On an Electro-Magneto-Elasto-Dynamic Transmission Problem.

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Abstract

We consider a coupled system of Maxwell’s equations and the equations of elasticity, where the coupling occurs not via material properties but through an interaction on an interface separating the two regimes. Evolutionary well-posedness in the sense of Hadamard well-posedness supplemented by causal dependence is shown for a natural choice of generalized interface conditions. The results are obtained in a Hilbert space setting incurring no regularity constraints on the boundary and the interface of the underlying regions.

1 Introduction

Similarities between various initial boundary value problems of mathematical physics have been noted as general observations throughout the literature. Indeed, the work by K. O. Friedrichs, [2, 3], already showed that the classical linear phenomena of mathematical physics belong – in the static case – to his class of symmetric positive hyperbolic partial differential equations, later referred to as Friedrichs systems, which are of the abstract form

\[(M_0 + A) u = f,\]  \hspace{1cm} (1)

with \( A \) at least formally, i.e. on \( C_\infty \)-vector fields with compact support in the underlying region \( \Omega \), a skew-symmetric differential operator and the \( L^\infty \)-matrix-valued multiplication-operator \( M_1 \) satisfying the condition

\[
\text{sym}(M_1) := \frac{1}{2} (M_1 + M_1^*) \geq c > 0
\]

for some real number \( c \). Indeed, a typical choice for the domain of \( A \) is to incorporate a boundary condition into \( D(A) \), so that \( A \) is skew-selfadjoint (\( A \) quasi-m-accretive would be sufficient). Problem (1) can be considered as the static problem associated with the dynamic problem (\( \partial_t \) denotes the time-derivative)

\[\partial_t M_0 + M_1 + A\]  \hspace{1cm} (2)

with \( M_0 \) a selfadjoint \( L^\infty \)-multiplication-operator and \( M_0 \geq 0 \), which were also addressed in [3]. It is noteworthy, that even the temporal exponential
weight factor, which plays a central role in our approach, is introduced as an ad-hoc formal trick to produce a suitable $M_1$ for a well-posed static problem. For the so-called time-harmonic case, where $\partial_0$ is replaced by $i\omega$, $\omega \in \mathbb{R}$, we replace $A$ simply by $i\omega M_0 + A$ to arrive at a system of the form.

Operators of the Friedrichs type $\mathcal{L}$, can be generalized to obtain a fully time-dependent theory allowing for operator-valued coefficients, indeed, in the time-shift invariant case, for systems of the general form

$$\left(\partial_0 M \left(\partial_0^{-1}\right) + A\right) U = F$$

(Evo-Sys)

where $A$ is -- for simplicity -- skew-selfadjoint and $M$ an operator-valued -- say -- rational function as an abstract coefficient. The meaning of $M \left(\partial_0^{-1}\right)$ is in terms of a suitable function calculus associated with the (normal) operator $\partial_0$, [12, Chapter 6]. We shall refer to such systems as evolutionary equations, 	extit{evo-systems} for short, to distinguish them from the special subclass of classical (explicit) evolution equations.

In this paper we intend to study a particular transmission problem between two physical regimes, electro-magneto-dynamics and elasto-dynamics, within this general framework and establish its well-posedness, which for evo-systems entails not only Hadamard well-posedness, i.e. uniqueness, existence and continuous dependence, but also the crucial property of causality.

The peculiarity of the problem we shall investigate is that the interaction between the two regimes is solely via the interface, not via material interactions as in piezo-electrics, compare e.g. [7] for the latter type of effects.

After properly introducing evo-systems in the next section, we shall establish the equations of electro-magneto-dynamics and elasto-dynamics respectively, as such systems in Section 3. Finally, in Section 4 we establish a particular interface coupling problem between the two regimes in adjacent regions via a mother-descendant mechanism, see the survey [15]. We emphasize that our setup allows for arbitrary open sets as underlying domains with no additional constraints on boundary regularity.

2 A Short Introduction to a Class of Evo-Systems

2.1 Basic Ideas

We shall approach solving (Evo-Sys) by looking at the equation as a space-time operator equation in a suitable Hilbert space setting. Without loss of generality we may\(^1\) and will assume that all Hilbert spaces are real.

\(^1\)Every complex Hilbert space $X$ is a real Hilbert space choosing only real numbers as multipliers and

$$(\phi, \psi) \mapsto \Re \langle \phi | \psi \rangle_X$$

as new inner product. Note that with this choice $\phi$ and $i\phi$ are always orthogonal. Moreover, for any skew-symmetric operator $A$ we have

$$x \perp Ax$$
Solutions will be discussed in a weighted $L^2$-space $H_\nu (\mathbb{R}, H)$, constructed by completion of the space $\tilde{C}_1 (\mathbb{R}, H)$ of differentiable $H$-valued functions with compact support w.r.t. $\langle \cdot | \cdot \rangle_{\nu,H}$ (norm: $| \cdot |_{\nu,H}$)

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi (t) | \psi (t) \rangle_H \exp (-2\nu t) \, dt.$$ 

Here $H$ denotes a generic real Hilbert space. We introduce time-differentiation $\partial_0$ as a closed operator in $H_\nu (\mathbb{R}, H)$ defined as the closure of $\tilde{C}_1 (\mathbb{R}, H) \subseteq H_\nu (\mathbb{R}, H)$, $\varphi \mapsto \varphi'$.

The operator $\partial_0$ is normal in $H_\nu (\mathbb{R}, H)$. For $\nu_0 \in ]0, \infty[ \subset [\nu, \infty[$, we have

$$\text{sym} (\partial_0) := \frac{1}{2} (\partial_0 + \partial_0^*) = \nu \geq \nu_0 > 0,$$

i.e. $\partial_0$ is a strictly (and uniformly w.r.t. $\nu \in ]\nu_0, \infty[$) positive definite (i.e. $m$-accretive) operator.

This core observation can be lifted to a larger class of more complex problems involving operator-valued coefficients and systems of the general form

$$\left( \partial_0 M (\partial_0^{-1}) + A \right) U = F \quad \text{(Evo-Sys)}$$

where $A$ is – for simplicity – skew-selfadjoint and $M$ an operator-valued – say – rational function as abstract coefficient.

