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On spectral gaps of growth-fragmentation semigroups with mass loss or death

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Abstract
We give a general theory on well-posedness and time asymptotics for growth fragmentation equations. These linear kinetic (integro-differential) equations arise in the modeling of various physical or biological phenomena involving concentration of aggregates which experience both growth and fragmentation. We prove first generation of $C_0$-semigroups $(V(t))_{t \geq 0}$ governing them for unbounded total fragmentation rate and fragmentation kernel $b(,)$ such that $\int_0^y xb(x, y)dx = y - y(y)y (0 \leq y(y) \leq 1$ expresses the mass loss) and continuous growth rate $r(\cdot)$ such that $\int_0^\infty \frac{1}{r(\tau)} d\tau = +\infty$. This is done in three natural functional spaces $L^1(\mathbb{R}_+, \mu(dx)) (\mu(dx) = dx, xdx$ or $(1 + x)dx$) which correspond respectively to finite number of aggregates, finite mass or finite mass and number of aggregates. The mass loss or death assumptions are needed only in the vicinity of points where the total fragmentation rate gets infinite. The analysis relies on unbounded perturbation theory peculiar to positive semigroups in $L^1$ spaces. Secondly, we show when the resolvent of the generator is compact and the semigroup has a spectral gap, i.e. $r_{ess}(V(t)) < r_\sigma(V(t)) (r_{ess}$ is the essential spectral radius), and an asynchronous exponential growth. The analysis relies on weak compactness tools and Frobenius theory of positive operators. A systematic functional analytic construction is provided.

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1 Introduction

This paper provides a general theory on well-posedness (in the sense of $C_0$-semigroups) and time asymptotics of growth-fragmentation equations

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} \left( r(x) u(x,t) \right) + \left( a(x) + d(x) \right) u(x,t)$$

$$= \int_x^{+\infty} a(y) b(x,y) u(y,t) dy, \quad u(x,0) = u_0(x), \quad x, t > 0 \quad (1)$$
with measurable nonnegative fragmentation kernel \( b(.,.) \) and positive growth rate \( r(.) \) satisfying the general structural assumptions

\[
\int_0^y xb(x, y)dx = y (1 - \eta(y)) , \quad 0 \leq \eta(y) \leq 1 \ (y \geq 0) \quad (2)
\]

\[
b(x, y) > 0 \text{ if } 0 \leq x < y \quad (3)
\]

\[
r(.) \in C (0, +\infty) , \quad \int_0^\infty \frac{1}{r(\tau)}d\tau = +\infty; \quad (4)
\]

while the total fragmentation rate \( a(.) \) and the death (or degradation) rate \( d(.) \) are measurable nonnegative and

\[
\beta := a + d \in L^1_{loc}(0, +\infty). \quad (5)
\]

Among the physical examples of growth rates we can find in the literature, note for instance the typical ones

\[
r(x) = 1 \text{ or } r(x) = x, \quad (x > 0). \quad (6)
\]

The kinetic equations (1) arise in the modeling of various physical or biological phenomena involving concentration of aggregates which experience both growth and fragmentation. Typical biological examples are provided by phytoplankton dynamics [1][2] or by prions dynamics [15]; we refer to [16] and references therein for a lot of contexts where these equations arise; see also the monographs [20][5][34] for more information. The unknown \( u(x, t) \) represents the concentration at time \( t \) of “aggregates” with mass \( x > 0 \) while \( b(x, y) \ (x < y) \) describes the distribution of mass \( x \) aggregates, called daughter aggregates, spawned by the fragmentation of a mass \( y \) aggregates. The local mass conservation in the fragmentation process corresponds to

\[
\frac{1}{y} \int_0^y xb(x, y)dx = 1
\]

i.e. to \( \eta(.) = 0 \). In this case, we say that the kernel \( b(.,.) \) is conservative. Most of the literature is concerned with conservative fragmentation kernels. On the other hand

\[
\eta(.) \neq 0 \quad (7)
\]

amounts to saying that a mass loss takes place in the fragmentation process, i.e.

\[
\frac{1}{y} \int_0^y xb(x, y)dx \leq 1
\]
where
\[ \eta(y) = 1 - \frac{1}{y} \int_0^y xb(x, y) dx \] (8)
quantifies this mass loss (Chapter 9).

Notice that these fragmentation kernels contain for example homogeneous kernels
\[ b(x, y) = \frac{1}{y} h\left(\frac{x}{y}\right) \text{ with } \int_0^1 zh(z)dz \leq 1 \] (9)
(for some \( h \in L^1_+(0, 1) dx \)) since
\[ \int_0^y xb(x, y)dx = \int_0^y y \frac{1}{y} h\left(\frac{x}{y}\right)dx = y \int_0^1 zh(z)dz = y(1 - \eta) \]
where
\[ \eta = 1 - \int_0^1 zh(z)dz. \]

More generally, for any conservative fragmentation kernel \( \tilde{b}(x, y) \) and
\[ 0 \leq \zeta(x, y) \leq 1, \] (10)
the kernel
\[ b(x, y) := \zeta(x, y) \tilde{b}(x, y) \] (11)
satisfies (2) with
\[ \eta(y) = \frac{1}{y} \int_0^y x (1 - \zeta(x, y)) \tilde{b}(x, y)dx. \] (12)
Conversely, any fragmentation kernel \( b(\cdot, \cdot) \) satisfying (2) is of the form (11) with
\[ \zeta(x, y) = \frac{1}{y} \int_0^y xb(x, y)dx \text{ and } \tilde{b}(x, y) = \frac{b(x, y)}{\zeta(x, y)}. \]

We have thus a full description of the fragmentation kernels considered in this paper by means of conservative kernels.

We point out that for a given concentration \( u(\cdot, \cdot) \),
\[ \int_0^{+\infty} u(x, t)x dx \text{ and } \int_0^{+\infty} u(x, t)dx \]
are respectively the total mass and the total number of aggregates at time \( t \geq 0 \). Thus, three natural functional spaces are of particular interest: the "finite mass" space
\[
X_1 := L^1(\mathbb{R}_+; xdx) \text{ with norm } \| \|_{X_1},
\]
the "finite aggregates number" space
\[
X_0 := L^1(\mathbb{R}_+; dx) \text{ with norm } \| \|_{X_0}
\]
and the "finite mass and aggregates number" space
\[
X_{0,1} := L^1(\mathbb{R}_+; (1 + x)dx) \text{ with norm } \| \|_{X_{0,1}}.
\]
Because of their physical relevance and for the sake of completeness, we consider the growth-fragmentation equations in each of the functional spaces above and for the different types of divergence (4) (see (21), (22) below). The technical details (and assumptions) may vary a bit from one context to another but overall the general mathematical arguments are similar.

We recall that semigroup generation for growth (i.e. transport) equations
\[
\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} (r(x)u(x,t)) + \beta(x)u(x,t) = 0
\]
(13)
is known in various functional settings and under various assumptions on the growth rate \(r\) (see e.g. [5][7][8] for a resolvent approach via Hille-Yosida theory). The first object of this paper is well-posedness of (1) in the sense of \(C_0\)-semigroups. To this end, we give first a direct and systematic construction of explicit growth semigroups \((U(t))_{t \geq 0}\) governing (13) with continuous growth rates \(r(\cdot)\) in the functional spaces above under "optimal" (i.e. sufficient and "necessary") assumptions. We consider then the fragmentation operator
\[
B : \varphi \in D(T) \rightarrow \int_x^{+\infty} a(y)b(x,y)\varphi(y)dy
\]
(14)
\((T\) is the generator of \((U(t))_{t \geq 0}\) as a perturbation and show a generation of a \(C_0\)-semigroup \((V(t))_{t \geq 0}\) by \(T + B\) with domain
\[
D(T + B) = D(T)
\]
(15)
under suitable assumptions depending on the functional space we consider. To this end, we use a perturbation theorem peculiar to positive \(C_0\)-semigroups in \(L^1\)-spaces by W. Desch [13]:

**Theorem 1** ([13]; see also [38] or [23] Chapter 8) Let \((U(t))_{t \geq 0}\) be a positive \(C_0\)-semigroup with generator \(T\) on some \(L^1(\mu)\)-space and let
\[
B : \varphi \in D(T) \subset L^1(\mu) \rightarrow L^1(\mu)
\]
be continuous on $D(T)$ (endowed with the graph norm) and positive (i.e. $B : D(T) \cap L^1_+(\mu) \to L^1_+(\mu)$). Then

$$T + B : D(T) \subset L^1(\mu) \to L^1(\mu)$$

generates a positive $C_0$-semigroup on $L^1(\mu)$ if and only if

$$\lim_{\lambda \to +\infty} r_\sigma \left( B (\lambda - T)^{-1} \right) < 1.$$

Based on a systematic use of weak compactness arguments, the second object of this paper is to analyze, in each of the above functional spaces and for different types of divergence (16) (see (21)(22) below), the existence of a spectral gap

$$r_{\text{ess}}(V(t)) < r_\sigma(V(t))$$

(16)

($r_{\text{ess}}$ and $r_\sigma$ refer respectively to the essential spectral radius and the spectral radius) or equivalently

$$\omega_{\text{ess}}(V) < \omega(V)$$

where $\omega_{\text{ess}}(V)$ and $\omega(V)$ denote respectively the essential type and the type of $(V(t))_{t \geq 0}$ (see e.g. [23] Chapter 2). This property is related to stability of essential type under suitable perturbations, (see below). Note that the expression "spectral gap" is widely used in the mathematical literature but has not always a univocal meaning; we use it here in the sense above. The combination of (16) with some irreducibility condition implies the so-called asynchronous exponential growth of $(V(t))_{t \geq 0}$

$$\left\| e^{-\lambda t} V(t) - P \right\| = O(e^{-\varepsilon t})$$

(17)

(for some $\varepsilon > 0$) where $P$ is a one-dimensional spectral projection relative to the leading isolated algebraically simple dominant eigenvalue $\lambda$ (Malthus parameter) of the generator, (see e.g. [37]). More precisely

$$P \varphi = \left( \int_{0}^{+\infty} \varphi(x) u_+(x) \mu(dx) \right) u$$

where $\mu(dx) = dx, xdx$ or $(1 + x) dx$ (depending on the choice of $X_0$, $X_1$ or $X_{0,1}$), $u$ is the nonnegative eigenfunction associated to the leading isolated eigenvalue $\lambda$ of the generator and $u_+$ is the dual nonnegative eigenfunction (associated to the leading isolated eigenvalue $\lambda$) with a normalization

$$\int_{0}^{+\infty} u(x) u_+(x) \mu(dx) = 1.$$
We point out that the existence of such Perron eigenelements is a consequence of the spectral gap \[16\].

The existence of Perron eigenvectors, \textit{regardless} of the occurrence of a spectral gap, and their asymptotic stability in weighted \(L^1\) spaces (the weight being the dual eigenvector) rely on different tools and have been the subject of rich works in the last decade. Without pretense to completeness, we refer e.g. to \[18\] \[15\] \[21\] \[16\] \[22\] \[7\] \[8\] \[9\] \[10\] where some results combine relative entropy techniques; see also \[11\] for a probabilistic approach. We refer to the introductions of \[22\] \[16\] \[8\] \[10\] for a comprehensive review of the existing tools and results. In particular, we point out that asymptotic stability need not be uniform with respect to initial data and, at least in suitable weighted spaces, we cannot expect the existence of a spectral gap for \textit{bounded} total fragmentation rates \(a(.)\) even if Perron eigenvectors can exist \[7\]. (We do not comment here on the case of bounded state spaces which goes back to the pioneer paper \[14\]; see \[4\] and references therein for more recent works in this direction.)

Our paper is rather in the same spirit as \[8\]. The latter deals with asynchronous exponential growth \(17\) under the divergence \(22\) below in higher moment spaces

\[
L^1(\mathbb{R}_+; (1+x)\alpha dx) \quad (\alpha > \alpha_* )
\]

for a suitable threshold \(\alpha_* \geq 1\) (see also \[22\] \[10\]). We deal here with the asynchronous exponential growth \(17\) in the natural spaces \(X_0, X_1, X_{0,1}\) under the general divergence \(4\) but at the expense of the additional assumption

\[
\eta(.) \neq 0 \text{ or } d(.) \neq 0
\]

(\textit{mass loss or death}) which does not occur in the literature on the subject. Assumption \(19\) opens new mathematical perspectives and allows a systematic functional analytic construction which is the object of this paper. This general theory is based on few structural assumptions only. Besides the main results on spectral gaps, many preliminary results of independent interest are also given and the role of \textit{unboundedness} of total fragmentation rates \(a(.)\) is fully highlighted.

Our construction, inspired by recent contributions to other structured models \[29\] \[30\] \[31\], relies on three key mathematical ingredients:

(i) The weak compactness tools, for absorption semigroups in \(L^1\) spaces, introduced in \[28\].

(ii) The convex (weak) compactness property of the strong operator topology in Banach spaces \[35\] (see also \[24\] for an elementary proof in \(L^1(\nu)\) spaces).
(iii) Strict comparison of spectral radii in Frobenius theory [19].

Among linear kinetic equations, growth-fragmentation equations present a very particular trait: the state-variable is one-dimensional. This gives them a local regularizing effect that does not exist in usual kinetic theory, e.g. in neutron transport, where the transport part has no local regularizing effect and the perturbation (the collision operator) is non-local with respect to another (velocity) \( v \)-variable; this second state variable has a regularizing (local compactness) effect with respect to space \( x \)-variable and induces the stability of the essential type \([25][26]\). On the other hand, for growth-fragmentation equations, the compactness results (i.e. the key point behind the spectral gap property) are consequences of the local regularizing effect we alluded to and of the confining role of singular absorptions \([28]\), hence the key role, in our construction, of unboundedness of total fragmentation rates \( a(\cdot) \). We point out that \([19]\) is needed only in the vicinity of points where \( a(\cdot) \) gets infinite. Finally, we note that for bounded total fragmentation rates, no spectral gap can exist in the weighted spaces

\[
L^1(\mathbb{R}_+; (1 + x)^\alpha \, dx) \quad (\alpha < 1),
\]

(20)

see \([7]\); we conjecture that we cannot expect spectral gaps in \( X_1, X_0 \) or \( X_{0,1} \) if the total fragmentation rate is bounded.

**Our paper is organized as follows:**

We provide first an explicit construction of growth \( C_0 \)-semigroups governing \([13]\) by the method of characteristics. Two different growth \( C_0 \)-semigroups occur according as

\[
\int_0^1 \frac{1}{r(\tau)} \, d\tau = +\infty, \quad \int_1^\infty \frac{1}{r(\tau)} \, d\tau = +\infty
\]

(21)

or

\[
\int_0^1 \frac{1}{r(\tau)} \, d\tau < +\infty, \quad \int_1^\infty \frac{1}{r(\tau)} \, d\tau = +\infty
\]

(22)

to cover e.g. the examples \([6]\). (For the sake of simplicity, we ignore the case \( \int_0^1 \frac{1}{r(\tau)} \, d\tau = +\infty, \int_1^\infty \frac{1}{r(\tau)} \, d\tau < +\infty \).) Note that (22) is complemented by a boundary condition, see (31) below. Our main results in the spaces \( X_1 \) and \( X_{0,1} \) under Assumption (21) are the following:

A transport \( C_0 \)-semigroup \((U(t))_{t \geq 0}\) governing \([13]\) exists in \( X_1 \) (resp. in \( X_{0,1} \)) and is given by

\[
U(t)f = e^{-\int_{X(y,t)}^{X(y,t)} \frac{\beta(p)}{\alpha(p)} \, dp} f(X(y,t)) \frac{\partial X(y,t)}{\partial y}
\]
\( (X(y, t) \text{ is defined by } \int_X^y \frac{1}{r(\tau)} d\tau = t ) \) provided that
\[
\alpha := \sup_{z > 0} \frac{r(z)}{z} < +\infty \quad (\text{resp. } \sup_{z > 1} \frac{r(z)}{z} < +\infty); \quad (23)
\]
(see Proposition 11 and Proposition 32). In addition, the assumptions (23) are "necessary" to a generation theory, (see Proposition 6 and Remark 26).

Note that under (21), the generation theory in \( X_{0,1} \) needs no condition on the growth rate at the origin. Note also that the \( C_0 \)-semigroup \( (U_0(t))_{t \geq 0} \) corresponding to \( \beta = 0 \) is not contractive, (see Remarks 5 and 27).

We also "compute" the spectral bound
\[
s(T) := \sup \{ \Re \nu ; \nu \in \sigma(T) \}
\]
of its generator \( T \); (see Proposition 13 and Proposition 34). We recall that \( s(T) \) coincides with the type of \( (U(t))_{t \geq 0} \) (this is a general property of positive semigroups in Lebesgue spaces, see e.g. [38]). The resolvent of \( T \) is given by
\[
((\lambda - T)^{-1} f)(y) = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda + \beta(r(\tau))}{r(\tau)} d\tau} f(x) dx \quad (\Re \lambda > s(T))
\]
in both spaces \( X_1 \) and \( X_{0,1} \); (see Proposition 12 and Proposition 33). Note the domination
\[
U(t) \leq U_0(t) \quad (t \geq 0) \text{ and } (\lambda - T)^{-1} \leq (\lambda - T_0)^{-1} \quad (\lambda > s(T_0))
\]
where \( T_0 \) is the generator of \( (U_0(t))_{t \geq 0} \). We show the pointwise a priori estimate in \( X_1 \)
\[
|((\lambda - T_0)^{-1} f)(y)| \leq \frac{1}{\sqrt{r(y)}} \|f\|_{X_1} \quad (f \in X_1) \quad (\lambda > \alpha);
\]
(see Lemma 10).

In \( X_{0,1} \), if we replace the natural condition \( \sup_{z > 1} \frac{r(z)}{z} < +\infty \) by the stronger one
\[
C := \sup_{z > 0} \frac{r(z)}{1 + z} < +\infty, \quad (24)
\]
we show the pointwise a priori estimate in \( X_{0,1} \)
\[
|((\lambda - T_0)^{-1} f)(y)| \leq \frac{1}{(1 + y) r(y)} \|f\|_{X_{0,1}} \quad (f \in X_{0,1}) \quad (\lambda > C),
\]
\[9\]
(see Lemma 30). Note that by domination, the pointwise estimates above are inherited by \((\lambda - T)^{-1}\).

