MULTIPLICITY BOUNDS AND THE SUBREPRESENTATION THEOREM FOR REAL SPHERICAL SPACES

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Abstract. Let $G$ be a real semi-simple Lie group and $H$ a closed subgroup which admits an open orbit on the flag manifold of a minimal parabolic subgroup. Let $V$ be a Harish-Chandra module. A uniform finite bound is given for the dimension of the space of $H$-fixed distribution vectors for $V$, and a related subrepresentation theorem is derived.

1. Introduction

Let $G$ be a connected real semi-simple Lie group and $P = MAN$ a minimal parabolic subgroup. Let $H < G$ be a closed and connected subgroup. We call $Z = G/H$ real spherical provided that there is an open $P$-orbit on $Z$. In [12] we have shown that this condition implies that there are only finitely many $P$-orbits on $Z$. The purpose of this paper is to explore the representation theoretic significance of real sphericity.

This paper relies in part on the forthcoming article [11] on the local structure of real spherical spaces, which forces us to assume that the Lie algebra of $H$ is an algebraic subalgebra of the Lie algebra of $G$.

For a Harish-Chandra module $V$ with smooth completion $V^\infty$, we show that if $G/H$ is real spherical with $PH$ open, then

$$\dim \text{Hom}_H(V^\infty, \mathbb{C}) \leq \dim(V/\bar{n}V)^{M\cap H},$$

with $\bar{n} = \text{Lie}(\bar{N})$ corresponding to an opposite parabolic subgroup.

In this context we recall that $V/\bar{n}V$ is finite dimensional, a consequence of the Casselman-Osborne lemma (see [14 Sect. 3.7]). For symmetric spaces (which are real spherical) finite dimensionality of $\text{Hom}_H(V^\infty, \mathbb{C})$ was originally established by van den Ban in [2 Cor. 2.2]. Finally we remark that certain bounds on $\dim \text{Hom}_H(V^\infty, \mathbb{C})$ were obtained with a different technique by Kobayashi and Oshima in [10 Thm. 2.4].

The bound in (1.1) is essentially sharp as equality is obtained for $H = \bar{N}$ and generic irreducible representations $V$. However, a statement is presented in Theorem 3.2 which in general can be stronger than (1.1).

The main part of the proof of (1.1) is elementary in the sense that it only invokes simple methods of ordinary differential equations, applied to generalized matrix coefficients on $Z$ (cf. the proof of the subrepresentation theorem in [14, Sect. 3.8]). However, these methods typically result in asymptotic expansions only. To prove

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that a matrix coefficient which is asymptotically zero (i.e. of super-exponential
decay) is in fact vanishing, we need more elaborate analytic methods. This is done
at the end of the paper, where we adapt some results from [5] and [1] to show that
the relevant system of differential equations has a regular singularity at infinity.

Suppose that \( PH \) is open. According to [11] there exists a parabolic subgroup
\( Q \supset P \) with a Levi decomposition \( Q = LU \) such that \( Q \cap H \subset L \) and \( L/(Q \cap H)Z(L) \)
is compact.

In the case that \( V \) is irreducible and \( H \)-spherical, i.e. \( \text{Hom}_H(V^\infty, \mathbb{C}) \neq \{0\} \), we prove the spherical subrepresentation theorem which asserts that \( V \) is a submodule of an induced module \( \text{Ind}^G_H \tau \), where \( \tau \) is an irreducible finite dimensional representation of \( L \) which is \( L \cap H \)-spherical. This was established for symmetric spaces by Delorme in [6]. For the group, it is the subrepresentation theorem of Casselman [4].

2. Some structure theory for real spherical spaces

Let \( G \) be a real reductive group and \( H \subset G \) a closed subgroup with finitely
many connected components. In what follows Lie algebras are always denoted by
corresponding lower case German letters, i.e. \( \mathfrak{g} \) is the Lie algebra of \( G \), \( \mathfrak{h} \) the Lie
algebra of \( H \), etc.

Let \( P \subset G \) be a minimal parabolic subgroup of \( G \). The homogeneous space
\( Z = G/H \) is called real spherical provided that \( P \) admits open orbits on \( Z \). This
means that by replacing \( P \) with a conjugate we can obtain that \( PH \) is open, that is,
\[
\mathfrak{g} = \mathfrak{p} + \mathfrak{h}.
\]

If \( \mathfrak{l} \) is a reductive Lie algebra, then we denote by \( \mathfrak{l}_n \) the sum of all simple non-
compact ideals of \( \mathfrak{l} \). We recall the following result from [11].

Proposition 2.1. Let \( H \subset G \) be algebraic groups over \( \mathbb{R} \), and assume \( Z = G/H \) is
real spherical. Let \( P \) be a minimal parabolic subgroup such that \( PH \) is open. Then
there exists a parabolic subgroup \( Q \) such that

1. \( Q \supset P \).
2. There is a Levi-decomposition \( Q = LU \) such that \( \mathfrak{l}_n \subset \mathfrak{q} \cap \mathfrak{h} \subset \mathfrak{l} \),
and in particular, \( L/(L \cap H)Z(L) \) is compact (here \( Z(L) \) denotes the center
of \( L \)).
3. \( L \cap H \) has finitely many components.
4. \( QH = PH \).

Throughout most of the paper we only need the properties (1)-(2), and for that
it clearly suffices to assume that \( Z \) is real spherical and locally algebraic, that is,
there are real algebraic groups \( H_1 \subset G_1 \) with Lie algebras \( \mathfrak{h} \) and \( \mathfrak{g} \). We assume
throughout that \( Z \) is locally algebraic, but emphasize that this assumption is only
used to obtain (1)-(2) above. In Remarks 3.3 and 4.3 we also need (3), which is a
consequence if \( G \) and \( H \) are algebraic since then \( L \) is algebraic. The last property
(4) is important for [11] but will not be needed here.

Note that in case \( H \) is symmetric, then the minimal \( \sigma \theta \)-stable parabolic sub-
groups ([3 Sect. 2]) satisfy (1)-(4). Here \( \theta \) is a Cartan involution which commutes
with the involution \( \sigma \) which defines \( \mathfrak{h} \).
If \( G/H \) is real spherical and \( P \subset Q = LU \) is as above, we let \( \theta \) be a Cartan involution of \( G \) which leaves \( L \) stable. The existence of \( \theta \) follows since \( L \) is a reductive subgroup of \( G \). Let \( \mathfrak{g} = \mathfrak{k} + \mathfrak{s} \) be the Cartan decomposition and \( K \subset G \) the corresponding maximal compact subgroup; then \( K_L = L \cap K \) is maximal compact in \( L \). Let \( a \) be a maximal abelian subspace in \( \mathfrak{k} \cap \mathfrak{s} \). We may assume \( A \) is contained in \( L \cap P \), since this intersection is a minimal parabolic subgroup in \( L \). Let \( L = K_LAN_L \) be an Iwasawa decomposition of \( L \) and put \( N = N_LU \). Note that \( a \) is maximal abelian in \( \mathfrak{s} \) as well. Let \( M \) be the centralizer of \( A \) in \( K \). Then \( P = MAN \), and it follows from \((2.2)\) above that \( a = a \cap \mathfrak{z}(l) + a \cap \mathfrak{h} \). Let \( a_Z \subset a \cap \mathfrak{z}(l) \) be a vector space complement to \( a \cap \mathfrak{h} \),

\[
a = a_Z \oplus (a \cap \mathfrak{h}),
\]

and \( m_Z \subset m \) a subspace such that \( a_Z + m_Z \) complements \( (a + m) \cap \mathfrak{h} \) in \( a + m \). Then we arrive at the direct sum decomposition

\[
(2.1) \quad \mathfrak{g} = \mathfrak{h} \oplus a_Z \oplus m_Z \oplus \mathfrak{u}.
\]

