THE INCREASING FAMILIES OF SETS GENERATED BY SELF-DUAL CLUTTERS

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Abstract. Using the KKS inequalities, we establish bounds on the numbers of \(k\)-sets in the increasing families generated by self-dual clutters (i.e., clutters \(A\) that coincide with the blockers \(\mathcal{B}(A)\)) on their ground set of even cardinality.

1. Introduction

We denote by \(E_t\), where \(t \geq 3\), the set of integers \([t] := \{1, \ldots, t\}\). A finite collection of sets \(F\) is called a family. The family \(2^{[t]}\) of all subsets of the set \(E_t\) is called the power set of \(E_t\). The empty set is denoted by \(\emptyset\), and the empty family containing no sets is denoted by \(\emptyset\). We denote by \(|\cdot|\) the cardinalities of sets, while the numbers of sets in families are denoted by \(\#\cdot\). Sometimes we say that the cardinality of a set \(A\) is the size of \(A\).

If \(F\) is a set family such that \(\emptyset \neq F \neq \{\emptyset\}\), then the union \(V(F) := \bigcup_{F \in F} F\) is called the vertex set of \(F\). Any finite set \(S\) such that \(S \supseteq V(F)\), fixed for someone’s research purposes, is called the ground set of the family \(F\).

Given a nonempty family \(F \subseteq 2^{[t]}\) on its ground set \(E_t\), we let \(F^\perp := \{F^c : F \in F\}\) denote the family of their complements, where \(F^c := E_t - F\). We also associate with the family \(F\) the family \(F^* := \{G^c : G \in 2^{[t]} - F\}\).

A family of sets \(A\), such that \(\emptyset \neq A \neq \{\emptyset\}\), is called a nontrivial clutter if for any indices \(i \neq j\) of members of the family \(\{A_1, \ldots, A_\alpha\} =: A\) we have \(A_i \nsubseteq A_j\).

For a subset \(A \subseteq E_t\), the principal increasing family of sets \(\{A\}^\perp \subseteq 2^{[t]}\), generated by the one-member clutter \(\{A\}\) on its ground set \(E_t\), is defined by \(\{A\}^\perp := \{B \subseteq E_t : B \supseteq A\}\). In particular, we have \(\{\emptyset\}^\perp = 2^{[t]}\), and \(\{E_t\}^\perp = \{E_t\}\).

Given a nonempty clutter \(A := \{A_1, \ldots, A_\alpha\} \subseteq 2^{[t]}\), the increasing family of sets \(A^\perp \subseteq 2^{[t]}\), generated by \(A\) on its ground set \(E_t\), is defined as the union \(A^\perp := \bigcup_{A \in A\} \{A\}^\perp\) of the principal increasing families \(\{A\}^\perp\).

A subset \(B \subseteq E_t\) is a blocking set of a nontrivial clutter \(A \subseteq 2^{[t]}\) if it holds \(|B \cap A| > 0\), for each member \(A \in A\). The blocker \(\mathcal{B}(A)\) of the clutter \(A\) is defined to be the family of all inclusion-minimal blocking sets of \(A\); see, e.g., the monographs [4, 7, 9, 10, 11, 12, 13, 15, 16, 19, 20, 21, 22, 24, 26, 28].
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31, 32, 33, 35, 36, 38]. Additional references can be found in [27, Pt 2]. The increasing family \( \mathcal{B}(\mathcal{A})^\vee \) is by definition the family of all blocking sets of the clutter \( \mathcal{A} \).

Recall that for a nontrivial clutter \( \mathcal{A} \) on its ground set \( E_t \) we have
\[
\mathcal{B}(\mathcal{A})^\vee = (\mathcal{A}^\vee)^*.
\]

Clutters \( \mathcal{A} \) with the property
\[
\mathcal{B}(\mathcal{A}) = \mathcal{A},
\]
or, equivalently,
\[
\mathcal{B}(\mathcal{A})^\vee = \mathcal{A}^\vee,
\]
are called self-dual or identically self-blocking; see, e.g., [1, 2][4, §2.1][24, Ch. 9][26, §5.7] on such clutters.

