Singular behavior of the velocity moments for inelastic Maxwell mixtures in the homogeneous cooling state

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The Boltzmann equation for \(d\)-dimensional inelastic Maxwell models is considered to determine the collisional moments of second, third and fourth degree in a granular binary mixture. These collisional moments are exactly evaluated in terms of the velocity moments of the distribution function of each species when diffusion is absent (mass flux of each species vanishes). The corresponding associated eigenvalues as well as cross coefficients are obtained as functions of the coefficients of normal restitution and the parameters of the mixture (masses, diameters and composition). The results are applied to the analysis of the time evolution of the moments (scaled with a thermal speed) in the homogeneous cooling state problem. It is shown that, for given values of the parameters of the system, the third and fourth degree moments could diverge in time. An exhaustive study on the influence of the parameter space of the mixture on the above singularity is carried out. The present work extends to binary mixtures previous theoretical results derived for monocomponent inelastic Maxwell models [V. Garzó and A. Santos, Third and fourth degree collisional moments for inelastic Maxwell models, J. Phys. A: Math. Theor. 40 (2007), 14927–14943].

I. INTRODUCTION

It is well established that when granular matter is externally excited it can be modeled as a gas of inelastic hard spheres (IHS). In the simplest version of the model, the spheres are assumed to be completely smooth (i.e., with no rotational degrees of freedom) so that the inelasticity of collisions is characterized by a (positive) constant coefficient of normal restitution \(\alpha \leq 1\). The case \(\alpha = 1\) corresponds to elastic collisions (molecular gases). In the low-density regime, all the relevant information on the state of the gas is given by the one-particle velocity distribution function. This distribution obeys the Boltzmann equation properly adapted to account for the inelastic nature of collisions [10].

One of the main difficulties for obtaining the collisional moments of the Boltzmann equation for hard spheres comes from the form of the collision rate, which is proportional to the relative velocity of the two colliding particles. This difficulty (which is also present for molecular gases) prevents the possibility of expressing the associated collisional moments in terms of a finite number of velocity moments, thus precluding the derivation of exact analytical results. In the case of elastic collisions, the above problem can be overcome by assuming that the gas is made of Maxwell molecules, namely, particles interacting via a repulsive potential inversely proportional to the fourth power of the distance. For this interaction potential, the collision rate is independent of the relative velocity and so, any collisional moment of degree \(k\) can be expressed in terms of velocity moments of a degree smaller than or equal to \(k\) [38]. Thanks to this property, nonlinear transport properties can be exactly obtained [10, 38] and, when properly reduced, they present a good agreement with results derived for other interaction potentials.

When the collisions are inelastic (\(\alpha < 1\)), one can still introduce the so-called inelastic Maxwell models (IMM) (see for instance Refs. [2, 6, 11] for some of the first papers where IMM were introduced). These models share with elastic Maxwell molecules the property that the collision rate is independent of the relative velocity but, on the other hand, their collision rules are the same as for IHS. In this sense, although IMM do not correspond to any microscopic interaction potential it has been widely shown in the past that the cost of sacrificing physical realism is in part compensated by the amount of exact analytical results. In particular, the IMM keeps the qualitatively correct structure and properties of the nonlinear macroscopic equations and obey Haff’s law [27]. It must be also remarked that in the framework of the Boltzmann equation one can introduce Maxwell models at the level of the cross section, without any reference to a specific interaction potential [12]. Recently [29], an inelastic rough Maxwell model has been also introduced.

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The simplifications introduced by IMM in the kernel of the Boltzmann collision operator has allowed in some cases to determine exact properties. For this reason, the Boltzmann equation for IMM has received a great attention of physicists and mathematicians in the last few years, specially in the study of overpopulated high energy tails in homogeneous states 1, 12, 13 and in the evaluation of the transport coefficients 18, 32. The existence of high energy tails of the Boltzmann equation is common for IHS and IMM, although this general qualitative agreement fails at a more quantitative level. The study of this problem has been mainly focused on the monodisperse gas case. Much less is known for inelastic Maxwell mixtures. In this case, Marconi and Puglisi have studied the high velocity moments in the free cooling 20 and driven 31 states for a one-dimensional gas (d = 1). For arbitrary dimensions and in the tracer limit, Ben-Naim and Krapivsky 2 have analyzed the velocity statistics of an impurity in a uniform granular gas while the fourth cumulant of the velocity distribution in the homogenous cooling state (HCS) has been also obtained 18. In addition, as for molecular gases, non-Newtonian transport properties have been exactly determined 17, 19, 21–24, 26, 28, 36 showing in some cases an excellent agreement with computer simulations for IHS 32.

Beyond the second degree velocity moments (which are directly related with the transport properties), Garzó and Santos 20 have computed all the third and fourth degree velocity moments of the Boltzmann collision operator for a monocomponent granular gas. In addition, the collisional rates associated with the isotropic velocity moments $\langle v^{2r} \rangle$ and the anisotropic moments $\langle v^{2r} v_i \rangle$ and $\langle v^{2r} (v_i v_j - \delta_{ij}) \rangle$ have been independently evaluated in Refs. 5, 13, 37. Here, $\langle h(v) \rangle = \int dv h(v) f(v)$, where $h(v)$ is an arbitrary function of the velocity $v$ and $f(v)$ is the one-particle velocity distribution function. All the above calculations have been made for an arbitrary number of dimensions $d$. To the best of our knowledge, the above papers are the only works where the computation of higher-degree collisional moments of IMM has been carried out.

The objective of the present paper is to extend the results displayed in Ref. 20 to the case of granular binary mixtures. Due to the long and complex algebra involved in the general problem, here we will consider situations where diffusion processes are present in the mixture (i.e., $U_1 = U_2 = U$). Although this limitation restricts the applicability of the present results to general nonequilibrium situations, they are still useful for contributing to the advancement in the knowledge of exact properties of IMM. Among the different problems, we can mention shearing problems and/or the relaxation of the third and fourth degree moments toward the HCS, starting from arbitrary initial conditions. This is likely one of the main motivations of the present work.

Some previous results derived for IMM in the monodisperse case 2, 13 have shown that for $d \geq 2$ the (scaled) velocity distribution function $\phi(c)$ has a high-velocity tail of the form $\phi(c) \sim c^{-d-\beta(\alpha)}$, where $c$ is the (scaled) velocity of the particle. The exponent $\beta(\alpha)$ obeys a transcendental equation whose solution is always larger than four ($\beta(\alpha) > 4$), except for $d = 1$ 1. Consequently, for any value of $\alpha$ and $d \geq 2$, the corresponding (scaled) velocity moments of degree $k$ equal to or less than four are convergent and tend towards well-defined values in the long-time limit. An interesting issue is explore whether or not the convergence of moments of degree $k \leq 4$ for the single gas case is also present for inelastic binary mixtures and, if so, to what extent. An indirect way of answering this question is through the knowledge of the high degree velocity moments (beyond the second ones) of the distribution function of each species. These moments play a prominent role in the high velocity region and can show in a clean way the tails of the distribution functions. Surprisingly, our results for binary mixtures show that the (anisotropic) third and fourth degree moments could diverge in time for given values of the parameters of the mixture. Therefore, in contrast to the findings for the monocomponent gases for $d \geq 2$, only the (scaled) moments of degree equal to or smaller than 2 are always convergent in the HCS for arbitrary values of the parameters of the mixture.

The plan of the paper is as follows. In section II the Boltzmann kinetic equation for inelastic Maxwell mixtures is presented. Next, the so-called Ikenberry polynomials 38 $Y_{2p|1,1,2,...i_2}^s(V)$ of degree $k = 2p + q$ are introduced and their collisional moments $J_{2p|1,1,2,...i_2}^s(V)$ with $k = 2, 3$ and 4 associated with the Boltzmann collision operators $J_{rs}[f_r, f_s]$ evaluated in section III. Some technical details involved in the calculations are relegated to the Appendix A. The time relaxation problem of the (scaled) moments toward their asymptotic values in the HCS is studied in section IV while an study of the regions of the parameter space where the third- and fourth-degree moments can be divergent is presented in section V. We close the paper in section VI with a brief discussion of the results derived in this paper.

