I. INTRODUCTION

Ground based gravitational wave detector networks (notably LIGO \cite{LIGO} and Virgo \cite{Virgo}) are sensitive to the relatively well understood signal from the lowest-mass compact binaries $M = m_1 + m_2 \leq 16 M_\odot$ \cite{LIGO}. Strong signals permit high-precision constraints on binary parameters, particularly when the binary precesses. Precession arises only from spin-orbit misalignment; occurs on a distinctive timescale between the inspiral and orbit; and produces distinctive polarization and phase modulations \cite{Precession}. As a result, the complicated gravitational wave signal from precessing binaries is unusually rich, allowing high-precision constraints on multiple parameters, notably the (misaligned) spin \cite{Spin}. Measurements of the spin orientations alone could provide insight into processes that misalign spins and orbits, such as supernova kicks \cite{Supernova}, or realign them, such as tides and post-Newtonian resonances \cite{Resonances}. More broadly, gravitational waves constrain the pre-merger orbital plane and total angular momentum direction, both of which may correlate with the presence, beaming, and light curve \cite{Beaming} of any post-merger ultrarelativistic blastwave (e.g., short GRB) \cite{ShortGRB}. Moreover, spin-orbit coupling strongly influences orbital decay and hence the overall gravitational wave phase: the accuracy with which most other parameters can be determined is limited by knowledge of BH spins \cite{Spin}. Precession is known to break this degeneracy \cite{Precession}. In sum, the rich gravitational waves emitted from a precessing binary allow higher-precision measurements of individual neutron star masses, black hole masses, and black hole spins, enabling constraints on their distribution across multiple events. In conjunction with electromagnetic measurements, the complexity of a fully precessing gravitational wave signal may enable correlated electromagnetic and gravitational wave measurements to much more tightly constrain the central engine of short gamma ray bursts.

Interpreting gravitational wave data requires systematically comparing all possible candidate signals to the data, constructing a Bayesian posterior probability distribution for candidate binary parameters \cite{Bayesian}. Owing to the complexity and multimodality of these posteriors, successful strategies adopt two elements: a
well-tested generic algorithm for parameter estimation, such as variants of Markov Chain Monte Carlo or nested sampling; and deep insight into the structure of possible gravitational wave signals, to ensure efficient and complete coverage of all possible options \cite{22,23}. Owing to both the relatively large number of parameters needed to specify a precessing binary’s orbit and to the seemingly-complicated evolution, Bayesian parameter estimation methods have only recently been able to efficiently draw inferences about gravitational waves from precessing sources \cite{23}. These improvements mirror and draw upon a greater theoretical appreciation of the surprisingly simple dynamics and gravitational waves from precessing binaries, both in the post-Newtonian limit \cite{16,17,44,45,46} and strong field \cite{17,52}. For our purposes, these insights have suggested particularly well-adapted coordinates with which to express the dynamics and gravitational waves from precessing BH-NS binaries, enabling more efficient and easily understood calculations. In particular, these coordinates have been previously applied to estimate how well BH-NS parameters can be measured by ground-based detectors \cite{18}. In this work, we will present the first detailed parameter estimation calculations which fully benefit from these insights into precessing dynamics. In short, we will review the natural parameters to describe the gravitational wave signal; demonstrate how well they can be measured, for a handful of selected examples; and interpret our posteriors using simple, easily-generalized analytic and geometric arguments.

As a concrete objective, following prior work \cite{18,27} we will explore whether higher harmonics break degeneracies and provide additional information about black hole-neutron star binaries. In the absence of precession, higher harmonics are known to break degeneracies and improve sky localization, particularly for LISA \cite{30,32}. That said, these and other studies also suggest that higher harmonics provide relatively little additional information about generic precessing binaries, over and above the leading-order quadrupole radiation \cite{18,32}. For example, for two fiducial nonprecessing and two fiducial precessing signals, Cho et al. \cite{15}, henceforth denoted COOKL, provide concrete predictions for how well detailed parameter estimation strategies should perform, for a specific waveform model. A previous work \cite{27}, henceforth denoted OFOCKL, demonstrated these simple predictions accurately reproduced the results of detailed parameter estimation strategies. In this work, we report on detailed parameter estimation for the two fiducial precessing signals described in COOKL. As with nonprecessing binaries, we find higher harmonics seem to provide significant insight into geometric parameters, in this case the projection of the orbital angular momentum direction on the plane of the sky. As this orientation could conceivably correlate with properties of associated electromagnetic counterparts, higher harmonics may have a nontrivial role in the interpretation of coordinated electromagnetic and gravitational wave observations.

This paper is organized as follows. In Section II we describe the gravitational wave signal from precessing BH-NS binaries, emphasizing suitable coordinates for the spins (i.e., defined at 100 Hz, relative to the total angular momentum direction) and the waveform (i.e., exploiting the corotating frame to decompose the signal into three timescales: orbit, precession, and inspiral). Our description of gravitational waves from precessing BH-NS binaries follows Brown et al. \cite{15}, henceforth denoted BLO, and \cite{16}, henceforth denoted LO. Next, in Section III we describe how we created synthetic data consistent with the two fiducial precessing signals described in COOKL in gaussian noise; reconstructed a best estimate (“posterior distribution”) for the possible precessing source parameters consistent with that signal; and compared those predictions with semianalytic estimates. These semianalytic estimates generalize work by COOKL, approximating the full response of a multidetector network with a simpler but more easily understood expression. Using simple analytic arguments, we describe how to reproduce our full numerical and semianalytic results using a simple separation of scales and physics: orbital cycles, precession cycles, and geometry. The success of these arguments can be extrapolated to regimes well outside its limited scope, allowing simple predictions for the performance of precessing parameter estimation. We conclude in Section IV.

