Combinatorial dynamics in quantum gravity

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ABSTRACT

We describe the application of methods from the study of discrete dynamical systems to the study of histories of evolving spin networks. These have been found to describe the small scale structure of quantum general relativity and extensions of them have been conjectured to give background independent formulations of string theory. We explain why the usual equilibrium second order critical phenomena may not be relevant for the problem of the continuum limit of such theories, and why the relevant critical phenomena analogue to the problem of the continuum limit is instead non-equilibrium critical phenomena such as directed percolation. The fact that such non-equilibrium critical phenomena may be self-organized implies the possibility that the classical limit of quantum theories of gravity may exist without fine tuning of parameters.

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1 Introduction

The idea that space and time are fundamentally discrete is very old and has often reappeared in the history of the search for a quantum theory of gravity\(^1\). However, it is only recently that concrete results from attempts to construct a quantum theory have gravity have been found which suggest very strongly that such a theory must be based on a discrete structure. These results come from the quantization of general relativity\(^2, 3\), string theory\(^4\) and the thermodynamics of black holes\(^5, 6, 7\). (For reviews see\(^8, 9, 10, 11\).)

If space and time are discrete, then the study of the dynamics of spacetime may benefit from our understanding of other discrete dynamical systems such as cellular automata\(^14\), froths\(^15\) and binary networks\(^16\). The importance of this may be seen once it is appreciated that a key problem in any discrete theory of quantum gravity must be the recovery of continuous space time and the fields that live on it as an approximation in an appropriate continuum limit. This continuum limit, which will be also related to the classical limit of the theory, (because the physical cutoff \(l_{\text{Planck}}\) which marks the transition between the discrete and continuous picture is proportional to \(\hbar\)) is then a problem in critical phenomena\(^17\). As one doesn’t want the existence of classical spacetime to rest on some fine tunings of parameters, this must presumably be some kind of spontaneous, or self-organized critical phenomena\(^18\).

However, there is a key element which which distinguishes quantum gravity from other kinds of quantum and statistical systems This is that the causal structure is dynamical. As a result, the usual second order equilibrium critical phenomena may not be relevant for the continuum limit of quantum theories of gravity, as its connection to quantum field theory relies on rotation from a Euclidean to Lorentzian metric and this is not well defined when the fluctuating degrees of freedom are the metric (or causal structure.) Instead, the relevant statistical physics analogue to the problem of the classical limit will be non-equilibrium critical phenomena\(^18\). To see why, let us consider the issue of critical behavior for a discrete dynamical systems whose only attribute is causal structure. Consider a set \(P\) of \(N\) events, such that for any two of them \(p\) and \(q\) one may have either \(p > q\), (meaning \(p\) is to the causal future of \(q\)), or \(q > p\), or neither, but never both. This gives the set \(P\) the structure of a partially ordered set, or poset. In

\(^1\)see, for example\(^1, 3\).

\(^2\)Indeed this is a general problem for particle physics, brought on by the hierarchy problem, which is the existence of several widely separated scales.
addition, if one assumes that there are no time like loops and that the poset is locally finite (which means that there are only a finite number of events in the intersection of the future of any event and the past of any other) one has what is called a causal set. One may then invent an action which depend on the causal relations and then study the quantum statistical physics of such a set, in the limit of large $N$.

This program has been pursued by physicists interested in using it as a model of quantum gravity, particularly by Myers, Sorkin, ‘tHooft and collaborators. This is motivated by the fact that the events of any Lorentzian spacetime form a poset, where $p < q$ is the causal relation arising from the lightcone structure of the metric. In fact, if the causal structure is given, the spacetime metric is determined up to an overall function.

Sorkin and collaborators have conjectured that the causal structure is sufficient to define a satisfactory quantum theory of spacetime. However, there is reason to believe that this may not be the case, and that additional structure associated with what may loosely be called the properties of space, must be introduced. One reason for this is that the models where the degrees of freedom are only causal structure do not seem, at least so far, to have yielded the kinds of results necessary to answer the key questions about the emergence of the classical limit.

As a result, recently, Markopoulou proposed adding structure to poset models of spacetime taken from results in other approaches to quantum gravity. Her idea has been to combine the discrete causal structure of poset construction with descriptions of a discrete quantum spatial geometry which has emerged from the study of non-perturbative quantum gravity. These descriptions are usually expressed in terms of spin networks, which are graphs whose edges are labeled with half-integers, $1/2, 1, 3/2, ...$ which represent quantum mechanical spins. Originally invented by Penrose, more recently they have been shown to represent faithfully a basis of exact non-perturbative states of the quantum gravitational field. Extensions of the spin network states have also been constructed that are relevant for supergravity and other extensions have been proposed in the context of a conjectured background independent formulation of string theory.

To show how the discrete causal structure of posets may be fitted to a discrete description of both spacetime and spatial geometry we may need to describe the structure of a causal set $P$ in more detail. The Alexandrov neighborhood of two events $p$ and $q$, $A(p,q)$, consist of all $x$ such that $p < x < q$. ‘t Hooft has proposed that the number of events in $A(p,q)$ should be a measure of its volume, in Planck units. If the poset is taken by events
picked randomly from a Lorentzian manifold, using the measure given by
the volume element, there is then exactly enough information in the poset
to reconstruct the metric, in the limit of an infinite number of events. Using
the Alexandrov neighborhoods of a poset, we may then construct a discrete
model of a spacetime geometry. When the theory has a good classical limit
that should approximate a continuous spacetime geometry.