A clutter \( \mathcal{A} \) on its ground set \( E_t \) is self-dual if and only if we have
\[
(\mathcal{A}^\vee)^* = \mathcal{A}^\vee, \tag{1.1}
\]
and although the (lack of) self-duality of clutters does not depend structurally on the cardinalities of their vertex sets and ground sets, relation (1.1) suggests the following criterion:

**Remark 1.1.** (see [26, Cor. 5.28(i)]) A clutter \( \mathcal{A} \subset 2^{[t]} \) on its ground set \( E_t \) is self-dual if and only if
\[
\#\mathcal{A}^\vee = 2^{t-1}. \tag{1.2}
\]

In Theorem 4.1 we consider condition (1.2) from a perspective of constraints that are satisfied by the numbers of \( k \)-sets in the increasing families \( \mathcal{A}^\vee \) generated by self-dual clutters \( \mathcal{A} \subset 2^{[t]} \) on their ground set \( E_t \) of even cardinality \( t \).

2. Long \( f \)- and \( h \)-vectors of set families

Let us associate with each family \( \mathcal{F} \subset 2^{[t]} \) on the ground set \( E_t \) its long \emph{f-vector}
\[
f(\mathcal{F};t) := (f_0(\mathcal{F};t), f_1(\mathcal{F};t), \ldots, f_t(\mathcal{F};t)) \in \mathbb{N}^{t+1},
\]
where
\[
f_k(\mathcal{F};t) := \#\{ F \in \mathcal{F}: |F| = k \}, \quad 0 \leq k \leq t.
\]
If ‘\( x \)’ is a formal variable, then the long \emph{h-vector}
\[
h(\mathcal{F};t) := (h_0(\mathcal{F};t), h_1(\mathcal{F};t), \ldots, h_t(\mathcal{F};t)) \in \mathbb{Z}^{t+1}
\]
of the family \( \mathcal{F} \) is defined by means of the relation
\[
\sum_{i=0}^{t} h_i(\mathcal{F};t) \cdot x^{t-i} := \sum_{i=0}^{t} f_i(\mathcal{F};t) \cdot (x-1)^{t-i}.
\]
Example 2.1. (cf. Example 3.1) For any element $a \in E_t$, the principal increasing family $\{\{a\}\}^\uparrow \subset 2^{[t]}$, generated by the self-dual clutter $\{\{a\}\}$ on its ground set $E_t$, is described by the vectors

$$f(\{\{a\}\}^\uparrow; t) = (0, \frac{(t-1)}{1-1}, \frac{(t-1)}{2-1}, \ldots, \frac{(t-1)}{t-1})$$

and

$$h(\{\{a\}\}^\uparrow; t) = (0, 1, 0, \ldots, 0).$$

Example 2.2. (i) (a) Given the self-dual clutter

$$\mathcal{A} := \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} \quad (2.1)$$

that generates on its ground set $E_3 = V(\mathcal{A})$ the increasing family

$$\mathcal{A}^\uparrow = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, E_3\},$$

we have

$$f(\mathcal{A}^\uparrow; 3) = (0, 0, 3, 1),$$

$$h(\mathcal{A}^\uparrow; 3) = (0, 0, 3, -2).$$

(b) If we choose the set $E_4 \supsetneq V(\mathcal{A})$ as the ground set of the self-dual clutter (2.1), instead of the set $E_3$, then $\mathcal{A}$ generates on $E_4$ the increasing family

$$\mathcal{A}^\uparrow = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, E_4\},$$

and we have

$$f(\mathcal{A}^\uparrow; 4) = (0, 0, 3, 4, 1),$$

$$h(\mathcal{A}^\uparrow; 4) = (0, 0, 3, -2, 0).$$

(ii) (a) For the self-dual clutter

$$\mathcal{A} := \{\{2\}\} \quad (2.2)$$

that generates on its ground set $E_3 \supsetneq V(\mathcal{A})$ the principal increasing family

$$\mathcal{A}^\uparrow = \{\{2\}, \{1, 2\}, \{2, 3\}, E_3\},$$

we have

$$f(\mathcal{A}^\uparrow; 3) = (0, 1, 2, 1),$$

$$h(\mathcal{A}^\uparrow; 3) = (0, 1, 0, 0).$$

(b) For the self-dual clutter (2.2) that generates on its ground set $E_4 \supsetneq V(\mathcal{A})$ the principal increasing family

