An anisotropic damage model for concrete structures under cyclic loading-uniaxial modeling

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Abstract. An anisotropic damage model is developed based on conventional rotating crack approach. It uses nonlinear unloading/linear reloading branches to model the hysteretic behavior of concrete. Two damage variables, determined by the ratio of accumulated dissipating energy to fracture energy, are introduced to represent the stiffness degradation in tension and compression. Three cyclic tests are simulated by this model and sensitivity analyses are conducted as well. The numerical responses calculated by the damage model are consistent with those obtained from the experiments. The numerical results reflect the nonlinear behavior observed in those tests, such as the damage-induced stiffness degradation, accumulation of residual deformation, energy dissipation caused by hysteretic behavior and stiffness recovery effect due to crack closure. Sensitivity analyses show that the damage exponents have significant influence on the computational accuracy. It is concluded that the anisotropic damage model is applicable to the nonlinear analyses of concrete structures subjected to cyclic loading.

1. Introduction
Analyzing the mechanical response of concrete structures subjected to general loading conditions is an essential yet challenging task for structural engineering, not to mention predicting the failure patterns of complex structures. The distinct failure mechanisms due to tensile cracking and compressive crushing also complicate the constitutive modeling of concrete. The failure process is accompanied by the initiation and propagation of cracks, which may be modeled macroscopically as softening behavior of the material.

The stiffness degradation is conveniently modeled by continuum damage mechanics. Mazars [1], [2] and Cervera [3] introduced a scalar damage factor to describe the relationship between stress and effective stress based on the assumption of isotropic damage. Papa [4], Al-Gadhib [5], Li Qingbin [6], Challamel [7], Badel [8] and Pröchtl [9] utilized a second-order or fourth-order damage tensor to model the anisotropic properties of concrete damage. Their models reflected the stiffness degradation. However, it is insufficient for those models to represent the irreversible deformation and inelastic volumetric expansion in compression.

Simo [10], Lubliner [11], Lucchini [12], Faria [13], Lee [14], Grassl [15], Wu Jianying [16], Voyiadjis [17] and Poh [18] developed plastic-damage models by combining the advantages of classical plasticity and damage mechanics. In this approach, the irreversible deformation and volumetric expansion in multiaxial compression is represented using the flow theory of plasticity and the stiffness degradation is modeled by introducing scalar damage variable or damage tensor. However, the plastic-damage models are very complicated and only a few simple ones have sufficient efficiency to conduct dynamic or cyclic analysis.
of complex concrete structure[19], [20]. Moreover, these models utilize linear stress branches to simulate the unloading/reloading responses, which is difficult to represent the hysteretic behavior of concrete.

Palermo[21] have proposed a cyclic concrete model which implements nonlinear unloading/linear reloading branches into rotating crack approach for modeling hysteretic behavior. In this model, stress branches are defined as functions of strain and stress along the damage axis and the key parameters are statistically derived from the concrete tests. Sima[22] developed a cyclic constitutive model using independently tensile and compressive damage parameters to simulate the stiffness degradation. The compressive hysteretic behavior is therein modeled by exponential unloading/linear reloading branches, whereas the tensile one is neglected.

An anisotropic damage model for cyclic loading was developed by introducing nonlinear unloading/linear reloading stress branches into conventional rotating crack approach[23]. The stiffness degradation was modeled by tensile and compressive damage variables which were determined by the ratio of accumulated dissipating energy to fracture energy. The stiffness recovery due to crack closure was also taken into account by using the approach proposed by Lee[14].

2. The constitutive relation of anisotropic damage model

Following presuppositions are used to model the nonlinear behavior of concrete.

(i) The strain is decomposed into elastic part and inelastic part for any material point.

(ii) There are three mutually orthogonal axes of material which always keep aligned with the directions of principal strain (and stress) during loading histories.

(iii) The damage constitutive relations are mutually decoupled and formulated as functions of principal stress and inelastic strain on the rotating damage axes.

2.1. Mathematical formulation of the damage model in incremental theory

As shown in Fig.1, \( y_1 \) axis is the principal direction on which damage occurs. The strain array \( \varepsilon \) in global coordinates \( o x_1 x_2 x_3 \) is decomposed into elastic part \( \varepsilon^e \) and inelastic part \( \varepsilon^{in} \) and linear elasticity is given by:

\[
\varepsilon = \varepsilon^e + \varepsilon^{in}, \quad \varepsilon^e = E_0^{-1}\sigma
\]

where \( \sigma \) denotes stress array and the initial elastic stiffness \( E_0 \) is a 6 x 6 matrix.

