Experimental Investigation of Nonideal Two-qubit Quantum-state Filter by Quantum Process Tomography

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We used quantum process tomography to investigate and identify the function of a nonideal two-qubit quantum-state filters subject to various degree of decoherence. We present a simple decoherence model that explains the experimental results and point out that a beamsplitter followed by a post-selection process is not, as commonly believed, a singlet-state filter. In the ideal case it is a triplet-state filter.

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Characterization of quantum devices is important, and the characterization of single- and two-qubit devices is particularly important because a single-qubit rotation gate and a two-qubit controlled-NOT (C-NOT) gate are two building blocks of the quantum computer[1]. The quantum systems in actual devices may undergo nonunitary state evolution because they interact with their environments, so identifying and understanding the function of such nonideal devices is of practical importance. Nonideal devices can be characterized by quantum process tomography (QPT)[2, 3, 4], and this method has been used to characterize single-qubit transmission devices and is called the process matrix in the standard operator space[16], where \( \hat{E}_{ij} \) or nonlocally. For example, the Kronecker products of a 16×16 positive Hermitean matrix \( \chi^S \) fully characterizes the function of quantum device and is called the process matrix in the standard operator basis[10].

We can transform \( \chi^S \) into the process matrix in any operator basis set \( \{ A_\alpha \}_{j=0}^{15} \) that spans two-qubit operator space. Note that \( A_\alpha \) may be defined either locally or nonlocally. For example, the Kronecker products of the Pauli operator basis \( \sum_{k,l} \hat{E}_{ij} \otimes \hat{E}_{kl} \) where \( \hat{E}_{ij} = \sum_{l} \hat{E}_{ij,l} \otimes \hat{E}_{kl} \) and \( \hat{E}_{ij,l} \) is an identity map and the superscript and subscript respectively denote the bipartite state, where \( \hat{E}_{ij,l} \) is a local operator basis, whereas the outer product \( \sum_{k,l} \hat{E}_{ij} \otimes \hat{E}_{kl} \) is a nonlocal one. The operator basis \( A_\alpha \) is, in general, unitarily related to the standard operator basis \( \hat{X}_k \otimes \hat{X}_l \). That is, \( A_\alpha = \sum_{k,l} \hat{U}_{\alpha} \otimes \hat{U}_{\alpha} \hat{X}_k \otimes \hat{X}_l \), where \( U \) is a 16×16 unitary matrix. Then \( \chi^S \) can be expanded using \( A_\alpha \) as follows:

\[
\chi^S = \sum_{\alpha, \beta} A_\alpha \rho A_\beta^\dagger, \tag{2}
\]

where \( \chi^A = U^\dagger \chi^S U \). Therefore, \( \chi^A \) is given by the unitary transform of \( \chi^S \).

To obtain \( \chi^S \), it is sufficient to determine the associated four-qubit states[17]. Let us consider a four-qubit system composed of the two two-qubit systems 1-2 and 3-4 and introduce the double-ket notation \( |ij\rangle := (I \otimes I) |\Phi\rangle \) to denote the bipartite state, where \( |\Phi\rangle = \sum_{i=0}^{1} |ii\rangle \) is the unnormalized maximally entangled state[4, 16]. Let us introduce the unnormalized four-qubit density operator \( D_E \) associated with \( E \) by \( D_E = E^{(13)} \otimes \bar{T}^{(24)}(|ij\rangle\langle ij|)_{12} \otimes (I \otimes |ij\rangle\langle ij|)_{34} \), where \( \bar{T} \) is an identity map and the superscript and subscript respectively denote the space on which each map acts and
the space to which the operator belongs. Using Eq. 1 and $|X_k\rangle := (\hat{X}_k \otimes \hat{I}) \Phi$, we can rewrite $\hat{D}_E$ as

$$\hat{D}_E = \sum_{k,l,m,n=0}^3 \left( \chi^S \right)_{[k][m,n]} |X_k\rangle \langle X_l| \langle X_m| \langle X_n|.$$  

If we note that $|X_{(ij)}\rangle \equiv |i, j\rangle$, Eq. 3 indicates that $\chi^S$ agrees exactly with the density matrix of the associated state $\hat{D}_E$ given in the standard basis. Evaluation of the process matrix is thus reduced to evaluation of the associated state $\hat{D}_E$.

