DOUBLE QUADRICS WITH LARGE AUTOMORPHISM GROUPS

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Abstract. We classify nodal Fano threefolds that are double covers of smooth quadrics branched over intersections with quartics acted on by finite simple non-abelian groups, and study their rationality.

1. Introduction

In this paper we study double covers of three-dimensional quadrics branched over intersections with quartics. For simplicity we will sometimes call such varieties just double quadrics. They are Fano threefolds of Picard rank 1, index 1 and anticanonical degree 4 (provided that their singularities are sufficiently good). Double quadrics are degenerations of quartic hypersurfaces (see e.g. [18, §12.2]), and share many birational properties with the latter. Rationality questions for double quadrics were studied in [27], [19, §2.2], [14], [15], and [29].

Motivated by a study of finite subgroups of Cremona groups, we are interested in the following problem. Given a finite group $G$ and a deformation family $\mathcal{X}$ of Fano varieties, we would like to be able to tell which of the varieties of $\mathcal{X}$ are acted on by $G$, and which of them are $G$-Fano varieties (see e.g. [23, §1] or [24, §1] for a definition). The case that is currently most challenging is when $\mathcal{X}$ is some family of Fano threefolds, and $G$ is a simple non-abelian group (cf. [26]). In [6] this question was studied for quartic double solids with an action of the icosahedral group $A_5$. The purpose of this paper is to study double quadrics from this point of view.

Our first result is a classification of possible finite simple non-abelian groups acting on double quadrics with mild singularities.

Proposition 1.1. Let $G$ be a finite simple non-abelian group. Suppose that $G$ acts by automorphisms of a threefold $X$ that is a double cover of an irreducible quadric branched over an intersection with a quartic. Then one has either $G \cong A_6$, or $G \cong A_5$. In the former case the variety $X$ is unique, its singularities are (isolated) ordinary double points, and $X$ is non-rational.

As one can see from Proposition 1.1, the most interesting group we have to deal with is the icosahedral group $A_5$. Our second result is a more refined classification of double quadrics with icosahedral symmetry.

Theorem 1.2. There exist a two-parameter family $X_{\mu,\nu}$, $\mu, \nu \in \mathbb{C}$, of threefolds, and a threefold $X_{irr}$ such that $X_{\mu,\nu}$ and $X_{irr}$ are acted on by the icosahedral group $A_5$, and the following properties hold.

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(i) Suppose that the group $A_5$ acts by automorphisms of a threefold $X$ that is a double cover of a smooth quadric branched over an intersection with a quartic. Suppose also that $X$ has at most isolated singularities. Then either there is an $A_5$-equivariant isomorphism $X \cong X_{\mu,\nu}$ for some $\mu$ and $\nu$, or there is an $A_5$-equivariant isomorphism $X \cong X_{\text{irr}}$.

(ii) The varieties $X_{\mu,\nu}$ are non-rational up to a finite number of possible exceptions.

(iii) A very general variety $X_{\mu,\nu}$ is not stably rational.

(iv) The variety $X_{\text{irr}}$ is non-rational.

Remark 1.3. We will see in §6 that the variety $X_{\text{irr}}$ from Theorem 1.2 is actually isomorphic to some variety $X_{\mu,\nu}$. However, there is no $A_5$-equivariant isomorphism between these two varieties, i.e. the corresponding two actions of the group $A_5$ are non-conjugate in the automorphism group of $X_{\text{irr}}$. To be more precise, for every $(\mu,\nu)$ the space $H^0(X_{\mu,\nu}, -K_{X_{\mu,\nu}})$ is the (reducible) five-dimensional permutation representation of the group $A_5$, while $H^0(X_{\text{irr}}, -K_{X_{\text{irr}}})$ is the irreducible five-dimensional representation of $A_5$. Moreover, we will see that there is an action of the group $A_5$ on $X_{\text{irr}}$, so that $X_{\text{irr}}$ also coincides with the variety described in Proposition 1.1. The two types of $A_5$-actions on $X_{\text{irr}}$ correspond to two non-conjugate embeddings of $A_5$ into $A_6$. As for the threefolds $X_{\mu,\nu}$, we will see in Remark 4.14 that they indeed form a two-parameter family up to isomorphism.

Theorem 1.2 gives a reasonable (although still not complete) answer to rationality questions for the family $X_{\mu,\nu}$ and for the variety $X_{\text{irr}}$ that were asked in [7, Example 1.3.6] and [7, Example 1.3.7], respectively. There are still some exceptional cases among the varieties $X_{\mu,\nu}$ one has to deal with (see Corollaries 4.16 and 6.6 and also Table 1). It is possible that some of these varieties are actually rational. However, we are not aware of any rationality constructions for double quadrics with ordinary double points that can be adapted for the case of large automorphism groups. This is somehow opposite to the situation with rational quartic threefolds that were extensively studied in the literature, see e.g. [30], [31], [32], [22], and [8]. Maybe some of the above exceptional cases could be a starting point for a search of rationality constructions for double quadrics.

The plan of the paper is as follows. In §2 we make some preliminary remarks about double quadrics and their automorphisms. In §3 we rule out most of the groups in question via their representation theory, and prove (most of) Proposition 1.1. In §4 we classify singular complete intersections of a smooth quadric and a quartic in the projectivization of the five-dimensional permutation representation of the group $A_5$, and make some conclusions about their non-rationality. In particular, in §4 we prove Theorem 1.2(i) and give the most essential part of the proof of Theorem 1.2(ii). Although the material of §4 is totally computational, this is the main technical part of the paper, modulo which all the rest boils down to well-known constructions. In §5 we produce a construction of double quadrics that are not stably rational similar to a famous Artin–Mumford construction of non-stably rational quartic double solids (see [1], or rather [2, Appendix] for an approach that we actually use), which proves Theorem 1.2(iii). Although we do not know any literature explicitly describing a construction like this for double quadrics, it was definitely known for a while, cf. [17, Remark 4.3], see also Remark 5.9 below for more references. Finally, in §6 we discuss non-rationality of the variety $X_{\text{irr}}$ using the approach of §3 and use it to make final conclusions about non-rationality of certain varieties $X_{\mu,\nu}$. This proves Theorem 1.2(iv) and completes the proofs of Proposition 1.1 and Theorem 1.2(ii).
Notation and conventions. We work over the field \( \mathbb{C} \) of complex numbers. By \( \mathfrak{S}_n \) and \( \mathfrak{A}_n \) we denote the symmetric and the alternating group on \( n \) letters, respectively. By \( \text{Cl}(X) \) we denote the group linear equivalence classes of Weil divisors on a variety \( X \). By an ordinary double point we always mean an isolated singularity that is locally (analytically) isomorphic to a singularity of a quadratic cone of an appropriate dimension. By a very general element of an algebraic family we mean an element in a complement to a countable union of Zariski closed subsets.

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2. Preliminaries

Let \( \tau : X \to Q \) be double cover of an irreducible three-dimensional quadric \( Q \) branched over a reduced surface \( S \) that is cut out on \( Q \) by a quartic. We will be interested in the case when \( X \) has terminal singularities. In particular, in this situation both \( Q \) and \( S \) have isolated singularities, so that \( S \) is irreducible. Note that the singularities of \( X \) lie either over the vertex of \( Q \) (if \( Q \) is a cone), or correspond to the singularities of \( S \).

Remark 2.1. Recall from \[27\] (or \[19, \S 2.2\]) that \( X \) is non-rational provided that \( Q \) and \( S \) are smooth. Also, if \( Q \) is a cone with an isolated singularity and \( S \) is a smooth surface that does not pass through the vertex of \( Q \) (so that \( X \) has two singular points), the threefold \( X \) is non-rational as well (see \[15\]). Note however that if \( Q \) is a cone with an isolated singularity then \( X \) cannot be a \( G \)-Fano variety with respect to any simple non-abelian group \( G \). Therefore, we will be mostly interested in the case when \( Q \) is smooth and \( S \) is singular.

We will need the following general auxiliary result.

Lemma 2.2. Let \( Y \subset \mathbb{P}^n \) be a linearly normal Gorenstein variety that is not contained in a hyperplane. Suppose that a group \( G \) acts on \( Y \) so that the class of the line bundle \( \mathcal{O}_Y(1) = \mathcal{O}_{\mathbb{P}^n}(1)|_Y \) in \( \text{Pic}(Y) \) is \( G \)-invariant. Suppose that \( \omega_Y \cong \mathcal{O}_Y(i) \) for some (non-zero) integer \( i \), where \( \omega_Y \) is the canonical line bundle on \( Y \), and suppose that the numbers \( i \) and \( n+1 \) are coprime. Then the line bundle \( \mathcal{O}_Y(1) \) has a \( G \)-equivariant structure.

Proof. Since the class of \( \mathcal{O}_Y(1) \) in \( \text{Pic}(Y) \) is \( G \)-invariant, there is an action of the group \( G \) on \( \mathbb{P}^n \cong \mathbb{P}(H^0(Y, \mathcal{O}_Y(1))^*) \) that agrees with the initial action of \( G \) on \( Y \). The line bundle \( \omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1) \) has a \( G \)-equivariant structure, and the same holds for its restriction to \( Y \). Also, the line bundle \( \omega_Y \) has a \( G \)-equivariant structure. By assumption there are integers \( a \) and \( b \) such that \( ai - b(n+1) = 1 \).

This gives \( \mathcal{O}_Y(1) \cong \omega_Y^a \otimes \omega_{\mathbb{P}^n}^b|_Y \). Therefore, \( \mathcal{O}_Y(1) \) has a \( G \)-equivariant structure as well. \(\square\)
Lemma 2.3 (cf. [10, Remark 2.2]). Suppose that $X$ admits a faithful action of a finite group $G$. Then there is a natural action of $G$ on $\mathbb{P}^4$ such that $Q$ is $G$-invariant, and the surface $S$ is cut out on $Q$ by a $G$-invariant quartic in $\mathbb{P}^4$. Vice versa, fix an action of $G$ on $Q$ such that the surface $S$ is invariant, and suppose that $S$ corresponds to a trivial subrepresentation of $G$ in $\text{Sym}^4(H^0(Q, \mathcal{O}_Q(H)))$, where $H$ is a hyperplane section of $Q$. Then $G$ acts on the threefold $X$.

Proof. To prove the first assertion note that the morphism $\tau$ is given by the linear system $|-K_X|$. Thus the group $G$ acts on $Q$ so that $S$ is $G$-invariant. One has $Q \subset \mathbb{P}^4$, and there is a $G$-equivariant identification of the linear systems $|\mathcal{O}_{\mathbb{P}^4}(1)|$ and $|\mathcal{O}_{\mathbb{P}^4}(1)|_Q$. In particular, $S$ is cut out on $Q$ by a $G$-invariant quartic in $\mathbb{P}^4$.

To prove the second assertion suppose that $G$ acts on $Q$ so that $S$ is $G$-invariant. One has $Q \subset \mathbb{P}^4$, and $\mathbb{P}^4$ is identified with a projectivization of the $G$-representation

$$U^\vee = H^0(Q, \mathcal{O}_Q(H))^V$$

by Lemma 2.2. The threefold $X$ has a natural embedding to the projectivization of the vector bundle

$$\mathcal{E} = \mathcal{O}_Q \oplus \mathcal{O}_Q(2H).$$

The vector bundle $\mathcal{E}$ has a $G$-equivariant structure by Lemma 2.2. By assumption the group $G$ acts trivially on the one-dimensional subspace of $\text{Sym}^4(U)$ corresponding to $S$, so that $X$ is given by an equation in the projectivization of $\mathcal{E}$. \hfill \Box

3. Representations

Lemma 3.1. Let $G$ be a finite simple non-abelian group. Suppose that $G$ acts by automorphisms of a threefold $X$ that is a double cover of an irreducible quadric branched over an intersection with a quartic. Then either $G \cong A_6$, or $G \cong A_5$. In the former case the variety $X$ is unique.

Proof. By Lemmas 2.3 and 2.2 there exists a faithful five-dimensional representation $V$ of the group $G$, and there is an irreducible (reduced) $G$-invariant quadric in $\mathbb{P}(V)$. Suppose that $G$ is not isomorphic to $A_5$. We know from [12, §8.5] that $G$ is one of the groups $\text{PSp}_4(F_3)$, $\text{PSL}_2(F_11)$, $\text{PSL}_3(F_7)$, or $A_6$.

