ON POINTS OF CONVERGENCE LATTICES AND SOBRIETY
FOR CONVERGENCE SPACES

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Abstract. We characterize the convergence spaces \((X, \xi)\) such that the space of points of \((\mathcal{P}X, \lim_\xi)\) in the category of convergence lattices is \((X, \xi)\). On the way, we study variants of sobriety and of the axiom \(T_D\) in convergence spaces. New phenomena appear when leaving the realm of topological spaces. We obtain new hindsight into the space of points of a convergence lattice and study a special quotient of it, which, in the case \(L = (\mathcal{P}X, \lim_\xi)\) for a topological space \((X, \xi)\), turns out to be homeomorphic to the sobrification of \(X\).

1. Preliminaries and introduction

1.1. Convergence Spaces. Let \(\mathcal{P}X\) denote the powerset of \(X\). If \(A \subset \mathcal{P}X\), 
\[A^\uparrow = \{B \subset X : \exists A \in A, A \subset B\}\]
and 
\[A^\# = \{B \subset X : \forall A \in A, A \cap B \neq \emptyset\}.\]
We also write \(A \# B\), and say that \(A\) and \(B\) mesh, if \(B \subset A^\#\), equivalently, \(A \subset B^\#\).

A convergence \(\xi\) on a set \(X\) is a relation between the set \(\mathcal{P}\mathcal{F}X\) of (set-theoretic) filters on \(X\) and the set \(X\), denoted \(x \in \lim_\xi F\) (or \(x \in \lim \xi F\) if there’s no risk of ambiguity) if \(F\) and \(x\) are \(\xi\)-related, subject to the following two axioms:

( monotone) \[F \subset G \Rightarrow \lim_\xi F \subset \lim_\xi G\]
(point axiom) \[x \in \lim_\xi \{x\}^\uparrow\]
for every \(x \in X\) and every \(F, G \in \mathcal{P}X\). A convergence is of finite depth if additionally

(finite depth) \[\lim_\xi (F \cap G) = \lim_\xi F \cap \lim_\xi G\]
for every \(F, G \in \mathcal{P}X\).

Continuity of a map \(f : (X, \xi) \to (Y, \tau)\), in symbols \(f \in C(\xi, \tau)\), is simply preservation of limits, that is,

( continuity) \(f(\lim_\xi F) \subset \lim_\tau f[F]\),
where \(f[F] = \{B \subset Y : f^{-1}(B) \in F\} \in \mathcal{P}Y\) is the image filter of \(F\) under \(f\).

Let \(\text{Conv}\) denote the category of convergence spaces and continuous maps. We denote by \(|\cdot| : \text{Conv} \to \text{Set}\) the forgetful functor, so that \(|\xi|\) denotes the underlying set of a convergence \(\xi\) and \(|f|\) is the underlying function of a morphism. If \(|\xi| = |\tau|\),

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we say that \( \xi \) is finer than \( \tau \) or that \( \tau \) is coarser than \( \xi \), in symbols, \( \xi \geq \tau \), if the identity map of their underlying set belongs to \( C(\xi, \tau) \). This order turns the set of convergences on a given set into a complete lattice whose greatest element is the discrete topology, least element is the antidiscrete topology, and whose suprema and infima are given by

\[
\lim_{\lor_{\ell \in \Xi} \xi} F = \bigcap_{\xi \in \Xi} \lim \xi F \quad \text{and} \quad \lim_{\land_{\ell \in \Xi} \xi} F = \bigcup_{\xi \in \Xi} \lim \xi F.
\]

**Conv** is a concrete topological category; in particular, for every \( f : X \to |\tau| \), there is the coarsest convergence on \( X \), called initial convergence for \( (f, \tau) \) and denoted \( f^{-} \tau \), making \( f \) continuous (to \( \tau \)), and for every \( f : |\xi| \to Y \), there is the finest convergence on \( Y \), called final convergence for \( (f, \xi) \) and denoted \( f\xi \), making \( f \) continuous (from \( \xi \)). Note that with these notations

\[
(1.2) \quad f \in C(\xi, \tau) \iff \xi \geq f^{-} \tau \iff f\xi \geq \tau.
\]

Moreover, the initial lift on \( X \) of a structured source \((f_i : X \to |\tau_i|)_{i \in I}\) turns out to be \( \lor_{i \in I} f_i^{-} \tau_i \) and the final lift on \( Y \) of a structured sink \((f_i : |\xi_i| \to Y)_{i \in I}\) turns out to be \( \land_{i \in I} f_i\xi_i \).

If \( f : |\xi| \to Y \) is surjective, \( f\xi \) is also called the quotient convergence. If \( A \subset |\xi| \), the subspace convergence \( i_A\xi \) or induced convergence on \( A \) is \( i_A^\ast \xi \) where \( i_A : A \to |\xi| \) is the inclusion map. If \( \Xi \) is a set of convergences, then the product convergence is

\[
(1.3) \quad \Pi_{\xi \in \Xi} \xi = \bigvee_{\xi \in \Xi} p_\xi \xi,
\]

where \( p_\xi : \Pi_{\xi \in |\xi|} \to |\xi| \) is the projection, that is, \( \Pi_{\xi \in \Xi} \xi \) is the initial convergence for the family of projections.

Convergences of finite depth form a concretely reflective subcategory of **Conv** with reflector \( L \) defined (on objects) by \( x \in \lim_{\xi} F \) if there is a finite set \( \mathbb{D} \) filters on \( X \) each converging to \( x \) for \( \xi \) such that \( F \supseteq \bigcap_{D \in \mathbb{D}} D \).

Of course, every topology \( \tau \) can be seen as a convergence given by \( x \in \lim_{\xi} F \) if and only if \( F \supseteq N_r(x) \), where \( N_r(x) \) denotes the neighborhood filter of \( x \) in the topology \( \tau \). This turns the category **Top** of topological spaces and continuous maps into a full (concretely reflective) subcategory of (the topological extensional and cartesian-closed category) **Conv**. If \( (X, \xi) \) is a convergence space, the topological modification \( T \xi \) of \( \xi \) is the topology on \( X \) whose set of closed sets are the \( \xi \)-closed subsets of \( X \), that is, the sets \( C \subset X \) with

\[
(1.4) \quad C \in F \implies \lim_{\xi} F \subset C.
\]

The \( \xi \)-open subsets are complements of \( \xi \)-closed sets, that is, subsets \( O \) satisfying

\[
\lim_{\xi} F \cap O \neq \emptyset \implies O \in F.
\]

\( T \xi \) is the finest among topologies coarser than \( \xi \). Note that if \( O_\xi \) denotes the set of \( \xi \)-open sets and \( O_\xi(x) = O_\xi \cap \{x\}^\uparrow \), then \( N_r(x) = (O_\xi(x))^\uparrow \).

A subspace \((A, \xi_A)\) of a convergence space \((X, \xi)\) is dense if for every \( x \in X \) there is \( F \in FX \) with \( A \in F \) and \( x \in \lim F \).

### 1.2. Convergence Lattices

A pointfree generalization was offered in \[5\] in which the function

\[
\lim_{\xi} : \mathcal{F} P X \to P X
\]
is abstracted away to a monotone function
\[ \lim : \mathbb{F}L \to L \]
from (order-theoretic) filters on a lattice \( L \) to \( L \). [point axiom] is not part of the axiomatic in this pointfree version of convergence spaces, though the notion can also be recovered (as so-called centered convergence lattices) in an abstract order-theoretic form.

**Definition 1.** Given a category \( C \) of lattices, a convergence \( C \)-object \((L, \lim)\) is a \( C \)-object \( L \) together with a monotone map \( \lim : \mathbb{F}L \to L \). The objects of the category \( C^{\text{conv}} \) are the convergence \( C \)-objects and the morphisms \( \varphi : L \to L' \) are the \( C \)-morphisms that are continuous in the sense that for every \( F \in \mathbb{F}L' \),

\[ (\text{ptfree continuity}) \quad \lim_{\mathbb{F}L} F \leq \varphi(\lim_{\mathbb{F}L} \varphi^{-1}(F)), \]

where \( \varphi^{-1}(F) = \{ \ell \in L : \varphi(\ell) \in F \} \).

The category \( \text{Conv} \) embeds coreflectively into \( (C^{\text{conv}})^{\text{op}} \) when \( C \) is the category of frames or of coframes: the powerset functor \( \mathbb{P} : \text{Conv} \to (C^{\text{conv}})^{\text{op}} \) sending \( (X, \xi) \) to \( (\mathbb{P}X, \lim_\xi) \) and \( f : (X, \xi) \to (Y, \tau) \) to \( \mathbb{P}f = f^{-1} : \mathbb{P}Y \to \mathbb{P}X \) (in \( C^{\text{conv}} \)) is then right-adjoint to the point-functor \( pt : (C^{\text{conv}})^{\text{op}} \to \text{Conv} \) (the coreflector) where the underlying set of \( pt L \), the set of “points” of \( L \), is the set of \( C^{\text{conv}} \)-morphisms from \( L \) to \( \mathbb{P}(1) \) (which is terminal in \( (C^{\text{conv}})^{\text{op}} \) [5, Lemma 2.7]), hence depends on the choice of \( C \), so we’ll write \( pt_C \) \( L \) when different options for the base category \( C \) of lattices are considered.

For a given \( C \), the convergence structure on \( pt L \) is given by

\[ (\text{Conv on pt}) \quad \lim_{pt L} F = (\lim_{\mathbb{F}L} F^\circ)^*, \]

where \( \ell^* = \{ \varphi \in pt L : \varphi(\ell) = \{1\} \} \) and \( F^\circ = \{ \ell \in L : \ell^* \in F \} \). Finally, if \( \varphi \in C^{\text{conv}}(L, L') \) then pt \( \varphi : pt L' \to pt L \) is defined by \( pt(\varphi)(f) = f \circ \varphi \).

As pointed out in [5, Remark 2.9], a continuous \( C \)-morphism \( \varphi : L \to \mathbb{P}(1) \) can be identified with the filter \( F = \varphi^{-1}(\{1\}) \) which satisfies \( \lim_{\mathbb{F}L} F \in F \) by continuity. Hence the set \( pt_C L \) of points of a \( C^{\text{conv}} \)-object \( L \) can be identified with sets of specific filters as follows [1]:

| cat. of \( C \) | \( pt_C L = \{ F \in \mathbb{F}L : \lim_{\mathbb{F}L} F \in F \text{ and...} \} \) | ref. |
|----------------|-------------------------------------------------|-----|
| lattices       | \( F \) is prime                                | [5, Remark 2.10] |
| frames          | \( F \) is completely prime                     | [5, Remark 2.11] |
| coframes        | \( F \) is principal of a \( \lor \)-prime element | [5, Remark 2.12] |

**Lemma 2.** [5, Lemma 2.19(6)] Let \( L \) be a convergence lattice and let \( U \in pt L \). Then \( (\{U\})^\circ = U \).

