RANDOM WALK ON LATTICE WITH AN ANTISYMMETRIC PERTURBATION IN ONE POINT

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Abstract. We study an homogeneous irreducible markovian random walk in \( \mathbb{Z}^\nu \), with an antisymmetric perturbation acting only in one point. We compute exactly spatial correction to the diffusive behaviour in the asymptotics of probability, in the spirit of local limit theorems for random walks.

1. Introduction, definitions and main result

In this work we are concerned about a Markov random walk in \( \mathbb{Z}^\nu \), with transition probabilities:

\[
\Pi(x, x + u) = P(u) + \delta_{x,0}c(u).
\]

Here \( P \) is the probability of a symmetric homogeneous walk, and the other term is a perturbation acting only in the origin. This is equivalent to a free random walk in which only the transition rates from the origin are modified. We are interested in finding the probability \( \Pi_n(x) \) of being at point \( x \) at time \( n \), when \( n \to \infty \).

It is well known that for the homogeneous case a local limit theorem holds: the probability distribution \( P_n(0, x) \) of being in \( x \) starting at the origin is asymptotically of the form:

\[
P_n(0, x) = \frac{1}{\sqrt{|B|(2\pi n)^{\nu/2}}} \exp \left( -\frac{B^{-1}x, x}{2n} \right) \left( 1 + \sum_{j=1}^{r-2} \frac{1}{n^{j/2}} Q_j \left( \frac{x}{\sqrt{n}} \right) \right) + r_n(x) o \left( \frac{1}{n^{(\nu r-2)/2}} \right),
\]

where \( P_n(0, x) \) is supposed to have the first \( r \geq 3 \) moments finite, \( B \) is the covariant matrix of the process (we have supposed no drift) and \( Q_j \) are certain polynomials of degree \( 3j \). Moreover, \( r_n(x) \) is a correction for small \( x \):

\[
r_n(x) = \frac{1}{1 + \left( \frac{|x|}{\sqrt{n}} \right)^r}.
\]

For a more detailed discussion see [4]. It is important to stress the range in which this classical result holds: of course it is meaningful only when the magnitude of the corrections is not bigger than the leading term; in this case up to the scale \( x = O(n^{\frac{1}{r}}) \).

Despite the rather simple formulation of the problem, it seems that no extensions of the previous theorem exist in the case in which the probability rate is changed only in one point of the lattice. However the obstacle problem also arises in the setting of random walk in random environment, see for instance [1], [9], [2].

In [6] is proven a general result about a random walk with an homogeneous part plus a perturbation \( V(x, y) \) acting in a finite neighborhood of the origin. In this case a correction to the diffusive behaviour arises. If we set

\[
\Delta_n(0, x) = \Pi_n(0, x) - P_n(0, x),
\]

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we have for $\nu = 1$
\[
\Delta_n(0, x) = \frac{1}{\sqrt{n}} \text{sign}(x)e^{-\frac{x^2}{2\sigma^2}} + \psi_n(x) + o\left(\frac{1}{n^{\nu/2}}\right).
\]

with $\psi_n(x)$ decreasing at least polynomially, and for $\nu \geq 2$,
\[
\Delta_n(0, x) \simeq \frac{1}{n^{\nu/2}} \frac{C}{1 + |x|^{(\nu-1)/2}} + o\left(\frac{1}{n^{\nu/2}}\right).
\]

This result is achieved under the general assumption that the perturbed random walk is irreducible (see [8]) and that there are no traps (i.e. a finite set $T \subset \mathbb{Z}^\nu$ such that a walk starting in $T$ will never leave this set with probability one). Furthermore in dimension $\nu = 2$ an additional and more technical hypothesis is needed, for which we refer the reader to the original paper [6]. Anyway the method used there is based on sophisticated functional analysis techniques, and so, as the same authors write, a direct probabilistic interpretation misses.

Other interesting works in the subject are [7], [10], [11].

In our work we analyze the problem in the simpler case in which the perturbation is concentrated in the origin, but by a method with a clear probabilistic interpretation, that is the Gihman-Skorohod expansion of the characteristic function of the random walk.

In order to clarify our argument, we will separate the symmetric and antisymmetric part of the potential function, $c(u) = \varepsilon s(u) + a(u)$, for a certain parameter $\varepsilon > 0$ ruling the strength of the symmetric part of the interaction.

Moreover we assume $P$ has only the first three moments finite, and $c$ the first two. Anyway, in principle we could relax this assumption, allowing more regularity both for $P$ and $c$; but it will be clear in a moment that for our purpose it is not useful at all, since at the diffusive scale higher moments play no role. Actually, since we will deal with $P$ a symmetric and $c$ an antisymmetric function, we will need only the covariance of the free walk and the drift given by the perturbation. Finally, we define a brownian scale, $x \simeq \sqrt{n}$, when $x = O(\sqrt{n}\phi_n)$, with $\phi_n$ an increasing sequence with $n$ growing slower than any power. For example, the result in [6] is valid up to $x = O(\sqrt{n \log n})$.

