NEW RESULTS ON SUM–PRODUCTS IN $\mathbb{R}$
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Abstract.

We improve a previous sum–products estimates in $\mathbb{R}$, namely, we obtain that
\[ \max\{|A+A|,|AA|\} \gg |A|^{\frac{4}{3}+\epsilon}, \]
where $c$ any number less than $\frac{5}{9813}$. New lower bounds for sums of sets with small the product set are found. Also we prove some pure energy sum–products results, improving a result of Balog and Wooley, in particular.

1 Introduction

Let $A, B \subset \mathbb{R}$ be finite sets. Define the sum set, the product set and quotient set of $A$ and $B$ as

\[ A + B := \{a + b : a \in A, b \in B\}, \]
\[ AB := \{ab : a \in A, b \in B\}, \]
and
\[ A/B := \{a/b : a \in A, b \in B, b \neq 0\}, \]
correspondingly. The Erdős–Szemerédi conjecture [2] says that for any $\epsilon > 0$ one has
\[ \max\{|A+A|,|AA|\} \gg |A|^{2-\epsilon}. \]
Roughly speaking, it asserts that an arbitrary subset of real numbers (or integers) cannot has good additive and multiplicative structure, simultaneously. Using some beautiful geometrical arguments Solymosi [9], proved the following

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Theorem 1 Let $A \subset \mathbb{R}$ be a set. Then

\[ |A + A|^2 |A/A|, \quad |A + A|^2 |AA| \geq \frac{|A|^4}{4 \log |A|}. \tag{1} \]

In particular

\[ \max \{|A + A|, |AA|\} \gg \frac{|A|^{4/3}}{\log^{1/3} |A|}. \tag{2} \]

Here and below we suppose that $|A| \geq 2$.

It is easy to see that bound (1) is tight up to logarithmic factors if the size of $A + A$ is small relatively to $A$. We will write $a \lesssim b$ or $b \gtrsim a$ if $a = O(b \cdot \log^c |A|)$, $c > 0$. If $a \lesssim b$ and $b \gtrsim a$ then we write $a \sim b$.

In paper [4] we improved bound (2).

Theorem 2 Let $A \subset \mathbb{R}$ be a set. Then

\[ \max \{|A + A|, |AA|\} \gtrsim |A|^{4/3 + c'}, \]

where $c' = \frac{1}{20598}$. The same is true if one replace $AA$ by $A/A$.

The main result of the article is the following.

Theorem 3 Let $A \subset \mathbb{R}$ be a set. Then

\[ \max \{|A + A|, |AA|\} \gtrsim |A|^{4/3 + c}, \]

where $c = \frac{5}{9813}$. The same is true if one replace $AA$ by $A/A$.

In paper [4] the case of sets with small the product/quotient sets was considered (sharper bounds for difference of two sets, having small multiplicative doubling can be found in [7]).

Theorem 4 Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ be a real number. Suppose that $|A/A| \leq K|A|$ or $|AA| \leq K|A|$. Then

\[ |A + A| \gtrsim |A|^\frac{15}{12} K^{-\frac{5}{6}} \tag{3} \]

and

\[ |A + A| \gtrsim |A|^\frac{49}{52} K^{-\frac{10}{13}}. \tag{4} \]
Inequality (4) is better than (3) for \( K \gtrsim |A|^{\frac{5}{23}} \).

We improve Theorem 4 for some range of parameters in the case of small quotient set.

**Theorem 5** Let \( A \subset \mathbb{R} \) be a finite set and \( K \geq 1 \) be a real number. Suppose that \( |A/A| \leq K|A| \). Then

\[
|A + A| \gtrsim \max\{|A|^{\frac{19}{12}} K^{-\frac{5}{6}}, |A|^{\frac{1313}{850}} K^{-\frac{336}{415}}\}. \tag{5}
\]

One can check that lower bound (5) coincides with (3) for \( K \lesssim |A|^{\frac{5}{23}} \) and is better than both estimates (3), (4) for \( |A|^{\frac{5}{23}} \lesssim K \lesssim |A|^{\frac{673}{2867}} \). If \( K \gtrsim |A|^{\frac{673}{2867}} \) then (4) gives better result.

Finally, in section 4 we prove sum–products results, which have deal just with the energies of sets but not with its sumsets or product sets. Similar results in the direction were obtained in [1], where the following Theorem was proved.

**Theorem 6** Let \( A \subset \mathbb{R} \) be a finite set and \( \delta = \frac{2}{33} \). Then there are two disjoint subsets \( B \) and \( C \) of \( A \) such that \( A = B \uplus C \) and

\[
\max\{E^+(B), E^x(C)\} \ll |A|^{3-\delta} (\log |A|)^{1-\delta}
\]

and

\[
\max\{E^+(B, C), E^x(B, C)\} \ll |A|^{3-\delta/2} (\log |A|)^{(1-\delta)/2}.
\]

Also it was proved in [1] that one cannot take \( \delta \) greater than 2/3. Our method gives an improvement of Theorem 6.

