ON A NEW GENERATING FUNCTIONS FOR THE FOX-WRIGHT FUNCTIONS
AND THEIRS APPLICATIONS

KHALED MEHREZ

Abstract. The main focus of the present paper is to investigate several generating functions for a certain classes of functions associated to the Fox-Wright functions. In particular, certain generating functions for a class of function involving the Fox-Wright functions will be expressed in terms of the H-function of two variables are investigated. As applications, some generating functions associated to the generalized Mathieu type power series and the extended Hurwitz-Lerch zeta function are established. Furthermore, some new double series identity are considered. A conjecture about the finite Laplace transform of a class of function associated to the Fox’s H-function is made.

1. Introduction

Throughout the present investigation, we use the following standard notations:

\[ \mathbb{N} := \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} \]

and

\[ \mathbb{Z}^{-} := \{-1, -2, -3, \ldots\}. \]

Also, as usual, \( \mathbb{Z} \) denotes the set of integers, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^+ \) denotes the set of positive numbers and \( \mathbb{C} \) denotes the set of complex numbers.

Here, and in what follows, we use \( p\Psi_q[[ \cdot ]] \) to denote the Fox-Wright generalization of the familiar hypergeometric \( pF_q \) function with \( p \) numerator and \( q \) denominator parameters, defined by [20, p. 4, Eq. (2.4)]

\[
p\Psi_q[(a_1, A_1), \ldots, (a_p, A_p), (b_1, B_1), \ldots, (b_q, B_q)] = \sum_{k=0}^{\infty} \prod_{l=1}^{p} \Gamma(a_l + kA_l) \prod_{l=1}^{q} \Gamma(b_l + kB_l) \frac{z^k}{k!}
\]

where,

\[ (A_l \geq 0, \ l = 1, \ldots, p; \ B_l \geq 0, \ l = 1, \ldots, q). \]

The convergence conditions and convergence radius of the series at the right-hand side of \((1.1)\) immediately follow from the known asymptotics of the Euler Gamma–function. The defining series in \((1.1)\) converges in the whole complex \( z \)-plane when

\[
\Delta = \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1.
\]

If \( \Delta = -1 \), then the series in \((1.1)\) converges for \( |z| < \rho \), and \( |z| = \rho \) under the condition \( \Re(\mu) > \frac{1}{2} \), (see [6] for details), where

\[
\rho = \left( \prod_{i=1}^{p} A_i^{-A_i} \right) \left( \prod_{j=1}^{q} B_j^{B_j} \right), \quad \mu = \sum_{j=1}^{q} b_j - \sum_{k=1}^{p} a_k + \frac{p-q}{2}
\]

Throughout this paper, we denote by

\[
p_{+1}\Psi_q[[\sigma, (a_1, A_1), (b_1, B_1), \ldots, (b_q, B_q)], \frac{1}{\Gamma(\sigma)}] = p_{+1}\Psi_q[[\sigma, (a_1, A_1), (b_1, B_1), \ldots, (b_q, B_q), \frac{1}{\Gamma(\sigma)}].
\]

1.2010 Mathematics Subject Classification. 30C45, 30D15, 33C10.

Key words and phrases. Fox-Wright function, Fox H-function, Hypergeometric function, Hurwitz-Lerch zeta function, Generating functions.
The generalized hypergeometric function $pF_q$ is defined by

\begin{equation}
\label{1.5}
pF_q\left[\begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_q
\end{array} \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{j=1}^q (b_j)_k} \frac{z^k}{k!}
\end{equation}

where, as usual, we make use of the following notation:

\begin{equation}
\label{1.10}
(\tau)_0 = 1, \quad (\tau)_k = (\tau + 1) \cdots (\tau + k - 1) = \frac{\Gamma(\tau + k)}{\Gamma(\tau)}, \quad k \in \mathbb{N},
\end{equation}

to denote the shifted factorial or the Pochhammer symbol. Obviously, we find from the definitions \[11\] and \[13\] that

\begin{equation}
\label{1.6}
p\Psi_q\left[\begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_q\end{array} \right] \mid z = \frac{\Gamma(a_1) \cdots \Gamma(a_p)}{\Gamma(b_1) \cdots \Gamma(b_q)} pF_q\left[\begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_q\end{array} \right] \mid z,
\end{equation}

Generating functions play an important role in the investigation of various useful properties of the sequences which they generate. They are used to find certain properties and formulas for numbers and polynomials in a wide variety of research subjects, indeed, in modern combinatorics. In this regard, in fact, a remarkable large number of generating functions involving a variety of special functions have been developed by many authors \[1, 2, 22, 3, 18, 19, 20, 21\].

In a recent papers \[8, 9, 10\], Mehrez have studied certain advanced properties of the Fox-Wright function including its new integral representations, the Laplace and Stieltjes transforms, Luke inequalities, Turán type inequalities and completely monotonicity property are derived. In particular, it was shown there that the following Fox-Wright functions are completely monotone:

\begin{align*}
p\Psi_q\left[\begin{array}{c}
\alpha_p, A_p \\
\beta_q, A_q\end{array} \right] - z, \quad s > 0, \\
p+1\Psi_q\left[\begin{array}{c}
\lambda_1, \alpha_p, A_p \\
\beta_q, A_q\end{array} \right] \mid 1/z \mid s > 0,
\end{align*}

and has proved that the Fox’s H-function $H_{p,q}^0[\cdot]$ constitutes the representing measure for the Fox-Wright function $p\Psi_q[\cdot]$, if $\mu > 0$, i.e., \[8, 10\] Theorem 1

\begin{equation}
\label{1.7}
p\Psi_q\left[\begin{array}{c}
\alpha_p, A_p \\
\beta_q, A_q\end{array} \right] \mid z = \int_0^\rho e^{zt} H_{p,q}^0\left( t \mid (B_q, \beta_q) \right) \frac{dt}{t},
\end{equation}

when $\mu > 0$. Here, and in what follows, we use $H_{p,q}^0[\cdot]$ to denote the Fox’s H-function, defined by

\begin{equation}
\label{1.8}
H_{p,q}^0\left( z \mid (B_q, \beta_q) \right) = \frac{1}{2\pi i} \int_{(c)} \prod_{j=1}^p \frac{\Gamma(A_j s + \alpha_j)}{\Gamma(B_k s + \beta_k)} z^{-s} ds,
\end{equation}

where $A_j, B_k > 0$ and $\alpha_j, \beta_k$ are real. The contour $L$ can be either the left loop $L_-$ starting at $-\infty + i\alpha$ and ending at $-\infty + i\beta$ for some $\alpha < 0 < \beta$ such that all poles of the integrand lie inside the loop, or the right loop $L_+$ starting $\infty + i\alpha$ at and ending $\infty + i\beta$ and leaving all poles on the left, or the vertical line $L_c$, $\Re(z) = c$, traversed upward and leaving all poles of the integrand on the left. Denote the rightmost pole of the integrand by $\gamma$:

\[ \gamma = \min_{1 \leq j \leq p} (\alpha_j / A_j). \]

The definition of the H-function is still valid when the $A_i$’s and $B_j$’s are positive rational numbers. Therefore, the H-function contains, as special cases, all of the functions which are expressible in terms of the G-function. More importantly, it contains the Fox-Wright generalized hypergeometric function defined in \[11\], the generalized Mittag-Leffler functions, etc. For example, the function $p\Psi_q[\cdot]$ is one of these special case of H-function. By the definition \[11\] it is easily extended to the complex plane as follows \[7, Eq. 1.31\],

\begin{equation}
\label{1.9}
p\Psi_q\left[\begin{array}{c}
\alpha_p, A_p \\
\beta_q, A_q\end{array} \right] \mid z = H_{p,q+1}^{1,q}\left. -z \right|_{(0,1),(B_q, \beta_q)} \mid (A_p, 1-\alpha_p)\right) .
\end{equation}

The representation \[13\] holds true only for positive values of the parameters $A_1$ and $B_1$.

