A MODEL FOR THE DECAY OF THE $D_s^+$ MESON INTO THREE PIONS

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Abstract

We propose a phenomenological two component model describing the decay amplitude of the process $D_s^+ \rightarrow 3\pi$, whose rate has been found surprisingly large. The first component is a constant background $F_{NR}$, and the second one is a Breit-Wigner type amplitude associated to a quasi two body $f_0(980)\pi^+$ state. We show that it is possible to reproduce the observed rate for $D_s^+ \rightarrow \pi^+\pi^+\pi^-$ as well as the two other measured branching ratios for the non resonant part and the resonant $f_0\pi^+$ one, with a common parameter $F_{NR}$.

Predictions are given for the $D_s^+ \rightarrow \pi^0\pi^0\pi^+$ rates, as well as for the various $\pi^+$ and $\pi^-$, or $\pi^0$ and $\pi^+$ energy distributions for the two possible final states.
I Introduction

We are interested, in this paper, in the decay of the $D_s^+$ meson into three pions. We observe that both flavoured constituents $c$ and $\bar{s}$ of the $D_s^+$ meson are absent in the final state. As a consequence, the decay mode $D_s^+ \rightarrow 3\pi$ can be described neither by a spectator, nor by a colour suppressed, nor by a penguin diagram, but only by a $W$ annihilation where the virtual time-like $W^+$ decays into three pions.

It is generally expected that $W$ annihilation amplitudes are small compared to spectator and colour suppressed ones. Surprisingly experimental data do not seem to support this belief. As an illustration, let us consider the three following decay modes of the $D_s^+$ meson involving the same Cabibbo, Kobayashi, Maskawa (CKM) factor $V_{cs}^* V_{ud}$ and compare their branching ratios [1]:

i) Spectator ;

$$Br(D_s^+ \rightarrow \phi + \pi^+) = (3.5 \pm 0.4)\%,$$

ii) Colour suppressed ;

$$Br(D_s^+ \rightarrow \overline{K^0} + K^+) = (3.5 \pm 0.7)\%,$$

iii) $W$ annihilation ;

$$Br(D_s^+ \rightarrow \pi^+ + \pi^+ + \pi^-) = (1.35 \pm 0.31)\%.$$

It is therefore interesting to understand the origin for such a large value of the $D_s^+ \rightarrow 3\pi$ branching ratio.

The transition of the virtual $W$ into three pions occurs also in the $\tau$ decay mode $\tau^+ \rightarrow \nu_\tau + 3\pi$. However, in this case, the full axial vector weak current contributes, and the amplitude depends on three independent structure functions. For the decay $D_s^+ \rightarrow 3\pi$, only one structure function associated to the divergence of the axial vector current is involved, because here the $W$ is in the $J^P = O^-$ state, like the $D_s$ meson. Therefore the study of $D_s \rightarrow 3\pi$ decay is the cleanest way to isolate the structure function $F_4$ of references [2, 3].

In Section II, we develop the formalism for $D_s^+ \rightarrow 3\pi$ decay. We study the kinematics and the three pion phase space. Experimental data as quoted in Reference [1] are listed.

By inspection of these experimental data, we observe that the quasi two body state $\rho^0\pi^+$ assumed in [4, 5] to be responsible of the structure function $F_4$ is negligible compared to the quasi two body state $f_0(980)\pi^+$. Moreover, since the non resonant rate is large, we propose a model in Section III where
the decay amplitude for $D_s^+ \to 3\pi$ has two components: the first one is associated to a background non resonant three pions, and the second one corresponds to a quasi two body $f_0 \pi^+\pi^-$ final state.

We compare our model with experiment in Section IV for the final state $\pi^+\pi^+\pi^-$ where data on rates are available and we make predictions for the $\pi^+$ and $\pi^-$ energy distributions.

For the final state $\pi^0\pi^0\pi^+$, using isospin analysis, we make predictions in Section V for the rates and the $\pi^0$ and $\pi^+$ energy distributions.

Our model is consistent with the rate measurements. However the serious test would be the observation of the various $\pi$ meson energy distributions and ultimately, if statistics is copious enough, of the Dalitz plots.

II Generalities and Kinematics

We study the decay of a $D_s^+$ meson of energy momentum $P$ into three pions of energy momenta $p_1, p_2, p_3$ with the relation $P = p_1 + p_2 + p_3$. We introduce the Mandelstam variables $s_1, s_2, s_3$ and the $\pi$ meson energies $E_1, E_2, E_3$ in the $D_s^+$ rest frame. Neglecting the mass difference between charged and neutral pions, we get

$$

g_1 = (p_2 + p_3)^2 = (P - p_1)^2 = m_{D_s}^2 + m_{\pi}^2 - 2m_{D_s}E_1 , \\
g_2 = (p_3 + p_1)^2 = (P - p_2)^2 = m_{D_s}^2 + m_{\pi}^2 - 2m_{D_s}E_2 , \\
g_3 = (p_1 + p_2)^2 = (P - p_3)^2 = m_{D_s}^2 + m_{\pi}^2 - 2m_{D_s}E_3 .
$$

Energy momentum conservation implies the relations

$$

s_1 + s_2 + s_3 = m_{D_s}^2 + 3m_{\pi}^2 , \quad E_1 + E_2 + E_3 = m_{D_s} . \quad (2)
$$

The double differential distribution is given by

$$

d\Gamma = \frac{1}{64\pi^3} \frac{1}{m_{D_s}} |<3\pi|T|D_s^+>|^2 \, dE_1 \, dE_2 , \quad (3)
$$

and the transition matrix element $<3\pi|T|D_s^+>$ involving only spinless particles is dimensionless.