We show that \(T\) has a smoothing effect in \(X_1\) for \(\lambda > \alpha\)

\[
\int_0^{+\infty} |(\lambda - T)^{-1}f(y, \beta(y) y dy \leq \int_0^{+\infty} |f(y)| y dy \quad (f \in X_1),
\]

(see Lemma 14).

In \(X_{0,1}\), if we replace the natural condition \(\sup_{z > 1} \frac{r(z)}{x} < +\infty\) by the stronger one (24), we show the smoothing effect for \(\lambda > C\)

\[
\int_0^{+\infty} |(\lambda - T)^{-1}f(y, \beta(y) (1 + y) dy \leq \int_0^{+\infty} |f(y)| (1 + y) dy \quad (f \in X_{0,1}),
\]

(see Lemma 35). The above estimates, combined to the general theory on compactness properties in \(L^1\) spaces induced by the confining effect of singular absorptions, show that if the sublevel sets of \(\beta\)

\[
\Omega_c = \{x > 0; \beta(x) < c\} \quad (c > 0)
\]

are "thin near zero and near infinity relatively to \(r\)" in the sense

\[
\int_0^{+\infty} \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty \quad (c > 0)
\]

(25)

where \(1_{\Omega_c}\) is the indicator function of \(\Omega_c\) (note that \(\frac{1}{r(z)} \notin L^1(0, +\infty)\)) then \(T\) is resolvent compact in both spaces \(X_1\) and \(X_{0,1}\), i.e. \((\lambda - T)^{-1}\) is compact in \(X_1\) and \(X_{0,1}\); (see Theorem 20 and Theorem 42). This occurs for instance if

\[
\lim_{y \to 0^+} \beta(y) = +\infty, \quad \lim_{y \to +\infty} \beta(y) = +\infty.
\]

Note that if \(\Omega_c\) has finite Lebesgue measure then \(\int_1^{+\infty} \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty\) provided that \(\frac{1}{r(\tau)} \in L^p(1, +\infty)\) for some \(p > 1\).

One shows that the fragmentation operator \(B\) given par (14) is \(T\)-bounded in \(X_1\) and

\[
\lim_{\lambda \to +\infty} \|B(\lambda - T)^{-1}\|_{\mathcal{L}(X_1)} \leq \lim_{a(y) \to +\infty} \sup_{\eta(y) \to +\infty} \frac{1 - \eta(y)}{1 + \frac{a(y)}{\eta(y)}},
\]

in particular, by W. Desch’s perturbation theorem (Theorem 1),

\[
T + B : D(T) \subset X_1 \to X_1
\]
generates a positive $C_0$-semigroup $(V(t))_{t \geq 0}$ on $X_1$ provided that

$$
\gamma := \lim \sup_{a(y) \to +\infty} \frac{1 - \eta(y)}{1 + \frac{d(y)}{a(y)}} < 1; \quad (26)
$$

(see Theorem 16). Note that (26) is satisfied e.g. if \( \lim \inf_{a(y) \to +\infty} \frac{d(y)}{a(y)} > 0 \)
or if

$$
\lim \inf_{a(y) \to +\infty} \eta(y) > 0. \quad (27)
$$

This explains why we need mass loss or death assumptions and why these are needed only in the vicinity of points where \( a(\cdot) \) gets infinite. We note that for homogeneous kernels (9), the condition (27) amounts to

$$
\int_0^1 z h(z) dz < 1.
$$

More generally, in the case (11), the condition (27) holds if

$$
\lim \sup_{a(y) \to +\infty} \zeta(x, y) < 1,
$$

(see Remark 18).

Under (26),

$$
T + B : D(T) \subset X_1 \to X_1
$$
is resolvent compact provided that \( T : D(T) \to X_1 \) is; (see Corollary 21). We build a $C_0$-semigroup \( \left( \hat{V}(t) \right)_{t \geq 0} \) on $X_1$ such that

$$
U(t) \leq \hat{V}(t) \leq V(t) \quad (t \geq 0).
$$

By using the convex (weak) compactness property of the strong operator topology \( 35 \) (see below), we show that \( \left( \hat{V}(t) \right)_{t \geq 0} \) and \( (V(t))_{t \geq 0} \) have the same essential type

$$
\omega_{ess}(\hat{V}) = \omega_{ess}(V).
$$

The resolvent compactness of their generators and the strict comparison results of spectral radii of positive compact operators in domination contexts \( 19 \) imply the strict comparison of the types

$$
\omega(\hat{V}) < \omega(V)
$$
and consequently \((V(t))_{t \geq 0}\) has a spectral gap \((16)\) and exhibits the asynchronous exponential growth \((17)\) in \(X_1\) provided that the support of \(a(.)\) is not bounded; (see Theorem \([23]\)).

The analysis in \(X_{0,1}\) is similar but needs a different assumption. Indeed, one shows that the fragmentation operator \((14)\) is \(T\)-bounded in \(X_{0,1}\) and

\[
\lim_{\lambda \to +\infty} \|B(\lambda - T)^{-1}\|_{\mathcal{L}(X_1)} \leq \lim_{a(y) \to +\infty} \sup_{a(y) \to +\infty} \left[ \frac{(y - \eta(y)y + n(y))}{(1 + y) \left(1 + \frac{d(y)}{a(y)}\right)} \right] < 1; \tag{28}
\]

provided that

\[
n(y) := \int_0^y b(x, y)dx,
\]

(the expected number of daughter aggregates spawned by a mother aggregates of mass \(y\)) is such that

\[
\sup_{y > 0} \frac{n(y)}{1 + y} < +\infty.
\]

By appealing again to W. Desch’s perturbation theorem,

\[
T + B : D(T) \subset X_{0,1} \to X_{0,1}
\]

generates a positive \(C_0\)-semigroup \((V(t))_{t \geq 0}\) on \(X_{0,1}\) provided that

\[
\lim_{a(y) \to +\infty} \sup_{a(y) \to +\infty} \left[ \frac{(y + \frac{n(y)}{1 - \eta(y)})}{(y + 1) \left(1 + \frac{d(y)}{a(y)}\right)} \right] < 1;
\]

(see Theorem \([38]\)). As previously, mass loss or death are needed only in the vicinity of points where \(a(.)\) gets infinite. Note that

\[
\frac{(y - \eta(y)y + n(y))}{(1 + y) \left(1 + \frac{d(y)}{a(y)}\right)} = \frac{y + \frac{n(y)}{1 - \eta(y)}}{(y + 1)} \left(1 - \eta(y)\right) \left(1 + \frac{d(y)}{a(y)}\right)
\]

so \((28)\) occurs provided that

\[
\lim_{a(y) \to +\infty} \sup_{a(y) \to +\infty} \left[ \frac{y + \frac{n(y)}{1 - \eta(y)}}{(y + 1)} \right] < \gamma^{-1}.
\]

In particular, if \(\gamma < 1\) (i.e. under the generation criterion in \(X_1\)) and if \(a(.)\) is unbounded at zero and at infinity only, then \((28)\) (i.e. the generation criterion in \(X_{0,1}\)) occurs provided that

\[
\max \left\{ \lim_{y \to 0} \sup_{y \to 0} \frac{n(y)}{1 - \eta(y)}, \ 1 + \lim_{y \to +\infty} \sup_{y \to +\infty} \frac{n(y)}{y \left(1 - \eta(y)\right)} \right\} < \gamma^{-1},
\]
(see Corollary 39). This is the case e.g. if \( \eta(.) = 0 \) and

\[
\max \left\{ \lim_{y \to 0} \sup_n (y), 1 + \lim_{y \to +\infty} \sup_n \frac{n(y)}{y} \right\} < 1 + \lim_{a(y) \to +\infty} \inf \frac{d(y)}{a(y)}
\]

or if \( d(.) = 0 \) and

\[
\max \left\{ \lim_{y \to 0} \sup_n \frac{n(y)}{1 - \eta(y)}, 1 + \lim_{y \to +\infty} \sup_n \frac{n(y)}{y(1 - \eta(y))} \right\} < \left( 1 - \lim_{a(y) \to +\infty} \eta(y) \right)^{-1}.
\]

We note that for homogeneous kernels (9),

\[
n(y) = \int_0^y \frac{1}{y} h\left(\frac{x}{y}\right) dx = \int_0^1 h(z) dz
\]

and the condition (29) amounts to

\[
\int_0^1 h(z) dz < 1.
\]

More generally, in the case (11), the condition (29) holds if \( \zeta^+ \leq 1 \) and

\[
\limsup_{y \to 0} \hat{n}(y) < \frac{\zeta^-}{(\zeta^+)^2} \quad \text{and} \quad \limsup_{y \to +\infty} \frac{\hat{n}(y)}{y} < \frac{1 - \zeta^+}{(\zeta^+)^2}
\]

where \( \hat{n}(y) = \int_0^y \tilde{b}(x, y) dx \) and

\[
\zeta^- = \lim_{a(y) \to +\infty} \inf \zeta(x, v) \quad \text{and} \quad \zeta^+ = \lim_{a(y) \to +\infty} \sup \zeta(x, v),
\]

(see Proposition 41).

Under Assumption (28),

\[
T + B : D(T) \subset X_{0,1} \to X_{0,1}
\]

is resolvent compact provided that \( T : D(T) \to X_{0,1} \) is; (see Corollary 43). As previously, we deduce that \( (V(t))_{t \geq 0} \) has a spectral gap (16) and exhibits the asynchronous exponential growth (17) in \( X_{0,1} \) provided that the support of \( a(.) \) is not bounded; (see Theorem 45). We conjecture that in \( X_{0,1} \), the results hold under the natural assumption \( \sup_{z > 1} \frac{r(z)}{z} < +\infty \) instead of (24), (see Remark 46).
Let us describe now very briefly the situation under Assumption (22); (see Section 3 for the different statements). First of all, we cannot expect a generation theory in $X_1$ under (22), (see Remark 51). A growth $C_0$-semigroup $(U(t))_{t \geq 0}$ governing (13) with boundary condition

$$\lim_{x \to 0} r(x)u(x, t) = 0$$

(31)

exists in the space $X_{0,1}$ and is given by

$$U(t)f = \mathcal{X}\left\{ \int_0^y \frac{1}{r(\tau)} d\tau > t \right\} e^{-\int_X^{y} \frac{1}{r(p)} dp} f(X(y,t)) \frac{\partial X(y,t)}{\partial y}$$

(32)

$(X(y,t)$ is defined by $\int_X^{y} \frac{1}{r(\tau)} d\tau = t$ for $\int_0^y \frac{1}{r(\tau)} d\tau > t$) provided that (24) is satisfied. This sufficient condition for a generation theory in $X_{0,1}$ under (22) is "partly necessary", (see Remark 50). The mathematical analysis is the same as in the previous case (21) in $X_{0,1}$. The only different result is that the resolvent compactness of $T$ holds once the sublevel sets of $\beta$

$$\Omega_c = \{ x > 0; \beta(x) < c \} \quad (c > 0)$$

are "thin near infinity relatively to r" in the sense

$$\int_{1}^{+\infty} \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty \quad (c > 0),$$

e.g. if $\lim_{y \to +\infty} \beta(y) = +\infty$; (no condition at $y = 0$ is needed). In particular, $(V(t))_{t \geq 0}$ has a spectral gap and exhibits the asynchronous exponential growth (17) in $X_{0,1}$ provided that the support of $a(.)$ is not bounded; (see Theorem 64).

Section 4 is devoted to the "finite aggregates number" space

$$X_0 = L^1(\mathbb{R}_+; \ dx)$$

under Assumption (22); (a similar construction could also be done under Assumption (21)). For simplicity, we restrict ourselves to

$$d(.) = 0.$$

A growth $C_0$-semigroup $(U(t))_{t \geq 0}$ governing (13) with boundary condition (31) exists in the space $X_0$ and is given by (32) but without any further condition on the growth rate $r(.)$. As in $X_1$ or $X_{0,1}$, its generator $T$ satisfies
a smoothing effect and the pointwise estimate and its resolvent is compact if the sublevel sets of $a(.)$ are "thin near infinity relatively to $r$", e.g. if

$$\lim_{y \to +\infty} a(y) = +\infty;$$

(see Theorem [67]). We show also that the fragmentation operator $[14]$ is $T$-bounded in $X_0$ and

$$\lim_{\lambda \to +\infty} \|B(\lambda - T)^{-1}\|_{\mathcal{L}(X_0)} \leq \lim \sup_{a(y) \to +\infty} n(y)$$

provided that

$$n(.) := \int_0^y b(x, .) dx \in L^\infty(0, +\infty).$$

In particular, by W. Desch’s perturbation theorem again,

$$T + B : D(T) \subset X_0 \to X_0$$

generates a positive $C_0$-semigroup $(V(t))_{t \geq 0}$ on $X_0$ provided that

$$\lim \sup_{a(y) \to +\infty} n(y) < 1;$$

(see Theorem [68]. Note that (33) cannot hold for conservative fragmentation kernels, hence the necessity of the mass loss condition. In particular, for homogeneous kernels (9), it amounts to $\int_0^1 h(z) dz < 1$. More generally, in the case (11), the condition (33) is satisfied if

$$\lim \sup_{a(y) \to +\infty} \hat{n}(y) < \frac{1}{\zeta_\infty};$$

(see Remark [70]).

We show that $(V(t))_{t \geq 0}$ has a spectral gap and exhibits the asynchronous exponential growth (17) in $X_0$ provided that the support of $a(.)$ is not bounded; (see Theorem [71]). If $a(.)$ is unbounded at zero or at infinity only, then (33) expresses a smallness condition on $n(.)$ at zero or at infinity.

As far as we know, our results are new and appear here for the first time. The role of mass loss or death assumptions appears in the spaces $X = X_1, X_0$ or $X_{0,1}$ at two key places: In the proof that

$$T + B : D(T) \to X$$

is a generator via W. Desch’s perturbation theorem and (consequently) in the fact that the resolvent compactness of $T$ implies the resolvent compactness of $T + B$. A priori, we can overcome the first point. Indeed, if we
consider for instance the space $X_1$, by adapting honesty theory (see e.g. [27]), without mass loss or death assumptions (i.e. $\eta(.) = d(.) = 0$), (34) need not be a generator but there exists a unique extension
\[ T_B \supset T + B \]
of (34) which generates a positive $C_0$-semigroup $(V(t))_{t \geq 0}$ in $X_1$. Unfortunately, even in the honest case (i.e. $T_B = T + B$), if $T + B$ is not closed, a priori we cannot infer that $T_B$ is resolvent compact when $T$ is. This is the main obstruction to build a general theory of asynchronous exponential growth in $X_1$ without mass loss or death conditions. The same observation can also be made for the other spaces.

In [8], under Assumption (22), where no mass loss or death condition is assumed and where $T_B = T + B$ (and a priori $T_B \neq T + B$), the asynchronous exponential growth is not obtained in the natural space $X_{0,1}$ but in higher moment spaces (18) ([8] Theorem 1.2). We note that a general construction similar to the present one holds without mass loss or death assumptions provided the growth-fragmentation equations are considered in higher moment spaces
\[ L^1(\mathbb{R}_+; (1 + x)^\alpha \, dx), \, L^1(\mathbb{R}_+; x^\alpha dx) \quad (\alpha > \alpha_*) \]
for a suitable threshold $\alpha_* \geq 1$ depending on the functional space [32]; (this is due to the fact that W. Desch’s perturbation theorem can apply in higher moment spaces without resorting to mass loss or death assumptions [6]). This strongly suggests the conjecture that we cannot expect the asynchronous exponential growth (17) in $X_1$, $X_0$ or $X_{0,1}$ without mass loss or death assumptions.

2 The first construction

We deal first with the case
\[ \int_0^1 \frac{1}{\tau(\tau)} \, d\tau = +\infty \quad \text{and} \quad \int_1^{\infty} \frac{1}{\tau(\tau)} \, d\tau = +\infty \quad (35) \]
and start with:

**Proposition 2** Let (35) be satisfied. The partial differential equation
\[ \frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} [r(x)u(x,t)] = 0, \quad (x,t > 0) \]
with initial condition \( u(x, 0) = f(x) \) has a unique solution given by

\[
u(y, t) = \frac{r(X(y, t))f(X(y, t))}{r(y)}
\]

where \( X(y, t) > 0 \) is defined by

\[
\int_{X(y,t)}^{y} \frac{1}{r(\tau)} d\tau = t \quad (t > 0).
\]  

(36)

**Proof.** We solve

\[
\frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} [r(x)u(x, t)] = 0
\]

with initial data \( u(x, 0) = f(x) \) by the method of characteristics. Making the change

\[
r(x)u(x, t) = \varphi(x, t)
\]

this amounts to solving

\[
\frac{1}{r(x)} \frac{\partial}{\partial t} \varphi(x, t) + \frac{\partial}{\partial x} [\varphi(x, t)] = 0; \quad \varphi(x, 0) = r(x)f(x).
\]

We introduce the characteristic equations

\[
\frac{dt}{ds} = \frac{1}{r(x(s))}, \quad \frac{dx}{ds} = 1
\]

with "initial" conditions

\[
x(0) = x, \quad t(0) = 0 \quad (x > 0)
\]

i.e. \( x(s) = s + x \) and

\[
t(s) = \int_{0}^{s} \frac{1}{r(\tau + x)} d\tau = \int_{x}^{s+x} \frac{1}{r(\tau)} d\tau \quad (s \geq 0).
\]

Thus

\[
[0, +\infty) \ni s \mapsto r(s + x)u(s + x, \int_{x}^{s+x} \frac{1}{r(\tau)} d\tau) \text{ is constant}
\]

and then

\[
r(s + x)u(s + x, \int_{x}^{s+x} \frac{1}{r(\tau)} d\tau) = r(x)u(x, 0) = r(x)f(x) \quad \forall s \geq 0.
\]
For \( t > 0 \) and \( y > 0 \) given, we set

\[
\int_x^{s+x} \frac{1}{r(\tau)} \, d\tau = t, \quad s + x = y
\]

i.e. \( \int_x^y \frac{1}{r(\tau)} \, d\tau = t \). Since \( x \in [0, y] \) \( \rightarrow \) \( \int_x^y \frac{1}{r(\tau)} \, d\tau \) is (a continuous function) strictly decreasing from \( +\infty \) to 0, then we denote by \( X(y, t) \) the unique \( x \in (0, y) \) such that \( \int_x^y \frac{1}{r(\tau)} \, d\tau = t \). Thus \( r(y)u(y, t) = r(X(y, t))f(X(y, t)) \) which ends the proof. 

