A class of non-linear fractional-order system stabilisation via fixed-order dynamic output feedback controller

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Abstract—This paper investigates the robust stabilisation of a class of fractional-order non-linear system with positive real uncertainty via fixed-order dynamic output feedback controller in terms of linear matrix inequalities (LMIs), the systematic stabilisation algorithm design for low-order controller based on direct Lyapunov approach is proposed for uncertain fractional-order systems. In the presented algorithm the conditions containing the bilinear variables are decoupled into separate conditions without imposing equality constraints or considering an iterative search of the controller parameters. There is no any limiting constraint on state space matrices and also we assumed the most complete output feedback controller. Simulations results are given to approve the effectiveness and the straightforwardness of the proposed design.

Keywords— Dynamic output feedback, Fractional-order systems, Linear matrix inequalities (LMIs), Nonlinear

I. INTRODUCTION

In recent decades, study of fractional-order systems has been expanded significantly. For example hereditary and long memory attribute of systems, such as viscoelastic polymers [1], biomedical applications [2], semi-infinite transmission lines with losses [3], dielectric polarization [4], have been described with fractional-order operators. More, stability analysis of fractional-order systems have been attracted considerable interests, where several literatures addressing this topic have been released [5–8] and subsequently fractional-order controllers design and implementation in system control field have become commonplace [9–11]. A fractional-order PI/sup Δ/ controller was proposed in [9]. Stability and stabilisation of fractional-order interval systems are studied in [11].

Design of fractional-order PIΔDμ controllers with an improved differential evolution is proposed in [12]. The analytical stability bound of delayed time invariant fractional-order systems is obtained by using Lambert function in [13]. Necessary and sufficient stability conditions of fractional-order interval linear systems are stabilised in [14]. In [14] the necessary and sufficient stability conditions of fractional-order systems are directly extended to the robust stability condition of fractional-order interval polynomial systems. Estimation of the system states and observer-based stabilisation were investigated in [15,16]. In [17] using continuous frequency distribution, the stability conditions of a class of Lipschitz nonlinear fractional-order systems based on indirect approach to Lyapunov stability are derived.

The definition of Mittag-Leffler stability definition was proposed in [18], and also fractional Lyapunov direct method was introduced. By using of Mittag-Leffler function, Laplace transform, and the generalized Gronwall inequality, a new sufficient condition ensuring local asymptotic stability and stabilization of a class of fractional-order nonlinear systems with fractional-order 1 < α < 2 is proposed in [19]. Stability analysis of fractional-order systems is studied in [20], in [20] an extension of Lyapunov direct method for fractional-order systems is proposed. Moreover the studies of Li, Wang and Lu [10] focuses on the observer-based stability problem of a class of non-linear fractional-order uncertain systems with admissible time-variant uncertainty. The proposed method therein is used for stabilisation of a class of nonlinear fractional-order system by assuming that input matrix of the system is of full row rank.

Note that most of the mentioned works focuses on stability study of linear fractional-order systems in which state feedback control law is the most existed control law. Existence of some technical and economic limitations makes it difficult to obtain the system states in practical applications. Output feedback controller eliminates mentioned problems of control and besides that among output feedback controllers, dynamic ones have more degrees of freedom in controller designing procedure and subsequently satisfying control objectives compared with static ones [21]. Due to achieving control objectives most of dynamic controller design methods lead to high order controllers, where High order controllers are not preferable along of costly implementation and maintenance, high fragility, and potential numerical errors [22]. Given that closed-loop performance could not be guaranteed through order reduction methods, it is worthwhile to have a solution to design a controller with low and fixed-order which can be as small as possible to satisfy control objectives [23-24].

