A Bayesian Nonparametric Test for Assessing Multivariate Normality

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\textbf{Abstract}

In this paper, a novel Bayesian nonparametric test for assessing multivariate normal models is presented. While there are extensive frequentist and graphical methods for testing multivariate normality, it is challenging to find Bayesian counterparts. The proposed approach is based on the use of the Dirichlet process and Mahalanobis distance. More precisely, the Mahalanobis distance is employed as a good technique to transform the \( m \)-variate problem into a univariate problem. Then the Dirichlet process is used as a prior on the distribution of the Mahalanobis distance. The concentration of the distribution of the distance between the posterior process and the chi-square distribution with \( m \) degrees of freedom

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is compared to the concentration of the distribution of the distance between the prior process and the chi-square distribution with \( m \) degrees of freedom via a relative belief ratio. The distance between the Dirichlet process and the chi-square distribution is established based on the Anderson-Darling distance. Key theoretical results of the approach are derived. The procedure is illustrated through several examples, in which the proposed approach shows excellent performance.

**Keywords:** Anderson-Darling distance, Dirichlet process, Mahalanobis distance, Multivariate hypothesis testing, Relative belief inferences.

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## 1 Introduction

The assumption of multivariate normality is a key assumption in many statistical applications such as pattern recognition (Dubes and Jain, 1980) and exploratory multivariate methods (Fernandez, 2010). The need to check this assumption is of special importance as if it does not hold, the obtained results based on this assumption may lead to an error. Specifically, for a given \( m \)-variate sampled data \( y_{m \times n} = (y_1, \ldots, y_n) \) with size of \( n \), where \( y_i \in \mathbb{R}^m \) for \( i = 1, \ldots, n \), the interest is to determine whether \( y_{m \times n} \) comes from a multivariate normal population.

Many tests and graphical methods have been considered to assess the multivariate normality assumption. Healy (1968) described an extension of normal plotting techniques to handle multivariate data. Mardia (1970) proposed a test based on the asymptotic distribution of measures of multivariate skewness and kurtosis. Tests based on transforming the multivariate problem into the univariate problem were established by Rincón-Gallardo et al. (1979), Royston (1983), Fattorini (1986), and Hasofer and Stein (1990). A class of invariant consistent
tests based on a weighted integral of the squared distance between the empirical characteristic function and the characteristic function of the multivariate normal distribution was suggested by Henze and Zirkler (1990). Holgersson (2006) presented a simple graphical method based on the scatter plot. Doornik and Hansen (2008) developed an omnibus test based on a transformed skewness and kurtosis. They showed that their test is more powerful than the Shapiro-Wilk test proposed by Royston (1983). Alva and Estrada (2009) proposed a multivariate normality test based on Shapiro-Wilks statistic for univariate normality and on an empirical standardization of the observations. Jönsson (2011) presented a robust test with powerful performance. Hanusz and Tarasińska (2012) proposed two tests based on Small’s and Srivastava’s graphical methods. A powerful test was offered by Zhou and Shao (2014) with good application in biomedical studies. Kim (2016) generalized Jarque-Bera univariate normality test to the multivariate case. Kim and Park (2018) derived several powerful omnibus tests based on the likelihood ratio and the characterizations of the multivariate normal distribution. Madukaife and Okafor (2018) proposed a powerful affine invariant test based on interpoint distances of principal components. Henze and Visagie (2019) derived a new test based on a partial differential equation involving the moment generating function.

It follows from the previous discussion that, while there are considerable frequentist and graphical methods for testing multivariate normality, it is difficult to find Bayesian counterparts. Most available Bayesian tests focused on employing Bayesian nonparametric methods for univariate data. See for example Al-Labadi and Zarepour (2013, 2014a). A remarkable work that covers the multivariate case was developed by Tokdar and Martin (2011), where they established a Bayesian test based on characterizing alternative models by Dirichlet process mixture distributions. In the current paper, a novel Bayesian nonpara-
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A metric test for assessing multivariate normality is proposed. The developed test is based on using Mahalanobis distance as a good technique to convert the $m$-variate problem into a univariate problem. Specifically, whenever $y_{m \times n}$ comes from a multivariate normal distribution, the distribution of the corresponding Mahalanobis distances, denoted by $P$, is approximately chi-square with $m$ degrees of freedom (Johnson and Wichern, 2007). This reduces the problem to test the hypothesis $H_0 : P = F_{(m)}$, where $F_{(m)}$ denotes the cumulative distribution function of the chi-square distribution with $m$ degrees of freedom. The heart of the proposed test is to consider the Dirichlet process as a prior on $P$. Then the concentration of the distribution of the distance between the posterior process and $F_{(m)}$ is compared to the concentration of the distribution of the distance between the prior process and $F_{(m)}$. The distance between the Dirichlet process and $F_{(m)}$ is developed based on the Anderson-Darling distance as an appropriate tool to detect the difference, especially when this difference is due to the tails. This comparison is made via a relative belief ratio. A calibration of the relative belief ratio is also presented. We point out that the comparison between the concentration of the posterior and the prior distribution of the distance was suggested by Al-Labadi and Evans (2018) for model checking of univariate distributions. The anticipated test is generic in the sense that it can be implemented when the mean or the covariance matrix is unknown or known. It is easy to implement with excellent performance and it does not require providing a closed form of the relative belief ratio. In addition, unlike the tests that use p-values, the proposed test permits us to state evidence for the null hypothesis.

The remainder of this paper is organized as follows. A general discussion about the relative belief ratio is given in Section 2. The definition and some fundamental properties of the Dirichlet process are presented in Section 3. An
explicit expression for calculating the Anderson-Darling distance between the Dirichlet process and its base measure is derived in Section 4. A Bayesian nonparametric test for assessing multivariate normality and some of its relevant properties are developed in Section 5. A computational algorithm to carry out the proposed test is discussed in Section 6. The performance of the proposed test is discussed in Section 7, where four simulated examples and two real data sets are considered. Finally, a summary of the findings is given in Section 8. All technical proofs are included in the Appendix.

2 Relative Belief Inferences

The relative belief ratio is a common measure of statistical evidence. It leads to a straightforward inference in hypothesis testing problems. For more details, let \( \{ f_\theta : \theta \in \Theta \} \) denote a collection of densities on a sample space \( \mathcal{X} \) and let \( \pi \) denote a prior on the parameter space \( \Theta \). Note that the densities may represent discrete or continuous probability measures but they are all with respect to the same support measure \( d\theta \). After observing the data \( x \), the posterior distribution of \( \theta \), denoted by \( \pi(\theta | x) \), is a revised prior and is given by the density \( \pi(\theta | x) = \pi(\theta)f_\theta(x)/m(x) \), where \( m(x) = \int_\Theta \pi(\theta)f_\theta(x) d\theta \) is the prior predictive density of \( x \). For a parameter of interest \( \psi = \Psi(\theta) \), let \( \Pi_\psi \) denote the marginal prior probability measure and \( \Pi_\psi(\cdot | x) \) denote the marginal posterior probability measure. It is assumed that \( \Psi \) satisfies regularity conditions so that the prior density \( \pi_\Psi \) and the posterior density \( \pi_\Psi(\cdot | x) \) of \( \psi \) exist with respect to some support measure on the range space for \( \Psi \) (Evans, 2015). The relative belief ratio for a value \( \psi \) is then defined by

\[
RB_\Psi(\psi | x) = \lim_{\delta \to 0} \Pi_\psi(N_\delta(\psi)|x)/\Pi_\psi(N_\delta(\psi)),
\]
where \( N_\delta(\psi) \) is a sequence of neighborhoods of \( \psi \) converging nicely to \( \psi \) as \( \delta \to 0 \) (Evans, 2015). When \( \pi_\Psi \) and \( \pi_\Psi(\cdot \mid x) \) are continuous at \( \psi \), the relative belief ratio is defined by

\[
RB_\Psi(\psi \mid x) = \frac{\pi_\Psi(\psi \mid x)}{\pi_\Psi(\psi)},
\]

the ratio of the posterior density to the prior density at \( \psi \). Therefore, \( RB_\Psi(\psi \mid x) \) measures the change in the belief of \( \psi \) being the true value from a priori to a posteriori. Note that a relative belief ratio is similar to a Bayes factor, as both are measures of evidence, but the latter measures evidence via the change in an odds ratio. In general, when a Bayes factor is defined via a limit in the continuous case, the limiting value equals the corresponding relative belief ratio. For a further discussion about the relationship between relative belief ratios and Bayes factors see, for instance, Chapter 4 of Evans, 2015.

