INTRINSICALLY HÖLDER SECTIONS IN METRIC SPACES

DANIELA DI DONATO

Abstract. We introduce a notion of intrinsically Hölder graphs in metric spaces. Following a recent paper of Le Donne and the author, we prove some relevant results as the Ascoli-Arzelà compactness Theorem, Ahlfors-David regularity and the Extension Theorem for this class of sections. In the first part of this note, thanks to Cheeger theory, we define suitable sets in order to obtain a vector space over \( \mathbb{R} \) or \( \mathbb{C} \), a convex set and an equivalence relation for intrinsically Hölder graphs. These last three properties are new also in the Lipschitz case. Throughout the paper, we use basic mathematical tools.

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1. Introduction

Starting to the seminal papers by Franchi, Serapioni and Serra Cassano [FSSC01, FSSC03b, FSSC03a] (see also [SCI6, FS16]), Le Donne and the author generalize the notion of intrinsically Lipschitz maps introduced in subRiemannian Carnot groups [ABB19, BLU07, CDPT07]. This concept was introduced in order to give a good definition of rectifiability in subRiemannian geometry after the negative result of Ambrosio and Kirchheim shown in [AK00] (see also [Mag04]) regarding the classical definition of rectifiability using Lipschitz maps given by Federer [Fed69]. The notion of rectifiable sets is a key one in Calculus of Variations and in Geometric Measure Theory. The reader can see [Pan04, CP06, Bat21, AM22a, AM22b, Mat93, Mar61, Mat75, NY18, DS91, DS93].

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In [DDLD22] we prove some relevant statements in metric spaces, like the Ahlfors-David regularity, the Ascoli-Arzelà Theorem, the Extension Theorem for the so-called intrinsically Lipschitz sections. Our point of view is to consider the graphs theory instead of the maps one. In a similar way, in this paper we give a natural definition of intrinsically Hölder sections which includes Lipschitz ones and we prove the following results using basic mathematical tools and getting short proofs.

1. Theorem 1.2, i.e., Compactness Theorem à la Ascoli-Arzelà for the intrinsically Hölder sections.
2. Theorem 1.3, i.e., Extension Theorem for the intrinsically Hölder sections.
3. Theorem 1.4, i.e., Ahlfors-David regularity for the intrinsically Hölder sections.
4. Proposition 3.4 states that the class of the intrinsically Hölder sections is a convex set.
5. Theorem 3.5 states that a suitable class of the intrinsically Hölder sections is a vector space over $\mathbb{R}$ or $\mathbb{C}$.
6. Theorem 4.2 gives an equivalence relation for a suitable class of the intrinsically Hölder sections.

The last three points are new results also in the context of Lipschitz sections.

Our setting is the following. We have a metric space $X$, a topological space $Y$, and a quotient map $\pi : X \to Y$, meaning continuous, open, and surjective. The standard example for us is when $X$ is a metric Lie group $G$ (meaning that the Lie group $G$ is equipped with a left-invariant distance that induces the manifold topology), for example a subRiemannian Carnot group, and $Y$ is the space of left cosets $G/H$, where $H < G$ is a closed subgroup and $\pi : G \to G/H$ is the projection modulo $H$, $g \mapsto gH$.

**Definition 1.1.** We say that a map $\varphi : Y \to X$ is an intrinsically $(L, \alpha)$-Hölder section of $\pi$, with $\alpha \in (0, 1]$ and $L > 0$, if
\[ \pi \circ \varphi = \operatorname{id}_Y, \]
and
\[ d(\varphi(y_1), \varphi(y_2)) \leq Ld(\varphi(y_1), \pi^{-1}(y_2))^{\alpha} + d(\varphi(y_1), \pi^{-1}(y_2)), \quad \text{for all } y_1, y_2 \in Y. \]

Here $d$ denotes the distance on $X$, and, as usual, for a subset $A \subset X$ and a point $x \in X$, we have $d(x, A) := \inf\{d(x, a) : a \in A\}$.

When $\alpha = 1$, a section $\varphi$ is intrinsically Lipschitz in the sense of [DDLD22]. In a natural way, the author introduced and studied other two classes of sections: the intrinsically quasi-symmetric sections [DD22a] and the intrinsically quasi-isometric sections [DD22c] in metric spaces.

In Proposition 2.3, we show that, when $Y$ is a bounded set, (1) is equivalent to ask that
\[ d(\varphi(y_1), \varphi(y_2)) \leq Kd(\varphi(y_1), \pi^{-1}(y_2))^{\alpha}, \quad \text{for all } y_1, y_2 \in Y, \]
for a suitable $K \geq 1$. This property will be useful in order to get Compactness Theorem à la Ascoli-Arzelà. Moreover, we underline that, in the case $\alpha = 1$ and $\pi$ is a Lipschitz quotient or submetry [BJL+99, VN88], the results trivialize, since in this case being intrinsically Lipschitz is equivalent to biLipschitz embedding, see Proposition 2.4 in [DDLD22].

The first series of our results is about the equicontinuity of intrinsically Hölder sections with uniform constant and consequently a compactness property à la Ascoli-Arzelà.
Theorem 1.2 (Equicontinuity and Compactness Theorem). Let $\pi : X \to Y$ be a quotient map between a metric space $X$ and a topological space $Y$.

(i) Every intrinsically Hölder section of $\pi$ is continuous.

Next, assume in addition that closed balls in $X$ are compact.

(ii) For all $K' \subset Y$ compact, $L \geq 1$, $\alpha \in (0, 1)$, $K \subset X$ compact, and $y_0 \in Y$ the set

$$\{ \varphi_{\pi, \alpha} : K' \to X \mid \varphi : Y \to X \text{ intrinsically } (L, \alpha)\text{-Hölder section of } \pi, \varphi(y_0) \in K \}$$

is equibounded, equicontinuous, and closed in the uniform-convergence topology.

(iii) For all $L \geq 1$, $\alpha \in (0, 1)$, $K \subset X$ compact, and $y_0 \in Y$ the set

$$\{ \varphi : Y \to X \mid \varphi \text{ intrinsically } (L, \alpha)\text{-Hölder section of } \pi, \varphi(y_0) \in K \}$$

is compact with respect to the topology of uniform convergence on compact sets.

Another crucial property of Hölder sections is that under suitable assumptions they can be extended. This property is much studied in the context of metric spaces if we consider the Hölder maps; the reader can see [AP20, LN05, Oht09] and their references. Our proof follows using the link between Hölder sections and level sets of suitable maps. This idea is widespread in the context of subRiemannian Carnot groups (see, for instance, ASCV06, ADDDLD20, DD20, Vit20). In next result, we say that a map $f$ on $X$ is $L$-biLipschitz on fibers (of $\pi$) if on each fiber of $\pi$ it restricts to an $L$-biLipschitz map.