To obtain $\hat{D}_E$, it is sufficient to evaluate all the maps $\mathcal{E}(\hat{X}_{(ij)} \otimes \hat{X}_{(kl)})$ for all the elements of the standard operator basis $\hat{X}_{(ij)} \otimes \hat{X}_{(kl)}$, since we can obtain the permutation equivalent of $\hat{D}_E$—that is, $\hat{D}_E = \hat{P}_{21}\hat{P}_{12}\hat{D}_E\hat{P}_{12}\hat{P}_{12}\hat{P}_{21}$—by using the following identity:

$$\hat{D}_E \equiv \sum_{k,l=0}^3 \hat{X}_k \otimes \hat{X}_l \otimes \mathcal{E}(\hat{X}_k \otimes \hat{X}_l),$$  

where $\hat{P}_{ij}$ is a permutation operator between the systems $i$ and $j$. To evaluate these maps, we need only to use state tomography to evaluate the 16 two-qubit output states $\{\mathcal{E}(\hat{\rho}_{ij}^{(1)} \otimes \hat{\rho}_{ij}^{(2)})\}_{i,j=0}^3$ associated with the set of 16 separable input states, where $\hat{\rho}_{ij} \in \{|00\rangle, |11\rangle, |+\rangle, |\pm\rangle\}$. We therefore can obtain the process matrix directly from the experimental data without using a matrix inversion procedure.

We used the above procedure to obtain the process matrices for nonideal two-qubit quantum-state filters. The experimental setup is shown in Fig. 1. A 0.3-mm-thick type-I crystal (BBO) was pumped by frequency-tripled pulses generated by 100-fs pulses from a mode-locked Ti:sapphire laser (repetition rate $\approx 82$ MHz). Pairs of horizontally polarized 532-nm photons were generated and those traveling in two directions $9^\circ$ from the pump beam were selected by 5-mm irises (1 m from the crystal) and identical 3-nm-width filters (IF) placed before the detectors. They were injected into a HOM interferometer having a broadband nonpolarizing cube BS (Suruga Seiki model S322-20-550). The horizontally ($H$) and vertically ($V$) polarized states (respectively $|0\rangle$ and $|1\rangle$) and the other two states required for the QPT were prepared by half- and quarter-wave plates (HWP and QWP) placed immediately before the BS. Down-converted photons were incident on the BS at $45^\circ$. After the BS, polarization analyzers consisting of a QWP and a HWP followed by a Glan-Thompson prism (POL) and a single-photon counting module (PMT) performed projective measurements onto the set of the 16 product states $\{\mathcal{E}(\hat{\rho}_{ij}^{(1)} \otimes \hat{\rho}_{ij}^{(2)})\}_{i,j=0}^3$. Only signals coincident at the two PMTs were recorded, and the associated 16 output states $\{\mathcal{E}(\hat{\rho}_{ij}^{(1)} \otimes \hat{\rho}_{ij}^{(2)})\}_{i,j=0}^3$ were evaluated by state tomography. Note that we used unnormalized output density matrices because the state filtering is trace-decreasing process whose output intensity depends on the input state.

FIG. 1: Experimental setup to investigate the beamsplitter.

We used optical trombones to equalize the two path lengths to within the coherence length $\ell_c$ of the down-converted photons ($\approx 200$ μm). Two-photon amplitudes contributing to coincidence detection—i.e., that both photons are reflected ($r-r$) or that both are transmitted ($t-t$)—were initially ($\tau = 0$) aligned to overlap both spatially and temporally, giving a HOM dip visibility of 85% as shown in Fig. 2. In this case, the BS is commonly thought to work as a singlet-state filter. To investigate the nonideal filter, we introduced a delay $\tau$ in one arm by moving one of the trombone prisms. The larger the $\tau$ is, the smaller the overlap of the two amplitudes is and the more the temporal and polarization information of the photon pair is entangled. We will show later that this results in decoherence if we observe only the polarization information.

FIG. 2: HOM dip observed in the coincidence record. Arrows indicate delays $\tau$ of (A) 0, (B) 100, and (C) 350 fs introduced in one of the two arms.