To the best of our knowledge, there is few result on designing dynamic output feedback controller for the stability of nonlinear fractional-order systems with positive real uncertainty in the literature, this motivate us for the study of this paper. This paper investigates the fixed predetermined order dynamic output feedback controller for the robust stabilisation of fractional-order nonlinear systems along with positive real uncertainty and Lipschitz nonlinearities in the states and inputs. It should be mentioned that positive real uncertainty is commonplace in many real systems and using positive real uncertainty to model the uncertainty is a suitable way for considering phase information [11,25]. No limiting constraints on the state space matrix, assumed in [10], are considered and also the complete model of dynamic output feedback controller is taken. Notwithstanding this, results are given in terms of linear matrix inequalities (LMIs) where the design parameters can be easily obtained by accessing the feasibility of LMI constraints through optimisation parsers and solvers.
The rest of this paper organised as follows: in Section 2, some preliminaries and problem formulation are presented. The proposed fixed-order dynamic output feedback controller with the design algorithm of controller for robust stabilisation of nonlinear fractional-order systems with positive real uncertainty are derived in Section 3. Some numerical examples are provided in Section 4 to illustrate the effectiveness of proposed method. Eventually section 5 draws the conclusion.

II. PROBLEM FORMULATION AND PRELIMINARIES

Some mathematical notations that are used throughout this paper, are defined here. \( A \otimes B \) represents the Kronecker product of matrices A and B. The transpose of M is denoted by \( M^T \) and \( \text{sym}(M) \) stands for \( M + M^T \). The notation * denotes symmetric matrix. Pseudo inverse of a given non-square matrix \( A_{nxm} \) is shown by \( A^\dagger \).

Fractional-order nonlinear system with the following dynamic is considered

\[
D^q x(t) = Ax(t) + Bu(t) + f(x(t), u(t))
\]

where

\[
\dot{\mathcal{A}} = A + \Delta A
\]

\[
\dot{\mathcal{B}} = B + \Delta B
\]

with initial condition

\[
x(0) = x_a
\]

where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p \) are pseudo state, input, measured output, respectively. \( \mathcal{A} \in \mathbb{R}^{nxn}, \mathcal{B} \in \mathbb{R}^{nxn} \) are known constant matrices, and \( \phi(\cdot) : [\mathbb{R}^n \times \mathbb{R}^m] \to \mathbb{R}^n \), is nonlinear function. \( \Delta \mathcal{A} \in \mathbb{R}^{nxn} \) and \( \Delta \mathcal{B} \in \mathbb{R}^{nxn} \) are time-invariant matrices, with parametric uncertainty. \( q \) is the fractional derivative order, there are several definitions for fractional-order derivative, among them Grünwald-Letnikov, Riemann-Liouville and Caputo are most commonly referred. However, since Caputo definitions initial condition is similar to integer orders one as a physical aspect, Caputo definition is used in this with the following definition

\[
D^q x(t) = \frac{1}{\Gamma(q - a)} \int_{t-a}^{t} (t-\eta)^{q-a-1} f(\eta) d\eta,
\]

where \( \Gamma(\cdot) \) is Gamma function defined by \( \Gamma(\epsilon) = \int_0^\infty e^{-t^r} t^{r-1} dt \) and \( \overline{n} \) is the smallest integer that is equal or greater than \( q \).

**Lemma 1** [10] Let \( f : \mathbb{R} \to \mathbb{R}^n \) be piecewise continuous respect to \( t \), where \( \mathbb{R}_e = \{(t,x) : 0 \leq t \leq a \text{ and } \|x-x_0\| \leq b\} \), \( f = [f_1, ..., f_m]^T \), \( x \in \mathbb{R}^n \) and \( \|f(t,x)\| \leq M \) on \( \mathbb{R}_e \). Then, there exists at least one solution for the system of fractional differential equations given by

\[
D^q x(t) = f(t,x(t))
\]

with the initial condition

\[
x(0) = x_a
\]

on \( 0 \leq t \leq \beta \) where \( \beta = \min(a, [(b/M)\Gamma(q+1)^{1/q}]) \), \( 0 < q < 1 \).