Theorem 1.3 (Extensions as level sets). Let $\pi : X \to Y$ be a quotient map between a metric space $X$ and a topological space $Y$.

(i) If $Z$ is a metric space, $z_0 \in z$ and $f : X \to Z$ is $(\lambda, \beta)$-Hölder and $\lambda$-biLipschitz on fibers, with $\lambda > 0$ and $\beta \in (0, 1)$, then there exists an intrinsically $(\lambda^2, \beta)$-Hölder section $\varphi : Y \to X$ of $\pi$ such that

$$\varphi(Y) = f^{-1}(z_0).$$

(ii) Vice versa, assume that $X$ is geodesic and that there exist $k \geq 1, \alpha \in (0, 1)$, $\rho : X \times X \to \mathbb{R}$ $k$-biLipschitz equivalent to the distance of $X$, and $\tau : X \to \mathbb{R}$ is $(k, \alpha)$-Hölder and $k$-biLipschitz on fibers such that

1. for all $\tau_0 \in \mathbb{R}$ the set $\tau^{-1}(\tau_0)$ is an intrinsically $(k, \alpha)$-Hölder graph of a section $\varphi_{\tau_0} : Y \to X$;
2. for all $x_0 \in \tau^{-1}(\tau_0)$ the map $X \to \mathbb{R}, x \mapsto \delta_{\tau_0}(x) := \rho(x_0, \varphi_{\tau_0}(\pi(x)))$ is $k$-Lipschitz on the set $\{ |\tau| \leq \delta_{\tau_0} \}$.

Let $Y' \subset Y$ a set and $L \geq 1$. Then for every intrinsically $(L, \alpha)$-Hölder section $\varphi : Y' \to \pi^{-1}(Y')$ of $\pi|_{\pi^{-1}(Y')}$, $\pi^{-1}(Y') \to Y'$, there exists a map $f : X \to \mathbb{R}$ that is $(K, \alpha)$-Hölder and $K$-biLipschitz on fibers, with $K \geq 1$, such that

$$\varphi(Y') \subseteq f^{-1}(0).$$

In particular, each ‘partially defined’ intrinsically Hölder graph $\varphi(Y')$ is a subset of a ‘globally defined’ intrinsically Hölder graph $f^{-1}(0)$.

We underline that an important point is that the constant $\beta$ in (1.3(i)) does not change. Following again [DDLD22], we can prove an Ahlfors-David regularity for the intrinsically Hölder sections. Recall that in Euclidean case $\mathbb{R}^s$, there are $(L, \alpha)$-Hölder maps such that the $(s + 1 - \alpha)$-Hausdorff measure of their graphs is not zero and the $(s + 1)$-Hausdorff measure of their graphs is zero, we give the following result.
Theorem 1.4 (Ahlfors-David regularity). Let \( \pi : X \to Y \) be a quotient map between a metric space \( X \) and a topological space \( Y \) such that there is a measure \( \mu \) on \( Y \) such that for every \( r_0 > 0 \) and every \( x, x' \in X \) with \( \pi(x) = \pi(x') \) there is \( C > 0 \) such that

\[
\mu(\pi(B(x, r))) \leq C\mu(\pi(B(x', r))), \quad \forall r \in (0, r_0).
\]

Let \( \ell \in (0, \infty) \). We also assume that there is an intrinsically \((L, \alpha)\)-Hölder section \( \varphi : Y \to X \) of \( \pi \) such that \( \varphi(Y) \) is locally \((\ell + 1 - \alpha)\)-Ahlfors-David regular with respect to the measure \( \varphi_*\mu \).

Then, for every intrinsically \((L, \alpha)\)-Hölder section \( \psi : Y \to X \) of \( \pi \), the set \( \psi(Y) \) is locally \( \alpha(\ell + 1 - \alpha)\)-Ahlfors-David regular with respect to the measure \( \psi_*\mu \).

Namely, in Theorem 1.4 locally \( Q \)-Ahlfors-David regularity means that the measure \( \varphi_*\mu \) is such that for each point \( x \in \varphi(Y) \) there exist \( r_0 > 0 \) and \( C > 0 \) so that

\[
C^{-1}r^Q \leq \varphi_*\mu(B(x, r) \cap \varphi(Y)) \leq Cr^Q, \quad \text{for all} \ r \in (0, r_0).
\]

The same inequality will hold for \( \psi_*\mu \) with a possibly different value of \( C \) and \( Q \). See Section 7.

Finally, following Cheeger theory [Che99] (see also [Kei04, KM16]), we give an equivalent property of Hölder section which we will prove in Section 2. Here it is fundamental that \( Y \) is a bounded set.

Proposition 1.5. Let \( X \) be a metric space, \( Y \) a topological and bounded space, \( \pi : X \to Y \) a quotient map, \( L \geq 1 \) and \( \alpha, \beta, \gamma \in (0, 1) \). Assume that every point \( x \in X \) is contained in the image of an intrinsic \((L, \alpha)\)-Hölder section \( \psi_x \) for \( \pi \). Then for every section \( \varphi : Y \to X \) of \( \pi \) the following are equivalent:

1. for all \( x \in \varphi(Y) \) the section \( \varphi \) is intrinsically \((L_1, \beta)\)-Hölder with respect to \( \psi_x \) at \( x \);
2. the section \( \varphi \) is intrinsically \((L_2, \gamma)\)-Hölder.

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2. Equivalent definitions for intrinsically Hölder sections

Definition 2.1 (Intrinsic Hölder section). Let \((X, d)\) be a metric space and let \( Y \) be a topological space. We say that a map \( \varphi : Y \to X \) is a section of a quotient map \( \pi : X \to Y \) if

\[ \pi \circ \varphi = \text{id}_Y. \]

Moreover, we say that \( \varphi \) is an intrinsically \((L, \alpha)\)-Hölder section with constant \( L > 0 \) and \( \alpha \in (0, 1) \) if in addition

\[
d(\varphi(y_1), \varphi(y_2)) \leq Ld(\varphi(y_1), \pi^{-1}(y_2))^\alpha + d(\varphi(y_1), \pi^{-1}(y_2)), \quad \text{for all} \ y_1, y_2 \in Y.
\]

Equivalently, we are requesting that

\[
d(x_1, x_2) \leq Ld(x_1, \pi^{-1}(\pi(x_2)))^\alpha + d(x_1, \pi^{-1}(\pi(x_2))), \quad \text{for all} \ x_1, x_2 \in \varphi(Y).
\]