**Proof.** \( \ell \in (\{U\})^\circ \) if and only if \( \ell^* \in \{U\}^\downarrow \), that is, if and only if \( U \in \ell^* \), equivalently, \( \ell \in U \). \(\square\)

\(^1\)Recall that a proper filter \( F \) on a lattice \( L \) is prime if \( \ell_1 \lor \ell_2 \in F \) implies that \( \ell_1 \lor \ell_2 \) belongs to \( F \) and completely prime if for every \( A \subseteq L \), \( \lor_{\ell \in A} \ell \in F \) implies that \( \ell \in F \) for some \( \ell \in A \). An element \( \ell \in L \) is join-prime if \( \ell \leq \ell_1 \lor \ell_2 \) implies \( \ell \leq \ell_1 \) or \( \ell \leq \ell_2 \).
It turns out that filters of the form \( \{x\}^\uparrow \) are always points of \( \mathcal{P}(X) \) in \( \mathbf{C}^{\text{conv}} \) provided that \( \mathbf{C} \) be an admissible category \(^2\) of lattices \(^3\) Remark 2.14. As a result \(^4\):

**Proposition 3.** A convergence space \((X, \xi)\) is always (homeomorphic to) a dense subspace of \( \text{pt}(\mathcal{P}(X), \text{lim}_\xi) \).

**Proof.** The map \( h : (X, \xi) \to \text{pt}(\mathcal{P}(X), \text{lim}_\xi) \) defined by \( h(x) = \{x\}^\uparrow \) is an embedding, i.e., an homeomorphism onto its image. The map \( h \) is clearly one-to-one. To see that it is an embedding, let \( L = \text{pt}(\mathcal{P}(X), \text{lim}_\xi) \) and recall that \( \lim_{\text{pt}_L} h(\mathcal{F}) = (\lim_{\xi} h(\mathcal{F}))^\circ \) where \( h(\mathcal{F})^\circ = \{A \in \mathcal{P}(X) : A^\bullet \in h(\mathcal{F})\} \) and \( A^\bullet = \{G \in \text{pt} L : A \in G\} \). Noting that \( A^\bullet \in h(\mathcal{F}) \) if and only if \( A \in \mathcal{F} \), we see that \( h(x) \in \lim_{\text{pt}_L} h(\mathcal{F}) \) if and only if \( x \in \lim_{\xi} \mathcal{F} \).

Suppose \( G \in \text{pt} L \). Then \( \lim_{\xi} G \in G \) (when \( G \) is seen as an element of \( \mathcal{P}(X) \)), that is, \( h(\mathcal{G}) \subseteq h(\mathcal{G})^\circ \). Moreover \( G \subseteq h(\mathcal{G})^\circ \) so that \( \lim_{\xi} G \subseteq \lim_{\xi} h(\mathcal{G})^\circ \) and thus \( G \subseteq (\lim_{\xi} G)^\circ \subseteq (\lim_{\xi} h(\mathcal{G}))^\circ \), that is, \( G \subseteq \text{pt}_{\text{lim}} h(\mathcal{G}) \). Hence \( h(X) \) is a dense subspace of \( \text{pt} L \). To see that \( G \subseteq h(\mathcal{G})^\circ \), note that if \( G \in \mathcal{G} \) then

\[
h(G) = \{ \{x\}^\uparrow : x \in G \} \subseteq G^\bullet
\]

so that \( G^\bullet \in h(G) \), that is, \( G \subseteq h(G)^\circ \).

As filters of the form \( \{x\}^\uparrow \) are exactly completely prime filters in \( \mathcal{P}(X) \) they are the only points in \( \mathbf{C}^{\text{conv}} \) if \( \mathbf{C} \) is the category of frames \(^5\) Remark 2.16]. Similarly, as join-prime elements of \( \mathcal{P}(X) \) are singletons, points of \( \mathcal{P}(X) \) in \( \mathbf{C}^{\text{conv}} \) are exactly singletons if \( \mathbf{C} \) is the category of coframes \(^5\) Remark 2.17]. Therefore, in both cases, \( \text{pt}(\mathcal{P}(X), \text{lim}_\xi) \) is homeomorphic to \((X, \xi)\).

On the other hand, prime filters of \( \mathcal{P}(X) \) are the ultrafilters on \( X \). Hence, when \( \mathbf{C} \) is the category \( \mathbf{Lat} \) of lattices and \((X, \xi)\) is a convergence space, the set of points of \((\mathcal{P}(X), \text{lim}_\xi)\) in \( \mathbf{C}^{\text{conv}} \) is

\[
\text{pt}_{\text{lim}} \mathcal{P}(X) = \{ \mathcal{U} \subseteq \mathcal{U} X : \text{lim}_\xi \mathcal{U} \in \mathcal{U}\},
\]

where \( \mathcal{U} X \) denotes the set of ultrfilters on \( X \) \(^5\) Remark 2.15]. Though it contains the “usual points” identified with their principal ultrafilter \( \{x\}^\uparrow \), there may in general be others (See Lemma \(^6\) below). One purpose of this note is to clarify when this does not happen, that is:

**Problem 4.** Under what condition on a convergence space \((X, \xi)\) are all ultrafilters \( \mathcal{U} \subseteq \mathcal{U} X \) with \( \text{lim}_\xi \mathcal{U} \in \mathcal{U} \) principal?

First we should point out that the question at hand was fully solved \(^6\) Theorem 2.2 with Addendum I] for the case where \((X, \xi)\) is topological. Though our motivations leading to Problem \(^4\) are distinct from those of \(^7\) and come from \(^5\), we are thus essentially exploring a generalization of \(^7\) Theorem 2.2] from topological spaces to convergence spaces. Such an extension may seem of little relevance, but in light of the question of finding out when \( \text{pt}_{\text{lim}} \mathcal{P}(X) \) are just the “usual points of \( X \)” in the context of convergence lattices, this is a most natural problem and

\(^2\)A category \( \mathbf{C} \) of lattices is admissible if \( \mathcal{P}(X) \) is a \( \mathbf{C} \)-object for every set \( X \) and there are two classes \( \mathcal{I} \) and \( \mathcal{J} \) of index sets such that for all pairs \( L, L' \) of \( \mathbf{C} \)-objects, the \( \mathbf{C} \)-morphisms from \( L \) to \( L' \) are exactly monotonic maps that preserve all \( \mathcal{I} \)-indexed infima that exist in \( L \) for \( I \in \mathcal{I} \) and all \( \mathcal{J} \)-indexed suprema that exist in \( L \) for \( J \in \mathcal{J} \). See \(^5\) Def. 2.5]

\(^3\)Save density, this was already observed in \(^5\) Lemma 2.26]
restricting ourselves to topological spaces is then highly undesirable. Corollary 47 gives a full answer to Problem 4 for a large class of convergence spaces.

It turns out that some interesting subtleties appear in this wider context of convergence spaces. On the way, we study variants of sobriety for convergence spaces and take the opportunity to correct a related misstatement in [5, Remark 2.15].

Though Problem 4 stems from a question on $pt\, L$ in the very particular case where $L = (\mathcal{P} X, \lim_\xi)$ for a convergence space $(X, \xi)$, we study more in depth the space $pt\, L$ for a general convergence lattice $L$. A particular natural quotient $pt'\, L$ turns out to be of importance. In particular, when $L = (\mathcal{P} X, \lim_\xi)$ for a topological space $(X, \xi)$, the space $pt'\, L$ is homeomorphic to the sobrification of $(X, \xi)$.

2. Irreducible filters, sobriety, and convergence variants

A filter $F$ on a convergence space $(X, \xi)$ is irreducible in the sense of [7] if $\lim_\xi F \in F$.

Hence, elements of $pt_\mathcal{C} \mathcal{P}(X)$ are specific types of irreducible filters. As we have seen, when $\mathcal{C}$ is either the category of frames or that of coframes, points turn out to be exactly principal ultrafilters without the irreducibility condition even playing a role. This is not the case for $pt_\mathcal{L} \mathcal{P}(X)$ whose elements are exactly irreducible ultrafilters. Note that if $F$ is irreducible and $G \supseteq F$ then $G$ is irreducible too and if $f : (X, \xi) \rightarrow (Y, \tau)$ is continuous and $F$ is irreducible on $X$ then $f[F]$ is irreducible on $Y$.

The notion is related to that of a compact filter in the sense of [2, 3, 11, 10]. A filter $F$ on a convergence space $(X, \xi)$ is compact (resp. compactoid) if

$$H \# F \Rightarrow \text{adh} H \# F \quad (\text{resp. adh} H \neq \emptyset),$$

for every $H \in \mathcal{F}X$.

**Lemma 5.** Let $(X, \xi)$ be a convergence space.

1. If $F$ is irreducible then $F$ is compact;
2. A compact ultrafilter is irreducible.

**Proof.** (1). If $F$ is irreducible and $H \# F$ then $\lim F \subseteq \text{adh} H$ so that $\text{adh} H \in F \subset F^\#$.

(2). If $U \in \mathcal{U}X$ is compact, then taking $H = U$ in (2.1) we obtain $\lim U \subseteq U^\# = U$ and thus $U$ is irreducible. \hfill $\square$

Hence an ultrafilter is irreducible if and only if it is compact. However, there are compact filters that are not irreducible. For instance, a neighborhood filter of the real line with its usual topology is compact but not irreducible.

Recall that a topological space $X$ is called Noetherian if every decreasing sequence of closed subsets is eventually constant. It is well-known that $X$ is Noetherian if and only if all of its subsets are compact. In [5] Remark 2.15, it is erroneously stated that a topological space is Noetherian if and only if every compact ultrafilter is principal. On the contrary:

**Lemma 6.** A topological space is Noetherian if and only if every filter, equivalently every ultrafilter, is compact. In particular, an infinite Noetherian topological space admits irreducible free ultrafilters.
Proof. If \( X \) is Noetherian, \( F \in \mathcal{F}X \) and \( H \) is a filter \( H \# F \) then \( \text{adh} \, H \cap F \neq \emptyset \) for every \( F \in \mathcal{F} \) because \( F \) is compact. Conversely, if every ultrafilter is compact and \( A \subset X \) then \( \lim \mathcal{U} \cap A \neq \emptyset \) whenever \( A \in \mathcal{U} \in \mathcal{U}X \) because \( \text{adh} \, \mathcal{U} = \lim \mathcal{U} \in \mathcal{U}^\# \).

Recall that a closed subset \( C \) of a topological space is irreducible if whenever \( C \subset D \cup F \) where \( D \) and \( F \) are closed, then either \( C \subset D \) or \( C \subset F \). A topological space \( X \) is sober if for every irreducible closed subset \( C \) of \( X \), there is a (necessarily unique) point \( x \in X \) with \( C = \text{cl}\{x\} \), which we call a generic point for \( C \). Note that we can equivalently formulate irreducibility of \( C \) in the following alternative terms: If \( O_1, O_2 \) are open sets intersecting with \( C \) then \( O_1 \cap O_2 \) also intersects with \( C \).