In the $(E_1, E_2)$ plane, the phase space is defined by the constraints

$$

m_{\pi} \leq E_1 \leq \frac{m_{D_s}^2 - 3m_{\pi}^2}{2m_{D_s}} , \\
E_-(E_1) \leq E_2 \leq E_+(E_1) , \quad (4)
$$

with

$$

E_\pm(E) = \frac{1}{2}(m_{D_s} - E) \pm \frac{1}{2} \left\{ \frac{(E^2 - m_{\pi}^2)(m_{D_s}^2 - 3m_{\pi}^2 - 2m_{D_s}E)}{m_{D_s}^2 + m_{\pi}^2 - 2m_{D_s}E} \right\}^{1/2} . \quad (5)
$$
Of course, the mass difference between charged and neutral pions being neglected, we have the same phase space in the two other planes : \((E_2, E_3)\) and \((E_1, E_3)\).

At the quark level, the decay \(D_s^+ \to 3\pi\) is described by a \(W\) annihilation diagram, and the transition matrix element is given by

\[
<3\pi|T|D_s^+> = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} a_1 f_{D_s} P^\mu <3\pi|A_\mu|0> .
\]

where \(a_1\) is the phenomenological parameter introduced by Bauer, Stech, Wirbel\(^4\), and \(f_{D_s}\) is the leptonic decay constant of the \(D_s\) meson. The matrix element of the divergence of the weak axial current between the vacuum and three pion final state is an unknown form factor \(F(E_1, E_2)\), function of two independent variables, taken as the pion energies.

\[
P^\mu <3\pi|A_\mu|0> = m_{D_s} F(E_1, E_2) .
\]

Since the transition matrix element is dimensionless, the form factor \(F(E_1, E_2)\)\(^5\) is also dimensionless.

In the \((E_1, E_2)\) plane, the Dalitz plot is given by:

\[
d\Gamma(D_s^+ \to 3\pi) = \frac{m_{D_s}}{64 \pi^3} \left(\frac{G_F}{\sqrt{2}}\right)^2 |V_{cs}|^2 |V_{ud}|^2 a_1^2 f_{D_s}^2 |F(E_1, E_2)|^2 dE_1 dE_2 .
\]

The decay rate is obtained by integration of the distribution (8) over the energies \(E_1\) and \(E_2\) inside the phase space described in Eq(4). The branching ratio can be written in the form:

\[
Br(D_s^+ \to 3\pi) = \frac{\tau_{D_s^+} m_{D_s}}{h} \frac{1}{64 \pi^3} \left(\frac{G_F m_{D_s}^2}{\sqrt{2}}\right)^2 |V_{cs}|^2 |V_{ud}|^2 a_1^2 \left(\frac{f_{D_s}}{m_{D_s}}\right)^2 I ,
\]

where the dimensionless integral \(I\) is defined by:

\[
I = \frac{1}{m_{D_s}^2} \int \int |F(E_1, E_2)|^2 dE_1 dE_2 .
\]

Using \(\tau_{D_s^+} = 4.67 \cdot 10^{-13} s\)\(^6\) and \(f_{D_s} = 280 MeV\)\(^5\), we get

\[
Br(D_s^+ \to 3\pi) = 1.31 \cdot 10^{-2} a_1^2 I .
\]

From the analysis of colour favoured \(D\) meson decay, a reasonable value for \(a_1\) is \(a_1 = 1.26\)\(^5\) and we obtain

\[
Br(D_s^+ \to 3\pi) = 2.08 \cdot 10^{-2} I .
\]

\(^1\)Our \(F(E_1, E_2)\) is related to the \(F_1\) of the Ref.\(^\#\) by \(F(E_1, E_2) = m_{D_s} F_4(s_1, s_2, Q^2 = m_{D_s}^2)\).

\(^2\)The leptonic decay \(D_s^+ \to \mu^+ + \nu_\mu\) has been measured by two groups and the extracted values of \(f_{D_s}\) are \(f_{D_s} = (232 \pm 69) MeV\) and \(f_{D_s} = (344 \pm 76) MeV\). Our choice of \(f_{D_s} = 280 MeV\) is consistent with data and with theoretical expectations.
We make the following choice of $\pi$-meson energy variables:

i) final state $\pi^+\pi^+\pi^-$

$$E_1(\pi^+), E_2(\pi^+), E_3(\pi^-),$$

ii) final state $\pi^0\pi^0\pi^+$

$$E_1(\pi^0), E_2(\pi^0), E_3(\pi^+).$$

Due to Bose-Einstein symmetry, the function $F(E_1, E_2)$ is symmetrical in the exchange between $E_1$ and $E_2$. Such a property extends to the Dalitz plot given by the double differential distribution represented by $|F(E_1, E_2)|^2$. It is probably premature to discuss the details of the Dalitz plot and it might be interesting to define one meson energy distribution $s$:

$$G(E_1) = \frac{1}{m_{D^+_s}} \int_{E_{\pi^+}(E_1)}^{E_{\pi^-}(E_1)} |F(m_{D^+_s} - E_2 - E_3, E_2)|^2 \, dE_2,$$

(13)

$$H(E_3) = \frac{1}{m_{D^+_s}} \int_{E_{\pi^-}(E_3)}^{E_{\pi^+}(E_3)} |F(m_{D^+_s} - E_2 - E_3, E_2)|^2 \, dE_2,$$

(14)

where $G(E_1)$ corresponds to the $\pi^+(\pi^0)$ for the final state $\pi^+\pi^+\pi^- (\pi^0\pi^0\pi^+)$, and $H(E_3)$ corresponds to the $\pi^-(\pi^+)$ for the final state $\pi^+\pi^+\pi^- (\pi^0\pi^0\pi^+)$. The presently available data concern the rates for the decay mode $D^+_s \to \pi^+\pi^+\pi^-$. They are given in Table 1, the last column indicates the value of the quantity $I$ determined from experiment [1] by using Eq.(12).

