Kostka Numbers and Longest Increasing Subsequences

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Abstract

A classical bijection relates certain Kostka numbers, the Catalan numbers, and permutations of length $n$ with longest increasing subsequence (LIS) of length at most 2. We generalize this bijection and find Kostka numbers which count the number of permutations of $n$ with LIS length at most $w$, the number of permutations with $(1, \ldots, w)$ as a LIS, and other similar subsets of permutations.

1 Introduction

The Kostka numbers $K_{\lambda \mu}$ appear naturally in symmetric function theory and in several counting problems related to the symmetric group $S_n$. They are indexed by a partition $\lambda$ of $n$ (written $\lambda \vdash n$) and a vector with non-negative integer entries $\mu$ such that $\sum \mu_i = n$. They count the number of Young tableaux with shape $\lambda$ that are filled with $\mu_i$ copies of the number $i$. $\mu$ is called the content (or weight) vector.

The Kostka numbers $K_{\lambda \mu}$ were originally defined as the monomial coefficients of the Schur functions $s_\lambda = \sum_{\mu \vdash n} K_{\lambda \mu} m_\mu$ where $\lambda \vdash n$, and $m_\mu$ is the monomial symmetric function. Littlewood later discovered the combinatorial interpretation we gave above.

Given $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n$, $(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_w})$ is an increasing subsequence in $\sigma$ if $i_1 < i_2 < \cdots < i_w$ and $\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_w}$. Consider the set $\mathcal{L}_n(w) \subset S_n$, whose elements have longest increasing subsequences (LIS) of length at most $w$. Let $u_n(w) = |\mathcal{L}_n(w)|$. It is a classical fact that $u_n(2) = C_n$, where $C_n$ is the $n^{th}$ Catalan number. $u_n(2)$ is related to a certain Kostka number through a classical operation on Young tableaux (discussed below). We generalize this bijection to count $\mathcal{L}_n(w)$ and other subsets such as $\{ \sigma \in S_n : (1, 2, \ldots, w) \text{ is an LIS} \}$ for $w > 2$ using related Kostka numbers. Computational evidence suggests that the latter Kostka numbers also count $r$-colored non-crossing partitions of $[n]$ [14], which are themselves counted by certain oscillating tableaux [5, 6] (see Conjecture 1).

Gessel provided a formula for $u_n(w)$ in terms of determinants of modified Bessel functions [10]; these formulas simplify for $w \leq 3$ and may be found in the same paper. By the Robinson-Schensted (RS) correspondence, $u_n(w)$ are counted by pairs of standard tableaux of the same shape with at most $w$ columns. Let $y_n(w)$ be the number of
standard tableaux with at most $w$ columns. Regev gives formulas for $y_n(w)$ for $w = 2$ and 3, and Gouyou-Beauchamps [7] gives combinatorial proofs and formulas for $2 \leq w \leq 5$. Wilf [4] gives a formula for $u_n(w)$ for $w = 2, 4, \ldots$ in terms of $y_j(w)$, $j = 1, \ldots, 2n$; thus formulas for $u_n(w)$ for $w = 4$ may be obtained from the formulas for $y_n(w)$. Goulden [8] gives formulas for both $y_n(w)$ and $u_n(w)$ for large $w$ (depending on $n$). Many of these facts and references may be found in Stanley’s book [8, pp 452, 493, problem 7.16].

The Robinson-Schensted (RS) correspondence is a bijective map between permutations in $S_n$ and pairs of standard Young tableau of the same shape $\lambda \vdash n$ [11, 12]. The length of the top row (or width) of $\lambda$ is the length of a LIS. Thus, via the RS correspondence, $L_n(2)$ is in bijection with pairs of tableaux $(P, Q)$ with width at most 2 and size $n$. There is an appealing, classical way to convert this pair of tableaux into a single rectangular tableau of shape $\square_{2 \times n}$, where $\square_{w \times n}$ denotes a rectangular diagram with width $w$ and height $n$ [13; 8, page 263, problem 6.19.ww]. First replace $i$ with $2n-i+1$ for $i \in [n]$ in the tableau $Q$. Then, rotate $Q$ by 180 degrees and glue it to $P$; the gluing is done so that the rows with length 1 in $P$ align with the rows of length 1 in $Q$. Thus we obtain a rectangular tableau with width 2 and height $n$. We illustrate this cutting, relabeling and gluing procedure in Example 1.

