Gravitational waves in a spatially closed de Sitter spacetime

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Abstract Perturbation of gravitational fields may be decomposed into scalar, vector and tensor components. In this paper we concern with the evolution of tensor mode perturbations in a spatially closed de Sitter background of Robertson–Walker form. It may be thought as gravitational waves in a classical description. The chosen background has the advantage of being maximally extended and symmetric. Spatially flat models commonly emerge from inflationary scenarios are not completely extended. We first derive the general weak field equations. Then the form of the field equations in two special cases, planar and spherical waves, are obtained and their solutions are presented. The radiation field from an isolated source is calculated. We conclude with discussing the significance of the results and their implications.

1 Introduction

Here we first investigate the freely propagating gravitational field requiring no local sources for their existence in a particular background. As an essential feature of the analysis of general theory of small fluctuations, we assume that all departures from homogeneity and isotropy are small, so that they can be treated as first order perturbations. We focus our analysis on an unperturbed metric that has maximal extension and symmetry by taking $K = 1$ and presence of a positive cosmological constant. The background is de Sitter spacetime in slicing such that the spatial section is a 3-sphere. In the previous works mostly the case $K = 0$ we investigated extensively [1–5]. Even though in some works $K$ is not fixed for demonstrating the general field equations, for solving them usually $K = 0$ is imposed [6, 7]. The study of this particular problem is interesting and relevant to present day cosmology for the following. Seven-year data from WMAP with imposed astrophysical data put constraints on the basic parameters of cosmological models. The dark energy equation of state parameter is $-1.1 \pm 0.14$, consistent with the cosmological constant value of $-1$. While WMAP data alone cannot constrain the spatial curvature parameter of the observable universe $\Omega_k$ very well, combining the WMAP data with other distance indicators such as $H_0$, BAO, or supernovae can constrain $\Omega_k$. Assuming $\omega = -1$, we find $\Omega_A = 0.73/\pm 0.04$ and $\Omega_{\text{total}} = 1.02/\pm 0.02$. Even though in WMAP seven-year data it has been concluded as evidence in support of a flat universe, but in no way the data does not rule out the case of $K = 1$ [8]. In the nine-year data from WMAP the reported limit on spatial curvature parameter is $\Omega_k = -0.0027/\pm 0.0039$ [9]. There is much hope that Planck data reports release will make the situation more promising to settle this dispute. But Planck 2013 results XXVI find no evidence for a multiply-connected topology with a fundamental domain within the last scattering surface. Further Planck measurement of CMB polarization probably provides more definitive conclusions [10]. In the analysis of gravitational waves commonly the Minkowski metric is taken as the unperturbed background. According to the mentioned observational data, the universe is cosmological constant dominated in our era. So in the analysis of gravitational waves we should replace the Minkowski background with de Sitter metric. The essential point is that spatially open and flat de Sitter spacetimes are subspaces of spatially closed de Sitter space. The first two are geodesically incomplete while the third is geodesically complete and maximally extended. From the singularity point of view the issue of completeness is crucial for a spacetime to be non-singular. Taking the issue of completeness seriously, we have no way except to choose $K = 1$. By choosing the maximally extended de Sitter metric as our unperturbed background we include both cosmological and curvature terms in discussion of gravitational waves [11–15]. We begin by deriving the required linear field equations.
Then the solution of the obtained equation are discussed. At the end an attempt is made to solve the field equation by source.

2 Linear weak field equations

Supposed unperturbed metric components in Cartesian coordinate system are [1]
\[ \tilde{g}_{00} = -1, \quad \tilde{g}_{0i} = 0, \quad \tilde{g}_{ij} = a^2(t)\tilde{g}_{ij} \]
\[ a(t) = a \cosh(t/\alpha), \quad \tilde{g}_{ij} = \delta_{ij} + K \frac{x^i x^j}{1 - K x^2}, \]  
(1)

with the inverse metric
\[ \tilde{g}^{00} = -1, \quad \tilde{g}^{0i} = 0, \quad \tilde{g}^{ij} = a^2(t)\tilde{g}^{ij}, \]
\[ \tilde{g}^{ij} = (\delta^{ij} - K x^i x^j), \]  
(2)

where \( K \) is curvature constant and \( \alpha = \sqrt{\frac{2}{\Lambda}} \). The non-zero components of the metric compatible connections are
\[ \tilde{\Gamma}_{ij}^0 = \frac{1}{a} \left( \delta_{ij} + K \frac{x^i x^j}{1 - K x^2} \right) = \tilde{a} \tilde{g}_{ij}, \]
\[ \tilde{\Gamma}_{0j}^i = \frac{\dot{a}}{a} \delta_{ij}, \]
\[ \tilde{\Gamma}_{jk}^i = \tilde{\Gamma}_{jk}^i = K \tilde{g}_{jk} x^i. \]  
(3)

A dot stands for derivative with respect to time. Since we are working in a holonomic basis, then the connection is torsion-free or symmetric with respect to lower indices. Let us decompose the perturbed metric as:
\[ \tilde{g}_{\mu \nu} = \tilde{g}_{\mu \nu} + h_{\mu \nu}, \]  
(4)

where \( \tilde{g}_{\mu \nu} \) is defined by Eq. (1) and \( h_{\mu \nu} \) is small symmetric perturbation term. The inverse metric is perturbed by
\[ \chi^{\mu \nu} = g^{\mu \nu} - \tilde{g}^{\mu \nu} = -\tilde{g}^{\mu \rho} \tilde{g}^{\nu \sigma} h_{\rho \sigma}, \]  
(5)

with components
\[ \chi^{00} = -h_{00}, \quad \chi^{0i} = a^2(h_{0i} - K x^i \dot{h}_{00}), \]
\[ \chi^{ij} = -a^{-2}(h_{ij} - K x^i \dot{x}^k h_{kj} - K x^j \dot{x}^k h_{ki}) + K^2 x^i x^j \dot{x}^k x^l h_{kl}. \]  
(6)

Perturbation of the metric produces a perturbation to the affine connection [15]
\[ \delta \Gamma_{\nu \lambda}^\mu = \frac{1}{2} \tilde{g}^{\mu \rho} \left[ -2h_{\rho \lambda} \tilde{\Gamma}_{\nu \lambda}^\rho + \partial_{\nu} h_{\rho \lambda} + \partial_{\rho} h_{\nu \lambda} - \partial_{\lambda} h_{\nu \rho} \right]. \]  
(7)

