Approximation of barter exchanges with cycle length constraints

Suiqian Luo and Pingzhong Tang and Chenggang Wu and Jianyang Zeng
Institute for Interdisciplinary Information Sciences, Tsinghua University

luosql3@mails.tsinghua.edu.cn kenshin@tsinghua.edu.cn
wuchenggang0316@gmail.com zengjy321@tsinghua.edu.cn

Abstract

We explore the clearing problem in the barter exchange market. The problem, described in the terminology of graph theory, is to find a set of vertex-disjoint, length-restricted cycles that maximize the total weight in a weighted digraph. The problem has previously been shown to be NP-HARD. We advance the understanding of this problem by the following contributions.

We prove three constant inapproximability results for this problem. For the weighted graphs, we prove that it is NP-HARD to approximate the clearing problem within a factor of $\frac{11}{14}$ under general length constraints and within a factor of $\frac{4}{3}$ when the cycle length is not longer than 3. For the unweighted graphs, we prove that this problem is NP-HARD to approximate within a factor of $\frac{693}{684}$.

For the unweighted graphs when the cycle length is not longer than 3, we design and implement two simple and practical algorithms. Experiments on simulated data suggest that these algorithms yield excellent performances.

1 Introduction

Over the past decade, kidney exchange and matching based market design in general, have become one of the most appealing applications at the interface of economics and computer science. In economics, designing desirable matching mechanism has been a topic of intensive research, ever since the seminal work on college admission and stable marriage problem [Gale and Shapley, 1962; Roth and Sotomayor, 1992]. In computer science and the multiagent system community, designing and fielding efficient clearing algorithms for such markets has been under close scrutiny lately [Abraham et al., 2007; Awasthi and Sandholm, 2009; Ashlagi et al., 2010; Dickerson et al., 2012a; Dickerson et al., 2013; Dickerson et al., 2014; Li et al., 2014].

In a typical kidney exchange system, a patient with renal disease teams up with a known but incompatible donor. While the pair donate a kidney to help some other compatible patient in the system, they obtain a compatible kidney in return. Both patients receive a compatible kidney in the end, resulting in social welfare improvement. Nowadays, kidney exchange serves as alternative solution besides cadaver donations and has been fielded successfully in a number of countries. For a comprehensive introduction of background, refer to [Roth et al., 2004; Roth et al., 2005] [Abraham et al., 2007] and the references therein.

In this paper, we explore the kidney exchange problem from a computational perspective. The problem, described in the terminology of graph theory, is to find a set of vertex-disjoint cycles that maximize the total weight in a weighted digraph. Each vertex in the graph represents a patient-donor pair and each arc represents compatibility between the pairs, with the arc weight denoting the payoff by performing the surgery. A cycle of length $L$ requires $2L$ people in simultaneous surgeries. To lessen the logistical pressure imposed by simultaneous surgeries, in practice, every cycle length is constrained to be less than or equal to $3$ [Abraham et al., 2007]. For $L \leq 3$, Abraham et. al. [Abraham et al., 2007] show that the kidney exchange problem is NP-HARD. They demonstrate an effective integer programming formulation with an advanced tree search algorithm that can solve a graph with 10000 nodes.

We further explore computational complexity of this problem, for both $L \leq 3$ and general $L$, in both weighted and unweighted graphs. Our conclusion is that the problem, under various definitions, is computationally hard to approximate. The seemingly-straightforward algorithms we propose can give good solutions on simulated data. In particular, we make the following contributions:

1.1 Our contribution

1. We prove that, for a weighted graph with general $L$, the kidney exchange problem is NP-HARD to approximate within a factor of $\frac{14}{15}$. The inapproximability result is obtained via a reduction from the inapproximability of

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1See, for example, http://www.unos.org/

2An alternative solution is to include “chains”, which start with a cadaver or altruistic donor that does not look for anything in return. Chain plays an important role in current kidney exchange systems. See, e.g., [Roth et al., 2004; Dickerson et al., 2012a; Ashlagi et al., 2012; Dickerson et al., 2014]. We do not consider chains in the current paper.
maximum 3-variable linear equations modular 2 satisfiability problem (aka. MAX-3LIN-2) by Håstad [Håstad, 2001].

2. We prove that, for a weighted graph with \( L \leq 3 \), the kidney exchange problem is NP-HARD to approximate within a factor of \( \frac{432}{433} \). The proof is via a reduction from the bounded occurrence version of the maximum 3-variable linear equations modular 2 satisfiability problem (aka. MAX-3LIN-2(3)).

