Dependence of vector fields and singular controls

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1 Introduction.

Systems of vector fields on manifolds are important objects in various area, say, in geometry, topology, global analysis and control theory. Let $X_1, X_2, \ldots, X_r$ be a system of $C^\infty$ vector fields over a $C^\infty$ manifold $M$ of dimension $n$, with $r \leq n$. We consider the control system

$$\dot{x} = \sum_{i=1}^{r} u_i X_i(x),$$

which is called a driftless control-affine system, generated by $X = (X_1, X_2, \ldots, X_r)$ with the control parameters $u = (u_1, \ldots, u_r)$ in an open set $\Omega \subset \mathbb{R}^r$.

In general, even generically, the dependence locus

$$\Sigma := \{ x \in M \mid X_1(x), X_2(x), \ldots, X_r(x) \text{ are linearly dependent} \}$$

is need not be void. We show, in this paper, an example providing a significance in geometric control theory of the existence of the dependence locus of the system. We show, in particular, the generic appearance of non-trivial singular trajectories embedded in the dependence locus.

An absolutely continuous curve $x: I \to M$ defined on a closed interval $I$ is called an $X$-trajectory if it is a solution curve of the differential equation $\dot{x} = \sum_{i=1}^{r} u_i(t)X_i(x)$ for some essentially bounded measurable function $u : I \to \Omega$ which is called a control. Note that the control is not unique where $X_1, X_2, \ldots, X_r$ are linearly dependent. A control $u : I \to \Omega$ and an initial point determine the trajectory $x = x(t)$ if it exists. Suppose a control $u : I \to \Omega$ defines an $X$-trajectory $x : I \to M$ on $I$ with an initial point $x_0 = x(a), I = [a, b]$. Then any $\tilde{u} : I \to \Omega$ in a neighborhood $\mathcal{C}$ of $u$, in the $L^\infty$ topology, defines uniquely an $X$-trajectory $\tilde{x} : I \to M$ on the same interval $I$ with the initial point $x_0 = \tilde{x}(a)$, so it defines the endpoint $\tilde{x}(b)$. Thus we have the endpoint mapping $\mathcal{C} \to M$, $u \mapsto \tilde{b}$. Then $u$ is called a singular control if it is a singular point of the endpoint mapping, and the corresponding trajectory is called a singular $X$-trajectory (see [1][3][4][5]).

An $X$-trajectory $x : I \to M$ is called non-trivial (resp. dependent) if for any subinterval $J \subset I, x|_J : J \to M$ is not constant (resp. $x(I) \subset \Sigma$).
Theorem. Let \( M \) be a \( C^\infty \) manifold of dimension 3. Then there exists a non-empty open subset \( \mathcal{U} \) in the set of systems of triple \( C^\infty \) vector fields \( X = (X_1, X_2, X_3) \) over \( M \) with the \( C^\infty \) topology such that, for any \( X \in \mathcal{U} \), there exists a one parameter family of non-trivial singular \( X \)-trajectories which are embedded in the dependent locus \( \Sigma \) of \( X \) and lifted to essentially unique singular controls.

The above Theorem means that there exists an example of a system with the property stated in Theorem such that any small perturbation of the system for \( C^\infty \) topology enjoys the same property. Therefore, in general, it is impossible to exclude generically the appearance of non-trivial singular trajectories embedded in the dependent locus.

The generic properties of driftless control-affine systems are established in [4] when the dependent locus is void. Moreover in [5] the theory was extended to the general case where the dependent locus may not be void. However a theorem (Theorem 2.13 of [5]) used there, which says that any \( X \) trajectory has essentially zero velocity on dependent locus for a generic \( X \) with \( r \leq n \), is incorrect and requires more conditions. We provide an example to the approach to the general theory based on the existence of dependent locus.

Regarding the existence of non-trivial dependent singular trajectories, the second author reformulates a theorem of [5] (Theorem 2.17 of [4]) and gives its proof in [7].

We concern with only small perturbations for \( C^\infty \) topology. However if \( M \) is not parallelizable, i.e. \( TM \) is not trivial, for instance if \( M = S^3 \), any system \( X = (X_1, X_2, X_3) \) on \( M \) has non-void dependence locus \( \Sigma \). Thus the phenomena of our Theorem are unavoidable by any means.

On the removability of dependence locus by large perturbations, see, for instance, the famous topological studies [2][6].