Since \( RB_\Psi(\psi \mid x) \) is a measure of the evidence that \( \psi \) is the true value, if \( RB_\Psi(\psi \mid x) > 1 \), then the probability of the \( \psi \) being the true value from a priori to a posteriori is increased, consequently there is evidence based on the data that \( \psi \) is the true value. If \( RB_\Psi(\psi \mid x) < 1 \), then the probability of the \( \psi \) being the true value from a priori to a posteriori is decreased. Accordingly, there is evidence against based on the data that \( \psi \) being the true value. For the case \( RB_\Psi(\psi \mid x) = 1 \) there is no evidence either way.

Obviously, \( RB_\Psi(\psi_0 \mid x) \) measures the evidence of the hypothesis \( \mathcal{H}_0 : \Psi(\theta) = \psi_0 \). For a large value of \( c (c \gg 1) \), \( RB_\Psi(\psi_0 \mid x) = c \) provides strong evidence in favor of \( \psi_0 \) because belief has increased by a factor of \( c \) after seeing the data. However, there may also exist other values of \( \psi \) that had even larger increases. Thus, it is also necessary, however, to calibrate whether this is strong or weak evidence for or against \( \mathcal{H}_0 \). A typical calibration of \( RB_\Psi(\psi_0 \mid x) \) is given by the
The value in (1) indicates that the posterior probability that the true value of \( \psi \) has a relative belief ratio no greater than that of the hypothesized value \( \psi_0 \). Noticeably, (1) is not a p-value as it has a very different interpretation. When \( RB_\Psi(\psi_0 \mid x) < 1 \), there is evidence against \( \psi_0 \), then a small value of (1) indicates strong evidence against \( \psi_0 \). On the other hand, a large value for (1) indicates weak evidence against \( \psi_0 \). Similarly, when \( RB_\Psi(\psi_0 \mid x) > 1 \), there is evidence in favor of \( \psi_0 \), then a small value of (1) indicates weak evidence in favor of \( \psi_0 \), while a large value of (1) indicates strong evidence in favor of \( \psi_0 \). For applications of the use of the relative belief ratio in different univariate hypothesis testing problems, see Evans (1997, 2015), Al-Labadi and Evans (2018) and Al-Labadi et al. (2017, 2018).

### 3 Dirichlet Process

The Dirichlet process prior, introduced by Ferguson (1973), is the most commonly used prior in the Bayesian nonparametric inferences. A substantial collection of theory has been devoted to this prior. Here we only present the most important definitions and properties of this prior. Consider a space \( \mathfrak{X} \) with a \( \sigma \)-algebra \( \mathcal{A} \) of subsets of \( \mathfrak{X} \), let \( H \) be a fixed probability measure on \( (\mathfrak{X}, \mathcal{A}) \), called the base measure, and \( a \) be a positive number, called the concentration parameter. A random probability measure \( P = \{P(A) : A \in \mathcal{A}\} \) is called a Dirichlet process on \( (\mathfrak{X}, \mathcal{A}) \) with parameters \( a \) and \( H \), denoted by \( P \sim DP(a, H) \), if for every measurable partition \( A_1, \ldots, A_k \) of \( \mathfrak{X} \) with \( k \geq 2 \), the joint distribution of the vector \( (P(A_1), \ldots, P(A_k)) \) has the Dirichlet distribution with parameter \( aH(A_1), \ldots, aH(A_k) \). Also, it is assumed that \( H(A_j) = 0 \) implies \( P(A_j) = 0 \).
with probability one. Consequently, for any \( A \in \mathcal{A} \), \( P(A) \sim \text{beta}(aH(A), a(1 - H(A))) \), \( E(P(A)) = H(A) \) and \( \text{Var}(P(A)) = H(A)(1 - H(A))/(1 + a) \). Accordingly, the base measure \( H \) plays the role of the center of \( P \) while the concentration parameter \( a \) controls variation of the process \( P \) around the base measure \( H \). One of the most well-known properties of the Dirichlet process is the conjugacy property. That is, when the sample \( x = (x_1, \ldots, x_n) \) is drawn from \( P \sim DP(a, H) \), the posterior distribution of \( P \) given \( x \), denoted by \( P_x \), is also a Dirichlet process with concentration parameter \( a + n \) and base measure

\[
H_x = a(a + n)^{-1}H + n(a + n)^{-1}F_n,
\]

where \( F_n \) denotes the empirical cumulative distribution function (cdf) of the sample \( x \). Note that, \( H_x \) is a convex combination of the base measure \( H \) and the empirical cdf \( F_n \). Therefore, \( H_x \to H \) as \( a \to \infty \) while \( H_x \to F_n \) as \( a \to 0 \). A detailed discussion about choosing the hyperparameters \( a \) and \( H \) will be presented in Section 5.

Following Ferguson (1973), \( P \sim DP(a, H) \) can be represented as

\[
P = \sum_{i=1}^{\infty} L^{-1}(\Gamma_i) \delta_{Y_i} / \sum_{i=1}^{\infty} L^{-1}(\Gamma_i),
\]

where \( \Gamma_i = E_1 + \cdots + E_i \) with \( E_i \overset{i.i.d.}{\sim} \text{exponential}(1) \), \( Y_i \overset{i.i.d.}{\sim} H \) independent of the \( \Gamma_i \), \( L^{-1}(y) = \inf\{x > 0 : L(x) \geq y \} \) with \( L(x) = a \int_x^{\infty} t^{-1}e^{-t}dt, x > 0 \), and \( \delta_n \) the Dirac delta measure. The series representation (3) implies that the Dirichlet process is a discrete probability measure even for the cases with an absolutely continuous base measure \( H \). Note that, by imposing the weak topology, the support of the Dirichlet process could be quite large. To be more precise, when the support of the base measure is \( \mathcal{X} \), then the space of all probability measures is the support of the Dirichlet process. In particular, when \( H \) is a
normal base measure, the corresponding Dirichlet process can choose any probability measure. Moreover, since data is always measured to finite accuracy, the true distribution being sampled from is discrete. This makes the discreteness property of $P$ has no practical significant limitation.

