Normal Subgroups of Profinite Groups of Non-negative Deficiency

Fritz Grunewald, Andrei Jaikin-Zapirain, Aline G.S. Pinto & Pavel A. Zalesski

October 11, 2008

Abstract

We initiate the study of profinite groups of non-negative deficiency. The principal focus of the paper is to show that the existence of a finitely generated normal subgroup of infinite index in a profinite group $G$ of non-negative deficiency gives rather strong consequences for the structure of $G$. To make this precise we introduce the notion of $p$-deficiency ($p$ a prime) for a profinite group $G$. This concept is more useful in the study of profinite groups than the notion of deficiency. We prove that if the $p$-deficiency of $G$ is positive and $N$ is a finitely generated normal subgroup such that the $p$-Sylow subgroup of $G/N$ is infinite and $p$ divides the order of $N$ then we have $\text{cd}_p(G) = 2$, $\text{cd}_p(N) = 1$ and $\text{vcd}_p(G/N) = 1$ for the cohomological $p$-dimensions; moreover either the $p$-Sylow subgroup of $G/N$ is virtually cyclic or the $p$-Sylow subgroup of $N$ is cyclic. A profinite Poincaré duality group $G$ of dimension 3 at a prime $p$ ($PD^2$-group) has deficiency 0. In this case we show that for $N$ and $p$ as above either $N$ is $PD^1$ at $p$ and $G/N$ is virtually $PD^2$ at $p$ or $N$ is $PD^2$ at $p$ and $G/N$ is virtually $PD^1$ at $p$. In particular if $G$ is pro-$p$ then either $N$ is infinite cyclic and $G/N$ is virtually Demushkin or $N$ is Demushkin and $G/N$ is virtually infinite cyclic. We apply this results to deduce structural information on the profinite completions of ascending HNN-extensions of free groups. We also give some implications of our theory to the congruence kernels of certain arithmetic groups.

2000. Mathematics Subject Classification Primary: 20E18.
1 Introduction

If a connected compact manifold $M$ admits a fibration over a compact base manifold $B$ with a compact manifold $F$ as a fiber its fundamental group $G = \pi_1(M)$ satisfies an exact sequence

$$\langle 1 \rangle \to N \to G = \pi_1(M) \to G/N \to \langle 1 \rangle$$

(1.1)

where $N$, being an image of $\pi_1(F)$, is a normal subgroup which is finitely generated as a group. Moreover there are many interesting situations where the quotient $G/N$ is the infinite cyclic group $\mathbb{Z}$. This for example happens when $M$ fibers over the circle. Our general intuition tells us that only somewhat special groups should have a finitely generated normal subgroup of infinite index. In fact J. Hempel and W. Jaco have proved
Theorem 1.1. (Hempel, Jaco [13]) Let $G$ be the fundamental group of compact 3-manifold $M^3$ (possibly with boundary) and $N$ a finitely generated normal subgroup of $G$ of infinite index. Then $N$ is isomorphic to the fundamental group of a compact, possibly bounded 2-manifold. In case $N$ is not infinite cyclic then the quotient $G/N$ has an infinite cyclic group of finite index.

Furthermore, Hempel and Jaco add some results on the geometric structure of the manifold $M^3$. The group theoretic result in Theorem 1.1 obtained a group theoretic proof by J. Hillman in [14] which also provided an important generalization. In fact J. Hillman proved Theorem 1.1 for Poincaré duality groups $G$ of dimension 3. This generalizes the case of a compact 3-manifold $M^3$ without boundary. Hillman [14] was also able to give the cases where $M^3$ is a compact 3-manifold with boundary an appropriate generalization.

In this paper we
- prove results analogous to Theorem 1.1 for profinite groups,
- study applications of our profinite results to discrete groups.

We for example establish in Section 4 the following profinite version of the result of Hillman:

**Theorem 1.2.** Let $G$ be a profinite $PD^3$-group at a prime $p$ and $N$ be a finitely generated normal subgroup of $G$ such that the $p$-Sylow $(G/N)_p$ is infinite and $p$ divides $|N|$. Then either $N$ is $PD^1$ at $p$ and $G/N$ is virtually $PD^2$ at $p$ or $N$ is $PD^2$ at $p$ and $G/N$ is virtually $PD^1$ at $p$.

The pro-$p$ version of this theorem as well as Theorem 1.1 reads as follows:

**Corollary 1.3.** Let $G$ be a pro-$p$ $PD^3$-group and $N$ be a finitely generated normal subgroup of $G$. Then either $N$ is infinite cyclic and $G/N$ is virtually Demushkin or $N$ is Demushkin and $G/N$ is virtually infinite cyclic.

To explain the content of this paper in more detail we have to introduce some concepts from group theory. The **deficiency** $\text{def}(G)$ of a group $G$ is the largest integer $k$ such that there exist a (finite) presentation of $G$ with the number of generators minus the number of relations equal to $k$. The groups of non-negative deficiency form an important class of finitely presented groups. It contains many important families of examples coming from geometry: fundamental groups of compact 3-manifolds, knot groups, arithmetic groups of rank 1, etc.
In this paper we initiate the study of profinite groups of non-negative deficiency. The deficiency for profinite groups is defined in the same way as for discrete groups. We consider presentations in the category of profinite groups, i.e., saying generators we mean topological generators. We note that any presentation of a group is a presentation of its profinite completion and so the profinite completion of a group of non-negative deficiency is a profinite group of non-negative deficiency. Therefore, the profinite completions of groups mentioned above are in the range of our study. Moreover, as was shown by Lubotzky [30, Corollary 1.2], all projective groups (i.e. profinite groups of cohomological dimension 1) have non-negative deficiency as well.

The principal focus of our study here is to show that the existence of a finitely generated normal subgroup of infinite index in a profinite group $G$ of non-negative deficiency gives rather strong consequences for the structure of $G$. We are ready to state the principal results for groups of positive deficiency.

**Theorem 1.4.** Let $G$ be a finitely generated profinite group with positive deficiency and $N$ a finitely generated normal subgroup such that the $p$-Sylow subgroup $(G/N)_p$ is infinite and $p$ divides the order of $N$. Then either the $p$-Sylow subgroup of $G/N$ is virtually cyclic or the $p$-Sylow subgroup of $N$ is cyclic. Moreover, $\text{cd}_p(G) = 2$, $\text{cd}_p(N) = 1$ and $\text{vcd}_p(G/N) = 1$, where $\text{cd}_p$ and $\text{vcd}_p$ stand for cohomological $p$-dimension and virtual cohomological $p$-dimension respectively.

To prove this theorem we introduce in Section 2.2 the concept of $p$-deficiency $\text{def}_p(G)$ for a prime $p$ and a profinite group $G$. These new invariants are more suitable to the study of profinite groups than just the deficiency. The example of pro-$p$ groups (which all have positive deficiency as profinite groups) already makes it clear that our result requires a more subtle approach. Section 2.2 contains also results interconnecting the deficiency of a profinite group with its various $p$-deficiencies.

Theorem 1.4 has the following immediate consequence.

**Corollary 1.5.** Let $G$ be a finitely generated profinite group of positive deficiency and $N$ a finitely generated normal subgroup of $G$ such that the $p$-Sylow subgroup $(G/N)_p$ is infinite whenever the prime $p$ divides $|N|$. Then $N$ is projective.

The abstract version of Theorem 1.4 is known only under further restrictions either on $N$ or on $G/N$ (see [1]) and its pro-$p$ version is proved in the paper [16] of Hillmann and Schmidt.
A group $G$ is called knot-like if $G/[G, G]$ is infinite cyclic and the deficiency satisfies $\text{def}(G) = 1$. These two properties are possessed by any knot group, i.e., the fundamental group of the complement of a knot in the 3-sphere $S^3$. It was conjectured by E. Rapaport-Strasser in [37] that if the commutator group $G' = [G, G]$ of a knot-like group $G$ is finitely generated then $G'$ should be free. This conjecture is true as it was recently proved by D.H. Kochloukova in [20].

The next corollary shows that the profinite version of the Rapaport-Strasser conjecture is also true. In fact our result is stronger then just the profinite version of the conjecture since we do not assume $G/[G, G]$ to be cyclic and assume positive deficiency rather than for deficiency one.

**Corollary 1.6.** Let $G$ be a finitely generated profinite group of positive deficiency whose commutator subgroup $[G, G]$ is finitely generated. Then $\text{def}(G) = 1$ and $[G, G]$ is projective. Moreover, $\text{cd}(G) = 2$ unless $G = \hat{\mathbb{Z}}$.

We observe that any pro-$p$ group has non-positive deficiency as a profinite group (see [30]). However it is worthwhile to note that a knot-like pro-$p$ group, i.e., a pro-$p$ group with infinite cyclic abelianization and deficiency equal to one as a pro-$p$ group, is generated by one element modulo its Frattini subgroup and therefore is cyclic.

The next class of groups where results apply are ascending HNN-extensions of free groups (also known as mapping tori of free group endomorphisms). Groups of this type often appear in group theory and topology and were extensively studied (see [9], [2] for example). In particular, many one-relator groups are ascending HNN-extensions of free groups and many of such groups are hyperbolic. Corollary 1.5 allows to establish the structure of the profinite completion of this important class of groups.

**Theorem 1.7.** Let $F = F(x_1, \ldots, x_n)$ be a free group of finite rank $n$ and $f : F \rightarrow F$ an endomorphism. Let $G = \langle F, t \mid x_i^t = f(x_i) \rangle$ be the HNN-extension. Then the profinite completion of $G$ is $\hat{G} = P \rtimes \hat{\mathbb{Z}}$, where $P$ is projective. $P$ is free profinite of rank $n$ if and only if $f$ is an automorphism.

Combining this theorem with [25, Corollary 4.16] we obtain the following surprising consequence.

**Theorem 1.8.** An ascending HNN-extension $G$ of a free group is good.
The group $G$ is called $p$-good if the homomorphism of cohomology groups

$$H^n(\hat{G}, M) \to H^n(G, M)$$

induced by the natural homomorphism $G \to \hat{G}$ of $G$ to its profinite completion $\hat{G}$ is an isomorphism for every finite $p$-primary $G$-module $M$. The group $G$ is called good if is $p$-good for every prime $p$. This important concept was introduced by J-P. Serre in [39, Section I.2.6]. In his book Serre explains the fundamental role that goodness plays in the comparison of properties of a group and its profinite completion. This theorem is surprising because the profinite topology on ascending HNN-extension is not strong (these groups are not subgroup separable for example), but goodness one usually expects for groups with the strong profinite topology.

Acknowledgements: We thank Wilhelm Singhof for conversations on the subject.

Our research was supported by the joint project CAPES/MECD 065/04 of the Brazilian and Spanish Governments and by the Deutsche Forschungsgemeinschaft in the realm of GRK 1150 (Homotopy and cohomology) and the Forschergruppe 790 (Classification of algebraic surfaces and compact complex manifolds). Andrei Jaikin-Zapirain was partially supported by the Spanish Ministry of Science and Education, grant MTM2007-65385 with FEDER funds. Aline G.S. Pinto was partially supported by CAPES, grant BEX4101/07-3. Pavel A. Zalesski was partially supported by “bolsa de produtividade de pesquisa” from CNPq, Brazil.

2 Preliminaries

This section contains certain preliminary lemmas which will be of use later. Also we fix the following standard notations for this paper.

- $\mathbb{F}_p$ - stands for the field of $p$ elements;
- $\mathbb{Q}_p$ - is the field of $p$-adic numbers;
- $\mathbb{Z}_p$ - is the ring of $p$-adic integers;
- $\hat{G}$ - is the the profinite completion of $G$;
2 PRELIMINARIES

- $G_{\hat{p}}$ - is the the pro-$p$ completion of $G$;
- $G_p$ - is the $p$-Sylow subgroup of $G$;
- $G_{[p]}$ - is the maximal pro-$p$ quotient of $G$;
- $\mathbb{Z}_p[[G]]$ - is the completed group ring, i.e., $\mathbb{Z}_p[[G]] = \varprojlim \mathbb{Z}_p[G_j]$ which is the inverse limit of ordinary group rings with $G_j$ ranging over all finite quotient groups of $G$;
- $[G : H]_p$ - is the largest power of $p$ dividing the index $[G : H]$ of $H$ in $G$.

2.1 Homology and cohomology of profinite groups

In this section we collect some notation and well known facts concerning the homology and cohomology of profinite groups. If we do not say the contrary module means left module.

Let $G$ be a profinite group and $B$ a profinite $\mathbb{Z}_p[[G]]$-module. The $i$th homology group $H_i(G, B)$ of $G$ with coefficients in $B$ is defined by

$$H_i(G, B) = \text{Tor}_i^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, B),$$

where $\text{Tor}_i^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, -)$ is the $i$th derived functor of $\mathbb{Z}_p \widehat{\otimes} \mathbb{Z}_p[[G]] -$. By [4, Corollary 4.3], homology commutes with inverse limits, that is, if $G = \varprojlim G_j$ is the inverse limit of profinite groups $G_j$, then $H_i(G, B) = \varprojlim H_i(G_j, B)$.

Similarly, given a discrete $\mathbb{Z}_p[[G]]$-module $A$, the $i$th cohomology group $H^i(G, M)$ of $G$ with coefficients in $A$ is defined by

$$H^i(G, A) = \text{Ext}^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, A),$$

where $\text{Ext}^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, -)$ is the $i$th derived functor of $\text{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, -)$. It can be calculated by using either projective resolutions of the trivial module $\mathbb{Z}_p$ in the category of profinite $\mathbb{Z}_p[[G]]$-modules or injective resolutions of the discrete $\mathbb{Z}_p[[G]]$-module $A$. Moreover, if $G = \varprojlim G_j$, we have

$$H^i(G, A) = \varinjlim H^i(G_j, A).$$

The categories of profinite and torsion discrete $\mathbb{Z}_p[[G]]$-modules are dual via the Pontryagin duality ([35, 5.1]) and so are $H_i(G, -)$ and $H^i(G, -^*)$, where
* stands for \( \text{Hom}(\cdot, \mathbb{Q}_p/\mathbb{Z}_p) \). Hence we have that \( H_i(G, \mathbb{F}_p) \) and \( H^i(G, \mathbb{F}_p) \) are vector spaces over \( \mathbb{F}_p \) of the same dimension.

By definition, a profinite group \( H \) is of type \( p\text{-FP}_m \) if the trivial profinite \( \mathbb{Z}_p[[H]] \)-module \( \mathbb{Z}_p \) has a profinite projective resolution over \( \mathbb{Z}_p[[H]] \) with all projective modules in dimensions \( \leq m \) finitely generated. (Note that this also implies the existence of a profinite projective resolution over \( \mathbb{Z}_p[[H]] \) for right \( \mathbb{Z}_p[[H]] \)-module \( \mathbb{Z}_p \).) We say that \( H \) is of type \( p\text{-FP}_\infty \) if \( H \) is of type \( p\text{-FP}_m \) for every \( m \).

Let now \( G \) be a profinite group of type \( p\text{-FP}_\infty \) and \( B \) a profinite \( \mathbb{Z}_p[[G]] \)-module. Then \( B = \lim_{\longrightarrow} B_j \), where each \( B_j \) is a finite discrete \( p \)-torsion \( \mathbb{Z}_p[[G]] \)-module, and so \( H^i(G, B_j) \) is finite for all \( i \) and \( j \). Thus we can define the \( i \)th continuous cohomology of \( G \) in the profinite module \( B \) by the profinite group

\[
H^i(G, B) = \lim_{\longrightarrow} H^i(G, B_j).
\]

Note that this definition coincides with the one given in [40, Thm. 3.7.2] where it is \( \text{Ext}^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, B) \) and \( \text{Ext}^i_{\mathbb{Z}_p}(\mathbb{Z}_p, -) \) is the \( i \)th continuous derived functor of \( \text{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, -) \).