3. We show that, for an unweighted graph with \( L \leq 3 \), the kidney exchange problem is NP-HARD to approximate within a factor of \( \frac{698}{697} \). The proof is via a reduction of the MAXIMUM-3-DIMENSIONAL-MATCHING problem.

4. For the unweighted graph with \( L = 3 \), we propose two algorithms which are easy to analyse and implement. We implement the algorithms and test them on simulated data. Both of them yield good experimental performance on these data.

For the three inapproximability results, these problems has previous been known to be APX-HARD [Biro and Cechlarova, 2006], i.e., there is no polynomial time approximation scheme for these problems (i.e., hard to find a \( 1 - \epsilon \) approximation). We advance the theoretical understandings on these problems by showing that it is hard to find a constant approximation.

These results have important implications. First, our hardness results complement the work by Abraham et. al. [Abraham et al., 2007] and can serve as a justification of their choice of integer programming implementation over approximation algorithm. Second, our proof techniques shed light on the close relations between the kidney exchange problem and several other landmark computational optimization problems, and thus can serve as technical basis for proving similar results in the kidney exchange literature. Third, the two simple and practical algorithms and experimental results suggest that there exist fast, near-optimal algorithms in practice.

2 Preliminary

2.1 The kidney exchange problem

We model kidney exchange as a directed weighted graph \( G = (V, E) \) where each vertex stands for a donor-patient pair and each edge represents a possible one-way kidney exchange. A cycle in the graph serves as a basic building block of the exchange outcome, where a patient receives a kidney from the donor in the preceding vertex along the cycle. The weight of each cycle is the sum of weights for each arc along the cycle. Depending on the specific application scenarios, edges can be both weighted [Abraham et al., 2007] or unweighted (with uniform weight on each arc) [Roth et al., 2005]. Our goal is then to find a collection of vertex-disjoint cycles that maximize the total weight in weighted graphs and the total size in unweighted graphs. In other words, the goal is to find a clearing algorithm that maximizes social welfare.

We denote the problem by MAX-WEIGHT-\( L \)-EXCHANGE for weighted graphs and MAX-SIZE-\( L \)-EXCHANGE for unweighted graphs, where \( L \) is the restriction of the exchange cycle length.

2.2 Gap problems and inapproximability

We first recall basic definitions of gap version optimization problems. For more information about the theory of gap problems and inapproximability of optimization problems, see, e.g., [Vazirani, 2001].

Let \( \mathcal{A} \) be any maximization problem, then the gap version of \( \mathcal{A} \) with parameter \( 0 < a < b \leq 1 \), denoted by \( \text{GAP-}\mathcal{A}[a,b] \), is the following decision problem: Given an instance \( I \) of \( \mathcal{A} \), distinguish whether the optimal solution has fractional size at least \( b \) or less than \( a \). When fractional size of the optimal solution is between \( a \) and \( b \), any output suffices.

Clearly, if there exists a polynomial time \( \frac{a}{b} \)-approximation algorithm of the original problem \( \mathcal{A} \), we can distinguish the two cases in polynomial time. Conversely, if the gap version problem \( \text{GAP-}\mathcal{A}[a,b] \) is NP-HARD, then the original problem \( \mathcal{A} \) is NP-HARD to approximate within a factor of \( \frac{b}{a} \).

Gap version of optimization problems plays a central role in proving inapproximability results. It has been widely studied in the theory community. Many landmark problems were shown to be NP-HARD to approximate.

Definition 1 MAX-3LIN-\( q \) is the following optimization problem:

**Input:** \( n \) variables \( x_1, x_2, \ldots, x_n \) with range \( \{0,1, \ldots, q-1\} \), and a set of \( m \) constraints \( c_1, c_2, \ldots, c_m \), where each \( c_i \) is a linear equation modular \( q \) with \( 3 \) variables, e.g., \( x_1 + x_4 + x_8 = 1 \) mod \( q \).

**Output:** Find an assignment satisfying the maximum number of constraints.

Following the monumental PCP-theorem [Arora et al., 1998; Arora and Safra, 1998], Håstad has proved the following celebrated theorem [Håstad, 2001], which serves as the start point of many inapproximability results:

**Theorem 1 (Håstad)** GAP-MAX-3LIN-\( q \)-[\( 1 + \epsilon \), \( 1 - \epsilon \)] is NP-HARD for any small constant \( \epsilon > 0 \). The theorem also holds when \( m = O(n) \) where the constant only depends on \( \epsilon \).

