A REVERSE ERGODIC THEOREM FOR INHOMOGENEOUS KILLED MARKOV CHAINS AND APPLICATION TO A NEW UNIQUENESS RESULT FOR REFLECTING DIFFUSIONS

BY CRISTINA COSTANTINI¹,a AND THOMAS G. KURTZ²,b

¹Department of Economic Studies and INdAM Local Unit, University of Chieti-Pescara, a c.costantini@unich.it
²Department of Mathematics and Department of Statistics, University of Wisconsin-Madison, b kurtz@math.wisc.edu

Bass and Pardoux (1987) deduce from the Krein-Rutman theorem a reverse ergodic theorem for a sub-probability transition function, which turns out to be a key tool in proving uniqueness of reflecting Brownian motion in cones in Kwon and Williams (1991) and Taylor and Williams (1993). By a different approach, we are able to prove an analogous reverse ergodic theorem for a family of inhomogeneous sub-probability transition functions.

This allows us to prove existence and uniqueness for a semimartingale diffusion process with varying, oblique direction of reflection, in a domain with one singular point that can be approximated, near the singular point, by a smooth cone, under natural, easily verifiable geometric conditions.

Along the way we also show that if the reflecting Brownian motion in a smooth cone is a semimartingale then the parameter \( \alpha \) of Kwon and Williams (1991) is strictly less than 1, thus partially extending the results of Williams (1985) to higher dimension.

1. Introduction. Let \( E \) be a compact metric space, and let \( Q(x, dy) \) be a sub-probability transition function on \( E \), that is \( Q(x, dy) \) satisfies all conditions for a probability transition function except that \( Q(x, E) \leq 1 \). As it is well known, \( Q(x, dy) \) is the transition function of a killed Markov chain. In the proof of their Theorem 5.4, Bass and Pardoux (1987) show that, if \( Q \) satisfies the conditions of the Krein-Rutman theorem (Theorems 6.1 and 6.3 of Krein and Rutman (1950)), then, for any pair of continuous functions \( f, g, g > 0 \), and any sequence of probability measures on \( E \), \( \{ \mu_k \} \),

\[
\lim_{k \to \infty} \frac{\int Q^k f(x) \mu_k(dx)}{\int Q^k g(x) \mu_k(dx)} = C(f, g),
\]

where the constant \( C(f, g) \) is independent of the sequence \( \{ \mu_k \} \). In particular, with \( g \equiv 1 \),

\[
\lim_{k \to \infty} \frac{\int Q^k f(x) \mu_k(dx)}{\int Q^k 1(x) \mu_k(dx)} = C(f).
\]

(1.1) can be viewed as a reverse ergodic theorem for killed Markov chains. Note that, typically, both the numerator and the denominator in (1.1) tend to zero. The result of Bass and Pardoux (1987) is a key element in the proof of uniqueness of reflecting Brownian motion in a smooth cone, with radially constant direction of reflection, by Kwon and Williams (1991), and in the proof of uniqueness of reflecting Brownian motion in a polyhedral cone, with constant directions of reflection on each face, by Taylor and Williams (1993) and Dai and Williams (1996).

MSC2020 subject classifications: Primary 60J60, 60H10; secondary 60J55, 60G17.

Keywords and phrases: Krein–Rutman theorem, subprobability transition function, reflecting diffusion, nonsmooth domain, constrained martingale problem.
Our first goal here is to extend the Bass and Pardoux (1987) result to a sequence of compact metric spaces $E_0, E_1, E_2, \ldots$ and a sequence of sub-transition functions $Q_1, Q_2, \ldots$, with $Q_k$ governing transitions from $E_k$ to $E_{k-1}$, and give conditions under which

\[ \lim_{k \to \infty} \frac{\int Q_k Q_{k-1} \cdots Q_1 f(x) \mu_k(dx)}{\int Q_k Q_{k-1} \cdots Q_1 g(x) \mu_k(dx)} = C(f, g), \]

where $C(f, g)$ is independent of $\{\mu_k\}$. We call (1.2) a reverse ergodic theorem for inhomogeneous killed Markov chains. Note that, even in the case when $E_k = E$ for all $k$ and $Q_k$ converges, as $k$ goes to infinity, to a sub-transition function $Q$ on $E$, it is not in general possible to obtain the limit in (1.2) from the Krein-Rutman theorem. In fact this would essentially require exchanging the limits

\[ \lim_{l \to \infty} \lim_{k \to \infty} \frac{\int Q_{l+k} Q_{l+k-1} \cdots Q_{l+1} f(x) \mu_k(dx)}{\int Q_{l+k} Q_{l+k-1} \cdots Q_{l+1} g(x) \mu_k(dx)} \]

and

\[ \lim_{k \to \infty} \lim_{l \to \infty} \frac{\int Q_{l+k} Q_{l+k-1} \cdots Q_{l+1} f(x) \mu_k(dx)}{\int Q_{l+k} Q_{l+k-1} \cdots Q_{l+1} g(x) \mu_k(dx)}. \]

Rather than trying to reinforce the conditions of the Krein-Rutman theorem, we provide new conditions under which (1.2) holds (Theorem 2.4). Our conditions are uniform lower bounds which have a clear probabilistic meaning and can be verified in many contexts.

In the second part of this paper we use our reverse ergodic theorem for inhomogeneous killed Markov chains to prove uniqueness for a semimartingale reflecting diffusion process with varying, oblique direction of reflection, in a curved domain with only one singular point, that, in a neighborhood of the singular point, can be approximated by a smooth cone. Although one expects that such a diffusion, for a short time after it leaves the singular point, can be approximated by a reflecting Brownian motion in the cone (we actually prove this in a rigorous sense: see Lemma 3.30), it is not clear how the uniqueness result we seek might follow from the Kwon and Williams (1991) result, essentially due to a limit exchange problem like the one mentioned above. We prove existence and uniqueness of a semimartingale reflecting diffusion process as above under very natural, geometric conditions (Conditions 3.1, 3.3 and 3.6): Besides mild regularity conditions, we only require that, at the singular point, where we have not a single normal direction and a single direction of reflection, but a cone of normal directions and a cone of directions of reflection, there is a direction of reflection that points strictly inside the approximating cone, and a normal vector that forms an angle of strictly less than $\pi/2$ with every direction of reflection (Conditions 3.3 (iii) and (iv)). Conditions 3.3 (iii) and (iv) are the analog of the well known completely-$S$ condition for an orthant with constant direction of reflection on each face.

Our argument follows the general outline of Kwon and Williams (1991), with two fundamental changes: We focus on semimartingale reflecting diffusions and characterize them as solutions of constrained martingale problems rather than submartingale problems; We replace the Krein-Rutman theorem by our reverse ergodic theorem for inhomogeneous killed Markov chains. Preliminary we apply the Markov selection results of Costantini and Kurtz (2019), so that we can reduce to proving uniqueness among strong Markov reflecting diffusions. The characterization of a semimartingale reflecting diffusion as a solution of a constrained martingale problem (more precisely a natural solution) is equivalent to the characterization as a solution of a stochastic differential equation used in Taylor and Williams (1993) and Dai and Williams (1996), but it avoids the need of the oscillation estimates required there. In order to apply our ergodic theorem, we need to prove our uniform lower bounds: (i) and (ii) in Theorem 2.4. We obtain the bound (i) by means of some auxiliary functions.
that we construct by elaborating on the functions $\psi_{\alpha}$ and $\chi$ introduced in Kwon and Williams (1991) (or the corresponding functions introduced in Varadhan and Williams (1985), in the 2-dimensional case). In order to do this, we have to prove that, under our conditions, the parameter $\alpha$ of Kwon and Williams (1991) (the one whose sign determines whether the reflecting Brownian motion hits the vertex of the cone) is strictly less than 1 (Theorem A.10). In fact, under our conditions, we can directly construct the reflecting Brownian motion as a semimartingale (Theorem A.9; By modifying the proof of Theorem 5 of Williams (1985), we show that this implies that $\alpha < 1$. Thus we partially extend the 2-dimensional results of Williams (1985) to higher dimension. In order to verify the bound (ii), we use a coupling argument based on Lemma 5.3 of Costantini and Kurtz (2018) (Lemma 3.31), the fact that, for any reflecting diffusion, $X$, the rescaled process $2^{2n}X(2^{-4n} \cdot)$ converges to a reflecting Brownian motion in the cone (the already mentioned Lemma 3.30), and the support theorem of Kwon and Williams (1991).

A recent paper (Costantini (2023)) shows that, under suitable assumptions, the results of Section 3 of this paper extend to obliquely reflecting Brownian motion in a nonpolyhedral, piecewise smooth cone. The uniqueness result of Costantini (2023) follows by applying Theorem 2.4. The proofs of the lemmas in Subsections 3.4 and 3.5, needed to verify the assumptions of Theorem 2.4, carry over essentially without any modification, except for Lemma 3.30, which requires an additional argument. The existence result of Costantini (2023) follows by modifying the proof of Theorem A.9. As it is well known, in at least two notable classes of stochastic networks, namely bandwidth sharing models under a weighted $\alpha$-fair bandwidth sharing policy (Kelly and Williams (2004) and Kang et al. (2009)) and input-queued switches under a maximum weight-$\alpha$ policy (Shah and Wischik (2012) and Kang and Williams (2012)), for all values different from 1 of the parameter $\alpha$, the diffusion approximation for the workload is conjectured to be an obliquely reflecting Brownian motion in a nonpolyhedral, piecewise smooth cone (note that $\alpha$ here is not the parameter of Kwon and Williams (1991).) The conjecture for the $\alpha \neq 1$ case could not be proved in particular due to the lack of a unique characterization for the conjectured limiting process. Now, at least in the $\alpha \geq 2$ case, Costantini (2023) provides conditions under which the stochastic differential equation that should characterize the limiting process has a unique solution. In Costantini (2023) these conditions are actually verified in a specific example of bandwidth sharing model. Some preliminary computations indicate that a refinement to the $\alpha > 1$ case should be possible.

In dimension 2, any piecewise $C^1$ domain, near each singular point that is not a cusp, looks like the domain studied in Section 3. In fact the results of Section 3, combined with a localization result for constrained martingale problems, and with Costantini and Kurtz (2018), yield new and optimal conditions for existence and uniqueness of obliquely reflecting diffusions in piecewise $C^1$ domains in dimension 2 (Costantini and Kurtz (2024)). These new conditions are strictly more general and more easily verifiable than the previously known conditions of Dupuis and Ishii (1993). Obliquely reflecting diffusions in 2-dimensional piecewise smooth domains arise naturally, for instance, in singular stochastic control problems. As an example, in the problem studied in Williams et al. (1994), an optimal control is found from a reflecting Brownian motion in a certain piecewise smooth domain, if such a process exists. In this example, when the dimension is 2, the conditions of Costantini and Kurtz (2024) seem to be satisfied, but some technical details still need to be verified. (See Dianetti and Ferrari (2023) for a discussion and a recent contribution on the issue of characterizing optimal singular stochastic controls.)

The outline of the paper is the following: In Section 2 we prove our reverse ergodic theorem for inhomogeneous killed Markov chains, while in Section 3 we prove our existence and uniqueness result for semimartingale reflecting diffusions. Section 3 is divided into
several subsections: In Subsection 3.1, we state the assumptions, formulate the constrained martingale problem and prove some preliminary results; In Subsection 3.2, we prove existence of a strong Markov, semimartingale reflecting diffusion; In Subsection 3.3, we show how uniqueness follows from the reverse ergodic theorem for inhomogeneous killed Markov chains; In Subsections 3.4 and 3.5, we prove the bounds required to apply the reverse ergodic theorem. Finally, in Appendix A, we summarize the results of Kwon and Williams (1991), Varadhan and Williams (1985) and Williams (1985) and we prove our new results for the cone; Appendix B contains the constructions of various auxiliary functions.

1.1. Notation. For a set $E$ (in a topological space), we denote by $\bar{E}$ the interior of $E$ and by $\overline{E}$ its closure.

For a metric space $E$, we denote by $\mathcal{B}(E)$ the $\sigma$-algebra of Borel subsets of $E$ and by $\mathcal{P}(E)$ and $\mathcal{M}(E)$ the sets of probability measures and of finite signed measures on $(E, \mathcal{B}(E))$ respectively. On $\mathcal{M}(E)$ we define the total variation norm by $||\mu||_{TV} := \sup_{C \in \mathcal{B}(E)} |\mu(C)|$. For $z \in E$, $\delta_z$ denotes the Dirac measure. We denote by $C(E)$ the space of continuous functions on $E$, endowed with the $\sup$ norm. $C_E(0,\infty)$ is the space of continuous functions from $[0, \infty)$ to $E$ and $D_E(0,\infty)$ is the space of right continuous functions with left hand limits from $[0, \infty)$ to $E$.

For a stochastic process $Z$, $\mathcal{F}^Z_t$ is the filtration generated by $Z$, that is $\mathcal{F}^Z_t := \sigma(Z(s), s \leq t)$, $t \geq 0$, and we use the notation

$$\mathbb{P}^z(\cdot) := \mathbb{P}(\cdot | Z(0) = z), \quad \mathbb{E}^z[\cdot] := \mathbb{E}[\cdot | Z(0) = z].$$

In $\mathbb{R}^d$, we denote by $B_r(0)$ the ball of radius $r$ centered at the origin and by $S^{d-1}$ the unit sphere.

2. A reverse ergodic theorem for inhomogeneous killed Markov chains. Let $E$ be a compact metric space, and let $Q(x, dy)$ be a sub-probability transition function on $E$, that is, for each $x \in E$, $Q(x, \cdot)$ is a finite measure on $E$ with $Q(x, E) \leq 1$ and for each $C \in \mathcal{B}(E)$, $Q(x, C)$ is Borel measurable in $x$. We will still denote by $Q$ the integral operator defined by $Q$. In the proof of their Theorem 5.4, Bass and Pardoux (1987) show that, if $Q$ satisfies the conditions of the Krein-Rutman theorem (Theorems 6.1 and 6.3 of Krein and Rutman (1950)), then, for all $f, g \in C(E), g > 0$ \footnote{The conditions of the Krein-Rutman theorem actually allow for $g \geq 0$ as long as $g$ is not identically zero, but $g > 0$ is enough for the application in Bass and Pardoux (1987) and for our purposes as well.}, $\{\mu_k\} \subset \mathcal{P}(E)$,

$$\lim_{k \to \infty} \frac{\int Q^k f(x) \mu_k(dx)}{\int Q^k g(x) \mu_k(dx)} = C(f, g),$$

where the constant $C(f, g)$ is independent of $\{\mu_k\}$.

Our goal is to extend this result to a sequence of compact metric spaces $E_0, E_1, E_2, \ldots$ and a sequence of sub-probability transition functions $Q_1, Q_2, \ldots$, with $Q_l$ governing transitions from $E_l$ to $E_{l-1}$, and give conditions under which (1.2), i.e.

$$\lim_{k \to \infty} \frac{\int Q_k Q_{k-1} \cdots Q_1 f(x) \mu_k(dx)}{\int Q_k Q_{k-1} \cdots Q_1 g(x) \mu_k(dx)} = C(f, g)$$

with $C(f, g)$ independent of $\{\mu_k\}$, holds. We may as well take $g \equiv 1$, and we will do so in the sequel.
Lemma 2.1. Assume

\[
\inf_{x \in E_l} Q_l(x, E_{l-1}) > 0, \quad \forall l,
\]

and set, for \( f \in C(E_0) \),

\[
T_k f(x) := \frac{Q_k Q_{k-1} \ldots Q_1 f(x)}{Q_k Q_{k-1} \ldots Q_1 1(x)}.
\]

If there exists a constant \( C(f) \) such that

\[
\sup_{x \in E_k} |T_k f(x) - C(f)| \to k \to \infty 0,
\]

then (1.2) holds for \( f \) and \( g = 1 \).

Proof. Divide and multiply by \( Q_k Q_{k-1} \ldots Q_1 1(x) \) inside the integral in the numerator of (1.2) (with \( g = 1 \)). □

Remark 2.2. Lemma 2.1, and hence Theorem 2.4 below, hold for a sequence of nonzero subprobability measures \( \{\mu_k\} \) as well.

Note that the operator \( T_k \) defined by (2.2) corresponds to a probability transition function from \( E_k \) to \( E_0 \) and can be written as

\[
T_k f(x) = P_k P_{k-1} \ldots P_1 f(x)
\]

where the \( P_l \) are the operators corresponding to the probability transition functions from \( E_l \) into \( E_{l-1} \) given by

\[
P_1(x, dy) := \frac{Q_1(x, dy)}{Q_1 1(x)} = \frac{Q_1(x, dy)}{\int_{E_0} Q_1(x, dz)}
\]

(2.4)

\[
P_l(x, dy) := \frac{Q_l(x, dy)[Q_{l-1} \ldots Q_1 1(y)]}{Q_l \ldots Q_1 1(x)} = \frac{Q_l(x, dy)[Q_{l-1} \ldots Q_1 1(y)]}{\int_{E_{l-1}} Q_{l-1} \ldots Q_1 1(z) Q_l(x, dz)} \quad l \geq 2.
\]

Lemma 2.3. Assume (2.1). Define

\[
f_l, \bar{x}(x, y) := \frac{dQ_l(x, \cdot)}{d(Q_l(x, \cdot) + Q_l(\bar{x}, \cdot))}(y)
\]

and

\[
\epsilon_l(x, \bar{x}) := \int (f_{l, \bar{x}}(x, y) \wedge f_{l, x}(\bar{x}, y)) (Q_l(x, dy) + Q_l(\bar{x}, dy)).
\]

Then, for \( P_l \) given by (2.4),

\[
\|P_l(\cdot, \cdot) - P_l(\bar{x}, \cdot)\|_{TV} \leq 1 - \epsilon_l(x, \bar{x}) \inf_{z \in E_l, y \in E_{l-1}} \left( \frac{Q_{l-1} \ldots Q_1 1(y)}{Q_l \ldots Q_1 1(z)} \right)
\]

(2.7)

\[
\leq 1 - \epsilon_l(x, \bar{x}) \inf_{z, y \in E_{l-1}} \left( \frac{Q_{l-1} \ldots Q_1 1(y)}{Q_{l-1} \ldots Q_1 1(z)} \right)
\]
PROOF. Observe that \( P_l(x, dy) \ll Q_l(x, dy) \) with density given by (2.4). Then

\[
\| P_l(x, \cdot) - P_l(\bar{x}, \cdot) \|_{TV} = \frac{1}{2} \int \left| \frac{f_{l, \bar{x}}(x, y)}{Q_l \ldots Q_1(x)} - \frac{f_{l,x}(\bar{x}, y)}{Q_l \ldots Q_1(\bar{x})} \right| (Q_l(x, dy) + Q_l(\bar{x}, dy))
\]

\[
= 1 - \int \left( \frac{f_{l,\bar{x}}(x, y)}{Q_l \ldots Q_1(x)} \cdot \frac{f_{l,x}(\bar{x}, y)}{Q_l \ldots Q_1(\bar{x})} \right) (Q_l(x, dy) + Q_l(\bar{x}, dy))
\]

\[
\leq 1 - \int \left( f_{l,\bar{x}}(x, y) \wedge f_{l,x}(\bar{x}, y) \right) \left( \frac{Q_l \ldots Q_1(y)}{Q_l \ldots Q_1(x) \lor Q_l \ldots Q_1(\bar{x})} \right) (Q_l(x, dy) + Q_l(\bar{x}, dy))
\]

\[
\leq 1 - \varepsilon_l(x, \bar{x}) \inf_{z \in E_l, y \in E_{l-1}} \left( \frac{Q_l \ldots Q_1(y)}{Q_l \ldots Q_1(z)} \right)
\]

The second inequality in (2.7) follows from the fact that

\[
Q_l \ldots Q_1(1) \leq Q_l(x, E_{l-1}) \sup_{z \in E_{l-1}} Q_l \ldots Q_1(z) \leq \sup_{z \in E_{l-1}} Q_l \ldots Q_1(z).
\]

\[\square\]

**Theorem 2.4.** For \( x, \bar{x} \in E_l \), let \( \varepsilon_l(x, \bar{x}) \) be defined as in Lemma 2.3. Assume \( Q_l \) is not identically zero, i.e. \( \sup_x Q_l(x, E_{l-1}) > 0 \), for all \( l \), and there exist \( c_0 > 0 \) and \( \varepsilon_0 > 0 \) such that

(i) \[ \inf_{x \in E_k} Q_k \ldots Q_1(x) \geq c_0 \sup_{x \in E_k} Q_k \ldots Q_1(x), \quad \forall k, \]

(ii) \[ \inf_k \inf_{x, \bar{x} \in E_k} \varepsilon_k(x, \bar{x}) \geq \varepsilon_0. \]

Then

\[
\inf_{x \in E_k} Q_k \ldots Q_1(x) > 0, \quad \forall k,
\]

and (1.2) holds for all \( f \in C(E_0) \) and \( g = 1 \).

