THE CHOW GROUP MOD $\ell$ OF A PRODUCT OF ELLIPTIC CURVES

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Abstract. Generalizing work of Schoen, we prove that the Chow group modulo $\ell$ of a product of 3 or more very general complex elliptic curves is infinite.

1. Introduction

A recent result of Totaro [13] shows that for any prime $\ell$ the Chow group modulo $\ell$ of a very general Abelian threefold is infinite, generalizing a result of Rosenschon and Srinivas in [8]. Our goal will be to prove the following analogous result, which extends work of Schoen in [11]:

**Theorem 1.1.** Let $E_1, E_2, \ldots, E_n$ be very general complex elliptic curves. Then,

$$\text{CH}^d(E_1 \times E_2 \times \cdots \times E_n) \otimes \mathbb{Z}/\ell$$

is infinite for all primes $\ell$ and $2 \leq d \leq n - 1$.

The key case is when $d = 2$ and $n = 3$ since the other cases may be obtained by pulling back along the projection to the triple product and then intersecting with a symmetric divisor. Our strategy in this case will be as follows. As noted by Totaro in [13], it suffices to prove the corresponding result modulo $\ell^r$ for $r \gg 0$. To this latter end, we first find a cycle which does not vanish modulo $\ell^r$. This will be an ersatz Ceresa cycle $\gamma$ which is a spreading out of the cycle considered in [1] and [4] to a certain 5-dimensional family of products of elliptic curves. It turns out that (using some rather general methods of Voisin in [14]) one can show that the geometry of the total space of this family is rather simple; this allows for a self-contained geometric proof of the non-vanishing of the normal function of $\gamma$ (in singular cohomology). More precisely, $\sim_{\text{rat}} = \sim_{\text{hom}}$ for this family so that proving homological non-triviality for $\gamma$ reduces to proving rational non-triviality for $\gamma$. It seems likely that this approach might work for other families of varieties for which the methods of [14] apply. This approach, however, does not work for the usual Ceresa cycle of genus 3 since this cycle is not defined on the universal Jacobian of genus 3 (with level $N$-structure), $J(N) \to \mathcal{M}_3(N)$ but only over some cover of the generic fiber. (In this case, the authors of [8] and [13] invoke a rather sophisticated result of Hain in [5] to prove the non-triviality of the normal function.) The results of [2] then give the required non-triviality mod $\ell^r$. In order to obtain infinitely many such cycles, we will modify Nori’s usual isogeny argument (using his suggestion in [7]).

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2. Construction of the main cycle

Let \( k \subset \mathbb{C} \) be algebraically closed and \( U \subset \mathbb{P}(H^0(P^2, O_{P^2}(2))) \) be an open subset consisting only of smooth conics. Then, we consider the universal family of conics over \( U \):

\[
\begin{array}{c}
Q \\ \downarrow \\
\mathbb{P}^2
\end{array}
\]

There is a finite morphism \( \phi : \mathbb{P}^2 \to \mathbb{P}^2 \) defined by

\[
[x, y, z] \mapsto [x^2, y^2, z^2]
\]

whose Galois group is \( G := \mathbb{Z}/2 \times \mathbb{Z}/2 \) and is generated by the involutions:

\[
[x, y, z] \overset{\sigma_1}{\longrightarrow} [-x, y, z], \quad [x, y, z] \overset{\sigma_2}{\longrightarrow} [x, -y, z]
\]

and denote by \( \sigma_3 := \sigma_1 \sigma_2 \). We then consider the Cartesian product:

\[
\begin{array}{c}
C \\ \downarrow \\
\mathbb{P}^2 \\ \downarrow \\
\mathbb{P}^2
\end{array}
\]

There is a natural morphism \( f : C \to U \), and after possibly shrinking \( U \), we may assume that all the fibers of \( f \) are smooth. The generic fiber is then a genus 3 curve in \( \mathbb{P}^2_{k(U)} \) defined by the following equation:

\[
ax_1^4 + bx_2^4 + cx_3^4 + dx_1^2x_2^2 + ex_2^2x_3^2 + fx_1^2x_3^2 = 0
\]

The involutions above then induce an action on \( C \xhookrightarrow{\bar{f}} U \), which gives rise to the quotients:

\[
p_j : C \to E_j := C/\sigma_j, \quad g_j : E_j \to U, \quad j = 1, 2, 3
\]

where \( g_j \circ p_j = f \). Note then that on \( \mathbb{P}^2 \) the involution \( \sigma_j \) fixes the line defined by \( x_j = 0 \) (as well as an isolated point). Again, upon shrinking \( U \), we may assume that none of the fibers of \( f \) pass through any of these isolated points. Thus, \( \sigma_j \) fixes precisely 4 points on the general fiber of \( f \). A Riemann-Hurwitz argument then shows that the fibers of \( g_j \) all have genus 1. Moreover, there are induced quotient maps:

\[
g_j : E_j \to Q = C/G
\]

as well as involutions, \((-1)_{E_j} : E_j \to E_j\), which induce the usual action by \(-1\) on the geometric generic fiber of \( g_j \). Then, we consider the fiber product

\[
g_1 \times_U g_2 \times_U g_3 : E_1 \times_U E_2 \times_U E_3 \to U
\]

and use the short-hand \( g := g_1 \times_U g_2 \times_U g_3 \) and \( A := E_1 \times_U E_2 \times_U E_3 \). There is then an action by

\[
\tilde{G} = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2
\]

on \( g : A \to U \) generated by involutions

\[
\tilde{\sigma}_j : A \to A, \quad j = 1, 2, 3
\]

which act by \((-1)_{E_j}\) on the \( j^{th} \) factor and the identity on the remaining factors. We also use the notation:

\[
\tilde{\sigma}_{jk} := \tilde{\sigma}_j \circ \tilde{\sigma}_k, \quad \tilde{\sigma}_{123} := \tilde{\sigma}_1 \circ \tilde{\sigma}_2 \circ \tilde{\sigma}_3
\]
Moreover, there is a morphism:
$$\rho : C \xrightarrow{\Delta^{(3)}} C^{(3)} := C \times_U C \times_U C \xrightarrow{p_2} A$$
where $p := p_1 \times_U p_2 \times_U p_3$ and
$$\rho_{jk} : C \xrightarrow{\rho} A \xrightarrow{\pi_{jk}} E_j \times_U E_k$$
Then, we have the following lemma:

**Lemma 2.1.** The generic fiber of $g : A \to U$ is geometrically isogenous to the Picard variety of the generic fiber of $f : C \to U$ and there is a commutative diagram:

$$
\begin{array}{ccc}
C & \xrightarrow{\sigma_i} & C \\
\rho \downarrow & & \rho \downarrow \\
A & \xrightarrow{\tilde{\sigma}_{jk}} & A
\end{array}
$$

for $i \notin \{j, k\}$. Further, $\rho$ is an imbedding; moreover, $\rho_{jk}$ is an immersion and is injective away from the fixed points of $\sigma_i$ for $i \notin \{j, k\}$.