In Section 2 we find the representation formula for the transition probabilities. It turns out that relevant simplifications occur by setting $\varepsilon = 0$, so in Section 3 we focus only on the antisymmetric case, and we find the asymptotics. Performing the same calculation in the symmetric case is harder from a technical point of view, due to the more involved representation formula. Nevertheless we believe that the method we use can be applied straightforwardly to this case too.

Our main result is the following:

**Theorem.** Be given an irreducible markovian random walk in $\mathbb{Z}^\nu$, with transition probabilities given by (1), with the perturbation antisymmetric, and

\[
\sum_{x \in \mathbb{Z}^\nu} P(x) = 1; \quad \sum_{x \in \mathbb{Z}^\nu} x^k P(x) = 0, \quad k = 1, 3; \quad \sum_{x \in \mathbb{Z}^\nu} x^2 P(x) = B < \infty;
\]

and

\[
\sum_{x \in \mathbb{Z}^\nu} xc(x) = d < \infty; \quad \sum_{x \in \mathbb{Z}^\nu} x^k c(x) = 0, \quad k = 0, 2.
\]

Then the following asymptotic form for the probability of being in the point $x$ at time $n$ up to the scale $x \simeq \sqrt{n}$ holds:

\[
\Pi_n(0, x) = \frac{1}{(|B|2\pi n)^{\nu/2}} \exp\left(-\frac{(B^{-1}x, x)}{2n}\right) + \Delta_n(x),
\]
where we have the following form for $\Delta'_n(x)$:

$$\Delta'_n(x) = |d| \cos(d, x) e^{-\frac{(B^{-1}x, x)}{2n^\nu}} \delta'_n(x) + o\left(\frac{1}{n^{\nu/2}}\right)$$

with

$$\delta'_n(x) = \frac{|x|}{|B|^{\nu/2} n^{\nu/2}} \int_{\frac{1}{n}}^\frac{1}{n} \frac{e^{-\frac{1-\alpha}{\nu} (B^{-1}x, x)}}{\alpha^{\nu/2+1} (1-\alpha)^{\nu/2}}$$

is a $O(1)$ function of $n$ when $x$ is fixed.

**Remark 1.** We will prove the following properties for $\delta'_n(x)$:

1. in dimension $\nu \geq 1$, $\delta'_n(x)$ is $O(1)$ in $n$ when $x$ is $O(1)$ in $n$, i.e. near the perturbation;
2. in dimension $\nu = 1$, $\delta'_n(x)$ gives a relevant contribution also for $x$ growing at the brownian scale $x \simeq \sqrt{n}$.

**Remark 2.** Although we use the same technique exposed in [4], we get that our theorem holds for a smaller range of values of $x$, i.e. up to the scale $\sqrt{n}$ (essentially as in [3]) rather than $n^{2/3}$. We believe that this is not a crucial feature of the problem, but it is rather due to the fact that we focus on the main term, and we estimate the higher order correction roughly with respect to the more careful analysis made in [4]. Namely we let the term $o\left(\frac{1}{n^{\nu/2}}\right)$ of unspecified form. For the same reason, for example, we have that the regularity of the probability, i.e. the number of finite moments of $P$ and $c$, play substantially no role in our proof, while of course it will be relevant in the exact form of the remainder $o\left(\frac{1}{n^{\nu/2}}\right)$.

Recently, a very similar theorem has been obtained with different methods for $\nu = 1$, but with a perturbation having no definite parity [4], up to the scale $x \simeq n^{3/4}$. We stress that the two approaches bring to analogous results.

### 2. The Representation Formula for the Transition Probability

The aim of this section is finding a representation for the transition probability (1).

It turns out that it is useful to define the probabilities $\Pi^0_n(x, y)$ and $P^0_n(x, y)$ of going to $y$ starting from $x$, without passing through the origin, respectively of the perturbed and free walk. We will concern especially about $P^0$, and we can represent such probability in terms of $P$, by using a classical argument, namely the inclusion-exclusion principle, as stated by the following

**Proposition 1.** We have

$$P^0_n(0, x) = P_n(0, x) - \rho_n(x),$$

where $\rho_n(x)$ is a positive function of $x$ for every $n$:

$$\rho_n(x) = \sum_{p=1}^{n-1} (-1)^p \sum_{k_1 > k_2 > \ldots > k_p \geq 1} P_{n-k_1}(0, 0) P_{k_1-k_2}(0, 0) \ldots P_{k_{p-1}-k_p}(0, 0) P_{k_p}(0, x)$$