**Proof.** First of all note that (i) above and the assumption that, for every \( l \), \( Q_l \) is never identically zero imply, by induction, (2.8), which, in turn, implies (2.1).

Next, for \( \mu \in \mathcal{P}(E_l) \), denote

\[
\mu P_l(dy) := \int_{E_l} P_l(x, dy) \mu(dx).
\]

Of course we can suppose \( \varepsilon_0 < 1\), \( c_0 < 1 \). For \( \mu, \bar{\mu} \in \mathcal{P}(E_l) \), by Lemma 2.3, for all \( l \),

\[
\| \mu P_l - \bar{\mu} P_l \|_{TV} = \sup_{C \in \mathcal{B}(E_{l-1})} \left| \int_{E_l} \int_{E_l} (P_l(x, C) - P_l(\bar{x}, C)) \mu(dx) \bar{\mu}(d\bar{x}) \right|
\]

\[
\leq \sup_{x, \bar{x} \in E_l} \| P_l(x, \cdot) - P_l(\bar{x}, \cdot) \|_{TV} \leq 1 - \varepsilon_0 c_0.
\]

Then, by Lemma 5.4 of Costantini and Kurtz (2018), for \( \mu, \bar{\mu} \in \mathcal{P}(E_k) \),

\[
\| \mu P_k P_{k-1} \ldots P_1 - \bar{\mu} P_k P_{k-1} \ldots P_1 \|_{TV} = \| (\mu P_k)(P_{k-1} \ldots P_1) - (\bar{\mu} P_k)(P_{k-1} \ldots P_1) \|_{TV}
\]

\[
\leq \| \mu P_k - \bar{\mu} P_k \|_{TV} \| P_{k-1} \ldots P_1 - \bar{\mu} P_{k-1} \ldots P_1 \|_{TV}
\]

\[
\leq (1 - \varepsilon_0 c_0) \| \mu P_{k-1} \ldots P_1 - \bar{\mu} P_{k-1} \ldots P_1 \|_{TV},
\]
and, by iterating,
\[ \| \mu P_k P_{k-1} \cdots P_1 - \bar{\mu} P_{k-1} \cdots P_1 \|_{TV} \leq (1 - \epsilon_0 c_0)^k. \]
In particular, for each \( f \in C(E_0) \), for an arbitrary \( \{ x_k \}, x_k \in E_k \) for each \( k \),
\[ |T_{k+1} f(x_{k+1}) - T_k f(x_k)| \leq \| (\delta_{x_k} P_{k+1} \cdots P_1) \bar{\mu} P_{k-1} \cdots P_1 - \delta_{x_k} P_k \cdots P_1 \|_{TV} f \]
\[ \leq (1 - \epsilon_0 c_0)^k \| f \|, \]
so that \( \{ T_k f(x_k) \} \) is a Cauchy sequence. If \( C(f) \) is its limit, we get, in an analogous manner,
\[ \sup_{x \in E_k} |T_k f(x) - C(f)| \leq (1 - \epsilon_0 c_0)^k \| f \| + |T_k f(x_k) - C(f)|, \]
which yields the assertion by Lemma 2.1. \( \square \)

3. Existence and uniqueness for reflecting diffusions in a domain with one singular point.

3.1. Formulation of the problem and preliminaries. We consider a connected domain \( D \subseteq \mathbb{R}^d \) that has a smooth boundary except at a single point, which we will take to be the origin, and that in a neighborhood of the singular point can be approximated by a cone. More precisely we assume the following condition (\( d_H \) denotes the Hausdorff distance).

**CONDITION 3.1.**

\begin{enumerate}[(i)]
  \item \( D \) is a bounded domain and \( \partial D - \{ 0 \} \) is of class \( C^1 \). There exist a nonempty domain, \( S \), in the unit sphere, \( S^{d-1} \), \( r_D > 0 \) and \( c_D > 0 \) such that, setting \( K := \{ x : x = rz, z \in S, r > 0 \} \),
  \begin{equation}
  d_H(\overline{D} \cap \partial B_r(0), \overline{K} \cap \partial B_r(0)) \leq c_D r^2, \tag{3.1}
  \end{equation}
  \begin{equation}
  \sup_{x \in \partial D \cap \partial B_r(0)} d(x, \partial K \cap \partial B_r(0)) \leq c_D r^2. \tag{3.2}
  \end{equation}
  \item For \( x \in \partial D - \{ 0 \} \) denote by \( n(x) \) the unit inward normal to \( \overline{D} \) and for \( x \in \partial K - \{ 0 \} \) denote by \( n^K(x) \) the unit inward normal to \( \overline{K} \). Note that \( n^K \) has the scaling property: \( n^K(rz) = n^K(z) \) for all \( z \in D, r > 0 \). For \( x \in \partial D - \{ 0 \} \), \( |x| \leq r_D \), and \( \tilde{z} \in \partial S \) such that \( \| \frac{x}{|x|} - \tilde{z} \| = d(\frac{x}{|x|}, \partial S) \),
  \begin{equation}
  |n(x) - n^K(\tilde{z})| \leq c_D |x|. \tag{3.3}
  \end{equation}
  \item For \( d \geq 3 \), the boundary \( \partial S \) of \( S \) in \( S^{d-1} \) is of class \( C^3 \).
\end{enumerate}

**REMARK 3.2.** Clearly (3.2) implies that, for every \( x \in \partial D, |x| \leq r_D \),
\[ d(\frac{x}{|x|}, \partial S) \leq c_D |x|, \]
and hence, by the smoothness of \( \partial S \) (Condition 3.1(iii)), for \( r_D \) sufficiently small and \( |x| \leq r_D \), there is a unique \( \tilde{z} \in \partial S \) such that
\[ \| \frac{x}{|x|} - \tilde{z} \| = d(\frac{x}{|x|}, \partial S). \]
The same for \( x \in \overline{D} - \overline{K} \).
We assume the following on the directions of reflection.

**Condition 3.3.**

(i) \( g : \mathbb{R}^d - \{0\} \rightarrow \mathbb{R}^d \) is a locally Lipschitz continuous vector field, of unit length on \( \partial D - \{0\} \), such that

\[
\inf_{x \in \partial D - \{0\}} g(x) \cdot n(x) > 0.
\]

There exists a unit vector field \( \bar{g} : \partial \mathcal{S} \rightarrow \mathbb{R}^d \) and \( c_g > 0 \) such that, for \( x \in \partial D - \{0\} \) and \( \bar{z} \) the closest point to \( \frac{x}{|x|} \) on \( \partial \mathcal{S} \),

\[
|g(x) - \bar{g}(\bar{z})| \leq c_g|x|, \quad \text{for } |x| \leq r_D.
\]

(ii) Extending the definition of \( \bar{g} \) to \( \partial \mathcal{K} - \{0\} \) by

\[
\bar{g}(x) := \bar{g}(\frac{x}{|x|}),
\]

\[
\inf_{x \in \partial \mathcal{K} - \{0\}} \bar{g}(x) \cdot n^\mathcal{K}(x) > 0.
\]

For \( d \geq 3 \), \( \bar{g} \) is of class \( C^2 \) and, for \( x \in \partial \mathcal{K} - \{0\} \), denoting by \( v^r(x) := \frac{x}{|x|} \), the radial unit vector, \( \frac{\bar{g} - \bar{v} - v^r \cdot \bar{n}^\mathcal{K}}{\bar{n}^\mathcal{K}} \) is of class \( C^4 \) (see Remark 3.5.)

(iii) For \( x \in \partial D - \{0\} \), let \( G(x) := \{ \eta g(x), \eta \geq 0 \} \), and let \( G(0) \) be the closed, convex cone generated by \( \{ \bar{g}(z), z \in \partial \mathcal{S} \} \). Assume

\[
G(0) \cap \mathcal{K} \neq \emptyset.
\]

(iv) Let \( N(0) \) denote the normal cone at the origin for \( \overline{D} \), that is,

\[
N(0) := \{ n \in \mathbb{R}^d : \liminf_{x \in \overline{D} - \{0\}, x \to 0} n \cdot \frac{x}{|x|} \geq 0 \}.
\]

There exists a unit vector \( e \in N(0) \) such that

\[
\inf_{g \in G(0), |g|=1} e \cdot g = c_e > 0,
\]

and, possibly by taking a smaller \( r_D \),

\[
e \cdot x \geq 0, \quad \forall x \in \overline{D}, |x| \leq r_D.
\]

(3.6) is always satisfied if \( D \) is convex or if \( N(0) \neq \emptyset \): See Remark 3.4 below.

**Remark 3.4.** \( N(0) \) is a closed convex cone. By Condition 3.1(i), \( N(0) \) is also the normal cone at the origin for \( \mathcal{K} \), i.e.

\[
N(0) = \{ n \in \mathbb{R}^d : n \cdot x \geq 0, \forall x \in \mathcal{K} \}.
\]

Condition 3.3(iv) implies that \( N(0) \neq \emptyset \), hence \( \mathcal{K} \) is always contained in a closed halfspace. If \( N(0) \) is nonempty, we can assume, without loss of generality, that \( e \in N(0) \). Then \( \mathcal{K} - \{0\} \) is contained in the open half space \( \{ x \in \mathbb{R}^d : x \cdot e > 0 \} \), and hence, by (3.1), possibly taking a smaller \( r_D \), \( (\mathcal{D} - \{0\}) \cap \overline{B_{r_D}(0)} \) is contained in the same open half space.

Assuming (3.6) instead of \( N(0) \neq \emptyset \) allows, for instance, to deal with same cases in which \( \partial D \) is actually smooth but \( g \) is discontinuous at the origin. (3.6) is used only in the proof of Lemma 3.20.
AN ERGODIC THEOREM AND UNIQUENESS FOR REFLECTING DIFFUSIONS

Remark 3.5. Conditions 3.1(iii) and 3.3(ii) are the assumptions of Kwon and Williams (1991), which we need because we will exploit some of their results.

Reflecting diffusions are often characterized as solutions of stochastic differential equations. Assume the following.

Condition 3.6.

(i) \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d} \) are Lipschitz continuous.
(ii) \( \sigma(0) \) is non singular.

Definition 3.7. A continuous process \( X \) is a solution of the stochastic differential equation with reflection in \( \overline{D} \) with coefficients \( b \) and \( \sigma \) and direction of reflection \( g \), if there exist a standard Brownian motion \( W \), a continuous, non decreasing process \( \lambda \) and a process \( \gamma \) with measurable paths, all defined on the same probability space as \( X \), such that \( W(t + \cdot) - W(t) \) is independent of \( \mathcal{F}_t^{X,W,\lambda,\gamma} \) for all \( t \geq 0 \) and

\[
X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \gamma(s)d\lambda(s), \quad t \geq 0,
\]

\[
\gamma(t) \in G(X(t)), \quad |\gamma(t)| = 1, \quad d\lambda - a.e., \quad t \geq 0,
\]

\[
X(t) \in \overline{D}, \quad \lambda(t) = \int_0^t 1_{\partial D}(X(s))d\lambda(s), \quad t \geq 0,
\]

is satisfied a.s..

Given an initial distribution \( \nu \in \mathcal{P}(\overline{D}) \), weak uniqueness or uniqueness in distribution holds if any two solutions of (3.8) with \( P\{X(0) = \cdot \} = \nu \) have the same distribution on \( C_{\overline{D}}[0,\infty) \).

A stochastic process \( \tilde{X} \) (for example a solution of an appropriate martingale problem or submartingale problem) is a weak solution of (3.8) if there is a solution \( X \) of (3.8) such that \( \tilde{X} \) and \( X \) have the same distribution.

We say that (3.8) admits a strong solution if, given a filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}) \), a standard \( \mathcal{F}_t \)-Brownian motion \( W \) and a \( \mathcal{F}_0 \)-measurable, \( \overline{D} \)-valued random variable \( X_0 \), there exist a continuous process \( X \), with \( X(0) = X_0 \), a continuous, non decreasing process \( \lambda \) and a process \( \gamma \) with measurable paths, all \( \mathcal{F}_t \)-adapted, such that (3.8) is satisfied a.s..

We say that strong (or pathwise) uniqueness holds for (3.8) if any two solutions defined on the same probability space with the same Brownian motion and the same initial condition coincide for all times a.s..

We denote by \( A \) the operator

\[
\mathcal{D}(A) := C^2(\overline{D}), \quad Af(x) := b(x) \cdot \nabla f(x) + \frac{1}{2} \text{tr}((\sigma \sigma^T)(x)D^2 f(x)).
\]

Remark 3.8. Since \( \sigma(0) \) is non singular, it is easy to check that \( X \) is a solution to (3.8) if and only if \( \sigma^{-1}(0)X \) is a solution of (3.8), with the appropriate coefficients, in the corresponding domain \( \sigma^{-1}(0)D \), with direction of reflection \( \frac{\sigma^{-1}(0)g \circ \sigma(0)}{|\sigma^{-1}(0)g \circ \sigma(0)|} \), and the new domain and direction of reflection satisfy Condition 3.1 and Condition 3.3 if and only if \( D \) and \( g \) do. Therefore, without loss of generality, we will take, from now on,

\[
\sigma(0) = I.
\]
Conditions 3.1, 3.3 and 3.6 will be our standing assumptions.

Reflecting diffusions can also be characterized as natural solutions of constrained martingale problems. Constrained martingale problems have been introduced by Kurtz (1990) and Kurtz (1991)) for operators on functions defined on metric spaces, and further studied in Kurtz and Stockbridge (2001), Costantini and Kurtz (2015) and Costantini and Kurtz (2019). Here we consider the specific constrained martingale problem corresponding to (3.8), which is formulated as follows.

Let $G(x), x \in \partial D - \{0\}$, and $G(0)$ be as in Condition 3.3 (iii), and let

$$U := S^{d-1},$$

(3.10) $\Xi := \{(x, u) \in \partial D \times U : u \in G(x)\}$,

$B : D(B) := C^2(\overline{D}) \to C(\Xi)$, $Bf(x, u) := \nabla f(x) \cdot u$

The following argument shows that $\Xi$ is closed: Suppose $\{(x^n, u^n)\} \subseteq \Xi, (x^n, u^n) \to (x, u)$ as $n \to \infty$. If $x \neq 0$, eventually $x^n \neq 0$, $u^n = g(x^n)$ and $u = g(x)$ by the continuity of $g$; if $x = 0$ and there exists a subsequence $\{x^{n_i}\}$ such that $x^{n_i} = 0$ for all $i$, then $u^{n_i} \in G(0)$ for all $i$, hence $u \in G(0)$ because $G(0)$ is closed; If eventually $x^n \neq 0$, then eventually $|g(x^n) - \bar{g}(\bar{z}^n)| \leq c_g |x^n|$, where $\bar{z}^n$ is the closest point to $x^n/|x^n|$ on $\partial S$, so that $\bar{g}(\bar{z}^n) \to u$ as $n \to \infty$ and again $u \in G(0)$ because $G(0)$ is closed.

Define $\mathcal{L}_U$ to be the space of measures $\mu$ on $[0, \infty) \times U$ such that $\mu([0, t] \times U) < \infty$ for all $t > 0$. $\mathcal{L}_U$ is topologized so that $\mu_n \in \mathcal{L}_U \to \mu \in \mathcal{L}_U$ if and only if

$$\int_{[0, \infty) \times U} f(s, u) \mu_n(ds \times du) \to \int_{[0, \infty) \times U} f(s, u)\mu(ds \times du)$$

for all continuous $f$ with compact support in $[0, \infty) \times U$. It is possible to define a metric on $\mathcal{L}_U$ that induces the above topology and makes $\mathcal{L}_U$ into a complete, separable metric space. Also define $\mathcal{L}_\Xi$ in the same way.

We will say that an $\mathcal{L}_U$-valued random variable $\Lambda_1$ is adapted to a filtration $\{\mathcal{G}_t\}$ if

$$\Lambda_1([0, \cdot] \times C) \subseteq \{\mathcal{G}_t\} \text{ adapted, } \forall C \in \mathcal{B}(U).$$

We define an adapted $\mathcal{L}_\Xi$-valued random variable analogously.