**Proof.** The second statement follows directly from the constructions of $\sigma_i$ and $\tilde{\sigma}_{jk}$.

For the third statement, suppose $\rho(x) = \rho(x')$ for $x \neq x'$; i.e., that

$$\sigma_1(x) = x', \sigma_2(x) = x', \sigma_3(x) = x'$$

But then one has $x' = \sigma_3(x) = \sigma_1(\sigma_2(x)) = \sigma_1(x') = x$, which is a contradiction.

For the fourth statement, the immersion part follows from noting that

$$d\rho_{jk} = (dp_j, dp_k) : TC_x \to TE_{j,x} \oplus TE_{k,x}$$

and this vanishes only where $dp_j$ and $dp_k$ vanish simultaneously, which does not happen. Indeed, possibly after shrinking $U$, we can assume that none of the fibers of $C \to U$ contain fixed points of both $\sigma_j$ and $\sigma_k$. For generic injectivity, note that if only (2) holds for only 2 out of the 3 involutions, $x$ must be a fixed point of the remaining involution. Finally for the first statement, let $x \in U$, $C_x$ and $A_x$ be fibers of $f$ and $g$, respectively. Then, using the universal property of the Albanese, there is a commutative diagram:

$$
\begin{array}{ccc}
C_x & \xrightarrow{\rho} & \text{Pic}^0(C_x) \\
\downarrow & & \downarrow \\
A_x & &
\end{array}
$$

So, it suffices to show that the map $\rho_x^* : H^0(A_x, \mathcal{O}_{A_x}^1) \to H^0(C_x, \mathcal{O}_{C_x}^1)$ is an isomorphism. However, since both sides have the same dimension, we are reduced to proving injectivity. To this end, $A_x = E_{1,x} \times E_{2,x} \times E_{3,x}$, so that

$$H^0(A_x, \mathcal{O}_{A_x}^1) = \pi_1^* H^0(E_{1,x}, \mathcal{O}_{E_{1,x}}^1) \oplus \pi_2^* H^0(E_{2,x}, \mathcal{O}_{E_{2,x}}^1) \oplus \pi_3^* H^0(E_{3,x}, \mathcal{O}_{E_{3,x}}^1)$$

Also, $\pi_j \circ \rho_x = p_j : C_x \to E_{j,x}$. So, injectivity will follow from the statement that

$$p_j^* H^0(E_{j,x}, \mathcal{O}_{E_{j,x}}^1) \cap p_k^* H^0(E_{k,x}, \mathcal{O}_{E_{k,x}}^1) = 0$$

for $j \neq k$. For this, observe that the left hand side of (3) is invariant under $\sigma_j$ and $\sigma_k$ and, hence, under $G$. However, $H^0(C_x, \mathcal{O}_{C_x}^1)^G = H^0(Q_x, \mathcal{O}_{Q_x}^1) = 0$, which gives the desired equality. \qed
The main cycle is then defined as an analogue of the Ceresa cycle:

\[ \gamma := \rho_*([C]) - \tilde{\sigma}_{123} \rho_*([C]) \in CH^2(A) \]

We observe that \( \gamma \) is anti-invariant under \( \tilde{\sigma}_{123} \) and invariant under \( \tilde{\sigma}_{jk} \) by Lemma 2.1 and, hence, is anti-invariant with respect to each \( \tilde{\sigma}_j \).

3. The normal function of \( \gamma \)

We now take \( k = C \) and let \( g_j : E_j \to U \) and \( g : A \to U \) be as in the previous section. Then, there is a decomposition of local systems over \( U \) given by:

\[ R^1 g_* Q \cong \bigoplus_{j=1}^3 R^1 g_j_* Q \]

Since \( \wedge^2 R^1 g_* Q \cong Q \), we also have

\[ R^1 g_* Q \cong \bigoplus_{j=1}^3 R^1 g_j_* Q \cong \bigoplus_{j=1}^3 R^1 g_j_* Q \oplus PR^3 g_* Q \]

where \( PR^3 g_* Q = R^1 g_1_* Q \otimes R^1 g_2_* Q \otimes R^1 g_3_* Q \). (The notation is suggestive of primitive cohomology, which is not an irreducible local system in the case at hand; so, \( PR^3 g_* Q \) will be considered in its place.) Now, we can find a relative correspondence:

\[ P \in CH^3(A \times_U A) \]

for which \( P_* (Rg_* Q) \cong PR^3 g_* Q [-3] \in D^b(S, Q) \), the derived category of \( Q \)-local systems over \( U \). Indeed, we can select

\[ \pi_{1,j} \in CH^1(E_j \times_U E_j) \]

for which \( \pi_{1,j_*} (Rg_j_* Q) = R^1 g_{j_*} Q [-1] \). (One can set \( \pi_{1,j} = \Delta E_j - \Gamma \sigma_j \) and this has the desired effect.) Then set

\[ P = \pi_{1,1} \times_U \pi_{1,2} \times_U \pi_{1,3} \in CH^3(A \times_U A) \]

With this definition, we have:

**Observation 3.1.** \( P_* CH^1(A)_Q \) is the subspace of \( CH^1(A)_Q \) on which \( \tilde{\sigma}_j \) acts by \( -id \).

Further, since \( P_* (Rg_* Q) \cong PR^3 g_* Q [-3] \), we can apply \( H^1 \) to obtain:

\[ P_* H^1(A, Q) \cong H^1(U, PR^3 g_* Q) \]

**Proposition 3.1.** \( P_* \gamma \neq 0 \in H^1(U, PR^3 g_* Q) \)

**Proof.** We will need to show that \( P_* \gamma \neq 0 \in H^1(A, Q) \); an important step towards this is the following lemma, which shows that the geometry of \( A \) is rather simple.

**Lemma 3.1.**

(a) \( C^{(3)} := C \times_U C \times_U C \) contains \( C^{(3)}_0 \), an open subset of a projective bundle over the blow-up of \( \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \) along a union of (transversely-intersecting) rational varieties.

