Modular Hamiltonian for (holographic) excited states

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Abstract

In this work we study the Tomita-Takesaki construction for a family of excited states that, in a strongly coupled CFT - at large $N$-, correspond to coherent states in an asymptotically AdS spacetime geometry. We compute the modular flow and modular Hamiltonian associated to these excited states in the Rindler wedge and for a ball shaped entangling surface. Using holography, one can compute the bulk modular flow and construct the Tomita-Takesaki theory for these cases. We also discuss generalizations of the entanglement regions in the bulk and how to estimate the modular Hamiltonian in a large $N$ approximation. Finally we present a holographic formula, based on the BDHM prescription, to compute the modular evolution of operators in the corresponding CFT algebra.

Contents

1 Introduction 2

2 On holographic states 3

2.1 Definition and geometric construction 3

2.2 Relation to coherent states 6

3 Modular Hamiltonian for excited states in CFT 8

3.1 Results in Conformal Field Theory for equipartite subsystems 8

3.2 Excited states in a Ball 12

3.3 On the Tomita-Takesaki theorem 15

4 The gravity dual of the modular Hamiltonians (at large $N$) 16

4.1 Expected results 17

4.2 Tomita - Takesaki formalism in the bulk 18

4.3 Computing the bulk modular Hamiltonian at large $N$ 21

5 Summary and conclusions 24

A TFD Basics 25

B Remarks on the TT construction for excited states in equipartite aAdS 27
1 Introduction

Modular Hamiltonians (also called entanglement Hamiltonians in the condensed matter community) are unbounded and hermitian operators properly defined in the context of Axiomatic Quantum Field theories (see [1, 2]) and in particular in the framework of the Tomita-Takesaki (TT) theorem [3]. Despite this it is useful in different areas of Physics as Quantum Information, Quantum Field Theory and in theories with gravity as well.

Given a theory on a spacetime $\mathcal{M}$ on a state defined through its density matrix $\rho$ we can define the reduced density matrix, $\rho_A$, on a subsystem $A \in \mathcal{M}$ as the partial trace on the complement of $A$ (denoted by $\bar{A}$). By definition this object is semi-definite positive and Hermitian and then can be always written as

$$\rho_A = \text{Tr}_{\bar{A}} \rho = e^{-K_A} \text{Tr} e^{-K_A},$$

(1.1)

where $K_A$ is the modular Hamiltonian. The denominator ensures $\text{Tr} \rho_A = 1$ but will not play an important role in the rest of the work.

In the condensed matter literature the spectrum of modular Hamiltonians is important because they have relevant information to characterize and identify topological states of matter. For example in [4] it was applied to Fractional Quantum Hall states and more recently it was used to analyze topological non-hermitian systems [5]. In Quantum Field Theory is a central object when one is interested in compute for example the relative entropy (which gives a measure of distinguishability) between two states (related works are [6]) or the capacity of entanglement [7] but as the modular Hamiltonian is typically non-local it is not easy to compute it. Despite of this some results were obtained when the subsystem is the Rindler wedge [8] (a very strong result that holds for any QFT on the vaccum state), Conformal Field Theories on a spherical entangling surface [9] and in time dependent situations after a quantum quench [10]. Out of conformal field theories some new analytical results were founded recently for free theories and many intervals on Minkoski spacetime [11] and on the torus [12]. Lastly, it is important to mention that due to the AdS/CFT correspondece [13, 14, 15] the study of the modular Hamiltonian was also useful for example to clarify the meaning of the Bekenstein bound [16] and is involved in the Averaged Null Energy and Quantum Null Energy Conditions [17] and bulk reconstruction [18]. Also the gravity dual of the modular Hamiltonian was studied in [19, 20].

In the present work we will study the modular Hamiltonian and its modular flow in the context on Conformal field theories that have an holographic dual for equipartite Hilbert spaces and on excited states. Some related works in the field theory context are [21, 22, 23] and from the axiomatic side [24]. In the present work we will study a particular set of excited states that we will call holographic states. They have the advantage that its precise holographic dual is known [25, 26] and that on the large $N$ limit in the gravity side are coherent. Due to this important property these states were extensively studied in different setups [27, 28, 29, 30] and extended to finite temperature cases [31, 32]. To be concrete, these excited states can be written as (notation will be made explicit is Sec. 2)

$$|\Psi_{\lambda}\rangle \equiv \mathcal{D} e^{-\int_{\tau<0} d\tau \phi(\tau) \cdot \lambda(\tau)} |0\rangle \quad \Leftrightarrow \quad \langle \phi_{\Sigma} |\Psi_{\lambda}\rangle \equiv \int \mathcal{D}_{\phi_{\Sigma};\Lambda} \Phi e^{-S_{\Phi}|\Phi|}$$

(1.2)

where the object on the left is an excited state on the CFT, and the object on the right is its wave function in the holographic dual.

In section 2 we will review how the states (1.2) are constructed and some of its properties. In section 3 we will study the modular Hamiltonian in the CFT context for the case of an equipartite system and any other that can be obtained from it through a conformal map (in particular we will show the example of the spherical entangling region using the Casini-Huerta-Myers map [9]). Moreover, we will show that in the $\lambda$ independent of $\tau$ case our result can be also framed in the TT formalism. Lastly, in section 4 we try to solve the problem for general $\lambda(\tau)$, in such a case we use the fact that in the large $N$ approximation the states on the gravity side (r.h.s in (1.2)) are coherent sates and using the Thermofield Double (TFD) formalism we showed that the TT theory in the bulk is satisfied by them (see also [33, 34]). In this way we showed that the QFT modular Hamiltonian can be identified with the Tomita modular operator in the strong coupling limit of the CFT. Moreover we compute the modular flow associated with the modular Hamiltonian. In section 5 we summarize our results and discuss some open problems.
Upon the completion of this work, we became aware of [35] where modular Hamiltonians and modular flows for these states are also studied with a perturbative technique.

2 On holographic states

On this section we will summarise some known results that will be on the basis of the present work. We will mention the particular states that we are going to consider and how them can be constructed in the Schwinger-Keldish formalism. These states were studied in many different contexts [25, 28, 30, 32, 36].

2.1 Definition and geometric construction

Consider a CFT defined by a conformally invariant action, \( S_C \), on a spacetime \( (R^{d+1}, \eta_{\mu\nu}) \). In the interaction picture the states defined in [25] have the form

\[
|\Psi_\lambda\rangle \equiv \mathcal{D} e^{-\int_{\text{reg}} d\tau \mathcal{O}(\tau,x^i) \cdot \lambda(\tau,x^i)} |\Psi_0\rangle.
\]  

(2.1)

Here, \( \tau \) represents the Euclidean time, and \( \mathcal{D} \) denotes integration on the spatial coordinates on a certain Cauchy surface. The function \( \lambda(\tau,x^i) \) parameterizes a family of states, and can be arbitrarily chosen on the asymptotic boundary of (a half of) an Euclidean aAdS spacetime (called \( E^- \)). The object \( \mathcal{D} \) is an operator \(^1\) of the theory and the operation denoted by \( \mathcal{D} \) is a path ordering. Along the work we will omit the \( x^i \) dependence of \( \mathcal{D} \) and \( \lambda \) in sake of clarity. By extending the source \( \lambda^\ast(\tau) \equiv \lambda(-\tau) \) to the region \( 0 < \tau < \infty \), one obtains the corresponding "bra"

\[
\langle \Psi_\lambda | \equiv \langle \Psi_0 | \mathcal{D} e^{-\int_{\text{reg}} d\tau \mathcal{O}(\tau,x^i) \cdot \lambda^\ast(\tau,x^i)}
\]  

(2.2)

According to the Hartle-Hawking construction, if \( \lambda \) is independent of \( \tau \) the states (2.1) can also be thought of as the ground state of a deformed Hamiltonian [29]

\[
H = H_0 + \int dx \mathcal{O}(x,0) \lambda(x)
\]  

(2.3)

if the source \( \lambda = \lambda(\tau,x) \) is vanishing when \( \tau \to 0 \) and decays to zero as \( \tau \to -\infty \), the state (2.1) can be interpreted as the (Wick rotated) time evolution of the fundamental state from the far past. This process provides excited states of the original Hilbert space [37].

The reason why these states are specially interesting is because of holography. In fact, there is a precise (non-perturbative) prescription for the states (2.1) and it corresponding bra (2.2) in the dual bulk theory:

\[
\Psi_\lambda |\Phi_\Sigma\rangle = \langle\Phi_\Sigma |\Psi_\lambda\rangle = \int_{\Phi_\Sigma = \lambda \cdot \Phi_0 \equiv \phi_\Sigma} [D\Phi] e^{-S(\Phi)}.
\]  

(2.4)

Here \( \partial E^- \) is an euclidean spacetime whose metric is locally AdS on the asymptotic boundary \( \partial E^- \), \( \Sigma \) is a space-like surface over which the state is defined, and \( S(\Phi) \) stands for the action for a bulk scalar field \( \Phi \), said the dual of a scalar (primary) operator \( \mathcal{O} \) in the CFT theory. Implicitly this action contains an Einstein-Hilbert term proportional to \( G_N^{-1} \sim N^2 \) that will contribute to the Ryu-Takayanagi area term, see Sec. 4.3.2. In the present study we will focus on the subleading \( (\propto C_N^0) \) terms, coming from the matter sector and backreaction, that shall describe the quantum corrections to the (bulk) modular Hamiltonian and entanglement entropies [38]. The expression (2.4) can be derived from the Skenderis - van Rees (SvR from now on) proposal [25, 39, 40], and generalizes the Hartle-Hawking wave functional of the gravitational vacuum to excited states. In what follows we will refer to these CFT states (2.1) simply as holographic states. We want to stress that there is a simple holographic dictionary that characterize them: Deformations of the CFT action on the euclidean times \( \tau < 0 \) correspond to deformations of the (Dirichlet) boundary conditions for the bulk fields on the Euclidean section of the dual aAdS spacetime. Notice that this rule captures the holographic correspondence between the Hartle-Hawking vacua of both (CFT and gravitational) theories in absence of sources (\( \lambda \to 0 \)). And as a by-product, in the large \( N \) limit, as gravity becomes semiclassical, the vacuum is given by a unique (classical) Euclidean aAdS spacetime which corresponds to the CFT vacuum state \( |\Psi_0\rangle \).

\(^1\)Originally in [25] only single trace operators were considered. Recently, [56] studied the effect of multi-trace deformations.
Figure 1: A depiction of the computation of \( \langle \phi^+ | \rho_A | \phi^- \rangle \) in (2.5) is shown. In the left it is shown the gluing of two states (2.1) and (2.2) by tracing over \( \bar{\Sigma}_A \), shown in grey, the complement of \( \Sigma_A \in \Sigma \). The CFT external source \( \lambda \) provides boundary conditions for the path integral in the bulk. The blue line corresponds to the extremal surface \( \gamma_A \). On the right, a case where a \( \zeta \) Killing vector that runs as an angle from \( \phi^- \) to \( \phi^+ \) pivoting around the blue point is shown. All but radial and euclidean time coordinates have been suppressed.

This prescription allows to obtain, at least formally, the holographic dual of the modular flow for arbitrary holographic states and arbitrary regions \( A \) of a Cauchy slice \( t = 0 \) of the boundary spacetime \( \partial \mathcal{E} \).

Something that will be important later is that with these kind of states we can study density matrices and partially reduced density matrices in a clean way. For instance, consider excitations (2.1) on pure AdS, one defines

\[
\rho_A[\phi^+, \phi^-] \equiv \langle \phi^+ | \rho_A | \phi^- \rangle = \int_{\lambda, \phi^\pm} \mathcal{D}\Phi e^{-S[\Phi]},
\]

(2.5)

where \( \phi^\pm \equiv \phi|_{\Sigma_A} \), represents two different data on the surface \( \Sigma_A \). Additionally, we have to impose the asymptotic conditions \( \Phi|_{\partial\mathcal{E}} = \lambda, \Phi|_{\partial\mathcal{E}^+} = \lambda^* \), and vanishing data for all the other bulk fields (including the graviton). This expression is straightforwardly obtained by gluing two halves of Euclidean AdS on the shaded regions of figure 1, which represents the operation \( \text{Tr}_{\bar{\Sigma}_A} |\Psi_A\rangle \langle \Psi_A| \) in a path integral description.

The Ryu-Takayanagi prescription (RT) determines the entangling surface in the bulk \( \gamma_A \), so as the spatial region \( \Sigma_A \) delimited by \( \gamma_A \cup A \). In this sense, this formula should be interpreted perturbatively at the Newton constant \( G_N \), in order to that the surface \( \gamma_A \) (and \( \Sigma_A \)) remains unchanged by the back-reaction effects to each order. In Sec 4 we will see that this prescription allows to obtain, at least formally, the holographic dual of the modular Hamiltonian (and its corresponding modular flow) for holographic states and arbitrary regions \( A \) of a Cauchy slice \( t = 0 \) of the boundary spacetime \( \partial \mathcal{E} \).

2.1.1 Open SK paths and In-Out formalism

Here we want to do more explicit the derivation of (2.1) and it relation with the SK contour. The SvR holographic prescription [39, 40] can be summarized in the following formula

\[
Z_{\text{CFT}}[\lambda(\mathcal{C})] = Z_{\text{grav}}[\Phi|_{\partial\mathcal{E}} = \lambda(\mathcal{C})]
\]

(2.6)

where the lhs is the generating function for correlation functions of CFT operators \( \Theta \) with the sources \( \lambda(\mathcal{C}) \) having support on a specific continuous path \( \mathcal{C} \) in the complex \( t \)-plane. The rhs is the partition function for the bulk field \( \Phi \), dual to \( \Theta \), on an aAdS spacetime with asymptotic boundary conditions \( \lambda(\mathcal{C}) \). This is a general path integral expression that applies to all contours \( \mathcal{C} \), concomitantly the dual spacetime combines both signatures, and in particular reduces to the purely Euclidean set up [14, 15] as the real-time intervals are removed, or Wick-rotated.