The special case for which the H-function reduces to the Meijer G-function is when $A_1 = \ldots = A_p = B_1 = \ldots = B_q = A, \quad A > 0$. In this case,

\begin{equation}
\label{1.10}
H_{m,n}^{1,1}\left( z \mid (A_p, 1) \right) = \frac{1}{A} G_{m,n}^{1,1}\left( z^{1/A} \mid A_p \right) .
\end{equation}
In our present investigation, certain generating functions for some classes of function related to the Fox-Wright functions will be evaluated in terms of the H-function of two variables. In order to present the results, we need the definition of the H-function of two complex variables introduced earlier by Mittal and Gupta [11]. The analysis developed here is based on the work of Saxena and Nishimoto [13], Saigo and Saxena [14]. The H-function of two variables is defined in terms of multiple Mellin-Barnes type contour integral as

\[
H\left[ \frac{x}{y} \right] = H^{0,n_1,m_2,n_2,m_3,n_3}_{F_1,F_2,F_3,F_4;F_5,F_6}\left[ \begin{array}{c}
\left( - \right) \\
\frac{x}{y} \end{array} \right] \left( \begin{array}{c}
\left( \alpha_p; A_{p_1}, a_{p_1}, \ldots, \beta_p; B_{p_2}, b_{p_2} \right) \\
\left( \gamma_p; E_{p_3}, e_{p_3} \right)
\end{array} \right)
\]

(1.11)

where \( x \) and \( y \) are not equal to zero. For convenience the parameters \( (\alpha_p; A_{p_1}, a_{p_1}, \ldots, \beta_p; B_{p_2}, b_{p_2}) \) and \( (\gamma_p; E_{p_3}, e_{p_3}) \) will abbreviate the sequence of the parameters \( (\alpha_1; A_1, a_1), \ldots, (\alpha_p; A_{p_1}, a_{p_1}) \) and \( (\gamma_1; E_1, e_1), \ldots, (\gamma_p; E_{p_2}, e_{p_2}) \) respectively, and similar meanings hold for the other parameters \( (\beta_q; B_{q_1}, b_{q_1}) \) and \( (\delta_q; \delta_{q_2}) \), etc. Here

\[
\phi(s,t) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - \alpha_i - \alpha_i s + A_i t)}{\prod_{i=n_1+1}^{n_p} \Gamma(\alpha_i - \alpha_i s - A_i t)} \frac{\prod_{j=n_1+1}^{n_2} \Gamma(1 - \beta_j + \beta_j s + \beta_j t)}{\prod_{i=n_1+1}^{n_p} \Gamma(1 - \beta_j - \beta_j s - \beta_j t)},
\]

(1.12)

\[
\phi_1(s) = \frac{\prod_{j=1}^{m_2} \Gamma(d_j - \delta_j s)}{\prod_{i=1}^{n_1} \Gamma(1 - \epsilon_i + \gamma_i s)} \frac{\prod_{j=n_2+1}^{n_3} \Gamma(1 - d_j + \delta_j s)}{\prod_{i=n_1+1}^{n_2} \Gamma(1 - \epsilon_i - \gamma_i s)},
\]

(1.13)

\[
\phi_2(t) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j t)}{\prod_{i=1}^{n_1} \Gamma(1 - f_j + F_j t)} \frac{\prod_{j=n_2+1}^{n_3} \Gamma(1 - f_j - F_j t)}{\prod_{i=n_1+1}^{n_2} \Gamma(1 - F_j - E_j t)},
\]

(1.14)

where \( \alpha_i, \beta_j, \gamma_i, \delta_j, \epsilon_i \) and \( f_j \) be complex numbers and associated coefficients \( a_i, A_i, b_j, B_j, \gamma_i, \delta_j, E_i \) and \( F_j \) be real and positive for the standardization purposes, such that

\[
\rho_1 = \sum_{i=1}^{n_1} a_i + \sum_{i=1}^{n_2} b_j - \sum_{j=1}^{n_2} \delta_j \leq 0,
\]

(1.15)

\[
\rho_2 = \sum_{i=1}^{n_1} A_i + \sum_{i=1}^{n_2} E_i - \sum_{j=1}^{n_2} B_j - \sum_{j=1}^{n_2} F_j \leq 0,
\]

(1.16)

\[
\Omega_1 = - \sum_{i=n_1+1}^{n_1} a_i - \sum_{j=1}^{n_1} b_j + \sum_{j=1}^{n_2} \delta_j - \sum_{j=n_2+1}^{n_2} \delta_j + \sum_{i=1}^{n_2} \gamma_i - \sum_{i=n_2+1}^{n_2} \gamma_i > 0,
\]

(1.17)

\[
\Omega_2 = - \sum_{i=n_1+1}^{n_1} A_i - \sum_{j=1}^{n_1} B_j + \sum_{j=1}^{n_2} F_j - \sum_{j=n_2+1}^{n_2} F_j + \sum_{i=1}^{n_2} E_i - \sum_{i=n_2+1}^{n_2} E_i > 0.
\]

(1.18)

The contour integral [11,11] converges absolutely under the conditions (1.15)–(1.18) and defines an analytic function of two complex variables \( x \) and \( y \) inside the sectors given by

\[
|\arg(x)| < \frac{\pi}{2} \Omega_1 \quad \text{and} \quad |\arg(y)| < \frac{\pi}{2} \Omega_2,
\]

the points \( x = 0 \) and \( y = 0 \) being tacitly excluded, for details the reader is referred to the book by Srivastava et al. [17].

This paper is organized as follows. In Section 2, we establish generating functions for some classes of functions related to the Fox-Wright function. In particular, certain generating functions for a class of function involving the Fox-Wright functions will be expressed in terms of the H-function of two variables. In Section 3, as applications of the main results in the Section 2, some generating functions associated to the generalized Mathieu type power series and the extended Hurwitz-Lerch zeta function are presented. Furthermore, some new double series identities are derived. In addition, we present an open conjecture, which may be of interest for further research.
2. Generating functions involving some classes of functions involving the Fox-Wright functions

Our aim in this section is to derive some new generating functions for some class of functions related to the Fox-Wright functions. We first recall that a generalized binomial coefficient \( \binom{\lambda}{n} \) may be defined (for real or complex parameters \( \lambda \) and \( \mu \)) by

\[
\binom{\lambda}{n} = \frac{\Gamma(\lambda + 1)}{\Gamma(\mu + 1)\Gamma(\lambda - \mu + 1)},
\]

so that, in the special case when \( \mu = n \in \mathbb{N}_0 \) we have

\[
\binom{\lambda}{n} = \frac{\lambda(\lambda-1)...(\lambda-n+1)}{n!}. \quad (n \in \mathbb{N}_0).
\]

Secondly, recall that the finite Laplace Transform of a continuous (or an almost piecewise continuous) function \( f(t) \) is denoted by

\[
\mathcal{L}_T f(t) = \int_0^T e^{-\xi t} f(\xi) d\xi, \quad t \in \mathbb{R}.
\]

Our first main result reads as follows.