| Mode                | Experimental Branching Ratios | Experimental values of $I$ |
|---------------------|-------------------------------|---------------------------|
| $\pi^+\pi^+\pi^-$   | $(1.35 \pm 0.31) \cdot 10^{-2}$ | $0.649 \pm 0.149$        |
| $(\pi^+\pi^+\pi^-)_{NR}$ | $(1.01 \pm 0.35) \cdot 10^{-2}$ | $0.486 \pm 0.168$        |
| $f_0\pi^+$          | $(1 \pm 0.4) \cdot 10^{-2}$   | $0.481 \pm 0.192$        |
| $\rho^0\pi^+$       | $\leq 0.28 \cdot 10^{-2}$     | $\leq 0.135$             |

Table 1

### III. Phenomenological Model for $D^+_s \to 3\pi$

In order to describe the decay mode $D^+_s \to 3\pi$, we use a simple phenomenological model where the function $F(E_1, E_2)$ is the sum of two contributions:

$$F(E_1, E_2) = F_{NR} + F_{RES}(E_1, E_2).$$

(15)

Obviously, this decomposition is guided by experimental data [1]. The first component $F_{NR}$ is assumed to be a real constant describing the non resonant part of the amplitude. The second component
\( F_{\text{RES}}(E_1, E_2) \) is associated to the quasi two body final state \( D^+_s \to f_0 + \pi^+ \) followed by the subsequent decays \( f_0 \to \pi^+\pi^- \) or \( \pi^0\pi^0 \).

1. At first, consider the charged case \( f_0 \to \pi^+\pi^- \). The \( \pi^- \) meson has an energy \( E_3 \) and the two \( \pi^+ \) mesons energies \( E_1 \) and \( E_2 \). We then have two possible resonant combinations and the general form of \( F_{\text{RES}}(E_1, E_2) \) is:

\[
F_{\text{RES}}(E_1, E_2) = D_c \{ BW(E_1) + BW(E_2) \} ,
\]

where the dimensionless Breit-Wigner function \( BW(E_j) \) is defined by:

\[
BW(E_j) = \frac{m^2_{f_0}}{m^2_{f_0} - s_j - i\sqrt{s_j} \Gamma_{f_0}} ,
\]

with \( s_j \) and \( E_j \) related by \( s_j = m^2_{D^+_s} + m^2_\pi - 2m_{D^+_s}E_j \ (j = 1, 2) \).

The complex constant \( D_c \) corresponds to the transition \( W^+ \to f_0\pi^+ \), for which we assume, by the Partial Conservation of the Axial Current (PCAC) in Eqs. (1) and (2), the existence of an intermediate state having the quantum number of a \( \pi^+ \) meson \([2, 3, 6]\) and which might be the \( \pi \) meson itself or its recurrence \( \pi(1300) \). It turns out that the \( \pi \) meson intermediate state will give a contribution many order of magnitude smaller than that of the \( \pi(1300) \) and only the later one is retained given the following expression for \( D_c \):

\[
D_c = \frac{m_{D^+_s}}{m_{f_0}} \frac{f_{\pi'}}{m^2_{\pi'}} - \frac{m^2_{D^+_s} - i m_{D^+_s} \Gamma_{\pi'}}{m_{\pi'}} g_{\pi'f_0\pi} g_{f_0\pi^+\pi^-} .
\]

In Eq. (18), \( \pi' \equiv \pi(1300) \), and the dimensionless coupling constants \( g_{M_0M_1M_2} \) describe the decay of a spinless meson \( M_0 \) into two spinless mesons \( M_1 \) and \( M_2 \):

\[
< M_1 M_2 \mid T \mid M_0 > = m_0 \ g_{M_0M_1M_2} .
\]

The term \( m_{D^+_s} \) in the numerator of Eq. (18) comes from the definition Eq. (5), the term \( m_{f_0} \) in the denominator of Eq. (18) when combined with Eq. (17) yields the definition Eq. (19). Also \( m_{\pi'} \) comes from Eq. (15) and \( f_{\pi'} \) comes from the coupling between \( \pi' \) and \( W \). The numerical value of \( g_{M_0M_1M_2} \) is obtained from the experimental decay rate \( \Gamma(M_0 \to M_1 + M_2) \) by the relation

\[
g^2_{M_0M_1M_2} = \frac{8 \pi \Gamma(M_0 \to M_1 + M_2)}{K} ,
\]

where \( K \) is the final momentum in the \( M_0 \) rest frame.

\footnote{We have used for width the simple energy dependence \( \Gamma(s) = \sqrt{s} \Gamma(m) \). In the case of the \( \pi(1300) \) width, more sophisticated dependence has been proposed \([4]\). Because of the not too large difference between the \( \pi(1300) \) and \( D^+_s \) masses, we expect the sensitivity of Eq. (18) to different forms of \( \Gamma_{\pi'}(s) \) to be relatively modest.}
Now we consider first the case \( f_0 \rightarrow 2\pi \). Using the experimental parameters [1]:

\[
m_{f_0} = 980 \text{ MeV} \quad , \quad \Gamma_{f_0} = (49 \pm 9) \text{ MeV} \quad ,
\]

\[
Br(f_0 \rightarrow 2\pi) = 0.781 \pm 0.024 \quad ,
\]

we obtain

\[
g_{f_0\pi^+\pi^-} = 1.144 \pm 0.109 \quad , \quad g_{f_0\pi^0\pi^0} = 0.809 \pm 0.077 \quad .
\]

The isoscalar property of the \( f_0 \) meson has been taken into account for relating both coupling constants.