Example 1.

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
3 & 2 & \\
\end{array} \rightarrow \begin{array}{c|c|c}
1 & 2 & 3 \\
1 & 3 & 5 \\
2 & 4 & 6 \\
\end{array} \rightarrow \begin{array}{c|c|c}
1 & 2 & 3 \\
1 & 2 & 3 \\
2 & 4 & 5 \\
\end{array}
\]

It is not hard to see that the steps of the transformation are invertible, and that it is a bijection. Let $\vec{a}_k$ be a vector of $k$ copies of the number $a$. The number of rectangular tableaux of shape $2 \times n$ is given by the Kostka number $K_{\square_{2 \times n}, \vec{1}_n}$. Since the rectangular tableaux are standard —all the weights are 1— we can use the hook length formula [11] to count them. Hence,

\[
 u_n(2) = K_{\square_{2 \times n}, \vec{1}_n} = \frac{(2n)!}{n!(n+1)!} = C_n.
\]

2 Main results

We first state a generalization of (1).

Theorem 2.1. Let $\mu_n^a = \vec{a}_n \oplus \vec{1}_n$ be the vector of $n$ copies of $a$ and $n$ copies of 1. Then,

\[
 K_{\square_{w \times n}, \mu_n^{a_{w-1}}} = u_n(w).
\]

The Kostka numbers in Theorem 2.1 count rectangular tableaux with particular weight vectors. The proof establishes a bijection between these rectangular tableaux
and $L_n(w)$ by generalizing the argument that proves (1). The generalization involves a standard involutive operation on columns [8, page 473, problem 7.41] (see section 4).

Next, we consider a further generalization of the algorithm by skewing the weight vector $\mu$. The skew weight vectors have an additional parameter, $k \in \mathbb{Z}$. We define

$$\mu_{a,n,k} = \vec{a}_{n-k} \oplus \vec{1}_{ak+n}$$

so that $\sum_i \left( \mu_{a,n,k}^i \right) = (a+1)n$; this last constraint ensures that the weights fill rectangular tableaux of width $a+1$ and height $n$. The magnitude of $k$ indicates how asymmetric the weight vector is. Clearly we must have $k \leq n$, and the constraint $n + ka \geq 0$ implies that $k$ must be an integer in the interval $[-\frac{n}{a}, n]$. When the number $n$ is understood, we will simply use $\mu_a^k$. If $k = 0$, we will drop $k$ from the notation and simply write $\mu_a$, since $\mu_{a,n}^0 = \mu_{a,0}$.

We will also use the following generalization of the Catalan numbers $A_{n,m}$ from Griffiths and Lord [14]. These count the number of standard rectangular tableaux of width $n$ and height $m$. When $m = 2$, again, these are the Catalan numbers. Using the hook-length formula, simple formulas for $A_{n,m}$ may be obtained:

$$A_{n,m} = \frac{(mn)!s(m-1)s(n-1)}{s(m+n-1)}$$

where $s$ denotes the super factorial function $s(k) = \prod_{i=1}^k i!$.

**Theorem 2.2.** For all $n$, and any $2 \leq w < n$, fix a nonzero integer $k$ in the interval $[-n/(w-1), n]$. Let

$$m = \begin{cases} n + k(w-1) & k > 0 \\ n - k & k < 0 \end{cases}$$

Then, $K_{\square_{w \times n}, \mu_{w-1}^n} A_{|k|,w}$ counts the number of permutations $\sigma \in S_m$ that have at least $|k|$ disjoint longest increasing subsequences of length $w$ using the numbers from $\{1, \ldots, |wk|\}$.

The $k = 0$ case is covered in Theorem 2.1. The proof follows the algorithm in Section 4 along with some tweaks to work with the skew weights (see Section 5). The $k = 1$ case gives the following corollary.