**Theorem 1.** Let \( \lambda > 0 \), then the following generating function

\[
\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} p \Psi_q \left( \binom{\alpha_p+kA_p}{\beta_q+kB_q} \right) t^k \mathcal{L}_P \left( \xi^{-1}(1-t\xi)^{-\lambda} H_{p,q}^{0,0} \left( \xi^{(B_q,B_p)}_{(A_q,A_p)} \right) \right) (-z),
\]

holds true for all \( |t| < 1 \) and \( z \in \mathbb{R} \). Furthermore, suppose that the following conditions

\[
(H_1): \mu > 0, \quad \gamma > 0, \quad \sum_{k=1}^{p} A_k = \sum_{j=1}^{q} B_j, \quad H_{p,q}^{0,0} \geq 0,
\]

are satisfied, then following inequalities

\[
\left( \frac{1}{\Gamma(\sigma)} \right) \cdot p+1 \Psi_q \left( \binom{\sigma,1,\alpha_p}{\beta_q} \right) t^k \leq \sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} p \Psi_q \left( \binom{\alpha_p+kA_p}{\beta_q+kB_q} \right) t^k 
\]

\[
\leq \left( \frac{e^{t\sigma}}{\Gamma(\sigma)} \right) \cdot p+1 \Psi_q \left( \binom{\sigma,1,\alpha_p}{\beta_q} \right) t^k
\]

hold true for all \( |t| < 1 \) and \( z > 0 \).

**Proof.** For convenience, let the left-hand side of (2.20) be denoted by \( \mathcal{I} \). Applying the integral expression [7, Property 1.5] to \( \mathcal{I} \), and we employ the formula [7, Property 1.5]

\[
H_{p,q}^{0,0} \left( \xi^{(B_q,B_p)}_{(A_q,A_p)} \right) = \xi^k H_{p,q}^{0,0} \left( \xi^{(B_q,A_p)}_{(A_q,A_p)} \right), \quad k \in \mathbb{C},
\]

we thus get

\[
\mathcal{I} = \sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} \left[ \int_0^t e^{t\xi} H_{p,q}^{0,0} \left( \xi^{(B_q,b_q+kA_q)}_{(A_q,a_p+kA_p)} \right) \frac{d\xi}{\xi} \right] t^k
\]

\[
= \sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} \left[ \int_0^t \xi^{k-1} e^{t\xi} H_{p,q}^{0,0} \left( \xi^{(B_q,b_q)}_{(A_q,a_p)} \right) d\xi \right] t^k
\]

\[
= \int_0^t e^{t\xi} H_{p,q}^{0,0} \left( \xi^{(B_q,b_q)}_{(A_q,a_p)} \right) \sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} (t\xi)^k \frac{d\xi}{\xi}
\]

Further, upon using the generalized binomial expansion

\[
\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} t^k = (1-t)^{-\lambda}, \quad |t| < 1,
\]

we thus arrive at (2.21) as required.
for evaluating the inner sum in (2.23), we obtain the desired formula (2.24). Now, let us focus on the inequalities (2.21). Since $\xi \in (0, \rho)$ it follows that
\[
\int_{0}^{\rho} H_{q,p}^{0} \left( \frac{(B_{q}, \beta_{q})}{(A_{p}, \alpha_{p})} \right) \frac{d\xi}{\xi(1-t\xi)^{\lambda}} \leq L_{\rho} \left( \frac{(B_{q}, \beta_{q})}{(A_{p}, \alpha_{p})} \right) \left( 1 - \frac{1}{1 - t\xi} \right)^{-\lambda} \left( 1 - \frac{1}{1 - t\xi} \right)^{-\lambda} \leq e^{\rho^{2}} \int_{0}^{\rho} H_{q,p}^{0} \left( \frac{(B_{q}, \beta_{q})}{(A_{p}, \alpha_{p})} \right) \frac{d\xi}{\xi(1-t\xi)^{\lambda}}.
\]
By means of the integral representation [5, Theorem 4]
\[
p_{+1} \Psi \left( (\sigma, 1), (\alpha_{p}, A_{p}) \right) \frac{z}{\lambda} = \Gamma(\sigma) \int_{0}^{\rho} H_{q,p}^{0} \left( \frac{(B_{q}, \beta_{q})}{(A_{p}, \alpha_{p})} \right) \frac{d\xi}{\xi(1-t\xi)^{\lambda}},
\]
where $\sigma > 0$ and $z \in \mathbb{C}$ such that $|z| < 1$, and the conditions $(H_{1})$ are satisfied. Therefore, the inequalities (2.25) transforms into the form
\[
\left( \frac{1}{\Gamma(\sigma)} \right) \cdot p_{+1} \Psi \left( (\sigma, 1), (\alpha_{p}, A_{p}) \right) \left| t \right| \leq L_{\rho} \left( \frac{(B_{q}, \beta_{q})}{(A_{p}, \alpha_{p})} \right) \left( 1 - \frac{1}{1 - t\xi} \right)^{-\lambda} \left( 1 - \frac{1}{1 - t\xi} \right)^{-\lambda} \leq e^{\rho^{2}} \left( \frac{1}{\Gamma(\sigma)} \right) \cdot p_{+1} \Psi \left( (\sigma, 1), (\alpha_{p}, A_{p}) \right) \left| t \right|.
\]
Hence, in view of (2.25) and (2.27) we deduce that the inequalities (2.21) hold true. The proof of Theorem 1 is complete.