**Lemma 2** [10] Consider initial fractional problem (4) and (5) with \( 0 < q < 1 \) and assume that Lemma 1 conditions hold. Let

\[
g(v,x,(u,v)) = f\left(t - (t^q - t^q(q + 1))^{1/q}, x(t - t^q(q + 1))^{1/q}\right)
\]

then \( x(t) \), is given by

\[
x(t) = x_a + \int_{0}^{t} x(t) dt
\]

where \( x_a(u) \) can be obtained by solving the following integer order differential equation

\[
\frac{dx(u)}{dt} = g(u,x,(u,v))
\]

(6)

System matrices \( \mathcal{A}, \mathcal{B}, \mathcal{C} \), nonlinear function \( \phi(\cdot) \) and uncertainty matrices \( \Delta \mathcal{A} \) and \( \Delta \mathcal{B} \) are assumed to satisfy the following assumptions.

**Assumption 1.** The pair of \( (\mathcal{A}, \mathcal{B}) \) and \( (\mathcal{A}, \mathcal{C}) \) are controllable and observable, respectively.

**Assumption 2.** \( \Delta \mathcal{A} \) and \( \Delta \mathcal{B} \) are time-invariant matrix of the following form:

\[
[\Delta \mathcal{A} \Delta \mathcal{B}] = \mathcal{M} \Delta(\sigma)[N_1 \ N_2]
\]

(7)

\[
\Delta(\sigma) = Z(\sigma)[I + JZ(\sigma)]^{-1}
\]

(8)

\[
\text{Sym}(\sigma) > 0,
\]

(9)

where \( \mathcal{M} \in \mathbb{R}^{nxm}, N_1 \in \mathbb{R}^{m_0 \times n}, N_2 \in \mathbb{R}^{m_0 \times m} \) are real known matrices. The uncertain matrix \( Z(\sigma) \in \mathbb{R}^{m_0 \times m_0} \) satisfies

\[
\text{Sym}(Z(\sigma)) \geq \mu
\]

(10)

where \( \sigma \in \Omega \), with \( \Omega \) being a compact set.

**Remark 1.** Condition (9) guarantees that \( I + JZ(\sigma) \) is invertible for all \( Z(\sigma) \) satisfying (10). Therefore \( \Delta(\sigma) \) in (7) is well defined (8).

**Assumption 3.** Nonlinear function \( \phi(x(t), u(t)) \) is Lipschitz on \( x(t) \) with Lipschitz constant \( \xi \)

\[
\|\phi(x_1(t), u_1(t)) - \phi(x_2(t), u_2(t))\| \leq \xi \|x_1(t) - x_2(t)\|
\]

(11)

for all \( x_1(t), x_2(t) \in \mathbb{R}^n \)

\[
\|\phi(0,0)\| = 0
\]

(12)

**Lemma 3** [6] Let \( \mathcal{A} \in \mathbb{R}^{nxn}, 0 < q < 1 \) and \( \theta = (1 - q)/2 \). The fractional-order system \( D^q x(t) = \mathcal{A} x(t) \) is asymptotically stable if and only if there exist a positive definite Hermitian matrices \( X = X^T > 0, X \in \mathbb{C}^{nxn} \) such that

\[
(rX + rX)^TA + A(rX + rX)^T < 0
\]

(13)

Where \( r = e^{\theta t} \).

Notations: In this paper \( A \otimes B \) denotes the kroneeker product of matrices A and B, and the symmetric matrix M will be shown by \( \text{sym}(\cdot) \), which is defined by \( \text{sym}(M) = M^T + M \), and also I is the symbol of pseudo inverse of matrix.

**Lemma 4** [7] Let

\[
\Omega = \{\Delta \in \mathbb{R}^{m_0 \times m_0} | \Delta \text{ is subject to (7) - (9)}\}
\]

Then

\[
\lim_{\delta \to 0} \Delta \notin \lim_{\delta \to 0} \Delta \text{sym} \Delta^T \leq \text{Sym}(\Delta)
\]

**III. MAIN RESULT**

In this work we will study the stability and asymptotically stabilisation of FOMASs composed of (1), with fixed-order dynamic output feedback controller.

