Manifold valued data analysis of samples of networks, with applications in corpus linguistics

Katie Severn, Ian L. Dryden and Simon P. Preston

K.E.SEVERN, I.L.DRYDEN and S.P.PRESTON,
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF NOTTINGHAM
UNIVERSITY PARK
NOTTINGHAM, NG7 2RD
UK

e-mail: katie.severn@nottingham.ac.uk; ian.dryden@nottingham.ac.uk; simon.preston@nottingham.ac.uk

Abstract: Networks can be used in many applications, such as in the analysis of text documents, social interactions and brain activity. We develop a general framework for extrinsic statistical analysis of samples of networks, motivated by networks representing text documents in corpus linguistics. We identify networks with their graph Laplacian matrices, for which we define metrics, embeddings, tangent spaces, and a projection from Euclidean space to the space of graph Laplacians. This framework provides a way of computing means, performing principal component analysis and regression, and carrying out hypothesis tests, such as for testing for equality of means between two samples of networks. We apply the methodology to the set of novels by Jane Austen and Charles Dickens.

MSC 2010 subject classifications: Primary 62H99, 62H15; secondary 62P99.

Keywords and phrases: Extrinsic mean, Graph Laplacian, Regression, Riemannian, Hypothesis test.

1. Introduction

The statistical analysis of networks dates back to at least the 1930’s, however interest has increased considerably in the 21st century (Kolaczyk, 2009). Networks are able to represent many different types of data, for example social networks, neuroimaging data and text documents. In this paper, each observation is a weighted network, denoted $G_m = (V, E)$, comprising a set of nodes, $V = \{v_1, v_2, \ldots, v_m\}$, and a set of edge weights, $E = \{w_{ij} : w_{ij} \geq 0, 1 \leq i, j \leq m\}$, indicating nodes $v_i$ and $v_j$ are either connected by an edge of weight $w_{ij} > 0$, or else unconnected (if $w_{ij} = 0$). An unweighted network is the special case with $w_{ij} \in \{0, 1\}$. We restrict attention to

*This work was supported by the Engineering and Physical Sciences Research Council [grant number EP/M02315X/1]. The authors are grateful to Michaela Mahlberg, Viola Wiegand and Anthony Hennessey for their help and discussions about the data.
networks that are undirected and without loops, so that \( w_{ij} = w_{ji} \) and \( w_{ii} = 0 \), then any such network can be identified with its graph Laplacian matrix \( L = (l_{ij}) \), defined as

\[
l_{ij} = \begin{cases} -w_{ij}, & \text{if } i \neq j \\ \sum_{k \neq i} w_{ik}, & \text{if } i = j \end{cases}
\]

for \( 1 \leq i, j \leq m \). The graph Laplacian can be written as \( L = D - A \), in terms of the adjacency matrix, \( A = (w_{ij}) \), and degree matrix \( D = \text{diag}(\sum_{j=1}^{m} w_{1j}, \ldots, \sum_{j=1}^{m} w_{mj}) = \text{diag}(A1_m) \), where \( 1_m \) is the \( m \)-vector of ones. The \( i \)th diagonal element of \( D \) equals the degree of node \( i \). The space of all graph Laplacians of dimension \( m \times m \) is

\[
\mathcal{L}_m = \{ L = (l_{ij}) : L = L^T; l_{ij} \leq 0 \forall i \neq j; L1_m = 0_m \},
\]

where \( 0_m \) is the \( m \)-vector of zeroes. The space \( \mathcal{L}_m \) is a manifold, in particular a convex subset of the cone of symmetric positive semi-definite matrices with corners (Ginestet et al., 2017).

For the tasks we address the data are a random sample \( L_1, \ldots, L_n \) from a population of networks, where each observation is a graph Laplacian \( L_k \in \mathcal{L}_m \) representing networks with a common node set \( V \). Graph Laplacians, as with most network representations, are not standard Euclidean data and so for typical statistical tasks, such as computing the mean, performing principal component analysis and regression, and testing equality of means based on two-samples, standard Euclidean methods need to be carefully adapted.

To perform statistical analysis on the manifold of graph Laplacians we must define suitable metrics. We will consider two general metrics between graph Laplacians:

- Euclidean power metric:
  \[
d_{\alpha}(L_1, L_2) = \| L_1^{\alpha} - L_2^{\alpha} \|,
\]

- Procrustes power metric:
  \[
d_{\alpha,S}(L_1, L_2) = \inf_{R \in O(m)} \| L_1^{\alpha} - L_2^{\alpha} R \|,
\]

where \( R \) is an orthogonal matrix for the ordinary Procrustes match of \( L_2^{\alpha} \) to \( L_1^{\alpha} \) (Dryden and Mardia, 2016, chapter 7) and \( \| A \| = \{ \text{trace}(A^T A) \}^{1/2} \) is the Frobenius norm, which is also known as the Euclidean norm. Common choices of Euclidean power metrics and Procrustes metrics are \( d_1, d_2 \) and \( d_1, S \), referred to as the Euclidean, square root Euclidean and Procrustes size-and-shape metrics respectively (Dryden, Koloydenko and Zhou, 2009). We provide more detail about these metrics in Section 3.

Analysing networks by representing them as elements of \( \mathcal{L}_m \) is an approach also used by Ginestet et al. (2017). The authors considered the Euclidean metric \( d_1 \) and derived a central limit theorem which they used to develop a test between two samples networks, driven by an application in neuroimaging. Motivation for our considering metrics other than \( d_1 \) includes evidence that interpolation of non-Euclidean data based on \( d_1 \) often has disadvantages, such as swelling (in the context of positive semi-definite matrices (Dryden, Koloydenko and Zhou, 2009)) and lack of interpretability (in the context of graph Laplacians (Bakker, Halapannavar and Sathanur, 2018)).
The Jane Austen and Charles Dickens novels from the CLiC database (Mahlberg et al., 2016)

2. Application: Jane Austen and Charles Dickens novels

In corpus linguistics, networks are used to model documents comprising a text corpus (Phillips, 1983). Each node represents a word, and edges indicate words that co-occur within some span—typically 5 words, which we use henceforth—of each other (Evert, 2008). Our dataset is derived from the full text in novels by Jane Austen and Charles Dickens, as listed in Table 1, obtained from CLiC (Mahlberg et al., 2016). For each of the 7 Austen and 16 Dickens novels, the “year written” refers to the year in which the author started writing the novel; see The Jane Austen Society of North America (2018) and Charles Dickens Info (2018). Our key statistical goals are to investigate the authors’ evolving writing styles, by regressing the networks on “year written”; to explore dominant modes of variability, by developing principal component analysis for samples of networks; and to test for significance of differences in Austen’s and Dickens’ writing styles, via a two-sample test of equality of mean networks.

