STATIC POTENTIALS ON ASYMPTOTICALLY FLAT MANIFOLDS

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Abstract. We discuss the question whether a static potential can have nonempty zero set near the infinity of an asymptotically flat 3-manifold. We prove that this does not occur if the metric is asymptotically Schwarzschild with nonzero mass. If the asymptotic assumption is relaxed to the usual assumption under which the total mass is defined, we prove that the static potential is unique up to scaling unless the manifold is flat. We also consider the rigidity question of complete, connected, asymptotically flat 3-manifolds that have a static potential. If the static potential is bounded, we prove that such a manifold must be isometric to a Schwarzschild manifold with positive mass or the Euclidean 3-space; if the static potential is unbounded, we prove that the manifold is isometric to the Euclidean 3-space if the topology of the underlying manifold is trivial or if the static potential has no critical points.

1. INTRODUCTION

In general relativity, a static vacuum spacetime is a 4-dimensional Lorentz manifold that is isometric to \((\mathbb{R}^1 \times M, -N^2dt^2 + g)\) where \((M, g)\) is a 3-dimensional Riemannian manifold, \(N\) is a positive function on \(M\), and the pair \((g, N)\) satisfies

\[
\nabla^2 N = NRic \quad \text{and} \quad \Delta N = 0.
\]

Here \(\nabla^2\), \(\Delta\) and \(\text{Ric}\) denote the Hessian, the Laplacian and the Ricci curvature of \(g\) respectively.

In [12], Corvino studied localized scalar curvature deformation of a Riemannian metric and introduced the following definition:

Definition 1 (Corvino [12]). A Riemannian metric \(g\) is called static on a manifold \(M\) if the linearized scalar curvature map at \(g\) has a

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nontrivial cokernel, i.e. if there exists a nontrivial function \( f \) on \( M \) such that
\[
-(\Delta f)g + \nabla^2 f - f\text{Ric} = 0. 
\]
We call a nontrivial solution \( f \) to (1.2) a static potential on \((M, g)\). In particular, a static potential \( f \) is not assumed to be always positive in this paper.

Let \( g \) be a metric that is static according to Definition 1, it is known that \( g \) necessarily has constant scalar curvature ([12, Proposition 2.3]). When this constant is zero, (1.2) is equivalent to
\[
\nabla^2 f = f\text{Ric} \quad \text{and} \quad \Delta f = 0. 
\]
Thus, the connection between a static Riemannian metric and a vacuum static spacetime is: if \( g \) and \( f \) satisfy (1.2) on a manifold \( M \) and the scalar curvature of \( g \) is zero, then \((\mathbb{R}^1 \times \tilde{M}, -f^2 dt^2 + g)\) is a static vacuum spacetime, where \( \tilde{M} = M \setminus f^{-1}(0) \).

In the literature, there are many important results concerning manifolds with a positive static potential (cf. [11, 1, 10] for example). The difference between those results and the discussion in this paper is that we allow static potentials to change sign.

The main motivation to our such consideration comes from Bartnik’s conjecture on minimal mass extensions ([5]). In [5], Bartnik defined the quasi-local mass of an extended body \( \Omega \) in a time-symmetric initial data set to be the infimum of the total mass of suitable asymptotically flat extensions of \( \Omega \). It was conjectured that this infimum is achieved and the corresponding minimal mass extension can be realized as a time-symmetric slice in an asymptotically flat, static vacuum spacetime. In [12, Theorem 8], as an application of his localized scalar curvature deformation theorem, Corvino proved that if a mass minimizer \((M, g)\) exists in a certain class of extensions, then \( g \) must be static in the sense of Definition 1. To go from knowing \( g \) is static on \( M \) to a conclusion that \((M, g)\) can arise as a time-symmetric slice in an asymptotically flat, static vacuum spacetime, one needs to know whether the static potential involved is free of zeros, at least in the asymptotic regime.

We recall the definition of an asymptotically flat 3-manifold.

**Definition 2.** A Riemannian 3-manifold \((M, g)\) is called asymptotically flat if for some compact set \( K \), \( M \setminus K \) consists of a finite number of components \( E_1, \ldots, E_k \), referred as the end of \((M, g)\), such that each \( E_i \) is diffeomorphic to \( \mathbb{R}^3 \) minus a ball and, under such diffeomorphisms, the metric \( g \) on \( E_i \) satisfies
\[
g_{ij} = \delta_{ij} + b_{ij} \quad \text{with} \quad b_{ij} = O_2(|x|^{-\tau})
\]
for some constant \( \tau > \frac{1}{2} \). Here \( x = (x_1, x_2, x_3) \) denotes the standard coordinate on \( \mathbb{R}^3 \) and a function \( \phi \) satisfies \( \phi = O(|x|^{-\tau}) \) provided \( |\partial^i \phi| \leq C|x|^{-\tau-i} \) for \( 0 \leq i \leq l \) and some constant \( C \).

Suppose \((M, g)\) is asymptotically flat and \( E \) is an end. If \( f \) is a static potential on \( E \), one can prove that \( f \) satisfies the following properties (see Proposition 3.1 and 3.2 in Section 3 and Proposition 2.1 in [8]): if \( f \) is unbounded, then it is a linear function plus a sublinear perturbation near infinity; in this case, the zero set of \( f \) penetrates the infinity of \( E \) and, outside a compact set, is given by the graph of a function with at most sublinear growth; on the other hand, if \( f \) is bounded, then \( f \) must not vanish outside a compact set and it tends to a nonzero constant at infinity. As a result, when \( f \) is bounded, it is always true that

\[
(1.5) \quad f = 1 - \frac{m}{|x|} + O(|x|^{-2}), \text{ as } x \to \infty
\]

for some constant \( m \) upon scaling.

It is well known (cf. [7, 11]) that (1.3) and (1.5) then imply that \((M, g)\) is asymptotically Schwarzchild in the sense that there exists another coordinate chart \( \{x_1, x_2, x_3\} \) near infinity in which the metric \( g \) satisfies

\[
(1.6) \quad g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + p_{ij}
\]

where \( |p_{ij}| = O_2(|x|^{-2}) \). From this one also knows that \( m \) in (1.5) is the ADM mass [2] of \((M, g)\) at the end \( E \).

We prove that the existence of a coordinate chart satisfying (1.6) indeed is a sufficient condition for the positivity of \( f \) near infinity.

**Theorem 1.1.** Let \((M, g)\) be an asymptotically flat 3-manifold with or without boundary. Let \( E \) be an end. Suppose \( g \) satisfies (1.6) in some coordinate chart on \( E \) and the constant \( m \), which is the mass at the end \( E \), is not zero. Then any static potential \( f \) on \( E \) must be bounded, and consequently either positive or negative outside a compact set.

We prove Theorem 1.1 using Proposition 3.2, which describes the asymptotic behavior of the zero set of \( f \), and an observation that the Ricci curvature of \( g \), when restricted to the zero set of \( f \), is a multiple of the induced metric, see Lemma 2.1 (iii).

In relation to the question of its positivity, we also ask “how many” static potentials may exist on an asymptotically flat manifold. We have

**Theorem 1.2.** Let \((M, g)\) be a connected, asymptotically flat 3-manifold with or without boundary. Let \( \mathcal{F} \) be the space of all solutions to (1.3). Then \( \dim(\mathcal{F}) \leq 1 \) unless \((M, g)\) is flat.
We note that important local properties for static metrics have been obtained by many authors (cf. [18, 6, 13] and references therein). In our proof of Theorem 1.2, we first prove a local result which asserts that, on any Riemannian 3-manifold \((M, g)\) of zero scalar curvature, if \(f_1\) and \(f_2\) are two linearly independent static potentials such that \(f_1^{-1}(0) \cap f_2^{-1}(0)\) is nonempty, then \((M, g)\) must be flat, see Theorem 2.1. This result, combined with our analysis of the asymptotical behavior of static potentials on an asymptotically flat end, yields the proof of Theorem 1.2.

Another question that we are interested in is the rigidity problem of complete, asymptotically flat manifolds which admit a static potential. In [11], Bunting and Masood-ul-Alam proved that an asymptotically flat manifold \((M, g)\) with nonempty boundary, with one end, which has a static potential \(f\) such that \(f \to 1\) at \(\infty\) and \(f = 0\) on \(\partial M\), must be isometric to a spatial Schwarzschild manifold outside its horizon. Using the method in [11] and Proposition 3.1 (ii), we prove that such manifolds are rigid if they have a bounded static potential.

**Theorem 1.3.** Let \((M, g)\) be a complete, connected, asymptotically flat 3-manifold without boundary. If there is a bounded static potential on \((M, g)\), then \((M, g)\) is isometric to either the Euclidean space \((\mathbb{R}^3, g_0)\) or a spatial Schwarzschild manifold \((\mathbb{R}^3 \setminus \{0\}, (1 + \frac{m}{|x|})^4 g_0)\) with \(m > 0\).

A direct corollary of Theorem 1.1, Theorem 1.3 and the positive mass theorem [16] is that

**Corollary 1.1.** Let \((M, g)\) be a complete, connected, asymptotically flat 3-manifold without boundary. Suppose \((M, g)\) is asymptotically Schwarzschild at each end. If there is a static potential on \((M, g)\), then \((M, g)\) is isometric to either the Euclidean space \((\mathbb{R}^3, g_0)\) or a spatial Schwarzschild manifold \((\mathbb{R}^3 \setminus \{0\}, (1 + \frac{m}{|x|})^4 g_0)\) with \(m > 0\).

If the static potential is unbounded, we prove the rigidity of \((M, g)\) for those \(M\) having simple topology.

**Theorem 1.4.** Let \((M, g)\) be a complete, connected, asymptotically flat 3-manifold without boundary on which there exists an unbounded static potential. If \(M\) is orientable and every 2-sphere in \(M\) is the boundary of a bounded domain, then \((M, g)\) is isometric to \((\mathbb{R}^3, g_0)\). In particular, if \(M\) is homeomorphic to \(\mathbb{R}^3\), then \((M, g)\) is isometric to \((\mathbb{R}^3, g_0)\).

Without the topological assumption, by analyzing the behaviors of integral curves of \(\nabla f\), we prove
Theorem 1.5. Let $(M, g)$ be a complete, connected, asymptotically flat 3-manifold without boundary. If $(M, g)$ admits an unbounded static potential that has no critical points, then $(M, g)$ is isometric to $(\mathbb{R}^3, g_0)$.

In the proof of Theorem 1.5, we also made use of the positive mass theorem (see Proposition 1.3 (b)). We conjecture that $(\mathbb{R}^3, g_0)$ is the unique, complete, connected, asymptotically flat manifold that admits an unbounded static potential.

The organization of the paper is as follows. In Section 2, we discuss local properties of static metrics which will be used later. In Section 3, we analyze static potentials on an asymptotically flat end and prove Theorems 1.1 and 1.2. In Section 4, we consider rigidity questions for complete asymptotically flat 3-manifolds which admits a static potential and prove Theorems 1.3, 1.4 and 1.5.

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2. Local properties of static metrics

In this section, we assume that $(M, g)$ is a 3-dimensional, connected, smooth Riemannian manifold whose scalar curvature $R$ is zero. Recall that a nontrivial function $f$ is called a static potential in this case if

$\nabla^2 f = f \text{Ric}$.  

In [18], Tod studied the question when a spatial metric could give rise to a static spacetime in more than one way. In our work, we often need to apply Proposition 2 (ii), Corollary 3 (i) and equation (15) in [18]. For convenience, we list these results of Tod in the next Proposition. We also sketch the proof.

Proposition 2.1 (Tod [18]). Let $\{e_1, e_2, e_3\}$ be an orthonormal frame that diagonalizes the Ricci curvature at a given point $p$.

(i) Suppose $f$ is a static potential. Then

$f(R_{33;1} - R_{31;3}) = (R_{22} - R_{33})f_{;1}$

$f(R_{11;2} - R_{12;1}) = (R_{33} - R_{11})f_{;2}$

$f(R_{22;3} - R_{23;2}) = (R_{11} - R_{22})f_{;3}$.  

