Almost periodic currents, chains and divisors in tube domains

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Theory of almost periodic functions in one real or complex variable was rapidly developing in the twenties-forties. The interest to the theory was provided by both its exceptional beauty and the significant role of almost periodic functions (uniform limits of exponential sums with imaginary exponents) in various branches of mathematics. The basis of the theory, its developments and applications were founded by Bohr, Wiener, Weyl, Jessen, Besikovich, Stepanov, Levitan, and numerous other mathematicians. Further developments of the theory were directed mainly to its extensions to abstract algebraic structures and applications to differential equations. At that time the natural problem of extension of theory of holomorphic almost periodic functions to several complex variables faced undeveloped proper multidimensional technique. The perspectives have been opened by theory of closed positive currents and Monge-Ampère operators. Recently, a series of results were obtained on holomorphic almost periodic functions and mappings in tube domains, and in particular on their zero set distribution (see [13, 14, 15, 16, 10, 11, 1, 18, 12]). A crucial point of the approach was an application of distribution theory machinery that had never been applied to holomorphic almost periodic functions before. Developing this method it is natural to consider more general almost periodic objects such as divisors and holomorphic chains, almost periodic distributions and currents. In [18] a notion of almost periodic distribution in a tube domain was introduced. A definition of almost periodic divisor based on that notion was given, and existence of density of such a divisor was proved. It was also shown that the divisor of an almost periodic holomorphic function is almost periodic. The converse relation is not true even in case of one variable [21].

In the present paper a notion of almost periodic current is introduced,
as well as a notion of almost periodic holomorphic chain proceeded from that definition. Such a chain can be defined either as a special case of almost periodic currents or as a holomorphic chain whose trace measure is an almost periodic distribution (as was done in [18] for almost periodic divisors). We prove that the both definitions are equivalent. From the other hand, it is shown that in general situation almost periodicity of the trace of a current does not imply that for the current itself, even if it is closed and positive.

The zero set (regarded as a holomorphic chain) of a holomorphic mapping can be represented as a Monge-Ampère type current, and one could expect that the zero set of an almost periodic holomorphic mapping should be almost periodic; however we construct an example of an almost periodic holomorphic mapping whose zero set is not almost periodic. Nevertheless, we prove almost periodicity of the Monge-Ampère currents corresponding to almost periodic holomorphic mappings with certain additional properties.

Then we restrict our attention to almost periodic divisors and construct functions that play the same role for almost periodic divisors as the so-called Jessen functions (see [4, 7, 14, 15]) for almost periodic holomorphic functions. In terms of Jessen function we give a sufficient condition for realizability of an almost periodic divisor as the divisor of a holomorphic almost periodic function; some necessary condition is obtained, too.

The paper is organized as follows. In section 1 preliminaries on almost periodic functions and distributions are given, and almost periodic currents are introduced and studied. Section 2 is devoted to almost periodic holomorphic currents. Jessen functions of almost periodic divisors are considered in section 3, and the problem of realizability of almost periodic divisors is studied in section 4.

1 Definition and properties of almost periodic currents

We first recall definitions of Bohr’s almost periodic function and almost periodic distribution in a tube domain $T_G = \{z = x + iy : x \in \mathbb{R}^n, y \in G\}, G \subset \mathbb{R}^n$ (see[18]).

Definition 1.1. A continuous function $f(z)$ in $T_G$ is called almost periodic if for any $\varepsilon > 0$ and every domain $G' \subset G$ the set $E_{\varepsilon,G'}(f) := \{\tau \in \mathbb{R}^n :$
\[ |f(z + \tau) - f(z)| < \varepsilon, \forall z \in T_{G'} \} \text{ is relatively dense in } \mathbb{R}^n, \text{ i.e. satisfies the condition} \]

\[ \exists L > 0, E_{\varepsilon,G'}(f) \cap \{ t \in \mathbb{R}^n : |t - a| < L \} \neq \emptyset, \forall a \in \mathbb{R}^n. \]

Almost periodicity on \( \mathbb{R}^n \) of a function \( f \in C(\mathbb{R}^n) \) is defined similarly by replacing the set \( E_{\varepsilon,G'}(f) \) with \( E_{\varepsilon}(f) := \{ \tau \in \mathbb{R}^n : |f(x + \tau) - f(x)| < \varepsilon, \forall x \in \mathbb{R}^n \}. \)

To introduce almost periodic distributions we need the following notation.

The space of functions of smoothness \( \geq p, 0 \leq p \leq \infty \), with compact support in a domain \( \Omega \) is denoted by \( \mathcal{D}(\Omega, p) \). The corresponding space of distributions (continuous linear functionals) is denoted by \( \mathcal{D}'(\Omega, p) \). When \( p = \infty \) we write as usual \( \mathcal{D}(\Omega) \) and \( \mathcal{D}'(\Omega) \) for \( \mathcal{D}(\Omega, p) \) and \( \mathcal{D}'(\Omega, p) \), respectively. The action of \( f \in \mathcal{D}'(\Omega, p) \) on a test function \( \varphi \in \mathcal{D}(\Omega, p) \) is denoted in a standard way by \( (f, \varphi) \).

**Definition 1.2.** A distribution \( f \in \mathcal{D}'(\Omega, p), p < \infty, \) is called almost periodic of order \( \leq p \) (or just of order \( p \)) if for any function \( \varphi \in \mathcal{D}(\Omega, p) \), the function \( (f(z), \varphi(z - t)) \) is almost periodic on \( \mathbb{R}^n \).

A notion of almost periodic current is defined in a similar way. Let \( \Phi \) be an outer differential form on \( \Omega \subset \mathbb{C}^n \) of bidimension \( (k, l) \), smoothness \( p \geq 0 \), and \( \text{supp } \Phi \subset \Omega \). The space of all such forms is denoted by \( \mathcal{D}_{k,l}(\Omega, p) \). The corresponding space of currents is denoted by \( \mathcal{D}'_{n-k,n-l}(\Omega, p) \). The action of a current \( F \) on a form \( \Phi \) is designated by \( (F, \Phi) \).

**Definition 1.3.** A current \( F \in \mathcal{D}'_{n-k,n-l}(T_G, p) \) is called almost periodic of order \( \leq p \) if for any form \( \Phi \in \mathcal{D}_{n-k,n-l}(T_G, p) \), the function \( (F(z), \Phi(z - t)) \) is almost periodic on \( \mathbb{R}^n \).

As is known, every current \( F \) can be considered as an outer differential form whose coefficients \( F_{I,J} \) are distributions. Comparing Definitions 1.2 and 1.3 one can see that a current \( F \) is almost periodic in \( T_G \) if and only if its coefficients are almost periodic distributions in \( T_G \).

It was proved in [18] that almost periodic distributions in \( T_G \) have properties similar to that of usual almost periodic functions. In particular, notions of the spectrum and the Fourier series is correctly defined, and uniqueness
and approximation theorems hold true. All this carried over to almost periodic currents trivially due to the mentioned connection between almost periodicity of a current and of its coefficients. We state here only the mean value theorems and a theorem on compactness of translations which we will need in future.

Usual almost periodic functions and almost periodic distributions can be described as functions whose translations form a compact family in the corresponding topology. For usual almost periodic functions, it is Bochner’s definition equivalent to the original Bohr’s definition; the case of almost periodic distributions is treated in [18]. A similar fact is true for currents.

**Theorem 1.1.** In order that a current \( F \in \mathcal{D}'_{k,l}(T_G, p) \) be almost periodic, it is necessary and sufficient that for any sequence of the currents \( F(z + h_j), \ h_j \in \mathbb{R}^n, \ j = 1, 2, \ldots \), one can choose a subsequence \( F(z + h_{j_k}) \) converging in \( \mathcal{D}'_{k,l}(T_G, p) \) uniformly on every collection of forms \( \{ \Phi(z + t) : t \in \mathbb{R}^n \}, \ \Phi \in \mathcal{D}_{n-k,n-l}(T_G, p) \).