**Theorem 7** Let \( A \subset \mathbb{R} \) be a finite set and \( \delta = \frac{1}{5} \). Then there are two disjoint subsets \( B \) and \( C \) of \( A \) such that \( A = B \uplus C \) and

\[
\max\{E^+(B), E^x(C)\} \ll |A|^{3-\delta}.
\]
In the proof of our results we use a combination of methods from [9], [6] and of course [4]. The main additional idea is to introduce some more flexible quantity $d_*(A)$ instead of quantity $d(A)$, see the definitions below. It allows us to avoid of using the Balog–Szemerédi–Gowers Theorem [11]. This usually provides better bounds and allows us, in addition, to obtain a series of pure energy results in section 4. We hope that our new quantity $d_*(A)$ will help in another problems of sum–products type.

2 Definitions and preliminary results

The additive energy $E^+(A, B)$ between two sets $A$ and $B$ is the number of the solutions of the equation (see [11])

$$E^+(A, B) = |\{a_1 + b_1 = a_2 + b_2 : a_1, a_2 \in A , b_1, b_2 \in B\}| .$$

The multiplicative energy $E^\times(A, B)$ between two sets $A$ and $B$ is the number of the solutions of the equation (see [11])

$$E^\times(A, B) = |\{a_1b_1 = a_2b_2 : a_1, a_2 \in A , b_1, b_2 \in B\}| .$$

In the case $A = B$ we write $E^+(A)$ for $E^+(A, A)$ and $E^\times(A)$ for $E^\times(A, A)$. Having $\lambda \in A/A$, we put $A_{\lambda} = A \cap \lambda A$. Clearly, if $0 \not\in A$ then

$$E^\times(A) = \sum_{\lambda \in A/A} |A_{\lambda}|^2$$

and, similarly, for the energy $E^+(A)$. Next, the Cauchy–Schwarz inequality implies for $0 \not\in A$, $A_1 \subset A$, $A_2 \subset A$ that

$$E^\times(A_1, A_2)|A_1/A_1| \geq |A_1|^2|A_2|^2 , \quad E^\times(A_1, A_2)|AA| \geq |A_1|^2|A_2|^2 .$$

(7)

In particular

$$E^\times(A)|A/A| \geq |A|^4 , \quad E^\times(A)|AA| \geq |A|^4 .$$

(8)

Finally, we will use the following inequality.
Lemma 8 Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{R}$. Then

\[
\left( E^+ \left( \bigcup_{i=1}^{n} A_i \right) \right)^{1/4} \leq \sum_{i=1}^{n} (E^+(A_i))^{1/4}.
\]

Similarly, if $A_1, \ldots, A_n$ are finite subsets of $\mathbb{R} \setminus \{0\}$, then

\[
\left( E^x \left( \bigcup_{i=1}^{n} A_i \right) \right)^{1/4} \leq \sum_{i=1}^{n} (E^x(A_i))^{1/4}.
\]

Proof. A similar result for subsets of finite abelian groups follows from inequality (4.18) and Exercise 4.2.1 from [11]. Subsets of $\mathbb{R}$ can be reduced to subsets of finite groups by Lemma 5.26 from [11].

We need in several auxiliary statements. The first one is the Szemerédi–Trotter Theorem [10], see also [11]. We call a set $\mathcal{L}$ of continuous plane curves a pseudo-line system if any two members of $\mathcal{L}$ share at most one point in common. Define the number of indices $I(P, \mathcal{L})$ between points and pseudo-lines as

\[
I(P, \mathcal{L}) = |\{(p, l) \in P \times \mathcal{L} : p \in l\}|.
\]

Theorem 9 Let $P$ be a set of points and let $\mathcal{L}$ be a pseudo-line system. Then

\[
I(P, \mathcal{L}) \ll |P|^{2/3}|\mathcal{L}|^{2/3} + |P| + |\mathcal{L}|.
\]

We need in a definition, see [8].

Definition 10 A finite set $A \subset \mathbb{R}$ is said to be of Szemerédi–Trotter type (abbreviated as SzT–type) with a parameter $D > 0$ if the inequality

\[
|\{ s \in A - B \mid |A \cap (B + s)| \geq \tau \}| \leq \frac{D|A||B|^2}{\tau^3},
\]

holds for every finite set $B \subset \mathbb{R}$ and every real number $\tau \geq 1$.

The quantity $D(A)$ can be considered as the infimum of numbers $D$ such that (9) takes place for any $B$ and $\tau \geq 1$ but, of course, the definition is applicable just for sets $A$ with small quantity $D(A)$.

Any SzT–type set has small number of solutions of a wide class of linear equations, see e.g. Corollary 8 from [4] (where nevertheless another quantity $D(A)$ was used) and Lemma 7, 8 from [8], say.
Corollary 11 Let $A_1, A_2, A_3 \subset \mathbb{R}$ be any finite sets and $\alpha_1, \alpha_2, \alpha_3$ be arbitrary nonzero numbers. Then the number of the solutions of the equation

$$\sigma(\alpha_1A_1, \alpha_2A_2, \alpha_3A_3) := |\{\alpha_1a_1 + \alpha_2a_2 + \alpha_3a_3 = 0 : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3\}|$$

(10)
does not exceed $O(D^{1/3}(A_1)|A_1|^{1/3}|A_2|^{2/3}|A_3|^{2/3})$. Further, $E^+(A_1, A_2) \ll D^{1/2}(A_1)|A_1||A_2|^{3/2}$.

