Reflection trees of graphs as boundaries of Coxeter groups

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I dedicate this paper to the memory of my Parents

Abstract. To any finite graph $X$ (viewed as a topological space) we associate some explicit compact metric space $X^r(X)$ which we call the reflection tree of graphs $X$. This space is of topological dimension $\leq 1$ and its connected components are locally connected. We show that if $X$ is appropriately triangulated (as a simplicial graph $\Gamma$ for which $X$ is the geometric realization) then the visual boundary $\partial_{\infty}(W, S)$ of the right angled Coxeter system $(W, S)$ with the nerve isomorphic to $\Gamma$ is homeomorphic to $X^r(X)$. For each $X$, this yields in particular many word hyperbolic groups with Gromov boundary homeomorphic to the space $X^r(X)$.

1. Introduction.

In [KK] the authors show that if the Gromov boundary $\partial G$ of a word-hyperbolic group $G$ is connected and has no local cut point (which corresponds to the fact that $G$ has no virtual splitting over a finite or 2-ended subgroup), and if this boundary has topological dimension 1, then it is homeomorphic to the Sierpiński curve or to the Menger curve. In this paper we introduce a big family of 1-dimensional topological spaces, each of which is either disconnected or has a local cut point, and each of which appears as Gromov boundary of many right angled hyperbolic Coxeter groups. We call these spaces the reflection trees of graphs.

The description of the above mentioned spaces is contained in Section 2, and here we only mention that to any finite graph $X$ (viewed as a topological space and thus called a topological graph) there is associated one such space, denoted $X^r(X)$, and called the reflection tree of graphs $X$. The space $X^r(X)$ is connected if and only if $X$ is connected and has no separating point, and if this is the case, the space $X^r(X)$ is also locally connected. $X^r(X)$ has topological dimension 1 if and only if $X$ contains a cycle (equivalently, iff $X$ is not a tree or a disjoint union of trees). Otherwise, except for the trivial cases when $X$ is a singleton or doubleton, the space $X^r(X)$ is homeomorphic to the Cantor set. For each $X$ there are many distinct topological graphs $Y$ such that the spaces $X^r(X)$ and $X^r(Y)$ are homeomorphic. The complete topological classification of the reflection trees of graphs is not known, and in this paper we make only few initial observations in this direction. One such observation consists of referring to the paper [DT] by Dani and Thomas, in which the authors study some aspects of the topology of boundaries of a class of hyperbolic Coxeter groups, including those which (in view of the present paper) yield connected reflection

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trees of graphs as boundaries. More precisely, Dani and Thomas completely determine the structure of local cut points (encoded in an object called the Bowditch’s JSJ tree) of the studied class of boundaries. In the light of the present paper, this gives in particular a complete description of the structure of local cut points of connected reflection trees of graphs, and this allows to distinguish many of the latter spaces up to homeomorphism. More detailed comment concerning the relationship between the results of [DT] and of the present paper is given in Remarks 2.5.(4) and 2.5.(5).

To state the main result of the paper, Theorem 1.1 below, we need some preparations. Recall that any right angled Coxeter system is uniquely determined by an object called the nerve of this system, which is an arbitrary finite flag simplicial complex. Moreover, to each Coxeter system \((W, S)\) there is canonically associated a compact metrisable space called the visual boundary (or boundary at infinity) of \((W, S)\), denoted \(\partial_{\infty}(W, S)\). In case when \(W\) is word-hyperbolic, this boundary coincides (up to homeomorphism) with the Gromov boundary \(\partial W\) of \(W\). We refer the reader to [D] for a detailed introduction to right angled Coxeter systems and their boundaries.

An essential vertex of a finite topological graph \(X\) is any point which locally splits \(X\) into different than 2 number of connected components. The set of essential vertices of \(X\) is obviously finite. An (open) essential edge of \(X\) is any connected component of the complement in \(X\) of the set of all essential vertices of \(X\). Note that loops and multiple edges typically occur as essential edges. We also view as an essential edge falling into the category of loop edges each connected component of \(X\) homeomorphic to the circle.

The main result of this paper is the following (compare Theorem 7.10 in the text, where a slightly more general statement is given).

1.1 Theorem. Let \(\Gamma\) be a finite simplicial graph which is flag, and denote by \(|\Gamma|\) the underlying topological graph. Suppose that each essential edge of \(|\Gamma|\) which is not a loop is the union of at least 3 edges of \(\Gamma\). Let \((W_\Gamma, S_\Gamma)\) be the right angled Coxeter system with nerve \(\Gamma\). Then the visual boundary \(\partial_{\infty}(W_\Gamma, S_\Gamma)\) is homeomorphic to the reflection tree of graphs \(|\Gamma|\), i.e. \(\partial_{\infty}(W_\Gamma, S_\Gamma) \cong X^r(|\Gamma|)\).

Remarks.

1. Note that flagness assumption for \(\Gamma\) implies that any essential loop edge of \(|\Gamma|\) is the union of at least 4 edges of \(\Gamma\). Note also that the no-empty square condition, which guarantees word-hyperbolicity of the group \(W_\Gamma\) (see Theorem 12.2.1 in [D]), holds for \(\Gamma\) satisfying the assumptions of Theorem 1.1 exactly when each essential loop edge of \(|\Gamma|\) is the union of at least 5 edges of \(\Gamma\). In particular, any finite topological graph \(X\) admits many triangulations \(\Gamma\) which satisfy all the assumptions of Theorem 1.1, and for which the group \(W_\Gamma\) is word-hyperbolic. Consequently, each space \(X^r(X)\) appears as the Gromov boundary of many right angled hyperbolic Coxeter groups.

2. Note that if a graph \(\Gamma\) is the simplicial suspension of a finite set consisting of at least 3 points (so that each essential edge of \(|\Gamma|\) consists of two edges of \(\Gamma\), then \(\partial_{\infty}(W_\Gamma, S_\Gamma)\) is homeomorphic to the suspension of the Cantor set, and this space is not homeomorphic to \(X^r(|\Gamma|)\) (e.g. because it is not locally connected). This example shows that the assumption that each essential edge of \(|\Gamma|\) consists of at least 3 edges of \(\Gamma\) cannot be omitted (even though it certainly can be weakened).
We finish the introduction with a brief description of the organization of the paper. In Section 2 we introduce reflection trees of graphs, the class of topological spaces mentioned in the title of the paper. In the same section we derive also the most basic properties of those spaces. In Section 3 we make a step towards the topological classification of reflection trees of graphs, by showing that the disconnected ones among them are fully understood in terms of connected ones, by means of the operation of the dense amalgam (as introduced in [Amalgam]). This observation, apart from being interesting on its own, is also an ingredient in the proof of the main result of the paper - Theorem 7.12 (which is a slightly stronger version of Theorem 1.1 stated above). In Section 4 we observe that reflection trees of graphs allow certain much more flexible description, in terms of so called violated reflection inverse systems. Section 5 deals with yet another relaxation in the description of reflection trees of graphs. We describe them as limits of the so called inverse sequences of X-graphs and X-blow-ups. The latter description is powerful enough to allow in Sections 7 and 8 identification of boundaries of some Coxeter groups as appropriate reflection trees of graphs (up to homeomorphism). Before turning to the proof of this identification, we recall in Section 6 the concept of the Coxeter-Davis complex of a Coxeter system. This is some CAT(0) geodesic metric space canincally associated to a Coxeter system, the visual boundary of which is, by definition, the boundary of the corresponding system. In the same section, we derive some geometric properties of such complexes, for the cases under our interest in this paper. In Section 7, we formulate (as Theorem 7.12) the main result of this paper in its full generality. We also state a crucial technical result, called Approximation Lemma, which relates geometry of the considered Coxeter-Davis complexes with the corresponding inverse sequences of X-graphs and X-blow-ups (introduced in Section 5). We show in the same Section 7 how the main result follows from Approximation Lemma. Finally, in the last Section 8, we give a proof of Approximation Lemma.

2. Reflection trees of graphs $X^r(X)$.

In this paper, by a graph (or topological graph) we mean the underlying topological space $X = |\Gamma|$ of a finite simplicial graph $\Gamma$. The natural cell structure of a graph $X$ is the coarsest cell structure on $X$. The vertices of this structure (also called the essential vertices of $X$) are these points $x \in X$ which locally split $X$ into different than 2 number of connected components. The open edges of this structure (called essential open edges) are the connected components of the complement in $X$ of the set of all essential vertices. Note that loops (loop edges) and multiple edges typically occur in the natural cell structure of a graph $X$. We also view as an essential loop edge each connected component of $X$ homeomorphic to the circle. Obviously, the natural cell structure of each graph $X$ consists of finitely many vertices and edges.

In this section, for any graph $X$, we describe a compact metric space $X^r(X)$, of topological dimension $\leq 1$, called the reflection tree of graphs $X$. To start describing this space, we need an auxiliary object, namely a countable dense subset $D \subset X$ containing all essential vertices. Such a subset is topologically unique, in the precise sense provided by the following easy observation.
2.1 Lemma. Let $D_1, D_2$ be two countable dense subsets of $X$, both containing all essential vertices of $X$. Then there is a homeomorphism $h : X \to X$ such that $h(D_1) = D_2$. Moreover, $h$ can be chosen to preserve all essential vertices and edges of $X$.

Given $X$ and $D \subset X$ as above, we describe now an inverse system $S(X,D)$ whose limit is, by definition, the reflection tree of graphs $X$.

The underlying poset of $S(X,D)$.

Let $T$ be a tree each vertex of which has valence equal to the cardinality of the subset $D$. Denote by $V_T, E_T$ the sets of vertices and (unoriented) edges of $T$, respectively. Let $F$ be the poset of all finite subtrees of $T$, ordered by inclusion. The inverse system $S(X,D)$ will have a form

$$S(X,D) = (\{X_F : F \in F\}, \{\pi_{F',F} : F \subset F'\}),$$

where each $X_F$ is a compact metric space (some graph formed of few copies of $X$), and each $\pi_{F',F} : X_{F'} \to X_F$ is a continuous map, as described below.

Blow-up of $X$ at a point or at a finite set.

For any point $x \in X$, denote by $X^\#(x)$ and call the blow-up of $X$ at $x$ the graph obtained by attaching to $X \setminus \{x\}$ as many points (which all become vertices of valence 1 in $X^\#(x)$) as the number of components into which $x$ splits its any normal neighbourhood in $X$. Denote the set of all those attached vertices by $P_x$. Denote also by $\rho_x : X^\#(x) \to X$ the blow down map which projects $P_x$ to $x$ and which is identical on the remaining part $X \setminus \{x\}$.

For any finite subset $J \subset X$, denote by $X^\#(J)$ the graph obtained from $X$ by performing blow-ups at all points $x \in J$ (the result does not depend on the order). Given any finite subsets $J, K$ of $X$ such that $J \subset K$, denote by $\rho_{K,J} : X^\#(K) \to X^\#(J)$ the blow down map which shrinks each of the subsets $P_x : x \in K \setminus J$ to the corresponding point $x$, and which is identical on the remaining parts of the graphs.

The spaces $X_F$.

To each vertex $t \in V_T$ associate a copy of the graph $X$, and denote it $X_t$. Equip also $T$ with a labelling $\lambda : E_T \to D$ such that for any $t \in V_T$, denoting by $A_t T$ the set of edges in $T$ adjacent to $t$, the restriction of $\lambda$ to $A_t T$ is a bijection on $D$. Such a labelling clearly exists, and is unique up to an automorphism of $T$.

Intuitively, for each edge $e = [t_1, t_2] \in E_T$ the label $\lambda(e)$ will represent (in the construction of the spaces $X_F$ given formally in the next paragraph) an operation of “doubling” $X$ at the point $\lambda(e)$, consisting of blowing up the copies $X_{t_1}, X_{t_2}$ at the point $\lambda(e)$ in each copy, and of gluing to each other the sets $P_{\lambda(e)}$ in the blown up copies through the identity map.

Fix a subtree $F \in F$, and denote by $V_F, E_F$ its vertex and edge set, respectively. For any $t \in V_F$, denote by $A_t F$ the set of edges in $F$ adjacent to $t$. Put

$$X_F = \bigsqcup_{t \in V_F} X^\#_{t}(\lambda(A_t F))/\sim,$$

where $\bigsqcup$ denotes the disjoint union, and where the equivalence relation $\sim$ is induced by the following equivalences: for each $e = [t_1, t_2] \in E_F$ and each $p \in P_{\lambda(e)}$, identify $p \in P_{\lambda(e)} \subset X^\#_{t_1}(\lambda(A_{t_1} F))$ with $p \in P_{\lambda(e)} \subset X^\#_{t_2}(\lambda(A_{t_2} F))$. 

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Note that for each $t \in V_F$ the set $X^#_t(\lambda(A_tF))$ is naturally a subspace in $X_F$.

The maps $\pi_{F',F}$.

Given any finite subtrees $F,F'$ of $T$ such that $F \subset F'$, define $\pi_{F',F} : X_{F'} \to X_F$ as follows. For $t \in V_F$, the restriction of $\pi_{F',F}$ to $X^#_t(\lambda(A_tF')) \subset X_{F'}$ coincides with the blow down map $\rho_{\lambda(A_tF'),\lambda(A_tF)}$ (whose image $X^#_t(\lambda(A_tF))$ is viewed as a subset in $X_F$). For $t \in V_{F'} \setminus V_F$, the subset $X^#_t(\lambda(A_tF'))$ is mapped by $\pi_{F',F}$ to the point $\lambda(e) \in X^#_s(\lambda(A_sF))$, where $s \in V_F$ is the closest to $t$ (in the polygonal metric in $T$) vertex of the subtree $F$, and where $e$ is the first edge on the path from $s$ to $t$ in $T$. Obviously, $X^#_s(\lambda(A_sF))$ is viewed in the previous sentence as a subset in $X_F$.

The above description of the inverse system $S(X,D)$, together with Lemma 2.1, immediately yield the following.

2.2 Lemma. Let $D_1,D_2$ be two countable dense subsets of $X$, both containing all essential vertices of $X$. Then the inverse systems $S(X,D_1)$ and $S(X,D_2)$ are isomorphic. Consequently, the inverse limits $\lim_\leftarrow S(X,D_1)$ and $\lim_\leftarrow S(X,D_2)$ are homeomorphic.

In view of Lemma 2.2, we will denote any inverse system $S(X,D)$ simply by $S_X$, and call it the standard reflection inverse system for $X$.

2.3 Definition. The reflection tree of graphs $X$, denoted $X^r(X)$, is the space obtained as inverse limit $\lim_\leftarrow S_X$ of the standard reflection inverse system for $X$.

At the end of this section, in Remark 2.6, we indicate an alternative way of describing reflection trees of graphs. Although we do not use this alternative description in the present paper, it exhibits connections of reflection trees of graphs with some other classes of spaces studied in the context of boundaries of groups, notably with trees of manifolds.

Next result collects some basic properties of the reflection trees of graphs.

2.4 Lemma.

(1) Reflection tree of a singleton graph is the empty space, and reflection tree of a doubleton is a doubleton.

(2) If $X$ is a tree or a disjoint union of trees, and if it is not a singleton and not a doubleton, then the space $X^r(X)$ is homeomorphic to the Cantor set.

(3) If $X$ is homeomorphic to the circle $S^1$ then $X^r(X) \cong S^1$.

(4) For any graph $X$ the topological dimension of the space $X^r(X)$ is $\leq 1$.

(5) If $Y$ is a subgraph of a graph $X$ (with respect to the natural cell structure in $X$), then the space $X^r(X)$ contains an embedded copy of $X^r(Y)$. Moreover, if $Y$ is a proper subgraph not reduced to a single vertex then there are countably infinitely many natural embeddings of $X^r(Y)$ in $X^r(X)$, with pairwise disjoint images.

(6) We have $\dim X^r(X) = 1$ iff $X$ is not a tree or a disjoint union of trees (i.e. $X$ contains a cycle).

(7) The space $X^r(X)$ is connected iff $X$ is connected and has no separating vertex.

(8) If $X^r(X)$ is connected, then it is also locally connected and contains local cut points.

Before giving the proof of the above lemma, we make few comments.
2.5 Remarks.

(1) Note that, due to part (8) of Lemma 2.4, the following 1-dimensional compacta do not appear among reflection trees of graphs:

- suspension of the Cantor set and the join of two Cantor sets (these spaces are not locally connected);
- Sierpiński curve and Menger curve (these spaces have no local cut points).

(2) Further topological properties of the spaces $\mathcal{X}^r(X)$ wait to be studied. In particular, it is an open problem to classify the spaces $\mathcal{X}^r(X)$ up to homeomorphism.

(3) In Section 3 we present a result (Proposition 3.8) which gives a full description of disconnected reflection trees of graphs, in terms of connected ones. Thus, the topological classification of the reflection trees of graphs reduces to the classification of the connected ones (which is anyway the most interesting case).

(4) Some further topological properties of reflection trees of graphs can be deduced from the results of Dani and Thomas in [DT]. In that paper the authors give, among others, a complete description of the structure of local cut points (the so called Bowditch’s JSJ tree) for Gromov boundaries of a class of right angled hyperbolic Coxeter groups, including all hyperbolic Coxeter groups appearing in Theorem 1.1 of the present paper. This gives the description of the structure of local cut points for all connected reflection trees of graphs. This structure (the Bowditch’s JSJ tree) is a powerful topological invariant, containing full information about degrees of local cut points, as well as of their mutual position inside the space. In our context, this invariant allows to distinguish many of the connected reflection trees of graphs, up to homeomorphism.

(5) As a complement to the previous remark (4), the reader should keep in mind that continua with the same Bowditch’s JSJ trees are not necessarily homeomorphic. Hence, Bowditch’s JSJ trees cannot be used to deduce that the boundaries $\partial_{\infty}(W_{\Gamma}, S_{\Gamma})$ as in Theorem 1.1, for distinct $\Gamma$ having the same underlying topological graph $|\Gamma|$, are homeomorphic.

Proof of Lemma 2.4:

The proof of part (1) is straightforward, and the proof of part (2) is an exercise on checking the conditions from the following well known characterization of the Cantor set: it is the unique compact metric space which is totally disconnected and has no isolated points. We skip the details.

To see part (3), note first that if $X$ is a circle then all of the spaces $X_F$ in the system $S_X$ are also homeomorphic to a circle. Moreover, the maps $\rho_{F',F}$ all have the following form: a circle is mapped to a circle by means of shrinking few pairwise disjoint subsegments in the source circle to points. Obviously, each such map $\rho_{F',F}$ can be approximated by a homeomorphism of the involved circles, i.e. it is a near-homeomorphism. By a result of Brown [Br], the inverse limit of a system with bonding maps which are near-homeomorphisms coincides with the spaces in the system, so in our case it is a circle, as required.

To get part (4), note that since each space $X_F$ is a graph, the well known estimate for the dimension of an inverse limit yields in our case the required inequality, as follows:

$$\dim \mathcal{X}^r(X) = \dim \lim_{\leftarrow} \left( \{X_F\}, \{\pi_{F',F}\} \right) \leq \sup \{\dim X_F : F \in \mathcal{F} \} \leq 1.$$
The proof of part (5) requires some preparations concerning morphisms of inverse systems. As a reference for this subject we use Section 2.5 in the book [En]. Recall that given inverse systems

\[ S_1 = (\{Y_\lambda : \lambda \in \Lambda\}, \{\pi^{1}_{\lambda, \lambda}\}) \quad \text{and} \quad S_2 = (\{X_\mu : \mu \in M\}, \{\pi^{2}_{\mu, \mu}\}), \]

with bonding maps \( \pi^{1}_{\lambda, \lambda} : X_\lambda \to X_\lambda \) and \( \pi^{2}_{\mu, \mu} : X_\mu \to X_\mu \), a morphism \( f : S_1 \to S_2 \) is a pair \( f = (\phi, \{f_\mu : \mu \in M\}) \) such that:

- \( \phi : \Lambda \to \Lambda \) is an order preserving map of underlying posets such that the image \( \phi(M) \) is a cofinal subset of \( \Lambda \);
- \( f_\mu : Y_\phi(\mu) \to X_\mu \) are continuous maps, one for each \( \mu \in M \), such that whenever \( \mu \leq \mu' \) then
  \[ f_\mu \pi^{1}_{\phi(\mu'), \phi(\mu)} = \pi^{2}_{\mu', \mu} f_\mu'. \]

A morphism \( f \) as above induces a continuous map \( f : \lim_{\leftarrow} S_1 \to \lim_{\leftarrow} S_2 \) such that if \( p \in \lim_{\leftarrow} S_1 \) is represented by a thread \( \{y_\lambda\}_{\lambda \in \Lambda} \) then its image \( f(p) \) is represented by

\[ \{f_\mu(y_\phi(\mu))\}_{\mu \in M}. \]

Moreover, we have the following result (see Lemma 2.5.9 in [En]).

**Fact A.** If all the maps \( f_\mu \) of a morphism \( f = (\phi, \{f_\mu\}) \) are injective, then the induced map \( f \) is also injective.

To proceed, under assumptions on graphs \( X \) and \( Y \) as in the statement of part (5), we introduce a uniform notation that indicates dependence of the involved objects on a graph \( Z \) from the set \( \{X, Y\} \). \( D_Z \) denotes a countable dense subset of \( Z \) containing all essential vertices of \( Z \). \( T_Z \) is a tree whose valence at every vertex is equal to the cardinality of the set \( D_Z \), and whose vertex and edge sets are denoted \( V_{T_Z}, E_{T_Z} \), respectively. \( T_Z \) is equipped with a labelling \( \lambda_Z : E_{T_Z} \to D_Z \) such that its restriction to any set of edges adjacent to a fixed vertex is a bijection on \( D_Z \). We denote by \( F_Z \) the poset of all finite subtrees of \( T_Z \), and by \( S_Z = S(Z, D_Z) \) the corresponding inverse system as described above in this section, consisting of the spaces \( Z_F : F \in F_Z \) and the maps \( \pi^{Z}_{F, F'} : Z_F \to Z_{F'} \), and such that \( \lim_{\leftarrow} S_Z = \mathcal{X}'(Z) \). Furthermore, we assume (without loss of generality) that \( D_Y = Y \cap D_X \).

Let \( S \) be any maximal subtree of \( T_X \) with the property that for each edge \( e \) of \( S \) we have \( \lambda_X(e) \in D_Y \). Note that if \( Y \) is a proper subgraph of \( X \) then there are countably infinitely many subtrees \( S \) as above, and the vertex sets of these subtrees form a partition of the vertex set \( V_{T_X} \). For each such \( S \) there is a label preserving isomorphism \( \psi : T_Y \to S \).

After fixing any \( S \) and \( \psi \) as above, denote by \( F^S_X \) the subposet of \( F_X \) consisting of those finite subtrees \( F \) of \( T_X \) which have nonempty intersection with \( S \). Note that \( F^S_X \) is cofinal in \( F_X \), and thus the restricted inverse system \( S^S_X := S_X |_{F^S_X} \) has the same limit as \( S_X \). We describe a morphism \( f^\psi = (\phi, \{f^\psi_\mu\}) : S_Y \to S^S_X \) as follows. Given \( F \in F^S_X \), we put \( \phi(F) := \psi^{-1}(F \cap S) \) and we note that this is a nonempty subtree of \( T_Y \). Obviously, \( \phi \) is order preserving and surjective, so its image is cofinal in \( F_Y \). For any \( F \in F^S_X \) define \( f^\psi_F : Y_{\psi^{-1}(F \cap S)} \to X_F \) as the injective map induced by the natural inclusions \( Y_s \to X_{\psi(s)} \).
for all vertices $s$ of the subtree $\psi^{-1}(F \cap S)$ We skip the straightforward details concerning the description of these maps, the verification that they are injective, and that they satisfy the appropriate commutativity equations, thus forming a morphism.

Let $f^\psi : \lim_\rightarrow S_T \rightarrow \lim_\rightarrow S^S_X$ be the map induced by the morphism $f^\psi$. Due to the above recalled Fact A, this map is injective. Since $\lim_\rightarrow S_T \cong \mathcal{X}^r(Y)$ and $\lim_\rightarrow S^S_X \cong \mathcal{X}^r(X)$, the map $f^\psi$ embeds a copy of $\mathcal{X}^r(Y)$ in $\mathcal{X}^r(X)$, as required in the first part of the assertion. To prove the second assertion of part (5), it is sufficient to show that the images of the embeddings $f^\psi$ corresponding to distinct subtrees $S$ are pairwise disjoint.

Fix two distinct subtrees $S, S'$ of $T_X$ equipped with label preserving isomorphisms $\psi : T_Y \rightarrow S$ and $\psi' : T_Y \rightarrow S'$. Consider any subtree $F \in F_X$ which has nonempty intersection with both $S$ and $S'$. Note that, since $S \cap S' = \emptyset$, we have

$$f^\psi_F(Y_{\psi^1(F \cap S)}) \cap f^\psi'_F(Y_{\psi'^1(F \cap S')}) = \emptyset,$$

and this obviously implies that the images of $f^\psi$ and $f^\psi'$ are disjoint, which completes the proof of part (5).

One implication in part (6) follows by observing that if $X$ contains a cycle then, due to parts (3) and (5), $\mathcal{X}^r(X)$ contains an embedded copy of $S^1$, and due to part (4) this means that $\dim \mathcal{X}^r(X) = 1$. The remaining implication follows from part (1) and (2), in view of the well known fact that the dimension of the Cantor set is 0.

To prove (7), suppose first that $X$ is not connected, or has a separating vertex. It is not hard to see that then some space $X_F$ from the inverse system $S_X$ is not connected. Since the bonding maps are all surjections (which is generally true in the inverse systems $S_X$, unless $X$ is a singleton), we get disconnectedness of the inverse limit (i.e. disconnectedness of the space $\mathcal{X}^r(X)$). To get the converse, note that if $X$ is connected and has no separating vertex then all the spaces $X_F$ are connected. Since in such a case the inverse limit is connected too, this completes the proof of part (7).

