PARITY BINOMIAL EDGE IDEALS

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Abstract. Parity binomial edge ideals of simple undirected graphs are introduced. Unlike binomial edge ideals, they do not have square-free Gröbner bases and are radical only if the characteristic of the ground field is not two. The minimal primes are determined and shown to encode combinatorics of even and odd walks in the graph. A mesoprimary decomposition is determined and shown to be a primary decomposition in characteristic two.

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1. Introduction

A binomial is a polynomial with at most two terms and a binomial ideal is a polynomial ideal generated by binomials. Binomial ideals appear frequently in mathematics and also applications to statistics and biology. This paper is about decompositions of binomial ideals which appear, for instance, in understanding the implications of conditional independence statements [4, Chapter 3], steady states of chemical reaction networks [17, 2], or combinatorial game theory [15, 16].

Decomposition theory of binomial ideals started with Eisenbud and Sturmfels’ fundamental paper [5] which proves the existence of binomial primary decomposition over algebraically closed fields. It can be seen, however, that the field assumption is not strictly necessary: a mesoprimary decomposition captures all combinatorial features and exists over any given field [13]. Separating the arithmetical
and combinatorial aspects of binomial ideals is important for applications where binomial primary decompositions over the complex numbers are often inadequate since they obscure combinatorics and prevent interpretations of the indeterminates as, say, probabilities or concentrations.

Actual primary decompositions have been computed almost exclusively of radical ideals. It is a general feature of (meso)primary decomposition that the embedded primes and components remain elusive. The partial decomposition of the Mayr-Meyer ideals by Swanson illustrates quite beautifully the mess one typically encounters when trying to determine components over embedded primes [19]. The minimal primes are often combinatorially fixed and thus much better behaved. For instance for lattice basis ideals they are entirely determined by the indeterminates they contain [10]. More examples of interesting combinatorial descriptions of minimal primes of binomial ideals appear for instance in [8, 11, 14]. In practice binomial (primary) decompositions can be found with computer algebra. For experimentation we used and recommend the package Binomials [12] in Macaulay2 [6].

This paper is about a class of ideals whose primary decomposition depends on the characteristic of the field and is in general different from the mesopprimary decomposition. We decompose these ideals using a new technique and hope to add to the toolbox for binomial decompositions. To define the key player, let $G$ be a simple undirected graph on $V(G)$ and with edge set $E(G)$. Let $k$ be any field and denote by $k[x, y] = k[x_i, y_i : i \in V(G)]$ the polynomial ring in $2|V(G)|$ indeterminates.

**Definition 1.1.** The parity binomial edge ideal of $G$ is

$$I_G := \langle x_ix_j - y_iy_j : \{i, j\} \in E(G) \rangle \subseteq k[x, y].$$

Parity binomial edge ideals share a number of properties with binomial edge ideals [8], but the combinatorics is more subtle. Various properties related to walks in $G$ depend on whether the walk has even or odd length (and hence the name). If $G$ is bipartite, then everything reduces to the results of [8] as follows.

**Remark 1.2.** Let $G$ be bipartite on the vertex set $V_1 \cup V_2$. Consider the ring automorphism of $k[x, y]$ which exchanges $x_i$ and $y_i$ if $i \in V_1$ and leaves all remaining indeterminates invariant. Under this automorphism, $I_G$ is the image of the binomial edge ideal of $G$.

Definition 1.1 was suggested by Rafael Villarreal at the MOCCA Conference 2014 in Levico Terme. He asked if parity binomial edge ideals are radical and Theorem 5.5 says that this is the case if $G$ is bipartite (by Remark 1.2), or $\text{char}(k) \neq 2$. We compute the minimal primes of $I_G$ in Section 4. In Proposition 5.4, we write $I_G$ as an intersection of binomial ideals whose combinatorics is simpler, since then a short induction shows that, under the field assumption, all occurring intersections are radical (Theorem 5.5) and hence $I_G$ is radical. In $\text{char}(k) = 2$ we determine a
primary decomposition (Theorem 5.9), which turns out to be also a mesoprimary decomposition (Theorem 5.10).

Our determination of the minimal primes goes a route that is familiar from [14]. We first determine generators of the distinguished component \( I_G : (\prod_{i \in V(G)} x_i y_i)^\infty \) (that is, a Markov basis) in Section 2. Binomials \( b \) that appear in the Markov basis but are not themselves contained in \( I_G \) have the property that \( mb \in I_G \) for some monomial \( m \). This means that \( I_G : b \) contains the monomial \( m \) and thus some minimal primes of \( I_G \) contain the indeterminates that constitute \( m \). In the most convenient case, the witness monomial \( m \) is just an indeterminate and knowledge about the Markov basis gives good knowledge of the minimal primes.

Just looking at Definition 1.1 one may hope that parity binomial edge ideals would deform to monomial edge ideals under the Gröbner deformation. This is not the case as already the simplest examples show, but nevertheless, the lexicographic Gröbner basis has combinatorial structure and we describe it completely in Section 3.

Shortly before first posting this paper on the arXiv, the authors became aware of [9]. That paper contains a different proof of radicality of parity binomial edge ideals in characteristic two since the linear transformation \( x_i \mapsto x_i - y_i, y_i \mapsto x_i + y_i \), maps the parity binomial edge ideal onto the permanental edge ideal \( \Pi_G \) defined there. Our approach here is different and was developed completely independently. In particular our proof of radicality cannot use the Gröbner basis by Remark 3.12. Additionally we can clarify the separation of combinatorics and arithmetics of \( I_G \) independent of \( \text{char}(\mathbb{k}) \) and determine its mesoprimary decomposition.

Conventions and Notation. For \( n \in \mathbb{N}_{>0} \), let \([n] := \{1, \ldots, n\}\). All graphs here are finite and simple, that is they have no loops or multiple edges. For any graph \( G \), \( V(G) \) is the vertex set. For any \( S \subseteq V(G) \), \( G[S] \) is the induced subgraph on \( S \) and for a sequence of vertices \( P = (i_1, \cdots, i_r) \in V(G)^r \), \( G[P] := G[\{i_1, \cdots, i_r\}] \). Throughout we assume that \( G \) is connected and in particular has no isolated vertices if \( |V(G)| \geq 2 \). According to Definition 1.1, if a graph is not connected then the parity binomial edge ideals of the connected components live in polynomial rings on disjoint sets of indeterminates such that the problem reduces to connected graphs. Despite this assumption, non-connected graphs appear. Thus, for any graph \( H \), let \( c(H) \) be the number connected components, \( c_0(H) \) the number of bipartite connected components, and \( c_1(H) \) the number of connected components which contain an odd cycle. We freely identify ideals of sub-polynomial rings of \( \mathbb{k}[x, y] \) with their images in \( \mathbb{k}[x, y] \). Likewise ideals of \( \mathbb{k}[x, y] \) that do not use some of the indeterminates are considered ideals of the respective subrings. A binomial is pure-difference if it equals the difference of two monomials.

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Figure 1. A graph with an even walk, but no even path from 4 to 6. The interior of the walk $(4, 3, 1, 2, 3, 5, 6)$ is $\{1, 2, 3, 5\}$.

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2. Markov bases

Markov bases where first defined for toric ideals, but the definition extends easily to other lattice ideals. In this paper, by a Markov basis we mean generators of $\mathcal{I}_G : (\prod_{i \in V(G)} x_i y_i)^\infty$, which is compatible with the extended notions of Markov bases used in [4, Section 1.3] and [18, Section 2.1].