II. BOLTZMANN KINETIC EQUATION FOR INELASTIC MAXWELL MIXTURES

We consider a granular binary mixture made of particles of diameters $\sigma_r$ and masses $m_r$ ($r = 1, 2$). In the absence of external forces and assuming molecular chaos, the one-particle velocity distribution function $f_r(r, v; t)$ of species $r$...
obeys the Boltzmann equation

\[ \frac{\partial f_r}{\partial t} + \mathbf{v} \cdot \nabla f_r = \sum_{s=1}^{2} J_{rs}[\mathbf{v}|f_r, f_s], \quad (r = 1, 2), \]  

(1)

where \( J_{rs}[f_r, f_s] \) is the Boltzmann collision operator for collisions between particles of species \( r \) and \( s \). In the case that the granular mixture is modeled as a gas of IHS then any collisional moment of \( J_{rs}[f_r, f_s] \) involves the full distributions \( f_r \) and \( f_s \) (namely, all their higher degree moments). This means that one has to resort to approximate forms of the distributions \( f_r \) and \( f_s \) to estimate the collisional moments of \( J_{rs} \). Usually the lowest order in a Sonine polynomial expansion of these distributions is considered \([25]\). This problem is also present in the conventional case of molecular binary mixtures (elastic collisions). However, if one assumes that the collision rate of the two colliding spheres is constant (IMM), the collisional moments of the operator \( J_{rs} \) can be expressed in terms of velocity moments of \( f_r \) and \( f_s \) without to know their explicit forms.

In the case of IMM, the form of the operator \( J_{rs} \) can be obtained from its corresponding form for IHS by replacing the collision rate of IHS by an effective mean-field collision rate independent of the velocity. With this simplification, the Boltzmann collision operator \( J_{rs}[f_r, f_s] \) for IMM is \([25]\)

\[ J_{rs}[\mathbf{v}_1|f_r, f_s] = \frac{\omega_{rs}}{n_s \Omega_d} \int d\mathbf{v}_2 \int d\hat{\sigma} \left[ \alpha_{rs}^{-1} f_r(\mathbf{v}_1^\prime) f_s(\mathbf{v}_2^\prime) - f_r(\mathbf{v}_1) f_s(\mathbf{v}_2) \right]. \]  

(2)

Here,

\[ n_r = \int d\mathbf{v} f_r(\mathbf{v}) \]  

(3)

is the number density of species \( r \), \( \omega_{rs} \) is an effective collision frequency (to be chosen later), \( \Omega_d = 2\pi^{d/2}/\Gamma(d/2) \) is the total solid angle in \( d \) dimensions, and \( \alpha_{rs} \leq 1 \) refers to the constant coefficient of restitution for \( r-s \) collisions. In addition, the double primes on the velocities denote the initial values \( \{\mathbf{v}_1^\prime, \mathbf{v}_2^\prime\} \) that lead to \( \{\mathbf{v}_1, \mathbf{v}_2\} \) following a binary collision:

\[ \mathbf{v}_1^\prime = \mathbf{v}_1 - \mu_{sr} \left( 1 + \alpha_{rs}^{-1} \right) (\hat{\sigma} \cdot \mathbf{g})\hat{\sigma}, \quad \mathbf{v}_2^\prime = \mathbf{v}_2 + \mu_{rs} \left( 1 + \alpha_{rs}^{-1} \right) (\hat{\sigma} \cdot \mathbf{g})\hat{\sigma}, \]  

(4)

where \( \mu_{rs} = m_r/(m_r + m_s) \), \( \mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2 \) is the relative velocity of the colliding pair and \( \hat{\sigma} \) is a unit vector directed along the centers of the two colliding spheres.

Apart from the densities \( n_r \), the granular temperature \( T \) is defined as

\[ T = \sum_r x_r T_r, \]  

(5)

where \( x_r = n_r/n \) is the mole fraction of species \( r \) (\( n = n_1 + n_2 \) is the total number density) and

\[ T_r = \frac{1}{dn_r} \int d\mathbf{v} m_r V^2 f_r(\mathbf{v}) \]  

(6)

is the partial temperature of species \( r \). In Eq. (6), we have introduced the peculiar velocity \( \mathbf{V} = \mathbf{v} - \mathbf{U} \), \( \mathbf{U} \) being the mean flow velocity defined as

\[ \rho \mathbf{U} = \sum_r \rho_r \mathbf{U}_r = \int d\mathbf{v} m_r \mathbf{v} f_r(\mathbf{v}). \]  

(7)

Here, \( \rho_r = m_r n_r \) is the mass density of species \( r \) and \( \rho = \rho_1 + \rho_2 \) is the total mass density. The second identity in Eq. (6) defines the partial mean flow velocities \( \mathbf{U}_r \). In addition, the mass flux of species \( r \) is given by \( \mathbf{J}_r = \rho_r (\mathbf{U}_r - \mathbf{U}) \). As said in the Introduction section, for the sake of simplicity, we will assume in this paper that the mass fluxes vanish (i.e., \( \mathbf{U}_r = \mathbf{U} \)).

To evaluate the collisional moments of the Boltzmann operator \( J_{rs}[f_r, f_s] \), a useful identity for an arbitrary function \( h(\mathbf{v}) \) is

\[ \int d\mathbf{v}_1 h(\mathbf{v}_1) J_{rs}[\mathbf{v}_1|f_r, f_s] = \frac{\omega_{rs}}{n_s \Omega_d} \int d\mathbf{v}_1 \int d\mathbf{v}_2 f_r(\mathbf{v}_1) f_s(\mathbf{v}_2) \int d\hat{\sigma} \left[ h(\mathbf{v}_1^\prime) - h(\mathbf{v}_1) \right], \]  

(8)
where
\[ v_1' = v_1 - \mu_{rs} (1 + \alpha_{rs}) (\hat{\sigma} \cdot g) \hat{\sigma} \]  
(9)
denotes the post-collisional velocity.

It is worthwhile mentioning that there is another more refined version of IMM where the collision rate has the same dependence on the scalar product \((\hat{\sigma} \cdot g)\) as in the case of IHS. The corresponding Boltzmann operator can be proved to be equivalent to the one given by Eq. (2) except that the scattering rules (4) and (9) must be changed \([7, 11]\). However, both versions of IMM yield similar results in problems such as the high energy tails. In this paper, for the sake of simplicity, we will consider the version provided by the collision rules (4) and (9).

A. Ikenberry polynomials

In the case of Maxwell models (both elastic and inelastic), it is convenient to introduce the Ikenberry polynomials \(Y_{2p|i_1 i_2 ... i_q}(V)\) of degree \(k = 2p + q\) \([38]\). The Ikenberry polynomials are defined as \(Y_{2p|i_1 i_2 ... i_q}(V) = V^{2p} Y_{i_1 i_2 ... i_q}(V)\), where \(Y_{i_1 i_2 ... i_q}(V)\) is obtained by subtracting from \(V_{i_1} V_{i_2} ... V_{i_q}\) that homogeneous symmetric polynomial of degree \(q\) in the components of \(V\) such as to annul the result of contracting the components of \(Y_{i_1 i_2 ... i_q}(V)\) on any pair of indices. The polynomials functions \(Y_{2p|i_1 i_2 ... i_q}(V)\) of degree smaller than or equal to four are

\[ Y_{000}(V) = 1, \quad Y_{0ij}(V) = V_i, \]  
(10)

\[ Y_{200}(V) = V^2, \quad Y_{0ij}(V) = V_i V_j - \frac{1}{d} V^2 \delta_{ij}, \]  
(11)

\[ Y_{2i}(V) = V^2 V_i, \quad Y_{0ij}(V) = V_i V_j - \frac{1}{d} V^2 (V_i \delta_{jk} + V_j \delta_{ik} + V_k \delta_{ij}), \]  
(12)

\[ Y_{400}(V) = V^4, \quad Y_{2ij}(V) = V^2 \left( V_i V_j - \frac{1}{d} V^2 \delta_{ij} \right), \]  
(13)

\[ Y_{0ijk\ell}(V) = V_i V_j V_k V_\ell - \frac{1}{d + 4} V^2 (V_i V_j \delta_{k\ell} + V_i V_k \delta_{j\ell} + V_i V_\ell \delta_{jk} + V_j V_k \delta_{i\ell} + V_j V_\ell \delta_{ik} + V_k V_\ell \delta_{ij}) + \frac{1}{(d + 2)(d + 4)} V^4 (\delta_{ij} \delta_{k\ell} + \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) \]  
\[ + V_j V_k V_\ell \delta_{ijk} + V_k V_\ell \delta_{ij} + \frac{1}{d + 4} \left[ Y_{2ij}(V) \delta_{k\ell} + Y_{2ik}(V) \delta_{j\ell} + Y_{2i\ell}(V) \delta_{jk} \right. \]  
\[ + Y_{2jk}(V) \delta_{i\ell} + Y_{2j\ell}(V) \delta_{ik} + Y_{2k\ell}(V) \delta_{ij} \]  
\[ - \frac{1}{d(d + 2)} Y_{400}(V) (\delta_{ij} \delta_{k\ell} + \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}). \]  
(14)

Let us introduce here the notation

\[ M^{(r)}_{2p|i_1 i_2 ... i_q} = \int dV Y_{2p|i_1 i_2 ... i_q}(V) f_r(V), \]  
(15)

\[ J^{(rs)}_{2p|i_1 i_2 ... i_q} = \int dV Y_{2p|i_1 i_2 ... i_q}(V) J_{rs}[f_r, f_s]. \]  
(16)

Equation (15) gives the definition of the velocity moments of the distribution \(f_r\) while Eq. (16) provides the definition of the collisional moments of the Boltzmann operator \(J_{rs}\).