**Corollary 2.3.** When $k = 1$, the algorithm used to prove Theorem 2.2 provides a bijection between rectangular tableaux of shape $\square_{w \times n}$ and weight $\mu_{w-1}^n$, and the set $\{\sigma \in S_n : (1,2,\ldots,w) \text{ is an LIS}\}$. Hence,

$$K_{\square_{w \times n}, \mu_{w-1}^n} = |\{\sigma \in S_n : (1,2,\ldots,w) \text{ is an LIS}\}|.$$

(3)

The following is a particular case of the symmetry noted by Briand, Orellana, and Rosas [13, Theorem 5].
Corollary 2.4.

\[ K_{\square_{w \times n}^{n,k}} = K_{\square_{w \times n+k(w-2)}^{n+k(w-2),-k}}. \]

The authors originally observed a relationship between the RHS of (3) and \( k \)-colored non-crossing partitions, originally defined by Marberg [16]. Let \( \text{NC}_2(n, k) \) be the number of noncrossing partitions of \([n]\) with \( k \) colors [16, Corollary 1.5].

Conjecture 1. \(|\{\sigma \in S_n: (1, 2, \ldots, w) \text{ is an LIS}\}| = \text{NC}_2(n - w + 1, w - 1).\)

2.1 Acknowledgements

Computer simulation indicated Conjecture 1. We are indebted to E. Marberg for pointing us towards these particular Kostka numbers, and this inspired our Theorem 2.2. He mentioned hearing the following conjecture from A. Tripathi:

\[ K_{\square_{3 \times n}^{n,1}} = \text{NC}_2(n, 2). \]

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3 Tableau Operations

We represent a Young diagram as \( \lambda = (\lambda_1, \ldots, \lambda_m) \) with \( \lambda_i \) being the number of boxes in the \( i \text{th column} \). This is nonstandard, since \( \lambda_i \) usually denotes to the length of the \( i \text{th row} \), but it makes the upcoming tableau complement operation easier to state. Recall that a Young tableau has each column strictly increasing and each row non-decreasing. We call a Young tableau a standard Young tableau if rows are strictly increasing as well. We use \([n]\) to denote the set \( \{1, \ldots, n\} \). If \( P \) is a Young tableau, we will use \( P_i \) to denote the \( i \text{th column} \).

We have two operations on tableaux that we need to introduce. One is a standard operation, but is phrased in a nonstandard way since it makes the second easier to state. If \( \lambda \) is a sub-diagram of \( \square_{w \times h} \) (\( \lambda \subset \square_{w \times h} \)), we define \( \square_{w \times h} - \lambda \) as the sub-diagram of \( \square_{w \times h} \) with a 180° rotated \( \lambda \) removed from the bottom right corner of \( \square_{w \times h} \). In more standard language, let \( \lambda' \) be a skew diagram such that \( \square_{w \times h}/\gamma = \lambda' \), where \( \gamma \subset \square_{w \times h} \). Let \( \lambda \) be a 180° rotated version of \( \lambda' \). Then, in our notation, \( \gamma = \square_{w \times h} - \lambda \).

Definition 3.1 (Column). A column is a column vector \( C = (c_1, \ldots, c_k) \) whose elements are strictly increasing. It will typically represent the column of a tableau. When we refer to a set as a column, we mean that its elements are arranged to be strictly increasing. Let \( |C| \) be its cardinality.

Definition 3.2 (Rectangular diagram/tableau subtraction). Let \( \lambda \subset \square_{w \times h} \). Then \( \gamma = \square_{w \times h} - \lambda \) is a Young diagram with columns \( \gamma_i \) that satisfy

\[ |\gamma_i| = h - |\lambda_{w-i+1}|. \]
When $R$ is a tableau of shape $\square_{w \times n}$ and $Q$ is a subtableau, we similarly remove a $Q$ shaped block from it, with $(R - Q)$ being composed of the $h - |Q_{w-i+1}|$ smallest elements in $R_i$.

For example

\[
\lambda = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{array} \\
\square_{3 \times 4} - \lambda = \begin{array}{ccc}
\hline
\hline
1
\hline
2
\hline
3
\hline
4
\hline
5
\hline
6
\hline
7
\hline
8
\hline
9
\hline
\end{array}
\]

**Definition 3.3 (Column complement).** The complement of a column $C$ with respect to a set $[n]$ consists of the column with elements $[n] \setminus C$, and is denoted $\overline{C}^n$. We omit the superscript $n$ if it is clear from context. The operation also applies to empty columns.