We further rephrase the definition as saying that \( \varphi(Y) \), which we call the graph of \( \varphi \), avoids some particular sets (which depend on \( \alpha, L \) and \( \varphi \) itself):
Proposition 2.2. Let \( \pi : X \to Y \) be a quotient map between a metric space and a topological space, \( \varphi : Y \to X \) be a section of \( \pi \), \( \alpha \in (0, 1) \) and \( L > 0 \). Then \( \varphi \) is intrinsically \((L, \alpha)\)-Hölder if and only if
\[
\varphi(Y) \cap R_{x,L} = \emptyset, \quad \text{for all } x \in \varphi(Y),
\]
where
\[
R_{x,L} := \left\{ x' \in X \mid Ld(x', \pi^{-1}(\pi(x))) + d(x', \pi^{-1}(x)) < d(x', x) \right\}.
\]

Proposition 2.2 is a triviality, still its purpose is to stress the analogy with the intrinsically Lipschitz sections theory introduced in [DDLD22] when \( \alpha = 1 \). In particular, the sets \( R_{x,L} \) are the intrinsic cones in the sense of Franchi, Serapioni and Serra Cassano when \( X \) is a subRiemannian Carnot group and \( \alpha = 1 \). The reader can see [DD22b] for a good notion of intrinsic cones in metric groups.

Definition 2.1 it is very natural if we think that what we are studying graphs of appropriate maps. However, in the following proposition, we introduce an equivalent condition of (7) when \( Y \) is a bounded set.

Proposition 2.3. Let \( \pi : X \to Y \) be a quotient map between a metric space \( X \) and a topological and bounded space \( Y \) and let \( \alpha \in (0, 1) \). The following are equivalent:

1. there is \( L > 0 \) such that
\[
d(\varphi(y_1), \varphi(y_2)) \leq Ld(\varphi(y_1), \pi^{-1}(y_2)) + d(\varphi(y_1), \pi^{-1}(y_2)), \quad \text{for all } y_1, y_2 \in Y.
\]
2. there is \( K \geq 1 \) such that
\[
d(\varphi(y_1), \varphi(y_2)) \leq Kd(\varphi(y_1), \pi^{-1}(y_2))^\alpha, \quad \text{for all } y_1, y_2 \in Y.
\]

**Proof.** (1) \( \Rightarrow \) (2). This is trivial when \( d(\varphi(y_1), \pi^{-1}(y_2)) \leq 1 \). On the other hand, if we consider \( y_1, y_2 \in Y \) and \( \bar{x} \in X \) such that \( d(\varphi(y_1), \pi^{-1}(y_2)) = d(\varphi(y_1), \bar{x}) > 1 \), then it is possible to consider \( \ell \) equidistant points \( x_1, \ldots, x_\ell \in X \) such that
\[
d(\varphi(y_1), \bar{x}) = d(\varphi(y_1), x_1) + \sum_{i=1}^{\ell-1} d(x_i, x_{i+1}) + d(x_\ell, \bar{x}),
\]
with \( d(\varphi(y_1), x_1) = d(x_1, x_{i+1}) = d(x_\ell, \bar{x}) \in (\frac{1}{\alpha}, 1) \). Here, \( \ell \leq \lceil d(\varphi(y_1), \bar{x}) \rceil + 1 \) depends on \( y_1, y_2 \). Here \( \lfloor k \rfloor \) is the integer part of \( k \). However, it is possible to choose \( k \in \mathbb{R}^+ \) defined as
\[
k := \sup_{y_1, y_2 \in Y} d(\varphi(y_1), \pi^{-1}(y_2)),
\]
such that \( k \) not depends on the points and \( \ell \leq k \). We notice that this constant is finite because, on the contrary, we get the contradiction \( \infty = d(\varphi(y_1), \pi^{-1}(y_2)) \leq d(\varphi(y_1), \varphi(y_2)) \). Hence,
\[
d(\varphi(y_1), \varphi(y_2)) \leq Ld(\varphi(y_1), \bar{x})^\alpha + d(\varphi(y_1), \bar{x})
\]
\[
= Ld(\varphi(y_1), \bar{x})^\alpha + d(\varphi(y_1), x_1) + \sum_{i=1}^{\ell-1} d(x_i, x_{i+1}) + d(x_\ell, \bar{x})
\]
\[
\leq Ld(\varphi(y_1), \bar{x})^\alpha + d(\varphi(y_1), x_1)^\alpha + \sum_{i=1}^{\ell-1} d(x_i, x_{i+1})^\alpha + d(x_\ell, \bar{x})^\alpha
\]
\[
\leq (L + 3([k] + 1))d(\varphi(y_1), \bar{x})^\alpha
\]
\[
=: Kd(\varphi(y_1), \bar{x})^\alpha.
\]
(2) $\Rightarrow$ (1). This is a trivial implication.

**Definition 2.4 (Intrinsic Hölder with respect to a section).** Given sections $\varphi, \psi : Y \to X$ of $\pi$. We say that $\varphi$ is intrinsically $(L, \alpha)$-Hölder with respect to $\psi$ at point $\hat{x}$, with $L > 0, \alpha \in (0, 1)$ and $\hat{x} \in X$, if

1. $\hat{x} \in \psi(Y) \cap \varphi(Y)$;
2. $\varphi(Y) \cap C_{\hat{x}, L}^\psi = \emptyset$,

where

$$C_{\hat{x}, L}^\psi := \{x \in X : d(x, \psi(\pi(x))) > Ld(\hat{x}, \psi(\pi(x)))^\alpha + d(\hat{x}, \psi(\pi(x)))\}.$$

**Remark 2.5.** Definition 2.4 can be rephrased as follows. A section $\varphi$ is intrinsically $(L, \alpha)$-Hölder with respect to $\psi$ at point $\hat{x}$ if and only if there is $\hat{y} \in Y$ such that $\hat{x} = \varphi(\hat{y}) = \psi(\hat{y})$ and

$$d(x, \psi(\pi(x))) \leq Ld(\hat{x}, \psi(\pi(x)))^\alpha + d(\hat{x}, \psi(\pi(x))), \quad \forall x \in \varphi(Y),$$

which equivalently means

$$d(\varphi(y), \psi(y)) \leq Ld(\hat{y}, \psi(y))^\alpha + d(\hat{y}, \psi(y)), \quad \forall y \in Y.$$

Notice that Definition 2.4 does not induce an equivalence relation because of lack of symmetry in the right-hand side of (11). In Section 4 we give a stronger definition in order to obtain an equivalence relation.

The proof of Proposition 2.6 is an immediately consequence of the following result.