For each Austen and Dickens novel we produce a network representing pairwise word co-occurrence. If the node set $V$ corresponded to every word in all the novels it would be very large, with $m = 48285$, but a relatively small number of words are used far more than others. The top $m = 50$ words cover 45.6% of the total word frequency, $m = 1000$ cover 79.6%, and $m = 10000$ cover 96.7%. We focus on a truncated set of the $m$ most frequent words and the $w_{ij}$’s are the pairwise co-occurrence counts between these words. In our analysis we choose $m = 1000$ as a sensible trade-off between having very large, very sparse graph Laplacians versus small graph Laplacians of just the most common words. For each novel and the truncated node set, the network produced is converted to a graph Laplacian. A pre-processing step for the novels is to normalise each graph Laplacian in order to remove the gross effects of different lengths of the novels by dividing each graph Laplacian by its own trace, resulting in a trace of

---

1 Christmas Carol and Lady Susan are short novellas rather than novels, but we shall use the term “novel” for each of the works for ease of explanation.
1 for each novel.

As an indication of the broad similarity of the most common words we list the top 25 words in the table in Appendix A. Of the top 25 words across all novels 22 appear in the most frequent 25 words for the Dickens novels and 23 for the Austen novels. The words *not*, *be*, *she* do not appear in Dickens’ top 25 and the words *mr* and *said* do not appear in Austen’s top 25. Some differences in relative rank are immediately apparent: *her*, *she*, *not* having higher relative rank in Austen and *he*, *his*, *mr*, *said* having relatively higher rank in Dickens.

We initially compare some choices of distance metrics on the Austen and Dickens data after constructing the graph Laplacians from the $m = 1000$ most frequent words across all 23 novels. Figure 1 (left column) shows the results of a hierarchical cluster analysis using Ward’s method (Ward, 1963), based on pairwise distances between novels using metrics $d_1$, $d_2$ and $d_3$. For computing the Procrustes metric we use the *shapes* package (Dryden, 2018) in R (R Core Team, 2018).

The dendrograms for square root and Procrustes separate the authors into two very distinct clusters, whereas for Euclidean distance Dickens’ *David Copperfield* and *Great Expectations* are clustered with Austen’s *Lady Susan* which is unsatisfactory. The next sub-division of the Dickens cluster using square root/Procrustes distance splits into groups of the earlier novels versus later novels, with the exception being the historical novel *A Tale of Two Cities* which is clustered with the earlier novels. There is not such a clear sub-division for Dickens using the Euclidean metric. In the Austen cluster for square root and Procrustes there is clearly a large distance between *Lady Susan* and the rest, where *Lady Susan* is her earliest work, a short novella published 54 years after Austen’s death.

Figure 1 (right column) shows corresponding plots of the first two multi-dimensional scaling (MDS) variables from a classical multi-dimensional scaling analysis. The square root and Procrustes MDS plots are visually identical, although they are slightly different numerically. We see that there is a clear separation in MDS space between Austen’s and Dickens’ works with a very strong separation in MDS1 using the square root and Procrustes distances, and less so for Euclidean distance.
Fig 1: Cluster analysis and MDS plots based on (from top to bottom) the Euclidean distance, $d_1$, square root distance, $d_{1/2}$, and Procrustes distance, $d_2$, each with $m = 1000$. The plots display Austen’s novels in blue and lower case, and Dickens’s novels in red and upper case.
3. Framework for the statistical analysis of graph Laplacians

3.1. Preliminary framework

The general framework we will define in this section for the statistical analysis of graph Laplacians involves embedding $L_m$, shown schematically in Figure 2. The identity projection, $Id$, illustrates that $L_m \subset PSD_m$, where

$$PSD_m = \{ S^{m \times m} : x^T S x \geq 0 \forall x \in \mathbb{R}^m ; S = S^T \}$$

is the space of symmetric positive semi-definite matrices of dimension $m \times m$. This is evident because any $L \in L_m$ is diagonally dominant, as $|l_{ii}| = \sum_{i \neq j} |l_{ij}|$, which is a sufficient condition for $L \in PSD_m$ (De Klerk, 2006, page 232).

Distance metrics such as (2) and (3) on manifolds are referred to as intrinsic or extrinsic. An intrinsic distance is the length of a shortest geodesic path in the manifold, whereas an extrinsic distance is one induced by a Euclidean distance in an embedding of the manifold (Dryden and Mardia, 2016, p112). On $L_m$, Euclidean distance $d_1$ is intrinsic, but in general $d_\alpha$ and $d_{\alpha,S}$ are extrinsic with respect to an embedding defined as follows.

First, we write $L = U \Lambda U^T$ by the spectral decomposition theorem, with $\Lambda = \text{diag}(\xi_1, \ldots, \xi_m)$ and $U = (u_1, \ldots, u_m)$, where $\{\xi_i\}_{i=1,\ldots,m}$ and $\{u_i\}_{i=1,\ldots,m}$ are the eigenvalues and corresponding eigenvectors of $L$. Since $L_m \subset PSD_m$ thus $\xi_i \geq 0$, hence for any $\alpha > 0$

$$F_\alpha(L) = L^\alpha = U \Lambda^\alpha U^T : PSD_m \rightarrow M_m.$$  \hspace{1cm} (5)

embeds $PSD_m$ into $M_m$. The embedding space $M_m$ is dependent on the choice of metric, and defined for specific metrics below.

Distance metrics (2) and (3) in terms of embedding $F_\alpha$, for $L_1, L_2 \in L_m$, are hence

$$d_\alpha(L_1, L_2) = \|F_\alpha(L_1) - F_\alpha(L_2)\|$$

$$d_{\alpha,S}(L_1, L_2) = \inf_{R \in O(m)} \|F_\alpha(L_1) - F_\alpha(L_2)R\|.$$

These distances in fact hold more generally for $L_1, L_2 \in PSD_m$. 
We consider three choices of $F^{-1}_\alpha$ for the reverse mapping back from the embedding space, which are suitable for different scenarios. The choice of $F^{-1}_\alpha$ is dependent on whether we want to project to $\text{PSD}$ before reversing the powering of $\alpha$.

When using the Euclidean power metric, the space $\mathcal{M}_m$ is the space of real symmetric $m \times m$ matrices with centred rows and columns, and we use

$$F^{-1}_\alpha(Q) = \begin{cases} (Q)^{\frac{1}{\alpha}}, \text{ when } \frac{1}{\alpha} \text{ is an odd integer} & : \mathcal{M}_m \to \mathcal{M}_m \\ \left(\frac{Q+Q^T+(Q+Q^T)^T(Q+Q^T)}{4}\right)^{\frac{1}{2\alpha}}, \text{ otherwise} & : \mathcal{M}_m \to \mathcal{PSD}_m \end{cases}$$

Note that the second expression before taking the power $\frac{1}{\alpha}$ is the closest symmetric positive semi-definite matrix to $Q$ in terms of Frobenius distance (Higham, 1988).