**Definition 2.1.** Let $G$ be a graph. A $(v, w)$-walk of length $r - 1$ is a sequence of vertices $v = i_1, i_2, \ldots, i_r = w$ such that $\{i_k, i_{k+1}\} \in E(G)$ for all $k \in [r - 1]$. The walk is odd (even) if its length is odd (even). A path is walk that uses no vertex twice. A cycle is a walk with $v = w$. The interior of a $(v, w)$-walk $P = (i_1, \ldots, i_r)$ is the set $\text{int}(P) = \{i_1, \ldots, i_r\} \setminus \{v, w\}$.

**Remark 2.2.** In this paper, a cycle is only defined with a marked start and end vertex. Consequently the interior of a cycle (in the usual graph theoretic sense) also depends on the choice of this vertex.

**Convention 2.3.** When no ambiguity can arise, for instance because the vertices are explicitly enumerated, we call a $(v, w)$-walk simply a walk.

**Lemma 2.4.** Let $P = (i_1, \ldots, i_r)$ be a walk in $G$ and $t_{ij} \in \{x_{ij}, y_{ij}\}$ arbitrary. If $P$ is odd, then

$$ (x_{i_1} x_{i_r} - y_{i_1} y_{i_r}) \prod_{j \in \text{int}(P)} t_{ij} \in \mathcal{I}_G. \tag{2.1} $$

If $P$ is even, then

$$ (x_{i_1} y_{i_r} - y_{i_1} x_{i_r}) \prod_{j \in \text{int}(P)} t_{ij} \in \mathcal{I}_G. \tag{2.2} $$
The Graver basis of $\mu_2(G)$ is $\pm(\mathcal{M}_G^{\text{odd}} \cup \mathcal{M}_G^{\text{even}})$.

**Proof.** According to Pottier’s termination criterion [3, Algorithm 3.3] it suffices to check that the sum of two elements of $\pm(\mathcal{M}_G^{\text{odd}} \cup \mathcal{M}_G^{\text{even}})$ can be reduced to zero sign-consistently. If there are no cancellations in the sum, for example if the two summands have disjoint support, the sum is reduced by either of the
summands. Cancellation among elements $e_{i_1} \pm e_{i_2}$ and $e_{j_1} \pm e_{j_2}$ can only occur if $|\{i_1, i_2, j_1, j_2\}| \leq 3$. Without loss of generality assume $i_2 = j_1$. Thus, if cancellation occurs, the sum of two proposed Graver elements must equal $\pm(e_{i_1} \pm e_{j_2})$ and this is either zero or another element in $\pm(M^\text{odd}_G \cup M^\text{even}_G)$ by concatenation of walks. □

Proposition 2.6 shows that the minimal Markov, or equivalently Graver, basis of the ideal saturation $J_G := I_G : (\prod_{i \in V(G)} x_i y_i)\infty$ at the coordinate hyperplanes consists of the following binomials:

**Proposition 2.7.**

\[ J_G = \langle x_ix_j - y_iy_j : \text{there is an odd } (i,j)\text{-walk in } G \rangle + \langle x_iy_j - y_jx_i : \text{there is an even } (i,j)\text{-walk in } G \rangle. \]  

(2.3)

*Proof.* This is Proposition 2.6 and [1, Proposition 1.1]. □

**Example 2.8.** Due to the odd cycle in the graph $G$ in Figure 1, for all pairs $(i, j)$ of vertices with $i \neq j$, both $x_ix_j - y_iy_j$ and $x_iy_j - x_jy_i$ are contained in $J_G$. Hence, the ideal $J_G$ has 15 generators for odd walks and 15 for even walks with disjoint endpoints. Since $G$ is not bipartite, $x_i^2 - y_i^2 \in J_G$ for all $i \in [6]$. In total, a minimal Markov basis of $J_G$ consists of 36 generators.

**Remark 2.9.** If $G$ is bipartite, the reachability of vertices with even or odd walks is determined by membership in the two groups of vertices. Consequently, for each spanning tree $T \subseteq G$ we have $J_T = J_G$. This is not true if $G$ has an odd cycle.

3. A Lexicographic Gröbner basis

For this section an ordering of $V(G)$ is necessary. Fix any bijection $\phi : [n] \rightarrow V(G)$ and let $\succ$ be the lexicographic ordering on $k[x, y]$ induced by $x_\phi(1) \succ \cdots \succ x_\phi(n) \succ y_\phi(1) \succ \cdots \succ y_\phi(n)$. For $i, j \in V(G)$ write $i \succ j$ if $x_{\phi^{-1}(i)} \succ x_{\phi^{-1}(j)}$.

**Definition 3.1.** An $(i, j)$-walk $P$ in $G$ is minimal if for no $k \in \text{int}(P)$ there is an $(i, j)$-walk with the same parity as $P$ in $G[P \setminus \{k\}]$.

We now describe the lexicographic Gröbner basis of the parity binomial edge ideal. Its binomials correspond to minimal walks, and differ depending on the parity of the walk. First, let $P$ be a minimal odd $(i, j)$-walk with $i \geq j$. The binomial

\[ (x_ix_j - y_iy_j) \prod_{k \in \text{int}(P)} y_k \]

is reduced if for all $k \in \text{int}(P)$:

- If there is an even $(i, k)$-walk in $G[P]$, then $k \succ i$,
- if there is an odd $(i, k)$-walk in $G[P]$, then $k \succ j$.
To define the even walks contributing binomials to the Gröbner basis, the monomial factor may have indeterminates $x_k$ or $y_k$ depending on combinatorial features of the walk. To see them, let $P$ be a minimal even $(i, j)$-walk with $i \succ j$, and for $k \in \text{int}(P)$, let $t_k \in \{x_k, y_k\}$ be arbitrary. The binomial

\begin{equation}
(x_i y_j - y_i x_j) \prod_{k \in \text{int}(P)} t_k
\end{equation}

is reduced if for all $k \in \text{int}(P)$:

- If there is an even $(i, k)$-walk in $G[P]$, then $t_k = y_k$ for all $k \succ i$ and $t_k = x_k$ for all $j \succ k$.
- If there is an odd $(i, k)$-walk in $G[P]$, then $t_k = y_k$.

The set of reduced binomials is $\mathcal{G}_\succ(G)$.

**Remark 3.2.** In the above definition for even $P$, let $k \in \text{int}(P)$ with $j \succ k$. If there is both an even and an odd $(i, k)$-walk, then it seems as if the two bullets in the definition would contradict each other. However, this case cannot happen, since $P$ is minimal—for each $k$ all $(i, k)$-walks have the same parity.

**Example 3.3.** In the parity binomial edge ideal for Figure 1, $(x_4 x_5 - y_4 y_5)y_3 y_2 y_1$ is reduced with respect to $\mathcal{G}_\succ(G)$ since the walk $(4, 3, 1, 2, 3, 5)$ is minimal. In particular, minimal walks can have odd cycles.

The first step is to see that reduced binomials have minimal leading terms among all binomials in Lemma 2.4 corresponding to walks, justifying their name.

**Lemma 3.4.** Let $P$ be an $(i, j)$-walk and $t_k \in \{x_k, y_k\}$ for $k \in \text{int}(P)$ arbitrary. Then $(x_i x_j - y_i y_j) \prod_{k \in \text{int}(P)} t_k$ if $P$ is odd and $(x_i y_j - y_i x_j) \prod_{k \in \text{int}(P)} t_k$ if $P$ is even, reduce to zero modulo $\mathcal{G}_\succ(G)$.