Note that \(M^{(r)}_{00} = n_r, J^{(rs)}_{00} = 0\) (conservation of mass), \(M^{(r)}_{0j} = 0\) (since we have assumed that \(U_r = U\)) and

\[ M^{(r)}_{2j} = \frac{a_{pr}}{m_r} = \frac{n_r T_r}{m_r}, \]  
(17)
where \( p_r = n_r T_r \) is the partial pressure of species \( r \). Moreover, \( M_{0ij}^{(r)} = (P_{r,ij} - p_r \delta_{ij})/m_r \), where

\[
P_{r,ij} = \int dV m_r V_i V_j f_r(v) \tag{18}
\]

is the partial pressure tensor of species \( r \) and \( M_{2|0}^{(r)} = 2q_r/m_r \), where

\[
q_r = \int dV m_r V^2 f_r(V) \tag{19}
\]

is the partial contribution to the total heat flux due to species \( r \).

The remaining third degree moments \( M_{0ijk}^{(r)} \) and the moments of degree \( k \geq 4 \) are not directly related to hydrodynamic quantities. However, they provide indirect information on the velocity distribution function \( f_r \).

### III. COLLISIONAL MOMENTS FOR INELASTIC MAXWELL MIXTURES

As in the elastic case \[16, 38\], the mathematical structure of the Maxwell collision kernel allows one to express a collisional moment of degree \( k \) of \( J_{rs}[f_r, f_s] \) in terms of velocity moments of \( f_r \) and \( f_s \) of degree less than or equal to \( k \). Let us now display the explicit expressions for the collisional moments \( J_{2p|2q}^{(rs)} \) for \( k = 2p + q \leq 4 \). Some technical details to obtain those collisional moments are provided in the Appendix A.

#### A. Second degree collisional moments

The second degree collisional moments were already evaluated in Refs. \[17, 21\]. They are given by

\[
J_{2|0}^{(rs)} \equiv -\frac{d}{m_r} \zeta_{rs} = -\frac{\omega_{rs}}{4dn_s} (1 + \beta_{rs}) \left[ (3 - \beta_{rs}) n_s M_{2|0}^{(r)} - (1 + \beta_{rs}) n_r M_{2|0}^{(s)} \right], \tag{20}
\]

\[
J_{0|ij}^{(rs)} = -\frac{\omega_{rs}}{2d(d+2)n_s} (1 + \beta_{rs}) \left[ (2d + 3 - \beta_{rs}) n_s M_{0|ij}^{(r)} - (1 + \beta_{rs}) n_r M_{0|ij}^{(s)} \right], \tag{21}
\]

where

\[
\beta_{rs} = 2\mu_{sr} (1 + \alpha_{rs}) - 1. \tag{22}
\]

The quantity \( \zeta_{rs} \) measures the rate of change of the partial temperature \( T_r \) of species \( r \) due to collisions with particles of species \( s \). The total cooling rate \( \zeta \) is

\[
\zeta = T^{-1} \sum_r x_r T_r \zeta_r, \quad \zeta_r = \sum_s \zeta_{rs}. \tag{23}
\]

According to Eq. (20), the parameters \( \zeta_{rs} \) are given by

\[
\zeta_{rs} = \frac{\omega_{rs}}{d} (1 + \beta_{rs}) \left[ 1 - \frac{(1 + \beta_{rs})(\theta_r + \theta_s)}{4\theta_s} \right], \tag{24}
\]

where

\[
\theta_r = \frac{m_r T_r}{m T_r}, \quad \overline{m} = \frac{m_1 + m_2}{2}. \tag{25}
\]

Note that Eq. (20) yields the result

\[
\sum_{r,s} m_r J_{2|0}^{(rs)} = -dn T \zeta = -\frac{\omega_{11}}{2} n_1 T_1 (1 - \alpha_{11}^2) - \frac{\omega_{22}}{2} n_2 T_2 (1 - \alpha_{22}^2)
- \omega_{12} n_1 \mu_{21} (\mu_{21} T_1 + \mu_{12} T_2) (1 - \alpha_{12}^2). \tag{26}
\]
For elastic collisions ($\alpha_{11} = \alpha_{22} = \alpha_{12} = 1$), Eq. (26) shows that the total kinetic energy is conserved by collisions regardless of the values of the masses and diameters of the mixture. This is the expected result.

As said before, the collision frequencies $\omega_{rs}$ can be seen as free parameters in the model. Its dependence on the parameter space of the mixture can be chosen to optimize the agreement with the results obtained from the Boltzmann equation for IHS. As in previous papers [17, 21], we adjust here the quantities parameter space of the mixture can be chosen to optimize the agreement with the results obtained from the Boltzmann equation for IHS. Since $\zeta_{rs}$ is not exactly known for IHS, we will estimate $\zeta_{rs}$ by considering Maxwellian distributions defined at temperatures $T_r$ and $T_s$ for the distributions $f_r$ and $f_s$, respectively. In this approximation, $\zeta_{rs}^{\text{IHS}}$ can be written as [25]

$$\zeta_{rs}^{\text{IHS}} \rightarrow \frac{4}{d} x_s \left( \frac{\sigma_{rs}}{\sigma_{12}} \right)^{d-1} \left( \frac{\theta_r + \theta_s}{\theta_s} \right)^{1/2} \left( 1 + \beta_{rs} \right) \nu_0 \left[ 1 - \frac{(1 + \beta_{rs})(\theta_r + \theta_s)}{4\theta_s} \right],$$ (27)

where $\sigma_{rs} = (\sigma_r + \sigma_s)/2$ and

$$\nu_0 = \frac{\Omega_d}{4\sqrt{\pi} \sigma_{12}^{d-1} v_0}$$ (28)

is an effective collision frequency. Here, $v_0 = \sqrt{2T/\pi}$ is a thermal speed defined in terms of the granular temperature of the mixture $T$. It must be noted that the expressions obtained for the collisional moments in this paper are independent of the specific choice of the collision rates $\omega_{rs}$.

B. Third degree collisional moments

The evaluation of the third degree collisional moments is performed in the Appendix [A]. The results are

$$J^{(rs)}_{2|i} = -\frac{1}{8d(d+2)} \frac{\omega_{rs}}{n_s} (1 + \beta_{rs}) \left\{ [3\beta_{rs}^2 - 2(2d+5)\beta_{rs} + 10d + 11] n_s M^{(r)}_{2|i} - 3(1 + \beta_{rs})^2 n_r M^{(s)}_{2|i} \right\},$$ (29)

$$J^{(rs)}_{0|ij} = -\frac{3}{4d(d+2)(d+4)} \frac{\omega_{rs}}{n_s} (1 + \beta_{rs}) \left\{ [\beta_{rs}^2 - 2(d+3)\beta_{rs} + 2d^2 + 10d + 9] n_r M^{(r)}_{0|ij} \right\} \times n_s M^{(s)}_{0|ij} - (1 + \beta_{rs})^2 n_r M^{(s)}_{0|ij},$$ (30)

Equation (29) is consistent with the expression derived in Ref. [17] when the mass flux $\dot{J}_r = 0$.