In the so-called in/out formalism, studied in [39, 40], the authors considered open contours, let us refer to these as \( \mathcal{C}_O \). The curve \( \mathcal{C}_O \equiv \{ t - i\tau \in \mathbb{C} \} = I_+ \cup I_L \cup I_- \) in the time complex plane is divided in three
Figure 2: An example of the SvR construction for the In-Out path. On the left, an open SK path in the complex time plane is shown, containing segments of definite signature. On the right, the holographic interpretation of the path is shown, built by gluing lorentzian and euclidean aAdS geometries with signatures matching the ones of the path.

pieces as depicted in figure 2(a), and the path ordering $\mathcal{P}$ follows the arrowed lines in each subset. Then, the prescription above takes the more explicit form

$$\langle \Psi_0 | U | \Psi_0 \rangle = \int_{\Phi|\equiv\lambda(\epsilon_O)} |\mathcal{P}| e^{iS(\Phi)} , \tag{2.7}$$

where $U$ is the evolution operator, given by the CFT Hamiltonian deformed with a single-trace operator $\mathcal{O} \equiv \mathcal{O}(x)$ multiplied by an arbitrary time dependent source $\lambda(x, \tau)$

$$U \equiv \mathcal{P} e^{-i \int_{\epsilon_O} d \theta \left( H + \mathcal{O}. \lambda(\theta) \right)} \tag{2.8}$$

and the in/out state $|\Psi_0 \rangle$ is the CFT vacuum, both expressed in the Schrödinger picture. The curve $\epsilon_O$ is parameterized so that $d \theta = -i d \tau$ on $I_-$, and $d \theta = d \tau$ on $I_+$. In its original form, the above proposal was studied in the semi-classical limit of the gravitational side, that corresponds to the large N limit in the standard AdS/CFT example [39]:

$$Z_{CFT} [\lambda(\epsilon_O)] = \langle \Psi_0 | \mathcal{P} e^{-i \int_{\epsilon_O} d \tau (H+\mathcal{O}. \lambda(\tau))} | \Psi_0 \rangle \approx e^{-S_0[\lambda(\epsilon_O)]} \tag{2.9}$$

with the boundary conditions $\lambda_{I_-} = \Phi_{I_-} \equiv 0$, although, it was also claimed that by imposing non vanishing asymptotic boundary conditions in the Euclidean regions $\phi_{I,F} \neq 0$, this formula should generalize to account for excited in/out states.

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This statement was explicitly verified in [25], splitting (2.8) as $U = U_I U_L U_F$ one gets the explicit formula for the holographic excitations

$$|\Psi_{\lambda} \rangle = U_\lambda |\Psi_0 \rangle = \mathcal{P} e^{-i \int_{\epsilon_O} d \tau (H+\mathcal{O}. \lambda(\tau))} |\Psi_0 \rangle \tag{2.10}$$

which become parametrized by the arbitrary source $\lambda(x, \tau)$ with compact support on the interval $\tau \in (-\infty, 0)$. In the interaction picture this state can be written as (2.1).

The correspondig duals ("bra") of these kets, are built by taking the Hermitian conjugate of the Euclidean evolution operator: $U_\lambda \rightarrow U_\lambda^\dagger \equiv U_{\lambda^*}$ in (2.10).

This operation defines the source $\lambda^* \equiv \lambda^*(x, -\tau)$ on the interval $(0, \infty)$ ($\tau \in (-\infty, 0)$).

Thus, in the Schrödinger picture it reads

$$\langle \Psi_{\lambda} | = \langle \Psi_0 | \mathcal{P} e^{-i \int_{\epsilon_O} d \tau (H+\mathcal{O}^\dagger(\tau). \lambda^*(\tau))} . \tag{2.11}$$
2.2 Relation to coherent states

It has been stressed that states of this form are holographic in the sense that correspond to well defined geometric duals [27, 36, 41]. One of the most interesting features of (2.10) is that, by canonically quantizing a (nearly) free non-backreacting field \( \Phi \) in the bulk, these states become coherent in the large \( N \) Hilbert space

\[
|\Psi_A\rangle \propto e^{\int d^k \lambda_a a^*_a}|\Psi_0\rangle,
\]

where \( a_k (a^*_k) \) are the annihilation (creation) operators associated to the canonically quantized bulk field \( \hat{\Phi} \) and \( \lambda_k \) are eigenvalues of \( a_k \), given by the Laplace transform of the Euclidean sources. This result can be achieved by using the so-called Banks, Douglas, Horowitz and Matinec (BDHM) prescription that relates the CFT with bulk field operators [42], i.e, the operators \( \Theta (2.10) \) are linearly expanded in terms of \( a_k, a^*_k \).

One can generalize this formalism to other more complex SK paths as we will see below, but the case presented already captures the essential aspects of this family of excited states. For example, by working in the Thermo Field Double (TFD) formalism one can extend the definition of these states also to closed SK paths. This was done in [31, 32] and in what follows we will use some of their results. One can relate these states with other mechanisms to build excited states in a CFT. Starting from (2.1) or (2.10), one can expand all the operator insertions around the origin of spacetime to get a (very complicated but) local operator acting on the vacuum, which more canonically defines an excited state in the CFT context, see [43]. The interested reader can see [27] for details.

When one considers a strongly coupled CFT in the approximation \( N \gg 1 \), or when the source \( \lambda \) is controlled by a dimensionless small parameter, the field \( \Phi \)-dual to the operator \( \Theta \) is nearly free, and holographic states become coherent in the bulk Hilbert space. This property has been explicitly proved in pure AdS [25] and Eternal black holes [32] and it has been also conjectured [36] that all the CFT states with a good semi-classical dual should have precisely the form (2.1). Conversely, it was shown that such states in fact have a good gravitational dual [27]. An immediate application of this property is that, in the free field approximation, these states work as generating states since by simple derivation with respect to the normal modes components \( \lambda_w \) we can obtain the expansion in a Fock basis. Although coherent states are an overcomplete basis, they are associated to (generate) a complete orthogonal basis of the Fock space. Schematically,

\[
|\Psi_A\rangle \equiv \prod_w e^{-\frac{\lambda_w^2}{2}} \sum_n \frac{(\lambda_w)^n}{\sqrt{n!}} (a^*_n)^n|\Psi_0\rangle.
\]

The product is on the (positive) frequencies \( w \) of the normal-modes, and \( \lambda_w \) are the components in the basis of functions on the Euclidean boundary induced from the normal-modes in the limit \( z \to 0 \), for instance in pure AdS, \( \lambda_w \) is the Laplace transform of the source in (2.1). The remarkable aspect of this expression is that expresses the holographic state as a linear combination of operators on the ground states, that according to the BDHM dictionary [42], can be translated to the CFT basis of states, \( (\Theta_w)^n |0\rangle \), where \( \Theta_w \) are nothing but the (normal) frequency components of the local primaries \( \Theta (\tau) \) on the Euclidean time \( \tau < 0 \) [25, 28, 32].

As said before, the SvR prescription is a generalization of the GKPW formula to combinations of both, Euclidean and Lorentzian regions of the spacetime, that interprets the euclidean pieces as states of the bulk theory, and the BDHM prescription is the rule to assign CFT operators to the operators of the bulk theory. Using both recipes, it has been shown that at large \( N \) the \( \lambda \)-states correspond to coherent excitations of a theory on a aAdS vacuum [25, 32], and the same construction relates this to an initial data \( f_A \) on the spacelike surface at \( t = i t = 0 \) through the relation:

\[
f_A(x) \equiv \langle \Psi_A|\hat{\Phi}(x, t = 0)|\Psi_A\rangle, \quad \forall x \in \Sigma,
\]

and for any other surface \( \Sigma_{t>0} \) of a foliation of the (aAdS) spacetime we have,

\[
\langle \Psi_A|\hat{\Phi}(x, t)|\Psi_A\rangle = f_A(x, t), \quad \forall x \in \Sigma_t,
\]

Notice that in the limit in which the Euclidean sources are removed (\( \lambda = 0 \)), we have \( f_0(x, t = 0) = f_0(x, \tau = 0) = 0 \). This implies that \( f_0(x, t) \) is a solution that vanishes asymptotically and that the operator \( \hat{\Phi}(x, t) \) shall be a linear combination of normalizable modes. So \( f_A(x, t) \) is the classical solution of the field equations.
Figure 3: In the figure, the holographic excited states construction is represented. In (a) a deformation \( \lambda \partial \) over a finite period of Euclidean time evolution (running rightward) is considered. The deformation turns off adiabatically before \( \Sigma \) so the result is the excited state (2.10) of the original theory. The holographic dual of the construction is presented in (b). The deformation now affects the classical evolution of the fields inside AdS, creating the excited state (2.4) in the bulk. The blue dot in the boundary represents the excitation on \( \Sigma \) drawn in (a).

with vanishing Dirichlet conditions on the asymptotic boundary and initial conditions (2.14). In fig. 3 a pictorial representation and summary of the excited state construction is presented.

The standard way of defining an excited state in QFT is by giving an initial data \( (f_\lambda, \partial_t f_\lambda) \) on \( \Sigma \), and these formulas connect this with giving sources \( \lambda(x, \tau) \) on the past of the euclidean asymptotic boundary \( \partial E^- \). Below we show this connection explicitly for a massive free scalar field in the large \( N \) approximation.

In order to do so, we start by writing the bulk vacuum wave-function,

\[
\langle \phi_\Sigma | \Psi_0 \rangle \equiv \int D\phi_\Sigma;0 \Phi e^{-S_E[\Phi]}, \quad S_E[\Phi] = \frac{1}{2} \int \sqrt{g} \partial \mu \Phi \partial^\mu \Phi + m^2 \Phi^2,
\]

which is just (2.4) with \( \lambda = 0 \). As before, the notation \( D\phi_\Sigma;0 \) means that we are integrating on configurations of the fields such that meet \( \phi_\Sigma \) on the Cauchy slice \( \Sigma \) and are null on the asymptotic boundary. The wavefunction on the excited state is (2.4), i.e.

\[
\langle \phi_\Sigma | \Psi_\lambda \rangle \equiv \int D\phi_{\Sigma;\lambda} \Phi e^{-S_E[\Phi]}.
\]

We now split the field as \( \Phi \to \Phi + f \), where the field \( f \) satisfies the equation of motion with the asymptotic boundary conditions corresponding to \( \lambda \). Note that the measure is invariant up to the boundary conditions, \( D\phi_{\Sigma;\lambda} \Phi \to D\phi_{\Sigma-f_\lambda;0} \Phi \), but the action is modified as

\[
S_E[\Phi] \to S_E[\Phi] + S_E[f] + \int_\partial \Phi \partial_n f + \int_\Sigma \Phi \partial_t f,
\]

where the bulk terms mixing \( \Phi \) and \( f \) are vanishing on shell by \( f \)'s EOMs. The third term above is the asymptotic boundary piece of the on shell action, which vanishes identically since \( \Phi_\partial = 0 \), whilst the fourth term does not vanish but comes out of the path integral, since both \( \Phi \) and \( \partial_t f \) are completely determined.

\[\text{For example on flat spacetime, the coherent excitation can be alternatively expressed as an unitary operator on the vacuum written in terms of operators on the Cauchy slice [44].}\]
on Σ and do not fluctuate. Thus, we get

\[ \langle \phi_\Sigma | \Psi_\lambda \rangle = \int \mathcal{D} \phi_{\Sigma,1} \Phi e^{-S_\Sigma[\Phi]} \]

\[ = e^{-S_\Sigma[f]} e^{-f_\Sigma \Phi_{2} \partial_{f} f} \int \mathcal{D} \phi_{\Sigma - f_\Sigma,0} \Phi e^{-S_\Sigma[\Phi]} \]

\[ = e^{-S_\Sigma[f]} e^{-f_\Sigma \Phi_{2} \partial_{f} f} \langle \phi_\Sigma - f_\Sigma | \Psi_0 \rangle \]

\[ = e^{-S_\Sigma[f]} \langle \phi_\Sigma | e^{f_\Sigma \Phi_{2} \partial_{f} f} | \Psi_0 \rangle \] (2.18)

where we have used that the boundary condition is reflected in the basis element, see (2.4), and that this displacement can be performed by the exponentiation of \( \hat{\Pi} \), the conjugated momentum of \( \hat{\Phi} \), in canonical quantization language. Finally, since \( \langle \phi_\Sigma | \) is an arbitrary element of a complete basis, we have proven that for a free theory,

\[ | \Psi_\lambda \rangle = e^{-S_\Sigma[f]} e^{-f_\Sigma \sqrt{\gamma} t \Phi_{2} \partial_{t} + f_\Sigma \sqrt{\gamma} \Phi_{2} \partial_{f} | \Psi_0 \rangle . \] (2.19)

From this expression one recovers (2.14) noticing that \( f_\Sigma = f_\lambda(x, t = 0) \), see Fig. 3(b). This expression is just the leading order contribution to the state in a \( 1/N \)-expansion of the bulk action, if one considers bulk interactions (\( 1/N \) corrections) the exact coherence is lost, see [28]. These results are similar to what one obtains by coupling an external source \( \lambda \) with a fundamental field of a free QFT [37].