Also we need in a result from [8] on connection between the sumsets and $D(A)$ for SzT–type sets $A$.

Theorem 12 Let $A$ has SzT type. Then

$$|A + A| \gtrsim |A|^{58/37}D(A)^{-21/37}.$$  

(11)

Now we can introduce a new characteristic of a set $A \subset \mathbb{R}$. Put

$$\text{Sym}_t^*(Q, R) = \{x : |Q \cap xR^{-1}| \geq t\},$$

and

$$d_*(A) = \min_{t>0} \min_{\emptyset \neq Q, R \subset \mathbb{R}\{0\}} \frac{|Q|^2|R|^2}{|A|^3t^3},$$

(12)

where the second minimum in (12) is taken over any $Q, R$ such that $A \subseteq \text{Sym}_t^*(Q, R)$ and $\max\{|Q|, |R|\} \geq |A|$.

Lemma 13 Let $A \subset \mathbb{R}$ be a finite set. Then $A$ is of Szemerédi–Trotter type with $O(d_*(A))$.

Proof. Let $R, Q$ be two sets and $t > 0$ be a real number such that $A \subseteq \text{Sym}_t^*(Q, R)$. Without loosing of generality assume that $|Q| = \max\{|Q|, |R|\} \geq |A|$. Let also

$$S_\tau := \{s \in A - B : |A \cap (B + s)| \geq \tau\}.$$

Our task is to estimate the size of $S_\tau$. It is easy to see that the bound

$$|S_\tau| \ll \frac{|Q|^2|R|^2|B|^2}{t^3\tau^3}.$$  

(13)
is enough. We have

$$\tau |S_{\tau}| \leq \sum_{s \in S_{\tau}} |A \cap (B + s)| = |\{a - b = s : a \in A, b \in B, s \in S_{\tau}\}| := \sigma.$$ 

Because $A \subseteq \text{Sym}_t^\times(Q, R)$, we obtain the following upper bound for the number of solutions $\sigma$

$$\sigma \leq t^{-1}|\{qr - b = s : q \in Q, r \in R, b \in B, s \in S_{\tau}\}| \quad (14)$$

First of all let us prove a trivial estimate for the size of $S_{\tau}$. Namely, dropping the condition $s \in S_{\tau}$ in (14), we get

$$\tau |S_{\tau}| t \leq |Q||R||B|$$

and hence inequality (13) should be checked in the range

$$t^2 \tau^2 \gg |Q||R||B| \quad (15)$$

only because otherwise

$$|S_{\tau}| \leq \frac{|Q||R||B|}{t\tau} \ll \frac{|Q||R||B|^2}{t^3 \tau^3}.$$ 

Further, consider the family $\mathcal{L}$ of $|R||S_{\tau}|$ lines $l_{r,s} = \{ (x, y) : ry - x = s \}$,

$r \in R, s \in S_{\tau}$ and the family of points $\mathcal{P} = Q \times B$. Applying Theorem 9 to the pair ($\mathcal{P}, \mathcal{L}$), we get

$$\sigma \leq t^{-1}J(\mathcal{P}, \mathcal{L}) \ll t^{-1} \left( (|\mathcal{P}||\mathcal{L}|)^{2/3} + |\mathcal{P}| + |\mathcal{L}| \right) \quad (16)$$

If the first term in (16) dominates then we obtain (13). Now suppose that required bound (13) does not hold. Then if the second term in (16) is the largest one, we obtain

$$\frac{|Q|^2|R|^2|B|^2}{t^2 \tau^2} \ll t\tau |S_{\tau}| \ll |\mathcal{P}| = |Q||B|.$$ 

But, clearly, $t \leq \min\{|Q|, |R|\} = |R|$ and $\tau \leq \min\{|A|, |B|\}$, thus we arrive to a contradiction in view of the assumption $|Q| \geq |A|$. Finally, we need to consider the case when the third term in (16) dominates. In the situation

$$t\tau |S_{\tau}| \ll |S_{\tau}||R|$$

...
and hence in view of (15)

\[ |R||Q||B| \ll |R|^2. \]

But this is a contradiction because \(|Q| \geq |R|\) and \(B\) is large enough. This completes the proof of the lemma. \(\Box\)

It is easy to see from the definition that \(1 \leq d_*(A) \leq |A|\). The second inequality can be obtained if one put \(Q = A\), \(R = \{1\}\), \(t = 1\).

**Remark 14** In paper [5], Lemma 7 (see also [6], Lemma 27) the same result was obtained for the quantity

\[ d(A) := \min_{C \neq \emptyset} \frac{|AC|^2}{|A||C|}. \]

Clearly, \(d_*(A) \leq d(A)\). Indeed, just take \(t = |C|\), \(Q = AC\), and \(R = C^{-1}\).