**Proof.** This is an easy induction on the length of the walk. It suffices to restrict to a minimal walk $P$. By induction, if $P$ is not minimal, its binomial is a monomial multiple of the binomial for a shorter walk. For a minimal walk, if its binomial is not reduced, then some $k \in \text{int}(P)$ violates one of the properties in the definition. In this case there exists two subwalks $(i, k)$ and $(k, j)$ whose binomials reduce to zero by the induction hypothesis, and which themselves reduce the original binomial. □

We now state the main theorem of this section. Its proof is by Buchberger’s criterion and splits into a couple of lemmas.

**Theorem 3.5.** The set $\mathcal{G}_\succ(G)$ of reduced binomials is the reduced Gröbner basis of $\mathcal{I}_G$ with respect to $\succ$. 
Remark 3.6. The Gröbner basis in [8] looks similar, but for the original binomial edge ideals there are no binomials corresponding to \( k \in \text{int}(P) \) with \( i \succ k \succ j \) in the Gröbner basis. The Gröbner basis there is also not a subset of our Gröbner basis. In fact, all elements of the Gröbner basis in [8] corresponding to odd \((i, k)\) -walks of length at least 2, with \( j \succ k \) are reduced by our odd moves. That this happens warrants the last bullet in the definition for reduced even walk binomials.

Remark 3.7. The reduced binomials for even minimal walks in (3.1) can be made more explicit using the following notation:

\[
P^x := \{ k \in \text{int}(P) : \text{there is an even } (i, k)\text{-walk in } G[P] \text{ and } j \succ k \} \quad \text{and} \quad P^y := \text{int}(P) \setminus P^x.
\]

By Remark 3.2,

\[
(x_i y_j - y_i x_j) \prod_{k \in \text{int}(P)} t_k = (x_i y_j - y_i x_j) \prod_{k \in P^x} x_k \prod_{k \in P^y} y_k.
\]

For the reduction of s-polynomials we use the following well-known fact.

Lemma 3.8. Let \( f, g \in k[x] \) and \( \succ \) a monomial ordering. If their leading monomials form a regular sequence. Then \( \text{spol}(uf, vg) \) reduces to zero for all monomials \( u, v \in k[x] \).

Lemma 3.9. Let \( g_P \) and \( g_Q \) be reduced binomials corresponding to even walks \( P \) and \( Q \). Then \( \text{spol}(g_P, g_Q) \) reduces to zero with respect to \( G\succ(G) \).

Proof. Let \( P \) be an even \((p_1, p_2)\)-walk with \( p_1 \succ p_2 \) and \( Q \) be an even \((q_1, q_2)\)-walk with \( q_1 \succ q_2 \). By Remark 3.7 we write

\[
g_P = (x_{p_1} y_{p_2} - y_{p_1} x_{p_2}) \cdot \prod_{i \in P^x} x_i \cdot \prod_{i \in P^y} y_i,
\]

\[
g_Q = (x_{q_1} y_{q_2} - y_{q_1} x_{q_2}) \cdot \prod_{i \in Q^x} x_i \cdot \prod_{i \in Q^y} y_i.
\]

If \(|\{p_1, p_2, q_1, q_2\}| = 4\), then \( x_{p_1} y_{p_2} \) and \( x_{q_1} y_{q_2} \) are coprime and thus form a regular sequence. Lemma 3.8 gives this case. If \( \{p_1, p_2, q_1, q_2\} = \{q_1, q_2\} \), then the s-polynomial is zero.

The only interesting case is when \( P \) and \( Q \) have precisely one endpoint in common. First, let that common endpoint be \( v := p_1 = q_1 \). Since \( p_1 \not\in Q^x \), \( q_1 \not\in P^x \), and we can assume that \( q_2 \succ p_2 \), the s-polynomial is

\[
(x_{q_2} y_{p_2} - y_{q_2} x_{p_2}) \cdot y_v \cdot \prod_{i \in P^x \cup Q^x} x_i \prod_{i \in (P^y \cup Q^y)\setminus\{q_2, p_2\}} y_i.
\]

This binomial is a monomial multiple of the binomial obtained from the \((q_2, p_2)\)-walk which might traverse the vertex \( v = p_1 = q_2 \). Hence, the s-polynomial reduces
to zero by Lemma 3.4. The case that \( p_2 = q_2 \) is similar and omitted. The last case
is (without loss of generality) \( q_1 > q_2 = p_1 > p_2 \). In this case \( x_{q_1}y_{q_2} \) and \( x_{p_1}y_{p_2} \)
form a regular sequence and due to Lemma 3.8 \( \text{spol}(g_P, g_Q) \) reduces to zero. □

**Lemma 3.10.** Let \( g_P \) and \( g_Q \) be reduced binomials corresponding to odd walks \( P \)
and \( Q \). Then \( \text{spol}(g_P, g_Q) \) reduces to zero with respect to \( G_{\prec}(G) \).

*Proof.* Assume that \( P \) is a \((p_1, p_2)\)-walk with \( p_1 > p_2 \) and \( Q \) is a \((q_1, q_2)\)-walk
with \( q_1 > q_2 \). Without loss of generality, let \( p_1 > q_1 \). By Lemma 3.8, we can assume
\( |\{p_1, p_2\} \cap \{q_1, q_2\}| \geq 1 \). Clearly, if \( \{p_1, p_2\} = \{q_1, q_2\} \), \( \text{spol}(g_P, g_Q) = 0 \). In total
assume that \( \{p_1, p_2\} \neq \{q_1, q_2\} \). Under this assumptions, in all remaining cases,
the \( s \)-polynomial is a monomial multiple of the binomial corresponding to the even
walk which arises from gluing \( P \) and \( Q \) along the vertex they have in common. □

**Lemma 3.11.** Let \( g_P \) and \( g_Q \) be reduced binomials corresponding, respectively, to an odd walk \( P \) and an even walk \( Q \). Then \( \text{spol}(g_P, g_Q) \) reduces to zero with respect to \( G_{\prec}(G) \).

*Proof.* Let \( P \) be an \((p_1, p_2)\)-walk with \( p_1 \geq p_2 \) and \( Q \) be an even \((q_1, q_2)\)-walk
with \( q_1 > q_2 \). By Remark 3.7 we write

\[
\begin{align*}
g_P &= (x_{p_1}x_{p_2} - y_{p_1}y_{p_2}) \prod_{i \in \text{int}(P)} y_i, \\
g_Q &= (x_{q_1}y_{q_2} - y_{q_1}x_{q_2}) \prod_{i \in Q^x} x_i \prod_{i \in Q^y} y_i.
\end{align*}
\]

By Lemma 3.8 it suffices to consider the case that \( q_1 \in \{p_1, p_2\} \). If \( p_1 = q_1 \), then

\[
\text{spol}(g_P, g_Q) = (x_{p_2}x_{q_2} - y_{q_2}y_{p_2})y_{p_1} \prod_{i \in \text{int}(P) \setminus \{q_1\}} y_i \prod_{i \in Q^y \setminus \{p_1\}} x_i.
\]

This \( s \)-polynomial is a monomial multiple of the binomial corresponding to some
\((p_2, q_2)\)-walk, traversing \( p_1 = q_1 \) if necessary. Thus it reduces by Lemma 3.4. The
case that \( p_2 = q_1 \) is similar and omitted. □

*Proof of Theorem 3.5.* According to Lemma 3.9, Lemma 3.10, and Lemma 3.11 the
set \( G_{\prec}(G) \) fulfills Buchberger’s criterion and hence is a Grobner basis of \( \mathcal{I}_G \). By
construction, the elements of \( G_{\prec}(G) \) are reduced with respect to \( \succ \).

Theorem 3.5 implies in particular that parity binomial edge ideals of bipartite
graphs are radical (which they must be by Remark 1.2). This, however, does not
require the square-free initial ideal: if \( \text{char}(\mathbb{k}) \neq 2 \) then all parity binomial edge
ideals are radical by Corollary 5.5.