For the Procrustes power metric, the space $\mathcal{M}_m$ is the reflection size-and-shape space, denoted $\mathcal{RS}\Sigma^{m-1}_m$ (Dryden, Koloydenko and Zhou, 2009; Dryden and Mardia, 2016, p67), and in this case we use

$$F^{-1}_\alpha(Q) = (QQ^T)^{\frac{1}{2\alpha}} : \mathcal{M}_m \to \mathcal{PSD}_m.$$

We choose this reverse map as it removes the orthogonal matrices from the Procrustes fits, which we will see in the next section are introduced from the exponential map.

3.2. Tangent space

To perform further statistical analysis the inverse exponential map, $\exp^{-1}_\nu$, is used to project into a tangent space to $\mathcal{M}_m$, in which standard statistical methods can be applied, where $\nu \in \mathcal{M}_m$ denotes the pole of the projection. Figure 3 shows a simple visualisation of a tangent space. The tangent space at $\nu$ is a Euclidean approximation touching the manifold in which a geodesic becomes a straight line preserving distance to the pole. In non-Euclidean spaces distances are the length of the shortest geodesic path between two points on a manifold. The exponential map provides a connection between the tangent space to the manifold and the inverse exponential map is the map from the manifold to the tangent space (Dryden and Mardia, 2016, Chapter 5).

Fig 3: A simple visualisation of the $\exp^{-1}_\nu$ map, mapping $X$ onto the tangent space $T_\nu$. 
As the graph Laplacian space has centering constraints on the rows and columns these constraints are also preserved in our choice of embedding $F_\alpha$ in $\mathcal{M}_m$. We can remove the centering constraints and reduce dimension when projecting to a tangent space by pre and post multiplying by the $m - 1 \times m$ Helmert sub-matrix $H$ and its transpose as a component of the projection. The Helmert sub matrix $H$, of dimension $m - 1 \times m$,

\[ h_j = -(j(j + 1))^{1/2}, \]

has $j$th row defined as $(h_j, \ldots, h_j, 0, \ldots, 0)$, $h_j = -j(j + 1)^{1/2}$, where $HH^T = I_{m-1}$ and $H^T H = C_m$, where $C_m = I_m - 1_m 1_m^T / m$ is the $m \times m$ centering matrix, $I_m$ is the $m \times m$ identity matrix and $1_m$ is the $m$-vector of ones.

For the Euclidean power metric we define the inverse exponential map $\exp^{-1}_\nu$ to the tangent space $T_\nu(\mathcal{M}_m) = \mathbb{R}^{m(m-1)/2}$ as

\[
\exp^{-1}_\nu(X) = \text{vech}\{H(X - \nu)H^T\}
\]

\[
\exp_\nu(Q) = \nu + H^T \text{vech}^{-1}(Q)H
\]

(6)

where $\text{vech}$ is the half vectorisation of a matrix including the diagonal. In this case $\mathcal{M}_m$ is actually Euclidean, with zero curvature, and analysis is unaffected by the choice of $\nu$, and hence we can take $\nu = 0$.

For the Procrustes power metric we define the map $\exp^{-1}_\nu$ to the tangent space $T_\nu(\mathcal{M}_m) = \mathbb{R}^{m-1 \times m-1}$ as

\[
\exp^{-1}_\nu(X) = \text{vec}\{HR(X - \nu)H^T\}
\]

\[
\exp_\nu(Q) = (\nu + H^T \text{vec}^{-1}(Q)H)\tilde{R}
\]

(7)

where $\text{vec}$ is the vectorise operator obtained from stacking the columns of a matrix, $\tilde{R}$ is the ordinary Procrustes match of $X$ to $\nu$ (Dryden and Mardia, 2016, chapter 7) and $R$ is the ordinary Procrustes match from $(\nu + H^T \text{vec}^{-1}(Q)H)$ to $\nu$. Note that the reflection size-and-shape space is a space with positive curvature (Kendall et al., 1999) and statistical analysis depends on the choice of $\nu$. A sensible choice for $\nu$ is the sample Fréchet mean.

### 3.3. Projection

The framework illustrated in Figure 2 involves a projection, $P_\mathcal{L}$, into the space of graph Laplacians. We seek a $P_\mathcal{L}$ that maps $Y = (y_{ij}) \in \mathcal{M}_m$ to the “closest point” in $\mathcal{L}_m$. For the Euclidean and Procrustes power metric intuitive projections are

\[
P_\alpha(Y) = \arg \inf_{L \in \mathcal{L}_m} d_\alpha(Y, L)
\]

\[
P_{\alpha,S}(Y) = \arg \inf_{L \in \mathcal{L}_m} d_{\alpha,S}(Y, L).
\]

(8)
It is desirable that optimisation involved in computing by the projection is convex, since convex optimisation problems have the useful characteristic that any local minimum must be the unique global minimum (Rockafellar, 1993).

**Result 1.** For $P_\alpha$ with $\alpha = 1$ then the projection can be found by solving a convex optimisation problem with a unique solution, by minimising

$$f(Y) = d_1^2(L, Y) = \sum_{i=1}^{m} \sum_{j=1}^{m} (l_{ij} - y_{ij})^2$$

subject to:

$$l_{ij} - l_{ji} = 0, \quad 0 \leq i, j \leq m$$

$$\sum_{j=1}^{m} l_{ij} = 0, \quad 0 \leq i \leq m$$

$$l_{ij} \leq 0, \quad 0 \leq i, j \leq m \text{ and } i \neq j.$$  \hspace{1cm} (9)

It is immediately clear that this is a convex optimization problem since the objective function is quadratic with Hessian $2I_{m(m-1)/2}$, which is strictly positive definite, and the constraints are linear. The unique global solution can be found using quadratic programming, and so for $Y_1, Y_2 \in \mathcal{M}_m$ if $Y_1 = Y_2$ then $P(Y_1) = P(Y_2)$.

Note that the choice of metric for projection does not need to be the same as the choice of metric for estimation. As the projection for the Euclidean power metric with $\alpha = 1$ involves convex optimisation we will use $P_L = P_1$ throughout for all our metrics. For $\alpha \neq 1$ the optimization is not in general convex. To implement this projection $P_1$ we can, for example, use either the CVXR (Fu et al., 2018) or rosqp (Anderson, 2018) packages in R (R Core Team, 2018) to solve the optimisation, and rosqp is particularly fast even for $m = 1000$.

### 3.4. Means

There are two main types of means on a manifold, the intrinsic mean and extrinsic mean (Dryden and Mardia, 2016, Chapter 6). We define the mean in the graph Laplacian space using extrinsic means, although the mean when the Euclidean power distance with $\alpha = 1$ is used is in fact an intrinsic mean.