To formulate the mean value theorems for currents, we introduce some more notation. Let

\[
\|x\| = \|(x_1, \ldots, x_n)\| = \max_{1 \leq i \leq n} |x_i|;
\]

\[
I^{(m)} = I = (i_1, \ldots, i_m), \ 1 \leq i_1 \leq n, \ldots, 1 \leq i_m \leq n;
\]

\[
J^{(m)} = J = (j_1, \ldots, j_m), \ 1 \leq j_1 \leq n, \ldots, 1 \leq j_m \leq n;
\]

\[
dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_m},
\]

\[
d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_m},
\]

\[
dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_m},
\]

and let \( \bar{I} \) denote the sample from the numbers 1, 2, \ldots, \( n \), complementary to \( I \). Given \( F \in \mathcal{D}'_{k,l}(\Omega, p) \), \( F_{I,J} \) denotes its coefficients considered as distributions from \( \mathcal{D}'(\Omega, p) \), so that the current \( F \) has the representation

\[
F = (i/2)^m \sum_{I,J} F_{I,J} dz^I \wedge d\bar{z}^J.
\]

We do it in the case \( F \in \mathcal{D}'_{k,l} \) as well.

The Lebesgue measure in \( \mathbb{R}^n \) is denoted by \( m_n \), and \( d^c \) denotes the operator \( \frac{1}{i}\frac{1}{2} (\partial - \bar{\partial}) \). Thus, \( dd^c = i\frac{1}{2} \partial \bar{\partial} \).
Theorem 1.2. Let \( F \in \mathcal{D}'_{m,m}(T_G, p) \) be an almost periodic current. Then there exists a current \( \mathcal{M} = \mathcal{M}(F) \in \mathcal{D}'_{m,m}(T_G, p) \) with the coefficients \( \mathcal{M}_{I,J} = e_{I,J} \otimes m_n, e_{I,J} \in \mathcal{D}'(G, p) \), such that

\[
\lim_{\nu \to \infty} \left( \frac{1}{2\nu} \right)^n \int_{\|t\|<\nu} (F(z + t), \Phi(z)) \, dx = (\mathcal{M}, \Phi),
\]

\( \forall \Phi \in \text{calD}_{n-m,n-m}(T_G) \).

If \( p = 0 \) and the coefficients of a form \( \Phi \) are functions from \( \mathcal{D}(G, 0) \), then in addition to (1), the equality holds

\[
\exists \lim_{\nu \to \infty} \left( \frac{1}{2\nu} \right)^n \int_{\|z\|<\nu, y \in G} F(z) \wedge \Phi(z) = (\mathcal{M}, \Phi).
\]

Another variant of the mean value theorem is given by

Theorem 1.3. Let \( F \) be an almost periodic current from \( \mathcal{D}'_{m,m}(T_G, 0) \) and

\[
F^{(\nu)} = (i/2)^{n-m} \sum_{I,J} F_{I,J}(\nu x + iy) dz^I \wedge d\bar{z}^J.
\]

Then the equality

\[
\lim_{\nu \to \infty} F^{(\nu)} = \mathcal{M}(F)
\]

takes place in \( \mathcal{D}'_{m,m}(T_G, 0) \).

Note also that the operator \( F \mapsto \mathcal{M}(F) \) keeps such properties of currents as positivity and closedness.

As was said above, Theorems 1.2 and 1.3 are trivial consequences of the corresponding results from [18] on almost periodic distributions. Non-trivial points appear when applying almost periodic currents to the study of almost periodic holomorphic mappings, divisors, and holomorphic chains. Such problems will be treated in the following sections.

2 Almost periodic holomorphic mappings and chains

1. Important examples of almost periodic currents appear from almost periodic holomorphic mappings.
Let $f : T_G \to \mathbb{C}^q$, $q \leq n$, be an almost periodic holomorphic mapping, that is $f = (f_1, \ldots, f_q)$ where $f_1, \ldots, f_q$ are almost periodic holomorphic functions in $T_G$. It is natural to expect that its zero set - or, more precisely, the holomorphic chain $Z_f$ generated by the mapping - inherits the property of almost periodicity. Actually the things are not so good, and we will show below that the zero set of an almost periodic holomorphic mapping need not be almost periodic. From the other hand, we will prove that under certain natural conditions on a mapping, the corresponding current of integration over $Z_f$ has to be almost periodic, as well as some other currents generated by $f$. Then we come to the notion of almost periodic holomorphic current and establish some its properties.

Recall some notions and facts concerning objects under consideration.

**Definition 2.1.** A holomorphic chain of dimension $m$ (or codimension $n - m$), $0 \leq m \leq n$, in a domain $\Omega \in \mathbb{C}^n$ is a pair $Z = (|Z|, \gamma_Z)$, where $|Z|$ is an analytic set in $\Omega$ of pure dimension $m$, and $\gamma_Z$ is a function on $|Z|$ that has constant integer value on each connected component of the set $\text{reg} |Z|$ of regular points of the set $|Z|$. If $|Z| = f^{-1}(0)$ for a holomorphic mapping $f$, and $\gamma_Z(z)$ is the multiplicity of the mapping $f$ in the point $z$, then the chain is said to be given by the mapping $f$ and is denoted by $Z_f$, as well as $\gamma_Z(z)$ is replaced with $\gamma_f(z)$. In case of $\dim Z = n - 1$ the holomorphic chain is called divisor.

It is clear that a holomorphic chain can also be considered as a mapping from $\Omega$ to $\mathbb{Z}$ with the specified properties of its support and values.

In the sequel we assume without special mentioning that $\gamma_Z > 0$. The current of integration over the chain $Z$ is denoted by $[Z]$. Such a current is defined by the equation

$$([Z], \Phi) = \int_{\text{reg} |Z|} \gamma_Z(z) \Phi |_{|Z|}, \quad \Phi \in D_{m,m}(\Omega, 0),$$

where $\Phi |_{|Z|}$ is the restriction of the form $\Phi$ to $\text{reg} |Z|$.

The space of holomorphic functions on a domain $\Omega$ is denoted as usual by $H(\Omega)$.

Let a holomorphic mapping $f : \Omega \to \mathbb{C}^q$, $q \leq n$, satisfy the condition $\dim |Z_f| \leq n - q$ (that is either $|Z_f| = \emptyset$ or $\dim_a |Z_f| = n - q$, $\forall a \in |Z_f|$). Then as is known (see [2]), the Monge-Ampère type currents $(dd^c \log |f|^2)^l$ and $\log |f|^2 (dd^c \log |f|^2)^l$, $l < q$, have locally summable coefficients, and the
current
\[(dd^c \log |f|^2)^q := dd^c (\log |f|^2 (dd^c \log |f|^2)^q)^{q-1})\]

coincides (up to a constant factor) with the current \([Z_f]\). The following statement was proved in [14] for \(l = q - 1\) and in [10] for \(l < q\): if a sequence of holomorphic mappings \(f_j : \Omega \to \mathbb{C}^q\) converges to the mapping \(f\) as \(j \to \infty\), uniformly on every compact subset of \(\Omega\), then

\[
\lim_{j \to \infty} \log |f_j|^2 (dd^c \log |f_j|^2)^l = \log |f|^2 (dd^c \log |f|^2)^l, \quad \forall l < q
\]
in the space \(D_{l,l}\).

Let \(f\) be an almost periodic holomorphic mapping of a domain \(T_G\) into \(\mathbb{C}^q\), \(q \leq n\). Then according to one of the equivalent definitions of almost periodic function (Bochner’s definition), for any sequence \(\{h_j\} \subset \mathbb{R}^n\) one can choose a subsequence \(\{h_{jk}\}\) such that \(f(z + h_{jk})\) converge as \(k \to \infty\), uniformly on every tube domain \(T_{G'}\), \(G' \subset \subset G\). The collection of all such limit mappings is denoted by \(\mathcal{F}_f\). The mapping \(f\) is called regular (see [14, 15]) if \(\dim |Z_f| \leq n - q\), \(\forall \tilde{f} \in \mathcal{F}_f\). The set of all regular almost periodic holomorphic mappings of the domain \(T_G\) into \(\mathbb{C}^q\) is denoted by \(R_q(G)\). A sufficient condition for an almost periodic holomorphic mapping to be regular, in terms of its spectrum, is given in [14] (see [15] also).

Now we have all necessary to formulate the mentioned statement on almost periodic currents connected with almost periodic holomorphic mappings.

**Theorem 2.1.** If \(f \in R_q(G)\) then the currents \(\log |f|^2 (dd^c \log |f|^2)^l\), \(l < q\), and \((dd^c \log |f|^2)^l\), \(l \leq q\), are almost periodic in \(T_G\).