**Remark 3.12.** The parity binomial edge ideal \( \mathcal{I}_{K_3} \) of the 3-cycle \( K_3 \), cannot have
a square-free initial ideal with respect to any monomial order. This follows from
the fact that \( \mathcal{I}_{K_3} \) is not radical in \( \mathbb{F}_2[x, y] \)—it contains \( x_i^2 - y_i^2 \) but not \( x_i - y_i \), for
i = 1, 2, 3 (see Example 5.1). If \( I_{K_3} \) had a square-free Gröbner basis over some field \( k \), its binomials must be pure-difference (since the generators of \( I_{K_3} \) are pure-difference). The pure-difference property yields that this Gröbner basis would also be a square-free Gröbner basis over every other field, in particular, over \( \mathbb{F}_2 \).

4. Minimal Primes

Generally, the minimal primes of a binomial ideal come in groups corresponding to the sets of indeterminates they contain. To start, we determine the minimal primes of \( I_G \) that contain no indeterminates, that is, the minimal primes of \( J_G \).

Lemma 4.1. Apart from zero rows, the Smith normal form of \( \begin{pmatrix} A_G \\ -A_G \end{pmatrix} \) is the diagonal matrix \( \text{diag}(1, \ldots, 1, 2, \ldots, 2) \) whose number of entries 1 is \( |V(G)| - c(G) \) and the number of entries 2 equals \( c_1(G) \).

Proof. See [7, Theorem 3.3].

The following ideals are the building blocks for the primary decomposition of \( J_G \). For any connected graph \( G \) with an odd cycle, let

\[
p^+(G) = \langle x_i + y_i : i \in V(G) \rangle \quad \text{and} \quad p^-(G) := \langle x_i - y_i : i \in V(G) \rangle.
\]

Proposition 4.2. Let \( G \) be a graph consisting of bipartite connected components \( B_1, \ldots, B_{c_0(G)} \) and non-bipartite connected components \( N_1, \ldots, N_{c_1(G)} \). If \( \text{char}(k) \neq 2 \), then \( J_G \) is radical, and its minimal primes are the \( 2^{c_1(G)} \) ideals

\[
\bigoplus_{i=1}^{c_0(G)} J_{B_i} + \bigoplus_{i=1}^{c_1(G)} p^{\sigma_i}(N_i),
\]

where \( \sigma \) ranges over \( \{+, -\}^{c_1(G)} \). On the other hand, if \( \text{char}(k) = 2 \), then

\[
J_G = \bigoplus_{i=1}^{c_0(G)} J_{B_i} + \bigoplus_{i=1}^{c_1(G)} J_{N_i}
\]

is primary of multiplicity \( 2^{c_1(G)} \) over the minimal prime \( \sum_{i=1}^{c_0(G)} J_{B_i} + \sum_{i=1}^{c_1(G)} p^+(N_i) \).

Proof. Assume first that \( k \) is algebraically closed. According to [5, Corollary 2.2] the primary decomposition \( J_G \) is determined by the saturations of the character that defines the lattice ideal \( J_G \). If a graph is disconnected, then its adjacency matrix has block structure according to the connected components. Therefore it suffices to assume that \( G \) is connected. If \( G \) is bipartite, then Lemma 4.1 and [5, Corollary 2.2] imply that the lattice ideal \( J_G \) is prime. We are thus left with the case that \( G \) is connected and not bipartite.
Assume first that \( \text{char}(k) \neq 2 \). Lemma 4.1 together with [5, Corollary 2.2] shows that \( J_G \) is radical and has two minimal primes. We show that these are \( p^+(G) \) and \( p^-(G) \). The first step is \( J_G \subseteq p^+(G) \) using Proposition 2.7. Let \( i, j \in V(G) \), then
\[
x_i x_j - y_i y_j = x_i \cdot (x_j + y_j) - y_j \cdot (x_i + y_i) \in p^+(G) \quad \text{and} \quad x_i y_j - x_j y_i = x_i \cdot (x_j + y_j) - x_j \cdot (x_i + y_i) \in p^+(G). \]
Similarly, \( J_G \subseteq p^-(G) \). Now let \( p \supseteq J_G \) be a prime ideal. If \( p \) contains \( x_i + y_i \) for all \( i \), then it is either equal to \( p^+(G) \) or not minimal over \( J_G \).
If there exists a vertex \( i \) such that \( x_i + y_i \notin p \), then since \( G \) has an odd cycle and is connected, for any vertex \( j \) there are both an odd and an even \((i, j)\)-walk. Thus
\[
(x_i + y_i) \cdot (x_j - y_j) = x_i x_j - y_i y_j + x_j y_i - x_i y_j \in p.
\]
Since \( p \) is prime, it contains \( x_j - y_j \) for each \( j \) and thus \( p^-(G) \subseteq p \). This shows that \( p^-(G) \) and \( p^+(G) \) are the minimal primes of \( J_G \).

If \( \text{char}(k) = 2 \), then [5, Corollary 2.2] gives that \( J_G \) is primary of multiplicity two over a minimal prime which equals \( p^+(G) = p^-(G) \) by the above computation. It is now evident that the algebraic closure assumption on \( k \) is irrelevant since all saturations of characters are defined over \( k \).

\[\square\]

**Remark 4.3.** The graph \( G \) is bipartite if and only if \( J_G \) is prime.

When decomposing a pure-difference binomial ideal, all components except those over the saturation \( J_G \) contain monomials (for a combinatorial reason see [13, Example 4.14]). Our next step is to determine the indeterminates in the minimal primes. To this end, for any \( S \subseteq V(G) \) let \( G_S \) be the induced subgraph of \( G \) on \( V(G) \setminus S \) and \( m_S := \langle x_s, y_s : s \in S \rangle \).

**Lemma 4.4.** Let \( p \) be a minimal prime of \( I_G \). Then there exists \( S \subseteq V(G) \) and a minimal prime \( p' \) of \( J_{G_S} \) such that \( p = m_S + p' \).

**Proof.** Let \( S := \{ s \in V(G) : x_s, y_s \in p \} \). We first show the inclusions
\[ I_G \subseteq m_S + J_{G_S} \subseteq p. \]
The first inclusion is clear, while for the second, it suffices to check that \( J_{G_S} \subseteq p \).
Generators of \( J_{G_S} \) correspond to \((i, j)\)-walks in \( G_S \) according to Proposition 2.7.
Let \( b \) be the binomial corresponding to any such walk and let \( \{ k_1, \ldots, k_r \} \subseteq V(G) \setminus S \) be its interior. By Lemma 2.4, \( t_{k_1} \cdots t_{k_r} \cdot b \in I_G \subseteq p \) for any choice of indeterminates \( t_{k_l} \in \{ x_{k_l}, y_{k_l} \} \), with \( 1 \leq l \leq r \). By the construction of \( S \), there exists some choice such that \( t_{k_1} \cdots t_{k_r} \notin p \). Since \( p \) is prime, \( b \in p \). The minimal primes of \( m_S + J_{G_S} \) arise as sums of \( m_S \) and minimal primes of \( J_{G_S} \). By minimality, \( p \) equals \( m_S + p' \) for some minimal prime \( p' \) of \( J_G \).

\[\square\]

Not all primes of the form \( m_S + p' \) are minimal over \( I_G \) (see Example 4.10). As for binomial edge ideals, cut points play a crucial role in determining the sets \( S \) which lead to minimal primes, but for parity binomial edge ideals we count connected components differently. The bipartite ones count double.
**Definition 4.5.** Let \( s(G) = c_0(G) + c(G) = 2c_0(G) + c_1(G) \). A set \( S \subseteq V(G) \) is a disconnector of \( G \) if \( s(G_S) > s(G_{S\setminus\{s\}}) \) for every \( s \in S \).