Thus Eq. (7) gives the components of the perturbed affine connection as
\[ \delta \Gamma_{jk}^i = \frac{1}{2a^2} \left[ -2a \dot{a}(h_{i0} - K x^i x^j h_{0j}) \left( \delta_{jk} + K \frac{x^i x^j}{1 - K x^2} \right) \right. \]
\[ -2K(h_{im} - K x^i x^j h_{lm}) \left( \delta_{jk} + K \frac{x^i x^k}{1 - K x^2} \right) x^m \]
\[ + \partial_k h_{ij} + \partial_j h_{ik} - \partial_i h_{jk} \right] \]
\[ - K x^i x^j (\partial_k h_{ij} + \partial_j h_{ik} - \partial_i h_{jk}). \]  
(8)

\[ \delta \Gamma_{j0}^i = \frac{1}{2a^2} \left[ -2a \dot{a} h_{ij} + \dot{h}_{ij} + \partial_j h_{00} - \partial_i h_{j0} \right] \]
\[ + 2K x^i x^k \dot{a} h_{kj} - K x^i x^k \dot{h}_{kj} - K x^j x^k \dot{a} h_{j0} \]
\[ + K x^i x^k \delta_{jk} h_{j0} \right], \]  
(9)

\[ \delta \Gamma_{00}^i = \frac{1}{2a^2} \left[ 2\dot{a} h_{i0} - \partial_i h_{00} - 2K x^i x^j \dot{h}_{0j} + K x^i x^j \partial_j h_{00} \right], \]  
(10)

\[ \delta \Gamma_{0i}^0 = -\frac{1}{2} \dot{a} h_{i0} - \frac{1}{2} \dot{h}_{00}, \]  
(11)

\[ \delta \Gamma_{0i}^0 = -\frac{1}{2} h_{00}. \]  
(12)

The tensor mode perturbation to the metric can be put in the form
\[ h_{00} = 0, \quad h_{i0} = 0, \quad h_{ij} = a^2 D_{ij}, \]  
(14)

where \( D_{ij} \)s are functions of \( \mathbf{X} \) and \( t \), satisfying the conditions
\[ \tilde{g}^{ij} D_{ij} = 0, \quad \tilde{g}^{ij} \tilde{\nabla}_i D_{jk} = 0. \]  
(15)

The perturbation to the affine connection in tensor mode are
\[ \delta \Gamma_{00}^0 = \delta \Gamma_{10}^0 = \delta \Gamma_{00}^1 = 0, \]  
(16)

\[ \delta \Gamma_{ij}^0 = a \ddot{a} D_{ij} + \frac{a^2}{2} \dot{D}_{ij}, \]  
(17)

\[ \delta \Gamma_{j0}^i = a \ddot{a} D_{ij} + \frac{a^2}{2} \dot{D}_{ij}, \]  
(18)
The Einstein field equation without matter source for the tensor mode of perturbation gives
\[
\delta R_{jk} = -\Lambda a^2 D_{jk},
\]
where
\[
\delta R_{jk} = -(2\ddot a + a\dot a) D_{jk} - \frac{3}{2} \dot D_{jk} - \frac{a^2}{2} \ddot D_{jk}
\]
\[+ \frac{1}{2} \delta_{ij} \partial_i D_{jk} - 4K D_{jk} - K \frac{1}{2} (\partial_i \partial_m D_{jk}) x^i x^m \]
\[- \frac{3}{2} K x^m \partial_m D_{jk} - K (\partial_i D_{mj} + \partial_j D_{mk}) x^m \]
\[+ K^2 D_{ml} (\delta_{jk} + K \frac{x^j x^k}{1 - Kx^2}) x^m x^l. \]

The scale factor \(a(t)\) satisfies the Friedmann equation, so we get
\[
2\ddot a + a\dot a = \Lambda a^2 - 2K.
\]
Inserting Eq. (22) in Eq. (21) and Eq. (21) in Eq. (20), we would have
\[
\frac{3}{2} a \dot a \dot D_{jk} - \frac{a^2}{2} \ddot D_{jk} - 2K D_{jk} + \frac{1}{2} \delta_{ij} \partial_i D_{jk}
\]
\[- \frac{K}{2} (\partial_i \partial_m D_{jk}) x^i x^m - \frac{3}{2} K x^m \partial_m D_{jk} \]
\[- K (\partial_i D_{mj} + \partial_j D_{mk}) x^m \]
\[+ K D_{ml} (\delta_{jk} + K \frac{x^j x^k}{1 - Kx^2}) x^m x^l = 0. \]

It is straightforward to show that
\[
\frac{1}{2} \nabla^2 D_{jk} = \frac{1}{2} g^{mn} \nabla_m \nabla_n D_{jk} = \frac{1}{2} \partial_i \partial^i D_{jk}
\]
\[- \frac{K}{2} (\partial_i \partial_m D_{jk}) x^i x^m - K D_{jk} \]
\[- K x^i (\partial_k D_{lj} + \partial_j D_{lk}) - \frac{3}{2} K x^i \partial_i D_{jk} \]
\[+ K x^i x^j (\delta_{jk} + K \frac{x^j x^k}{1 - Kx^2}) D_{li}. \]

It remains to put Eq. (24) in Eq. (23), then we obtain the final equation:
\[
\frac{1}{2} \nabla^2 D_{jk} - \frac{3}{2} a \dot a \dot D_{jk} - \frac{a^2}{2} \ddot D_{jk} - K D_{jk} = 0.
\]

Our first task to establish the field equations is fulfilled. Next we look for special solutions of this field equation analogue to plane and spherical waves. For the plane wave like solutions, using Cartesian coordinate systems is suitable while for the spherical waves, polar coordinates \((\chi, \theta, \phi)\) are convenient.

### 3 Plane-wave analogue

In the case of flat models i.e. \(K = 0\) condition (15) reduces to
\[
D_{ij} = 0, \quad \delta_{ij} \partial_i D_{kj} = 0.
\]
Looking for a wave propagating in \(z\)-direction, Eq. (26) simply gives
\[
D_{ij} = 0,
\]
with two independent modes
\[
D_{+ij} = D_+ (z, t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and}
\]
\[
D_{xij} = D_x (z, t) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
where \(D(z, t)\) satisfies \(\Box^2 D(z, t) = 0\) with the well-known plane-wave solution.

