)^n \neq 0$. We say that $\alpha$ is a contact form, and the pair $(\Xi, \xi)$ is a contact manifold.
Example 9.1 (Standard structure) For $J^1\mathbb{R}^n \simeq T^\ast\mathbb{R}^n \times \mathbb{R}$, we can choose $\alpha = dz - ydx$ in local coordinates $(x,y,z)$. It is straightforward to verify that $J^1(M)$ is a contact manifold and such $\alpha$ is a contact form: Consider the 1-form $\alpha$ in local coordinates, i.e.,

$$\alpha = dz - ydx = dz - \sum_{i=1}^{n} y_i dx_i.$$  \hfill (46)

Then

$$d\alpha = -dy \wedge dx = -\sum_{i=1}^{n} dy_i \wedge dx_i,$$

and therefore

$$(d\alpha)^n = (-1)^n n! \bigwedge_{i=1}^{n} (dy_i \wedge dx_i).$$

Thus

$$\alpha \wedge (d\alpha)^n = (-1)^n n! dz \bigwedge_{i=1}^{n} (dy_i \wedge dx_i) \neq 0.$$  

Remark 24 Indeed, along this work we will just consider the contact structure in $J^1\mathbb{R}^n$ induced by $\alpha = dz - ydx$. In the next paragraphs we will explain why it is enough to choose this contact structure.

Notice also that if $\alpha$ is a contact form, then $\alpha \wedge (d\alpha)^n$ is a volume form and therefore, $\Xi$ is orientable. However, some authors define $\xi = \ker \alpha$ just locally and in that case $\Xi$ is not necessary an orientable manifold anymore.

If there were another form such that $\xi = \ker \beta$, then we would have $\beta = \lambda \alpha$, for some function $\lambda \in C^\infty(\Xi, \mathbb{R} \setminus \{0\})$, because

$$\text{codim} \xi = \dim \text{Im} \alpha = 1.$$  

Since

$$\beta \wedge (d\beta)^n = (\lambda \alpha) \wedge (d(\lambda \alpha)) = \lambda \alpha \wedge (\lambda d\alpha + d\lambda \wedge \alpha)^n = \lambda^{n+1} (\alpha \wedge (d\alpha)^n) \neq 0$$

the definition of a contact structure $\xi$ does not depend on a particular choice of $\alpha$.

Claim 25 [37, Rem. 1.3.7] An equivalent definition of $\xi \subset T\Xi$ as a contact structure over a manifold of dimension $(2n + 1)$ is as follows: For any local 1-form $\alpha$ with $\xi = \ker \alpha$, $(d\alpha)^n|_p|_\xi$ is non-degenerate, that is, $(\xi_p, d\alpha|_{\xi_p})$ is a symplectic space for every $p \in \Xi$.

As in symplectic topology, the following result states that there are not local invariants in contact topology:
Theorem 26 (Darboux) [10, Th. 2.5.1] Let \((\Xi, \xi)\) be a contact structure and \(p \in \Xi\). Thus there exists a coordinate system 

\[(U, x_1, y_1, \ldots, x_n, y_n, z)\]
centered around \(p\) such that in \(U\) and a diffeomorphism \(f : U \rightarrow f(U) \subset J^1\mathbb{R}^n\) such that \(Df (\xi) = \ker \alpha\) by defining

\[\alpha = dz - ydx.\] (47)

A submanifold is called integral if the tangent plane at each point is a subspace of the contact plane. For example, a 1−graph is always an integral submanifold of \(J^1(M)\), of the same dimension as \(M\). For \(\sigma(x) = (x, \sigma_y(x), \sigma_z(x)) : \)

\[\sigma = j^1\varphi \Rightarrow \sigma_z(x) = \varphi(x), \sigma_y(x)dx = d\varphi(x)\]

\[\Rightarrow \sigma^*\alpha = 0.\]

Since \(\alpha (D\sigma(x)(v)) = 0\), we have

\[\text{Im } D\sigma(x) \subset \ker \alpha|_{\sigma(x)},\] (48)

that is, every tangent plane of every 1−graph at a given point lies in the same hyperplane.

Indeed, there are not integral submanifolds of higher dimension.

Definition 9.2 Let \((\Xi, \xi)\) be a contact manifold. A submanifold \(L\) of \((\Xi, \xi)\) is called isotropic if \(T_pL \subset \xi_p\) for every \(p \in L\).