**Definition 3.9.** Let $A, \Xi$ and $B$ be as in (3.9) and (3.10). A process $X$ in $D_{\overline{D}}[0, \infty)$ is a solution of the constrained martingale problem for $(A, D, B, \Xi)$ if there exists a random measure $\Lambda$ with values in $\mathcal{L}_\Xi$ and a filtration $\{\mathcal{F}_t\}$ such that $X$ and $\Lambda$ are $\{\mathcal{F}_t\}$-adapted and for each $f \in C^2(\overline{D})$,

(3.11) $f(X(t)) - f(X(0)) - \int_0^t A f(X(s))ds - \int_{[0, t] \times \Xi} Bf(x, u)\Lambda(ds \times dx \times du)$

is a $\{\mathcal{F}_t\}$-local martingale. By the continuity of $f$, we may assume, without loss of generality, that $\{\mathcal{F}_t\}$ is right continuous.

Given $\nu \in \mathcal{P}(\overline{D})$, we say that there is a unique solution of the constrained martingale problem for $(A, D, B, \Xi)$ with initial distribution $\nu$ if any two solutions with $P\{X(0) \in \cdot\} = \nu$ have the same distribution on $D_{\overline{D}}[0, \infty)$.

**Remark 3.10.** Since $f(x) \equiv x_i$ $i = 1, ..., d$, belongs to $\mathcal{D}(A) = \mathcal{D}(B)$, every solution of the constrained martingale problem for $(A, D, B, \Xi)$ is a semimartingale.
In general, an effective way of constructing solutions of a constrained martingale problem is by time-changing solutions of the corresponding controlled martingale problem which is a "slowed down" version of the constrained martingale problem.

**Definition 3.11.** Let $A, U, \Xi$ and $B$ be as in (3.9) and (3.10). $(Y, \lambda_0, \Lambda_1)$ is a solution of the controlled martingale problem for $(A, D, B, \Xi)$, if $Y$ is a process in $D_{\overline{D}[0, \infty)}$, $\lambda_0$ is nonnegative and nondecreasing, $\Lambda_1$ is a random measure with values in $L_U$ such that

$$\lambda_1(t) := \Lambda_1([0, t] \times U) = \int_{[0, t] \times U} \mathbf{1}_{\Xi}(Y(s), u) \Lambda_1(ds \times du),$$

and there exists a filtration $\{G_t\}$ such that $\lambda_0$, $\lambda_1$, and $\Lambda_1$ are $\{G_t\}$-adapted and

$$f(Y(t)) - f(Y(0)) - \int_0^t A f(Y(s))d\lambda_0(s) - \int_{[0, t] \times U} B f(Y(s), u) \Lambda_1(ds \times du)$$

is a $\{G_t\}$-martingale for all $f \in C^2(\overline{D})$. By the continuity of $f$, we may assume, without loss of generality, that $\{G_t\}$ is right continuous.

**Remark 3.12.** It can be easily verified (e.g. by Proposition 3.10.3 of Ethier and Kurtz (1986)) that, for every solution of the controlled martingale problem for $(A, D, B, \Xi)$, $Y$ is continuous.

**Definition 3.13.** Let $A, U, \Xi$ and $B$ be as in (3.9) and (3.10). A solution, $X$, of the constrained martingale problem for $(A, D, B, \Xi)$ is called natural if, for some solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for $(A, D, B, \Xi)$ with filtration $\{G_t\}$,

$$X(t) = Y(\lambda_0^{-1}(t)), \quad \mathcal{F}_t = G_{\lambda_0^{-1}(t)},$$

$$\Lambda([0, t] \times C) = \int_{[0, \lambda_0^{-1}(t)] \times U} \mathbf{1}_C(Y(s), u) \Lambda_1(ds \times du), \quad C \in \mathcal{B}(\Xi),$$

where $\lambda_0^{-1}(t) = \inf\{s : \lambda_0(s) > t\}, \quad t \geq 0$.

Given $\nu \in \mathcal{P}(\overline{D})$, we say that there is a unique natural solution of the constrained martingale problem for $(A, D, B, \Xi)$ with initial distribution $\nu$ if any two natural solutions with $P\{X(0) \in \cdot\} = \nu$ have the same distribution on $D_{\overline{D}[0, \infty)}$.

Given a solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for $(A, D, B, \Xi)$, the time changed process $X$ defined by (3.14) will not always be a solution of the corresponding constrained martingale problem. In fact it may be impossible to stop (3.13), after the time change by $\lambda_0^{-1}$, in such a way that the stopped process is a local martingale. Conditions under which it is possible are given in Costantini and Kurtz (2019), Corollary 3.9, and the following lemma guarantees that they are satisfied under our standing assumptions.

**Lemma 3.14.** There exists a function $F \in C^2(\overline{D})$ such that

$$\inf_{x \in \partial D} \inf_{g \in G(x), |g(x)| = 1} \nabla F(x) \cdot g := c_F > 0.$$

**Proof.** See Appendix B. □
PROPOSITION 3.15. Let $A$, $U$, $\Xi$ and $B$ be as in (3.9) and (3.10) and assume Conditions 3.1, 3.3 and 3.6.

For every solution of the controlled martingale problem for $(A, D, B, \Xi)$, $\lambda_0(t) \to \infty$ as $t \to \infty$ almost surely and the time changed process $X$ defined by (3.14) is a natural solution of the corresponding constrained martingale problem. (3.11) is a martingale.

PROOF. By Lemma 3.14, Lemma 3.1 of Costantini and Kurtz (2019) and Corollary 3.9 a) of Costantini and Kurtz (2019), $\lim_{t \to \infty} \lambda_0(t) = \infty$ a.s. and, after the time change by $\lambda_0^{-1}$, (3.13) is a martingale.

Theorem 3.18 below shows that, under our standing assumptions, the two characterizations of a reflecting diffusion in $\overline{D}$ with coefficients $b$ and $\sigma$ and direction of reflection $g$ as a solution of (3.8) and as a natural solution of the constrained martingale problem defined by (3.9) and (3.10) are equivalent. This result parallels the equivalence of the two characterizations of a reflecting diffusion as as a solution of (3.8) and as a solution of the corresponding submartingale problem proved in Theorem 1 of Kang and Ramanan (2017).

The proof of Theorem 3.18 relies on the following lemma.

LEMMA 3.16. For every solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for $(A, D, B, \Xi)$, $\lambda_0(t) > 0$ for all $t > 0$, a.s.

Moreover, $\lambda_0$ is strictly increasing, a.s..

PROOF. The first assertion follows essentially from Condition 3.3(iv) and Remark 3.12. The proof is analogous to that of Lemma 6.8 of Costantini and Kurtz (2019) and Lemma 3.1 of Dai and Williams (1996). The second assertion follows from Lemma 3.4 of Costantini and Kurtz (2019).

REMARK 3.17. It follows from Remark 3.12 and Lemma 3.16 that every natural solution of the constrained martingale problem for $(A, D, B, \Xi)$ is a.s. continuous and so is the corresponding process $\Lambda([0, \cdot] \times \Xi)$.

THEOREM 3.18. Let $A$, $U$, $\Xi$ and $B$ be as in (3.9) and (3.10) and assume Conditions 3.1, 3.3 and 3.6.

Every solution of (3.8) is a natural solution of the constrained martingale problem for $(A, D, B, \Xi)$.

Every natural solution of the constrained martingale problem for $(A, D, B, \Xi)$ is a weak solution of (3.8).

PROOF. The proof is the same as for Theorem 6.12 of Costantini and Kurtz (2019). It relies essentially on Lemma 3.16 and Remark 3.17.

REMARK 3.19. The first assertion of Proposition 3.15 could have been proved, alternatively, by Lemma 3.16 and Lemmas 3.3 and 3.4 of Costantini and Kurtz (2019).

We conclude this section with two important properties of a natural solution $X$ of the constrained martingale problem for $(A, D, B, \Xi)$. For $\delta > 0$, define

$$\tau^{X, \delta} := \inf\{t \geq 0 : |X(t)| = \delta\}.$$  

Whenever there is no risk of confusion, we will omit the superscript $X$. 
Lemma 3.20. There exists $\delta > 0$, $\delta \leq \tau_D$, $\bar{c} > 0$, depending only on the data of the problem, such that, for $\delta \leq \hat{\delta}$, for every natural solution $X$ of the constrained martingale problem for $(A, D, B, \Xi)$ starting at $0$,

$$\mathbb{E}[r^{X, \delta}] \leq \bar{c}\delta^2.$$ 

Proof. The assertion follows essentially from Condition 3.3(iv). The proof is analogous to that of Lemma 4.2 of Costantini and Kurtz (2018) and Lemma 6.4 of Taylor and Williams (1993).

Lemma 3.21. For every natural solution $X$ of the constrained martingale problem for $(A, D, B, \Xi)$,

$$\int_0^\infty \mathbf{1}_{\{0\}}(X(t))\,dt = 0, \text{ a.s.}$$

Proof. The proof uses the same argument as Lemma 2.1 of Taylor and Williams (1993). Fix an arbitrary unit vector $v$. Then, by Remark 3.17,

$$m(t) := v \cdot X(t) - v \cdot X(0) - \int_0^t v \cdot b(X(s))\,ds - \int_{[0,t] \times \Xi} v \cdot u \Lambda(ds \times du)$$

is a continuous semimartingale with

$$[m, m](t) = \int_0^t |\sigma(X(s))^T v|^2\,ds.$$

Therefore, by Tanaka’s formula (see, e.g., Protter (2004), Corollary 1 to Theorem 51, Chapter IV, Section 5), for each $t > 0$,

$$\int_0^t \mathbf{1}_{\{0\}}(X(s) \cdot v) |\sigma(X(s))^T v|^2\,ds = \int_\mathbb{R} \mathbf{1}_{\{0\}}(a) L_t(a)\,da = 0, \text{ a.s.,}$$

$L_t(a)$ being the local time of $m$ at $a$. Hence the set of times $\{s \leq t : X(s) \cdot v = 0 \text{ and } |\sigma(X(s))^T v| \neq 0\}$ has zero Lebesgue measure, a.s., which yields the assertion by Condition 3.6(ii).

3.2. Existence. In this subsection we show that there exists a strong Markov, natural solution of the constrained martingale problem for $(A, D, B, \Xi)$, and hence, by Theorem 3.18, of the stochastic differential equation with reflection (3.8). The strong Markov property will be crucial in our argument to prove uniqueness of the solution. Our arguments are very similar to those of Taylor and Williams (1993), but using the constrained martingale problem approach we do not need the oscillation estimates (4.5) and (4.6) of Taylor and Williams (1993).

Lemma 3.22. Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, a standard $\{\mathcal{F}_t\}$-Brownian motion $W$ and a $\mathcal{F}_0$-measurable random variable $\xi_0$ with compact support in $\overline{\mathcal{T}} - \{0\}$, there exist a continuous process $\xi$ and a continuous, nondecreasing process $\ell$ such that, setting

$$(3.16) \quad \theta := \inf\{t \geq 0 : \xi(t) = 0\},$$

$\xi$ and $\ell$ satisfy, a.s.,

$$\xi(t) = \xi_0 + \int_0^t b(\xi(s))\,ds + \int_0^t \sigma(\xi(s))\,dW(s) + \int_0^t g(\xi(s))\,d\ell(s), \quad 0 \leq t < \theta,$$

$$(3.17) \quad \xi(t) \in \overline{\mathcal{T}} - \{0\}, \quad \int_0^t \mathbf{1}_{\partial D - \{0\}}(\xi(s))\,d\ell(s) = \ell(t), \quad 0 \leq t < \theta,$$
and, on the set \( \{ \theta < \infty \} \), \( \xi(t) = 0 \), \( \ell(t) = \ell(\theta) \) for \( t \geq \theta \).

Moreover \( \xi \) is pathwise unique.

**Proof.** Let \( \{ \delta_k \} \) be a strictly decreasing sequence of positive numbers converging to zero, and let \( \{ D^k \} \) be a sequence of bounded domains with \( C^1 \) boundary such that \( D^k \subset D^{k+1} \subset D \), \( D^k \cap B_{\delta_k}(0)^c = \overline{D} \cap B_{\delta_k}(0)^c \) and \( D^k \cap B_{\delta_k}(0) \subset D^{k+1} \). Also let \( g^k : \mathbb{R}^d - \{ 0 \} \rightarrow \mathbb{R}^d \), \( k \in \mathbb{N} \), be a locally Lipschitz vector field, of unit length on \( \partial D^k \), such that \( g^k(x) = g(x) \) for \( x \in \partial D \cap B_{\delta_k}(0)^c \) and that, denoting by \( n^k(x) \) the unit, inward normal at \( x \in \partial D^k \), \( \inf_{x \in \partial D^k} g^k(x) \cdot n^k(x) > 0 \). Then we know, by the results of Dupuis and Ishii (1993), that there is a strong solution to (3.8) in \( \overline{D}^k \) with direction of reflection \( g^k \) and it is pathwise unique. Given a Brownian motion \( W \), for each random variable \( \xi_0 \) with compact support in \( \overline{D} - \{ 0 \} \), independent of \( W \), for all \( k \) large enough that \( \xi_0 \) is supported in \( \overline{D}^k \), let \( \xi^k \) be the strong solution of (3.8) in \( \overline{D}^k \) with direction of reflection \( g^k \), with Brownian motion \( W \) and initial condition \( \xi_0 \), and \( \ell^k \) be the corresponding nondecreasing process. The sequence of stopping times \( \{ \theta^k \} \),

\[
\theta^k := \inf\{ t \geq 0 : \xi^k(t) \in \partial D^k \cap B_{\delta_k}(0) \}, \]

is strictly increasing and, setting

\[
\xi(t) := \xi^k(t), \quad \ell(t) := \ell^k(t), \quad 0 \leq t \leq \theta^k, \quad \theta := \lim_{k \to \infty} \theta^k, \]

\( \xi \) and \( \ell \) satisfy (3.17).

We will now show that, on the set \( \{ \theta < \infty \} \),

\[
(3.18) \quad \sup_{0 \leq t < \theta} \ell(t) < \infty, \quad \lim_{t \to \theta^-} \xi(t) = 0, \quad \text{a.s.} \]

In fact, let \( F \) be the function of Lemma 3.14. Then, by Ito’s formula, on the set \( \{ \theta < \infty \} \), we have

\[
\ell(\theta^k) \leq \frac{1}{c_F} \left\{ 2\|F\| + \|AF\| \theta^k + \left| \int_0^{\theta^k} \nabla F(\xi(s)^T \sigma(\xi(s)))dW(s) \right| \right\}
\]

\[
= \frac{1}{c_F} \left\{ 2\|F\| + \|AF\| \theta^k + \left| \int_0^{\theta^k} 1_{\{s < \theta\}} \nabla F(\xi(s)^T \sigma(\xi(s)))dW(s) \right| \right\}.
\]

Since the process \( 1_{\{s < \theta\}} \nabla F(\xi(s)^T \sigma(\xi(s))) \) is predictable and bounded, the process \( \int_0^s 1_{\{ s < \theta \}} \nabla F(\xi(s)^T \sigma(\xi(s)))dW(s) \) is a.s. continuous. Therefore, on the set \( \{ \theta < \infty \} \), the limit \( \lim_{k \to \infty} \int_0^{\theta^k} 1_{\{ s < \theta \}} \nabla F(\xi(s)^T \sigma(\xi(s)))dW(s) \) exists and is finite a.s.. This yields the first assertion in (3.18) and allows to define, on the set \( \{ \theta < \infty \} \), \( \ell(\theta) := \sup_{0 \leq t < \theta} \ell(t) \). The second assertion follows by observing that, on the set \( \{ \theta < \infty \} \), all the terms on the r.h.s. of the first equation in (3.17) are uniformly continuous on \([0, \theta] \). \( \square \)

**Theorem 3.23.** Under Conditions 3.1, 3.3 and 3.6, for each \( \nu \in \mathcal{P}(\overline{D}) \), there exists a strong Markov, natural solution of the constrained martingale problem for \((A, D, B, \Xi)\), with initial distribution \( \nu \).

**Proof.** We will first construct a solution of the controlled martingale problem. Let \( g^0 \) be a unit vector in \( G(0) \cap K \) (Condition 3.3 (iii)). Then, by Condition 3.1(i), for \( \rho \) small enough
\( \rho g^0 \in G(0) \cap D \). Let \( \{ \rho_n \} \) be a decreasing sequence of positive numbers such that \( \rho_n \to 0 \) and \( \rho_n g^0 \in G(0) \cap D \) for all \( n \). Consider the stochastic differential equation with reflection

\[
X^n(t) = X^n_0 + \int_0^t b(X^n(s))ds + \int_0^t \sigma(X^n(s))dW(s) + \int_0^t g(X^n(s))dl^n(s) + \rho_n g^0 L^n(t),
\]

(3.19) \( X^n(t) \in \overline{D} - \{0\} \), \( l^n \) non decreasing, \( \int_0^t 1_{\partial D - \{0\}}(X^n(s))dl^n(s) = l^n(t) \),

\[
L^n(t) = \# \{ s \leq t : X^n(s^-) = 0 \},
\]

where \( \# \) denotes cardinality, \( W \) is a standard Brownian motion, \( X^n_0 \) is a random variable independent of \( W \), with law \( \nu_n \) defined as

\[
\nu_n(C) := \nu(\{0\}) \delta_{x^n}(C) + \nu(\overline{D} - \{0\}) \frac{\nu(C \cap K_n)}{\nu(K_n)}, \quad \forall C \in \mathcal{B}(\overline{D}),
\]

where \( \{ x^n \} \) is a sequence of points in \( \overline{D} - \{0\} \) converging to 0 and \( \{ K_n \} \) is an increasing sequence of compact sets of \( \mathbb{R}^d \) such that \( \bigcup_{n=1}^\infty K_n = \overline{D} - \{0\} \). Then \( \nu_n \) has compact support in \( \overline{D} - \{0\} \) and \( \{ \nu_n \} \) converges weakly to \( \nu \). Existence of \( X^n \) follows from Lemma 3.22.