In the case of \(K = 1\) an analogue solution for Eq. (15) exists. Following a lengthy calculation due to non-diagonal components of \(\tilde g_{ij}\) we obtain
\[
D_{+ij} = \frac{D_+ (z, t)}{\sqrt{1 - X^2}} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
\[
D_{xij} = \frac{D_x (z, t)}{\sqrt{1 - X^2(1 - y^2 - z^2)}} \times \begin{pmatrix} 1 & 2y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
By inserting Eqs. (29) and (30) in Eq. (25) with some manipulation we conclude that each mode, × and +, satisfies
\[
(1 - z^2) \frac{\partial^2}{\partial z^2} D(z, t) + 3z \frac{\partial}{\partial z} D(z, t) - D(z, t) + \frac{6D(z, t)}{1 - z^2} - 3 az \ddot{D}(z, t) - a^2 \ddot{D}(z, t) - 2D(z, t) = 0. \tag{31}
\]

We use method of separation of variables to find the solutions of Eq. (31). Then we may write
\[
D(z, t) = D(z) \hat{D}(t). \tag{32}
\]

Using Eq. (32), Eq. (31) leads to
\[
\frac{1 - z^2}{D_q(z, \partial_{z^2} D_q(z) + \frac{3z}{D_q(z)} \partial z D_q(z) - 1 + \frac{6}{1 - z^2}} = a^2 \frac{\ddot{D}(t)}{D(t)} + 3a \ddot{D}(t) + 2. \tag{33}
\]

Equation (33) may hold merely if each side is equal to a constant, i.e. we have
\[
\frac{1 - z^2}{D_q(z)} \partial_{z^2} D_q(z) + \frac{3z}{D_q(z)} \partial z D_q(z) - 1 + \frac{6}{1 - z^2} = -q^2, \tag{34}
\]
\[
a^2 \frac{\ddot{D}(t)}{D(t)} + 3a \ddot{D}(t) + 2 = -q^2, \tag{35}
\]

where \(q^2\) is an arbitrary positive constant. We should take it positive since we are looking for a periodic wave. Equations (34) and (35) can be written as
\[
(1 - z^2) \frac{\partial^2}{\partial z^2} D_q(z) + 3z \frac{\partial}{\partial z} D_q(z) + \left(q^2 - 1 + \frac{6}{1 - z^2}\right) D_q(z) = 0, \tag{36}
\]
and
\[
a^2 \ddot{D}_q(t) + 3a \ddot{D}_q(t) + (q^2 + 2) \ddot{D}_q(t) = 0. \tag{37}
\]

To solve Eq. (36) and finding \(D_q(z)\) we define
\[
D_q(z) = (1 - z^2) U_q(z). \tag{38}
\]

inserting Eq. (38) in Eq. (36) we get the following equation for \(U_q(z)\):
\[
(1 - z^2) \frac{d^2}{dz^2} U_q(z) - z \frac{d}{dz} U_q(z) + (q^2 + 3) U_q(z) = 0. \tag{39}
\]

We notice that the solutions of Eq. (39) may be written as Chebyshev polynomial of type I provided we take, \(q^2 = n^2 - 3\), where \(n\) is integer and \(U_n(z)\) is
\[
U_n(z) \propto \exp(\pm in \arccos z). \tag{40}
\]

So we have
\[
D_n(z) = (1 - z^2) \exp(\pm in \arccos z), \tag{41}
\]
and
\[
\int_{-1}^{1} (1 - z^2) \frac{d^2}{dz^2} D_n(z) D_m^*(z) = \frac{1}{\pi} \delta_{nm}. \tag{42}
\]

Next we examine the temporal dependence of this mode. Putting Eq. (41) in Eq. (37) gives
\[
a^2 \ddot{D}_n(t) + 3a \ddot{D}_n(t) + (n^2 - 1) \ddot{D}_n(t) = 0. \tag{43}
\]

It is convenient to define the conformal time \(\tau\) by
\[
d\tau = \frac{dt}{a(t)} \quad \text{where} \quad a(t) = a \cosh(t/\alpha). \tag{44}
\]

Integrating Eq. (44) gives
\[
\exp(t/\alpha) = \tan(\tau/2). \tag{45}
\]

Notice that \(t = -\infty, 0, +\infty\) corresponds to \(\tau = 0, \pi/2, \pi\), respectively. So while the domain of the coordinate time is \(-\infty < t < +\infty\), the domain of the conformal time is \(0 < \tau < \pi\).

We may recast Eq. (43) in terms of conformal time as
\[
\frac{d^2}{d\tau^2} \tilde{D}_n(\tau) - 2 \cot \tau \frac{d}{d\tau} \tilde{D}_n(\tau) + (n^2 - 1) \tilde{D}_n(\tau) = 0, \tag{46}
\]
where \(\tilde{D}_n(t) = \tilde{D}_n(\tau)\). To solve Eq. (46), let us define a new parameter \(Y = \cos \tau\) with domain \(-1 < Y < +1\) and
\(t = -\infty, 0, +\infty\) correspond to \(Y = 1, 0, -1\), respectively. In terms of the new parameter \(Y\), Eq. (46) becomes
\[
(1 - Y^2) \frac{d^2}{dY^2} \tilde{D}_n(Y) + Y \frac{d}{dY} \tilde{D}_n(Y) + (n^2 - 1) \tilde{D}_n(Y) = 0, \tag{47}
\]
where \(\tilde{D}_n(\tau) = \tilde{D}_n(Y)\).

If we define \(W_n(Y) = \frac{d}{dY} \tilde{D}_n(Y)\) and differentiate Eq. (47) with respect to \(Y\), this gives
\[
(1 - Y^2) \frac{d^2}{dY^2} W_n(Y) - Y \frac{d}{dY} W_n(Y) + n^2 W_n(Y) = 0. \tag{48}
\]
Again Eq. (48) is Chebyshev of the first kind and its solutions are
\[
W_n(Y) = \exp(\pm in \arccos Y). \tag{49}
\]

For last step we should solve
\[
\frac{d}{dY} \tilde{D}_n(Y) = \exp(\pm in \arccos Y). \tag{50}
\]
We have previously defined $Y = \cos \tau$, so we have $\tau = \arccos Y$. Let us recast Eq. (50) in terms of $\tau$, so it becomes
\[
\frac{d}{d\tau} \tilde{D}_n(\tau) = -\sin \tau e^{\pm i n \tau},
\]
with
\[
\tilde{D}_n(\tau) = \frac{1}{1 - n^2} (\cos \tau \mp i \sin \tau) e^{\pm i n \tau}. \tag{52}
\]