The cohomological \( p \)-dimension of a profinite group \( G \) is the lower bound of the integers \( n \) such that for every discrete torsion \( G \)-module \( A \), and for every \( i > n \), the \( p \)-primary component of \( H^i(G, A) \) is null. We shall use the standard notation \( \text{cd}_p(G) \) for cohomological \( p \)-dimension of the profinite group \( G \). The cohomological dimension \( \text{cd}(G) \) of \( G \) is defined as the supremum \( \text{cd}(G) = \sup_p(\text{cd}_p(G)) \) where \( p \) varies over all primes \( p \).

The next proposition gives a well-known characterization for \( \text{cd}_p \).

**Proposition 2.1.** Let \( G \) be a profinite group, \( p \) a prime and \( n \) an integer. The following properties are equivalent:

1. \( \text{cd}_p(G) \leq n \);
2. \( H^i(G, A) = 0 \) for all \( i > n \) and every discrete \( G \)-module \( A \) which is a \( p \)-primary torsion module;
3. \( H^{n+1}(G, A) = 0 \) when \( A \) is simple discrete \( G \)-module annihilated by \( p \);
4. $H^{n+1}(H, \mathbb{F}_p) = 0$ for any open subgroup $H$ of $G$.

Note that if $G$ is pro-$p$ then there is only one simple discrete $G$-module annihilated by $p$, namely the trivial module $\mathbb{F}_p$.

The Lyndon-Hochschild-Serre spectral sequence will be the most important tool in this paper. We will give its brief description and the most important consequences. For the details see [38].

**Theorem 2.2** ([38 Thm. 7.2.4]). Let $N$ be a normal closed subgroup of a profinite group $G$, and let $A$ be a discrete $G$-module. Then there exists a spectral sequence $E = (E^{r,s}_t)$ such that

$$E^{r,s}_2 \cong H^r(G/N, H^s(N, A)) \Rightarrow H^n(G, A).$$

**Corollary 2.3.** Let $N$ be a normal closed subgroup of a profinite group $G$, and let $A$ be a discrete $G$-module. Then the following holds.

1. There exists always a five term exact sequence
   $$0 \to H^1(G/N, A^N) \to H^1(G, A) \to H^1(N, A)^{G/N} \to H^2(G/N, A^N) \to H^2(G, A).$$

2. If $H^i(N, A) = 0$ for all $i \geq 1$, then $H^i(G, A) \cong H^i(G/N, A^N)$.

3. If $H^1(N, A) = 0$, then $H^1(G/N, A) \cong H^1(G, A)$ and $H^2(G/N, A) \hookrightarrow H^2(G, A)$.

4. If $H^i(N, A) = 0$ for all $i \geq 2$, then there exists the following exact sequence
   $$\ldots \to H^i(G/N, H^1(N, A)) \to H^{i+2}(G/N, A^N) \to H^{i+2}(G, A) \to$$
   $$\to H^{i+1}(G/N, H^1(N, A)) \to H^{i+3}(G/N, A^N) \to \ldots$$

5. If $\text{cd}_p(G/N) = 1$ and $pA = 0$, then
   $$H^{i+1}(G, A) \cong H^1(G/N, H^i(N, A)) \oplus H^{i+1}(N, A)^{G/N}.$$

6. If $G/N^p[N, N]$ splits as the direct product $G/N \times N/N^p[N, N]$ then
   $$H^2(G/N, \mathbb{F}_p) \oplus H^1(G/N, H^1(N, \mathbb{F}_p)) \hookrightarrow H^2(G, \mathbb{F}_p).$$
Proof. Items (1) and (2) can be found in [38, Corollary 7.2.5]; (3) follows directly from (1); (4) can be found in [36, Exercise 2.1.5]. We sketch the proofs of items (5) and (6).

(5) Consider the Lyndon-Hochschild-Serre spectral sequences \((E_\bullet^\bullet, d_\bullet)\) for the \(G\)-module \(A\) associated to the extension \(1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1\). From the hypothesis it follows that \(E_t^0 = 0\) for \(t \geq 2\). Hence the spectral sequence \((E_\bullet^\bullet, d_\bullet)\) collapses at the \(E_2\)-term.

(6) Consider the Lyndon-Hochschild-Serre spectral sequences \((E_\bullet^\bullet, d_\bullet)\) for the trivial \(G\)-module \(\mathbb{F}_p\) associated to the extension \(1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1\), and \((\bar{E}_\bullet^\bullet, d_\bullet)\) for the trivial \(G/Np[N, N]\)-module \(\mathbb{F}_p\) associated to the extension \(1 \rightarrow N/Np[N, N] \rightarrow G/Np[N, N] \rightarrow G/N \rightarrow 1\). Note that by construction, the second extension is a direct product, and thus the spectral sequence \((\bar{E}_\bullet^\bullet, d_\bullet)\) collapses at the \(\bar{E}_2\)-term, i.e., \(\bar{d}_t = 0\) for all \(t \geq 2\) ([36, p. 96, Exercise 7]). Denote by \(\pi^{n,k}_2\) the natural map \(E^{n,k}_2 \rightarrow E^{n,k}_2\). Then for each pair \(n, k\) we have the following commutative diagram

\[
\begin{array}{ccc}
E^{n,k}_2 & \xrightarrow{0} & \bar{E}^{n+2,k-1}_2 \\
\downarrow \pi^{n,k}_2 & & \downarrow \pi^{n+2,k-1}_2 \\
E^{n,k}_2 & \xrightarrow{d^{n,k}_2} & \bar{E}^{n+2,k-1}_2
\end{array}
\]

Since \(\pi^{0,1}_2\) is an isomorphism, it follows from the diagram that \(d^{0,1}_2 = 0\) and so \(E^3_2 = E^3_2,0 = H^2(G/N, \mathbb{F}_p)\). In the same way, as \(\pi^{1,1}_2\) is an isomorphism, we get that \(d^{1,1}_2 = 0\) and so \(E^3_2 = E^3_2,1 = H^1(G/N, H^1(N, \mathbb{F}_p))\).

We will use the following result that relates the cohomological \(p\)-dimensions of \(G, N\) and \(G/N\).

**Theorem 2.4** ([42, Thm. 1.1]). Let \(G\) be a profinite group of finite cohomological \(p\)-dimension \(cd_p(G) = n\) and let \(N\) be a closed normal subgroup of \(G\) of cohomological \(p\)-dimension \(cd_p(N) = k\) such that \(H^k(N, \mathbb{F}_p)\) is nonzero and finite. Then \(G/N\) is of virtual cohomological \(p\)-dimension \(n - k\).

In the case when \(N\) is of cohomological \(p\)-dimension 0 or 1 we have the following corollary.

**Corollary 2.5.** Let \(G\) be a profinite group of finite cohomological \(p\)-dimension \(cd_p(G)\) and let \(N\) be a finitely generated closed normal subgroup of \(G\) of cohomological \(p\)-dimension \(cd_p(N) \leq 1\). Then \(G/N\) is of virtual cohomological \(p\)-dimension \(cd_p(G) - cd_p(N)\).
Proof. If \( p \) does not divide \(|N|\), then Corollary 2.3(2) implies the isomorphisms \( H^k(U/N, \mathbb{F}_p) \cong H^k(U, \mathbb{F}_p) \) for all open subgroups \( U \) of \( G \) which contain \( N \) and all \( k \). Now from Proposition 2.1 follows that \( cd_p(G/N) = cd_p(G) \).

If \( p \) divides \(|N|\) then, by Proposition 2.1, \( cd_p(N) = 1 \) and so there exists an open subgroup \( V \) of \( N \) such that \( H^1(V, \mathbb{F}_p) \neq 0 \). Find an open subgroup \( U \) of \( G \) such that \( U \cap N = V \) and apply Theorem 2.4 to \( U \) and \( V \). To finish the proof we observe that cohomological \( p \)-dimension of a profinite group and its open subgroups coincide (when the cohomological \( p \)-dimension of the group is finite) (cf. [38], Theorem 7.3.7). \( \square \)

2.2 The deficiency

If \( G \) is a finitely generated group (profinite group), then we say that \( G \) is of deficiency \( k \) if there exist a presentation of \( G \) of deficiency \( k \), i.e. such that the number of generators minus the number of relations is \( k \). Of course, by our definition a group of deficiency \( k + 1 \) is also of deficiency \( k \). We denote by \( \text{def}(G) \) the greatest \( k \) such that \( G \) is of deficiency \( k \). Note that for any group \( G \) one has \( \text{def}(G) \leq \text{def}(\hat{G}) \).

Below we shall introduce other invariants that help to describe more precisely properties of profinite groups; some of them are taken from [30]. The necessity of this already can be seen from the fact that any pro-\( p \) group (including free pro-\( p \) groups) as a profinite group is of non-positive deficiency.

Let \( G \) be a finitely generated profinite group. Denote by \( d(G) \) its minimal number of generators. If \( M \) is a non zero finite \( G \)-module we denote by \( \dim M \) the length of \( M \) as \( \mathbb{Z} \)-module and put

\[
\overline{\chi}_2(G, M) = \frac{-\dim H^2(G, M) + \dim H^1(G, M) - \dim H^0(G, M)}{\dim(M)}.
\]

Also we introduce a cohomological variation of the minimal number of generators. We put

\[
\overline{\chi}_1(G, M) = \frac{\dim H^1(G, M) - \dim H^0(G, M)}{\dim(M)}.
\]

If \( M = \{0\} \), then we agree that \( \overline{\chi}_2(G, M) = +\infty \) and \( \overline{\chi}_1(G, M) = -\infty \). In fact, \( \overline{\chi}_k(G, M) = -\chi_k(G, M) / \dim(M) \), where \( \chi_k(G, M) \) is the partial Euler characteristic of the \( G \)-module \( M \).
Example 2.6. Let $F$ be a free profinite group of finite rank. Then
\[
\overline{\chi}_2(G, M) = \overline{\chi}_1(G, M) = \text{rk}(F) - 1.
\]
Indeed, the first equality we obtain from $H^2(F, M) = 0$. Now write a free $\mathbb{Z}_p[[F]]$-resolution
\[
\mathcal{F} : 0 \to \mathbb{Z}_p[[F]]^{\text{rk}(F)} \to \mathbb{Z}_p[[F]] \to \mathbb{Z}_p \to 0
\]
and apply $\text{Hom}_{\mathbb{Z}_p[[F]]}(*, M)$ to the complex $\mathcal{F}_{\text{del}}$ obtained by suppressing $\mathbb{Z}_p$. Thus we get that cohomology of $F$ is the cohomology of the following complex
\[
\text{Hom}_{\mathbb{Z}_p[[F]]}(\mathcal{F}_{\text{del}}, M) : 0 \to M \xrightarrow{\phi} M^{\text{rk}(F)} \xrightarrow{\psi} 0.
\]
Then $\dim(M)\overline{\chi}_1(G, M) = \dim H^1(F, M) - \dim H^0(F, M) = \dim \ker(\psi) - \dim \text{Im}(\phi) - \dim \ker(\phi) = \text{rk}(F) \dim(M) - \dim(M) = \dim(M)(\text{rk}(F) - 1)$.

Lemma 2.7. Let $G$ be a finitely generated profinite group. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finite $G$-modules. Then
\[
\overline{\chi}_2(G, M) \geq \min\{\overline{\chi}_2(G, M'), \overline{\chi}_2(G, M'')\}
\]
and
\[
\overline{\chi}_1(G, M) \leq \max\{\overline{\chi}_1(G, M'), \overline{\chi}_1(G, M'')\}.
\]

Proof. We will prove the first inequality. The second one is proved using the same method. The short exact sequence
\[
0 \to M' \to M \to M'' \to 0
\]
gives a long exact sequence in cohomology
\[
0 \to H^0(G, M') \to H^0(G, M) \to \ldots \to H^2(G, M) \to H^2(G, M'') \xrightarrow{\delta_3} \ldots
\]
Counting dimension we have
\[
\dim H^0(G, M) = \dim H^0(G, M') - \dim H^0(G, M'') + \dim H^1(G, M') - \dim H^1(G, M) + \dim H^1(G, M'') - \dim H^2(G, M') + \dim H^2(G, M'') - \dim \text{Im} \delta_3
\]
from where we obtain
\[
\dim(M')\overline{\chi}_2(G, M') - \dim(M)\overline{\chi}_2(G, M) + \dim(M'')\overline{\chi}_2(G, M'') \leq 0.
\]
Hence,
\[
\overline{\chi}_2(G, M) \geq \frac{\dim(M')\overline{\chi}_2(G, M') + \dim(M'')\overline{\chi}_2(G, M'')}{\dim(M)} \geq \min\{\overline{\chi}_2(G, M'), \overline{\chi}_2(G, M'')\}.
\]

Let \(G\) be a finitely generated profinite group and let
\[
1 \to R \to F \to G \to 1
\]
be a presentation of \(G\), where \(F\) is a \(d(G)\)-generated free profinite group. Then \(\bar{R} := R/[R, R] \cong H_1(R, \hat{Z})\) as \(G\)-modules and it is called the relation module of \(G\). By Shapiro’s lemma \(\bar{R} \cong H_1(F, \hat{Z}[[G]])\) and, as it is shown in \([30]\), it does not depend on the presentation.

If we decompose \(\bar{R}\) as the product of its \(p\)-primary components \(\bar{R} = \prod_p \bar{R}_p\), one gets \(\bar{R}_p \cong H_1(F, Z_p[[G]])\). Since the augmentation ideal of \(Z_p[[F]]\) is a free profinite \(Z_p[[F]]\)-module of rank \(d(G)\), we have the exact sequence
\[
0 \to Z_p[[F]]^{d(G)} \to Z_p[[F]] \to Z_p \to 0
\]
of free \(Z_p[[F]]\)-modules. Applying the functor \(Z_p[[F/R]] \hat{\otimes} Z_p[[F]]\), we obtain the following exact sequence
\[
0 \to \bar{R}_p \to Z_p[[G]]^{d(G)} \xrightarrow{\varphi} Z_p[[G]] \to Z_p \to 0, \tag{2.1}
\]
since \(\ker(\varphi) = H_1(F, Z_p[[G]])\). If \(M\) is a \(Z_p[[G]]\)-module, we denote by \(d_G(M)\) its minimal number of generators. Thus, the last exact sequence may be rewritten as follows.
\[
Z_p[[G]]^{d_G(\bar{R}_p)} \to Z_p[[G]]^{d(G)} \to Z_p[[G]] \to Z_p \to 0. \tag{2.2}
\]

The following result is proved in \([30]\).

**Theorem 2.8.** Let \(G\) be a finitely generated profinite group. Then, for a fixed prime \(p\),
\[
d_G(\bar{R}_p) = \min_M\{d(G) - 1 - \overline{\chi}_2(G, M)\}
\]
where $M$ runs over all irreducible $\mathbb{Z}_p[[G]]$-modules, and
\[ d_G(\bar{R}) = \max_p d_G(\bar{R}_p). \]

Moreover, $\text{def}(G) = d(G) - d_G(\bar{R})$ unless $\bar{R} = 0$ and $G$ is not free, in which case $\text{def}(G) = d(G) - 1$.