Define

\[
\lambda^n_0(t) := \inf \{ s : s + l^n(s) + \rho^n L^n(s) > t \},
\]

\[
\Lambda^n_1([0,t] \times C) := \int_0^t 1_C(g(X^n(\lambda^n_0(s)))dl^n(\lambda^n_0(s))) + 1_{C}(g^0(\rho_n L^n(\lambda^n_0(s))),
\]

\[
B^n f(x,u) := u \cdot \nabla f(x) 1_{\partial D - \{0\}}(x) + (\rho_n)^{-1}[f(x + \rho_n u) - f(x)]1_{\{0\}}(x),
\]

and \( Y^n(t) := X^n(\lambda^n_0(t)) \), in particular \( Y^n(0) = X^n_0 \). Then for each \( f \in \mathcal{C}^2(\overline{D}) \),

\[
f(Y^n(t)) - f(Y^n(0)) - \int_0^t A f(Y^n(s))d\lambda^n_0(s) - \int_{[0,t] \times U} B^n f(Y^n(s^-),u)\Lambda^n_1(ds \times du)
\]

is a martingale with respect to \( \{\mathcal{F}^{Y^n,\lambda^n_0,\Lambda^n_1}_t\} \). \( (Y^n, \lambda^n_0, \Lambda^n_1) \) is not a solution of the controlled martingale problem for \( (A,D,B^n,\Xi) \) in the precise sense of Costantini and Kurtz (2019) and Kurtz (1991), because \( B^n f \) is not continuous on \( \Xi \) and because we can only say

\[
t \leq \lambda^n_0(t) + \Lambda^n_1([0,t] \times U) \leq t + \rho_n.
\]

However, the same relative compactness arguments as, for example, in Lemma 2.8 of Costantini and Kurtz (2019) apply and, since the law of \( Y^n(0) \), \( \nu_n \), converges to \( \nu \), any limit point of \( \{(Y^n, \lambda^n_0, \Lambda^n_1)\} \) will be a solution of the controlled martingale problem for \( (A,D,B,\Xi) \) with initial distribution \( \nu \).

Then the assertion follows from Lemma 3.14, Lemma 3.16, Lemma 3.1 of Costantini and Kurtz (2019) and Corollary 4.12 a) of Costantini and Kurtz (2019).

3.3. Uniqueness. We state preliminarily the following lemma, which holds for constrained martingale problems in general.

**Lemma 3.24.** Suppose that for each \( \nu \in \mathcal{P}(\overline{D}) \) any two strong Markov, natural solutions of the constrained martingale problem for \( (A,D,B,\Xi) \) with initial distribution \( \nu \) have the same one-dimensional distributions. Then, for each \( \nu \in \mathcal{P}(\overline{D}) \), any two strong Markov, natural solutions of the constrained martingale problem for \( (A,D,B,\Xi) \) with initial distribution \( \nu \) have the same distribution.
PROOF. The proof of point (a) of Theorem 4.4.2 of Ethier and Kurtz (1986) carries over. \(\square\)

Our uniqueness result is formulated in the following theorem. By Theorem 3.18, it is equivalent to uniqueness in distribution for the solution of (3.8).

**THEOREM 3.25.** Under Conditions 3.1, 3.3 and 3.6, for every \(\nu \in \mathcal{P}(\mathcal{D})\), there is a unique natural solution of the constrained martingale problem for \((A, D, B, \Xi)\) with initial distribution \(\nu\). The solution is a strong Markov process.

The proof of Theorem 3.25 requires several steps. The first building block is the following lemma. For \(\delta^*, 0 < \delta^* \leq \tilde{\delta}\), where \(\tilde{\delta}\) is as in Lemma 3.20, let

\[
D_n := \overline{D} \cap B_{\delta^* 2^{-2n}}, \quad E_n := \{x \in \overline{D} : |x| = \delta^* 2^{-2n}\}, \quad n \geq 0.
\]

Let \(X\) be a solution of the constrained martingale problem for \((A, D, B, \Xi)\). Define recursively

\[
\tau^0_n := 0, \quad \vartheta^0_n := \vartheta := \inf\{t \geq 0 : X(t) = 0\}, \quad \tau^n := \inf\{t \geq 0 : X(t) \in E_n\}, \quad n \geq 0,
\]

\[
(3.21) \quad \tau^n_l := \inf\{t \geq \vartheta^n_{l-1} : X(t) \in E_n\}, \quad \vartheta^n_l := \inf\{t \geq \tau^n_l : X(t) = 0\}, \quad l \geq 1, \quad n \geq 0.
\]

**LEMMA 3.26.** Suppose that, for some \(\delta^*, 0 < \delta^* \leq \bar{\delta}\), the hitting distributions \(\{\mu_n\}\) defined by

\[
\mu_n(C) := \mathbb{P}\{X(\tau^n) \in C\}, \quad C \in \mathcal{B}(E_n), \quad n \geq 0,
\]

are the same for any strong Markov, natural solution \(X\) of the constrained martingale problem for \((A, D, B, \Xi)\) starting at 0.

Then, for each \(\nu \in \mathcal{P}(\overline{D})\), there is a unique natural solution of the constrained martingale problem for \((A, D, B, \Xi)\) with initial distribution \(\nu\).

**PROOF.** Lemma 3.16 allows to apply Corollary 4.13 of Costantini and Kurtz (2019). Therefore it is enough to prove uniqueness among strong Markov, natural solutions of the constrained martingale problem for \((A, D, B, \Xi)\). Let \(X\) be such a solution with initial distribution \(\nu\).

For \(\eta > 0\) and \(f \in \mathcal{C}(\overline{D})\) that vanishes in a neighborhood of the origin, define, for each \(n \geq 0\),

\[
R_{\eta}^n f := \mathbb{E} \left[ \int_0^\vartheta e^{-\eta t} f(X(t)) dt \right] + \mathbb{E} \left[ \sum_{l=1}^{\infty} \prod_{m=0}^{l-1} e^{-\eta(\vartheta^m_l - \tau^m_l)} \int_{\tau^m_l}^{\vartheta^m_l} e^{-\eta(t - \tau^m_l)} f(X(t)) dt \right],
\]

with the convention \(e^{-\infty} = 0\). The hypotheses ensure that the distribution of \(X(\tau^m_l)\) is \(\mu_n\) for all \(l\). Then, by the strong Markov property, Lemma 3.22 and Theorem 3.18, for each \(l\) the factors of the product in the infinite sum are independent, with distributions determined by the initial distribution \(\nu\) and by the unique distribution of \(X^n(\cdot \land \theta^n)\), where \(X^n\) is a solution of the constrained martingale problem with initial distribution \(\mu_n\) and \(\theta^n\) is the first time \(X^n\) hits zero. Consequently, each term on the right hand side is uniquely determined. The independence and the fact that \(\{\vartheta^m_l - \tau^m_l\}_{m \geq 1}\) are identically distributed imply also that the series is convergent and has finite expectation.
Let $\mathcal{T}^n := [0, \vartheta) \cup \bigcup_{i=1}^{\infty} [T^n_i, \vartheta^n_i)$. Then $R^n_\eta f$ can be written as

$$R^n_\eta f = \mathbb{E} \left[ \int_0^\infty 1_{\mathcal{T}^n}(t) e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} f(X(t)) dt \right].$$

We are going to show that

$$R^n_\eta f \to_{n \to \infty} \mathbb{E} \left[ \int_0^\infty e^{-\eta t} f(X(t)) dt \right] := R_\eta f.$$

First of all note that, for $n$ large enough, depending only on $f$,

$$\int_0^\infty 1_{\mathcal{T}^n}(t) e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} f(X(t)) dt = \int_0^\infty e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} f(X(t)) dt, \quad a.s.,$$

$$\int_0^\infty 1_{\mathcal{T}^n}(t) e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} |f(X(t))| dt = \int_0^\infty e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} |f(X(t))| dt, \quad a.s..$$

By Lemma 3.21, for each $t \geq 0$,

$$e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} \to_{n \to \infty} e^{-\eta t}, \quad a.s.,$$

therefore we only need to show that we can pass to the limit in the time integral and under the expectation. Now we have

$$\int_0^\infty 1_{\mathcal{T}^n}(s) ds = \vartheta + \sum_{i=1}^{\infty} (\vartheta^n_i - T^n_i) = \infty, \quad a.s., \forall n,$$

because the random variables in the infinite sum in the right hand side are positive i.i.d. random variables. Hence, by (3.23),

$$\lim_{t_0 \to \infty} \limsup_{n \to \infty} \int_{t_0}^\infty e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} |f(X(t))| dt$$

$$= \lim_{t_0 \to \infty} \limsup_{n \to \infty} \int_{t_0}^\infty 1_{\mathcal{T}^n}(t) e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} |f(X(t))| dt$$

$$\leq \lim_{t_0 \to \infty} \limsup_{n \to \infty} \frac{\|f\|}{\eta} e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} = 0 \quad a.s.,$$

so that, almost surely, the sequence $\{e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} f(X(t))\}$ is uniformly integrable in time and we can pass to the limit in the time integral. Analogously

$$\int_0^\infty e^{-\eta \int_0^t 1_{\mathcal{T}^n}(s) ds} |f(X(t))| dt \leq \frac{\|f\|}{\eta}, \quad a.s.,$$

and we can also pass to the limit under the expectation.

The class of continuous functions on $\overline{D}$ that vanish in a neighborhood of the origin is separating for the probability measures on $\overline{D}$, therefore it follows that the one-dimensional distributions of $X$ are uniquely determined, and hence the assertion, by Lemma 3.24.

The next lemma shows how Theorem 2.4 comes into play in verifying the assumption of Lemma 3.26. Recall that, by Lemma 3.22, for all $n \geq 0$, $x \in \overline{D} - \{0\}$, $f \in \mathcal{C}(E_n)$, the expectation

$$\mathbb{E}[f(X(\tau^n)) 1_{\tau^n < \vartheta} | X(0) = x]$$

is the same for all natural solutions $X$ of the constrained martingale problem for $(A, D, B, \Xi)$. 


Lemma 3.27. For $k \geq 1$, let $Q_k$ be the subprobability transition operator defined by

\begin{equation}
Q_k f(x) := \mathbb{E}[f(X(\tau_{k-1}))1_{\tau_{k-1}<\vartheta}X(0) = x], \quad x \in E_k, \quad f \in \mathcal{C}(E_{k-1}).
\end{equation}

Then, for every strong Markov, natural solution $X$ of the constrained martingale problem for $(A,D,B,\Xi)$ starting at 0, for every $n \geq 0$,

\begin{equation}
\mathbb{E}[f(X(\tau^n))] = \frac{\int Q_{n+k} \cdots Q_{n+1} f(x) \mu_{n+k}(dx)}{\int Q_{n+k} \cdots Q_{n+1} 1(x) \mu_{n+k}(dx)}, \quad \forall f \in \mathcal{C}(E_n), \quad \forall k \geq 1,
\end{equation}

where $\mu_{n+k}$ is defined by (3.22).

Proof. Note that, since $X$ starts at 0, $\tau^n = \tau_1^n$ for all $n \geq 0$. By the strong Markov property, we have, for $n \geq 0$,

\begin{align*}
\mathbb{E}[f(X(\tau^n))] &= \mathbb{E}[f(X(\tau^n))1_{\vartheta_1^n < \tau_{n+k-1}}] \\
&= \mathbb{E}[f(X(\tau^n))1_{\vartheta_1^n < \tau_{n+k-1}}1_{\vartheta_1^{n+k-1} < \tau_{n+k-2}}] \\
&\quad + \mathbb{E}[f(X(\tau^n))1_{\vartheta_1^{n+k} > \tau_{n+k-1}}1_{\vartheta_1^{n+k-1} < \tau_{n+k-2}}] \\
&\quad + \cdots \\
&\quad + \mathbb{E}[f(X(\tau^n))1_{\vartheta_1^{n+k} > \tau_{n+k-1}} \cdots 1_{\vartheta_1^{n+2} > \tau_{n+1}}1_{\vartheta_1^{n+1} < \tau_n}] \\
&\quad + \mathbb{E}[f(X(\tau^n))1_{\vartheta_1^{n+k} > \tau_{n+k-1}} \cdots 1_{\vartheta_1^{n+2} > \tau_{n+1}}1_{\vartheta_1^{n+1} > \tau_n}] \\
&= \mathbb{E}[f(X(\tau^n))\mathbf{1}_{\vartheta_1^n < \tau_{n+k-1}}1_{\vartheta_1^{n+k-1} < \tau_{n+k-2}}] \\
&\quad + \mathbb{E}[f(X(\tau^n))\mathbf{1}_{\vartheta_1^{n+k} > \tau_{n+k-1}} \cdots 1_{\vartheta_1^{n+2} > \tau_{n+1}}1_{\vartheta_1^{n+1} < \tau_n}] \\
&\quad + \mathbb{E}[f(X(\tau^n))\mathbf{1}_{\vartheta_1^{n+k} > \tau_{n+k-1}} \cdots 1_{\vartheta_1^{n+2} > \tau_{n+1}}1_{\vartheta_1^{n+1} > \tau_n}]
\end{align*}

Proof of Theorem 3.25 Suppose that, for some $\delta^*, 0 < \delta^* \leq \tilde{\delta}$ and for each fixed $n \geq 0$, the sequence of subprobability transition functions $\{Q_{n+k}\}_{k \geq 1}$ defined in Lemma 3.27 satisfies the assumptions of Theorem 2.4. Then, for every $f \in \mathcal{C}(E_n)$ there exists a constant $C_n(f)$ such that

\[ \mathbb{E}[f(X(\tau^n))] = C_n(f) \]
for any strong Markov, natural solution \( X \) of the constrained martingale problem for \( (A, D, B, \Xi) \) starting at 0, and the theorem follows from Lemma 3.26. Therefore we are reduced to verifying that the sequence of subprobability transition functions \( \{Q_{n+k}\}_{k \geq 1} \) satisfies the assumptions of Theorem 2.4. This is the object of the next two subsections. More precisely, in Section 3.4 we show that the subprobability transition functions \( \{Q_{n+k}\}_{k \geq 1} \) are not identically zero and that condition (i) is verified and in Section 3.5 we show that condition (ii) is verified. \( \square \)

3.4. Estimates on hitting times. In this subsection we prove that, for \( \delta^* \) small enough, for each \( n \geq 0 \), the subtransition functions \( \{Q_{n+k}\}_{k \geq 1} \) defined by (3.24), (3.22) and (3.20) are not identically zero and verify assumption (i) of Theorem 2.4.

We have, for \( x \in E_{n+k} \),

\[
Q_{n+k}1(x) = \mathbb{P}(\tau^{n+k-1} < \theta | X(0) = x),
\]

(3.25)

\[
Q_{n+k} \cdots Q_{n+1}1(x) = \mathbb{P}(\tau^n < \theta | X(0) = x),
\]

where \( X \) is a strong Markov, natural solution of the constrained martingale problem for \( (A, D, B, \Xi) \) and the right hand sides in the above equalities are uniquely determined by Lemma 3.22. Recall that we use the notation

\[
\mathbb{P}^x(\cdot) := \mathbb{P}(\cdot | X(0) = x), \quad \mathbb{E}^x[\cdot] := \mathbb{E}[\cdot | X(0) = x].
\]

The fact that \( Q_{n+k} \) is not identically zero holds if, more generally, for \( 0 < \delta \leq \delta^* \),

(3.26)

\[
\mathbb{P}^x(\tau^\delta < \theta) > 0, \quad \forall x \in \overline{D}, \quad 0 < |x| < \delta,
\]

where \( \tau^\delta := \tau^{X,\delta} \) is defined by (3.15). Analogously, supposing \( \mathbb{P}^x(\tau^\delta < \theta) > 0 \) for all \( x \in \overline{D}, \ 0 < |x| < \delta, \ 0 < \delta \leq \delta^* \), assumption (i) of Theorem 2.4 is verified if there exists \( c_0 > 0 \) such that, for all \( 0 < \delta \leq \delta^* \),

(3.27)

\[
\inf_{x,y \in \overline{D}: 0 < |x| = |y| < \delta} \frac{\mathbb{P}^x(\tau^\delta < \theta)}{\mathbb{P}^y(\tau^\delta < \theta)} \geq c_0.
\]

The proof of (3.26) and (3.27) is based on estimating \( \mathbb{P}^x(\tau^\delta < \theta) \) by means of suitable auxiliary functions (Lemmas 3.29 and 3.28). These auxiliary functions are constructed by elaborating on some functions introduced by Varadhan and Williams (1985) and Kwon and Williams (1991) in the study of the reflecting Brownian motion in a cone with radially constant direction of reflection. (See Appendices A and B.)

Let \( \alpha^* \) be defined as in Theorems A.2 and A.4 for \( K \) and \( g \) given by Conditions 3.1 and 3.3.

**Lemma 3.28.** There exists \( \delta^* > 0 \) such that:

(i) If \( \alpha^* \leq 0 \), there exists a function \( V \in C^2(\overline{D} - \{0\}) \) such that

(3.28)

\[
\lim_{x \in \overline{D}, x \to 0} V(x) = \infty,
\]

(3.29)

\[
\nabla V(x) \cdot g(x) \leq 0, \quad \forall x \in (\partial D - \{0\}) \cap \overline{B_\delta}(0)
\]

(3.30)

\[
A V(x) \leq 0, \quad \forall x \in (\overline{D} - \{0\}) \cap \overline{B_\delta}(0).
\]
(ii) If $0 < \alpha^* < 1$, there exist two functions $V_1, V_2 \in C^2(\overline{D} - \{0\})$ such that
\[
V_1(x) > 0, \quad V_2(x) > 0, \quad \text{for } x \in (\overline{D} - \{0\}) \cap B_\delta(0),
\]
\[
\lim_{x \in \overline{D}, x \to 0} V_1(x) = \lim_{x \in \overline{D}, x \to 0} V_2(x) = 0,
\]
\[
\inf_{0 < \delta \leq \delta^*} \inf_{|x| = \delta} V_1(x) > 0, \quad \inf_{0 < \delta \leq \delta^*} \inf_{|x| = \delta} V_2(x) > 0
\]
\[
\nabla V_1(x) \cdot g(x) \geq 0, \quad \nabla V_2(x) \cdot g(x) \leq 0, \quad \forall x \in (\partial D - \{0\}) \cap \overline{B_\delta(0)}
\]
\[
AV_1(x) \geq 0, \quad AV_2(x) \leq 0, \quad \forall x \in (\overline{D} - \{0\}) \cap \overline{B_\delta(0)}.
\]

**Proof.** See Appendix B. \(\Box\)

**Lemma 3.29.** Assume Conditions 3.1, 3.3 and 3.6. Then $\alpha^* < 1$.

For a natural solution, $X$, of the constrained martingale problem for $(A, D, B, \Xi)$, let $\vartheta$ be defined by (3.22) and $\delta^*$ be defined by (3.15). Then there exists $\delta^* > 0$ such that, for every $x \in D, 0 < |x| < \delta \leq \delta^*, \mathbb{P}^x(\tau^\delta < \infty) = 1$ and:

(i) if $\alpha^* \leq 0$,
\[
\mathbb{P}^x(\tau^\delta < \vartheta) = 1.
\]

(ii) if $0 < \alpha^* < 1$,
\[
\mathbb{P}^x(\tau^\delta < \vartheta) > 0.
\]

Moreover there exists $c_0 > 0$ such that, for all $0 < \delta \leq \delta^*$,
\[
\inf_{x, y \in D, 0 < |x| = |y| < \delta} \frac{\mathbb{P}^x(\tau^\delta < \vartheta)}{\mathbb{P}^y(\tau^\delta < \vartheta)} \geq c_0.
\]

