SOME CONNECTIONS BETWEEN VARIOUS SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS INVOLVING PASCAL DISTRIBUTION SERIES

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ABSTRACT. In the present paper, we investigate connections between various subclasses of harmonic univalent functions by using a convolution operator involving the Pascal distribution series.

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1. Introduction

Let $\mathcal{H}$ denote the family of continuous complex valued harmonic functions of the form $f = h + \overline{g}$ defined in the open unit disk $U = \{ z : |z| < 1 \}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (1)$$

are analytic in $U$.

A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $U$ is that $|h'(z)| > |g'(z)|$ in $U$ (see [4]).

Denote by $\mathcal{SH}$ the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g}$ which are harmonic, univalent and sense-preserving in $U$ and normalized by $f(0) = f_z(0) - 1 = 0$. One can easily show that the sense-preserving property implies that $|b_1| < 1$. The subclass $\mathcal{SH}^0$ of $\mathcal{SH}$ consisting of all functions in $\mathcal{SH}$ which have the additional property $b_1 = 0$. Note that $\mathcal{SH}$ reduces to the class $S$ of normalized analytic univalent functions in $U$, if the co-analytic part of $f$ is identically zero.

A function $f \in \mathcal{SH}$ is said to be harmonic starlike of order $\alpha$ ($0 \leq \alpha < 1$) in $U$ if and only if

$$\Re \left\{ \frac{zf_z(z) - \overline{zf}_z(z)}{f(z)} \right\} > \alpha, \quad (z \in U) \quad (2)$$
and is said to be harmonic convex of order $\alpha$ ($0 \leq \alpha < 1$) in $U$ if and only if
\[
\Re \left\{ \frac{z^2 f_{zz}(z) + zf_z(z) + z^2 f_{zz}(z) + \bar{z} f_{\bar{z}}(z)}{zf_z(z) - \bar{z} f_{\bar{z}}(z)} \right\} > \alpha, \quad (z \in U). \tag{3}
\]

These classes represented by $SH^*(\alpha)$ and $KH(\alpha)$, respectively, were extensively studied by Jahangiri [8]. Denote by $SH^*$ and $KH$ the classes $SH^*(0)$ and $KH(0)$, respectively. For definitions and properties of these classes, one may refer to [9, 10] or [3].

The elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial have been partially studied in the Geometric Function Theory from a theoretical point of view (see [1, 2, 5, 7]).

Let us consider a non-negative discrete random variable $X$ with a Pascal probability generating function
\[
P(X = n) = \binom{n + r - 1}{r - 1} p^n (1 - p)^r, \quad n \in \{0, 1, 2, 3, \ldots\}
\]
where $p, r$ are called the parameters.

Now we introduce a power series whose coefficients are probabilities of the Pascal distribution, that is
\[
P^p_r(z) = z + \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} p^{n-1} (1 - p)^r a_n z^n. \quad (r \geq 1, \ 0 \leq p \leq 1, \ z \in U) \tag{4}
\]

Note that, by using ratio test we conclude that the radius of convergence of the above power series is $1/p$. Now, for $r, s \geq 1$ and $0 \leq p, q \leq 1$, we introduce the operator $P_{p,q}^{r,s}: \mathcal{H} \to \mathcal{H}$ by
\[
P_{p,q}^{r,s}(f)(z) = P^p_r(z) * h(z) + P^s_q(z) * g(z) = H(z) + G(z)
\]
where
\[
H(z) = z + \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} p^{n-1} (1 - p)^r a_n z^n \tag{5}
\]
\[
G(z) = b_1 z + \sum_{n=2}^{\infty} \binom{n + s - 2}{s - 1} q^{n-1} (1 - q)^s b_n z^n
\]
and "$*$" denotes the convolution (or Hadamard product) of power series.
2. Preliminary Lemmas

To prove our theorems we will use the following lemmas.