Combining this theorem with Lemma 2.7, we obtain that if $\bar{R} \neq 0$, then
\[ \text{def}(G) = \min \{ 1 + \chi_2(G, M) | M \text{ is a finite } G \text{-module} \}. \quad (2.3) \]

If $N$ is a closed subgroup of $G$ and $\mathcal{M}_p(N)$ is the set of all finite $\mathbb{Z}_p[G]$-modules on which $N$ acts trivially, we introduce the following invariants:
\[ \text{def}_p(G, N) = \min_{M \in \mathcal{M}_p(N)} \{ 1 + \chi_2(G, M) \} \]
and
\[ d_p(G, N) = \max_{M \in \mathcal{M}_p(N)} \{ 1 + \chi_1(G, M) \}. \]

For simplicity, we put
\[ \text{def}_p(G) = \text{def}_p(G, \{1\}) \quad \text{and} \quad d_p(G) = d_p(G, \{1\}). \]

Comparing this invariant with the deficiency of $G$ we observe that
\[ \text{def}(G) \leq \text{def}_p(G) \leq \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p), \quad (2.4) \]
so
\[ \text{def}(G) \leq \text{def}_p(G) \leq \chi_2(G, \mathbb{F}_p) - 1. \]

When $G$ is pro-$p$, we obtain from $[2,3]$ that $\text{def}_p(G) = \chi_2(G, \mathbb{F}_p) - 1$; by [30, Corollary 5.5] $d_G(\bar{R}) = \max \{ d, \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) \}$ from where we deduce that
\[ \text{def}(G) = \min \{ 0, \text{def}_p(G) \}. \]

Note that from Lemma 2.7 it follows that if $N_1 \leq N_2$ are two normal subgroups of $G$ and $N_2/N_1$ is a pro-$p$ group, then $\text{def}_p(G, N_1) = \text{def}_p(G, N_2)$ and $d_p(G, N_1) = d_p(G, N_2)$.

**Lemma 2.9.** Let $G$ be a profinite group and $G_p$ its maximal pro-$p$ quotient, then $\chi_2(G, \mathbb{F}_p) \leq \chi_2(G_p, \mathbb{F}_p)$. 
Proof. Let \( N \) be the kernel of the natural map \( G \to G_p \). Then \( H^1(N, \mathbb{F}_p) = 0 \). Thus, the lemma follows from Corollary \ref{c23}\). \(\square\)

**Lemma 2.10.** Let \( G \) be a finitely presented profinite group, \( N \) a closed subgroup and \( H \) an open subgroup containing \( N \). Then

\[
def_p(H, N) - 1 \geq [G : H](\text{def}_p(G, N) - 1)
\]

and

\[
d_p(H, N) - 1 \leq [G : H](d_p(G, N) - 1).
\]

Proof. Let \( M \) be a finite \( \mathbb{F}_p[H] \)-module. By Shapiro’s lemma \( H^i(H, M) = H^i(G, \text{Coind}_H^G(M)) \) for all \( i \), so we obtain that

\[
\chi_i(H, M) = [G : H]\chi_i(G, \text{Coind}_H^G(M)).
\]

Thus,

\[
\text{def}_p(H, N) - 1 = \min_{M \in \mathcal{M}_p(N)} \left\{ \chi_2(H, M) \right\}
\]

\[
= [G : H]\min_{M \in \mathcal{M}_p(N)} \left\{ \chi_2(G, \text{Coind}_H^G(M)) \right\}
\]

\[
\geq [G : H](\text{def}_p(G, N) - 1)
\]

and

\[
d_p(H, N) - 1 = \max_{M \in \mathcal{M}_p(N)} \left\{ \chi_1(H, M) \right\} \leq [G : H](d_p(G, N) - 1). \quad \square
\]

We shall finish the section with two general technical results about the numerical invariants introduced here.

**Proposition 2.11.** Let \( G_1 \) be a finitely generated profinite group, \( N \) a normal subgroup, \( G_2 \) an open normal subgroup of index a power of \( p \) in \( G_1 \) containing \( N \) and \( M \) a non zero finite \( \mathbb{F}_p[[G_1]] \)-module. Denote by \( \beta_i \) (\( i = 1, 2 \)) the restrictions maps \( H^2(G_i, M) \to H^2(N, M) \) respectively. Then

\[
\dim(\text{Im} \beta_2) \geq (\chi_2(G_1, M) \dim(M) + \dim(\text{Im} \beta_1))[G_1 : G_2] - \chi_2(G_2, M) \dim(M)
\]

and, in particular,

\[
\chi_1(G_2, M) \geq [G_1 : G_2]\left( \frac{\dim(\text{Im} \beta_1)}{\dim(M)} + \chi_2(G_1, M) \right).
\]
Proof. Let
\[ 1 \rightarrow R \rightarrow F_1 \xrightarrow{\phi} G_1 \rightarrow 1 \]
be a presentation for \( G_1 \) with \( d(F_1) = d(G_1) \). We have the following presentation for \( G_2 \):
\[ 1 \rightarrow R \rightarrow F_2 \xrightarrow{\phi} G_2 \rightarrow 1, \]
where \( F_2 = \phi^{-1}(G_2) \).

From Corollary 2.3(1), we obtain the following two exact sequences \((i = 1, 2):\)
\[ 0 \rightarrow H^1(G_i, M) \rightarrow H^1(F_i, M) \rightarrow H^1(R, M)^{G_i} \rightarrow H^2(G_i, M) \rightarrow 0. \]  \( (2.5) \)

Note that \( \chi_1(F_i, M) = d(F_i) - 1 \) (see Example 2.6). Therefore, using the Schreier formula for \( F_i \) and the equality \( H^0(F_i, M) = H^0(G_i, M) \) one has
\[
\dim H^1(R, M)^{G_i} = \dim H^1(F_i, M) - \chi_2(G_i, M) \dim(M) - \dim H^0(G_i, M)
\]
\[
= (\chi_1(F_i, M)[G_1 : G_i] - \chi_2(G_i, M)) \dim(M).
\]

Let \( \alpha_i \) be the composition of the map \( H^1(R, M)^{G_i} \rightarrow H^2(G_i, M) \) from (2.5) and the restriction map \( \beta_i : H^2(G_i, M) \rightarrow H^2(N, M) \). Put
\[ M_i = \ker \alpha_i. \]

Since the following diagram is commutative
\[
\begin{array}{ccc}
H^1(R, M)^{G_1} & \xrightarrow{\alpha_1} & H^2(N, M) \\
\searrow & & \| \\
H^1(R, M)^{G_2} & \xrightarrow{\alpha_2} & H^2(N, M)
\end{array}
\]
we obtain that \( M_2^{G_1} \leq M_1 \). Recall that
\[
\dim(M_1) = \dim H^1(R, M)^{G_1} - \dim(\text{Im} \alpha_1)
\]
\[
= \dim H^1(R, M)^{G_1} - \dim(\text{Im} \beta_1)
\]
\[
= (\chi_1(F_1, M) - \chi_2(G_1, M)) \dim(M) - \dim(\text{Im} \beta_1). \]
Since \( G_1/G_2 \) is a finite \( p \)-group, we have \( \dim(M_2) \leq \dim(M_2^{G_1})[G_1 : G_2] \) (see page 7 in [18] for \( G \) cyclic, the general case follows by an obvious induction) we get

\[
\dim(M_2) \leq \dim(M_2^{G_1})[G_1 : G_2] \leq \dim(M_1)[G_1 : G_2] \leq ((\chi_1(F_1, M) - \chi_2(G_1, M)) \dim(M) - \dim(\text{Im} \beta_1))[G_1 : G_2]).
\]

Thus, \( \dim(\text{Im} \beta_2) = \dim(\text{Im} \alpha_2) = \dim H^1(R, M)^{G_2} - \dim(M_2) \geq (\chi_2(G_1, M) \dim(M) + \dim(\text{Im} \beta_1))[G_1 : G_2] - \chi_2(G_2, M) \dim(M). \)

Since

\[
\chi_1(G_2, M) = \frac{\chi_2(G_2, M) \dim(M) + \dim H^2(G_2, M)}{\dim(M)} \quad \text{and} \quad \dim H^2(G_2, M) \geq \dim(\text{Im} \beta_2),
\]

we obtain

\[
\chi_1(G_2, M) \geq \frac{\dim(\text{Im} \beta_2) + \chi_2(G_2, M) \dim(M)}{\dim(M)} \geq (\chi_2(G_1, M) + \frac{\dim(\text{Im} \beta_1)}{\dim(M)})[G_1 : G_2].
\]

\[\square\]

**Proposition 2.12.** Let \( G \) be a finitely generated profinite group and \( N \) a normal subgroup such that \( \chi_1(N, \mathbb{F}_p) \) is non-negative and finite. Then there exists an open subgroup \( V \) of \( G \) containing \( N \) such that for any open subgroup \( U \) of \( V \) containing \( N \)

\[
\chi_2(U, \mathbb{F}_p) \leq -\chi_1(U/N, \mathbb{F}_p)\chi_1(N, \mathbb{F}_p) - \dim H^2(U/N, \mathbb{F}_p)
\]

\[
= - (\chi_1(U, \mathbb{F}_p) - \chi_1(N, \mathbb{F}_p) - 1)\chi_1(N, \mathbb{F}_p) - \dim H^2(U/N, \mathbb{F}_p).
\]

**Proof.** Since \( N^p[N, N] \) is open in \( N \) there exists an open subgroup \( J \) of \( G \) such that \( J \cap N = N^p[N, N] \). Put \( V = JN \) and let \( U \) be an open subgroup of \( V \). Then \( \chi_1(U, \mathbb{F}_p) = \chi_1(U/N, \mathbb{F}_p) + \chi_1(N, \mathbb{F}_p) + 1 \). Thus using Corollary
2.3(6), we obtain that

$$\chi_2(U, \mathbb{F}_p) = \chi_1(U, \mathbb{F}_p) - \dim H^2(U, \mathbb{F}_p)$$

$$\leq (\chi_1(U/N, \mathbb{F}_p) + \chi_1(N, \mathbb{F}_p) + 1) - (\chi_1(U/N, \mathbb{F}_p) + 1)(\chi_1(N, \mathbb{F}_p) + 1) - \dim H^2(U/N, \mathbb{F}_p)$$

$$= -\chi_1(U/N, \mathbb{F}_p)\chi_1(N, \mathbb{F}_p) - \dim H^2(U/N, \mathbb{F}_p)$$

$$= -(\chi_1(U, \mathbb{F}_p) - \chi_1(N, \mathbb{F}_p) - 1)\chi_1(N, \mathbb{F}_p) - \dim H^2(U/N, \mathbb{F}_p).$$

\[\square\]

2.3 The number of generators of modules over a profinite group

In this subsection we describe an easy way to calculate the number of generators of a profinite $G$-module. This will be used several times in the paper. As an application we obtain a characterization of a profinite group to be of type $pFP_m$ similar to Lubotzky’s characterization of a profinite group to have finite deficiency.

For any irreducible $\mathbb{Z}_p[[G]]$-module $M$, denote by $I_M$ the annihilator of $M$ in $\mathbb{Z}_p[[G]]$. If $K$ is a $\mathbb{Z}_p[[G]]$-module, then $K/I_M K \cong (\mathbb{Z}_p[[G]]/I_M) \widehat{\otimes} \mathbb{Z}_p[[G]]$ is the maximal quotient of $K$ isomorphic to a direct sum of copies of $M$. Thus, $\mathbb{F}_p[[G]]/I_M$ is the maximal cyclic $\mathbb{Z}_p[[G]]$-module isomorphic to a direct sum of copies of $M$.

Note that the Jacobson radical $J(K)$ of $K$ is equal to the intersection of $I_M K$ and so

$$K/J(K) \cong \prod_{M \text{ is irreducible}} K/I_M K.$$ 

Thus

$$d_G(K) = d_G(K/J(K)) = \max_{M \text{ is irreducible}} d_G(K/I_M K),$$

and so we obtain that

$$d_G(K) = \max_{M \text{ is irreducible}} \left[ \frac{\dim(K/I_M K)}{\dim(\mathbb{Z}_p[[G]]/I_M)} \right].$$
Since $K/I_MK$ and $\mathbb{Z}_p[[G]]/I_M$ are direct sums of copies of $M$, we conclude that

$$\left\lceil \frac{\dim(K/I_MK)}{\dim(\mathbb{Z}_p[[G]]/I_M)} \right\rceil = \left\lceil \frac{\dim \text{Hom}_{\mathbb{Z}_p[[G]]}(K/I_MK, M)}{\dim \text{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G]]/I_M, M)} \right\rceil.$$

Note that

$$\text{Hom}_{\mathbb{Z}_p[[G]]}(K/I_MK, M) \cong \text{Hom}_{\mathbb{Z}_p[[G]]}(K, M)$$

and

$$\dim \text{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G]]/I_M, M) = \dim \text{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G]], M) = \dim M.$$

Thus, we conclude that

$$d_G(K) = \max_{M \text{ is irreducible}} \left\lceil \frac{\dim \text{Hom}_{\mathbb{Z}_p[[G]]}(K, M)}{\dim M} \right\rceil. \quad (2.6)$$

For example, as a first application of this formula we obtain.

**Proposition 2.13.** Let $A$ be a projective $\mathbb{Z}_p[[G]]$-module. Assume that for any irreducible $\mathbb{Z}_p[[G]]$-module $M$ $\dim \text{Hom}_{\mathbb{Z}_p[[G]]}(A, M) = \dim M$. Then $A$ is isomorphic to $\mathbb{Z}_p[[G]]$.

**Proof.** By (2.6) $A$ is generated by 1 element. Since $A$ is projective, we have $\mathbb{Z}_p[[G]] = A \oplus B$, for some $B$. If $B$ is not trivial there exists an irreducible $\mathbb{Z}_p[[G]]$-module $M$ such that $\text{Hom}_{\mathbb{Z}_p[[G]]}(B, M)$ is not trivial. Since

$$\text{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G]], M) = \text{Hom}_{\mathbb{Z}_p[[G]]}(A, M) \oplus \text{Hom}_{\mathbb{Z}_p[[G]]}(B, M)$$

we obtain a contradiction with

$$\dim \text{Hom}_{\mathbb{Z}_p[[G]]}(A, M) = \dim \text{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G]], M) = \dim M.$$

Hence $B = 0$ and $A$ is isomorphic to $\mathbb{Z}_p[[G]]$. \qed

The next theorem is inspired by a theorem of Lubotzky [30, Theorem 0.3] that says that a finitely generated profinite group is finitely presented if and only if there exists $C$ such that $\dim H^2(G, M) \leq C \dim M$ for any irreducible $\hat{\mathbb{Z}}[[G]]$-module $M$.

**Theorem 2.14.** Let $G$ be a profinite group. Then $G$ is of type $p$-$FP_m$ if and only if there exists a constant $C$ such that $\dim H^i(G, M) \leq C \dim M$ for any irreducible $\mathbb{Z}_p[[G]]$-module $M$ and any $0 \leq i \leq m$. 

Proof. We prove the theorem by induction on $m$. The case $m = 0$ is trivial. Assume that theorem holds for $m - 1$.