### 3 Modular Hamiltonian for excited states in CFT

In this section we will study the modular Hamiltonian in the excited holographic states introduced in the previous section. In order to do so we will consider that these states lives in a bigger TFD Hilbert space. In Appendix A we review TFD definitions and notations that will be useful. In order to treat modular Hamiltonians we have to specify which spacetime region inside a Cauchy surface and which state we are considering. Then, we will start studying the situation where we can take the subsystem and its complement in an equipartite way i.e. the same number of operators on each of them. An example of this is the Rindler wedge, where we can think the total Hilbert space as \( \mathcal{H}_{\text{tot}} = \mathcal{H}_{L} \otimes \mathcal{H}_{R} \) with \( \mathcal{A}_{L,R} \) denoting the algebra of operators on the left (right) wedges respectively. But, the situation considered here will be more general than equipartite systems and then the results can be extended to any subregion that can be obtained from a conformal map from the Rindler result. As an example we will follow the CHM map [9] to obtain the modular Hamiltonian in holographic excited states for a spherical entangling surface.

#### 3.1 Results in Conformal Field Theory for equipartite subsystems

Let us apply the construction reviewed in section 2 for a CFT defined on a globally hyperbolic spacetime \( M \sim \Sigma \times \mathbb{R} \) with a lorentzian metric \( \eta \) that we assume flat for simplicity. Consider first the case of an equipartition of the degrees of freedom in two equal sides \( \Sigma \equiv \Sigma_{L} \cup \Sigma_{R} \), and the causal domain of both sides are called \( W_{R/L} \). The equipartition requirement is not essential to study entanglement, but here we are motivated by an ingredient of the TFD formalism which supposes that the system described in \( W_{L} \) (and the corresponding algebra of operators) is a copy of the system that lives on \( W_{R} \). The more familiar example of that is the Rindler spacetime and Rindler wedges. In that case the gravity dual has also two wedges connected by the horizon of a black hole, but there are also examples where each side could be compact and disconnected to each other, e.g. \( \Sigma_{L/R} \sim S^{d-1} \), which is holographically related to an extended black hole [31, 32, 45]. The results of this section apply to both possibilities. If we consider the algebra of operators \( \mathcal{A} \) restricted only to one wedge, say \( W_{R} \), all the expectation values in a pure state \( | \Psi_\lambda \rangle \) can be computed through a reduced density matrix \( \rho_\lambda (R) \) by

\[ Tr_{\mathcal{R}} (\rho_\lambda (R) \mathcal{O}(X_1) \mathcal{O}(X_2) \ldots \mathcal{O}(X_n)) = \langle \Psi_\lambda | \mathcal{O}(X_1) \mathcal{O}(X_2) \ldots \mathcal{O}(X_n) | \Psi_\lambda \rangle, \quad \forall X_i \in \Sigma_{R}. \] (3.1)

The vacuum state \( (\lambda = 0) \) is thermal with respect to the Hamiltonian \( K_\lambda (R) \equiv -\log \rho_\lambda (R) \), that coincides with the generator of the time translations for accelerated observers [8].

In the context of the present work, it will be useful to consider the Schwinger- Keldysh formalism and the extension of the Rindler time parameter to a closed time contour \( \mathcal{C} \) in the complex plane [31, 32]. The
Figure 4: (a) Closed symmetric Schwinger-Keldysh (sSK) path in the complex $t$-plane. The horizontal lines represent real time evolution. The vertical lines give imaginary time evolution, and the regions $I_{\pm}$ have identical lengths equal to $\beta/2$. The insertion of sources in the vertical lines generate excitations over the (vacuum) thermal state. (b) The radial Rindler coordinate is shown so that one can see that the L and R pieces are connected. Sources $\lambda_{\pm}$ are turned on in the corresponding Euclidean regions $E_{\pm}$. Notice that the red point in (a) is now a red line, representing a spacelike $\Sigma$ surface.

left wedge of the Minkowski spacetime $W_L$ can be identified with the other real-time component of the SK contour and the corresponding algebra of operators with the commutant of $\mathcal{A} : \bar{\mathcal{A}}$. The initial (and final) pure global state is described by the Euclidean intervals (see figure 4).

Then, we will compute the modular Hamiltonian for a thermal (in the sense mentioned before) excited state using the ingredients reviewed in the previous section and the TFD formalism shortly explained in appendix A. Note that the symmetric SK path shown in figure 4(a), that involves two imaginary path of equal length $\beta/2$ is equivalent to the TFD formalism [46, 47, 48, 49, 50].

The evolution operator in this case is generated by the CFT Hamiltonian $H \equiv K_0$ slightly deformed by external (local) sources $\lambda(x, \tau)$. The operators $U_{\pm}$ on both imaginary time intervals, univocally describe the initial/final excited states in terms of the local sources $\lambda$ [32]:

$$U_{\lambda} \equiv \mathcal{P} e^{\int_{\tau_L} d\tau (K_0 + \rho, \lambda(\tau))}.$$

(3.2)

where the Euclidean time $\tau$ runs on the interval $L_\tau \equiv (-\beta/2, 0)$, that joined to $I_L \equiv (0, \beta/2)$ completes a (un-closed) circle $S^1_\beta$ of radius $\beta/2\pi$. In fact, in the context of TFD these operators are equivalent to pure states, rearranged as $\text{kets}$ in the duplicated space.

Therefore, the global state is given in terms of this operator by [31] (see Appendix A)

$$|\Psi_{\lambda}\rangle = (U_{\lambda} \otimes 1)|1\rangle = U_{\lambda} |1\rangle,$$

(3.3)

and one can define an Hermitian (reduced) density matrix as

$$\rho_{\lambda} \equiv \text{Tr}_{\bar{\mathcal{A}}} |\Psi_{\lambda}\rangle \langle \Psi_{\lambda}| = \text{Tr}_{\bar{\mathcal{A}}} U_{\lambda} |1\rangle \langle 1| U^\dagger_{\lambda} = U_{\lambda} U^\dagger_{\lambda},$$

(3.4)

where we have used

$$\text{Tr}_{\bar{\mathcal{A}}} |1\rangle \langle 1| = \sum_n |n\rangle \langle n| = \mathbb{1}_{\bar{\mathcal{A}}}.$$

(3.5)

These expressions explicitly show the connection between the pure state (3.3) in the TFD setup and the mixed density matrix (3.4) in a single Hilbert space, both univocally determined by the evolution operator $U_{\lambda}$. The main result of this section is that in this case: for a thermal (but excited) state, one can find that the reduced density matrix on $W_R$ is
\[ \rho_\lambda = \mathcal{P} \ e^{-\int_{\lambda_0}^{\lambda} (K_0 + \Theta \cdot \lambda(\tau))} \]  

(3.6)

where \( \beta = 2\pi \) for simplicity. Although this is non-local by virtue of the integration on the interval \( S^1 \), its form is very simple.

Because of eq. (3.4) this result can be rephrased as

\[ \rho_\lambda = U_\lambda(-\pi, \pi) \]  

(3.7)

where the r.h.s. is the evolution operator valued on a imaginary time interval that covers the complete circle \( S^1 \), provided that the source satisfies \( \lambda(\tau) = \lambda(-\tau) \).

A special case to be considered is whether the source \( \lambda \) does not depend on \( \tau \). Thus, since the path ordering play no role, the (local) modular Hamiltonian on \( W_R \) results

\[ K_\lambda \equiv 2\pi(K_0 + \Theta \cdot \lambda) = K_0 + \int_{\Sigma_R} \lambda(X)\Theta(X) \sqrt{g_{\Sigma_R}} dX^{d-1} \]  

(3.8)

where \( K_0 \) can be expressed in terms of the energy-momentum tensor, the (timelike) Killing vector and the unit \( n^\mu \) future pointing normal to \( \Sigma_R \),

\[ K_0 = \int_{\Sigma_R} T^{\mu\nu} n_\mu n_\nu \sqrt{g_{\Sigma_R}} dX^{d-1}. \]  

(3.9)

Since it will be important in what follows, we would to conclude this section by showing that the states (3.3) are cyclic and separating, see also [51]. Assume that there exists an operator \( B \in \mathcal{A} \) such that

\[ B |\Psi_\lambda\rangle = (U_\lambda \otimes B) |1\rangle = 0 , \]  

(3.10)

multiply this by \( U_\lambda^\dagger \), and since \( \rho_\lambda = U_\lambda (i\pi, 0) U_\lambda^\dagger (i\pi, 0) \equiv U_\lambda (i\pi, 0) U_\lambda (0, -i\pi) \) (see Appendix A), we get

\[ (\rho_\lambda \otimes B) |1\rangle = 0. \]  

(3.11)

Since \( \rho_\lambda \) is Hermitian (and positive), it is invertible and can removed from this equation:

\[ B |1\rangle = 0. \]  

(3.12)

Recalling that the object \( |1\rangle = \sum_n |n\rangle |\bar{n}\rangle \) has been defined in terms of an orthonormal basis of \( \mathcal{H} \otimes \bar{\mathcal{H}} \), we project this equation on an arbitrary element \( |m\rangle \langle k| \) and obtain

\[ \langle k|B|\bar{n}\rangle = 0 \text{ for all } n, k, \text{i.e all the matrix elements of the operator } B \text{ vanish. This shows that (3.3) is cyclic.} \]

On the other hand, recalling that the state can also be equivalently written in terms of an operator \( \tilde{U}_\lambda \in \mathcal{A} \) as

\[ |\Psi_\lambda\rangle = |\otimes \tilde{U}_\lambda |1\rangle = \tilde{U}_\lambda |1\rangle \]  

(3.13)

we can repeat the argument for an operator \( A \in \mathcal{A} \) to show that the state is separating.

### 3.1.1 On the geometry of the modular operators

In the modular theory, the modular operator \( \Delta \), and the modular conjugation \( J \), are considered more fundamental objects since they lead to well defined quantities. The Tomita-Takesaki theory (see section 3.3) is formulated in terms of them. In the Rindler spacetime, the modular conjugation is nothing but the "tilde" operation of TFD. Interestingly, the SK extension of the geometry captures the modular flow \( \Delta^{is} \) and the global state for purely imaginary values of the time. Let us see how it appears naturally in a symmetric Schwinger-Keldysh (sSK) complexification of the evolution parameter (figure 4).

For instance, let us consider the formula for a (initial) global state (3.3):
such that the modular Hamiltonian coincides with the generator of boosts and writes as (3.8). For accelerated observers it generates the time translations and the state lies on the surface corresponding to $T^-$ (see fig 5); on the other hand the same state, lying on the Cauchy surface with modular parameter $T^- + s$, also obeys the prescription (3.14) and writes as

$$|\Psi_\lambda(s)\rangle = U_\lambda(s) |\Psi_\lambda\rangle,$$

where $K_L \equiv \tilde{K}_R$ for the excited states considered, and then $\Delta \equiv e^{K_R - K_L}$.

Denoting $X^\mu \in W_R$ any point of the right wedge on the Rindler spacetime and using the notation $X^0 \equiv t$ and $X^d \equiv r$, the flat metric in conventional Rindler coordinates writes

$$ds^2 = -\frac{r^2}{R^2} dt^2 + dr^2 + dX^i dX_i, \quad i = 1 \ldots d - 1$$

and then

$$e^{-isK_R} \Theta_R(X^\mu) e^{isK_R} \equiv \Theta_R(\gamma(s)) = \Theta_R(r, X^i, t + s),$$

with $\gamma(t)$ denoting the curve referred as geometric flow and $\Theta_R \in W_R$. Therefore a constant $t$ defines a particular foliation of $W_R$ in surfaces $\Sigma_R(t)$ that are homologous to $\Sigma_R$.

In the case of the Rindler spacetime it is useful to construct the sSK extension from the Rindler wedge as follows. One takes the standard Minkowski spacetime, that covered by Rindler coordinates splits in four regions or patches. Then, let us take only the left and right sides $W \equiv W_L \cup W_R$ whose boundaries $\Sigma_{\pm}$ are homologous to an extended foliation $\Sigma(t)$ of $W$ that corresponds to the parameter $t \rightarrow \pm \infty$.

In this and other cases, the associated algebra of operators to $W_L$ is the commuting algebra of $W_R$. 

\[11\]
Then take two halves of the analytical extension of the Rindler spacetime to purely imaginary time coordinate $t \rightarrow -i\tau$.

The rank of the coordinate $t$ must be $[0, 2\pi R]$ in order to avoid the conical singularity at the origin. Now we split this geometry in two (past and future) halves $E^\pm$ by the intervals $\tau \in [0, \pi R]$ and $\tau \in [\pi R, 2\pi R]$ respectively so we can define the closed complexified (Rindler) space time, denoted by $W_\epsilon$ by smoothly gluing $E^\pm$ with $W$ through the surfaces $\Sigma^\pm$. This construction is similar to $[31, 32]$ and the smoothness conditions implies the continuity of the metric and the extrinsic curvature along the parameter $\tau$. The total geometry can be seen as a fibration $W_\epsilon = \Sigma_R \times \mathcal{C}$.

An important remark is that that complexified geometric flow $\gamma^\mu(\theta)$ where $\theta \in \mathcal{C}$ is the complexified parameter of the closed symmetric SK path of figure 4, describes a closed curve that intersects $\Sigma_R(t)$ only in the point $X^\mu$, and the union of all these covers completely $W_\epsilon$ (except the points with $r = 0$). Interestingly, the modular flow evolves the local operators along these curves according to
\[
(\theta, X^i, t + \theta) \equiv e^{-i\theta K_\epsilon} \theta(X^\mu) e^{i\theta K_\epsilon} \quad \theta \in \mathcal{C}, X^\mu \in \Sigma(t)
\]
and defines a sort of extension of the operator algebra $\mathcal{A}(W_\epsilon)$ to local operators on all the points of $W_\epsilon$. Then the Tomita-Takesaki theory might be interpreted as a constraint, relating the operators (3.19) with certain operators $\tilde{\theta}$ of the commuting algebra $\mathcal{A}$ when they act on a specific state. This point of view will become clear in the following Sections.