**Lemma 1.** (See [6]) If \( f = h + g \in \mathcal{K}\mathcal{H}^0 \) where \( h \) and \( g \) are given by (1) with \( b_1 = 0 \), then
\[
|a_n| \leq \frac{n + 1}{2}, \quad |b_n| \leq \frac{n - 1}{2}.
\]

**Lemma 2.** (See [8]) Let \( f = h + g \) be given by (1). If for some \( \alpha \) (0 \( \leq \alpha < 1 \)) and the inequality
\[
\sum_{n=2}^{\infty} (n - \alpha) |a_n| + \sum_{n=1}^{\infty} (n + \alpha) |b_n| \leq 1 - \alpha
\]
(6) is hold, then \( f \) is harmonic, sense-preserving, univalent in \( U \) and \( f \in \mathcal{S}\mathcal{H}^* (\alpha) \).

Define \( T\mathcal{S}\mathcal{H}^* (\alpha) = \mathcal{S}\mathcal{H}^* (\alpha) \cap \mathcal{T}^2 \) and \( T\mathcal{K}\mathcal{H} (\alpha) = \mathcal{K}\mathcal{H} (\alpha) \cap \mathcal{T}^1 \) where \( \mathcal{T}^k \), \((k = 1, 2)\) consisting of the functions \( f = h + g \) in \( \mathcal{S}\mathcal{H} \) so that \( h(z) \) and \( g(z) \) are of the form
\[
h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = (-1)^k \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1 \ (k = 1, 2).
\]
(7)

**Remark 1.** (See [8]) Let \( f = h + g \) be given by (7). Then \( f \in T\mathcal{S}\mathcal{H}^* (\alpha) \) if and only if the coefficient condition (6) is satisfied. Also, if \( f \in T\mathcal{S}\mathcal{H}^* (\alpha) \), then
\[
|a_n| \leq \frac{1 - \alpha}{n - \alpha}, \quad n \geq 2, \quad |b_n| \leq \frac{1 - \alpha}{n + \alpha}, \quad n \geq 1.
\]
(8)

**Lemma 3.** (See [8]) Let \( f = h + g \) be given by (1). If for some \( \alpha \) (0 \( \leq \alpha < 1 \)) and the inequality
\[
\sum_{n=2}^{\infty} n (n - \alpha) |a_n| + \sum_{n=1}^{\infty} n (n + \alpha) |b_n| \leq 1 - \alpha
\]
(9) is hold, then \( f \) is harmonic, sense-preserving, univalent in \( U \) and \( f \in \mathcal{K}\mathcal{H} (\alpha) \).

**Remark 2.** (See [8]) Let \( f = h + g \) be given by (7). Then \( f \in T\mathcal{K}\mathcal{H} (\alpha) \) if and only if the coefficient condition (9) holds. Also, if \( f \in T\mathcal{K}\mathcal{H} (\alpha) \), then
\[
|a_n| \leq \frac{1 - \alpha}{n(n - \alpha)}, \quad n \geq 2, \quad |b_n| \leq \frac{1 - \alpha}{n(n + \alpha)}, \quad n \geq 1.
\]
(10)

**Lemma 4.** (See [6]) If \( f = h + g \in \mathcal{S}\mathcal{H}^* \) where \( h \) and \( g \) are given by (1) with \( b_1 = 0 \), then
\[
|a_n| \leq \frac{(2n + 1)(n + 1)}{6}, \quad |b_n| \leq \frac{(2n - 1)(n - 1)}{6}, \quad n \geq 2.
\]

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3. Main Results

**Theorem 5.** Let \( r, s \geq 1 \) and \( 0 \leq p, q < 1 \). Also, let \( f = h + g \in \mathcal{H} \) is given by (1). If the inequalities

\[
\sum_{n=2}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \leq 1, \quad (|b_1| < 1) \tag{11}
\]

and

\[
(1 - p)^r + (1 - q)^s \geq 1 + |b_1| + \frac{rp}{1-p} + \frac{sq}{1-q} \tag{12}
\]

are hold, then the operator \( P_{p,q}^{r,s} \) is harmonic, sense-preserving, univalent and maps \( \mathcal{H} \) into \( S\mathcal{H}^* \).

