Methods for the construction of generators of algebraic curvature tensors

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Dedicated to the memory of Professor Brian G. Wybourne.

Abstract. We demonstrate the use of several tools from Algebraic Combinatorics such as Young tableaux, symmetry operators, the Littlewood-Richardson rule and discrete Fourier transforms of symmetric groups in investigations of algebraic curvature tensors.

In [10, 12, 13] we constructed and investigated generators of algebraic curvature tensors and algebraic covariant derivative curvature tensors. These investigations followed the example of the paper [14] by S.A. Fulling, R.C. King, B.G.Wybourne and C.J. Cummins and applied tools from Algebraic Combinatorics such as Young tableaux, symmetry operators (in particular Young symmetrizers), the Littlewood-Richardson rule, but also discrete Fourier transforms of symmetric groups. The present paper is a short summary of [10, 12, 13] in which we want to demonstrate the use of these methods.

1. The problem

Let \( V \) be a finite dimensional \( \mathbb{K} \)-vector space, \( \mathbb{K} = \mathbb{R}, \mathbb{C} \), and let \( T_r V \) denote the \( \mathbb{K} \)-vector space of covariant tensors of order \( r \) over \( V \).

DEFINITION 1. Algebraic curvature tensors \( \mathfrak{R} \in T_4 V \) and algebraic covariant derivative curvature tensors \( \mathfrak{R}' \in T_5 V \) are tensors of order 4 or 5 whose coordinates satisfy

\[
\begin{align*}
\mathfrak{R}_{ijkl} &= -\mathfrak{R}_{jikl} = \mathfrak{R}_{klij} \\
\mathfrak{R}_{ijkl} + \mathfrak{R}_{iklj} + \mathfrak{R}_{iljk} &= 0 \\
\mathfrak{R}'_{ijklm} &= -\mathfrak{R}'_{jiklm} = \mathfrak{R}'_{klijm} \\
\mathfrak{R}'_{ijklm} + \mathfrak{R}'_{ikljm} + \mathfrak{R}'_{iljkm} &= 0 \\
\mathfrak{R}'_{ijklm} + \mathfrak{R}'_{ijlkm} + \mathfrak{R}'_{ijmkl} &= 0.
\end{align*}
\]

They are tensors which possess the same symmetry properties as the Riemannian curvature tensor \( R_{ijkl} \) and its covariant derivative \( R_{ijkl;m} \) of a Levi-Civita connection \( \nabla \).

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The vector space of algebraic curvature tensors $\mathcal{R} \in \mathcal{T}_4 V$ is spanned by each of the following sets of tensors (P. Gilkey [14] pp.41-44, B. Fiedler [9])

$$\gamma(S)_{ijkl} := S_{il}S_{jk} - S_{ik}S_{jl} ; \quad S \text{ symmetric}$$

$$\alpha(A)_{ijkl} := 2A_{ij}A_{kl} + A_{ik}A_{jl} - A_{il}A_{jk} ; \quad A \text{ skew-symmetric}.$$  

The vector space of algebraic covariant derivative curvature tensors $\mathcal{R}' \in \mathcal{T}_5 V$ is spanned by the following set of tensors (P. Gilkey [16] p.236, B. Fiedler [10])

$$\hat{\gamma}(S, \hat{S})_{ijkl} := S_{il}\hat{S}_{jk} - S_{jl}\hat{S}_{ik} + S_{jk}\hat{S}_{ils} - S_{ik}\hat{S}_{jls} ; \quad S, \hat{S} \text{ symmetric}.$$  

PROBLEM 2. In the present paper we search for generators of algebraic curvature tensors $\mathcal{R}$ or algebraic covariant derivative curvature tensors $\mathcal{R}'$ which can be formed by a suitable symmetry operator from the following types of tensors

$$\mathcal{R} : \quad U \otimes w , \quad U \in \mathcal{T}_3 V , w \in \mathcal{T}_1 V ,$$

$$\mathcal{R}' : \quad U \otimes W , \quad U \in \mathcal{T}_3 V , W \in \mathcal{T}_2 V ,$$

where $W$ and $U$ belong to symmetry classes of $\mathcal{T}_2 V$ and $\mathcal{T}_3 V$ which are defined by minimal right ideals $\mathcal{r} \subset \mathbb{K}[S_2]$ and $\mathcal{t} \subset \mathbb{K}[S_3]$, respectively.

