Core-compactness of Smyth powerspaces

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Abstract

We prove that the Smyth powerspace $Q(X)$ of a topological space $X$ is core-compact if and only if $X$ is locally compact. As a straightforward consequence we obtain that the Smyth powerspace construction does not preserve core-compactness generally.

Keywords: core-compact; locally compact; Smyth powerspace; prime-continuous
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1. Introduction

Given a topological space $X$, the Smyth powerspace $Q(X)$ is the set of compact saturated subsets of $X$ with the upper Vietoris topology. In domain theory, the Smyth powerspace coincides with the Smyth powerdomain for continuous domains with the Scott topology, where the latter construction is used in modelling non-deterministic computation, see for example [11, 4]. The Smyth powerspace construction has many nice properties and useful applications. For example, it was proved by Schalk [10], Heckmann and Keimel [6] that a space is sober if and only if its Smyth powerspace is sober. Xu, Xi and Zhao [12] proved that a similar result holds for well-filtered spaces. That is, a space is well-filtered if and only if its Smyth powerspace is well-filtered. In the same paper, the Smyth powerspace construction was heavily employed in giving a spatial frame which is not sober in its Scott topology.

In this note, we consider another important topological property, core-compactness, and investigate whether it can be preserved by the Smyth powerspace construction. A topological space is core-compact if and only if the lattice of its open subsets (under set inclusion) is a continuous lattice in the sense of domain theory. Core-compact spaces are of great importance in topology and domain theory since these spaces are precisely exponentiable objects in the category of $T_0$ topological spaces and continuous functions. We prove that for a topological space $X$, its Smyth powerspace $Q(X)$ is core-compact if and only if $Q(X)$ is locally compact if and only if $X$ is locally compact. Since there exists core-compact spaces which are not locally compact [8], it follows that core-compactness cannot be preserved by the Smyth powerspace construction in general.

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2. Preliminaries

We refer to [2, 1, 3] for the standard definitions and notations of order theory, topology and domain theory. For a topological space $X$, we use $O(X)$ to denote the lattice of open subsets of $X$. A topological space is a c-space if for any $x \in X$ and any open neighbourhood $U$ of $x$, there is a point $y \in U$ such that $x \in \text{int}(\uparrow y)$, where the symbol $\uparrow$ is the saturation operator. For a subset $A$ of space $X$, $\uparrow A$ is the the intersection of all open neighbourhood of $A$ and called the saturation of $A$. A set $A$ is called saturated if and only if $A = \uparrow A$. A set $A$ is compact if and only if its saturation $\uparrow A$ is compact. For a topological space $X$, we denote the set of all compact saturated sets of $X$ by $Q(X)$. We consider the upper Vietoris topology on $Q(X)$, generated by the sets $\square U = \{K \in Q(X) : K \subseteq U\}$, where $U$ ranges over the open subsets of $X$. One sees that $\square U$’s form a base of the upper Vietoris topology since $\square U \cap \square V = \square (U \cap V)$ for open sets $U, V$. For a compact saturated set $G$, we use $\uparrow_v G$ to denote the saturation of the singleton $\{G\}$ with respect to the upper Vietoris topology on $Q(X)$. Note that $\uparrow_v G = \{K \in Q(X) \mid K \subseteq G\}$.

Let $P$ be a poset and $B$ be a subset of $P$, we say that $B$ is a basis of $P$, if $a = \bigvee(\downarrow a \cap B)$ for all $a \in P$, where $\downarrow a$ is the set of all elements that are below $a$. For a subset $A$ of $P$, we fix $\downarrow A = \bigcup_{a \in A} \downarrow a$.

Let $L$ be a complete lattice, we define the way-way-below relation $\ll$ on $L$ by $x \ll y$ if for any $A \subseteq L$ with $y \leq \bigvee A$, there is $a \in A$ such that $x \leq a$. We call $L$ prime-continuous if for any $x \in L$, $x = \bigvee\{y \in L : y \ll x\}$ holds.

Every prime-continuous complete lattice is a continuous lattice. The following proposition provides a criteria for a continuous lattice to be prime-continuous.

**Proposition 2.1.** Let $L$ be a continuous lattice with a basis $B$. If for any $b \in B$ and finite $F \subseteq B$, $b \leq \bigvee F$ implies that $b \in \downarrow F$, then $L$ is prime-continuous.

