Tails of Triangular Flows

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Abstract

Triangular maps are a construct in probability theory that allows the transformation of any source density to any target density. We consider flow based models that learn these triangular transformations which we call triangular flows and study the properties of these triangular flows with the goal of capturing heavy tailed target distributions. In one dimension, we prove that the density quantile functions of the source and target density can characterize properties of the increasing push-forward transformation and show that no Lipschitz continuous increasing map can transform a light-tailed source to a heavy-tailed target density. We further precisely relate the asymptotic behavior of these density quantile functions with the existence of certain function moments of distributions. These results allow us to give a precise asymptotic rate at which an increasing transformation must grow to capture the tail properties of a target given the source distribution. In the multivariate case, we show that any increasing triangular map transforming a light-tailed source density to a heavy-tailed target density must have all eigenvalues of the Jacobian to be unbounded. Our analysis suggests the importance of source distribution in capturing heavy-tailed distributions and we discuss the implications for flow based models.

1 Introduction

Increasing triangular maps are a recent construct in probability theory that can transform any source density to any target probability density [3]. The Knothe-Rosenblatt transformation [30; 18], [36, Ch.1], gives a heuristic construction of an increasing triangular map for transporting densities that is unique (up to null sets) [3]. These transformations provide a unified framework to study popular neural density estimation methods like normalizing flows [33; 32; 29] and autoregressive models [26; 14; 17; 33; 19] which provide a tractable method for evaluating a probability density [15]. Indeed, these methods are becoming increasingly attractive for task of multivariate density estimation in unsupervised machine learning.

This work is devoted to studying the properties of triangular flows that learn increasing triangular transformations when the target density is a heavy-tailed distribution. Heavy tailed analysis studies the phenomena governed by large movements and encompasses both statistical inference and probabilistic modelling [28]. Indeed, heavy-tail analysis is extensively used in diverse applications...
We summarize our main contributions as follows: 1) We show that density quantiles precisely characterize the slope of an (unique) increasing transformation. Subsequently, we give an exact characterisation of degree of heavy-tailedness of a distribution based on the asymptotic properties of the density quantile function. This allows us to clearly characterize the properties of an increasing transformation required to push a source density to any target density with varying tail behaviour respectively. Finally, we make precise the connection between the asymptotics of the density quantile function and existence of higher-order moments of a distribution. We use this to give a precise rate (which accounts for the relative heaviness of source and target densities) at which an increasing transformation must grow to capture the tail behaviour of the target density.

In §3 we extend these results for higher dimensions. We define multivariate heavy-tailed distributions as distributions whose marginals are heavy tailed in all directions and show that any increasing triangular map from a light-tailed distribution to a heavy-tailed distribution must have all diagonal entries of the Jacobian matrix (and hence all eigenvalues and the determinant) to be unbounded. We discuss the implications of our findings for neural density estimation in §3. We highlight the trade-off between choosing an appropriate source density and the “complexity” of the transformation required to learn a target density. We provide all the proofs in §A.

We summarize our main contributions as follows: 1) We show that density quantiles precisely capture the properties of a push-forward transformation, 2) We relate the properties of density quantiles to existence of functional moments and tail-properties allowing us to provide asymptotic rates for transformations required to capture heavy-tailed behaviour, 3) We reveal properties of density quantiles for certain classes of distributions both for one dimensions and higher-dimensions that might be of independent interest, 4) We precisely study the properties of increasing maps required to capture heavy-tailed behaviour, 5) We reveal the trade-off between choosing a “complex” source density and an “expressive” transformation for representing target densities and its implications for flow based models.

2 Preliminaries

Consider two probability density functions $p$ and $q$ (with respect to the Lebesgue measure) over the source domain $Z \subseteq \mathbb{R}^d$ and the target domain $X \subseteq \mathbb{R}^d$, respectively. There always exists a deterministic transformation $T : Z \rightarrow X$ (cf. [Ch.1, 36]) such that for all (measurable) set $B \subseteq X$,

$$\int_B q(x) \, dx = \int_{T^{-1}(B)} p(z) \, dz. \quad (1)$$

Specifically, by using the change of variables formula, i.e. $x = T(z)$, a diffeomorphic function $T$ can push forward a base random variable $z \sim p$ to a target random variable $x \sim q$ such that $q$ is the push forward of $p$ i.e. $q(x) = p(T^{-1}x)|\nabla T(T^{-1}x)|^{-1} =: T\#p$, where $|\nabla T(z)|$ is the absolute value of the determinant of the Jacobian of $T$.

Fortuitously, it is always possible to construct such a transformation $T$: we call a mapping $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ triangular if its $j$-th component $T_j$ only depends on the first $j$ variables $z_1, \ldots, z_j$. The name “triangular” comes from the fact that the Jacobian of $T$ is a triangular matrix function. We call $T$ increasing if for all $j \in [d]$, $T_j$ is an increasing function of $z_j$.

**Theorem 1** ([3]). For any two densities $p$ and $q$ over $Z = X = \mathbb{R}^d$, there exists a unique (up to null sets of $p$) increasing triangular map $T : Z \rightarrow X$ so that $q = T\#p$. 

Like financial risk-modelling wherein the financial returns and risk-management calculations require heavy-tailed analysis [3, 10], in data-networks where heavy-tailed distributions are observed for file sizes, transmission rates, transmission duration and network traffic [24, 8, 20], and in modelling insurance claim sizes and frequencies in order to set premiums efficiently to quantify the risk to the company [5, 10].