**Remark 15** Let \(A\) be a set and \(\Pi = AA\) or \(A/A\). By Katz–Koester inclusion [3] that is \(|\Pi \cap \lambda \Pi| \geq |A|\) for any \(\lambda \in A/A\) one has \(d_*(\Pi) \leq |\Pi|^3/|A|^3\). The last estimate is usually better than ordinary \(|\Pi\Pi^2/|\Pi|^2\) even if one applies Plünnecke–Ruzsa inequality [11] (even for large subsets of \(A\)).

One can easily prove an analog of Lemma 13 in a dual form. In the case for any sets \(Q, R\) and a real number \(t > 0\) put

\[ \text{Sym}^+_t(Q, R) := \{x : |Q \cap (x - R)| \geq t\} \]

and consider the following quantity

\[ d_+(A) := \min_{t>0} \min_{\emptyset \neq Q, R \subseteq \mathbb{R} \setminus \{0\}} \frac{|Q|^2|R|^2}{|A|^t}, \tag{17} \]

where the second minimum in (18) is taken over any \(Q, R\) such that \(A \subseteq \text{Sym}^+_t(Q, R)\) and \(\max\{|Q|, |R|\} \geq |A|\). After that repeating the proof of Lemma 13, we need to estimate the cardinality of the set

\[ S_\tau := \{s \in AB^{-1} : |A \cap sB| \geq \tau\}. \]
So, we have arrived to the equation $ab^{-1} = s$, $s \in S_\tau$, $a \in A$, $b \in B$ and, further, to the equation $q + r = sb$, $s \in S_\tau$, $b \in B$, $q \in Q$, $r \in R$. It corresponds to the lines $l_{r,s} = \{(x,y) : y + r = sx\}$ and Theorem 9, combining with the calculations of the rest of Lemma 13 gives the result.

Thus, we have obtained an analog of Lemma 13.

**Lemma 16** Let $A, B \subset \mathbb{R}$ be two finite sets. Then for any real number $\tau \geq 1$ one has

$$\{ s \in AB^{-1} : |A \cap sB| \geq \tau \} \ll \frac{d_+(A)|A||B|^2}{\tau^3}. \quad (18)$$

So, one can define a set $A \subset \mathbb{R}$ to be of (multiplicative) Szemerédi–Trotter type if inequality (18) holds for any $B \subset \mathbb{R}$ and every real number $\tau \geq 1$.

We will consider further generalizations of the quantities $d_+(A), d_+(A)$ in our forthcoming paper.

### 3 The proof of the main results

We need in two technical lemmas from [4].

Let $A \subset \mathbb{R}$, $0 \not\in A$ be a finite set and $\tau > 0$ be a real number. Let also $S'_\tau$ be a set

$$S'_\tau \subset S_\tau := \{ \lambda : \tau < |A\lambda| \leq 2\tau \} \subseteq A/A$$

and for any nonzero $\alpha_1, \alpha_2, \alpha_3$ and different $\lambda_1, \lambda_2, \lambda_3 \in S'_\tau$ one has

$$\sigma(\alpha_1 A\lambda_1, \alpha_2 A\lambda_2, \alpha_3 A\lambda_3) \leq \sigma.$$ 

**Lemma 17** Let $A \subset \mathbb{R}$, $0 \not\in A$ be a finite set, $\tau > 0$ be a real number,

$$32\sigma \leq \tau^2 \leq |A + A|\sqrt{\sigma}, \quad (19)$$

and $S'_\tau$, $\sigma$ are defined above. Then

$$|A + A| \geq \frac{\tau^3|S'_\tau|}{128\sqrt{\sigma}}. \quad (20)$$
Lemma 18 Let $A \subset \mathbb{R}$, $0 \notin A$ be a finite set and $L \geq 1$ be a real number. Suppose that
\[ |A + A|^2 |A/A| \leq L |A|^4. \] (21)
Then there is $\tau \geq E^x(A)/(2|A|^2)$ and some sets $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau| \tau^2 \geq E^x(A)$, $|S'_\tau| \geq |S_\tau|/2$ such that for any element $\lambda$ from $S'_\tau$ one has
\[ |\lambda A/A| \gtrsim \tau^2 L^{-16}. \] (22)
Similarly, if
\[ |A + A|^2 |AA| \leq L |A|^4 \] (23)
then there exists $\tau \geq E^x(A)/(2|A|^2)$ and some sets $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau| \tau^2 \geq E^x(A)$, $|S'_\tau| \geq |S_\tau|/2$ such that for any $\lambda \in S'_\tau$, we have
\[ |\lambda A_A| \gtrsim \tau^2 L^{-16}. \] (24)