Representation (3) presents a simple way to generate sample form the Dirichlet process. However, complex calculations may be encountered because no closed form exists for $L(\cdot)$. This issue motivates researchers to propose several methods in order to approximately simulate the Dirichlet process. One such efficient method was presented by Zarepour and Al-Labadi (2012). They showed that the Dirichlet process $P \sim DP(a, H)$ can be approximated by

$$P_N = \sum_{i=1}^{N} J_i \delta_{Y_i}, \quad (4)$$

with the monotonically decreasing weights $J_i = \frac{G_{a/N}(\Gamma_i/\Gamma_{N+1})}{\sum_{i=1}^{N} G_{a/N}(\Gamma_i/\Gamma_{N+1})}$, where $\Gamma_i$ and $Y_i$ are defined as before, $N$ is a positive large integer and $G_{a/N}$ denotes the complement-cdf of the gamma($a/N, 1$) distribution. Note that, $G_{a/N}^{-1}(p)$ is the $(1-p)$-th quantile of the gamma($a/N, 1$) distribution. Also, Zarepour and Al-Labadi (2012) showed that $P_N$ converges almost surely to (3) as $N$ goes to infinity with a fast rate of convergence. The following algorithm describes how the approximation (4) can be used to generate a sample from $DP(a, H)$.

Algorithm A (Approximately generating a sample from $DP(a, H)$):

(i) Fix a large positive integer $N$ and generate i.i.d. $Y_i \sim H$ for $i = 1, \ldots, N$.

(ii) For $i = 1, \ldots, N + 1$, generate i.i.d. $E_i$ from the exponential distribution with rate 1, independent of $(Y_i)_{1 \leq i \leq N}$ and put $\Gamma_i = E_1 + \cdots + E_i$.

(iii) Compute $G_{a/N}^{-1}(\Gamma_i/\Gamma_{N+1})$ for $i = 1, \ldots, N$ and return $P_N$.

For other simulation methods of the Dirichlet process, see Bondesson (1982), Sethuraman (1994), Wolpert and Ickstadt (1998) and Al-Labadi and Zarepour.
4 Statistical Distance

Measuring the distance between two distributions is an essential tool in model checking. In this section, two well-known statistical distances, namely Anderson-Darling distance and Mahalanobis distance, are considered.

4.1 Anderson-Darling Distance

The Anderson-Darling distance between two cdf’s $F$ and $G$ is given by

$$d_{AD}(F, G) = \int_{-\infty}^{\infty} \frac{(F(t) - G(t))^2}{G(t)(1-G(t))} dG(t).$$

Anderson-Darling distance can be viewed as a modification of the Cramér-von Mises distance that gives more weight to data points in the tails of the distribution. The next lemma provides an explicit formula to compute the Anderson-Darling distance between a discrete cdf and a continuous cdf. Throughout this paper, “log” denotes the natural logarithm.

**Lemma 1** Let $G$ be a continuous cdf and $P_N = \sum_{i=1}^{N} J_i \delta_{Y_i}$ be a discrete distribution, where $Y_{(1)} \leq \cdots \leq Y_{(N)}$ are the order statistics of $(Y_i)_{1 \leq i \leq N}$ and $J_1', \ldots, J_N'$ are the associated jump sizes such that $J_i = J_j'$ when $Y_i = Y_{(j)}$. Then the Anderson-Darling distance between $P_N$ and $G$ is given by

$$d_{AD}(P_N, G) = 2 \sum_{i=1}^{N-1} \sum_{j=1}^{i} J_j' J_k' C_{i,i+1} + \sum_{i=1}^{N-1} \sum_{j=1}^{i} J_j'^2 C_{i,i+1} + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{i} J_j' C_{i,i+1}^* - \sum_{i=1}^{N-1} C_{i,i+1}^* - \log[G(Y_{(N)})][1 - G(Y_{(1)})] - 1,$$

where $C_{i,i+1} = \log \frac{G(Y_{(i+1)})[1-G(Y_{(i)})]}{G(Y_{(i)})(1-G(Y_{(i+1)})]}$ and $C_{i,i+1}^* = \log \frac{1-G(Y_{(i+1)})}{1-G(Y_{(i)})}$. 
Proof. The proof is given in Appendix A.

The next corollary indicates that the distribution of $d_{AD}(P_N, G)$ is independent of $G$.

**Corollary 2** Suppose that $(Y_i)_{1 \leq i \leq N}$ are i.i.d. from $G$, independent of $(J_i)_{1 \leq i \leq N}$ and $P_N = \sum_{i=1}^{N} J_i \delta_{Y_i}$. Then $d_{AD}(P_N, G) \overset{d}{=} 2 \sum_{i=1}^{N-1} \sum_{j=1}^{i} J'_i U^*_{i,i+1} - \sum_{i=1}^{N-1} U^*_{i,i+1} + \sum_{i=1}^{N-1} \sum_{j=1}^{i} J'_j U_{i,i+1} + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{i} \sum_{k=j+1}^{i} J'_j J'_k U_{i,i+1} - \log[\frac{U(N)(1-U(1))}{U(1)(1-U(N))}] - 1$, where $U_{i,i+1} = \log \frac{U(i+1)(1-U(i))}{U(i)(1-U(i+1))}$, $U^*_{i,i+1} = \log \frac{1-U_{i,i+1}}{1-U(i)}$ and $U(i)$ is the $i$-th order statistic for $(U_i)_{1 \leq i \leq N}$ i.i.d. uniform[0,1].

**Proof.** Since for $1 \leq i \leq N$, $U_i = G(Y_i)$ and $Y_i$ is a sequence of i.i.d. random variable with continuous distribution $G$, the probability integral transformation theorem implies that $(U_i)_{1 \leq i \leq N}$ is a sequence of i.i.d. random variable uniformly distributed on the interval [0,1]. The proof is completed by considering the order statistic of $(U_i)_{1 \leq i \leq N}$ in Lemma 1.

**Lemma 3** Let $G$ be a continuous cdf, $P \sim DP(\alpha, H)$ and $P_N$ be the approximation of the Dirichlet process in (4), then $d_{AD}(P_N, G) \overset{a.s.}{\longrightarrow} d_{AD}(P, G)$ as $N \to \infty$.

**Proof.** Since $P_N(t)$ converges monotonically to $P(t)$ as $N \to \infty$ (Zarepour and Al-Labadi, 2012), then the monotonically of $(\frac{P_N(t)-G(t)}{G(t)(1-G(t))})^2$ is concluded. Hence, the result follows by the monoton convergence theorem.

### 4.2 Mahalanobis Distance

Mahalanobis distance measures the distance of $m$-variate point $Y$ generated from a known distribution $F_\theta$ to the mean $\mu_m = E_{\theta}(Y)$ of the distribution. Let $\Sigma_{m \times m}$ be the covariance matrix of the $m$-variate distribution $F_\theta$, the Ma-
halanobis distance is defined as

\[ D_M(Y) = \sqrt{(Y - \mu_m)^T \Sigma_m^{-1} (Y - \mu_m)}. \]  (5)

Note that, (5) is limited to the cases when both \( \mu_m \) and \( \Sigma_{m \times m} \) are known. However, in most cases, the mean and covariance of \( F_0 \) exist but unknown. For such cases, the Mahalanobis distance is defined based on the measuring of distance between a subject’s data and the mean of all observation in an observed sample. To be more precise, given a sample of \( n \) independent \( m \)-variate \( y_1, \ldots, y_n \), the sample Mahalanobis distance of \( y_i \) to the sample mean \( \bar{y} \), denoted by \( d_M(y_i) \), is defined by

\[ d_M(y_i) = \sqrt{(y_i - \bar{y})^T S_y^{-1} (y_i - \bar{y})}, \]  (6)

where \( S_y \) denotes the sample covariance matrix. Some interesting properties of the Mahalanobis distance are presented in Johnson and Wichern (2007). For example, when the parent population is \( m \)-variate normal, for large enough values of \( n \) and \( n - m \) (\( n - m > 30 \)), each of the squared distance \( d^2_M(y_1), \ldots, d^2_M(y_n) \) behave like a chi-square random variable with \( m \) degrees of freedom.