$$\mathcal{A}^\uparrow = \{\{2\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, E_4\},$$

we have

$$f(\mathcal{A}^\uparrow; 4) = (0, 1, 3, 3, 1),$$

$$h(\mathcal{A}^\uparrow; 4) = (0, 1, 0, 0, 0).$$
(iii) The self-dual clutter
\[ A := \{\{1,2,3,4\}, \{1,5\}, \{2,5\}, \{3,5\}, \{4,5\} \}, \]
that generates on its ground set \( E_5 = V(A) \) the increasing family
\[ A^\uparrow = \{\{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}, \{1,2,5\}, \{1,3,5\}, \{1,4,5\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, E_5 \}, \]
is described by the vectors
\[ f(A^\uparrow; 5) = (0, 0, 4, 6, 5, 1), \]
\[ h(A^\uparrow; 5) = (0, 0, 4, -6, 5, -2). \]

Remark 2.3. Given a family \( F \subseteq 2^{[d]} \) on its ground set \( E_t \), we have:

(i) (see [26, Prop. 2.1(iii)(a)])
\[ h_t(F; t) = (-1)^t \sum_{k=0}^t (-1)^k \binom{t-k}{t-k} f_k(F; t), \quad 0 \leq t \leq t; \quad (2.3) \]
\[ f_t(F; t) = \sum_{k=0}^t (-1)^k \binom{t-k}{t-k} h_k(F; t), \quad 0 \leq t \leq t. \quad (2.4) \]

(ii) (see [26, Prop. 2.1(iii)(b)])

\[ h_0(F; t) = f_0(F; t); \]
\[ h_1(F; t) = f_1(F; t) - tf_0(F; t); \]
\[ h_{t-1}(F; t) = (-1)^{t-1} \sum_{k=0}^{t-1} (-1)^k (t-k) f_k(F; t); \]
\[ h_t(F; t) = (-1)^t \sum_{k=0}^t (-1)^k f_k(F; t); \quad (2.5) \]
\[ \sum_{k=0}^t h_k(F; t) = f_t(F; t). \]

(iii) (see [26, Prop. 2.1(iii)(c)])
\[ \sum_{k=0}^t f_k(F; t) = \sum_{k=0}^t 2^{t-k} h_k(F; t) = \# F. \]

(iv)
\[ h(F; t) + h(2^{[d]} - F; t) = (1, 0, 0, \ldots, 0) = h(2^{[d]}; t). \]

Being redundant analogues of standard \( h \)-vectors of abstract simplicial complexes [8, 23, 28, 34, 37, 40], long \( h \)-vectors (see, e.g., [29, p. 170][30, p. 265]) demonstrate their usefulness below in relations (2.6), where they describe a curious enumerative connection between the families \( F \) and \( F^* \).
Remark 2.4. Given a family $\mathcal{F} \subseteq 2^{[t]}$ on its ground set $E_t$, we have:

(i) $\#\mathcal{F}^* + \#\mathcal{F} = 2^t$.

More precisely, 

$$f_\ell(\mathcal{F}^*; t) + f_{t-\ell}(\mathcal{F}; t) = \binom{t}{\ell}, \quad 0 \leq \ell \leq t. $$

(ii) (see [26, Prop. 2.1(iii)(d)])

$$h_\ell(\mathcal{F}^*; t) + (-1)^\ell \sum_{k=\ell}^t \binom{k}{\ell} h_k(\mathcal{F}; t) = \delta_{\ell,0}, \quad 0 \leq \ell \leq t, \quad (2.6)$$

where $\delta_{\ell,0} := 1$ if $\ell = 0$, and $\delta_{\ell,0} := 0$ otherwise. Note that

$$h_t(\mathcal{F}^*; t) = (-1)^{t+1} h_t(\mathcal{F}; t).$$

3. Abstract simplicial complexes $\Delta$, such that $\Delta^* = \Delta$, on their ground sets $E_t$ of even cardinality $t$

Although in the present paper we are concerned with enumerative properties of specific increasing families of sets, in this section we turn to specific abstract simplicial complexes (i.e., “decreasing” families of “faces”) because the numbers of faces of size $k$ in complexes are known to satisfy the classical Kruskal–Katona–Schützenberger (KKS) inequalities; see, e.g., [34, §10.3] and [3, 4, 6, 8, 14, 17, 18, 23, 24, 25, 37, 38, 39, 40].