2.1 Theory in the space \( X_1 \)

Within assumption (35), we first develop a general theory on well-posedness and spectral analysis in the "finite mass" space

\[
X_1 := L^1(\mathbb{R}_+; \, xdx).
\]

2.1.1 An unperturbed semigroup \((U_0(t))_{t \geq 0}\)

We put now the solution given in Proposition 2 in the functional space \( X_1 \).

Let

\[
(U_0(t)f)(y) := \frac{r(X(y, t))f(X(y, t))}{r(y)}
\]

where \( X(y, t) \) (\( t > 0 \)) is defined by (36).

**Theorem 3** Let (35) be satisfied. Then

\[
(U_0(t)f)(y) = f(X(y, t)) \frac{\partial X(y, t)}{\partial y}
\]

and \((U_0(t))_{t \geq 0}\) is a positive \( C_0 \)-semigroup on the space \( X_1 \) if and only if

\[
\sup_{x > 0} \frac{y(x, t)}{x} < +\infty \quad \forall t \geq 0 \quad \text{and}
\]

\[
[0, +\infty) \ni t \rightarrow \sup_{x > 0} \frac{y(x, t)}{x} \quad \text{is locally bounded} \quad (37)
\]

where \( y(x, t) \) is defined by

\[
\int_x^{y(x, t)} \frac{1}{r(\tau)} \, d\tau = t \quad (t > 0) .
\]

If

\[
\alpha := \sup_{z > 0} \frac{r(z)}{z} < +\infty \quad (39)
\]

then (37) is satisfied; more precisely \( \frac{y(x, t)}{x} \leq e^{\alpha t} \).
Proof. Let us check that $U_0(t)$ is a bounded operator on $X_1$. Note that (36) shows that (for $t > 0$ fixed) $X(y, t)$ is strictly increasing in $y$ and tends to 0 as $y \to 0$ (because $X(y, t) \leq y$). Note that

$$(0, +\infty) \ni y \to X(y, t) \in (0, +\infty)$$

is continuous. Since (for $t > 0$ fixed)

$$U(y, z) := \int_z^y \frac{1}{r(\tau)} d\tau - t$$

is of class $C^1$ in $(y, z)$ with

$$\frac{\partial U(y, z)}{\partial z} = -\frac{1}{r(z)} \neq 0$$

then the implicit function theorem shows that $X(y, t)$ is a $C^1$ function in $y \in (0, +\infty)$ so that differentiating (36) in $y \in (0, +\infty)$

$$\frac{1}{r(y)} - \frac{1}{r(X(y, t))} \frac{\partial X(y, t)}{\partial y} = 0$$

i.e.

$$\frac{\partial X(y, t)}{\partial y} = \frac{r(X(y, t))}{r(y)}$$

and

$$(U_0(t)f)(y) = f(X(y, t)) \frac{\partial X(y, t)}{\partial y} ; \quad y \in (0, +\infty).$$

Thus

$$\|u(., t)\|_{X_1} = \int_0^{+\infty} |u(y, t)| y dy = \int_0^{+\infty} |f(X(y, t))| \frac{\partial X(y, t)}{\partial y} y dy.$$

Note that $\int_y^\infty \frac{1}{r(\tau)} d\tau = t$ shows that $\lim_{y \to +\infty} X(y, t) = +\infty$. The change of variable $x = X(y, t)$ gives

$$\|u(., t)\|_{X_1} = \int_0^{+\infty} |f(x)| y(x, t) dx$$

where $y(x, t)$ is the unique $y > x$ such that $x = X(y, t)$, see (38). Thus

$$\|u(., t)\|_{X_1} = \int_0^{+\infty} \frac{y(x, t)}{x} |f(x)| x dx$$
and $f(.) \rightarrow u(.,t)$ defines a bounded linear operator on $X_1$ if and only if
\[ \sup_{x>0} \frac{y(x,t)}{x} < +\infty. \]
Hence
\[ U_0(t) : X_1 \ni f \rightarrow \frac{r(X(y,t))f(X(y,t))}{r(y)} \in X_1 \]
is bounded with
\[ \|U_0(t)\|_{\mathcal{L}(X_1)} = \sup_{x>0} \frac{y(x,t)}{x} \]
and then $[0, +\infty) \ni t \to U_0(t) \in \mathcal{L}(X_1)$ is locally bounded if and only if
\[ \sup_{x>0} \frac{y(x,t)}{x} \]
is. It follows (see e.g. [12]) that $(U_0(t))_{t\geq 0}$ is exponentially bounded. In this case, to show that $(U_0(t))_{t\geq 0}$ is strongly continuous on $X_1$ it suffices to check that
\[ U_0(t)f \rightarrow f \text{ in } L^1(\mathbb{R}_+; \, x \, dx) \text{ as } t \to 0 \]
on a dense subspace of $L^1(\mathbb{R}_+; \, x \, dx)$, e.g. for $f$ continuous with compact support in $(0, +\infty)$. Note that (36) shows that $X(y,t) \rightarrow y$ as $t \to 0$ uniformly on compact sets of $(0, +\infty)$. Let the support of $f$ be included in a set $[c, c^{-1}]$ with $0 < c < 1$. Since
\[ X\left(\frac{c}{2}, t\right) \rightarrow \frac{c}{2} \text{ and } X(2c^{-1}, t) \rightarrow 2c^{-1} \text{ as } t \to 0 \]
there exists $t_c > 0$ such that
\[ X\left(\frac{c}{2}, t\right) < c, \quad X(2c^{-1}, t) > c^{-1} \]
once $t < t_c$ and consequently, for $t < t_c$,
\[ X(y,t) < c \forall y \leq \frac{c}{2}, \quad X(y,t) > c^{-1} \forall y > 2c^{-1} \]
because $y \to X(y,t)$ is strictly increasing. Hence
\[ U_0(t)f = 0 \text{ on } (0, \frac{c}{2}) \cup (2c^{-1}, +\infty), \quad t < t_c. \]
Since $X(y,t) \rightarrow y$ as $t \to 0$ uniformly on $\left[\frac{c}{2}, 2c^{-1}\right]$, $r(X(y,t))f(X(y,t))$ is uniformly bounded in $y \in \left[\frac{c}{2}, 2c^{-1}\right]$ as $t \to 0$ and $\frac{1}{r}$ is integrable on $\left[\frac{c}{2}, 2c^{-1}\right]$ so
\[ \frac{r(X(y,t))f(X(y,t))}{r(y)} \rightarrow f \text{ in } L^1\left(\left[\frac{c}{2}, 2c^{-1}\right], \, dx\right) \]
as \( t \to 0 \) by the dominated convergence theorem. In particular \( U_0(t)f \to f \) in \( L^1(\mathbb{R}_+; \, xdx) \) as \( t \to 0 \).

It follows from (39) that

\[
\frac{r(z)}{z} \leq \alpha z \quad \forall z > 0.
\]

(40)

We differentiate (38) in \( t \) to obtain

\[
\frac{\partial y(x,t)}{\partial t} = r(y(x,t)) \quad \forall t > 0
\]

(41)

so

\[
y(x,t) = x + \int_{t}^{0} r(y(x,s))ds \leq x + \int_{0}^{t} \alpha y(x,s)ds.
\]

Gronwall’s lemma gives

\[
y(x, t) \leq xe^{\alpha t} \quad \text{so} \quad \|u(\cdot, t)\|_{X_1} \leq e^{\alpha t} \int_{0}^{+\infty} |f(x)| \, xdx \quad \text{and finally} \quad \|U_0(t)\|_{\mathcal{L}(X_1)} \leq e^{\alpha t}.
\]

Summarising:

**Corollary 4** Let (35)–(39) be satisfied. Let \( X(y,t) \) be defined by (36). Then

\[
(U_0(t)f)(y) := \frac{r(X(y,t))f(X(y,t))}{r(y)} = f(X(y,t)) \frac{\partial X(y,t)}{\partial y}
\]

defines a \( C_0 \)-semigroup \( (U_0(t))_{t \geq 0} \) on \( X_1 \) such that \( \|U_0(t)\|_{\mathcal{L}(X_1)} \leq e^{\alpha t} \) where \( \alpha = \sup_{z > 0} \frac{r(z)}{z} \).

**Remark 5** Note that the fact that \( x < y(x, t) \) (for \( t > 0 \)) implies that

\[
\sup_{x} \left( x^{-1}y(x, t) \right) > 1 \quad \text{and then} \quad \|U_0(t)\|_{\mathcal{L}(X_1)} > 1 \quad \text{for} \quad t > 0, \quad \text{i.e.} \quad (U_0(t))_{t \geq 0} \text{ is not contractive}.
\]

We strongly suspect that the sufficient condition (39) for (37) is actually necessary. Indeed, we have:

**Proposition 6** Let (35) be satisfied. If

\[
\lim_{z \to 0} \frac{r(z)}{z} = +\infty \quad \text{or} \quad \lim_{z \to +\infty} \frac{r(z)}{z} = +\infty
\]

then \( \sup_{x > 0} \frac{y(x, t)}{x} = +\infty \). where \( y(x, t) \) is defined by (38). In particular, the generation theory in \( X_1 \) fails.
Proof. We have \( t = \int_{x}^{y(t)} \frac{1}{r(t)} dt \) so the change of variable \( \frac{x}{s} = t \) gives

\[
\int_{1}^{C} \frac{x}{r(xs)} \frac{1}{s} ds = t \quad (t > 0).
\]

Arguing by contradiction, suppose that \( C := \sup_{x > 0} \frac{y(x,t)}{x} < +\infty \). Then

\[
0 < t \leq \int_{1}^{C} \frac{x}{r(xs)} \frac{1}{s} ds \quad (x > 0).
\]  \( (42) \)

If \( \lim_{z \to +\infty} \frac{r(z)}{z} = +\infty \) then \( \lim_{z \to +\infty} \frac{z}{r(z)} = 0 \) and consequently, for any sequence \( (x_n) \) such that \( x_n \to +\infty \) there exists a positive constant \( C \) such that

\[
\frac{x_n}{r(x_n)} \frac{1}{s} \leq \frac{C}{s} \quad (s \in (1, C))
\]

for \( n \) large enough. Since \( \frac{x_n}{r(x_n)} \frac{1}{s} \to 0 \quad (n \to \infty) \quad (s \in (1, C)) \) then the dominated convergence theorem implies

\[
\int_{1}^{C} \frac{x}{r(xs)} \frac{1}{s} ds \to 0
\]

which contradicts \( (42) \). We argue similarly if \( \lim_{z \to -\infty} \frac{r(z)}{z} = +\infty \). \( \blacksquare \)

2.1.2 On the generator of \( (U_0(t))_{t \geq 0} \)

We identify now the resolvent of the generator.

**Proposition 7** Let \( (35)(39) \) be satisfied. Let \( T_0 \) be the generator of \( (U_0(t))_{t \geq 0} \) in \( X_1 \). Then

\[
((\lambda - T_0)^{-1} f)(y) = \frac{1}{r(y)} \int_{0}^{y} e^{-\int_{y}^{x} \frac{\lambda}{r(s)} ds} f(x) dx, \quad \text{Re} \lambda > s(T_0)
\]

where \( s(T_0) \) is the spectral bound of \( T_0 \)

**Proof.** We recall that the spectral bound of \( T_0 \) is nothing but the type of \( (U_0(t))_{t \geq 0} \), see e.g. [33]. Note first that

\[
((\lambda - T_0)^{-1} f)(y) = \int_{0}^{\infty} e^{-\lambda t} \frac{r(X(y, t)) f(X(y, t))}{r(y)} dt \quad (\text{Re} \lambda > s(T_0)).
\]

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Note that (36) shows that \( t^2(0; 1) \) is strictly decreasing from \( y \) to 0. Differentiating (36) in \( t \) we get

\[
-\frac{1}{r(X(y, t))} \frac{\partial X(y, t)}{\partial t} = 1
\]

so the change of variable \( x = X(y, t) \) gives

\[
dx = -r(X(y, t)) dt
\]

and

\[
\int_0^{+\infty} e^{-\lambda t} \frac{r(X(y, t)) f(X(y, t))}{r(y)} dt = \frac{1}{r(y)} \int_y^y e^{-\lambda X^{-1}(y, x)} f(x) dx
\]

where \( X^{-1}(y, x) \) is the inverse of \( t \to x = X(y, t) \). Observe that this inverse is nothing but

\[
x \to t = -\int_y^x \frac{1}{r(\tau)} d\tau
\]

so

\[
((\lambda - T_0)^{-1} f)(y) = \frac{1}{r(y)} \int_y^{y} e^{-\lambda f_x} \frac{1}{r(x)} dx f(x) dx
\]

and this ends the proof. \( \blacksquare \)

**Remark 8** It is possible to characterize \( T_0 \)

\[
D(T_0) = \left\{ f \in X_1; \frac{\partial (rf)}{\partial y} \in X_1 \right\}, \quad T_0 = -\frac{\partial (rf)}{\partial y}
\]

where \( \frac{\partial (rf)}{\partial y} \) is the derivative (in the sense of distributions on \((0, +\infty))\) of the locally integrable function \( rf \) on \((0, +\infty))\); see [5].

We characterize the spectral bound of \( T_0 \).

**Proposition 9** We assume that (35)-(39) are satisfied. The spectral bound of \( T_0 \) (or equivalently the type of \((U_0(t))_{t \geq 0}\)) is given by

\[
s(T_0) = \lim_{t \to +\infty} \frac{1}{t} \ln \left( \sup_x (x^{-1}y(x, t)) \right) = \inf_{t > 0} \frac{1}{t} \ln \left( \sup_x (x^{-1}y(x, t)) \right) \geq 0
\]

where \( y(x, t) \) is defined by (38).
Proof. In the proof of Theorem 3, \( \|U_0(t)\|_{\mathcal{L}(X_1)} = \sup_{x > 0} \frac{y(x,t)}{x} \) and the type of \((U_0(t))_{t \geq 0}\) is given by \( \lim_{t \to +\infty} \frac{1}{t} \ln \|U_0(t)\|_{\mathcal{L}(X_1)} \).

We give now a key pointwise estimate in \( X_1 \).

Lemma 10 Let \((35)(39)\) be satisfied. Let \( \lambda > \alpha \). Then

\[
|\langle \lambda - T_0^{-1}f \rangle(y) | \leq \frac{1}{yr(y)} \|f\|_{X_1} \quad (f \in X_1).
\]

Proof. Since \( r(x) \leq \alpha x \) then

\[
|\langle \lambda - T_0^{-1}f \rangle | \leq \frac{1}{r(y)} \int_{0}^{y} e^{-\lambda \int_{x}^{y} \frac{1}{r} dr} |f(x)| dx = \frac{1}{r(y)} \int_{0}^{y} \frac{1}{x} e^{-\lambda \int_{x}^{y} \frac{1}{r} dr} |f(x)| x dx.
\]

On the other hand, if \( \lambda > \alpha \) then \( \frac{\lambda}{\alpha} - 1 > 0 \) and

\[
\frac{1}{x} e^{-\lambda \int_{x}^{y} \frac{1}{r} dr} = \frac{1}{x} e^{-\lambda \ln(y) \frac{1}{y}} = \frac{1}{x} e^{\frac{\ln(y)}{y}} = \frac{1}{x} e^{\frac{\ln(y)}{y}} = \frac{x^{-\lambda}}{y^\lambda} \leq \frac{y^{-\lambda}}{y^\lambda} = \frac{1}{y} \quad (\forall x \leq y)
\]

so that

\[
|\langle \lambda - T_0^{-1}f \rangle | \leq \frac{1}{yr(y)} \int_{0}^{y} |f(x)| x dx \leq \frac{1}{yr(y)} \|f\|_{X_1}.
\]

This ends the proof.

2.1.3 A first perturbed semigroup \((U(t))_{t \geq 0}\)

Arguing as previously, we solve

\[
\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} \left[ r(x)u(x,t) \right] + \beta(x)u(x,t) = 0, \quad u(x,0) = f(x)
\]

by the method of characteristics and get

\[
u(y,t) = e^{-\int_{X(y,t)}^{y} \frac{\beta(p)}{r(p)} dp \int X(x,t) f(X(y,t))}
\]

where \( X(y,t) \) is defined by \((36)\).
Proposition 11  Let (35)(39) be satisfied. Let \( X(y, t) \) be defined by (36). Then

\[
U(t)f = e^{-\int_{X(y, t)}^y \frac{\beta(p)}{r(p)} dp \frac{r(X(y, t)) f(X(y, t))}{r(y)}} f(X(y, t)) \frac{\partial X(y, t)}{\partial y} = e^{-\int_{X(y, t)}^y \frac{\beta(p)}{r(p)} dp U_0(t) f}
\]

defines a positive \( C_0 \)-semigroup \((U(t))_{t \geq 0}\) on \( X_1 \).

Proposition 12  Let (35)(39) be satisfied. Let \( T \) be the generator of \((U(t))_{t \geq 0}\). Then

\[
((\lambda - T)^{-1} f)(y) = \frac{1}{r(y)} \int_0^y \frac{\lambda + \beta(r)}{r(r)} dr f(x) dx.
\]

for \( \text{Re} \lambda > s(T) \), where \( s(T) \) is the spectral bound of \( T \).