(b) The cycle class map \( CH^2(A)_Q \to H^4(A, Q) \) is injective.
Proof. For (a) we adapt an argument of Voisin (Proposition 3.11 in [14]), although the case at hand is less technical. We note that the universal conic over $U \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is given by

$$Q = \{(Q, x) \in U \times \mathbb{P}^2 \mid Q \in U, \, Q(x) = 0\}$$

We set $(\mathbb{P}^2)^3 := \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ and consider the triple fiber product over $U$:

$$(\mathbb{P}^2)^3 := Q \times_U Q \times_U Q = \{(Q, x, y, z) \in U \times (\mathbb{P}^2)^3 \mid Q \in U, \, Q(x) = Q(y) = Q(z) = 0\}$$

There is the natural map $p^{(3)} : Q^{(3)} \to (\mathbb{P}^2)^3$ defined as the projection onto the last $3$ factors. Now, we consider the closed subvarieties $\Delta_{ij} \subset (\mathbb{P}^2)^3$ defined as the set of $(x_1, x_2, x_3) \in (\mathbb{P}^2)^3$ such that $x_i = x_j$ and the blow-up $\epsilon : (\mathbb{P}^2)^3 \to (\mathbb{P}^2)^3$ along

$$\Delta_{12} \cup \Delta_{23} \cup \Delta_{13}$$

Since $Q \to U$ is a relative curve, we see that $(p^{(3)})^{-1}(\Delta_{ij})$ is of codimension $1$ and thus by the universal property of blow-up $p^{(3)} : Q^{(3)} \to (\mathbb{P}^2)^3$ factors through $\epsilon : (\mathbb{P}^2)^3 \to (\mathbb{P}^2)^3$, giving a morphism $\tilde{p}^{(3)} : Q^{(3)} \to (\mathbb{P}^2)^3$. Now, we let

$$Q^{(3)}_0 = Q^{(3)} \setminus \Delta^{(3)}_Q, \quad (\mathbb{P}^2)^3_0 = (\mathbb{P}^2)^3 \setminus \epsilon^{-1}(\Delta^{(3)}_{p^{(3)}})$$

where $\Delta^{(3)}_Q$ and $\Delta^{(3)}_{p^{(3)}}$ denote the small diagonals in $Q^{(3)}$ and $(\mathbb{P}^2)^3$, respectively. By construction, $(\mathbb{P}^2)^3_0$ consists of length $3$ subschemes of $\mathbb{P}^2$ which are supported on at least $2$ distinct points. Then, there is a map

$$p^{(3)}_0 : Q^{(3)}_0 \to (\mathbb{P}^2)^3_0$$

such that for any length $3$ subscheme $Z \in (\mathbb{P}^2)^3_0$ the fiber $(p^{(3)}_0)^{-1}(Z)$ consists of all conics $Q \in U$ such that $Q$ vanishes on $Z$. It is an easy result that such a subscheme imposes $3$ independent conditions on conics, so that the dimension of any fiber $(p^{(3)}_0)^{-1}(Z)$ is an open subset of $\mathbb{P}^2$. In fact, $Q^{(3)}_0$ is an open subscheme of a projective bundle over $(\mathbb{P}^2)^3_0$, for which the fiber over $Z \in (\mathbb{P}^2)^3_0$ is $H^0(\mathbb{P}^2, \mathcal{I}_Z(2))$.

Recall the morphism $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ defined in the previous section by $[x, y, z] \mapsto [x^2, y^2, z^2]$ and now consider the triple product of this map, $\phi^3 : (\mathbb{P}^2)^3 \to (\mathbb{P}^2)^3$. Now, $(\phi^3)^{-1}(\Delta_{jk})$ consists of $(x_1, x_2, x_3) \in (\mathbb{P}^2)^3$ satisfying

$$x_j^2 - x_k^2 = (x_j - x_k)(x_j + x_k) = 0$$

So, we let $\tilde{\epsilon} : (\mathbb{P}^2)^3 \to (\mathbb{P}^2)^3$ be the blow-up of $(\mathbb{P}^2)^3$ along the subvariety

$$(\phi^3)^{-1}(\Delta_{12}) \cup (\phi^3)^{-1}(\Delta_{23}) \cup (\phi^3)^{-1}(\Delta_{13})$$

We then obtain a morphism $\tilde{\phi}^3 : (\mathbb{P}^2)^3 \to (\mathbb{P}^2)^3$. From the Cartesian diagram (11), it is straightforward to see that there is a Cartesian diagram:

$$\begin{array}{ccc}
C^{(3)} & \xrightarrow{q^{(3)}} & Q^{(3)} \\
\downarrow & & \downarrow \\
(\mathbb{P}^2)^3 & \xrightarrow{p^{(3)}} & (\mathbb{P}^2)^3 \\
\end{array}$$

Thus, the open set $C^{(3)}_0 := (\tilde{\phi}^{(3)})^{-1}(Q^{(3)}_0) \subset C^{(3)}$ is an open subset of a projective bundle over $(\mathbb{P}^2)^3_0 := (\tilde{\phi}^{(3)})^{-1}((\mathbb{P}^2)^3_0) \subset (\mathbb{P}^2)^3$, which gives the proof of (a).
For (b), by applying Lemmas 2.1, 2.2, and 2.3 of \cite{14} to $C^{(3)}_0$, it follows that the cycle class map:

$$CH^2(C^{(3)}_0, \mathbb{Q}) \to H^4(C^{(3)}_0, \mathbb{Q})$$

is injective. Since the complement of $C^{(3)}_0$ in $C^{(3)}$ has codimension $\geq 2$, a Gysin exact sequence argument shows that

$$CH^2(C^{(3)} \times \mathbb{Q}) \to H^4(C^{(3)} \times \mathbb{Q})$$

is also injective. Finally, since $A = \mathcal{E}_1 \times_U \mathcal{E}_2 \times_U \mathcal{E}_3$ is a quotient of $C^{(3)}$ by $H = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$, applying the functor $(\ )^H$ to (4), we obtain the statement of the lemma.