One natural variation of the MAX-3LIN-2 problem is to restrict the number of occurrence of each variable in all the equations, we denote this problem by MAX-3LIN-2(\( t \)). For \( t = 3 \), the problem has been shown to be NP-HARD to approximate within some constant factor [Berman and Karpinski, 1998].

**Theorem 2** The MAX-3LIN-2(\( 3 \)) problem is NP-HARD to approximate within a factor of \( \frac{62}{61} \). More precisely, the GAP-MAX-3LIN-2(\( 3 \))-[\( 1 + \epsilon \), \( 1 - \epsilon \)] problem is NP-HARD for any small constant \( \epsilon > 0 \).

3-DIMENSIONAL-MATCHING is one of Karp’s 21 NP-complete problems [Karp, 1972]. MAXIMUM-3-DIMENSIONAL-MATCHING is a generalization of maximum matching in bipartite graphs to maximum hyperedge matching in tripartite graphs and also has been shown to be NP-HARD to approximate:

**Theorem 3** (Berman and Karpinski 2003) GAP-MAXIMUM-3-DIMENSIONAL-MATCHING-[\( 697/698 \), \( 698/697 \)] is NP-HARD for any small constant \( \epsilon > 0 \).
3 Max-Weight-L-Exchange

In this section, we prove that the MAX-WEIGHT-L-EXCHANGE problem is NP-HARD to approximate within factor $\frac{14}{13}$. To our best knowledge, this problem is only known to be APX-HARD. Here, we give the first explicit constant factor inapproximability result.

**Theorem 4** The MAX-WEIGHT-L-EXCHANGE problem is NP-HARD to approximate within a factor $\frac{14}{13}$. More precisely, the GAP-MAX-WEIGHT-L-EXCHANGE-$[\frac{13}{14} + \epsilon, 1 - \epsilon]$ problem is NP-HARD.

**Proof.** We reduce from the GAP-MAX-3LIN-2-$[\frac{13}{14} + \epsilon, 1 - \epsilon]$ problem.

**Construction:** Given any MAX-3LIN-2 system with $n$ variables and $m$ equations (or constraints), we first construct a variable gadget for each variable $x_i$ with $2m + 1$ nodes as illustrated in Figure 1. The node in the center is named the super node for $x_i$. If $x_i$ appears in the $j$-th equation (as shown in boxes with bold frames in Figure 1), then the arc entering the $j$-th row has weight 2, otherwise the weight will be 0. We call the cycles passing through the super node variable cycles. The weight for any variable cycle associated with $x_i$ is $m_i$, where $m_i$ is the total number of equations that variable $x_i$ appears.

Second, for each equation, we use the six nodes in the corresponding row to construct equation gadget. Figure 2 illustrates an example of constraint gadget construction for the equation $x_1 + x_2 + x_3 \equiv 1 \mod 2$. If the equation is equal to 0, we can construct the graph symmetrically.

In each equation gadget, each arc is associated with weight $\frac{1}{3}$. A property of the graph gadget is that there are exactly four cycles with length 3 in total for each equation and the weight of each cycle is exactly 1. There exists cycles which length is greater than 3 in this gadget, but the total weights is at most 2. Each cycle with length 3 corresponds to one possible satisfiable assignment of the equation. We call these cycles equation cycles.

At last, we set $L = m + 1$. The total number of vertices is $N = (2m + 1)n = O(n^2)$. The reduction can be computed in polynomial time.

**Completeness:** We want to show that if there exists an assignment $\sigma$ satisfying at least $(1 - \epsilon)m$ constraints, we can always construct a collection $C$ of cycles with a total weight at least $6m + (1 - \epsilon)m$. The proof is straightforward. For each $x_i$, if $x_i$ is assigned with 1, then we add the variable cycle passing through nodes labeled with $x_i = 0$; otherwise we add the other variable cycle involving nodes labeled with $x_i = 1$. For each satisfiable constraint, we add the corresponding constraint cycle in the corresponding constraint gadget. All the cycles in $C$ are vertex-disjoint. The sum of weight of variable cycles is $6m$, and the sum of weight of equation cycles is at least $(1 - \epsilon)m$. Therefore the total weight is at least $6m + (1 - \epsilon)m$.