Let \( L_n \) be the analytic subgroup of \( L \) with Lie algebra \( \mathfrak{l}_n \). Since \( \mathfrak{l} = \mathfrak{m} + \mathfrak{a} + \mathfrak{l}_n \) and \( \mathfrak{l}_n \subset \mathfrak{h} \), and since \( M \) meets every component of \( L \) (see \([9, \text{Prop. 7.33}]\)), we conclude

\[
L = MAL_n
\]

and \( L_n \subset H \). For any Lie group \( J \) we denote by \( J_0 \) its identity component.

Lemma 2.2. There exists a vector subgroup \( A_h \subset MA \) such that

\[
(2.2) \quad (L \cap H)_0 = (M \cap H)_0 A_h L_n.
\]

Moreover, if \( L \cap H \) has finitely many connected components, then

\[
(2.3) \quad L \cap H = (M \cap H) A_h L_n.
\]

In this case \( L \cap H \) is a real reductive group and \( L \cap H \cap K \) is a maximal compact subgroup.

Proof. We have \( L \cap H = (MA \cap H)L_n \). Since the Lie algebra \( \mathfrak{m} + \mathfrak{a} \) is compact, then so is its intersection with \( \mathfrak{h} \). It follows that \( (MA \cap H)_0 \) is the direct product of a compact group \( S \) and a vector group \( A_h \). The compact group projects trivially to \( A \) along \( M \); hence \( S \subset M \) (whereas it need not be the case that \( A_h \subset A \)). This implies \((2.2)\). The argument for \((2.3)\) is the same. The last statement follows from \((2.3)\). \( \square \)

3. Finite multiplicity

We assume throughout this section that \( G/H \) is spherical with \( PH \) open and that a Cartan involution \( \theta \) of \( G \) has been chosen as described in the preceding section. Accordingly we write \( P = MAN \). Let \( \Sigma^+(\mathfrak{g}, \mathfrak{a}) \) be the set of positive roots of \( \mathfrak{a} \) in \( \mathfrak{g} \), and let \( \bar{P} = \theta(P) = MAN \) be the opposite parabolic subgroup.

For a Harish-Chandra module \( V \) we denote by \( V^\infty \) its unique smooth moderate growth Fréchet completion. Note that \( V^\infty \) is a \( G \)-Fréchet module, and we set \( V^{-\infty} := (V^\infty)' \) for its strong dual. Our goal is to provide a bound for the dimension of the space of \( H \)-invariants

\[
(V^{-\infty})^H := \text{Hom}_H(V^\infty, \mathbb{C})
\]

where \( \text{Hom} \) stands for continuous linear homomorphisms.
For \( v \in V^\infty \) and \( \eta \) a continuous \( H \)-invariant functional on \( V^\infty \), we form the matrix coefficient

\[
m_{v,\eta}(g) := \eta(\pi(g^{-1})v) \quad (g \in G).
\]

Note that \( m_{v,\eta} \) is a smooth function on \( G \), even analytic for \( v \in V \). We start with a general lemma which we prove later. We shall say that \( f : \mathbb{R} \to \mathbb{C} \) is of super-exponential decay for \( t \to \infty \) if \( f(t) = O(e^{\lambda t}) \) for all \( \lambda \in \mathbb{R} \).

**Lemma 3.1.** Let \( V \) be a Harish-Chandra module. Let \( X \in \mathfrak{a} \) be any element which is strictly anti-dominant with respect to \( P \) and for which \( \alpha(X) \in \mathbb{Q} \) for each \( \alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \). Fix \( \eta \in (V^{-\infty})^H \). Suppose that for all \( v \in V \) the function

\[
\mathbb{R} \ni t \mapsto m_{v,\eta}(\exp(tX)) \in \mathbb{C}
\]

is of super-exponential decay for \( t \to \infty \). Then \( \eta = 0 \).

**Proof.** Let \( v \in V \) and set \( F_v(t) = m_{v,\eta}(\exp(tX)) \). We may assume that \( \alpha(X) \) is an integer for each root \( \alpha \). With the new variable \( z = e^{-t} \) we will show in Section 5 that \( F_v \) then admits an expansion

\[
(3.1) \quad F_v(t) = \sum_{j=1}^N \sum_{k=0}^M z^{\lambda_j}(\log z)^k f_{j,k}(z) \quad (t \gg 0),
\]

where \( \lambda_j \in \mathbb{C} \) and the \( f_{j,k} \) are holomorphic functions in a neighborhood of \( z = 0 \) in \( \mathbb{C} \). The fact that \( F_v \) is of super-exponential decay then forces \( F_v = 0 \) (this follows for example from [14] Lemma 4.A.1.2]). Then \( F_v(0) = \eta(v) = 0 \), and hence \( \eta = 0 \) since \( v \in V \) was arbitrary. \( \square \)

For a Harish-Chandra module \( V \) we recall that \( V/\bar{\mathfrak{n}}V \) is a finite dimensional module for \( P = MAN \) with \( N \) acting trivially. Recall from Proposition 2.1 the parabolic subgroup \( Q = LU \) and its subalgebra \( \mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subset \mathfrak{p} \). Let \( \mathfrak{q} = \theta(\mathfrak{q}) = \mathfrak{l} + \bar{\mathfrak{u}} \) and define a subalgebra \( \bar{\mathfrak{q}}_1 \) of \( \bar{\mathfrak{q}} \) by

\[
(3.2) \quad \bar{\mathfrak{q}}_1 := (\mathfrak{l} \cap \mathfrak{h}) + \bar{\mathfrak{u}} \subset \mathfrak{h} + \mathfrak{u}.
\]

Note that \( \bar{\mathfrak{n}} \subset \mathfrak{l}_n + \bar{\mathfrak{u}} \subset \mathfrak{q}_1 \). Hence the quotient \( V/\bar{\mathfrak{q}}_1V \) is finite dimensional. Moreover, it carries a natural action of \( L \cap K \cap H \) since \( \bar{\mathfrak{q}}_1 \) is \( L \cap H \)-invariant.

**Theorem 3.2.** Let \( V \) be a Harish-Chandra module and \( V^\infty \) its unique smooth Fréchet completion. Then

\[
\dim \text{Hom}_H(V^\infty, \mathbb{C}) \leq \dim(V/\bar{\mathfrak{q}}_1V)^{L \cap K \cap H}.
\]

In particular (1.1) from the introduction is valid.