In classical general relativity it is possible to define an infinite number
of spatial slices, which have defined on them three dimensional Reimannian
geometries. There are an infinite number of ways to slice a spacetime into
a sequence of spatial slices, each of which may be associated with surfaces
of simultaneity defined by a family of observers and clocks moving in the
spacetime. Because the choice of how to slice spacetime into a series of
spatial geometries is arbitrary time in general relativity is referred to as
being “many-fingered”.

A completely analogous notion of spatial geometry can be defined strictly
in terms of a poset. To do this we consider a set of events \( \Sigma \subset P \) which
consists of events \( y_i \) such that no two of them are causally related (i.e.
neither \( y_i < y_j \) or \( y_j < y_i \) for all pairs in \( \Sigma \).) These may be called “spacelike
related”. If no event of \( P \) may be added to \( \Sigma \) preserving the condition of
no causal relations it is a maximal set of spacelike related events. Such
sets are called antichains or discrete spacelike slices of \( P \). The basic idea
of [21] is then to endow the antichains of causal sets with the properties of
discrete quantum geometries represented by spin networks. The result gives
a notion of a quantum spacetime, which is discrete but which has many of
the attributes of continuous spacetime, including causal structure, spacelike
slices and many-fingered time. As described in [21] discrete sets having these
properties can be constructed by beginning with a spin network and then
altering it by a series of local moves.

The purpose of this paper is to raise several key issues involved in the
study of the continuum limits in this kind of formulation of quantum gravity.
It is written for statistical physicists, relativists and quantum field theorists.
Our intention in writing it is mainly to point the attention of people in these
fields to the existence of a class of problems in which methods used to study
non-equilibrium critical phenomena may play an important role in studies
of quantum gravity.

In the next section we describe the basic structure of a causally evolving
spin network, in language we hope is accessible to statistical physicists. We
do not give any details about how these structures are related to general
relativity or its quantization, these may be found elsewhere[8, 11, 22, 23, 1].
Section 3 and 4 then discuss the problem of the classical limit of this theory. In section 5 some structures are defined on the set of quantum states of the theory, which are then used in sections 6 and 7, in the context of a simplified model, to argue for the existence of a classical limit that may reproduce general relativity. Section 8 then introduces a new question, which is how the dynamics of the theory is to be chosen. We suggest that it may be reasonable for the dynamics to evolve as the spacetime does, leading to the classical limit as a kind of self-organized critical phenomena.

2 Combinatorial descriptions of quantum spacetime

There are actually several closely related versions of the spin network description of quantum spatial geometry\[24, 25, 26\]. As our interest here is on the analysis of their dynamics, we will consider only one kind of model, which is the easiest to visualize. This is associated with combinatorial triangulations\[21\].

We describe first the quantum geometry of space, then how these evolve to make combinatorial spacetimes.

2.1 Combinatorial description of spatial geometry

A combinatorial \( m \)-simplex is a set of \( m \) points, \( e_1, \ldots, e_m \) called the vertices, together with all the subsets of those points. Those subsets with two elements, \( e_{12} = \{e_1, e_2\} \) are called edges, those with three \( e_{123} = \{e_1, e_2, e_3\} \) faces and so on. A combinatorial tetrahedron is a combinatorial 4-simplex.

A three dimensional simplicial psuedomanifold, \( T \), consists of a set of \( N \) combinatorial tetrahedra joined such that each face is in exactly two tetrahedra. Many such psuedomanifolds define manifolds, in which case the neighborhoods of the edges and nodes are homeomorphic to the neighborhoods of edges and nodes in triangulations of Euclidean three space. These are constraints on the construction of the psuedomanifold, which are called the manifold conditions. When they are not satisfied, we have a more general structure of a psuedomanifold. Many psuedomanifolds can be constructed from manifolds by identifying two or more edges or nodes.

\[3\] Its exact relationship to the spin network states which arise in canonical quantum gravity is complicated, due to some subtleties which need not concern us here. These are discussed in \[27\].
The sets on which the manifold conditions fail to be satisfied constitute defects in the topology defined by the combinatorial triangulation. Under suitable choices of the evolution rules these defects propagate in time, forming extended objects, with dimension up to two less than the dimension of the spacetime. When the discrete spacetime has a dynamics, as we will describe below, laws of motion for the extended objects are induced. It is very interesting that string theory in its present form has in it extended objects of various dimensions; the relationship between those “branes” and the defects in pseudomanifolds is under investigation\cite{27}.

A psuedomanifold may be labeled by attaching suitable labels to the faces and tetrahedra. For quantum gravity it is useful to consider labels that come from the representation theory of some algebra $G$, which may be a Lie algebra, a quantum Lie algebra, a supersymmetry algebra, or something more general. Such algebras are characterized by a set of representations, $i, j, k...$ and by product rules for decomposing products of representations, $j \otimes k = \sum_l f_{jk}^l l$, where the $f_{jk}^l$ are integers. Each such algebra has associated to it linear vector spaces $V_{ijkl}$, which consists of the linear maps $\mu : i \otimes j \otimes k \otimes l \to 1$, where 1 is the one dimensional identity representation. It is then usual to label a model of quantum gravity with algebra $G$ by associating a representation $k$ with each face and an intertwinor $\mu \in V_{ijkl}$ to each tetrahedra, where $i, j, k, l$ label its four faces. The pseudomanifold $T$, together with a set of labels is denoted $\Gamma$ and called a labeled pseudomanifold.