Suppose, first, that $G$ is of type $p$-$FP_m$. By induction, there exists $C'$ such that $\dim H^i(G, M) \leq C' \dim M$ for any irreducible $\mathbb{Z}_p[[G]]$-module $M$ and any $1 \leq i \leq m - 1$. Since $G$ is of type $p$-$FP_m$, there exists an exact sequence of finitely generated projective modules

$$
\mathcal{R} : \quad P_m \to P_{m-1} \to \ldots \to P_0 \to \mathbb{Z}_p \to 0.
$$

So, for an irreducible $\mathbb{Z}_p[[G]]$-module $M$, from the sequence $\text{Hom}_{\mathbb{Z}_p[[G]]}(\mathcal{R}, M)$, we obtain

$$
\sum_{i=0}^{m} (-1)^{m-i} \dim \text{Hom}_{\mathbb{Z}_p[[G]]}(P_i, M) \geq \sum_{i=0}^{m} (-1)^{m-i} \dim H^i(G, M), \quad (2.7)
$$

since $H^i(G, M)$ is a subquotient of $\text{Hom}_{\mathbb{Z}_p[[G]]}(P_i, M)$. Therefore

$$
\dim H^m(G, M) \leq \sum_{i=0}^{m} \dim \text{Hom}_{\mathbb{Z}_p[[G]]}(P_i, M) + \sum_{i=0}^{m-1} \dim H^i(G, M)
$$

$$
\leq \left( \sum_{i=0}^{m} d_G(P_i) + mC' \right) \dim M.
$$

Thus we may put $C = \sum_{i=0}^{m} d_G(P_i) + mC'$.

Suppose now that there exists a constant $C$ such that $\dim H^i(G, M) \leq C \dim M$ for any irreducible $\mathbb{Z}_p[[G]]$-module $M$ and any $0 \leq i \leq m$. By inductive assumption, there exists an exact sequence

$$
\mathcal{R} : \quad 0 \to A \to P_{m-1} \to \ldots \to P_0 \to \mathbb{Z}_p \to 0
$$

with $P_i$ finitely generated projective for $0 \leq i \leq m - 1$. We want to show that $A$ is finitely generated, since then we can cover it by a finitely generated free module. Let $M$ be an irreducible $\mathbb{Z}_p[[G]]$-module. Considering the sequence $\text{Hom}_{\mathbb{Z}_p[[G]]}(\mathcal{R}, M)$ we obtain that

$$
\dim \text{Hom}_{\mathbb{Z}_p[[G]]}(A, M) \leq - \sum_{i=0}^{m-1} (-1)^{m-i} \dim \text{Hom}_{\mathbb{Z}_p[[G]]}(P_m, M)
$$

$$
+ \sum_{i=0}^{m} \dim(-1)^{m-i}H^i(G, M)
$$

$$
\leq \left( \sum_{i=0}^{m-1} d_G(P_i) + (m + 1)C \right) \dim M.
$$

Therefore, by (2.6), $A$ is finitely generated. \qed
We will need the following application of the previous theorem.

**Corollary 2.15.** Let $G$ be a profinite group of type $p$-$FP_2$ and $N$ a normal subgroup of $G$. Assume that $N$ is finitely generated as a normal subgroup. Then $G/N$ is of type $p$-$FP_2$.

**Proof.** Let $r$ denote the minimal number of generators of $N$ as a normal subgroup of $G$. Let $M$ be an irreducible $\mathbb{Z}_p[[G/N]]$-module. From Corollary 2.3 we obtain that

\[
\dim H^1(G/N, M) \leq \dim H^1(G, M)
\]

and

\[
\dim H^2(G/N, M) \leq \dim H^1(N, M)^{G/N} + \dim H^2(G, M) \\
\leq r \dim M + \dim H^2(G, M).
\]

Since $G$ is of type $p$-$FP_2$, the previous theorem implies that $G/N$ is also of type $p$-$FP_2$. \qed

### 3 Normal subgroups of profinite groups of positive deficiency

This section consists of main results on profinite groups of positive deficiency. We divide our results in subsections by the reverse order on deficiency.

#### 3.1 Groups of deficiency $\geq 2$

**Proposition 3.1.** Let $G$ be a finitely generated profinite group, $N$ a normal subgroup and $H$ and $J$ two open subgroups containing $N$. If $\def_p(H, N) \geq 2$, then

\[
\frac{d_p(J, N) - 1}{[G : J]} \geq \frac{1}{[G : H]}.
\]

**Proof.** Using Lemma 2.10 we obtain that

\[
d_p((J \cap H), N) \geq \def_p((J \cap H), N) \geq [H : (J \cap H)](\def_p(H, N) - 1) + 1 \\
\geq [H : (J \cap H)] + 1
\]
Hence, by Lemma 2.10,
\[ d_p(J, N) - 1 \geq \frac{d_p((J \cap H), N) - 1}{[J : J \cap H]} \geq \frac{[H : (J \cap H)]}{[J : J \cap H]} \geq \frac{[G : J]}{[G : H]}. \]

**Proposition 3.2.** Let \( G \) be a finitely generated profinite group and \( N \) a normal subgroup of infinite index such that \( H^1(N, \mathbb{F}_p) \neq 0 \). If \( \text{def}_p(G, N) \geq 2 \), then \( H^1(N, \mathbb{F}_p) \) is infinite.

**Proof.** If \( H^1(N, \mathbb{F}_p) \) is finite, then by Proposition 2.12 there exists an open subgroup \( U \) containing \( N \) such that

\[ \chi_2(U, \mathbb{F}_p) \leq -\chi_1(U/N, \mathbb{F}_p)\chi_1(N, \mathbb{F}_p) - \dim H^2(U/N, \mathbb{F}_p). \]

Since by Proposition 3.1 we have \( \chi_1(U/N, \mathbb{F}_p) \) non negative, it follows that \( \chi_2(U, \mathbb{F}_p) \leq 0 \). Thus \( \text{def}_p(U, N) = \min_{M \in \mathcal{M}_p(N)} \{1 + \chi_2(U, N)\} \leq 1 \). But this contradicts Lemma 2.10 because \( \text{def}_p(G, N) \geq 2 \) implies \( \text{def}_p(U, N) \geq 2 \).

**Corollary 3.3.** Let \( G \) be a finitely generated profinite group and \( N \) a normal subgroup of infinite index such that \( p \) divides \( |N| \). If \( \text{def}_p(G) \geq 2 \), then some open subgroup of \( N \) has infinite \( p \)-abelianization. In particular, \( N \) is infinitely generated.

**Proof.** Find an open subgroup \( U \) of \( G \) such that \( H^1(U \cap N, \mathbb{F}_p) \neq 0 \) and apply the previous proposition.

### 3.2 Groups of deficiency 1

**Theorem 3.4.** Let \( G \) be a finitely generated profinite group, \( K \leq N \) two normal subgroups of \( G \) such that \( |G/N|_p \) is infinite and \( \text{def}_p(G, K) \geq 1 \). Let \( M \) be a non zero finite \( \mathbb{F}_p[[G]] \)-module on which \( K \) acts trivially. Suppose that

\[ \inf \left\{ \frac{d_p(H, K) - 1}{[G : H]_p} \mid N < H \leq G \right\} = 0, \]

where \( [G : H]_p \) is the greatest power of \( p \) dividing \( [G : H] \). Then

(1) \( \text{def}_p(U, K) = 1 \) for any open subgroup \( U \) containing \( N \) and

(2) \( H^2(N, M) = \{0\} \).
Proof. Let $U$ be an open subgroup of $G$ containing $N$. Then by Lemma 2.10, $\text{def}_p(U, K) \geq 1$ and by Proposition 3.1, $\text{def}_p(U, K) \leq 1$. Thus, $\text{def}_p(U, K) = 1$.

Now, by way of contradiction let us assume that $H^2(N, M) \neq \{0\}$. Let $G = H_1 > H_2 > \ldots$ be a chain of open normal subgroups such that $\cap_i H_i = N$. Thus we have that

$$N = \varprojlim H_i.$$ 

Hence $H^2(N, M) = \varprojlim H^2(H_i, M)$. Since $H^2(N, M) \neq 0$, we obtain that there exists $j$ such that the image of the restriction map $\beta : H^2(H_j, M) \to H^2(N, M)$ is not zero.

Let $H$ be a open normal subgroup of $G$ contained in $H_j$ which contains $N$ and $P$ a subgroup of $H_j$ containing $H$ such that $P/H$ is a $p$-Sylow subgroup of $H_j/H$. By (1) we have $\text{def}_p(P, K) = 1$, so $\chi_2(P, M) \geq 0$. Therefore by Proposition 2.11

$$\chi_1(H, M) \geq (\chi_2(P, M) + \frac{\dim(\text{Im} \beta)}{\dim(M)})[P : H] \geq \frac{[P : H]}{\dim(M)} = \frac{[G : H]_p}{[G : H_j]_p \dim(M)}.$$

Hence $d_p(H, K) \geq 1 + \frac{[G : H]_p}{[G : H_j]_p \dim(M)}$. Now, if $H$ is an arbitrary open subgroup containing $N$, applying Lemma 2.10 for $H$ and $H \cap H_j$ and the above inequality for $H \cap H_j$, we get

$$\frac{d_p(H, K) - 1}{[G : H]_p} \geq \frac{d_p((H \cap H_j), K) - 1}{[H : (H \cap H_j)][G : H_j]_p} \geq \frac{1}{[G : H_j]_p[H : (H \cap H_j)]_p \dim(M)}.$$

But it is a contradiction with the hypothesis, because $H_j$ is fixed. Hence, $H^2(N, M) = 0$. 

Corollary 3.5. Let $G$ be a finitely generated pro-$p$ group and $N$ a normal subgroup of infinite index. Suppose that

$$\inf \left\{ \frac{d(H) - 1}{[G : H]} \mid N < H \leq_0 G \right\} = 0.$$

Then $N$ is a free pro-$p$ group.

Proof. The previous theorem implies that $H^2(N, \mathbb{F}_p) = 0$. 

\qed
3.3 Finitely generated normal subgroups of profinite groups of deficiency 1

We need the following criterion for a group of positive deficiency to have cohomological \( p \)-dimension 2.

**Proposition 3.6.** Let \( G \) be a finitely generated profinite group with \( \text{def}_p(G) = 1 \). Suppose that for any open subgroup \( V \) of \( G \) there exist an open subgroup \( U \) of \( V \) such that \( \overline{X}_2(U, \mathbb{F}_p) = 0 \). Then \( \text{cd}_p(G) \leq 2 \).

**Proof.** Since \( \text{def}_p(G) = 1 \), there exists an exact sequence of right modules
\[
\mathcal{R} : 0 \to M \to \mathbb{F}_p[[G]]^{d-1} \to \mathbb{F}_p[[G]]^d \to \mathbb{F}_p[[G]] \to \mathbb{F}_p \to 0,
\]
where \( d = d(G) \) (see (2.2)). Let \( V \) be an open subgroup of \( G \) and let \( U \trianglelefteq V \) be such that \( \overline{X}_2(U, \mathbb{F}_p) = 0 \). Applying the functor \( -\otimes_{\mathbb{F}_p[[U]]}\mathbb{F}_p \) to \( \mathcal{R} \) we obtain the complex
\[
0 \to M \otimes_{\mathbb{F}_p[[U]]}\mathbb{F}_p \xrightarrow{h} \mathbb{F}_p[[G/U]]^{d-1} \xrightarrow{g} \mathbb{F}_p[[G/U]]^d \xrightarrow{f} \mathbb{F}_p[[G/U]] \to \mathbb{F}_p \to 0.
\]
Let \( n = |G/U| \). Counting \( \mathbb{F}_p \)-dimension one gets
\[
\dim H_1(U, \mathbb{F}_p) = \dim H_1(\mathcal{R}_U)
= \dim(\ker f) - \dim(\text{Im} g)
= nd - n + 1 - [n(d - 1) - (\dim(\text{Im} h) + \dim H_2(\mathcal{R}_U))]
= 1 + \dim(\text{Im} h) + \dim H_1(U, \mathbb{F}_p) - 1.
\]
It follows that \( \text{Im} h = 0 \) and so \( M = 0 \). Thus, \( \text{cd}_p(G) \leq 2 \) (cf. Proposition 2.1). \( \square \)

**Remark 3.7.** Note that the hypothesis from the previous proposition are equivalent to \( \overline{X}_2(G, M) = 0 \) for any non zero finite \( \mathbb{F}_p[[G]] \)-module \( M \).

Now we are ready to prove Theorem 1.4.

**Theorem 3.8.** Let \( p \) be a prime. Let \( G \) be a finitely generated profinite group with \( \text{def}_p(G) \geq 1 \) and \( N \) a finitely generated normal subgroup such that \( |G/N|_p \) is infinite and \( p \) divides \( |N| \). Then \( \text{def}_p(G) = 1 \) and either the \( p \)-Sylow subgroup of \( G/N \) is virtually cyclic or the \( p \)-Sylow subgroup of \( N \) is cyclic. Moreover, \( \text{cd}_p(G) = 2 \), \( \text{cd}_p(N) = 1 \) and \( \text{vcd}_p(G/N) = 1 \).
3 NORMAL SUBGROUPS OF PROFINITE GROUPS OF POSITIVE DEFICIENCY

Proof. First observe that Corollary 3.3 implies def\(_p(G) = 1\).

If the \(p\)-Sylow subgroup of \(N\) is not cyclic, then we can find an open subgroup \(J \triangleleft G\) such that \(\overline{\chi}_1(J \cap N, \mathbb{F}_p) \geq 1\). By Proposition 2.12, there exists an open subgroup \(V \triangleleft J\) containing \(J \cap N\) such that for any open subgroup \(U \triangleleft V\) containing \(J \cap N\),

\[
\overline{\chi}_2(U, \mathbb{F}_p) \leq -\overline{\chi}_1(U/N \cap J, \mathbb{F}_p) \overline{\chi}_1(N \cap J, \mathbb{F}_p) - \dim H^2(U/N, \mathbb{F}_p) \tag{3.1}
\]

Therefore we get \(\overline{\chi}_1(U/N \cap J, \mathbb{F}_p) \leq 0\), because \(\overline{\chi}_2(U, \mathbb{F}_p) \geq \text{def}_p(U) - 1 \geq 0\). This means that the \(p\)-Sylow subgroup of \(G/N\) is virtually cyclic.

Let \(W\) be an open subgroup of \(G\). Since \(|N|\) and \(|G/N|\) are infinite, we can find an open subgroup \(U \triangleleft W\) such that \(\overline{\chi}_1(U \cap N, \mathbb{F}_p) \) and \(\overline{\chi}_1(U, \mathbb{F}_p)\) are non-negative. Then using Proposition 2.12 together with the equality \(\overline{\chi}_2(U/N, \mathbb{F}_p) = \overline{\chi}_1(U/N, \mathbb{F}_p) - \dim H^2(U/N, \mathbb{F}_p)\), we get \(\overline{\chi}_2(U, \mathbb{F}_p) \leq 0\). Hence, by Proposition 3.6, \(\text{cd}_p(G) \leq 2\).

Theorem 3.8 is the profinite version of [16, Theorem 4], where \(G\) was assumed pro-\(p\).

**Corollary 3.9.** Let \(G\) be a finitely generated profinite group of positive deficiency and \(N\) a finitely generated normal subgroup of \(G\) such that for every prime \(p\) dividing \(|N|\) the \(p\)-Sylow subgroup \((G/N)_p\) is infinite. Then \(N\) is projective. Moreover, if none of the non-trivial \(p\)-Sylow subgroups \((G/N)_p\) is virtually cyclic then \(N\) is solvable of type \(\mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho\) where \(\pi\) and \(\rho\) are disjoint sets of primes.