**Proof.** $\alpha^* < 1$ by Theorem A.10. Let $\delta^*, V, V_1, V_2$ be as in Lemma 3.28. Let $\alpha^* \leq 0$. Of course we can always suppose that $V$ is nonnegative on $(\overline{D} - \{0\}) \cap \overline{B_\delta(0)}$. By applying Ito’s formula to the function $V$, we have, for $\delta \leq \delta^*$, for every fixed $x \in (\overline{D} - \{0\}) \cap B_\delta(0)$ and $\epsilon < |x|$,
\[
\mathbb{E}^x[V(X(\tau^\epsilon \wedge \tau^\delta))] \leq V(x),
\]
which yields
\[
\inf_{|y| = \epsilon} \mathbb{E}^y[V(Y(\tau^\epsilon < \tau^\delta))] \leq V(x),
\]
and hence, by letting $\epsilon \to 0$, by (3.28),
\[
\mathbb{P}^x(\vartheta < \tau^\delta) = 0.
\]

If $0 < \alpha^* < 1$, by applying Ito’s formula to $V_1$, we obtain, for $x \in (\overline{D} - \{0\}) \cap B_\delta(0)$ and $\epsilon < |x|$,
\[
\mathbb{E}^x[V_1(X(\tau^\epsilon \wedge \tau^\delta))] \geq V_1(x),
\]
which yields
\[
\sup_{|u| = \delta} V_1(u) \mathbb{P}^x(\tau^\delta < \tau^\epsilon) + \sup_{|u| = \epsilon} V_1(u) \mathbb{P}^x(\tau^\epsilon < \tau^\delta) \geq V_1(x),
\]
and hence, by letting $\epsilon \to 0$,

$$
\sup_{|u| = \delta} V_1(u) \mathbb{P}^x(\tau^\delta < \vartheta) \geq V_1(x),
$$

which yields the first assertion in part (ii) of the thesis. Analogously, by applying Ito’s formula to $V_2$ we get,

$$
\inf_{|u| = \delta} V_2(u) \mathbb{P}^x(\tau^\epsilon < \vartheta) + \inf_{|u| = \epsilon} V_2(u) \mathbb{P}^x(\tau^\epsilon < \tau^\delta) \leq V_2(x),
$$

and hence, by letting $\epsilon \to 0$,

$$
\inf_{|u| = \delta} V_2(u) \mathbb{P}^x(\tau^\delta < \vartheta) \leq V_2(x).
$$

Combining (3.34) and (3.35), we get, for $x, y \in (\mathcal{D} - \{0\}) \cap B_\delta(0)$ with $|x| = |y|$, 

$$
\frac{\mathbb{P}^x(\tau^\delta < \vartheta)}{\mathbb{P}^y(\tau^\delta < \vartheta)} \geq \frac{V_1(x) \inf_{|u| = \delta} V_2(u)}{V_2(y) \sup_{|u| = \delta} V_1(u)} \geq \inf_{0 < \delta \leq \delta'} \inf_{0 < \delta' \leq \delta} \inf_{|u| = \delta} \frac{V_2(u)}{V_1(u)} \geq 0.
$$

3.5. Estimates on hitting distributions. In this subsection we verify assumption (ii) of Theorem 2.4 (Lemma 3.31). Lemma 3.31 follows essentially from the fact that, for any $x, \bar{x} \in E_n$, we can construct, on the same probability space, two strong Markov, natural solutions of the constrained martingale problem for $(A, D, B, \Xi)$, starting at $x$ and $\bar{x}$, such that the probability that they hit $E_{n-1}$ before the origin and that they couple before hitting $E_{n-1}$ (i.e. that their paths agree, up to a time shift, for some time before they hit $E_{n-1}$) is larger than some $c_0 > 0$ independent of $x$ and $\bar{x}$ and of $n$. The construction is based on a coupling result of Costantini and Kurtz (1988) and on a uniform lower bound on the probability that a natural solution of the constrained martingale problem for $(A, D, B, \Xi)$ starting on $E_n$ hits the intermediate layer $\{x \in \mathcal{D} : |x| = 2^{-2n+1} \delta^*\}$ in the set $O^n := \{x \in D : 2^{2n} x \in \mathcal{O}\}$, where $\mathcal{O}$ is an arbitrary open set such that $\mathcal{O} \cap \mathcal{K} \cap \partial B_{2\delta^*}(0) \neq \emptyset$. In turn this uniform lower bound is proved by showing that, for any natural solution of the constrained martingale problem for $(A, D, B, \Xi)$, $X$, the rescaled process $Z^2 X(2^{-4n})$ converges to the reflecting Brownian motion in $\bar{\mathcal{K}}$ with direction of reflection $\bar{g}$ (Lemma 3.30) and by the support theorem of Kwon and Williams (1991). Existence and uniqueness of the reflecting Brownian motion in $\bar{\mathcal{K}}$ with direction of reflection $\bar{g}$ has been proved in Varadhan and Williams (1985) and Kwon and Williams (1991), assuming only Condition 3.1 and Conditions 3.3 (i) and (ii).

We show in Appendix A (Theorem A.9) that, if Conditions 3.3 (iii) and (iv) are verified, this reflecting Brownian motion is the unique natural solution of the constrained martingale problem for $(\frac{1}{2} \Delta, \mathcal{K}, B, \Xi)$, where $\Delta$ is the Laplacian operator, $D(\Delta) := C^2_b(\mathcal{K})$, and

$$
\Xi := \{(x, u) \in \partial \mathcal{K} \times S^{d-1} : u \in \bar{G}(x)\},
$$

with

$$
\bar{G}(x) := \{\eta \bar{g}(x), \eta \geq 0\}, \quad \bar{G}(0) := G(0),
$$

$$
B : C^2_b(\bar{\mathcal{K}}) \to C(\Xi), \quad Bf(x, u) := \nabla f \cdot u.
$$

In particular the reflecting Brownian motion is a semimartingale (see Remark 3.10.) The reflecting Brownian motion in $\bar{\mathcal{K}}$ with direction of reflection $\bar{g}$ will be denoted by $\bar{X}$.

Recall that we are assuming Conditions 3.1, 3.3 and 3.6 and that, for a stochastic process $Z$, we use the notation

$$
\mathbb{P}^x(\cdot) := \mathbb{P}(\cdot | Z(0) = z), \quad \mathbb{E}^x[\cdot] := \mathbb{E}[\cdot | Z(0) = z].
$$
Let $X$ be a natural solution of the constrained martingale problem for $(A,D,B,\Xi)$. Define,

$$\sigma^{n-1} := \inf \{ t \geq 0 : |X(t)| = 2^{-2n+1}\delta^* \}, \quad n \geq 1,$$

and note that $\sigma^{n-1}$ is the hitting time of the surface “halfway” between $E_n$ and $E_{n-1}$. Recall that $\vartheta$ is defined by (3.22).

**Lemma 3.30.** For any sequence $\{x^n\} \subseteq \mathcal{O} \setminus \{0\}$ such that $\{2^{2n}x^n\}$ converges to some $\bar{x} \in \overline{\mathcal{K}} \setminus \{0\}$, let $X^{x^n}$ be a natural solution of the constrained martingale problem for $(A,D,B,\Xi)$ starting at $x^n$ and $\bar{X}^{\bar{x}}$ be the reflecting Brownian motion in $\overline{\mathcal{K}}$ with direction of reflection $\bar{g}$, starting at $\bar{x}$. Then

$$2^{2n} X^{x^n}(2^{-4n} \cdot) \overset{\mathcal{L}}{\rightarrow} \bar{X}^{\bar{x}}(\cdot).$$

In particular, for any open set $\mathcal{O}$ such that $\mathcal{O} \cap \mathcal{K} \cap \partial B_{2\delta^*}(0) \neq \emptyset$, there exists $\eta_0 = \eta_0(\mathcal{O}) > 0$ such that, for $|x^n| = 2^{-2n}\delta^*$, $\{2^{2n}x^n\}$ converging to $\bar{x}$, and $\mathcal{O}^n := \{x : 2^{2n}x \in \mathcal{O}\}$,

$$\limsup_{n \to \infty} \mathbb{P}^{x^n}(\sigma^{n-1} < \vartheta, X(\sigma^{n-1}) \in \mathcal{O}^n) \geq \eta_0.$$

**Proof.** The convergence in (3.38) follows from time change and compactness arguments similar, for instance, to those used in Lemma 4.5 and Theorem 4.1 of Costantini and Kurtz (2018) and from the fact that $X$ is the unique natural solution of the constrained martingale problem for $(\Xi, \mathcal{D}, \mathcal{K})$ (Theorem A.9).

In order to prove (3.39), fix $\bar{x}^0 \in \mathcal{O} \cap \mathcal{K} \cap \partial B_{2\delta^*}(0)$, $\epsilon < \delta^*$ such that $B_{\epsilon}(\bar{x}^0) \subseteq \mathcal{O} \cap \mathcal{K}$ and $t_0 > 0$, and let $\zeta : [0,t] \to \mathbb{R}^d$ be a continuous function such that

$$\zeta(0) = \bar{x}, \quad \zeta\left(\frac{t_0}{2}\right) = \frac{\bar{x}^0}{2}, \quad |\zeta(s)| = \delta^*, \quad \frac{\zeta(s)}{|\zeta(s)|} \in \mathcal{S}, \quad \text{for } 0 < s < \frac{t_0}{2},$$

$$\frac{\zeta(s)}{|\zeta(s)|} = \frac{\bar{x}^0}{|\bar{x}^0|}, \quad \frac{\zeta(s)}{|\zeta(s)|} = \delta^* \left(\frac{2}{t_0}(t_0 - s) + \left(2\delta^* + \epsilon\right)^{\frac{2}{t_0}}(s - \frac{t_0}{2})\right), \quad \text{for } \frac{t_0}{2} < s \leq t_0.$$

Then

$$\limsup_{n \to \infty} \mathbb{P}\left(\sigma^{n-1} < \vartheta, X^{x^n}(\sigma^{n-1}) \in \mathcal{O}^n\right)$$

$$\geq \limsup_{n \to \infty} \mathbb{P}\left(\sup_{0 \leq s \leq t_0} \left|2^{2n} X^{x^n}(2^{-4n} s) - \zeta(s)\right| > \frac{\epsilon}{2}\right)$$

$$\geq \mathbb{P}\left(\sup_{0 \leq s \leq t_0} \left|\bar{X}^{\bar{x}}(s) - \zeta(s)\right| > \frac{\epsilon}{2}\right)$$

$$= \mathbb{P}\left(\sup_{0 \leq s \leq t_0} \left|\xi^\mathcal{K}(s) - \zeta(s)\right| < \frac{\epsilon}{2}\right),$$

where $\xi$ is the absorbed process in the proof of Theorem A.9 and has a uniquely determined distribution. Then the assertion follows from Theorem 3.1 of Kwon and Williams (1991) and the Feller property of $\xi$. \[\square\]

**Lemma 3.31.** Let $\{E_n\}$ be defined by (3.20) and $\{Q_n\}$ be given by (3.24) and (3.22). For $x, \bar{x} \in E_n$, let

$$\epsilon_n(x, \bar{x}) := \int \left(\tilde{f}_n(x,y) \wedge f_n(\bar{x},y)\right)(Q_n(x,dy) + Q_n(\bar{x},dy)), \quad n \geq 1,$$

where

$$\tilde{f}_n(x, \cdot) := \frac{dQ_n(x, \cdot)}{d(Q_n(x, \cdot) + Q_n(\bar{x}, \cdot))}, \quad f_n(\bar{x}, \cdot) := \frac{dQ_n(\bar{x}, \cdot)}{d(Q_n(x, \cdot) + Q_n(\bar{x}, \cdot))}. $$
Then there exists $\epsilon_0 > 0$ such that

$$\inf_{n \geq 1} \inf_{x, \bar{x} \in E_n} \epsilon_n(x, \bar{x}) \geq \epsilon_0.$$ 

PROOF. By Lemma 3.30 and Lemma 5.3 in Costantini and Kurtz (2018), we can construct, on the same probability space, two strong Markov, natural solutions, $X$ and $\bar{X}$, of the constrained martingale problem for $(A, D, B, \Xi)$, starting at $x$ and $\bar{x}$ respectively, such that, denoting by $\tau_n^{-1}$ and $\bar{\theta}_0$ the analogs of $\tau_n^{-1}$ and $\theta$ for $\bar{X}$, and by $\mathcal{E}$ the event

$$\mathcal{E} := \{ \exists t, \bar{t}, 0 \leq t < \tau_n^{-1} \land \theta, 0 \leq \bar{t} < \tau_n^{-1} \land \bar{\theta}_0 : X(t + s) = \bar{X}(\bar{t} + s), 0 \leq s \leq (\tau_n^{-1} - t) \},$$

for some positive constant $\epsilon_0$ independent of $x$, $\bar{x}$ and $n$. This implies, for every $C \in \mathcal{B}(E_{n-1})$,

$$Q_n(x, C) \leq \mathbb{P}(\{\tau_n^{-1} < \theta\} \cap \{X(\tau_n^{-1}) \in C\} \cap \mathcal{E}) + \mathbb{P}(\tau_n^{-1} < \theta) - \epsilon_0 \leq Q_n(x, C) + Q_n(x, E_{n-1}) - \epsilon_0,$$

and the same exchanging $x$ and $\bar{x}$, so that

$$||Q_n(x, \cdot) - Q_n(\bar{x}, \cdot)||_{TV} \leq Q_n(x, E_{n-1}) \lor Q_n(\bar{x}, E_{n-1}) - \epsilon_0.$$ 

On the other hand, we have

$$Q_n(x, E_{n-1}) = \int_{\{y: \tilde{f}_n(x, y) > f_n(\bar{x}, y)\}} (\tilde{f}_n(x, y) - \tilde{f}_n(x, y) \land f_n(\bar{x}, y)) (Q_n(x, dy) + Q_n(\bar{x}, dy))$$

$$+ \int (\tilde{f}_n(x, y) \land f_n(\bar{x}, y)) (Q_n(x, dy) + Q_n(\bar{x}, dy))$$

$$= \int_{\{y: \tilde{f}_n(x, y) > f_n(\bar{x}, y)\}} (\tilde{f}_n(x, y) \lor f_n(\bar{x}, y) - \tilde{f}_n(x, y) \land f_n(\bar{x}, y)) (Q_n(x, dy) + Q_n(\bar{x}, dy))$$

$$+ \epsilon_n(x, \bar{x}) \leq ||Q_n(x, \cdot) - Q_n(\bar{x}, \cdot)||_{TV} + \epsilon_n(x, \bar{x}),$$

and the same exchanging $x$ and $\bar{x}$.

\[\Box\]

APPENDIX A: RESULTS ON THE CONE

Let $\mathcal{K}$ be the cone in Condition 3.1 and $\bar{g}$ be the vector field in Condition 3.3. The reflecting Brownian motion in $\mathcal{K}$ with direction of reflection $\bar{g}$ has been studied by Varadhan and Williams (1985) and Williams (1985), for $d = 2$, and by Kwon and Williams (1991) for $d \geq 3$, without assuming Conditions 3.3 (iii) and (iv). We summarize below the main results of Varadhan and Williams (1985) and Kwon and Williams (1991).

If Conditions 3.3 (iii) and (iv) are satisfied, a modification of Theorem 3.23 (Theorem A.9 below) yields that the reflecting Brownian motion in $\mathcal{K}$ with direction of reflection $\bar{g}$ is a semimartingale. In dimension $d = 2$, Williams (1985) proves that this is equivalent to the fact that the parameter $\alpha^*$ (defined in (A.3) below) is strictly less than 1. In dimension $d \geq 3$, the issue of when the reflecting Brownian motion is a semimartingale is not discussed in Kwon and Williams (1991). We prove here one of the two implications, namely that the
reflecting Brownian motion being a semimartingale implies that the parameter $\alpha^*$ (defined by Kwon and Williams (1991) as in Theorem A.4) is strictly less than 1 (Theorem A.10). Beyond the intrinsic interest, this allows us to use some results for the cone $\mathcal{K}$ to obtain some of the estimates we need to prove uniqueness. (See the proof of Lemma 3.28.)

Let $\mathcal{K}$ be a cone as in Condition 3.1, $\bar{g}$ be a vector field as in Conditions 3.3 (i) and (ii) and $\Delta$ denote the Laplacian operator. In Varadhan and Williams (1985), Williams (1985) and Kwon and Williams (1991), the reflecting Brownian motion in $\mathcal{K}$ with direction of reflection $\bar{g}$ is viewed as a solution to the following submartingale problem.

**Definition A.1.** A stochastic process $X$ with paths in $C_0([0,\infty))$ is a solution of the submartingale problem for $(\frac{1}{2}\Delta, \bar{g} \cdot \nabla, \partial \mathcal{K})$, if there exists a filtration $\{\mathcal{F}_t\}$, on the space on which $X$ is defined, such that $X$ is $\{\mathcal{F}_t\}$-adapted and

$$f(X(t)) - f(X(0)) - \frac{1}{2} \int_0^t \Delta f(X(s)) \, ds$$

is an $\{\mathcal{F}_t\}$-submartingale for all $f \in C^2_{\bar{g}}(\mathcal{K})$ such that $f$ is constant in a neighborhood of the origin and

$$\bar{g} \cdot \nabla f \geq 0 \quad \text{on } \partial \mathcal{K} - \{0\}.$$

The solution to the submartingale problem for $(\frac{1}{2}\Delta, \mathcal{K}, \bar{g} \cdot \nabla, \partial \mathcal{K})$ is unique if any two solutions have the same distribution.