Now, suppose by way of contradiction that $P_\gamma \gamma = 0 \in H^4(A, \mathbb{Q})$; then, by the previous lemma, we must have $P_\gamma \gamma = 0 \in CH^2(A, \mathbb{Q})$. Since $\gamma$ is anti-invariant under $\tilde{\sigma}$, it follows from Observation 3.1 that $\gamma = 0 \in CH^2(A, \mathbb{Q})$ and, hence, by specialization:

$$\gamma_x = 0 \in CH^2(E_{1,x} \times E_{2,x} \times E_{3,x}, \mathbb{Q})$$

for each $x \in U$. Now, since $x$ varies in a 5-dimensional family, we may fix $E_{2,x} = E$ and $E_{3,x} = E'$ and allow $E_{1,x} = E_t$ to vary in a 1-parameter family indexed by $t$. The desired contradiction is obtained using the following lemma:

**Lemma 3.2.** If (4) holds, then $CH^2(K, \mathbb{Q})$ has rank 1, where $K$ is the Kummer surface of $E \times E'$.

**Proof of Lemma.** By Lemma 2.1, one has:

$$C_t \subset E_t \times E'$$

and the projection to the last two factors gives an immersed curve $\rho_{23}(C_t) \subset E \times E'$ whose only singularities are nodes contained in the set of 2-torsion points of $E \times E'$ (i.e., the fixed points of $\tilde{\sigma}_{23}$). Now, we let $\tilde{E} \times E'$ denote the blow-up of $E \times E'$ along the 2-torsion points. There is a lift $C_t \subset \tilde{E}_t \times E' \times E'$ for which $\rho_{23}(C_t)$ is smooth (since its only singular points are nodes in the set of blown-up points). The action of $\tilde{\sigma}_{23} = (-1)_{E \times E'}$ lifts to $\tilde{E} \times E'$ and the corresponding action on $E_t \times \tilde{E} \times E'$ induces the action of $\sigma_1$ on $C_t$ by Lemma 2.1. Taking the quotient by $\sigma_1$ gives

$$E_t \subset E_t \times K$$

where $K$ is the Kummer surface of $E \times E'$. By construction, the projection of $E_t$ onto $K$ is an isomorphism, and so we have an inclusion $j_t : E_t \hookrightarrow K$. Now, since $K$ is a $K3$ surface, $|j_t(E_t)|$ is a 1-dimensional linear system and induces the structure of an elliptic surface on $K$. In fact, we could have chosen the 1-parameter family indexed by $t$ so that $|j_t(E_t)|$ does not depend on $t$. Thus, $|j_t(E_t)|$ induces the structure of an elliptic surface on $K$. Finally, the action of $\tilde{\sigma}_1$ on $E_t \times E \times E'$ descends to $E_t \times K$, acting as $(-1)_{E_t}$ on $E_t$ and the identity on $K$.

By our assumption, the main cycle $\gamma_t \in CH^2(E_t \times E \times E')$ vanishes. We note that

$$\gamma_t = \rho_*(C_t) - \tilde{\sigma}_{123}^* \rho_*(C_t) = \rho_*(C_t) - \tilde{\sigma}_1^* \rho_*(C_t)$$
since \( \rho(C_t) \) is \( \delta_{23} \)-invariant. Pulling \( \gamma_t \) back via the blow-up and pushing it forward to \( E_t \times K \), we obtain
\[
\varpi_t = (id \times j_t)(\Delta_{E_t}) - ((-1)_{E_t} \times j_t)(\Delta_{E_t}) = (id \times j_t)_*(\Delta_{E_t} - \Gamma_{(-1)_{E_t}}) \in CH^2(E_t \times K)_Q
\]
Now, we may view \( \varpi_t \) as a correspondence between \( E_t \) and \( K \). By Liebermann’s lemma, we have
\[
\varpi_t = (id \times j_t)_*(\Delta_{E_t} - \Gamma_{(-1)_{E_t}}) = \Gamma_{j_t} \circ (\Delta_{E_t} - \Gamma_{(-1)_{E_t}}) \in Cor^1(E_t \times K)
\]
Let \( e_t \in CH^1(E_t)_Q \) denote the class of the identity element of \( E_t \); then, we have the decomposition:
\[
CH^1(E_t)_Q = Q \cdot e_t \oplus CH^1_{(1)}(E_t)_Q
\]
where \( CH^1_{(1)}(E_t)_Q \) is the summand on which \( (-1)_{E_t} \) acts by \(-id\). Since \( \gamma_t \) is 0 for all \( t \), it follows that for any \( \alpha \in CH^1(E_t)_Q \) of degree \( d \) we have
\[
0 = j_{t*}(\alpha) = j_{ts}(\alpha - de_t)
\]
from which it follows that \( j_{ts}(CH^1(E_t)_Q) = Qj_{ts}(e_t) \subset CH^2(K) \). Moreover, \( j_{ts}(e_t) = j_{ts}(e_s) \) for all \( s, t \) (since both lie on a rational curve). Thus, the image of the direct sum of pushforwards:
\[
\oplus_t CH^1(E_t)_Q \xrightarrow{\oplus j_{ts}} CH^2(K)_Q
\]
has rank 1. By our construction, \( K \) is an elliptic surface whose fibers are \( E_t \) and the \( E_t \) account for all but finitely many of the fibers. This would imply that \( CH^2(K)_Q \)
has rank 1, as desired.

\[\square\]

Corollary 3.1. Let \( k_0 \subset C \) be an algebraically closed field such that \( k_0(U) \subset C \) and let \( G_{k_0(U)} \) be the absolute Galois group and
\[
P_\gamma \in CH^2(U_{\text{et}}(A_{k_0(U)}))
\]
be the cycle constructed above (on the geometric generic fiber of \( A \to U \)). Then, for each prime \( \ell \) there is some \( r \gg 0 \) for which the cycle class
\[(7) \quad P_\gamma \neq 0 \in H^1(G_{k_0(U)}, PH^3_{et}(A_{k_0(U)}, \mathbb{Z}/\ell^r(2)))
\]
Moreover, the class \( \tilde{\sigma}_{123} \) is invariant with respect to \( \tilde{\sigma}_{23} \) and anti-invariant with respect to \( \tilde{\sigma}_{123} \).

Proof. From the Gysin sequence, the natural map
\[
H^1_{et}(U_{k_0}, PH^3_{et}(A_{k_0(U)}, \mathbb{Z}/\ell^r(2))) \to H^1(G_{k_0(U)}, PH^3_{et}(A_{k_0(U)}, \mathbb{Z}/\ell^r(2)))
\]
is injective. Moreover, since étale cohomology is unchanged under taking algebraically closed extensions, it suffices to show that
\[
P_\gamma \neq 0 \in H^1_{et}(U_C, PH^3_{et}(A_{C}(U), \mathbb{Z}/\ell^r(2)))
\]
Then, using the compatibility isomorphism with singular cohomology, it suffices to show that this is true for singular cohomology. On the other hand, from the above proposition, we have that
\[
P_\gamma \neq 0 \in H^1(U_C, PH^3(A_C, \mathbb{Q}(2)))
\]
which implies \( P_\gamma \in H^1(U_C, PH^3(A_C, \mathbb{Z}(2))) \) is non-torsion, which means that
\[
P_\gamma \neq 0 \in H^1(U_C, PH^3(A_C, \mathbb{Z}(2))) \otimes \mathbb{Z}/\ell^r
\]
for \( r \gg 0 \). Since \( H^3(A_C, \mathbb{Z}) \) is torsion-free, the universal coefficient theorem then gives the desired result.

\[ \square \]

4. **Nontriviality of \( \gamma \)**

We would like to realize the above construction as a cycle on the general product of three elliptic curves. For this, we will need the following result on the period map associated to \( A \rightarrow U \).