We define the population mean for graph Laplacians as

$$\mu = P_L(\eta), \quad \text{where} \quad \eta = \inf_{u \in \mathcal{PSD}_m} \mathbb{E}[d^2(L, u)], \quad \text{(10)}$$

assuming $\mu$ exists, and the sample mean for a set of graph Laplacians as

$$\hat{\mu} = P_L(\hat{\eta}), \quad \text{where} \quad \hat{\eta} = \inf_{u \in \mathcal{PSD}_m} \frac{1}{n} \sum_{k=1}^{n} d^2(L_k, u). \quad \text{(11)}$$
For the Euclidean power distance we have
\[ \eta = F_\alpha^{-1}(\mathbb{E}[F_\alpha(L)]) \]
\[ \hat{\eta} = F_\alpha^{-1}\left(\frac{1}{n} \sum_{k=1}^{n} F_\alpha(L_k)\right) = \left(\frac{1}{n} \sum_{k=1}^{n} (L_k)^\alpha\right)^{\frac{1}{\alpha}}, \]

\( \mu \) and \( \hat{\mu} \) are unique in this case. For the Procrustes power distance \( \mu \) and \( \hat{\mu} \) may be sets, and the conditions for uniqueness rely on the curvature of the space (Le, 1995). In particular the support of the distribution is a geodesic ball \( B_r \) such that \( B_{2r} \) is regular. We will assume uniqueness exists throughout. For the Euclidean power metric when \( \alpha = 1 \), we have \( \hat{\mu} = \hat{\eta} \) and the mean is a Fréchet intrinsic mean (Fréchet, 1948; Ginestet et al., 2017) in this case.

**Result 2.** Let \( L_k \) be a random sample of i.i.d. observations from a distribution with population mean \( \mu \) in (10). For the power Euclidean distance \( d_\alpha \) the estimator \( \hat{\mu} \), in (11), is a consistent estimator of \( \mu \).

The proof of this result can be found in Appendix B. Note that a similar result holds for \( d_{\alpha,S} \) where stronger conditions for consistency of \( \hat{\eta} \) are given in Bhattacharya and Patrangenaru (2003), but the same projection argument used in the proof for \( d_\alpha \) holds.

Figure 4 shows an illustration of the sample means for (a) Austen and (b) Dickens novels using \( d_1 \), with the 1000 words arranged in a grid and edges drawn between words which co-occur with adjacency weight at least \( 10^{-5} \) of the sum of the nodes.
Plots for the square root Euclidean and Procrustes metric, which are not shown, are visually similar to those for the Euclidean mean. Plots (a) and (b) are very similar, perhaps unsurprisingly as approximately half of the words in each novel are represented by the first 50 words. Figure (c) shows edges present in the Austen mean but not in the Dickens mean, and (d) the edges present in the Dickens mean but not in the Austen mean, to highlight the differences between the two networks. These illustrate more co-occurrences of she, her by Austen and the, his, don’t by Dickens, among many others. These plots are drawn using the program Cytoscape (Shannon et al., 2003) and more detail can be seen by magnifying the view to a large extent. We shall explore the differences in more detail later in Section 4.5.

3.5. Interpolation and extrapolation

We now consider an interpolation path, $L(c)$, for $c$ being the position along the path, $0 \leq c \leq 1$, between the graph Laplacians at $L(0)$ and $L(1)$. For $c < 0$ and $c > 1$ the path $L(c)$ is extrapolating from the graph Laplacians, at $L(0)$ and $L(1)$. The interpolation and extrapolation path between graph Laplacians for each metric is defined by first finding the geodesic path in the embedding space between the embedded graph Laplacians, which is then projected to $L_m$. The minimal geodesic passing through $L_1 = P_L(F^{-1}_\alpha(\nu))$ and $L_2$ is

$$L(c) = P_L(F^{-1}_\alpha(\exp_\nu\{c\exp_\nu^{-1}(F_\alpha(L_2))\})).$$

(12)

For the Euclidean power this simplifies to

$$L(c) = P_L(F^{-1}_\alpha(F_\alpha(L_1) + c(F_\alpha(L_2) - F_\alpha(L_1)))).$$

(13)

Figure 5 shows the interpolation and extrapolation paths, for the 25 nodes, corresponding to the most frequent words, out of $m = 1000$ nodes, between the mean Austen and Dickens novels, when using $d_1$. At $c = 6$ the feminine words have larger degrees and their edges have larger weights, for example her to to, of and she to to. For $c = -5$ the nodes for she and her are actually removed indicating they have degree 0, which is further evidence of the fact Austen used female words more than Dickens.

4. Further inference

4.1. Principal component analysis

There are several generalisations of PCA to manifold data, and the following approach is similar to Fletcher et al. (2004) in computing PCA in the tangent space and projecting back to the manifold. See also earlier approaches of PCA in tangent spaces in shape analysis include Kent (1994) and Cootes et al. (1994).
Fig 5: Interpolation \((c = 0.5)\) and extrapolation \((c = -5, c = 6)\) networks between Dickens' and Austen's mean novels using \(d_1\). The top 25 words are displayed where the mean novels for the authors are estimated using \(d_1\) and \(m = 1000\).

Let \(v_k = \text{vec}(\exp^{-1}(F_{\alpha}(L_k))))\), where \(\nu = F_{\alpha}(\hat{\eta})\) for either the Euclidean or Procrustes power metric, then \(S = \frac{1}{2} \sum_{k=1}^{n} v_k v_k^T\) is an estimated covariance matrix. Suppose \(S\) is of rank \(r\) with non-zero eigenvalues \(\lambda_1, \ldots, \lambda_r\), then the corresponding eigenvectors \(\gamma_1, \ldots, \gamma_r\) are the principal components (PCs) in the tangent space, and the PC scores are

\[
s_{kj} = \gamma_j^T v_k, \quad \text{for } k = 1, \ldots, n, \quad j = 1, \ldots, r. \tag{14}
\]

The path of the \(j\)th PC in \(L_m\) is

\[
L(c) = P_C(F_{\alpha}^{-1}(\exp_{\nu}(c \frac{1}{2} \text{vec}^{-1}(\gamma_j))))), \quad c \in \mathbb{R}. \tag{15}
\]

When for the Euclidean case when \(\alpha = 1\) is chosen, the importance of the \(i\)th word in the principal component \(\gamma\) is given by

\[
\frac{\exp_{\nu}(\text{vec}^{-1}(\gamma))_{ii}}{\left(\sum_{j=1}^{m} \exp_{\nu}(\text{vec}^{-1}(\gamma))_{jj}\right)}, \quad \text{for } 1 \leq i \leq m. \tag{16}
\]

We now apply the methods of PCA to the Austen and Dickens text data, for \(m = 1000\). The first and second PC scores are plotted in Figure 6 for the Euclidean and square root Euclidean metric. The Procrustes metric is not included as it gave visually identical results to the square root Euclidean. The extrinsic regression lines are included which we will define and explain below. The variance explained by PC 1 and PC 1 and 2 together was 49\% and 70\%, 37\% and 46\% and 37\% and 46\% for the Euclidean, square root Euclidean and Procrustes size-and-shape respectively. A benefit of the square root Euclidean metric is clear here as it separates the Austen and Dickens novels with a large gap on PC1 where as \textit{David Copperfield} (DC) and \textit{Persuasion} (PE) are very close in PC1 for the Euclidean. We now analyse the Euclidean PCs in more detail.