We use Boerner’s definition [11] p.127 of symmetry classes of tensors. An element $a = \sum_{p \in S_r} a(p)p \in \mathbb{K}[S_r]$ of the group ring of the symmetric group $S_r$ can be considered a symmetry operator for covariant tensors $T \in \mathcal{T}_r V$. The action of $a$ on $T$ is defined by:

$$(aT)(v_1, \ldots, v_r) := \sum_{p \in S_r} a(p) T(v_{p(1)}, \ldots, v_{p(r)}) , \quad v_i \in V .$$

DEFINITION 3. Let $\mathcal{r} \subset \mathbb{K}[S_r]$ be a right ideal of $\mathbb{K}[S_r]$ for which an $a \in \mathcal{r}$ and a $T \in \mathcal{T}_r V$ exist such that $aT \neq 0$. Then the tensor set

$$\mathcal{T}_r := \{ aT \mid a \in \mathcal{r} , T \in \mathcal{T}_r V \}$$

is called the symmetry class of tensors defined by $\mathcal{r}$.

Boerner [11] p.127 showed: If $e \in \mathbb{K}[S_r]$ is a generating idempotent of $\mathcal{r}$, i.e. $\mathcal{r} = e \cdot \mathbb{K}[S_r]$, then it holds

$$T \in \mathcal{T}_r V \text{ belongs to } \mathcal{T}_r \Leftrightarrow eT = T .$$

[11] uses also tools of Algebraic Combinatorics, in particular plethysms.
2. Young symmetrizers

Young symmetrizers are important symmetry operators. In particular the symmetries of the Riemann tensor $R$ and its covariant derivatives are characterized by a Young symmetrizer. First we define Young tableaux.

A *Young tableau* $t$ of $r \in \mathbb{N}$ is an arrangement of $r$ boxes such that

1. the numbers $\lambda_i$ of boxes in the rows $i = 1,\ldots,l$ form a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$ with $\lambda_1 + \ldots + \lambda_l = r$,
2. the boxes are fulfilled by the numbers $1, 2,\ldots,r$ in any order.

For instance, the following graphics shows a Young tableau of $r = 16$.

\[
\begin{align*}
\lambda_1 &= 5 & 11 & 2 & 5 & 4 & 12 \\
\lambda_2 &= 4 & 9 & 6 & 16 & 15 \\
\lambda_3 &= 4 & 8 & 14 & 1 & 7 \\
\lambda_4 &= 2 & 13 & 3 \\
\lambda_5 &= 1 & 10 \\
\end{align*}
\]

Obviously, the unfilled arrangement of boxes, the *Young frame*, is characterized by a partition $\lambda = (\lambda_1,\ldots,\lambda_l) \vdash r$ of $r$.

If a Young tableau $t$ of a partition $\lambda \vdash r$ is given, then the *Young symmetrizer* $y_t$ of $t$ is defined by

\[
y_t := \sum_{p \in \mathcal{H}_t} \sum_{q \in \mathcal{V}_t} \text{sign}(q) \ p \circ q
\]  

(8)

where $\mathcal{H}_t, \mathcal{V}_t$ are the groups of the horizontal or vertical permutations of $t$ which only permute numbers within rows or columns of $t$, respectively. The Young symmetrizers of $\mathbb{K}[S_r]$ are essentially idempotent and define decompositions

\[
\mathbb{K}[S_r] = \bigoplus_{\lambda \vdash r} \bigoplus_{t \in ST_\lambda} \mathbb{K}[S_r] \cdot y_t , \quad \mathbb{K}[S_r] = \bigoplus_{\lambda \vdash r} \bigoplus_{t \in ST_\lambda} y_t \cdot \mathbb{K}[S_r]
\]  