We obtained the process matrices $\chi^E$ associated with three-delay conditions ($\tau = 0$, 100, and 350 fs). To make the results clear, we represent them here in the nonlocal operator basis $\hat{F}_{[ij]} = (\hat{E}_i \otimes \hat{E}_j)\hat{U}_S$, where $\hat{U}_S$ is a swap operator. They are shown in Fig. 3 where space is saved by showing only the real parts. What is most interest-
ing is that the contribution of \((\chi^F)_{1515}\) increases with increasing delay \(\tau\).

This result is consistent not with the common belief but with what was pointed out by Mitchell et al.\[9\]. Indeed, we found that our results are well reproduced if we assume that \(E(\hat{\rho})\) has an operator-sum representation with two Kraus operators \(\hat{P}^-\) and \(\hat{P}^+\):\[1,14\]

\[
E(\hat{\rho}) = (1 - p)\hat{P}^-\hat{\rho}\hat{P}^{-1} + p\hat{P}^+\hat{\rho}\hat{P}^{+1},
\]

\[
\hat{P}^\pm = \mathcal{T}\hat{U}_3(\pm i\mathcal{R}_\Gamma, \theta_\Sigma)\hat{U}_S,
\]

where \(p\) (\(0 \leq p \leq 1/2\)) is a parameter that can be interpreted as the degree of decoherence and where \(\hat{U}_3(\theta_\Sigma, \theta_\Gamma) = e^{i\theta_\Sigma/2\sigma_3} \otimes e^{-i\theta_\Gamma/2\sigma_3}\), which explains the large contribution of \((\chi^F)_{1515}\) instead of \((\chi^F)_{00}\) as would be expected for the nonideal singlet-state filter. The Kraus operators \(\hat{P}^-\) and \(\hat{P}^+\) correspond respectively to the operator for the ideal singlet-state filter and the noise operator. Note that Kraus operators are not unique and are not necessarily orthogonal\[1\].

Figure 4 shows the process matrices calculated using the measured value \(\mathcal{R}/\mathcal{T} = 0.76\) and appropriate values of the parameters \(\theta_\Sigma\) and \(\theta_\Gamma\). As shown in Fig. 4, Eqs. 5 and 6 well reproduced the experimental results. It follows that the operator \(\hat{P}^-\) describing ideal operation deviates from the singlet-state because of the factor \(\hat{U}_3(\theta_\Sigma, \theta_\Gamma)\), which is consistent with Ref. \[9\].

To understand these results, we need a decoherence model. Let us first call our attention to the function of the BS in the polarization space. Let \(\hat{a}\) and \(\hat{b}\) (\(\hat{c}\) and \(\hat{d}\)) denote input (output) field operators of the BS, and the polarization be indicated by subscripts (\(e.g., \)by \(\hat{a}_H\) and \(\hat{a}_V\) for the \(H\)- and \(V\)-polarized field operators). If we use a right-handed coordinate system defined by the positive directions of \(H\)-, \(V\)-polarized field (\(p\) and \(s\) components) and propagation vector (Fig. 4) to describe the input and output fields, their operator equations are given by:

\[
\begin{align*}
\hat{e}_H &= \sqrt{T}a_H + e^{i\pi/4}\sqrt{\mathcal{R}}b_H, \\
\hat{d}_H &= e^{-i\pi/4}\sqrt{\mathcal{R}}a_H + \sqrt{T}b_H, \\
\hat{e}_V &= \sqrt{T}a_V + e^{i\Delta}\sqrt{\mathcal{R}}b_V, \\
\hat{d}_V &= e^{-i\Delta}\sqrt{\mathcal{R}}a_V + \sqrt{T}b_V,
\end{align*}
\]