Proposition 27 [10, Prop. 1.5.12] Let \((\Xi, \xi)\) be a contact manifold of dimension \(2n + 1\) and \(L \subset (\Xi, \xi)\) a isotropic submanifold. Then \(\dim L \leq n\).

Definition 9.3 An isotropic submanifold \(L \subset (\Xi, \xi), \dim \Xi = 2n + 1\) of maximal dimension \(n\) is called Legendrian.

Claim 28 Let \((\Xi, \xi) = (J^1\mathbb{R}^n, \ker \alpha)\). It follows form (48) and the fact that

\[\dim (j^1f) = \dim (M)\]

that 1−graphs are Legendrian submanifolds of \(J^1(M)\).

Another two important concepts in the study of geometry of our Cauchy problem are contactomorphism and contact isotopy. The first one is a diffeomorphism preserving contact structures and the second one is a family of contactomorphisms varying in the time. Our most important example of a contact isotopy is the flow generated by \((H1)\,(H3)\). At the end of the section, we will give a result about how to recovery a classical solution for \((HJ)\) from this flow. From here and because Theorem 26 we will just consider contact manifolds \((J^1\mathbb{R}^n, \ker \alpha)\) with \((x, y, z)\) local coordinates for \(J^1\mathbb{R}^n \simeq T^*\mathbb{R}^n \times \mathbb{R}\) and \(\alpha = dz - ydx\).
Let $\xi = \ker \alpha$ a contact structure for contact manifold $J^1M$. A diffeomorphism $\psi : J^1M \to J^1M$ is called a contactomorphism if $\psi$ preserves the oriented hyperplane field $\xi$. This is equivalent to the condition $$\psi^* \alpha = e^h \alpha$$ for some function $h : J^1M \to \mathbb{R}$. A contact isotopy is a smooth family $\psi_t : J^1M \to J^1M$ of contactomorphism such that $$\psi_t^* \alpha = e^{ht} \alpha.$$ 

A vector field $X : J^1M \to TJ^1M$ which satisfies $L_X \alpha = g\alpha$ for some function $g : J^1M \to \mathbb{R}$ is called a contact vector field.

For every contact form $\alpha$, we define the Reeb vector field $R_\alpha$ as the unique one for which the following equations hold
\begin{align*}
\{ &d\alpha(R_\alpha, v) = 0, \quad v \in T_pJ^1M \\
&\alpha(R_\alpha) = 1. \tag{Reeb}
\}
\end{align*}

Such field exists because for every $p \in J^1M$, $\ker d\alpha|_{T_pJ^1M}$ is one-dimensional, and $R_\alpha$ is defined except for a rescaling and the second condition allows us choose it uniquely. The importance of this vector field is given by the following result:

**Lemma 29 [8, lemma 3.49]** Let $(\Xi, \xi = \ker \alpha)$ be a contact structure with Reeb field $R_\alpha$. Thus:

(i) $X : \Xi \to T\Xi$ is a contact vector field if and only if there exist a function $H : \Xi \to \mathbb{R}$ such that
\begin{align*}
\begin{cases}
  i(X)\alpha = -H, \\
  i(X)d\alpha = dH - (i(R_\alpha)dH)\alpha.
\end{cases}
\end{align*}

(ii) For every function $H : \Xi \to \mathbb{R}$, there exists an unique contact vector field $X_H : \Xi \to T\Xi$ which satisfies (49).

**Example 9.2** The Reeb vector field for the standard contact 1–form $\alpha = dz - ydx$ on $J^1\mathbb{R}^n$ is $\partial_z$. Let’s verify this claim: If $R = (R_x, R_y, R_z)$, then
$$0 = i(R)d\alpha = -R_ydx + R_xdy,$$
implies $R_x = R_y = 0$. Finally
$$1 = \alpha(R) = R_z - yR_x = R_z.$$
We conclude that $R = (0, 0, 1) = \partial_z$. So $X_H = Z - H R_\alpha = (Z_x, Z_y, Z_z) - H(0, 0, 1) = (\partial_y H, -\partial_z H - y\partial_x H, y\partial_y H - H)$, and therefore, the flow generated by $X_H$ is given precisely by [H1], [H2] and [H3].
For a given function $H \in C^2_c([0,T] \times J^1\mathbb{R}^k)$, the following statement relates solutions $S : [0,T] \times \mathbb{R}^k \to \mathbb{R}$ for Cauchy problem of First Order (HJ) (being $S_0 : \mathbb{R}^k \to \mathbb{R}$ the initial condition) and solutions for characteristic equations (H1)-(H3) starting at the 1-graph of $S_0$.