Specifically, we study triangular flows to represent multivariate heavy-tailed elliptical distributions often used for modeling financial data and in the theory of portfolio optimization. Indeed, the basis of modern portfolio optimization relies on the Gaussian distribution hypothesis [23, 34, 31]. However, as demonstrated by multiple studies [9, 12, 16], Gaussian distribution hypothesis cannot be justified for financial modelling and elliptical distributions are the suggested alternative particularly because they allow to retain certain desirable practical properties of normal distribution.

We begin our exposition in §3 where we show that in one-dimension, the density quantile functions of the source and the target probability density precisely characterize the properties of a push-forward transformation. Consider two probability density functions $p$ and $q$ (with respect to the Lebesgue measure) over the source domain $Z \subseteq \mathbb{R}^d$ and the target domain $X \subseteq \mathbb{R}^d$, respectively. There always exists a deterministic transformation $T : Z \rightarrow X$ (cf. [Ch.1, 36]) such that for all (measurable) set $B \subseteq X$,

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Before proceeding further, let us first give an example of a construction of an increasing triangular transformation $T$ to help better understand Theorem 1. This example will subsequently form the basis of our theoretical exposition in the paper.

**Example 1 (Increasing Rearrangement).** Let $p$ and $q$ be univariate probability densities with distribution function $F$ and $G$, respectively. One can define the increasing map $T = G^{-1} \circ F$ such that $q = T \# p$, where $G^{-1} : [0, 1] \to \mathbb{R}$ is the quantile function of $q$:

$$G^{-1}(u) := \inf \{ t : G(t) \geq u \}. \quad (2)$$

Indeed, if $Z \sim p$, one has that $F(Z) \sim$ uniform. Also, if $U \sim$ uniform, then $G^{-1}(U) \sim q$. Theorem 1 is a rigorous iteration of this univariate argument by repeatedly conditioning (a construction popularly known as the Knothe-Rosenblatt transformation [30; 18]). Note that the increasing property is essential for claiming the uniqueness of $T$ [1].

Thus, triangular mappings constitute an appealing function class to learn a target density. Indeed, many recent generative models in unsupervised machine learning are precisely special cases of this approach [15]. In this paper, we characterize the properties of such increasing triangular mappings $T$ required to learn a target density $q$ that is heavy-tailed from a source density $p$.

### 3 Properties of Univariate Transformations

Increasing Rearrangement is a unique increasing transformation between two densities. (cf. Example 1). Conveniently, we can analyze the slope of this transformation analytically. For a probability density $p$ over a domain $Z \subseteq \mathbb{R}$, let $F_p : Z \to [0, 1]$ denote the cumulative distribution function of $p$, $Q_p : [0, 1] \to Z$ be the quantile function given by $Q_p = F_p^{-1}$ and $f_Q_p : [0, 1] \to \mathbb{R}_+$ be the density quantile function with a functional form as $f_Q_p = 1/q'_p$. It is further given by the reciprocal of the derivative of the quantile function i.e. $f_Q_p = 1/q'_p$. The slope of $T$ such that $q := T \# p$ where $p, q$ are two densities is given by the ratio of the density quantile function of the source and the target distribution respectively, i.e.

$$T'(z) = \frac{p(z)}{q(T(z))} = \frac{p(F_p^{-1}(u))}{q(F_q^{-1}(u))}, \text{ where } u = F_p(z), \text{ i.e. } T'(z) = \frac{f_{Q_p}(u)}{f_{Q_q}(u)}. \quad (3)$$

**Theorem 2.** Let $p$ and $q$ be two densities and $T$ be an increasing map such that $q := T \# p$. If the density quantile $f_{Q_p}$ of $p$ shrinks to 0 at a rate slower than the density quantile $f_{Q_q}$ of $q$, then $T'(z)$ is asymptotically unbounded.

Clearly, the density quantile functions precisely characterize the slope of an increasing transformation. Moreover, we can further characterise the asymptotic properties of an increasing transformation using the asymptotics of density quantiles of distributions following [28, 1], who proved that the limiting behaviour of any density quantile function as $u \to 1^-$ (corresponding to right tails) is:

$$f_{Q_p}(u) \sim (1 - u)^\alpha, \quad \alpha > 0 \quad (4)$$

where $g(u) \sim h(u)$ implies that $\lim_{u \to 1^-} g(u)/h(u)$ is a finite constant.

**Example 2.** Let $p \sim \mathcal{N}(0, 1)$ and $q \sim t_1(0, 1)$. Then, $T$ such that $q := T \# p$ is given by:

$$T(z) = G^{-1} \circ F = \tan \left( \frac{\pi}{2} \text{erf} \left( \frac{z}{\sqrt{2}} \right) \right) \quad \& \quad T'(z) = \sqrt{\pi} e^{-\frac{z^2}{2}} \sec^2 \left( \frac{\pi}{2} \text{erf} \left( \frac{z}{\sqrt{2}} \right) \right) \quad (5)$$

where $\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds$ is the error function. Furthermore, $f_{Q_p}(u) \sim (1 - u) \left( -2 \log(1 - u) \right)^{1/2}$ and $f_{Q_q}(u) \sim (1 - u)^2$ and hence, $\lim_{z \to \infty} T'(z) = \lim_{u \to 1^-} (1 - u)^{-1} \left( -2 \log(1 - u) \right)^{1/2}$. Similarly, for $p \sim$ uniform[0,1]:

$$T(z) = G^{-1} \circ F = \tan \left( \pi \left( z - \frac{1}{2} \right) \right) \quad \& \quad T'(z) = \pi \sec^2 \left( \pi \left( z - \frac{1}{2} \right) \right) \quad (6)$$

and $f_{Q_p}(u) = 1$. Therefore, $\lim_{z \to \infty} T'(z) = \lim_{u \to 1^-} (1 - u)^{-2}$.