It is particularly convenient to work with a quantum group at a root of unity, as the label sets in these cases are finite. In canonical quantum gravity, the quantum deformation is related to the cosmological constant\cite{29, 30}.

To each labeled pseudomanifold $\Gamma$ we associate a basis state $|\Gamma\rangle$ of a quantum theory of gravity. The set of such states spans the state space of the theory, $\mathcal{H}$, whose inner product is chosen so that the topologically distinct $|\Gamma\rangle$’s comprise an orthonormal basis.

Each labeled pseudomanifold is also dual to a spin network, which is a combinatorial graph constructed by drawing an edge going through each face and joining the four edges that enter every tetrahedra at a vertex\cite{1, 4}. The edges are then labeled by representations and the nodes by intertwinors.$^4$

$^4$ Note that the pseudomanifolds have more information than the spin networks, for a given spin network may come from several combinatorial triangulations. The spin network structure may be extended so as to code this additional information, for example by extending the edges into tubes as in \cite{25, 26}. For simplicity in this paper we stick to pseudomanifolds. In some papers these are also called “dual spin networks”\cite{21}. 

6
If one wants a simpler model one may simply declare all labels to be identical and leave them out. These are called "frozen models\[27\]". Frozen models are like the dynamical triangulation models of Euclidean quantum gravity, except that there are different kinds of simplices, corresponding to causal ordering. We may also consider "partly frozen" models in which the spins on the faces are all equal, but the intertwinors are allowed to vary over a set of allowed values.

One of the results of the canonical quantization of general relativity is a geometrical interpretation for the spins and intertwinors of spin networks. Given the correspondence of labeled triangulations to spin networks, this interpretation may be applied directly to the simplices of the labeled spin networks. Doing this, we find that each face $f_{abc}$ of the combinatorial triangulation has an area, which is related to the spin $j_{abc}$ on the face by the formula\[3\],

$$A_{abc} = l_{\text{Planck}}^2 \sqrt{j_{abc}(j_{abc}+1)}$$

There are also quanta of volume associated with the combinatorial tetrahedras of the combinatorial triangulations. This correspondence is more complicated, and is motivated as well from canonical quantum gravity. Associated with the finite dimensional space of intertwinors $\mathcal{H}_{j_\alpha}$ at each node, where the spins of the 4 incident edges are fixed to be $j_\alpha$, is a volume operator $V_{j_\alpha}\[10,3\]$. These operators are constructed in canonical quantum gravity\[10,3\] and shown to be hermitian\[31\]. They are also finite and diffeomorphism invariant, when constructed through an appropriate regularization procedure\[10,3\]. Their spectra have been computed\[31\], yielding a set of eigenvalues $\{v^I_{j_\alpha}\}$ and eigenstates $|v^I_{j_\alpha}\rangle \in \mathcal{H}_{j_\alpha}$. These eigenvalues are given, in units of $l_{\text{Planck}}^3$ by certain combinatorial expressions found in \[31\]. Thus, a combinatorial triangulation represents a quantum geometry where the faces have areas and the tetrahedra volumes, which depend on the labelings in the way we have described.

### 2.2 Causal evolution of quantum geometries

We now follow the proposal of \[21\] and construct combinatorial quantum spacetimes by applying a set of evolution rules to the states we have just described. A basis state $|\Gamma_0\rangle \in \mathcal{H}$ may evolve to one of a finite number of possible successor states $|\Gamma'_0\rangle$. Each $|\Gamma'_0\rangle$ is derived from $|\Gamma_0\rangle$ by application of one of four possible moves, called Pachner moves\[1\]. These moves modify the state $|\Gamma_0\rangle$ in a local region involving one to four adjacent
tetrahedra.

Consider any subset of $\Gamma$ consisting of $n$ adjacent tetrahedra, where $n$ is between 1 and 4, which make up $n$ out of the 5 tetrahedra of a four-simplex $S_4$. Then there is an evolution rule by which those $n$ tetrahedra are removed, and replaced by the other $5 - n$ tetrahedra in the $S_4$. This is called a Pachner move. The different possible moves are called $n \rightarrow (5 - n)$ moves (Thus, there are $1 \rightarrow 4$, $2 \rightarrow 3$, etc. moves. The new tetrahedra must be labeled, by new representations $j$ and intertwiners $k$. For each move there are 15 labels involved, 10 representations on the faces and 5 intertwinors on the tetrahedra. This is because the labels involved in the move are exactly those of the four simplex $S_4$. For each $n$ there is then an amplitude $A_{n \rightarrow 5 - n}$ that is a function of the 15 labels. A choice of these amplitudes for all possible labels, for the four cases $1 \rightarrow 4$, $2 \rightarrow 3$, etc., then constitutes a choice of the dynamics of the theory.

The application of one of the possible Pachner moves to $\Gamma_0$, together with a choice of the possible labelings on the new faces and tetrahedra the move creates, results in a new labeled pseudomanifold state $\Gamma_1$. This differs from $\Gamma_0$ just in a region which consisted of between 1 and 4 adjacent tetrahedra. The process may be continued a finite number of times $N$, to yield successor labeled pseudomanifold states $\Gamma_2, \ldots, \Gamma_N$.