### 3.2 Excited states in a Ball

Let us see, first in a naive way, which should be the explicit form of the modular Hamiltonian for the excited state on the spherical entangling surface. The strategy will be to do the conformal transformation that maps $\tilde{\theta}$ becomes clear in the following Sections.

Let us derive this expression from a path integral approach for a general source $\lambda(x, \tau)$. The previous construction of the sSK path allows to compute time ordered $n$-point functions in arbitrary points of the extension $W_\epsilon$, and then one can also construct the corresponding sSK extension for the ball, $D_\epsilon$, by applying the CHM transformation to each component of $W_\epsilon$, and glue them. Here $I = W_R, E^-, W_L, E^+$ refers to all the pieces of the symmetric SK complexified spacetime. In particular $W_R, W_L$ map into $D, D(\tilde{V})$ respectively (see figs 6). Since the analytical extension of modular flow $x(-i\epsilon)$ is ill defined for the center of the ball $x^i = 0 \quad i = 1 \ldots d$, it is convenient to define $D_\epsilon$ as the foliation $\{V_0(\theta)\}|_{\mathcal{C}} \sim V_0 \times \mathcal{C}$ where $V_0$ is the ball minus this point.

Consider so the sSK extension of the result (3.7)
\[
Z(\lambda) = \text{Tr} \quad U \equiv \mathcal{P} e^{-i \int \mathcal{D}\theta (K_\epsilon + \theta^\epsilon \lambda(\theta))},
\]
then the $n$-points correlation functions in the Rindler wedge can be computed from
\[
\langle \Psi_{0\epsilon}| \theta(X_1) \theta(X_2) \ldots \theta(X_n)| \Psi_{0\epsilon} \rangle = (-i)^n \left. \frac{\partial^n}{\partial \lambda(X_1) \partial \lambda(X_2) \ldots} \right|_{\lambda = 0} Z_R(\lambda)
\]
for all set of (arbitrary) \( n \) points \( X^\mu_1, \ldots, X^\mu_n \in W_R \).

Now we will apply the CHM map, which is nothing but a conformal transformation \( W_R \to D \equiv D(B_R) \) implemented by the unitary transformation \( \mathcal{U} \) on the Hilbert spaces. In particular the (spinless) primary operators transform as

\[
\mathcal{O}_D(x) = \Omega(X)^\Delta \mathcal{O}(X) \mathcal{U}^{-1}; \quad \forall X^\mu \in W_R.
\]

Using that \( \mathcal{U}|0\rangle = |0\rangle \) we obtain

\[
0 \langle \mathcal{O}_D(x_1) \mathcal{O}_D(x_2) \ldots \mathcal{O}_D(x_n) |0 \rangle = \prod_{i=1}^n \Omega(X_i)^\Delta \Omega_0 \langle \mathcal{O}(X_1) \mathcal{O}(X_2) \ldots \mathcal{O}(X_n) |0 \rangle
\]

for any set of (arbitrary) \( n \) points \( X^\mu_i(\theta), \ldots, X^\mu_n(\theta) \), at the same hypersurface \( \theta = \text{constant} \). The left hand side is nothing but

\[
(-i)^n \frac{\partial^n}{\partial \lambda(x_1) \partial \lambda(x_2) \ldots} Z_D(\lambda) \bigg|_{\lambda=0},
\]

so one can think these relations as *probing* the generating function for both theories in both extended spaces. In fact they imply that the expansions (in powers of \( \lambda \)) of both functionals coincide

\[
Z_D(\lambda) = Z_R(\lambda \to \Omega^{-\Delta} \lambda, W_\mathcal{E} \to D_\mathcal{E}),
\]

where

\[
Z_D = \int [D\phi] \ e^{i \int_{D_\mathcal{E}} d\theta \sqrt{-g} \sqrt{\gamma} d^4x \left( \mathcal{L}_{\mathcal{E}} + \Omega^{-\Delta} \Omega^{\Delta} \lambda(x) \mathcal{O}_D(x) \right)}
\]

satisfies all the relations (3.25). Since \( D_\mathcal{E} = V_0 \times \mathcal{E} \), this is defined on fields with periodic conditions in \( \theta \), and we can also express this as

\[
Z_D(\lambda) = \text{Tr} \ U_D \quad \quad U_D \equiv \mathcal{D} e^{-i \int_{D_\mathcal{E}} d\theta \left( K_{\mathcal{E}} + \int_{V_0} \sqrt{-g} \sqrt{\gamma} d^4x \beta(x) \Omega^{-\Delta} \mathcal{O}_D(x) \right)}
\]

where the exponent corresponds to the canonical energy for each slice \( V^0(\theta) \) computed from the deformed Lagrangian of (3.28). \(^6\)

\(^5\)On the other hand, one also have an extension associated the commuting algebra \( \mathcal{D} \).

\(^6\)It is the component \( T^{\mu \nu}_0 n_\mu(\theta) r^{(V)}(\theta) \) of the energy-momentum tensor derived from (3.28), where \( n^\mu(\theta) \) is the unit vector, orthonormal to \( V_0(\theta) \) and recalling that \( \theta \) is the analytically extended Rindler time, \( \beta(x) \) is locally defined by \( \frac{\beta}{2\pi} r^{(V)}(\theta) = \partial_\theta x \).
The same argument holds for the matrix elements of \( U(\lambda) \) (and \( U_D(\lambda) \)). In fact, one can remove the periodic boundary condition from this path integral, and to consider the evolution operator between two hypersurfaces \( \Sigma(\theta_1) \) and \( \Sigma(\theta_2) \) by imposing arbitrary field configurations on each one. Then the (dynamical) evolution operators relate by

\[
U_D(\theta_1, \theta_2)|\lambda\rangle = \mathcal{U} U(\theta_1, \theta_2)|\lambda\rangle \mathcal{U}^{-1}
\]

and the matrix elements can be computed with the following path integral:

\[
\langle \phi_1|U_D(\lambda)|\phi_2\rangle = \int_{\phi_1}^{\phi_2} [D\phi] e^{i\int_{\phi_1}^{\phi_2} \mathcal{L}\tilde{E}_0 d\theta \int_{\phi_1}^{\phi_2} \sqrt{-\tilde{g}} d^4x (\mathcal{L}_{CFT} + \Omega^{-1}(x^\mu)\lambda(x)\mathcal{L}_D(x))}
\]

where \( \phi_1, \phi_2 \) are two arbitrarily specified configurations of the fields on the surface \( V_0(\theta_1), V_0(\theta_2) \) respectively. Since the CHM conformal transformation maps one-to-one the points of \( B_R(\theta) \) into \( \Sigma_R(\theta) \) and \( V(= V_0(-i\pi)) \) into \( \Sigma_L(\theta) \) these correspond to the configurations on \( \Sigma(\theta_1), \Sigma(\theta_2) \) of the sSK extension of the Rindler wedge. Finally, by taking \( \theta_{1,2} \) to be the red points in Fig 2 (a), and choosing \( \phi_1 = \phi_2 \), we obtain the main formula (3.27).

By virtue of (3.30), we have

\[
U_D^I(\lambda) = \mathcal{U} U^I(\lambda) \mathcal{U}^{-1}
\]

where \( I = W_R, E^-, W_L, E^+ \) label the pieces of the symmetric SK complexified spacetime.

The reduced matrix density of excited global (pure) states can be obtained by taking the limit \( |T^+ - T^-| \to 0 \) and removing the real time components of the geometry, then the entire sSK gemoetry is nothing but \( E \equiv E^+ \cup E^- \). In fact, the analytical extension to purely imaginary values of the parameter \( \tau \to -i\tau, \tau \in [0, 2\pi) \) evolves the operators in the manifold \( \Sigma \times S^1 \). The (pure) global state is built with the evolution operator on the interval \( (0, \pi) \), so the excited states can be systematically constructed by deforming the CFT action with a source \( \lambda(X, \tau) \), therefore by virtue of (3.4), we must extend the source to all the manifold \( E \) demanding \( \lambda(X, \tau) = \lambda(X, -\tau) \). Because of the CHM map, all these remarks can be transplanted to the description of the CFT on \( D_{E^0} \) with the DOF within a sphere \( V \), see fig. 6(b).

Using the result (3.4), and using (3.31), we obtain the (unnormalized) reduced density matrix for any \( \lambda \)-state in the ball shaped region

\[
\rho_D(\lambda) = U_D(0, 2\pi)(\lambda).
\]

Finally by virtue of (3.30) the matrix elements are

\[
\langle \phi_1|\rho_D(\lambda)|\phi_2\rangle = \int_{\phi_1}^{\phi_2} [D\phi] e^{-i\int_{\phi_1}^{\phi_2} \mathcal{L}\tilde{E}_0 d\theta \int_{\phi_1}^{\phi_2} \sqrt{-\tilde{g}} d^4x (\mathcal{L}_{CFT} + \Omega^{-1}(x^\mu)\lambda(x)\mathcal{L}_D(x))}
\]

Although we have found the reduced matrix density for a ball-shaped region in an excited state, in CFT, a similar formula and construction can be obtained for any region obtained from the Rindler spacetime by some conformal mapping. Actually, \( \rho(\lambda) \) is even a candidate to define a good modular flow and, in what follows, we shall show that it fulfills the conditions of the Tomita-Takesaki theorem.

It is worth noticing that the previous result capture all the excited states of the Hilbert space (often referred to as the vacuum sector of the Hilbert space [2]) defined as \( \mathcal{H}_0 = \mathcal{A}(0) \). The holographic dual of this space is the Fock space associated to quantized fields in the bulk spacetime [42, 62]. So for instance, the reduced density matrix corresponding to single-particle state, created at the point \((-i\tau, X) \in E^- \ (0 \leq \tau \leq \pi)\), is

\[
\rho_1 = U_0(-i\pi, -i\tau)\phi(-i\tau, X)U_0(-i\tau, 0) U_0(0, i\tau)\phi(i\tau, X)U_0(i\tau, -i\pi)
\]

\[
e^{-i\pi\mathcal{L}_O} K_0(-i\tau, X) K_0(0, -i\tau)\phi(i\tau, X) e^{i\pi\mathcal{L}_O} K_0
\]

(3.33)

where \( X^\mu \equiv (-i\tau, X) \) denotes a point on the surface \( \Sigma_R(-i\tau) \) of the foliation of \( E^- \). This expression can be derived from eqs. (3.3) and (3.4) since in the sSK extension of the Rindler spacetime, the pure state writes\(^7\)

\[
|\Psi_1\rangle = \frac{\delta|\Psi_{\lambda}\rangle}{\delta\lambda(X, \tau)}|_{\lambda=0} = U_0(-i\pi, -i\tau)\phi(-i\tau, X)U_0(-i\tau, 0)|1\rangle
\]

(3.34)

\(^7\)For a \( n \)-particle state we have to take \( n \) derivatives.
and using that $\Theta^t(-it, X) = \Theta(it, X)$ and $U^t(it_j, it_f) = U(-it_f, -it_j)$. Expression (3.33) for the unnormalized density matrix can be generalized to any other region conformally related to the Rindler wedge, say $\mathcal{D}$, by inserting the prefactor $\Omega^{-\Lambda}(x(-it))\Omega^{-\Lambda}(x(it))$. Here $x(it)$ denotes the geometric flow in the transformed space $\mathcal{D}$, such that $x(0)$ stands for the CHM map of the point $(0, X) \in \Sigma_R(0)$.

### 3.3 On the Tomita-Takesaki theorem

The Tomita-Takesaki [3] theorem is one of the most important theorems in the algebraic quantum field theory setup. But despite being a very formal tool, it has many applications in physics and mathematics (see [2] for a nice review and references therein). Let $\mathcal{A}$ be a von Neumann algebra on a certain Hilbert space $\mathcal{H}$ which contains a vector $|\Omega\rangle$ that is cyclic and separating on $\mathcal{A}$. Let us now define and operator $S$ on $\mathcal{H}$ by the following relationship

$$SA|\Omega\rangle = A^1|\Omega\rangle,$$

(3.35)

for all $A$ belonging to the von Neumann algebra. The operator $S$ is called the Tomita operator with respect to $(\mathcal{A}, \Omega)$ and it has a unique polar decomposition

$$S = f\Delta^{1/2} = \Delta^{-1/2}f.$$

(3.36)

The operator $\Delta = S^\dagger S$ is called the modular operator and the anti-unitary $f$ is called modular conjugation.

From the definition can be seen that $f = f^\dagger$ and $f^2 = 1$. The TT theorem states that for $A$ a subalgebra of $\mathcal{A}$ and $|\Omega\rangle$ a separating and cyclic vector on $\mathcal{A}$ then the following properties hold

$$J|\Omega\rangle = \Delta|\Omega\rangle = |\Omega\rangle, \quad JA = A^\prime, \quad \Delta^{it}A\Delta^{-it} = A, \quad \forall t \in \mathbb{R}.$$

Where the subalgebra $A^\prime$ belongs to the commutant of $\mathcal{A}$ (called $\mathcal{A}'$). This theorem ensures that there exist an uniparametric group of automorphisms $\sigma_t(A) = \Delta^{it}A\Delta^{-it}$ that will be called in our context as modular flow of the vector $|\Omega\rangle$. This flow can be local, when there is a geometric curve that the operator $A$ follow, or non local. Typical examples of local flows are those produced by modular operators in the vacuum state and for the Rindler wedge [8], or the vacuum of a CFT on a sphere [9] and more recently the single interval case of fermions on a torus [12]. Typical examples of a non local flow is when the region is on the real line and made of disjoint intervals [11] and as well when we study fermions in disjoint intervals on the torus [12].