5 Bayesian Nonparametric Approach for Assessing Multivariate Normality

In this section, a new test for assessing multivariate normality is presented. For this purpose, consider the family of \( m \)-variate normal distribution \( \mathcal{F} = \{ N_m(\mu_m, \Sigma_{m \times m}) : \mu_m \in \mathbb{R}^m, \det(\Sigma_{m \times m}) > 0 \} \). Let \( Y_1, \ldots, Y_n \) be a random sample from \( m \)-variate distribution \( F \). The problem under consideration is to
test the hypothesis

\[ \mathcal{H}_0 : F \in \mathcal{F}, \quad (7) \]

using the Bayesian nonparametric framework. The first step is to reduce the multivariate problem to a univariate problem. One way to accomplish that is through the Mahalanobis distance. For this, assume that \( \hat{\mu}_m = \bar{Y} \) and \( \hat{\Sigma}_{m \times m} = S_Y \) are the sample mean and sample covariance matrix, respectively. Then, \( N_m(\hat{\mu}_m, \hat{\Sigma}_{m \times m}) \) is the best representative of the family \( \mathcal{F} \) to compare with distribution \( F \).

Define

\[ D_M(Y_i) = \sqrt{(Y_i - \hat{\mu}_m)^T \hat{\Sigma}_{m \times m}^{-1} (Y_i - \hat{\mu}_m)} \quad \text{for} \quad 1 \leq i \leq n. \]

Assume that \( F_m \) is the cdf of the chi-square distribution with \( m \) degrees of freedom and \( (D_M^2(Y_i))_{1 \leq i \leq n} \) is a sequence of random variables with continuous distribution function \( P \). From Johnson and Wichern (2007), if \( \mathcal{H}_0 \) is true, then we expect that \( P \) is (approximately) the same as \( F_m \). Thus, testing \( (7) \) is equivalent to testing

\[ \mathcal{H}_0 : P = F_m. \quad (8) \]

Note that, when \( \mu_m \) or \( \Sigma_{m \times m} \) in \( \mathcal{H}_0 \) is known, we consider its known value in \( D_M(Y_i) \).

For testing \( \mathcal{H}_0 \), let \( y_{m \times n} = (y_1, \ldots, y_n) \) be an observed sample from \( F \) and \( d = (d_M^2(y_1), \ldots, d_M^2(y_n)) \) be the corresponding observed squared Mahalanobis distance. If \( P \sim DP(a, F_m) \), for a given choice of \( a \), by \( (2) \),

\[ P_d = P|d \sim DP(a + n, H_d). \]

From \( (6) \), if \( \mathcal{H}_0 \) is true, then the sampled observations \( d_M^2(y_1), \ldots, d_M^2(y_n) \) would be approximately independent chi-square with \( m \) degrees of freedom. Thus, the posterior distribution of the distance between \( P_d \) and \( F_m \) should be more concentrated around zero than the prior
distribution of the distance between $P$ and $F_{(m)}$. A good reason for choosing $H = F_{(m)}$ is to avoid prior-data conflict. Prior-data conflict means that there is a tiny overlap between the effective support regions of $P$ and $H$. It can lead to errors in the computation of the posterior distribution of the distance between $P_d$ and $F_{(m)}$ when $H_0$ is true. For more discussion about prior-data conflict see Evans and Moshonov (2006), Al-Labadi and Evans (2017, 2018), and Nott et al. (2019). Another crucial reason for choosing $H = F_{(m)}$ is the independence of $d_{AD}$ from the data, which is immediately followed from Corollary 2.

In the proposed test, we use Lemma 1 to compute the Anderson-Darling distances $d_{AD}(P, F_{(m)})$ and $d_{AD}(P_d, F_{(m)})$. Then, the relative belief ratio is used to compare the concentration of the posterior and the prior distribution of $d_{AD}(P_d, F_{(m)})$, respectively, at zero. By Lemma 3 we approximate the posterior distribution of $d_{AD}(P_d, F_{(m)})$ by $d_{AD}(P_{dn}, F_{(m)})$ where $P_{dn}$ is the approximation of the Dirichlet process $P_d \sim DP(a + n, H_d)$.

Another significant concern to validate the test is determining suitable values of $a$. The following Lemma presents the expectation and variance of the prior distribution of the Anderson-Darling distance. For the Cramér-von Mises distance $d_{CvM}$, Al-Labadi and Evans (2018) showed that $E_P (d_{CvM}(P, H)) = 1/6(a + 1)$.

**Lemma 4** Let $P \sim DP(a, H)$ and $d_{AD}(P, H)$ be the Anderson-Darling distance between $P$ and $H$, then

(i) $E_P (d_{AD}(P, H)) = \frac{1}{a + 1}$,

(ii) $Var_P (d_{AD}(P, H)) = \frac{2 \left( (\pi^2 - 9)a^2 + (30 - 2\pi^2)a - 3\pi^2 + 36 \right)}{3(a + 1)^2(a + 2)(a + 3)}$,

where $E_P (d_{AD}(P, H))$ and $Var_P (d_{AD}(P, H))$ are the expectation and variance of the prior distribution of $d_{AD}(P, H)$ with respect to the $P$, respectively.

**Proof.** The proof is given in Appendix B. ■
The next corollary highlights the effect of the value $a$ on the prior distribution of the Anderson-Darling distance.

**Corollary 5** Let $P \sim DP(a, H)$ and $H = F_{(m)}$ be the cdf of the chi-square distribution with $m$ degrees of freedom. Suppose that $a \to \infty$, then

(i) $E_P (d_{AD}(P, H)) \to 0$, and, $Var_P (d_{AD}(P, H)) \to 0$.

(ii) $E_P ((a + 1)d_{AD}(P, H)) \to 1$, and, $Var_P ((a + 1)d_{AD}(P, H)) \to \frac{2}{3}(\pi^2 - 9)$.

(iii) $d_{AD} \xrightarrow{\text{qm}} 0$, and, $d_{AD} \xrightarrow{\text{a.s}} 0$.

Where “$\xrightarrow{\text{qm}}$” denotes convergence in quadratic mean.

**Proof.** The proof of (i) and (ii) are followed immediately by letting $a \to \infty$ in part (i) and (ii) of Lemma 4. For (iii), the convergence in quadratic mean is followed by letting $a \to \infty$ in $E_P (d_{AD}(P, H))^2 = \frac{a(2\pi^2 - 15) - 6(\pi^2 - 15)}{3(a+3)(a+2)(a+1)}$; see the proof of Lemma 4 in Appendix B. To prove the almost surely convergence, assume that $a = kc$, for $k \in \{1, 2, \ldots, \}$ and a fixed positive number $c$. Then, for any $\epsilon > 0$, $Pr \{d_{AD}(P, H) > \epsilon\} \leq \frac{E_P(d_{AD}(P, H))^2}{\epsilon^2}$. Since, $\sum_{k=1}^{\infty} Pr \{d_{AD}(P, H) > \epsilon\} < \infty$, then, by the first Borel-Catelli lemma, $d_{AD} \xrightarrow{\text{a.s}} 0$, as $k \to \infty$ ($a \to \infty$).

Interestingly, the limit of the expectation and variance in part (ii) of Corollary 5 coincide with the limit of $E_P (n d_{AD}(F_n, H))$ and $Var_P (n d_{AD}(F_n, H))$, given by Anderson and Darling (1954), as $n \to \infty$, where $F_n$ is the empirical cdf of the sample $d$.