Figure 7 contains plots representing the importance and sign of each word in the first and second Euclidean PC. From Figure 6 a more positive PC 1 score is indicative
Fig 6: Plot of PC 1 and PC 2 scores for the Austen and Dickens novels, coloured in time order (red to violet) with extrinsic regression lines for Dickens novels (blue) and Austen novels (red) using the a) Euclidean and b) square root Euclidean metric.

Fig 7: The importance of each word given by (16) in (left) PC 1 and (right) PC 2. The red bar represents the importance of each word in the difference of means, \( D = \hat{\mu}^{\text{Austen}}_E - \hat{\mu}^{\text{Dickens}}_E \), given by \( (D)_{ii} / (\sum_{j=1}^{1000} (D)_{jj}) \), \( 1 \leq i \leq 1000 \).
of an Austen novel whilst a more negative one a Dickens novel. For a positive PC1 score the nodes *her* and *she* have importance whilst for a negative score words such as *his*, and *he* have more importance, which is expected as Austen writes with more female characters. The second PC actually is similar to a fitted regression line which we describe in the next section. An interesting point to note is that the Austen novels over time have the second PC increasing, as *Lady Susan* (LS) and *Persuasion* (PE) are her earliest and latest novels respectively. This is the opposite to Dickens where PC2 decreases with time. *Pickwick papers* (PP) is Dickens earliest and *The Mystery of Edwin Drood* (ED) his latest. The second PC has feminine words like *her* and *she* as the most positive words, but more first and second person words, such as *I*, *my* and *you* as negative words. This is consistent with Austen increasingly using a stylistic device called free indirect speech in her later novels novels (Shaw, 1990). Free indirect speech has the property the third person pronouns, such as *she* and *her* are used instead of first person pronouns, such as *I* and *my*.

### 4.2. Regression

Here we assume the data are the pairs $\{L_k, t_k\}$, for $1 \leq k \leq n$ in which the $L_k \in \mathcal{L}_m$ are graph Laplacians to be regressed on covariate vectors $t_k = (t_{1k}, \ldots, t_{uk})$, and consider the regression error model

$$\vech^* (\exp^{-1}_\nu(F_\alpha(L_k))) = \vech^* (D_0 + \sum_{w=1}^{u} t_{wk}^* D_w) + \epsilon, \quad \epsilon \sim \mathcal{N}_{m(m-1)/2}(0, \Omega),$$

where $\vech^*$ is the $\vech$ operator but with $\sqrt{2}$ multiplying the terms corresponding to the off-diagonal. In general $\Omega$ has a large number of elements, so in practice it is necessary to restrict $\Omega$ to be diagonal or even isotropic, $\Omega = \omega^2 I_{m(m-1)/2}$.

When using the power Euclidean metric we take $\nu = 0$ and the parameters $\{\hat{D}_0, \ldots, \hat{D}_u\}$ in (4.2) are the least squares solution

$$(\hat{D}_0, \ldots, \hat{D}_u) = \arg \min_{D_0, \ldots, D_u} \sum_{k=1}^{n} \| \exp^{-1}_\nu(F_\alpha(L_k)) - (D_0 + \sum_{w=1}^{u} t_{wk}^* D_w) \|^2, \quad (17)$$

and the fitted values are

$$f(t_k) = \hat{L}_k = P_L \left( F_{\alpha}^{-1} \left( \exp_\nu \left( \hat{D}_0 + \sum_{w=1}^{u} t_{wk}^* \hat{D}_w \right) \right) \right) \in \mathcal{L}_m, \quad (18)$$

and so $\hat{L}_k$ predicts a graph Laplacian with covariates $t_k$. A similar model can be used for the Procrustes power metric but with $\nu = F_\alpha(\hat{\eta})$. The optimisation in (18) is convex and the parameters of the regression line are found using the standard least squares approach in the tangent space. This optimisation reduces element-wise for $1 \leq i, j \leq m$, to $(m(m-1))$ independent optimisations.
A test for the significance of covariate $t^w$ involves the hypotheses $H_0 : D_w = 0$ and $H_1 : D_w \neq 0$. By Wilks’ Theorem (Wilks, 1962), if $H_0$ is true then the likelihood ratio test statistic is

$$T^\ell = -2 \log \Delta = -2 \left( \sup_{D \cdot D_w = 0} \ell(D) - \sup_{D \cdot D_w \neq 0} \ell(D) \right) \sim \chi^2_{m(m-1)},$$

approximately when $n$ is large, where $D = \{D_0, \ldots, D_u, \Omega\}$ and $\ell$ is the log-likelihood function of $\phi(\exp^{-1}(F_\alpha(L_k)))$ under the distribution from (4.2). We assume $\Omega$ is a diagonal matrix. Using equation (19) $H_0$ is rejected in favour of $H_1$ at the 100$\alpha$% significance level if $T^\ell$ is greater than the $(1 - \alpha)$ quantile of $\chi^2_{m(m-1)}$.

For the Austen and Dickens data, each novel, represented by a graph Laplacian $L_k$ is paired with the year, $t_k$, the novel was written. We regress the $\{L_k\}$ on the $\{t_k\}$ using the method above with $u = 1$ for each author. To visualise the regression lines in Figure 6 we find $f(t_k)$ for many values of $t_k$ for the specific metrics, and project these to the PC1 and PC2 space. For each metric the regression lines seem to fit the data well, and could be used to see how writing styles have changed over time. When the test for regression was performed on the novels the p-values were extremely small ($< 10^{-16}$) for both the Austen and Dickens regression lines, for both the Euclidean and square root Euclidean metrics. Hence there is very strong evidence to believe that the writing style of both authors changes with time, regardless of which metric we choose.

### 4.3. A central limit theorem

Consider independent random samples $A_k$ where $F_\alpha(A_k)$ have a distribution with mean $\mathbb{E}(F_\alpha(A))$. As the extrinsic mean is based on the arithmetic mean for the power Euclidean metrics, a central limit holds for the sample mean graph Laplacian, under the condition $\text{var}(F_\alpha(A))_{ij}$ is finite.

**Result 3.** For any power Euclidean metric

$$\sqrt{n} \left( \psi(F_\alpha(\hat{\eta})) - \psi(F_\alpha(\eta)) \right) \xrightarrow{D} \mathcal{N}_{m(m-1)}(0, \Sigma),$$

as $n \to \infty$, where $\psi(X) = \text{vech}^*(HXH^T)$ and recall $\text{vech}^*$ is the vech operator but with $\sqrt{2}$ multiplying the terms corresponding to the off-diagonal, and $\Sigma$ is a finite variance matrix.

When $\alpha = 1$ this result is similar to that in Ginestet et al. (2017) although they work directly in $L_m$ whereas we work in the embedding space.