**Remark 1.** Mehrez [5] Formula (4.70)] have further shown the following Luke type inequalities for the Fox-Wright function $p_{+1} \Psi_{q}[\cdot]$, that is
\[
\frac{\psi_{0,0}}{1 + \frac{\psi_{0,0}}{\psi_{0,0}} t^{\lambda}} \leq p_{+1} \Psi \left( (\lambda, 1), (\alpha_{p}, A_{p}) \right) - z \leq \left[ \frac{\psi_{0,0}}{1 - \frac{\psi_{0,0}}{\psi_{0,0}} t^{\lambda}} \right]^{\frac{1}{\rho}} \left( 1 - \frac{1}{1 + \rho t^{\lambda}} \right)^{1}, \quad z \in \mathbb{R}, \quad \lambda > 0,
\]
which holds under the conditions $(H_{1})$, where
\[
\psi_{0,0} = \prod_{j=1}^{p} \Gamma(a_{j}), \quad \text{and} \quad \psi_{0,1} = \prod_{j=1}^{p} \Gamma(a_{j} + A_{j}).
\]
In view of (2.27) and (2.28) we get the following inequalities
\[
\frac{\psi_{0,0}}{1 - \frac{\psi_{0,0}}{\psi_{0,0}} t^{\lambda}} \leq \sum_{k=0}^{\infty} \left( \frac{\lambda + k - 1}{k} \right) \cdot \frac{\psi_{0,0}}{1 - \frac{\psi_{0,0}}{\psi_{0,0}} t^{\lambda}} \leq e^{\rho^{2}} \left[ \psi_{0,0} - \psi_{0,1} t^{\lambda} \right]^{\frac{1}{\rho}} \left( 1 - \frac{1}{1 + \rho t^{\lambda}} \right)^{1}
\]

**Conjecture 1.** Motivated by the previous Theorem, we ask the following question: Proved the finite Laplace transform of the function
\[
\xi \mapsto \frac{1}{\xi(1-t\xi)^{\lambda}} H_{q,p}^{0} \left( \frac{(B_{q}, \beta_{q})}{(A_{p}, \alpha_{p})} \right), \quad (0 < t, \xi < 1),
\]
or, did you express the finite Laplace transform of the above function in terms of the Fox H-function?

On setting $p = q = 1, A_{1} = B_{1} = A > 0$ in (2.20), in view of the following formula
\[
H_{q,p}^{0} \left( \frac{(B_{q}, \beta_{q})}{(A_{p}, \alpha_{p})} \right) = \frac{\xi^{\beta - \alpha - 1}}{\Gamma(\beta - \alpha)} \left( 1 - \frac{\xi^{\beta - \alpha - 1}}{\Gamma(\beta - \alpha)} \right), \quad A > 0, \quad \beta > \alpha > 0,
\]
we get the following results as follows:

**Corollary 1.** If $b > a \geq A$, then the following relation holds true:
\[
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + k) \Gamma(\alpha + (k + n)A)}{\Gamma(\beta + (k + m)A)} \frac{z^{n} t^{k}}{n! k!} = \frac{\Gamma(\lambda)}{\Gamma(\beta - \alpha)} \left( \xi^{\beta - \alpha - 1} \right) \left( 1 - \frac{\xi^{\beta - \alpha - 1}}{\Gamma(\beta - \alpha)} \right), \quad |t| < 1.
\]
On taking $A_{1} = B_{1} = A > 0$ in (2.20), from (1.10) we compute the following results as follows:

**Corollary 2.** Let $\lambda > 0$, then the following generating function
\[
\sum_{k=0}^{\infty} \left[ \frac{\psi_{0,0} + k A_{1}}{(B_{q} + k A_{1})} \right]^{k} \frac{z^{k}}{k!} = \frac{1}{A} L_{\rho} \left( \frac{(B_{q}, \beta_{q})}{(A_{p}, \alpha_{p})} \right) \left( 1 - \frac{1}{1 + \rho t^{\lambda}} \right)^{1}, \quad z \in \mathbb{R}, \quad \min(a_{i}/A_{1}) \geq 1 \quad \text{and} \quad t > 0.
\]
holds true for all $|t| < 1$ and $z \in \mathbb{R}$ such that $\min(a_{i}/A_{1}) \geq 1 \quad \text{and} \quad t > 0$. 

If we set \( A_i = B_i = 1 \) in (2.26), we get the following result as follows:

**Corollary 3.** Let \( \lambda > 0 \), then the following generating function

\[
(2.33) \quad \sum_{k=0}^{\infty} \frac{(\lambda + k - 1)}{k} \rho \frac{\Gamma_{q,k}[a_1+k,\ldots,a_p+k]}{(b_1+k,\ldots,b_q+k)} |z|^k = \mathcal{L}_\rho \left( \xi^{-1}(1-t\xi)^{-\lambda} \mathcal{G}_{q,p,0}^\rho \left( \xi^{b_q/a_p}, \frac{b_q}{a_p} \right) \right) (-z),
\]

holds true for all \( |t| < 1 \) and \( z \in \mathbb{R} \) such that \( \min(a_i) \geq 1 \) and \( \mu > 0 \).

We will need the following Lemma which is considered the main tool to arrive at the next Theorem and Theorem 3.

**Lemma 1.** [7, Exercice 2.3, p. 72] Let \( \alpha, \beta, \gamma_1 \in \mathbb{C} \), either \( \alpha > 0, |\arg y| < \frac{\pi}{2} \pi \) or \( \alpha = 0, \Re(\mu) > 1 \). Further, let \( \eta \geq 0, b \neq a_i, \frac{b-a_i}{a+i} < 1, \frac{y(b-a_i)^{\alpha}}{(a+i)^{\beta}} < 1, \frac{\arg \left( \frac{d\phi+a}{\phi+i} \right)}{\pi} < \pi \) be such that \( \Re(\alpha) + \sigma \min_{1 \leq i \leq n} \left( \frac{\Re(\alpha)}{\lambda_i} \right) > 0, \Re(\alpha) + \eta \min_{1 \leq i \leq n} \left( \frac{\Re(\alpha)}{\lambda_i} \right) > 0 \) for \( \alpha > 0 \) or \( \alpha = 0, \Delta \leq 0 \) and \( \Re(\alpha) + \sigma \min_{1 \leq i \leq n} \left( \frac{\Re(\alpha)}{\lambda_i} \right) > 0, \Re(\alpha) + \eta \min_{1 \leq i \leq n} \left( \frac{\Re(\alpha)}{\lambda_i} \right) > 0 \) for \( \alpha > 0 \) or \( \alpha = 0, \Delta > 0 \), then there holds the formula

\[
(2.34) \quad \int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(\alpha+1)H_{q,p}^{m,n} \left( y(x-a)^\sigma(b-x)^\tau(\alpha+1)(\beta+1) \frac{(x,a,b_1,a_1)}{(b_1,a)} \right) dx = (b-a)^{\alpha+\beta-1}(\alpha+1)(\beta+1)H_{q,p}^{m,n} \left( y(x-a)^\sigma(b-x)^\tau(\alpha+1)(\beta+1) \frac{(x,a,b_1,a_1)}{(b_1,a)} \right).
\]

**Theorem 2.** Let \( \lambda > 0 \), and assume that the hypotheses (H1) are satisfied. If \( \mu > 1 \), then there holds the generating function

\[
(2.35) \quad \sum_{k=0}^{\infty} \frac{(\lambda + k - 1)}{k} \rho \frac{\Gamma_{q,k}[a_1+k,\ldots,a_p+k]}{(b_1+k,\ldots,b_q+k)} |1-t\rho| \rho^k = (1-t \mu)^{\lambda} H_{q,p}^{m,n} \left( y(x-a)^\sigma(b-x)^\tau(\alpha+1)(\beta+1) \frac{(x,a,b_1,a_1)}{(b_1,a)} \right),
\]

for all \( 1-t < \rho < 1 \).