For the benefit of experts, in Appendix A we discuss the numerical stability and separability of our effective Fisher matrix.

II. KINEMATICS AND GRAVITATIONAL WAVES FROM PRECESSING BH-NS BINARIES

A. Kinematics and dynamics of precessing binaries

The kinematics of precessing binaries are well described in \cite{17, BLO, LO}; see, e.g., Eq. (10) in BLO. In brief, the orbit contracts in the instantaneous orbital plane on a long timescale \(1/\Omega_{\text{rad}}\) over many orbital periods \(1/\Omega_\phi\). On an intermediate timescale \(1/\Omega_{\text{prec}}\), due to spin-orbit coupling the angular momenta precess around the total angular momentum direction, which remains nearly constant. On timescales \(1/\Omega_{\text{prec}}\) between \(1/\Omega_\phi\) and \(1/\Omega_{\text{rad}}\), the orbital angular momentum traces out a “precession cone”. For this reason, we adopt coordinates at 100 Hz which describe the orientation of all angular momenta relative to \(\hat{J}\); our coordinates are identical to those used in BLO, LO, and COOKL. Relative to a frame with \(\hat{z}\) oriented along the line of sight, the total, orbital,
TABLE I: Fiducial source parameters for precessing binaries. We adopt two fiducial binaries A, C similar to those used in COOKL. All parameters are specified when twice the orbital frequency is 100 Hz. The post-Newtonian signals used in the text terminate at an orbital $f_{\text{MECO}}/2$, where $f_{\text{MECO}}$ is the smaller of the “minimum energy circular orbit” (hence the acronym) and the frequency at which $\dot{\omega} < 0$; the values shown are derived from the same lalsimulation output used in our simulations, estimated from data evaluated at a 32 kHz sampling rate. Comparing with model waveforms that include inspiral, merger, and ringdown, we anticipate this abrupt termination causes relatively little mismatch between our model and the physical signal; see OFOCKL.

![Coordinate system for the precessing binary.](image)

**FIG. 1:** Coordinate system for the precessing binary. The left coordinate corresponds to the conventional GW radiation frame. $\theta_{JN}$ ($\phi_{JN}$) is a polar (azimuthal) angle of the total angular momentum ($J$) with respect to the radiation vector ($N$). In the right coordinate $\beta_{JL}$ ($\alpha_{JL}$) is a polar (azimuthal) angle of the orbital angular momentum ($L$) with respect to the total angular momentum ($J$). In the right coordinate, $N$, $J$, and $x'$ are coplanar and the shaded region indicates the orbital plane.

| Name | $m_1$ | $m_2$ | $\chi_1$ | $\psi_J$ | $\psi_L$ | $\iota$ | $\beta_{JL}$ | $\theta_{JN}$ | $\alpha_{JL}$ | $\theta_{LS}$ | $\phi_{rad}$ | $f_{\text{MECO}}$ | Source location |
|------|-------|-------|----------|----------|----------|---------|-------------|-------------|-------------|-------------|--------------|----------------|----------------|
| A    | 10    | 1.4   | 1.0      | 1.25     | 0.72     | 1.512   | $\pi/4$    | 0.730       | 2.95        | 1.176       | 0.93         | 810 Hz        | (x, y, z)      |
| C    | 10    | 1.4   | 1.0      | 1.09     | 2.23     | 0.411   | $\pi/4$    | 0.891       | 5.75        | 1.176       | 1.65         | 810 Hz        | (x', y', z')   |

TABLE II: Source location: Source geocenter event time and sky location. For a sense of scale, this table also provides the time differences between different detector sites, implied by that sky location and event time.

| $d$ | $t$ | DEC | RA | $\Delta t_{LH}$ | $\Delta t_{VH}$ |
|-----|-----|-----|----|----------------|-----------------|
| Mpc | s   | ms  | ms |                |                 |
| 23.1| 894.38 | 679.0 | 0.5747 | 0.6485 | -3.93 | 5.98 |

and spin angular momenta are described by the vectors:

1. $\hat{J} = \sin \theta_{JN} \cos \psi_J \hat{x} + \sin \theta_{JN} \sin \psi_J \hat{y} + \cos \theta_{JN} \hat{z}$  
2. $\hat{L} = \sin \iota \cos \psi_L \hat{x} + \cos \iota \sin \psi_L \hat{y} + \cos \iota \hat{z}$
3. $\hat{S}_1 = \sin \theta_1 \cos (\psi_L + \phi_1) \hat{x} + \sin \theta_1 \sin (\psi_L + \phi_1) \hat{y} + \cos \theta_1 \hat{z}$

where in this and subsequent expressions we restrict to a binary with a single spin (i.e., $\tilde{S}_2 = 0$). Because the orbital angular momentum evolves along a cone, precessing around $J$, we prefer to describe the orbital and spin angular momenta in frame aligned with the total angular momentum $\hat{z}' = \hat{J}$:

$\hat{L} = \sin \beta_{JL} \cos \alpha_{JL} \hat{x}' + \sin \beta_{JL} \sin \alpha_{JL} \hat{y}' + \cos \beta_{JL} \hat{z}'$  

where the frame is defined so $\hat{y}'$ is perpendicular to $\hat{N}$ as in Figure 1.[6]

$\hat{y}' = -\frac{\hat{N} \times \hat{J}}{|\hat{N} \times \hat{J}|} \cdot \hat{z}' = \hat{y}' \times \hat{J} = \frac{\hat{N} - \hat{J} \cdot \hat{N}}{|\hat{N} \times \hat{J}|}$

In this phase convention for $\alpha_{JL}$, the zero of $\alpha_{JL}$ is one of the two points when $\hat{L}, \hat{J}, \hat{N}$ are all in a common plane, sharing a common direction in the plane of the sky. Transforming between these two representations for $\hat{L}$ is straightforward. For example, given $\hat{N}, \hat{L}$ and $\hat{J}$, we identify $\alpha$ and $\beta_{JL}$ via

$\beta_{JL} = \cos^{-1} \frac{\hat{J} \cdot \hat{L}}{}$  
$\alpha_{JL} = \text{arg} \left[ \frac{\hat{L} \times (\hat{x}' + i\hat{y}')}{} \right]$  

The spin angular momentum direction is determined from the direction of $\hat{L}$, the direction of $\hat{J}$, and the angle $\theta_{LS}$ between $\hat{S}_1$ and $\hat{L}$:

$\hat{S}_1 = \sin (\beta_{JL} - \theta_{LS}) \cos \alpha_{JL} \hat{x}' + \sin (\beta_{JL} - \theta_{LS}) \sin \alpha_{JL} \hat{y}'$  
$\cos (\beta_{JL} - \theta_{LS}) \hat{z}'$  

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1 Strictly speaking, the total angular momentum $\vec{J}$ precesses [17]. For the results described in this work, we always adopt the total angular momentum direction evaluated at $f = 100$ Hz.
Using this parameter, the opening angle $\beta_{JL}$ of the precession cone (denoted $\lambda_L$ in ACST) can be expressed trigonometrically as

$$\beta_{JL}(t) \equiv \arccos \hat{J} \cdot \hat{L} = \arccos \frac{1 + \kappa \gamma}{\sqrt{1 + 2\kappa \gamma + \gamma^2}}$$ (10)

where $\kappa = \cos \theta_{LS} = \hat{L} \cdot \hat{S}_1$. Most BH-NS binaries’ angular momenta evolve via simple precession: $\alpha$ increases nearly uniformly on the precession timescale, producing several precession cycles in band [Eq. (9) in BLO], while $\beta_{JL}$ increases slowly on the inspiral timescale, changing opening angle only slightly [Fig. 1 in BLO]; see Figure 2.

As described in OFOCKL, we evolve the angular momenta according to expressions derived from general relativity in the post-Newtonian, adiabatic, orbit-averaged limit, an approximation presented in [17] and described [53]. Though some literature adopts a purely Hamiltonian approach to characterize spin precession [54 [59], this orbit-averaged approach is usually adopted when simulating gravitational waves from precessing binaries [6 [13] 22].

In this work we adopt two fiducial precessing BH-NS binaries, with intrinsic and extrinsic parameters specified in Tables I and II. Figure 2 shows how each binary precesses around the total angular momentum direction $\hat{J}$ and in the plane of the sky. For this mass ratio, the opening angle $\beta_{JL}$ adopted in consistent with randomly-oriented BH spin; see, e.g., Eq. (17) and Fig. 4 of LO. For this sky location, our simplified three-detector gravitational wave network has comparable sensitivity to both linear (or both circular) polarizations.

B. Gravitational waves from precessing binaries

Precession introduces modulations onto the “carrier signal” produced by the secular decay of the orbit over time. BLO and LO provide a compact summary of the associated signal, in the time and frequency domain. In a frame aligned with the total angular momentum, several harmonics $h_{lm}$ are significant:

$$h_+ - i h_\times = \sum_{lm} h_{lm} Y_{lm}^{(-2)}$$ (11)

where the harmonics $h_{lm}$ are provided and described in the literature [13]. By “significant”, we mean that harmonics have nontrivial power $p_{lm}$:

$$p_{lm}^2 = 2 \int_{-\infty}^{\infty} \left| \tilde{h}_{lm} \right|^2 S_h(f)$$ (12)

where $S_h$ is the fiducial initial LIGO design noise power spectrum. These precession-induced modulations are most easily understood in a corotating frame, as in LO [13] [47] [39] [52] [60]:

$$h_{lm} = \sum_{m'} D_{mm'}^l (\alpha_{JL}, \beta_{JL}, \gamma) h_{lm'}^{ROT}$$ (13)
where \( \gamma = - \int d\alpha \cos \beta_{\text{JL}} \) and where \( D^l_{m,m} \) is a Wigner D-matrix. In this expression, \( h^\text{ROT}_{lm} \) is the gravitational wave signal emitted by a binary with instantaneous angular momentum along the \( L \) axis. In the low-velocity limit, \( h^\text{ROT}_{lm} \) is dominated by leading-order radiation and hence by equal-magnitude \((l,m) = (2, \pm 2)\) modes. Due to spin-orbit precession with \( \beta_{\text{JL}} \neq 0 \), however, these harmonics are mixed. When \( \beta_{\text{JL}} \) is greater than tens of degrees, then in the simulation frame all \( h_{lm} \) are generally present and significant.