Proof of Theorem 3. Consider the situation with $A/A$, because the case of $AA$ is similar. By $\Pi$ denote $A/A$. Without losing of generality, suppose that $0 \notin A$. Now assume that inequality (21) holds with some parameter $L$. Let $|A/A|^3 \leq L'|A|^4$. Our task is to find a lower bound for the quantities $L$, $L'$. Using Lemma 18, we find a number $\tau \geq E^x(A)/(2|A|^2)$ and a set $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau| \tau^2 \geq E^x(A)$, $|S'_\tau| \geq |S_\tau|/2$ such that for any element $\lambda$ from $S'_\tau$ one has $|\lambda A_A| \gtrsim \tau^2 L^{-16}$. Using Katz-Koester inclusion, namely, $A_A \cap \lambda \Pi^{-1}$, $\lambda \in \Pi$ (see [3]), we get for any $\lambda \in S'_\tau$ that
\[ |\Pi \cap \lambda \Pi^{-1}| \geq |\lambda A_A| \gtrsim \tau^2 L^{-16} = t. \]
In particular, $S'_\tau \subseteq \text{Sym}_t^x(\Pi, \Pi)$. Because of $S'_\tau \subseteq S_\tau$, we obtain
\[ \sum_{a \in A} |A \cap aS'_\tau| = \sum_{\lambda \in S'_\tau} |A \cap \lambda \Pi| \gg \tau|S'_\tau| \]
and hence there is $a \in A$ such that for the set $A' := A \cap aS'_\tau$ one has
\[ |A'| \gg \tau|S'_\tau||A|^{-1}. \] (25)
We know that $S'_\tau \subseteq \text{Sym}_t^x(\Pi, \Pi)$. Hence $A' \subseteq \text{Sym}_t^x(a\Pi, \Pi)$. Applying formula (12) with $Q = a\Pi$, $R = \Pi$, we obtain
\[ d_s(A') \lesssim \frac{|\Pi|^4}{|A'|t^3} \ll \frac{|\Pi|^4 L_{48}^4}{|A'| \tau^6} \ll \frac{L_{48}^4 |\Pi|^4}{|S_\tau| \tau^7}. \] (26)
Using Theorem 12 and Lemma 13 as well as inequalities (8), (25), (26), we get

\[ |A + A| \geq |A' + A'| \geq |A'|^{\frac{56}{37}} d_s(A')^{\frac{21}{37}} \geq (\tau |S_\tau| |A|^{-1})^{\frac{56}{27}} (|S_\tau| \tau^7 L^{-48} |A|^{-1} |\Pi|^{-4})^{\frac{23}{37}} \]

\[ \geq (E^\tau(A))^\frac{23}{37} \tau^{\frac{37}{27}} L^{-\frac{1008}{27}} |A|^{-\frac{56}{27}} |\Pi|^{-\frac{21}{27}} \geq (E^\tau(A))^\frac{120}{37} L^{-\frac{1008}{37}} |A|^{-\frac{172}{37}} |\Pi|^{-\frac{84}{37}} \geq \]

\[ \geq |A|^{\frac{10}{37}} L^{\frac{1008}{37}} |\Pi|^{\frac{210}{37}} \geq L^{\frac{1008}{37}} (L')^{\frac{126}{37}} |A|^{\frac{23}{37}}. \]

The last estimate is greater than $|A|^{4/3}$ by some power of $|A|$. Easy calculations show that one can take any number less than $\frac{5}{10813}$ for the constant $c$. This concludes the proof. \(\square\)

**Proof of Theorem 5.** Let $\Pi = A/A$, $|\Pi| = K|A|$. In the proof we can restrict ourselves considering just the case $|A|^{5/23} \leq K \leq \gamma |A|^{1/4}$ where $\gamma > 0$ is a small constant. Without losing of generality, suppose that $0 \not\in A$. Using Dirichlet principle we find $\tau \geq |A|/(2K)$ with $|S_\tau| \tau \geq |A|^2$. Consider two subsets $S'_\tau$, $S''_\tau$ of $S_\tau$ such that $|S'_\tau| = |S''_\tau| \geq |S_\tau|/2$ and for some parameter $\kappa \in (0, 1]$ the following holds $|A_\lambda/A| \leq \kappa |\Pi|$ for all $\lambda \in S'_\tau$ and $|A_\lambda/A| \geq \kappa |\Pi|$ for any $\lambda \in S''_\tau$. For any $\lambda \in S'_\tau$ one has

\[ d_s(A_\lambda) \leq d(A_\lambda) \leq \frac{\kappa^2 |\Pi|^2}{|A_\lambda/A|} \leq \kappa^2 |\Pi|^2 \tau^{-1} |A|^{-1}. \quad (27) \]

Thus, applying Corollary 11 and Lemma 17, we see that

\[ |A + A|^2 \gg \tau^3 |S_\tau| (\kappa^2 |\Pi|^2 \tau^{-1} |A|^{-1})^{-1/6} \tau^{-5/6} = \tau^{7/3} |S_\tau| |A|^{1/6} |\Pi|^{-1/3} \kappa^{-1/3}, \]

provided that conditions (19) hold. Using inequalities $\tau \geq |A|/(2K)$ and $|S_\tau| \tau \geq |A|^2$, we obtain

\[ |A + A|^2 \gtrsim |A|^2 (|A|/K)^{4/3} |A|^{1/6} (|A|K)^{-1/3} \kappa^{-1/3} \gg |A|^{19/6} K^{-5/3} \kappa^{-1/3}. \]