Secondly, for the decay \( \pi(1300) \rightarrow f_0 + \pi \), experimental data are poor [1] and we shall use :

\[
m_{\pi'} = 1300 \text{ MeV} \quad , \quad \Gamma_{\pi'} = (400 \pm 200) \text{ MeV} \quad ,
\]

\[
Br(\pi' \rightarrow f_0\pi) = 0.68 \quad .
\]

The result is

\[
g_{\pi'f_0\pi} = 5.2 \pm 1.2 \quad .
\]

The last parameter entering in Eq.(18) is the leptonic constant \( f_{\pi'} \). The estimate of reference [3] is

\[
f_{\pi'} = 40 \text{ MeV}
\]

Now we are in a position to compute the numerical value of the dimensionless coupling constant \( D_c \) written in the form \( D_c = |D_c| \exp(i \Phi_D) \). Retaining only the large error due to the poor knowledge of the total \( \pi(1300) \) width, we obtain for different \( \Gamma_{\pi'} \) values :

\[
\begin{array}{|c|c|c|}
\hline
\Gamma_{\pi'} & |D_c| & \Phi_D \\
600 \text{ MeV} & 0.3068 & 152^\circ \\
400 \text{ MeV} & 0.2679 & 160^\circ \\
200 \text{ MeV} & 0.1981 & 170^\circ \\
\hline
\end{array}
\]

2). In the second case \( f_0 \rightarrow \pi^0\pi^0 \), we call \( E_1 \) and \( E_2 \) the energies of the \( \pi^0 \)’s and \( E_3 \) the energy of the \( \pi^\pm \). The general form of \( F_{RES}(E_1, E_2) \) is

\[
F_{RES}(E_1, E_2) = D_N \text{ BW}(E_3) \quad ,
\]

and because of Eq.(22), \( D_N = D_c/\sqrt{2} \).

IV. Results of the model for \( D_s^+ \rightarrow \pi^+\pi^+\pi^- \)

The results of the model for the one pion energy distributions and for the rates are now given in the three following cases :

\footnote{Different values of \( f_{\pi'} \) have been proposed in the literature. See for instance references [4] and [5].}
1) non resonant $\pi^+\pi^+\pi^-$ state : $F_{NR}$

2) quasi two body $f_0\pi^+ \to \pi^+\pi^-\pi^+$ state : $F_{RES}$

3) superposition of the non resonant and resonant amplitudes : $F_{NR} + F_{RES}$

1). In the non resonant case, we have an uniform Dalitz plot and for a constant $F_{NR}$, we get from Eqs.(13) and (14) :

$$G_{NR}(E) = H_{NR}(E) = F_{NR}^2 \frac{1}{m_{Ds}} \left\{ \frac{(E^2 - m_\pi^2)(m_{Ds}^2 - 3m_\pi^2 - 2m_{Ds}E)}{m_{Ds}^2 + m_\pi^2 - 2m_{Ds}E} \right\}^{1/2}.$$ (28)

Here $E$ stands for $E_1$, $E_2$, $E_3$ indifferently.

The energy distribution is represented on Figure 1 for $F_{NR} = 1$. The integral $I_{NR}$ given by

$$I_{NR} = \frac{1}{m_{Ds}} \int_{m_\pi}^{m_{Ds} - 3m_\pi^2} G_{NR}(E) \ dE ,$$ (29)

has the numerical value

$$I_{NR} = 0.1053 \ F_{NR}^2 .$$ (30)

Using, for $I_{NR}$, the experimental value quoted in Table 1 in the non resonant case, we obtain

$$|F_{NR}| = 2.15 \pm 0.34 \ .$$ (31)

2). Now we consider the quasi two body case $D_s^+ \to f_0 + \pi^+ \to \pi^+\pi^-\pi^+$ for which the amplitude $F_{RES}$ is given by Eq.(16). It is convenient to write

$$|F_{RES}(E_1, E_2)|^2 = |D_c|^2 \ K_c(E_1, E_2) ,$$ (32)

where the function $K_c(E_1, E_2)$ is the sum of three terms ;

$$K_c(E_1, E_2) = |BW(E_1)|^2 + |BW(E_2)|^2 + 2Re \{BW(E_1)BW(E_2)^*\} .$$ (33)

After integration of $K_c(E_1, E_2)$ over $E_2$ at fixed $E_1$, as indicated in Eq.(13), we obtain the $\pi^+$ meson energy distribution in the form :

$$G_{RES}(E_1) = |D_c|^2 \ K_c^{(+)}(E_1) .$$ (34)

The function $K_c^{(+)}(E_1)$ is represented on Figure 2. Clearly we see the narrow peak due to the Breit-Wigner term in $E_1$ and a quasi-constant background corresponding to the Breit-Wigner term in $E_2$.

The interference between the two Breit-Wigner terms gives a very small contribution to $K_c^{(+)}(E_1)$.

**In the limit $m_\pi = 0$, the calculation of $I_{NR}$ is trivial and the result is $I_{NR} = 0.125 \ F_{NR}^2$. Correction due to $m_\pi \neq 0$ is large and of the order of 16% in spite of the very small value of $m_\pi/m_{Ds} \simeq 0.07$.**
Integrating now $K_c(E_1, E_2)$ over $E_2$ at fixed $E_3$, as indicated in Eq.(14), we obtain the $\pi^-$ meson energy distribution:

$$H_{RES}(E_3) = |D_c|^2 K_c^{(-)}(E_3) \ .$$

(35)

The function $K_c^{(-)}(E_3)$ is also represented on Figure 2. The result is a quasi-constant plateau in most of the allowed phase space domain of $E_3$.

