# A REMARK ON HOPKINS’ CHROMATIC SPLITTING CONJECTURE

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**Abstract.** Ravenel [4 §8.10] proved the remarkable fact that the $K$-theoretic localization $L_K S^0$ of the sphere spectrum has

$$\pi_{-2} L_K S^0 \cong \mathbb{Q}/\mathbb{Z}.$$ 

Hopkins’ chromatic splitting conjecture [2] implies, more generally, that there are $3^{n-1}$ copies of $(\mathbb{Q}/\mathbb{Z})_p$ in the homotopy groups of the $E(n)$-localization of $S^0$; but where these copies occur can be confusing. We try here to simplify this book-keeping.

## 1. Introduction

This document is a footnote to Mark Hovey’s account of Mike Hopkins’ chromatic splitting conjecture, but some of the notation here differs slightly from his:

I’ll write $[X, Y]$ for the spectrum of maps from $X$ to $Y$, and $[X, Y]_s$ for its $s$th homotopy group; $S X$ is the suspension of $X$. The Bousfield localization of the sphere spectrum $S^0$ with respect to $E(n)$ will be our main concern (the relevant prime $p$ will be suppressed) and for a general spectrum $E$ I’ll simplify $L_E S^0$ to $L_E$ and write $L_n$ for $L_{E(n)}$. Since $E(n)$ is smashing, $L_n X = L_n \wedge X$ is a homology theory.

There are standard homomorphisms $L_n \to L_{n-k}$ of ringspectra for $n \geq k$, and a fiber product square

$$
\begin{array}{ccc}
L_n & \to & L_{K(n)} \\
\downarrow & & \downarrow \\
L_{n-1} & \to & L_{n-1} L_{K(n)}
\end{array}
$$

as well as a cofiber sequence [2 §4.1]

$$S^{-1} L_{K(n)} \to [L_{n-1}, L_n] \to L_n.$$ 

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This work was motivated by Hans-Werner Henn’s talk on [1] at the August 2011 Hamburg conference on structured ringspectra.
The splitting conjecture [II §4.2] asserts

- the existence of a monomorphism of an exterior algebra

\[ E^*_W(F_p)(\zeta_{2i+1} \mid 0 \leq i \leq n - 1) \to H^*_c(S_n, W(F_p)) \]

[cf [3 §2.2.5]; \( \zeta_i \) corresponds to Hovey’s \( x_{i+1} \)] which becomes an isomorphism after rationalization, such that the products

\[ \zeta^I := \zeta_{2i_1+1} \cdots \zeta_{2i_l+1} \]

(here \( I = i_1, \ldots, i_l, \ 0 \leq i_1 < \cdots < i_l < n \) has weight \( w(I) = n \) and degree \( |I| = 2 \sum i_k + l \)) survive the descent spectral sequence

\[ H^*_c(S_n, W(F_p))^\text{Gal}(\overline{F}_p/F_p) \Rightarrow \pi_* L_{K(n)} \]

to define maps

\[ \zeta^I : S^{-|I|} \to L_{K(n)} \]

such that

- the resulting composites

\[
\begin{array}{cccc}
S^{-|I|} & \xrightarrow{S^{-1}\zeta^I} & S^{-1} L_{K(n)} & \xrightarrow{[L_{n-1}, L_n]} \\
\downarrow & & \downarrow S^{-1}\zeta^I & \\
S^{-|I|} L_{n-1-i_l} & & & \\
\end{array}
\]

factor as shown, yielding an equivalence

\[ \vee S^{-1} \zeta^I : \vee_{I,i_l>0} S^{-|I|} L_{n-1-i_l} \xrightarrow{\cong} [L_{n-1}, L_n]. \]

2. Generating functions

The vector space dual

\[ L_n^Q(X) := [L_0, L_n \wedge X]_D^s := \text{Hom}([L_0, L_n \wedge X]_{-s}, Q) \]

of \([L_0, L_n \wedge X]\) defines a cohomology theory which, following Hovey, is convenient for tracking the divisible groups in the homotopy of \( L_n \). Let

\[ L_n^Q(T) := \sum_s [L_0, L_n]_{-s}^D \cdot T^s, \]

where we have identified a \( \mathbb{Q}_p \)-vector space with its dimension, eg

\[ L_0^Q(T) = 1, \ L_1^Q(T) = T^2, \ L_2^Q(T) = 2T^4 + T^5. \]
Proposition:

\[
\sum_{n \geq 0} L_n^Q(T) u^n = (1 - \sum_{k \geq 1} \epsilon_{k-1}(T) (uT^2)^k)^{-1}
\]

where

\[
\epsilon_k(T) = \prod_{0 \leq i \leq k-1} (1 + T^{2i+1})
\]

is the Poincaré series for \( E^*(k) := H_c^*(S_k, \mathbb{Q}_p) \) (with \( \epsilon_0 = 1 \)).