Proof. We note that for \( \text{Re} \lambda > s(T) \)

\[
((\lambda - T)^{-1} f)(y) = \int_0^{+\infty} e^{-\lambda t} e^{-\int_{X(y, t)}^y \frac{\beta(p)}{r(p)} dp \frac{r(X(y, t)) f(X(y, t))}{r(y)}} dt
\]

where \( X(y, t) \) is defined by (36). Arguing as in the proof of Proposition 7, the change of variable \( x = X(y, t) \) gives

\[
((\lambda - T)^{-1} f)(y) = \frac{1}{r(y)} \int_0^y e^{-\lambda t} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp \frac{r(X(y, t)) f(X(y, t))}{r(y)}} dt
\]

and ends the proof. \( \blacksquare \)

We study the spectral bound of \( T \).

Proposition 13  Let (35)(39) be satisfied. The spectral bound of \( T \) is given by

\[
s(T) = \lim_{t \to \infty} \sup_{x > 0} \left[ -t^{-1} \int_x^{y(x, t)} \frac{\beta(p)}{r(p)} dp + t^{-1} \left( \ln \frac{y(x, t)}{x} \right) \right].
\]

In particular

\[
s(T) \leq - \lim_{t \to \infty} \inf_{x > 0} t^{-1} \int_x^{y(x, t)} \frac{\beta(p)}{r(p)} dp + s(T_0).
\]

If \( \bar{\alpha} := \inf \frac{\beta(p)}{r(p)} > 0 \) (e.g. if \( \inf \beta > 0 \)) then

\[
s(T) \leq (1 - \bar{\alpha}) s(T_0).
\]
Proof. We have
\[ \|U(t)\|_{L_1} = \int_0^{+\infty} e^{-\int_X \frac{\partial(y,t)}{\partial y} \ dp} f(X(y,t)) \frac{\partial X(y,t)}{\partial y} y \ dy. \]

The change of variable \( x = X(y,t) \) (i.e. \( \int_x \frac{y(x,t)}{r(p)} dp = t \)) gives
\[ \|U(t)\|_{L_1} = \int_0^{+\infty} e^{-\int_x y(x,t) \frac{\beta(p)}{r(p)} dp} f(x) y(x,t) dx = \int_0^{+\infty} e^{-\int_x y(x,t) \frac{\beta(p)}{r(p)} dp} y(x,t) \frac{\partial}{\partial x} f(x) dx \]
and
\[ \|U(t)\|_{L_1} = \sup_{x > 0} \left( e^{-\int_x y(x,t) \frac{\beta(p)}{r(p)} dp} \frac{y(x,t)}{x} \right). \]

Hence
\[ \ln \left( \|U(t)\|_{L_1} \right) = \ln \left( \sup_{x > 0} \left( e^{-\int_x y(x,t) \frac{\beta(p)}{r(p)} dp} \frac{y(x,t)}{x} \right) \right) = \sup_{x > 0} \ln \left( e^{-\int_x y(x,t) \frac{\beta(p)}{r(p)} dp} \frac{y(x,t)}{x} \right) \]
\[ = \sup_{x > 0} \left[ -\int_x y(x,t) \frac{\beta(p)}{r(p)} dp + \ln \left( \frac{y(x,t)}{x} \right) \right] \]
\[ \leq \sup_{x > 0} \left[ -\int_x y(x,t) \frac{\beta(p)}{r(p)} dp + \sup_{x > 0} \left( \ln \left( \frac{y(x,t)}{x} \right) \right) \right] \]
\[ = -\inf_{x > 0} \int_x y(x,t) \frac{\beta(p)}{r(p)} dp + \sup_{x > 0} \left( \ln \left( \frac{y(x,t)}{x} \right) \right). \]

This ends the first claim.

Note that if \( \beta > 0 \) then \( \alpha := \inf_{x > 0} \frac{\beta(p)}{r(p)} > 0 \) since \( \sup_{x > 0} \frac{\beta(p)}{r(p)} < +\infty \). Thus
\[ \int_x y(x,t) \frac{\beta(p)}{r(p)} dp \geq \alpha \int_x y(x,t) \frac{1}{p} dp = \alpha \ln \left( \frac{y(x,t)}{x} \right) \]
and
\[ \ln \left( \|U(t)\|_{L_1} \right) \leq \sup_{x > 0} \left( -\alpha \ln \left( \frac{y(x,t)}{x} \right) + \ln \left( \frac{y(x,t)}{x} \right) \right) \]
\[ = (1 - \alpha) \sup_{x > 0} \left( \ln \left( \frac{y(x,t)}{x} \right) \right) \]
\[ = (1 - \alpha) \ln \left( \sup_{x > 0} \left( \frac{y(x,t)}{x} \right) \right) = (1 - \alpha) \ln \left( \|U_0(t)\|_{L_1} \right) \]
implies
\[ s(T) = \lim_{t \to +\infty} t^{-1} \ln \left( \|U(t)\|_{L_1} \right) \leq (1 - \alpha) \lim_{t \to +\infty} t^{-1} \ln \left( \|U_0(t)\|_{L_1} \right) \]
and ends the proof. ±
2.1.4 A smoothing effect of the perturbed resolvent

We give now another key estimate in $X_1$.

**Lemma 14** Let \([35],[39]\) be satisfied. Let $\lambda > \alpha$. Then

$$\int_0^{+\infty} |((\lambda - T)^{-1} f) (y)| \beta(y) y dy \leq \int_0^{+\infty} |(f(y))| y dy, \ (f \in X_1).$$

**Proof.** We note first that

$$e^{-\lambda \int_y^x \frac{1}{r(\tau)} d\tau} \leq e^{-\frac{\lambda}{\alpha} \int_y^x \frac{1}{\tau} d\tau} = e^{-\frac{\lambda}{\alpha} \ln(y) - x} = \left(\frac{x}{y}\right)^{\frac{\lambda}{\alpha}}$$

so

$$\int_0^{+\infty} |((\lambda - T)^{-1} f)(y)| \beta(y) y dy \leq \int_0^{+\infty} \frac{\beta(y) y}{r(y)} \left( \int_0^y e^{-\lambda \int_y^{\tau} \frac{1}{r(\tau)} d\tau} e^{-\int_y^\tau \frac{\beta(p)}{r(p)} dp} |f(x)| \, d\tau \right) y dy$$

$$= \int_0^{+\infty} \left( \int_x^y \frac{1}{y} \frac{\lambda}{\alpha} e^{-\int_y^\tau \frac{\beta(p)}{r(p)} dp} \beta(y) y \frac{1}{r(y)} \, dy \right) |f(x)| \, x dx$$

$$= \int_0^{+\infty} \left( \int_x^y \frac{1}{y} \frac{\lambda}{\alpha} e^{-\int_y^\tau \frac{\beta(p)}{r(p)} dp} \beta(y) \frac{1}{r(y)} \, dy \right) |f(x)| \, x dx$$

$$\leq \int_0^{+\infty} \left( \int_x^y e^{-\int_y^\tau \frac{\beta(p)}{r(p)} dp} \beta(y) \frac{1}{r(y)} \, dy \right) |f(x)| \, x dx.$$ 

On the other hand

$$\int_x^{+\infty} e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \beta(y) \frac{1}{r(y)} \, dy = - \int_x^{+\infty} \frac{d}{dy} \left( e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \right) \, dy$$

$$= - \left[ e^{-\int_x^y \frac{\beta(p)}{r(p)} dp} \right]_{y=x}^{y=+\infty} \leq 1$$

whence

$$\int_0^{+\infty} |((\lambda - T)^{-1} f)(y)| \beta(y) y dy \leq \int_0^{+\infty} |f(x)| \, x dx$$

and we are done. 

**Remark 15** One can deduce from Lemma 14 that

$$D(T) = \{ f \in D(T_0); \beta f \in X_1 \}, \ T f = T_0 f - \beta f.$$
2.1.5 On the full perturbed semigroup \((V(t))_{t \geq 0}\)

We give now a second perturbation theorem in \(X_1\).

**Theorem 16** Let \((35)(39)\) be satisfied. Then the fragmentation operator \((14)\) is \(T\)-bounded in \(X_1\) and

\[
\lim_{\lambda \to +\infty} \|B(\lambda - T)^{-1}\|_{\mathcal{L}(X_1)} \leq \lim_{a(y) \to +\infty} \sup_{a(y) \to +\infty} \frac{(1 - \eta(y))}{1 + \frac{d(y)}{a(y)}}.
\]

In particular,

\[T + B : D(T) \subset X_1 \to X_1\]

generates a positive semigroup \((V(t))_{t \geq 0}\) in \(X_1\) if

\[
\lim_{a(y) \to +\infty} \sup_{a(y) \to +\infty} \frac{(1 - \eta(y))}{1 + \frac{d(y)}{a(y)}} < 1. \tag{43}
\]

**Proof.** We observe that

\[
\|B\varphi\|_{X_1} \leq \int_0^{+\infty} \left( \int_x^{+\infty} a(y)b(x, y) |\varphi(y)| dy \right) dx
\]

\[
= \int_0^{+\infty} a(y) \left( \int_y^{+\infty} b(x, y) dx \right) |\varphi(y)| dy
\]

\[
= \int_0^{+\infty} a(y) (y - \eta(y)y) |\varphi(y)| dy
\]

\[
= \int_0^{+\infty} a(y) (1 - \eta(y)) |\varphi(y)| ydy
\]

so

\[
\|B(\lambda - T)^{-1}f\|_{X_1}
\]

\[
= \int_0^{+\infty} a(y) (1 - \eta(y)) \left( (\lambda - T)^{-1} |f| \right) ydy
\]

\[
= \int_{\{a \leq c\}} a(y) (1 - \eta(y)) \left( (\lambda - T)^{-1} |f| \right) ydy
\]

\[
+ \int_{\{a > c\}} a(y) (1 - \eta(y)) \left( (\lambda - T)^{-1} |f| \right) ydy.
\]

We have

\[
\int_{\{a \leq c\}} a(y) (1 - \eta(y)) \left( (\lambda - T)^{-1} |f| \right) ydy \leq c \| (\lambda - T)^{-1}f \|_{X_1}
\]

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and
\[
\int_{\{a>c\}} a(y) (1 - \eta(y)) \left((\lambda - T)^{-1} |f|\right) ydy = \int_{\{a>c\}} \frac{a(y) (1 - \eta(y))}{a(y) + d(y)} (a(y) + d(y)) \left((\lambda - T)^{-1} |f|\right) ydy \\
\leq \sup_{\{a>c\}} \frac{a(y) (1 - \eta(y))}{a(y) + d(y)} \int_0^{+\infty} (a(y) + d(y)) \left((\lambda - T)^{-1} |f|\right) ydy \\
= \sup_{\{a>c\}} \frac{1 - \eta(y)}{1 + \frac{d(y)}{a(y)}} \int_0^{+\infty} (a(y) + d(y)) \left((\lambda - T)^{-1} |f|\right) ydy.
\]

On the other hand, according to Lemma 14 for \(\lambda > \alpha\)
\[
\int_0^{+\infty} \|((\lambda - T)^{-1} f) (x)\| (a(x) + d(x)) xdx \leq \int_0^{+\infty} |f(x)| xdx
\]
whence
\[
\|B(\lambda - T)^{-1}\|_{\mathcal{L}(X_1)} \leq c \|\lambda - T\|_{\mathcal{L}(X_1)} + \sup_{\{a>c\}} \frac{(1 - \eta(y))}{1 + \frac{d(y)}{a(y)}}
\]
and
\[
\lim_{\lambda \to +\infty} \|B(\lambda - T)^{-1}\|_{\mathcal{L}(X_1)} \leq \sup_{\{a>c\}} \frac{(1 - \eta(y))}{1 + \frac{d(y)}{a(y)}} \quad (\forall c > 0).
\]
Finally
\[
\lim_{\lambda \to +\infty} \|B(\lambda - T)^{-1}\|_{\mathcal{L}(X_1)} \leq \lim_{a(y) \to +\infty} \sup_{a(y) \to +\infty} \frac{(1 - \eta(y))}{1 + \frac{d(y)}{a(y)}}
\]
and this ends the proof by invoking W. Desch’s theorem (i.e. Theorem 1) since \(T\) generates a positive semigroup \((U(t))_{t \geq 0}\). ■

**Remark 17** The well posedness via W. Desch’s theorem depends on the existence of an amount of mass loss or death in the system. For instance (43) is satisfied if
\[
d = 0 \text{ and } \lim_{a(y) \to +\infty} \eta(y) > 0 \text{ or } \eta = 0 \text{ and } \lim_{a(y) \to +\infty} \frac{d(y)}{a(y)} > 0.
\]

**Remark 18** For homogeneous kernels (9), \(\lim_{a(y) \to +\infty} \eta(y) > 0\) amounts to \(\int_0^1 zh(z)dz < 1\). More generally, in the case (11),
\[
\lim_{a(y) \to +\infty} \eta(y) \geq 1 - \zeta_+^\infty
\]
where

\[ \zeta^+ = \lim_{a(y) \to +\infty} \sup \zeta(x, y) := \lim_{m \to +\infty} \sup_{\{(x,v); \, a(y) \geq m\}} \zeta(x,y) \]

and \( \liminf_{a(y) \to +\infty} \eta(y) > 0 \) holds if \( \zeta^+ < 1 \).

**2.1.6 Compactness results in \( X_1 \)**

Let

\[ \Omega_c = \{ x > 0; \beta(x) < c \} \quad (c > 0) \]

be the sublevel sets of \( \beta \).

**Definition 19** If \( \int_0^1 \frac{1}{r(\tau)} d\tau = +\infty \) we say that the sublevel sets of \( \beta \) are thin near zero relatively to \( r \) (thin near zero for short) if

\[ \int_0^1 \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty \quad (c > 0) \]

where \( 1_{\Omega_c} \) is the indicator function of \( \Omega_c \). In particular, if \( \lim_{y \to 0} \beta(y) = +\infty \) then the sublevel sets of \( \beta \) are automatically thin near zero.

Similarly, if \( \int_1^\infty \frac{1}{r(\tau)} d\tau = +\infty \) we say that the sublevel sets of \( \beta \) are thin near infinity relatively to \( r \) (thin near infinity for short) if

\[ \int_1^\infty \frac{1_{\Omega_c}(\tau)}{r(\tau)} d\tau < +\infty \quad (c > 0). \]

In particular, if \( \lim_{y \to +\infty} \beta(y) = +\infty \) then the sublevel sets of \( \beta \) are automatically thin near infinity.

The confining role of singular absorption potentials in compactness properties of (perturbed) positive contraction semigroups in abstract \( L^1 \) spaces has been systematically analyzed in [28]. Note that growth-semigroups are not contractive and we provide here a direct analysis adapted to them.

**Theorem 20** Let \( (35)-(39) \) be satisfied. If

the sublevel sets of \( \beta \) are thin near zero and infinity \hspace{1cm} (44)

then \( T \) is resolvent compact on \( X_1 \).
Proof. Let $\lambda > \alpha$ and $f$ in the unit ball of $X_1$, i.e.

$$\int_0^{+\infty} |f(x)| \, dx \leq 1.$$ 

According to Lemma 14

$$\int_0^{+\infty} |((\lambda - T)^{-1} f)(x)| \beta(x) \, dx \leq 1.$$ 

Let $c > 0$ and $\varepsilon > 0$ be arbitrary. We have

$$1 \geq \int_0^{\varepsilon} |((\lambda - T)^{-1} f)(x)| \beta(x) \, dx = \int_0^{\varepsilon} 1_{\{\beta < c\}} \left|((\lambda - T)^{-1} f)(x)\right| \beta(x) \, dx$$

$$+ \int_0^{\varepsilon} 1_{\{\beta \geq c\}} \left|((\lambda - T)^{-1} f)(x)\right| \beta(x) \, dx$$

so

$$\sup_{\|f\|_{X_1} \leq 1} \int_0^{\varepsilon} 1_{\{\beta \geq c\}} \left|((\lambda - T)^{-1} f)(x)\right| \beta(x) \, dx \leq \frac{1}{c}.$$ 

On the other hand, according to Lemma 10

$$\left|((\lambda - T)^{-1} f)(x)\right| \leq \frac{1}{r(x)} \quad (x > 0)$$

uniformly in $\|f\|_{X_1} \leq 1$ so

$$\int_0^{\varepsilon} 1_{\{\beta < c\}} \left|((\lambda - T)^{-1} f)(x)\right| \beta(x) \, dx \leq \int_0^{\varepsilon} 1_{\{\beta < c\}} \frac{1}{r(x)} \, dx = c \int_0^{\varepsilon} 1_{\{\beta < c\}} \frac{1}{r(x)} \, dx$$

and

$$\sup_{\|f\|_{X_1} \leq 1} \int_0^{\varepsilon} \left|((\lambda - T)^{-1} f)(x)\right| \beta(x) \, dx \leq \frac{1}{c} + c \int_0^{\varepsilon} 1_{\{\beta < c\}} \frac{1}{r(x)} \, dx$$

can be made arbitrarily small by choosing first $c$ large enough and then $\varepsilon$ small enough.