Assume that $G \cong \text{PSp}_4(F_3)$. Then $V$ is one of the two irreducible five-dimensional representations of $G$ (see [10 p. 27]). However, there are no $G$-invariant quadrics in $\mathbb{P}(V)$, see e.g. [3].

Assume that $G \cong \text{PSL}_2(F_{11})$. Then $V$ is one of the two irreducible five-dimensional representations of $G$, see [10 p. 7]. As in the previous case, there are no $G$-invariant quadrics in $\mathbb{P}(V)$.

Assume that $G \cong \text{PSL}_2(F_7)$. Then $V \cong I \oplus I \oplus V_3$, where $I$ is a trivial representation and $V_3$ is one of the two irreducible three-dimensional representations of $G$, see [10 p. 3]. There are no one-dimensional $G$-subrepresentations in $\text{Sym}^2(V_3^\vee)$, see e.g. [20]. Thus, every $G$-invariant quadric in $\mathbb{P}(V)$ is reducible (or non-reduced).

Finally, assume that $G \cong A_6$. Then $G$ has two faithful five-dimensional representations, and both of them are irreducible (see e.g. [10 p. 5]). Moreover, the images of $G$ in $\text{PGL}_5(\mathbb{C})$ under these two representations are conjugate. One can check that $\text{Sym}^2(V^\vee)$ contains a unique trivial subrepresentation, and

$$\dim \text{Hom}(I, \text{Sym}^4(V^\vee)) = 2.$$
there is a unique form of degree 2 in \( \mathfrak{A}_6 \) namely:

\[ I \oplus I \oplus W_3, \quad I \oplus I \oplus W'_3, \quad I \oplus W_4, \quad \text{and} \quad W_5. \]

Remark 3.2. Let \( X \) be the (unique) threefold that is a double cover of an irreducible quadric branched over an intersection with a quartic such that \( X \) has a non-trivial action of the group \( \mathfrak{A}_6 \) (see Lemma 3.1). We will see later in Section \( \Box \) that the singularities of \( X \) are ordinary double points.

Keeping in mind Lemma 3.1, in the rest of the paper we will work with double quadrics that admit a faithful action of the icosahedral group \( \mathfrak{A}_5 \). Denote by \( I \) the trivial representation of the group \( \mathfrak{A}_5 \). Let \( W_3 \) and \( W'_3 \) be the two irreducible three-dimensional representations of \( \mathfrak{A}_5 \), and let \( W_4 \) and \( W_5 \) be the irreducible four-dimensional and five-dimensional representations of \( \mathfrak{A}_5 \), respectively (see e.g. [10, p. 2]). Note that \( I, W_4, \) and \( W_5 \) can be also considered as representations of the group \( \mathfrak{S}_5 \).

There are four faithful five-dimensional representations of the icosahedral group \( \mathfrak{A}_5 \), namely: \( I \oplus I \oplus W_3, \quad I \oplus I \oplus W'_3, \quad I \oplus W_4, \) and \( W_5 \).

Remark 3.3. Put \( U = I \oplus I \oplus W_3 \) or \( U = I \oplus I \oplus W'_3 \). Then

\[
\dim \text{Hom}(I, \text{Sym}^2(U^\vee)) = 4, \quad \dim \text{Hom}(I, \text{Sym}^4(U^\vee)) = 9.
\]

Let \( x_0 \) and \( x_1 \) be coordinates in the \( \mathfrak{A}_5 \)-subrepresentation \( I \oplus I \), and let \( y_0, y_1, y_2 \) be coordinates in the \( \mathfrak{A}_5 \)-subrepresentation \( W_3 \) or \( W'_3 \). It is well-known that up to scaling there is a unique form of degree 2 in \( y_0, y_1, y_2 \) that is preserved by \( \mathfrak{A}_5 \). We may assume that \( y_0, y_1, y_2 \) are chosen so that this form is \( y_0^2 + y_1^2 + y_2^2 \). One can consider \( x_0, x_1, y_0, y_1, y_2 \) as homogeneous coordinates on \( \mathbb{P}^4 = \mathbb{P}(U) \). Every \( \mathfrak{A}_5 \)-invariant quadric in \( \mathbb{P}^4 \) is given by equation

\[ F_2(x_0, x_1) + \alpha(y_0^2 + y_1^2 + y_2^2) = 0, \]

where \( F_2 \) is a form of degree 2 and \( \alpha \in \mathbb{C} \). In particular, this quadric contains the curve \( C \) given by equations

\[ x_0 = x_1 = y_0^2 + y_1^2 + y_2^2 = 0. \]

Every \( \mathfrak{A}_5 \)-invariant quartic in \( \mathbb{P}^4 \) is given by equation

\[ G_4(x_0, x_1) + G_2(x_0, x_1)(y_0^2 + y_1^2 + y_2^2) + \beta(y_0^2 + y_1^2 + y_2^2)^2 = 0, \]

where \( G_i \) is a form of degree \( i \) and \( \beta \in \mathbb{C} \). In particular, it is singular along the curve \( C \). Therefore, every complete intersection of a quadric and a quartic in \( \mathbb{P}(U) \) has non-isolated singularities.

By Remark 3.3 to classify three-dimensional double quadrics with isolated singularities that admit an icosahedral symmetry it remains to consider the \( \mathfrak{A}_5 \)-representations \( I \oplus W_4 \) and \( W_5 \). We will do this in the next sections.

Remark 3.4. In general, it is an interesting problem to classify Fano varieties with an action of some relatively large finite group, for example the icosahedral group \( \mathfrak{A}_5 \). Apart from applications to classification of finite subgroups of Cremona groups, this often leads to really beautiful geometric constructions. We do not expect many examples like this among Fano threefolds of large anticanonical degree. However, it is possible that some interesting cases can arise among \( \mathbb{Q} \)-Fano threefolds of large index (cf. [25]) or smooth Fano fourfolds of large anticanonical degree (cf. [21]).
4. Projectivization of the permutation representation

Consider the vector space \( I \oplus W_4 \) as a permutation representation of the groups \( A_5 \) and \( S_5 \), and put \( \mathbb{P}^4 = \mathbb{P}(I \oplus W_4) \). Let \( x_0, \ldots, x_4 \) be homogeneous coordinates in \( \mathbb{P}^4 \) that are permuted by \( S_5 \). Put

\[
\sigma_k(x_0, \ldots, x_4) = x_0^k + \ldots + x_4^k.
\]

It is easy to see that every reduced \( A_5 \)-invariant quadric in \( \mathbb{P}^4 \) is given by equation

\[
(4.1) \quad \sigma_2(x_0, \ldots, x_4) + \lambda \sigma_1(x_0, \ldots, x_4)^2 = 0
\]

for some \( \lambda \in \mathbb{C} \).

**Lemma 4.1.** A quadric \( Q \) given by equation (4.1) is singular if and only if \( \lambda = -\frac{1}{5} \). If it is non-singular, there is an \( S_5 \)-equivariant linear change of coordinates such that after it \( Q \) is given by equation (4.1) with \( \lambda = 0 \).

**Proof.** The quadratic form associated to the quadric (4.1) is given by the matrix

\[
M = \begin{pmatrix}
\lambda + 1 & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda + 1 & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda + 1 & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda + 1 & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda + 1
\end{pmatrix}.
\]

One has \( \det M = 5\lambda + 1 \), so the matrix is degenerate only for \( \lambda = -\frac{1}{5} \), and this is the only case when the quadric is singular. Let \( \lambda \neq -\frac{1}{5} \) and let \( \alpha \) be a root of the equation

\[
(4.2) \quad 5\alpha^2 + 2\alpha = \lambda.
\]

Consider a linear change of coordinates putting

\[
x'_i = x_i + \alpha \sigma_1(x_0, \ldots, x_4).
\]

Note that it is indeed an invertible change of coordinates, i.e. the corresponding matrix is non-degenerate. One has

\[
\sigma_2(x'_0, \ldots, x'_4) = \sigma_2(x_0, \ldots, x_4) + 2\alpha \sigma_1^2(x_0, \ldots, x_4) + 5\alpha^2 \sigma_1^2(x_0, \ldots, x_4) = \\
= \sigma_2(x_0, \ldots, x_4) + \lambda \sigma_1^2(x_0, \ldots, x_4).
\]

□

Keeping in mind Lemma 4.1 and Remark 2.1 in the rest of this section we will ignore the case \( \lambda = -\frac{1}{5} \) and will denote by \( Q \) the quadric given by equation

\[
(4.3) \quad \sigma_2(x_0, \ldots, x_4) = 0,
\]

i.e. by equation (4.1) with \( \lambda = 0 \). Every irreducible reduced \( A_5 \)-invariant intersection of \( Q \) with a quartic in \( \mathbb{P}^4 \) is given by equations (4.3) and

\[
(4.4) \quad \sigma_4(x_0, \ldots, x_4) + 4\mu \sigma_3(x_0, \ldots, x_4) \sigma_1(x_0, \ldots, x_4) + \nu \sigma_1(x_0, \ldots, x_4)^4 = 0
\]

for some \( \mu, \nu \in \mathbb{C} \). We will denote such intersection by \( S_{\mu,\nu} \).

**Remark 4.2.** Double covers of \( Q \) branched over \( S_{\mu,\nu} \) form a two-parameter family of double quadrics. Thus, Lemma 3.1 and Remark 3.3 imply Theorem 1.2(i).
Writing down partial derivatives, we see that a point $P \in \mathbb{P}^4$ is a singular point of the surface $S_{\mu,\nu}$ if and only if the form $\sigma_2$ and the determinants of the matrices

$$M_{ij} = \begin{pmatrix} x_i^3 + \mu \sigma_3 + 3\mu x_i^2 \sigma_1 + \nu \sigma_1^3 & x_i^3 + \mu \sigma_3 + 3\mu x_i^2 \sigma_1 + \nu \sigma_1^3 \\ x_j & x_j \end{pmatrix}$$

for all $0 \leq i < j \leq 4$ vanish at $P$.

**Lemma 4.3.** Let $P = (1 : x_1 : x_2 : x_3 : x_4)$ be a singular point of a surface $S_{\mu,\nu}$. Then one of the following cases occurs.

(i) Up to permutation of coordinates one has $P = (1 : 1 : i : i : 0)$.

(ii) Up to permutation of coordinates one has $P = (1 : 0 : 0 : 0 : i)$.

(iii) Up to permutation of coordinates one has $P = (1 : 1 : 1 : a : b)$, for some $a, b \in \mathbb{C}$ such that $a^2 + b^2 + 3 = 0$.

(iv) Up to permutation of coordinates one has $P = (1 : a : a : b : b)$, for some $a, b \in \mathbb{C}$ such that $2a^2 + 2b^2 + 1 = 0$.

**Proof.** Rewrite (4.5) as

$$M_{0,i} = (x_i - 1) \cdot ((\sigma_3(P) - 3x_i \sigma_1(P))\mu + \sigma_1(P)^3 \nu - x_i(x_i + 1)).$$

We are going to show that $|\{1, x_1, x_2, x_3, x_4\}| \leq 3$, i.e. the number of non-equal coordinates that are also not equal to 1 among $x_1, x_2, x_3, x_4$ is at most 2. Indeed, suppose that $x_1, x_2,$ and $x_3$ are three non-equal coordinates that are also not equal to 1. Consider the following cases.

- One has $\sigma_1(P) = \sigma_3(P) = 0$. From the vanishing of the determinant of $M_{0,i}$ one gets either $x_i = 1$ or $x_i(x_i + 1) = 0$ for every $i = 1, \ldots, 4$. This means that $x_i \in \{0, 1, -1\}$, so that condition (4.3) fails.

- One has $\sigma_1(P) = 0, \sigma_3(P) \neq 0$. From the vanishing of the determinant of $M_{0,i}$ one gets

$$\mu = \frac{x_i(x_i + 1)}{\sigma_3(P)}$$

for every $i = 1, 2, 3$. This implies

$$0 = x_i(x_i + 1) - x_j(x_j + 1) = (x_i - x_j)(x_i + x_j + 1)$$

for $i, j = 1, 2, 3$, so that $x_i + x_j + 1 = 0$, which gives a contradiction with the assumption that all $x_k, k = 1, 2, 3,$ are different.