#### 3.3.1 TFD formalism and the Tomita-Takesaki theory for holographic states

The TFD formalism can be applied to study the entanglement in systems separated in two identical subsystems and reduced to one of them, such as the the vacuum of a QFT on a Rindler spacetime (or a black hole [31, 32]) reduced to a wedge $X^1 \geq 0$.

According to this formalism, the condition that defines the (thermal) ground state and fixes the Bogoliubov transformation that relates this state with the (disentangled) vacuum of inertial observers [48, 49], is usually expressed as a constraint on the fields $\Phi \in \mathcal{A}$ [32] for a initial time $t \in (T^-, T^+)$:

$$\left(\tilde{\Phi}(\tilde{X}, t) - \Phi^t(X, t - i\beta/2)\right) |\Psi_0\rangle = 0.$$

(3.37)

where $\tilde{\Phi}(\tilde{X}) \equiv \Phi(\tilde{X})$ is defined by the tilde conjugation rule [48] (see appendix A) \(^8\). In the Rindler space, this represents the field in the algebra $\mathcal{A}$ of operators on the left wedge $\mathcal{W}_l$\(^9\). This is known as the thermal state condition in the TFD context [48, 49], and in particular has been studied in space times with event horizons, and interpreted as the quantum/operator formulation of the Unruh-trick [32] (see Appendix). The geometric interpretation of this equation is that, in the state $|\Psi_0(t)\rangle$, the field on the right can be related to the left ones by an analytically continued time evolution in $i\beta/2$.

---

\(^8\)This equation shall thought to be supplemented with that for the canonical momentum $\Pi_\Phi$, this ensures that the constraint can be written for any operator $\mathcal{O}(\Phi, \Pi_\Phi)$.

\(^9\)Formally, they are obtained by applying a CPT, composed with certain rotation in a $\pi$ angle, on the fields $\Phi$.
Then, by identifying the tilde conjugation with the action of the operator $J$ of the polar decomposition (3.36), and taking
\[ \Delta^{1/2} \equiv U_0 (i \beta/2) \otimes U_0 (-i \beta/2) = e^{-\beta (K_0 - \bar{K}_0)/2} \] (3.38)
and $|\Psi_0(t)\rangle$ is given by (3.3) with $\lambda = 0$. We can verify that (3.37) is equivalent to the Tomita-Takesaki relation
\[ S \Phi |\Psi_0\rangle = \Phi^\dagger |\Psi_0\rangle \] (3.39)
which, being $|\Psi_0\rangle$ cyclic and separable, guarantees that the operator $\Delta^{1/2} \equiv (SS^\dagger)^{1/2}$ defines the modular flow for the state $\Psi_0$.

Therefore, we have shown with this simple example that in some cases, we can translate the Tomita-Takesaki theory to the TFD analysis, and interpret eq. (3.39) as a constraint defining a state.

The TFD constraint (3.37) can be generalized to excited states as [32]10:
\[ \left( \Phi(X, t) - U_A (-i \beta/2) \Phi^\dagger(X, t) U_A (i \beta/2) \right) |\Psi_A\rangle = 0. \] (3.40)

Let us see, for instance, that this constraint is trivially satisfied for the (holographic) excited states constructed with a time-independent source $\lambda \equiv \lambda(X)$, in an arbitrary QFT.

In this case the modular Hamiltonian writes, see (3.8),
\[ K_0 \rightarrow K_A \equiv K_0 + \int_{\Sigma_S} dX \lambda(X) \Theta(X). \] (3.41)
and, because of the theorem of Bisognano and Wichmann [8], the time evolution coincides with the modular evolution generated by this operator. Consequently, the state defined as
\[ |\Psi_A\rangle = U_A (-i \beta/2) |1\rangle = e^{-\beta K_A/2} |1\rangle \] (3.42)
is the new TFD-vacuum for the deformed Hamiltonian (3.41), although it is an excited state of the original (undeformed) theory. In other words, the constraint (3.40) reduces to (3.37).

Thus, if we define $\Delta^{1/2}$ by substituting $U_0 \rightarrow U_A$ in (3.38) it can be verified that
\[ S_A \Phi |\Psi_A\rangle = \Phi^\dagger |\Psi_A\rangle, \] (3.43)
with
\[ S_A \equiv J \Delta_A^{1/2}. \] (3.44)
Notice that $J$ remains unchanged under the deformation. This follows from the fact that the deformation can be equivalently created by operators acting only on either side of the DOF splitting, cf. (3.3) and (3.13). This type of deformations are known to preserve the $J$ operator and are contained in what is called the “standard cone”, see [1]. From a TFD perspective, the formalism naturally admits excited states, see [52, 53] for example, without deforming the tilde map between the duplicated theories.

We would like to close this Section by pointing out that in CFT, this construction can be straightforwardly extended to regions bounded by a sphere through the CHM map, or regions conformally related to a Rindler wedge. We will see later that for arbitrary excited states, generated with sources $\lambda \equiv \lambda(X, t)$ in strongly coupled CFT, this analysis shall be carried to the dual (bulk) spacetime to reproduce the Tomita-Takesaki theory, and obtain the holographic modular flow.

4 The gravity dual of the modular Hamiltonians (at large $N$)

In the previous section we were computing the modular Hamiltonian and modular flows for the particular excited states introduced in Section 2 from a CFT point of view. We concluded that the result can be written as (3.6) for an equipartite subsystem and (3.21) for the spherical entangling surface. But, we can’t ensure that this result is the one related to the modular operator $\Delta$ of the TT theory in the general case with explicit $\tau$-dependence of the source $\lambda$. In the present section we will use the fact that we know the precise holographic dual of the state (3.6) and of the condition (3.37) then we will show that it satisfies the TT constraint in the bulk. Then, we will show in this way that the modular Hamiltonian computed in the previous section in the context of a CFT is the one associated with $\Delta$ in the strong coupling limit of the field theory.

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10This has been shown for free theories on maximally extended black holes but the result also applies in this context.
4.1 Expected results

The cases that we have studied can be considered dual to spacetimes with a Killing vector in the bulk: \( \zeta^\mu = \partial^\mu / \partial t \), such as AdS-Black Holes, Rindler-AdS, or isometric mappings of these spaces (e.g. regions whose asymptotic boundaries are ball-shaped [9, 54]). In these cases, \( \zeta \) is bifurcating on the entangling surface, given by the RT recipe (see [9]). Therefore, according to the prescription discussed in section 3, the expectation is that the (bulk) modular flow shall also be determined by the euclidean evolution operator \( U_{\text{bulk}}(0, i2\pi) \), but the \( \lambda \)-deformations shall be described as no vanishing Dirichlet asymptotic BCs on the bulk fields [25, 28, 32].

In the following analysis of the bulk theory we will assume a large \( N \) approximation, which in particular, supposes non-backreacting and weakly coupled QFT in the gravity side, i.e. the existence of ladder operators. For simplicity we will also consider a scalar field \( \Phi \).

Then, for the canonical Hamiltonian \( H = H(\Phi, \Pi) \) derived from the action \( S[\Phi] \), is the operator that generates the \( t \)-translations on the entanglement wedge (see red lines in fig. 7). Thus, for the vacuum state \( \lambda = 0 \) one can write (2.5) as

\[
\rho_0[\phi^+, \phi^-] \equiv \langle \phi^+ | e^{-2\pi H} | \phi^- \rangle ,
\]

and so the modular Hamiltonian in this case is simply \( K_0 \equiv 2\pi H \), and the modular flow will be defined by

\[
U(t) = e^{-i t 2\pi H}.
\]

The simple form of the holographic formula (2.5) suggests an ansatz for holographic excited states defined by non trivial asymptotic conditions on the Euclidean sections of the spacetime. Thus, we expect that at large \( N \), the backreaction is negligible and the Killing vector remains \( \zeta^\mu \), and thus \( K_\lambda \) will depend on \( \lambda \) in the following way

\[
K_\lambda \propto H[\phi + f_\lambda, \pi - \partial_t f_\lambda]
\]

where \( f_\lambda \) is a particular solution of the classical e.o.m. satisfying non vanishing asymptotic boundary conditions \( \lambda; \lambda^* = f_\lambda|_{\partial E} \) on the euclidean pieces of the spacetime. This is nothing but the canonical Hamiltonian under the substitution \( \phi \to \phi_\lambda \equiv \phi + f_\lambda \) in the action.

Although the explicit computation will be done later, a simple path integral argument for it is that this field redefinition transforms the expression (2.5) to
\[
\rho_\lambda |\phi^+,\phi^-\rangle = \langle \phi^+|\rho_\lambda|\phi^-\rangle = \langle \phi^+|\mathcal{D} e^{-\int_{t_0}^{t_1} dt H_1} |\phi^-\rangle = \int_{\lambda=0,\phi^+} [D\Phi] e^{-S[\Phi + f_\lambda]},
\]
where the sum is over fields \( \Phi \) with vanishing asymptotic b.c.s. So this can be thought as a new theory that under certain conditions, in particular at large \( N \), does not break the time-translational symmetry and the Hamiltonian canonically derived from the action \( S[\Phi + f_\lambda] \) coincides with (4.3). Furthermore, it would generate the modular/dynamic flow in the bulk, and (4.4) can be expressed as

\[
\rho_\lambda = e^{-2\pi H_1}\]

Thus, one can show this anzatz by interpreting the above substitution as a canonical transformation of the fields (and its canonically conjugated momenta), one can see that the equations of motion

\[
\partial_t \phi_\lambda = i [H_1, \phi_\lambda], \quad \partial_t \pi_\lambda = i [H_1, \pi_\lambda]
\]
are preserved, as the fields are promoted to operators according to the rule

\[
[\phi_\lambda(x_1), \pi_\lambda(x_2)] = i \delta_{1,2} \quad x_{1,2} \in \Sigma.
\]

Then, the quantization of this theory in the Heisenberg picture consists in finding the general solution to the equations (4.6) (while the state keep the same) provided (4.7). Assuming that \( \phi \) is the most general solution of the problem with vanishing (asymptotic) boundary conditions, e.g. at large-\( N \) approximation, it is a linear combination of the normalizable modes. Therefore, \( \phi_\lambda, \pi_\lambda \) is nothing but the most general field (solution) of the equations of motion, and the condition (4.7) is automatically satisfied, by demanding that the particular solution \( f_\lambda \) be a c-number.

Then, the time-evolution for any operator \( A(\phi_\lambda, \pi_\lambda) \) of the theory is given by

\[
A(t) = e^{itH_1} A(0) e^{-itH_1}
\]
and in particular

\[
\phi(t) = e^{itH_1} \phi(0) e^{-itH_1} - \left[f_\lambda(x, t) - f_\lambda(x, 0)\right],
\]
\[
\pi(t) = e^{itH_1} \pi(0) e^{-itH_1} - \left[\dot{f}_\lambda(x, t) - \dot{f}_\lambda(x, 0)\right]
\]
Then (4.3) is the modular Hamiltonian corresponding to the excited (holographic) state \( |\Psi_\lambda\rangle \), since it generates the modular flow. We will see in Sec. 4.2 that by virtue of large-\( N \) approximation, the field equations can be assumed to be linear, and then one can exactly consider the canonical quantization.

### 4.2 Tomita - Takesaki formalism in the bulk

Here we will give the proof in the bulk of the formulae derived in Sec. 3.3.1 for the modular Hamiltonians for the excited states. The central point of the argument resides again in the relation between TFD and Tomita-Takesaki theory. Starting from the TFD vacuum thermal equilibrium condition, we will show explicitly how to map it to the Tomita-Takesaki equation and how to extract the exact \( \Delta_0 \) and \( J \). From there, we will deform the original thermal equilibrium condition to include the holographic excited states but we will still be able to perform the mapping to the Tomita-Takesaki equation. Thus, we will be able to identify the excited \( \Delta_1 \) and \( j \) for the bulk theory. In Sec. 4.3.2 we will connect these results with the ones in the CFT via holography, using implicitly the JLMS proposal.

#### 4.2.1 TFD to Tomita-Takesaki: Vacuum State

We will extract from the thermal state condition of the TFD vacuum state both the modular and \( J \) operators. We start from the bulk analog of (3.37),

\[
[\Phi_R(t) - \Phi_L(t - i\beta/2)] |\Psi_0\rangle = (\Phi_R(t) - U_0(-i\pi, 0)\Phi_L(t) U_0(0, i\pi)) |\Psi_0\rangle = 0,
\]
and show that this can be rewritten as (cf. with (3.35))

\[
S\Phi_R(t) |\Psi_0\rangle = \Phi_R^\dagger(t) |\Psi_0\rangle = \Phi_R(t) |\Psi_0\rangle
\]
where we have used the fact that the fields $\Phi$ are Hermitian.

Recall that the TFD formalism readily provides an antiunitary tilde operation which map the operators from R to L and viceversa, $\Phi_R(t) = \Phi_L(t)$ and $\Phi_L(t) = \Phi_R(t)$. This can be represented as an operator $J = f^{-1}$ such that $\Phi_R(t) \equiv f\Phi_R(t)f^{-1} = \Phi_L(t)$. Notice that $J$ does not factorize into L and R pieces. The specific form of $J$ is however not important for our purposes but the interested reader can see [2] for details.