**Proof.** Note that \( P_{p,q}^{r,s}(f) = H(z) + \overline{G(z)} \), where \( H(z) \) and \( G(z) \) are given by (5). To prove that \( P_{p,q}^{r,s}(f) \) is locally univalent and sense-preserving it suffices to prove that \( |H'(z)| - |G'(z)| > 0 \) in \( U \). Using (11), we compute

\[
|H'(z)| - |G'(z)| > 1 - \sum_{n=2}^{\infty} n \left( \frac{n + r - 2}{r - 1} \right) p^{n-1} (1 - p)^r
\]

\[- |b_1| - \sum_{n=2}^{\infty} n \left( \frac{n + s - 2}{s - 1} \right) q^{n-1} (1 - q)^s
\]

\[= 1 - |b_1| - \sum_{n=2}^{\infty} (n - 1 + 1) \left( \frac{n + r - 2}{r - 1} \right) p^{n-1} (1 - p)^r
\]

\[- \sum_{n=2}^{\infty} (n - 1 + 1) \left( \frac{n + s - 2}{s - 1} \right) q^{n-1} (1 - q)^s
\]

\[= 1 - |b_1| - rp (1 - p)^r \sum_{n=2}^{\infty} \left( \frac{n + r - 2}{r} \right) p^{n-2}
\]

\[- (1 - p)^r \sum_{n=2}^{\infty} \left( \frac{n + r - 2}{r - 1} \right) p^{n-1} - sq (1 - q)^s \sum_{n=2}^{\infty} \left( \frac{n + s - 2}{s} \right) q^{n-2}
\]

\[- (1 - q)^s \sum_{n=2}^{\infty} \left( \frac{n + s - 2}{s - 1} \right) q^{n-1}
\]

\[= 1 - |b_1| - rp (1 - p)^r \sum_{n=0}^{\infty} \left( \frac{n + r}{r} \right) p^n
\]

\[- (1 - p)^r \sum_{n=0}^{\infty} \left( \frac{n + r - 1}{r - 1} \right) p^n + (1 - p)^r
\]

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\[-sq (1 - q)^s \sum_{n=0}^{\infty} \binom{n + s}{s} q^n\]

\[-(1 - q)^s \sum_{n=0}^{\infty} \binom{n + s - 1}{s - 1} q^n + (1 - q)^s\]

\[= (1 - p)^r + (1 - q)^s - 1 - |b_1| - \frac{rp}{1 - p} - \frac{sq}{1 - q} \geq 0.\]

To prove \(P_{r,s}^{p,q}(f)\) is univalent in \(U\), referring Theorem 1 in [8], for \(z_1 \neq z_2\) in \(U\), we need to show that

\[\Re \left( \frac{P_{r,s}^{p,q}(f)(z_2) - P_{r,s}^{p,q}(f)(z_1)}{z_2 - z_1} \right) > \int_0^1 \left( \Re(H'(z(t))) - |G'(z(t))| \right) dt. \tag{13}\]

By (11), we have

\[\Re(H'(z(t))) - |G'(z(t))| > 1 - \sum_{n=2}^{\infty} n \binom{n + r - 2}{r - 1} p^{n-1} (1 - p)^r\]

\[-|b_1| - \sum_{n=2}^{\infty} n \binom{n + s - 2}{s - 1} q^{n-1} (1 - q)^s.\]

Using (12), we obtain that the inequality above is nonnegative. Therefore, from the inequality (13) we have

\[\Re \left( \frac{P_{r,s}^{p,q}(f)(z_2) - P_{r,s}^{p,q}(f)(z_1)}{z_2 - z_1} \right) > 0.\]

Hence univalency of \(P_{r,s}^{p,q}(f)\) is proved.

In order to show that \(P_{r,s}^{p,q}(f) \in \mathcal{SH}^r\), we need to prove \(\Phi_1 \leq 1\), by Lemma 2, where

\[\Phi_1 = \sum_{n=2}^{\infty} n \binom{n + r - 2}{r - 1} p^{n-1} (1 - p)^r |a_n| + |b_1| + \sum_{n=2}^{\infty} n \binom{n + s - 2}{s - 1} q^{n-1} (1 - q)^s |b_n|.\]

Since \(|a_n| \leq 1, |b_n| \leq 1, \forall n \geq 2\) because of (11), we have

\[\Phi_1 \leq rp (1 - p)^r \sum_{n=0}^{\infty} \binom{n + r}{r} p^n + (1 - p)^r \sum_{n=0}^{\infty} \binom{n + r - 1}{r - 1} p^n\]

\[-(1 - p)^r + |b_1| + sq (1 - q)^s \sum_{n=0}^{\infty} \binom{n + s}{s} q^n\]

\[= \frac{rp}{1 - p} - \frac{sq}{1 - q} \geq 0.\]
\[ + (1 - q)^s \sum_{n=0}^{\infty} \binom{n + s - 1}{s - 1} q^n (1 - q)^s \]
\[ = |b_1| + \frac{r p}{1 - p} + 1 - (1 - p)^r + \frac{s q}{1 - q} + 1 - (1 - q)^s \]
\[ \leq 1 \]

from (12). Thus proof of Theorem 5 is complete.