**Remark 3.3.** If \( L \cap H \) has finitely many components, then it follows from Lemma 2.2 that the trivial action of \( \mathfrak{l} \cap \mathfrak{h} \) on \( V/\bar{\mathfrak{q}}_1V \) lifts to an action of \( L \cap H \) which agrees with the natural action of \( L \cap H \cap K \). In this case then \( (V/\bar{\mathfrak{q}}_1V)^{L \cap K \cap H} = (V/\bar{\mathfrak{q}}_1V)^{L \cap H} \) for this action.

**Proof.** Set \( \Upsilon = (V^{-\infty})^H \). It is almost immediate from the definitions (see [13] eq. (3.7)) that there exists \( \delta \in \mathfrak{a}^* \) such that for all \( v \in V^\infty, \eta \in \Upsilon \) there is a constant \( C_{v,\eta} > 0 \) such that

\[
(3.3) \quad |m_{v,\eta}(a)| \leq C_{v,\eta}a^\delta \quad (a \in A^\infty).
\]
Fix an element $X \in \mathfrak{a}$ as in Lemma 3.1. After rescaling, we may assume that $\min_{\alpha \in \Sigma_{-}(\mathfrak{g},\mathfrak{a})} \alpha(X) = 1$. For all $v \in V^{\infty}$ and $\eta \in \Upsilon$ we define
\[
F_{v,\eta}(t) := m_{v,\eta}(\exp(tX)) \quad (t \geq 0).
\]
We say that $\Lambda \in \mathbb{R} \cup \{-\infty\}$ bounds $\eta \in \Upsilon$, provided for all $\Lambda' > \Lambda$ and all $v \in V$,\begin{equation}
(3.4) \quad \sup_{t \geq 0} e^{-t\Lambda'} |F_{v,\eta}(t)| < \infty.
\end{equation}
Let $\Upsilon_{\Lambda} \subset \Upsilon$ denote the space of elements bounded by $\Lambda$; then $\Upsilon_{\Lambda_{1}} \subset \Upsilon_{\Lambda_{2}}$ for $\Lambda_{1} \leq \Lambda_{2}$. It follows from (3.3) that
\begin{equation}
(3.5) \quad \Upsilon = \cup_{\Lambda} \Upsilon_{\Lambda},
\end{equation}
and from Lemma 3.1 that
\begin{equation}
(3.6) \quad \Upsilon_{-\infty} = \cap_{\Lambda} \Upsilon_{\Lambda} = \{0\}.
\end{equation}
The element $X$ acts on the space $(V/\bar{q}_{1}V)^{L \cap K \cap H}$. First note that $I \cap \mathfrak{h} = I_{n} + ((\mathfrak{m} + \mathfrak{a}) \cap \mathfrak{h})$ is normalized by $\mathfrak{a}$. Hence so is $\bar{q}_{1}$ and thus $X$ acts on $V/\bar{q}_{1}V$. Likewise, since $\mathfrak{a} = \mathfrak{a} \cap \mathfrak{z}(l) + \mathfrak{a} \cap \mathfrak{h}$ we find that $\text{Ad}(l)X = X \mod I \cap \mathfrak{h}$ for $l \in L \cap K \cap H$, and hence $X$ preserves the space of $L \cap K \cap H$-invariant vectors in the quotient.

We denote by $\Xi$ the set of values $-\text{Re} \lambda$ where $\lambda \in \mathbb{C}$ is an eigenvalue for $X$ on $(V/\bar{q}_{1}V)^{L \cap K \cap H}$, and write $\Xi = \{\mu_{1}, \ldots, \mu_{l}\}$ where
\[
\mu_{l+1} := -\infty < \mu_{l} < \cdots < \mu_{1} < \mu_{0} := +\infty.
\]
Let $m_{1}, \ldots, m_{l}$ denote the sums of the algebraic multiplicities of the corresponding eigenvalues $\lambda$. Then
\[
m_{1} + \cdots + m_{l} = n_{0} := \dim(V/\bar{q}_{1}V)^{L \cap K \cap H}.
\]
We shall prove:\begin{enumerate}[(1)]  
(1) $\Upsilon_{\Lambda} = \Upsilon_{\mu_{k+1}}$ for $\mu_{k+1} \leq \Lambda < \mu_{k}$ and $k = 0, \ldots, l$.  
(2) The codimension of $\Upsilon_{\mu_{k+1}}$ in $\Upsilon_{\mu_{k}}$ is at most $m_{k}$ ($k = 1, \ldots, l$).
\end{enumerate}
It is easily seen that these statements together with (3.5) and (3.6) imply the theorem. Before proving (1) and (2) we need some preparations.

Let $\bar{w}_{1}, \ldots, \bar{w}_{n_{0}}$ be a basis for $(V/\bar{q}_{1}V)^{L \cap K \cap H}$ and let $B$ denote the corresponding $n_{0} \times n_{0}$-matrix defined by
\[
X \bar{w}_{j} = \sum_{k} b_{jk} \bar{w}_{k}
\]
for $j = 1, \ldots, n_{0}$. We choose a representative $w_{j} \in V$ for each $\bar{w}_{j}$ and define
\[
u_{j} = Xw_{j} - \sum_{k} b_{jk}w_{k} \in \bar{q}_{1}V.
\]
We can arrange that $B$ consists of block matrices $B_{1}, \ldots, B_{l}$ along the diagonal such that each $B_{k}$ is an $m_{k} \times m_{k}$ matrix all of whose eigenvalues have real part $-\mu_{k}$. In the following we shall make the identification
\[
\mathbb{C}^{n_{0}} = \mathbb{C}^{m_{1}} \times \cdots \times \mathbb{C}^{m_{l}}
\]
and write elements $x \in \mathbb{C}^{n_{0}}$ accordingly as $x = (x_{1}, \ldots, x_{l})$.

For a given $\eta \in \Upsilon$ let
\[
F(t) = (F_{1}(t), \ldots, F_{l}(t)) \in \mathbb{C}^{n_{0}}
\]
where $F_k(t) \in \mathbb{C}^{m_k}$ is the $m_k$-tuple with entries $F_{w_j,\eta}(t)$, corresponding to the $k$-th block of $B$. Likewise we put

$$R(t) = (R_1(t), \ldots, R_l(t)) \in \mathbb{C}^{n_0}$$

where $R_k(t)$ has entries $F_{u_j,\eta}(t)$. Then for each $k$,

$$F_k'(t) = -B_k F_k(t) - R_k(t).$$

Hence

$$F_k(t) = e^{-tB_k}c_0 - e^{-tB_k} \int_0^t e^{sB_k} R_k(s) \, ds$$

where $c_0 = F_k(0)$.