C. Fourth degree collisional moments

The calculation of the fourth degree collisional moments is more involved. The collisional moments $J^{(rs)}_{4|0}$ and $J^{(rs)}_{2|ij}$ can be written as

$$J^{(rs)}_{4|0} = -\frac{1}{d(d+2)} \frac{\omega_{rs}}{n_s} (1 + \beta_{rs}) \left\{ (3 - \beta_{rs})(3\beta_{rs}^2 - 6\beta_{rs} + 8d + 7) \frac{n_r M_{4|0}^{(r)}}{16} - \frac{3}{16} (1 + \beta_{rs})^3 n_r M_{4|0}^{(s)} \right\} - \frac{(1 + \beta_{rs})(3\beta_{rs}^2 - 6\beta_{rs} + 4d + 7) d^2 p_r p_s}{m_r m_s},$$ (31)
\[ J^{(rs)}_{2|ij} = \frac{1}{d(d+2)(d+4)} \frac{\omega_{rs}}{n_s} (1 + \beta_{rs}) \left\{\frac{3}{4} (1 + \beta_{rs})^3 n_r M^{(s)}_{2|ij} \right. \]
\[ + \frac{3}{4} (d+5) \beta_{rs}^2 + \beta_{rs} (d^2 + 14d + 25) - 7d^2 - 31d - 21 \frac{n_s M^{(r)}_{ij}}{m_r m_s} \]
\[ + \frac{6}{4} (d+2) \beta_{rs}^2 - 2(d+7) \beta_{rs} + d - 2 \frac{(P_{r,ik} P_{s,kj} + P_{r,jk} P_{s,ki})}{m_r m_s} \]
\[ + \frac{(1 + \beta_{rs}) (3 \beta_{rs}^2 + 2d + 5) dp_r}{m_r m_s} P_{s,ij} \]
\[ + \frac{1 + \beta_{rs}}{4} \frac{3 d^3 - 3(d+4) \beta_{rs} + d^2 + 7d + 9}{m_r m_s} \frac{dp_s}{m_r m_s} P_{r,ij} \]
\[ - \frac{(1 + \beta_{rs}) [6 \beta_{rs}^2 - 3(d+4) \beta_{rs} + d^2 + 9d + 14]}{4} \frac{dp_s}{m_r m_s} P_{r,ij} \]
\[ - \frac{(1 + \beta_{rs}) [6 \beta_{rs}^2 - 3(d+4) \beta_{rs} + d - 2]}{2d} \frac{P_{r,ij} P_{s,ij}}{m_r m_s} \right\}, \quad (32) \]

The expression of \( J^{(rs)}_{0|ijkl} \) is rather large. For the sake of concreteness, we will display here two representative collisional moments of this class: \( J^{(rs)}_{0|xxxx} \) and \( J^{(rs)}_{0|xxxy} \). They are given by

\[ J^{(rs)}_{0|xxxx} = \frac{1}{2d(d+2)(d+4)(d+6)} \frac{\omega_{rs}}{n_s} (1 + \beta_{rs}) \]
\[ \times \left\{ \left[ 3 \beta_{rs}^3 - (6d + 27) \beta_{rs}^2 + 3 \beta_{rs} (2d^2 + 16d + 27) - 4d^3 - 42d^2 \right] \right. \]
\[ - 122d - 81 n_s M^{(r)}_{0|xxxx} + 3 (1 + \beta_{rs})^3 n_r M^{(s)}_{0|xxxx} \]
\[ + 6 \beta_{rs}^3 - 3 \beta_{rs}^2 (d+4) + d^2 + 7d + 9 (1 + \beta_{rs}) \]
\[ \left. \frac{(d+4)}{m_r m_s} \right\} \times \left[ (d+2)^{-1} (2 P_{r,ij} P_{s,ij} + d^2 p_r p_s) - d(p_s P_{r,xx} + p_r P_{s,xx}) \right. \]
\[ - 4 P_{r,ik} P_{s,kj} + (d+4) P_{r,xx} P_{s,xx} \right\}, \quad (33) \]

\[ J^{(rs)}_{0|xxxy} = \frac{1}{2d(d+2)(d+4)(d+6)} \frac{\omega_{rs}}{n_s} (1 + \beta_{rs}) \]
\[ \times \left\{ \left[ 3 \beta_{rs}^3 - (6d + 27) \beta_{rs}^2 + 3 \beta_{rs} (2d^2 + 16d + 27) - 4d^3 - 42d^2 \right] \right. \]
\[ - 122d - 81 n_s M^{(r)}_{0|xxxy} + 3 (1 + \beta_{rs})^3 n_r M^{(s)}_{0|xxxy} \]
\[ + 3 \beta_{rs}^3 - 3 \beta_{rs}^2 (d+4) + d^2 + 7d + 9 (1 + \beta_{rs}) \]
\[ \left. \frac{(d+4)}{m_r m_s} \right\} \times \left[ (d+4) (P_{r,xx} P_{s,xy} + P_{r,xy} P_{s,xx}) - d(p_s P_{r,xy} + p_r P_{s,xy}) \right. \]
\[ - 2 (P_{r,ik} P_{s,kj} + P_{r,jk} P_{s,ki}) \right\}. \quad (34) \]

For mechanically equivalent particles \((m_1 = m_2, \sigma_1 = \sigma_2, \alpha_{11} = \alpha_{22} = \alpha_{12})\), all the expressions reduce to known results for molecular \((\alpha = 1)\) three-dimensional \((d = 3)\) Maxwell gases \([16, 38]\). Regarding inelastic collisions \((\alpha < 1)\) and mechanically equivalent particles, the expressions for the collisional moments are consistent with previous results \([20]\) derived for monodisperse gases. Moreover, in the one-dimensional inelastic case \((d = 1, \alpha < 1)\) our expressions for binary mixtures of the isotropic collisional moments \( J^{(rs)}_{0|ij} \) and \( J^{(rs)}_{0|ik} \) agree with previous results reported by Marconi and Puglisi \([30]\). This shows the consistency of our expressions with known results published in the granular literature.

IV. RELAXATION TO THE HOMOGENEOUS COOLING STATE

The results derived in the preceding section can be applied to several interesting situations. Here, we will consider the most basic problem in a granular mixture: the time evolution of the moments of degree less than or equal to four
where we recall that \( \zeta \) for a three-dimensional binary mixture \((d = 3)\) monotonically decays in time. In this case, the set of Boltzmann kinetic equations (1) for \( f_1 \) and \( f_2 \) becomes

\[
\partial_t f_1(v, t) = J_{11}[v, f_1, f_1] + J_{12}[v, f_1, f_2],
\]

and a similar equation for \( f_2 \). Since the collisions are inelastic, the granular temperature \( T(t) \) monotonically decays in time and so a steady state does not exist. In the context of IMM, it has been proven \cite{8, 9} that, provided that \( f_r(v, 0) \) has a finite moment of some degree higher than two, \( f_r(v, t) \) asymptotically tends toward a self-similar solution of the form

\[
f_r(v, t) \to n_r[v_0(t)]^{-d} \phi_r(V/v_0(t)),
\]

where \( \phi_r(c) \) is an isotropic distribution.

At a hydrodynamic level, the only relevant balance equation is that of the temperature \( T(t) \). Its time evolution equation can be easily obtained from the moments \( M_{210}^{(1)} \) and \( M_{220}^{(2)} \) and it is given by

\[
\partial_t T = -\zeta T.
\]

However, on a kinetic level, it is also interesting to discuss energy transfer through the partial temperatures \( T_r \). The time evolution of the temperatures \( T_r \) can be derived from the velocity moment \( M_{220}^{(r)} \):

\[
\partial_t T_r = -\zeta_r T_r,
\]

where we recall that \( \zeta_r = \sum_s \zeta_{rs} \) and \( \zeta_{rs} \) is given by Eq. (24). The time evolution of the temperature ratio \( \gamma \equiv T_1(t)/T_2(t) \) follows from Eq. (55) as

\[
\partial_t \ln \gamma = \zeta_2 - \zeta_1.
\]

After a transient period, it is expected that the mixture achieves a hydrodynamic regime where all the time dependence of \( f_r \) only occurs through the granular temperature \( T(t) \). This necessarily implies that the three temperatures \( T_1(t) \), \( T_2(t) \) and \( T(t) \) are proportional to each other and their ratios are all constant. This does not mean that all three temperatures are equal (as in the case of elastic collisions) since both partial temperatures are in general different \cite{15}. According to Eq. (29), if \( \gamma \equiv \text{const} \) then \( \zeta_1(t) = \zeta_2(t) \) in the hydrodynamic regime. The condition of equal partial cooling rates provides the dependence of the temperature ratio on the parameters of the mixture.

