

ON A SUM INVOLVING GENERAL ARITHMETIC FUNCTIONS AND
THE INTEGRAL PART FUNCTION

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Abstract. Let $f$ be an arithmetic function satisfying some simple conditions. The aim
of this paper is to establish an asymptotical formula for the quantity

$$S_f(x) := \sum_{n \leq x} \frac{f([x/n])}{[x/n]}$$

as $x \to \infty$, where $[t]$ is the integral part of the real number $t$. This generalizes some
recent results of Bordellès, Dai, Heyman, Pan and Shparlinski.

1. Introduction

As usual, we denote by $\varphi(n)$ the Euler totient function, by $[t]$ the integral part of the
real number $t$, by $\log_2$ the iterated logarithm and by $\gamma$ the Euler constant, respectively.
Motivated by the following well-known results

$$\sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor = x \log x + (2\gamma - 1)x + O(x^{1/3}),$$

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x(\log x)^{2/3}(\log_2 x)^{4/3})$$

as $x \to \infty$, using Bourgain’s new exponent pair \cite{3}. Bordellès Dai, Heyman, Pan and Sh-
parlinski \cite{2} proposed to investigate the asymptotical behaviour of the summation function

$$\sum_{n \leq x} \varphi\left(\left\lfloor \frac{x}{n} \right\rfloor\right)$$

as $x \to \infty$. They proved

$$\left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + o(1)\right) x \log x \leq \sum_{n \leq x} \varphi\left(\left\lfloor \frac{x}{n} \right\rfloor\right) \leq \left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + \frac{1380}{4009} + o(1)\right) x \log x$$

and conjectured that

$$\sum_{n \leq x} \varphi\left(\left\lfloor \frac{x}{n} \right\rfloor\right) \sim \frac{6}{\pi^2} x \log x$$

as $x \to \infty$.

The bounds in \eqref{1.3} have been sharpened by Wu \cite{14} using the van der Corput inequality
\cite{5}. Zhai \cite{16} resolved the conjecture \eqref{1.4} by combining the Vinogradov method with an

\begin{flushright}
\textit{Date:} March 2, 2023.
\textit{Key words and phrases.} General arithmetic function, Integral part, Asymptotic formula, Multiple
exponential sums.
\end{flushright}
idea of Goswami [4]. He established the following asymptotic formula

\[
\sum_{n \leq x} \varphi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \frac{6}{\pi^2} x \log x + O(x (\log x)^{2/3} (\log_2 x)^{1/3})
\]

and showed that the error term in (1.5) is \( \Omega(x) \). For part of further related works see [1, 6–8, 11, 12, 15–17].

Furthermore, for the Euler totient function \( \varphi(n) \), Bordellès-Dai-Heyman-Pan-Shparlinski [2, Corollary 2.4] also derived that

\[
S_{\varphi}(x) := \sum_{n \leq x} \varphi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) \left\lfloor \frac{x}{n} \right\rfloor = x \sum_{m \geq 1} \frac{\varphi(m)}{m^2(m+1)} + O(x^{1/2}).
\]

Subsequently, Wu [15] sharpened the error term of [2] and proved

\[
S_{\varphi}(x) = x \sum_{m \geq 1} \frac{\varphi(m)}{m^2(m+1)} + O(x^{1/3} \log x).
\]

Ma and Sun [9,10] generalized Wu’s work by showing that

\[
S_{f}(x) := \sum_{n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) \left\lfloor \frac{x}{n} \right\rfloor = x \sum_{m \geq 1} f(m) m^2(m+1) + O(x^{1/3} \log x)
\]

for \( f \in \{ \varphi, \Psi, \sigma, \beta \} \), where \( \varphi \) denotes the Euler totient function, \( \Psi \) denotes the Dedekind function, \( \sigma \) denotes the sum-of-divisors function, and \( \beta \) denotes the alternating sum-of-divisors function.

In this paper, we will consider a more general case of (1.7), and give a uniform treatment. Define \( \text{id}(n) = n \) and \( \mathbb{1}(n) = 1 \) for \( n \geq 1 \). Let \( f \) be an arithmetic function and \( g \) be the arithmetic function such that \( f = \text{id} \ast g \). Our main result is as follows.

**Theorem 1.1.** Let \( f = \text{id} \ast g \) be an arithmetic function with

\[
g(n) \ll 1
\]

for \( n \geq 1 \). Then,

\[
\sum_{n \leq x} \frac{f \left( \left\lfloor \frac{x}{n} \right\rfloor \right)}{\left\lfloor \frac{x}{n} \right\rfloor} = x \sum_{m \geq 1} \frac{f(m)}{m^2(m+1)} + O(x^{1/3} \log x)
\]

as \( x \to \infty \).