(9)

of $\mathbb{K}[S_r]$ into minimal left or right ideals $\mathbb{K}[S_r] \cdot y_t, y_t \cdot \mathbb{K}[S_r]$. In (9), the symbol $ST_\lambda$ denotes the set of all standard tableaux of the partition $\lambda$. Standard tableaux are Young tableaux in which the entries of every row and every column form an increasing number sequence.\(^1\)

The inner sums of (9) are minimal two-sided ideals

\[
\mathfrak{a}_\lambda := \bigoplus_{t \in ST_\lambda} \mathbb{K}[S_r] \cdot y_t = \bigoplus_{t \in ST_\lambda} y_t \cdot \mathbb{K}[S_r]
\]  

(10)

of $\mathbb{K}[S_r]$. The set of all Young symmetrizers $y_t$ which lie in $\mathfrak{a}_\lambda$ is equal to the set of all $y_t$ whose tableau $t$ has the frame $\lambda \vdash r$. Furthermore two minimal left ideals

\[^1\]About Young symmetrizers and Young tableaux see for instance [1, 14, 15, 17, 18, 20, 21].
$l_1, l_2 \subseteq \mathbb{K}[S_r]$ or two minimal right ideals $r_1, r_2 \subseteq \mathbb{K}[S_r]$ are equivalent iff they lie in the same ideal $a_\lambda$. Now we say that a symmetry class $T_r$ belongs to $\lambda \vdash r$ iff $r \subseteq a_\lambda$.

S.A. Fulling, R.C. King, B.G. Wybourne and C.J. Cummins showed in [14] that the symmetry classes of the Riemann tensor $R$ and its symmetrized covariant derivatives

$$\left(\nabla^{(u)} R\right)_{ijkl\ldots s_u} := \nabla_{(s_1, s_2, \ldots, s_u)} R_{ijkl} = R_{ijkl; (s_1\ldots s_u)}$$

(11)

are generated by special Young symmetrizers\(^2\).

**PROPOSITION 4.** (Fulling, King, Wybourne, Cummins)

Let $\nabla$ be the Levi-Civita connection of a pseudo-Riemannian metric $g$. For $u \geq 0$ the symmetrized covariant derivatives $\nabla^{(u)} R$ fulfil

$$e_t^* \nabla^{(u)} R = \nabla^{(u)} R$$

(12)

where $e_t := y_t(u+1)/(2 \cdot (u+3)!)$ is an idempotent which is formed from the Young symmetrizer $y_t$ of the standard tableau

$$t = \begin{array}{cccccc}
1 & 3 & 5 & \ldots & \ldots & (u+4) \\
2 & 4
\end{array}$$

(13)

The $'*$' in (12) is the mapping $*: a = \sum_{p \in S_r} a(p) p \mapsto a^* := \sum_{p \in S_r} a(p) p^{-1}$.

We see from Proposition 4 that the tensor fields $\nabla^{(u)} R$ belong to the symmetry class which is defined by the symmetrizer $y_t^*$ of (13), more precisely, by the right ideal $r = y_t^* \cdot \mathbb{K}[S_{u+4}]$. In the special case of algebraic tensors $\mathcal{R}$, $\mathcal{R}'$ we have the following corollary (see [10]):

**COROLLARY 5.** Let us denote by $t$ and $t'$ the standard tableaux

$$t = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4
\end{array}, \quad t' = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4
\end{array}$$

(14)

Then a tensor $T \in \mathcal{T}_4 V$ or $\tilde{T} \in \mathcal{T}_5 V$ is an algebraic curvature tensor or an algebraic covariant derivative curvature tensor iff these tensors satisfy

$$y_t^* T = 12 T , \quad y_{t'}^* \tilde{T} = 24 \tilde{T} ,$$

(15)

respectively. Thus the symmetry classes of the algebraic tensors $\mathcal{R}$, $\mathcal{R}'$ are defined by the minimal right ideals $y_t^* \cdot \mathbb{K}[S_4]$, $y_{t'}^* \cdot \mathbb{K}[S_5]$ which belong to the partitions $(2 2) \vdash 4$, $(3 2) \vdash 5$.

\(^2\)(\ldots) denotes the symmetrization with respect to the indices $s_1, \ldots, s_u$.