We may write
\[
\tilde{D}_n(z, \tau)
\]

\[
D_{+ij}(x, y, z, t) = \frac{D_x(x, t)}{\sqrt{1 - X^2}} \left( \begin{array}{ccc}
\frac{x^2(x^2-y^2)}{(1-x^2)(1-x^2-z^2)} & \frac{-xy}{(1-x^2)(1-x^2-z^2)} & \frac{xz}{(1-x^2)(1-x^2-z^2)} \\
\frac{-xy}{(1-x^2)(1-x^2-z^2)} & \frac{1}{(1-x^2-z^2)} & 0 \\
\frac{xz}{(1-x^2)(1-x^2-z^2)} & 0 & \frac{1}{(1-x^2-z^2)} \\
\end{array} \right), \tag{54}
\]

\[
D_{xij}(x, y, z, t) = \frac{D_x(x, t)}{(1-x^2-y^2)\sqrt{1-X^2}} \left( \begin{array}{ccc}
\frac{xy}{(1-x^2-y^2)} & \frac{xz(1-x^2+y^2)}{(1-x^2)(1-x^2-y^2)} & \frac{xy}{1-x^2} \\
\frac{xz(1-x^2+y^2)}{(1-x^2)(1-x^2-y^2)} & \frac{1}{(1-x^2-y^2)} & 1 \\
\frac{xy}{1-x^2} & 1 & 0 \\
\end{array} \right), \tag{55}
\]

\[
D_{+ij}(x, y, z, t) = \frac{D_y(y, t)}{\sqrt{1 - Y^2}} \left( \begin{array}{ccc}
\frac{1}{(1-y^2-z^2)} & \frac{xy}{(1-y^2-z^2)} & 0 \\
\frac{xy}{(1-y^2-z^2)} & \frac{1}{(1-y^2-z^2)} & \frac{-yz}{(1-y^2-z^2)} \\
0 & \frac{-yz}{(1-y^2-z^2)} & \frac{1}{(1-y^2-z^2)} \\
\end{array} \right), \tag{56}
\]

\[
D_{xij}(x, y, z, t) = \frac{D_y(y, t)}{(1-y^2-z^2)\sqrt{1-Y^2}} \left( \begin{array}{ccc}
0 & \frac{yz}{1-y^2} & \frac{xy}{1-y^2} \\
\frac{yz}{1-y^2} & 1 & \frac{xy}{1-y^2} \\
\frac{xy}{1-y^2} & \frac{1}{1-y^2} & \frac{2yz}{1-y^2} \\
\end{array} \right). \tag{57}
\]

where $D(x, t)$ and $D(y, t)$ are given by
\[
\tilde{D}_n(x, \tau) = \frac{1 - x^2}{1 - n^2} (\cos \tau \mp i \sin \tau)
\times \begin{cases}
\exp[\pm i n (\arccos x + \tau)] & n \neq 1 \\
\exp[\pm i n (\arccos x - \tau)] & n = 1
\end{cases} \tag{58}
\]

and
\[
\tilde{D}_n(y, \tau) = \frac{1 - y^2}{1 - n^2} (\cos \tau \mp i \sin \tau)
\times \begin{cases}
\exp[\pm i n (\arccos y + \tau)] & n \neq 1 \\
\exp[\pm i n (\arccos y - \tau)] & n = 1
\end{cases} \tag{59}
\]

It is important to notice that Eq. (58) and Eq. (59) may be obtained from Eq. (29) and Eq. (30), respectively, by the coordinate transformation $x \to z$, $y \to y$, and $z \to -x$. This leads us to write the solution of waves moving in an arbitrary direction. Let us assume this arbitrary direction is
\[
\hat{n}_1 = \sin \theta \sin \varphi, \quad \hat{n}_2 = \sin \theta \cos \varphi, \quad \hat{n}_3 = \cos \theta. \tag{60}
\]

The result can be obtained from Eqs. (29) and (30) by the coordinate transformation introduced by
\[
\begin{align*}
z & \to \hat{n} \cdot \mathbf{X}, \\
y & \to \frac{\hat{n}_3(\hat{n} \cdot \mathbf{X}) - z}{\sqrt{1 - \hat{n}_3^2}}, \\
x & \to \frac{\hat{n}_2 x - \hat{n}_1 y}{\sqrt{1 - \hat{n}_3^2}}.
\end{align*} \tag{61}
\]
We get

\[ D_{n+}^{\chi ij}(X, t) = A_{n+}^{\chi ij}(X, \hat{n})(1 - (\hat{n} \cdot X)^2) \frac{1}{1 - n^2}(\cos \tau \mp i n \sin \tau) \]

\[ \times \begin{cases} \exp[\pm i n(\arccos(\hat{n} \cdot X) + \tau)] & n \neq 1, \\ \exp[\pm i n(\arccos(\hat{n} \cdot X) - \tau)] & n = 1, \end{cases} \]

(62)

where the explicit forms of \( A_{n+}^{\chi ij}(X, \hat{n}) \) are listed in the Appendix.

This result may be used to expand a general function as linear superposition of these eigenfunctions, i.e. it should be replaced with \( \exp(ik_{\mu}x^\mu) \) in Fourier transformations.

4 Spherical wave analogue

To consider this case it is more suitable to work in polar coordinates, \( x^i = (\chi, \theta, \phi) \). In this basis the non-zero components of the unperturbed metric are

\[ \tilde{g}_{11} = 1, \quad \tilde{g}_{22} = \sin^2 \chi, \quad \tilde{g}_{33} = \sin^2 \chi \sin^2 \theta, \]

with the inverse

\[ \tilde{g}^{11} = 1, \quad \tilde{g}^{22} = \sin^{-2} \chi, \quad \tilde{g}^{33} = \sin^{-2} \chi \sin^{-2} \theta, \]

(63)

(64)

The non-zero components of the unperturbed connections are

\[ \Gamma_{22}^1 = -\sin \chi \cos \chi, \quad \Gamma_{33}^1 = -\sin \chi \cos \chi \sin^2 \theta, \]

\[ \Gamma_{21}^2 = \cot \chi, \quad \Gamma_{33}^2 = -\sin \theta \cos \theta, \]

\[ \Gamma_{31}^3 = \cot \chi, \quad \Gamma_{32}^3 = \cot \theta. \]

(65)

In this case \( \tilde{g}_{ij} \) is diagonal and the conditions (15) for a transverse wave give

\[ D_{ij} = 0. \]

