Abstract

High-dimensional depth separation results for neural networks show that certain functions can be efficiently approximated by two-hidden-layer networks but not by one-hidden-layer ones in high-dimensions $d$. Existing results of this type mainly focus on functions with an underlying radial or one-dimensional structure, which are usually not encountered in practice. The first contribution of this paper is to extend such results to a more general class of functions, namely functions with piece-wise oscillatory structure, by building on the proof strategy of (Eldan and Shamir, 2016).

A common theme in the proof of such results is the fact that one-hidden-layer fail to approximate high-energy functions whose Fourier representation is spread in the domain. On the other hand, existing approximation results of a function by one-hidden-layer neural networks rely on the function having a sparse Fourier representation. The choice of the domain also represents a source of gaps between upper and lower approximation bounds. Focusing on a fixed approximation domain, namely the sphere $S^{d-1}$ in dimension $d$, we provide a characterization of both functions which are efficiently approximable by one-hidden-layer networks and of functions which are provably not, in terms of their Fourier expansion.

Keywords: Neural networks, Depth separation, Approximation theory.

1. Introduction

Learning in high-dimensions is a challenging task for computational, statistical and approximation reasons. Even in the classic supervised learning setup, current empirical successes of Deep Learning algorithms remain largely out of reach for existing theories, despite phenomenal recent progress. Amongst the algorithmic aspects enabling this success, depth remains a major non-negotiable element. Depth in structured neural networks such as Convolutional Neural Networks provides a multiscale processing of information, but more generally it defines an intricate function class with powerful approximation biases.

The benefits of depth for approximating certain functions of interest is a long-standing question. The classic result of the universal approximation theorem ensures approximation by neural networks of any continuous function, but it is limited to shallow (that is, one-hidden-layer) models and comes with possibly exponential (in the dimension) rates. The seminal work (Barron, 1994) provides dimension-free quadratic approximation rates under a condition of sparsity of the Fourier transform. Recent works (Telgarsky, 2016; Eldan and Shamir, 2016; Daniely, 2017) suggest that this property is somehow also necessary to have polynomial approximation rates, by constructing examples of deep networks having slowly decaying Fourier coefficients. In the high-dimensional regime, this entails having energy sufficiently spread in direction and away from zero. On the other hand, these results are currently limited to univariate or essentially radial functions.
In this work we extend the previous results cited above, further cementing Barron’s intuition. We describe rates of approximation by one-hidden-layer networks in terms of the number of units $N$ of the network, by looking at the Fourier representation of the function to be approximated. We consider two type of rates, inspired by (Safran et al., 2019): (i) the rate of approximation is polynomial in both the input dimension $d$ and the error estimation $\epsilon$, that is $N \sim \text{poly}(d, \epsilon^{-1})$ - we refer to this rate of approximation as universal approximation (ii) for any fixed error threshold $\epsilon$, the number of units $N$ needed for approximation depends at most polynomially on $d$, that is $N \sim \text{poly}(d)$ for any fixed error threshold $\epsilon$ - we refer to this rate of approximation as fixed-threshold approximation. We distinguish two fundamentally different regimes of approximation: relative to a ‘heavy-tailed’, unbounded data distribution, or relative to a concentrated distribution. Whereas the former captures the most general setup, the latter is motivated by practical ML applications. Our contributions are as follows.

- We consider a broad class of two-hidden-layer networks exhibiting piece-wise oscillatory behavior, namely functions of the form

$$f_{r,w,v} : x \in \mathbb{R}^d \mapsto e^{2\pi ir(v^T x + w^T x)}.$$  

In Section 3, we show that, under appropriately heavy-tailed data distributions, approximation at a rate $N \sim \text{poly}(d)$ cannot hold (unconditionally on the weights of the approximant network), as long as the rate of oscillations $r$ grows faster than $d$. On the other hand, $f_{r,w,v}$ can be universally represented (that is, at a rate $\text{poly}(d, \epsilon^{-1})$) as a two-hidden-layer network with any practical activation of choice. The proof of this result (Theorem 2) extends the main idea introduced by the results of (Eldan and Shamir, 2016).

- In Section 4, we show that the $\text{poly}(d)$-oscillatory aspect and the heavy-tailed data distributions are jointly necessary in the depth-separation result mentioned above. More specifically, we show that any deep network, with $O(1)$-bounded weights and $O(1)$-Lipschitz activation, can be fixed-threshold approximated by one-hidden-neural networks over a compact set of radius $O(1)$ (Theorem 9). This extends the results of (Safran et al., 2019).

- Aforementioned depth separation results consider functions whose Fourier representation is spread in high frequencies. On the other hand, universal approximation results often require the function to be approximated to be, in some sense, sparse in the Fourier domain. Unfortunately, there are currently many gaps between these two types of results, one of them being the definition of approximation domain. In order to reduce the gap between the two results above, we consider approximation on a fixed compact domain, namely the unit sphere $S^{d-1}$, where Fourier analysis can be done using spherical harmonics. We individuate two conditions on the spherical harmonics decomposition of a function $f \in C(S^{d-1})$. The first is a sparsity condition on the decomposition, which we show to be sufficient to prove universal approximation (that is, at a rate $N \sim \text{poly}(d, \epsilon^{-1})$) of $f$ by one-hidden-layer networks. The second is a high-energy spreadness condition on the spherical harmonics decomposition of $f$, which we show to imply that universal approximation of $f$ by one-hidden-layer networks cannot hold. This is the content of Section 5, whose main results are summarized in the introduction of the section.

**Related Works** There is a huge literature of approximation results for neural networks. Early approximation results provided upper and lower bounds on the approximation of some functional spaces such as Sobolev spaces (Maiorov and Meir, 2000) or $L^p$ spaces (Pinkus, 1999) by neural networks. For high input dimensions $d$, such results hold for functions with smoothness proportional to $d$, or require an approximation rate that scales as $N \sim \epsilon^{-d}$ (see e.g. (Petersen, 2020) for a review), where $N$ denotes the number of units of the network and $\epsilon$ the error threshold.
In more recent years, quite a few works pointed out the benefits of deep networks versus their shallow counterparts from the point of view of approximation rates. For example, this has been shown for sawtooth function (Telgarsky, 2016), functions with positive curvature (Liang and Srikant, 2016; Yarotsky, 2017; Safran and Shamir, 2017), functions with a compositional structure (Poggio et al., 2017), piecewise smooth functions (Petersen and Voigtlaender, 2018), functions in $L^2(\mathbb{R}^d)$ ( Bölcskei et al., 2019), Gaussian mixture models (Jalali et al., 2019) or model reduction models (Rim et al., 2020). It must be noticed that most of the cited works show depth separation that is independent of the dimension $d$ and that increases exponentially with the depth of the network. Another line of works (Eldan and Shamir, 2016; Daniely, 2017; Safran et al., 2019) on the other hand shows depth separation exponential in the dimension $d$, between networks with one and two hidden layers. This is the framework of this work. It was also shown recently that depth separation results between fixed depths greater than this are arguably difficult to prove (Vardi and Shamir, 2020; Vardi et al., 2021).

Many works tried to understand the reasons behind depth-separation, by considering some complexity measure over functions. These complexity measures encompass the notions of global curvature (Poole et al., 2016), fractals (Malach and Shalev-Shwartz, 2019), tensor decomposition (Cohen et al., 2016), Betti numbers (Bianchini and Scarselli, 2014), periodicity (Chatziarfratis et al., 2019) or number of linear regions (Pascun et al., 2013; Montufar et al., 2014; Rashu et al., 2017; Hanin and Rolnick, 2019a,b). Another related question is also whether depth-separation holds from a learnability (therefore, not solely approximation) point of view as well (Malach and Shalev-Shwartz, 2019; Malach et al., 2021). In this work we focus on approximation and we consider the number of oscillations or, equivalently, the Fourier representation as a complexity measure. This is the approach followed by e.g. (Telgarsky, 2016; Eldan and Shamir, 2016; Daniely, 2017), which construct examples of deep neural networks, whose Fourier energy is exponentially higher than those of shallow neural networks with a moderate number of units.

On the other hand, sparsity of the Fourier transform has been used to show polynomial rates of approximation of functions by neural networks (Klusowski and Barron, 2018; Ongie et al., 2019; Bresler and Nagaraj, 2020). In the last part of the paper, we show that an equivalent condition can be described in terms of spherical harmonics decomposition.

2. Preliminaries

Neural networks Throughout the paper, we consider feed-forward neural networks defined as follows. For $L \geq 1$, an $L$-hidden-layer neural network is a function 
\[ f : x \in \mathbb{R}^d \to x_L(x) \in \mathbb{C} \]
where $x_L$ is defined by recursion by $x_0(x) = x$,
\[ x_k(x) = \sigma^k(A^k x_{k-1}(x)) \quad \text{for } k \in [L] \quad \text{and} \quad x_{L+1}(x) = a^T L+1 x_L(x), \]
where $A^k = [a_{k,1} \cdots a_{k,d_k}]^T \in \mathbb{R}^{d_k \times d_{k-1}}$ for $k \in [L]$ with $d_0 = d$, $a_{L+1} \in \mathbb{C}^{d_{L+1}}$ and $\sigma^k : \mathbb{R}^{d_k} \to \mathbb{R}^{d_k}$ are component-wise activation functions, that is $(\sigma^k(x))_i = \sigma^k_i(x_i)$ for some function $\sigma_i^k$. The number of units of $f$ is given by $N(f) = \sum_{k=1}^L d_k$. We say that a network has activation $\sigma$ if $\sigma^k_i(x) = \sigma(x+b^k_i)$ for some $b^k_i \in \mathbb{R}$ for all $k,i$. In particular, we denote the space of one-hidden-layer networks with at most $N$ units by $F_N$, and the space of one-hidden-layer networks with at most $N$ units and given activation $\sigma$ by $F^\sigma_N$. Some of the following results also require a control on the value of the weights; such controls are expressed in terms of $m_p(f) = \max_{k,i} \|a_{k,i}\|_p$ for some $p \in [1, \infty]$.

Neural network approximation rates We measure the approximation error between two functions $f, g : \Omega \subset \mathbb{R}^d \to \mathbb{C}$ in terms of the $L^2_\mu$ (with respect to a probability measure or density $\mu$) or $L^\infty$ norm,
for some one-hidden-layer
threshold approximable
if for any
universally approximable
by one-hidden-layer networks if it is approximable at a
sigmoid( also refer to as
shallow
to establish upper and lower bounds for certain function classes by one-hidden-layer networks (which we
also refer to as shallow), for high dimensions d. We say that a sequence \( \{ f^{(d)} : \Omega_d \subseteq \mathbb{R}^d \to \mathbb{R} \}_{d \geq 2} \) is universally approximable by one-hidden-layer networks if it is approximable at a \( \text{poly}(d, e^{-1}) \) rate; that is
if there exists some constants \( \alpha > 0 \) and \( \beta > 0 \) such that for every \( \epsilon > 0 \) it holds \( D(f^{(d)}, f_N; \Omega_d)_{\infty} \leq \epsilon \)
for some one-hidden-layer \( f_N \in \mathcal{F}_N \) satisfying \( N + m_{\infty}(f_N) \leq \alpha(d e^{-1})^\beta \). We say that \( \{ f^{(d)} \}_d \) is fixed-threshold approximable if for any \( \epsilon \in (0, 1) \) it is \( \epsilon \)-approximable at a \( \text{poly}(d) \) rate; that is if for any \( \epsilon > 0 \) there exists some constants \( \alpha > 0 \) and \( \beta > 0 \) such that for every \( \epsilon > 0 \) it holds \( D(f^{(d)}, f_N; \Omega_d)_{\infty} \leq \epsilon \)
for some one-hidden-layer \( f_N \in \mathcal{F}_N \) satisfying \( N + m_{\infty}(f_N) \leq \alpha d^\beta \). These approximation schemes were introduced in (Safran et al., 2019). To ensure significance of the approximation rates, in the following upper
and lower bounds are stated for objective functions \( f \) normalized such that \( \| f \|_{\infty} \leq 1 \).

**Infinitely-wide limits and Barron spaces** When considering the spaces \( \mathcal{F}^1_N \) of one-hidden-layer networks with a given activation \( \sigma \) with increasing number of units \( N \), it is natural to consider the continuous analogous variational space. This is defined as the space
\[
\mathcal{F}^1_{\sigma} = \left\{ f : \mathbb{R}^d \to \mathbb{C} : f = f^\sigma_\pi \text{ for some signed Radon measure } \pi \right\}
\]
where \( f^\sigma_\pi(x) = \int_{\mathbb{R}^d \times \mathbb{R}} \sigma(w^T x + b) \, d\pi(w, b) \). The space \( \mathcal{F}^1_{\sigma} \) is a Banach space if equipped with the norm
\[
\gamma^\sigma(f) = \inf_{\pi : f = f^\sigma_\pi} \| \pi \|_1
\]
where \( \| \pi \|_1 \) denotes the total variation norm. Many proofs of approximation rates actually first construct an approximation in \( \mathcal{F}^1_{\sigma} \); the rate of approximation by finite neural networks can then be controlled as follows\(^1\).

**Proposition 1 (Informal)** Let \( f = f_\pi \in \mathcal{F}^1_{\sigma} \) where \( \pi \) has compact support. Then it holds that
\[
\inf_{f_N \in \mathcal{F}^1_N} \| f - f_N \|_{\infty} \lesssim \text{poly}(d) \cdot \frac{\gamma^\sigma(f)}{N^{1/2}}.
\]
Moreover, \( f_N \) satisfying the bound can be chosen to satisfy \( \gamma^\sigma(f_N) \lesssim \gamma^\sigma(f) \).

**Activation assumptions** Finally, the results in the next sections generally hold for activations satisfying the following assumption, which are satisfied by common activation such as the ReLU \( \text{ReLU}(x) = x_+ \)
or the sigmoid \( \text{sigmoid}(x) = \frac{1}{1 + e^{-x}} \) (Eldan and Shamir, 2016). Most of the results can be easily
generalized to hold under less strict conditions, but we take these assumptions for sake of simplicity.

**Assumption 1** Given an activation \( \sigma : \mathbb{R} \to \mathbb{R} \), there exist constants \( \iota_\sigma \) and \( \nu_\sigma \) such that
1. it is \( \iota_\sigma \)-linearly bounded: \( \sigma(x) \leq \iota_\sigma (1 + |x|) \) for all \( x \in \mathbb{R} \);
2. for any \( L \)-Lipschitz function \( f : \mathbb{R} \to \mathbb{R} \) constant outside of an interval \([-R, R] \) and any \( \epsilon > 0 \) there exits \( f_N \in \mathcal{F}^1_N \) with \( D_{\infty}(f, f_N; \mathbb{R}) \leq \epsilon \) such that \( N + w_{\infty}(f_N) \leq \nu_\sigma LR \epsilon^{-1} \).

\(^1\) Different versions of similar statements are available in the literature, for different choices of activations and integration do-

1989; Barron, 1993; Yukich et al., 1995; Cheung and Barron, 2000; Klusowski and Barron, 2018)
3. Depth separation for functions with piece-wise structure

Our starting point for the study of depth-separation is to consider a generic data distribution $\mu$ with ‘adversarial’ properties against shallow approximations. In the seminal work (Eldan and Shamir, 2016), the authors establish an unconditional depth-separation result (with no restrictions on the weight norms) by considering a density $\mu$ in $\mathbb{R}^d$ with tails $\mu(\|x\|_2) \sim \|x\|_2^{(d+1)/2}$ and a radial function $f_d(x) = h_d(\|x\|_2)$ with $h_d : \mathbb{R} \to \mathbb{R}$ a carefully chosen oscillating function with compact support. The proof of Eldan and Shamir reveals the limitations of shallow neural networks at approximating high-dimensional functions via a powerful harmonic analysis insight, that is particularly convenient in the setting of radial functions. In this section, we show that their proof strategy can be extended to include more diverse function classes, namely those arising naturally from deeper ReLU networks. Specifically, we consider networks of the form

$$f_{r,w,v} : x \in \mathbb{R}^d \mapsto \sigma_r(v^T x + w^T x_+) \quad (1)$$

where $x_+$ denotes the element-wise ReLU activation, $v, w \in \mathbb{R}^d$ and $\sigma_r(t) = e^{2\pi i rt}$. We are thus considering a function which is piece-wise oscillatory, with constant envelope $|f_{r,w,v}(x)| = 1$, and where the frequency of oscillations is controlled by $r$. The main result of this section can be summarized as follows.

**Theorem 2 (Informal)** Assume that $\|w\|_2, \|v\|_2 = \Theta(1)$ and that $r = \Theta(d^k)$ for some $k > 0$. Then there exists a (low-decay) product measure $\mu$ on $\mathbb{R}^d$ such that the function $f_{r,w,v}$ is universally approximable by two-hidden-layer networks but it is not fixed-threshold approximable by one-hidden-layer networks.

We now give the full detail of the results. Let $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\|\psi\|_2 = 1$, and such that its Fourier transform $\hat{\psi}$ is compactly supported in $[-K, K]$, for some $K > 0$. Assume also that

$$\|\psi\|_1 < \sqrt{8/K} \quad (2)$$

The condition ensure that the density $\psi$ is sufficiently spread away from zero (see Remark 5 for an example). Our first objective is to establish depth separation for the approximation of $f_{r,w,v}$ under the $L^2$ metric defined by the probability density $\varphi^2$, where $\varphi : x \in \mathbb{R}^d \mapsto \prod_{j=1}^d \psi(x_j)$.

**Theorem 3** For a fixed $\gamma > 0$, define

$$\tau \doteq \sup \|v + w_S\|_\infty, \quad \Omega \doteq \{ j \in [d] : |r_j| > \gamma d^2 \} \quad \text{and} \quad \eta \doteq \frac{|\Omega|}{d},$$

where $w_S \in \mathbb{R}^d$ is defined by $w_{S,i} = w_i \mathbbm{1}_{\{i \in S\}}$. Assume that

(i) oscillations grow as $\text{poly}(d)$, that is $\tau \cdot r = \Omega(d^k)$ for some constant $k \geq 1$;

(ii) the density $\varphi^2$ is sufficiently spread, i.e. $K \|\psi\|_1^2 < 2^{2\eta+1}$.