\(^3\)A proof of this result of [14] can be found in [5, Sec.6], too. See also [10] for more details.
3. Symmetry classes of $T_3V$ belonging to $\lambda = (2 1)$

The group ring $\mathbb{K}[S_3]$ contains the minimal 2-sided ideals $a_{(3)}, a_{(2 1)}, a_{(1 3)}$. The 2-sided ideals $a_{(3)}, a_{(1 3)} \subset \mathbb{K}[S_3]$ have dimension $1$ and define consequently unique symmetry classes of $T_3V$. The 2-sided ideal $a_{(2 1)} \subset \mathbb{K}[S_3]$, however, has dimension $4$ and contains an infinite set of minimal right ideals $\hat{r}$ (of dimension $2$) which lead to an infinite set of possible symmetry classes for the tensor $U \in T_3V$.

We use discrete Fourier transforms to determine a generating idempotent for every such $\hat{r}$.

DEFINITION 6. A discrete Fourier transform $D$ for $S_r$ is an isomorphism

$$D : \mathbb{K}[S_r] \rightarrow \bigotimes_{\lambda: r} \mathbb{K}^{d_{\lambda} \times d_{\lambda}}$$

where $\sum_{p \in S_r} a(p) \mapsto \left( \begin{array}{ccc} A_{\lambda_1} & 0 \\ 0 & A_{\lambda_2} \\ \vdots & \vdots \\ 0 & A_{\lambda_k} \end{array} \right)$

according to Wedderburn’s theorem which maps the group ring $\mathbb{K}[S_r]$ onto an outer direct product $\bigotimes_{\lambda: r} \mathbb{K}^{d_{\lambda} \times d_{\lambda}}$ of full matrix rings $\mathbb{K}^{d_{\lambda} \times d_{\lambda}}$.

In (16) the matrix ring $\mathbb{K}^{d_{\lambda} \times d_{\lambda}}$ corresponds to the minimal two-sided ideal $a_{\lambda}$ of $\mathbb{K}[S_r]$. For $S_3$ we have a mapping

$$D : a = \sum_{p \in S_r} a(p) \mapsto \left( \begin{array}{cc} A_{(3)} & 0 \\ 0 & A_{(2 1)} \\ \end{array} \right),$$

where $A_{(3)}$ and $A_{(1 3)}$ are $1 \times 1$-matrices and $A_{(2 1)}$ is a $2 \times 2$-matrix. It holds $a \in a_{(2 1)}$ iff $A_{(3)} = A_{(1 3)} = 0$. In [10] we proved

PROPOSITION 7. Every minimal right ideal $r \subset \mathbb{K}^{2 \times 2}$ is generated by exactly one of the following (primitive) idempotents

$$X_\infty := \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), \quad X_\nu := \left( \begin{array}{cc} 1 & 0 \\ \nu & 0 \end{array} \right), \quad \nu \in \mathbb{K}.$$  \(18\)

From (18) we obtain the generating idempotents for the right ideals $\hat{r} \subset a_{(2 1)} \subset \mathbb{K}[S_3]$ by

$$\xi_\nu = D^{-1} \left( \begin{array}{c} 0 \\ X_\nu \\ 0 \end{array} \right) = \begin{cases} \frac{1}{3} \{[1, 2, 3] - [2, 1, 3] - [2, 3, 1] + [3, 2, 1]\}, & \nu = \infty \\ \frac{1}{3} \{[1, 2, 3] + \nu[1, 3, 2] + (1 - \nu)[2, 1, 3] - \nu[2, 3, 1] + (-1 + \nu)[3, 1, 2] - [3, 2, 1]\}, & \text{else}. \end{cases}$$  \(19\)

1. The dimension of a minimal left or right ideal can be calculated from the Young frame belonging to it by the hook length formula (see e.g. [1], [14], [17], [18].)

2. Three discrete Fourier transforms are known for symmetric groups $S_r$: Young’s natural representation [1] pp.102-108, Young’s seminormal representation [2], [3] p.130, [18] p.76 and Young’s orthogonal representation [2, 3, 18] of $S_r$. 