![Diagram](attachment:figure1.png)

**Figure 1.** Local coordinates determined by the principal directions of damaged concrete

The orthotropic axes of material, represented by the local coordinates \( o' y_1 y_2 y_3 \) in Fig.1, keep aligned with the directions of principal strain and stress. Thus, the shear components of stress and strain array vanish in the coordinate \( o' y_1 y_2 y_3 \). Additionally, the strain \( \varepsilon \), elastic part \( \varepsilon^e \) and inelastic part \( \varepsilon^{in} \) in the local coordinates \( o' y_1 y_2 y_3 \) observe the strain decomposition assumption as well.

\[
\varepsilon = \varepsilon^e + \varepsilon^{in}
\]
Herein the relation between local and global inelastic strains follows:

\[ \varepsilon^{in} = T_r e^{in} \]  

where \( T_r \) is the transformation matrix of strain from the global coordinate \( o\text{x}_1\text{x}_2\text{x}_3 \) to the local one \( o'\text{y}_1\text{y}_2\text{y}_3 \). Similarly, the relation between local (or principal) stress \( s \) and global one \( \sigma \) satisfies:

\[ s = T_r^T \sigma \]

The relation between principal stress \( s \) and inelastic strain \( e^{in} \) in \( o'\text{y}_1\text{y}_2\text{y}_3 \) is formulated as follows:

\[ e^{in} = G s \]

\[ G = \begin{bmatrix} g_1(e_1^{in}) & 0 & 0 \\ 0 & g_2(e_2^{in}) & 0 \\ 0 & 0 & g_3(e_3^{in}) \end{bmatrix} \]

in which \( G \) is the flexibility matrix of damage in local coordinates \( o'\text{y}_1\text{y}_2\text{y}_3 \) and \( g_i(e_i^{in}) \) denotes the secant flexibility on damage axis \( i \).

Therefore, the constitutive relations are derived as follows from equation (1)~(5):

\[ \varepsilon = D \varepsilon \]

\[ D = (E + T_r G T_r^T)^{-1} \]

where \( D \) is the constitutive matrix between stress and strain after damage occurs. The transformation matrix \( T_r \) and damage flexibility one \( G \) are determined by current strain and its inelastic part respectively. \( T_r^T \) is the transposed matrix of \( T_r \).

Because the constitutive relation in Equation (6) is nonlinear, the linearization concept is employed to derive the incremental form of constitutive relation. The deriving process is similar to that of the rotating crack model which may refer to the work of [24] and [25]. Finally, the constitutive relation in incremental form is formulated as follows:

\[ \Delta \sigma = D_{tan}\Delta \varepsilon = T^T D' T \Delta \varepsilon \]

in which \( D_{tan} \) and \( D' \) denote the tangent material stiffness matrices in global and local coordinates, respectively; and \( T \) is the transformation matrix of engineering strain from \( o\text{x}_1\text{x}_2\text{x}_3 \) to \( o'\text{y}_1\text{y}_2\text{y}_3 \). The formulation of matrix \( D' \) is given as follows:

\[ D' = \begin{bmatrix} \hat{D}_n & 0 \\ 0 & \hat{D}_s \end{bmatrix} \]

\[ \hat{D}_n = (\hat{C}_e + \hat{C}_c)^{-1} \]

\[ \hat{C}_e = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \]

\[ \hat{C}_c = \begin{bmatrix} \left( \frac{d\sigma_1}{de_1^{in}} \right)^{-1} & 0 & 0 \\ 0 & \left( \frac{d\sigma_2}{de_2^{in}} \right)^{-1} & 0 \\ 0 & 0 & \left( \frac{d\sigma_3}{de_3^{in}} \right)^{-1} \end{bmatrix} \]
\[ \mathbf{D}_s = \begin{bmatrix} \frac{s_1 - s_2}{2(e_1 - e_2)} & 0 & 0 \\ 0 & \frac{s_1 - s_3}{2(e_1 - e_3)} & 0 \\ 0 & 0 & \frac{s_2 - s_3}{2(e_2 - e_3)} \end{bmatrix} \]

where \( E \) is the initial elastic modulus and \( \nu \) denotes Poisson ratio. Equation (7) and (8) show that the constitutive relations are only dependent on the function \( g_i(e_i^{in}) \) and the derivative \( ds_i/de_i^{in} \) in the local coordinates. Consequently, establishing the mathematical expression of cyclic stress branches is the key issue of the model.