(66)

We may distinguish two independent polarizations as

\[ D_{+ij}(\chi, \theta, t) = \frac{D_{ij}(\chi, t)}{\sin^2 \theta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\sin^2 \theta \end{pmatrix}, \]

(67)

\[ D_{\times ij}(\chi, \theta, t) = \frac{D_{ij}(\chi, t)}{\sin \theta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \]

(68)

Inserting Eqs. (64) and (65) in Eq. (25) and expressing \( \nabla^2 \) in polar coordinates, with a rather lengthy but straightforward calculation it can be shown that both \( D_{+}(\chi, t) \) and \( D_{\times}(\chi, t) \) must satisfy the same equation as

\[ \frac{\partial^2}{\partial \chi^2} D(\chi, t) - 2 \cot \chi \frac{\partial}{\partial \chi} D(\chi, t) + \frac{2}{\sin^2 \chi} D(\chi, t) \]

\[ - 3a \frac{\partial D(\chi, t)}{\partial \chi} - a^2 \frac{\partial^2 D(\chi, t)}{\partial \chi^2} = 0. \]

(69)

To solve Eq. (69) we may assume that

\[ D(\chi, t) = D(\chi) \frac{\dot{D}(t)}{D(t)}. \]

(70)

Then we have

\[ \frac{1}{D(\chi)} \frac{\partial^2}{\partial \chi^2} D(\chi) - 2 \cot \chi \frac{\partial}{\partial \chi} D(\chi) + \frac{2}{\sin^2 \chi} D(\chi) = -\left( \frac{\partial^2}{\partial \chi^2} \right) \]

\[ = a^2 \frac{\ddot{D}(t)}{D(t)} + 3a \frac{\dot{D}(t)}{D(t)}. \]

(71)

Equation (71) holds provided that each side is equal to a constant, i.e.

\[ \frac{1}{D(\chi)} \frac{\partial^2}{\partial \chi^2} D(\chi) - 2 \cot \chi \frac{\partial}{\partial \chi} D(\chi) + \frac{2}{\sin^2 \chi} D(\chi) = -(q^2 - 1), \]

(72)

\[ \frac{\partial^2}{\partial \chi^2} D(\chi) + 3a \frac{\partial}{\partial \chi} D(\chi) + (q^2 - 1) D(\chi) = 0. \]

(73)

where \( q^2 \) is an arbitrary positive constant. So we have

\[ \frac{\partial^2}{\partial \chi^2} D(\chi) - 2 \cot \chi \frac{\partial}{\partial \chi} D(\chi) + \left( q^2 + \frac{2}{\sin^2 \chi} \right) D(\chi) = 0, \]

(74)

\[ a^2 \frac{\ddot{D}(t)}{D(t)} + 3a \frac{\dot{D}(t)}{D(t)} + (q^2 - 1) \dot{D}(t) = 0. \]

(75)

To solve Eq. (74) for \( D_q(\chi) \) we may define a new parameter \( X = \cos \chi \) and \( D(\chi) = \hat{D}(X) \), then Eq. (74) gives

\[ (1 - X^2) \frac{d^2}{dX^2} \hat{D}(X) + X \frac{d}{dX} \hat{D}(X) \]

\[ + \left( q^2 - 1 + \frac{2}{1 - X^2} \right) \hat{D}(X) = 0. \]

(76)

Equation (76) has a solution as

\[ \hat{D}_q(X) = (1 - X^2)^{1/2} \frac{d}{dX} U_q(X), \]

(77)

where \( U_q(X) \) satisfy the following equation:

\[ (1 - X^2) \frac{d^2}{dX^2} U_q(X) + X \frac{d}{dX} U_q(X) + (q^2 - 1) U_q(X) = 0. \]

(78)
Now we take \( V_q(X) = \frac{d}{dX} U_q(X) \), which satisfies
\[
(1 - X^2) \frac{d^2}{dX^2} V_q(X) - X \frac{d}{dX} V_q(X) + (q^2 - 1) V_q(X) = 0.
\] (79)

Equation (79) is a Chebyshev type I provided that we take \( q = n \). Then we have
\[
V_n(x) = \exp(\pm in \arccos x),
\] (80)
and
\[
D_n(\chi) = \sin \chi \exp(\pm in \chi).
\] (81)

The temporal part is the same as plane-wave analogue equation (52) and we have
\[
D_n(\chi, t) = \frac{\cos(\tau \pm in \sin \tau)}{1 - n^2} \sin \chi \left( \exp(\pm in(\chi + \tau)) \right. \left. \exp(\pm in(\chi - \tau)) \right).
\] (82)

It is interesting to note that in the case of flat models, i.e. \( K = 0 \), Eq. (74) in the \((r, \theta, \phi)\) basis takes the form
\[
\frac{d^2}{dr^2} D(r) - \frac{2}{r} \frac{d}{dr} D(r) + \frac{2}{r^2} D(r) = -q^2 D(r),
\] (83)
which has the solution
\[
D_q(r) \propto r e^{\pm iq r},
\] (84)
where \( q \) can be any arbitrary real number. If we consider the ratio \( \frac{h_{22}}{g_{22}} \) we get
\[
\frac{h_{22}}{g_{22}} \propto \frac{1}{\sin \chi} \left\{ \exp(\pm in(\chi + \tau)) \right. \left. \exp(\pm in(\chi - \tau)) \right\} \quad \text{for} \quad K = 1
\] (85)
\[
\frac{h_{22}}{g_{22}} \propto \frac{1}{r} \exp(\pm iq r), \quad \text{for} \quad K = 0.
\] (86)

Equations (85) and (86) both decrease by increasing the radial coordinate. The radiation fields (EM or GW) always decreases as inverse of radial coordinate regardless it is dipole or quadrupole field. Its dependence on the source properly appears as a factor which could be dipole or quadrupole.