**Remark 4.6.** The empty set is a disconnector of any graph and disconnectors cannot contain isolated vertices.

**Remark 4.7.** If a graph \( G \) has no isolated vertices, then \( s(G_{\{s\}}) \geq s(G) \) for all \( s \in V(G) \) and according to Definition 4.5 a vertex \( s \) is a disconnector of \( G \) exactly if the inequality is strict. Moreover, one can conclude from the following proposition that \( s \) is a disconnector of \( G \) if and only if \( J_G \not\subseteq m_{\{s\}} + J_{G_{\{s\}}} \).

**Proposition 4.8.** Let \( G \) be a graph and \( S \subseteq V(G) \). Then \( J_G \subseteq m_S + J_{G_S} \) if and only if for all \((i, j)\)-walks in \( G \) with \( i, j \in V(G_S) \), there is an \((i, j)\)-walk in \( G_S \) of the same parity.

*Proof.* Let \( J_G \subseteq m_S + J_{G_S} \). Let \( m \in J_G \) be a Graver move corresponding to an \((i, j)\)-walk in \( G \) with \( i, j \notin S \). Since \( m \in k[x_i, x_j, y_i, y_j] \), and no polynomial in \( J_{G_S} \) uses indeterminates from \( S \), we find \( m \in J_{G_S} \). It follows that \( m \) is an element of the Graver basis of \( J_{G_S} \) and thus corresponds to an \((i, j)\)-walk in \( G_S \) of the same parity.

On the other hand, let \( m \in J_G \) be a move corresponding to a \((i, j)\)-walk in \( G \). If \( i \in S \) or \( j \in S \), then \( m \in m_S \). If otherwise \( i, j \in V(G_S) \), then \( m \in J_{G_S} \) by assumption. \( \square \)

The next lemma states that the indeterminates contained in a minimal prime correspond to a disconnector of \( G \), and Theorem 4.15 below says when the converse is true as well.

**Lemma 4.9.** Let \( p \) be a minimal prime of \( I_G \). There exists a disconnector \( S \subseteq V(G) \) of \( G \) and a minimal prime \( p' \) of \( J_{G_S} \) such that \( p = m_S + p' \).

*Proof.* Let \( S \) and \( p' \) be as in Lemma 4.4. We prove that \( S \) is a disconnector. Assume the converse, i.e., there exists \( s \in S \) such that \( \{s\} \) is not a disconnector of \( G_{S\setminus\{s\}} \).

According to Remark 4.7 and Proposition 4.8

\[
J_{G_{S\setminus\{s\}}} \subseteq m_{\{s\}} + J_{G_S} \subseteq m_{\{s\}} + p'.
\]

Hence, since the ideal on the right-hand side is prime, choose a minimal prime \( p'' \) of \( J_{G_{S\setminus\{s\}}} \) such that \( J_{G_{S\setminus\{s\}}} \subseteq p'' \subseteq m_{\{s\}} + p' \). This give rise to

\[
I_G \subseteq m_{S\setminus\{s\}} + p'' \subseteq m_S + p' = p
\]

which contradicts the minimality of \( p \). \( \square \)
Let $S \subseteq V(G)$ be a disconnector of $G$. The induced subgraph $G_S$ splits into bi-partite components $B_1, \ldots, B_{c_0(G_S)}$ and non-bipartite components $N_1, \ldots, N_{c_1(G_S)}$. By Proposition 4.2 the minimal primes of $J_{G_S}$ are

$$p = \sum_{i=1}^{c_0(G_S)} J_{B_i} + \sum_{i=1}^{c_1(G_S)} p^{\sigma_i}(N_i),$$

where

$$\sigma_i \in \{+, -\}, \quad \text{if } \text{char}(k) \neq 2,$$

$$\sigma_i = +, \quad \text{if } \text{char}(k) = 2.$$

Not all of these primes lead to minimal primes of $I_G$ because of the following effect.

**Example 4.10.** Let $G$ be the graph in Figure 2. The vertex 4 is a disconnector and $G_{\{4\}}$ consists of the two triangles $N_1 = \{1, 2, 3\}$ and $N_2 = \{5, 6, 7\}$. Choosing for both triangles the positive sign component, we obtain the prime ideal

$$m_{\{4\}} + p^+(N_1) + p^+(N_2) = m_{\{4\}} + \langle x_i + y_i : i \in [7] \setminus \{4\} \rangle$$

which is not minimal over $I_G$ since it contains the prime ideal $p^+(G)$. On the other hand, both ideals with the binomial part $m_{\{4\}} + p^+(N_1) + p^-(N_2)$, each having different signs on the triangles, are minimal over $I_G$.

A combinatorial condition on $\sigma$ in (4.1) guarantees that a minimal prime of $J_{G_S}$ is the binomial part of a minimal prime of $I_G$ (the monomial part being $m_S$). To see it, let $s \in S$ be such that $c(G_S) > c(G_{S\setminus\{s\}})$, i.e., when adding $s$ back to $G_S$ some of its connected components are joined. Denote by $C_{G_S}(s)$ the set of only those connected components of $G_S$ which are joined when adding $s$.

**Definition 4.11.** Let $S \subseteq V(G)$ be a disconnector of $G$. A minimal prime $p$ of $J_{G_S}$ is sign-split if for all $s \in S$ such that $C_{G_S}(s)$ contains no bipartite graphs, the prime summands of $p$ corresponding to connected components in $C_{G_S}(s)$ are not all equal to $p^+$ or all equal to $p^-$.

**Remark 4.12.** If $C_{G_S}(s)$ contains at least one bipartite graph, then Definition 4.11 imposes no restriction and every choice of prime summands is sign-split.

**Remark 4.13.** If char($k$) = 2, then all signs $\sigma$ in (4.1) are fixed. In this case Definition 4.11 can only be satisfied if $C_{G_S}(s)$ contains a bipartite component for each $s \in S$.
Example 4.14. Not every disconnector $S \subseteq V(G)$ of $G$ admits a sign-split minimal prime for $\mathcal{J}_{G_S}$, and thus not every disconnector contributes minimal primes to $\mathcal{I}_G$. Consider the graph in Figure 3. The set of blue square vertices is a disconnector that does not contribute minimal primes. Adding one of the squares back yields the requirement that the primes on the two now connected triangles have different signs, but these three requirements cannot be satisfied simultaneously.

![Figure 3. A disconnector whose binomial parts cannot be sign-split.](image)

**Theorem 4.15.** The minimal primes of $\mathcal{I}_G$ are the ideals $\mathfrak{m}_S + \mathfrak{p}$ where $S \subseteq V(G)$ is a disconnector of $G$ and $\mathfrak{p}$ is a sign-split minimal prime of $\mathcal{J}_{G_S}$.

**Proof.** According to Lemma 4.9, all minimal primes of $\mathcal{I}_G$ have the form $\mathfrak{m}_S + \mathfrak{p}$ where $S \subseteq V(G)$ is a disconnector and $\mathfrak{p}$ is a minimal prime of $\mathcal{J}_{G_S}$. We first show that if $\mathfrak{p}$ is sign-split, this ideal is minimal over $\mathcal{I}_G$. Assume not, then by Lemma 4.4 there exists a set $T \subseteq V(G)$ and a minimal prime $\tilde{\mathfrak{p}}$ of $\mathcal{J}_{G_T}$ such that

\begin{equation}
\mathcal{I}_G \subseteq \mathfrak{m}_T + \tilde{\mathfrak{p}} \subseteq \mathfrak{m}_S + \mathfrak{p}.
\end{equation}

This implies $T \subset S$, since if $T = S$ then, by Lemma 4.4 also $\tilde{\mathfrak{p}} = \mathfrak{p}$. Let $s' \in S \setminus T$, then $G_S \subseteq G_{S \setminus \{s'\}} \subseteq G_T$. Since $s'$ is a disconnector of $G_{S \setminus \{s'\}}$, $s(G_S) > s(G_{S \setminus \{s'\}})$.