2.2. Formula of stress branches on the rotating damage axis

Stress branches under cyclic tension or compression consist of loading, unloading, reloading, partial unloading and partial reloading. The stress branches under uniaxial cyclic loading are shown in Fig.2, with \( s_1 > 0, e_1 > 0 \) for tensile response and \( s_1 < 0, e_1 < 0 \) for compressive one.

![Figure 2. Stress branches under uniaxial cyclic loading: a) tension; b) compression](image)

According to Equation (5)–(7), the relations between principal stress and inelastic strain are decoupled for all material axes. Furthermore, it is suitable to define stress branches in the form of principal stress and inelastic strain because there is no energy dissipation caused by the relations between principal stress and elastic strain.

2.2.1. Loading branch

A piecewise linear function which connects characteristic data points of experimental response is utilized to define the loading stress branch.

\[ |s_i| = \begin{cases} \sigma_i^k + k_i^k(e_i^{in} - a_i^k) & a_i^k \leq e_i^{in} \leq a_i^{k+1} (i = 0, n - 1) \\ \sigma_i^k & \text{if } a_i^k < e_i^{in} < a_i^{k+1} \end{cases} \]

\[ k_i^k = \frac{\sigma_i^{k+1} - \sigma_i^k}{a_i^{k+1} - a_i^k}, \quad \kappa = \begin{cases} t & \text{if } e_i^{in} > 0, s_i^{in} > 0 \\ c & \text{if } e_i^{in} < 0, s_i^{in} < 0 \end{cases} \]

where \( \kappa \in (t, c) \) is state variable with \( \kappa = t \) for tension and \( \kappa = c \) for compression, \((\sigma_i^k, \sigma_i^c)\) are the absolute values of inelastic strain and corresponding stress of the \( i \)th point. Fig.3 shows the loading stress branch defined by equation (9). However, equation (9) is merely applicable to uniaxial loading or the conditions with low lateral stress because it doesn’t account for the influence of multiaxial stress state on tensile or compressive strength. Therefore, it is necessary to develop a general function of loading branch under multiaxial stress state in the further study.
2.2.2. Unloading branch

Concrete tests conducted by Karsan[26], Buyukozturk[27], Bahn[28], Gopalaratnam[29] and Yankelevsky[30] show that irrecoverable deformation occurs even completely removing external loads. The results of confined tests from Buyukozturk[27] also indicate that the irrecoverable deformation was not remarkably affected by confining stresses or strains. The residual strain, called plastic offset strain by Palermo[21], is essentially the amount of irrecoverable damage resulting from concrete crushing, compressing of internal voids and micro-crack opening.

The unloading stress branch shown in Fig. 4 is given as follows.

\[
\begin{align*}
|s_1| &= |\sigma_m| + B^\kappa |e^{in}_{m} - e^{in}_{1}| + C^\kappa |e^{in}_{m} - e^{i\alpha}_{1}| \\
B^\kappa &= \beta^\kappa [\alpha^\kappa |\sigma_m| \cdot |e^{in}_{m} - e^{p}_{\kappa}|^{\alpha^\kappa - 1} - H^{\kappa}_{1} |e^{in}_{m} - e^{p}_{\kappa}|] \\
C^\kappa &= \beta^\kappa [H^{\kappa}_{1} |e^{in}_{m} - e^{p}_{\kappa}| - |\sigma_m|] \\
\beta^\kappa &= \frac{1}{(1 - \alpha^\kappa) |e^{in}_{m} - e^{p}_{\kappa}|^{\alpha^\kappa}}
\end{align*}
\]

in which \(e^{in}_{1}\) and \(s_1\) are the present inelastic strain and principal stress on damage axis \(y_1\), \(e^{in}_{m}\) and \(\sigma_m\) denote the previously maximum inelastic strain and corresponding stress at the onset of unloading, \(e^{p}_{\kappa}\) is the plastic offset strain, \(H^{\kappa}_{1}\) represents the tangent modulus at a zero stress and exponent \(\alpha^\kappa\) (\(0 < \alpha^\kappa < 1\)) is material constant determined by experimental response. Here \(\kappa \in (t, c)\) denotes tensile or compressive state respectively.