**Theorem 6.** Let \( 0 \leq \alpha < 1 \), \( r, s \geq 1 \) and \( 0 \leq p, q < 1 \). If the inequality
\[ \frac{r (r + 1) p^2}{(1 - p)^2} + \frac{4 - \alpha}{1 - p} + \frac{s (s + 1) q^2}{(1 - q)^2} + \frac{2 + \alpha}{1 - q} \]
\[ \leq 2 (1 - \alpha) (1 - p)^r \]
is hold, then \( P_{p,q}^{r,s} (K^0_H) \subset S^{\alpha,0} (H) \).

**Proof.** Suppose that \( f = h + \overline{g} \in K^0_H \) where \( h \) and \( g \) are given by (1) with \( b_1 = 0 \). It suffices to show that \( P_{p,q}^{r,s} (f) = H + G \in S^{\alpha,0} (H) \), where \( H \) and \( G \) are given by (5) with \( b_1 = 0 \) in \( U \). Using Lemma 2, we need to prove that \( \Phi_2 \leq 1 - \alpha \), where
\[ \Phi_2 = \sum_{n=2}^{\infty} (n - \alpha) \binom{n + r - 2}{r - 1} (1 - p)^r p^{n-1} |a_n| \quad (14) \]
\[ + \sum_{n=2}^{\infty} (n + \alpha) \binom{n + s - 2}{s - 1} (1 - q)^s q^{n-1} |b_n|. \quad (15) \]

Using Lemma 1, we compute
\[ \Phi_2 \leq \frac{1}{2} \left\{ \sum_{n=2}^{\infty} (n - \alpha) (n + 1) \binom{n + r - 2}{r - 1} (1 - p)^r p^{n-1} \right. \]
\[ + \left. \sum_{n=2}^{\infty} (n + \alpha) (n - 1) \binom{n + s - 2}{s - 1} (1 - q)^s q^{n-1} \right\} \]
\[ = \frac{1}{2} \left\{ \sum_{n=2}^{\infty} [(n - 1) (n - 2) + (4 - \alpha) (n - 1) + 2 (1 - \alpha)] \binom{n + r - 2}{r - 1} (1 - p)^r p^{n-1} \right. \]
\[ + \left. \sum_{n=2}^{\infty} [(n - 1) (n - 2) + (2 + \alpha) (n - 1)] \binom{n + s - 2}{s - 1} (1 - q)^s q^{n-1} \right\} \]
\[
\begin{align*}
&= \frac{1}{2} \left\{ r (r + 1) p^2 (1 - p)^r \sum_{n=3}^{\infty} \frac{(n + r - 2)}{r + 1} p^{n-3} \\
&\quad + (4 - \alpha) r p (1 - p)^r \sum_{n=2}^{\infty} \frac{(n + r - 2)}{r} p^{n-2} \\
&\quad + 2 (1 - \alpha) (1 - p)^r \sum_{n=2}^{\infty} \frac{(n + r - 2)}{r - 1} p^{n-2} \\
&\quad + s (s + 1) q^2 (1 - q)^s \sum_{n=3}^{\infty} \frac{(n + s - 2)}{s + 1} q^{n-3} \\
&\quad + (2 + \alpha) s q (1 - q)^s \sum_{n=2}^{\infty} \frac{(n + s - 2)}{s} q^{n-2} \right\} \\
&= \frac{1}{2} \left\{ r (r + 1) p^2 (1 - p)^r \sum_{n=0}^{\infty} \frac{(n + r + 1)}{r + 1} p^n \\
&\quad + (4 - \alpha) r p (1 - p)^r \sum_{n=0}^{\infty} \frac{(n + r)}{r} p^n \\
&\quad + 2 (1 - \alpha) (1 - p)^r \sum_{n=0}^{\infty} \frac{(n + r - 1)}{r - 1} p^n - 2 (1 - \alpha) (1 - p)^r \\
&\quad + s (s + 1) q^2 (1 - q)^s \sum_{n=0}^{\infty} \frac{(n + s + 1)}{s + 1} q^n \\
&\quad + (2 + \alpha) s q (1 - q)^s \sum_{n=0}^{\infty} \frac{(n + s)}{s} q^n \right\} \\
&= \frac{1}{2} \left\{ \frac{r (r + 1) p^2}{(1 - p)^2} + \frac{(4 - \alpha) r p}{1 - p} + 2 (1 - \alpha) \\
&\quad - 2 (1 - \alpha) (1 - p)^r + \frac{s (s + 1) q^2}{(1 - q)^2} + \frac{(2 + \alpha) s q}{1 - q} \right\}.
\end{align*}
\]