In order to achieve the objectives on system (1), we use the following non-fragile control protocol

\[
D^q x_a(t) = A x_a(t) + B_e y(t),
\]

\[
u = C x_a + D_e y(t),
\]

\[
x_a(0) = x_{a0}
\]

(14)
where \( x_e \in \mathbb{R}^{n_c} \) is controller pseudo state in which \( n_c \) is the controller order and \( A_c, B_c, C_c \) and \( D_c \) are controller matrices to be designed.

The closed-loop system is achieved as follows

\[
D^2X(t) = \phi(X(t), t) = A_{cl,d}X(t) + \begin{bmatrix} \phi(x(t), u(t)) \end{bmatrix} \\
X(0) = [x_0^T \ x_0^{T}]^T
\]  
\[
(15)
\]

where

\[
X(t) = [x_1^T(t) \ x_2^T(t)]^T, \quad A_{cl,d} = A_d - A_d
\]

\[
A_d = \begin{bmatrix} A + B D_c C & B_c \\ B_c & A_c \end{bmatrix}, \quad A_c = \begin{bmatrix} \Delta \bar{M} N \\ \bar{N} = \begin{bmatrix} M_f^T & 0 \\ 0 & 0 \end{bmatrix}, \quad 
(16)
\]

**Theorem 1.** Consider the nonlinear fractional-order multi-agent system (1) with output dynamic controller (14) is stabilised if there exist positive\(\tau, \mu\) and positive definite matrix \( P \in \mathbb{R}^{(n+n_c) \times (n+n_c)} \) such that the following matrix inequality holds

\[
\begin{bmatrix} A_{cl} & \rho \psi \phi & \rho \psi \phi \end{bmatrix} - \begin{bmatrix} P \phi & \rho \psi \phi \phi \end{bmatrix} - \begin{bmatrix} P \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi 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\phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi 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\[ -Q'Q \leq -\left( W_{\frac{1}{2}}(M^T + N^T) + W_{\frac{1}{2}}(M^T + N^T) - W_{\frac{1}{2}}(M^T) + W_{\frac{1}{2}}(N^T) \right) \]

\[ + \frac{1}{2} \Delta'(\sigma)M^T \quad \frac{1}{2} \Delta'(\sigma)N^T < 0 \]

\[ \Leftrightarrow -\text{sym} \left( W_{\frac{1}{2}}(M^T) + W_{\frac{1}{2}}(N^T) \right) \]

\[ -\Delta'(\sigma)M^T - P^T N W_{\frac{1}{2}}^T N P \quad + \text{Sym}(M^T(M^T) + N^T(N^T) \right) \]

\[ \Leftrightarrow -\Delta'(\sigma)M^T - P^T N W_{\frac{1}{2}}^T N P \quad + \text{Sym}(M^T(M^T) + N^T(N^T) \right) \]

\[ \leq 0 \]

(34)

it follows from Lemma 4 that \( \text{Sym}(\Delta(\sigma)) = -\Delta'(\sigma)W\Delta(\sigma) > 0 \), and the following inequality holds

\[ \text{Sym}(\Delta(\sigma)N^T) \leq \text{Sym}(\Delta(W^T)N^T) + \]

\[ P^T N W_{\frac{1}{2}}(M^T) + \frac{1}{2} \Delta'(\sigma)N^T \]

Inequality (35) is equivalent to that there exist \( \mu > 0 \) such that

\[ \text{Sym}(\Delta(\sigma)N^T) \leq \text{Sym}(\Delta(W^T)N^T) + \]

\[ P^T N W_{\frac{1}{2}}(M^T) + \frac{1}{2} \Delta'(\sigma)N^T \]

which is equivalent to that there exist and \( \mu > 0 \) such that

\[ \text{Sym}(\Delta(\sigma)N^T) \leq \]

\[ \left[ \begin{array}{ll}
\frac{1}{2} \Delta'(\sigma)N^T & P^T N W_{\frac{1}{2}}(M^T) \\
& \frac{1}{2} \Delta'(\sigma)N^T
\end{array} \right] \]

\[ \leq \]

\[ \left[ \begin{array}{ll}
\mu & -\mu \\
-\mu & \mu
\end{array} \right] \]

