$C_0$-semigroups of $m$-isometries on Hilbert spaces

T. Bermúdez, A. Bonilla and H. Zaway *

October 18, 2018

Abstract

Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on a separable Hilbert space $H$. We characterize that $T(t)$ is an $m$-isometry for every $t$ in terms that the mapping $t \in \mathbb{R}^+ \to \|T(t)x\|^2$ is a polynomial of degree less than $m$ for each $x \in H$. This fact is used to study $m$-isometric right translation semigroup on weighted $L^p$-spaces. We characterize the above property in terms of conditions on the infinitesimal generator operator or in terms of the cogenerator operator of $\{T(t)\}_{t \geq 0}$. Moreover, we prove that a non-unitary 2-isometry on a Hilbert space satisfying the kernel condition, that is,

$$T^*T(Ker T^*) \subset Ker T^*,$$

then $T$ can be embedded into a $C_0$-semigroup if and only if $\dim(Ker T^*) = \infty$.

1 Introduction

Let $H$ be a complex Hilbert space and $B(H)$ denote the $C^*$-algebra of all bounded linear operators on $H$.

A one parameter family $\{T(t)\}_{t \geq 0}$ of bounded linear operators from $H$ into $H$ is a $C_0$-semigroup if:

1. $T(0) = I$.
2. $T(s + t) = T(t)T(s)$ for every $t, s \geq 0$.
3. $\lim_{t \to 0^+} T(t)x = x$ for every $x \in H$, in the strong operator topology.

The linear operator $A$ defined by

$$Ax := \lim_{t \to 0^+} \frac{T(t)x - x}{t},$$

*The first and second authors were supported by MINECO and FEDER, Project MTM2016-75963-P. The third author was supported in part by Departamento de Análisis Matemático of Universidad de La Laguna and by a grant of Université de Gabès, UNG 933989527.
for every

\[ x \in D(A) := \{ x \in H : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \} \]

is called the *infinitesimal generator* of the semigroup \( \{T(t)\}_{t \geq 0} \). It is well-known that \( A \) is a closed densely defined linear operator.

If \( 1 \) is in the resolvent set of \( A \), \( \rho(A) \), then the Caley transform of \( A \) defined by

\[ V := (A + I)(A - I)^{-1} \]

is called the *cogenerator* of the \( \mathcal{C}_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \).

Notice that the point \( 1 \) could be on the spectrum of \( V \), \( \sigma(V) \).

For a positive integer \( m \), an operator \( T \in B(H) \) is called an *\( m \)-isometry* if

\[ \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \|T^k x\|^2 = 0 , \]

for any \( x \in H \).

We say that \( T \) is a *strict \( m \)-isometry* if \( T \) is an \( m \)-isometry but it is not an \((m-1)\)-isometry.

**Remark 1.1.**

1. For \( m \geq 2 \), the strict \( m \)-isometries are not power bounded. If \( T \) is an \( m \)-isometry, then \( \|T^n x\|^2 \) is a polynomial at \( n \) of degree at most \( m - 1 \), for every \( x \), [6] Theorem 2.1. In particular, \( \|T^n\| = O(n) \) for 3-isometries and \( \|T^n\| = O(n^{1/2}) \) for 2-isometries.

2. There are no strict \( m \)-isometries on finite dimensional spaces for even \( m \). See [2] Proposition 1.23.

3. If \( T \) is an \( m \)-isometry, then \( \sigma(T) = \overline{D} \) or \( \sigma(T) \subseteq \partial D \), [2] Lemma 1.2.

The remainder of the paper is organized as follows. In Section 2, we characterize that \( T(t) \) is an \( m \)-isometry for every \( t \) if the mapping \( t \in \mathbb{R}^+ \to \|T(t)x\|^2 \) is a polynomial of degree less than \( m \) for each \( x \in H \). This fact is used in the last section in order to study \( m \)-isometric right translation semigroup on weighted \( L^p \)-spaces. Also, we characterize the above property in terms of conditions on the generator operator or in terms that the cogenerator operator be an \( m \)-isometry. Moreover, we obtain that \( \{T(t)\}_{t \geq 0} \) is an \( m \)-isometry for all \( t > 0 \) if and only if \( T(t) \) is an \( m \)-isometry, for all \( t \in [0, t_1] \) with \( t_1 > 0 \) or on \([t_1, t_2]\) with \( 0 < t_1 < t_2 \).

Section 3 is devoted to embed \( m \)-isometries into \( \mathcal{C}_0 \)-semigroups. That is, given an \( m \)-isometry \( T \) finds a \( \mathcal{C}_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) such that \( T(1) = T \). Using the model for 2-isometries we conclude that a non-unitary 2-isometry on a Hilbert space such that satisfies the *kernel condition*, that is

\[ T^*T(Ker(T^*)) \subset Ker(T^*) , \]

can be embedded into a \( \mathcal{C}_0 \)-semigroup if and only if \( \dim(KerT^*) = \infty \).

Finally, in Section 4, we obtain a characterization of the right translation \( \mathcal{C}_0 \)-semigroup on some weighted space, to be a semigroup of \( m \)-isometries for all \( t > 0 \).
2 \textit{C}_0\text{-semigroups of } m\text{-isometries}

Recall that any \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) have some real quantities associated as:

1. The \textit{spectral bound}, \( s(A) \), by setting
   \[
   s(A) := \sup \{ \Re \lambda : \lambda \in \sigma(A) \} .
   \]

2. \textit{Growth bound}, \( w_0 \), given by
   \[
   w_0 := \inf \{ w \in \mathbb{R} : \exists M_w \geq 1 : \|T(t)\| \leq M_w e^{wt}, \forall t \geq 0 \} .
   \]

The above quantities are related in the following way, \( s(A) \leq w_0 \) and also
\[
  w_0 = \frac{1}{t} \log r(T(t)),
\]
where \( r(T(t)) \) denotes the spectral radius of the operator \( T(t) \).