Then there exists a constant $\alpha \in (0, 1)$ such that

$$\inf_{f_N \in \mathcal{F}_N} D_2(f_{r,w,v}, f_N; \varphi^2) \geq 1 - N \cdot \alpha^d \cdot O(d^{k+1}) \quad (3)$$

**Proof [Sketch]** We follow the same strategy as in (Eldan and Shamir, 2016), which expresses the approximation error in the Fourier domain, as

$$D(f_{r,w,v}, g; \varphi^2) = \|f_{r,w,v} \cdot \varphi - g \cdot \varphi\|_2 = \|\hat{f}_{r,w,v} \ast \hat{\varphi} - \hat{g} \ast \hat{\varphi}\|_2,$$

and then leverages a key property of the target function $f_{r,w,v}$, namely that its Fourier transform has its energy sufficiently spread in the high-frequencies, after the convolution by $\hat{\varphi}$. Since $\hat{\varphi}$ is compactly supported
and the Fourier transform of a single-unit network is localised in a frequency ray, the Fourier transform of $f_{r,w,v} \cdot \varphi$ is a collection of ‘tubes’. Rather than coming from the radial property, in our case such frequency spread is caused by the ‘shattering’ of the first ReLU layer, which effectively creates $\Theta(2^d)$ different frequencies. The piece-wise structure arising from the ReLU is handled in the Fourier domain by the Hilbert transform of the function $\psi$, which has sufficient decay thanks to the assumptions on $\psi$. The detailed proof is deferred to Appendix A.

Remark 4 Theorem 3 asks for two main conditions to hold. First, the magnitude of oscillations of the function (parametrised by $r$) must grow at least polynomially with $d$, similarly as (Eldan and Shamir, 2016) and (Daniely, 2017). Second, the data distribution $\mu$ with density $\varphi^2$ should be heavy-tailed, in order for its Fourier transform to be sufficiently localised. Notice that this lower bound is unconditional on the weights of the neurons $m_\infty(f_N)$, as in Eldan and Shamir (2016). When $r$ does not grow fast enough with $d$, the energy starts piling up at the low frequencies, creating an important roadblock to establish approximation lower-bounds, and leaving open the possibility of efficient shallow approximation. Similarly, when $\mu$ concentrates too quickly, the proof strategy also fails, due to the fact that in that case $\hat{\varphi}$ is too spread in the Fourier domain, creating full overlap of the energies.

Remark 5 The admissibility condition (2) is necessary since $\eta \leq 1$ by definition. The choice $\psi(t) = \sqrt{3/2} \text{sinc}^2(\pi t)$ corresponds to $K = 1$, $\|\psi\|_1 = \sqrt{3/2}$ and $\|\psi\|_2^2 = 1$, which verifies (2). In that case, from condition (ii) we need $\eta > \frac{\log_2 3}{2} \approx 0.79$. However, the choice $\psi(t) = C \text{sinc}(\pi t)$ (the equivalent separable version of the of density considered in (Eldan and Shamir, 2016)) is not admissible, since $\psi$ is not in $L^1(\mathbb{R})$. The lower bound is optimized by finding compactly supported windows with an optimal $L^1$ to $L^2$ ratio of their Fourier transforms. This condition thus amounts to a form of Heisenberg uncertainty principle, since the $L^1$-to-$L^2$ ratio is a measure of spatial localisation and $K$ is a measure of spectral localisation.

Remark 6 The assumptions of the theorem are easily verified by generic weights. For instance, if the weights $w, v$ are drawn iid from $\mathcal{N}(0, 1/d)$ and $r_d := d^3$, we have $\tau_d \lesssim \log d$ with high probability and $(r_d w_j) / d^d \overset{d}{\sim} Z \sim \mathcal{N}(0, 1)$, which implies that for any $\rho < 1$, we can pick $\gamma > 0$ such that $\Pr_{\mathcal{Z}}(|\mathcal{Z}| > \gamma) \geq \rho$, which by the law of large numbers implies that $\eta \to \rho$ with high probability.

Remark 7 The theorem considers a separable ReLU transform $x \mapsto x_+$, combined with a separable data distribution $\mu$ with density $\varphi^2$. One could expect a similar lower bound to apply in the more general case of a layer of the form $x \mapsto (Ux + b)_+$, $U \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d$. Such general case replaces the Hilbert transform of $\psi$ with the Fourier transform of indicators of convex polytopes, which has been used in the context of ReLU networks to characterize spectral properties (Rahaman et al., 2019).

According to the definition of neural networks we gave before, the function $f_{r,w,v}$ is naturally a two-hidden-layer neural network. Although, while there are cases of sinusoidal activations being used in practice, activations such as ReLU or sigmoid are more relevant to practical applications. The following theorem, proved in Appendix C, shows that we can efficiently represent the function $f_{r,w,v}$ in the hypothesis of the Theorem 3 as a two-hidden-layer neural network with fixed activation, such as the ReLU or the sigmoid.

Theorem 8 Let $\sigma$ be an activation satisfying Assumption 1. Assume that there exist constants $k \geq 1$ such that $m_\infty(f_{r,w,v}) \leq O(d^k)$ and assume that $\psi$ is such that $|\psi(x)| = O(|x|^{-2})$. Then, for every $\epsilon > 0$, there exists $f_N \in \mathcal{F}_N^r$ such that

$$N + m_\infty(f_N) \leq O\left(d^{(5k+1)/3} \epsilon^{-5/3}\right)$$

such that $D_2(f_N; f_{r,w,v}; \varphi^2) \leq \epsilon$.
Theorems 3 and 8 therefore establish a depth separation result. If \( f^{(d)} = f_{r_d, w_d, v_d} \) are defined for \( d \geq 2 \) with \( r_d, w_d, v_d \) satisfying the assumptions of both theorems with some fixed constant (that is, independent from \( d \)), then Theorem 3 says that \( \{ f^{(d)} \}_{d \geq 2} \) is not fixed-threshold approximable by one-hidden-layer networks, while Theorem 8 shows that it is universally approximable by two-hidden-layer networks with a fixed activation satisfying Assumption 1.

We thus identify two key aspects responsible for such depth separation: heavy-tailed data and oscillations growing with dimension. In the next sections we want to understand how necessary these two conditions are. The next section shows that if these two condition do not hold anymore, then a lower bound such as the one in Theorem 3 is not achievable; more specifically we show that the objective function is fixed-threshold approximable.

4. Approximation of deep networks by shallow ones

In this section, we show that any deep neural network \( f \) (which include the target functions considered in the previous section) can be approximated by shallow ones at a rate which is polynomial in \( d \), as long as the rate of oscillation in the inner layers of \( f \) is constant in \( d \) and the metric is concentrated in a ball of constant radius. We start by reporting the result in a general form for two-hidden-layer networks and we discuss some consequences and extensions afterwards.

Consider a two-hidden-layers neural network \( f \) defined as

\[
\begin{align*}
f : x \in \mathbb{R}^d \mapsto & \gamma^T g(W^T h(U^T x)) \in \mathbb{C},
\end{align*}
\]

where \( h : \mathbb{R}^p \to \mathbb{R}^p \) and \( g : \mathbb{R}^o \to \mathbb{R}^o \) are, respectively, component-wise 1-Lipschitz and \( (1, \alpha) \)-Holder activation functions, and \( U \in \mathbb{R}^{d \times p}, W \in \mathbb{R}^{p \times o}, \gamma \in \mathbb{C}^o \). We wish to approximate \( f \) with a one-hidden-layer neural network with a given activation.

**Theorem 9** Let \( K \) be a compact set of radius \( \sup_{x \in K} \|x\|_\infty = O(1) \) and assume that the weights of \( f \) defined at (4) are bounded (that is \( m_1(f) = O(1) \)), as \( d \to \infty \). Then, for every activation \( \sigma \) satisfying Assumption 1.2 and every \( \epsilon \in (0, 1) \) it holds that

\[
\inf_{f_N^\sigma \in F_N^\sigma} D_\infty(f, f_N^\sigma; K) \leq \epsilon \quad \text{for some } N \leq \exp \left( O \left( \epsilon^{-1-2/\alpha} \log(p/\epsilon) \right) \right).
\]

Moreover, it is possible to choose \( f_N^\sigma \) attaining (5) with \( m_\infty(f_N^\sigma) \) satisfying a bound similar to the one on \( N, \) for example \( m_\infty(f_N^\sigma) \leq (1 + N^2). \)

**Proof** [Sketch] The proof is constructive. Each of the non-linearities applied to the first hidden layer can be approximated by a trigonometric polynomial at a polynomial rate on the interval of interest. Similarly, every non-linearity applied to the second hidden layer can be approximated by a polynomial at a linear (in the degree of the polynomial) rate on the interval of interest. The composition of the two approximation following the structure of the target network result in a one-hidden-layer Fourier network (that is with activation \( \sigma_1(t) = e^{2\pi i t} \)) which size is controlled by the width of the first hidden layer and the degrees of the polynomial approximations. Moreover, it is also possible to control the size of the coefficients appearing in the final approximation. With this, we can approximate each summand in the one-hidden-layer Fourier network by a one-hidden-layer network with activation \( \sigma \) with a controlled number of units, thanks to Assumption 1.2. Fully detailed statement and proof are reported in Appendix D.2.

**Remark 10** A result similar to Theorem 9 can be shown for approximation of \( L \)-hidden-layers networks, which gives a rate scaling as \( \exp(O(\epsilon^{-L} \log(p/\epsilon))) \). See Proposition 36 in the appendix for a formal
statement. While it has been shown that generic $O(1)$-Lipschitz function can not be represented by neural networks with $N \sim \text{poly}(d)$ units (Vardi et al., 2021), an interesting related conjecture is whether our result can be generalized to any generic $O(1)$-Lipschitz function which is poly($d$)-computable.

**Remark 11** Notice that the approximation rate shown in Theorem 9 is actually polynomial in $p$ (the size of the first hidden layer of $f$) rather than in $d$. Although, up to choosing a worst (yet constant) exponent in $\epsilon$, we can replace $p$ by $d$ in the statement, by considering the function as a $(L+1)$-hidden-layer network, where the first layer is the identity.

**Remark 12** The result in Theorem 9 concerns uniform approximation over a compact set of constant radius. Using this result it is easy to prove analogous ones for the approximation in the $L^2$ metric, as long as the metric concentrates on a compact set of constant radius as $d \to \infty$. We give an example of this in the appendix (Proposition 35), showing a result on the lines of Corollary 14 for approximation in the $L^2$ metric under the Gaussian metric with covariance $d^{-1}I$.

The above theorem allows to recover, for any fixed threshold $\epsilon > 0$, a poly($d$) rate for the approximation of $f_{r,W,V}$ by one-hidden-layer networks and it can be seen as a generalization of Theorem 1 in (Safran et al., 2019). This is the content of the following corollaries. In the following $B_{r,p}$ denotes the $\ell^p$-ball of radius $r$ in $\mathbb{R}^d$.

**Corollary 13 (Radial functions)** Let $f(x) = \varphi(||x||_2)$, where $\varphi : [-1,1] \to \mathbb{R}$ is 1-Lipschitz. Then, it holds that

$$\inf_{f_N^\varphi \in F_{\varphi}^N} D_\infty(f_N^\varphi, f; B_{1,2}^d) \leq \epsilon \quad \text{for some } N \leq \exp(O(\epsilon^{-5} \log(d/\epsilon))).$$

Moreover, $f_N^\varphi$ can be chosen so that $m_\infty(f_N^\varphi) \leq \exp(O(\epsilon^{-5} \log(d/\epsilon))).$

Consider $f_{w,U} : x \in \mathbb{R}^d \mapsto e^{i w^T (U x)}$ for some $w \in \mathbb{R}^p$, $U \in \mathbb{R}^{p \times d}$. This is a more general version of the function $f_{r,w,v}$ considered in Section 3. If the weights are bounded as $d \to \infty$ (that is $m_1(f_{w,U}) = O(1)$), then Theorem 9 implies the following.

**Corollary 14 (Shallow approximation of (1))** If $r = O(1)$ as $d \to \infty$, it holds that

$$\inf_{f_N^\sigma \in F_{\sigma}^N} D_\infty(f_N^\sigma, f_{r,w,u}; B_{1,2}^d) \leq \epsilon \quad \text{for some } N \leq \exp(O(\epsilon^{-2} \log(p/\epsilon))).$$

Moreover, $f_N^\sigma$ can be chosen so that $m_\infty(f_N^\sigma) \leq \exp(O(\epsilon^{-2} \log(p/\epsilon))).$

In essence, in the above proposition, we show that it is possible to approximate a three-layer neural network with constant($d$) oscillations at a poly($d$) rate over a compact set of constant($d$) radius. On the other hand, it easy to show that it is also possible to obtain approximation at a poly($\epsilon^{-1}$) rate (see Appendix D.7), for fixed $d$. Finally, existing results in the literature (see (Safran et al., 2019)) show that universal approximation is not possible, the counterexample being essentially a radial function.

Interestingly, the upper bound in Theorem 9 does not depend on the number of units in the second layer of the objective function. This parameter is hidden in the control we impose on the $\ell^1$ norm of the objective weights. The proof technique of this upper bound highlights how the difficulty of approximating at poly($d$, $\epsilon^{-1}$) rate stems from the high-energy of the second layer, which requires the shallow network used for approximation to have a (potentially) exponential (in $d$) number of directions. Notice that the lower bound in Theorem 3 actually tells that the function is not fixed-threshold approximable. High oscillations in the lower bound (3) essentially ensure that an exponential (in $d$) number of neurons are necessary. An open questions at this point is whether a low-decaying measure is necessary for such a result to hold.
5. Approximation by shallow networks on the sphere: a spherical harmonics analysis

As already discussed, difficulties in approximating functions in high dimension by shallow networks appear when the function has a Fourier transform spread in a (exponential) number of directions in (polynomial) high energy. On the other hand, the presence of only one of these two condition is not enough to prevent efficient approximability. While the previous results highlight this, the lower bound presented above (Theorem 3) also uses a specific choice of error measure, with (polynomially) slowly decaying tails. The choice of a specific measure in the lower bound is also a source of gaps between the two regimes.

In this section, we aim to disentangle the role of the measure and understand how the Fourier representation can tell whether a function is efficiently approximable by a one-hidden-layer network or not. In particular, we focus on approximation results for functions defined over the \((d - 1)\)-dimensional sphere \(S^{d-1}\), for which a rich literature of Fourier analysis is available.

First, we give a sufficient condition on the target function in terms of its spherical harmonics decomposition to be not efficiently approximable by shallow one-hidden-layer networks. This condition captures a slowly decaying and sufficiently spread spherical harmonic expansion. We also show that certain symmetry properties imply this condition. On the other hand, one may ask if a reverse statement holds. In this direction, building on existing theory, we provide a sufficient condition for approximation by one-hidden-layer networks.

**Spherical harmonics decomposition**  
Let \(d \geq 2\) and let \(dS\) denote the uniform measure over \(S^{d-1}\). The spherical harmonics are a particular orthonormal basis for \(L^2_S\). They consist of

\[
\bigcup_{k=0}^{\infty} \text{span}\{Y_{k,i}^d\}_{i=1}^{N_k^d} = \bigcup_{k=0}^{\infty} H_k^d
\]

where \(Y_{k,i}^d\) is a restriction to \(S^{d-1}\) of an homogeneous harmonic polynomial of degree \(k\). We denote by \(\mathcal{P}_k^d\) the projection operator over \(H_k^d\) and by \(\mathcal{P}_I\) the operator \(\oplus_{i \in I} \mathcal{P}_k^d\), for any \(I \subseteq \mathbb{N}\). The function \(f_k = \mathcal{P}_k^d(f)\) is referred as the degree \(k\) spherical harmonic (or Fourier representation) component of the function \(f\). The spaces \(H_k^d\) are \(N_k^d\)-dimensional (where \(N_k\) grows as \(k^{d-2}\)) and have a RKHS structure with kernel given by

\[
(v, w) \in S^{d-1} \times S^{d-1} \mapsto N_k^d P_k^d(v^T w)
\]

where \(P_k^d\) are the Gegenbauer polynomials with parameter \((d - 2)/2\). See Appendix E for more details about spherical harmonics.

**Concentration and spreadness in \(H_k^d\)**  
Intuitively, we can say that a function \(f \in C(S^{d-1})\) is concentrated over \(S^{d-1}\) if there is an area \(\Omega \subset S^{d-1}\) such that the mass of \(f\) is concentrated over \(\Omega\). On the other hand we say that \(f\) is spread if it assumes non-negligible values uniformly over the sphere. The spread/concentration of the function \(f\) can be quantified by looking at ratios of the type

\[
\ell_{q,p}(f) \doteq \frac{\|f\|_q}{\|f\|_p}
\]

for \(1 \leq p < q \leq \infty\). Since the norms above are with respect to a probability measure, it holds that \(\ell_{q,p} \geq 1\). Intuitively, the closest this ratio is to 1, the more spread is the function. On the other hand, the largest this ratio, the more concentrated the function is. Consider the case of a function \(f_k \in H_k^d\). Then, it holds that

\[
\ell_{\infty,2}(f_k) \leq \sqrt{N_k^d} \quad \text{and} \quad \ell_{2,1}(f_k) \leq \sqrt{N_k^d}\]  \hspace{1cm} (6)

The equality in the first equation is attained for functions of the type \(f_k(x) = \alpha P_k^d(w^T x)\) for some \(\alpha \in \mathbb{C}\) and \(w \in S^{d-1}\), i.e. zonal harmonics. In this sense, zonal harmonics could be considered as the most concentrated functions in \(H_k^d\). Nevertheless, zonal harmonics do not attain equality in the second equation.
Main result We show that these notion of concentration and spreadness can be used to determine whether a function \( f \in C(S^{d-1}) \) is universally approximable or not by one-hidden-layer networks. Assuming that \( \| f_k \|_2 \simeq \text{poly}(d, k^{-1}) \), the results can be summarized as follows:

- If the spherical components of \( f \) are (polynomially) concentrated in \( \ell_{2,1}(f) \) sense, that is, for example,
  \[
  \ell_{2,1}(f_k) \gtrsim \text{poly}(d^{-1}, k^{-1}) \cdot \sup_{f \in H_k^d} \ell_{2,1}(f)
  \]
  then \( f \) is universally approximable by one-hidden-layer networks.

- If the spherical components of \( f \) are (exponentially) spread in \( \ell_{\infty,2} \) sense, that is, for example,
  \[
  \ell_{\infty,2}(f_k) \gtrsim e^k \cdot \sup_{f \in H_k^d} \ell_{\infty,2}(f) \quad \text{for some } \epsilon \in (0, 1)
  \]
  then \( f \) is provably not universally approximable by one-hidden-layer networks.

Notice that, on the other hand, if \( \| f_k \|_2 \) decreases exponentially fast then universal approximation follows, and similarly if \( \| f_k \|_2 \) decreases exponentially slowly then universal approximation can not hold. The first of the two conditions above expresses concentration of the Fourier decomposition, while the second expresses spreadness of the same. We notice at least two gaps between the two conditions. The first one is the expression of the concentration phenomena: one is w.r.t. \( \ell_{\infty,2} \), while the other one is w.r.t. \( \ell_{2,1} \). Second, the two regimes above do not include many other possible ones. For example, we suspect the existence of a regime which prevents universal approximability but allows for fixed-threshold one, a topic worth of future study. These results are properly formalized and stated in the next sections 5.1 and 5.2. Proofs and details are reported in appendices E, F and G.

