Light-cone path integral approach to the Landau–Pomeranchuk–Migdal effect and the SLAC data on bremsstrahlung from high energy electrons

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Abstract

We analyze the recent data of the SLAC E-146 collaboration [11] on the Landau-Pomeranchuk-Migdal effect for bremsstrahlung from 8 and 25 GeV electrons within a rigorous light-cone path integral approach previously developed in Refs. [14, 16]. Numerical calculations have been carried out treating rigorously the Coulomb effects and including the inelastic processes. Comparison with the experimental data is performed taking into account multi-photon emission and photon absorption. For most of the targets our predictions are in excellent agreement with the experimental data.
1 Introduction

Landau and Pomeranchuk [1] showed in 1953 that at high energies multiple scattering suppresses radiation process in matter. Within classical electrodynamics they obtained for a high energy electron the bremsstrahlung spectrum \( \propto 1/\sqrt{k} \) (\( k \) is the photon momentum) in contrast to the \( 1/k \) Bethe-Heitler spectrum for an isolated atom. Later this prediction was corroborated by Migdal [2] who developed a quantum-mechanical theory of bremsstrahlung and pair production in medium.

Since the works by Landau and Pomeranchuk [1] and Migdal [2] suppression of the radiation processes in medium, called in the current literature the Landau-Pomeranchuk-Migdal (LPM) effect, has been attracting much attention (for a complete list of references see [3, 4, 5, 6]). The LPM effect has been qualitatively corroborated in the Serpukhov experiment on bremsstrahlung from 40 GeV electrons [7] and in the experiments with cosmic rays [8, 9]. However, only recently the first quantitative measurement of the LPM effect for bremsstrahlung from 8 and 25 GeV electrons has been performed by the SLAC E-146 collaboration [10, 11]. This experiment stimulated a new theoretical activity [12, 13, 14, 15, 16, 17] on the LPM effect in QED. Nevertheless, a detailed theoretical analysis of the SLAC data is still lacking.

In the present paper we analyze the experimental data [11] within a rigorous theory of the LPM effect developed in Ref. [14] (see also [16]) which is applicable for both QED and QCD. The approach of Ref. [14] is based on the light-cone path integral formalism in the coordinate representation and the technique of statistical averaging over the medium potential previously developed in Ref. [18]. In Ref. [14] evaluation of the radiation rate has been reduced to solving a two-dimensional Schrödinger equation with an imaginary potential. In QED this potential is proportional to the dipole cross section for scattering of \( e^+e^- \) pair off an atom. From the conceptual viewpoint the approach of Ref. [14] is equivalent to Migdal’s analysis within time-ordered perturbation theory in the momentum representation. However, in Migdal’s formalism a simple expression for the radiation rate can be obtained only within the Fokker-Planck approximation, in which the...
Coulomb effects are treated to logarithmic accuracy. It works well in the limit of strong LPM suppression in an infinite medium, but it is not good for real situations due to the uncertainty in the value of the Coulomb logarithm. Previously, in Ref. [16], we compared the theoretical predictions with the experimental spectrum for 25 GeV beam on a gold target with thickness \( L = 0.7\% X_0 \) (\( X_0 \) is the radiation length) presented in the first SLAC publication [10]. In the present paper we carry out a detailed comparison of the theoretical predictions with the complete data taken by the SLAC E-146 collaboration [11] for 8 and 25 GeV electrons in a variety of materials. In our analysis we take into account multi-photon emission which plays an important role for the targets with thickness \( L \sim 2 - 6\% X_0 \) used in the SLAC experiment. We also take into account photon absorption. As compared with the analysis of Ref. [16] we use a parametrization of the dipole cross section which is more accurate for heavy elements.

The presentation is organized as follows. In section 2 we give our basic formulas for the radiation rate obtained neglecting multi-photon emission. In section 3 we discuss relationship between the probability distribution in the radiated energy, measured at the SLAC experiment, and the probability of one-photon emission. Including one- and two-photon emission we derive a simple relation between these quantities. In section 4 we present the numerical results for LPM suppression factors for a variety of materials and compare the theoretical predictions with the SLAC data. The results are summarized in section 5.

## 2 Expression for the probability of photon emission

In Ref. [14] we have expressed the cross section for the radiation process \( a \rightarrow bc \) through the Green’s function of a two-dimensional Schrödinger equation in impact-parameter space for which the longitudinal coordinate \( z \) plays the role of time. This equation describes evolution of the light-cone wave function of a fictitious three-body \( \bar{abc} \) state. In this state the transverse coordinate of the particle \( a \) coincides with the center-of-mass of the \( bc \)
system, and the only dynamic spatial variable is the transverse separation between the particles \( b \) and \( c \). For radiation of a photon with momentum \( k \) from an electron with high energy \( E_e \gg m_e \) (\( m_e \) is the electron mass) incident on an amorphous target the corresponding Hamiltonian, describing the \( e^+ e^- \gamma \) system, is given by

\[
H = \frac{q^2}{2\mu(x)} + v(\rho, z),
\]

\[
v(\rho, z) = -i \frac{n(z)\sigma(|\rho|x)}{2},
\]

where \( x = k/E_e \) is the photon fractional longitudinal momentum, the Schrödinger mass is \( \mu(x) = E_e x(1 - x) \), \( n(z) \) is the number density of the target (assumed to be independent of the transverse coordinate), and \( \sigma(\rho) \) is the dipole cross section for scattering of \( e^+ e^- \) pair of the transverse size \( \rho \) off an atom. The transverse coordinate \( \rho \) in Eqs. (1), (2) is the transverse distance between electron and photon in the \( e^+ e^- \gamma \) system. In terms of \( x \) and \( \rho \) the electron-positron and photon-positron transverse separations are \( \rho_{ee} = -x \rho \) and \( \rho_{\gamma e} = (1 - x) \rho \), respectively.