To get the local connectedness assertion in part (8), note that if $\mathcal{X}^r(X)$ is connected then (due to part (7)) $X$ is connected and has no separating vertex. As it was already observed, each space $X_F$ is then connected. For a similar reason, point preimages for all bonding maps of the system $S_X$ are also connected. Consequently, the assertion follows by the general fact that limit of a monotone inverse system of locally connected continua is locally connected, see [Capel].

To get existence of local cut points (i.e. the second assertion of part (8)), note that under the assumption of this part the graph $X$ has at least one edge, and hence $D$ contains points which are distinct from essential vertices. Consider a point $x \in D$ not being an essential vertex of $X$, and choose an edge $e = [t_1, t_2]$ in $T$ labelled with $x$ (i.e. such that $\lambda(e) = x$). Let $F$ be the subtree of $T$ spanned on the vertices $t_1, t_2$ (coinciding with the edge $e$). Then any point $p \in P_F$ (viewed as a point of $X_F$ under the canonical embedding of $X^F_{t_1}(x)$ or $X^F_{t_2}(x)$ in $X_F$) is a local cut point of $X_F$. Moreover, it is not hard to realize that the preimage of $p$ in the inverse limit $\lim_\rightarrow S(X, D) = \mathcal{X}^r(X)$ is a singleton, and we denote the unique point in this preimage by $\tilde{p}$. Now, let $U$ be any connected neighbourhood in $X_F$ which is disconnected by $p$, and let $A, B$ be some nonempty open and closed subsets of $U \setminus \{p\}$ forming a partition of this subset. Denoting by $\pi_F : \lim_\rightarrow S(X, D) \rightarrow X_F$ the canonical projection associated to $S(X, D)$, we claim that
(1) $\pi_F^{-1}(U)$ is a connected neighbourhood of $\tilde{p}$, and
(2) the sets $\pi_F^{-1}(A), \pi_F^{-1}(B)$ form a nontrivial open and closed partition of $\pi_F^{-1}(U) \setminus \{\tilde{p}\}$.

Property (1) above follows by observing (similarly as in the proof of part (7)) that for each $F' \supset F$ the preimage $\pi_F^{-1}(U)$ is connected. Property (2) follows from the fact that $\pi_F^{-1}(p) = \{\tilde{p}\}$, and from surjectivity of $\pi_F$ (which in turn follows from surjectivity of the bonding maps in the system $\mathcal{S}(X,D)$). Properties (1) and (2) mean that $\tilde{p}$ is a local cut point in $X^r$, hence the assertion.

2.6 Remark. Each of the spaces $X^r(X)$ can be alternatively described using the setting of the paper [Tr-metr], as limit of some tree system $\Theta^r(X)$ associated to a graph $X$ in a natural way. We explain the details below, referring the reader to [Tr-metr] for the general introduction concerning tree systems, as well as details of terminology and notation. However, it should be made clear that in the present paper we make no essential use of this alternative description of reflection trees of graphs.

Given any $X$, consider a family $\mathcal{D}$ of (closed) normal neighbourhoods $\Delta(x)$ of points $x$ in $X$ such that:

(b1) the sets from $\mathcal{D}$ are pairwise disjoint, and their union is dense in $X$;
(b2) for each essential vertex $v$ in $X$ there is a set of form $\Delta(v)$ in $\mathcal{D}$.

It is not hard to observe that a family $\mathcal{D}$ as above is unique up to a homeomorphism of $X$ preserving all essential vertices and edges.

The reflection tree system of graphs $X$ is the tree system $\Theta^r(X)$ described uniquely up to an isomorphism of tree systems by the following conditions:

(r1) each vertex space is homeomorphic to the space $X \setminus \bigcup_{\Delta \in \mathcal{D}} \text{int}(\Delta)$, where “int” denotes here the topological interior in $X$;
(r2) for each vertex space $X \setminus \bigcup_{\Delta \in \mathcal{D}} \text{int}(\Delta)$ the family of peripheral subsets in this space coincides with the family $\{\text{bd}(\Delta) : \Delta \in \mathcal{D}\}$, where “bd” denotes here the topological boundary in $X$;
(r3) the glueing maps in the system are the identity maps $\text{bd}(\Delta) \to \text{bd}(\Delta)$ between the copies of the same peripheral set $\text{bd}(\Delta)$ in the two adjacent vertex spaces.

Note that, after shrinking to points the peripheral subsets $\text{bd}(\Delta) : \Delta \in \mathcal{D}$ of the space $X \setminus \bigcup_{\Delta \in \mathcal{D}} \text{int}(\Delta)$, we get a space homeomorphic to $X$, and the set of points obtained by shrinking the subsets $\text{bd}(\Delta)$ is a countable dense subset in the quotient, containing its all essential vertices. Using this fact, it is straightforward to verify that the inverse system $\mathcal{S}_X$ is isomorphic to the system $\mathcal{S}_{\Theta^r(X)}$ associated to the tree system $\Theta^r(X)$ as in Section 1.D of [Tr-metr]. (The latter system is called in [Tr-metr] the standard inverse system associated to the tree system $\Theta^r(X)$.) It follows then from Proposition 1.D.1 in [Tr-metr] that the reflection tree of graphs $X^r(X)$ is homeomorphic to the limit $\lim \Theta^r(X)$ of the tree system $\Theta^r(X)$.

The above observation allows to make the following comment, which puts the spaces $X^r(X)$ in a wider perspective. The pair of data

$$(X \setminus \bigcup_{\Delta \in \mathcal{D}} \text{int}(\Delta), \{\text{bd}(\Delta) : \Delta \in \mathcal{D}\})$$

which could be called the densely punctured graph $X$, is fairly analogous to an object $M^\circ$ described in section 1.E.2 of [Tr-metr], called there the densely punctured manifold $M$. 

The latter object leads (in a way similar as in the above description of the tree system $\Theta^r(X)$) to a tree system $\mathcal{M}(M)$ called the dense tree system of manifolds $M$. (In fact, if $X = S^1$ then the tree systems $\Theta^r(X)$ and $\mathcal{M}(S^1)$ are just isomorphic.) Consequently, reflection trees of graphs $\mathcal{X}^r(X)$ are analogous to the spaces $\mathcal{X}(M)$, called trees of manifolds, obtained as limits of the systems $\mathcal{M}(M)$ (or alternatively as inverse limits of the standard inverse systems $\Theta_{\mathcal{M}(M)}$ associated to $\mathcal{M}(M)$). The latter spaces are known in topology for many years (see e.g. [AS], [J], [St]), and more recently they have been studied in the context of boundaries of groups (see [Fi], [PS] and [Tr-mfld]). In this context, our main result (Theorem 1.1) may be viewed as analogon of Fisher’s result [Fi] saying that Coxeter groups with PL manifold nerves have boundaries homeomorphic to the appropriate trees of manifolds (see Theorem 3.A.3 in [Tr-metr] for correct statement of this result).

3. Disconnected spaces $\mathcal{X}^r(X)$ as dense amalgams.

Recall that the space $\mathcal{X}^r(X)$ is connected if and only if $X$ is connected and has no separating essential vertex. If this is not the case, $X$ decomposes naturally into a family of subgraphs, called blocks of $X$, which are connected and which have no separating essential vertices (we describe this decomposition in detail below). A block is nontrivial if it is not a single (isolated) vertex of $X$ and not a single non-loop edge of $X$.

The aim of this section is to show that if the space $\mathcal{X}^r(X)$ is not connected, then it is homeomorphic to some uniquely determined combination of the connected spaces $\mathcal{X}^r(Y)$ (whose copies appear as the nontrivial connected components in $\mathcal{X}^r(X)$), where $Y$ runs through the family of nontrivial blocks of $X$. The above mentioned “combination of spaces” corresponds to the operation of the dense amalgam, introduced in [Amalgam] and recalled below in Definition 3.6. See Proposition 3.8 below for a precise statement of the above announced main result of the section.

We start with recalling the concepts of block decomposition of a graph and dense amalgam of a family of compact metric spaces.

**Block decomposition of a graph $X$.**

**3.1 Definition.** We say that a topological graph $X$ is non-separable if it is connected and has no separating essential vertex. (In particular, the circle is non-separable.) A block of $X$ is any subgraph (for the natural cell structure consisting of essential vertices and edges) which is maximal for the inclusion in the family of all non-separable subgraphs of $X$, and which is not homeomorphic to a segment, and not a single isolated vertex of $X$.

We now describe the set of blocks of a graph $X$ in terms of a sequence of operations called splittings. We will need this description in the arguments both in this section, and in Section 7 (in the proof of Theorem 7.12).

**3.2 Definition.** A natural subgraph of a graph $X$ is its any subgraph for the natural cell structure (consisting of essential edges and essential vertices). A splitting of a connected graph $X$ is a pair of connected natural subgraphs $X_1, X_2$, both distinct from a singleton, such that $X_1 \cup X_2 = X$ and $X_1 \cap X_2$ is a single essential vertex of $X$. The subgraphs $X_1$ and $X_2$, viewed as topological graphs (equipped with their new natural cell structures), are then called the parts of this splitting.
Observe that a connected graph has no splitting if and only if it is non-separable.

3.3 Definition. A split decomposition of a connected graph \( X \) is any sequence of splittings having the following recursive description:

- the empty sequence of splittings forms the trivial split decomposition of \( X \), and the set of factors of this decomposition is \( \{ X \} \);
- a single splitting of \( X \) is a split decomposition, and its set of factors is the set consisting of the two parts of the splitting;
- if some sequence of splittings is a split decomposition of \( X \), and if a collection \( \{ X_1, \ldots, X_m \} \) of natural subgraphs of \( X \) is the set of factors of this split decomposition, then adding to this sequence a splitting of one of those factors, say \( X_m \), we also get a split decomposition of \( X \); moreover, if \( X'_m, X''_m \) are the parts of the above splitting of \( X_m \), the set of factors of the new split decomposition is \( \{ X_1, \ldots, X_{m-1}, X'_m, X''_m \} \).

A split decomposition of \( X \) is terminal if its every factor is non-separable. A factor of a split decomposition is trivial if it is a singleton or if it is homeomorphic to a segment. Otherwise a factor is nontrivial.

Every finite connected topological graph admits a terminal split decomposition. This follows by observing that, after adding a new splitting to a split decomposition, the total number of essential edges in the factors does not increase (actually, it may happen that this number strictly decreases, since the natural cell structure of a subgraph might be coarser than the structure induced from the initial graph). Moreover, each factor of any split decomposition of \( X \) is a natural subgraph of \( X \).

It is not hard to observe that any block of a connected graph \( X \) is contained in precisely one factor of any split decomposition of \( X \). As an easy consequence, the following result holds.

3.4 Lemma. Let \( X \) be a finite connected topological graph. The set of nontrivial factors of any terminal split decomposition of \( X \) coincides with the set of blocks of \( X \).

Lemma 3.4 obviously implies the following.

3.5 Corollary. Let \( X \) be any finite topological graph (not necessarily connected). the set of blocks of \( X \) coincides with the union of the sets of nontrivial factors of any terminal split decompositions of all connected components of \( X \).

The dense amalgam of metric compacta.

We recall the concept of the dense amalgam of metric compacta, as introduced in [Amalgam].

3.6 Definition. The dense amalgam is an operation \( \sqcup \) which to any finite tuple \( B_1, \ldots, B_k \) of nonempty metric compacta associates a metric compactum

\[ A = \sqcup (B_1, \ldots, B_k) \]

determined uniquely up to homeomorphism by the following. The space \( A \) can be equipped with a countable infinite family \( A \) of subsets, partitioned as \( A = A_1 \sqcup \ldots \sqcup A_k \), such that:
(a1) the subsets in \( \mathcal{A} \) are pairwise disjoint and for each \( i \in \{1, \ldots, k\} \) the family \( \mathcal{A}_i \) consists of embedded copies of the space \( B_i \);

(a2) the family \( \mathcal{A} \) is null, i.e. for any metric on \( \mathcal{A} \) compatible with the topology the diameters of sets in \( \mathcal{A} \) converge to 0;

(a3) each subset from the family \( \mathcal{A} \) is a boundary subset of \( \mathcal{A} \) (i.e. its complement is dense in \( \mathcal{A} \));

(a4) for each \( i \), the union of the family \( \mathcal{A}_i \) is dense in \( \mathcal{A} \);

(a5) any two points of \( \mathcal{A} \) which do not belong to the same subset of \( \mathcal{A} \) can be separated from each other by an open and closed subset \( H \subset \mathcal{A} \) which is \( \mathcal{A} \)-saturated (i.e. such that any element of \( \mathcal{A} \) is either contained in or disjoint with \( H \)).

The uniqueness claim appearing in the above definition is proved in [Amalgam], as Theorem 0.2. Section 1 of the same paper contains an effective construction of the space \( \tilde{\sqcup}(B_1, \ldots, B_k) \), for any given \( B_1, \ldots, B_k \).

We call the result \( \tilde{\sqcup}(B_1, \ldots, B_k) \) of the above described operation the dense amalgam of the spaces \( B_1, \ldots, B_k \). Obviously, the dense amalgam of any family \( B_1, \ldots, B_k \) of spaces is a disconnected perfect compact metric space. Moreover, it follows easily from the condition (a5) above that if the spaces \( B_1, \ldots, B_k \) are connected, then the connected components of their dense amalgam are precisely the subsets from the family \( \mathcal{A} \) and the singletons from the complement of the union \( \cup \mathcal{A} \).

The following result concerning “algebraic” properties of the operation of dense amalgam is proved as Proposition 0.1 in [Amalgam].

**3.7 Proposition.** For any nonempty metric compacta \( B, B_1, \ldots, B_k \) the following equalities hold, up to homeomorphism:

1. \( \tilde{\sqcup}(B_1, \ldots, B_k) = \tilde{\sqcup}(B_1 \sqcup \ldots \sqcup B_k) \);
2. \( \tilde{\sqcup}(B_1, \ldots, B_k) = \tilde{\sqcup}(B_1, \ldots, B_{i-1}, \tilde{\sqcup}(B_i, \ldots, B_k)) \) for any \( k \geq 1 \) and any \( 1 \leq i \leq k \);
3. \( \tilde{\sqcup}(B, B_1, \ldots, B_k) = \tilde{\sqcup}(B, B_1, \ldots, B_k) \) for any \( k \geq 0 \);
4. for any totally disconnected nonempty compact metric space \( Q \), and any \( k \geq 1 \), we have
   \[ \tilde{\sqcup}(B_1, \ldots, B_k, Q) = \tilde{\sqcup}(B_1, \ldots, B_k) \];
5. for any totally disconnected space \( Q \) (including the case when \( Q \) is a singleton) we have \( \tilde{\sqcup}(Q) = C \), where \( C \) is the Cantor set.

In consistency with the properties from Proposition 3.7, we use the convention that \( \tilde{\sqcup}(\emptyset) := C \) and \( \tilde{\sqcup}(\emptyset, X_1, \ldots, X_k) := \tilde{\sqcup}(X_1, \ldots, X_k) \). Moreover, if \([\emptyset] \) denotes the empty family of metric compacta, we set \( \tilde{\sqcup}([\emptyset]) := C \).

**Main result of the section, and its proof.**

Now we are ready to state the main result of the section.

**3.8 Proposition.** Let \( X \) be a finite topological graph distinct from a singleton and from a doubleton, and let \( Y_1, \ldots, Y_m \) be the (possibly empty) family of homeomorphism types of all blocks of \( X \). Then

\[ \mathcal{X}^\tau(X) \cong \tilde{\sqcup}(\mathcal{X}^\tau(Y_1), \ldots, \mathcal{X}^\tau(Y_m)) \].
Our proof of Proposition 3.8 is based on the following.

3.9 Lemma. Suppose that for a finite topological graph $X$ one of the following conditions holds:
(a) $X$ is the disjoint union of its nonempty subgraphs $X_1$ and $X_2$, at least one of which is not a singleton;
(b) $X$ is connected and has a splitting with parts $X_1$ and $X_2$.
Then
$$\mathcal{X}^r(X) \cong \sqcup(\mathcal{X}^r(X_1), \mathcal{X}^r(X_2)).$$

We now give a proof of Proposition 3.8, assuming Lemma 3.9, and we postpone the proof of Lemma 3.9 until the last part of the section.

Proof of Proposition 3.8 (assuming Lemma 3.9): Suppose that $Z$ is a cycle in $X$, i.e. a subgraph homeomorphic to the circle, and let $X_0$ be the connected component of $X$ containing $Z$. Observe that for any split decomposition of $X_0$, $Z$ is contained in precisely one of the factors of this splitting. By Lemma 3.4, $Z$ is then contained in a block of $X_0$, which is also a block of $X$. In particular, this shows that $X$ contains some block if and only if it contains a cycle.

Consider first the less interesting case when the family of blocks of $X$ is empty. By the discussion of the previous paragraph, this happens exactly when $X$ contains no cycle. By Lemma 2.4(2), the reflection tree $\mathcal{X}^r(X)$ is then homeomorphic to the Cantor set $C$. Since, by our convention, $C$ is the dense amalgam of the empty family of spaces, the proposition follows in this case for “conventional” reasons.

We now pass to the real case of interest, when $X$ has at least one block, i.e. the parameter $m$ from the statement satisfies $m \geq 1$. Let $E_1, \ldots, E_k$ be the family of all trivial factors of some terminal split decompositions of all components of $X$. We then have $k \geq 0$ (i.e. the family may be empty), and each $E_j$ is homeomorphic either to a segment or to a singleton (the latter appear only when $X$ contains a connected component which is a singleton). Let $X_1, \ldots, X_p$ be the family of all connected components of $X$. By applying recursively Lemma 3.9(a) together with Proposition 3.7(2) (and using parts (1) and (2) of Proposition 2.4 and our conventions concerning the empty set in the case when all $X_i$ are singletons), we get
$$\mathcal{X}^r(X) \cong \sqcup(\mathcal{X}^r(X_1), \ldots, \mathcal{X}^r(X_p)).$$

By referring to the fact that blocks of $X$ are exactly the nontrivial factors in terminal split decompositions of the components $X_1, \ldots, X_p$ (Corollary 3.5), and applying again Proposition 3.7(2), and then Proposition 3.7(3) (to exclude repetitions of homeomorphism types among the amalgamated nontrivial blocks), we get
$$\mathcal{X}^r(X) \cong \sqcup(\mathcal{X}^r(Y_1), \ldots, \mathcal{X}^r(Y_m), \mathcal{X}^r(E_1), \ldots, \mathcal{X}^r(E_k)).$$

Now, each of the spaces $\mathcal{X}^r(E_j)$ is either homeomorphic to the Cantor set $C$ (by Lemma 2.4(2)) or is empty (by Lemma 2.4(1)). Then, applying Proposition 3.7(4) and one of our conventions concerning the empty set, we get
$$\mathcal{X}^r(X) \cong \sqcup(\mathcal{X}^r(Y_1), \ldots, \mathcal{X}^r(Y_m)).$$
as required. This finishes the proof.

**Proof of Lemma 3.9:** The proof needs to be split according to the following two subcases:

(i) either $X = X_1 \sqcup X_2$ and no one of $X_i$ is a singleton, or $X$ is connected and has a splitting with parts $X_1, X_2$;

(ii) $X = X_1 \sqcup X_2$ and one of the subgraphs $X_i$, say $X_2$, is a singleton.

In the subcase (i) both spaces $\mathcal{X}^r(X_i)$ are nonempty and thus we need to check that the space $\mathcal{X}^r(X)$ can be equipped with a family $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2$ of subsets satisfying all the conditions of Definition 3.6, for $k = 2$, with $\mathcal{X}^r(X_i)$ substituted for $B_i$. We deal with this subcase carefully below. In the subcase (ii) we have $\mathcal{X}^r(X_2) = \emptyset$, so that our assertion reads

$$\mathcal{X}^r(X) \cong \tilde{\mathcal{U}}(\mathcal{X}^r(X_1), \emptyset) = \tilde{\mathcal{U}}(\mathcal{X}^r(X_1)),$$

and thus we need to equip $\mathcal{X}^r(X)$ with a family $\mathcal{A} = \mathcal{A}_1$ of subsets satisfying all the conditions of Definition 3.6, for $k = 1$, with $\mathcal{X}^r(X_1)$ substituted for $B_1$. The arguments necessary to deal with this subcase are either the same or very similar to those corresponding to the subcase (i), and we omit them.

We turn to the setting of the subcase (i). Put $\mathcal{A}_i$ to be the family of images of the natural embeddings of $\mathcal{X}^r(X_i)$ in $\mathcal{X}^r(X)$ described in the proof of part (5) of Lemma 2.4 (with $X_i$ substituted for $Y$). Since each $X_i$ is a proper subgraph of $X$ not reduced to a vertex, both families $\mathcal{A}_i$ are countable infinite. We need to check conditions (a1)-(a5) of Definition 3.6.

**Condition (a1).**

In view of the above description of the families $\mathcal{A}_i$ and the second assertion of Lemma 2.4(5), to verify this condition, it remains to show that for any $E_1 \in \mathcal{A}_1$ and $E_2 \in \mathcal{A}_2$ we get $E_1 \cap E_2 = \emptyset$. To see this, recall that both $E_i$ have the following description. There is an appropriate subtree $S_i \subset T_X$, and a label preserving isomorphism $\psi_i : T_{X_i} \to S_i$, which determine the morphism $f^{\psi_i} : S_{X_i} \to S^{S_i}_X$, such that for the induced map $f^{\psi_i}$ we have $E_i = f^{\psi_i}(\lim_{\leftarrow} S_{X_i})$.

Consider first the case when either assumption (a) of the lemma holds (i.e. $X = X_1 \sqcup X_2$) or the subtrees $S_1, S_2$ are disjoint. Let $F$ be any finite subtree of $T_X$ which has nonempty intersection with both $S_i$. It is not hard to see that then

$$f_F^{\psi_1}((X_1)_{\psi_1^{-1}(F \cap S_1)}) \cap f_F^{\psi_2}((X_2)_{\psi_2^{-1}(F \cap S_2)}) = \emptyset,$$

and hence the image sets $E_i$ of the maps $f^{\psi_i}$ are disjoint too, as required.

We are now left with the case when assumption (b) of the lemma holds and the subtrees $S_i$ intersect. Denote by $v$ the common vertex of $X_1$ and $X_2$, which is an essential vertex of $X$. Since we assume, without loss of generality, that $D_{X_i} = X_i \cap D_X$ for both $i$, we obviously get that $D_{X_1} \cap D_{X_2} = \{v\}$. The intersection of the subtrees $S_i$ consists then of a single edge $e$ such that $\lambda_X(e) = v$. We denote this intersection by $F_e$, and we observe that

$$X_{F_e} = f_{F_e}^{\psi_1}((X_1)_{\psi_1^{-1}(F_e)}) \sqcup f_{F_e}^{\psi_2}((X_2)_{\psi_2^{-1}(F_e)}),$$

where $\sqcup$ denotes the disjoint union. As in the previous case, this implies that the image sets $E_i$ of the maps $f^{\psi_i}$ are disjoint, which completes the verification of condition (a1).
Condition (a2).

We will show that for \( i = 1, 2 \) the family \( A_i \) of subsets is null, which obviously implies the condition. We refer to the description of the subsets \( E_i \in A_i \) as recalled above, in the part of proof concerning condition (a1).

Given any subtree \( F \in \mathcal{F}_X \), for a fixed \( i \in \{1, 2\} \) only finitely many of the subtrees \( S_i \) (out of the countable infinite family of such subtrees) intersect \( F \). Consequently, denoting by \( \pi^X_F : \lim_\leftarrow S_X \to X_F \) the projection canonically associated to the system, we get that only finitely many of the images \( \pi^X_F(E_i) \) are not singletons. This easily implies that the family \( A_i = \{E_i\} \) is null, as required.

Condition (a3).

For any \( i \in \{1, 2\} \), fix a subset \( E_i \in A_i \), described as \( E_i = f^{\psi_i}(\lim_\leftarrow S_{X_i}) \), as explained in the earlier parts of this proof. We need to show that any open set in \( \mathcal{A}''(X) = \lim_\leftarrow S_X \) contains a point from the complement of \( E_i \). Obviously, it is sufficient to show this for subsets \( U \) from some basis of topology, e.g. for subsets of form

\[
U = (\pi^X_F)^{-1}(W) : F \in \mathcal{F}_X, \ W \text{ is a nonempty open subset of } X_F
\]

where \( \pi^X_F : \lim_\leftarrow S_X \to X_F \) are the canonical projections associated to the system.

For any \( F \) and \( W \subset X_F \) as above, there is a vertex \( t \) of \( F \) such that, if we view the blow-up space \( (X_t)\#(\lambda_X(A_t F)) \) naturally as a subspace in \( X_F \), then \( W \) has nonempty intersection with this subspace. Consequently, viewing \( D_X \setminus \lambda_X(A_t F) \) as a subset in \( (X_t)\#(\lambda_X(A_t F)) \), there is \( x \in D_X \setminus \lambda_X(A_t F) \) which belongs to \( W \). Denote by \( F' \) the subtree of \( T_X \) equal to the union of \( F \) and the edge \( e \) adjacent to \( t \) and labelled with \( x \). Denote by \( s \) the vertex of \( e \) other than \( t \), and choose any point \( x' \) in the space \( X_s \) distinct from \( x \) and not belonging to \( X_i \) (here we canonically view \( X_i \) as a subset of \( X_s \), since the latter is a copy of \( X \)). Now, viewing \( x' \) as a point of \( X_{F'} \), and \( x \) as a point of \( X_F \), we make the following easy observations. First, since the bonding maps in the system \( S_X \) are all surjective, it follows that the preimage \( (\pi^X_F)^{-1}(x') \) is nonempty. Since \( x' \notin X_i \), we get that this preimage is disjoint with \( E_i \). Finally, since \( \pi^F,F(x') = x \), we get that \( (\pi^X_F)^{-1}(x') \subset (\pi^X_F)^{-1}(x) \subset (\pi^X_F)^{-1}(W) = U \). Consequently, \( U \) contains points from the complement of \( E_i \), which completes the verification of condition (a3).