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1For instance, if $p$ is symmetric, then both $T = \text{id}$ and $T = -\text{id}$ push $p$ to itself.
Additionally, we can also define the limiting behaviour of the quantile function \( Q(u) \) as \( u \to 1^- \) as:

\[
Q(u) \sim (1 - u)^{-\gamma}, \quad \gamma = \alpha - 1.
\]  
(7)

The parameter \( \alpha \) is called the tail-exponent and defines the (right) tail-area of a distribution. Indeed, if for two distributions with tail exponents \( \alpha_1 \) and \( \alpha_2 \), if \( \alpha_1 > \alpha_2 \), the corresponding distribution has heavier tails relative to the other. The tail exponent \( \alpha \) allows us to define distributions based on their degree of heaviness as follows:

\[
\mathcal{H}_\alpha := \left\{ p : f Q_p \sim (1 - u)^\alpha \text{ as } u \to 1^- \right\},
\]  
(8)

Following [27], if \( 0 < \alpha < 1 \) the distributions are short-tailed, e.g. Uniform distribution. Here, we further show that a distribution has support bounded from above if and only if the right density quantile function has tail-exponent \( 0 < \alpha < 1 \).

**Proposition 1.** Let \( p \) be a density with \( f Q_p \sim (1 - u)^\alpha \) as \( u \to 1^- \). Then, \( 0 < \alpha < 1 \) iff \( \text{supp}(p) = [a, b] \) where \( b < \infty \) i.e. \( p \) has a support bounded from above.

\( \mathcal{H}_1 \) corresponds to a family of distributions for which all higher order moments exist. However, these distributions are relatively heavier tailed than short-tailed distributions and were termed as medium tailed distributions in [27], e.g. normal and exponential distribution. Additionally, for \( \alpha = 1 \), a more refined description of the asymptotic behaviour of quantile function can be given in terms of the shape parameter \( \beta \):

\[
f Q(u) \sim (1 - u) \left( \log \frac{1}{1 - u} \right)^{1-\beta}, \quad \text{and} \quad Q(u) \sim \left( \log \frac{1}{1 - u} \right)^{\beta}, \quad 0 \leq \beta \leq 1.
\]  
(9)

\( \beta \) determines the degree of heaviness in medium tailed distributions; the smaller the value of \( \beta \), the heavier the tails of the distribution e.g. exponential distribution has \( \beta = 1 \), and normal distribution has \( \beta = 0.5 \). Based on this, we can define

\[
\mathcal{H}_{1,\beta} = \left\{ p : f Q_p \sim (1 - u) \left( \log \frac{1}{1 - u} \right)^{1-\beta}, \quad 0 \leq \beta \leq 1 \right\}
\]  
(10)

Therefore, we have \( \mathcal{H}_1 = \bigcup_{0 \leq \beta \leq 1} \mathcal{H}_{1,\beta} \), and \( L = \bigcup_{0 < \alpha \leq 1} \mathcal{H}_\alpha \). Finally, heavy tailed distributions have \( \alpha > 1 \) e.g. Cauchy and \( t_\nu \). We next give a precise characterisation of asymptotic properties of a diffeomorphic transformation from one distribution to the other with varying tail behaviour in the following corollary of Theorem [2].

**Corollary 1.** Let \( p \in \mathcal{H}_{\alpha_p} \) be a source distribution, \( q \in \mathcal{H}_{\alpha_q} \) be a target distribution and \( T \) be an increasing transformation such that \( q := T_p \). Then,

- if \( \alpha_p > \alpha_q \), the slope of \( T \) converges asymptotically to 0
- if \( \alpha_p = \alpha_q \neq 1 \), the slope of \( T \) converges asymptotically to a finite constant
- if \( \alpha_p < \alpha_q \), the slope of \( T \) asymptotically diverges to infinity
- if \( \alpha_p = \alpha_q = 1 \),
  - if \( \beta_p > \beta_q \), the slope of \( T \) diverges to infinity asymptotically
  - if \( \beta_p = \beta_q \), the slope of \( T \) converges to a finite constant
  - if \( \beta_p < \beta_q \), the slope of \( T \) converges to zero asymptotically

Let us give another example to underscore the importance of using density quantiles to define tail-behaviour and the increasing push-forward transformations.

**Example 3 (Pushing uniform to normal).** Let \( p \) be uniform over \([0, 1]\) and \( q \sim N(\mu, \sigma^2) \) be normal distributed. The unique increasing transformation

\[
T(z) = G^{-1} \circ F = \mu + \sqrt{2\sigma} \cdot \text{erf}^{-1}(2z - 1) = \mu + \sqrt{2\sigma} \cdot \sum_{k=0}^{\infty} \frac{\pi^{k+1/2}c_k}{2k + 1} (z - \frac{1}{2})^{2k+1},
\]  
(11)

where \( \text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} \, ds \) is the error function, which was Taylor expanded in the last equality. The coefficients \( c_0 = 1 \) and \( c_k = \sum_{m=0}^{k-1} \frac{c_m c_{k-m-1}}{(m+1)(2m+1)} \). We observe that the derivative of \( T \) is an
We recall that there exists a unique bijective increasing triangular map
These definitions allow us to finally give the rate an increasing transformation must emulate to
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Theorem 4 Properties of Multivariate Transformations

We recall that there exists a unique bijective increasing triangular map \( T : \mathbb{R}^d \to \mathbb{R}^d \) that transforms a high-dimensional joint source density \( p \) to a target density \( q \). The \( j \)-th component \( T_j \) of \( T \) is given by \( x_j = T_j(x_1, \ldots, x_{j-1}, x_j) = F_{p,j|<j}^{-1} \circ F_{q,j|<j}(z_j) \) where \( F_{q,j|<j} \) is the cdf of the conditional distribution of \( X_j \) given \( X_{<j} := (X_1, \ldots, X_{j-1}) \). Analogous to our results in §3 we shall characterise the properties of \( T \) by studying the properties of \( |\nabla T| \) required to push \( p \) to \( q \) with varying tail properties. Evidently, for a triangular transformation \( T \), the determinant of the Jacobian i.e. \( |\nabla T| \) is just the product of the diagonals where each diagonal entry is given by \( \frac{\partial T_j}{\partial x_j} = \frac{f_{q,j|<j}}{f_{p,j|<j}} \).