### 5.1 Inapproximability of functions with spread Fourier representation

From the point of view of spherical harmonic decomposition, one-hidden-layer functions have a zonal structure. If \( h(x) = \sigma(w^T x + b) \) for some \( w \in S^{d-1} \) and \( b \in \mathbb{R} \), then it is easy to see that

\[
 h_k(x) = s_k \| h_k \|_2 \sqrt{\frac{d}{k} N_k^d P_k^d(w^T x)}
\]

with \( s_k \in \{ \pm 1 \} \). In particular, it follows that \( \| h_k \|_\infty = | h_k(\pm w) | = (N_k^d)^{1/2} \| h_k \|_2 \). Following the discussion in the previous section, this can be interpreted by saying that the Fourier components of single neurons are most 'concentrated' (along the neuron direction) in space. Therefore, it is natural to expect that functions with 'spread' Fourier decomposition are difficult to approximate by neural networks. The proposition below formalizes this fact. The proof follows a technique similar to the one used in (Daniely, 2017) (see Appendix G.3 for a comparison) and essentially upper bounds the scalar product between the objective function and the network.

**Proposition 15** Let \( \{ f^{(d)} \}_{d \geq 2} \) a sequence of functions such that \( f^{(d)} \in C(S^{d-1}) \). Assume that for every \( d \) there exists \( I_d \subseteq \mathbb{N} \) such that

1. It holds that \( \| f^{(d)} \|_2 \leq O(d^M) \cdot \| P_{I_d} f^{(d)} \|_2 \) for some \( M > 0 \);

2. There exists a non-negative sequence \( \{ c_{d,k} \}_{k \in I_d} \) such that \( \| f^{(d)}_k \|_\infty \leq c_{d,k} \sqrt{N_k^d \| f^{(d)} \|_2} \) for all \( k \in I_d \) and such that \( \left( \sum_{k \in I_d} c_{d,k}^2 \right)^{1/2} \leq c^{d^\alpha} \cdot O(d^M) \) for some \( \epsilon \in (0, 1) \) and \( \alpha > 0 \).
Moreover, assume that \( \|f^{(d)}\|_\infty = O(1) \) and \( \|f^{(d)}\|_2 = \Omega(\mathcal{d}^{-M}) \). Then the sequence \( \{f^{(d)}\}_{d \geq 2} \) is not universally approximable by one-hidden-neural networks.

The two conditions in the proposition above can be interpreted as follows. The first one asks for the energy of the function to decay slowly enough, that is polynomially in \( \mathcal{d}^{-1} \). The second condition asks for the Fourier components to be exponentially (in \( \mathcal{d} \)) spread, with respect to the first bound in (6), at high energies. This resembles the conditions of the lower bound in Section 3. An interesting family of functions for which condition 2. holds are Rademacher-symmetric functions: \( f \in C(S^{d-1}) \) such that \( f(\varepsilon \circ x) = f(x) \) for every \( \varepsilon \in \{\pm 1\}^d \) and \( x \in S^{d-1} \).

Lemma 16 Let \( f \in C(S^{d-1}) \). If \( \|f_k\|_\infty = \sup_{\varepsilon \in \{\pm 1\}^d} |f_k(\varepsilon)| \) for some \( k = k_d \geq 16d^2 \) even and \( f \) is Rademacher-symmetric then it holds
\[
\|f_{k_d}\|_\infty \leq 2 \cdot 2^{-d/2} \sqrt{N_{k_d}^d} \|f_{k_d}\|_2.
\]

5.2 Efficient approximation under a sparsity condition of the Fourier decomposition

Works by Barron (Barron, 1993; Klusowski and Barron, 2018) essentially show that efficient approximation holds under a sparsity condition on the Fourier transform of the function to approximate; more specifically, for \( f \in L^1(\mathbb{R}^d) \), the rate of approximation is controlled by the quantity \( \int_{\mathbb{R}^d} \|\hat{f}(w)\|_1^2 \hat{f}(w) \) \( dw \). In this section we show that an equivalent control can be determined for approximation on the sphere, in terms of spherical harmonics decomposition. For technical reason, the result is established for functions in \( H' = H^d_1 \oplus \bigoplus_{k=1}^{\infty} H^d_{2k} \) (which correspond to the space of function whose odd part is linear) and mainly for ReLu activation. Nevertheless, thanks to Assumption 1, it is in fact sufficient to show universal approximation by any of the activation \( \sigma \) satisfying the assumption. Then, thanks to Proposition 1, universal approximation by \( F_\mathcal{N}^\infty \) is equivalent to universal approximation by \( F_\mathcal{N}^1 \). Up to linear terms, this is implied by universal approximation by
\[
F^1 = \left\{ f : x \in S^{d-1} \mapsto \int_{S^{d-1}} |w^T x|^2 \ d\pi(w) : \pi \text{ is a signed Radon measure} \right\}.
\]

Let \( \gamma_1 \) the variation-norm on \( F^1 \). Following the approach of (Ongie et al., 2019), it is possible to express this norm in a functional form, by considering the transformation
\[
T : \varphi \in C(S^{d-1}) \mapsto \int_{S^{d-1}} |x^T y| \varphi(y) \ dS(y).
\]

The operator \( T \) can be shown (Rubin, 1998) to be an automorphism of \( C^\infty_{\text{even}}(S^{d-1}) \) (the subset of \( C^\infty(S^{d-1}) \) composed by even functions) and can be described in terms of spherical harmonics as \( T \varphi = \sum_{k=0}^{\infty} \sigma_{2k} \varphi_{2k} \) for some \( \sigma_k \) (see Appendix F).

Proposition 17 Let \( f \in C(S^{d-1}) \) even. Then \( \gamma_1(f) = \sup_{\varphi \in C^\infty_{\text{even}}(S^{d-1})} \|\varphi\|_{\infty,1} \langle T^{-1} \varphi, f \rangle < \infty \) if and only if \( f \in F^1 \).

Given \( f \in C(S^{d-1}) \) even, the condition above is implied by the (weak) convergence (as \( N \to \infty \)) of the series \( S_N = \sum_{k=0}^{N} \sigma_{2k} f_{2k} \) to a finite signed measure \( \pi \). A stronger condition is that \( \sum_{k \geq 0 \text{ even}} |\sigma_{2k}|^{-1} \|f_k\|_1 < \infty \). Using these observations it is easy to prove the following.

Proposition 18 Let \( \{f^{(d)}\}_{d \geq 2} \) a sequence of even functions in \( C(S^{d-1}) \). Assume that there exist some constant \( M, N > 0 \) constant such that
\[
\sqrt{N_k^d} \|f_k^{(d)}\|_1 \leq O(k^M d^N) \cdot \|f_k^{(d)}\|_2 \quad \text{and} \quad \sum_{k=0}^{\infty} k^{M+2} \|f_k^{(d)}\|_2 = O(d^N).
\]
Then the sequence \( \{ f^{(d)} \}_{d \geq 2} \) is universally approximable by the space \( \mathcal{F}_N \).

The proposition above requires essentially two conditions to hold. First, that the energy of the functions decreases fast enough (yet polynomially in \( k \) and \( d \)). The second condition is that the Fourier components of the function are concentrated enough, that is they are polynomially close to the second in bound (6).

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Appendix A. Proof of Theorem 3

The proof of the lower bound follows the same strategy as (Eldan and Shamir, 2016). Let $S \subseteq [d]$ a subset and let $I_S$ be the truncated identity matrix defined as

$$I_S := \sum_{s \in S} e_s e_s^\top.$$

Moreover, we define the function $H(S, x)$ as

$$H(S, x) := \prod_{i : i \in S} 1_{x_i > 0} \prod_{j : j \in [d] \setminus S} 1_{x_j \leq 0}. \tag{7}$$

Lastly, for a subset $S \subseteq [d]$, we introduce the vector $v_S := v + I_S w$ and define the function $\sigma_{r,S}(x) := \sigma_r(v_S^\top x)$. Therefore, the expression of $f_{r,w,v}$ can be rewritten as:

$$f_{r,w,v}(x) = \sum_{S \subseteq [d]} g_S(x) = \sum_{S \subseteq [d]} H(S, x) \sigma_{d,S}(x)$$

where $g_S(x) := H(S, x) \sigma_{r,S}(x).$
A.1 Formal result

Let the space of $N$-units one-hidden-layer networks be

$$\mathcal{F}_N = \left\{ f : x \in \mathbb{R}^n \mapsto \sum_{k=1}^{N} \sigma_k(a_k^T x) : a_k \in \mathbb{R}^d, \sigma_k \text{ are 1-Lipschitz activations} \right\}.$$

Assume that

(A1) it holds that $\tau \cdot r = \Omega(d^k)$ for some constant $k \geq 1$;

(A2) it holds that $\eta > \log_2 \left( \| \psi \|_{1} \sqrt{K/2} \right)$

Then it holds

$$\inf_{f \in \mathcal{F}_N} \| f_{r, w, v} - f \|_{\phi}^2 \geq 1 - N \left( \frac{2^{-2\eta K \| \psi \|_{1}^2}}{2} \right)^d O(d \cdot \tau \cdot r), \quad (9)$$

where we denote

$$\| g \|_{\phi}^2 = \int_{\mathbb{R}^d} |g(x)|^2 \phi(x) \, dx$$

for $g \in L^2_{\phi}$. In particular, if $N \simeq \text{poly}(d)$, then the error (9) tends to 1 as $d \to \infty$.

A.2 Proof

Let $\mathcal{F} = \{ \hat{f} \varphi : f \in \mathcal{F}_1 \}$, and denote by $F := \varphi \cdot \hat{f}_{r, w, v} = \hat{f}_{r, w, v} * \varphi$. Since $\varphi$ has compact support in $[-K, K]^d$ and the Fourier transform of a one-unit shallow network $f(x) = \sigma(x^T a)$ has support in the line $\{ \xi : \xi = \alpha a, \alpha \in \mathbb{R} \}$, it follows that any function in $\mathcal{F}$ is supported in a tube $T = \{ \xi : \xi = \alpha a + [-K, K]^d, \alpha \in \mathbb{R} \}$ of radius $K$. For each tube $T$ of radius $K$, we consider $\mathcal{T}_T = \{ \phi \in L^2 : \text{supp}(\phi) \subseteq T \}$

and

$$\kappa = \sup_{T \text{ tube of radius } K} \| P_{T_T}(F) \,$$

where $P_{T_T}(F) = \text{argmin}_{h \in \mathcal{T}_T} \| h - F \|_2^2$. We claim that

$$\inf_{f \in \mathcal{F}_N} \| f_{r, w, v} - f \|_{\phi}^2 \geq 1 - N \kappa^2. \quad (10)$$

Indeed, given $f \in \mathcal{F}_N$, denote by $T_1, \ldots, T_N$ the associated $N$ tubes, and by $\mathcal{T}_{T_1, \ldots, T_N} = \bigoplus_{k \in [N]} \mathcal{T}_{T_k}$ the corresponding subspace spanned by $\mathcal{T}_{T_k}, k \in [N]$. Then, by using the isometry of the Fourier transform, we have that

$$\inf_{f \in \mathcal{F}_N} \| f - f_{r, w, v} \|_{\phi}^2 = \inf_{f \in \mathcal{F}_N} \| \hat{f} \varphi - F \|_2^2$$

$$\geq \inf_{T_1, \ldots, T_N} \inf_{h \in \mathcal{T}_{T_1, \ldots, T_N}} \| h - F \|_2^2$$

$$= \inf_{T_1, \ldots, T_N} \| P_{T_{T_1, \ldots, T_N}}(F) - F \|_2^2$$

$$= \inf_{T_1, \ldots, T_N} \left( \| F \|_2^2 - \| P_{T_{T_1, \ldots, T_N}}(F) \|_2^2 \right). \quad (11)$$

Now, observe that $\sup_{T_1, \ldots, T_N} \| P_{T_{T_1, \ldots, T_N}}(F) \|_2^2 \leq N \sup_T \| P_{T_T}(F) \|_2^2$. Equation (11) therefore becomes

$$\inf_{f \in \mathcal{F}_N} \| f - f_{r, w, v} \|_{\phi}^2 \geq \| F \|_2^2 - N \sup_T \| P_{T_T}(F) \|_2^2$$
which proves (10) by plugging-in the definition of \( \kappa \) and recalling that \( \|F\|_2^2 = \|f_{r,w,v}\|_\psi^2 = 1 \) by Parseval. To establish (9), it is therefore sufficient to prove that
\[
\kappa^2 \leq (\|\psi\|_2^2 2^{-2\eta - 1} K)^d O(d \cdot \tau \cdot r) .
\] (12)
The rest of the proof will be devoted to establishing a sufficiently sharp upper bound for \( \|P_T(F)\| \). Observe that \( P_T(F) \) is simply obtained by setting to zero all frequencies of \( F \) outside \( T \). We start by computing an upper bound on \( |F(\xi)| \).

**Lemma 19** It holds that
\[
|F(\xi)| \leq \frac{\|\varphi\|_1}{2^d} \sum_{S \subseteq [d]} \prod_{j=1}^d \min \left( 1, \frac{1}{\pi(\|\xi_j - \xi_{S,j}\| - K)_+} \right) .
\] (13)
Let \( D(\xi) = \sum_S D_S(\xi) \), with \( D_S(\xi) = \prod_{j=1}^d \min \left( 1, \frac{1}{\pi(\|\xi_j - \xi_{S,j}\| - K)_+} \right) \), so that from (13) we have
\[
|F(\xi)| \leq 2^{-d} \|\varphi\|_1 D(\xi) .
\] (14)
Recall that \( \tau = \sup_{S \subseteq [d]} \|v_S\|_\infty \). Given \( \xi \) non-zero, we claim the following.

**Lemma 20** It holds that
\[
D(\xi) \lesssim 2^{d(1-\eta)} \min \left\{ 1, \left( \pi(\|\xi\|_\infty - r\tau - K)_+ \right)^{-1} \right\} .
\] (15)
Now, pick any arbitrary non-zero direction \( \nu \) and assume w.l.o.g. that \( \|\nu\|_\infty = 1 \). Let
\[
T = \{ \xi : \inf_{\alpha \in \mathbb{R}} \|\xi - \alpha\nu\|_\infty \leq K \}
\] (16)
denote the tube of radius \( K \) in the direction \( \nu \) and pick \( \vartheta > 1 \). It holds that
\[
\int_T D(\xi)^2 d\xi = \int_{T \cap \{\|\xi\|_\infty \leq \vartheta \tau r\}} D(\xi)^2 d\xi + \int_{T \cap \{\|\xi\|_\infty > \vartheta \tau r\}} D(\xi)^2 d\xi .
\] (17)
In order to control the two terms \( t_1 \) and \( t_2 \), we use the following lemma to upper bound the measure of a \( \ell_\infty \)-cylinder.

**Lemma 21** Let \( T \) be an \( \ell_\infty \)-tube of radius \( K \) as defined in (16). If \( \mu \) denotes the \( d \)-dimensional Lebesgue measure, then
\[
\mu \left( T \cap [-R, R]^d \right) \leq 2eR(d - 1)(2K)^{d-1} .
\] (18)
Moreover, if \( g : \mathbb{R} \to \mathbb{R} \) is in \( L^1(\mathbb{R}) \) and non-increasing, then
\[
\int_T g(\|\xi\|_\infty) d\xi \leq 2e(d - 1)(2K)^{d-1} \int_0^\infty g(u)du .
\] (19)
From (15) and (18), the first term of (17) is bounded as
\[
t_1 \lesssim 2^{2d(1-\eta)} (2K)^{d-1} \vartheta \tau r
\]
\[
\lesssim (2^{(1-\eta)+1} K)^d \vartheta \tau r
\] (20)
and from (19) the second term \( t_2 \) in turn is bounded as
\[
\begin{align*}
t_2 & \lesssim 2^{2d(1-\eta)} (2K)^{d-1} d \int_{\partial T} (\pi(s - \tau d) - K)^{-2} ds \\
& \lesssim 2^{2d(1-\eta)} (2K)^{d-1} ((\vartheta - 1)\tau r - K)^{-1} \\
& \lesssim (2^{(1-\eta)+1}K)^d d(\tau r)^{-1}.
\end{align*}
\] (21)

Thus, collecting (20) and (21) and using (14), we obtain
\[
\int T \frac{F(\xi)}{\varphi} d\xi \lesssim \|\varphi\|^2_1 \left( \frac{2^{(1-\eta)+1}K}{4} \right)^d O(d\tau r).
\]

Therefore, it holds
\[
\|P_T(F)\|^2_2 = \int_T |F(\xi)|^2 d\xi \lesssim (\|\varphi\|^2_1 2^{-2\eta - 1} K)^d O(d\tau r).
\]

We have just established (12), and this concludes the proof of the theorem.

**Appendix B. Lemmas of Section A.1**

**B.1 Proof of Lemma 19**

We compute first \( \hat{f}_{r,w,v} \). From the definition of \( \sigma_r \), it follows that
\[
\hat{\sigma}_{r,S}(\xi) = \delta(\xi - rv_S),
\] (22)

which combined with the definition (8) yields
\[
\hat{f}_{r,w,v}(\xi) = \sum_{S \in [d]} \mathcal{H}(S; \cdot) * \hat{\sigma}_{d,S}(\xi) = \sum_{S \in [d]} \mathcal{H}_S(\xi - rv_S).
\]

Let \( \xi_S = rv_S \). We write
\[
F(\xi) = \int_{\mathbb{R}^d} \hat{f}_{r,w,v}(\nu) \hat{\varphi}(\xi - \nu) d\nu = \sum_{S \in [d]} \int_{\mathbb{R}^d} \mathcal{H}_S(\nu - \xi_S) \hat{\varphi}(\xi - \nu) d\nu
\]
\[
= \sum_{S \in [d]} \int_{\mathbb{R}^d} \mathcal{H}_S(\nu) \hat{\varphi}(\xi - \xi_S - \nu) d\nu.
\] (23)

Let us bound each term \( F_S \) separately. We have
\[
F_S(\xi) = \int \mathcal{H}_S(\nu) \hat{\varphi}(\xi - \nu) d\nu
\]
\[
= \int H_S(x) e^{2i\pi \xi^T x} \varphi(x) dx = \prod_{j=1}^d F_j(\xi_j),
\] (24)

where
\[
F_j(t) = \int_{\mathbb{R}} 1\{\epsilon_j x > 0\} e^{2i\pi \epsilon_j x} \psi(x) dx,
\] (25)

with \( \epsilon_j = \pm 1 \). Assume w.l.o.g. that \( \epsilon_j = 1 \). Observe that \( F_j = \hat{Q} \), where
\[
Q(u) = 1\{u > 0\} \psi(u).
\] (26)
Since $\psi \in L_1(\mathbb{R})$ and its Fourier transform $\hat{\psi}$ has compact support in $[-K, K]$, we have that
\begin{equation}
|\hat{\psi}(\tau)| \leq \|\psi\|_1 \quad \text{for } \tau \in [-K, K] \quad \text{and} \quad \hat{\psi}(\tau) = 0 \quad \text{for } |\tau| > K .
\end{equation}
(27)

On the one hand, since $\psi$ is even, we have, by directly bounding (25), that
\begin{equation}
|F_j(t)| \leq \frac{1}{2} \int_{-K}^{K} |\psi(u)| du = \frac{1}{2} \|\psi\|_1 \quad \text{for all } t ,
\end{equation}
and from (27) and the Hilbert transform of $Q$ we deduce on the other hand that
\begin{equation}
|F_j(t)| = \frac{1}{2\pi} \left| \int_{-K}^{K} \frac{\hat{\psi}(\tau)}{t-\tau} d\tau \right| \leq \frac{2K\|\psi\|_1}{(2\pi)(|t|-K)} \quad \text{for } |t| > K ,
\end{equation}
so we obtain that
\begin{equation}
|F_j(t)| \leq \frac{\|\psi\|_1}{2} \min\left(1, \frac{1}{\pi(|t|-K)_+}\right) .
\end{equation}
(29)

Thus, from equations (23), (24) and (29) we deduce that
\begin{equation}
|F(\xi)| \leq \sum_{S \in [d]} |F_S(\xi - \xi_s)|
\leq \frac{\|\varphi\|_1}{2^d} \sum_{S \in [d]} \prod_{j=1}^{d} \min\left(1, \frac{1}{\pi(|\xi_j - \xi_{s,j}| - K)_+}\right) ,
\end{equation}
which proves Lemma 19.