In many practical cases skew-selfadjointness of $A$ is evident from its structure as a block operator matrix of the form

$$A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix},$$

with $H = H_0 \oplus H_1$ and $C : D (C) \subseteq H_0 \to H_1$ a densely defined, closed linear operator.

### 2.2 Well-Posedness for Evo-Systems.

Since reasonable well-posedness requires closed operators we describe our problem class more rigorously as of the form

$$\left( \partial_0 M (\partial_0^{-1}) + A \right) U = F \quad \text{(Evo-Sys)}$$

for all $x \in D (A)$.

Indeed, since $\langle x | y \rangle - \langle y | x \rangle = 0$ (symmetry) we have

$$\langle x | Ax \rangle - \langle Ax | x \rangle = 0$$

or by skew-symmetry

$$0 = 2 \langle x | Ax \rangle$$

for all $x \in D (A)$.
For a convenient special class, more than sufficient for our purposes here, we record the following general well-posedness result, see [10, 11, 15].

**Theorem 2.1.** Let \( z \mapsto M(z) \) be a rational \( \mathcal{L}(H,H) \)-valued function in a neighborhood of 0 such that \( M(0) \) is selfadjoint and\(^2\)

\[
\nu M(0) + \text{sym} \left( M'(0) \right) \geq \eta_0 > 0
\]

for some \( \eta_0 \in \mathbb{R} \) and all \( \nu \in ]\nu_0, \infty[ , \nu_0 \in ]0, \infty[ \) sufficiently large, and let \( A \) be skew-selfadjoint. Then well-posedness of (Evo-Sys) follows for all \( \nu \in ]\nu_0, \infty[ \). Moreover, the solution operator \( (\partial_0 M (\partial_0^{-1}) + A)^{-1} \) is causal in the sense that

\[
\chi_{] -\infty, 0[} (\partial_0 M (\partial_0^{-1}) + A)^{-1} = \chi_{] -\infty, 0[} (\partial_0 M (\partial_0^{-1}) + A)^{-1} \chi_{] -\infty, 0[}.
\]

Indeed, apart from occasional side remarks we will simply have

\[
M (\partial_0^{-1}) = M_0 + \partial_0^{-1} M_1
\]

and since \( \partial_0 \), \( A \) can be continuously extended to suitable extrapolation spaces, it is justified\(^3\) to drop the closure bar, which we shall do henceforth.

## 3 Maxwell’s Equations and the Equations of Linear Elasticity as Evo-Systems

### 3.1 Maxwell’s Equations as an Evo-Systems.

James Clerk Maxwell developed his new ideas on electro-magnetic waves in 1861-64 resulting in his famous two volume publication: A Treatise on Electricity and Magnetism, [6]. His ingenious contribution to what we nowadays call Maxwell’s equations is to amend Ampere’s law with a so-called *displacement current* term. Heaviside and Gibbs have given the system in its now familiar form as

\[
\begin{align*}
\partial_0 D + \sigma E - \text{curl} H &= -j_{\text{ext}}, \quad \text{(Ampere’s law)} \quad (\text{Faraday’s law of induction}) \\
\partial_0 B + \text{curl} E &= 0, \\
D &= \varepsilon E, \\
B &= \mu H.
\end{align*}
\]

\(^2\)Here we use sym in an analogous meaning to (3), i.e.

\[
\text{sym} \left( B \right) := \frac{1}{2} (B + B^*),
\]

which is equal to \( \frac{1}{2} (B + B^*) \) since \( B \) is continuous.

\(^3\)Albeit this being sometimes confusing and misleading, it is a common practice in the field of partial differential equations. E.g. one frequently writes

\[
\Delta = \partial_1^2 + \partial_2^2
\]

although \( \phi \in D(\Delta) \) does in general not – as the notation appears to suggest – allow for \( \phi \in D(\partial_1^2) \cap D(\partial_2^2) \).
The usually included divergence conditions are redundant, since the two equations together with the material relations can be seen to be leading already to a well-posed initial boundary value problem. The so-called six-vector block matrix form:

$$
\begin{pmatrix}
\partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\text{curl} \\ \text{curl} & 0 \end{pmatrix}
\end{pmatrix}
\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -j_{\text{ext}} \\ 0 \end{pmatrix}
$$

brings us already close to our initial goal to formulate the equations as an evo-system. Here curl denotes the $L^2$-closure of the classical curl defined on $C_1(\mathbb{R}^3)$-vector fields vanishing outside closed, bounded subsets of $\mathbb{R}^3$. Moreover, $\text{curl} := \text{curl}$ so the spatial Maxwell operator is skew-selfadjoint in $L^2(\mathbb{R}^3, \mathbb{R}^6)$. In case of a domain $\Omega$ with boundary we take curl constructed analogously with $C_1(\Omega)$-vector fields vanishing outside closed, bounded sets contained in $\Omega$, where $\Omega$ is a non-empty open set in $\mathbb{R}^3$ (strong definition of curl) and define

$$\text{curl} := \text{curl}^*$$

(weak definition of curl). Thus we arrive indeed at the evo-system

$$
\begin{pmatrix}
\partial_0 M \left( \partial_0^{-1} \right) + A
\end{pmatrix}
\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -j_{\text{ext}} \\ k_{\text{ext}} \end{pmatrix}
$$

with $M \left( \partial_0^{-1} \right) = M(0) + \partial_0^{-1} M'(0)$ and here specifically

$$M(0) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}, \quad M'(0) = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -\text{curl} \\ \text{curl} & 0 \end{pmatrix}.$$  \hspace{0.5cm} (6)

which satisfies the well-posedness constraint if we assume $\varepsilon, \mu$ selfadjoint and (compare (3) and (2))

$$\nu \varepsilon + \text{sym} \left( \sigma \right), \mu \geq \eta_0 > 0,$$  \hspace{0.5cm} (7)

for all sufficiently large $\nu \in [0, \infty[$. Note that with this assumption also $\varepsilon$ having a non-trivial null space, the so-called eddy current problem, can be handled without further adjustments. Of course, in the spirit of Theorem 2.1 we could consider more general media. More recently, so-called electro-magnetic metamaterials have come into focus, which are media, where $M'' \neq 0$ or $M(z)$ is not block-diagonal. To classify some prominent cases, there are for example:

- Bi-anisotropic media, characterized by

$$M(0) = \begin{pmatrix} \varepsilon & \kappa^* \\ \kappa & \mu \end{pmatrix}, \quad \kappa \neq 0.$$  

\hspace{0.5cm} (4)
Since, due to (4), we must have $M(0) \geq 0$, we get $\varepsilon \geq 0$ and
\[
\left| \mu^{-1/2} \kappa \varepsilon^{-1/2} \right| \leq 1.
\]

Note that this is a strong smallness constraint on the off-diagonal entry $\kappa$. For example in homogeneous, isotropic media $c_0 = \varepsilon^{-1/2} \mu^{-1/2}$ is the speed of light and the above condition yields
\[
|\kappa| \leq \frac{1}{c_0}.
\]

- **Chiral media:**
  \[
  M'(0) = \begin{pmatrix}
  0 & -\chi \\
  \chi & 0
  \end{pmatrix}, \quad \chi \neq 0 \text{ selfadjoint.}
  \]

- **Omega media:**
  \[
  M'(0) = \begin{pmatrix}
  0 & \chi \\
  \chi & 0
  \end{pmatrix}, \quad \chi \neq 0 \text{ skew-selfadjoint.}
  \]

### 3.2 The Equations of Linear Elasto-Dynamics as an Evo-System

Linear elasto-dynamics is usually discussed in a symmetric tensor-valued $L^2$-setting for the stress $T$, i.e. $T \in L^2(\Omega, \text{sym} [\mathbb{R}^{3 \times 3}])$, and a vector $L^2$-setting for the displacement $u \in L^2(\Omega, \mathbb{R}^3)$. Here sym is the (orthogonal) projector onto real-symmetric-matrix-valued $L^2$-functions. More precisely, we extend sym to the matrix-valued case by letting
\[
sym : L^2(\Omega, \mathbb{R}^{3 \times 3}) \to L^2(\Omega, \mathbb{R}^{3 \times 3}),
\]
\[
W \mapsto \frac{1}{2} (W + W^*),
\]
where the adjoint $W^*$ is taken point-wise by the standard Frobenius inner product
\[
(T, S) \mapsto \text{trace} \left( T^\top S \right)
\]
for $3 \times 3$-matrices, such that
\[
\mathbb{R}^{3 \times 3} \to \mathbb{R}^6
\]
\[
\begin{pmatrix}
  T_{00} & T_{01} & T_{02} \\
  T_{10} & T_{11} & T_{12} \\
  T_{20} & T_{21} & T_{22}
\end{pmatrix}
\mapsto
\begin{pmatrix}
  T_{00} \\
  T_{11} \\
  T_{22} \\
  T_{12} \\
  T_{20} \\
  T_{01} \\
  T_{21} \\
  T_{02} \\
  T_{10}
\end{pmatrix}
\]
is unitary. Then with
\[
\iota_{\text{sym}} : L^2(\Omega, \text{sym} [\mathbb{R}^{3 \times 3}]) \to L^2(\Omega, \mathbb{R}^{3 \times 3})
\]
\[
T \mapsto T,
\]

denoting the canonical embedding of the subspace $L^2 (\Omega, \text{sym } [\mathbb{R}^{3 \times 3}])$ in $L^2 (\Omega, \mathbb{R}^{3 \times 3})$ we have

$$\iota_{\text{sym}}^*: L^2 (\Omega, \mathbb{R}^{3 \times 3}) \rightarrow L^2 (\Omega, \text{sym } [\mathbb{R}^{3 \times 3}])$$

$$W \mapsto \text{sym} W$$

and so we have the useful factorization

$$\text{sym} = \iota_{\text{sym}}^* \iota_{\text{sym}}.$$

With this observation we can now approach the standard equations of elasticity theory. The dynamics of elastic processes is commonly captured in a second order formulation for the displacement $u$ by

$$\varrho \partial_0^2 u - \text{Div} C \text{Grad} u = f,$$

where

$$\text{Grad} u := \iota_{\text{sym}}^* (\nabla u)$$

$$\text{Div} T := \left( \nabla^\top T \right)^\top$$

for symmetric $T$, i.e. $T \in L^2 (\Omega, \text{sym } [\mathbb{R}^{3 \times 3}])$. The elasticity ‘tensor’, i.e. rather the mapping

$$C : L^2 (\Omega, \text{sym } [\mathbb{R}^{3 \times 3}]) \rightarrow L^2 (\Omega, \text{sym } [\mathbb{R}^{3 \times 3}])$$

and the mass density operator

$$\varrho : L^2 (\Omega, \mathbb{R}^3) \rightarrow L^2 (\Omega, \mathbb{R}^3)$$

are assumed to be selfadjoint and strictly positive definite.

The origin, from which the above second order system is derived, is naturally a system of algebraic and first order differential equations. The original system can be easily reconstructed by re-introducing the relevant physical quantities velocity $v := \partial_0 u$ and stress $T := C \text{Grad} u$. Thus, we arrive at the system

$$\varrho \partial_0 v - \text{Div} T = f,$$

$$T = C \text{Grad} \partial_0^{-1} v,$$

in the unknowns $v$ and $T$. Differentiating the second equation with respect to time, we end up with a system of the block operator matrix form

$$(\partial_0 \begin{pmatrix} \varrho & 0 \\ 0 & C^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -\text{Div} \\ -\text{Grad} & 0 \end{pmatrix}) \begin{pmatrix} v \\ T \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$ 

Choosing now for example a homogeneous Dirichlet boundary condition, i.e. we replace $\text{Grad}$ by the projection $\varrho_{\text{sym}}^* \text{grad}$, Korn’s inequality shows that the closure bar is superfluous

$$\text{Grad} = \iota_{\text{sym}}^* \text{grad}.$$
where \( \text{grad} \) is the closure of differentiation for vector fields (the Jacobian matrix) with compact support in \( \Omega \) as a mapping from \( L^2(\Omega, \mathbb{R}^3) \) to \( L^2(\Omega, \mathbb{R}^{3 \times 3}) \), and

\[
\text{Div} := \text{div}_{\text{sym}}
\]

so that

\[
\text{Grad} = -\text{Div}^*.
\]

we are led to consider an evo-system of the form

\[
\begin{pmatrix}
\partial_t 
& \begin{pmatrix} 0 & 0 \\ 0 & C^{-1} \end{pmatrix}
\end{pmatrix}
+ \begin{pmatrix}
0 
& \begin{pmatrix} 0 & -\text{Div} \\ -\text{Grad} & 0 \end{pmatrix}
\end{pmatrix}
\begin{pmatrix} v \\ T \end{pmatrix}
= \begin{pmatrix} f \\ g \end{pmatrix}.
\]

(8)

Remark 3.1. We note that also here we have “weak equals strong” following the same rationale as in the electro-magneto-dynamics case, compare Footnote 4.