Any particular set of $N$ moves beginning with a state $\Gamma_0$ and ending with a state $\Gamma_N$ defines a four-dimensional combinatorial structure, which we will call a history, $\mathcal{M}$ from $\Gamma_0$ to $\Gamma_N$. Each history consists of $N$ combinatorial four simplices. The boundary of $\mathcal{M}$, is a set of tetrahedra which fall into two connected sets so that $\partial \mathcal{M} = \Gamma_0 \cup \Gamma_1$. All tetrahedra not in the boundary of $\mathcal{M}$ are contained in exactly two four simplices of $\mathcal{M}$.

Each history $\mathcal{M}$ is a causal set, whose structure is determined as follows. The tetrahedra of each four simplex, $S_4$ of $\mathcal{M}$ are divided into two sets, which are called the past and the future set. This is possible because each four simplex contains tetrahedra in two states $\Gamma_i$ and $\Gamma_{i+1}$ for some $i$ between 0 and $N$. Those in $\Gamma_i$ were in the group that were wiped out by the Pachner move, which were replaced by those in $\Gamma_{i+1}$. Those that were wiped out are called the past set of that four simplex, the new ones, those in $\Gamma_{i+1}$ are called the future set. With the exception of those in the boundary, every tetrahedron is in the future set of one four simplex and the past set of another.

The causal structure of $\mathcal{M}$ is then defined as follows. The tetrahedra of $\mathcal{M}$ make up a causal set defined as follows. Given two tetrahedra $T_1$ and $T_2$ in $\mathcal{M}$, we say $T_2$ is to the future of $T_1$ (written $T_2 \succ T_1$) iff there is a
sequence of causal steps that begin on $T_1$ and end on $T_2$. A causal step is a step from a tetrahedron which is an element of the past set of some four simplex, $S_4$ to any tetrahedron which is an element of the future set of the same four simplex. By construction, there are no closed causal loops, so the partial ordering gives a causal set.

Each history $\mathcal{M}$ may also be foliated by a number of spacelike slices $\Gamma$. These are the anitchains that we defined in section 1.

Each $\Gamma_i$ in the original construction of $\mathcal{M}$ constitutes a spacelike slice of $\mathcal{M}$. But there are also many other spacelike slices in $\mathcal{M}$ that are not one of the $\Gamma_i$. In fact, given any spacelike slice $\Gamma$ in $\mathcal{M}$ there are a large, but finite, number of slices which are differ from it by the application of one Pachner move. Because of this, there is in this formulation a discrete analogue of the many fingered time of the canonical picture of general relativity.

### 2.3 How the dynamics are specified

We have now defined quantum spatial geometry and quantum spacetime histories, both completely combinatorially. To turn this structure into a physical theory we must invent some dynamics. Although it is not the only possible starting point (and we will discuss another in section 8) it is best to begin by being conservative and using the standard notion of the path integral. We then assign to each history $\mathcal{M}$ an amplitude $A[\mathcal{M}]$ given by

$$A[\mathcal{M}] = \prod_i A[i]$$

where the product is over the moves, or equivalently the 4-simplices, labeled by $i$. $A[i]$ is the amplitude for that four simplex, which will be a function of its causal structure ($1 \to 4$ or the others) and the labels on its faces and tetrahedra. The dynamics is specified by giving the complex function $A[i]$, which depends on the possible causal structures and labels, a choice of such a function is equivalent to a choice of an action.

The amplitude for the transition from an initial state $|i>$ to a final state $|f>$, both in $\mathcal{H}$ is then given by

$$T[i, f] = \sum_{\mathcal{M} : \partial \mathcal{M} = |i \cup |f}> A[\mathcal{M}]$$

where the sum is over all histories from the given initial and final state.

The theory is then specified by giving the kinematics, which is the algebra from which the label set is chosen and the dynamics, which is the choice
of functions $A[i]$. One important question, which we will now discuss, is whether there are choices that lead to theories that have a good classical limit.

3 The problem of the classical limit and its relationship to critical phenomena

Having defined the class of models we will study, we now turn to our main subject, which is the problem of the classical limit and its relation to problems in non-equilibrium critical phenomena. We begin by making the following observation: Suppose that the amplitudes of each move were real numbers of the form,

$$A[i] = e^{-S(i)}$$

Then the sum over histories can be considered to define a statistical system, whose partition function is of the form,

$$Z[i, f] = \sum_{M | \partial M = i \cup f} e^{-\sum_i S(i)}$$

Thus we have a statistical average over histories, each weighed by a probability, just as in non-equilibrium systems such as percolation problems. In fact, there is an exact relationship with directed percolation problems, as the following example shows.

In Figure (1) we show the setup of a 1 + 1 directed percolation problem. The degrees of freedom are the arrows, each of which points to the future, which is upwards in the picture. The value or state of an arrow is whether it is on or off. A history, $M$ of a directed percolation problem is a record of which arrows are on. One such history is shown in Figure (2).

In the simplest version of directed percolation, each arrow is turned on with a probability $p$. There is a critical probability $p^*$ at which the percolation phase transition takes place. Below $p^*$ the on arrows make up disconnected clusters of finite size, whereas for $p > p^*$ the on arrows almost always form a single connected cluster. At $p^*$ the system is just barely connected. At this point correlation functions are scale invariant.