To illustrate that gravitational wave emission from a precessing binary requires several harmonics \( h_{lm} \) to describe it when \( \beta_{\text{JL}} > 0 \), we evaluate \( \rho_{2m} \), conservatively assuming only the \((2, \pm 2)\) corotating-frame modes are nonzero:

\[
\rho_{2m}^2 \approx \left| \rho_{2,2,2}^{\text{ROT}} \right|^2 |d_{2,m}(\beta_{\text{JL}})|^2 + \left| \rho_{2,2,2}^{\text{ROT}} \right|^2 |d_{2,m}(\beta_{\text{JL}})|^2 - \rho_{2,2}^{\text{ROT}} |d_{2,m}(\beta_{\text{JL}})|^2 + |d_{2,m}(\beta_{\text{JL}})|^2
\]

where we use orthogonality of the corotating-frame \((2, \pm 2)\) modes. Figure 3 shows that except for a small region \( \beta_{\text{JL}} \approx 0 \), several harmonics contribute significantly to the amplitude along generic lines of sight, with \( \rho_{2m}/\rho_{22} \gtrsim 0.1 \). At this level, these harmonics change the signal significantly, both in overall amplitude \((\rho^2/2 + 0.12 \approx \rho^20.03)\) and in fit to candidate data.

Gravitational waves from precessing BH-NS binaries are modulated in amplitude, phase, and polarization. A generic precessing source oscillates between emitting preferentially right-handed and preferentially left-handed radiation along any line of sight; see [47]. For the scenario adopted here, however, the orbital angular momentum almost always preferentially points towards the observer \((L \cdot N \gtrsim 0)\), so gravitational waves emitted along the line of sight are principally right handed for almost all time.

### III. PARAMETER ESTIMATION OF PRECESSING BINARIES

To construct synthetic data containing a signal, to interpret that signal, and to compare interpretations from different simulations to each other and to theory, we adopt the same methods as used in OFOCKL. Specifically, to determine the shape of each posterior, we employ the lalsimulation and lalinference [18, 33] code libraries developed by the LIGO Scientific Collaboration and Virgo collaboration. As in OFOCKL, we adopt a fiducial 3-detector network: initial LIGO and Virgo, with analytic gaussian noise power spectrum provided by their Eqs. (1-2).

In contrast to the simplified, purely single-spin discussion adopted in Section II to describe the kinematics of the physical signal in the data, the model used to interpret the data allows for nonzero, generic spin on both compact objects. That said, because compact object spin scales as the mass squared times the dimensionless spin parameter \((S = m^2 \chi)\), in our high-mass-ratio systems

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FIG. 3: **Harmonic amplitude versus opening angle**:
A plot of \( \rho^2_{2m}/(\rho_{22}^{\text{ROT}})^2 \) predicted by Eq. (14) for \( m = 2 \) (blue), 1 (red), and 0 (yellow).
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Source} & \text{Harmonics} & \text{Seed} & \rho & \rho_{\text{rec}} & \ln Z / \ln V/V_{\text{prior}} & N_{\text{eff}} \\
\hline
A & \text{no} & - & 19.86 & 20.13 & 165. & -37.9 & 10037 \\
A & \text{no} & 1234 & 19.86 & 21.26 & 186. & -40.1 & 10042 \\
A & \text{no} & 56789 & 19.86 & 20.59 & 176. & -36.4 & 10110 \\
C & \text{no} & -^* & 19.12 & 19.39 & 146. & -41.6 & 101600 \\
C & \text{no} & 1234^* & 19.12 & 20.64 & 169. & -43.9 & 105941 \\
C & \text{no} & 56789^* & 19.12 & 18.83 & 138. & -37.9 & 105042 \\
C & \text{with} & 1234^* & 19.73 & 21.05 & 144. & -41.8 & 42701 \\
C & \text{with} & 56789^* & 19.73 & 19.32 & 142. & -41.9 & 9814 \\
\hline
\end{array}
\]

**TABLE III: Simulations used in this work**

Table of distinct simulations performed. The first set of columns indicate the simulated binary, whether higher harmonics were included, and random seed choice used to generate noise (a “-” means no noise was used; the asterisk indicates a different noise and MCMC realization). The two quantities \(\rho_{\text{rec}}\) provide the injected and best-fit total signal amplitude in the network [Eqs. (19) and (22) in OFOCKL]. The latter quantity depends on the noise realization of the network. The columns for \(\ln Z\) and \(\ln V/V_{\text{prior}}\) provide the evidence [Eq. (17) in OFOCKL] and volume fraction [Eq. (17) of OFOCKL]; the evidence, volume fraction, and signal amplitude are related by \(\rho_{\text{rec}}/2 = \ln Z/(V/V_{\text{prior}})\).

The small neutron star’s spin has minimal dynamical impact. Our simulations show gravitational waves provide almost no information about the neutron star’s spin magnitude or direction. For the purposes of simplicity, we will omit further mention of the smaller spin henceforth.