**Lemma 4.1.** Let \( k \subset \mathbb{C} \) be an algebraically closed field \( X(1) \) the \( j \)-line over \( k \). Then, the period map \( U \rightarrow X(1)^{\times 3} \) induced by \( g : A \rightarrow U \) is dominant.

**Proof.** Indeed, note that by Lemma 2.1 \( A_x \) is isogenous to \( Pic^0(C_x) \), so it suffices to prove the corresponding statement for the period map \( U \rightarrow A_3 \) induced by \( Pic^0(C) \rightarrow U \) (where \( A_3 \) is the coarse moduli space of ppav’s of dimension 3). But by the Torelli theorem, it suffices to prove the statement for the map \( U \rightarrow \mathcal{M}_3 \) (induced by \( C \rightarrow U \)), which will be done once we show that the image of \( U \rightarrow \mathcal{M}_3 \) has dimension 3 or, equivalently, that the fibers are generally of dimension 2. This latter is precisely the claim below:

**Claim 4.1.** For the general \( x \in U \), \( \{ y \in U \mid C_y \cong C_x \} \) has dimension 2.

**Proof of Claim.** First note that an isomorphism \( C_y \cong C_x \) is induced by an automorphism \( \phi : \mathbb{P}^2 \rightarrow \mathbb{P}^2 \) that maps \( C_y \) to \( C_x \). We would like to show that \( \phi \) must lie in an algebraic group of dimension 2. To this end, note that for the general \( x \in U \), \( Aut(C_x) = G \), (one can see this by specializing to the Fermat quartic curve whose automorphism group is the wreath product \( \mathbb{Z}/4 \wr \mathbb{S}_3 \) and then noting that only the elements of \( G \) spread out to automorphisms of the general fiber of \( C \rightarrow U \)). Thus, such a \( \phi \in PGL_3(k) \) must lie in \( N(G) \), the normalizer of \( G \leq PGL_3(k) \). Then, using a dimension argument, it is easy to see that \( k[G]^\times \rightarrow GL_3(k) \) is the group of diagonal matrices \( D \), so that \( N(G) = N(D) \). However, since \( D \) is a torus over \( k \), it is a well-known fact that the identity component of \( N(D) \) is the centralizer of \( D \), \( C(D) = D \). Thus, \( N(D) \) has dimension 2, as desired.

\[ \square \]

**Corollary 4.1.** Let \( k_0 \subset \mathbb{C} \) be an algebraically closed field for which \( k_0(U) \subset \mathbb{C} \). Then, \( A_{k_0(U)} \times_{k_0(U)} \mathbb{C} \) is isomorphic as a scheme to the very general product of 3 complex elliptic curves, \( E_1 \times E_2 \times E_3 \). Moreover, for each prime \( \ell \) and \( r \in \mathbb{N} \), we have

\[ CH^2(A_{k_0(U)}) \otimes \mathbb{Z}/\ell^r \cong CH^2(E_1 \times E_2 \times E_3) \otimes \mathbb{Z}/\ell^r \]

**Proof.** The first statement follows from the previous lemma and the second statement follows from Suslin rigidity \([12]\) in the form of \([9]\).

In order to prove the nontriviality of \( \gamma \), we will need the following lemma, whose proof relies on a deep result of Bloch-Esnault in \([2]\) as well as the Merkurjev-Suslin theorem. Similar results for more general Abelian varieties have been obtained by Schoen in \([9], [11]\) and Totaro in \([13]\).

**Lemma 4.2.** For each prime \( \ell \) and \( r \in \mathbb{N} \) and very general complex elliptic curves \( E_1, E_2 \) and \( E_3 \),

\[ P_\ast CH^2(E_1 \times E_2 \times E_3)[\ell^r] = 0 \]
Proof. While general results of this type are well-known to the experts, it is possible to determine explicitly the $\ell'$-torsion in this case. We refer the reader to the appendix for this. \hfill \Box

Corollary 4.2. Let $\ell$ be a prime, $m \in \mathbb{N}$ and $E_1, E_2, E_3$ be very general elliptic curves over $\mathbb{C}$. Then, there exists $\gamma \in CH^2(E_1 \times E_2 \times E_3)$ for which

$$\gamma \neq 0 \in CH^2(E_1 \times E_2 \times E_3) \otimes \mathbb{Z}/\ell'$$

is anti-invariant with respect to $(-1)_E$, for $r >> 0$.

Proof. By Corollary 3.1 we have

$$CH^2(A_{k_0(U)}) \otimes \mathbb{Z}/\ell' \cong CH^2(E_1 \times E_2 \times E_3) \otimes \mathbb{Z}/\ell'$$

so it suffices to prove the corresponding result for $A_{k_0(U)}$. Suppose by way of contradiction that for $r >> 0$, there is some $\delta \in CH^2(A_{k_0(U)})$ such that

$$\gamma = \ell' \cdot \delta$$

Then, following the argument of [2] Proposition 4.1 (or [9] Proposition 6.2 or [13] §2), we would like to show that $P_\ast \delta$ is defined over $k_0(U)$. To this end, let $G_{k_0(U)}$ be the absolute Galois group and $g \in G_{k_0(U)}$. Since $\gamma$ is defined over $k_0(U)$, we have $g(\gamma) = \gamma$ for all $g \in G_{k_0(U)}$. Thus,

$$0 = m(\gamma - g(\gamma)) = \ell' (\delta - g(\delta))$$

So, $\delta - g(\delta)$ is $\ell'$-torsion, which means that by the previous lemma,

$$0 = P_\ast (\delta - g(\delta)) = P_\ast (\delta - g(\delta))$$

So, $P_\ast \delta$ descends to a cycle over $k_0(U)$ which implies that

$$P_\ast \gamma = 0 \in CH^2(A_{k_0(U)})$$

Applying the cycle class map, we obtain a contradiction to Corollary 3.1. As in [13], the exact same argument applied to $2 \cdot \gamma$ shows that $\gamma$ satisfies the required anti-invariance property. \hfill \Box

5. Proof of Theorem 1.1

Beginning with $\gamma$, we will use Nori’s isogeny argument in [7] to obtain the required infinitely many cycles mod $\ell'$. We let $N \geq 3$ be a positive integer such that $\ell \nmid N$ and let $X(N)$ be the fine moduli space of elliptic curves with full level-$N$ structure, let $E(N) \to X(N)$ be the universal elliptic curve. Then, consider the triple product

$$(8) \quad A(N) := E(N) \times E(N) \times E(N) \to W(N) := X(N) \times X(N) \times X(N)$$