B.2 Proof of Lemma 20

Let us first define for any $\xi \in \mathbb{R}^d$ and $\lambda > 0$
\begin{equation}
n(\xi, \lambda) = |\{ j \in [d] : |\xi_j| > \lambda \}| .
\end{equation}

Recall that $v_S = v + I_S w$ and $\xi_S = r v_S$. Observe that $\xi_S - \xi_{S'} = r(I_S - I_{S'})w$, so
\begin{equation}
|\xi_{S,j} - \xi_{S',j}| = \begin{cases} r|w_j| & \text{if } j \in (S \cup S') \setminus (S \cap S') \\ 0 & \text{otherwise} \end{cases} .
\end{equation}
(30)

If $d(S, S')$ denotes the Hamming distance between two subsets $S, S'$, then for all $S, S'$, the following holds.

**Lemma 22** It holds that
\begin{equation}
n(\xi_S - \xi_{S'}, \gamma d^2) = d(S \cap \Omega, S' \cap \Omega) .
\end{equation}
(31)

This immediately implies that
\begin{equation}
n\left(\xi - \xi_S, \gamma d^2 \right) + n\left(\xi - \xi_{S'}, \gamma d^2 \right) \geq d(S \cap \Omega, S' \cap \Omega) \quad \text{for all } \xi \text{ and } S \neq S' .
\end{equation}
(32)

Indeed, if that was not the case, applying the triangle inequality coordinate-wise would contradict equation (31). The first upper bound is obtained by first noticing that
\begin{equation}
D_S(\xi) \leq (\pi(\gamma d^2/2 - K))^{-n(\xi-\xi_S, \gamma d^2/2)} \quad \text{for all } S \text{ and } \xi .
\end{equation}
Now, if we define \( S_\xi^* = \arg \min_{S \in [d]} n(\xi - \xi_S, \gamma d^2 / 2) \), from (32) we obtain that

\[
n(\xi - \xi_S, \gamma d^2 / 2) \geq \frac{d(S \cap \Omega, S' \cap \Omega)}{2} \quad \text{for all } S \neq S_\xi^*
\]

and thus

\[
D(\xi) = D_{S_\xi^*}(\xi) + \sum_{S \neq S_\xi^*} D_S(\xi)
\]

\[
\leq D_{S_\xi^*}(\xi) + \sum_{s=1}^{[\Omega]} \sum_{S : d(S \cap \Omega, S_\xi^* \cap \Omega) = s} (\pi(\gamma d^2 / 2 - K))^{-s/2}
\]

\[
\leq D_{S_\xi^*}(\xi) + 2^{d-\Omega} \sum_{s=1}^{[\Omega]} \left( \frac{|\Omega|}{s} \right) (\pi(\gamma d^2 / 2 - K))^{-s/2}
\]

\[
\leq 1 + 2^{d-\Omega} \left( 1 + \frac{1}{\sqrt{\pi(\gamma d^2 / 2 - K)}} \right)^{\Omega}
\]

\[
\lesssim 2^{d(1-\eta)}
\]

(33)

since \(|\{S : d(S \cap \Omega, S_\xi^* \cap \Omega) = s\}| \leq 2^{d-\Omega} \langle \Omega \rangle_s\). The second upper bound is obtained using an induction step as follows. Let \( q_\xi = \arg \max_j |\xi_j| \). Then

\[
D(\xi) \leq \sum_{S \in [d]} \frac{1}{\pi(|\xi_q_\xi - \xi_{S,j} - K|_+)} \cdot \prod_{j \neq q_\xi} \min \left( 1, \frac{1}{\pi(|\xi_j - \xi_{S,j} - K|_+)} \right)
\]

\[
\leq (\pi(\|\xi\|_\infty - \tau r - K)_+)^{-1} \sum_{S \neq q_\xi} \prod_{j \neq q_\xi} \min \left( 1, \frac{1}{\pi(|\xi_j - \xi_{S,j} - K|_+)} \right)
\]

\[
\lesssim (\pi(\|\xi\|_\infty - \tau r - K)_+)^{-1} \cdot 2^{d(1-\eta)}
\]

(34)

by noticing that the argument leading to (33) can now be repeated for the \((d - 1)\)-dimensional vector \( \tilde{\xi} = (\xi_1, \ldots, \xi_{q_\xi - 1}, \xi_{q_\xi + 1}, \ldots, \xi_d) \), so that

\[
n(\tilde{\xi} - \xi_S, \gamma d^2 / 2) \geq \frac{d((S \cap \Omega) \setminus \{q_\xi\}, (S' \cap \Omega) \setminus \{q_\xi\})}{2} \quad \text{for all } S \neq S_\xi^*
\]

(35)

which proves (34) and concludes the proof of Lemma 20. □

**B.2.1 Proof of Lemma 22**

In fact, we show that the two sets \( A_1 := \{j \in [d] : |\xi_{S,j} - \xi_{S',j}| \geq \gamma d^2\} \) and \( A_2 := \{j \in [d] : j \in (S \cap \Omega) \setminus (S' \cap \Omega)\} \) are equal. Let \( j \in A_1 \). Then \( |\xi_{S,j} - \xi_{S',j}| \geq \gamma d^2 \). Since this quantity is nonzero, equation (30) indicates that therefore \( j \in S'\setminus S \) w.l.o.g. Moreover, \( |\xi_{S,j} - \xi_{S',j}| = r|w_j| \) which implies that \( r|w_j| > \gamma d^2 \) and \( j \in \Omega \). We conclude that \( j \in (S \cap \Omega) \setminus (S' \cap \Omega) \) which implies that \( j \in A_2 \). Now, let \( j \in A_2 \). Then, w.l.o.g. \( j \in (S \cap \Omega) \setminus (S' \cap \Omega) \). Then, we have \( r|w_j| > \gamma d^2 \) since \( j \in S \setminus S' \) according to (30) and \( |\xi_{S,j} - \xi_{S',j}| = r|w_j| \). Combining these two facts, we obtain \( |\xi_{S,j} - \xi_{S',j}| > \gamma d^2 \) which means that \( j \in A_2 \).
B.3 Proof of Lemma 21

Let \( T_R(\nu) = T(\nu) \cap [-R, R]^d = \{ \xi : \inf_{\alpha \in \mathbb{R}} \sup_{j \in [d]} |\xi_j - \alpha \nu_j| \leq K \text{ and } \|\xi\|_\infty \leq R \} \). We would like to upper bound the volume of \( T_R(\nu) \) for any \( \nu \). Assume w.l.o.g. that \( \|\nu\|_\infty = 1 \). We will cover \( T_R(\nu) \) with \( \ell_\infty \)-balls of radius \( K' = \vartheta K \) centered along the ray defined by \( \nu \), that is

\[
T_R(\nu) \subseteq \bigcup_{j=1}^{\lfloor R/s \rfloor} \left( js\nu + [-\theta K, \theta K]^d \right) .
\]  

(36)

In other words, we want to optimize both the oversampling rate \( s/K \leq 1 \) and the radius ratio \( \vartheta \geq 1 \) while satisfying (36). Given \( s \), let us first compute the smallest admissible \( \vartheta \). Any \( x \in T_R(\nu) \) satisfies

\[
\|x - (j + y)s\nu\|_{\infty} \leq K
\]

for some \( j \in \mathbb{N} \) and \( 0 \leq y < 1 \). In other words, for any \( i \in [d], |x_i - (j + y)\nu_is| < K \) implies that \( |x_i - j\nu_is| \leq K + s\nu_ig \). We need to satisfy for any \( i \in [d] \) that

\[
|x_i - (j + 1/2)\nu_is| < \vartheta K
\]

which holds if \( K + s/2 \leq \vartheta K \) and therefore \( \vartheta = 1 + s(2K)^{-1} \). Now, the volume of

\[
S_R = \bigcup_{j=1}^{\lfloor R/s \rfloor} \left( js\nu + \left( -1 + \frac{s}{2K} \right) K, \left( 1 + \frac{s}{2K} \right) K \right]^d
\]

is given by

\[
l(s) = \frac{R(2K)^d}{s} \left( 1 + \frac{s}{2K} \right)^d .
\]

Let’s compute the minimum of this function with respect to the oversampling factor \( s \). We have that

\[
l'(s) = 0 \iff -\frac{R}{s^2}(2K)^d \left( 1 + \frac{s}{2K} \right)^d + \frac{Rd}{s}(2K)^{d-1} \left( 1 + \frac{s}{2K} \right)^{d-1} = 0
\]

\[
\iff -\frac{2K}{s} \left( 1 + \frac{s}{2K} \right) + d = 0
\]

\[
\iff \frac{2K}{s} + 1 = d
\]

\[
\iff s = \frac{2K}{d-1} .
\]

Therefore, we have for all \( \nu \in \mathbb{R}^d \), it holds \( T_R(\nu) \leq R(d-1)(2K)^{d-1} \left( 1 + \frac{1}{d-1} \right)^d \), which proves (18). Finally, equation (19) is established analogously, by observing that

\[
\int_{T_R} g(\|\xi\|_{\infty})d\xi \leq \left[ (2K)^d \left( 1 + \frac{1}{d-1} \right)^d \right] \left[ \frac{R(2K)^{d-1}}{(d-1)!} \sum_{j=1}^{\lfloor R(2K)/(d-1) \rfloor} g((j + 1/2)2K/(d-1)) \right] \leq \left( d-1 \right)(2K)^{d-1} \left( 1 + \frac{1}{d-1} \right)^d \int_0^R g(u)du
\]

since \( g \) is non-increasing.
Appendix C. Proof of Theorem 8

We approximate \( \sigma_r \) with 1-Lipschitz functions \( \sigma_k \). Since \( \sigma_r \) is \( (2\pi r) \)-Lipschitz, we obtain a uniform error over an interval \([-Q, Q]\) of the form

\[
\sup_{|t| \leq Q} \left| \sigma_r(t) - \sum_{k=1}^{N} \alpha_k \sigma_k(t - \beta_k) \right| \leq \frac{2Qr}{N},
\]
as well as

\[
\left| \sum_{k=1}^{N} \alpha_k \sigma_k(t - \beta_k) \right| \leq C \quad \text{for } t \in \mathbb{R}.
\]

Now, let \( \gamma = \|v\|_1 + \|w\|_1 \) and \( \bar{Q} = \frac{Q}{\gamma} \), so that by definition when \( \|x\|_\infty \leq \bar{Q} \) we have

\[
|v^T x + w^T x| \leq Q.
\]

We decompose the error as follows:

\[
\int_{\mathbb{R}^d} (f_{r,w,v}(x) - f(x))^2 \varphi(x)^2 \, dx =
\]

\[
= \int_{\|x\|_\infty \leq \bar{Q}} (f_{r,w,v}(x) - f(x))^2 \varphi(x)^2 \, dx + \int_{\|x\|_\infty > \bar{Q}} (f_{r,w,v}(x) - f(x))^2 \varphi(x)^2 \, dx
\]

\[
\leq \frac{4Qr^2}{N^2} \|\varphi\|_2^2 + C \left( \|\varphi\|_2^2 - \|\varphi\|_2^2 - C' \bar{Q}^{-3d} \right)
\]

\[
\lesssim \|\varphi\|_2^2 \left( \frac{4\bar{Q}^2 \gamma^2 r^2}{N^2} + \bar{C} d \bar{Q}^{-3} \right) = \|f_{r,w,v}\|_\varphi^2 \left( \frac{4\bar{Q}^2 \gamma^2 r^2}{N^2} + \bar{C} d \bar{Q}^{-3} \right),
\]

since \( |\varphi(x)| = O(|x|^{-2}) \). Optimizing this upper bound with respect to \( \bar{Q} \) gives

\[
\bar{Q} = \left( \frac{3\bar{C} d N^2}{8r^2 \gamma^2} \right)^{1/5}
\]

resulting in

\[
\|f_{r,w,v} - f\|_\varphi^2 \lesssim d^{\frac{2}{5}} \gamma^2 r^2 N^{-6/5},
\]

which concludes the proof.

Appendix D. Proofs of Section 4

D.1 Preliminary lemmas

D.1.1 Approximation by polynomials

We start by reporting two known results.

Lemma 23 (Jackson’s Theorem) Let \( f : [a, b] \to \mathbb{R} \) with modulus of continuity \( \omega \). Then there exists a polynomial \( p_n(t) = \sum_{k=0}^{n} p_k t^k, p_k \in \mathbb{R}, \) such that

\[
\sup_{t \in [-r, r]} |f(t) - p_n(t)| \leq 6 \omega \left( \frac{b - a}{2n} \right).
\]
**Proof** See Theorem 1.4 in (Rivlin, 1981).

**Lemma 24** Let \( k \geq 0 \) and \( f : [-1, 1] \rightarrow \mathbb{R} \) whose \( k \)-derivative has modulus of continuity \( \omega_k \). Then there exists a polynomial \( p_n(t) = \sum_{k=0}^{n} p_k t^k, p_k \in \mathbb{R} \), such that

\[
\sup_{t \in [-r, r]} |f(t) - p_n(t)| \leq \left(\frac{6e}{n}\right)^k \frac{6}{1+k} \omega_k \left(\frac{1}{n-k}\right)
\]

for \( n \geq k \).

**Proof** See Theorem 1.5 in (Rivlin, 1981).

The next lemma yields a worst approximation rate but allows us to control the coefficients of the polynomial. It is a small modification of Lemma 4 in (Safran et al., 2019).

**Lemma 25** Let \( f : [-r, r] \rightarrow \mathbb{R} \), \( (1, \alpha) \)-Holder. Then for any \( \epsilon > 0 \) there exists a polynomial \( p_n(t) = \sum_{k=0}^{n} p_k t^k, p_k \in \mathbb{R} \), of degree \( n = \left\lceil \frac{1}{\alpha^2} \frac{r^\alpha}{\epsilon} \right\rceil \) such that

\[
\sup_{t \in [-r, r]} |f(t) - p_n(t)| \leq \epsilon.
\]

Moreover, \( p_n \) can be chosen such that \( |p_k| \leq 2^{n-k} r^\alpha, k \in [n], \) and \( |r_0| \leq r^\alpha + |f(0)| \).

**Proof** Notice that we can assume \( f(0) = 0 \) w.l.o.g. Define \( g(t) = f(r(2t-1)) \) for \( t \in [0, 1] \) and notice that \( g \) is \( (2r^\alpha, \alpha) \)-Holder. Also, define the \( n \) Bernstein polynomial \( b_{n,i}, i \in [0, n] \), as

\[
b_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}
\]

for \( t \in [0, 1] \). Notice that they form a partition of unity. We define

\[
g_n(t) = \sum_{i=0}^{n} g \left(\frac{i}{n}\right) b_{n,i}(t).
\]

We have that

\[
|g_n(t) - g(t)| \leq \sum_{i=0}^{n} b_{n,i}(t) \left| g(t) - g \left(\frac{i}{n}\right) \right|
\]

\[
= \sum_{i : \left|\frac{i}{n} - t\right| < \epsilon} b_{n,i}(t) \left| g(t) - g \left(\frac{i}{n}\right) \right| + \sum_{i : \left|\frac{i}{n} - t\right| \geq \epsilon} b_{n,i}(t) \left| g(t) - g \left(\frac{i}{n}\right) \right|
\]

\[
\leq \epsilon^\alpha + 2r^\alpha \sum_{i : \left|\frac{i}{n} - t\right| \geq \epsilon} b_{n,i}(t) \leq \epsilon^\alpha + \frac{r^\alpha}{2n \epsilon^2}.
\]

In particular \( \frac{r^\alpha}{2n \epsilon^2} \leq \epsilon^\alpha \) if

\[
n \geq \frac{r^\alpha}{2 \epsilon^{2+\alpha}}.
\]

If we define \( p_n(t) = g_n \left(\frac{t}{2r} + \frac{1}{2}\right) \), then we have that

\[
\sup_{x \in [-r, r]} |f(t) - p_n(t)| \leq \epsilon
\]
if

\[ n \geq \frac{4 \pi^2 \alpha}{\epsilon^{1+\frac{2}{\alpha}}} \).

Finally, we want to upper bound the coefficients of \( p_n \). Notice that we have

\[ p_n(t) = (2r)^{-n} \sum_{i=0}^{n} \binom{n}{i} g \left( \frac{i}{n} \right) (t + r)^i (t - r)^{n-i} \. \]

It follows that the coefficients of \( p_n \) can be bounded by those of

\[ (2r)^{-n} \sum_{i=0}^{n} \binom{n}{i} \left| g \left( \frac{i}{n} \right) \right| (t + r)^n \leq r^{\alpha-n}(t + r)^n \. \]

Let \( r_k \) the \( k \)-th coefficients of \( r^{\alpha-n}(t + r)^n \). Then

\[ r_k = r^{\alpha-n} \binom{n}{k} r^{n-k} \leq 2^n r^{\alpha-k} \. \]

This concludes the proof.

D.1.2 APPROXIMATION BY SHALLOW FOURIER NEURAL NETWORKS

We start by reporting a known result.