Next we recall from [13, proof of Thm. 3.2] (with $a^+, \bar{a}$ interchanged by $a^-, \bar{a}$) that the elements in $\bar{a} \cap V$ satisfy improved bounds as follows. Assume $\eta \in \Upsilon_{\Lambda}$ and $\Lambda' > \Lambda$. Then (3.4) is valid for all $v \in V$, and it follows for all $u \in \bar{a} \cap V$ that

$$\sup_{t \geq 0} e^{-t(\Lambda'-1)} |F_{u,\eta}(t)| < \infty.$$  

The bound (3.8) is valid for $u \in \bar{q}_1 V$ as well. To see this we note first that since $a = a \cap \bar{q}(l) + a \cap h$ we have

$$m_{v,\eta}(\exp(tX)) = m_{v,\eta}(\exp(tY)) \quad (v \in V),$$

for some $Y \in a$ which centralizes $l$. Then $u = \sum_j X_j v_j + u'$ with $X_j \in l \cap h$, $v_j \in V$ and $u' \in \bar{a} \cap V$, and it follows that $Y X_j v_j = X_j Y v_j \in h \cap V$. Hence $m_{u,\eta}(\exp(tY)) = m_{u',\eta}(\exp(tY'))$. We conclude that

$$F_{u,\eta} = F_{u',\eta}$$

and hence (3.8) is valid as claimed. We conclude the existence of a constant such that

$$|R_k(t)| \leq C e^{t(\Lambda'-1)}.$$  

Based on (3.10) we shall provide the following two estimates for $F_k(t)$:

$$\sup_{t \geq 0} e^{-t\gamma} |F_k(t)| < \infty, \quad \forall \gamma > \max\{\mu_k, \Lambda - 1\}$$

for all $k = 1, \ldots, l$, and

$$\sup_{t \geq 0} e^{-t\gamma} |F_k(t)| < \infty, \quad \forall \gamma > \Lambda - 1$$

for those $k = 1, \ldots$ for which $\mu_k > \Lambda$.

Let $\gamma > \max\{\mu_k, \Lambda - 1\}$ and let $\Lambda' \in (\Lambda, \gamma + 1)$. It is clear that the first term in (3.7) is bounded by a polynomial times $e^{t\mu_k}$. Applying (3.10) we see that the integrand $e^{sB_k} R_k(s)$ of the second term is dominated by a polynomial times $e^{t\max\{\mu_k, \Lambda' - 1\}}$. It follows that $|F_k(t)|$ is dominated by a polynomial times $e^{t\max\{\mu_k, \Lambda' - 1\}}$, and this implies (3.11).

Before we prove the second estimate we note the following fact. If $\mu_k > \Lambda' - 1$, then $e^{sB_k} R_k(s)$ is integrable to infinity. Hence if $\mu_k > \Lambda - 1$, we can replace the solution formula (3.7) by

$$F_k(t) = e^{-tB_k}c_0 + e^{-tB_k} \int_t^\infty e^{sB_k} R_k(s) \, ds$$

where $c_0 = F_k(0)$. Hence

$$F_k(t) = e^{-tB_k}c_0 - e^{-tB_k} \int_0^t e^{sB_k} R_k(s) \, ds$$

where $c_0 = F_k(0)$. Hence

$$F_k(t) = e^{-tB_k}c_0 - e^{-tB_k} \int_0^t e^{sB_k} R_k(s) \, ds$$

where $c_0 = F_k(0)$. Hence

$$F_k(t) = e^{-tB_k}c_0 - e^{-tB_k} \int_0^t e^{sB_k} R_k(s) \, ds$$

where $c_0 = F_k(0)$.
where \( c_\infty = \lim_{s \to \infty} e^{sB_k}F_k(s) \). Note that with (3.10) the equation (3.13) implies (3.12) for every \( k \) for which the limit \( c_\infty \) vanishes.

For the proof of (3.12) we assume \( \mu_k > \Lambda \). Then it follows from (3.11) with \( \Lambda < \Lambda' < \mu_k \) that \( \lim_{s \to \infty} e^{sB_k}F_k(s) = 0 \). Hence \( c_\infty = 0 \) and (3.14) implies (3.12).

We are now ready to prove our claims (1) and (2). Let \( \Lambda = \max \Xi \cap \mathbb{R}_\leq \Lambda \) (with max 0 = -\( \infty \)). Let \( \eta \in \Upsilon_{\Lambda} \). We will show

(3.14) \[ \eta \in \Upsilon_{\max(\mu_\Lambda, \Lambda - 1)}. \]

By iteration this will then imply that \( \Upsilon_{\Lambda} = \Upsilon_{\mu_\Lambda} \) as stated in (1). Indeed, if \( \Lambda - 1 \leq \mu_\Lambda \), then this follows immediately. Otherwise (3.14) implies \( \Upsilon_{\Lambda} = \Upsilon_{\Lambda - 1} \).

Since \( \Lambda \) was arbitrary we reach \( \Upsilon_{\Lambda} = \Upsilon_{\mu_\Lambda} \) after finitely many steps.

Let \( \lambda' > \max\{\mu_\Lambda, \Lambda - 1\} \). We need to prove

(3.15) \[ \sup_{t \geq 0} e^{-t\lambda'} |F_{v,\eta}(t)| < \infty \]

for all \( v \in V \).

Note that for \( m \in L \cap K \cap H \) we have \( F_{mv,\eta} = F_{v,\eta} \) (see (3.9)) and hence \( F_{v,\eta} = F_{v_0,\eta} \), where \( v_0 \) is the projection of \( v \) to \( VL \cap K \cap H \). Hence for every \( v \in V \) we have that \( F_{v,\eta} \) is a linear combination of the functions \( F_{w,\eta} \) plus \( F_{u,\eta} \) for some \( u \in q_0V \). Thus it follows from (3.8) that it suffices to establish (3.15) for \( v = w_j \) for all \( j \), and hence it suffices to show

(3.16) \[ \sup_{t \geq 0} e^{-t\lambda'} |F_k(t)| < \infty \]

for \( k = 1, \ldots, l \). If \( \mu_k \leq \mu_\Lambda \), then this is clear from (3.11). On the other hand, if \( \mu_k > \mu_\Lambda \), then \( \mu_k > \Lambda \) and (3.16) follows from (3.12).

Next we show (2). Let \( 1 \leq k_0 \leq l \) and \( \eta \in \Upsilon_{\Lambda} \) where \( \Lambda = \mu_{k_0} \). Then \( \mu_k > \Lambda - 1 \) and according to (3.13) the limit \( \lim_{s \to \infty} e^{sB_k}F_k(s) \) exists for every \( k \leq k_0 \). Moreover, for \( k < k_0 \) we have \( \mu_k > \Lambda \), and the limit vanishes as seen in the proof of (3.12) above. If we assume that this limit is zero also for \( k = k_0 \), then (3.12) is valid for all \( k \leq k_0 \), from which we conclude as above that (3.14) holds, with \( \mu_\Lambda \) now replaced by \( \mu_{k_0 + 1} \). By step (1) this implies that \( \eta \in \Upsilon_{\mu_{k_0+1}} \).

Hence

\[ \Upsilon_{\mu_{k_0+1}} = \{ \eta \in \Upsilon_{\mu_{k_0}} \mid \lim_{s \to \infty} e^{sB_{k_0}}F_{k_0}(s) = 0 \}, \]

and as \( \eta \mapsto \lim_{s \to \infty} e^{sB_{k_0}}F_{k_0}(s) \) is linear into \( C^{m_{k_0}} \), (2) follows.