Neglecting multi-photon radiation, the probability of photon emission can be written in the form

\[
\frac{dP_\gamma}{dx} = 2 \text{Re} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \exp \left(-\frac{i\Delta \xi}{L_f}\right) g(\xi_1, \xi_2, x) \left[ \mathcal{K}(0, \xi_2|0, \xi_1) - \mathcal{K}_v(0, \xi_2|0, \xi_1) \right].
\]

Here \( \mathcal{K} \) is the Green’s function for the Hamiltonian (1), \( \mathcal{K}_v \) is the vacuum Green’s function for the Hamiltonian (1) with \( v(\rho, z) = 0 \),

\[
L_f = \frac{2E_e(1 - x)}{m_e^2 x}
\]

is the so-called photon formation length. The vertex operator \( g(\xi_1, \xi_2, x) \) is given by

\[
g(\xi_1, \xi_2, x) = \frac{\alpha[4 - 4x + 2x^2]}{4x} q(\xi_2) \cdot q(\xi_1) + \frac{\alpha m_e^2 x}{2E_e^2(1 - x)^2},
\]

where \( \alpha = 1/137 \). The two terms on the right-hand side of Eq. (5) correspond to the \( e \to e' \gamma \) transitions conserving and changing the electron helicity.
Treating the potential (2) as a perturbation one can represent the radiation rate (3) in the following form [16]

\[
\frac{dP}{dx} = \frac{dP^BH}{dx} + \frac{dP^{abs}}{dx},
\]

(6)

\[
\frac{dP^BH}{dx} = T \frac{d\sigma^BH}{dx},
\]

(7)

\[
\frac{d\sigma^BH}{dx} = \int d\rho W^{e\gamma}_e(x, \rho) \sigma(\rho x).
\]

(8)

\[
W^{e\gamma}_e(x, \rho) = \frac{1}{2} \sum_{\{\lambda_i\}} |\Psi(x, \rho, \{\lambda_i\})|^2,
\]

(9)

\[
\frac{dP^{abs}}{dx} = -\frac{1}{4} \text{Re} \sum_{\{\lambda_i\}} \int_0^L dz_1 n(z_1) \int_{z_1}^L dz_2 n(z_2) \int d\rho \Psi^\ast(x, \rho, \{\lambda_i\})
\]

\[
\times \sigma(\rho x) \Phi(x, \rho, \{\lambda_i\}, z_1, z_2) \exp \left[ -\frac{i(z_2 - z_1)}{L_f} \right].
\]

(10)

Here \( T = \int_0^L dz n(z) \) is the optical thickness of the target (we assume that \( n(z) = 0 \) at \( z < 0 \) and \( z > L \)), \( \Psi(x, \rho, \{\lambda_i\}) \) is the light-cone wave function for the transition \( e \rightarrow e^\prime \gamma \), \( \{\lambda_i\} \) is the set of the helicity variables. In Eq. (1) the function \( \Phi(x, \rho, \{\lambda_i\}, z_1, z_2) \) is the solution of the two-dimensional Schrödinger equation

\[
i \frac{\partial \Phi(x, \rho, \{\lambda_i\}, z_1, z_2)}{\partial z_2} = H \Phi(x, \rho, \{\lambda_i\}, z_1, z_2)
\]

(11)

with the Hamiltonian (1). The boundary condition for \( \Phi(x, \rho, \{\lambda_i\}, z_1, z_1) \) is

\[
\Phi(x, \rho, \{\lambda_i\}, z_1, z_1) = \Psi(x, \rho, \{\lambda_i\}) \sigma(\rho x).
\]

The light-cone wave function \( \Psi(x, \rho, \lambda, \lambda^\prime, \lambda^\gamma) \) for \( \lambda^\prime = \lambda \) is given by

\[
\Psi(x, \rho, \lambda, \lambda^\prime, \lambda^\gamma) = \frac{1}{2\pi} \sqrt{\frac{\alpha x}{2}} [\lambda^\gamma(2 - x) + 2\lambda x] \exp(-i\lambda^\gamma \varphi) m_e K_1(\rho m_e x),
\]

(12)

for \( \lambda^\prime = -\lambda \) the only nonzero component is the one with \( \lambda^\gamma = 2\lambda \)

\[
\Psi(x, \rho, \lambda, -\lambda, 2\lambda) = \frac{i}{2\pi} \sqrt{2\alpha x^3} m_e K_0(\rho m_e x),
\]

(13)

where \( \varphi \) is the azimuthal angle, \( K_0 \) and \( K_1 \) are the modified Bessel functions.
The first term on the right-hand side of Eq. (6), coming from the term $\propto v$ in expansion of the Green’s function $K$ in the potential, corresponds to the impulse approximation. From this follows that Eq. (8) gives the Bethe-Heitler cross section. Note that this representation for the Bethe-Heitler cross section can be also derived directly for bremsstrahlung on an isolated atom using the light-cone approach developed in Ref. [19], where the heavy quark production was discussed. In Eq. (6) LPM suppression is described by the second term which is analogous to the Glauber absorptive correction in hadron-nucleus collisions. In the present paper we will use for numerical calculations representation of the radiation rate in form (6). It allows one to bypass evaluation of the singular Green’s function. This makes it convenient for accurate computations with a rigorous treatment of the Coulomb effects.