Similarly,

$$1 \geq \int_{\varepsilon^{-1}}^{+\infty} |((\lambda - T)^{-1} f)(x)| \beta(x) \, dx = \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta < c\}} \left|((\lambda - T)^{-1} f)(x)\right| \beta(x) \, dx$$

$$+ \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta \geq c\}} \left|((\lambda - T)^{-1} f)(x)\right| \beta(x) \, dx$$

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We have also
\[
\int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta < c\}} \left| \left( (\lambda - T)^{-1} f \right) (x) \right| \, dx \leq c \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta < c\}} \frac{1}{x \tau(x)} \, dx = c \int_{\varepsilon^{-1}}^{+\infty} 1_{\{\beta < c\}} \frac{1}{\tau(x)} \, dx
\]
and
\[
\sup_{\|f\|_{X_1} \leq 1} \int_{0}^{\varepsilon} \left| \left( (\lambda - T)^{-1} f \right) (x) \right| \, dx \leq \frac{1}{x \tau(x)} \quad (x > 0) \quad (\|f\|_{X_1} \leq 1)
\]
can be made arbitrarily small by choosing first $c$ large enough and then $\varepsilon$ small enough. Finally, the uniform estimate
\[
\left| \left( (\lambda - T)^{-1} f \right) (x) \right| \leq \frac{1}{x \tau(x)} \quad (x > 0) \quad (\|f\|_{X_1} \leq 1)
\]
gives a uniform domination by $\frac{1_{\{\varepsilon, \varepsilon^{-1}\}}}{x \tau(x)} \in X_1$
\[
1_{\{\varepsilon, \varepsilon^{-1}\}} \left| \left( (\lambda - T)^{-1} f \right) (x) \right| \leq \frac{1_{\{\varepsilon, \varepsilon^{-1}\}}}{x \tau(x)} \quad (\|f\|_{X_1} \leq 1)
\]
so
\[
\left\{ 1_{\{\varepsilon, \varepsilon^{-1}\}} \left| \left( (\lambda - T)^{-1} f \right) \right| ; \|f\|_{X_1} \leq 1 \right\} \text{ is relatively weakly compact.}
\]
Finally, $\left\{ (\lambda - T)^{-1} f ; \|f\|_{X_1} \leq 1 \right\}$ is as close to a relatively weakly compact set as we want and consequently is weakly compact. Hence $(\lambda - T)^{-1}$ is weakly compact operator and consequently (see [28] Lemma 14) $(\lambda - T)^{-1}$ is compact. ■

We can state:

**Corollary 21** Let (35)(39)(44)(43) be satisfied. Then
\[
T + B : D(T) \subset X_1 \to X_1
\]
is resolvent compact in $X_1$.

**Proof.** Theorem 16 implies $\sum_{j=0}^{+\infty} (B(\lambda - T)^{-1})^j \in \mathcal{L}(X_1)$ and
\[
(\lambda - T - B)^{-1} = (\lambda - T)^{-1} \sum_{j=0}^{+\infty} (B(\lambda - T)^{-1})^j
\]
so Theorem 20 ends the proof. ■
2.1.7 Spectral gap of the full semigroup \((V(t))_{t \geq 0}\) in \(X_1\)

We start with:

**Lemma 22** Let \(35\) be satisfied. We assume that the support of \(a(\cdot)\) is not bounded. Then \((-\lambda + T - B)^{-1}\) is positivity improving, i.e.

\[
(-\lambda + T - B)^{-1} f > 0 \text{ a.e.}
\]

for any nontrivial nonnegative \(f \in X_1\), or equivalently \((V(t))_{t \geq 0}\) is irreducible in \(X_1\).

**Proof.** Note that

\[
(-\lambda + T - B)^{-1} f = (-\lambda + T)^{-1} \sum_{n=0}^{+\infty} \left[ B(-\lambda + T)^{-1} \right]^n f
\]

and

\[
\left( (-\lambda + T)^{-1} \varphi \right) (y) = \frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} \, d\tau} e^{-\int_x^y \frac{\beta(\rho)}{r(\rho)} \, d\rho} \varphi(x) \, dx.
\]

Note also that

\[
B(-\lambda + T)^{-1} f
\]

\[
= \int_0^{+\infty} 1_{\{x < y\}} a(y) b(x, y) \left[ \frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} \, d\tau} e^{-\int_x^y \frac{\beta(\rho)}{r(\rho)} \, d\rho} f(x) \, dx \right] \, dy
\]

\[
= \int_0^{+\infty} \left[ \int_0^{+\infty} \frac{1}{r(y)} 1_{\{x < y\}} a(y) b(x, y) e^{-\lambda \int_x^y \frac{1}{r(\tau)} \, d\tau} e^{-\int_x^y \frac{\beta(\rho)}{r(\rho)} \, d\rho} \, dy \right] f(x) \, dx
\]

\[
= \int_0^{+\infty} \left[ \int_x^{+\infty} \frac{1}{r(y)} a(y) b(x, y) e^{-\lambda \int_x^y \frac{1}{r(\tau)} \, d\tau} e^{-\int_x^y \frac{\beta(\rho)}{r(\rho)} \, d\rho} \, dy \right] f(x) \, dx
\]

and

\[
\int_x^{+\infty} \frac{1}{r(y)} a(y) b(x, y) e^{-\lambda \int_x^y \frac{1}{r(\tau)} \, d\tau} e^{-\int_x^y \frac{\beta(\rho)}{r(\rho)} \, d\rho} \, dy > 0 \forall x > 0
\]

so that \(B(-\lambda + T)^{-1} f > 0\) a.e. for any nontrivial nonnegative \(f\). It follows that

\[
\sum_{n=1}^{+\infty} \left[ B(-\lambda + T)^{-1} \right]^n f > 0 \text{ a.e.}
\]
and consequently
\[(\lambda - T)^{-1} \sum_{n=1}^{+\infty} [B(\lambda - T)^{-1}]^n f > 0 \text{ a.e.}\]

for any nontrivial nonnegative \(f\). This ends the proof. ■

We are now ready to show the main result of Subsection 2.1.

**Theorem 23** We assume that (35)(39)(43)(44) are satisfied and that the support of \(a(.)\) is not bounded. Then \((\hat{V}(t))_{t \geq 0}\) has a spectral gap in \(X_1\), i.e.
\[\text{r}_{\text{ess}}(\hat{V}(t)) < \text{r}_\sigma(\hat{V}(t)),\]
and satisfies the asynchronous exponential growth.

**Proof.** Let
\[k(x, y) := 1_{(x < y)}a(y)b(x, y)\]
be the kernel of \(B\). Let
\[\overline{k}(x, y) := k(x, y) \wedge 1\]
and
\[\overline{k}_c(x, y) := \overline{k}(x, y)p(x)p(y)\]
where \(p \in C(0, +\infty)\) has a compact support in \((0, +\infty)\) and \(0 \leq p(x) \leq 1\). Note that \(k(x, y) \geq \overline{k}_c(x, y)\) and
\[k(x, y) = (k(x, y) - \overline{k}_c(x, y)) + \overline{k}_c(x, y) = \hat{k}(x, y) + \overline{k}_c(x, y)\]
where \(\hat{k}(x, y) := k(x, y) - \overline{k}_c(x, y)\). Let \(\overline{B}\) be the integral operator with kernel \(\overline{k}_c(x, y)\) and \(\hat{B}\) be the integral operator with kernel \(\hat{k}(x, y)\). Since \(\hat{k}(x, y) \leq k(x, y)\) then
\[\|\hat{B} (\lambda - T)^{-1}\|_{L(X_1)} \leq \|B (\lambda - T)^{-1}\|_{L(X_1)} < 1\]
for \(\lambda\) large enough so \(T + \hat{B} : D(T) \rightarrow X_1\) generates a positive semigroup \((\hat{V}(t))_{t \geq 0}\). Note that \((V(t))_{t \geq 0}\) is generated by
\[(T + \hat{B}) + \overline{B}\]
where $\mathcal{B}$ is a bounded operator on $X$. Actually the kernel of $\mathcal{B}$ is compactly supported in $(0, +\infty) \times (0, +\infty)$ and bounded and consequently $\mathcal{B}$ is a weakly compact operator on $X$. On the other hand

$$V(t) = \hat{V}(t) + \int_0^t \hat{V}(t-s)\mathcal{B}\hat{V}(s)ds$$

and, by the convex (weak) compactness property of the strong operator topology (see [35] or [24]), the strong integral (not a Bochner integral)

$$\int_0^t \hat{V}(t-s)\mathcal{B}\hat{V}(s)ds$$

defined just strongly, i.e. by

$$X_1 \ni \varphi \to \int_0^t \hat{V}(t-s)\mathcal{B}\hat{V}(s)\varphi ds \in X_1,$$

is a weakly compact operator. It follows that $\hat{V}(t)$ and $V(t)$ have the same essential spectrum [17] and therefore

$$r_{ess}(\hat{V}(t)) = r_{ess}(V(t)) \ (t > 0) \ (45)$$

or, equivalently, the identity of their essential types

$$\omega_{ess}(\hat{V}) = \omega_{ess}(V).$$

On the other hand $\hat{V}(t) \leq V(t)$

$$(\lambda - T - \hat{B})^{-1} \leq (\lambda - T - B)^{-1}$$

and

$$(\lambda - T - \hat{B})^{-1} \neq (\lambda - T - B)^{-1}$$

because $\mathcal{B} \neq 0$. Since, by Lemma [22] $(\lambda - T - B)^{-1}$ is positivity improving (and thus irreducible) and compact (by Corollary [21]) then

$$r_\sigma [(\lambda - T - B)^{-1}] < r_\sigma [(\lambda - T - B)^{-1}]$$

see [19]. Moreover

$$r_\sigma [(\lambda - T - B)^{-1}] = \frac{1}{\lambda - s(T + B)} \text{ and } r_\sigma [(\lambda - T - B)^{-1}] = \frac{1}{\lambda - s(T + B)}$$

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(see e.g. [33]) whence $s(T + \hat{B}) < s(T + B)$. This implies in particular that

$$s(T + B) > -\infty.$$ 

Note that the type of a positive semigroup on $L^1$ coincides with the spectral bound of its generator so that

$$r_\sigma(\hat{V}(t)) = e^{s(T+\hat{B})t} < e^{s(T+B)t} = r_\sigma(V(t)).$$ 

Since $r_{\text{ess}}(\hat{V}(t)) \leq r_\sigma(\hat{V}(t))$ then (45) gives $r_{\text{ess}}(V(t)) < r_\sigma(V(t))$, i.e. $(V(t))_{t \geq 0}$ has a spectral gap. Finally, the asynchronous exponential growth follows from the irreducibility of $(V(t))_{t \geq 0}$. 

**Remark 24** Note that if the sublevel sets of $a(.)$ are thin at infinity then the support of $a(.)$ is not bounded.

### 2.2 Theory in the space $X_{0,1}$

Within assumption (35), we develop now a general theory on well-posedness and spectral analysis in the "finite mass and number of aggregates" space

$$X_{0,1} := L^1([0, +\infty), (1 + x)dx).$$

#### 2.2.1 First generation result in $X_{0,1}$

We put now the solution given in Proposition 2 in the functional space $X_{0,1}$.

**Theorem 25** Let (35) be satisfied and $X(y, t)$ be given by (36). Then

$$(U_0(t)f)(y) := \frac{r(X(y, t))f(X(y, t))}{r(y)} = f(X(y, t)) \frac{\partial X(y, t)}{\partial y}$$

defines a stongly continuous semigroup $(U_0(t))_{t \geq 0}$ on $X_{0,1}$ if and only if

$$\sup_{x > 0} \frac{1 + y(x, t)}{1 + x} < +\infty \quad \forall t \geq 0$$

(46)

and

$$[0, +\infty) \ni t \rightarrow \sup_{x > 0} \frac{1 + y(x, t)}{1 + x} \text{ is locally bounded}$$

(47)

where $y(x, 0) = x$ and $y(x, t)$ is defined for $t > 0$ by (38). If

$$\alpha_1 := \sup_{z > 1} \frac{r(z)}{z} < +\infty$$

(48)

then (47) is satisfied.

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Proof. We have
\[ \|U_0(t)f\|_{X_{0,1}} = \int_0^{+\infty} |u(y,t)| (1 + y) \, dy = \int_0^{+\infty} |f(X(y,t))| \frac{\partial X(y,t)}{\partial y} (1 + y) \, dy. \]
By the change of variable \( x = X(y,t) \), we have
\[ \|U_0(t)f\|_{X_{0,1}} = \int_0^{+\infty} |f(x)| (1 + y(x,t)) \, dx = \int_0^{+\infty} \frac{1 + y(x,t)}{1 + x} |f(x)| (1 + x) \, dx \]
so \( U_0(t) \) is a bounded operator on \( X_{0,1} \) if and only if (46) holds; in this case,
\[ \|U_0(t)\|_{\mathcal{L}(X_{0,1})} = \sup_{x > 0} \frac{1 + y(x,t)}{1 + x}. \] (49)
This shows the first claim. Note that under (35), \( \int_x^{y(x,t)} \frac{1}{r(\tau)} \, d\tau = t \) implies that \( \lim_{x \to 0} y(x,t) = 0 \) uniformly in \( t \) bounded so
\[ \lim_{x \to 0} \frac{1 + y(x,t)}{1 + x} = 1 \quad (t \geq 0) \]
uniformly in \( t \) bounded. In particular (46) holds if and only if
\[ \sup_{x > 1} \frac{1 + y(x,t)}{1 + x} < +\infty \quad (t \geq 0) \]
and
\[ [0, +\infty) \ni t \to \sup_{x > 1} \frac{1 + y(x,t)}{1 + x} \] is locally bounded
or equivalently if
\[ \sup_{x > 1} \frac{y(x,t)}{x} < +\infty \quad (t \geq 0) \]
and
\[ [0, +\infty) \ni t \to \sup_{x > 1} \frac{y(x,t)}{x} \] is locally bounded.
It follows from (41) that
\[ y(x,t) = x + \int_0^t r(y(x,s)) \, ds. \]
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Since \( y(x,t) > x \) then (48) gives
\[
r(y(x,s)) \leq \alpha_1 y(x,s) \quad (x \geq 1)
\]
so
\[
y(x,t) \leq x + \int_0^t \alpha_1 y(x,s)ds, \quad (x \geq 1)
\]
and Gronwall’s lemma gives \( \sup_{x>1} \frac{y(x,t)}{x} \leq e^{\alpha_1 t} \). The strong continuity at the origin can be dealt with as in the space \( X_1 \). ■

Remark 26 By arguing as in Proposition 6 one can check that if
\[
\lim_{z \to +\infty} \frac{r(z)}{z} = +\infty
\]
then \( \sup_{x>1} \frac{y(x,t)}{x} = +\infty \) and consequently the generation theory in \( X_{0,1} \) fails.

Remark 27 We observe that in contrast to the \( X_1 \)-generation theory, we need no assumption on the growth rate function at the origin. The fact that \( y(x,t) > x \) and (49) show that \( (U_0(t))_{t \geq 0} \) is not contractive in \( X_{0,1} \).

2.2.2 On the generator of \( (U_0(t))_{t \geq 0} \) in \( X_{0,1} \)

As in \( X_1 \), the resolvent of the generator \( T_0 \) in \( X_{0,1} \) is characterized by:

Proposition 28 Let (35)(48) be satisfied. Let \( T_0 \) be the generator of \( (U_0(t))_{t \geq 0} \) in \( X_{0,1} \). Then
\[
\left( (\lambda - T_0)^{-1} f \right)(y) = \frac{1}{r(y)} \int_y^\infty e^{-\int_y^u \frac{\lambda}{r(s)}ds} f(x)dx, \quad \text{Re} \lambda > s(T_0)
\]
where \( s(T_0) \) is the spectral bound of \( T_0 \). Moreover,
\[
D(T_0) = \left\{ f \in X_{0,1}; \frac{\partial (rf)}{\partial y} \in X_{0,1} \right\}, \quad T_0 = -\frac{\partial (rf)}{\partial y}
\]
where \( \frac{\partial (rf)}{\partial y} \) is the derivative (in the sense of distributions on \( (0,+\infty) \)) of the locally integrable function \( rf \) on \( (0,+\infty) \).

By using (49) we get:
Proposition 29 Let (48) be satisfied. The spectral bound of $T_0$ (or equivalently the type of $(U_0(t))_{t>0}$) in $X_{0,1}$ is given by

$$\widehat{s}(T_0) = \lim_{t \to +\infty} \frac{1}{t} \ln \left[ \sup_{x>0} \frac{1 + y(x,t)}{1 + x} \right] = \inf_{t>0} \frac{1}{t} \ln \left[ \sup_{x>0} \frac{1 + y(x,t)}{1 + x} \right] \geq 0$$

where $y(x,t)$ is defined by (38).

We can recover the previous pointwise estimate but under an assumption stronger than (48).

Lemma 30 Let (35) be satisfied. If $C := \sup_{z>0} \frac{r(z)}{1+z} < +\infty$ then

$$|\lambda - T_0|^{-1} f(y) \leq \frac{1}{(1+y) r(y)} \|f\|_{X_{0,1}}, \quad (f \in X_{0,1}), \quad (\lambda > C).$$

Proof. Since $r(z) \leq C(z+1) \quad (\forall z > 0)$ then

$$\frac{1}{r(z)} \geq \frac{C^{-1}}{z+1}$$

and

$$e^{-\lambda} \int_y^{y+1} \frac{1}{r(y)} dy \leq e^{-\frac{\lambda}{C}} \int_y^{y+1} \frac{1}{r(y)} \ln(y+1) \, dy = \frac{(x+1)}{(y+1)} \frac{\lambda}{C}. \quad (50)$$

It follows that

$$|\lambda - T_0|^{-1} f(y) \leq \frac{1}{r(y)} \int_0^y e^{-\lambda} \int_0^y \frac{1}{r(y)} \, dx \, |f(x)| \, dx$$

$$\leq \frac{1}{r(y)} \int_0^y \frac{(x+1)}{(y+1)} \frac{\lambda}{C} \ |f(x)| \, dx$$

$$= \frac{1}{(1+y) r(y)} \int_0^y \frac{(x+1)}{(y+1)} \frac{\lambda}{C} - 1 \ |f(x)| \, (1+x) \, dx$$

$$\leq \frac{1}{(1+y) r(y)} \int_0^{+\infty} |f(x)| \, (1+x) \, dx$$

because $\frac{x+1}{y+1} \leq 1$ and $\frac{\lambda}{C} - 1 > 0$. This ends the proof. \[\blacksquare\]

Remark 31 (Open question) We suspect that a similar statement should hold under the general assumption (48).
2.2.3 A first perturbed semigroup in $X_{0,1}$

As previously in $X_1$ we have:

**Proposition 32** Let (35)(48) be satisfied. Let $X(y,t)$ be defined by (36). Then

$$U(t)f = e^{-\int_{y}^{u} \frac{\beta(p)}{r(p)} dp} \frac{r(X(y,t))f(X(y,t))}{r(y)} = e^{-\int_{y}^{u} \frac{\beta(p)}{r(p)} dp} f(X(y,t)) \frac{\partial X(y,t)}{\partial y} = e^{-\int_{y}^{u} \frac{\beta(p)}{r(p)} dp} U_0(t)f$$

defines a positive $C_0$-semigroup $(U(t))_{t \geq 0}$ on $X_{0,1}$.

and

**Proposition 33** Let (35)(48) be satisfied. Let $T$ be the generator of $(U(t))_{t \geq 0}$ in $X_{0,1}$. Then

$$((\lambda - T)^{-1}f)(y) = \frac{1}{r(y)} \int_{0}^{y} e^{-\lambda \int_{x}^{y} \frac{1}{r(r)} dr} e^{-\int_{x}^{y} \frac{\beta(r)}{r(r)} dr} f(x) dx$$

$$= \frac{1}{r(y)} \int_{0}^{y} e^{-\int_{x}^{y} \frac{\lambda + \beta(r)}{r(r)} dr} f(x) dx.$$ 

for $\Re \lambda > s(T)$, where $s(T)$ is the spectral bound of $T$.