According to part (i) of Corollary 5, it seems that when increasing the value of $a$, the result of testing (8) will be more accurate. As recommended in Al-Labadi and Zarepour (2017), the value of $a$ should be at most $0.5n$, where $n$ is the sample size, as otherwise the prior may become too influential. On the other hand, if $\mathcal{H}_0$ is true, with increasing the sample size $n$, the expectation of the posterior distribution of the distance between $P_d$ and $F_{(m)}$ converges to zero. Also, if $\mathcal{H}_0$ is not true, this expectation converges to a positive value. The following Lemma indicates the effect of increasing the value of $a$ and sample size.
A BNP Test for Multivariate Normality

Lemma 6 If $P_d \sim DP(a+n, H_d)$ and $H_d$ is given by (2) with $H = F(m)$, the cdf of the chi-square distribution with $m$ degrees of freedom, then

(i) $E_{P_d}(d_{AD}(P_d, F(m))) \to 0$ as $a \to \infty$.
(ii) If $\mathcal{H}_0$ is true, then, $E_{P_d}(d_{AD}(P_d, F(m))) \to 0$ as $n \to \infty$.
(iii) If $\mathcal{H}_0$ is not true, then there exists a positive value $c$ such that $E_{P_d}(d_{AD}(P_d, F(m))) \to c$ as $n \to \infty$.

Proof. The proof is given in Appendix C. ■

Part (i) of Lemma 6 confirms that for a too large value of $a$ (relative to $n$), we may accept $\mathcal{H}_0$ even when $\mathcal{H}_0$ is not true.

6 Computational Algorithm

The steps for the proposed test is detailed in this section. Note that, since no closed forms of the densities of $D = d_{AD}(P, F(m))$ and $D_d = d_{AD}(P_d, F(m))$ are available, simulation is used to approximate relative belief ratio. The following gives a computational algorithm to test $\mathcal{H}_0$. This algorithm is a revised version of Algorithm B of Al-Labadi and Evans (2018).

Algorithm B (Relative belief algorithm for testing multivariate normality):
1. Use Algorithm A to (approximately) generate a $P$ from $DP(a, F(m))$.
2. Compute $d_{AD}(P, F(m))$ as in Lemma 1.
3. Repeat steps (1)-(2) to obtain a sample of $r_1$ values from the prior of $D$.
4. Use Algorithm A to (approximately) generate a $P_d$ from $DP(a+n, H_d)$.
5. Compute $d_{AD}(P_d, F(m))$.
6. Repeat steps (4)-(5) to obtain a sample of $r_2$ values of $D_d$. 

$n$ on the expectation of the posterior distribution of $d_{AD}(P_d, F(m))$. 

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(i) $E_{P_d}(d_{AD}(P_d, F(m))) \to 0$ as $a \to \infty$.
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(i) $E_{P_d}(d_{AD}(P_d, F(m))) \to 0$ as $a \to \infty$.
(ii) If $\mathcal{H}_0$ is true, then, $E_{P_d}(d_{AD}(P_d, F(m))) \to 0$ as $n \to \infty$.
(iii) If $\mathcal{H}_0$ is not true, then there exists a positive value $c$ such that $E_{P_d}(d_{AD}(P_d, F(m))) \to c$ as $n \to \infty$.

Proof. The proof is given in Appendix C. ■

Part (i) of Lemma 6 confirms that for a too large value of $a$ (relative to $n$), we may accept $\mathcal{H}_0$ even when $\mathcal{H}_0$ is not true.
7. Let $M$ be a positive number. Let $\hat{F}_D$ denote the empirical cdf of $D$ based on the prior sample in (3) and for $i = 0, \ldots, M$, let $\hat{d}_{i/M}$ be the estimate of $d_{i/M}$, the $(i/M)$-th prior quantile of $D$. Here $\hat{d}_0 = 0$, and $\hat{d}_1$ is the largest value of $d_{AD}$. Let $\hat{F}_{D}(\cdot | d)$ denote the empirical cdf of $D$ based on the posterior sample in 6. For $q \in [\hat{d}_{i/M}, \hat{d}_{(i+1)/M})$, estimate $RB_D(q | d) = \pi_D(q | d)/\pi_D(q)$ by

$$\hat{R}B_D(q | d) = M\{\hat{F}_D(\hat{d}_{(i+1)/M} | d) - \hat{F}_D(\hat{d}_{i/M} | d)\},$$

the ratio of the estimates of the posterior and prior contents of $[\hat{d}_{i/M}, \hat{d}_{(i+1)/M})$. It follows that, we estimate $RB_D(0 | d) = \pi_D(0 | d)/\pi_D(0)$ by $\hat{R}B_D(0 | d) = M\hat{F}_D(\hat{d}_{p_0} | d)$ where $p_0 = i_0/M$ and $i_0$ is chosen so that $i_0/M$ is not too small (typically $i_0/M \approx 0.05$).

8. Estimate the strength $DP_D(RB_D(q | d) \leq RB_D(0 | d) | d)$ by the finite sum

$$\sum_{\{i \geq i_0: \hat{R}B_D(\hat{d}_{i/M} | d) \leq \hat{R}B_D(0 | d)\}} (\hat{F}_D(\hat{d}_{(i+1)/M} | d) - \hat{F}_D(\hat{d}_{i/M} | d)).$$

For fixed $M$, as $r_1 \to \infty, r_2 \to \infty$, then $\hat{d}_{i/M}$ converges almost surely to $d_{i/M}$ and (9) and (10) converge almost surely to $RB_D(q | d)$ and $DP_D(RB_D(q | d) \leq RB_D(0 | d) | d)$, respectively.

As recommended in the next section, one should try different values of $a$ to make sure the right conclusion has been obtained. The consistency of the proposed test is achieved by Proposition 6 of Al-Labadi and Evans (2018).

7 Examples

In this section, the performance of the proposed test is illustrated through six examples. Examples 1-4 are devoted to assess the bivariate normality for samples of sizes $n = 35$ generated from a variety of distributions. The following
notations have been used: \( t_r \) for the \( t \)-Student distribution with \( r \) degrees of freedom and \( \text{EXP}(\lambda) \) for the exponential distribution with rate \( \lambda \). In Examples 1-3, we consider \( \mu_2 = 0_2 \) and \( \Sigma_{2 \times 2} = I_2 \), where \( 0_2 \) and \( I_2 \) are, respectively, 2-dimensional zero vector and \( 2 \times 2 \) identity matrix. Two real data sets are presented in Examples 5 and 6. For all cases, we set \( r_1 = r_2 = 1000 \) and \( M = 20 \) in Algorithm B. To study the sensitivity of the approach to the choice of the chosen concentration parameter, various values of \( a \) are considered.

**Example 1. Fixed mean vector and fixed covariance matrix**

In this example, we check whether the sampled data come from a bivariate normal distribution with (fixed) mean vector \( 0_2 \) and (fixed) covariance matrix \( I_2 \). That is, \( H_0 : F = N_2(0_2, I_2) \). Recall that, when \( H_0 \) is true, we expect that \( RB > 1 \) and the strength close to 1. Also, when \( H_0 \) is not true, \( RB < 1 \) and the strength close to 0 . The results are reported in Table 1. They show the excellent performance of the approach in both accepting and rejecting \( H_0 \).