Figure 3. The stress branch of loading defined by principal stress and inelastic strain

Figure 4. The nonlinear unloading branch defined in the form of principal stress and inelastic strain
2.2.3. Reloading branch  In this model, reloading is modeled using a linear response combined with a damage factor $d$ representing stiffness degradation.

$$
|s_1| = |\sigma_a| + H |e_i^{\text{in}} - e_a^{\text{in}}| \tag{11} 
$$

$$
H = \frac{1-d}{d} E, \quad 0 < d < 1
$$

where $e_a^{\text{in}}$ and $\sigma_a$ are the inelastic strain and corresponding stress at the onset of reloading. $H$ is the slope of reloading branch. Damage factor $d$ is defined as a function of tensile and compressive damage variables, which is formulated in section 2.4.

![Figure 5. The linear reloading branch defined in the form of principal stress and inelastic strain](image)

Fig. 5 shows the linear reloading branch defined by equation (11). It should be noted that reloading concludes once the stress updated on the reloading branch exceeds the envelope defined by equation (9). This means that concrete undergoes loading again and therefore the principal stress $s_1$ and its derivative $ds_1/de_i^{\text{in}}$ should be calculated according to equation (9).

2.2.4. Partial unloading branch  Most of the models available in the literature ignores the behavior of concrete for the case of partial unloading/reloading because there is a lack of experimental information considering the general case where partial unloading is followed by partial reloading to strains less than the previously maximum unloading strain. Herein the proposed rule for the partial unloading response is similar to that assumed for full unloading. Nevertheless, the previously maximum inelastic strain $e_i^{\text{in}}$ and corresponding stress $\sigma_i$ at the onset of full unloading are replaced by the inelastic strain $e_i^{\text{in}}$ and stress $\sigma_i$ at the onset of partial unloading,

$$
|s_1| = |\sigma_i| + B^\xi |e_i^{\text{in}} - e_i^{\text{ex}}| + C^\xi |e_i^{\text{in}} - e_i^{\text{in}}| + H^\xi |e_i^{\text{in}} - e_i^{\text{ex}}| \tag{12} 
$$

$$
B^\xi = \beta_i^\xi \left[ \sigma_i \cdot |e_i^{\text{in}} - e_p^{\text{ex}}|^{\alpha^\xi - 1} - H_i^\xi |e_i^{\text{in}} - e_p^{\text{ex}}| \right] 
$$

$$
C^\xi = \beta_i^\xi \left[ H_i^\xi |e_i^{\text{in}} - e_p^{\text{ex}}| - |\sigma_i| \right] 
$$

$$
\beta_i^\xi = \frac{1}{(1 - \alpha^\xi)|e_i^{\text{in}} - e_p^{\text{ex}}|^{\alpha^\xi}} 
$$

in which parameter $H_i^\xi$ and $\alpha^\xi$ are identical to those used in equation (10). Fig. 4 also shows the partial reloading branch defined by equation (12).
2.2.5. **Partial reloading branch** The stress branch of partial reloading is also assumed as a linear response similar to that of full reloading:

\[
|s_1| = |\sigma_b| + H |e_1^i - e_b^i| \tag{13}
\]

\[
H = \frac{1 - d}{d} E, \quad 0 < d < 1
\]

where \(e_b^i\) and \(\sigma_b\) are the inelastic strain and stress at the onset of partial reloading. The slope \(H\) and damage factor \(d\) are identical to those used in equation (11).

Based on the derivation of anisotropic damage model in section 2.1 and the descriptions of stress branches in section 2.2, it is convenient to calculate the stress and tangent material stiffness matrix. Then, the nonlinear equations of equilibrium can be solved using the Newton-Raphson method.

### 2.3. The definitions of damage variables

Herein, the definition of damage variable \(d_\kappa, \kappa \in (t, c)\) depends on the density of accumulated dissipating energy \(\gamma^\kappa, \kappa \in (t, c)\) and that of fracture energy \(g_f^\kappa, \kappa \in (t, c)\) in pure tension or compression. As shown in Fig. 6a, the density of fracture energy is defined as the area of the shadow region:

\[
g_f^\kappa = \int_{0}^{e_{1f}^\kappa} s_1 de_1^i = \frac{G_f^\kappa}{l^K} \tag{14}
\]

in which \(e_{1f}^\kappa, \kappa \in (t, c)\) denotes the inelastic strain corresponding to complete damage in tension or compression. \(G_f^\kappa\) is a material property representing the tensile fracture energy or its counterpart in the compressive state, and \(l^K\) is the size of localization zone determined by the characteristic length of element to maintain objective results at the structural level. However, it is still an open issue whether the counterpart of the fracture energy in compressive failure is a material property.