**Proof.** By the previous theorem \(N\) is projective. By [38, Exercise 2.3.18], a profinite group with cyclic Sylow subgroups is of type \(\mathbb{Z}_\pi \rtimes \mathbb{Z}_\rho\) where \(\pi\) and \(\rho\) are disjoint sets of primes.
Corollary 3.10. Let $G$ be a finitely generated profinite group of positive deficiency whose commutator subgroup $[G,G]$ is finitely generated. Then $\text{def}(G) = 1$ and $[G,G]$ is projective. Moreover, $\text{cd}(G) = 2$ unless $G = \mathbb{Z}$. 

Proof. First let us suppose that $G$ is abelian. We want to show that $G \cong \mathbb{Z} \times \mathbb{Z}_\pi$ for some set of primes $\pi$ (possibly empty).

If $d(G) = 1$ then positive deficiency means 0 relations, so the result is obvious in this case.

Let $G_{[p]}$ be a maximal pro-$p$ quotient of $G$. Then $d(G) = d(G_{[p]})$ for some $p$, and so $\text{def}_p(G_{[p]}) \geq \text{def}(G)$ because any presentation of $G$ serves as a presentation for $G_{[p]}$ as a pro-$p$ group. Since $\text{def}_p(G_{[p]}) = \dim H_1(G_{[p]},\mathbb{F}_p) - \dim H_2(G_{[p]},\mathbb{F}_p)$ we have $\text{def}_p(G_{[p]}) \leq 0$ for $d(G) > 2$. Therefore $\text{def}(G) \leq 0$ for $d(G) > 2$.

Suppose $d(G) = 2$. It suffices to prove that $G_{[p]}$ is non-trivial for every $p$. But this is clear since otherwise $0 \leq \text{def}(G) \leq \text{def}_p(G) \leq \text{def}_p(G_{[p]}) = 0$, a contradiction.

Suppose now that $G$ is not abelian. Let $G_{[p]}$ denote again the maximal pro-$p$ quotient of $G$. Then $\text{def}_p(G_{[p]}) \geq \text{def}(G) > 0$ and so, by [31, Window 5, Sec.1, Lemma 3], $G_{[p]}$ has infinite abelianization. Hence $G$ has $\mathbb{Z}_p$ as an epimorphic image for every $p$ and therefore has $\hat{\mathbb{Z}}$ as a quotient. Then Theorem 3.8 implies that $\text{def}(G) = 1$, $\text{cd}(G) = 2$, $[G,G]$ is projective.

Remark 3.11. The groups considered in Subsection 5.1 show that $[G,G]$ does not have to be free profinite.

3.4 Pro-$p$ groups of subexponential subgroup growth

Let $G$ be a profinite group. Denote by $a_n(G)$ the number of open subgroups of $G$ of index $n$. If $G$ is finitely generated then $a_n(G)$ is finite for all $n$. We say that a group $G$ is of subexponential subgroup growth if $\limsup_{n \to \infty} a_n(G)^{1/n} = 1$. The following characterization of pro-$p$ groups of subexponential subgroup growth is given by Lackenby.

Proposition 3.12. ([23, Theorem 1.7]) Let $G$ be a finitely generated pro-$p$ group. Then $G$ is of subexponential subgroup growth if and only if

$$\limsup_{[G:U] \to \infty} \frac{d(U)}{[G:U]} = 0.$$
For example, since $p$-adic analytic profinite groups have finite rank, they are of subexponential subgroup growth (in fact, they are of polynomial subgroup growth).

We believe that the following conjecture holds.

**Conjecture 1.** Let $G$ be a finitely generated pro-$p$ group of subexponential subgroup growth with $\chi_2(G, \mathbb{F}_p) = 0$. Then $G$ is $\mathbb{Z}_p$ or $\mathbb{Z}_p \rtimes \mathbb{Z}_p$.

We can prove the following result.

**Theorem 3.13.** Let $G$ be a finitely generated pro-$p$ group of subexponential subgroup growth. If $\chi_2(G, \mathbb{F}_p) = 0$, then $G$ is free pro-$p$ by cyclic and all finitely generated subgroups of infinite index are free pro-$p$ groups.

**Proof.** Since $G$ is a pro-$p$ group $\text{def}_p(G) = \chi_2(G, \mathbb{F}_p) + 1 = 1$. Hence there exists a map of $G$ onto $\mathbb{Z}_p$. Let $N$ be the kernel of this map. If $N$ is infinitely generated, then $G/\Phi(N)$ maps onto $C_p \times \mathbb{Z}_p$. But the last group has exponential subgroup growth. Hence $N$ should be finitely generated. Applying Corollary 3.5, we obtain that $N$ is a free pro-$p$ group. In particular $\text{cd}(G) = 2$ and $\chi_2(U, \mathbb{F}_p) = 0$ for all open subgroups.

**Claim** Let $V$ be an open subgroup of $G$ and $U$ an open normal subgroup of $V$. Assume that the image of $H_2(U, \mathbb{F}_p)$ in $H_2(V, \mathbb{F}_p)$ is not trivial. Then $\chi_1(U, \mathbb{F}_p) \geq \chi_1(V, \mathbb{F}_p) - 1 + [V : U]$.

Let $d = d(V)$. Since $V$ is pro-$p$ of cohomological dimension 2 and $p$-deficiency is 1, we have the following exact sequence of right modules:

$$0 \to \mathbb{F}_p[[V]]^{d-1} \to \mathbb{F}_p[[V]]^d \to \mathbb{F}_p[[V]] \to \mathbb{F}_p \to 0. \quad (3.2)$$

Applying the functor $- \otimes_{\mathbb{F}_p[[V]]} \mathbb{F}_p$ we obtain the complex

$$0 \to \mathbb{F}_p[V/U]^{d-1} \xrightarrow{\alpha} \mathbb{F}_p[V/U]^d \xrightarrow{\beta} \mathbb{F}_p[V/U] \to \mathbb{F}_p \to 0, \quad (3.3)$$

where $H_2(U, \mathbb{F}_p) \cong \ker \alpha$ and $H_1(U, \mathbb{F}_p) \cong \ker \beta/\text{Im} \alpha$. Note that we can calculate $H_1(V, \mathbb{F}_p)$ and $H_2(V, \mathbb{F}_p)$ either tensoring (3.3) with $- \otimes_{\mathbb{F}_p[[V/U]]} \mathbb{F}_p$ or tensoring (3.2) with $- \otimes_{\mathbb{F}_p[[V]]} \mathbb{F}_p$, once the obtained complexes are isomorphic.

Let $T$ be a transversal of $U$ in $V$. Denote by $a$ the element $\sum_{t \in T} t$ of $\mathbb{F}_p[V]$. Since $d = d(V)$, the rank of

$$(\mathbb{F}_p[V/U]^d / \text{Im} \alpha) \otimes_{\mathbb{F}_p[[V]]} \mathbb{F}_p \cong H_1(V, \mathbb{F}_p)$$
is $d$. Thus, $\text{Im} \alpha$ is contained in the Jacobson radical of $V$-module $F_p[V/U]^d$ and so $a \text{Im} \alpha = 0$. Thus $a(F_p[V/U]^{d-1}) \leq \ker \alpha$ which implies that $(\ker \alpha)^V$ has rank $d - 1$.

Since the image of $H^2(U, F_p)$ in $H^2(V, F_p)$ is not trivial, the rank of $(F_p[V/U]^{d-1}/\ker \alpha) \otimes_{F_p[[V]]} F_p$ is at most $d - 2$. Hence $\ker \alpha$ is not in the Jacobson radical of $V$-module $F_p[V/U]^{d-1}$. Since any element outside the Jacobson radical of $F_p[V/U]^{d-1}$ generates a $V$-submodule isomorphic to $F_p[V/U]$, we conclude that the $F_p[[V]]$-module $\ker \alpha$ contains a submodule $Z$ isomorphic to $F_p[V/U]$. Thus, since $\dim Z^V = 1$ we obtain that $\dim H^2(U, F_p) = \dim \ker \alpha \geq \dim Z + \dim(\ker \alpha)^V - \dim Z^V = |V/U| + d - 2$.

This proves Claim.

Now, let $K$ be a finitely generated subgroup of infinite index in $G$. By way of contradiction assume $H^2(K, F_p) \neq 0$. Then there exists an open subgroup $V$ of $G$ containing $K$ such that for any open subgroup $U$ of $V$ containing $K$ the image or restriction map $H^2(K, F_p) \to H^2(U, F_p)$ is not trivial. Let $U_0 = V$, and let $U_{i+1}$ $(i = 0, 1, \ldots)$ be a subgroup of index $p$ in $U_i$ containing $K$. Applying Claim, we obtain that $\chi_1(U_i, F_p) \geq \chi_1(U_0, F_p) + (p - 1)i$. Hence we can find an open subgroup $U$ of $V$ containing $K$ such that $\chi_1(U, F_p) - 1 = d(U)$ is arbitrary large and in particular, there exists $K < U \leq V$ such that

$$d(U) - 1 + d(K) < \frac{(d(U) - d(K))^2}{4}.$$ 

Let $N$ be a normal subgroup of $U$ generated by $K$. Then $U/N$ does not satisfy the Golod-Shafarevich inequality (see, for example, [7, Interlude D]):

$$\dim H^2(U/N, F_p) \leq \dim U - 1 + d(K) < \frac{(d(U) - d(K))^2}{4} \leq \frac{(d(U/N))^2}{4},$$

and so $N$ is of infinite index in $U$. Since $N$ is not free pro-$p$, Corollary 3.5 implies that $U$ is not of subexponential subgroup growth. We have a contradiction. \qed
4 Poincaré duality groups of dimension 3

As defined in [40], a profinite group $G$ is called a Poincaré duality group at $p$ of dimension $n$ if $G$ is of type $p$-$FP_{\infty}$ with $cd_p(G) = n$ and

\[
\begin{align*}
H^i(G, \mathbb{Z}_p[[G]]) &= 0, & \text{if } i \neq n, \\
H^n(G, \mathbb{Z}_p[[G]]) &\cong \mathbb{Z}_p & (\text{as abelian groups}).
\end{align*}
\]

We use the notation profinite $PD^n$-group at $p$ for a Poincaré duality profinite group $G$ at $p$ of dimension $n$. If $G$ is a profinite group with $cd_p(G) < \infty$ and $U$ is an open subgroup of $G$, then $G$ is a profinite $PD^n$-group at $p$ if and only if $U$ is a profinite $PD^n$-group at $p$.

By a result of Lazard, compact $p$-adic analytic groups $G$ are virtual Poincaré duality groups of dimension $n = \text{dim}(G)$ at a prime $p$ ([40, Thm. 5.9.1]). The Demushkin pro-$p$ groups are exactly the pro-$p$ $PD^2$-groups ([39, I.4.5 Example 2]) and $\mathbb{Z}_p$ is the only pro-$p$ $PD^1$-group ([40, Example 4.4.4]).

In this section we are interested in finitely generated $PD^3$-groups at $p$. The important consequences of the Poincaré duality are recollected in the following proposition.

**Proposition 4.1.** Let $G$ be a finitely generated $PD^3$-group at $p$.

1. For trivial $G$-module $\mathbb{F}_p$

\[
\bar{\chi}_2(G, \mathbb{F}_p) = -\dim H^3(G, \mathbb{F}_p) = -1.
\]

2. If $M$ is a non-trivial irreducible $\mathbb{F}_p[[G]]$-module, then

\[
\bar{\chi}_2(G, M) = \dim H^3(G, M) = 0.
\]

3. The $p$-deficiency of $G$ is equal to 0 and $d_G(\bar{R}_p) = d(G)$.

4. There exists the following exact sequence

\[
0 \to \mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G]]^{d(G)} \to R_p \to 0.
\]

**Proof.** The first and second statements follow from [40, Proposition 4.5.1]. The third statement is a consequence of Theorem 2.8.
Let us prove (4). By (3) there exists a submodule $A$ of $\mathbb{Z}_p[[G]]^{d(G)}$ such that $\bar{R}_p \cong \mathbb{Z}_p[[G]]^{d(G)}/A$. Hence we have the following exact sequence.

$$0 \rightarrow A \rightarrow \mathbb{Z}_p[[G]]^{d(G)} \rightarrow \mathbb{Z}_p[[G]]^{d(G)} \rightarrow \mathbb{Z}_p[[G]] \rightarrow \mathbb{Z}_p \rightarrow 0. \quad (4.1)$$

By [38, Proposition 7.1.4(f)], $A$ is projective. Let $M$ be an irreducible $\mathbb{Z}_p[[G]]$-module. Since $H^4(G, M) = 0$, $\dim H^3(G, M) = \dim H^0(G, M) = \dim \text{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, M)$ and $\dim H^2(G, M) = \dim H^1(G, M)$, applying the functor $\text{Hom}_{\mathbb{Z}_p[[G]]}(-, M)$ in (4.1) we obtain that

$$\dim \text{Hom}_{\mathbb{Z}_p[[G]]}(A, M) = \dim M - \dim H^2(G, M) + \dim H^1(G, M) + \dim H^0(G, M) - \dim \text{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, M) = \dim M.$$

Hence, by Proposition 2.13, $A \cong \mathbb{Z}_p[[G]]$.

**Theorem 4.2.** Let $G$ be a finitely generated profinite $PD^3$-group at $p$, $L \leq N$ two normal subgroup such that $|G/N|_p$ is infinite and $M$ is a finite $\mathbb{F}_p[[G]]$-module on which $L$ acts trivially. Suppose that

$$\inf \left\{ \frac{d_p(H, L)}{|G : H|_p} \bigg| N < H \leq_0 G \right\} = 0.$$

Then, $\dim H^2(N, M) \leq \sigma_M \dim M$, where $\sigma_M = 0$ if $M$ does not have a section isomorphic to a trivial $N$-module and $\sigma_M = 1$ otherwise. Moreover, if $\dim H^2(N, \mathbb{F}_p) = 1$, then the $p$-Sylow subgroup of $G/N$ is virtually cyclic.

**Proof.** First, let us show that $\dim H^2(N, M) \leq \sigma_M \dim M$. By way of contradiction let us assume that $\dim H^2(N, M) > \sigma_M \dim M$. Let $G = H_1 > H_2 > \ldots$ be a chain of open normal subgroups such that $\cap_i H_i = N$. Thus we have that

$$N = \varinjlim H_i.$$

Hence $H^2(N, M) = \varinjlim H^2(H_i, M)$. Since $\dim H^2(N, M) > \sigma_M \dim(M)$, we obtain that there exists $j$ such that if $\beta$ denotes the restriction map $H^2(H_j, M) \rightarrow H^2(N, M)$, then $\dim \text{Im} \beta > \sigma_M \dim(M)$.