The following lemma allow us to define the cogenerator of a \( C_0 \)-semigroup of \( m \)-isometries.

\textbf{Lemma 2.1.} Let \( \{T(t)\}_{t \geq 0} \) be a \( C_0 \)-semigroup on a separable Hilbert space \( H \) consisting of \( m \)-isometries and \( A \) its generator. Then \( 1 \in \rho(A) \) and therefore the cogenerator \( V \) of \( \{T(t)\}_{t \geq 0} \) is well-defined.

\textit{Proof.} If \( T(t) \) is an \( m \)-isometry for all \( t \), then the spectral radius of \( T(t) \), \( r(T(t)) \) is 1 for all \( t \). Using equality (1), then \( s(A) \leq w_0 = 0 \) and then \( 1 \in \rho(A) \). \( \square \)

The following combinatorial result will be necessary for the proof of Theorem 2.1.

\textbf{Lemma 2.2.} Let \( m \) be a positive integer and \( p, q \) be integers such that \( 0 \leq p, q \leq m \).

1. If \( p + q \neq m \), then
   \[
   \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \left\{ \sum_{i=0}^{p} \binom{m-k}{i} \binom{k}{p-i} (-1)^i \right\} \left\{ \sum_{j=0}^{q} \binom{m-k}{j} \binom{k}{q-j} (-1)^j \right\} = 0
   \]

2. If \( p + q = m \), then
   \[
   \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{i} \binom{k}{q-i} (-1)^i \right\}^2 = 2^m \binom{m}{q} = 2^m \binom{m}{p} .
   \]
Proof. We define the polynomials $r$ and $s$ by $r(x, y) := 2^m(x + y)^m$ and $s(x, y) := ((x + 1)(y + 1) - (x - 1)(y - 1))^m$.

Then
\[
\begin{align*}
r(x, y) &= 2^m \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \\
s(x, y) &= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (x + 1)^k (x - 1)^{m-k} (y + 1)^k (y - 1)^{m-k} \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \sum_{i=0}^{p} \binom{m-k}{i} \binom{k}{i} \sum_{j=0}^{m-k} \binom{m-k}{j} \sum_{l=0}^{i} \binom{m-k}{l} (-1)^{i+j} x^{i+j} y^{i+l} \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \left\{ \sum_{i=0}^{p} \binom{m-k}{i} \binom{k}{i} \binom{m-k}{j} (-1)^{j} \right\} x^{i+j} y^{i+l} \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \left\{ \sum_{i=0}^{p} \binom{m-k}{i} \binom{k}{p-i} (-1)^{i} \right\} x^{i+j} y^{i+l}.
\end{align*}
\]

Denote $\overrightarrow{xy}^q$ the coefficient of the power $x^p y^q$ on the polynomial $f$. Since $s(x, y) = r(x, y)$, then $\overrightarrow{xy}^q = \overrightarrow{xy}^r$ for every $p$ and $q$. So, if $p + q \neq m$, then $\overrightarrow{xy}^q = \overrightarrow{xy}^r = 0$. If $p + q = m$, then
\[
\overrightarrow{xy}^q = \overrightarrow{xy}^r = \overrightarrow{xy}^{m-p} = 2^m \binom{m}{m-p} = 2^m \binom{m}{m-p} = 2^m \binom{m}{q}.
\]

By other hand, it is straightforward to verify that, if $p + q = m$, then
\[
\overrightarrow{xy}^q = \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{i} \binom{k}{i} (-1)^{i} \right\}^2 \\
&= \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{i} \binom{k}{q-i} (-1)^{i} \right\}^2.
\]

This complete the proof. \(\square\)

Given a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$, we have two different operators associated to $\{T(t)\}_{t \geq 0}$, the infinitesimal generator $A$ and the cogenerator $V$, if $1 \in \rho(A)$. This two operators will be useful in the next result, where we obtain a natural generalization to $C_0$-semigroup of $m$-isometries of [12, Proposition 2.2]. See also [13, Proposition 2.6] and [21, Theorem 2].
Theorem 2.1. Let \( \{T(t)\}_{t \geq 0} \) be a \( C_0 \)-semigroup on a separable Hilbert space \( H \). Then the following conditions are equivalent:

(i) \( T(t) \) is an \( m \)-isometry for every \( t \).

(ii) The mapping \( t \in \mathbb{R}^+ \to \|T(t)x\|^2 \) is a polynomial of degree less than \( m \) for each \( x \in H \).

(iii) The operator inequality
\[
\sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}x, A^kx \rangle = 0 ,
\]
for any \( x \in D(A^m) \).

(iv) The cogenerator \( V \) of \( \{T(t)\}_{t \geq 0} \) exists and is an \( m \)-isometry.

Proof. (i) \(\Leftrightarrow\) (ii) If \( T(t) \) is an \( m \)-isometry for every \( t \), then
\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \|T(t+k\tau)x\|^2 = 0 ,
\]
for every \( t, \tau > 0 \) and \( x \in H \). From the assumption on the semigroup, it is clear that the function \( t \in \mathbb{R}^+ \to f(t) := \|T(t)x\|^2 \) is continuous. By [15, Theorem 13.7], the function \( f(t) \) is a polynomial of degree less than \( m \) for each \( x \in H \).

Conversely, if the mapping \( t \in \mathbb{R}^+ \to \|T(t)x\|^2 \) is a polynomial of degree less than \( m \) for each \( x \in H \), then \( T(t) \) is an \( m \)-isometry for every \( t \), [15, page 271].