(37)

the only remaining challenge is to match matrices dimensions. Considering the equation (32) we rewrite the equation (37) as follows

\[ \left[ \begin{array}{ll}
\text{Sym}(\Delta(\sigma)N^T) & 0 \\
0 & \text{Sym}(\Delta(\sigma)N^T)
\end{array} \right] \leq \]

\[ \left[ \begin{array}{ll}
\mu & -\mu \\
-\mu & \mu
\end{array} \right] \]

(38)

Substituting (38) into inequality (31), and applying Schur complement completes the proof. ■

Since \( \mathcal{A}_{\phi} \) and \( \mathcal{N} \) containing varying terms, is multiplied by \( P \) the inequality (17) is bilinear matrix inequality (BMI).

To deal with this obstacle, the following theorem investigates the consensus problem of system (1) in term of LMI (linear matrix inequality).

**Theorem 2.** The output feedback controller (14) solves the stability problem of the system (1) with \( 0 < \alpha < 1 \), if there exist positive constants \( \tau \), \( \mu \) and positive definite matrices \( P_u \in \mathbb{R}^{m \times m} \), \( P_d \in \mathbb{R}^{n \times n} \) and matrices \( \mathcal{F} \in \mathbb{R}^{m \times n}, \mathcal{G} \in \mathbb{R}^{n \times m}, \mathcal{H} \in \mathbb{R}^{m \times p}, \mathcal{D} \in \mathbb{R}^{m \times p} \) such that the following matrix inequality holds

\[ \begin{bmatrix}
\lambda_{11} & P_u & P_d & N_u \\
\lambda_{12} & N_d & \lambda_{22} & 0
\end{bmatrix} < 0, \]

(39)

where

\[ \lambda_{11} = \left[ \begin{array}{c}
y_{11} + N_d y_{11} \quad P_u & \lambda_{11} \lambda_{12} \quad P_d & N_u
\end{array} \right] \quad \lambda_{22} = \left[ \begin{array}{c}
y_{22} \quad \lambda_{22}
\end{array} \right] \quad y_{11} = [q_1 \quad q_2 \quad 0]^T \]

and \( \lambda_{11}, \lambda_{12}, \lambda_{22} \) are as follows

\[ q_1 = N_1 P_u + N_2 D, \quad q_2 = N_3 \mathcal{E} \]

moreover the controller matrices \( \mathcal{A}_c, \mathcal{B}_c, \mathcal{C}_c \) and \( \mathcal{D}_c \) are as follows

\[ \mathcal{A}_c = \mathcal{W} P_u, \quad \mathcal{B}_c = \mathcal{W} P_u \mathcal{E} \]

\[ \mathcal{C}_c = \mathcal{C} P_u, \quad \mathcal{D}_c = \mathcal{D} P_u \mathcal{E} \]

(41)

**Proof.** According to the proof of Theorem 1, the output feedback controller (14) solves the consensus problem of the system (1) if the inequality (17) holds. To deal with multiplication of variables, according to \( P = \text{diag}(P_u, P_d) \) we expand the matrix \( \mathcal{P}_\mathcal{A}_{\mathcal{C}t} + \mathcal{A}_{\mathcal{C}t} P \)

\[ \mathcal{P}_\mathcal{A}_{\mathcal{C}t} + \mathcal{A}_{\mathcal{C}t} P = \]

\[ \left[ \begin{array}{ccc}
\lambda_{11} & \lambda_{12} & q_1 \\
\lambda_{21} & \lambda_{22} & q_2
\end{array} \right] \]