Condition (a4).

We need to show that in any open set \( U = (\pi^X_F)^{-1}(W) \) from the basis of topology (as described in the previous part of the proof) there is a point \( p \) belonging to the union \( \bigcup A_i \).

Similarly as in the previous part of the proof, consider a vertex \( t \) of the subtree \( F \) such that \( W \cap (X_t)\#(A_t F) \neq \emptyset \), and then consider a point \( x \) from this intersection which is also a point of \( D_X \setminus \lambda_X(A_t F) \). Again, denote by \( e \) this edge of \( T_X \) adjacent to \( t \) for which \( \lambda_X(e) = x \), and denote by \( s \) the vertex of \( e \) other than \( t \). Denote by \( F' \) the subtree of \( T_X \) being the union of \( F \) and \( e \). Moreover, let \( x' \) be a point of \( X_s \) which, under the identification of \( X_s \) with \( X \), is distinct from \( x \) and belongs to \( X_1 \). We then view this \( x' \) as a point of \( X_{F'} \).

Consider the set \( E_i \in A_i \) which corresponds to this subtree \( S_i \) of \( T_X \) which contains \( s \). Consider also a label preserving isomorphism \( \psi_i : T_{X_i} \to S_i \). Note that \( x' \) is in the image of the map \( f^{\psi_i} \) from the morphism \( f^{\psi_i} \), so that for some \( x'' \in (X_i)_{\psi_i^{-1}(F' \cap S_i)} \) we
have \( f_{F'}^{(i)}(x'') = x' \). By surjectivity of the bonding maps, there is \( q \in (\pi_{\Psi_i}^{-1}(F'_i \cap S_i))^{-1}(x'') \), and then \( p := f_{\Psi}(q) \) belongs to both \((\pi_{F_i}^{-1})^{-1}(x')\) and to \( E_i \). Since \( \pi_{F',F}(x') = x \), we get

\[
p \in (\pi_{F_i}^{-1})^{-1}(x') \subset (\pi_{F_i}^{-1})^{-1}(x) \subset (\pi_{F_i}^{-1})^{-1}(W) = U,
\]

which shows that \( E_i \) intersects \( U \), as required.

**Claim.** If for each \( E \) of \( \mathcal{A} \), Condition (a5).

We start with describing a family of open, closed and \( \mathcal{A} \)-saturated subsets of the space \( \mathcal{X}'(X) = \lim_\rightarrow S_X \).

Suppose first that \( X_1, X_2 \) are the factors of a splitting of \( X \), and let \( v \) be the vertex of \( X \) at the intersection of these factors. Let \( e \) be any edge of the tree \( T_X \) labelled with \( v \), and denote by \( F_e \) the subtree of \( T_X \) coinciding with \( e \). It is not hard to see that the space \( X_{F_e} \) splits as the disjoint union of two subgraphs, which can be canonically identified as \((X_1)_{F_e} \) (if we view \( e \) as an edge in the labelled trees \( T_{X_1} \), respectively). Denoting by \( \pi_{F_e} : \lim_\rightarrow S_X \rightarrow X_{F_e} \) the map canonically associated to the inverse limit, we put

\[ H^e_i := (\pi_{F_e})^{-1}(v) \]

as \((X_1)_{F_e} \) and \((X_2)_{F_e} \) (if we view \( e \) as an edge in the labelled trees \( T_{X_1} \), respectively). We then put

\[ H^e_1 := (\pi_{F_e})^{-1}(v) \] \[ H^e_2 := (\pi_{F_e})^{-1}(v) \]

and denote by \( F \) the subtree of \( X \) at the intersection of these factors. Let \( e \) be any edge of the tree \( T_X \) labelled with \( v \), and consider any point \( v \in D_{X_1} \), and consider any edge \( e \) and the related subtree \( F_e \) as in the previous paragraph. Denote by \( s, s' \) the vertices of the edge \( e \). Then the space \( X_{F_e} \) splits as the disjoint union of two subgraphs, which can be canonically identified as \((X_1)_{F_e} \) and \((X_2)_{F_e} \) (if we view \( e \) as an edge in the labelled trees \( T_{X_1} \), respectively). We then put

\[ H^e_1 := (\pi_{F_e})^{-1}(v) \] \[ H^e_2 := (\pi_{F_e})^{-1}(v) \]

and we next show that the subsets \( H^e_i \) as above are \( \mathcal{A} \)-saturated. Actually, it is obviously sufficient to show this for the subsets \( H^e_1 \). Fix one of the subsets \( H^e_1 \), and let \( E \) be an arbitrary set from the family \( \mathcal{A} \). Then \( E = f_{\Psi}(\lim_\rightarrow S_X) \) for appropriate \( i \in \{1,2\} \) and appropriate label preserving isomorphism \( \psi : T_{X_i} \rightarrow S \), where \( S \) is a subtree of \( T_X \). Consider first the case when \( e \) is disjoint with \( S \). Let \( e' \) be the first edge on the shortest path from \( e \) to \( S \) in \( T_X \). Obviously, \( \lambda_X(e') \neq \lambda_X(e) = v \). It follows that if \( \lambda_X(e') \neq X_1 \) then \( E \subset H^e_1 \), and if \( \lambda_X(e') \neq X_2 \) then \( E \cap H^e_1 = \emptyset \). In the remaining case, when \( e \) intersects \( S \), \( e \) is in fact contained in \( S \). Thus, if \( i = 1 \), we easily see that \( E \subset H^e_1 \), and if \( i = 2 \) then \( E \cap H^e_1 = \emptyset \). This shows that \( H^e_1 \) is \( \mathcal{A} \)-saturated, as required.

We now turn to the required separation property. Let \( p = (p_F) \) and \( q = (q_F) \) be two distinct points of \( \mathcal{X}'(X) = \lim_\rightarrow S_X \). Put \( E^v_X = \{ e \in E_X : \lambda_X(e) = v \} \). Observe that if for some \( e \in E^v_X \) exactly one of the points \( p_{F_e}, q_{F_e} \) belongs to the subgraph \((X_1)_{F_e} \subset X_{F_e} \), then the set \( H^e_1 \) separates \( p \) from \( q \). Thus, we need to prove the following.

**Claim.** If for each \( e \in E^v_X \) the alternative of the following two conditions holds:

(a) \( p_{F_e}, q_{F_e} \in (X_1)_{F_e} \), or

(b) \( p_{F_e} \notin (X_1)_{F_e} \) and \( q_{F_e} \notin (X_1)_{F_e} \),

then there is \( E \in \mathcal{A} \) which contains both \( p \) and \( q \).

To prove the claim, we first make the following observation. In its statement, we view singletons \( \{ t \} \) for \( t \in V_X \) as subtrees of \( T_X \).
Subclaim 1. There is \( t \in V_X \) such that \( p\{t\} \neq q\{t\} \).

To justify the subclaim, suppose that its assertion is not true. It is not hard to prove inductively (using the arguments similar as in the more difficult proof of Subclaim 2 below) that for each \( F \in \mathcal{F}_X \) we have then \( p_F = q_F \), which yields \( p = q \), a contradiction.

Coming back to the proof of Claim, let \( t \) be as in the assertion of Subclaim 1, and let \( e \) be the edge from \( E_X' \) adjacent to \( t \). We then obviously have \( p_F \neq q_F \). Suppose that condition (a) as in the claim holds, i.e. \( p_F, q_F \in (X_1)_F \). Denote by \( F_X' \) this maximal subtree of \( T_X \) with all edge labels in \( D_X \), which contains \( e \). Denote by \( \psi : T_X \to S \) any label preserving isomorphism. Put \( F = f^{\psi}(\lim_{\leftarrow} S_X) \) and note that \( E \in \mathcal{A}_1 \subset \mathcal{A} \).

Subclaim 2. In the setting as above, we have \( p, q \in E \).

To get Subclaim 2, by cofinality of the subposet of \( \mathcal{F}_X \) appearing below, it is sufficient to show that for those \( F \in \mathcal{F}_X \) which contain \( e \) we have

\[
(+) \quad p_F, q_F \in f^{\psi}(\{(X_1)_{\psi^{-1}(S\cap F)}\}).
\]

We will do this by induction. Note that due to condition (a) of the claim (under assumption of which we work), assertion (++) holds true for \( F = F_e \). Suppose it holds for some \( F_0 \) containing \( e \), and let \( F_0' \) be a subtree containing \( F_0 \) and having one more vertex than \( F_0 \). We will show that (++) holds true for \( F = F_0' \).

Denote by \( t_0 \) the vertex of \( F_0' \) not contained in \( F_0 \), and by \( e_0 = [t_0, s_0] \) the edge connecting \( t_0 \) to \( F_0 \). Put also \( x_0 = \lambda_X(e_0) \). By our assumptions, \( p_{F_0} \) and \( q_{F_0} \) are two distinct points belonging to the subset \( f^{\psi}_{F_0}(\{(X_1)_{\psi^{-1}(S\cap F_0)}\}) \subset X_{F_0} \). As a consequence, at least one of the points \( p_{F_0'} \) and \( q_{F_0'} \), say \( p_{F_0'} \), is not contained in the part \( X_{t_0}^#(x_0) \) of \( X_{F_0'}^# \). Consequently, identifying canonically \( X_{t_0} \) with \( X \), we get \( p\{t_0\} = x_0 \).

If \( x_0 \) happens to belong to \( (X_1)_{t_0} \subset X_{t_0} \), by the assumption of the claim we get that \( q\{t_0\} \in (X_1)_{t_0} \) as well. This obviously implies assertion (++) for \( F = F_0' \). If \( x_0 = \lambda_X(e_0) \notin (X_1)_{t_0} \), we get that \( p_{F_0'} \) cannot be contained in \( X_{t_0}^#(x_0) \), since otherwise \( p_{F_0} = \pi_{X_{t_0}^#}^X F_0(p_{F_0'}) \) would be equal to \( x_0 \) viewed as belonging to the part \( X_{s_0}^#(\mathcal{A}_{s_0} F_0) \) of \( X_{F_0} \), contradicting the assumption that \( p_{F_0} \in f^{\psi}_{F_0}(\{(X_1)_{\psi^{-1}(F_0\cap S)}\}) \). Thus, both \( p_{F_0'} \) and \( q_{F_0'} \) are then not contained in the part \( X_{t_0}^#(x_0) \), and hence they must belong to the subset \( f^{\psi}_{F_0'}(\{(X_1)_{\psi^{-1}(F_0'\cap S)}\}) \subset X_{F_0'} \), as required. This completes the proof of Subclaim 2.

To conclude the proof of Claim, it remains to consider the case when condition (b) of the claim holds. In this case, an argument similar as above shows that for some \( E \in \mathcal{A}_2 \) we have \( p, q \in E \). We omit further details, hence finishing the proof of Claim, the verification of condition (a5), and the proof of Lemma 3.9.

4. Violated reflection trees of graphs.

In this section we show that reflection trees of graphs allow a bit more flexible description than the one given in Section 2. We will need this description in the later arguments in the paper, notably in the proof of the main result of the next section (Proposition 5.5) which is then used in the proof of the main result of the paper - Theorem 7.12.

Let \( X \) be a finite topological graph. A violated reflection inverse system for \( X \) is any inverse system \( S_X \) described as follows. Fix a countable dense subset \( D \) of \( X \) containing
all essential vertices. Let \( T \), \( V_T \), \( E_T \), \( \mathcal{F} \) and \( \lambda : E_T \to D \) be as described in Section 2. To each vertex \( t \in V_T \) associate a copy \( X_t \) of the graph \( X \). Denote by \( \mathcal{E}_T \) the set of all oriented edges of \( T \), and for each \( \epsilon \in \mathcal{E}_T \) denote by \( \alpha(\epsilon), \omega(\epsilon) \) the initial and the terminal vertex of \( \epsilon \), respectively, and by \( |\epsilon| \) the underlying unoriented edge of \( T \). Fix any family \( \beta_\epsilon : \epsilon \in \mathcal{E}_T \) of maps \( P_{\lambda(|\epsilon|)} \to P_{\lambda(|\epsilon|)} \) (where the source and the target blow-up locus \( P_{\lambda(|\epsilon|)} \) is viewed as a subset in any appropriate blow-ups of the copies \( X_{\alpha(\epsilon)} \) and \( X_{\omega(\epsilon)} \), respectively) such that

(i) \( \beta_\epsilon \) is the identity if \( \lambda(|\epsilon|) \) is an essential vertex of \( X \), and it is either the identity or the transposition of the doubleton \( P_{\lambda(|\epsilon|)} \) otherwise;

(ii) for each \( \epsilon \in \mathcal{E}_T \), if \( \bar{\epsilon} \) denotes the oppositely oriented edge \( \epsilon \), we have \( \beta_{\bar{\epsilon}} = \beta_\epsilon^{-1} \).

The underlying poset of the now being described system \( \mathcal{S}_X^v \) is \( \mathcal{F} \), and its spaces \( Z_F : F \in \mathcal{F} \) are as follows. Fix a subtree \( F \in \mathcal{F} \), and use the notation \( V_F, E_F \) and \( A_t F : t \in V_F \) as in Section 2. Denote also by \( \mathcal{E}_F \) the set of all oriented edges of \( F \). Put

\[
Z_F := \bigsqcup_{t \in V_T} X_{t \#}(\lambda(A_t F))/\simeq,
\]

where the equivalence relation \( \simeq \) is induced by the following equivalences: for each \( \epsilon \in \mathcal{E}_F \) and each \( p \in P_{\lambda(|\epsilon|)} \) identify \( p \in P_{\lambda(|\epsilon|)} \subset X_{\alpha(\epsilon)} \#(\lambda(A_{\alpha(\epsilon)} F)) \) with \( \beta_\epsilon(p) \in P_{\lambda(|\epsilon|)} \subset X_{\omega(\epsilon)} \#(\lambda(A_{\omega(\epsilon)} F)) \). The maps \( \pi_{F', F} : Z_{F'} \to Z_F \) of the system \( \mathcal{S}_X^v \) are then defined in the same way as the corresponding maps \( \pi_{F', F} \) of the system \( \mathcal{S}_X \), see Section 2.

We now state, as a proposition below, the main result of this section. The assumptions which we put on \( X \) are not essential, but they simplify the argument, and correspond to the case that we need for our later purposes in the paper.

4.1 Proposition. Let \( X \) be a finite connected topological graph without essential loop edges, and let \( \mathcal{S}_X^v \) be any violated reflection inverse system for \( X \). Then \( \lim_\prec \mathcal{S}_X^v \cong \mathcal{X}^v(X) \).

The remaining part of the section is devoted to the proof of Proposition 4.1. Choose a function \( \kappa \) which to any essential edge \( a \) of \( X \) associates an arbitrarily selected essential vertex adjacent to \( a \). Denote by \( X_a^2 \) the graph obtained by gluing the two copies of the blow-up \( X_a^\#(\kappa(a)) \) through the identity map along the blow-up locus \( P_{\kappa(a)} \). Denote also by \( D_a^2 \subset X_a^2 \) the union of two copies of the set \( D \setminus \{\kappa(a)\} \subset X_a^\#(\kappa(a)) \). Denote by \( \bar{\epsilon} \) this essential edge of \( X_a^2 \) which is made of the two copies of this edge of \( X_a^\#(\kappa(a)) \) which corresponds to the edge \( a \) of \( X \) (we denote this edge of \( X_a^\#(\kappa(a)) \) also by \( a \)). To prove Proposition 4.1, we will need the following easy to observe result, whose proof we omit.

4.2 Lemma. For any points \( x_1, x_2 \in D_a^2 \cap \bar{\epsilon} \) which are not essential vertices of \( X_a^2 \) (i.e. no one of them is an endpoint of \( \bar{\epsilon} \)), including the case \( x_1 = x_2 \), consider the blow-ups \((X_a^2)^\#(x_i)\) with their corresponding blow-up loci \( P_{x_i} \), which are doubletons. Then for any bijection \( \alpha : P_{x_1} \to P_{x_2} \) there is a homeomorphism \( h : (X_a^2)^\#(x_1) \to (X_a^2)^\#(x_2) \) such that:

1. \( h \) restricted to the complement of the union of two edges in \((X_a^2)^\#(x_1)\) intersecting the blow-up locus \( P_{x_1} \) coincides either with the identity or with the relevant restriction of the canonical involution of \( X_a^2 \) which exchanges identically the two copies of \( X_a^\#(\kappa(a)) \) contained in \( X_a^2 \);

2. \( h \) restricted to \( P_{x_1} \) coincides with \( \alpha \);
We now turn to the essential part of the proof of Proposition 4.1. We will show that for some appropriately chosen cofinal subposets $G, G' \subset F$ the restricted inverse systems $S_X|_G$ and $S_X'|_{G'}$ are isomorphic, thus yielding

$$\lim_{\leftarrow} S^v_X = \lim_{\leftarrow} S^v_X|_G \cong \lim_{\leftarrow} S_X|_G = \lim_{\leftarrow} S_X = \lambda^r(X).$$

Actually, we will find appropriate $G$ and $G'$ in the form of increasing sequences $F_n : n \geq 0$ and $F'_n : n \geq 0$ of finite subtrees of $T$ such that $\bigcup F_n = \bigcup F'_n = T$. Thus, the restricted to $G$ and $G'$ inverse systems mentioned above will be in fact inverse sequences.

We start with describing an appropriate sequence $F_n$. Put $F_0$ to be any single vertex of $T$. Recursively, having described $F_n$, we choose a vertex $w \in V_T$ not contained in $F_n$ and such that it is connected to $F_n$ by a single edge of $T$, which we denote $e$. If $\lambda(e)$ is an essential vertex of $X$, we put $F_{n+1}$ to be the subtree of $T$ spanned on $F_n \cup \{w\}$. If $\lambda(e)$ is not an essential vertex of $X$, let $a_e$ be the essential edge of $X$ containing $\lambda(e)$ in its interior. Let $e'$ be the edge of $T$ adjacent to $w$ and such that $\lambda(e') = \kappa(a_e)$, and let $z$ be the vertex of $e'$ other than $w$. We then put $F_{n+1}$ to be the subtree of $T$ spanned on $F_n \cup \{w, z\}$. It is clear that choices of the vertices $w$ at corresponding steps of the above construction can be made so that $\bigcup F_n = T$, and after making these choices in such a way we get a sequence $G = (F_n)_{n \geq 0}$ as required.

We now turn to describing inductively subtrees $F'_n$ and homeomorphisms $g_n : X_{F_n} \to Z_{F'_n}$ which commute with the bonding maps $\pi_{F_i,F_j}$ and $\pi^v_{F'_i,F'_j}$ in both inverse sequences $S_X|_G$ and $S_X'|_{G'}$, respectively. We start with putting $F'_0 = F_0$ and, under canonical identification of both spaces $X_{F_0}$ and $Z_{F'_0}$ with $X$, we take $g_0$ to be the identity of $X$.

To proceed with the induction, suppose that for some $n \geq 0$ and for all $i = 0, \ldots, n$ subtrees $F'_i$ and homeomorphisms $g_i : X_{F_i} \to Z_{F'_i}$ have been already defined, and they satisfy the following properties:

(a) $F'_0 \subset \ldots \subset F'_n$;

(b) the commutativity equation $g_j \pi_{F_i,F_j} = \pi^v_{F'_i,F'_j} g_i$ holds for all $0 \leq j < i \leq n$;

(c) for $x \in X_{F_n}$ we have that $x \in D \setminus \lambda(A_t F_n) \subset X^# \setminus (\lambda(A_t F_n)) \subset X_{F_n}$ for some $t \in V_{F_n}$ if and only if there is $s \in V_{F'_n}$ such that $g_n(x) \in D \setminus \lambda(A_s F'_n) \subset X^# \setminus (\lambda(A_s F'_n)) \subset Z_{F'_n}$;

(d) if $x$ as in (c) is an essential vertex of $X$ then $x$ and $g_n(x)$, viewed as points of $X$, coincide;

(e) if $x$ as in (c) is not an essential vertex of $X$ then $g_n(x)$ is also not an essential vertex of $X$; moreover, $x$ and $g_n(x)$ (viewed as points of $X$) belong to the same essential edge of $X$.

Note that for $n = 0$ (and thus for the already defined subtree $F'_0$ and map $g_0$) properties (a)-(e) are indeed satisfied. To describe the subtree $F_{n+1}$ and the map $g_{n+1} : X_{F_{n+1}} \to Z_{F'_{n+1}}$, suppose first that, in the description of the sequence $F_k : k \geq 0$, the subtree $F_{n+1}$ is spanned on $F_n \cup \{w\}$, i.e. $\lambda(e)$ is an essential vertex of $X$. Denote by $t$ the vertex of $F_n$ adjacent to $e$ and view $x = \lambda(e)$ as a point in $X_t$, and hence also in $X_{F_n}$. Let $s \in V_{F'_n}$ be this vertex for which, due to property (c) above, $g_n(x) \in D \setminus \lambda(A_s F'_n) \subset X^# \setminus (\lambda(A_s F'_n)) \subset Z_{F'_n}$. Let $d$ be this edge of $T$ adjacent to $s$ for which $\lambda(d) = g_n(x)$ (where $g_n(x)$ is viewed here
as an element of $D \subset X$). Note that, by property (d), we have $x = g_n(x)$ (as points of $X$), and hence $\lambda(e) = \lambda(d)$ (as points of $D$). Let $u$ be the vertex of $d$ other than $s$. Note that then $u \notin V_{F'_n}$ and put $F'_{n+1}$ to be the subtree of $T$ spanned on $F'_n \cup \{u\}$. Define a map $g_{n+1} : X_{F_{n+1}} \to Z_{F'_{n+1}}$ as follows. Viewing $\lambda(e)$ again as a point of $X_{F_n}$ (as above), and at the same time as a point of $X_w$, we have $X_{F_{n+1}} = X^\#(\lambda(e)) \cup X^\#(\lambda(e))$, where the intersection corresponds to the blow-up loci $P_{\lambda(e)}$ in both summands, which are identified with each other by the identity map. Similarly, we have $Z_{F'_{n+1}} = Z^\#(\lambda(d)) \cup Z^\#(\lambda(d))$, with the subsets $P_{\lambda(d)}$ in both summands glued to each other also by the identity map (due to condition (i) in the description of a violated reflection inverse system for $X$).

Let $\hat{g}_n : X_{F_n}^\#(\lambda(e)) \to Z_{F_n}^\#(\lambda(d))$ be the map induced by $g_n$ (which makes sense since $g_n(\lambda(e)) = \lambda(d)$). Put $g_{n+1}$ to be equal to $\hat{g}_n$ on the part $X^\#_{F_n}(\lambda(e))$, and put it to be equal to the identity map $X^\#_{F_n}(\lambda(e)) \to X^\#_{F_n}(\lambda(d))$ (after identifying canonically both $X_w$ and $X_u$ with $X$, and remembering that $\lambda(e) = \lambda(d)$).

We now describe the subtree $F'_{n+1}$ and the map $g_{n+1} : X_{F_{n+1}} \to Z_{F'_{n+1}}$ in the second case, when $F_{n+1}$ is spanned on $F_n \cup \{w, z\}$, i.e. $\lambda(e)$ is not an essential vertex of $X$. As in the previous case, denote by $t$ the vertex of $F_n$ adjacent to $e$ and view $x_1 = \lambda(e)$ as a point in $X_t$, and hence also in $X_{F_n}$. Let $s \in V_{F'}$ be this vertex for which, due to property (c) above, $x_2 := g_n(x_1) \in D \setminus \lambda(A_{\kappa}F'_n) \subset X^\#(\lambda(A_{\kappa}F'_n)) \subset Z_{F'_n}$. Let $d$ be this edge of $T$ adjacent to $s$ for which $\lambda(d) = x_2$ (where $x_2$ is viewed here as an element of $D \subset X$). Note that, by property (e), we have that the points $x_1, x_2$ (viewed as points of $X$) belong to the interior of the same essential edge $a_e$ of $X$. Let $u$ be the vertex of $d$ other than $s$. Note that then $u \notin V_{F'_n}$. Let $d'$ be the edge of $T$ adjacent to $u$ and such that $\lambda(d') = \kappa(a_e)$, and let $v$ be the vertex of $d'$ other than $u$. Put $F'_{n+1}$ to be the subtree of $T$ spanned on $F'_n \cup \{u, v\}$. Define a map $g_{n+1} : X_{F_{n+1}} \to Z_{F'_{n+1}}$ as follows. Recall that we denote by $e'$ the edge connecting the vertices $w$ and $z$ of the subtree $F_{n+1}$. Denote by $F'_e$ and $F'_d$ the subtrees of $T$ consisting of single edges $e'$ and $d'$, respectively. View $x_1 = \lambda(e)$ again as a point of $X_{F_e}$ (as above), and at the same time as a point of $X_u$, and hence also of $X_{F'_d}$. We then have $X_{F_{n+1}} = X_{F'_e}(x_1) \cup X_{F'_d}(x_1)$, where the intersection corresponds to the blow-up loci $P_{x_1}$ in both summands, which are identified with each other by the identity map. Similarly, we have $Z_{F'_{n+1}} = Z_{F'_e}(x_2) \cup Z_{F'_d}(x_2)$, with the subsets $P_{x_2}$ in both summands glued to each other by a map $\beta$ which is either the identity or the transposition (due to condition (ii) in the description of a violated reflection inverse system for $X$). Note that since both $x_1, x_2$ (viewed as points of $X$) belong to the interior of the same essential edge $a_e$ of $X$, and since $\lambda(e') = \lambda(d') = \kappa(a_e)$, we can canonically identify the spaces $X^\#_{F'_e}(x_1)$ and $Z^\#_{F'_d}(x_2)$ with the spaces $(X_{a_e}^2)^\#(x_1)$ and $(X_{a_e}^2)^\#(x_2)$ (in the notation used in Lemma 4.2), respectively. Let $h : X_{F'_e}(x_1) \to Z_{F'_d}(x_2)$ be a homeomorphism as in Lemma 4.2, for $\alpha = \beta \hat{g}_n|_{P_{x_1}}$ (under the above mentioned identifications of the source and the target with the appropriate blow-ups of $X_{a_e}^2$). Let $\hat{g}_n : X_{F'_e}(x_1) \to Z_{F'_n}(x_2)$ be the map induced by $g_n$ (which makes sense since $g_n(x_1) = x_2$). Put $g_{n+1}$ to be equal to $\hat{g}_n$ on the part $X_{F'_e}(x_1)$, and put it to be equal to $h$ on the remaining part $X_{F'_e}(x_1)$.