Hence, by being able to characterize the properties of the conditional density quantiles, we shall be able to characterize the properties of \( T \). However, we first define the notion of tail-behaviour in multivariate distributions: A multivariate distribution is heavy-tailed if the marginal distributions in every direction on the (high-dimensional) sphere are heavy-tailed i.e. a distribution \( F(x), \ x \in \mathbb{R}^d \) is said to be heavy-tailed if for all vectors \( v \in B^1 \) where \( B_r = \{ v \in \mathbb{R}^d : \| v \| = r \} \) and \( \forall \lambda > 0, \int_{B_r} e^{\lambda v^T x} F(dx) = \infty \). This definition automatically implies that the univariate random variable \( |x| \) is heavy tailed. In particular, we will consider the class of elliptical distributions since they admit the same tail-behaviour in every direction.

Definition 4 (Elliptical distribution, §3). A random vector \( X \subseteq \mathbb{R}^d \) is said to be elliptically distributed denoted by \( X \sim \mathcal{E}_d(\mu, \Sigma, F_R) \) with \( \text{rank}(\Sigma) = r \) if and only if there exists a \( \mu \in \mathbb{R}^d \),

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Footnote: This condition takes the left-tail into account as well. Note that it is not necessary for both tails to have the same behaviour and our analysis will extend to such cases easily.
a matrix \( A \in \mathbb{R}^{d \times r} \) with maximal rank \( r \) and, a non-negative random variable \( R \), such that \( X = \mu + R A U^{(r)} \), where the random \( r \)-vector \( U \) is independent of \( R \) and is uniformly distributed over the unit sphere \( \mathcal{B}_{r-1} \), \( \Sigma = A^T A \) and \( F_R \) is the cumulative distribution function of the variate \( R \).

For ease in developing our results, we consider only full rank elliptical distributions i.e. \( \text{rank}(\Sigma) = d \). The spherical random vector \( U^{(r)} \) produces elliptically contoured density surfaces due to the transformation \( A \). The density function of an elliptical distribution as defined above\(^3\) is given by:

\[
    f_X(x) = |\det \Sigma|^{-\frac{1}{2}} g_R((x - \mu)^T \Sigma^{-1} (x - \mu)),
\]

where the function \( g_R(t) : [0, \infty) \to [0, \infty) \) is related to \( f_R \), the density of \( R \), by the equation:

\[
    f_R(r) = s_d r^{d-1} g_R(r^2), \quad \forall r \geq 0,
\]

here \( s_d = \frac{2^d \pi^{d/2}}{\Gamma(d/2)} \) is the area of a unit sphere. Thus, the tail properties of a random variable with an elliptical distribution i.e. \( X \sim \varepsilon_d(\mu, \Sigma, F_R) \) is determined by the generating random variable \( R \). Indeed, \( X \) is heavy-tailed in all directions if the univariate generating random variable \( R \) is heavy-tailed. Define

\[
    \forall l \in \mathbb{R}_+, \quad \mu_l = \int_0^\infty r^{l+d-1} g_R(r^2) \, dr = \frac{1}{s_d} \int_0^\infty r^l f_R(r) \, dr \quad (13)
\]

Intuitively, \( \mu_l \) is the \( l \)-th order moment of \( f_R \) when \( l \) is integer-valued. We can now generalize Definition\(^3\) to the multivariate case: the distribution \( \varepsilon_d(\mu, \Sigma, F_R) \) is \( \omega^{-1} \)-heavy iff \( \mu_l \) is finite for all \( l < \omega \) iff \( F_R \) is \( \omega^{-1} \)-heavy. Similarly, from Definition\(^2\) one has that \( \varepsilon_d(\mu, \Sigma, F_R) \) is \( \omega \)-heavy iff \( F_R \) is \( \omega \)-heavy.

Elliptical distributions have certain convenient properties: an affinely transformed elliptical random vector is elliptical. Let \( A \in \mathbb{R}^k \) and \( B \in \mathbb{R}^{k \times d} \). Consider the transformed vector \( Y = a + BX \) where \( X \Rightarrow \mu + R A U^{(r)} \). Then, \( Y \Rightarrow (a + B \mu) + R B A U^{(r)} \). In particular, if \( P \in \{0, 1\}^{d \times d} \) is a permutation matrix then \( \tilde{Y} := PX \) is also elliptically distributed and belongs to the same location-scale family as \( X \). Additionally, the marginal and conditional distributions of an elliptical distributions are also elliptical.