**Proposition 2.6.** Let $X$ be a metric space, $Y$ a topological and bounded space, and $\pi : X \to Y$ a quotient map. Let $L \geq 1$ and $y_0 \in Y$. Assume $\varphi_0 : Y \to X$ is an intrinsically $(L, \alpha)$-Hölder section of $\pi$. Let $\varphi : Y \to X$ be a section of $\pi$ such that $x_0 := \varphi(y_0) = \varphi_0(y_0)$. Then the following are equivalent:

1. For some $L_1 \geq 1$ and $\beta \in (0, 1)$, $\varphi$ is intrinsically $(L_1, \beta)$-Hölder with respect to $\varphi_0$ at $x_0$;
2. For some $L_2 \geq 1$ and $\gamma \in (0, 1)$, $\varphi$ satisfies

$$d(x_0, \varphi(y)) \leq L_2d(x_0, \pi^{-1}(y))^\gamma, \quad \forall y \in Y.$$

Moreover, the constants $L_1$ and $L_2$ are quantitatively related in terms of $L$.

**Proof.** We begin recall that, by Proposition 2.3, (1) is equivalent to (8).

(1) $\Rightarrow$ (2). For every $y \in Y$, it follows that

$$d(\varphi(y), x_0) \leq d(\varphi(y), \varphi_0(y)) + d(\varphi_0(y), x_0)$$

$$\leq L_1d(\varphi(y), x_0)^\beta + d(\varphi_0(y), x_0)$$

$$\leq L_1Ld(x_0, \pi^{-1}(y))^{\beta\alpha} + Ld(x_0, \pi^{-1}(y))^\alpha$$

$$\leq L(L_1 + 1) \max\{d(x_0, \pi^{-1}(y))^{\beta\alpha}, d(x_0, \pi^{-1}(y))^\alpha\}$$

where in the first inequality we used the triangle inequality, and in the second one the intrinsic Hölder property (1) of $\varphi$. Then, in the third inequality we used the intrinsic Hölder property of $\varphi_0$. Here, noticing that $\beta\alpha < \alpha$, we have that if $d(x_0, \pi^{-1}(y)) \leq 1$, then

$$\max\{d(x_0, \pi^{-1}(y))^{\beta\alpha}, d(x_0, \pi^{-1}(y))^\alpha\} = d(x_0, \pi^{-1}(y))^{\beta\alpha}.$$
1, then using a similar technique using in Proposition 2.3 we obtain the same maximum with additional constant \( K := L + 3([k] + 1) \) where \( k \in \mathbb{R} \) is given by (9). Definitely,
\[
d(\varphi(y), x_0) \leq LK(L_1 + 1)d(x_0, \pi^{-1}(y))^{\beta \alpha}.
\]

(2) \implies (1). For every \( y \in Y \), we have that
\[
d(\varphi(y), \varphi_0(y)) \leq d(\varphi(y), x_0) + d(x_0, \varphi_0(y))
\]
\[
\leq L_2d(\varphi_0(y), x_0)^\gamma + d(\varphi_0(y), x_0),
\]
where in the first equality we used the triangle inequality, and in the second one we used (12). (Here it is better to use the first definition of intrinsically Hölder sections.) \( \square \)

It is easy to see that if \( \alpha = 1 \), then we get \( \beta = \gamma \) and so we have the following corollary.

**Corollary 2.7.** Let \( X \) be a metric space, \( Y \) a topological space, \( \pi : X \to Y \) a quotient map, \( L \geq 1 \) and \( \beta \in (0,1) \). Assume that every point \( x \in X \) is contained in the image of an intrinsic \( L \)-Lipschitz section \( \psi_x \) for \( \pi \). Then for every section \( \varphi : Y \to X \) of \( \pi \) the following are equivalent:

1. For all \( x \in \varphi(Y) \) the section \( \varphi \) is intrinsically \((L_1, \beta)\)-Hölder with respect to \( \psi_x \) at \( x \);
2. The section \( \varphi \) is intrinsically \((L_2, \beta)\)-Hölder.

### 2.1. Continuity

An intrinsically \((L, \alpha)\)-Hölder section \( \varphi : Y \to X \) of \( \pi \) is a continuous section. Indeed, fix a point \( y \in Y \). Since \( \pi \) is open, for every \( \varepsilon \in (0,1) \) and every \( x \in X \) such that \( x = \varphi(y) \) we know that there is an open neighborhood \( U_x \) of \( \pi(x) = y \) such that
\[
U_x \subset \pi(B(x, [\varepsilon/(L + 1)]^{1/\alpha})).
\]
Hence, if \( y' \in U_x \) then there is \( x' \in B(x, [\varepsilon/(L + 1)]^{1/\alpha}) \) such that \( \pi(x') = y' \). That means \( x' \in \pi^{-1}(y') \) and, consequently,
\[
d(\varphi(y), \varphi(y')) \leq Ld(\varphi(y), \pi^{-1}(y'))^\alpha + d(\varphi(y), \pi^{-1}(y')) \leq (L + 1)d(x, x')^\alpha \leq \varepsilon,
\]
i.e., \( \varphi(U_x) \subset B(x, \varepsilon) \).

### 3. Properties of linear and quotient map

In order to give some relevant properties as convexity and being vector space over \( \mathbb{R} \) we need to ask that \( \pi \) is also a linear map. Notice that this fact is not too restrictive because in our idea \( \pi \) is the "usual" projection map. More precisely, throughout the section we will consider \( \pi \) a linear and quotient map between a normed space \( X \) and a topological space \( Y \).

#### 3.1. Basic properties

In this section we give two simple results in the particular case when \( \pi \) is a linear map.

**Proposition 3.1.** Let \( \pi : X \to Y \) be a linear and quotient map between a metric space \( X \) and a topological space \( Y \). The set of all section of \( \pi \) is a convex set.

**Proof.** Fix \( t \in [0, 1] \) and let \( \varphi, \psi : Y \to X \) sections of \( \pi \). By the simply fact
\[
\pi(t\varphi(y) + (1-t)\psi(y)) = t\pi(\varphi(y)) + (1-t)\pi(\psi(y)) = y,
\]
we get the thesis. \( \square \)
Proposition 3.4. Let $\pi: X \to Y$ be a linear and quotient map between a normed space $X$ and a topological space $Y$. If $\varphi: Y \to X$ is an intrinsically Hölder section of $\pi$, then for any $\lambda \in \mathbb{R} - \{0\}$ the section $\lambda \varphi$ is also intrinsic Hölder for $1/\lambda \pi$ with the same Lipschitz constant up to the constant $|\lambda|^{1-\alpha}$.