**Proof.** Differentiation evidently keeps almost periodicity of a current. So the equality

\[(dd^c \log |f|^2)^l = dd^c (\log |f|^2 (dd^c \log |f|^2)^l)^{l-1})\]

shows that it suffices to prove the statement of the theorem for the currents

\[A_f^{(l)} := \log |f|^2 (dd^c \log |f|^2)^l, \quad l < q.\]

Let \(\Phi \in D_{n-l,n-l}(T_G, 0)\). Denote \(Q(t) = (A_f^{(l)}(z + t), \Phi(z))\), and let \(\{h_j\}\) be an arbitrary sequence in \(\mathbb{R}^n\). By almost periodicity of the mapping \(f\), one may take without restricting any generality \(f(z + h_j)\) converging as \(j \to \infty\) to some mapping \(\tilde{f} \in \mathcal{F}_f\), uniformly on every domain \(T_{G'}\), \(G' \subset \subset G\). Since
\[ f \in R_q(G), \dim |Z_f| \leq n - q \] and thus the current \( A^{(l)}_f \) is well defined. We claim that for these \( h_j \),

\[ Q(t + h_j) \rightarrow \tilde{Q}(t) := (A^{(l)}_f(z + t), \Phi(z)). \]

uniformly in \( \mathbb{R}^n \).

Supposing the contrary, there exist a number \( C_0 \) and a subsequence \( \{t_j\} \subset \mathbb{R}^n \) such that \( |Q(t_j + h_j) - \tilde{Q}(t_j)| \geq C_0, \forall j \). Passing if necessary to a subsequence one may take that, uniformly in \( T_{G'}, \forall G' \subset \subset G \), there exists the limit

\[ \lim_{j \to \infty} \tilde{f}(z + t_j) =: \hat{f}(z). \]

It is clear that then \( f(z + t_j + h_j) \rightarrow \hat{f}(z) \) and \( \hat{f} \in R_q(G) \). By the mentioned convergence theorem for the Monge-Ampère type currents, \( A^{(l)}_{f(z + t_j)} \rightarrow A^{(l)}_f(z) \) and \( A^{(l)}_{f(z + h_j + t_j)} \rightarrow A^{(l)}_f(z) \). Therefore

\[
\limsup_{j \to \infty} |Q(t_j + h_j) - \tilde{Q}(t_j)| \leq \limsup_{j \to \infty} |(A^{(l)}_{f(z + t_j + h_j)}, \Phi) - (A^{(l)}_f, \Phi)| + \limsup_{j \to \infty} |(A^{(l)}_{f(z + t_j)}, \Phi) - (A^{(l)}_f, \Phi)| = 0.
\]

that contradicts to the choice of the sequence \( \{t_j\} \). So, our claim is proved, and then the function \( Q(t) \) is almost periodic on \( \mathbb{R}^n \) and it is true for every form \( \Phi \in \mathcal{D}_{n-l,n-l}(T_G, 0) \). Thus the current \( A^{(l)}_f \) is almost periodic. The theorem is proved.

The following two results are easy consequences of Theorem 2.1 and the foregoing exposition.

**Corollary 2.1.** The integration current \([Z_f]\) over the holomorphic chain generated by a mapping \( f \in R_q(G) \), is almost periodic.

**Corollary 2.2.** If \( f \in R_q(G) \) then the currents \( A^{(l-1)}_f \) and \( a^{(l)}_f := (dd^c \log |f|^2)^l, l \leq q \), have mean values

\[
\mathcal{M}(A^{(l-1)}_f) =: \tilde{A}^{(l-1)}_f
\]

and

\[
\mathcal{M}(a^{(l)}_f) =: \tilde{a}^{(l)}_f.
\]
The currents $\tilde{a}_f^{(l)}$ inheriting properties of the current $a_f^{(l)}$, are evidently positive, closed, and connected with the currents $\tilde{A}_f^{(l-1)}$ by the relation

$$dd^c \tilde{A}_f^{(l)} = \tilde{a}_f^{(l)}.$$  \hspace{1cm} (3)

Corollary 2.2 was proved earlier as an original result not exploiting almost periodicity of the corresponding currents, in [14] for $l = q - 1$ and in [10] for any $l$.

Now we construct an example of almost periodic holomorphic mapping showing that the condition $f \in \mathbb{R}_q(G)$ is essential for these results.

For $k \in \mathbb{N}$ set

$$g_k(\zeta) = \frac{\sin \pi \zeta \sin \pi \zeta}{k}.$$  

It is an entire periodic function in the complex plane, and for an integer $n$, $g_k(n) = 0$ if and only if $k$ is not a divisor of $|n|$ and $n \neq 0$. Choose a sequence $\{a_k\} \subset \mathbb{R}$ such that

$$0 < a_k \sup \{|g_k(\zeta)| : |\text{Im} \zeta| < 3\} \cdot \sup \{|\sin \pi \zeta| : |\text{Im} \zeta| < 3\} < k^{-2}$$

and consider the mapping $f(z_1, z_2) = (f_1, f_2)$:

$$f_1(z_1, z_2) = \sin \frac{\pi(z_1 - 2)}{5},$$

$$f_2(z_1, z_2) = \sum_{k \geq 2} a_k g_k(z_1) \sin \pi k z_2.$$  

By the choice of $a_k$, the function $f_2$ is defined by the series that converges uniformly in the tube domain $T_G = \{z \in \mathbb{C}^2 : |\text{Im} z_j| < 2, j = 1, 2\}$ and thus is an almost periodic holomorphic function there. So, $f$ is an almost periodic holomorphic mapping from $T_G$ to $\mathbb{C}^2$.

First we show that $|Z_f|$ is discrete. To this end it suffices to prove that $|Z_f|$ contains no set of the form $\{z : z_1 = 5n + 2\}, n \in \mathbb{Z}$. Let $m > 1$ be an integer, then $g_k(m) = 0 \forall k > m$, and so

$$f_2(m, z_2) = \sum_{k \in A(m)} a_k g_k(m) \sin \pi k z_2,$$

where $A(m)$ is the set of all divisors $k > 1$ of the number $m$. Since $g_k(m) \neq 0 \forall k \in A(m)$, $f_2(m, z_2) \neq 0$ for any $m > 1$. It remains true for $m < -1$ as $f_2(-m, z_2) = f_2(m, z_2)$. It proves our claim because $|5n + 2| > 1, \forall n \in \mathbb{Z}$.  

9
Now we show that the current \([Z_f]\) is not almost periodic. Let \(p > 1\) be a prime number, then \(A(p) = \{p\}\) and
\[
f_2(p, z_2) = a_p g_p(p) \sin \pi p z_2.
\]
Therefore \(f_2\) vanishes in the points \((p, l/p), l = 0, \pm 1, \ldots, \pm p\). Let \(P\) denote the set of all primes of the form \(5n + 2\). Then we have
\[
\{(p, l/p) : p \in P, l = 0, \pm 1, \ldots, \pm p\} \subset Z_f.
\]
Hence any open ball of radius one and center at \((p, 0)\), \(p \in P\), contains at least \(2p - 1\) points of the set \(Z_f\). Since every such ball is a translation of the ball \(B = \{z : |z| < 1\}\) and the set \(P\) is infinite, the function \(([Z_f], \varphi(z - t))\) is not bounded on \(\mathbb{R}^2\) for a non-negative test function \(\varphi\), \(\text{supp} \varphi \supset B\), so that the current \([Z_f]\) is not almost periodic.

2. Now we introduce the following

**Definition 2.2.** A holomorphic chain \(Z\) in a domain \(T_G\) is called almost periodic if the current \([Z]\) of integration over \(Z\) is almost periodic\(^1\).

Corollary 2.2 implies that the chain \(Z_f\) associated to a mapping \(f \in R_q(G)\) is almost periodic.

By the mentioned properties of almost periodic currents, the integration current over an almost periodic holomorphic chain has the mean value. Now we study the geometrical sense of this mean value.