Note that the above connection between \( \nu \) and \( \xi_\nu \) depends on the concrete discrete Fourier transform \( D \) which is used in (19). The above formula (19) was determined by means of the Mathematica package PERMS \( ^7 \) whose discrete Fourier transform \( D \) is based on Young’s natural representation of \( S_r \).

Examples of tensor fields with a \((21)\)-symmetry can be constructed from covariant derivatives of symmetric or alternating 2-fields.

PROPOSITION 8. Let \( \nabla \) be a torsion-free covariant derivative on a \( C^\infty \)-manifold \( M \), \( \dim M \geq 2 \). Further let \( \psi, \omega \in T^2_M \) be differentiable tensor fields of order 2 which are symmetric or skew-symmetric, respectively. Then for every point \( p \in M \) the tensors

\[
(\nabla \psi - \text{sym}(\nabla \psi))|_p \Rightarrow \nu = 0 , \quad (\nabla \omega - \text{alt}(\nabla \omega))|_p \Rightarrow \nu = 2
\]

lie in \((21)\)-symmetry classes whose generating idempotents \( \xi_\nu \) belong to the above \( \nu \)-values 0 or 2. (’sym’ = symmetrization, ’alt’ = anti-symmetrization.)

In a future paper we will show that tensor fields with a \((21)\)-symmetry also occur in curvature formulas connected with stationary and static space-times.

4. Our main results

Now we formulate our main results which were proved in \([10, 13]\). The next section will give some ideas of the proofs.

THEOREM 9. A solution of Problem 2 can be constructed at most from such products \( ^4 \) or \( ^5 \) whose factors belong to the following symmetry classes:

| product | partitions of the symm. classes |
|---------|---------------------------------|
| \( \mathcal{R} : U \otimes w \) | \( (a) \quad U \to (21) \) |
| \( \mathcal{R}' : U \otimes W \) | \( (a') \quad U \to (3) , W \to (2) \) | \( U \) and \( W \) symmetric \( ^2 \).
| | \( (b') \quad U \to (21) , W \to (2) \) | \( W \) symmetric \( ^2 \).
| | \( (c') \quad U \to (21) , W \to (1^2) \) | \( W \) skew-symmetric \( ^2 \).

The case \( (a') \) of Theorem 9 is realized in formula \( ^3 \). The cases \( (a) \), \( (b') \) and \( (c') \) of Theorem 9 lead to

THEOREM 10. The products \( (a) \), \( (b') \), \( (c') \) lead to generators

\[
y^*_t(U \otimes w) , \quad y^t(U \otimes S) , \quad y^*_t(U \otimes A)
\]

of the spaces of algebraic tensors \( \mathcal{R} \), \( \mathcal{R}' \) if and only if the generating idempotent \( \xi_\nu \) of the symmetry class of \( U \) fulfills

\[
\nu \neq \frac{1}{2} .
\]

Here \( t \) and \( t' \) are the Young tableaux \( ^{14} \).
5. Ideas of the proofs

5.1. Use of the Littlewood-Richardson rule (Proof of Theorem 10)

When we consider a right ideal \( r \) that defines the symmetry class of a product \( T_1 \otimes T_2 \) of tensors of order \( r_1, r_2 \), then we can determine information about the decomposition of \( r \) into minimal right ideals by means of Littlewood-Richardson products (see [20, 14, 6, 8]). Let \( r_1, r_2 \) be the right ideals defining the symmetry classes of \( T_1, T_2 \). We consider the left ideals \( l_i := \tau_i^* \) representation spaces of subrepresentations \( \alpha_i \) of the natural representation of \( S_{r_i} \). Then we have \( r = l^* \) where the left ideal \( l \) is the representation space of the Littlewood-Richardson product \( \alpha_1 \alpha_2 := (\alpha_1 \neq \alpha_2) \uparrow S_{r_1 + r_2} \). (\( \neq \) denotes the outer tensor product and \( \uparrow \) the forming of the induced representation.)