In the light of (4) the well-posedness results from assuming that

\[
ge_*, C \geq \eta_0 > 0
\]

for some real constant \( \eta_0 \).

4 An Interface Coupling Mechanism.

After the above preliminary considerations, we are now ready to consider the situation, where the electro-magnetic field in one region interacts with elastic media in another region via some common interface. Rather than basing our choice of transmission constraints on the interface by physical arguments, we shall explore a deep connection between electro-magneto-dynamics and elasto-dynamics to arrive at natural transmission conditions built into the construction of the evo-system. This construction will utilize the idea of a mother-descendant construction introduced in [13], see [14] for a more viable version, which we will briefly recall.

4.1 Mother Operators and their Descendants

We recall from [13] the following simple but crucial lemma.

Lemma 4.1. Let \( C : D(C) \subseteq H \rightarrow Y \) be a closed densely-defined linear operator between Hilbert spaces \( H, Y \). Moreover, let \( B : Y \rightarrow X \) be a continuous linear operator into another Hilbert space \( X \). If \( C^* B^* \) is densely defined, then

\[
\overline{BC} = (C^* B^*)^*.
\]

Proof. It is

\[
C^* B^* \subseteq (BC)^*.
\]

If \( \phi \in D((BC)^*) \) then

\[
\langle BCu|\phi \rangle_X = \langle u| (BC)^* \phi \rangle_H
\]

for all \( u \in D(C) \). Thus, we have

\[
\langle Cu|B^* \phi \rangle_Y = \langle BCu|\phi \rangle_X = \langle u| (BC)^* \phi \rangle_H
\]
for all \( u \in D(C) \) and we read off that \( B^* \phi \in D(C^*) \) and
\[
C^* B^* \phi = (BC)^* \phi.
\]
Thus we have
\[
(BC)^* = C^* B^*.
\]
If now \( C^* B^* \) is densely defined, we have for its adjoint operator
\[
(C^* B^*)^* = BC.
\]

As a consequence we have that the descendant
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & B \\
\end{pmatrix}
\begin{pmatrix}
0 & -C^* \\
C & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & B^* \\
\end{pmatrix}
= \begin{pmatrix}
0 & -C^* \\
C & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & B^* \\
\end{pmatrix}
\]
indeed inherits its skew-selfadjointness from its mother \( \begin{pmatrix}
0 & -C^* \\
C & 0 \\
\end{pmatrix} \)
(with \( C \) replaced by \( BC \)). Moreover, we record the following result on the stability of well-posedness in the mother-descendant process.

**Theorem 4.2.** Let \( C : D(C) \subseteq H \to Y \) be a closed densely-defined linear operator between Hilbert spaces \( H, Y \). Moreover, let \( B : Y \to X \) be a continuous linear operator into another Hilbert space \( X \) with a closed range \( B[Y] \) such that \( C^* B^* \) is densely defined. Then, if
\[
\begin{pmatrix}
\partial_0 M (\partial_0^{-1}) + \begin{pmatrix}
0 & -C^* \\
C & 0 \\
\end{pmatrix} \end{pmatrix}
\begin{pmatrix}
U_0 \\
U_1 \\
\end{pmatrix}
= \begin{pmatrix}
F_0 \\
F_1 \\
\end{pmatrix}
\]
with data \( \begin{pmatrix}
F_0 \\
F_1 \\
\end{pmatrix} \) \( \in H_\nu (\mathbb{R}, H \oplus X) \) and a solution \( \begin{pmatrix}
U_0 \\
U_1 \\
\end{pmatrix} \) \( \in H_\nu (\mathbb{R}, H \oplus X) \)
is a well-posed evo-system (satisfying in particular \( [\text{1}] \)), so is the descendant problem
\[
\begin{pmatrix}
\partial_0 \tilde{M} \left( \partial_0^{-1} \right) + \tilde{A} \end{pmatrix}
U = \begin{pmatrix}
F_0 \\
G_1 \\
\end{pmatrix} \in H_\nu (\mathbb{R}, H \oplus B[Y]),
\]
where
\[
\tilde{M} \left( \partial_0^{-1} \right) = \begin{pmatrix}
1 & 0 & 0 \\
0 & B \\
\end{pmatrix}
M \left( \partial_0^{-1} \right) \begin{pmatrix}
1 & 0 \\
0 & B^* \\
\end{pmatrix},
\]
\[
\tilde{A} = \begin{pmatrix}
1 & 0 & 0 \\
0 & B \\
\end{pmatrix}
\begin{pmatrix}
0 & -C^* \\
C & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & B^* \\
\end{pmatrix}.
\]

**Proof.** The positive-definiteness condition \([\text{1}]\) carries over to the new material law operator in the following way. If
\[
\nu M(0) + \text{sym} \left( M'(0) \right) \geq c_\nu > 0
\]
for all \( \nu \in [\nu_0, \infty[ \) and some \( \nu_0 \in ]0, \infty[ \), then
\[
\nu \tilde{M}(0) + \text{sym} \left( \tilde{M}'(0) \right) = \nu \begin{pmatrix}
1 & 0 & 0 \\
0 & B \\
\end{pmatrix}
M(0) \begin{pmatrix}
1 & 0 \\
0 & B^* \\
\end{pmatrix} +
+ \text{sym} \left( \begin{pmatrix}
1 & 0 & 0 \\
0 & B \\
\end{pmatrix}
M'(0) \begin{pmatrix}
1 & 0 \\
0 & B^* \\
\end{pmatrix} \right).
\]
and we estimate for \((V_0, V_1) \in H \oplus B[Y]\)

\[
\nu \left< \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} M(0) \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right>_{H \oplus B[Y]} + \\
\nu \left< \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} \text{sym}(M'(0)) \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right>_{H \oplus B[Y]} = \\
= \nu \left< \begin{pmatrix} 1 & 0 \\ B^* & V_1 \end{pmatrix} | M(0) \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right>_{H \oplus Y} + \\
\geq c_\nu \left< \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} | \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right>_{H \oplus Y} + \\
\geq \bar{c}_\nu \left< \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} | \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right>_{H \oplus B[Y]}
\]