Let again $\mathcal{C}_{G_S}(s')$ be the set of connected components in $G_S$ that are joined to $s'$ in $G_{S \setminus \{s'\}}$. If $\mathcal{C}_{G_S}(s')$ contains at least one bipartite component, adding $s'$ to $G_S$ either this component becomes non-bipartite in $G_{S \setminus \{s'\}}$ or it is joined to another bipartite component of $G_S$. In the first case, let $B$ be a bipartite component which becomes non-bipartite. There exists $i \in V(B)$ such that $x_i^2 - y_i^2 \in \mathcal{J}_{G_{S \setminus \{s'\}}} \subseteq \mathcal{J}_{G_T} \subseteq \tilde{\mathfrak{p}}$, but $x_i^2 - y_i^2 \notin \mathcal{J}_B$. Since $\mathcal{J}_B$ is a summand of $\mathfrak{p}$, $x_i^2 - y_i^2 \notin \mathfrak{m}_S + \mathfrak{p}$, in contradiction to (4.2).

In the second case, let $B_1$ and $B_2$ be the bipartite components of $G_S$ which are joined to $s'$. There are $i_1 \in V(B_1)$ and $i_2 \in V(B_2)$ such that there exists an $(i_1, i_2)$-walk in $G_{S \setminus \{s'\}}$. Independent of the parity of this walk, the corresponding Markov move is not contained in $\mathcal{J}_{B_1} + \mathcal{J}_{B_2}$ since there is no applicable move from the Graver basis. Since $\mathcal{J}_{B_1}$ and $\mathcal{J}_{B_2}$ are summands of $\mathfrak{p}$ involving the indeterminates $i_1$ and $i_2$, there is a binomial which is not in $\mathfrak{m}_S + \mathfrak{p}$ but in $\mathcal{J}_{G_{S \setminus \{s'\}}} \subseteq \tilde{\mathfrak{p}}$ contradicting (4.2).

Assume now that all components in $\mathcal{C}_{G_S}(s')$ are non-bipartite (there are at least two of them since $s'$ is a disconnector). By assumption, $\mathfrak{p}$ is sign-split, i.e., there exist distinct components $N_1, N_2 \in \mathcal{C}_{G_S}(s)$ such that $\mathfrak{p}^+(N_1)$ and $\mathfrak{p}^-(N_2)$ are summands of $\mathfrak{p}$. There is an odd walk from a vertex $i_1 \in V(N_1)$ to a vertex $i_2 \in V(N_2)$
in $G_{S\setminus\{s\}}$ and therefore $x_i x_2 - y_i y_2 \in \mathcal{J}_{G_{S\setminus\{s\}}} \subseteq \mathfrak{p}$. Since
$$x_i x_2 - y_i y_2 \not\in \mathfrak{p}^+(N_1) + \mathfrak{p}^-(N_2),$$
also $x_i x_2 - y_i y_2 \not\in \mathfrak{p}$. By construction, $i_1, i_2 \not\in S$ and thus
$$x_i x_2 - y_i y_2 \not\in \mathfrak{m}_S + \mathfrak{p}$$
which contradicts (4.2). This shows minimality of $\mathfrak{m}_S + \mathfrak{p}$.

Let now $\mathfrak{m}_S + \mathfrak{p}$ be a minimal prime of $\mathcal{I}_G$. The set $S$ is a disconnector by Lemma 4.9 and thus it remains to prove that $\mathfrak{p}$ is sign-split. To the contrary, assume there is a vertex $s \in S$ with $c(G_{S\setminus\{s\}}) > c(G_S)$ such that $\mathcal{C}_{G_S}(s) = \{N_1, \ldots, N_k\}$ consists exclusively of non-bipartite components, $k \geq 2$, and all summands of $\mathfrak{p}$ corresponding to $N_i$ have the same sign, say +. When adding $s$ back to $G_S$, the components in $\mathcal{C}_{G_S}(s)$ are joined to a single, non-bipartite connected component $H$ in $G_{S\setminus\{s\}}$, whereas all other components of $G_S$ coincide with connected components of $G_{S\setminus\{s\}}$. Since
$$\mathfrak{p}^+(H) = \mathfrak{m}_S + \sum_{i=1}^k \mathfrak{p}^+(N_i) \subseteq \mathfrak{m}_S + \sum_{i=1}^k \mathfrak{p}^+(N_i),$$
choosing on all other components of $G_{S\setminus\{s\}}$ the same prime component as in $G_S$, we obtain a prime ideal that is strictly smaller than $\mathfrak{m}_S + \mathfrak{p}$.

**Remark 4.16.** Example 4.10 and Definition 4.11 are valid independent of char($k$). In the above proof, the case of char($k$) = 2 could be simplified, but everything works in general without the need for a case distinction.

5. Radicality and mesoprimary decomposition

The intersection of the minimal primes of $\mathcal{I}_G$ depends on char($k$) so that we do not attempt to compute it directly. Theorem 5.5 below says that $\mathcal{I}_G$ is radical exactly if the characteristic is not two. Here is the principal source of field dependence.

**Example 5.1.** The parity binomial edge ideal of the triangle $K_3$ is not radical in characteristic two. According to Lemma 2.4, $((x_3 - y_3) y_1 y_2)^2 = (x_3^2 - y_3^2) y_1^2 y_2^2 \in \mathcal{I}_{K_3}$ while $(x_3 - y_3) y_1 y_2 \not\in \mathcal{I}_{K_3}$.

**Remark 5.2.** The ideal $\mathcal{I}_G$ is homogeneous with respect to the multigrading $\text{deg}(x_i) = \text{deg}(y_i) = e_i$, where $e_i$ is the $i$-th standard basis vector of $\mathbb{R}^{V(G)}$.

**Lemma 5.3.** Let $i \in V(G)$ and $m \in \mathcal{I}_G + \mathfrak{m}_{\{i\}}$ be a monomial. Then $m \in \mathfrak{m}_{\{i\}}$.

**Proof.** Since it is generated by pure difference binomials, $\mathcal{I}_G$ does not contain any monomials. Thus any monomial in $\mathcal{I}_G + \mathfrak{m}_{\{i\}}$ is equivalent to one in $\mathfrak{m}_{\{i\}}$ modulo term replacements using binomials in $\mathcal{I}_G$, but these do not change membership in $\mathfrak{m}_{\{i\}}$ by Remark 5.2. □
Proposition 5.4. For any graph $G$, $\mathcal{I}_G = \mathcal{J}_G \cap \bigcap_{i \in V(G)} (\mathcal{I}_G + \mathfrak{m}_{(i)})$.