In order to obtain the rate, we must perform a second energy integration. The quantity $I_{RES}$ is written in the form:

$$I_{RES} = |D_c|^2 K_c \ ,$$

(36)

where $K_c$ is given by:

$$K_c = \frac{1}{m_{D_s}} \int_{m_s}^{m_{D_s}^2} K_c^{(+)}(E_1) \ dE_1 = \frac{1}{m_{D_s}} \int_{m_s}^{m_{D_s}^2} K_c^{(-)}(E_3) \ dE_3 \ .$$

(37)

The numerical value obtained for $K_c$ is $K_c = 5.7217$.

Using now the results of Eq.(26) for $|D_c|$, we obtain the numerical value in our model of $I_{RES}$ defined in Eq.(36). The result is $I_{RES} = 0.411 \pm 0.128$,$^{+0.186}_{-0.186}$, (38)

where, as previously explained, the errors correspond to $\Gamma_{\pi'} = 400 \pm 200$MeV.

The theoretical value of $I_{RES}$ in Eq.(38) is consistent with the measured value of $0.481 \pm 0.192$ quoted in Table 1. If we consider seriously the experimental one standard deviation limit, $I_{RES} \geq 0.289$, we get a lower bound for $\Gamma_{\pi'}$, $\Gamma_{\pi'} \geq 263$MeV (assuming $f_{\pi'} = 40$MeV). There is no upper bound constraint on $\Gamma_{\pi'}$.

3). Finally we consider the full amplitude written in Eq.(15). As the constant $F_{NR}$ is real, we get

$$|F(E_1, E_2)|^2 = F_{NR}^2 + 2 F_{NR} \ Re \{F_{RES}(E_1, E_2)\} + |F_{RES}(E_1, E_2)|^2 \ .$$

(39)

We integrate over $E_1$ and $E_2$ in order to obtain the quantity $I$ of Eq.(10).

Using the previous results Eqs.(30) and (36), we obtain

$$I = 0.1053 \ F_{NR}^2 + 2 \ F_{NR} \ |D_c| \ |J| \ \ Cos(\Phi_D + \Phi_J) + 5.7217 \ |D_c|^2 \ ,$$

(40)

where the complex integral $J$ is defined by:

$$J = \frac{1}{m_{D_s}^2} \int \int \{BW(E_1) + BW(E_2)\} \ dE_1 \ dE_2 = |J| \ e^{i\Phi_J} \ .$$

(41)

The results are

$$|J| = 0.2677 \ , \quad \Phi_J = 89.60^0 \ .$$

(42)
Using the experimental constraints for $I_{NR}$ and $I$ given in Table 1, from Eqs.(30) and (40), we can check the consistency of our model by determining the parameter $F_{NR}$ and compare to Eq.(31). The results are presented on Table 2 for three value of $\Gamma_{\pi'}$, $\Gamma_{\pi'} = 600, 400, 263$ MeV. For large value of $\Gamma_{\pi'}$, we obtain only positive solutions for $F_{NR}$. When $\Gamma_{\pi'}$ decreases, $|D_c|$ decreases and the interference term in Eq.(40) becomes smaller, then we obtain both positive and negative solutions for $F_{NR}$.

| $\Gamma_{\pi'}$ (MeV) | $F_{NR}$ | $I_{NR}$ | $I$ | $I_{RES}$ |
|-----------------------|----------|----------|-----|-----------|
| 600                   | 1.7378   | 1.9898   | 0.318 0.417 | 0.719 0.798 0.5385 |
| 400                   | 1.7378   | 2.1680   | 0.318 0.495 | 0.642 0.798 0.4105 |
| 263                   | 1.7378   | 2.3385   | 0.318 0.576 | 0.557 0.798 0.2890 |
| -1.7378 -2.0671       | 0.318 0.576 | 0.318 0.456 | 0.657 0.798 0.2890 |
| Experiment            | -        | 0.318 − 0.654 | 0.500 − 0.798 0.289 − 0.673 |

Table 2

The full pion energy distributions coming from the total amplitude in Eq.(39) depend on the two quantities $F_{NR}$ and $D_c$. Writing the squared modulus of the total amplitude in the form :

$$|F(E_1, E_2)|^2 = F_{NR}^2 + 2 F_{NR} Re \{D_c J_c(E_1, E_2)\} + |D_c|^2 K_c(E_1, E_2) \ ,$$

where $K_c(E_1, E_2)$ is given by Eq.(33) and $J_c(E_1, E_2)$ is simply given by :

$$J_c(E_1, E_2) = BW(E_1) + BW(E_2) \ .$$

Of course, $K_c(E_1, E_2) = |J_c(E_1, E_2)|^2$.