Arguing as in Proposition 13 we get:

**Proposition 34** Let (35)(48) be satisfied. The spectral bound of $T$ in $X_{0,1}$ is given by

$$\tilde{s}(T) = \lim_{t \to \infty} \sup_{x > 0} \left[ -t^{-1} \int_{x}^{y(x,t)} \frac{\beta(p)}{r(p)} dp \right] + t^{-1} \left( \ln \frac{1 + y(x,t)}{1 + x} \right).$$

In particular

$$\tilde{s}(T) \leq - \lim_{t \to \infty} \inf_{x > 0} t^{-1} \int_{x}^{y(x,t)} \frac{\beta(p)}{r(p)} dp + \tilde{s}(T_0).$$

If $\tilde{\alpha} := \inf \frac{(1+p)\beta(p)}{r(p)} > 0$ (e.g. if $\inf \beta > 0$) then

$$\tilde{s}(T) \leq (1 - \tilde{\alpha}) \tilde{s}(T_0).$$
2.2.4 A smoothing effect of the perturbed resolvent in $X_{0,1}$

As in $X_1$, we show now a smoothing effect in $X_{0,1}$ but we have to replace the natural assumption \[48\] by a stronger one.

**Lemma 35** Let \[45\] be satisfied. If $C := \sup_{z > 0} \frac{r(z)}{z} < +\infty$ then, for $\lambda > C$,

$$
\int_0^{+\infty} |((\lambda - T)^{-1} f)(y)| \beta(y) (1 + y) dy \leq \int_0^{+\infty} |(f(y)) (1 + y) dy, \ (f \in X_{0,1}).
$$

**Proof.** By using \[40\] we have for $\lambda > C$

$$
\int_0^{+\infty} |((\lambda - T)^{-1} f)(y)| \beta(y) (1 + y) dy \\
\leq \int_0^{+\infty} \frac{\beta(y) (1 + y)}{r(y)} \left( \int_0^{y} e^{-\lambda \int_0^y \frac{1}{r(z)} dz} e^{-\int_x^y \frac{p(z)}{r(z)} dp} |f(x)| dx \right) dy \\
= \int_0^{+\infty} \left( \int_x^{+\infty} \frac{x + 1}{y + 1} \beta e^{-\int_x^y \frac{p(z)}{r(z)} dp} \beta(y) (1 + y) dy \right) |f(x)| dx \\
= \int_0^{+\infty} \left( \int_x^{+\infty} \frac{x + 1}{y + 1} \beta e^{-\int_x^y \frac{p(z)}{r(z)} dp} \beta(y) dy \right) |f(x)| (1 + x) dx \\
\leq \int_0^{+\infty} \left( \int_x^{+\infty} e^{-\int_x^y \frac{p(z)}{r(z)} dp} \beta(y) dy \right) |f(x)| (1 + x) dx
$$

where we have used in the last step that $\frac{x}{y} \leq 1$ and $\frac{x}{y} - 1 > 0$. We already know that

$$
\int_x^{+\infty} e^{-\int_x^y \frac{p(z)}{r(z)} dp} \beta(y) dy = - \int_x^{+\infty} \frac{d}{dy} \left( e^{-\int_x^y \frac{p(z)}{r(z)} dp} \right) dy = - \left[ e^{-\int_x^y \frac{p(z)}{r(z)} dp} \right]_{y=x}^{y=+\infty} \leq 1
$$

whence

$$
\int_0^{+\infty} |((\lambda - T)^{-1} f)(y)| \beta(y) (1 + y) dy \leq \int_0^{+\infty} |f(x)| (1 + x) dx
$$

and we are done. ■

**Remark 36 (Open question)** We suspect that a similar smoothing effect should hold under \[48\].

**Remark 37** One can deduce from Lemma 35 that

$$
D(T) = \{f \in D(T_0); \ \beta f \in X_{0,1}\}, \ T f = T_0 f - \beta f.
$$
2.2.5 The full perturbed semigroup in $X_{0,1}$

We give now a second perturbation theorem in $X_{0,1}$.

**Theorem 38** Let (35) (48) be satisfied. We assume that

$$n(y) := \int_0^y b(x, y) \, dx$$

is such that $\tilde{C} := \sup_{y>0} \frac{n(y)}{1+y} < +\infty$. Then the fragmentation operator (14) is $T$-bounded in $X_{0,1}$ and

$$\lim_{\lambda \to +\infty} \| B(\lambda - T)^{-1} \|_{L(X_{0,1})} \leq \lim_{a(y) \to +\infty} \sup_{a(y) \to +\infty} \frac{[(y - \eta(y)y) + n(y)]}{(1 + y) \left(1 + \frac{d(y)}{a(y)}\right)}.$$ 

In particular, if

$$\lim_{a(y) \to +\infty} \sup_{a(y) \to +\infty} \frac{[(y - \eta(y)y) + n(y)]}{(1 + y) \left(1 + \frac{d(y)}{a(y)}\right)} < 1$$

then

$$T + B : D(T) \subset X_{0,1} \to X_{0,1}$$

generates a positive semigroup $(V(t))_{t \geq 0}$ in $X_{0,1}$.

**Proof.** We have

$$\| B \varphi \|_{X_{0,1}} \leq \int_0^{+\infty} \left( \int_x^{+\infty} a(y) b(x, y) |\varphi(y)| \, dy \right) (1 + x) \, dx$$

$$= \int_0^{+\infty} a(y) \left( \int_0^y (1 + x) b(x, y) \, dx \right) |\varphi(y)| \, dy$$

$$= \int_0^{+\infty} a(y) [(y - \eta(y)y) + n(y)] |\varphi(y)| \, dy$$

$$= \int_0^{+\infty} a(y) \frac{[(y - \eta(y)y) + n(y)]}{1 + y} |\varphi(y)| (1 + y) \, dy$$

so

$$\| B(\lambda - T)^{-1} f \|_{X_{0,1}} \leq$$

$$= \int_{\{a \leq \epsilon\}} a(y) \frac{[(y - \eta(y)y) + n(y)]}{1 + y} |((\lambda - T)^{-1} f)| (1 + y) \, dy$$

$$+ \int_{\{a > \epsilon\}} a(y) \frac{[(y - \eta(y)y) + n(y)]}{1 + y} |((\lambda - T)^{-1} f)| (1 + y) \, dy.$$
Note that
\[
\int_{\{a \leq c\}} \frac{a(y) [(y - \eta(y)y) + n(y)]}{1 + y} \left| ((\lambda - T)^{-1}f) \right| (1 + y) dy \\
\leq c \left(1 + \bar{C}\right) \|(\lambda - T)^{-1}f\|_{X_0,1}
\]
while
\[
\int_{\{a > c\}} \frac{a(y) [(y - \eta(y)y) + n(y)]}{1 + y} \left| ((\lambda - T)^{-1}f) \right| (1 + y) dy \\
= \int_{\{a > c\}} \frac{a(y) [(y - \eta(y)y) + n(y)]}{(1 + y) (a(y) + d(y))} (a(y) + d(y)) \left( (\lambda - T)^{-1} |f| \right) (1 + y) dy \\
\leq \sup_{\{a > c\}} \frac{a(y) [(y - \eta(y)y) + n(y)]}{(1 + y) (a(y) + d(y))} \int_0^{+\infty} (a(y) + d(y)) \left( (\lambda - T)^{-1} |f| \right) (1 + y) dy \\
\leq \sup_{\{a > c\}} \frac{a(y) [(y - \eta(y)y) + n(y)]}{(1 + y) (a(y) + d(y))} \int_0^{+\infty} |f(y)| (1 + y) dy
\]

(Lemma 35 is used in the last step) so
\[
\|B(\lambda - T)^{-1}\|_{L(X_0,1)} \leq c \left(1 + \bar{C}\right) \|(\lambda - T)^{-1}\|_{L(X_0,1)} + \sup_{\{a > c\}} \frac{[(y - \eta(y)y) + n(y)]}{(1 + y) \left(1 + \frac{d(y)}{a(y)}\right)}
\]

for arbitrary \(c > 0\) and consequently
\[
\lim_{\lambda \to +\infty} \|B(\lambda - T)^{-1}\|_{L(X_0,1)} \leq \lim_{a(y) \to +\infty} \sup_{\{a > c\}} \frac{[(y - \eta(y)y) + n(y)]}{(1 + y) \left(1 + \frac{d(y)}{a(y)}\right)}
\]
which ends the proof by invoking W. Desch’s theorem (i.e. Theorem [1]).

Let us check Assumption (51).

Corollary 39 We assume that \(a(.) \in L^{\infty}_{loc}(0, +\infty)\) and
\[
\gamma := \lim_{a(y) \to +\infty} \sup_{\{a > c\}} \frac{(1 - \eta(y))}{1 + \frac{d(y)}{a(y)}} < 1.
\]

Then Assumption (51) is satisfied in the following cases:
(i) \(a(.)\) is unbounded at zero and at infinity and
\[
\max \left\{ \limsup_{y \to 0} \frac{n(y)}{(1 - \eta(y))}, \; 1 + \limsup_{y \to +\infty} \frac{n(y)}{y (1 - \eta(y))} \right\} < \gamma^{-1}.
\]
(ii) $a(.)$ is unbounded at zero only and
\[
\limsup_{y \to 0} \frac{n(y)}{(1 - \eta(y))} < \gamma^{-1}.
\]

(iii) $a(.)$ is unbounded at infinity only and
\[
1 + \limsup_{y \to +\infty} \frac{n(y)}{y (1 - \eta(y))} < \gamma^{-1}.
\]

**Proof.** Note that
\[
\frac{[(y - \eta(y)y) + n(y)]}{(1 + y) \left( 1 + \frac{d(y)}{a(y)} \right)} = \frac{y + \frac{n(y)}{(1 - \eta(y))} (1 - \eta(y))}{(y + 1) \left( 1 + \frac{d(y)}{a(y)} \right)}
\]
so that (51) is satisfied if
\[
\limsup_{a(y) \to +\infty} \frac{y + \frac{n(y)}{(1 - \eta(y))}}{(y + 1)} < \gamma^{-1}.
\]

This ends the proof since
\[
\limsup_{y \to 0} \frac{y + \frac{n(y)}{(1 - \eta(y))}}{(y + 1)} = \limsup_{y \to 0} \frac{n(y)}{(1 - \eta(y))}
\]
and
\[
\limsup_{y \to +\infty} \frac{y + \frac{n(y)}{(1 - \eta(y))}}{(y + 1)} = 1 + \limsup_{y \to +\infty} \frac{n(y)}{y (1 - \eta(y))}.
\]

**Remark 40** As noted in the Introduction, for homogeneous fragmentation kernels (9), the above conditions are satisfied if $\int_0^1 h(z)dz < 1$. Let us give more general examples.

**Proposition 41** Let the fragmentation kernel be given by (11). Let $a(.)$ be unbounded at zero and at infinity only. Then (52) holds if $\zeta_\infty < 1$ and
\[
\limsup_{y \to 0} \hat{n}(y) < \frac{\zeta^-}{(\zeta_\infty^+)^2} \quad \text{and} \quad \limsup_{y \to +\infty} \frac{\hat{n}(y)}{y} < \frac{(1 - \zeta_\infty^+) \zeta^-}{(\zeta_\infty^+)^2},
\]
where $\hat{n}(y) = \int_y^1 \hat{n}(x, y)dx$ and $\zeta^-$, $\zeta_\infty^+$ are given by (30).
Proof. Note that
\[ 1 - \eta(y) = \frac{1}{y} \int_0^y x \zeta(x, y) \hat{b}(x, y) \, dx \]
so
\[ 1 - \lim_{a(y) \to +\infty} \inf \eta(y) \leq \lim_{a(y) \to +\infty} \sup \frac{1}{y} \int_0^y x \zeta(x, y) \hat{b}(x, y) \, dx \leq \zeta_+^+ \]
and
\[ 1 - \lim_{a(y) \to +\infty} \sup \eta(y) \geq \lim_{a(y) \to +\infty} \inf \frac{1}{y} \int_0^y x \zeta(x, y) \hat{b}(x, y) \, dx \geq \zeta_-^- \]
Since
\[ \left( 1 - \lim_{a(y) \to +\infty} \inf \eta(y) \right)^{-1} \geq \left( \zeta_+^+ \right)^{-1} \]
and
\begin{align*}
\max \left\{ \lim_{y \to 0} \sup \frac{n(y)}{1 - \eta(y)}, \ 1 + \lim_{y \to +\infty} \sup \frac{n(y)}{y (1 - \eta(y))} \right\} \\
\leq \max \left\{ \lim_{y \to 0} \frac{\zeta_+^+ \hat{n}(y)}{\zeta_-^-}, \ 1 + \lim_{y \to +\infty} \frac{\zeta_+^+ \hat{n}(y)}{y \zeta_-^-} \right\}
\end{align*}
then \((52)\) holds if
\[ \max \left\{ \frac{\zeta_+^+}{\zeta_-^-} \lim_{y \to 0} \hat{n}(y), \ 1 + \frac{\zeta_+^+}{\zeta_-^-} \lim_{y \to +\infty} \frac{\hat{n}(y)}{y} \right\} < \frac{1}{\zeta_+^+} \]
which is equivalent to \(\zeta_+^+ < 1\) and
\[ \lim_{y \to 0} \hat{n}(y) < \frac{\zeta_-^-}{(\zeta_+^+)^2} \quad \text{and} \quad \lim_{y \to +\infty} \frac{\hat{n}(y)}{y} < \frac{(1 - \zeta_+^+ \zeta_-^-)}{(\zeta_+^+)^2} \]

2.2.6 Compactness results in \(X_{0,1}\)

By replacing the natural assumption \((48)\) by a stronger one, we can show:

**Theorem 42** Let \((35),(44)\) be satisfied. If \(C := \sup_{z>0} \frac{r(z)}{1+z} < +\infty\) then \(T\) is resolvent compact on \(X_{0,1}\).
Proof. Let \( \lambda > C \) and \( f \) be in the unit ball of \( X_{0,1} \), i.e.
\[
\int_0^{+\infty} |f(x)| (1 + x) dx \leq 1.
\]
According to Lemma 35
\[
\int_0^{+\infty} |((\lambda - T)^{-1} f)(x)| \beta(x) (1 + x) dx \leq 1.
\]
Let \( c > 0 \) and \( \varepsilon > 0 \) be arbitrary. We have
\[
1 \geq \int_{\varepsilon}^{+\infty} \left| ((\lambda - T)^{-1} f)(x) \right| \beta(x) (1 + x) dx =
\int_{\varepsilon}^{+\infty} 1_{\{\beta < c\}} \left| ((\lambda - T)^{-1} f)(x) \right| \beta(x) (1 + x) dx
+ \int_{\varepsilon}^{+\infty} 1_{\{\beta \geq c\}} \left| ((\lambda - T)^{-1} f)(x) \right| \beta(x) (1 + x) dx
\]
so
\[
\sup_{\|f\|_{X_{0,1}} \leq 1} \int_{\varepsilon}^{+\infty} 1_{\{\beta < c\}} \left| ((\lambda - T)^{-1} f)(x) \right| (1 + x) dx \leq \frac{1}{c}.
\]
On the other hand, according to Lemma 30
\[
|((\lambda - T)^{-1} f)| \leq |((\lambda - T_0)^{-1} f)| \leq \frac{1}{(1 + x) r(x)} \|f\|_{X_{0,1}}
\]
so
\[
\int_{\varepsilon}^{+\infty} 1_{\{\beta < c\}} \left| ((\lambda - T)^{-1} f)(x) \right| \beta(x) (1 + x) dx \leq c \int_{\varepsilon}^{+\infty} 1_{\{\beta < c\}} \frac{1}{r(x)} dx
\]
and then
\[
\sup_{\|f\|_{X_{0,1}} \leq 1} \int_{\varepsilon}^{+\infty} \left| ((\lambda - T)^{-1} f)(x) \right| (1 + x) dx \leq \frac{1}{c} + c \int_{\varepsilon}^{+\infty} 1_{\{\beta < c\}} \frac{1}{r(x)} dx
\]
can be made arbitrarily small by choosing first \( c \) large enough and then \( \varepsilon \) small enough.

Similarly, we have
\[
1 \geq \int_0^{\varepsilon} \left| ((\lambda - T)^{-1} f)(x) \right| \beta(x) (1 + x) dx
= \int_0^{\varepsilon} 1_{\{\beta < c\}} \left| ((\lambda - T)^{-1} f)(x) \right| \beta(x) (1 + x) dx
+ \int_0^{\varepsilon} 1_{\{\beta \geq c\}} \left| ((\lambda - T)^{-1} f)(x) \right| \beta(x) (1 + x) dx
\]
so
\[
\sup_{\|f\|_{X_0,1} \leq 1} \int_0^\varepsilon 1_{\{\beta \geq \varepsilon\}} \left| \left( (\lambda - T)^{-1} f \right)(x) \right| (1 + x) \, dx \leq \frac{1}{c}.
\]
As previously
\[
\int_0^\varepsilon 1_{\{\beta < \varepsilon\}} \left| \left( (\lambda - T)^{-1} f \right)(x) \right| \beta(x) (1 + x) \, dx \leq c \int_0^\varepsilon 1_{\{\beta < \varepsilon\}} \frac{r(x)}{r(x)} \, dx
\]
so
\[
\sup_{\|f\|_{X_0,1} \leq 1} \int_0^\varepsilon \left| \left( (\lambda - T)^{-1} f \right)(x) \right| (1 + x) \, dx \leq \frac{1}{c} + c \int_0^\varepsilon 1_{\{\beta < \varepsilon\}} \frac{r(x)}{r(x)} \, dx
\]
can be made arbitrarily small by choosing first \( c \) large enough and then \( \varepsilon \) small enough.