(ii) \(\Leftrightarrow\) (iii) Let \( y \in D(A^m) \). The function \( t \in \mathbb{R}^+ \to \|T(t)y\|^2 \) has \( m \)-th derivative and it is given by
\[
\sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}T(t)y, A^kT(t)y \rangle .
\]
By (ii) and (3) at \( t = 0 \) we have that
\[
\sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}x, A^kx \rangle = 0 ,
\]
for any \( x \in D(A^m) \).

Conversely, if (3) holds on \( D(A^m) \), then the \( m \)-th derivative of the function \( t \in \mathbb{R}^+ \to \|T(t)x\|^2 \) agree with
\[
\sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}T(t)x, A^kT(t)x \rangle ,
\]
for every \( t > 0 \) and \( x \in D(A^m) \). Since \( D(A^m) \) is dense by [17, Theorem 2.7], we obtain the result.
(iii) ⇔ (iv) It will be sufficient to prove that
\[
2^m \sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}x, A^kx \rangle = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \langle (A+I)^k(A-I)^{m-k}x, (A+I)^{k}(A-I)^{m-k}x \rangle
\]
for all \(x \in D(A^m)\), since (4) is equivalent to
\[
2^m \sum_{k=0}^{m} \langle A^{m-k}(A-I)^{-m}y, A^k(A-I)^{-m}y \rangle
= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \langle (A+I)^{k}(A-I)^{-k}y, (A+I)^{k}(A-I)^{-k}y \rangle
= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \|V^k y\|^2,
\]
for all \(y \in R(A-I)^m\).

Note that the second part of equality (4) is given by
\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \sum_{\ell, h=0}^{k} \binom{k}{\ell} \binom{k}{h} \sum_{i,j=0}^{m-k} \binom{m-k}{i, j} \binom{m-k}{j} (-1)^{i+j} \langle A^{\ell+i}x, A^{h+j}x \rangle.
\]
We denote \(\hat{A}_{i,j}\) the numerical coefficient of \(\langle A^i x, A^j x \rangle\). It is straightforward to verify that if \(p + q = m\), then
\[
\hat{A}_{q,m-q} = \hat{A}_{m,q}
= \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{m-k-i} \binom{k}{k+i-q} (-1)^i \right\}^2
= \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{i} \binom{k}{q-i} (-1)^i \right\}^2 = 2^m \binom{m}{q},
\]
where the last equality is obtained by part (2) of Lemma 2.2.

If \(p + q \neq m\), then
\[
\hat{A}_{p,q} = \hat{A}_{m-p,m-q}
= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \left\{ \sum_{i=0}^{p} \binom{m-k}{m-k-i} \binom{k}{k+i-p} (-1)^i \right\}
\left\{ \sum_{j=0}^{q} \binom{m-k}{m-k-j} \binom{k}{k+j-q} (-1)^j \right\}
= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \left\{ \sum_{i=0}^{p} \binom{m-k}{i} \binom{k}{p-i} (-1)^i \right\} \left\{ \sum_{j=0}^{q} \binom{m-k}{j} \binom{k}{q-j} (-1)^j \right\}
= 0,
\]
where the last equality is obtained by part (1) of Lemma 2.2. So we get the result. \(\square\)
Corollary 2.1. Let \( \{T(t)\}_{t \geq 0} \) be a \( C_0 \)-semigroup on a separable Hilbert space \( H \). Then \( T(t) \) is a strict \( m \)-isometry for every \( t > 0 \) if and only if the cogenerator \( V \) of \( \{T(t)\}_{t \geq 0} \) is a strict \( m \)-isometry.

In the following corollary, we give an example of \( m \)-isometric semigroup.

Corollary 2.2. Let \( Q \) be a nilpotent operator of order \( n \) on a separable Hilbert space. Then the \( C_0 \)-semigroup generated by \( Q \) is a strict \((2n-1)\)-isometric semigroup.

**Proof.** Since \( Q \) is the generator of \( \{T(t)\}_{t \geq 0} \) and \( 1 \in \rho(Q) \), then the cogenerator is well-defined and given by

\[
V := (Q + I)(Q - I)^{-1}.
\]

Thus \( V = -(Q + I)(I + Q + \cdots + Q^{n-1}) = -I - 2Q(I + Q + \cdots + Q^{n-2}) \), that is, the sum of an isometry and a nilpotent operator of order \( n \). By \([7, \text{Theorem 2.2}]\), the cogenerator is a strict \((2n-1)\)-isometry. Then \( \{T(t)\}_{t \geq 0} \) is a strict \((2n-1)\)-isometric semigroup by Corollary 2.1.

For a positive integer \( m \), a linear closed operator \( A \) defined on a dense set \( D(A) \subset H \) is called an \( m \)-symmetry if

\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \langle A^{m-k}x, A^kx \rangle = 0 ,
\]

for all \( x \in D(A^m) \). We say that \( A \) is a strict \( m \)-symmetry if \( A \) is an \( m \)-symmetry but it is not an \((m-1)\)-symmetry. There is not strict \( m \)-symmetry bounded for even \( m \). See \([1, \text{Page 7}]\).

In the next result, we present the connection between the condition (iii) of Theorem 2.1 and the \( m \)-symmetric operators.

Corollary 2.3. Let \( A \) be an \( m \)-symmetry on a separable Hilbert space. Then the \( C_0 \)-semigroup generated by \( iA \) is an \( m \)-isometric semigroup.

**Proof.** If \( A \) is an \( m \)-symmetry, then

\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \langle (iA)^{m-k}x, (iA)^kx \rangle = (-i)^m \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \langle A^{m-k}x, A^kx \rangle = 0 ,
\]

for all \( x \in D(A^m) \). Thus the \( C_0 \)-semigroup generated by \( iA \) is an \( m \)-isometric semigroup. The result is completed by Theorem 2.1.