Proof. According to [5, Corollary 1.5] the intersection is binomial. Let $b$ be any binomial in the intersection. For each $i \in V(G)$, there are three cases: either no term of $b$ is individually contained in $\mathcal{I}_G + \mathfrak{m}_{(i)}$, exactly one is, or both are. In the first case [5, Proposition 1.10] implies $b \in \mathcal{I}_G$. In the second case it implies that the other monomial is contained in $\mathcal{I}_G$ which is impossible. Thus it suffices to consider binomials $b$ both of whose monomials are contained in $\mathcal{I}_G + \mathfrak{m}_{(i)}$ for all $i \in V(G)$. By Lemma 5.3 both monomials of $b$ are contained in $\mathfrak{m}_{(i)}$ for each $i \in V(G)$. Since $b \in \mathcal{J}_G$, there exist Markov moves $m_{s_1t_1}, \ldots, m_{s_r t_r}$ corresponding to $(s_1, t_1), \ldots, (s_r, t_r)$-walks, respectively, such that

$$b = x^{h_1}y^{h'_1}m_{s_1t_1} + \cdots + x^{h_r}y^{h'_r}m_{s_r t_r}$$

with $h_i, h'_i \in \mathbb{N}^n$. We can assume that one monomial of $b$ equals one of the monomials of $x^{h_1}y^{h'_1}m_{s_1t_1}$. Thus both monomials of $x^{h_1}y^{h'_1}m_{s_1t_1}$ are divisible by at least one indeterminate for each $i \in V(G)$ and, by Lemma 2.4, $x^{h_1}y^{h'_1}m_{s_1t_1} \in \mathcal{I}_G$. Replacing $b$ by $b - x^{h_1}y^{h'_1}m_{s_1t_1}$ and iterating the argument eventually yields $b \in \mathcal{I}_G$. □

Theorem 5.5. Let $G$ be a graph. If $\text{char}(\mathbb{k}) \neq 2$, then $\mathcal{I}_G$ is a radical ideal.

Proof. The proof is by induction on the number of vertices of $G$. If $G$ has at most one vertex, then $\mathcal{I}_G = 0$ and the claim holds. Remark 5.2 shows that $\mathcal{I}_G + \mathfrak{m}_{(i)} = \mathcal{I}_{G_{(i)}} + \mathfrak{m}_{(i)}$ for all $i \in V(G)$. Thus Theorem 5.4 reads as $\mathcal{I}_G = \mathcal{J}_G \cap \bigcap_{i=1}^n (\mathcal{I}_{G_{(i)}} + \mathfrak{m}_{(i)})$. By the induction hypothesis, $\mathcal{I}_{G_{(i)}}$ is radical and thus $\mathcal{I}_{G_{(i)}} + \mathfrak{m}_{(i)}$ is radical. Remark 4.3 says that $\mathcal{J}_G$ is radical if and only if $\text{char}(\mathbb{k}) \neq 2$ which yields the result. □

Theorem 5.9 below contains a primary decomposition of $\mathcal{I}_G$ in the case $\text{char}(\mathbb{k}) = 2$. It uses the following lemma, which allows to transport decompositions between different characteristics. Recall that the combinatorics of any binomial ideal $I$ is encoded in its congruence $\sim_I$ which identifies monomials $m_1, m_2$, whenever $m_1 - \lambda m_2 \in I$ for some non-zero $\lambda \in \mathbb{k}$. A binomial ideal is unital if it is generated by monomials and pure differences of monomials. Then each congruence is the congruence of a unital binomial ideal, though not uniquely.

Lemma 5.6. If a decomposition $I = J_1 \cap \cdots \cap J_s$ of a unital binomial ideal $I$ into unital binomial ideals $J_i$, $i = 1, \ldots, s$ is valid in some characteristic, then it is valid in any characteristic.

Proof. The congruence $\sim_I$ induced by $I$ is the common refinement of the congruences $\sim_{J_i}$ induced by the $J_i$, $i = 1, \ldots, s$. Thus, in any characteristic, [13, Theorem 9.12] implies that $I$ and $J_1 \cap \cdots \cap J_s$ can only differ if one of them contains monomials, but the other does not. This cannot happen since unital binomial ideals contain monomials if and only if they have monomials among the generators. □
According to Example 4.14, not all disconnectors contribute minimal primes. From Definition 4.11 it may seem that this is an arithmetic effect. It is not; the primary decomposition of \( I_G \) in characteristic two also witnesses it. For the following definition, recall that a hypergraph is \( k \)-colorable if the vertices can be colored with \( k \) colors so that no edge is monochromatic.

**Definition 5.7.** Let \( S \subseteq V(G) \) be a disconnector and let \( s_1, \ldots, s_r \in S \) be the vertices such that \( C_{G_S}(s_i) \) consists exclusively of non-bipartite components of \( G_S \). Let \( H \) be the hypergraph whose vertex set consists of the connected components \( C_{G_S}(s_1) \cup \ldots \cup C_{G_S}(s_r) \) and with edge set \( \{ C_{G_S}(s_1), \ldots, C_{G_S}(s_r) \} \). The disconnector \( S \) is effective if \( H \) is 2-colorable.

**Remark 5.8.** A disconnector is effective if and only if, in characteristic zero, it admits sign-split minimal primes.

**Theorem 5.9.** Let \( S \) be the set of effective disconnectors of \( G \). Then

\[
I_G = \bigcap_{S \in S} (m_S + J_{G_S}).
\]

If \( \text{char}(k) = 2 \), then (5.1) is a primary decomposition of \( I_G \).

**Proof.** For each disconnector \( S \in S \), let \( B^S_1, \ldots, B^S_{c_0(G_S)} \) be the bipartite components and \( N^S_1, \ldots, N^S_{c_1(G_S)} \) the non-bipartite components of \( G_S \). Let \( \Sigma^S \subseteq \{+, -\}^{c_1(G_S)} \) denote the set of sign patterns that are sign-split. In characteristic zero, by Theorems 4.15 and 5.5, \( I_G \) decomposes as

\[
I_G = \bigcap_{S \in S} \bigcap_{\sigma \in \Sigma^S} \left( m_S + \sum_{i=1}^{c_0(G_S)} J_{B^S_i} + \sum_{i=1}^{c_1(G_S)} p^{\sigma_i}(N^S_i) \right).
\]

The intersection remains valid when intersecting over additional ideals containing \( I_G \). In particular, the sign-split requirement can be dropped and \( \Sigma^S \) replaced by \( \{+, -\}^{c_1(G_S)} \). Carrying out this inner intersection yields the ideals \( m_S + J_{G_S} \) by Proposition 4.2, and hence (5.1) is valid in characteristic zero. Since all involved ideals are unital, Lemma 5.6 yields that (5.1) is valid in any characteristic. The ideals under consideration are primary if \( \text{char}(k) = 2 \) according to Proposition 4.2 and thus the second statement follows. □

The technique of adding “phantom components” to a primary decomposition so that it faithfully exists over some other field (as we did in the proof of Theorem 5.9) was mentioned in the introduction of [13] as one way to arrive at more combinatorially accurate decompositions of binomial ideals. The upshot of the rest of the paper is that the primary decomposition in characteristic two is an example of a mesoprimary decomposition. This means not only that all ideals in (5.1) are mesoprimary, but additionally each of the intersectands witnesses a combinatorial
feature of the graph that the binomials of $\mathcal{I}_G$ induce on the monomials of $\mathbb{k}[x, y]$. Generally it can be quite challenging to determine a mesoprimary decomposition, exactly because of the stringent combinatorial conditions that it has to meet. Here it is mostly a translation of the (involved) definitions, essentially because all ideals are unital and the ambient ring is the polynomial ring $[13, \text{Remark 12.8}]$. We refrain from introducing too much of the machinery from $[13]$ here, but do employ their notation. In the following we give explicit references to all relevant definitions.