**Proof.** Upon setting the left hand-side of the formula (2.35) by \( \mathcal{J} \). Then, by substituting the integral representation (2.24) into \( \mathcal{J} \), we find that

\[
(2.36) \quad \mathcal{J} = \sum_{k=0}^{\infty} \frac{(\lambda + k - 1)}{k} \left[ \int_0^\rho H_{q,p}^{m,n} \left( \left( \frac{y(x-a)^\sigma(b-x)^\tau(\alpha+1)(\beta+1)}{(x,a,b_1,a_1)} \right) \right) \frac{d\xi}{\xi(1-\xi(1-t)/\rho)\lambda+k} \right] t^k,
\]

which, upon changing the order of sum and integral and after a little simplification when we make use of (2.24) yields

\[
(2.37) \quad \mathcal{J} = \int_0^\rho H_{q,p}^{m,n} \left( \left( \frac{y(x-a)^\sigma(b-x)^\tau(\alpha+1)(\beta+1)}{(x,a,b_1,a_1)} \right) \right) \frac{d\xi}{\xi(1-\xi(1-t)/\rho)\lambda} = \left( \frac{\rho}{1-t} \right)^\lambda \int_0^\rho \xi^{-1}(\rho-\xi)^{-\lambda} H_{q,p}^{m,n} \left( \left( \frac{y(x-a)^\sigma(b-x)^\tau(\alpha+1)(\beta+1)}{(x,a,b_1,a_1)} \right) \right) d\xi.
\]

Now, let us put \( \alpha = \eta = \alpha = \nu = \gamma_1 = 0, \beta = 1-\lambda, d = c = y = 1 \) and \( b = \rho \) in Lemma 1 then we obtain

\[
(2.38) \quad \int_0^\rho \xi^{-1}(\rho-\xi)^{-\lambda} H_{q,p}^{m,n} \left( \left( \frac{y(x-a)^\sigma(b-x)^\tau(\alpha+1)(\beta+1)}{(x,a,b_1,a_1)} \right) \right) d\xi = \rho^{-\lambda} H_{q,p}^{m,n} \left( \left( \frac{y(x-a)^\sigma(b-x)^\tau(\alpha+1)(\beta+1)}{(x,a,b_1,a_1)} \right) \right) d\xi.
\]

Finally, in view of (2.37) and (2.38), we get the desired assertion (2.35) of Theorem 2.

\[\square\]
Remark 2. By using (1.4) and under the hypotheses of Theorem 3 the formula (2.39) can be rewritten as follows:

\[
\sum_{k=0}^{\infty} p+1 \Psi_q \left[ \frac{(\lambda+k,1,1)}{(b_q,b_q)} \right] \left( 1 - t \right) \frac{k^h}{\rho} k! = \frac{\Gamma(\lambda)}{(1-t)^\lambda} \times H^{0,2}_{1,1}(1;1,0) \left( \frac{(1,1,1,1,1,0,0)}{(b_q,b_q,0,0,\lambda)} \cdot \frac{(\lambda,1,1,1,1,0,0)}{(\lambda,1,1,1,1,0,0)(1,0)} \right).
\]

(2.39)

Theorem 3. Let \( \lambda > 0, \tau > 0 \) and assume that the hypotheses (H1) are satisfied. Moreover, suppose that the following hypotheses

\[ (H_2) : \tau + \min_{1 \leq j \leq p} (a_j/A_j, (\mu - 1/2)/\Delta) > 0, \]

are verified. Then the following generating function

\[
\sum_{k=0}^{\infty} \binom{\lambda + k - 1}{k} \rho^{\tau} \Psi_q \left[ \frac{(\lambda+k,1,1,1)}{(b_q+B_q,b_q)} \right] \left( 1 - t \right) \frac{k^h}{\rho} k! = \frac{\rho^\tau}{(1-t)^\lambda} \times H^{0,2}_{1,1}(1;1,0) \left( \frac{(1-\tau,1,1,1,0,0)}{(b_q,b_q,0,0,\lambda)} \cdot \frac{(\lambda-\tau,1,1,1,1,0,0)}{(\lambda-\tau,1,1,1,1,0,0)(1,0)} \right),
\]

(2.40)

holds true for all \( 1 - t < \rho < 1 \).

Proof. Making use of (2.22), (2.24) and (2.26), and then changing the order of integration and summation, the left-hand side of the result (2.40) (say \( K \)) it follows that

\[
K = \left( \frac{\rho}{1-t} \right)^\lambda \int_0^\rho \xi^{\tau-1} (\rho - \xi)^{-\lambda} H_{q,p}^{p,0} \left( \xi \frac{(b_q,b_q)}{(a_q,a_q)} \right) d\xi.
\]

Now, let us put \( \eta = a = \gamma_1 = \nu = 0, \alpha = \tau, \beta = 1 - \lambda, d = c = y = 1 \) and \( b = \rho \) in Lemma 1, then we have

\[
\int_0^\rho \xi^{\tau-1} (\rho - \xi)^{-\lambda} H_{q,p}^{p,0} \left( \xi \frac{(b_q,b_q)}{(a_q,a_q)} \right) d\xi = \rho^\tau \Gamma(\lambda) \frac{(1-\tau,1,1,1,0,0)}{(b_q,b_q,0,0,\lambda)} \cdot \frac{(\lambda-\tau,1,1,1,1,0,0)}{(\lambda-\tau,1,1,1,1,0,0)(1,0)}.
\]

(2.41)

(2.42)

Now on taking (2.41) and (2.42) into account, one can easily arrive at the desired result (2.40). This completes the proof. \( \square \)

Remark 3. By using (1.4) and under the hypotheses of Theorem 3 the formula (2.40) can be rewritten as follows:

\[
\sum_{k=0}^{\infty} p+1 \Psi_q \left[ \frac{(\lambda+k,1,1,1)}{(b_q+B_q,b_q)} \right] \left( 1 - t \right) \frac{k^h}{\rho} k! = \frac{\Gamma(\lambda)}{(1-t)^\lambda} \times H^{0,2}_{1,1}(1;1,0) \left( \frac{(1-\tau,1,1,1,0,0)}{(b_q,b_q,0,0,\lambda)} \cdot \frac{(\lambda-\tau,1,1,1,1,0,0)}{(\lambda-\tau,1,1,1,1,0,0)(1,0)} \right),
\]

(2.43)

Theorem 4. Let \( \lambda > 0 \). Then the following generating function

\[
\sum_{k=0}^{\infty} p+1 \Psi_q \left[ \frac{(\lambda+k,1,1,1)}{(b_q,b_q)} \right] \left( 1 - t \right) \frac{k^h}{\rho} k! = \left( 1 - t \right)^{-\lambda} \frac{1}{p+1} \Psi_q \left[ \frac{(a_{p-1},A_{p-1})}{(b_{q-1},b_{q-1})} \right] \frac{z}{(1-t)^A},
\]

(2.44)

holds true for all \( |t| < 1 \) and \( z \in \mathbb{C} \).