(ii) Suppose $\{R_{11}, R_{22}, R_{33}\}$ are distinct and suppose $N, V$ are two positive static potentials. Then $V = cN$ for some constant $c$.

(iii) Suppose $R_{11} = R_{22} \neq R_{33}$ and suppose $N$ is a positive static potential. If $f$ is another static potential, then $Z = N^{-1}f$ satisfies $Z_{;1} = Z_{;2} = 0$.  

Proof. (i) Let \( \{a, b, c, \ldots\} \) denote indices that run through \( \{1, 2, 3\} \). Differentiating the static equation, one has

\[
\frac{f_{abc}}{f_{abc}} = \frac{f_{ac}}{f_{ac}} R_{ab} + f R_{abc}.
\]

Let \( R_{dabc} \) be the curvature tensor. (In our notation, \( R_{dabc} \) is given by

\[
\nabla_{\partial_a} \nabla_{\partial_b} \partial_c - \nabla_{\partial_b} \nabla_{\partial_c} \partial_a = R_{dabc} \partial_d
\]

in a local coordinate chart.) Then

\[
R_{dabc} f_{d} = f_{abc} - f_{acb} = f_{c} R_{ab} - f_{b} R_{ac} + f (R_{abc} - R_{acb}).
\]

(2.2)

In 3-dimension, the curvature tensor and the Ricci curvature are related by

\[
R_{dabc} = \delta_{d} R_{ac} - \delta_{b} R_{ac} + g_{ac} R_{d} - g_{ab} R_{c} + \frac{1}{2} R (\delta_{c} g_{ab} - \delta_{b} g_{ac}).
\]

(2.3)

It follows from (2.2), (2.3) and the fact \( R = 0 \) that

\[
2(f_{b} R_{ac} - f_{c} R_{ab}) + g_{ac} f_{d} R_{b} - g_{ab} f_{d} R_{c} = f (R_{abc} - R_{acb}).
\]

(2.4)

Take \( a = b \neq c \) and use the fact \( \{e_1, e_2, e_3\} \) diagonalizes Ric, one has

\[
f_{c}(\delta_{c} R_{ab} - \delta_{b} R_{ac}) = f (R_{abc} - R_{acb}).
\]

(2.5)

Now (i) follows from (2.5) and the fact \( R = 0 \).

(ii) The assumption on Ric implies that Ric has distinct eigenvalues in an open set \( U \). Hence, \( \nabla \log N = \nabla \log V \) on \( U \) by (i), which shows \( V = cN \) for some constant \( c \) on \( U \). Since \( V, N \) are both harmonic functions, \( V = cN \) on \( M \) by unique continuation.

(iii) Apply (i) to \( N \) and \( f = ZN \), one has

\[
(R_{22} - R_{33}) NZ_{,1} = (R_{33} - R_{11}) NZ_{,2} = (R_{11} - R_{22}) NZ_{,3} = 0.
\]

The claim then follows from the fact \( N \neq 0 \) and \( R_{11} = R_{22} \neq R_{33} \).

The zero set of a static potential, if nonempty, was known to be a totally geodesic hypersurface (cf. [12, Proposition 2.6]). In the next lemma, we give more geometric properties of such a zero set.

**Lemma 2.1.** Suppose \( f \) is a static potential with nonempty zero set. Let \( \Sigma = f^{-1}(0) \).

(i) \( \Sigma \) is a totally geodesic hypersurface and \( |\nabla f| \) is a positive constant on each connected component of \( \Sigma \).

(ii) At any \( p \in \Sigma \), \( \nabla f \) is an eigenvector of Ric.

(iii) At any \( p \in \Sigma \), let \( \{e_1, e_2, e_3\} \) be an orthonormal frame that diagonalizes Ric such that \( e_3 \) is normal to \( \Sigma \). Then \( R_{11} = R_{22} \).
(iv) Let $K$ be the Gaussian curvature of $\Sigma$ at $p$. Using the same notations in (iii), one has $K = 2R_{11} = 2R_{22}$. In particular, $K$ is zero if and only if $(M, g)$ is flat at $p$.

Proof. (i) Let $p \in \Sigma$. If $\nabla f(p) = 0$, then along any geodesic $\gamma(t)$ emanating from $p$, $f(\gamma(t))$ satisfies $f'' = \text{Ric}(\gamma', \gamma')f$ and $f(0) = f'(0) = 0$. This implies $f$ is zero near $p$. By unique continuation, $f = 0$ on $M$, thus a contradiction. Hence, $\nabla f(p) \neq 0$, which implies that $\Sigma$ is an embedded surface. On $\Sigma$, the static equation shows $\nabla^2 f(X, Y) = 0$ and $\nabla^2 f(X, \nabla f) = 0$ for any tangent vectors $X, Y$ tangential to $\Sigma$, which readily implies that $\Sigma$ is totally geodesic and $\nabla_X|\nabla f|^2 = 0$.

(ii) Since $\Sigma$ is totally geodesic, it follows from the Codazzi equation that $\text{Ric}(\nu, X) = 0$ for all $X$ tangent to $\Sigma$, where $\nu$ is the unit normal of $\Sigma$. Therefore, $\nabla f = \partial f/\partial \nu \nu$ is an eigenvector of $\text{Ric}$.

(iii) Apply Proposition 2.1 (iii), one has

$$(R_{11} - R_{22})f_{;3} = f(R_{22;3} - R_{23;2}) = 0.$$  

Since $|f_{;3}| = |\nabla f| > 0$, one concludes $R_{11} = R_{22}$.

(iv) It follows from the Gauss equation, the fact $R = 0$ and (iii) that $K = -R_{33} = 2R_{11} = 2R_{22}$. As a result, $K = 0$ if and only if $\text{Ric} = 0$ at $p$. \qed

In what follows, we let $\mathcal{F} = \{ f \mid \nabla^2 f = f \text{Ric} \}$.

**Lemma 2.2.** If the Ricci curvature of $g$ has distinct eigenvalues at a point, then $\text{dim}(\mathcal{F}) \leq 1$.

**Proof.** The assumption on $\text{Ric}$ implies there is an open set $U$ such that $\text{Ric}$ has distinct eigenvalues everywhere in $U$. By Lemma 2.1 (iii), a static potential $f$ is either positive or negative in $U$. The claim now follows from Proposition 2.1 (ii). \qed

Given two static potentials, if one of them is positive, one can look at their quotient.

**Lemma 2.3.** Suppose $f$ and $N$ are two static potentials. Suppose $N$ is positive. Let $Z = M/N$. Then either $Z$ is a constant or $\nabla Z$ never vanishes. In the latter case, one has

(i) each level set of $Z$ is a totally geodesic hypersurfaces.

(ii) $N^2 |\nabla Z|^2$ equals a constant on each connected component of the level set of $Z$.

(iii) $(M, g)$ is locally isometric to $((-\epsilon, \epsilon) \times \Sigma, N^2 dt^2 + g_0)$ where $\Sigma$ is a 2-dimensional surface, $Z$ is a constant on each $\Sigma_t = \{ t \} \times \Sigma$ and $g_0$ is a fixed metric on $\Sigma$. 


Proof. Let \( \{ x_i \} \) be local coordinates on \( M \). Since \( N \) and \( f = NZ \) both are solutions to (2.1), we have
\[
NZR_{ij} = (NZ)_{;ij} = NZR_{ij} + NZ_{;ij} + N_{;i}Z_{;j} + N_{;j}Z_{;i}.
\]
Therefore, \( NZ_{;ij} = -N_{;i}Z_{;j} - N_{;j}Z_{;i} \) or equivalently
\[
(2.6)
\]
\[
N\nabla^2 Z(v, w) = -\langle \nabla N, v \rangle \langle \nabla Z, w \rangle - \langle \nabla N, w \rangle \langle \nabla Z, v \rangle
\]
for any tangent vectors \( v, w \).

Suppose \( \nabla Z = 0 \) at some point \( p \). Similar to the proof of Lemma 2.1 (i), we consider an arbitrary geodesic \( \gamma(t) \) emanating from \( p \). Taking \( v = w = \gamma' \) in (2.6), we have \(NZ(\gamma(t))'' = -2N(\gamma(t))'Z(\gamma(t))'\). As \( N > 0 \) and \( Z(\gamma(t))'|_{t=0} = 0 \), we have \( Z(\gamma(t))' = 0 \) \( \forall t \), hence \( Z \) is a constant near \( p \). By unique continuation [3], \( Z \) is a constant on \( M \).

Next, suppose \( \nabla Z \neq 0 \) everywhere. In this case, every level set \( Z^{-1}(t) \), if nonempty, is an embedded hypersurface. Let \( v \) and \( w \) be tangent vectors tangent to \( Z^{-1}(t) \), (2.6) implies \( N\nabla^2 Z(v, w) = 0 \). As \( N > 0 \) and \( \nabla^2 Z(v, w) = \langle \nabla_v(\nabla Z), w \rangle = |\nabla Z|\Pi(v, w), \) where \( \Pi(\cdot, \cdot) \) is the second fundamental form of \( Z^{-1}(t) \) with respect to \( \nu = \nabla Z/|\nabla Z| \), we have \( \Pi = 0 \). Hence \( Z^{-1}(t) \) is totally geodesic, which proves (i).

To prove (ii), let \( v = \nabla Z \) and \( w \) be tangent to \( Z^{-1}(t) \) in (2.6), we have \( Nw(|\nabla Z|^2) = -2w(N)|\nabla Z|^2 \), which implies \( w(N^2|\nabla Z|^2) = 0 \). Hence \( N^2|\nabla Z|^2 \) equals a constant on each connected component of \( Z^{-1}(t) \).

For (iii), let \( X = \nabla Z/|\nabla Z|^2 \) which is a nowhere vanishing vector field. Given any point \( p \in M \), let \( \Sigma \) be a connected hypersurface passing \( p \) on which \( Z \) is a constant. By considering the integral curves of \( X \) starting from \( \Sigma \) and shrinking \( \Sigma \) if necessary, one knows there exists an open neighborhood \( U \) of \( p \), diffeomorphic to \( (-\epsilon, \epsilon) \times \Sigma \) for some \( \epsilon > 0 \), on which the metric \( g \) takes the form
\[
g = \frac{1}{|\nabla Z|^2} dt^2 + gt
\]
where \( \partial_t = X \), \( Z \) is a constant on each \( \Sigma_t = \{ t \} \times \Sigma \) and \( gt \) is the induced metric on \( \Sigma_t \). Consider a background metric
\[
\bar{g} = dt^2 + gt
\]
on \( U = (-\epsilon, \epsilon) \times \Sigma \). Let \( \Pi, \bar{\Pi} \) be the second fundamental form of \( \Sigma_t \) in \((U, g), (U, \bar{g})\) respectively with respect to \( \partial_t \). Then \( \Pi = |\nabla Z|\bar{\Pi} \). Since \( \bar{\Pi} = 0 \) by (i), we have \( \Pi = 0 \). Hence \( \frac{d}{dt} gt = 0 \) by the fact \( \bar{\Pi} = \frac{1}{2} \frac{d}{dt} gt \). This shows, for each \( t, gt = g_0 \) which is a fixed metric on \( \Sigma \). By (ii),
N|∇Z| is a constant on $\Sigma_t$. Let $\phi(t) = N|\nabla Z|$. Then

$$g = \frac{N^2}{\phi(t)^2}dt^2 + g_0.$$  

Replacing $t$ by $\int \frac{1}{\phi(t)} dt$, we have $g = N^2 dt^2 + g_0$. This proves (iii).  