Proof. Fix $\lambda \in \mathbb{R} - \{0\}$. The fact that $\lambda \varphi$ is a section is trivial using the similar argument of Proposition 3.1. On the other hand, for any $y_1, y_2 \in Y$
\[
\|\lambda \varphi(y_1) - \lambda \varphi(y_2)\| \leq |\lambda|Ld(\varphi(y_1), \pi^{-1}(y_2))^{\alpha} = |\lambda|^{1-\alpha}Ld(\lambda \varphi(y_1), (1/\lambda \pi)^{-1}(y_2))^{\alpha},
\]
i.e., the thesis holds. This fact follows by these observations:

1. if $d(\varphi(y_1), \pi^{-1}(y_2)) = d(\varphi(y_1), a)$ then $|\lambda|^\alpha d(\varphi(y_1), \pi^{-1}(y_2)) = \|\lambda \varphi(y_1) - \lambda a\|^\alpha$.
2. $\lambda a \in \pi^{-1}(\lambda y)$.
3. $\pi^{-1}(\lambda y) = (1/\lambda \pi)^{-1}(y)$.

The second point is true because using the linearity of $\pi$ we have that $\pi(\lambda a) = \lambda \pi(a) = \lambda y$. Finally, the third point holds because
\[
\pi^{-1}(\lambda y) = \{x \in X : \pi(x) = \lambda y\} = \{x \in X : 1/\lambda \pi(x) = y\} = (1/\lambda \pi)^{-1}(y),
\]
as desired. \hfill \Box

3.2. Convex set. In this section we show that the set of all intrinsically Hölder sections is a convex set. We underline that the hypothesis on boundness of $Y$ is not necessary.

Definition 3.3 (Intrinsic Hölder set with respect to $\psi$). Let $\alpha \in (0, 1]$ and $\psi: Y \to X$ a section of $\pi$. We define the set of all intrinsically Hölder sections of $\pi$ with respect to $\psi$ at point $\hat{x}$ as
\[
H_{\psi, \hat{x}, \alpha} := \{\varphi: Y \to X \text{ a section of } \pi : \varphi \text{ is intrinsically } (\hat{L}, \alpha)-\text{Hölder w.r.t. } \psi \text{ at point } \hat{x}\}
\]
for some $\hat{L} > 0$.

Proposition 3.4. Let $\pi: X \to Y$ be a linear and quotient map with $X$ a normed space and $Y$ a topological space. Assume also that $\alpha \in (0, 1]$, $\psi: Y \to X$ a section of $\pi$ and $\hat{x} \in \psi(Y)$. Then, the set $H_{\psi, \hat{x}, \alpha}$ is a convex set.

Proof. Let $\varphi, \eta \in H_{\psi, \hat{x}, \alpha}$ and let $t \in [0, 1]$. We want to show that
\[
w := t\varphi + (1-t)\eta \in H_{\psi, \hat{x}, \alpha}.
\]
Notice that, by Proposition 3.1, $w$ is a section of $\pi$ and $w(\bar{y}) = \varphi(\bar{y}) = \eta(\bar{y}) = \hat{x}$ for some $\bar{y} \in Y$. On the other hand, for every $y \in Y$ we have
\[
\|w(y) - \psi(y)\| = \|t(\varphi(y) - \psi(y)) + (1-t)(\eta(y) - \psi(y))\|,
\]
and so
\[
\|w(y) - \psi(y)\| \leq t\|\varphi(y) - \psi(y)\| + (1-t)\|\eta(y) - \psi(y)\|.
\]
Hence,
\[
d(w(y), \psi(y)) \leq tL_\varphi d(\psi(\bar{y}), \psi(y))^{\alpha} + (1-t)L_\eta d(\psi(\bar{y}), \psi(y))^{\alpha} + d(\psi(\bar{y}), \psi(y))
\]
and so
\[
\|w(y) - \psi(y)\| \leq t\|\varphi(y) - \psi(y)\| + (1-t)\|\eta(y) - \psi(y)\|.
\]
Hence,
\[
d(w(y), \psi(y)) \leq tL_\varphi d(\psi(\bar{y}), \psi(y))^{\alpha} + (1-t)L_\psi d(\psi(\bar{y}), \psi(y))^{\alpha} + d(\psi(\bar{y}), \psi(y))
\]
for every $y \in Y$, as desired. \hfill \Box
3.3. Vector space. In this section we show that a ‘large’ class of intrinsically Hölder sections is a vector space over $\mathbb{R}$ or $\mathbb{C}$. Notice that it is no possible to obtain the statement for $H_{\lambda,\hat{x},\alpha}$ since the simply observation that if $\psi(y) = \hat{x}$ then $\psi(y) + \psi(y) \neq \hat{x}$.

**Theorem 3.5.** Let $\pi: X \to Y$ is a linear and quotient map from a normed space $X$ to a topological space $Y$. Assume also that $\psi: X \to X$ is a section of $\pi$ and $\{\lambda\hat{x}: \lambda \in \mathbb{R}^+\} \subset X$ with $\hat{x} \in \psi(Y)$.

Then, for any $\alpha \in (0, 1]$, the set $\bigcup_{\lambda \in \mathbb{R}^+} H_{\lambda,\psi,\lambda\hat{x},\alpha} \cup \{0\}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$.

**Proof.** Let $\varphi, \eta \in \bigcup_{\lambda \in \mathbb{R}^+} H_{\lambda,\psi,\lambda\hat{x},\alpha}$ and $\beta \in \mathbb{R} - \{0\}$. We want to show that

1. $w = \varphi + \eta \in \bigcup_{\lambda \in \mathbb{R}^+} H_{\lambda,\psi,\lambda\hat{x},\alpha}$.
2. $\beta \varphi \in \bigcup_{\lambda \in \mathbb{R}^+} H_{\lambda,\psi,\lambda\hat{x},\alpha}$.

(1). If $\varphi \in H_{\delta_1\psi,\delta_1\hat{x},\alpha}$ and $\eta \in H_{\delta_2\psi,\delta_2\hat{x},\alpha}$ for some $\delta_1, \delta_2 \in \mathbb{R}^+$ it holds

$$w \in H_{(\delta_1+\delta_2)\psi,(\delta_1+\delta_2)\hat{x}}.$$ For simplicity, we choose $\varphi, \eta \in H_{\psi,\hat{x},\alpha}$ and so it remains to prove

$$w \in H_{2\psi,2\hat{x},\alpha}.$$

By linear property of $\pi$, $w$ is a section of $1/2\pi$. On the other hand, if $\psi(y) = \hat{x}$, then $w(y) = \varphi(y) + \eta(y) = 2\psi(y) \in X$. Moreover, using (11), we deduce

$$\|w(y) - 2\psi(y)\| = \|\varphi(y) + \eta(y) - 2\psi(y)\|$$

$$\leq \|\varphi(y) - \psi(y)\| + \|\eta(y) - \psi(y)\|$$

$$\leq 2 \max\{L_{\varphi}, L_{\eta}\}\|\psi(y) - \psi(y)\| + 2\|\psi(y) - \psi(y)\|$$

$$= 2^{1-\alpha} \max\{L_{\varphi}, L_{\eta}\}\|\psi(y) - \psi(y)\| + \|\psi(y) - \psi(y)\|,$$

for any $y \in Y$, as desired. (2). In a similar way, it is possible to show the second point. □

**Remark 3.6.** Theorem 3.5 holds also if we consider $\lambda \in \mathbb{R}^-$ or $\lambda \in \mathbb{R}$ instead of $\mathbb{R}^+$.