(42)

which completes the proof. ■

**Corollary 1.** [26] Consider fractional-order multi-agent system (1) without nonlinear term. The output dynamic controller makes the system in (1) asymptotically stable if there exist positive definite Hermitian matrix \( \mathcal{P} = \mathcal{P}^* \) in the form of

\[ \mathcal{P} = \text{diag}(\mathcal{P}_u, \mathcal{P}_d) \]

with \( \mathcal{P}_u \in \mathbb{R}^{n \times n}, \mathcal{P}_d \in \mathbb{R}^{n \times n} \), and matrices \( \mathcal{T}_i, i = 1, \ldots, 4 \) and a real constant \( \mu > 0 \) such that the following LMI constraint become feasible:

\[ \left[ \begin{array}{cc}
\lambda_{11} & \lambda_{13} \\
\lambda_{22} & -\mu
\end{array} \right] < 0, \]

(45)

where

\[ \lambda_{11} = \left[ \begin{array}{ccc}
\lambda_{11} & \lambda_{12} & q_1 \\
\lambda_{21} & \lambda_{22} & q_2
\end{array} \right] \quad q_1 = (1 - \mu) \pi / 2 \]

and \( \lambda_{11}, \lambda_{12}, \lambda_{22} \) are as follows

\[ \mathcal{A}_c = \mathcal{W} P_u, \quad \mathcal{B}_c = \mathcal{W} P_u \mathcal{E} \]

\[ \mathcal{C}_c = \mathcal{C} P_u, \quad \mathcal{D}_c = \mathcal{D} P_u \mathcal{E} \]

(47)

**Corollary 2.** Proposed methods in Theorem 1, Theorem 2 and Corollary 1 are applicable to the certain form of FO-LTI system (1) by solving the inequalities \( \lambda_{11} < 0, \lambda_{11} < 0 \) and \( \lambda_{11} \) respectively.

**Proof.** Assuming \( \mathcal{M} = 0 \) in the proof Theorem 1, Theorem 2 and Corollary 1, it can be easily obtained.
**Remark 1.** Special case of static output feedback controller for the stabilisation of system (1) can be obtained by solving proposed LMIs for $n_c = 0$.

### IV. SIMULATION

**A. Example 1**

We consider the following non-linear fractional-order system which is available in [10]

\[
D^q x(t) = A x(t) + B \dot{u}(t) + \phi(x(t), u(t)) \\
g(t) = C x(t)
\]

(48)

with the fractional-order $q = 0.9$ and

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix}, \quad B = [1 \ 0.5]^T, \quad C = [1 \ 1], \\
\phi(x(t), u(t)) = \begin{bmatrix} \sin(x_2) \\ -\sin(x_1) + 0.5\sin(x_2u(t)) \end{bmatrix}
\]

(49)

and the uncertainty parameters with

\[
M = \begin{bmatrix} 0.5 & 1 \\ -0.4 & 0.2 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0.5 & 1.5 \\ 0 & 0.5 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & -0.5 \end{bmatrix}^T, \quad J = I_2
\]

(50)

The time response of the system (48) without control input and $x_0 = [-0.3, 0.3]^T$, is demonstrated in Fig. 1 which shows that states are not convergent and the system (48) is not asymptotically stable.

![Fig. 1 Time response of 5 random systems (48) with positive real uncertainty with u(t) = 0](image)

Using Theorem 2, dynamic output feedback controllers of arbitrary orders parameters that stabilise the unstable nonlinear system (48), tabulated in Table 1. Time responses of uncertain nonlinear system (48) via controllers resulted in Table 1 are illustrated in Fig. 2. Results show that unstable system is stabilisable, even with lower orders of dynamic output feedback controllers and all the states asymptotically converge to zero. Nevertheless, comparing the dynamic controller result with the static one indicates that oscillation and settling time of the response of the nonlinear system via dynamic feedback controllers are better than the static output feedback.