We skip a straightforward verification that in both cases above conditions (a)-(e) are
still satisfied, for \( n \) replaced with \( n + 1 \). We also note that, due to condition (c) kept during the inductive construction, we have \( \bigcup F_n' = T \), and hence the subposet \( G' = (F_n')_{n \geq 0} \) is cofinal in \( F \). Moreover, the commuting sequence of homeomorphisms \( (g_n) \) obviously yields an isomorphism of the restricted inverse systems \( S_X|_G \) and \( S_X^{v'}|_{G'} \), as required. This finishes the proof of Proposition 4.1.

5. Inverse sequences of \( X \)-graphs and \( X \)-blow-ups.

In this section, for any topological graph \( X \) without loop edges and without vertices of degree 1, we describe a useful class of inverse sequences of topological graphs, called null and dense inverse sequences of \( X \)-graphs and \( X \)-blow-ups, such that for any inverse sequence \( \mathcal{R} \) in this class we have \( \lim_{\to} \mathcal{R} \cong X''(X) \). One may think of these sequences as of another (quite flexible and convenient) way of expressing the spaces \( X''(X) \). We will use such sequences as a tool in the proof of the main result of this paper (Theorem 7.12), to recognize visual boundaries of some Coxeter systems as (being homeomorphic to) the trees of graphs \( X''(X) \).

Let \( X \) be a (finite) topological graph, equipped with its natural stratification into essential vertices and edges. For all of this section we assume that \( X \) contains no essential loop edges (but it may have multiple edges), and no vertices of degree 1. Let \( V, E \) be the sets of (essential) vertices and edges of \( X \), respectively. For each \( v \in V \) consider the induced labelling of the elements of the link \( X_v \) with the elements of \( E \). Clearly, \( X \) itself is tautologically an \( X \)-graph. Note that bijections \( \{ \beta_v \} \) are in fact unique, and we view them as part of the structure of an \( X \)-graph.

We now describe, in Definitions 5.2 and 5.3, a class of maps between \( X \)-graphs that we call \( X \)-blow-ups.

5.1 Definition. Given a graph \( X \) as above, with the sets \( V, E \) of vertices and edges, an \( X \)-graph is a finite topological graph \( X' \) having no essential loop edges, equipped with labellings of its vertices and edges with elements of the sets \( V, E \) respectively, such that if \( v' \) is a vertex of \( X' \) labelled with \( v \in V \) then there is a bijection \( \beta_{v'} : X' \to X_v \) preserving the induced labels from the set \( E \). Clearly, \( X \) itself is tautologically an \( X \)-graph. Note that bijections \( \{ \beta_{v'} \} \) are in fact unique, and we view them as part of the structure of an \( X \)-graph.

We now describe, in Definitions 5.2 and 5.3, a class of maps between \( X \)-graphs that we call \( X \)-blow-ups.

5.2 Definition. Let \( X' \) be an \( X \)-graph and \( v' \) a vertex of \( X' \) labelled with \( v \in V \). The \( X \)-blow-up at \( v' \) is a pair \( (X'', f) \) consisting of an \( X \)-graph \( X'' \) and a map \( f : X'' \to X' \) defined as follows:

- delete \( v' \) from \( X' \) and complete the resulting space naturally so that to distinct edges issuing from \( v' \) distinct points are attached in place of \( v' \); label the attached points \( p \) naturally with elements of the link \( X'_v \) and denote these labels by \( \lambda(p) \) (this labelling plays only auxiliary role); delete from \( X \) the open star of the vertex \( v \), i.e. \( v \) and the interiors of the edges issuing from \( v \); glue the above modified \( X' \) to the above modified \( X \) through the map, which to any point \( p \) attached to \( X' \setminus \{ v' \} \) associates the terminal vertex (other than \( v \)) of this edge deleted from \( X \) which coincides with the label \( \lambda(p) \) (this gluing map is not necessarily a bijection); view the obtained space as a topological graph (with natural stratification), and denote it by \( X'' \); consider
also the obvious labelling of vertices and edges of $X''$, induced from the corresponding labellings in $X'$ and $X$, and note that with this labelling $X''$ is an $X$-graph;

• define $f : X'' \to X'$ to be the identity on the part of $X''$ corresponding to $X' \setminus \{v'\}$, and to be the constant map with value $v'$ on the remaining part of $X''$.

If $f : X'' \to X'$ is an $X$-blow-up at a vertex $v'$ of $X'$, we call $v'$ the blow-up point of $f$.

5.3 Definition. Let $X'$ be an $X$-graph, $e'$ an edge of $X'$ labelled with $e \in E$, and let $L$ be a (closed) segment contained in the interior of $e'$. The $X$-blow-up at $L$ is a pair $(X'', f)$ consisting of an $X$-graph $X''$ and a map $f : X'' \to X'$ defined as follows:

• delete the interior of $L$ from $X'$ and the interior of $e$ from $X$; choose any bijection between the sets of endpoints of $L$ and of $e$ and glue the remaining parts of $X'$ and $X$ accordingly with this bijection; view the obtained space as a topological graph (with natural stratification), and denote it by $X''$; consider the obvious labelling of vertices and edges of $X''$ induced from those in $X'$ and $X$ and note that with this labelling $X''$ is an $X$-graph;

• define $f : X'' \to X'$ to be the identity on the part of $X''$ corresponding to $X' \setminus \text{int}(L)$; if $\tilde{e}$ is an edge of $X$ distinct from $e$ and connecting the endpoints of $e$, let $f$ map $\tilde{e}$ homeomorphically on $L$, consistently with the identification of the endpoints; to define $f$ on the remaining part of $X''$, choose a point $p$ in the interior of $L$ and consider two half-segments into which $p$ splits $L$; for any edge $\varepsilon$ issuing from an endpoint of $e$ and terminating not at the other endpoint of $e$, let $f$ map $\varepsilon$ homeomorphically on the appropriate half-segment of $L$; finally, let $f$ map the remaining part of $X''$ to $p$.

If $f : X'' \to X'$ is an $X$-blow-up at a segment $L$, as above, we call $L$ the blow-up segment of $f$.

Note that in fact we have two possible operations of $X$-blow-up at a segment $L$, related to the two choices of the gluing bijection in the above description. These two operations are in general essentially distinct.

Note also that, by the assumption that $X$ has no vertex of degree 1, the $X$-blow-up maps at segments are surjective, and thus all $X$-blow-up maps are surjective.

To unify terminology, we use the term blow-up-locus for both blow-up points and blow-up segments. If $f : X'' \to X'$ is an $X$-blow-up with blow-up locus $\Lambda$, we view all points in $X' \setminus \Lambda$ as points of $X''$, and we say that these are the points of $X'$ unaffected by $f$.

In the main result of this section, Proposition 5.5, we deal with inverse sequences

$$X_1 \xleftarrow{\pi_1} X_2 \xleftarrow{\pi_2} \ldots$$

of the form presented in the following definition.

5.4 Definition. We call an inverse sequence $\mathcal{R} = (\{X_n\}_{n \geq 1}, \{\pi_n\}_{n \geq 1})$ a null and dense inverse sequence of $X$-graphs and $X$-blow-ups if it satisfies the following conditions:

(i) $X_1 = X$, each $X_i$ is an $X$-graph, and each map $\pi_i : X_{i+1} \to X_i$ is an $X$-blow-up (either at some vertex of $X_i$, or at some segment $L$ contained in the interior of some edge of $X_i$);
(i2) for each $i$ every vertex $v'$ in $X_i$ eventually blows-up; more precisely, there is $k \geq i$ such that $v'$ is unaffected by the maps $\pi_i, \pi_{i+1}, \ldots, \pi_{k-1}$ (i.e. it is naturally a vertex of $X_k$) and the map $\pi_k : X_{k+1} \to X_k$ is the $X$-blow-up at $v'$.

(i3) for each $i$ the family of the images in $X_i$ (through appropriate compositions of the maps $\pi_k$) of all blow-up segments of the maps $\pi_m$ with $m > i$, is null, i.e. the diameters of those images converge to 0; more precisely, after choosing any metric in $X_i$, for any $\epsilon > 0$ the set of indices $m$ such that $m > i$, $\pi_m$ is a blow-up at a segment, and the image of this blow-up segment of $\pi_m$ in $X_i$ has diameter greater than $\epsilon$, is finite;

(i4) for each $i$ the union of the images in $X_i$ of all blow-up segments of the maps $\pi_m$ with $m > i$ is dense in $X_i$.

5.5 Proposition. Let $X$ be a finite topological graph whose natural stratification contains no loops and no vertices of degree 1, and let $R$ be any inverse sequence satisfying conditions (i1)–(i4), i.e. a null and dense inverse sequence of $X$-graphs and $X$-blow-ups. Then the inverse limit $\lim_\leftarrow R$ is homeomorphic to the reflection tree $X^{r}(X)$.

Before starting the proof of Proposition 5.5, we present some fairly general result which describes a method of modifying an inverse sequence of metric compacta without affecting its limit. The modification consists of quotiening the spaces in the sequence subject to their appropriate partitions. We call the sequence resulting from any such modification a good quotient of the initial sequence. We will use a modification of this kind as a tool in our proof of Proposition 5.5.

Recall that an inverse sequence of compact metric spaces is a tuple $P = (\{P_i\}, \{\pi_i\})$, where each $P_i$ is a compact metric space and each $\pi_i : P_{i+1} \to P_i$ is a continuous map. For each $i > j$ we denote by $\pi_{i,j} : P_i \to P_j$ the composition $\pi_{i,j} := \pi_j \circ \cdots \circ \pi_i$. We also use the convention that for each $i$ the map $\pi_{i,i}$ is the identity of $P_i$. Recall that the limit $\lim_\leftarrow P$ is, by definition, a subspace in the product $\prod_i P_i$, with the induced product topology, consisting of the sequences $(x_i) \in \prod_i P_i$ such that $\pi_i(x_{i+1}) = x_i$ for each $i$. Sequences $(x_i)$ as in the previous sentence are called threads of $P$.

We refer the reader to [Dav] for the introduction and basic results concerning partitions of the spaces (called also decompositions) and their associated quotients.

5.6 Lemma. Let $P = (\{P_i\}_{i \geq 1}, \{\pi_i\}_{i \geq 1})$ be an inverse sequence of compact metric spaces, and for each $i \geq 1$ let $Q_i$ be an upper-semicontinuous partition of $P_i$. Suppose that the following further conditions are satisfied:

(p1) (compatibility) for each $i \geq 1$ and for each $Q \in Q_{i+1}$ there is $Q' \in Q_i$ such that $\pi_i(Q) \subset Q'$;

(p2) (fineness) for any sequence $(Q_i)_{i \geq k}$ of subsets $Q_i \in Q_i$ such that $\pi_i(Q_{i+1}) \subset Q_i$ for each $i \geq k$, the diameters of the images $\pi_{i,k}(Q_i) : i \geq k$ converge to 0.

Then the induced inverse sequence of quotients $P^* = (\{P_i/Q_i\}_{i \geq 1}, \{\pi_i^* : P_{i+1}/Q_{i+1} \to P_i/Q_i\}_{i \geq 1})$ is well defined, consists of compact metric spaces and continuous maps, and

$$\lim P^* \cong \lim P.$$
Proposition 4.1 and Lemma 5.6, this will imply that a cofinal subsequence \( P \) so defined (quotient)

Proof of Proposition 5.5: To prove the proposition, we will show that some good quotient \( \rho \) of subsets obtained from \( \pi \) of subsets (condition (p2), the diameters of these subsets converge to 0. Hence, the intersection that the assertion of this claim directly means that \( \rho_k \) for all \( k \) there is a set \( Q_k \) such that \( z_k \) and \( w_k \) are not contained both in the same \( Q \in Q_k \).

Observe further, that the family \( (\rho_i) \) of the quotient maps yields a morphism \( \mathcal{P} \to \mathcal{P}^* \) of the inverse sequences, due to the commutativity relations \( \pi_i^* \rho_{i+1} = \rho_i \pi_i \) mentioned in the previous paragraph. This morphism induces then a continuous map

\[
\rho : \lim \mathcal{P} \to \lim \mathcal{P}^*
\]

given by \( \rho((x_i)) := (\rho_i(x_i)) \). We will show that \( \rho \) is a homeomorphism. Since both limits are compact, it suffices to show that \( \rho \) is injective and surjective.

To prove injectivity of \( \rho \), consider \( z, w \in \lim \mathcal{P}, z \neq w, z = (z_i) \) and \( w = (w_i) \). We claim that there is \( k \) such that \( z_k \) and \( w_k \) are not contained both in the same \( Q \in Q_k \). Suppose this is not true. Then for each \( i \geq 1 \) there is a set \( Q_i \in Q_i \) with \( z_i, w_i \in Q_i \). By compatibility condition (p1), we then have the inclusions \( \pi_i(Q_{i+1}) \subset Q_i \) for all \( i \). By condition (p2), for each \( k \geq 1 \) we have

\[
(+) \quad \lim_{i \to \infty} \text{diam}(\pi_{i,k}(Q_i)) = 0.
\]

It follows from our a contrario assumption that for each \( i \geq k \) both \( z_k = \pi_{i,k}(z_i) \) and \( w_k = \pi_{i,k}(w_i) \) belong to the set \( \pi_{i,k}(Q_i) \). As a consequence of (++) we then have \( z_k = w_k \) for all \( k \), contradicting the assumption that \( z \neq w \). Thus our claim follows. Note however that the assertion of this claim directly means that \( \rho_k(z_k) \neq \rho_k(w_k) \), and this obviously implies that \( \rho(z) \neq \rho(w) \), as required.

To prove surjectivity of \( \rho \), consider any \( x^* = (x^*_i) \in \lim \mathcal{P}^* \). We need to construct a thread \( (x_i) \) for \( \mathcal{P} \) such that \( \rho_i(x_i) = x^*_i \) for each \( i \). For any \( i \geq 1 \) put \( Q_i = \rho_i^{-1}(x^*_i) \), and note that \( Q_i \in Q_i \) and \( \pi_i(Q_{i+1}) \subset Q_i \) for all \( i \). Note also that, for each \( k \geq 1 \) the sequence of subsets \( (\pi_{i,k}(Q_i))_{i \geq k} \) of \( P_k \) is nested, i.e., \( \pi_{i+1,k}(Q_{i+1}) \subset \pi_{i,k}(Q_i) \), and, by fineness condition (p2), the diameters of these subsets converge to 0. Hence, the intersection \( \bigcap_{i \geq k} (\pi_{i,k}(Q_i)) \) is a singleton, and we take it as \( x_k \). It is then straightforward to check that so defined \( (x_i) \) is a thread, and that \( \rho_i(x_i) = x^*_i \).

This completes the proof of the lemma.

5.7 Definition. A good quotient of an inverse system \( \mathcal{P} \) is any inverse sequence \( \mathcal{P}^* \) obtained from \( \mathcal{P} \) as in Lemma 5.6.

Proof of Proposition 5.5: To prove the proposition, we will show that some good quotient \( \mathcal{R}^* \) of the sequence \( \mathcal{R} \) is isomorphic, as inverse sequence of topological spaces, to a cofinal subsequence \( \mathcal{P} \) in some violated reflection inverse system \( \mathcal{S}_X^\mathcal{R} \) for \( X \). In view of Proposition 4.1 and Lemma 5.6, this will imply that

\[
\lim \mathcal{R} \cong \lim \mathcal{R}^* \cong \lim \mathcal{P} = \lim \mathcal{S}_X^\mathcal{R} \cong \mathcal{X}^\mathcal{R}(X).
\]

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To clarify the exposition, we split this rather long proof into steps.

**Step 1.**
We start with some useful terminology and observations.

Recall that \( \mathcal{R} = \{(X_i)_{i \geq 1}, \{\pi_i\}_{i \geq 1}\} \) is a null and dense inverse sequence of \( X \)-graphs and \( X \)-blow-ups. For any \( i \geq 1 \), we call a point \( x \in X_i \) a *direct blow-up point* of \( X_i \) if \( x \) is a blow-up point of \( \pi_i \) or if for some \( n \geq i + 1 \) the point \( x \) is unaffected by the blow-ups \( \pi_i, \pi_{i+1}, \ldots, \pi_{n-1} \) and, when viewed as a point of \( X_n \), \( x \) is a blow-up point of \( \pi_n \). We define similarly a *direct blow-up segment* in \( X_i \) as a closed segment \( L \) contained in the interior of some edge of \( X_i \) such that, either \( L \) is the blow-up segment of \( \pi_i \), or for some \( n \geq i + 1 \) the blow-ups \( \pi_i, \pi_{i+1}, \ldots, \pi_{n-1} \) do not affect \( L \) (so that it can be viewed as a segment contained in the interior of some edge of \( X_n \)), and \( L \) is a blow-up segment of \( \pi_n \). We use the term *direct blow-up locus* to denote both any direct blow-up point and any direct blow-up segment. It is not hard to make the following observations, which hold for each \( i \geq 1 \):

- **(d1)** if \( \Lambda \) is the blow-up locus of \( \pi_i \), then its preimage \( \pi_i^{-1}(\Lambda) \) in \( X_{i+1} \), and hence also its preimage in any \( X_j \) with \( j > i \), is a subgraph of \( X_j \) (i.e. the union of some family of closed essential edges of \( X_j \) and some essential vertices of \( X_j \));
- **(d2)** if \( \Lambda \) is the blow-up locus of \( \pi_i \), then the image in \( X_i \) (through the appropriate composition of the maps \( \pi_k \)) of the blow-up locus of any map \( \pi_j \) with \( j > i \) is either contained in or disjoint with \( \Lambda \);
- **(d3)** the direct blow-up loci in \( X_i \) are pairwise disjoint (this can be deduced e.g. from property (d2));
- **(d4)** the image in \( X_i \) (through the appropriate composition of the maps \( \pi_k \)) of the blow-up locus of any map \( \pi_j \) with \( j > i \) is contained in some direct blow-up locus of \( X_i \) (this can be deduced e.g. from properties (d2) and (d3));
- **(d5)** each essential vertex of \( X_i \) is a direct blow-up point (this follows from condition (i2)), and there are no more direct blow-up points in \( X_i \);
- **(d6)** the union of all direct blow-up loci in \( X_i \) is a dense subset of \( X_i \) (this follows from condition (i4) in view of the above observation (d4));
- **(d7)** each point in \( x \in X_i \) which is not an essential vertex and does not belong to a direct blow-up segment of \( X_i \), is not affected by any of the \( X \)-blow-ups \( \pi_k \) with \( k \geq i \), and thus it can be canonically identified with a point in the limit \( \lim_{\leftarrow} \mathcal{R} \); we call any such point \( x \) a *stable* point of \( X_i \);
- **(d8)** the essential vertices of \( X_i \) and the ends of all direct blow-up segments of \( X_i \) belong to the closure of the set of all stable points of \( X_i \); moreover, each stable point is an accumulation point of the set of all stable points, and it can be approached by the stable points from each its side in the edge of \( X_i \) in the interior of which it is contained.

For each \( k \geq 1 \), denote by \( \Lambda_k \) the blow-up locus of the map \( \pi_k : X_{k+1} \to X_k \) in the sequence \( \mathcal{R} \). We make the following easy observation, the proof of which we skip.

**5.10 Fact.** *For each \( k \geq 1 \), if \( \Lambda \) is a direct blow-up locus in \( X_k \) distinct from \( \Lambda_k \), then*
(1) the restriction $\pi_k|_{\pi_k^{-1}(\Lambda)} : \pi_k^{-1}(\Lambda) \to \Lambda$ is a bijection, so that we can view $\Lambda$ as a subset in $X_{k+1}$ (by identifying it with its preimage via the above mentioned bijection);

(2) the preimage $\pi_k^{-1}(\Lambda)$ is a direct blow-up locus in $X_{k+1}$.

We say that a direct blow-up locus of $X_{k+1}$ is \textit{inherited} from $X_k$ if it has a form $\pi_k^{-1}(\Lambda)$ for some direct blow-up locus $\Lambda$ of $X_k$ distinct from $\Lambda_k$. Note that, in each of the graphs $X_{k+1}$ we have two kinds of blow-up loci $\Lambda$:

- the ones inherited from $X_k$, and
- those for which $\pi_k(\Lambda) \subset \Lambda_k$.

The latter will be called \textit{non-inherited} blow-up loci of $X_{k+1}$. Locus $\Lambda_{k+1}$ can be of any of the above two kinds. We artificially view all the blow-up loci of $X_1$ as inherited.

Step 2.

In this step we describe some good quotient $R^*$ of the sequence $R$, by choosing appropriate partitions $Q_i$ of the spaces $X_i$ in $R$. We also verify the assumptions (p1) and (p2) of Lemma 5.6 for the chosen partitions.

We start with choosing a partition $Q_1$ of $X_1$. Let $J_1$ be a family consisting of closed normal neighbourhoods of vertices of $X_1$ and of closed segments contained in the interiors of edges of $X_1$, such that

(j1) the elements in $J_1$ are pairwise disjoint,

(j2) for any direct blow-up locus $\Lambda$ of $X_1$ there is $J \in J_1$ such that $\Lambda \subset \text{int}(J)$.

(j3) for each $J \in J_1$ the boundary $\partial J$ consists of stable points of $X_1$.

Such a family $J_1$ can be constructed recursively, as follows:

- put $J_1$ to be some closed normal neighbourhood in $X_1$ of the blow-up locus $\Lambda_1$ (the blow-up locus of $\pi_1$), such that the boundary $\partial J_1$ consists of stable points of $X_1$;
- having already defined the neighbourhoods $J_1, \ldots, J_n$, let $i_n$ be the smallest integer such that $\Lambda_{i_n}$ (the blow-up locus of the map $\pi_{i_n}$) is a direct blow-up locus of $X_1$ not contained in $J_1 \cup \ldots \cup J_n$; note that, since the boundaries of $J_1, \ldots, J_n$ consist of stable points, they are disjoint with $\Lambda_{i_n}$, and since the latter is connected and not contained in any $J_i$, it is disjoint with all these $J_i$; by this and by property (d8), one may choose some closed normal neighbourhood of $\Lambda_{i_n}$ in $X_1$ disjoint with $J_1 \cup \ldots \cup J_n$ and such that $\partial J_{n+1}$ consists of stable points of $X_1$; we take this neighbourhood as $J_{n+1}$.

We take as $J_1$ the family $\{J_i : i \geq 1\}$, noting that it indeed satisfies conditions (j1)-(j3). We then take as the partition $Q_1$ of $X_1$ all the sets of $J_1$ and all singletons in the complement of their union. Note that, since the diameters of the sets in the family $J_1$ obviously converge to 0, the partition $Q_1$ is upper semi-continuous (see Proposition 3 on page 14 in [Dav]).

Suppose know, that for some $k \geq 1$, and for each $i \leq k$, we have already chosen the family $J_i$ of subsets $J \subset X_i$, such that the following conditions are satisfied:

(jj0) each $J \in J_i$ is either a closed normal neighbourhood of a vertex of $X_i$, or a segment contained in the interior of an edge of $X_i$, and its boundary $\partial J$ consists of stable points of $X_i$,

(jj1) the sets of the family $J_i$ are pairwise disjoint,
We describe a family $J_k$ of subsets of $X_{k+1}$ which also satisfies conditions (jj0)-(jj4), with $k + 1$ substituted for $i$. It will be a (disjoint) union of two subfamilies $J'_k$ and $J''_k$ described as follows. As $J'_k$ take the family of preimages $\pi_{k}^{-1}(J)$ for all $J \notin J_{k}$, $J \neq J(\Lambda_k)$. Note that, since the map $\pi_k : X_{k+1} \to X_k$ is injective on the preimage $\pi_{k}^{-1}(X_k \setminus \Lambda_k)$ (because $X_k \setminus \Lambda_k$ is viewed as part of $X_{k+1}$ and $\pi_k$ is viewed as the identity map on this part), this map is also injective on any preimage $\pi_{k}^{-1}(J)$ for any $J \in J_k \setminus \{J(\Lambda_k)\}$. Thus, we can identify each set $\pi_{k}^{-1}(J)$ as above with the corresponding $J$. From this point of view, $J'_k$ can be identified with the family $J_k \setminus \{J(\Lambda_k)\}$. To describe $J''_k$, we perform a recursive construction similar to that in the description of the family $J_1$. More precisely, the construction consists of the following steps:

- let $i_0 \geq k + 1$ be the smallest integer such that $\Lambda_{i_0}$ is contained in $X_{k+1} \setminus (X_k \setminus J(\Lambda_k))$; note that then $\Lambda_{i_0}$ is a direct blow-up locus in $X_{k+1}$; put $J''_1$ to be a closed normal neighbourhood of $\Lambda_{i_0}$ in $X_{k+1}$, with $\partial J''_1$ consisting of stable points of $X_{k+1}$, and so close to $\Lambda_{i_0}$ that
  (1) $J''_1 \subset X_{k+1} \setminus (X_k \setminus J(\Lambda_k))$, and
  (2) for any $j < k + 1$ we have
      \[ \text{diam}(\pi_{k+1,j}(J''_1)) < \max[2 \cdot \text{diam}(\pi_{k+1,j}(\Lambda_{i_0})), 1/k + 1] \]

- having already defined the neighbourhoods $J''_1, \ldots, J''_n$, let $i_n$ be the smallest integer such that $\Lambda_{i_n}$ is a direct blow-up locus of $X_k + 1$ contained in $X_{k+1} \setminus (X_k \setminus J(\Lambda_k))$, but not contained in $J''_1 \cup \ldots \cup J''_n$; then $\Lambda_{i_n}$ is actually disjoint with $J''_1 \cup \ldots \cup J''_n$; put $J''_{n+1}$ to be a closed normal neighbourhood of $\Lambda_{i_n}$ in $X_{k+1}$, with $\partial J''_{n+1}$ consisting of stable points of $X_{k+1}$, and so close to $\Lambda_{i_n}$ that
  (1) $J''_{n+1} \subset X_{k+1} \setminus (X_k \setminus J(\Lambda_k))$ and $J''_{n+1} \cap (J''_1 \cup \ldots \cup J''_n) = \emptyset$, and
  (2) for any $j < k + 1$ we have
      \[ \text{diam}(\pi_{k+1,j}(J''_{n+1})) < \max[2 \cdot \text{diam}(\pi_{k+1,j}(\Lambda_{i_n})), 1/k + 1] \]

We take as $J''_i$ the family $\{J''_i : i \geq 1\}$. We skip further details.