**Proposition 30** [2, Prop. 1.3] Suppose that $S \in C^2([0,T] \times \mathbb{R}^k)$ is a solution for the given Cauchy problem (HJ). Then, if a solution for the equation

$$\dot{x} = \partial_y H(t, x, \partial_x S(t, x), S(t, x)), \quad (50)$$

on $[0,T]$ is given, the curve defined by

$$(x(t), y(t), z(t)) = (x(t), \partial_x S(t, x(t)), S(t, x(t))) \quad (51)$$

is a solution for characteristic equations (H1)-(H3).

Conversely, $S_0 \in C^1(M), T$ is small enough and

$$(X_{(t,x)}, Y_{(t,x)}, Z_{(t,x)}) : [0,T] \to J^1\mathbb{R}^k$$

is the unique solution for characteristic equations (H1)-(H3) with boundary conditions

$$\begin{cases} 
X_{(t,x)}(t) = x, \\
Y_{(t,x)}(0) = \partial_x S_0 X_{(t,x)}(0)), \\
Z_{(t,x)}(0) = S_0(X_{(t,x)}(0)), 
\end{cases}$$

then

$$S(t, x) = Z_{(t,x)}(t) \quad (52)$$

defines the solution for the Cauchy problem with initial condition $S(0, x) = S_0(x)$.

If $(\varphi^t)_{t \in [0,T]}$ is the isotopy related to $H$ that is,

$$\frac{d}{dt} \varphi^t = X_H(\varphi^t); \quad (53)$$

proposition 30 says that if $S(x, t) = S_t(x)$ is a solution for (HJ), then

$$j^1 S_t = \varphi^t (j^1 S_0), \quad (54)$$

and conversely, if $t$ is small enough, then the solution for the Cauchy problem (HJ) with initial condition $S_0$ is defined by (54).

However, there could happen some issues; first of all, contact transformation $\varphi^t$ could no longer be well-defined at each point of $j^1 S_0$. In addition, although $\varphi_t(j^1 S_0)$ was well-defined, it could happen that it is not a section of $\pi : J^1\mathbb{R}^k \to \mathbb{R}^k$, $(x, y, z) \to x$ anymore.
10 Clarke Calculus

In this part, we will show some results from Clarke calculus for generalized gradients, used in parts above.

We say that \( \mathbb{R}^m \to \mathbb{R} \) Lipschitz of rank \( K \) near a given point \( x \in \mathbb{R}^m \), if for some point \( \epsilon > 0 \), we have

\[
|f(y) - f(z)| \leq K \|y - z\|,
\]

for all \( y, z \in B(x, \epsilon) \).

The generalized directional derivative of \( f \) at \( x \) in the direction \( v \) is defined as

\[
f^\circ(x; v) := \limsup_{y \to x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t},
\]

for \( y \in \mathbb{R}^m \) and \( t > 0 \).

A function \( g \) is called positively homogeneous if \( g(\lambda v) = \lambda g(v) \) for \( \lambda \geq 0 \), and subadditive if for every \( v, w \):

\[
g(v + w) \leq g(v) + g(w).
\]

**Definition 10.1** A function \( F : X \to \mathbb{R} \) is called upper semicontinuous if:

\[
v_i \to v \in X = \limsup_{i \to \infty} F(v_i) \leq F(v).
\]

**Proposition 31** [7, Prop. 10.2] Let \( f \) be a Lipschitz function of rank \( K \) near \( x \). Hence:

(a) \( v \mapsto f^\circ(x; v) \) is finite, positively homogeneous and subadditive function on \( \mathbb{R}^m \), and for all \( v \in \mathbb{R}^m \):

\[
|f^\circ(x; v)| \leq K \|v\|.
\]

(b) For all \( v \in \mathbb{R}^m \), the map \( (u, w) \mapsto f^\circ(u; w) \) is upper semicontinuous at \( (x; v) \), and \( w \mapsto f^\circ(x; v) \) is Lipschitz of rank \( K \) on \( \mathbb{R}^m \).