Let \(Z\) be an almost periodic holomorphic chain of dimension \(l\) in \(T_G\). Denote \(\beta_l = \frac{1}{l!}(dd^c|z|^2)^l\), and let \(\chi_E = \chi_E(z)\) be the indicator function of a Borelian set \(E\). Then (see for example [9]) the value
\[
V_Z(\Omega) := (\chi_\Omega[Z], \beta_l) = \int \chi_\Omega[Z] \wedge \beta_l = \int_{|Z|} \gamma_Z(z) \beta_l,
\]
where \(\Omega \subset T_G\), is equal to the \(2l\)-dimensional volume of the chain \(Z\) in the domain \(\Omega\). By \(\text{tr}[Z]\) denote the trace measure of the current \([Z]\); in other

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\(^1\)In [18] for the case of codim \(Z = 1\), i.e. when \(Z\) is a divisor, a definition of almost periodicity was given, different from Definition 2.2 and based on the notion of almost periodic distribution. For the relation between the two definitions, see Theorem 2.3 below. In the case \(n = 1\) a definition of almost periodic divisor (almost periodic set) was given in [5, 7] under certain restrictions on divisors, and in [20] in general situation; that definition is based on an approach different from considered here.
words,  
\[ \text{tr}[Z] = \sum_{I}[Z]_{I,I}. \]
Evidently,  
\[ [Z] \wedge \beta_t = \text{tr}[Z] \beta_n. \]
According to Theorem 1.3 we construct the currents \([Z]^{(\nu)}\) and then the measures  
\[ \text{tr}[Z]^{(\nu)} =: \mu_Z^{(\nu)}. \]
Since the current \([Z]\) is positive, the measures \(\mu_Z^{(\nu)}\) are positive, too. Denote  
\[ \Pi_{t,E} = \{z : \|x\| < t, y \in G'\}, \quad E \subset G, \]
and observe that for \(G' \subset \subset G\)  
\[ \mu_Z^{(\nu)}(\Pi_{1,G'}) = \int_{\Pi_{1,G'}} \text{tr}[Z]^{(\nu)} \beta_n = \nu^{-n} \int_{\Pi_{\nu,G'}} [Z] \wedge \beta_t = \nu^{-n} V_Z(\Pi_{\nu,G'}). \quad (4) \]
By Theorem 1.3, the currents \([Z]^{(\nu)}\) converge in \(D'_{l,0}(T_G, 0)\) to a current \(\mathcal{M}([Z])\) of the form  
\[ \mathcal{M}([Z]) = (i/2)^{n-l} \sum_{I,J} b_{I,J} \otimes dz^I \wedge d\bar{z}^J, \]
where \(I = I^{(n-l)}, J = J^{(n-l)}, b_{I,J} \in D'(G, 0), \) and \(m_n\) is the Lebesgue measure in \(\mathbb{R}^n_{(y)}\).
Define a measure \(\mu_Z\) on \(G\) by setting for Borelian sets \(E \subset G\)  
\[ \mu_Z(E) = \int_{\Pi_{1,E}} \text{tr} \mathcal{M}([Z]) \beta_n. \]
so that  
\[ \mu_Z = \sum_{I} b_{I,I} \]
and \(\mu_Z \otimes m_n = \text{tr} \mathcal{M}([Z]).\)

The convergence of the currents \([Z]^{(\nu)}\) to the current \(\mathcal{M}([Z])\) implies that the measures \(\mu_Z^{(\nu)}\) converge weakly to the measure \(\mu_Z \otimes m_n\) as \(\nu \to \infty\). Therefore (see for example [6, 15]) if a domain \(G' \subset \subset G\) satisfies the conditions  
\[ (\mu_Z \otimes m_n)(\partial \Pi_{1,G'}) = 0 \]
or, that is the same, \( \mu_Z(\partial G') = 0 \), then
\[
\lim_{\nu \to \infty} \mu_Z^{(\nu)}(\Pi_{1,G'}) = (\mu_Z \otimes m_n)(\Pi_{1,G'}) = 2^n \mu_Z(G').
\]

Combined with (4) it gives us the following statement.

**Theorem 2.2.** Let \( Z \) be an almost periodic holomorphic chain of a dimension \( l < n \) in a domain \( T_G \) and let a domain \( G' \subset \subset G \) satisfy \( \mu_Z(\partial G') = 0 \). Then there exists the limit
\[
\lim_{\nu \to \infty} (2\nu)^{-n} V_Z(\Pi_{\nu,G'})
\]
and the equation takes place
\[
\lim_{\nu \to \infty} (2\nu)^{-n} V_Z(\Pi_{\nu,G'}) = \mu_Z(G').
\]

It is natural to call the measure \( \mu_Z \) from Theorem 2.2 *the density of the chain* \( Z \), so that the theorem can be viewed as a theorem on existence and evaluation of the density of an almost periodic chain. In the case of \( Z = Z_f, f \in R_q(G) \), the result was obtained earlier (with no regard to almost periodicity of the current \( [Z_f] \)) in [13] for \( q = 1 \) and in [14] for \( 1 \leq q \leq n \) (see [15] also). Those results were preceded by the classical Jessen theorem ([4], see also [7, 8]) on density of the zero set of an almost periodic holomorphic function of one complex variable.

The following useful property of almost periodic holomorphic chains is connected with their densities.

**Theorem 2.3.** Let \( Z \) be an almost periodic holomorphic chain in \( T_G \) and let \( \mu_Z(G') = 0 \) for some domain \( G' \subset G \). Then \( |Z| \cap T_{G'} = \emptyset \).

**Proof.** Suppose the contrary. Then there exists a form \( \Phi \in D_{l,l}(T_G,0) \) with \( l = \dim Z \) such that \( ([Z], \Phi) \neq 0 \). It follows from almost periodicity of the function \( ([Z], \Phi(z - t)) \) that for a certain \( L \) the function is identically zero on no domain \( \{ t \in R^n : ||t - kL|| < L \}, k \in Z^n. \) Therefore if a domain \( G'' \) satisfies the condition \( \supp \Phi \subset G'' \subset \subset G' \) then for any \( k \in Z^n \) there is a point \( z(k) \in |Z| \cap \{ z \in C^n : ||x - kL|| < L, y \in G'' \}. \) Choose \( \varepsilon = \dist (G'', \partial G') \) and set \( \omega_{\varepsilon}(z(k)) = \{ z \in C^n : |z - z(k)| < \varepsilon \}. \) By the Lelong theorem on the lower bound for the mass of a closed positive current (see [9]),
\[
V_Z(\omega_{\varepsilon}(z(k))) \geq \text{const } \varepsilon^{2l},
\]
and thus for \( \nu \) sufficiently great,
\[
V_Z(\Pi_{\nu,G'}) \geq \text{const } \varepsilon^{2l}(2\nu)^l.
\]
In view of (5) it contradicts to the assertion \( \mu_Z(G') = 0 \). The theorem is proved.

3. As was mentioned, almost periodicity of a current \( F \) is equivalent to that of its coefficients \( F_{I,J} \). In case of \( F = [Z] \), \( Z \) being a holomorphic chain, a weaker assumption is sufficient for almost periodicity of the current (and so for the chain).

**Theorem 2.4.** Let \( Z \) be a holomorphic chain of dimension \( l \) in \( T_G \), and let the distribution \( \text{tr} [Z] \) is almost periodic in \( T_G \). Then the chain \( Z \) is almost periodic, too.

This theorem allows us to define almost periodic holomorphic chain as such a chain that the corresponding integration current has almost periodic trace. Actually, for \( l = n - 1 \) it is the mentioned definition of almost periodic divisor from [18] based on consideration of the trace of the current of integration over the divisor.

**Proof.** Denote by \([Z]_t\) the "translated" current, i.e. the current acting on a form \( \Phi \in \mathcal{D}'_{l,l}(T_G,0) \) as \( ([Z]_t, \Phi) = ([Z], \Phi(z-t)) \). The theorem would be proved if we show that for any sequence \( \{t_j\} \subset \mathbb{R}^n \) one can choose a subsequence \( \{t_{j_k}\} \) such that \( \{[Z]_{t_{j_k}+t}\} \) converges as \( k \rightarrow \infty \), uniformly on \( \mathbb{R}^n, \forall \Phi \in \mathcal{D}'_{l,l}(T_G,0) \).