**Proposition 2.2.** If $(M,g)$ is not flat at a point, then $\dim(F) \leq 2$.

**Proof.** Suppose $\dim(K) > 2$. Let $f_1, f_2, f_3$ be three linearly independent static potentials. Let $U$ be an open set such that $g$ is not flat at every point in $U$. By Lemma 2.1, $U \setminus \bigcup_{i=1}^3 f_i^{-1}(0)$ is nonempty. Hence one can find a connected open set $V \subset U$ such that each $f_i$ is nowhere vanishing on $V$. Let $\{\lambda_1, \lambda_2, \lambda_3\}$ denote the eigenvalues of Ric in $V$. \{\lambda_1, \lambda_2, \lambda_3\} can not be distinct by Proposition 2.1 (ii). The fact $g$ is not flat and $R = 0$ shows $\{\lambda_1, \lambda_2, \lambda_3\}$ can not be identical. Therefore, one may assume $\lambda_1 = \lambda_2 \neq \lambda_3$ in $V$. Let $Z_1 = f_1/f_3$, $Z_2 = f_2/f_3$. By Proposition 2.1(iii), both $\nabla Z_1$ and $\nabla Z_2$ are parallel to the eigenvector of Ric with eigenvalue $\lambda_3$. Therefore, at a point $q \in V$, $\nabla Z_1 + \alpha \nabla Z_2 = 0$ for some constant $\alpha$. By Lemma 2.3, $\nabla Z_1 + \alpha \nabla Z_2 \equiv 0$ in $V$. So $Z_1 + \alpha Z_2$ is a constant in $V$. Hence, $f_1 + \alpha f_2 = \beta f_3$ for some constant $\beta$, which is a contradiction.  

When the zero set of a given static potential is not empty, we can consider the behavior of another static potential along such a set.

**Lemma 2.4.** Suppose $f$ and $\tilde{f}$ are two static potentials. Suppose $\tilde{f}$ has nonempty zero set. Let $\Sigma = \tilde{f}^{-1}(0)$. Then

$$\nabla^2 f = \frac{1}{2} K f \gamma$$  

along $\Sigma$. Here $\nabla^2 f$ is the Hessian on $\Sigma$, $\gamma$ is the induced metric on $\Sigma$, and $K$ is the Gaussian curvature of $(\Sigma, \gamma)$. Consequently, $K f^3$ equals a constant along each connected component of $\Sigma$.

**Proof.** By Lemma 2.1(iii), $\text{Ric}(X,Y) = \lambda \gamma(X,Y)$ for all $X, Y$ tangent to $\Sigma$, where $2\lambda + \text{Ric}(\nu, \nu) = 0$ and $\nu$ is a unit normal to $\Sigma$. Therefore, $\nabla^2 f(X,Y) = f \lambda \gamma(X,Y)$ along $\Sigma$. On the other hand, $\nabla^2 f(X,Y) = \nabla^2 \tilde{f}(X,Y)$ since $\Sigma$ is totally geodesic. Hence $\nabla^2 f = f \lambda \gamma = \frac{1}{2} f K \gamma$, where we have used $K = 2\lambda$ by Lemma 2.1(iv).

Let $\{x_\alpha\}$ be local coordinates on $\Sigma$. Taking divergence and trace of (2.7), we have

$$\Delta_\Sigma \tilde{f} + K f_{;\alpha} = \frac{1}{2} (K f)_{;\alpha} \text{ and } \Delta_\Sigma f = K f$$
where $\Delta_\Sigma$ is the Laplacian on $(\Sigma, \gamma)$. It follows from (2.8) that
\[ K_{i\alpha} f + 3K_{f\alpha} = 0, \]
which implies $(K f^3)_{i\alpha} = 0$. Hence, $K f^3$ is a constant on each connected component of $\Sigma$.

To prove the main result in this section, we need an additional lemma in connection with Lemma 2.3 (iii).

**Lemma 2.5.** Suppose $(\Sigma_0, g_0)$ is a flat surface. If $\dim(\mathcal{F}) \geq 2$ on $(M, g) = ((-\epsilon, \epsilon) \times \Sigma, N^2 dt^2 + g_0)$ where $N$ is a positive function on $M$ and $g$ has zero scalar curvature, then $(M, g)$ is flat.

**Proof.** Take any $(t, q) \in (-\epsilon, \epsilon) \times \Sigma$, the surface $\Sigma_t = \{t\} \times \Sigma$ has zero Gaussian curvature and is totally geodesic in $(M, g)$. Let $\{e_1, e_2, e_3\}$ be an orthonormal frame at $(t, q)$ which diagonalizes the Ricci curvature and satisfies $e_3 \perp \Sigma_t$. Then $R_{33} = 0$ by the Gaussian equation. Hence, $R_{11} + R_{22} = 0$. If $R_{11} \neq R_{22}$, then Ric has distinct eigenvalues at $(t, q)$ and Lemma 2.2 implies $\dim(\mathcal{F}) \leq 1$, contradicting to the assumption $\dim(\mathcal{F}) \geq 2$. Therefore $R_{11} = R_{22} = 0$ by Lemma 2.3 (iii). We conclude that $g$ has zero curvature at $(t, q)$.

**Proposition 2.3.** Suppose $\dim(\mathcal{F}) \geq 2$. Let $f_1$ and $f_2$ be two linearly independent static potentials. Let $P_1$, $P_2$ be a connected component of $f_1^{-1}(0)$, $f_2^{-1}(0)$ respectively. If $P_1 \cap P_2 \neq \emptyset$, then

1. $(M, g)$ is flat along $P_1 \cup P_2$.
2. $(M, g)$ is flat in an open set which contains $P_1 \setminus f_2^{-1}(0)$ and $P_2 \setminus f_1^{-1}(0)$.

**Proof.** First we note that $f_1^{-1}(0) \cap f_2^{-1}(0)$ is an embedded curve (hence a geodesic since both $P_1$ and $P_2$ are totally geodesic). This is because $f_1$ and $f_2$ are linearly independent, which implies $\nabla f_1$ and $\nabla f_2$ are linearly independent at any point in $f_1^{-1}(0) \cap f_2^{-1}(0)$.

Now let $K_1$, $K_2$ be the Gaussian curvature of $P_1$, $P_2$ respectively. By Lemma 2.4, $K_1 f_1^3 = C$ for some constant $C$ on $P_1$ and $K_2 f_2^3 = D$ for some constant $D$ on $P_2$. Since $f_1 = f_2 = 0$ on $P_1 \cap P_2$, we have $C = D = 0$. As $P_1 \cap f_2^{-1}(0)$, $P_2 \cap f_1^{-1}(0)$ consists of embedded curves, we conclude $K_1 = 0$ on $P_1$ and $K_2 = 0$ on $P_2$. Consequently $g$ is flat along $P_1 \cup P_2$ by Lemma 2.1 (iv). This proves (i).

To prove (ii), let $p$ be an arbitrary point in $P_1 \setminus f_2^{-1}(0)$, then $f_2$ does not vanish in an open set $U$ containing $p$. Consider $Z = f_1/f_2$ on $U$. We have $Z = 0$ on $P_1 \cap U$. By Lemma 2.3 (iii), there exists an open neighborhood $W$ of $p$, diffeomorphic to $(-\epsilon, \epsilon) \times \Sigma$, where $\Sigma$ is a small
piece of \( P_1 \) containing \( p \), and \( Z \) is a constant on each \( \{ t \} \times \Sigma \), such that on \( W \) the metric \( g \) takes the form of
\[
g = f^2 \, dt^2 + g_0
\]
where \( g_0 \) is the induced metric on \( \Sigma \). By (i), \((\Sigma, g_0)\) has zero Gaussian curvature. Since \( \dim(F) \geq 2 \) on \((W, g)\), Lemma 2.5 implies that \( g \) is flat in \( W \). Similarly, we know \( g \) is flat in an open neighborhood of any point in \( P_2 \setminus f^{-1}_1(0) \). Therefore, (ii) is proved. \( \square \)

To end this section, we apply the analyticity of a static metric to improve Proposition 2.3. It is known that, if \((M, g)\) admits a static potential \( f \), then \( g \) is analytic in harmonic coordinates around any point \( p \) with \( f(p) \neq 0 \) (cf. [12, Proposition 2.8]).

**Theorem 2.1.** Suppose \( \dim(F) \geq 2 \). Let \( f_1 \) and \( f_2 \) be two linearly independent static potentials. If \( f_1^{-1}(0) \cap f_2^{-1}(0) \) is nonempty, then \((M, g)\) is flat.

**Proof.** Let \( S = f_1^{-1}(0) \cap f_2^{-1}(0) \). Given any \( p \in M \setminus S \), either \( f_1(p) \neq 0 \) or \( f_2(p) \neq 0 \), hence there exists an open set containing \( p \) in which \( g \) is analytic. As \( f_1 \) and \( f_2 \) are linearly independent, \( S \) is an embedded curve. In particular \( M \setminus S \) is path-connected. Therefore, by Proposition 2.3 (ii), we conclude that \( g \) is flat in \( M \setminus S \), hence flat in \( M \). \( \square \)

**Remark 2.1.** We note that a much stronger analytic property of static metrics was shown by Chruściel in [13, Section 4]. Theorem 2.1 also follows from Proposition 2.3 and the result of Chruściel in [13].

3. **STATIC POTENTIALS ON AN ASYMPOTICALLY FLAT END**

In this section, unless otherwise stated, we assume that \( M \) is diffeomorphic to \( \mathbb{R}^3 \setminus B(p) \), where \( B(p) \) is an open Euclidean ball centered at the origin with radius \( \rho > 0 \), and \( g \) is a smooth metric on \( M \) such that with respect to the standard coordinates \( \{x_i\} \) on \( \mathbb{R}^3 \), \( g \) satisfies
\[
g_{ij} = \delta_{ij} + b_{ij} \quad \text{with} \quad b_{ij} = O_2(|x|^{-\tau})
\]
for some constant \( \tau \in (\frac{1}{2}, 1] \). We also assume that \( g \) has zero scalar curvature.

On such an \((M, g)\), if \( f \) is a static potential, then [12] and the assumption that \( g \) is smooth up to \( \partial M \) imply that \( f \) is necessarily smooth up to \( \partial M \) (cf. [12, Proposition 2.5]).

**Lemma 3.1.** Suppose \( f \) is a static potential on \((M, g)\). Then \( f \) has at most linear growth, i.e. there exists \( C > 0 \) such that \( |f(x)| \leq C|x| \).
Proof. Let $R_m$ denote the Riemann curvature tensor of $g$. By the AA condition (3.1), we have

$$(3.2) \quad r^{2+r}|R_m| = O(1)$$

where $r = |x|$. Therefore, given any $\epsilon > 0$, there is $r_0 > \rho$ such that

$$|R_m(x)| \leq \frac{1}{2} \epsilon |x|^{-2} \leq \epsilon(d(x) + r_0)^{-2}$$

if $|x| > r_0$. Here $d(x) = \text{dist}(x, S_{r_0})$, where $S_{r_0} = \partial B(r_0)$, the Euclidean sphere with radius $r_0$. Given any $x$ outside $S_{r_0}$, let $\gamma(t), t \in [r_0, T]$, be a minimal geodesic parametrized by arc length connecting $x$ and $S_{r_0}$ with $\gamma(r_0) \in S_{r_0}$ and $\gamma(T) = x$. Then $f(t) = f(\gamma(t))$ satisfies

$$f''(t) = h(t)f(t),$$

where $h(t) = \text{Ric}(\gamma'(t), \gamma'(t))$ satisfies

$$|h(t)| \leq \epsilon t^{-2}.$$

Let $\alpha = \frac{1}{2}(1 + \sqrt{1 + 4\epsilon})$ and $\alpha = \sup_{S_{r_0}}(|f| + |\nabla f|)$. Define $w(t) = At^\alpha$, where $A > 0$ is chosen so that $Ar_0^\alpha > a$ and $A\alpha r_0^{\alpha-1} > a$, then $w(t)$ satisfies

$$w''(t) = \epsilon t^{-2}w, \quad |f(r_0)| < w(r_0) \quad \text{and} \quad |f'(r_0)| < w'(r_0).$$