**A. Intrinsic parameters**

As shown in Figure 4, the intrinsic parameters of our relatively loud (\(\rho \simeq 20\)) fiducial binaries are extremely well-constrained. For example, the neutron star’s mass, black hole’s mass, and black hole spin are all relatively well-measured, compared to the accuracy of existing measurements and hypothesized distributions of these parameters [61–63]. Higher harmonics provide relatively little additional information about these parameters.

Applied to an even simpler idealized problem – a similar source known to be directly overhead two orthogonal detectors – the effective fisher matrix procedure of COOKL produces qualitatively similar results, notably reproducing relatively minimal impact from higher harmonics. Given the simplifications adopted, the effective fisher matrix predictions inevitably disagree quantitatively with our detailed Monte Carlo calculations, particularly regarding multidimensional correlations. We nonetheless expect the effective fisher matrix to correctly identify scales and trends in parameter estimation; moreover, being amenable to analysis, this simple construct allows us to develop and validate simple interpretations for why some parameters can be measured as well as they are. As a concrete example, we can explain Figure 4.

The time-dependent orbital phase depends on the black hole spin, principally through the “aligned component” \(\hat{L} \cdot \hat{S}_1\) [28]. As discussed in COOKL and OFOCKL, the “aligned component” cannot be easily distinguished from the mass ratio in the absence of precession. Spin-orbit precession breaks this degeneracy, allowing significantly tighter constraints on the mass ratio of our precessing binary. In our particular example, comparing Fig 3 in OFOCKL to our Figure 4, we can measure \(\eta\) and hence the smaller mass roughly three times more accurately at the same signal amplitude. As seen in the bottom panel of Figure 4 both the spin-orbit misalignment \(\hat{L} \cdot \hat{S}_1\) and spin magnitude \(\chi_1\) remain individually poorly-constrained. As a concrete example, our ability to measure \(\chi_1\) for this precessing binary is comparable to the accuracy possible for a similar nonprecessing binary [OFOCKL].

One correlated combination of \(\hat{L} \cdot \hat{S}_1\) and \(\chi_1\) is well-constrained: the combination that enters into the precession rate. In Figure 4 we show contours of constant precession cone opening angle (\(\beta_{JL}\)) and constant precession rate [LO Eq. (7–8)]

\[
\Omega_p = \frac{|J|}{2r^3} = \eta \left(2 + \frac{3m_2}{2m_1}\right) v^5 \sqrt{1 + 2\kappa \gamma + \gamma^2}
\]

When evaluating these expressions, we estimate \(\gamma \simeq 1.85\chi_1\) [Eq. (9)], so the contours shown correspond to \(\cos \beta_{JL} = 0.65, 0.7, 0.75\) and \(\sqrt{1 + 2\kappa \gamma + \gamma^2} = 2.2, 2.4, 2.6\). As expected, the presence of several precession cycles allows us to relatively tightly constrain the precession rate. Future gravitational wave detectors, being sensitive to longer signals and hence more precession cycles, can be expected to even more tightly constrain this combination. By contrast, as described below, the precession geometry \(\beta_{JL}\) is relatively poorly constrained, with error independent of the number of orbital or precession cycles.\(^2\)

\(^2\) With relatively few precession cycles in our study, the discrepancy between these two measurement accuracies is fairly small. However, when advanced instruments with longer waveforms can probe more precession cycles, we expect this simple argument will explain dominant correlations.
TABLE IV: One-dimensional parameter errors: Measurement accuracy $\sigma_x$ for $x$ one of several intrinsic ($M_c, \eta, \chi_1$), extrinsic ($\psi_\pm, t, RA, DEC$), and precession-geometry ($\alpha_{JL}, \beta_{JL}, \theta_{JN}$) parameters. The extrinsic parameters are the event time $t$; the sky position measured in RA and DEC; and the sky area $A$, estimated using the $2 \times 2$ covariance matrix $\Sigma_{ab}$ on the sky via $\pi|\Sigma|$. The precession cone parameters are as described in Figure 1: the precession phase $\alpha_{JL}$ at the reference frequency; the precession cone opening angle $\beta_{JL}$; and the viewing angle $\theta_{JN}$.