Note that

\[
\overline{\overline{C}^n} = C,
\]

that is, the operation is involutive. If we complement all the columns of a tableau and reverse the column order, we obtain a new tableau. (See Lemma 3.5.)

**Definition 3.4 (Tableau complement).** The complement of a tableau $P$ with respect to a width $w$ and set $[n]$ is denoted by $P^{w,n}$, where

\[
(P^{w,n})_i = P_{w-i+1}^n \quad 1 \leq i \leq w.
\]

Again, we occasionally omit the number $n$ when it is clear from context. A column $P_{w-i+1}$ may be empty, and in this case $(P^{w,n})_i = [n]$.

**Example 2.**

\[
P = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 4 \\
3 & 4 & \end{array} \\
\overline{P}^{3,4} = \begin{array}{ccc}
1 & 2 & 4 \\
\hline
3
\end{array}
\]

**Lemma 3.5.** If $P$ is a tableau then so is $P^{w,n}$ for all $w, n \in \mathbb{N}$.

Lemma 3.5 follows from Claim 4.5 in Section 4.

**Lemma 3.6.** The map $P \mapsto P^{w,n}$ is invertible, and is its own inverse.

**Proof.** If $M = P^{w,n}$, the complement of the $w - i + 1$ th column of $M$ is equal to the $i$ th column of $P$. That is, for $1 \leq i \leq w$,

\[
M_{w-i+1}^n = ([n] - M_{w-i+1}) = ([n] - ([n] - P_{w-(w-i+1)+1})) = ([n] - ([n] - P_i)) = P_i.
\]

\[\square\]
4 Algorithm

We now provide an invertible algorithm which takes a $\square_{w \times n}$ shaped tableau with weights $\mu_{w-1}^n$, and returns a pair of tableaux $(P, Q)$ with shape $\lambda$ and width at most $w$. The triple $(P, Q, \lambda)$ in turn corresponds to a permutation with LIS of length at most $w$ by the RS correspondence.

Let $R$ be a semistandard tableau of shape $\square_{w \times n}$ with weight $\mu_{w-1}^n$.

1. The elements $[n + 1, 2n]$ in $R$ must form a contiguous skew tableau in the bottom right corner of $R$. If we rotate this skew tableau by 180 deg and replace the numbers $k$ with $2n - k + 1$ for $k \in [n + 1, 2n]$, we obtain a standard tableau $Q$ of shape $\lambda$. For all coordinates $(i, j)$ in the diagram $\lambda$ we have $Q_{i,j} = 2n - R_{w-i+1,n-j+1} + 1$.

2. To get $P$, we remove the elements $[n + 1, 2n]$ from $R$, and then take its tableau complement:

$$P = R - Q^w,n$$

Example 3. If

$$R = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 4 \\
3 & 4 & 5 \\
6 & 7 & 8 
\end{array}$$

First, we split the $R$ into $R - \lambda$ and a piece that will eventually become $Q$.

$$\begin{array}{ccc}
1 & 1 & 1 \\
2 & 3 & 4 \\
3 & 4 & 5 \\
6 & 7 & 8 
\end{array}$$

Then, we take the complement of the first tableau to get $P$. To get $Q$, we rotate and replace the numbers as specified above. The algorithm returns:

$$P = \begin{array}{ccc}
1 & 2 & 4 \\
3 & 4 & 3 
\end{array} \quad Q = \begin{array}{ccc}
1 & 2 & 3 \\
4 & & 
\end{array}$$

Lemma 4.1. The $P$ given by the algorithm is a standard tableau, and has the same shape $\lambda$ as $Q$.

Proof. The proof follows from Claims 4.3, 4.4 and 4.5 below. They prove that the complement operation on $R - Q$ gives the correct shape $\lambda$ for $P$ (by 4.3), $P$ has the right entries and that $P$ is a standard tableau (by Claim 4.4 when $k = 0$ and Claim 4.5).

Lemma 4.2. The algorithm is a bijection from rectangular tableau of shape $\square_{w \times n}$ and weight $\mu_{w-1}^n$ to pairs of standard tableau of the same shape having $n$ elements and width at most $w$. 