**Lemma 1** (Marginal distributions of an elliptical distribution are elliptical. \([11]\)). Let \( X = (X_1, X_2) \sim \varepsilon_d(\mu, \Sigma, F_R) \) where \( X_1 \subseteq \mathbb{R}^{d_1} \) and \( X_2 \subseteq \mathbb{R}^{d_2} \) partition \( X \) such that \( d_1 + d_2 = d \). Let \( \mu_1 \in \mathbb{R}^{d_1} \), \( \mu_2 \in \mathbb{R}^{d_2} \) and \( \Sigma_{i1} \in \mathbb{R}^{d_1 \times d_1} \), \( \Sigma_{12} \in \mathbb{R}^{d_1 \times d_2} \), \( \Sigma_{22} \in \mathbb{R}^{d_2 \times d_2} \) be the corresponding partitions of \( \mu \) and \( \Sigma \) respectively. Then, \( X_i \sim \varepsilon_d(\mu_i, \Sigma_{ii}, F_{R_i}) \), \( i \in \{1, 2\} \).

**Lemma 2** (Conditional distributions of an elliptical distribution are elliptical. \([4, 11]\)). Let \( X \sim \varepsilon_d(\mu, \Sigma, F_R) \) where \( \mu = (\mu_1, \mu_2) \in \mathbb{R}^d \) and \( \Sigma \in \mathbb{R}^{d \times d} \) is p.s.d with \( \text{rank}(\Sigma) = r \) and \( \Sigma = A^T A \) where \( X \Rightarrow \mu + R A U^{(r)} \). Further, let \( X_1 \subseteq \mathbb{R}^{d_1} \) and \( X_2 \subseteq \mathbb{R}^{d_2} \) partition \( X \) such that \( d_1 + d_2 = d \). Let \( \mu_1 \in \mathbb{R}^{d_1} \), \( \mu_2 \in \mathbb{R}^{d_2} \) and \( \Sigma_{i1} \in \mathbb{R}^{d_1 \times d_1} \), \( \Sigma_{12} \in \mathbb{R}^{d_1 \times d_2} \), \( \Sigma_{22} \in \mathbb{R}^{d_2 \times d_2} \) be the corresponding partitions of \( \mu \) and \( \Sigma \) respectively. If the conditional random vector \( X_2 | (X_1 = x_1) \) exists then

\[
    X_2 | (X_1 = x_1) \sim \mu^* + R^* \Sigma^* U^{(d_2)}
\]

where \( \mu^* = \mu_2 + \Sigma_{21} \Sigma_{11}^{-1} (x_1 - \mu_1), \quad \Sigma^* = \Sigma_{22} + \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}, \quad R^* = \left( (R^2 - h(x_1))^{1/2} | X_1 = x_1 \right) \),

where \( h(x_1) = (x_1 - \mu_1) \Sigma_{11}^{-1} (x_1 - \mu_1)^T \).

In our next result we analyze the tail-properties of the conditional distributions of heavy-tailed elliptical distribution.

**Proposition 2.** Under the same assumptions as in Lemma\(^2\) if \( X \sim \varepsilon_d(0, I, F_R) \) is \( \omega^{-1} \)-heavy, then the conditional distribution of \( X_2 | (X_1 = x_1) \) is \( (\omega + d_1)^{-1} \)-heavy.

We now state the main result of this section: an increasing triangular map \( T \) that transforms a light-tailed elliptical distribution to a heavy-tailed elliptical distribution has all diagonal entries of \( \nabla T \) to be unbounded.
Theorem 5. Let \( Z \sim \varepsilon_d(0, \mathcal{I}, F_S) \) and \( X \sim \varepsilon_d(0, \mathcal{I}, F_R) \) be two random variables with elliptical distributions with densities \( p \) and \( q \) respectively where \( F_R \) is heavier tailed than \( F_S \). If \( T : Z \rightarrow X \) is an increasing triangular map such that \( q := T_{\#} p \), then all diagonal entries of \( |\nabla T| \) are unbounded. Moreover, the determinant of the Jacobian of \( T \) is also unbounded.

We next give a general result for any transformation.

**Proof.** We provide the proof in Appendix A.

5 Triangular Flows and Approximation

Neural density estimation methods like autoregressive models [25, 2, 19, 35] and normalizing flows [29, 33, 32] provide a tractable way to evaluate the exact density and are increasingly being used for the purpose of multivariate density estimation in machine learning [17, 7, 24, 55, 14]. Invariably, these methods aim to learn a bijective, invertible and increasing transformation \( T \) from a simple, known source density to a desired target density such that the inverse \( T^{-1} \) and the Jacobian \( |\nabla T| \) are easy to compute.

As discussed in [15], most autoregressive models and normalizing flows at their core implement exactly a triangular map i.e. they learn a transformation \( T \) such that \( x_j = T_j(z_1, \cdots, z_{j-1}, z_j), \forall j \in [d]. \) [17] considered the affine map \( T_j(z_1, \cdots, z_{j-1}, z_j) = \mu_j(z_{<j}) + \sigma_j(z_{<j}) \cdot z_j. \) [14] alternatively replaced the affine form of [17] with a univariate neural network and [15] proposed to use the primitive of a univariate sum-of-squares of polynomials as the approximation of an increasing function. [13] and [28] proposed efficient implementations of these methods based on affine maps using binary masks that compute all the parameters of the transformation in a single pass of the network. Interestingly, all these methods compose several triangular maps in the hope that this composition of functions is “complex” enough to approximate any generic triangular map.