A solution $X$ is said to spend zero time at the origin if

$$\mathbb{E}\left[\int_0^\infty 1_{\{0\}}(X(s))\, ds\right] = 0.$$

For $d = 2$, in polar coordinates,

(A.1) \[ \mathcal{K} = \{(r, z) : r > 0, 0 < z < \zeta\}, \quad 0 < \zeta < 2\pi. \]

Let $\partial_1 \mathcal{K} := \{(r, z) : r > 0, z = 0\}$, $\partial_2 \mathcal{K} := \{(r, z) : r > 0, z = \zeta\}$ and denote by $n^1$ and $n^2$ the unit inward normal vectors on $\partial_1 \mathcal{K}$ and $\partial_2 \mathcal{K}$. Conditions 3.3 (i) and (ii) reduce simply to

(A.2) \[ \bar{g}(x) := \begin{cases} \bar{g}^1, & \text{for } x \in \partial_1 \mathcal{K}, \\ \bar{g}^2, & \text{for } x \in \partial_2 \mathcal{K}. \end{cases} \]

**Theorem A.2.** (Varadhan and Williams (1985))

Let $d = 2$, and let $\mathcal{K}$ and $\bar{g}$ be as in (A.1) and (A.2). Let $\zeta_1$ and $\zeta_2$ denote the angles between $\bar{g}^1$ and $n^1$, and between $\bar{g}^2$ and $n^2$, respectively, taken to be positive if $\bar{g}^1$ ($\bar{g}^2$) points towards the origin. Set

(A.3) \[ \alpha^* := \frac{\zeta_1 + \zeta_2}{\zeta}, \]

and

(A.4) \[ \psi_{\alpha^*}(z) := \cos(\alpha^* z - \zeta_1), \quad \text{if } \alpha^* \neq 0, \]

\[ \psi^0(z) := -\tan \zeta_1, \quad \text{if } \alpha^* = 0. \]

Then the function

(A.5) \[ \Psi(r, z) := \begin{cases} r^{\alpha^*} \psi_{\alpha^*}(z), & \text{if } \alpha^* \neq 0, \\ -\ln r + \psi^0(z), & \text{if } \alpha^* = 0, \end{cases} \]

satisfies

(A.6) \[ \Delta \Psi = 0 \quad \text{in } \mathcal{K}, \]

\[ \bar{g} \cdot \nabla \Psi = 0 \quad \text{on } \partial \mathcal{K} - \{0\}. \]
THEOREM A.3. (Varadhan and Williams (1985), Williams (1985))
Let \( d = 2 \), and let \( K, \bar{g}^1, \bar{g}^2, \alpha^* \) be as in Theorem A.2. For \( \alpha^* < 2 \), for each \( x \in \overline{K} \), there exists one and only one solution to the submartingale problem for \( (\frac{1}{2} \Delta, K, \bar{g} \cdot \nabla, \partial K) \) starting at \( x \) that spends zero time at the origin and it is a strong Markov process and a Feller process. This solution is a semimartingale if and only if \( \alpha^* < 1 \). For \( \alpha^* \geq 2 \), for each \( x \in \overline{K} \), there exists one and only one solution to the submartingale problem for \( (\frac{1}{2} \Delta, K, \bar{g} \cdot \nabla, \partial K) \) starting at \( x \), and it is absorbed at the origin after the first time it hits it.

Now let \( d \geq 3 \). Recall that \( v^r \) denotes the unit radial vector, i.e. \( v^r(x) := \frac{x}{|x|} \), and \( n^K(x) \) denotes the unit inward normal to \( \overline{K} \) at \( x \neq 0 \).

THEOREM A.4. (Kwon and Williams (1991))
Let \( d \geq 3 \), \( K \) be a cone as in Condition 3.1, \( \bar{g} \) be a vector field as in Conditions 3.3 8i) and (ii). For each \( \alpha \in \mathbb{R} \), there exist \( \lambda(\alpha) \in \mathbb{R} \) and \( \psi_\alpha \in C^2(\overline{S}) \) such that
\[
\begin{align*}
\lambda(\alpha) \psi_\alpha + \Delta_{S^{d-1}} \psi_\alpha &= 0 \quad \text{in } S, \\
\alpha \bar{g}_r \psi_\alpha + \bar{g}_T \cdot \nabla_{S^{d-1}} \psi_\alpha &= 0 \quad \text{on } \partial S,
\end{align*}
\]
where \( \bar{g}_r v^r \) and \( \bar{g}_T \) are the radial component and the component tangential to \( S^{d-1} \) of \( \bar{g} \). \( \psi_\alpha \) is strictly positive. \( \lambda \) and \( \alpha \mapsto \psi_\alpha \in C^2(\overline{S}) \) are analytic functions. \( \lambda \) is concave, \( \lambda(0) = 0 \) and
\[
\lambda'(0) = -\int_{\partial S} \frac{1}{\bar{g} \cdot n^K} \bar{g}_r \bar{\psi}^*,
\]
where \( \bar{\psi}^* \) is the unique solution of
\[
\begin{align*}
\Delta_{S^{d-1}} \bar{\psi}^* &= 0 \quad \text{in } S, \\
n^K \cdot \nabla_{S^{d-1}} \bar{\psi}^* - \text{div}_{\partial S} \left( \bar{\psi}^* \left( \frac{|n^K \cdot \bar{g}_T - n^K|}{\bar{g} \cdot n^K} \right) \right) &= 0 \quad \text{on } \partial S,
\end{align*}
\]
such that \( \bar{\psi}^* \) is strictly positive and \( \int_{\overline{S}} \bar{\psi}^* = 1 \).

If \( \lambda'(0) \neq d - 2 \), there exists a unique \( \alpha^* \neq 0 \) such that
\[
\lambda(\alpha^*) = \alpha^*(\alpha^* + d - 2),
\]
and the function \( \Psi \) defined as in (A.5) for \( \alpha^* \neq 0 \) satisfies (A.6). \( \alpha^* > 0 \) if \( \lambda'(0) > d - 2 \), \( \alpha^* < 0 \) if \( \lambda'(0) < d - 2 \).

If \( \lambda'(0) = d - 2 \), there exists a solution \( \psi^0 \in C^2(\overline{S}) \) to
\[
\begin{align*}
-(d - 2) + \Delta_{S^{d-1}} \psi^0 &= 0 \quad \text{in } S, \\
-\bar{g}_r + \bar{g}_T \cdot \nabla_{S^{d-1}} \psi^0 &= 0 \quad \text{on } \partial S,
\end{align*}
\]
and the function \( \Psi \) defined as in (A.5) for \( \alpha^* = 0 \) satisfies (A.6). In this case, we set \( \alpha^* := 0 \).

THEOREM A.5. (Kwon and Williams (1991))
Let \( d \geq 3 \) and let \( K \) and \( \bar{g} \) be as in Theorem A.4. For \( \alpha^* < 2 \), for each \( x \in \overline{K} \), there exists a unique solution to the submartingale problem for \( (\frac{1}{2} \Delta, K, \bar{g} \cdot \nabla, \partial K) \), starting at \( x \), that spends zero time at the origin and it is a strong Markov process and a Feller process. For \( \alpha^* \geq 2 \), for each \( x \in \overline{K} \), there exists a unique solution to the submartingale problem for \( (\frac{1}{2} \Delta, K, \bar{g} \cdot \nabla, \partial K) \) starting at \( x \), and it is absorbed at the origin after the first time it hits it.

REMARK A.6. In the case \( \alpha^* = 0 \), both for \( d = 2 \) and \( d \geq 3 \), the function used in Kwon and Williams (1991) is actually \( -\Psi \), but we prefer to have \( \Psi(v, z) \rightarrow r \rightarrow 0 \infty \).

REMARK A.7. For \( d = 2 \), \( \psi_{\alpha^*}, \psi^0 \in C^\infty(\overline{S}) \). For \( d \geq 3 \), a careful inspection of the proofs of Kwon and Williams (1991) shows that \( \psi_{\alpha^*}, \psi^0 \in C^{2+\beta}(\overline{S}) \) for every \( 0 < \beta < 1 \) (see Theorem 6.31 of Gilbarg and Trudinger (1983) and the Remark following it.)
The function defined in (2.7) of Kwon and Williams (1991):

\[
\Phi(x) := \begin{cases} 
\Psi(x)^{-1}, & \text{if } \alpha^* < 0, \\
\exp(-\Psi(x)), & \text{if } \alpha^* = 0, \\
\Psi(x), & \text{if } \alpha^* > 0,
\end{cases}
\]

(A.9)

gives a way of measuring the distance from the origin and satisfies

\[ \bar{g} \cdot \nabla \Phi = 0 \text{ on } \partial \mathcal{K} - \{0\}. \]

\( \Psi \) and \( \Phi \) will be used both to localize and to construct auxiliary functions (see Appendix B).

Let \( G, \bar{\Xi} \) and \( B \) be as in (3.36). \( \mathcal{K} \) is unbounded, but the definitions of constrained martingale problem, controlled martingale problem and natural solution of the constrained martingale problem carry over to \( (\frac{\partial}{\partial} \mathcal{K}, B, \bar{\Xi}) \) without any modification, except that \( C^2(\mathcal{D}) \) is replaced by \( C^2_{\bar{\mathcal{K}}} \) everywhere.

**Lemma A.8.** There exists a function \( F \in C^2_{\bar{\mathcal{K}}} \) such that

\[ \inf_{x \in \partial \mathcal{K}, \ g \in G(x), \ |g| = 1} \nabla F(x) \cdot g := c_F > 0. \]

**Proof.** See Appendix B.

**Theorem A.9.** Let \( \mathcal{K} \) and \( \bar{\xi} \) be as in Theorem A.2, for \( d = 2 \), and as in Theorem A.4, for \( d \geq 3 \), and, in addition, assume that Conditions 3.3 (iii) and (iv) are satisfied.

Then, for each \( \nu \in \mathcal{P}(\mathcal{K}) \), there exists one and only one natural solution, \( \mathcal{X} \), to the constrained martingale problem for \( (\frac{\partial}{\partial} \mathcal{K}, B, \bar{\Xi}) \) with initial distribution \( \nu \) and it is the unique solution of the submartingale problem for \( (\frac{\partial}{\partial} \mathcal{K}, \bar{\xi}, \bar{\nabla}, \partial \mathcal{K}) \) that spends zero time at the origin. \( \mathcal{X} \) is a semimartingale and a strong Markov process. The associated random measure, \( \Lambda \), satisfies \( \mathbb{E}[\Lambda([0, t] \times \Xi)] < \infty \) for all \( t \geq 0 \), and, for every \( f \in C^2(\mathcal{K}) \), (3.11) is a martingale.

**Proof.** Let \( \{\delta_k\} \) be a strictly decreasing sequence of positive numbers converging to zero, with \( \delta_1 < 1 \), and let \( \{D^k\} \) be a sequence of domains with \( C^1 \) boundary such that \( D^k \subset D^{k+1} \subset \mathcal{K}, \ \overline{D^k \cap B_{\delta_k}(0)} = \overline{\mathcal{K} \cap B_{\delta_k}(0)} \) and \( \overline{D^k \cap B_{\delta_k}(0)} \subset D^{k+1} \). Also let \( g^k : \mathbb{R}^d - \{0\} \rightarrow \mathbb{R}^d, \ k \in \mathbb{N} \), be a locally Lipschitz vector field, of unit length on \( \partial D^k \), such that \( g^k(x) = \bar{\xi}(x) \) for \( x \in \partial \mathcal{K} \cap B_{\delta_k}(0) \) and that, denoting by \( n^k(x) \) the unit, inward normal at \( x \in \partial D^k \), it holds \( \inf_{x \in \partial D^k} g^k(x) \cdot n^k(x) > 0 \).

For each \( k \), consider a sequence of bounded domains \( \{D^{k,N}\}, \ N \in \mathbb{N} \), with \( C^1 \) boundary, such that \( D^{k,N} \subset D^{k,N+1} \subset D^k, \ \overline{D^{k,N} \cap B_N(0)} = \overline{D^k \cap B_N(0)} \) and \( \overline{D^{k,N} \cap (\overline{B_N(0)})^c} \subset D^{k,N+1} \). Also let \( g^{k,N} \) be a locally Lipschitz vector field, of unit length on \( \partial D^{k,N} \), such that \( g^{k,N}(x) = g^k(x) \) for \( x \in \partial D^k \cap \overline{B_N(0)} \) and that, denoting by \( n^{k,N}(x) \) the unit, inward normal at \( x \in \partial D^{k,N} \), it holds \( \inf_{x \in \partial D^{k,N}} g^{k,N}(x) \cdot n^{k,N}(x) > 0 \).

Let \( \xi_0 \) be a random variable with compact support \( \text{supp}(\xi_0) \subset \overline{\mathcal{K} - \{0\}} \). For \( k \) and \( N \) large enough that \( \text{supp}(\xi_0) \subset \overline{D^{k,N}} \), let \( \xi^{k,N} \) be the strong solution of (3.8) in \( D^{k,N} \) with direction of reflection \( g^{k,N} \) and initial condition \( \xi_0 \), and let \( I^{k,N} \) be the corresponding nondecreasing process. Define

\[ \Theta^{k,N} := \inf\{t \geq 0 : |\xi^{k,N}(t)| \geq N\}. \]

Let \( \varphi \in C^2(\overline{\mathcal{K}}) \) be defined by:

\[
\varphi(x) := \chi(\Phi(x)),
\]

(A.10)
where $\Phi$ is defined in (A.9) and $\chi: \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth, nondecreasing function such that $\chi(u) = 0$, for $u \leq \sup_{|x| \leq 1} \Phi(x)$ and $\chi(u) = u$, for $u \geq \inf_{|x| \geq \delta} \Phi(x)$, for $\delta > 1$ such that $0 < \sup_{|x| \leq 1} \Phi(x) < \inf_{|x| \geq \delta} \Phi(x)$. Then

$$
\nabla \varphi(x) \cdot \theta^k(x) = 0, \text{ for } x \in \partial D^{k,N}, \quad |x| \leq N,
$$

$$
\lim_{|x| \to \infty} \varphi(x) = \infty, \quad \Delta \varphi(x) \leq c(1 + \varphi(x)), \text{ for } x \in K, \quad |x| \leq N,
$$

for some $c > 0$, and, by applying Ito’s formula to $\varphi$, we obtain

(A.11) 
$$
\lim_{N \to \infty} \sup_{k: \text{supp}(\xi_0) \subset D^k} \mathbb{P}(\Theta^{k,N} \leq t) = 0.
$$

From (A.11), by a standard argument, we see that, for $\text{supp}(\xi_0) \subset \overline{D^k}$, there is one and only one strong solution, $\xi^k$, to (3.8) in $\overline{D^k}$ with direction of reflection $\theta^k$ and initial condition $\xi_0$, and it is defined for all times. Moreover, setting

$$
\theta^k := \inf\{t \geq 0 : \xi^k(t) \in \partial D^k \cap B_{\delta_k}(0)\}, \quad \theta := \lim_{k \to \infty} \theta^k,
$$

(A.11) yields that

(A.12) 
$$
\mathbb{P}(\theta < \infty, \quad \sup_{k: \text{supp}(\xi_0) \subset \overline{D^k}} \sup_{t \leq \theta^k} |\xi^k(t)| = \infty) = 0.
$$

Hence, as in Lemma 3.22, we can define a pair of stochastic processes $\xi$ and $l$ that satisfies (3.17) for $0 \leq t < \theta$, and almost every path of $\xi$ such that $\theta < \infty$ is bounded. For each $N \in \mathbb{N}$, for each path such that $\theta < \infty$ and $\sup_{t < \theta} |\xi(t)| \leq N$, we can repeat the argument of Lemma 3.22 and obtain (3.18), so that (3.18) holds for almost every path such that $\theta < \infty$. Therefore the solution of (3.17) is well defined, up to $\theta$ included if $\theta$ is finite.

We can now proceed as in Theorem 3.23 and construct a sequence $\{(Y^n, \lambda_0^n, \Lambda_1^n)\}$ such that, for each $f \in C_b^2(K)$,

$$
f(Y^n(t)) - f(Y^n(0)) - \frac{1}{2} \int_0^t \Delta f(Y^n(s)) d\lambda_0^n(s) - \int_{[0,t] \times U} B^n f(Y^n(s), u) \Lambda_1^n(ds \times du)
$$

is a martingale with respect to $\{F_t^{Y^n, \lambda_0^n, \Lambda_1^n}\}$, where

$$
B^n f(x, u) := u \cdot \nabla f(x) 1_{\partial K}^{-1}(x) + (\rho_n)^{-1}[f(x + \rho_n u) - f(x)] 1_{(0)}(x),
$$

and the law of $Y^n(0)$ has compact support in $\overline{K} - \{0\}$ and converges weakly to $\nu$. By employing again the function $\varphi$ defined in (A.10), we can see that $\{Y^n\}$ satisfies the compact containment condition. Then the same relative compactness arguments as in Theorem 3.23 apply and any limit point of $\{(Y^n, \lambda_0^n, \Lambda_1^n)\}$ is a solution of the controlled martingale problem for $\left(\frac{1}{2} \Delta, \mathcal{K}, B, \overline{\Xi}\right)$ with initial distribution $\nu$.

Lemma 3.1 in Costantini and Kurtz (2019) holds for non compact state spaces as well, provided $f$ and $A f$ in its statement are bounded, and Lemma A.8 ensures that its assumptions are verified. In addition Lemma 3.16 carries over to the present context. Therefore, by Corollary 4.12 a) of Costantini and Kurtz (2019), for each $\nu \in \mathcal{P}(\overline{K})$ there exists a strong Markov, natural solution of the constrained martingale problem for $\left(\frac{1}{2} \Delta, \mathcal{K}, B, \overline{\Xi}\right)$ with initial distribution $\nu$. Moreover, it can be easily checked, in the same way as in Remark 3.12, that all solutions to the controlled martingale problem for $\left(\frac{1}{2} \Delta, \mathcal{K}, B, \overline{\Xi}\right)$ are continuous and that Remark 3.17 and Lemma 3.21 carry over to the present context. Thus every natural solution of the constrained martingale problem for $\left(\frac{1}{2} \Delta, \mathcal{K}, B, \overline{\Xi}\right)$ is a solution of the submartingale.
problem for \((\frac{1}{2} \Delta, \mathcal{K}, \bar{g} \cdot \nabla, \partial \mathcal{K})\) that spends zero time at the origin, hence there is only one solution, \(X\).

Finally, by Lemma 3.1 of Costantini and Kurtz (2019), the random measure associated to \(X, \Lambda\), satisfies \(\mathbb{E}[\Lambda([0, t] \times \mathbb{Z})] < \infty\) for all \(t \geq 0\), and, for every \(f \in \mathcal{C}_b^2(\mathcal{K})\), (3.11) is a martingale. Since the function \(f(x) := x_i, i = 1, \ldots, d\), can be approximated, uniformly over compact sets, by functions in \(\mathcal{C}_b^2(\mathcal{K})\), \(X\) is a semimartingale. \(\square\)

**Theorem A.10.** Let \(\mathcal{K}\) and \(\bar{g}\) be as in Theorem A.2, for \(d = 2\), and as in Theorem A.4, for \(d \geq 3\), and, in addition, assume that Conditions 3.3 (iii) and (iv) are satisfied. Let \(\alpha^*\) be the parameter defined in Theorems A.2 and A.4. Then it holds

\[ \alpha^* < 1. \]

**Proof.** In dimension \(d = 2\), the assertion follows immediately from Theorems A.9 and A.3. By adapting, in a nontrivial way, an argument of Williams (1985), we are able to prove that it holds in dimension \(d \geq 3\) as well.