Indeed, since by the closed range assumption \(B[Y]\) and \(B^*[X]\) are Hilbert spaces and by the closed graph theorem the operator

\[
\begin{pmatrix} 1 & 0 \\ B^* \iota_{B[Y]} \end{pmatrix} : H \oplus B[Y] \rightarrow H \oplus B^*[X]
\]

\[
\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \mapsto \begin{pmatrix} V_0 \\ B^* V_1 \end{pmatrix}
\]

has a continuous inverse, we have

\[
\left< \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right>_{H \oplus B[Y]} = \\
= \left< \begin{pmatrix} 1 & 0 \\ 0 & B^* \iota_{B[Y]} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right>_{H \oplus B[Y]} \\
\leq \left< \begin{pmatrix} 1 & 0 \\ 0 & B^* \iota_{B[Y]} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right>_{H \oplus Y}
\]

and so we may choose

\[
\bar{c}_\nu = c_\nu \left< \begin{pmatrix} 1 & 0 \\ 0 & B^* \iota_{B[Y]} \end{pmatrix}^{-1} \right>^{-2}
\]

to confirm that

\[
\nu \tilde{M}(0) + \text{sym}(\tilde{M}'(0)) \geq \bar{c}_\nu > 0
\]

for all \(\nu \in [\nu_0, \infty]\) and some \(\nu_0 \in ]0, \infty[\). \(\square\)

As a particular instance of this construction we can take \(B\) specifically as \(i_S^*\), where \(\iota_S : S \rightarrow H, x \mapsto x\), is the canonical embedding of the closed subspace \(S\) in \(H\). Then

\[
\begin{pmatrix} 1 & 0 \\ 0 & \iota_S^* \end{pmatrix} \begin{pmatrix} 0 & -C \\ C^* & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \iota_S \end{pmatrix} = \begin{pmatrix} 0 & -C \iota_S \\ \iota_S^* C^* & 0 \end{pmatrix}
\]

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is skew-selfadjoint if $C : D(C) \cap S \subseteq S \to Y$, the restriction of $C : D(C) \subseteq H \to Y$ to the closed subspace $S \subseteq H$ is densely defined in $S$. This is the construction we shall employ to approach our specific problem. First we observe that both physical regimes do indeed have the same mother.

### 4.2 Two Descendants of Non-symmetric Elasticity

As a convenient mother to start from we take the theory of non-symmetric elasticity, W. Nowacki, [8, 9], leading to an evo-system of the form

$$
\left( \partial_0 M_0 + M_1 + \begin{pmatrix} 0 & -\text{div} \\ -\text{grad} & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ T \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.
$$

We shall now discuss two particular descendants.

1. Classical symmetric elasticity theory can be considered as a descendant of the form

$$
\left( \partial_0 \begin{pmatrix} 1 & 0 \\ 0 & \iota_{\text{sym}} \end{pmatrix} \right) M_0 \begin{pmatrix} 1 & 0 \\ 0 & \iota_{\text{sym}} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \iota_{\text{sym}} \end{pmatrix} M_1 \begin{pmatrix} 1 & 0 \\ 0 & \iota_{\text{sym}} \end{pmatrix} +
$$

$$
+ \left( \begin{pmatrix} 0 & -\text{div} \\ -\text{Grad} & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ T_{\text{sym}} \end{pmatrix} = \begin{pmatrix} f \\ g_{\text{sym}} \end{pmatrix},
$$

where

$$\text{Grad} := \iota_{\text{sym}} \text{grad}$$

and

$$\text{Div} := \text{div}_{t\text{sym}}.$$

Note that the assumptions of Theorem 4.2 are clearly satisfied since smooth elements with compact support are already a dense subdomain of $\text{div}_{t\text{sym}}$. In the classical situation, which we shall assume for simplicity, we have $M_1 = 0$ and

$$M_0 = \begin{pmatrix} \theta & 0 \\ 0 & C^{-1} \end{pmatrix}.$$

2. Maxwell’s equation are obtained in a sense by the opposite construction.

If we denote analogously

$$\text{skew} : L^2 (\Omega, \mathbb{R}^{3 \times 3}) \to L^2 (\Omega, \mathbb{R}^{3 \times 3}),$$

$$W \mapsto \frac{1}{2} (W - W^*),$$

then with

$$t_{\text{skew}} : L^2 (\Omega, \text{skew} \mathbb{R}^{3 \times 3}) \to L^2 (\Omega, \mathbb{R}^{3 \times 3}),$$

$$T \mapsto T,$$
denoting the canonical embedding of $L^2 (\Omega, \text{ skew } [\mathbb{R}^{3 \times 3}])$ in $L^2 (\Omega, \text{ skew } [\mathbb{R}^{3 \times 3}])$ we find

\[ \iota^* : L^2 (\Omega, \text{ skew } [\mathbb{R}^{3 \times 3}]) \to L^2 (\Omega, \text{ skew } [\mathbb{R}^{3 \times 3}]) \]

\[ W \mapsto \text{ skew } W. \]

With this we may now construct the Maxwell evo-system as

\[
\begin{pmatrix}
\partial_0 \\
0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -\sqrt{2} I^* \iota^* \text{ skew } I
\end{pmatrix}
M_0
\begin{pmatrix}
1 & 0 & -\sqrt{2} I^* \iota^* \text{ skew } I
\end{pmatrix}
\]

\[ + \begin{pmatrix}
0 & -\text{ curl } \\
\text{ curl } & 0
\end{pmatrix}
\begin{pmatrix}
E \\
H
\end{pmatrix} = \left( -I^* g_{\text{ skew}} \right), \]

where

\[ I : \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -\alpha_3 & \alpha_2 \\
\alpha_3 & 0 & -\alpha_1 \\
-\alpha_2 & \alpha_1 & 0
\end{pmatrix}
\]

is a unitary transformation and so is its inverse

\[ I^* : \begin{pmatrix}
0 & -\alpha_3 & \alpha_2 \\
\alpha_3 & 0 & -\alpha_1 \\
-\alpha_2 & \alpha_1 & 0
\end{pmatrix} \mapsto \sqrt{2} \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix} \]