Observe that $W(N)$ is the fine moduli space parametrizing products of 3 elliptic curves with full level-$N$ structure and $A(N)$ is the universal triple product. Further, denote by

$$K_N = \mathbb{C}(W(N))$$

the function field of $W(N)$ and let $A_{K_N}$ be the generic fiber of $A$. Note that there are 3 involutions, $\sigma_1, \sigma_2$, and $\sigma_3$, on $A_{K_N}$ (induced by the action of $-1$ on each of the three factors). Then, by Corollary 4.2 there exists $\gamma \in CH^2(A_{K_N})$ for which

$$\gamma \neq 0 \in CH^2(A_{K_N}) \otimes \mathbb{Z}/\ell'$$
is anti-invariant with respect to each $\tilde{\sigma}_j$ for $r >> 0$. In order to make use of Nori's argument, we need a result that gives a characterization of the minimum field over which $\gamma$ is defined.

**Definition 5.1.** Given the function field $K = \mathbb{C}(V)$ of a variety over $\mathbb{C}$, we say that a finite extension $K \subset L$ is unramified if there exists an étale cover $\tilde{V} \to V$ such that $L = \mathbb{C}(\tilde{V})$.

**Lemma 5.1.** Let $L_N$ be an unramified extension of $K_N$ and let $Y$ be an étale cover $\rho : Y \to W(N)$ such that $L_N = \mathbb{C}(Y)$. Then, there exists étale covers $\rho_j : X_j \to X(N)$ such that $\rho_1 \times \rho_2 \times \rho_3 : X = X_1 \times X_2 \times X_3 \to W(N)$ factors through $\rho : Y \to W(N)$.

**Proof.** We have

$$\pi_1(Y) \leq \pi_1(X(N)) \times \pi_1(X(N)) \times \pi_1(X(N)) = \Gamma(N)^{\times 3}$$

which means that $\pi_1(Y)$ is a lattice in the semi-simple Lie group $SL_2(\mathbb{R})^{\times 3}$. By [15] Proposition 4.3.3 (a consequence of the Borel Density theorem), there exist lattices $\Gamma_j \subset SL_2(\mathbb{R})$ for $j = 1, 2, 3$ such that $\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3$ is of finite index in $\pi_1(Y)$. It follows that there are finite étale covers $\rho_j : X_j \to X(N)$ for $j = 1, 2, 3$ such that $\pi_1(X_j) = \Gamma_j$. Since

$$\Gamma \leq \pi_1(Y) \leq \pi_1(W(N))$$

there is a finite étale cover $X \to Y$ for which the composition

$$X \to Y \to W(N)$$

is $\rho_1 \times \rho_2 \times \rho_3$. \hfill $\square$

**Corollary 5.1.** In the notation of the above lemma, let $A_Y := A(N) \times_{W(N)} Y$. Then, there is an isomorphism of $X$-schemes,

$$A_Y \times_X Y \cong \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$$

where $\mathcal{E}_j := \mathcal{E}(N) \times_{X(N)} X_j$.

**Proof.** This follows from the fineness of the moduli space $W(N)$ and the fact that the composition $X \to Y \to W(N)$ coincides $\rho_1 \times \rho_2 \times \rho_3$. \hfill $\square$

**Proposition 5.1.** Let $L_N$ be an unramified extension of $K_N$. Then, for $r >> 0$

$$\gamma \not\in (CH^2(\overline{A_L}) \otimes \mathbb{Z}/\ell^r)^{\text{Gal}(\overline{L}/L_N)}$$

**Proof.** Let $K_N \subset K'_N$ be a finite extension over which

$$\gamma \in CH^2(\overline{A_L})$$

is defined (i.e., for which $\gamma$ lies in the image of $CH^2(A_{K'_N}) \to CH^2(\overline{A_L})$) and let $L'_N := K'_N L_N$. Our goal will be to show that

$$\gamma \not\in (CH^2(\overline{A_L}) \otimes \mathbb{Z}/\ell^r)^{\text{Gal}(\overline{L}/L_N)}$$

For this, we consider the cycle class:

$$(9) \quad P_* \gamma \in H^1(\text{Gal}(\overline{L}/L'_N), PH^3(\overline{A_{L_N}}, \mathbb{Z}/\ell^r(2)))$$
We need to show that (3) is not $Gal(L_N'/L_N)$-invariant. To this end, let $V_r := H^1(A_{\overline{L_N'}/\mathbb{Z}/\ell^r})$ and as at the beginning of §3 we have

$$W_r := V_r^\otimes 3 = PH^3(A_{\overline{L_N}}, \mathbb{Z}/\ell^r)$$

Using inflation-restriction, there is an exact sequence:

$$0 \to H^1(Gal(L_N'/L_N), W_r^{Gal(\overline{L_N}/L_N')} ) \to H^1(\Gal(\overline{L_N}/L_N), W_r ) \to H^1(\Gal(\overline{L_N'/L_N}), W_r )$$

Claim 5.1. $W_r^{Gal(\overline{L_N}/L_N')} = 0$

Proof of claim. Since $Gal(\overline{L_N}/L_N') \leq Gal(\overline{L_N}/K_N)$ has finite index, it suffices to show that $W_r$ is irreducible as a $Gal(\overline{L_N}/K_N)$-module. To this end, let $x \in W(N)(\mathbb{C})$ be very general; then, the problem is reduced to showing that $PH^3(A_{\overline{L_N}}, \mathbb{Z}/\ell^r)$ is irreducible as a $\pi_1(W(N))$-module. This is done in the Appendix.

Suppose by way of contradiction that (3) is $Gal(L_N'/L_N)$-invariant. Then, by the previous claim, this means that for $r >> 0$

$$P_* \gamma \in H^1(Gal(\overline{L_N}/L_N), W_r ) \cong H^1_\pi(W(N), PR^3g_\gamma^*\mathbb{Z}/\ell^r(2))$$

Lemma 5.2. $P_* \gamma \in \lim_{r \to \infty} H^1_\pi(W(N), PR^3g_\gamma^*\mathbb{Z}/\ell^r(2))$ is non-torsion.