Note that as follows from the properties of almost periodic distributions (see [18]), the value

\[
\|Z\|_{G^0} := \sup_{a \in \mathbb{R}^n} \int_{a+\Pi_{1,G^0}} \text{tr} [Z] \beta_n
\]

is finite for every domain \( G^0 \subset \subset G \). Therefore the masses of the "translated" currents are uniformly bounded on each compact subset \( K \) of \( T_G \). Hence the family \( \{[Z]_t\} \) is weakly compact and thus, given a sequence \( \{t_j\} \subset \mathbb{R}^n \), one can choose a subsequence \( \{t_{j_k}\} \) such that \( \{[Z]_{t_{j_k}+t}\} \) converges as \( k \rightarrow \infty \), uniformly on \( \mathbb{R}^n, \forall \Phi \in \mathcal{D}'_{l,l}(T_G,0) \).

At the same time the assumed almost periodicity of \( \text{tr} [Z] \) provides the existence of a subsequence \( \{t_{j^*}\} \) of \( \{t_{j_k}\} \) such that the functions \( (\text{tr} [Z]_{t+t_{j^*}}, \varphi) \) or, that is the same, the functions \( ([Z]_{t+t_{j^*}}, \varphi \beta_l) \), converge as \( j^* \rightarrow \infty \), uniformly on \( \mathbb{R}^n, \forall \varphi \in \mathcal{D}(T_G,0) \). For the sake of brevity we will write \( t_j \) for \( t_{j^*} \). So, we deal with the sequence \( \{t_j\} \subset \mathbb{R}^n \).
satisfying the conditions

\[ \exists \lim_{j \to \infty} [Z]_{t_j} =: F; \tag{6} \]

\[ \lim_{j \to \infty} \sup_{t} |([Z]_{t + t_j} - F_t, \varphi)| = 0, \quad \forall \varphi \in \mathcal{D}(T_G, 0); \]

here \( F_t = F(z + t) \).

We claim that these conditions imply the equality

\[ \lim_{j \to \infty} \sup_{t \in \mathbb{R}^n} \left| ([Z]_{t + t_j} - F_t, \Phi) \right| = 0, \quad \forall \Phi \in \mathcal{D}_{l,l}(T_G, 0). \]

Suppose the contrary. Then for some \( \varepsilon > 0 \) and some form \( \Phi \in \mathcal{D}_{l,l}(T_G, 0) \) there exists a sequence \( \{h_j\} \in \mathbb{R}^n \) such that

\[ \left| ([Z]_{t_j + h_j} - F_{h_j}, \Phi) \right| > \varepsilon, \quad \forall j. \tag{7} \]

Choose a subsequence \( \{j'\} \subset \{j\} \) in a way the equalities

\[ \exists \lim_{j' \to \infty} [Z]_{t_j' + h_j'} := \tilde{F} \tag{8} \]

\[ \exists \lim_{j' \to \infty} F_{h_j'} = \hat{F} \]

hold. It is possible due to the mentioned weak compactness of the family \( \{[Z]_t\}_{t \in \mathbb{R}^n} \) and so of the family \( \{F_t\}_{t \in \mathbb{R}^n} \).

Passing to the limit in (7) as \( j' \to \infty \) we get

\[ |(\tilde{F} - \hat{F}, \Phi)| > \varepsilon. \tag{9} \]

However it is impossible. Indeed,

\[ |([Z]_{t_j + h_j, \beta_{t_j}} - F_{h_j}, \varphi)| = |([Z]_{t_j} - F, \beta_t(z) \varphi(z - h_j))| \leq \sup_{t \in \mathbb{R}^n} |([Z]_{t_j} - F, \beta_t(z) \varphi(z - t))| \]

for all \( j \), for all \( \varphi \in \mathcal{D}(T_G, 0) \). Together with (6) it implies that for \( j' \to \infty \)

\[ |([Z]_{t_{j'} + h_{j'}, \beta_{n-t}} - (F_{h_{j'}}, \beta_{n-t}, \varphi)| \to 0. \]
In its turn it gives us by (8)

\[(\hat{F} \wedge \beta_{n-l}, \varphi) = (\hat{F} \wedge \beta_{n-l}, \varphi) \quad \forall \varphi \in \mathcal{D}(T_G, 0)\].

Therefore

\[\text{tr} \hat{F} = \text{tr} \tilde{F}.\] (10)

Each of the currents \([Z]_t\) is the integration current over the corresponding holomorphic chain. Since their masses are uniformly bounded, their limits are also currents of integration over holomorphic chains due to the known Federer-Bishop theorem (see for example [3]). So, there exist holomorphic chains \(\tilde{Z}\) and \(\hat{Z}\) of dimension \(l\) such that \(\tilde{F} = [\tilde{Z}]\) and \(\hat{F} = [\hat{Z}]\). By (10), these chains satisfy the condition \(\text{tr} [\tilde{Z}] = \text{tr} [\hat{Z}]\). However for integration currents over holomorphic chains the coincidence of the traces implies coincidence of the currents themselves. Therefore \((\tilde{F} - \hat{F}, \Phi) = 0 \forall \Phi \in \mathcal{D}_{l,l}(T_G, 0)\) that contradicts to (9). The theorem is proved.

In conclusion of the section we give an example showing that the trace of a closed positive current can be almost periodic while the current itself need not be almost periodic.

Set \(u(z) = u(z_1, z_2) = y_1^2 + y_2^2 + x_1 \Re e^{x_2^2}\). Then

\[\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) = \begin{pmatrix} 2 & -2z_2e^{x_2^2} \\ -2z_2e^{x_2^2} & 2 \end{pmatrix}\]

and

\[\det \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) = 4 \left(1 - (x_2^2 + y_2^2)e^{-2(x_2^2-y_2^2)}\right).\]

Elementary calculations with a fixed \(y_2 \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\) show that

\[\sup_{-\infty < x_2 < \infty} (x_2^2 + y_2^2)e^{-2(x_2^2-y_2^2)} = \frac{1}{2}e^4y_2^2 - 1.\]

So for a fixed \(y_2 \in \left(-\frac{1}{2}, \frac{1}{2}\right)\),

\[\det \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) > 0,\]
and thus the Levy form of the function $u$ is positively defined in the domain $T_G \subset \mathbb{C}^2$ with the base $G = \{(y_1, y_2) : -\infty < y_1 < \infty, -\frac{1}{2} < y_2 < \frac{1}{2}\}$.

Let $F = (dd^c u)^2$. It is a closed positive current in $T_G$. Its trace $\text{tr} F = 4m_4$ and hence is an almost periodic measure. At the same time, if $\Phi = \frac{i}{2} \psi dz_1 \wedge d\bar{z}_2$ with $\psi \in \mathcal{D}(T_G, 0)$, then the function $(F(z + t), \Phi(z))$, $t \in \mathbb{R}^2$, is not almost periodic. Indeed,

$$(F(z + t), \Phi(z)) = \int (z_2 + t_2)^2 e^{-(z_2 + t_2)^2} \psi(z) dm_4$$

so that $(F(z + t), \Phi(z)) \to 0$ as $t_2 \to \infty$. If the function $(F(z + t), \Phi(z))$ were almost periodic it would imply it vanishes everywhere, that is deliberately not the case. Therefore the current $F$ is not almost periodic.

### 3 Jessen functions of almost periodic divisors

Let $f$ be an almost periodic holomorphic function in $T_G$. By Theorem 2.2 and Corollary 2.1, its divisor $Z_f$ has the density $\mu_f := \mu_{Z_f}$ connected by the mean value $\mathcal{M}(Z_f)$ by the relation

$$\mu_f \otimes m_n = \text{tr} \mathcal{M}(Z_f)$$

At the same time, $\mathcal{M}(Z_f) = \tilde{a}_f^{(1)}$, so that (see (2))

$$\mathcal{M}(Z_f) = dd^c \tilde{A}_f^{(0)}$$

It was proved in [13], see also [15], that, given an almost holomorphic function $f(z)$, there exists the limit

$$\lim_{\nu \to \infty} \left(\frac{1}{2\nu}\right)^n \int_{|x|<\nu} \log |f(x + iy)| dx =: A_f(y)$$

and the function $A_f(y)$ is convex. Following [13, 15] we will call it the Jessen function of the function $f(z)$.\(^2\)

\(^2\)B. Jessen who introduced such a function in [4] for $n = 1$, called it Jensen’s function, referring to the case where the function $f$ is periodic and $A_f$ can be transformed into the corresponding term of the Jensen formula. We believe, however, that due to the deep Jessen’s investigations of specific features of the function $A_f$ in case of his own name.
Since $A_f(y) = \frac{1}{2} \tilde{A}_f^{(0)}$, relations (11) and (12) imply that
\[
\mu_f = 2 A_f(y),
\]
so that $\mu_f$ coincides up to a constant factor with the measure associated by Riesz (the Riesz measure) of a convex and thus subharmonic, function $A_f(y)$. Note that, given a positive measure, there is a family of subharmonic functions whose Riesz measures coincide with it (each of the functions differs from other by a harmonic function). However there may be no convex function in the family. So, relation (13) can be considered as a certain characteristics of the density of a divisor $Z_f$.