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Proof. The algorithm is reversible at every stage: the splitting of $R$ into $R - \lambda$ and $Q$ is invertible; the relabeling of the elements in $Q$ is a bijection from \{1, \ldots, n\} to \{n + 1, \ldots, 2n\}; and the complement operation is an invertible operation (Lemma 3.6) that sends a tableau of shape $\square_{w \times n - \lambda}$ and weight $\mu_{w-1}$ to a standard tableau of shape $\lambda$. We may invert the algorithm by reversing the steps: given a pair of tableaux $(P, Q)$ of shape $\lambda \vdash n$ and width at most $w$, we can reconstruct a $\square_{w \times n}$ shaped tableau $R$ with weight $\mu_{w-1}$. Since each of the stages of the algorithm are bijections, the algorithm itself is a bijection.

Claim 4.3 (Complement gives the right shape). Let $\lambda$ be a Young diagram. Let $A$ be a tableau with shape $\square_{w \times n - \lambda}$. Then, $\overline{A}^{w,n}$ has shape $\lambda$.

Proof. By Definition 3.2 there are exactly $n - |\lambda_{w-i+1}|$ distinct elements in column $A_i$, since $A$ is formed by subtracting a rotated $\lambda$ from the square tableau of height $n$. The elements are distinct because columns in a tableau must be strictly increasing. Therefore,

$$|\overline{(A^{w,n})}_{w-i+1}| = n - |A_i| = \lambda_{w-i+1}.$$ 

This shows that all the columns of $\overline{A}^{w,n}$ have the correct height. \hfill \Box

The following claim is stated in a general form so that it applies to the skew tableau considered in Section 5. We only need the case $k = 0$ for Lemma 4.1.

Claim 4.4 (Complement gives the right entries). For integers $n, w > 1$, and $k$ such that $-n/(w-1) \leq k \leq n$, let $\lambda \vdash n + k(w - 1)$ be a diagram with width at most $w$ and let $A$ be a tableau with shape $\square_{w \times n - \lambda}$ with weight $\mu_{w-1}$. Then, $\overline{A}^{w,n-k}$ contains exactly one copy of each element in $[n - k]$.

Proof. Since there are at most $w$ columns in $A$, and no column can have a duplicate element (if a column did then it would not be strictly increasing), there can be at most one column that does not contain some element $v$. Let $\rho(v)$ be the only column of $A$ such that $A_{\rho(v)}$ does not contain $v \in [n - k]$. This means that $v \in (\overline{A}^{w,n-k})_{w-\rho(v)+1}$, and so all elements $[n - k]$ appear in $\overline{A}^{w,n-k}$. Since $\rho(v)$ is the unique column that does not contain each $v \in [n - k]$, this also shows that it appears exactly once in $\overline{A}^{w,n-k}$; since the complement is with respect to $n - k$, these are the only elements that appear in $\overline{A}^{w,n-k}$.

For 2 columns $w, v$, we say $w \preceq v$ if $|w| \geq |v|$ and $\forall i \leq |v|, w_i \leq v_i$. Thus $P$ is a semistandard tableau with columns $P_i$ if for all $i \leq j$ we have $P_i \preceq P_j$.

Claim 4.5 (Column complement reverses $\preceq$ order). Let $w \preceq v$ be two columns. Then, $\overline{w} \preceq \overline{v}$. 

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Proof. Let $U = [n]$ be a set of integers such that $v, w \subset U$. We omit the superscript in $\pi^i$ in the following. Let $w$ and $v$ be two columns such that $w \preceq v$. The proof proceeds by induction on the elements of $U$. We imagine that we are growing $\pi$ and $\overline{\pi}$ by examining each element of $U$ sequentially and adding it to either $\overline{(\pi)}_q$ or $(\overline{\pi})_q$, where $(\overline{\pi})_q := \{ x \in U \setminus v, x \leq a_q \}$ to be the elements of $U$ added to $(\overline{\pi})_q$ by step $q$ and $(\overline{\pi})_q$ similarly.