Let \( w \) be a weighted function defined on \( \mathbb{T} \). It is defined

\[
L_w^2(\mathbb{T}) := \{ f : \mathbb{T} \to \mathbb{C} : \int_{\mathbb{T}} |f(z)|^2 w(z) dz < \infty \}.
\]

Let \( \{T(t)\}_{t \geq 0} \) be a \( C_0 \)-group defined on \( L_w^2(\mathbb{T}) \) with non-constant weighted function \( w \) by \( T(t)f(z) := f(e^{it}z) \). Then \( T(t) \) is an isometry if and only if \( t \) is a multiple of \( 2\pi \). See \([3, \text{page 8}]\).
Proposition 2.1. Let \( \{T(t)\}_{t \in \mathbb{R}} \) be a \( C_0 \)-group on a Hilbert space \( H \). Then the following conditions are equivalent:

(i) \( T(t) \) is an \( m \)-isometry for every \( t \).

(ii) \( T(t) \) is an \( m \)-isometry for \( t_1 \) and \( t_2 \) where \( \frac{t_1}{t_2} \) is irrational.

Proof. For every \( t \), we have that

\[
\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T(t + kt_i)x\|^2 = 0 \quad \text{for} \quad i = 1, 2.
\]

Montel’s Theorem ([15, Theorem 13.5] & [19, Theorem 1.1]) gives that \( \|T(t)x\|^2 \) is a polynomial of degree less than \( m \) for each \( x \in H \), since \( \frac{t_1}{t_2} \) is irrational. Thus by Theorem 2.1 we have that \( T(t) \) is an \( m \)-isometry for every \( t \).

If \( T \) is an \( m \)-isometry, then any power \( T^r \) is also an \( m \)-isometry, [14, Theorem 2.3]. In general, the converse is not true. However, if \( T^r \) and \( T^{r+1} \) are \( m \)-isometries for a positive integer \( r \), then \( T \) is an \( m \)-isometry (see, [5, Corollary 3.7]). The stability of the class of \( m \)-isometry with powers is fundamental to give necessary and sufficient conditions for a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) to be \( T(t) \) an \( m \)-isometry for all \( t \geq 0 \).

Theorem 2.2. Let \( \{T(t)\}_{t \geq 0} \) be a \( C_0 \)-semigroup on a Hilbert space \( H \). Then the following properties are equivalent:

(i) \( T(t) \) is an \( m \)-isometry for every \( t \geq 0 \).

(ii) \( T(t) \) is an \( m \)-isometry for every \( t \in [0, t_1] \) for some \( t_1 > 0 \).

(iii) \( T(t) \) is an \( m \)-isometry on an interval of the form \([t_1, t_2]\) with \( t_1 < t_2 \).

Proof. The implications (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii) are clear.

(ii) \( \Rightarrow \) (i). Fixed \( t > 0 \), there exist \( n \in \mathbb{N} \) and \( t' \in [0, t_1] \) such that \( t = nt' \). Since \( T(t') \) is an \( m \)-isometry, then any power of \( T(t') \) is an \( m \)-isometry by [5, Theorem 3.1].

So, \( T(t) \) is an \( m \)-isometry.

(iii) \( \Rightarrow \) (i). Let us prove that \( T(t) \) is an \( m \)-isometry for every \( t \in (0, \frac{t_2 - t_1}{4}] \).

Choose \( k := \lceil \frac{t}{t_1} \rceil + 1 \). Then \( kt \) and \( (k + 1)t \) belong to \([t_1, t_2]\). Thus \( T(t)^k \) and \( T(t)^{k+1} \) are \( m \)-isometries. Hence \( T(t) \) is an \( m \)-isometry by [5, Corollary 3.7].

3 Embedding \( m \)-isometries into \( C_0 \)-semigroups

We are interested to study: When can we embed an \( m \)-isometry into a continuous \( C_0 \)-semigroup?, that is, given an \( m \)-isometry \( T \), to find a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) such that \( T(1) = T \).

Recall that an isometry \( T \) on a Hilbert space can be embedded into a \( C_0 \)-semigroup if and only if \( T \) is unitary or \( \text{codim}(R(T)) = \infty \), where \( R(T) \) denotes the range of \( T \). In this case, it is also possible to embed \( T \) into an isometric \( C_0 \)-semigroup [10, Theorem V.1.19].

Note that, if \( T \) can be embedded into \( C_0 \)-semigroup, then \( \text{dim}(\text{Ker}T) \) and \( \text{dim}(\text{Ker}T^*) \) are zero or infinite [10, Theorem V.1.7].
Proposition 3.1.  
(i) All $m$-isometries in finite dimensional space are embeddable on a $C_0$-group.

(ii) All normal $m$-isometries are embeddable.

(iii) The weighted forward shift $m$-isometries are not embeddable.

Proof. (i) On a finite-dimensional space an operator can be embeddable if and only if its spectrum does not contain 0 (see, [10, page 166]). Moreover, on finite-dimensional space the spectrum of $m$-isometries is contained in the unit circle. Hence any $m$-isometry on finite dimensional space is embeddable in a group.

(ii) If $T$ is a normal $m$-isometry, then $T$ is invertible and by [10, Theorem V.1.14] $T$ is embeddable.

(iii) Assume that $T$ is a weighted forward shift $m$-isometry. Then $\dim \ker (T^*) = 1$, (3 & 6).

Let $M_z$ be the multiplication operator on the Dirichlet space $D(\mu)$ for some finite non-negative Borel measure on $\mathbb{T}$ defined by

$$D(\mu) := \{ f : \mathbb{D} \to \mathbb{C} \text{ analytic} : \int_{\mathbb{D}} |f'(z)|^2 \varphi_{\mu}(z) dA(z) < \infty \} ,$$

where $A$ denotes the normalized Lebesgue area measure on $\mathbb{D}$ and $\varphi_{\mu}$ is defined by

$$\varphi_{\mu}(z) := \frac{1}{2\pi} \int_{[0,2\pi]} \frac{1 - |z|^2}{|e^{it} - z|^2} d\mu(t) ,$$

for $z \in \mathbb{D}$.