2 Proof of Theorem.

Let \( M \) be a manifold of dimension 3. We consider a generic distribution \( D = \langle X_1, X_2, X_3 \rangle \subset TM \) with singularities. In fact we impose on \( X = (X_1, X_2, X_3) \) the following generic conditions:

1. The dependence locus \( \Sigma \) is a smooth surface. Moreover,
2. Outside of a curve \( \gamma \subset \Sigma \), \( D_x \) is transverse to \( T_x\Sigma \) \( (x \in \Sigma \setminus \gamma) \).

The above conditions are achieved by a transversality of 1-jets of \( X \). Moreover for any \( M \), there exists a system \( X \) satisfying the above transversality condition with \( \Sigma \neq \emptyset \).
Take any $x_0 \in \Sigma \setminus \gamma$. Then there exists a system of local coordinates $x_1, x_2, x_3$ around $x_0$ on an open set $U \subset M$ such that $\Sigma = \{x_1 = 0\}$ and $D$ is generated by

\[
\left\{ \begin{array}{l}
X_1 = \frac{\partial}{\partial x_1} + P \frac{\partial}{\partial x_2} \\
X_2 = \frac{\partial}{\partial x_2} + Q \frac{\partial}{\partial x_3}
\end{array} \right.,
\]

where $P = P(x_2, x_3), Q = Q(x_2, x_3)$. Then we have

\[
D = \{(x_1, x_2, x_3; u_1, u_2, P(x_2, x_3)u_1 + Q(x_2, x_3)u_2 + x_1u_3) \mid (x_1, x_2, x_3) \in U, u_1, u_2, u_3 \in \mathbb{R}\}.
\]

and the dependence locus $\Sigma = \{x_1 = 0\}$. Then we have

\[
D \cap T\Sigma = \{(0, x_2, x_3; 0, u_2, Q(x_2, x_3)u_2)\},
\]

which is a line field on $\Sigma$ defined by $dx_3 - Q(x_2, x_3)dx_2 = 0$.

The Hamiltonian function $H : T^*\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ of the system is given by

\[
H = \langle p, u_1X_1 + u_2X_2 + u_3X_3 \rangle \\
= u_1(p_1 + Pp_3) + u_2(p_2 + Qp_3) + u_3x_1p_3 \\
= u_1p_1 + u_2p_2 + (u_1P + u_2Q + u_3x_1)p_3
\]

The constrained Hamiltonian system is given by

\[
\left\{ \begin{array}{l}
\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = u_1P + u_2Q + u_3x_1, \\
\dot{p}_1 = -u_3p_3, \quad \dot{p}_2 = -(u_1P_x + u_2Q_x)p_3, \quad \dot{p}_3 = -(u_1P_x + u_2Q_x)p_3, \\
p_1 + Pp_3 = 0, \quad p_2 + Qp_3 = 0, \quad x_1p_3 = 0, \quad p = (p_1, p_2, p_3) \neq 0.
\end{array} \right.
\]

If $x_1 \neq 0$, then $p = 0$. Therefore we suppose $x_1 = 0$. Then $u_1 = 0$. Then the system is reduced to the system on $\Sigma$:

\[
\left\{ \begin{array}{l}
\dot{x}_2 = u_2, \quad \dot{x}_3 = u_2Q, \\
\dot{p}_2 = -u_2Q_xp_3, \quad \dot{p}_3 = -u_2Q_xp_3, \\
p_2 = -Qp_3, \quad p_3 \neq 0,
\end{array} \right.
\]

with additional conditions $p_1 = -Pp_3, u_3 = -\dot{p}_1/p_3, x_1 = 0$.

Take any solution $x_2 = x_2(t), x_3 = x_3(t)$ of the equation $dx_3 - Q(x_2, x_3)dx_2 = 0$. Then we have

\[
p_3(t) = a \exp \int (-\dot{x}_2(t)Q_x(x_2(t), x_3(t))) \, dt, \quad a \neq 0,
\]

and therefore $\dot{p}_2(t)$ and $p_1(t) = -P(x_2(t), x_3(t))p_3(t)$ are determined. Then we have

\[
(p_2 + Qp_3) = \dot{p}_2 + Q_x\dot{x}_2p_3 + Q_{xx}\dot{x}_3p_3 + Qp_3 = 0.
\]

If we choose any initial value of $p_2$, then we have $p_2 = -Qp_3$. Therefore, for any solution curve $(x_2(t), x_3(t))$ of the equation $dx_3 - Q(x_2(t), x_3(t))dx_2 = 0$, the curve $(0, x_2(t), x_3(t))$ on the dependence locus $\Sigma$ is a singular trajectory of the control system. Moreover the corresponding singular control is unique to it. 

\[\square\]
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