For the tensor products \([4], [5]\) we have to calculate the following Littlewood-Richardson products by means of the Littlewood-Richardson rule\(^1\):

\[
\mathcal{R} : \quad [3][1] \sim [4] + [3][1] \\
[2][1][1] \sim [3][1] + [2][2] + [2][1^2] \\
[1^3][1] \sim [2][1^2] + [1^4]
\]

\[
\mathcal{R}': \quad [3][2] \sim [5] + [3][2] + [4][1] \\
[3][1^2] \sim [4][1] + [3][1^2] \\
[2][1][2] \sim [3][2] + [4][1] + [2^2][1] + [3][1^2] \\
[2][1][1^2] \sim [3][2] + [2^2][1] + [3][1^2] + [2][1^3] \\
[1^3][2] \sim [3][1^2] + [2][1^3] \\
[1^3][1^2] \sim [2^2][1] + [2][1^3] + [1^5]
\]

Only the products \([2][1][1]\) for \( \mathcal{R} \) and \([3][2], [2][1][2], [2][1][1^2]\) for \( \mathcal{R}' \) contain minimal right ideals that belong to the partitions \((22)\) for \( \mathcal{R} \) and \((32)\) for \( \mathcal{R}' \). We will definitely obtain \( y^*_r(U \otimes w) = 0 = y^*_{r'}(U \otimes W) \) if the ideal \( r \) or \( r' \) of \( W \otimes w \) or \( U \otimes W \) does not possess a subideal belonging to \((22)\) or \((32)\), since then \( y^*_r \in a_{(22)}, y^*_{r'} \in a'_{(32)} \) but \( r \cap a_{(22)} = 0, r' \cap a'_{(32)} = 0 \).

5.2. A step of the proof of Theorem 10

Let us consider the example of expressions \( y^*_r(U \otimes S) \) and \( y^*_{r'}(U \otimes A) \). To treat such expressions we form the following group ring elements of \( \mathbb{K}[S_5] \):

\[
\sigma_{\nu, \epsilon} := y^*_r \cdot \xi'_\nu \cdot \zeta''_\epsilon \quad (21)
\]

\[
\zeta''_\epsilon := \text{id} + \epsilon (45), \quad \epsilon \in \{1, -1\} \quad (22)
\]

\[
\xi'_\nu \mapsto \xi'_\nu \in \mathbb{K}[S_5] \quad (23)
\]

Formula \((28)\) denotes the embedding of the group ring elements \( \xi'_\nu \in \mathbb{K}[S_3] \) into \( \mathbb{K}[S_5] \) which is induced by the mapping \( S_3 \rightarrow S_5, [i_1, i_2, i_3] \mapsto [i_1, i_2, i_3, 4, 5] \). The symmetry operator \( \xi'_\nu \cdot \zeta''_\epsilon \) maps arbitrary product tensors \( T'' \otimes T'' \) to products \( U \otimes S \) or \( U \otimes A \). Using our Mathematica package PERMS \([7]\) we verified \( \sigma_{\nu, \epsilon} \neq 0 \Leftrightarrow \nu \neq \frac{1}{2} \). The value \( \nu = \frac{1}{2} \) has to be excluded since \( \sigma_{\nu, \epsilon} = 0 \) and \( \mathcal{R}' = 0 \) in this case.

\(^1\)See [20, 14, 6].
6. Shortest formulas for generators of $R$ and $R'$

In this section we want to construct generators (4), (5) whose coordinate representation has a minimal number of summands. To this end we determine systems of linear identities which are satisfied by the coordinates of all tensors from the symmetry class of $U$. In [6, Sec.III.4.1] we proved

**PROPOSITION 11.** Let $\tau \subset \mathbb{K}[S_\tau]$ be a $d$-dimensional right ideal that defines a symmetry class $T_\tau$ of tensors $T \in T_\tau V$. If a basis $\{h_1, \ldots, h_d\}$ of the left ideal $l = \tau^*$ is known, then every solution $x_p$ of the linear $(d \times r!)$-equation system

$$\sum_{p \in S_\tau} h_i(p) x_p = 0 \quad (i = 1, \ldots, d). \tag{24}$$

yields the coefficients for a linear identity

$$\sum_{p \in S_\tau} x_p T_{\bar{i}(1)i_p(2)i_p(3)} = 0 \tag{25}$$

fulfilled by the coordinates of all $T \in T_\tau$.