Proposition 3.2. $M_z$ on $D(\mu)$ can not be embedded into $C_0$-semigroup.

Proof. By Richter’s Theorem [20], $M_z$ is an analytic 2-isometry with $\dim \ker (T^*) = 1$, then $M_z$ can not be embedded into $C_0$-semigroup.

Let $Y$ be an infinite dimensional Hilbert space. Denote by $\ell_2^Y$ the Hilbert space of all vector sequences $(h_n)_{n=1}^{\infty}$ such that $\sum_{n \geq 1} \|h_n\|^2 < \infty$ with the standard inner product. If $(W_n)_{n=1}^{\infty} \subset B(Y)$ is an uniformly bounded sequence of operators, then the operator $S_W \in B(\ell_2^Y)$ defined by

$$S_W(h_1, h_2, \cdots) := (0, W_1 h_1, W_2 h_2, \cdots) ,$$

for any $(h_1, h_2, \cdots) \in \ell_2^Y$, is called the operator valued unilateral forward weighted shifts with weights $W := (W_n)_{n \geq 1}$.

Following some ideas of [10, Proposition V.1.18], we obtain the next result.

Lemma 3.1. Let $S_W$ be the valued unilateral forward weighted shift operator with weights $W = (W_n)_{n \geq 1}$ on $\ell_2^Y$ where $Y$ be an infinite dimensional Hilbert space. Then $S_W$ can be embedded into $C_0$-semigroup.
Proof. The operator $S_W$ is unitarily equivalent to the valued unilateral forward weighted shift operator on $\ell^2_{L^2([0,1), Y)}$. Moreover, $\ell^2_{L^2([0,1), Y)}$ can be identified with $L^2(\mathbb{R}^+, Y)$ by

$$(f_1, f_2, \ldots) \mapsto (s \to f_n(s - n), \ s \in [n, n + 1)) .$$

For $0 < t \leq 1$, define the following family of operators on $L^2(\mathbb{R}^+, Y)$ by

$$(T(t)f)(s) := \begin{cases} 
0 & s < t \\
f(s - t) & n - 1 + t \leq s < n \\
W_n f(s - t) & n \leq s < n + t ,
\end{cases}$$

that is,

$$(T(t)f)(s) = \begin{cases} 
0 & s < t \\
\sum_{n \geq 1} (f(s - t)\chi_{[n-1+t,n)}(s) + W_n f(s - t)\chi_{[n,n+t)}(s)) & s \geq t .
\end{cases}$$

In particular, for $t = 1$, we have that

$$(T(1)f)(s) = \begin{cases} 
0 & s < 1 \\
\sum_{n \geq 1} W_n f(s - 1)\chi_{[n,n+1)}(s) & s \geq 1 .
\end{cases}$$

Hence $T(1)$ is unitarily equivalent to $S_W$.

For $t > 1$, define $T(t) := T^{[t]}(1)T(t - [t])$, where $[t]$ denotes the greatest integer less than or equal to $t$.

Let us prove that $T(t)$ is a $C_0$-semigroup. Given any $f \in L^2(\mathbb{R}^+, Y)$, we have that

$$\lim_{t \to 0^+} T(t)f(s) = \sum_{n \geq 1} \chi_{[n-1,n)}f(s) = f(s) ,$$

in the strong topology of $L^2(\mathbb{R}^+, Y)$.

Let $t, t' \in [0,1)$. Then $T(t)T(t')f(s) = T(t)\tilde{f}(s)$, where

$$\tilde{f}(s) := T(t')f(s) = \begin{cases} 
0 & s < t' \\
\sum_{n \geq 1} (\chi_{[n-1+t',n)}(s) + W_n \chi_{[n,n+t')}(s))f(s - t') & s \geq t' .
\end{cases}$$

Then

$$T(t)T(t')f(s) = \begin{cases} 
0 & s < t + t' \\
\sum_{m \geq 1} \left( \sum_{n \geq 1} (\chi_{[n-1+t',n)}(s - t) + W_n \chi_{[n,n+t')}(s - t)) \chi_{[m-1+t,m)}(s) \right) + \\
W_m \sum_{n \geq 1} (\chi_{[n-1+t',n)}(s - t) + W_n \chi_{[n,n+t')}(s - t)) \\
\chi_{[m,m+t]}(s) & s \geq t + t' 
\end{cases}$$
\[
T(t) = \begin{cases}
0 & s < t + t' \\
\sum_{m \geq 1} \left\{ \sum_{n \geq 1} \left( \chi_{[n-1+t', n+1]}(s) + W_n \chi_{[n+t, n+t'+t]}(s) \right) \right\} \chi_{[m-1+t, m]}(s) + \\
W_m \sum_{n \geq 1} \left( \chi_{[n-1+t', n+1]}(s) + W_n \chi_{[n+t, n+t'+t]}(s) \right) & s \geq t + t'
\end{cases}
\]

If \( t' + t < 1 \), then (6) is given by

\[
T(t)T(t')f(s) = \begin{cases}
0 & s < t' + t \\
\sum_{n \geq 1} \left( \chi_{[n-1+t', n]}(s) + W_n \chi_{[n+n', t]}(s) \right) f(s - t' - t) & s \geq t' + t
\end{cases}
\]

\[
= T(t + t')f(s) .
\]