It was shown in the previous section that not only the divisor of an almost periodic holomorphic function but also any almost periodic divisor has the density. And the following question is quite natural: do such densities possess the above property of the densities of the divisors $A_f$? The affirmative answer is a consequence of the following result.

**Theorem 3.1.** Let $F$ be a closed positive almost periodic current from $\mathcal{D}'(\mathbb{R}^{n-1},\mathbb{R}^{n-1})$. Then there exist a convex function $A(y)$ in $G$ and constants $c_{j,k} \in \mathbb{R}$, $j, k = 1, \ldots, n$, such that
\[
\mathcal{M}(F) = \frac{1}{2\pi} \int \left( A(y) + \sum_{j<k} 2 c_{j,k} (x_i y_j - y_i x_j) \right) dx dy.
\]

**Proof.** Denote $\theta = \mathcal{M}(F)$. As was mentioned above, the property of a current to be closed and positive is inhereted by its mean value. So, the current $\theta$ is closed and positive. It can be represented in the form $\theta = \theta' + i \theta''$, where $\theta'$ and $\theta''$ are the currents whose coefficients $\theta'_{j,k}$ and $\theta''_{j,k}$ are real measures. Since the current $\theta$ is positive, $\theta'_{j,k} = \theta'_{k,j}$, $\theta''_{j,j} = 0$, $\theta''_{j,k} = -\theta''_{k,j}$, $\theta''_{j,j} = 0$. Furthermore, as the measures $\theta_{j,k} = \theta'_{j,k} + i \theta''_{j,k}$ have the form $\theta_{j,k} = \hat{\theta}_{j,k} \otimes m_n$, where $\hat{\theta}_{j,k} = \hat{\theta}'_{j,k} + i \hat{\theta}''_{j,k}$ are complex measures in $G$, then
\[
\frac{\partial \hat{\theta}_{j,k}}{\partial x_s} = 0,
\]
so that
\[
\frac{\partial \hat{\theta}_{j,k}}{\partial z_m} = -i/2 \frac{\partial \hat{\theta}_{j,k}}{\partial y_m}, \quad \frac{\partial \hat{\theta}_{j,k}}{\partial \bar{z}_m} = i/2 \frac{\partial \hat{\theta}_{j,k}}{\partial y_m} \quad (14)
\]
The corresponding relations are certainly valid for the measures \( \theta'_{j,k} \) and \( \theta''_{j,k} \), too. Notice further that the currents \( \theta' \) and \( \theta'' \) are closed. By (14), it gives us

\[
\frac{\partial \theta'_{j,k}}{\partial y_l} = \frac{\partial \theta'_{k,l}}{\partial y_j} = \frac{\partial \theta'_{l,j}}{\partial y_k},
\]

(15)

\[
\frac{\partial \theta''_{j,k}}{\partial y_l} = \frac{\partial \theta''_{k,l}}{\partial y_j} = \frac{\partial \theta''_{l,j}}{\partial y_k}, \quad \forall j, k, l.
\]

(16)

Since \( \theta''_{j,k} = -\theta''_{k,j} \), it follows from (16) that

\[
\frac{\partial \theta''_{j,k}}{\partial y_l} = -\frac{\partial \theta''_{k,l}}{\partial y_j} = -\frac{\partial \theta''_{l,j}}{\partial y_k} = -\frac{\partial \theta''_{k,j}}{\partial y_l}, \quad \forall j, k, l.
\]

Therefore

\[
\frac{\partial \theta''_{j,k}}{\partial y_l} = 0, \quad \forall j, k, l,
\]

so that \( \theta''_{j,k} = c_{j,k} m_n \otimes m_n \), where \( c_{j,k} \) are some real constants. Hence

\[
\theta'' = idd^k(\sum_{j<k} c_{j,k}(z_j \bar{z}_k - z_k \bar{z}_j)).
\]

Now to prove the theorem, it is sufficient to find a convex function \( A(y) \) in \( G \) such that

\[
\frac{\partial^2 A}{\partial y_j \partial y_k} = 4\theta'_{j,k}, \quad \forall j, k.
\]

(17)

We will first solve this problem in the balls \( B(x_0, R) = \{ x \in \mathbb{R}^n : |x - x_0| < R \} \subset G \) and then will ”glue” the solutions. To simplify notation, we take \( x_0 = 0 \) and denote \( B(0, R) \) by \( B(R) \).

The desired function will be constructed by a method close to one of the proof of Theorem 2.28 from [9].

We first consider the function

\[
A_1(y) = -\int_{B(R+\varepsilon)} |\zeta - y|^{2-n} d\sigma(\zeta),
\]
subharmonic in $\mathbb{R}^n$, where $\varepsilon$ is such that $B(R + 2\varepsilon) \subset G$, the measure $\sigma(\zeta)$ is defined by the equality

$$\sigma(\zeta) = \kappa_n \sum_{l=1}^{n} \theta'_{l,l},$$

and the norming constant $\kappa_n$ is determined by the condition

$$\Delta A_1(y) = 4 \sum_{l=1}^{n} \theta'_{l,l}(y).$$

Set

$$v_{j,k}(y) = \theta'_{j,k}(y) - \frac{1}{4} \frac{\partial^2 A_1(y)}{\partial y_j \partial y_k}. \quad (18)$$

In view of (18) and (15), we have in $B(R + \varepsilon)$

$$\sum_{l=1}^{n} \frac{\partial^2 v_{j,k}}{\partial y_l^2} = \sum_{l=1}^{n} \frac{\partial^2 \theta'_{l,l}}{\partial y_j \partial y_k} - \frac{1}{4} \frac{\partial^2 \Delta A_1(y)}{\partial y_j \partial y_k} = 0,$$

that is the functions $v_{j,k}$ are harmonic in $B(R + \varepsilon)$. Therefore, in this ball they can be represented as

$$v_{j,k} = \sum_{s=0}^{\infty} P_{s,j,k}(y)$$

where $P_{s,j,k}$ are homogeneous polynomials of degree $s$. It follows directly from the definition of the functions $v_{j,k}$ that

$$\frac{\partial v_{j,k}}{\partial y_l} = \frac{\partial v_{k,l}}{\partial y_j} = \frac{\partial v_{l,j}}{\partial y_k}, \quad \forall j, k, l.$$

It is easy to see that the same relations have place for the functions $P_{s,j,k}(y)$. Together with the Euler equation

$$\sum_{l=1}^{n} \frac{\partial P_{s,j,k}}{\partial y_l} y_l = s P_{s,j,k}$$

it implies that the function

$$A_2(y) = 4 \sum_{j,k=1}^{n} \sum_{s=0}^{\infty} (s + 1)^{-1}(s + 2)^{-1} P_{s,j,k}(y) y_j y_k$$

19
satisfies the equations

\[ \frac{\partial^2 A_2(y)}{\partial y_j \partial y_k} = 4v_{j,k}, \quad \forall j, k, \]

in the ball \( B(R + \varepsilon) \). Then the function \( A(y) := A_1(y) + A_2(y) \) satisfies the equations (17) in the ball \( B(R) \). Note also that the function \( A(y) \) is convex. It follows from the fact that in view of the positivity of the current \( \theta \), the measure

\[ 4 \sum_{j,k} \theta'_{j,k} t_j t_k = \sum_{j,k=1}^n \frac{\partial^2 A}{\partial y_j \partial y_k} t_j t_k \]

is positive for any \( t_j, t_k \in \mathbb{R} \).