where \(T\) and \(\mathcal{R}\) are respectively the transmission and reflection coefficients of the BS. We have assumed that the \(p\) and \(s\) components of the field respectively acquire phase shifts \(\Gamma + \pi\) and \(\Delta\) upon reflection. The factor \(\pi\) accounts for abrupt change in \(p-s\) phase shift due to nonadiabatic changes in the \(k\) vector upon reflection, which has been frequently noted in the papers concerning the topological phase effect in optics\[20,21,22\]. Note that the helicity of a photon in a state \(|\psi\rangle\) changes sign upon reflection from the BS but not upon transmission through the BS if \(\Delta - \Gamma = 0\). If we consider the output fields that can contribute to coincidence measurement, we conclude from Eqs. 4 that the BS projects the input state onto \(\hat{P}^-\) with \(\theta_1 = \theta_2 = \Delta - \Gamma\), which may be finite for a non-polarizing BS. Note that the projector \(\hat{P}^-\) is the coherent sum of the identity operator \(\hat{U}_I\) associated with transmission and the operator \(\hat{U}_S(\theta_1, \theta_2)\hat{U}_S\) associated with reflection, where \(\hat{U}_3(\theta_1, \theta_2)\) accounts for the helicity reversal and the extra \(p-s\) phase shift upon reflection.

Although the above description may suffice when \(\tau = 0\), the temporal (\(T\)) space of the two-photon state should be taken into account when \(\tau \neq 0\). Concerning the temporal state space, it would be reasonable to assume that the BS leaves the state untouched on transmission and swaps the state on reflection. We thus assume that the function of the BS is described by the following operator entangled with respect to the polarization and temporal state spaces:

\[
\hat{P}^{PT} = \mathcal{T}\hat{U}_P^\dagger \otimes \hat{U}_T^\dagger - \mathcal{R}\hat{U}_P^\dagger(\theta_1, \theta_2)\hat{U}_S^\dagger \otimes \hat{U}_S^\dagger,
\]

where the superscripts indicate the space on which each operator acts. We further assume that input state is unentangled, \(i.e.,\) that it is a product state of polarization \(\hat{\rho}\) and temporal state \(\hat{\omega}\). Then, by partially tracing over the temporal degree of freedom, we obtain the following output polarization state \(E(\hat{\rho})\) consistent with Eqs. 5 and 6:

\[
E(\hat{\rho}) = \mathcal{T}\tau\hat{P}^{PT} (\hat{\rho} \otimes \hat{\omega}) (\hat{P}^{PT})^\dagger = (1 - p)\hat{P}^-\hat{\rho}\hat{P}^{-1} + p\hat{P}^+\hat{\rho}\hat{P}^{+1},
\]

where the parameter \(p\) depends on the overlap of the two-photon amplitudes contributing to coincidence detection (\(r-r\) or \(t-t\)) and thus on the delay \(\tau\). Ideally, \(p\) is zero if \(\tau = 0\), increases as \(\tau\) increases, and approaches \(1/2\) when \(\tau \to \infty\). Figure 4 shows that this model largely explains the experimental results, except that \(\theta_1\) is not actually equal to \(\theta_2\) as in the model. The reason is uncertain, but this inequality may be due to polarization-dependent \(\mathcal{R}/\mathcal{T}\) ignored in the model.

We conclude that the function of the ideal BS is described by \(\hat{P}^-\), which depends on the \(p-s\) phase shift as well as the coefficients \(\mathcal{R}\) and \(T\). These parameters obviously depend on the detailed design of the BS. Properly designed BS would satisfy \(\mathcal{R}/\mathcal{T} = 1\) and \(\Delta - \Gamma = 0\) simultaneously, at least for photons that have a particular wavelength and are incident at 45°. In this idealized case, \(\hat{P}^-\) agrees with one of the four Bell states: a triplet state. This shows that the common belief that an ideal BS behaves as a singlet-state filter is a misconception due to overlooking the helicity reversal of the photon upon reflection at the BS.

In summary, we used QPT to investigate the nonideal two-qubit quantum-state filter and identify its function. We showed that our results are satisfactorily explained by a simple model with several reasonable assumptions that a common belief about the function of a BS followed by post-selection process is mistaken.

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FIG. 3: The process matrices of the state filter obtained for delays $\tau$ of (A) 0, (B) 100, and (C) 350 fs. Only the real parts of the matrices are shown (in arbitrary units). Vertical arrows indicate the diagonal elements $(\chi^T)_{15,15}$.

FIG. 4: Calculated process matrices corresponding to Fig. 3 (see text). The following parameters were used: (A) $p = 0.14$, (B) $p = 0.325$, (C) $p = 0.5$ and the common parameters $R/T = 0.76$, $\theta_1 \approx 0.41\pi$, and $\theta_2 \approx 0.076\pi$.

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