**Lemma 26** Let \( g : [-\pi, \pi] \to \mathbb{R} \) \( 2\pi \)-periodic with modulus of continuity \( \omega \). Then there exists a trigonometric polynomial \( q_n(t) = \sum_{k=-n}^{n} q_k e^{ikt}, q_k \in \mathbb{C} \), with real values (i.e. \( q_n(t) \in \mathbb{R} \) for all \( t \in [-\pi, \pi] \)), such that

\[ \sup_{t\in[-\pi,\pi]} |g(t) - q_n(t)| \leq \frac{2}{\pi} \omega \left( \frac{2}{n} \right) \left[ 2 + \omega(\pi) - \log \omega \left( \frac{2}{n} \right) \right] \. \]

Moreover, it holds that

\[ |q_k| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t)| dt \. \]

**Proof** The polynomial \( q_n \) is given by the Fejer sum of the Fourier series of \( g \), that is

\[ q_n(t) = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=-j}^{j} \hat{g}_k e^{ikt} = \sum_{k=-n}^{n-1} \frac{n - |k|}{n} \hat{g}_k e^{ikt} \]

where

\[ \hat{g}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{-ikt} dt \. \]

The proof of the upper bound can be found in (Burkill, 1959), Theorem 18. Finally, notice that \( q_n \) is real-valued since

\[ \hat{g}_k e^{ikt} \overline{\hat{g}_k e^{-ikt}} = 2\text{Re} \left( \hat{g}_k e^{ikt} \right) \]

since \( \hat{g}_{-k} = \overline{\hat{g}_k} \) since \( g \) takes values in \( \mathbb{R} \).

The above result immediately implies a convergence rate for univariate approximation by shallow Fourier networks.
Lemma 27 Let $f : [-r, r] \to \mathbb{R}$ be $L$-Lipschitz. Then there exists a real-valued Fourier shallow network $q_n(t) = \sum_{k=-n}^{n} q_k e^{ikw_k t}$, $q_k \in \mathbb{C}$, $w_k \in \mathbb{R}$, such that
\[
\sup_{x \in [-r,r]} |f(x) - q_n(x)| \leq 3 \left( 1 + 2L^2 r^2 \right) \frac{\log n}{n}
\]
for any $n \geq 2$. Moreover $q_n$ can be chosen such that $|w_k| \leq \frac{\pi|k|}{2r}$ and $|b_k| \leq \|f\|_{\infty}$ for any $k \in [-n, n]$.

Proof Assume, w.l.o.g., that $f(r) \leq f(-r)$ (otherwise we can consider $f(-x)$ in place of $f(x)$). First, we want to transform $f$ into a 2-pi periodic function on $[-\pi, \pi]$. To do this we consider $\tilde{g}$ defined as
\[
\tilde{g}(x) = \begin{cases} 
L(x + r) + f(-r) & \text{if } x \in [-r - \frac{2\pi}{L}, -r] \\
f(x) & \text{if } x \in [-r, r] \\
L(x - r) + f(r) & \text{if } x \in [r, r + \frac{2\pi}{L}]
\end{cases}
\]
where $c = f(-r) - f(r)$. Notice that $\tilde{g}$ is $L$-Lipschitz and $2\left(r + \frac{2\pi}{L}\right)$-periodic. Finally, let $g : [-\pi, \pi] \to \mathbb{R}$ defined as
\[
g(x) = \tilde{g}\left(\frac{2Lr + c}{2L\pi} x\right).
\]
We have that $g$ is $2\pi$-periodic and $\ell$-Lipschitz for
\[
\ell = \frac{2Lr + c}{2\pi} \leq \frac{2Lr}{\pi}.
\]
Therefore, we can apply Lemma 26 to $g$. This gives us a (real-valued) trigonometric polynomial $r_n(t) = \sum_{k=-n}^{n} r_k e^{ikt}$ such that
\[
\sup_{x \in [-\pi,\pi]} |g(x) - r_n(x)| \leq \frac{4\ell}{\pi n} \left[ 2 + \ell \pi - \log \frac{2\ell}{n} \right] \leq 3 \left( 1 + 2L^2 r^2 \right) \frac{\log n}{n}
\]
for $n \geq 2$. Since
\[
\sup_{x \in [-r,r]} \left| f(x) - r_n\left(\frac{L}{\ell} x\right) \right| \leq \sup_{x \in [-r - \frac{2\pi}{L}, -r + \frac{2\pi}{L}]} \left| \tilde{g}(x) - r_n\left(\frac{L}{\ell} x\right) \right| = \sup_{x \in [-\pi,\pi]} |g(x) - r_n(x)|
\]
the thesis follows.

To conclude we make some remarks about shallow Fourier networks. Note that a generic shallow Fourier network can be represented as
\[
f(x) = \sum_{k=1}^{N} u_k e^{iw_k^T x}.
\]
Indeed we have that
\[
\sum_{k=1}^{N} u_k e^{i(w_k^T x + b_k)} + b = \sum_{k=1}^{N} \left( u_k e^{ib_k} \right) e^{i\omega_k^T x} + b \cdot e^{\omega^T x}
\]
for any $b, b_k \in \mathbb{C}$. Let $\mathcal{F}_N$ be the space of networks as in (37). Also notice that a universal approximation theorem holds for shallow Fourier networks as well. This is because the universal approximation theorem holds for shallow networks with activation $\sigma(t) = \cos(t)$ and since $\cos(t) = (e^{it} + e^{-it})/2$, the thesis follows. Finally, the following lemma will be used in the proof of convergence rates.
Lemma 28  If $f$ is a (real-valued) shallow Fourier neural network, then so is $f^k$, for $k$ non-negative integer. Moreover, if $f$ has $n$ units, then the number of units of $f^k$ is upper bounded by

$$\binom{n + k - 1}{k}.$$  

Proof  Let $f(x) = \sum_{j=1}^{n} u_j e^{i w_j^T x}$ be a shallow Fourier neural network. Then, by the multinomial formula, we have that

$$f^k(x) = \left( \sum_{j=1}^{n} u_j e^{i w_j^T x} \right)^k = \sum_{p_1 + \cdots + p_n = k} \binom{k}{p_1, \ldots, p_n} \prod_{j=1}^{n} u_j^{p_j} e^{i \left( \sum_{j=1}^{n} p_j w_j \right)^T x}.$$  

Clearly, if $f$ is real-valued, so is $f^k$. Finally notice that by the formula above, the number of units of $f^k$ is upper bounded by $\{(p_1, \ldots, p_n) : p_1 + \cdots + p_n = k\}$.

D.2 poly($d$) upper bounds for two-hidden-layers networks

Consider a two-hidden-layers neural network $f$ defined as

$$f : x \in \mathbb{R}^d \mapsto \gamma^T g(W^T h(U^T x)) \in \mathbb{C},$$

where $h : \mathbb{R}^p \to \mathbb{R}^p$ and $g : \mathbb{R}^o \to \mathbb{R}^o$ are, respectively, component-wise 1-Lipschitz and $(1, \alpha)$-Holder activation functions, and $U \in \mathbb{R}^{d \times p}$, $W \in \mathbb{R}^{p \times o}$, $\gamma \in \mathbb{C}^o$. We wish to approximate $f$ with a one-hidden-layer neural network with a given activation $\sigma$ satisfying Assumption 1.2, for some constant $\nu_\sigma > 0$.

We start by proving a result for poly($d$) approximation by shallow Fourier networks. In the following, for a matrix $A \in \mathbb{R}^{n \times m}$, and $p, q \in [1, \infty]$, we use the following norm notations: $\|A\|_{F,p} = \|\text{vec}(A)\|_p$, $\|A\|_{p,q} = \sup_{x \in \mathbb{R}^m : \|x\|_p=1} \|Ax\|_q$ and $\|A\|_p = \|A\|_{p,p}$.

Proposition 29  Let $K \subset \mathbb{R}^d$ be a compact set. It holds that

$$\inf_{f_N \in \mathcal{F}_K^d} \sup_{x \in K} \left| f(x) - f_N^f(x) \right| \leq \varepsilon$$

as long as

$$N \geq \left( 18 \cdot 4^\frac{1}{p} \frac{\gamma^2 \|W\|_2^2}{\epsilon^{\frac{1}{\alpha}}} \left( 1 + \frac{2C^2}{C} \right)^2 \right)^{\frac{1}{\alpha}} \left[ 1 + M \left( \frac{2\gamma \delta}{\varepsilon} \right)^{\frac{1}{\alpha}} \right]^\frac{1}{p}$$

where we denoted

$$C = \sup_{x \in K} \|U^T x\|_\infty \quad \text{and} \quad M = \sup_{x \in K} \|W^T h(U^T x)\|_\infty.$$  

Moreover $f_N^f$ can be chosen such that

$$f_N^f(x) = \sum_{\nu=1}^{N} b_e e^{i w_e^T x}.$$  

2. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is $(\gamma, \alpha)$-Holder if it holds that $|f(x) - f(y)| \leq \gamma |x - y|^{\alpha}$ for any $x, y \in \mathbb{R}$.  

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We have that
\[
N = (2np + 1)^m
\]
if
\[
n \geq 9 \cdot 4^{\frac{1}{\alpha}} \|\gamma\|_2^2 \|W\|_2^2 (1 + 2C^2)^2 \frac{2}{\epsilon} \frac{1}{\epsilon^{1+\frac{1}{\alpha}}} \|\gamma\|_1^{\frac{1}{\alpha}} \left( \left( \frac{\epsilon}{2\|\gamma\|_1} \right)^{\frac{1}{\alpha}} + M \right)^{\alpha}
\]
and such that it holds
\[
\sup_{x \in K} |v_T^k x| \leq \pi mn \quad \text{and} \quad |b_v| \leq 2\|\gamma\|_1 \left[ 1 + \left( \frac{\epsilon}{2\|\gamma\|_1} \right)^{\frac{1}{\alpha}} + M \right]^{\alpha} (4nH\|W\|_{F,\infty})^m
\]
where \( H = \sup_{x \in [-C,C]^d} \|h(x)\|_{\infty} \).

**Proof** Let \( q_n^j \) given by Lemma 27 to approximate \( h_j \) over \([-C,C]\) and

\[
q_n^j(x) = \sum_{j=1}^{p} w_{k,j} q_n^j(u_j^T x)
\]

for \( k \in [0] \). We have that

\[
|q_n^j(x) - w_k^T h(U^T x)| \leq \sum_{j=1}^{p} |w_{k,j}| |q_n^j(u_j^T x) - h_j(u_j^T x)|
\]

\[
\leq 3\|W\|_{\infty} (1 + 2C^2) \frac{\log n}{n} \|x\|_{\infty} (1 + 2C^2) \epsilon_n
\]

for \( x \in [-R,R]^d \). Moreover, \( q_n^j \) is a real-valued shallow Fourier network with \((2n - 1)p\) terms and first layers weights given by \( \frac{n}{k!} \) for \( k \in [-(n - 1), n - 1] \). Moreover, we have that

\[
|q_n^j(x)| \leq |q_n^j(x) - w_k^T h(U^T x)| + |w_k^T h(U^T x)| \leq \|W\|_{\infty} (1 + 2C^2) \epsilon_n + M \equiv L.
\]

Let \( p_m^k(t) = \sum_{h=0}^{m} \beta_h^k t^h \) given by Corollary 3 to approximate \( g_k \) over the interval \([-L,L]\) and \( \epsilon_m \) the relative error. Let then

\[
f_{n,m}(x) = \sum_{k=1}^{o} \gamma_k p_m^k(q_k^n(x)).
\]

We have that

\[
|f(x) - f_{n,m}(x)| \leq \sum_{k=1}^{o} |\gamma_k| \left| g_k(w_k^T h(U^T x)) - p_m^k(q_k^n(x)) \right|
\]

\[
\leq \sum_{k=1}^{o} |\gamma_k| \left| g_k(w_k^T h(U^T x)) - g_k(q_k^n(x)) \right| + \sum_{k=1}^{o} |\gamma_k| \left| g_k(q_k^n(x)) - p_m^k(q_k^n(x)) \right|
\]

\[
\leq \|\gamma\|_1 \sup_{k \in [0]} |w_k^T h(U^T x) - q_k^n(x)|^{\alpha} + \|\gamma\|_1 \epsilon_m
\]

\[
\leq \|\gamma\|_1 \|W\|_{\infty}^{\alpha} (1 + 2C^2)^{\alpha} \epsilon_n^{\alpha} + \|\gamma\|_1 \epsilon_m.
\]

We have that

\[
\|\gamma\|_1 \|W\|_{\infty}^{\alpha} (1 + 2C^2)^{\alpha} \epsilon_n^{\alpha} \leq \frac{\epsilon}{2}
\]
as long as
\[ n \geq \frac{9 \cdot 4^\frac{1}{\alpha} \gamma_1^2 \| W \|_2^2 (1 + 2C^2)^2}{\epsilon} . \] (40)

Similarly
\[ \| \gamma \|_1 \epsilon_m \leq \frac{\epsilon}{2} \]
as long as
\[ m \geq L \left( \frac{12 \| \gamma \|_1}{\epsilon} \right)^{\frac{1}{\alpha}} \left( \frac{12 \| \gamma \|_1}{\epsilon} \right)^{\frac{1}{\alpha}} [\| W \|_\infty (1 + 2C^2) \epsilon_n + M] . \]

Moreover, by Lemma 25, \( p^k_m(t) = \sum_{h=0}^{m} \beta_h^k \) can be chosen with
\[ m \geq \frac{2 \cdot 16^\frac{1}{\alpha} \gamma_1^2 L^\alpha}{\epsilon^{1 + \frac{1}{\alpha}}} \left( \frac{12 \| \gamma \|_1}{\epsilon} \right)^{\frac{1}{\alpha}} [\| W \|_\infty (1 + 2C^2) \epsilon_n + M]^\alpha \]
such that its coefficients \( \beta_h^k, k \in [m] \), are bounded by
\[ |\beta_h^k| \leq \max \left\{ 2^m L^{\alpha - k}, L^\alpha + |g(0)| \right\} \leq 2^m (1 + L^\alpha) + |g(0)| \]
\[ = 2^m (1 + [\| W \|_\infty (1 + 2C^2) \epsilon_n + M]^\alpha) + |g(0)| . \]

Notice that we can assume \( g(0) = 0 \) w.l.o.g. Therefore
\[ \sup_{x \in K} |f(x) - f_{n,m}(x)| \leq \epsilon \] (41)
as long as (40) holds and
\[ m \geq \left( \frac{12 \| \gamma \|_1}{\epsilon} \right)^{\frac{1}{\alpha}} \left( \frac{\epsilon}{2 \| \gamma \|_1} \right)^{\frac{1}{\alpha}} + M \right] = 6^\frac{1}{\alpha} \left( 1 + M \left( \frac{2 \| \gamma \|_1}{\epsilon} \right)^{\frac{1}{\alpha}} \right) . \] (42)

If we further assume that
\[ m \geq \frac{2 \cdot 16^\frac{1}{\alpha} \gamma_1^2 L^\alpha}{\epsilon^{1 + \frac{1}{\alpha}}} \left( \frac{\epsilon}{2 \| \gamma \|_1} \right)^{\frac{1}{\alpha}} + M \right] \]
we can also assume that
\[ |\beta_h^k| \leq 2^1 \frac{16^\frac{1}{\alpha} \gamma_1^2}{\epsilon^{1 + \frac{1}{\alpha}}} \left[ \left( \frac{\epsilon}{2 \| \gamma \|_1} \right)^{\frac{1}{\alpha}} + M \right] \]
\[ \left( 1 + \left[ \left( \frac{\epsilon}{2 \| \gamma \|_1} \right)^{\frac{1}{\alpha}} + M \right] \right) \]
for \( k \in [m] \). Finally, notice that, by Lemma 28, \( f_{n,m} \) is a shallow Fourier neural network with number of units upper bounded by
\[ N = \sum_{k=0}^{m} \binom{(2n - 1)p + k - 1}{m} = \binom{(2n - 1)p + m}{m} \]
\[ = \frac{1}{m!}((2n - 1)p + k + m) \cdots ((2n - 1)p + 1) \]
\[ \leq ((2n - 1)p + 1)^m . \]

Therefore, it holds that
\[ \inf_{f_N \in \mathcal{F}_N} \sup_{x \in K} |f(x) - f_N(x)| \leq \epsilon \]
as long as
\[ N \geq (2np + 1)^m \]
with \( n \) and \( m \) given by (40) and (42) respectively. Finally, notice that the first layer weights of \( f_{n,m} \) are given by
\[
\sum_{j=1}^{p} \sum_{k=-(n-1)}^{n-1} s_{k,j} \frac{\pi k}{C} u_{j} \]
over all non-negative integers \( s_{k,j} \) such that \( \sum_{j=1}^{p} \sum_{k=-(n-1)}^{n-1} s_{k,j} \leq m \). Therefore, if
\[
f_{n,m}(x) = \sum_{\nu=1}^{N} b_{\nu} e^{i\nu^T x},
\]
then
\[
|\nu^T x| \leq m \frac{\pi(n-1)}{C} \max_{j \in [p]} |u_j^T x| \leq mn\pi.
\]
On the other hand, the coefficients \( b_k \) have the form
\[
b_{\nu} = \left( \frac{h}{s} \right)^{\nu} \sum_{k=1}^{\nu} \gamma_k \beta_{h}^k (w_{k,j}(q_{n}^j)^{s_{l,j}})
\]
for all non-negative integers \( s = (s_{l,j})_{l,j} \) such that \( \sum_{j=1}^{p} \sum_{l=-(n-1)}^{n-1} s_{l,j} = h \leq m \), where \( (q_{n}^j)_{l} \) denotes the \( l \)-th coefficients of \( q_{n}^j \). By Lemma 26, we know that
\[
|\nu^T x| \leq m \frac{\pi(n-1)}{C} \max_{j \in [p]} |u_j^T x| \leq mn\pi.
\]
This concludes the proof.

We can now conclude with a detailed version of Theorem 9.

**Theorem 30** Let \( K \) be a compact set and
\[
C = \sup_{x \in K} \|U^T x\|_{\infty}, \quad M = \sup_{x \in K} \|W^T h(U^T x)\|_{\infty} \quad \text{and} \quad H = \sup_{x \in [-C,C]^d} \|h(x)\|_{\infty}.
\]
It holds that
\[
\inf_{f_N \in F_N} \sup_{x \in K} |f(x) - f_N^x(x)| \leq \epsilon
\]
for some
\[
N \leq \frac{16\pi \nu_{\sigma}}{\epsilon} \|\gamma\|_1 mn(4np + 1)^{2m}(H\|W\|_{F,\infty})^{m} \left[ 1 + \left( \frac{\epsilon}{2\|\gamma\|_1} \right)^{\frac{1}{\alpha}} + M \right]^{\alpha}.
\]
where
\[ n = \frac{9 \cdot 4^\frac{1}{\alpha} ||\gamma||_1^2 ||W||_\infty^2 (1 + 2C)^2}{\epsilon^{\frac{2}{\alpha}}} \quad \text{and} \quad m = \frac{2 \cdot 16^\frac{1}{\alpha} ||\gamma||_1^2 \left( \left( \frac{\epsilon}{2 ||\gamma||_1} \right)^{\frac{1}{\alpha}} + M \right)^\alpha}{\epsilon^{1 + \frac{2}{\alpha}}}. \]

Moreover, it is possible to choose \( f_N^\gamma \) attaining (44) with \( m_{\infty}(f_N^\gamma) \) satisfying a bound similar to the one on \( N \), for example \( m_{\infty}(f_N^\gamma) \leq (1 + N^2) \).