More generally, we have

**Theorem 1.2.** Let \( f = \text{id} \ast g \) be an arithmetic function with

\[
g(n) \ll \varepsilon n^\varepsilon
\]

for \( n \geq 1 \). Then,

\[
\sum_{n \leq x} \frac{f \left( \left\lfloor \frac{x}{n} \right\rfloor \right)}{\left\lfloor \frac{x}{n} \right\rfloor} = x \sum_{m \geq 1} \frac{f(m)}{m^2(m+1)} + O(x^{1/3+\varepsilon})
\]

as \( x \to \infty \) for any \( \varepsilon > 0 \).
Notice that
\[ \varphi = \text{id} * \mu, \quad \sigma = \text{id} * 1, \quad \beta = \text{id} * (-1)^{\Omega(n)}, \quad \Psi = \text{id} * \mu^2, \]
and each of \( \mu(n), 1(n), (-1)^{\Omega(n)} \) and \( \mu^2(n) \) satisfies (1.10). Therefore, Theorem 1.1 implies the main theorem in [9, 10].

Let \( P \) be the set of all primes and define \( \mathbb{1}_P(n) = \begin{cases} 1, & \text{if } n \in P, \\ 0, & \text{otherwise}. \end{cases} \)
(1.12)
Then \( \mathbb{1}_P(n) \ll 1 \). Let \( f = \text{id} * \mathbb{1}_P \). Then Theorem 1.1 implies the following corollary.

**Corollary 1.3.** Let \( f = \text{id} * \mathbb{1}_P \) and \( \mathbb{1}_P(n) \) is defined in (1.12). Then
\[ \sum_{n \leq x} \frac{f([x/n])}{[x/n]} = x \sum_{m \neq 1} \frac{f(m)}{m^2(m+1)} + O(x^{1/3} \log x) \]
as \( x \to \infty \).

**Notations.** We write \( e(t) \) for \( e^{2\pi it} \), \( m \sim M \) for \( M < m \leq 2M \) and \( \psi(t) = \{t\} - \frac{1}{2} \), where \( \{t\} \) denotes the fractional part of the real number \( t \).

## 2. Some Lemmas

In this section, we will introduce two lemmas which will be needed in our proof. The first one is due to Valaler ( [2 Theorem A.6]).

**Lemma 2.1.** For \( x \geq 1 \) and \( H \geq 1 \), we have
\[ \psi(x) = - \sum_{1 < |h| < H} \Phi \left( \frac{h}{H+1} \right) e(hx) + R_H(x), \]
where \( \Phi(t) := \pi t(1 - |t|) \cot(\pi t) + |t|, \) and the error term \( R_H(x) \) satisfies
\[ |R_H(x)| \leq \frac{1}{2H^2} \sum_{|h| < H} \left( 1 - \frac{|h|}{H+1} \right) e(hx). \]
(2.1)

The second one is a direct corollary of [6 Proposition 3.1].

**Lemma 2.2.** Let \( \alpha, \beta, \gamma > 0 \) and \( \delta \in \mathbb{R} \) be some constants. For \( X > 0 \), \( H, M, N \geq 1 \), define
\[ S_\delta = S_\delta(H, M, N) := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_{h,m} b_n \epsilon \left( X \frac{M^\beta N^\gamma}{H^\alpha} \frac{k^\alpha}{m^\beta n^\gamma + \delta} \right), \]
where \( a_{h,m}, b_n \in \mathbb{C} \) such that \( |a_{h,m}|, |b_n| \leq 1 \). For any \( \epsilon > 0 \), we have
\[ S_\delta \ll (X^\kappa H^{2+\kappa} M^{1+\kappa+\lambda} N^{2+\kappa+1/(2+2\kappa)} + HM^{1/2} N + H^{1/2} MN^{1/2} + X^{-1/2} HMN) X^{\epsilon} \]
uniformly for \( H \leq N^{\gamma-1} M^\beta \) and \( 0 \leq \delta \leq 1/\epsilon \), where \( (\kappa, \lambda) \) is an exponent pair and the implied constant depends on \( (\alpha, \beta, \gamma, \epsilon) \) only.
3. A KEY ESTIMATE

Define

\[ \mathcal{G}_\delta^f(x, D) := \sum_{D < d < 2D} \frac{f(d)}{d} \psi \left( \frac{x}{d + \delta} \right). \]

From \( f = \text{id} \ast g \), we get

\[ \frac{f(d)}{d} = \sum_{m|d} \frac{g(m)}{m}, \]

so that we can decompose \( \mathcal{G}_\delta^f(x, D) \) into bilinear forms

\[ \mathcal{G}_\delta^f(x, D) = \sum_{D < mn < 2D} \frac{g(m)}{m} \psi \left( \frac{x}{mn + \delta} \right). \]

In this section we will get the following estimation of \( \mathcal{G}_\delta^f(x, D) \) which will play a key role in the proof of the main theorem.

