A NECESSARY AND SUFFICIENT CONDITION FOR A
SELF-DIFFEOMORPHISM OF A SMOOTH MANIFOLD TO BE
THE TIME-1 MAP OF THE FLOW OF A DIFFERENTIAL
EQUATION

JEFFREY J. ROLLAND

ABSTRACT. In topological dynamics, one considers a topological space $X$ and a
self-map $f : X \to X$ of $X$ and studies the self-map’s properties. In global analysis
and geometric mechanics, one considers a smooth manifold $M^n$ and a differential
equation $\xi : M \to TM$ on $M$ and studies the flow $\Phi_t : M \times \mathbb{R} \to M$ of the
differential equation. In this paper, we consider a necessary and sufficient condition
for a self-diffeomorphism $f$ of a manifold $M$ to be the time-1 map $\Phi_1$ of the flow of
a differential equation on $M$.

1. Introduction and Main Result

Let $M^n$ be a smooth manifold, let $h : M \to \mathbb{R}^n$ be a smooth map, and let $p \in M$
be an isolated zero of $h$. Then we might attempt to use Newton’s method to
find $p$, that is, we might choose $x_0$ in a neighborhood of $p$, find a solution $v_i$ of
$h(x_i) + Dh(x_i)(v) = 0$, then set $x_{i+1} = \exp(v_i)$ and hope $\lim_{i \to \infty} (x_i) = p$. (A modified
version of this might be used to find an isolated critical point of a differential equation
on a Riemannian manifold $M$.) If $Dh(p) : T_p(M) \to \mathbb{R}^n$ is nonsingular for all $p \in M$,
define $f : M \to M$ by $f(p) = \exp\{Dh(p)^{-1}[-h(p)]\}$. Then $f$ is a self-map of $M$
of the kind that might be considered in smooth dynamics. One wants to know the
behavior of the sequences $(x_i) = f^i(x_0)$ for various choices of $x_0$, in particular, when
they converge to such an isolated zero $p$, when they tend towards a “cyclic sequence”
of points $(p_1, p_2, \ldots, p_k)$, and when and how quickly they diverge to infinity. Various
sets, such as the Fatou set and the Julia set of $f$, can then be defined.

Let $M^n$ be a smooth manifold, let $\mathfrak{X}(M) = \{\zeta : M \to TM \mid \pi_M \circ \zeta = \text{id}_M\}$ be
the set of all the differential equations (also known as a tangent vector fields) on $M$, and let $\xi \in \mathfrak{X}(M)$. (We are most interested in the case that $M = TQ$ for $Q$
a closed, Riemannian manifold – the configuration space [or “c-space”] of a robot
arm, for instance; see [4] – and $\xi$ is the vector field corresponding to an “elementary”
Lagrangian on $TQ$ [elementary with respect to a specific coordinate patch on $Q$]; see,
for instance, [3]). Then, if $\xi$ is complete, $\xi$ has a flow, $\Phi_t : M \times \mathbb{R} \to M$; see, for
instance, \([1]\). Note 1) that \(\xi\) is always complete if \(M\) is closed, and 2) \(\Phi_1 = f\) is a self-diffeomorphism of \(M\).

We are interested in when a self-diffeomorphism \(f\) of a manifold \(M\) is the time-1 map \(\Phi_1\) of the flow \(\Phi_t\) of a differential equation \(\xi\) on \(M\). It is known that the set of such self-diffeomorphisms of a manifold \(M\) form a Baire first category set from \([5]\).

Here is the statement of the main results.

**Theorem 1.1** (A Necessary and Sufficient Condition for a Self-Diffeomorphism of a Manifold to be the Time-1 Map of the Flow of a Differential Equation). Let \(n \in \mathbb{Z}^+, M^n\) be a closed, smooth manifold, and \(f\) be a self-diffeomorphism of \(M\). Then \(f = \Phi_1\) for \(\Phi_t\) the flow of a differential equation on \(M\) if and only if \(f\) is smoothly rootable to the identity. In this case, for a smooth root system to the identity \(\left(g^b\right)\) and corresponding flow \(\Psi_t\), the differential equation \(\xi\) inducing \(\Psi_t\) is given by \(\xi = \frac{\partial}{\partial t}\{\Psi_t\}\big|_{t=0}\).