Again, for simplicity we focus on the classical choice of (6). We calculate

\[ I^* \iota^* \text{ skew } \text{ grad } v = \]

\[ = \frac{1}{2} I^* \begin{pmatrix}
0 & \partial_2 v_1 - \partial_1 v_2 \\
\partial_1 v_1 - \partial_2 v_3 & 0 \\
\partial_3 v_1 - \partial_1 v_3 & \partial_2 v_3 - \partial_3 v_2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\partial_2 v_1 - \partial_1 v_2 \\
\partial_1 v_1 - \partial_2 v_3 \\
\partial_3 v_1 - \partial_1 v_3
\end{pmatrix} = \sqrt{2} \begin{pmatrix}
\partial_2 v_1 - \partial_1 v_2 \\
\partial_1 v_1 - \partial_2 v_3 \\
\partial_3 v_1 - \partial_1 v_3
\end{pmatrix}
\]

and also confirm that

\[ \text{ div } \iota_{\text{ skew } I} = -\frac{1}{\sqrt{2}} \text{ curl } v. \]

In other terms, we have the congruence to a descendant

\[ = \begin{pmatrix}
1 & 0 \\
0 & -\sqrt{2} I^*
\end{pmatrix} \begin{pmatrix}
0 & -\text{ curl } \\
\text{ curl } & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & -\sqrt{2} I
\end{pmatrix}, \]

where we have used that
Note that again smooth elements with compact support are a dense sub-domain of $\text{div}_\text{skew}$ and so the assumptions of Theorem 4.2 are clearly satisfied. Motivated by the observation that Maxwell’s equations and the (symmetric) elasto-dynamic equations are both descendants from the asymmetric elasto-dynamics equations of Novacki, [8, 9], we will now discuss boundary interactions between both systems.

### 4.3 An Application to Interface Coupling

Motivated by a paper of F. Cakoni & G.C. Hsiao, [1], where the time-harmonic isotropic homogeneous case of electro-dynamics and elasticity, respectively, is studied via transmission conditions across a separating interface, we consider the corresponding time-dependent case. We assume $\Omega_0 \cup \Omega_1 \subseteq \Omega$, such that the orthogonal decompositions

\[
L^2(\Omega, \mathbb{R}^{3 \times 3}) = L^2(\Omega_0, \mathbb{R}^{3 \times 3}) \oplus L^2(\Omega_1, \mathbb{R}^{3 \times 3})
\]

\[
L^2(\Omega, \mathbb{R}^3) = L^2(\Omega_0, \mathbb{R}^3) \oplus L^2(\Omega_1, \mathbb{R}^3)
\]

hold, and let $I_0 := \left( \begin{array}{c} \iota_{L^2(\Omega_0, \text{sym}[\mathbb{R}^{3 \times 3}])} \\ -\sqrt{2} \iota_{L^2(\Omega_1, \text{skew}[\mathbb{R}^{3 \times 3}])} \end{array} \right)$, i.e.

\[
I_0 \left( \begin{array}{c} S \\ v \end{array} \right) = \iota_{L^2(\Omega_0, \text{sym}[\mathbb{R}^{3 \times 3}])} S - \sqrt{2} \iota_{L^2(\Omega_1, \text{skew}[\mathbb{R}^{3 \times 3}])} v
\]

with the respective canonical embeddings into $L^2(\Omega, \mathbb{R}^{3 \times 3})$. Then

\[
I_0^* : L^2(\Omega, \mathbb{R}^{3 \times 3}) \rightarrow L^2(\Omega, \text{sym}[\mathbb{R}^{3 \times 3}]) \oplus L^2(\Omega, \mathbb{R}^3),
\]

\[
T \mapsto \left( \begin{array}{c} \iota_{L^2(\Omega_0, \text{sym}[\mathbb{R}^{3 \times 3}])}^* T \\ -\sqrt{2} \iota_{L^2(\Omega_1, \text{skew}[\mathbb{R}^{3 \times 3}])}^* T \end{array} \right),
\]

and so

\[
I_0^* = \left( \begin{array}{c} \iota_{L^2(\Omega_0, \text{sym}[\mathbb{R}^{3 \times 3}])}^* \\ -\sqrt{2} \iota_{L^2(\Omega_1, \text{skew}[\mathbb{R}^{3 \times 3}])}^* \end{array} \right).
\]

With this we get a congruence to a descendant construction as

\[
A = \left( \begin{array}{cc} 1 & 0 \\ 0 & I_0^* \end{array} \right) \left( \begin{array}{cc} 0 & -\text{div} \\ -\text{grad} & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & I_0 \end{array} \right)
\]

\[
\subseteq \left( \begin{array}{cc} 0 & -\text{Div}_{\Omega_0} \\ -\text{Grad}_{\Omega_0} & 0 \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)
\]

and

\[
M(0) = \left( \begin{array}{cc} \vartheta_{*,\Omega_0} + \varepsilon_{\Omega_1} & 0 \\ 0 & C_{\Omega_0}^{-1} \end{array} \right)
\]

\[
M'(0) = \left( \begin{array}{cc} \sigma_{\Omega_1} & 0 \\ 0 & 0 \end{array} \right)
\]

and

\[
M'(0) = \left( \begin{array}{cc} \sigma_{\Omega_1} & 0 \\ 0 & 0 \end{array} \right).
\]
The indexes $\Omega_k$, $k = 0, 1$, are used to denote the respective supports of the quantities. The coefficients are – as a matter of simplification labeled in the same meaning as in (6) and (8), just with the support information added. The unknowns are now

$$
\begin{pmatrix}
  v_{\Omega_0} + E_{\Omega_1} \\
  T_{\Omega_0} \\
  H_{\Omega_1}
\end{pmatrix}
\in H = L^2(\Omega, \mathbb{R}^3) \oplus L^2(\Omega_0, \text{sym} [\mathbb{R}^{3 \times 3}]) \oplus L^2(\Omega_1, \mathbb{R}^3),
$$