Proof. It suffices to show that $P_* \gamma \neq 0$ after tensoring with $\mathbb{Q}/\ell$. Using the comparison isomorphism with singular cohomology, we reduce to showing that the cycle class:

$$P_* \gamma \neq 0 \in H^1(Y, PR^3g_\gamma^*\mathbb{Q}(2))$$

As in §3, it suffices to show that $P_* \gamma \neq 0 \in H^4(A_Y, \mathbb{Q}(2))$, which will follow from the following claim:

Claim 5.2. $P_* \gamma \neq 0 \in H^4(A_Y, \mathbb{Q}(2))$

Proof of Claim. By Proposition 5.1 this is true for $A$; one may then pass to some cover $W \to U$ for which $A_W := A \times_U W$ has the full level-$N$ structure so that (by fineness of $W(N)$) there is a (dominant) morphism $W \to W(N)$ with

$$A_W \cong A(N) \times_{W(N)} W$$

Then, we have a dominant morphism $W' : W \times_{W(N)} Y \to Y$ for which

$$A_W \times_{W} W' \cong A_Y \times_{Y} W'$$

By pulling back $\gamma$ via $A_W \times_{W} W' \to A_W$, it follows that

(10) $P_* \gamma \neq 0 \in H^4(A_W \times_{W} W', \mathbb{Q}(2)) \cong H^4(A_Y \times_{Y} W', \mathbb{Q}(2))$

Thus, if $P_* \gamma$ is defined over $A_Y$, its pullback to $A_W \times_{W} W'$ is precisely the cycle in (10).
To obtain a contradiction, observe that by pulling back via \(A_Y \times_X X \to A_Y\) and using Corollary 5.1
\[ P_*\gamma \neq 0 \in H^4(\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3, \mathbb{Z}/\ell^r(2)) \]
which is anti-invariant with respect to each of the three involutions \(\tilde{\sigma}_j\) on \(\mathcal{E}_j\). However, by the Künneth theorem, we have
\[ H^4(\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3, \mathbb{Z}/\ell^r) \cong \bigoplus_{i_1+i_2+i_3=4} H^{i_1}(\mathcal{E}_1, \mathbb{Z}/\ell^r) \otimes H^{i_2}(\mathcal{E}_2, \mathbb{Z}/\ell^r) \otimes H^{i_3}(\mathcal{E}_3, \mathbb{Z}/\ell^r) \]
It is easy to see that the anti-invariant part of \(H^k(\mathcal{E}_j, \mathbb{Z}/\ell^r) = 0\) unless \(k = 2\). Thus, none of the summands above contains a cycle which is anti-invariant with respect to all three involutions. Hence, the lemma. \(\square\)

**Remark 5.1.** Since we are working over \(SL_2(\mathbb{Z})\), we note that the lattices \(\Gamma_j\) in the above proof need not be congruence subgroups.

Now, \(L := \varinjlim K_n\) \(\to \mathcal{N} | n\)
We consider the absolute Galois group \(G_L\), and, following Nori’s suggestion at the end of [7], we notice that to every representation:
\[ G_L \xrightarrow{\phi} GL_n(\mathbb{Z}/\ell^r) \]
we may associate a (possibly empty) subset \(R(\phi)\) in the following way. Let \(H\) be the kernel of \(\phi\); then there is some finite Galois extension \(L \subset L'\) whose Galois group is the image of \(\phi\). This means for each \(m | N\), there is a compatible system of finite covers \(V(n) \to W(n)\) such that \(L'\) is the direct limit of \(K_n' := \mathcal{C}(V(N))\); let \(R_n(\phi)\) be the branch locus of this cover. The branch locus pulls back via \(\mathbb{H}^3 \to W(n)\) (where \(\mathbb{H}^3\) is the triple product of the upper half plane), and this pullback does not depend on the \(n\) chosen. The result of pulling back is \(R(\phi)\).

**Definition 5.2.** We say that \(\phi\) is a ramified representation if \(R(\phi) \neq \emptyset\).

Now, we consider the \(\mathbb{Z}/\ell^r[G_L]\)-submodule of \(CH^2(A_T) \otimes \mathbb{Z}/\ell^r\) generated by a \(\beta \in CH^2(A_T)\); we denote the module by \(M_{\beta, r}\) and the corresponding representation by \(\phi_{\beta}\). Then, we have the following lemma:

**Lemma 5.3.** \(\phi_{\gamma}\) is a ramified representation.

**Proof of lemma.** This follows directly from Proposition 5.1. Indeed, if \(R(\phi_{\gamma}) = \emptyset\), then \(\gamma \in CH^2(A_T) \otimes \mathbb{Z}/\ell^r\) would be invariant under \(Gal(L/F)\), where \(F\) is the direct limit of unramified extensions \(F_n\) of \(K_n\), which would contradict Proposition 5.1. \(\square\)

The remainder of the proof will be a sketch, since very similar arguments can be found in [7] and in [13]. There is an action on the field \(L\) by a group \(G\) which dominates \(GL_2(\mathbb{Q}) \times 3\), which can be described as follows. Indeed, for \(g \in (GL_2(\mathbb{Q}) \cap M_2(\mathbb{Z})) \times 3\),
there is the usual action on $\mathbb{H}^3$ and this action extends to $\mathbb{H}^3 \times \mathbb{C}^3$. Then, for each $n$, there is some $m \mid n$ and a commutative diagram:

$$
\begin{array}{ccc}
A(n) & \xrightarrow{g} & A(m) \\
\downarrow & & \downarrow \\
W(n) & \xrightarrow{g} & W(m)
\end{array}
$$

Taking the direct limit over $n$ gives an action of $g$ on $L$ which lifts to an action on the generic fiber, $A_L$ (which plays the role of an isogeny). Using the approach of Nori in [13], one constructs a group $\mathcal{G} \subset \text{Aut}(\mathcal{L}) \times G$ such that for $\tilde{g} = (\tau, g) \in \mathcal{G}$, there is an action on the Chow group

$$
\tilde{g}^*: CH^2(A_L) \rightarrow CH^2(A_L)
$$

which satisfies the following compatibility with the usual action of $G_L$ on the Chow group:

$$
\sigma \tilde{g}^* = \tilde{g}^* \sigma^{-1} : CH^2(A_L) \rightarrow CH^2(A_L)
$$

and, so long as $\tilde{g}^*: CH^2(A_L) \otimes \mathbb{Z}/\ell \rightarrow CH^2(A_L) \otimes \mathbb{Z}/\ell$ is injective, we have

$$
R(\phi_{\tilde{g}_j}) = g^{-1} R(\phi_{\gamma})
$$

(The injectivity can be achieved by ensuring that none of the denominators of the entries of $g \in G$ are divisible by $\ell$; see the Isogeny lemma in [13]). Now, view $SL_2(\mathbb{Q})$ as a subgroup of $G$ via the diagonal imbedding, and we select $g_1, g_2, \ldots \in SL_2(\mathbb{Q})$ so that $g_i$ represent different cosets modulo $SL_2(\mathbb{Z})$ (and so that the denominators of the entries are not divisible by $\ell$). Then, select corresponding lifts $\tilde{g}_j = (\tau, g_i) \in \mathcal{G}$. We would like to show that there are infinitely many elements in

$$
\{ \tilde{g}_j^* \gamma, \tilde{g}_2^* \gamma, \ldots \}
$$

Since $\tilde{g}_j^*$ were chosen to be injective, it suffices to show the representations $\phi_{\tilde{g}_j} \gamma$ are all distinct, which can be done by showing that the $R_j := R(\phi_{\tilde{g}_j} \gamma)$ are all distinct. To this end, we have the following lemma, as in [7] and [13]:

**Lemma 5.4.** The closed subgroup

$$
G_j := \{ \beta \in SL_2(\mathbb{Q}) \mid \beta(R_j) = R_j \}
$$

is a subgroup of $SL_2(\mathbb{Z})$.

**Proof.** Because $R_j$ is $SL_2(\mathbb{Z})$-stable, the Lie algebra of $G_j$ is stable under the adjoint action $SL_2(\mathbb{Z})$ (and, hence, also that of $SL_2(\mathbb{Q})$). Since $sl_2$ is simple, this forces the Lie algebra of $G_j$ to be 0. Using the Borel density theorem ([13] Theorem 7), it then follows that this group lies inside $SL_2(\mathbb{Z})$.

Thus, $gR_j = R_j \Rightarrow g \in SL_2(\mathbb{Z})$. Since $g_j$ were chosen to have distinct cosets modulo $SL_2(\mathbb{Z})$, the result of Theorem [13] now follows.

6. Appendix

**Proposition 6.1.** Let $E_1$, $E_2$ and $E_3$ be complex elliptic curves for which the set of $j$-invariants $\{j(E_1), j(E_2), j(E_3)\}$ has transcendence degree 3 over $\mathbb{Q}$ (so that the product $A = E_1 \times E_2 \times E_3$ is very general in the moduli of products of elliptic curves). Also, let

$$
\iota_{12} : E_1 \times E_2 \hookrightarrow A, \ \iota_{23} : E_2 \times E_3 \hookrightarrow A, \ \iota_{13} : E_1 \times E_3 \hookrightarrow A
$$
be the inclusions. Then, the $\ell$ torsion group:

$$CH^2(A)[\ell] = \bigoplus_{i \neq k} CH^1(E_i \times E_k)[\ell]$$

for all primes $\ell$.

Proof. Fix a prime $\ell$ and let $N$ be a positive integer for which $2\ell \nmid N$. Then, there exists a fine moduli space of elliptic curves with full level $N$ structure, $X(N)_{\mathbb{C}}$; let

$$f : \mathcal{E}(N)_{\mathbb{C}} \to X(N)_{\mathbb{C}}$$

be the corresponding universal elliptic curve and its triple product. It is well-known that $X(N)$ and $\mathcal{E}(N)_{\mathbb{C}}$ admit models over $\mathbb{Q}(\zeta_N)$ which have good reduction modulo $\ell$. In particular, the generic fiber of $g$, which we denote by $A$ can be viewed as an Abelian threefold over $\mathbb{Q}(\zeta_N)(W(N)) \subset \mathbb{Q}_{\ell}$ with good ordinary reduction. It follows that the main result of [2] applies and we deduce that

$$N^1 H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell(2)) \neq H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell(2))$$

where $N^*$ denotes the usual coniveau filtration on étale cohomology and, using the comparison isomorphism between étale and singular cohomology, we obtain

$$N^1 H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell) \neq H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell)$$

Now, we observe that

$$H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell) \cong \bigwedge^3 H^1(B_{\mathbb{C}}, A_{\mathbb{C}}, \mathbb{Z}/\ell) \cong V_1^{\oplus 2} \oplus V_2^{\oplus 2} \oplus V_3^{\oplus 2} \oplus V_1 \otimes V_2 \otimes V_3$$

where $V_i = H^1(E_i_{\mathbb{C}}, \mathbb{Z}/\ell)$ in the notation of the statement of the proposition. We consider the monodromy action of the fundamental group

$$\pi_1(W(N)) = \pi_1(X(N)) \times \pi_1(X(N)) \times \pi_1(X(N)) = \Gamma(N) \times \Gamma(N) \times \Gamma(N)$$

on (12), for which the $i$th $\Gamma(N)$ factor of $\pi_1(W(N))$ acts on $V_i$ via (the reduction modulo $\ell$ of) the standard representation. Since $\ell \nmid N$, the reduction map of $\Gamma(N) \twoheadrightarrow SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/\ell)$ is surjective. Thus, $\pi_1(X(N))$ acts on $V_i$ as $SL_2(\mathbb{Z}/\ell)$ and, hence, the $V_1 \otimes V_2 \otimes V_3$ component of (12) is an irreducible $\pi_1(W(N))$-module. Moreover, the first six summands of $H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell)$ in (12) lie in $N^1 H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell)$; in fact, first six summands correspond to the image of

$$\bigoplus_{j,k} H^1(E_{j,\mathbb{C}} \times E_{k,\mathbb{C}}, \mathbb{Z}/\ell) \xrightarrow{\boxtimes_{j,k}} H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell)$$

Now, $N^1 H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell)$ is invariant under the monodromy action of $\pi_1(U)$ for $U \subset W(N)$ open. Since the natural map $\pi_1(U) \to \pi_1(X)$ is surjective, we may view $H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell)$ as a $\pi_1(U)$-module, and the action of $\pi_1(U)$ induces the action of $\pi_1(W(N))$ on $H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell)$. As $\pi_1(U)$-modules, we have $N^1 H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell) \neq H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell)$ and $V_1 \otimes V_2 \otimes V_3$ irreducible. Thus, it follows that

$$N^1 H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell) = V_1^{\oplus 2} \oplus V_2^{\oplus 2} \oplus V_3^{\oplus 2}$$

We deduce from [2] that Bloch’s cycle class map

$$CH^2(A_{\mathbb{C}})[\ell] \to H^3_B(A_{\mathbb{C}}, \mathbb{Z}/\ell)$$
is an isomorphism whose image is $V_1^{\oplus 2} \oplus V_2^{\oplus 2} \oplus V_3^{\oplus 2}$. However, these summands correspond to the image of

$$\bigoplus_{jk} CH^1(E_j,C \times E_k,C)[\ell] \xrightarrow{\oplus \psi_{jk}*} CH^2(A_C)[\ell]$$

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