- One has $\sigma_1(P) \neq 0$. From the equalities $\det M_{0,i} = \det M_{0,j}$ for $i, j = 1, 2, 3$ one gets

$$\mu = \frac{-x_i + x_j + 1}{3\sigma_1(P)}.$$ 

Hence

$$x_1 + x_2 + 1 = x_1 + x_3 + 1 = x_2 + x_3 + 1,$$

which again gives a contradiction with the condition that all $x_k, k = 1, 2, 3,$ are different.

Thus one can assume that $x_1 = a, x_2 = b,$ and each of the coordinates $x_3$ and $x_4$ equals either 1, or $a$, or $b$. This together with equation $\sigma_2(P) = 0$ gives possibilities (i)–(iv) for $P$.  

\[\square\]
Corollary 4.4. Let \( P = (x_0, \ldots, x_4) \) be a point such that \( \sigma_1(P) = 0 \). Then \( P \) is not a singular point of any surface \( S_{\mu, \nu} \).

Proof. Suppose that \( P \) is a singular point of a surface \( S_{\mu, \nu} \). Recalling equations (4.3) and (4.4), we see that

\[
\sigma_1(P) = \sigma_2(P) = \sigma_4(P) = 0.
\]

On the other hand, the point \( P \) must be of one of the types listed in Lemma 4.3. For the cases (i) and (ii) the assumption \( \sigma_1(P) = 0 \) fails. In the remaining two cases it is straightforward to check that the system of equations (4.8) has no solutions. \( \square \)

Corollary 4.5. Any surface \( S_{\mu, \nu} \) has isolated singularities.

Proof. If \( S_{\mu, \nu} \) has non-isolated singularities, then there is a singular point \( P \) of \( S_{\mu, \nu} \) such that \( \sigma_1(P) = 0 \). The latter is impossible by Corollary 4.4. \( \square \)

We introduce the following notation:

- \( \Sigma_5^+ \) denotes the \( S_5 \)-orbit of the point \( P = (1 : 1 : 1 : 2i) \);
- \( \Sigma_5^- \) denotes the \( S_5 \)-orbit of the point \( P = (1 : 1 : 1 : -2i) \);
- \( \Sigma_{10}^+ \) denotes the \( S_5 \)-orbit of the point \( P = (1 : 1 : 1 : \sqrt{\frac{2}{2i}} : \sqrt{\frac{2}{2i}}) \);
- \( \Sigma_{10}^- \) denotes the \( S_5 \)-orbit of the point \( P = (1 : 1 : 1 : -\sqrt{\frac{2}{2i}} : -\sqrt{\frac{2}{2i}}) \);
- \( \Sigma_{20} \) denotes the \( S_5 \)-orbit of the point \( P = (1 : 0 : 0 : 0 : 1) \);
- \( \Sigma_{20}^{a,b} \) denotes the \( S_5 \)-orbit of the point \( P = (1 : 1 : 1 : a : b) \), where \( a, b \in \mathbb{C} \) are such that \( a \neq b \), \( a \neq 1 \), and \( b \neq 1 \); note that the pairs \((a, b)\) and \((b, a)\) give the same \( S_5 \)-orbit, and thus only a non-ordered pair is important here;
- \( \Sigma_{30} \) denotes the \( S_5 \)-orbit of the point \( P = (1 : 1 : i : 1 : 0) \);
- \( \Sigma_{30}^{a,b} \) denotes the \( S_5 \)-orbit of the point \( P = (1 : a : a : b : b) \), where \( a, b \in \mathbb{C} \) are such that \( a \neq b \), \( a \neq 1 \), and \( b \neq 1 \); as before, the pairs \((a, b)\) and \((b, a)\) give the same \( S_5 \)-orbit, and thus only a non-ordered pair is important here.

Remark 4.6. One has

\[
|\Sigma_k^+| = |\Sigma_k^-| = k, \quad |\Sigma_k| = k, \quad |\Sigma_k| = k.
\]

Moreover, one can check that apart from the unique \( S_5 \)-fixed point in \( \mathbb{P}^4 \) that corresponds to the trivial subrepresentation (and that is not contained in the quadric \( Q \)), the orbits \( \Sigma_{15}^+, \Sigma_{15}^-, \Sigma_{10}^+, \Sigma_{10}^-, \Sigma_{20}, \Sigma_{20}^{a,b}, \Sigma_{30}, \) and \( \Sigma_{30}^{a,b} \) are the only \( S_5 \)-orbits of length less than 60 in \( \mathbb{P}^4 \). Note also that \( \Sigma_{5}^+, \Sigma_{5}^-, \Sigma_{10}^+, \Sigma_{10}^-, \) and \( \Sigma_{20} \) are contained in the closure of the union of the \( S_5 \)-orbits \( \Sigma_{20}^{a,b} \). Similarly, \( \Sigma_{5}^+, \Sigma_{5}^-, \Sigma_{10}^+, \Sigma_{10}^-, \) and \( \Sigma_{30} \) are contained in the closure of the union of the \( S_5 \)-orbits \( \Sigma_{30}^{a,b} \).

Corollary 4.7. Let \( P = (x_0, \ldots, x_4) \in S_{\mu, \nu} \) be a point such that \( \sigma_1(P) = 0 \). Then \( P \) is singular on \( S_{\mu, \nu} \) if and only if one of the following cases occurs.

(i) Up to permutation of coordinates one has \( P = (1 : 0 : 0 : 0 : i) \), so that \( P \in \Sigma_{20} \); in this case

\[
\mu = -\frac{1}{3}, \quad \nu = -\frac{1}{6}.
\]

(ii) Up to permutation of coordinates one has \( P = (1 : 1 : i : i : 0) \), so that \( P \in \Sigma_{30} \); in this case

\[
\mu = -\frac{1}{6}, \quad \nu = -\frac{1}{48}.
\]
(iii) Up to permutation of coordinates one has \( P = (1 : 1 : 1 : 1 : 2i) \), so that \( P \in \Sigma^+_5 \). Then \( P \) is a singular point of a surface \( S_{\mu,\nu} \) if and only if

\[
\nu = \left( \frac{8}{25} + \frac{6}{25}i \right) \mu + \left( \frac{7}{500} + \frac{6}{125}i \right).
\]

(iii') Up to permutation of coordinates one has \( P = (1 : 1 : 1 : 1 : -2i) \), so that \( P \in \Sigma^-_5 \). Then \( P \) is a singular point of a surface \( S_{\mu,\nu} \) if and only if

\[
\nu = \left( \frac{8}{25} - \frac{6}{25}i \right) \mu + \left( \frac{7}{500} - \frac{6}{125}i \right).
\]

(iv) Up to permutation of coordinates one has \( P = \left(1 : 1 : 1 : \sqrt{\frac{3}{2}}i : \sqrt{\frac{3}{2}}i \right) \), so that \( P \in \Sigma^+_{10} \). Then \( P \) is a singular point of a surface \( S_{\mu,\nu} \) if and only if

\[
\nu = \left( \frac{8}{25} + \frac{2\sqrt{6}}{75}i \right) \mu + \frac{23}{750} + \frac{2\sqrt{6}}{375}i.
\]

(iv') Up to permutation of coordinates one has \( P = \left(1 : 1 : 1 : -\sqrt{\frac{3}{2}}i : -\sqrt{\frac{3}{2}}i \right) \), so that \( P \in \Sigma^-_{10} \). Then \( P \) is a singular point of a surface \( S_{\mu,\nu} \) if and only if

\[
\nu = \left( \frac{8}{25} - \frac{2\sqrt{6}}{75}i \right) \mu + \frac{23}{750} - \frac{2\sqrt{6}}{375}i.
\]

(v) Up to permutation of coordinates one has \( P = (1 : 1 : 1 : a : b) \) or \( P = (1 : a : a : b : b) \) with \( a \neq 1, b \neq 1, a \neq b \); in this case

\[
\mu = -\frac{a + b + 1}{3\sigma_1(P)}, \quad \nu = \frac{\sigma_3(P)(a + b + 1) - 3ab\sigma_1(P)}{3\sigma_1(P)^4}.
\]

Proof. The value of \( \mu \) is given by (4.7), and the value of \( \nu \) can be found from (4.6). □

Using Corollary 4.7, we derive the following facts.

**Corollary 4.8.** Suppose that \( P = (1 : 1 : 1 : a : b) \) and \( \sigma_2(P) = a^2 + b^2 + 3 = 0 \). Suppose also that \( a \neq 1, b \neq 1, a \neq b \), i.e. one has \( P \in \Sigma^{a,b}_{20} \). Then one of the following cases occurs.

(i) Up to permutation of coordinates one has

\[
P = \left(1 : 1 : 1 : -\frac{3}{2} + \frac{\sqrt{15}}{2}i : -\frac{3}{2} - \frac{\sqrt{15}}{2}i \right).
\]

Then \( \sigma_1(P) = 0 \), and \( P \) is not a singular point of any surface \( S_{\mu,\nu} \) (cf. Corollary 4.4).

(ii) One has \( \sigma_1(P) \neq 0 \). Then \( P \) is a singular point of a surface \( S_{\mu,\nu} \) if and only if

\[
\mu = -\frac{a + b + 1}{3(a + b + 3)}, \quad \nu = \frac{(a^3 + b^3 + 3)(a + b + 1) - 3ab(a + b + 3)}{3(a + b + 3)^4}.
\]

**Remark 4.9.** In case (ii) of Corollary 4.8 one has

\[
\nu = -\frac{405}{4} \mu^4 - 81\mu^3 - 27\mu^2 - \frac{1}{4}.
\]
Corollary 4.10. Suppose that \( P = (1 : a : a : b : b) \), and \( \sigma_2(P) = 2a^2 + 2b^2 + 1 = 0 \). Suppose also that \( a \neq 1, b \neq 1, \) and \( a \neq b, \) i.e. one has \( P \in \Sigma_{30}^{a,b} \). Then one of the following cases occurs.

(i) Up to permutation of coordinates one has

\[
P = \left( -1 : \frac{1}{4} + \frac{\sqrt{5}}{4}i : \frac{1}{4} + \frac{\sqrt{5}}{4}i : \frac{1}{4} - \frac{\sqrt{5}}{4}i : \frac{1}{4} - \frac{\sqrt{5}}{4}i \right).
\]

Then \( \sigma_1(P) = 0 \), and \( P \) is not a singular point of any surface \( S_{\mu,\nu} \) (cf. Corollary 4.4).

(ii) One has \( \sigma_1(P) \neq 0 \). Then \( P \) is a singular point of a surface \( S_{\mu,\nu} \) if and only if

\[
\mu = -\frac{a + b + 1}{3(2a + 2b + 1)}, \quad \nu = \frac{(2a^3 + 2b^3 + 1)(a + b + 1) - 3ab(2a + 2b + 1)}{3(2a + 2b + 1)^4}.
\]

Remark 4.11. In case (ii) of Corollary 4.10 one has

\[
\nu = 405\mu^4 + 324\mu^3 + 99\mu^2 + 14\mu + \frac{3}{4}.
\]

Now we are ready to give a complete description of singular loci of the surfaces \( S_{\mu,\nu} \) and the corresponding double covers. To do this we will need some additional notation.