Let $\Delta \subseteq 2^{[t]}$, where $\emptyset \neq \Delta \neq \{\emptyset\}$, be an abstract simplicial complex on its ground set $E_t$, with its vertex set $V(\Delta) \subseteq E_t$. By definition of complex, the following implications hold: $(B \in \Delta, A \subset B) \implies A \in \Delta$. The inclusion-maximal faces of $\Delta$ are called its facets. Sometimes one says that a complex with one facet (i.e., the power set of a set) is a simplex. The long $f$-vector of the complex $\Delta$ has the form

$$f(\Delta; t) := (1, f_1(\Delta; t), \ldots, f_{d(\Delta)}(\Delta; t), 0, \ldots, 0),$$

where $d(\Delta) := \max\{k \in [t] : f_k(\Delta; t) > 0\}$.

Example 3.1. (cf. Example 2.1) For any element $a \in E_t$, the simplex $\Delta := \{F : F \subseteq (E_t - \{a\})\} = \Delta^*$, whose facet is the subset $(E_t - \{a\}) \subset E_t$ of size $(t - 1)$, is described by the vectors

$$f(\Delta; t) = \left(\binom{t-1}{0}, \ldots, \binom{t-1}{t-2}, \binom{t-1}{t-1}, 0\right)$$

and

$$h(\Delta; t) = (1, -1, 0, \ldots, 0).$$
Remark 3.2. An abstract simplicial complex $\Delta$ with vertex set $V(\Delta)$, such that $\Delta$ coincides with its Alexander dual complex $\Delta^\vee$, defined by

$$\Delta^\vee := \{V(\Delta) - F : F \in 2^{V(\Delta)} - \Delta\}$$

when $V(\Delta^\vee) = V(\Delta)$, is known as an Alexander self-dual complex. Note that

$$\Delta = \Delta^\vee \iff \#\Delta = 2^{|V(\Delta)| - 1}.$$  

See, e.g., [38] and [5] on combinatorial Alexander duality.

Remark 3.3. Suppose that $\Delta \subset 2^t$ is an abstract simplicial complex, such that $\Delta^* = \Delta$, on its ground set $E_t$. Then we have:

(i) $\#\Delta = \sum_{k=0}^{t} f_k(\Delta; t) = \sum_{k=0}^{d(\Delta)} f_k(\Delta; t) = \sum_{k=0}^{t} 2^{t-k} h_k(\Delta; t) = 2^t - 1$.

(ii) $f_\ell(\Delta; t) + f_{t-\ell}(\Delta; t) = \binom{t}{\ell}, \quad 0 \leq \ell \leq t$.  \hspace{1cm} (3.1)

In particular, if $t$ is even, then

$$f_{t/2}(\Delta; t) = \frac{1}{2} \binom{t/2}{t/2}.$$  

Example 3.4. (i) The abstract simplicial complex

$$\Delta := \{\hat{0}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\},$$

such that $\Delta^* = \Delta$, on its ground set $E_4 = V(\Delta)$, is described by the vectors

$$f(\Delta; 4) = (1, \quad 4, \quad 3, \quad 0, \quad 0),$$

$$h(\Delta; 4) = (1, \quad 0, -3, \quad 2, \quad 0).$$

The complex $\Delta$ is an Alexander self-dual complex, that is, $\Delta = \Delta^\vee$, because $8 = \#\Delta = 2^{|V(\Delta)| - 1} = 2^4 - 1$.

(ii) The simplex

$$\Delta := \{\hat{0}, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\},$$

such that $\Delta^* = \Delta$, on its ground set $E_4 \supsetneq V(\Delta)$, is described by the vectors

$$f(\Delta; 4) = (1, \quad 3, \quad 3, \quad 1, \quad 0),$$

$$h(\Delta; 4) = (1, -1, \quad 0, \quad 0, \quad 0).$$

• Suppose again that $\Delta \subset 2^t$ is an abstract simplicial complex, such that $\Delta^* = \Delta$, on its ground set $E_t$ of even cardinality $t$. We would like to apply the KKS inequalities to the complex $\Delta$.