It was shown in [7] that a topological space \( X \) is sober if and only if for every irreducible filter \( F \) on \( X \) there is a unique point \( x \in X \) with \( \lim F = \lim \{x\} \). Following [7], we can take this as a definition of a sober convergence space and call weakly sober a convergence space in which for every irreducible ultrafilter \( U \) there is a unique \( x \) with \( \lim \mathcal{U} = \lim \{x\} \). The two notions (sober and weakly sober) coincide for topological spaces, but not for convergence spaces. Though [7] Remark 1.9 mentions this, it provides neither a proof nor an example and does not explore the conditions on a convergence space for weak sobriety and sobriety to coincide. We fill this gap here (Example 14, Proposition 8).

In this context, it is useful to point out that the condition

\[(S_0) \quad (x \in \lim \mathcal{F} \text{ and } t \in \lim \{x\}) \implies t \in \lim \mathcal{F}\]

is automatically true in a topological space because in this case \( t \in \lim \{x\} \) implies \( O(t) \subset O(x) \). However condition \((S_0)\) is not generally true in an arbitrary convergence space. This (vacuous for topological spaces) weak diagonal axiom is called \( S_0 \) in [14].

Recall (e.g., [4]) that a convergence space is \( T_0 \) if points can be distinguished by the convergence structure, that is, if

\[x \neq t \implies \{ \mathcal{F} \in \mathcal{F}X : x \in \lim \mathcal{F} \} \neq \{ \mathcal{F} \in \mathcal{F}X : t \in \lim \mathcal{F} \},\]

and \( T_1 \) if singletons are closed. Of course, a topological space is \( T_0 \) (resp. \( T_1 \)) in the usual sense if and only if it is in the convergence sense.

Note that if a convergence has closed limits (that is, \( \lim \mathcal{F} \) is closed for every filter) then it is \( S_0 \), but not conversely. Of course, in a topological space limits are closed. It might be useful to have Figure 2 below in mind when considering the next few results:
Lemma 7. (1) If limit sets of principal ultrafilters are closed (in particular if the convergence is S₀), then
\[ \{x, y\} \subseteq \lim\{x\} \cap \lim\{y\} \iff \lim\{x\} = \lim\{y\}. \]

(2) If X is T₀ and S₀ then
\[ \{x, y\} \subseteq \lim\{x\} \cap \lim\{y\} \implies x = y. \]

Proof. It is clear that if \( \lim\{x\} = \lim\{y\} \) then in particular \( \{x, y\} \subseteq \lim\{x\} \cap \lim\{y\} \). Conversely, if \( \lim\{x\} \) is closed and belongs to \( \{y\} \), then \( \lim\{y\} \subseteq \lim\{x\} \) and similarly for the reverse inclusion if \( \lim\{y\} \) belongs to \( \{x\} \).

For the second part if \( x \neq y \), there is a filter \( F \) with \( \text{card}(\lim F \cap \{x, y\}) = 1 \) because the convergence is T₀. Say \( x \in \lim F \). Then \( y \notin \lim\{x\} \) for otherwise \( y \in \lim F \) by S₀, and similarly \( x \notin \lim\{y\} \) if \( x \in \lim F \), so that \( \{x, y\} \not\subseteq \lim\{x\} \cap \lim\{y\} \). \( \square \)

In view of Lemma 7 in a T₀ topological space a generic point of an irreducible closed set is necessarily unique. However,

(2.3) \[ \lim\{x\} = \lim\{y\} \implies x = y \]

requires both S₀ and T₀ for general convergence spaces as shown by Examples 9 and 10 below.

Proposition 8. A weakly sober convergence space in which limits of irreducible filters are closed (in particular, a topological space) is sober.

Proof. If \( X \) is a weakly sober convergence space and \( F \) is an irreducible filter on \( X \), then an ultrafilter \( U \) finer than \( F \) is also irreducible, so that \( \lim U = \lim\{x_U\} \) for a unique point \( x_U \in X \). Moreover, \( \lim F \subseteq \lim U \) because \( U \geq F \) and as \( \lim F \in F \subseteq U \) and \( \lim F \) is closed, \( \lim U \subseteq \lim F \). Hence \( \lim U = \lim F \) for every
\( \mathcal{U} \in \mathcal{U}(\mathcal{F}) \). Hence all points \( x_{\mathcal{U}} \) coincide and this point is the unique generic point for \( \mathcal{F} \).

It turns out that all new phenomena in the realm of general convergences (as opposed to topological spaces) can be observed with finite examples. Filters on a finite sets are all principal and a finitely deep convergence on a finite set is entirely determined by its restriction to principal ultrafilters. Hence such convergences can easily be described by diagrams in which \( x \to y \) means that \( y \in \lim\{x\}^\uparrow \) and we include such pictures in each relevant example, using this convention and systematically omitting the loops (i.e., we do not depict the automatic convergence relation \( x \in \lim\{x\}^\uparrow \)).

**Example 9** (A \( T_0 \) convergence with \( x \neq y \) and \( \lim\{x\}^\uparrow = \lim\{y\}^\uparrow \)). Let \( X = \{x, y, z\} \) with the convergence of finite depth given by \( \lim\{x\}^\uparrow = \lim\{y\}^\uparrow = \lim\{x, y\}^\uparrow = \{x, y\} \) and \( \lim\{z\}^\uparrow = \{z, x\} \). By finite depth, \( \lim\{x, y, z\}^\uparrow = \lim\{y, z\}^\uparrow = \{y, z\}^\uparrow = \{x\} \).

**Example 10** (A (non-\( T_0 \)) \( S_0 \) convergence space with \( x \neq y \) and \( \lim\{x\}^\uparrow = \lim\{y\}^\uparrow \)). Let \( X = \{x, y, z\} \) the convergence of finite depth given by \( \lim\{x\}^\uparrow = \lim\{y\}^\uparrow = \lim\{x, y\}^\uparrow = \{x, y\} \) and \( \lim\{z\}^\uparrow = \{z\} \). By finite depth, \( \lim\{x, y, z\}^\uparrow = \lim\{y, z\}^\uparrow = \lim\{x, z\}^\uparrow = \{x\} \).

Note that weakly sober implies (2.3): if there are points \( x \neq y \) with \( \lim\{x\}^\uparrow = \lim\{y\}^\uparrow \) then \( \{x\}^\uparrow \) is an irreducible ultrafilter with two different generic points, so that the space is not weakly sober. Hence the easy fact that finite \( T_0 \) topological spaces are sober fails to extend to convergence spaces, because of the uniqueness requirement on generic points, though this requirement is vacuous in topological spaces. Therefore, it makes sense to also consider the variants without this requirement:

A convergence space \( X \) is quasi-sober (resp. weakly quasi-sober) if for every irreducible filter \( \mathcal{F} \) (respectively, for every irreducible ultrafilter \( \mathcal{F} \)) on \( X \) there is a (not necessarily unique) point \( x \) of \( X \) with \( \lim\mathcal{F} = \lim\{x\}^\uparrow \). Example 9 is \( T_0 \) and quasi-sober but not weakly sober. Example 10 is \( S_0 \) and quasi-sober but not weakly sober. A quasi-sober space does not need to be \( T_0 \) for the antidiscrete topology is quasi-sober. Of course, if a convergence space satisfies (2.3) and is quasi weakly sober, it is weakly sober. Lemma 7 states that this is the case in a \( T_0 \) space with closed limits of principal ultrafilters. More generally, let us call a convergence space antisymmetric if (2.2) holds and almost antisymmetric, or aas for short, if (2.3) holds. Note that:

**Proposition 11.** (1) A convergence space is weakly sober if and only if it is weakly quasi-sober and aas.

(2) An antisymmetric space is \( T_0 \).
(3) In a convergence in which limits of principal ultrafilters are closed, the following are equivalent:
(a) weakly sober;
(b) weakly quasi-sober and antisymmetric;
(c) weakly quasi-sober and aas.

In particular a weakly sober space in which limits of principal ultrafilters are closed is $T_0$.

**Proof.** (1) If the space is weakly sober and $\lim\{x\}^\uparrow = \lim\{y\}^\uparrow$ then $x = y$ by uniqueness of generic points for limits of irreducible ultrafilters, hence the space is aas. Conversely, aas ensure uniqueness of generic points in a weakly quasi-sober space.

(2) Suppose $\{F \in FX : x \in \text{lim } F\} = \{F \in FX : y \in \text{lim } F\}$. In particular $\{x, y\} \subset \lim\{x\}^\uparrow \cap \lim\{y\}^\uparrow$ so that $x = y$ by aas.

For (3), to see $(a) \implies (b)$, suppose $\{x, y\} \subset \lim\{x\}^\uparrow \cap \lim\{y\}^\uparrow$. Then $\lim\{x\}^\uparrow \subset \lim\{y\}^\uparrow$ because $\lim\{y\}^\uparrow \in \{x\}^\uparrow$ and $\lim\{y\}^\uparrow$ is closed, and similarly for the reverse inclusion. $(b) \implies (c)$ is clear and $(a) \iff (c)$ is (1). \qed

A topological sober space is always $T_0$ but things are slightly more complicated for the general case:

**Example 12** (A sober convergence that is not $T_0$, whose modification of finite depth is not sober). Let $X = \{x, y, s, t\}$ with the convergence given by $\lim\{x\}^\uparrow = \{x, y, s\}$, $\lim\{y\}^\uparrow = \{x, y, t\}$, $\lim\{s\}^\uparrow = \{s\}$ and $\lim\{t\}^\uparrow = \{t\}$, and all other filters do not converge (note that this convergence is not of finite depth).

\[
\begin{array}{ccc}
  x & \longrightarrow & s \\
  \downarrow & & \\
  y & \longrightarrow & t
\end{array}
\]

Then $\lim^{-1}(x) = \lim^{-1}(y) = \{\{x\}^\uparrow, \{y\}^\uparrow\}$ but $\lim\{x\}^\uparrow \neq \lim\{y\}^\uparrow$ and the condition of sobriety is satisfied. Note that the modification of finite depth $L \xi$ would add the irreducible filter $\{x, y\}^\uparrow$ because the $\lim_{L \xi}(x, y)^\uparrow = \lim_{L \xi}\{x\}^\uparrow \cap \lim_{L \xi}\{y\}^\uparrow = \{x, y\}$ but this limit has no generic point, hence sobriety would fail.