- Let \( \mu_{5,1}^+ \) and \( \mu_{5,2}^+ \) be two different roots of the equation
  \[
  894825\mu^2 + (126510 + 149670i)\mu - 1249 + 9382i = 0
  \]
  and define \( \nu_{5,1}^+ \) and \( \nu_{5,2}^+ \) by formula (4.9).
- Let \( \mu_{5}^{ab,+} = -\frac{1}{5} - \frac{1}{15}i, \quad \nu_{5}^{ab,+} = -\frac{17}{500} - \frac{8}{375}i. \)
- Let \( \mu_{5,1}^- \) and \( \mu_{5,2}^- \) be two different roots of the equation
  \[
  894825\mu^2 + (126510 - 149670i)\mu - 1249 - 9382i = 0
  \]
  and define \( \nu_{5,1}^- \) and \( \nu_{5,2}^- \) by formula (4.10).
- Let \( \mu_{5}^{ab,-} = -\frac{1}{5} + \frac{1}{15}i, \quad \nu_{5}^{ab,-} = -\frac{17}{500} + \frac{8}{375}i. \)
- Let \( \mu_{10,1}^+ \) and \( \mu_{10,2}^+ \) be two different roots of the equation
  \[
  216090\mu^2 + (34140 + 1485\sqrt{6}i)\mu + 1556 + 93\sqrt{6}i = 0
  \]
  and define \( \nu_{10,1}^+ \) and \( \nu_{10,2}^+ \) by formula (4.11).
- Let \( \mu_{20,10}^{+} = -\frac{1}{5} - \frac{2\sqrt{5}}{45}i, \quad \nu_{20,10}^{+} = -\frac{59}{2250} - \frac{16\sqrt{5}}{1125}i. \)
- Let \( \mu_{30,10}^{+} = -\frac{1}{5} + \frac{2\sqrt{5}}{45}i, \quad \nu_{30,10}^{+} = -\frac{59}{2250} + \frac{16\sqrt{5}}{1125}i. \)
- Let \( \mu_{10,1}^- \) and \( \mu_{10,2}^- \) be two different roots of the equation
  \[
  216090\mu^2 + (34140 - 1485\sqrt{6}i)\mu + 1556 - 93\sqrt{6}i = 0
  \]
  and define \( \nu_{10,1}^- \) and \( \nu_{10,2}^- \) by formula (4.12).
- Let \( \mu_{20,10}^{-} = -\frac{1}{5} + \frac{2\sqrt{5}}{45}i, \quad \nu_{20,10}^{-} = -\frac{59}{2250} + \frac{16\sqrt{5}}{1125}i. \)
- Let \( \mu_{30,10}^{-} = -\frac{1}{5} - \frac{2\sqrt{5}}{45}i, \quad \nu_{30,10}^{-} = -\frac{59}{2250} - \frac{16\sqrt{5}}{1125}i. \)
- Let \( a_{20,i} \) be the roots of the equation
  \[
  (4.14) \quad 4a_{20,i}^4 + 16a_{20,i}^7 + 56a_{20,i}^6 + 116a_{20,i}^5 + 217a_{20,i}^4 + 266a_{20,i}^3 + 257a_{20,i}^2 + 172a_{20,i} + 52 = 0
  \]
and put
\[
b_{20,i} = \frac{98}{215}a_{20,i}^7 + \frac{314}{215}a_{20,i}^6 + \frac{1094}{215}a_{20,i}^5 + \frac{376}{43}a_{20,i}^4 + \frac{1401}{86}a_{20,i}^3 + 6\frac{67}{430}a_{20,i}^2 + \frac{2874}{215}a_{20,i} + \frac{1271}{215}.
\]
Note that \(b_{20,i}\) is also a root of equation (4.14), and there are only four possible non-ordered pairs \((a_{20,i}, b_{20,i})\); thus we will assume that \(i = 1, \ldots, 4\). Define \(\mu_{20,i}\) and \(\nu_{20,i}\) by formula (4.13).

- Let \(a_{30,i}\) be the roots of equation
\[
(4.15) \quad 512a_{30,i}^8 + 2048a_{30,i}^7 + 4608a_{30,i}^6 + 5952a_{30,i}^5 + 5698a_{30,i}^4 + 3740a_{30,i}^3 + 1765a_{30,i}^2 + 602a_{30,i} + 163 = 0
\]
and put
\[
b_{30,i} = -\frac{22544}{12639}a_{30,i}^7 - \frac{26992}{4213}a_{30,i}^6 - \frac{174304}{12639}a_{30,i}^5 - \frac{204442}{67408}a_{30,i}^4 - \frac{1036395}{404448}a_{30,i}^3 - \frac{1670095}{12639}a_{30,i}^2 - \frac{1606787}{404448}a_{30,i} - \frac{480445}{404448}.
\]
Note that \(b_{30,i}\) is also a root of equation (4.15), and there are only four possible non-ordered pairs \((a_{30,i}, b_{30,i})\); thus we will assume that \(i = 1, \ldots, 4\). Define \(\mu_{30,i}\) and \(\nu_{30,i}\) by formula (4.13).

We also use the following notation.

- Let
\[
a_{20}^+ = -\frac{19 + 2\sqrt{95}}{26} + \frac{30 + 3\sqrt{95}}{26}i, \quad b_{20}^+ = -\frac{19 + 2\sqrt{95}}{26} - \frac{30 + 3\sqrt{95}}{26}i.
\]
- Let
\[
a_{20}^- = -\frac{19 + 2\sqrt{95}}{26} + \frac{30 + 3\sqrt{95}}{26}i, \quad b_{20}^- = -\frac{19 + 2\sqrt{95}}{26} + \frac{30 - 3\sqrt{95}}{26}i.
\]
- Let \(a_{20,1}^+\) and \(b_{20,1}^+\) be two different roots of the equation
\[
x^2 + (1 + \sqrt{6}i)x + \sqrt{6} - 1 = 0.
\]
- Let \(a_{20,2}^+\) and \(b_{20,2}^+\) be two different roots of the equation
\[
49x^2 + (-63 + 35\sqrt{6}i)x - 45\sqrt{6}i + 39 = 0.
\]
- Let \(a_{20,1}^-\) and \(b_{20,1}^-\) be two different roots of the equation
\[
x^2 + (1 - \sqrt{6}i)x - \sqrt{6}i - 1 = 0.
\]
- Let \(a_{20,2}^-\) and \(b_{20,2}^-\) be two different roots of the equation
\[
49x^2 + (-63 - 35\sqrt{6}i)x + 45\sqrt{6}i + 39 = 0.
\]
- Let \(a_{30,1}^+\) and \(b_{30,1}^+\) be two different roots of the equation
\[
4x^2 + (4 - 4i)x - 4i + 1 = 0.
\]
- Let \(a_{30,2}^+\) and \(b_{30,2}^+\) be two different roots of the equation
\[
676x^2 - (52 + 260i)x + 20i + 121 = 0.
\]
Let $a_{30,1}^+$ and $b_{30,1}^-$ be two different roots of the equation
\[ 4x^2 + (4 + 4i)x + 4i + 1 = 0. \]
Let $a_{30,2}^+$ and $b_{30,2}^-$ be two different roots of the equation
\[ 676x^2 - (52 - 260i)x - 20i + 121 = 0. \]
Let $a_{30}^+$ and $b_{30}^-$ be two different roots of the equation
\[ 98x^2 + (14 + 35\sqrt{6}i)x + 5\sqrt{6}i - 12 = 0. \]
Let $a_{30}^-$ and $b_{30}^+$ be two different roots of the equation
\[ 98x^2 + (14 - 35\sqrt{6}i)x - 5\sqrt{6}i - 12 = 0. \]

We denote by $X_{\mu,\nu}$ the double cover of $Q$ branched over $S_{\mu,\nu}$. Recall that the singularities of $X_{\mu,\nu}$ are ordinary double points if and only if the same holds for the singularities of $S_{\mu,\nu}$. Define the \textit{defect} $\delta(X_{\mu,\nu})$ of the threefold $X_{\mu,\nu}$ as
\[ \delta(X_{\mu,\nu}) = |\text{Sing}(S_{\mu,\nu})| + \dim H^0(\mathbb{P}^4, \mathcal{I}(3)) - 35, \]
where $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^4}$ is the ideal sheaf of the set $\text{Sing}(S_{\mu,\nu})$. Corollaries 4.7, 4.8, and 4.10 provide a complete information about singularities of the surfaces $S_{\mu,\nu}$ and thus also of the threefolds $X_{\mu,\nu}$. Note that given a singular point $P$ of $S_{\mu,\nu}$ one can check whether $P$ is an ordinary double point of $S_{\mu,\nu}$ by making an appropriate (analytic) change of coordinates so that $Q$ is given in a neighborhood of $P$ by vanishing of some new coordinate $z$, and then finding the rank of the matrix of the second derivatives of the restriction of the quartic (4.1) to the subspace $z = 0$. Collecting all this information, we obtain the following result.

**Corollary 4.12.** Suppose that the surface $S_{\mu,\nu}$ is singular. Then one of the cases given in Table 7 occurs. In the first column of Table 7 we place the number of singular points $|\text{Sing}(S_{\mu,\nu})|$. In the second column we describe the set $\text{Sing}(S_{\mu,\nu})$ itself, namely, we list the $\mathfrak{S}_5$-orbits whose union gives $\text{Sing}(S_{\mu,\nu})$. Note that in the 12th line we refer to non-ordered pairs $a, b$ such that $a^2 + b^2 + 3 = 0$, $a \neq 1$, $b \neq 1$, $a \neq b$ and $a + b + 3 \neq 0$; similarly, in the 20th line we refer to non-ordered pairs $a, b$ such that $2a^2 + 2b^2 + 1 = 0$, $a \neq 1$, $b \neq 1$, $a \neq b$, and $2a + 2b + 1 \neq 0$. The corresponding pairs $(\mu, \nu)$ are in the third column. In the fourth column there are all pairs $(\mu, \nu)$ for which the singularities of $S_{\mu,\nu}$ are not just ordinary double points. Finally, in the last column there is the defect $\delta(X_{\mu,\nu})$.

**Table 1: Singularities of $S_{\mu,\nu}$**

| \# | $\text{Sing}(S_{\mu,\nu})$ | $(\mu, \nu)$ | non-ODP $(\mu, \nu)$ | $\delta$ |
|----|-----------------|-------------|-----------------|--------|
| 5  | $\Sigma_5^+$   | $(\mu, (\frac{9}{25} + \frac{6}{25}i) \mu + \frac{7}{500} + \frac{6}{125}i)$ | $(\mu_{5,1}^+, \nu_{5,1}^+), (\mu_{5,2}^+, \nu_{5,2}^+), (\mu_{ab}, \nu_{ab})$ | 0 |
| 5  | $\Sigma_5^-$   | $(\mu, (\frac{8}{25} - \frac{6}{25}i) \mu + \frac{7}{500} - \frac{6}{125}i)$ | $(\mu_{5,1}^-, \nu_{5,1}^-), (\mu_{5,2}^-, \nu_{5,2}^-), (\mu_{ab}, \nu_{ab})$ | 0 |
| 10 | $\Sigma_5^+, \Sigma_5^-$ | $(- \frac{1}{5}, - \frac{1}{25})$ | $\emptyset$ | 0 |
### Table 1: Singularities of $S_{\mu, \nu}$