By Theorems A.9 and A.5, \(\alpha^* < 2\). Suppose, by contradiction, that \(1 \leq \alpha^* < 2\). Let \(\mathbf{n}^{\mathcal{K}}\) be the inward, unit normal to \(\mathcal{K}\), \(\mathbf{n}^T(x) := \frac{x}{|x|}\). In the following, it is convenient to normalize \(\bar{g}\) so that

\[ \bar{g}(x) \cdot \mathbf{n}^{\mathcal{K}}(x) = 1, \]

rather than \(|\bar{g}(x)| = 1\). Of course this does not affect equation (A.7) and Condition 3.3. It can be easily checked that, for \(\epsilon > 0\) less than a threshold determined by the data of the problem, the vector

\[ g^\epsilon(z) := \bar{g}(z) - \epsilon \mathbf{n}^T, \quad z \in \partial \mathcal{S}, \tag{A.13} \]

satisfies all points of Condition 3.3. Then, by Theorems A.9 and A.5, \(\alpha^{\epsilon*}\), defined as in Theorem A.4 with \(\bar{g}\) replaced by \(g^\epsilon\), satisfies

\[ \alpha^{\epsilon*} < 2. \tag{A.14} \]

Let us show that

\[ \alpha^{\epsilon*} > \alpha^*. \tag{A.15} \]

For \(\alpha > 0\), let \((\lambda(\alpha), \psi_\alpha)\) be as in Theorem A.4 and \((\lambda^\epsilon(\alpha), \psi_\alpha^\epsilon)\) be the corresponding objects with \(\bar{g}\) replaced by \(g^\epsilon\). With the notation of (A.7), since \(g^\epsilon_T = \bar{g}_T\), and \(g^\epsilon_r = \bar{g}_r - \epsilon, (\lambda^\epsilon(\alpha), \psi_\alpha^\epsilon)\) satisfies

\[ \lambda^\epsilon(\alpha) \psi_\alpha^\epsilon + \Delta_{S^{a-1}} \psi_\alpha^\epsilon = 0, \quad \text{in } \mathcal{S}, \]

\[ \alpha(\bar{g}_r - \epsilon) \psi_\alpha^\epsilon + \bar{g}_T \cdot \nabla_{S^{a-1}} \psi_\alpha^\epsilon = 0, \quad \text{on } \partial \mathcal{S}. \tag{A.16} \]

Consider the function \(\psi_\alpha/\psi_\alpha^\epsilon\). Straightforward computations show that (A.7) and (A.16) imply that, for \(z \in \mathcal{S}\),

\[ \Delta_{S^{a-1}}(\psi_\alpha/\psi_\alpha^\epsilon)(z) = [\lambda^\epsilon(\alpha) - \lambda(\alpha)](\psi_\alpha/\psi_\alpha^\epsilon)(z) - 2[(\psi_\alpha^\epsilon)^{-1} \nabla_{S^{a-1}}(\psi_\alpha/\psi_\alpha^\epsilon) \cdot \nabla_{S^{a-1}}(\psi_\alpha^\epsilon)](z), \]

and, for \(z \in \partial \mathcal{S}\),

\[ \left( \nabla_{S^{a-1}}(\psi_\alpha/\psi_\alpha^\epsilon) \cdot \bar{g}_T \right)(z) = -\alpha \epsilon (\psi_\alpha/\psi_\alpha^\epsilon)(z). \]

Let \(z^0\) be a point of global minimum for \(\psi_\alpha(\psi_\alpha^\epsilon)^{-1}\). If \(z^0 \in \partial \mathcal{S}\), since \(\bar{g}_T \cdot \mathbf{n}^{\mathcal{K}}(z) = 1\), it must hold

\[ \left( \nabla_{S^{a-1}}(\psi_\alpha/\psi_\alpha^\epsilon) \cdot \bar{g}_T \right)(z^0) \geq 0, \]
while
\[-\alpha \epsilon (\psi_\alpha / \psi_\alpha^*(z^0)) < 0, \quad \forall \alpha > 0,\]
because \(\psi_\alpha\) and \(\bar{\psi_\alpha}\) are strictly positive. Therefore it must be \(z^0 \in \mathcal{S}\) and
\[\nabla_{\mathcal{S}^{e-1}}(\psi_\alpha / \psi_\alpha^*)(z^0) = 0, \quad \Delta_{\mathcal{S}^{e-1}}(\psi_\alpha / \psi_\alpha^*)(z^0) > 0,\]
which yields
\[(A.17) \quad \lambda'(\alpha) > \lambda(\alpha), \quad \forall \alpha > 0.\]
Then \((\lambda')'(0) \geq \lambda'(0) > d - 2\), so that \(\alpha^{e*} > 0\). Hence, taking into account that \(\lambda(\alpha) - \alpha(\alpha + d - 2)\) vanishes for \(\alpha = 0\) and \(\alpha = \alpha^{e*}\) and is strictly concave, \((A.17)\) gives \((A.15)\).

The function \(\Psi^e\) defined by \((A.7)\) with \(\psi_\alpha\) replaced by \(\psi_\alpha^e\) and \(\alpha^*\) replaced by \(\alpha^{e*}\) has the following properties:
\[\Psi^e \in C^2(\mathcal{K} - \{0\}) \cap C^1(\mathcal{K}), \quad \nabla \Psi^e(0) = 0, \quad \Delta \Psi^e(x) = 0, \quad x \in \mathcal{K},\]
\[(A.18)\]
\[c_1 \Psi(x)^{(\alpha^{e*} - 1) / \alpha^{e*}} \leq (\bar{g} \cdot \nabla \Psi^e)(x) \leq c_2 \Psi(x)^{(\alpha^{e*} - 1) / \alpha^{e*}}, \quad \text{on } \partial \mathcal{K} - \{0\},\]
\[c_1 \Psi(x)^{\alpha^{e*} / \alpha^{e*}} \leq \Psi^e(x) \leq c_2 \Psi(x)^{\alpha^{e*} / \alpha^{e*}}, \quad \text{in } \mathcal{K},\]
where the constants \(c_1\) and \(c_2\) can be taken independent of \(\epsilon\) because the map \(\alpha \to \psi_\alpha\) is continuous and \(1 \leq \alpha^{e*} \leq 2\). Of course \((A.18)\) still holds if we revert to the usual normalization of \(\bar{g}, \|\bar{g}\| = 1\), as we will do for the rest of the proof.

Let \(X\) be the solution of the submartingale problem for \((\frac{1}{2} \Delta, \mathcal{K}, \bar{g} \cdot \nabla, \partial \mathcal{K})\) spending zero time at the origin and starting at \(x = 0\). The rest of the proof uses essentially the same argument as Theorem 5 of Williams (1985), but our proof is simpler because we can take advantage of the fact that \(X\) is the solution of the constrained martingale problem for \((\frac{1}{2} \Delta, \mathcal{K}, B, \mathbb{E})\). Fix \(0 < \delta < 1\), and let
\[T^1 := \inf\{t \geq 0 : \Psi(X(t)) \geq 1\},\]
\[X^1(t) := X(t \wedge T^1).\]
Define
\[\vartheta^1_0 := \inf\{t \geq 0 : X^1(t) = 0\} = 0,\]
\[\theta^1_n := \inf\{t \geq \vartheta^1_{n-1} : \Psi(X^1(t)) = \delta\}, \quad n \geq 1,\]
\[\vartheta^1_n := \inf\{t \geq \theta^1_n : X^1(t) = 0\}, \quad n \geq 1,\]
with the usual convention that the infimum of the empty set is \(\infty\). By the continuity of \(X\), \(\theta^1_n \uparrow \infty\) and \(\vartheta^1_n \uparrow \infty\) as \(n \to \infty\). We have
\[(A.19)\]
\[\Psi^e(X^1(t)) = \sum_{n=0}^{\infty} \mathbf{1}_{[\vartheta^1_n \leq t]} [\Psi^e(X^1(t \wedge \theta^1_{n+1})) - \Psi^e(X^1(t \wedge \vartheta^1_n))] \]
\[+ \sum_{n=1}^{\infty} \mathbf{1}_{[\theta^1_n \leq t]} [\Psi^e(X^1(t \wedge \vartheta^1_n)) - \Psi^e(X^1(t \wedge \theta^1_n))].\]
As far as the first summand is concerned, we have, by (A.18), on the set \( \{ \vartheta_1^{1} \leq t \} \),
\[
|\Psi^\epsilon(X^1(t \land \vartheta_1^{n+1}) - \Psi^\epsilon(X^1(\vartheta_1^n))] = \Psi^\epsilon(X^1(t \land \vartheta_1^{n+1}) \leq c_2 \delta^{\alpha^*+\alpha^*}. \]
In addition, it can be easily checked that the argument used to prove (52) in Williams (1985), combined with Lemma 2.8 of Kwon and Williams (1991), still works, that is
\[
(A.20) \quad \mathbb{E}^\theta \left[ \sum_{n=1}^{\infty} 1_{\{ \vartheta_1^{1} \leq t \}} \right] \leq \mathbb{E}^\theta \left[ \sum_{n=1}^{\infty} 1_{\{ \vartheta_1^{1} \leq t \}} \right] \leq c \delta^{-1} \frac{t + 1}{1 - \delta^{2/\alpha^*}},
\]
where \( c \) depends only on \( \alpha^* \) and \( \psi_{\alpha^*} \). Thus, for each \( \epsilon \), by (A.15), the expectation of the first summand in (A.19) vanishes as \( \delta \to 0 \). As for the second summand, by (A.20) and the definition of \( X^1 \), it is bounded above by an integrable random variable. Moreover, taking into account that \( \theta_1^{1} \leq t \) implies \( \theta_1^{1} \leq T^1 \), hence \( \{ \theta_1^{1} \leq t \} \in \mathcal{F}_{\theta_1^{1} \land T^1} \), we have
\[
\mathbb{E}^\theta \left[ \sum_{n=1}^{\infty} 1_{\{ \vartheta_1^{1} \leq t \}} \right] \mathbb{E}^\theta \left[ \Psi^\epsilon(X^1(t \land \vartheta_1^{1}) - \Psi^\epsilon(X^1(\vartheta_1^{1}))) \right]
= \mathbb{E}^\theta \left[ \sum_{n=1}^{\infty} 1_{\{ \vartheta_1^{1} \leq t \}} \right] \mathbb{E}^\theta \left[ \Psi^\epsilon(X(t \land \vartheta_1^{1} \land T^1)) - \Psi^\epsilon(X(\vartheta_1^{1} \land T^1)) \right]
= \mathbb{E}^\theta \left[ \sum_{n=1}^{\infty} 1_{\{ \vartheta_1^{1} \leq t \}} \right] \mathbb{E}^\theta \left[ \Psi^\epsilon(X(t \land \vartheta_1^{1} \land T^1)) - \Psi^\epsilon(X(\vartheta_1^{1} \land T^1)) \mid \mathcal{F}_{\theta_1^{1} \land T^1} \right].
\]
Recall that \( \Xi \) and \( B \) are defined by (3.36). Note that, by (A.18), \( B \Psi^\epsilon \) is a continuous function on \( \Xi \) and \( 0 \leq B \Psi^\epsilon(x, u) \leq c_2 \Psi(x) \) for all \( (x, u) \in \Xi \), and that, by the definition of natural solution,
\[
\mathbb{E} \left[ \int_{[\theta_1^{1} \land T^1, t \land \vartheta_1^{1} \land T^1]} B \Psi^\epsilon(x, u) \Lambda(ds \times dx \times du) \right]
= \mathbb{E} \left[ \int_{[\theta_1^{1} \land T^1, t \land \vartheta_1^{1} \land T^1]} B \Psi^\epsilon(X(s), u) \Lambda(ds \times dx \times du) \right]
= \mathbb{E} \left[ \int_{[\theta_1^{1} \land T^1, t \land \vartheta_1^{1} \land T^1]} (B \Psi^\epsilon(x, u) \Lambda(ds \times dx \times du) \right]
\]
Then, taking into account that \( \mathbb{E} \left[ \Lambda([0, t] \times \Xi) \right] < \infty \) (Theorem A.9), although \( \Psi^\epsilon \notin C^2_b(\Xi) \) the above chain of equalities can be continued as
\[
= \mathbb{E} \left[ \sum_{n=1}^{\infty} 1_{\{ \vartheta_1^{1} \leq t \}} \int_{[\theta_1^{1} \land T^1, t \land \vartheta_1^{1} \land T^1]} (B \Psi^\epsilon(x, u) \Lambda(ds \times dx \times du) \mid \mathcal{F}_{\theta_1^{1} \land T^1} \right]
\leq \mathbb{E} \left[ \sum_{n=1}^{\infty} 1_{\{ \vartheta_1^{1} \leq t \}} \int_{[\theta_1^{1} \land T^1, t \land \vartheta_1^{1} \land T^1]} (B \Psi^\epsilon(x, u) \Lambda(ds \times dx \times du) \right]
\leq \mathbb{E} \left[ \int_{[0, t \land T^1]} (B \Psi^\epsilon(X(s), u) \Lambda(ds \times dx \times du) \right].
\]
Summing up, we have proved that, for each \( \epsilon \),
\[
\mathbb{E}^\theta \left[ \Psi^\epsilon(X(t \land T^1)) \right] \leq \mathbb{E} \left[ \int_{[0, t \land T^1]} (B \Psi^\epsilon(x, u) \Lambda(ds \times dx \times du) \right].
\]
By (A.18), for $\Psi(x) \leq 1$, $\Psi^\epsilon(x) \geq c_1 \Psi(x)^2$, so that, in the limit as $\epsilon \to 0$, we find

$$\mathbb{E}^0 \left[ \Psi(X(t \land T^1)) \right] = 0, \quad \forall t > 0,$$

which contradicts the fact that $X$ spends zero time at the origin. \hfill \square

**APPENDIX B: AUXILIARY FUNCTIONS**

**Lemma B.1.** There exists $\delta^* > 0$ and a function $V \in \mathcal{C}^2(\overline{D} - \{0\})$ such that

$$V(x) > 0, \quad \text{for } x \in D - \{0\}, \quad \lim_{x \in D, x \to 0} V(x) = 0,$$

$$\nabla V(x) \cdot g(x) \leq 0, \quad \text{for } x \in (\partial D - \{0\}) \cap \overline{B_{\delta^*}(0)}.$$

Of course we can always define $V(0) := 0$.

**Proof.** It is enough to prove that there exists $\delta^* > 0$ and a function $V \in \mathcal{C}^2((D - \{0\}) \cap \overline{B_{\delta^*}(0)})$ such that $V(x) > 0$ for $x \in (D - \{0\}) \cap \overline{B_{\delta^*}(0)}$, the limit in (B.1) holds and (B.2) is satisfied.

Let $\alpha^*, \psi_{\alpha^*}$ and $\psi^0$ be as in Theorem A.2, for $d = 2$, and in Theorem A.4, for $d \geq 3$. Since $\partial \mathcal{S}$ is smooth, by Condition 3.1 (ii), we can extend $\psi_{\alpha^*}$ to a $\mathcal{C}^2$ function on some open neighborhood $\mathcal{S}^*$ of $\overline{\mathcal{S}}$ such that

$$\inf_{z \in \mathcal{S}^*} \psi_{\alpha^*}(z) > 0.$$

Analogously we can extend $\psi^0$ to a $\mathcal{C}^2$ function on some open neighborhood $\mathcal{S}^*$ of $\overline{\mathcal{S}}$ such that $\inf_{z \in \mathcal{S}^*} e^{-\psi^0(z)} > 0$. Let $K^* := \{x : x = rz, z \in \mathcal{S}^*, r > 0\}$.

Let $\Phi$ be the function defined in (A.9). We have

$$\Phi(x) > 0 \text{ in } K^*, \quad \lim_{x \in K^*, x \to 0} \Phi(x) = 0,$$

and, for some $c_\Phi \geq c'_{\Phi} > 0$,

$$\frac{c_\Phi \Phi(x)}{|x|} \leq |\nabla \Phi(x)| \leq \frac{c_\Phi \Phi(x)}{|x|}, \quad |D^2 \Phi(x)| \leq \frac{c_\Phi \Phi(x)}{|x|^2}, \quad x \in K^*. $$

We will look for a function $V$ of the form

$$V(x) := f(\Phi(x)) - c_V e \cdot x,$$

for some $f \in \mathcal{C}^2((0, \infty))$ such that $\lim_{u \to 0^+} f(u) = 0$ and some $c_V > 0$. Then, by Condition 3.3(iv),

$$\nabla V(x) \cdot g(x) \leq -c_V c_e, \quad \text{for } x \in \partial K - \{0\}.$$

By Condition 3.1 (i), there is $\delta^*, 0 < \delta^* \leq r_D$, such that $(D - \{0\}) \cap B_{\delta^*}(0) \subset K^* \cap \overline{B_{\delta^*}(0)}$. Then, for $x \in (\partial D - \{0\}) \cap \overline{B_{\delta^*}(0)}$, letting $\tilde{z}$ be the closest point on $\partial \mathcal{S}$ to $\frac{x}{|x|}$, by Condition 3.1 (i) and Condition 3.3(i), we have

$$\nabla V(x) \cdot g(x)$$

$$\leq \nabla V(|x| \tilde{z}) \cdot \dot{g}(|x| \tilde{z}) + |\nabla V(x) - \nabla V(|x| \tilde{z})| + |\nabla V(|x| \tilde{z})| |g(x) - \dot{g}(|x| \tilde{z})|$$

$$\leq -c_V c_e + d \sup_{0 < t < 1} |D^2 V(tx + (1-t)|x| \tilde{z})|c_D |x|^2 + |\nabla V(|x| \tilde{z})|c_g |x|. $$
Since \( \inf_{0 < t < 1} \left| tx + (1 - t)|x| \right| \geq \frac{1}{2}|x| \) for \( |x| \leq \delta^* \), \( \delta^* \leq \sqrt{3}/c_D \), one way to ensure (B.2), is to choose \( f \) in (B.5) so that
\[
\lim_{x \in K^*, x \to 0} |\nabla V(x)| |x| = 0, \quad \lim_{x \in K^*, x \to 0} |D^2 V(x)| |x|^2 = 0,
\]
that is
\[
\lim_{x \in K^*, x \to 0} |f'(\Phi(x))||\nabla \Phi(x)||x| = 0, \quad \lim_{x \in K^*, x \to 0} |f''(\Phi(x))| |\nabla \Phi(x)|^2 |x|^2 = 0, \quad \lim_{x \in K^*, x \to 0} |f'(\Phi(x))||D^2 \Phi(x)||x|^2 = 0.
\]
In view of (B.4), this is implied by
\[
\lim_{x \in K^*, x \to 0} f'(\Phi(x))\Phi(x) = 0, \quad \lim_{x \in K^*, x \to 0} f''(\Phi(x))\Phi(x)^2 = 0.
\]
If, in addition,
\[
\inf_{x \in (\overline{D} - \{0\}) \cap \overline{B_{\delta^*}(0)} \cap \overline{B_{\delta^*}(0)}} \frac{f(\Phi(x))}{|x|} > 0,
\]
then, by choosing \( c_V < \inf_{x \in (\overline{D} - \{0\}) \cap \overline{B_{\delta^*}(0)}} \frac{f(\Phi(x))}{|x|} \), we will obtain \( V(x) > 0 \) for \( x \in (\overline{D} - \{0\}) \cap \overline{B_{\delta^*}(0)} \).
Therefore we can take, for instance,
\[
f(u) := u^{1/|\alpha^*|}, \text{ for } \alpha^* \neq 0, \quad f(u) := u, \text{ for } \alpha^* = 0.
\]
\[\square\]

**Proof of Lemma 3.14.**

Let \( \delta^* \) and \( V \) be as in Lemma B.1.