So, the problem of finding a convex function satisfying conditions (17) is solved for each ball \( B(x_0, R) \subset G \). Denote all such balls by \( B_\alpha \) and the corresponding functions by \( A_\alpha \). Observe that if \( B_\alpha \cap B_{\alpha'} \neq \emptyset \), the function

\[ A_{\alpha',\alpha} := A_{\alpha'} - A_\alpha \]  \hspace{1cm} (19)

is defined on that intersection and satisfies the equations

\[ \frac{\partial^2 A_{\alpha',\alpha}}{\partial y_j \partial y_k} = 0, \quad \forall j, k \]

in the distribution space, so that is a linear function. We also set \( A_{\alpha,\alpha'} = 0 \) if \( B_\alpha \cap B_{\alpha'} = \emptyset \). It is evident that \( A_{\alpha,\alpha'} = -A_{\alpha',\alpha} \) and \( A_{\alpha',\alpha} + A_{\alpha''},\alpha + A_{\alpha'\alpha''} = 0 \). Thus the functions \( A_{\alpha,\alpha'} \) form a cocycle in the space of 1-cochains of the covering \( \{ B_\alpha \} \) with the values in the bundle of germs of linear functions on \( G \). As the domain \( G \) is convex, the corresponding cohomology group equals zero, so that every cocycle is a coboundary. Therefore there exist linear functions \( h_\alpha(y) \) such that

\[ h_\alpha - h_{\alpha'} = A_{\alpha',\alpha}, \quad \forall \alpha', \alpha. \] \hspace{1cm} (20)

Set

\[ A(y) = A_\alpha(y) + h_\alpha(y), \quad y \in B_\alpha. \]

The function \( A(y) \) is well defined in the whole domain \( G \) due to (19) and (20). Furthermore, it is convex in \( G \) because the functions \( A_\alpha(y) \) are convex
in \( B_\alpha \) and the functions \( h_\alpha(y) \) are linear. Finally, by the construction of the functions \( A_\alpha(y) \), the equations

\[
\frac{\partial^2 A_\alpha}{\partial y_j \partial y_k} = 4\theta'_{j,k}, \quad \forall j, k,
\]

have place in \( B_\alpha \), so that the equation (17) holds in \( G \). Therefore, the function \( A(y) \) is what we sought. The theorem is proved.

Remark 3.1. Applying Theorem 3.1 to the current of integration over an almost periodic divisor \( Z \) and remembering the preceding observations about the density of the divisor, we get the following statement.

**Theorem 3.2.** Let \( Z \) be an almost periodic divisor in \( T_G \). Then there exists a convex function \( A_Z(y) \) in \( G \) (the Jessen function of the divisor \( Z \)) such that its Riesz measure \( \mu_A \) coincides with the density of the divisor \( Z \).

Note that the function \( A_Z \) is not uniquely defined by the divisor \( Z \), namely up to a linear term. Besides, it is clear that in the case of \( Z = Z_f \), where \( f \) is an almost periodic function from \( H(T_G) \), \( A_Z = A_f + L \), \( L \) being a linear function.

Remark 3.2. An analysis of the proof of Theorem 3.1 shows that incidentally the following result has been proved.

**Proposition 3.1.** Let \( F \) be a closed positive current with the coefficients \( F_{j,k} = \hat{F}_{j,k} \otimes m_\alpha \), where \( \hat{F}_{j,k} \) are measures on \( G \). Then there exist a convex function \( A(y) \) in \( G \) and constants \( c_{j,k} \in \mathbb{R} \), \( j, k = 1, \ldots, n \), such that

\[
F = dd^c(u(y) + \sum_{j<k} 2c_{j,k}(x_i y_j - y_i x_j)),
\]

Remark 3.3. If \( f \) is an almost periodic holomorphic function in \( T_G \), then according to (3) and (2),

\[
\mathcal{M}(dd^c \log |f|^2) = \tilde{a}_{f}^{(1)} = dd^c \tilde{A}_{f},
\]

so that the corresponding constants \( c_{j,k} = 0 \). Therefore vanishing of the constants \( c_{j,k} \) is a necessary condition for realizability of an almost periodic divisor \( Z \) as the divisor of an almost periodic holomorphic function. Using of this condition allows us to show that the periodic divisor

\[
\tilde{Z} = \sum_{(k_1, k_2) \in \mathbb{Z}^2} (Z - k_1 - ik_2)
\]
where $Z = (|Z|, \gamma_Z)$, $|Z| = \{(z_1, z_2) : z_1 - iz_2 = 0\}$, $\gamma_Z \equiv 1$, cannot be realized not only as the divisor of an entire periodic function (as was shown in [17]), but also as the divisor of an entire almost periodic function. Really, simple calculations give us

$$[\tilde{Z}] = i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 - dz_1 \wedge d\bar{z}_2 + dz_2 \wedge d\bar{z}_1) = dd^c(4(x_1^2 + y_1^2) + 4(x_1y_2 - x_2y_1))$$

so that $c_{1,2} \neq 0$.  

### 4 Divisors with piecewise linear Jessen function

Here we consider convex piecewise linear functions in a convex domain $G$, i.e. convex functions $A(y)$, $y \in G$, such that

1. the set $\Lambda_A$ of the points of $G$ in whose neighbourhoods the function $A(y)$ is linear, is dense in $G$;

2. the intersection of $\Lambda_A$ with every domain $G' \subset \subset G$ consists of a finite number of connected components.

The Jessen function of an almost periodic divisor is by definition convex, however not every convex function can be the Jessen function of some divisor. A description was given in [16] for convex piecewise linear functions that are the Jessen functions of almost periodic holomorphic functions. The proof is based on the connection between the Jessen function of an almost periodic holomorphic function and the density of its divisor. As follows from Theorem 3.2, the similar connection exists between an almost periodic divisor and its Jessen function, too. Therefore the following theorem is true, and the proof repeats the corresponding arguments from [16] practically word to word and thus is omitted here.

---

3One can show that for a periodic divisor $Z$, the matrix $(C_{j,k})$ coincides (up to a factor $2\pi$) with the matrix $\tilde{N}$ introduced in [19]. It was proved there that vanishing of the matrix $\tilde{N}_Z$ is equivalent to realizability of $Z$ as the divisor of a periodic function. It follows thus from Remark 3.3 that if a periodic divisor can be realized as the divisor of no periodic holomorphic function, then it can be realized as the divisor of no almost periodic holomorphic function, too.
Theorem 4.1 In order that a convex piecewise linear function \( A(y) \) in \( G \) be the Jessen function of some almost periodic divisor in \( T_G \), it is necessary and sufficient that, up to a linear term,

\[
A(y) = \sum_{j=1}^{\omega} \gamma_j \langle y, \lambda^{(j)} \rangle - h_j^+, \tag{21}
\]

where \( \omega \leq \infty \), \( \lambda^{(j)} \in \mathbb{R}^n \), \( |\lambda^{(j)}| = 1 \), \( h_j \in \mathbb{R} \), \( \gamma_j > 0 \), \( (\cdot)^+ = \max(\cdot; 0) \). Furthermore, the support \( |Z| \) of the divisor is the union of complex hyperplanes of the form

\[
L_{j,p} = \{ z : \langle z, \lambda^{(j)} \rangle - ih_j - \alpha_{j,p} = 0 \} \quad \alpha_{j,p} \in \mathbb{R}. \tag{22}
\]

By the help of the results from [12] concerning divisors in \( \mathbb{C} \), this theorem can be supplemented by the statement of realizability of an almost periodic divisor with piecewise linear Jessen function as the divisor of an almost periodic holomorphic function from a specified class.

Theorem 4.2 In order that the Jessen function \( A_Z(y) \) of an almost periodic divisor \( Z \in T_G \) be piecewise linear, it is necessary and sufficient that the divisor \( Z \) be the divisor of a function \( f \in H(T_G) \) of the form

\[
f(z) = \prod_{j=1}^{\omega} f_j(\langle z, \lambda^{(j)} \rangle - ih_j), \tag{23}
\]

where \( \omega \leq \infty \), \( \lambda^{(j)} \in \mathbb{R}^n \), \( |\lambda^{(j)}| = 1 \), \( h_j \in \mathbb{R} \), and \( f_j(w) \) are entire almost periodic functions of \( w \in \mathbb{C} \) with real zeros.