| #  | $\text{Sing}(S_{\mu, \nu})$ | $(\mu, \nu)$ | non-ODP $(\mu, \nu)$ | $\delta$ |
|----|-----------------------------|---------------|----------------------|--------|
| 10 | $\Sigma^+_{10}$            | $(\mu, \left(\frac{8}{25} + \frac{2\sqrt{75}}{75}\right) \mu + \frac{23}{750} + \frac{2\sqrt{75}}{375} i)$ | $(\mu^+_{10,1}, \nu^+_{10,1}), (\mu^+_{10,2}, \nu^+_{10,2}), (\mu^+_{20 \to 10}, \nu^+_{20 \to 10}), (\mu^+_{30 \to 10}, \nu^+_{30 \to 10})$ | 0      |
| 10 | $\Sigma^-_{10}$            | $(\mu, \left(\frac{8}{25} - \frac{2\sqrt{75}}{75}\right) \mu + \frac{23}{750} - \frac{2\sqrt{75}}{375} i)$ | $(\mu^-_{10,1}, \nu^-_{10,1}), (\mu^-_{10,2}, \nu^-_{10,2}), (\mu^-_{20 \to 10}, \nu^-_{20 \to 10}), (\mu^-_{30 \to 10}, \nu^-_{30 \to 10})$ | 0      |
| 15 | $\Sigma^+_{10}, \Sigma^+_{10}$ | $(-\frac{1}{5} - \frac{(9 + \sqrt{6})}{120} i, \frac{-16 + \sqrt{6}}{500} + \frac{-9 + \sqrt{6}}{375} i)$ | $\emptyset$ | 0      |
| 15 | $\Sigma^-_{10}, \Sigma^+_{10}$ | $(-\frac{1}{5} + \frac{(9 + \sqrt{6})}{120} i, \frac{-16 - \sqrt{6}}{500} + \frac{-9 + \sqrt{6}}{375} i)$ | $\emptyset$ | 0      |
| 15 | $\Sigma^-_{10}, \Sigma^+_{10}$ | $(-\frac{1}{5} + \frac{(9 - \sqrt{6})}{120} i, \frac{-16 - \sqrt{6}}{500} + \frac{-9 - \sqrt{6}}{375} i)$ | $\emptyset$ | 0      |
| 15 | $\Sigma^-_{10}, \Sigma^+_{10}$ | $(-\frac{1}{5} + \frac{(9 + \sqrt{6})}{120} i, \frac{-16 + \sqrt{6}}{500} + \frac{-9 + \sqrt{6}}{375} i)$ | $\emptyset$ | 0      |
| 20 | $\Sigma^+_{10}, \Sigma^-_{10}$ | $(-\frac{1}{5}, -\frac{1}{30})$ | $\emptyset$ | 0      |
| 20 | $\Sigma_{20}$              | $(-\frac{1}{3}, \frac{1}{6})$ | $\emptyset$ | 0      |
| 20 | $\Sigma^{a,b}_{20}$        | $\left(\frac{(a+b+1)}{3(a+b+3)}, \frac{(a^3+b^3+3(a+b+1)-3ab(a+b+3))}{3(a+b+3)^2}\right)$ | $(\mu_{20,1}, \nu_{20,1}), 1 \leq i \leq 4$ | 0      |
| 25 | $\Sigma^+_{5}, \Sigma^+_{20}, \Sigma^+_{20}$ | $(-\frac{1}{5} + \frac{1}{15} i, -\frac{49}{500} + \frac{8}{125} i)$ | $\emptyset$ | 0      |
| 25 | $\Sigma^-_{5}, \Sigma^+_{20}, \Sigma^+_{20}$ | $(-\frac{1}{5} - \frac{1}{5} i, -\frac{49}{500} - \frac{8}{125} i)$ | $\emptyset$ | 0      |
| 30 | $\Sigma^-_{10}, \Sigma^+_{20,1}, \Sigma^+_{20}$ | $(-\frac{1}{5} + \frac{\sqrt{6}}{13} i, -\frac{11}{250} + \frac{8\sqrt{6}}{375} i)$ | $\emptyset$ | 0      |
| 30 | $\Sigma^-_{10}, \Sigma^+_{20,2}, \Sigma^+_{20}$ | $(-\frac{1}{5} + \frac{\sqrt{6}}{45} i, -\frac{83}{2250} + \frac{8\sqrt{6}}{1125} i)$ | $\emptyset$ | 0      |
| 30 | $\Sigma^-_{10}, \Sigma^+_{20,1}, \Sigma^+_{20}$ | $(-\frac{1}{5} - \frac{\sqrt{6}}{13} i, -\frac{11}{250} - \frac{8\sqrt{6}}{375} i)$ | $\emptyset$ | 0      |
| 30 | $\Sigma^-_{10}, \Sigma^+_{20,2}, \Sigma^+_{20}$ | $(-\frac{1}{5} - \frac{\sqrt{6}}{45} i, -\frac{83}{2250} - \frac{8\sqrt{6}}{1125} i)$ | $\emptyset$ | 0      |
| 30 | $\Sigma^+_{30}$            | $(-\frac{1}{6}, \frac{1}{6})$ | $\emptyset$ | 5      |
| 30 | $\Sigma^+_{30}$            | $\left(\frac{(a+b+1)}{3(2a+2b+1)}, \frac{(2a^3+2b^3+1)(a+b+1)-3ab(2a+2b+1)}{3(2a+2b+1)^2}\right)$ | $(\mu_{30,1}, \nu_{30,1}), 1 \leq i \leq 4$ | 5      |
| 35 | $\Sigma^+_{5}, \Sigma^+_{30,1}, \Sigma^+_{30,1}$ | $(-\frac{2}{15} + \frac{1}{15} i, -\frac{67}{1500} + \frac{14}{375} i)$ | $\emptyset$ | 5      |
| 35 | $\Sigma^-_{5}, \Sigma^+_{30,2}, \Sigma^+_{30,2}$ | $(-\frac{4}{15} + \frac{1}{15} i, -\frac{131}{1500} + \frac{2}{375} i)$ | $\emptyset$ | 5      |
| 35 | $\Sigma^-_{5}, \Sigma^+_{30,1}, \Sigma^+_{30,1}$ | $(-\frac{2}{15} - \frac{1}{15} i, -\frac{67}{1500} - \frac{14}{375} i)$ | $\emptyset$ | 5      |
| 35 | $\Sigma^-_{5}, \Sigma^+_{30,2}, \Sigma^+_{30,2}$ | $(-\frac{4}{15} - \frac{1}{15} i, -\frac{131}{1500} - \frac{2}{375} i)$ | $\emptyset$ | 5      |
| 40 | $\Sigma^+_{10}, \Sigma^+_{30}, \Sigma^+_{30}$ | $(-\frac{1}{5} - \frac{\sqrt{6}}{30} i, -\frac{7}{250} + \frac{4\sqrt{6}}{375} i)$ | $\emptyset$ | 10     |
| 40 | $\Sigma^-_{10}, \Sigma^+_{30}, \Sigma^+_{30}$ | $(-\frac{1}{5} + \frac{\sqrt{6}}{30} i, -\frac{7}{250} + \frac{4\sqrt{6}}{375} i)$ | $\emptyset$ | 10     |
Remark 4.13. Let \( C_5^+, C_5^-, C_{10}, C_{20}, C_{30} \) be curves in the plane \( \text{Spec } \mathbb{C}[\mu, \nu] \cong \mathbb{A}^2 \) parameterizing pairs of \((\mu, \nu)\) that correspond to surfaces \( S_{\mu, \nu} \) with singularities at the points of the \( \mathfrak{S}_5 \)-orbits \( \Sigma_5^+, \Sigma_5^- \) \( \Sigma_{10}, \Sigma_{20}, \Sigma_{30} \), and \( \Sigma_{5a}^b \), respectively. Let \( C_{20}, C_{30} \) be closures of \( C_{20}^0, C_{30}^0 \) in \( \text{Spec } \mathbb{C}[\mu, \nu] \), respectively. Then as suggested by notation one has

\[
\begin{align*}
C_5^+ \cap C_{20} &= C_5^+ \cap C_{30} = \left( \mu_5^{ab,+}, \nu_5^{ab,+} \right), \\
C_5^- \cap C_{20} &= C_5^- \cap C_{30} = \left( \mu_5^{ab,-}, \nu_5^{ab,-} \right), \\
C_{10}^+ \cap C_{20} &= \left( \mu_{20 \to 10}^{+}, \nu_{20 \to 10}^{+} \right), \\
C_{10}^+ \cap C_{30} &= \left( \mu_{30 \to 10}^{+}, \nu_{30 \to 10}^{+} \right), \\
C_{10}^- \cap C_{20} &= \left( \mu_{20 \to 10}^{-}, \nu_{20 \to 10}^{-} \right), \\
C_{10}^- \cap C_{30} &= \left( \mu_{30 \to 10}^{-}, \nu_{30 \to 10}^{-} \right).
\end{align*}
\]

Corollary 4.7, Remark 4.9, and Remark 4.11 show that the locus of pairs \((\mu, \nu)\) corresponding to singular threefolds \( X_{\mu, \nu} \) is a union of four lines and two quartic curves in \( \text{Spec } \mathbb{C}[\mu, \nu] \).

Remark 4.14. The affine plane \( \mathbb{A}^2 = \text{Spec } \mathbb{C}[\mu, \nu] \) is a parameter space for the threefolds \( X_{\mu, \nu} \), but not a moduli space: there do exist different pairs \((\mu, \nu)\) and \((\mu', \nu')\) such that \( X_{\mu, \nu} \cong X_{\mu', \nu'} \), and moreover the latter isomorphism is \( \mathfrak{A}_5 \)-equivariant (see Remark 5.2 below for such examples). However, there is still a two-parameter family of threefolds \( X_{\mu, \nu} \) and a one-parameter family of singular threefolds \( X_{\mu, \nu} \) up to isomorphism. Indeed, every isomorphism between \( X_{\mu, \nu} \) and \( X_{\mu', \nu'} \) gives rise to an element of \( \text{Aut}(\mathbb{P}^4) \cong \text{PGL}_5(\mathbb{C}) \) that gives an isomorphism \( S_{\mu, \nu} \cong S_{\mu', \nu'} \) provided that \( S_{\mu, \nu} \) and \( S_{\mu', \nu'} \) have sufficiently nice singularities (which is the case for a general singular \( S_{\mu, \nu} \) by Corollary 4.12). Choose a surface \( S_{\mu, \nu} \) that has at most ordinary double points as singularities, and suppose that there is a one-parameter family of automorphisms

\[ A_t \in \text{PGL}_5(\mathbb{C}) \]

such that \( A_t(S_{\mu, \nu}) = S_{\mu_t, \nu_t} \) for some \((\mu_t, \nu_t)\). Then the action of \( \mathfrak{A}_5 \) is normalized by \( A_t \) since the automorphism group of \( S_{\mu, \nu} \) is finite, and thus \( A_t \) actually commutes with the action of \( \mathfrak{A}_5 \) since the automorphism group of \( \mathfrak{A}_5 \) itself is finite. On the other hand, since \( I \oplus W_4 \) is a sum of two irreducible \( \mathfrak{A}_5 \)-representations, there is only one one-parameter family of automorphisms of \( \mathbb{P}^4 \) that commute with the action of \( \mathfrak{A}_5 \). Moreover, we already know such one-parameter family, i.e. the family of coordinate changes used in the proof of Lemma 4.1. But a general automorphism from this family does not preserve the quadric \( Q \), and thus does not map \( S_{\mu, \nu} \) to any of the surfaces \( S_{\mu', \nu'} \) since \( Q \) is the unique quadric passing through the surface \( S_{\mu', \nu'} \).

Remark 4.15. Vanishing of the defect for the threefolds \( X_{\mu, \nu} \) described in the first five lines of Table 1 can be obtained not only by a direct computation, but also from [29 Proposition 1.5].

Now we are ready to make conclusions on (non-)rationality of the threefolds \( X_{\mu, \nu} \). It follows from [11 Theorem 2] (or rather from the proof of this theorem) that

\[ \text{rk } \text{Cl}(X_{\mu, \nu}) = 1 + \delta(X_{\mu, \nu}) \]

provided that \( X_{\mu, \nu} \) has only ordinary double points as singularities. On the other hand, by [29 Theorem 1.1] the variety \( X_{\mu, \nu} \) is non-rational provided that the singularities of \( X_{\mu, \nu} \) are ordinary double points, and \( \text{rk } \text{Cl}(X_{\mu, \nu}) = 1 \). Therefore, Corollary 4.12 implies the following result.
Corollary 4.16. The varieties $X_{\mu,\nu}$ with $|\text{Sing}(X_{\mu,\nu})| \neq 30$ listed in Table 1 are non-rational up to a finite number of (possible) exceptions.

Proof. Note that there is only a finite number of pairs $(\mu, \nu)$ such that the singularities of $X_{\mu,\nu}$ are at worse than ordinary double points (they are listed in the fourth column of Table 1). Moreover, among $(\mu, \nu)$ such that $|\text{Sing}(X_{\mu,\nu})| \neq 30$ there is only a finite number of cases when the defect of $X_{\mu,\nu}$ does not vanish (see the fifth column of Table 1). □

We will see later in Corollary 6.6 that the varieties $X_{\mu,\nu}$ with $|\text{Sing}(X_{\mu,\nu})| = 30$ are non-rational up to a finite number of possible exceptions as well.

5. Artin–Mumford-type double covers

In this section we show that certain double quadrics with an action of the group $\mathfrak{A}_5$ are not stably rational. We use the notation of §4.