\textbf{Proof} Let \( f_N \) given by Proposition 29 such that
\[ \sup_{x \in K} |f(x) - f_N(x)| \leq \frac{\epsilon}{2}. \]

We know that
\[ f_N(x) = \sum_{k=1}^{N} b_k e^{i\gamma^T x} = f_N^c(x) + i f_N^s(x) \]

where
\[ f_N^c(x) = \sum_{k=1}^{N} b_k \cos(\gamma^T x) \quad \text{and} \quad f_N^s(x) = \sum_{k=1}^{N} b_k \sin(\gamma^T x) \]

and \(|b_k| \leq B \) and \(|\gamma^T x| \leq V \) for \( x \in K \), where \( B \) and \( V \) are given by (39). Using the assumption on \( \sigma \), we know that, for each \( k \in [N] \), there exist shallow networks \( f_k^c \) and \( f_k^s \) with activation \( \sigma \) and number of units
\[ n \leq c_\sigma 4VBN \]

such that
\[ \sup_{x \in K} |f_k^c(x) - \cos(\gamma_k^T x)| \leq \frac{\epsilon}{4NB} \quad \text{and} \quad \sup_{x \in K} |f_k^s(x) - \sin(\gamma_k^T x)| \leq \frac{\epsilon}{4NB}. \]

Letting \( f_N(x) = \sum_{k=1}^{N} b_k f_k^c(x) + i \sum_{k=1}^{N} b_k f_k^s(x) \) it holds that
\[ \sup_{x \in K} |f_N(x) - f_N(x)| \leq \sup_{x \in K} \left| \sum_{k=1}^{N} b_k (f_k^c(x) - \cos(\gamma_k^T x)) \right| + \sup_{x \in K} \left| \sum_{k=1}^{N} b_k (f_k^s(x) - \sin(\gamma_k^T x)) \right| \leq \sum_{k=1}^{N} |b_k| \sup_{x \in K} |f_k^c(x) - \cos(\gamma_k^T x)| + \sum_{k=1}^{N} |b_k| \sup_{x \in K} |f_k^s(x) - \sin(\gamma_k^T x)| \leq NB \frac{\epsilon}{4NB} + NB \frac{\epsilon}{4NB} = \frac{\epsilon}{2} \]

which implies that
\[ \sup_{x \in K} |f_N(x) - f(x)| \leq \epsilon. \]

Moreover notice that we can assume that all second layer weights of \( f_N \) are real; indeed, if this is not the case, one can replace them by the real part, and upper bound above can only get better. Finally, we have that the number of units of \( f_N \) is given by
\[ N \leq \frac{8 c_\sigma}{\epsilon} \cdot V \cdot B \cdot N. \]

Applying Proposition 29 concludes the proof. \( \blacksquare \)
D.3 Radial functions

Let \( f(\mathbf{x}) = \varphi(\|\mathbf{x}\|) \) with \( \varphi \) 1-Lipschitz. Then we have that \( f(\mathbf{x}) = g(1^T \mathbf{h}(\mathbf{x})) \) where \( g(t) = \varphi(\sqrt{t}) \) and \( \mathbf{h} : \mathbb{R}^d \to \mathbb{R}^d \) is defined as \( h_i(\mathbf{x}) = x_i^2 \). Clearly, \( \sup_{\mathbf{x} \in B^d_{1,2}} \|\mathbf{x}\|_{\infty} = 1 \), \( \sup_{\mathbf{x} \in B^d_{1,2}} |1^T \mathbf{h}(\mathbf{x})| = \sup_{\mathbf{x} \in B^d_{1,2}} \|\mathbf{x}\|^2 = 1 \) and \( \sup_{\mathbf{x} \in [-1,1]^d} \|\mathbf{h}(\mathbf{x})\|_{\infty} = \sup_{\mathbf{x} \in [-1,1]} |x|^2 = 1 \). Moreover, \( \|1\|_1 = d \) and \( g \) is \((1, 1/2)\)-Holder. Then, by applying Theorem 30 we get the following.

Corollary 31 (Radial functions) It holds that

\[
\inf_{f_N^* \in F_N^*} D_{\infty}\left(f_N^*, f; B^d_{1,2}\right) \leq \epsilon
\]

for some

\[
N \leq \nu_\alpha \alpha \cdot d^2 \cdot \frac{(4 + \epsilon)^2}{\epsilon^{10}} \left(\frac{d^3}{\epsilon^4} + 1\right)^{\frac{5}{2}} \leq (2 + \epsilon)
\]

where \( \alpha > 0 \) is a numerical constant.

D.4 Shallow approximation of (1)

Consider \( f_{\mathbf{w}, \mathbf{U}} : \mathbf{x} \in \mathbb{R}^d \mapsto e^{i \mathbf{w}^T (\mathbf{U} \mathbf{x})} \) for some \( \mathbf{w} \in \mathbb{R}^p \), \( \mathbf{U} \in \mathbb{R}^{p \times d} \). Then Theorem 30 implies the following.

Corollary 32 (Approximation of (1) by shallow networks) It holds that

\[
\inf_{f_N^* \in F_N^*} D_{\infty}\left(f_{\mathbf{w}, \mathbf{U}}, f_N^*; B^d_{r,p}\right) \leq \epsilon
\]

for some

\[
N \leq \frac{\nu_\beta \beta}{\epsilon} \cdot (2 + \epsilon + 2r \|\mathbf{w}\|_1 \|\mathbf{U}\|_{1,\infty})^2 \cdot \left[r \|\mathbf{w}\|_{\infty} \|\mathbf{U}\|_{p,\infty} \left(\frac{4p\beta}{\epsilon^2} + 1\right)^2\right]^{\frac{5}{2}} \leq (2 + \epsilon)
\]

where \( \beta = \alpha \|\mathbf{w}\|_1^2 \cdot (1 + 2r^2 \|\mathbf{U}\|_{2,\infty}^2)^2 \) and \( \alpha \) is a numerical constant.

D.5 Approximation bounds under the Gaussian metric

For sake of simplicity in this section we consider approximation bounds for the function of interest

\( f_{\mathbf{w}, \mathbf{U}} : \mathbf{x} \in \mathbb{R}^d \mapsto e^{i \mathbf{w}^T (\mathbf{U} \mathbf{x})} \)

for some \( \mathbf{w} \in \mathbb{R}^p \), \( \mathbf{U} = [\mathbf{u}_1; \cdots; \mathbf{u}_d] \in \mathbb{R}^{p \times d} \). Notice that the following results can be naturally extended to any three-layer network target. We are interested in upper bounding the error

\[
\inf_{f_N^* \in F_N^*} \left( \mathbb{E}_x \left| f_{\mathbf{w}, \mathbf{U}}(\mathbf{x}) - f_N^* (\mathbf{x}) \right|^2 \right)^{\frac{1}{2}}
\]

where the expectation is taken over \( \mathbf{x} \sim N(0, \sigma^2 I)^d \). For sake of simplicity of notation, we denote

\[
\|f - g\|_{2,\sigma} = \left( \mathbb{E}_x |f - g|^2 \right)^{\frac{1}{2}}.
\]

It is a well known fact that Gaussian vector concentrates in a ball of radius \( \sqrt{d} \). We recall a quantitative version of this fact in the following.
Lemma 33 Let \( x \sim N(0, \sigma^2)^d \). Then it holds that
\[
P\left( \|x\|_2 \geq \sigma \sqrt{d} + t \right) \leq e^{-\frac{t^2}{2\sigma^2}}. \tag{45}
\]

Thanks to Proposition 29, we have the following.

Lemma 34 Let \( r > 0 \). Then it holds that
\[
\inf_{f_N' \in \mathcal{F}_N'} \sup_{x \in B_{r,2}^d} \left| f_N'(x) - f_{w,U}(x) \right| \leq \delta
\]
as long as
\[
N \geq (2np + 1)^m
\]
where
\[
n = \frac{36}{\delta^2} \left\| w \right\|^2_1 \left( 1 + r^2 \left\| U \right\|_{2,\infty}^2 \right)^2 \quad \text{and} \quad m \geq \frac{16}{\delta^3} (\delta + 2r \left\| w \right\|_1 \left\| U \right\|_{2,\infty}).
\]

Moreover, under the same assumption, we can also assume that the function \( f_N' \) that satisfies (45) also satisfies
\[
\|f_N'\|_\infty \leq N(2 + \delta + 2r \|w\|_1 \|U\|_{2,\infty})(4np \|w\|_\infty \|U\|_{2,\infty})^m.
\]

Using these lemmas, we can prove the following proposition.

Proposition 35 Let \( \sigma = \frac{1}{\sqrt{d}} \) and assume that \( \|U\|_{2,\infty} \leq 1 \). Then it holds
\[
\inf_{f_N' \in \mathcal{F}_N'} \left\| f_N' - f_{w,U} \right\|_{2,\sigma} \leq \epsilon
\]
as long as
\[
N \geq \left[ KP \left( 1 + \frac{1}{\epsilon^2} \right) (1 + \|w\|_1) \right]^{K\left(1+(\frac{\log p}{\delta^2})^s\right)} \left(1+(\frac{\log p}{\delta^2})^s\right)
\]
where \( K > 0 \) and \( s \geq 1 \) are some numerical constants.

Proof Let \( c = \|w\|_1 \). First, notice that \( \|f_{w,U}\|_\infty = 1 \). Let \( \chi_r(x) = \mathbb{1}\{\|x\| \leq r\} \) and \( f_N \) given by Lemma 34 for a certain \( \delta > 0 \). Then we have that
\[
\|f_N - f_{w,U}\|_{2,\sigma} \leq \|(f_N - f_{w,U})(1 - \chi_r)\|_{2,\sigma} + \|(f_N - f_{w,U})\chi_r\|_{2,\sigma}
\]
\[
\leq \sup_{x \in B_{r,2}^d} |f_N(x) - f_{w,U}(x)| + P(\|x\|_2 > r(\|f_N\|_\infty + \|f_{w,U}\|_\infty)).
\]

If we take \( r = 1 + t \) for \( t > 0 \), we get
\[
\|f_N - f_{w,U}\|_{2,\sigma} \leq \delta + e^{-\frac{at^2}{2}} (1 + \|f_N\|_\infty)
\]
as long as
\[
N \geq \left( \frac{72p}{\delta^2} e^2 (1 + r^2)^2 + 1 \right)^{\frac{1}{2} (\delta + 2rc)}.
\]

Moreover, we can assume
\[
\|f_N\|_\infty \leq (2 + \delta + 2r\omega) \left( \frac{72p}{\delta^2} e^2 (1 + r^2)^2 + 1 \right)^{\frac{16}{9} (\delta + 2rc)} \left( 144 \frac{p}{\delta^2} \omega^3 (1 + r^2)^2 + 1 \right)^{\frac{16}{9} (\delta + 2r\omega)}
\]
\[
\leq (2 + \delta + 2r\omega) \left( 144 \frac{p}{\delta^2} \omega^3 (1 + r^2)^2 + 1 \right)^{\frac{16}{9} (\delta + 2r\omega)}
\]
where we denoted \( \omega = \max(1, c) \). Now, let’s take \( \delta = \frac{\omega}{2} \). If we take \( t \geq 1 \), we have that

\[
\|f_N\|_\infty \leq (4\omega + \epsilon + 2\omega t) \left( \frac{576p}{e^2} \omega^3 (1 + t)(1 + (1 + t)^2)^2 + 1 \right)^{\frac{Kp}{e^2} (\epsilon + 2\omega + 2\omega t)} \leq K(\epsilon + \omega + \omega t) \left( \frac{Kp}{e^2} \omega^5 + 1 \right)^{\frac{Kp}{e^2} (\epsilon + \omega + \omega t)}.
\]

In the equation above above and in the following, we use \( K \) to denote a (large enough) numerical constant.

Therefore

\[
e^{-\frac{dt^2}{2}(1 + \|f_N\|_\infty)} \leq \frac{e}{2} \tag{47}
\]

as long as

\[
\frac{dt^2}{2} - \log \left( 1 + K(\epsilon + \omega + \omega t) \left( \frac{Kp}{e^2} \omega^5 + 1 \right)^{\frac{Kp}{e^2} (\epsilon + \omega + \omega t)} \right) + \log \frac{\epsilon}{2} \geq 0.
\]

Since \( \log(1 + C s^\alpha) \leq \log(1 + C) + \alpha \log(s) \) if \( s \geq 1, C > 0 \) and \( \alpha > 0 \), the above is implied by

\[
\frac{dt^2}{2} - \log(1 + K(\epsilon + \omega + \omega t)) - \frac{K}{e^3}(\epsilon + \omega + \omega t) \log \left( \frac{Kp}{e^2} \omega^5 + 1 \right) + \log \frac{\epsilon}{2} \geq 0.
\]

Since

\[
\log(1 + K(\epsilon + \omega + \omega t)) \leq K(\epsilon + \omega + \omega t)
\]

and

\[
\log \left( \frac{Kp}{e^2} \omega^5 + 1 \right) \leq \log \left( 1 + K \frac{p\omega^2}{e^2} \right) + 5 \log t \leq \log \left( 1 + K \frac{p\omega^2}{e^2} \right) + 5\sqrt{t}
\]

equation (47) holds if

\[
\frac{dt^2}{2} - \alpha - \beta t^{1/2} - \gamma t - \eta t^{3/2} \geq 0
\]

where

\[
\alpha = K(\epsilon + \omega) + \frac{K}{e^3}(\epsilon + \omega) \log \left( 1 + K \frac{p\omega^2}{e^2} \right) - \log \frac{\epsilon}{2} > 0,
\]

\[
\beta = \frac{K}{e^3}(\epsilon + \omega) > 0,
\]

\[
\gamma = K \omega t + \frac{K}{e^3} \omega \log \left( 1 + K \frac{p\omega^2}{e^2} \right) > 0,
\]

\[
\eta = \frac{K}{e^3} \omega t > 0.
\]

It follows that eq. (47) holds if

\[
t \geq 1 + 4 \left( \frac{\alpha + \beta + \gamma + \eta}{d} \right)^2.
\]

It follows that the error bound (46) holds as long as

\[
N \geq \left( \frac{Kp}{e^2} (1+c)^2 \left( 1 + 4 \left( \frac{\alpha + \beta + \gamma + \eta}{d} \right)^2 \right)^2 + 1 \right)^{\frac{Kp}{e^2} (\epsilon + c(1 + (\frac{\alpha + \beta + \gamma + \eta}{d})^2))}.
\]

The thesis follows.
D.6 Extension to generic L-layers networks

The results presented in the previous section can be generalized to hold for approximating generic multi-layer neural networks. In this section we present analogous results for the more general case. Consider a multi-layer neural network \( f \) defined as

\[
f : x \in \mathbb{R}^d \rightarrow x_L(x) \in \mathbb{C}
\]

where \( x_L \) is defined by recursion as

\[
x_1(x) = A^1x,
\]

\[
x_k(x) = A^k\rho^k(x_{k-1}(x)) \quad \text{for} \quad k \in [2, L - 1],
\]

\[
x_{L+1}(x) = a_{L+1}^T \rho^{L+1}(x_L(x)).
\]

where \( A^k \in \mathbb{R}^{d_{k+1} \times d_k} \) for \( k \in [1, L] \), \( a_{L+1} \in \mathbb{C}^{d_{L+1}} \), \( d_0 = d \), and \( \rho^{k+1} : \mathbb{R}^{d_{k+1}} \rightarrow \mathbb{R}^{d_{k+1}} \) are \( \frac{1}{k} \)-Lipschitz diagonal functions and verify \( \rho^{k+1}(0) = 0 \) for \( k \in [L] \). In the following we also assume that \( \| A^k \|_\infty \leq 1 \) for \( k \in [L] \) and \( \| a_{L+1} \|_1 \leq 1 \). Note that these assumption can easily be relaxed, but we adopt them here for sake of simplicity.

**Proposition 36** Let \( K = [-1, 1]^d \) and \( f \) as above. It holds that

\[
\inf_{f_N' \in F_N'} \sup_{x \in K} |f_N(x) - f(x)| \leq \epsilon \quad (48)
\]

as long as

\[
N \geq \left( 2^L C \left( 1 + \frac{1}{\epsilon^2} \right) d_1 \right) \frac{CL(1+\frac{1}{\epsilon})^{L-1}}{d_1}
\]

where \( C \) is a numerical constant.

Before proving the above proposition, we prove two preliminary lemmas.

**Lemma 37** Let \( W = \{w_t\}_{t \in [K]} \subset \mathbb{R}^d \) and \( h : \mathbb{R}^d \rightarrow \mathbb{R}^p \) is such that \( h_j \) is a shallow Fourier neural networks with first layer weight given by \( W \) for each \( j \in [p] \). Consider \( q : \mathbb{R}^p \rightarrow \mathbb{R}^m \) is of the form

\[
q(x) = B\rho(x)
\]

where \( \rho : \mathbb{R}^p \rightarrow \mathbb{R}^p \) is a diagonal polynomial function of degree at most \( D \) and \( B \in \mathbb{C}^{m \times p} \). Then there exists \( V \subset \mathbb{R}^d \) finite such that \( f = q \circ h \) is such that \( f_j \) is a Fourier neural nets with first layer weights given by \( V \) for each \( j \in [p] \) and such that

\[
|V| \leq (2K)^D.
\]

**Proof** The functions \( f_j \) have the form

\[
f_j(x) = \sum_{k=1}^p b_{jk} \sum_{l=0}^D \alpha_{k,l}(h_k(x))^l = \sum_{k=1}^p b_{jk} \sum_{l=0}^D \alpha_{k,l} \left( \sum_{\nu=1}^K \beta_{k,\nu} e^{i\nu w_{k}\cdot x} \right)^l.
\]

By Lemma 28, we see that each \( f_j \) is a Fourier neural network with the same set of first layer weights of size at most

\[
\sum_{l=0}^D \binom{K+l-1}{l} = \binom{K+D}{D} \leq (K+1)^D \leq (2K)^D.
\]

This concludes the proof. 

\[\blacksquare\]
Lemma 38  Consider the same assumption as Proposition 36. Then, there exists a polynomial

\[ f_{N_1,\ldots,N_L} : x \in \mathbb{R}^d \to y_L(x) \in \mathbb{C} \]

given by the recursion

\[
\begin{align*}
    y_1(x) &= A^1x \\
y_k(x) &= A^k p_{N_k}^k(y_{k-1}(x)) & \text{ for } k \in [2, L] \\
y_{L+1}(x) &= a_T^{L+1} p_{N_L}^L(y_L(x))
\end{align*}
\]

where \( p_{N_k}^k \) are diagonal polynomial functions of degree \( N_k \), such that

\[
    \sup_{x \in \mathcal{R}} |f(x) - f_{N_1,\ldots,N_L}(x)| \leq \epsilon
\]

(49)
as long as \( N_k \geq \frac{L}{\epsilon} + (L - 1) \) for \( k \in [L] \). In particular, \( f \) is a polynomial of degree \( \prod_{k=1}^L N_k \).