Figure 1 shows the dependence of the temperature ratio \( T_1/T_2 \) on the (common) coefficient of restitution \( \alpha_{rs} \equiv \alpha \) for a three-dimensional binary mixture \((d = 3)\) for \( \sigma_1/\sigma_2 = 1 \), \( x_1 = \frac{1}{2} \), and three different values of the mass ratio. Theoretical results for IMM are compared against Monte Carlo simulations carried out in Ref. \cite{14} for IHS. Figure 1 highlights one of the most characteristic features of granular mixtures (as compared with molecular mixtures): the

\[\text{FIG. 1: Temperature ratio } T_1/T_2 \text{ versus the (common) coefficient of restitution } \alpha \text{ for a three-dimensional binary mixture } (d = 3) \text{ with } \sigma_1/\sigma_2 = 1, \ x_1 = \frac{1}{2}, \text{ and three different values of the mass ratio: } m_1/m_2 = 5 \text{ (circles), } m_1/m_2 = 5 \text{ (triangles), and } m_1/m_2 = 0.5 \text{ (squares). The lines correspond to the theoretical results obtained here for IMM while symbols refer to the results obtained from the DSMC method for IHS.}\]
partial temperatures are different even in homogenous states. We observe that the departure of $\gamma$ from 1 (breakdown of energy equipartition) increases with increasing the disparity in the mass ratio. In general, the temperature of the lighter species is smaller than that of the heavier species. It is also important to remark the excellent agreement found between theory (developed for IMM) and computer simulations (performed for IHS), even for quite strong inelasticity.

A. Eigenvalues for inelastic Maxwell mixtures

![Graph](A)

**FIG. 2:** Panel A: Dependence of the eigenvalues $\nu_{10}^{(11)}$ (a), $\nu_{0ij}^{(11)}$ (b), $\nu_{2ij}^{(11)}$ (c), and $\nu_{0ijk}^{(11)}$ (d) for a three-dimensional binary mixture constituted by particles of the same mass density $[m_1/m_2 = (\sigma_1/\sigma_2)^3]$ with $x_1 = \frac{1}{2}$ and $m_1/m_2 = 2$. Panel B: The same as panel A for the eigenvalues $\nu_{10}^{(12)}$ (a), $\nu_{0ij}^{(12)}$ (b), $\nu_{2ij}^{(12)}$ (c), and $\nu_{0ijk}^{(12)}$ (d).

![Graph](B)

**FIG. 3:** Panel A: Dependence of the eigenvalues $\nu_{40}^{(11)}$ (a), $\nu_{2ij}^{(11)}$ (b), and $\nu_{0ijkl}^{(11)}$ (c) for a three-dimensional binary mixture constituted by particles of the same mass density $[m_1/m_2 = (\sigma_1/\sigma_2)^3]$ with $x_1 = \frac{1}{2}$ and $m_1/m_2 = 2$. Panel B: The same as panel A for the eigenvalues $\nu_{40}^{(12)}$ (a), $\nu_{2ij}^{(12)}$ (b), and $\nu_{0ijkl}^{(12)}$ (c).

Apart from the partial temperatures, it is worthwhile analyzing the time evolution of the higher-degree velocity moments in the HCS. To get this equation, one takes velocity moments in both sides of Eq. (35) and achieves the equation

$$\partial_t M_r^{(1)} = J_{2p|\bar{q}}^{(11)} + J_{2p|\bar{q}}^{(12)},$$

and a similar equation for species 2. Here, we have introduced the short-hand notation $\bar{q} \equiv i_1 i_2 \ldots i_q$. To study the time evolution of the moments it is convenient to introduce the \textit{scaled} moments

$$M_r^{(r)}(t) \equiv n_r^{-1} [v_0(t)]^{-(2p+q)} M_{2p|\bar{q}}^{(r)}(t).$$
In accordance with Eq. (26), one expects that after a transient regime the dimensionless moments $M^{(r)}_{2p/q}$ (scaled with
the time-dependent thermal velocity) reach an asymptotic steady value.

The time evolution of the scaled moments $M^{(r)}_{2p/q}$ can be obtained when one takes into account the time evolution
equation (37) for the temperature $T(t)$. In that case, from Eqs. (10) and (37) one simply gets
\[
\partial_t M^{(1)}_{2p/q} = J^{(1)}_{2p/q} + J^{(1)}_{2p/q} + \frac{2p+q}{2} \zeta^* M^{(1)}_{2p/q},
\]
where $\zeta^* = \zeta/\nu_0$,
\[
J^{(r,s)}_{2p/q} = J^{(r,s)}_{2p/q}/\nu_0 n_{2p/q},
\]
and
\[
\tau = \int_0^t dt' \nu_0(t').
\]
Here, $\nu_0(t) \propto nT^{1/2}$ is defined by Eq. (28). The parameter $\tau$ measures time as the number of (effective) collisions per
particle.

It is quite apparent from Eqs. (20), (21), and (29)-(34) that a collisional moment of degree $k$ of the operator $J_{rs}[f_r, f_s]$ can be expressed in terms of velocity moments of a degree less than or equal to $k$. More specifically, the
combination $J^{(11)}_{2p/q} + J^{(12)}_{2p/q}$ has the structure
\[
J^{(11)}_{2p/q} + J^{(12)}_{2p/q} = -\nu^{(11)}_{2p/q} M^{(1)}_{2p/q} - \nu^{(12)}_{2p/q} M^{(2)}_{2p/q} + C^{(11)}_{2p/q} + C^{(12)}_{2p/q},
\]
where the terms $\nu^{(rs)}_{2p/q}$ are bilinear combinations of moments of degree less than $2p+q$. Since the first two terms on
the right-hand side of Eq. (15) are linear, then the quantities $\nu^{(11)}_{2p/q}$ and $\nu^{(12)}_{2p/q}$ can be considered as the eigenvalues (or
collisional rates) of the linearized collision operators corresponding to the eigenfunctions $Y_{2p/q}$. Their explicit forms for
velocity moments of degree less than or equal to four are given in the Appendix [13].

As an illustration, the dependence of the eigenvalues (collision rates) associated with the second, third, and fourth
degree moments on the (common) coefficient of restitution $\alpha_{rs} \equiv \alpha$ is plotted in Figs. 2 and 3 for a binary mixture
constituted by particles of the same mass density. Here, $d = 3$, $x_1 = \frac{1}{2}$, and $m_1/m_2 = 2$. While the eigenvalues $\nu^{(11)}_{ij}$
and $\nu^{(11)}_{0ij}$ decrease with increasing inelasticity, the other two eigenvalues ($\nu^{(11)}_{2i}$ and $\nu^{(11)}_{0i}$) exhibit a non-monotonic
dependence on $\alpha$. The eigenvalues of the second- and third-degree moments associated to cross-collisions ($\nu^{(12)}_{2i0}$, $\nu^{(12)}_{0ij}$, $\nu^{(12)}_{2i}$, and $\nu^{(12)}_{0ijk}$) are negative and they increase with decreasing $\alpha$. A similar behavior can be found for
the eigenvalues associated with the fourth-degree moments, as Fig. 4 shows. In general, we can conclude that the
influence of inelasticity on those eigenvalues is in general important, specially in the case of the ones associated with
the self-collisions (i.e., those of the form $\nu^{(11)}_{2p/q}$).

**B. Time evolution of the velocity moments**

Let us evaluate the time evolution of the velocity moments. Thus, inserting the expression (13) into Eq. (12), in
matrix form one finally gets
\[
(\delta_{ss'} \partial_t + L_{ss'}) M_{ss'} = C_s,
\]
where $M$ is the column matrix defined by the set
\[
\left\{M^{(1)}_{2p/q}, M^{(2)}_{2p/q}\right\},
\]
and the square matrix $L$ is given by
\[
L = \begin{pmatrix}
\nu^{(11)}_{2p/q} & \nu^{(12)}_{2p/q} \\
\omega^{(21)}_{2p/q} & \omega^{(22)}_{2p/q}
\end{pmatrix},
\]
The solution of Eq. (46) can be written as
\[ \sigma_p = \frac{2p + q}{2} \zeta^* + \omega_{2p}^{(22)} - \frac{2p + q}{2} \zeta^* + \omega_{2p}^{(2)} - \frac{2p + q}{2} \zeta^* . \] (49)

The quantities \( \omega_{2p}^{(rs)} \) can be seen as shifted collisional rates associated with the scaled moments \( M_{2p}^{(rs)} \). Moreover, the column matrix \( \mathbf{C} \) is
\[ \mathbf{C} = \begin{pmatrix} C_{2p}^{(11)} + C_{2p}^{(12)} \\ C_{2p}^{(22)} + C_{2p}^{(21)} \end{pmatrix}. \] (50)

The solution of Eq. (46) can be written as
\[ \mathbf{M}(\tau) = e^{-\mathbf{L}\tau} \cdot [\mathbf{M}(0) - \mathbf{M}(\infty)] + \mathbf{M}(\infty), \] (51)
where the asymptotic steady value \( \mathbf{M}(\infty) \) is given by
\[ \mathbf{M}(\infty) = \mathbf{L}^{-1} \cdot \mathbf{C}. \] (52)