![Figure 6](image-url)

**Figure 6.** The definition of fracture energy and accumulated dissipating energy under pure tension or compression: a) the density of fracture energy; b) the density of accumulated dissipating energy

The accumulated dissipating energy per unit volume \(\gamma^\kappa\) is defined as the summation of energy dissipation during previously cyclic loading. As illustrated in Fig.6b, the \(\gamma^\kappa\) is calculated as:

\[
\gamma^\kappa = \gamma_{1f}^\kappa + \gamma_2^\kappa + \gamma_3^\kappa + \cdots \tag{15}
\]

where \(\gamma_1^\kappa, \gamma_2^\kappa\) and \(\gamma_3^\kappa\) denote the energy dissipation corresponding to loading, unloading/reloading and partial unloading/reloading responses respectively. Thus, the damage variable \(d_\kappa\) is defined as:

\[
d_\kappa = (\kappa_\kappa)^{\gamma^\kappa}, \quad \kappa = \frac{\gamma^\kappa}{g_f^\kappa}, \quad \kappa \in (t, c) \tag{16}
\]
in which $\kappa_c$ is a dimensionless variable defined as the ratio of accumulated dissipating energy to fracture energy, and $\theta_c$ denotes damage exponent which is material property. Obviously, $\kappa_c = 0$ represents intact state with $d_c = 0$ while $\kappa_c = 1$ denotes completely damage with $d_c = 1$. Moreover, the damage factor $d$ used in equation (11) and (13) equals the damage variable $d_c$ for pure tension or compression.

2.4. The effect of stiffness recovery

The approach proposed by [14] is used to model the effect of stiffness recovery due to crack opening/closing. Herein, the damage factor $d$ on the rotating damage axis is defined as follows by combining the tensile damage variable $d_t$ with the compressive one $d_c$:

$$d = 1 - (1 - S_t d_t) (1 - S_c d_c)$$

(17)

where $S_t$ and $S_c$ are functions of the stress state which are introduced to model stiffness recovery effects associated with stress reversals. They are defined as

$$S_t = 1 - w_t r^s (s_t) \quad 0 \leq w_t \leq 1$$
$$S_c = 1 - w_c [1 - r^s (s_t)] \quad 0 \leq w_c \leq 1$$
$$r^s (s_t) = \begin{cases} 1 & \text{if } s_t > 0 \\ 0 & \text{if } s_t < 0 \end{cases}$$

The weight factors $w_t$ and $w_c$ are assumed to be material properties which controls the recovery of tensile and compressive stiffness upon load reversal. Concrete tests show that the compressive stiffness is recovered upon crack closure as the load changes from tension to compression, whereas the tensile one is not recovered as the load reverses from compression to tension. Therefore, $w_t = 0$ and $w_c = 1$ representing the above-mentioned behavior are the default values used in this model.

This approach is an approximate modeling of stiffness recovery effect because it assumes that the recovery of stiffness accomplishes immediately at point $(\varepsilon^p_c, 0)$ once stress reversal occurs. In fact, concrete tests conducted by [31] and [32] show that complete closing of cracks requires a certain amount of compression and the stiffness is not affected by accumulated damage in tension once the cracks are closed. Therefore, its accuracy should be carefully investigated in the further study.

3. Numerical simulations of cyclic concrete tests

3.1. Cyclic compressive test conducted by Karsan and Jirsa

The specimens tested by [26] were $82.6 \times 82.6 \times 82.6$ mm in dimension. Material properties are: Young’s modulus $E = 3.17 \times 10^4$ MPa, Poisson’s ratio $\nu = 0.2$, compressive strength $f_c = -27.6$ MPa, the density of compressive fracture energy $g^c_f = 0.20$ N/mm$^2$ and the exponent of unloading branch $\kappa^c = 0.5$. Fig.7 shows the finite element mesh and loading branch used to model the test.

The compressive plastic offset is given as

$$\frac{\varepsilon^p}{\varepsilon^0_c} = c_c + k_c \left( \frac{\varepsilon^1}{\varepsilon^0_c} \right) + m_c \left( \frac{\varepsilon^1}{\varepsilon^0_c} \right)^2$$

(18)

where $\varepsilon^0_c$ denotes the strain at the peak stress under uniaxial monotonic compression. Based on the experimental data of Karsan’s test, $\varepsilon^0_c = 2.5 \times 10^{-3}$, $c_c = -0.1175$, $k_c = 0.5060$, $m_c = 0.0989$ and the damage exponent $\theta_c = 0.2678$ are obtained by an inverse analysis.