Proof  We can show this by induction over \( L \). First, consider the case \( L = 1 \). By Lemma 23 for each \( j \in [d_1] \), there exist polynomials \( p_{j,N} : \mathbb{R} \to \mathbb{R} \) of degree \( N \) which verify

\[
    |p_{j,N}(a_j^1x) - \rho_j^2(a_j^1x)| \leq \frac{1}{N}
\]
since \( |a_j^1x| \leq 1 \) by assumption. Since \( ||a_2||_1 \leq 1 \), it follows that

\[
    |a_T^1 p_N(A^1x) - a_T^2 \rho^2(A^1x)| \leq \frac{1}{N}.
\]

This implies the thesis for the case \( L = 1 \). Now consider the induction step, that is, assume that, for every \( \delta > 0 \) and \( j \), there exists a certain \( f_{N_1,\ldots,N_{L-1}}^j \) such that

\[
    |x_{L-1,j}(x) - f_{N_1,\ldots,N_{L-1}}^j(x)| \leq \delta
\]
as long as \( N_k \geq \frac{L-1}{\delta} + (L - 2) \) for \( k \in [L-1] \). Notice that this implies that

\[
    |f_{N_1,\ldots,N_{L-1}}^j(x)| \leq 1 + \delta.
\]

Therefore for each \( j \in [d_L] \), by Lemma 23, we can find polynomials \( p_{j,N} \) of degree \( N \) such that

\[
    |p_{j,N}(f_{N_1,\ldots,N_{L-1}}^j(x)) - \rho_j^{L+1}(f_{N_1,\ldots,N_{L-1}}^j(x))| \leq \frac{1+\delta}{N}.
\]

Let then \( f_{N_1,\ldots,N_{L-1},N} \) be defined as

\[
    f_{N_1,\ldots,N_{L-1},N}(x) = \sum_{j=1}^N a_{L+1,j} p_{j,N}(f_{N_1,\ldots,N_{L-1}}^j(x)).
\]

Since \( ||a_{L+1}||_1 \leq 1 \), we have that

\[
    |f_{N_1,\ldots,N_{L-1},N}(x) - f(x)| \leq |f_{N_1,\ldots,N_{L-1},N}(x) - a_T^{L+1}(f_{N_1,\ldots,N_{L-1},N}(x))| \\
    + |a_T^{L+1}(f_{N_1,\ldots,N_{L-1},N}(x)) - f(x)| \\
    \leq \frac{1+\delta}{N} + \delta.
\]

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If we take \( \delta = \frac{L-1}{L} \varepsilon \) then eq. (49) holds as long as
\[
N \geq \frac{1 + \frac{L-1}{L} \varepsilon}{\varepsilon} = \frac{L}{\varepsilon} + (L - 1)
\]
and
\[
N_k \geq \frac{L - 1}{L} + (L - 2) = \frac{L}{\varepsilon} + (L - 2).
\]
This concludes the proof of the lemma.

**Proof** [Proof of Proposition 36] We have that
\[
f(x) = g(\rho^2(A^1 x))
\]
where \( g \) is a \( L \)-layers neural network with input dimension \( d_1 \). By Lemma 26 for every \( \delta > 0 \) and \( j \in [d_1] \), there exists Fourier networks \( q_{j, N_1}(x) \) with \( 2N_1 - 1 \) units such that
\[
|\rho_j^2(a_j^1 x) - q_{j, N_1}(a_j^1 x)| \leq \frac{C}{\sqrt{N_1}}
\]
where \( C > 0 \) is a numerical constant. Notice that this implies that, for \( N_1 \geq 4C^2 \),
\[
\|q_{N_1}(A^1 x)\|_\infty \leq 1.
\]
Now, we can approximate \( g \) with a polynomial neural network \( g_{N_1, \ldots, N_2} \) as given by Lemma 38. In particular, for any \( \delta > 0 \), we can take \( g_{N_1, \ldots, N_2} \) such that
\[
\sup_{x \in K} |g_{N_1, \ldots, N_2}(x) - g(x)| \leq \delta
\]
as long as \( N_k \geq \frac{L-1}{\delta} + (L - 2) \) for \( k \in [2, L] \). It follows that
\[
|g_{N_1, \ldots, N_2}(q_{N_1}(A^1 x)) - f(x)| \leq \delta + \frac{C}{\sqrt{N_1}}.
\]
Let \( f_N(x) = g_{N_1, \ldots, N_2}(q_{N_1}(A^1 x)) \). By choosing \( \delta = \varepsilon/2 \), it holds that
\[
\sup_{x \in K} |f_N(x) - f(x)| \leq \varepsilon
\]
as long as \( N_k \geq \frac{2L-1}{\varepsilon} + (L - 2) \) for \( k \in [2, L] \) and \( N_1 \geq C^2(1 + \frac{4}{\varepsilon^2}) \). We claim that \( f_N \) is a Fourier network with at most
\[
N = (2L-N_1d_1) \prod_{k=2}^{L} N_k
\]
units. We can prove this by induction over \( L \geq 2 \). Remember that \( g_{N_1, \ldots, N_2} \) is is the form
\[
g_{N_1, \ldots, N_2}(x) = a_{L+1}^T \mathbf{g}_{N_L}^L \left( A^L \mathbf{g}_{N_{L-1}}^{L-1} \left( \cdots \mathbf{g}_{N_2}^2 \left( A^2 \mathbf{q}_{N_1}^0 \right) \right) \right)
\]
where \( \mathbf{g}_{N_k}^k \) is a diagonal polynomial of degree at most \( N_k \), for \( k \in [2, L] \). We start by the case \( L = 2 \). Notice that each component of \( \mathbf{A}^2 \mathbf{q}_{N_1}^0 \) is a Fourier network with the same set of first layer weights, of size at most \( (2N-1) d_1 \). Then, by Lemma 37, we have that each component of
\[
f_{N_2, N_1}^2(x) = \mathbf{A}_1^3 \mathbf{g}_{N_2}^2 \left( \mathbf{A}^2 \mathbf{q}_{N_1} \right)
\]
is a Fourier network with the same set of first layer weights of size at most
\[(2(2N_1 - 1)d_1)^{N_2}.
\]
Finally, consider the induction step. By the assumption hypothesis, the function
\[f_{L-1,...,N_1}(x) = A^L g_{N_{L-1}}^{L-1} (A_{L-1} \cdot \cdots \cdot g_{N_2}^2 (A^2 q_{N_1} (A^1 x)))
\]
is such that each component is a Fourier network with the same set of first layer weights of size at most
\[(2^{L-2}(2N_1 - 1)d_1) \prod_{k=2}^{L-1} N_k.
\]
Then, by Lemma 37, the function
\[f_N(x) = \mathbf{a}_{L+1}^T g_{N_L}^{L-1} (N_{L-1} \ldots N_1 (x))
\]
is a Fourier network with at most
\[2 \cdot (2^{L-2}(2N_1 - 1)d_1) \prod_{k=2}^{L-1} N_k)^{N_L} = 2^{N_L} 2^{(L-2)} \prod_{k=2}^{L-1} N_k ((2N_1 - 1)d_1) \prod_{k=2}^{L-1} N_k
\]
which implies equation (50). Plugging in the lower bounds on \(N_k\) in terms of \(\epsilon\), we get the thesis.

D.7 Fixed-dimension approximation

The results of Section D.7 on fixed-threshold approximation can be complemented by the following result on fixed-dimension approximation. The proposition below is a straight-forward generalization of Theorem 3 in (Safran et al., 2019).

**Proposition 39** Let \(\sigma\) be an activation satisfying Assumption 1. Then there exists a constant \(\beta > 0\) such that for any \(f : B_{1/2}^d \to \mathbb{C}\) 1-Lipschitz function and \(\epsilon > 0\) there exists a network \(f_N \in F_N^d\) such that
\[D_{\infty}(f, f_N; [-1, 1]^d) \leq \epsilon
\]
for some \(N \leq 2 + \beta d^7 (\beta \epsilon^{-1})^d \epsilon^{-6}.
\]

**Proof** The result is proved by noticing that the proof of Theorem 3 in (Safran et al., 2019) actually holds for any function \(f\) as in the statement. Moreover, using Assumption 1, \(f_N\) can also be chosen so that an equivalent bound holds for \(m_{\infty}(f_N)\).

Appendix E. Facts about spherical harmonics decomposition

We start by introducing the notation and summarizing some facts about spherical harmonics decomposition. Let \(d \geq 2\). We denote by \(dS\) the Lebesgue measure over \(S^{d-1}\). The area of the sphere is given by
\[
\omega_d = |S^{d-1}| = \int_{S^{d-1}} dS(\xi)
\]
and we denote by $dS$ the normalized measure $dS(\xi) = \omega_d^{-1} d\hat{S}(\xi)$. In particular

$$\omega_d = \omega_{d-1} \frac{\sqrt{\pi}}{\Gamma\left(\frac{d+1}{2}\right)} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} = \Theta\left(\frac{(2\pi e)^{d/2}}{d^{d/2-1/2}}\right) = \Theta\left(\frac{\sqrt{d}}{d^{d/2}}\right).$$

The spherical harmonics are a particular orthonormal basis for $L^2_S$. They consists of

$$\bigcup_{k=0}^{\infty} \text{span} \left( \{ Y^d_{k,i} \}_{i=1}^{N^d_k} \right) = \bigcup_{k=0}^{\infty} H^d_k$$

where $Y^d_{k,i}$ is a restriction to $S^{d-1}$ of an homogeneous harmonic polynomial of degree $k$. We denote by $\mathcal{P}^d_k$ the projection operator over $H^d_k$, given by

$$\mathcal{P}^d_k : f \in L^2_{S^{d-1}} \mapsto f_k = \sum_{i=1}^{N^d_k} \langle f, Y^d_{k,i} \rangle Y^d_{k,i}.$$ 

Similarly, we use notations such as $\mathcal{P}^d_{k \geq k}$ to indicate the operator $\oplus_{i=1}^{\infty} \mathcal{P}^d_i$. The function $f_k$ is referred as the degree $k$ spherical harmonic component of the function $f$. Since the spherical harmonic form an orthonormal basis of $L^2_S$, it holds that $f = \sum_{k \geq 0} f_k$ and $\| f \|_2^2 = \sum_{k \geq 0} \| f_k \|_2^2$ for every $f \in L^2_S$, where $\| \cdot \|_2$ denotes the norm in $L^2_S$. As spherical harmonics decomposition can be seen as a generalization of Fourier series to dimensions $d \geq 3$, in the following we refer to the spherical decomposition of a function as its Fourier representation, interchangeably. The kernel associated with $\mathcal{P}_k$ is given by

$$\sum_{i=1}^{N^d_k} Y^d_{k,i}(x) Y^d_{k,i}(y) = \frac{N^d_k}{|S^{d-1}|} P^d_k(x^T y),$$

where

$$N^d_k = \frac{(2k + d - 2)(k + d - 3)!}{k!(d-2)!} = \Theta\left(\frac{(k+1)^{k+1}}{kd^k} \frac{d^2}{(k+1)^2}\right)$$

is the dimension of $H^d_k$ and $P^d_k$ is the $((d-2)/2)$-Gegenbauer polynomial defined as

$$P^d_k(x) = k! \Gamma\left(\frac{d-1}{2}\right) \sum_{j=0}^{[k/2]} (-1)^j \frac{(1-t^2)^j x^{k-2j}}{4^j j! (k-2j)! \Gamma\left(j + \frac{d-1}{2}\right)}.$$ 

The polynomials $\{ \sqrt{N^d_k} P^d_k \}_{k \geq 0}$ form a basis of orthonormal polynomials for $L^2_d \cong L^2_{\mu_d}$, where $\mu_d$ is the finite measure on $[-1,1]$ defined by

$$d\mu_d(t) = (1-t^2)^{(d-3)/2} dt.$$ 

In the following we also denote $d\tilde{\mu}_d(t) = \alpha_d^{-1} \mu_d(t)$ where

$$\alpha_d = \mu_d([-1,1]) = \frac{\omega_{d-1}}{\omega_d} = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)} = \Theta\left(\sqrt{d}\right).$$ 

Notice that, given a function $f \in L^2_{S^{d-1}}$, it holds

$$f_k(x) = N^d_k \int_{S^{d-1}} f(y) P^d_k(x^T y) dS(y).$$

In the following we sometimes omit the dimension $d$ from the notation, when this is clear from the context.
**Funk-Hecke formula** This formula is crucial in the analysis of spherical harmonics. For every \( \sigma : [-1, 1] \to \mathbb{C} \) such that \( x \in \mathbb{S}^{d-1} \mapsto \sigma(x_1) \) is in \( L^2_\mathbb{S} \), and for every \( w \in \mathbb{S}^{d-1} \), it holds that

\[
\int_{\mathbb{S}^{d-1}} \sigma(w^T x) P^d_k(\xi^T x) \, dS(x) = \alpha_d \lambda_k P^d_k(\xi^T w)
\]

where

\[
\lambda_k = \int_{-1}^{1} \sigma(t) P^d_k(t)(1 - t^2)^{\frac{d-3}{2}} \, dt
\]

and

\[
\alpha_d = \frac{\omega_{d-1}}{\omega_d} = \Theta\left(\sqrt{d}\right).
\]

**Zonal harmonics and RKHS structure of \( H_k \)** We call zonal harmonics the functions of the form

\[
x \in \mathbb{S}^{d-1} \mapsto \alpha P^d_k(w^T x)
\]

for some \( \alpha \in \mathbb{R} \) and \( w \in \mathbb{S}^{d-1} \). By the Funk-Hecke formula it follows that

\[
\int_{\mathbb{S}^{d-1}} P^d_k(w^T x) P^d_k(\nu^T x) \, dS(x) = (N^d_k)^{-1} P^d_k(w^T \nu)
\]

for any \( w, \nu \in \mathbb{S}^{d-1} \). This implies that \( H_k^d \) has an RKHS structure with kernel given by

\[
K(v, w) = N^d_k P^d_k(v^T w).
\]

In particular, zonal harmonics actually span \( H_k^d \). Moreover, it can be shown that there exists \( w_1, \ldots, w_{N^d_k} \in \mathbb{S}^{d-1} \) such that \( H_k^d = \text{span}\{P^d_k(w_i^T \cdot)\}_{i=1}^{N^d_k} \).

**Low-coherence Zonal frames** In this paragraph, we wish to quantify how much incoherent can a frame composed of zonal harmonics be. More specifically, we wish to find a lower bound for

\[
N(d, k, \epsilon) = \sup \left\{ N \geq 1 : \exists w_1, \ldots, w_N \in \mathbb{S}^{d-1} : \sup_{i \neq j} \left| P^d_k(w_i^T w_j) \right| \leq \epsilon \right\}
\]

for \( \epsilon \in (0, 1) \).

**Lemma 40** It holds that

\[
N(d, k, \epsilon) \geq \sup \left\{ N \geq 1 : \exists w_1, \ldots, w_N \in \mathbb{S}^{d-1} : \sup_{i \neq j} \left| w_i^T w_j \right| \leq \sqrt{1 - \frac{d}{k \epsilon^{d/4}}} \right\}
\]

for \( k > d \geq 5 \) and \( \left( \frac{d}{2} \right)^{d/4} \leq \epsilon < 1 \).

**Proof** We recall that it holds

\[
\left| P^d_k(t) \right| \leq \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{d-1}{2}\right) \left(\frac{4}{k(1 - t^2)}\right)^{(d-2)/2}
\]

for \( d \geq 2 \) and \( t \in (-1, 1) \) (cfr. eq. (2.117) in (Atkinson and Han, 2012)) and that

\[
\Gamma(x) \leq \left(\frac{x}{2}\right)^{x-1}
\]
for $x \geq 2$. Therefore we get

$$
|P^d_k(t)| \leq \frac{1}{\sqrt{\pi}} \left(\frac{d}{\frac{d}{4}}\right)^{(d-3)/2} \left(\frac{4}{k(1-t^2)}\right)^{(d-2)/2}
$$

$$
\leq \frac{1}{\sqrt{\pi}} \left(\frac{d}{\frac{d}{4}}\right)^{-1/2} \left(\frac{d}{k(1-t^2)}\right)^{(d-2)/2} \leq \left(\frac{d}{k(1-t^2)}\right)^{(d-2)/2}
$$

for $d \geq 5$ and $|t| < 1$. In particular, for $\epsilon \in (0, 1)$, it holds that $|P^d_k(t)| \leq \epsilon$ if $d \geq 5$ and $|t| < 1$. In particular, for $\epsilon \in (0, 1)$, it holds that $|P^d_k(t)| \leq \epsilon$ if

$$
\frac{d}{k(1-t^2)} \leq \epsilon^{4/d}
$$

that is if

$$
|t| \leq \sqrt{1 - \frac{d}{k\epsilon^{4/d}}}.
$$

The thesis follows.

\begin{itemize}
  \item Let’s define
    \begin{align*}
      N(d, \delta) &= \sup \left\{ N \geq 1 : \exists \mathbf{w}_1, \ldots, \mathbf{w}_N \in \mathbb{S}^{d-1} : \sup_{i \neq j} |\mathbf{w}_i^T \mathbf{w}_j| \leq \delta \right\}
    \end{align*}

  \end{itemize}

for $\delta \in (0, 1)$. The previous lemma says that

$$
N(d, k, \epsilon) \geq N \left( d, \sqrt{1 - \frac{d}{k\epsilon^{4/d}}} \right).
$$

\textbf{Example 1} Taking

$$
\{w_i\}_{i=1}^N = \left\{ \epsilon \in \left\{ \pm \frac{1}{\sqrt{d}} \right\}^d : \epsilon_1 > 0 \right\}
$$

we have that $N = 2^{d-1}$ and

$$
\max_{i \neq j} |\mathbf{w}_i^T \mathbf{w}_j| = 1 - \frac{2}{d}.
$$

Therefore

$$
N \left( d, 1 - \frac{2}{d} \right) \geq 2^{d-1}.
$$

Taking $\epsilon = 2^{-d}$, we have that, if $k \geq 8d^2$, then

$$
N \left( d, k, 2^{-d} \right) \geq 2^{d-1}.
$$

\textbf{Appendix F. ReLU variational space}

In this section, we are interested in understanding approximation by the spaces

$$
\mathcal{F}^{\text{ReLU}}_N = \left\{ f : x \in \mathbb{S}^{d-1} \mapsto \sum_{k=1}^N u_k(\mathbf{w}_k^T x)_+ : u \in \mathbb{R}^N, \mathbf{w}_k \in \mathbb{R}^d \right\}.
$$

We must notice that since

$$
(w^T x) = \frac{1}{2} |w^T x| + \frac{1}{2} w^T x
$$
it holds that $\mathcal{F}_N \subset H^d_1 \oplus \bigoplus_{k=1}^\infty H^d_{2k}$, which consists of the subset of $L^2$ formed by function which are sum of a linear component plus an even component. Since a linear function can always be represented exactly by functions in $\mathcal{F}_{ReLU}^2$, we can equivalently consider the problem of approximating even functions by the space

$$\mathcal{F}_{abs}^N = \left\{ f : x \in S^{d-1} \mapsto \sum_{k=1}^N u_k |w_k^T x| : u \in \mathbb{R}^N, w_k \in \mathbb{R}^d \right\}$$

The variational space associated to $\mathcal{F}_{abs}^N$ is

$$\mathcal{F}^1 = \left\{ f_{\pi}^{abs} : \pi \text{ is a even signed Radon measure} \right\}$$

where we define

$$f_{\pi}^{abs} : x \in S^{d-1} \mapsto \int_{S^{d-1}} |w^T x| d\pi(w).$$

Endowed with the norm

$$\gamma_1(f) = \min_{\pi : f = f_{\pi}^{abs}} ||\pi||_1$$

the space $\mathcal{F}^1$ is a Banach space; the norm $\gamma_1(f)$ controls the rate of approximation of a function $f \in \mathcal{F}^1$ by $\mathcal{F}_{abs}^N$, as shown by the following result (see (Bourgain et al., 1989) for a proof).