Figure 1 provides plots of the prior and posterior density of the Anderson-Darling distance for Case 1 and Case 2 when \( a = 15 \). From part (a) of Figure 1, it is obvious that, when \( H_0 \) is true, the posterior density of the distance is more concentrated about 0 than the prior densities of the distance. On the other hand, From part (b) of Figure 1 when \( H_0 \) is not true, the prior density of the distance is more concentrated about 0 than the posterior densities of the distance.

It is interesting to consider the effect of the value of \( a \) on the results of the procedure. Clearly, when \( H_0 \) is true, increasing \( a \) keeps the values of \( RB \) greater than 1. On the other hand, when \( H_0 \) is not true, increasing \( a \) drops the \( RB \) below 1. The value of \( a \) should be at most 0.5 \( n \) (Al-Labadi and Zarepour, 2017).
Example 1

| Case | Sample | \( \alpha \) | \( RB(\text{Strength}) \) |
|------|--------|--------------|-----------------------------|
| 1    | \( y_{2 \times n} \sim N_2(\mathbf{0}_2, I_2) \) | 1             | 19.180 (1.000)              |
|      |        | 5             | 13.620 (1.000)              |
|      |        | 10            | 9.100 (1.000)               |
|      |        | 15            | 5.660 (1.000)               |
| 2    | \( y_{11}, \ldots, y_{1n} \sim \text{Cauchy}(0, 1) \) \( y_{21}, \ldots, y_{2n} \sim N(0, 1) \) | 1             | 0.160 (0.025)               |
|      |        | 5             | 0.000 (0.000)               |
|      |        | 10            | 0.000 (0.000)               |
|      |        | 15            | 0.000 (0.000)               |
| 3    | \( y_{2 \times n} \sim N_2 (\mathbf{0}_2, (\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix})) \) | 1             | 1.920 (1.000)               |
|      |        | 5             | 0.240 (0.012)               |
|      |        | 10            | 0.060 (0.003)               |
|      |        | 15            | 0.040 (0.002)               |
| 4    | \( y_{2 \times n} \sim N_2 (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, I_2) \) | 1             | 0.970 (0.253)               |
|      |        | 5             | 0.120 (0.015)               |
|      |        | 10            | 0.060 (0.004)               |
|      |        | 15            | 0.020 (0.001)               |

Table 1: Relative belief ratios and strengths for testing the bivariate normality assumption with various alternatives and choices of \( \alpha \) in Example 1.

Example 2. **Unknown mean vector and fixed covariance matrix**

In this example, we examine whether the sampled data arise from a multivariate normal distribution with fixed covariance matrix \( I_2 \). That is, \( H_0 : F = N_2(\mathbf{\mu}_2, I_2) \). The results presented in Table 2 lead to conclude that the proposed
approach performs extremely well in all cases.

| Example 2 | Sample | a  | RB(Strength) |
|-----------|--------|----|--------------|
| 1         |        |    |              |
| 2         |        |    |              |

Table 2: Relative belief ratios and strengths for testing the bivariate normality assumption with various alternatives and choices of \( a \) in Example 2.

**Example 3. Fixed mean vector and unknown covariance matrix**

In this example, the goal is to assess whether the sampled data originated from a bivariate normal distribution with fixed mean vector \( \mathbf{0}_2 \). That is, we would like to test the null hypothesis \( \mathcal{H}_0 : F = N_2(\mathbf{0}_2, \Sigma_{2 \times 2}) \). The results are reported in Table 3. It is seen that in all cases the methodology gives the correct answer.

| Example 3 | Sample | a  | RB(Strength) |
|-----------|--------|----|--------------|
| 1         |        |    |              |
| 2         |        |    |              |

Table 3: Relative belief ratios and strengths for testing the bivariate normality assumption with various alternatives and choices of \( a \) in Example 3.

**Example 4. Unknown mean vector and unknown covariance matrix**

In this example, we test for the general bivariate normality. That is, \( \mathcal{H}_0 : F = N_2(\mu_2, \Sigma_{2 \times 2}) \). The results are presented in Table 4. Clearly, the proposed method provides the correct conclusion.

In the next two examples, we look at the performance of the methodol-
Example 4

| Case | Sample | $a$ | $RB$(Strength) |
|------|--------|----|----------------|
| 1    | $y_{2 \times n} \sim N_2 \left( \left( \begin{array}{c} 1 \\ 0 \\ 2 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right) \right)$ | 1   | 19.520(1.000) |
|      |        | 5   | 14.220(1.000) |
|      |        | 10  | 9.340(1.000)  |
|      |        | 15  | 7.060(1.000)  |
| 2    | $y_{11}, \ldots, y_{1n} \sim EXP(1/2)$, $y_{21}, \ldots, y_{2n} \sim Cauchy(0,1)$ | 1   | 0.240(0.012)  |
|      |        | 5   | 0.040(0.004)  |
|      |        | 10  | 0.000(0.000)  |
|      |        | 15  | 0.000(0.000)  |
| 3    | $y_{11}, \ldots, y_{1n} \sim t_1$, $y_{21}, \ldots, y_{2n} \sim N(0,1)$ | 1   | 1.300(0.366)  |
|      |        | 5   | 0.100(0.004)  |
|      |        | 10  | 0.020(0.001)  |
|      |        | 15  | 0.000(0.000)  |

Table 4: Relative belief ratios and strengths for testing the bivariate normality assumption with various alternatives and choices of $a$ in Example 4.

ogy by using two real data sets. For comparison purposes, the results of the Doornik-Hanson (DH) test (Doornik and Hansen, 2008) are also presented. The R package asbio is used to compute p-values of the DH test.

Example 5. Two measurements of stiffness for boards.

In this example, we consider the data of 30 boards with two different measurements of stiffness presented in Johnson and Wichern (2007). The first measurement involves sending a shock wave down the board, and the second measurement is determined while vibrating the board. The problem is to check whether the data come from a bivariate normal distribution. The DH’s p-value is 0.3235, which implies not to reject the normality assumption. The results of the proposed test are presented in Table 5 for various values of $a$. It is seen that the proposed test supports the normality assumption of this data set. Figure 2 part (a), confirms these results as the posterior density of the Anderson-Darling distance is more concentrated about 0 than the prior density of the Anderson-Darling distance. The chi-square Q-Q plot of this data set is presented in Appendix D, which favors the multivariate normality.

Example 6. National track records for women.

In this example, we consider data from Johnson and Wichern (2007) which
Table 5: Relative belief ratios and strengths for testing the bivariate normality assumption of two measurements of stiffness for boards (Example 5) with various choices of $a$.

| $a$ | $RB$(Strength) |
|-----|----------------|
| 1 | 17.800(1.000) |
| 5 | 10.340(1.000) |
| 10 | 6.710(1.000) |
| 15 | 3.810(1.000) |

reprints women’s athletic records for 54 countries with seven variables. The variables are 100, 200, 400 meters in seconds, 800, 1500, 3000 meters in minutes and the marathon. The problem is to assess the seven-variate normality assumption for this data set. Atkinson et al. (2004) provided the Q-Q plot (see Appendix D) of ordered Mahalanobis distances against the square root of the percentage points of the chi-square distribution with seven degrees of freedom and then suggested the non-normality of this data set. On the other hand, DH’s p-value for this data is $4.042382 \times 10^{-68}$, which shows strong evidence to reject the multivariate normality assumption. The results of the proposed test are given in Table 6. It follows that the methodology also presents strong evidence to reject the underlying distribution. Figure 2 part (b), shows the good performance of the methodology in this data set.