Here, we argue that there are two ways to learn a target density \( q \): First, as we discussed in Section 3.4 we can choose an appropriate base density \( p \) such that the resulting triangular transformation from \( p \) to \( q \) can be represented using simpler triangular transformations that are Lipschitz continuous, or, we can choose a base density \( p \) from a simple class of distributions (say Gaussian with identity covariance) and learn a “complex” triangular transformation via composition of several triangular transformations. However, we note here that composing several triangular maps is essentially tantamount to converting the source density to more complex base density such that the final composition of the triangular map transforms this to the target density. We propose an alternative that allows for simpler transformations to target density \( q \) by considering a more flexible class of source densities than the Gaussian distribution. One way would be to parametrize the source density as an elliptical distribution where the generating variate \( R \) is from a student-t distribution \( t_\nu \) with \( \nu \) degrees of freedom where \( \nu \) is a parameter to be learned along with the parameters of the transformation. It is evident from our exposition in Section 4 that such a model would require a Lipschitz continuous triangular map when learning heavy-tailed distributions.

Related question is how well one can approximate a distributions \( q \) with another distribution \( \tilde{q} := T_{\#} p \), where \( q \) is heavy tailed, \( \tilde{q} \) is light tailed, but \( T \) is not flexible enough to push tails, so that \( q \) is heavier than \( \tilde{q} \). One can consider several similarity metrics for this task. Let us start with Wasserstein distance, most natural for the flow theory. We wish to find a lower bound on the approximation error for \( W_1(q, \tilde{q}) = \inf_{\Pi \in \Pi(q, \tilde{q})} \int |x - y| \, d\gamma(x, y) \), where \( \Pi(q, \tilde{q}) \) is a set of all measures on \( \mathbb{R}^d \times \mathbb{R}^d \) with marginals \( q \) and \( \tilde{q} \) on the first and the second factors. Here we have two situations. First, assume that \( q \) doesn’t have the \( l \)-th moment. Then, because \( q \) is lighter than \( \tilde{q} \), \( q \neq \tilde{q} \). And because \( q \) doesn’t have \( l \)-th moments, \( W_1(q, \tilde{q}) = \infty \). The only possibility to have a finite distance in this case is exactly if \( q = \tilde{q} \), and the distance is zero. Alternatively, assume that \( q \) has the \( l \)-th moment. Then \( W_1(q, \tilde{q}) \) is finite. The measure \( q \) is Radon (as a finite measure on a second-countable space). Because the set of finite-supported Radon measures is dense in the metric space of Radon measures with \( W_1 \)-distance [22], one can approximate \( q \) arbitrary well with any finite-supported Radon measure \( q_0 \). Hence, varying \( T \) one can find \( \tilde{q} \) arbitrary close to \( q_0 \).
One can do similar analysis with $f$-divergence. The existence of the integral $D_f(\tilde{q}||q) = \int_{\mathbb{R}} f(\tilde{q}/q) \, d\tilde{x}$ depends on tail behaviour of both distributions among other properties. However, if the integral exists and is finite, one writes it as an integral over a compact set $C$ plus an integral over tails, and make the latter as small as wanted by simply increasing $C$. Hence, heaviness determines the possibility of approximation. In case when the target distribution $q$ has very heavy tails, the approximation reduces to representation problem, and one needs a flexible enough transformation $T$ in order to make $T \# p$ as heavy as $q$.

6 Conclusion

We studied the properties of triangular flows for capturing heavy-tailed distributions. We showed that density quantile functions play a central role in characterising the properties of increasing push-forward maps. Subsequently, we proved that for a triangular flow all eigenvalues of the Jacobian are unbounded when pushing a light-tailed distribution to a heavy-tailed distribution. We revealed properties of quantile and density quantile functions and related it to both existence of functional moments and heavy-tailedness of a distribution that can be of independent interest. As a by-product of our analysis, we demonstrated the trade-off between the complexity of source distribution and expressivity of transformations in capturing target densities in generative models. This work opens the possibility for multiple future directions: an interesting line of research will be to conduct holistic experiments to systematically analyze our results for example by considering flexible source distributions with parameters that can be trained along with the model. Another direction will be to analyze general flows that are non-triangular. Further, application of these insights into real-world problems of finance, insurance and networks might also be interesting.

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\[ \text{For example, topological properties: for KL-divergence, supp}(\tilde{q}) \text{ must be contained in supp}(q). \]
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A  Proofs

**Proposition 1.** Let \( p \) be a density with \( fQ_p \sim (1-u)^\alpha \) as \( u \to 1^- \). Then, \( 0 < \alpha < 1 \) iff \( \text{supp}(p) = [a,b] \) where \( b < \infty \) i.e. \( p \) has a support bounded from above.

**Proof.** Let \( 0 < \alpha < 1 \).

\[
fQ_p(u) \sim (1-u)^\alpha \iff Q(u) \sim (1-u)^\delta + c, \quad 0 < \delta < 1, \ c \text{ is a finite constant} \quad (14)
\]

\[
\iff \lim_{u \to 1^-} Q(u) \to c \quad (15)
\]

\[
\iff F_p^{-1}(1) = c \iff p \text{ has support bounded from above.} \quad (16)
\]

A similar argument proves the reverse direction. \( \square \)

**Theorem 3.** Let \( p \) be a distribution with \( Q_p(u) \sim (1-u)^{-\gamma} \) as \( u \to 1^- \). Then, \( \int_0^\infty z^\omega p(z) \, dz \) exists and is finite for some \( z_0 \) iff \( \omega < \frac{1}{\gamma} \).