In other words, the modification of finite depth of a sober convergence may fail to be sober. On the other hand, the example above is easily modified to make it of a finite depth:

**Example 13** (A sober convergence of finite depth that is not $T_0$). Let $X = \{x, y, s, t\}$ with the convergence of finite depth given by $\lim\{x\}^\uparrow = \{x, y, z, s\}$, $\lim\{y\}^\uparrow = \{x, y, z, t\}$, $\lim\{z\}^\uparrow = \{x, y, z\}$, $\lim\{s\}^\uparrow = \{s\}$ and $\lim\{t\}^\uparrow = \{t\}$.

\[
\begin{array}{ccc}
  x & \longrightarrow & s \\
  \downarrow & & \\
  z & \longrightarrow & y \longrightarrow t
\end{array}
\]

This convergence is not $T_0$ as the same filters converge to $x, y$ and $z$. By finite depth,

\[
\lim\{x, y\}^\uparrow = \lim\{x, z\}^\uparrow = \lim\{y, z\}^\uparrow = \lim\{x, y, z\}^\uparrow = \{x, y, z\},
\]
so that these four filters are irreducible and all have the same unique generic point $z$.

**Example 14** (A $T_0$ weakly sober convergence of finite depth that is not sober). On $X = \{x, y, t, s, w, z\}$ consider the convergence of finite depth given by $\lim\{x\}^\uparrow = \{x, y, s, t\}$, $\lim\{y\}^\uparrow = \{x, y, t, w\}$, $\lim\{t\}^\uparrow = \{t\}$, $\lim\{s\}^\uparrow = \{s\}$, $\lim\{w\}^\uparrow = \{z\}$, and $\lim\{z\}^\uparrow = \{z, x\}$.

\[
\begin{array}{cccc}
& & z & \\
& x & \nearrow & t \\
y & \nearrow & w
\end{array}
\]

This is a $T_0$ weakly sober convergence of finite depth that is not sober, because $\{x, y\}^\uparrow$ is irreducible but does not have a generic point.

Note that if $\xi \geq \tau$ and $\tau$ is (weakly) sober then $\xi$ is (weakly) sober. In particular, if $T \xi$ is (weakly) sober, so is $\xi$ but the converse is false:

**Example 15** (A $T_0$ sober convergence of finite depth with a non-sober topological modification). Take on $X = \{x, y, z\}$ the finitely deep convergence given by $\lim\{x\}^\uparrow = \{x, y\}$, $\lim\{y\}^\uparrow = \{x, y, z\}$ and $\lim\{z\}^\uparrow = \{y, z\}$. Then by finite depth, $\lim\{x, y\}^\uparrow = \{x, y\}$, $\lim\{x, z\}^\uparrow = \lim\{x, y, z\}^\uparrow = \{y\}$, and $\lim\{y, z\}^\uparrow = \{y, z\}$.

\[
\begin{array}{cccc}
& & x & \\
& z & \nearrow & y \\
& & & \nearrow
\end{array}
\]

The irreducible filters are $\{x\}^\uparrow$, $\{y\}^\uparrow$, $\{z\}^\uparrow$, $\{x, y\}^\uparrow$, and $\{y, z\}^\uparrow$, $x$ is the only generic point for $\lim\{x\}^\uparrow$ and for $\lim\{x, y\}^\uparrow$, $y$ is the only generic point for $\lim\{y\}^\uparrow$, $z$ is the only generic point for $\lim\{z\}^\uparrow$ and for $\lim\{y, z\}^\uparrow$. Hence this convergence is sober, but its topological modification is antidiscrete, hence is not even $T_0$.

As observed in [7], an arbitrary product of (weakly) sober convergence spaces is (weakly) sober and a subspace of a (weakly) sober convergence does not need to be (weakly) sober, though a closed subspace does.

### 3. More on the convergence space $\text{pt} \ L$

**Proposition 16.** For every convergence lattice $L$ the convergence space $\text{pt} \ L$ is weakly quasi-sober.

**Proof.** Suppose $\lim_{\text{pt} \ L} \mathcal{U} \subseteq \mathcal{U}$ for some $\mathcal{U} \subseteq \text{U}(\text{pt} \ L)$. Then $\mathcal{U}^\circ \subseteq \text{pt} \ L$ and moreover $\lim_{\text{pt} \ L} \{\mathcal{U}^\circ\}^\uparrow = \lim_{\text{pt} \ L} \mathcal{U}$. Indeed, $\mathcal{U}^\circ$ is prime if for $\ell \vee m \in \mathcal{U}^\circ$ then, since elements of pt $L$ are prime filters, $(\ell \vee m)^* = \ell^* \cup m^* \in \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter so either $\ell^*$ or $m^*$ belongs to $\mathcal{U}$, that is, either $\ell \in \mathcal{U}^\circ$ or $m \in \mathcal{U}^\circ$. Moreover, as $\lim_{\text{pt} \ L} \mathcal{U} = (\lim_{\xi} \mathcal{U}^\circ)^* \in \mathcal{U}$ we conclude that $\lim_{\xi} \mathcal{U}^\circ \in \mathcal{U}^\circ$, by definition of $\mathcal{U}^\circ$. The fact that $\lim_{\text{pt} \ L} \{\mathcal{U}^\circ\}^\uparrow = \lim_{\text{pt} \ L} \mathcal{U}$ follows from Lemma 2. \qed
Proposition 17. Let \((L, \lim_L)\) be a convergence lattice satisfying
\[
(3.1) \quad \lim_L x \land \lim_L y \in x \cap y \implies x = y
\]
for every \(x, y \in \text{pt } L\). Then the convergence space \(\text{pt } L\) is aas, hence weakly sober.

Proof. Assume (3.1) and suppose \(x, y \in \text{pt } L\) with \(\lim_{\text{pt } L}\{x\}^\uparrow = \lim_{\text{pt } L}\{y\}^\uparrow\), that is,
\[
(\lim_L\{x\})^\circ = (\lim_L\{y\})^\circ.
\]
In view of Lemma 2, \((\lim_L x)^\bullet = (\lim_L y)^\bullet\) so that \(\lim_L x \land \lim_L y \in x \cap y\). As a result of (3.1), \(x = y\).

In view of Propositions 16 and 11, \(\text{pt } L\) is not only quasi weakly sober but also weakly sober. \(\Box\)

An element \(\ell\) of a convergence lattice \((L, \lim)\) is closed (e.g., \([12, 13]\)) if \(\lim F \leq \ell\) whenever \(\ell \in F\) (equivalently whenever \(\ell \in F^\#\) where \(F^\# = \{m \in L : \forall f \in F \ m \land f > \perp\}\)) and open if \(\lim F \land \ell > \perp\) implies \(\ell \in F\) \((4)\). This is an abstraction of the case \(L = (\mathcal{P}X, \lim_L)\) in which closed elements are exactly the closed subsets of \((X, \xi)\) and open elements are exactly the open subsets. Let \(\mathcal{O}_L\) and \(\mathcal{C}_L\) denote the sublattices of \(L\) formed by open elements and closed elements respectively.

Lemma 18. If \(\ell \in L\) is closed then \(\ell^\bullet\) is a closed subset of \(\text{pt } L\). If \(\ell \in L\) is open, then \(\ell^\circ\) is an open subset of \(\text{pt } L\).

Proof. Let \(x \in \lim_{\text{pt } L} F\) with \(\ell^\bullet \in F\). We need to show that \(x \in \ell^\bullet\), equivalently, that \(\ell \in x\). Since \(\lim_L F^\circ \in x\) and \(\ell \in F^\circ\) for \(\ell\) closed, we conclude that \(\lim_L F^\circ \leq \ell\), so that \(\ell \in x\).

Let \(F \in \mathbb{F}\text{pt } L\) with \(x \in \lim_{\text{pt } L} F \land \ell^\bullet\), that is, \(\ell \in x\) and \(\lim_L F^\circ \in x\) so that \(\ell \land \lim_L F^\circ \neq \perp\) and \(\ell\) is open, hence \(\ell \in F^\circ\), that is, \(\ell^\circ \in F\). \(\Box\)

Proposition 19. If \((L, \lim_L)\) is a convergence lattice in which
\begin{enumerate}
\item \(\lim_L F\) is closed for every \(F \in \mathbb{F}\) \(L\) then limits sets are closed in the convergence space \(\text{pt } L\).
\item \(\lim_L x\) is closed for every \(x \in \text{pt } L\) then \(\lim_{\text{pt } L}\{x\}^\uparrow\) is closed for every \(x \in \text{pt } L\).
\end{enumerate}

Proof. Let \(\lim_{\text{pt } L} F \in G \subset \mathbb{F}\text{pt } L\). By definition \(\lim_{\text{pt } L} F = (\lim_L F^\circ)^\bullet\) is closed because \(\lim_L F^\circ\) is closed, by Lemma 18. The second part is the particular case where \(F = \{x\}^\uparrow\) for \(x \in \text{pt } L\), with the observation that in this case \(F^\circ = x\) by Lemma 2. \(\Box\)

Corollary 20. If \((L, \lim_L)\) is a convergence lattice in which \(\lim_L x\) is closed whenever \(x \in \text{pt } L\), the following are equivalent:
\begin{enumerate}
\item \(\text{pt } L\) is weakly sober;
\item \(\text{pt } L\) is antisymmetric;
\item \(\text{pt } L\) is aas;
\item \(\lim_L x = \lim_L x\) for every \(x, y \in \text{pt } L\);
\item \(\lim_L x = \lim_L y \implies x = y\) for every \(x, y \in \text{pt } L\).
\end{enumerate}

\(^4\)called quasi-closed in \([5]\).
\(^5\)called fully open in \([9]\) and a weaker notion than the notion of open element in \([5]\).
Proof. By Propositions [16] and [19] pt \( L \) is a weakly quasi-sober space in which limits of principal ultrafilters are closed. In view of Proposition [11] (1), (2) and (3) are equivalent. That (4) \( \Rightarrow \) (5) is clear and (4) \( \Rightarrow \) (3) is Proposition [17]. To see (5) \( \Rightarrow \) (4) suppose that \( \lim L x \wedge \lim L y \in x \cap y \). Then \( \lim L y \leq \lim L x \) because \( \lim L x \) is closed and belongs to \( y \), and similarly for the reverse inequality. Hence by (5), \( x = y \).

To see that (1) \( \Rightarrow \) (5) suppose \( \lim L x = \lim L y \) so that, in view of Lemma [2] \( \lim L(\{x\})^o = \lim L(\{y\})^o \) so that \( \lim pt L \{x\}^1 = \lim pt L \{y\}^1 \). By uniqueness of generic point of irreducible ultrafilters in \( pt L, x = y \). \( \square \)

In the case \( L = (\mathcal{P} X, \lim \xi) \), this means:

**Corollary 21.** If \((X, \xi)\) is a convergence space in which limits of irreducible ultrafilters are closed then \((X, \xi)\) is a dense subspace of \((\mathcal{P} X, \lim \xi)\) which is weakly sober if and only if

\[
\lim \xi \mathcal{U} = \lim \xi \mathcal{W} \implies \mathcal{U} = \mathcal{W}
\]

for every pair \( \mathcal{U}, \mathcal{W} \) of irreducible ultrafilters.