5 The effect of gravitational waves

To obtain a measure of the waves effect in this background, we consider the rotation of the nearby particles as described by the geodesic deviation equation. For some nearly particles with four-velocity \( U^\mu(x) \) and separation vector \( S^\mu \), we have
\[
\frac{D^2 S^\mu}{dt^2} = R_{\nu\rho\sigma} U^\mu U^\nu S^\sigma.
\] (87)
Since the Riemann tensor is already first order for test particles that are moving slowly we may write \( U^\mu = (1, 0, 0, 0) \) in Eq. (87). So in computing Eq. (87) we only need \( R_{00j} \) which is
\[
R_{00j} = \frac{\tilde{a}}{a} \delta_{ij} + \frac{1}{2} \tilde{g}^{ik} \tilde{D}_{kj} + \frac{\tilde{a}}{a} \tilde{g}^{ik} \tilde{D}_{kj}.
\] (88)

For slowly moving particles we have \( \tau = \dot{x} = t \) to lowest order so the geodesic deviation equation becomes
\[
\frac{\partial^2 S^i}{\partial t^2} = R_{i00} S^j.
\] (89)
The first term in Eq. (88) will become an exponential expansion which is a general characteristic of de Sitter space. Of course this is not the effect of gravitational waves and we may ignore it. Making use of Eq. (43) and Eq. (52) and noticing that \( n \gg 1 \). The contribution of last term in Eq. (89) vanishes, so we have
\[
\frac{\partial^2 S^i}{\partial t^2} = \frac{1}{2} \frac{\partial^2}{\partial t^2} D^i S^j.
\] (90)

To be specific we choose Eq. (29), as a wave moving in \( z \)-direction, so we have
\[
D^i_{+j} = \frac{D_{+}(z, t)}{\sqrt{1 - X^2}(1 - z^2)} \times \left( \begin{array}{cccc}
1 & \frac{x y}{1 - y^2 - z^2} & \frac{x z}{1 - y^2 - z^2} & 0 \\
-x y & 1 & 0 & 0 \\
-x z & 0 & 1 & 0 \\
0 & 0 & 0 & 1 - x^2 - z^2
\end{array} \right).
\] (91)

The eigenvalues of matrix equation (91) are
\[
\lambda = 0,
\lambda_\pm = \pm \sqrt{\frac{D_{+}(z, t)}{1 - X^2}(1 - z^2)} \times \sqrt{1 - \frac{x^2 y^2}{(1 - y^2 - z^2)(1 - x^2 - z^2)}}.
\] (92)

Certainly the \( z \)-component of the separation vector is not affected by this gravitational waves. As the case of flat space a circle of particles hit by a gravitational wave has an oscillatory motion but in closed spaces the normal axis of oscillation rotate at different location of spacetime.

6 Generation of gravitational waves

With presenting plane-wave solutions to the linearized vacuum field equation, it remains to discuss the generation of gravitational radiation by source. For this purpose it is necessary to consider the equation coupled to matter. Making
use of completeness relation of the eigenfunction of vacuum equation we may write the solution of the field equation with source as

\[ D_{ij}(x, \tau) = +16\pi G \sum_n \sum_M \int d^2\hat{n} \int d^3 x' \frac{d^3 x'}{\sin^2 \tau'} \]

\[ \times \frac{d\tau'}{\sin^2 \tau'} (1 - (\hat{n} \cdot x)^2)(1 - (\hat{n} \cdot x')^2) \]

\[ \times \exp[i(n \cdot \arccos(\hat{n} \cdot x) - \arccos(\hat{n} \cdot x'))] \]

\[ \times (\cos \tau + im \sin \tau) \]

\[ \times (\cos \tau' - im \sin \tau') \exp[-im(\tau - \tau')] \]

\[ \times \frac{T_{ij}(x', \tau')}{(1 - m^2)^2(n^2 - m^2)}. \]

(93)

We assume the source is isolated, far away, and slowly moving. This implies \(|x'| \ll |x|\). Also we take the distance to the source is not of cosmic scale so that \(|x| \ll 1\). With these approximations Eq. (93) may be written as

\[ D_{ij}(x, \tau) = +16\pi G \sum_n \sum_M \int d^2\hat{n} \int d^3 x' \frac{d^3 x'}{\sin^2 \tau'} \]

\[ \times \exp[in(x - x') \cdot \hat{n}] \]

\[ \times [\cos(\tau - \tau') + im \sin(\tau - \tau')] \]

\[ + (m^2 - 1) \sin \tau \sin \tau' \]

\[ \times \exp[-im(\tau - \tau')] \]

\[ \frac{T_{ij}(x', \tau')}{(1 - m^2)^2(n^2 - m^2)}. \]

(94)

Performing summation on \(m\)

\[ \sum_m \frac{\cos(\tau - \tau') + im \sin(\tau - \tau') + (m^2 - 1) \sin \tau \sin \tau'}{(1 - m^2)^2(n^2 - m^2)} \]

\[ \times \exp(-im(\tau - \tau')) \]

\[ = \frac{2\pi \theta(\tau - \tau')}{(1 - n^2)2n} \left[ -\cos(\tau - \tau') \sin(n(\tau - \tau')) + n \sin(\tau - \tau') \cos(n(\tau - \tau')) + (n^2 - 1) \sin \tau \sin \tau' \right]. \]

(95)

Integrating on \(\hat{n}\) and putting Eq. (95) in Eq. (94) gives

\[ D_{ij}(x, \tau) = -16\pi G \sum_n \frac{\sin(nR)}{n^2(1 - n^2)} \int d\tau' \theta(\tau - \tau') \]

\[ \times \frac{\cos(\tau - \tau') \sin(n(\tau - \tau')) - n \sin(\tau - \tau') \cos(n(\tau - \tau')) + (n^2 - 1) \sin \tau \sin \tau'}{\sin \tau^2} \]

\[ \int d^3 x' T_{ij}(x', \tau'). \]

(96)

Since the source is localized and spacetime locally looks flat, we have

\[ T_{\mu\nu}^{\mu\nu} = 0. \]

(97)

Putting \(\mu = 0\) in Eq. (97) and differentiating with respect to \(x^0\) gives

\[ T_{00}^{00} = -T_{0i}^{0i}. \]

(98)

Then putting \(\mu = k\) in Eq. (97) gives

\[ T_{ik}^{00} + T_{ij}^{ik} = 0. \]

(99)

Combining Eq. (98) and Eq. (99) leads to

\[ T_{00}^{00} + T_{ik}^{ik} = 0. \]

(100)

Now we multiply both sides of Eq. (100) by \(x^m x^n\) and integrate by parts two times over all space, ignoring the surface terms, finishing up with

\[ \int T_{mn}(x, \tau) d^3 x' = \frac{1}{2} \frac{\partial^2}{\partial t^2} \int T_{0m} x^n x^n d^3 x \]

\[ = \frac{1}{2} \ddot{I}_{nn}(t), \]