Now, for each $i \geq 2$, we define the partition $Q_i$ of the space $X_i$ as consisting of all sets from the family $J_i$, and all singletons from the complement of the union of these
sets. It follows fairly directly from conditions (jj0) and (jj1) that diameters of the sets $J \in \mathcal{J}_i$ converge to 0 (i.e. the family $\mathcal{J}_i$ is null), and hence the partition $\mathcal{Q}_i$ is upper semi-continuous.

It follows directly from condition (jj3) that the family $\mathcal{Q}_i : i \geq 1$ of partitions satisfies the compatibility condition (p1) of Lemma 5.6. We will show that it satisfies also condition (p2) of this lemma.

To verify the fineness condition (p2), let $(Q_i)_{i \geq k}$ be any sequence as in this condition. It follows from the incusions $\pi_i(Q_{i+1}) \subset Q_i$ that the sequence of diameters $(\text{diam}(\pi_i(k(Q_i))))_{i \geq k}$ is nonincreasing. Note that, if some of the sets $Q_i \in \mathcal{Q}_i$ is a singleton, the condition (p2) is satisfied trivially. So we assume that for each $i \geq k$ we have $Q_i \in \mathcal{J}_i$. Under this assumption, we have the following claim.

**Claim.** For infinitely many $i \geq k$ we have $Q_i = J(\Lambda_i)$.

To prove the claim, we note first that, as a consequence of observation (d6), for each $i \geq k$ the neighbourhood $Q_i$ contains at least one direct blow-up locus of $X_i$. Let $n_i$ be the smallest among the numbers $n \geq i$ such that $\Lambda_n$ is a direct blow-up locus of $X_i$ contained in $Q_i$. Then the points of $Q_i$ are successively unaffected by the maps $\pi_i, \pi_{i+1}, \ldots, \pi_{n_i-1}$, and hence we view $Q_i$ also as a subset of $X_{n_i}$. Moreover, by the construction of the families $\mathcal{J}_n$, we have $Q_i \in \mathcal{J}_{n_i}$ (actually, $Q_i \in \mathcal{J}'_{n_i}$), and hence $J(\Lambda_{n_i}) = Q_i$. It is also not hard to observe that $Q_i$, as a subset of $X_{n_i}$, coincides with the preimage $\pi_{n_i,i}^{-1}(Q_i)$. It follows that for any $J \in \mathcal{J}_{n_i}, J \neq Q_i$, the preimage $\pi_{n_i,i}^{-1}(J)$ is disjoint with $Q_i \subset X_i$. Since we have $\pi_{n_i,i}(Q_{n_i}) \subset Q_i$, we get that $Q_{n_i} = Q_i$, and consequently $Q_{n_i} = J(\Lambda_{n_i})$. Thus, for each $i \geq k$ there is $n \geq i$ such that $Q_n = J(\Lambda_n)$. This obviously implies the claim.

Denote by $(i_m)_{m \geq 1}$ the infinite increasing sequence consisting of all $i \geq k + 1$ such that $Q_{i-1} = \Lambda_{i-1}$. Obviously, for those $i$ we also have $\pi_{i-1}(Q_i) \subset J(\Lambda_{i-1})$. By condition (jj4), for each $m$ we have

\[(*) \quad \text{diam}(\pi_{i,m}(Q_{i,m})) < \max[2 \cdot \text{diam}(\pi_{i,m}(\Lambda_{i,m})), 1/i_m].\]

Since, by condition (i3) in Definition 5.4, we have $\lim_{m \to \infty} \text{diam}(\pi_{i,m}(\Lambda_{i,m})) = 0$, we deduce from the estimate (*), and from the fact that the full sequence $(\text{diam}(\pi_{i,k}(Q_i)))_{i \geq k}$ is nonincreasing, that $\lim_{i \to \infty} \text{diam}(\pi_{i,k}(Q_i)) = 0$ as well. Thus, the partitions $\mathcal{Q}_i$ satisfy the fineness condition (p2).

Denote by $\pi_i^* : X_{i+1}/\mathcal{Q}_{i+1} \to X_i/\mathcal{Q}_i$ the maps naturally induced by the maps $\pi_i$ on the quotients (which are well defined due to compatibility condition (p1)). Put $\mathcal{R}^*$ to be the good quotient of $\mathcal{R}$ given by

$$\mathcal{R}^* := \{X_i/\mathcal{Q}_i\}_{i \geq 1}, \{\pi_i^*\}_{i \geq 1}.$$
Step 3.
For the remaining part of the proof of Proposition 5.5, fix some countable dense subset $D \subset X$ containing all essential vertices of $X$. Recall that $V$ and $E$ denote the sets of all essential vertices and essential edges of $X$, respectively.

In this step of the proof we investigate the quotient spaces $X_1/Q_1$ appearing in the sequence $R^*$ constructed in the previous step. We start with the following observation concerning $X_1/Q_1$. In its statement, we denote by $\rho : X_1 \to X_1/Q_1$ the quotient map.

5.11 Lemma. There is a homeomorphism $h_1 : X_1/Q_1 \to X$ satisfying the following conditions:

(1) $h_1$ respects labels of vertices and edges, in the following sense: if $v_1$ is a vertex of $X_1$ labelled with $v \in V$ then $h_1(\rho(v_1)) = v$; similarly, if $x$ is a point contained in an edge $e_1$ of $X_1$ labelled with $e \in E$ then $h_1(\rho(x))$ is contained in the edge $e$ of $X$;

(2) if we denote by $D_1 \subset X_1/Q_1$ the subset consisting of all points corresponding to the shrunk subsets $J \in J_1$, then $h_1(D_1) = D$.

Proof: We view existence of a homeomorphism $h_1$ satisfying (1) as rather obvious. It easily follows from the fact that, after taking the quotient of a segment given by shrinking to points its subsegments from any pairwise disjoint family, we get a space homeomorphic to a segment. Additional property (2) can be achieved by the following observations. First, note that the union of the family $J_1$ is dense in $X_1$. This is a consequence of observation (d6) and condition (j2). It follows that $D_1$ is a countable dense subset of $X_1/Q_1$. Moreover, by condition (i2) in Definition 5.4, $D_1$ contains all vertices of $X_1/Q_1$. The fact that $h_1$ can be chosen to satisfy additionally condition (2) follows then from Lemma Δ.1.

Recall that we denote by $\Lambda_n$ the blow-up locus of the bonding map $\pi_n : X_{n+1} \to X_n$ from the inverse sequence $R$. It follows from the description of $X$-blow-up maps in Definitions 5.2 and 5.3 that we can view $X_n \setminus \Lambda_n$ as a subset of $X_{n+1}$. Under this perspective, we also view $X_n \setminus \text{int}(J(\Lambda_n))$ as a subset of $X_{n+1}$. Furthermore, it follows from our previous considerations that any set $J \in J_{n+1}$ is contained either in $X_n \setminus J(\Lambda_n)$ (viewed as a subset of $X_{n+1}$) or in $X_{n+1} \setminus ([X_n \setminus \text{int}(J(\Lambda_n))].$ Actually, the sets $J \subset X_n \setminus J(\Lambda_n)$ form a subfamily in $J_{n+1}$ which coincides with the family $J'_{n+1}$ in our earlier recursive description of the family $J_{n+1}$ in Step 2, while the remaining subsets $J$ constitute the subfamily $J''_{n+1}$. In particular, there is a well defined quotient space $(X_{n+1} \setminus (X_n \setminus J(\Lambda_n)))/J''_{n+1}$, and we denote by

$$\rho_{n+1} : X_{n+1} \setminus (X_n \setminus J(\Lambda_n)) \to (X_{n+1} \setminus (X_n \setminus J(\Lambda_n)))/J''_{n+1}$$

the quotient map. Under the just recalled notations and conventions, and by the arguments similar as in the proof of Lemma 5.11, we get the following result, the proof of which we skip.

5.12 Lemma.

(0) For each $n \geq 1$ the quotient space $X_n/Q_n$ is homeomorphic to $X_n$, and it naturally inherits from $X_n$ the structure of an $X$-graph.

(1) For each $n \geq 1$, view $X_n \setminus \text{int}(J(\Lambda_n))$ as a subspace of $X_{n+1}$, and the family $J'_{n+1}$ as the partition of this subspace (which consists of the sets from the family $J''_{n+1}$ and of the singletons). Denote also by $p$ the point in the quotient space $X_n/Q_n$ corresponding
Suppose that \( \Lambda \) is semi-continuous, and there is a homeomorphism \( \Lambda \) satisfying the following conditions:

(2) Suppose that \( \Lambda \) is a blow-up segment. Denote by \( e' \) this edge of \( X \) which contains \( \Lambda \), and by \( e \in E \) its label (in the \( X \)-graph structure of \( X \)). Let \( p \in D \subset X \) be any point from the interior of the edge \( e \) of \( X \), and let \( X^\#(p) \) be the blow-up of \( X \) at \( p \), as described in Section 2. Then \( \mathcal{J}'_{n+1} \), viewed as a partition of the subspace \( X_{n+1} \setminus (X_n \setminus J(\Lambda_n)) \) consisting of the subsets from \( \mathcal{J}'_{n+1} \) and of singletons, is upper semi-continuous, and there is a homeomorphism

\[
h_{n+1} : (X_{n+1} \setminus (X_n \setminus J(\Lambda_n))) / \mathcal{J}'_{n+1} \rightarrow X^\#(p)
\]

satisfying the following conditions:

(a) \( h_{n+1} \) respects the labels of vertices and edges in the following sense: if \( v' \) is a vertex of \( X_{n+1} \) labelled with \( v \in V \), and contained in \( X_{n+1} \setminus (X_n \setminus J(\Lambda_n)) \), then \( h_{n+1} \rho_{n+1}(v') = v \); similarly, if \( x \) is a point contained in an edge \( \varepsilon' \) of \( X_{n+1} \) labelled with \( \varepsilon \in E \), and such that \( x \in X_{n+1} \setminus (X_n \setminus J(\Lambda_n)) \), then \( h_{n+1} \rho_{n+1}(x) \) is contained in the edge \( \varepsilon \) of \( X^\#(p) \) (or in the blown-up edge \( e^\#(p) \) if \( \varepsilon = e \));

(b) if we denote by \( D_{n+1} \subset (X_{n+1} \setminus (X_n \setminus J(\Lambda_n))) / \mathcal{J}'_{n+1} \) the subset consisting of all points corresponding to the shrinked subsets \( J \in \mathcal{J}'_{n+1} \), then \( h_{n+1}(D_{n+1}) = D \setminus \{p\} \), where \( D \setminus \{p\} \) is naturally understood as a subset in \( X^\#(p) \).

(3) Suppose \( \Lambda \) is a blow-up point, i.e. \( \Lambda = \{v'\} \) for some vertex \( v' \) of \( X \). Let \( v \in V \) be the label of \( v' \) (in the \( X \)-graph structure of \( X \)), and let \( X^\#(v) \) be the blow-up of \( X \) at \( v \), as described in Section 2. Then \( \mathcal{J}_{n+1}'' \), viewed as a partition of the subspace \( X_{n+1} \setminus (X_n \setminus J(\Lambda_n)) \) consisting of the subsets from \( \mathcal{J}_{n+1}'' \) and of singletons, is upper semi-continuous, and there is a homeomorphism

\[
h_{n+1} : (X_{n+1} \setminus (X_n \setminus J(\Lambda_n))) / \mathcal{J}_{n+1}'' \rightarrow X^\#(v)
\]

satisfying the following conditions:

(a) \( h_{n+1} \) respects the labels of vertices and edges in the following sense: if \( w' \) is a vertex of \( X_{n+1} \) labelled with \( w \in V \), and contained in \( X_{n+1} \setminus (X_n \setminus J(\Lambda_n)) \), then \( h_{n+1} \rho_{n+1}(w') = w \); similarly, if \( x \) is a point contained in an edge \( \varepsilon' \) of \( X_{n+1} \) labelled with \( \varepsilon \in E \), and such that \( x \in X_{n+1} \setminus (X_n \setminus J(\Lambda_n)) \), then \( h_{n+1} \rho_{n+1}(x) \) is contained in the edge \( \varepsilon \) of \( X^\#(v) \) (under the obvious interpretation of edges of \( X \) as edges of \( X^\#(v) \));

(b) if we denote by \( D_{n+1} \subset (X_{n+1} \setminus (X_n \setminus J(\Lambda_n))) / \mathcal{J}_{n+1}'' \) the subset consisting of all points corresponding to the shrinked subsets \( J \in \mathcal{J}_{n+1}'' \), then \( h_{n+1}(D_{n+1}) = D \setminus \{v\} \), where \( D \setminus \{v\} \) is naturally understood as a subset in \( X^\#(v) \).
Step 4.

In this last step of the proof of Proposition 5.5 we give a description of a violated reflection inverse system $\mathcal{S}_X^v$, its cofinal subsequence $\mathcal{P}$, and an isomorphism $\phi : \mathcal{R}^* \to \mathcal{P}$ of inverse sequences. A subsequence $\mathcal{P}$ of $\mathcal{S}_X^v$ will have the following form (under notation as in Section 4): for some increasing sequence $H_n : n \geq 1$ of finite subtrees of $T$ such that each $H_n$ has exactly $n$ vertices, and such that $\bigcup_{n \geq 1} H_n = T$, it holds that $\mathcal{P} = (\{Z_{H_n}\}_{n \geq 1}, \{\pi_{H_{n+1}}^{H_n}\}_{n \geq 1})$. An isomorphism $\phi : \mathcal{R}^* \to \mathcal{P}$ will be described as a sequence $\phi_n : X_n/Q_n \to Z_{H_n}$ of homeomorphisms which commute with the bonding maps of the sequences, i.e. satisfy the equations $\phi_n \pi_n^* = \pi_{H_n+1}^* \phi_{n+1}$.

We pass to the description of the tree $T$ and the sequence $H_n$ of its subtrees. For each $n \geq 1$, denote by $k_n$ the smallest number $k$ such that the blow-up locus $\Lambda_n$ of the map $\pi_n$ is a direct blow-up locus in $X_k$. Obviously, we have $1 \leq k_n \leq n$. For the vertex set $V_T$ of $T$ take a countable infinite set $t_1, t_2, \ldots$ ordered into a sequence. For the edge set $E_T$ take the sequence $e_n : n \geq 1$, where each $e_n$ connects $t_{n+1}$ with $t_n$. For each $n \geq 1$, take as $H_n$ the subtree of $T$ spanned on the vertex set $\{t_1, \ldots, t_n\}$; note that then $\{t_1, \ldots, t_n\}$ is the vertex set of $H_n$, and $\{e_1, \ldots, e_{n-1}\}$ is its edge set.

Recall that at the beginning of Step 3 of this proof we have fixed a countable dense subset $D \subset X$ containing all vertices of $X$. Recall also that, in order to describe a violated inverse system $\mathcal{S}_X^v$, we need:

(a) to associate to any edge $e_n$ of the above described tree $T$ a label $\lambda(e_n)$ from the set $D$, so that the label function $\lambda : E_T \to D$ satisfies the requirements described in Section 4, and

(b) for each edge $e_n$ whose label $\lambda(e_n)$ is not a vertex of $X$, to decide whether the corresponding gluing map $\beta_{e_n} : P_{\lambda(e_n)} \to P_{\lambda(e_n)}$ (and $\beta_{e_n}$ which is the inverse of $\beta_{e_n}$), where $|e| = e_n$, is the identity or the transposition.

We will describe the data (a) and (b) as above recursively, together with the description of the homeomorphisms $\phi_n$ constituting an isomorphism $\phi$.

Start with a choice of any homeomorphism $h_1$ as in Lemma 5.11. Recall that $J(\Lambda_1)$ is this set from the family $\mathcal{J}_1$ which contains the blow-up locus $\Lambda_1$ of the map $\pi_1$. Label the edge $e_1$ of $T$ with the element $h_1([J(\Lambda_1)]) \in D_1$, where $[J(\Lambda_1)]$ is the point in $X_1/Q_1$ corresponding to the shrunked set $J(\Lambda_1)$. Since, by the description of Section 4, the space $Z_{H_1}$ is canonically equal to $X$, we can take as $\phi_1 : X_1/Q_1 \to Z_{H_1}$, simply the homeomorphism $h_1$.

Now, assume that for some $n$ we have already associated the labels $\lambda(e_i)$ to the edges $e_1, \ldots, e_{n-1}$, and that we have already described the gluing maps $\beta_{e_i}$ related to the same edges $e_1, \ldots, e_{n-1}$. This portion of data determines uniquely the spaces $Z_{H_i}$ for all $i \leq n$, and the maps $\pi_{H_{i+1}}^{H_{i+1}}$ for all $i \leq n - 1$. Assume also that we have already defined the homeomorphisms $\phi_i : X_i/Q_i \to Z_{H_i}$ for all $i \leq n$, and that they satisfy the appropriate earlier mentioned commutativity equations. Finally, suppose that for some appropriately chosen homeomorphisms $h_1$ as in Lemma 5.11 and $h_i : 2 \leq i \leq n$ as in parts (2) or (3) of Lemma 5.12, the homeomorphisms $\phi_i$ and the labels $\lambda(e_i)$ above satisfy the following:

(f1) under the natural identification of $Z_{H_i}$ with $X$, we have $\phi_i = h_i$;

(f2) for all $1 \leq i \leq n - 1$, we have $\lambda(e_i) = h_k_i([J(\Lambda_i)])$ and $h_{i+1}(D_{i+1}) = D \setminus \{\lambda(e_i)\}$;
(f3) if \(2 \leq i \leq n\), then under the natural identification of \(X_{t_i}\) with \(X\), consider the subset \(X_{t_i}^*(\lambda(e_{i-1})) \subset Z_{H_i}\); consider also the subset \((X_i \setminus (X_{i-1} \setminus J(\Lambda_{i-1}))) / J''_i \subset X_i / Q_i\); then the image of the restriction of \(\phi_i\) to \((X_i \setminus (X_{i-1} \setminus J(\Lambda_{i-1}))) / J''_i\) coincides with \(X_{t_i}^*(\lambda(e_{i-1}))\), and the restriction itself coincides with \(h_i\), i.e.

\[\phi_i|_{(X_i \setminus (X_{i-1} \setminus J(\Lambda_{i-1}))) / J''_i} = h_i.\]

Note that all the above assumptions are indeed satisfied in the already described case of \(n = 1\), mostly trivially. To accomplish an inductive step of our construction, we need to determine the label \(\lambda(e_n)\), the form of the gluing map \(\beta_\varepsilon\) for \(|\varepsilon| = e_n\), and the shape of the homeomorphism \(\phi_{n+1}\).

To determine the label \(\lambda(e_n)\), observe that, by definition of the number \(k_n\), the set \(\Lambda_n\) is a direct blow-up locus in \(X_{k_n}\). Consequently, the set \(J(\Lambda_n)\) is also a subset of \(X_{k_n}\). Moreover, by minimality of \(k_n\) in the definition of \(k_n\), we have that \(J(\Lambda_n) \subset J''_n\) (where we use the convention that \(J''_1 := J_1\)). We also have \(J(\Lambda_n) \subset X_{k_n} \setminus (X_{k_n-1} \setminus J(\Lambda_{k_n-1}))\) if \(k_n \geq 2\). We put \(\lambda(e_n) = h_{k_n}(J(\Lambda_n)))\), where \([J(\Lambda_n)]\) denotes the point corresponding to the shrinked subset \(J(\Lambda_n)\) in \(X_1 / Q_1\) if \(k_n = 1\), and in \((X_{k_n} \setminus (X_{k_n-1} \setminus J(\Lambda_{k_n-1}))) / J''_{k_n}\) if \(k_n \geq 2\).

Note that if \(\Lambda_n\) is not a segment then \(\lambda(e_n)\) is not a vertex of \(X\). In this case we need to determine the form of the gluing map \(\beta_\varepsilon : P(\lambda(e_n)) \rightarrow P(\lambda(e_n))\) for \(|\varepsilon| = e_n\). Denote by \(e'\) the edge of \(X_{k_n}\) which contains \(\Lambda_n\), and let \(e \in E\) be the label of \(e'\) in the \(X\)-graph structure of \(X_{k_n}\). Denote by \(u, v \in V\) the endpoints of \(e\) (which are the vertices of \(X\)), and note that the endpoints of \(e'\) (which are the vertices of \(X_{k_n}\)) are labeled with \(u\) and \(v\). Denote by \(u', v'\) the endpoints of \(e'\) labeled with \(u, v\), respectively. Denote also by \(a_u\) this endpoint of \(\Lambda_n\) which is closer to \(u'\) in \(e'\) than the other endpoint. Referring to the description of \(X_{n+1}\) as obtained from \(X \setminus \text{int}(\Lambda_n)\) and \(X \setminus \text{int}(\varepsilon)\) by gluing the endpoints of \(\Lambda_n\) to the endpoints of \(e\) (as in Definition 5.3), for \(|\varepsilon| = e_n\), put \(\beta_\varepsilon\) to be

- the identity of \(P(\lambda(e_n))\), if \(a_u\) is identified in \(X_{n+1}\) with \(u\);
- the transposition of \(P(\lambda(e_n))\), if \(a_u\) is identified with \(v\).

To describe a homeomorphism \(\phi_{n+1} : X_{n+1} / Q_{n+1} \rightarrow Z_{H_{n+1}}\) as required, choose any homeomorphism \(h_{n+1} : (X_{n+1} \setminus (X_n \setminus J(\Lambda_n))) / J''_{n+1} \rightarrow X^*(\lambda(e_n))\) as in part (2) or (3) of Lemma 5.12, where \(\lambda(e_n)\) plays the role of \(p\) or \(v\). Viewing \((X_{n+1} \setminus J(\Lambda_n)) / J''_n\) naturally as a subspace in both \(X_{n+1} / Q_{n+1}\) and \(X_n / Q_n\), and under the identification of \(X\) with \(X_{t_{n+1}}\) (and hence also of \(X^*(\lambda(e_n))\) with \(X_{t_{n+1}}^*(\lambda(e_n))\)), we put

- \(\phi_{n+1}(x) = \phi_n(x)\) if \(x \in (X_n \setminus J(\Lambda_n)) / J''_n\), and
- \(\phi_{n+1}(x) = h_{n+1}(x)\) if \(x \in (X_{n+1} \setminus (X_n \setminus J(\Lambda_n))) / J''_n\).

We skip the straightforward verification that

- \(\phi_{n+1} : X_{n+1} / Q_{n+1} \rightarrow Z_{H_{n+1}}\) as above is a well defined homeomorphism, and that it satisfies the commutativity equation \(\phi_n \pi^*_{n} = \pi^*_n \phi_{n+1}\), and
- the data \(\phi_i : 1 \leq i \leq n + 1, \lambda(e_i) : 1 \leq i \leq n,\) and \(\beta_\varepsilon\) for \(|\varepsilon| = e_j\) and \(1 \leq i \leq n\), satisfy conditions (f1)-(f3), with \(n + 1\) substitutes for \(n\).

We also skip the straightforward verification of other details allowing to conclude that, by iterating a step of the inductive construction as above, we obtain a well defined violated reflection system \(S^*_X\), its cofinal subsequence \(\mathcal{P}\), and an isomorphism of inverse sequences \(\phi : \mathcal{R}^* \rightarrow \mathcal{P}\).
6. The Coxeter-Davis complexes of systems \((W_\Gamma, S_\Gamma)\).

In this section we describe the Coxeter-Davis complexes \(\Sigma_\Gamma\) associated to the right-angled Coxeter systems \((W_\Gamma, S_\Gamma)\) whose nerves are simplicial graphs \(\Gamma\). We then focus on a class of graphs \(\Gamma\) satisfying conditions \((\ast 1)\) and \((\ast 2)\) below, and for this class we introduce some useful stratification of complexes \(\Sigma_\Gamma\) into strata called sectors and branch components. This stratification will be essentially used in Section P, to analyze the geodesic inverse system in the complex \(\Sigma_\Gamma\) for such graphs \(\Gamma\) (the limit of this system is, by definition, the visual boundary \(\partial_\infty(W_\Gamma, S_\Gamma)\) of the corresponding Coxeter system).