Proof. For convenience, let the left-hand side of the formula (2.44) of Theorem 3 be denoted by \( S \). Then, by substituting the series expression from (1.4) into \( S \) and applying the binomial expansion (2.24) we
where it is tacitly assumed that the positive sequence generalizes Mathieu-type series and the extended Hurwitz–Lerch zeta function. Here, and in what follows, we get for all $\lambda > 0$

\[
S = \sum_{k=0}^{\infty} \left[ \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + k + n\lambda)}{\Gamma(\lambda + n\lambda)} \prod_{m=1}^{n-1} \Gamma(b_m + nB_m) \frac{z^k}{k!} \right]^t
\]

\[
= \sum_{n=0}^{\infty} \prod_{m=1}^{n-1} \Gamma(\lambda + k + n\lambda) \prod_{m=1}^{n-1} \Gamma(b_m + nB_m) \left[ \sum_{k=0}^{\infty} \frac{\Gamma(\lambda + k + n\lambda)}{\Gamma(\lambda + n\lambda)} \frac{t^k}{k!} \right] z^n
\]

\[
= \left(1 - t\right)^{-\lambda} \sum_{n=0}^{\infty} \frac{\prod_{m=1}^{n-1} \Gamma(a_m + n\lambda)}{n!} \prod_{m=1}^{n-1} \Gamma(b_m + nB_m) \left( \frac{z}{(1-t)^{\alpha}} \right)^n
\]

which shows that the generating function (2.44) holds true. This completes the proof of Theorem 3. □

Remark 4. If we setting $p = q = 1$ in the above Theorem and the same steps as in the proof of Theorem 4 we get for all $\lambda > 0$

\[
\sum_{k=0}^{\infty} \prod_{j=1}^{k} \left[ \frac{(\lambda + k, A)}{(\lambda, A)} \cdot |z| \right] \frac{t^k}{k!} = \left(1 - t\right)^{-\lambda} \exp \left( \frac{z}{(1-t)^{\lambda}} \right),
\]

where $|t| < 1$. In particular, for all $\lambda > 0$ we have

\[
\sum_{k=0}^{\infty} \prod_{j=1}^{k+1} \left[ \frac{(\lambda + k, A)}{(\lambda, A)} \cdot |z| \right] \frac{t^k}{k!} = \frac{\Gamma(\lambda)}{(1-t)^{\lambda}} \exp \left( \frac{z}{1-t} \right),
\]

where $|t| < 1$.

Remark 5. Setting $A_i = B_j = 1$ we obtain for all $a_i, b_j > 0$ and $|t| < 1$ the following generating function (known or new) for the hypergeometric function

\[
\sum_{k=0}^{\infty} \prod_{j=1}^{k} \left[ \frac{(a_1 + k, a_2, \ldots, a_p)}{(b_1, b_2, \ldots, b_q)} \cdot |z| \right] \frac{t^k}{k!} = \frac{\Gamma(a_1)}{(1-t)^{a_1}} \prod_{j=1}^{p-1} \frac{\Gamma(a_2, \ldots, a_p)}{\Gamma(b_2, \ldots, b_q)} \left( \frac{z}{1-t} \right).
\]

3. Applications: Generating functions for a certain classes of the generalized Mathieu-type series and the extended Hurwitz–Lerch zeta functions

The main object of this section is to investigate several generating functions for a certain class of generalized Mathieu-type series and the extended Hurwitz–Lerch zeta function. Here, and in what follows, the generalized Mathieu-type series is defined by [10]:

\[
S^{(\alpha, \beta)}(r; a) = S^{(\alpha, \beta)}(r; \{a_k\}) = \sum_{k=0}^{\infty} \frac{2a^\alpha_k}{(r^2 + a^\alpha_k)^\alpha}, \quad (r, \alpha, \beta, \mu > 0),
\]

where it is tacitly assumed that the positive sequence

\[
a = (a_k), \text{ such that } \lim_{k \to \infty} a_k = \infty,
\]

is chosen so that the infinite series in the definition (3.49) converges, that is, that the following auxiliary series:

\[
\sum_{k=0}^{\infty} \frac{1}{a^\mu_k a^\beta_k}
\]

is convergent.

Theorem 5. Let $\alpha > 0$ and $\mu > 1$. Then for $r > 0$ and $x > 0$ there holds the formula

\[
\sum_{k=0}^{\infty} \Gamma(\mu + k) S^{(\alpha, \beta)}(r; \{n^\alpha\}) \frac{t^k}{k!} = \frac{2\Gamma(\mu)}{(1-t)^{\mu}} \left( 1 + \frac{r^2}{1-t} \right),
\]
where \( \zeta(s, a) \) is the Hurwitz Zeta Function defined by:

\[
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}, \quad \Re(s) > 1.
\]

Proof. We make use the representation integral for the Mathieu’s series [23],

\[
S^{(\alpha, \beta)}_{\mu, k}(r; \{k^\nu\}_{k=1}^{\infty}) = \frac{2}{\Gamma(\mu)} \int_{0}^{\infty} \frac{x^{\mu-1}}{e^x - 1} \left[ \sum_{k=0}^{\infty} \left( \frac{1}{L_{\mu}^{(\nu)}(\nu \mu - \beta, \nu), \nu} \right) r^2 x^\alpha \right] dx,
\]
combining with [2.46], we have

\[
\sum_{k=0}^{\infty} \Gamma(\mu + k)S^{(\alpha, k)}_{\mu + k}(r; \{n^\frac{1}{\alpha}\}_{n=1}^{\infty}) \frac{k!}{k!} = \int_{0}^{\infty} \frac{x^{\mu-1}}{e^x - 1} \left[ \sum_{k=0}^{\infty} \left( \frac{1}{L_{\mu}^{(1)}(\nu, 1)} \right) r^2 x^\alpha \right] dx
\]

\[
= (1 - t)^{-\lambda} \int_{0}^{\infty} \frac{x^{\mu-1}}{e^x - 1} \left[ \sum_{k=0}^{\infty} \left( \frac{1}{L_{\mu}^{(1)}(\nu, 1)} \right) r^2 x^\alpha \right] dx
\]

\[
= (1 - t)^{-\lambda} \int_{0}^{\infty} \frac{x^{\mu-1}}{e^x - 1} \left( \frac{2^\mu}{(1 - t)^{\lambda}} \right) dx.
\]

We now make use of the following known formula [5, Eq. (10), p. 144]

\[
\int_{0}^{\infty} \frac{x^{\nu-1} e^{-\alpha x}}{(1 - e^{-x})} \, dx = \alpha^\nu \Gamma(\nu) \zeta(\nu, \alpha), \quad (\nu > 1, \alpha > 0),
\]

Inserting the above result with the help of (3.50), the results (3.50) readily follows. \( \square \)

Corollary 4. Let \( \alpha > 0 \) and \( \mu > 1 \). Then the following formula

\[
\sum_{k=0}^{\infty} \Gamma(\mu + k)S^{(\alpha, \mu)}_{\mu + k}(\sqrt{1 - t}, \{n^\frac{1}{\mu}\}_{n=1}^{\infty}) = \frac{2\Gamma(\mu)}{(1 - t)^{\lambda}} \zeta(\mu, 2),
\]
	holds true for all \( |t| < 1 \). Moreover, the following double series identity holds true:

\[
\sum_{n=1}^{\infty} \left( \frac{k + 1}{(1 + mn)^2} \right) ^k \left( \frac{m - 1}{m} \right) ^k = \pi^2 - 6,
\]

holds true for \( m \geq 2 \).