where the first component is to be understood in the sense of (10). Note that the assumptions of Theorem 4.2 are clearly satisfied since smooth elements with compact support in $\Omega_0$ and $\Omega_1$, respectively, are already a dense sub-domain as in the separate cases of Subsection 4.2. From the inclusion (11), (12), we read off that the resulting evo-system

$$
(\partial_0 M (0) + M' (0) + A)
\begin{pmatrix}
  v_{\Omega_0} + E_{\Omega_1} \\
  T_{\Omega_0} \\
  H_{\Omega_1}
\end{pmatrix}
= \begin{pmatrix}
  f_{\Omega_0} - \text{ext.} \Omega_1 \\
  g_{\text{sym.} \Omega_0} \\
  k_{\text{ext.} \Omega_1}
\end{pmatrix}
$$

indeed yields

$$
\partial_0 (\varphi_+, \Omega_0 + \varepsilon \Omega_1) (v_{\Omega_0} + E_{\Omega_1}) - \text{Div}_{\Omega_0} T_{\Omega_0} - \text{curl}_{\Omega_1} H_{\Omega_1} = f_{\Omega_0} - \text{ext.} \Omega_1,
$$

which in turn splits into

$$
\begin{align*}
\partial_0 \varphi_+ \Omega_0 v_{\Omega_0} - \text{Div}_{\Omega_0} T_{\Omega_0} &= f_{\Omega_0}, \\
\partial_0 \varepsilon \Omega_1 E_{\Omega_1} - \text{curl}_{\Omega_1} H_{\Omega_1} &= - \text{ext.} \Omega_1.
\end{align*}
$$

The second block row yields another pair of equations

$$
\begin{align*}
\partial_0 C^{-1} T_{\Omega_0} - \text{Grad} v_{\Omega_0} &= g_{\text{sym.} \Omega_0}, \\
\partial_0 \mu_{\Omega_1} H_{\Omega_1} + \text{curl} E_{\Omega_1} &= k_{\text{ext.} \Omega_1}.
\end{align*}
$$

The actual system models now natural transmission conditions on the common boundary part $\Omega_0 \cap \Omega_1$ and the homogeneous Dirichlet boundary condition on $\Omega_0 \setminus \Omega_1$ and the standard homogeneous electric boundary condition on $\Omega_1 \setminus \Omega_0$ without assuming any smoothness of the boundary.

On the contrary, assuming sufficient regularity of the boundary one can see that the model yields a generalization of the classical transmission conditions on $\Omega_0 \cap \Omega_1$:

$$
\begin{align*}
T_{\Omega_0} n &= n \times H_{\Omega_1}, \\
n \times v_{\Omega_0} &= n \times E_{\Omega_1},
\end{align*}
$$

with for example

$$
m_{11} = \begin{pmatrix}
  \ast_{\text{sym.} \Omega_1} & \ast_{\text{skew.} \Omega_0} & \ast_{\text{skew.} \Omega_0} & \ast_{\text{skew.} \Omega_1}
\end{pmatrix}
$$

Then the described mother-descendant mechanism would lead to a descendant, which in turn would be congruent to the described interface system.

---

7Although we consider for convenience and physical relevance this evo-system in its own right, a formal mother material law – without physical meaning – could be easily given:

$$
\begin{pmatrix}
  \varphi_+ \Omega_0 + \varepsilon \Omega_1 + \partial_0 \sigma \Omega_1 & 0 \\
  0 & m_{11}
\end{pmatrix}
$$

with for example

$$
m_{11} = \begin{pmatrix}
  \ast_{\text{sym.} \Omega_1} & \ast_{\text{sym.} \Omega_1} & \ast_{\text{skew.} \Omega_0} & \ast_{\text{skew.} \Omega_1} & \ast_{\text{sym.} \Omega_1} & \ast_{\text{skew.} \Omega_1}
\end{pmatrix}
$$

Then the described mother-descendant mechanism would lead to a descendant, which in turn would be congruent to the described interface system.
where \( n \) is a smooth unit normal field on \( \tilde{\Omega}_0 \cap \tilde{\Omega}_1 \). Indeed, with
\[
\begin{pmatrix}
v_{\Omega_0} + E_{\Omega_1} \\
T_{\Omega_0} \\
H_{\Omega_1}
\end{pmatrix} \in D(A)
\]
we have (noting for the smooth exterior unit normal vector fields \( n_{\Omega_0}, n_{\Omega_1} \) on the boundaries of \( \Omega_0 \) and \( \Omega_1 \), respectively, that \( n_{\Omega_0} = -n_{\Omega_1} \) on \( \tilde{\Omega}_0 \cap \tilde{\Omega}_1 \) with
\[
\tilde{A} = \begin{pmatrix}
0 & (\text{Div}_{\Omega_0} - \text{curl}_{\Omega_1}) \\
-\text{Grad}_{\Omega_0} & 0 \\
\text{curl}_{\Omega_0} & 0
\end{pmatrix},
\]
that
\[
0 = \left\langle \begin{pmatrix} v_{\Omega_0} + E_{\Omega_1} \\
T_{\Omega_0} \\
H_{\Omega_1}
\end{pmatrix} \mid \tilde{A} \begin{pmatrix} v_{\Omega_0} + E_{\Omega_1} \\
T_{\Omega_0} \\
H_{\Omega_1}
\end{pmatrix} \right\rangle_H
= -\left\langle v_{\Omega_0} \mid \text{Div} \, T_{\Omega_0} \right\rangle_{L^2(\Omega_0, \mathbb{R}^3)} - \left\langle T_{\Omega_0} \mid \text{Grad}_{\Omega_0} v_{\Omega_0} \right\rangle_{L^2(\Omega_0, \mathbb{R}^3)} +
+ \left\langle H_{\Omega_1} \mid \text{curl}_{\Omega_1} E_{\Omega_1} \right\rangle_{L^2(\Omega_1, \mathbb{R}^3)} - \left\langle E_{\Omega_1} \mid \text{curl}_{\Omega_1} H_{\Omega_1} \right\rangle_{L^2(\Omega_1, \mathbb{R}^3)}
= -\int_{\tilde{\Omega}_0 \cap \tilde{\Omega}_1} v_{\Omega_0}^T T_{\Omega_0} n_{\Omega_0} \, do + \int_{\tilde{\Omega}_0 \cap \tilde{\Omega}_1} n_{\Omega_0}^T (E_{\Omega_1} \times H_{\Omega_1}) \, do
= -\int_{\tilde{\Omega}_0 \cap \tilde{\Omega}_1} v_{\Omega_0}^T T_{\Omega_0} n_{\Omega_0} \, do + \int_{\tilde{\Omega}_0 \cap \tilde{\Omega}_1} E_{\Omega_1}^T (n_{\Omega_0} \times H_{\Omega_1}) \, do.
\]
Since \( (v_{\Omega_0} + E_{\Omega_1}) \in D(\text{grad}) \) is by construction admissible we may choose \( v_{\Omega_0} = E_{\Omega_1} \) on the interface and conclude that
\[
T_{\Omega_0} n_{\Omega_0} = n_{\Omega_0} \times H_{\Omega_1} \quad (17)
\]
is a needed transmission condition. In particular, we see
\[
n_{\Omega_0}^T T_{\Omega_0} n_{\Omega_0} = 0.
\]
Inserting the explicit transmission condition (17) now yields
\[
-\int_{\tilde{\Omega}_0 \cap \tilde{\Omega}_1} (n_{\Omega_0} \times (n_{\Omega_0} \times (v_{\Omega_0} - E_{\Omega_1})))^T (n_{\Omega_0} \times H_{\Omega_1}) \, do =
= \int_{\tilde{\Omega}_0 \cap \tilde{\Omega}_1} (v_{\Omega_0} - E_{\Omega_1})^T (n_{\Omega_0} \times H_{\Omega_1}) \, do = 0,
\]
which, with \( n_{\Omega_0} \times H_{\Omega_1} \) for \( H_{\Omega_1} \in D(\text{curl}_{\Omega_1}) \) being sufficiently arbitrary, now implies
\[
n_{\Omega_0} \cdot v_{\Omega_0} = n_{\Omega_0} \times E_{\Omega_1}.
\]
i.e. the continuity of the tangential components