**Soundness:** We need to show that if there is a collection $C$ of cycles in the constructed graph with total weight $\geq 6m + (\frac{1}{2} + \epsilon)m$, we can resolve assignments to the variables such that at least $(\frac{1}{2} + \epsilon)m$ constraints are satisfied.

For each valid collection $C$ of cycles, if for each variable $x_i$, there is exactly one variable cycle in $C$, then we call $C$ a good collection. Otherwise we call $C$ a bad collection.

Our first claim is that for any bad collection $C$ of cycles, we can always adjust $C$ to some good collection $C'$ without decreasing the total weight. Suppose that neither of the variable cycles passing through the super node for $x_i$ is included in $C$. Recall that the number of appearance of $x_i$ is $m_i$. We know that there will be at most $m_i$ cycles passing through the nodes associated with the variable $x_i$. Adding the variable cycle will break up at most $m_i$ equation cycles associated with $x_i$. We add that variable cycle and delete all the broken equation cycles. As the weight of the variable cycle is $m_i$, the total weight will not decrease. By repeating this process, we will get a good collection.

Therefore we can assume that there is a good collection of cycles with total weight $\geq 6m + (\frac{1}{2} + \epsilon)m$. The contribution of variable cycles is $3m$ exactly. Therefore there are at least $(\frac{1}{2} + \epsilon)m$ equation cycles in $C$. We assign $x_i = 0$ if the variable cycle passing through nodes on the right column is included in $C$ and assign $x_i = 1$ otherwise. As $C$ is a valid collection of cycles, all the constraints with equation cycles in $C$ are satisfied by the above assignments. Therefore at least $(\frac{1}{2} + \epsilon)m$ constraints are satisfied.

**Inapproximability ratio:** The total weight of cycles is at most $7m$. Together with Theorem 1, it is NP-HARD to distinguish whether the optimal solution is at least $\frac{6m + (1 - \epsilon)m}{7m} = 1 - \epsilon'$ or at most $\frac{6m + (\frac{1}{2} + \epsilon)m}{7m} = \frac{13}{14} + \epsilon'$, where $\epsilon' = \epsilon - \frac{1}{4}$. Therefore the MAX-WEIGHT-L-EXCHANGE problem is NP-HARD to approximate within $\frac{14}{13}$.

4 Max-Weight-3-Exchange

In this section, we prove that the MAX-WEIGHT-3-EXCHANGE problem is NP-HARD to approximate within a constant factor:

**Theorem 5** The MAX-WEIGHT-3-EXCHANGE problem is NP-HARD to approximate within $\frac{333}{334}$. More precisely, the GAP-MAX-WEIGHT-3-EXCHANGE-$[\frac{333}{334} + \epsilon, 1 - \epsilon]$ is NP-HARD for any small constant $\epsilon > 0$.

**Proof.** We reduce from a bounded occurrence version of the MAX-3LIN-2(3) problem:

**Construction:** Given an instance of the MAX-3LIN-2(3) with $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ constraints
The four cycles with length $L$ need not be considered since the constraint involving the exactly twice as the number of appearances of $x$. Note that the total weight of cycles with thin edges illustrated in Figure 3 depending on the number of appearances. As shown in [Berman and Karpinski, 1998], it is NP-HARD to distinguish whether the optimal solution satisfies $\geq (1 - \epsilon)m$ constraints or $\leq (\frac{61 - \epsilon}{62} + \epsilon)m$ constraints for any constant $\epsilon > 0$.

For each variable $x$, we construct a variable gadget as illustrated in Figure 3 depending on the number of appearances. Note that the total weight of cycles with thin edges and the total weight of cycles with thick edges are both exactly twice as the number of appearances of $x_i$.

For each equation constraint, we construct a constraint gadget involving the 6 corresponding nodes. The constraint gadget is the same as before, illustrated in Figure 4. Each of the four cycles with length 3 has weight 1 and the longer cycles need not be considered since the constraint $L = 3$. The reduction can be computed in polynomial time.

**Completeness:** We show that if there is a good assignment to the MAX-3LIN-2(3) instance with more than $(1 - \epsilon)m$ satisfiable constraints, then we can construct a collection of cycles $C$ as follows: If $x_i = 0$, then we add the cycles with thick edges in the variable gadget, otherwise we add the cycles with thin edges; for each constraint, if it is satisfied, then add the corresponding cycle in the constraint gadget. As the sum of appearance of all variables is $3m$, the weight contributed by the cycles from variable gadget is $6m$. Moreover, there are at least $(1 - \epsilon)m$ cycles from the constraint gadgets. Therefore the total weight is at least $6m + (1 - \epsilon)m$.