A more complicated version of directed percolation can be described as follows. Each diagonal link is turned on or off according to a rule which depends on several parameters. To do this one introduces a time coordinate, which is a label attached to the nodes which is increasing in the direction
Figure 1: A $1 + 1$ dimensional directed percolation problem.

Figure 2: One history of a directed percolation system.
the arrows point and so that all nodes that share a common time coordinate are causally unrelated. We then apply the rule to each node at a given time, successively in time, generating the evolution of the history from some initial state.

Each node has two arrows pointing towards it, which we will call the node’s past arrows and two arrows leaving it, which we will call its future arrows. The rule governs whether one or both of the future pointing arrows at the node are on, as a function of the state of the past arrows. For our purposes the exact form of the rules is not important, what matters is that there is a critical surface in the space of parameters at which the behavior of the system is critical, corresponding to the percolation phase transition. At the critical point the system is in the same universality class as simple directed percolation depending on the one parameter $p$. This second model will be called the dynamical model, as the histories evolve in time, by applying the rule to the nodes at later and later times. A dynamical model may be probabilistic or deterministic, depending on the nature of the rule applied at each node.

Notice that a history $\mathcal{M}$ of a directed percolation problem is a causal set. We will say that a node $p$ is to the future of a node $q$ (and write $p > q$) in a given history $\mathcal{M}$ if there is a chain of on arrows beginning at $q$ and ending at $p$. A model of directed percolation in $d + 1$ dimensions is then a model of dynamical causal structure for a discrete $d + 1$ dimensional spacetime. A history $\mathcal{M}$ of a directed percolation model then has a causal structure and all its acutraments, including discrete spacelike surfaces, light cones, future causal domains, past causal domains, etc. In a percolation problem based on a fixed spacetime lattice as in Figure (1), we may define the background causal structure to be the one defined by the history in which all the arrows are on.

In particular, the values of the arrows (on or off) at one time $t$ make a state $|\psi >$. If the model has $n$ arrows in each constant time surface, the state space is $4^n$ dimensional. In the deterministic models an initial state $|\psi_0 >$ evolves to a unique history $\mathcal{M}$. Thus a deterministic model of directed percolation is a cellular automata, called a Doman-Kunsel cellular automata model[32].

One way to understand what happens at the directed percolation critical point is to use the concept of damage[10]. In a deterministic model of directed percolation pick an initial state $|\psi_0 >$. Evolve the system to a history $\mathcal{M}_0$. Then change one arrow $a_0$ in the initial state and evolve to the corresponding history $\mathcal{M}_1$. Label any arrow whose value is different in the
two histories as *damaged*. The damaged arrows make a connected set \( D \), called the damaged set, which lie in the future causal domain of the arrow \( a_0 \) according to the background causal structure.

Hence, we see that damage corresponds to a perturbation of the discrete causal structure. It is interesting to ask how the morphology of the damaged region depends on the phase of the percolation system. Below the percolation phase transition the causal domains are finite and isolated, and the same is true for the damaged sets. Just at the phase transition point, damage is able to propagate arbitrarily far, for the first time. However, the damage is constrained to follow the background causal structure, which is the causal structure of the unperturbed history. Thus, if the theory has a continuum limit, the spread of the damage will correspond to the propagation of some causal effect. But if there is a continuum limit associated with the phase transition, then the correlation functions that measure the spread of damage will be power-law. In this case they should correspond in the continuum limit to the propagation of massless particles. Thus, if we think of the damage as the propagation of a perturbation in the causal structure, it must correspond in the continuum limit to the propagation of a graviton, which is how the propagation of a change in the causal structure is described in the perturbative theory. If the theory has a good continuum limit then the gravitons must travel arbitrarily far up the lightcones of the background causal structure. We see that this will only be possible at the critical point of the directed percolation model.

Thus, by identifying a directed percolation model with a dynamical theory of causal structure, we see that if that theory is to have a continuum limit corresponding to general relativity in 4 or more spacetime dimensions, the only possibility for the existence of such a limit is at the critical point of the directed percolation model. Thus we see that directed percolation critical phenomena must play the same role for discrete models of dynamical causal structure that ordinary second order critical behavior plays in Euclidean quantum field theory.

## 4 Is there quantum directed percolation?

There is however an important difference between what is required for a theory of quantum gravity and the directed percolation models so far studied by statistical physicists. In a discrete model of quantum gravity each history \( \mathcal{M} \) is assigned an amplitude \( \mathcal{A}[\mathcal{M}] \), which is generally a complex
number. All directed percolation models so far studied (to the authors’ knowledge) are either deterministic or probabilistic. In the latter case a probability $p[M]$ is assigned to each history $M$, which is of course a real number between 0 and 1. It is only in this case, in which each history has a probability, that we know anything about the critical phenomena associated to directed percolation. However, in quantum mechanics paths are weighed by amplitudes, which are complex numbers. Thus, it would thus be very interesting to know whether there are analogous critical phenomena in models which are set up as directed percolation models, (for example as in Figure [1], except that a complex amplitude $A[e]$, rather than a probability, is assigned to the state at each node. We may call such a model a quantum directed percolation model. We believe that the study of such models could be very useful for understanding the conditions required for discrete models of quantum gravity to have good continuum limits.

One issue that must be stressed is that very little is actually known about the continuum limit for Lorentzian path integrals where the histories are weighed by complex phases rather than probabilities. In quantum mechanics and conventional quantum field theory the path integrals are normally defined by analytic continuation from Euclidean field theory, where the weights can be considered probabilities. In the absence of such a definition, one might try to define the sum over histories directly. However, one faces a serious question of whether the sums converge at all.