**Table 1 controller parameters obtained by Theorem 2 for the system (48)**

| $n_c$ | $A_c$ | $B_c$ | $C_c$ | $D_c$ |
|-------|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     | -1.6  |
| 1     | -1.3  | -2.8  | 0.6   | -2.3  |

![Fig. 2 Time response of closed-loop system defined in (48) via obtained controllers in Table 1, with $n_c = 2$ (blue), $n_c = 1$ (red), and ordinary static output feedback controller (black)](image)

To study the robustness of the proposed method in Theorem 2, the resulted dynamic output feedback controller resulted through the Theorem 2 with $n_c = 2$ is utilized for 50 random systems with positive real uncertainties defined in (48). The output time responses depicted in Fig. 3. Results ensure that the proposed method for the control of uncertain systems is reliable and control protocol is effectively robust for the positive definite uncertainty defined in (7) to (9).

![Fig. 3 Output time response of 50 random system with positive real uncertainties defined in (48) via dynamic controller proposed in Theorem 2 with $n_c = 2$](image)

**B. Example 2**

The system (48) with the following system matrices is considered

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix}, \quad B = [1 \ 0.5]^T, \quad C = \begin{bmatrix} 1 & 2 \\ 0.5 & 1 \end{bmatrix}, \\
\phi(x(t), u(t)) = \begin{bmatrix} \sin(x_2) \\ -\sin(x_1) + 0.5\sin(x_2u(t)) \end{bmatrix}
\]

(51)

and the uncertainty parameters are

\[
M = \begin{bmatrix} 0.5 & 1 \\ -0.4 & 0.2 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0.5 & 1.5 \\ 0 & 0.5 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & -0.5 \end{bmatrix}^T, \quad J = I_2
\]

(52)

In this example we considered a system with rank It's noteworthy that for the abovementioned unstable system no observer-based feedback controller, using the method proposed in [10] can be designed because of the rank constraint on the system matrix $C$. According to Theorem 2 it can be concluded that system (51) is stabilisable by the

**Table 2 controllers obtained by Theorem 2 for the system of Example 2**

| $n_c$ | $A_c$ | $B_c$ | $C_c$ | $D_c$ |
|-------|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     | -0.4  |
| 1     | -1.4  | 0.1   | -0.3  | -1.5  |
| 2     | -1.3  | 0.4   | 0.1   | -1.5  |

![Figure showing the output time response of Example 2](image)
dynamic output feedback controllers with arbitrary orders tabulated in Table 2.

![Fig. 4 Time response of closed-loop system defined in (51) via obtained controllers in Table 2, with $n_c = 2$ (blue), $n_c = 1$ (red), and ordinary static output feedback controller (black).](image)

Results show that all the states asymptotically converge to zero. Also the effectiveness of proposed dynamic output feedback controller is clear in higher orders which responses have slighter oscillation and shorter settling time. The robustness of the proposed dynamic output feedback controller is studied for the 50 random system (51) with positive real uncertainty via controller in the Table 2 with $n_c = 2$, where the results are depicted in Fig. 5.

![Fig. 5 Output time response of 50 random system with positive real uncertainties defined in (51) via dynamic controller proposed in Theorem 2 with $n_c = 2$.](image)

V. CONCLUSION

In this paper, first fixed-order dynamic output feedback controller has been applied to a class of uncertain fractional-order nonlinear system with positive real uncertainty. Then the sufficient conditions for the robust stability of the nonlinear fractional-order system with dynamic output feedback controller with predetermined order, through the direct Lyapunov approach, are derived. The dynamic output feedback controller benefits are accessible, which its order can be set as ideal value in order to reach the desire performance. Note that there are no limitative constraints on the state space matrices of system and the most complete form of dynamic output feedback controller strategy is assumed in our design procedure. Moreover, the result of robust stabilisation is presented in term of LMI, which is straightforward to be utilised. Eventually, some numerical examples are presented to illustrate the effectiveness and advantages of the proposed method.

VI. REFERENCES

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