The last expression is bounded above by \((1 - \alpha)\) by the given condition.

Thus the proof of Theorem 6 is completed.
Theorem 7. Suppose $0 \leq \alpha < 1$, $r, s \geq 1$ and $0 \leq p, q < 1$. If the inequality
\[
\begin{align*}
2r (r + 1) (r + 2) p^3 & + (15 - 2\alpha) r (r + 1) p^2 \frac{(24 - 9\alpha) r p}{1 - p} \\
+ 2s (s + 1) (s + 2) q^3 & + (9 + 2\alpha) s (s + 1) q^2 \frac{(6 + 3\alpha) s q}{1 - q} \\
\leq 6 (1 - \alpha) (1 - p)^r
\end{align*}
\]
is hold then $P_{p,q}^{r,s} (\mathcal{S}^{\alpha,0}) \subset \mathcal{S}^{\alpha,0}$.

Proof. Suppose $f = h + g \in \mathcal{S}^{\alpha,0}$ where $h$ and $g$ are given by (1) with $b_1 = 0$. It suffices to show that $P_{p,q}^{r,s} (f) = H + G \in \mathcal{S}^{\alpha,0}$ where $H$ and $G$ are given by (5) with $b_1 = 0$. By Lemma 2, we need to prove that $\Phi_2 \leq 1 - \alpha$, where
\[
\Phi_2 = \sum_{n=2}^{\infty} (n - \alpha) \binom{n + r - 2}{r - 1} (1 - p)^r p^{n-1} |a_n| + \sum_{n=2}^{\infty} (n + \alpha) \binom{n + s - 2}{s - 1} (1 - q)^s q^{n-1} |b_n|.
\]
Using Lemma 4, we have
\[
\Phi_2 \leq \frac{1}{6} \left\{ \sum_{n=2}^{\infty} (n - \alpha) (2n + 1) (n + 1) \binom{n + r - 2}{r - 1} (1 - p)^r p^{n-1} + \sum_{n=2}^{\infty} (n + \alpha) (2n - 1) (n - 1) \binom{n + s - 2}{s - 1} (1 - q)^s q^{n-1} \right\}
\]
\[
= \frac{1}{6} \left\{ 2 \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} (n - 1) (n - 2) (n - 3) (1 - p)^r p^{n-1} + (15 - 2\alpha) \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} (n - 1) (n - 2) (1 - p)^r p^{n-1} + (24 - 9\alpha) \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} (n - 1) (1 - p)^r p^{n-1} + 6 (1 - \alpha) \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} (1 - p)^r p^{n-1} + 2 \sum_{n=2}^{\infty} \binom{n + s - 2}{s - 1} (n - 1) (n - 2) (n - 3) (1 - q)^s q^{n-1} \right\}
\]
+ (9 + 2\alpha) \sum_{n=2}^{\infty} \binom{n + s - 2}{s - 1} (n - 1) (n - 2) (1 - q)^s q^{n-1}

+ (6 + 3\alpha) \sum_{n=2}^{\infty} \binom{n + s - 2}{s - 1} (n - 1) (1 - q)^s q^{n-1}

= \frac{1}{6} \left\{ 2r (r + 1) (r + 2) p^3 (1 - p)^r \sum_{n=4}^{\infty} \binom{n + r - 2}{r + 2} p^{n-4} \right. 