Alternately, \( L_n^Q(S^0) \) is isomorphic, as a graded vector space, to

\[
\oplus (r_1, \ldots, r_m) (\otimes_{1 \leq i \leq m} E^*(k_i-1)^{\otimes r_i}) [2n],
\]

summed over sequences \( 0 < r_i \) and \( 0 \leq k_1 < \cdots < k_m \) such that \( \sum r_i = r \) and \( \sum r_i k_i = n \).

Proof: It follows from the splitting conjecture that

\[
[L_0, L_n] = [L_0, [L_{n-1}, L_n]]
\]

[2 Prop 5.1], so when \( n \geq 1 \),

\[
L_n^Q(T) = \sum_{i,w(I)=n} [L_0, L_{n-1-i}]^D_{[I]+1-s} T^s = \sum_{w(I)=n} L_{n-1-i}^Q(T) \cdot T^{[I]+1}.
\]

This can be rearranged as

\[
\sum_{0 \leq k \leq n-1} L_{n-1-k}^Q(T) \cdot \sum_{i_k=k} T^{[I]+1};
\]

but the right-most term can be written as the sum

\[
T \cdot T^{2k+1} \sum T^{[\tilde{I}]} = T^{2(k+1)} \epsilon_k(T)
\]

over sequences \( \tilde{I} \) of the form \( \{0 \leq i_1 < \cdots < k\} \).

We thus have the recursion relation

\[
\tilde{L}_n^Q(T) := T^{-2n} L_n^Q(T) = \sum_{0 \leq k \leq n-1} \epsilon_{n-k-1}(T) \tilde{L}_k^Q(T)
\]

and if \( \epsilon(u) := 1 + \sum_{i \geq 0} \epsilon_i u^{i+1} \) then

\[
\tilde{L}^Q := \sum_{n \geq 0} \tilde{L}_n^Q(T) u^n = \sum_{n-1 \geq k \geq 0, n \geq 0} \epsilon_{n-k-1}(T) \tilde{L}_k^Q(T)
\]

\[
= \sum_{n-1 \geq k \geq 0, n \geq 0} \epsilon_{n-k-1} u^{n-k} \cdot \tilde{L}_k^Q u^k = 1 + (\epsilon - 1) \tilde{L}^Q,
\]
\( \bar{L}^Q = (2 - \epsilon)^{-1} \). Replacing \( u \mapsto uT^2 \), we have

\[
\sum_{n \geq 0} L_n^Q(T) \ u^n = (1 - \sum_{k \geq 1} \epsilon_{k-1}(T) (uT^2)^k)^{-1}
\]
as asserted. □

For example, if we specialize \( T \to 1 \) then \( \epsilon_k \to 2^k \) and

\[
\epsilon \to 1 + \sum_{0 \leq k} 2^k u^{k+1} = \frac{1 - u}{1 - 2u},
\]
so

\[
\bar{L}^Q \to \frac{1 - 2u}{1 - 3u} = 1 + \sum_{1 \leq k} 3^{k-1} u^k,
\]
ie \( L_n^Q(1) \to 3^{n-1} \) if \( n \geq 1 \) [H §5]. Similarly, we recover

\[
L_3^Q = T^6(1 + 2\epsilon_1 + \epsilon_2) = 4T^6 + 3T^7 + T^9 + T^{10}.
\]

The alternate form of the proposition above implies that

\[
X \mapsto \bigoplus_{n \geq 0} L_n^{Q*}(X)
\]
defines a bigraded rational cohomology theory \( L_n^{Q*} \). Its coefficient ring \( L_\ast^{Q*}(S^0) \) has an obvious graded-commutative algebra structure, which lacks any very obvious geometric interpretation.

Mark Behrens asks if or how Gross-Hopkins duality fits into this story. This suggests the remark, that the proposition above looks like it might have a reformulation in terms of some kind of Koszul duality; but I can’t imagine what that might be . . .

References

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