4. THE SPHERICAL SUBREPRESENTATION THEOREM

Let \( Z = G/H \) be real spherical with \( PH \) open, and recall the definition (3.2) of the subalgebra \( \bar{\mathfrak{q}}_1 \subset \bar{\mathfrak{q}} \). The algebraic version of the subrepresentation theorem is:

**Corollary 4.1.** Let \( V \) be a Harish-Chandra module and assume that \( \text{Hom}_H(V^\infty, \mathbb{C}) \neq \{0\} \). Then \( (V/\bar{\mathfrak{q}}_1V)^L \cap K \cap H \neq \{0\} \).

**Proof.** Immediate from Theorem 3.2. \( \square \)

Let \( \bar{Q} = \theta(Q) \) denote the parabolic subgroup opposite to \( Q \), with Levi decomposition \( \bar{Q} = L \bar{U} \). If \( \tau \) is a finite dimensional representation of \( \bar{Q} \), then we write \( \text{Ind}_{\bar{Q}}^G \) for the induced \((g, K)\)-module of \( K \)-finite smooth sections of the \( G \)-equivariant vector bundle \( \tau \times Q G \to \bar{Q} \backslash G \).
Theorem 4.2. Let $V$ be an irreducible Harish-Chandra module such that $\text{Hom}_G(V^\infty, C) \neq \{0\}$.

Then there exists a finite dimensional irreducible representation $\tau$ of $\bar{Q}$, trivial on $\bar{U}$, such that $\text{Hom}_{(l \cap h, L \cap H \cap K)}(\tau, C) \neq \{0\}$, and an embedding

$$V \hookrightarrow \text{Ind}^G_Q \tau.$$ 

Remark 4.3. Assume that $L \cap H$ has finitely many connected components, and recall (see Proposition 2.1) that this assumption is valid for example when $G$ and $H$ are algebraic. Then it follows from Lemma 2.2 that $\text{Hom}_{L \cap H}(\tau, C) \neq \{0\}$ for the representation $\tau$ above.

Proof. Let $\bar{q}_0 = l_n + \bar{u}$, then $\bar{q}_1 = \bar{q}_0 + (l \cap h)$. The space $V/\bar{q}_0 V$ is finite dimensional since $\bar{u} \subset \bar{q}_0$, and it carries compatible actions of $\bar{q}$ and $L \cap K$ since $\bar{q}_0$ is an $L \cap K$-invariant ideal in $\bar{q}$. Since $L$ is reductive the action of the pair $(l, L \cap K)$ lifts uniquely to a representation of $L$ on $V/\bar{q}_0 V$. We can then extend to an action of $Q = L\bar{U}$, which is trivial on $\bar{U}$ and compatible with the action of $\bar{q}$.

The quotient map $V/\bar{q}_0 V \to V/\bar{q}_1 V$ is clearly a homomorphism for the pair $(l \cap h, L \cap H \cap K)$. By Corollary 4.1 the module $V/\bar{q}_1 V$ has a non-zero vector fixed by the compact group $L \cap K \cap H$. Hence also its dual admits such a vector, and by composing with the quotient map we obtain that $\text{Hom}_{(l \cap h, L \cap H \cap K)}(V/\bar{q}_0 V, C) \neq \{0\}$.

Let $\tau$ be an irreducible $\bar{Q}$-subrepresentation of $V/\bar{q}_0 V$ for which

$$\text{Hom}_{(l \cap h, L \cap H \cap K)}(\tau, C) \neq \{0\}.$$ 

Since the quotient map $V/\bar{u} V \to V/\bar{q}_0 V$ is $L$-equivariant, we have

$$\text{Hom}_L(V/\bar{u} V, \tau) \neq \{0\}.$$ 

As $V$ is irreducible, the desired embedding follows by Frobenius reciprocity (see [7] Theorem 4.9), and note that our induction is not normalized. \hfill \Box

5. Regular singularities

The goal of this section is to provide a proof for the expansion (3.1). To begin with let us first recall that matrix coefficients on $Z$ satisfy certain systems of differential equations. For that we fix a Harish-Chandra module $V$ and a $K$-type $\tau$ occurring in $V$. We denote by $V[\tau]$ the $\tau$-isotypical part of $V$, and consider for $\eta \in (V^{-\infty})^H$ and $v \in V[\tau]$ the matrix coefficient

$$f(a) = m_{\nu, \eta}(a) = \eta(\pi(a^{-1})v)$$ 

on $A$ (where $A$ originates from the Iwasawa decomposition $G = KAN$ chosen in Section 2).

For simplicity we assume that $V$ is irreducible and obtain that the center $Z(g)$ of $\mathcal{U}(g)$ acts by scalars on $V$ (otherwise we replace the annihilating ideal of $V$ in $Z(g)$ with an ideal of finite co-dimension, and proceed as before). The theory of $\tau$-radial parts (see Remark 5.4 below) then gives a system of differential equations for $f$ on a subcone of $A$.

Let $R_1, \ldots, R_n$ be a basis of root vectors of $u$; say $R_j \in g^{\alpha_j}$ corresponds to the root $\alpha_j$. Set $Q_j := \theta(R_j) \in g^{-\alpha_j}$. For $t \in \mathbb{R}$ we let

$$a_t = \{X \in a \mid \alpha_j(X) < -t, \ j = 1, \ldots, n\}$$
and $A_t := \exp(a_t)$. Let $D_{\epsilon} := \{|z| < \epsilon\}^n \subset \mathbb{C}^n$ for $\epsilon > 0$, and define

$$
u : A_t \to \mathbb{C}^n, \quad a \mapsto \nu(a) := (a^{\alpha_j})_{j=1,\ldots,n}$$

where $a^{\alpha_j} = e^{\alpha_j(X)}$ for $a = \exp(X)$; then $\nu(A_t) \subset D_{\epsilon^{-1}}$.

**Lemma 5.1.** There exists $t \geq 0$ such that for all $a \in A_t$ one has

$$g = \text{Ad}(a^{-1})\mathfrak{t} + \mathfrak{a}_Z + \mathfrak{h}.$$

For symmetric spaces this is obtained with $t = 0$ in [2, Lemma 1.5].