The stress-strain response calculated by the model is compared with that obtained by the test in Fig.8a. Numerical responses such as residual deformation, nonlinear unloading branches and degraded stiffness are in close agreement with the experimental ones. The hysteretic behavior of concrete is also represented via the nonlinear unloading/linear reloading branches. Moreover, the numerical reloading branch approaches the experimental one and models the stiffness degradation. This indicates that it
3.2. Cyclic tensile test conducted by Gopalartnam and Shah

Cyclic tests conducted by [29] are analyzed by the present model to verify its capability for simulating the nonlinear behavior of concrete in tension. The material properties are given as follows: Young’s modulus $E = 3.10 \times 10^4$ MPa, Poisson’s ratio $\nu = 0.2$, tensile strength $f_t = 3.48$ MPa, tensile fracture energy density $g_t^f = 6.80 \times 10^{-4}$ N/mm$^2$ and the exponent of unloading branch $a^t = 0.75$. Fig.9 shows the finite element mesh and loading branch used in the simulation.

The tensile plastic offset is given as

$$\frac{\varepsilon^p}{\varepsilon^0} = c_t + k_t \left( \frac{\varepsilon^{in}}{\varepsilon^0} \right)$$

(19)
where $\varepsilon_0^t$ denotes the strain at the peak stress under uniaxial monotonic tension. $\varepsilon_0^t = 1.12 \times 10^{-4}$, $c_t = -0.119$, $k_t = 0.824$ and the damage exponent $\theta_t = 0.824$ are calculated from an inverse analysis.

The numerical result is shown in Fig.10, compared with the experimental one. The stress-strain response is comparable to that obtained in test, such as the stiffness degradation, irrecoverable deformation and hysteretic behavior. The effect of $\theta_t$ on the response is also investigated by modeling the test with $\theta_t = 0.75, 0.8242$ and $0.90$. Under displacement loadings, the unloading branches obtained in these three cases are identical and the reloading ones are slightly different.

### Figure 9. Numerical simulation of Gopalaratnam’s test: a) the finite element mesh; b) the stress branch of loading

3.3. Cyclic tensile-low compressive test conducted by Reinhardt
The test conducted by Reinhardt [31] is simulated herein. The material properties are as follows: Young’s modulus $E = 3.90 \times 10^4$ MPa, Poisson’s ratio $\nu = 0.20$, tensile strength $f_t = 3.20$ MPa, compressive strength $f_c = -40.0$ MPa, the density of tensile fracture energy $g_f^t = 1.13 \times 10^{-3}$ N/mm$^2$, its counterpart under compression $g_f^c = 0.294$ N/mm$^2$, and the tensile and compressive exponents of unloading branch $\alpha^t = 0.75$ and $\alpha^c = 0.50$. The tensile plastic offset is given by Equation (19), with $\varepsilon_0^t = 0.82 \times 10^{-4}$, $c_t = 0.2019$, $k_t = 0.6098$ obtained from an inverse analysis. Fig.11 shows the finite element mesh and loading branch used in this simulation.
Figure 11. Numerical model of Reinhardt’s test: a) the finite element mesh; b) the loading branch

The recovery of compressive stiffness is modeled with $w_c = 1.0$, 0.5, 0.3 and 0.0 and that of the tensile stiffness is assumed as $w_t = 0.0$. Fig.12 shows the stress-strain responses obtained from the numerical simulations and experiment.

Figure 12. The comparison between numerical and experimental results: a) $w_c = 1.0$; b) $w_c = 0.0$; c) $w_c = 0.5$; d) $w_c = 0.3$

Compared with the experimental result, the four cases obtain similar responses in tension which reflect the stiffness degradation, accumulation of residual deformation and hysteretic behavior of concrete under cyclic loading. However, the variation of $w_c$ causes significant difference in the numerical response, especially in the compressive domain.

In the case of $w_c = 1.0$, the compressive stiffness is fully recovered once the load reverses from tension to compression and is not influenced by the tensile damage factor $d_t$. Linear loading/unloading
branches, whose stiffness is equal to the initial elastic stiffness \( E \), are obtained in the compressive domain because the stress is much lower than the compressive strength of concrete. As shown in Fig.12a, the compressive stress-strain response obtained with \( w_c = 1.0 \) is overestimated compared to the experimental result and it does not reflect the hysteretic behavior in compression.