Let $H$ be a open normal subgroup of $G$ contained in $H_j$ and $P$ a subgroup of $H_j$ containing $H$ such that $P/H$ is a $p$-Sylow subgroup of $H_j/H$. Since $P$ is
also a PD3-group at \( p \), by Proposition 4.1 and Lemma 2.7. Therefore by Proposition 2.11

\[
\chi_1(H, M) \geq (\chi_2(P, M) + \frac{\dim(\text{Im} \beta)}{\dim(M)}[P : H]) = \frac{[G : H]_p}{[G : H_j]_p \dim(M)}.
\]

Now, if \( H \) is an arbitrary open subgroup containing \( N \), applying Lemma 2.10 for \( H \) and \( H \cap H_j \) and the above inequality for \( H \cap H_j \), we get

\[
\frac{d_p(H, L) - 1}{[G : H]_p} \geq \frac{d_p((H \cap H_j), L) - 1}{[G : (H \cap H_j)]_p \dim(M)} = \frac{1}{[G : H_j]_p [H : (H \cap H_j)]_p'} \dim(M).
\]

But it is a contradiction with hypothesis. Hence, \( \dim H^2(N, M) \leq \sigma_M \dim M \).

Let us now analyze the case \( \dim H_2(N, \mathbb{F}_p) = 1 \). Without loss of generality we may assume that \( H_2(N, \mathbb{F}_p) \) is embedded injectively in \( H_2(G, \mathbb{F}_p) \) (if not we may replace \( G \) by some open subgroup containing \( N \)). Applying \( \mathbb{F}_p[[G/N]] \otimes_{\mathbb{Z}_p[[G]]} \to \) to the sequence

\[
\mathcal{P} : 0 \to \bar{R}_p \to \mathbb{Z}_p[[G]]^{d(G)} \to \mathbb{Z}_p[[G]] \to \mathbb{Z}_p \to 0,
\]

we obtain that \( H_2(N, \mathbb{F}_p) \leq \mathbb{F}_p[[G/N]] \otimes_{\mathbb{Z}_p[[G]]} \bar{R}_p \) and we denote \( H_2(N, \mathbb{F}_p) \) by \( B \) when viewing this way. Thus \( B \) is a trivial submodule. Put

\[
K = (\mathbb{F}_p[[G/N]] \otimes_{\mathbb{Z}_p[[G]]} \bar{R}_p)/B.
\]

If \( M \) is an irreducible \( \mathbb{Z}_p[[G/N]] \)-module, by counting dimensions of the complex \( \text{Hom}_{\mathbb{Z}_p[[G]]}(\mathcal{P}, M) \) we obtain

\[
\dim \text{Hom}_{\mathbb{Z}_p[[G]]}(\bar{R}_p, M) = (d(G) - 1 - \chi_2(G, M)) \dim M.
\]

Thus, if \( M \) is non-trivial, using Proposition 4.1(2) we obtain that

\[
\dim \text{Hom}_{\mathbb{Z}_p[[G/N]]}(K, M) = \dim \text{Hom}_{\mathbb{Z}_p[[G]]}(\bar{R}_p, M) = (d(G) - 1) \dim M.
\]

Since \( H_2(N, \mathbb{F}_p) \) is embedded in \( H_2(G, \mathbb{F}_p) \), we have that the image of \( H_2(N, \mathbb{F}_p) \) in \( \mathbb{F}_p \otimes_{\mathbb{Z}_p[[G]]} \bar{R}_p \) is not trivial. Thus

\[
\dim \text{Hom}_{\mathbb{Z}_p[[G/N]]}(K, \mathbb{F}_p) = \dim \text{Hom}_{\mathbb{Z}_p[[G]]}(\bar{R}_p, \mathbb{F}_p) - 1 = (d(G) - 1 - \chi_2(G, \mathbb{F}_p)) - 1 = d(G) - 1,
\]

since \( \chi_2(G, \mathbb{F}_p) = 0 \) by Lemma 2.7.
where the last equality follows from Proposition "Proposition 1.1". Then, by "Proposition 2.0" combined with the above calculations of \( \dim \text{Hom}_{\mathbb{Z}_p[[G]]}(K, M) \), we obtain \( d_{G/N}(K) \leq d(G) - 1 \).

Note that, for any \( N \leq U \triangleleft G \), by tensoring \( P \) over \( \mathbb{Z}_p[[G]] \) with \( \mathbb{F}_p[G/U] \), we obtain the complex

\[
\mathbb{F}_p[G/U] \otimes_{\mathbb{Z}_p[[G]]} \tilde{R}_p \to \mathbb{F}_p[G/U]^{d(G)} \to \mathbb{F}_p[G/U] \to \mathbb{F}_p \to 0,
\]

from where it follows that

\[
\dim \mathbb{F}_p[G/U] \otimes_{\mathbb{Z}_p[[G]]} \tilde{R}_p = (d(G) - 1)|G/U| - \chi_2(U, \mathbb{F}_p).
\]

Since \( U \) is \( PD^3 \), we get

\[
\dim \mathbb{F}_p[G/U] \otimes_{\mathbb{Z}_p[[G]]} \tilde{R}_p = (d(G) - 1)|G/U| + 1.
\]

Hence

\[
\dim \mathbb{F}_p[G/U] \otimes_{\mathbb{Z}_p[[G/N]]} K = (d(G) - 1)|G/U|,
\]

and so

\[
\mathbb{F}_p[G/U] \otimes_{\mathbb{Z}_p[[G/N]]} K \cong \mathbb{F}_p[[G/U]]^{d(G)-1}.
\]

Passing to the inverse limit, we obtain that \( K \cong \mathbb{F}_p[[G/N]]^{d(G)-1} \). In particular \( K \) is projective \( \mathbb{F}_p[[G/N]] \)-module, and so

\[
\mathbb{F}_p[[G/N]] \otimes_{\mathbb{Z}_p[[G]]} \tilde{R}_p \cong \mathbb{F}_p[[G/N]]^{d(G)-1} \oplus B \quad (4.2)
\]

as \( \mathbb{F}_p[[G/N]] \)-module.

Note that if \( U \) is an open subgroup, then the corestriction map \( H^3(U, \mathbb{F}_p) \to H^3(G, \mathbb{F}_p) \) is an isomorphism (see proof of Proposition 30, item (5), in [39, I.§4.5]). Hence \( H^3(N, \mathbb{F}_p) = 0 \) because \( |G/N|_p \) is infinite. Therefore, tensoring the exact sequence from Proposition "Proposition 4.1.4" over \( \mathbb{Z}_p[[G]] \) with \( \mathbb{F}_p[[G/N]] \), from (4.2) and the fact that \( \dim B = \dim H^2(N, \mathbb{F}_p) = 1 \), we obtain the sequence

\[
0 \to \mathbb{F}_p[[G/N]] \overset{\alpha}{\to} \mathbb{F}_p[[G/N]]^{d(G)} \overset{\beta}{\to} \mathbb{F}_p[[G/N]]^{d(G)-1} \oplus \mathbb{F}_p \to 0,
\]

where \( \alpha \) is injective, \( \beta \) is surjective and \( \ker \beta / \text{Im} \alpha \cong H_2(N, \mathbb{F}_p) \cong \mathbb{F}_p \).

We can find a \( \mathbb{F}_p[[G/N]] \)-submodule \( L \) of \( \mathbb{F}_p[[G/N]]^{d(G)} \) isomorphic to \( \mathbb{F}_p[[G/N]]^{d(G)-1} \) such that \( \beta \) sends \( L \) to the summand \( \mathbb{F}_p[[G/N]]^{d(G)-1} \). Put \( C = \mathbb{F}_p[[G/N]]^{d(G)}/(L + \text{Im} \alpha) \). Since \( \dim C = 2 \) we may assume that \( G \) acts
trivially on $C$ (if not we pass to an open subgroup). Hence $C \cong \mathbb{F}_p^2$. Since $C$ has projective dimension 2,

$$\text{Ext}^2_{\mathbb{F}_p[[G/N]]}(\mathbb{F}_p, A) \oplus \text{Ext}^2_{\mathbb{F}_p[[G/N]]}(\mathbb{F}_p, A) \cong \text{Ext}^2_{\mathbb{F}_p[[G/N]]}(C, A) = 0$$

for any discrete $\mathbb{F}_p[[G]]$-module $A$, and so by [38, Proposition 7.1.4(d)], $\text{cd}_p(G/N) = 1$.

If the $p$-Sylow subgroup of $G/N$ is not cyclic, then there exists an open subgroup $U$ of $G$ containing $N$ such that $\chi_1(U/N, \mathbb{F}_p) \geq 1$ and $U$ acts trivially on $H^2(N, \mathbb{F}_p)$. Thus, by Corollary 2.3(5),

$$1 = \text{dim} H^3(G, \mathbb{F}_p) = \text{dim} H^1(U/N, H^2(N, \mathbb{F}_p)) > 1,$$

a contradiction. Hence, the $p$-Sylow subgroup of $G/N$ is cyclic.

Now we need the following.

**Lemma 4.3.** Let $G$ be a profinite group of type $p$-$FP_\infty$ and $A$ a free profinite $\mathbb{Z}_p[[G]]$-module over a profinite space $X$. Then

$$H^q(G, A) \cong H^q(G, \mathbb{Z}_p[[G]]) \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[X]].$$

**Proof.** Decompose $X = \varprojlim X_j$ as inverse limit of finite discrete spaces $X_j$. Then

$$H^q(G, A) \cong \varprojlim_j H^q(G, \bigoplus X_j \mathbb{Z}_p[[G]])$$

$$\cong \varprojlim_j \bigoplus_{X_j} H^q(G, \mathbb{Z}_p[[G]]) \hat{\otimes}_{\mathbb{Z}_p} (\bigoplus X_j \mathbb{Z}_p)$$

$$\cong H^q(G, \mathbb{Z}_p[[G]]) \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[X]].$$

**Theorem 4.4.** Let $G$ be a profinite $PD^3$-group at a prime $p$ and $N$ be a finitely generated normal subgroup of $G$ such that $|G/N|_p$ is infinite and $p$ divides $|N|$. Then either $N$ is $PD^1$ at $p$ and $G/N$ is virtually $PD^2$ at $p$ or $N$ is $PD^2$ at $p$ and $G/N$ is virtually $PD^1$ at $p$.

**Proof.** During this proof when we write $PD^n$ we shall mean $PD^n$ at $p$.

If the $p$-Sylow subgroup of $N$ is not cyclic, then we can find an open subgroup $J$ of $G$ such that $\chi_1(J \cap N, \mathbb{F}_p) \geq 1$. Applying Proposition 2.12 and repeating the same argument as in (3.1), we obtain that there exists an
open subgroup $V$ of $J$ containing $J \cap N$ such that for any open subgroup $U$ of $J$ containing $J \cap N$, $\chi(U/N \cap J, \mathbb{F}_p) \leq 0$. This means that the $p$-Sylow subgroup of $G/N$ is virtually cyclic. For simplicity let us assume that the $p$-Sylow subgroup of $G/N$ is cyclic, and so $\text{cd}_p(G/N) = 1$.

Let $K$ be an irreducible finite $\mathbb{F}_p[[N]]$-module. There exists a finite irreducible $\mathbb{F}_p[[G]]$-module $M$ isomorphic to a direct sum of copies of $K$ as a $\mathbb{F}_p[[N]]$-module. Note that since $[G/N]_p$ is infinite we have $\text{cd}_p(N) \leq 2$ (see [39 I.§4.5 Exercise 5]). Thus, using Corollary 2.3(5), we obtain that

$$H^3(G, M) \cong H^1(G/N, H^2(N, M)).$$

Note that if $U$ is an open subgroup of $G$ containing $N$ and $M$ a finite $\mathbb{Z}_p[[U]]$-module, then by Corollary 2.3(1),

$$\dim H^1(U, M) \leq \dim H^1(U/N, M^N) + \dim H^1(N, M)^{G/N}.$$

Thus, $d_p(U) \leq 1 + d_p(N)$ and so we may apply Theorem 4.2 for $L = N$.

If $K$ is not trivial, then by Theorem 4.2, $\dim H^2(N, M) = 0$ and so $H^2(N, K) = 0$. If $K = \mathbb{F}_p$, then by Corollary 2.3(5),

$$H^1(G/N, H^2(N, \mathbb{F}_p)) \cong H^3(G, \mathbb{F}_p) \cong \mathbb{F}_p$$

and so $\dim H^2(N, \mathbb{F}_p) = 1$.

Thus $\dim H^2(N, K) \leq \dim K$, for every irreducible finite $\mathbb{F}_p[[N]]$-module $K$. Therefore since $N$ has cohomological $p$-dimension 2, it follows from Theorem 2.14 that $N$ is of type $pFP_{\infty}$.

As $\mathbb{Z}_p[[G]]$ is a free profinite $\mathbb{Z}_p[[N]]$-module over the profinite space $G/N$, by Lemma 4.3 one has

$$H^i(N, \mathbb{Z}_p[[G]]) \cong H^i(N, \mathbb{Z}_p[[N]]) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G/N]].$$

Let $E$ be a closed basis of $H^i(N, \mathbb{Z}_p[[N]])$ as an elementary abelian profinite group. Then

$$H^i(N, \mathbb{Z}_p[[G]]) \cong H^i(N, \mathbb{Z}_p[[N]]) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G/N]] \cong \mathbb{Z}_p[[E]] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G/N]]$$

is a free $\mathbb{Z}_p[[G/N]]$-module over $E$. Applying Lemma 4.3 once more we obtain

$$H^3(G/N, H^i(N, \mathbb{Z}_p[[G]])) \cong H^3(G/N, \mathbb{Z}_p[[G/N]]) \otimes_{\mathbb{Z}_p} H^i(N, \mathbb{Z}_p[[N]]).$$

(4.4)
From Corollary 2.3(5) we get $H^1(G/N, H^2(N, A)) \cong H^3(G, A)$ for each finite discrete $\mathbb{Z}_p[[G]]$-module $A$. Thus, since $G, N$ and $G/N$ are $p$-$FP_\infty$, it follows that

$$H^1(G/N, H^2(N, \mathbb{Z}_p[[G]])) \cong H^3(G, \mathbb{Z}_p[[G]]) \cong \mathbb{Z}_p,$$

On the other hand, from (4.4) one gets

$$\mathbb{Z}_p \cong H^1(G/N, H^2(N, \mathbb{Z}_p[[G]])) \cong H^1(G/N, \mathbb{Z}_p[[G/N]]) \otimes_{\mathbb{Z}_p} H^2(N, \mathbb{Z}_p[[N]]).$$

Thus $H^1(G/N, \mathbb{Z}_p[[G/N]]) \cong \mathbb{Z}_p$ and $H^2(N, \mathbb{Z}_p[[N]]) \cong \mathbb{Z}_p$, so $G/N$ is $PD^1$ and by [21] Theorem 3 $N$ is $PD^2$.

Assume now that the $p$-Sylow subgroup of $N$ is cyclic. By Corollary 2.5, $\nu_{cd}(G/N) = 2$. Let us assume for simplicity that $\nu_{cd}(G/N) = 2$. Since $N$ is finitely generated, $G/N$ is of type $p$-$FP_2$ by Corollary 2.13 and so it is also of type $p$-$FP_\infty$.

Corollary 2.3(4) implies the isomorphism

$$H^3(G, A) \cong H^2(G/N, H^1(N, A)),$$

for any finite $\mathbb{Z}_p[[G]]$-module $A$. Since $G, N$ and $G/N$ are $p$-$FP_\infty$, so that their cohomology commutes with inverse limits of finite modules, we obtain

$$H^3(G, \mathbb{Z}_p[[G]]) \cong H^2(G/N, H^1(N, \mathbb{Z}_p[[G]])).$$

Since $G$ is $PD^3$, $H^3(G, \mathbb{Z}_p[[G]]) \cong \mathbb{Z}_p$ and so taking into account (4.4) we get $H^1(N, \mathbb{Z}_p[[N]]) = \mathbb{Z}_p$ and $H^2(G/N, \mathbb{Z}_p[[G/N]]) \cong \mathbb{Z}_p$. Thus, $N$ is $PD^1$ and by [36] Theorem 3.7.4 $G/N$ is $PD^2$.