**Proof.** Recall the decomposition (2.1). As $\mathfrak{m}_Z$ is centralized by $A$, we have $\mathfrak{m}_Z \subset \text{Ad}(a^{-1})\mathfrak{t}$ for every $a \in A$. Hence it suffices to show that each $R_j \in \mathfrak{u}$ decomposes according to (5.2). For all $1 \leq j \leq n$ we obtain from (2.1)

$$Q_j = X_{m,j} + X_{a,j} + X_{h,j} + \sum_{i=1}^n c_{ij}R_i$$

with $X_{m,j} \in \mathfrak{m}_Z$, $X_{a,j} \in \mathfrak{a}_Z$ and $X_{h,j} \in \mathfrak{h}$. For $a \in A$ we have

$$R_j = a^{\alpha_j} \text{Ad}(a^{-1})(Q_j + R_j) - a^{2\alpha_j}Q_j,$$

and inserting (5.3) for the last term in (5.4) we obtain

$$R_j + a^{2\alpha_j}\sum_{i=1}^n c_{ij}R_i$$

$$= a^{\alpha_j} \text{Ad}(a^{-1})\underbrace{(Q_j + R_j)}_{\in \mathfrak{t}} - a^{2\alpha_j}(X_{m,j} + X_{a,j} + X_{h,j}).$$

Consider the $n \times n$-matrix

$$1 + (c_{ij}a^{2\alpha_j})_{ij}.$$ 

It is invertible for $a \in A_t$ for sufficiently large $t$. This proves (5.2). □

In fact, a more precise version of the decomposition (5.2) is obtained as follows. Let $(X_j)_j$, $(Y_k)_k$ and $(Z_l)_l$ be fixed bases for $\mathfrak{t}$, $\mathfrak{a}_Z$ and $\mathfrak{h}$, respectively. Then the previous proof shows (see (5.1) and (5.5)) the following.

**Lemma 5.2.** There exists $t > 0$ such that for every $X \in \mathfrak{g}$ there exist holomorphic functions $f_j, g_k, h_l \in \mathcal{O}(D_{\epsilon^{-1}})$ such that for all $a \in A_t$ one has

$$X = \sum_{j=1}^{\dim \mathfrak{t}} f_j(\nu(a)) \text{Ad}(a^{-1})X_j + \sum_{k=1}^{\dim \mathfrak{a}_Z} g_k(\nu(a))Y_k + \sum_{l=1}^{\dim \mathfrak{h}} h_l(\nu(a))Z_l.$$

Moreover, if $X \in \mathfrak{u}$, then this can be attained with

$$f_j, g_k, h_l \in \mathcal{O}_0(D_{\epsilon^{-1}}) = \{\varphi \in \mathcal{O}(D_{\epsilon^{-1}}) | \varphi(0) = 0\}.$$

Let $(U_j)_j$ be a fixed homogeneous basis of $\mathcal{U}(\mathfrak{t})$, $(V_k)_k$ one of $\mathcal{U}(\mathfrak{a}_Z)$ and $(W_l)_l$ of $\mathcal{U}(\mathfrak{h})$. We then obtain the following from Lemma 5.1 and Lemma 5.2

**Lemma 5.3.** There exists $t > 0$ such that for every $u \in \mathcal{U}(\mathfrak{g})$ there exist $f_{j,k,l} \in \mathcal{O}(D_{\epsilon^{-1}})$ such that

$$u = \sum_{j,k,l} f_{j,k,l}(\nu(a)) \text{Ad}(a^{-1})U_jV_kW_l,$$

for all $a \in A_t$, where $j, k, l$ extend over all sets of indices with

$$\deg(U_j) + \deg(V_k) + \deg(W_l) \leq \deg(u).$$
Proof. Let $t$ be as in Lemma \[5.2\]. We proceed by induction on $\text{deg}(u)$, the case of degree zero being clear. Let $u \in \mathcal{U}(\mathfrak{g})$ and assume $u = Xv$ with $X \in \mathfrak{g}$ and $v \in \mathcal{U}(\mathfrak{g})$ of degree one less. Applying Lemma \[5.2\] to $X$, we see that it suffices to treat the case where $X \in a_Z + \mathfrak{h}$. We write $Xv = [X, v] + vX$ and apply the induction hypothesis to $[X, v]$. For the term $vX$ we apply the induction hypothesis to $v$ and write it as a linear combination of elements $\langle \text{Ad}(a^{-1})U_j \rangle V_k W_l$ with holomorphic coefficients and $\text{deg}(U_j) + \text{deg}(V_k) + \text{deg}(W_l) \leq \text{deg}(v)$. Next we write

$$\langle \text{Ad}(a^{-1})U_j \rangle V_k W_l X = \langle \text{Ad}(a^{-1})U_j \rangle V_k ([W_l, X] + XW_l).$$

The induction hypothesis applies to $V_k[W_l, X]$. The terms with $V_kXW_l$ already have the form asserted in \[5.6\]. \[\square\]

Remark 5.4. The decomposition \[5.6\] allows one to define the $\tau$-radial part of an element $u \in \mathcal{U}(\mathfrak{g})^M \cap H$. This goes as follows: We let $\mathcal{U}(\mathfrak{g})$ act on $C^\infty(G)$ by right differentiation:

$$(X \cdot f)(g) = \frac{d}{dt} \bigg|_{t=0} f(g \exp(tX)) \quad (f \in C^\infty(G), g \in G, X \in \mathfrak{g}).$$

Henceforth we regard smooth functions on $Z$ as right $H$-invariant functions on $G$. For a $K$-type $(\tau, W_\tau)$ we denote by $C^\infty(Z)_\tau$ the space of functions which are of left $K$-type $\tau$, for example the matrix coefficients $m_{v, \eta}$ with $v \in V[\tau]$.

It is easily seen from the Peter-Weyl theorem that for every $f \in C^\infty(Z)_\tau$ there exist a unique $\Phi \in C^\infty(Z, W_\tau \otimes W_{\tau^*})$ such that

$$f(gH) = \text{Tr}(\Phi(gH)), \quad gH \in Z,$$

where $\text{Tr}$ stands for the contraction $W_\tau \otimes W_{\tau^*} \to \mathbb{C}$. Moreover $\Phi(kgH) = (1 \otimes \tau^*(k))\Phi(gH)$ for $k \in K$. In particular, $\Phi(aH) \in W_\tau \otimes W_{\tau^*}^{M \cap H}$ for $a \in A$. Hence we have a finite sum

$$(5.7) \quad f(gH) = \sum_j \langle F_j(gH), w_j \rangle,$$

with $F_j \in C^\infty(Z, W_{\tau^*})$ and $w_j \in W_\tau$. Moreover, $F_j(kgH) = \tau^*(k)(gH)$ for all $k \in K, g \in G$.

Let $u \in \mathcal{U}(\mathfrak{g})^M \cap H$ and let $t > 0$ be as above. According to \[5.6\] we can write $u$ as a sum of $\langle \text{Ad}(a^{-1})U \rangle V W$ with $U \in \mathcal{U}(\mathfrak{f})^M \cap H$, $V \in \mathcal{U}(\mathfrak{a}_Z)$ and $W \in \mathcal{U}(\mathfrak{h})$ and coefficients depending holomorphically on $\iota(a)$ for $a \in A_t$. In order to compute $u \cdot f|_{A_t}$ we may assume that $W = 1$ as $f$ is right $H$-invariant. For $u = (\text{Ad}(a^{-1}))UV$ we then have

$$u \cdot f(a) = \sum_j \langle (V \cdot F_j)(a), U^t \cdot w_j \rangle.$$

We finally arrive at an action of $\mathcal{U}(\mathfrak{g})^M \cap H$ on $C^\infty(A_t, W_{\tau^*}^{M \cap H})$ given by

$$\text{rad}(u)(F)(a) := \sum \tau^*(U)(V \cdot F)(a),$$

a differential operator with $\text{End}(W_{\tau^*}^{M \cap H})$-valued coefficients. With this definition we find

$$uf(a) = \sum_j \langle \text{rad}(u)(F_j)(a), w_j \rangle$$

when $f$ is given by \[5.7\].
In particular for \( f = m_{w,0} \) the functions \( F_j \in C^\infty(A_t, W_r^{M \cap H}) \) obtained as above satisfy the system of differential equations

\[
\text{rad}(z)(F) = \chi_V(z)F \quad (z \in \mathcal{Z}(\mathfrak{g}))
\]

with \( \chi_V : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C} \) the infinitesimal character of \( V \).