We will see with Remark [38] below that the condition (3.2) turns out to be very restrictive.

An alternative approach to force (3.1) is to consider the equivalence relation \( \sim \) on \( pt L \) defined by \( x \sim y \) if \( \lim L x = \lim L y \) and consider the quotient set

\[ pt' L := pt L/ \sim, \]

endowed with the quotient convergence. To be explicit, with \( q : pt L \to pt' L \) the canonical surjection, this means

\[
\lim_{pt' L} \mathcal{F} = \bigcup_{q[\mathcal{G}] \subseteq \mathcal{F}} q(\lim_{pt L} \mathcal{G})
\]

\[
= \bigcup_{q[\mathcal{G}] \subseteq \mathcal{F}} q((\lim_{pt L} \mathcal{G})^\bullet).
\]

**Remark 22.** Note that we can identify \( pt' L \) with \( \{\lim L x : x \in pt L\} \subset L \) and in this interpretation, the canonical surjection \( q \) is the restriction \( lim' : pt L \to pt' L \) of \( lim L \) to points, when points of \( L \) are interpreted as filters on \( L \). Hence we can consider the order induced by \( L \) on \( pt' L \) [3] which we also denote \( \leq \). Note that \( pt' L \) is not a sublattice of \( L \) in general. In contrast the subset \( C_L \) of closed elements of \( L \) is a sublattice of \( L \).

Note that

\[
\forall c \in C_L \forall x \in pt L \ (c \in x \iff \lim L x \leq c),
\]

because \( c \in x \implies \lim L x \leq c \) follows from \( c \) closed, and the converse is true whether \( c \) is closed or not, because \( \lim L x \in x \). Hence \( x \cap C_L = (\uparrow \lim L x) \cap C_L \) whenever \( x \in pt L \).

6namely, if \( x, y \in pt' L \), let \( x \sqsubseteq y \) denote \( \lim L t \leq \lim L s \) where \( t \in q^{-1}(x) \) and \( s \in q^{-1}(y) \), which is well-defined by definition of \( \sim \), but with the interpretation

\[ pt' L = \{\lim L x : x \in pt L\} \subset L \]

There is no need to distinguish \( \sqsubseteq \) from the order \( \leq \) of \( L \).
Corollary 24. If \((L, \lim_L)\) is a convergence lattice with closed limits, that is, \(\lim_L : FL \to C_L\), then
\[ x \in \lim_{pt} L F \iff \lim_L x \leq \lim_L F^\circ, \]
that is,
\[ (3.4) \quad \lim_{pt} L F = q^{-1}(pt' L \cap \lim_L F^\circ). \]
Thus
\[ (3.5) \quad \lim_{pt'} L \mathcal{H} = \bigcup_{q[F] \leq \mathcal{H}} pt' L \cap \lim_L F^\circ. \]
In particular, \(\lim_{pt'} L \mathcal{H} = \downarrow \lim_{pt'} L \mathcal{H}\).

Proof. By definition, \(x \in \lim_{pt} L F\) if and only if \(\lim_L F^\circ \subseteq x\), which is equivalent to \(\lim_L x \leq \lim_L F^\circ\) by Lemma 23. The formula (3.4) is a rephrasing from which (3.5) follows because \(q\) is onto and thus \(qq^{-1} = \text{id}_{pt'}\). □

In particular, if \(\mathcal{H} = \{\lim_L x\}^\dagger\) is a principal ultrafilter on \(pt' L\) then:

Corollary 24. If \((L, \lim_L)\) is a convergence lattice with closed limits, then
\[ (3.6) \quad \lim_{pt'} L \{\lim_L x\}^\dagger = pt' L \cap \lim_L x. \]
Thus \(pt' L\) is an aas convergence space.

Proof. Note that \(\lim_{pt'} L \{\lim_L x\}^\dagger = \bigcup_{q[F] \leq \{\lim_L x\}^\dagger} pt' L \cap \lim_L F^\circ\) by Lemma 23 and that \(q[F] \leq \{\lim_L x\}^\dagger\) means \(\{\lim_L x\}^\bullet \subseteq F\) by Lemma 18, so that \(\lim_{pt} L F = (\lim_L F^\circ)^\bullet \subseteq (\lim_L x)^\bullet\). Hence if \(\lim_L y \in pt' L \cap \lim_L F^\circ\) then \(y \in (\lim_L F^\circ)^\bullet \subseteq (\lim_L x)^\bullet\), that is, \(\lim_L x \subseteq y\) and \(\lim_L x\) is closed thus \(\lim_L y \leq \lim_L x\). Consequently, \(\lim_L y \in pt' L \cap \lim_L x\). Conversely, if \(\lim_L y \leq \lim_L x\) then \(\lim_L x \subseteq y\) and \(\lim_L x \subseteq \lim_L y\) because \(F^\circ = y\) by Lemma 18 which completes the proof of (3.6).

As a result of (3.6), if \(\lim_{pt'} L \{\lim_L x\}^\dagger = \lim_{pt'} L \{\lim_L x\}^\dagger\) then \(\lim_L x \leq \lim_L y\) and \(\lim_L y \leq \lim_L x\) so that \(\lim_L x = \lim_L y\). Hence, \(pt' L\) is aas. □

Lemma 25. If \(U\) is an open or a closed subset of \(pt L\) then \(U\) is saturated, that is, \(U = q^{-1}(q(U))\). In particular, if \(\ell\) is either an open or a closed element of \(L\) then \(\ell^\bullet\) is saturated.

Proof. Suppose \(U\) is open and \(x \in q^{-1}(q(U))\) that is, \(\lim_L x \in q(U)\), that is, there is \(t \in U\) with \(\lim_L t = \lim_L x\). As
\[ \lim_{pt} L \{t\}^\dagger = (\lim_L \{t\})^\dagger = (\lim_L x)^\bullet = \lim_{pt} L \{x\}^\dagger, \]
we conclude that \(\lim_{pt} L \{x\}^\dagger \cap U \neq \emptyset\) so that \(U \in \{x\}^\dagger\) because \(U\) is open, that is, \(x \in U\). If \(U\) is closed instead, then \(\lim_{pt} L \{t\}^\dagger = \lim_{pt} L \{x\}^\dagger \subseteq U\) and we conclude similarly. The case of \(\ell^\bullet\) follows from Lemma 18. □

Corollary 26. \(q = \lim_L : pt L \to pt' L\) is an open and closed map (sends open sets to open sets and closed sets to closed sets). Moreover, the initial topology for \(\lim_L\) is the topological modification of \(pt L\):
\[ T(pt L) = q^- (T pt' L) = q^- (T q(pt L)). \]
Lemma 27. If $c$ is a closed element of $(L, \lim_L)$ then

$$q^{-1}\left(\text{pt}' L \downarrow \downarrow c\right) = \text{pt} L \downarrow c^*.$$

Proof. Assume that $x \in q^{-1}(\text{pt}' L \downarrow \downarrow c)$ if and only if $\lim_L x \not\in c$, which, in view of (3.3), is equivalent to $c \not\in x$, that is, to $x \in \text{pt} L \downarrow c^*$. □

Note that if $L = (\mathbb{P}X, \lim_\xi)$ then for every open subset $U$ of $\text{pt} L$ there is an open element $\ell$ of $L$ with $U = \ell^*$ and similarly for closed sets. Indeed:

Proposition 28. Let $L = (\mathbb{P}X, \lim_\xi)$. Every subset $U$ of $\text{pt} L$ that is either open or closed satisfies $U = (U \cap X)^*$. 

Proof. Assume that $U$ is open. Let $U \in (U \cap X)^*$, that is, $(U \cap X) \in U$. As $\lim_{\text{pt} L}\{U\}^\uparrow = (\lim_\xi(\{U\}^\uparrow))^* = (\lim_\xi U)^\uparrow$ and there is $x \in \lim_\xi U \cap U \cap X$, we conclude that $\{x\}^\uparrow \subset (\lim_\xi U)^\uparrow \cap U$. Hence $\lim_{\text{pt} L}\{U\}^\uparrow \cap U \neq \emptyset$ and $U$ is $\text{pt} L$-open so that $U \in \{U\}^\uparrow$, that is, $U \in U$. Conversely, if $U \in U$, since $U \in \lim_{\text{pt} L} h[U]$ where $h : X \to \text{pt} L$ is $h(x) = \{x\}^\uparrow$ and $U$ is open, hence $U \cap X \in h[U]$, that is, $U \in (U \cap X)^*$. Assume now that $U$ is closed. If $U \in U$ then $\lim_{\text{pt} L}\{U\}^\uparrow = (\lim_\xi U)^\uparrow \subset U$ and thus $\lim_\xi U \subset U \cap X$ so that $U \cap X \in U$, is $U \in (U \cap X)^*$. Conversely, if $U \in (U \cap X)^*$, that is, $U \cap X \in U$, then $U \in \lim_{\text{pt} L} h[U] \subset U$ so that $U \in U$. □

Let us say that a convergence lattice $(L, \lim_L)$ has enough closed elements if for every closed subset $C$ of $\text{pt} L$ there is a closed element $c \in L$ with $C = c^*$. We similarly say that $(L, \lim_L)$ has enough open elements if every open subset $O$ of $\text{pt} L$ is of the form $u^*$ for some open element $u$ of $L$. By Proposition 28 $L = (\mathbb{P}X, \lim_\xi)$ always has enough closed elements and enough open elements.

Theorem 29. If $(L, \lim_L)$ is a convergence lattice with enough closed elements, then the topological modification of $\text{pt}' L$ is given by the open sets

$$\{\text{pt} L' \downarrow \downarrow c : c \in C_L\}.$$

Proof. In view of Lemma 27 sets of the form $\text{pt} L' \downarrow \downarrow c$ for a closed element $c \in L$ are $\text{pt}' L$-open, because their preimages under $q$ are $L$-open by Lemma 27. If now $V$ is $\text{pt}' L$-open, that is, $q^{-1}(V)$ is $\text{pt} L$-open, then $\text{pt} L' \downarrow q^{-1}(V)$ is closed, hence there is a closed element $c$ of $L$ with $q^{-1}(V) = \text{pt} L \downarrow c^*$. In view of Lemma 27 $q^{-1}(V) = q^{-1}(\text{pt} L' \downarrow \downarrow c)$ and $q$ is surjective, so that $V = \text{pt} L' \downarrow \downarrow c$. □

Remark 30. Recall that the upper topology of a poset $(X, \preceq)$ is that in which sets $\{X \downarrow x : x \in X\}$ form a subbase of open sets (e.g., [4]). Note that when $L$ has closed limits (that is, $\lim_L : \mathbb{F}L \to C_L$), then $\text{pt}' L \subset C_L$ and the topology induced on $\text{pt}' L$ by the upper topology of $C_L$ is given by the subbase $\{\text{pt} L' \downarrow \downarrow c : c \in C_L\}$, so that, in view of Theorem 29, it coincides with the quotient topology induced by $\lim_L' : \text{pt} L \to \text{pt}' L$ where $\text{pt} L$ carries the topological modification of its standard convergence.