Corollary 4.5. Let $G$ be a finitely generated pro-$p$ $PD^3$-group at $p$ and $N$ a normal subgroup of infinite index. Suppose that

$$\inf \left\{ \frac{d(H) - 1}{[G : H]} \right\} \quad N < H \leq G = 0.$$

Then $N$ is either free pro-$p$ or a Demushkin group.

Proof. By Theorem 4.2 $\dim H^2(N, \mathbb{F}_p) \leq 1$. If $H^2(N, \mathbb{F}_p) = 0$ then $N$ is free pro-$p$. If $H^2(N, \mathbb{F}_p) = \mathbb{F}_p$ then Theorem 1.2 implies that $G/N$ is virtually cyclic. Hence the condition

$$\inf \left\{ \frac{d(H) - 1}{[G : H]} \right\} \quad N < H \leq G = 0$$

implies that $G/\Phi(N)$ is not of exponential subgroup growth and so $N/\Phi(N)$ is finite. Thus, $N$ is finitely generated. By the previous theorem, $N$ is a Demushkin group. 

$\square$
5 Applications for discrete groups

In this section we shall describe applications of our profinite results to discrete finitely generated groups.

5.1 Ascending HNN-extensions

In this section we study mapping tori of injective endomorphisms of free groups. For a free group $F_n := \langle x_1, \ldots, x_n \rangle$ ($n \in \mathbb{N}$) let $\phi : F_n \to F_n$ be an endomorphism. The HNN-extension

$$M_\phi := \langle x_1, \ldots, x_n, t \mid t^{-1}x_it = \phi(x_i) \text{ for } i = 1, \ldots, n \rangle$$

is traditionally called the mapping torus of $\phi$. Sometimes we shall also say that $M_\phi$ is the ascending HNN-extension of $\phi$ or of the free group $F_n$. Groups of this type often appear in group theory and topology and were extensively studied see [9], [2], [17] for example. In particular, many one-relator groups are ascending HNN-extensions of free groups and many of such groups are hyperbolic. Our methods are applicable to the study of mapping tori because

- the group $M_\phi$ has positive deficiency,
- the cohomological dimension of $M_\phi$ is less or equal to 2.

The second property follows by an application of the Mayer-Vietoris of cohomology while the first property is obvious.

Let us discuss a simple example. Let the endomorphism $\phi_1 : F_1 = \langle x \rangle \to F_1$ be given by $\phi_1(x) := x^2$. It is elementary to see that the corresponding mapping torus is metabelian and in fact isomorphic to a split extension

$$M_{\phi_1} = \langle x, t \mid x^t = x^2 \rangle \cong \mathbb{Z} \left[ \frac{1}{2} \right] \rtimes \mathbb{Z}.$$

Where the generator 1 of the infinite cyclic group acts by multiplication by 2 on the group $\mathbb{Z}[1/2]$ of rational numbers with a 2-power denominator. The commutator subgroup of $M_{\phi_1}$ is equal to $\mathbb{Z}[1/2]$ appropriately embedded in $M_{\phi_1}$. From here it is easy to detect the profinite completion of $M_{\phi_1}$.

**Proposition 5.1.** We have

$$\hat{M}_{\phi_1} = \hat{\mathbb{Z}}[1/2] \times \hat{\mathbb{Z}}.$$

The profinite completion $\hat{\mathbb{Z}}[1/2]$ is a projective but not a free profinite group.
In fact the isomorphism

\[ \widehat{\mathbb{Z}}[1/2] \cong \prod_{p \neq 2} \mathbb{Z}_p \]

shows that all \( p \)-Sylow subgroups of \( \widehat{\mathbb{Z}}[1/2] \) are free pro-\( p \) groups which implies that \( \widehat{\mathbb{Z}}[1/2] \) is projective. Our example also shows that the projectivity of \([G,G]\) in Corollary 3.10 can not be replaced by freeness.

Corollary 3.9 allows to establish the structure of the profinite completion of this important class of groups.

**Theorem 5.2.** Let \( M_\phi \) be an ascending HNN-extension of a finitely generated free group \( F = F_n \) of rank \( n \in \mathbb{N} \) with respect to an endomorphism \( \phi : F \rightarrow F \). Let \( P \) be the closure of \( F \) in the profinite completion \( \widehat{M}_\phi \). Then \( P \) is normal in \( \widehat{M}_\phi \) and the profinite completion of \( M_\phi \) is isomorphic to the split extension \( \widehat{M}_\phi = P \rtimes \widehat{\mathbb{Z}} \), where \( P \) is a projective profinite group. The group \( P \) is free profinite of rank \( n \) if and only if \( \phi \) is an automorphism.

**Proof.** Let \( P \) be the closure of the image of \( F \) in \( \widehat{M}_\phi \). To see that \( P \) is normal consider any finite quotient \( H \) of \( M_\phi \). Then observe that the images of \( F \) and \( F^t \) in \( H \) coincide because in a finite group conjugate subgroups have to be of the same order. Thus \( \widehat{M}_\phi = P \rtimes \widehat{\mathbb{Z}} \).

The group \( M_\phi \) has positive deficiency and therefore so has its profinite completion \( \widehat{M}_\phi \). Corollary 3.9 implies that \( P \) is projective.

If \( \phi \) is an automorphism clearly \( P = \widehat{F} \). Suppose now that \( \phi \) is not an isomorphism. Let \( F_0 \) be the image of \( \phi \) it is a finitely generated subgroup of \( F \) distinct from \( F \). By a result of M. Hall \( F_0 \) is not dense in the profinite completion of \( F \), see [33], Proposition 3.10. To see that \( P \) is not free profinite of rank \( n \), it suffices to show that the profinite topology of \( M_\phi \) does not induce the full profinite topology on \( F \). But this follows from the theorem of Hall since as was just observed \( F \) coincides with \( F_0 \) in every finite image of \( M_\phi \).

**Remark 5.3.** If \( f \) is not an automorphism the theorem in principal allows that \( P \) is a free profinite group of rank less than \( n \). It is, however, easy to give criteria when this does not happen. Indeed, if \( F \) admits a finite quotient \( F/N \) modulo a characteristic subgroup \( N \) such that \( d(G/N) = n \) and \( FN = f(F)N \) then \( d(P) = n \). So for instance, if \( f(x_i) = x_i^p \) for \( i = 1, \ldots, n \) for some prime \( p \) then \( G/N \) can be taken to be elementary \( q \)-group of rank \( n \) where \( q \) is coprime to \( p \).
Problem 2. Describe the projective groups $P$ obtained in this manner.

The following interesting example was communicated to us by I. Kapovich.

**Example 5.4.** Let $F := F_3 = \langle a, b, c \rangle$ be a free group of rank 3. Let the endomorphism $\phi : F \to F$ be given by

$$\phi(a) = a, \quad \phi(b) = a^{-1}ca, \quad \phi(c) = a^{-1}bab^{-1}.$$ 

Let $G := M_\phi = \langle a, b, c, t | tat^{-1} = a, tbt^{-1} = a^{-1}ca, tct^{-1} = a^{-1}bab^{-1} \rangle$ be the corresponding ascending HNN-extension. Then $\phi(F) = \langle a, c, bab^{-1} \rangle$ is a proper subgroup of $F$, so that $G$ is a strictly ascending HNN-extension.

However, we can also rewrite the defining relations for $G$ as follows:

$$a^{-1}ta = t, \quad a^{-1}ca = tbt^{-1}, \quad a^{-1}ba = tct^{-1}b.$$ 

Thus $G$ is an HNN-extension of $H := \langle t, b, c \rangle$ with respect to $\psi : H \to H$ where

$$\psi(t) = t, \quad \psi(c) = tbt^{-1}, \quad \psi(b) = tct^{-1}b$$

and with stable letter $a$. We have $\psi(H) = \langle t, tbt^{-1}, tct^{-1}b \rangle = \langle t, b, c \rangle = H$. Thus $\psi$ is an automorphism of $H$ and hence $H$ is normal in $G$.

This example combined with Remark 5.3 shows that the profinite completion $\hat{G}$ can be written as semidirect product $P \rtimes \hat{\mathbb{Z}}$ of a projective (non-free) finitely generated profinite group and as a semidirect product $F \rtimes \hat{\mathbb{Z}}$ of a free profinite group of rank 3 and $\hat{\mathbb{Z}}$.

We note also that ascending HNN-extensions can be linear (see [5]).

The next result that comes as an application of Corollary 3.9 is that $G$ is good. Let $\Gamma$ be a group, $\hat{\Gamma}$ its profinite completion. The group $\Gamma$ is called good if the homomorphism of cohomology groups

$$H^n(\hat{\Gamma}, M) \to H^n(\Gamma, M)$$

induced by the natural homomorphism $\Gamma \to \hat{\Gamma}$ is an isomorphism for every finite $\Gamma$-module $M$.

**Theorem 5.5.** An ascending HNN-extension $G$ of a finitely generated free group is good.
Proof. By Theorem 5.2, \( cd(\hat{G}) = 2 \). The main result of [25] states that
\[
H^n(\hat{G}, M) \to H^n(G, M)
\]
is an isomorphism for \( n = 1, 2 \). The result follows.

We finish the subsection with a similar construction in the profinite setting that not necessarily arises from the profinite completion of the respective abstract construction, but gives a similar result. The proof is analogous to the above.

**Proposition 5.6.** Let \( F(x_1, \ldots, x_n) \) be a free profinite group of finite rank \( n \) and \( f : F \to F \) an endomorphism. Form a profinite HNN-extension \( G = \langle F, t \mid x_i^t = f(x_i) \rangle \). Then \( G = P \rtimes \hat{\mathbb{Z}} \), where \( P \) is projective. Here \( P \) is the image of the natural homomorphism \( F \to G \).

Since many one relator groups are ascending HNN-extensions, Theorem 5.2 is reminiscent of the problem of Serre (corrected by D. Gildenhuys, [10]) asking whether torsion free one relator pro-\( p \) groups have cohomological dimension 2. For an ascending pro-\( p \) HNN-extensions this follows from Theorem 3.8 and also can be proved directly using Frattini quotients. We state it as

**Proposition 5.7.** Let \( F(x_1, \ldots, x_n) \) be a free pro-\( p \) group of finite rank \( n \) and \( f : F \to F \) an endomorphism. Form a pro-\( p \) HNN-extension \( G = \langle F, t \mid x_i^t = f(x_i) \rangle \). Then \( G = P \rtimes \mathbb{Z}_p \), where \( P \) is free pro-\( p \). Here \( P \) is the image of the natural homomorphism \( F \to G \).

A group \( \Gamma \) is called pro-\( p \) good if the homomorphism of cohomology groups
\[
H^n(\hat{\Gamma}, \mathbb{F}_p) \to H^n(\Gamma, \mathbb{F}_p)
\]
induced by the natural homomorphism \( \Gamma \) to its pro-\( p \) completion \( \hat{\Gamma} \) is an isomorphism for all \( n \). The notion attracted attention recently in geometric group theory (see [24]). As a corollary we deduce that an ascending HNN-extension of a free group is pro-\( p \) good.

**Corollary 5.8.** An ascending HNN-extension \( G \) of a finitely generated free group \( F \) is pro-\( p \) good.
5 APPLICATIONS FOR DISCRETE GROUPS

Proof. Since the pro-$p$ completion of $G$ is an ascending (possibly not strictly) HNN-extension of a free pro-$p$ group (the pro-$p$ completion of $F$) by Proposition 5.7 it has cohomological dimension 2. Then the result follows from the fact, that first cohomology (in $\mathbf{F}_p$) of a finitely generated group coincides with the first cohomology of the pro-$p$ completion. In particular, the Mayer-Vietoris sequence for $G$ and $\hat{G}_p$ coincide until $H^2$. The result follows. \hfill $\Box$

5.2 More examples

This section contains some more constructions for groups for which our techniques are applicable.

Let $F = F(x_1, \ldots, x_n, y_1 \ldots y_m)$ be a free group of finite rank $n + m$. Let

$$f_1 : F(x_1 \ldots x_n) \longrightarrow F(x_1 \ldots x_n), \quad f_2 : F(y_1 \ldots y_m) \longrightarrow F(y_1 \ldots y_m)$$

be injective endomorphisms.

Theorem 5.9. Let $G = \langle F, t \mid x_i^t = f_1(x_i), y_j^{t^{-1}} = f_2(y_j) \rangle$. Then the profinite completion of $G$ is $\hat{G} = P \rtimes \hat{\mathbb{Z}}$, where $P$ is projective. $P$ is free profinite of rank $n + m$ if and only if $f_1, f_2$ are automorphism.

Proof. The group $G$ has positive deficiency and therefore so is $\hat{G}$. Let $P$ be the closure of the image of $F$ in $\hat{G}$. To see that $P$ is normal fix any finite quotient $\overline{G}$ of $G$ and use bar to denote the images in $\overline{G}$. Then one observes that $\overline{F}(x_1 \ldots x_n) = \overline{F}(x_1 \ldots x_n)^t$ and $\overline{F}(y_1 \ldots y_m) = \overline{F}(y_1 \ldots y_m)^t$ because finite conjugate groups have to be of the same order. Thus $\hat{G} = P \rtimes \hat{\mathbb{Z}}$ and so by Corollary 3.9 $P$ is projective. \hfill $\Box$

Theorem 5.10. Let $G = \langle F, t \mid x_i^t = f_1(x_i), y_j^{t^{-1}} = f_2(y_j) \rangle$. Then $G$ is residually finite and good.

Proof. Put $F_1 = F(x_1 \ldots, x_n)$ and $F_2 = F(y_1 \ldots y_m)$. The group $G$ can be represented as an amalgamated free product of ascending HNN-extensions $G_1 *_Z G_2$, where $G_1 = \langle F_1, t \mid x_i^t = f_1(x_i) \rangle$, $G_2 = \langle F_2, t \mid y_j^{t^{-1}} = f_2(y_j) \rangle$ and $Z = \langle t \rangle$. Then by Exercise 9.2.7 in [38] $G$ is residually finite and by Proposition 3.5 in [11] combined with Theorem 5.5 is good. \hfill $\Box$
5.3 Implications for the congruence kernel

Here we give some results relating to the structure of the congruence kernels of certain arithmetic groups.

Recall that a lattice in $\text{SL}_2(\mathbb{C})$ is a discrete group (Kleinian group) of finite covolume. One particular family of lattices is a family of arithmetic groups. We recall the definition of an arithmetic group in this case; see [8], [34] for more details. Let $k$ be a number field with exactly one pair of complex places and let $A$ be a quaternion algebra over $k$ which is ramified at all real places. Let $\rho$ be a $k$-embedding of $A$ into the algebra $M_2(\mathbb{C})$ of two by two matrices over $\mathbb{C}$ (using one of the complex places). Let $\mathcal{O}$ be the ring of integers of $k$ and let $\mathcal{R}$ be a $\mathcal{O}$-order of $A$. Let $A^1(\mathcal{R})$ be the corresponding group of elements of norm one. It is well known that $\rho(A^1(\mathcal{R}))$ is a lattice in $\text{SL}_2(\mathbb{C})$. Then a subgroup $\Gamma$ of $\text{SL}_2(\mathbb{C})$ is an arithmetic Kleinian group if it is commensurable with some such a $\rho(A^1(\mathcal{R}))$ (groups are commensurable if they have $\text{SL}_2(\mathbb{C})$-conjugate subgroups of finite index). The quotient $\text{SL}_2(\mathbb{C})/\rho(A^1(\mathcal{R}))$ is not compact if $k$ is an imaginary quadratic number field and if $A = M_2$.