Proof. Letting \( r = \sqrt{1 - t} \) in (3.50) we easily get the formula (3.52). Next, setting \( t = 1 - \frac{1}{m} \), \( m \geq 2 \) and \( \mu = 2 \) in (3.52) we find

\[
\sum_{n=1}^{\infty} \left( \frac{k + 1}{2m^2} \right) ^k \left( \frac{m - 1}{m} \right) ^k = \zeta(2) = \zeta(2) - 1,
\]

where \( \zeta(s) \) is the Riemann zeta function defined by

\[
\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad s > 1.
\]

Now, combining (3.53) with the definition of the generalized Mathieu series (3.30), and using the fact that \( \zeta(2) = \frac{\pi^2}{6} \), we obtain the desired formula (3.55) and consequently the proof of Corollary 4 is complete. \( \square \)

Remark 6. Setting in the formula (3.55), \( m = 2, m = 3 \) and \( m = 4 \) respectively, we get the following double series identities

\[
\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \left( \frac{k + 1}{1 + 2n} \right) ^k \left( \frac{n}{1 + 2n} \right) ^k = \frac{\pi^2}{6} - 6,
\]

\[
\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \left( \frac{k + 1}{1 + 3n} \right) ^k \left( \frac{2n}{1 + 3n} \right) ^k = \frac{\pi^2}{6} - 6,
\]

\[
\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \left( \frac{k + 1}{1 + 4n} \right) ^k \left( \frac{3n}{1 + 4n} \right) ^k = \frac{\pi^2}{6} - 6.
\]
Remark 7. If we set $\mu = 3$ (respectively $m = 4$) and $t = 1 - \frac{1}{m}$, $m \geq 2$ in the formula (3.52) we obtain the following formulas

$$\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(k+1)(k+2)}{(1+mn)^3} \left( \frac{(m-1)n}{1+mn} \right)^k = \zeta(3, 2) = \zeta(3) - 1 \approx 0.2056903. \tag{3.58}$$

and

$$\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(k+1)(k+2)(k+3)}{(1+mn)^4} \left( \frac{(m-1)n}{1+mn} \right)^k = \zeta(4, 2) = \zeta(4) - 1 = \frac{\pi^4 - 90}{90}. \tag{3.59}$$

The extended Hurwitz-Lerch zeta function

$$\Phi^{(\mu_j, \rho_j, p)} (z, s, a) = \Phi^{(\mu_j, \rho_j, \sigma_j, q)} (z, s, a) \tag{3.60}$$

holds true for all $|z| < \nabla^*$.

Moreover, the extended Hurwitz-Lerch zeta function possesses the following integral representation

$$\Phi^{(\mu_j, \rho_j, p)} (z, s, a) = \left( \prod_{j=1}^{p} \Gamma(\mu_j) \prod_{j=1}^{\sigma_j} \Gamma(\lambda_j + k\rho_j) \right) \int_0^{\infty} \xi^{s-1} e^{-a\xi} \psi \left[ \frac{\lambda_1, \rho_1, \ldots, \lambda_p, \rho_p}{\mu_1, \sigma_1, \ldots, \mu_q, \sigma_q} \right] e^{-\xi} d\xi \tag{3.61}$$

Theorem 6. The following generating function

$$\sum_{k=0}^{\infty} \Phi^{(\mu_j, \rho_j, p)} (z, s, a) \frac{\Gamma(\lambda_j + k)}{k!} = \frac{\Gamma(\lambda_1)}{\Gamma(s)(1-t)^{\lambda_1}} \Phi^{(\mu_j, \rho_j, p)} (z(1-t)^{-\rho_1}, s, a) \tag{3.62}$$

holds true for all $|t| < 1$.

Proof. In virtue of (3.61) and (2.14) we get

$$\sum_{k=0}^{\infty} \Phi^{(\mu_j, \rho_j, p)} (z, s, a) \frac{\Gamma(\lambda_j + k)}{k!} = \frac{\Gamma(\lambda_1)}{\Gamma(s)(1-t)^{\lambda_1}} \Phi^{(\mu_j, \rho_j, p)} (z, s, a) \times \int_0^{\infty} \xi^{s-1} e^{-a\xi} \left[ \sum_{k=0}^{\infty} \psi \left[ \frac{\lambda_1 + k\rho_1, \ldots, \lambda_p + k\rho_p}{\mu_1, \sigma_1, \ldots, \mu_q, \sigma_q} \right] e^{-\xi} \right] \frac{\xi^k}{k!} d\xi \tag{3.63}$$

Combining this with (3.61) yields to the desired assertion (3.62) of Theorem 6. \qed
The Hurwitz-Lerch zeta function $\Phi(z, s, a)$ is defined by
\begin{equation}
\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s}
\end{equation}

\((H_3): a \in \mathbb{C} \setminus \mathbb{Z_0}^\ast; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1\).

The Hurwitz-Lerch zeta function itself reduces not only to the Riemann zeta function $\zeta(s)$, the Hurwitz zeta function $\zeta(s, a)$, but also to such other important functions of Analytic Number Theory as as the Polylogarithm function (or de Jonqui`ere’s function) $L_i(z)$, the Lipschitz-Lerch zeta function $L(\xi, a, s)$ and the Lerch zeta function $l_s(\xi)$ defined by [4, Chapter 1, p. 27-31]

\begin{align}
\text{(3.64)} & \quad L_i(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad (\Re(s) > 0; z \in \mathbb{C} \text{ when } |z| < 1) \\
\text{(3.65)} & \quad L(\xi, a, s) = \sum_{n=0}^{\infty} e^{2in\pi \xi} \frac{1}{(n + a)^s}, \quad (\Re(s) > 1; \xi \in \mathbb{R}; 0 < a \leq 1).
\end{align}

and

\begin{equation}
\text{(3.66)} \quad l_s(\xi) = \sum_{n=0}^{\infty} e^{2in\pi \xi} \frac{1}{(n + 1)^s}, \quad (\Re(s) > 1; \xi \in \mathbb{R}).
\end{equation}

**Corollary 5.** The following generating function
\begin{equation}
\sum_{k=0}^{\infty} \Phi^{(\rho_1,1,1)}_{\lambda_1+k,\lambda_1}(z,s,a) \frac{\Gamma(\lambda_1+k)k^k}{k!} = \frac{\Gamma(\lambda_1)}{(1-t)^{\lambda_1}} \Phi(z(1-t)^{-\rho_1}, s, a)
\end{equation}

holds true for all $-1 < t < 0$. Furthermore, the following generating function involving the Lipschitz-Lerch zeta function $L(\xi, a, s)$
\begin{equation}
\sum_{k=0}^{\infty} \Phi^{(\rho_1,1,1)}_{\lambda_1+k,1,\lambda_1}(e^{2n\pi \xi}, s, a) \frac{\Gamma(\lambda_1+k)k^k}{k!} = \frac{\Gamma(\lambda_1)}{(1-t)^{\lambda_1}} L(\xi, a, s),
\end{equation}

holds true for all $-1 < t < 0, 0 < a \leq 1, \Re(s) > 1$ and $\xi \in \mathbb{R}$.