Suppose $|f(t)| > w(t)$ for some $t \in [r_0, T]$. Let

$$t_1 = \inf\{t \in [r_0, T] \mid |f(t)| > w(t)\}.$$

Then $t_1 > r_0$ and $|f(t_1)| = w(t_1)$. On $[r_0, t_1]$, we have

$$|f''(t)| = |h(t)f(t)| \leq \epsilon t^{-2}w = w''(t).$$

Therefore, $\forall t \in [r_0, t_1]$,

$$-w'(t) + w'(r_0) \leq f'(t) - f'(r_0) \leq w'(t) - w'(r_0)$$

which implies $-w'(t) < f'(t) < w'(t)$ because $|f'(r_0)| < w'(r_0)$. Integrating again, we have

$$-w(t) + w(r_0) < f(t) - f(r_0) < w(t) - w(r_0),$$

which shows $-w(t) < f(t) < w(t)$ because $|f(t_0)| < w(t_0)$. Therefore, $|f(t_1)| < w(t_1)$, which is a contradiction. Hence we have

$$(3.3) \quad |f(t)| \leq At^\alpha, \forall t.$$

Now choose $\epsilon$ such that $\alpha < 1 + \frac{\tau}{2}$. It follows from (3.2) and (3.3) that

$$|f''(t)| = |h(t)f(t)| \leq A|h(t)|t^{1+\frac{\tau}{2}}.$$
where $|h(t)| \leq C_1 t^{-2-\tau}$ for some $C_1$ independent on $x$ and $t$. This shows $|f'(t)| \leq C_2$ for some constant $C_2$ independent on $x$. Hence

$$|f(x)| \leq a + C_2(|x| - r_0),$$

which proves that $f$ has at most linear growth.

\[ \square \]

**Proposition 3.1.** Suppose $f$ is a static potential on $(M,g)$. Then

(i) there exists a tuple $(a_1, a_2, a_3)$ such that

$$f = a_1 x_1 + a_2 x_2 + a_3 x_3 + h$$

where $h$ satisfies $\partial h = O_1(|x|^{-\tau})$ and

$$|h| = \begin{cases} 
O(|x|^{1-\tau}) & \text{when } \tau < 1, \\
O(\ln |x|) & \text{when } \tau = 1.
\end{cases}$$

(ii) $(a_1, a_2, a_3) = (0, 0, 0)$ if and only if $f$ is bounded. In this case, either $f > 0$ near infinity or $f < 0$ near infinity; moreover, upon rescaling,

$$f = 1 - \frac{m}{|x|} + o(|x|^{-1})$$

for some constant $m$.

**Proof.** By (3.1) and Lemma 3.1, $|\nabla^2 f| = |f \text{Ric}| = O(r^{-1-\tau})$ where $r = |x|$. Let $\phi = |\nabla f|^2$, then

$$|\nabla \phi|^2 \leq 4|\nabla^2 f|^2 \phi \leq C_1 r^{-2-2\tau} \phi$$

for some constant $C_1$. By considering $\phi$ restricted to a minimal geodesic emanating from the boundary, as in the proof of Lemma 3.1, it is not hard to see that (3.4) implies $\phi$ is bounded. Hence

$$|\partial_{x_i} \partial_{x_j} f| = |f_{ij} + \Gamma^k_{ij} \partial_{x_k} f| = O(r^{-1-\tau}),$$

where “ ; ” denotes covariant derivative and $\Gamma^k_{ij}$ are the Christoffel symbols. It follows from (3.5) that, for each $i$, $\lim_{x \to \infty} \partial_{x_i} f$ exists and is finite. Let $a_i = \lim_{x \to \infty} \partial_{x_i} f$ and define $\lambda = \sum_{i=1}^3 a_i x_i$, then

$$|\partial_{x_i} \partial_{x_j} (f - \lambda)| = |\partial_{x_i} \partial_{x_j} f| = O(r^{-1-\tau})$$

and $\lim_{x \to \infty} \partial_{x_i} (f - \lambda) = 0$. This implies

$$|\partial_{x_i} (f - \lambda)| = O(r^{-\tau}),$$

which then shows

$$f - \lambda = \begin{cases} 
O(r^{1-\tau}) & \text{when } \tau < 1, \\
O(\ln r) & \text{when } \tau = 1.
\end{cases}$$

Let $h = f - \lambda$. This proves (i).
To prove (ii), first suppose \( a_1 = a_2 = a_3 = 0 \). Let \( \tau' \) be any fixed constant with \( \tau > \tau' > \frac{1}{2} \). Then \( |f| = |h| = O(r^{1-\tau'}) \), hence \( |\nabla^2 f| = |f \text{Ric}| = O(r^{-1-2\tau'}) \). This combined with \( |\partial_x f| = O(r^{-\tau}) \) implies \( |\partial_x, \partial_x, f| = O(r^{-1-2\tau}) \), which in turns shows \( |\partial_x f| = O(r^{-2\tau}) \). Since \( 2\tau' > 1 \), we conclude that \( f \) has a finite limit as \( x \to \infty \). In particular, \( f \) is bounded.

Next, suppose \( f \) is bounded. Then \( a_1, a_2, a_3 \) must be zero since \( h \) grows slower than a linear function. Moreover, \( \lim_{x \to \infty} \phi = 0 \) since \( |\partial_x f| = O(r^{-\tau}) \). Let \( \Sigma = f^{-1}(0) \). By Lemma \( \text{[2.1]}(i) \), \( \Sigma \) is an embedded totally geodesic surface and \( \phi \) is a positive constant on any connected component of \( \Sigma \). Let \( P \) be any connected component of \( \Sigma \), then \( P \) must be bounded (hence compact), for otherwise contradicting to the fact \( \lim_{x \to \infty} \phi = 0 \) and \( \phi \) is a positive constant on \( P \). However, there is \( R_0 > 0 \) such that \( \partial B(R), \forall R \geq R_0 \), has positive mean curvature in \((M, g)\). Therefore, \( P \cap \{|x| > R_0\} = \emptyset \) by the maximum principle and the fact that \( P \) is an embedded minimal surface. This proves that either \( f > 0 \) or \( f < 0 \) on \( \{|x| > R_0\} \).

To complete the proof, let \( a = \lim_{x \to \infty} f \) (which was shown to exists). Since \( \Delta f = 0 \), we have \( f = a + A|x|^{-1} + o(|x|^{-1}) \) for some constant \( A \) (cf. [4]). We want to show \( a \neq 0 \). Suppose \( a = 0 \). By what we have proved, we may assume \( f > 0 \) near infinity. Let \( R > 0 \) be a constant such that \( f > 0 \) on \( S_R = \partial B(R) \). Let \( \psi \) be a harmonic function outside \( S_R \) such that \( \psi = \inf_{S_R} f > 0 \) on \( S_R \) and \( \lim_{x \to \infty} \psi = 0 \). Then \( f \geq \psi \) by the maximum principle. Since \( \psi \) behaves like the Green’s function which has a decay order of \( \frac{1}{|x|} \), we have \( A > 0 \). On the other hand, the assumption \( a = 0 \) implies \( f = O(|x|^{-1}) \), hence \( |\nabla^2 f| = O(r^{-3-\tau}) \). Since \( |\partial_x f| = O(r^{-2\tau}) \), we have \( |\partial_x, \partial_x, f| = O(r^{-3-\tau}) + O(r^{-1-\tau-2\tau}) \) which implies \( |\partial_x f| = O(r^{-3\tau}) \). Iterating this argument and using the fact \( \tau' \) can be chosen arbitrarily close to \( \tau \), we conclude \( |\partial_x, \partial_x, f| = O(r^{-3-\tau}) \) and \( |\partial_x f| = O(r^{-2-\tau}) \). This together with \( a = 0 \) shows \( |f| = O(r^{-1-\tau}) \), contradicting the fact \( A > 0 \). Therefore, \( a \neq 0 \). Multiplying \( f \) by a nonzero constant, we conclude \( f = 1 - m|x|^{-1} + o(|x|^{-1}) \) for some constant \( m \). This complete the proof of (ii).

\[ \square \]

Remark 3.1. We thank Justin Corvino and Marc Mars for informing us that Proposition \([3.1]\) was also given in a more general setting by Beig and Chruściel in [8 Proposition 2.1] for Killing initial data sets (also see [9]).

The next proposition describes the zero set of a static potential \( f \) near infinity in the case that \( f \) is unbounded.
Proposition 3.2. Suppose $f$ is an unbounded static potential on $(M, g)$. There exists a new set of coordinates $\{y_i\}$ on $\mathbb{R}^3 \setminus B(\rho)$ obtained by a rotation of $\{x_i\}$ such that, outside a compact set, $f^{-1}(0)$ is given by the graph of a smooth function $q = q(y_2, y_3)$ over

$$\Omega_C = \{(y_2, y_3) \mid y_2^2 + y_3^2 > C^2\}$$

for some constant $C > 0$, where $q$ satisfies

$$\partial q = O_1(|\bar{y}|^{-\tau})$$

and $|q| = \begin{cases} O(|\bar{y}|^{1-\tau}) & \text{when } \tau < 1 \\ O(\ln |\bar{y}|) & \text{when } \tau = 1. \end{cases}$

Here $\bar{y} = (y_2, y_3)$. As a result, if $\gamma_R \subset f^{-1}(0)$ is the curve given by

$$\gamma_R = \{(q(y_2, y_3), y_2, y_3) \mid y_2^2 + y_3^2 = R^2\}$$

and $\kappa$ is the geodesic curvature of $\gamma_R$ in $f^{-1}(0)$, then

$$\lim_{R \to \infty} \int_{\gamma_R} \kappa = 2\pi.$$

Proof. Let $(a_1, a_2, a_3)$ and $h$ be given by Proposition 3.1 such that $f = \sum_{i=1}^3 a_i x_i + h$. As $f$ is unbounded, $(a_1, a_2, a_3) \neq (0, 0, 0)$. We can rescale $f$ so that $\sum_{i=1}^3 a_i^2 = 1$. Hence, there exists new coordinates $\{y_i\}$ obtained by a rotation of $\{x_i\}$ such that

$$f = y_1 + h(y_1, y_2, y_3)$$

where $h$ satisfies

$$\partial h = O_1(|y|^{-\tau})$$

and $|h| = \begin{cases} O(|y|^{1-\tau}) & \text{when } \tau < 1 \\ O(\ln |y|) & \text{when } \tau = 1. \end{cases}$

It follows from (3.9) and (3.10) that

$$\frac{\partial f}{\partial y_1} = 1 + \frac{\partial h}{\partial y_1} = 1 + O(|y|^{-\tau}).$$

Therefore there exists a constant $C > 0$ such that

$$\frac{\partial f}{\partial y_1} > \frac{1}{2}, \quad \forall (y_2, y_3) \in \Omega_C = \{(y_2, y_3) \mid |\bar{y}| > C\}.$$

For any fixed $(y_2, y_3) \in \Omega_C$, (3.9) and (3.10) imply

$$\lim_{y_1 \to -\infty} f = -\infty, \quad \lim_{y_1 \to \infty} f = \infty.$$

Hence the set $f^{-1}(0) \cap \{(y_2, y_3) \mid (y_2, y_3) \in \Omega_C\} \neq \emptyset$ and is given by the graph of some function $q = q(y_2, y_3)$ defined on $\Omega_C$. Since $\nabla f \neq 0$ on $f^{-1}(0)$, $q$ is a smooth function by the implicit function
Theorem. Given the constant \( C \), (3.9) and (3.10) imply there exists another constant \( C_1 > 0 \) such that
\[
|f| \geq \frac{1}{2} |y_1| > 0 \text{ whenever } |\bar{y}| \leq C \text{ and } |y_1| > C_1.
\]
Therefore,
\[
f^{-1}(0) \cap \{(y_1, y_2, y_3) \mid (y_2, y_3) \in \Omega_C \} = f^{-1}(0) \setminus \{(y_1, y_2, y_3) \mid |y_1| \leq C_1, |\bar{y}| \leq C \}.
\]
This proves that, outside a compact set, \( f^{-1}(0) \) is given by the graph of \( q \) over \( \Omega_C \).