4. $\mathcal{Z}$-regularity and structure of $\text{pt}' L$

If $\mathcal{Z} \subset L$ we say that $(L, \lim_L)$ is $\mathcal{Z}$-regular if

$$\lim_L \mathcal{F} = \lim_L \uparrow (\mathcal{F} \cap \mathcal{Z})$$

for every $\mathcal{F} \in \mathbb{F}L$. This is a generalization of the notion defined for convergence spaces in [4] Section VII.2, namely, if $\mathcal{Z} \subset \mathbb{P}X$, then $(X, \xi)$ is $\mathcal{Z}$-regular if $\lim_\xi \mathcal{F} = \lim_\xi (\mathcal{F} \cap \mathcal{Z})^{\uparrow}$ for every filter $\mathcal{F} \in \mathbb{F} \mathbb{P}X$. 

Proposition 31. Let \( f : (X, \xi) \to (Y, \tau) \) be a continuous, onto, and open map. If \( \xi \) is topological, so is \( \tau \).

Proof. We show \( \tau \) is \( \mathcal{O} \)-regular. To this end, let \( y \in \lim_\tau \mathcal{G} \). In view of [4, Exercise XV.1.29], \( \mathcal{G} \geq f[\mathcal{N}_\xi(x)] \) for every \( x \in f^{-1}(y) \). Because \( f \) is open \( f[\mathcal{N}_\xi(x)] \subset (\mathcal{G} \cap \mathcal{O}_\tau)^\uparrow \) and \( y \in \lim_\tau f[\mathcal{N}_\xi(x)] \) by continuity. Hence, \( y \in \lim_\tau (\mathcal{G} \cap \mathcal{O}_\tau)^\uparrow \). \( \square \)

Similarly a convergence space \((X, \xi)\) is topologically regular (in the sense of [4, Section VI.4]) if and only if it is \( \mathcal{C}_\xi \)-regular, equivalently, the convergence lattice \( L = (\mathcal{P}X, \lim_\xi) \) is \( \mathcal{C}_L \)-regular.

In [5], a convergence lattice \( L \) is called classical if it is regular with respect to the set of complemented elements of \( L \) (see [5, Lemma 2.29]). In [12] and [9], \( \ast \ast \)-regular convergence frames are considered. They are those convergence frames that are \( \mathcal{Z} \)-regular for the set \( \mathcal{Z} = \{ \ell \in L : \ell = \ell^* \} \), where \( \ell^* \) denotes the pseudocomplement of \( \ell \) (note that every element of a frame has a pseudocomplement).

Lemma 32. Let \( (L, \lim_0 L) \) be a convergence lattice. Then \( \mathrm{pt} L \) is a \( \mathcal{Z} \)-regular convergence space for \( \mathcal{Z} = \{ A \subset \mathrm{pt} L : \exists \ell \in L \ (A = \ell^\bullet) \} \). Moreover \( \mathrm{pt}' L \) is \( q(\mathcal{Z}) \)-regular where \( q(\mathcal{Z}) = \{ q(A) : A \in \mathcal{Z} \} \).

Proof. \( \mathcal{Z} \)-regularity of \( \mathrm{pt} L \) follows from \( \lim_{\mathrm{pt} L} \mathcal{F} = (\lim_0 \mathcal{F}^\circ)^\bullet \) and \( \mathcal{F}^\circ = ((\mathcal{F} \cap \mathcal{Z})^\downarrow)^\circ \). Now,

\[
\lim_{\mathrm{pt}' L} \mathcal{F} = \bigcup_{q(\mathcal{G}) \subseteq \mathcal{F}} q(\lim_{\mathrm{pt} L} \mathcal{G}) = \bigcup_{q(\mathcal{G}) \subseteq \mathcal{F}} q(\lim_{\mathrm{pt} L} (\mathcal{G} \cap \mathcal{Z})^\uparrow)
\]

and \( q(\mathcal{G} \cap \mathcal{Z}) \subset \mathcal{F} \cap \mathcal{Z} \subset \mathcal{F} \) so that

\[
\lim_{\mathrm{pt}' L} \mathcal{F} = \bigcup_{q(\mathcal{G} \cap \mathcal{Z}) \subseteq \mathcal{F} \cap \mathcal{Z}} q(\lim_{\mathrm{pt} L} (\mathcal{G} \cap \mathcal{Z})^\uparrow) \subset \lim_{\mathrm{pt}' L} (\mathcal{F} \cap \mathcal{Z})^\uparrow.
\]

\( \square \)

Lemma 33. If \( (L, \lim_0 L) \) is \( \mathcal{Z} \)-regular where \( \ell^\bullet \) is saturated for every \( \ell \in \mathcal{Z} \), then

\[
\lim_{\mathrm{pt}' L} \mathcal{U} = q(\lim_{\mathrm{pt} L} q^{-1}[\mathcal{U}]) = q(\lim_{\mathrm{pt} L} \mathcal{W}),
\]

for every ultrafilter \( \mathcal{U} \) on \( \mathrm{pt}' L \) on every \( \mathcal{W} \in U(q^{-1}[\mathcal{U}]) \).

Proof. Let \( L \) be \( \mathcal{Z} \)-regular. Because \( \mathcal{U} \) is an ultrafilter, \( \lim_{\mathrm{pt}' L} \mathcal{U} = \bigcup_{\mathcal{W} \in U(q^{-1}[\mathcal{U}])} q((\lim_0 \mathcal{W}^\circ)^\bullet) \).

Since \( \lim_0 \mathcal{W}^\circ = \lim_0 (\mathcal{W}^\circ \cap \mathcal{Z}) \) by \( \mathcal{Z} \)-regularity, it is enough to show that \( \mathcal{W}^\circ \cap \mathcal{Z} = \mathcal{Z} \cap (q^{-1}[\mathcal{U}])^\circ \) whenever \( \mathcal{W} \in U(q^{-1}[\mathcal{U}]) \). This follows from the observation that \( \ell \in \mathcal{W}^\circ \cap \mathcal{Z} \) if and only if \( \ell \in \mathcal{Z} \) and \( \ell^\bullet \in \mathcal{W} \), and \( \ell^\bullet \) is saturated and thus \( \ell^\bullet \in q^{-1}[\mathcal{W}] = q^{-1}[\mathcal{U}] \). \( \square \)

\( ^7 \)Indeed, it is clear that a topological space is \( \mathcal{O}_\xi \)-regular, and conversely, if \( \xi \) is \( \mathcal{O}_\xi \)-regular, then

\[
x \in \lim(x)^\uparrow = \lim (\{x\}^\uparrow \cap \mathcal{O}_\xi)^\uparrow = \lim_\xi \mathcal{N}_\xi(x),
\]

so that \( \xi \) is topological.
Corollary 35. If Proposition 31.

Proof. Let

\[ \lim_{pt} L \cap \{ u^* : u \in O_L \} \]

because \( L \) is \( O_L \)-regular. Hence \( pt \) is \( O \)-regular, that is, topological. As \( q \) is a continuous, onto, and open map by Corollary 26. If \( pt' \) is topological, so is \( pt \).

Corollary 35. If \( L \) is \( O_L \)-regular, has enough open sets, and has closed limits then \( pt' \) is a sober topological space.

Proof. Let \( U \) be an irreducible ultrafilter on \( pt' \), that is, \( \lim_{pt'} L U \in U \). In view of Lemma 25. \( \lim_{pt'} L U = q(\lim_{pt} L W) \) for every \( W \in \mathcal{U} \), so that

\[ q^{-1}(\lim_{pt'} L U) = q^{-1}(q(\lim_{pt} L W)) = \lim_{pt} L W \in q^{-1}(U) \subset W \]

because each \( \lim_{pt} L W \) is closed (because \( L \) is topological, hence also \( pt' \) has closed limits by Proposition 19). Hence \( L \) is irreducible ultrafilter. In other words, each \( W \) is an irreducible ultrafilter so that \( \lim_{pt} L W = \lim_{pt} L \{ x \} = (\lim L x)^* \) for some \( x \in pt \) because, in view of Proposition 25, \( pt' \) is weakly quasi-sober. Thus \( \lim_{pt'} L U = q((\lim L x)^*) = \lim_{pt'} L \{ \lim L x \} \) by Corollary 24. Hence \( pt' \) is weakly quasi-sober and aas by Corollary 24 hence weakly sober. In view of Proposition 11 it is sober because it is topological.

In view of Lemma 18 and Proposition 28 (\( P X, \lim \xi \)) always has enough open elements and enough closed elements. Moreover, it has closed limits when \( \xi \) has closed limits, in particular if \( \xi \) is topological. Hence, Proposition 14 and Corollary 35 apply to the effect that:

Corollary 36. If \( (X, \xi) \) is a topological space and \( L = (P X, \lim \xi) \) then \( pt L \) is topological and weakly quasi-sober and \( pt' L \) is a topological sober space.

In fact, in this case \( pt' L \) is exactly the sobrification of the topological space \( X \), as we will see in Section 5 below.

5. Sobrification and the space \( pt(P X, \lim \xi) \)

Recall that a closed subset \( C \) of a convergence space is irreducible if \( C \subset D \) or \( C \subset F \) whenever \( C \subset D \cup F \) where \( D \) and \( F \) are closed. Let us call \( C \) c-irreducible if it satisfies this property even if \( C \) is not necessarily closed. Note that \( C \) is c-irreducible if and only if for every open sets \( O \) and \( U \) intersecting \( C \), \( O \cap U \cap C \neq \emptyset \). Obviously, \( C \) is c-irreducible if and only if \( \text{cl} C \) is an irreducible closed set.

Lemma 37. In a convergence space, limits of irreducible filters are c-irreducible and the closure of every c-irreducible set is the limit of a T\( \xi \)-irreducible ultrafilter.
irreducible ultrafilters are principal; note that if there is a free irreducible ultrafilter on a topological space, then

\[ \text{being both sober and } \] and more generally for (some) convergence spaces in Corollary 47 below to

\[ \text{T} \]

is a filter base because \( X \)

\[ \text{if } \]

\[ \text{and the filter base } \]

\[ \text{non-maximal filter-base } \]

\[ \text{filter-base.} \]

□

Remark 38. In particular, in a topological space every irreducible closed set \( C \) is the limit of an irreducible ultrafilter—indeed, of many, for every ultrafilter containing the filter base \( \{ O \in O_\xi : O \cap C \neq \emptyset \} \) satisfies this condition. As a result, if \( X \) is a \( T_0 \) topological space then \( [\text{3.2}] \) is equivalent to every irreducible ultrafilters being principal, a property characterized for topological spaces in \( [\text{7} \) Theorem 2.2] and more generally for (some) convergence spaces in Corollary 47 below to be equivalent with \( X \) being both sober and \( T_D \). To see that \( [\text{3.2}] \) implies that irreducible ultrafilters are principal, note that if there is a free irreducible ultrafilter on a topological space, then \( C = \lim U \) is an infinite irreducible closed set and the non-maximal filter-base \( \{ O \in O_\xi : O \cap C \neq \emptyset \} \) admits one, hence many finer free ultrafilters all of which admit \( C \) as their limit, so that \( [\text{3.2}] \) fails. Conversely, if all irreducible ultrafilters are principal, then every irreducible closed set has a generic point, which is necessarily unique in a \( T_0 \) topological space, so that \( [\text{3.2}] \) is satisfied.