This problem cannot be avoided in a case such as the present, in which the system is discrete. Of course, the usual wisdom is that the classical limit will exist because the phases from histories which are far-from-classical paths interfere destructively, leaving only the contributions near-classical histories, which add constructively. The problem is that in a finite system, in which there are a finite number of histories in the sum, the cancellation coming from the destructive interference will not be complete. There will be a residue coming from the sum, with a random phase and an absolute value of order $\sqrt{n}$, if there are $n$ far-from-classical histories. This contribution must be much smaller than those coming from close to classical paths, which will have an absolute value of order $m$, where $m$ is the number of close to classical paths. Thus, the existence of the classical limit seems to require that $m >> \sqrt{n}$, which means that there are many more near classical paths than far-from-classical paths. Of course, in any standard quantum system the actual situation is the opposite, there are many more far-from-classical paths.

5This has been verified in a numerical computation by Sameer Gupta.
than near-classical paths.

This argument suggests that the existence of the classical limit may require that a continuum limit has been taken in which the number of histories diverges. In this case it may be possible to tune parameters to define a limit in which the non-classical contribution to the amplitude cancels completely. In essence, this is what is forced by defining the theory in terms of an analytic continuation from a Euclidean field theory.

In the absence of a definition by an analytic continuation, the sums over causal histories may fail to have a good classical limit because they lack both an infinite sum over histories and a suitable definition of a corresponding Euclidean theory. This is perhaps the key question concerning the classical limit of such theories.

5 Discrete superspace and its structure

Having raised several issues concerned with the evaluation of the path integrals that arise in studies of evolving spin networks, we would now like to describe here a formalism and a language which may be useful for addressing them. It is convenient to consider a superspace $\Omega$ consisting of all 3 dimensional pseudomanifolds constructed with a finite number of tetrahedra. Associated to this is $\Omega^G$, which is the space of all labeled pseudomanifolds based on the algebra $G$. These spaces have intrinsic structure generated by the evolution under the Pachner moves.

Consider an initial pseudomanifold $\Gamma^0$, with a finite number of tetrahedra. We then consider all pseudomanifolds $\gamma^1_\alpha$ that can be reached from $\Gamma^0$ by one instance of any of the 4 allowed moves $n \rightarrow 5 - n$. They are finite in number, and labeled by an arbitrary integer $\alpha$. We will call this set $S^1_{\gamma^0}$. Generalizing this, it is natural then to consider the set $S^N_{\gamma^0}$ of all pseudomanifolds that can be reached from $\Gamma^0$ in $N$ or less moves. Clearly we have $S^N_{\gamma^0} = S^{N-1}_{\gamma^0} \subset S^N_{\gamma^0}$. We will also want to speak about the “boundary” of $S^N_{\gamma^0}$, which is $B^N_{\gamma^0}$, the set of all four valent graphs that can be reached from $\gamma^0$ in $N$ moves, but cannot be reached from $\gamma^0$ by any path in fewer than $N$ moves. A pseudomanifold in $B^N_{\gamma^0}$ will be labeled $\gamma^N_{\alpha_1, \ldots, \alpha_N}$ where, for example, $\gamma^2_{\alpha_1, \alpha_2}$ is the $\alpha_2$’th labeled pseudomanifold that can be reached from $\gamma^1_{\alpha_1}$.

It is also convenient to use the following terminology, borrowed from considerations of combinatorial chemistry\cite{33}. We will call the set $S^1_{\gamma^0}$ the adjacent possible set of $\gamma_0$, as it consists of all the possible states that could directly follow $\gamma_0$. More generally, for any $N$, the set $B^N_{\gamma^0}$ will be called the
$N$’th adjacent possible, since it contains all the possible new states available to the universe after $N$ steps that were not available after $N - 1$ steps.

It is clear that the for states composed of a large number of labeled tetrahedra, the $N$’th adjacent possible sets grow quickly, as is typical for combinatorial systems.

We may make some straightforward observations about the sets $S^N_{y_0}$.

- Given two pseudomanifolds $\alpha$ and $\beta$ in $S^N_{y_0}$, we will say that $\alpha$ generates $\beta$ if there is a single move that takes $\alpha$ to $\beta$. (For example $\gamma_{\alpha_1}^1$ generates $\gamma_{\alpha_1,\alpha_2}^2$.) $S^N_{y_0}$ then has the structure of a supergraph $G^N_{y_0}$, which is a directed graph whose nodes consist of the elements of $S^N_{y_0}$, connected by directed edges that represent generation.

- A path $p$ in $S^N_{y_0}$ is a list of pseudomanifolds $\gamma_1, ..., \gamma_m$, each of whom generates the next. If there exists a path $p$ that runs from $\alpha$ to $\delta$, both elements of $S^N_{y_0}$ then we may say that $\alpha \leq \delta$, or “$\alpha$ precedes $\delta$”. $S^N_{y_0}$ thus has the structure of a partially ordered set.