**Proof.** Letting $p = 2, q = 1$ and $\lambda_2 = 1$ in (3.62), we obtain
\begin{align}
\sum_{k=0}^{\infty} \Phi^{(\rho_1,1,1)}_{\lambda_1+k,1,\lambda_1}(z,s,a) \frac{\Gamma(\lambda_1+k)k^k}{k!} = \frac{\Gamma(\lambda_1)}{(1-t)^{\lambda_1}} \Phi^{(1,-)}(z(1-t)^{-\rho_1}, s, a)
\end{align}

\begin{align}
&= \frac{\Gamma(\lambda_1)}{(1-t)^{\lambda_1}} \sum_{n=0}^{\infty} \frac{(z(1-t)^{-\rho_1})^n}{(n + a)^s} \\
&= \frac{\Gamma(\lambda_1)}{(1-t)^{\lambda_1}} \Phi(z(1-t)^{-\rho_1}, s, a),
\end{align}

and consequently the formula (3.67) holds true. Finally, setting $z = e^{2n\pi \xi}$ in (3.67) we find (3.68), which completes the proof of Corollary 5.

**Corollary 6.** Assume that the Hypotheses $(H_3)$ are satisfied. Then the following double series identities
\begin{equation}
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(n+k+1)(1-t)^{n+1}z^{n+1}}{n! (n + a)^s} = \Phi(z, s, a),
\end{equation}

holds true for all $-1 < t < 0$. In particular, the following double series identities
\begin{equation}
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(n+k+1)(1-t)^{n+1}e^{2n\pi \xi}z^{n+1}}{n! (n + a)^s} = L(\xi, a, s),
\end{equation}

holds true for all $-1 < t < 0, \xi \in \mathbb{R}, \Re(s) > 1$ and $0 < a \leq 1$.

**Proof.** Upon setting $\lambda_1 = \rho_1 = 1$ and replace $z$ by $(1-t)z$ in (3.67) and straightforward calculation would yield to the formula (3.72). Now, letting $z = e^{2n\pi \xi}$ in (3.72) we obtain (3.71).
Remark 8. Using the fact that \( \text{Li}_s(z) = z \Phi(z, s, 1) \) we get

\[
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(n+k+1)(1-t)^{n+k}z^n}{n! (n+1)^s} = \frac{\text{Li}_s(z)}{z},
\]

holds true for all \(-1 < t < 0, \Re(s) > 0\) and \(z \in \mathbb{C}\) when \(|z| < 1\). In particular, by using some particular expressions of the Polylogarithm function we obtain for \(-1 < t < 0\):

\[
\begin{align*}
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(n+k+1)(1-t)^{n+k}z^n}{(n+1)!} &= -\log(1-z), 0 < z < 1 \\
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(n+k+1)(1-t)^{n+k+1}z^n}{(n+1)!2n} &= \log^2(2) \\
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(n+k+1)(1-t)^{n+k}}{(n+1)!(n+1)^22^n} &= \frac{\log^3(2)}{3} - \frac{\log(2)}{6} + \frac{\log(2)}{6} + \frac{7}{8} \zeta(3).
\end{align*}
\]

References

[1] P. Agarwal, C. L. Kou, On generating functions, J Rajasthan Acad Phys Sci., 2(3) (2003), 173–180.
[2] P. Agarwal, M. Chand, S. D. Purohit, A Note on Generating Functions Involving the Generalized Gauss Hypergeometric Function, Natl. Acad. Sci. Lett., 37 (5) (2014), 457–459.
[3] M. P. Chen, H. M. Srivastava, Orthogonality relations and generating functions for Jacobi polynomials and related hypergeometric functions, Appl. Math. Comput., 68 (1995), 153–188.
[4] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G Tricomi, Higher transcendental functions, vol. I, McGraw-Hill Book Company, New York (1953).
[5] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, Tables of Integral Transforms, Vol. I, McGraw–Hill Book Company, New York and London, (1954).
[6] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, (2006).
[7] A.M. Mathai, R. K. Saxena H. J. Haubold The H-functions: Theory and applications, Springer (2010).
[8] K. Mehrez, New Integral representations for the Fox-Wright functions and its applications, J. Math. Anal. Appl. 468 (2018), 650–673.
[9] K. Mehrez, S. M. Sitnik, Functional inequalities for the Fox-Wright functions, The Ramanujan J., 50 (2) (2019), 263–287.
[10] K. Mehrez, New properties for several classes of functions related to the Fox-Wright functions, Journal of Computational and Applied Mathematics 362 (2019), 161–171.
[11] P. K. Mittal, K. C. Gupta, An integral involving generalized function of two variables, Proc Indian Acad Sci Sect A, 75 (1972), 117–123.
[12] T. R. Prabhakar, A singular integral equation with a generalized Mittag–Leffler function in the kernel, Yokohama J. Math. J. 19 (1971), 7–15.
[13] R. K. Saxena RK, K. Nishimoto, Fractional integral formula for the H-function, J Fract Calc., 6 (1994), 65–75.
[14] M. Saigo, R. K. Saxena, Applications of generalized fractional calculus operators in the solution of an integral equation, J Fract Calc., 14 (1998), 53–63.
[15] A. D. Sokal, Real-variables characterization of generalized Stieljes functions, Expo. Math., 28 (2010), 179–185.
[16] H. M. Srivastava, Z. Tomovski, Some problems and solutions involving Mathieu’s series and its generalizations, J. Inequal. Pure Appl. Math. 5 (2) (2004), Article 45, 1–13 (electronic).
[17] H. M. Srivastava, K. C. Gupta, S. P. Goval, The H-functions of one and two variables with applications, South Asian Publishers, New Delhi (1982).
[18] H. M. Srivastava, H. L. Manocha, A treatise on generating functions. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto (1984).
[19] H. M. Srivastava, Certain generating functions of several variables, Z. Angew. Math. Mech. 57 (1977), 339–340.
[20] H. M. Srivastava, Generating relations and other results associated with some families of the extended Hurwitz-Lerch Zeta functions, SpringerPlus 2 (67) (2013), 1–14 (Article ID 2).
[21] H. M. Srivastava, M.A. Özzarslan, C. Kaanoglu, Some families of generating functions for a certain class of three-variable polynomials, Integral Transforms Spec. Funct., 21 (2010) 885–896.
[22] H. M. Srivastava, P. Agarwal, S. Jain, Generating functions for the generalized Gauss hypergeometric functions, Appl. Math. Comput., 247 (2014), 348–352.
[23] Z. Tomovski, K. Mehrez, Some families of generalized Mathieu-type power series, associated probability distributions and related inequalities involving complete monotonicity and log-convexity, Math. Ineq. and Applications, 20 (4) (2017), 973–986.
[24] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, (1922).
[25] D. V. Widder, The Laplace Transform, Princeton University Press, (1946).
[26] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, *Journal London Math. Soc.* 10 (1935), 287–293.

Khaled Mehrez. Département de Mathématiques, Faculté de Sciences de Tunis, Université Tunis El Manar, Tunisia.

E-mail address: k.mehrez@yahoo.fr