| Source Harmonics | Seed | $\rho$ | $\dot{\rho}$ | $\sigma_{M_c}$ | $\sigma_{\eta}$ | $\sigma_{\chi_1}$ | $\sigma_t$ | $\sigma_{RA}$ | $\sigma_{DEC}$ | $\sigma_{A}$ | $\sigma_{\alpha_{JL}}$ | $\sigma_{\beta_{JL}}$ | $\sigma_{\theta_{JN}}$ | $A$ | $N_{\text{eff}}$ |
|------------------|------|--------|---------------|----------------|----------------|----------------|-----------|-------------|----------------|---------|----------------|----------------|----------------|-----|--------------|
| A no             | -    | 19.86  | 20.13         | 5.16           | 5.62           | 0.040         | 0.505     | 0.493       | 0.747         | 1.06    | 0.094         | 0.0880         | 0.0594         | 10037|
| A no             | 1234 | 19.86  | 21.26         | 5.06           | 4.33           | 0.041         | 0.491     | 0.542       | 0.733         | 1.21    | 0.100         | 0.0891         | 0.0643         | 10042|
| A no             | 56789| 19.86  | 20.59         | 5.08           | 4.31           | 0.033         | 0.313     | 0.572       | 0.714         | 1.27    | 0.105         | 0.0807         | 0.0523         | 10110|
| C no             | -*   | 19.12  | 19.39         | 4.81           | 4.39           | 0.032         | 0.276     | 0.464       | 0.739         | 0.967   | 0.105         | 0.0741         | 0.0484         | 101600|
| C no             | 1234*| 19.12  | 20.64         | 4.76           | 3.72           | 0.032         | 0.247     | 0.385       | 0.647         | 0.709   | 0.0937        | 0.0624         | 0.0473         | 105941|
| C no             | 56789*| 19.12  | 18.83         | 5.40           | 4.23           | 0.039         | 0.221     | 0.469       | 0.717         | 0.960   | 0.113         | 0.0784         | 0.0622         | 105042|
| C with           | 1234*| 19.73  | 21.05         | 4.80           | 3.49           | 0.030         | 0.191     | 0.309       | 0.551         | 0.474   | 0.087         | 0.0580         | 0.0450         | 42701|
| C with           | 56789*| 19.73  | 19.32         | 4.87           | 4.20           | 0.039         | 0.191     | 0.393       | 0.661         | 0.651   | 0.112         | 0.0752         | 0.0603         | 9814 |

B. Geometry

As expected analytically and demonstrated by Figure 5, precession-induced modulations encode the orientation of the various angular momenta relative to the line of sight. For our loud fiducial signal, the individual spin components can be well-constrained. Equivalently, because our fiducial source performs many precession cycles about a wide precession cone and because that source is viewed along a generic line of sight, we can tightly constrain the precession cone’s geometry: its opening angle; its orientation relative to the line of sight; and even the precise precession phase, measured either by $\cos \lambda$ or $\alpha_{JL}$. The effective Fisher matrix provides a reliable estimate of how well these parameters can be measured; see Table IV and Figure 5.

C. Comparison to and interpretation of analytic predictions

COOKL presented an effective Fisher matrix for two fiducial precessing binaries, adopting a specific post-Newtonian model to evolve the orbit. Following OFOCKL, we adopt a refined post-Newtonian model, including higher-order spin terms. In the Supplementary Material, available online, we provide a revised effective Fisher matrix, including the contribution from these terms. Table V summarizes key features of this seven-dimensional effective Fisher matrix for case A. As noted above, the two-dimensional marginalized predictions are in good qualitative agreement. The one-dimensional marginalized predictions agree surprisingly well with our simulations [Table IV]. Since the ingredients of the effective Fisher matrix are fully under our analytic control, we can directly assess what factors drive measurement accuracy in each parameter.

First and foremost, as in COOKL, this effective Fisher matrix has a hierarchy of scales and eigenvalues, with decreasing measurement error: $M_c, \eta, \chi_1, \ldots$. Unlike nonprecessing binaries, this hierarchy does not clearly split between well-constrained intrinsic parameters ($M_c, \eta, \chi_1$) and poorly-constrained geometric parameters (everything else); for example, as seen in Table V, the eigenvalues of the Fisher matrix span a continuous range of scales.

The scales in the Fisher matrix are intimately tied to timescales and angular scales in the outgoing signal. The largest eigenvalues of the Fisher matrix are set by the shortest timescales: the orbital timescale, and changes to the orbital phase versus time. These scales control measurement of $M_c, \eta, L, a$ and set the reference event time and phase. Qualitatively speaking, we measure these parameters well because good matches require the orbital phase to be aligned over a wide range in time. We measure the reference waveform phase reliably because each waveform must be properly aligned. For this reason, parameters related to orbital phase (i.e., $M_c$) can be mea-

![Table IV: One-dimensional parameter errors](image)

TABLE V: Properties of precessing effective Fisher matrix: Quantities derived from the normalized effective Fisher matrix $\hat{\Gamma}$, as provided in the supplementary information: the eigenvalues $\lambda_k$ and one-dimensional parameter measurement accuracies $\sqrt{\hat{\Gamma}^{-1}/\rho^2}$ evaluated for $\rho = 20$. (As we only compute Fisher matrices after marginalizing over $\psi$ or $\phi_{ref}$, we provide only 7 eigenvalues and independent parameter measurement errors at a time.)
FIG. 4: Estimating astrophysical parameters (C): For our fiducial binary C, the solid and dotted lines show an estimated 90% confidence interval with and without higher harmonics, respectively; colors indicate different noise realizations; and the (nearly indistinguishable) thick solid and dashed lines show an approximate effective Fisher matrix resulting, with and without higher harmonics, not accounting for the constraint imposed by $\chi_s$ versus $\hat{\eta}$: The precession phase at some reference frequency (i.e., $\eta$) will be measured to a relative accuracy $1/\sqrt{N_p}$, Applied to the mass ratio, this estimate leads to the surprisingly successful estimate

$$\sigma_\eta \approx 0(1) \times \frac{\eta}{\sqrt{N_p} \rho} \approx O(1) \times 1.6 \times 10^{-3}$$

i.e., roughly $1/\sqrt{N_p}$ times smaller than the measurement accuracy possible without breaking the spin-mass ratio degeneracy.