4. AN EQUIVALENCE RELATION

In this section $X$ is a metric space, $Y$ a topological space and $\pi: X \to Y$ a quotient map (we do not ask that $\pi$ is a linear map). We stress that Definition 2.4 does not induce an equivalence relation, because of lack of symmetry in the right-hand side of (11). As a consequence we must ask a stronger condition in order to obtain an equivalence relation.

**Definition 4.1** (Intrinsic Hölder with respect to a section in strong sense). Given sections $\varphi, \psi: Y \to X$ of $\pi$. We say that $\varphi$ is intrinsically $(L, \alpha)$-Hölder with respect to $\psi$ at point $\hat{x}$ in strong sense, with $L > 0, \alpha \in (0, 1]$ and $\hat{x} \in X$, if

1. $\hat{x} \in \psi(Y) \cap \varphi(Y)$;
2. it holds

(13) $d(\varphi(y), \psi(y)) \leq \min\{Ld(\psi(y), \psi(y))^{\alpha} + d(\psi(y), \psi(y)), Ld(\psi(y), \varphi(y))^{\alpha} + d(\psi(y), \varphi(y))\}$,

for every $y \in Y$.

Now we are able to give the main theorem.
Theorem 4.2. Let $\alpha \in (0,1]$ and $\pi : X \to Y$ be a quotient map from a metric space $X$ to a topological space $Y$. Assume also that $\psi : Y \to X$ is a section of $\pi$ and $\hat{x} \in X$. Then, being intrinsically Hölder with respect to $\psi$ at point $\hat{x}$ in strong sense induces an equivalence relation. We will write the class of equivalence of $\psi$ at point $\hat{x}$ as

$$[H_{\psi,\hat{x},\alpha}] := \{ \varphi : Y \to X \mid \varphi \text{ is intrinsically } (\hat{L},\alpha)\text{-Hölder with respect to } \psi \text{ at point } \hat{x} \text{ in strong sense, for some } \hat{L} > 0 \}.$$ 

An interesting observation is that, considering $H_{\psi,\hat{x},\alpha}$, the intrinsic constants $L$ can be change but it is fundamental that the point $\hat{x}$ is a common one for the every section. 

Proof. We need to show:

(1) reflexive property;
(2) symmetric property;
(3) transitive property;
(1). It is trivial that $\varphi \sim \varphi$.
(2). If $\varphi \sim \psi$, then $\psi \sim \varphi$. This follows from [13].
(3). We know that $\varphi \sim \psi$ and $\psi \sim \eta$. Hence, $\hat{x} = \varphi(\hat{y}) = \psi(\hat{y}) = \eta(\hat{y})$. Moreover, by [13], it holds

$$d(\varphi(y), \psi(y)) \leq \min\{ L_1 d(\psi(\hat{y}), \psi(y))^\alpha + d(\psi(\hat{y}), \psi(y)), L_1 d(\psi(\hat{y}), \varphi(y))^\alpha + d(\psi(\hat{y}), \varphi(y)) \},$$

for any $y \in Y$ and, consequently, if $\hat{L} = 2 \max\{ L_1, L_2 \}$, then

$$d(\varphi(y), \psi(y)) \leq d(\varphi(y), \psi(y)) + d(\psi(y), \eta(y))$$

$$\leq \min\{ \hat{L} d(\eta(\hat{y}), \eta(y))^\alpha + d(\eta(\hat{y}), \eta(y)), \hat{L} d(\psi(\hat{y}), \varphi(y))^\alpha + d(\psi(\hat{y}), \varphi(y)) \},$$

for any $y \in Y$. This means that $\varphi \sim \eta$, as desired. \qed

5. An Ascoli-Arzelà compactness theorem

In this section, we talk about compact subset of a metric space and so we can use Proposition 2.3 (and, in particular, (3)).

Theorem 5.1 (Compactness Theorem). Let $\pi : X \to Y$ be a quotient map between a metric space $X$ for which closed balls are compact and a topological space $Y$. Then,

(i) For all $K' \subset Y$ compact, $L \geq 1$, $\alpha \in (0,1)$, $K \subset X$ compact, and $y_0 \in Y$ the set

$$\mathcal{A}_0 := \{ \varphi_{|K'} : K' \to X \mid \varphi : Y \to X \text{ intrinsically } (L,\alpha)\text{-Hölder section of } \pi, \varphi(y_0) \in K \}$$

is equibounded, equicontinuous, and closed in the uniform convergence topology.

(ii) For all $L \geq 1$, $\alpha \in (0,1)$, $K \subset X$ compact, and $y_0 \in Y$ the set

$$\{ \varphi : Y \to X : \varphi \text{ intrinsically } (L,\alpha)\text{-Hölder section of } \pi, \varphi(y_0) \in K \}$$

is compact with respect to the uniform convergence on compact sets.

We need the following remark proved in [DDLD22, Remark 2.1].
Remark 5.2. Let $\pi : X \to Y$ be an open map, $K \subset X$ be a compact set and $y \in Y$. Then $\pi$ is uniformly open on $K \cap \pi^{-1}(y)$, in the sense that, for every $\varepsilon > 0$ there is a neighborhood $U_{\varepsilon}$ of $y$ such that

$$U_{\varepsilon} \subset \pi(B(x, \varepsilon)), \quad \forall x \in K \cap \pi^{-1}(y).$$

Proof. (i). We shall prove that for all $K' \subset Y$ compact, $L \geq 1$, $\alpha \in (0,1)$, $K \subset X$ compact, and $y_0 \in Y$ the set $A_0$ is

(a): equibounded;
(b): equicontinuous;
(c): closed.

(a). Fix a compact set $K' \subset Y$ such that $y_0 \in K'$. We shall prove that for any $y \in K'$

$$A := \{\varphi(y) : \varphi|_{K'} : K' \to X \text{ an intrinsically } (L, \alpha) \text{-Hölder section of } \pi \text{ and } \varphi(y_0) \in K\}$$

is relatively compact in $X$. Fix a point $x_0 \in K$ and let $k := \text{diam}_{d}(K)$ which is finite because $K$ is compact in $X$. Then, for every $\varphi$ belongs to $A$, we have that

$$d(x_0, \varphi(y)) \leq d(x_0, \varphi(y_0)) + d(\varphi(y_0), \varphi(y))$$

$$\leq k + Ld(\pi^{-1}(y), \varphi(y_0))^\alpha$$

$$\leq k + L \max_{x \in K} d(\pi^{-1}(y), x)^\alpha,$$

where in the first equality we used the triangle inequality, and in the second one we used the fact that $\varphi \in A$ and $x_0 \in K$. Finally, in the last inequality we used again $\varphi(y_0) \in K$ and that the map $X \ni x \mapsto d(\pi^{-1}(y), x)^\alpha$ is a continuous map and so admits maximum on compact sets. Since closed balls on $X$ are compact, we infer that the set $A$ is relatively compact in $X$, as desired.