For our tensors $U$ the rank of the equation system (24) is equal to $\dim \tau = 2$. The columns of (24) are numbered by the permutations $p \in S_3$.

Now we determine identities (25) for $U$ by the following procedure. We form the system (24) from the idempotent $\xi_\nu$ by the determination of a basis $\{p \cdot \xi_\nu \mid p \in S_3\}$ of $l = \tau^*$. Then for every subset $\mathcal{P} = \{p_1, p_2\} \subset S_3$ we check the determinant $\Delta_{\mathcal{P}}$ of the corresponding $(2 \times 2)$-submatrix of (24). If $\Delta_{\mathcal{P}} \neq 0$, then we determine identities (25) of the special form

$$0 = \sum_{\bar{p} \in S_3 \setminus \mathcal{P}} x_{\bar{p}}(p) U_{i_{\bar{p}(1)}i_{\bar{p}(2)}i_{\bar{p}(3)}} + U_{i_{\bar{p}(1)}i_{\bar{p}(2)}i_{\bar{p}(3)}} \quad (\bar{p} \in S_3 \setminus \mathcal{P}). \tag{26}$$

For instance, the set $\mathcal{P} = \{[1, 2, 3], [1, 3, 2]\}$ leads to the determinant $\Delta_{\mathcal{P}}(\nu) = \frac{1}{\nu^2} (1 - \nu)(1 + \nu)$ which has the roots $\nu_1 = 1$ and $\nu_2 = -1$. For $\nu \notin \{1, -1\}$ we obtain the identities

$$-\frac{\nu^2 - \nu + 1}{\nu^3 - 1} U_{ijk} + \frac{2\nu - 1}{\nu^2 - 1} U_{ikj} + U_{kji} = 0 \quad U_{kji} = 0 \tag{27}$$

There exist 15 subsets $\mathcal{P} = \{p_1, p_2\} \subset S_3$ and consequently 15 systems (27) for $U$.

\footnote{Faster algorithms which determine a basis also for a large $S_\tau$ by means of discrete Fourier transforms were developed in [1].}
THEOREM 12. Let \( \frac{1}{24}(y^*_\nu(U \otimes S))_{ijklr} \) of generators for \( \mathfrak{H} \). If we use \([27]\) to express all coordinates of \( U \) by \( U_{ijk} \) and \( U_{ikj} \) we obtain the following sum of 16 terms.

\[
\begin{align*}
-\frac{1}{24}(-1+\nu)(1+\nu) & \quad U_{jlr}S_{ik} + \quad \nu \frac{1}{24}(-1+\nu)(1+\nu) \quad U_{jl}S_{ik} + \\
-\frac{1}{24}(-1+\nu)(1+\nu) & \quad U_{jkl}S_{ir} - \quad \nu \frac{1}{24}(-1+\nu)(1+\nu) \quad U_{jkl}S_{ir} + \\
-\frac{1}{24}(-1+\nu)(1+\nu) & \quad U_{il}S_{jr} - \quad \nu \frac{1}{24}(-1+\nu)(1+\nu) \quad U_{il}S_{jr} + \\
-\frac{1}{24}(-1+\nu)(1+\nu) & \quad U_{ik}S_{jr} + \quad \nu \frac{1}{24}(-1+\nu)(1+\nu) \quad U_{ik}S_{jr} + \\
-\frac{1}{24}(-1+\nu)(1+\nu) & \quad U_{ijkl}{S_{kr}} + \quad \nu \frac{1}{24}(-1+\nu)(1+\nu) \quad U_{ijkl}{S_{kr}} + \\
-\frac{1}{24}(-1+\nu)(1+\nu) & \quad U_{ij}{S_{klr}} + \quad \nu \frac{1}{24}(-1+\nu)(1+\nu) \quad U_{ij}{S_{klr}} + \\
\end{align*}
\]

This sum has the structure

\[
\mathfrak{p}_{\text{red}}^{t_1...t_5} = \sum_{q \in S_5} \frac{P_q^P(\nu)}{Q_q^P(\nu)} U_{i_{q(1)}i_{q(2)}i_{q(3)}}S_{i_{q(4)}i_{q(5)}} .
\] (28)

where \( P_q^P(\nu) \) and \( Q_q^P(\nu) \) are polynomials. If we determine the set \( N_\mathcal{P} \) of all roots \( \nu \neq \frac{1}{2} \) of the \( P_q^P(\nu) \), for which \( \Delta_\mathcal{P}(\nu) \neq 0 \), and set the \( \nu \in N_\mathcal{P} \) into (28), the length of (28) will decrease.