Corollary 39. If \( \mathcal{F} \) is an irreducible filter on a convergence space \((X, \xi)\) then

\[ \text{cl}_\xi(\lim_\xi \mathcal{F}) = \lim_\xi \mathcal{F}. \]

Proof. We only need to show that \( \lim_\xi \mathcal{F} \subseteq \text{cl}_\xi(\lim_\xi \mathcal{F}) \) for the reverse inclusion is always true. In view of Lemma 37 \( \lim_\xi \mathcal{F} \) is c-irreducible and thus, for every ultrafilter \( U \) of the filter-base \( \{ O \in O_\xi : O \cap \lim_\xi \mathcal{F} \neq \emptyset \} \), \( \text{cl}_\xi(\lim_\xi \mathcal{F}) = \lim_\xi \mathcal{U} \). Note that the condition \( O \cap \lim_\xi \mathcal{F} \neq \emptyset \) implies that \( O \in \mathcal{F} \), so that \( \mathcal{F} \) contains this filter-base. □

Recall (e.g., [8]) that the sobrification of a \( T_0 \) topological space \( X \) is the set \( ^*X \) of irreducible closed subsets of \( X \) endowed with the topology \{ \{ ^*O : O \in O_\xi \} \} where \( ^*O = \{ ^*C \in ^*X : C \cap O \neq \emptyset \} \). Because \( X \) is \( T_0 \) the map \( e : X \to ^*X \) defined by \( e(x) = \text{cl}\{x\} \) is one-to-one. It is a dense embedding because \( ^*O \cap e(X) = e(O) \).

Note that in view of Lemma 37 we can identify elements of \( \text{pt}' L \) (equivalence classes of irreducible ultrafilters with the same limit) and elements of \( ^*X \). In fact, since the topology of \( ^*X \) is the topology induced on \(^*X \) by the upper topology of the space of closed subsets of \( X \) ordered by inclusion because

\[ ^*O = \{ ^*C \in ^*X : C \cap O \neq \emptyset \} = ^*X \setminus \{ X \setminus O \}, \]

we conclude via Theorem 29 that \( ^*X \) is homeomorphic to \( \text{pt}' L \) endowed with the quotient topology induced by \( q = \lim_\xi : \text{pt} L \to \text{pt}' L \), but this is the the structure of \( \text{pt}' L \), since \( \text{pt}' L \) is topological by Corollary 36

\[ ^* \text{to see that this defines indeed a topology, note that } ^* \left( \bigcup_{O \in B} O \right) = \bigcup_{O \in B} ^* O \text{ and that } ^* (O \cap U) = ^* O \cap ^* U \text{ because an irreducible closed sets intersects } O \cap U \text{ whenever it intersects open sets } O \text{ and } U. \]
Theorem 40. Let \((X, \xi)\) be a topological and let \(L = (\mathcal{P}X, \text{lim}_\xi)\). Then \(pt' L\) is the sobrification \(^*X\) of \(X\), and \(e = q \circ h\) is a dense embedding in the diagram below whenever \((X, \xi)\) is \(T_0\):

\[
\begin{array}{ccc}
X & \xrightarrow{h:x \mapsto \{x\}^+} & \text{pt}(\mathcal{P}X, \text{lim}_\xi) \\
\downarrow & & \Downarrow q \\
*X & \xrightarrow{c:x \mapsto \text{cl}\{x\}} & \text{pt}'(\mathcal{P}X, \text{lim}_\xi)
\end{array}
\]

6. The Axiom \(T_D\) in Convergence Spaces

Recall that a topological space \(X\) is \(T_D\) if for every \(x \in X\) there is \(U \in \mathcal{O}(x)\) with \(U \setminus \{x\}\) open (e.g., [13]).

Lemma 41. In a topological space \(X\), the following are equivalent:

1. \(X\) is \(T_D\);
2. for every \(x \in X\) there is \(U \in \mathcal{O}(x)\) with
   \[
   \text{lim}\{x\}^+ \cap U = \{x\};
   \]
3. for every \(x \in X\) and every filter \(\mathcal{F}\) on \(X\) that converges to \(x\), there is \(A \in \mathcal{F}\) with
   \[
   \text{lim}\{x\}^+ \cap A \subset \{x\};
   \]
4. for every \(x \in X\) and every ultrafilter \(\mathcal{U}\) on \(X\) that converges to \(x\), there is \(A \in \mathcal{U}\) with
   \[
   \text{lim}\{x\}^+ \cap A \subset \{x\}.
   \]

Proof. (1) \(\implies\) (2): If \(X\) is \(T_D\) and \(x \in X\), pick \(U \in \mathcal{O}(x)\) with \(U \setminus \{x\}\) open. Then \(\text{lim}\{x\} \cap U = \{x\}\) because \(t \in \text{lim}\{x\}^+ \cap (U \setminus \{x\})\) would imply \(t \in \{x\}^+\) which is impossible.

(2) \(\implies\) (3): If \(x \in \text{lim} \mathcal{F}\), then \(\mathcal{F} \geq \mathcal{O}(x)\) and we can take the \(U\) from (1) as \(A\).

(3) \(\implies\) (4) is obvious.

(4) \(\implies\) (1): For every \(U \in \mathcal{U}(\mathcal{N}(x))\), there is \(A_U \in \mathcal{U}\) with \(\text{lim}\{x\} \cap A_U \subset A_U\). Hence there is a finite subset \(F\) of \(U(\mathcal{N}(x))\) with \(\bigcup_{U \in F} A_U \in \mathcal{N}(x)\). Let \(U \in \mathcal{O}(x)\) with \(U \subset \bigcup_{U \in F} A_U\). Then \(U \setminus \{x\}\) is open. Indeed, if \(t \in \text{lim} \mathcal{U} \cap (U \setminus \{x\})\) for some ultrafilter \(\mathcal{U}\) then \(U \in \mathcal{U}\) because \(U\) is open. If \(U \setminus \{x\} \notin \mathcal{U}\) then \(U \cap \{x\} \notin \mathcal{U}\), hence \(\mathcal{U} = \{x\}\) and (4) ensures a contradiction. \(\square\)

Note that (3) and (4) make sense for general convergences and are equivalent \(\square\). Thus we call a convergence space \(T_D\) if it satisfies (3), equivalently (4). As far as I know, this is the first time this axiom is introduced for convergence spaces.

Just as for topological space, \(T_D\) lies between \(T_0\) and \(T_1\): \(T_1\) obviously implies \(T_D\) as in this case \(\text{lim}\{x\}^+ = \{x\}\) for every \(x\). On the other hand, if \(X\) is \(T_D\) and

\[
\text{lim}\{x\} \cap A = \bigcup_{U \in F} \text{lim}\{x\} \cap A_U \subset \{x\}.
\]

\(\square\)
\{F \in \mathcal{F} X : x \in \lim F \} = \{x \in \lim \mathcal{F} : t \in \lim \mathcal{F} \}

Then \{x, t\} \subset \lim \{x\} \cap \lim \{t\}

so by \(T_D\) there is \(A \in \{x\}\) with \(\lim \{t\} \cap A \subset \{t\}\) which ensures \(x = t\) because \(x \in \lim \{t\} \cap A\).

Note that in all the examples of Section 2 we have \(x \neq y\) with \(\{x, y\} \in \lim \{x\} \cap \lim \{y\}\) so that \(\{y\}\) is an ultrafilter converging to \(x\) and \(y \in \lim \{x\}\). Hence \(y \in \lim \{x\} \cap A\) for every \(A \in \{y\}\) and the space is not \(T_D\).

Preservation properties for \(T_D\) easily extend from topologies to convergences with this definition:

**Proposition 42.** If \(\xi \leq \tau\) and \(\xi = T_D\), so is \(\tau\); a subspace of a \(T_D\) convergence space is \(T_D\); a finite product of \(T_D\) convergence spaces is \(T_D\).

**Proof.** If \(x \in \lim \xi \subset \lim \xi \mathcal{F}\) there is \(A \in \mathcal{F}\) with \(\lim \{x\} \cap A \subset \{x\}\) and the conclusion follows from \(\lim \xi \{x\} \subset \lim \xi \{x\}\) if \(A \subset X\) with inclusion map \(i : A \to X\) and \(x \in A \cap \lim \xi \mathcal{F},\ x \in \lim \xi \mathcal{F}\) so that \(F \in \mathcal{F}\) with \(\lim \{x\} \cap F \subset \{x\}\) so that \(\lim \xi \mathcal{F} \cap (A \cap F) \subset \{x\}\) and \((A, \xi_A)\) is \(T_D\). Suppose \((X_1, \xi_1), \ldots, (X_n, \xi_n)\) are \(T_D\) convergence spaces and let \((x_1, \ldots, x_n) \in \lim \{x\}\mathcal{F}\), that is \(x_i \in \lim \xi_i \mathcal{F}\) for every \(i\), where \(p_i : \Pi_{n=1}^\infty X_i \to X_i\) is the projection. There is \(A_i \in \Pi_{n=1}^\infty \mathcal{F}\) with \(\lim \xi_i \{x_i\} \cap A_i \subset \{x_i\}\) so that \(\Pi_{n=1}^\infty A_i \in \Pi_{n=1}^\infty \mathcal{F} \subset \mathcal{F}\) and then \(\lim \Pi_{n=1}^\infty \{x_1, \ldots, x_n\} \cap \Pi_{n=1}^\infty A_i \subset \{x_1, \ldots, x_n\}\).

On the other hand, a countable product of topological \(T_D\) spaces (e.g., countably many copies of the Sierpiński space) may fail to be \(T_D\) (See e.g., [1]). However:

**Lemma 43.**

1. A \(T_D\) convergence space is antisymmetric.

2. Moreover, if the space is finite the converse is true, that is, \(\to\) is antisymmetric then the convergence is \(T_D\).

3. If \(\lim \{x\}\) is closed for every \(x\) then \(\to\) is transitive. In particular if the convergence is \(S_0\) then \(\to\) is transitive, and conversely if the space is finite and of finite depth.