Table 6: Relative belief ratios and strengths for testing the seven-variate normality assumption of national track records for women (Example 6) with various choices of $a$.

| $a$ | $RB$(Strength) |
|-----|----------------|
| 1 | 7.480(0.626) |
| 5 | 1.140(0.405) |
| 6 | 0.700(0.133) |
| 8 | 0.480(0.063) |
| 10 | 0.240(0.012) |
| 15 | 0.120(0.006) |
8 Conclusion

A novel Bayesian nonparametric test for assessing multivariate normality has been proposed. The suggested test is developed by using Mahalanobis distance as a good technique to convert the $m$-variate problem into the univariate problem. The Dirichlet process has been considered as a prior on the distribution of the Mahalanobis distance. Then, the concentration of the distribution of the distance between posterior process and chi-square distribution with $m$ degrees of freedom is compared to the concentration of the distribution of the distance between prior process and chi-square distribution with $m$ degrees of freedom via relative belief ratio. The distance between the Dirichlet process and the chi-square distribution is developed based on the Anderson-Darling distance. Several theoretical results including consistency have been discussed for the proposed test. The test has been examined by several simulation studies to show the good performance of the methodology. Finally, applications including two real data sets have been presented.
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**Appendix A  Proof of Lemma 1**

Consider $P_N(x)$ as

$$
P_N(x) = \begin{cases} 
0 & x < Y(1) \\
P_N(Y(i)) & Y(i) \leq x < Y(i+1), (i = 1, \ldots, N - 1) \\
1 & x \geq Y(N) \end{cases}
$$

Let $g(x) = \frac{dG(x)}{dx}$, then

$$
d_{AD}(P_N, G) = \int_{-\infty}^{\infty} \frac{(P_N(x) - G(x))^2}{G(x) \left(1 - G(x)\right)} g(x) \, dx \\
= \int_{-\infty}^{Y(1)} \frac{G(x)^2}{G(x) \left(1 - G(x)\right)} g(x) \, dx + \int_{Y(N)}^{\infty} \frac{(1 - G(x))^2}{G(x) \left(1 - G(x)\right)} g(x) \, dx \\
+ \sum_{i=1}^{N-1} \int_{Y(i)}^{Y(i+1)} \frac{(P_N(Y(i)) - G(x))^2}{G(x) \left(1 - G(x)\right)} g(x) \, dx.
$$

Substituting $y = G(x)$, $G(Y(i)) = U(i)$, $G(-\infty) = 0$ and $G(\infty) = 1$, gives
\[ d_{AD}(P_N, G) = \sum_{i=1}^{N-1} \int_{U(i)}^{U(i+1)} \frac{(P_N(Y(i)) - y)^2}{y(1-y)} \, dy + \int_0^{U(1)} y \, dy + \int_{U(N)}^1 \frac{1-y}{y} \, dy \]
\[ = \sum_{i=1}^{N-1} \left[ P_N^2(Y(i)) \log(y) - \left[ (P_N(Y(i)) - 1)^2 \log(1-y) \right] - y \right]_{U(i)}^{U(i+1)} \]
\[ + \left[ -y - \log(1-y) \right]_{0}^{U(1)} + \left[ \log(y) - y \right]_{U(N)}^{1} \]
\[ = I_1 + I_2 + I_3. \]

Note that,

\[ I_1 = - \sum_{i=1}^{N-1} (U_{i+1} - U_i) \sum_{i=1}^{N-1} (P_N(Y(i)) - 1)^2 \left( \log(1 - U_{i+1}) - \log(1 - U_i) \right) \]
\[ + \sum_{i=1}^{N-1} P_N^2(Y(i)) \left( \log(U_{i+1}) - \log(U_i) \right) \]
\[ = \sum_{i=1}^{N-1} P_N^2(Y(i)) \log \frac{U_{i+1}(1-U_i)}{U_i(1-U_{i+1})} + \sum_{i=1}^{N-1} (2P_N(Y(i)) - 1) \log \frac{1-U_{i+1}}{1-U_i} \]
\[ - (U_N - U_1). \]

Also, \( I_2 = -U_{(1)} - \log \left( 1 - U_{(1)} \right) \) and \( I_3 = -1 - \log U_{(N)} + U_{(N)} \). Therefore, adding \( I_1, I_2 \) and \( I_3 \), gives

\[ d_{AD}(P_N, G) = \sum_{i=1}^{N-1} P_N^2(Y(i)) \log \frac{U_{i+1}(1-U_i)}{U_i(1-U_{i+1})} + \sum_{i=1}^{N-1} (2P_N(Y(i)) - 1) \log \frac{1-U_{i+1}}{1-U_i} \]
\[ - 1 - \log \left( U_{(N)}(1-U_{(1)}) \right). \] (11)

The proof is completed by substituting \( P_N(Y(i)) = \sum_{j=1}^i J_j^i \), \( P_N^2(Y(i)) = \sum_{j=1}^i J_j^i \), \( \sum_{j=1}^{i-1} \sum_{k=j+1}^i J_k^j J_j^i \) and \( U_{(i)} = G(Y_{(i)}) \) in terms on the right-hand side of (11).
Appendix B  Proof of Lemma 4

To prove (i), note that, from the property of the Dirichlet process, for any $t \in \mathbb{R}$, $E_P (P(t) - H(t))^2 = \frac{H(t)(1-H(t))}{a+1}$.

$$E_P (d_{AD}(P,H)) = \int_{-\infty}^{\infty} E_P (P(t) - H(t))^2 \frac{dH(t)}{H(t)(1-H(t))} = \frac{1}{a+1}.$$ 

To prove (ii), it is enough to compute $E_P (d_{AD}(P,H))^2$. According to the Corollary 2, we consider $H$ to be the cdf of the Uniform distribution on $[0, 1]$. Then

$$E_P (d_{AD}(P,H))^2 = E_P \left( \int_{-\infty}^{\infty} (P(t) - H(t))^2 \frac{dH(t)}{H(t)(1-H(t))} \right)^2$$

$$= E_P \left( 1 \int_{0}^{1} \frac{(P(t) - t)^2}{t(1-t)} dt \int_{0}^{1} \frac{(P(s) - s)^2}{s(1-s)} ds \right)$$

$$= \frac{x}{y} \int_{0}^{1} \frac{(P(t) - t)^2}{t(1-t)} dt \int_{0}^{1} \frac{(P(s) - s)^2}{s(1-s)} ds dt$$

$$+ \int_{0}^{1} \int_{0}^{s} \frac{(P(t) - t)^2}{t(1-t)s(1-s)} ds dt$$

$$= 2 \int_{0}^{1} \int_{0}^{t} \frac{(P(t) - t)^2}{t(1-t)t(1-s)} ds dt$$

$$= 2 \int_{0}^{1} \int_{0}^{t} \frac{(P(s) + P((s,t])) - t^2}{t(1-t)t(1-s)} ds dt.$$ 