On \( (\varepsilon, \varepsilon^{-1}) \) we have the uniform domination
\[
\left| (\lambda - T)^{-1} f \right| \leq \left| (\lambda - T_0)^{-1} f \right| \leq \frac{1_{(\varepsilon, \varepsilon^{-1})}(x)}{(1 + x) r(x)} \in X_{0,1} \quad (\|f\|_{X_{0,1}} \leq 1)
\]
so the restriction of the set \( \left\{ \left| (\lambda - T)^{-1} f \right|, \|f\|_{X_{0,1}} \leq 1 \right\} \) to the set \( (\varepsilon, \varepsilon^{-1}) \)
is is relatively weakly compact on \( X_{0,1} \). Finally
\[
\left\{ (\lambda - T)^{-1} f ; \|f\|_{X_{0,1}} \leq 1 \right\}
\]
is as close to a relatively weakly compact set as we want and consequently is weakly compact. This shows that \( (\lambda - T)^{-1} \) is weakly compact operator and consequently (see [28] Lemma 14) \( (\lambda - T)^{-1} \) is compact. ■

As in Corollary 21 we have:

**Corollary 43** Let \( [33][51][44] \) be satisfied. If \( C := \sup_{z > 0} \frac{r(z)}{1+z} < +\infty \) then
\[
T + B : D(T) \subset X_{0,1} \to X_{0,1}
\]
is resolvent compact.

**2.2.7 Spectral gap of the full semigroup \( (V(t))_{t \geq 0} \) in \( X_{0,1} \)**
The same arguments as in the proof of Lemma 22 give:
Lemma 44 Let (35),(48) be satisfied. We assume that the support of $a(.)$ is not bounded. Then $(\lambda - T - B)^{-1}$ is positivity improving, i.e. 

$$(\lambda - T - B)^{-1} f > 0 \ a.e.$$ 

for any nontrivial nonnegative $f \in X_{0,1}$, or equivalently $(V(t))_{t \geq 0}$ is irreducible in $X_{0,1}$.

Arguing as in the proof of Theorem 23 we get the main result of Subsection 2.2.

Theorem 45 Let (35),(51),(44) be satisfied. If

$$\sup_{z > 0} r(z) \frac{1}{1+z} < +\infty$$

and the support of $a(.)$ is not bounded then $(V(t))_{t \geq 0}$ has a spectral gap in $X_{0,1}$, i.e.

$$r_{\text{ess}}(V(t)) < r_{\sigma}(V(t)),$$

and satisfies the asynchronous exponential growth.

Remark 46 (Open questions) Following Remarks 31 and 36, we suspect that the different statements of this subsection 2.2 should hold under Assumption (48) instead of $\sup_{z > 0} r(z) \frac{1}{1+z} < +\infty$.

3 The second construction

We consider now the case

$$\int_0^1 \frac{1}{r(\tau)} d\tau < +\infty \ \text{and} \ \int_1^\infty \frac{1}{r(\tau)} d\tau = +\infty. \ \ (53)$$

It turns out that we cannot expect a generation theory in the space $X_1 = L^1(\mathbb{R}_+, xdx)$, see Remark 51 below. So we restrict ourselves to the "finite mass and number of aggregates" space

$$X_{0,1} = L^1(\mathbb{R}_+, (1 + x)dx).$$

We start with:

Proposition 47 Let (53) be satisfied. Then the partial differential equation

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} [r(x)u(x,t)] = 0, \ (x, t > 0)$$

with initial condition $u(x,0) = f(x)$ and boundary condition

$$\lim_{y \to 0} r(y)u(y,t) = 0 \ (t > 0)$$
has a unique solution given by

\[ u(y, t) = \begin{cases} \frac{r(X(y, t))f(X(y, t))}{r(y)} & \text{if } \int_0^y \frac{1}{r(\tau)} d\tau > t \\ = 0 & \text{if } \int_0^y \frac{1}{r(\tau)} d\tau < t \end{cases} \]

where \( X(y, t) \) \((t > 0)\) is defined by

\[ \int_X^y \frac{1}{r(\tau)} d\tau = t, \quad \left( \int_0^y \frac{1}{r(\tau)} d\tau > t \right). \]

**Proof.** We solve

\[ \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} [r(x)u(x, t)] = 0 \]

with initial data

\[ u(x, 0) = f(x) \]

and boundary condition

\[ \lim_{x \to 0} r(x)u(x, t) = 0 \quad (t > 0) \]

by the method of characteristics. This amounts to solving

\[ \frac{1}{r(x)} \frac{\partial}{\partial t} \varphi(x, t) + \frac{\partial}{\partial x} [\varphi(x, t)] = 0; \quad \varphi(x, 0) = r(x)f(x). \]

We introduce the characteristic equations

\[ \frac{dt}{ds} = \frac{1}{r(x(s))}, \quad \frac{dx}{ds} = 1 \]

with "initial" conditions

\[ x(0) = x, \quad t(0) = 0 \quad (x > 0) \]

i.e. \( x(s) = s + x \) and

\[ t(s) = \int_0^s \frac{1}{r(\tau + x)} d\tau = \int_x^{s+x} \frac{1}{r(\tau)} d\tau \quad (s \geq 0). \]

Thus

\[ [0, +\infty) \ni s \to r(s + x)u(s + x, \int_x^{s+x} \frac{1}{r(\tau)} d\tau) \text{ is constant} \]
and then
\[ r(s + x)u(s + x, \int_x^{s+x} \frac{1}{r(\tau)} d\tau) = r(x)u(x, 0) = r(x)f(x) \quad \forall s \geq 0. \]

For \( t > 0 \) and \( y > 0 \) given, we set
\[ \int_x^{s+x} \frac{1}{r(\tau)} d\tau = t, \quad s + x = y \]
i.e. \( \int_x^y \frac{1}{r(\tau)} d\tau = t \). Since \( \int_0^y \frac{1}{r(\tau)} d\tau < +\infty \) let \( y_0(t) > 0 \) be defined by
\[ \int_0^{y_0(t)} \frac{1}{r(\tau)} d\tau = t. \quad (54) \]

Hence there exists a unique \( X(y, t) < y \) such that
\[ \int_y^{y_0(t)} \frac{1}{r(\tau)} d\tau = t, \quad (y > y_0(t)). \quad (55) \]

We denote by \( X(y, \cdot) \) the continuous function which gives \( x \in (0, y) \) from \( t \) (given \( y > 0 \)). Thus, for \( y > y_0(t) \)
\[ r(y)u(y, t) = r(X(y, t))f(X(y, t)) \]
i.e.
\[ u(y, t) = \frac{r(X(y, t))f(X(y, t))}{r(y)} \quad (y > y_0(t)). \]

On the other hand, for \( y < y_0(t) \)
\[ \int_0^y \frac{1}{r(\tau)} d\tau < t. \]

We introduce the characteristic equations
\[ \frac{dt}{ds} = \frac{1}{r(x(s))}, \quad \frac{dx}{ds} = 1 \]
with "initial" conditions
\[ x(0) = 0, \quad t(0) = \bar{t} > 0 \]
i.e. \( x(s) = s \) and
\[ t(s) = \bar{t} + \int_0^s \frac{1}{r(\tau)} d\tau. \]
Note the constancy of
\[ r(s)u(s, \bar{t}) + \int_0^s \frac{1}{r(\tau)} d\tau \quad (s \geq 0) \]
amounts to
\[ r(y)u(y, \bar{t}) + \int_0^y \frac{1}{r(\tau)} d\tau \quad (y > 0) \text{ is constant} \]
i.e. (formally)
\[ r(y)u(y, \bar{t}) + \int_0^y \frac{1}{r(\tau)} d\tau = r(0)u(0, \bar{t}) = 0 \]
Thus
\[ r(y)u(y, \bar{t}) + \int_0^y \frac{1}{r(\tau)} d\tau = 0 \quad \forall y > 0, \; \bar{t} > 0 \]
Thus for any \( t > 0 \) and \( y > 0 \) such that
\[ \int_0^y \frac{1}{r(\tau)} d\tau < t \]
we can choose \( \bar{t} > 0 \) such that
\[ \bar{t} + \int_0^y \frac{1}{r(\tau)} d\tau = t \]
namely
\[ \bar{t} = t - \int_0^y \frac{1}{r(\tau)} d\tau. \]
Finally, \( u(y, t) = 0 \) if \( \int_0^y \frac{1}{r(\tau)} d\tau < t \).  

3.1 Theory in the space \( X_{0,1} \)

As previously, we develop a general theory on well-posedness and spectral analysis.

3.1.1 The first generation result

**Theorem 48** Let \( (53) \) be satisfied. Let \( X(y, t) \) be defined by \( (55) \). Then

\[ U_0(t)f := X \left\{ \int_0^y \frac{1}{r(\tau)} d\tau > t \right\} \left( r(X(y, t))f(X(y, t)) \right) \frac{r(y)}{r(0)} \]
defines a positive $C_0$-semigroup on $X_{0,1}$ if and only if

$$[0, +\infty) \ni t \rightarrow \sup_{x>0} \frac{1 + y(x, t)}{1 + x}$$

is locally bounded where $y(x, t)$ is defined by (59). This occurs if

$$r(z) \leq C(z + 1) \quad (\forall z > 0);$$

in this case

$$\frac{1 + y(x, t)}{x + 1} \leq e^{Ct} \quad (x > 0).$$

**Proof.** Let us check that $U_0(t)$ is a bounded operator on $X_{0,1}$. Note that for $\int_0^y \frac{1}{r(\tau)} d\tau > t$ (i.e. if $y > y_0(t)$) we have

$$\int_y^{y_0(t)} \frac{1}{r(\tau)} d\tau = t$$

which shows that (for $t > 0$ fixed) $X(y, t)$ is strictly increasing in $y$ and tends to 0 as $y \rightarrow y_0(t)$. Note that

$$(y_0(t), +\infty) \ni y \rightarrow X(y, t) \in (0, +\infty)$$

is continuous. By arguing as previously we show that

$$\frac{1}{r(y)} = \frac{1}{r(X(y, t))} \frac{\partial X(y, t)}{\partial y}$$

and

$$(U_0(t)f)(y) = f(X(y, t)) \frac{\partial X(y, t)}{\partial y}; \quad y \in (y_0(t), +\infty).$$

Thus

$$\|U_0(t)f\|_{X_{0,1}} = \int_0^{+\infty} |(U_0(t)f)(y)| (1+y)dy = \int_{y_0(t)}^{+\infty} |f(X(y, t))| \frac{\partial X(y, t)}{\partial y} (1+y)dy$$

and the change of variable $x = X(y, t)$ gives

$$\|U_0(t)f\|_{X_{0,1}} = \int_0^{+\infty} |f(x)| (1 + y(x, t))dx$$

where $y(x, t)$ is the unique $y > x$ such that $x = X(y, t)$ i.e.

$$\int_x^{y(x, t)} \frac{1}{r(\tau)} d\tau = t.$$

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Since
\[ \|U_0(t)f\|_{X_{0,1}} = \int_0^{+\infty} \frac{1 + y(x,t)}{1 + x} |f(x)| (1 + x) \, dx \]
then \( U_0(t) \) is a bounded linear operator in \( X_{0,1} \) if and only if
\[ \sup_{x > 0} \frac{1 + y(x,t)}{1 + x} < +\infty. \]
In such a case
\[ \|U_0(t)\|_{\mathcal{L}(X_{0,1})} = \sup_{x > 0} \frac{1 + y(x,t)}{1 + x} \]
and
\[ [0, +\infty) \ni t \mapsto U_0(t) \in \mathcal{L}(X_{0,1}) \]
is locally bounded if and only if
\[ [0, +\infty) \ni t \mapsto \sup_{x > 0} \frac{1 + y(x,t)}{1 + x} \]
is. It follows (see e.g. [12]) that \( (U_0(t))_{t \geq 0} \) is exponentially bounded. As previously, to show that \( (U_0(t))_{t \geq 0} \) is strongly continuous on \( X_{0,1} \) it suffices to check that
\[ U_0(t)f \to f \text{ in } L^1(\mathbb{R}_+; (1 + x) \, dx) \text{ as } t \to 0 \]
on a dense subspace of \( L^1(\mathbb{R}_+; (1 + x) \, dx) \), e.g. for \( f \) continuous with compact support in \( (0, +\infty) \). Note that for any compact set \([c, c^{-1}]\)
\[ \int_0^y \frac{1}{r(\tau)} d\tau > t \]
for \( t \) small enough uniformly in \( y \in [c, c^{-1}] \) so
\[ (U_0(t)f)(y) = f(X(y,t)) \frac{\partial X(y,t)}{\partial y} \quad \forall y \in [c, c^{-1}] \]
for \( t \) small enough. In particular
\[ \int_{X(y,t)}^y \frac{1}{r(\tau)} d\tau = t \quad \forall y \in [c, c^{-1}] \]
and
\[ U_0(t)f = \frac{r(X(y,t))f(X(y,t))}{r(y)} \quad \forall y \in [c, c^{-1}] . \]
for $t$ small enough. We note that $X(y, t) \to y$ as $t \to 0$ for any $y > 0$ and uniformly in $y \in \left[\frac{c}{2}, 2e^{-1}\right]$. Hence

$$U_0(t)f = \frac{r(X(y, t))f(X(y, t))}{r(y)} \to f(y) \quad (t \to 0)$$

and, by the dominated convergence theorem, $U_0(t)f \to f$ in $L^1(\mathbb{R}_+; (1 + x) \, dx)$ as $t \to 0$. It follows from (59) that

$$\frac{\partial y(x, t)}{\partial t} = r(y(x, t)) \quad \forall t > 0.$$

so, using (57),

$$y(x, t) = x + \int_0^t r(y(x, s)) \, ds \leq x + \int_0^t C (y(x, s) + 1) \, ds$$

and

$$y(x, t) + 1 \leq x + 1 + \int_0^t C (y(x, s) + 1) \, ds.$$

Gronwall’s lemma gives $y(x, t) + 1 \leq (x + 1) e^{Ct}$. Finally $\frac{1+y(x,t)}{x+1} \leq e^{Ct}$ $(x > 0)$ and $\|U_0(t)\|_{L(X_0, 1)} \leq e^{Ct}$. □

**Remark 49** The proof above shows that $(U_0(t))_{t \geq 0}$ is not contractive in $X_{0,1}$.

**Remark 50** As in Remark 26 on checks that if $\lim_{x \to +\infty} \frac{r(x)}{x} = +\infty$ then the generation theory fails. Hence the sufficient condition (57) is partly necessary. The necessity of boundedness of the growth rate at the origin is unclear.

**Remark 51** Note that $\|U_0(t)f\|_{X_1} = \int_0^{+\infty} \frac{y(x,t)}{x} |f(x)| \, dx$ so that the boundedness of $U_0(t)$ on $X_1$ amounts to

$$\sup_{x > 0} \frac{y(x,t)}{x} < +\infty. \quad (60)$$

But (54) and (59) imply that $\lim_{x \to 0} y(x, t) = y_0(t) > 0$ $(t > 0)$ so that (60) is violated and we cannot expect a generation theory in $X_1$ under assumption (59).
3.1.2 On the generator of \((U_0(t))_{t \geq 0}\)

We identify now the resolvent of the generator.

**Proposition 52** Let \([53],[57]\) be satisfied. Let \(T_0\) be the generator of \((U_0(t))_{t \geq 0}\).

Then

\[
((\lambda - T_0)^{-1}f)(y) = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda}{r(\tau)} d\tau} f(x) dx, \quad \text{Re } \lambda > s(T_0)
\]

where \(s(T_0)\) is the spectral bound of \(T_0\)

**Proof.** We know that

\[U_0(t)f = X \left\{ \int_0^y \frac{1}{r(\tau)} d\tau > t \right\} \frac{r(X(y,t)) f(X(y,t))}{r(y)}\]

so

\[
((\lambda - T_0)^{-1}f)(y) = \int_0^{+\infty} e^{-\lambda t} \chi \left\{ \int_0^y \frac{1}{r(\tau)} d\tau < t \right\} \frac{r(X(y,t)) f(X(y,t))}{r(y)} dt
\]

\[= \int_0^y \frac{1}{r(\tau)} e^{-\lambda t} \frac{r(X(y,t)) f(X(y,t))}{r(y)} dt\]

Note that for any fixed \(y > 0\) and \(t < \int_0^y \frac{1}{r(\tau)} d\tau\) we have \(\int_0^y \frac{1}{r(\tau)} d\tau = t\) and

\[- \frac{1}{r(X(y,t))} \frac{\partial X(y,t)}{\partial t} = 1.\]

One sees that

\[t \in (0, \int_0^y \frac{1}{r(\tau)} d\tau) \rightarrow x := X(y,t)\]

is strictly decreasing from \(y\) to 0 so the change of variable \(t \rightarrow x = X(y,t)\) gives

\[\int_0^y \frac{1}{r(\tau)} e^{-\lambda t} \frac{r(X(y,t)) f(X(y,t))}{r(y)} dt = \int_0^y e^{-\lambda X^{-1}(y,x)} \frac{f(x)}{r(y)} dx\]

where \(X^{-1}(y,x)\) is the inverse of \(t \rightarrow x = X(y,t)\). This inverse is nothing but \(x \rightarrow t = \int_x^y \frac{1}{r(\tau)} d\tau\) so

\[
((\lambda - T_0)^{-1}f)(y)) = \frac{1}{r(y)} \int_0^y e^{-\lambda \int_x^y \frac{1}{r(\tau)} d\tau} f(x) dx
\]

and this ends the proof. ■
Remark 53  Note that $X_{0,1} \subset L^1(\mathbb{R}_+; \, dx)$. We can check that
\[ D(T_0) = \left\{ f \in X_{0,1}; \quad \frac{\partial (rf)}{\partial y} \in X_{0,1}, \lim_{y \to 0} r(y)f(y) = 0 \right\}, \quad T_0 = -\frac{\partial (rf)}{\partial y} \]
where $\frac{\partial (rf)}{\partial y}$ is the derivative (in the sense of distributions on $(0, +\infty)$) of the function $rf \in L^1(\mathbb{R}_+; \, dx)$. Note that $rf \in W^{1,1}(\mathbb{R}_+)$ so that $\lim_{y \to 0} r(y)f(y)$ exists.