+ (15 - 2\alpha) r (r + 1) p^2 (1 - p)^r \sum_{n=3}^{\infty} \binom{n + r - 2}{r + 1} p^{n-3} 

+ (24 - 9\alpha) rp (1 - p)^r \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} (1 - p)^r p^{n-1} 

+2s (s + 1) (s + 2) q^3 (1 - q)^s \sum_{n=4}^{\infty} \binom{n + s - 2}{s + 2} q^{n-4} 

+ (9 + 2\alpha) s (s + 1) q^2 (1 - q)^s \sum_{n=3}^{\infty} \binom{n + s - 2}{s + 1} q^{n-3} 

\left. + (6 + 3\alpha) sq (1 - q)^s \sum_{n=2}^{\infty} \binom{n + s - 2}{s} q^{n-2} \right\}

= \frac{1}{6} \left\{ 2r (r + 1) (r + 2) p^3 (1 - p)^r \sum_{n=0}^{\infty} \binom{n + r + 2}{r + 2} p^n \right. 

+ (15 - 2\alpha) r (r + 1) p^2 (1 - p)^r \sum_{n=0}^{\infty} \binom{n + r + 1}{r + 1} p^n 

+ (24 - 9\alpha) rp (1 - p)^r \sum_{n=0}^{\infty} \binom{n + r}{r} p^n 

+6 (1 - \alpha) (1 - p)^r \sum_{n=0}^{\infty} \binom{n + r - 1}{r - 1} p^n - 6 (1 - \alpha) (1 - p)^r 

+2s (s + 1) (s + 2) q^3 (1 - q)^s \sum_{n=0}^{\infty} \binom{n + s + 2}{s + 2} q^n 

+ (9 + 2\alpha) s (s + 1) q^2 (1 - q)^s \sum_{n=0}^{\infty} \binom{n + s + 1}{s + 1} q^n
by the given condition.

**Theorem 8.** If \(0 \leq \alpha < 1\), \(r, s \geq 1\) and \(0 \leq p, q < 1\) then \(P_{p,q}^{r,s}(TSH^* (\alpha)) \subset TSH^* (\alpha)\) if and only if the inequality

\[
(1 - p)^r + (1 - q)^s \geq 1 + \frac{(1 + \alpha) |b_1|}{(1 - \alpha)}
\]

is hold.

**Proof.** Suppose \(f = h + g \in TSH^* (\alpha)\) where \(h\) and \(g\) are given by (7). We need to prove that the operator

\[
P_{p,q}^{r,s}(f)(z) = z - \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} (1 - p)^r p^{n-1} |a_n| z^n + |b_1| z + \sum_{n=2}^{\infty} \binom{n + s - 2}{s - 1} (1 - q)^s q^{n-1} |b_n| z^n
\]

is in \(TSH^* (\alpha)\) if and only if \(\Phi_3 \leq 1 - \alpha\), where

\[
\Phi_3 = \sum_{n=2}^{\infty} (n - \alpha) \binom{n + r - 2}{r - 1} (1 - p)^r p^{n-1} |a_n| + (1 + \alpha) |b_1| + \sum_{n=2}^{\infty} (n + \alpha) \binom{n + s - 2}{s - 1} (1 - q)^s q^{n-1} |b_n|.
\]

By Remark 1, we have

\[
\Phi_3 \leq (1 - \alpha) \left\{ \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} (1 - p)^r p^{n-1} + \sum_{n=2}^{\infty} \binom{n + s - 2}{s - 1} (1 - q)^s q^{n-1} \right\} + (1 + \alpha) |b_1|
\]
\[
\begin{align*}
(1 - \alpha) \left\{ (1 - p)^r \sum_{n=0}^{\infty} \left( \frac{n + r - 1}{r - 1} \right) p^n - (1 - p)^r \right. & \\
+ (1 - q)^s \sum_{n=0}^{\infty} \left( \frac{n + s - 1}{s - 1} \right) q^n - (1 - q)^s \left\} + (1 + \alpha) |b_1| \\
\leq 1 - \alpha
\end{align*}
\]

by the given condition and thus the proof of the theorem is completed.

We next explore a sufficient condition which guarantees that \( P_{p,q}^{r,s} \) maps \( KH_0^{\alpha} \) into \( KH_0^{\alpha} \).