Proof. If an almost periodic function \( F(z) \neq 0 \) in a domain \( T_{G'} \), \( G' \subset G \), the Jessen function \( A_F(y) \) is evidently linear in \( G' \). Then the Jessen function of the function \( f(z) \) from (23) and thus the Jessen function of its divisor is linear out of the union of the planes

\[
\hat{L}_j = \{ y \in \mathbb{R}^n : \langle y, \lambda^{(j)} \rangle - h_j = 0 \}, \quad j = 1, 2, \ldots, \omega \tag{24}
\]

Furthermore, given a point \( y \in G \), there exists its neighbourhood which is intersecting with finite many planes (24) only. That proves the sufficiency part of the statement.
Let now $Z$ be an almost periodic divisor in $T_G$ with piecewise linear Jessen function $A_Z(y)$. By Theorem 4.1, the function $A_Z(y)$ has the representation (21) and $|Z|$ is the union of complex hyperplanes $L_{j,p}$ of the form (22). As was mentioned above, vanishing of the density of an almost periodic divisor in a domain $G'$ implies the relation $Z \cap T_{G'} = \emptyset$, so the projections of any hyperplane $L_{j,p}$ to $\mathbb{R}^n_{(y)}$ with a fixed $j$ have the form $\hat{L}_j$ from (24) (with the same $j$).

Note that a transform of the form $z' = Bz + ia$ where $B$ is an orthogonal matrix with real coefficients and $a \in \mathbb{R}^n$, affects neither almost periodicity of the divisor, nor piecewise linearity of the function $A_Z$. Hence one can take $\hat{L}_j = \{y : y_1 = 0\}$ and, respectively, $L_{j,p} = \{z : z_1 = \alpha_{j,p}\}$, where $\alpha_{j,p}$ are the same as before. One can also take $0 \in \hat{L}_j$ belongs to no other plane $\hat{L}'_j$, $j' \neq j$.

Let $\gamma_{j,p}$ be the density of the divisor in a regular point of $L_{j,p}$. It remains evidently the same for all the points that are not the points of intersection of $L_{j,p}$ with $L_{j',p}$, $j' \neq j$. Choose $\varepsilon > 0$ in a way that the interval $I = \{y \in \mathbb{R}^n : |y_1| < \varepsilon, y_2 = \ldots = y_n = 0\}$ is contained in $G$ and intersects no plane $L_{j',p}$, $j' \neq j$. Consider the divisor $Z_j$ in the strip $\Pi_\varepsilon = \{w \in \mathbb{C} : |\text{Im } w| < \varepsilon\}$ formed by the points $\alpha_{j,p}$ with the multiplicities $\gamma_{j,p}$. We claim that this divisor is almost periodic. To prove it, notice that in the one-dimensional situation the integration current over any divisor $Z$ is the measure $\delta_Z$ concentrated on the support of the divisor and such that the measure of each point equals its multiplicity. Therefore it suffices to prove that the function

$$(\delta_{Z_j}(w), \psi(w - t)) = \sum_p \gamma_{j,p} \psi(\alpha_{j,p} - t)$$

is almost periodic on $\mathbb{R}$ for any function $\psi \in \mathcal{D}(\Pi_\varepsilon)$ with the support in a small enough neighbourhood of the origin.

By almost periodicity of the divisor $Z$, the function

$$(\text{tr } [Z], \varphi(z - t)) = \sum_{j,p} \gamma_{j,p} \int_{L_{j,p}} \varphi(z - t)dV_{j,p}(z), \quad t \in \mathbb{R}^n$$ (25)

where $dV_{j,p}$ is the element of $(2n - 2)$-volume on $L_{j,p}$, is an almost periodic function on $\mathbb{R}^n$ for any $\varphi \in \mathcal{D}(T_G)$. Choose a function $\varphi(z)$ of the form $\varphi(z) = \varphi_1(z_1) \varphi_2'(z)$, where $'z = (z_2, \ldots, z_n)$, such that supp $\varphi_1 \subset \Pi_\varepsilon$ and
\[ \text{supp} \varphi_2 \subset \{ z \in \mathbb{C}^{n-1} : |z| < \varepsilon_1 \} \] with \( \varepsilon_1 \) small enough that the projection of the set \( \text{supp} \varphi_1 \times \text{supp} \varphi_2 \) to \( \mathbb{R}^n \) is out of the planes \( \hat{L}_{j'} \) with \( j' \neq j \). Then the function in (25) takes the form
\[
F_\varphi(t_1, t) = \sum_{p=1}^{\infty} \gamma_{j,p} \int_{L_{j,p}} \varphi_1(z_1 - t_1) \varphi_2(z - t')dV_{j,p}(z),
\]
where \( t' = (t_2, \ldots, t_n) \). Since this function is almost periodic in \( \mathbb{R}^n \), the function
\[
F_\varphi(t_1, 0) = C \sum_{p=1}^{\infty} \gamma_{j,p} \varphi_1(\alpha_{j,p} - t_1),
\]
where
\[
C = \int_{\mathbb{C}^{n-1}} \varphi_2(z)dV_{2n-2}(z).
\]
is an almost periodic function on \( \mathbb{R} \). As the function \( \varphi_1 \in D(\Pi^\varepsilon) \) has been chosen arbitrarily, almost periodicity of the divisor \( Z_j \) is proved.

Observe now that since \( \text{Card} \{ j : G' \cap \hat{L}_j \neq \emptyset \} < \infty \) for every \( G' \subset \subset G \), one can choose a sequence of domains \( G_k \subset \subset G_{k+1}, \ k = 1, 2, \ldots, G = \bigcup G_k \), such that \( \bar{G}_k \cap \hat{L}_k = \emptyset \). Replacing if necessary \( \lambda^{(k)} \) and \( h_k \) with \( -\lambda^{(k)} \) and \( -h_k \) one may take
\[
\bar{G}_k \subset \{ y \in \mathbb{R}^m : \langle y, \lambda^{(k)} \rangle - h_k > 0 \}
\]
and thus, for some \( 0 < r_k < R_k < \infty \) and any \( l \leq k \),
\[
\bar{G}_l \subset \bar{G}_k \subset \{ y \in \mathbb{R}^m : r_k < \langle y, \lambda^{(k)} \rangle - h_k < R_k \} \tag{26}
\]
Now we can use Theorem 2 from [12] resulting that every almost periodic divisor in \( \mathbb{C} \) with the support in \( \mathbb{R} \) is realizable as the divisor of an entire almost periodic function. So, let \( \tilde{f}_j(w) \) be an entire almost periodic function in \( \mathbb{C} \) with \( Z_{\tilde{f}_j} = Z_j \). Since \( \tilde{f}_j(w) \) does not vanish out of the real axis, one has in \( \{ w : \text{Im} w > 0 \} \) the form (see for example [8])
\[
\tilde{f}_j(w) = \exp\{ic_j w + g_j(w)\},
\]
where \( c_j \in \mathbb{R} \), and \( g_j(w) \) are almost periodic holomorphic functions in the upper half-plane. By the approximation theorem for almost periodic functions, there exist finite exponential sums \( q_j(s) \) such that
\[
\sup\{|g_j(w) - q_j(w)| : r_j < \text{Im} w < R_j \} \leq j^{-2}. \tag{27}
\]
Now set
\[ f_j(w) = \tilde{f}_j(w) \exp\{-ic_jw - q_j(w)\}. \]

It follows from (27) and (26) that for any fixed \( k \), the inequality
\[ \sum_{j=k}^{\omega} \mid \log f_j(\langle \lambda^{(j)}, z \rangle - ih_j) \mid \leq \sum_{j=k}^{\omega} j^{-2} \]
takes place in the domain \( G_k \). Therefore the product (23) converges uniformly on every domain \( T_{G'} \) with \( G' \subset \subset G \), and is an almost periodic function in \( T_G \) because each function \( f_j(\langle \lambda^{(j)}, z \rangle - i h_j) \) is almost periodic in \( \mathbb{C}^n \).

The rest follows from the fact that the zero set of the function \( f_j(\langle \lambda^{(j)}, z \rangle - ih_j) \) coincides with \( \bigcup_{p=1}^{\infty} L_{j,p} \), and if a point \( z \in L_{j,p} \) is a regular point of the zero set, the multiplicity of this point as a zero of the function equals \( \gamma_{j,p} \). Therefore the divisor \( Z_f \) coincides with the given divisor \( Z \). The proof is complete.

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