○ Since $\frac{1}{2} \binom{t}{t/2} = \binom{t-1}{t/2}$, we have the simplest $(t/2)$-binomial expansion

$$f_{t/2}(\Delta; t) = \binom{t}{t/2}$$

of the middle component of the long $f$-vector $f(\Delta; t)$, where

$$\alpha_{t/2}(t/2) = t - 1.$$
Then relations (3.1) and the KKS inequalities (lower and upper bound versions) [34, Cor. 10.1, Cor. 10.2] imply that

\[
\left(\frac{t}{(t/2) - 1}\right) - f(t/2 + 1)(\Delta; t) = f(t/2 - 1)(\Delta; t) \geq \left(\alpha(t/2, t/2)\right) = \left(\frac{t-1}{(t/2) - 1}\right) = \left(\frac{t}{(t/2)}\right),
\]

\[
\left(\frac{t}{(t/2) + 1}\right) - f(t/2 - 1)(\Delta; t) = f(t/2 + 1)(\Delta; t) \leq \left(\alpha(t/2, t/2)\right) = \left(\frac{t-1}{(t/2) + 1}\right) = \left(\frac{t}{(t/2) - 2}\right).
\]

Further, note that if our complex \( \Delta \) had the minimum possible number of its faces of size \((t/2 - 1)\), namely \((\frac{t-1}{(t/2) - 1})\) faces, then the following implication would hold:

\[
f(t/2 - 1)(\Delta; t) := \left(\frac{t-1}{(t/2) - 1}\right) = \left(\frac{t}{t/2}\right) \implies f(t/2 + 1)(\Delta; t) = \left(\frac{t}{(t/2) + 1}\right) - \left(\frac{t-1}{(t/2) - 1}\right) = \left(\frac{t}{(t/2) - 1}\right) = \left(\frac{t-1}{(t/2) - 2}\right).
\]

Note also that \((\frac{t}{(t/2) - 1}) - \left(\frac{t-1}{(t/2) - 1}\right) = \left(\frac{t-1}{(t/2) - 2}\right) = \left(\frac{t-1}{(t/2) + 1}\right).

\(\circ\) If \((\frac{t}{(t/2) - 1}) > 1\), then suppose that our complex \( \Delta \) indeed has the minimum possible number of faces of size \((t/2 - 1)\), that is, \(f(t/2 - 1)(\Delta; t) := \left(\frac{t-1}{(t/2) - 1}\right)\). In this case we have the \((\frac{t}{(t/2) - 1})\)-binomial expansion

\[
f(t/2 - 1)(\Delta; t) = \binom{\alpha(t/2 - 1)(t/2 - 1)}{(t/2) - 1}
\]

of the component \(f(t/2 - 1)(\Delta; t)\) of the vector \(f(\Delta; t)\), where \(\alpha(t/2 - 1)(t/2 - 1) = t - 1\).

Relations (3.1) and the KKS inequalities (lower bound version) [34, Cor. 10.1] imply that

\[
f(t/2 - 2)(\Delta; t) = f(t/2 + 2)(\Delta; t) \geq \binom{\alpha(t/2 - 1)(t/2 - 1)}{(t/2) - 2} = \left(\frac{t-1}{(t/2) - 2}\right).
\]

On the other hand, suppose that the complex \( \Delta \) has the maximum possible number of faces of size \((t/2 + 1)\), that is, \(f(t/2 + 1)(\Delta; t) := \left(\frac{t-1}{(t/2) + 1}\right)\). We have the \((\frac{t}{(t/2) + 1})\)-binomial expansion

\[
f(t/2 + 1)(\Delta; t) = \binom{\alpha(t/2 + 1)(t/2 + 1)}{(t/2) + 1}
\]

of the component \(f(t/2 + 1)(\Delta; t)\) of the vector \(f(\Delta; t)\), where \(\alpha(t/2 + 1)(t/2 + 1) = t - 1\).

Now relations (3.1) and the KKS inequalities (upper bound version) [34, Cor. 10.2] imply that

\[
\left(\frac{t}{(t/2) + 2}\right) - f(t/2 - 2)(\Delta; t) = f(t/2 + 2)(\Delta; t) \leq \binom{\alpha(t/2 + 1)(t/2 + 1)}{(t/2) + 2} = \left(\frac{t-1}{(t/2) + 2}\right).
\]

Note that \((\frac{t}{(t/2) + 2}) - \left(\frac{t-1}{(t/2) - 2}\right) = \left(\frac{t-1}{(t/2) - 3}\right) = \left(\frac{t-1}{(t/2) + 2}\right).