The following results allows us to prove the above one:

**Theorem 32** [7, Theorem 4.25] Let \( g \in \mathbb{R}^m \to \mathbb{R} \cup \{ \infty \} \) a lower semicontinuous, subadditive and positively homogeneous function such that \( g(0) = 0 \). Then there exists a unique convex subset \( \Sigma \subset \mathbb{R}^m \) such that \( g \) is the support function \( H_\sigma \) of \( \Sigma \), that is, for every \( x \in \mathbb{R}^m \):

\[
g(x) = H_\sigma(x) := \sup_{\sigma \in \Sigma} \langle \sigma, x \rangle.
\]

The set \( \Sigma \) is characterized by

\[
\Sigma = \{ \eta \in \mathbb{R}^m | g(v) \geq \langle \eta, v \rangle, v \in \mathbb{R}^m \},
\]

and it is a compact one if and only if \( g \) is bounded on the unitary disc.
**Definition 10.2** The generalized gradient of a function \( f \) on \( x \), denoted by \( \partial f(x) \) in the unique compact convex non-empty subset of \( \mathbb{R}^m \) whose support function is \( f^o(x; \cdot) \). Therefore:

\[
\left\{ \zeta \in \partial f(x) \iff \forall v \in \mathbb{R}^m : f^o(x; v) \geq \zeta \cdot v \right. \\
\forall v \in \mathbb{R}^m : f^o(x; v) = \max \{ \zeta \cdot v | \zeta \in \partial f(x) \}
\]

As corollary, the generalized gradient is well-defined.

**Proposition 33** Let \( f \) be a Lipschitz function of rank \( K \) near \( x \). Then \( \partial f(x) \subset B(0, K) \).

**Proof.** For any \( \eta \in \partial f(x) \), by proposition \[31\] we have that for all \( v \in \mathbb{R}^m \)

\[
\langle \eta, v \rangle \leq K \|v\|.
\]

If \( f \) is Lipschitz near \( x \), and differentiable at \( x \), then \( f'(x) \subset \partial f(x) \), and the following result allows to generalize the concept of a critical point:

**Lemma 34 (Fermat Rule)** Let \( f : \mathbb{R}^m \to \mathbb{R} \) be locally Lipschitz:

(a) If \( f \) has a local maximum or minimum at \( x \), then \( 0 \in \partial f(x) \).

(b) If \( 0 \notin \partial f(x) \), there exists a decreasing direction \( v \) for \( f \) at \( x \), that is,

\[
\limsup_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t} < 0.
\]

We can obtain a version of the following classical result for generalized calculus:

**Theorem 35 (Mean Value Theorem)** [7, Theorem 10.17] Given \( x, y \in \mathbb{R}^m \) such that \( f \) is Lipschitz in a neighborhood of interval

\[
[x, y] = \{ \lambda x + (1 - \lambda)y, \lambda \in [0, 1] \}.
\]

Hence, there exist a point at \( (x, y) = \text{Interior}[x, y] \) such that

\[
f(y) - f(x) \in \langle \partial f(z), y - x \rangle.
\]

We will use the following special case of the chain rule:

**Lemma 36** Suppose that \( f \) is Lipschitz in a neighborhood of interval \( [x, y] \). Define \( x_t = x + t(y - x), t \in [0, 1] \) and

\[
g : [0, 1] \to \mathbb{R}, g(t) = f(x_t).
\]

Then \( \partial g(t) \subset \langle \partial f(x_t), y - x \rangle \).
**Proof.** Since both sets in the inclusion are indeed intervals on \( \mathbb{R} \), it is enough to show that for every \( v = \pm 1 \):

\[
\max \{ \partial g(t) v \} \leq \max \{ \langle \partial f(x_t), y - x \rangle v \}.
\]

However, in this case \( g^o(t; v) = \partial g(t) \). Hence

\[
g^o(t; v) = \limsup_{s \to t, \lambda \downarrow 0} \left\{ \frac{g(s + \lambda v) - g(s)}{\lambda} \right\} v \leq \limsup_{s \to t, \lambda \downarrow 0} \left\{ \frac{f(x + [s + \lambda v](y - x)) - f(x + s(y - x))}{\lambda} \right\} \leq \limsup_{z \to x_t, \lambda \downarrow 0} \left\{ \frac{f(z + \lambda v(y - x)) - f(z)}{\lambda} \right\} = f^o(x_t; v(y - x)) = \max \{ \partial f(x_t), v(y - x) \}.
\]

**Proof of the Mean Value Theorem**

Consider the function : \([0, 1] \to \mathbb{R}\) defined by

\[
\theta(t) = f(x_t) + t (f(x) - f(y)).
\]

Note that \( \theta(0) = \theta(1) = f(x) \), so that there exists \( t^* \in [0, 1] \) at which \( \tau \) attains an extremal value. In this case, \( 0 \in \partial \theta(t^*) \), so that

\[
0 \in f(x) - f(y) + \langle \partial f(x_{t^*}), y - x \rangle,
\]

and we have proved our Theorem when \( z = x_{t^*} \).