**Proof.** The unperturbed probability to reach a point $x$ from the origin can be decomposed by the following expression:

$$P_n(0, x) = \sum_{k=1}^{n-1} P_{n-k}(0, 0) P^0_k(0, x),$$
Proposition 2. Let $\Pi$ stated and $P$ be defined as in (11), but considering only the antisymmetric part of the point potential i.e. $\varepsilon = 0$ and $c(u) = a(u)$. The following representation formula holds:

$$\Pi_n(0, x) = P_n(0, x) + \sum_{k=0}^{n-1} P_k(0, 0)(a \ast P_{n-k-1}^0(x)).$$

Proof. We have

$$\Pi_n^0(0, 0) = \sum_{y \in \Sigma_0} \Pi(0, y)\Pi_{n-1}^0(y, 0)$$

$$= \sum_{y \in \Sigma_0} (P(0, y) + a(y))P_{n-1}^0(y, 0)$$

$$= \sum_{y \in \Sigma_0} P(0, y)P_{n-1}^0(y, 0) + \sum_{y \in \Sigma_0} a(y)P_{n-1}^0(0, -y)$$

$$= P_n^0(0, 0),$$

because of the antisymmetry of $a(y)$. Hence the probability of returning in zero for the first time in $k$ steps is equal to the unperturbed one. Furthermore, for an arbitrary time $n$, we have

$$P_n(0, 0) = \sum_k P_k^0(0, 0)P_{n-k}(0, 0),$$

$$\Pi_n(0, 0) = \sum_k \Pi_k^0(0, 0)\Pi_{n-k}(0, 0) = \sum_k P_k^0(0, 0)\Pi_{n-k}(0, 0).$$

This is a recursive formula for $P_n(0, 0)$ and $\Pi_n(0, 0)$, with the same initial condition $P(0, 0) = \Pi(0, 0)$, that implies $P_n(0, 0) = \Pi_n(0, 0)$. Now we are ready to find a form for the probability $\pi_n(0, x)$. We use a decomposition with respect to the time of first return at the origin, such that

$$\Pi_n(0, x) = \sum_{k=0}^{n-1} \Pi_k(0, 0)\Pi_{n-k}^0(0, x) = \sum_{k=0}^{n-1} P_k(0, 0) \sum_{y \in \mathbb{Z}_0} (P(0, y) + a(y))P_{n-k-1}^0(y, x)$$

$$= \sum_{k=0}^{n-1} P_k(0, 0)(P_{n-k}(0, x) + \sum_{y \in \mathbb{Z}_0} a(y)P_{n-k-1}(0, y - x))$$

$$= P_n(0, x) + \sum_{k=0}^{n-1} P_k(0, 0)(a \ast P_{n-k-1}^0(x)).$$

Remark 3. It is easily seen that the formula (5) can be simplified: in fact the paths returning into the origin, due to the antisymmetry of $a(x)$, give a vanishing contribution to the convolution, i.e. $a \ast P_{n-k-1}^0(x) = a \ast P_{n-k-1}(x)$ and thus we can rewrite the previous representation formula
Therefore, putting (8) in the representation for \(\Pi_n\), we get

\[
\Pi_n(0, x) = P_n(0, x) + \sum_{k=0}^{n-1} P_k(0, 0)(a \ast P_{n-k-1}(x)).
\]

Now we can concern about the symmetric part of the perturbation. In this case, due to the symmetric nature of the interaction, formulae easily become very complicated; for this reason in this work we will treat in full detail only the case \(\varepsilon = 0\).

We can state and prove the following

**Proposition 3.** Be \(\Pi\) and \(P\) defined as before in (1), but considering only the symmetric part of the one point potential, \(c(x) = \varepsilon s(x)\). Thus the following representation formula holds:

\[
\Pi_n(0, x) = P_n(0, x) + \varepsilon \sum_{l=0}^{n-1} P_l(0, 0)(s \ast P_{n-l-1}^0)(x) + \varepsilon \sum_{l=0}^{n-1} \sum_{k=1}^{l} P_{l-k}(0, 0)(s \ast P_{k-1}^0)(0) P_{n-l}(0, x) \\
+ \sum_{l=0}^{n-1} \sum_{\alpha=2}^{\mu} \varepsilon^\alpha \sum_{K(\alpha)=1} P_{l-K(\alpha)}(0, 0) P_{n-l}^0(0, x) \prod_{i=1}^{\alpha} (s \ast P_{k_i-1}^0)(0) \\
+ \sum_{l=0}^{n-1} \sum_{\alpha=1}^{\mu} \varepsilon^{1+\alpha} \sum_{K(\alpha)=1} P_{l-K(\alpha)}(0, 0) (s \ast P_{n-l-1}^0)(x) \prod_{i=1}^{\alpha} (s \ast P_{k_i-1}^0)(0).
\]