After integration of Eq.(33) over $E_2$ at fixed $E_1$, we obtain the $\pi^+$ meson energy distribution $G_c(E_1) :

$$G_c(E_1) = F_{NR}^2 G_{NR}(E_1) + 2 F_{NR} Re \{D_c J_c^+(E_1)\} + |D_c|^2 K_c^+(E_1) \ .$$

The quantities $G_{NR}(E_1)$ and $K_c^+(E_1)$ have been represented on Figures 1 and 2, respectively, and $J_c^+(E_1)$ is defined by :

$$J_c^+(E_1) = \frac{1}{m_{D_c}} \int_{E_-(E_1)}^{E_+(E_1)} J_c(E_1, E_2) \ dE_2 \ .$$

In a similar way, integrating over $E_2$ at fixed $E_3$, we obtain the $\pi^-$ meson energy distribution in the form :

$$H_c(E_3) = F_{NR}^2 G_{NR}(E_3) + 2 F_{NR} Re \{D_c J_c^-(E_3)\} + |D_c|^2 K_c^-(E_3) \ .$$


In Eq.(47), $G_{NR}(E_3)$ as given by Eq.(28) is represented on Figure 1. The quantity $K_c^-(E_3)$ is represented on Figure 2 and $J_c^-(E_3)$ is defined by:

$$J_c^-(E_3) = \frac{1}{m_{D_s}} \int_{E_-(E_3)}^{E_+(E_3)} J_c(m_{D_s} - E_2 - E_3, E_2) \, dE_2 \ .$$

As mentioned above, the full $\pi^+$ and $\pi^-$ energy distributions $G_{c}(E_1)$ and $H_{c}(E_3)$ are functions of $F_{NR}$ and $D_c$. They are represented respectively in Figures 3 and 4.

V. Prediction of the model for $D_s^+ \rightarrow \pi^0\pi^0\pi^+$

We have no experimental data on the decay mode $D_s^+ \rightarrow \pi^0\pi^0\pi^+$. However our model can make predictions for the three following cases:

1) non resonant $\pi^0\pi^0\pi^+$ state,
2) quasi two body $f_0\pi^+ \rightarrow \pi^0\pi^0\pi^+$ state,
3) superposition of the non resonant and resonant amplitude.

1). In the non resonant case of a constant function $F_{NR}(E_1, E_2)$, we obtain the same shape for one pion energy distribution as shown on Figure 1.

However the constant $F_{NR}$ has no reason to be the same for the two modes $\pi^+\pi^+\pi^-$ and $\pi^0\pi^0\pi^-$. We now discuss this problem. By assumption, with a constant function $F_{NR}(E_1, E_2)$, we have a full symmetry in space between the three pions. From the Bose-Einstein symmetry, the isospin configuration has also to be totally symmetric. Consider now a third rank fully symmetric tensor in a 3 dimensional space. It has 10 independent components. With respect to the isospin SO(3) orthogonal group, such a tensor is reducible into an isospin $I = 3$ part with 7 components and an isospin $I = 1$ part with 3 components. In our specific problem, only the later part contributes, the $ud\overline{7}$ weak current being an isovector. By inspection of the relevant Clebsch-Gordan coefficients, we obtain the result :

$$F_{NR}(\pi^+\pi^+\pi^-) = 3 \ F_{NR}(\pi^0\pi^0\pi^+) \ .$$

As a consequence, in our model, we obtain for the non resonant part :

$$\frac{\Gamma(D_s^+ \rightarrow \pi^0\pi^0\pi^+)_{NR}}{\Gamma(D_s^+ \rightarrow \pi^+\pi^+\pi^-)_{NR}} = 11.1 \% \ ,$$

and the non resonant branching ratio $Br(D_s^+ \rightarrow \pi^0\pi^0\pi^+)_{NR}$ is expected to occur only at the $10^{-3}$ level.

2). For the quasi two body decay of $D_s^+ \rightarrow f_0\pi^+ \rightarrow \pi^0\pi^0\pi^+$, the amplitude $F_{RES}$ is given by Eq.(27):

$$|F_{RES}(E_1, E_2)|^2 = |D_N|^2 \ K_N(E_1, E_2) \ .$$
where the function $K_N(E_1, E_2)$ is simply given by:

$$K_N(E_1, E_2) = |BW(E_3)|^2 = |BW(m_{D_s} - E_1 - E_2)|^2$$

(52)

After integration over $E_2$ at fixed $E_1$, as indicated in Eq.(13), we obtain the $\pi^0$ meson energy distribution in the form:

$$G_{RES}(E_1) = |D_N|^2 K_N^{(0)}(E_1)$$

(53)

The function $K_N^{(0)}(E_1)$ is a quasi flat distribution represented on Figure 5.

Integrating now over $E_2$ at fixed $E_3$, as indicated in Eq.(14), we have the $\pi^+$ energy distribution in the form:

$$H_{RES}(E_3) = |D_N|^2 K_N^{(+)}(E_3)$$

(54)

The function $K_N^{(+)}(E_3)$ is represented on Figure 5 and the result exhibits a narrow Breit-Wigner peak similar to the one drawn on Figure 2 for $K_s^+(E_1)$.

The decay rate is now obtained by performing a second energy integration. The quantity $I_{RES}$ is written in the form:

$$I_{RES} = |D_N|^2 K_N$$

(55)

where $K_N$ is given by:

$$K_N = \frac{1}{m_{D_s}} \int_{m_{\pi}}^{m_w^2 - 3m_{\pi}^2} K_N^{(0)}(E_1) \, dE_1 = \frac{1}{m_{D_s}} \int_{m_{\pi}}^{m_w^2 - 3m_{\pi}^2} K_N^{(+))(E_3)} \, dE_3$$

(56)

The numerical value of $K_N$ is $K_N = 2.7469$.

Using the relation between $D_N$ and $D_c$, we make the prediction

$$\frac{\Gamma(D_s^+ \to f_0\pi^+ \to \pi^0\pi^0\pi^+)}{\Gamma(D_s^+ \to f_0\pi^+ \to \pi^+\pi^-\pi^+)} = 24\%$$

(57)

The departure of this ratio from 25 % is simply due to the interference between the two Breit-Wigner contributions in $D_{s}^{+} \to f_0\pi^{+} \to \pi^{+}\pi^{-}\pi^{+}$.