**Theorem 5.10.** The decomposition (5.1) is a mesoprimary decomposition of $\mathcal{I}_G$.

**Proof.** For $S \subseteq \mathcal{S}$, let $J_S := \mathfrak{m}_S + \mathcal{J}_{GS}$ be the intersectand corresponding to $S$. The ideal $J_S$ is $P_S$-mesoprimary where $P_S \subseteq \mathbb{N}^{2|V(G)|}$ is the monoid prime ideal $\langle e_{x_i}, e_{y_i} : i \in S \rangle$. In fact (like any ideal that equals a lattice ideal plus monomials in a disjoint set of indeterminates), $J_S$ is mesoprime $[13, \text{Definition 10.4}]$ since it equals the kernel of the monomial homomorphism

$$\mathbb{k}[x, y] \rightarrow \mathbb{k}[x_i^\pm, y_i^\pm : i \notin S]/\mathcal{J}_{GS} = \mathbb{k}[\mathbb{Z}^{2|V(G)\setminus S}]/L_S$$

which maps $x_i, y_i$ to zero if $i \in S$ and to their images in the Laurent ring if $i \notin S$. Here $L_S$ is the image in $\mathbb{Z}^{2|V(G)\setminus S}$ of the adjacency matrix of $G_S$.

According to $[13, \text{Definition 13.1}]$, it remains to show that at each cogenerator $[13, \text{Definitions 7.1 and 12.16}]$ of $J_S$, the $P_S$-mesoprimes of $\mathcal{I}_G$ and $J_S$ agree. Let $J_S^\pm$ be the image of $J_S$ in $R^S = \mathbb{k}[x_i, y_i, i \in S, x_j^\pm, y_j^\pm, j \notin S]$. The cogenerators of $J_S$ are monomials in $\mathbb{k}[x, y]$ whose images in $R_S/J_S^\pm$ are annihilated by $\mathfrak{m}_S$. Since $J_S$ contains $\mathfrak{m}_S$, the cogenerators are simply all monomials in the indeterminates $x_i, y_i$ for $i \notin S$. Now the $P_S$-mesoprime of the mesoprime $J_S$ (at any monomial) is just $J_S$. Thus it remains to compute the $P_S$-mesoprime of $\mathcal{I}_G$ at any cogenerator. Translating $[13, \text{Definition 11.1}]$ to $\mathbb{k}[x, y]$, this mesoprime is given by $(\mathcal{I}_G + \mathfrak{m}_S) : (\prod_{t \in S} x_i y_i)^\infty$. The result now follows by Lemma 5.3. \qed

The stringent combinatorial conditions that guarantee a canonical mesoprimary decomposition require additional knowledge about the witness structure of $\mathcal{I}_G$.

**Conjecture 5.11.** The mesoprimary decomposition in (5.1) is combinatorial and characteristic.

To prove Conjecture 5.11 one needs precise control over the various witnesses that contribute to coprincipal decompositions $[13, \text{Theorems 8.4 and 16.9}]$. Experiments with MACAULAY2 indicate that the mesoprimary decomposition of the congruence $\sim_{\mathcal{I}_G}$ differs significantly from that of the ideal $\mathcal{I}_G$. For example, if $G$ is a path $G = 1 - 2 - 3 - 4 - 5$, then $\mathcal{I}_G$ has the following mesoprimary decomposition:

$$\mathcal{I}_G = \mathcal{J}_G \cap (\mathfrak{m}_{\{4\}} + \mathcal{J}_{1-2-3}) \cap (\mathfrak{m}_{\{2\}} + \mathcal{J}_{3-4-5}) \cap (\mathfrak{m}_{\{3\}} + \mathcal{I}_{1-2} \cap \mathcal{I}_{4-5}) \cap \mathfrak{m}_{\{2,4\}}.$$
Intersecting all but the last ideal yields the ideal

\[ \mathcal{I}_G + \langle x_1 x_3 y_3 y_5 - x_1 x_5 y_3^2 - x_3^2 y_1 y_5 + x_3 x_5 y_1 y_3 \rangle. \]

This ideal has the same binomials as \( \mathcal{I}_G \) and thus induces the same congruence. The monomial ideal that was omitted does not influence the congruence. Its sole purpose is to cut away non-binomials. The monoid prime \( \langle e_{x_2}, e_{y_2}, e_{x_4}, e_{y_4} \rangle \subseteq \mathbb{N}^{2|V(G)|} \) contributes only non key witnesses to the case. Nevertheless these witnesses are essential in the sense of [13, Definition 12.1].

References

1. Dave Bayer, Sorin Popescu, and Bernd Sturmfels, *Syzygies of unimodular Lawrence ideals*, Journal für die Reine und Angewandte Mathematik 534 (2001), 169–186.
2. Carsten Conradi and Thomas Kahle, *Detecting binomiality*, preprint, arXiv:1502.04893 (2015).
3. Jesús A. De Loera, Raymond Hemmecke, and Matthias Köppe, *Algebraic and geometric ideas in the theory of discrete optimization*, MPS-SIAM Series on Optimization, SIAM, 2013.
4. Mathias Drton, Bernd Sturmfels, and Seth Sullivant, *Lectures on algebraic statistics*, Oberwolfach Seminars, vol. 39, Springer, Berlin, 2009, A Birkhäuser book.
5. David Eisenbud and Bernd Sturmfels, *Binomial ideals*, Duke Mathematical Journal 84 (1996), no. 1, 1–45.
6. Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.
7. Jerrold W. Grossman, Devadatta M. Kulkarni, and Irwin E. Schochetman, *On the minors of an incidence matrix and its Smith normal form*, Linear Algebra and its Applications 218 (1995), no. 1995, 213–224.
8. Jürgen Herzog, Takayuki Hibi, Freyja Hreinsson, Thomas Kahle, and Johannes Rauh, *Binomial edge ideals and conditional independence statements*, Advances in Applied Mathematics 45 (2010), no. 3, 317–333.
9. Jürgen Herzog, Antonio Macchia, Sara Saeedi Madani, and Volkmar Welker, *On the ideal of orthogonal representations of a graph in \( \mathbb{R}^2 \)*, preprint, arXiv:1411.3674 (2014).
10. Serkan Hoşten and Jay Shapiro, *Primary decomposition of lattice basis ideals*, Journal of Symbolic Computation 29 (2000), no. 4-5, 625–639.
11. Serkan Hoşten and Seth Sullivant, *Ideals of adjacent minors*, Journal of Algebra 277 (2004), no. 2, 615–642.
12. Thomas Kahle, *Decompositions of binomial ideals*, Journal of Software for Algebra and Geometry 4 (2012), 1–5.
13. Thomas Kahle and Ezra Miller, *Decompositions of commutative monoid congruences and binomial ideals*, Algebra and Number Theory 8 (2014), no. 6, 1297–1364.
14. Thomas Kahle, Johannes Rauh, and Seth Sullivant, *Positive margins and primary decomposition*, Journal of Commutative Algebra 6 (2014), no. 2, 173–208.
15. Ezra Miller, *Theory and applications of lattice point methods for binomial ideals*, Proceedings of the Abel Symposium held at Voss, Norway, 2009, Springer, 2011, pp. 99–154.
16. Ezra Miller, *Affine stratifications from finite misère quotients*, Journal of Algebraic Combinatorics 37 (2013), no. 1, 1–9.
17. Mercedes Pérez Millán, Alicia Dickenstein, Anne Shiu, and Carsten Conradi, *Chemical reaction systems with toric steady states*, Bulletin of Mathematical Biology 74 (2012), 1027–1065.
18. Johannes Rauh and Seth Sullivant, *Lifting Markov bases and higher codimension toric fiber products*, preprint, arXiv:1404.6392 (2014), 1–36.

19. Irena Swanson, *On the embedded primes of the Mayr-Meyer ideals*, Journal of Algebra **275** (2004), 143–190.

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