Next we estimate \( q \) and its derivatives. The equation (3.11)
\[
q + h(q, y_2, y_3) = 0
\]
and (3.10) imply that, if \( |\bar{y}| \) is large,
\[
|q| = |h(q, y_2, y_3)| \leq \left\{ \begin{array}{ll}
C_2 (|q| + |\bar{y}|)^{1-\tau}, & \tau < 1 \\
C_2 \ln (|q| + |\bar{y}|), & \tau = 1
\end{array} \right.
\]
for some constant \( C_2 > 0 \). This in turn implies, as \( |\bar{y}| \to \infty \),
\[
|q| = O(|\bar{y}|^{1-\tau}) \text{ if } \tau < 1 \text{ and } |q| = O(\ln |\bar{y}|) \text{ if } \tau = 1.
\]
Let \( \alpha, \beta \in \{2, 3\} \). Taking derivative of (3.11), we have
\[
(3.12) \quad \frac{\partial q}{\partial y_\alpha} = -\frac{\partial h}{\partial y_\alpha} 1 + \frac{\partial h}{\partial y_\beta} = O(|\bar{y}|^{-\tau}).
\]
Similarly, by taking derivative of (3.12), we have
\[
\frac{\partial^2 q}{\partial y_\beta y_\alpha} = O(|\bar{y}|^{-1-\tau}).
\]
To verify (3.8), we consider the pulled back metric \( \sigma = F^*(g) \) on \( \Omega_C \) where \( F : \Omega_C \to \mathbb{R}^3 \) is given by \( F(y_2, y_3) = (q(y_2, y_3), y_2, y_3) \). It follows from (3.1) and (3.7) that
\[
(3.13) \quad \sigma_{\alpha\beta} = \delta_{\alpha\beta} + h_{\alpha\beta}
\]
where \( h_{\alpha\beta} = \sigma(\partial_{y_\alpha}, \partial_{y_\beta}) \) and \( h_{\alpha\beta} \) satisfies
\[
(3.14) \quad |h_{\alpha\beta}| + |\bar{y}| |\partial h_{\alpha\beta}| = O(|\bar{y}|^{-\tau}).
\]
Direct calculation using (3.13) and (3.14) then shows
\[
(3.15) \quad \kappa = R^{-1} + O(R^{1-\tau})
\]
while the length of \( C_R \) is \( 2\pi R + O(R^{1-\tau}) \). From this, we conclude that (3.8) holds. \( \square \)
Remark 3.2. In [10], Beig and Schoen solved static $n$-body problem in the case that there exists a closed, noncompact, totally geodesic surface disjoint from the bodies. One may compare Proposition 3.2 with Proposition 2.1 in [10].

Now we are ready to prove the main results of this section.

Theorem 3.1. Let $(M,g)$ be a connected, asymptotically flat 3-manifold with or without boundary. If $\dim(F) \geq 2$, then $(M,g)$ is flat.

Proof. It suffices to prove this result on an end of $(M,g)$. So we assume $M$ is diffeomorphic to $\mathbb{R}^3$ minus an open ball. Suppose $f$ and $\tilde{f}$ are two linearly independent static potentials. We have the following three cases:

Case 1. Suppose both $f$ and $\tilde{f}$ are bounded. By Proposition 3.1 (ii), after rescaling, we have
\[
 f = 1 - \frac{m}{|x|} + o(|x|^{-1}), \quad \tilde{f} = 1 - \frac{\tilde{m}}{|x|} + o(|x|^{-1})
\]
for some constants $m, \tilde{m}$. Therefore, $f - \tilde{f}$ is a bounded static potential satisfying $f - \tilde{f} = -\frac{m-\tilde{m}}{|x|} + o(|x|^{-1})$. This contradicts Proposition 3.1 (ii). Hence, this case does not occur.

Case 2. Suppose $f$ is bounded and $\tilde{f}$ is unbounded. By Proposition 3.1, upon a rotation of coordinates and scaling, we may assume that $\tilde{f} = x_1 + h$, where $h$ satisfies the properties in Proposition 3.1(i), and $f = 1 - \frac{m}{|x|} + o(|x|^{-1})$ for some constant $m$. Let $r_0 > \rho$ be a fixed constant such that $f > \frac{1}{2}$ on $\{|x| \geq r_0\}$, and $S_r = \partial B(r)$ has positive mean curvature $\forall \ r \geq r_0$. Let $\lambda_0 > 0$ be another constant such that if $\lambda > \lambda_0$, then $\tilde{f}_\lambda := \tilde{f} - \lambda f$ will be negative on $S_{r_0}$. For each $\lambda > \lambda_0$, let $\Sigma_\lambda = \{x \ | \ \tilde{f}_\lambda(x) = 0, \ |x| \geq r_0\}$. Then $\Sigma_\lambda \neq \emptyset$ by Proposition 3.2. As $\tilde{f}_\lambda < 0$ on $S_{r_0}$, $\Sigma_\lambda$ does not intersect $S_{r_0}$. Hence $\Sigma_\lambda$ is a surface without boundary. Let $P$ be any connected component of $\Sigma_\lambda$, then $P$ can not be compact because $(M,g)$ is foliated by positive mean curvature surfaces $\{S_r\}$ outside $S_{r_0}$ and $P$ is a minimal surface. Hence $P$ is non-compact. By Proposition 3.2, we have $P = \Sigma_\lambda$. Let $K$ be the Gaussian curvature of $\Sigma_\lambda$. By Lemma 2.4, $Kf^3 = C$ for some constant $C$ along $\Sigma_\lambda$. Note that $\lim_{x \to \infty} K = 0$ because $g$ is asymptotically flat and $\Sigma_\lambda$ is totally geodesic. This implies $C = 0$ since $f$ is bounded. Hence $Kf^3 = 0$ on $\Sigma_\lambda$. As $f > 0$ outside $S_{r_0}$, we conclude $K = 0$. Hence, $(M,g)$ is flat along $\Sigma_\lambda$ by Lemma 2.1(iv).

Thus we have proved that $(M,g)$ is flat at every point in the set
\[
 U = \bigcup_{\lambda > \lambda_0} \{x \ | \ \tilde{f}(x) - \lambda f(x) = 0, \ |x| > r_0\}.
\]
By the growth condition on \( h \), we know that there exists a constant \( a > 0 \) such that for all \( x_1 > a \) and all \((x_2, x_3) \in \mathbb{R}^2\) with \( x_1^2 + x_2^2 < 1 \),
\[
\tilde{f}(x_1, x_2, x_3) > \lambda_0 f(x_1, x_2, x_3) > 0.
\]
Clearly this implies that these points \((x_1, x_2, x_3) \in U\) and \( U \) contains a nonempty interior. Let \( \hat{M} = M \setminus (f^{-1}(0) \cap \tilde{f}^{-1}(0)) \). \( \hat{M} \) is either \( M \) itself or \( M \) minus an embedded curve, hence \( \hat{M} \) is path-connected. Since \( g \) is analytic on \( \hat{M} \) which intersects \( U \), we conclude that \( g \) is flat on \( \hat{M} \), hence flat everywhere in \( M \).

Case 3. Suppose both \( f \) and \( \tilde{f} \) are unbounded. By the proof of Proposition 3.2 upon a rotation of coordinates and scaling, we may assume \( f = x_1 + h \), \( \tilde{f} = a_1 x_1 + a_2 x_2 + a_3 x_3 + \tilde{h} \), where \( h = O(|x|^\theta) \), \( \tilde{h} = O(|x|^\theta) \) for some constant \( 0 < \theta < 1 \), and \( a_i, i = 1, 2, 3 \), are some constants. Moreover, we can assume that \( f^{-1}(0) \), outside a compact set, is given by the graph of \( q = q(x_2, x_3) \) where \( q = O(|x_2|^\theta + |x_3|^\theta) \).

Replacing \( \tilde{f} \) by \( \tilde{f} - a_1 f \), we may assume \( a_1 = 0 \). In this case, if \( a_2 = a_3 = 0 \), then Proposition 3.1 (ii) implies that \( \tilde{f} \) is bounded and we are back to Case 2. Therefore we may assume \( (a_2, a_3) \neq (0, 0) \). Without loss of generality, we can assume \( a_2 = 1 \) upon rescaling \( \tilde{f} \) so that \( \tilde{f} = x_2 + a_3 x_3 + \tilde{h} \). Given any large positive number \( a \), consider the point \( x_+ = (q(a, 0), a, 0) \) which lies in \( f^{-1}(0) \). We have
\[
\tilde{f}(x_+) = a + \tilde{h}(q(a, 0), a, 0)
= a + O(|a|^\theta + |a|^{2\theta}).
\]
Hence \( \tilde{f}(x_+) > 0 \) if \( a \) is sufficiently large. Similarly, we have \( \tilde{f}(x_-) < 0 \), where \( x_- = (q(-a, 0), -a, 0) \), for large \( a \). Since \( x_+ \) and \( x_- \) can be joint by a curve that is contained in the graph of \( q \), hence in \( f^{-1}(0) \), we conclude
\[
f^{-1}(0) \cap \tilde{f}^{-1}(0) \neq \emptyset.
\]
Therefore \((M, g)\) is flat by Theorem 2.1.

**Theorem 3.2.** Let \( g \) be a smooth metric on \( M = \mathbb{R}^3 \setminus B(\rho) \), where \( B(\rho) \) is an open ball, such that
\[
g_{ij}(x) = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + p_{ij}
\]
where \( p_{ij}(x) = O_2(|x|^{-2}) \) and \( m \neq 0 \) is a constant. If \( f \) is a static potential of \((M, g)\), then \( f \) does not vanish outside a compact set.

**Proof.** By Proposition 3.1 (ii), it suffices to prove that \( f \) is bounded. Suppose \( f \) is unbounded, by Proposition 3.2 there exists a new set of coordinates \( \{y_i\} \), obtained by a rotation of \( \{x_i\} \), such that the zero
set of \( f \) which we denote by \( \Sigma \), outside a compact set, is given by the graph of a smooth function \( q = q(y_2, y_3) \) defined on

\[
\Omega_C = \{(y_2, y_3) \mid y_2^2 + y_3^2 > C^2\}
\]

for some constant \( C > 0 \). Here \( q \) satisfies (3.7) with \( \tau = 1 \).

Since \( \{y_i\} \) differs from \( \{x_i\} \) only by a rotation, the asymptotically Schwarzschild condition (3.17) is preserved in the \( \{y_i\} \) coordinates, i.e.

\[
g_{ij}(y) = \left(1 + \frac{m}{2|y|}\right)^4 \delta_{ij} + p_{ij}
\]

where \( p_{ij}(y) = O_2(|y|^{-2}) \). The Ricci curvature of \( g \) now can be estimated explicitly in terms of \( y \). By [15, Lemma 1.2], (3.18) implies

\[
(3.18) \quad \text{Ric}(\partial_{y_i}, \partial_{y_j}) = \frac{m}{|y|^3} \phi(y)^{-2} \left( \delta_{ij} - \frac{3y_iy_j}{|y|^2} \right) + O(|y|^{-4}),
\]

where \( \phi(y) = 1 + \frac{m}{2|y|} \).