**Proposition 3.1.** Let \( f = \text{id} \ast g \) be an arithmetic function, \( \delta \geq 0 \) be a fixed constant, \((\kappa, \lambda)\) be an exponent pair. Then

\[ \mathcal{G}_\delta^f(x, D) \ll (x^\kappa D^{-\kappa+\lambda})^{1/(1+\kappa)} + x^\kappa D^{-2\kappa+\lambda} \log x + x^{-1} D^2 \]

uniformly for \( 1 \leq D \leq x \), if \( g(n) \ll 1 \);

\[ \mathcal{G}_\delta^f(x, D) \ll D^\varepsilon \left( (x^\kappa D^{-\kappa+\lambda})^{1/(1+\kappa)} + x^\kappa D^{-2\kappa+\lambda} \log x + x^{-1} D^2 \right) \]

uniformly for \( 1 \leq D \leq x \), if \( g(n) \ll 1 \).

**Proof.** From (3.1), then we can be written (3.2) as

\[ \mathcal{G}_\delta^f(x, D) = \sum_{m \leq 2D} \frac{g(m)}{m} \sum_{D/m < n \leq 2D/m} \psi \left( \frac{x}{mn + \delta} \right). \]

Applying lemma 2.1 and noticing that \( 0 < \Phi(t) < 1 \) for \( 0 < |t| < 1 \) and \( g(m) \ll m^\varepsilon \) for \( m \geq 1 \), we can derive

\[ \mathcal{G}_\delta^f(x, D) \ll \sum_{m \leq 2D} \frac{g(m)}{m} \left( \frac{(D/m)}{H} + \sum_{1 < h < H} \frac{1}{h} \sum_{D/m < n < 2D/m} e \left( \frac{hx}{mn + \delta} \right) \right) \]

for \( 1 \leq H \leq D/m \). Applying the exponent pair \((\kappa, \lambda)\) to the sum over \( n \), we find that

\[ \mathcal{G}_\delta^f(x, D) \ll \sum_{m \leq 2D} \frac{g(m)}{m} \left( \frac{(D/m)}{H} + \sum_{1 < h < H} \frac{1}{h} \left\{ \left( \frac{hx}{D^2/m} \right)^\kappa \left( \frac{D/m}{h} \right)^\lambda + \left( \frac{D^2/m}{hx} \right) \right\} \right) \]

\[ \ll \sum_{m \leq 2D} \frac{g(m)}{m} \left( \frac{(D/m)}{H} + x^\kappa H^\kappa (D^2/m)^{-\kappa} (D/m)^\lambda + x^{-1} (D^2/m) \right) \]

for all \( H \in [1, D/m] \). Optimising the parameter \( H \) over \([1, D/m]\) we obtain

\[ \mathcal{G}_\delta^f(x, D) \ll \sum_{m \leq 2D} \frac{g(m)}{m} \left( (x^\kappa D^{\lambda-\kappa} m^{-\lambda})^{1/(1+\kappa)} + x^\kappa (D^2/m)^{-\kappa} (D/m)^\lambda + x^{-1} (D^2/m) \right), \]

which implies (3.3) and (3.3). \( \square \)
4. Proof of Theorem 1.1

Let \( f = \text{id} \ast g \) and \( N_f \in [1, x^{1/3}] \) be a parameter to be chosen later. Firstly, we write

\[
\sum_{n \leq x} \frac{f([x/n])}{[x/n]} = S_f^\dagger(x) + S_f^\sharp(x)
\]

with

\[
S_f^\dagger(x) := \sum_{n \leq N_f} \frac{f([x/n])}{[x/n]}, \quad S_f^\sharp(x) := \sum_{N_f < n \leq x} \frac{f([x/n])}{[x/n]}.
\]

Secondly, we bound \( S_f^\sharp(x) \). Put \( m = \lfloor x/n \rfloor \). Then \( x/(m+1) < n \leq x/m \). Thus

\[
S_f^\sharp(x) = \sum_{m \leq x/N_f} \frac{f(m)}{m} \left( \frac{x}{m} - \psi \left( \frac{x}{m} \right) - \frac{x}{m+1} + \psi \left( \frac{x}{m+1} \right) \right)
\]

\[
= x \sum_{m \geq 1} \frac{f(m)}{m^2(m+1)} - R_f^\delta(x) + O(N_f),
\]

where

\[
R_f^\delta(x) := \sum_{N_f < m \leq x/N_f} \frac{f(m)}{m} \psi \left( \frac{x}{m + \delta} \right)
\]

for \( \delta = 0, 1 \), and we have used the following bounds

\[
x \sum_{m \geq x/N_f} \frac{f(m)}{m^2(m+1)} \ll N_f, \quad \sum_{m \leq N_f} \frac{f(m)}{m} \psi \left( \frac{x}{m + \delta} \right) \ll N_f.
\]