Note that, from the property of the Dirichlet process, for any $s < t$ and $i, j \in \mathbb{N}$, $E_P \left( P^i(s)P^j((s,t]) \right) = \frac{\Gamma(a)}{\Gamma(a+i+j)} \frac{\Gamma(a+i)}{\Gamma(a+i+j)} \frac{\Gamma(a+j)}{\Gamma(a+i+j)}$ and $E_P (P^i(s)) = \prod_{k=0}^{i-1} \frac{a+k}{a+k+i}$. Then
\[ E_P(d_{AD}(P, H))^2 = \int_0^1 \int_0^t \frac{1}{ts(1-t)(1-s)} \left\{ \frac{2as(a(t - s) + 1)(as + 1)(t - s)}{(a+3)(a+2)(a+1)} ight. \\
+ \frac{4as(as + 2)(as + 1)(t - s)}{(a+3)(a+2)(a+1)} + \frac{2s(as + 3)(as + 2)(as + 1)}{(a+3)(a+2)(a+1)} \\
- \frac{4as(2s + t)(as + 1)(t - s)}{(a+2)(a+1)} - \frac{4as^2(a(t - s) + 1)(t - s)}{(a+2)(a+1)} \\
- \frac{4s(as + 2)(as + 1)(s + t)}{(a+2)(a+1)} + \frac{2s(s^2 + 4st + t^2)(as + 1)}{a+1} \\
+ \frac{4as^2(2s + t)(t - s)}{a+1} + \frac{2s^2(t - s)(a(t - s) + 1)}{a+1} - 4s^2(t - s) \\
- 4s^2(t + s + 2s^2) \right\} ds \, dt. \]

After simplification, we get

\[ E_P(d_{AD}(P, H))^2 = \int_0^1 \int_0^t \frac{2((a - 6)((3t - 2)s - t) - 6)}{t(s - 1)(a + 3)(a + 2)(a + 1)} ds \, dt \\
= \int_0^1 \frac{2}{t(a + 3)(a + 2)(a + 1)} \left\{ (a - 6)(3t - 2)t - 2i\pi ((a - 6)t - a + 3) \\
+ 2(a(t - 1) - 6t + 3) \log(t - 1) \right\} dt \\
= \frac{a(2\pi^2 - 15) - 6(\pi^2 - 15)}{3(a + 3)(a + 2)(a + 1)}, \]

for \( \text{Re}(t) < 1 \) or \( t \not\in \mathbb{R} \), where \( \text{Re}(t) \) denotes the real part of \( t \) and \( i \) is the imaginary unit. Then, the variance of \( d_{AD}(P, H) \) is given by

\[ \text{Var}_P(d_{AD}(P, H)) = E_P(d_{AD}(P, H))^2 - E_P^2(d_{AD}(P, H)) = \frac{2((\pi^2 - 9)a^2 + (30 - 2\pi^2)a - 3\pi^2 + 36)}{3(a + 1)^2(a + 2)(a + 3)}. \]

Hence, the proof is completed.
Appendix C  Proof of Lemma 6

Assume that $d = (d_M^2(y_1), \ldots, d_M^2(y_n))$ is the observed square Mahalanobis distance from $P$ where $P \sim DP(a, F_m)$. We consider the Anderson-Darling distance formula for $P_d$ and $F_m$ as $d_{AD}(P_d, F_m) = \int_{-\infty}^{\infty} \frac{(P_d(t) - F_m(t))^2}{F_m(t)(1 - F_m(t))} dF_m(t)$.

Then

$$E_{P_d} (d_{AD}(P_d, F_m)) = \int_{-\infty}^{\infty} \frac{E_{P_d} (P_d(t) - F_m(t))^2}{F_m(t)(1 - F_m(t))} dF_m(t)$$

$$= \int_{-\infty}^{\infty} \frac{E_{P_d} (P_d(t) - H_d(t))^2 + E_{P_d} (H_d(t) - F_m(t))^2}{F_m(t)(1 - F_m(t))} dF_m(t)$$

$$+ 2 \int_{-\infty}^{\infty} \frac{E_{P_d} \{ (P_d(t) - H_d(t))(H_d(t) - F_m(t)) \}}{F_m(t)(1 - F_m(t))} dF_m(t).$$

Since for any $t \in \mathbb{R}$, $E_{P_d} (P_d(t)) = H_d(t)$ and $E_{P_d} (P_d(t) - H_d(t))^2 = \frac{H_d(t)(1-H_d(t))}{a+n+1}$, then

$$E_{P_d} (d_{AD}(P_d, F_m)) = \int_{-\infty}^{\infty} \frac{H_d(t)(1-H_d(t))}{(a+n+1)F_m(t)(1 - F_m(t))} dF_m(t)$$

$$+ \int_{-\infty}^{\infty} \frac{(H_d(t) - F_m(t))^2}{F_m(t)(1 - F_m(t))} dF_m(t).$$

(12)

To prove (i), note that from [2], for any $t \in \mathbb{R}$, $H_d(t) \to F_m(t)$ as $a \to \infty$. Also, $|H_d(t)(1-H_d(t))| \leq 1$ and $(H_d(t) - F_m(t))^2 \leq 1$. Then, by using DCT for the integral on the right of (12), we have $E_{P_d} (d_{AD}(P_d, F_m)) \to 0$ as $a \to \infty$. To prove (ii) and (iii), note that, for any $t \in \mathbb{R}$, $H_d(t) \to P(t)$ as $n \to \infty$, where $P$ is the true distribution of the sample $d$. Letting $n \to \infty$ and using the DCT on the right of (12), then

$$E_{P_d} (d_{AD}(P_d, F_m)) \to \int_{-\infty}^{\infty} \frac{(P(t) - F_m(t))^2}{F_m(t)(1 - F_m(t))} dF_m(t).$$

(13)

According to the right of (13), if $H_0$ is true, then for any $t \in \mathbb{R}$, $P(t) = F_m(t)$ and finally $E_{P_d} (d_{AD}(P_d, F_m)) \to 0$ as $n \to \infty$; otherwise, for some $t \in \mathbb{R}$, $P(t) \neq F_m(t)$.
$F_{(m)}(t)$ and $E_{P_d}(d_{AD}(P_d, F_{(m)})) \to c$, where $c := \int_{-\infty}^{\infty} \frac{(P(t) - F_{(m)}(t))^2}{F_{(m)}(t)(1 - F_{(m)}(t))} dF_{(m)}(t)$, $c > 0$, and the proof is completed.

Appendix D  Chi-square Q-Q plot of real data sets

Chi-square Q-Q plot shows the relationship between the distribution of data points and chi-square distribution. It is a common graphical method to assess multivariate normality. Specifically, for a given $m$-variate sampled data $y_{m \times n} = (y_1, \ldots, y_n)$ with the size of $n$, where $y_i \in \mathbb{R}^m$ for $i = 1, \ldots, n$, chi-square Q-Q plot provide the plot of square ordered Mahalanobis distances against the quantile points of the chi-square distribution with $m$ degrees of freedom. From Johnson and Wichern (2007), if $y_{m \times n}$ comes from an $m$-variate normal distribution, we should see the points of square ordered Mahalanobis distances are nearly a straight line with unit slope (reference line).

In the main paper, we showed that the proposed test accepts the bivariate normality assumption for the real data of Example 5 (two measurements of stiffness for boards) and rejects the seven-variate normality assumption for the real data set of Example 6 (national track records for women). Here, we show the chi-square Q-Q plot confirms the results of the proposed test. For this, we provide the chi-square Q-Q plot with its approximate confidence envelope around the reference line for real data sets of Examples 5 and 6 in Figure 3. Note that, the confidence envelope is plotted to show the central intervals for each quantile of the chi-square distribution with two and seven degrees of freedom in part (a) and part (b) of Figure 3, respectively. In fact, the envelope in part (a) and part (b) of Figure 3 provides a density estimate of the quantiles drawn from the chi-square distribution with two and seven degrees of freedom, respectively.
For more detailed studies about the general methods for Q-Q plot see Oldford (2016). From Figure 3 it is clear that the bivariate assumption for the real data set of Example 5 is accepted and the seven-variate normality assumption for the real data set of Example 6 is rejected.

Figure 3: Chi-square Q-Q plot of real data sets of Examples 5 and 6 in the main paper with 95% envelope.