Denote \( t'' := t' + t - [t' + t] \). If \( t' + t > 1 \), then \( t'' = t' + t - 1 \) and

\[
T(t + t')f(s) = T^{[t+t']}(1)(T(t + t' - [t + t']))f(s) = T(1)T(t + t' - 1)f(s) = T(1)T(t'')f(s)
\]

\[
= \begin{cases}
0 & s < t'' \\
T(1) \sum_{n \geq 1} \left( \chi_{[n-1+t''', n]}(s) + W_n \chi_{[n+n'', t']} (s) \right) f(s - t'') & s \geq t''
\end{cases}
\]

\[
= \begin{cases}
0 & s - t'' < 1 \\
\sum_{m \geq 1} W_m \sum_{n \geq 1} \left( \chi_{[n-1+t''', n]}(s - 1) + W_n \chi_{[n+n'', t']} (s - 1) \right) \chi_{[m,m+1]}(s)f(s - t'') - 1) & s - t'' \geq 1
\end{cases}
\]

\[
= \begin{cases}
0 & s - t'' < 1 \\
\sum_{m \geq 1} W_m \sum_{n \geq 1} \left( \chi_{[n+t''', n+1]}(s) + W_n \chi_{[n+1,n+t'''+1]}(s) \right) \chi_{[m,m+1]}(s)f(s - t'' - 1) & s \geq 1 + t''
\end{cases}
\]

\[
= \begin{cases}
0 & s < t + t' \\
\sum_{n \geq 1} \left( W_n \chi_{[n,t''', n+1]}(s) + W_{n+1} W_n \chi_{[n+1,n+t'''+1]}(s) \right) f(s - t' - t) & s \geq t + t'.
\end{cases}
\]

By other hand, by (6) we have that

\[
T(t)T(t')f(s) =
\]

11
\[
\begin{cases}
0 & s < t + t' \\
\sum_{n \geq 1} (W_n \chi_{n+t'+t-1,n+1})(s) + W_{n+1} W_n \chi_{n+1,n+t+t}(s)) f(s-t'-t) & s \geq t + t'
\end{cases}
\]

This completes the proof. \(\square\)

We say that an operator \(T \in B(H)\) satisfies the \textit{kernel condition} if
\[
T^*(\text{Ker} T^*) \subset \text{Ker} T^*.
\]

**Corollary 3.1.** A non-unitary 2-isometry on a Hilbert space satisfying the kernel condition can be embedded into \(C_0\)-semigroup if and only if \(\text{dim} (\text{Ker} T^*) = \infty\).

**Proof.** If \(T\) is a non-unitary 2-isometry on a Hilbert space satisfying the kernel condition as consequence of [4, Theorem 3.8], we obtain that \(T \cong U \oplus W\) with \(U\) unitary and \(W\) a operator valued unilateral forward weighted shifts operator in \(\ell^2_M\) with \(\text{dim} M = \text{dim} (\text{Ker} T^*)\). Thus by [10, Theorem V.1.19] and Lemma 3.1, \(T\) can be embedded into \(C_0\)-semigroup. \(\square\)

By the Wold Decomposition Theorem for 2-isometries (see [16]), all 2-isometries can be decomposed as directed sum of unitary operator and an analytic 2-isometries.

**Question 3.1.** An analytic 2-isometry on a Hilbert space can be embedded into \(C_0\)-semigroup if and only if \(\text{dim} (\text{Ker} T^*) = \infty\)?

## 4 Translation semigroups of \(m\)-isometries

In this section, we discuss examples of semigroups of \(m\)-isometries.

**Definition 4.1.** By a \textit{right admissible weighted function} in \((0, \infty)\), we mean a measurable function \(\rho : (0, \infty) \to \mathbb{R}\) satisfying the following conditions:

1. \(\rho(\tau) > 0\) for all \(\tau \in (0, \infty)\),

2. there exist constants \(M \geq 1\) and \(\omega \in \mathbb{R}\) such that \(\rho(t + \tau) \leq Me^{\omega t} \rho(\tau)\) holds for all \(\tau \in (0, \infty)\) and all \(t > 0\).

For a right admissible weighted function, we define the \textit{weighted space} \(L^2(\mathbb{R}^+, \rho)\) of measurable functions \(f : \mathbb{R}^+ \to \mathbb{C}\) such that
\[
\|f\|_{L^2(\mathbb{R}^+, \rho)} = \int_0^\infty |f(s)|^2 \rho(s) ds < \infty.
\]

Then the \textit{right translation semigroup} \((S(t))_{t \geq 0}\) given for \(t \geq 0\) and \(f \in L^2(\mathbb{R}^+, \rho)\) by
\[
(S(t)f)(s) := \begin{cases}
0 & \text{if } s \leq t \\
(f(s-t)) & \text{if } s > t
\end{cases},
\]
is a strongly continuous semigroup and straightforward computation shows that for \(s, t \geq 0\) and \(f \in L^2(\mathbb{R}^+, \rho)\)
\[
(S^*(t)f)(s) = \frac{\rho(s+t)}{\rho(s)} f(s+t).
\]


Theorem 4.1. Let \( \{S(t)\}_{t \geq 0} \) be the right translation \( C_0 \)-semigroup on \( L^2(\mathbb{R}^+, \rho) \) with \( \rho \) a continuous function. The semigroup \( \{S(t)\}_{t \geq 0} \) is an \( m \)-isometry for every \( t > 0 \) if and only if \( \rho(s) \) is a polynomial of degree less than \( m \).