In the limit $L_f \to 0$ the second term in Eq. (6) vanishes and the Bethe-Heitler regime obtains. A simple expression for the radiation rate can be also obtained in the limit $L_f \gg L$. One can easily show that in this regime the kinetic term in the Hamiltonian (1) can be neglected. This means that the transverse variable $\rho$ is approximately frozen, and $\Phi(x, \rho, \{\lambda_i\}, z_1, z_2)$ can be written in the eikonal form

$$
\Phi(x, \rho, \{\lambda_i\}, z_1, z_2) \approx \exp \left[ -\frac{\sigma(\rho x)}{2} \int_{z_1}^{z_2} dz n(z) \right] \Psi(x, \rho, \{\lambda_i\}) \sigma(\rho x). \tag{14}
$$

Using Eq. (14) one can easily obtain from Eqs. (6), (7), (8) and (9) for the radiation rate in the frozen-size approximation

$$
\frac{dP^{fr}_{\gamma}}{dx} = 2 \int d\rho W_{e^+}^{\gamma}(x, \rho) \left\{ 1 - \exp \left[ -\frac{\sigma(\rho x) T}{2} \right] \right\}. \tag{15}
$$

Note that Eq. (15) is analogous to the formula for the cross section for heavy quark production in hadron-nucleus collision obtained in Ref. [19].

In the momentum space the spectrum (15) after the Fourier transform can be rewritten in the form

$$
\frac{dP^{fr}_{\gamma}}{dx} = \int dq P(x, q) I(q), \tag{16}
$$
where

\[ I(q) = \frac{1}{(2\pi)^2} \int d\rho \exp \left[ i q \rho - \frac{\sigma(\rho)}{2} T \right] \]

is the distribution function in momentum transfer for the electron after passing through the target \([18]\), and the function

\[ P(x, q) = \int d\rho W_{e\gamma}(x, \rho) [1 - \exp(i q x)] \]

describes the probability of photon emission for scattering of the electron with momentum transfer equals \(q\). The factorized form of the integrand in Eq. (16) reflects the fact that for the photons with \(L_f \gg L\) the target acts as a single radiator. In the limit \(x \to 0\) from Eq. (16) one can obtain the spectrum of Ref. [12] evaluated within classical electrodynamics.

The dipole cross section, entering the imaginary potential (2), for an atom with the atomic number \(Z\) can be written in the form

\[ \sigma(\rho) = \rho^2 C(\rho), \quad (17) \]

\[ C(\rho) = Z^2 C_{el}(\rho) + ZC_{in}(\rho). \quad (18) \]

In Eq. (18) the terms \(\propto Z^2\) and \(\propto Z\) correspond to elastic and inelastic intermediate states in interaction of \(e^+e^-\) pair with an atom. The components of the light-cone wave function \(\Psi(x, \rho, \{\lambda_i\})\) defined by Eqs. (12), (13) decrease steeply at \(|\rho| \gtrsim 1/m_e x\). As a consequence, the Bethe-Heitler cross section (8) is dominated by the region \(\rho \sim 1/m_e x\). For this reason the bremsstrahlung rate is only sensitive to the behavior of \(\sigma(\rho)\) at \(\rho \lesssim 1/m_e \ll a\), where \(a \sim r_B Z^{-1/3}\) is the atomic size. In this region both the \(C_{el}\) and \(C_{in}\) have only weak logarithmic dependence on \(\rho\), and we can parametrize them in the form

\[ C_{el}(\rho) = 4\pi\alpha^2 \left[ \log \left( \frac{2a_{el}}{\rho} \right) + \frac{1 - 2\gamma}{2} - f(Z\alpha) \right], \quad (19) \]

\[ f(y) = y^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2 + y^2)}, \quad (20) \]
\[ C_{\text{in}}(\rho) = 4\pi\alpha^2 \left[ \log \left( \frac{2a_{\text{in}}}{\rho} \right) + \frac{(1 - 2\gamma)}{2} \right] , \]  

(21)

where \( \gamma = 0.577 \) is Euler’s constant. Eq. (19) defines \( C_{\text{el}}(\rho) \) for \( \rho \gg R_A \), here \( R_A \) is the nucleus radius. At \( \rho \ll R_A \) on the right-hand side of Eq. (19) \( \rho \) must be replaced by \( R_A \).

The parametrization (19) of the elastic component corresponds to the result of calculation of \( C(\rho) \) for scattering of \( e^+e^- \) pair on the atomic potential \( \phi(r) = (Ze/4\pi r) \exp(-r/a_{\text{el}}) \).

The first two terms in the square brackets on the right-hand side of Eq. (19) give the contribution of the Born approximation while the last one is related to the Coulomb correction. It is expressed through function (20) which was introduced in the well-known analysis of pair production and bremsstrahlung by Davies, Bethe and Maximon [20].

This correction becomes important only for heavy elements. For \( Z \sim 70 - 90 \) it decreases \( C_{\text{el}}(\rho \sim 1/m_e) \), and accordingly the Bethe-Heitler cross section, by \( \sim 7 - 10\% \). Using the parametrizations (19), (21) and formulas (12), (13) for the light-cone wave function one can obtain from Eq. (8) for the cross section of photon emission on an isolated atom

\[ \frac{d\sigma^{BH}}{dx} = \frac{d\sigma^{BH}_{nf}}{dx} + \frac{d\sigma^{BH}_{sf}}{dx} , \]  

(22)

\[ \frac{d\sigma^{BH}_{nf}}{dx} = \frac{4\alpha^3}{3m_e^2} \left\{ Z^2[F_{\text{el}} - f(Z\alpha)] + ZF_{\text{in}} + \frac{Z(Z + 1)}{12} \right\} , \]  

(23)

\[ \frac{d\sigma^{BH}_{sf}}{dx} = \frac{4\alpha^3}{3m_e^2} \left\{ Z^2[F_{\text{el}} - f(Z\alpha)] + ZF_{\text{in}} - \frac{Z(Z + 1)}{6} \right\} , \]  

(24)

where \( F_i = \log(a_i m_e \exp(1/2)) \). The two terms in Eq. (22) correspond to transitions conserving (nf) and changing (sf) the electron helicity. We have adjusted \( a_{\text{el}} \) and \( a_{\text{in}} \) to reproduce the Bethe-Heitler cross section evaluated in the standard approach with realistic atomic formfactors \( F_{\text{el}} \approx \log(184/Z^{1/3}) \), and \( F_{\text{in}} \approx \log(1194/Z^{2/3}) \), obtained within the Thomas-Fermi-Molier model [21]. This gives \( a_{\text{el}} = 0.81 \ r_B Z^{-1/3} \) and \( a_{\text{in}} = 5.3 \ r_B Z^{-2/3} \).