**Soundness:** We claim that if there is a collection $C$ of cycles in the constructed graph with total weight $\geq 6m + (\frac{61}{62} + \epsilon)m$, we can find assignments to the variables such that $(\frac{61}{62} + \epsilon)m$ constraints are satisfied.

For any collection $C$ of cycles, if for all variable gadgets, either all the cycles with thin edges are in $C$ or all the cycles with thick edges are in $C$, then we call $C$ a good collection. Otherwise we say $C$ is bad.

We claim that for any bad collection $C$, we can adjust $C$ to a good collection $C'$, while the total weight will not decrease. We assume that there is a good collection of cycles with weight $\geq 6m + (\frac{61}{62} + \epsilon)m$. As the contribution by cycles from variable gadgets is exactly $6m$, there are at least $(\frac{61}{62} + \epsilon)m$ cycles from constraint gadgets in $C$. We construct an assignment as following: for each $x_i$, if all the thin edges are in $C$, then we assign $x_i = 1$; otherwise we assign $x_i = 0$. For each cycle from constraint gadgets in $C$, the corresponding constraint is satisfiable under the above assignment. So at least $(\frac{61}{62} + \epsilon)m$ constraints are satisfied.

**Inapproximability ratio:** The total weight is at most $7m$. Together with Theorem 2, it is NP-HARD to distinguish whether the optimal solution is at least $\frac{6m + (1 - \epsilon)m}{7m} = 1 - \epsilon'$ or at most $\frac{6m + (\frac{61}{62} + \epsilon)m}{7m} = \frac{433}{434} + \epsilon'$, where $\epsilon' = \frac{\epsilon}{7}$. Therefore the MAX-WIGHT-3-EXCHANGE problem is NP-HARD to approximate within $\frac{434}{433}$.

**5 Max-Size-3-Exchange**

In this section we prove an inapproximability result for the MAX-SIZE-3-EXCHANGE problem. The unweighted exchange problem is a special case of the weighted exchange problem with equal weights on each edge, therefore the weighted exchange problem is even harder. Thus this inapproximability result also holds for MAX-WIGHT-3-EXCHANGE.

**Theorem 6** The MAX-SIZE-3-EXCHANGE problem is NP-HARD to approximate within $\frac{696}{700}$. More precisely, the GAP-MAX-SIZE-3-EXCHANGE-$\frac{696}{700} + \epsilon, \frac{696}{700} - \epsilon$ is NP-HARD for any small constant $\epsilon > 0$.

**Proof.** We reduce from the GAP-MAXIMUM-3-DIMENSIONAL-MATCHING problem.

**Construction:** In [Berman and Karpinski, 2003], a family of 3-DIMENSIONAL-MATCHING instances have been constructed, where $|X| = |Y| = |Z| = 100k$ for some integer $k$, with $m = 200k$ triples, and each element in $X \cup Y \cup Z$ appears in exactly 2 triples. It is NP-HARD to
Figure 4: The construction for a triple \( t = \{x_a, y_b, z_c\} \)

distinguish whether the size of the maximum matching is at least \((98 - \epsilon)k\) or at most \((97 + \epsilon)k\).

Given such an instance, we construct a graph \( G \) as follows: for each element in \( X \cup Y \cup Z \), we have a node for the element with the same label; for each triple \( t = \{x_a, y_b, z_c\} \in T \), we add a gadget as shown in Figure 4. There are 7 cycles in the gadget, we call the three cycles at the bottom down cycles, the three cycles in the middle upper cycles, and the cycle on the top triple cycle. All cycles has uniform weight. The reduction can be computed in polynomial time.

Completeness: If there is a matching \( T' \subseteq T \) with size \((98 - \epsilon)k\), we construct a collection \( C \) of cycles as followings: for each triple \( t = \{x_a, y_b, z_c\} \), if \( t \in T' \), then we add the triple cycle and the three down cycles to \( C \). Otherwise we add the three upper cycles to \( C \). As \( T' \) is a matching, all the cycles in \( C \) are vertex disjoint. The total number of cycles in \( C \) is at least \( 200k + 3 + (98 - \epsilon)k = (698 - \epsilon)k \).

Soundness: If there is a collection \( C \) of disjoint cycles of size at least \((697 + \epsilon)k\), we construct a matching \( T' \subseteq T \) with size at least \((97 + \epsilon)k\).