Proof. By definition, \( S(t) \) is an \( m \)-isometry for every \( t > 0 \) if

\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \|S^k(t)f\|^2 = 0,
\]

for all \( t \geq 0 \) and \( f \in L^2(\mathbb{R}^+, \rho) \). That is,

\[
\int_0^\infty \left( \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{\rho(s + kt)}{\rho(s)} \right) |f(s)|^2 \rho(s) ds = 0,
\]

for all \( t \geq 0 \) and \( f \in L^2(\mathbb{R}^+, \rho) \). Fixed \( t \geq 0 \), define

\[
g(s) := \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{\rho(s + kt)}{\rho(s)}.
\]

If \( g(s) \neq 0 \), we can suppose without lost of generality that, \( g(s) > 0 \), for some \( s \geq 0 \). Then by continuity of \( \rho \) and \( \rho(\tau) > 0 \) for all \( \tau \geq 0 \), we obtain that there exits an interval \( I \subset \mathbb{R}^+ \), with finite measure, such that there exists \( M > 0 \) such that \( g(s) > M \) for all \( s \in I \).

Let \( f_1(s) := \frac{1}{\sqrt{\rho(s)}} \chi_I(s) \in L^2(\mathbb{R}^+, \rho) \). Then by (7) we have that

\[
0 = \int_0^\infty g(s)|f_1(s)|^2 \rho(s) ds = \int_I g(s) ds > M \mu(I),
\]

which it is an absurd. So,

\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{\rho(s + kt)}{\rho(s)} = 0,
\]

for all \( s \geq 0 \) and \( t \geq 0 \). Then by [15, Theorem 13.5], the function \( \rho \) is a polynomial of degree less than \( m \).

Corollary 4.1. Let \( \{S(t)\}_{t \geq 0} \) be the right translation \( C_0 \)-semigroup on \( L^2(\mathbb{R}^+, \rho) \) with \( \rho \) a continuous function. Then \( \{S(t)\}_{t \geq 0} \) is a strict \( m \)-isometry for every \( t > 0 \) if and only if \( \rho(s) \) is a polynomial of degree \( m - 1 \).

Consider the right weighted translation \( C_0 \)-semigroup, \( S_\rho \), defined on \( L^2(\mathbb{R}^+) \) as

\[
(S_\rho(t)f)(s) := \begin{cases} 0 & \text{if } s \leq t \\ \frac{\rho(s)}{\rho(s-t)} f(s-t) & \text{if } s > t \end{cases}.
\]
Then \( \{S_\rho(t)\}_{t \geq 0} \) is a strongly continuous semigroup if and only if \( \rho \) is a right admissible weight [9] and for \( s, t \geq 0 \) and \( f \in L^2(\mathbb{R}^+) \)
\[
(S_\rho^*(t)f)(s) = \frac{\rho(s + t)}{\rho(s)} f(s + t).
\]

We now improve part (2) of [18, Corollary 3.3].

**Theorem 4.2.** Let \( \{S_\rho(t)\}_{t \geq 0} \) be the right weighted translation \( C_0 \)-semigroup on \( L^2(\mathbb{R}^+) \) with \( \rho \) a continuous function. Then \( \{S_\rho(t)\}_{t \geq 0} \) is an \( m \)-isometry for every \( t > 0 \) if and only if \( \rho(s)^2 \) is a polynomial of degree less than \( m \).

**Proof.** Consider \( M_\rho : L^2(\mathbb{R}^+, \rho^2) \rightarrow L^2(\mathbb{R}^+) \) defined by \( M_\rho f = \rho f \). Then \( S_\rho(t) = M_\rho S(t) M_\rho^{-1} \). Thus \( \{S_\rho(t)\}_{t \geq 0} \) is an \( m \)-isometry for every \( t > 0 \) on \( L^2(\mathbb{R}^+, \rho^2) \) if and only if \( \{S(t)\}_{t \geq 0} \) is an \( m \)-isometry for every \( t > 0 \) on \( L^2(\mathbb{R}^+, \rho^2) \) if and only if \( \rho(s)^2 \) is a polynomial of degree less than \( m \). \( \square \)

**Corollary 4.2.** Let \( \{S_\rho(t)\}_{t \geq 0} \) be the right weighted translation \( C_0 \)-semigroup on \( L^2(\mathbb{R}^+) \) with \( \rho \) a continuous function. Then \( \{S_\rho(t)\}_{t \geq 0} \) is a 2-isometry for every \( t > 0 \) if and only if \( \rho(s)^2 = as + b \) for some constants \( a \) and \( b \).

At continuation we characterize the weighted spaces where the adjoint of translation operator is an \( m \)-isometry.

**Definition 4.2.** By a left admissible weighted function in \((0, \infty)\), we mean a measurable function \( w : (0, \infty) \rightarrow \mathbb{R} \) satisfying the following conditions:

1. \( w(\tau) > 0 \) for all \( \tau \in (0, \infty) \),

2. there exist constants \( M \geq 1 \) and \( \alpha \in \mathbb{R} \) such that \( w(\tau) \leq Me^{\alpha \tau}w(t + \tau) \) holds for all \( \tau \in (0, \infty) \) and all \( t > 0 \).

For a left admissible weighted function we define the weighted space \( L^2(\mathbb{R}^+, w) \) as the measurable functions \( f : \mathbb{R}^+ \rightarrow \mathbb{C} \) such that
\[
\|f\|_{L^2(\mathbb{R}^+, w)} = \int_0^\infty |f(s)|^2 w(s) ds < \infty.
\]

Let \( \{T(t)\}_{t \geq 0} \) be the left shift semigroup given for \( t \geq 0 \) and \( f \in L^2(\mathbb{R}^+, w) \) by
\[
(T(t)f)(s) := f(s + t).
\]

Then \( \{T(t)\}_{t \geq 0} \) is a strongly continuous semigroup [11]. For \( f \in L^2(\mathbb{R}^+, w) \),
\[
(T^*(t)f)(s) = \begin{cases} 
0 & \text{if } s \leq t \\
\frac{w(s - t)}{w(s)} f(s - t) & \text{if } s > t.
\end{cases}
\]