**Theorem 9.** Suppose \( 0 \leq \alpha < 1 \), \( r, s \geq 1 \) and \( 0 \leq p, q < 1 \). If the inequality

\[
\frac{r (r + 1) (r + 2) p^3}{(1 - p)^3} + \frac{(7 - \alpha) r (r + 1) p^2}{(1 - p)^2} + \frac{(10 - 4\alpha) rp}{1 - p} \\
+ \frac{s (s + 1) (s + 2) q^3}{(1 - q)^3} + \frac{(5 + \alpha) s (s + 1) q^2}{(1 - q)^2} + \frac{(4 + 2\alpha) sq}{1 - q}
\leq 2 (1 - \alpha) (1 - p)^r
\]

is hold, then \( P_{p,q}^{r,s} (KH_0) \subset KH_0^{\alpha} \).

**Proof.** Let \( f = h + g \in KH_0 \) where \( h \) and \( g \) are given by (1) with \( b_1 = 0 \). It suffices to show that \( P_{p,q}^{r,s} (f) = H + G \in KH_0^{\alpha} \) where \( H \) and \( G \) are given by (5) with \( b_1 = 0 \). Referring Lemma 1, we need to prove that \( \Phi_4 \leq 1 - \alpha \), where

\[
\Phi_4 = \sum_{n=2}^{\infty} n (n - \alpha) \left( \frac{n + r - 2}{r - 1} \right) (1 - p)^r p^{n-1} |a_n| \\
+ \sum_{n=2}^{\infty} n (n + \alpha) \left( \frac{n + s - 2}{s - 1} \right) (1 - q)^s q^{n-1} |b_n|.
\]

Using Lemma 1, we have

\[
\Phi_4 \leq \frac{1}{2} \left\{ \sum_{n=2}^{\infty} (n - 1) (n - 2) (n - 3) \left( \frac{n + r - 2}{r - 1} \right) (1 - p)^r p^{n-1} \\
+ \sum_{n=2}^{\infty} (7 - \alpha) (n - 1) (n - 2) \left( \frac{n + r - 2}{r - 1} \right) (1 - p)^r p^{n-1} \right\}
\]

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\[ + \sum_{n=2}^{\infty} (10 - 4\alpha) (n - 1) \left( \frac{n + r - 2}{r - 1} \right) (1 - p)^r p^{n-1} \]
\[ + \sum_{n=2}^{\infty} (n - 1) (n - 2) (n - 3) \left( \frac{n + s - 2}{s - 1} \right) (1 - q)^s q^{n-1} \]
\[ + \sum_{n=2}^{\infty} (5 + \alpha) (n - 1) (n - 2) \left( \frac{n + s - 2}{s - 1} \right) (1 - q)^s q^{n-1} \]
\[ + \sum_{n=2}^{\infty} (4 + 2\alpha) (n - 1) \left( \frac{n + s - 2}{s - 1} \right) (1 - q)^s q^{n-1} \]
\[ = \frac{1}{2} \left\{ \frac{r (r + 1) (r + 2) p^3}{(1 - p)^3} + \frac{(7 - \alpha) r (r + 1) p^2}{(1 - p)^2} + \frac{(10 - 4\alpha) rp}{1 - p} \right. \]
\[ + 2 \frac{1 - \alpha}{1 - \alpha} - 2 \frac{1 - \alpha}{1 - \alpha} (1 - p)^r \]
\[ + \frac{s (s + 1) (s + 2) q^3}{(1 - q)^3} + \frac{(5 + \alpha) s (s + 1) q^2}{(1 - q)^2} + \frac{(4 + 2\alpha) sq}{1 - q} \left\} \right. \]
\[ \leq 1 - \alpha \]

by the given condition.

The proofs of following theorems are similar to previous theorems so we omit them.

**Theorem 10.** Let \( 0 \leq \alpha < 1, r, s \geq 1 \) and \( 0 \leq p, q < 1 \). If the inequality
\[ (1 - p)^r + (1 - q)^s \geq 1 + \frac{rp}{1 - p} + \frac{sq}{1 - q} + \frac{(1 + \alpha) |b_1|}{1 - \alpha} \] (17)
is hold, then \( P_{p,q}^{r,s} (TSH^* (\alpha)) \subset \mathcal{K}H (\alpha) \).