In the decomposition (5.6) of \( u \in \mathcal{U}(\mathfrak{g}) \) we would like to restrict the middle parts \( V_k \) from \( \mathcal{U}(\mathfrak{a}_Z) \) to a fixed finite set of elements, independent of \( u \). This will be done at the cost of enlarging the product with an extra factor from \( \mathcal{Z}(\mathfrak{g}) \) (which act by scalars on \( V \)).

For that we need to recall parts of the construction of the Harish-Chandra homomorphism. The decomposition \( \mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{m} + \mathfrak{n} \) results in a direct sum decomposition

\[
\mathcal{U}(\mathfrak{g}) = (\mathfrak{n}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}) + \mathcal{U}(\mathfrak{a} + \mathfrak{m})
\]

and allows for a linear projection

\[
\mu_1 : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{a} + \mathfrak{m})
\]

Recall ([1, Lemma 3.6]) that \( \mu_1 \) restricts to an algebra homomorphism \( \mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{m} + \mathfrak{a}) \) and that for \( z \in \mathcal{Z}(\mathfrak{g}) \) of degree \( d \) one has

\[
z - \mu_1(z) \in \mathfrak{n}\mathcal{U}(\mathfrak{g})_{d-1}
\]

(where \( \mathcal{U}(\mathfrak{g})_{d-1} \) signifies elements of degree \( \leq d - 1 \)). It follows from the Harish-Chandra isomorphism theorem that \( \mathcal{Z}(\mathfrak{m} + \mathfrak{a}) \) is finitely generated over \( \mu_1(\mathcal{Z}(\mathfrak{g})) \).

More precisely (see [1, Lemma 3.7]), there exist elements \( v_1, \ldots, v_r \in \mathcal{Z}(\mathfrak{m} + \mathfrak{a}) \) such that every \( v \in \mathcal{Z}(\mathfrak{m} + \mathfrak{a}) \) can be written as

\[
v = \sum_{j=1}^r \mu_1(z_j)v_j
\]

with \( z_j \in \mathcal{Z}(\mathfrak{g}) \) and \( \deg(z_j) + \deg(v_j) \leq \deg(v) \) for each \( j \).

Further, as \( \mathfrak{a} \) and \( \mathfrak{m} \) commute and as \( \mathfrak{a} = \mathfrak{a}_Z + (\mathfrak{a} \cap \mathfrak{h}) \), we obtain an algebra homomorphism

\[
p : \mathcal{U}(\mathfrak{m} + \mathfrak{a}) \to \mathcal{U}(\mathfrak{a}_Z)
\]

with kernel \( (\mathfrak{m} + (\mathfrak{a} \cap \mathfrak{h}))\mathcal{U}(\mathfrak{m} + \mathfrak{a}) \). We compose with \( \mu_1 \) and obtain a linear map

\[
\mu_2 : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{a}_Z)
\]

The restriction to \( \mathcal{Z}(\mathfrak{g}) \) is an algebra homomorphism and will be denoted by \( \mu \). It follows from the above that

\[
z - \mu(z) \in \mathfrak{n}\mathcal{U}(\mathfrak{g})_{d-1} + (\mathfrak{m} + (\mathfrak{a} \cap \mathfrak{h}))\mathcal{U}(\mathfrak{m} + \mathfrak{a})_{d-1}
\]

for \( z \in \mathcal{Z}(\mathfrak{g}) \). Furthermore by applying \( p \) to (5.8) we see that \( \mathcal{U}(\mathfrak{a}_Z) \) is finitely generated over \( \mu(\mathcal{Z}(\mathfrak{g})) \) with generators \( p(v_j) \in \mathcal{U}(\mathfrak{a}_Z) \). Let \( \mathcal{Y} \) denote the finite set

\[
\mathcal{Y} = \{p(v_1), \ldots, p(v_r)\} \subset \mathcal{U}(\mathfrak{a}_Z).
\]

**Lemma 5.5.** For all \( n \in \mathbb{Z}_{\geq 0} \) there exists \( t = t_n > 0 \) such that for all \( u \in \mathcal{U}(\mathfrak{g}) \) with \( \deg(u) \leq n \) there exist

\[
U_j \in \mathcal{U}(\mathfrak{t}), V_j \in \mathcal{Y} \subset \mathcal{U}(\mathfrak{a}_Z), W_j \in \mathcal{U}(\mathfrak{h}), z_j \in \mathcal{Z}(\mathfrak{g})
\]

with

\[
\deg U_j + \deg V_j + \deg W_j + \deg z_j \leq n,
\]
and holomorphic functions $f_j \in \mathcal{O}(D_{e^{-s}})$ such that

\begin{equation}
(5.10) \quad u = \sum_{j=1}^{p} f_j(\iota(a))(\text{Ad}(a^{-1})U_j)V_jW_jz_j
\end{equation}

for all $a \in A_t$.

**Proof.** Note the dependence of $t$ on the degree of $u$, contrary to what was the case in Lemma 5.3. The proof will again be by induction on $n$, and the case $n = 0$ is again clear. Furthermore, we see from Lemma 5.3 that it suffices to establish the decomposition for $u \in \mathcal{U}(a_Z)_n$, since if $\deg(U_j) > 0$ or $\deg(W_i) > 0$ in (5.9), then $\deg(V_k) < n$ and the induction hypothesis applies.

Using (5.8) we write $u = \sum \mu(z_i)Y_i$ with $z_i \in Z(\mathfrak{g})$ and $Y_i \in \mathcal{Y}$, such that $\deg(z_i) + \deg(Y_i) \leq n$ for all $1 \leq i \leq r$. With $n = u + n_L$ and $n_L \subset \mathfrak{h}$ we obtain from (5.9) for each $i$,

$$
\mu(z_i) = z_i + \sum_{\mathfrak{m}} R_m u_{m,i} + \sum_{m'} S_{m',i} u'_m + \sum_{m''} T_{m'',i} u''_{m''} i
$$

with $u_{m,i}, u'_{m',i}, u''_{m'',i} \in \mathcal{U}(\mathfrak{g})$ all of degree $< \deg z_i$. Hence

$$
u = \sum_i z_iY_i + \sum_{i,m} R_m u_{m,i}Y_i + \sum_{i,m'} S_{m',i} u'_mY_i + \sum_{i,m''} T_{m'',i} u''_{m''}Y_i.
$$

The terms in the first sum already have the desired form. The terms with $S_{m',i}$ are dealt with directly by the induction hypothesis, and the terms with $T_{m'',i}$ are dealt with similarly after commuting $T_{m'',i}$ and $u''_{m'',i}Y_i$. Hence only the second sum remains, which is a sum of terms $Rv$ with $R \in u$ and $v \in \mathcal{U}(\mathfrak{g})_{n-1}$.