### 4.4. Hypothesis tests

Consider two populations $A$ and $B$ of $m \times m$ graph Laplacians with corresponding population means $\mu_A$ and $\mu_B$ defined in (10). Given two samples $\{A_1, A_2, \ldots, A_{n_A}\}$
and \( \{B_1, B_2, \ldots, B_{n_B}\} \) respectively from \( A \) and \( B \), the goal is to test the hypotheses

\[
H_0 : \mu_A = \mu_B \quad \text{and} \quad H_1 : \mu_A \neq \mu_B.
\]

We define the test statistic as

\[
T = d(\hat{A}, \hat{B})^2,
\]

where \( \hat{A} \) and \( \hat{B} \) are defined by \( \hat{\eta} \) in (11) for the sets \( A \) and \( B \) respectively and using a suitable metric. Any Euclidean or Procrustes power metric is suitable to use, we however will just consider the Euclidean

\[
T_E = d_1(\hat{A}_E, \hat{B}_E)^2 ;
\]

the square root Euclidean

\[
T_H = d_1^2(\hat{A}_H, \hat{B}_H)^2 ;
\]

and the Procrustes size-and-shape

\[
T_S = d_1^2(\hat{A}_S, \hat{B}_S)^2 ,
\]

where the subscripts \( \{E, H, S\} \) refer to whether the Euclidean, square root or Procrustes size-and-shape means have been used, respectively.

Using Result 3 the distribution of the test statistics for large \( n \) is given as follows.

**Result 4.** Consider independent random samples of networks of size \( n_A \) and \( n_B \). For the power Euclidean metrics under the null hypothesis, \( H_0: \mu_A = \mu_B \), as \( n_A, n_B \rightarrow \infty \), such that \( n_A/n_B \rightarrow r \in (0, \infty) \):

\[
\frac{n_A n_B}{n_A + n_B} T = \frac{n_A n_B}{n_A + n_B} \frac{d(\hat{A}, \hat{B})^2}{D^2} \sum_{i=1}^{m(m-1)/2} \delta_i \chi_1^2 ,
\]

(20)

in which each \( \chi_1^2 \) is independent and \( \delta_i \) are the \( m(m-1)/2 \) non-zero eigenvalues of \( \Sigma \).

For the Procrustes power metric similar central limit theorem results follow providing the more stringent conditions of Bhattacharya and Patrangenaru (2005) hold. In practice \( \Sigma \) needs to be estimated, which can be very high dimensional. In our application with \( m = 1000 \) this is a symmetric matrix with \( M(M + 1)/2 \) parameters where \( M = m(m-1)/2 = 499500. \) One approach is to use the shrinkage estimator from Schäfer and Strimmer (2005), as employed by Ginestet et al. (2017), but this is impractical for our application with \( m = 1000 \). If we assume a diagonal matrix \( \Sigma = \Lambda^* \) then the \( \delta_i \) correspond to the variances of individual components of the difference in means, and these can be estimated consistently from method of moments estimators. A further very simple model would be to have an isotropic covariance matrix with covariance matrix \( \Sigma = \sigma^2 I_{m(m-1)/2} \), which requires estimation of a single variance parameter \( \sigma^2 \). Note that the likelihood ratio test for regression with test statistic \( -2 \log \Delta \) in Section 4.2 gives an alternative test for equality of means when the covariates are group labels, but the additional assumption of normality for the observations needs to be made in that case.

An alternative non-parametric test, which does not depend on large sample asymptotics is a random permutation test, similar to Preston and Wood (2010) as follows.
deviation, and in particular the population pooled variance is estimated by

We estimate the variance in our sample with a weighted average of the sample variances

The histograms of the off-diagonal individual graph Laplacians are

For the Austen and Dickens data have test statistics

A limitation of using the permutation test is it assumes exchangeability of the observations under the null hypothesis (Amaral, Dryden and Wood, 2007). This means under the null hypothesis the populations \( A \) and \( B \) are assumed identical. A test based on the bootstrap is an alternative possibility, which requires weaker assumptions about \( A \) and \( B \); see for example Amaral, Dryden and Wood (2007).

For the Austen and Dickens data have test statistics \( T_E = 0.0011, T_H = 0.2759, T_S = 0.0091 \). We compute the p-value from the permutation test with \( r = 199 \) permutations for each of \( T_E, T_H, T_S \) and in each case all permuted values were less than the observed statistics for the data. Hence, in each case the estimated p-value is zero, indicating very strong evidence for a difference in mean graph Laplacian.

4.5. Exploring differences between authors

Given that the Austen and Dickens novels are significantly different in mean we would like to explore how they differ. In particular we examine the off-diagonal elements of \( P_C(\hat{\eta}_{\text{Dickens}}) - P_C(\hat{\eta}_{\text{Austen}}) \), i.e. the differences in the mean weighted adjacency matrix, and compare them to appropriate measures of standard error of the differences using a \( z \)-statistic. The histograms of the off-diagonal individual graph Laplacians are heavy tailed, and a plot of sample standard deviations versus sample means show an overall average linear increase with approximate slope \( \beta = 0.2 \), but with a large spread. We shall use this relationship in a regularised estimate of our choice of standard error.

For a particular co-occurrence pair of words we have weighted adjacency values \( x_i, \beta = 1, \ldots, n_A \) and \( y_j, j = 1, \ldots, n_B \) with sample means \( \bar{x} \) and \( \bar{y} \), and sample standard deviations \( s_x \) and \( s_y \). For our analysis here we use the Euclidean mean graph Laplacians. We estimate the variance in our sample with a weighted average of the sample variance and an estimate based on the linear relationship between the mean and standard deviation, and in particular the population pooled variance is estimated by

\[
s_p^2 = \frac{n_A(w_A s_x^2 + (1 - w_A)\beta^2 \bar{x}^2) + n_B(w_B s_y^2 + (1 - w_B)\beta^2 \bar{y}^2)}{(n_A + n_B - 2)},
\]

where the weights are taken as \( w_A = n_A/N, w_B = n_B/N \), where we take \( N = 200 \). Note that if all values in one of the samples are 0 (due to no word co-occurrence

**Algorithm 1** Random permutation test to test the equality of means for two sets of graph Laplacians, \( A \) and \( B \), using the test statistic \( T \).

1. Calculate the test statistics between \( A \) and \( B \), given by \( T = T(A, B) \).
2. Generate random sets \( A^* \) and \( B^* \) of size \( |A| \) and \( |B| \) respectively, by randomly sampling without replacement from \( A \cup B \).
3. Compute the test statistic of sets \( A^* \) and \( B^* \), given by \( T^* = T(A^*, B^*) \).
4. Repeat steps 2 and 3 \( r \) times, to give test statistics \( T_{1}^*, T_{2}^*, \ldots, T_{r}^* \).
5. Order the test statistics \( T^*(1) \leq T^*(2) \leq \ldots \leq T^*(r) \).
6. Calculate the p-value, which is \( 1 - \frac{1}{r} \) for the minimum \( 1 \leq j \leq r \) satisfying \( T^*(j) < T \leq T^*(j + 1) \), unless \( T \leq T^*(1) \), in which case the p-value is 1 or if \( T > T^*(r) \), in which case the p-value is 0.
pairings in any of that author’s books) then we drop that word pairing from further analysis, as we are only interested in the relative usage of the word occurrences that are actually used by both authors. A univariate $z$-statistic for comparing adjacencies is then

$$z = \frac{\bar{x} - \bar{y}}{(q + s_p)\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}},$$

where we include the regularizing offset $q > 0$ to avoid highlighting very small differences in mean adjacency with very small standard errors. The value for $q$ is chosen as the median of all $s_p$ values under consideration.