Given any \( \bar{y} = (y_2, y_3) \in \Omega_C \), let \( y = (q(\bar{y}), y_2, y_3) \) and \( T_y\Sigma \) be the tangent space to \( \Sigma \) at \( y \). As a subspace in \( T_y\mathbb{R}^3 \), \( T_y\Sigma \) is spanned by

\[
v = (\partial_{y_2}, q)\partial_{y_1} + \partial_{y_2}, \quad w = (\partial_{y_3}, q)\partial_{y_1} + \partial_{y_3}.
\]

Let \( |v|_g, |w|_g \) be the length of \( v, w \) with respect to \( g \) respectively. Define \( \tilde{v} = |v|_g^{-1}v, \tilde{w} = |w|_g^{-1}w \), we want to compare

\[
\text{Ric}(\tilde{v}, \tilde{v}) \text{ and } \text{Ric}(\tilde{w}, \tilde{w})
\]

when \( |\bar{y}| \) is large. By (3.7) and (3.19), we have

\[
(3.20) \quad \text{Ric}(v, v) = \frac{m}{|y|^3} \phi(y)^{-2} \left[ 1 + (\partial_{y_2}, q)^2 - \frac{3}{|y|^2} \left[(\partial_{y_2}, q)q + y_2\right]^2 \right] + O(|\bar{y}|^{-4})
\]

\[
= \frac{m}{|y|^3} \phi(y)^{-2} \left( 1 - \frac{3y_2^2}{|y|^2} \right) + O(|\bar{y}|^{-4}).
\]

Similarly,

\[
(3.21) \quad \text{Ric}(w, w) = \frac{m}{|y|^3} \phi(y)^{-2} \left( 1 - \frac{3y_3^2}{|y|^2} \right) + O(|\bar{y}|^{-4}).
\]

On the other hand, (3.7) and (3.18) imply

\[
(3.22) \quad |v|_g^2 = \phi(y)^4 + O(|\bar{y}|^{-2}), \quad |w|_g^2 = \phi(y)^4 + O(|\bar{y}|^{-2}).
\]

Therefore, it follows from (3.20) – (3.22) that

\[
(3.23) \quad \text{Ric}(\tilde{v}, \tilde{v}) - \text{Ric}(\tilde{w}, \tilde{w}) = \frac{3m}{\phi(y)^6} \frac{(y_3^2 - y_2^2)}{|y|^5} + O(|\bar{y}|^{-4}).
\]
Together with (3.7), this shows that there exists \((y_2, y_3)\) such that \(\text{Ric}(\tilde{v}, \tilde{v}) \neq \text{Ric}(\tilde{w}, \tilde{w})\) when \(|\tilde{y}|\) is large. For instance, let \(y_2 = 0\) and \(y_3 \to +\infty\), then

\[
|y_3|^2 (\text{Ric}(\tilde{v}, \tilde{v}) - \text{Ric}(\tilde{w}, \tilde{w})) \to 3m \neq 0.
\]

This is a contradiction to Lemma 2.1 (iii). We conclude that \(f\) must be bounded. \(\square\)

4. RIGIDITY OF STATIC ASYMPTOTICALLY FLAT MANIFOLDS

In this section, we consider a complete, asymptotically flat 3-manifold without boundary, with finitely many ends, on which there exists a static potential \(f\). Two basic examples are

**Example 1.** The Euclidean space \((\mathbb{R}^3, g_0)\). Here \(f = a_0 + \sum_{i=1}^3 a_i x_i\) and \(\{a_i\}\) are constants.

**Example 2.** A spatial Schwarzschild manifold with mass \(m > 0\), i.e. \((\mathbb{R}^3 \setminus \{0\}, (1 + m^2 |x|)^{-4} g_0)\). In this case, \(f = 1 - \frac{m^2}{1 + \frac{m^2}{|x|^2}}\).

A natural question is whether these are the only examples of such manifolds? We start by showing that \(f\) must have nonempty zero set unless the manifold is \((\mathbb{R}^3, g_0)\).

**Lemma 4.1.** Let \((M, g)\) be a complete, connected, asymptotically flat 3-manifold without boundary. If \((M, g)\) has a static potential \(f\), then \(f^{-1}(0)\) is nonempty unless \((M, g)\) is isometric to \((\mathbb{R}^3, g_0)\).

**Proof.** By Bochner’s formula and the static equation (1.3),

\[
\frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2 + f^{-1} \nabla^2 f (\nabla f, \nabla f)
\]

\[
= |\nabla^2 f|^2 + \frac{1}{2} f^{-1} \nabla f (|\nabla f|^2)
\]

wherever \(f \neq 0\). Suppose \(f^{-1}(0)\) is empty, then Proposition 3.2 implies \(f\) is bounded. By Proposition 3.1 (ii), \(\lim_{r \to \infty} |\nabla f| = 0\) at each end of \((M, g)\). Hence there is \(p \in M\) such that \(|\nabla f|^2(p) = \sup_M |\nabla f|^2\).

By (1.1) and the strong maximum principle, \(|\nabla f|^2\) must be a constant and hence is identically zero. Therefore, \(f\) is a nonzero constant and \(\text{Ric} = 0\) everywhere. This shows \((M, g)\) is flat and hence isometric to \((\mathbb{R}^3, g_0)\) by volume comparison as \((M, g)\) is asymptotically flat. \(\square\)

In [11], Bunting and Masood-ul-Alam proved that if \((M, g)\) is an asymptotically flat 3-manifold with nonempty boundary, with one end, on which there exists a static potential \(f\) which tends to \(1\) at \(\infty\) and is \(0\) on \(\partial M\), then \((M, g)\) is isometric to a spatial Schwarzschild manifold...
with positive mass outside its horizon. By examining the proof in [11], we observe that the result in [11] holds on manifolds with any number of ends.

**Proposition 4.1.** Let \((M, g)\) be a complete, connected, asymptotically flat 3-manifold with nonempty boundary, with possibly more than one ends. Suppose \(f\) is a static potential such that \(f > 0\) in the interior and \(f = 0\) on \(\partial M\). Then \((M, g)\) is isometric to a spatial Schwarzschild manifold with positive mass outside its horizon.

**Proof.** Since \(f > 0\) away from the boundary, \(f\) must be bounded by Proposition 3.2. Upon scaling, we may assume \(\sup_M f = 1\). Suppose \((M, g)\) has \(k\) ends \(E_1, \ldots, E_k\), \(k \geq 1\). For each \(1 \leq i \leq k\), Proposition 3.1 (ii) implies \(\lim_{x \to \infty, x \in E_i} f(x) = a_i\) for some constant \(0 < a_i \leq 1\). By the maximal principal, \(a_i = 1\) for some \(i\). Without losing generality, we may assume \(a_1 = 1\).

We proceed as in [11]. Define \(\gamma^+ = (1 + f)^4 g\) and \(\gamma^- = (1 - f)^4 g\). Then the following are true:

- \(\gamma^+\) and \(\gamma^-\) have zero scalar curvature (cf. Lemma 1 in [11]).
- If \(a_j = 1\), then \(E_j\) is an asymptotically flat end in \((M, \gamma^+)\) and the mass of \((M, \gamma^+)\) at \(E_j\) is zero; on the other hand, \(E_j\) gets compactified in \((M, \gamma^-)\) in the sense that if \(p_j\) is the point of infinity at \(E_j\), then there is a \(W^{2,q}\) extension of \(\gamma^-\) to \(E_j \cup \{p_j\}\) (cf. Lemma 2 and 3 in [11]).
- If \(a_j < 1\), then clearly \(E_j\) is an asymptotically flat end in both \((M, \gamma^+)\) and \((M, \gamma^-)\).

Glue \((M, \gamma^+)\) and \((M, \gamma^-)\) along \(\partial M\) to obtain a manifold \((\tilde{M}, \tilde{g})\), then \(\tilde{g}\) is \(C^{1,1}\) across \(\partial M\) in \(\tilde{M}\) (cf. Lemma 4 in [11]). Apply the Riemannian positive mass theorem as stated in [11, Theorem 1] and use the fact that the mass of \(E_1\) in \((\tilde{M}, \tilde{g})\) is zero, we conclude that \((\tilde{M}, \tilde{g})\) is isometric to \((\mathbb{R}^3, g_0)\). In particular, this shows that \((M, g)\) only has one end. The rest now follows from the main theorem in [11]. \(\square\)

Proposition 4.1 can be used to answer the rigidity question in the case that \(f\) is bounded.

**Theorem 4.1.** Let \((M, g)\) be a complete, connected, asymptotically flat 3-manifold without boundary, with finitely many ends. If there exists a bounded static potential on \((M, g)\), then \((M, g)\) is isometric to either \((\mathbb{R}^3, g_0)\) or a spatial Schwarzschild manifold \(\left(\mathbb{R}^3 \setminus \{0\}, \left(1 + \frac{m}{2|x|}\right)^4 g_0\right)\) with \(m > 0\).
Proof. Let $f$ be a bounded static potential. If $(M,g)$ has only one end, then $f$ must be a constant by Proposition 3.1 (ii) and the fact $\Delta f = 0$. Hence, $(M,g)$ is flat and is isometric to $(\mathbb{R}^3, g_0)$.

Next suppose $(M,g)$ has more than one ends, in particular $(M,g)$ is not isometric to $(\mathbb{R}^3, g_0)$. By Lemma 4.1, $f^{-1}(0) \neq \emptyset$. By Lemma 2.1 (i) and Proposition 3.1 (ii), $f^{-1}(0)$ is a closed totally geodesic hypersurface (possibly disconnected); moreover $f$ changes sign near $f^{-1}(0)$. Let $N_1$ be a component of $\{f > 0\}$, then $N_1$ is unbounded as $f = 0$ on $\partial N$.

Since $f$ is either positive or negative near the infinity of each end of $(M,g)$, $N_1$ must be asymptotically flat, with possibly more than one ends, with nonempty boundary $\Sigma$ on which $f = 0$. By Proposition 4.1 and [11], $(N_1, g)$ is isometric to $\left(\{x \in \mathbb{R}^3 \mid |x| > \frac{m_1}{2}\}, \left(1 + \frac{m_1}{2|x|}\right)^4 \delta_{ij}\right)$ with some constant $m_1 > 0$.

Similarly, let $N_2$ be the component of $\{f < 0\}$ whose boundary contains $\Sigma$. By the same argument, we know that $(N_2, g)$ is isometric to $\left(\{y \in \mathbb{R}^3 \mid 0 < |y| < \frac{m_2}{2}\}, \left(1 + \frac{m_2}{2|y|}\right)^4 \delta_{ij}\right)$ for some $m_2 > 0$. Since $M$ is connected, we conclude that $M = N_1 \cup N_2 \cup \Sigma$.

Now we have $\Sigma = \{|x| = 2m_1\} = \{|y| = 2m_2\}$. As the area of $\Sigma$ is given by $16\pi m^2_1$ and $16\pi m^2_2$ respectively, we have $m_1 = m_2$. This proves that $(M,g)$ is isometric to a spatial Schwarzschild manifold with positive mass. \hfill $\square$

Next, we consider the rigidity question without the boundedness assumption of $f$. We recall that, by Proposition 3.1 (ii) and Proposition 3.2, the zero set of a static potential on an asymptotically flat manifold has only finitely many components.

**Proposition 4.2.** Let $(M,g)$ be a complete, connected, asymptotically flat 3-manifold without boundary, with finitely many ends $E_1, \ldots, E_k$. Suppose there exists a static potential $f$ on $(M,g)$. Then

(i) $\int_M f |\text{Ric}|^2 = 0$.

(ii) $\int_M |f| |\text{Ric}|^2 = 4\pi \left[\sum_{\alpha} c_{\alpha}(\chi(\Sigma_\alpha) - k_\alpha) + \sum_{\beta} \tilde{c}_{\beta}\chi(\tilde{\Sigma}_\beta)\right]$. Here $\{\Sigma_\alpha \mid 0 \leq \alpha \leq m\}$ and $\{\tilde{\Sigma}_\beta \mid 0 \leq \beta \leq n\}$ are the sets of unbounded components and bounded components of $f^{-1}(0)$ respectively. $c_{\alpha} > 0$ and $\tilde{c}_{\beta} > 0$ are the constants which equal $|\nabla f|$ on $\Sigma_\alpha$ and $\tilde{\Sigma}_\beta$ respectively. For each $\alpha$, $k_\alpha \geq 1$ is the
number of ends $E_i$ with $E_i \cap \Sigma_\alpha \neq \emptyset$. $\chi(\Sigma_\alpha)$ and $\chi(\tilde{\Sigma}_\beta)$ denote the Euler characteristic of $\Sigma_\alpha$ and $\tilde{\Sigma}_\beta$.