*Proof.* Following the same scheme adopted in the proof of the previous lemma, we start by computing the probability for returns in the origin in \(n\) steps:

\[
\Pi_n(0, 0) = \sum_{k_1+\ldots+k_{\mu}=n} \Pi_{k_1}^0(0, 0) \ldots \Pi_{k_{\mu}}^0(0, 0)
\]

with of course \(\mu = n/2\). Again we have

\[
\Pi_k^0(0, 0) = P_k^0(0, 0) + \varepsilon (s \ast P_{k-1}^0)(0),
\]

and

\[
\Pi_k^0(0, x) = P_k^0(0, x) + \varepsilon (s \ast P_{k-1}^0)(x).
\]

Therefore, putting (5) in the representation for \(\Pi_n(0, 0)\), and naming \(K(\alpha) = \sum_{i=1}^{\alpha} k_i\), we have

\[
\Pi_n(0, 0) = \sum_{k_1+\ldots+k_{\mu}=n} (P_{k_1}^0(0, 0) + \varepsilon (s \ast P_{k_1-1}^0)(0)) \ldots (P_{k_{\mu}}^0(0, 0) + \varepsilon (s \ast P_{k_{\mu}-1}^0)(0)) \\
= P_n(0, 0) + \sum_{\alpha=1}^{\mu} \varepsilon^\alpha \sum_{K(\alpha)=1} P_{n-K(\alpha)}(0, 0) \prod_{i=1}^{\alpha} (s \ast P_{k_i-1}^0)(0).
\]
Now we are ready to write down the representation formula for the transition probability from the origin to a given point:

\[
\Pi_n(0,x) = \sum_{l=0}^{n-1} \Pi_l(0,0)\Pi_{n-l}^0(0,x)
\]

\[
= \sum_{l=0}^{n-1} P_l(0,0)\Pi_{n-l}^0(0,x) + \sum_{l=0}^{n-1} \sum_{\alpha=1}^{\mu} \varepsilon^{\alpha} \sum_{K(\alpha)=1}^{L} P_{l-K(\alpha)}(0,0)\Pi_{n-l}^0(0,0) \prod_{i=1}^{\alpha} (s \ast P_{k_{i-1}}^0)(0)
\]

\[
= P_n(0,x) + \varepsilon \sum_{l=0}^{n-1} P_l(0,0)(s \ast P_{n-l-1}^0)(x) + \varepsilon \sum_{l=0}^{n-1} \sum_{k=1}^{L} P_{l-k}(0,0)(s \ast P_{k-1}^0)(0)P_{n-l}^0(0,x)
\]

\[
+ \sum_{l=0}^{n-1} \sum_{\alpha=1}^{\mu} \varepsilon^{\alpha} \sum_{K(\alpha)=1}^{L} P_{l-K(\alpha)}(0,0)\Pi_{n-l}^0(0,x) \prod_{i=1}^{\alpha} (s \ast P_{k_{i-1}}^0)(0)
\]

\[
+ \sum_{l=0}^{n-1} \sum_{\alpha=1}^{\mu} \varepsilon^{1+\alpha} \sum_{K(\alpha)=1}^{L} P_{l-K(\alpha)}(0,0)(s \ast P_{n-l-1}^0)(x) \prod_{i=1}^{\alpha} (s \ast P_{k_{i-1}}^0)(0).
\]

Putting together these three propositions, we obtain the subsequent

**Lemma 1.** The following representation formula for the transition probabilities holds

\[
\Pi_n(0,x) = P_n(0,x) + \sum_{k=0}^{n} P_k(0,0) (c \ast P_{n-k-1})(x)
\]

\[
+ \varepsilon \sum_{k=0}^{n-1-n-k} \sum_{h=1}^{n-k} P_{n-h-k}(0,0)(s \ast P_{h-1})(0)P_k(0,x)
\]

\[
+ \sum_{l=0}^{n-1} \sum_{\alpha=2}^{\mu} \varepsilon^{\alpha} \sum_{K(\alpha)=1}^{L} P_{l-K(\alpha)}(0,0) \prod_{i=1}^{\alpha} (s \ast P_{k_{i-1}}^0)(0)P_{n-l}^0(0,x)
\]

\[
+ \sum_{l=0}^{n-1} \sum_{\alpha=1}^{\mu} \varepsilon^{1+\alpha} \sum_{K(\alpha)=1}^{L} P_{l-K(\alpha)}(0,0) \prod_{i=1}^{\alpha} (s \ast P_{k_{i-1}}^0)(0)(s \ast P_{n-l-1}^0)(x)
\]