3). We finally consider the full amplitude written in the form of Eq.(15). An equation similar to Eq.(40) gives the total rate, and going from the mode $\pi^+\pi^+\pi^-$ to the mode $\pi^0\pi^0\pi^+$ implies the changes:

$$F_{NR} \to \frac{1}{3} F_{NR} \ , \ D_c \to \frac{1}{\sqrt{2}} D_c \ , \ J \to \frac{1}{2} J \ , \ K_c \to K_N$$

(58)

and we obtain instead of Eq.(40):

$$I_{\pi^0\pi^0\pi^+} = 0.0117 F_{NR}^2 + 0.2357 F_{NR} |D_c| |J| \cos(\Phi_D + \Phi_J) + 1.3735 |D_c|^2$$

(59)
Using the solutions for $F_{NR}$ given in the Table 2, we can compute the decay rate for the $\pi^0\pi^0\pi^+$ mode as a function of $F_{NR}$ and of the $\pi'$ width $\Gamma_{\pi'}$ via the quantity $D_c$. In Table 3, we show the results for the ratio $R$ defined by:

$$R = \frac{\Gamma(\pi^0\pi^0\pi^+)}{\Gamma(\pi^+\pi^+\pi^-)} . \quad (60)$$

| $\Gamma_{\pi'}$ (MeV) | $F_{NR}$ | $R$ |
|------------------|---------|-----|
| 600              | 1.7378  | 1.9898 | 20.2 ± 0.5 % |
| 400              | 1.7378  | 2.1680 | 18.5 ± 0.8 % |
| 263              | 1.7378  | 2.3385 | 16.7 ± 1 % |
|                  | -1.7378 | -2.0671 | 16.3 ± 0.5 % |

Table 3

The pion energy distributions depend on $F_{NR}$ and $D_c$, and we have for the squared amplitude an expression similar to Eq.(43) with the appropriated changes indicated on Eq.(58):

$$|F(E_1, E_2)|^2 = \frac{1}{9} F_{NR}^2 G_{NR}(E_1) + \frac{2}{3\sqrt{2}} F_{NR} Re \left\{ D_c J_N(E_1, E_2) \right\} + \frac{1}{2} |D_c|^2 K_N(E_1, E_2) , \quad (61)$$

where $J_N(E_1, E_2)$ contains only one Breit-Wigner amplitude:

$$J_N(E_1, E_2) = BW(E_3) = BW(m_{D_s} - E_1 - E_2) , \quad (62)$$

and $K_N(E_1, E_2) = |J_N(E_1, E_2)|^2$.

The full $\pi^0$ energy distribution is obtained after integration of $|F(E_1, E_2)|^2$ over $E_2$ at fixed $E_1$, and the full $\pi^+$ energy distribution after integration over $E_2$ at fixed $E_3$:

$$G_N(E_1) = \frac{1}{9} F_{NR}^2 G_{NR}(E_1) + \frac{2}{3\sqrt{2}} F_{NR} Re \left\{ D_c J_N^{(0)}(E_1) \right\} + \frac{1}{2} |D_c|^2 K_N^{(0)}(E_1) , \quad (63)$$

$$H_N(E_3) = \frac{1}{9} F_{NR}^2 G_{NR}(E_3) + \frac{2}{3\sqrt{2}} F_{NR} Re \left\{ D_c J_N^{(+)}(E_3) \right\} + \frac{1}{2} |D_c|^2 K_N^{(+)}(E_3) , \quad (64)$$

The functions $K_N^{(0)}(E_1)$ and $K_N^{(+)}(E_3)$ have been represented on Figure 5. The functions $J_N^{(0)}$ and $J_N^{(+)}$ are defined by:

$$J_N^{(0)}(E_1) = \frac{1}{m_{D_s}} \int_{E_-(E_1)}^{E_+(E_1)} BW(m_{D_s} - E_1 - E_2) dE_2 , \quad (65)$$

$$J_N^{(+)}(E_3) = \frac{1}{m_{D_s}} \int_{E_-(E_3)}^{E_+(E_3)} BW(E_3) dE_2 , \quad (66)$$

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The $\pi^0$ and $\pi^+$ energy distributions $G_N(E_1)$ and $H_N(E_3)$ are functions of $F_{NR}$ and $D_c$. They are represented on Figures 6 and 7 with the same values for $\Gamma_{a''}$ and $F_{NR}$ as those used in Figures 3 and 4. The first observation is the minor role now played by the constant non resonant component essentially due to the isospin factor Eq.(49). However the sign of the interference between the two components can be observed at large values of $E_3$ as illustrated in Figures 7-a and 7-b.

V. Summary and Concluding Remarks

The large value experimentally observed for the branching ratio of the decay mode $D_{s}^+ \rightarrow \pi^+\pi^+\pi^-$ is a very interesting problem which necessitates a better understanding of the role played by the $W$ annihilation mechanism in $D$ meson decay, and which is probably more important than usually expected.

We have proposed to construct the unique structure function $F(E_1, E_2)$ as the sum of a constant non resonant term described by a real parameter $F_{NR}$, and a Breit-Wigner resonant term associated to a quasi two body $f_0(980) + \pi^+$ state which seems to play an important role as indicated by experiments. The quasi two body $\rho^0 + \pi$ state which is commonly considered by previous authors $[2, 3, 6]$ is experimentally depressed and has been disregarded here.

It is possible to determine the resonant amplitude using parameters experimentally known, even if a large uncertainty remains on their precise experimental values. We have only retained, for simplicity, the uncertainty due to the $\pi(1300)$ width. Our model depends only on one free parameter, and we show that it is possible to find values of $F_{NR}$ fitting the three observed rates: total $D_{s}^+ \rightarrow \pi^+\pi^+\pi^-$, non resonant ($D_{s}^+ \rightarrow \pi^+\pi^+\pi^-$)$_{NR}$, and resonant $D_{s}^+ \rightarrow f_0\pi^+ \rightarrow \pi^+\pi^+\pi^-$. This result is obviously non trivial.