Proof. At each end $E_i$, $1 \leq i \leq k$, let $\{y_1, y_2, y_3\}$ be a set of coordinates in which $g$ satisfies (3.1). If $f$ is unbounded in $E_i$, we require that $\{y_1, y_2, y_3\}$ be given by Proposition 3.2. For any large $r > 0$, let $S^i_r$ be the coordinate sphere $\{|y| = r\}$ in $E_i$. Let $U_r$ be the region bounded by $S^1_r, \ldots, S^k_r$ in $M$.

By Lemma 3.1 and (3.1), $|f| = O(r)$ and $|\text{Ric}| = O(r^{-2-\tau})$ in each $E_i$. Hence, the integrals in (i) and (ii) exist and are finite. The static equation (1.3) implies

\begin{equation}
|f|\text{Ric}|^2 = \langle \nabla^2 f, \text{Ric} \rangle.
\end{equation}

Integrating (4.2) over $U_r$, and doing integration by parts, we have

\begin{equation}
\int_{U_r} f|\text{Ric}|^2 = \sum_{i=1}^k \int_{S^i} \text{Ric}(\nabla f, \nu)
\end{equation}

where $\nu$ is the unit outward normal to $S^i_r$ and we also have used the fact $g$ has zero scalar curvature. Since $|\nabla f|$ is bounded by Proposition 3.1, $|\text{Ric}| = O(r^{-2-\tau})$, and the area of $S^i_r$ is of order $r^2$, we conclude that (i) holds by letting $r \to \infty$ in (4.3).

To prove (ii), we first choose $r$ sufficiently large so that $\tilde{\Sigma}_\beta \subset U_r, \forall \beta$. If $f$ is unbounded, we assume it is unbounded in the ends $E_1, \ldots, E_l$, $1 \leq l \leq k$, and bounded in the other ends. We then choose $r$ large enough so that outside each $S^i_r$ in $E_i$, $1 \leq i \leq l$, $f^{-1}(0)$ is the graph of some function $q = q(\bar{y})$ given by Proposition 3.2; moreover, by (3.7) we can assume the graph of $q(\bar{y})$ always intersects $S^i_r$ transversally. Hence, the set $U^+_r = U_r \cap \{f > 0\}$ has Lipschitz boundary. Integrating (4.2) over $U^+_r$ gives

\begin{equation}
\int_{U^+_r} f|\text{Ric}|^2 = \int_{U_r \cap (\cup_{i=1}^m \Sigma_\alpha)} \text{Ric}(\nabla f, \nu) + \int_{\cup_{\beta=0}^n \tilde{\Sigma}_\beta} \text{Ric}(\nabla f, \nu) + \int_{\partial U_r \cap \{f > 0\}} \text{Ric}(\nabla f, \nu).
\end{equation}

Here $\nu$ denotes the outward unit normal to $\partial U^+_r$. As in (i),

\begin{equation}
\lim_{r \to \infty} \int_{\partial U_r \cap \{f > 0\}} \text{Ric}(\nabla f, \nu) = 0.
\end{equation}

On each $\tilde{\Sigma}_\beta$ or $\Sigma_\alpha$, by the fact $\nu = -\nabla f |\nabla f|$, we have

\begin{equation}
\text{Ric}(\nabla f, \nu) = -|\nabla f|\text{Ric}(\nu, \nu) = |\nabla f| K,
\end{equation}

where $K$ is the scalar curvature of $M$.
where $K$ is the Gaussian curvature of $\tilde{\Sigma}_\beta$ or $\Sigma_\alpha$ by Lemma 2.1 (iv). Hence,

$$\int_{U^\beta_{i=0}} \text{Ric}(\nabla f, \nu) = 2\pi \sum_{\beta=0}^{n} \tilde{c}_\beta \chi(\tilde{\Sigma}_\beta),$$

by the Gauss-Bonnet theorem, and

$$\int_{U_r \cap (\cup_{\alpha=1}^{m} \Sigma_\alpha)} \text{Ric}(\nabla f, \nu) = \sum_{\alpha=0}^{m} c_\alpha \int_{U_r \cap \Sigma_\alpha} K.$$  

Note that $\Sigma_\alpha$ is totally geodesic, hence (3.1) implies that $|K|$ decays on $\Sigma_\alpha$ in the order of $O(|y|^{-2-\tau})$ in each end $E_i$ with $\Sigma_\alpha \cap E_i \neq \emptyset$. But (3.7) implies that, on $\Sigma_\alpha \cap E_i$, $|y|$ is equivalent to the intrinsic distance function to a fixed point in $\Sigma_\alpha$. Therefore,

$$\int_{\Sigma_\alpha} |K| < \infty.$$

Let $C^i_R$ be the curve in $\Sigma_\alpha \cap E_i$ which is the graph of $q$ over the circle $\{|y| = R\}$ (see the definition of $C^i_R$ in Proposition 3.2). Let $\kappa$ denote the geodesic curvature of $C^i_R$ in $\Sigma_\alpha$. By the Gauss-Bonnet theorem and Proposition 3.2, we have

$$\int_{\Sigma_\alpha} K = \lim_{R \to \infty} \left( 2\pi \chi(\Sigma_\alpha) - \sum_{i \in \Lambda_\alpha} \int_{C^i_R} \kappa \right) = 2\pi \chi(\Sigma_\alpha) - 2\pi k_\alpha,$$

where $\Lambda_\alpha$ is the set of indices $i$ such that $\Sigma_\alpha \cap E_i \neq \emptyset$. It follows from (4.7) – (4.9) that

$$\lim_{r \to \infty} \int_{U_r \cap (\cup_{\alpha=1}^{m} \Sigma_\alpha)} \text{Ric}(\nabla f, \nu) = 2\pi \sum_{\alpha=0}^{m} c_\alpha (\chi(\Sigma_\alpha) - k_\alpha).$$

By (4.4) – (4.6) and (4.10), we conclude that

$$\int_{\{f > 0\}} f |\text{Ric}|^2 = 2\pi \sum_{\alpha=0}^{m} c_\alpha (\chi(\Sigma_\alpha) - k_\alpha) + 2\pi \sum_{\beta=0}^{n} \tilde{c}_\beta \chi(\tilde{\Sigma}_\beta).$$

(ii) now follows from (4.11) and (i).

Remark 4.1. From (3.13) and (3.14), one can show $\lim_{r \to \infty} \frac{A(r)}{r^2} = \pi$, where $A(r)$ is the area of $D(r) \cap E_i$, $i \in \Lambda_\alpha$, for a geodesic ball $D(r)$ with radius $r$ in $\Sigma_\alpha$. Therefore, the fact $\int_{\Sigma_\alpha} K = 2\pi (\chi(\Sigma_\alpha) - k_\alpha)$ also follows from results in [14, 17].
Proposition 4.2 implies that \((M, g)\) must be \((\mathbb{R}^3, g_0)\) if \(M\) has simple topology.

**Theorem 4.2.** Let \((M, g)\) and \(f\) be given as in Proposition 4.2. If \(M\) is orientable and every 2-sphere in \(M\) is the boundary of a bounded domain, then \((M, g)\) is isometric to \((\mathbb{R}^3, g_0)\). In particular, if \((M, g)\) is homeomorphic to \(\mathbb{R}^3\), then \(M\) is isometric to \((\mathbb{R}^3, g_0)\).

**Proof.** Suppose \(\Sigma_{\beta}\) is a compact component of \(f^{-1}(0)\). Since \(M\) is orientable and \(\Sigma_{\beta}\) is two-sided (with a nonzero normal \(\nabla f\)), \(\Sigma_{\beta}\) is orientable. If \(\chi(\Sigma_{\beta}) > 0\), then \(\Sigma_{\beta}\) is a 2-sphere. Hence \(\Sigma_{\beta} = \partial \Omega\) for some bounded domain \(\Omega\) in \(M\) by the assumption. This implies \(f \equiv 0\) in \(\Omega\) by the maximum principal and therefore \(f \equiv 0\) in \(M\) by unique continuation [3]. Thus, \((M, g)\) is flat and is isometric to \((\mathbb{R}^3, g_0)\). Since \((\mathbb{R}^3, g_0)\) does not contain a closed minimal surface, this case does not occur.

Therefore, we may assume \(\chi(\Sigma_{\beta}) \leq 0\) for any compact component of \(f^{-1}(0)\) if such a component exists. On the other hand, if \(\Sigma_{\alpha}\) is a noncompact component of \(f^{-1}(0)\), then \(\chi(\Sigma_{\alpha}) \leq 1\). By Proposition 4.2 (ii), we have

\[
\int_M |f| |\text{Ric}|^2 \leq 0.
\]

This implies \(\text{Ric} \equiv 0\) and therefore \((M, g)\) is isometric to \((\mathbb{R}^3, g_0)\). \(\square\)

In what follows, we replace the topological assumption imposed in Theorem 4.2 by an assumption that \(f\) has no critical points. For this purpose, we analyze the behavior of integral curves of the gradient of a static potential. We formulate the results in a setting similar to that in Proposition 4.1.

**Proposition 4.3.** Let \((M, g)\) be a complete, connected, asymptotically flat 3-manifold with nonempty boundary, with finitely many ends \(E_1, \ldots, E_k\). Suppose there exists a static potential \(f\) with \(f = 0\) on \(\partial M\). Given any point \(p \in \text{Int}(M)\), the interior of \(M\), let \(\gamma_p(t)\) be the integral curve of \(\nabla f\) with \(\gamma_p(0) = p\). Let \((\alpha, \beta)\) be the maximal interval of existence of \(\gamma_p\) inside \(\text{Int}(M)\).

(a) If \(\beta < \infty\), then \(\lim_{t \to \beta} \gamma_p(t) = x\) for some \(x \in \partial M\); if \(\alpha > -\infty\), then \(\lim_{t \to \alpha} \gamma_p(t) = y\) for some \(y \in \partial M\). Consequently, either \(\alpha = -\infty\) or \(\beta = \infty\).

(b) If \(\beta = \infty\), then \(\lim_{t \to \infty} f(\gamma_p(t)) = b > -\infty\). Moreover,

(i) if \(b = \infty\), then, as \(t \to \infty\), \(\gamma_p(t)\) tends to infinity in an end \(E_i\) on which \(f\) is unbounded;
(ii) if \( b < \infty \), then \( b \neq 0 \) and \( \lim_{t \to \infty} |\nabla f(\gamma_p(t))| = 0 \); 

(iii) if \( b < 0 \), then \( \bigcap_{t>0} \{ \gamma_p(s) \mid s > t \} \neq \emptyset \) and consists of critical points of \( f \).

(c) If \( \alpha = -\infty \), then \( \lim_{t \to -\infty} f(\gamma_p(t)) = a < \infty \). Moreover,

(i) if \( a = -\infty \), then, as \( t \to -\infty \), \( \gamma_p(t) \) tends to infinity in an end \( E_i \) on which \( f \) is unbounded;

(ii) if \( a > -\infty \), then \( a \neq 0 \) and \( \lim_{t \to -\infty} |\nabla f(\gamma_p(t))| = 0 \);

(iii) if \( a > 0 \), then \( \bigcap_{t<0} \{ \gamma(s) \mid s < t \} \neq \emptyset \) and consists of critical points of \( f \).

Proof. If \( p \) is a critical point of \( f \), then \( \gamma_p(t) = p \), \( \forall \ t \in (-\infty, \infty) \). Also \( f(p) \neq 0 \) by Lemma \( 2 \) (i). The proposition is obviously true in this case. In the following, we assume \( \nabla f(p) \neq 0 \). Then \( \nabla f(\gamma_p(t)) \neq 0 \) for all \( t \) and

\[
(4.12) \quad \frac{d}{dt}f(\gamma_p(t)) = |\nabla f|^2(\gamma_p(t)) > 0.
\]

By Proposition \( 3.1 \) \( \lim_{x \to \infty} |\nabla f| \) exists and is finite at each end \( E_i \).