We determine the minimal length of (28) by this procedure for \( y^*_\nu(U \otimes w) \), \( y^*_\nu(U \otimes S) \), \( y^*_\nu(U \otimes A) \) and for every of the 15 identity systems of type \( [27] \) in the case \( \nu \neq \infty \). Table 1 shows the results for \( y^*_\nu(U \otimes S) \), \( y^*_\nu(U \otimes A) \). (In the column for \( \mathcal{P} = \{p_1, p_2\} \) the \( p_i \) are denoted by their numbers in the lexicographically ordered \( S_3 \).) Furthermore we calculate the lengths of (28) for the 15 systems \([27]\) in the case \( \nu = \infty \). Altogether, the number of calculations comes to (3 generator types) \( \times \) (2 \( \nu \)-cases) \( \times \) (15 systems \([27]\)) = 75.

We obtain (see \([12, 13]\))

**THEOREM 12.** Let \( \dim V \geq 3 \). Then the coordinates of \( y^*_\nu(U \otimes w) \), \( y^*_\nu(U \otimes S) \), \( y^*_\nu(U \otimes A) \) are sums of the following lengths

| \( \nu \) | \( y^*_\nu(U \otimes w) \) | \( y^*_\nu(U \otimes S) \) | \( y^*_\nu(U \otimes A) \) |
|---|---|---|---|
| (a)-generic case for \( \nu \)| 8 | 16 | 20 |
| (b)- \( \nu \) producing minimal length | 4 | 12 | 10 |

The computer calculations were carried out by means of the Mathematica packages Ricci \([19]\) and PERMS \([7]\). Notebooks of the calculations are available on the web page \([4]\).

It is very remarkable that \( U \) admits an index commutation symmetry if the coordinates of \( y^*_\nu(U \otimes w) \), \( y^*_\nu(U \otimes S) \), \( y^*_\nu(U \otimes A) \) have the minimal lengths of case (b) in Theorem \([12]\) (see \([12, 13]\)).
| $\mathcal{P}$ | $y_t^\ast(U \otimes S)$ roots of $P_\mathcal{T}(\nu)$ with $\Delta_{\mathcal{T}}(\nu) \neq 0, \nu \neq 1/2$ | length of $|\mathscr{P}_{11,15}^{\text{red}}|$ | $y_t^\ast(U \otimes A)$ roots of $P_\mathcal{T}(\nu)$ with $\Delta_{\mathcal{T}}(\nu) \neq 0, \nu \neq 1/2$ | length of $|\mathscr{P}_{11,15}^{\text{red}}|$ |
|---|---|---|---|---|
| 12 | 0 | 12 | \(-1\) | 12 |
| 13 | -1 | 14 | -1 | 18 |
| 14 | -1 | 14 | -1 | 18 |
| 15 | -1 | 12 | -1 | 10 |
| 16 | -1 | 12 | -1 | 10 |
| 23 | -1 | 14 | -1 | 18 |
| 24 | -1 | 14 | -1 | 18 |
| 25 | -1 | 12 | -1 | 10 |
| 26 | -1 | 12 | -1 | 10 |
| 34 | 0 | 12 | 2 | 12 |
| 35 | -1 | 14 | -1 | 12 |
| 36 | -1 | 14 | -1 | 12 |
| 45 | -1 | 14 | -1 | 12 |
| 46 | -1 | 14 | -1 | 12 |
| 56 | 2 | 14 | 2 | 18 |

**TABLE 1.** The lengths of $|\mathscr{P}_{11,15}^{\text{red}}|$ for $y_t^\ast(U \otimes S), y_t^\ast(U \otimes A)$ and $\nu \neq \infty$.  

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