The strength of the LPM effect can be characterized by the suppression factor, \( S(k, L) \), defined as (hereafter we assume that the target is homogeneous)

\[ S(k, L) = \frac{dP}{dx} \left( nL \frac{d\sigma^{BH}}{dx} \right)^{-1} . \]  

(25)
In terms of the suppression factors $S_{nf}$ and $S_{sf}$, defined by relations analogous to Eq. (25) for transitions conserving and changing the electron helicity, $S(k, L)$ is given by

$$S(k, L) = \left[ \frac{d\sigma_{nM}^{BH}}{dx} S_{nf}(k, L) + \frac{d\sigma_{sM}^{BH}}{dx} S_{sf}(k, L) \right] \left( \frac{d\sigma_{BH}^{M}}{dx} \right)^{-1}. \quad (26)$$

In the kinematical domain of the SLAC experiment $x \ll 1$, and the spin-flip transitions give a negligible contribution to the radiation rate. As a result, $S(k, L)$ turns out to be very close to $S_{nf}(k, L)$.

The edge effects vanish, and $S(k, L)$ becomes close to the suppression factor for infinite medium, $S_{inf}(k)$, for sufficiently large target thickness (or small $x$) $L \gg L_f'$, here $L_f'$ is the medium-modified photon formation length. In terms of the representation (3) $L_f'$ is the typical value of $|\xi_2 - \xi_1|$ dominating the integral on the right-hand side of Eq. (3). The medium-modified formation length characterizes the nonlocality of photon emission connected with interference effects. It is important from the viewpoint of applicability limits for the probabilistic treatment of multi-photon effects which will be discussed in the next section.

To get an idea about the value of $L_f'$ and the strength of LPM suppression one can use the results of evaluation of the radiation rate in the oscillator approximation [14]. It corresponds to replacement of the function $C(\rho)$ (18) by its value at $\rho \sim \rho_{eff} x$, where $\rho_{eff}$ is the typical photon-electron separation dominating the Green’s function $K$ (the path integral representation is assumed) in Eq. (3). For the case of not very strong LPM suppression, which will be interesting for analysis of the SLAC data, one can take $\rho_{eff} \sim 1/m_e x$. Then, the Hamiltonian (1) takes the oscillator form with the frequency

$$\Omega = \frac{(1 - i)}{\sqrt{2}} \left( \frac{n C_{osc} x}{E_e (1 - x)} \right)^{1/2},$$

where

$$C_{osc} = C(\rho \sim 1/m_e) \approx 4\pi \alpha^2[Z^2(F_d - f(Z\alpha)] + ZF_{in}].$$

In this approximation the radiation rate in form (3) can be evaluated using the known oscillator Green’s function. For an infinite medium the oscillator model suppression factors
depend on the dimensionless parameter \[14\]
\[
\eta = L_f | \Omega | = 2 \left[ \frac{n E_e (1 - x) C_{osc}}{m_e^2 x} \right]^{1/2}
\approx \left\{ \frac{1.3 E_e^2 (GeV) [1 - 10^{-3} k (MeV)/E_e (GeV)]}{k (MeV) X_0 (mm)} \right\}^{1/2}.
\] (27)

Here we expressed \(C_{osc}\) through the radiation length defined as in Ref. [21]. For weak suppression \((\eta \ll 1)\) \(S^{\text{nf}}_{nf} \approx 1 - 16 \eta^4 / 21, S^{\text{nf}}_{sf} \approx 1 - 31 \eta^4 / 21\), and for regime of strong suppression \((\eta \gg 1)\) \(S^{\text{nf}}_{nf} \approx 3 / \eta \sqrt{2}, S^{\text{nf}}_{sf} \approx 3 \pi / 2 \eta^2\) [14]. In terms of the photon momentum LPM suppression becomes significant for \(k \approx k_{\text{LPM}}\) where \(k_{\text{LPM}}\), corresponding to \(\eta = 1\), is given by
\[
k_{\text{LPM}} \approx \frac{1.3 E_e^2 (GeV)}{X_0 (mm) [1 + 0.0013 E_e (GeV) / X_0 (mm)]}.
\] (28)

In Table 1 we give the values of \(k_{\text{LPM}}\) for the target materials and electron energies used in the SLAC experiment [11]. We are also give in this table the radiation lengths.

Closer inspection of the expression (3) in the oscillator approximation allows one to obtain the following estimate for the medium-modified photon formation length for the transitions conserving the electron helicity \(L'_f \sim L_f / \max(1, \eta)\) [14]. Using Eqs. (4), (27) one can rewrite it at \(x \ll 1\), which will subsequently be interesting for analysis of the SLAC data [11], in the form
\[
L'_f (mm) \sim 10^{-3} \cdot \min \left\{ \frac{1.5 E_e^2 (GeV)}{k (MeV)}, 1.32 E_e (GeV) \sqrt{\frac{X_0 (mm)}{k (MeV)}} \right\}.
\] (29)

We will use this approximate formula for estimate of \(L'_f\) for the SLAC data.