**Conjecture 1** (Rolland Rigidity). Suppose \(M\) is a locally symmetric space, \(\rho\) is the natural Riemannian metric on \(M\), and \(\nabla\) is the Levi-Civita connection associated to \(\rho\) on \(M\). We suspect that there is a class of self-diffeomorphisms \(f\) of \(M\) with the property that if there are two distinct smooth root systems to the identity \(\left(g^b\right)\) and \(\left(g^b\right)\) inducing distinct vector fields \(\xi_1\) and \(\xi_2\), then for each \(p \in M\), \(p\) and \(f(p)\) are conjugate points of \(\nabla\). For this class of self-diffeomorphisms, there should be a kind of rigidity of \(M\) determined by \(f\), that is, there should be a differential equation \(\xi_b\) on \(M\) (called a base differential equation of \(f\)), positive integers \(k_1\) and \(k_2\), and inversion symmetries \(P_1\) and \(P_2\) of \((M, \rho)\) with \(k_1DP_1 \circ \xi_1 = \xi_2 \circ P_1\), where \(DP_1 : TM \to TM\) is the induced map on the tangent bundle. In particular, if \(\nabla\) is nonpositively curved, \((M, \nabla)\) has no conjugate points, and for any self-diffeomorphism \(f\) of this class, there should be a unique differential equation \(\xi\) inducing a necessarily unique flow \(\Phi_t\) with \(f = \Phi_{t=1}\).

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[https://math.stackexchange.com](https://math.stackexchange.com)

2. Proof of the Main Result

**Definition 1.** Let \(M^n\) be a smooth manifold. Let \(f : M \to M\) be a self-diffeomorphism of \(M\). We say \(f\) is rootable to the identity if \(f\) is isotopic to the identity (note this implies \(f\) is orientation-preserving if \(M\) is orientable) and there is a sequence of self-diffeomorphisms \(\left(g^b\right)\) of \(M\) with

1) each \(g^b\) is isotopic to the identity (note this implies each \(g^b\) is orientation-preserving
if \(M\) is orientable)

2) \(\left(g^b\right)^b = f\) (that is, each \(g^b\) is a "\(b\)th root of \(f\)"")

3) (The commutativity condition) for \(b_1, b_2 \in \mathbb{N}\), \(g_{b_1}^b[g_{b_2}^b(p)] = g_{b_1 + b_2}^b(p) = g_{b_1}^b[g_{b_2}^b(p)]\)
4) (The coherency condition) for \( a \in \mathbb{Z} \), \((g_b)^a = \left( g_{\frac{b}{b^p+\frac{a^p}{b^p}}} \right) ^a \).

5) \( \lim_{b \to \infty} (g_b) = \text{id}_M \).

We call such a sequence \((g_b)\) a root system to the identity. Note one should only need to determine \((g_b)\) on some cofinal subset of the naturals that leads to a dense subset of the rationals, e.g., \( b = 2^c \), leading to the dyadic rationals.

Set \( \Phi_{\frac{b}{b^p}} : M \times \mathbb{Q} \to M \) by \( \Phi_{\frac{b}{b^p}}(p) = (g_b)^a(p) \). We say \( f \) is continuously rootable to the identity if \( \Phi_{\frac{b}{b^p}} \) extends to a continuous function \( \Phi_t : M \times \mathbb{R} \to M \) and call the root system to the identity \((g_b)\) a continuous root system to the identity in this case. Note that if such a continuous extension \( \Phi_t \) exists, it is unique. We say \( f \) is smoothly rootable to the identity if \( \Phi_t \) is smooth and call the root system to the identity \((g_b)\) a smooth root system to the identity in this case.

**Lemma 2.1.** Let \((g_b)\) be a root system to the identity, let \( a_1, a_2 \in \mathbb{Z} \), and let \( b_1, b_2 \in \mathbb{Z}^+ \). Then \( g^{a_2}_{b_2}[g^{a_1}_{b_1}(p)] = (g_{b_1b_2})^{a_1b_2+a_2b_2}(p) \)

**Proof** \( g^{a_2}_{b_2}[g^{a_1}_{b_1}(p)] = g^{a_2b_2}_{b_2}[g^{a_1b_2}_{b_1}(p)] \) by the coherency condition, so \( g^{a_2}_{b_2}[g^{a_1}_{b_1}(p)] = (g_{b_1b_2})^{a_2b_1+a_1b_2}(p) \) by the definition of composition of functions.

**Theorem 1.1** (A Necessary and Sufficient Condition for a Self-Diffeomorphism of a Manifold to be the Time-1 Map of the Flow of a Differential Equation). Let \( n \in \mathbb{Z}^+ \), \( M^n \) be a closed, smooth manifold, and \( f \) be a self-diffeomorphism of \( M \). Then \( f = \Phi_1 \) for \( \Phi_t \) the flow of a differential equation on \( M \) if and only if \( f \) is smoothly rootable to the identity. In this case, for a smooth root system to the identity \((g_b)\) and corresponding flow \( \Psi_t \), the differential equation \( \xi \) inducing \( \Psi_t \) is given by \( \xi = \frac{\partial}{\partial t} \{ \Psi_t \} \big|_{t=0} \).

**Proof** Let \( n \in \mathbb{Z}^+ \), \( M^n \) be a closed, smooth manifold, and \( f \) be a self-diffeomorphism of \( M \).