We use Lemma 5.2 to decompose $R$ as a linear combination of terms $\text{Ad}(a^{-1})X$, with $X \in \mathfrak{k}$, and basis vectors of $\mathfrak{a}_Z$ or $\mathfrak{h}$, and with coefficients from $\mathcal{O}_0(D_{e^{-s}})$ (for some fixed $s > 0$). By the induction hypothesis applied to $v$, the terms $(\text{Ad}(a^{-1})X)v$ again have the desired form (5.10). Here $a \in A_{t_{n-1}}$. We thus reach the conclusion that for each such $a$, our element $u \in \mathcal{U}(a_Z)_n$ is of form (5.10) plus elements of the form $g(\iota(a))w$ with $g \in \mathcal{O}_0(D_{e^{-s}})$ and $w \in \mathcal{U}(\mathfrak{g})_n$ independent of $a$. The same conclusion then applies to every element $u \in \mathcal{U}(\mathfrak{g})_n$, as observed in the beginning of the proof.

We do the above for a basis $u_1, \ldots, u_N$ of $\mathcal{U}(\mathfrak{g})_n$ and finally use the fact that an $N \times N$-matrix of the form $\mathbf{1}_{N \times N} + F(z)$ with $F$ holomorphic and $F(0) = 0$ is invertible for sufficiently small $z$. \qed

We can now give the proof of (3.1). For $v \in V[\tau]$ we consider the matrix coefficient $f(gH) = m_{v,\eta}(gH) = \eta(\pi(g)^{-1}v)$, which we recall is a joint eigenfunction for $Z(\mathfrak{g})$. As explained in Remark 5.4, $f$ is a sum of functions of the form $\langle F(gH), w \rangle$ where $w \in W_\tau$ and where $F \in C^\infty(Z, W_\tau)$ is $\tau$-spherical, that is, $F(kgH) = \tau^*(k)F(gH)$ for all $k \in K$, $g \in G$. Note that this implies $F(aH) \in W_{\tau}^M \cap H$ for $a \in A$. Furthermore, the action of elements from $\mathcal{U}(\mathfrak{g})^M \cap H$ is computed by taking radial parts, and $F$ is a joint eigenfunction for $Z(\mathfrak{g})$.

Let $\Pi \subset \mathfrak{a}^*$ denote the set of simple roots. For simplicity we assume $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ in the remainder of this section, so that $\Pi$ is a basis for $\mathfrak{a}^*$. Extension to the reductive case is elementary. Let $d = |\Pi| = \dim \mathfrak{a}$.
Proposition 5.6. Let $F \in C^\infty(Z,W_\tau^*)$ be a $\tau^*$-spherical joint eigenfunction for $Z\langle g \rangle$. There exist a neighborhood $D$ of 0 in $\mathbb{C}^d$, a number $M \in \mathbb{N}$, a finite set $S \subset \mathbb{C}^d$, and for each $s \in S$ and each multi-index $0 \leq |m| \leq M$ a holomorphic $W_{\tau^*}^{M\cap H}$-valued function $h_{s,m}$ on $D$ such that

$$F(aH) = \sum_{s \in S} \sum_{0 \leq |m| \leq M} z^s (\log z)^m h_{s,m}(z),$$

for all $a \in A$ such that $z = (a^\alpha)_{\alpha \in \Pi} \in D$.

Proof. Let $X_1, \ldots, X_d$ be the basis for $a$ which is dual to $\Pi$, and let $D$ be the maximal degree of the finite set of operators from $\mathcal{V} \subset \mathcal{U}(a\mathbb{Z})$. For each multi-index $k = (k_1, \ldots, k_d)$ with $|k| \leq D$ we define

$$F_k(gH) = X_1^{k_1} \cdots X_d^{k_d} F(gH), \quad gH \in Z,$$

a $W_{\tau^*}$-valued function. For each $j = 1, \ldots, d$ and each multi-index $l$ with $|l| \leq D$, the function $X_j F_l$ can then be determined by giving the monomial $X_j X_1^{k_1} \cdots X_d^{k_d} \in \mathcal{U}(a)$ an expression (5.10) according to Lemma 3.1, with $t = t_{D+1}$, and applying the theory of radial parts, as explained in Remark 5.4. It follows that on $A_1$, the function $X_j F_l$ is a combination of the $F_k$'s, with $|k| \leq D$ and with $\text{End}(W_{\tau^*}^{M\cap H})$-valued coefficients depending on $a \in A_1$.

We thus see that the vector-valued function $F(a) = (F_k(aH))_k$ satisfies a first order ordinary differential system

$$X_1 F(a) = M(a) F(a)$$

on $A_1$ with an operator $M(a)$. Furthermore, it follows from (5.10) that $M(a)$ extends to a holomorphic function of $z = (a^\alpha)_{j=1,\ldots,n} \in \mathbb{C}^n$ in a neighborhood of 0. In particular, $M(a)$ is then a holomorphic function of the smaller tuple $(a^\alpha)_{\alpha \in \Pi} \in \mathbb{C}^d$ in a neighborhood of 0, since every positive root is a sum of simple roots. With coordinates on $A$ given by $(a^\alpha)_{\alpha \in \Pi} \in \mathbb{C}^d$ we thus obtain a system of equations, which has a simple singularity at 0 according to [8] p. 702, Example 2]. The proposition now follows from [8] Thm. B.16].

By taking inner products with elements from $W_\tau$ we obtain a similar expansion of the matrix coefficients $m_{v,\eta}(a)$ for $a \in A_1$, for all $v \in V[\tau]$.

Corollary 5.7. Let $\eta \in (V^{-\infty})^H$ and $v \in V[\tau]$. There exist a neighborhood $D$ of 0 in $\mathbb{C}^d$, a number $M \in \mathbb{N}$, a finite set $S \subset \mathbb{C}^d$, and for each $s \in S$ and each multi-index $0 \leq |m| \leq M$ a holomorphic function $f_{s,m}$ on $D$ such that

$$m_{v,\eta}(a) = \sum_{s \in S} \sum_{0 \leq |m| \leq M} z^s (\log z)^m f_{s,m}(z),$$

for all $a \in A$ such that $z = (a^\alpha)_{\alpha \in \Pi} \in D$.

Now let $X \in a$ be such that $\alpha(X) \in \mathbb{Z}_-$ for each $\alpha \in \Pi$ and consider $m_{v,\eta}(a_s)$ for $s \to \infty$, where $a_s = \exp(sX)$. Note that $(a_s^\alpha)_{\alpha \in \Pi} \in D$ for $s \gg 0$ and that $a_s^\alpha$ is a positive integral power of $e^{-s}$ for each $\alpha \in \Pi$. Hence $f_{s,m}(a_s^\alpha)_{\alpha \in \Pi}$ extends to a holomorphic function of $\zeta = e^{-s}$. This completes the proof of (5.7) and hence of Lemma 3.1.
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