**Proof.** 1. Assume there are points \(x \neq y\) with \(\{x, y\} \subset \lim \{x\} \cap \lim \{y\}\). Then \(y \in \lim \{x\} \cap A\) for every \(A \in \{y\}\) and the convergence is not \(T_D\).

2. Conversely for a finite space, if the convergence is not \(T_D\) there is a (principal because the space is finite) ultrafilter \(\{y\}\) converging to \(x\) such that \(\lim \{x\} \cap A \subset \{x\}\) for every \(A \in \{y\}\), in particular for \(A = \{y\}\). Hence \(y \neq x\) and \(y \in \lim \{x\}\) and thus \(\{x\} \subset \lim \{x\} \cap \lim \{y\}\).

3. That \(\to\) is transitive if \(\lim \{x\}\) is closed for every \(x\) is clear. For the converse for a finite space, assume \(\mathcal{F} = \{F\}\) converges to \(x\) and \(\{x\}\) converges to \(y\). For every \(t \in F\), \(t \to x\) and \(x \to y\), so \(t \to y\) by transitivity. By finite depth, \(y \in \cap \lim \{t\}\) converges to \(x\) and \(\{x\}\) converges to \(y\). Hence, the space is \(S_0\) and the limits of principal ultrafilters are closed.

Note that in view of Lemma 43 a finite, weakly sober, and non-sober convergence like Example 14 cannot be \(T_D\). This is in fact general:

**Proposition 44.** In a \(T_D\) weakly sober space every irreducible ultrafilter is principal. If moreover the convergence is of finite depth, then the only irreducible filters are principal ultrafilters and thus the convergence is also sober.
Proof. If \( \xi \) is weakly sober and \( \mathcal{U} \) is an irreducible filter, then there is a (unique) \( x \) with \( \lim \mathcal{U} = \lim \{ x \}^\uparrow \in \mathcal{U} \). As \( \xi \) is \( T_D \) there is \( U \in \mathcal{U} \) with \( \lim \{ x \}^\uparrow \cap U \subset \{ x \} \). As a result, \( \{ x \} \in \mathcal{U} \) and \( \mathcal{U} \) is principal. Hence if \( \mathcal{F} \) is irreducible, every ultrafilter finer than \( \mathcal{F} \) is also irreducible, hence principal. As a result, \( \mathcal{F} \) is the principal filter of a finite set \( F \). If the convergence is moreover of finite depth,

\[
\lim \mathcal{F} = \lim \{ F \}^\uparrow = \bigcap_{x \in F} \lim \{ x \}^\uparrow \in \mathcal{F} = \{ F \}^\uparrow.
\]

By Lemma 43 (1), \( F \) is a singleton. The space is then sober. \( \square \)

In view of Proposition 42 if \( T \xi \) is \( T_D \) so is the finer convergence \( \xi \), but the converse is false:

Example 45 (A finitely deep \( T_D \) convergence whose topological modification is not \( T_D \)). Consider on \( X = \{ x, y, z \} \) the finitely deep convergence determined by \( \lim \{ x \}^\uparrow = \{ x, y \} \), \( \lim \{ y \}^\uparrow = \{ y, z \} \) and \( \lim \{ z \}^\uparrow = \{ z, x \} \).

This convergence is \( T_D \) By Lemma 43 but its topological modification is antidiscrete, hence not even \( T_0 \).

7. Problem 41

Theorem 46. Let \( (X, \xi) \) be a convergence space.

1. If \( \xi \) is weakly sober and \( T_D \) then every irreducible filter is principal.
2. If \( \xi \) is \( S_0 \) and \( T_0 \) the converse is true, that is, \( \xi \) is weakly sober and \( T_D \) whenever every irreducible ultrafilter is principal.

Proof. (1) is Proposition 44

(2) If \( \xi \) is \( T_0 \) and \( S_0 \) and every irreducible ultrafilter is principal then \( \xi \) is weakly sober: if \( \lim \mathcal{U} \in \mathcal{U} \) there is \( x \) with \( \mathcal{U} = \{ x \}^\uparrow \) and thus \( \lim \mathcal{U} = \lim \{ x \}^\uparrow \). If \( t \neq x \) satisfies \( \lim \{ t \}^\uparrow = \lim \{ x \}^\uparrow \), by \( T_0 \) there is \( \mathcal{F} \in \mathcal{FX} \) with card(\( \lim \mathcal{F} \cap \{ t, x \} \)) = 1, which is not compatible with \( S_0 \). Moreover, \( \xi \) is \( T_D \). Suppose to the contrary that there is \( x \in X \) and \( \mathcal{U} \in \mathcal{UX} \) with \( x \in \lim \mathcal{U} \) such that \( \lim \{ x \}^\uparrow \cap U \cap (X \setminus \{ x \}) \neq \emptyset \) for every \( U \in \mathcal{U} \). In particular, \( \lim \{ x \}^\uparrow \in \mathcal{UX} = \mathcal{U} \) and \( \lim \{ x \}^\uparrow \subset \lim \mathcal{U} \) because \( \xi \) is \( S_0 \). Hence \( \lim \mathcal{U} \in \mathcal{U} \) and \( \mathcal{U} \) is irreducible, hence principal, that is, \( \mathcal{U} = \{ t \}^\uparrow \) and \( \lim \{ x \}^\uparrow \subset \lim \{ t \}^\uparrow \). Hence \( x \in \lim \{ t \}^\uparrow \) and \( t \in \lim \{ x \}^\uparrow \) by taking \( U = \{ t \} \) in \( \lim \{ x \}^\uparrow \cap U \cap (X \setminus \{ x \}) \neq \emptyset \), the argument above for the uniqueness of a generic point applies to the effect that \( x = t \). But that is not compatible with \( \lim \{ x \}^\uparrow \cap U \cap (X \setminus \{ x \}) \neq \emptyset \) for \( U = \{ t \} = \{ x \} \) and we have a contradiction. \( \square \)

Corollary 47. Let \( (X, \xi) \) be a \( S_0 \) and \( T_0 \) convergence space of finite depth. The following are equivalent:

1. Every irreducible ultrafilter is principal;
2. Every irreducible filter is principal;
3. \( \xi \) is weakly sober and \( T_D \);
4. \( \xi \) is sober and \( T_D \);
5. Every subspace of \( \xi \) is sober;
Corollary 47.

(6) If $\theta \geq \xi$ then $\theta$ is sober;

(7) $pt_{\text{Lat}}(\mathcal{P}X, \lim_{\xi})$ is homeomorphic to $(X, \xi)$.

Proof. Equivalence between points 1 through 4 follows directly from Theorem 46 Proposition 44.

(1) $\iff$ (5): By Proposition 44 sober and weakly sober are equivalent in a $T_D$ convergence of finite depth, hence in all its subspaces. Let $A \subset X$ with inclusion map $i : A \to X$, and let $\mathcal{U}$ be an ultrafilter on $A$ with $\lim_{\xi} \mathcal{U} \in \mathcal{U}$. Hence $i(\lim_{\xi} \mathcal{U}) \in i[\mathcal{U}]$ and $i(\lim_{\xi} \mathcal{U}) \subset \lim_{\xi} i[\mathcal{U}]$ so that the ultrafilter $i[\mathcal{U}]$ on $(X, \xi)$ is irreducible. By (1), there is $x \in X$ with $i[\mathcal{U}] = \{x\}^\uparrow$. As $A \in i[\mathcal{U}]$, $x \in A$ and thus $\lim_{\xi} \mathcal{U} = \lim_{\xi} \{x\}$ and $A$ is weakly sober.

Conversely, if there is a free ultrafilter $\mathcal{U}$ that is irreducible, if $\lim \mathcal{U}$ does not have a generic point, then the space is not weakly sober. If $\lim \mathcal{U}$ has a generic point, it is unique by Lemma 7, say, $\lim \mathcal{U} = \lim \{x_0\}$. Let $A = \lim \mathcal{U} \setminus \{x_0\}$ with the induced convergence. Then $A \in \mathcal{U}$ (because $\lim \mathcal{U} \in \mathcal{U}$ and $X \setminus \{x_0\} \in \mathcal{U}$ as $\mathcal{U}$ is free) and $\lim_{\xi} \mathcal{U} = A \in \mathcal{U}$ but $\lim_{\xi} \mathcal{U}$ has not generic point. Hence, $(A, \xi_A)$ is a non weakly sober subspace.

(1) $\iff$ (6): If $\mathcal{U} \in \mathcal{UX}$ with $\lim_{\xi} \mathcal{U} \in \mathcal{U}$ then $\lim_{\xi} \mathcal{U} \in \mathcal{U}$ because $\lim_{\xi} \mathcal{U} \subset \lim_{\xi} \mathcal{U}$, so that $\mathcal{U}$ is $\xi$-irreducible, hence principal. Hence it has a unique generic point, because $\xi$ is $T_0$ and $T_0$, hence almost antisymmetric. Thus $\theta$ is sober.

Conversely, suppose there is a free $\xi$-irreducible ultrafilter $\mathcal{U}$. Let $\theta$ be defined by $x \in \lim_{\xi} \mathcal{F}$ if and only if $\mathcal{F} = \{x\}^\uparrow$ whenever $x \notin \lim_{\xi} \mathcal{U}$ and $x \in \lim_{\xi} \mathcal{F}$ if and only if $\mathcal{F} = \mathcal{U}$ or $\mathcal{F} = \{x\}^\uparrow$ whenever $x \in \lim_{\xi} \mathcal{U}$. Then $\theta \geq \xi$ and $\theta$ is not sober: $\mathcal{U}$ is $\theta$-irreducible because $\lim_{\xi} \mathcal{U} = \lim_{\xi} \mathcal{U} \in \mathcal{U}$ but does not have a generic point because singletons are closed and $\lim_{\xi} \mathcal{U}$ is infinite.

(1) $\iff$ (7): Let $L = (\mathcal{P}X, \lim_{\xi})$. By (1) the map the map $h : pt_{\text{Lat}} L \to (X, \xi)$ defined by $h(\{x\}^\uparrow) = x$ is onto and thus is an homeomorphism by Proposition 3.

Conversely, if $pt L$ is homeomorphic to $(X, \xi)$ but there are free $\xi$-irreducible ultrafilters then, in view of Proposition 3 $pt L$ contains an homeomorphic copy of itself as a dense proper subset via an homeomorphism $j : pt L \to j(pt L) \subset pt L$, which is not possible, for $j(pt L)$ is closed as an homeomorphic image of a closed subset of $pt L$, hence $pt L \setminus j(pt L)$ is open, making it impossible for $j(pt L)$ to be dense in $pt L$.

Note that in view of Remark 35, when $(X, \xi)$ is topological, we can add 35 for every pair $\mathcal{U}, \mathcal{W}$ of irreducible ultrafilters as an additional equivalent condition in Corollary 47.

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