For the gadget for any triple, if the corresponding triple cycle is in \( C \), then the three upper cycles are not in \( C \); if any of the down cycles is not in \( C \), then there are at most 3 cycles in \( C \) within this gadget, therefore we can replace these cycles with the three upper cycles, and the number of cycles will not decrease. If the triple cycle is not in \( C \), then we can just choose the 3 upper cycles, making no effects on other gadgets, and the number of cycles will not decrease. After the adjusting processes, within each gadget, either the three upper cycles are chosen, or the triple cycle and the three down cycles are chosen. Therefore there are at least \((97 + \epsilon)k\) triple cycles in \( C \). All the corresponding triples are pairwise disjoint, otherwise the down cycles will not be disjoint. So there is a matching of size at least \((97 + \epsilon)k\).

Inapproximability ratio: The total number of cycles is at most \( 700k \). Together with Theorem 3, it is NP-HARD to distinguish whether the optimal solution is at least \( \frac{698}{700} k \) or at most \( \frac{697}{700} k \), where \( \epsilon' = \frac{\epsilon}{700} \). Therefore it is NP-HARD to approximate the MAX-SIZE-3-EXCHANGE problem within \( \frac{698}{697} \).

6 Algorithms for Max-Size-3-Exchange

In this section, we mainly focus on the MAX-SIZE-3-EXCHANGE problem. Previously, we have shown that it is NP-HARD to approximate within a factor of \( \frac{698}{697} \). Now we propose two simple and practical algorithms.

6.1 Algorithm 1 via greedy search

We present the basic greedy algorithm first. For the MAX-SIZE-3-EXCHANGE problem, we figure out all the cycles with length 2 or 3 and initialize our solution to be an empty set. Then we pick up the cycles with length 2 or 3 one by one and check whether it is available to be appended in the solution. After this greedy process, the solution is the final result of the algorithm.

Theorem 7 The approximation ratio of the basic greedy algorithm is 3.

Proof. For any given graph \( G \) in the MAX-SIZE-3-EXCHANGE problem, suppose the optimal solution is \( T \). Assume that there are \( d \) and \( t \) cycles with length 2 and 3 respectively in the optimal solution so that the size of solution is \( |T| = 2 \times d + 3 \times t \). We denote the approximation result by \( T' \). It is clear that for any cycles with length 2 or 3, there is at least one node which is in \( T' \). As a result, we have \( |T'| \geq d + t \) and

\[
\frac{|T|}{|T'|} \leq \frac{2 \times d + 3 \times t}{d + t} \leq 3.
\]

So the approximation ratio is 3.

We now add an important heuristics that improves the performance of the basic greedy algorithm. We consider the degree of each node in the graph. Intuitively, the smaller the degree of node is, the more difficult it is to be chosen in a cycle. The strategy is that we pick the cycles with length 3 first and pick up the cycles with smaller degrees in priority by ordering the nodes according to the number of degrees.

Given a graph \( G \), calculate the number of in-degree and out-degree for each node. Sort the order of nodes increasingly by \#(indegree) \times \#(outdegree). Find out all the cycles with length 3 in \( G \) and sort them with lexicographical order. From the smallest number to the biggest number, pick up the cycle if it is available, and delete the nodes in \( G \). After that, using the same method, find out all the cycles with length 2 in the remaining graph \( G \), then sort them and pick up one by one if possible. At last, output all the cycles which we have picked up as final results. In the experiments section, we will implement the algorithm above and test its performance using simulated data.
6.2 Algorithm 2 via maximum matching

Based on the fact that given a graph, the optimal solution of the MAX-SIZE-2-EXCHANGE is close to the solution of MAX-SIZE-EXCHANGE problem [Abraham et al., 2007], we propose a straightforward method to improve the results of MAX-SIZE-2-EXCHANGE. The idea is that, after we figure out the maximum matching, we use the remaining nodes and the matching edges to constitute cycles with length 3 as far as possible. For this purpose, we create a weighted graph and compute the maximum matching for the second time in order to improve the opportunity of using the remaining nodes.