The long time behavior of \( M_\sigma \) (\( \sigma = 1, 2 \)) is governed by the smallest eigenvalue \( \ell_{2p}^{\text{min}} \) of the matrix \( \mathbf{L} \). Given that the eigenvalues \( \ell \) of the matrix \( \mathbf{L} \) obey the quadratic equation
\[ \ell^2 - (\omega_{2p}^{(11)} + \omega_{2p}^{(22)})\ell + \left( \omega_{2p}^{(11)}\omega_{2p}^{(22)} - \nu_{2p}^{(12)}\nu_{2p}^{(21)} \right) = 0, \] (53)
the smallest eigenvalue \( \ell_{2p}^{\text{min}} \) is
\[ \ell_{2p}^{\text{min}} = \frac{\omega_{2p}^{(11)} + \omega_{2p}^{(22)} - \sqrt{\left( \omega_{2p}^{(11)} + \omega_{2p}^{(22)} \right)^2 - 4 \left( \omega_{2p}^{(11)}\omega_{2p}^{(22)} - \nu_{2p}^{(12)}\nu_{2p}^{(21)} \right)}}{2}. \] (54)

If \( \ell_{2p}^{\text{min}} > 0 \), then all the scaled moments of degree \( 2p + q \) tend asymptotically to finite values. Otherwise, for given values of the parameters of the mixture, if \( \ell_{2p}^{\text{min}} \) becomes negative for \( \alpha_{rs} \) smaller than a certain critical value \( \alpha_c \), then the moments of degree \( 2p + q \) exponentially grow in time for \( \alpha_{rs} < \alpha_c \). The critical value \( \alpha_c \) can be obtained from the condition \( \ell = 0 \) which implies
\[ \omega_{2p}^{(11)}\omega_{2p}^{(22)} - \nu_{2p}^{(12)}\nu_{2p}^{(21)} = 0. \] (55)

V. SINGULAR BEHAVIOR OF THE THIRD AND FOURTH DEGREE MOMENTS

The purpose of this section is to analyze the relaxation of the second, third and fourth degree moments to the HCS. However, a full analysis is difficult due to the many parameters involved in the problem: \( \alpha_{11}, \alpha_{12}, \alpha_{22}, m_1/m_2, x_1, \) and \( \sigma_1/\sigma_2 \). For the sake of concreteness, henceforth we will consider the particular case \( \alpha_{11} = \alpha_{12} = \alpha_{22} = \alpha \). With respect to the second degree moments (those related to the elements of the pressure tensor), as expected our results show that these moments are convergent and tend asymptotically to well-defined values. In this context, it is important to remark that the reliability of the second degree moments of inelastic Maxwell mixtures have been tested against Monte Carlo simulations for IHS in the uniform shear flow problem \[17\]. The comparison between theory and simulations shows an excellent agreement between both approaches, even for quite strong dissipation and/or disparate values of the mass and diameter ratios. Let us analyze now the behavior of the third and fourth degree velocity moments.

A. Third-degree moments

In the case of the third degree moments, Eqs. (29) and (30) show that all the moments tend to zero for sufficiently long times provided the corresponding eigenvalues are negative. However, our analysis shows that while the moments \( M_{2i,j,k}^{(rs)} \) appear to be convergent (at least in the cases studied), the moments \( M_{2i}^{(rs)} \) (which are related to the heat flux) can be divergent since the corresponding eigenvalue \( \ell_{2i}^{\text{min}} \) can be negative. As an illustration, in Fig. 4 weplot \( \ell_{2i}^{\text{min}} \) for
Given values of the parameters of the mixture, the three-dimensional system ($d = 3$) can be sensitive to the values of the parameters of the mixture. In particular, when the (scaled) moments develop an algebraic high velocity tail in the long time limit of the form $f_r(V) \sim V^{-d-s}$. The exponent $s$ is quite sensitive to the values of the parameters of the mixture. In particular, when the (scaled) moments $M_{2|i}^{(r)}$ diverge for given values of the control parameters, then $s \leq 3$.

An exhaustive analysis of the solutions to Eq. (55) shows that the third degree moments $M_{2|i}^{(r)}$ appear usually to be convergent (i) when $\sigma_1 > \sigma_2$ and $m_1 > m_2$, regardless the value of the mole fraction $x_1$, or (ii) when $\sigma_1 < \sigma_2$, $m_1 > m_2$ but the mole fraction $x_1$ is not small enough (say, $x_1 \gtrsim 0.2$). To illustrate these trends, Fig. 5 shows a phase diagram associated with the singular behavior of the third-degree moments $M_{2|i}^{(r)}$. Here, $d = 3$, $x_1 = 0.1$ and two different values of the size ratio $\sigma_1/\sigma_2$. The curves $\alpha_c(m_1/m_2)$ split the parameter space into two regions: the region below the curve corresponds to states $(\alpha, m_1/m_2)$ where the third-degree moments $M_{2|i}^{(r)}$ diverge in time while the region above defines the states where those moments are convergent (and so they go to zero). It is apparent that the region of divergent moments grows as the size of the defect species decreases with respect to that of the excess component.
B. Fourth-degree moments

In the HCS state, \( p_{r,ij} = p_r \delta_{ij} \) and so, the inhomogeneous terms \( C_{249/0}^{(rs)} \) appearing in the right hand side of the expressions (31)–(34) become

\[
C_{249/0}^{(rs)} = \frac{\omega^{*}_{rs}}{32} (1 + \beta_{rs})^2 \frac{3\beta_{rs}^2 - 6\beta_{rs} + 4d - 1}{\theta_r \theta_s}, \quad C_{249/0}^{(rs)} = C_{249/0}^{(rs)} = 0. \tag{56}
\]

Consequently, if the eigenvalues of the matrix \( \mathcal{L} \) corresponding to the fourth-degree moments are positive, Eq. (56) shows that all the moments, except \( M_{400}^{(r)} \) tend to zero for \( \tau \to \infty \). The asymptotic expression of \( M_{400}^{(r)} \) is

\[
M_{400}^{(r)} = \frac{\omega_{400}^{(11)} C_{400}^{(11)} + C_{400}^{(12)}}{\omega_{400}^{(22)} - \nu_{400}^{(12)}} \frac{\omega_{400}^{(22)}}{C_{400}^{(22)}}. \tag{57}
\]

The expression for \( M_{400}^{(2)} \) can be easily obtained from (57) by just making the changes 1 \( \leftrightarrow \) 2. Note that when the condition (55) applies, the fourth-degree moments \( M_{400}^{(r)} \) tend to infinite as expected.

An analysis on the possible divergence of the fourth-degree moments clearly shows that these moments can be also divergent in some regions of the parameter space of the system, especially the (isotropic) moment \( M_{400}^{(r)} \). Figure 6 illustrates this fact for \( d = 3, x_1 = 0.1 \) and \( \sigma_1 / \sigma_2 = 0.2 \). In this case, while the moments \( M_{400}^{(r)} \) are convergent (and...

\[ \text{FIG. 6: Phase diagram in the (} \alpha, m_1/m_2 \text{)–plane for the asymptotic long time behavior of the fourth-degree moments } M_{400}^{(r)} \text{ and } M_{249}^{(r)} \text{ for a three-dimensional system (} d = 3 \text{) with } x_1 = 0.1 \text{ and } \sigma_1 / \sigma_2 = 0.2. \]

\[ \text{FIG. 7: Phase diagram in the (} \alpha, \sigma_1 / \sigma_2 \text{)–plane for the asymptotic long time behavior of the fourth-degree moments } M_{400}^{(r)} \text{ for a three (} d = 3 \text{) and two (} d = 2 \text{) dimensional system with } x_1 = 0.5 \text{ and } m_1/m_2 = 2. \]
tend to zero for long times), the moments $M_{4|0}^{s(r)}$ and $M_{2|ij}^{s(r)}$ can be divergent (regions below the curves). It is also apparent that for a given value of the mass ratio, the critical value of the coefficient of restitution of $M_{4|0}^{s(r)}$ is larger than the one found for $M_{2|ij}^{s(r)}$ so that the divergent region of the former is bigger than the latter one. Figures 7 and 8 complement the results shown in Fig. 6. We plot the phase diagram of $M_{4|0}^{s(r)}$ for a three ($d=3$) and two ($d=2$) dimensional system with $x_1 = 0.5$ and $m_1/m_2 = 0.5$.

FIG. 8: Phase diagram in the $(\alpha, \sigma_1/\sigma_2)$-plane for the asymptotic long time behavior of the fourth-degree moments $M_{4|0}^{s(r)}$ for a three ($d = 3$) and two ($d = 2$) dimensional system with $x_1 = 0.5$ and $m_1/m_2 = 0.5$.