By Condition 3.3 (i) and (iv), possibly by taking a smaller \( \delta^* \), we can always suppose that
\[
\inf_{g \in G(x), |g| = 1, x \in \partial D, |x| \leq \delta^*} e \cdot g > 0.
\]
Let \( 0 < p^* < 1 \) be such that
\[
\sup_{x \in \overline{D}, |x| \leq p^* \delta^*} V(x) < \inf_{x \in \overline{D}, |x| \geq \delta^*} V(x).
\]
Let \( \widetilde{D} \) be a bounded domain with \( C^1 \) boundary such that \( \overline{D} \subset D \) and \( \overline{D} \cap \overline{B_{p^* \delta^*}(0)}^c = \overline{D} \cap B_{p^* \delta^*}(0)^c \) and let \( \widetilde{g} : \mathbb{R}^d \to \mathbb{R}^d \) be a locally Lipschitz vector field, of unit length on \( \partial \overline{D} \), such that \( \widetilde{g}(x) = g(x) \) for \( x \in \partial \overline{D} \cap \overline{B_{p^* \delta^*}(0)}^c \) and, denoting by \( \tilde{n}(x) \) the unit, inward normal at \( x \in \partial \overline{D} \), it holds \( \inf_{x \in \partial \overline{D}} \widetilde{g}(x) \cdot \tilde{n}(x) > 0 \). There exists a function \( \widetilde{F} \in C^2(\overline{D}) \) such that
\[
\inf_{x \in \partial \overline{D}} \nabla \widetilde{F}(x) \cdot \widetilde{g}(x) > 0,
\]
(see Crandall et al. (1992), Lemma 7.6). Of course we can always assume that
\[
\sup_{x \in \overline{D}, p^* \delta^* \leq |x| \leq \delta^*} \widetilde{F}(x) \leq -\delta^*.
\]
Now let \(\chi: \mathbb{R} \to [0,1]\) be a nonincreasing, \(C^\infty\) function such that \(\chi(u) = 1\) for \(u \leq \sup_{x \in \overline{D}, |x| \leq r^\delta} V(x)\) and \(\chi(u) = 0\) for \(u \geq \inf_{x \in \overline{D}, |x| \geq \delta} V(x)\). Defining

\[ F(x) := \chi(V(x)) e \cdot x + (1 - \chi(V(x))) \tilde{F}(x), \]

we have

\[ \nabla F(x) = [\chi(V(x)) e + (1 - \chi(V(x))) \nabla \tilde{F}(x)] + (e \cdot x - \tilde{F}(x)) \chi'(V(x)) \nabla V(x) \]

so that, for all \(x \in \partial D\), \(g \in G(x), |g| = 1\),

\[ \nabla F(x) \cdot g \geq \inf_{y \in G(y), |g| = 1, y \in \partial D, |y| \leq \delta^*} e \cdot g \wedge \inf_{y \in \partial D} \nabla \tilde{F}(y) \cdot g(y). \]

Proof of Lemma 3.28.

Let \(\alpha^*, \psi_{\alpha^*}\) and \(\psi^0\) be as in Theorem A.2, for \(d = 2\), and in Theorem A.4, for \(d \geq 3\), and let \(\Psi\) be given by (A.5). By Theorem A.10, we can fix \(\beta\) such that

\[(B.6) \quad \alpha^* \lor 0 < \beta < 1.\]

Then, by Remark A.7, we can extend \(\psi_{\alpha^*}\) to a \(C^{2+\beta}\) function on some open neighborhood \(S^*\) of \(\overline{S}\) such that

\[ \inf_{z \in S^*} \psi_{\alpha^*}(z) > 0. \]

Analogously we can extend \(\psi^0\) to a \(C^{2+\beta}\) function on some open neighborhood \(S^*\) of \(\overline{S}\). Let \(K^* := \{x: x = rz, z \in S^*, r > 0\}\). We will choose \(S^*\) such that

\[(B.7) \quad e \cdot x \geq -c_e^* |x|, \quad 0 < c_e^* < 1, \quad x \in K^*. \]

The derivatives of \(\Psi\) satisfy, for some \(c_\Psi > c_\Psi^* > 0\),

\[(B.8) \quad \frac{c_\Psi^*}{|x|} \leq |\nabla \Psi(x)| \leq \frac{c_\Psi}{|x|}, \quad |D^2 \Psi(x)| \leq \frac{c_\Psi}{|x|^2}, \quad x \in K^*, \quad \text{if } \alpha^* = 0, \]

\[(B.9) \quad \frac{c_\Psi}{|x|} \leq |\nabla \Psi(x)| \leq \frac{c_\Psi^*}{|x|}, \quad |D^2 \Psi(x)| \leq \frac{c_\Psi^*}{|x|^2}, \quad x \in K^*, \quad \text{if } \alpha^* \neq 0. \]

Let \(\delta^*, 0 < \delta^* \leq r_D\), be such that \((D - \{0\}) \cap B_{\delta^*}(0) \subset K^* \cap B_{\delta^*}(0)\). The fact that \(\psi^0, \psi_{\alpha^*} \in C^{2+\beta}(S^*)\), combined with (A.6) and Condition 3.1 (i), implies that

\[(B.10) \quad |\Delta \Psi(x)| \leq \frac{c_\Psi}{|x|^{2-\beta}}, \quad x \in (D - \{0\}) \cap B_{\delta^*}(0), \quad \text{if } \alpha^* = 0, \]

\[(B.11) \quad |\Delta \Psi(x)| \leq \frac{c_\Psi}{|x|^{2+\beta}}, \quad x \in (D - \{0\}) \cap B_{\delta^*}(0), \quad \text{if } \alpha^* \neq 0. \]

Consider first the case \(\alpha^* \leq 0\). We look for \(V\) of the form

\[ V(x) := f(\Psi(x)) - e \cdot x, \]

with \(f \in C^2((0,\infty))\) such that

\[ \lim_{x \to 0, x \in (D - \{0\})} f(\Psi(x)) = \infty. \]
By the same computations as in the proof of Lemma B.1 and by (B.8), (B.9), we see that (3.29) is verified as soon as

\[(B.12) \quad \lim_{x \in K^*, x \to 0} f'(\Psi(x)) = 0, \quad \lim_{x \in K^*, x \to 0} f''(\Psi(x)) = 0, \quad \text{if } \alpha^* = 0,\]

\[(B.13) \quad \lim_{x \in K^*, x \to 0} f'(\Psi(x)) \Psi(x) = 0, \quad \lim_{x \in K^*, x \to 0} f''(\Psi(x)) \Psi(x)^2 = 0, \quad \text{if } \alpha^* \neq 0.\]

As far as (3.30) is concerned, we have, by Condition 3.6, (B.8) and (B.12), or (B.9), and (B.13),

\[2AV(x)\]

\[(B.14) = \Delta V(x) + |x|^{-1}o(1)\]

\[= f''(\Psi(x)) |\nabla \Psi(x)|^2 + f'(\Psi(x)) \Delta \Psi(x) + |x|^{-1}o(1)\]

\[= |x|^{\beta-2} \left( f''(\Psi(x)) |\nabla \Psi(x)|^2 |x|^{2-\beta} + f'(\Psi(x)) \Delta \Psi(x) |x|^{2-\beta} + |x|^{1-\beta}o(1) \right).\]

Hence, taking into account (B.8) and (B.10) or (B.9) and (B.11), (3.30) holds for instance if, in addition to (B.12) or (B.13),

\[\sup_{x \in (D-\{0\}) \cap \overline{B_r(0)}} f''(\Psi(x)) |x|^{-\beta} < 0, \quad \text{if } \alpha^* = 0,\]

\[\sup_{x \in (D-\{0\}) \cap \overline{B_r(0)}} f''(\Psi(x)) \Psi(x)^2 |x|^{-\beta} < 0, \quad \text{if } \alpha^* < 0.\]

Therefore, for \(\delta^*\) small enough that \(\Psi(x) > 1\), we can take

\[f(u) := \ln(u), \quad \text{for } \alpha^* = 0, \quad f(u) := \ln(\ln(u)), \quad \text{for } \alpha^* < 0.\]

In the case \(0 < \alpha^* < 1\), we look for \(V_1\) and \(V_2\) of the form

\[V_1(x) := f_1(\Psi(x)) + e \cdot x, \quad V_2(x) := f_2(\Psi(x)) - e \cdot x,\]

with \(f_1, f_2 \in C^2((0, \infty))\) such that

\[\lim_{x \in \overline{D}, x \to 0} f_1(\Psi(x)) = 0, \quad \liminf_{x \in \overline{D}, x \to 0} f_1(\Psi(x)) |x|^{-1} \geq 0,\]

\[\lim_{x \in \overline{D}, x \to 0} f_2(\Psi(x)) = 0, \quad \liminf_{x \in \overline{D}, x \to 0} f_2(\Psi(x)) |x|^{-1} > 1.\]

Consider \(V_2\). If

\[(B.15) \quad \sup_{x \in (K^* \setminus \{0\}) \cap \overline{B_r(0)}} |f_2'(\Psi(x))| < \infty, \quad \sup_{x \in (K^* \setminus \{0\}) \cap \overline{B_r(0)}} |f_2''(\Psi(x))| |\Psi(x)| < \infty,\]

we have, by the same computations as for (B.14),

\[2AV_2(x)\]

\[= f_2''(\Psi(x)) |\nabla \Psi(x)|^2 + f_2'(\Psi(x)) \Delta \Psi(x) + |x|^{-1}\Psi(x)O(1)\]

\[= |x|^{\beta-2}\Psi(x) \left( f_2''(\Psi(x)) |\nabla \Psi(x)|^2 |x|^{-1} |x|^{2-\beta} + f_2'(\Psi(x)) \Delta \Psi(x) |\Psi(x)|^{-1} |x|^{2-\beta} + |x|^{1-\beta}O(1) \right).\]
If, in addition to (B.15), $f_2$ satisfies
\[
\lim_{x \in D, x \to 0} f''_2(\Psi(x)) |x|^{-\beta} = -\infty,
\]
then, taking into account (B.9), $V_2$ satisfies (3.33). Note that (B.15) implies (B.13), so that (3.32) is verified as well. By analogous computations we see that if $f_1$ satisfies (B.15) and
\[
\lim_{x \in D, x \to 0} f''_1(\Psi(x)) |x|^{-\beta} = \infty,
\]
then $V_1$ will satisfy (3.32) and (3.33). With the choice
\[
f_1(u) = \exp(u) - 1, \quad f_2(u) = \ln(u + 1),
\]
$V_1$ and $V_2$ will satisfy the third condition in (3.31) as well. \( \square \)

**Proof of Lemma A.8.** Since $\partial S$ is of class $C^3$, there exists $F^S \in C^2(\overline{S})$ such that $F^S(z) \geq 0$ and, on $\partial S$, $F^S(z) = 0$, $\nabla_{S^d-1} F^S(z) \cdot \tilde{g}(z) \geq 1$. Let $\chi^* \in C^\infty_b([0, \infty))$ be a nondecreasing function such that $\chi^*(u) = u$ for $u \leq 1/2$, $\chi^*(u) = 1$ for $u \geq 1$. Define
\[
F^*(x) := \chi^*(|x| F^S(\frac{x}{|x|})).
\]
Then $F^* \in C^2_b(\overline{K} - \{0\})$ and, for $x \in \partial K - \{0\}$,
\[
\nabla F^*(x) \cdot \tilde{g}(x) = \nabla_{S^d-1} F^S(\frac{x}{|x|}) \cdot \tilde{g}(\frac{x}{|x|}) \geq c^*.
\]

Now let $\delta > 1$ be such that $\sup_{x \in K, |x| \leq 1} \Phi(x) < \inf_{x \in K, |x| \geq \delta} \Phi(x)$. Let $D$ be a bounded domain such that $D \subset K \cap B_{\delta+1}(0)$, $\overline{D} \cap B_\delta(0) = \overline{K} \cap B_\delta(0)$ and $\partial D - \{0\}$ is of class $C^1$. Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be a locally Lipschitz vector field, of unit length on $\partial D$, such that $g(x) = \tilde{g}(x)$ for $x \in \partial D \cap B_\delta(0)$ and, denoting by $n(x)$ the unit, inward normal at $x \in \partial D$, it holds $\inf_{x \in \partial D - \{0\}} g(x) \cdot n(x) > 0$. Then Lemma 3.14 ensures the existence of a function $F^D \in C^2(\overline{K} \cap B_\delta(0))$ such that $\nabla F^D(x) \cdot g \geq c_F > 0$ for every $g \in G(x)$, $x \in \partial K \cap B_\delta(0)$. Let $\chi : \mathbb{R} \to [0,1]$ be a nonincreasing, $C^\infty$ function such that $\chi(u) = 1$ for $u \leq \sup_{x \in K, |x| \leq \delta} \Phi(x)$ and $\chi(u) = 0$ for $u \geq \inf_{x \in K, |x| \geq \delta} \Phi(x)$. Then the function
\[
F(x) := \chi(\Phi(x)) F^D(x) + (1 - \chi(\Phi(x))) F^*(x)
\]
has the desired properties. \( \square \)

**REFERENCES**

BASS, R. F. and PARDOUX, É. (1987). Uniqueness for diffusions with piecewise constant coefficients. *Probab. Theory Related Fields* **76** (4) 557–572.

COSTANTINI, C. (2023). Existence and uniqueness of obliquely reflecting Brownian motion in nonpolyhedral, piecewise smooth comers, with an example of application to diffusion approximation of bandwidth sharing queues. arXiv:2308.06745

COSTANTINI, C. and KURTZ, T. G. (2015). Viscosity methods giving uniqueness for martingale problems. *Electron. J. Probab.* **20** no. 67, 27.

COSTANTINI, C. and KURTZ, T. G. (2018). Existence and uniqueness of reflecting diffusions in cusps. *Electron. J. Probab.* **23** no. 84, 21.
COSTANTINI, C. and KURTZ, T. G. (2019). Markov selection for constrained martingale problems. *Electron. J. Probab.* **24** no. 135, 31.

COSTANTINI, C. and KURTZ, T. G. (2024). Localization for constrained martingale problems and optimal conditions for uniqueness of reflecting diffusions in 2-dimensional domains. *Stochastic Process. Appl.* doi: 10.1016/j.spa.2024.104295.

CRANDALL, M. G., ISHII, H. and LIONS, P. L. (1992). User’s guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N. S.)* **27** (1) 1–67.

DAI, J. and WILLIAMS, R. J. (1996). Existence and uniqueness of semimartingale reflecting Brownian motions in convex polyhedrons. *Theory Probab. Appl.* **40** (1) 1–40.

DIANIETTI, J. and FERRARI, G. (2023). Multidimensional singular control and related Skorokhod problem: sufficient conditions for the characterization of optimal controls. *Stochastic Process. Appl.* **162** (1) 547–592.

DU PUIS, P. and ISHII, H. (1993). SDEs with oblique reflection on nonsmooth domains. *Ann. Probab.* **21** (1) 554–580.

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York.

GILBARG, D. and TRUDINGER, N. S. (1983). *Elliptic partial differential equations of second order*, 2nd ed. Springer-Verlag, Berlin.

KANG, W. N., KELLY, F. P., LEE, N. H. and WILLIAMS, R. J. (2009). State space collapse and diffusion approximation for a network operating under a fair bandwidth sharing policy. *Ann. Appl. Probab.* **19** (5) 1719–1780.

KANG, W. N. and RAMANAN, K. (2017). On the submartingale problem for reflected diffusions in domains with piecewise smooth boundaries. *Ann. Probab.* **45** (1) 404–468.

KANG, W. N. and WILLIAMS, R. J. (2012). Diffusion approximation for an input-queued switch operating under a maximum weight matching policy. *Stochastic Systems* **2** (2) 277–321.

KELLY, F. P. and WILLIAMS, R. J. (2004). Fluid model for a network operating under a fair bandwidth-sharing policy. *Ann. Appl. Probab.* **14** (3) 1055–1083.

KREINFELT, M. G. and RUTMAN, M. A. (1950). Linear operators leaving invariant a cone in a Banach space. *Amer. Math. Soc. Transl.* **26** 128.

KURTZ, T. G. (1990). Martingale problems for constrained Markov problems. In *Recent advances in stochastic calculus (College Park, MD, 1987)* 151–168. Springer, New York.

KURTZ, T. G. (1991). A control formulation for constrained Markov processes. In *Mathematics of random media (Blacksburg, VA, 1989)* 139–150. Amer. Math. Soc., Providence, RI.

KURTZ, T. G. and STOCKBRIDGE, R. H. (2001). Stationary solutions and forward equations for controlled and singular martingale problems. *Electron. J. Probab.* **6** no. 17, 52.

KWON, Y. and WILLIAMS, R. J. (1991). Reflected Brownian motion in a cone with radially homogeneous reflection field. *Trans. Amer. Math. Soc.* **327** (2) 739–780.

POTTER, P. E. (1990). *Stochastic integration and differential equations*. Springer-Verlag, Berlin.

SHAH, D. and WISCHIK, D. (2012). Switched networks with maximum weight policies: fluid approximation and multiplicative state space collapse. *Ann. Appl. Probab.* **22** (1) 70–127.

TAYLOR, L. M. and WILLIAMS, R. J. (1993). Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. *Probab. Theory Related Fields* **96** (3) 283–317.

VARADHAN, S. R. S. and WILLIAMS, R. J. (1985). Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **38** (4) 405–443.

WILLIAMS, R. J. (1985). Reflected Brownian motion in a wedge: semimartingale property. *Z. Wahrschein. Verw. Gebiete* **69** (2) 161–176.

WILLIAMS, S. A., CHOW, P. L. and MENALDI, J. L. (1994). Regularity of the free boundary in singular stochastic control. *J. Differential Equations* **111** (1) 175–201.