**Proposition 41** Let $f \in \mathcal{F}^1$. Then it holds that

$$\inf_{f_N \in \mathcal{F}_{abs}^N} ||f - f_N||_\infty \leq c \frac{\gamma_1(f)}{N^{1/3}}$$

where $c > 0$ is a numerical constant. Moreover, $f_N$ satisfying the bound can be chosen to satisfy $\gamma_1(f_N) \leq \gamma_1(f)$.

**Blaschke–Levy representations** Consider the transformation

$$T\varphi = \int_{S^{d-1}} |x^T y| \varphi(y) d\hat{S}(y)$$

for functions $\varphi \in C(S^{d-1})$. $T$ can be described in terms of spherical harmonics (Rubin, 1998) as

$$T\varphi = \sum_{k \geq 0 \text{ even}} \sigma_k \varphi_k$$

where

$$\sigma_k = \frac{(-1)^{1+k/2} \Gamma((k-1)/2)\Gamma(d/2)}{2\pi \Gamma((k+d+1)/2)}.$$

In particular, it holds that (Rubin, 1998) the functional $T$ is an automorphism of $C_{even}^\infty(S^{d-1})$. Clearly, its inverse can be defined in terms of spherical harmonics by

$$T^{-1} : \varphi \in C_{even}^\infty(S^{d-1}) \mapsto \sum_{k \geq 0 \text{ even}} \sigma_k^{-1} \varphi_k.$$

The following is immediate.

**Proposition 42** For any $\varphi \in C_{even}^\infty(S^{d-1})$ it holds that $\varphi \in \mathcal{F}_1$ and

$$\gamma_1(\varphi) = \|T^{-1}\varphi\|_1.$$
Using these results, we can proceed similarly to (Ongie et al., 2019) to prove Proposition 17.

**Proof** [Proof of Proposition 17] Assume first that \( f \in \mathcal{F}^1 \). Then \( f = f_\pi \) for some \( \pi \) even signed Radon measure. Thus

\[
\gamma_1(f) = \|\pi\|_1 = \sup_{\varphi \in C(\mathbb{S}^{d-1}) : \|\varphi\|_\infty \leq 1} \int_{\mathbb{S}^{d-1}} \varphi(w) d\pi(w)
\]

\[
= \sup_{\varphi \in C_{\text{even}}(\mathbb{S}^{d-1}) : \|\varphi\|_\infty \leq 1} \int_{\mathbb{S}^{d-1}} \varphi(w) d\pi(w)
\]

\[
= \sup_{\varphi \in C_{\text{even}}(\mathbb{S}^{d-1}) : \|\varphi\|_\infty \leq 1} \int_{\mathbb{S}^{d-1}} T(T^{-1}\varphi)(w) d\pi(w)
\]

\[
= \sup_{\varphi \in C_{\text{even}}(\mathbb{S}^{d-1}) : \|\varphi\|_\infty \leq 1} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |w^T x| (T^{-1}\varphi)(x) dS(x) d\pi(w)
\]

\[
= \sup_{\varphi \in C_{\text{even}}(\mathbb{S}^{d-1}) : \|\varphi\|_\infty \leq 1} \langle T^{-1}\varphi, f \rangle .
\]

This shows one side of the statement. On the other hand, assume that

\[
\sup_{\varphi \in C_{\text{even}}(\mathbb{S}^{d-1}) : \|\varphi\|_\infty \leq 1} \langle T^{-1}\varphi, f \rangle < \infty .
\] (52)

Then, the transformation

\[
S_f(\varphi) = \langle T^{-1}\varphi, f \rangle
\]

defines a bounded linear operator \( S_f : C_{\text{even}}(\mathbb{S}^{d-1}) \to \mathbb{R} \). Since \( C_{\text{even}}(\mathbb{S}^{d-1}) \) is dense in \( C_{\text{even}}(\mathbb{S}^{d-1}) \), \( S_f \) can be extended to a bounded linear operator on \( C_{\text{even}}(\mathbb{S}^{d-1}) \). By setting

\[
S_f(\varphi) = S_f(\varphi_{\text{even}})
\]

we can extend it on \( C(\mathbb{S}^{d-1}) \). By the Riesz representation theorem, there exists a signed Radon measure \( \pi \) on \( \mathbb{S}^{d-1} \) such that

\[
S_f(\varphi) = \int_{\mathbb{S}^{d-1}} \varphi(w) d\pi(w)
\]

for every \( \varphi \in C(\mathbb{S}^{d-1}) \). Moreover, since \( S_f(\varphi) = 0 \) for every odd \( \varphi \), we can assume that \( \pi \) is even. Let \( f_\pi \) be the function in \( \mathcal{F}_1 \) defined by \( \pi \). Then it holds that

\[
\langle T^{-1}\varphi, f \rangle = \|\pi\|_{TV} = \langle T^{-1}\varphi, f_\pi \rangle
\]

for every \( \varphi \in C_{\text{even}} \). Since \( T \) is an automorphism over \( C_{\text{even}}^\infty \), then it holds

\[
\langle \varphi, f \rangle = \langle \varphi, f_\pi \rangle
\]

for every \( \varphi \in C_{\text{even}}^\infty \). Since \( f \) and \( f_\pi \) are even, this implies that \( f = f_\pi \).

Functions that satisfy equation (52) include all even functions in \( C^{d+2}(\mathbb{S}^{d-1}) \) if \( d \) is even and all even functions in \( C^{d+3}(\mathbb{S}^{d-1}) \) if \( d \) is odd (Weil, 1976). This is in line with existing results that show approximability by neural networks for functions whose regularity is proportional to the dimension \( d \) (see e.g. (Petersen, 2020)). It follows by the proof above that if \( f \in C(\mathbb{S}^{d-1}) \) is even, then \( f \in \mathcal{F}_1 \) if

\[
S_N = \sum_{k=0}^{N} \sigma_{2k}^{-1} f_{2k}
\]
converges weakly (as $N \to \infty$) to a finite signed measure $\pi$. In this case $f = f^\text{abs}_\pi$. In particular, a stronger condition is convergence in $L^1_S$. This is implied if it holds that
\[
\sum_{k \geq 0 \text{ even}} |\sigma_k|^{-1} \|f_k\|_1 < \infty.
\]

Appendix G. Proofs of Section 5.2

G.1 Proof of Proposition 15

Let $f_N : \mathbb{R}^d \to \mathbb{R}$ a one-hidden-layer network defined by
\[
f_N(x) = \sum_{i=1}^{N} u_i f_{\sigma_i, w_i}^i(x) = \sum_{i=1}^{N} u_i \sigma_i(w_i^T x)
\]
where $u \in \mathbb{R}^N$, $w_i \in S^{d-1}$, and $\sigma_i$ are $\ell$-linearly bounded activations. Thanks to Parseval’s formula, we get that
\[
\|f_N - f(d)\|_2^2 \geq \|P_{I_d} f_N - P_{I_d} f(d)\|_2^2 \\
\geq \|P_{I_d} f(d)\|_2^2 - 2 \sum_{j \in I_d} \sum_{i=1}^{N} u_i \langle f_{\sigma_i, w_i}^j, f(d)_j \rangle \\
\geq \|P_{I_d} f(d)\|_2^2 - 2 \sum_{j \in I_d} \sum_{i=1}^{N} \frac{1}{\sqrt{N_j}} |u_i| \|f(d)_j\|_\infty \|f_{\sigma_i, w_i}^j\|_2 \\
\geq \|P_{I_d} f(d)\|_2^2 - 2 \sum_{i=1}^{N} |u_i| \|f_{\sigma_i, w_i}^j\|_2 \left( \sum_{j \in I_d} c_{d,j}^2 \right)^{1/2} \\
\geq \|P_{I_d} f(d)\|_2^2 - 2c \|f(d)\|_2 \sum_{i=1}^{N} |u_i| \|f_{\sigma_i, w_i}^j\|_2 \\
\]
Finally, notice that it holds that
\[
\|f_{\sigma_i, w_i}^j\|_2 \leq 2 m_\infty(f_N)
\]
and therefore
\[
\|f_N - f(d)\|_2^2 \geq \beta \gamma d^{-2M} - 4 \gamma d^M \|f(d)\|_\infty |u|_\infty \cdot N \\
\geq \frac{1}{d^{3M}} \left[ 1 - 4 \beta^{-1} \cdot d^\alpha \cdot \left( N m_\infty^2(f_N) \right) \right].
\]
This concludes the proof.

Two particular cases We discuss here two particular cases where the assumptions of Proposition 15 hold. Let $\{f(d)\}_{d \geq 2}$ be a sequence of functions $f(d) \in C(S^{d-1})$.

Case 1 Assume that assumption 1 in Proposition 15 holds with $I_d = \{ k \in \mathbb{N} : k \geq d^2 \}$ and that $\|f(d)\|_2 \geq c d^{-M}$ for some constants $c > 0$, $M > 0$. If it holds that
\[
\ell_\infty,2(f_k(d)) \leq \bar{\ell}
\]
for all \( k \geq d^2 \) for some constant \( \tilde{\ell} \geq 1 \), then it is easy to check that Proposition 15 holds. This condition could be thought as the spherical harmonic components of the function \( f^{(d)} \) being uniformly spread for high energy \( (k \geq d^2) \). Indeed assumption 2 holds with 

\[
c_{d,k} = \frac{\tilde{\ell}}{\sqrt{N^d_k}} \frac{\|f^{(d)}_k\|_2}{\|f^{(d)}\|_2}
\]

since 

\[
\sum_{k=d^2}^\infty c_{d,k}^2 \leq \frac{\tilde{\ell}^2}{N d^2} \leq \gamma d^{3-d}
\]

for some constant \( \gamma > 0 \). This is similar to the condition used in (Daniely, 2017) (see Appendix G.3).

**Case 2** Assume that assumption 1 in Proposition 15 holds with 

\[
I_d = \{ k \in \mathbb{N} : k \geq \rho d^\beta \}
\]

for some \( \rho > 0 \), \( \beta > 0 \) and that \( \|f^{(d)}\|_2 \geq cd^{1-\alpha} \) for some constants \( c > 0 \), \( \alpha > 0 \). If it holds that 

\[
\ell_{\infty,2}(f^{(d)}_k) \leq \gamma e^{kM} \sqrt{N^d_k}
\]

for all \( k \geq \rho d^\beta \) for some constant \( \gamma > 0 \), \( M > 0 \), then Proposition 15 holds, since 

\[
\sum_{k=\rho d^\beta}^\infty \epsilon^k = \frac{e^{\rho d^\beta}}{1 - \epsilon}.
\]

This condition could also be thought as the spherical harmonic components of the function \( f^{(d)} \) being uniformly spread for high energy \( (k \geq d^2) \), although in this case the spreadness is required to increase exponentially, as the degree increases, with respect to the maximum spreadness achievable (that is \((N^d_k)^{1/2}\)).

**G.2 Proof of Lemma 16**

Notice that since \( f \) is Rademacher-symmetric, so is \( f_k \). Consider the function 

\[
P : x \in S^{d-1} \mapsto 2^{-d} N^d_k \sum_{\epsilon \in \{\pm 1\}^d} P^d_k(\epsilon^T x).
\]

We claim that the function \( P \) satisfies \( \|P\|_2 \leq 2 \cdot 2^{-d/2} \sqrt{N^d_k} \). Let \( \epsilon \in \{\pm 1\}^d \). Then it holds 

\[
\|f_k\|_\infty = |f_k(\epsilon)| = |\langle f_k, P \rangle| \leq \|P\|_2 \|f_k\|_2 \leq 2 \cdot 2^{-d/2} \sqrt{N^d_k} \|f_k\|_2.
\]

This concludes the proof.

**Proof of the claim** Take \( k \geq 16d^2 \) even and let 

\[
\hat{P}(x) = \beta_d \sum_{i=1}^{2^{d-1}} (N^d_k)^{1/2} P^d_k(w_i^T x)
\]

with \( \beta_d = 2(2^d + 2)^{-1/2} \) and \( w_i \) as in 51. Then it holds that 

\[
\|\hat{P}\|^2_2 = \beta_d^2 \left[ 2^{d-1} + \sum_{i \neq j} P^d_k(w_i^T w_j) \right]
\]

\[
\leq \frac{2}{2^{d-1} + 1} \left[ 2^{d-1} + \left( 2^{2d-2} - 2^{d-1} \right) 2^{-d} \right]
\]

\[
= \frac{2}{2^{d-1} + 1} \left[ 2^{d-1} + 2^{d-2} - 2^{-1} \right] \leq 3
\]
and that
\[
\|\hat{P}\|_2^2 \geq \frac{2}{2^{d-1} + 1} \left[2^{d-1} - \left(2^{2d-2} - 2^{d-1}\right)2^{-d}\right]
\]
\[= \frac{2}{2^{d-1} + 1} \left[2^{d-1} - 2^{d-2} + 2^{-1}\right] \geq 1.
\]

On the other hand, it holds that
\[
\|\hat{P}\|_\infty \leq \beta_d (N_k^d)^{1/2} \sup_{x \in S^{d-1}} \sum_{i=1}^{2^{d-1}} \left|P_k^d(w_i^T x)\right|.
\]

By definition of the vectors \(\{w_i\}_{i=1}^{2^{d-1}}\), it holds
\[
\sup_{x \in S^{d-1}} \sum_{i=1}^{2^{d-1}} \left|P_k^d(w_i^T x)\right| = \frac{1}{2} \sup_{x \in S^{d-1}, x > 0} \sum_{\epsilon \in \{\pm 1/2\}^d} \left|P_k^d(x^T \epsilon)\right|
\]
\[\leq 1 + \frac{1}{2} \sup_{x \in S^{d-1}, x > 0} \sum_{\epsilon \in \{\pm 1/2\}^d, x^T \epsilon < \sqrt{d}} \left(\frac{1}{16d \left(1 - |x^T \epsilon|^2\right)}\right)^{(d-2)/2}
\]
\[\leq 1 + \frac{1}{2} (2^d - 2) \left(\frac{1}{16d \left(1 - \frac{1}{2d}\right)}\right)^{(d-2)/2} \leq 1 + \frac{2^{d-1} - 1}{4d^2} \leq 2.
\]

This proof the claim.

**G.3 (Daniely, 2017)’s proof**

(Daniely, 2017) shows a depth separation result using a results similar to Proposition 15. The difference in this case is that the author considers functions defined on \(S^{d-1} \times S^{d-1}\). Although, since \(L^2(S^{d-1} \times S^{d-1}) = L^2(S^{d-1}) \otimes L^2(S^{d-1})\), the space \(L^2(S^{d-1} \times S^{d-1})\) admits a decomposition in spherical harmonics
\[
L^2(S^{d-1} \times S^{d-1}) = \bigoplus_{j,k=0}^{\infty} H_j^d \otimes H_k^d.
\]

In particular, (Daniely, 2017) considers functions of the type
\[
f^{(d)} : (x, y) \in S^{d-1} \times S^{d-1} \mapsto h^{(d)}(x^T y)
\]
for some \(h^{(d)} \in C([-1, 1])\). Such functions belong to \(\bigoplus_{k=0}^{\infty} H_k^d \otimes H_k^d\) and satisfy
\[
\ell_{\infty, 2}(f^{(d)}) \leq \ell \cdot \left[N_k^d\right]^{1/2} = \ell \cdot \left[N_k^d\right]^{-1/2} \cdot \ell_{k, k}^d
\]
where \(\ell_{k, k}^d = \max_{f \in H_k^d \otimes H_k^d} \ell_{\infty, 2}(f)\). The equation above resembles condition 2 in Proposition 15, since it implies that
\[
\left\|f^{(d)}_{k, k}\right\|_{\infty} \leq \frac{\ell \cdot \left|f^{(d)}_{k, k}\right|_2}{\sqrt{N_k^d} \cdot \left|f^{(d)}\right|_2} \cdot \ell_{k, k}^d \cdot \left\|f^{(d)}\right\|_2
\]
and since
\[ c_d = \left[ \sum_{k \geq k_d} \left( \frac{\bar{\ell}}{\sqrt{N^d_k \|f^{(d)}\|_2}} \right)^2 \right]^{1/2} \leq \frac{\bar{\ell}}{\sqrt{N^d_{k_d}}} \]

which, for \( k_d \geq d^2 \) implies that \( c_d \lesssim d^{3 - d} \). The proof is then concluded by choosing \( I_d = \{(k, k) : k \geq k_d\} \), since (using the same notations as in Section G.1), it holds
\[ \|f_N - f^{(d)}\|_2^2 \geq \|P_{I_d} f^{(d)}\|_2^2 - 2 \sum_{(j,j) \in I_d} \sum_{i=1}^N (\ell^*_j)^{-1} |u_i| \|f_j^{(d)}\|_\infty \|f^{\sigma_i, w_i}_j\|_2 \]

which is an equivalent of formula (53).

**G.4 Proof of Proposition 18**

Some computations show that
\[ |\sigma_k|^{-1} \leq \Theta \left( d^{3/4} k^2 \sqrt{N^d_k} \right). \]

By Proposition 42 and the observations in Section F above we get that
\[ \gamma_1(f^{(d)}) \leq \sum_{k \geq 0 \text{ even}} |\sigma_k|^{-1} \|f_k^{(d)}\|_1 \leq \Theta \left( d^{3/4} \right) \sum_{k \geq 0 \text{ even}} k^2 s_{d,k} \|f_k^{(d)}\|_2 \leq \Theta (s_d d^{3/4}) \|f^{(d)}\|_2. \]

The application of Proposition 41 concludes the proof.