VI. DISCUSSION

The use of collision models where the collision rate of two colliding particles is independent of their relative velocity allows one to determine exactly the collisional moments without to know the one-particle velocity distribution function. In the case of molecular gases (i.e., particles colliding elastically), the model of Maxwell molecules (namely, when particles collide with a potential inversely proportional to the fourth power of the distance) is very useful to obtain exactly transport properties in far from equilibrium states [16, 38]. When the collisions are inelastic and characterized by a constant coefficient of normal restitution $\alpha \leq 1$, the use of IMM opens up the possibility of obtaining exact results (the elastic limit $\alpha = 1$ is a special limit). This is likely the main reason for which IMM has attracted the attention of physicists and mathematicians in the past few years.

In the case of monocomponent granular gases, the choice of the Ilenberry polynomials $Y_{2p|q}$ [38] of degree $2p + q$ allows one to express the corresponding collisional moment $J_{2p|q}$ as an eigenvalue $-\nu_{2p|q}$ times the velocity moment $M_{2p|q}$ plus a bilinear combination of moments of degree less than $2p + q$. All the third and fourth degree collisional moments of IMM for monocomponent gases were evaluated in Ref. [20]. We have extended in this paper the above results to the interesting case of binary granular mixtures. Due to the intricacy of the general problem, we have considered here situations where diffusion processes are absent. This means that the mean flow velocities $U_\tau$ of each species are equal to the mean flow velocity $U$ of the mixture ($U_1 = U_2 = U$). Apart from this simplification, the results reported here for the third and fourth degree collisional moments are exact for arbitrary values of the masses $m_\tau$, diameters $\sigma_\tau$, concentration $x_\tau$, and coefficients of restitution $\alpha_\tau$. In addition, all the derived expressions apply for any dimensionality $d$. Known results for three-dimensional molecular gases [16, 38] and for $d$-dimensional monocomponent granular gases [20] are recovered. In the one-dimensional case ($d = 1$) for binary granular mixtures, our results for the (isotropic) collisional moments $J_{2|0}^{(rs)}$ and $J_{4|0}^{(rs)}$ agree with the ones obtained by Marconi and Puglisi [30].

As for monocomponent gases [20], we have observed that some of the eigenvalues $\nu_{2p|q}^{(rs)}$ exhibit a non-monotonic dependence on the coefficients of restitution $\alpha_{rs}$ at a given values of the mass and diameter ratios and the concentration. We also see that the impact of the inelasticity in collisions on the eigenvalues is in general important, specially in the case of the eigenvalues associated with the self-collision terms. Although the above observations are restricted to the moments of degree $2p + q \leq 4$, we expect that they extend to moments of higher degree.
As a simple application of the results derived in section III, we have analyzed the time evolution of the moments of degree equal to or less than 4 in the HCS. In this state, given that the granular temperature $T$ decreases in time, one has to scale the moments with the thermal speed $v_{th}(t) = \sqrt{2T(t)/m}$ to reach steady values in the long time limit. Our analysis shows that while all the second degree moments tend toward finite values in the long time limit, the third degree moments $M_3^{(r)}$ (which are related to the heat flux) can diverge in a region of the parameter space of the mixture. This sort of divergence also appears in all (isotropic and anisotropic) fourth degree moments. The above conclusions contrast with the ones achieved for monocomponent granular gases [20] where all the moments of degree 2 + $p + q \leq 4$ are convergent for $d \geq 2$. The singular behavior of the third degree moments is consistent with an algebraic high velocity tail of the form $f_r(V) \sim V^{-(d+s)}$, where $s \leq 3$ when the moments $M_3^{(r)}$ are divergent. We plan to explore this possibility in the near future.

The explicit results provided in this paper can be employed to analyze different nonequilibrium problems. One of them is the so-called simple or uniform shear flow; which is intrinsically a non-Newtonian state. Apart from the rheological properties [17], it would be interesting to study the time evolution of the fourth degree velocity moments toward their steady values and investigate whether these moments can be divergent as occurs for elastic collisions [34]. Another interesting application of the present results is to determine some of the generalized transport coefficients characterizing small perturbations around the simple shear flow problem. Work along these lines will be carried out in the near future.

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Appendix A: Some technical details in the evaluation of the collisional moments

In this Appendix we give some technical details on the derivation of the collisional moments $J^{(rs)}_{2|i_1,i_2...i_q}$ associated with the Ilenberry polynomials of third and fourth degree when the mass flux of each species vanishes.

We consider for the sake of concreteness the (anisotropic) third degree collisional moment

$$J^{(rs)}_{ij} = \int d\mathbf{V}[V_i V_j V_k - \frac{1}{d+2} V^2 (V_i \delta_{jk} + V_j \delta_{ik} + V_k \delta_{ij})] J_{rs}[f_r, f_s]$$

$$= J^{(rs)}_{ijk} - \frac{1}{d+2} \left( J^{(rs)}_{2|i} \delta_{jk} + J^{(rs)}_{2|j} \delta_{ik} + J^{(rs)}_{2|k} \delta_{ij} \right),$$  \hspace{1cm} (A1)

where

$$J^{(rs)}_{ijk} = \int d\mathbf{V} V_i V_j V_k J_{rs}[f_r, f_s],$$  \hspace{1cm} (A2)

$$J^{(rs)}_{2|i} = \int d\mathbf{V} V^2 V_i J_{rs}[f_r, f_s].$$  \hspace{1cm} (A3)

Let us evaluate first the third degree (canonical) collisional moment $J^{(rs)}_{ijk}$. Taking into account the property [3], Eq. (A2) can be written as

$$J^{(rs)}_{ijk} = \frac{\bar{\omega}_{rs}}{n_s \Omega_d} \int d\mathbf{V}_1 \int d\mathbf{V}_2 f_r(\mathbf{V}_1) f_s(\mathbf{V}_2) \int d\hat{\mathbf{S}} \ (V_i' V_j' V_k' - V_i V_j V_k),$$  \hspace{1cm} (A4)

where Eq. [3] gives the relationship between the post-collisional velocity $\mathbf{V}_1$ and the pre-collisional velocity $\mathbf{V}_2$. The fact that $\mathbf{U}_1 = \mathbf{U}_2 = \mathbf{U}$ allows one to map some of the results obtained for monodisperse gases [20] by making the change

$$\alpha \rightarrow \beta_{rs} = 2 \mu_{sr} (1 + \alpha_{rs}) - 1.$$  \hspace{1cm} (A5)
Therefore, from Eqs. (A11) and (A13)–(A15) one finally obtains

\[ V'_{i_1}V'_{i_2}V_{k} - V_{i_1}V_{i_2}V_{k} = -\frac{1+\beta_{rs}}{2} (\hat{\sigma} \cdot g) \left\{ V_{i_1}V_{i_2}\hat{\sigma}_k + V_{i_1}V_{i_2}\hat{\sigma}_j + V_{i_1}V_{i_2}\hat{\sigma}_i \right\} \]

\[ -\frac{1+\beta_{rs}}{2} (\hat{\sigma} \cdot g) \left[ V_{i_1}\hat{\sigma}_k + V_{i_1}\hat{\sigma}_j + V_{i_1}\hat{\sigma}_i \right] \]

To carry out the angular integrations in (A4) one needs the results

\[ \int d\hat{\sigma} (\hat{\sigma} \cdot g)^{2k+1} \hat{\sigma}_i = B_{k+1}g^{2k}g_i, \quad (A7) \]

\[ \int d\hat{\sigma} (\hat{\sigma} \cdot g)^{2k} \hat{\sigma}_j = \frac{B_k}{2k+1} g^{2(k-1)} (2kg_i g_j + g^2 \delta_{ij}), \quad (A8) \]

\[ \int d\hat{\sigma} (\hat{\sigma} \cdot g)^{2k+1} \hat{\sigma}_i \hat{\sigma}_j = \frac{B_{k+1}}{2k+1} g^{2(k-1)} [2kg_i g_j + g^2 (\delta_{ij} + \delta_{ik} g_j + \delta_{jk} g_i)], \quad (A9) \]

Here, the coefficients \( B_k \) are

\[ B_k = \int d\hat{\sigma} (\hat{\sigma} \cdot g)^{2k} = \Omega_{d} \pi^{-1/2} \frac{\Gamma(\frac{d}{2}) \Gamma(k + \frac{1}{2})}{\Gamma(k + \frac{d}{2})}. \quad (A10) \]