The Coxeter-Davis complex \(\Sigma_\Gamma\).

For this part of exposition, let \(\Gamma\) be any finite simplicial graph which is flag as simplicial complex (i.e. does not contain cycles of length 3). Denote by \(V_\Gamma, E_\Gamma\) the sets of all vertices and edges of \(\Gamma\), respectively. Let \((W_\Gamma, S_\Gamma)\) be the right angled Coxeter system with nerve \(\Gamma\). This means that \(W_\Gamma\) is a group and \(S_\Gamma\) is a generating set for \(W_\Gamma\) such that
- the generators in \(S_\Gamma\) are identified with the vertices of \(\Gamma\);
- \(W_\Gamma\) has the presentation \(\langle S_\Gamma \mid \{s^2 : s \in S_\Gamma\} \cup \{(ss')^2 : [s,s']\text{ is an edge of } \Gamma\}\rangle\).

The next fact provides description and basic properties of an object called the Coxeter-Davis complex for the system \((W_\Gamma, S_\Gamma)\). We refer the reader to [D] for more detailed introduction to Coxeter systems and their Coxeter-Davis complexes. It is worth keeping in mind that for right angled Coxeter systems \((W_\Gamma, S_\Gamma)\) under our consideration the Coxeter-Davis complex of any such system coincides with the Cayley complex of the group \(W_\Gamma\), for the canonical presentation mentioned above. (This is explained with more details in remark right after Fact 6.1.)

Recall that a square complex is a 2-dimensional cubical complex.

6.1 Definition/Fact. Suppose that \(\Gamma\) is a simplicial graph which is flag, and let \(\Sigma_\Gamma\) denote the Coxeter-Davis complex of the system \((W_\Gamma, S_\Gamma)\).

(1) \(\Sigma_\Gamma\) is the unique labelled square complex satisfying the following properties:
   - \(\Sigma_\Gamma\) is simply connected.

(2) \(\Sigma_\Gamma\) is a CAT(0) cubical complex. More precisely, by this we mean that \(\Sigma_\Gamma\) comes equipped with the so called standard piecewise euclidean metric (for which each cubical cell is isometric with the unit euclidean cube of the corresponding dimension). With this metric, \(\Sigma_\Gamma\) is a geodesic metric space satisfying the CAT(0) condition. We refer the reader to [BH] for an exhausting exposition of the CAT(0) concept, and below in this section we recall few of its aspects that will be used in this paper.

(3) The group \(W_\Gamma\) acts on \(\Sigma_\Gamma\) by combinatorial automorphisms preserving the labels. The action is proper discontinuous and cocompact. In fact, \(W_\Gamma\) coincides with the group of all
label preserving combinatorial automorphisms of $\Sigma_\Gamma$, and its action is simply transitive on the vertices.

**Remark.** The above description of $\Sigma_\Gamma$, as a labelled square complex, can be given alternatively as follows. Let $C_\Gamma$ be the Cayley graph of the group $W_\Gamma$ with respect to the generating set $S_\Gamma$. By the convention which we use in this paper, $C_\Gamma$ has the vertex set which coincides with $W_\Gamma$, and it has one unoriented edge for any pair $\{g, gs\}$ of elements that differ by a multiplication on the right by a generator. We label such an edge with the corresponding generator $s$. For any edge $[s, s'] \in E_\Gamma$, consider the special subgroup $W_{\{s, s'\}} < W_\Gamma$, and note that it is canonically isomorphic to the dihedral group of order 4. To each left coset $gW_{\{s, s'\}}$ in $W_\Gamma$ there is associated a subgraph in $C_\Gamma$ which is a cycle of length 4. For each such coset, we attach to $C_\Gamma$ a square 2-cell along the corresponding 4-cycle. We label this 2-cell with the corresponding edge $[s, s']$. Doing this for all edges of $\Gamma$, we obtain a square 2-complex that we denote by $C^2_{\Gamma}$ and call the Cayley complex of $(W_\Gamma, S_\Gamma)$. It is a well known fact that (under our assumptions on $\Gamma$), as a labelled complex, $C^2_{\Gamma}$ coincides with the Coxeter-Davis complex $\Sigma_\Gamma$, as described in above Fact 6.1.

Recall that the visual boundary of $(W_\Gamma, S_\Gamma)$ is, by definition, the visual boundary of the Coxeter-Davis complex $\Sigma_\Gamma$. We denote it $\partial_\infty(W_\Gamma, S_\Gamma)$. If $W_\Gamma$ happens to be word-hyperbolic (as it is the case for all $\Gamma$ satisfying conditions $(\ast 1)$ and $(\ast 2)$ below), $\partial_\infty(W_\Gamma, S_\Gamma)$ coincides with the Gromov boundary of $W_\Gamma$ (see Proposition III.3.7(2) in [BH]).

**The stratification of $\Sigma_\Gamma$: sectors and branch components.**

Recall that for a simplicial graph $\Gamma$ we denote by $|\Gamma|$ the underlying topological graph. In the remaining part of the section we restrict our attention to the class of graphs $\Gamma$ that are connected, not reduced to a single vertex, and satisfy the following properties:

$(\ast 1)$ each essential edge of $|\Gamma|$ contains no essential loop edges;

$(\ast 2)$ each essential edge of $|\Gamma|$ consists of at least 3 edges of the corresponding simplicial graph $\Gamma$.

**6.2 Definition.** Given an essential edge $\varepsilon$ of $|\Gamma|$ (which, by our assumptions $(\ast 1)$ and $(\ast 2)$, is not a loop and consists of at least 3 edges of $\Gamma$), a sector of $\Sigma_\Gamma$ labelled with $\varepsilon$ (shortly, an $\varepsilon$-sector) is any maximal connected subcomplex of $\Sigma_\Gamma$ which is the union of squares labelled with edges $e \in E_\Gamma$ contained in $\varepsilon$.

**Remark.** Alternatively, $\varepsilon$-sectors can be described as follows. Consider the special subgroup $W_\varepsilon < W_\Gamma$ with its standard generating set $S_\varepsilon = S_\Gamma \cap \varepsilon$. Viewing $\Sigma_\Gamma$ as the Cayley complex of $(W_\Gamma, S_\Gamma)$, we have a natural identification of the Cayley complex $C^2_{\varepsilon}$ of $(W_\varepsilon, S_\varepsilon)$ with a subcomplex in $\Sigma_\Gamma$. The $\varepsilon$-sectors are then the translates in $\Sigma_\Gamma$, under the action of $W_\Gamma$, of the subcomplex $C^2_{\varepsilon}$.

The next fact describes the combinatorial types of sectors of $\Sigma_\Gamma$, as square complexes. We omit its straightforward proof.

**6.3 Fact.** Let $k \geq 3$ be the number of edges of $E_\Gamma$ contained in an essential (non-loop) edge $\varepsilon$ of $|\Gamma|$. Then each $\varepsilon$-sector of $\Sigma_\Gamma$ is isomorphic to the square complex $\Omega_k$ uniquely determined by the following properties:

$(k1)$ each vertex link of $\Omega_k$ is a polygonal arc consisting of $k$ edges;
(k2) $\Omega_k$ is simply connected;
(k3) for each square $Q$ in $\Omega_k$ each of the two reflections in $Q$ through a midsegment of $Q$ parallel to a pair of its opposite sides extends to a combinatorial automorphism of the whole $\Omega_k$.

It is a consequence of conditions (k1) and (k2) that $\Omega_k$ is topologically a noncompact planar surface with infinitely many boundary components which are all noncompact.

**Remark.** It is also not hard to realize that each $\varepsilon$-sector $\Omega$ in $\Sigma\Gamma$ is combinatorially isomorphic, as labelled square complex with labels inherited from $\Sigma\Gamma$, to the Coxeter-Davis complex $\Sigma_\varepsilon$ of the right angled Coxeter system $(W_\varepsilon,S_\varepsilon)$ corresponding to the parabolic subgroup of $W_\Gamma$ spanned by the standard generators corresponding to the vertices in $\varepsilon$. The boundary $\partial\Omega$ is the subcomplex of $\Omega$ consisting of all edges which are labelled with the two essential vertices of $|\Gamma|$ contained in $\varepsilon$ (i.e. the endpoints of $\varepsilon$). Each component $L$ of $\partial\Omega$ is an infinite polygonal line, and the edges in $L$ are labelled alternately with the two endpoint vertices of $\varepsilon$.

Some useful properties of sectors, as subcomplexes of $\Sigma\Gamma$, are gathered in the next fact. Recall that a subset $A$ of a geodesic metric space $X$ is convex if for any two points of $A$, any geodesic in $X$ connecting these points is contained in $A$. If $A$ is a subcomplex of a CAT(0) cubical complex $X$, then convexity of $A$ has the following characterization: $A$ is convex iff it is connected and for any vertex $v$ of $A$ the link of $A$ at $v$ is a full subcomplex in the link of $X$ at $v$. In the statement below we use the term strict convexity for the following property, which is clearly stronger than convexity: a subcomplex $A$ of a CAT(0) square complex $X$ is strictly convex if it is connected and for any vertex $v$ of $A$ the link of $A$ at $v$ is a full subcomplex of $X$ at $v$. The latter means that $A_v$ is a full subgraph of $\Sigma_v$, and that any polygonal path of $\Sigma_v$ intersecting $A_v$ only at its endpoints has length at least 3.

**6.4 Fact.**

1. Each sector $\Omega$, as well as its any boundary component $L$, is a strictly convex subcomplex of $\Sigma\Gamma$.
2. Each connected component of the complement $\Sigma\Gamma \setminus \text{int}(\Omega)$ is a strictly convex subcomplex of $\Sigma\Gamma$.
3. The inclusion provides a bijective correspondence between the boundary components of $\Omega$ and the connected components of $\Sigma\Gamma \setminus \text{int}(\Omega)$.

**Proof:** Suppose that $\Omega$ is an $\varepsilon$-sector.

To prove part (1), note that for any vertex $v$ of $\Omega$ the pair of links $((\Sigma\Gamma)_v, \Omega_v)$ is isomorphic to the pair $(\Gamma, \varepsilon)$. Strict convexity of $\Omega$ follows then from condition $(\ast 2)$. Similarly, if $L$ is a boundary component of $\Omega$, then for any vertex $v$ of $L$ the pair $((\Sigma\Gamma)_v, L_v)$ is isomorphic to the pair $(\Gamma, \{a, b\})$, where $a, b$ are the endpoints of $\varepsilon$. Strict convexity of $L$ follows then again from condition $(\ast 2)$.

To prove part (2), consider a connected component $Q$ of $\Sigma\Gamma \setminus \text{int}(\Omega)$, and note that it is a subcomplex of $\Sigma\Gamma$. Moreover, if $v$ is a vertex of $Q$ not contained in $\Omega$, we have $Q_v = (\Sigma\Gamma)_v$, and if $v$ is contained in $\Omega$ then the pair of links $((\Sigma\Gamma)_v, Q_v)$ is isomorphic to the pair $(\Gamma, \Gamma \setminus \text{int}(\varepsilon))$. Again, strict convexity of $Q$ follows easily from condition $(\ast 2)$.

To get part (3), we need to show two things about any connected component $Q$ of $\Sigma\Gamma \setminus \text{int}(\Omega)$: first, that $Q$ intersects $\Omega$, and second, that $Q$ does not intersect more than
one boundary component of $\Omega$. To see the first assertion above, note that if it were not true, $Q$ would be a connected component of $\Sigma\Gamma$, which contradicts connectedness of the latter. The second assertion follows immediately from convexity of $Q$ and of $\Omega$.

This completes the proof of Fact 6.4.

We pass to the discussion of objects that we call branch components of $\Sigma\Gamma$. Denote by $V$ the set of essential vertices of the underlying topological graph $|\Gamma|$.

**6.5 Definition.** A branch component of $\Sigma\Gamma$ is any maximal connected subcomplex of $\Sigma\Gamma$ which is the union of edges of $\Sigma\Gamma$ labelled with the essential vertices of $|\Gamma|$ (i.e. with the elements of $V$). A branch locus of $\Sigma\Gamma$ is the union of all branch components, and we denote it by $C\Gamma$.

**Remark.** An alternative description of branch components can be given as follows. Consider the special subgroup $W_V < W\Gamma$ spanned on those generators from $S\Gamma$ that correspond to all essential vertices of $|\Gamma|$. View the Cayley complex $C^2_V$ (which actually coincides with the Cayley graph $C_V$) naturally as the subcomplex of the Cayley complex $C^2\Gamma$. After identifying the latter with $\Sigma\Gamma$, the branch components of $\Sigma\Gamma$ are the translates under the action of $W\Gamma$ of the subcomplex $C^2_V$.

**6.6 Fact.**
1. Each branch component $C$ of $\Sigma\Gamma$ is isomorphic to the regular tree of degree equal to the cardinality of the set $V$. Moreover, for each vertex $x \in C$ the labelling restricted to the edges of $C$ issuing from $x$ is a bijection on $V$.
2. Every edge of a branch component $C$ is contained either in exactly one or in at least three squares of $\Sigma\Gamma$.
3. Each boundary component of any sector in $\Sigma\Gamma$ is contained in some branch component.
4. Each branch component $C$ is a strictly convex subcomplex of $\Sigma\Gamma$.
5. The connected components of the complement $\Sigma\Gamma \setminus C$ of any branch component $C$ are in the natural bijective correspondence (via inclusion) with (the interiors of) the sectors of $\Sigma\Gamma$ adjacent to $C$.

**Proof:** Note that for each vertex $v$ of any branch component $C$ the vertex link $C_v$, as a subcomplex in the vertex link $(\Sigma\Gamma)_v$, corresponds to the set $V$ viewed as a subcomplex of $\Gamma$. In view of the condition $(*)2$, it follows that $C$ is a strictly convex subcomplex of $\Sigma\Gamma$. In particular, it is a convex subcomplex, and therefore it must be simply connected, and hence a tree. The assertions (2) and (3) follow even more directly from the above observation concerning links.

To prove (5) note that, since $\Sigma\Gamma$ is connected, the closure of each connected component $U$ of $\Sigma\Gamma \setminus C$ intersects $C$. It follows that $U$ contains the interior of at least one sector $\Omega$ adjacent to $C$. To see that there is only one such sector $\Omega$, consider the metric completion $\overline{U}$ of $U$, which can be viewed as obtained by attaching to $U$, disjointly (!), the boundary components $L\Omega$ contained in $C$ of all sectors $\Omega$ as above. The natural map $\iota : \overline{U} \to \Sigma\Gamma$ induced by the inclusion $\iota : U \to \Sigma\Omega$ is then easily seen to be the local isometry, since all the vertex links of $\overline{U}$ embed onto full subcomplexes in the corresponding vertex links of $\Sigma\Gamma$. Suppose that we have at least two sectors $\Omega$ as above, say $\Omega_1$ and $\Omega_2$. Consider
any points \( p_i \in L^{\Omega_i} \subset \overline{U} \), for \( i = 1, 2 \), and let \( \Gamma \) be a geodesic in \( \overline{U} \) connecting \( p_1 \) with \( p_2 \). Since the subcomplexes \( L^{\Omega_i} \) of \( \overline{U} \) are disjoint, \( \gamma \) necessarily passes through \( U \). On the other hand, since \( \bar{i} \) is a local isometry, the image \( \bar{i}(\gamma) \) is a geodesic in \( \Sigma_\Gamma \), and it connects the points \( \bar{i}(p_i) \), which both belong to \( C \). Since \( C \) is convex in \( \Sigma_\Gamma \), the geodesic \( \bar{i}(\gamma) \) is contained in \( C \), which contradicts the earlier observation that \( \Gamma \) passes through \( U \). This completes the proof.

The separation properties of the sectors and the branch components in \( \Sigma_\Gamma \) are nicely represented by a dual object, the adjacency graph \( A_\Gamma \), which we now introduce.

### 6.7 Definition.

1. A sector \( \Omega \) is adjacent to a branch component \( C \) of \( \Sigma_\Gamma \) if some boundary component of \( \Omega \) is contained in \( C \).
2. An adjacency graph \( A_\Gamma \) is the bipartite graph whose vertices represent all sectors and all branch components of \( \Sigma_\Gamma \), and whose edges correspond to the adjacency relation defined in (1).

### 6.8 Fact. For any \( \Gamma \) as above, the adjacency graph \( A_\Gamma \) is a tree.

**Proof:** If \( |\Gamma| \) is a single essential edge then \( \Sigma_\Gamma \) reduces to a single sector, and in this case the assertion is obvious. Otherwise, it is still a fairly straightforward observation that the graph \( A_\Gamma \) is connected. Thus, it is sufficient to show that each vertex of \( A_\Gamma \) separates this graph. For vertices represented by sectors of \( \Sigma_\Gamma \) this follows easily from Fact 6.4(3), while for vertices represented by branch components - from Fact 6.6(5).

**Remark.** In view of Fact 6.8, we will call \( A_\Gamma \) the adjacency tree of \( \Sigma_\Gamma \). Due to Fact 6.4(3), this tree has infinite order at every vertex corresponding to a sector. It also has infinite order at every vertex corresponding to a branch component, except the case when \( |\Gamma| \) is a single essential edge.

An argument similar as in the proof of Fact 6.8 yields also the following.

### 6.9 Fact. Suppose that each of the three distinct objects \( S, S_1, S_2 \) is either the interior of some sector or a branch component of \( \Sigma_\Gamma \), and let \( s, s_1, s_2 \) be the vertices in the adjacency tree \( A_\Gamma \) corresponding to \( S, S_1, S_2 \), respectively. Then \( S \) separates \( S_1 \) from \( S_2 \) in \( \Sigma_\Gamma \) if and only if \( s \) separates \( s_1 \) from \( s_2 \) in \( A_\Gamma \).

### 7. Approximation Lemma and the proof of Main Theorem.

In this section we formulate a lemma (Lemma 7.11 below), which we call Approximation Lemma, and which, together with Proposition 5.5, immediately verifies Theorem 1.1 of the introduction for topological graphs \( X = |\Gamma| \) that are connected, have no essential loop edges and no vertices of degree 1. We also state the main result of this paper in its full generality, as Theorem 7.12 below (which slightly extends Theorem 1.1 of the introduction). We show how Theorem 7.12 follows from Approximation Lemma, in view of the results of Sections 3 and 5. The proof of Approximation Lemma is postponed until the next section (Section 8).

We start with describing the setting, and with some terminological preparations. Let \( \Gamma \) be a finite connected simplicial graph which is flag, and which satisfies the assumptions
7.1 Definition. A strict star domain in $(\Sigma_\Gamma, x_0)$ is any closed bounded subset $D$ of $\Sigma_\Gamma$ distinct from the singleton $\{x_0\}$ such that for each $x \in D \setminus \{x_0\}$ the geodesic segment $[x_0, x]$ is contained in $D$, and its part $[x_0, x) = [x_0, x] \setminus \{x\}$ is contained in the interior $\text{int}(D)$ of $D$.

Note that balls centered at $x_0$ are strict star domains in $(\Sigma_\Gamma, x_0)$. We mention without proof the following obvious fact.

7.2 Fact. Let $D$ be a strict star domain in $(\Sigma_\Gamma, x_0)$. Then for any point $y \in \Sigma_\Gamma \setminus \text{int}(D)$ the geodesic segment $[x_0, y]$ intersects the boundary $\partial D$ in precisely one point.

7.3 Definition. Given two strict star domains $D \subset D'$ in $(\Sigma_\Gamma, x_0)$ the geodesic projection $g : \partial D' \to \partial D$ is the map which to any point $x' \in \partial D'$ associates the unique point $x \in \partial D$ lying on the geodesic segment $[x_0, x']$.

The above notion of geodesic projection generalizes the one for concentric spheres. It also has the following property.

7.4 Lemma For any strict star domains $D \subset D'$ in $(\Sigma_\Gamma, x_0)$ the geodesic projection $g : \partial D' \to \partial D$ is continuous with respect to the topologies induced from $\Sigma_\Gamma$.

Proof: Fix any $x' \in \partial D'$, and any sequence $(x'_n)$ of points of $\partial D'$ converging to $x'$. Put $x = g(x')$, $x_n = g(x'_n)$, and suppose that $(x_n)$ does not converge to $x$. Since $\partial D$ is compact, there is a subsequence of $(x_n)$ which converges to some $y \in \partial D$ distinct from $x$. Since geodesics in CAT(0) spaces depend continuously on their endpoints (see Proposition 2.2 on p. 176 in [BH]), the point $y$ must belong to the geodesic $[x_0, x']$. Since $x$ also belongs to this geodesic, and since both $x$ and $y$ are contained in $\partial D$, we get a contradiction with the definition of a strict star domain, and hence $(x_n)$ converges to $x$. This finishes the proof.

7.5 Definition. An exhausting sequence in $(\Sigma_\Gamma, x_0)$ is a sequence $D_n : n \geq 1$ of strict star domains in $(\Sigma_\Gamma, x_0)$ such that
  
  (es1) for each $n \geq 1$ we have $D_n \subset D_{n+1}$;
(es2) for each \( r > 0 \) there is \( n \) such that the ball of radius \( r \) in \( \Sigma_\Gamma \) centered at \( x_0 \) is contained in \( D_n \) (equivalently, the union \( \bigcup_{n=1}^{\infty} D_n = \Sigma_\Gamma \)).

To any exhausting sequence \((D_n)\) in \((\Sigma_\Gamma, x_0)\) there is associated an inverse sequence \((\{\partial D_n\}, \{g_n\})\), where \( g_n : \partial D_{n+1} \rightarrow \partial D_n \) is the geodesic projection. We mention without proving the following easy observation.

**7.6 Fact.** For any exhausting sequence \((D_n)\) in \((\Sigma_\Gamma, x_0)\) and the associated sequence of geodesic projections \( g_n : \partial D_{n+1} \rightarrow \partial D_n \) we have

\[
\lim_{\leftarrow} (\{\partial D_n\}, \{g_n\}) = \lim_{\leftarrow} (\{S_r\}, \{g_r'\}) = \partial_\infty (W_\Gamma, S_\Gamma).
\]

**7.7 Definition.** A strict star domain \( D \) in \((\Sigma_\Gamma, x_0)\) is regular if the following conditions are satisfied

1. The boundary \( \partial D \) intersects the branch locus of \( \Sigma_\Gamma \) at finitely many points;
2. If \( x \in \partial D \) belongs to the interior of an edge \( e \) of \( \Sigma_\Gamma \) contained in the branch locus then \( \partial D \) is transversal to \( e \) at \( x \);
3. If \( x \in \partial D \) is a vertex of \( \Sigma_\Gamma \), then for some (small) open neighbourhood \( U \) of \( x \) in \( \Sigma_\Gamma \) the intersection with \( U \) of the interior of precisely one edge of the branch locus that issues from \( x \) is contained in \( \text{int}(D) \), and intersections with \( U \) of the interiors of the remaining such edges are contained in \( \Sigma_\Gamma \setminus D \).

Observe that the boundary \( \partial D \) of any regular strict star domain is naturally a finite topological graph whose vertex set coincides with the intersection of \( \partial D \) with the branch locus, and whose every essential edge coincides with a component of the intersection of \( \partial D \) with some sector of \( \Sigma_\Gamma \). Moreover, we equip \( \partial D \) (viewed as a topological graph) with the labelling of its essential vertices and edges described as follows:

- If \( u \) is a vertex of \( \partial D \) which lies in the interior of some branch edge \( e \) of \( \Sigma_\Gamma \), we associate to \( u \) the label \( v \in V \) equal to the label of \( e \);
- If \( u \) is a vertex of \( \partial D \) which is a vertex of \( \Sigma_\Gamma \), we associate to it the label \( v \) of this unique branch edge issuing from \( u \) whose interior intersects \( \text{int}(D) \) arbitrarily close to \( u \) (existence and uniqueness of such an edge follows from condition (r3) in Definition 7.7);
- Each edge \( d \) of \( \partial D \) is contained in a unique sector \( \Omega \) of \( \Sigma_\Gamma \), and we label \( d \) with this \( \varepsilon \in E \) for which \( \Omega \) is an \( \varepsilon \)-sector.

We make a record of the following easy observation.

**7.8 Lemma.** Let \( D \) be a regular strict star domain in \((\Sigma_\Gamma, x_0)\), and view \( \partial D \) as a topological graph equipped with the labelling of its essential vertices and edges, as described above. Then \( \partial D \) is an \( X \)-graph.

Given any two continuous maps \( f, g : Y \rightarrow Z \) between compact metric spaces, denote by \( \|f - g\| \) their uniform distance, i.e. the number \( \|f - g\| := \sup_{y \in Y} d_Z(f(y), g(y)) \). In order to formulate Approximation Lemma (the main technical result of this section), we need to recall a result of M. Brown (see Theorem 2 in [Br]).

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7.9 Lemma (Brown’s Lemma). Let $\mathcal{S} = (\{X_n\}, \{h_n\})$ be an inverse sequence of compact metric spaces. Then for any $k \geq 0$ and for any sequence $s_k = (f_1, \ldots, f_k)$ of continuous maps $f_i : X_{i+1} \to X_i$ (where by $s_0$ we mean the empty sequence) there is a positive real number $\epsilon(s_k) = \epsilon(f_1, \ldots, f_k) > 0$ such that, if $(h'_i)_{i \geq 1}$ is a sequence of continuous maps $h'_i : X_{i+1} \to X_i$ satisfying the estimates

$$||h'_{i+1} - h_{i+1}|| < \epsilon(h'_1, \ldots, h'_i) \quad \text{for all } i \geq 0$$

then for the modified inverse sequence $\mathcal{S}' = (\{X_n\}, \{h'_n\})$ we have

$$\lim\leftarrow \mathcal{S}' = \lim\leftarrow \mathcal{S}.$$ 

We will call any inverse sequence $\mathcal{S}'$ obtained from a sequence $\mathcal{S}$ subject to the condition (B) as in the above lemma a Brown’s approximation of $\mathcal{S}$.