Therefore,

\[
(4.13) \quad |\nabla f|(x) < B, \ \forall \ x \in M
\]

for some constant \( B > 0 \). Suppose \( \beta < \infty \), then for \( t_2 > t_1 > 0 \),

\[
d(\gamma_p(t_1), \gamma_p(t_2)) \leq \int_{t_1}^{t_2} |\gamma'_p(s)|ds \leq (t_2 - t_1)B,
\]

where \( d(\cdot, \cdot) \) denotes the distance on \((M, g)\). Hence \( \lim_{t \to \beta} \gamma_p(t) = x \) for some \( x \in M \). Since \((\alpha, \beta)\) is the maximal interval of existence of \( \gamma_p(t) \) in \( \text{Int}(M) \), we conclude \( x \in \partial M \). Similarly, if \( \alpha > -\infty \), then \( \lim_{t \to \beta} \gamma_p(t) = y \), for some \( y \in \partial M \). If \( \alpha > -\infty \) and \( \beta < \infty \), then \( f(x) = 0 = f(y) \), which contradicts \((4.12)\). This proves (a).

To prove (b), we note that \((4.12)\) implies \( f(\gamma_p(t)) \) is increasing, hence \( \lim_{t \to \infty} f(\gamma_p(t)) = b \) exists and \( b > -\infty \). If \( b = \infty \), then there exists \( t_n \to \infty \) such that \( \gamma_p(t_n) \to \infty \) in some end \( E_i \) on which \( f \) is unbounded. Let \( \{t'_n\} \) be any other sequence with \( t'_n \to \infty \). We claim that \( \gamma_p(t'_n) \) must tend to infinity in \( E_i \) as well. Otherwise, passing to subsequence, we may assume that \( \gamma_p(t'_n) \) tends to infinity in another end \( E_j \) with \( j \neq i \). But this implies that, for large \( n \), there exists \( t''_n \) between \( t_n \) and \( t'_n \) such that \( \gamma_p(t''_n) \) lies in a fixed compact set \( K \) of \( M \) (for instance the set \( K \) used in Definition \( 2 \) ). This contradicts the fact \( \lim_{n \to \infty} f(\gamma_p(t''_n)) \to b = \infty \). Therefore, \( \gamma_p(t) \) tends to infinity in \( E_i \) as \( t \to \infty \), which proves (i) in (b).
Next, suppose \( b < \infty \). Let \( \{t_n\} \) be any sequence such that \( t_n \to \infty \). Given any fixed number \( 0 < \delta < \frac{1}{B} \), we have

\[
\int_{t_n - \delta}^{t_n + \delta} |\nabla f|^2 (\gamma_p(t)) dt = f(\gamma_p(t_n + \delta)) - f(\gamma_p(t_n - \delta)) \to 0, \quad n \to \infty.
\]

Hence there exists \( t'_n \in [t_n - \delta, t_n + \delta] \) such that \( |\nabla f| (\gamma_p(t'_n)) \to 0 \). Define \( B_{\gamma_p(t_n)}(1) = \{q \in M \mid d(q, \gamma_p(t_n)) < 1\} \). For large \( n \), (4.13) implies \( |f| < 2|b| + 2B \) on \( B_{\gamma(t_n)} (1) \). This together with the fact \( \nabla^2 f = f \text{Ric} \) and \( (M, g) \) is asymptotically flat implies

\[
(4.14) \quad |\nabla^2 f| \leq C_1
\]

on \( B_{\gamma(t_n)} (1) \) for some constant \( C_1 \) independent on \( n \) and \( \delta \). Now let \( \phi = |\nabla f|^2 \), then \( \nabla \phi \) is dual to the 1-form \( 2\nabla^2 f (\nabla f, \cdot) \). By (4.13) and (4.14), we conclude

\[
|\nabla \phi| \leq C_2
\]

on \( B_{\gamma(t_n)} (1) \) by a constant \( C_2 \) independent on \( n \) and \( \delta \). Note that \( d(\gamma(t_n), \gamma(t'_n)) \leq \delta B < 1 \), we therefore have

\[
\phi(\gamma_p(t_n)) \leq \phi(\gamma_p(t'_n)) + 2\delta BC_2.
\]

Since \( \phi(\gamma_p(t'_n)) \to 0 \) and \( \delta \) can be arbitrarily chosen, we conclude that \( \phi(\gamma_p(t_n)) \to 0 \) as \( n \to \infty \).

We also want to show \( b \neq 0 \). Let \( \{t_n\} \) be given as above. Suppose \( \{\gamma_p(t_n)\} \) is unbounded, then passing to a subsequence we may assume \( \gamma_p(t_n) \to \infty \) in some end \( E_j \). If \( f \) is unbounded in \( E_j \), we would have \( |\nabla f| (\gamma_p(t_n)) \geq C_3 \) for some \( C_3 > 0 \) independent of \( n \) by Proposition 3.1 (i), contradicting to the fact \( |\nabla f| (\gamma_p(t_n)) \to 0 \). Hence, \( f \) is bounded in \( E_j \). By Proposition 3.1 (ii), we have \( b = \lim_{x \to \infty, x \in E_j} f \neq 0 \). Next, suppose \( \{\gamma_p(t_n)\} \) is bounded. Passing to a subsequence, we may assume \( \gamma_p(t_n) = q \in M \). Then \( q \) is a critical point of \( f \) since \( |\nabla f| (\gamma_p(t_n)) \to 0 \). Therefore, \( b = f(q) \neq 0 \) by Lemma 2.1 (i). This completes the proof of (ii) in (b).

To prove (iii) of (b), it is sufficient to prove that if \( b < 0 \) and if \( \{t_n\} \) is a sequence tending to \( \infty \), then \( \{\gamma_p(t_n)\} \) must be bounded, hence containing a subsequence converging to a critical point in \( M \). Suppose \( \{\gamma(t_n)\} \) is unbounded, then passing to a subsequence we may assume \( \gamma(t_n) \to \infty \) in an end \( E_j \) where \( f \) is bounded by the proof in (ii) above. On \( E_j \), Proposition 3.1 (ii) implies

\[
(4.15) \quad f = b - \frac{A}{|x|} + o(|x|^{-1}), \quad |x| \to \infty
\]
where $A$ is a constant such that 

$$\frac{A}{b} = m$$

which is the mass of $(M, g)$ at the end of $E_j$ (cf. \[7\] \[11\]). By the positive mass theorem \[16\] \[19\], we have $m > 0$ (which can be seen by reflecting $(M, g)$ through $\partial M$ since $\partial M$ is totally geodesic). Therefore, $A < 0$ because $b < 0$. As a result, $f(\gamma_p(t_n)) > b$ for large $n$ by \[4.15\]. But this leads to a contradiction to the fact that $b = \lim_{n \to \infty} f(\gamma_p(t_n))$ and $f(\gamma_p(t))$ is strictly increasing in $t$. Therefore, $\{\gamma_p(t_n)\}$ must be bounded. This proves (iii) of (b).

Claim (c) follows from (b) by replacing $f$ by $-f$. \hfill $\Box$

Using Proposition \[4.3\] we obtain an analogue of Proposition \[4.1\] with the assumption $f > 0$ replaced by an assumption $f$ has no critical points.

**Corollary 4.1.** Let $(M, g)$ be a complete, connected, asymptotically flat 3-manifold with nonempty boundary $\partial M$, with finitely many ends. Suppose there exists a static potential $f$ without critical points such that $f = 0$ on $\partial M$. Then $(M, g)$ is isometric to a spatial Schwarzschild manifold with positive mass outside its horizon.

**Proof.** By definition, $\partial M$ is compact. Let $\Sigma$ be a component of $\partial M$. Since $\nabla f \neq 0$ at $\Sigma$ by Lemma \[2.1\] (i), we may assume that $\nabla f$ is inward pointing at $\Sigma$. Consider the map $F : \Sigma \times (0, \infty) \to \text{Int}(M)$ given by $F(x, t) = \gamma_x(t)$ which is the integral curve of $\nabla f$ such that $\gamma_x(0) = x \in \Sigma$. By Proposition \[4.3\] (a), $\gamma_x$ is defined on $[0, \infty)$. The fact $f = 0$ and $\nabla f \neq 0$ at $\Sigma$ implies that $F$ is one-to-one. Hence, by the invariance of domain, the image $N$ of $F$ is open in $\text{Int}(M)$. We want to prove that $N$ is also closed in $\text{Int}(M)$.

Let $y \in \text{Int}(M)$ be a point that lies in the closure of $N$ in $\text{Int}(M)$. Let $\gamma_y(t)$ be the integral curve of $\nabla f$ with $\gamma_y(0) = y$. Then there exist $x_i \in \Sigma$ and $t_i > 0$ such that $\tilde{x}_i = \gamma_{x_i}(t_i)$ converge to $y$. Passing to a subsequence, we may assume that $x_i \to x \in \Sigma$ and $t_i \to a$ with $0 \leq a \leq \infty$. We claim that $a < \infty$. If this is true, we will have $y = \lim_{i \to \infty} \gamma_{x_i}(t_i) = \gamma_x(a) \in N$. Suppose $a = \infty$. Consider the integral curve $\gamma_{x_i}(t) = \gamma_{x_i}(t + t_i)$, which is defined on $(-t_i, 0]$. Since $t_i \to \infty$, $\{\gamma_{\tilde{x}_i}(t)\}$ converge uniformly to $\gamma_y(t)$ on $[-n, 0]$ for any $n > 0$. In particular, $\gamma_y(t)$ is defined on $(-\infty, 0]$. On the other hand, $f(\gamma_{x_i}(t))$ is strictly increasing in $t$ for all $i$. Hence, $f(\gamma_{x_i}(t)) > 0$ on $(-t_i, 0]$, which implies $f(\gamma_y(t)) \geq 0$ on $(-\infty, 0]$. By Proposition \[4.3\] (c), there exists a critical point of $f$ in $M$, contradicting the assumption that $f$ has no critical points.
Therefore, \( N \) is closed in \( \text{Int}(M) \) and hence \( N = \text{Int}(M) \). Since \( f > 0 \) along each \( \gamma_x(t) \) on \((0, \infty)\), we conclude that \( f > 0 \) in \( N = \text{Int}(M) \). Hence, \((M, g)\) is isometric to a spatial Schwarzschild manifold with positive mass outside its horizon by [11] or Proposition 4.1.

Corollary 4.1 implies the following rigidity result for asymptotically flat manifolds without boundary.

**Theorem 4.3.** Let \((M, g)\) be a complete, connected, asymptotically flat 3-manifold without boundary, with finitely many ends. If there exists a static potential \( f \) on \((M, g)\) which has no critical points, then \((M, g)\) is isometric to either \((\mathbb{R}^3, g_0)\) or a spatial Schwarzschild manifold \((\mathbb{R}^3 \setminus \{0\}, (1 + \frac{m}{2|x|})^4 g_0)\) with \( m > 0 \).

**Proof.** If \( f^{-1}(0) \) has no compact component, then \((M, g)\) is isometric to \((\mathbb{R}^3, g_0)\) by Proposition 4.2 (ii) (cf. the proof of Theorem 4.2). Next, suppose \( f^{-1}(0) \) has a compact component \( \Sigma \). Cutting \( M \) along \( \Sigma \), and let \((\tilde{M}, \tilde{g})\) be the metric completion of \((M \setminus \Sigma, g)\). Then either \( \tilde{M} \) has two components whose boundary is isometric to \( \Sigma \), or \( \tilde{M} \) is connected with two boundary components that are isometric to \( \Sigma \). Applying Corollary 4.1 to each component of \((\tilde{M}, \tilde{g})\) shows that \((\tilde{M}, \tilde{g})\) can not be connected, and hence has two components each of which is isometric to a spatial Schwarzschild manifold with positive mass outside its horizon. Since their boundaries are isometric, we conclude that \((M, g)\) itself is isometric to a complete spatial Schwarzschild manifold with positive mass.

**Remark 4.2.** If Corollary 4.1 holds in general (that is without assuming the static potential has no critical points), then it is clear from the proof of Theorem 4.3 that the conclusion of Theorem 4.3 holds in general.

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