While some parameters change the rate at which orbital and precession phase accumulate, other reference phases simply fix the geometry. For example, a shift in the precession phase at some reference frequency (i.e., $\alpha(f = 100 Hz)$) leads to a correlated shift in the precession and hence gravitational wave phase in each precession cycle. In other words, like our ability to measure the orbital phase at some time, our ability to measure the reference precession phase is essentially independent of the number of orbital or precession cycles, solely reflecting geometric factors. We expect the accuracy with which these purely geometric parameters $x$ can be determined can be estimated from first principles. To order of magnitude, we expect Fisher matrix components $\Gamma_{xx}$ comparable to $\Delta x^2$, where $\Delta x$ is the parameter’s range. For example, the angular parameters $(\beta_{1L}, \theta_{1N}, \alpha, \phi)$ should be measured to within

$$\sigma_{\text{angle}} \approx \frac{(2\pi)}{\sqrt{12} \rho} \approx 0.09 \text{ rad}$$

where the factor $\sqrt{12}$ is the standard deviation of a uniform distribution over $[0, 1]$. This simple order-of-magnitude estimate compares favorably to the Fisher matrix results shown in Table [V] and to our full numerical
FIG. 5: Source geometry: Angular momenta (C,A): For case C (top panels) and case A (bottom panels), the posterior for the “precession cone” (path of the angular momentum direction), expressed using the precession cone representation. This figure demonstrates that both the path ($\theta_{JN}, \beta_{JL}$) and instantaneous orientation ($\alpha_{JL}, \iota$) of the orbital angular momentum can be well-determined. As in Figure 4, colors indicate different noise realizations; solid and dotted lines indicate the neglect or use of higher harmonics; the green point shows the actual value; and the solid gray path shows the trajectory of $L$ over one precession cycle. Left panels: The precession angle $\alpha_{JL}$ of $L$ around $J$. For comparison, the green points show the simulated values; when present, the solid blue path shows variables covered in one precession cycle. Roughly speaking, the precession phase can be measured with relative accuracy tens of percent at this signal amplitude $\rho$. Right panels: Illustration that both the opening angle $\beta_{JL}$ of the precession cone and the angle $\theta_{JN}$ between the line of sight and $\hat{J}$ can be measured accurately.

simulations [Figure 6 and Table IV]. This naive estimate ignores all dependence on precession geometry; in general, all geometric factors are tied directly to the magnitude of precession-induced modulations, which grow increasingly significant for larger misalignment, roughly in proportion to $\cos \beta_{JL}$. This estimate for how well geometric angles can be measured should break down for nearly end-over-end precession ($\beta_{JL} \rightarrow \pi/2$). Nearly end-over-end precession requires extreme fine tuning; is associated with transitional precession; and is correlated with rapid change in $\beta_{JL}$ [BLO]. We anticipate a different set of approximations will be required to address this limit.

D. Relative role of higher harmonics

To this point, both our analytic and numerical calculations suggest higher harmonics provide relatively little additional information about intrinsic and extrinsic parameters. That said, as illustrated by Figure 6, higher harmonics do break a discrete degeneracy, determining the orientation of $\hat{L}$ on the plane of the sky at $f = 100$ Hz up to a rotation by $\pi$.

OFOCKL used the evidence to demonstrate conclusively that higher harmonics had no additional impact, beyond improving knowledge of one parameter. Given expected systematic uncertainties in the evidence, at the present time we do not feel we can make as robust and global a statement. That said, all of our one- and two-dimensional marginalized posteriors support the same conclusion: higher harmonics provide little new information, aside from breaking one global degeneracy.
FIG. 6: Angular momentum direction on the sky (C): Projection of the orbital angular momentum direction ($\hat{L}$) on the plane of the sky at $f = 100$ Hz; compare to Figure 2. This figure demonstrates that the individual angular momenta to be well-constrained to two discrete regions and that higher harmonics allow us to distinguish between the two alternatives; and (3) that the precession cone is well-determined, at the accuracy level expected from the number of precession cycles. As in Figure 3 colors indicate different noise realizations; solid and dotted lines indicate the neglect or use of higher harmonics; and the green point shows the expected solution.

FIG. 7: Distance and inclination degeneracy broken (C): Posterior probability contours in distance and inclination.

E. Timing, sky location, and distance

As seen in Figure 7, precessing binaries do not have the strong source orientation versus distance degeneracy that plagues nonprecessing binaries: because they emit distinctively different multi-harmonic signals in each direction, both the distance and emission direction can be tightly constrained.

Conversely, the sky location of precessing binaries can be determined to little better than the sky location of a nonprecessing binary with comparable signal amplitude; compare, for example, Table IV and Figures 3 against the corresponding figures in OFOCKL.

Finally, the event time can be marginally better determined for a precessing than for a nonprecessing binary. This accuracy may be of interest for multimessenger observations of gamma ray bursts.

F. Advanced versus initial instruments

All the discussion above assumed first-generation instrumental sensitivity. For comparison and to further validate our estimates, we have also done one calculation using the expected sensitivity of second-generation instruments [64, 65]. In this calculation, the source (event C) has been placed at a larger distance ($d = 298.7$ Mpc) to produce the same network SNR. Also unlike the analysis above, we have for simplicity assumed the smaller compact object has no spin.