A central piece of this argument is the fact that one can build the TFD vacuum out of an operator with support only on one of the sides, i.e. $U_0(0,-i\pi)$, on the identity operator $|1\rangle$ introduced before, see Appendix A,

$$|\Psi_0\rangle \equiv U_0(0,-i\pi) \otimes |1\rangle = I \otimes U_0(0,-i\pi)|1\rangle.$$  \hfill (4.13)

This is a consequence of the highly entangled nature of the $|1\rangle$ state.

Let’s demonstrate the connection between (4.11) and (4.12) starting from a more explicit version of (4.11)

$$I \otimes \Phi_R |\Psi_0\rangle = (U_0(-i\pi,0) \Phi_L U_0(0,i\pi)) \otimes |\Psi_0\rangle \equiv |\Psi_0\rangle$$ \hfill (4.14)

where in the second line we inserted $1 = U_0(0,i\pi)U_0(0,-i\pi,0)$ and in the last equality we used $S|\Psi_0\rangle = J^{\frac{1}{2}}|\Psi_0\rangle = |\Psi_0\rangle$ which is trivial if $S$ is the correct Tomita operator, but from our perspective this is still left to prove. Actually, in order to meet the Tomita-Takesaki theorem conditions, we need to prove both $J^{\frac{1}{2}}|\Psi_0\rangle = J|\Psi_0\rangle = |\Psi_0\rangle$ independently.

The condition on $J$ follows trivially from the fact that $U_0(-i\pi,0) = e^{-\frac{i}{2}H}$, where $H$ is an hermitian Hamiltonian which can be diagonalised with real eigenfunctions and the fact that it acts trivially on $|1\rangle$ by definition. The demonstration then follows as

$$J|\Psi_0\rangle = JU_0(0,-i\pi) \otimes |1\rangle = J(U_0(0,-i\pi) \otimes I)J|1\rangle = I \otimes U_0(0,-i\pi)|1\rangle = |\Psi_0\rangle.$$ \hfill (4.15)

As for $J^{\frac{1}{2}}$, this is also immediate

$$J^{\frac{1}{2}}|\Psi_0\rangle = (U_0(0,i\pi) \otimes U_0(-i\pi,0))(U_0(-i\pi,0) \otimes I)|1\rangle = I \otimes U_0(0,-i\pi)|1\rangle = |\Psi_0\rangle$$ \hfill (4.16)

which completes the demonstration. Notice that (4.13) was crucial for both demonstrations.

### 4.2.2 TFD to Tomita-Takesaki: Excited States

Once proven for the vacuum, we will consider the holographic excited states, 

$$|\Psi_\lambda\rangle \equiv U_\lambda(0,-i\pi) \otimes |1\rangle = I \otimes U_\lambda(0,-i\pi)|1\rangle.$$ \hfill (4.17)

Notice again that the state admits two equivalent ways of defining it via operators acting only on either L or R. One could readily argue that the $J$ operation should not be deformed, see discussion below (3.43). The constraint on the excited state is (see Appendix B)

$$(\Phi_R(t) - U_\lambda(-i\pi,0)\Phi_L(t)U_\lambda(0,i\pi)) |\Psi_\lambda\rangle = 0,$$ \hfill (4.18)

which will lead to

$$S_\lambda \Phi_R(t)|\Psi_\lambda\rangle = \Phi_R^\dagger(t)|\Psi_\lambda\rangle = \Phi_R(t)|\Psi_\lambda\rangle,$$ \hfill (4.19)

where we have used again hermiticity of the fields $\Phi$.

The demonstration of (4.19) from (4.18) follows as in (4.14). As before, one has to prove that

$$S_\lambda |\Psi_\lambda\rangle = J |\Psi_\lambda\rangle = J^{\frac{1}{2}}|\Psi_\lambda\rangle = |\Psi_\lambda\rangle,$$
which also follow analogously from the vacuum computation. This completes the demonstration.

As a summary, the main result is that for the set of excited states and systems described above, we get a closed expression for the Modular operator, which can be written as

$$\Delta_1 = \rho_A^{-1} \otimes \rho_1, \quad \langle \phi_+ | \rho_1 | \phi_- \rangle = \int \mathcal{D} \Phi \lambda_1 \phi_+ e^{-S_1[\Phi]}.$$  \hspace{1cm} (4.20)

Notice again that the theory is undeformed, and only the boundary conditions are affected. From here we can compute a reduced modular Hamiltonian, which coincides with the one found in [55] and [56] for these type of states at first order in $1/N$.

At this point we clarify that the excited constraint (4.18) is not an extra hypothesis but can actually be proven from (4.11), i.e. (4.20) is a direct consequence of (4.11), which is a standard TFD equation. This can be done to any order in perturbation theory, a $1/N$ expansion in this context, and is overall a good candidate as a non perturbative constraint as well. A leading order demonstration can be found in [32] and Baker-Campbell-Hausdorff dissentangling theorems can be used to prove the general statement.\footnote{The proof follows from writing $U_A(0,i\pi)$ in terms of an exponential of ladder operators via a semiclassical expansion of (2.19), see also [32], and then via dissentangling theorems as $U_A(0,i\pi) = A_1 U_0(0,i\pi)$. It is then immediate to prove that the excited constraint (4.18) follows from (4.11) as

$$[\Phi_R(t) - U_A(-i\pi,0)\Phi_L(t)U_A(0,i\pi)] | \Psi_\lambda \rangle = (A_1 \otimes 1) (\Phi_R(t) - U_0(-i\pi,0)\Phi_L(t)U_0(0,i\pi)) | \Psi_0 \rangle = 0.$$}

To conclude this subsection we observe that, once (4.20) is known, one can also obtain the modular operator for any other (that can be non equipartite) subsystems connected to (4.20) via an isometry of AdS. Since we are considering an excited state, both the region and the state are affected by the transformation. As the excited states are created by a perturbation localized only on one the subsystems, the AdS isometries will still map the deformation inside of the transformed subsystem.

The canonical example would be that of the equipartite subsystems being the Rindler-AdS patches of the complete pure AdS geometry. Via AdS isometries, one can both move and rotate the Rindler horizons of the subsystems, see [9]. From a CFT point of view, one is performing conformal transformations that, in this example, either moves or dilates the sphere under consideration.

Consider such a transformation embodied by the unitary operator $T$. We can apply the transformation to the constraint (4.11) to get,

$$T \otimes \Phi_R(t) | \Psi_\lambda \rangle = T S_A (\otimes \Phi_R(t)) | \Psi_\lambda \rangle \Rightarrow 1 \otimes \Phi_R^T(t) | \Psi_\lambda^T \rangle = S_A^T (\otimes \Phi_R^T(t)) | \Psi_\lambda^T \rangle,$$  \hspace{1cm} (4.21)

where we have defined

$$T \otimes \Phi_R(t) T^{-1} = \otimes \Phi_R^T(t); \quad T S_A T^{-1} = S_A^T; \quad T | \Psi_\lambda \rangle = | \Psi_\lambda^T \rangle,$$

the super-index $T$ referring to objects after transforming.

Both the state and the Tomita operator are transformed according to the standard relations for unitary transformations, i.e.

$$| \Psi_\lambda^T \rangle \equiv T U_A(-i\pi,0) \otimes | (T^{-1} | 1 \rangle = U_A(-i\pi,0) \otimes | 1 \rangle, \quad S_A^T = T J_0 T^{-1} \Delta_1 T^{-1} = J_0^T \Delta_1^T,$$  \hspace{1cm} (4.22)

where $| 1 \rangle$ is the transformed identity state. It is relevant to stress that the $J_0$ operation does necessarily change after $T$ into $J_0^T$, for it needs now to map the transformed subsystems into each other. The modular operator is thus,

$$\Delta_1^T = (\rho_A^T)^{-1} \otimes \rho_1^T, \quad \langle \phi_+ | \rho_1^T | \phi_- \rangle = \int \mathcal{D} \Phi \lambda_1 \phi_+ e^{-S_1[\Phi]}.$$  \hspace{1cm} (4.23)

where the $T$ superindex in the integrand is a reminder that the Euclidean time evolution must correspond to the Wick rotated version of the transformed time Killing vector $\zeta_T^\mu = T \zeta^\mu T^{-1}$ where $\zeta^\mu$ is the original reduced timelike killing vector that generated $\rho_0$ in the first place, see discussion near (4.3).
4.3 Computing the bulk modular Hamiltonian at large $N$

Consider an equipartite bulk spacetime with a Killing vector $\zeta$, which is bifurcating on the entangling surface, and it is holographically dual to the Rindler spacetime studied in the previous Sections. We can describe this with the following metric:

$$ds^2 = -u^2 dt^2 + \frac{du^2}{1 + u^2} + (1 + u^2) \left( d\chi^2 + \sinh^2 \chi d\Omega_{d-2}^2 \right)$$  \hfill (4.24)

where the coordinate $y \equiv (\chi, \Omega_{d-2})$ involves a non-compact component $\chi$, while $\Omega_{d-2}$ describes a $(d-2)$-sphere. The holographic coordinate $u$ can be extended to take all the real values (e.g. [59]); thus $u > 0$ stands for the wedge that, after a suitable change of coordinates: $u^2 - u^2 - 1$, is dual to the a hyperbolic cylinder on the boundary [54], that can be conformally mapped to a ball shaped region, or to one of the two wedges of the flat Rindler spacetime [9]. Figure 7 illustrates how the Killing vector $\zeta \equiv \partial_t$, asymptotically coincides (up to a conformal map) with the vector $\partial_t$ of the boundary metric (3.17). The sSK extension of this geometry is similar to Fig. 4, but the theory here is to be sourced by a Dirichlet BC at the asymptotic boundary of the euclidean regions $\partial \mathcal{E}^\pm$, that corresponds to the sources $\lambda$ given on the euclidean extension of the manifold described in Sec. 3.1.1 $E^\pm$.

So consider again a canonically quantized free scalar field $\Phi$ in the bulk. This is essentially the behaviour of all the fields of the bulk theory in the large $N$ approximation. The general solution on the entanglement wedge (one of the two sides of the bulk spacetime, say $u > 0$) writes

$$\Phi(u, y, t) = \sum_n a_n \phi_n(u, y, t) + h.c.$$ \hfill (4.25)

The eigenfunctions $\phi_n(u, y, t)$ are assumed to be an orthonormal basis of the space of (positive energy) solutions of the e.o.m., and the subindex $n$ collectively denote its quantum numbers.

Then the global state in the bulk theory, can be computed through the formula

$$|\Psi_\lambda\rangle = U_\lambda(0, i\pi)|1\rangle$$ \hfill (4.26)

where $U_\lambda(0, i\pi)$ is the (euclidean) evolution operator in the Schrödinger picture. A convenient trick is to transform this to the Interaction Picture, then this state can be expressed as

$$|\Psi_\lambda\rangle = D(\lambda)|\Psi_0\rangle = D(\lambda)e^{-i\pi H}|1\rangle$$ \hfill (4.27)

where $D(\lambda) = \prod_n D(\lambda_n)$ is the (unitary) displacement operator such that $D(\lambda_n)a_n D^\dagger(\lambda_n) = a_n + \lambda_n$. Then using (3.4) we get

$$\rho_\lambda = D(\lambda)\rho_0 D^\dagger(\lambda) = D(\lambda)e^{-i\pi H} D^\dagger(\lambda)$$ \hfill (4.28)

which is nothing but a thermal coherent state. Then, by expressing the Hamiltonian as $K_0 \equiv 2\pi H = 2\pi \sum_n w_n : a_n a_n^\dagger :$ and certain algebraic work using the BCH formulas one obtains

$$\rho_\lambda = e^{-2\pi H_\lambda}$$ \hfill (4.29)

where

$$H_\lambda = D(\lambda)HD^\dagger(\lambda) = \sum_n w_n D(\lambda_n) : a_n a_n^\dagger : = D^\dagger(\lambda) = \sum_n w_n : (a_n + \lambda_n)(a_n^\dagger + \lambda_n^*) : .$$ \hfill (4.30)

Here we stand for $\lambda$ the decomposition of the source in (euclidean) normal modes [25, 32]:

$$\lambda_n \equiv \lim_{|n| \to \infty} \int_{\mathbb{R}} dy \int_0^\pi dt \lambda(y, t) \phi_n(u, y, -it) ,$$ \hfill (4.31)

This is nothing but the expected expression (4.3) in terms of the frequency components of the fields and momenta, and the displacement operator realizes the canonical transformation in these variables. In fact, one of the results of this analysis is that, at large $N$, the holographic excitations consist of a family of canonical transformations, parameterized by the holographic source $\lambda(y, t)$. It is worth emphasizing that here, the source $\lambda$ can depend arbitrarily on the coordinates of the half euclidean boundary $\partial \mathcal{E}^-$.

$^{12}$The AdS radius and temperature scales are hidden in the coordinates.
4.3.1 Modular flow on the bulk

One of the results of this analysis is that, at large $N$, the holographic excitations consist of a family of canonical transformations, parameterized by the holographic source $\lambda(x, \tau)$

Note that the map

$$H \rightarrow H_\lambda \quad a_\lambda = a + \lambda \quad a_\lambda^\dagger = a^\dagger + \lambda^* \tag{4.32}$$

is a canonical transformation. The label $n$ for each normal frequency mode from (4.31) is left implicit. In fact, one can verify that the canonical commutation relations are preserved for the new set of ladder operators and therefore, the e.o.m for the Heisemberg operators

$$\dot{a}_\lambda = [a_\lambda, H] = w a_\lambda, \quad \dot{a}_\lambda^\dagger = [a_\lambda^\dagger, H] = -w a_\lambda^\dagger \tag{4.33}$$

Here we stand for $\lambda$ the decomposition of the source in (euclidean) normal modes, eq. (4.31).