Given a graph $G$, find the maximum matching in the bidirectional edge of $G$ first. Suppose $A$ is the set of nodes which are not in the maximum matching and $B$ is the nodes which are in the maximum matching. As shown in Figure 5, the colours of set $A$ and $B$ are green and yellow respectively. For each node $u$ in $B$, we construct a new node $u'$ in $G'$. For any cycle $(u, v)$ in $B$, we add a edge $(u', v')$ into $G'$ with a constant weight $w$ ($w = 10$ in Figure 5). For any node $a$ in $A$, if node $a, u, v$ can constitute a cycle, then add a edge $(u', v')$ with weight 1. The weights of the same edge will be added up. Then find the maximum matching of $G'$ as illustrated by the red edges. Assume the matching is $M$. Construct a bipartite graph $H$. The nodes in the left side stand for the nodes of $A$ while the right side represent the edges in $M$. If the node and the edge can constitute a cycle in the original graph $G$, then add the edge in $H$. At last, find the maximum matching of $H$ as shown by the blue edges. For any matching edge in $H$, output the corresponding cycle with length 3 in $G$. For any node in the right side of $H$ but not in the matching edge, output the corresponding cycle with length 2 if exists.

Both algorithms 1 and 2 are easy to implement and run in polynomial time. We will show in the next section that the two simple algorithms yield good experimental performance.

7 Experimental results

In this section, we implement the algorithms proposed in the previous section and compare their performance both in terms of running time and solution quality.

7.1 Experiments setup

All our experiments are performed in Linux (openSUSE 13.1), using a PC with four 3.2GHz Intel i5-3470 processors, and 4GB of RAM. Our experimental data is carefully simulated based on the statistics of US and China populations. We simulate the US data according to UNOS waiting list and living donors, and simulate China data based on the transplant researches [Tan et al., 2006; Tu et al., 2005].

Since it is NP-HARD to figure out the optimal solution in polynomial time, we use the following method to analyse the experimental performance. Both MAX-SIZE-2-EXCHANGE and MAX-SIZE-EXCHANGE can be solved in polynomial time using the maximum matching technique [Abraham et al., 2007]. For any given graph, since the size of MAX-SIZE-3-EXCHANGE will not be smaller than the MAX-SIZE-2-EXCHANGE and the size of MAX-SIZE-EXCHANGE will be the largest among them, the results of MAX-SIZE-2-EXCHANGE and MAX-SIZE-EXCHANGE can be considered as the lower bound and the upper bound of MAX-SIZE-3-EXCHANGE problem respectively. We compare the solution of our algorithms with these bounds in order to analyse the performance.

For a particular size of nodes, we randomly generate 10 copies of the simulated graph and calculate the average of solutions of each algorithm.

7.2 Experimental results

Figure 6 and 7 show the quality of solutions for each size of graph in proportion to MAX-SIZE-EXCHANGE in the US data and the China data respectively. Figure 6 and 7 show the running time for each size of nodes in the US data and the China data respectively.

The difference between the US data and China data is that the simulated graph of the China data is sparser than the US data. This leads to the result that the running time in the US data is more than in the China data while the result of the US data is higher than the China data. The approximation ratio of algorithm 1 is 3, while algorithm 2 runs faster and yields better solution.
As shown in Figure 8 and 9, the solutions of our algorithms are extremely close to the upper bound. In particular, algorithm 2 has even figured out the optimal solution for 5 times out of 10 test points in the case that the size of nodes is 2000 in the US data. Both algorithms return within two minutes.

To sum up, both algorithm 1 and 2 are easy to implement and yield good experimental performance on simulated data.

8 Conclusion and future work

We explore computational complexity of the clearing problem in the kidney exchange market. Our inapproximability results, in comparison to the best existing ones, are summarized in Table 1. We have proposed two algorithms which run in polynomial time and perform well on the simulated data as illustrated in Figure 8. Both of these two algorithms are easy to implement, and give the solution which are very close to the optimal. We make a conclusion that, the kidney exchange problem in practice can be solved by a satisfactory solution using the practical algorithms.

There are several exciting directions for future research. First of all, we are interested in closing the gap between approximation and inapproximability. In particular, we are interested in improving the $\frac{14}{13}$ result. In addition, we are interested in the complexity of the clearing problems in similar matching markets. From a practical perspective, to test the average performance, we are going to experimentally test the new algorithms using real data.

References

[Abraham et al., 2007] David J. Abraham, Avrim Blum, and Tuomas Sandholm. Clearing algorithms for barter exchange markets: enabling nationwide kidney exchanges. In ACM EC, pages 295–304, 2007.

[Arora and Safra, 1998] Sanjeev Arora and Shmuel Safra. Probabilistic checking of proofs: A new characterization of np. Journal of the ACM (JACM), 45(1):70–122, 1998.