7.10 Remarks.

(1) It is not hard to observe that a Brown’s approximation can be always constructed recursively so that it additionally satisfies the following condition:

for any $n \geq 2$, let $\delta_n > 0$ be a uniform continuity constant for $\epsilon_n = \frac{1}{n}$, common for all maps $g'_i \circ g'_{i+1} \circ \ldots \circ g'_{n-1}$ with $i < n$; then for each $m \geq n$ we have

$$||g'_n \circ g'_{n+1} \circ \ldots \circ g'_m - g_n \circ g_{n+1} \circ \ldots \circ g_m|| < \delta_n.$$ 

We will demand and use the above condition (A) in our later arguments in Section 8.

(2) The proof of Brown’s Lemma in [Br] gives also the following property, to which we will refer in our later arguments in Section 8. For any approximation $\mathcal{S}'$ satisfying condition (B), if $x \in \lim\leftarrow \mathcal{S}'$ is represented by a thread $(x'_n)$ then for each $k$ the limit

$$y_k := \lim_{n \to \infty} g_k \circ \ldots \circ g_{n-1}(x'_n)$$

exists, and the sequence $(y_k)$ is a thread of the inverse sequence $\mathcal{S}$. Moreover, if we denote by $y \in \lim\leftarrow \mathcal{S}$ the element induced by $(y_k)$, then the map given by $x \mapsto y$ describes a homeomorphism between $\lim\leftarrow \mathcal{S}'$ and $\lim\leftarrow \mathcal{S}$.

7.11 Lemma (Approximation Lemma). Let $\Gamma$ be a finite connected simplicial graph which is flag, not reduced to a single vertex, which satisfies the assumptions $(\ast 1)$ and $(\ast 2)$ from Section 6 (i.e. $|\Gamma|$ has no essential loop edge, and each essential edge of $|\Gamma|$ consists of at least 3 edges of $\Gamma$), and which has no vertex of degree 1. Then there exists an exhausting sequence $D_n : n \geq 1$ of regular strict star domains in $(\Sigma_{\Gamma}, x_0)$ satisfying the following property: the inverse sequence $((\partial D_n), \{g_n\})$ (where $g_n : \partial D_n \to \partial D_{n-1}$ are the geodesic projections, and where we view each $\partial D_n$ as a $|\Gamma|$-graph) admits a Brown’s approximation $((\partial D_n), \{g'_n\})$ which is isomorphic to some null and dense inverse sequence $\mathcal{R}$ of $|\Gamma|$-graphs and $|\Gamma|$-blow-ups.
We postpone the proof of the above lemma until the next section, and we present now how this lemma implies the main result of the paper stated below in its full generality, as Theorem 7.12.

Recall that a topological graph \( X \) is non-separable if it is connected and has no separating essential vertex. (In particular, the circle is non-separable.) A block of \( X \) is any subgraph of \( X \) (for the natural cell structure consisting of essential vertices and edges) which is maximal for the inclusion in the family of all non-separable subgraphs of \( X \), and which is not (homeomorphic to) a segment or a singleton. We view each block as a topological graph, and we note that its natural cell structure (consisting of its own essential vertices and edges) is not necessarily induced from the natural cell structure of \( X \), and in general it might be “coarser” than the structure induced from \( X \). Note that any block has no vertices of degree 1 and, if it is not a circle, it also has no loop edges.

7.12 Theorem (Main Theorem). Let \( \Gamma \) be a finite simplicial graph which is flag, and denote by \(|\Gamma|\) the underlying topological graph. Suppose that for any block \( Y \) of \(|\Gamma|\) which is not a circle, each essential edge of \( Y \) (which automatically is not a loop) is the union of at least 3 edges of \( \Gamma \). Let \((W_\Gamma,S_\Gamma)\) be the right angled Coxeter system with nerve \( \Gamma \). Then the visual boundary \( \partial_\infty(W_\Gamma,S_\Gamma) \) is homeomorphic to the reflection tree of graphs \(|\Gamma|\), i.e. \( \partial_\infty(W_\Gamma,S_\Gamma) \cong \mathcal{X}(|\Gamma|) \).

Proof: Suppose first that \( \Gamma \) is connected, distinct from a singleton, satisfies assumptions \((*)1\) and \((*)2\) from Section 6, and contains no vertex of degree 1. Let \((D_n)\) be an exhausting sequence for \((\Sigma_\Gamma,x_0)\) as asserted by Approximation Lemma (Lemma 7.11). By Lemma 7.6, the visual boundary \( \partial_\infty(W_\Gamma,S_\Gamma) \) is homeomorphic to the inverse limit \( \lim_{\leftarrow} \{\partial D_n,\{g_n\}\} \). It follows from Approximation Lemma that the inverse sequence \( \{\partial D_n,\{g_n\}\} \) has a Brown’s approximation \( \{\partial D_n,\{g'_n\}\} \) which is isomorphic to some null and dense inverse sequence \( \mathcal{R} \) of \( \Gamma \)-graphs and \( \Gamma \)-blow-ups. By Proposition 5.5 we get

\[
\partial_\infty(W_\Gamma,S_\Gamma) \cong \lim_{\leftarrow} \{\partial D_n,\{g_n\}\} \cong \lim_{\leftarrow} \{\partial D_n,\{g'_n\}\} \cong \lim_{\leftarrow} \mathcal{R} \cong \mathcal{X}(|\Gamma|)
\]  

for graphs \( \Gamma \) considered in this case.

Now, let \( \Gamma \) be any graph satisfying the assumptions of the theorem, and let \( Y_1,\ldots,Y_k \) be the family of factors of some terminal split decompositions of all connected components of \(|\Gamma|\). Suppose that the first \( m \) of these factors \( Y_i \) form the set of all blocks of \(|\Gamma|\), while the others are all trivial. For \( i = 1,\ldots,k \) denote by \((W_i,S_i)\) the Coxeter system of the special subgroup of \( W_\Gamma \) spanned on the subset \( S_i \) consisting of those generators of \( S_\Gamma \) which correspond to the vertices of \( \Gamma \) contained in \( Y_i \).

If \( m = 0 \) (i.e. \(|\Gamma|\) has no blocks) then \( \Gamma \) is a tree or a disjoint union of trees. If \( \Gamma \) is a singleton or a doubleton, the theorem holds true by a direct inspection. Otherwise, it is easy to verify (e.g. by referring to [D], Proposition 8.8.2 and Theorem 8.7.4) that \( W_\Gamma \) is then virtually free non-abelian, and thus its visual boundary is homeomorphic to the Cantor set. On the other hand, by Lemma 2.4(2) the space \( \mathcal{X}(|\Gamma|) \) is then also homeomorphic to the Cantor set. This verifies the assertion of the theorem in the case \( m = 0 \).

If \( m = k = 1 \) and \(|\Gamma|\) is a circle \( S^1 \), it is well known that \( W_\Gamma \) is a cocompact reflection group in the euclidean or the hyperbolic plane, see Example 14.2.2 in [D]. As a
consequence, we have $\partial_\infty(W_\Gamma, S_\Gamma) \cong S^1$. Since by Lemma 2.4(3) we have also $X^r(|\Gamma|) = X^r(S^1) \cong S^1$, the assertion follows in this case. Furthermore, if $m = k = 1$ and $|\Gamma| = Y_1$ is not a circle, we are in the setting of the first paragraph of the proof (because any block distinct from a circle has no loop edges and no vertices of degree 1), and by (7.12.1) we get the assertion as well.

Finally, we are left with the case when $m \geq 1$ and $k \geq 2$. In this case $W_\Gamma$ has infinitely many ends (see [D], Theorem 8.7.4). Note that the subgroups $W_i$ form a family of special subgroups of $W_\Gamma$ corresponding to a split decomposition of the graph $\Gamma$ (i.e. a decomposition as described in Definition 3.3), and this coincides in our case with a more general decomposition of simplicial complexes as described in Subsection 7.1 of the paper [Amalgam]. It is shown in [Amalgam] that if a Coxeter group is not 2-ended then its visual boundary is homeomorphic to the dense amalgam of the visual boundaries of its special subgroups corresponding to any decomposition of its nerve (see Proposition 7.3.2 in that paper). In particular, in our setting we get that

$$\partial_\infty(W_\Gamma, S_\Gamma) \cong \tilde{\cup}(\partial_\infty(W_1, S_1), \ldots, \partial_\infty(W_k, S_k)).$$

Observe that for $i > m$ each of the groups $W_i$ is either finite or virtually free (including the 2-ended case), and consequently its visual boundary is either empty, or homeomorphic to a doubleton or to the Cantor set. It follows then from Proposition 3.7(4), and from the conventions concerning the dense amalgam of a tuple containing the empty space, that

$$\partial_\infty(W_\Gamma, S_\Gamma) \cong \tilde{\cup}(\partial_\infty(W_1, S_1), \ldots, \partial_\infty(W_m, S_m)).$$

Since by (7.12.1) we have that $\partial_\infty(W_i, S_i) \cong X^r(Y_i)$ for all $i \leq m$, denoting by $Y'_1, \ldots, Y'_p$ the homeomorphism types of the blocks $Y_1, \ldots, Y_m$ and applying Propositions 3.7(3) and 3.8, we get

$$\partial_\infty(W_\Gamma, S_\Gamma) \cong \tilde{\cup}(X^r(Y_1'), \ldots, X^r(Y_m')) \cong \tilde{\cup}(X^r(Y'_1), \ldots, X^r(Y'_p)) \cong X^r(|\Gamma|),$$

which completes the proof.

8. The proof of Approximation Lemma.

This section is devoted entirely to the proof of Approximation Lemma (Lemma 7.11). The proof is long and requires many observations and partial constructions.

Geodesic rays and the branch locus of $\Sigma_\Gamma$.

We begin with some observations concerning the interactions between branch components and geodesic rays in $\Sigma_\Gamma$ started at the base vertex $x_0$. Recall that all branch components of $\Sigma_\Gamma$ are trees (see Fact 6.6(1)), and that they are strictly convex in $\Sigma_\Gamma$ (Fact 6.6(4)). Denote by $C_0$ the branch component of $\Sigma_\Gamma$ containing $x_0$. The next claim follows directly from convexity of $C_0$ in $\Sigma_\Gamma$.

8.1 Claim. Each ray in the tree $C_0$ issuing from $x_0$ is a geodesic ray of $\Sigma_\Gamma$.

Recall that for any closed convex subset $A$ of a CAT(0) space $Z$ and any point $x \in Z \setminus A$ there is a unique point $x' \in A$ which is closest to $x$ (see [BH], Proposition II.2.4). This
point is called the \textit{projection} of \(x\) to \(A\). Since branch components are closed and convex in \(\Sigma_\Gamma\), we may speak of projections of the point \(x_0\) to them, and we have the following property of such projections.

8.2 Claim. \textit{For each branch component} \(C \neq C_0\) \textit{the projection of} \(x_0\) \textit{to} \(C\), \textit{denoted} \(x_C\), \textit{is a vertex of} \(\Sigma_\Gamma\).

\textbf{Proof:} If \(x_C\) were not a vertex, it would be an interior point of some edge \(e\) of \(C\). By the property of projection, the geodesic segment \([x_0, x_C]\) would be then orthogonal to \(e\) at \(x_C\). It is not hard to see that in any square complex a geodesic hitting some edge orthogonally at an interior point does not pass through any vertex of the complex. This contradicts the fact that \(x_0\) is a vertex, thus proving the claim.

For each branch component \(C \neq C_0\) denote by \(\Omega_C\) this sector of \(\Sigma_\Gamma\) from which the geodesic \([x_0, x_C]\) approaches \(x_C\). Denote also by \(e_{C,1}\) and \(e_{C,2}\) the two edges in \(C\) issuing from \(x_C\) which are adjacent to (i.e. contained in the boundary of) \(\Omega_C\). Denote by \(x_{C,1}, x_{C,2}\), respectively, the endpoints of the edges \(e_{C,1}, e_{C,2}\) other than \(x_C\).

8.3 Claim. \textit{If} \(C \neq C_0\) \textit{is a branched component of} \(\Sigma_\Gamma\), \textit{then}

\begin{enumerate}
\item \(\text{any ray in} \ C \ \text{started at} \ x_C \ \text{and not passing along} \ e_{C,1} \ \text{or} \ e_{C,2} \ \text{is a part of a geodesic ray of} \ \Sigma_\Gamma \ \text{started at} \ x_0;\)
\item \(\text{for} \ i = 1, 2, \ \text{any ray in} \ C \ \text{started at} \ x_{C,i} \ \text{and not passing along} \ e_{C,i} \ \text{is a part of a geodesic ray of} \ \Sigma_\Gamma \ \text{started at} \ x_0.\)
\end{enumerate}

\textbf{Proof:} We sketch the proof of part (2), skipping that of part (1), which is analogous. It is not hard to deduce from Claim 6.4 that the interior \(\text{int}(\Omega)\) separates \(x_0\) from \(C\). It follows that, for \(i = 1, 2\), the geodesic \([x_0, x_{C,i}]\) approaches \(x_{C,i}\) either along the edge \(E_{C,i}\) or directly from the interior of the sector \(\Omega_C\). Moreover, in the latter case the angle at \(x_{C,i}\) between \([x_0, x_{C,i}]\) and \(e_{C,i}\) is strictly less than \(\pi/2\). As a consequence, by strict convexity of \(C\) in \(\Sigma_\Gamma\), the angle at \(x_{C,i}\) between the geodesic \([x_0, x_{C,i}]\) and any edge \(e\) of \(C\) distinct from \(e_{C,i}\) is strictly larger than \(\pi\). It follows that \([x_0, x_{C,i}]\) extends as a geodesic along \(e\). By this, and by convexity of \(C\) in \(\Sigma_\Gamma\), any extension of \([x_0, x_{C,i}]\) by a ray as in the statement yields a geodesic ray in \(\Sigma_\Gamma\). This completes the proof.

\textit{Bifurcations and shadows of geodesics.}

Next observations deal with bifurcations of geodesics in \(\Sigma_\Gamma\), i.e. the phenomenon when a geodesic has a non-unique extension. Recall that in a piecewise euclidean CAT(0) complex \(K\) the possible local extensions of a geodesic \(\alpha = [x, y]\) behind its end \(y\) are described in terms of a \textit{shadow} of \(\alpha\) at \(y\), denoted \(\text{Sh}_y \alpha\). By definition, this is the subset of all points in the spherical link \(\text{Lk}_y K\) lying at distance \(\geq \pi\) from the point \(a \in \text{Lk}_y K\) representing \(\alpha\). Each \(b \in \text{Sh}_y \alpha\) represents uniquely a germ of a geodesic \(\beta = [y, z]\) which yields an extension of \(\alpha\). (We refer the reader to [BH], Section I.7.14, for the description of spherical links in piecewise euclidean complexes, and to Lemma (2d.1) in [DJ] for the proof of the above mentioned characterization of geodesic extensions in terms of shadows.)

An endpoint \(y\) of a geodesic segment \(\alpha\) is a \textit{bifurcation point} if the cardinality of the shadow \(\text{Sh}_y \alpha\) is greater than 1. It is not hard to observe that, under our assumptions on \(\Gamma\), the shadow of any geodesic segment in \(\Sigma_\Gamma\) at its any endpoint is nonempty. For geodesics with an endpoint at a vertex \(v\) of \(\Sigma_\Gamma\) (at which the spherical link \(\text{Lk}_v \Sigma_\Gamma\) is isometric with
Γπ/2), this follows e.g. by observing that any point in Γπ/2 lies on a cycle of perimeter \( \geq 2\pi \). For geodesics with other endpoints, this is even more obvious. Thus, the endpoint of any geodesic segment in ΣΓ is either a point of unique local extension, or a bifurcation point. As a consequence, any geodesic segment in ΣΓ can be extended to a geodesic ray (i.e. ΣΓ is geodesically complete).

**8.4 Claim.** Geodesics in ΣΓ have no bifurcations at interior points of the sectors.

**Proof:** At each interior point \( y \) of a sector in ΣΓ the spherical link \( Lk_y \Sigma Γ \) is the standard circle (of perimeter 2\( \pi \)). Consequently, any shadow is a singleton, and hence geodesics have unique extensions.

**8.5 Claim.** Geodesics in ΣΓ started at \( x_0 \) have no bifurcations at interior points of all edges \( e \) of the following two kinds:

1. edges contained in the branch component \( C_0 \) containing \( X_0 \);
2. edges distinct from \( e_{C,1} \) and \( e_{C,2} \) contained in any other branch component \( C \).

**Proof:** Let \( y \) be an interior point in any edge \( e \) as in the statement. By Claim 8.3, the geodesic \( α = [x_0, y] \) approaches \( y \) along \( e \) (more precisely, along this part of \( e \) which is closer to \( x_0 \) or \( x_C \) in the corresponding branch component). In the spherical link \( Lk_y \Sigma Γ \) the only point at distance \( \geq \pi \) from the point \( a \) representing \( α \) is the point \( b \) representing the geodesic issuing from \( y \) and going along the other part of \( e \) (more distant from \( x_0 \) or \( x_C \)). This completes the proof.

**Shadows at vertices of ΣΓ.**

Recall that link of ΣΓ at any vertex \( x \) is, by definition of ΣΓ, canonically isomorphic to the graph Γ. When viewed as spherical link \( Lk_x \Sigma Γ \), it is canonically isometric to Γ equipped with the length metric for which each edge has length \( \pi/2 \). We denote the resulting metric space by \( Γ_{\pi/2} \), to emphasise the above mentioned metric with which it is equipped. We will view the spherical vertex links of ΣΓ (under their identifications with \( Γ_{\pi/2} \)) naturally as \( |Γ| \)-graphs, for the tautological labellings of their essential vertices and edges. We will also view as \( |Γ| \)-graphs, with the induced labellings, various subspaces of the vertex links, notably the shadows of geodesics terminating at these vertices.

We now describe the shadows of geodesics \([x_0, x]\), as subsets of \( Γ_{\pi/2} = Lk_x \Sigma Γ \), for all vertices \( x \neq x_0 \) in ΣΓ. We use the following notation: if \( a \in Γ_{\pi/2} \) then

\[
\text{Sh}(a) := \{ b \in Γ_{\pi/2} : d(a, b) \geq \pi \},
\]

where \( d \) is the metric in \( Γ_{\pi/2} \). The following claim is obvious.

**8.6 Claim.** Let \( x \neq x_0 \) be any vertex in ΣΓ and let \( C \) be the component of the branch locus containing \( x \). Suppose that the geodesic \([x_0, x]\) terminates with an edge \( e \) contained in \( C \). Let \( v \in V \) be the label of \( e \) (and at the same time the point of \( Γ_{\pi/2} \)). Then \( \text{Sh}_v[x_0, x] = \text{Sh}(v) \) and thus, as a \(|Γ|\)-graph, this shadow is isomorphic to \(|Γ| \setminus U_v \), where \( U_v \) is some open normal neighbourhood of \( v \) in \(|Γ|\).

**8.7 Remark.** Note that, in view of Fact 8.3, Claim 8.6 applies to

1. all vertices \( x \neq x_0 \) contained in \( C_0 \), and
(2) all vertices in any other branch component $C$ which are distinct from $x_{C,1}$ and $x_{C,2}$.

Slightly less obvious is the following observation concerning vertices $x = x_{C,i}$.

**8.8 Claim.** Let $x$ be any of the vertices of form $x_{C,i}$, for any branch component $C \neq C_0$, and let $v \in V$ be the label of the edge $e_{C,i}$ (and at the same time the point of $\Gamma_{\pi/2}$). Then the shadow $Sh_x[x_0,x] \subset \Gamma_{\pi/2}$, viewed as a $|\Gamma|$-graph, is isomorphic to $|\Gamma| \setminus U_v$, where $U_v$ is some open normal neighbourhood of $v$ in $|\Gamma|$.

**Proof:** Let $a_0$ be the point in the spherical link $Lk_x \Sigma_\Gamma = \Gamma_{\pi/2}$ representing the geodesic $[x_0,x]$, and let $\varepsilon \in E$ be such that the sector $\Omega_C$ (as defined right before Claim 8.3) is an $\varepsilon$-sector. As it was shown in the first part of proof of Claim 8.3, $a_0$ either coincides with $v$, or it is an interior point of the essential edge $\varepsilon$ lying at distance less than $\pi/2$ from $v$. In any case the shadow $Sh_x[x_0,x] = Sh(a_0)$ has the form $|\Gamma| \setminus U_v$, as required, which completes the proof.

Finally, we deal with the shadows of the geodesics $[x_0,x_C]$ at their endpoints $x_C$. Recall that $\Omega_C$ is the sector of $\Sigma_\Gamma$ as described right before Claim 8.3.

**8.9 Claim.** Let $x$ be any of the vertices of form $x_{C}$, for any branch component $C \neq C_0$, and let $\varepsilon \in E$ be such that the sector $\Omega_C$ is an $\varepsilon$-sector. Then the shadow $Sh_x[x_0,x] \subset \Gamma_{\pi/2}$, viewed as a $|\Gamma|$-graph, is isomorphic to $|\Gamma| \setminus U$, where $U \subset |\Gamma|$ has one of the following forms:

1. $U$ is an open segment contained in the interior of $\varepsilon$ (in fact, this case consists of three subcases: 0, 1 or 2 endpoints of $U$ are the endpoints of $\varepsilon$);
2. $U$ is an open normal neighbourhood of one of the endpoints of $\varepsilon$ in $|\Gamma|$;
3. $U$ is the union of the interior of $\varepsilon$ and an open normal neighbourhood of one of the endpoints of $\varepsilon$ in $|\Gamma|$;
4. $U$ is an open normal neighbourhood of $\varepsilon$ in $|\Gamma|$.

**Proof:** Let $a_0$ be the point in the spherical link $Lk_x \Sigma_\Gamma = \Gamma_{\pi/2}$ representing the geodesic $[x_0,x]$, and let $v_i \in V$, for $i = 1, 2$, be the labels of the edges $e_{C,i}$ (and at the same time the endpoints of the essential edge $\varepsilon$ in $|\Gamma|$). Since $x$ is the projection of $x_0$ to $C$, the distances in $\Gamma_{\pi/2}$ between $a_0$ and $v_i$ are both $\geq \pi/2$. The assertion follows then directly by considering the cases whether these distances are smaller, equal or larger than $\pi$. We omit the details.

**Filtration of the branch locus $C_\Gamma$.**

In order to describe an appropriate sequence $D_n$ of strict star domains in $(\Sigma_\Gamma,x_0)$, as asserted in Approximation Lemma, we use certain auxiliary object that we call a filtration of $C_\Gamma$, which we now introduce. To do this, we need more terminology, notation and some observations. For any sector $\Omega$ of $\Sigma_\Gamma$ let $\partial_{x_0}\Omega$ be this component of $\partial \Omega$ which either contains $x_0$ or separates $\text{int}(\Omega)$ from $x_0$. Existence and uniqueness of such component of $\partial \Omega$ follows from Facts 6.4(3) and 6.9. Next, denote by $H_\Omega$ the closure of this connected component of $\Sigma_\Gamma \setminus \partial_{x_0}\Omega$ which contains $\text{int}(\Omega)$. Finally, let $g_\Omega : H_\Omega \rightarrow \partial_{x_0}\Omega$ be the geodesic projection (i.e. the map which, to any point $y \in H_\Omega$ associates the point closest to $y$ in the intersection of the geodesic $[x_0,y]$ with $\partial_{x_0}\Omega$). In general, $g_\Omega$ is not continuous, but it satisfies the following.
8.10 Claim. Let $\Omega$ be a sector of $\Sigma_{\Gamma}$ and let $C$ be any branch component adjacent to $\Omega$ and not containing $\partial x_0 \Omega$. Then $g_{\Omega}(C) = g_{\Omega}(e_{C,1} \cup e_{C,2})$. Moreover, $g_{\Omega}(C)$ is a bounded subset of $\partial x_0 \Omega$.

Proof: The first assertion follows easily from Claim 8.3. In view of the first assertion, the second one follows by observing that the union $\bigcup \{[x_0,y] : y \in e_{C,1} \cup e_{C,2}\}$ is a bounded subset of $\Sigma_{\Gamma}$, and the image $g_{\Omega}(C)$ is its subset.

8.11 Definition. A filtration of the branch locus $C_{\Gamma}$ is any sequence $F_n : n \geq 1$ of finite subgraphs of $C_{\Gamma}$ satisfying the following conditions:

1. $F_1 = \{x_0\}$; $F_{n+1}$ has exactly one more vertex than $F_n$ for all $n \geq 1$, and $\bigcup F_n = C_{\Gamma}$;
2. for each $n$, the intersection of $F_n$ with any branch component $C$ is either empty or a subtree containing the vertex $x_C$ (we use the convention that $x_{C_0} = x_0$);
3. for any branch component $C \neq C_0$, if $i$ is the smallest index for which $x_C \in F_i$, then $g_{\Omega_C}(C) \subset F_{i-1}$ (here again $\Omega_C$ denotes the sector of $\Sigma_{\Gamma}$ described right before Claim 8.3).

We skip an easy argument for showing that a filtration of $C_{\Gamma}$ always exists (in fact, there are lots of choices and plenty of filtrations). We only note that, in order to construct a filtration, it is convenient to order the components of the branch locus $C_{\Gamma}$ into a sequence $C_0, C_1, \ldots$ such that:

- $C_0$ is the component containing $x_0$;
- for each $n \geq 0$, the subtree of the adjacency tree $A_{\Gamma}$ (described in Definition 6.7 and in Fact 6.9) spanned on the vertices corresponding to the branch components $C_0, \ldots, C_n$ contains no other vertices corresponding to branch components.