(10)

where

\[
R_n(x) = \varepsilon \sum_{l=0}^{n} \sum_{k=1}^{L} P_{l-k}(0,0)(s \ast P_{k-1}^0)(0)\rho_{n-l}(0,x)
\]

\[
+ \sum_{l=0}^{n-1} \sum_{\alpha=2}^{\mu} \varepsilon^{\alpha} \sum_{K(\alpha)=1}^{L} P_{l-K(\alpha)}(0,0) \prod_{i=1}^{\alpha} (s \ast P_{k_{i-1}}^0)(0)\rho_{n-l}(0,x)
\]

\[
+ \sum_{l=0}^{n-1} \sum_{\alpha=1}^{\mu} \varepsilon^{1+\alpha} \sum_{K(\alpha)=1}^{L} P_{l-K(\alpha)}(0,0) \prod_{i=1}^{\alpha} (s \ast P_{k_{i-1}}^0)(0)(s \ast \rho_{n-l-1})(x).
\]

(11)
3. Proof of the Theorem

Hereafter we will set $\varepsilon = 0$, so that the representation formula for the transition probability reduces to (6).

In order to prove the Theorem, it is convenient to split the proof in two steps. The first be essentially a convenient way to write the inhomogeneous term of the transition probability, following the method in [4]. At first it is useful to define the return probability in the perturbed point at time step $n$ in dimension $\nu$ as

$$ p_n^\nu = P_n(0,0). $$

**Remark 4.** Throughout the whole paper we will use the well known behaviour of the returns probabilities of the free walk $p_n^\nu \simeq n^{-\nu/2}$ [8].

Therefore we are ready to state the sequent crucial lemma:

**Lemma 2.** The correction term $\Delta_n(x)$ at the leading order in $n$ has the form

$$ (12) \quad \Delta_n(x) = \sum_{k=1}^{n-1} \frac{p_{n-k-1}^\nu}{k^{\nu/2}} e^{-\frac{(B^{-1}x,x)}{2k}}(d \cdot x), $$

where $d = \sum_x x a(x)$.

**Remark 5.** Before the proof, it must be noticed that the order in time of $\Delta_n$ is the right one, that it $n^{-\frac{\nu}{2}}$. This is easily seen for instance by an expansion of the function $e^{-\frac{(B^{-1}x,x)}{2k}}$:

$$ e^{-\frac{(B^{-1}x,x)}{2k}} = 1 - \frac{1}{2} \frac{(B^{-1}x,x)^2}{4k^2} - \frac{1}{6} \frac{(B^{-1}x,x)^3}{8k^3} + \ldots $$

and, for $\nu > 0$, we have that for every $q > 0$ [5]

$$ \sum_{k=1}^{n-1} \frac{p_{n-k-1}^\nu}{k^{\nu+2+1/2q}} \simeq \frac{\text{Const}}{n^{\nu/2}}. $$

**Proof.** In primis we calculate the characteristic function of the transition probabilities starting by a rearrangement of formula (6):

$$ \phi_n(\lambda) = \phi_0^\nu(\lambda) + \sum_{k=1}^{n-1} p_{n-k-1}^\nu \tilde{a}(\lambda) \tilde{P}^k(\lambda), $$

where $\lambda \in [-\pi,\pi]^\nu$, $\phi_0^\nu(\lambda)$ is the characteristic function of the free random walk, and

$$ \tilde{a}(\lambda) = \sum_{x \in \mathbb{Z}^\nu} a(x) e^{i\lambda x}, $$

$$ \tilde{P}^k = \sum_{x \in \mathbb{Z}^\nu} P_k(0,x) e^{i\lambda x}. $$

Then we write

$$ \Pi_n(0,x) = \int_{[-\pi,\pi]^\nu} d^\nu \lambda \frac{\phi_n(\lambda)}{(2\pi)^\nu} e^{-i\lambda x} \phi_n(\lambda) = \int_{[-\pi,\pi]^\nu} d^\nu \lambda \frac{\tilde{a}(\lambda) \tilde{P}^k(\lambda)}{(2\pi)^\nu} e^{-i\lambda x}, $$

and we will focus on the inhomogeneous part of the characteristic function, and on the sum of integrals

$$ (13) \quad \Delta_n(0,x) = \sum_{k=1}^{n-1} p_{n-k-1}^\nu \int_{[-\pi,\pi]^\nu} d^\nu \lambda \frac{\tilde{a}(\lambda) \tilde{P}^k(\lambda)}{(2\pi)^\nu} e^{-i\lambda x}. $$
The main idea is very simple \[4\]: to perform a Taylor expansion of the Fourier transform of the potential, that is supposed to be $C^2$, and of the free walk transition probability, that has three moments finite. It is