(b). We shall to prove that for every $y \in K'$ and every $\varepsilon > 0$ there is an open neighborhood $U_y \subset K' \subset Y$ such that for any $\varphi \in A$ and any $y' \in U_y$, it follows

$$d(\varphi(y), \varphi(y')) \leq \varepsilon. \tag{15}$$

Because of equiboundedness, we have that for any $\varphi \in A, y \in Y$ the set $\varphi(y)$ lies within a compact set $K_y$ and so, by Remark 5.2, $\pi$ is uniformly open on $K_y \cap \pi^{-1}(y)$. Now let $U_{\varepsilon}$ an neighborhood of $y$ such that $U_{\varepsilon} \subset \pi(B(x, (\varepsilon/L)^{1/\alpha}))$ for every $x \in K_y \cap \pi^{-1}(y)$. Then we want to show that such neighborhood $U_{\varepsilon}$ of $y$ is the set that we are looking for. Take $y' \in U_{\varepsilon}$ and let $x = \varphi(y)$. Hence, there is $x' \in B(x, (\varepsilon/L)^{1/\alpha})$ with $\pi(x') = y'$ and, consequently, $x' \in \pi^{-1}(y')$. Thus we have that for all $\varphi$ belongs to $A$

$$d(\varphi(y), \varphi(y')) \leq Ld(\varphi(y), \pi^{-1}(y'))^\alpha \leq Ld(x, x')^\alpha \leq \frac{\varepsilon}{L} \leq \varepsilon,$$

i.e., (15) holds. Finally, since the bound is independent on $\varphi$, we proved the equicontinuity.

(c). By (a) and (b) we can apply Ascoli-Arzela Theorem to the set $A$. Hence, every sequence in it has a converging subsequence. Moreover, this set is closed since if $\varphi_h$ is a sequence in it converging pointwise to $\varphi$, then $\varphi \in A$. Indeed, taking the limit of

$$d(\varphi_h(y), \varphi_h(y')) \leq Ld(\pi^{-1}(y), \varphi_h(y'))^\alpha,$$

one gets

$$d(\varphi(y), \varphi(y')) \leq Ld(\pi^{-1}(y), \varphi(y'))^\alpha.$$

Finally, it is trivial that the condition $\varphi_h(y_0) \in K$ passes to the limit since $K$ is compact.

(ii). Follows from the latter point (i) using Ascoli-Arzela Theorem. \qed
6. Level sets and extensions

In this section we prove Theorem 1.3. The proof of this statement is similar to the one of [DLD20, Theorem 1.4] regarding Lipschitz sections theory (see also [Vit20, Theorem 1.5]). We need to mention several earlier partial results on extensions of Lipschitz graphs in the context of Carnot groups, as for example in [FSSC06, DDF21, Mon14, Proposition 4.8], [Vit12, Proposition 3.4], [PS16, Theorem 4.1].

Regarding extension theorems in metric spaces, the reader can see [AP20, LN05, Oht09] and their references.

Proof of Theorem 1.3 (1.3.i). We begin noting that for any \( y \in Y \) the map \( f_{x^{-1}(y)} : \pi^{-1}(y) \to Z \) is biHölder and so for any \( y \in Y \) there is a unique \( x \in \pi^{-1}(y) \) such that \( f(x) = z_0 \). Hence, it is natural to define \( \varphi(y) := x \) and so (3) holds. Moreover, \( \varphi : Y \to X \) intrinsically \((\lambda^2, \beta)\)-Hölder; indeed, if we consider \( x_1 \in \pi^{-1}(y_1) \cap f^{-1}(z_0) \) and \( x_2 \in \pi^{-1}(y_2) \cap f^{-1}(z_0) \), then we would like to prove that

\[
(16) \quad d(x_1, x_2) \leq \lambda^2 d(x_1, \pi^{-1}(y_2))^\beta + d(x_1, \pi^{-1}(y_2)).
\]

Let \( \tilde{x}_2 \in \pi^{-1}(y_2) \) such that \( d(x_1, \tilde{x}_2) = d(x_1, \pi^{-1}(y_2)) \), then it follows that

\[
(17) \quad d(x_1, x_2) \leq d(x_1, \tilde{x}_2) + d(\tilde{x}_2, x_2)
\]

\[
\leq d(x_1, \tilde{x}_2) + \lambda d(f(\tilde{x}_2), f(x_2))
\]

\[
= d(x_1, \tilde{x}_2) + \lambda d(f(\tilde{x}_2), f(x_1))
\]

\[
\leq d(x_1, \pi^{-1}(y_2)) + \lambda^2 d(x_1, \pi^{-1}(y_2))^\beta,
\]

where in the first inequality we used the triangle inequality, and in the second inequality we used the biLipschitz property of \( f \) on the fibers. In the first equality we used the fact that \( f(x_1) = f(x_2) = z_0 \) and finally we used again the biHölder property of \( f \). Consequently, (16) is true and the proof is complete. \( \square \)

Proof of Theorem 1.3 (1.3.ii). Fix \( x_0 \in \tau^{-1}(\tau_0) \). We consider the map \( f_{x_0} : X \to \mathbb{R} \) defined as

\[
f_{x_0}(x) = \begin{cases} 
2(\tau(x) - \tau(x_0)) - \gamma(\delta_{\tau_0}(x) + \delta_{\tau_0}(x)) & \text{if } |\tau(x) - \tau(x_0)| \leq 2\gamma|\delta_{\tau_0}(x) + \delta_{\tau_0}(x)| \\
\tau(x) - \tau(x_0) & \text{if } \tau(x) - \tau(x_0) > 2\gamma|\delta_{\tau_0}(x) + \delta_{\tau_0}(x)| \\
3(\tau(x) - \tau(x_0)) & \text{if } \tau(x) - \tau(x_0) < -2\gamma|\delta_{\tau_0}(x) + \delta_{\tau_0}(x)| 
\end{cases}
\]

where \( \gamma := 2kL + 1 \). We prove that \( f_{x_0} \) satisfies the following properties:

(i): \( f_{x_0} \) is \((K, \alpha)\)-Hölder;
(ii): \( f_{x_0}(x_0) = 0 \);
(iii): \( f_{x_0} \) is biLipschitz on fibers.

where \( K = \max\{3k, 2k + 4\gamma k\} = 2k + 4\gamma k \) because \( \gamma > 1 \). The property (i) follows using that \( \tau, \delta_{\tau_0} \) are Hölder and Lipschitz, respectively, and \( X \) is a geodesic space. On the other hand, (ii) is true noting that \( \delta_{\tau_0}(x_0) = \rho(x_0, \varphi_{\tau_0}(\pi(x_0))) = 0 \) because \( x_0 \in \varphi_{\tau_0}(Y) \).