**Proof.**

\[
\int_{z_0}^{\infty} z^\omega p(z) \, dz \text{ exists } \iff \int_{u_0}^{1} Q_p^\omega(u) \, du \text{ exists for some } u_0 > 0 \quad (17)
\]

\[
\iff \int_{u_0}^{1-\epsilon} Q_p^\omega(u) \, du \text{ exists } \& \int_{1-\epsilon}^{1} Q_p^\omega(u) \, du \text{ exists} \quad (18)
\]

The first integral is finite because the integrand is non-singular. For the second integrand, we can use the asymptotic behaviour of the quantile function by choosing \( \epsilon \) very close to 1. Subsequently, the integral exists and converges if and only if \( 1 - \omega \gamma > 0 \iff \omega < \frac{1}{\gamma} \). \( \square \)

**Theorem 4.** Let \( p \) be a \( \omega_p^{-1} \)-heavy distribution, \( q \) be a \( \omega_q^{-1} \)-heavy distribution and \( T \) be a diffeomorphism such that \( q \coloneqq T \# p \). Then for small \( \epsilon > 0 \), \( T(z) = o(\sqrt[\omega q]{z^{\omega_q}}) \).

**Proof.** The integral

\[
\mathbb{E}_q[|x|^{\omega_q-\epsilon}] = \int_{\mathbb{R}} |x|^{\omega_q-\epsilon} q(x) \, dx \\
= \int_{\mathbb{R}} |T(z)|^{\omega_q-\epsilon} p(z) \, dz \quad (19)
\]

converges for \( 0 < \epsilon < \omega_p \), because \( q \) is \( \omega_q^{-1} \)-heavy. Because \( T \) is a univariate diffeomorphism, it is a strictly monotone function. Without loss of generality, let us consider \( T \) to be positive increasing function and investigate the right asymptotic. Consider the function \( T(z)^{\omega_q-\epsilon} / z^{\omega_p} \) for big positive \( z \). Assume there is a sequence \( \{z_i\}_{i=1}^\infty \), such that \( \lim_{i} z_i = +\infty \) and the sequence \( T(z_i)^{\omega_q-\epsilon} / z_i^{\omega_p} \) does not converge to zero. In other words, there exists \( \alpha > 0 \), such that for any \( N > 0 \) there exists \( z_j > N \), such that \( T(z_j)^{\omega_q-\epsilon} / z_j^{\omega_p} > \alpha \). Let us work with this infinite sub-sequence \( \{z_j\} \). Because \( T(z) \) is increasing function, we can estimate its integral from the left by its left Riemann sum with respect to the sequence of points \( \{z_j\} \):

\[
\int_{N}^{\infty} T(z)^{\omega_q-\epsilon} p(z) \, dz \geq \sum_{j} T(z_j)^{\omega_q-\epsilon} p(\Delta z_j) > \alpha \sum_{j} z_j^{\omega_p} p(\Delta z_j).
\]

Since, \( p \) is \( \omega_p^{-1} \)-heavy, the series on the right hand side diverges as a left Riemann sum of a divergent integral. But this contradicts to the convergence of the integral on the left hand side. Hence, our assumption was wrong and for all sequences \( \{z_i\} \) we have: \( \lim_{i} T(z_i)^{\omega_q-\epsilon} / z_i^{\omega_p} = 0 \). Hence, \( |T(z)|^{\omega_q-\epsilon} = o(\sqrt[\omega_p]{z^{\omega_q}}) \) which leads to the desired result that \( |T(z)| = o(\sqrt[\omega_q]{z^{\omega_q}}) \). \( \square \)

**Proposition 2.** Under the same assumptions as in Lemma 2, if \( X \sim c_{d}(0, I, F_R) \) is \( \omega^{-1} \)-heavy, then the conditional distribution of \( X_2 | (X_1 = x_1) \) is \( (\omega + d_1)^{-1} \)-heavy.
Proof. The density function of the conditional \( p(x|X_1 = x_1) \) is proportional to \( g_R((x - \mu^*)^T \Sigma^{*-1}(x - \mu^*)) \), where \( x \in \mathbb{R}^{d_2} \) and \( g_R \) is the same function as for the distribution of \( X \) (see [4]). Then, because it is a \( d_2 \)-dimensional elliptical distribution, it is \( \alpha \)-heavy iff \( \mu_1 = \int_0^\infty r^{l+d_2-1} g_R(r^2) dr < \infty \) for all \( 0 < l < \alpha \). It is given that \( X \) is \( \omega^{-1} \)-heavy, which is equivalent to \( \int_0^\infty r^{l+d_2-1} g_R(r^2) dr < \infty \), \( \forall 0 < l < \omega \). Because \( d = d_1 + d_2 \), one gets that \( \int_0^\infty r^{l+d_2-1} g_R(r^2) dr < \infty \), \( \forall 0 < \tilde{l} < \omega + d_1 \), hence \( X_2|X_1 = x_1 \) is \( (\omega + d_1)^{-1} \)-heavy. \( \square \)

Theorem 5. Let \( Z \sim \varepsilon_d(0, I, F_Z) \) and \( X \sim \varepsilon_d(0, I, F_R) \) be two random variables with elliptical distributions with densities \( p \) and \( q \) respectively where \( F_R \) is heavier tailed than \( F_S \). If \( T : Z \to X \) is an increasing triangular map such that \( q := T \# p \), then all diagonal entries of \( |\nabla T| \) are unbounded. Moreover, the determinant of the Jacobian of \( T \) is also unbounded.

Proof. We need to show that
\[
\lim_{z_j \to \infty} \frac{\partial T_{ij}}{\partial z_j} = \lim_{z_j \to \infty} \frac{\partial Q_{ij}}{\partial z_j} \to \infty, \quad \forall j \in [d] \quad (21)
\]
Thus, all we need to show is that the generating variate \( R^* \) of the conditional distribution for the target is heavier than the generating variate \( S^* \) of the conditional distribution of the source. From [3] we know that the tail exponent in the asymptotics of the density quantile function characterize the degree of heaviness. Furthermore, we also know that asymptotical behaviour of the density quantile function is directly related to the asymptotical behaviour of the density function since if \( f \) is a density function, the cdf is given by \( F(x) = \int f(x) \, dx \), the quantile function therefore is \( Q = F^{-1} \) and the density quantile function is the reciprocal of the derivative of the quantile function i.e. \( f_Q = 1/Q' \). Hence, we need to ensure that asymptotically, the density of \( R^* \) is heavier than the density of \( S^* \).