Table VI shows the resulting one-dimensional measurement accuracies, compared against a concrete simulation. All results agree with the expected scalings, as described previously. First and foremost, all geometric quantities ($RA, DEC, \alpha_{JL}, \beta_{JL}, \theta_{JN}$) and time can be measured to the same accuracy as in initial instruments, at fixed SNR. Second, quantities that influence the orbital decay – chirp mass, mass ratio, and spin – are all measured more precisely, because more gravitational wave cycles contribute to detection with advanced instruments. Finally, as illustrated by Figure 8, quantities that reflect precession-induced modulation – the precession rate $\Omega_p$ and precession cone angle $\beta_{JL}$ misalignment – are at best measured marginally more accurately, reflecting the relatively small increase in number of observationally-accessible precession cycles for advanced detectors [Eqs. (16 and 17)]. As shown by the bottom panel of Figure 4, this small increase in sensitivity is comparable to the typical effect of different noise realizations.

IV. CONCLUSIONS

In this work we performed detailed parameter estimation for two selected BH-NS binaries, explained several features in terms of the binary’s kinematics and geome-
TABLE VI: Parameter estimation with initial and advanced instruments: Like Table IV measurement accuracy $\sigma_x$ for several intrinsic and extrinsic parameters. The first row provides results for initial-scale instruments, duplicating an entry in Table IV. The second row provides results for advanced detectors, operating at design sensitivity. At fixed signal amplitude, most geometric quantities can be measured to fixed accuracy, independent of detector sensitivity. Quantities impacting the orbital phase versus time (mass, mass ratio, and spin) are more accurately measured with advanced instruments, with their access to lower frequencies and hence more cycles.

| Source | Instrument | Harmonics | Noise |
|--------|------------|-----------|-------|
| C      | Initial    | no        | no    |
|        |            |           |       |
| C      | Advanced   | no        | yes   |
|        |            |           |       |

Due to the relatively limited calculations of spin effects in post-Newtonian theory, all inferences regarding black hole spin necessarily come with significant systematic limitations. For example, Nitz et al. [68] imply that poorly-constrained spin-dependent contributions to the orbital phase versus time could significantly impact parameter estimation of nonprecessing black hole-neutron star binaries. Fortunately, the leading-order precession equations and physics are relatively well-determined. For example, the amplitude of precession-induced modulations is set by the relative magnitude and misalignment of $\vec{L}$ and $\vec{S}_1$. In our opinion, the leading-order symmetry-breaking effects of precession are less likely to be susceptible to systematic error than high-order corrections to the orbital phase. Significantly more study would be needed to validate this hypothesis.

FIG. 8: Estimating astrophysical parameters with advanced detectors: Like the bottom panel of Figure 4 but using advanced instruments; see Table VI.
predictions about both spin magnitude and misalignment might therefore be put to a strong test with gravitational wave measurements.

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Appendix A: More properties of the effective Fisher matrix

1. Separation of scales and mutual information

On physical grounds, we expect the timescales and modulations produced by precession to separate, allowing roughly independent measurements of orbital- and precession-rate-related parameters (i.e., $M_c, \eta, a, \beta_{JL}$) and purely geometric parameters ($\alpha, \phi, \theta$). To assess this hypothesis quantitatively, we evaluate the mutual information between the two subspaces. For a gaussian distribution described by a covariance matrix $\Gamma$, the mutual information between two subspaces $A, B$ is [OFOCKL Eq. (31)]:

$$I(A, B) = -\frac{1}{2} \ln |\frac{\Gamma}{\Gamma_A \Gamma_B}|$$

Table VII shows that after marginalizing out orbital phase ($\phi_{\text{ref}}$), the mutual information between orbital-phase-related parameters ($M_c, \eta, a, \beta_{JL}$) and geometric parameters is small but nonzero (0.31): the two subspaces are weakly correlated. By comparison, the mutual information $I(a,c | B)$ between two intrinsic parameters $a,c$ in $A = \{M_c, \eta, \chi_1\}$ is large. Finally, after marginalizing out all other parameters, the mutual information between $\alpha_{JL}$ and $(\theta_{JN})$ is small, as expected given the different forms in which these quantities enter into the outgoing gravitational wave signal.

2. Regularizing calculations with a prior

Due to the wide range of eigenvalues and poor condition number, all Fisher matrices are prone to numerical instability in high dimension. Additionally, due to physical near-degeneracies, the error ellipsoid derived from the Fisher matrix alone may extend significantly outside the prior range; see, e.g., examples in [74].

Following convention, to insure our results are stable to physical limitations, we derive parameter measurements accuracies $\Sigma = \Gamma^{-1}/\rho^2$ by combining the signal amplitude $\rho$, the normalized effective Fisher matrix $\hat{\Gamma}_{\text{eff}}$ provided above, and a prior $\Gamma_{\text{prior}}$:

$$\Gamma_{\text{eff}} \equiv \rho^2 \hat{\Gamma}_{\text{eff}} + \lambda \Gamma_{\text{prior}}$$

As expected given the eigenvalues and signal amplitude, this prior has no significant impact on our calculations. In particular, the eigenvalues and parameter measurement accuracies reported in the text are unchanged if this weak prior is included.