There are corresponding statements for $\Omega_G$, the space of all finite labeled pseudomanifolds. We may define the set $M^N_{\gamma_0}$, an element of which is a labeled pseudomanifold $\Gamma$. This corresponds to all elements of $\Omega_G$ which may be reached in $N$ steps from an initial labeled pseudomanifold $\gamma_0$. We may extend the relations just defined to the elements of $M^N_{\gamma_0}$. Thus, given two labeled pseudomanifolds $\Gamma$ and $\Delta$, we may say $\Gamma$ generates $\Delta$ if the graph $\gamma$ of $\Gamma$ generates the graph $\delta$ of $\Delta$, with the obvious extensions to the notion of a path. Thus, $M^N_{\gamma_0}$ has as well the structure of a partially ordered set. In addition, we have the “boundary” of $M^N_{\gamma_0}$, consisting of all the labelings of the elements of $B^N_{\gamma_0}$, which we may call $A^N_{\gamma_0}$.

We may note that neither $M^N_{\gamma_0}$ nor $S^N_{y_0}$ are causal sets, as for $N$ large enough there will be closed paths that may begin and end on a graph $\gamma \in S^N_{y_0}$.

We may note that there is an obvious map $r : \Omega_G \rightarrow \Omega$ in which labels are erased.

We consider $M^N_{\gamma_0}$ to be then the discrete analogue of Wheelers superspace. This is suggested by the fact that the labeled pseudomanifolds diagonalize observables that measure the three geometry. We may note that just as in the continuum case we may put a metric on $M^N_{\gamma_0}$. If $\alpha > \beta$ or $\beta > \alpha$ then we may say that $\alpha$ and $\beta$ are causally related. In this case, the metric $g(\alpha, \beta) = n$, the length of the shortest path that connects them. Thus, as in the continuum case, the metric gives the superspaces a poset structure.
6 Some simple models

We will now illustrate some of the issues involved in the continuum limit, using the frozen model as an example. This model is similar to dynamical triangulation models of Euclidean quantum gravity, but it differs from those because of the role of the causal structure. To write it down more explicitly, we let the index $c$ take values over the four types of causal structure: $c \in \{1 \to 4, 2 \to 3, 3 \to 2, 4 \to 1\}$.

There are then four amplitudes $\mathcal{A}[c]$ that must be specified. We may write them in terms of amplitudes and phases as,

$$\mathcal{A}[c] = a_c e^{i\theta_c}$$  \hspace{1cm} (6)

The amplitude for a history is then given by

$$\mathcal{A}[\mathcal{M}] = \prod_c (\mathcal{A}[c])^{N_c}$$  \hspace{1cm} (7)

where $N_c$ is the number of occurrences of the $c$'th causal structure in the history. These of course satisfy

$$N = \sum_c N_c.$$  \hspace{1cm} (8)

The model has four parameters, which are the four complex numbers $\mathcal{A}[c]$. It can be further simplified so that it depends only on fewer parameters. One way to do this is to insist that the amplitude are pure phases, so that all four moves have equal probability, but with certain phases,

$$\mathcal{A}[c] = e^{i\theta_c}$$  \hspace{1cm} (9)

We can further simplify by insisting that each of the pair of moves that are time reversals of each other have the same phase, this means that\footnote{The reader may wonder why we assign the time reversed amplitude to be equal to the original, rather than its complex conjugate. The answer is that we want a process followed, by its time reversal, to be distinct from the process where nothing happens. In general relativity a process and its time reversal are related by a diffeomorphism and thus have equal actions, thus in the quantum theory they are given by equal amplitudes.}

$$\mathcal{A}[1 \to 4] = \mathcal{A}[1 \to 4] = e^{i\alpha}$$  \hspace{1cm} (10)

$$\mathcal{A}[2 \to 3] = \mathcal{A}[3 \to 2] = e^{i\beta}$$  \hspace{1cm} (11)
To write the amplitude let us then define \( \lambda = \frac{1}{2}(\alpha + \beta) \) and \( \mu = \frac{1}{2}(\alpha - \beta) \).

The total amplitude of a history \( \mathcal{M} \) is then,

\[
\mathcal{A}[\mathcal{M}] = e^{i(\lambda N_{\text{total}} + \mu N_{\text{diff}})}
\]  

\[(12)\]

where

\[
N_{\text{diff}} = N[1 \to 4] + N[4 \to 1] - N[2 \to 3] - N[3 \to 2]
\]  

\[(13)\]

We see that as \( N_{\text{total}} \) is proportional to the four volume, \( \lambda \) plays the role of a cosmological constant. It is interesting to compare this to the action for dynamical triangulations, which is of the form \( S^{DT} = \lambda N_{\text{total}} + \kappa N_2 \) where \( N_2 \) is the number of two simplices which is a measure of the averaged spacetime scalar curvature. This suggests that if there is a continuum limit \( N_{\text{diff}} \) might also be a measure of the averaged spacetime curvature scalar, suitable for spacetimes of Minkowskian signature.

7 The classical limit of the frozen models

Of course, the actual behavior of the evolution described by the theory will depend on the details of the amplitudes \( \mathcal{A}[c] \). However, it is useful to ask whether any conclusions can be drawn about the evolution in the case that we have no information about the actual forms of the amplitudes. Let us make the simplest possible assumption, which is that all the amplitudes are given by some random real phase, so that \( \mathcal{A}[c] = e^{i\theta} \). Then the amplitude for any path \( p \) is \( \exp[i\theta n(p)] \).