To define the congruence kernels let us (without loss of generality) in the following consider the arithmetic Kleinian group $\Gamma = \rho(A^1(\mathcal{R}))$. The congruence kernel $\mathcal{C}(A, \mathcal{R})$ is the kernel of the canonical map from the profinite completion $\widehat{\Gamma}$ of $\Gamma$ to $\rho(A^1(\widehat{\mathcal{R}}))$. Here $\widehat{\mathcal{R}}$ stands for the profinite completion of the ring $\mathcal{R}$. The congruence subgroup problem (in general, for arithmetic groups) asks whether the congruence kernel is trivial. If the congruence kernel is finite, i.e. the congruence subgroup problem has almost positive solution, one says that $\Gamma$ has a congruence subgroup property. It is proved by Lubotzky [28] that the congruence kernels $\mathcal{C}(A, \mathcal{R})$ of the arithmetic lattices in $\text{SL}_2(\mathbb{C})$ are infinite. But for some of the arithmetic Kleinian groups, like for example for the $\text{SL}_2(\mathcal{O})$ ($\mathcal{O}$ the ring of integers in some imaginary quadratic number field), some further information has been obtained. For these arithmetic Kleinian groups there are subgroups of finite index which map onto non-abelian free groups. For the $\text{SL}_2(\mathcal{O})$ this was proved in [12], many more cases are treated in [29]. The fact that $\Gamma$ has a subgroup of finite index which maps onto non-abelian free group, leads to an embedding of the free profinite group on countably many generators $\widehat{\mathcal{F}_\omega}$ into the corresponding congruence kernel, see [27].

Led by a result of Melnikov [35] in the case $\Gamma = \text{SL}_2(\mathbb{Z})$ we ask:

**Question 3.** Is the congruence kernel of an arithmetic Kleinian group isomorphic to $\widehat{\mathcal{F}_\omega}$? Or more vaguely, what can be said about the congruence
kernel in this case?

Of course the answer to question 3 is negative if the cohomological dimension of the congruence kernel is not one. We shall describe in the following an interesting connection between question 3 and certain cohomological problems.

In [26] it is proved that if $\Gamma$ is a lattice in $\text{SL}_2(\mathbb{C})$, then for any chain of normal subgroups $\Gamma_i$ of finite index of $\Gamma$ with trivial intersection the numbers

$$\frac{\dim H^1(\Gamma_i, \mathbb{C})}{[\Gamma : \Gamma_i]}$$

tend to zero when $i$ tends to infinity. Let us formulate the following analogous problem for the dimensions of the first cohomology groups over $\mathbb{F}_p$.

**Question 4.** Let $\Gamma$ be an arithmetic Kleinian group and $p$ a prime number. Do the numbers

$$\frac{\dim H^1(\Gamma_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]}$$

tend to zero when $i$ tends to infinity for any chain of normal subgroups $\Gamma_i$ of $\Gamma$ with trivial intersection of index a power of $p$.

There is an interesting case studied by Calegari, Dunfield [6] and Boston, Ellenberg [3] where the answer to Question 4 is positive. The paper [6] contains the description of a subgroup $\Gamma_{CD} \leq \text{SL}_2(\mathbb{C})$ with the following properties (proofs in [6], [3]).

- $\Gamma_{CD}$ is a cocompact arithmetic Kleinian lattice in $\text{SL}_2(\mathbb{C})$,
- $\Gamma_{CD}$ is a pro-3 group,
- the pro-3 completion of $\Gamma_{CD}$ is analytic,
- $\dim H^1(\Delta, \mathbb{F}_3) = 3$ for any normal subgroup of 3-power index in $\Gamma_{CD}$.

We shall now describe a connection between the problems posed in Questions 3 and 4. Our methods are applicable since the discrete subgroups of $\text{SL}_2(\mathbb{C})$ have a very restrictive structure. If $\Gamma$ is a finitely generated torsion free discrete subgroup of $\text{SL}_2(\mathbb{C})$, then the deficiency of $\Gamma$ is 0 or 1 depending whether the quotient space $\text{SL}_2(\mathbb{C})/\Gamma$ is compact or not. Moreover, if $\text{SL}_2(\mathbb{C})/\Gamma$ is compact, then $\Gamma$ is Poincaré duality groups of dimension 3. We prove:
Theorem 5.11. Let $\Gamma$ be an arithmetic Kleinian group in $\text{SL}_2(\mathbb{C})$ and $p$ be a prime number. If the answer to the Question 4 is positive for any subgroup of $\Gamma$ of finite index, then

1. $\Gamma$ is $p$-good and

2. the $p$-cohomological dimension of the congruence kernel of $\Gamma$ is 1.

Proof. Passing, if necessary to a subgroup of finite index of $\Gamma$, we may assume that $\Gamma$ is residually $p$. If $\Gamma$ is a non-cocompact lattice, then $\Gamma$ is commensurable to a Bianchi group, see [8]. In this case we already know that $\Gamma$ is $p$-good (see [11]). So, let us assume that $\Gamma$ is cocompact. Note that by Proposition 3.1, the deficiency of all open subgroups of $\hat{\Gamma}$ is at most 1, and so by [22, Corollary 4.3], $\hat{\Gamma}$ is a Poincaré duality group of dimension 3. Thus, by [22, Theorem A], $\Gamma$ is pro-$p$ good. Applying, [41, Proposition 3.1], we obtain that $\Gamma$ is $p$-good. Thus, $G = \hat{\Gamma}$ is a Poincaré duality profinite group of dimension 3 or has $p$-deficiency 1.

Let $C$ denote the congruence kernel corresponding to $\Gamma$. Suppose the cohomological dimension of $C$ is not 1. Then by Proposition 2.1 there exists an open subgroup $C_0$ of $C$ with $H^2(C_0, \mathbb{F}_p) \neq 0$ for some $p$. Further, there exists a subgroup $\Gamma_0$ of finite index in $\Gamma$ which congruence kernel is equal to $C_0$. Hence without loss of generality we may assume that $C = C_0$ and $\Gamma = \Gamma_0$.

Note that

$$H^2(C, \mathbb{F}_p) = \lim_{C \leq G} H^2(H, \mathbb{F}_p)$$

where $G$ is the profinite completion of $\Gamma$. Hence there exist a subgroup $C \leq H \triangleleft G$ such that the image of the restriction map $H^2(H, \mathbb{F}_p) \to H^2(C, \mathbb{F}_p)$ is not trivial. Note that $H \cap \Gamma$ is a congruence subgroup of $\Gamma$ and $H \cong H \cap \Gamma$. Hence without loss of generality we may assume that $H = G$.

Since $\Gamma$ is an arithmetic Kleinian subgroup of $\text{SL}_2(\mathbb{C})$ there exists a normal subgroup $N$ of $G$ such that $C \leq N$ and $G/N$ is $p$-adic analytic group. Moreover $\Gamma$ is embedded in $G/N$. Replacing $\Gamma$ by one of its congruence subgroups we may also assume that $G/N$ is a pro-$p$ group.

Note that the image of compositions of two restriction maps

$$H^2(G, \mathbb{F}_p) \to H^2(N, \mathbb{F}_p) \to H^2(C, \mathbb{F}_p)$$

is not zero. Hence $H^2(N, \mathbb{F}_p) \neq 0$. Thus, we may apply Theorem 3.4 and Theorem 4.2 and obtain that there exists $c > 0$ such that for any open
subgroup \( N \leq H \leq G \)

\[
\dim H^1(\Gamma \cap H, \mathbb{F}_p) = d_p(H, N) \geq c[G : H] = c[\Gamma : (\Gamma \cap H)].
\]

But this contradicts the assumption that the answer to Question \ref{question:4} is positive for \( \Gamma \). \qed

References

[1] R. Bieri, *Deficiency and the geometric invariants of a group. With an appendix by Pascal Schweitzer*, J. Pure Appl. Algebra 208 (2007) 951–959.

[2] A. Borisov and M. Sapir, *Polynomial maps over finite fields and residual finiteness of mapping tori of group endomorphisms*. Invent. Math. 160, 341-356.

[3] N. Boston, J. Ellenberg, *Pro-p groups and towers of rational homology spheres*. Geom. Topol. 10 (2006), 331–334.

[4] A. Brumer, *Pseudocompact algebras, profinite groups and class formations*, J. Algebra 4 (1966), 442–470.

[5] D. Calegari, N.M. Dunfield, *An ascending HNN extension of a free group inside \( \text{SL}_2 \mathbb{C} \)*. Proc. Amer. Math. Soc. 134, (2006), no. 11, 3131–3136.

[6] D. Calegari, N.M. Dunfield, *Automorphic forms and rational homology 3-spheres*. Geom. Topol. 10 (2006), 295–329.

[7] J. Dixon, M. du Sautoy, A. Mann, D. Segal, *Analytic pro-p groups*. Cambridge University Press, Cambridge 1999.

[8] J. Elstrodt, F. Grunewald, J. Mennicke, *Groups acting on hyperbolic space. Harmonic analysis and number theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, (1998), 524 pp.

[9] M. Feighn, M. Handel, *Mapping tori of free group automorphisms are coherent*. Ann. Math 149, (1999), 1061–1077.
REFERENCES

[10] D. Gildenhuys, On pro-p-groups with a single defining relator, Invent. Math. 5 1968 357–366.

[11] F. Grunewald, A. Jaikin-Zapirain, P.A. Zalesskii Cohomological goodness and the profinite completion of Bianchi groups. Duke Math. J. 144 (2008), 53–72.

[12] F. Grunewald, J. Schwermer, Free nonabelian quotients of $SL_2$ over orders of imaginary quadratic numberfields. J. Algebra 69 (1981), no. 2, 298–304.

[13] J. Hempel, W. Jaco, Fundamental groups of 3-manifolds which are extensions. Ann. of Math. (2) 95 (1972) 86–98.

[14] J. Hillman, Three-dimensional Poincaré duality groups wich are extensions, Math. Z. 195 (1987), 89–92.

[15] J. Hillman, Four-Manifolds, geometries and knots Geometry and topology publications, University of Warwick 2002. Revision 2007.

[16] J. Hillman, A. Schmidt, Pro-p groups of positive deficiency, Bulletin of LMS (to appear).

[17] I. Kapovich, Mapping tori of endomorphisms of free groups. Communications in Algebra, 28 (6), (2000) 2895–2917.

[18] E.I. Khukhro, Nilpotent groups and their automorphisms. de Gruyter Expositions in Mathematics, 8. Walter de Gruyter and Co., Berlin, 1993, 252 pp.

[19] D. Kochloukova, On a conjecture of E. Rapaport Strasser about knot-like groups and its pro-p version, J. Pure Appl. Algebra 204 (2006), 536–554.

[20] D. Kochloukova, Some Novikov rings that are von Neumann finite and knot-like groups, Comm. Math Helvetici 81 (2006), 931–943.

[21] D. Kochloukova, A. Pinto, Finiteness conditions on subgroups of profinite p-Poincaré duality groups, to appear in Israel J. Math.
[22] D. Kochloukova, P. Zalesskii, Profinite completions of Poincaré duality
groups of dimension 3. Transactions of the American Mathematical
Society 360 (2008) 1927-1949.

[23] M. Lackenby, Large groups, Property (tau) and the homology growth of
subgroups, preprint.

[24] P. Linnell, Schick, T. Finite group extensions and the Atiyah conjecture,
J. Amer. Math. Soc. 20 (2007), no. 4, 1003–1051.

[25] K. Lorensen, Groups with the same cohomology as their profinite com-
pletions, to appear in J. Algebra.

[26] J. Lott, W. Lück, $L^2$-topological invariants of 3-manifolds. Invent.
Math. 120, (1995), 15-60.

[27] A. Lubotzky, Free quotients and the congruence kernel of $SL_2$. J. Al-
gebra 77 (1982), no. 2, 411–418.

[28] A. Lubotzky, Group presentation, $p$-adic analytic groups and lattices
in $SL_2(C)$. Ann. of Math. (2) 118 (1983), no. 1, 115–130.

[29] A. Lubotzky, Free quotients and the first Betti number of some hyper-
bolic manifolds. Transform. Groups 1 (1996), no. 1-2, 71–82.

[30] A. Lubotzky, Pro-finite presentations. J. Algebra 242, (2001), 672–690.

[31] A. Lubotzky, D. Segal, Subgroup growth, Progress in Mathematics, 212.
Birkhäuser Verlag, Basel, 2003.

[32] W. Lück, $L^2$-invariants: theory and applications to geometry and $K$-
theory. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.
44. Springer-Verlag, Berlin, 2002. xvi+595 pp.

[33] R.C. Lyndon, P.E. Schupp, Combinatorial group theory. Ergebnisse der
Mathematik und ihrer Grenzgebiete, Band 89, Springer-Verlag, Berlin-
New York, (1977).

[34] C. Maclachlan, A.W. Reid, The arithmetic of hyperbolic 3-manifolds.
Graduate Texts in Mathematics, 219. Springer-Verlag, New York,
(2003). xiv+463 pp.
REFERENCES

[35] O.V. Melnikov, *Congruence kernel of the group SL_2(Z)*. (Russian) Dokl. Akad. Nauk SSSR 228 (1976), no. 5, 1034–1036.

[36] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, Grundlehren der Mathematischen Wissenschaften, 323. Springer-Verlag, Berlin, 2000.

[37] E. Rapaport-Strasser, *Knot-like groups*. in Knots, groups and 3-manifolds, in: L.P. Neuwirt (Ed.), Annals of Mathematical Study, vol. 84, Princeton University Press, Princeton, NJ, 1975, pp. 119–133.

[38] L. Ribes, P. Zalesskii, *Profinite groups*. Springer-Verlag, Berlin, (2000).

[39] J-P. Serre, *Galois cohomology*. Berlin, Springer-Verlag 1997.

[40] P. Symonds and T. Weigel, *Cohomology of p-adic analytic groups*. New horizons in pro-p groups, Progr. Math. 184, 349–410. Birkhäuser Boston, Boston, MA, 2000.

[41] T. Weigel, *On profinite groups with finite abelinization*. Selecta Mathematica 13, (2007), 175–181.

[42] T. Weigel, P. Zalesski, *Profinite groups of finite cohomological dimension*. C. R. Acad. Sci. Paris, Ser. I 338, (2004), 353–358.

Author’s Adresses:

Fritz Grunewald
Mathematisches Institut,
Heinrich-Heine-Universität,
D-40225 Düsseldorf
Germany

Andrei Jaikin-Zapirain
Departamento de Matematicas, Facultad de Ciencias,
Universidad Autónoma de Madrid,
Cantoblanco Ciudad Universitaria,
28049 Madrid
Spain
and: Instituto de Ciencias Matematicas-CSIC, UAM, UCM,UC3M
REFERENCES

Aline G.S. Pinto
Departamento de Matemática,
Universidade de Brasília,
70910-900 Brasília DF,
Brazil

Pavel A. Zalesski
Departamento de Matemática,
Universidade de Brasília,
70910-900 Brasília DF
Brazil