3 Multi-photon emission and the probability distribution in the radiated energy

The experimental spectra of Ref. [11] were obtained by measuring in a calorimeter the total energy of the photons radiated by the electron. This means that, up to a small
correction connected with photon absorption, the spectra of Ref. [11] correspond to the probability distribution in the electron energy loss, $dP_e/dx$, called in the literature the electron struggling function. For sufficiently thin targets, when multi-photon effects can be neglected, the electron struggling function is close to the probability of emission of a single photon, considered in previous section. For the SLAC data [11] multi-photon emission is small for the gold target with $L = 0.7\% X_0$, but for other targets with $L \sim 2 - 6\% X_0$ it becomes important. A rigorous quantum-mechanical analysis of the LPM effect including the multi-photon radiation in the kinematical region where $L_f' \gtrsim L$ is a complicated task requiring evaluation of higher order diagrams. We will compare the theoretical prediction with the data of Ref. [11] for $k > 5\text{ MeV}$ (here $k$ is viewed as the total radiated energy). In this region of $k$ for the targets with $L \sim 2 - 6\% X_0$ the medium-modified photon formation length for 8 and 25 GeV electrons used in the SLAC experiment turns out to be considerably smaller than the target thickness. At $L_f' \ll L$ one can neglect the edge effects and the nonlocality of photon emission. This allows us to use the probabilistic approach to the multi-photon effects in which the probability of photon emission from an electron per unit length can be written in terms of the medium-modified Bethe-Heitler cross section

$$\frac{d\sigma^{eff}}{dx} = S_{inf}(k = xE)\frac{d\sigma^{BH}}{dx}.$$  

(30)

Then, the electron struggling function can be obtained by solving the standard diffusion equation (see for instance [22, 23]). However, for analysis of the SLAC data we need $dP_e/dx$ only at small $x$ and for sufficiently small target thicknesses $L \lesssim 6\% X_0$. In this case we can neglect the effect of the electron energy loss on the probability of photon emission and restrict ourselves to the one- and two-photon processes. This allows us to bypass solving the diffusion equation and to write the electron struggling function in the following simple form

$$\frac{dP_e}{dx} = \int_0^L dz U(0, z) n \frac{d\sigma^{eff}}{dx} U(z, L)$$
\[ + \int_0^L \int_{z_1}^L \int_{x_{\text{min}}}^x dx_1 dx_2 \delta(x_1 + x_2 - x) U(0, z_1)n \frac{d\sigma^{\text{eff}}}{dx_1} U(z_1, z_2)n \frac{d\sigma^{\text{eff}}}{dx_2} U(z_2, L), \] 

where

\[ U(z_1, z_2) = \exp \left[ -(z_2 - z_1) n \int_{x_{\text{min}}}^1 dx_1 \frac{d\sigma^{\text{eff}}}{dx_1} \right] \] 

is the attenuation factor for propagation of the electron in the target from \( z_1 \) to \( z_2 \) without photon emission. In Eqs. (31), (32) we introduced an infrared regulator \( x_{\text{min}} \) which can be chosen from the condition \( L'_{f(x_{\text{min}})} \sim L \). This suggests that the electron struggling function (31) includes the processes with an arbitrary number of soft photons with \( x < x_{\text{min}} \) radiated from initial and final electrons. As will be seen later, our final expression for the electron struggling function, similarly to the solution of the diffusion equation [22], is infrared stable, and \( x_{\text{min}} \) will be set equal to zero. Note that within the light-cone path integral approach to the LPM effect of Ref. [14], the attenuation factor (32) emerges as a result of evaluation of the radiative corrections to the transverse electron propagator connected with the chain diagrams within the dilute gas approximation. This approximation corresponds neglecting the space overlapping of different electron-photon loops, i.e., it gives the attenuation factor to leading (zeroth) order in \( L'/L \).

Neglecting small surface effects we can rewrite Eq. (31) in terms of the probability of photon emission evaluated neglecting multi-photon effects as

\[ \frac{dP_e}{dx} = \exp \left[ -\int_{x_{\text{min}}}^1 dx_1 \frac{dP_\gamma}{dx_1} \right] \left\{ \frac{dP_\gamma}{dx} + \frac{1}{2} \int_{x_{\text{min}}}^x dx_1 dx_2 \delta(x_1 + x_2 - x) \frac{dP_\gamma}{dx_1} \frac{dP_\gamma}{dx_2} \right\}. \] 

For \( L \ll X_0 \) the exponential factor on the right-hand side of Eq. (33) can be written as

\[ \exp \left[ -\int_{x_{\text{min}}}^1 dx_1 \frac{dP_\gamma}{dx_1} \right] \approx \exp \left[ -\int_{x_{\text{min}}}^x dx_1 \frac{dP_\gamma}{dx_1} \right] \left\{ 1 - \int_{x_{\text{min}}}^x dx_1 \frac{dP_\gamma}{dx_1} \right\}. \] 

Then, using Eqs. (33) and (34) we obtain

\[ \frac{dP_e}{dx} = \frac{dP_\gamma}{dx} K(x), \] 

where

\[ K(x) = \exp \left[ -\int_{x_{\text{min}}}^1 dx_1 \frac{dP_\gamma}{dx_1} \right] \]
\[ \times \left\{ 1 - \int_{x_{\min}}^{x} dx_1 \frac{dP_\gamma}{dx_1} + \frac{1}{2} \int_{x_{\min}}^{x} dx_1 dx_2 \delta(x_1 + x_2 - x) \frac{dP_\gamma}{dx_1} \frac{dP_\gamma}{dx_2} \left( \frac{dP_\gamma(x)}{dx} \right)^{-1} \right\}. \]  

(36)