Hence

\[ |A + A| \gtrsim |A|^{19/12} K^{-5/6} \kappa^{-1/6}. \quad (28) \]

For the set $S''_\tau$ we use the arguments as in the proof of Theorem 3. Using Katz–Koester inclusion, namely, $A_\lambda/A \subseteq \Pi \cap \lambda \Pi^{-1}$, we get for any $\lambda \in S''_\tau$ that

\[ |\Pi \cap \lambda \Pi^{-1}| \geq |A_\lambda/A| \geq \kappa |\Pi| := t. \]
In particular, $S''_\tau \subseteq \text{Sym}^\chi(\Pi, \Pi)$. Because of $S''_\tau \subseteq S_\tau$, we obtain
\[
\sum_{a \in A} |A \cap aS''_\tau| = \sum_{\lambda \in S''_\tau} |A \cap \lambda A| \gg \tau |S''_\tau| \gg \eta \tau |S_\tau|
\]
and hence there is $a \in A$ such that for the set $A' := A \cap aS''_\tau$ one has
\[
|A'| \gg \tau |S_\tau||A|^{-1}.
\]
We know that $S''_\tau \subseteq \text{Sym}^\chi(\Pi, \Pi)$. Hence $A' \subseteq \text{Sym}^\chi(a\Pi, \Pi)$. Applying formula (12) with $Q = a\Pi$, $R = \Pi$, we obtain
\[
d_s(A') \leq \frac{|\Pi|^4}{|A'|t^3} = \frac{|\Pi|}{|A'|^{\kappa^3}} \ll \frac{|A| |\Pi|}{\kappa^3 |S_\tau| \tau}.
\]
Using Theorem 12 and Lemma 13 as well as inequalities (8), (29), (30), we get
\[
|A + A| \geq |A' + A'| \gtrsim |A'\bar{d}_s(A') - \frac{3}{2} \gtrsim (\tau |S_\tau||A|^{-1})^{\frac{4}{5}} \left(\kappa^3 |S_\tau||A|^{-1}|\Pi|^{-1}\right)^{\frac{4}{5}} = (|S_\tau|\tau^{\frac{4}{5}} |A|^{-\frac{4}{5}} |\Pi|^{-\frac{4}{5}} K^{-\frac{6}{5}} \kappa^{\frac{6}{5}} \gtrsim |A|^{\frac{415}{21}} K^{-\frac{37}{21}} K^{\frac{63}{21}}.
\]
Combining bound (31) with (28), we find that the optimal choice of $\kappa$ is
\[
\kappa = |A|^{\frac{415}{21}} K^{-\frac{37}{21}} \leq 1
\]
because $|A|^{5/23} \leq K$. Substituting the last inequality into (28), we obtain
\[
|A + A| \gtrsim |A|^{19/12} K^{-5/6} (|A|^{\frac{415}{21}} K^{-\frac{37}{21}})^{-1/6} = |A|^{\frac{1313}{336}} K^{-\frac{21}{336}}.
\]
The only we need to check conditions (19). The inequality $\tau^2 \geq 32\sigma$ easily follows from (27) and inequality $K \lesssim |A|^{1/4}$. Indeed by Corollary 11 and bounds (27), $\tau \geq |A|/(2K)$, we have
\[
\sigma \leq (K^2 |A|\tau^{-1})^{1/3} \tau^{5/3} \ll \gamma^{2/3} \tau^2,
\]
and $\sigma \leq \tau^2/2$ if $\gamma$ is small enough. It remains to check $\tau^2 \leq |A + A| \sqrt{\sigma}$. We have taken $\sigma = \tau^{4/3} K^{2/3} |A|^{1/3} \kappa^{2/3}$. Thus we need to insure in the inequality
\[
\tau^8 \leq |A + A|^6 K^2 |A| \kappa^2.
\]
By bound (1) one has $|A + A|^2 \gg |A|^3 K^{-1} \log^{-1} |A|$. In addition, in view of (32) and because of $K \ll |A|^{1/4}$, we have $\kappa \gg |A|^{-3/2} \geq |A|^{-\frac{3}{100}}$. Thus,
\[
|A + A|^6 K^2 |A| \kappa^2 \gg |A|^{10-3/50} K^{-1} \log^{-3} |A| \gg |A|^9,
\]
and (33) is true for large $|A|$ since $\tau \leq |A|$. This concludes the proof. \(\square\)
4 Sum–products results with energies

In the section we prove sum–products results, which have deal just with the energies of sets but not with its sumsets or product sets.

Let us start with a lemma which can be interesting in its own right.

Lemma 19 Let $A, P \subset \mathbb{R}$ be two sets. Put

$$\sigma_* := \sum_{x \in P} |A \cap xA|.$$ 

Then there is $A' \subseteq A$ such that $A'$ has SzT–type with $d_*(A') \lesssim \frac{|P|^2|A|^2|A'|^2}{q^3}$ and $|A'| \gtrsim \sigma_*|P|^{-1}$. Similarly, put

$$\sigma_+ := \sum_{x \in P} |A \cap (x + A)|.$$ 

Then there exists $A'' \subseteq A$ such that $A''$ has SzT–type with $d_+(A'') \lesssim \frac{|P|^2|A|^2|A''|^2}{\sigma_+^3}$ and $|A''| \gtrsim \sigma_+|P|^{-1}$.