\[ v_{\Omega_0, t} = E_{\Omega_1, t}, \]

as a complementing transmission condition. These more or less heuristic considerations motivate to take the above evo-system as an appropriate generalization to cases, where the boundary does not have a reasonable normal vector field.

All in all, we summarize our findings in the following well-posedness result.

**Theorem 4.3.** The evo-system (15) is well-posed if \( q \cdot \alpha_0, C_{\Omega_0} \text{ and } \varepsilon_{\Omega_1}, \mu_{\Omega_1} \) are self-adjoint, non-negative, continuous operators on \( L^2(\Omega_0, \mathbb{R}^3) \), \( L^2(\Omega_0, \text{sym} [\mathbb{R}^{3 \times 3}]) \) and on \( L^2(\Omega_1, \mathbb{R}^3) \), respectively, \( \sigma_{\Omega_1} \) is continuous and linear on \( L^2(\Omega_1, \mathbb{R}^3) \) and such that

\[ q \cdot \alpha_0, C_{\Omega_0}, \mu_{\Omega_1} \geq \eta_0 > 0, \]

as well as

\[ \nu \varepsilon_{\Omega_1} + \text{sym}(\sigma_{\Omega_1}) \geq \eta_0 > 0 \]

for some real number \( \eta_0 \) and all sufficiently large \( \nu \).

**Remark 4.4.**

1. If we formally transcribe the time-harmonic case into its time-dependent form, the transmission conditions of (1) are actually

\[ T_{\Omega_0} n = n \times \partial_0^{-1} H_{\Omega_1}, \]

\[ n \times \partial_0^{-1} v_{\Omega_0} = n \times E_{\Omega_1}. \]

Although these obviously differ from (16), we give preference to our choice above for several reasons. For one, the energy balance requirement of (1 formula (5)), which reads as

\[ v_{\Omega_0}^\top T_{\Omega_0} n = n^\top (H_{\Omega_1} \times E_{\Omega_1}), \]

is satisfied by (16) but not by (18). With the latter transmission conditions we obtain instead

\[ v_{\Omega_0}^\top T_{\Omega_0} n = (\partial_0 E_{\Omega_1})^\top (n \times (\partial_0^{-1} H_{\Omega_1})) = n^\top ((\partial_0^{-1} H_{\Omega_1}) \times (\partial_0 E_{\Omega_1})). \]

The problem seems to be that the difference to (19) becomes unnoticeable in the formal time-harmonic transcription of (1), since there \( \partial_0 \) is formally replaced by \( i \omega \sqrt{\varepsilon_0 \mu_0} \) and so algebraic cancellation essentially makes the product rule for differentiation disappear, erroneously suggesting that the energy balance is satisfied.

2. In the notation above, (11), (13), (14), if \( M(0) \) is already strictly positive definite, we can construct a fundamental solution as a small perturbation of the fundamental solution of \( \partial_0 + \sqrt{M(0)^{-1}} A \sqrt{M(0)^{-1}} \), which in turn is obtained from the unitary group

\[ \left( \exp \left( -t \sqrt{M(0)^{-1}} A \sqrt{M(0)^{-1}} \right) \right)_{t \in \mathbb{R}} \]

\[ ^8 \text{The correct energy balance in the time-harmonic case would actually involve temporal convolution products.} \]
by cut-off as
\[
\left( \chi_{[0, \infty]}(t) \exp \left( -t \sqrt{M(0)^{-1} A \sqrt{M(0)^{-1}}} \right) \right)_{t \in \mathbb{R}}.
\]
The restriction of the fundamental solution to \([0, \infty[\) yields the family
\[
\left( \exp \left( -t \sqrt{M(0)^{-1} A \sqrt{M(0)^{-1}}} \right) \right)_{t \in [0, \infty[}
\]
commonly referred to as the associated one-parameter semi-group.

In general, however, a fundamental solution may be complicated or impossible to construct.

3. We note that beyond eddy current type behavior, which is actually a change of type situation from hyperbolic to parabolic, and beyond the possibility of including for example piezo-electric effects via a more complex material law, we may actually allow for completely general rational material laws as long as condition (4) is warranted.

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