In the case of \( w_c = 0.0 \), the effect of stiffness recovery is completely ignored and the compressive stiffness simultaneously takes into account the tensile and compressive damage. Due to the degraded stiffness, the linear reloading branch yields inelastic strain which causes the occurrence of plastic offset strain and nonlinear unloading behavior in compression. Thus, this case reflects the hysteretic behavior in the compressive domain based on the linear reloading/nonlinear unloading branches. However, the result analyzed with \( w_c = 0.0 \) remarkably underestimates the compressive stress-strain response observed in the cyclic test.

In the cases of \( w_c = 0.5 \) and \( 0.3 \), the compressive stiffness is partially recovered to \((1 - d_c) [1 - (1 - w_c)d_c] E \) and the stiffness of reloading branch increases with the value of \( w_c \). As shown in Fig.12, the compressive response calculated with \( w_c = 0.3 \) is in the closest agreement with the experimental result while that obtained with \( w_c = 0.5 \) is slightly overestimated.

These results show that the stiffness recovery factor \( w_c \) has significant influence on the computational accuracy of the present model. With appropriate \( w_c \), this model can obtain consistent response with the experimental one, which reflect the stiffness degradation, accumulation of residual deformation and hysteretic behavior in tension and compression. Therefore, it is necessary to study the calibration of \( w_c \).

4. Conclusions

An anisotropic damage model is developed for analyzing the response of concrete structures subjected to cyclic loading. Nonlinear unloading/linear reloading branches are implemented into the model to represent the hysteretic behavior. Two independent damage variables, one for tensile and the other for compressive load, are determined by the ratio of the accumulated dissipating energy to the fracture energy to model the stiffness degradation. In addition, this study presents the calibrating procedures of the key parameters which govern the constitutive behavior. Using this model, cyclic tests are numerically analyzed and the results draw the following conclusions:

(i) The stress-strain response calculated by the numerical analysis is similar to that obtained by the experiment. The numerical response reflects the damage-induced degradation of stiffness, the accumulation of residual deformation, energy dissipation due to hysteretic behavior and the recovery of stiffness upon crack closure. From an engineering point of view, the proposed damage model is applicable to simulating the nonlinear behavior of concrete under cyclic loading.

(ii) The calculated responses of unloading and reloading are consistent with the experimental ones. This indicates that the hysteretic behavior can be approximately modeled using the nonlinear unloading/linear reloading branches and it is feasible to define the tensile or compressive damage variable \( d_c \) as a function of the ratio of the accumulated dissipating energy to the fracture energy.

(iii) The plastic offset strain \( \varepsilon_p^0 \), tangent modulus at a zero stress of unloading \( H_p^0 \) and damage exponent \( \theta_c \) are the key parameters which govern the accuracy of the model. The response calculated with the calibrated parameters is in close agreement with that obtained by the experiment, which means that the calibrating procedures proposed in this study is suitable to determine these key parameters.

(iv) The effect of stiffness recovery is approximately modeled using a sudden recovery of stiffness with weight factor \( w_c \). This approach oversimplifies the simulation of crack closure and the value of \( w_c \) has significant influence on the computational accuracy. A refined approach which reasonably represents the crack closure mechanism should be implemented into the present model in the further study.
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References