Making use of Eqs. (A6)–(A9), one gets

\[ J^{(rs)}_{ij} = \frac{1}{2d(d+2)} \omega_{rs} (1+\beta_{rs}) \left\{ (d+2) \langle V_{i_1}V_{i_2}g_k + V_{i_1}V_{i_2}g_j + V_{i_1}V_{i_2}g_i \rangle \right. \]

\[ -\frac{1+\beta_{rs}}{2} \left( 2(g_i g_j V_{i_2} + g_i g_k V_{i_1} + g_j g_k V_{i_1}) \right. \]

\[ + g^2 (V_{i_1} \delta_{jk} + V_{i_1} \delta_{ik} + V_{i_1} \delta_{ij})) \left] + \frac{3}{4(d+4)} (1+\beta_{rs})^2 \right. \]

\[ \times \left[ 2(g_i g_j g_k + g^2 (g_i \delta_{jk} + g_j \delta_{ik} + g_k \delta_{ij})) \right] \}, \quad (A11) \]

where the brackets are defined as

\[ \langle h(V_1, V_2) \rangle \equiv \int dV_1 \int dV_2 h(V_1, V_2) f_r(V_1) f_s(V_2). \quad (A12) \]

The integrations over velocities give the relations

\[ \langle V_{i_1}V_{i_2}V_{i_k} \rangle = \langle g_i g_j V_k \rangle = n_s M^{(rs)}_{ijk}, \quad (g^2 V_{i_1}) = n_s M^{(rs)}_{2ij}, \quad (A13) \]

\[ \langle g_i g_j g_k \rangle = n_s M^{(rs)}_{ijk} - n_r M^{(rs)}_{ijk}, \quad (A14) \]

where

\[ M^{(rs)}_{ijk} = \int dVV_{i_1}V_{i_2}V_{k} f_r(V). \quad (A15) \]

Therefore, from Eqs. (A11) and (A13)–(A15) one finally obtains

\[ J^{(rs)}_{ij} = -\frac{3}{4d(d+2)(d+4)} \omega_{rs} (1+\beta_{rs}) \left( \left[ \beta_{rs} - 2(d+3)\beta_{rs} + 2d^2 + 10d + 9 \right] \right) \]

\[ \times n_s M^{(rs)}_{ijk} - (1+\beta_{rs})^2 n_r M^{(s)}_{ijk} \]

\[ + \frac{(1+\beta_{rs})}{6} \left( 3\beta_{rs} - 2d - 5 \right) n_s \left( M^{(rs)}_{2ii} \delta_{ij} + M^{(rs)}_{2ij} \delta_{ik} + M^{(rs)}_{2ik} \delta_{ij} \right) \]

\[ - \frac{(1+\beta_{rs})}{2} n_r \left( M^{(s)}_{2ii} \delta_{ij} + M^{(s)}_{2ij} \delta_{ik} + M^{(s)}_{2ik} \delta_{ij} \right). \quad (A16) \]
If one makes \( j = k \) and sum over \( j \) one obtains the expression (29) for \( J^{rs}_{2ji} \). Also, by subtracting \( (J^{rs}_{2ii} \delta_{jk} + J^{rs}_{2ij} \delta_{jk} + J^{rs}_{2ji} \delta_{jk}) / (d + 2) \) from both sides of Eq. (A16) one gets Eq. (30) for \( J^{rs}_{0ijjk} \).

The calculations for the fourth degree collisional moments are similar to those carried out for the third degree moments. In addition, most of the mathematical steps followed to get them can be easily mapped from those made for monodisperse gases in Ref. [20] by replacing \( \alpha \rightarrow \beta_{rs} \). After a long and tedious algebra one obtains the results displayed along the Section II.

### Appendix B: Explicit forms of the eigenvalues

In this Appendix we give the explicit expressions of the eigenvalues \( \nu_{2p|q}^{(11)} \) and \( \nu_{2p|q}^{(12)} \) with \( 2p + q \leq 4 \) of the collisional moments \( J^{rs}_{2pi} \). In the case of the second degree collisional moments, the eigenvalues are

\[
\nu_{2i0}^{(11)} = \omega_{11}^{*} \frac{1 - \alpha_{11}^{2}}{2d} + \frac{\omega_{12}^{*}}{4d} (1 + \beta_{12})(3 - \beta_{12}), \quad \nu_{2i0}^{(12)} = -\frac{\omega_{12}^{*}}{4d} (1 + \beta_{12})^{2},
\]

\[
\nu_{0ij}^{(11)} = \frac{\omega_{11}^{*}}{d(d + 2)} (1 + \alpha_{11})(d + 1 - \alpha_{11}) + \frac{\omega_{12}^{*}}{2d(d + 2)} (1 + \beta_{12})(2d + 3 - \beta_{12}),
\]

\[
\nu_{0ij}^{(12)} = -\frac{\omega_{12}^{*}}{2d(d + 2)} (1 + \beta_{12})^{2},
\]

where \( \omega_{ij}^{*} = \omega_{ij} / \nu_{0} \). The corresponding eigenvalues in the case of the third degree collisional moments are given by

\[
\nu_{2i1}^{(11)} = \frac{\omega_{11}^{*}}{4d(d + 2)} (1 + \alpha_{11})(4 + 5d - (d + 8)\alpha_{11}) + \frac{\omega_{12}^{*}}{8d(d + 2)} (1 + \beta_{12}) \times [3\beta_{12}^{2} - 2(d + 5)\beta_{12} + 10d + 11],
\]

\[
\nu_{2i1}^{(12)} = -3\frac{\omega_{12}^{*}}{8d(d + 2)} (1 + \beta_{12})^{3},
\]

\[
\nu_{0ij2}^{(11)} = \frac{3}{2d(d + 2)} (1 + \alpha_{11})(d + 1 - \alpha_{11}) + \frac{3\omega_{12}^{*}}{4d(d + 2)(d + 4)} (1 + \beta_{12}) \times [\beta_{12}^{2} - 2(d + 3)\beta_{12} + 2d^{2} + 10d + 9],
\]

\[
\nu_{0ij2}^{(12)} = -3\frac{\omega_{12}^{*}}{4d(d + 2)(d + 4)} (1 + \beta_{12})^{3}.
\]

Finally, the eigenvalues of the fourth degree collisional moments are

\[
\nu_{4i0}^{(11)} = \frac{\omega_{11}^{*}}{8d(d + 2)} (1 + \alpha_{11})(9 + 12d - (4d + 17)\alpha_{11} + 3\alpha_{11}^{2} - 3\alpha_{11}^{3}) + \frac{\omega_{12}^{*}}{16d(d + 2)} (1 + \beta_{12})(3 - \beta_{12})[3\beta_{12}^{2} - 6\beta_{12} + 8d + 7],
\]

\[
\nu_{4i0}^{(12)} = -3\frac{\omega_{12}^{*}}{16d(d + 2)} (1 + \beta_{12})^{3},
\]

\[
\nu_{2i2}^{(11)} = \frac{\omega_{11}^{*}}{4d(d + 2)(d + 4)} (1 + \alpha_{11})[7d^{2} + 31d + 18 - (d^{2} + 14d + 34)\alpha_{11}] + 3(d + 2)\alpha_{11}^{2} - 6\alpha_{11}^{3} + \frac{\omega_{12}^{*}}{4d(d + 2)(d + 4)} (1 + \beta_{12}) \times [7d^{2} + 31d + 21 - (d^{2} + 14d + 25)\beta_{12} + 3(d + 5)\beta_{12}^{2} - 3\beta_{12}^{3}],
\]
\[ \nu_{2(12)}^{(11)} = -3 \frac{\omega_{12}^2}{4d(d+2)(d+4)}(1 + \beta_{12})^4, \]  

(B11)

\[ \nu_{0(12)}^{(11)} = \frac{\omega_{11}^2}{2d(d+2)(d+4)(d+6)}(1 + \alpha_{11})[2d^3 + 21d^2 + 61d + 39 

- 3(d + 3)(d + 5)\alpha_{11} + 3(d + 3)\alpha_{11}^2 - 3\alpha_{11}^3] 

+ \frac{\omega_{12}^4}{2d(d+2)(d+4)(d+6)}(1 + \beta_{12})[4d^3 + 42d^2 + 122d + 81 

- 3(2d^2 + 16d + 27)\beta_{12} + (6d + 27)\beta_{12}^2 - 3\beta_{12}^3], \]  

(B12)

\[ \nu_{0(12)}^{(12)} = -3 \frac{\omega_{12}^2}{2d(d+2)(d+4)(d+6)}(1 + \beta_{12})^4. \]  

(B13)

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