\(\circ\) Proceeding by induction, we arrive at the following counterpart of Theorem 4.1:
**Theorem 4.1.** (see also Example 3.1) Let $\Delta \subset 2^{|t|}$ be an abstract simplicial complex on its ground set $E_t$ of even cardinality $t$. If

$$\Delta^* = \Delta,$$

then we have

$$f_0(\Delta; t) = 1, \quad \text{and} \quad f_t(\Delta; t) = 0;$$

$$f_{t/2}(\Delta; t) = \frac{1}{2} \binom{t}{t/2} = \binom{t-1}{t/2};$$

$$f_{(t/2)-i}(\Delta; t) \geq \binom{t-1}{(t/2)-i} = \binom{t}{(t/2)+i-1}, \quad 1 \leq i \leq (t/2) - 1;$$

$$f_{(t/2)+j}(\Delta; t) \leq \binom{t-1}{(t/2)+j} = \binom{t}{(t/2)-j-1}, \quad 1 \leq j \leq (t/2) - 1;$$

$$f_{(t/2)-k}(\Delta; t) + f_{(t/2)+k}(\Delta; t) = \binom{t}{(t/2)-k} = \binom{t}{(t/2)+k}, \quad 1 \leq k \leq (t/2) - 1.$$

**4. The increasing families of sets generated by self-dual clutters on their ground sets $E_t$ of even cardinality $t$**

Let $\mathcal{F} \subset 2^{|t|}$ be a family on its ground set $E_t$. The implication

$$\mathcal{F}^{*} = \mathcal{F} \iff (2^{|t|} - \mathcal{F})^{*} = 2^{|t|} - \mathcal{F},$$

allows us to obtain from Lemma 3.5 the following result:

**Theorem 4.1.** (see also Example 2.1) Let $\mathcal{A} \subset 2^{|t|}$ be a clutter on its ground set $E_t$ of even cardinality $t$. If the clutter $\mathcal{A}$ is self-dual, that is,

$$\mathfrak{B}(\mathcal{A}) = \mathcal{A},$$

then we have

$$f_0(\mathcal{A}^*; t) = 0, \quad \text{and} \quad f_t(\mathcal{A}^*; t) = 1;$$

$$f_{t/2}(\mathcal{A}^*; t) = \frac{1}{2} \binom{t}{t/2} = \binom{t-1}{t/2};$$

$$f_{(t/2)-i}(\mathcal{A}^*; t) \leq \binom{t-1}{(t/2)-i} = \binom{t}{(t/2)+i-1}, \quad 1 \leq i \leq (t/2) - 1;$$

$$f_{(t/2)+j}(\mathcal{A}^*; t) \geq \binom{t-1}{(t/2)+j} = \binom{t}{(t/2)-j-1}, \quad 1 \leq j \leq (t/2) - 1;$$

$$f_{(t/2)-k}(\mathcal{A}^*; t) + f_{(t/2)+k}(\mathcal{A}^*; t) = \binom{t}{(t/2)-k} = \binom{t}{(t/2)+k}, \quad 1 \leq k \leq (t/2) - 1.$$

**5. Appendix: Remarks on set families $\mathcal{F}$ such that $\mathcal{F}^{*} = \mathcal{F}$**

**Remark 5.1.** Let $\mathcal{F} \subset 2^{|t|}$ be a family, such that $\mathcal{F}^{*} = \mathcal{F}$, on its ground set $E_t$.