Finally, the following result says that the generalized gradient is closed:

**Theorem 37** [19, prop. A.2] Let \( f \) be Lipschitz of rank \( K \) near \( x \) and \( x_i, v_i \) sequences on \( \mathbb{R}^m \) such that

\[
\begin{cases}
x_i \to x \\
\zeta_i \in \partial f(x_i).
\end{cases}
\]

If \( \zeta_i \to \zeta \), then \( \zeta \in \partial f(x) \).

**Proof.** Fix \( v \in \mathbb{R}^m \). For every \( i \), we have that

\[
f^o(x_i; v) \geq \zeta_i \cdot v.
\]

The sequence \( \zeta_i \cdot v \) is bounded on \( \mathbb{R} \) and it contains term which are arbitrarily close to \( \zeta \cdot v \). Extract a subsequence of \( \zeta_i \) (which we denote in the same way as the original one) such that \( \zeta_i \cdot v \to \zeta \cdot v \). Taking the limit in the above inequality and whereas \( f^o \) is upper semicontinuous en \( x \), we conclude that

\[
f^o(x; v) \geq \zeta \cdot v.
\]

Since \( v \) is arbitrary, it follows that \( \zeta \in \partial f(x) \). ■
11 The Minimax Principle

Definition 11.1  1. A sequence \((u_m)\) in a manifold \(M\) is called of Palais-Smale for \(E \in C^1(M)\) if \(|E(u_m)| \leq c\) uniformly in \(m\) and
\[
\lim_{m \to \infty} \|DE(u_m)\| = 0.
\]

2. We say that \(E\) satisfies Palais-Smale compactness condition \((PS)\) if every \(P-S\) sequence for \(E\) has a strongly convergent subsequence.

Palais-Smale condition allows us to distinguish certain family of neighborhood of critical point of a given functional \(E\); and hence, it will useful to characterize regular values of \(E\).

For \(\beta \in \mathbb{R}\), \(\delta > 0\), \(\rho > 0\) define
\[
E_\beta = \{u \in V : E(u) < \beta\}
\]
\[
K_\beta = \{u \in V : E(u) = \beta, DE(u) = 0\}
\]
\[
N_{\beta,\delta} = \{u \in V : |E(u) - \beta| < \delta, \|DE(u)\| < \delta\}
\]
\[
U_{\beta,\rho} = \bigcup_{u \in K_\beta} \{v \in V : \|v - u\| < \rho\}.
\]

Proposition 38 \([13\text{ lemma 2.3]}\) Suppose that \(E\) satisfies \((P.S)\). Then for every \(\beta \in \mathbb{R}\), it follows that:

1. \(K_\beta\) is a compact subset;
2. both \(\{U_{\beta,\rho}\}_{\rho > 0}\) and \(\{N_{\beta,\delta}\}_{\delta > 0}\) are fundamental systems of neighborhoods for \(K_\beta\).

Remark 39 In particular, if \(K_\beta = \emptyset\) for some \(\beta \in \mathbb{R}\) there exists \(\delta > 0\) such that \(N_{\beta,\delta} = \emptyset\); that is, the differential \(DE(u)\) is uniformly bounded in norm for all \(u \in V\), far enough from origin but with \(E(u)\) close to \(\beta\).

Denote by
\[
\bar{V} = \{u \in V : DE(u) \neq 0\}
\]
the set of regular points of \(E\). Instead of a gradient, which requires the existence of well-defined inner product, we will use the following:

Definition 11.2 A pseudogradient vector field for \(E\) is a locally Lipschitz continuous one \(v \in \bar{V} \to V\) for which the following conditions hold:

1. \(\|v(u)\| < 2 \min \{\|DE(u)\|, 1\}\);
2. for all \(u \in \bar{V} : \langle v(u), DE(u) \rangle > \min \{\|DE(u)\|, 1\} \|DE(u)\|\).
Lemma 40 \[\text{[15, lemma 3.2]}\] Any functional \(E \in C^1(V)\) admits a pseudogradient vector field \(v : V \rightarrow V\).