Proof. We have

$$\sigma_* = \sum_{x \in A} |P \cap xA^{-1}|$$ 

and thus by the pigeonholing principle there is a set $A' \subseteq A$ and a number $q \leq |A|$ such that $|A'|q \sim \sigma_*$ and $q < |P \cap xA^{-1}| \leq 2q$ for any $x \in A'$. Because $q \leq |P|$, we have $|A'| \gtrsim \sigma_*|P|^{-1}$. Using Lemma 13 with $Q = P$ and $R = A$ we see the set $A'$ has SzT–type such that $d_*(A')$ does not exceed

$$d_*(A') \ll \frac{|P|^2|A|^2}{q^3|A'|} \lesssim \frac{|P|^2|A|^2|A'|^2}{\sigma_*^3}$$

as required. By similar arguments and an application of Lemma 16 instead of Lemma 13, we obtain the existence of the set $A''$. This completes the proof. \square

Now we are ready to formulate the main result of the section, which shows that any set either has multiplicative energy or there is a large subset with small additive energy and visa versa. Similar results were obtained in [1] but as we said in the introduction we do not use the Balog–Szemerédi–Gowers Theorem in the proof.
Theorem 20 Let $A \subset \mathbb{R}$ be a set. Then there is $A_1 \subseteq A$ such that $|A_1| \gtrsim E^\times(A)|A|^{-2}$ and
\[ E^+(A_1)E^\times(A) \lesssim |A_1|^{7/2}|A|^2. \] (34)
Similarly, there is $A_2 \subseteq A$ such that $|A_2| \gtrsim E^+(A)|A|^{-2}$ and
\[ E^\times(A_2)E^+(A) \lesssim |A_2|^{7/2}|A|^2. \] (35)

Proof. Put
\[ E^\times(A) := \sum_{x} |A \cap xA|^3. \]
By the pigeonhole principle there is $P \subseteq A/A$ and a number $\Delta$ such that $\Delta^3 |P| \sim E^\times(A)$ and $\Delta < |A \cap xA| \leq 2 \Delta$ for any $x \in P$. Applying Lemma 19 with $\sigma_* \sim \Delta |P|$, we find a set $A_1 \subseteq A$, $|A_1| \gtrsim \Delta$ such that $d^*(A_1) \lesssim \frac{|A_1|^2|A|^2}{|P|\Delta^3}$. We have $\Delta \gtrsim E^\times(A) \Delta^{-2}|P|^{-1}$ and hence by the Cauchy–Schwarz inequality, we get $|A_1| \gtrsim E^\times(A)^2|A|^{-2} \Delta^{-2}|P|^{-1}$. Next,
\[ E^\times(A) \gtrsim \sum_{x \in P} |A \cap xA|^2 \geq \Delta^2 |P|. \]
Therefore, $|A_1| \gtrsim E^\times(A)|A|^{-2}$. Using Corollary 11, we get
\[ (E^+(A_1))^2 E^\times(A) \lesssim (E^+(A_1))^2 |P|\Delta^3 \ll |A_1|^7|A|^2. \]
Finally, applying the Cauchy–Schwarz inequality again, we obtain
\[ E^+(A_1)E^\times(A) \lesssim |A_1|^{7/2}|A|^2 \]
as required. By similar arguments we obtain the existence of the set $A_2$. This completes the proof. \( \Box \)

Now we can prove Theorem 7 from the introduction.

Corollary 21 Let $A \subset \mathbb{R}$ be a finite set and $\delta = 1/5$. Then there are two disjoint subsets $B$ and $C$ of $A$ such that $A = B \sqcup C$ and
\[ \max\{E^+(B), E^\times(C)\} \lesssim |A|^{3-\delta}. \]
Proof. Let $M \geq 1$ be a parameter which we will choose later. Our arguments is a sort of an algorithm. We construct a decreasing sequence of sets $C_1 = A \supseteq C_2 \supseteq \cdots \supseteq C_k$ and an increasing sequence of sets $B_0 = \emptyset \subseteq B_1 \subseteq \cdots \subseteq B_{k-1} \subseteq A$ such that for any $j = 1, 2, \ldots, k$ the sets $C_j$ and $B_{j-1}$ are disjoint and moreover $A = C_j \cup B_{j-1}$. If at some step $j$ we have $E^\times(C_j) \leq |A|^3/M$ then we stop our algorithm putting $C = C_j$, $B = B_{j-1}$, and $k = j - 1$. In the opposite situation where $E^\times(C_j) > |A|^3/M$ we apply Theorem 20 to the set $C_j$, finding the subset $D_j$ of $C_j$ such that $|D_j| \gtrsim |A|/M$ and

\[ E^+(D_j) \lesssim |D_j|^{7/2}M|A|^{-1}. \] (36)