It is also helpful to note that if $\partial x_0 \Omega_{C_n} \subset C_i$ then $i < n$, and to refer to Claim 8.10. We omit further details.

Description of the domains $D_n$ and the inverse sequence induced by a filtration.

We now describe an exhausting sequence of regular strict star domains $(D_n)$ in $(\Sigma_{\Gamma}, x_0)$, as required in Approximation Lemma. It will be determined essentially uniquely by a choice of a filtration of the branch locus $C_{\Gamma}$.

Given a filtration $(F_n)$ of $C_{\Gamma}$, for each $n$ consider a regular strict star domain $D_n$ satisfying the following properties:

(d1) for each branch component $C$, $C$ intersects $\text{int}(D_n)$ iff $C \cap F_n \neq \emptyset$, and if this is the case then $C \cap \text{int}(D_n)$ is a connected subset of $C$ containing $C \cap F_n$ and containing no other vertices of $C$ than those of $C \cap F_n$;
(d2) if for some $C \neq C_0$ we have $F_n \cap C = \{x_C\}$ then $x_{C,i} \in \partial D_n$ for $i = 1, 2$.

It is fairly obvious that domains $D_n$ as above exist. Moreover, we have the following uniqueness result.

8.12 Claim. If $D_n \subset D'_n$ are two regular strict star domains satisfying conditions (d1)-(d2) then the geodesic projection $g : \partial D'_n \to \partial D_n$ is a homeomorphism which is also an isomorphism of $|\Gamma|$-graphs. As a consequence, any two regular strict star domains $D_n$ satisfying conditions (d1)-(d2) are isomorphic as $|\Gamma|$-graphs.

Proof: The first assertion follows from the fact that geodesics started at $x_0$ have no bifurcations in the set $\text{int}(D'_n) \setminus \text{int}(D_n)$ (this is a direct consequence of Claims 8.4 and
and from the fact (following from Claims 8.1, 8.3 and 8.5) that for each \( y' \in \partial D_n' \) its image \( y = g(y') \) satisfies the following:

- if \( y' \in \text{int}(\Omega) \) for some sector \( \Omega \) then \( y \in \text{int}(\Omega) \);
- if \( y' \) lies in the interior of some branch edge of \( \Sigma \) then \( y \) belongs to the interior of the same edge;
- if \( y' \) is a vertex of \( \Sigma \) then \( y = y' \).

To prove the second assertion, note that for any two regular strict star domains \( D_n, D_n' \) satisfying (d1)-(d2), their intersection is a domain of the same kind. Thus the second assertion follows from the first one.

We also obviously have the following.

8.13 Claim. The \( |\Gamma| \)-graph \( \partial D_1 \) is isomorphic to \( |\Gamma| \) (with the tautological labelling of its essential vertices and edges).

We next observe that a sequence of domains \((D_n)\) as above can easily be chosen so that it forms an increasing exhausting sequence. It follows from Claim 8.12 that the induced inverse sequence \((\{\partial D_n\}, \{g_n\})\), where \( g_n : \partial D_{n+1} \to D_n \) are the geodesic projections, is then unique up to isomorphism of inverse sequences of \( |\Gamma| \)-graphs. We will call this sequence the inverse sequence induced by the filtration \((F_n)\).

To conclude the proof of Approximation Lemma, it remains to show that the maps \( g_n \) in the above inverse sequence can be approximated by appropriate \( |\Gamma| \)-blow-ups. To do this, we will use, among others, the previous analysis of shadows.

Approximating geodesic projections by \( |\Gamma| \)-blow-ups.

Given a filtration \((F_n)\) of the branch locus \( C_\Gamma \), let \((\{\partial D_n\}, \{g_n\})\) be the inverse sequence of \( |\Gamma| \)-graphs induced by this filtration. For each \( n \geq 2 \), denote by \( x_n \) the unique vertex in \( F_n \setminus F_{n-1} \). We construct the required approximations of the maps \( g_{n-1} \) separately in the three cases corresponding to the three types of the vertices \( x_n \).

Case 1: \( x_n \in C_0 \) or \( x_n \in C \setminus \{x_C, x_{C,1}, x_{C,2}\} \) for some \( C \neq C_0 \).

Note that, since \( n \geq 2 \), we have \( x_n \neq x_0 \). Thus, in view of Remark 8.7, \( x_n \) satisfies the assumption of Claim 8.6. We pass to the notation as in Claim 8.6, with \( x = x_n \) and with \( e \) coinciding with the unique edge in \( F_n \) adjacent to \( x_n \). Denote by \( v_e \) the vertex in the \( |\Gamma| \)-graph \( \partial D_{n-1} \) corresponding to the intersection point \( \partial D_{n-1} \cap e \). It follows from Claim 8.6, and from the arguments as in the proof of Claim 8.12, that the \( |\Gamma| \)-graph \( \partial D_n \) and the map \( g_{n-1} \) have the following form. The graph \( \partial D_n \) can be naturally viewed as obtained from \( \partial D_{n-1} \) by:

- first, deleting \( v_e \) and replacing it with as many points as edges adjacent to \( v_e \);
- then, gluing to the so obtained graph the shadow \( \text{Sh}(v) = \Gamma \setminus U_v \), through the map which associates to the points which have replaced \( v_e \) the points of \( \partial U_v \), in the way respecting labels of the edges containing these points.

Under this perspective on \( \partial D_n \), the map \( g_{n-1} \) has the following description:

- the points of \( \partial D_n \) corresponding to \( \partial D_{n-1} \setminus \{v_e\} \) are mapped identically to the same points in \( \partial D_{n-1} \);
- the points of \( \partial D_n \) corresponding to the shadow \( \Gamma \setminus U_v \) are all mapped to \( v_e \).
The map \( g_{n-1} \) can be approximated, arbitrarily close with respect to the uniform distance, by maps \( g'_{n-1} \) of the following form:

- for any edge \( \eta \) of \( \partial D_{n-1} \) issuing from \( v_e \), choose a point \( p_\eta \) on \( \eta \) sufficiently close to \( v_e \); denote by \( q_\eta \) the new point at the end of \( \eta \) that has replaced \( v_e \) in the description of \( \partial D_n \) above; let \( q'_\eta \) be the point in \( \Gamma \setminus U_v \) to which \( q_\eta \) is glued, and let \( v_\eta \) be the endpoint other than \( v \) in the edge of \( \Gamma \) containing \( q'_\eta \);
- put \( g'_{n-1} \) to coincide with \( g_{n-1} \) on the part of \( \partial D_n \) corresponding to \( \partial D_{n-1} \setminus \bigcup_{\eta \in \Gamma} [p_\eta, v_e] \) and on the part corresponding to the complement of the open star of the vertex \( v \) in \( \Gamma \) (i.e. on the subgraph of \( \Gamma \), for the natural stratification, spanned on all essential vertices except \( v \)); moreover, for each \( \eta \) as before put \( g'_{n-1} \) to map the segment \( [p_\eta, q_\eta] \cup [q'_\eta, v_\eta] \) in \( \partial D_n \) homeomorphically to the segment \( [p_\eta, v_e] \) in \( \partial D_{n-1} \).

Any map \( g'_{n-1} \) described above clearly has the form of a map associated to the \( [\Gamma] \)-blow-up of \( \partial D_{n-1} \) at \( v_e \), and \( g_{n-1} \) can be approximated by such a map as close as necessary.

**Case 2:** \( x_n = x_{C,i} \) for some \( C \neq C_0 \) and some \( i \in \{1, 2\} \).

The argument in this case is the same as in Case 1, except that the role of the vertex \( v_e \) is now played by \( x_n \) itself, the roles of \( e \) and \( v \) are played by the edge \( e_{C,i} \) and by its label \( v \in V \), respectively, and we use Claim 8.8 in place of Claim 8.6. We omit further details.

**Case 3:** \( x_n = x_C \) for some \( C \neq C_0 \).

As we show below, in this case, which is slightly different from the previous ones, the map \( g_{n-1} \) can be approximated by maps \( g'_{n-1} \) corresponding to blow-ups at a segment rather than at a vertex. Accordingly with the variety of forms for the set \( U \) in Claim 8.9, this case splits into many subcases. We deal in detail with only one of those subcases, leaving the remaining ones (in which the argument is very similar) to the reader.

We start with the setting for the case under consideration. We use the notation from Claim 8.9 and from its proof. Let \( a_0 \) be the point in the spherical link \( \text{Lk}_{x_n} \Sigma_\Gamma = \Gamma_{\pi/2} \) representing the geodesic \( [x_0, x_n] \), and let \( v_i \) be the points in this link representing the edges \( e_{C,i} \). We assume that in the considered subcase the distance in the link from \( a_0 \) to \( v_1 \) is smaller than \( \pi \), while the distance to \( v_2 \) is greater than \( \pi \).

Fix the following notation. Put \( z_0 := [x_0, x_n] \cap \partial D_{n-1} \) and \( z_1 := [x_0, x_C] \cap \partial D_{n-1} \) and note that, since the distance from \( a_0 \) to \( v_1 \) in the link at \( x_n \) is less than \( \pi \), these two points are distinct. Assume that the sector \( \Omega_C \) is labelled with \( \varepsilon \in E \). Then the graph \( \partial D_{n-1} \) contains an edge (labelled with \( \varepsilon \)) corresponding to its part contained in \( \Omega_C \), and we denote this edge by \( \varepsilon_{n-1} \). Note that both \( z_0 \) and \( z_1 \) belong to the interior of this edge. Further, denote by \( a_2 \) the point in the link \( \text{Lk}_{x_n} \Sigma_\Gamma \), lying at distance \( \pi \) from \( a_0 \) on the segment \( [a_0, v_2] \). This is one of the "extremal" points in the shadow \( \text{Sh}(a_0) \). Consider the geodesic issuing from \( x_n \) which corresponds to \( a_2 \) and note that it does not bifurcate before reaching \( \partial D_n \); denote by \( y_2 \) the point of intersection of this geodesic with \( \partial D_n \). Similarly, for each essential edge \( \eta \) other than \( \varepsilon \) issuing from \( v_1 \) in \( \text{Lk}_{x_n} \Sigma_\Gamma = \Gamma_{\pi/2} \), denote by \( a_\eta \) the interior point of \( \eta \) having distance \( \pi \) from \( a_0 \). Points of the form \( a_\eta \) are the remaining "extremal" points in the shadow \( \text{Sh}(a_0) \). For each \( \eta \), consider the geodesic issuing from \( x_n \) which corresponds to \( a_\eta \) and note that it does not bifurcate before reaching \( \partial D_n \); denote by \( y_\eta \) the point of intersection of this geodesic with \( \partial D_n \).

We make the following observations, which fully describe the form of the map \( g_{n-1} \):

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(1) points of the preimage $g_{n-1}^{-1}(z_0)$ are in the natural bijective correspondence (which is also an isomorphism of $|\Gamma|$-graphs) with the points of the shadow $\text{Lk}_{x_n}\Sigma_{\Gamma} = \text{Sh}(a_0)$; this correspondence is given by associating to a point $y \in g_{n-1}^{-1}(z_0)$ the point in the link induced by the geodesic $[x_n, y]$;

(2) for each $\eta$ as above, there is a segment $[y_\eta, x_{C,1}]$ in the graph $\partial D_n$, and each such segment is mapped by $g_n$ homeomorphically on the segment $[z_0, z_1]$ in $\partial D_{n-1}$;

(3) the complement in $\partial D_n$ of the set $g_{n-1}^{-1}(z_0) \cup \bigcup_\eta [y_\eta, x_{C,1}]$ is mapped by $g_n$ isomorphically (by isomorphism of $|\Gamma|$-graphs) on the complement $\partial D_{n-1} \setminus [x_0, x_1]$.

Referring to the above description, we can approximate the map $g_{n-1}$, arbitrarily close, by maps $g'_{n-1}$ of the following form:

- consider the segment $[y_2, x_{C,2}]$ in the graph $\partial D_n$ and let $y_3 \in \partial D_n$ be a point lying outside this segment as close to $y_2$ as necessary; similarly, let $y_4 \in \partial D_n$ be a point lying outside the segment $[y_3, x_{C,2}]$ as close to $y_3$ as necessary, and put $z_3 := g_{n-1}(y_3)$ and $z_4 := g_{n-1}(y_4)$; finally, let $z_2$ be a point in the interior of the segment $[z_0, z_1]$ contained in the edge $\varepsilon_{n-1}$ of $\partial D_{n-1}$, and assume that it is as close to $z_0$ as necessary;

- for any edge $\xi$ of $\partial D_n$ adjacent to $x_{C,2}$ and not containing $y_2$, choose a point $y_\xi$ in the interior of $\xi$ as close to $x_{C,2}$ as necessary;

- $g'_{n-1}$ maps the segment $[y_2, y_\xi] \subset \partial D_n$ homeomorphically onto the segment $[z_2, z_3] \subset \partial D_{n-1}$, similarly it maps $[y_3, x_{C,2}]$ onto $[z_3, z_0]$, $[x_{C,2}, y_\xi]$ onto $[z_0, z_2]$ for all $\xi$ as above, $[y_\eta, x_{C,1}]$ onto $[z_2, z_1]$ for all $\eta$ as above, it also maps $g_{n-1}^{-1}(z_0) \setminus [y_2, x_{C,2}]$ to the point $z_2$, and it coincides with $g_{n-1}$ on the remaining part of $\partial D_n$.

It is not hard to note that any map $g'_{n-1}$ above has the form of a map associated to the $|\Gamma|$-blow-up of $\partial D_{n-1}$ at the segment $[z_0, z_1]$ contained in the interior of the edge $\varepsilon_{n-1}$. Moreover, $g_{n-1}$ can be clearly approximated by such a map as close as necessary. In one of the later arguments we will also use the following property of the above described map $g'_{n-1}$ (in the statement of which $\pi_{D_n}$ denotes the geodesic projection towards $x_0$ from the complement of $\text{int}(D_n)$ to the boundary $\partial D_n$):

\begin{equation}
(8.1) \quad \text{the blow-up segment of } g'_{n-1} \text{ coincides with the image } \pi_{D_{n-1}}(e_{C,1} \cup e_{C,2}).
\end{equation}

In all other subcases of Case 3 the construction of approximations $g'_{n-1}$ can be also performed so that property (8.1) holds. We skip further details.

We now construct recursively a Brown’s approximation (\{(\partial D_n), \{g'_n\}\}) as required. Suppose that for some $n \geq 0$ we have already chosen appropriate modified maps $g'_1, \ldots, g'_n$. Choose $g'_{n+1} : \partial D_{n+2} \to \partial D_{n+1}$ to be a map as described in the appropriate corresponding case above such that

\[ ||g'_{n+1} - g_{n+1}|| < \epsilon(g'_1, \ldots, g'_n), \]

where $\epsilon(g'_1, \ldots, g'_n)$ is the appropriate constant from Brown’s Lemma (Lemma 7.9).

It is clear from the description of the maps $g'_i$ given in the three cases above that these maps are $|\Gamma|$-blow-ups.

An approximating inverse sequence (\{(\partial D_n), \{g'_n\}\}) is null.

To conclude the proof of Approximation Lemma we need to show that a Brown’s approximation (\{(\partial D_n), \{g'_n\}\}) as constructed above is null and dense (i.e. satisfies conditions (i3) and (i4) of Definition 5.4).
We first deal with condition (i3), i.e. nullness. To verify it, we need the following.

8.14 Claim. For arbitrary $i \geq 1$ consider the family of images through geodesic projections on $\partial D_i$ of the sets $e_{C,1} \cup e_{C,2}$, for all branch components $C$ which are disjoint with $D_i$. Then the diameters of these images converge to zero.

The claim follows easily from convexity of the CAT(0) metric in $\Sigma_\Gamma$ (see Proposition II.2.2 in [BH]), since the sets $e_{C,1} \cup e_{C,2}$ have diameters uniformly bounded by 2, and their distances from $x_0$ (and from $\partial D_i$) diverge to infinity.

Now, observe that for each $i$ the projections to $\partial D_i$ of the blow up segments of the sequence $\{(\partial D_n), \{g'_n\}\}$ satisfy the following conditions:

- they are all connected;
- they form a nested family, i.e. any two sets are either disjoint or one of them is contained in the other (see properties (d2)-(d4) in Step 1 of the proof of Proposition 5.5, in Section 5).

Note that, given a countable nested family $A_n : n \geq 1$ of connected subsets in a finite graph, the only reason for such a family not to be null is that there is an increasing infinite sequence $n_k$ such that:

- for all $k \geq 1$ we have $A_{n_{k+1}} \subset A_{n_k}$, and
- $\lim_{k \to \infty} \text{diam}(A_{n_k}) > 0$.

To prove that the appropriate families of projections of blow-up segments, as in condition (i3), are null, it is thus sufficient to exclude the phenomenon as above.

Suppose on the contrary that for some $i_0 \geq 1$ we are given a sequence $L_k$ of blow-up segments, with $L_k \subset \partial D_{n_k}$ and $n_k > i_0$, such that, denoting by $L'_k$ the projections of $L_k$ onto $\partial D_{i_0}$ through appropriate compositions of the maps $g'_i$, we have:

1. $L'_{k+1} \subset L'_k$ for all $k \geq 1$;
2. $\lim_{k \to \infty} \text{diam}(L'_k) > 0$.

Since all the sets $L'_k$ are compact, it follows from property (2) above that the intersection $\cap_{k \geq 1} L'_k$ consists of at least two points. Let $x^1_0, x^2_0$ be any two distinct points of this intersection. Note also that, without loss of generality, we can assume that for each $k \geq 1$ we have $n_{k+1} > n_k$, and consequently $g'_{n_k} \circ \ldots \circ g'_{n_{k+1}-1}(L_{k+1}) \subset L_k$. It follows that there exist strings $(x^1_n)$ and $(x^2_n)$ for the inverse sequence $S' = \{(\partial D_n), \{g'_n\}\}$ such that for $j = 1, 2$ we have $x^j_{i_0} = x^j_0$ and $x^j_{n_k} \in L_k$ for all $k \geq 1$.

Recall that by (8.1), for each $k$ there is a branch component $C_k$ in $\Sigma_\Gamma$ such that $L_k = \pi_{D_{n_k}}(e_{C_k,1} \cup e_{C_k,2})$. Moreover, it is not hard to observe that the components $C_k$ are then pairwise distinct. For each $k \geq 1$ and for $j = 1, 2$ choose $z^j_k \in e_{C_{k,1}} \cup e_{C_{k,2}}$ such that $\pi_{D_{n_k}}(z^j_k) = x^j_{n_k}$. Note also that, by Remark 7.10(2), the limits $y^j = \lim_{k \to \infty} g_{k_0} \circ \ldots \circ g_{n_k-1}(x^j_{n_k})$ exist and are distinct. On the other hand, for each $k$ we have that $y^j = g_{k_0} \circ \ldots \circ g_{n_k-1} \circ \pi_{D_{k_0}}(z^j_k)$, which simply means that $y^j = \pi_{D_{k_0}}(z^j_k)$. Since it follows then from Claim 8.14 that, as $k \to \infty$, the distance between the projections $\pi_{D_{k_0}}(z^j_k)$ converges to 0, we conclude that $y^1 = y^2$, contradicting an earlier observation that these points are distinct.

This completes the proof of nullness.
An approximating inverse sequence \( (\{\partial D_n\}, \{g'_n\}) \) is dense.

To prove denseness (condition (i4)), we will need the following.

**8.15 Claim.** The set \( M = \{x_C : C \neq C_0\} \) is a net in \( \Sigma_\Gamma \), i.e. there is \( r > 0 \) such that each ball of radius \( r \) in \( \Sigma_\Gamma \) intersects \( M \).

To prove the claim, we will show that for any sector \( \Omega \) in \( \Sigma_\Gamma \) the intersection \( \partial \Omega \cap M \) is a net in \( \Omega \). Denote by \( L_0 \) this boundary component of \( \Omega \) through which geodesics started at \( x_0 \) enter \( \Omega \). Note that each point of \( \Omega \) lies at distance at most 2 from some square \( Q \) of \( \Omega \) disjoint with \( L_0 \). Furthermore, any such square \( Q \) separates \( L_0 \) from some other boundary component \( L \subset \partial \Omega \) disjoint from \( Q \) and lying at distance at most 2 from \( Q \). It is also not hard to see that, if \( C \) denotes the branch component in \( \Sigma_\Gamma \) containing \( L \) then \( x_C \in L \) and the distance from \( x_C \) to \( Q \) is also \( \leq 2 \). Since \( \text{diam}(Q) \leq 2 \), the number \( r = 6 \) is as required, which completes the proof of Claim 8.15.

As a consequence of Claim 8.15, we get the following.

**8.16 Corollary.** For any \( n \geq 1 \), let \( \pi_{D_n} : \Sigma_\Gamma \setminus \text{int}(D_n) \to \partial D_n \) be the geodesic projection towards the base vertex \( x_0 \). Then the set

\[
\pi_{D_n}(\{x_C : C \text{ is a branch component such that } C \cap D_n = \emptyset\})
\]

is dense in \( \partial D_n \).

To justify the corollary, recall that under our assumptions on \( \Gamma \), the Coxeter-Davis complex \( \Sigma_\Gamma \) is geodesically complete, and hence each of the maps \( \pi_{D_n} \) is surjective. The corollary thus follows from Claim 8.15 due to convexity of the CAT(0) metric in \( \Sigma_\Gamma \).

In order to show that our inverse sequence \( (\{\partial D_n\}, \{g'_n\}) \) of \( |\Gamma| \)-graphs and \( |\Gamma| \)-blow-ups is dense, we assume, without loss of generality, that it satisfies the additional condition described in Remark 7.10(1), and in particular the inequalities (A) mentioned there.

Suppose that \( L \subset \partial D_i \) is a blow-up segment (which means that the map \( g'_i : \partial D_{i+1} \to \partial D_i \) is a \( |\Gamma| \)-blow-up at \( L \)). It follows from the description of \( g'_i \) in Case 3 above, and more precisely from (8.1), that then for some branch component \( C \) we have \( x_C \in \partial D_{i+1} \) and \( g'_i(x_C) \in L \). In view of this, to prove that the inverse sequence \( (\{\partial D_n\}, \{g'_n\}) \) is dense, it is sufficient to show the following.

**8.17 Claim.** For any branch component \( C \) of \( \Sigma_\Gamma \), denote by \( n_C \) this number for which \( x_C \in \partial D_{n_C} \). Then for each \( i \geq 1 \), the set

\[
\{g'_i \circ g'_{i+1} \circ \cdots \circ g'_{n_C-1}(x_C) : C \cap D_i = \emptyset\}
\]

is dense in \( \partial D_i \).

To prove Claim 8.17, fix any point \( p \in \partial D_i \), and any \( \varepsilon > 0 \). We will show that there is \( C \) such that the distance in \( \partial D_i \) from \( p \) to the point \( g'_i \circ g'_{i+1} \circ \cdots \circ g'_{n_C-1}(x_C) \) is less than \( 2\varepsilon \). Pick \( n > i \) such that \( \frac{1}{n} < \varepsilon \), and let \( \delta_n \) be a constant as in Remark 7.10(1). Let \( q \in \partial D_n \) be any point such that \( g'_i \circ g'_{i+1} \circ \cdots \circ g'_{n-1}(q) = p \) (such \( q \) exists since all the maps \( g'_i \) are surjective, which follows from the form of \( |\Gamma| \)-blow-up maps in view of the assumption that
Γ has no vertices of degree 1). By Corollary 8.16, there is \( m > n \) and a point \( x_C \in \partial D_m \) such that
\[
d_{\partial D_n}(q, g_n \circ g_{n+1} \circ \ldots \circ g_{m-1}(x_C)) < \delta_n,
\]
where \( d_{\partial D_n} \) is the metric in \( \partial D_n \) (this is true because \( g_n \circ g_{n+1} \circ \ldots \circ g_{m-1}(x_C) = \pi_{D_n}(x_C) \)). By condition (A) of Remark 7.10(1), we have
\[
d_{\partial D_n}(g'_n \circ g'_{n+1} \circ \ldots \circ g'_{m-1}(x_C), g_n \circ g_{n+1} \circ \ldots \circ g_{m-1}(x_C)) < \delta_n.
\]
Since \( \delta_n \) is a uniform continuity constant for \( \epsilon_n = \frac{1}{n} \) (for the map \( g'_i \circ g'_{i+1} \circ \ldots \circ g'_{n-1} \)), we get
\[
d_{\partial D_i}(p, g'_i \circ g'_{i+1} \circ \ldots \circ g'_{n-1} \circ g'_n \circ \ldots \circ g'_{m-1}(x_C)) =
\]
\[
= d_{\partial D_i}(g'_i \circ g'_{i+1} \circ \ldots \circ g'_{n-1}(q), g'_i \circ g'_{i+1} \circ \ldots \circ g'_{n-1} \circ g'_n \circ \ldots \circ g'_{m-1}(x_C)) \leq
\]
\[
= d_{\partial D_i}(g'_i \circ \ldots \circ g'_{n-1}(q), g'_i \circ \ldots \circ g'_{n-1} \circ g_n \circ \ldots \circ g_{m-1}(x_C)) +
\]
\[
+ d_{\partial D_i}(g'_i \circ \ldots \circ g'_{n-1} \circ g_n \circ \ldots \circ g_{m-1}(x_C), g'_i \circ \ldots \circ g'_{n-1} \circ g'_n \circ \ldots \circ g'_{m-1}(x_C)) <
\]
\[
< \frac{1}{n} + \frac{1}{n} < 2\epsilon.
\]
This finishes the proof of the claim, and thus also of Approximation Lemma.

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