Since the parameter of the evolution generated by $H_\lambda$ is often called $s$, from (4.33) one gets the equations of evolution

$$i \frac{d a_\lambda}{d s} = w a_\lambda, \quad i \frac{d a_\lambda^\dagger}{d s} = -w a_\lambda^\dagger \tag{4.34}$$

which can integrated to obtain the explicit modular evolution:

$$a_\lambda(s) \equiv e^{-i s H_\lambda} (a_\lambda) e^{i s H_\lambda} = e^{i s w} a_\lambda \quad a_\lambda^\dagger(s) \equiv e^{-i s H_\lambda} (a_\lambda^\dagger) e^{i s H_\lambda} = e^{-i s w} a_\lambda^\dagger. \tag{4.35}$$

Define the (deformed) field operator as

$$\Phi_\lambda(r, x, 0) \equiv D(\lambda) \Phi(u, y, 0) D^\dagger(\lambda) = \sum_n D(\lambda_n) a_{\lambda, n}^\dagger D^\dagger(\lambda_n) \phi_n(u, y, t = 0) + h.c. =$$

$$= \sum_n (a_{\lambda, n}^\dagger + \lambda^*_n) \phi_n(u, y, t = 0) + h.c. = \sum_n (a_{\lambda, n}^\dagger) \phi_n(u, y, 0) + h.c. \tag{4.36}$$

therefore, we can compute the $s$-evolution of this operator

$$\rho_\lambda^{is} \Phi_\lambda(u, y, 0) \rho_\lambda^{-is} = \sum_n e^{-i s w} \phi_n(u, y, 0) + h.c. = \sum_n (a_{\lambda, n}^\dagger + \lambda^*_n) \phi_n(u, y, s) + c.c. = \Phi(u, y, s) + f_\lambda(u, y, s) \tag{4.37}$$

that by virtue of (4.35), takes the form:

$$\sum_n (a_{\lambda, n}^\dagger) e^{-i s w} \phi_n(u, y, 0) + h.c. = \sum_n (a_{\lambda, n}^\dagger + \lambda^*_n) \phi_n(u, y, s) + c.c. = \Phi(u, y, s) + f_\lambda(u, y, s) \tag{4.38}$$

where

$$\Phi(u, y, s) \equiv \sum_n a_{\lambda, n}^\dagger \phi_n(u, y, s) + h.c. \quad f_\lambda(u, y, s) \equiv \sum_n \lambda^*_n \phi_n(u, y, s) + c.c. \tag{4.40}$$

$\Phi(u, y, s)$ is the canonically quantized field, and $f_\lambda(u, y, s)$ is a (classical) solution of the eom as the time coordinate $t$ is interpreted as the parameter $s$ (recall the relation (2.14)).

On the other hand, from (4.36) notice that

$$\Phi_\lambda(u, y, 0) = \Phi(u, y, 0) + f_\lambda(u, y, 0), \tag{4.41}$$

thus,

$$e^{-i s H_\lambda} \Phi_\lambda(u, y, 0) e^{i s H_\lambda} = e^{-i s H_\lambda} \Phi(u, y, 0) e^{i s H_\lambda} + f_\lambda(u, y, 0). \tag{4.42}$$

Comparing finally with (4.39), we obtain the modular evolution of the original field

$$e^{-i s H_\lambda} \Phi(u, y, 0) e^{i s H_\lambda} = \Phi(u, y, s) + f_\lambda(u, y, s) - f_\lambda(u, y, 0). \tag{4.43}$$

$$= \Phi(\gamma^\mu(s)) + f_\lambda(\gamma^\mu(s)) - f_\lambda(\gamma^\mu(0)) \tag{4.44}$$

where $\gamma^\mu(s) \equiv \tau^\mu$ is the timelike Killing vector of the AdS-Rindler spacetime. In these coordinates $\gamma^\mu(s) = (u, y, t + 2\pi s)$. This result resembles the one obtained in [24] in the axiomatic Quantum Field Theory context on flat spacetime.
Finally, it is worth emphasizing that by virtue of the BDHM prescription [42]:

\[ \Theta(y,0) = \lim_{u \to \infty} |u|^\Delta \Phi(u,y,0) \]  

(4.45)

and by assuming the holographic duality between the (QFT/bulk) modular flows, one can compute the modular evolution of operators \( \Theta(s) \in \mathcal{A} \) in the dual CFT by

\[ \Theta(y,s) = \lim_{u \to \infty} |u|^\Delta e^{-isK_{\lambda}} \Phi(u,y,0) e^{isK_{\lambda}}, \]  

(4.46)

that in the case studied here (at the large \( N \)) gives:

\[ \Theta(y,s) = \lim_{u \to \infty} |u|^\Delta \Phi(u,y,s) + \lim_{u \to \infty} |u|^\Delta \left[ f_{\lambda}(u,y,s) - f_{\lambda}(u,y,0) \right]. \]  

(4.47)

Notice that the last terms does not vanish in the \( u \to \infty \) limit. This (radial) limit generates a set of operators that are included in the boundary algebra \( \mathcal{A} \).

We would like to conjecture that the formula (4.46), to compute modular evolutions of operators in a holographic field theory from its gravity dual, has general validity for arbitrary regions \( A \) and states (see next Section).

### 4.3.2 The gravity dual of modular Hamiltonians for arbitrary entangling surfaces

As we mentioned many times the states studied here are particularly relevant in holography since they are closely related to coherent states in the bulk (see section 2). The AdS/CFT conjecture prescribes that the respective Hilbert spaces are equal; therefore, one actually has a single object \( |\Psi\rangle \) in a Hilbert space representing the same state. Of course, this state can have very different representations in one or other theory. This hypotesis has been useful to obtain the explicit descriptions of holographic excitations in both theories and to obtain conclusions on their coherence in the bulk at large \( N \) [25, 31, 32].

Consequently, by tracing carefully to both sides of the correspondence (subtleties with the entangle-
ment wedge and the rule to separate in direct products in the bulk should be taken into account [57]), one obtains that the reduced density matrices also coincide. Thus one concludes that as an operator equation

\[ K_{\text{CFT}} = K_{\text{bulk}} \]  

(4.48)

holds, even thought that the bulk modular Hamiltonian has a non trivial structure that comes from an expansion in the Newton constant [38], and the purely gravitational contribution \( o(G^{-1}_N) \) involves an area operator [19, 44].

The objective of this section is to take advantage of this formula and use our previous knowledge on excited states in order to compute the contribution of the deformation (3.2) at \( o(G^0_N) \). In principle, this can be used to compute the leading contributions to the (bulk) matrix elements of \( \rho_{\lambda}[\Sigma_A] \) for any set \( A \equiv \partial \Sigma_A \), although in absence of a bulk killing vector one cannot describe the whole (euclidean) space time as \( S^1 \times \Sigma_A \) and it is hard to check important symmetry features of the modular Hamiltonian.

The JLMS prescription for the modular Hamiltonian in a theory consisting of gravity and a nearly free real field \( \phi \) is [19, 60]

\[ K^\lambda_{\text{bulk}} = \frac{\hat{A}}{4G} + K^\text{grav} + K^\text{matter} \]  

(4.49)

where \( \hat{A} \) is the area operator.

This formula can be obtained from a saddle point approximation (large \( N \)) of the path integral (2.5) (see Fig. 1). The (first) area term can be explained from an additional contribution to the boundary term of the gravitational action called Hayward term\(^\text{13}\) [61], in particular, it was recently shown that the holographic gravitational entropy can be obtained from this term using replica calculations [60]. We leave the study of this term in the calculus of the modular Hamiltonian for a forthcoming work.

\(^{13}\text{It is the contribution associated to the blue line in Fig 1.}\)
Interestingly, even though the whole euclidean spacetime $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^-$ cannot be foliated as $S^1_\tau \times \Sigma_\Lambda$, as in the previous subsections, the matrix elements of the second and third term can be evaluated as

$$
\langle + | K^{grav}_\Lambda | - \rangle = \frac{1}{8\pi G} \int_{\Sigma_+} K^+ \sqrt{h^+} + \frac{1}{8\pi G} \int_{\Sigma_-} K^- \sqrt{h^-} 
$$

(4.50)

$$
\langle + | K^{matter}_\Lambda | - \rangle = \int_{\Sigma_+} \phi^+ \Pi^+ \sqrt{h^+} + \int_{\Sigma_-} \phi^- \Pi^- \sqrt{h^-} + \int_{\delta\mathcal{E}} \lambda \partial_n \lambda \sqrt{h}.
$$

(4.51)

For concreteness, these expressions are understood in the set up of section 2, where $|\pm\rangle \equiv |\phi^\pm, h^\pm\rangle$ are arbitrary configurations of the fields and induced metrics on the surfaces $\Sigma^\pm$, that are two homologous copies of $\Sigma_\Lambda$, as shown in Fig 1b. The asymptotic source $\lambda$ is a smooth function defined on $E^- = \delta\mathcal{E}^-$ (vanishing on $\tau = 0$ and $\tau = -\infty$ for technical issues) and extended to $\partial\mathcal{E}^+$ with reflection symmetry with respect to $\tau = 0$, and $n$ is the normal vector to the asymptotic boundary. The solution for the field is

$$
\Phi(x) = \int_{\Sigma^+} G_\pm(x-y) \phi^\pm(y) dy + \int_{\delta\mathcal{E}} G_\delta(x-z) \lambda(z) dz
$$

(4.52)

where $x$ is any point in the bulk and $z \equiv (\tau, \Omega) \in (-\infty, \infty) \times S^d = \delta\mathcal{E}$. Here $G_\pm$ and $G_\delta$ differ from the standard bulk-to-bulk and bulk-to-boundary propagators. They are solutions to be determined by demanding the following consistency (boundary) conditions.

Denote by $\hat{\mathcal{E}}$ the euclidean manifold of Fig. 1(b), then $B_{i+}, i = \Sigma_-, \Sigma_+, \partial\mathcal{E}$ denotes the three different components of $\partial\hat{\mathcal{E}}$, and the solution can be expressed as

$$
\Phi(x) = \sum_{i} \int_{B_i} G_i(x-y) \phi_i(y) dy
$$

(4.53)

where $\phi_{\partial\mathcal{E}}(z) \equiv \lambda(z)$, thus, the consistency condition adopts the simple form of boundary conditions

$$
G_i(x_j - y_i) = \delta_{ij} \delta(x_i - y_i).
$$

(4.54)

Since $\pm\partial_\tau$ is orthonormal to $\Sigma^\pm$ respectively, using (4.52), one defines

$$
\Pi^\pm(x) \equiv \pm\partial_\tau \Phi(x)|_{\Sigma^\pm}
$$

(4.55)

Finally, substituting by this and $\Phi^\pm(x) \equiv \Phi(x)|_{\Sigma^\pm}$ into eq. (4.51) we obtain the explicit matrix element in the large $N$ approximation. Observe that this is a quadratic form in the input functions $\Phi^\pm(x), \Pi^\pm(x), \lambda(x)$. For a non backreacting field the formula (4.50) can be explicitly calculated in the same way, from the aAdS solution of the Einstein equation on the manifold $\hat{\mathcal{E}}$ with boundary conditions $h^\pm$ on $\Sigma^\pm$, and then $K^\pm$ are the (trace of) extrinsic curvatures on these surfaces.

We can see these formulas as providing the natural candidate to the gravity dual of modular Hamiltonians (up to $o(\mathcal{G}_A^N)$) for arbitrary regions $\Lambda$ and states $\{|\Psi_\Lambda\rangle\}$. Nevertheless, in absence of a Killing vector associated to the modular evolution in the bulk, it is difficult to meet the Tomita-Takesaki structure.

## 5 Summary and conclusions

In this work we have studied entanglement properties of excited states of a QFT on the Rindler spacetime in a systematic way, say, studying a family of excitations holographically related to coherent states $[25, 31, 32]$. In particular we have studied the modular flow and the details of corresponding Tomita-Takesaki theory. In this way we have captured the states of the vacuum sector of the Hilbert space, which are holographically associated to the single- (and n-)particle excitations.

These generating (holographic) states can be constructed geometrically by analytically extending the spacetime to Euclidean times in a Hartle-Hawking fashion, and sourcing the theory with operators on these regions. We have shown that when one considers the (extended) modular flow $\Delta^{ext}$, a nice geometric structure combining both spacetime signatures emerges, and the TT theory can be interpreted geometrically. The case of CFTs is particularly interesting because using the CHM map $[9]$, one can extend this (geometric) construction and results to ball shaped regions of the space, and more interestingly: one can move this
analysis to bulk spacetimes to study the TT theory in certain strong coupling regime \((N \gg 1)\). This method implicitly assumes the dual map between the objects (operators) of the TT theorem; consequently, the TT construction in aAdS spacetimes implies that in the field theory.

We have also shown the connection between the TT theory and the TFD formalism, where the so-called thermal state condition is a constraint defining the (thermal) vacuum \([58, 59]\), and can be generalized to the holographic states considered here \([32]\). At a more practical level, this plays an important role in formulating the Unruh problem correctly, and to find the Bogoliubov transformation. This suggests a novel way of interpreting the TT formalism: as a constraint between operators of an algebra \(\mathcal{A}\) and its commuting \(\tilde{\mathcal{A}}\), as they act on a specific state. In this sense, it would be interesting to study if the construction shown in Appendix B applies to other systems and partitions where the TT theory be known.