Note that

$$\text{Sym}^2(W_4^\vee) \cong I \oplus W_4 \oplus W_5.$$  

Let $W$ be the (unique) subrepresentation of $\text{Sym}^2(W_4^\vee)$ isomorphic to $I \oplus W_4$. Then $W$ is identified with the vector subspace in the space of quadratic forms in variables $x_0, \ldots, x_4$ subject to the relation

$$x_0 + \ldots + x_4 = 0,$$

that is spanned by the forms $x_i^2$, $0 \leq i \leq 4$. Let $\xi_{ii}$, $0 \leq i \leq 4$, be coordinates in $W$ that are permuted by the groups $\mathfrak{A}_5$ and $\mathfrak{S}_5$. Put

$$\sigma_k(\xi_{00}, \ldots, \xi_{44}) = \xi_{00}^k + \ldots + \xi_{44}^k.$$  

Any (reduced) $\mathfrak{A}_5$-invariant quadric in $\mathbb{P}^4$ is given by

$$\sigma_2(\xi_{00}, \ldots, \xi_{44}) + \lambda \sigma_1(\xi_{00}, \ldots, \xi_{44})^2 = 0$$  

for some $\lambda \in \mathbb{C}$, cf. (4.1).

Let $Y$ be the quartic in $\mathbb{P}(W)$ given by the vanishing of the determinant of a quadratic form. Then the equation of $Y$ can be written as

$$\det \begin{pmatrix} \xi_{11} + \xi_{00} & \xi_{00} & \xi_{00} & \xi_{00} \\ \xi_{00} & \xi_{22} + \xi_{00} & \xi_{00} & \xi_{00} \\ \xi_{00} & \xi_{00} & \xi_{33} + \xi_{00} & \xi_{00} \\ \xi_{00} & \xi_{00} & \xi_{00} & \xi_{44} + \xi_{00} \end{pmatrix} =$$

$$= \xi_{11} \xi_{22} \xi_{33} \xi_{44} + \xi_{00} \xi_{22} \xi_{33} \xi_{44} + \xi_{00} \xi_{11} \xi_{33} \xi_{44} + \xi_{00} \xi_{11} \xi_{22} \xi_{44} + \xi_{00} \xi_{11} \xi_{22} \xi_{33} = 0.$$  

Denote by $Q_\lambda$ the quadric given by equation (5.2), and denote by $S_\lambda$ the intersection of $Y$ with $Q_\lambda$. Suppose that $Q_\lambda$ is smooth. By Lemma 4.1 this happens if and only if $\lambda \neq -\frac{1}{5}$. Let $\alpha$ be a root of the equation

$$5\alpha^2 + 2\alpha = \lambda,$$

and put

$$\xi_{ii}' = \xi_{ii} + \alpha \sigma_1(\xi_{00}, \ldots, \xi_{44}).$$
In particular, one has
\[ \xi_{ii} = \xi'_{ii} - \frac{\alpha}{5\alpha + 1}\sigma_1(\xi_{00}, \ldots, \xi_{44}), \]
so the change of variables (5.3) is invertible. The quadric \( Q_\lambda \) is given by the equation
\[ \sigma_2(\xi_{00}, \ldots, \xi_{44}) = 0 \]
(cf. the proof of Lemma 4.1), while the quartic \( Y \) is given by the equation
\[
\sigma_4(\xi_{00}, \ldots, \xi_{44}) - 4 \cdot \frac{3\alpha + 1}{3(5\alpha + 1)} \cdot \sigma_3(\xi_{00}, \ldots, \xi_{44})\sigma_1(\xi_{00}, \ldots, \xi_{44}) - \\
- 165\alpha^4 + 164\alpha^3 + 66\alpha^2 + 12\alpha + 1 \cdot \sigma_1(\xi_{00}, \ldots, \xi_{44})^4 + \\
+ \frac{11\alpha^2 + 6\alpha + 1}{(5\alpha + 1)^2} \cdot \sigma_2(\xi_{00}, \ldots, \xi_{44})\sigma_1(\xi_{00}, \ldots, \xi_{44})^2 = 0.
\]
Therefore, the surface \( S_\lambda \) is isomorphic to the surface \( S_{\mu, \nu} \) in the notation of §4 for
\[ \mu = -\frac{3\alpha + 1}{3(5\alpha + 1)}, \quad \nu = -\frac{165\alpha^4 + 164\alpha^3 + 66\alpha^2 + 12\alpha + 1}{6(5\alpha + 1)^4}. \]
Applying Corollary 4.12, we obtain the following result.

**Lemma 5.1.** The singular locus of the surface \( S_\lambda \) is a single \( \mathfrak{S}_5 \)-orbit of twenty ordinary double points, provided that

\[ \lambda \in \mathbb{C} \setminus \{-\frac{3}{5}, -1, -\frac{7}{2}\}. \]

**Remark 5.2.** Using Table 1 one can describe singularities of the surfaces \( S_\lambda \) for exceptional values \( \lambda \in \{-1, -\frac{3}{2}\} \). Taking \( \alpha = -\frac{1}{5} - \frac{2}{5}i \) and \( \alpha = -\frac{1}{5} + \frac{2}{5}i \) gives isomorphisms
\[ S_{\mu_{5}^{ab,+}, \nu_{5}^{ab,+}} \cong S_{-1} \cong S_{\mu_{5}^{ab,-}, \nu_{5}^{ab,-}} \]
respectively, so that \( S_{-1} \) has exactly five singular points, and all of them are worse than ordinary double ones. Taking \( \alpha = -\frac{1}{5} - \sqrt{\frac{19}{10}}i \) and \( \alpha = -\frac{1}{5} + \sqrt{\frac{19}{10}}i \) gives isomorphisms
\[ S_{\mu_{20\rightarrow 10}^{+}, \nu_{20\rightarrow 10}^{+}} \cong S_{-\frac{1}{3}} \cong S_{\mu_{20\rightarrow 10}^{-}, \nu_{20\rightarrow 10}^{-}} \]
respectively, so that \( S_{-\frac{1}{3}} \) has exactly ten singular points, and all of them are worse than ordinary double ones.

Let \( \Delta \) be the determinantal hypersurface in
\[ \mathbb{P}^9 = \mathbb{P}(\text{Sym}^2(W^\vee)), \]
i.e. the hypersurface parameterizing singular quadrics in \( \mathbb{P}^3 = \mathbb{P}(W_4) \). Let \( \Delta_i \subset \Delta \), \( i = 1, 2 \), be the subvariety parameterizing quadrics of rank at most \( 3 - i \). Then \( \Delta \) is singular along \( \Delta_1 \), the singularity of \( \Delta \) at every point \( P \in \Delta_1 \setminus \Delta_2 \) is locally isomorphic to a product of a germ of a surface ordinary double point with \( \mathbb{A}^6 \), and the singularity of \( \Delta \) at every point of \( \Delta_2 \) has multiplicity 3. Applying Lemma 5.1 we obtain the following result.

**Corollary 5.3.** Suppose that \( \lambda \) is like in (5.4). Then the surface \( S_\lambda \) intersects the subvariety \( \Delta_1 \subset \Delta \) transversally at 20 points, and has no singular points outside \( \Delta_1 \). In particular, \( Q_\lambda \) intersects \( \Delta \) transversally at the points of \( \Delta \setminus \Delta_1 \), intersects \( \Delta_1 \) transversally at the points of \( \Delta_1 \setminus \Delta_2 \), and is disjoint from \( \Delta_2 \).
Let $T$ be a subvariety of $\mathbb{P}^9 \cong \mathbb{P}(\text{Sym}^2(W_4))$. Let $\phi: T \to T$ be the restriction of the tautological quadric bundle over $\mathbb{P}^9$ to $T$. Our current goal is to find conditions on $T$ to guarantee that $T$ is smooth. This is due to the fact that in our construction we restrict ourselves to a certain subfamily of quadric threefolds in $\mathbb{P}^9$; so that we have to check smoothness of (total spaces of) corresponding bundles explicitly as opposed to more standard approaches that make use of generality assumptions, see e.g. [13, Exercise 7.3.2(i)].

**Lemma 5.4.** Let $P$ be a point of $T$, and $\mathcal{T}_P$ be the fiber of $\phi$ over $P$. The following assertions hold.

(i) Suppose that $P \notin \Delta$. Then $T$ is smooth at every point of $\mathcal{T}_P$ if and only if $T$ is smooth at $P$.

(ii) Suppose that $P \in \Delta \setminus \Delta_1$. Then $T$ is smooth at every point of $\mathcal{T}_P$ if and only if $T$ is smooth at $P$, and $T$ intersects $\Delta$ transversally at $P$.

(iii) Suppose that $P \in \Delta_1 \setminus \Delta_2$. Then $T$ is smooth at every point of $\mathcal{T}_P$ provided that $T$ is smooth at $P$, and $T$ intersects $\Delta_1$ transversally at $P$.

**Proof.** Choose homogeneous coordinates $z_0, \ldots, z_3$ in $\mathbb{P}^3$ and homogeneous coordinates $\zeta_{ij}, 0 \leq i \leq j \leq 3$, in $\mathbb{P}^9$. Suppose that $T$ is given in $\mathbb{P}^9$ by equations

$$F_1(\zeta_{ij}) = \ldots = F_k(\zeta_{ij}) = 0.$$  

Then $T$ is given in $\mathbb{P}^9 \times \mathbb{P}^3$ by equations (5.5) and the equation

$$\sum_{0 \leq i \leq j \leq 3} \zeta_{ij}z_iz_j = 0.$$  

Let $r$ be the codimension of $T$ in $\mathbb{P}^9$, so that $T$ has codimension $r + 1$ in $\mathbb{P}^9 \times \mathbb{P}^3$. The variety $T$ is non-singular at a point $P \in \mathcal{T}_P$ if and only if the matrix $M$ of partial derivatives of equations (5.5) and (5.6) with respect to the variables $\zeta_{ij}$ and $z_i$ has rank $r + 1$ at $P$. If $T$ is singular at $P$, then the matrix of partial derivatives of equations (5.5) has rank at most $r - 1$ at $P$, so that $T$ is singular at every point of the fiber $\mathcal{T}_P$.

From now on we assume that $T$ is non-singular at $P$. Note that the partial derivatives of (5.5) with respect to variables $z_i$ vanish at every point, while for a smooth point of $\mathcal{T}_P$ there exists a partial derivative of (5.6) with respect to some variable $z_i$ that does not vanish at that point. Thus we see that the matrix $M(\mathcal{P})$ has rank $r + 1$ provided that the quadric $\mathcal{T}_P$ is non-singular at $\mathcal{P}$. In particular, this proves assertion (i).

Suppose that $P \in \Delta \setminus \Delta_1$. We have already seen that $T$ is smooth at $\mathcal{P}$ provided that $\mathcal{P}$ is a smooth point of $\mathcal{T}_P$. So we suppose that $\mathcal{P}$ is the (unique) singular point of $\mathcal{T}_P$. After a suitable change of coordinates $z_0, \ldots, z_3$ we may also assume that the point $P$ corresponds to the quadratic form

$$z_0^2 + z_1^2 + z_2^2 = 0,$$

and $\mathcal{P}$ corresponds to the point $(0 : 0 : 0 : 1)$ in $\mathbb{P}^3$. The only partial derivative of (5.6) that does not vanish at $\mathcal{P}$ is the one with respect to $\zeta_{33}$. Similarly, the only partial derivative of the equation of $\Delta$ that does not vanish at the point $P$ is the one with respect to $\zeta_{33}$. Therefore, singularity of $T$ at $\mathcal{P}$ and transversality of $T$ and $\Delta$ at $P$ are given by the same condition. This proves assertion (ii).

Now suppose that $P \in \Delta_1 \setminus \Delta_2$, and $\mathcal{P}$ is a singular point of $\mathcal{T}_P$. After a suitable change of coordinates $z_0, \ldots, z_3$ we may also assume that the point $P$ corresponds to the
a quadratic form
\[ z_0^2 + z_1^2 = 0, \]
and \( \mathcal{P} \) corresponds to the point \((0 : 0 : \alpha : \beta)\) in \( \mathbb{P}^3 \). The partial derivatives of \((5.6)\) at \( \mathcal{P} \) with respect to variables \( \zeta_{22}, \zeta_{23}, \) and \( \zeta_{33} \) equal \( \alpha^2, \beta^2 \) and \( \alpha \beta \), respectively, while all other partial derivatives of \((5.6)\) vanish at \( \mathcal{P} \). The subvariety \( \Delta_1 \subset \mathbb{P}^6 \) is given by vanishing of the \( 3 \times 3 \)-minors of a matrix of a quadratic form. All partial derivatives of these minors except those with respect to \( \zeta_{22}, \zeta_{23}, \) and \( \zeta_{33} \) vanish at the point \( P \). Therefore, transversality of \( T \) and \( \Delta_1 \) at \( P \) implies that \( T \) is non-singular at \( \mathcal{P} \). This proves assertion (iii). \( \square \)

**Remark 5.5.** It would be interesting to have an “if and only if” condition for smoothness of the variety \( T \) in terms of the varieties \( T, \Delta \) and \( \Delta_1 \).