**Proof.** Give $x \in L$ and $b \in B$, we prove that $b \ll x$ if and only if $b \ll x$. The “if ” direction is obvious. For the converse we assume that $b \ll x$ and let $A$ be any subset of $L$ with $x \leq \bigvee A$. Since $B$ is a basis of $L$, we know that $\bigvee A = \bigvee(\downarrow A \cap B)$. This means that we can find a finite subset $F$ of $\downarrow A \cap B$ such that $b \leq \bigvee F$ as $b \ll x$. Notice that $b \in B$ and $F \subseteq B$, by assumption there exists an element $f \in F \subseteq \downarrow A \cap B$ such that $b \leq f$. Hence $b$ is below some point of $A$, and this implies that $b \ll x$. $\square$

The following result about c-spaces and prime-continuity is well-known in domain theory, and the proof can be found in [3], for example.

**Theorem 2.2.** Let $X$ be a topological space. Then $X$ is a c-space iff $O(X)$ is prime-continuous.

3. Main results

We arrive at the main result of this note.

**Theorem 3.1.** Let $X$ be a topological space. The following statements are equivalent:

1. $X$ is locally compact;
2. $Q(X)$ is a c-space;
3. $Q(X)$ is locally compact;
4. \( Q(X) \) is core-compact.

Remark 3.2. The equivalence between (1) and (2) is folklore among domain theorists and the proof can also be found in \cite{7}.

Proof. (1) \( \Rightarrow \) (2): Let \( U \) be an open set of \( X \) and \( K \) be a compact saturated set in \( \square U \). This means that \( K \subseteq U \). Since \( X \) is locally compact and \( K \) is compact, we can find an open set \( V \) and a compact saturated set \( G \) such that \( K \subseteq V \subseteq G \subseteq U \). It follows that \( K \in \square V \subseteq \uparrow_{v} G \subseteq \square U \). This implies that \( Q(X) \) is a c-space.

(2) \( \Rightarrow \) (1): For any \( x \in X \) and any open neighbourhood \( U \) of \( x \), it is clear that \( \uparrow_{x} \in \square U \). Since \( Q(X) \) is a c-space, there are \( K \in Q(X) \) and \( V \in O(X) \) such that \( \uparrow_{x} \in \square V \subseteq \uparrow_{v} K \subseteq \square U \). It follows that \( x \in V \subseteq K \subseteq U \). Therefore \( X \) is locally compact.

(2) \( \Rightarrow \) (3): Obvious.

(3) \( \Rightarrow \) (4): Obvious.

(4) \( \Rightarrow \) (2): In light of Theorem 2.2 we prove \( Q(X) \) is a c-space by showing that its open set lattice \( O(Q(X)) \) is prime-continuous. Since \( Q(X) \) is core-compact, \( O(Q(X)) \) is a continuous lattice. Moreover, the set \( \{ \square U \mid U \in O(X) \} \) is a base of the upper Vietoris topology on \( Q(X) \), then it is a basis of the continuous lattice \( O(Q(X)) \). By Proposition 2.1 without loss of generality, we only need to prove that \( \square U \subseteq \square V \) or \( \square U \subseteq \square W \) whenever \( \square U \subseteq \square V \cup \square W \), where \( U, V, W \) are opens in \( X \). This is just a small variant of \cite{3} Lemma 4.2]; we speak the proof in full, nevertheless. Assume this is not true. Then we can find compact sets \( K_{i} \subseteq U, i = 1, 2 \), such that \( K_{1} \nsubseteq V \) and \( K_{2} \nsubseteq W \). So the union \( K_{1} \cup K_{2} \), which is compact saturated, is not in \( \square V \cup \square W \). However this is impossible since \( K_{1} \cup K_{2} \subseteq U \) and \( \square U \subseteq \square V \cup \square W \).

The following also appears as Exercise V-5.25 of \cite{2}.

Theorem 3.3. \cite{9} There exists a core-compact topological space which is not locally compact.

Combining the above theorems, we get our final result.

Corollary 3.4. Let \( X \) be a core-compact space but not locally compact. Then \( Q(X) \) is not core-compact.

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