We see that, as was said above, the electron struggling function defined by Eqs. (35), (36) is an infrared stable quantity. Therefore, we can set \( x_{\min} = 0 \) and rewrite the multi-photon \( K \)-factor in the form

\[ K(x) = \exp \left[ - \int_{x}^{1} dx_1 \frac{dP_\gamma}{dx_1} \right] \times \left\{ 1 - \frac{1}{2} \int_{0}^{x} dx_1 \left[ \frac{dP_\gamma}{dx_1} + \frac{dP_\gamma}{dx_2} - \frac{dP_\gamma}{dx_1} \frac{dP_\gamma}{dx_2} \left( \frac{dP_\gamma(x)}{dx} \right)^{-1} \right] \right\}, \]  

(37)

where \( x_2 = x - x_1 \). The major \( x \)-dependence of the \( K \)-factor comes from the exponential factor which reflects a simple fact that emission of the photons with the fractional momentum bigger than \( x \) is forbidden. We checked the accuracy of the relation (37) using as a test solution the exact expression for the electron struggling function

\[ \frac{dP_e}{dx} = \left[ \log \left( \frac{1}{1-x} \right) \right]^{bt-1} \Gamma(bt)^{-1} \]  

(38)

(here \( t = L/X_0 \), and \( \Gamma \) is the Euler Gamma-function) obtained by Bethe and Heitler (24) (see also [23]) for the model bremsstrahlung cross section

\[ \frac{d\sigma}{dx} = b \left[ \log \left( \frac{1}{1-x} \right) \right]^{-1}. \]  

(39)

This theoretical experiment shows that for the kinematical domain \( 5 < k < 500 \) MeV, which will subsequently be interesting, Eq. (35) with the \( K \)-factor has inaccuracy \( \lesssim 0.5\% \).

The \( K \)-factor was obtained ignoring the edge effects. In principle, in the probabilistic approach one can obtain a formula for the \( K \)-factor including the boundary radiation. However, as was above mentioned, the probabilistic approach itself is justified only to zeroth order in \( L'/L \). On the other hand, the boundary radiation is an effect of the order of \( \sim L'/L \). For this reason an evaluation of the \( K \)-factor including the edge effects does not make much sense, and we will compare our predictions with experiment...
using Eqs. (35), (37). For the targets with \( L \sim 2 - 6\% X_0 \) at \( E_e = 25 \) GeV the corresponding errors cannot exceed a few percent at \( k \sim 5 - 10 \) MeV, and become negligible for \( k \gtrsim 20 - 30 \) MeV. For \( E_e = 8 \) GeV they are negligible in the whole range of \( k \).

Due to the possibility of absorption of the radiated photons in the target the spectra of Ref. [11] do not correspond exactly to the electron struggling function. In our analysis, bearing in mind dominance of the one-photon emission, we take into account the photon absorption by multiplying the theoretical electron struggling function by the averaged one-photon absorption factor \( \langle K_{abs} \rangle \approx 1 - L/2\lambda_{ph} \), where \( \lambda_{ph} \) is the photon attenuation length. This factor decreases our theoretical predictions by \( \lesssim 1 - 3\% \) for the carbon and aluminum targets used in [11]. For other targets the effect is even smaller.

It is appropriate here to comment also about the status of Eqs. (3), (6) when multi-photon effects become important. It is clear that for \( L \gtrsim X_0 \) the formulas of previous section are inapplicable. Nonetheless, Eqs. (3), (6) will give approximately right predictions for the intensity of radiation of soft photons with \( L' \ll L \) on the targets with \( L \ll X_0 \). In this case one can neglect the possibility of radiation of hard photons, and evaluate the bremsstrahlung rate at \( x \ll 1 \) ignoring the electron energy loss and energy correlations for emitted photons. Then, one can easily show that multi-photon contribution completely cancels the effect of the attenuation factor for the one-photon emission, and the intensity of bremsstrahlung is given by the formula obtained for emission of a single photon.

### 4 Numerical results and comparison with the SLAC data

We will compare our predictions with the SLAC data [11] for the targets with \( L \sim 0.7 - 6\% X_0 \). We exclude from our analysis the data for the gold target with \( L = 0.1\% X_0 \) for which there is a problem with normalization of the experimental spectrum [11]. The measurements of Ref. [11] were performed with 8 and 25 GeV electron beams for the total
radiated energy from 200 keV to 500 MeV. In the present paper we restrict ourselves to the region above 5 MeV. In this case one can neglect the dielectric effect which was not included in our analysis. On the other hand, this also justifies, as was argued in section 3, the probabilistic treatment of the multi-photon effects for the targets with \( L \sim 2 - 6\% X_0 \). At \( E_e = 8 \) GeV this approach can be also used for the 0.7\% \( X_0 \) gold target. To illustrate the degree of nonlocality in photon emission we show in Table 2 the ratio \( L/L' \) at \( k = 5 \) and 100 MeV for the targets with \( L \lesssim 3\% X_0 \) used in the SLAC experiment [11] evaluated using Eq. (29) (we adopt for the targets the notations of Ref. [11]). Table 2 demonstrates that even for the lower bound of our kinematical domain the inequality \( L'/L \ll 1 \) is satisfied for all the targets with \( L \gtrsim 2\% X_0 \), and at \( E_e = 8 \) GeV also for the 0.7\% \( X_0 \) gold target. The only exception is the 0.7\% \( X_0 \) gold target at \( E_e = 25 \) GeV. In this case the probabilistic approach becomes applicable for \( k \gtrsim 50 - 100 \) MeV. However, for the 0.7\% \( X_0 \) target the \( K \)-factor is close to unity. For this reason the inaccuracy of the probabilistic approach cannot lead to considerable errors in the region \( k \lesssim 50 - 100 \) MeV, and we will use the probabilistic \( K \)-factor (37) in this case as well.

In Fig. 1 we have plotted the infinite medium suppression factor as a function of the photon momentum for carbon, aluminum, iron, tungsten and uranium for 8 and 25 GeV electrons. It is seen that the LPM effect is considerably stronger for 25 GeV electrons. The upper bound of the region of \( k \) where LPM suppression becomes significant for the results shown in Fig. 1 agrees with the values of \( k_{LPM} \) given in Table 1.