[Arora et al., 1998] Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy. Proof verification and the hardness of approximation problems. Journal of the ACM (JACM), 45(3):501–555, 1998.

[Ashlagi et al., 2010] Itai Ashlagi, Felix A. Fischer, Ian A. Kash, and Ariel D. Procaccia. Mix and match. In ACM EC, pages 305–314, 2010.

[Ashlagi et al., 2012] Itai Ashlagi, David Gamarnik, Michael A. Rees, and Alvin E. Roth. The need for (long) chains in kidney exchange. Working Paper 18202, National Bureau of Economic Research, July 2012.

[Awasthi and Sandholm, 2009] Pranjal Awasthi and Tuomas Sandholm. Online stochastic optimization in the large: Application to kidney exchange. In IJCAI, pages 405–411, 2009.

[Berman and Karpiński, 2003] Piotr Berman and Marek Karpiński. Improved approximation lower bounds on small occurrence optimization. Inst. für Informatik, 2003.

[Berman and Karpinski, 1998] Piotr Berman and Marek Karpiński. On some tighter inapproximability results, further improvements. 1998.

[Biro and Cechlarova, 2006] Peter Biro and Katarina Cechlarova. Inapproximability of the kidney exchange problem. Institute of Mathematics, PJ Safarik University, Slovakia. Mimeo, 2006.

[Dickerson et al., 2012a] John P. Dickerson, Ariel D. Procaccia, and Tuomas Sandholm. Dynamic matching via weighted myopia with application to kidney exchange. In AAAI, 2012.

| Problem               | Our ratio | Previous   |
|-----------------------|-----------|------------|
| MAX-WEIGHT-L-EXCHANGE | 14/13     | APX-HARD   |
| MAX-WEIGHT-3-EXCHANGE | 434/433   | APX-HARD   |
| MAX-SIZE-3-EXCHANGE   | 098/697   | APX-HARD   |
[Dickerson et al., 2012b] John P. Dickerson, Ariel D. Procaccia, and Tuomas Sandholm. Optimizing kidney exchange with transplant chains: theory and reality. In AAMAS, pages 711–718, 2012.

[Dickerson et al., 2013] John P. Dickerson, Ariel D. Procaccia, and Tuomas Sandholm. Failure-aware kidney exchange. In ACM Conference on Electronic Commerce, pages 323–340, 2013.

[Dickerson et al., 2014] John P Dickerson, Ariel D Procaccia, and Tuomas Sandholm. Price of fairness in kidney exchange. In Proceedings of the 2014 international conference on Autonomous agents and multi-agent systems, pages 1013–1020. International Foundation for Autonomous Agents and Multiagent Systems, 2014.

[Gale and Shapley, 1962] D. Gale and L.S. Shapley. College admissions and the stability of marriage. American Mathematical Monthly, 69(1):9–15, 1962.

[Håstad, 2001] Johan Håstad. Some optimal inapproximability results. Journal of the ACM (JACM), 48(4):798–859, 2001.

[Karp, 1972] Richard M Karp. Reducibility among combinatorial problems. Springer, 1972.

[Li et al., 2014] Jian Li, Yicheng Liu, Lingxiao Huang, and Pingzhong Tang. Egalitarian pairwise kidney exchange: fast algorithms via linear programming and parametric flow. In International conference on Autonomous Agents and Multi-Agent Systems, AAMAS ’14, Paris, France, May 5-9, 2014, pages 445–452, 2014.

[Roth and Sotomayor, 1992] A.E. Roth and M.A.O. Sotomayor. Two-sided matching: A study in game-theoretic modeling and analysis, volume 18. Cambridge University Press, 1992.

[Roth et al., 2004] A.E. Roth, T. Sönmez, and M. Utku Ünver. Kidney exchange. Quarterly Journal of Economics, 119:457–488, 2004.

[Roth et al., 2005] A.E. Roth, T. Sönmez, and M. Utku Ünver. Pairwise kidney exchange. Journal of Economic Theory, 125(2):151–188, 2005.

[Tan et al., 2006] J.-m. Tan, Y.-c. Zhou, and X.-d. Tang. Tissue typing technology and clinical application. People’s Medical Publishing House, 2006.

[Tu et al., 2005] B. Tu, J.-g. Chen, and F. Wang. The distribution of abo bloodtype in kidney transplantation cases. Journal of Chinese eugenics and heredity, 13(15):110–111, 2005.

[Vazirani, 2001] Vijay V Vazirani. Approximation algorithms. springer, 2001.