**Theorem 11.** If \( 0 \leq \alpha < 1, r, s \geq 1 \) and \( 0 \leq p, q < 1 \) then \( P_{p,q}^{r,s} (TKH (\alpha)) \subset TKH (\alpha) \) if and only if the inequality
\[ (1 - p)^r + (1 - q)^s \geq 1 + \frac{(1 + \alpha) |b_1|}{1 - \alpha} \]
is hold.

**Example 1.** Consider the harmonic polynomial \( f_1(z) = z - \frac{1}{2}z^2 \). If we take \( s = 10 \) and \( q = 0.1 \) then from (5), we have
\[ P_{p,0.1}^{r,10} (f_1)(z) = z - 0.17z^2. \]
One can easily see that coefficients of $f_1(z)$ satisfy condition (11). Condition (12) is also hold for $s = 10, q = 0.1$ and specific choices of $r$ and $p$ such as when $r = 1$ $p$ can be chosen from 0 to 0.49 and when $r = 2$ $p$ can be chosen from 0 to 0.31. Then, using Theorem 5, $P_{p,0.1}^{r,10}(f_1) \in \mathcal{SH}^*$. Images of concentric circles inside $\mathcal{U}$ under the functions $f_1$ and $P_{p,0.1}^{r,10}(f_1)$ are shown in Figures 1 and 2.

![Figure 1: Image of $f_1$](image1.png)  
![Figure 2: Image of $P_{p,0.1}^{r,10}(f_1)$](image2.png)

**Example 2.** Consider the harmonic right half plane mapping $f_0(z) = \frac{z}{(1-z)^2} + \frac{-\frac{1}{2}z^2}{(1-z)^2} \in \mathcal{KH}^0$. If we take $r = 2, s = 2, p = 0.01$ and $q = 0.01$ then from (5), we have

\[
P_{p,0.01}^{r,2}(f_0)(z) = z + \sum_{n=2}^{\infty} \frac{n(n+1)}{2} \frac{(0.01)^{n-1}(0.99)^2}{2} z^n + \sum_{n=2}^{\infty} \frac{n(-n+1)}{2} \frac{(0.01)^{n-1}(0.99)^2}{2} z^n.
\]

Then, according to the Theorem 9, $P_{p,0.01}^{r,2}(f_0)(z) \in \mathcal{KH}^0(\alpha)$ for $0 \leq \alpha < 1$. Images of concentric circles inside $\mathcal{U}$ under the functions $f_0$ and $P_{p,0.01}^{r,2}(f_0)$ are shown in Figures 3 and 4.

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Figure 3: Image of $f_0$

Figure 4: Image of $P_{0.01,0.01}^{2,2}(f_0)$

REFERENCES

[1] Ş. Altınkaya, S. Yalçın, Poisson distribution series for certain subclasses of starlike functions with negative coefficients, Annals of Oradea University Mathematics Fascicola 24, (2017), 5-8.

[2] Ş. Altınkaya, S. Yalçın, Poisson distribution series for analytic univalent functions, Complex Analysis and Operator Theory 12, (2018), 1315-1319.

[3] Y. Avcı, E. Zlotkiewicz, On harmonic univalent mappings, Ann. Univ. Mariae Curie Sklodowska Sect. A 44, (2009), 1-7.

[4] J. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A I Math. 9, (1984), 3-25.

[5] S. Çakmak, S. Yalçın, Ş. Altınkaya, Some connections between various classes of analytic functions associated with the power series distribution, Sakarya University Journal of Science 23, (2019), 982-985.

[6] P. Duren, Harmonic Mappings in the Plane, Cambridge University Press, Cambridge (2004).

[7] S. M. El-Deeb, T. Bulboaca, J. Dziok, Pascal distribution series connected with certain subclasses of univalent functions, Kyungpook Math. J. 59, (2019), 301-314.

[8] J. M. Jahangiri, Harmonic functions starlike in the unit disk, J. Math. Anal. Appl. 235, (1999), 470-477.

[9] H. Silverman, Harmonic univalent function with negative coefficients, J. Math. Anal. Appl. 220, (1998), 283-289.
[10] H. Silverman, E. M. Silvia, *Subclasses of harmonic univalent functions*, New Zealand J. Math. 28, (1999), 275-284.

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