Writing \( D_j := x/(2^j N_f) \), we have \( N_f \leq D_j \leq x/N_f \leq x \) for \( 0 \leq j \leq \log(x/N_f^2)/\log 2 \). Thus we can apply (3.3) of Proposition 3.1 with \((\kappa, \lambda) = (1/2, 1/2)\) to get

\[
|R_f^\delta(x)| \ll \sum_{0 \leq j \leq \log(x/N_f^2)/\log 2} |\mathcal{G}_f^\delta(x, D_j)|
\]

\[
\ll \sum_{0 \leq j \leq \log(x/N_f^2)/\log 2} (x^{1/3} + x^{1/2} D_j^{-1/2} \log x + x^{-1} D_j^2)
\]

\[
\ll (x^{1/3} + x^{1/2} N_f^{-1/2} + x N_f^{-2}) \log x.
\]

Inserting this into (4.2) and taking \( N_f = x^{1/3} \), we derive

\[
S_f^\sharp(x) = x \sum_{m \geq 1} \frac{f(m)}{m^2(m+1)} + O(x^{1/3} \log x).
\]

Finally, \( f(d)/d = \sum_{m=d} g(m)/m \leq \sum_{m|d} \frac{1}{m} \leq \log d \) implies \( S_f^\dagger(x) \ll N_f \log x \). Inserting this and (4.3) into (4.1), we get (1.9).
5. Proof of Theorem 1.2

Let \( f = id \ast g \) and \( N_f \in [1, x^{1/3}) \) be a parameter to be chosen later. Firstly, we write

\[
\sum_{n \leq x} \frac{f([x/n])}{[x/n]} = S_f^1(x) + S_f^2(x)
\]

with

\[
S_f^1(x) := \sum_{n \leq N_f} \frac{f([x/n])}{[x/n]}, \quad S_f^2(x) := \sum_{N_f < n \leq x} \frac{f([x/n])}{[x/n]}.
\]

Secondly, we bound \( S_f^2(x) \). Put \( m = [x/n] \). Then \( x/(m+1) < n \leq x/m \). Thus

\[
S_f^2(x) = \sum_{N_f < n \leq x} \frac{f([x/n])}{[x/n]} = \sum_{m \leq N_f \atop x/(m+1) < n \leq x/m} \frac{f(m)}{m} \sum_{1 \leq n \leq x/m} 1
\]

where

\[
R_f^\delta(x) := \sum_{N_f < n \leq x} \frac{f(m)}{m} \psi \left( \frac{x}{m+\delta} \right)
\]

for \( \delta = 0, 1 \), and we have used the following bounds

\[
x \sum_{m > x/N_f} \frac{f(m)}{m^2(m+1)} \ll \varepsilon x \sum_{m > x/N_f} \frac{m^{1+\varepsilon}}{m^2(m+1)} \ll N_f x^\varepsilon,
\]

\[
\sum_{m \leq N_f} \frac{f(m)}{m} \psi \left( \frac{x}{m+\delta} \right) \ll \varepsilon \sum_{m \leq N_f} \frac{f(m)}{m} \ll N_f x^\varepsilon.
\]

Writing \( D_j := x/(2^j N_f) \), we have \( N_f \leq D_j \leq x/N_f \leq x \) for \( 0 \leq j \leq \log(x/N_f^2)/\log 2 \). Thus we can apply (3.4) of Proposition 3.1 with \( (\kappa, \lambda) = (1/2, 1/2) \) to get

\[
|R_f^0(x)| \leq \sum_{0 \leq j < \log(x/N_f^2)/\log 2} |G_f^j(x, D_j)| \ll \sum_{0 \leq j < \log(x/N_f^2)/\log 2} (x^{1/3} + x^{1/2} D_j^{-1/2} \log x + x^{-1} D_j^2) D_j^\varepsilon \ll (x^{1/3} + x^{1/2} N_f^{-1/2} + x N_f^{-2}) x^\varepsilon.
\]

Inserting this into (5.2) and taking \( N_f = x^{1/3} \), we derive

\[
S_f^2(x) = x \sum_{m \geq 1} \frac{f(m)}{m^2(m+1)} + O(x^{1/3+\varepsilon}).
\]

Finally, \( f(d)/d = \sum_{m=1}^{d} g(m)/m \leq \sum_{m|d} \frac{m^\varepsilon}{m} \leq d^\varepsilon \) implies \( S_f^1(x) \ll N_f x^\varepsilon \). Inserting this and (5.3) into (5.1), we get (1.11).
Acknowledgments This work is supported by the National Natural Science Foundation of China (Grant No. 11771252).

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