**Theorem 4.3.** Let \( \{T^*(t)\}_{t \geq 0} \) be the adjoint of left weighted translation \( C_0 \)-semigroup on \( L^2(\mathbb{R}^+, w) \) such that \( w \) is a continuous function. Then \( \{T^*(t)\}_{t \geq 0} \) is an \( m \)-isometry for every \( t > 0 \) if and only if \( w(s) = \frac{1}{p(s)} \) for some polynomial \( p \) of degree less than \( m \).
Proof. $T^*(t)$ is an $m$-isometry for every $t > 0$ if and only if
\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \|T^k(t)f\|^2 = 0 ,
\]
for all $t > 0$ and $f \in L^2(\mathbb{R}^+, w)$. Then
\[
0 = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} (-1)^{m-k} \int_{kt}^{\infty} \left| \frac{w(s-kt)}{w(s)} f(s-kt) \right|^2 w(s) ds
\]
\[
= \int_{0}^{\infty} \left( \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{w(u)}{w(u+kt)} \right) |f(u)|^2 w(u) du ,
\]
for all $f \in L^2(\mathbb{R}^+, w)$. As in the proof of Theorem 4.1 we get that $\frac{1}{w(s)}$ is a polynomial of degree less than $m$.

Corollary 4.3. Let $\{T^*(t)\}_{t \geq 0}$ be the adjoint of left weighted translation $C_0$-semigroup on $L^2(\mathbb{R}^+, w)$ such that $w$ is a continuous function. Then $\{T^*(t)\}_{t \geq 0}$ is a strict $m$-isometry for every $t > 0$ if and only if $w(s) = \frac{1}{p(s)}$ for some polynomial $p$ of degree $m - 1$. \hfill $\Box$

Acknowledgements

The authors thank to Almira and Hernández-Abreu for calling their attention over Montel’s theorems.

References

[1] J. Agler, A disconjugacy theorem for Toeplitz operators. Amer. J. Math., 112 (1990), no. 1, 1-14.

[2] J. Agler, M. Stankus, $m$-isometric transformations of Hilbert space. I, Integral Equations Operator Theory, 21 (1995), no. 4, 383-429.

[3] J. M. Almira and A. J. López-Moreno, On solutions of the Fréchet functional equation. J. Math. Anal. Appl. 332 (2007), no. 2, 1119-1133.

[4] A. Anand, S. Chavan, Z. J. Jablonski and J. Stochel, A solution to the Cauchy dual subnormality problem for 2-isometries, arXiv:1702.01264v3.

[5] T. Bermúdez, C. Díaz and A. Martinón, Powers of $m$-isometries, Studia Math., 208 (2012), no. 3, 249–255.

[6] T. Bermúdez, A. Martinón and E. Negrín, Weighted shift operators which are $m$-isometries, Integral Equations Operator Theory, 68 (2010), no. 3, 301–312.
[7] T. Bermúdez, A. Martinón and J. A. Noda, An isometry plus a nilpotent operator is an $m$-isometry, Applications. J. Math. Anal. Appl., 407 (2013), no. 2, 505-512.

[8] I. Chalendar, J. R. Partington, Compactness, differentiability and similarity to isometry of composition semigroups. Problems and recent methods in operator theory, 67–73, Contemp. Math., 687, Amer. Math. Soc., Providence, RI, 2017.

[9] M.R. Embry and A. Lambert, Weighted translation semigroups, Rocky Mountain J. of Math., 7 (1977), no. 2, 333-344.

[10] T. Eisner, Stability of operators and operator semigroups, Operator Theory: Advances and Applications, 209. Birkhäuser Verlag, Basel, 2010.

[11] K.-G. Grosse-Erdmann and A. Peris, Linear Chaos, Springer, London, 2011.

[12] E. A. Gallardo-Gutiérrez and J. R. Partington, $C_0$-semigroups of 2-isometries and Dirichlet spaces, Revista Matematica Iberoamericana, to appear.

[13] B. Jacob, J. R. Partington, S. Pott, and A. Wynn, $\beta$-admissibility of observation and control operators for hypercontractive semigroups, J. Evolution Equations, 18 (2018) no. 1, 153-170.

[14] Z. J. Jablonski, Complete hyperexpansivity, subnormality and inverted boundedness conditions. Integral Equations Operator Theory 44 (2002), no. 3, 316-336.

[15] M. Kuczma, Functional equations in a single variable, PWN-Polish Scientific Publishers, Warszawa, 1968.

[16] A. Olofsson, A von Neumann-Wold decomposition of two-isometries, Acta Sci. Math. (Szeged), 70 (2004), no. 3-4, 715–726.

[17] A. Pazy, Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

[18] G. M. Phatak and V. M. Sholapurkar, Hyperexpansive weighted translation semigroups, arXiv:1803.08623v1.

[19] D. Popa and I. Rasa, The Fréchet functional equation with application to the stability of certain operators. J. Approx. Theory, 164 (2012), no. 1, 138-144.

[20] S. Richter, A representation theorem of cyclic analytic two-isometries, Trans. Amer. Math. Soc., 328 (1991), no. 1, 325-349.

[21] E. Rydhe, An Agler-type model theorem for $C_0$-semigroups of Hilbert space contractions, J. Lond. Math. Soc., 93 (2016), no. 2, 420-438.

T. Bermúdez
Departamento de Análisis Matemático, Universidad de La Laguna, 38271, La Laguna (Tenerife), Spain.
e-mail: tbermude@ull.es
A. Bonilla
Departamento de Análisis Matemático, Universidad de La Laguna, 38271, La Laguna (Tenerife), Spain.
e-mail: abonilla@ull.es

H. Zaway
Departamento de Análisis Matemático, Universidad de La Laguna, 38271, La Laguna (Tenerife), Spain.
Department of Mathematics, Faculty of Sciences, University of Gabes, 6072, Tunisia.
e-mail: hajer.zaway@live.fr