\((\Rightarrow)\) Suppose \( f = \Phi_1 \) for \( \Phi_t \) the flow of a differential equation \( \xi \) on \( M \). Define \( H : M \times \mathbb{I} \to M \) by \( H(p,t) = \Phi(p,1-t) \). Then \( H(p,0) = \Phi_0(p) = \Phi_1(p) = f(p) \) and \( H(p,1) = \Phi_1(p) = \Phi_0(p) = \text{id}_M(p) \), so \( f \) is isotopic to \( \text{id}_M \). Let \( b \in \mathbb{Z}^+ \) and set \( g_b(p) = \Phi_{\frac{b}{b^p}}(p) \).

1) Define \( H_b : M \times \mathbb{I} \to M \) by \( H_b(p,t) = \Phi(p,\frac{1-t}{b}) \). Then \( H(p,0) = \Phi_{\frac{b}{b^p}}(p) = \Phi_1(p) = g_b(p) \) and \( H(p,1) = \Phi_{\frac{b}{b^p}}(p) = \Phi_0(p) = \text{id}_M(p) \), so each \( g_b \) is isotopic to \( \text{id}_M \).

2) \( (g_b)^b(p) = \Phi_{\frac{b}{b^p}}(p) = \Phi_{\frac{b}{b^p}}(p) = \Phi_1(p) = f(p) \), so each \( g_b \) is a “\( b \)th root of \( f \)”.

3) Let \( b_1, b_2 \in \mathbb{Z}^+ \). Then \( g_{b_1}[g_{b_2}(p)] = \Phi_{\frac{b}{b_2}}[\Phi_{\frac{b_1}{b_2}}(p)] = \Phi_{\frac{b_1+b_2}{b_2}}(p) \) as \( \Phi_t \) is a flow = \( \Phi_{\frac{b_1}{b_1+b_2}}(p) = \Phi_{\frac{b_1}{b_1}}[\Phi_{\frac{b_2}{b_2}}(p)] = g_{b_2}[g_{b_1}(p)] \), so \( (g_b) \) obeys the commutativity condition.
4) For $a \in \mathbb{Z}$, $(g_b)^a(p) = (\Phi_p^a)^a(p) = \Phi^a(p) = \Phi_{\mathbb{C}C P[a,b]}^a(p) = (g_{\mathbb{C}C P[a,b]}^a)^a(p)$ as

$\Phi_t$ is a flow, so $(g_b)$ obeys the coherency condition.

5) $\lim_{b \to \infty} (g_b)(p) = \lim_{b \to \infty} \Phi^a_p(p) = \Phi_0(p) = \text{id}_M$.

Hence, $(g_b)$ is a smooth root system to the identity, and $f$ is smoothly rootable to the identity.

$(\Leftrightarrow)$ Suppose $f$ is smoothly rootable to the identity. Let $(g_b)$ be a smooth root system to the identity for $f$. Define $\Psi^a_p : M \times \mathbb{Q} \to M$ by $\Psi^a_p(p) = (g_b)^a(p)$. Then

$$\left( g_{\mathbb{C}C P[a,b]}^a \right)(p) = \Psi_{\mathbb{C}C P[a,b]}^a(p)$$

by the coherency condition, so $\Psi^a_p$ is well-defined. Moreover, as $(g_b)$ is a smooth root system to the identity, it is a continuous root system to the identity, so $\Psi^a_p$ extends uniquely to a continuous function $\Psi^a_t : M \times \mathbb{R} \to M$. Next, as $(g_b)$ is a smooth root system to the identity, $\Psi^a_t : M \times \mathbb{R} \to M$ is smooth. We must show $\Psi^a_t$ is a flow on $M$.

a) Note $\Psi^a_0 = \Psi^a_p = (g_b)^0 = \text{id}_M$.

b) (i) Let $a_1, a_2 \in \mathbb{Z}$ and $b_1, b_2 \in \mathbb{Z}^+$. Then

$$\Psi^a_{a_2, b_2} \left( \Psi^a_{a_1, b_1} (p) \right) = (g_{a_1}((g_{a_2})(p))) = \Psi^a_{a_1 + a_2, b_1 + b_2} (p)$$

by Lemma 2.1

(ii) It then follows that $\Psi^a_{t_2} \left( \Psi^a_{t_1} (p) \right) = \Psi^a_{t_1 + t_2} (p)$ for $t_1, t_2 \in \mathbb{R}$ by continuity as $(g_b)$ is a continuous root system to the identity.

(iii) The fact that $\Psi^a_t$ is smooth follows from the fact that $\Psi^a_t$ is a smooth root system to the identity.

This shows that $\Psi^a_t$ is a flow and $f = \Psi^a_1$.