(101)

where \(I_{nn}\) is the quadrupole moment of the mass distribution of the source. Inserting Eq. (101) in Eq. (96) gives

\[ D_{ij}(x, \tau) = -16\pi G \sum_n \frac{\sin(nR)}{n^2(1 - n^2)} \int d\tau' \theta(\tau - \tau') \]

\[ \times \left[ \cos(\tau - \tau') \sin(n(\tau - \tau')) + n \sin(\tau - \tau') \cos(n(\tau - \tau')) + (n^2 - 1) \sin \tau \sin \tau' \right] \ddot{I}_{ij}(\tau'). \]

(102)
The result of summation on \( n \) gives the retarded solution as
\[
D_{ij}(x, \tau) = \frac{8\pi G}{R} \int \frac{d\tau'}{\sin^2 \tau'} \theta(\tau - \tau') \theta(\tau - R') \times 
\frac{2\pi i}{4} \left[ \frac{1}{2} \cos(\tau - \tau') \cos(\tau' - R) + \frac{1}{2} \sin(\tau - \tau') \sin(\tau - \tau' - R) \right] \tilde{I}_{ij}.
\]
Therefore we have
\[
D_{ij}(x, \tau) = \frac{2\pi^2 G}{R} i \int \frac{d\tau'}{\sin^2 \tau'} \tilde{I}_{ij}(\tau - \tau') \theta(\tau - R') \times 
\frac{2\pi^2 G}{R} i \left[ I_{ij}(\tau - R) \right].
\]
The result depends on the first time derivative of the moment at retarded time.

7 Discussion

Our investigations show that in analysis of gravitational waves the background of de Sitter with \( K = +1 \) fundamentally differs from the scale-free de Sitter with \( K = 0 \). We found the wave numbers should be discrete as already it has been realized that the spectrum of the Laplacian in spherical space is always discrete [16]. Another relevant feature is the existence of cut off on the long-wavelength of gravitational waves. This may be tested by the measurement of dipole and higher multi-pole moments of the CMBR anisotropy which contains information about the long-wavelength portion of the spectrum of energy density produced the large scale galactic structure of the universe. These are sensitive to the presence of long-wavelength perturbations. The obtained eigenmodes are the fundamental tools of analysis of cosmic evolution of perturbations in spatially closed models. In the formation of large scale structure and study of anisotropies of CMBR we should use these eigenmodes to expand the perturbations. Another feature of the obtained eigenmodes is that they are effectively transverse in zone near the origin and at far distances, in contrast to flat spacetimes, this is not so. For example a wave moving in \( z \)-direction its \( 3j \) components are not vanishing exactly but in near zone may be approximated to zero. It has been shown that \( h_{ij} = a^2 D_{ij} \propto \frac{\delta h_{(x,t)}}{\sin^2 \tau} \). This means that in a collapsing phase i.e. in the time interval \(-\infty \leq t \leq 0 \) corresponding to conformal time \( 0 \leq \tau \leq \pi/2 \), the perturbation is decaying while in the expanding phase, i.e. in the time interval \( 0 \leq t \leq +\infty \) corresponding to conformal time \( \pi/2 \leq \tau \leq \pi \), it is growing. Of course always the smallness conditions of the perturbation with respect to the unperturbed metric holds. The amplitude of the gravitational wave changes with time as \( h_{ij} \propto \frac{1 + (\tau^2 - 1) \sin^2 \tau}{(\tau^2 - 1) \sin^2 \tau} \). So its growth is significant for the modes that satisfy the condition \( \sqrt{n^2 - 1} \sin^2 \tau \ll 1 \), where it changes as \( |h_{ij}| \propto \frac{1}{\sqrt{n^2 - 1} \sin^2 \tau} \) and changes are relatively smooth. Singularities appearing in the solutions are of coordinate type, where we may see them in the unperturbed metric too. At last the obtained results are crucial to expand any perturbation appears in different contexts of the spatially closed cosmological models in terms of them.

Appendix

The components of the amplitude of gravitational waves moving in an arbitrary direction:
\[
A_{ij} \times_{11} (X, \hat{n}) = D_{ij} \times_{11} (X, \hat{n}) \times_{11} \left( \frac{\hat{n}_2^2}{1 - \hat{n}_3^2} \right) + D_{ij} \times_{22} (X, \hat{n}) \times_{22} \left( \frac{\hat{n}_2^2 \hat{n}_3^2}{1 - \hat{n}_3^2} \right)
\]
\[
D_{ij} \times_{33} (X, \hat{n}) \hat{n}_1^2 \hat{n}_3 \times_{11} \left( \frac{\hat{n}_2^2}{1 - \hat{n}_3^2} \right) + D_{ij} \times_{12} (X, \hat{n}) \times_{12} \left( \frac{\hat{n}_1^2 \hat{n}_2^2}{1 - \hat{n}_3^2} \right)
\]
\[
2D_{ij} \times_{23} (X, \hat{n}) \times_{23} \left( \frac{\hat{n}_2^2 \hat{n}_3^2}{1 - \hat{n}_3^2} \right)
\]
\[
A_{ij} \times_{22} (X, \hat{n}) = D_{ij} \times_{11} (X, \hat{n}) \times_{11} \left( \frac{\hat{n}_2^2}{1 - \hat{n}_3^2} \right) + D_{ij} \times_{22} (X, \hat{n}) \times_{22} \left( \frac{\hat{n}_2^2 \hat{n}_3^2}{1 - \hat{n}_3^2} \right)
\]
\[
+ D_{ij} \times_{33} (X, \hat{n}) \hat{n}_2^2 - 2D_{ij} \times_{12} (X, \hat{n}) \times_{12} \left( \frac{\hat{n}_1^2 \hat{n}_2 \hat{n}_3^2}{1 - \hat{n}_3^2} \right)
\]
\[
- 2D_{ij} \times_{13} (X, \hat{n}) \times_{13} \left( \frac{\hat{n}_1 \hat{n}_2}{1 - \hat{n}_3^2} \right)
\]
\[
+ 2D_{ij} \times_{23} (X, \hat{n}) \times_{23} \left( \frac{\hat{n}_2 \hat{n}_3^2}{1 - \hat{n}_3^2} \right),
\]
\[
A_{ij} \times_{33} (X, \hat{n}) = D_{ij} \times_{22} (X, \hat{n}) \times_{22}(1 - \hat{n}_3^2) + D_{ij} \times_{33} (X, \hat{n}) \times_{33} \left( \frac{\hat{n}_2 \hat{n}_3^2}{1 - \hat{n}_3^2} \right)
\]
\[
- 2D_{ij} \times_{23} (X, \hat{n}) \times_{23} \left( \frac{\hat{n}_2 \hat{n}_3^2}{1 - \hat{n}_3^2} \right),
\]
\[
A_{ij} \times_{12} (X, \hat{n}) = -D_{ij} \times_{11} (X, \hat{n}) \times_{11} \left( \frac{\hat{n}_1 \hat{n}_2}{1 - \hat{n}_3^2} \right) + D_{ij} \times_{22} (X, \hat{n}) \times_{22} \left( \frac{\hat{n}_1 \hat{n}_2 \hat{n}_3^2}{1 - \hat{n}_3^2} \right)
\]
\[
+ D_{ij} \times_{33} (X, \hat{n}) \times_{33} \left( \frac{\hat{n}_2 \hat{n}_3^2}{1 - \hat{n}_3^2} \right)
\]
\[
+ D_{ij} \times_{12} (X, \hat{n}) \times_{12} \left( \frac{\hat{n}_2^2 - \hat{n}_1^2}{1 - \hat{n}_3^2} \right),
\]
\[ + D_+ \times_{13} (X, \hat{n}) \left( \frac{n_1^2 - n_3^2}{1 - n_3^2} \right) \]