The same proof as in Lemma 30 gives a pointwise estimate in $X_{0,1}$.

Lemma 54  Let (53)–(57) be satisfied. Let $\lambda > C$. Then
\[ \|(\lambda - T_0)^{-1}f\|_y \leq \frac{1}{(1 + y) r(y)} \|f\|_{X_{0,1}} \quad (f \in X_{0,1}). \]

3.1.3  The first perturbed semigroup

We build now a second explicit $C_0$-semigroup by the method of characteristics. We solve
\[ \frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} \left[ r(x)u(x,t) \right] + \beta(x)u(x,t) = 0 \]
with boundary condition $\lim_{x \to 0} r(x)u(x,t) = 0$ and initial data $u(x,0) = f(x)$. By arguing as in subsection 2.2 we show that the solution is given by
\[ U(t)f = \chi_{\{ \int_0^y \frac{1}{r(p)} dp \geq t \}} e^{-\int_0^y \frac{\partial (rf)}{\partial y} dp \int X(y,t) f(X(y,t)) r(y)} = e^{-\int_x^y \frac{\partial (rf)}{\partial y} dp U_0(t)f} \]
where $U(t)$ is given by (36) and defines a $C_0$-semigroup $(U(t))_{t \geq 0}$ on $X_{0,1}$ while the resolvent of its generator is given by:

Proposition 55  Let (53)–(57) be satisfied. The resolvent of the generator $T$ of $(U(t))_{t \geq 0}$ in $X_{0,1}$ is given by
\[ (\lambda - T)^{-1}f = \frac{1}{r(y)} \int_0^y e^{-\int_x^y \frac{\lambda + \beta(p)}{r(p)} dp} f(x) dx. \]

Remark 56  We can "compute" the spectral bound of $T_0$ and $T$ in $X_{0,1}$ as in Proposition 29 and Proposition 34.

The same proof as in Lemma 35 gives a smoothing effect in $X_{0,1}$.
Lemma 57 Let (53)-(57) be satisfied. Let $\lambda > C$. Then
\[
\int_0^{+\infty} |((\lambda - T)^{-1} f)(y)| \beta(y)(1 + y) dy \leq \int_0^{+\infty} |(f(y))| (1 + y) dy, \ (f \in X_{0,1}).
\]

Remark 58 One can deduce from Lemma 57 that
\[
D(T) = \{ f \in D(T_0); \beta f \in X_{0,1} \}, \ T f = T_0 f - \beta f.
\]

3.1.4 The second perturbed semigroup

The proof of the following theorem relying on W. Desch’s perturbation theorem is identical to that of Theorem 58.

Theorem 59 Let (53)-(57) be satisfied. We assume that
\[
n(y) := \int_0^y b(x,y) dx
\]
is such that $\sup_{y > 0} \frac{n(y)}{1 + y} < +\infty$. Then the fragmentation operator (14) is $T$-bounded in $X_{0,1}$ and
\[
\lim_{\lambda \to +\infty} \| B(\lambda - T)^{-1} \|_{L(X_{0,1})} \leq \lim_{a(y) \to +\infty} \sup_{y \to +\infty} \frac{[(y - \eta(y)y) + n(y)]}{(1 + y) \left(1 + \frac{d(y)}{a(y)}\right)}.
\]

In particular, if
\[
\lim_{a(y) \to +\infty} \sup_{y \to +\infty} \frac{[(y - \eta(y)y) + n(y)]}{(1 + y) \left(1 + \frac{d(y)}{a(y)}\right)} < 1.
\]

then
\[
T + B : D(T) \subset X_{0,1} \to X_{0,1}
\]
generates a positive semigroup $(V(t))_{t \geq 0}$ in $X_{0,1}$.

Remark 60 We can state results similar to those given in Proposition 41.

3.1.5 Compactness results in $X_{0,1}$

We are ready to show:
Theorem 61 Let \( \{ 53 \}, \{ 57 \} \) be satisfied. Let the sublevel sets of \( \beta \) be thin at infinity in the sense that
\[
\int_1^{+\infty} \frac{1_{\{ \beta < c \}}}{r(x)} \, dx < +\infty \quad (c > 0)
\] (e.g. let \( \lim_{x \to +\infty} \beta(x) = +\infty \)). Then \( T \) is resolvent compact on \( X_{0,1} \).

Proof. Let \( \lambda > C \) and \( f \) be in the unit ball of \( X_{0,1} \), i.e.
\[
\int_0^{+\infty} |f(x)| (1 + x) \, dx \leq 1.
\]
According to Lemma \( \{ 57 \} \)
\[
\int_0^{+\infty} \left| (\lambda - T)^{-1} f \right| \beta(x) (1 + x) \, dx \leq 1.
\]
Let \( c > 0 \) and \( \epsilon > 0 \) be arbitrary. We have
\[
1 \geq \int_{\epsilon - 1}^{+\infty} \left| (\lambda - T)^{-1} f \right| \beta(x) (1 + x) \, dx
\]
\[
= \int_{\epsilon - 1}^{+\infty} 1_{\{ \beta < c \}} \left| (\lambda - T)^{-1} f \right| \beta(x) (1 + x) \, dx
\]
\[
+ \int_{\epsilon - 1}^{+\infty} 1_{\{ \beta \geq c \}} \left| (\lambda - T)^{-1} f \right| \beta(x) (1 + x) \, dx
\]
so
\[
\sup_{\| f \|_{X_{0,1}} \leq 1} \int_{\epsilon - 1}^{+\infty} 1_{\{ \beta \geq c \}} \left| (\lambda - T)^{-1} f \right| (1 + x) \, dx \leq \frac{1}{c}.
\]
On the other hand, according to Lemma \( \{ 54 \} \)
\[
| (\lambda - T)^{-1} f | \leq | (\lambda - T_0)^{-1} f | \leq \frac{1}{(1 + x) r(x)} \| f \|_{X_{0,1}}
\]
so
\[
\int_{\epsilon - 1}^{+\infty} 1_{\{ \beta < c \}} \left| (\lambda - T)^{-1} f \right| \beta(x) (1 + x) \, dx \leq c \int_{\epsilon - 1}^{+\infty} \frac{1_{\{ \beta < c \}}}{r(x)} \, dx
\]
and then
\[
\sup_{\| f \| \leq 1} \int_{\epsilon - 1}^{+\infty} \left| (\lambda - T)^{-1} f \right| (1 + x) \, dx \leq \frac{1}{c} + c \int_{\epsilon - 1}^{+\infty} \frac{1_{\{ \beta < c \}}}{r(x)} \, dx
\]
can be made arbitrarily small by choosing first $c$ large enough and then $\varepsilon$ small enough.

On the other hand on $(0, \varepsilon^{-1})$ we have the uniform domination

$$\|(\lambda - T)^{-1}f\| \leq \frac{1_{(0,\varepsilon^{-1})}(x)}{(1 + x) r(x)} \in X_{0,1} \quad (\|f\|_{X_{0,1}} \leq 1)$$

because

$$\int_0^{\varepsilon^{-1}} \frac{1}{r(x)} \, dx < +\infty.$$ 

Finally $\{(\lambda - T)^{-1}f; \|f\|_{X_{0,1}} \leq 1\}$ is as close to a relatively weakly compact set as we want and consequently is weakly compact so $(\lambda - T)^{-1}$ is weakly compact operator and consequently (see [28] Lemma 14) $(\lambda - T)^{-1}$ is compact. \(\blacksquare\)

As in Corollary 43 we have:

**Corollary 62** Let (53)(57)(61)(62) be satisfied. Then $T+B : D(T) \to X_{0,1}$ is resolvent compact.

### 3.1.6 Spectral gap of the full semigroup $(V(t))_{t \geq 0}$ in $X_{0,1}$

The same arguments as in Lemma 22 give:

**Lemma 63** Let (53)(57)(61) be satisfied. We assume that the support of $a(.)$ is not bounded. Then $(\lambda - T - B)^{-1}$ is positivity improving, i.e.

$$(\lambda - T - B)^{-1}f > 0 \text{ a.e.}$$

for any nontrivial nonnegative $f \in X_{0,1}$, or equivalently $(V(t))_{t \geq 0}$ is irreducible in $X_{0,1}$.

The same arguments as in Theorem 23 give the main result of Section 3

**Theorem 64** Let (53)(57)(61)(62) be satisfied. We assume that the support of $a(.)$ is not bounded. Then $(V(t))_{t \geq 0}$ has a spectral gap in $X_{0,1}$, i.e.

$$r_{ess}(V(t)) < r_{\sigma}(V(t)),$$

and satisfies the asynchronous exponential growth.
4 Theory in the space $X_0$

This last section is devoted to growth-fragmentation equations in the "finite agregates number" space

$$ X_0 := L^1 (\mathbb{R}_+, dx) $$

under (53). For simplicity, we restrict ourselves to the case

$$ d(\cdot) = 0. $$

By resuming the proof of Theorem 48 (the sublinearity condition (57) is no longer necessary here) one sees that

$$(U_0(t)f)(y) = \begin{cases} f(X(y, t)) \frac{\partial X(y, t)}{\partial y}; & y \in (y_0(t), +\infty) \\ 0 & (y < y_0(t)) \end{cases}$$

so

$$ ||U_0(t)f||_{X_0} = \int_0^{+\infty} |(U_0(t)f)(y)| dy = \int_{y_0(t)}^{+\infty} |f(X(y, t))| \frac{\partial X(y, t)}{\partial y} dy $$

and the change of variable $x = X(y, t)$ gives

$$ ||U_0(t)f||_{X_0} = \int_0^{+\infty} |f(x)| dx = ||f||_{X_0}. $$

Hence we have:

**Theorem 65** Let (53) be satisfied. Let $X(y, t)$ be defined by (55). Then

$$ U_0(t)f := \chi\left\{ \int_0^y \frac{r(X(y, t))f(X(y, t))}{r(y)} dx > 0 \right\} $$

defines a stochastic $C_0$-semigroup $(U_0(t))_{t \geq 0}$ on $X_0$.

As previously, we show that

$$ U(t)f = \chi\left\{ \int_0^y \frac{r(X(y, t))f(X(y, t))}{r(y)} dx > 0 \right\} e^{-\int_0^y \frac{a(p)}{r(p)} dp r(X(y, t))f(X(y, t))} = e^{-\int_0^y \frac{a(p)}{r(p)} dp U_0(t)f} $$

(X(y, t) is given by (55)) defines a contraction $C_0$-semigroup $(U(t))_{t \geq 0}$ on $X_0$ and the resolvent of its generator $T$ is given by

$$ (\lambda - T)^{-1} f = \frac{1}{r(y)} \int_0^y e^{-\int_y^u \frac{\lambda a(p)}{r(p)} dr} f(x)dx. \quad (63) $$

We have a smoothing effect in $X_0$. 

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Lemma 66 Let (53) be satisfied. Then
\[ \int_0^{+\infty} \left| (\lambda - T)^{-1} f \right| (y) \, a(y) \, dy \leq \int_0^{+\infty} |f(x)| \, dx \quad (f \in X_0), \quad (\lambda > 0). \]

Proof. One sees that for \( \lambda > 0 \)
\[
\int_0^{+\infty} \left| (\lambda - T)^{-1} f \right| (y) \, a(y) \, dy \\
\leq \int_0^{+\infty} \frac{a(y)}{r(y)} \left( \int_0^y e^{-\lambda \int_0^y \frac{1}{r(\tau)} \, d\tau} \, e^{-\int_0^y \frac{a(\tau)}{r(\tau)} \, d\tau} \, |f(x)| \, dx \right) \, dy \\
= \int_0^{+\infty} \left( \int_x^{+\infty} \frac{a(y)}{r(y)} \, e^{-\lambda \int_x^y \frac{1}{r(\tau)} \, d\tau} \, e^{-\int_x^y \frac{a(\tau)}{r(\tau)} \, d\tau} \, dy \right) \, |f(x)| \, dx \\
\leq \int_0^{+\infty} \left( \int_x^{+\infty} \frac{a(y)}{r(y)} \, e^{-\int_x^y \frac{a(\tau)}{r(\tau)} \, d\tau} \, dy \right) \, |f(x)| \, dx \leq \int_0^{+\infty} |f(x)| \, dx
\]
since
\[
\int_x^{+\infty} \frac{a(y)}{r(y)} \, e^{-\int_x^y \frac{a(\tau)}{r(\tau)} \, d\tau} \, dy = - \int_x^{+\infty} \frac{d}{dy} \left( e^{-\int_x^y \frac{a(\tau)}{r(\tau)} \, d\tau} \right) \, dy = - \left[ e^{-\int_x^y \frac{a(\tau)}{r(\tau)} \, d\tau} \right]_{y=x}^{y=+\infty} \leq 1.
\]

A trivial consequence of (63) is the pointwise estimate
\[ \left| (\lambda - T)^{-1} f \right| (y) \leq \frac{1}{r(y)} \int_0^{+\infty} |f(x)| \, dx \quad (f \in X_0), \quad (\lambda > 0). \quad (64) \]

By combining the above smoothing effect and (64) and arguing as in the proof of Theorem 61, we get.

Theorem 67 Let (53) be satisfied. Let the sublevel sets of \( a \) be thin at infinity in the sense that
\[ \int_1^{+\infty} 1 \{a < c\} \frac{1}{r(x)} \, dx < +\infty \quad (c > 0) \]
(e.g. let \( \lim_{x \to +\infty} a(x) = +\infty \)). Then \( T \) is resolvent compact on \( X_0 \).

We give now the full generation result.

Theorem 68 Let (53) be satisfied and let
\[ n(.) := \int_0^y b(x, .) \, dx \in L^\infty(0, +\infty). \quad (66) \]
Then the fragmentation operator \(T\) is \(T\)-bounded in \(X_0\) and

\[
\lim_{\lambda \to +\infty} \|B(\lambda - T)^{-1}\|_{\mathcal{L}(X_0)} \leq \lim_{a(y) \to +\infty} n(y).
\]

In particular, if

\[
\lim_{a(y) \to +\infty} n(y) < 1 \tag{67}
\]

then

\[
T + B : D(T) \subset X_0 \to X_0
\]

generates a positive semigroup \((V(t))_{t \geq 0}\) in \(X_0\).

**Proof.** We note that

\[
\|B\varphi\|_{X_0} \leq \int_0^{+\infty} \left( \int_x^{+\infty} a(y)b(x, y) |\varphi(y)| dy \right) dx
\]

\[
= \int_0^{+\infty} a(y) \left( \int_0^y b(x, y) dx \right) |\varphi(y)| dy
\]

\[
= \int_0^{+\infty} a(y)n(y) |\varphi(y)| dy
\]

and

\[
\|B(\lambda - T)^{-1} f\|_{X_0} = \int_0^{+\infty} a(y)n(y) |(\lambda - T)^{-1} f| (y) dy
\]

\[
= \int_{\{a \leq c\}} a(y)n(y) |(\lambda - T)^{-1} f| (y) dy
\]

\[
+ \int_{\{a > c\}} a(y)n(y) |(\lambda - T)^{-1} f| (y) dy.
\]

Since

\[
\int_{\{a \leq c\}} a(y)n(y) |(\lambda - T)^{-1} f| (y) dy \leq c \|n\|_{L^\infty} \|(\lambda - T)^{-1}\|_{\mathcal{L}(X_0)} \|f\|_{X_0}
\]

and

\[
\int_{\{a > c\}} a(y)n(y) |(\lambda - T)^{-1} f| (y) dy \leq \sup_{\{a > c\}} n(y) \int_0^{+\infty} a(y) |(\lambda - T)^{-1} f| (y) dy
\]

\[
\leq \sup_{\{a > c\}} n(y) \int_0^{+\infty} |f(x)| dx
\]

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(the smoothing effect is used in the last step), then
\[ \|B(\lambda - T)^{-1}\|_{\mathcal{L}(X_0)} \leq c \|n\|_{L^\infty} \|\lambda - T\|_{\mathcal{L}(X_0)} + \sup_{\{a>c\}} n(y) \ (c > 0) \]
and
\[ \lim_{\lambda \to +\infty} \|B(\lambda - T)^{-1}\|_{\mathcal{L}(X_0)} \leq \sup_{\{a>c\}} n(y) \ (\forall c > 0) \).

Finally, W. Desch’s theorem ends the proof.

**Remark 69** Note that \( n(y) \geq 1 \) for conservative fragmentation kernels. This shows the key role of the mass loss assumption.

**Remark 70** For homogeneous kernels (9) with mass loss, (67) amounts to \( \int_0^1 b(z)dz < 1 \). More generally, for a fragmentation kernel given by (11) the condition (67) holds if \( \lim_{\lambda \to +\infty} \sup_{a(y) \to +\infty} \hat{n}(y) < 1 \) where \( \hat{n}(y) = \int_0^y b(x,y)dx \).

As previously, \( T + B \) is resolvent compact and, arguing as in the proof of Theorem 23, we obtain the main result of Section 4.

**Theorem 71** Let (53) (65) (66) (67) be satisfied. We assume that the support of \( a(.) \) is not bounded. Then \( (V(t))_{t \geq 0} \) has a spectral gap in \( X_0 \), i.e.
\[ r_{\text{ess}}(V(t)) < r_\sigma(V(t)), \]
and satisfies the asynchronous exponential growth in \( X_0 \).

**Remark 72** We could build a similar theory in \( X_0 \) under (35). We leave the details to the interested reader.

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