A proof of the fact that for a complete, time-independent vector field $\xi$, the differential equation $\xi$ inducing $\Psi^a_t$ is given by $\xi = \frac{\partial}{\partial t} \{ \Psi^a_t \} |_{t=0}$ may be found in, for instance, [2].

3. Examples

Example 1. Let $M = S^1$ and let $f$ be the antipodal map, $f(p) = -p = e^{i\pi}p = e^{-i\pi}p$. Then there are two obvious smooth root systems to the identity, $g_{1, b}(p) = e^{i\xi}p$ and $g_{2, b}(p) = e^{-i\xi}p$. Note that the isometry $P$ of $S^1$ with its standard Riemannian metric defined by $P(p) = \overline{p}$ (complex conjugation, picturing $S^1$ as being the unit circle in $\mathbb{C}$) has $D P \circ \xi_1 = \xi_2 \circ P$, for $\xi_1$ the differential equation determined by $(g_{1, b})$, where $D P : T S^1 \to T S^1$ is the induced map on the tangent bundle.
Example 2. (A generalization of an example from Jason Devito) With $S^3$, thinking of $S^3$ as a Lie group, the antipodal map (left multiplication by -1, $L_{-1}$) has uncountably many square roots: left multiplication by any purely imaginary unit quaternion. Every imaginary quaternion has exactly two quaternion square roots, $u_3$ and $-u_3$ with $(\pm u_3)^2 = q$. Only one of the $\pm u_3$’s at level 3 will have a angle smaller that $q$ with 1, the other one will be $-u_3$ and will have a smaller angle than $q$ with -1. This pattern continues with $u_{c-1}$ has exactly two quaternion square roots, $u_c$ and $-u_c$ with $(\pm u_c)^2 = u_{c-1}$. Only one of the $\pm u_c$’s at level c will have a angle smaller that $u_{c-1}$ with 1, the other one will be $-u_c$ and will have a smaller angle than $-u_{c-1}$ with -1. If $g_2^c = L_{u_c}$, then $(g_2^c)$ is a sequence of $2^c$ th roots of $f = L_{-1}$ defined on a cofinal subset of the naturals with each $(g_2^c)$ a smooth root system to the identity. Hence, we have a case where we have uncountably many different differential equations $\xi_q$ with $\Phi_{q,t=1} = f$. Note that if $q_1$ and $q_2$ are distinct imaginary quaternions with $q_1^2 = q_2^2 = -1$, the isometry $P$ of $S^3$ defined by $P(p) = L_{q_2(q_1^{-1})(p)}$ has $DP \circ \xi_1 = \xi_2 \circ P$, for $\xi_i$ the differential equation determined by $(g_i,b)$, where $DP : TS^3 \rightarrow TS^3$ is the induced map on the tangent bundle.

Conjecture [Rolland Rigidity]. Suppose $M$ is a locally symmetric space, $\rho$ is the natural Riemannian metric on $M$, and $\nabla$ is the Levi-Civita connection associated to $\rho$ on $M$. We suspect that there is a class of self-diffeomorphisms $f$ of $M$ with the property that if there are two distinct smooth root systems to the identity $(g_{1,b})$ and $(g_{2,b})$ inducing distinct vector fields $\xi_1$ and $\xi_2$, then for each $p \in M$, $P$ and $f(p)$ are conjugate points of $\nabla$. For this class of self-diffeomorphisms, there should be a kind of rigidity of $M$ determined by $f$, that is, there should be a differential equation $\xi_b$ on $M$ (called a base differential equation of $f$), positive integers $k_1$ and $k_2$, and inversion symmetries $P_1$ and $P_2$ of $(M,\rho)$ with $k_iDP_i \circ \xi_c = \xi_i \circ P_1$, where $DP_i : TM \rightarrow TM$ is the induced map on the tangent bundle. In particular, if $\nabla$ is nonpositively curved, $(M,\nabla)$ has no conjugate points, and for any self-diffeomorphism $f$ of this class, there should be a unique differential equation $\xi$ inducing a necessarily unique flow $\Phi_t$ with $f = \Phi_{t=1}$.

Note that the set $G_f = \{P_i \mid 1DP_i \circ \xi_b = \xi_i \circ P_i\}$, for $\xi_i$ a differential equation on $M$ inducing $f$ with coefficient 1 for its inversion symmetry conjugating it to the base differential equation $\xi_b$, is a group under composition of inversion symmetries. For any choice of basepoint $*$ of $M$, $G_f$ embeds in $GL[T_*(M)]$. A choice of base differential equation $\xi_b$ from any of the $\xi_i$’s with $1DP_i \circ \xi_b = \xi_i \circ P_i$ for $P_i \in G_f$ used to define $G_f$ should be thought of as a choice similar to the choice of basepoint in defining the fundamental group.

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Unaffiliated

Email address: rollandj@uwm.edu