\[ + 2D_+ \times_{23} (X, \hat{n}) \frac{\hat{n}_1 \hat{n}_3 n_3}{1 - n_3^2} \]  

\[ A_+ \times_{13} (X, \hat{n}) = -D_+ \times_{22} (X, \hat{n}) \hat{n}_1 n_3 + D_+ \times_{33} (X, \hat{n}) \hat{n}_1 \hat{n}_3 \]

\[ - D_+ \times_{12} (X, \hat{n}) \hat{n}_3 \]

\[ D_+ \times_{13} (X, \hat{n}) \frac{\hat{n}_3}{1 - n_3^2} \]

\[ \text{where } D_+ \times_{ij} (X, \hat{n}) \text{ are as follows:} \]

\[ D_{+11}(X, \hat{n}) = \left[ \sqrt{1 - X^2} \left( 1 - \frac{(\hat{n}_2 X - \hat{n}_1 y)^2}{1 - \hat{n}_3^2} - (\hat{n} \cdot X)^2 \right) \right]^{-1}, \quad D_{\times 11}(X, \hat{n}) = 0, \quad (A.7) \]

\[ D_{\times 12}(X, \hat{n}) = \left[ \sqrt{1 - X^2} \left[ 1 - \frac{(\hat{n}_3 (\hat{n} \cdot X) - z)^2}{1 - \hat{n}_3^2} - (\hat{n} \cdot X)^2 \right] \right]^{-1}, \quad D_{+12}(X, \hat{n}) = 0, \quad (A.8) \]

\[ D_{+13}(X, \hat{n}) = \frac{\hat{n} \cdot X)(\hat{n}_2 \hat{n} - \hat{n}_1 y)}{\sqrt{1 - X^2} \sqrt{1 - \hat{n}_3^2} (1 - (\hat{n} \cdot X)^2)[1 - \frac{(\hat{n}_2 X - \hat{n}_1 y)^2}{1 - \hat{n}_3^2} - (\hat{n} \cdot X)^2]} \]

\[ D_{\times 13}(X, \hat{n}) = \frac{(\hat{n} \cdot X)(\hat{n}_3 (\hat{n} \cdot X) - z)}{\sqrt{1 - X^2} \sqrt{1 - \hat{n}_3^2} [1 / 1 - \hat{n}_3^2] - (\hat{n} \cdot X)^2}(1 - (\hat{n} \cdot X)^2)^2, \quad (A.9) \]

\[ D_{+22}(X, \hat{n}) = \left[ 1 - \frac{(\hat{n}_3 (\hat{n} \cdot X) - z)^2}{1 - \hat{n}_3^2} - (\hat{n} \cdot X)^2 \right]^{-1}, \quad (A.10) \]

\[ D_{\times 22}(X, \hat{n}) = \frac{2(\hat{n}_2 X - \hat{n}_1 y)(\hat{n}_3 (\hat{n} \cdot X) - z)}{(1 - \hat{n}_3^2) \sqrt{1 - X^2} \sqrt{[1 - \frac{(\hat{n}_3 (\hat{n} \cdot X) - z)^2}{1 - \hat{n}_3^2} - (\hat{n} \cdot X)^2]}} \]

\[ D_{+23}(X, \hat{n}) = \frac{(-\hat{n}_3 (\hat{n} \cdot X) - z)(\hat{n} \cdot X)}{\sqrt{1 - X^2} (1 - \hat{n}_3^2)(1 - \hat{n}_3^2)(1 - (\hat{n} \cdot X)^2)[1 - \frac{(\hat{n}_3 (\hat{n} \cdot X) - z)^2}{1 - \hat{n}_3^2} - (\hat{n} \cdot X)^2]} \]

\[ D_{\times 23}(X, \hat{n}) = \frac{D_{\times 23}(X, \hat{n})}{\sqrt{1 - X^2} \sqrt{1 - \hat{n}_3^2} (1 - (\hat{n} \cdot X)^2)[1 - \frac{(\hat{n}_3 (\hat{n} \cdot X) - z)^2}{1 - \hat{n}_3^2} - (\hat{n} \cdot X)^2]} \]

\[ D_{+33}(X, \hat{n}) = \frac{(\hat{n} \cdot X)^2 \left[ (\hat{n}_2 X - \hat{n}_1 y)^2 - (\hat{n}_3 (\hat{n} \cdot X) - z)^2 \right]}{(1 - \hat{n}_3^2)^2}\left[ (1 - \hat{n}_3^2)^2 - (\hat{n} \cdot X)^2 \right]^{-1}, \quad (A.11) \]

\[ D_{\times 33}(X, \hat{n}) = \frac{2(\hat{n}_2 X - \hat{n}_1 y)(\hat{n}_3 (\hat{n} \cdot X) - z)(\hat{n} \cdot X)^2}{\sqrt{1 - X^2} \sqrt{1 - \hat{n}_3^2} (1 - \hat{n}_3^2)(1 - (\hat{n} \cdot X)^2)[1 - \frac{(\hat{n}_3 (\hat{n} \cdot X) - z)^2}{1 - \hat{n}_3^2} - (\hat{n} \cdot X)^2]} \]
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