(i) We have

$$\# \mathcal{F} = 2^{t-1}.$$ More precisely,

$$f_{\ell}(\mathcal{F}; t) + f_{t-\ell}(\mathcal{F}; t) = \binom{t}{\ell}, \quad 0 \leq \ell \leq t. \quad (5.1)$$

(ii) Relations (5.1) and (2.4) imply that

$$\sum_{k=0}^{\ell} \binom{\ell-k}{t-\ell} h_k(\mathcal{F}; t) + \sum_{j=0}^{t-\ell} \binom{t-j}{\ell} h_j(\mathcal{F}; t) = \binom{t}{\ell}, \quad 0 \leq \ell \leq t.$$
Given a family $\mathcal{F} \subset 2^{[t]}$, such that $\mathcal{F}^* = \mathcal{F}$, on its ground set $E_t$, from (2.3) we have for any $\ell$, where $0 \leq \ell \leq t$:

$$h_\ell(\mathcal{F}; t) = (-1)\ell \sum_{k=0}^\ell (-1)^k \binom{t-k}{t-\ell} f_k(\mathcal{F}; t)$$

$$= (-1)\ell \sum_{k=0}^\ell (-1)^k \binom{t-k}{t-\ell} \left( \binom{t}{k} - f_{t-k}(\mathcal{F}; t) \right).$$

**Remark 5.2.** Let $\mathcal{F} \subset 2^{[t]}$ be a family, such that $\mathcal{F}^* = \mathcal{F}$, on its ground set $E_t$.

(i) We have

$$h_\ell(\mathcal{F}; t) = \delta_{\ell,0} - (-1)^{t-\ell} \sum_{j=0}^{t-\ell} (-1)^j \binom{t-j}{t-\ell} f_j(\mathcal{F}; t), \quad 0 \leq \ell \leq t; \quad (5.2)$$

$$f_\ell(\mathcal{F}; t) = \binom{t}{\ell} - \sum_{j=0}^{t-\ell} \binom{t-j}{t-\ell} h_j(\mathcal{F}; t), \quad 0 \leq \ell \leq t. \quad (5.3)$$

(ii) Relations (5.2) and (2.3) yield

$$(-1)\ell \sum_{k=0}^\ell (-1)^k \binom{t-k}{t-\ell} f_k(\mathcal{F}; t)$$

$$+ (-1)^{t-\ell} \sum_{j=0}^{t-\ell} (-1)^j \binom{t-j}{t-\ell} f_j(\mathcal{F}; t) = \delta_{\ell,0}, \quad 0 \leq \ell \leq t.$$

**Remark 5.3.** Let $\mathcal{F} \subset 2^{[t]}$ be a family, such that $\mathcal{F}^* = \mathcal{F}$, on its ground set $E_t$ of odd cardinality $t$. From (2.5) we have

$$h_t(\mathcal{F}; t) = (-1)^t \sum_{k=0}^t (-1)^k f_k(\mathcal{F}; t)$$

$$= - \sum_{k=0}^{\lfloor t/2 \rfloor} (-1)^k \left( f_k(\mathcal{F}; t) - \binom{t}{k} - f_k(\mathcal{F}; t) \right)$$

$$= \binom{t-1}{(t-1)/2} - 2 \sum_{k=0}^{(t-1)/2} (-1)^k f_k(\mathcal{F}; t).$$
and
\[
 h_\ell(\mathcal{F}; t) = (-1)^t \sum_{j=0}^{t} (-1)^j f_j(\mathcal{F}; t)
\]
\[
 = - \sum_{j=[t/2]}^{t} (-1)^j \left( f_j(\mathcal{F}; t) - \binom{t}{j} - f_j(\mathcal{F}; t) \right)
\]
\[
 = -\binom{t-1}{(t-1)/2} - 2 \sum_{j=(t+1)/2}^{t} (-1)^j f_j(\mathcal{F}; t) .
\]

As a consequence, we see that
\[
\sum_{k=0}^{(t-1)/2} (-1)^k f_k(\mathcal{F}; t) = \binom{t-1}{(t-1)/2} + \sum_{j=(t+1)/2}^{t} (-1)^j f_j(\mathcal{F}; t) .
\]

**Remark 5.4.** Let \( \mathcal{F} \subset 2^{[t]} \) be a family, such that \( \mathcal{F}^* = \mathcal{F} \), on its ground set \( E_t \).