Let $(\overline{\pi})_0$ and $(\overline{\pi})_0$ be empty, and suppose the first $q$ numbers have been examined. As the induction hypothesis, suppose $|\overline{(\pi)}_q| \geq |\overline{(\pi)}_q|$ and $\overline{\pi}_i \leq \overline{\pi}_i$ for $i = 1, \ldots, |(\overline{\pi})_q|$. Then, there are four possibilities for the $(q + 1)^{th}$ element:

1. $q + 1 \in v \cap w$, then $q + 1$ is not added to either $(\overline{\pi})_q$ or $(\overline{\pi})_q$.
2. $q + 1 \in v \cap \overline{w}$, then $q + 1$ is added to $(\overline{\pi})_q$.
3. $q + 1 \in v \cap \overline{w}$, then $q + 1$ is added to $(\overline{\pi})_q$.
4. $q + 1 \in v \cap \overline{\pi}$, then $q + 1$ is added to both $(\overline{\pi})_q$ and $(\overline{\pi})_q$.

Then, in case 1, $q + 1$ is not added to either set, and hence $|\overline{(\pi)}_{q + 1}| = |\overline{(\pi)}_q| \geq |\overline{(\pi)}_{q + 1}|$ and the induction hypothesis holds. In case 2, $|\overline{(\pi)}_{q + 1}| \geq |\overline{(\pi)}_{q + 1}|$ and since no element was added to $(\overline{\pi})_{q + 1}$, $\overline{\pi}_{(\overline{\pi})_{q + 1}} \leq \overline{\pi}_{(\overline{\pi})_{q + 1}}$. If $(\overline{\pi})_{q + 1}$ is added eventually).

If we are in case 3 then we claim that $|\overline{(\pi)}_q| > |(\overline{\pi})_q|$. Suppose for contradiction that $|\overline{(\pi)}_q| = |\overline{(\pi)}_q|$, then for $r = q - |(\overline{\pi})_q|$ we have $q + 1 = v_{r + 1}$, and hence $w_{r + 1} > v_{r + 1}$ or $|v| > |w|$. This contradicts $w \preceq v$. Case 4 is trivial, and hence the induction step is proved. To establish the $q = 0$ case, we essentially repeat the above argument with empty $(\overline{\pi})_0$ and $(\overline{\pi})_0$. It is easy to see here that case 3 cannot occur. The claim is proved when $q = n$, as then $(\overline{\pi})_q = \overline{\pi}$ and likewise for $w$. \qed

5 Skewed Weights

Algorithm 5.1. Generalized Algorithm

Let $2 \leq w \leq n$.

1. Let $R$ be a tableau of shape $\square_{w \times n}$ and content $\mu_{w-k}^{n-k}$. It contains $n-1$ copies of the numbers $1$ through $n-k$ and $1$ copy of the numbers $n-k + 1$ through $2n + k(w-2)$. As before, define the skew tableau $Q'$ by choosing cells in $R$ containing the numbers $\{n-k+1, n-k+2, \ldots, 2n+k(w-2)\}$. Let $Q'$ be the standard tableau obtained by first rotating $Q'$ by 180 degrees and then applying the map $x \mapsto 2n + k(w-2) - x + 1$ to each entry in $Q'$. Let $\lambda$ be the shape of $Q'$.

2. Let $P'' = R - Q''_{w,n-k}$.

The following three steps are new.

3. If $k > 0$, let $(P', Q) = (P'', Q')$; otherwise, let $(P', Q) = (Q', P'')$. That is, if $k < 0$ we swap the two tableaux before proceeding.
4. We apply the map \( x \mapsto x + |kw| \) to each element in \( P' \) so that it is a standard tableau with values starting at \( |kw| + 1 \).

5. Let \( M \) be a fixed tableau of shape \( \square_{|k| \times w} \) containing the elements \([|k|w]\). Let \( P \) be the standard tableau obtained by placing \( M \) on top of \( P' \) such that the first columns of \( P' \) and \( M \) are aligned.

Let \( P' \) and \( Q \) have shapes \( \lambda' \) and \( \lambda \) respectively. In Lemma 5.2 we show that each column of \( \lambda' \) has \(|k|\) fewer entries than the corresponding column in \( \lambda \), and step 5 produces a standard tableau of shape \( \lambda \). There are \( A_{|k|w} \) choices for the rectangular standard tableau \( M \) and hence it appears in Theorem 2.2.