Theorem 41 (Deformation Lemma) \[\text{[15, Theorem 3.4]}\] Suppose that \(E \in C^1(V)\) satisfies (PS). Fix \(\beta \in \mathbb{R}, \epsilon > 0\) and \(N\) some neighborhood of \(K_\beta\). Thus there exist \(\epsilon \in (0, \bar{\epsilon})\) and a uniparametric continuous family of homeomorphisms

\[\Phi(\cdot, t) : V \rightarrow V, \ 0 \leq t < t\]

with the following properties:

1. \(\Phi(u, t) = u\), if \(t = 0\) or \(DE(u) = 0\) or \(|E(u) - \beta| \geq \bar{\epsilon}\);
2. \(E(\Phi(u, t))\) is non-decreasing on \(t\) for all \(u \in V\);
3. \(\Phi(E_{\beta+\epsilon} \setminus N, 1) \subset E_{\beta-\epsilon}\), and \(\Phi(E_{\beta+\epsilon}, 1) \subset E_{\beta-\epsilon} \cup N\).

Moreover \(\Phi : V \times [0, \infty) \rightarrow V\) has semigroup property, that is,

\[\forall s, t \geq 0 : \Phi(\cdot, t) \circ \Phi(\cdot, s) = \Phi(\cdot, s + t)\).

Since \(\Phi : V \times [0, \infty)\) is obtained integrating a truncated pseudogradient vector field in suitable manner, \(\Phi\) is called local pseudogradient flow.

Definition 11.3 Let \(\Phi : M \times [0, \infty) \rightarrow M\) be a semiflow in a manifold \(M\). A family \(F\) of subsets of \(M\) is called positively \(\Phi\)-invariant if \(\Phi(F, t) \in F\) for every \(F \in F, t \geq 0\).

Theorem 42 (Minimax Principle) \[\text{[15, Theorem 4.2]}\] Suppose that \(M\) is a complete Finsler manifold of class \(C^{1,1}\) and \(E \in C^3(M)\) satisfies the (PS) condition. Also suppose that \(\mathcal{F} \subset \mathcal{P}(M)\) is a collection of subsets invariant with respect to every semiflow \(\Phi : M \times [0, \infty) \rightarrow M\) satisfying

(a) \(\Phi(\cdot, 0) = \text{Id}\),
(b) \(\Phi(\cdot, t)\) is a homeomorphism of \(M\) for every \(t \geq 0\), and
(c) \(E(\Phi(u, t))\) is non-decreasing in \(t\) for every \(u \in M\).

Therefore, if

\[\beta = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u)\]

is finite, \(\beta\) is a critical value of \(E\).

Example 11.1 Let \(X\) be a topological space and \([X, M]\) the set of free-homotopy classes \([f]\) of continuous maps \(f : X \rightarrow M\). For \([f] \in [X, M]\) define

\[\mathcal{F} = \{g(X) | g \in [f]\}\].

Since \([\Phi \circ f] = [f]\) for any homeomorphism \(\Phi\) de \(M\) homotopic to identity, the family \(\mathcal{F}\) is invariant under such maps \(\Phi\). Hence if

\[\beta = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u)\]

is finite, \(\beta\) is a critical value.
Definition 11.4 A point $\xi \in X$ is called critical for $f$ if $0 \in \partial f(\xi)$; the value $f(\xi)$ is called critical for $f$. Note that the critical set $\text{crit}(f)$ of $f$, consisting of every critical point is closed on $X$.

Define

$$\lambda(\xi) = \min_{w \in \partial f(\xi)} \|w\|_{X^*}.$$ 

We say that $f$ satisfies the Palais-Smale condition (PS) if every subsequence $(\xi_n)$ such that $f(\xi_n)$ is bounded and $\lambda(\xi_n) \to 0$ has a convergent subsequence whose limit is a critical point of $f$ and thus there exists $y_n \in \partial f(\xi_n)$ such that $y_n \to 0$.

Proposition 43 (C.f. [19], example A.4) (PS) condition holds where $\|f - Q\|_{\text{Lip}} \leq \infty$ for some non-degenerate quadratic form $Q : X \to \mathbb{R}$. In this case $\text{crit}(f)$ is compact.

Proof. If we define $\psi = f - Q$, each subset $\partial f(\xi) = \partial \psi(\xi) + dQ(\xi)$ consists of vectors whose norm is at least $\|dQ(\xi)\| - \|\psi\|_{\text{Lip}}$, hence

$$\lambda(x) \geq \|dQ(\xi)\| - \|\psi\|_{\text{Lip}},$$

so that $\lambda(x) \to \infty$ where $\|x\| \to \infty$. Therefore, there exists $R > 0$ such that each sequence $(x_n)$ with $\lim \lambda(x) = 0$ satisfies $\|x_n\| \leq R$ for $n$ large enough, and this follows both (PS) condition and compactness of $\text{crit}(f)$. □

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