After that we put $C_{j+1} = C_j \setminus D_j$, $B_j = B_{j-1} \cup D_j$ and repeat the procedure. Clearly, $B_k = \bigcup_{j=1}^k D_j$ and because of $|D_j| \gtrsim |A|/M$, we have $k \lesssim M$. Finally, by the Hölder inequality, Lemma 8 and (36), we get

\[
(E^+(B_k))^{1/4} \leq \sum_{j=1}^k (E^+(D_j))^{1/4} \lesssim (M|A|^{-1})^{1/4} \sum_{j=1}^k |D_j|^{7/8} \leq (M|A|^{-1})^{1/4} \left( \sum_{j=1}^k |D_j| \right)^{7/8} \lesssim (M|A|^{-1})^{1/4} |A|^{7/8} M^{1/8} = M^{3/8} |A|^{5/8}.
\]

Hence

\[ E^+(B_k) \lesssim M^{3/2} |A|^{5/2}. \]

Optimizing over $M$, that is choosing $M = |A|^{1/5}$, we obtain the result. This completes the proof. \qed

We immediately get from Theorem 20 that

\[ E^+(A_1)E^\times(A) \lesssim |A|^{11/2} \]

and

\[ E^\times(A_2)E^+(A) \lesssim |A|^{11/2}. \]

If $E^\times(A)$ (respectively, $E^+(A)$) is not too big, then it is not difficult to construct larger sets $A_1$ and $A_2$ satisfying these inequalities.

**Corollary 22** Let $A \subset \mathbb{R}$ be a set. Then there is $A_1 \subseteq A$ such that $|A_1| \gg (E^\times(A))^{1/3}$ and

\[ E^+(A_1)E^\times(A) \lesssim |A|^{11/2}. \] (37)
Similarly, there is \( A_2 \subseteq A \) such that \( |A_2| \gg (E^+(A))^{1/3} \) and

\[
E^x(A_2)E^+(A) \lesssim |A|^{11/2}. \tag{38}
\]

**Proof.** We proceed as in the proof of Theorem 7.

We construct a decreasing sequence of sets \( C_1 = A \supseteq C_2 \supseteq \cdots \supseteq C_k \) and an increasing sequence of sets \( B_0 = \emptyset \subseteq B_1 \subseteq \cdots \subseteq B_{k-1} \subseteq A \) such that for any \( j = 1, 2, \ldots, k \) the sets \( C_j \) and \( B_{j-1} \) are disjoint and moreover \( A = C_j \cup B_{j-1} \). If at some step \( j \) we have \( |B_{j-1}| > (E^x(A))^{1/3}/2 \) then we stop our algorithm putting \( A_1 = B_{j-1} \) and \( k = j - 1 \). In the opposite situation where \( |B_{j-1}| \leq (E^x(A))^{1/3}/2 \) we apply Theorem 20 to the set \( C_j \), finding the subset \( D_j \) of \( C_j \) such that \( |D_j| \gtrsim E^x(C_j)/|C_j|^2 \) and

\[
E^+(D_j) \lesssim |D_j|^{7/2}|C_j|^2E^x(C_j)^{-1}.
\]

We observe, however, that the inequality \( E^x(B_{j-1}) \leq E^x(A)/8 \) due to \( |B_{j-1}| \leq (E^x(A))^{1/3}/2 \) implies \( E^x(C_j) \gg E^x(A) \). Therefore,

\[
|D_j| \gtrsim E^x(A)/|A|^2
\]

and

\[
E^+(D_j) \lesssim |D_j|^{7/2}|A|^2E^x(A)^{-1}.
\]

Next we put \( C_{j+1} = C_j \setminus D_j, B_j = B_{j-1} \cup D_j \) and repeat the procedure.

By Lemma 8, we have

\[
(E^+(B_{k-1}))^{1/4} \leq \sum_{j=1}^{k-1} (E^+(D_j))^{1/4} \lesssim |A|^{1/4}E^x(A)^{-1/4} \sum_{j=1}^{k-1} |D_j|^{7/8}
\]

\[
\lesssim |A|^{1/4}E^x(A)^{-1/4} \sum_{j=1}^{k-1} |D_j| (E^x(A)/|A|^2)^{-1/8}
\]

\[
\leq |A|^{1/2}E^x(A)^{-1/4}E^x(A)^{1/3} (E^x(A)/|A|^2)^{-1/8}
\]

\[
= |A|^{3/4}E^x(A)^{-1/24} \leq |A|^{11/8}E^x(A)^{-1/4}.
\]

So,

\[
E^+(B_{k-1}) \lesssim |A|^{11/2}E^x(A)^{-1}. \tag{39}
\]

Next,

\[
E^+(D_k) \lesssim |D_k|^{7/2}|A|^2E^x(A)^{-1} \leq |A|^{11/2}E^x(A)^{-1}. \tag{40}
\]

Since \( A_1 = B_k = B_{k-1} \cup D_k \), we combine (39) and (40) to complete the proof of the first claim of the corollary. The proof of the second claim is similar.

\[\square\]
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