(i) From (2.6) we have
\[
 h_\ell(\mathcal{F}; t) = \delta_{\ell,0} + (-1)^{\ell+1} \sum_{k=\ell}^{t} \binom{k}{\ell} h_k(\mathcal{F}; t) , \quad 0 \leq \ell \leq t . \tag{5.4}
\]

If \( t \) is even, then we see that
\[
 h_t(\mathcal{F}; t) = 0 . \tag{5.5}
\]

(ii) For even indices \( \ell \), where \( 2 \leq \ell \leq t \), relations (5.4) imply that
\[
\sum_{k=\ell}^{t} \binom{k}{\ell-1} h_k(\mathcal{F}; t) = 0 , \tag{5.6}
\]

that is,
\[
\ell h_\ell(\mathcal{F}; t) + \sum_{k=\ell+1}^{t} \binom{k}{\ell-1} h_k(\mathcal{F}; t) = 0 ;
\]

we also have
\[
2 h_\ell(\mathcal{F}; t) + \sum_{k=\ell+1}^{t} \binom{k}{\ell} h_k(\mathcal{F}; t) = 0 .
\]

If \( t \) is odd, then we have
\[
(t - 1) h_{t-1}(\mathcal{F}; t) + \binom{t}{2} h_t(\mathcal{F}; t) = 0 ,
\]

that is,
\[
2 h_{t-1}(\mathcal{F}; t) + t h_t(\mathcal{F}; t) = 0 .
\]
If \( t \) is even, then we have

\[
\frac{t}{2} h_{t/2}(\mathcal{F}; t) + \sum_{k=(t/2)+1}^{t-1} \binom{k}{(t/2)-1} h_k(\mathcal{F}; t) = 0 ,
\]

and

\[
h_{t/2}(\mathcal{F}; t) + \frac{1}{2} \sum_{k=(t/2)+1}^{t-1} \binom{k}{(t/2)} h_k(\mathcal{F}; t) = 0 .
\] (5.7)

**Remark 5.5.** Let \( \mathcal{F} \subset 2^{[t]} \) be a family, such that \( \mathcal{F}^* = \mathcal{F} \), on its ground set \( E_t \).

If \( t \) is even, then relations (5.5) and (5.6) in the case \( \ell := 2 \) yield

\[
h_t(\mathcal{F}; t) = 0 , \quad \text{and} \quad \sum_{k=2}^{t-1} k h_k(\mathcal{F}; t) = 0 .
\]

Similarly, if \( t \) is odd, then from (5.6) we have

\[
\sum_{k=2}^{t} k h_k(\mathcal{F}; t) = 0 .
\]

**Remark 5.6.** Let \( \mathcal{F} \subset 2^{[t]} \) be a family, such that \( \mathcal{F}^* = \mathcal{F} \), on its ground set \( E_t \) of even cardinality \( t \).

(i) On the one hand, since

\[
f_{t/2}(\mathcal{F}; t) = \frac{1}{2} \binom{t}{t/2} ,
\] (5.8)

relations (2.4) imply that

\[
\sum_{k=0}^{t/2} \binom{t-k}{t/2} h_k(\mathcal{F}; t) = f_{t/2}(\mathcal{F}; t)
\]

\[
= h_{t/2}(\mathcal{F}; t) + \sum_{k=0}^{(t/2)-1} \binom{t-k}{t/2} h_k(\mathcal{F}; t) = \frac{1}{2} \binom{t}{t/2} .
\] (5.9)

On the other hand, (5.8) and relations (5.3) imply that

\[
\binom{t}{t/2} - \sum_{j=0}^{t/2} \binom{t-j}{t/2} h_{t/2}(\mathcal{F}; t) = f_{t/2}(\mathcal{F}; t) = \frac{1}{2} \binom{t}{t/2} ,
\]

that is,

\[
\sum_{j=0}^{t/2} \binom{t-j}{t/2} h_{t/2}(\mathcal{F}; t) = \frac{1}{2} \binom{t}{t/2} .
\]
(ii) Relations (5.7) and (5.9) imply that

\[
\sum_{k=0}^{(t/2)-1} \binom{t-k}{t/2} h_k(F; t) - \frac{1}{2} \sum_{j=(t/2)+1}^{t-1} \binom{j}{t/2} h_j(F; t) = \frac{1}{2} \binom{t}{t/2},
\]

and

\[
h_{t/2}(F; t) = \frac{1}{4} \binom{t}{t/2} - \frac{1}{2} \sum_{k=0}^{(t/2)-1} \binom{t-k}{t/2} h_k(F; t) - \frac{1}{4} \sum_{j=(t/2)+1}^{t-1} \binom{j}{t/2} h_j(F; t).
\]

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