Our numerical calculations show that at \( E_e = 8 \) GeV for all the targets used in [11] the finite-size effects are negligible and the exact suppression factor is close to that for infinite medium. At \( E_e = 25 \) GeV they become sizeable at \( k \lesssim 10 \) MeV for the targets with \( L \sim 2 - 3\% X_0 \). To illustrate the role of the finite-size effects we have plotted in Fig. 2 the results for the suppression factor for the 0.7\% \( X_0 \) gold and 2\% \( X_0 \) lead targets (solid line). We also show in this figure the results for infinite medium (dashed line) and predictions of the frozen-size approximation (dotted line) [15]. One can see from Fig. 2 that at \( E_e = 8 \) GeV the finite-size effects are negligible. For the lead target at \( E_e = 25 \) GeV at \( k \sim 5 \) MeV.
MeV the edge effects increase the radiation rate by $\sim 10\%$ and become negligible for $k \gtrsim 15 - 20$ MeV. In the case of the 0.7% $X_0$ gold target for 25 GeV electrons the exact suppression factor differs strongly from that for infinite medium at $k \lesssim 10 - 15$ MeV. It is seen that in this region the frozen-size approximation works well. The transition between the infinite medium and the frozen-size regime occurs at $k \sim 20$ MeV. In terms of the photon formation length (for 25 GeV electrons $L_f$ (mm) $\approx 0.94 \cdot (1\text{MeV}/k(\text{MeV}))$ in the region of $k$ shown in Fig. 2) the borderline between the two regimes is at $L_f/L \sim 2$ (for the 0.7% $X_0$ gold target $L = 0.023$ mm), and in terms of the medium-modified formation length ($L'_f/L \sim 0.6$). Thus, Fig. 2b shows that the transition between the infinite-medium and the frozen-size regime occurs in a sufficiently narrow region in the vicinity of $k$ corresponding to $L'_f \sim L/2$.

To illustrate the role of multi-photon emission we show in Fig. 3 $k$-dependence of the $K$-factor (37) for some of the targets used in Ref. [11]. It is seen that multi-photon effects are important for the targets with $L \gtrsim 2\% X_0$. However, we see that, as was above said, they are marginal for the 0.7% $X_0$ gold target. Note that the decrease of the $K$-factor at small $k$ seen from Fig. 3 is connected with the $x$-dependence of the exponential attenuation factor on the right-hand side of Eq. (37).

In Figs. 4-10 we compare our predictions (solid line) with the experimental spectra of Ref. [11] (the theoretical predictions and experimental data presented in the same form as in Ref. [11]). The theoretical curves have been obtained taking into account multi-photon emission and photon absorption. To demonstrate the role of the LPM effect better we also show the Bethe-Heitler spectrum (dashed line). The theoretical curves in Figs. 4-10 were multiplied by the normalization constants, $C_{\text{norm}}$, which were adjusted to minimize $\chi^2$ for our predictions. Their values and the corresponding $\chi^2$ per degree of freedom are given in Table 3. For most of the targets the fit quality is quite good $\chi^2/N \sim 1$ (the value of $\chi^2/N$ averaged over all the targets is $\sim 1.5$). This says that our predictions describe well the shape of the experimental spectra. This is also seen directly from Figs. 4-10. For the 0.7% $X_0$ gold target at 25 GeV (Fig. 8b) we have also depicted the spectrum
obtained with the infinite medium suppression factor (dot-dashed line). It is seen that
for this version the curve goes below the experimental points for \( k \lesssim 30 \text{ MeV} \), while the
curve obtained including the finite-size effects is close to the experimental spectrum in
the whole range of \( k \). It is worth to emphasize that our theoretical predictions do not
contain fitting parameters except the normalization constants.

From Table 3 one can see that for most of the targets the agreement of our predictions
with experimental data in the absolute cross section is within \( \sim 5\% \). This is close to
estimate of the systematic error \( \sim 3.5 - 4.6\% \) given by the authors of Ref. [11]. However,
the disagreement in normalization is rather big for the uranium targets \( (C_{\text{norm}} \approx 0.86 - \\
0.89 \text{ for both 8 and 25 GeV beams}) \), and for the 0.7\% \( X_0 \) gold target for 8 GeV beam
\( (C_{\text{norm}} \approx 1.17) \). The origin of the above disagreement in normalization is not clear.
Note that the normalization constant for the 0.7\% \( X_0 \) gold target for \( E_e = 25 \text{ GeV} \)
obtained in the present paper is bigger than that of our previous analysis [16] by \( \sim 13\% \).
This discrepancy is connected with neglecting the Coulomb correction to the dipole cross
section and multi-photon emission in Ref. [16] which, however, practically do not affect
the shape of the the spectrum.

5 Conclusion

In the present paper we have carried out a detailed theoretical analysis of the recent SLAC
data [11] on the LPM effect for bremsstrahlung from 8 and 25 GeV electrons in a variety
of materials. The calculations have been performed within a rigorous light-cone path
integral approach to the LPM effect previously developed in Refs. [14, 16], which reduces
evaluation of the radiation rate to solution of a two-dimensional Schrödinger equation
with an imaginary potential. This potential is proportional to the dipole cross section for
scattering of \( e^+e^- \) pair off an atom. In our calculations we treat rigorously the Coulomb
effects and include the inelastic processes. We have compared the theoretical prediction
with the SLAC data taking into account multi-photon emission and photon absorption.
For most of the targets our predictions are in very good agreement with the experimental data. In particular, we describe well the spectrum for the 0.7% $X_0$ gold target at $E_e = 25$ GeV for which the finite-size effects play an important role.