In the remaining part of this section we suppose that \( \lambda \) is like in \((5.4)\). Let \( X_\lambda \) be a double cover of \( Q_\lambda \) branched over \( S_\lambda \). Put
\[ X^o_\lambda = X_\lambda \setminus \text{Sing}(X_\lambda), \quad Q^o_\lambda = Q_\lambda \setminus \text{Sing}(S_\lambda). \]
Then \( X^o_\lambda \) is a double cover of \( Q^o_\lambda \). Let \( \psi: Q_\lambda \to Q_\lambda \) be the restriction of the tautological quadric bundle over \( \mathbb{P}(\text{Sym}^2(W^\vee)) \) to \( Q_\lambda \). Corollary \((5.3)\) shows that \( X^o_\lambda \) can be identified with a variety parameterizing families of lines on quadrics corresponding to the points of \( Q^o_\lambda \), and that there is a natural \( \mathbb{P}^1 \)-bundle \( \pi: \mathcal{P}^o_\lambda \to X^o_\lambda \) such that the points of \( \mathcal{P}^o_\lambda \) parameterize the lines on such quadrics. The following result is obtained in a way identical to a well-known approach to rationality of double solids (cf. [2] Appendix, [17] Lemma 3.2, [13] Exercise 7.3.2).

**Lemma 5.6** (cf. [17] Lemma 3.2]). The \( \mathbb{P}^1 \)-bundle \( \pi \) is not a projectivization of a vector bundle.

**Proof.** Suppose that \( \pi \) is a projectivization of some vector bundle. Then there exists a rational section \( \sigma: X^o_\lambda \to \mathcal{P}^o_\lambda \) of \( \pi \). Note that \( \sigma \) defines a family \( \Sigma \) of lines on quadrics corresponding to the points of \( Q^o_\lambda \), and for a (smooth) quadric \( M \) corresponding to a general point of \( Q^o_\lambda \) there are exactly two lines in \( \Sigma \) that are contained in \( M \), one of them in each of the two one-parameter families of lines on \( M \). Taking an intersection point of the latter two lines, we define a rational section \( \sigma' \) of the quadric bundle \( \psi \). Let \( R \subset Q_\lambda \) be the closure of the image of \( \sigma' \). Then for a fiber \( F \) of \( \psi \) one has an intersection \( R \cdot F = 1 \) on \( Q_\lambda \).

The fivefold \( Q_\lambda \) is naturally embedded into the sixfold \( Q_\lambda \times \mathbb{P}^3 \) as a divisor of bidegree \((1, 2)\). Moreover, \( Q_\lambda \) is smooth by Lemma \((5.4)\) and Corollary \((5.3)\).

By the Lefschetz hyperplane section theorem, there is an element
\[ \bar{R} \in H^4(Q_\lambda \times \mathbb{P}^3, \mathbb{Z}) \]
that restricts to the element \( R \in H^4(Q_\lambda, \mathbb{Z}) \). The fiber \( F \) of \( \psi \) can be considered both as an element of \( H^6(Q_\lambda, \mathbb{Z}) \) and of \( H^8(Q_\lambda \times \mathbb{P}^3, \mathbb{Z}) \). In the latter group it is divisible by 2. The intersection of \( F \) with \( R \) on \( Q_\lambda \) equals the intersection of \( F \) with \( \bar{R} \) on \( Q_\lambda \times \mathbb{P}^3 \). On the other hand, the former intersection equals 1 while the latter intersection is even, which gives a contradiction. \( \square \)

**Corollary 5.7** (cf. [17] Theorem 3.3]). Let \( \tilde{X}_\lambda \) be a blow up of the singular points of \( X_\lambda \). Then there is a non-trivial torsion element in the cohomology group \( H^4(\tilde{X}_\lambda, \mathbb{Z}) \). In particular, the varieties \( \tilde{X}_\lambda \) and \( X_\lambda \) are not stably rational.
Proof. By Lemma 5.6 the $\mathbb{P}^1$-bundle $\pi$ defines a non-trivial element in the Brauer group
$$\text{Br}(X_\lambda^0) \cong \text{Br}(\tilde{X}_\lambda).$$
The latter gives a non-trivial torsion element in $H^3(\tilde{X}_\lambda, \mathbb{Z})$, see e.g. [2] Appendix. By [1] Proposition 1] such element provides an obstruction to stable rationality of $\tilde{X}_\lambda$, and thus also of $X_\lambda$.

Applying Corollary 5.7 together with [33, Theorem 0.1], we prove Theorem 1.2(iii). In fact, this gives the following more general result (that is actually well-known to experts).

**Proposition 5.8.** Let $X$ be a very general double cover of a (smooth) three-dimensional quadric branched over an intersection with a quartic. Then $X$ is not stably rational.

**Remark 5.9.** Since double covers of quadrics branched over intersections with quartics are degenerations of quartic hypersurfaces, Corollary 5.7 implies that a very general quartic threefold is stably non-rational. This result is known from [9], see also [16], where another approach was used to obtain it. Moreover, the approach to stable non-rationality of quartic threefolds via double quadrics was used in [28], and is actually not affected by a gap in [28] that becomes crucial only in higher dimensions.

### 6. Projectivization of the irreducible representation

In this section we study the unique double quadric with an action of the group $\mathfrak{A}_5$ such that the corresponding quadric is embedded into the projectivization of the irreducible five-dimensional representation of the group $\mathfrak{A}_5$, i.e. the variety $X_{\text{irr}}$ in the notation of Theorem 1.2. This will also appear to be the unique double quadric with an action of the group $\mathfrak{A}_6$.

Recall that $W_5$ denotes the unique five-dimensional irreducible representation of the group $\mathfrak{S}_6$. Note that $W_5$ can be also considered as a representation of the groups $\mathfrak{A}_6$ and $\mathfrak{S}_6$. Put $\mathbb{P}^4 = \mathbb{P}(W_5)$. One has

$$\dim \text{Hom}(I, \text{Sym}^2(W_5^*)) = 1, \quad \dim \text{Hom}(I, \text{Sym}^4(W_5^*)) = 2,$$

where $I$ and $W_5$ are considered as representations of either of the groups $\mathfrak{A}_5$, $\mathfrak{A}_6$, or $\mathfrak{S}_6$. In particular, in $\mathbb{P}^4$ there is a unique quadric $Q$ invariant under the group $\mathfrak{A}_5$ (or $\mathfrak{A}_6$, or $\mathfrak{S}_6$, respectively), and a unique intersection $S$ of $Q$ with a quartic invariant under the group $\mathfrak{A}_5$ (or $\mathfrak{A}_6$, or $\mathfrak{S}_6$, respectively). They can be described as follows.

Let $W \cong I \oplus W_5$ be the six-dimensional permutation representation of the group $\mathfrak{S}_6$. Put $\mathbb{P}^5 = \mathbb{P}(W)$, and let $y_0, \ldots, y_5$ be the homogeneous coordinates in $\mathbb{P}^5$ that are permuted by $\mathfrak{S}_6$. Put

$$\sigma_k(y_0, \ldots, y_5) = y_0^k + \ldots + y_5^k.$$

Equation $\sigma_1 = 0$ defines the linear subspace $\mathbb{P}^4 \subset \mathbb{P}^5$. The quadric $Q$ is defined by an equation

$$\sigma_1 = \sigma_2 = 0,$$

and reduced $\mathfrak{S}_6$-invariant (respectively, $\mathfrak{A}_6$-invariant, $\mathfrak{A}_5$-invariant) quartics are defined by equations

$$\sigma_1 = \sigma_4 - \zeta \sigma_2^2 = 0, \quad \zeta \in \mathbb{C}.$$
In particular, the surface $S$ is defined by equations
\[ \sigma_1 = \sigma_2 = \sigma_4 = 0. \]

**Remark 6.1.** The quartics given by equation (6.3) were studied in [3] and [8].

Consider the point
\[ P = (1 : 1 : \omega : \omega^2 : \omega^2 : \omega^2) \in \mathbb{P}^5, \]
where $\omega$ is a non-trivial cubic root of 1. Let $\Xi$ be the $\mathfrak{S}_6$-orbit of the point $P$. Then one has $|\Xi| = 30$. Choose a subgroup $\mathfrak{A}'_5 \subset \mathfrak{S}_6$ that is isomorphic to $\mathfrak{A}_5$ but is not conjugate to the initial $\mathfrak{A}_5 \subset \mathfrak{S}_6$. The restriction of the representation $W_5$ of $\mathfrak{S}_6$ to $\mathfrak{A}'_5$ is isomorphic to $I \oplus W_4$, and one can assume that the coordinates $y_1, \ldots, y_5$ are permuted by $\mathfrak{A}'_5$. In these coordinates $\Xi$ is the $\mathfrak{A}'_5$-orbit of the point
\[ P' = (1 : \omega : \omega^2 : \omega^2) \in \mathbb{P}^4, \]
and the quadric $Q$ is given by equation (4.11) with $\lambda = 1$. Therefore, in the notation of §4 the $\mathfrak{A}'_5$-orbit $\Xi$ is identified with $\Sigma_{a,b}^{30}$ for
\[ a = \frac{-2 + \sqrt{6}}{4} + \frac{2\sqrt{3} - \sqrt{2}}{4}i, \quad b = \frac{-2 + \sqrt{6}}{4} + \frac{-2\sqrt{3} + \sqrt{2}}{4}i, \]
and the surface $S$ is identified with $S_{\mu,\nu}$ for
\[ \mu = \frac{-6 + \sqrt{6}}{30}, \quad \nu = \frac{-3 + 8\sqrt{6}}{750}. \]

In particular, by Corollary 4.12 one has $\text{Sing}(S) = \Xi$, and the singularities of the surface $S$ are ordinary double points.

Let $X$ be a double cover of $Q$ branched over $S$.

**Remark 6.2.** By Corollary 4.12 the singularities of $X$ are 30 ordinary double points. The threefold $X$ is the unique double quadric with an action of the group $\mathfrak{A}_6$, see Lemma 3.1. In the notation of §4 there is an isomorphism $X \cong X_{\mu,\nu}$ for $\mu$ and $\nu$ given by (6.4), but this isomorphism is not $\mathfrak{A}_5$-equivariant.

Below we will need the following notation. Consider the weighted projective space $\mathbb{P} = \mathbb{P}(1^5, 2)$ with weighted homogeneous coordinates $y_1, \ldots, y_5, u$, where $y_i$ have weight 1, and $u$ has weight 2. The group $\mathfrak{S}_6$ acts on $\mathbb{P}$ so that $H^0(\mathcal{O}_\mathbb{P}(1))$ is identified with the $\mathfrak{S}_6$-representation $W_5$. Put
\[ y_0 = -(y_1 + \ldots + y_5), \]
and define the forms $\sigma_i(y_1, \ldots, y_5) = \sigma_i(y_0, y_1, \ldots, y_5)$ by formula (6.2).

**Proposition 6.3** (cf. [3]). The intermediate Jacobian of $X$ is not isomorphic to a product of Jacobians of curves as a principally polarized abelian variety. In particular, $X$ is non-rational.