We have shown that the excited constraint in Tomita-Takesaki formalism (4.18) allows to find the modular Hamiltonian for the holographic states to any order in a ladder operator expansion of the bulk fields. In particular, for bulk fields linear and quadratic in ladder operators one can find a closed expression for the Tomita operator and modular Hamiltonian. These last two cases are specially interesting from a holography perspective as they correspond to large \(N\) and leading order \(1/N\) corrections in the CFT dual \([25, 28, 56]\). Finally, we have computed the modular propagation of bulk fields and correlation functions on the entanglement wedge, and our result match/agree with those of \([24]\) in a weakly coupled field theory. The modular Hamiltonian for the excited states consists of certain canonical transformation of the original fields and momenta. These results hold for general two sided AdS black holes.

We also found a formula, in terms of the the holographic dual of the modular Hamiltonian for arbitrary spacelike regions \(A\) and for arbitrary coherent excitation \(\lambda\). Even though that in most cases a timelike symmetry does not exists, this would be the natural candidate to modular Hamiltonian in the bulk to large \(N\). The final formula is a non-local expression, involving bulk propagators, quadratic in the field on the bulk entanglement region \(\Sigma_A\) and in the parameter \(\lambda\), which resembles the previous results for free QFTs on a Minkowski spacetime (see \([6]\) and references therein).

Finally, the statement (4.48) allows to argue (using the BDHM prescription \([42]\)) that the formula (4.46) might be considered a holographic prescription to compute modular evolution of operators in a field theory, it would be interesting to check if the results of Sec. 4.3.1, agrees with an explicit computation with the modular flow in the field theory.

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**A TFD Basics**

The thermofield dynamics (TFD) formalism was originally built to study finite temperature systems in real time using zero temperature techniques \([52]\). In this appendix we will present the relevant aspects of the TFD formalism for this work.

Consider a quantum field theory, whose states belong to the Hilbert space \(\mathcal{H}\). In the TFD formalism, one constructs a second copy of the system \([52]\), namely \(\tilde{\mathcal{H}}\), so that the total new system consist of the original CFT and its TFD copy living on disconnected asymptotic boundaries of the gravity dual, whose total states space is \(\mathcal{H} \otimes \tilde{\mathcal{H}}\). Thus, given an operator \(A\), acting on \(\mathcal{H}\), one builds \([63]\) the corresponding operator \(\tilde{A}\) on \(\tilde{\mathcal{H}}\) using the so-called “tilde” conjugation map \([48, 53]\),

\[
[A, \tilde{B}] = 0 \quad (AB) = \tilde{A}\tilde{B} \quad (c_1 A + c_2 B)^\dagger = c_1^* \tilde{A} + c_2^* \tilde{B} \quad (A^\dagger) = \tilde{A}^\dagger. \quad (A.1)
\]

Alternatively, one can denote the extended operators as \(A_L\) and \(A_R\) respectively:

\[
A \otimes \mathbb{1} \equiv A_L, \quad \tilde{(A \otimes \mathbb{1})} = \mathbb{1} \otimes \tilde{A} \equiv A_R. \quad (A.2)
\]
Figure 8: (a) A piece of Euclidean evolution cut at regions $\Sigma_1$ and $\Sigma_2$ understood depicted as the matrix element $\langle n|\rho_\lambda|m \rangle$ of a density matrix $\rho_\lambda$. (b) The same geometry can be instead understood as the coefficient $\langle \langle \tilde{m}|\langle \lambda |\tilde{\Psi}\rangle\rangle$ of a ket $|\tilde{\Psi}\rangle$ defined in the TFD Hilbert space $\mathcal{H} \otimes \tilde{\mathcal{H}}$.

We will use this notation throughout this work.

One can now build the so called TFD vacuum, denoted $|\Psi_0\rangle \in \mathcal{H} \otimes \tilde{\mathcal{H}}$ as follows. We start from the identity state

$$|1\rangle = \sum_n |n\rangle \otimes |n\rangle = e^{a_L^+ a_L^0} |0\rangle \otimes |0\rangle$$

which is a divergent norm state built as a maximally entangled state of the energy eigenfunctions of the spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$. The (unnormalized) TFD vacuum can be built as,

$$|\Psi_0\rangle = \sum_n e^{-\frac{E_n}{\beta}} |n\rangle \otimes |n\rangle \; , \; \text{ (A.3)}$$

where $\beta^{-1} = T$ is the temperature of the system and $E_n$ are its energy eigenvalues. Conformal invariance will allow us to work with $\beta = 2\pi$ without loss of generality. The relevance of the TFD vacuum resides in that it allows to compute expectation values at finite temperature of the original system $\mathcal{H}$ as a vacuum expectation value in the doubled space $\mathcal{H} \otimes \tilde{\mathcal{H}}$. It can explicitly be checked that $[52],

$$\langle A \rangle_\beta \equiv \text{Tr} (\rho A) = \langle \Psi_0 | A \otimes I | \Psi_0 \rangle \; . \; \text{ (A.4)}$$

The vacuum character of $|\Psi_0\rangle$ can be understood in terms of the global Hamiltonian $(H_R - H_L) \in \mathcal{H} \otimes \tilde{\mathcal{H}}$, for which it is immediate to check

$$(H_R - H_L)|\Psi_0\rangle = 0 \; .$$

The equation above suggests a physical interpretation in terms of two systems evolving in opposite time directions. This interpretation has found holographic support especially in the eternal BH solutions [61]. It has also been observed that the DOFs splitting of a system into two Rindler patches can be understood as a TFD doubled space.

In this work, we exploit the fact that the TFD vacuum can also be thought as an Euclidean time evolution operator $U_0(0, i\beta/2)$ acting on the identity state,

$$|\Psi_0\rangle = U_0(0, -i\beta/2) \otimes |1\rangle = |1\rangle \otimes U_0(0, -i\beta/2)|1\rangle \; ,$$

and study excitations of the TFD vacuum defined as in (3.3),

$$|\Psi_\lambda\rangle = U_\lambda(0, -i\beta/2) \otimes |1\rangle = |1\rangle \otimes U_\lambda(0, -i\beta/2)|1\rangle \; .$$

26
This equation, projected into an energy eigenstate basis, can be also geometrically understood as shown in figure 8: \( U_\lambda \) is depicted on the left as an evolution operator on a single Hilbert space, the corresponding TFD-ket \(|\Psi_\lambda\rangle\) is illustrated on the right with the two cylinder’s ends now representing the doubled TFD DOFs at some spacial surface at a fixed time \( t \).

**B  Remarks on the TT construction for excited states in equipartite aAdS**

Consider an example of an equipartite system on \( 2 + 1 \) dimensional aAdS spacetime, say, a free scalar field \( \Phi \) on a metric given by

\[
ds^2 = -u^2 dt^2 + \frac{du^2}{1 + u^2} + d\chi^2 \tag{B.1}
\]

if the coordinate \( \chi \) is non-compact this metric describes the Rindler-AdS spacetime, while if it is compact \((\chi \in (0, 2\pi))\) and periodic, one is describing a maximally extended BTZ black hole \([32]\). The remarks below apply to both cases. Observe that if the coordinate is extended to cover all the real interval \( u \in (-\infty, \infty) \) this metric captures both sides \( L \) \((u \leq 0)\) and \( R \) \((u \geq 0)\), and the event horizon is placed on \( u = 0 \). Clearly, the sSK extension of this geometry is similar to Fig. 4 \([31]\), and the ground states and holographic excitations correspond to the euclidean pieces as explained in the paper.

Let us express the general solution on the lorentzian regions as

\[
\Phi(u, \chi, t) = \Phi_L(u, \chi, t) \Theta(-|u|) + \Phi_R(u, \chi, t) \Theta(|u|) \tag{B.2}
\]

Notice that \( \partial/\partial t \) is a Killing vector, and \( \Phi_R \) (and \( \Phi_L \)) can be canonically quantized in terms of (positive-energy) normalizable modes as in equation \((4.25)\)

Notice that in the TFD context the dual left (quantum) fields \( \Phi_L \) can be obtained from the right ones \( \Phi_R \) by the tilde conjugation rules \((A.1)\), and although they are independent operators in commuting algebras, their respective action on vacuum state are related by an imaginary time translation though the Euclidean piece \( E^- \) (see Figs 4(b) and 6(a)), i.e.

\[
\Phi_L(-|u|, t = T_-, \chi)|\Psi_0\rangle \equiv \Phi_R(|u|, t = T_- - i\pi, \chi)|\Psi_0\rangle, \forall u, \chi \tag{B.3}
\]

which must be complemented with a similar condition for the canonically conjugated momentum fields \( \Pi_{R/L}(u, t, \chi) \), and so for any operator \( \Theta(\Phi, \Pi) \) of the theory. These equations constitute a constraint to be imposed on the (initial) state at the spacelike surface \( t = T_- \). Recall that the (imaginary) time translation is realized by the operator \( U_0(-i\pi) \), that in the Rindler space is the boost generator \([8]\), analytically extended to a purely imaginary parameter.

We have shown in Sec. 3 that this constraint (on the vacuum) is equivalent to the Tomita-Takesaki formalism. In this example we want see how this also determines the Bogoliubov transformation relating the particle notion for inertial/accelerated observers, and also captures the so-called Unruh trick.

In fact, by substituting by the (2nd quantized) solution \((4.25)\), and using the orthonormality relations of the eigenfunctions \( \phi_n(u, t, \chi) \), one obtains the following constraint equations

\[
\hat{d}_n^{(1)}|\Psi_0\rangle \equiv C_1 \left( \hat{a}_n - e^{-i\omega n} \hat{a}^+_n \right)|\Psi_0\rangle = 0, \quad \hat{d}_n^{(2)}|\Psi_0\rangle \equiv C_2 \left( \hat{a}^+_n - e^{+i\omega n} \hat{a}_n \right)|\Psi_0\rangle = 0, \forall n, \tag{B.4}
\]

where \( a_n \) and \( \tilde{a}_n \) denote the L and R independent ladder operators in \( \mathcal{A} \) and \( \mathcal{A}^\dagger \) respectively, and \( C_{1,2} \) are numeric factors. Since these equations can be viewed as annihilating the global vacuum, this automatically defines the Bogoliubov transformation between the \( R/L \) ladder operators and the new set \( d_n^{(1,2)} \), associated to particles for (inertial) observers that have access to the global spacetime. Finally, one can easily verify that the state \(|\Psi_0\rangle = U_0|1\rangle\) (eq. \((4.13)\)) solves these equations.

A useful consequence of this formulation is that eigenfunctions associated to these operators, are the precise linear combinations appearing in \((B.4)\) of the original \( \phi_n, \tilde{\phi}_n \) solutions, which are analytic at the throat \( u = 0 \). This is the precise meaning of the Unruh trick.

---

\(^{14}\) \( \Phi_R(u, \chi, t) \) is what we call \( \Phi \) in the calculations of Sec. 3

\(^{15}\) Different formulations of the thermal state condition as a constraint in the string context can be found in \([58, 59]\)
The last important remark is how this constraint/TT theory can be generalized to the excited states studied in the paper. If one realizes the time translation in $-i\pi$ of the R fields with the sourced evolution operator $U_A(-i\pi)$ in place of $U_0$,

$$\Phi_R(T_--i\pi) \equiv U_A(-i\pi) \Phi_R(T_-) U_A(i\pi), \quad \text{(B.5)}$$

the constraint (B.4) generalizes to

$$[\Phi_L(-|u|, t = T_-, \chi) - U_A(-i\pi) \Phi_R(T_-) U_A(i\pi)] |\Psi_\lambda\rangle = 0, \quad \forall u, \chi \quad \text{(B.6)}$$

As shown in Sec. 3, this equation has the ingredients to construct the Tomita-Takesaki theory for excited states. It decomposes in two linearly independent set of equations:

$$\left( \hat{a}_n - e^{-\omega \pi} a_n^\dagger - e^{-\omega \pi} \lambda_n^{} \right) |\Psi_\lambda\rangle = 0; \quad \left( \hat{a}_n^\dagger - e^{+\omega \pi} a_n - e^{+\omega \pi} \lambda_n^* \right) |\Psi_\lambda\rangle = 0, \quad \forall n, \quad \text{(B.7)}$$

where we have used that the operator $U_\lambda$ act on ladder operator as a displacement, composed with time translation: $U_\lambda(-i\pi) a_n U_\lambda(i\pi) = e^{+\omega \pi} (a_n^{} + \lambda_n^{})$ (and its h. c.), where the numbers $\lambda_n$ are given by (4.31). It is straightforward to verify that the solution of this equation is the state (4.17), that can also be expressed as (4.27).

Notice finally that these equations can be expressed as equations of eigenvalues for the new (global) annihilation operators. Multiplying them by $C_{1,2}$ respectively, we obtain

$$\left( \hat{d}^{(1,2)}_n - \lambda^{(1,2)}_n \right) |\Psi_\lambda\rangle = 0, \quad \text{(B.8)}$$

where the eigenvalues are given by $\lambda^{(1)}_n = C_1 e^{-\omega \pi} \lambda_n^{}$ and $\lambda^{(2)}_n = C_2 e^{\omega \pi} \lambda_n$. This is nothing but the condition solved by a coherent state of $d$-particles.

It would be interesting to study this construction in other partitions of the system where the TT theory be known, for instance, one could apply an isometry of this spacetime such that the entanglement wedge be dual to the causal development of a ball shaped region in the boundary [9].

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