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Table 1: The values of $k_{LPM}$ obtained using Eq. (28) for the target materials used in the SLAC experiment [11]. In the third column we give the radiation lengths used in Ref. [11] and in the present paper.

| Material | Z  | $X_0$ (mm) | $k_{LPM}$ (MeV) $E_e = 8$ GeV | $k_{LPM}$ (MeV) $E_e = 25$ GeV |
|----------|----|------------|-----------------------------|-------------------------------|
| C        | 6  | 196        | 0.42                        | 4.1                           |
| Al       | 13 | 89         | 0.93                        | 9.1                           |
| Fe       | 26 | 17.6       | 4.7                         | 46.1                          |
| W        | 74 | 3.5        | 23.8                        | 230                           |
| Au       | 79 | 3.3        | 25.1                        | 244                           |
| Pb       | 82 | 5.6        | 14.8                        | 144                           |
| U        | 92 | 3.5        | 23.7                        | 230                           |
Table 2: The ratio $L/L'_f$ for the targets with $L \lesssim 3\% \times 0$ used in Ref. [11] estimated with the help of Eq. (25).

| Target | $E_e = 8$ GeV | | $E_e = 25$ GeV | |
|--------|---------------|---------------|---------------|---------------|
|        | $k = 5$ MeV  | $k = 100$ MeV | $k = 5$ MeV  | $k = 100$ MeV |
| 2%C    | 213 | 4270 | 22 | 437 |
| 3%Al   | 162 | 3250 | 22 | 333 |
| 3%Fe   | 25.5 | 510 | 8 | 52 |
| 2%W    | 10 | 92 | 3.3 | 15 |
| 0.7%Au | 2.7 | 24 | 0.86 | 3.85 |
| 2%Pb   | 13.5 | 156 | 4.3 | 19.3 |
| 3%U    | 9 | 82 | 2.8 | 12.6 |
Table 3: List of the normalization constants adjusted to match our predictions (solid line in Figs. 4-10) with the SLAC data [11]. The corresponding $\chi^2$ per degree of freedom are also given. The second column gives the target thicknesses in mm.

| Target | $L$ (mm) | $E_e = 8$ GeV | $E_e = 25$ GeV |
|--------|---------|---------------|---------------|
|        | $C_{\text{norm}}$ | $\chi^2/N$ | $C_{\text{norm}}$ | $\chi^2/N$ |
| 2%C   | 4.1     | 0.943 ± 0.004 | 0.98 | 0.957 ± 0.003 | 4.2 |
| 6%C   | 11.7    | 0.964 ± 0.004 | 1.14 | 0.964 ± 0.002 | 3.46 |
| 3%Al  | 3.12    | 0.985 ± 0.003 | 1.02 | 0.981 ± 0.003 | 1.41 |
| 6%Al  | 5.3     |               |     | 0.982 ± 0.003 | 1.6 |
| 3%Fe  | 0.49    | 1.00 ± 0.005  | 0.79 | 0.972 ± 0.002 | 1.85 |
| 6%Fe  | 1.08    |               |     | 0.96 ± 0.002  | 1.55 |
| 2%W   | 0.088   | 0.942 ± 0.003 | 1.14 | 0.953 ± 0.003 | 2.8 |
| 6%W   | 1.08    |               |     | 1.007 ± 0.003 | 1.45 |
| 0.7%Au| 0.023   | 1.174 ± 0.007 | 1.44 | 1.056 ± 0.004 | 0.8 |
| 6%Au  | 0.2     | 1.014 ± 0.003 | 1.15 | 1.031 ± 0.002 | 0.89 |
| 2%Pb  | 0.15    | 1.032 ± 0.004 | 1.01 | 1.009 ± 0.002 | 0.94 |
| 3%U   | 0.079   | 0.875 ± 0.003 | 1.0  | 0.886 ± 0.002 | 2.63 |
| 5%U   | 0.147   | 0.865 ± 0.004 | 1.04 | 0.877 ± 0.003 | 1.63 |
Figure 1: The infinite medium suppression factor for bremsstrahlung from 8 (a) and 25 (b) GeV electrons for carbon (solid line), aluminum (dashed line), iron (long-dashed line), tungsten (dot-dashed line) and uranium (dotted line).
Figure 2: The LPM suppression factor for the 0.7\% X_0 gold and the 2\% X_0 lead targets (solid line). The dashed line shows the results for infinite medium, and the dotted line corresponds to the frozen-size approximation (15).
Figure 3: The multi-photon $K$-factor (37) for the carbon (a), gold (b), lead (c) and uranium (d) targets. The solid and dashed line correspond to 8 and 25 beams, respectively.
Figure 4: The spectrum in the radiated energy for the 2% $X_0$ (a, b) and 6% $X_0$ (c, d) carbon targets. The experimental data from Ref. [11]. The solid line shows our results obtained using Eq. (6). The dashed line shows the Bethe-Heitler spectrum. In both these cases the multi-photon emission and photon absorption are taken into account.
Figure 5: The same as in Fig. 4 but for the 3% $X_0$ (a, b) and 6% $X_0$ (c) aluminum targets.

Figure 6: The same as in Fig. 4 but for the 3% $X_0$ (a, b) and 6% $X_0$ (c) iron targets.
Figure 7: The same as in Fig. 4 but for the 2% $X_0$ (a, b) and 6% $X_0$ (c) tungsten targets.

Figure 8: The same as in Fig. 4 but for the 0.7% $X_0$ (a, b) and 6% $X_0$ (c, d) gold targets.
Figure 9: The same as in Fig. 4 but for the 2% $X_0$ lead target.

Figure 10: The same as in Fig. 4 but for the 3% $X_0$ (a, b) and 5% $X_0$ (c, d) uranium targets.