The exploratory graphical displays in Figure 8 illuminate striking differences between the novelists. For Austen there are very common pairings of words with her, she, herself, which form important hubs in this network. Austen also pairs these hubs with more emotional words feelings, felt, feel, kindness, happiness, affection, pleasure and stronger words power, attention, must, certainly, advantage and opinion. Also we see more use of letter in Austen, which is a literary device often used by the author. For Dickens there are more common uses of abbreviations, especially don’t which is an important hub, and also it’s, i’ll and that’s. In contrast the Austen network highlights not. Dickens also more prominently pairs body parts arm, arms, eyes, feet, hair, hand, hands, head, mouth, face, shoulder, legs in combination with the strong hubs his and the. These hubs are also paired with other objects, such as door, chair, glass. Finally, Dickens has the more prominent use of pairs with a sombre word, such as dark, black and dead, which might have been expected.

5. Conclusion

We have developed a general framework for extrinsic statistical analysis of graph Laplacians and considered in particular the distances $d_1$, $d_2$ and $d_\infty$. Other metrics fit in our framework and could be considered. One example is the log metric used in Bakker, Halappanavar and Sathanur (2018) which uses the embedding $F_\log(L) = \sum_{i=1}^{l} \log(\xi_i)u_iu_i^T$ and it easy to see $F_\log(L) = \lim_{\alpha \to 0} \frac{1}{\alpha}(F_\alpha(L) - F_0(L))$ where we define $0^\lambda = 0$ in $F_0$ and $l$ is the rank of $L$. The metric is then $d_{\log}(L_1, L_2) = ||F_\log(L_1) - F_\log(L_2)||$. The log embedding is the limit of the the Box-Cox transform, $F_\alpha(L) = \frac{\alpha L - I}{\alpha}$, when $\alpha \to 0$, the reverse is given as $F_{\alpha}^{-1}(Q) = (\alpha Q + I)^{1/\alpha}$.

Another metric to consider is the element-wise metric of the form $d_{\rho}(L_1, L_2) = (\sum_i \sum_j |(L_1)_{ij} - (L_2)_{ij}|^\rho)^{1/\rho}$. Of particular interest would be comparing $\rho = 2$, which is the Frobenius/Euclidean norm $d_2$, with $\rho = 1$ which can be similar to the square root norm (and is identical for diagonal matrices).

Our methodology gives appropriate results for comparing co-occurrence networks for Jane Austen and Charles Dickens novels, but the methodology is widely applicable, for example to neuroimaging networks and social networks, and such applications will be explored in further work.
Fig 8: Networks displaying the top 100 pairs of words ranked according to the $z$-statistic in (21), with more prominent co-occurrences used by Austen (left, in blue) and the more prominent co-occurrences used by Dickens for (right, in yellow).
References

AMARAL, G. J. A., DRYDEN, I. L. and WOOD, A. T. A. (2007). Pivotal Bootstrap Methods for k-Sample Problems in Directional Statistics and Shape Analysis. Journal of the American Statistical Association 102 695-707.

ANDERSON, E. (2018). rosqp: Quadratic Programming Solver using the OSQP Library R package version 0.1.0.

BAKKER, C., HALAPPAHANAVAR, M. and SATHANUR, A. V. (2018). Dynamic graphs, community detection, and Riemannian geometry. Applied Network Science 3 3.

BHATTACHARYA, R. and PATRANGENARU, V. (2003). Large sample theory of intrinsic and extrinsic sample means on manifolds. Ann. Statist. 31 1–29.

BHATTACHARYA, R. and PATRANGENARU, V. (2005). Large sample theory of intrinsic and extrinsic sample means on manifoldsII. Ann. Statist. 33 1225–1259.

COOTES, T. F., TAYLOR, C. J., COOPER, D. H. and GRAHAM, J. (1994). Image search using flexible shape models generated from sets of examples. In Statistics and Images: Vol. 2 (K. V. Mardia, ed.) 111-139. Carfax, Oxford.

DE KLERK, E. (2006). Aspects of semidefinite programming: interior point algorithms and selected applications 65. Springer Science & Business Media.

DRYDEN, I. L. (2018). shapes package R Foundation for Statistical Computing, Vienna, Austria Contributed package, Version 1.2.4.

DRYDEN, I. L., KOLOYDENKO, A. and ZHOU, D. (2009). Non-Euclidean Statistics for Covariance Matrices, with Applications to Diffusion Tensor Imaging. The Annals of Applied Statistics 3 1102-1123.

DRYDEN, I. L. and MARDIA, K. V. (2016). Statistical shape analysis with applications in R, second ed. Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester. MR3559734

EVERT, S. (2008). Corpora and collocations. Corpus linguistics. An international handbook 2 1212–1248.

FLETCHER, P. T., LU, C., PIZER, S. M. and JOSHI, S. (2004). Principal geodesic analysis for the study of nonlinear statistics of shape. IEEE transactions on medical imaging 23 995–1005.

FRÉCHET, M. (1948). Les éléments aléatoires de nature quelconque dans un espace distancié. Annales de l’institut Henri Poincaré 10 215-310.

FU, A., NARASIMHAN, B., DIAMOND, S. and MILLER, J. (2018). CVXR: Disciplined Convex Optimization R package version 0.99.

GINESTET, C. E., LI, J., BALACHANDRAN, P., ROSENBERG, S., KOLACZYK, E. D. et al. (2017). Hypothesis testing for network data in functional neuroimaging. The Annals of Applied Statistics 11 725–750.

HIGHAM, N. J. (1988). Computing a nearest symmetric positive semidefinite matrix. Linear algebra and its applications 103 103–118.

CHARLES DICKENS INFO (2018). Charles Dickens Timeline. https://www.charlesdickensinfo.com/life/timeline/, Last accessed on 2018-11-12.

JOYCE, D. (2009). On manifolds with corners. arXiv preprint arXiv:0910.3518.

KENDALL, D. G., BARDEN, D., CARNE, T. K. and LE, H. (1999). Shape and Shape Theory. Wiley, Chichester.
KENT, J. T. (1994). The complex Bingham distribution and shape analysis. *Journal of the Royal Statistical Society. Series B (Methodological)* 285–299.

KOLACZYK, E. D. (2009). *Statistical analysis of network data: methods and models.* Springer Science & Business Media.

LE, H. (1995). Mean Size-and-Shapes and Mean Shapes: A Geometric Point of View. *Advances in Applied Probability* 27 44–55.

MAHLBERG, M., STOCKWELL, P., de JOODE, J., SMITH, C. and O’DONNELL, M. B. (2016). CLiC Dickens: novel uses of concordances for the integration of corpus stylistics and cognitive poetics. *Corpora* 11 433-463.