Using the result of the cdf of a conditional distribution as given by Eq.(15) in [4] we have that asymptotically
\[
f_{R^*}(x) = C_d x^{d_1-d} f_R(x) \quad (22)
\]
where \( d_1 \) is the dimension of the partition that is being conditioned upon. Since, \( R \) is heavier tailed than \( S \), we have that \( R^* \) is heavier tailed than \( S^* \) for all the conditional distributions. \( \square \)

Theorem 6. Let \( Z \subseteq \mathbb{R}^d \) be a random variable with density function \( p(z) \) that is light-tailed and \( X \subseteq \mathbb{R}^d \) be a target random variable with density function \( q(x) \) that is heavy-tailed. If \( T(z) = (T_1(z), T_2(z), \ldots, T_d(z)) \) pushes forward \( p(z) \) to \( q(x) \) i.e. \( T : Z \to X \) such that \( q = T \# p \), then there exists an index \( i \in [d] \) such that \( \|\nabla_z T_i\| \) is unbounded.

Proof. We will prove this using contradiction; assume that \( \forall (i, j) \in [d]^2, \frac{\partial T_{ij}}{\partial z_j} \leq M < \infty \). Assume for simplicity that \( T(0) = c < \infty \). Therefore, we have
\[
T_i(z) - T_i(0) = \int_{r(0 \to z):0}^z \nabla T_i \cdot dr' \quad (23)
\]
\[
\implies |T_i(z) - T_i(0)| \leq M \sum_{i=1}^d |z_i| \quad (24)
\]
Since, \( q(z) \) is heavy tailed, \( \exists u \in B_1 \) such that \( \forall \kappa > 0 \)
\[
\int_{\mathbb{R}^d} e^{\kappa u^T z} q(z) \, dz = \infty \quad (25)
\]
i.e.
\[
\int_{\mathbb{R}^d} e^{\kappa u^T T(z)} p(z) \, dz = \infty \quad [\text{change of variables}] \quad (26)
\]
We have
\[
\int_{\mathbb{R}^d} e^{\kappa u^T T(z)} p(z) \, dz = \int_{\mathbb{R}^d} \prod_{i=1}^d e^{\kappa u_i T_i(z)} p(z) \, dz = \int_{\mathbb{R}^d} \prod_{i=1}^d \sum_{z_i} e^{\kappa u_i |T_i(z)|} p(z) \, dz
\]
(27)
\[
\leq C \int_{\mathbb{R}^d} \prod_{i=1}^d e^{\kappa u_i |T_i(z)|} p(z) \, dz, \quad [C = \text{finite constant}]
\]
(28)
\[
\leq C \int_{\mathbb{R}^d} \prod_{i=1}^d e^{\kappa M \sum_{i=1}^d |u_i| z_i} p(z) \, dz, \quad [u = \max |u_i|]
\]
(29)
\[
\leq \tilde{C} \int_{\mathbb{R}^d} e^{\kappa M \sum_{i=1}^d |u_i| z_i} p(z) \, dz
\]
(30)
\[
= \tilde{C} \int_{\mathbb{R}^d} e^{\kappa M \sum_{i=1}^d \text{sign}(z_i) |u_i| z_i} p(z) \, dz
\]
(31)
Partition $\mathbb{R}^d$ into $2^d$ sets $U_k$, $k \in [2^d]$, i.e. $\mathbb{R}^d = \bigcup_{k=1}^{2^d} U_k$ such that if $a = (a_1, a_2, \cdots, a_d) \in U_i$, and $b = (b_1, b_2, \cdots, b_d) \in U_j$, $i \neq j$, then there exists at least one index $m \in [d]$ such that $\text{sign}(a_m) \neq \text{sign}(b_m)$. Subsequently, we can rewrite the integral above as
\[
\tilde{C} \int_{\mathbb{R}^d} e^{\kappa M \sum_{i=1}^d \text{sign}(z_i) |u_i| z_i} p(z) \, dz = \tilde{C} \sum_{k=1}^{2^d} \int_{U_k} e^{\kappa M \sum_{i=1}^d \text{sign}(z_i) |u_i| z_i} p(z) \, dz
\]
(32)
\[
= \tilde{C} \sum_{k=1}^{2^d} \int_{U_k} e^{\kappa M w^T z} p(z) \, dz, \quad w_i = \text{sign}(z_i) \cdot |u_i|
\]
(33)
We will prove that each integral over the set $U_k$ is finite.
\[
\int_{U_k} e^{\kappa M w^T z} p(z) \, dz \leq \int_{\mathbb{R}^d} e^{\kappa M w^T z} p(z) \, dz
\]
(34)
Since $p(z)$ is light-tailed, we know that for any $u \in B_1$, there exists a $\lambda > 0$ such that $\int_{\mathbb{R}^d} e^{\lambda u^T z} p(z) \, dz < \infty$. Choose any $u \in B_1$, then for $\lambda = \kappa M/\|w\|$ we have that the above integral is finite. This directly implies that
\[
\sum_{k=1}^{2^d} \int_{U_k} e^{\kappa M w^T z} p(z) \, dz < \infty
\]
(36)
Hence, we have our contradiction. \qed