\begin{equation}
\tilde{a}(\lambda) = i(d \cdot \lambda) + r_a(\lambda)
\end{equation}

where $d$ is a kind of drift term given by the first moment of $a(x)$ and

\begin{align*}
\tilde{P}(\lambda)^k &= \left(1 - \frac{1}{2}(B\lambda, \lambda) + r_P(\lambda)\right)^k \\
&= e^{k \log(1 - \frac{1}{2}(B\lambda, \lambda) + r_P(\lambda))} \\
&= e^{-\frac{k}{4}(B\lambda, \lambda) - kr_P(\lambda)},
\end{align*}

where $B$ is the covariance matrix of $P$. Moreover the reminders $r_a(\lambda), r_P(\lambda)$ are bounded functions such that

\begin{align*}
\lim_{\lambda \to 0} r_a(\lambda) \frac{1}{|\lambda|^2} &= 0, \\
\lim_{\lambda \to 0} r_P(\lambda) \frac{1}{|\lambda|^3} &= 0.
\end{align*}

We split the quantity of interest (13) into two pieces:

\begin{align*}
I &= \sum_{k=1}^{n-1} \sum_{k=1}^{n-1} \nu \left( \int_{[-\pi, \pi]^\nu} \frac{d^\nu x}{(2\pi)^\nu} e^{-ix\lambda} e^{-\frac{k}{4}(B\lambda, \lambda) i(d \cdot \lambda)} e^{-kr_P(\lambda)} \right. \\
II &= \sum_{k=1}^{n-1} \sum_{k=1}^{n-1} \nu \left( \int_{[-\pi, \pi]^\nu} \frac{d^\nu x}{(2\pi)^\nu} e^{-ix\lambda} e^{-\frac{k}{4}(B\lambda, \lambda) o(|\lambda|^2)} e^{-kr_P(\lambda)} \right).
\end{align*}

We define also $\overline{I}$ and $\overline{II}$ as $I, II$ with the integral taken in the complement to $\mathbb{R}^\nu$ of the cube. Now we have to notice two crucial features:

(1) Inside $[-\pi, \pi]^\nu$, $e^{-kr_P(\lambda)}$ can be replaced by 1 with an error $o(1)$;

(2) $\overline{I}$ and $\overline{II}$ are exponentially vanishing.

To prove point 1 is useful to rescale $\lambda \to \lambda \sqrt{n}$ in such a way the integration domain becomes a cube of size $\frac{1}{\sqrt{n}}$ and observe that

\[ r_P(\lambda) \simeq \left(\frac{\lambda}{\sqrt{n}}\right)^3 \psi(\lambda, n), \quad \psi(\lambda, n) \to 0 \text{ if } n \to \infty, \]

so

\[ e^{-kn^\frac{3}{2}r_P(\lambda)} \simeq e^{-k\psi(\lambda, n)} = 1 + o(1) \quad \text{in} \quad \left[-\pi, \pi\right]^\nu \]

at most, i.e. when $k$ is finite, while if $k \simeq n$ in the sum the terms are exponentially vanishing because of the presence of $e^{-\frac{k}{4}(B\lambda, \lambda)}$.

The proof of point 2 is a bit longer. It is useful to rescale the modes: $\lambda \to \frac{\lambda}{\sqrt{n}}$ in such a way the integration domain becomes a cube of size $\sqrt{n}$, $[-\sqrt{n}\pi, \sqrt{n}\pi]^\nu$. We will focus on the lattice of translations:

\[ [-\sqrt{n}\pi, \sqrt{n}\pi]^\nu + \lambda_0, \quad \lambda_0 \equiv h2\pi \sqrt{n}, \quad h \in \mathbb{Z}^\nu. \]
This of course is a partition of $\mathbb{R}^\nu$, so
\[
\int_{\mathbb{R}^\nu} (\cdot) = \int_{[-\sqrt{n}\pi, \sqrt{n}\pi]^\nu} (\cdot) + \sum_{\lambda_0 \neq 0} \int_{[-\sqrt{n}\pi, \sqrt{n}\pi]^\nu + \lambda_0} (\cdot).
\]
Now we expand $\tilde{a}$ and $\tilde{P}$ around each $\lambda_0$. It is
\[
\tilde{a}(\lambda) = \tilde{a}(\lambda_0) + id(\lambda_0) \cdot \lambda + r_a(\lambda),
\]
with
\[
\tilde{a}(\lambda_0) = \sum_{x \in \mathbb{Z}^\nu} e^{i\lambda_0 x} a(x),
\]
\[
d(\lambda_0) = \sum_{x \in \mathbb{Z}^\nu} e^{i\lambda_0 x} xa(x),
\]
and
\[
\tilde{P}^k(\lambda) = e^{k \log \tilde{P}(\lambda_0)} e^{-\frac{1}{2} \tilde{B}(\lambda_0) \lambda, \lambda} e^{-kr_p(\lambda)},
\]
where
\[
\tilde{P}(\lambda_0) = \sum_{x \in \mathbb{Z}^\nu} e^{i\lambda_0 x} P(x),
\]
\[
\tilde{B}(\lambda_0) = \sum_{x \in \mathbb{Z}^\nu} e^{i\lambda_0 x} x^2 P(x) \frac{\tilde{P}(\lambda_0)}{\tilde{P}(\lambda_0)}.
\]
So we can write for instance for $\tilde{I}$:
\[
\tilde{I} = \sum_{k=1}^{n-1} p_{n-k-1} \sum_{\lambda_0 \neq 0} \int_{[-\pi, \pi]^\nu + \lambda_0} d(\lambda) \sum_{x \in \mathbb{Z}^\nu} e^{i\lambda_0 x} \frac{\tilde{P}(\lambda_0)}{(2\pi \sqrt{n})^\nu} e^{-\frac{i}{\sqrt{n}} (\tilde{B}(\lambda_0) \lambda, \lambda) (\tilde{a}(\lambda_0) + id(\lambda_0) \cdot \lambda)}
\]
and it is immediate to show that
\[
\frac{\tilde{I}}{\tilde{I}} \simeq \sum_{k=1}^{n-1} p_{n-k-1} \sum_{\lambda_0 \neq 0} e^{k \log \tilde{P}(\lambda_0)}
\]
\[
= \sum_{k=1}^{n-1} p_{n-k-1} \sum_{\lambda_0 \neq 0} e^{-\frac{1}{2} \tilde{B}(\lambda_0, \lambda_0) e^{-kr_p(\lambda_0)}}
\]
\[
\leq \sum_{k=1}^{n-1} p_{n-k-1} \sum_{|\lambda_0| \geq 2\pi \sqrt{n}} e^{-\frac{1}{2} \tilde{B}(\lambda_0, \lambda_0)},
\]
where the main contribution in the sum is given by small $k$, and we have the same superexponential decay as in a gaussian tail starting by $2\pi \sqrt{n}$.

Of course the same is valid for $\tilde{II}$.
Moreover, we claim that even $\tilde{II}$ gives a correction of small order with respect to the diffusive scale. Infact inside the rescaled cube $[\frac{[-\pi, \pi]^\nu}{\sqrt{n}}$ we have
\[
\tilde{II} \simeq n^{\nu/2} \sum_{k=1}^{n-1} p_{n-k-1} \int_{[-\pi, \pi]^\nu} \frac{d^\nu \lambda}{(2\pi)^\nu} e^{-i\lambda x} e^{-\frac{k\lambda}{2} \tilde{B}(\lambda, \lambda)} |\lambda|^2 o(1).
\]
On the other hand we have seen that we can extend the integration to the whole space with a very little error. So we have
Since we are interested in the asymptotic behaviour of the correction term, we can replace δ(20)

\[ \delta \] (17)

Now we can rewrite the correction (12) as

\[ \delta \] (18)

Therefore we say that the mean contribution is given only by:

\[ \delta \] (19) + \text{a small error. So we have}

\[ \delta \] (20)

This concludes the proof of the theorem.
For sake of completeness we analyse the integral in (18) separately in the case of dimension \( \nu = 1, 2, 3 \) in order to get exact formulas for the asymptotics.

**Case \( \nu = 1 \):**
In one dimension we put \( B = \sigma^2 \) and the integral turns out to be

\[
\int_{\frac{1}{n}}^{1 - \frac{1}{n}} d\alpha \frac{e^{-x^2/2\alpha^2}}{\alpha \sqrt{\alpha(1 - \alpha)}} = \frac{\sqrt{2\pi n}}{x} e^{-\frac{x^2}{2\sigma^2 n}} \left( \sigma \text{erf} \left( \frac{|x|}{\sqrt{2\sigma} \sqrt{n(1 - 1/n)}} \right) + \sigma \text{erfc} \left( \frac{|x|}{\sqrt{2\sigma} \sqrt{n(n - 1)}} \right) \right),
\]

where \( \text{erf} \) is as usually the error function and \( \text{erfc} \) its complementary with respect to 1. The second addendum in the brackets gives non vanishing contribution only when \( x \) grows like \( n \), therefore we can ignore it. What remains is a gaussian plus a correction that is significant when \( x \) is near to the perturbation. In fact defining