THE JANE AUSTEN SOCIETY OF NORTH AMERICA (2018). Jane Austen’s Works. [http://jasna.org/austen/works/](http://jasna.org/austen/works/). Last accessed on 2018-11-12.

PHILLIPS, M. K. (1983). *Lexical Macrostructure in Science Text.* University of Birmingham.

PRESTON, S. P. and WOOD, A. T. A. (2010). Two-Sample Bootstrap Hypothesis Tests for Three-Dimensional Labelled Landmark Data. *Scandinavian Journal of Statistics* 37 568–587.

ROCKAFELLAR, R. T. (1993). Lagrange multipliers and optimality. *SIAM review* 35 183–238.

SCHÄFER, J. and STRIMMER, K. (2005). A shrinkage approach to large-scale covariance matrix estimation and implications for functional genomics. *Statistical applications in genetics and molecular biology* 4 Article32.

SHANNON, P., MARKIEL, A., OZIER, O., BALIGA, N. S., WANG, J. T., Ramage, D., Amin, N., SCHWIKOWSKI, B. and IDEKER, T. (2003). Cytoscape: a software environment for integrated models of biomolecular interaction networks. *Genome Research* 13 2498–2504.

SHAW, N. (1990). Free Indirect Speech and Jane Austen’s 1816 Revision of Northanger Abbey. *Studies in English Literature, 1500-1900* 30 591–601.

R CORE TEAM (2018). R: A Language and Environment for Statistical Computing R Foundation for Statistical Computing, Vienna, Austria.

WARD, J. H. JR. (1963). Hierarchical grouping to optimize an objective function. *J. Amer. Statist. Assoc.* 58 236–244. MR0148188 (26 #5696)

WILKS, S. S. (1962). *Mathematical Statistics.* Wiley, New York.
Appendix A: Most common words

| Word | Rank in all novels | Rank in Dickens novels | Rank in Austen novels |
|------|------------------|-----------------------|----------------------|
| the  | 1                | 1                     | 1                    |
| and  | 2                | 2                     | 3                    |
| to   | 3                | 3                     | 2                    |
| of   | 4                | 4                     | 4                    |
| a    | 5                | 5                     | 5                    |
| i    | 6                | 6                     | 7                    |
| in   | 7                | 7                     | 8                    |
| that | 8                | 8                     | 15                   |
| it   | 9                | 11                    | 10                   |
| he   | 10               | 10                    | 16                   |
| his  | 11               | 9                     | 20                   |
| was  | 12               | 13                    | 9                    |
| you  | 13               | 12                    | 15                   |
| with | 14               | 14                    | 21                   |
| her  | 15               | 16                    | 6                    |
| as   | 16               | 15                    | 18                   |
| had  | 17               | 17                    | 17                   |
| for  | 18               | 20                    | 19                   |
| at   | 19               | 21                    | 25                   |
| me   | 20               | 18                    | 36                   |
| not  | 21               | 26                    | 12                   |
| be   | 22               | 28                    | 14                   |
| she  | 23               | 31                    | 11                   |
| said | 24               | 19                    | 58                   |
| have | 25               | 25                    | 23                   |

Table 2: The most common 25 words in the Austen and Dickens novels

Appendix B: Proof for result 2

For an estimator $\hat{\mu}$ to be consistent for a population mean $\mu$, it must converge in probability to $\mu$. Let $\{\hat{\mu}_n\}$ be a sequence of estimates from a sample set $\{L_1, \ldots, L_n\}$, for this to converge in probability to $\mu$ then for any $\epsilon > 0$ and any $\delta > 0$ there exists a number $N$ such that for all $n \geq N$ $P_n < \delta$, where $P_n = P(|\hat{\mu}_n - \mu| > \epsilon)$.

We can see $\exp^{-1}(F_{\alpha}(\hat{\eta}))$ is a consistent estimator as it converges in probability to $\exp^{-1}(F_{\alpha}(\eta))$ from the law of large numbers, and so by the continuous mapping theorem $\hat{\eta}$ converges in probability to $\eta$, as long as $\eta$ exists and is unique.
We now need to show the convergence in probability holds when we project \( \hat{\eta} \) and \( \eta \) to \( L_m \). As \( M_m \subset L_m \) then the projection will always be on the boundary of \( L_m \) denoted \( B(L_m) \). To have convergence in probability of \( \hat{\eta} \), for any \( \epsilon > 0 \) and \( \delta > 0 \) there must exist an \( N_1 \) such that for \( n \geq N_1 \) then \( P(|\hat{\mu} - \mu| > \epsilon) < \delta \). We know from the convergence in probability of \( \hat{\eta} \), that for any \( \epsilon > 0 \) and \( \delta > 0 \) there exists an \( N_2 \) such that for \( n \geq N_2 \), \( P(|\hat{\eta} - \eta| > \epsilon) < \delta \). We choose an \( \epsilon \) small enough so that the boundary of the graph Laplacian space can be thought to have 0 curvature. From Ginestet et al. (2017) we know \( L_m \) is a manifold with corners, and stated briefly a \( d \)-dimensional manifold with corners can be locally modelled by \( [0, \infty)^k \times \mathbb{R}^{d-k} \), for full details see Joyce (2009). Let \( |\hat{\eta} - \eta| = \epsilon \) and \( |\hat{\mu} - \mu| = \zeta \). This leads to two cases:

- **Case 1:** \( \mu \) is not on a corner of \( B(L_m) \). In this case the estimator behaves as in Figure 9a. The estimator \( \hat{\eta} \) is orthogonally projected to \( \hat{\mu} \), hence due to Pythagoras’ theorem it is clear \( \zeta \leq \epsilon \).

- **Case 2:** \( \mu \) is on a corner of \( B(L_m) \). In this case the estimator behaves as in Figure 9b. Clearly \( \frac{\pi}{2} \leq \vartheta \leq \pi \). We consider a point \( q \) along the line between \( \hat{\eta} \) and \( \eta \) such that the angle between \( \hat{\mu}, \mu \) and \( q \) is \( \frac{\pi}{2} \). Note \( \zeta \leq |\hat{\eta} - q| \) following identical arguments as in case 1, and clearly \( |\hat{\eta} - q| \leq \epsilon \). Hence \( \zeta \leq \epsilon \).

We do not consider \( \hat{\mu} \) on a corner when \( \mu \) is not as for small enough \( \epsilon \) this will not occur. We now have for \( n \geq N_2 \) that \( \zeta \leq \epsilon \), hence

\[
\delta > P(|\hat{\eta} - \eta| > \epsilon) = P(|\hat{\mu} - \mu| > \zeta) \geq P(|\hat{\mu} - \mu| > \epsilon).
\]

Therefore when \( n \geq \max(N_1, N_2) \) then \( P(|\hat{\mu} - \mu| > \epsilon) < \delta \) and so \( \{\hat{\mu}\} \) converges in probability to \( \mu \) as \( n \to \infty \), i.e. \( \hat{\mu} \) is a consistent estimator.