Of course, the Dalitz plots are determined by the squared modulus of the structure function $|F(E_1, E_2)|^2$ and they are predicted by our model. Before the double differential quantities could be observed for the reason of statistics, it would be easier to measure the one pion energy distributions which are nothing but projections of the Dalitz plots on one energy axis. The figures shown in this paper present the various situations to be compared with experimental results when available.

The decay mode $D_{s}^+ \rightarrow \pi^0\pi^0\pi^+$ has not been experimentally observed, however the rates, the pion energy distributions, and the Dalitz plots can be computed in our model. We compare the $\pi^0\pi^0\pi^+$ and $\pi^+\pi^+\pi^-$ integrated widths through the ratio $R$ in Eq.(60) and Table 3, and we also draw the $\pi^0$ and $\pi^+$ meson energy distributions. The quantity $R$ is predicted to be in the $16\% - 20\%$ range and that might be the reason why the mode $D_{s}^+ \rightarrow \pi^0\pi^0\pi^+$ has not yet been detected, its branching ratio being expected to be few $10^{-3}$.
We wish to emphasize that in the absence of additional experimental informations, our crude model is the simplest one accounting for the observed rates. In particular, for the resonant component, the normalization is estimated theoretically and constrained experimentally. Fortunately both approaches overlap, a fact a priori not evident, and they can be improved when experimental uncertainty will be reduced.

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Figure captions

1. **Figure 1**: The shape of the pion energy distributions for a uniform Dalitz plot. $E$ stands for $E_1$, $E_2$, $E_3$ indifferently.

2. **Figure 2**: The shapes of the $\pi^+$ and $\pi^-$ meson energy distributions for $D_s^+ \to f_0 \pi^+ \to \pi^+\pi^-\pi^+$. $K_+(E)$ represents the energy distribution of $\pi^+$ meson and $K_-(E)$ represents that of $\pi^-$ meson. The quantity $E_0 = 7.45$ GeV is associated to the $f_0$ resonance.

3. **Figure 3**: The fully normalized $\pi^+$ meson energy distribution for $D_s^+ \to \pi^+\pi^-\pi^+$.
   
   3-a) for $\Gamma_{\pi'} = 600$ MeV, $1.74 \leq F_{NR} \leq 1.99$

   3-b) for $\Gamma_{\pi'} = 263$ MeV, $-2.07 \leq F_{NR} \leq -1.74$

   The two lines in each curve correspond to the extremum values of $F_{NR}$. The quantity $E_0 = 7.45$ GeV is associated to the $f_0$ resonance.

4. **Figure 4**: The fully normalized $\pi^-$ meson energy distribution for $D_s^+ \to \pi^+\pi^-\pi^+$.
   
   The parameters are the same as in Figure 3.

5. **Figure 5**: The shapes of the $\pi^0$ and $\pi^+$ meson energy distributions for $D_s^+ \to f_0 \pi^+ \to \pi^0\pi^0\pi^+$. The quantity $E_0 = 7.45$ GeV is associated to the $f_0$ resonance.

6. **Figure 6**: The fully normalized $\pi^0$ meson energy distribution for $D_s^+ \to \pi^0\pi^0\pi^+$.
   
   The parameters are the same as in Figure 3.

7. **Figure 7**: The fully normalized $\pi^+$ meson energy distribution for $D_s^+ \to \pi^0\pi^0\pi^+$.
   
   The parameters are the same as in Figure 3. The quantity $E_0 = 7.45$ GeV is associated to the $f_0$ resonance.
Figure 1: The shape of the pion energy distributions for a uniform Dalitz plot. \( E \) stands for \( E_1, E_2, E_3 \) indifferently.
Figure 2: The shapes of the $\pi^+$ and $\pi^-$ meson energy distributions for $D_s^+ \rightarrow f_0 \pi^+ \rightarrow \pi^+\pi^-\pi^+$. $K_{\pi}^+(E)$ represents the energy distributions of $\pi^+$ meson, and $K_{\pi^-}^-(E)$ represents that of $\pi^-$ meson. The quantity $E_0 = 7.45 \text{ GeV}$ is associated to the $f_0$ resonance.
Figure 3: The fully normalized $\pi^+$ meson energy distributions for $D_s^+ \to \pi^+\pi^-$\textsuperscript{+}. \textbf{3-a)} for $\Gamma_{\pi^+} = 600 \ MeV$, $1.74 \leq F_{NR} \leq 1.99$, \textbf{3-b)} for $\Gamma_{\pi^+} = 263 \ MeV$, $-2.07 \leq F_{NR} \leq -1.74$. The two lines in each curve correspond to the extremum values of $F_{NR}$. The quantity $E_0 = 7.45 \ GeV$ is associated to the $f_0$ resonance.
Figure 4: The fully normalized $\pi^-$ meson energy distributions for $D_s^+ \to \pi^+\pi^-\pi^+$. The parameters are the same as in Figure 3.
Figure 5: The shapes of the $\pi^0$ and $\pi^+$ meson energy distributions for $D^+_s \rightarrow f_0\pi^+ \rightarrow \pi^0\pi^0\pi^+$. The quantity $E_0 = 7.45$ GeV is associated to the $f_0$ resonance.
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Figure 7: The fully normalized $\pi^+$ meson energy distribution for $D_s^+ \to \pi^0\pi^0\pi^+$. The parameters are the same as in Figure 3. The quantity $E_0 = 7.45$ GeV is associated to the $f_0$ resonance.
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