\[
\text{erf}_\sigma(x) \equiv \int_0^x dt e^{-t^2/\sigma^2},
\]

we have that

\[
\frac{\text{erf}_\sigma(x)}{\sigma} = \text{erf} \left( \frac{x}{\sigma} \right),
\]

so we can rewrite our correction as

\[
\Delta_1^n(x) = \frac{\sqrt{2\pi} |d| \text{sign}(x)}{\sigma \sqrt{n}} e^{-\frac{x^2}{2\sigma^2 n}} \left( 1 - \text{erfc}_\sigma \left( \frac{|x|}{\sqrt{2} \sqrt{n(1 - 1/n)}} \right) \right).
\]

**Case \( \nu = 2 \):** In dimension two it is

\[
\int_{\frac{1}{n}}^{1 - \frac{1}{n}} d\alpha \frac{e^{-x^2/2\alpha^2}}{\alpha^2 (1 - \alpha)} = \frac{2n|B|^2}{|x|^2} e^{-\frac{(B - 1, x)^2}{2n}} \left( e^{-\frac{2n^2}{2n(n-1)}} - e^{-\frac{x^2 n^{-1}}{2}} + o(1, x) \right),
\]

where \( o(1, x) \) is a function of \( x \) vanishing when \( n \to \infty \) when \( x \) does not grow like \( n \). Thus we can write

\[
\Delta_2^n(x) = \frac{2|B| |d| \cos(d, x)}{n|x|} e^{-\frac{(B - 1, x)^2}{2n}} \left( e^{-\frac{2n^2}{2n(n-1)}} - e^{-\frac{x^2 n^{-1}}{2}} + o(1, x) \right).
\]

We notice that this term gives contribution only near the perturbation. Infact if \( x \) is fixed, i.e. not growing with \( n \), \( \Delta_n(x) \) is a correction of the same order of diffusion. Moreover when \( x \) approaches to the origin it vanishes, since for large \( n e^{-\frac{x^2 n^{-1}}{2}} \approx 1 \)

\[
\lim_{x \to 0} \frac{1 - e^{-\frac{x^2 n^{-1}}{2}}}{|x|} = 0.
\]

The correction disappears for larger \( x \). If for instance \( x \) grows like \( \sqrt{n} \) we easily see that \( \Delta_n(x) \) gives no contribution to the scale \( 1/n \).

**Case \( \nu = 3 \):**
The three dimensional case is qualitatively analogous to the two dimensional one. By an elementary change of variables $e^{-y} = e^{-(B^{-1}x,x)/2n\alpha}$ we can rewrite our integral as

$$\left(\frac{2n}{(B^{-1}x,x)}\right)^{3/2} \int \left(\frac{(B^{-1}x)}{2(n-1)}\right) dy e^{-y/2n}$$

$$= n^{3/2} e^{-(B^{-1}x,x)/2n} \sqrt{2\pi} \text{erf} \left(\frac{x}{\sqrt{2}}\right) - 2\sqrt{|B^{-1}|}|x|e^{-\frac{(B^{-1}x,x)}{2}},$$

up to corrections $o(n^{3/2})$, and we notice that the limiting value of the last integral for $x \to 0$ is $2n^{3/2}$. In this way we can write at the leading order in $n$

$$\Delta^3_n(x) = \frac{1}{n^{3/2}} \frac{\cos(d,x)}{|x|^2} e^{-(B^{-1}x,x)/2n} \left(\sqrt{2\pi} \text{erf} \left(\frac{x}{\sqrt{2}}\right) - 2\sqrt{|B^{-1}|}|x|e^{-\frac{(B^{-1}x,x)}{2}}\right).$$

Finally we see that the correction term is relevant only when $x$ is near the perturbation (though it vanishes at the perturbation point), but it gives higher order contributes in time to the asymptotics when $x$ is growing with $n$.

These are expected results in high dimension. Of course a similar analysis can be pursued starting by (18) for any value of the dimension $\nu$.

Conclusions and outlooks

In this paper we have studied a random walk on $\mathbb{Z}^\nu$ perturbed in one point. We have found corrections to the diffusion in the spirit of local limit theorem in the case of antisymmetric perturbation.

Natural extension of the result is the symmetric case. As we mentioned, technical difficulties arise due to more involved form of the transition probability. Although we cannot show details about that, it seems clear that the behaviour of the correction should remain qualitatively the same. This is supported also by comparison to other known results [3], [6].

Another interesting perspective could be the analysis of the more general case of several points in which a perturbation is present, sparse, or all confined in a finite region. This would make a connection with the studies about obstacles in random walk in random environment [9].

We hope to report soon on these topics.

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