[1] Mazars J., 1986. A description of micro- and macroscale damage of concrete structures. Eng. Fract. Mech. 25(5-6), 729–37.
[2] Mazars J., Pijaudier-Cabot G., 1989. Continuum damage theory – application to concrete. ASCE J. Eng. Mech. 115(2), 345–65.
[3] Cervera M., Oliver J., Faria R., 1995. Seismic evaluation of concrete dams via continuum damage models. Earthq. Eng. Struct. D. 24(9), 1225–45.
[4] Papa E., Taliercio A., 1996. Anisotropic damage model for the multiaxial static and fatigue behavior of plain concrete. Eng. Fract. Mech. 55(2), 163–79.
[5] Al-Gadhib A.H., Baluch M.H., Shaalan A., Khan A.R., 2000. Damage model for monotonic and fatigue response of high strength concrete. Int. J. Damage Mech. 9(1), 57–78.
[6] Li Q.B., Zhang L.X., Ansari F. 2002. Damage constitutive for high strength concrete in triaxial cyclic compression. Int. J. Solids Struct. 39(15), 4013–25.
[7] Challamel N., Lanois C., Casandjian C. 2005. Stress-based anisotropic damage modelling and unilateral effects. Int. J. Mech. Sci. 47(3), 459–73.
[8] Badel P., Godard V., Leblond J.B., 2007. Application of some anisotropic damage model to the prediction of the failure of some complex industrial concrete structure. Int. J. Solids Struct. 44(18-19), 5848–74.
[9] Pröchtel P., Händler-Combe U., 2008. On the dissipative zone in anisotropic damage models for concrete. Int. J. Solids Struct. 45(16), 4384–406.
[10] Simo J.C., Ju J.W. 1987. Strain- and stress-based continuum damage models–I. Formulation. Int. J. Solids Struct. 23(7), 821–40.
[11] Lubliner J., Oliver J., Oller S., Onate E. 1998. A plastic-damage model for concrete. Int. J. Solids Struct. 25(3), 299–326.
[12] Luccioni B., Oller S., Danesi R., 1996. Coupled plastic-damaged model. Comput. Methods Appl. Mech. Eng. 129(1-2), 81–9.
[13] Faria R., Oliver J., Cervera M. 1998. A strain-based plastic viscous-damage model for massive concrete structures. Int. J. Solids Struct. 35(14), 1533–58.
[14] Lee J., Fenves G.L., 1998. Plastic-damage model for cyclic loading of concrete structure. ASCE J. Eng. Mech. 124(8), 892–900.
[15] Grassl P., Jirásk M., 2006. Damage-plastic model for concrete failure. Int. J. Solids Struct. 43(22-23), 7166–96.
[16] Wu J.Y., Li J., 2006. An energy release rate-based plastic-damage model for concrete. Int. J. Solid Struct. 43(3-4), 583–612.
[17] Voyiadis G.Z., Taqieddin Z.N., Kattan P.I., 2008. Anisotropic damage-plasticity model for concrete. Int. J. Plasticity 24(10), 1946–65.
[18] Poh L.H., Svaddiwudhipong S., 2009. Over-nonlocal gradient enhanced plastic-damage model for concrete. Int. J. Solids Struct. 46(25-26), 4369–78.
[19] Lee J., Fenves G.L., 1998. Plastic-damage concrete model for earthquake analysis of dams. Earthq. Eng. Struct. D. 27(9), 937–56.
[20] Faleiro J., Oller S., Barbat A.H., 2008. PlasticDamage seismic model for reinforced concrete frames. Comput. Struct. 86(7-8), 581–97.
[21] Palermo D., Vecchio F.J., 2003. Compression field modeling of reinforced concrete subjected to reversed loading: formulation. ACI Struct. J. 100(5), 616–25.
[22] Sima J.F., Roca P., Molins C., 2008. Cyclic constitutive model for concrete. Eng. Struct. 30(3), 695–706.
[23] Rois J., Blauwendraad J., 1989. Crack models for concrete: discrete or smeared? Fixed, multi-directional or rotating? Heron 34(1), 3–59.
[24] Bazant, Z.P., 1983. Comment on orthotropic models for concrete and geomaterials. ASCE J. Eng. Mech. 109, 849–65.
[25] Willam K., Pramono E., Sture S., 1987. Fundamental issues of smeared crack models. In: Shah S.P., Swartz S.E.(Eds.), Proceedings SEM/RILEM International Conference of Fracture of Concrete and Rock. Springer-Verlag, Heidelberg, Berlin, New York, pp. 142–157.
[26] Karsan I.K., Jirsa J.O., 1969. Behaviour of concrete under compressive loadings. ASCE J. Struct. Div. 95(12), 2543–63.
[27] Buyukozturk O., Tieng T.M., 1984. Concrete in biaxial cyclic compression. ASCE J. Struct. Eng. 110(3), 461–76.
[28] Bahn B.Y., Hsu C.T.T., 1998. Stress-strain behaviour of concrete under cyclic loading. ACI Mater. J. 95(2), 178–93.
[29] Gopalaratnam V.S., Shah S.P., 1985. Softening response of plain concrete in direct tension. J. ACI 82(2), 310–23.
[30] Yankelevsky D.S., Reinhardt H.W., 1989. Uniaxial behaviour of concrete in cyclic tension. ASCE J. Struct. Eng. 115(1), 166–82.
[31] Reinhardt H.W., 1984. Fracture mechanics of an elastic softening material like concrete. Heron 29(2), 1–42.
[32] Ramtani S., Berthaud Y., Mazars J, 1992. Orthotropic behavior of concrete with direction aspects: Modeling and experiments. Nucl. Eng. Des. 133(1), 97–111.