**Lemma 5.2.** Placing a standard tableau \( M \) of shape \( \square_{|k| \times w} \) on top of \( P' \) such that the first columns of \( P' \) and \( M \) are aligned, produces a standard tableau \( P \) of the same shape as \( Q \).

**Proof of Lemma 5.2.** Consider \( P'' \) and \( Q' \) before the swapping in step 3. Recall that \( P''_i = (R - Q')_{w-i+1}^{w,n-k} \) from the algorithm. Also recall that \(|Q'_i| + |(R - Q')_{w-i+1}| = n\) since \( R \) has \( n \) rows. Then, from step 2:

\[
|P''_i| = |(R - Q')_{w-i+1}^{w,n-k}| = n - k - |(R - Q')_{w-i+1}|
\]

\[= n - k - (n - |Q_i|) = |Q_i| - k.\]

This works when both \( k < 0 \), and \( k > 0 \), and implies that either \( P'' \) has \(-k\) more rows than \( Q \), or that \( P'' \) has \( k \) fewer rows than \( Q \). This holds for all the columns \( 1 \leq i \leq w \) and all \( k \) in the range specified in Theorem 2.2. So this implies that if we attach a square tableau of shape \( \square_{|k| \times w} \) to the top of \( P'' \), then the columns of \( P'' \) and \( Q' \) have the same sizes. Since step 4 ensures that \( P'' \) does not contain the numbers in \([|k|w]\), \( P \) is a standard tableau.

**Proof of Theorem 2.2.** Let \( P \) be the tableau obtained by attaching \( M \) to \( P' \) in step 5. By Lemma 5.2, we have a pair \((P, Q)\) with the same shape \( \lambda \). By the RSK algorithm, this corresponds to a permutation \( \sigma \) of size \( m = |\lambda| \), where \( m \) is defined in the statement of the theorem.

Since \( M \) contains the numbers 1 through \(|k|w\), it is a well known fact that there must be \( k \) disjoint increasing subsequences in \( \sigma \) each of length \(|k|\) using the numbers 1 through \(|k|w\) [17, Chapter 3, Lemma 1]. Therefore \( P \) must also have at least \(|k|\) such disjoint increasing subsequences.

As before, it is easy to see that all the steps of the algorithm are invertible. The parameters \( k, w \) and \( n \) are fixed from earlier. The reverse algorithm starts with a permutation in \( S_m \) with \(|k|\) disjoint LIS of length \( w \) made up of the numbers \([|k|w]\). We use the RSK algorithm to form the pairs of tableau \((P, Q)\), and drop the first \(|k|\) rows of \( P \) and relabel its elements to form \( P' \). Then, depending on the sign of \( k \), we swap the tableau \( P' \) and \( Q \) to obtain \( P'' \) and \( Q' \).
We then apply the (invertible) complement operation to $P''$ and then relabel the elements of $Q$ by inverting the map in step 1 to form $Q'$. Then $\overline{P^w, n-k}$ can be joined to a rotated $Q'$ to obtain $R$, a rectangular tableau of shape $\square_{w \times n}$ and content $\mu_{w-1}^{n-k}$.

**Proof of Corollary 2.3.** Note that when $k = 1$ then the block $M$ has only one choice: it must consist of the elements $[w]$ in a diagram of shape $\square_{1 \times n}$, and we immediately get a bijection between permutations where $(1, 2, \ldots, w)$ is an LIS.

**Proof of Corollary 2.4.** Without loss of generality we may assume $0 < k \leq n$, (for $k < 0$, replacing $k$ with $-k$ permutes the left and right hand sides of the equation) and $w > 1$. Fix a $\square_{w \times k}$ shaped tableau $M$.

Theorem 2.2 shows that $K_{\square_{w \times n}}^{n-k}$ counts the number of pairs $P, Q$ of Young tableaux having the same shape with width $w$, $M$ occupying the first $k$ rows of $P$, and $|P| = m = n + k(w - 1)$. Analogously, it shows that $K_{\square_{w \times n+k(w-2)}}^{n-k}$ counts the exact same set, since the width and $M$ are unchanged, and $|P| = n + k(w - 2) + k = m$.

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