In this case we can draw some simple conclusions as follows. Consider the amplitudes \( \mathcal{A}[\Gamma_0 \to \Gamma_f] \) for all labeled pseudomanifolds \( \Gamma_f \in \mathcal{M}_N^{\mathcal{H}} \). It is clear that for \( \Gamma_f \in \mathcal{A}_{N_0}^{\mathcal{H}} \) the amplitudes \( \mathcal{A}[\Gamma_0 \to \Gamma_f] = W e^{i\theta N} \), where \( W \) is the number of inequivalent ways to reach \( \Gamma_f \) in \( N \) steps. Thus, the amplitudes evolve in such a way that the amplitudes for the states on the boundary is always a coherent phase.

On the other hand, consider a \( \Gamma_t \) which is in the interior of \( \mathcal{M}_N^{\mathcal{H}} \). Let this be an element that is in \( \mathcal{A}_M^{\mathcal{H}} \) for some \( M << N \). There will typically be a number of different paths that reach \( \Gamma_t \), with a variety of different path lengths. The number of such paths will grow rapidly with \( N \), as long as \( M << N \). The total amplitudes for such labeled pseudomanifolds to be reached after \( N \) steps then will be \( \mathcal{A}[\Gamma_0 \to \Gamma_f] = \sum_r e^{i\theta r} \) with \( r \) a finite set of integers \( M \geq r \geq N \). As \( N \) grows large this set grows, and there are typically no interesting correlations amongst them. In this case, as \( N \) grows large then \( \mathcal{A}[\Gamma_0 \to \Gamma_f] \approx 0 \).
This means that for $N$ large, most of the amplitude predicted by the path integral \(^3\) with these assumptions will be concentrated on $A_{\gamma_0}^M$ and a narrow shell trailing it.

This may be considered to be a form of the classical limit, because as $N$ grows, the amplitude to have evolved from $\Gamma_0$ to a state $\Gamma_f$ by an $N$ step path is concentrated on those states that can be reached in $N$ steps, but no fewer. This means that as $N$ increases the amplitude is evolving along geodesics of the metric $G$ defined in the discrete superspace.

\section{Dynamics including the parameters}

In the class of theories we have formulated here the dynamics of the theory is given by four functions $A[c]$ which give the amplitude for each four simplex which is added to a history as the result of a Pachner move. These functions depend on the causal structure $c$ and labels on the 4-simplices. By using the requirement that the functions are invariant under permutations of the elements of the four simplex that do not change the causal structure, we can reduce the functions $A[c,p]$ to particular forms which depend on a set of parameters, $p$, which live in a parameter space $\mathcal{P}$. The main dynamical problem is to find the set $\mathcal{P}^* \subset \mathcal{P}$ such that the amplitudes defined by the sum over histories \(^3\) has a good classical limit.

However there is clearly something unsatisfactory about this formulation. No fundamental theory can be considered acceptable if it has a large number of parameters which must be finely tuned to some special values in order that the theory reproduces the gross features of our world. Instead, we would prefer a theory in which the critical behavior necessary for the existence of the classical limit was achieved automatically. Theories of this kind are called “self-organized critical systems”.

One possibility is that the parameters $p$ which determine the amplitudes for the different evolution moves are themselves dynamical variables which evolve during the course of the evolution of the system to values which define a critical system with a good continuum limit.

Here is one form of such a theory. We associate to each tetrahedron in the model, $T_i$ a value of the parameters $p_i$. When a move is made it involves $n < 5$ tetrahedra. We will assume that the amplitude of the move is given by $A[c, < p >]$ where $< p >$ is the average of the $p_i$ among the $n$ tetrahedra involved in the move. The move creates $5 - n$ new tetrahedra. We assign to each of them the new parameters $< p >$. This rule guarantees that those
choices of parameters that spread the most widely through the population of tetrahedra govern the most amplitudes. In this way, the system itself may discover and select the parameters that lead to criticality, and hence a classical limit.

Other rules for the new parameters may be contemplated. Another choice is the following. The set of parameters $p_\alpha$ are divided randomly into $n$ sets. The new $p_\alpha$’s in each of these sets are taken from the corresponding values in one of the $n$ “parent” tetrahedra that were input into the move. This distribution of the parameters is made separately for each of the $5 - n$ new tetrahedra.

The reader may object that the possibility for giving different rules for the choices of parameters violates our intention that the system choose its own laws. However, this is not the point. There is no way to avoid making a choice in giving rules to the system. What we want to avoid is the circumstance that the rules which result in a classical limit are so unlikely that it seems a miracle that they be chosen properly. What would be more comfortable is an evolution rule that has no sensitive dependence on a choice of parameters that results in the system naturally having a classical limit. By making the system choose the parameters itself, on the basis of a rule that selects those that lead to the most efficient propagation of information, we may make it possible for the system to tune itself to criticality.

9 Concluding remarks

In this paper we have explored the suggestion[18] that the problem of the continuum limit of a certain class of quantum theories of gravity may be explored through methods developed to study non-equilibrium critical phenomena. We have identified two key obstacles to the realization of this suggestion. First, techniques must be developed which allow efficient computations to be done on the class of theories described here. Second, those techniques must allow us to identify critical phenomena in real quantum mechanical path integrals, in which one has a problem like quantum directed percolation, in which histories are weighed by complex amplitudes. When these obstacles are overcome we will have the tools we need to discover which non-perturbative quantum theories of gravity have good continuum limits which reproduce classical general relativity interacting with quantized matter degrees of freedom, in a way that does not rely assumptions about the applicability of Euclidean methods to theories of dynamical causal structure,
that are unjustified and likely false.

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