Finally, for any \( x \in \pi^{-1}(y) \) we have that \( \rho(x_0, \varphi_{\tau_0}(\pi(x))) = \rho(x_0, \varphi_{\tau_0}(\pi(x'))) \) and so \( f \) is biLipschitz on fibers because \( \tau \) is so too. Hence (iii) holds.

Now we consider the map \( f : X \to \mathbb{R} \) given by

\[
f(x) := \sup_{x_0 \in \varphi(Y)} f_{x_0}(x), \quad \forall x \in X,
\]

where \( \varphi : Y \to X \) and \( \varphi(Y) \) is a compact set in \( X \). We prove that \( f_{x_0} \) satisfies the following properties:
and we want to prove that it is the map we are looking for. The Hölder property and the Lipschitz property on the fiber are true recall that the function $\delta_{x_0}$ is constant on the fibers. Consequently, the only non-trivial fact to show is (4). Fix $\bar{x}_0 \in \varphi(Y')$. By (ii) we have that $f_{\bar{x}_0}(\bar{x}_0) = 0$ and so it is sufficient to prove that $f_{\bar{x}_0}(\bar{x}_0) \leq 0$ for $x_0 \in \varphi(Y')$. Let $x_0 \in \varphi(Y')$. Then using in addition that $\tau$ is $k$-Lipschitz, and that $\varphi$ is intrinsically $L$-Lipschitz, we have

$$|\tau(\bar{x}_0) - \tau(x_0)| \leq k d(\bar{x}_0, x_0) \leq k L d(x_0, \varphi_{\tau_n}(\pi(\bar{x}_0))]^\alpha + k d(x_0, \varphi_{\tau_n}(\pi(\bar{x}_0))) < \gamma (\delta_{\tau_n}(\bar{x}_0)]^\alpha + \delta_{\tau_n}(\bar{x}_0),$$

and so

$$f_{\bar{x}_0}(\bar{x}_0) = 2(\tau(\bar{x}_0) - \tau(x_0)) - \gamma (\delta_{\tau_n}(\bar{x}_0)]^\alpha + \delta_{\tau_n}(\bar{x}_0)) < 0,$$

i.e., (4) holds.

7. Ahlfors-David regularity

In this section we prove Ahlfors-David regularity for the intrinsically Hölder sections. The proof of this statement is similar to the one of [DDLD22, Theorem 1.3] (see also [DD22c] for the intrinsically quasi-isometric case).

In order to give the proof, we need the following statement

**Lemma 7.1.** Let $X$ be a metric space, $Y$ a topological space, and $\pi : X \to Y$ a quotient map. If $\varphi : Y \to X$ is an intrinsically $(L, \alpha)$-Hölder section of $\pi$ with $\alpha \in (0, 1)$ and $L > 0$, then

$$\pi(B(p, r)) \subset \pi(B(p, (L + 1)r^\alpha) \cap \varphi(Y)) \subset \pi(B(p, (L + 1)r^\alpha)), \forall p \in \varphi(Y), \forall r > 0.$$  

**Proof.** Regarding the first inclusion, fix $p \in \varphi(Y), r > 0$ and $q \in B(p, r)$. We need to show that $\pi(q) \in \pi(\varphi(Y) \cap B(p, (L + 1)r^\alpha))$. Actually, it is enough to prove that

$$\varphi(\pi(q)) \in B(p, (L + 1)r^\alpha),$$

because if we take $g := \varphi(\pi(q))$, then $g \in \varphi(Y)$ and

$$\pi(g) = \pi(\varphi(\pi(q))) = \pi(q) \in \pi(\varphi(Y) \cap B(p, (L + 1)r^\alpha)).$$

Hence in a similar way to the point (i), we get that for any $p, q, g \in \varphi(Y)$ with $g = \varphi(\pi(q))$,

$$d(p, g) \leq L d(p, \pi^{-1}(\pi(g))^\alpha + d(\varphi(y_1), \pi^{-1}(\pi(g)))) \leq L d(p, q)^\alpha + d(p, q) \leq (L + 1)r^\alpha,$$

i.e., (19) holds, as desired. Finally, the second inclusion in (18) follows immediately noting that $\pi(\varphi(Y)) = Y$ because $\varphi$ is a section and the proof is complete.

Now we are able to give the proof of Theorem [1.4]

**Proof of Theorem [1.4]** Let $\varphi$ and $\psi$ intrinsically $(L, \alpha)$-Hölder sections, with $L > 0$ and $\alpha \in (0, 1)$. Fix $y \in Y$. By Ahlfors-David regularity of $\varphi(y)$, we know that there is $c_1 > 0$ such that

$$\varphi_* \mu(B(\varphi(y), r) \cap \varphi(Y)) \leq c_1 r^{\ell + 1 - \alpha},$$

for all $0 \leq r \leq 1$. We would like to show that there is $c_2 > 0$ such that

$$\psi_* \mu(B(\psi(y), r) \cap \psi(Y)) \leq c_2 r^{\alpha(\ell + 1 - \alpha)},$$
for every $0 \leq r \leq 1$. We begin noticing that

\begin{equation}
\psi_* \mu(B(\psi(y), r) \cap \psi(Y)) = \mu(\psi^{-1}(B(\psi(y), r) \cap \psi(Y))) = \mu(\pi(B(\psi(y), r) \cap \psi(Y))),
\end{equation}

and, consequently,

\begin{align*}
\psi_* \mu(B(\psi(y), r) \cap \psi(Y)) &\leq \mu(\pi(B(\psi(y), r))) \\
&\leq C \mu(\pi(B(\varphi(y), (L+1)r^\alpha) \cap \varphi(Y))) \\
&= C \varphi_* \mu(B(\varphi(y), (L+1)r^\alpha) \cap \varphi(Y)) \\
&\leq C_1 (L+1)^{\alpha(\ell+1-\alpha)} r^{\alpha(\ell+1-\alpha)},
\end{align*}

where in the first inequality we used the second inclusion of (18) with $\psi$ in place of $\varphi$, and in the second one we used (5). In the third inequality we used the first inclusion of (18) and in the fourth one we used (22) with $\varphi$ in place of $\psi$. \qed

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