**Proof.** One can write equations of the threefold $X$ in $\mathbb{P}$ as
\[ \sigma_2 = u^2 - \sigma_4 = 0. \]

Consider the equations
\[ \sigma_2 - \theta u = u^2 - \sigma_4 = 0, \quad \theta \in \mathbb{C}. \]
If $\theta \neq 0$, we can rewrite (6.6) as
\begin{equation}
(6.7) \quad u - \theta^{-1}\sigma_2 = \sigma_4 - \theta^{-2}\sigma_2^2 = 0.
\end{equation}

The latter equations define a threefold $X_\theta$ that is isomorphic to a quartic given by (6.3) for $\zeta = \theta^{-2}$. On the other hand, if $\theta = 0$, then (6.6) is rewritten as (6.3). This shows that the quartic threefolds given by (6.3) or (6.7) degenerate to the double quadric $X$. All varieties $X_\theta$, $\theta \neq 0$, and $X$ are $\mathfrak{S}_6$-invariant. Note that the singularities of a general threefold $X_\theta$, as well as the singularities of $X$, are 30 ordinary double points that form the $\mathfrak{S}_6$-orbit $\Xi''$ of the point $P'' \in \mathbb{P}$ with coordinates
\[ y_1 = 1, \, y_2 = y_3 = \omega, \, y_4 = y_5 = \omega^2, \, u = 0, \]
see [3] for details.

Put $\Pi = \mathbb{P} \times \mathbb{A}^1$, and let $\mathcal{X} \subset \Pi$ be the subvariety
\[ \mathcal{X} = \{ (P, \theta) \mid \sigma_2(P) - \theta u = u^2 - \sigma_4(P) = 0 \}. \]

Let $\pi: \Pi \rightarrow \mathbb{A}^1$ be the natural projection, and put $\mathcal{X}_\theta = \pi^{-1}(\theta)$. Then $\mathcal{X}_\theta \cong X_\theta$ for $\theta \neq 0$, and $\mathcal{X}_0 \cong X$.

Let $\varphi: \bar{\Pi} \rightarrow \Pi$ be the blow up of the locus $\Xi'' \times \mathbb{A}^1 \subset \Pi$, and let $\bar{\mathcal{X}}$ be the proper transform of $\mathcal{X}$ on $\bar{\Pi}$. Denote by $\bar{\mathcal{X}}_\theta$ the proper transform of $\mathcal{X}_\theta$ on $\bar{\Pi}$. Then $\bar{\mathcal{X}}_\theta$ is smooth for a general $\theta \in \mathbb{A}^1$, and $\bar{\mathcal{X}}_0$ is smooth as well. Put
\[ C = \mathbb{A}^1 \setminus \{ \theta \mid \bar{\mathcal{X}}_\theta \text{ is singular} \}, \]
and denote by $\bar{\mathcal{X}}^0$ the preimage of $C$ in $\bar{\mathcal{X}}$.

The fibration $\bar{\mathcal{X}}^0 \rightarrow C$ defines a vector bundle $\mathcal{W} \rightarrow C$ whose fiber $\mathcal{W}_\theta$ over $\theta \in C$ is identified with the vector space $H^{2,1}(\bar{\mathcal{X}}_\theta)$, see e.g. [34, §10.2.1]. Moreover, there is a fiberwise action of the group $\mathfrak{S}_6$ on $\bar{\mathcal{X}}^0$. It gives rise to a fiberwise action of $\mathfrak{S}_6$ on $\mathcal{W}$. By [3, Proposition] one has $\mathcal{W}_\theta \cong W_5$ for $\theta \neq 0$. Note that representations of finite groups do not vary in families, since there is only a finite number of (characters of) representations of fixed dimension of a given group. Therefore, we have $\mathcal{W}_0 \cong W_5$. Now the assertion of the proposition follows by an argument of [4, §3]. \hfill \Box

Remark 6.4. An alternative proof of Proposition 6.3 could be given following the method of [3] step by step, since the singular loci of the variety $X$ and the quartics considered in [4, §2], i.e. the quartics given by (6.3), arise from the same $\mathfrak{S}_6$-orbit in $\mathbb{P}^4$.

Remark 6.5. Recall from (5.1) that there is a unique $\mathfrak{A}_5$-subrepresentation isomorphic to $W_5$ in $\text{Sym}^2(W_4^*)$. Thus we can conclude from (6.1) that $X$ is a double cover of the $\mathfrak{A}_5$-invariant quartic $Q$ branched over an intersection with the determinantal quartic $\Delta \subset \mathbb{P}(\text{Sym}^2(W_4^*))$. However, the approach of §5 is not applicable to prove non-rationality of $X$ since its branch surface $S \subset Q$ has singularities outside the singular locus $\Delta_1 \subset \Delta$, cf. Lemma 5.4(ii).

Proposition 6.3 proves Theorem 1.2(iv). Also, Lemma 3.1, Remark 6.2, and Proposition 6.3 imply Proposition 1.1. Another consequence of Proposition 6.3 is the following result.

Corollary 6.6. The varieties $X_{\mu, \nu}$ with $|\text{Sing}(X_{\mu, \nu})| = 30$ listed in Table 2 are non-rational up to a finite number of (possible) exceptions.
Proof. Every double quadric $X_{\mu,\nu}$ is naturally embedded into the weighted projective space $\mathbb{P}$, so that $X_{\mu,\nu} \subset \mathbb{P}$ is given by equations

$$\sigma_2 = u^2 - \sigma_4(y_1, \ldots, y_4) + 4\mu\sigma_3(y_1, \ldots, y_4)\sigma_1(y_1, \ldots, y_4) + \nu\sigma_1(y_1, \ldots, y_4)^4 = 0.$$ 

Define a (locally closed) curve $B \subset \text{Spec} \mathbb{C}[\mu, \nu] \cong \mathbb{A}^2$ as

$$B = \left\{(\mu, \nu) \mid \text{there are } a, b \in \mathbb{C} \text{ such that } \text{Sing}(S_{\mu,\nu}) = \Sigma_{a,b}, \right. \left. \text{and the singularities of } S_{\mu,\nu} \text{ are ordinary double points} \right\}.$$ 

By Remark 6.2 the curve $B$ contains the point $(\mu, \nu)$ given by (6.4).

Put $\Upsilon = \mathbb{P} \times B$. Define $Z \subset \Upsilon$ as

$$Z = \{(P, \mu, \nu) \mid P \in X_{\mu,\nu}\}.$$ 

Let $\phi: \tilde{\Upsilon} \to \Upsilon$ be the blow up of the locus

$$\Sigma = \{(P, \mu, \nu) \mid P \in \text{Sing}(X_{\mu,\nu})\} \subset \Upsilon,$$

and let $\tilde{Z}$ be the proper transform of $Z$ on $\tilde{\Upsilon}$. Denote by $\tilde{Z}_{\mu,\nu}$ the fiber of the natural projection $\tilde{Z} \to B$ over a point $(\mu, \nu) \in B$. Then $\tilde{Z}_{\mu,\nu}$ is smooth.

We see that $\tilde{Z}$ is a family of resolutions of singularities of the threefolds $X_{\mu,\nu}$ degenerating to a resolution of singularities of the threefold $X$. On the other hand, we know from Proposition 6.3 that the intermediate Jacobian of $X$ is not isomorphic to a product of Jacobians of curves as a principally polarized abelian variety. Now the required assertion follows from [4, Lemme 5.6.1] applied to the family $\tilde{Z}$. 

Corollaries 4.16 and 6.6 prove Theorem 1.2(ii).

References

[1] M. Artin, D. Mumford, Some elementary examples of unirational varieties which are not rational, Proc. Lond. Math. Soc. 25 (1972), 75–95.
[2] P. S. Aspinwall, D. R. Morrison, Stable singularities in string theory. With an appendix by Mark Gross, Comm. Math. Phys., 178, no. 1 (1996), 115–134.
[3] A. Beauville, Non-rationality of the $\mathbb{G}_a$-symmetric quartic threefolds, Rend. Sem. Mat. Univ. Politec. Torino 71 (2013), 385–388.
[4] A. Beauville, Variétés de Prym et jacobienes intermediaires, Ann. Sci. Ecole Norm. Sup. (4) 10 (1977), 309–391.
[5] H. Burkhardt, Untersuchungen auf dem Gebiete der hyperelliptischen Modulfunktionen, Mathematische Annalen 38 (1891), 161–224.
[6] I. Cheltsov, V. Przyjalkowski, C. Shramov, Quartic double solids with icosahedral symmetry, European J. of Math. 2 (2016), no. 1, 96–119.
[7] I. Cheltsov, C. Shramov, Cremona groups and the icosahedron, CRC Press, 2015.
[8] I. Cheltsov, C. Shramov, Two rational nodal quartic threefolds, preprint arXiv:1511.07508 (2015).
[9] J.-L. Colliot-Thélène, A. Pirutka, Hypersurfaces quartiques de dimension 3: non rationalité stable, to appear in Annales Sc. Ec. Norm. Sup.
[10] J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson, Atlas of finite groups, Clarendon Press, Oxford, 1985.
[11] S. Cynk, Defect of a nodal hypersurface, Manuscripta Math., 104:3 (2001), 325–331.
[12] W. Feit, The current situation in the theory of finite simple groups, Actes du Congres International des Mathematiciens, Gauthier-Villars, Paris (1971), 55–93.
[13] S. Gorchinskiy, C. Shramov, Unramified Brauer group and its applications (in Russian), preprint arXiv:1512.00874 (2015).
[14] M. Grinenko, *Birational automorphisms of a three-dimensional double quadric with an elementary singularity*, Sbornik: Mathematics, **189**:1 (1998), 97–114.

[15] M. Grinenko, *Birational automorphisms of a three-dimensional double cone*, Sbornik: Mathematics, **189**:7 (1998), 991–1007.

[16] B. Hassett, Yu. Tschinkel, *On stable rationality of Fano threefolds and del Pezzo fibrations*, preprint, arXiv:1601.07074 (2016).

[17] A. Iliev, L. Katzakov, V. Przyjalkowski, *Double solids, categories and non-rationality*, Proceedings of the Edinburgh Mathematical Society **57** (2014), 145–173.

[18] V. Iskovskikh, Yu. Prokhorov, *Fano varieties*, Encyclopaedia of Mathematical Sciences **47**, Springer, Berlin, 1999.

[19] V. Iskovskikh, A. Pukhlikov, *Birational automorphisms of multidimensional algebraic manifolds*, Journal of Mathematical Sciences, **82**:4 (1996), 3528–3613.

[20] F. Klein, *Ueber die Transformation siebenter Ordnung der elliptischen Functionen*, Math. Ann. **14** (1879), 428–471.

[21] A. Kuznetsov, *On Küchle varieties with Picard number greater than 1*, Izvestiya: Mathematics, **79**:4 (2015), 698–709.

[22] K. Pettersen, *On nodal determinantal quartic hypersurfaces in \(P^4\)*, Ph.D. Thesis, University of Oslo, 1998.

[23] Yu. Prokhorov, *On G-Fano threefolds*, Izvestiya: Mathematics, **79**:4 (2015), 795–808.

[24] Yu. Prokhorov, *Singular Fano threefolds of genus 12*, preprint, arXiv:1508.04371 (2015).

[25] Yu. Prokhorov, *Q-Fano varieties of index 7*, Proceedings of the Steklov Institute of Mathematics, **294** (2016).

[26] Yu. Prokhorov, *Simple finite subgroups of the Cremona group of rank 3*, J. Alg. Geom. **21** (2012), 563–600.

[27] A. Pukhlikov, *Birational automorphisms of a double space and a double quadric*, Mathematics of the USSR-Izvestiya, **32**:1 (1989), 233–243.

[28] S. Schreieder, L. Tasin, *A very general quartic or quintic fivefold is not stably rational*, preprint, arXiv:1510.02011 (2015).

[29] C. Shramov, *Birational rigidity and \(Q\)-factoriality of a singular double quadric*, Mathematical Notes, **84**:2 (2008), 280–289.

[30] J. Todd, *Configurations defined by six lines in space of three dimensions*, Math. Proc. of the Cambridge Phil. Soc. **29** (1933), 52–68.

[31] J. Todd, *A note on two special primals in four dimensions*, Q. J. Math. **6** (1935), 129–136.

[32] J. Todd, *On a quartic primal with forty-five nodes, in space of four dimensions*, Q. J. Math. **7** (1935), 168–174.

[33] C. Voisin, *Unirational threefolds with no universal codimension 2 cycle*, Invent. Math. **201** (2015), 207–237.

[34] C. Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics **76**, Cambridge University Press, 2007.